What's Decidable about Discrete Linear Dynamical Systems?

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Abstract. We survey the state of the art on the algorithmic analysis of discrete linear dynamical systems, focussing in particular on reachability, model-checking, and invariant-generation questions, both unconditionally as well as relative to oracles for the Skolem Problem.

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1 Introduction

Dynamical systems are a fundamental modelling paradigm in many branches of science, and have been the subject of extensive research for many decades. A *(rational) discrete linear dynamical system (LDS)* in ambient space $\mathbb{R}^d$ is given by a square $d \times d$ matrix M with rational entries, together with a starting point $x \in \mathbb{Q}^d$.[3] The *orbit* of (M, x) is the infinite trajectory $\mathcal{O}(M, x) := \langle x, Mx, M^2x, \dots \rangle$. An example of a four-dimensional LDS is given in Figure 1. Our main focus in the present paper is on delineating the class of assertions on the orbits of LDS that can be algorithmically decided.

$$M \stackrel{\mathsf{def}}{=} \begin{pmatrix} 3 & 2 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 4 & 3 & 13 \\ 3 & 11 & 6 & 24 \end{pmatrix} \qquad x \stackrel{\mathsf{def}}{=} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

Fig. 1. A four-dimensional discrete linear dynamical system.

[3] All of the results we present in this paper carry over to the more general setting of *real-algebraic* LDS, whose entries are allowed to be real algebraic numbers. Nevertheless, we stick here to rationals for simplicity of exposition.

One of the most natural and fundamental computational questions concerning linear dynamical systems is the *Point-to-Point Reachability Problem*, also known as the *Kannan-Lipton Orbit Problem*: given a d-dimensional LDS (M, x) together with a point target $y \in \mathbb{Q}^d$, does the orbit of the LDS ever hit the target? The decidability of this question was settled affirmatively in the 1980s in the seminal work of Kannan and Lipton [35,36]. In fact, Kannan and Lipton showed that this problem is solvable in polynomial time, answering an earlier open problem of Harrison from the 1960s on reachability for linear sequential machines [33].

Interestingly, one of Kannan and Lipton's motivations was to propose a line of attack to the well-known *Skolem Problem*, which had itself been famously open since the 1930s. The Skolem Problem remains unsolved to this day, although substantial advances have recently been made—more on this shortly. Phrased in the language of linear dynamical systems, the Skolem Problem asks whether it is decidable, given (M, x) as above, together with a $(d-1)$-dimensional subspace H of $\mathbb{R}^d$, to determine if the orbit of (M, x) ever hits H. Kannan and Lipton suggested that, in ambient space $\mathbb{R}^d$ of arbitrary dimension, the problem of hitting a low-dimensional subspace might be decidable. Indeed, this was eventually substantiated by Chonev *et al.* for linear subspaces of dimension at most 3 [21,23].

Subsequent research focussed on the decidability of hitting targets of increasing complexity, such as half-spaces [32,39,47,46,48], polytopes [52,22,5], and semialgebraic sets [6,7]. It is also worth noting that discrete linear dynamical systems can equivalently be viewed as linear (or affine) simple, branching-free while loops, where reachability corresponds to loop termination. There is a voluminous literature on the topic, albeit largely focussing on heuristics and semi-algorithms (via spectral methods or the synthesis of ranking functions, in particular), rather than exact decidability results. Relevant papers include [54,19,11,12,18,20,24,50,51,34,13,10]. Several of these approaches have moreover been implemented in software verification tools, such as Microsoft's Terminator [26,27].

In recent years, motivated in part by verification questions for stochastic systems and linear loops, researchers have begun investigating more sophisticated decision problems than mere reachability: for example, the paper [1] studies approximate LTL model checking of Markov chains (which themselves can be viewed as particular kinds of linear dynamical systems), whereas [38] focusses on LTL model checking of low-dimensional linear dynamical systems with semialgebraic predicates.[4] In [4], the authors solve the semialgebraic model-checking problem for diagonalisable linear dynamical systems in arbitrary dimension against prefix-independent MSO[5] properties, whereas [37] investigates

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[4] Semialgebraic predicates are Boolean combinations of polynomial equalities and inequalities.

[5] Monadic Second-Order Logic (MSO) is a highly expressive specification formalism that subsumes the vast majority of temporal logics employed in the field of automated verification, such as Linear Temporal Logic (LTL). "Prefix independence" is

semialgebraic MSO model checking of linear dynamical systems in which the dimensions of predicates are constrained. To illustrate this last approach, recall the dynamical system (M, x) from Figure 1, and consider the following three semialgebraic predicates:

$$P_1(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} x_1 + x_2 + x_3 - x_4 = 0 \wedge (x_1^3 = x_2^2 \vee x_4 \geq 3x_1^2 + x_2)$$
$$P_2(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} x_1 + x_2 + 2x_3 - 2x_4 = 0 \wedge x_1^3 + x_3^2 + x_3 > x_4$$
$$P_3(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} x_1^4 - x_2^2 = 3 \wedge 2x_3^2 = x_4 \wedge x_1^2 - 2x_2^3 = 4x_3\,.$$

Recall that the ambient space is $\mathbb{R}^4$. We identify the above predicates with the corresponding subsets of $\mathbb{R}^4$, and wish to express assertions about the orbit of (M, x) as it traces a trajectory through $\mathbb{R}^4$. For example (in LTL notation),

$$\mathbf{G}(P_1 \Rightarrow \mathbf{F}\neg P_2) \wedge \mathbf{F}(P_3 \vee \neg P_1)$$

asserts that whenever the orbit visits P_1, then it must eventually subsequently visit the complement of P_2, and moreover that the orbit will eventually either visit P_3 or the complement of P_1. The reader will probably agree that whether or not the above assertion holds for our LDS (M, x) is not immediately obvious to determine (even, arguably, in principle). Nevertheless, this example falls within the scope of [37], as the semialgebraic predicates P_1, P_2, and P_3 are each either contained in some three-dimensional subspace (this is the case for P_1 and P_2), or have intrinsic dimension at most 1 (this is the case of P_3, which is 'string-like', or a curve, as a subset of $\mathbb{R}^4$). Naturally, we shall return to these notions in due course, and articulate the relevant results in full details.

A recent and closely related line of inquiry concerns the study of algebraic model checking of linear dynamical systems [42]. The setting is similar to the above, the only difference being that the allowable predicates are the *constructible* ones, i.e., built from *algebraic* sets[6] using arbitrary Boolean operations (including complementation). The paper [42] introduces in addition the key notion of *Skolem oracle*, which we discuss next.

1.1 Skolem Oracles

There is an intimate connection between linear dynamical systems and linear recurrence sequences. An *(integer) linear recurrence sequence (LRS)* $\boldsymbol{u} = \langle u_n \rangle_{n=0}^{\infty}$ is an infinite sequence of integers satisfying

$$u_{n+d} = c_1 u_{n+d-1} + \cdots + c_{d-1} u_{n+1} + c_d u_n \tag{1}$$

for all $n \in \mathbb{N}$, where the coefficients $c_1, \ldots, c_d$ are integers and $c_d \neq 0$. We say that the above recurrence has *order* d. We moreover say that an LRS is *simple*

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a quality of properties that are *asymptotic* in nature—we provide a precise definition shortly.

[6] Algebraic sets correspond to positive Boolean combinations of polynomial equalities.

if the characteristic polynomial[7] of its minimal-order recurrence has no repeated roots. The sequence of Fibonacci numbers $\langle f_n \rangle_{n=0}^{\infty} = \langle 0, 1, 1, 2, 3, 5, \ldots \rangle$, which obeys the recurrence $f_{n+2} = f_{n+1} + f_n$, is perhaps the most emblematic LRS, and also happens to be simple.

The celebrated theorem of Skolem, Mahler, and Lech (see [31]) describes the structure of the set $\{n \in \mathbb{N} : u_n = 0\}$ of zero terms of an LRS as follows:

Theorem 1. *Given a linear recurrence sequence $\boldsymbol{u} = \langle u_n \rangle_{n=0}^{\infty}$, its set of zero terms is a semilinear set, i.e., it consists of a union of finitely many full arithmetic progressions,[8] together with a finite set.*

As shown by Berstel and Mignotte [14], in the above one can effectively extract all of the arithmetic progressions; we refer herein to the corresponding procedure as the 'Berstel-Mignotte algorithm'. Nevertheless, how to compute the leftover finite set of zeros remains open, and is easily seen to be equivalent to the *Skolem Problem*: given an LRS $\boldsymbol{u}$, does $\boldsymbol{u}$ contain a zero term?

The paper [42] therefore introduces the notion of a *Skolem oracle*: given an LRS $\boldsymbol{u} = \langle u_n \rangle_{n=0}^{\infty}$, such an oracle returns the finite set of indices of zeros of $\boldsymbol{u}$ that do not already belong to some infinite arithmetic progression of zeros. Likewise, a *Simple-Skolem oracle* is a Skolem oracle restricted to simple LRS.

As mentioned earlier, the decidability of the Skolem Problem is a longstanding open question [31,49], with a positive answer for LRS of order at most 4 known since the mid-1980s [53,55]. Very recently, two major conditional advances on the Skolem Problem have been made, achieving decidability subject to certain classical number-theoretic conjectures: in [40], Lipton *et al.* established decidability for LRS of order 5 assuming the *Skolem Conjecture* (also known as the *Exponential Local-Global Principle*); and in [15], Bilu *et al.* showed decidability for simple LRS of arbitrary order, subject to both the Skolem Conjecture and the *p-adic Schanuel Conjecture* (we refer the reader to [15] for the precise definitions and details). It is interesting to note that in both cases, the procedures in question rely on the conjectures *only* for termination; correctness is unconditional. In fact, these procedures are *certifying algorithms* (in the sense of [45]) in that, upon termination, they produce an independent certificate (or witness) that their output is correct. Such a certificate can be checked algorithmically by a third party with no reliance on any unproven conjectures. The authors of [15] have implemented their algorithm within the SKOLEM tool, available online.[9]

In view of the above, Simple-Skolem oracles *can* be implemented with unconditional correctness, and guaranteed termination subject to the Skolem and p-adic Schanuel conjectures. Whether full Skolem oracles can be devised is the subject of active research (see, e.g., [41,43]); at the time of writing, to the best of our knowledge, no putative procedure is even conjectured in the general (non-simple) case.

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[7] The characteristic polynomial associated with recurrence (1) is $X^d - c_1 X^{d-1} - \ldots - c_d$.

[8] A full arithmetic progression is a set of non-negative integers of the form $\{a + bm : m \in \mathbb{N}\}$, with $a, b \in \mathbb{N}$ and $a < b$.

[9] https://skolem.mpi-sws.org/ .

1.2   Paper Outline

Questions of reachability and model checking for linear dynamical systems constitute one of the central foci of this paper. In Section 2, we cover the state of the art and beyond, both unconditionally and relative to Skolem oracles. We paint what is essentially a complete picture of the landscape, in each situation establishing either decidability (possibly conditional on Skolem oracles), or hardness with respect to longstanding open problems. An important theme in the classical theory of dynamical systems concerns the study of *asymptotic* properties (e.g., stability, convergence, or divergence of orbits), and we therefore consider both MSO along with its *prefix-independent* fragment piMSO. Section 3 then focusses on questions of robustness through the notion of pseudo-orbit. In Section 4, we discuss the algorithmic synthesis of inductive invariants for linear dynamical systems, and Section 5 examines the situation in which orbits originate from an initial set rather than a single point. Finally, Section 6 concludes with a brief summary and a glimpse of several research directions.

2   Model Checking

Throughout this section, we assume familiarity with the rudiments of Monadic Second-Order Logic (MSO); an excellent reference is the text [16].

Let us work in fixed ambient space $\mathbb{R}^d$, and consider a d-dimensional LDS (M, x) (i.e., $M \in \mathbb{Q}^{d \times d}$ and $x \in \mathbb{Q}^d$). Recall that the orbit $\mathcal{O} = \mathcal{O}(M, x)$ of our LDS is the infinite sequence $\langle x, Mx, M^2 x, \ldots \rangle$ in $\mathbb{Q}^d$. Let us write $\mathcal{O}[n]$ for the nth term of the orbit.

Given an MSO formula φ over the collection of semialgebraic predicates $\mathcal{P} = \{P_1, \ldots, P_m\}$, where each $P_i \subseteq \mathbb{R}^d$, the model-checking problem consists in determining whether the orbit (more precisely, the characteristic word $\alpha \in (2^{\mathcal{P}})^{\omega}$ of the orbit $\mathcal{O}(M, x)$ with respect to $\mathcal{P}$, where $P_i \in \alpha[n]$ iff $\mathcal{O}[n] \in P_i$) satisfies φ. Reachability problems for LDS constitute special cases of the model-checking problem, and already the questions of determining whether a given orbit reaches a hyperplane (Skolem Problem) or a halfspace (the *Positivity Problem* [46]) are longstanding open problems in number theory, couched in the language of linear dynamical systems. Recent research has, however, succeeded in uncovering several important decidable subclasses of the model-checking problem and in demarcating the boundary between what is decidable and what is hard with respect to longstanding open mathematical problems.

In order to present the main results, we require some further definitions to specify the classes of predicates that are allowed within MSO formulas. Let us write $\mathcal{S} \subseteq 2^{\mathbb{R}^d}$ and $\mathcal{C} \subseteq 2^{\mathbb{R}^d}$ to denote respectively the collections of all semialgebraic subsets of $\mathbb{R}^d$ and of all constructible subsets of $\mathbb{R}^d$.[10] We also define the collection $\mathcal{T} \subseteq 2^{\mathbb{R}^d}$ of *tame* sets as follows: $\mathcal{T}$ comprises all semialgebraic subsets

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[10] Recall that $\mathcal{C}$ is the smallest set containing all algebraic subsets of $\mathbb{R}^d$, and which is closed under finite union, finite intersection, and complement. (The terminology of "constructible" originates from algebraic geometry.)

of $\mathbb{R}^d$ that are either contained in a three-dimensional subspace of $\mathbb{R}^d$, or that have intrinsic dimension at most one.[11] Moreover, $\mathcal{T}$ is defined to be the smallest such set which is in addition closed under finite union, finite intersection, and complement. Finally, we define the set $\mathcal{T} \oplus \mathcal{C}$ to be the smallest superset of $\mathcal{T} \cup \mathcal{C}$ that is closed under finite union, finite intersection, and complement.

Note that all of $\mathcal{S}$, $\mathcal{C}$, $\mathcal{T}$, and $\mathcal{T} \oplus \mathcal{C}$ are closed under all Boolean operations; this is in keeping with their intended use as collections of predicates for MSO formulas, bearing in mind that MSO itself possesses all Boolean operators.

The motivation for considering the collection $\mathcal{T}$ of tame predicates has origins in the results of [21,38,6,9]. A common theme is that for tame predicates, the proofs that establish how to decide reachability also provide one with a means of representing, in a finitary manner, all the time steps at which the orbit of a given LDS is in a particular predicate set T. The authors of [37] show how to combine these representations (one for each predicate) to obtain structural information about the characteristic word α that is sufficient for determining whether a deterministic automaton $\mathcal{A}$ accepts α, leading to the following.

Theorem 2. *Let (M, x) be an LDS, $\mathcal{P} = \{T_1, \ldots, T_m\} \subseteq \mathcal{T}$ be a collection of tame predicates and φ be an MSO formula over $\mathcal{P}$. It is decidable whether $(M, x) \vDash \varphi$ (i.e., whether the characteristic word α of the orbit $\mathcal{O}(M, x)$ with respect to $\mathcal{P}$ satisfies φ).*

Let us note in passing that Theorem 2 subsumes the decidability of the Kannan-Lipton Point-to-Point Reachability Problem, since points are singleton sets and the latter are evidently tame.

It is also worth pointing out, absent other restrictions, that this delineation of the decidable fragment of the model-checking problem is tight as trying to expand the definition of tame predicates runs into open problems already for formulas that describe mere reachability properties. In particular, the Skolem Problem in dimension 5 is open and can be encoded (i) as a reachability problem with a four-dimensional LDS and a three-dimensional affine subspace [21] (that is, in general, not contained in a three-dimensional linear subspace) and (ii) as a reachability problem with a target of intrinsic dimension two [9].

To sidestep these obstacles, in [4] the authors restrict φ to formulas that define *prefix-independent* properties. A property is prefix-independent if the infinite words that satisfy it are closed under the operations of insertion and deletion of finitely many letters.[12] Such properties capture behaviours that are intrinsically

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[11] The intrinsic dimension of a semialgebraic set is formally defined via cell decomposition; intuitively, one-dimensional semialgebraic sets can be viewed as 'strings' or 'curves', whereas zero-dimensional semialgebraic sets are finite collections of singleton points.

[12] It is interesting to note that whether an MSO formula φ is prefix-independent or not is decidable. To see this, for $A = (Q, q_0, \Sigma, \Delta, F)$ a deterministic Müller automaton, define $A(q)$, for $q \in Q$, to be the same as A, except that the initial state of $A(q)$ is q (rather than q_0). We say that a deterministic Müller automaton A (as above) is *prefix-independent* if, for all $q \in Q$ that are reachable from q_0, $A(q)$ recognises the

asymptotic in nature (for example: "does the orbit enter P infinitely often?"); note that the property of reachability is *not* prefix-independent. The main theorem of [4] in this direction concerns *diagonalisable* linear dynamical systems:[13]

Theorem 3. *Let (M, x) be a diagonalisable LDS and φ be a prefix-independent MSO formula over a collection of semialgebraic predicates $S_1, \ldots, S_m \in \mathcal{S}$. It is decidable whether $(M, x) \vDash \varphi$.*

Note in the above that the semialgebraic predicates are entirely unrestricted (in particular, not required to be tame). However, the restrictions to prefix-independent formulas and to diagonalisable systems both again turn out to be essential. Since the Skolem Problem is open for diagonalisable systems (in dimensions $d \geq 5$), the (non-prefix-independent) model-checking problem for diagonalisable LDS is Skolem-hard already for four-dimensional systems and affine subspace targets, as discussed earlier. On the other hand, if we allow non-diagonalisable systems, then the problem of determining whether the orbit of an LDS is eventually trapped in a given half-space H (known as the *Ultimate Positivity Problem*, corresponding to the prefix-independent formula $\varphi = \mathbf{F}\,\mathbf{G}\,H$) is hard with respect to certain longstanding open problems in Diophantine approximation [46].

This last observation however suggests that it might be possible to orchestrate a trade-off between the type of LDS under consideration and the class of allowable specification predicates. Indeed, it turns out that one can lift the restriction to diagonalisable LDS if one agrees to restrict the class of predicates:

Theorem 4. *Let (M, x) be an LDS and φ be a prefix-independent MSO formula over a collection of predicates $P_1, \ldots, P_m \in \mathcal{T} \oplus \mathcal{C}$. It is decidable whether $(M, x) \vDash \varphi$.*

Theorem 4 goes beyond both Theorem 2 (in that constructible predicates are allowed into the mix) as well as [42, Thm. 7.3] (in that tame predicates are allowed).

Let us provide a brief proof sketch of Theorem 4. Let (M, x) and φ be as above. Observe first (as a straightforward exercise) that any $P \in \mathcal{T} \oplus \mathcal{C}$ can be written in conjunctive normal form, i.e., as an expression of the form $P = \bigcap_{i=1}^{a} \bigcup_{j=1}^{b} B_{i,j}$, where each $B_{i,j}$ is either a tame predicate or a constructible predicate. Without loss of generality, one may therefore assume that each predicate appearing in φ is either tame or constructible.

We now invoke [42, Prop. 5] to conclude that, for each constructible predicate B_ℓ appearing in φ, the Boolean-valued word α_ℓ tracking the passage of the orbit

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same language as A. Write $L(A)$ to denote the language recognised by A. It is now straightforward to show that A is prefix-independent iff $L(A)$ is prefix-independent. Since any MSO formula is encodable as a deterministic Müller automaton, and equality of ω-regular languages is decidable, the desired decidability result follows.

[13] An LDS (M, x) is *diagonalisable* if the matrix M is diagonalisable (over $\mathbb{C}$). In a measure-theoretic sense, most LDS are diagonalisable.

of (M, x) through B_ℓ is ultimately periodic. Moreover, thanks to the Berstel-Mignotte algorithm, the attendant arithmetic progressions can all be effectively elicited. For each such α_ℓ, one can therefore construct a (fully) periodic word α'_ℓ which differs from α_ℓ in at most finitely many places. In other words, for all sufficiently large n, $M^n x \in B_\ell$ iff $\alpha'[n] = true$. Being periodic, α'_ℓ can in turn be described by an MSO subformula ψ_ℓ. Let us therefore replace within φ every occurrence of a constructible predicate B_ℓ by the subformula ψ_ℓ, obtaining in this process a new MSO formula φ' that comprises exclusively tame predicates. As φ is prefix-independent, it is immediate that $(M, x) \vDash \varphi$ iff $(M, x) \vDash \varphi'$. But the latter is of course decidable thanks to Theorem 2, concluding the proof sketch of Theorem 4.

Once again, Theorem 4 is tight: in Appendix A, we show that the ability to solve the model-checking problem for prefix-independent MSO specifications making use of semialgebraic predicates in ambient space $\mathbb{R}^4$ would necessarily entail major breakthroughs in Diophantine approximation.

Let us now turn to the question of the extent to which the above results can be enhanced through the use of Skolem oracles. The key result is as follows, in effect enabling us to drop the restriction of prefix-independence from Theorem 4:

Theorem 5. *Let (M, x) be an LDS and φ be an MSO formula over a collection of predicates $P_1, \ldots, P_m \in \mathcal{T} \oplus \mathcal{C}$. It is decidable whether $(M, x) \vDash \varphi$, subject to the existence of a Skolem oracle.*

The same result also holds for diagonalisable LDS, assuming the existence of a Simple-Skolem oracle.

The proof of Theorem 5 is similar to that of Theorem 4; we provide a brief sketch below.

Let (M, x) and φ be as above, and assume, thanks to the representation of $(\mathcal{T} \oplus \mathcal{C})$-predicates in conjunctive normal form, that every predicate occurring in φ is either tame of constructible. Thanks to [42, Cor. 6], for each constructible predicate B_ℓ appearing in φ, the Boolean-valued word α_ℓ tracking the passage of the orbit of (M, x) through B_ℓ is effectively ultimately periodic (this requires the use of a Skolem oracle, or a Simple-Skolem oracle if M is diagonalisable). In other words, we have a finitary exact representation of α_ℓ, and can therefore describe it via an MSO subformula ψ_ℓ.

We can now replace within φ every occurrence of a constructible predicate B_ℓ by its corresponding subformula ψ_ℓ, obtaining in this process an equivalent MSO formula φ' that comprises exclusively tame predicates. The desired result then immediately follows from Theorem 2, concluding the proof sketch of Theorem 5.

As noted earlier, Simple-Skolem oracles can be implemented into provably correct certifying procedures which terminate subject to classical number-theoretic conjectures [15]. Let us therefore separately record an important corollary:

Corollary 1. *Let (M, x) be a diagonalisable LDS and φ be an MSO formula over a collection of predicates $P_1, \ldots, P_m \in \mathcal{T} \oplus \mathcal{C}$. It is decidable whether*

$(M, x) \vDash \varphi$, *assuming the Skolem Conjecture and the p-adic Schanuel Conjecture. Moreover, correctness of the attendant procedure is unconditional, and independent correctness certificates can be produced upon termination.*

Let us point out that Theorem 5 is, once again, tight. For arbitrary LDS, a similar argument as that put forth in Appendix A applies, since it is not known (or even believed) that Skolem oracles are of any use in tackling Ultimate Positivity problems. For diagonalisable LDS, we can invoke the order-10 Simple Positivity Problem, which remains open to this day (see [46]); it can be modelled straightforwardly as a half-space reachability problem in ambient space $\mathbb{R}^{10}$ (or even $\mathbb{R}^9$, by considering an affine half-space). In fact, critical unsolved cases of order-10 Positivity can even be formulated as semialgebraic reachability problems in ambient space $\mathbb{R}^4$; we omit the details in the interests of space and simplicity of exposition.

We summarise the main results of this section in Figure 2 below.

Diagonalisable LDS:

	unconditional	Simple-Skolem oracle
piMSO	$\mathcal{S}$ (Thm. 3)	$\mathcal{S}$ (Thm. 3)
MSO	$\mathcal{T}$ (Thm. 2)	$\mathcal{T} \oplus \mathcal{C}$ (Thm. 5)

Arbitrary LDS:

	unconditional	Skolem oracle
piMSO	$\mathcal{T} \oplus \mathcal{C}$ (Thm. 4)	$\mathcal{T} \oplus \mathcal{C}$ (Thm. 4 or Thm. 5)
MSO	$\mathcal{T}$ (Thm. 2)	$\mathcal{T} \oplus \mathcal{C}$ (Thm. 5)

Fig. 2. A summary of the decidable model-checking fragments for both diagonalisable and arbitrary linear dynamical systems. The prefix-independent fragment of MSO is denoted piMSO. $\mathcal{S}$ is the collection of semialgebraic predicates, $\mathcal{T}$ is the collection of tame predicates (Boolean closure of semialgebraic sets that either are contained in a three-dimensional subspace, or have intrinsic dimension at most one), $\mathcal{C}$ is the collection of constructible predicates (Boolean closure of algebraic sets), and $\mathcal{T} \oplus \mathcal{C}$ is the Boolean closure of $\mathcal{T} \cup \mathcal{C}$. The right-hand columns in both tables assume access to Skolem or Simple-Skolem oracles.

Taken together, Theorems 2–5, along with Corollary 1, not only subsume—to the best of our knowledge—all existing results regarding model-checking and reachability problems for discrete linear dynamical systems, but moreover paint an essentially complete picture of what is (even in principle) feasible, barring major breakthroughs in longstanding open problems. It is noteworthy that, in this characterisation, there appears to be very little difference between being able to decide mere reachability for a given class of predicates, and being able to decide the whole of MSO over the same class of predicates.

3   Pseudo-Reachability and Robustness

In this section we discuss decision problems about pseudo-orbits that are related to robustness of computation. Given an LDS (M, x), recall that the orbit of x under M is the sequence $\langle x, Mx, M^2x, \ldots \rangle$. We say that the sequence $\langle x_n : n \in \mathbb{N} \rangle$ is an ε-*pseudo-orbit* of x under M if $x_0 = x$ and $x_{n+1} = Mx_n + d_n$ for some perturbation d_n with $||d_n|| < \epsilon$. The *pseudo-orbit* of x under M is then defined as the set of points that are reachable from x via an ε-pseudo-orbit for every $\epsilon > 0$. This notion of an $(\varepsilon$-)pseudo-orbit, introduced and studied by Anosov [8], Bowen [17] and Conley [25], is an important conceptual tool in dynamical systems. From the computational perspective, an ε-pseudo-orbit can be viewed as a trajectory after a rounding error of magnitude at most ϵ is applied at each step.

Given these definitions, we can consider the reachability and model-checking problems for pseudo-orbits. A natural analogy to the Kannan-Lipton Orbit Problem is the *Pseudo-Orbit Problem*, which is to determine whether a target point y belongs to the pseudo-orbit of x under M. In [28] the authors show that, just like the Orbit Problem, the Pseudo-Orbit Problem is decidable in polynomial time. Generalising from points to sets, let us say that a target set T is pseudo-reachable if for every $\epsilon > 0$ there exists an ε-pseudo-orbit of x under M that reaches T. We can then define the *Pseudo-Skolem Problem* and the *Pseudo-Positivity Problem* to be the pseudo-reachability problems with a hyperplane and a halfspace as target sets, respectively. Surprisingly, [28] shows that both of these problems are in fact decidable! Moreover, [29] establishes decidability of pseudo-reachability with arbitrary semialgebraic targets for diagonalisable linear dynamical systems.

Inspired by the above results, one might consider the model-checking problem for pseudo-orbits, namely the problem of determining, given (M, x) and a formula φ, whether for every $\varepsilon > 0$, there exists an ε-pseudo-orbit that satisfies φ. After all, as discussed in the preceding section, for genuine orbits the fragments of the reachability problem and the full MSO model-checking problem that are known to be decidable (i.e., the restrictions on the class of predicates and the property φ that make the problems decidable) are essentially the same. This optimism is, however, quickly shattered by the following observation. Let H be a closed halfspace and φ be the property $\mathbf{G}\,H$ ("the trajectory always remains inside H"). Then the pseudo-orbits satisfy φ (in the sense defined above) if and only the (genuine) orbit satisfies φ. The problem of determining whether the orbit $\mathcal{O}(M, x)$ always remains in H, is however, equivalent to the problem of determining whether the orbit ever hits an open halfspace, which itself is the Positivity Problem (a longstanding open question).

4   Invariant Generation

In the absence of fully general algorithms to decide whether the orbit of a given LDS reaches targets of arbitrary forms, much effort has been expended on sound—but possibly incomplete—techniques, and particularly on constructing

certificates of (non-)reachability. This splits into two broad lines of attack: ranking functions and invariants. The former are certificates of reachability, demonstrating that progress is being made towards the target. Inductive invariants are, on the other hand, certificates of non-reachability, establishing that the orbit will not reach the target by enclosing the former within a set that is itself disjoint from the latter. We focus in this section on the algorithmic generation of invariants.

More precisely, a set $\mathcal{I} \subseteq \mathbb{R}^d$ is said to be an *inductive invariant* of (M, x) if it contains x ($x \in \mathcal{I}$), and is stable under M, that is:

$$M\mathcal{I} \stackrel{\text{def}}{=} \{My : y \in \mathcal{I}\} \subseteq \mathcal{I}.$$

Clearly there are some trivial invariants, such as $\mathbb{R}^d$ and the orbit $\mathcal{O}(M, x)$ itself. They are not particularly useful in the sense that the ambient space $\mathbb{R}^d$ is never disjoint from whatever target might be under consideration, whereas for various classes of targets (such as hyperplanes or half-spaces; or more generally arbitrary semialgebraic sets) we do not in general know how to decide whether $\mathcal{O}(M, x)$ is disjoint from the target. Hence one does not seek *any* invariant, but rather an invariant that can be algorithmically established to be disjoint from the target.
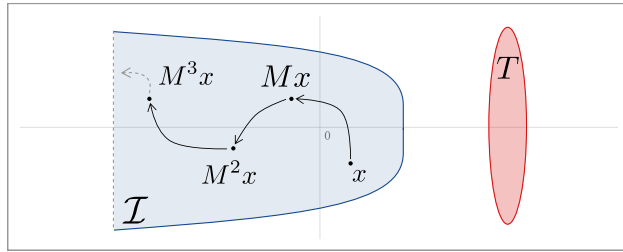
We therefore seek a sufficiently large, or expressive, class of invariants $\mathcal{F}$ which moreover exhibits favourable algorithmic properties. A natural family to consider is the collection of all semialgebraic sets. We now have:

Theorem 6 ([3]). *Given an LDS (M, x) in ambient space $\mathbb{R}^d$, together with a semialgebraic target $T \subseteq \mathbb{R}^d$, it is decidable whether there exists a semialgebraic invariant of (M, x) that is disjoint from T.*

Furthermore, the algorithm explicitly constructs the invariant when it exists, in the form of a Boolean combination of polynomial inequalities.

Theorem 6 holds for an even larger class $\mathcal{F}$, namely that of o-minimal sets. We give an informal definition. Recall the contents of Tarski's quantifier-elimination theorem, to the effect that semialgebraic subsets of $\mathbb{R}^d$ are closed under projections. Moreover, semialgebraic subsets of $\mathbb{R}$ are quite simple: they are finite unions of intervals. Other families of sets that enjoy these two properties exist, notably those definable in the first-order theory of the real numbers augmented with a symbol for the exponential function, an important result due to Wilkie [56]. Structures of $\mathbb{R}^d$ that are induced by such logical theories are called

o-minimal, and an *o-minimal set* is a set that belongs to such a structure [30]. These include semialgebraic sets, as well as sets definable in the first-order theory of the reals with restricted analytic functions.

In [2], it is shown that it is decidable whether there exists an o-minimal invariant (for a given LDS (M, x)) that is disjoint from a semialgebraic target T; and moreover, when such an invariant exists, it is always possible to exhibit one that is in fact semialgebraic [3]. Once again, these results are effective, and the invariants can always be explicitly produced.

Varying the class of invariants and the class of targets gives rise to a number of natural questions that— for the most part—remain unexplored. Let us however mention a further desirable property enjoyed both by the class of o-minimal sets and that of semialgebraic sets: in either case, they admit *minimal families* of invariants, a notion which we now explain. Let $\mathcal{F}$ be a class of sets—either the class of o-minimal sets or that of semialgebraic sets. It is easy to verify that, in general, $\mathcal{F}$ does not possess minimal invariants. Nevertheless, [2,3] show how to produce a sequence of (M, x)-invariants $\langle \mathcal{C}_k : k \in \mathbb{N} \rangle$, all belonging to $\mathcal{F}$, and such that $\mathcal{C}_{k+1} \subset \mathcal{C}_k$. It can moreover be shown that, given any (M, x)-invariant $\mathcal{I} \in \mathcal{F}$, it is always the case that $\mathcal{I}$ contains one of the $\mathcal{C}_k$, and *ipso facto* also all $\mathcal{C}_j$ for $j \geq k$.

5   Semialgebraic Initial Sets

Up until now we have exclusively considered problems concerning the orbit $\mathcal{O}(M, x)$ of a *single* initial point x. It is natural to ask whether the algorithmic problems which we have discussed remain solvable if one instead considers an entire *set* of initial points $S \subseteq \mathbb{R}^d$. Unfortunately the answer is negative. We sketch below the proof of the undecidability of a natural model-checking problem.

Theorem 7. *The following problem is undecidable. Given a natural number $k \in \mathbb{N}$, a semialgebraic set $S \subseteq \mathbb{R}^d$, a $d \times d$ rational matrix M, and a hyperplane H in $\mathbb{R}^d$ (having rational normal vector), determine whether there exists $x \in S$ such that the orbit generated by (M, x) hits H at least k times.*

In symbols, whether there is some $x \in S$ such that

$$|\mathcal{O}(M, x) \cap H| \geq k$$

is undecidable.

It is worth noting that all the problems that we have discussed so far (including the Skolem and Positivity Problems) are not known to be undecidable, and are in fact widely conjectured to be decidable. It is therefore perhaps somewhat

surprising that this natural generalisation of our setting immediately leads to undecidability.

The proof of Theorem 7 proceeds by reduction from a variant of Hilbert's tenth problem. Recall that Hilbert's tenth problem asks whether a given polynomial $P \in \mathbb{Z}[Y_1, Y_2, \ldots, Y_{d-1}]$ with integer coefficients and $d - 1$ variables has a root with all unknowns taking integer values. This problem is undecidable, as shown by Davis, Putnam, Robinson, and Matiyasevich [44].

The variant that we will reduce from asks whether the polynomial has roots with the unknowns being distinct natural numbers. It is straightforward to show that this variant is also undecidable.

Let $d \in \mathbb{N}$, $d > 1$, and $P \in \mathbb{Z}[Y_1, Y_2, \ldots, Y_{d-1}]$ be an arbitrary polynomial. We define the subset $S \subseteq \mathbb{R}^d$ via a formula of the first-order theory of the reals:

$$S(x_1, x_2, \ldots, x_d) \stackrel{\text{def}}{=} \exists y_1, y_2, \ldots y_{d-1} \begin{cases} 0 & = P(y_1, y_2, \ldots, y_{d-1}), \\ x_1 & = (1 - y_1)(1 - y_2) \cdots (1 - y_{d-1}), \\ x_2 & = (2 - y_1)(2 - y_2) \cdots (2 - y_{d-1}), \\ & \vdots \\ x_d & = (d - y_1)(d - y_2) \cdots (d - y_{d-1}). \end{cases}$$

A point $x := (x_1, \ldots, x_d) \in \mathbb{R}^d$ is in the set S if and only if one can find real numbers $y_1, \ldots, y_{d-1}$ for which the above equations hold. The idea behind this definition comes from the fact that, for $x, y_1, \ldots, y_{d-1}$ as above, one can construct a $d \times d$ matrix M with rational entries such that

$$(M^n x)_1 = (n - y_1)(n - y_2) \cdots (n - y_{d-1}),$$

for all $n \in \mathbb{N}$, where $(\cdot)_1$ refers to the first entry of the vector. Admitting the existence of such a matrix, let H be the hyperplane having normal vector $(1, 0, \ldots, 0)$ and going through the origin. Then clearly $\mathcal{O}(M, x)$ enters H at least $d - 1$ times if and only if the reals $y_1, \ldots, y_{d-1}$ are distinct natural numbers, because only then is the first entry of $M^n x$—the polynomial $(n - y_1) \cdots (n - y_{d-1})$—equal to zero.

The existence of the matrix M rests on the fact that the expression $u_n = (n - y_1) \cdots (n - y_{d-1})$ (for fixed $y_1, \ldots, y_{d-1}$) can be obtained as a linear recurrence sequence of order d, and in turn such a linear recurrence sequence can be represented as the sequence of fixed-position entries of increasing powers of a fixed $d \times d$ matrix M. In the standard construction of this matricial representation, one must in addition set $x_1 = u_1$, $x_2 = u_2$, $\ldots$, $x_d = u_d$, which is achieved through the definition of our initial semialgebraic set S.

It is worth noting that Theorem 7 holds even if k is fixed, due to the fact that Hilbert's tenth problem remains undecidable for a fixed number of variables. Furthermore, if k is fixed to be 1, then the problem becomes decidable in low dimensions, however even in the case where the ambient space has dimension 2 and $k = 2$, the problem does not seem to be trivially decidable.

6   Research Directions and Open Problems

We have presented an overview of the state of the art regarding decidability and solvability of a range of algorithmic problems for discrete linear dynamical systems, focussing on reachability, model-checking, and invariant-generation questions. In the case of model checking in particular, we have painted an essentially complete picture of what is achievable, both unconditionally as well as relative to Skolem oracles. We pointed out that extending the existing results further runs up against formidable mathematical obstacles (longstanding open problems in number theory); the work presented here therefore appears to lie at the very frontier of what is attainable, barring major breakthroughs in mathematics.

   A central open question is whether Skolem oracles can actually be devised and implemented. As remarked earlier, a certifying algorithm for the Simple-Skolem Problem has recently been proposed, with termination relying on classical (and widely believed) number-theoretic conjectures. Whether similar algorithms can be obtained for the general Skolem Problem is the subject of active research.

   An even more difficult question is whether oracles for the Positivity Problem (or Simple-Positivity Problem) can be devised. This would enable one to circumvent the mathematical obstacles mentioned earlier, and allow one to substantially extend the scope of semialgebraic model checking for linear dynamical systems. For the time being, this goal appears to be well out of reach.

   In the present paper we have entirely confined ourselves to matters of decidability, largely on account of space limitations, but also because the complexity-theoretic picture is not nearly as clear-cut as its decidability counterpart. Providing a comprehensive account of the complexity landscape for linear dynamical systems would be an interesting and promising research direction.

A   Prefix-Independent Model Checking for LDS

The goal of this appendix is to exhibit boundaries on the extent to which Theorem 4 can be improved. More precisely, we show that the ability to solve the model-checking problem for arbitrary LDS against prefix-independent MSO specifications making use of semialgebraic predicates in ambient space $\mathbb{R}^4$ would necessarily entail major breakthroughs in Diophantine approximation.

   We build upon the framework developed in [47, Sec. 5]. To this end, consider the class of order-6 rational LRS of the form

$$u_n = -n + \frac{1}{2}(n - ri)\lambda^n + \frac{1}{2}(n + ri)\overline{\lambda}^n = r\operatorname{Im}(\lambda^n) - n(1 - \operatorname{Re}(\lambda^n)),$$

where $\lambda \in \mathbb{Q}(i)$ and $|\lambda| = 1$, and $r \in \mathbb{Q}$. Let us write $\mathcal{L}$ to denote this class of LRS.

   It is shown in [47] that solving the *Ultimate Positivity Problem* for LRS in $\mathcal{L}$, i.e., providing an algorithm which, given an LRS $\langle u_n \rangle_{n=0}^{\infty} \in \mathcal{L}$, determines whether there exists some integer N such that, for all $n \geq N$, $u_n \geq 0$, would necessarily entail major breakthroughs in the field of Diophantine approximation.

The purpose of the present section is to reduce the Ultimate Positivity Problem for LRS in $\mathcal{L}$ to the prefix-independent semialgebraic MSO model-checking problem for 4-dimensional LDS.

Given λ and r as above, let

$$
M = \begin{bmatrix} \mathrm{Re}(\lambda) & -\mathrm{Im}(\lambda) & 1 & 0 \\ \mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) & 0 & 1 \\ 0 & 0 & \mathrm{Re}(\lambda) & -\mathrm{Im}(\lambda) \\ 0 & 0 & \mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.
$$

Observe that M has rational entries. We have that

$$
M^n x = \begin{bmatrix} \mathrm{Re}(\lambda^n) - \mathrm{Im}(\lambda^n) + n\,\mathrm{Re}(\lambda^{n-1}) - n\,\mathrm{Im}(\lambda^{n-1}) \\ \mathrm{Im}(\lambda^n) + \mathrm{Re}(\lambda^n) + n\,\mathrm{Im}(\lambda^{n-1}) + n\,\mathrm{Re}(\lambda^{n-1}) \\ \mathrm{Re}(\lambda^n) - \mathrm{Im}(\lambda^n) \\ \mathrm{Im}(\lambda^n) + \mathrm{Re}(\lambda^n) \end{bmatrix}.
$$

As a semialgebraic target consider the set $S = \{x : p(x) > 0\}$, where

$$
p(x_1, x_2, x_3, x_4) = \frac{r}{2}(x_4 - x_3) - \frac{x_1 - x_3}{\mathrm{Re}(\lambda^{-1})x_3 - \mathrm{Im}(\lambda^{-1})x_4}\left(1 - \frac{x_3 + x_4}{2}\right).
$$

We now have that $p(M^n x) = r\,\mathrm{Im}(\lambda^n) - n(1 - \mathrm{Re}(\lambda^n))$, and that $\langle u_n \rangle_{n=0}^{\infty}$ is ultimately positive if and only if the orbit of x under M eventually gets trapped in S. This can be expressed by the prefix-independent LTL formula $\varphi = \mathbf{F}\,\mathbf{G}\,S$.

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