

A characterization of definability in the theory of real addition

Alexis Bès

Univ. Paris Est Creteil, LACL, 94000, Creteil, France

`bes@u-pec.fr`

Christian Choffrut

IRIF (UMR 8243), CNRS and Université Paris 7 Denis Diderot, France

`Christian.Choffrut@irif.fr`

Abstract

Given a subset of $X \subseteq \mathbb{R}^n$ we can associate with every point $x \in \mathbb{R}^n$ a vector space V of maximal dimension with the property that for some ball centered at x , the subset X coincides inside the ball with a union of hyperplanes parallel with V . A point is singular if V has dimension 0.

In an earlier paper we proved that a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation X is actually definable in $\langle \mathbb{R}, +, <, 1 \rangle$ if and only if the number of singular points is finite and every rational section of X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, where a rational section is a set obtained from X by fixing some component to a rational value.

Here we show that we can dispense with the hypothesis of X being $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable by assuming that the components of the singular points are rational numbers. This provides a topological characterization of first-order definability in the structure $\langle \mathbb{R}, +, <, 1 \rangle$.

1 Introduction

This paper continues the line of research started in [1]. Consider the structure $\langle \mathbb{R}, +, <, 1 \rangle$ of the additive ordered group of reals along with the constant 1. It is well-known that the subgroup \mathbb{Z} of integers is not first-order-definable. Add the predicate $x \in \mathbb{Z}$ resulting in the structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$. In [1] we prove a topological characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relations in the family of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations, and use it to derive, on the one hand, that it is decidable whether or not a relation on the reals definable in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ can be defined in $\langle \mathbb{R}, +, <, 1 \rangle$, and on the other hand that there is no intermediate structure between $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ and $\langle \mathbb{R}, +, <, 1 \rangle$ (since then the latter result was generalized by Walsberg [4] to a large class of *o*-minimal structures).

The topological characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$, can be described as follows. We say that the neighborhood of a point $x \in \mathbb{R}^n$ relative to a relation $X \subseteq \mathbb{R}^n$ has *strata* if there exists a direction such that the intersection of all sufficiently small neighborhoods around x with X is the trace of a union of lines parallel to the given direction. When X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, all points have strata, except finitely many which we call singular. In [1] we give necessary and sufficient conditions for a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation to be $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, namely (FSP): it has finitely many singular points and (DS): all intersections of X with arbitrary hyperplanes parallel to $n - 1$ axes and having rational components on the remaining axis are $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. These conditions were inspired by Muchnik's characterization of definability in Presburger Arithmetic [3] (see [1] for details).

In [1] we asked whether it is possible to remove the assumption that X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable in our characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ -definability. In the present paper we prove that the answer is positive provided an additional assumption is required: (RSP) all singular points of X have rational components.

Let us explain the structure of the proof. The necessity of the two conditions (FSP) and (DS) of our characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ in [1] is trivial. The difficult part was their sufficiency and it used very specific properties of the $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations, in particular the fact that $\langle \mathbb{R}, +, <, 1 \rangle$ - and $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations are locally indistinguishable. In order to construct a $\langle \mathbb{R}, +, <, 1 \rangle$ -formula for a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation we showed two intermediate properties, (RB): for every nonsingular point x , the set of strata at x is a subspace which can be generated by a set of vectors with rational coefficients, and (FI): there are finitely many "neighborhood types" (i.e., the equivalence relation $x \sim y$ on \mathbb{R}^n which holds if there exists $r > 0$ such that $(x + w \in X \leftrightarrow y + w \in X$ for every $|w| < r$) has finite index). For general relations, the same intermediate properties are used but (RB) and (FI) are far from being obvious and are actually insufficient since we need the extra condition (RSP).

We give a short outline of our paper. Section 2 gathers basic notations and definitions. In Section 3 we recall the main useful definitions and results from [1]. In Section 4 we show how the condition " X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable" can be replaced by the conjunction of conditions (RSP), (RB) and (FI), then state and prove the main result. We also provide an alternative formulation of this result in terms of generalized projections of X .

2 Preliminaries

Throughout this work we assume the vector space \mathbb{R}^n is provided with the metric L_∞ (i.e., $|x| = \max_{1 \leq i \leq n} |x_i|$). The open ball centered at $x \in \mathbb{R}^n$ and of radius $r > 0$ is denoted by $B(x, r)$. Given $x, y \in \mathbb{R}^n$ we denote $[x, y]$ (resp. (x, y)) the closed segment (resp. open segment) with extremities x, y . We use also notations such as $[x, y)$ or $(x, y]$ for half-open segments.

Let us specify our logical conventions and notations. We work within first-order predicate calculus with equality. We confuse formal symbols and their interpretations.

We are mainly concerned with the structures $\langle \mathbb{R}, +, <, 1 \rangle$ and $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$. Given a structure \mathcal{M} with domain D and $X \subseteq D^n$, we say that X is definable in \mathcal{M} , or \mathcal{M} -definable, if there exists a formula $\varphi(x_1, \dots, x_n)$ in the signature of \mathcal{M} such that $\varphi(a_1, \dots, a_n)$ holds in \mathcal{M} if and only if $(a_1, \dots, a_n) \in X$ (this corresponds to the usual notion of definability without parameters).

The $\langle \mathbb{R}, +, <, 1 \rangle$ -theory admits quantifier elimination in the following sense, which can be interpreted geometrically as saying that a $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relation is a finite union of closed and open polyhedra.

Theorem 1. [2, Thm 1] *Every formula in $\langle \mathbb{R}, +, <, 1 \rangle$ is equivalent to a Boolean combination of inequalities between linear combinations of variables with coefficients in \mathbb{Z} (or, equivalently, in \mathbb{Q}).*

3 Local properties

Most of the notions and results in this section are taken from [1]. We only give formal proofs for the new results.

3.1 Strata

The following clearly defines an equivalence relation.

Definition 2. Given $x, y \in \mathbb{R}^n$ we write $x \sim_X y$ or simply $x \sim y$ when X is understood, if there exists a real $r > 0$ such that the translation $w \mapsto w + y - x$ is a one-to-one mapping from $B(x, r) \cap X$ onto $B(y, r) \cap X$.

Example 3. Consider a closed subset of the plane delimited by a square. There are 10 equivalence classes: the set of points interior to the square, the set of points interior to its complement, the four vertices and the four open edges.

Let $\mathcal{Cl}(x)$ denote the \sim -equivalence class of x .

Definition 4. 1. Given a non-zero vector $v \in \mathbb{R}^n$ and a point $y \in \mathbb{R}^n$ we denote $L_v(y)$ the line passing through y in the direction v . More generally, if $X \subseteq \mathbb{R}^n$ we denote $L_v(X)$ the set $\bigcup_{x \in X} L_v(x)$.

2. A non-zero vector $v \in \mathbb{R}^n$ is an X -stratum at x (or simply a stratum when X is understood) if there exists a real $r > 0$ such that

$$B(x, r) \cap X = B(x, r) \cap L_v(X) \quad (1)$$

This can be seen as saying that inside the ball $B(x, r)$, the relation X is a union of lines parallel to v . By convention the zero vector is also considered as a stratum.

3. The set of X -strata at x is denoted by $\text{Str}_X(x)$ or simply $\text{Str}(x)$.

Proposition 5. [1, Proposition 9] For all $X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ the set $\text{Str}(x)$ is a vector subspace of \mathbb{R}^n .

Definition 6. Given a relation $X \subseteq \mathbb{R}^n$, the dimension $\dim(x)$ of a point $x \in \mathbb{R}^n$ is the dimension of the subspace $\text{Str}(x)$. We say that x is a d -point if $d = \dim(x)$. Moreover if $d = 0$ then x is said to be X -singular, or simply singular, and otherwise it is nonsingular.

Example 7. (Example 3 continued) Let $x \in \mathbb{R}^2$. If x belongs to the interior of the square or of its complement, then $\text{Str}(x) = \mathbb{R}^2$. If x is one of the four vertices of the square then we have $\text{Str}(x) = \{0\}$, i.e x is singular. Finally, if x belongs to an open edge of the square but is not a vertex, then $\text{Str}(x)$ has dimension 1, and two points of opposite edges have the same one-dimensional subspace, while two points of adjacent edges have different one-dimensional subspaces.

It can be shown that all strata at x can be defined by a common value r in expression (1).

Proposition 8. [1, Proposition 14] If $\text{Str}(x) \neq \{0\}$ then there exists a real $r > 0$ such that for every $v \in \text{Str}(x) \setminus \{0\}$ we have

$$B(x, r) \cap X = B(x, r) \cap L_v(X).$$

Definition 9. A X -safe radius (or simply a safe radius when X is understood) for x is a real $r > 0$ satisfying the condition of Proposition 8. Clearly if r is safe then so are all $0 < s \leq r$. By convention every real is a safe radius if $\text{Str}(x) = \{0\}$.

Example 10. (Example 3 continued) For an element x of the interior of the square or the interior of its complement, let r be the (minimal) distance from x to the edges of the square. Then r is safe for x . If x is a vertex then $\text{Str}(x) = \{0\}$ and every $r > 0$ is safe for x . In all other cases r can be chosen as the minimal distance of x to a vertex.

Remark 11. If $x \sim y$ then $\text{Str}(x) = \text{Str}(y)$ therefore given an \sim -equivalence class E , we may define $\text{Str}(E)$ as the set of common strata of all $x \in E$.

Observe that the converse is false. Indeed consider, e.g., $X = \{(x, y) \mid y \leq 0\} \cup \{(x, y) \mid y = 1\}$ in \mathbb{R}^2 . The points $(0, 0)$ and $(0, 1)$ have the same subspace of strata, namely that generated by $(1, 0)$, but $x \not\sim y$.

It is possible to combine the notions of strata and of safe radius.

Lemma 12. [1, Lemma 18] Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ and r be a safe radius for x . Then for all $y \in B(x, r)$ we have $\underline{\text{Str}}(x) \subseteq \underline{\text{Str}}(y)$.

Along a stratum all points inside a ball of a safe radius are \sim -equivalent.

Lemma 13. Let x be non-singular, $v \in \text{Str}(x) \setminus \{0\}$, and r be safe for x . For every $z \in B(x, r)$ we have $L_v(z) \cap B(x, r) \subseteq \text{Cl}(z)$.

Proof. Let z' be a point on $L_v(z) \cap B(x, r)$ and $s > 0$ be such that both $B(z, s), B(z', s)$ are included in $B(x, r)$. For every $w \in B(0, s)$ we have $z + w \in X \Leftrightarrow z' + w \in X$. \square

Example 14. (Example 3 continued) Consider a point x on an (open) edge of the square and a safe radius r for x . For every point y in $B(x, r)$ which is not on the edge we have $\text{Str}(x) \subset \text{Str}(y) = \mathbb{R}^2$. For all other points we have $\text{Str}(x) = \text{Str}(y)$.

3.2 Relativization to affine subspaces

We relativize the notion of singularity and strata to an affine subspace $S \subseteq \mathbb{R}^n$. The next definition should come as no surprise.

Definition 15. Given a subset $X \subseteq \mathbb{R}^n$, an affine subspace $S \subseteq \mathbb{R}^n$ and a point $x \in S$, we say that a vector $v \neq \{0\}$ parallel to S is an (X, S) -stratum for the point x if for all sufficiently small $r > 0$ it holds

$$S \cap X \cap B(x, r) = S \cap L_v(X) \cap B(x, r)$$

The set of (X, S) -strata of x is denoted $\text{Str}_{(X, S)}(x)$. We define the equivalence relation $x \sim_{(X, S)} y$ on S as follows: $x \sim_{(X, S)} y$ if and only if there exists a real $r > 0$ such that $x + w \in X \Leftrightarrow y + w \in X$ for every w parallel to S and such that $|w| < r$. A point $x \in S$ is (X, S) -singular if it has no (X, S) -stratum. For simplicity when S is the space \mathbb{R}^n we will still stick to the previous terminology and speak of X -strata and X -singular points.

Remark 16. Singularity and nonsingularity do not go through restriction to affine subspaces. E.g., in the real plane, let $X = \{(x, y) \mid y < 0\}$ and $S = \{(x, y) \mid x = 0\}$. Then the origin is not X -singular but it is (X, S) -singular. All other elements of S admit $(0, 1)$ as an (X, S) -stratum thus they are not (X, S) -singular. The opposite situation may occur. In the real plane, let $X = \{(x, y) \mid y < 0\} \cup S$. Then the origin is X -singular but it is not (X, S) -singular.

3.2.1 Relativization of the space of strata

Lemma 17. Let S be an affine subspace and $x \in S$. Let V the subspace generated by $(\text{Str}_X(x) \setminus \text{Str}_{(X, S)}(x)) \cup \{0\}$. If $V \neq \{0\}$ then $\text{Str}_X(x) = V + \text{Str}_{(X, S)}(x)$, and otherwise $\text{Str}_X(x) \subseteq \text{Str}_{(X, S)}(x)$.

Proof. It is clear that if $V \neq \{0\}$ then every X -stratum of S is an (X, S) -stratum.

Now assume there exists $v \in \text{Str}_X(x) \setminus \text{Str}_{(X, S)}(x)$. It suffices to prove that for all $w \in \text{Str}_{(X, S)}(x)$ we have $w \in \text{Str}_X(x)$. Let $s > 0$ be simultaneously (X, S) -safe and X -safe for x . Let $0 < s' < s$ be such that $L_v(z) \cap S \subseteq B(x, s)$ for every $z \in B(x, s')$. Let $y_1, y_2 \in B(x, s')$ be such that $y_1 - y_2$ and w are parallel. It suffices to prove the equivalence $y_1 \in X \Leftrightarrow y_2 \in X$. Let y'_1 (resp. y'_2) denote the intersection point of $L_v(y_1)$ and S (resp. $L_v(y_2)$ and S). We have $y_1, y'_1 \in B(x, s)$, $v \in \text{Str}_X(x)$, and s is X -safe for x , thus $y_1 \in X \Leftrightarrow y'_1 \in X$. Similarly we have $y_2 \in X \Leftrightarrow y'_2 \in X$. Now $y'_1, y'_2 \in B(x, s)$, $y'_1 - y'_2$ and w are parallel, and $w \in \text{Str}_{(X, S)}(x)$, which implies $y'_1 \in X \Leftrightarrow y'_2 \in X$. \square

Corollary 18. *Let S be an affine subspace with underlying subspace V , and let $x \in S$ be non-singular. If $\text{Str}_X(x) \setminus V$ is nonempty then $\text{Str}_{(X,S)}(x) = \text{Str}_X(x) \cap V$.*

3.2.2 Relativization of the \sim -relation

Lemma 19. *Let $X \subseteq \mathbb{R}^n$, S be an affine subspace of dimension $n-1$, $y, z \in S$, and $v \neq \{0\}$ be a common X -stratum of y, z not parallel to S . If $y \sim_{(X,S)} z$ then $y \sim_X z$.*

Proof. Assume $y \sim_{(X,S)} z$, and let $r > 0$ be (X, S) -safe both for y and z . Let $0 < r' < r$ be X -safe both for y and z . Since v is not parallel to S , there exists $s > 0$ such that for every $w \in \mathbb{R}^n$ with $|w| < s$, the intersection point of $L_v(y+w)$ (resp. $L_v(z+w)$) and S exists and belongs to $B(y, r')$ (resp. $B(z, r')$).

It suffices to show that for every $w \in \mathbb{R}^n$ with $|w| < s$ we have $y+w \in X \leftrightarrow z+w \in X$. Let w' be such that $y+w'$ is the intersection point of $L_v(y+w)$ and S .

By our hypothesis on s , $y+w'$ belongs to $B(y, r')$. Moreover r' is X -safe for y , $v \in \text{Str}_X(y)$, and $w'-w$ is parallel to v , therefore $y+w \in X \leftrightarrow y+w' \in X$. Similarly we have $z+w \in X \leftrightarrow z+w' \in X$. Now $|w'| < r' < r$, thus by our assumptions on y, z and r we have $y+w' \in X \leftrightarrow z+w' \in X$. \square

We consider here a particular case for S which plays a crucial role in expressing the characterisation stated in the main theorem and in our reasoning by induction in Section 4.3.

Definition 20. Given an index $0 \leq i < n$ and a real $c \in \mathbb{R}$ consider the hyperplane

$$H = R^i \times \{c\} \times \mathbb{R}^{n-i-1}$$

The intersection $X \cap H$ is called a section of X . It is a rational section if c is a rational number. We define π_H as the projection $R^i \times \{c\} \times \mathbb{R}^{n-i-1} \rightarrow \mathbb{R}^{n-1}$.

The following facts are easy consequences of the above definitions: for all $x, y \in H$ and v a vector parallel to H we have:

1. $x \sim_{(X,H)} y$ if and only if $\pi_H(x) \sim_{\pi_H(X)} \pi_H(y)$
2. $v \in \text{Str}_{(X,H)}(x)$ if and only if $\pi_H(v) \in \text{Str}_{\pi_H(X)}(\pi_H(x))$. In particular x is (X, H) -singular if and only if $\pi_H(x)$ is $\pi_H(X)$ -singular.

3.3 Intersection of lines and equivalence classes

In this section we describe the intersection of a \sim -class E with a line parallel to some $v \in \text{Str}(E)$. It introduces the notion of compatibility of \sim -classes.

Lemma 21. *[1, Lemma 34 and Corollary 36] Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$, $E = \text{Cl}(x)$ and let $v \in \text{Str}(x) \setminus \{0\}$. The set $L_v(x) \cap E$ is a union of disjoint open segments (possibly infinite in one or both directions) of $L_v(x)$, i.e., of the form $(y - \alpha v, y + \beta v)$ with $0 < \alpha, \beta \leq \infty$ and $y \in E$.*

If $\alpha < \infty$ (resp. $\beta < \infty$) then the point $y - \alpha v$ (resp. $y + \beta v$) belongs to a \sim -class $F \neq E$ such that $\dim(F) < \dim(E)$, and we say that F is v -compatible (resp. $(-v)$ -compatible) (or simply compatible when v is understood) with E .

4 Main result

4.1 The characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ - in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations

We recall our previous characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ -definable among $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations. Recall that the notion of section is defined in Definition 20.

Theorem 22. [1, Theorem 37] Let $n \geq 1$ and let $X \subseteq \mathbb{R}^n$ be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable. Then X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable if and only if the following two conditions hold:

(FSP) There exist only finitely many singular points;

(DS) Every rational section of X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

The necessity of condition (FSP) is proved by Proposition 27 of [1] and that of (DS) is trivial since a rational section is the intersection of two $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relations. The proof that conditions (FSP) and (DS) are sufficient uses several properties of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations which are listed in the form of a proposition below.

Proposition 23. Let $n \geq 1$ and $X \subseteq \mathbb{R}^n$ be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable. The following holds.

(RSP) The components of the singular points are rational numbers [1, Proposition 27].

(FI) The equivalence relation \sim has finite index and thus the number of different spaces $\text{Str}(x)$ is finite when x runs over \mathbb{R}^n [1, Corollary 25].

(RB) For all nonsingular points x , the vector space $\text{Str}(x)$ has a rational basis in the sense that it can be generated by a set of vectors with rational coefficients [1, Proposition 28].

Along with the two properties (FSP), (DS) of Theorem 22, condition (RSP) proves the “only if” direction of Theorem 24. These three properties are also instrumental in the proof of the “if” direction when, using property (DS), the induction on the dimension of the space reduces an arbitrary relation to an $\langle \mathbb{R}, +, <, 1 \rangle$ -relation.

4.2 The general case

Here we show that we may remove the condition “ X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable”, i.e., state a result for arbitrary relations, at the (modest) price of adding condition (RSP).

Theorem 24. Let $n \geq 1$ and $X \subseteq \mathbb{R}^n$. Then X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable if and only if it satisfies the three conditions (FSP), (DS), (RSP).

(FSP) It has only finitely many singular points.

(DS) Every rational section of X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

(RSP) Every singular point has rational components.

Observe that the three conditions are needed, as shown by the following relations which are not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

- Consider the binary relation $X = \{(x, x) \mid x \in \mathbb{Z}\}$. The singular elements of X are precisely the elements of X , thus X satisfies (RSP) but not (FSP). It satisfies (DS) because every rational section of X is either empty or equal to the singleton $\{(x, x)\}$ for some $x \in \mathbb{Z}$, thus is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.
- The binary relation $X = \mathbb{R} \times \mathbb{Z}$ has no singular point thus it satisfies (FSP) and (RSP). However it does not satisfy (DS) since, e.g., the rational section $\{0\} \times \mathbb{Z}$ is not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.
- The unary relation $X = \{\sqrt{2}\}$ admits $\sqrt{2}$ as its unique singular point, thus it satisfies (FSP) but not (RSP). It satisfies (DS) since every rational section of X is empty.

Proof. The necessity of the first two conditions is a direct consequence of Theorem 22, that of the third condition is due to Proposition 23.

The proof in the other direction is based on two claims 25 and 26 which show that (RB) and (FI) respectively are consequences of conditions (FSP), (DS) and (RSP).

Claim 25. *If X satisfies conditions (FSP), (DS) and (RSP) then it satisfies condition (RB).*

Proof. We prove that for every non-singular point $x \in \mathbb{R}^n$, $\text{Str}(x)$ has a rational basis. If $n = 1$ this follows from the fact that for every $x \in \mathbb{R}$ the set $\text{Str}(x)$ is either equal to $\{0\}$ or equal to \mathbb{R} , thus we assume $n \geq 2$.

For every $i \in \{1, \dots, n\}$ let $H_i = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = 0\}$, and let us call rational i -hyperplane any hyperplane S of the form $S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = c\}$ where $c \in \mathbb{Q}$ (note that the direction of S is H_i).

Let x be a d -point with $d \geq 1$, i.e., a point for which $V = \text{Str}(x)$ has dimension d . For $d = n$ the result is obvious. For $1 \leq d < n$ we prove the result by induction on d .

Case $d = 1$: It suffices to show that every 1-point x has a stratum in \mathbb{Q}^n . Let $v \in \text{Str}(x) \setminus \{0\}$, and let $r > 0$ be safe for x . We can find i and two distinct rational i -hyperplanes S_1 and S_2 , not parallel to v , and such that $L_v(x)$ intersects S_1 (resp. S_2) inside $B(x, r)$, say at some point y_1 (resp. y_2). By Lemma 13 we have $y_1 \sim x$. By Corollary 18 it follows that

$$\text{Str}_{(X, S_1)}(y_1) = \text{Str}_X(y_1) \cap H_i = \text{Str}_X(x) \cap H_i$$

and the rightmost expression is reduced to $\{0\}$ since $d = 1$ and $v \notin H_i$. This implies that y_1 is (X, S_1) -singular, i.e., that $\pi_{S_1}(y_1)$ is $\pi_{S_1}(X)$ -singular. Similarly y_2 is (X, S_2) -singular, i.e., $\pi_{S_2}(y_2)$ is $\pi_{S_2}(X)$ -singular.

By condition (DS) the rational sections $X \cap S_1$ (resp. $X \cap S_2$) are $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, thus the $(n-1)$ -ary relations $\pi_{S_1}(X)$ (resp. $\pi_{S_2}(X)$) are also $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, and by point (RSP) of Proposition 23 this implies that $\pi_S(y_1)$ (resp. $\pi_S(y_2)$) has rational components. Thus the same holds for y_1 and y_2 , and also for $y_1 - y_2$, and the result follows from the fact that $y_1 - y_2 \in \text{Str}_X(x)$.

Case $2 \leq d < n$: Let $I \subseteq \{1, \dots, n\}$ denote the set of indices i such that $V \not\subseteq H_i$. We have $V \subseteq \bigcap_{i \in \{1, \dots, n\} \setminus I} H_i$ thus $\dim(V) \leq n - (n - |I|) = |I|$, and it follows from our assumption $\dim(V) = d \geq 2$ that $|I| \geq 2$.

Now we prove that $V = \sum_{i \in I} (V \cap H_i)$. It suffices to prove $V \subseteq \sum_{i \in I} (V \cap H_i)$, and this in turn amounts to prove that $\dim(\sum_{i \in I} (V \cap H_i)) = d$. For every $1 \leq i \leq n$ we have

$$\dim(V + H_i) = \dim(V) + \dim(H_i) - \dim(V \cap H_i)$$

Now if $i \in I$ then $\dim(V + H_i) > \dim(H_i)$ i.e. $\dim(V + H_i) = n$, which leads to $\dim(V \cap H_i) = d + (n - 1) - n = d - 1$. Thus in order to prove $\dim(\sum_{i \in I} (V \cap H_i)) = d$ it suffices to show that there exist $i, j \in I$ such that $V \cap H_i \neq V \cap H_j$. Assume for a contradiction that for all $i, j \in I$ we have $V \cap H_i = V \cap H_j$. Then for every $i \in I$ we have

$$V \cap H_i = V \cap \bigcap_{j \in I} H_j \subseteq \bigcap_{j \notin I} H_j \cap \bigcap_{j \in I} H_j = \{0\}$$

which contradicts the fact that $\dim(V \cap H_i) = d - 1 \geq 1$.

We proved that $V = \sum_{i \in I} (V \cap H_i)$, thus it suffices to prove that for every $i \in I$, $V \cap H_i$ has a rational basis. Let $v \in V \setminus H_i$, and let r be safe for x . We can find a rational i -hyperplane S not parallel to v and such that the intersection point of S and $L_v(x)$, say y , belongs to $B(x, r)$. By Lemma 13 (applied to $z = x$) we have $y \sim x$. Corollary 18 then implies

$$\text{Str}_{(X,S)}(y) = \text{Str}_X(y) \cap H_i = \text{Str}_X(x) \cap H_i = V \cap H_i$$

which yields

$$\text{Str}_{\pi_S(X)}(y) = \pi_S(V \cap H_i)$$

Now by condition (DS), $X \cap S$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, and $\pi_S(X)$ as well. By condition (RB) this implies that $\pi_S(V \cap H_i)$ has a rational basis, and this implies that $V \cap H_i$ also has a rational basis. \square

Claim 26. *If X satisfies conditions (FSP), (DS) and (RSP) then it satisfies condition (FI).*

Proof. Before proving the claim we need a simple definition.

Definition 27. A subset Z is X -isolated (or simply isolated when X is understood) if there exists a \sim_X -class E such that Z is the subset of elements x of E such that $L_v(x) \subseteq Z$ for all $v \in \text{Str}(E)$.

Lemma 28. *Let $X \subseteq \mathbb{R}^n$ satisfy (FSP) and (DS). We have*

1. *let E be a \sim -class and $Z \subseteq E$ be isolated.*

- (a) *if $\text{Str}(E) = \{0\}$ then Z is a finite union of points with rational components.*
- (b) *if $\text{Str}(E) \neq \{0\}$ then Z is a finite union of parallel affine subspaces with direction $\text{Str}(E)$ each having a point with rational components*

2. *There exist finitely many isolated subsets.*

Proof. By induction on n . For $n = 1$ if X is equal to \mathbb{R} or to the empty set, the only isolated set is X and it obviously satisfies (1b). Otherwise every nonempty isolated set Z belongs to a \sim -class E such that $\text{Str}(E) = \{0\}$, i.e. is a union of singular points. Now by (FSP) and (DS) there exist finitely many such points and they have rational components, which implies (1a) and (2).

Now let $n \geq 1$. Using the same argument as above, we know that all isolated sets Z such that Z belongs to a \sim -class E with $\text{Str}(E) = \{0\}$ satisfy (1a), and moreover there are finitely many such sets Z . Thus in order to prove (2) it suffices to consider the case where $Z \neq \emptyset$ and $\text{Str}(E) \neq \{0\}$. In this case there exist $v \in \text{Str}(E)$ and $i \in \{1, \dots, n\}$ such that Z intersects the hyperplane H_i . All elements of $Z \cap H_i$ are \sim_X -equivalent thus they are also $\sim_{(X, H_i)}$ -equivalent. Furthermore for every $x \in Z \cap H_i$ we have $\text{Str}_{(X, H_i)}(x) = \text{Str}_X(x) \cap H_i$ by Corollary 18, and the fact that $x \in Z$ implies that for every $w \in \text{Str}_X(x) \cap H_i$ we have $L_w(x) \subseteq Z \cap H_i$. This shows that $\pi_{H_i}(x)$ belongs to a $\pi_{H_i}(X)$ -isolated class, hence $\pi_{H_i}(Z)$ is included in a $\pi_{H_i}(X)$ -isolated class, say W .

Now by (DS) $\pi_{H_i}(X)$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, thus by Theorem 22 it satisfies also (FSP) and (DS). By our induction hypothesis it follows that W can be written as $W = \bigcup_{j=1}^p W_j$, where either all W_j 's are parallel affine subspaces with direction $\pi_{H_i}(E)$ each having one rational point (by (1b)), or each W_j is reduced to a point with rational components (by (1a)). Every W_j which intersects $\pi_{H_i}(Z)$ satisfies $W_j \subseteq \pi_{H_i}(Z)$, which shows that $\pi_{H_i}(Z) = \bigcup_{j \in J} W_j$ for some $J \subseteq \{1, \dots, p\}$. That is, we have $Z \cap H_i = \bigcup_{j \in J} W'_j$ where each W'_j denotes the subset of H_i such that $\pi_{H_i}(W'_j) = W_j$. Observe that if x is a rational point in W_j then $x' \in H_j$ is also a rational point if $\pi_{H_i}(x') = x$. Now $Z = (Z \cap H_i) + \text{Str}(E)$ thus $Z = \bigcup_{j \in J} (W'_j + \text{Str}(E))$. Since the direction of each W'_j is included in $\text{Str}(E)$, this proves (1).

For (2) we observe that Z is completely determined by $Z \cap H_i$, i.e. $\pi_{H_i}(Z)$. By our induction hypothesis there are finitely many $\pi_{H_i}(X)$ -isolated subsets hence finitely many possible sets W_j , and finitely many possible union of such sets. \square

Now we turn to the proof of Claim 26. Lemma 28 shows that the number of \sim -classes having a non-empty isolated subset is finite. It thus suffices to prove that for every $0 \leq d \leq n$ there exist finitely many d -classes whose isolated subset is empty. In particular, if the dimension of such a \sim -class E is non-zero there exists a \sim -class F such that F is compatible with E and $\dim(E) > \dim(F)$. Remember from Lemma 21 that this means that there exists $y \in F$ and $x \in E$ such that $[x, y] \subseteq Cl(x)$ where $x - y \in \text{Str}(E)$.

For $d = n$ there exist at most two d -classes, which correspond to elements in the interior of X or the interior of its complement.

For $0 \leq d < n$ we reason by induction on d and show that the number of d -classes compatible with d' -classes for some $d' < d$ is finite. Since by induction the number of classes of dimension less than d is finite, this will prove the claim. For $d = 0$ the result follows from (FSP) and the fact that each 0-class is a union of singular points.

Now we assume $0 < d < n$. By induction hypothesis there exist finitely many d' -classes for $d' < d$. Thus in order to meet a contradiction we assume that there exists a d' -class F which is compatible with infinitely many d classes $E_j, j \in J$. We may furthermore assume that for each class E_j there is no integer $d'' > d'$ and no d'' -class which is compatible with E_j .

We first consider the case $d' = 0$.

Let $y \in F$. Because of condition (FSP), for some real $s > 0$ the point y is the unique singular point in $B(y, s)$. Moreover for every $j \in J$, F is compatible with E_j , thus there exists a point $x_j \in E_j$ such that $[x_j, y] \subseteq E_j$. Let HL_j denote the open halfline with endpoint y and containing x_j . Observe that we necessarily have $HL_j \cap B(y, s) \subseteq Cl(x_j)$. Indeed, by Lemma 21 the condition $HL_j \cap B(y, s) \subsetneq Cl(x_j)$ implies that there exists a point $z = y + \alpha(x_j - y) \in B(y, s)$ such that $\alpha > 1$ and $\dim(z) < d$. Since y is the unique singular point in $B(y, s)$ this implies $\dim(z) > 0$ but then because of $[x_j, z] \subseteq Cl(x_j)$ the maximality condition stipulated for d' is violated.

Thus, let z_j be the point on HL_j at distance $\frac{s}{2}$ from y and let z be adherent to the set of these z_j 's. The point z is nonsingular since y is the unique singular point in the ball $B(y, s)$. Let $v \in \text{Str}(z) \setminus \{0\}$. Consider some $\ell \in \{1, \dots, n\}$, some rational ℓ -hyperplane S such that $z \notin S$ and some $t < \frac{s}{2}$ such that $L_v(B(z, t)) \cap S \subseteq B(z, \frac{s}{2})$. The ball $B(z, t)$ contains infinitely many non \sim -equivalent points, and by Lemma 19 their projections on S in the direction v are non $\sim_{(X, S)}$ -equivalent. But by condition (DS) the relation $X \cap S$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, thus $\pi_S(X)$ satisfies condition (FI) of Proposition 23, a contradiction.

Now we consider the case where $d' > 0$.

Let $y \in F$. By definition y is adherent to all d -classes E_j for $j \in J$. Choose some $v \in \text{Str}(y)$ and let r be a safe radius for y . We can find $0 < s < r$, $k \in \{1, \dots, n\}$ and some k -hyperplane S not parallel to v such that $L_v(B(y, s)) \cap S \subseteq B(y, r)$. By definition of y , $B(y, s)$ intersects infinitely many pairwise distinct d -classes. Given two non \sim -equivalent d -points $z_1, z_2 \in B(y, s)$, and w_1, w_2 their respective projections over S along the direction v , we have $w_1 \not\sim_{(X, S)} w_2$ by Lemma 19. This implies that there exist infinitely many $\sim_{(X, S)}$ -classes. However by condition (DS), the relation $X \cap S$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, thus $\pi_S(X)$ satisfies condition (FI) of Proposition 23, a contradiction. \square

Observe that X is equal to the union of \sim -classes of its elements, thus by Claim 26, in order to prove that X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable it suffices to prove that all \sim_X -classes are $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

We prove that each \sim -class E is definable from \sim -classes F with smaller dimension, i.e. that E is definable in the expansion of $\langle \mathbb{R}, +, <, 1 \rangle$ obtained by adding a predicate for each such F . We proceed by induction on the dimension d of $\text{Str}(E)$.

If $d = 0$ then E is a union of singular points, and by (FSP) and (RSP) it follows that E is a finite subset of \mathbb{Q}^n thus is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

Assume now $0 < d \leq n$. By Claim 25 there exists a rational basis $V(E) = \{v_1, \dots, v_d\}$ of $\text{Str}(E)$. Let $Z \subseteq E$ be isolated and let $Z' = E \setminus Z$. By Lemma 28 (1b), Z is a finite union of parallel affine subspaces with direction $V(E)$ each having a point with rational components, thus Z is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

It remains to prove that Z' is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. We use the following characterization of Z' .

Lemma 29. *For every $x \in \mathbb{R}^n$, we have $x \in Z'$ if and only if there exist $1 \leq p \leq d$ and a sequence of pairwise distinct elements $x_0, \dots, x_p \in \mathbb{R}^n$ such that $x_0 = x$ and*

1. *for every $0 \leq k \leq p-1$, $[x_k, x_{k+1})$ is parallel to some $u_{k+1} \in V(E)$ and does not intersect any \sim -class F such that $\dim(F) < \dim(E)$*
2. *if we denote $F = \text{Cl}(x_p)$ then $\dim(F) < \dim(E)$. Moreover if $x_p = x_{p-1} + \alpha u_p$ with $\alpha \neq 0$ then F is $(\text{sgn}(\alpha)u_p)$ -compatible with E .*

Proof. We first prove that the conditions are sufficient. We prove by backward induction that $[x_k, x_{k+1}) \subseteq E$ for every $0 \leq k \leq p-1$. This will imply that $x = x_0 \in E$, and the fact that $x_p - x$ belongs to $\text{Str}(E)$ and $\dim(F) < \dim(E)$ will lead to $x \in Z'$. If $k = p-1$ then by our hypotheses $[x_{p-1}, x_p)$ is parallel to u_p , and $\text{Cl}(x_p)$ is $(\text{sgn}(\alpha)u_p)$ -compatible with E , thus $[x_{p-1}, x_p)$ intersects E . Moreover $[x_{p-1}, x_p)$ does not intersect any \sim -class F such that $\dim(F) < \dim(E)$, thus by Lemma 21 we have $[x_{p-1}, x_p) \subseteq E$. For $0 \leq k < p-1$, by our induction hypothesis we have $x_{k+1} \in E$. Moreover $[x_k, x_{k+1})$ does not intersect any \sim -class F such that $\dim(F) < \dim(E)$, thus $[x_k, x_{k+1}) \subseteq E$ by Lemma 21.

We prove the necessity. By definition of Z' and Lemma 21 there exist $v \in \text{Str}(E)$ and $y \in L_v(x)$ such that $[x, y) \subseteq E$ and $y \notin E$. Decompose $v = \alpha_1 v_{i_1} + \dots + \alpha_t v_{i_p}$ where $0 < i_1 < \dots < i_p \leq d$ and $\alpha_1 \dots \alpha_t \neq 0$. We can assume w.l.o.g that v is chosen such that p is minimal. For $0 \leq k < p$ set $x_k = x + \alpha_1 v_{i_1} + \dots + \alpha_k v_{i_k}$. By minimality of p , the segments $[x_0, x_1), \dots, [x_{p-2}, x_{p-1})$ intersect no class of dimension less than $\dim(E)$. Moreover by Lemma 21 there exists $\alpha \neq 0$ such that $[x_{p-1}, x_{p-1} + \alpha v_p)$ intersects no class of dimension less than $\dim(E)$, and if we set $x_p = x_{p-1} + \alpha v_{i_p}$ then x_p belongs to a class F of dimension less than E and which is $(\text{sgn}(\alpha)v_{i_p})$ -compatible with E . \square

In order to prove that Z' is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable it suffices to show that we can express in $\langle \mathbb{R}, +, <, 1 \rangle$ the existence of a sequence $x_0, \dots, x_p \in \mathbb{R}^n$ which satisfies both conditions of Lemma 29. Observe that $V(E)$ is finite and each of its element is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, thus we can express in $\langle \mathbb{R}, +, <, 1 \rangle$ the fact that a segment is parallel to some element of $V(E)$. Moreover by (FI) there exist finitely many \sim -classes F such that $\dim(F) < \dim(E)$, and all such classes are $\langle \mathbb{R}, +, <, 1 \rangle$ -definable by our induction hypothesis. This allows to express condition (1) in $\langle \mathbb{R}, +, <, 1 \rangle$. For (2) we use again the fact that there are only finitely many classes F to consider and all of them are $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. \square

4.3 An alternative formulation of Theorem 24

In this section we re-formulate Theorem 24 in terms of (generalized) projections of X .

We extend the notion of section by allowing to fix several components.

Definition 30. A *generalized section* of X is a relation of the form

$$X_{s,a} = \{(x_1, \dots, x_n) \in X \mid x_{s_1} = a_{s_1}, \dots, x_{s_r} = a_{s_r}\} \quad (2)$$

where $r > 0$, $(s)_{1,\dots,r} = 1 \leq s_1 < \dots < s_r \leq n$ is an increasing sequence, and $a = (a_{s_1}, \dots, a_{s_r})$ is an r -tuple of reals. When $r = 0$ we define $X_{s,a} = X$ by convention, i.e. X is a generalized section

of itself. If $r > 0$ then the section is said to be proper. If all elements of a are rationals then $X_{s,a}$ is called a rational generalized section of X .

In the above definition, each $X_{s,a}$ is a subset of \mathbb{R}^n . If we forget the r fixed components x_{s_1}, \dots, x_{s_r} we can see $X_{s,a}$ as a subset of \mathbb{R}^{n-r} , which will be called a generalized projection of X (resp. a rational generalized projection of X if $X_{s,a}$ is a rational generalized section of X).

Proposition 31. *For every $n \geq 1$ and every relation $X \subseteq \mathbb{R}^n$, X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable if and only if every generalized rational projection of X has finitely many singular points and they have rational components.*

Proof. The proof goes by induction on n . The case $n = 1$ is obvious. Assume now $n > 1$.

Let X be $\langle \mathbb{R}, +, <, 1 \rangle$ -definable and let Y be a generalized rational projection of X . If $Y = X$ then the result follows from Theorem 24. If Y is proper then Y is definable in $\langle \mathbb{R}, +, <, 1, X \rangle$ thus it is also $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, and the result follows from our induction hypothesis.

Conversely assume that every generalized rational projection of X has finitely many singular points and they have rational components. We show that X satisfies all three conditions of Theorem 24. Conditions (FSP) and (RSP) follow from our hypothesis and the fact that X is a generalized rational projection of itself. It remains to prove condition (DS) namely that every rational section of X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. This amounts to prove that every rational projection Z of X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Now Z is a generalized projection of X , and every generalized projection Y of Z is also a generalized projection of X , thus by our induction hypothesis Y has finitely many singular points and they have rational components. Since Z is a proper projection of X , by our induction hypothesis it follows that Z is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. \square

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