$$\geqslant \frac{\cos\alpha_{2l}}{l^2 \cdot 2^{\left(\frac{1}{2}-\frac{1}{p}\right)4l}} \left| \|\varphi_{4l+1}(x)\|_p \sin\alpha_{2l} - \|\varphi_{4l}(x)\|_p \cos\alpha_{2l} \right| \geqslant$$

$$\geqslant \frac{\cos\alpha_{2l}}{l^2 \cdot 2^{\left(\frac{1}{2}-\frac{1}{p}\right)4l}} \left[2^{\left(\frac{1}{2}-\frac{1}{p}\right)4s} \sin\alpha_{2l} - 2^{\left(\frac{1}{2}-\frac{1}{p}\right)4l} \cos\alpha_{2l} \right] = \frac{\cos\alpha_{2l} \cos\frac{\alpha_{2l}}{2}}{l^2 \cdot 2^{\left(\frac{1}{2}-\frac{1}{p}\right)4l}} \varepsilon_{2l} \cdot 2^{\left(\frac{1}{2}-\frac{1}{p}\right)4s_l} - \frac{\cos^2\alpha_{2l}}{l^2}. \tag{35}$$

By (27) this expression does not tend to zero and the series (33) does not converge in $L^p(0, 1)$ after the brackets are expanded. Thus, the system $\{\psi_\nu(x)\}$ is not a basis in L^p, $2 < p < \infty$. But since the system $\{\psi_\nu(x)\}$ is orthonormal, it is not a basis in the dual space $L^{p'}$, $1 < p' < 2$, either. This completes the proof of Theorem 2.

Thus, the answer to the question posed by P. L. Ul'yanov is negative.

LITERATURE CITED

1. P. L. Ul'yanov, "Divergent series with respect to a Haar system and with respect to bases," Dokl. Akad. Nauk SSSR, 138, No. 3, 556-559 (1961).
2. A. M. Olevskii, "Divergent series in L^2 with respect to complete systems," Dokl. Akad. Nauk SSSR, 138, No. 3, 545-548 (1961).
3. A. I. Rubinshtein, "Rearrangements of complete systems of convergence," Sib. Mat. Zh., 13, No. 2, 420-428 (1972).
4. S. Kaczmarz and H. Steinhaus, Theory of Orthogonal Series [in German], Chelsea Publ.
5. I. M. Gel'fand, Lectures on Linear Algebra [in Russian], Nauka, Moscow (1966).

PRESBURGERNESS OF PREDICATES REGULAR IN TWO NUMBER SYSTEMS

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In this paper we study the relations between classes of predicates regular in different number systems and we prove in detail the theorems from [1].

A number-theoretic predicate is said to be regular in a p-ary number system ($p \geq 2$) if it is recognized by a multitape finite automaton working with the p-ary notation for positive integers. We shall also say that such predicates are p-regular. A number-theoretic predicate is named Presburger if we express it in an elementary theory of the addition of positive integers. Büchi showed that for any p every Presburger predicate is p-regular and among p-regular predicates there are no Presburger predicates; for example, a one-place predicate "is of degree p" [2], Positive integers m and n are said to be multiplicatively dependent if $m^k = n^l$ is fulfilled for some positive integers k and l and multiplicatively independent otherwise. If m and n are multiplicatively dependent, then every m-regular predicate is n-regular (Büchi's result [2]).

In the present paper we prove that every predicate regular in two number systems with multiplicatively independent bases are Presburger.

Cobham [3] established the corresponding theorem for one-place predicates.

First of all we introduce some notation: N is the set of positive numbers $\{0, 1, \ldots\}$, Z is the set of integers, Q_+ is the set of rational nonnegative numbers, Q is the set of rational numbers.

For $k \geq 1$, by Q^k we denote the k-dimensional affine space over Q; the k-th Cartesian power of sets N, Z, and Q_+ are denoted N^k, Z^k, and Q^k_+. We take it that they lie in Q^k. Further, 0 is a vector all of whose components are zero, $N_k = N^k \setminus \{0\}$, $x_{(i)}$ is the i-th component of vector x, $\|x\|$ is the maximum of the absolute values of the components of x. When x, $y \in Q^k$, we write $x \geq y$ if $x - y \in Q^k_+$. For $R \subseteq Q$ and $H = \{h_1, \ldots, h_l\} \subseteq Q^k$

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by $R\langle H\rangle$ we denote the set of elements from Q^k representable in the form $\sum_{i=1}^{n} r_i h_i$, where $r_i \in R$. We define Ω as the set of points x from Q_+^k for which $\|x\| \leq 1$. We shall denote the elements of Ω by the letters ξ and η. For $a \in Q^k$ and $b \in Q_+$ the set $a + b\Omega$ is called a cube. For $a \in Q^k$ and $d \in Q_+^k \setminus \{0\}$ the set $a + Q_+ d$ is called a ray and the set $a + Nd$ is called an arithmetic progression. Let U, $V \subseteq Q^k$ and $r \in Q$; then the fact that $\|u - v\| \leq r$ is fulfilled for some $u \in U$ and $v \in V$ will be written as $\rho(U, V, r)$. For a, $b \in Q$ we set $[a, b] \rightleftharpoons \{t \in Q \mid a \leq t \leq b\}$ and $(a, b) \rightleftharpoons \{t \in Q \mid a < t < b\}$. In the formulas we shall often replace a one-element set by this element if this does not lead to ambiguity.

Predicates expressible in a first-order theory of addition of positive numbers are called Presburger predicates; k-place Presburger predicates specify subsets of N^k; such subsets are called polylinear sets. Every infinite polylinear set contains an arithmetic progression [4].

By $|x|$ we shall denote the length of word x and by λ, the empty word. Let $\Sigma = \{a_1, \ldots, a_p\}$ be an alphabet. A first-order theory with the following elementary predicates on the set of all words in alphabet Σ is called a theory of regular predicates over Σ: word x is the start of word y, $|x| = |y|$, word x ends with the letter a_i, $i = 1, \ldots, p$. Predicates expressible in this theory are called regular predicates over Σ. For every Σ the theory of regular predicates over Σ is solvable [5].

Let $p \geq 2$. We define D_p as an alphabet of p-ary digits $\{0, 1, \ldots, p-1\}$. By $(x)_p$ we denote the p-ary notation for the positive integer x. We shall take it that the first letter of the notation is nonzero and that the notation of zero is the empty word. If x is a word in alphabet D_p, by $[x]_p$ we denote the number whose notation is x with the initial zeros discarded.

With every predicate $p(x_1, \ldots, x_k)$, given on a set of words in alphabet D_p, there corresponds a predicate on the set of positive integers, true on the collection $a_1, \ldots, a_k$ if and only if P is true on the collection $(a_1)_p, \ldots, (a_k)_p$. If predicate P is regular over D_p, then the predicate thus obtained is said to be regular in the p-ary number system or to be a p-regular predicate. An equivalent definition of p-regular predicates can be given in terms of finite automata [2, 5].

A subset of N^k, specified by a p-regular predicate, will be called a p-regular set or simply a p-set. We take it that p-regular predicates and p-sets are given by corresponding formulas of the theory of regular predicates over D_p. A predicate is said to be m, n-regular if it is m-regular and n-regular. The concept of an m, n-set is defined analogously. By D_p^k we denote a alphabet consisting of k-element collections of p-ary digits and by $\mathbf{O}$, the collection consisting of zeros. Let x be a word in alphabet D_p^k. For $i = 1, \ldots, k$ we define the i-th component of x, denoted $x_{(i)}$, as a word in alphabet D_p, in which the j-th letter is the i-th element of the j-th letter in x. For $x \in N_k$ we define the p-ary notation of x, denoted $(x)_p$, as a word in alphabet D_p^k such that the first letter of $(x)_p$ is different from $\mathbf{O}$ and $[(x)_{p(i)}] = x_{(i)}$ for $i = 1, \ldots, k$. For example, let x be a vector with the components 12, 7, and 895. Then its Cartesian notation is the word (0, 0, 8)(1, 0, 9)(2, 7, 5). The p-ary notation of zero is the empty word. For a word x in alphabet D_p^k we define $[x]_p$ as a vector from N^k, whose notation is x with the initial letters $\mathbf{O}$ discarded.

We call the equivalence $\sim$ partitioning N_k into a finite number of p-sets a p-equivalence if for any vector x, $y \in N_k$ and any word u in alphabet D_p^k, from $x \sim y$ follows $[(x)_p u]_p \sim [(y)_p u]_p$.

Let $\mathfrak{M}$ be a p-set. Elements x, $y \in N_k$ are said to be p-equivalent relative to $\mathfrak{M}$ if and only if for every word u in alphabet D_p^k the membership of $([x]_p u)_p$ to set $\mathfrak{M}$ is equivalent to the membership of $([y]_p u)_p$ to set $\mathfrak{M}$. Using standard methods of working with regular predicates [2, 5], it is easy to show that p-equivalence relative to $\mathfrak{M}$ is p-equivalence.

In what follows we assume that m and n are fixed multiplicatively independent numbers, $m, n \geq 2$, $\mathfrak{A}$ is a fixed n-set, $\mathfrak{A} \subseteq N^k$. By Δ we denote the set of all words in alphabet D_n^k and we define Δ_+ as $\Delta \setminus \{\lambda\}$. We identify the elements of N^k with their n-ary notations and instead of $[x]_n$ we shall write $[x]$. Let $x, y \in \Delta$, where $xy \in N^k$ (i.e., the first digit of xy is different from $\mathbf{O}$ or xy is the empty word). Then, obviously, $xy = n^{|y|} x + [y]$. We shall call the n-equivalence relative to $\mathfrak{A}$ simply an equivalence and denote it $\sim$. Equivalence classes relative to $\mathfrak{A}$ will be simply called equivalence classes. When considering relations of equivalence other than $\sim$ we shall always indicate precisely which relation we have in mind.

In what follows we shall use the following notation connected with equivalence: $\sigma(x)$ is the equivalence class containing x and ν is the number of equivalence classes. The notation $x \succcurlyeq y$ signifies the existence of u from Δ such that $x \sim yu$. In order to establish that x and y possess such a property, it is sufficient to consider all u with a length less than ν. The notation $x \succ y$ signifies $x \succcurlyeq y$ and that $y \succcurlyeq x$ is false. Let s, s_1, and s_2 be equivalence classes and let $y \in N^k$. Then $s_1 \succcurlyeq s_2$ signifies that for some y_1 from s_1 and y_2 from s_2 there is fulfilled $y_1 \succcurlyeq y_2$; analogously, $s \succcurlyeq y$ signifies that for some z from s there is fulfilled $z \succcurlyeq y$.

The following lemmas are needed in the proof of the main result.

LEMMA 1. If $\mathfrak{A}$ is an m, n-set, then every equivalence class is an m-set.

Proof. Let $x = (x_1, \ldots, x_k)$, $P(x)$ be a predicate specifying $\mathfrak{A}$, and $P_1(x), \ldots, P_\nu(x)$ be predicates specifying different equivalent classes. By virtue of the properties of equivalence relative to $\mathfrak{A}$, for any unequal i and j in Δ we can find a word u such that

$$\forall x,\ y(P_i(x) \& P_j(y) \supset (P(xu) \not\equiv P(yu))).$$

We define the predicate $T_{i,j}(x)$ as $P(xu)$ if $\exists z(P_i(z) \& P(zu))$ is true and as $\neg P(xu)$ if $\exists z(P_i(z) \& \neg P(zu))$ is true. Then $P_i(x) \supset T_{i,j}(x)$ and $T_{i,j}(x) \supset \neg P_j(x)$. Consequently, $P_i(x) \equiv \bigwedge_{i \neq j} T_{i,j}(x)$. We now note that $P(xu)$ is equivalent to $\exists z(z = n^{|u|}x + [u] \& P(z))$. For a fixed u the predicate $z = n^{|u|}x + [u]$ is Presburger and by virtue of Büchi's results is an m-predicate. Since $P(z)$ is an m-predicate, $P(xu)$ is an m-predicate and $T_{i,j}(x)$ is an m-predicate. Consequently, $P_1(x), \ldots, P_\nu(x)$ are m-predicates. The lemma is proved.

Remark 1. As is easy to see, the proof of Lemma 1 is effective in the sense that from the formulas specifying the set $\mathfrak{A}$ in an n-ary and an m-ary number system, we can construct formulas specifying the equivalence classes. In all the subsequent arguments the effectiveness of the construction of the formulas, vectors, and numbers is just as easily verified and will not be mentioned.

The proof of the theorem will be carried out by induction over the number k (the dimension of the space N^k in which $\mathfrak{A}$ lies). For $k = 1$ Theorem 1 is Cobham's theorem [3]. In what follows we assume that Theorem 1 is true for dimensions of the space less than k. The next statement stems from this assumption.

LEMMA 2. Let $\mathfrak{A}$ be an m, n-set, s be an equivalence class, $G \subseteq Q^k$, card $G < k$, $a \in Q^k$, $\pi = a + Q\langle G \rangle$. Then the set $\pi \cap s$ is polylinear.

Proof. From the lemma's hypothesis it follows that the projection onto some coordinate hyperplane is one-to-one onto π. From Lemma 1 it follows that the image of $\pi \cap s$ under this projection is an m, n-set. The lemma's statement now is easily obtained from the inductive assumption.

LEMMA 3. For any $x \in N^k$ and $u \in \Delta_+$ the points $x, xu, xu^2, \ldots$ lie on one ray of the form $Q_+ d - \xi$, where $d \in Q_+^k$, $\xi \in \Omega$. If i is such that $d_{(i)} = 0$, then $x_{(i)} = u_{(i)} = \xi_{(i)} = 0$.

Proof. As a matter of fact, for $j = 0, 1, 2, \ldots$

$$xu^j = n^{j|u|}x + \sum_{l=0}^{j-1} n^{l|u|}[u] \in Q_+\left(x + \frac{[u]}{n^{|u|} - 1}\right) - \frac{[u]}{n^{|u|} - 1}.$$

The lemma is proved.

Let x and u satisfy the condition $xu \sim x$. The ray constructed in Lemma 3 for x and u is called a cyclic ray for x and u or a cyclic ray for x. Sometimes, for convenience of calculations, we shall call the part of a cyclic ray, started at point x, the cyclic ray. By virtue of Lemma 3 a cyclic ray contains an infinite set $\{x, xu, \ldots\}$ of points equivalent to x. If $\mathfrak{A}$ is an m, n-set, then by Lemma 2 it follows that this ray contains an arithmetic progression of the form $a + Nd$, a, $d \in N_k$, all terms of which are equivalent to x. Such a progression will be called a cyclic progression for x and u (or for x).

For $x \in N_k$, i, and $A \in N_1$ we define

$$\gamma_i(x, A) \rightleftharpoons n^i x + (n^i - n^{i-A} - 1)\Omega,$$

$$\Gamma(x, A) \rightleftharpoons \bigcup_{i=1}^{\infty} \gamma_i(x, A).$$

Some of the properties of the cube $\gamma_i(x, A)$, connected with the operation of attribution, are established in the next lemma.

LEMMA 4. All points of $\gamma_i(x, A) \cap N_k$ have the form xu, where $u \in \Delta$, $|u| = i$. For $j \in N$ the cube $\gamma_{i+j}(x, A)$ contains all points of the form yu, where $y \in \gamma_i(x, A) \cap N_k$, $u \in \Delta$, $|u| = j$.

Proof. The first statement follows directly from the definition. To prove the second we note that all points of form $y\delta$, where $y \in \gamma_i(x, A)$, $\delta \in D_n^k$, lie in $n\gamma_i(x, A) + (n-1)\Omega = n^{i+1}x + (n^{i+1} - n^{i+1-A} - n)\Omega + (n-1)\Omega = \gamma_{i+1}(x, A)$. The lemma is proved.

LEMMA 5. $\gamma_i(x, A) + n^{i-(A+1)}\Omega \subseteq \gamma_i(x, A+1)$.

Proof. Let $z \in \gamma_i(x, A) + n^{i-(A+1)}\Omega$. Then $z \geq n^i x$ and $\| z - n^i x \| \leq (n^i - n^{i-A} - 1) + n^{i-(A+1)} \leq n^i - n^{i-(A+1)} - 1$. The lemma is proved.

LEMMA 6. Every ray of the form $x + Q_+ d$, intersecting with some cube $\gamma_j(x, A)$, intersects with all cubes $\gamma_i(x, A)$ when $i \geq j$.

Proof. Let $(x + Q_+ d) \cap \gamma_i(x, A) \neq \emptyset$ be fulfilled for some i, i.e., $r \in Q_+$ exists such that $x + rd \in \gamma_i(x, A)$. It can be verified that $x + (n^{i+1} - 1)rd/(n^i - 1) \in x + Q_+ d \cap \gamma_{i+1}(x, A)$. The lemma's statement is easily obtained from this.

In the following arguments an essential role will be played by points satisfying a condition stronger than the existence of a cyclic ray for them. A point x is called nonsingular if $\Gamma(x, A) \cap \sigma(x) \neq \emptyset$ is fulfilled for some A. Points which are not nonsingular are called singular. It is obvious that x is a nonsingular point if and only if there exists $u \in \Delta_+$ such that $xu \sim x$ and $\| u \| < n^{|u|} - 1$. An equivalence class containing a nonsingular point is called a nonsingular class.

A positive integer b is said to be representative for a point x if for all $i \geq b$, $y \succcurlyeq x$ there is fulfilled $\gamma_i(x, b) \cap \sigma(y) \neq \emptyset$. An important property of singular points is established in the next lemma.

LEMMA 7. Let $\mathfrak{A}$ be an m, n-set and x be a nonsingular point. Then a representative number for x exists.

Proof. Since x is nonsingular, then $\gamma_{i_0}(x, A) \cap \sigma(x) \ni xu$ is fulfilled for some A, i_0, and u. Let $x + Q_+ d$ be a cyclic ray and $a + Nd$ be a cyclic progression for x and u. By Lemma 6, for every $i \geq i_0$ we can find $r_i \in Q_+$ such that $a + r_i d \in \gamma_i(x, A)$.

We choose $i_1 \geq i_0$ such that $r_i \geq 0$ and $[0, 1]d \subseteq n^{i-(A+1)}\Omega$ are fulfilled for $i \geq i_1$. By Lemma 5, when $i \geq i_1$ the cube $\gamma_i(x, A+1)$ contains the segment $a + [r_i, r_i + 1]d$ and, hence, intersects $a + Nd$. When $i \geq i_1$, by x_i we denote some point of $(a + Nd) \cap \gamma_i(x, A+1)$. Then $x_i \sim x$. We note further that for every $y \succcurlyeq x$ we can find a word v from Δ such that $|v| < \nu$ and $x_i v \sim y$. By Lemma 4, $x_i v \in \gamma_{i+|v|}(x, A+1)$. Hence it is clear that the maximum one of $i_1 + \nu$ and $A+1$ is representative for x.

LEMMA 8. Let $\mathfrak{A}$ be an m, n-set and x be a singular point. A $c_0 \in N$ exists such that for any $i \geq c_0$ and for any x' and x'' from N_k such that $x' \preccurlyeq x \preccurlyeq x''$, we can find v from Δ for which $|v| = i$ and $x'v \sim x''$.

Proof. As the desired c_0 we can take $b + \nu$, where b is a representative number for x, and we can make use of Lemma 7.

Let $L = a + Nd$ be some cyclic progression for point x. By D we denote the length of the m-ary notation for vector d and for each positive integer $q > |(a)_m|$ we pick out in L the subprogression $L(q) = a + (Nm^D + 1)m^q d$. We define f(q) as $\max\{t \in N \mid n^t \leq m^{q-1}\}$. We say that a point t from L(q) is suitable if $t = zv$, $z \in Nd$, is fulfilled for some z and v. We define R(q) as the number of different v such that $|v| = f(q)$ and the point zv is suitable for some z.

In the next lemma we establish that the n-ary notation for the elements of L(q) have sufficiently many different ends of length f(q). At the same time, we see subsequently, the m-ary notation for the elements of L(q) have like ends of length q. These two facts make possible the important construction in Lemma 10.

LEMMA 9. $\lim_{q \to \infty} R(q) = \infty$.

Proof. Let δ and q_1 be such that $0 < \delta < 1$ and the inequality $n^{f(q)} - 1 \geq \delta n^{f(q)} \| d \| + \| a \|$ holds when $q > q_1$. From now on we take it that $q > q_1$. Then for every positive integer p, satisfying the condition $p \leq \delta n^{f(q)}$, there is fulfilled $a + pd \in (n^{f(q)} - 1)\Omega$. Consequently, if for some h from N_1 the positive integer p satisfies the condition $hn^{f(q)} \leq p \leq (h+\delta)n^{f(q)}$, then $a + pd \in hn^{f(q)}d + (n^{f(q)} - 1)\Omega$. We set $\tilde{L}(q) = (Nm^D + 1)m^q$.

Let us represent $a + pd$ in the form zv, where $|v| = f(q)$. Then $z = hd \in Nd$. If, in addition, $p \in \tilde{L}(q)$, then $a + pd$ is a suitable point. We define r as the remainder from the division of p by $n^{f(q)}$. Then $v = rd + a$. Therefore, R(q) is not smaller than the number of different r such that $r \leq \delta n^{f(q)}$ and r is the remainder from the division of some element of $\tilde{L}(q)$ by $n^{f(q)}$.

By $\omega(q)$ we denote the greatest common divisor of the numbers $n^{f(q)}$ and m^{D+q}. Every positive integer from zero to $n^{f(q)}$, congruent to m^q modulo $\omega(q)$, is a remainder from the division of some element of $\widetilde{L}(q)$ by $n^{f(q)}$. Thus, to prove the lemma's statement it is sufficient to show that $\lim\limits_{q\to\infty} n^{f(q)}/\omega(q)=\infty$. This is easy to obtain by using the multiplicative independence of m and n and by considering the exponents with which the different primes occur in the expansion of the numbers $n^{f(q)}$ and m^{D+Q} into prime factors. The lemma is proved.

Let x be a point, c be a number, and g be be a vector. We assume that for an arbitrary $x'\succcurlyeq x$ and for any u_1 and u_2 from Δ, if $|u_1|=|u_2|\le c$ and $[u_2]=[u_1]+g$, we have $x'u_1\sim x'u_2$. Then the vector g is called a c-admissible shift for x.

LEMMA 10. Let $\mathfrak{A}$ be an m,n-set, x be a nonsingular point, and $x+Q_+d$ be a cyclic ray for x. For every c_1 from N there exists c_2 from N such that for any $c_3\ge c_2$ we can find g from N_k with the following properties:

a) $|g|<c_3-c_1$;

b) $g\in Q_+d$;

c) g is a c_3-admissible shift for x.

Proof. By Lemma 1 every equivalence class is an m-set. By $\approx$ we denote a relation which is the intersection of m-equivalences relative to all classes. It is easy to see that $\approx$ is an m-equivalence.

Let $L=a+Nd$ be a cyclic progression for x, lying on the ray $x+Q_+d$. Earlier we constructed the progression L(q) for every q greater than $|(a)_m|$. From the construction it follows that the m-ary notation for its elements have the form $(y)_m(d)_m O^{q-|(a)_m|}(a)_m$ for certain y from N_k.

We represent the n-ary notation for the elements of L(q) in the form zv, where $z\in N_k$, $v\in\Delta$, $|v|=f(q)$. By a_1 and a_2 we denote two terms of the progression and by y_α, z_α, v_α $(\alpha=1,2)$, the corresponding words in their notation. Let us prove that for some q_0, for all $q\ge q_0$ we can find a_1 and a_2 satisfying the conditions:

1) $z_1\sim z_2$,

2) $y_1\approx y_2$;

3) z_1, $z_2\in Nd$,

4) $v_1\ne v_2$ and $|[v_2]-[v_1]|<f(q)-(c_0+c_1+h)$, where c_0 is the constant from Lemma 8 for point x and h is the integral part of $\log_n m$.

Indeed, conditions 1 and 2 partition L(q) into a finite bounded constant independent of q and a number of subsets any pair of elements from each of which satisfies conditions 1 and 2. The cardinality of the set any two elements v_1 and v_2 of which do not satisfy condition 4 also is bounded by a constant independent of q. Condition 3 signifies that a_1 and a_2 are suitable points. By virtue of what we have said above and of Lemma 9 we can find q_0 such that when $q\ge q_0$ there exist a_1 and a_2 satisfying conditions 1-4.

We set $c_2\rightleftharpoons f(q_0)$. Since function f(q) is monotonic and increases unboundedly and since $f(q+1)-f(q)\le h+1$, for every $c_3\ge c_2$ we can find a number $q\ge q_0$ such that $c_3+c_0\le f(q)\le c_3+c_0+h$ is fulfilled with c_0 and h from the fourth condition.

For this q we select vectors a_1 and a_2 satisfying the four conditions written out above so that $v_1-v_2\notin N_k$. We set $g=v_2-v_1$. Let us prove that g possesses properties a)-c). From condition 4 it follows that $|g|<f(q)-(c_0+c_1+h)\le c_3-c_1$. Thus, property a) holds. Property b) follows from the fact that $g=v_2-v_1=a_2-a_1-n^{f(q)}(z_2-z_1)$, the vectors a_2-a_1 and z_2-z_1 belong to **Z**d, and $(-g)\notin N_k$.

Let us prove property c). Let u_1 and u_2 satisfy the conditions $[u_2]=[u_1]+g$, $|u_1|=|u_2|\le c_3\le f(q)-c_0$; $x'\succcurlyeq x$. Since point a_1 belongs to a cyclic progression for x, $z_1v_1\sim x$; therefore, $z_1\preccurlyeq x$ and by Lemma 8 a word w exists such that $|w|=f(q)-|u_1|$, $z_1w\sim x'$. Since $z_1\sim z_2$, $z_2w\sim x'$. Thus, to prove the equivalence of x'_1 and $x'u_2$ it is sufficient to verify that $z_1wu_1\sim z_2wu_2$. For this we find the form of the m-ary notation for $z_\alpha wu_\alpha$ $(\alpha=1,2)$. Since $wu_2-wu_1=g=v_2-v_1$, $wu_1-v_1=wu_2-v_2$. We set $r\rightleftharpoons wu_1-v_1$. Then $z_\alpha wu_1=z_\alpha v_\alpha+r$ $(\alpha=1,2)$. From the definition of f(q) it follows that $\|r\|<m^{q-1}$. At first let i be such that $d_{(i)}\ne 0$. Then $0\le r_{(i)}+a_{(i)}+m^q d_{(i)}<m^q(d_{(i)}+1)$. Hence it is clear that when r is added the digits standing to the left of the digits $(d)_m$ in the i-th component of the m-ary notation for $z_\alpha v_\alpha$, i.e., for the word $(y)_m(d)_m O^{q-|(a)_m|}(a)_m$, do not change. However, if $d_{(i)}=0$, then by Lemma 3, $(z_\alpha v_\alpha)_{(i)}=0$ and $(z_\alpha wu_\alpha)_{(i)}=r_{(i)}$. Thus, in both cases the m-ary notation for $(z_\alpha wu_\alpha)_{(i)}$ has the form $((y_\alpha)_m)_{(i)}t_{(i)}$, where t is independent of α. By virtue of condition 2, $y_1\approx y_2$.

Consequently, $[(y_1)mt]_m \approx [(y_2)mt]_m$, and, in particular, $[(y_1)mt]_m$ and $[(y_2)mt]_m$ lie in one n-equivalence class relative to $\mathfrak{A}$. From the definition on n-equivalence $z_1wu_1 \sim z_2wu_2$. Property c), and hence, the whole lemma, are proved.

For $x \in \mathbf{N}_k$ and $i \in \mathbf{N}$ we set

$$\gamma_i(x) \rightleftharpoons n^i y + (n^i - 1)\Omega, \ \Gamma(x) = \bigcup_{j=0}^{\infty} \gamma_j(x).$$

It is easy to see that the cube $\gamma_i(x)$ contains those and only those points of $\mathbf{N}_k$ which have the form yu, where $u \in \Delta$ and $|u| = i$. It is also obvious that $\gamma_i(x, A) \subseteq \gamma_i(x)$ is fulfilled for any i and A.

Let $V = \{d_1, \ldots, d_\varkappa\}$ be a linearly independent system of vectors. We call V the cyclic base for point x if $x + \mathbf{Q}_+ d_1, \ldots, x + \mathbf{Q}_+ d_\varkappa$ are cyclic rays for x and $\sigma(x) \cap \Gamma(x) \subseteq x + \mathbf{Q}\langle V\rangle$.

LEMMA 11. A cyclic base exists for every point.

Proof. Obviously, from a point x and vectors $d_1, \ldots, d_j$ we can establish that the set $(\Gamma(x) \cap s) \setminus (x + \mathbf{Q}\langle\{d_1, \ldots, d_j\}\rangle)$ is empty, or we can find a point in this set. We can draw a cyclic ray through point x and the point found. We obtain the required set by starting from the empty set of cyclic rays. The lemma is proved.

Lemma 12 following is rather trivial geometrically and does not use the properties of n-sets. In Lemmas 13-17, x is a nonsingular point, b is a representative number for x, and G is a cyclic base of c-admissible shifts for x with $c = 2b + |1\langle G\rangle| + k + 1$.

Let us consider the case when $\mathfrak{A}$ is an m, n-set. The existence of the representative number b for x follows from Lemma 7. Let us prove the existence of a cyclic base with the property indicated. By Lemma 11 we construct the cyclic base $\{d_1, \ldots, d_\varkappa\}$ for x. By Lemma 10 we find c_2 for c_1 equal to $2b + 2k + 1$ for each of the rays $x + \mathbf{Q}_+ d_i$, $i = 1, \ldots, \varkappa$. We take c_3 equal to the maximum of the numbers c_2 found and, following Lemma 10, we construct the vectors $g_1, \ldots, g_\varkappa$. We set $G \rightleftharpoons \{g_1, \ldots, g_\varkappa\}$, $c \rightleftharpoons 2b + |1\langle G\rangle| + k + 1$. Then G is the cyclic base of c-admissible vectors for x

For $z_1, z_2 \in \mathbf{Z}^k$ we write $z_1 \equiv z_2 (\mathrm{mod}\, G)$ if $z_1 - z_2 \in \mathbf{Z}\langle G\rangle$. By $\mathbf{1}$ we denote the vector all of whose components equal unity.

LEMMA 12. Let R be a convex of $\mathbf{Q}^k$, H be a linearly independent subset of $\mathbf{Q}^k$, and the points z_1 and z_2 be such that $z_\alpha + [0, 2]\langle H\rangle \subseteq R$ ($\alpha = 1, 2$). Then there exists the sequence $y_0 = z_1, y_1, \ldots, y_p = z_2$ such that for every $j = 0, 1, \ldots, p-1$ there is fulfilled $y_j \in R$ and $y_{j+1} = y_j + \delta_j h_j$ for some $\delta_j \in \{-1, 1\}$, $h_j \in H$.

Proof. It is obvious that it is sufficient to indicate a sequence with the required properties for the points $z'_\alpha = z_\alpha + 1\langle H\rangle$ ($\alpha = 1, 2$) for which $z'_\alpha + [-1, 1]\langle H\rangle \subseteq R$. Let $\pi = z'_1 + \mathbf{Q}\langle H\rangle$ and $\bar{R} = R \cap \pi$; then $\bar{R}$ is convex and $z'_\alpha + [-1, 1]\langle H\rangle \subseteq \bar{R}$. Thus, in the proof of the lemma we can restrict ourselves to the case when the number of elements of H equals the dimension of space $\mathbf{Q}^k$. Obviously we can take it that H is the base of the coordinate system and $z'_1 = 0$. We denote z'_2 by z. By a change of sign in certain elements of base H we can achieve that $z \in \mathbf{N}^k$. Let $t_0 = 0$ and let $\{t_1, \ldots, t_q\}$ be a set of all points from $[0, 1]z$ at least one coordinate of which is a positive integer and let these points be ordered such that $0 \le t_{j+1} - t_j \le 1$ $(0 \le j \le q-1)$. We define y_j as a point whose coordinates are the integral parts of the coordinates of t_j. Then $0 \le t_j - y_j \le 1$ and $-1 \le (t_j - y_{j+1}) \le 1$. We fix some value $j < q$. Let $i_1, \ldots, i_r$ be the numbers of the coordinates at which the vector y_{j+1} differs from vector y_j. It is clear that $(y_{j+1})_{(i_l)} + (y_j)_{(i_l)} + 1$ for $l = 0, \ldots, r-1$. We define $y_j^{(l)}$, $l = 0, \ldots, r$, such that $(y_j^{(0)} = y_j, y_j^{(l+1)} = y_j^{(l)} + h_{i_l}$ (h_{i_l} is the base vector of the i_l-th coordinate). We make this construction for all j from 0 to $q-1$. To prove that $y_0^{(0)}, y_0^{(1)}, \ldots, y_1^{(0)}, \ldots, y_{q-1}^{(0)}, \ldots, y_q$ form the needed sequence, it remains to verify that $y_j^{(l)} \in R$ for any j and l. By construction $\langle\Omega\rangle y_j^{(l)} = t_j + \tau$ for some τ from $[-1, 1]$; $t_j = \varepsilon z$ for some ε from $[0, 1]$. Consequently, $y_j^{(l)} = (1-\varepsilon)\tau + \varepsilon(z+\tau) \subseteq (1-\varepsilon)[-1, 1]\langle\Omega\rangle + \varepsilon(z + [-1, 1]\langle\Omega\rangle)$. From the lemma's hypothesis and from the convexity of R we obtain $y_j^{(l)} \in R$. The lemma is proved.

Let $H \subseteq \mathbf{N}_k$; a set $U \subseteq \mathbf{Q}^k$ is said to be H-invariant if $x_1 \sim x_2$ follows from $x_1 \equiv x_2 (\mathrm{mod}\, H)$ for any $x_1, x_2 \in U \cap \mathbf{N}_k$.

The further proof consists, roughly speaking, in obtaining all the "large" G-invariant sets. An important step in this direction is Lemma 17. Lemmas 13-16 are preparatory for it.

LEMMA 13. The set $\gamma_c(x, b)$ is G-invariant.

Proof. Since $c > b + |1\langle G\rangle| + 2$ and $[0, 2]\langle G\rangle \subseteq n^{r+1}\Omega$, by Lemma 5, $\gamma_c(x, b) + [0, 2]\langle G\rangle \subseteq \gamma_c(x)$. Let $x_1, x_2 \in \gamma_c(x, b)$, $x_1 \equiv x_2 (\mathrm{mod}\, G)$. Then by Lemma 12 we can construct a sequence $y_0, \ldots, y_p \in \gamma_c(x)$ such that

$y_0 = x_1$, $y_p = x_2$ and that the difference $y_{j+1} - y_j$ belongs to $G \cup (-G)$ for $j = 0, 1, \ldots, p-1$. Since every element of G is a c-admissible shift for x, $y_{j+1} \sim y_j$. Hence $x_1 \sim x_2$. The lemma is proved.

LEMMA 14. Let $y \in \Gamma(x)$, $i \leq b + |1\langle G\rangle| + k + 1$. Then the set $\gamma_i(y)$ is G-invariant.

Proof. Since $c - i \geq b$, by Lemma 7 we can find y' such that $y' \sim y$, $y' \in \gamma_{c-i}(x, b)$. By Lemma 4, $\gamma_i(y') \subseteq \gamma_c(x, b)$; consequently, by Lemma 13, $\gamma_i(y')$ is G-invariant. The lemma is proved.

LEMMA 15. Let H be a linearly independent subset of N_k, $q = |1\langle H\rangle|$, $a_0 \in N_k$. Then a, $y \in N_k$ exist such that $(a_0 + n^k[0, 1]\langle H\rangle) \cap \gamma_{q+k}(y) \supseteq a + [0, 1]\langle H\rangle$.

Proof. We set $z_j = a_0 + j\langle H\rangle$ $(j = 0, 1, \ldots, n^k)$. We define $w \in N^k$ such that $n^{q+k}(w - 1) \leq z_j \leq n^{q+k}(w+1)$ is fulfilled for $j = 0, 1, \ldots, n^k$, i.e., for any i and j there holds one of the two inequalities $n^{q+k}(w_{(i)} - 1) \leq (z_j)_{(i)} < n^{q+k}w_{(i)}$ or $n^{q+k}w_{(i)} \leq (z_j)_{(i)} < n^{q+k}(w_{(i)} + 1)$. For each j we define $y_j \in N_k$ such that $z_j \in \gamma_{q+k}(y_j)$. Then for any i and j there is fulfilled $(y_j)_{(i)} = w_{(i)} - 1$ or $(y_j)_{(i)} = w_{(i)}$. Consequently, when $j = 0, 1, \ldots, n^k$ the point y_j takes no more than 2^k different values and, since the number of values of j, equal to $n^k + 1$ exceeds 2^k, we can find j_1 and j_2 such that $j_1 < j_2$ and $y_{j_1} = y_{j_2}$. We set $a = z_{j_1}$, $y = y_{j_1}$. Then the points a and $a + (j_2 - j_1)\langle H\rangle$ lie in $\gamma_{q+k}(y)$ and, since $H \subseteq Q_+^k$, $a + [0, 1]\langle H\rangle \subseteq \gamma_{q+k}(y)$. The lemma is proved.

LEMMA 16. Let G_1 and G_2 be linearly independent subsets of N_k, where card G_1 = card G_2 and $Z\langle G_1\rangle \subseteq Z\langle G_2\rangle$. Further, let $a \in Q_+^k$, $T = a + [0, 1]\langle G_1\rangle$, $U \supseteq T$. If T is G_2-invariant and U is G_1-invariant, then U is G_2-invariant.

Proof. From an analysis of the coordinate system with origin at point a and with a base which is the complement of G_1 with respect to the base of Q^k, it follows that for any points $z_\alpha \in U$, $\alpha = 1, 2$, we can find points $z'_\alpha \in T$ satisfying the conditions $z'_\alpha \equiv z_\alpha (\mathrm{mod}\, G_1)$. Since U is G_1-invariant, $z'_\alpha \sim z_\alpha$; since $Z\langle G_1\rangle \subseteq Z\langle G_2\rangle$, $z'_\alpha \equiv z_\alpha (\mathrm{mod}\, G_2)$. Now let $z_1 \equiv z_2 (\mathrm{mod}\, G_2)$; then $z'_1 \equiv z'_2 (\mathrm{mod}\, G_2)$. By virtue of the G_2-invariance of T we have $z'_1 \sim z'_2$. Consequently, $z_1 \sim z_2$. The lemma is proved.

Let s be a nonsingular class. A linearly independent set H is called a system of periods for s if:

1) for any y such that $y \succcurlyeq s$ and for any i the set $\gamma_i(y)$ is H-invariant;

2) $s \subseteq y + Q\langle H\rangle$ is fulfilled for any $y \in s$.

LEMMA 17. G is a system of periods for $\sigma(x)$.

Proof. Let us show that the first property of a system of periods is fulfilled for G. Since for any $y \in N^k$, $u_1, u_2 \in \Delta$, $|u_1| = |u_2|$ the congruence $yu_1 \equiv yu_2 (\mathrm{mod}\, G)$ is equivalent to the congruence $[u_1] \equiv [u_2] (\mathrm{mod}\, G)$, from the G-invariance of $\gamma_i(x)$ for all i follows the G-invariance of $\gamma_i(x')$ for all $x' \succcurlyeq x$ and all i. We shall prove the G-invariance of $\gamma_i(x)$ by induction over i. For $i \leq b + |1\langle G\rangle| + 1$ the G-invariance of $\gamma_i(x)$ has been established in Lemma 14. Now let $r = |1\langle G\rangle|$, $i > b + r + k + 1$. Since b is a representative number for x, the set $\gamma_b(x, b) \cap \sigma(x)$ is not empty. Let y lie in it. Then $\gamma_i(y) \subseteq \gamma_{b+i}(x, b)$. Let us consider an arbitrary nonempty set V of the form $\gamma_i(y) \cap (a_0 + Q\langle G\rangle) \cap N_k$, $a_0 \in N_k$. By Lemma 5, U, equal to $V + n^{b+k}[0, 1]\langle G\rangle$, is contained in $\gamma_{b+i}(x)$. By the inductive assumption the set $\gamma_{i-1}(x)$ is G-invariant. Hence there easily follows the $n^{b+1}G$-invariance of set U lying in $\gamma_{b+i}(x)$. By Lemma 15 there exist a and $\bar{y}$ such that $T = a + [0, 1]\langle n^{b+1}G\rangle \subseteq U \cap \gamma_{r+b+k+1}(\bar{y})$. Since $U \subseteq \Gamma(x)$, $\bar{y} \in \Gamma(x)$ and T is G-invariant by Lemma 14. Hence, U is G-invariant by Lemma 16. Since V is a subset of U and is chosen arbitrarily in $\gamma_i(y)$, the G-invariance of $\gamma_i(y)$ and $\gamma_i(x)$ has been proved.

Let us prove that the second property from the definition of a system of periods is fulfilled for G. For this it is enough to verify that $\sigma(x) \subseteq x + Q\langle G\rangle$. Since G is a cyclic base for x, $\Gamma(x) \cap \sigma(x) \subseteq x + Q\langle G\rangle$. We set $i = b + r + 1$. In the cube $\gamma_i(x, b)$ we select a point t such that $t \sim x$. Then by Lemma 5 the set $U = t + (G \cup \{0\})$ lies in $\gamma_i(x, b+1)$ and by the proved first property G consists of points equivalent to x. We denote card G by $\varkappa$. By virtue of the linear independence of G the set U is contained in a single $\varkappa$-dimensional space Q^k, viz., in $x + Q\langle G\rangle$. Let $y \in \sigma(x)$; we consider the set $U' = U - n^i x + n^i y$. Since $\gamma_i(y, b+1) = \gamma_i(x, b+1) - n^i x + n^i y$, U' is contained in $\gamma_i(y, b+1)$ and consists of points belonging to $\sigma(x)$. These points are contained in a single $\varkappa$-dimensional subspace, viz., in $n^i(y) + (n^i - 1)x + Q\langle G\rangle$. If y does not lie in this subspace, then we can construct $\varkappa + 1$ cyclic rays passing through y and the points of U' and containing $\varkappa + 2$ points $yu_1, \ldots, yu_{\varkappa+2}$ of $\sigma(y)$, lying in some cube $\gamma_j(y)$ and not lying in any $\varkappa$-dimensional subspace. Having considered the points $xu_1, \ldots, xu_{\varkappa+2}$, we obtain a contradiction with the fact that $\Gamma(x) \cap \sigma(x)$ is contained in some $\varkappa$-dimensional subspace. However, if $y \in n^i y - (n^i - 1)x + Q\langle G\rangle$, then $(n^i - 1)y - (n^i - 1)x \in Q\langle G\rangle$ and $y \in x + Q\langle G\rangle$. The lemma is proved.

The G-invariance of any infinite set of points does not follow directly from Lemma 17. Points congruent modulo G but lying in $\gamma_i(x)$ and $\gamma_j(x)$ for $i \neq j$ or in $\gamma_i(x)$ and $\gamma_j(y)$ can, in general, belong to different classes.

Our immediate problem is to establish in Lemma 23 the semilinear conditions under which congruence modulo G implies equivalence.

In what follows we shall use another type of geometric objects, viz., (polyhedral) cones "almost described" around $\Gamma(x)$ and $\Gamma(x, A)$. For $x \in N_k$, $A \in N_1$ we set

$$\Theta(x) \rightleftharpoons Q_+(x+\Omega),$$
$$\Theta(x, A) \rightleftharpoons Q_+(x+(1-n^{-A})\Omega).$$

Obviously, $\Theta(x) \supseteq \Gamma(x)$, $\Theta(x, A) \supseteq \Gamma(x, A)$.

In Lemmas 18-20 we investigate the geometric properties of cones $\Theta(x)$ and $\Theta(x, A)$. Lemma 18 is used frequently in what follows in the proof of the fact that the shift of some point y from $\Theta(x, A)$ yields a point from $\Theta(x, A+1)$. From this point of view this lemma is analogous to Lemma 5.

LEMMA 18. Let $x \in N_k$, $t \in Q_+^k$, $l \in N$, $\|t\| \le n^l$, y be a point of cone $\Theta(x, A)$ such that $\|y\| \ge n^{2|x|+A+l+2}$, $y+t \in Q_+^k$. Then $y+t \in \Theta(x, A+1)$.

Proof. By hypothesis $y = r(x+(1-n^{-A})\xi)$ is fulfilled for some r from Q_+ and ξ from Ω. The lemma's statement is that for every $t \in Q^k$, from $\|t\| \le n^l$, $y+t \in Q_+^k$ follows $y+t \in \Theta(x, A+1)$. We set $q \rightleftharpoons n^l$, $w \rightleftharpoons (y+t)/(r-q) - x$. Let us show that $w \in (1-n^{-(A+1)})\Omega$. The lemma's statement will follow hence.

At first we verify that $\|w\| \le 1 - n^{-(A+1)}$. Indeed, since $\|y\| \ge n^{2|x|+A+l+2}$, $r \ge \|y\|/(\|x\|+1) \ge n^{2|x|+A+l+2} n^{-|x|} = n^{|x|+A+l+2}$. Hence $r - q > {}^3/_4 r$ and $q/(r-q) < ({}^4/_3)n^{-(|x|+A+2)}$. Let us represent w in the form $w = (q/r-q) \cdot (x+(1-n^{-A})\xi + t/q) + (1-n^{-A})\xi$. We note that $\|x+(1-n^{-A})\xi + t/q\| < \|x\|+2 \le ({}^3/_2)n^{|x|}$. Thus, $\|w\| < ({}^4/_3)n^{-(|x|+A+l+2)}({}^3/_2)n^{|x|} + 1 - n^{-A} \le n^{-(A+1)} + 1 - n^{-A} \le 1 - n^{-(A+1)}$.

We now note that $w_{(i)} \ge 0$ for all i. For i such that $x_{(i)} = 0$, this follows from the condition $w_{(i)} = ((y+t)/(r-q))_{(i)}$, for other i, from $x_{(i)} \ge 1 \ge \|t/q\|$. The lemma is proved.

LEMMA 19. Let x and y be points of N_k such that $|y| > 2|x|+A+2$, $\Theta(x, A) \cap \Theta(y) \ne \emptyset$. Then $\Theta(y) \subseteq \Theta(x, A+1)$ and $\Gamma(y) \subseteq \Theta(x, A+1)$.

Proof. From the definition of $\Theta(x, A)$ and $\Theta(y)$ and from the lemma's hypothesis it follows that $\Theta(x, A) \cap \Theta(y)$ contains some straight line passing through zero. On this straight line we choose the point $y' \in y+\Omega$. Since $\|y'\| \ge \|y\| \ge n^{2|x|+A+2}$, y' satisfies the hypothesis of Lemma 18 with $l = 0$. Any point from $y+\Omega$ has the form $y'+t$, where $\|t\| \le 1$. Therefore, $y+\Omega \subseteq \Theta(x, A+1)$ and, consequently, $\Theta(y) \subseteq \Theta(x, A+1)$, $\Gamma(y) \subseteq \Theta(x, A+1)$. The lemma is proved.

From the next lemma it follows that an arithmetic progression having an infinite intersection with cone $\Theta(x, A)$ intersects all cubes $\gamma_i(x, A+1)$, beginning with a certain one. Thus, the cubes $\gamma_i(x, A+1)$ sort of "partition off" this cone.

LEMMA 20. Let a, $d \in N_k$ and $(a+Q_+d) \cap \Theta(x, A)$ be an unbounded set. Then we can find b such that $(a+Nd) \cap \gamma_i(x, A+1) \ne \emptyset$ is fulfilled for each $i \ge b$.

Proof. We select b so as to fulfill the inequality $n^{b-(A+1)} \ge \|a\|+\|d\|+1$. Let $i \ge b$. By the lemma's hypothesis we can find r and q from Q_+ such that $q \ge n^i$ and $a+rd \in q(x+(1-n^{-A})\Omega)$. Then $n^i(a+rd)/q \in n^i(x+(1-n^{-A})\Omega)$. Hence, from the definition of $\gamma_i(x, A)$ and from the fact that $0 \le 1 - n^i/q \le 1$, we obtain $n^i(a+rd)/q - n^i a/q + a + [0, 1]d \subseteq \gamma_i(x, A) + (\|a\|+\|d\|+1)\Omega$. By Lemma 5 the last set is contained in $\gamma_i(x, A+1)$. Finally, the set $n^i rd/q + a + [0, 1]d$ intersects $a+Nd$. The lemma is proved.

In Lemma 21 we prove that the structure of the cones in Q_+^k is in some sense compatible with the structure of the equivalence classes. This yields the possibility of proving in succeeding lemmas the G-invariance of certain semilinear subsets in the cones.

We shall say that a point $y \in N_k$ is cyclic if $yu \sim y$ is fulfilled for some $u \in \Delta_+$.

LEMMA 21. Let $\mathfrak{A}$ be an m,n-set, $A \in N$, y be a cyclic point such that $|y| > 2|x|+A+2$, $\Theta(y) \cap \Theta(x, A) \ne \emptyset$. Then $y \succcurlyeq x$. If, in addition, x is singular, then $y \succ x$.

Proof. By the definition of a cyclic point, for y there exists a cyclic ray having an unbounded intersection with $\Gamma(y)$ and, consequently, with $\Theta(y)$. By Lemma 19 it has an unbounded intersection with $\Theta(x, A+1)$ and by Lemma 20 the cyclic progression for y, lying on this ray, intersects some $\gamma_i(x, A+2)$. Hence right away we obtain the lemma's statements.

In Lemmas 22-27 we fix some nonsingular class s, the system of periods G for this class, and a point x from s. For brevity we shall sometimes drop the x in the symbols $\Gamma(x)$, $\Gamma(x, A)$, $\gamma_i(x)$, $\gamma_i(x, A)$, $\Theta(x)$, $\Theta(x, A)$, i.e., we shall write $\Gamma(A)$ instead of $\Gamma(x, A)$, etc. A coset of the additive group $\mathbf{Z}^k$ by the additive subgroup $\mathbf{Z}\langle G\rangle$ is called a G-net. By $\Phi(A)$ we denote the intersection of a G-net Φ with $\Theta(x, A)$. We say that a G-net Φ is initial if $\rho(\Phi, Q\langle G\rangle, 2)$. It is easy to see that there is a finite number of initial G-nets. Obviously, every G-net is initial when card $G = k$.

By $T|_B$ we shall denote the set $\{t \subseteq T \mid \|t\| \geq n^B\}$ when $T \in Q^k$ and $B \in N$.

LEMMA 22. Let $\mathfrak{A}$ be an m, n-set. For any A there exists B such that for every initial G-net Φ, if $\Phi(A)|_B \neq \emptyset$, the set $\Phi(A+2)|_B$ lies in some equivalence class, while the set $\Phi(A)$ intersects $\gamma_i(x, A+1)$ with $i \geq B$.

Proof. Since there is a finite number of initial G-nets, there exists B_0 such that for every initial G-net Φ the infiniteness of $\Phi(A)$ follows from the nonemptiness of $\Phi(A)|B_0$. Let $\Phi(A)|_{B_0} \neq \emptyset$. Then $\Phi(A)|B_0$ contains some arithmetic progression. By Lemma 20 there exists $B_1 \geq B_0$ such that this progression intersects all $\gamma_i(A+1)$ for $i \geq B_1$. Thus, if $\Phi(A)|_{B_1} \neq \emptyset$, then $\Phi(A)$ intersects $\gamma_i(A+1)$ for $i \geq B_1$. We consider further two cases: card $G < k$ and card $G = k$.

In the first case, for $B \geq B_1$ and for a nonempty $\Phi(A)|_B$ we can, with the aid of Lemma 2, represent $\Phi(A+2)$ in the form $\Phi(A+2) = F \cup \left(\bigcup_{\alpha=1}^{M} L_\alpha\right)$, where F is finite and the L_α are infinite semilinear sets each of which is contained in some equivalence class. Using Lemma 20 we can construct j such that all the L_α intersect with γ_j. Since G is a system of periods for s, all points of $\Phi \cap \gamma_j$ belong to one equivalence class. Since $L_\alpha \subseteq \Phi$, all sets L_α are contained in one and the same equivalence class. We select $B \geq B_1$ such that for every initial G-net Φ the set F constructed for Φ would lie in $n^B\Omega$. Then $\Phi(A+2)|_B \subseteq \bigcup_{\alpha=1}^{M} L_\alpha$ and, consequently, $\Phi(A+2)|_B$ lies in some equivalence class. The lemma has been proved in the first case.

Now let card $G = k$. We note that for every D, in every point of $N_k|_{D+\nu}$ there is a beginning which is a cyclic point of length greater than D and less than $D+\nu$. We set D equal to $2|x|+A+4$; then $\Phi(A+2)|_{D+\nu}$ is contained in a finite union of $\Gamma(y)$ over such cyclic y that $D < |y| < D+\nu$ and $\Gamma(y) \cap \Phi(A+2)$ is nonempty. We fix one of these y. From the last condition and the restruction on the length of y, by Lemma 19 it follows that $\Theta(y) \subseteq \Theta(x, A+3)$. We construct an arbitrary point $\in \Gamma(y) \cap \Phi(A+2)$ and through this point we pass a ray lying in $\Theta(y, |z|)$. Since card $G=k$ and $z \in \Phi$, this ray has an infinite semilinear intersection with Φ. Using Lemma 2, in this intersection we pick out an arithmetic progression lying in some equivalence class and we denote it $L(y)$. Since $L(y) \subseteq \Theta(x, A+3)$, by Lemma 20 there exists b such that $L(y)$ intersects γ_i for $i \geq b$. By Lemma 21, by virtue of the cyclicity of y, we have $y \succcurlyeq x$. Since G is a system of periods for s and $L(y) \subseteq \Phi$, when $i \geq b$ the set $\gamma_i(y) \cap \Phi$, as also $\gamma_i \cap \Phi$, is contained in the class to which all elements of $L(y)$ belong. This class depends only on x and Φ, but not on y. Hence, all points of $\Phi(A+2)|_{|y|+b}$ belong to one class.

The choice of the needed value of B is made subsequently analogously to the way it was done in the analysis of the first case. The lemma is proved.

When card $G < k$ the number of G-nets intersecting $\Theta(x, A)$ can be infinite and Lemma 22 does not enable us to indicate a neighborhood of zero outside which the cone $\Theta(x, A)$ is G-invariant. However, Lemma 22 can be used for proving in Lemma 23 the G-invariance of a sufficiently large subset of $\Theta(x, A)$.

LEMMA 23. Let $\mathfrak{A}$ be an m, n-set. For every $A \in N$ we can find a number B, greater than $|x|+A+4$, such that for any G-net Φ and any $R \in N$ from $\rho(\Phi, Q\langle G\rangle, n^R)$, $\Phi(A)|_{R+B} \neq \emptyset$ it follows that $\Phi(A+2)|_{R+B}$ lies in some equivalence class and intersects $\gamma_i(x, A+2)$ for $i \geq R+B$.

Proof. In Lemma 22 let us substitute the numbers $A+1$ and $A+3$ in the place of the number A. The values obtained for B we denote B_1 and B_2. We set B equal to the maximum of B_1, B_2, and $2|x|+A+5$. Let Φ be a G-net such that $\rho(\Phi, Q\langle G\rangle, n^R)$, $\Phi(A)|_{R+B} \neq \emptyset$. We consider an arbitrary point z of $\Phi(A+2)|_{R+B}$. By z' we denote the beginning of z of length $|z| - R$. Let us prove that $z' \in \Phi'(A+3)$, where Φ' is some initial G-net. It is obvious that $\|n^{-R}z - z'\| \leq 1$. Since $\rho(z, Q\langle G\rangle, n^R)$, $\rho(n^{-R}z, Q\langle G\rangle, 1)$ and $z' \in t + Q\langle G\rangle$ for some $t \in Q^k$, $\|t\| \leq 2$. We note that $|z'| > 2|x|+A+4$ and since $\Gamma(z') \cap \Theta(x; A+2) \neq \emptyset$, by Lemma 19, $z' \in \Theta(x, A+3)$. Thus, $\Phi(A+2)|_{R+B}$ is contained in a finite union of sets of the form $\Psi_\alpha = n^R\Phi'_\alpha(A+3)|_B + [u_\alpha]$, where Φ'_α is an initial G-net and $|u_\alpha| = R$. By Lemma 22 and by the choice of B all points of the set $\Phi'_\alpha(A+3)|_B$ are mutually equivalent and, hence, all points of the set Ψ_α are mutually equivalent. By Lemma 22 every nonempty set $\Phi'_\alpha(A+3)|_B$ intersects γ_i for $i \geq B$; consequently, every set Ψ_α intersects γ_i for $i \geq B+R$. Since G is a system of periods for $\sigma(x)$ and all Ψ_α lie in one G-net, all Ψ_α lie in one equivalence class and the first statement of the lemma has been proved.

Let us prove that when $\Phi(A)|_{R+B} \neq \emptyset$ there is fulfilled $\Phi(A+2)|_{R+B} \cap \gamma_i(A+2) \neq \emptyset$. Analogously to what we did in the first part of the proof, from $\Phi(A)$ we construct a G-net Φ' such that $z' \in \Phi'(A+1)$, $|u| = R$, $z'u \in \Phi(A)|_{R+B}$ would be fulfilled for some z' from N^k and some u from Δ. Since $\Phi'(A+1)|_B \neq \emptyset$, by Lemma 22, $\Phi'(A+1)$ intersects all $\gamma_i(A+2)$ for $i \geq B$. We fix an arbitrary $i \geq B$. Let $t \in \gamma_i(A+2) \cap \Phi'(A+1)$. We set $\bar{t} = tu$. Since $t \in \Phi'(A+1)$, from the definition of Φ' it follows that $\bar{t} \in \Phi$. By Lemma 4, from $t \in \gamma_i(A+2)$ follows $\bar{t} \in \gamma_{i+R}(A+2)$. Therefore $\bar{t} \in \Phi(A+2) \cap \gamma_{i+R}(A+2)$. The lemma is proved.

The next lemma enables us to reduce the proof of the semilinearity of every equivalence class to the proof of the semilinearity of the intersection of this class with certain cones $\Theta(y, A)$. In what follows it turns out that here it is sufficient to limit the analysis to nonsingular y.

We introduce some notation. For $X \in N_k$, A, $B \in N$ we set

$$\Theta(X) \rightleftharpoons \bigcup_{x \in X} \Theta(x),\ \Theta(X, A) \rightleftharpoons \Theta(X \times \{A\}) \rightleftharpoons \bigcup_{x \in X} \Theta(x, A),$$

$$\hat{B} \rightleftharpoons \{t \in N_k \| ut \| \geqslant n^B\}.$$

LEMMA 24. For every $E \in N$ there exists $C \in N$ such that $\Theta(\hat{E}, C) \supseteq Q_+^k$.

Proof. It is obvious that $y + (1 - n^{-C})\Omega \subseteq \Theta(\hat{E}, C)$ is fulfilled for every y from $\hat{E}$. Thus, to prove the lemma it is enough to find C such that for every d from Q_+^k the ray Q_+d intersects some cube of the form $y + (1 - n^{-C})\Omega$, where $y \in \hat{E}$. We denote n^{-C} by ε and we select C so that $\varepsilon < 1/2(n^E+1)k$. Let Q_+d be a ray not possessing the above-mentioned property; let z and z' be points of this ray such that $\|z\| = n^E$ and $\|z'\| = n^E + 1 - \varepsilon$. It is obvious that $z_{(i)} = n^E$ and $z'_{(i)} = n^E + 1 - \varepsilon$ are fulfilled for some i. We set $\bar{d} = z' - z$, then $\|\bar{d}\| = \bar{d}_{(i)} = 1 - \varepsilon$. By virtue of the choice of ray Q_+d, for every point of the segment $z + [0, 1]\bar{d}$ we can find j such that the j-th coordinate of this point lies in $N_1 - (0, \varepsilon)$. For $j = 1, \ldots, k$ we denote by T_j a subset of $[0, 1]$, consisting of t, for which $(z + t\bar{d})_{(j)} \in N_1 - (0, \varepsilon)$. Obviously, $T_i = \emptyset$. Let us consider another value of j. If $\bar{d}_{(j)} < 1/2(n^E+1)$, then $z'_{(j)} = (z'_{(i)}/\bar{d}_{(i)})\,\bar{d}_{(j)} < (n^E + 1 - \varepsilon)/(1-\varepsilon)\,(2(n^E+1)) < 1 - \varepsilon$. The set T_j is empty for such j. Let $\bar{d}_{(j)} \geq 1/2(n^E+1) > k\varepsilon$. Since $\bar{d}_{(j)} \leq 1 - \varepsilon$, in this case set T_j is empty or is either an interval or a semi-interval whose length does not exceed $\varepsilon/\bar{d}_{(j)} < \varepsilon/k\varepsilon = 1/k$. Since the union of the T_j must cover the segment $[0, 1]$, we obtain a contradiction, and that proves the lemma.

LEMMA 25. For any $y \in N_k$, A, $B \in N$ there exist a finite set V and a number A' greater than A and B such that $\Theta(V, A') \supseteq \Theta(y, A)$. Here if y is singular and $w \in V$, then $w \succ y$.

Proof. In Lemma 24 we set E equal to $2|y| + A + B + \nu + 2$ and we construct the corresponding C. We set A' equal to $C + \nu$. For every point z of E we construct a cyclic point t of greatest length, being the beginning of z. Let $z = tu$. Then $Q_+(t + (1 - n^{-C-\nu})\Omega) \supseteq n^{|u|}t + (n^{|u|} - n^{-C})\Omega \supseteq n^{|u|}t + [u] + (1 - n^{-C})\Omega \supseteq z + (1 - n^{-C})\Omega$. Consequently, $\Theta(t, A') \supseteq \Theta(z, C)$. We denote the set of such points t by V_1. From the preceding and from Lemma 24 we have $\Theta(V_1, A') \supseteq Q_+^k$. We define V as the set of points w from V_1 such that $\Theta(w, A') \cap \Theta(y, A) \neq \emptyset$. It is obvious that $\Theta(V, A') \supseteq \Theta(y, A)$. Now we note that $|w| > 2|y| + A + 2$ is fulfilled for every w from V_1. Hence from the construction of V_1, by Lemma 21 it follows that $w \succ y$ if y is singular. The lemma is proved.

We now construct a covering of Q_+^k by cones of the form $\Theta(y, A)$, where y is nonsingular.

Let B be a positive integer. For $i = 1, \ldots, \nu$ we inductively define U_i, viz., a finite subset of N_k, and a number C_i in the following manner. We set $U_1 \rightleftharpoons \hat{B}$ and we choose C_1 by Lemma 24 such that $\Theta(U_1, C_1) \supseteq Q_+^k$. Suppose that U_i and C_i have been constructed. For each of the singular points of U_i we construct by Lemma 25 the set V and the number A', we define U_{i+1} as the union of U_i and all the sets V constructed, and we define C_{i+1} as the maximum of C_i and all the numbers A' constructed. By $Y_0(B)$ we denote the set U_ν constructed in this way from the number B, by $Y_1(B)$ we denote the number C_ν, and we set $Y(B) \rightleftharpoons (Y_0(B), Y_1(B))$. Using Lemma 25 it is easy to show that $\Theta(Y(B)) \supseteq Q_+^k$, that all points of $Y_0(B)$ are nonsingular, and that $Y_0(B) \subseteq N_k|_B$.

We define $Y_0(B, x, A)$ as the set of points y of $Y_0(B)$ such that $\Theta(y) \cap \Theta(x, A) \neq \emptyset$ and we set $Y_1(B, x, A)$ equal to $Y_1(B)$, $\bar{Y}_0(B, x, A)$ equal to $Y_0(B, x, A) \setminus (x + Q\langle G \rangle)$, $\bar{Y}_1(B, x, A)$ equal to $Y_1(B)$. By Lemma 21, when $B \geq 2|x| + A + 2$, $y \in Y_0(B, x, A)$ there is fulfilled $y \succcurlyeq x$. Hence from the definition of a system of periods it follows that where $B \geq 2|x| + A + 2$, $y \in \bar{Y}_0(B, x, A)$ there is fulfilled $y \succ x$.

In the next lemma we establish the Presburger conditions under which a point from $\Theta(x, A)$ lies in $\Theta(\bar{Y}(B, x, A))$.

LEMMA 26. Let $z \in \Theta(x, A)$, $B \in \mathbf{N}$, and $\neg\rho(z, \mathbf{Q}\langle G\rangle, n^{-B}\|z\|)$; then $z \in \Theta(\bar{Y}(B, x, A))$.

Proof. Since $\Theta(x, A) \subseteq \Theta(Y(B, x, A))$, it is sufficient to show that from $z \in \Theta(x, A) \cap \Theta(Y_0(B, x, A) \cap (x + \mathbf{Q}\langle G\rangle), Y_1(B))$ follows $\rho(z, \mathbf{Q}\langle G\rangle, n^{-B}\|x\|)$. Let $y \in Y_0(B, x, A) \cap (x + \mathbf{Q}\langle G\rangle)$. Then $\|y\| \geq n^B$ and $y \in \mathbf{Q}\langle G\rangle - \xi$ is fulfilled for some $\xi \in \Omega$. We consider an arbitrary point $z \in \Theta(y)$. We find r and η such that $z = r(y + \eta) \in r(\mathbf{Q}\langle G\rangle - \xi + \eta)$. Since $\|\eta - \xi\| \leq 1$, $\rho(z, \mathbf{Q}\langle G\rangle, r)$. It is obvious that $r \leq \|z\| / \|y\|$, and since $\|y\| \geq n^B$, $\rho(z, \mathbf{Q}\langle G\rangle, n^{-B}\|z\|)$. The lemma is proved.

The next lemma established that the semilinearity of the intersection of $\Theta(x, A)$ with every equivalence class follows from the semilinearity of the intersection with every equivalence class of certain cones $\Theta(y, B)$ for $y \succ x$.

LEMMA 27. Assume that $\mathfrak{A}$ is an m, n-set. For every number A there exists a number D such that from the semilinearity of the intersection of $\Theta(\bar{Y}(D, x, A+2))$ with every equivalence class follows the semilinearity of the intersection of $\Theta(x, A)$ with every equivalence class and that from $y \in Y_0(D, x, A+2)$ follows $y \succ x$.

Proof. Let B be the number constructed from x and A in Lemma 23. We set $D \rightleftharpoons B + |x| + 1$. Then $D > 2|x| + A + 4$; therefore, for every y from $\bar{Y}_0(D, x, A+2)$ there is fulfilled $y \succ z$.

We fix an arbitrary equivalence class s. We define the predicates

$$P_1(z) \rightleftharpoons z \in \Theta(x, A),$$
$$P_2(z) \rightleftharpoons z \in \Theta(\bar{Y}(D, x, A+2)),$$
$$P(z) \rightleftharpoons z \in s.$$

Let us prove that the predicate $P_1(z) \& P(z)$ is Presburger under the assumption that predicate $P_2(z) \& P(z)$ is Presburger. The lemma's statement obviously follows from this.

We denote $\mathbf{Q}\langle G\rangle$ by π. We introduce additional predicates:

$$P_3(z) \rightleftharpoons P_1(z) \& \rho(z, \pi, 1),$$
$$P_4(z) \rightleftharpoons P_1(z) \& \neg P_3(z) \& \rho(z, \pi, n^{-D}\|z\|),$$
$$P_5(z) \rightleftharpoons P_1(z) \& \neg P_3(z) \& \neg P_4(z).$$

The predicates $P_i(z)$, $i = 1, \ldots, 5$, are Presburger. In order to prove that $P_1(z) \& P(z)$ is a Presburger predicate it is sufficient to establish that the predicates $P_i(z) \& P(z)$ are Presburger for $i = 3, 4, 5$. Predicate $P_3(z) \& P(z)$ is Presburger by Lemma 22. By Lemma 26 the truth of $P_2(z)$ follows from the the truth of $P_5(z)$. Hence by the assumption made it follows that predicate $P_5(z) \& P(z)$ is Presburger. Further, let the predicate $P_4(z)$ be true. We select a positive integer R so as to fulfill the conditions $\neg\rho(z, \pi, n^{R-1})$ and $\rho(z, \pi, n^R)$. Then $\|z\| > n^{R-1+D}$. By Φ we denote the G-net containing z. The hypotheses of Lemma 23 are fulfilled for Φ and R, i.e., $\Phi(A)|_{R+B} \neq \emptyset$ and $\rho(\Phi, \pi, n^R)$. By Lemma 23, $\Phi(A+2)$ contains a point z' such that $|z'| = |x| + R + B = R + D - 1$. For z' there is fulfilled $n^{R+D-2} \leq \|z'\| < n^{R+D-1}$. By virtue of the conditions $\neg\rho(z, \pi, n^{R-1})$ and $\rho(z, \pi, n^R)$ we obtain $\neg\rho(z', \pi, n^{-D}\|z'\|)$ and $\rho(z', \pi, n^{-D+2}\|z'\|)$. Thus, the Presburger predicate

$$P_6(z') \rightleftharpoons z' \in \Theta(x, A+2) \& \neg\rho(z', \pi, n^{-D}\|z'\|) \& \rho(z', \pi, n^{-D+2}\|z'\|)$$

is true for the point z' selected.

If predicate $P_6(z')$ is true for some point of Φ, then $n^{-D}\|z'\| > n^{R-1}$. Thus, point z' belongs to $\Phi(A+2)|_{R+B}$. Hence by Lemma 23 it follows that all points z' of Φ, for which predicate $P_6(z')$ is true, are mutually equivalent and are equivalent to z.

By Lemma 26 the truth of predicate $P_2(z')$ follows from the truth of predicate $P_6(z')$. Consequently, predicate $P_6(z') \& P(z')$ is Presburger. By virtue of what has been proved, predicate $P_4(z) \& P(z)$ is equivalent to the predicate $P_1(z) \& \exists z'(z' \equiv z (\mathrm{mod}\, G) \& P_6(z') \& P(z'))$. Thus, $P_4(z) \& P(z)$ is a Presburger predicate.

In the next lemma we prove with the aid of Lemma 27 the semilinearity of every equivalence class.

LEMMA 28. Let $\mathfrak{A}$ be an m, n-set. Then every equivalence class is semilinear.

Proof. We construct a sequence $W_0, \ldots, W_\nu$ by elements which will be finite subsets of $N_k \times N$. We set $W_0 \rightleftharpoons Y(0)$. Suppose that W_i has been defined; then

$$W_{i+1} \rightleftharpoons \bigcup_{(x,A)\in W_i} \overline{Y}(D(x, A), x, A+2)),$$

where D(x, A) is the number corresponding to x and A by Lemma 27. From the construction of Y(0) we have $\Theta(W_0) \supseteq N_k$. By Lemma 27 the semilinearity of the intersection of every equivalence class with $\Theta(W_i)$ follows from the semilinearity of the intersection of every equivalence class with $\Theta(W_{i+1})$. From the construction of the W_i and from Lemma 27 it follows that for every pair $(x, A) \in W_i$ there exists a sequence $x_0, \ldots, x_i = x$ such that $x_{j+1} > x_j$, $j = 0, \ldots, i-1$. Therefore, $W_\nu \neq \emptyset$. The lemma is proved.

The paper's main result is easily obtained from the last lemma.

THEOREM 1. Let m and n be multiplicatively independent. Then every m,n-regular predicate is Presburger.

Proof. A subset of N^k, specifiable by the predicate being examined, is the union of a certain number of equivalence classes and, perhaps, of the set $\{0\}$. By Lemma 28 all equivalence classes are semilinear. Hence we obtain the theorem's assertion.

LITERATURE CITED

1. A. L. Semenov, "Presburgerness of sets recognizable by finite automata in two number systems," Third All-Union Conference of Mathematical Logic. Reports Abstracts [in Russian], Izd. Inst. Math. Sib. Otdel. Akad. Nauk SSSR, Novosibirsk (1974), pp. 201-203.
2. J. R. Büchi, "Weak second-order arithmetic and finite automata," Z. Math. Logik Grundl. Math., 6, No. 1, 66-92 (1960); Kiberneticheskii Sb., No. 8, 42-77 (1964).
3. A. Cobham, "On the base-dependence of sets of numbers, recognizable by finite automata," Math. Systems Theory, 3, No. 2, 186-192 (1969); Kiberneticheskii Sb., Nov. Ser., No. 8, 62-71 (1971).
4. S. Ginsburg and E. H. Spanier, "Semigroups, Presburger formulas and languages," Pac. J. Math., 13, No. 4, 570-581 (1966).
5. J. W. Thatcher, '" Decision problems for multiple successor arithmetics," J. Symbolic Logic, 31, No. 2, 182-190 (1966).

VARIETIES AND SHEAVES OF SEMIGROUPS

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The set of all subvarieties of an arbitrary variety of algebras is a lattice relative to set-theoretic inclusion. In the literature there are now a rather large number of papers devoted to the study of the properties of this lattice for various concrete varieties. However, it is natural to examine other operations and relations on the set of varieties. In [1] Mal'tsev introduced the following operation of multiplication of classes of algebras.† Let $\mathfrak{M}$ be an arbitrary fixed class of algebras of some signature and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be subclasses of it. The product $\mathfrak{M}_1 \cdot \mathfrak{M}_2$ of classes $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in $\mathfrak{M}$ is the collection of all $\mathfrak{M}$-algebras A such that a congruence α exists on A, by which all classes which are subalgebras lie in $\mathfrak{M}_1$ and the quotient algebra A/α lies in $\mathfrak{M}_2$.

As $\mathfrak{M}$ we shall take the variety $\mathfrak{S}$ of all semigroups. A product of two varieties will always be a prevariety, i.e., a class closed relative to subalgebras and direct products and containing a one-element algebra. But this product is not necessarily a variety. For this reason it is natural to examine the following operation on a set of varieties: the variety generated by the class $\mathfrak{M}_1 \cdot \mathfrak{M}_2$ is called the product $\mathfrak{M}_1 * \mathfrak{M}_2$ of varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in variety $\mathfrak{M}$. This natural modification of the preceding definition was also noted in [1]. The set L of all

†By a class of algebras we always understand an abstract class, i.e., one which is closed relative to isomorphic images.

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