Fixed Point Logics on Hemimetric Spaces

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***Abstract*—The μ-calculus can be interpreted over metric spaces and is known to enjoy, among other celebrated properties, variants of the McKinsey-Tarski completeness theorem and of Dawar and Otto's modal characterization theorem. In its topological form, this theorem states that every topological fixed point may be defined in terms of the *tangled derivative,* a polyadic generalization of Cantor's perfect core. However, these results fail when spaces not satisfying basic separation axioms are considered, in which case the base modal logic is not the well-known $\mathsf{K4}$, but the weaker $\mathsf{wK4}$.**

In this paper we show how these shortcomings may be overcome. First, we consider semantics over the wider class of *hemimetric spaces*, and obtain metric completeness results for $\mathsf{wK4}$ and related logics. In this setting, the Dawar-Otto theorem still fails, but we argue that this is due to the tangled derivative not being suitably defined for general application in arbitrary topological spaces. We thus introduce the *hybrid tangle,* which coincides with the tangled derivative over metric spaces but is better behaved in general. We show that only the hybrid tangle suffices to define *simulability* of finite structures, a key 'test case' for an expressively complete fragment of the μ-calculus.

I. Introduction

The modal μ-calculus and topological semantics for non-classical logics are two paradigms that have cemented their place in computer science; the first is a decidable framework embedding many logics used in program verification, such as PDL, CTL and CTL^*, and the second a central component in the logical modelling of constructive reasoning. In recent years these have combined to produce the topological μ-calculus. In many ways it combines the best of both worlds, since it inherits many classical results both from the μ-calculus and from the topological semantics literature. Some of the most notable properties of the topological μ-calculus are the following.

A. Decidability and Topological Completeness

The μ-calculus enjoys natural axiomatizations over the class of all topological spaces, as well as some important subclasses, such as the class of T_0 spaces and T_D spaces, both with respect to the closure semantics and the more expressive derivational semantics, as shown by Baltag et al. [1]. These logics are extensions of $\mathsf{wK4}$, a weakening of the well-known modal logic $\mathsf{K4}$. The logic $\mathsf{K4}$ is characterized by the 'transitivity' axiom $\Diamond\Diamond p \to \Diamond p$. However, when $\Diamond$ is interpreted as the Cantor derivative d (where $\mathrm{d}A$ is the set of limit points of A), this axiom is only valid over spaces that satisfy the T_D condition, which states that every singleton is the intersection of an open and a closed set (see Proposition II.8). Spaces that fail to be T_D may still have T_0 separation, i.e., two points never have exactly the same neighbourhoods. In arbitrary topological spaces, only the weaker property $\Diamond\Diamond p \to p \vee \Diamond p$ holds in general. This 'weak transitivity' axiom is the characteristic axiom of the logic $\mathsf{wK4}$, which together with the standard inductive axioms for fixed points axiomatize the topological μ-calculus [1] (see Definitions II.1 and II.3). These completeness results also apply to many extensions of $\mathsf{wK4}$, including the logic $\mathsf{wK4T_0}$ of T_0 spaces or $\mathsf{K4}$ itself.

B. Metric Completeness Theorems

Recall that a metric space is a pair $(X, \triangle)$, where X is a set of points and $\triangle \colon X \times X \to \mathbb{R}_{\geq 0}$ is such that $\triangle(x, y) = 0$ iff $x = y$ (separation), $\triangle(x, y) = \triangle(y, x)$ (symmetry) and $\triangle(x, z) \leq \triangle(x, y) + \triangle(y, z)$ (the triangle inequality). The space X is *crowded* if for every $x \in X$ and $\varepsilon > 0$ there is $y \neq x$ such that $\triangle(x, y) < \varepsilon$. The McKinsey-Tarski theorem states that, when $\Diamond$ is interpreted as topological closure, the modal logic of any crowded metric space is $\mathsf{S4}$. This result was proven in [19] for the separable case, and extended by Rasiowa and Sikorski [22] to arbitrary crowded spaces. But when $\Diamond$ is interpreted as derivative, $\mathsf{K4}$ is not complete for an arbitrary metric space, as for example the logics of the rationals [5], of the real line [18], and of the plane [25] are all distinct.

Instead, Goldblatt and Hodkinson [13] exhibited a version of the μ-calculus whose validities are exactly those formulas that are valid on *some* crowded metric space. In a modern presentation, such results are proven by showing that any suitable Kripke frame is a *d-morphic* image of some crowded metric space; here, d-morphisms are strictly open and strictly continuous maps (see Section IV). While Kripke models are not topological spaces *per se,* one can make sense of notions such as *continuity* when dealing with Kripke models, by viewing them as instances of the wider class of *derivative spaces* (Definition II.2).

C. Expressive Completeness of Tangled Fragments

Bisimulations between models of modal logic are binary relations that preserve truth of modal formulas, and more generally μ-calculus formulas, and as such provide the adequate notion of 'equivalence' between modal structures (see Section IV). This is supported by Van Benthem's modal characterization theorem [2], which states that modal logic is precisely the bisimulation-invariant fragment of first order logic (FOL) – a result that still holds when restricted to finite Kripke models [23]. Likewise, Janin and Walukiewicz [16] showed that the μ-calculus is the bisimulation-invariant fragment of monadic second order logic (MSO).

The situation for finite, transitive frames is somewhat different: in this case, Dawar and Otto [7] proved a modal characterization theorem, which shows that the bisimulation-invariant fragments of FOL and MSO are equal and a proper extension of basic modal logic. A corollary of this result is that modal logic enriched with the *tangled derivative,* a polyadic generalization of Cantor's perfect core, is expressively equivalent to the full μ-calculus over the class of metric spaces. In this case, if $(X, \triangle)$ is any metric space and $A \subseteq X$, the perfect core of A is denoted $\mathrm{d}^\infty(A)$ and is the greatest perfect subset of X in which A is dense, i.e. $\mathrm{d}^\infty(A) \subseteq \mathrm{d}(A \cap \mathrm{d}^\infty(A))$. The tangled derivative extends this notion to tuples of sets $A_1, \ldots, A_n$ (see Definition V.1).

However, only I-A applies to wK4 or other logics between wK4 and K4. In the case of I-B, completeness of such logics for metric spaces seems to be out of the question, since these spaces already validate K4. For I-C, Baltag et al. [1] showed that the tangled derivative is no longer expressively complete over the class of all topological spaces.

In this paper we address these shortcomings, and partially extend I-B and I-C to logics below K4. With respect to metric completeness, while standard metric spaces do validate K4, we observe that *hemimetric* spaces have been overlooked in the topological modal logic literature and provide semantics for wK4. Hemimetric spaces weaken the definition of metric spaces in two ways: first, it is not assumed that the distance between two points is symmetric, in that $\triangle(x, y) \neq \triangle(y, x)$ is allowed. Second, two distinct points can be at distance zero from each other. While arbitrary non-T_D spaces remain of somewhat specialist interest, hemimetric spaces are more accessible, and thus provide a natural 'playground' to derivational modal logic. They are also found in applications to fuzzy reasoning [24], [27] and stochastic processes [14], [21]. In between hemimetric and metric spaces are *quasimetric spaces,* where two points cannot be at total distance zero, in the sense that $\triangle(x, y) + \triangle(y, x) = 0$ implies $x = y$. Quasimetric spaces provide semantics for $\mathsf{wK4T_0}$, the logic of all T_0 spaces. In fact we prove that these logics, and their μ-calculus extensions, are complete for these two classes, thus obtaining a variant of the McKinsey-Tarski theorem for logics below K4.

As for expressive completeness, our techniques show that the Dawar and Otto theorem also fails for quasimetric spaces. One necessary (if not sufficient) condition for expressive completeness is the definability of *simulability,* which is the appropriate notion of substructure in models for modal logics – and indeed a crucial ingredient in a completeness proof technique for dynamic logics [8], [11]. In contrast to bisimulation, it has been shown by Fernández-Duque [9] that the modal language does not suffice to characterize finite models up to simulability in the context of transitive frames or closure spaces (i.e. topological spaces with $\Diamond$ interpreted as closure). This led to the introduction of the *tangled closure*, a close kin of the tangle derivative, which succeeds at defining simulability over reflexive and transitive frames.

However, once the T_D assumption is dropped, we will see that the tangle operators no longer compete with the expressive power of the μ-calculus. In fact, the content of [1] already shows that simulability is not definable in terms of the tangled derivative, even over the class of T_0 spaces. We also show that simulability is not definable over the class of T_D spaces in terms of the tangled closure, and that even the two tangled operators combined do not suffice to define simulability.

This raises the question of whether it is possible to define simulability in anything short of the full μ-calculus. As an upper bound, it is known that the topological μ-calculus collapses to its alternation-free fragment [20], but this fragment is substantially more complex from a syntactic perspective than the tangled fragments. Can something similar be obtained in the general topological setting?

We give a positive answer to this question. By analyzing the semantics of the tangled derivative, we argue that it does not adequately capture its intuitive meaning on spaces that do not validate K4. Following this observation, we propose an alternative operator and dub it the *hybrid tangle.* The hybrid tangle is equivalent to the tangled derivative over T_D spaces (and in particular over metric spaces), but over arbitrary spaces it behaves better. Moreover, the hybrid tangle is more expressive than the tangled closure and derivative, and indeed we show that it defines simulability over all spaces.

The paper is structured as follows: in Section II, we recall some relevant material regarding derivative spaces and the μ-calculus. In Section III, we introduce hemimetric spaces and their variants. In Section IV, we introduce the notion of simulation on derivative spaces. In Section V, we present the tangled closure and tangled derivative, as well as their respective limitations. In Section VI, we show that simulability is not definable by means of the tangled closure and derivative. In Section VII, we introduce the hybrid tangle and study its properties. In Section VIII, we prove that simulability is definable with the hybrid tangle, and thus deduce that the hybrid tangle is strictly more expressive than its counterparts. In Section IX we explain how to lift a Kripke model to a hemimetric space, and what this result implies in terms of completeness and definable simulability. In Section X, we comment our results and propose directions for future work.

II. Background

In this section we review the syntax and semantics of the topological μ-calculus. Following [1], [12], we present our semantics in the general setting of *derivative spaces,* and work in a language with ν (rather than μ) as primitive.

Definition II.1. We fix a finite non-empty set P of *atomic propositions*. The language $\mathcal{L}_\mu$ of the modal μ-calculus is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \nu p.\varphi$$

where $p \in P$ and in the construct $\nu p.\varphi$, the formula φ is *positive in* p, that is, every occurrence of p lies under the scope of an even number of negations. Abbreviations such as $\varphi \vee \psi$, $\Box\varphi$ or $\varphi \to \psi$ are defined as usual, and we write $\Diamond^+\varphi := \varphi \vee \Diamond\varphi$. Finally, the *basic modal language* $\mathcal{L}_\Diamond$ is the fragment of $\mathcal{L}_\mu$ without occurrences or ν.

Definition II.2. A *derivative space* is a pair $\mathcal{X} = (X, \mathrm{d})$, where X is a set of *points* and $\mathrm{d} : 2^X \to 2^X$ is an operator on subsets of X, satisfying for all $A, B \subseteq X$:

- $\mathrm{d}\varnothing = \varnothing$,
- $\mathrm{d}(A \cup B) = \mathrm{d}A \cup \mathrm{d}B$,
- $\mathrm{d}\mathrm{d}A \subseteq A \cup \mathrm{d}A$.

On occasion, we will write d as d_X to avoid all risk of ambiguity. A *derivative model* based on $\mathcal{X}$ is a tuple of the form $\mathfrak{M} = (X, \mathrm{d}, V)$ with $V : P \to 2^X$ a *valuation*. Given $x \in X$ we then call $(\mathfrak{M}, x)$ a *pointed derivative model*. If $p \in P$ and $A \subseteq X$, we define the valuation $V[p := A]$ by $V[p := A](p) := A$ and $V[p := A](q) := V(q)$ if $q \neq p$. We then write $\mathfrak{M}[p := A] := (X, \mathrm{d}, V[p := A])$.

Definition II.3. Given a derivative model $\mathfrak{M} = (X, \mathrm{d}, V)$, we define by induction on a formula $\varphi \in \mathcal{L}_\mu$ the *extension* $[\![\varphi]\!]_\mathfrak{M}$ of φ in $\mathfrak{M}$ by:

- $[\![p]\!]_\mathfrak{M} := V(p)$,
- $[\![\neg\varphi]\!]_\mathfrak{M} := X \setminus [\![\varphi]\!]_\mathfrak{M}$,
- $[\![\varphi \wedge \psi]\!]_\mathfrak{M} := [\![\varphi]\!]_\mathfrak{M} \cap [\![\psi]\!]_\mathfrak{M}$,
- $[\![\Diamond\varphi]\!]_\mathfrak{M} := \mathrm{d}[\![\varphi]\!]_\mathfrak{M}$,
- $[\![\nu p.\varphi]\!]_\mathfrak{M} := \bigcup\left\{A \subseteq X : A \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]}\right\}$.

We then write $\mathfrak{M}, x \vDash \varphi$ whenever $x \in [\![\varphi]\!]_\mathfrak{M}$. If $[\![\varphi]\!]_\mathfrak{M} = X$, we write $\mathfrak{M} \vDash \varphi$. If $\mathfrak{M} \vDash \varphi$ for all models $\mathfrak{M}$ based on $\mathcal{X}$, then we write $\mathcal{X} \vDash \varphi$ and we say that φ is *valid* on $\mathcal{X}$. If $\mathcal{C}$ is a class of derivative spaces such that $\mathcal{X} \vDash \varphi$ holds for all $\mathcal{X} \in \mathcal{C}$, we write $\mathcal{C} \vDash \varphi$ and we say that φ is *valid* on $\mathcal{C}$.

Derivative spaces are useful, as they generalise both weakly transitive Kripke frames and topological spaces (either with the closure or the derivative operator).

Definition II.4. A Kripke frame is a pair $\mathfrak{F} = (W, \rhd)$ where $\rhd$ is a binary relation over W. We write $w \unrhd u$ whenever $w \rhd u$ or $w = u$, and $w \triangleq u$ whenever $w \unrhd u$ and $u \unrhd w$. An element w is said to be *reflexive* if $w \rhd w$, and *irreflexive* otherwise. We say that $\mathfrak{F}$ is *weakly transitive* if $w \rhd u$ and $u \rhd v$ implies $w \unrhd v$. In this case $\mathfrak{F}$ induces a derivative space $(W, \mathrm{d}_\rhd)$ with $\mathrm{d}_\rhd$ defined by $\mathrm{d}_\rhd A := \{w \in W : \exists u \in A, w \rhd u\}$. Slightly abusing terminology, we will identify $\mathfrak{F}$ and $(W, \mathrm{d}_\rhd)$, since one can be constructed from the other. A derivative model based on a Kripke frame will be called a *Kripke model*.

If $\mathfrak{F}$ is weakly transitive, then $\triangleq$ is an equivalence relation, and its equivalence classes are called *clusters*. A cluster $\mathcal{C}$ is said to be reflexive if all its elements are reflexive. If $w \in W$ is such that $w \unrhd u$ for all $u \in \mathcal{C}$, then we write $w \unrhd \mathcal{C}$.

Kripke frames are valuable not only because they are simple structures, but also because they completely describe the class of finite derivative spaces.

Proposition II.5. *For all finite derivative spaces* (X, d)*, there exists a weakly transitive relation* $\rhd$ *on* X *such that* $\mathrm{d}_\rhd = \mathrm{d}$.

Proof. We define $\rhd$ by $x \rhd y$ iff $x \in \mathrm{d}\{y\}$. If $x \in \mathrm{d}\{y\}$ and $y \in \mathrm{d}\{z\}$, we have $x \in \mathrm{d}\mathrm{d}\{z\} \subseteq \{z\} \cup \mathrm{d}\{z\}$, and thus either $x = z$ or $x \in \mathrm{d}\{z\}$. Therefore, $\rhd$ is weakly transitive. Then, given $A \subseteq X$, we prove that $\mathrm{d}_\rhd A = \mathrm{d}A$. First, if $x \in \mathrm{d}_\rhd A$, then $x \rhd y$ for some $y \in A$, that is, $x \in \mathrm{d}\{y\} \subseteq \mathrm{d}A$. Conversely, suppose that $x \in \mathrm{d}A$. Since A is finite and d commutes with unions, we have $\mathrm{d}A = \bigcup_{y \in A} \mathrm{d}\{y\}$, and so $x \in \mathrm{d}\{y\}$ for some $y \in A$. Thus $x \rhd y$, and since $y \in A$ we obtain $x \in \mathrm{d}_\rhd A$, as desired. ☐

Now we turn our attention to topological spaces.

Definition II.6. Let X be a set of *points*. A *topology* on X is a set $\tau \subseteq 2^X$ containing $\varnothing$ and X, closed under arbitrary unions, and closed under finite intersections. The pair (X, τ) is then called a *topological space*. The elements of τ are called the *open* sets of X. The complement of an open set is called a *closed* set. Slightly abusing notation, we will often keep τ implicit and let X refer to the space (X, τ).

Let $A \subseteq X$. The *closure* $\mathrm{c}A$ of A is the smallest closed set containing A. The *derivative* of A is then defined as the set $\mathrm{d}A := \{x \in X : x \in \mathrm{c}(A \setminus \{x\})\}$.

Given a topological space X, the identity $\mathrm{c}A = A \cup \mathrm{d}A$ is easily verified. Further, we can check that the pair (X, d) is a derivative space. Conversely, the topology τ can be recovered from d since the closed sets are exactly the sets $A \subseteq X$ such that $\mathrm{d}A \subseteq A$. For this reason we choose, again, to identify (X, τ) and (X, d). Derivative models based on topological spaces will be called *topological models*.

Definition II.7. Let $\mathcal{X} = (X, \mathrm{d})$ be a derivative space. We say that $\mathcal{X}$ is:

- a *closure space* if $A \subseteq \mathrm{d}A$ for all $A \subseteq X$,
- a T_D *space* if $\mathrm{d}\mathrm{d}A \subseteq \mathrm{d}A$ for all $A \subseteq X$,
- a T_0 *space* if $A \cap \mathrm{d}(B \cap \mathrm{d}A) \subseteq \mathrm{d}A \cup \mathrm{d}(B \cap \mathrm{d}B)$ for all $A, B \subseteq X$.

Accordingly, a derivative model based on a T_D (resp. T_0) space is said to be T_D (resp. T_0) as well.

Observe that all closure spaces are also T_D, and that all T_D spaces are also T_0 (this is an immediate consequence of [4, Th. 2]). Below we characterize these conditions in the case of Kripke frames and topological spaces.

Proposition II.8. *Let* $\mathfrak{F}$ *be a weakly transitive Kripke frame, and* X *be a topological space.*

1) $\mathfrak{F}$ *is* T_D *iff* $\mathfrak{F}$ *is transitive* [6, Sect. 3].
2) $\mathfrak{F}$ *is* T_0 *iff every cluster in* $\mathfrak{F}$ *contains at most one irreflexive point* [4, Lemma 1].

TABLE I
AXIOMATIZATION OF μ-wK4

PL	Axioms and rules of propositional logic
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
N	$\neg\Diamond\bot$
w4	$\Diamond\Diamond\varphi \to \varphi \vee \Diamond\varphi$
Fix	$\nu x.\theta \to \theta(\nu x.\theta)$
Mon	From $\varphi \to \psi$ infer $\Diamond\varphi \to \Diamond\psi$
Ind	From $\varphi \to \theta(\varphi)$ infer $\varphi \to \nu x.\theta$

3) *X is T_D iff every singleton in X is the intersection of some open set and some closed set* [3, Sect. 4.1].
4) *X is T_0 iff for all $x, y \in X$, there exists an open set U such that $U \cap \{x, y\}$ is a singleton* [4, Th. 1].

The μ-calculus is decidable and enjoys a natural axiomatization in each of the above-mentioned classes of spaces. As usual, a logic $\mathsf{L} \subseteq \mathcal{L}_\mu$ is said to be *sound and complete* for a class $\mathcal{C}$ of spaces if we have $\mathsf{L} = \{\varphi \in \mathcal{L}_\mu : \forall \mathcal{X} \in \mathcal{C}, \mathcal{X} \vDash \varphi\}$.

Definition II.9. The logic μ-wK4 is the least set of formulas of $\mathcal{L}_\mu$ containing the axioms and closed under the rules presented in Table I where φ, ψ are arbitrary formulas and $\theta = \theta(x)$ is positive in x. In addition, let 4 be the axiom $\Diamond\Diamond\varphi \to \Diamond\varphi$, and T_0 be the axiom $\varphi \wedge \Diamond(\psi \wedge \Diamond\varphi) \to \Diamond\varphi \vee \Diamond(\psi \wedge \Diamond\psi)$. We then introduce the logics μ-K4 $:=$ μ-wK4 $+ 4$ and μ-wK4T_0 $:=$ μ-wK4 $+ \mathsf{T}_0$.

Note the inclusion of N and Mon, which are needed since $\Diamond$ is taken as primitive.

Theorem II.10 ([1, Th. III.5])**.**
1) *The logic μ-wK4 is sound and complete for the class of all derivative spaces, as well as for the class of all finite weakly transitive frames.*
2) *The logic μ-wK4T_0 is sound and complete for the class of all T_0 topological spaces, as well as for the class of all finite T_0 frames.*
3) *The logic μ-K4 is sound and complete for the class of all T_D topological spaces, as well as for the class of all finite transitive frames.*

Item 3 can be improved using a result of Bezhanishvili and Lucero-Bryan [5], which implies that any finite transitive frame is the image of a subspace of the rational numbers. We will also obtain this version from our construction below, which uniformly yields completeness for the three logics interpreted on hemimetric spaces – which we discuss next.

III. HEMIMETRIC SPACES

Topological spaces can be seen as a generalization of *metric spaces,* where the closure and Cantor derivative are defined in terms of the distance between points. In the metric space setting, the space X is equipped with a function $\vartriangle : X \times X \to \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ denotes the set of non-negative reals. The intuition is that $\vartriangle(x, y)$ is the distance between the points x and y, and it is assumed that $\vartriangle(x, y) > 0$ if x, y are distinct, and $\vartriangle(x, y) = \vartriangle(y, x)$, as one would expect when (say) measuring the distances between two points in space using a tape measure. However, if we are measuring the energy required to get from x to y, we might have $\vartriangle(x, y) < \vartriangle(y, x)$ if (say) x is at the top of a hill but y is at the bottom. On occasion it may even make sense to have $\vartriangle(x, y) = 0$: if we are concerned with the amount of fuel a speedboat needs to get from x to y, this could be zero if the current is flowing toward y. By considering different combinations of these conditions, we arrive at the following (see e.g. [15]).

Definition III.1. Let X be any set. A *hemimetric on X* is a function $\vartriangle : X \times X \to \mathbb{R}_{\geq 0}$ such that, for all $x, y, z \in X$:
- $\vartriangle(x, x) = 0$,
- $\vartriangle(x, z) \leq \vartriangle(x, y) + \vartriangle(y, z)$.

If $\vartriangle$ moreover is such that $\vartriangle(x, y) + \vartriangle(y, x) = 0$ implies $x = y$ then $\vartriangle$ is a *quasimetric,*[1] and if $\vartriangle$ is a quasimetric that is symmetric in the sense that $\vartriangle(x, y) = \vartriangle(y, x)$, then $\vartriangle$ is a *metric.* In the above cases, the pair $(X, \vartriangle)$ is a *hemimetric space, quasimetric space* or *metric space,* respectively.

Given $x \in X$ and $\varepsilon > 0$, the *open ball* centered in x and of radius ε is the set $B(x, \varepsilon) := \{y \in X : \vartriangle(x, y) < \varepsilon\}$. The collection of all open balls is a base for a topology that we denote $\tau_\vartriangle$ [15], and we will freely identify $(X, \vartriangle)$ to $(X, \tau_\vartriangle)$. Topological models based on hemimetric spaces (resp. quasimetric spaces, metric spaces) will be called *hemimetric models* (resp. *quasimetric models*, *metric models*).

The following is proven in [15], in terms of the characterizations in Proposition II.8.

Proposition III.2. *Let $(X, \vartriangle)$ be a hemimetric space.*
1) *If $\vartriangle$ is a quasimetric, then $(X, \tau_\vartriangle)$ is a T_0 space.*
2) *If $\vartriangle$ is a metric, then $(X, \tau_\vartriangle)$ is a T_D space.*[2]

Thus, hemimetric spaces provide semantics for wK4 and extensions, while being more concrete than derivative spaces or even topological spaces. As we will see, the completeness results of Baltag et al. [1] also specialize to this class of spaces. In order to obtain this result from known results for Kripke semantics, we first need to review morphisms between derivative spaces.

IV. SIMULATIONS ON DERIVATIVE SPACES

While we may speak of homeomorphisms or embeddings between topological spaces, these notions are too fine when regarding them as models for the μ-calculus: two spaces may be far from homeomorphic and still validate the same formulas. In the Kripke semantics setting, equivalence between models is given by *bisimulation,* and the appropriate notion of 'substructure' is given by *simulation* (see e.g. [6, Sect. 2.7]). Here we adapt these notions to the setting of derivative spaces. First, let us set some notations. Let $R \subseteq X \times Y$ be a binary

[1]Note that there is no universally agreed definition of quasimetrics: some authors like Seebach and Steen [26] require quasimetrics to satisfy the stronger condition that $\vartriangle(x, y) = 0$ implies $x = y$. However, this version would enforce T_1 separation on quasimetric spaces, whereas we are more interested in the T_0 condition (see Proposition III.2).

[2]Metric spaces actually satisfy the stronger T_2 condition.

relation. As usual $x\ R\ y$ stands for $(x,y) \in R$. If $A \subseteq X$ we write $R(A) := \{y \in Y : \exists x \in A, x\ R\ y\}$. If $x \in X$ we write $R(x) := R(\{x\})$. The inverse of R is defined as $R^{-1} := \{(y,x) : (x,y) \in R\}$.

Definition IV.1. If $\mathfrak{M} = (X, \mathrm{d}_X, V_X)$ and $\mathfrak{N} = (Y, \mathrm{d}_Y, V_Y)$ are derivative models, a *simulation* is a binary relation $S \subseteq X \times Y$ such that for all $p \in P$, $(x,y) \in S$ and $A \subseteq X$ we have:

- $x \in V_X(p) \iff y \in V_Y(p)$, *(atom preservation)*
- $S(\mathrm{d}_X A) \subseteq \mathrm{d}_Y(S(A))$. *(forth condition)*

Given $x_0 \in X$ and $y_0 \in Y$, S is a simulation between the pointed models $(\mathfrak{M}, x_0)$ and $(\mathfrak{N}, y_0)$ if it is a simulation and moreover satisfies $x_0\ S\ y_0$. In this case we write $(\mathfrak{M}, x_0) \rightleftharpoons (\mathfrak{N}, y_0)$ and say that $(\mathfrak{N}, y_0)$ *simulates* $(\mathfrak{M}, x_0)$. If both S and S^{-1} are simulations, then S is a *bisimulation.* A bisimulation that is also a function is a *d-morphism.*

Proposition IV.2. *The relation $\rightleftharpoons$ is reflexive and transitive.*

Proof. The identity (on any set) is obviously a simulation, so reflexivity is clear. Now let $\mathfrak{M} = (X, \mathrm{d}_X, V_X)$, $\mathfrak{N} = (Y, \mathrm{d}_Y, V_Y)$ and $\mathfrak{P} = (Z, \mathrm{d}_Z, V_Z)$ be three derivative models. Suppose that S is a simulation between $(\mathfrak{M}, x)$ and $(\mathfrak{N}, y)$, and that S' is a simulation between $(\mathfrak{N}, y)$ and $(\mathfrak{P}, z)$. Let $S'' := \{(x,z) : (x,y) \in S \text{ and } (y,z) \in S'\}$ be the composition of S and S'. We claim that S'' is a simulation between $(\mathfrak{M}, x)$ and $(\mathfrak{P}, z)$. First, let $p \in P$ and $(x,z) \in S''$. Then there exists $y \in Y$ such that $x\ S\ y$ and $y\ S'\ z$. It follows that $x \in V_X(p) \iff y \in V_Y(p) \iff z \in V_Z(p)$. For the forth condition, consider $A \subseteq X$. Then we have $S''(\mathrm{d}_X A) = S'(S(\mathrm{d}_X A)) \subseteq S'(\mathrm{d}_Y(S(A))) \subseteq \mathrm{d}_Z(S'(S(A))) = \mathrm{d}_Z(S''(A))$, and this concludes the proof. $\square$

While bisimulations preserve the full language of the μ-calculus, simulations preserve only the existential fragment.

Proposition IV.3. *We define the* existential fragment $\mathcal{L}_\mu^\Diamond$ *of* $\mathcal{L}_\mu$ *by the following grammar:*

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \nu p.\varphi.$$

Let $\varphi \in \mathcal{L}_\mu^\Diamond$ and suppose that $\mathfrak{M}, x \vDash \varphi$ and $(\mathfrak{M}, x) \rightleftharpoons (\mathfrak{N}, y)$. Then $\mathfrak{N}, y \vDash \varphi$.

Proof. We prove by induction on $\varphi \in \mathcal{L}_\mu^\Diamond$ that for all simulations S between two derivative models $\mathfrak{M} = (X, \mathrm{d}_X, V_X)$ and $\mathfrak{N} = (Y, \mathrm{d}_Y, V_Y)$, we have $S([\![\varphi]\!]_\mathfrak{M}) \subseteq [\![\varphi]\!]_\mathfrak{N}$. For atomic and Boolean formulas, this is clear. Assume that it holds for φ. Then we have $S([\![\Diamond\varphi]\!]_\mathfrak{M}) = S(\mathrm{d}_X [\![\varphi]\!]_\mathfrak{M}) \subseteq \mathrm{d}_Y S([\![\varphi]\!]_\mathfrak{M}) \subseteq \mathrm{d}_Y [\![\varphi]\!]_\mathfrak{N} = [\![\Diamond\varphi]\!]_\mathfrak{N}$, as desired. Now let us prove that $S([\![\nu p.\varphi]\!]_\mathfrak{M}) \subseteq [\![\nu p.\varphi]\!]_\mathfrak{N}$. We have

$$S([\![\nu p.\varphi]\!]_\mathfrak{M}) = \bigcup \big\{S(A) : A \subseteq X \text{ and } A \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]}\big\}.$$

Let $A \subseteq X$ such that $A \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]}$. Then we have $S(A) \subseteq S\big([\![\varphi]\!]_{\mathfrak{M}[p:=A]}\big)$. Further, since S is a simulation between $\mathfrak{M}$ and $\mathfrak{N}$, it is easy to check that it is also a simulation between $\mathfrak{M}[p := A]$ and $\mathfrak{N}[p := S(A)]$, so by the induction hypothesis we obtain $S\big([\![\varphi]\!]_{\mathfrak{M}[p:=A]}\big) \subseteq [\![\varphi]\!]_{\mathfrak{N}[p:=S(A)]}$. Thus $S(A) \subseteq [\![\varphi]\!]_{\mathfrak{N}[p:=S(A)]}$, whence $S(A) \subseteq [\![\nu p.\varphi]\!]_\mathfrak{N}$ by definition. Since A was arbitrary, it follows that $S([\![\nu p.\varphi]\!]_\mathfrak{M}) \subseteq [\![\nu p.\varphi]\!]_\mathfrak{N}$, and we are done. $\square$

It will be useful to characterize simulations on Kripke models in terms of the Kripke relations.

Proposition IV.4. *If $\mathfrak{M} = (X, \rhd_X, V_X)$ and $\mathfrak{N} = (Y, \rhd_Y, V_Y)$ are weakly transitive models, then $S \subseteq X \times Y$ is a simulation between $\mathfrak{M}$ and $\mathfrak{N}$ if and only if S preserves atoms and whenever $x\ S\ y$ and $x \rhd_X x'$, there exists $y' \in Y$ such that $x'\ S\ y'$ and $y \rhd_Y y'$ (forward confluence).*

Proof. Suppose that S is a simulation. Assume that $x\ S\ y$ and that $x \rhd_X x'$. Then by definition we have $y \in S(\mathrm{d}_{\rhd_X}\{x'\})$, whence $y \in \mathrm{d}_{\rhd_Y}(S(x'))$, that is, there exists $y' \in Y$ such that $y \rhd_Y y'$ and $y' \in S(x')$. It follows that $x'\ S\ y'$, as desired. Conversely, suppose that S preserves atoms and satisfies the condition of forward confluence. Let $A \subseteq X$ and $y \in S(\mathrm{d}_{\rhd_X} A)$. Then $x\ S\ y$ for some $x \in \mathrm{d}_{\rhd_X} A$, which means that $x \rhd_X x'$ for some $x' \in A$. Then by assumption there exists $y' \in Y$ such that $x'\ S\ y'$ and $y \rhd_Y y'$. As a result $y \in \mathrm{d}_{\rhd_Y}(S(A))$, and this proves the claim. $\square$

An important simulation on Kripke models is obtained by the process of dereflexivation, which consists in duplicating every reflexive point so as to eliminate reflexive edges. The construction is adapted from [4, Sect. 6].

Definition IV.5. Let $\mathfrak{M} = (W, \rhd, V)$ be a weakly transitive Kripke model. We denote by W^r the set of reflexive points of W, and by W^i the set of irreflexive points of W. Let $W_\bullet := (W^\mathrm{i} \times \{0\}) \cup (W^\mathrm{r} \times \{0,1\})$. Let $\pi : W_\bullet \to W$ be defined by $\pi(w,k) := w$. Then the *dereflexivation* of $\mathfrak{M}$ is the Kripke model $\mathfrak{M}_\bullet = (W_\bullet, \rhd_\bullet, V_\bullet)$, where $V_\bullet(p) := \pi^{-1}(V(p))$ for all $p \in P$, and $x \rhd_\bullet y$ iff $\pi(x) \rhd \pi(y)$ and $x \neq y$.

Proposition IV.6 ([4, Sect. 6]). *The model $\mathfrak{M}_\bullet$ is irreflexive and weakly transitive, and π is a d-morphism from $\mathfrak{M}_\bullet$ to $\mathfrak{M}$.*

The following characterization will be particularly helpful, as we will make extensive use of d-morphims from hemimetric models to Kripke models.

Proposition IV.7. *If $\mathfrak{M} = (X, \vartriangle, V_X)$ is a hemimetric model and $\mathfrak{N} = (Y, \rhd, V_Y)$ is a weakly transitive model, then $f \colon X \to Y$ is a d-morphism between $\mathfrak{M}$ and $\mathfrak{N}$ iff:*

1) *f preserves atoms;*
2) *f is strictly continuous in the sense that for every $x \in X$ there exists $\varepsilon > 0$ such that if $x' \neq x$ and $\vartriangle(x,x') < \varepsilon$, then $f(x) \rhd f(x')$;*
3) *f is strictly open in the sense that whenever $x \in X$, $\varepsilon > 0$, and $y \lhd f(x)$, then there exists $x' \neq x$ such that $f(x') = y$ and $\vartriangle(x,x') < \varepsilon$.*

Proof. Suppose that f preserves atoms, is strictly continuous, and is strictly open. Let $A \subseteq X$ and $y \in f(\mathrm{d}_X A)$. Then $y = f(x)$ for some $x \in \mathrm{d}_X A$. Let $\varepsilon > 0$ be as given by the continuity condition at x. Since $x \in \mathrm{d}_X A$ there exists $x' \in A \setminus \{x\}$ such that $\vartriangle(x,x') < \varepsilon$. Then $y \rhd f(x')$, which

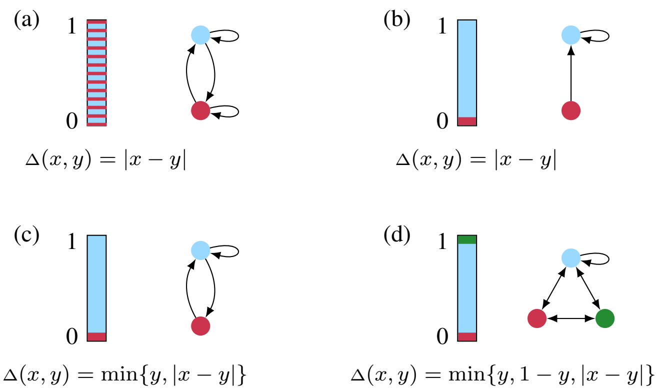

Fig. 1. Some d-morphisms from the unit interval to finite Kripke models. Reals map to the point with the same colour.

yields $y \in \mathrm{d}_{\rhd}(f(A))$. Now let $B \subseteq Y$ and $x \in f^{-1}(\mathrm{d}_{\rhd} B)$. Then $f(x) \rhd y$ for some $y \in B$. Let $\varepsilon > 0$. By openness, there exists $x' \neq x$ such that $\vartriangle(x, x') < \varepsilon$ and $f(x') = y$, whence $x' \in f^{-1}(B)$. Therefore $x \in \mathrm{d}_X(f^{-1}(B))$. This proves that f is a d-morphism.

For the other direction, we reason by contraposition. So first assume that the continuity condition fails on some $x \in X$. Then for all $n \in \mathbb{N}$, there is $x_n \neq x$ such that $\vartriangle(x, x_n) < \frac{1}{n+1}$ and $f(x) \not\trianglerighteq f(x_n)$. Setting $A := \{x_n : n \in \mathbb{N}\}$, we obtain $x \in \mathrm{d}_X A$ and so $f(x) \in f(\mathrm{d}_X(A))$. Yet we also have $f(x) \notin \mathrm{d}_{\rhd}(f(A))$ by construction, and thus $f(\mathrm{d}_X(A)) \not\subseteq \mathrm{d}_{\rhd}(f(A))$. Now assume that the openness condition fails. Then there are $x \in X$, $\varepsilon > 0$ and $y \lhd f(x)$ such that $y \notin f(B(x, \varepsilon) \setminus \{x\})$. Setting $B := \{y\}$, it follows that $f^{-1}(B) \cap B(x, \varepsilon) \setminus \{x\} = \varnothing$, and thus $x \notin \mathrm{d}_X(f^{-1}(B))$. Yet since $f(x) \rhd y$, we have $x \in f^{-1}(\mathrm{d}_{\rhd} B)$. Therefore $f^{-1}(\mathrm{d}_{\rhd} B) \not\subseteq \mathrm{d}_X(f^{-1}(B))$. So in both cases, f fails to be a d-morphism. $\square$

Example IV.8. In Figure 1, we see some d-morphisms from the unit interval with various hemimetrics. In (a), we have a map that sends (say) rationals to the top point and irrationals to the bottom point; the frame is transitive, and, accordingly, the hemimetric is a metric, namely the standard metric defined by $\vartriangle(x, y) = |x - y|$. In (b) we use the same metric but we now have an irreflexive point followed by a reflexive point; in this case, we note that only 0 maps to the irreflexive point. Generally speaking, irreflexive points correspond to discrete subspaces, and reflexive points to crowded subspaces.

The bottom figures use different hemimetrics. In (c), we have $\vartriangle(x, y) = \min\{y, |x - y|\}$ for all points x and y, and in particular $\vartriangle(x, 0) = 0$. This can no longer be a metric space but it does satisfy $\vartriangle(x, y) + \vartriangle(y, x) > 0$ whenever $x \neq y$, so it is a quasimetric space. Accordingly, it maps to a T_0 frame, since there is only one irreflexive point (hence, at most one irreflexive point per cluster). The metric in (d) is defined similarly: we have $\vartriangle(x, y) = \min\{y, 1 - y, |x - y|\}$ for all x and y, and so $\vartriangle(0, 1) + \vartriangle(1, 0) = 0$; this is now a hemimetric and not a quasimetric. Accordingly, the respective frame has more than one irreflexive point in the same cluster.

V. Tangled Closure and Derivative

In this paper we are largely concerned with expressivity of sub-languages of the topological μ-calculus. Given $\mathcal{L}, \mathcal{L}' \subseteq \mathcal{L}_\mu$ and $\mathcal{C}$ a class of derivative spaces, we say that $\mathcal{L}'$ *is at least as expressive as* $\mathcal{L}$ *over* $\mathcal{C}$ if for all $\varphi \in \mathcal{L}$, there exists $\varphi' \in \mathcal{L}'$ such that $\mathcal{C} \models \varphi \leftrightarrow \varphi'$. Specifically, we will work with 'tangled' fragments of the μ-calculus.

Definition V.1. Let $(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}_\mu^n$ be a n-tuple of formulas. The *tangled derivative* is defined by $\Diamond_\infty(\varphi_1, \ldots, \varphi_n) := \nu p. \bigwedge_{i=1}^n \Diamond(\varphi_i \wedge p)$. The *tangled closure* is defined by $\Diamond_\infty^+(\varphi_1, \ldots, \varphi_n) := \nu p. \bigwedge_{i=1}^n \Diamond^+(\varphi_i \wedge p)$. We then define:

- $\mathcal{L}_{\Diamond_\infty}$ to be the basic modal language extended with $\Diamond_\infty$,
- $\mathcal{L}_{\Diamond_\infty^+}$ to be the basic modal language extended with $\Diamond_\infty^+$,
- $\mathcal{L}_{\Diamond_\infty \Diamond_\infty^+}$ to be the basic modal language extended with both operators $\Diamond_\infty$ and $\Diamond_\infty^+$.

Observe that the tangled closure and derivative are definable within the existential fragment, that is, if $\Phi \in \mathcal{L}_\mu^{\Diamond\, n}$ then we have $\Diamond_\infty \Phi \in \mathcal{L}_\mu^\Diamond$ and $\Diamond_\infty^+ \Phi \in \mathcal{L}_\mu^\Diamond$. Thus, by Proposition IV.3, the truth of $\Diamond_\infty \Phi$ and $\Diamond_\infty^+ \Phi$ is preserved by simulation.

The tangled closure was introduced and axiomatized by Fernández-Duque [10], and the tangled derivative by Goldblatt and Hodkinson [13], who observed that the tangled derivative is more expressive than the tangled closure over the class of T_D spaces. These operations are topological generalizations of a polyadic modality introduced by Dawar and Otto [7] in the Kripke semantics setting, who showed it to be coexpressive with the full μ-calculus over the class of transitive frames. As we will see, none of these claims regarding expressivity remain true over arbitrary spaces.

It will be convenient to define these operations without appealing to the μ-calculus. Let (X, d) be a derivative space, $\mathcal{A} \subseteq 2^X$ and $B \subseteq X$. We say that $\mathcal{A}$ is *tangled* in B if every $A \in \mathcal{A}$ is *dense* in B, in the sense that $B \subseteq \mathrm{d}(A \cap B)$. Then, $\mathrm{d}^\infty \mathcal{A}$ is the union of all $B \subseteq X$ such that $\mathcal{A}$ is tangled in B. Equivalently, it is the largest subset B of X such that $\mathcal{A}$ is tangled in B. From this point of view, in any model $\mathfrak{M}$, we have that $[\![\Diamond_\infty(\varphi_1, \ldots, \varphi_n)]\!]_{\mathfrak{M}} = \mathrm{d}^\infty\{[\![\varphi_1]\!]_{\mathfrak{M}}, \ldots, [\![\varphi_1]\!]_{\mathfrak{M}}\}$. The tangled closure is defined similarly, except that we use c instead of d.

The special case of a finite Kripke frame will be particularly important for us, and here the definitions simplify quite a bit, at least in the transitive setting. Let $(W, \rhd)$ be a finite transitive frame and $A_1, \ldots, A_n \subseteq 2^W$. Then, $w \in \mathrm{d}_{\rhd}^\infty\{A_1, \ldots, A_n\}$ iff there are $w_1, \ldots, w_n$ such that $w_i \in A_i$, $w \rhd w_i$ and $w_i \rhd w_j$ for all $i, j \in [1, n]$; note that $i = j$ is included, i.e. the points must be reflexive. In other words, $w \in \mathrm{d}_{\rhd}^\infty\{A_1, \ldots, A_n\}$ if there is a reflexive cluster $\mathcal{C}$ accessible from w such that each A_i intersects $\mathcal{C}$. The tangled closure $\mathrm{c}_{\rhd}^\infty\{A_1, \ldots, A_n\}$ is defined exactly the same way, except that we replace $\rhd$ by $\trianglerighteq$, so that the reflexivity condition holds 'for free'.

The issue with the tangled derivative is that the definition becomes less natural if we pass to finite irreflexive, weakly transitive frames. Then, unravelling the definition of the tangled derivative shows that $w \in \mathrm{d}_{\rhd}^\infty\{A_1, \ldots, A_n\}$ if and only if there are $w_1, w_1', \ldots, w_n, w_n'$ such that we have $w_i, w_i' \in A_i$ and $w_i \neq w_i'$ for all $i \in [1,n]$, and $w_i \triangleq w_i' \triangleq w_j$ for all $i,j \in [1,n]$. In other words, there exists a cluster $\mathcal{C}$ such that $A_i \cap \mathcal{C}$ has at least two elements for each $i \leq n$, which is arguably an odd condition. As we will see in Section VI, clusters where each A_i occurs exactly once cannot always be 'recognized' by the tangled derivative. In Section VII, we will propose a new tangle operator which amends this issue.

VI. Inexpressivity Results

As motivated in the introduction, we are interested in languages that can describe whether one derivative model simulates another, by means of logical formulas. Those languages are said to have *definable simulability*. Obviously, here we only ask for simulability of *finite* models, as encoding the shape of infinite structures in finite formulas would be far too demanding. This leads to the following definition.

Definition VI.1. Let $\mathcal{L} \subseteq \mathcal{L}_\mu$ and let $\mathcal{C}$ be a class of derivative spaces. We say that simulability is definable in $\mathcal{L}$ over $\mathcal{C}$ if for all pointed derivative models $(\mathfrak{M}, x)$, there exists a formula $\varphi \in \mathcal{L}$ such that we have $\mathfrak{N}, y \vDash \varphi \iff (\mathfrak{M}, x) \rightleftharpoons (\mathfrak{N}, y)$ for all pointed models $(\mathfrak{N}, y)$ based on some derivative space in $\mathcal{C}$. In this case φ *defines simulability of* $(\mathfrak{M}, x)$ over $\mathcal{C}$.

Remark VI.2. Observe that if simulability is definable in $\mathcal{L}$ over $\mathcal{C}'$, and $\mathcal{C} \subseteq \mathcal{C}'$, then simulability is also definable in $\mathcal{L}$ over $\mathcal{C}$. Thus, statements of the form "simulability is definable in $\mathcal{L}$ over $\mathcal{C}$" are stronger when $\mathcal{C}$ is larger. Conversely, statements of the form "simulability is *not* definable in $\mathcal{L}$ over $\mathcal{C}$" are stronger when $\mathcal{C}$ is narrower.

In this section we show that the tangled fragments from the literature do not suffice to define topological simulability. The first result we present follows from the proof of Baltag et al. [1, Sect. IV], who showed that the tangled closure is not definable in terms of the tangled derivative over the class of T_0 spaces. Since their proof also yields our result regarding simulability, we briefly recall their construction.

We define a 'spine' model $\mathfrak{S}^1$ based on the ordinal $\omega + 3$, and depicted in Figure 2. We briefly recall that ω denotes the first infinite ordinal, and follow the set-theoretic convention that each ordinal is identified with its set of predecessors. With this in mind, we set $\mathfrak{S}^1 := (\omega+3, \rhd, V)$, where $\alpha \rhd \beta$ if one of the following occurs:

- $\alpha > \beta$,
- $\alpha = \beta$ and α is odd (including $\omega + 1$), or
- $\alpha = \omega + 1$ and $\beta = \omega + 2$;

and $\alpha \in V(p)$ iff α is odd, and $V(q) := \varnothing$ for all $q \neq p$.

Lemma VI.3 ([1, Sect. IV])**.** *$\mathfrak{S}^1$ is a T_0 model in which ω and $\omega + 2$ satisfy the same formulas of $\mathcal{L}_{\Diamond_\infty}$.*

Proposition VI.4. *Simulability is not definable in $\mathcal{L}_{\Diamond_\infty}$ over the class of T_0 Kripke frames.*

Proof. Let $\mathfrak{C}$ be a two-element cluster with one reflexive point w satisfying p, and one irreflexive point u not satisfying p. Clearly $(\mathfrak{C}, u)$ is only simulated by $\omega + 2$ in $\mathfrak{S}^1$. However, ω

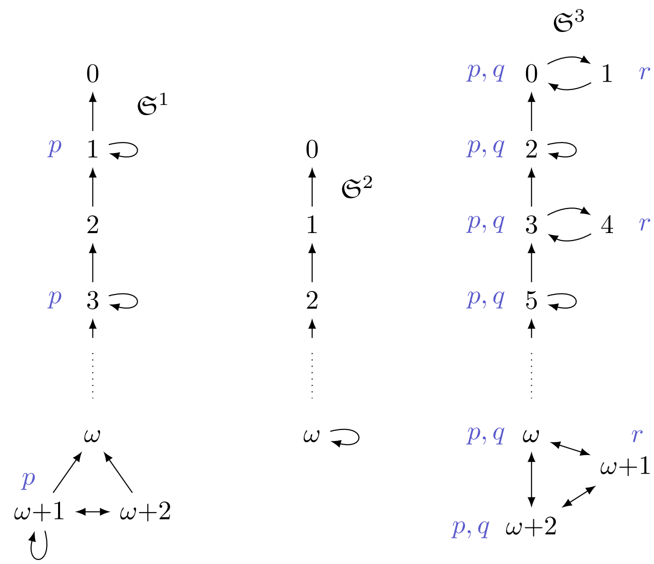

Fig. 2. An infinite spine with a two-cluster (left), one with a reflexive point (middle), and one with a three-cluster (right)

and $\omega + 2$ satisfy the same formulas of $\mathcal{L}_{\Diamond_\infty}$, meaning that no formula of that language defines simulability of $(\mathfrak{C}, u)$. $\square$

Simulability is not definable in $\mathcal{L}_{\Diamond^+_\infty}$ either. To prove this, we introduce the Kripke model $\mathfrak{S}^2 := (\omega + 1, \rhd, V)$ where $\alpha \rhd \beta$ iff $\beta < \alpha$ or $\alpha = \beta = \omega$, and $V(p) := \varnothing$ for all $p \in P$.

Lemma VI.5. *For all formulas $\varphi \in \mathcal{L}_{\Diamond^+_\infty}$, there exists $n_\varphi < \omega$ such that $n_\varphi \leq \alpha, \beta \leq \omega$ implies $\mathfrak{S}^2, \alpha \vDash \varphi \iff \mathfrak{S}^2, \beta \vDash \varphi$.*

Proof. By induction on φ:

- For an atomic proposition q, we simply have $n_q := 0$.
- If this holds for φ, then it is clear that $n_{\neg\varphi} := n_\varphi$ works for $\neg\varphi$.
- If this holds for $\varphi \wedge \psi$, then it is clear that $n_{\varphi\wedge\psi} := \max\{n_\varphi, n_\psi\}$ works for $\varphi \wedge \psi$.
- Suppose that this holds for φ. We set $n_{\Diamond\varphi} := n_\varphi + 1$. Suppose that $n_{\Diamond\varphi} \leq \alpha, \beta \leq \omega$ and that $\mathfrak{S}^2, \alpha \vDash \Diamond\varphi$. Then there exists $\xi \leq \omega$ such that $\alpha \rhd \xi$ and $\mathfrak{S}^2, \xi \vDash \varphi$. If $\xi < \beta$ we are done, otherwise $n_\varphi < \beta \leq \xi$. Then by the induction hypothesis, $\mathfrak{S}^2, \xi \vDash \varphi$ entails $\mathfrak{S}^2, n_\varphi \vDash \varphi$, and therefore $\mathfrak{S}^2, \beta \vDash \Diamond\varphi$.
- Suppose that this holds for $\varphi_1, \ldots, \varphi_m$ and let $\Phi := (\varphi_1, \ldots, \varphi_m)$ and $\varphi := \Diamond^+_\infty \Phi$. We set $n_\varphi := \max\{n_{\varphi_i} : 1 \leq i \leq m\}$ and $\varphi := \Diamond^+_\infty \Phi$. Suppose that $n_\varphi \leq \alpha, \beta \leq \omega$ and that $\mathfrak{S}^2, \alpha \vDash \Diamond^+_\infty \Phi$. Let α_0 be the smallest ordinal such that $\mathfrak{S}^2, \alpha_0 \vDash \Diamond^+_\infty \Phi$. For all $i \in [1, m]$, there exists ξ_i such that $\alpha_0 \trianglerighteq \xi_i$ and $\mathfrak{S}^2, \xi_i \vDash \varphi_i \wedge \Diamond^+_\infty \Phi$. Then $\xi_i = \alpha_0$ by minimality of α_0, and therefore $\mathfrak{S}^2, \alpha_0 \vDash \bigwedge_{i=1}^m \varphi_i$. If $n_\varphi < \alpha_0$, then by the induction hypothesis we obtain $\mathfrak{S}^2, n_\varphi \vDash \bigwedge_{i=1}^m \varphi_i$ and thus $\mathfrak{S}^2, n_\varphi \vDash \Diamond^+_\infty \Phi$, contradicting the minimality of α_0. Therefore $\alpha_0 \leq n_\varphi \leq \beta$, and it follows that $\mathfrak{S}^2, \beta \vDash \Diamond^+_\infty \Phi$.

$\square$

Proposition VI.6. *Simulability is not definable in $\mathcal{L}_{\Diamond^+_\infty}$ over the class of transitive Kripke frames.*

Proof. Let $(\mathfrak{M}, w)$ be a pointed Kripke model consisting of a single reflexive point with all atoms false. Observe that in $\mathfrak{S}^2$, the point ω is the only point of $\mathfrak{S}^2$ simulating $(\mathfrak{M}, w)$. Toward a contradiction, suppose that there is a formula φ such that $(\mathfrak{S}^2, x) \models \varphi$ iff x simulates $(\mathfrak{M}, w)$. Let n_φ be the integer given by Lemma VI.5. Since $\mathfrak{S}^2, \omega \vDash \varphi$ we obtain $\mathfrak{S}^2, n_\varphi \vDash \varphi$ as well, so that n_φ simulates $(\mathfrak{M}, w)$, which is absurd. $\square$

This raises the question of whether the combined language $\mathcal{L}_{\Diamond_\infty \Diamond^+_\infty}$ *does* suffice to define simulability. Unfortunately, the answer is also negative. Define a model $\mathfrak{S}^3 := (\omega + 3, \rhd, V)$ where $\alpha \rhd \beta$ if one of the following occurs:

- $\alpha > \beta$,
- $\alpha = \beta$ and $\alpha < \omega$ and $\alpha \equiv_3 2$,
- $\beta = \alpha + 1$ and $\alpha < \omega$ and $\alpha \equiv_3 0$,
- $\alpha, \beta \in \{\omega, \omega + 1, \omega + 2\}$ and $\alpha \neq \beta$;

and V is defined by

- $V(p) := V(q) := \{\alpha < \omega + 3 : \alpha \equiv_3 0 \text{ or } \alpha \equiv_3 2\}$,
- $V(r) := \{\alpha < \omega + 3 : \alpha \equiv_3 1\}$,
- $V(s) := \varnothing$ for all $s \notin \{p, q, r\}$

(see Figure 2). Here the relation $\equiv_3$ is the classical equivalence modulo 3 extended to ordinals, that is, given $\alpha, \beta \in \{0, \omega\}$ and $n, m \in \omega$ we write $\alpha + n \equiv_3 \beta + m$ whenever there exists $k \in \mathbb{Z}$ such that $n = m + 3k$.

Lemma VI.7. *For every formula $\varphi \in \mathcal{L}_{\Diamond_\infty \Diamond^+_\infty}$, there exists $n_\varphi < \omega$ such that $n_\varphi \leq \alpha, \beta < \omega + 3$ and $\alpha \equiv_3 \beta$ implies $\mathfrak{S}^3, \alpha \vDash \varphi \iff \mathfrak{S}^3, \beta \vDash \varphi$.*

Proof. By induction on φ:

- For an atomic proposition p', we simply have $n_{p'} := 0$.
- If this holds for φ, then it is clear that $n_{\neg\varphi} := n_\varphi$ works for $\neg\varphi$.
- If this holds for $\varphi \wedge \psi$, then it is clear that $n_{\varphi\wedge\psi} := \max\{n_\varphi, n_\psi\}$ works for $\varphi \wedge \psi$.
- Suppose that this holds for φ. We set $n_{\Diamond\varphi} := n_\varphi + 3$. Suppose that $n_{\Diamond\varphi} \leq \alpha, \beta < \omega + 3$ and that $\mathfrak{S}^3, \alpha \vDash \Diamond\varphi$. Then there exists $\xi < \omega + 3$ such that $\alpha \rhd \xi$ and $\mathfrak{S}^3, \xi \vDash \varphi$. If $\xi < \beta$ we are done, otherwise $n_\varphi \leq \beta \leq \xi$. Let k be the integer in $\{n_\varphi, n_\varphi + 1, n_\varphi + 2\}$ satisfying $k \equiv_3 \xi$. Since $n_\varphi + 3 \leq \beta$, we have $k < \beta$ and thus $\beta \rhd k$. Then by the induction hypothesis, $\mathfrak{S}^3, \xi \vDash \varphi$ entails $\mathfrak{S}^3, k \vDash \varphi$, and therefore $\mathfrak{S}^3, \beta \vDash \Diamond\varphi$.
- Suppose that this holds for $\varphi_1, \ldots, \varphi_m$ and let $\Phi := (\varphi_1, \ldots, \varphi_m)$ and $\varphi := \Diamond^+_\infty \Phi$. We set $n_\varphi := \max\{n_{\varphi_i} : 1 \leq i \leq m\} + 2$. Suppose that $n_\varphi \leq \alpha, \beta < \omega + 3$ and that $\mathfrak{S}^3, \alpha \vDash \Diamond^+_\infty \Phi$. Let α_0 be the smallest ordinal such that $\mathfrak{S}^3, \alpha_0 \vDash \Diamond^+_\infty \Phi$. We consider two cases.

  Suppose that $\alpha_0 < \omega$. For all $i \in [1, m]$, there exists ξ_i such that $\alpha_0 \unrhd \xi_i$ and $\mathfrak{S}^3, \xi_i \vDash \varphi_i \wedge \Diamond^+_\infty \Phi$. Then $\xi_i \in \{\alpha_0, \alpha_0 + 1\}$ by minimality of α_0. Let k be the integer in $\{n_\varphi - 2, n_\varphi - 1, n_\varphi\}$ satisfying $k \equiv_3 \alpha_0$. Suppose that $\alpha_0 > n_\varphi$. Then for all $i \in [1, m]$ we have $\alpha_0, k \geq n_{\varphi_i}$, as well as $\mathfrak{S}^3, \alpha_0 \vDash \varphi_i$ or $\mathfrak{S}^3, \alpha_0 + 1 \vDash \varphi_i$, so $\mathfrak{S}^3, k \vDash \varphi_i$ or $\mathfrak{S}^3, k + 1 \vDash \varphi_i$ by the induction hypothesis. We consider three cases. If $\alpha_0 \equiv_3 0$, we have $k \equiv_3 0$ as well so $k \triangleq k + 1$, and therefore $\mathfrak{S}^3, k \vDash \Diamond^+_\infty \Phi$. The case $\alpha_0 \equiv_3 1$ cannot occur because then $\mathfrak{S}^3, \alpha_0 - 1 \vDash \Diamond^+_\infty \Phi$ as well, contradicting the minimality of α_0. Finally, if $\alpha_0 \equiv_3 2$, then $\alpha_0 \not\unrhd \alpha_0 + 1$, so $\xi_i = \alpha_0$. Therefore $\mathfrak{S}^3, \alpha_0 \vDash \bigwedge_{i=1}^m \varphi_i$, whence $\mathfrak{S}^3, k \vDash \bigwedge_{i=1}^m \varphi_i$, and it follows that $\mathfrak{S}^3, k \vDash \Diamond^+_\infty \Phi$. So in all cases we have $\mathfrak{S}^3, k \vDash \Diamond^+_\infty \Phi$, and since $k < \alpha_0$, this contradicts the minimality of α_0. Hence $\alpha_0 \leq n_\varphi \leq \beta$, so $\beta \unrhd \alpha_0$ and it follows that $\mathfrak{S}^3, \beta \vDash \Diamond^+_\infty \Phi$.

  Otherwise, we have $\alpha_0 \geq \omega$. Then for all $i \in [1, m]$ there exists ξ_i such that $\mathfrak{S}^3, \xi_i \vDash \varphi_i \wedge \Diamond^+_\infty \Phi$. By minimality of α_0 we have $\xi_i \in \{\omega, \omega + 1, \omega + 2\}$, and since ω and $\omega + 2$ are bisimilar we can assume $\xi_i \in \{\omega, \omega + 1\}$. Then, if we take $k < \omega$ such that $k \geq n_\varphi$ and $k \equiv_3 0$, we obtain by the induction hypothesis that either $\mathfrak{S}^3, k \vDash \varphi_i$ or $\mathfrak{S}^3, k + 1 \vDash \varphi_i$. Therefore $\mathfrak{S}^3, k \vDash \Diamond^+_\infty \Phi$, contradicting the minimality of α_0.
- Suppose that this holds for $\varphi_1, \ldots, \varphi_m$ and let $\Phi := (\varphi_1, \ldots, \varphi_m)$ and $\varphi := \Diamond_\infty \Phi$. We set $n_\varphi := \max\{n_{\varphi_i} : 1 \leq i \leq m\} + 3$. Suppose that $n_\varphi \leq \alpha, \beta < \omega + 3$ and that $\mathfrak{S}^3, \alpha \vDash \Diamond_\infty \Phi$. Let α_0 be the smallest ordinal such that $\mathfrak{S}^3, \alpha_0 \vDash \Diamond_\infty \Phi$. First it is clear that $\alpha_0 \not\equiv_3 1$, for otherwise we have $\mathfrak{S}^3, \alpha_0 - 1 \vDash \Diamond_\infty \Phi$, contradicting the minimality of α_0. We consider two cases.

  Suppose that $\alpha_0 < \omega$. For all $i \in [1, m]$, there exists ξ_i such that $\alpha_0 \rhd \xi_i$ and $\mathfrak{S}^3, \xi_i \vDash \varphi_i \wedge \Diamond_\infty \Phi$. Then $\xi_i \in \{\alpha_0, \alpha_0 + 1\}$ by minimality of α_0. If $\alpha_0 \equiv_3 2$ then $\xi_i = \alpha_0$. Otherwise $\alpha_0 \equiv_3 0$ and $\xi_i = \alpha_0 + 1$; we also have $\mathfrak{S}^3, \alpha_0 + 1 \vDash \Diamond_\infty \Phi$ so there exists $\xi'_i < \alpha_0 + 1$ such that $\mathfrak{S}^3, \xi'_i \vDash \varphi_i \wedge \Diamond_\infty \Phi$; again by minimality of α_0 we obtain $\xi'_i = \alpha_0$. So in all cases we have $\mathfrak{S}^3, \alpha_0 \vDash \varphi_i$. Let k be the integer in $\{n_\varphi - 3, n_\varphi - 2, n_\varphi - 1\}$ satisfying $k \equiv_3 \alpha_0$. Suppose that $\alpha_0 \geq n_\varphi$. We consider two cases. If $\alpha_0 \equiv_3 0$ then for all $i \in [1, m]$ we have $\alpha_0, k \geq n_{\varphi_i}$, as well as $\mathfrak{S}^3, \alpha_0 \vDash \varphi_i$ and $\mathfrak{S}^3, \alpha_0 + 1 \vDash \varphi_i$, so $\mathfrak{S}^3, k \vDash \varphi_i$ and $\mathfrak{S}^3, k + 1 \vDash \varphi_i$ by the induction hypothesis. Since $k \rhd k + 1$ and $k + 1 \rhd k$, it follows that $\mathfrak{S}^3, k \vDash \Diamond_\infty \Phi$. If instead $\alpha_0 \equiv_3 2$ then for all $i \in [1, m]$ we have $\alpha_0, k \geq n_{\varphi_i}$, as well as $\mathfrak{S}^3, \alpha_0 \vDash \varphi_i$, so $\mathfrak{S}^3, k \vDash \varphi_i$ by the induction hypothesis. Since k is reflexive, we obtain $\mathfrak{S}^3, k \vDash \Diamond_\infty \Phi$. All cases contradict the minimality of α_0, hence $\alpha_0 < n_\varphi \leq \beta$. As a result $\beta \rhd \alpha_0$, and thus $\mathfrak{S}^3, \beta \vDash \Diamond_\infty \Phi$.

  Otherwise, we have $\alpha_0 \geq \omega$. Then in particular $\mathfrak{S}^3, \omega + 1 \vDash \Diamond_\infty \Phi$, so for all $i \in [1, m]$ there exists ξ_i such that $\omega + 1 \rhd \xi_i$ and $\mathfrak{S}^3, \xi_i \vDash \varphi_i \wedge \Diamond_\infty \Phi$. By minimality of α we must have $\xi_i \in \{\omega, \omega + 2\}$, and since ω and $\omega + 2$ are bisimilar we can assume $\xi_i = \omega + 2$. Then, if we take $k < \omega$ such that $k \geq n_\varphi$ and $k \equiv_3 2$, we obtain by the induction hypothesis that $\mathfrak{S}^3, k \vDash \varphi_i$. Therefore $\mathfrak{S}^3, k \vDash \Diamond_\infty \Phi$, contradicting the minimality of α_0.

$\square$

Proposition VI.8. *Simulability is not definable in $\mathcal{L}_{\Diamond_\infty \Diamond^+_\infty}$ over the class of weakly transitive Kripke frames.*

Proof. Let $\mathfrak{C}$ be a model consisting of a single cluster with two points u, v, where u is reflexive and satisfies p and v is irreflexive and does not satisfy p. First it is clear that $(\mathfrak{S}^3, \omega)$ simulates $(\mathfrak{C}, u)$. We also prove by induction on $k < \omega$ that $(\mathfrak{S}^3, k)$ does not simulate $(\mathfrak{C}, u)$. To this end, let $S \subseteq \{u, v\} \times (\omega + 3)$ be an arbitrary simulation; we show that the range of S is contained in $\{\omega, \omega + 1, \omega + 2\}$. Toward a contradiction, let n be the least element of $\mathbb{N} \cap S(u)$. Note that by atom preservation, $n \not\equiv_3 1$. Consider two cases.

If $n \equiv_3 2$, then by the forth condition, there is $m \leq n$ such that $v\ S\ m$. Then, atom preservation yields $m < n$. By the forth condition again, there exists m' such that $m \rhd m'$ and $u\ S\ m'$. But then $m' < n$, contradicting our choice of n.

Otherwise, $n \equiv_3 0$. By the forth condition, there is m such that $n \rhd m$ and $u\ S\ m$. By atom preservation, once again $m \not\equiv_3 1$, and so we must have $m < n$, a contradiction.

Now suppose that there exists a formula $\varphi \in \mathcal{L}_{\Diamond_\infty \Diamond^+_\infty}$ such that $[\![\varphi]\!]_{\mathfrak{S}^3}$ is the set of points in $\mathfrak{S}^3$ that simulate $(\mathfrak{C}, u)$. Let n_φ be the integer given by Lemma VI.7, and consider some $k < \omega$ such that $k \geq n_\varphi$ and $k \equiv_3 0$. Yet we have seen that $(\mathfrak{S}^3, \omega)$ simulates $(\mathfrak{C}, u)$ and $(\mathfrak{S}^3, k)$ does not, so φ cannot define simulability of $(\mathfrak{C}, u)$, and we are done. □

VII. The Hybrid Tangle

We have seen that neither the tangled derivative nor the tangled closure suffice to define topological simulability. In this section we introduce the *hybrid tangled operator* $\blacklozenge_\infty$, which coincides with the tangled derivative over T_D spaces but has more expressive power over arbitrary topological spaces. Given a n-tuple of formulas $(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}^n_\mu$, we define

$$\blacklozenge_\infty(\varphi_1, \ldots, \varphi_n) := \nu p. \bigvee_{j=1}^{n} \left(\Diamond^+(\varphi_j \wedge p) \wedge \bigwedge_{i \neq j} \Diamond(\varphi_i \wedge p) \right).$$

We then denote by $\mathcal{L}_{\blacklozenge_\infty}$ the basic modal language extended with $\blacklozenge_\infty$. The hybrid tangle is a sort of a mix of the tangled closure and the tangled derivative, hence its name. Again, we can see that $\blacklozenge_\infty$ is definable within the existential fragment, and so by Proposition IV.3 it is preserved by simulation. It will be convenient to elucidate its semantics. Below, if f is a partial function then $y \neq f(x)$ means that either $f(x)$ is defined and distinct from y, or else $f(x)$ is undefined.

Proposition VII.1. *Let $\mathfrak{M} = (X, \mathrm{d}, V)$ be a derivative model and $\Phi = (\varphi_1, \ldots, \varphi_n)$ be a tuple of formulas. Then, $[\![\blacklozenge_\infty \Phi]\!]_\mathfrak{M}$ is the greatest subset S of X for which there exists a partial assignment $i_\cdot \colon S \rightharpoonup [1, n]$ such that for every $x \in S$,*

1) *if i_x is defined, then $x \in [\![\varphi_{i_x}]\!]_\mathfrak{M}$, and*
2) *if $i \in [1, n]$ and $i \neq i_x$, then $x \in \mathrm{d}([\![\varphi_i]\!]_\mathfrak{M} \cap S)$.*

Proof. First we prove that $[\![\blacklozenge_\infty \Phi]\!]_\mathfrak{M}$ has such a partial assignment. So let $x \in [\![\blacklozenge_\infty \Phi]\!]_\mathfrak{M}$. We set $i_x := j$ for some arbitrary $j \in [1, n]$ satisfying $\mathfrak{M}, x \vDash \varphi_j \wedge \bigwedge_{i \neq j} \Diamond(\varphi_i \wedge \blacklozenge_\infty \Phi)$, provided that such a j exists; otherwise, i_x is left undefined. Then by construction, 1 is satisfied. We prove that 2 is also satisfied. So let $k \in [1, n]$ such that $k \neq i_x$. If i_x is defined, then $\mathfrak{M}, x \vDash \Diamond(\varphi_k \wedge \blacklozenge_\infty \Phi)$ is immediate. Otherwise, since $\mathfrak{M}, x \vDash \blacklozenge_\infty \Phi$, there exists $j \in [1, n]$ such that $\mathfrak{M}, x \vDash \Diamond^+(\varphi_j \wedge \blacklozenge_\infty \Phi) \wedge \bigwedge_{i \neq j} \Diamond(\varphi_i \wedge \blacklozenge_\infty \Phi)$. However, $\mathfrak{M}, x \vDash \varphi_j \wedge \blacklozenge_\infty \Phi$ is not an option, since this would make j a possible value for i_x. So we have in fact $\mathfrak{M}, x \vDash \Diamond(\varphi_j \wedge \blacklozenge_\infty \Phi)$, whence $\mathfrak{M}, x \vDash \bigwedge_{i=1}^{n} \Diamond(\varphi_i \wedge \blacklozenge_\infty \Phi)$. Therefore $\mathfrak{M}, x \vDash \Diamond(\varphi_k \wedge \blacklozenge_\infty \Phi)$, or equivalently $x \in \mathrm{d}([\![\varphi_k]\!]_\mathfrak{M} \cap S)$, as desired.

Now let S be any subset of X with such a partial assignment $i_\cdot$, and let $x \in S$. If i_x is defined, then from 1 and 2 we obtain $\mathfrak{M}[p := S], x \vDash \varphi_{i_x} \wedge p \wedge \bigwedge_{i \neq i_x} \Diamond(\varphi_i \wedge p)$. If i_x is undefined, then from 2 we obtain $\mathfrak{M}[p := S], x \vDash \bigwedge_{i=1}^{n} \Diamond(\varphi_i \wedge p)$, and so $\mathfrak{M}[p := S], x \vDash \Diamond^+(\varphi_j \wedge p) \wedge \bigwedge_{i \neq j} \Diamond(\varphi_i \wedge p)$ for an arbitrary j. Thus $S \subseteq [\![\bigvee_{j=1}^{n} (\Diamond^+(\varphi_j \wedge p) \wedge \bigwedge_{i \neq j} \Diamond(\varphi_i \wedge p))]\!]_{\mathfrak{M}[p:=S]}$, and by the semantics of ν, it follows that $S \subseteq [\![\blacklozenge_\infty \Phi]\!]_\mathfrak{M}$. □

Note that if Φ is a one-element tuple (φ), then $\blacklozenge_\infty \Phi$ is simply equivalent to $\Diamond^+ \varphi$. Next we elucidate the semantics of $\blacklozenge_\infty$ on finite Kripke models, which will be the most useful case to us. The following fact is easily obtained from Proposition VII.1 by taking $\mathcal{C}$ to be a $\rhd$-maximal cluster intersecting $[\![\blacklozenge_\infty \Phi]\!]$.

Proposition VII.2. *Let $\mathfrak{M} = (W, \rhd, V)$ be a finite weakly transitive Kripke model, $w \in W$ and $\Phi = (\varphi_1, \ldots, \varphi_n)$ be a tuple of formulas. Then, $\mathfrak{M}, w \vDash \blacklozenge_\infty \Phi$ if and only if there exists a cluster $\mathcal{C}$ and a partial assignment $i_\cdot \colon \mathcal{C} \rightharpoonup [1, n]$ such that $w \unrhd \mathcal{C}$ and for every $u \in \mathcal{C}$:*

1) *if i_u is defined, then $u \in [\![\varphi_{i_u}]\!]_\mathfrak{M}$, and*
2) *if $i \in [1, n]$ and $i \neq i_u$, then there is $v \in [\![\varphi_i]\!]_\mathfrak{M} \cap \mathcal{C}$ such that $u \rhd v$.*

Despite being somewhat more elaborate, the hybrid tangle avoids some unnatural behavior of the tangled derivative. Indeed, consider a model $\mathfrak{M}$ consisting of a trivial topological space $X := \{x_1, \ldots, x_n\}$, where the only open sets are $\varnothing$ and X, and such that the atomic proposition q_i holds only in x_i. Then, $\mathfrak{M} \vDash \neg \Diamond_\infty(q_1, \ldots, q_n)$, simply because it is not possible to have $\mathfrak{M}, x_i \vDash \Diamond(q_i \wedge p)$, regardless of the interpretation of p. If we wanted to 'fix' this example so that $\Diamond_\infty(q_1, \ldots, q_n)$ were satisfied, we could instead define $X' := \{x_1, x'_1, \ldots, x_n, x'_n\}$, and have $V(q_i) := \{x_i, x'_i\}$. Conversely, we do have that $\mathfrak{M} \vDash \Diamond^+_\infty(q_1, \ldots, q_n)$, but on the other hand if $\mathfrak{M}_0$ consisted of a single point x_0 satisfying $q_1 \wedge \ldots \wedge q_n$, then we would also have $\mathfrak{M}_0 \vDash \Diamond^+_\infty(q_1, \ldots, q_n)$. Thus both the tangled closure and the tangled derivative are unsuitable for describing clusters: the tangled derivative because it is too stringent, the tangled closure because it is too lax. On the other hand, it can be verified that $\mathfrak{M} \vDash \blacklozenge_\infty(q_1, \ldots, q_n)$ but $\mathfrak{M}_0 \nvDash \blacklozenge_\infty(q_1, \ldots, q_n)$. Indeed, if we set $V(p) := X$, each x_j satisfies $\Diamond^+(q_j \wedge p) \wedge \bigwedge_{i \neq j} \Diamond(q_i \wedge p)$; the conjunct $\Diamond^+(q_j \wedge p)$ serves as a 'permission' for x_j to not satisfy $\Diamond(q_j \wedge p)$, provided it *does* satisfy $q_j \wedge p$. Moreover, the hybrid tangle subsumes the other two operators.

Proposition VII.3. *The modalities $\Diamond_\infty$ and $\Diamond^+_\infty$ can be expressed in $\mathcal{L}_{\blacklozenge_\infty}$.*

Proof. Let $\Phi = (\varphi_1, \ldots, \varphi_n)$ be a tuple of formulas. Then each disjunct in $\blacklozenge_\infty(\varphi_1, \varphi_1, \ldots, \varphi_n, \varphi_n)$ must contain the term $\lozenge(\varphi_i \wedge p)$ for every $i \in [1,n]$, and therefore $\blacklozenge_\infty(\varphi_1, \varphi_1, \ldots, \varphi_n, \varphi_n)$ is equivalent to $\lozenge_\infty(\varphi_1, \ldots, \varphi_n)$.

For the tangled closure, we define

$$\varphi := \bigvee_{(I_1,\ldots,I_m)\in \mathrm{Part}([1,n])} \blacklozenge_\infty \left(\bigwedge_{i\in I_j} \varphi_i : 1 \leq j \leq m\right)$$

where $\mathrm{Part}([1,n])$ denotes the set of partitions of $[1,n]$. We show that φ is equivalent to $\lozenge^+_\infty\Phi$. By Theorem II.10, it suffices to prove that $\mathfrak{M}, w \vDash \varphi \leftrightarrow \lozenge^+_\infty\Phi$ for any pointed finite weakly transitive model $(\mathfrak{M}, w)$. We write $\mathfrak{M} = (W, \rhd, V)$. That $\mathfrak{M}, w \vDash \varphi$ implies $\mathfrak{M}, w \vDash \lozenge^+_\infty\Phi$ is easy to check. Conversely, suppose that $\mathfrak{M}, w \vDash \lozenge^+_\infty\Phi$. Then there exist a cluster $\mathcal{C}$ in $\mathfrak{M}$ reachable from w and for all $i \in [1,n]$, a point $w_i \in \mathcal{C}$ such that $\mathfrak{M}, w_i \vDash \varphi_i$. However, the w_i's are not necessarily pairwise distinct, so we write $\{w_1, \ldots, w_n\} = \{u_1, \ldots, u_m\}$ where the u_i's are pairwise distinct. This induces a partition $(I_1, \ldots, I_m)$ of $[1,n]$, where $I_j := \{i \in [1,n] : w_i = u_j\}$ for all j. Then given $j \in [1,m]$ we have $\mathfrak{M}, u_j \vDash \bigwedge_{i\in I_j}\varphi_i$, and whenever $j' \neq j$ we have $u_j \rhd u_{j'}$ and thus $\mathfrak{M}, u_j \vDash \lozenge \bigwedge_{i\in I_{j'}}\varphi_i$. Therefore $\mathfrak{M}, w \vDash \blacklozenge_\infty(\bigwedge_{i\in I_j}\varphi_i : 1 \leq j \leq m)$. □

On the other hand, the hybrid tangle is a natural modification of the tangled derivative, in the sense that they coincide over the class of T_D spaces:

Proposition VII.4. *Let Φ be a tuple of formulas.*

1) *$\blacklozenge_\infty\Phi$ is equivalent to $\lozenge^+_\infty\Phi$ over the class of closure spaces.*
2) *If Φ has at least two elements then $\blacklozenge_\infty\Phi$ is equivalent to $\lozenge_\infty\Phi$ over the class of T_D derivative spaces.*

Proof. The first item is clear since the semantics of $\lozenge$ and $\lozenge^+$ coincide over closure spaces. For the second item, it suffices to show that $\mathfrak{M}, w \vDash \blacklozenge_\infty\Phi \leftrightarrow \lozenge_\infty\Phi$ for all pointed finite transitive models $(\mathfrak{M}, w)$, and then we conclude by Theorem II.10. The implication from right to left is clear. Conversely, suppose that $\mathfrak{M}, w \vDash \blacklozenge_\infty\Phi$. Let $\mathcal{C}$ be the cluster given by Proposition VII.2. Since Φ has at least two elements, we see that $\mathcal{C}$ is either a reflexive singleton, or contains at least two elements. Since $\mathfrak{M}$ is transitive, the cluster $\mathcal{C}$ is reflexive in all cases. Therefore $\mathfrak{M}, w \vDash \lozenge_\infty\Phi$. □

The completeness result of Baltag et al. [1] is stated not only for the μ-calculus, but also for *natural sublanguages*, and for several extensions of $\mathsf{wK4}$ including $\mathsf{wK4T_0}$. In particular, it follows that the hybrid tangle can be axiomatized via its standard fixed point axioms.

Definition VII.5. Given a tuple $\Phi = (\varphi_1, \ldots, \varphi_n)$, define

$$\mathrm{HT}_\Phi(p) := \bigvee_{j=1}^n \left(\lozenge^+(\varphi_j \wedge p) \wedge \bigwedge_{i\neq j} \lozenge(\varphi_j \wedge p)\right).$$

Let $\mathsf{Fix}_{\blacklozenge_\infty}$ be the axiom $\blacklozenge_\infty\Phi \to \mathrm{HT}_\Phi(\blacklozenge_\infty\Phi)$, and $\mathsf{Ind}_{\blacklozenge_\infty}$ be the rule

from $\varphi \to \mathrm{HT}_\Phi(\varphi)$ infer $\varphi \to \blacklozenge_\infty\Phi$.

We then define $\mathsf{wK4}^\infty$ to be the extension of $\mathsf{wK4}$ over $\mathcal{L}_{\blacklozenge_\infty}$ with $\mathsf{Fix}_{\blacklozenge_\infty}$ and $\mathsf{Ind}_{\blacklozenge_\infty}$, as well as $\mathsf{wK4T}_0^\infty := \mathsf{wK4}^\infty + \mathsf{T_0}$ and $\mathsf{K4}^\infty := \mathsf{wK4}^\infty + \mathsf{4}$.

Theorem VII.6.

- *The logic $\mathsf{wK4}^\infty$ is sound and complete for the class of all topological spaces, as well as the class of all finite weakly transitive frames.*
- *The logic $\mathsf{wK4T}_0^\infty$ is sound and complete for the class of all T_0 topological spaces, as well as the class of all finite T_0 frames.*
- *The logic $\mathsf{K4}^\infty$ is sound and complete for the class of all T_D topological spaces, as well as the class of all finite transitive frames.*

We note that by Proposition VII.4, the axioms for $\mathsf{K4}^\infty$ can be replaced by those for the tangled derivative, which are a bit simpler [13]. Similarly, when working with transitive, reflexive frames, we may use the axioms of the tangled closure [9].

VIII. Encoding Simulation

Although simulability is not definable in the basic modal language, over $\mathcal{L}_{\blacklozenge_\infty}$ it is – a fact that we undertake to prove in this section. Recall (from Definition VI.1) that a language $\mathcal{L}$ defines simulability if it is expressive enough to characterize the simulability of any pointed finite derivative model. But combining Proposition II.5 and Proposition IV.6, we see that simulating a pointed finite derivative model amounts to simulating some pointed irreflexive, weakly transitive, finite model $(\mathfrak{M}, w)$. So we can in fact restrict our attention to these models. Following this observation, we are going to construct a formula $\mathrm{Sim}(\mathfrak{M}, w)$ in $\mathcal{L}_{\blacklozenge_\infty}$ defining simulability of $(\mathfrak{M}, w)$.

Let us write $\mathfrak{M} = (W, \rhd, V)$. We define $\mathrm{Sim}(\mathfrak{M}, w)$ for $w \in W$ by induction on the height of w. More precisely, we denote by $>$ the strict accessibility relation, i.e. $w > v$ if $w \rhd v$ but $v \not\rhd w$. Then since $\mathfrak{M}$ is finite, the relation $>$ is suitable for backward induction. We also denote by $\tau(w)$ the conjunction of all literals true at $(\mathfrak{M}, w)$. First define, for $v \triangleq w$, the formula $\delta(v) := \tau(v) \wedge \bigwedge_{v>u} \lozenge\mathrm{Sim}(\mathfrak{M}, u)$. We then set $\mathrm{Sim}(\mathfrak{M}, w) := \delta(w) \wedge \blacklozenge_\infty\big(\delta(v) : v \triangleq w\big)$ to be the *simulability formula* of w.

Proposition VIII.1. *Let $(\mathfrak{N}, x)$ be a pointed derivative model. Then, $(\mathfrak{N}, x)$ simulates $(\mathfrak{M}, w)$ iff $\mathfrak{N}, x \vDash \mathrm{Sim}(\mathfrak{M}, w)$.*

Proof. We write $\mathfrak{M} = (W, \rhd, V)$ and $\mathfrak{N} = (X, \mathrm{d}, V')$. First, it is readily proved by backward induction on $>$ that $\mathfrak{M}, w \vDash \mathrm{Sim}(\mathfrak{M}, w)$ for all $w \in W$. Thus, if $(\mathfrak{N}, x)$ simulates $(\mathfrak{M}, w)$, it follows by Proposition IV.3 that $\mathfrak{N}, x \vDash \mathrm{Sim}(\mathfrak{M}, w)$.

Conversely, assume that $\mathfrak{N}, x \vDash \mathrm{Sim}(\mathfrak{M}, w)$. Define a binary relation S given by $u\ S\ x$ if and only if $\mathfrak{N}, x \vDash \mathrm{Sim}(\mathfrak{M}, u)$. We show that S is a simulation. It is easy to see that S preserves atoms because x satisfies $\tau(w)$ whenever $\mathfrak{N}, x \vDash \mathrm{Sim}(\mathfrak{M}, w)$. What remains to check is that $S(\mathrm{d}_\rhd A) \subseteq \mathrm{d}_X S(A)$ for every $A \subseteq W$. Since A is finite and $\mathrm{d}_\rhd$ commutes with unions, we may assume that $A = \{v\}$ for some v. So, suppose that $x \in S(\mathrm{d}_\rhd\{v\})$, i.e. $w\ S\ x$ for some $w \in W$ with $w \rhd v$. Consider two cases. If $w > v$, then from

$\mathfrak{N}, x \vDash \mathrm{Sim}(\mathfrak{M}, w)$, we immediately obtain $\mathfrak{N}, x \vDash \delta(w)$ and hence $\mathfrak{N}, x \vDash \Diamond\mathrm{Sim}(\mathfrak{M}, v)$, i.e. $x \in \mathrm{d}_X S(v)$, as needed.

Otherwise, w and v belong to the same cluster $\mathcal{C}$. Then since $\mathfrak{N}, x \vDash \blacklozenge_\infty\big(\delta(u) : u \triangleq w\big)$, there is a set T and a partial assignment $u_\cdot \colon T \rightharpoonup \mathcal{C}$ such that $x \in T$ and for all $y \in T$, $y \in [\![\delta(u_y)]\!]_{\mathfrak{N}}$ when u_y is defined, and for all $u' \in \mathcal{C} \setminus \{u_y\}$, we have $y \in \mathrm{d}_X[\![\delta(u') \wedge \blacklozenge_\infty\big(\delta(u) : u \triangleq w\big)]\!]_{\mathfrak{N}}$. Note that $\delta(u') \wedge \blacklozenge_\infty\big(\delta(u) : u \triangleq w\big)$ is equal to $\mathrm{Sim}(\mathfrak{M}, u')$, since $u \triangleq w$ is equivalent to $u \triangleq u'$.

Consider two cases. If $u_x \neq v$ (possibly because it is undefined), then we immediately obtain $x \in \mathrm{d}_X[\![\mathrm{Sim}(\mathfrak{M}, v)]\!]_{\mathfrak{N}}$. Otherwise, $u_x = v$, and thus $u_x \neq w$ since v is irreflexive and $w \rhd v$. Therefore $x \in \mathrm{d}_X[\![\mathrm{Sim}(\mathfrak{M}, w)]\!]_{\mathfrak{N}}$. Further, x satisfies both $\delta(w)$ and $\delta(u_x) = \delta(v)$, and thus also $\tau(w)$ and $\tau(v)$. It follows that $\tau(w) = \tau(v)$. Since $w \triangleq v$, it is then clear that $\mathrm{Sim}(\mathfrak{M}, w) = \mathrm{Sim}(\mathfrak{M}, v)$. So in all cases we have $x \in \mathrm{d}_X[\![\mathrm{Sim}(\mathfrak{M}, v)]\!]_{\mathfrak{N}}$, that is, $x \in \mathrm{d}_X S(v)$, as needed. □

Combining Propositions VII.4 and VIII.1, we obtain the following.

Theorem VIII.2. *Simulability is definable:*

1) *in $\mathcal{L}_{\Diamond_\infty^+}$ over the class of closure spaces,*
2) *in $\mathcal{L}_{\Diamond_\infty}$ over the class of T_D derivative spaces,*
3) *in $\mathcal{L}_{\blacklozenge_\infty}$ over the class of all derivative spaces.*

Proof. Item 3 is a direct consequence of Proposition VIII.1. By Proposition VII.4, the language $\mathcal{L}_{\Diamond_\infty}$ is at least as expressive as $\mathcal{L}_{\blacklozenge_\infty}$ over the class of T_D derivative spaces, and Item 2 follows. Item 1 is obtained similarly. □

Putting this together with our results from Section VI, we obtain the following results of relative non-expressivity. These are depicted in Figure 3, where a language $\mathcal{L}_1$ is placed above a language $\mathcal{L}_2$ whenever $\mathcal{L}_1$ is more expressive than $\mathcal{L}_2$.

Corollary VIII.3.

1) *$\mathcal{L}_{\Diamond_\infty}$ is not as expressive as $\mathcal{L}_{\Diamond_\infty^+}$ over the class of T_0 Kripke frames* [1, Sect. IV].
2) *$\mathcal{L}_{\Diamond_\infty^+}$ is not as expressive as $\mathcal{L}_{\Diamond_\infty}$ over the class of transitive frames.*
3) *$\mathcal{L}_{\Diamond_\infty\Diamond_\infty^+}$ is not as expressive as $\mathcal{L}_{\blacklozenge_\infty}$ over the class of weakly transitive frames.*

Proof. Item 2 stems from Proposition VI.6 and Item 2 of Theorem VIII.2. Item 3 stems from Proposition VI.8 and Item 3 of Theorem VIII.2. □

We also obtain undefinability results for classes of finite frames.

Corollary VIII.4. *Simulability is not definable:*

1) *in $\mathcal{L}_{\Diamond_\infty}$ over the class of finite T_0 Kripke frames,*
2) *in $\mathcal{L}_{\Diamond_\infty^+}$ over the class of finite transitive frames,*
3) *in $\mathcal{L}_{\Diamond_\infty\Diamond_\infty^+}$ over the class of finite weakly transitive frames.*

Proof. This stems from our previous results combined with the finite model property of the weakly transitive μ-calculus. Here we spell out the proof of Item 3. So let $(\mathfrak{M}, w)$ be a pointed irreflexive, weakly transitive, finite model, and let $\varphi \in \mathcal{L}_{\Diamond_\infty\Diamond_\infty^+}$. By Proposition VI.8, φ does not define simulability of $(\mathfrak{M}, w)$ over the class of weakly transitive frames, whence $\varphi \leftrightarrow \mathrm{Sim}(\mathfrak{M}, w) \notin \mu\text{-}\mathsf{wK4}$. Thus, by Theorem II.10, $\varphi \leftrightarrow \mathrm{Sim}(\mathfrak{M}, w)$ is falsified on some finite weakly transitive model, and so φ does not define simulability of $(\mathfrak{M}, w)$ over the class of finite weakly transitive frames. Since φ was arbitrary, this proves the claim. Similarly, we derive Item 1 from Proposition VI.4, and Item 2 from Proposition VI.6. □

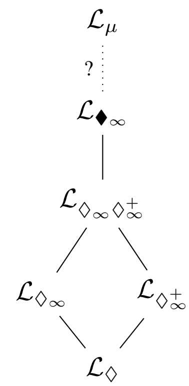

Fig. 3. Relative expressivity of the tangled fragments

IX. Hemimetric Lifting

Next we adapt Theorem II.10 to the setting of hemimetric spaces. For this it suffices, given a weakly transitive model $\mathfrak{M}$, to construct a hemimetric model $\vec{\mathfrak{M}}$ and a d-morphism π from $\vec{\mathfrak{M}}$ to $\mathfrak{M}$ (as characterized by Proposition IV.7), in such a way that if $\mathfrak{M}$ is T_0 then $\vec{\mathfrak{M}}$ will be a quasimetric model, and if $\mathfrak{M}$ is transitive then $\vec{\mathfrak{M}}$ will be a metric model. The construction is a variant of a classic unwinding, but dealing with irreflexive points has its subtleties.

To get some intuition, consider a simple example where $\mathfrak{M} = (W, \rhd, V)$ consists of two points $0, 1$ and $\rhd$ is the total relation; i.e., our frame is a reflexive cluster (see Figure 1(a)). Our new space, $\vec{W}$, will consist of sequences of elements of W, i.e. infinite binary sequences. To each sequence of $\vec{w} \in \vec{W}$, we will assign an element $\ell(\vec{w}) \in W$ (ℓ stands for 'last'). For a sequence such as $\vec{w} = 1^\omega$ (i.e., a sequence consisting of only ones), or even $0^3 \frown 1^\omega$ (i.e., three zeroes followed by only ones), we have that $\ell(\vec{w}) := 1$ unequivocally. However, for sequences such as $(0, 1, 0, 1, \ldots)$ that alternate between 0 and 1, ℓ might not be uniquely defined. There are a few ways to deal with this (see e.g. [17] for a nice algebraic approach), but let us work only with eventually constant sequences, which we call *stabilizing*. For our purposes, we can define $\vec{W}$ to be the set of all such sequences.

Binary sequences can be viewed as elements of the Cantor set, which is itself a metric space, and we can define $\vartriangle(\vec{w}, \vec{v}) := 2^{-n}$ if n is the least natural number such that $w_n \neq v_n$; if no such n exists, the sequences are equal and their distance is 0. Then $\ell \colon \vec{W} \to W$ is a d-morphism, provided we use the valuation $\vec{V} := \ell^{-1} V$ on $\vec{W}$.

However, this choice of $\mathfrak{M}$ was particularly convenient, and a few things could go wrong. The first issue that may arise is in the case of a reflexive singleton. If instead we have $W = \{0\}$ with $0 \rhd 0$, then $\vec{W}$ consists of the singleton 0^ω, and ℓ cannot be a d-morphism since $\mathrm{d}_\rhd\{0\} = \{0\}$ but $\mathrm{d}\{0^\omega\} = \varnothing$. However, this issue is easily solved: we only work with models $\mathfrak{M}$ that have the property that, if w is a reflexive point, then there is $v \neq w$ which is also reflexive and such that $v \triangleq w$. Let us call such models *abundant.* To see that we can always ensure this condition, we use a mild variant of the dereflexivation of $\mathfrak{M}$.

Definition IX.1. Let $\mathfrak{M} = (W, \rhd, V)$ be a weakly transitive Kripke model. We define a new model $\mathfrak{M}_\circ := (W_\circ, \rhd_\circ, V_\circ)$ with $W_\circ := W_\bullet$, $V_\circ := V_\bullet$ and $\pi\colon W_\circ \to W$ as given by Definition IV.5, and $x \rhd_\circ y$ if and only if $\pi(x) \rhd \pi(y)$.

It is easy to check that π is still a d-morphism, and that the model $\mathfrak{M}_\circ$ is weakly transitive and abundant. Moreover, if $\mathfrak{M}$ is T_0 then so is $\mathfrak{M}_\circ$, and if $\mathfrak{M}$ is transitive then so is $\mathfrak{M}_\circ$.

The next issue that arises is with irreflexive points. In fact the construction will mostly work fine as long as $\rhd$ is transitive. In this case, any irreflexive point has the property that its cluster is a singleton. The trouble arises when a cluster contains an irreflexive point and another point. To see this, consider now the case where $W = \{0, 1\}$ and the two are in the same cluster, but now they are irreflexive. The sequence 1^ω maps to 1, and $1 \not\rhd 1$, so there should exist an open set U containing 1^ω and such that the image of $U \setminus \{1^\omega\}$ does not contain 1. But if we allow for arbitrary stabilizing paths, this will not be the case: the sequence $(0) \frown 1^\omega$, $(1, 0) \frown 1^\omega$, $(1, 1, 0) \frown 1^\omega$... converges to 1^ω, and every element of this sequence maps to 1. Thus we need to avoid there being too many sequences in a given cluster which stabilize on an irreflexive point. The way we enforce this is by only allowing sequences to either remain on a single irreflexive point when they enter a cluster, or else avoid irreflexive points altogether.

Definition IX.2. Let $\mathfrak{M} = (W, \rhd, V)$ be any abundant weakly transitive model. We define $\vec{W}$ to be the set of all stabilizing sequences $\vec{w} = (w_0, w_1, \dots)$ of elements of W such that for all $i \in \mathbb{N}$, $w_i \trianglerighteq w_{i+1}$, and if $w_{i+1} \rhd w_i$, then w_{i+1} is reflexive. We call these sequences *superstable,* and $\ell(\vec{w})$ denotes the element of W that $\vec{w}$ stabilizes on.

Define a binary relation $\circleddash$ on W by $w \circleddash v$ if $w \triangleq v$ and v is irreflexive. Then set $\vartriangle(\vec{w}, \vec{v}) := \max D(\vec{w}, \vec{v})$ where

$$D(\vec{w}, \vec{v}) := \{0\} \cup \{2^{-n-\delta} : w_n \neq v_n \text{ and } w_n \not\circleddash v_{n+\delta}\}.$$

Finally, we define $\vec{V}$ by $\vec{V}(p) := \ell^{-1}(V(p))$, and define the *hemimetric lifting* of $\mathfrak{M}$ as $\vec{\mathfrak{M}} := (\vec{W}, \vartriangle, \vec{V})$.

In other words, $\vec{w}$ is superstable if it is stable (i.e., it is monotone on $\trianglerighteq$ and eventually constant), and whenever v_n is irreflexive, v_{n+1} can either be in a different cluster from v_n, or be in the same cluster, but in this case, v_{n+1} must be reflexive or equal to v_n. Intuitively, this means that sequences can enter a cluster through an irreflexive point, but they cannot jump to a different irreflexive point once they are already in its cluster. This ensures that we do not get too many copies of irreflexive points, and thus the set of their copies will be discrete, i.e. each one will be isolated from the others.

Example IX.3. In the cluster on Figure 1(c), we have one reflexive point (which we will call r) and one irreflexive point (which we will call i). Due to superstability, sequences can begin on i and stay there arbitrarily long, but cannot return to i once they reach r. Thus $\vec{W} = \{i^n \frown r^\omega : n \in \mathbb{N}\} \cup \{i^\omega\}$. Note that $\ell(i^\omega) = i$, so this sequence essentially plays the role of 0 in Figure 1(c): this point is isolated, but $\vartriangle(\vec{x}, i^\omega) = 0$ for all $\vec{w} \in \vec{W}$, much like the corresponding quasimetric on $[0, 1]$ satisfies $\vartriangle(x, 0) = 0$. The relation $\circleddash$ helps model this: $r \circleddash i$ says that i is 'infinitely close' to r, and thus the distance between $\vec{x}$ and i^ω should be zero unless $x_n \not\circleddash i$ for some n, which in this example will never happen.

Contrast this to (b), where $\vec{W}$ is identical as a set of points, but has a different metric. For example, in the sequence $\vec{x} := i^3 \frown r^\omega$, we see that $x_3 \not\circleddash i$. Accordingly, $\vartriangle(\vec{x}, i^\omega) = 2^{-3}$; this is consistent with the standard metric on $[0, 1]$, where every point is at a positive distance from 0.

Note that the statement $w \circleddash v$ depends only on the cluster of w, i.e if $u \triangleq w$ and $w \circleddash v$, then $u \circleddash v$. Since $w \circleddash v$ implies $w \triangleq v$, we also obtain that if $w \circleddash v$, then $w \circleddash u$ if and only if $v \circleddash u$. We will use these properties freely below.

Our hemimetric ensures that copies of irreflexive points are isolated, given the following.

Lemma IX.4. *Let $\mathfrak{M} = (W, \rhd, V)$ be an abundant weakly transitive model and let n and $\vec{w}, \vec{v} \in \vec{W}$ be such that $w_k = v_k$ for all $k \geq n$. Then, if $\vec{w} \neq \vec{v}$, we have $\vartriangle(\vec{w}, \vec{v}) \geq 2^{-n}$.*

Proof. Since $\vec{w} \neq \vec{v}$, there must be $m < n$ such that $v_m \neq w_m$, and we may choose m maximal. We show that this leads to $\vartriangle(\vec{w}, \vec{v}) \geq 2^{-n}$. If $w_m \not\circleddash v_{m+\delta}$ for some $\delta \leq 1$, we are done, since then $\vartriangle(\vec{w}, \vec{v}) \geq 2^{-m-\delta} \geq 2^{-n}$. Otherwise, v_m and v_{m+1} are irreflexive and in the same cluster as w_m, so by superstability, $v_m = v_{m+1}$. Also by superstability, either $w_{m+1} \not\rhd w_m$, which yields $w_{m+1} \neq v_m = v_{m+1}$ (since $v_m \rhd w_m$), or else $w_{m+1} \rhd w_m$ and w_{m+1} is reflexive, again yielding $w_{m+1} \neq v_{m+1}$. In either case, this contradicts the maximality of m. □

Proposition IX.5. *If $\mathfrak{M} = (W, \rhd, V)$ is any abundant weakly transitive Kripke model, then $\vec{\mathfrak{M}}$ is a hemimetric model and $\ell\colon \vec{W} \to W$ is a d-morphism.*

Proof. We first check that $\vartriangle$ is a hemimetric on $\vec{w}$. Clearly $\vartriangle(\vec{w}, \vec{v}) \geq 0$ always and $\vartriangle(\vec{w}, \vec{w}) = 0$ by definition. For the triangle inequality, we trivially have $\vartriangle(\vec{u}, \vec{w}) \leq \vartriangle(\vec{u}, \vec{v}) + \vartriangle(\vec{v}, \vec{w})$ if $\vartriangle(\vec{u}, \vec{w}) = 0$, so we assume otherwise. Let $\vartriangle(\vec{u}, \vec{w}) = 2^{-n-\delta}$ with $u_n \neq w_n$ and $u_n \not\circleddash w_{n+\delta}$. Consider the following cases.

1) If $u_n = v_n$, then we also have $v_n \neq w_n$ and $v_n \not\circleddash w_{n+\delta}$. Thus $2^{-n-\delta} \in D(\vec{v}, \vec{w})$, and so $\vartriangle(\vec{v}, \vec{w}) \geq 2^{-n-\delta}$.
2) If $u_n \circleddash v_{n+\delta}$, then u_n and $v_{n+\delta}$ are in the same cluster, hence from $u_n \not\circleddash w_{n+\delta}$ we obtain $v_{n+\delta} \not\circleddash w_{n+\delta}$, as well as $v_{n+\delta} \neq w_{n+\delta}$. But setting $n' = n + \delta$ and $\delta' = 0$, we see that $v_{n'} \neq w_{n'}$ and $v_{n'} \not\circleddash w_{n'+\delta'}$, so $\vartriangle(\vec{v}, \vec{w}) \geq 2^{-n'-\delta'} = 2^{-n-\delta}$.

3) If $u_n \neq v_n$ and $u_n \not\circledcirc v_{n+\delta}$, we have $\Delta(\vec{u}, \vec{v}) \geq 2^{-n-\delta}$.

Next we show that ℓ is a d-morphism. Clearly it preserves atoms. For continuity, let $\vec{w} \in \vec{W}$ and let n be least so that $w_k = \ell(\vec{w})$ for all $k \geq n$. Let $\varepsilon = 2^{-n}$. Then, if $\vec{v} \neq \vec{w}$ and $\Delta(\vec{w}, \vec{v}) < \varepsilon$, we have that either $w_n = v_n$ or $w_n \circledcirc v_n$, which in either case yields $w_n \triangleq v_n$. Transitivity of $\trianglerighteq$ readily implies that $v_n \trianglerighteq v_k$ for all $k \geq n$. It follows that $\ell(\vec{w}) \trianglerighteq \ell(\vec{v})$. If $\ell(\vec{w}) \rhd \ell(\vec{v})$, we are done, so assume toward a contradiction that $\ell(\vec{w}) = \ell(\vec{v})$ and $\ell(\vec{w}) \not\rhd \ell(\vec{v})$. Then $\ell(\vec{w})$ is irreflexive.

Since $\vec{v}$ is superstable and $v_n \triangleq \ell(\vec{w})$, we claim that $v_k = \ell(\vec{w})$ for all $k \geq n$. Otherwise, if $v_k \neq \ell(\vec{w})$ for some $k \geq n$, we have that $\ell(\vec{w}) \rhd v_k$ by weak transitivity, and then also $v_{k+1} \neq \ell(\vec{w})$ since this would violate superstability. By induction on k we obtain that $v_k \neq \ell(\vec{w})$ for all large enough k, which contradicts $\ell(\vec{v}) = \ell(\vec{w})$. But then $\vec{v} \neq \vec{w}$ and $v_k = \ell(\vec{w}) = w_k$ for all $k \geq n$, so by Lemma IX.4 we obtain $\Delta(\vec{w}, \vec{v}) \geq \varepsilon$, a contradiction.

For openness, let $\vec{w} \in \vec{W}$, $\varepsilon > 0$, and $v \lhd \ell(\vec{w})$. Let n be large enough so that $2^{-n} < \varepsilon$, and also so that $w_n = \ell(\vec{w})$. First assume that either v is reflexive or $v \not\rhd w_n$. If $v \neq \ell(\vec{w})$, then define $\vec{v} := (w_0, \ldots, w_n, v, v, v, \ldots)$. This is readily checked to be a superstable path. Clearly $\vec{v} \neq \vec{w}$, $\ell(\vec{v}) = v$ and $\Delta(\vec{w}, \vec{v}) < \varepsilon$ because the two agree on the first n elements. If instead $v = \ell(\vec{w})$, then since $\ell(\vec{w}) \rhd v$, it follows that v is reflexive. Since $\mathfrak{M}$ is abundant, there is $v' \neq v$ which is also reflexive and in the same cluster. Define $\vec{v} := (w_0, \ldots, w_n, v', v, v, v, \ldots)$. As before, it is easy to check that $\vec{v}$ has the desired properties.

Finally, we consider the case where v is irreflexive and $v \rhd w_n$. Recall that $w_n = \ell(\vec{w}) \rhd v$, whence $v \triangleq w_n$. Let m be the least integer such that $w_m \triangleq w_n$, and define $\vec{v} := (w_0, \ldots, w_{m-1}, v, v, v, \ldots)$. For every k, either $v_k = w_k$, or else $k \geq m$ so that $w_k = \ell(\vec{w})$ and for all δ, $v_{k+\delta} = v$ and thus $w_k \circledcirc v_{k+\delta}$. It follows that $\Delta(\vec{w}, \vec{v}) = 0$. Further, we have $w_n \rhd v$ and v irreflexive, so $w_n \neq v$, whence $\vec{w} \neq \vec{v}$. There remains to check that $\vec{v}$ is superstable. In case that $m > 0$, if $i+1 < m$ then we already had that $v_i = w_i$ and $v_{i+1} = w_{i+1}$ are such that $v_i \trianglerighteq v_{i+1}$ and if $v_{i+1} \rhd v_i$ then v_{i+1} is reflexive. For $i = m-1$, we have $w_{m-1} \trianglerighteq w_m \triangleq w_n \rhd v$, whence $w_{m-1} \trianglerighteq v$. Further, if $v \rhd w_{m-1}$, then from $w_n \rhd v$ we obtain $w_n \trianglerighteq w_{m-1}$, and from $w_{m-1} \trianglerighteq w_m \triangleq w_n$ we obtain $w_{m-1} \trianglerighteq w_n$; this leads to $w_{m-1} \triangleq w_n$, in contradiction with the minimality of m. Therefore $v \not\rhd w_{m-1}$. Finally, for $i \geq m$ we have that $v_{i+1} \not\rhd v_i$ since both are equal to v, which is irreflexive. □

This already tells us that any satisfiable formula is satisfiable on a hemimetric space. However, our construction gives us more: if our Kripke model is T_0 or transitive, then our hemimetric model will be quasimetric or metric, repectively.

Proposition IX.6. *Let $\mathfrak{M} = (W, \rhd, V)$ be an abundant weakly transitive Kripke model and $\vec{\mathfrak{M}} = (\vec{W}, \Delta, \vec{V})$ be its associated hemimetric lifting.*

1) *If $\mathfrak{M}$ is T_0, then $(\vec{W}, \Delta)$ is a quasimetric space.*
2) *If $\mathfrak{M}$ is transitive, then $(\vec{W}, \Delta)$ is a metric space.*

Proof. For the first item, assume that $\mathfrak{M} = (W, \rhd, V)$ is a T_0 model, and that $\vec{w} \neq \vec{v}$; we show that $\Delta(\vec{w}, \vec{v}) + \Delta(\vec{v}, \vec{w}) > 0$. We may assume that $\Delta(\vec{w}, \vec{v}) = 0$. Let n be least such that $v_n = \ell(\vec{v})$. Suppose that there is $m \geq n$ such that $v_m \neq w_m$. Since $\Delta(\vec{w}, \vec{v}) = 0$ we must have that $w_m \circledcirc v_m$ and v_m is irreflexive, hence since $\mathfrak{M}$ is T_0, the point w_m is reflexive. But then $v_m \not\circledcirc w_m$ and $\Delta(\vec{v}, \vec{w}) \geq 2^{-m}$. Otherwise, $v_m = w_m$ for all $m \geq n$, so that by Lemma IX.4, $\Delta(\vec{v}, \vec{w}) \geq 2^{-n}$.

For the second item, suppose that $\mathfrak{M}$ is transitive. Then in particular $\mathfrak{M}$ is T_0, so Δ is a quasimetric, and it remains to prove that Δ is symmetric. Observe that the relation $\circledcirc$ simplifies to $w \circledcirc v$ iff $w = v$ and v is irreflexive. Thus, $w \neq v$ implies $w \not\circledcirc v$. As a result, the value of $\Delta(\vec{w}, \vec{v})$ is 2^{-n} if n is the least integer such that $w_n \neq v_n$, and 0 if no such integer exists. Therefore $\Delta(\vec{w}, \vec{v}) = \Delta(\vec{v}, \vec{w})$ for all $\vec{w}, \vec{v} \in \vec{W}$. □

Combining this with Theorem II.10 and Theorem VII.6, we obtain the following completeness results.

Theorem IX.7.

1) *The logics μ-$\mathsf{wK4}$ and $\mathsf{wK4}^\infty$ are sound and complete for the class of hemimetric spaces.*
2) *The logics μ-$\mathsf{wK4T_0}$ and $\mathsf{wK4T_0^\infty}$ are sound and complete for the class of quasimetric spaces.*
3) *The logics μ-$\mathsf{K4}$ and $\mathsf{K4}^\infty$ are sound and complete for the class of metric spaces.*

As before, we notice that $\mathsf{K4}^\infty$ can be simplified, since the hybrid tangle coincides with the tangled derivative in this context [13]. Similarly, our expressivity and inexpressivity results also apply to classes of hemimetric spaces.

Theorem IX.8. *Simulability is definable:*

1) *in $\mathcal{L}_{\Diamond^+_\infty}$ over the class of hemimetric closure spaces (i.e. closure spaces whose operator is the topological closure of some hemimetric space),*
2) *in $\mathcal{L}_{\Diamond_\infty}$ over the class of metric spaces,*
3) *in $\mathcal{L}_{\blacklozenge_\infty}$ over the class of hemimetric spaces.*

Proof. Follows from Theorem VIII.2. □

Theorem IX.9. *Simulability is not definable:*

1) *in $\mathcal{L}_{\Diamond^+_\infty}$ over the class of metric spaces,*
2) *in $\mathcal{L}_{\Diamond_\infty}$ over the class of quasimetric spaces,*
3) *in $\mathcal{L}_{\Diamond_\infty \Diamond^+_\infty}$ over the class of hemimetric spaces.*

Proof. We prove the first item only, as the others follow the same pattern. Let $(\mathfrak{M}, w)$ be a pointed reflexive singleton with all atoms false. Recall that $\omega + 1$ is the domain of the model $\mathfrak{S}^2$, where ω is the only reflexive point. Write $\mathfrak{S}^2 = (W, \rhd, V)$ and let $\mathfrak{S}^2_\circ = (W_\circ, \rhd_\circ, V_\circ)$ and $\pi : W_\circ \to W$ be as given by Definition IX.1. Note that $W_\circ$ has one copy of n for each $n < \omega$ and two copies of ω, both of which are reflexive. The model $\mathfrak{S}^2_\circ$ is transitive, hence its hemimetric lifting $\vec{\mathfrak{S}}^2_\circ$ is a metric model by Proposition IX.6. Moreover, $\pi\ell\colon \vec{W}_\circ \to W$ is a surjective d-morphism, hence a bisimulation. By Proposition IV.2, it follows that $(\vec{\mathfrak{S}}^2_\circ, \vec{\xi})$ simulates $(\mathfrak{M}, w)$ if and only if $(\mathfrak{S}^2, \pi\ell(\vec{\xi}))$ simulates $(\mathfrak{M}, w)$, i.e. if and only if $\pi\ell(\vec{\xi}) = \omega$. On the other hand, by Lemma VI.5, any formula φ of $\mathcal{L}_{\Diamond^+_\infty}$

that is true on $(\mathfrak{S}^2, \omega)$ is also true on $(\mathfrak{S}^2, n)$ for all $n \geq n_\varphi$. In particular, suppose that φ defines simulability of $(\mathfrak{M}, w)$ over metric spaces and $\vec{\omega}$, $\vec{n_\varphi}$ are chosen so that $\pi\ell(\vec{\omega}) = \omega$ and $\pi\ell(\vec{n_\varphi}) = n_\varphi$. Then $\vec{\mathfrak{S}}^2_\circ, \vec{\omega} \vDash \varphi$, and so $\vec{\mathfrak{S}}^2_\circ, \vec{n_\varphi} \vDash \varphi$, yet $\vec{\omega}$ simulates $(\mathfrak{M}, w)$ but $\vec{n_\varphi}$ does not. This proves the claim. $\square$

X. Conclusion

We have introduced semantics based on hemimetric spaces for wK4 and its extensions and shown that topological completeness results smoothly specialize to this setting. This grants the topological μ-calculus, and weakly transitive modal logics in general, a semantics which is more concrete and closer to many real-world applications. We have considered the logic of all hemimetric spaces, of all quasimetric spaces, and of all metric spaces. Note that the results of Baltag et al. [1] apply to many modal logics above wK4, namely the *canonical cofinal subframe logics,* many of which have natural topological semantics. This opens a line of research devoted to the classification of logics of classes of hemimetric spaces. The McKinsey-Tarski theorem applies to crowded spaces, a property which may readily apply to hemimetric spaces; in fact, our method should already yield completeness for classes of crowded hemimetric spaces. Metric completeness, on the other hand, is more involved, as there are various notions of completeness for hemimetric spaces [15], and it is an interesting question whether the notions of hemimetric completeness affect the underlying modal logic.

The notion of simulability is the natural notion of substructure in the context of modal logic, and readily extends to the general setting of derivative spaces. We have shown that standard tangle languages do not suffice to characterize finite structures up to simulability, contrary to the well-studied T_D case [13]. This stems from arguably unintended behavior in the setting of non-T_D spaces, and we have shown how it may be amended by modifying the definition of tangle operators to better extend to this setting. Our proposal thus provides an extension of tangled spatial logics that maintains many of their desirable properties for T_D spaces.

Logics over $\mathcal{L}_{\blacklozenge_\infty}$ inherit many valuable properties from the topological μ-calculus [1], like decidability of the validity problem and a natural axiomatization for many natural classes of spaces, including all topological spaces, T_0 spaces, and T_D spaces. Moreover, the facts that simulability is expressible and that the μ-calculus collapses to its alternation-free fragment are suggestive that $\mathcal{L}_{\blacklozenge_\infty}$ may be expressively complete with respect to the μ-calculus. We indeed conjecture this to be the case, but leave it for future work.

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