DEFINABILITY AND INTERPOLATION WITHIN DECIDABLE FIXPOINT LOGICS *

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ABSTRACT. We look at characterizing which formulas are expressible in rich decidable logics such as guarded fixpoint logic, unary negation fixpoint logic, and guarded negation fixpoint logic. We consider semantic characterizations of definability, as well as effective characterizations. Our algorithms revolve around a finer analysis of the tree-model property and a refinement of the method of moving back and forth between relational logics and logics over trees.

1. INTRODUCTION

A major line of research in computational logic has focused on obtaining extremely expressive decidable logics. The guarded fragment (GF) [1], the unary negation fragment (UNF) [36], and the guarded negation fragment (GNF) [8] are rich decidable fragments of first-order logic. Each of these has extensions with a fixpoint operator that retain decidability: GFP [27], UNFP [36], and GNFP [8] respectively. In each case the argument for satisfiability relies on "moving to trees". This involves showing that the logic possesses the tree-like model property: whenever there is a satisfying model for a formula, it can be taken to be of tree-width that can be effectively computed from the formula. Such models can be coded by trees, thus reducing satisfiability of the logic to satisfiability of a corresponding formula over trees, which can be decided using automata-theoretic techniques. This method has been applied for decades (e.g. [39, 25]).

A question is how to recognize formulas in these logics, and more generally how to distinguish the properties of the formulas in one logic from another. Clearly if we start with a formula in an undecidable logic, such as first-order logic or least fixed point logic (LFP), we have no possibility for effectively recognizing any non-trivial property. But we could still hope for an insightful semantic characterization of the subset that falls within the decidable logic. One well-known example of this is van Benthem's theorem [38] characterizing modal

logic within first-order logic: a first-order sentence is equivalent to a modal logic sentence exactly when it is bisimulation invariant. For fixpoint logics, an analogous characterization is the Janin-Walukiewicz theorem [30], stating that the modal μ-calculus (L_μ) captures the bisimulation-invariant fragment of monadic second-order logic (MSO). If we start in one decidable logic and look to characterize another decidable logic, we could also hope for a characterization that is effective. For example, Otto [32] showed that if we start with a formula of L_μ, we can determine whether it can be expressed in modal logic.

In this work we will investigate both kinds of characterizations. We will begin with GFP. Grädel, Hirsch, and Otto [25] have already provided a characterization of GFP-definability within a very rich logic extending MSO called guarded second-order logic (GSO). The characterization is exactly analogous to the van Benthem and Janin-Walukiewicz results mentioned above: GFP captures the "guarded bisimulation-invariant" fragment of GSO. The characterization makes use of a refinement of the method used for decidability of these logics, which moves *back and forth between relational structures and trees*:
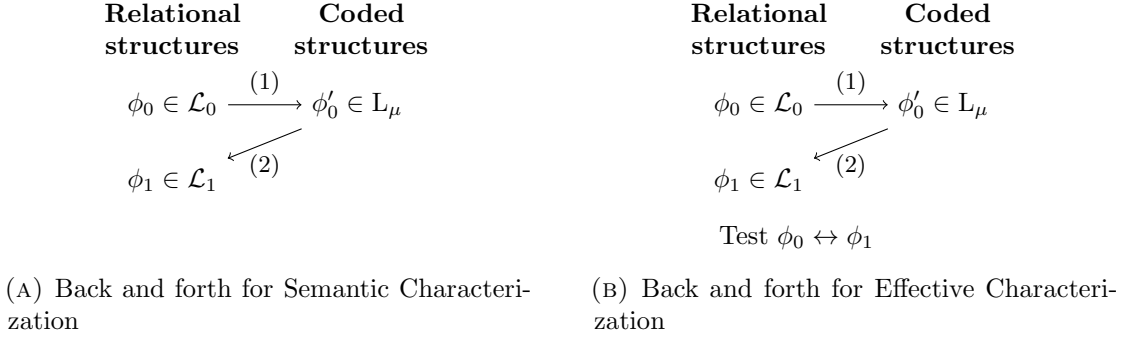
(1) define a *forward mapping* taking a formula ϕ_0 in the larger logic $\mathcal{L}_0$ (e.g. GSO invariant under guarded bisimulation) over relational structures to a formula ϕ_0' that describes the trees that code structures satisfying ϕ_0; and
(2) define a *backward mapping* based on the invariance going back to some ϕ_1 in the restricted logic $\mathcal{L}_1$ (e.g. GFP).

The method is shown in Figure 1a.

Our first main theorem is an effective version of the above result: if we start with a formula in certain richer decidable fixpoint logics, such as GNFP, we can decide whether the formula is in GFP. At the same time we provide a refinement of [25] which accounts for two signatures, the one allowed for arbitrary relations and the one allowed for "guard relations" that play a key role in the syntax of all guarded logics. We extend this result to deciding membership in the "k-width fragment", $GNFP^k$; roughly speaking this consists of formulas built up from guarded components and positive existential formulas with at most k variables. We provide a semantic characterization of this fragment within GSO, as the fragment closed under the corresponding notion of bisimulation (essentially, the GN^k-bisimulation of [8]). As with GFP, we show that the characterization can be made effective, provided that one starts with a formula in certain larger decidable logics. The proof also gives an effective characterization for the k-width fragment of UNFP.

These effective characterizations also rely on a back-and-forth method. The revised method is shown schematically in Figure 1b. We apply a forward mapping to move from a formula ϕ_0 in a larger logic $\mathcal{L}_0$ (e.g. $GNFP^k$) on relational structures to a formula ϕ_0' on tree encodings. But then we can apply a *different backward mapping*, tuned towards the smaller logic $\mathcal{L}_1$ (e.g. GFP) and the special properties of its tree-like models. The backward mapping of a tree property ϕ_0' is always a formula ϕ_1 in the smaller logic $\mathcal{L}_1$. But it is no longer guaranteed to be "correct" unconditionally—i.e. to always characterize structures whose codes satisfy ϕ_0'. Still, we show that *if* the original formula ϕ_0 is definable in the smaller logic $\mathcal{L}_1$, then the backward mapping applied to the forward mapping gives such a definition. Since we can check the equivalence of two sentences in our logic effectively, this property suffices to get decidability of definability.

The technique above has a few inefficiencies; first, it translates forward to sentences in a rich logic on trees, for which analysis is non-elementary. Secondly, it implicitly moves between relational structures and tree structures *twice*: once to construct the formula ϕ_0', and a second time to check that ϕ_0 is equivalent to ϕ_1, which in turn requires first forming

| **Relational** **Coded** | **Relational** **Coded** |
| structures structures | structures structures |

$$\phi_0 \in \mathcal{L}_0 \xrightarrow{\;(1)\;} \phi_0' \in \mathrm{L}_\mu \qquad\qquad \phi_0 \in \mathcal{L}_0 \xrightarrow{\;(1)\;} \phi_0' \in \mathrm{L}_\mu$$

$$\phi_1 \in \mathcal{L}_1 \overset{(2)}{\swarrow} \qquad\qquad\qquad \phi_1 \in \mathcal{L}_1 \overset{(2)}{\swarrow}$$

Test $\phi_0 \leftrightarrow \phi_1$

(A) Back and forth for Semantic Characteri-zation

(B) Back and forth for Effective Characteri-zation

a formula over trees ϕ_1' via a forward mapping and then checking its equivalence with ϕ_0'. We show that in some cases we can perform an optimized version of the process, allowing us to get tight bounds on the equivalence problem.

We show that our results "restrict" to fragments of these guarded logics, including their first-order fragments. In particular, our results give effective characterizations of GF definability. They can be thus seen as a generalization of well-known effective character-izations of the conjunctive existential formulas in GF, the *acyclic queries*. We show that we can apply our techniques to the problem of transforming conjunctive formulas to a well-known efficiently-evaluable form (acyclic formulas) relative to GF theories. These results complement previous results on query evaluation with constraints from [9, 22].

This refined back-and-forth method can be tuned in a number of ways, allowing us to control the signature as well as the sublogic. We show that this machinery can be adapted to give an approximation of the formula ϕ_0 within the logic $\mathcal{L}_1$, which is a kind of *uniform interpolant*.

Related work. The immediate inspiration for our work are characterizations of definabil-ity in the guarded fragment within first-order logic [1], and characterization of definability in guarded fixpoint logic within guarded second-order logic [25]. Neither of these character-izations can be effective, since the larger logics in question are too expressive.

Identifying formulas in definable sublogics has been studied extensively in the context of regular word and tree languages ([33, 34]), and the corresponding characterizations are effective. These techniques do not lift easily to the setting of relational languages, even those with tree-like models, since one would require decidability over infinite trees, and the few results there (e.g. [16]) do not map back to natural logics over decodings. Although we know of no work on effectively identifying formulas definable in a fixpoint logic, there are a number of works on identifying sufficient conditions for a decidable fixpoint logic formula to be convertible into a formula without recursion (e.g. [32, 12]).

Our work is also inspired by prior automata-theoretic approaches to uniform inter-polation. The key result here is D'Agostino and Hollenberg's [19], which shows uniform interpolation for the modal μ-calculus. We make use of this result in our proofs. Craig interpolation for guarded logics has been considered in the past (e.g. [14]), but we know of no other work considering uniform interpolation for logics on arbitrary arity signatures.

Organization. Section 2 defines the logics we study in this paper and reviews their prop-erties. It also introduces tree encodings, bisimulation games, and unravelling constructions

that will be the basis for several of our proofs. It concludes with a description of automata that can operate on the tree codes. We would encourage readers to consult this section as needed, particularly the section on automata.

Section 3 presents an overview of the back-and-forth technique, and how it can be used to answer definability questions. Section 4 presents characterization results for GFP, which provides a first example of the technique in the action. Section 5 extends this technique to GNFP^k and UNFP^k. Section 6 presents applications of the technique to interpolation, while Section 7 gives conclusions.

2. Preliminaries

We work with finite relational signatures σ. We use $\boldsymbol{x}, \boldsymbol{y}, \ldots$ (respectively, $\boldsymbol{X}, \boldsymbol{Y}, \ldots$) to denote vectors of first-order (respectively, second-order) variables. For a formula ϕ, we write $\text{free}(\phi)$ to denote the free first-order variables of ϕ, and write $\phi(\boldsymbol{x})$ to indicate that these free variables are among $\boldsymbol{x}$. If we want to emphasize that there are also free second-order variables $\boldsymbol{X}$, we write $\phi(\boldsymbol{x}, \boldsymbol{X})$. We often use α to denote atomic formulas, and if we write $\alpha(\boldsymbol{x})$ then we assume that the free first-order variables in α are precisely $\boldsymbol{x}$. The *width* of ϕ, denoted $\text{width}(\phi)$, is the maximum number of free variables in any subformula of ϕ, and the width of a signature σ is the maximum arity of its relations.

2.1. **Guardedness.** An atomic formula α is a *guard* for variables $\boldsymbol{x}$ if α uses every variable in $\boldsymbol{x}$. We say α is a guard for a formula ϕ if it is a guard for the free variables in ϕ. This means $\text{free}(\alpha) \supseteq \text{free}(\phi)$. Guards can take the form $\top$ (if ϕ is a sentence) or the form $x = x$ (if ϕ has one free variable x). A *strict guard* for a formula ϕ is a guard such that the free variables of α are identical to the free variables in ϕ; that is $\text{free}(\alpha) = \text{free}(\phi)$. For example, Rxy could serve as a strict guard for $\exists z.(Ryz \wedge Rzx)$.

We can also talk about guardedness within a structure $\mathfrak{A}$. Any set of elements of size at most 1 is considered to be both guarded and strictly guarded. Otherwise, we say a set U of elements in the domain of $\mathfrak{A}$ is *guarded* in $\mathfrak{A}$ if there is some atom $\alpha(\boldsymbol{a})$ such that every element in U appears in $\boldsymbol{a}$. In the special case that this atom uses precisely the elements in U and no more, then we say U is *strictly guarded* in $\mathfrak{A}$.

If we want to emphasize that the guards come from a certain signature σ_g, then we will say σ_g-guarded or strictly σ_g-guarded.

2.2. **Basics of guarded logics.** The *Guarded Negation Fragment* of FO [8] (denoted GNF) is built up inductively according to the grammar $\phi ::= R\,\boldsymbol{x} \mid \exists x.\phi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \alpha(\boldsymbol{x}) \wedge \neg\phi(\boldsymbol{x})$ where R is either a relation symbol or the equality relation, and α is a guard for ϕ. If we restrict α to be an equality, then each negated formula can be rewritten to use at most one free variable; this is the *Unary Negation Fragment*, UNF [36]. GNF is also related to the *Guarded Fragment* [1] (GF), typically defined via the grammar $\phi ::= R\,\boldsymbol{x} \mid \exists \boldsymbol{x}.\big(\alpha(\boldsymbol{xy}) \wedge \phi(\boldsymbol{xy})\big) \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi(\boldsymbol{x})$ where R is either a relation symbol or the equality relation, and α is a guard for ϕ. Here it is the quantification that is guarded, rather than negation. GNF subsumes GF sentences and UNF formulas. GNF also subsumes GF formulas in which the free variables are guarded.

The fixpoint extensions of these logics (denoted GNFP, UNFP, and GFP) extend the base logic with formulas $[\mathbf{lfp}_{X,\boldsymbol{x}} . \alpha(\boldsymbol{x}) \wedge \phi(\boldsymbol{x}, X, \boldsymbol{Y})](\boldsymbol{x})$ where (i) $\alpha(\boldsymbol{x})$ is a guard for $\boldsymbol{x}$, (ii) X only appears positively in ϕ, (iii) second-order variables like X cannot be used as guards.

Some alternative (but equi-expressive) ways to define the fixpoint extension are discussed in [4]; in all of the definitions, the important feature is that tuples in the fixpoint are guarded by an atom in the original signature. In UNFP, there is an additional requirement that only unary or 0-ary predicates can be defined using the fixpoint operators. GNFP subsumes both GFP sentences and UNFP formulas. These logics are all contained in LFP, the fixpoint extension of FO.

In this work we will be interested in varying the signatures considered, and in distinguishing more finely which relations can be used in guards. If we want to emphasize the relational signature σ being used, then we will write, e.g., GNFP$[\sigma]$. For $\sigma_g \subseteq \sigma$, we let GNFP$[\sigma, \sigma_g]$ denote the logic built up as in GNFP but allowing only equality or relations $R \in \sigma$ at the atomic step and only guards α using equality or relations $R \in \sigma_g$. We define GFP$[\sigma, \sigma_g]$ similarly. Note that UNFP$[\sigma]$ is equivalent to GNFP$[\sigma, \emptyset]$, since if the only guards are equality guards, then the formula can be rewritten to use only unary negation and monadic fixpoints.

Fixpoint semantics and notation. We briefly review the semantics of the fixpoint operator. Take some $\alpha(\boldsymbol{x}) \wedge \phi(\boldsymbol{x}, X, \boldsymbol{Y})$ where X appears only positively. Then it induces a monotone operator $U \mapsto \mathcal{O}_\phi^{\mathfrak{A}, \boldsymbol{V}}(U) := \{\boldsymbol{a} : \mathfrak{A}, U, \boldsymbol{V} \models \alpha(\boldsymbol{a}) \wedge \phi(\boldsymbol{a}, X, \boldsymbol{Y})\}$ on every structure $\mathfrak{A}$ with valuation $\boldsymbol{V}$ for $\boldsymbol{Y}$, and this operator has a unique least fixpoint.

One way to obtain this least fixpoint is based on fixpoint approximants. Given some ordinal β, the *fixpoint approximant* $\phi^\beta(\mathfrak{A}, \boldsymbol{V})$ of ϕ on $\mathfrak{A}, \boldsymbol{V}$ is defined such that

$$\phi^0(\mathfrak{A}, \boldsymbol{V}) := \emptyset$$
$$\phi^{\beta+1}(\mathfrak{A}, \boldsymbol{V}) := \mathcal{O}_\phi^{\mathfrak{A}, \boldsymbol{V}}(\phi^\beta(\mathfrak{A}, \boldsymbol{V}))$$
$$\phi^\beta(\mathfrak{A}, \boldsymbol{V}) := \bigcup_{\beta' < \beta} \phi^{\beta'}(\mathfrak{A}, \boldsymbol{V}) \quad \text{where } \beta \text{ is a limit ordinal.}$$

We let $\phi^\infty(\mathfrak{A}, \boldsymbol{V}) := \bigcup_\beta \phi^\beta(\mathfrak{A}, \boldsymbol{V})$ denote the least fixpoint based on this operation. Thus, $[\mathbf{lfp}_{X,\boldsymbol{x}} . \alpha(\boldsymbol{x}) \wedge \phi(\boldsymbol{x}, X, \boldsymbol{Y})]$ defines a new predicate named X of arity $|\boldsymbol{x}|$, and $\mathfrak{A}, \boldsymbol{V}, \boldsymbol{a} \models [\mathbf{lfp}_{X,\boldsymbol{x}} . \alpha(\boldsymbol{x}) \wedge \phi(\boldsymbol{x}, X, \boldsymbol{Y})](\boldsymbol{x})$ iff $\boldsymbol{a} \in \phi^\infty(\mathfrak{A}, \boldsymbol{V})$. If $\boldsymbol{V}$ is empty or understood in context, we just write $\phi^\infty(\mathfrak{A})$.

It is often convenient to allow *simultaneous fixpoints* (also known as *vectorial fixpoints*). These are fixpoints of the form $[\mathbf{lfp}_{X_i, \boldsymbol{x}_i} . S](\boldsymbol{x})$ where S is a system of equations

$$\begin{cases} X_1, \boldsymbol{x}_1 := \alpha_1(\boldsymbol{x}_1) \wedge \phi_1(\boldsymbol{x}_1, X_1, \ldots, X_j, \boldsymbol{Y}) \\ \vdots \\ X_j, \boldsymbol{x}_j := \alpha_j(\boldsymbol{x}_j) \wedge \phi_j(\boldsymbol{x}_j, X_1, \ldots, X_j, \boldsymbol{Y}) \end{cases}$$

where $X_1, \ldots, X_j$ occur only positively. Such a system can be viewed as defining a monotone operation on a vector of j valuations, where the i-th component in the vector is the set of tuples satisfying X_i (i.e. the i-th component is the valuation for X_i). The formula $[\mathbf{lfp}_{X_i, \boldsymbol{x}_i} . S](\boldsymbol{x})$ expresses that $\boldsymbol{x}$ is a tuple in the i-th component of the least fixpoint defined by this operation. Simultaneous fixpoints can be eliminated in favor of traditional fixpoints using what is known as the Bekič principle [2]. This can be done using a recursive procedure that eliminates a component of the simultaneous fixpoint by in-lining this formula in the

other expressions. This in-lining process preserves any guardedness properties of the fixpoints, so we can allow simultaneous fixpoints in GNFP, UNFP, and GFP without changing the expressivity of these logics.

Expressivity. These guarded fixpoint logics are expressive: the μ-calculus (see Section 2.3) is contained in each of these logics, and so are many description logics [3]. GF, and hence all of the logics defined previously, can express many standard integrity constraints on database tables, such as *inclusion dependencies*, first-order sentences of the form $\forall \boldsymbol{x}. [R(\boldsymbol{x}) \to \exists \boldsymbol{y}. (S(\boldsymbol{x}))]$. Furthermore, every positive existential formula is expressible in UNF and GNF. More specifically, UNF and GNF can express conjunctive queries and unions of conjunctive queries. A *conjunctive query* (CQ) is a formula of the form $\exists x_1 \ldots x_j. \left(\bigwedge_{i \leq n} A_i \right)$ where each A_i is an atomic formula, and a *Union of Conjunctive Queries* (UCQ) is a disjunction of CQs. A *Boolean CQ* is a CQ that is a sentence. Further, since UNF and GNF are closed under boolean combinations of sentences, they can express that a Boolean CQ Q_2 is implied by another CQ Q_1 conjoined with a set of sentences Σ of the logic. This allows satisfiability of these guarded logics to be utilized to solve implication problems of interest in database theory, such as the *certain answer problem.* For more details on these applications, the reader can check [5, 6]. The fixpoint extensions such as GFP and GNFP allow one to express many queries involving reachability; some examples of this can be found in Section 6.2.

Nevertheless, these logics are decidable and have nice model theoretic properties. In particular satisfiability and finite satisfiability is 2-ExpTime-complete for GNF and GNFP [7]. The same holds for UNFP and GFP [36, 27].

Normal form. It is often helpful to consider the formulas in a normal form. *Strict normal form* GNFP$[\sigma, \sigma_g]$ formulas can be generated using the following grammar:

$$\phi ::= \bigvee_i \exists \boldsymbol{x}_i. \left(\bigwedge_j \psi_{ij} \right)$$

$$\psi ::= \top \mid \bot \mid R\,\boldsymbol{x} \mid X\,\boldsymbol{x} \mid \alpha(\boldsymbol{x}) \wedge \phi(\boldsymbol{x}) \mid \alpha(\boldsymbol{x}) \wedge \neg\phi(\boldsymbol{x}) \mid [\mathbf{lfp}_{X,\boldsymbol{x}} . \alpha(\boldsymbol{x}) \wedge \phi(\boldsymbol{x}, X, \boldsymbol{Y})](\boldsymbol{x})$$

where R is either a relation symbol in σ or the equality relation, and α is a strict σ_g-guard for ϕ.

We will sometimes refer to a formula like $\bigvee_i \exists \boldsymbol{x}_i. \left(\bigwedge_j \psi_{ij} \right)$ as a *UCQ-shaped formula*, and each disjunct $\exists \boldsymbol{x}_i. \left(\bigwedge_j \psi_{ij} \right)$ as a *CQ-shaped formula*. If $\boldsymbol{x}_i$ is non-empty, then we say $\exists \boldsymbol{x}_i. \left(\bigwedge_j \psi_{ij} \right)$ is a *CQ-shaped formula with projection.* Note that UCQ-shaped and CQ-shaped formulas generalize UCQs and CQs, respectively. The normal form captures the notion that formulas in the logic are built up by combining the usual guarded logic constructors and UCQ-constructors.

Every GNFP formula can be converted to this normal form.

Proposition 2.1. *Let θ be a formula in* GNFP$[\sigma, \sigma_g]$. *We can construct an equivalent formula* convert$(\theta) \in$ GNFP$[\sigma, \sigma_g]$ *in strict normal form such that* $|\text{convert}(\theta)| \leq 2^{f(|\theta|)}$ *and* width$(\text{convert}(\theta)) \leq |\theta|$, *where f is a polynomial function independent of θ.*

Proof. We proceed by induction on θ. The output convert(θ) is a UCQ-shaped formula in strict normal form, with the same free variables as θ.

- If θ is atomic or is an equality, then $\mathrm{convert}(\theta) := \theta$.
- Suppose $\theta = \alpha \wedge \neg\psi$ where α is a σ_g-guard for $\mathrm{free}(\psi)$. Then $\mathrm{convert}(\theta) := \alpha \wedge \neg(\mathrm{convert}(\alpha \wedge \psi))$. Note that the resulting formula is strictly σ_g-guarded.
- Suppose $\theta = \exists y.(\psi)$. If $\mathrm{convert}(\psi)$ is a UCQ-shaped formula of the form $\bigvee_i \exists \boldsymbol{z}_i.\left(\bigwedge_j \psi_{ij}\right)$, then $\mathrm{convert}(\theta) := \bigvee_i \exists y\boldsymbol{z}_i.\left(\bigwedge_j \psi_{ij}\right)$.
- Suppose $\theta = [\mathbf{lfp}_{Y,\boldsymbol{y}}.\alpha(\boldsymbol{y}) \wedge \psi(\boldsymbol{y})](\boldsymbol{x})$. Then $\mathrm{convert}(\theta) := [\mathbf{lfp}_{Y,\boldsymbol{y}}.\alpha(\boldsymbol{y}) \wedge \mathrm{convert}(\psi(\boldsymbol{y}))](\boldsymbol{x})$.
- Suppose $\theta = \psi_1 \vee \psi_2$. Then $\mathrm{convert}(\theta)$ is the UCQ-shaped formula $\mathrm{convert}(\psi_1) \vee \mathrm{convert}(\psi_2)$.
- Suppose $\theta = \psi_1 \wedge \psi_2$. Assume that $\mathrm{convert}(\psi_1) = \bigvee_i \exists \boldsymbol{x}_i.\chi_i$ and $\mathrm{convert}(\psi_2) = \bigvee_{i'} \exists \boldsymbol{x}'_{i'}.\chi'_{i'}$. Then $\mathrm{convert}(\theta) := \bigvee_{i,i'} \exists \boldsymbol{y}_i\boldsymbol{y}'_{i'}.(\chi_i[\boldsymbol{y}_i/\boldsymbol{x}_i] \wedge \chi'_{i'}[\boldsymbol{y}'_{i'}/\boldsymbol{x}'_{i'}])$ where the variables in every $\boldsymbol{y}_i$ and $\boldsymbol{y}'_{i'}$ are fresh.

It is straightforward to check that the new formula $\mathrm{convert}(\theta)$ is of size at most $2^{f(k)}$ for $k = |\theta|$ and f some polynomial function independent of θ. Moreover, the number of free variable names needed in any subformula is at most k, so $\mathrm{width}(\mathrm{convert}(\theta)) \leq k$, and hence $\mathrm{convert}(\theta) \in \mathrm{GNFP}^k[\sigma, \sigma_g]$. $\qquad\square$

Later, we will need another version of this conversion process that preserves the width, assuming the input satisfies some additional properties (this is not needed until the proof of Lemma 5.5). We say a formula starting with a block of existential quantifiers is *strictly σ_g-answer-guarded* if it is of the form $\exists \boldsymbol{y}.(\alpha(\boldsymbol{x}) \wedge \chi(\boldsymbol{x}, \boldsymbol{y}))$.

Proposition 2.2. *Let θ be a formula in $\mathrm{GNFP}[\sigma, \sigma_g]$ such that any subformula starting with an existential quantifier and not directly below another existential quantifier is strictly σ_g-answer-guarded and any negation is strictly σ_g-guarded.*

Then we can construct an equivalent formula $\mathrm{convert}(\theta) \in \mathrm{GNFP}[\sigma, \sigma_g]$ that is in strict normal form and satisfies $|\mathrm{convert}(\theta)| \leq 2^{f(|\theta|)}$ and $\mathrm{width}(\mathrm{convert}(\theta)) = \mathrm{width}(\theta)$, where f is a polynomial function independent of θ.

Proof. We assume that each subformula in θ that starts with an existential quantifier and is not directly below another existential quantifier is a strictly σ_g-answer-guarded formula, and every negation is strictly σ_g-guarded. We proceed by induction on the structure of the formula θ, ensuring that the output $\mathrm{convert}(\theta)$ is a UCQ-shaped formula in strict normal form, with the same free variables as θ, and where every CQ-shaped formula with projection (i.e. every CQ-shaped formula that uses existential quantification) is strictly σ_g-answer-guarded.

- If θ is atomic or is an equality, then $\mathrm{convert}(\theta) := \theta$.
- Suppose $\theta = \alpha \wedge \neg\psi$ where α is a strict σ_g-guard for $\mathrm{free}(\psi)$. Then $\mathrm{convert}(\theta) := \alpha \wedge \neg\,\mathrm{convert}(\psi)$ which is strictly σ_g-answer guarded since $\mathrm{free}(\psi) = \mathrm{free}(\mathrm{convert}(\psi))$.
- Suppose $\theta = \exists \boldsymbol{y}.(\beta(\boldsymbol{x}) \wedge \psi)$, a strictly σ_g-answer-guarded formula starting with a block of existential quantifiers. Let $\mathrm{convert}(\psi)$ be the UCQ-shaped formula $\bigvee_i \exists \boldsymbol{z}_i.\left(\bigwedge_j \psi_{ij}\right)$, and let $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ be the subset of $\boldsymbol{x}$ and $\boldsymbol{y}$ used in $\bigwedge_j \psi_{ij}$. Then $\mathrm{convert}(\theta) := \bigvee_i \exists \boldsymbol{y}_i\boldsymbol{z}_i.(\alpha_i \wedge \bigwedge_j \psi_{ij})$ where α_i is the strict σ_g-guard for $\boldsymbol{x}_i$ in $\bigwedge_j \psi_{ij}$ if $\boldsymbol{z}_i$ is non-empty, and $\alpha_i = \beta(\boldsymbol{x})$ if $\boldsymbol{z}_i$ is empty (since we need to add a σ_g-guard to ensure strict σ_g-answer-guardedness for this new CQ-shaped formula with projection). Note that this process does not increase the width.

- Suppose $\theta = [\mathbf{lfp}_{Y,\boldsymbol{y}} . \alpha(\boldsymbol{y}) \wedge \psi(\boldsymbol{y})](\boldsymbol{x})$. Then we have $\mathrm{convert}(\theta) := [\mathbf{lfp}_{Y,\boldsymbol{y}} . \alpha(\boldsymbol{y}) \wedge \mathrm{convert}(\psi(\boldsymbol{y}))](\boldsymbol{x})$.
- Suppose $\theta = \psi_1 \vee \psi_2$. Then $\mathrm{convert}(\theta) := \mathrm{convert}(\psi_1) \vee \mathrm{convert}(\psi_2)$.
- Suppose $\theta = \psi_1 \wedge \psi_2$. Let $\mathrm{convert}(\psi_1) = \bigvee_i \chi_i$ and $\mathrm{convert}(\psi_2) = \bigvee_{i'} \chi'_{i'}$. Let α_i be the strict σ_g-answer-guard for χ_i if χ_i is a CQ-shaped formula with projection, and $\top$ otherwise. Similarly for $\alpha'_{i'}$. Then $\mathrm{convert}(\theta) := \bigvee_i \bigvee_{i'}((\alpha_i \wedge \chi_i) \wedge (\alpha'_{i'} \wedge \chi'_{i'}))$. The outer level UCQ now only has CQ-shaped formulas without projection of the form $(\alpha_i \wedge \chi_i) \wedge (\alpha'_{i'} \wedge \chi'_{i'})$. $\qquad\square$

Second-order logic. Guarded second-order logic over a signature σ (denoted $\mathrm{GSO}[\sigma]$) is a fragment of second-order logic in which second-order quantification is interpreted only over guarded relations, i.e. over relations where every tuple in the relation is guarded by some predicate from σ. We refer the interested reader to [25] for more background and some equivalent definitions of this logic. The logics UNFP, GNFP, and GFP can all be translated into GSO.

Proposition 2.3. *Given* $\phi \in \mathrm{GNFP}[\sigma]$, *we can construct an equivalent* $\phi' \in \mathrm{GSO}[\sigma]$.

Proof. The translation is straightforward. The interesting case is for the least fixpoint. If $\phi(\boldsymbol{y}) = [\mathbf{lfp}_{X,\boldsymbol{x}} . \alpha(\boldsymbol{x}) \wedge \psi(X, \boldsymbol{x})](\boldsymbol{y})$ then

$$\phi'(\boldsymbol{y}) := \forall X.[(\forall \boldsymbol{x}.((\alpha(\boldsymbol{x}) \wedge \psi'(X, \boldsymbol{x})) \rightarrow X\boldsymbol{x})) \rightarrow X\boldsymbol{y}]$$

where second-order quantifiers range over guarded relations. $\qquad\square$

2.3. Transition systems and their logics.
A special kind of signature is a *transition system signature* Σ consisting of a finite set of unary predicates (corresponding to a set of propositions) and binary predicates (corresponding to a set of actions). A structure for such a signature is a *transition system*. Trees allowing both edge labels and node labels have a natural interpretation as transition systems.

We will be interested in two logics over transition system signatures. One is *monadic second-order logic* (denoted MSO)—where second-order quantification is only over unary relations. MSO is contained in GSO, because unary relations are trivially guarded. While MSO and GSO can be interpreted over arbitrary signatures, there are logics like *modal logic* that have syntax specific to transition system signatures. Another is the *modal μ-calculus* (denoted L_μ), an extension of modal logic with fixpoints. Given a transition system signature Σ, formulas $\phi \in \mathrm{L}_\mu[\Sigma]$ can be generated using the grammar $\phi ::= P \mid X \mid \phi \wedge \phi \mid \neg\phi \mid \langle \rho \rangle \phi \mid \mu X.\phi$ where P is a unary relation in Σ and ρ is a binary relation in Σ. The formulas $\mu X.\phi$ are required to use the variable X only positively in ϕ. The meaning of a μ-calculus formula can be seen by the following straightforward conversion into LFP:

Proposition 2.4. *For every* $\phi \in \mathrm{L}_\mu[\Sigma]$ *there is a formula* $\psi(x) \in \mathrm{LFP}[\Sigma]$ *such that for every transition system* $\mathfrak{M}$ *over* Σ *and every node* v *in* $\mathfrak{M}$, *the formula* ϕ *holds at* v *in* $\mathfrak{M}$ *iff* $\mathfrak{M}, v \models \psi(x)$.

Proof. We proceed by induction on the structure of ϕ. We let ψ' denote the inductive translation of ϕ'.
- If $\phi = P$, then $\psi(x) = P(x)$.
- If $\phi = X$, then $\psi(x) = X(x)$.

- The translation commutes with $\vee$, $\wedge$, and $\neg$.
- If $\phi = \langle\rho\rangle\phi'$, then $\psi(x) = \exists y.\big(\rho(x,y) \wedge \psi'(y)\big)$.
- If $\phi = [\rho]\phi'$, then $\psi(x) = \forall y.\big(\rho(x,y) \rightarrow \psi'(y)\big)$.
- If $\phi = \mu X.\phi'$, then $\psi(x) = [\mathbf{lfp}_{X,z}.\psi'(z)](x)$. $\qquad\qquad\square$

We say a L_μ-formula ϕ holds from a position v in a transition system $\mathfrak{M}$ if $\mathfrak{M}, v \models \psi(x)$, where $\psi(x)$ is the LFP formula given by the previous proposition. We say that an L_μ-formula holds in a tree iff it holds at the root of the tree.

It is well-known that L_μ can also be translated into MSO [2].

2.4. **Tree-like model property and tree codes.** The guarded logics that we consider in this paper exhibit interesting model theoretic properties. For example, GNF (and hence UNF and GF) has the finite-model property [7]: if ϕ is satisfiable, then ϕ is satisfiable in a finite structure. This finite model property does not hold for the fixpoint extensions of these logics. In this paper we will be concerned only with equivalence over all structures, not just finite structures.

Our work relies heavily on a different model theoretic property, called the tree-like model property. We review now what it means for a relational structure to be "tree-like". Roughly speaking, these are structures that can be decomposed into a tree form. Formally, a *tree decomposition* of a structure $\mathfrak{M}$ consists of a tree (V, E) and a function λ assigning to each vertex $v \in V$ a subset $\lambda(v)$ of elements in the domain of $\mathfrak{M}$, so that the following hold:

- For each atom $Rc_1 \ldots c_n$ that holds in $\mathfrak{M}$, there is a v such that $\lambda(v)$ includes each element of $c_1 \ldots c_n$.
- For each domain element e in the domain of $\mathfrak{M}$, the set of nodes

$$\{v \in V : e \in \lambda(v)\}$$

is a connected subset of the tree. In other words, for any two vertices v_1, v_2 such that $e \in \lambda(v_1)$ and $e \in \lambda(v_2)$, there is a path between v_1 and v_2 such that $e \in \lambda(u)$ for every node u on this path.

The *width* of a decomposition is one less than the maximum size of $\lambda(v)$ over any element $v \in V$. The subsets $\lambda(v)$ of $\mathfrak{M}$ are called *bags* of the decomposition, so structures of tree-width $k-1$ have bags of size at most k.

GNFP (and hence UNFP and GFP) has the *tree-like model property* [7]: if ϕ is satisfiable, then ϕ is satisfiable over structures with tree decompositions of some bounded tree-width. In fact satisfiable GNFP^k formulas have satisfying structures of tree-width $k-1$. Satisfiable GFP sentences have an even stronger property: each bag in the decomposition describes a guarded set of elements, so the width of the tree decomposition is bounded by the maximum arity of the relations.

It is well-known that structures of tree-width $k-1$ can be encoded by labelled trees over an alphabet that depends only on the signature σ of the structure and k. Our encoding scheme will make use of trees with both node and edge labels, i.e. trees over a transition system signature $\Sigma_{\sigma,k}^{\mathrm{code}}$. Each node in a tree code represents atomic information over at most k elements, so the signature $\Sigma_{\sigma,k}^{\mathrm{code}}$ includes unary predicates to indicate the number of elements represented at that node, and the atomic relations that hold of those elements. The signature includes binary predicates that indicate the overlap and relationship between

the names of elements encoded in neighboring nodes of the tree. Formally, $\Sigma_{\sigma,k}^{\text{code}}$ contains the following relations:

- There are unary relations $D_n \in \Sigma_{\sigma,k}^{\text{code}}$ for $n \in \{0, \ldots, k\}$, to indicate the number of elements represented at each node. We call these *domain predicates* since they are used to specify the number of domain elements encoded at a given node.
- For every relation $R \in \sigma$ of arity n and every sequence $\boldsymbol{i} = i_1 \ldots i_n$ over $\{1, \ldots, k\}$, there is a unary relation $R_{\boldsymbol{i}} \in \Sigma_{\sigma,k}^{\text{code}}$ to indicate that the tuple of elements coded by $\boldsymbol{i}$ is a tuple of elements in R. For example, if T is a ternary relation in σ and a_i is the element coded by name i in some node, then $T_{3,1,3}$ indicates that $T(a_3, a_1, a_3)$ holds.
- For every partial 1-1 map ρ from $\{1, \ldots, k\}$ to $\{1, \ldots, k\}$, there is a binary relation $E_\rho \in \Sigma_{\sigma,k}^{\text{code}}$ to indicate the relationship between the names of elements in neighboring nodes. For example, if $(u, v) \in E_\rho$ and $\rho(3) = 1$, then the element with name 3 in u is the same as the element with name 1 in v.

For a unary relation $R_{\boldsymbol{i}}$, we write $\text{names}(R_{\boldsymbol{i}})$ to denote the set of elements from $\{1, \ldots, k\}$ appearing in $\boldsymbol{i}$. We will refer to the elements of $\{1, \ldots, k\}$ as *indices* or *names*.

For nodes u, v in a $\Sigma_{\sigma,k}^{\text{code}}$-tree $\mathcal{T}$ and names i, j, we will say (u, i) is *equivalent* to (v, j) if there is a simple undirected path $u = u_1 u_2 \ldots u_n = v$ in $\mathcal{T}$, and $\rho_1, \ldots, \rho_{n-1}$ such that $(u_i, u_{i+1}) \in E_{\rho_i}^{\mathcal{T}}$ or $(u_{i+1}, u_i) \in E_{\rho_i^{-1}}^{\mathcal{T}}$, and $(\rho_{n-1} \circ \cdots \circ \rho_1)(i) = j$. In words, the i-th element in node u corresponds to the j-th element in node v, based on the composition of edge labels (or their inverses) on the simple path between u and v. We write $[u, i]$ for the equivalence class based on this equivalence relation.

Given some subsignature $\sigma_g \subseteq \sigma$ and some set of indices $I \subseteq \{1, \ldots, k\}$, we say that $R_{\boldsymbol{i}} \in \Sigma_{\sigma,k}^{\text{code}}$ is a σ_g-*guard* for I if $\text{names}(R_{\boldsymbol{i}}) \supseteq I$ and $R \in \sigma_g$. Likewise, $R_{\boldsymbol{i}} \in \Sigma_{\sigma,k}^{\text{code}}$ is a *strict σ_g-guard* for I if $\text{names}(R_{\boldsymbol{i}}) = I$ and $R \in \sigma_g$. Given a set τ of unary relations from $\Sigma_{\sigma,k}^{\text{code}}$ we say I *is σ_g-guarded in τ* if $|I| \leq 1$ or there is some $R_{\boldsymbol{i}} \in \tau$ that is a σ_g-guard for I. Similarly, we say I *is strictly σ_g-guarded in τ* if $|I| \leq 1$ or there is some $R_{\boldsymbol{i}} \in \tau$ that is a strict σ_g-guard for I. These definitions are analogous to the definitions of guardedness and strict guardedness that were given in Section 2.1, but adapted to encodings. For example, if I is strictly σ_g-guarded in τ, then the set τ is encoding some relation that would strictly guard the elements encoded by I.

Given some $\Sigma_{\sigma,k}^{\text{code}}$-tree $\mathcal{T}$, we say $\mathcal{T}$ is *consistent* if it satisfies certain natural conditions that ensure that the tree actually corresponds to a code of some tree decomposition of a σ-structure:

(1) there is exactly one domain predicate D_i that holds at each node, and the root v_0 is in $D_0^{\mathcal{T}}$;

(2) edge labels respect the domain predicates: if $u \in D_m^{\mathcal{T}}$, $v \in D_n^{\mathcal{T}}$, and $(u, v) \in E_\rho^{\mathcal{T}}$, then $\text{dom}(\rho) \subseteq \{1, \ldots, m\}$ and $\text{rng}(\rho) \subseteq \{1, \ldots, n\}$;

(3) node labels respect the domain predicates: if $v \in D_n^{\mathcal{T}}$ and $v \in R_{\boldsymbol{i}}^{\mathcal{T}}$, then $\text{names}(R_{\boldsymbol{i}}) \subseteq \{1, \ldots, n\}$;

(4) neighboring node labels agree on shared names: if $u \in R_{\boldsymbol{i}}^{\mathcal{T}}$, $(u, v) \in E_\rho^{\mathcal{T}}$, and $\text{names}(R_{\boldsymbol{i}}) \subseteq \text{dom}(\rho)$, then $v \in R_{\rho(\boldsymbol{i})}^{\mathcal{T}}$; similarly, if $v \in R_{\boldsymbol{i}}^{\mathcal{T}}$, $(u, v) \in E_\rho^{\mathcal{T}}$, and $\text{names}(R_{\boldsymbol{i}}) \subseteq \text{rng}(\rho)$, then $u \in R_{\rho^{-1}(\boldsymbol{i})}^{\mathcal{T}}$;

where $P^{\mathcal{T}}$ denotes the interpretation of relation P in $\mathcal{T}$.

It is now easy to verify the fact mentioned at the beginning of this subsection: tree decompositions of every σ-structure of tree-width $k-1$ can be encoded in consistent $\Sigma_{\sigma,k}^{\text{code}}$-trees.

The next step is to describe how a consistent $\Sigma_{\sigma,k}^{\text{code}}$-tree can be decoded to an actual σ-structure. The *decoding* of $\mathcal{T}$ is the σ-structure $\mathfrak{D}(\mathcal{T})$ where the universe is the set

$$\{[v,i] : v \in \text{dom}(\mathcal{T}) \text{ and } i \in \{1,\ldots,k\}\}$$

and a tuple $([v_1,i_1],\ldots,[v_r,i_r])$ is in $R^{\mathfrak{D}(\mathcal{T})}$ iff there is some node $w \in \text{dom}(\mathcal{T})$ such that $w \in R_{j_1\ldots j_r}$ and $[w,j_m] = [v_m,i_m]$ for all $m \in \{1,\ldots,r\}$.

Finally, we introduce some notation related to $\Sigma_{\sigma,k}^{\text{code}}$. We often use τ to denote a node label, and $\tau(v)$ to denote the label at some node v in a tree. We write EDGES for the set of functions ρ such that the binary predicate E_ρ is in $\Sigma_{\sigma,k}^{\text{code}}$. We write NODELABELS for the set of *internally consistent* node labels, i.e. the set consisting of sets of unary predicates from $\Sigma_{\sigma,k}^{\text{code}}$ that satisfy properties (1) and (3) in the definition of consistency above.

2.5. **Bisimulations and unravellings.** The logic L_μ over transition system signatures lies within MSO. Similarly the guarded logics GFP, UNFP, and GNFP all lie within GSO and apply to arbitrary-arity signatures. It is easy to see that these containments are proper. In each case, what distinguishes the smaller logic from the larger is *invariance* under certain equivalences called *bisimulations*, each of which is defined by a certain player having a winning strategy in a two-player infinite game played between players Spoiler and Duplicator.

For L_μ, the appropriate game is the classical *bisimulation game* between transition systems $\mathfrak{A}$ and $\mathfrak{B}$: the definition of the game and the basic results about it can be found in [26]. It is straightforward to check that $L_\mu[\Sigma]$-formulas are Σ-bisimulation-invariant, i.e. $L_\mu[\Sigma]$-formulas cannot distinguish between Σ-bisimilar transition systems. We will make use of a stronger result of Janin and Walukiewicz [30] that the μ-calculus is the bisimulation-invariant fragment of MSO (we state it here for trees because of how we use this later): A class of trees is definable in $L_\mu[\Sigma]$ iff it is definable in MSO$[\Sigma]$ and closed under Σ-bisimulation within the class of all Σ-trees. The proof of this result is effective in the following sense: given an MSO$[\Sigma]$ sentence ϕ it is possible to construct a μ-calculus formula ϕ' such that, if ϕ is bisimulation-invariant, ϕ' holds from the root of a tree $\mathcal{T}$ iff $\mathcal{T}$ satisfies ϕ.

We now describe a generalization of these games between structures $\mathfrak{A}$ and $\mathfrak{B}$ over a signature σ with arbitrary arity relations, parameterized by some subsignature σ' of the structures. Each position in the game is a partial σ' homomorphism h from $\mathfrak{A}$ to $\mathfrak{B}$, or vice versa. The *active structure* in position h is the structure containing the domain of h. The game starts from the empty partial map from $\mathfrak{A}$ to $\mathfrak{B}$. In each round of the game, Spoiler chooses between one of the following moves:

- *Extend:* Spoiler chooses some set X of elements in the active structure such that $X \supseteq \text{dom}(h)$, and Duplicator must then choose h' extending h (i.e. such that $h(c) = h'(c)$ for all $c \in \text{dom}(h)$) such that h' is a partial σ' homomorphism; Duplicator loses if this is not possible. Otherwise, the game proceeds from the position h'.
- *Switch:* Spoiler chooses to switch active structure. If h is not a partial σ' isomorphism, then Duplicator loses. Otherwise, the game proceeds from the position h^{-1}.
- *Collapse:* Spoiler selects some $X \subseteq \text{dom}(h)$ and the game continues from position $h \restriction_X$.

Duplicator wins if she can continue to play indefinitely.

We will consider several variants of this game. For $k \in \mathbb{N}$ and $\sigma_g \subseteq \sigma'$:

- **k-width guarded negation bisimulation game**: The $\mathrm{GN}^k[\sigma', \sigma_g]$-game is the version of the game where the domain of every position h is of size at most k, and Spoiler can only make a switch move at h if $\mathrm{dom}(h)$ is strictly σ_g-guarded in the active structure.
- **block k-width guarded negation bisimulation game**: The $\mathrm{BGN}^k[\sigma', \sigma_g]$-game is like the $\mathrm{GN}^k[\sigma', \sigma_g]$-game, but additionally Spoiler is required to alternate between extend/switch moves and moves where he collapses to a strictly σ_g-guarded set. We call it the "block" game since Spoiler must select all of the new extension elements in a single block, rather than as a series of small extensions. The key property is that the game alternates between positions with a strictly σ_g-guarded domain, and positions of size at most k. The restriction mimics the alternation between formulas of width k and strictly σ_g-guarded formulas within normalized GNFP^k formulas.
- **guarded bisimulation game**: The $\mathrm{G}[\sigma', \sigma_g]$-game is the version of the game where the domain of every position must be strictly σ_g-guarded in the active structure. Note that in such a game, every position h satisfies $|\mathrm{dom}(h)| \le \mathrm{width}(\sigma_g)$.

We say $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathrm{GN}^k[\sigma', \sigma_g]$-*bisimilar* if Duplicator has a winning strategy in the $\mathrm{GN}^k[\sigma', \sigma_g]$-game starting from the empty position. We say a sentence ϕ is $\mathrm{GN}^k[\sigma', \sigma_g]$-*invariant* if for any pair of $\mathrm{GN}^k[\sigma', \sigma_g]$-bisimilar σ'-structures, $\mathfrak{A} \models \phi$ iff $\mathfrak{B} \models \phi$. A logic $\mathcal{L}$ is $\mathrm{GN}^k[\sigma', \sigma_g]$-invariant if every sentence in $\mathcal{L}$ is $\mathrm{GN}^k[\sigma', \sigma_g]$-invariant. When the guard signature is the entire signature, we will write, e.g., $\mathrm{GN}^k[\sigma']$ instead of $\mathrm{GN}^k[\sigma', \sigma']$. We similarly talk about $\mathrm{G}[\sigma', \sigma_g]$-*invariance* where we replace the $\mathrm{GN}^k[\sigma', \sigma_g]$-game by the $\mathrm{G}[\sigma', \sigma_g]$-game.

It is known that the bisimulation games characterize certain fragments of FO: $\mathrm{GF}[\sigma']$ is the $\mathrm{G}[\sigma']$-invariant fragment of $\mathrm{FO}[\sigma']$ [1] and $\mathrm{GNF}^k[\sigma']$ can be characterized as either the $\mathrm{BGN}^k[\sigma']$-invariant or the $\mathrm{GN}^k[\sigma']$-invariant fragment of $\mathrm{FO}[\sigma']$ [7]. Likewise, for fixpoint logics and fragments of GSO, $\mathrm{GFP}[\sigma']$ is the $\mathrm{G}[\sigma']$-invariant fragment of $\mathrm{GSO}[\sigma']$ [25], while $\mathrm{UNFP}^k[\sigma']$ is the $\mathrm{BGN}^k[\sigma', \emptyset]$-invariant fragment of $\mathrm{GSO}[\sigma']$ [13].

In this paper, we will prove a corresponding characterization for $\mathrm{GNFP}^k[\sigma']$ in terms of $\mathrm{BGN}^k[\sigma']$-invariance: $\mathrm{GNFP}^k[\sigma']$ is the $\mathrm{BGN}^k[\sigma']$-invariant fragment of $\mathrm{GSO}[\sigma']$ (in fact we will refine this to also talk about the guard signature; see Theorem 5.15). Note that for fixpoint logics, $\mathrm{GN}^k[\sigma']$-invariance is strictly weaker than $\mathrm{BGN}^k[\sigma']$-invariance. For example, [10] gives another decidable logic within GSO which is $\mathrm{GN}^k[\sigma']$-invariant but not $\mathrm{BGN}^k[\sigma']$ invariant.

Unravellings. Given a σ-structure $\mathfrak{A}$ and $k \in \mathbb{N}$ and $\sigma_g \subseteq \sigma' \subseteq \sigma$, we would like to construct a structure that is $\mathrm{GN}^k[\sigma', \sigma_g]$-bisimilar to $\mathfrak{A}$ but has a tree-decomposition of bounded tree-width. A standard construction achieves this, called the $\mathrm{GN}^k[\sigma', \sigma_g]$-unravelling of $\mathfrak{A}$. Let Π_k be the set of finite sequences of the form $Y_0 Y_1 \ldots Y_m$ such that $Y_0 = \emptyset$ and Y_i is a set of elements from $\mathfrak{A}$ of size at most k. Each such sequence can be seen as the projection to $\mathfrak{A}$ of a play in the $\mathrm{GN}^k[\sigma', \sigma_g]$-bisimulation game between $\mathfrak{A}$ and some other structure. For Y a set of elements from $\mathfrak{A}$, let $\mathrm{AT}_{\mathfrak{A}, \sigma'}(Y)$ be the set of atoms that hold of the elements in Y: $\{R(a_1, \ldots, a_l) : R \in \sigma', \{a_1, \ldots, a_l\} \subseteq Y, \mathfrak{A} \models R(a_1, \ldots, a_l)\}$. Now define a $\Sigma_{\sigma', k}^{\mathrm{code}}$-tree $\mathcal{U}_{\mathrm{GN}^k[\sigma', \sigma_g]}(\mathfrak{A})$ where each node corresponds to a sequence in Π_k, and the sequences are arranged in prefix order. The node label of every $v = Y_0 \ldots Y_{m-1} Y_m$ is an encoding of $\mathrm{AT}_{\mathfrak{A}, \sigma'}(Y_m)$, and the edge label between its parent u and v indicates the relationship between the shared elements $Y_{m-1} \cap Y_m$ encoded in u and v. We define $\mathfrak{D}(\mathcal{U}_{\mathrm{GN}^k[\sigma', \sigma_g]}(\mathfrak{A}))$ to

be the $\mathrm{GN}^k[\sigma', \sigma_g]$-unravelling of $\mathfrak{A}$. By restricting the set Π_k to reflect the possible moves in the games, we can define unravellings based on the other bisimulation games in a similar fashion. We summarize the two unravellings that will be most relevant later on:

- **block k-width guarded negation unravelling**: The $\mathrm{BGN}^k[\sigma', \sigma_g]$-unravelling is denoted $\mathfrak{D}(\mathcal{U}_{\mathrm{BGN}^k[\sigma', \sigma_g]}(\mathfrak{A}))$. Its encoding $\mathcal{U}_{\mathrm{BGN}^k[\sigma', \sigma_g]}(\mathfrak{A})$ is obtained by considering only sequences $Y_0 \ldots Y_m \in \Pi_k$ such that for all *even* i, $Y_{i-1} \supseteq Y_i \subseteq Y_{i+1}$ and Y_i is strictly σ_g-guarded in $\mathfrak{A}$. The tree $\mathcal{U}_{\mathrm{BGN}^k[\sigma', \sigma_g]}(\mathfrak{A})$ is consistent and is called a σ_g-*guarded-interface tree* since it alternates between *interface nodes* with strictly σ_g-guarded domains—these correspond to collapse moves in the game—and *bag nodes* with domain of size at most k that are not necessarily σ_g-guarded.
- **guarded unravelling**: The $\mathrm{G}[\sigma', \sigma_g]$-unravelling is denoted $\mathfrak{D}(\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{A}))$ and its encoding $\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{A})$ is obtained by considering only sequences $Y_0 \ldots Y_m \in \Pi_k$ such that for all i, Y_i is strictly σ_g-guarded in $\mathfrak{A}$. The tree $\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{A})$ is consistent and is called a σ_g-*guarded* tree since the domain of every node in the tree is strictly σ_g-guarded.

It is straightforward to check that:

Proposition 2.5. *Let $\mathfrak{A}$ be a σ-structure, and let $k \in \mathbb{N}$ and $\sigma_g \subseteq \sigma' \subseteq \sigma$. Then*
- *$\mathfrak{A}$ is $\mathrm{BGN}^k[\sigma', \sigma_g]$-bisimilar to $\mathfrak{D}(\mathcal{U}_{\mathrm{BGN}^k[\sigma', \sigma_g]}(\mathfrak{A}))$, the block k-width guarded negation unravelling;*
- *$\mathfrak{A}$ is $\mathrm{G}[\sigma', \sigma_g]$-bisimilar to the guarded unravelling $\mathfrak{D}(\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{A}))$.*

Because these unravellings have tree codes of some bounded tree-width, this implies that these guarded logics have tree-like models. The structural differences in the tree decompositions will be exploited for our definability decision procedures.

Note that it is also possible (and more standard) to define the bisimulation games and unravellings by replacing every occurrence of "strictly guarded" by "guarded". The games would still preserve the corresponding logic, and the analog of Proposition 2.5 would still hold. Further, our forward mapping results, saying that we can translate a formula in the logic into a formula running over the encoding of the unravelling (see e.g. Lemma 3.1), would still hold. Our use of strict guards will come into play only in simplifying the definition of the backward mappings (e.g. Lemma 5.4).

2.6. **Automata.** We will make use of automata on trees for the optimized decision procedures in Section 4. We suggest that readers skip this section until it is needed there.

Our goal in this section is to define two automaton models that can function on trees that have unbounded (possibly infinite) branching degree. This is because the tree codes derived from the unravellings described earlier may have this unbounded branching. We describe these automata below, but will assume familiarity with standard automata theory over infinite structures (see, e.g., [37]).

Fix a transition system signature Σ consisting of unary relations Σ_p and binary relations Σ_a (for the node labels and edge labels, respectively).

A *2-way alternating μ-automaton* $\mathcal{A}$ is a tuple $\langle \Sigma, Q_E, Q_A, q_0, \delta, \Omega \rangle$ where $Q := Q_E \cup Q_A$ is a finite set of states partitioned into states Q_E controlled by Eve and states Q_A controlled by Adam, and $q_0 \in Q$ is the initial state. The transition function has the form

$$\delta : Q \times \mathcal{P}(\Sigma_p) \to \mathcal{P}(\mathrm{Dir} \times \Sigma_a \times Q)$$

where $\mathrm{Dir} = \{\uparrow, 0, \downarrow\}$ is the set of possible directions (up $\uparrow$, stay 0, down $\downarrow$). The acceptance condition is a parity condition specified by $\Omega : Q \to \mathrm{Pri}$, which maps each state to a priority in a finite set of priorities Pri.

Let $\mathcal{T}$ be a tree over Σ, and let $\mathcal{T}(v)$ denote the set of unary propositions in Σ_p that hold at v.

The notion of acceptance of $\mathcal{T}$ by $\mathcal{A}$ starting at node $v_0 \in \mathrm{dom}(\mathcal{T})$ is defined in terms of a game $\mathcal{G}(\mathcal{A}, \mathcal{T}, v_0)$. The arena is $Q \times \mathrm{dom}(\mathcal{T})$, and the initial position is (q_0, v_0). From a position (q, v) with $q \in Q_E$ (respectively, $q \in Q_A$), Eve (respectively Adam) selects $(d, a, r) \in \delta(q, \mathcal{T}(v))$, and an a-neighbor w of v in direction d (note if $d = 0$, then v is considered the only option, and we sometimes write just $(0, r)$ instead of $(0, a, r)$). The game continues from position (r, w).

A *play* in $\mathcal{G}(\mathcal{A}, \mathcal{T}, v_0)$ is a sequence $(q_0, v_0), (q_1, v_1), (q_2, v_2), \ldots$ of moves in the game. Such a *play is winning* for Eve if the *parity condition* is satisfied: the maximum priority that occurs infinitely often in $\Omega(q_0), \Omega(q_1), \ldots$ is even.

A *strategy* for one of the players is a function that returns the next choice for that player given the history of the play. If the function depends only on the current position (rather than the full history), then it is *positional*. Choosing a strategy for both players fixes a play in $\mathcal{G}(\mathcal{A}, \mathcal{T}, v_0)$. A play π is *compatible* with a strategy ζ if there is a strategy for the other player such that ζ and ζ' yield π. A *strategy is winning* for Eve if every play compatible with it is winning.

We write $L_{v_0}(\mathcal{A})$ for the set of trees $\mathcal{T}$ such that Eve has a winning strategy in $\mathcal{G}(\mathcal{A}, \mathcal{T}, v_0)$. If v_0 is the root of $\mathcal{T}$, then we just write $L(\mathcal{A})$ to denote the *language* of $\mathcal{A}$.

The *dual* of a 2-way alternating μ-automaton $\mathcal{A}$ is the automaton $\mathcal{A}'$ obtained from $\mathcal{A}$ by switching Q_A and Q_E, and incrementing each priority by 1 (i.e. $\Omega'(q) := \Omega(q) + 1$). This has the effect of switching the roles of the two players, so the resulting automaton accepts the complement of $L(\mathcal{A})$.

These 2-way alternating μ-automata are essentially the same as the automata used in [27]; we use slightly different notation here and allow directions stay, up, and down, rather than just stay and 'move to neighbor'.

We are also interested in a type of automaton on trees with arbitrary branching that operates in a 1-way, nondeterministic fashion. These automata were introduced by Janin-Walukiewicz [30, 31]; we follow the presentation given in [19]. A μ-*automaton* $\mathcal{M}$ is a tuple $\langle \Sigma_p, \Sigma_a, Q, q_0, \delta, \Omega \rangle$. where the transition function now has the form

$$\delta : Q \times \mathcal{P}(\Sigma_p) \to \mathcal{P}(\mathcal{P}(\Sigma_a \times Q)).$$

Again, the acceptance condition is a parity condition specified by Ω. As before, we define acceptance of $\mathcal{T}$ from a node $v_0 \in \mathrm{dom}(\mathcal{T})$ based on a game $\mathcal{G}(\mathcal{A}, \mathcal{T}, v_0)$. The arena is $Q \times \mathrm{dom}(\mathcal{T})$, and the initial position is (q_0, v_0). From a position (q, v), Eve selects some $S \in \delta(q, \mathcal{T}(v))$, and a marking of every successor of v with a set of states such that (i) for all $(a, r) \in S$, there is some a-successor whose marking includes r, and (ii) for all a-successors w of v, if r is in the marking of w, then there is some $(a, r) \in S$. Adam then selects some successor w of v and a state r in the marking of w chosen by Eve, and the game continues from position (r, w). A winning play and strategy is defined as above.

Properties of μ-automata. These automata are bisimulation invariant on trees.

Proposition 2.6 ([30]). *Let $\mathcal{A}$ be a 2-way alternating μ-automaton or a μ-automaton. For all trees $\mathcal{T}$, if $\mathcal{T} \in L(\mathcal{A})$ and $\mathcal{T}'$ is bisimilar to $\mathcal{T}$, then $\mathcal{T}' \in L(\mathcal{A})$.*

These automata models also have nice closure properties.

Proposition 2.7. *2-way alternating μ-automata are closed under:*
- *Intersection: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be 2-way alternating μ-automata. Then we can construct a 2-way alternating μ-automaton $\mathcal{A}$ such that $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$, and the size of $\mathcal{A}$ is linear in $|\mathcal{A}_1| + |\mathcal{A}_2|$.*
- *Union: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be 2-way alternating μ-automata. Then we can construct a 2-way alternating μ-automaton $\mathcal{A}$ such that $L(\mathcal{A}) = L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$, and the size of $\mathcal{A}$ is linear in $|\mathcal{A}_1| + |\mathcal{A}_2|$.*
- *Complement: Let $\mathcal{A}$ be a 2-way alternating μ-automaton. Then we can construct a 2-way alternating μ-automaton $\mathcal{A}'$ of size at most $|\mathcal{A}|$ such that $L(\mathcal{A}')$ is the complement of $L(\mathcal{A})$.*

Proof. These are standard constructions for alternating automata.

For intersection, we can just take the disjoint union of the two automata, and create a new initial state q_0 controlled by Adam with moves to stay in the same position and go to state $q_0^{\mathcal{A}_1}$, or stay in the same position and go to state $q_0^{\mathcal{A}_2}$. Depending on this initial choice, the automaton then simulates either $\mathcal{A}_1$ or $\mathcal{A}_2$. The construction for the union is similar, but the initial choice is given to Eve, rather than Adam.

For complement, we use the dual automaton, which requires switching Q_A and Q_E, and incrementing the priority mapping by 1. □

It is straightforward to construct a 2-way alternating μ-automaton that is equivalent to a given μ-automaton. Moreover, it is known that μ-automata, 2-way alternating μ-automata and the μ-calculus are equivalent over trees (this follows from [30]).

Theorem 2.8 ([30]). *Given $\phi \in \mathrm{L}_\mu[\Sigma]$, we can construct a μ-automaton $\mathcal{A}$ such that $L(\mathcal{A})$ is the set of Σ-trees such that $\mathcal{T} \models \phi$.*

Likewise, given a μ-automaton or 2-way alternating μ-automaton $\mathcal{A}$ over signature Σ, we can construct $\phi \in \mathrm{L}_\mu[\Sigma]$ such that $L(\mathcal{A})$ is the set of Σ-trees such that $\mathcal{T} \models \phi$.

3. Decidability via back-and-forth method and equivalence

We now describe the main components of our approach, and explain how they fit together.

3.1. **Forward mapping.** The first component is a *forward mapping*, translating an input GSO formula ϕ to a formula over tree codes, holding on precisely the codes that correspond to tree-like models of ϕ. We start in the most general way we can, with GSO formulas ϕ that are GN^l-invariant for some l. In this case, we can define a forward mapping that produces a μ-calculus formula that holds in a tree code iff ϕ holds in its decoding.

Lemma 3.1 (Fwd, adapted from [25]). *Given a $\mathrm{GN}^l[\sigma]$-invariant sentence $\phi \in \mathrm{GSO}[\sigma]$ and given some $n \geq \max\{\mathrm{width}(\sigma), l\}$, we can construct $\phi^\mu \in \mathrm{L}_\mu[\Sigma_{\sigma,n}^{code}]$ such that for all consistent $\Sigma_{\sigma,n}^{code}$-trees $\mathcal{T}$, $\mathcal{T} \models \phi^\mu$ iff $\mathfrak{D}(\mathcal{T}) \models \phi$.*

Note that when moving to trees, we must specify the size of the bags (i.e. the tree width of the corresponding tree decompositions). The n in the lemma can be seen as the desired size of the bags in the tree codes. To prove Lemma 3.1, we use an inductive translation that produces a formula in MSO, and then apply the Janin-Walukiewicz Theorem [30] to convert this to the required formula in L_μ. Applying this conversion from MSO to L_μ requires that the trees are (at least) $\Sigma_{\sigma,l}^{code}$-bisimilar, which is why we must use $n \geq \max\{\text{width}(\sigma), l\}$ for the size of the bags in the tree codes.

This inductive translation must deal with formulas with free variables, and hence must use codes that include valuations for these variables. A valuation for a first-order variable x can be encoded by a valuation of second-order variables $x^\rightarrow = (Z_i^x)_{i \in \{1,\dots,n\}}$. The set Z_i^x consists of the nodes v in the tree code where the i-th element in v corresponds to the element identified by x. Likewise, a valuation for a second-order variable X corresponding to an r-ary σ-guarded relation (a relation that only includes tuples guarded in σ) can be encoded by a sequence of second-order variables $X^\rightarrow = (Z_{\boldsymbol{i}}^X)_{\boldsymbol{i} \in \{1,\dots,n\}^r}$. The set $Z_{\boldsymbol{i}}^X$ consists of the nodes v in the tree code where the tuple of elements coded by $\boldsymbol{i}$ in v are in the valuation for X.

It is straightforward to construct the following auxiliary formulas that check whether a tree is consistent, and whether some tuple of second-order variables actually encodes a valuation for a first-order variable or a σ-guarded relation in the way we have just described.

Lemma 3.2. *Given σ and n, we can construct the following* $\mathrm{MSO}[\Sigma_{\sigma,n}^{code}]$ *formulas:*

- *a formula $\phi_{\text{consistent}}$ such that for all $\Sigma_{\sigma,n}^{code}$-trees $\mathcal{T}$, $\mathcal{T} \models \phi_{\text{consistent}}$ iff $\mathcal{T}$ is a consistent $\Sigma_{\sigma,n}^{code}$-tree.*
- *a formula $\text{correct}(x^\rightarrow)$ such that for all consistent $\Sigma_{\sigma,n}^{code}$-trees $\mathcal{T}$ and for all $j^\rightarrow = (J_i)_{i \in \{1,\dots,n\}}$, $\mathcal{T} \models \text{correct}(j^\rightarrow)$ iff there is some element a in $\mathfrak{D}(\mathcal{T})$ such that for all i, we have $J_i = \{v \in \mathcal{T} : [v, i] = a\}$.*
- *a formula $\text{correct}_r(X^\rightarrow)$ such that for all consistent $\Sigma_{\sigma,n}^{code}$-trees $\mathcal{T}$ and for all $J^\rightarrow = (J_{\boldsymbol{i}})_{\boldsymbol{i} \in \{1,\dots,n\}^r}$, $\mathcal{T} \models \text{correct}_r(J^\rightarrow)$ iff there is some σ-guarded relation J of arity r on $\mathfrak{D}(\mathcal{T})$ and for all $\boldsymbol{i} = i_1 \dots i_r$, $J_{\boldsymbol{i}} = \{v \in \mathcal{T} : ([v, i_1], \dots, [v, i_r]) \in J\}$.*

Using these auxiliary formulas, we can perform the forward translation to $\mathrm{MSO}[\Sigma_{\sigma,n}^{code}]$.

Lemma 3.3. *Let ψ be a formula in $\mathrm{GSO}[\sigma]$ with free first-order variables among $x_1, \dots, x_n$, and free second-order variables among $X_1, \dots, X_m$. We can construct a formula*

$$\psi^\rightarrow(x_1^\rightarrow, \dots, x_n^\rightarrow, X_1^\rightarrow, \dots, X_m^\rightarrow) \in \mathrm{MSO}[\Sigma_{\sigma,n}^{code}]$$

such that for all consistent $\Sigma_{\sigma,n}^{code}$-trees $\mathcal{T}$, for all elements $a_1, \dots, a_n$ in $\mathfrak{D}(\mathcal{T})$ encoded by $j_1^\rightarrow, \dots, j_n^\rightarrow$ and for all sets of σ-guarded relations $J_1, \dots, J_m$ encoded by $J_1^\rightarrow, \dots, J_m^\rightarrow$,

$$\mathfrak{D}(\mathcal{T}), a_1, \dots, a_n, J_1, \dots, J_m \models \psi \quad \text{iff} \quad \mathcal{T}, j_1^\rightarrow, \dots, j_n^\rightarrow, J_1^\rightarrow, \dots, J_m^\rightarrow \models \psi^\rightarrow.$$

Proof. The proof is by induction on the structure of ψ.

- Assume $\psi = R x_{i_1} \dots x_{i_r}$. Then

$$\psi^\rightarrow := \exists z. \left(\bigvee_\rho \left(z \in R_{\rho(i_1)\dots\rho(i_r)} \wedge \bigwedge_{i \in \{i_1, \dots, i_r\}} z \in Z_{\rho(i)}^{x_i} \right) \right)$$

where ρ ranges over maps from $\{1, \dots, r\}$ to $\{1, \dots, n\}$. This expresses that there is some node in the coded structure where R holds for elements coded by $\rho(i_1) \dots \rho(i_r)$, and these elements are precisely $x_{i_1} \dots x_{i_r}$.

Similarly for $\psi = X x_{i_1} \ldots x_{i_r}$.

- Assume $\psi = (x_{i_1} = x_{i_2})$. Then

$$\psi^{\rightarrow} := \forall z. \left(\bigwedge_{j \in \{1, \ldots, n\}} \left(z \in Z_j^{x_{i_1}} \leftrightarrow z \in Z_j^{x_{i_2}} \right) \right).$$

This expresses that the valuations for the variables x_{i_1} and x_{i_2} are identical.

- The translation commutes with $\vee$, $\wedge$, and $\neg$.
- Assume $\psi = \exists x.(\chi)$. Then

$$\psi^{\rightarrow} := \exists x^{\rightarrow}. \left(\mathrm{correct}(x^{\rightarrow}) \wedge \chi^{\rightarrow} \right).$$

- Assume $\psi = \exists X.(\chi)$ for X an r-ary relation. Then

$$\psi^{\rightarrow} := \exists X^{\rightarrow}. \left(\mathrm{correct}_r(X^{\rightarrow}) \wedge \chi^{\rightarrow} \right).$$

The proof of correctness is straightforward. $\qquad\qquad\square$

We now return to the proof of Lemma 3.1. Recall that we have a sentence $\phi \in \mathrm{GSO}[\sigma]$ that is $\mathrm{GN}^l[\sigma]$-invariant and $n \geq \max \{\mathrm{width}(\sigma), l\}$.

We can apply Lemma 3.3 to produce a sentence $\phi^{\rightarrow} \in \mathrm{MSO}[\Sigma_{\sigma,n}^{\mathrm{code}}]$ such that for all consistent $\Sigma_{\sigma,n}^{\mathrm{code}}$-trees $\mathcal{T}$, $\mathcal{T} \models \phi^{\rightarrow}$ iff $\mathfrak{D}(\mathcal{T}) \models \phi$. This is the property required in Lemma 3.1, but the sentence $\phi^{\rightarrow}$ is in MSO rather than L_μ.

Consider the sentence $\phi^{\rightarrow} \wedge \phi_{\mathrm{consistent}}$. We claim that this is $\Sigma_{\sigma,n}^{\mathrm{code}}$-bisimulation invariant. Let $\mathcal{T}$ and $\mathcal{T}'$ be $\Sigma_{\sigma,n}^{\mathrm{code}}$-bisimilar. If there is any inconsistency in one tree, then bisimilarity implies that the other is also inconsistent, and $\phi^{\rightarrow} \wedge \phi_{\mathrm{consistent}}$ does not hold in either tree. In the case that $\mathcal{T}$ and $\mathcal{T}'$ are both consistent, then their bisimilarity implies that their decodings are both $\mathrm{GN}^n[\sigma]$-bisimilar and hence $\mathrm{GN}^l[\sigma]$-bisimilar (since we have ensured that $n \geq l$). But this $\mathrm{GN}^l[\sigma]$-invariance and the property in Lemma 3.3, imply that they agree on $\phi^{\rightarrow}$. Therefore, $\mathcal{T}$ and $\mathcal{T}'$ agree on $\phi^{\rightarrow} \wedge \phi_{\mathrm{consistent}}$ which is enough to conclude that $\phi^{\rightarrow} \wedge \phi_{\mathrm{consistent}}$ is $\Sigma_{\sigma,n}^{\mathrm{code}}$-bisimulation invariant.

This means that we can apply the Janin-Walukiewicz Theorem to $\phi^{\rightarrow} \wedge \phi_{\mathrm{consistent}}$ to produce an equivalent formula $\phi^{\mu} \in \mathrm{L}_\mu[\Sigma_{\sigma,n}^{\mathrm{code}}]$ for the forward mapping. This completes the proof of Lemma 3.1.

3.2. **Backward mapping.** The second component will depend on our target sublogic $\mathcal{L}_1$. It requires an operation (not necessarily effective) taking a σ-structure $\mathfrak{B}$ to a tree structure $\mathcal{U}_{\mathcal{L}_1}(\mathfrak{B})$ such that $\mathfrak{D}(\mathcal{U}_{\mathcal{L}_1}(\mathfrak{B}))$ agrees with $\mathfrak{B}$ on all $\mathcal{L}_1$ sentences. Informally, $\mathcal{U}_{\mathcal{L}_1}(\mathfrak{B})$ will be the encoding of some unravelling of $\mathfrak{B}$ appropriate for $\mathcal{L}_1$, perhaps with additional properties. A *backward mapping for $\mathcal{L}_1$* takes $\phi_0' \in \mathrm{L}_\mu$ describing tree codes to a sentence $\phi_1 \in \mathcal{L}_1$ such that: for all σ-structures $\mathfrak{B}$, $\mathfrak{B} \models \phi_1$ iff $\mathcal{U}_{\mathcal{L}_1}(\mathfrak{B}) \models \phi_0'$.

The formula ϕ_1 will depend on simplifying the formula ϕ_0' based on the fact that one is working on an unravelling. For $\mathcal{L}_1 = \mathrm{GFP}[\sigma', \sigma_g]$ over subsignatures σ', σ_g of the original signature σ, $\mathcal{U}_{\mathcal{L}_1}(\mathfrak{B})$ will be the appropriate guarded unravelling; we will see that results of [25] can easily be refined to give the backward mapping formula ϕ_1 in $\mathrm{GFP}[\sigma', \sigma_g]$. For GNFP^k, providing both the appropriate unravelling and the formula in the backward mappings will require more work.

3.3. **Definability problem.** The $\mathcal{L}_1$ *definability problem for logic* $\mathcal{L}$ asks: given some input sentence $\phi \in \mathcal{L}$, is there some $\psi \in \mathcal{L}_1$ such that ϕ and ψ are logically equivalent?

The forward and backward method of Figure 1b gives us a generic approach to this problem. The algorithm consists of applying the forward mapping to get ϕ_0', applying the backward mapping to ϕ_0' and obtaining the formula component of the mapping, ϕ_1, and then checking if ϕ_1 is equivalent to ϕ_0. We claim ϕ_0 is $\mathcal{L}_1$ definable iff ϕ_0 and ϕ_1 are equivalent. If ϕ_0 and ϕ_1 are logically equivalent then ϕ_0 is clearly $\mathcal{L}_1$ definable using ϕ_1. In the other direction, suppose that ϕ_0 is $\mathcal{L}_1$-definable. Fix $\mathfrak{B}$, and let $\mathcal{U}_{\mathcal{L}_1}(\mathfrak{B})$ be given by the backward mapping. Then

$$\mathfrak{B} \models \phi_0 \Leftrightarrow \mathfrak{D}(\mathcal{U}_{\mathcal{L}_1}(\mathfrak{B})) \models \phi_0 \text{ since } \phi_0 \text{ is equivalent to an } \mathcal{L}_1 \text{ sentence and}$$

$$\mathfrak{D}(\mathcal{U}_{\mathcal{L}_1}(\mathfrak{B})) \text{ agrees with } \mathfrak{B} \text{ on } \mathcal{L}_1 \text{ sentences}$$

$$\Leftrightarrow \mathcal{U}_{\mathcal{L}_1}(\mathfrak{B}) \models \phi_0' \text{ by Lemma Fwd}$$

$$\Leftrightarrow \mathfrak{B} \models \phi_1 \text{ by Backward Mapping for } \mathcal{L}_1$$

Hence, ϕ_0 and ϕ_1 are logically equivalent, as required. Thus, we get the following general decidability result:

Proposition 3.4. *Let* $\mathcal{L}_1$ *be a subset of* $\mathrm{GN}^l[\sigma]$*-invariant* $\mathrm{GSO}[\sigma]$ *such that we have an effective backward mapping for* $\mathcal{L}_1$*. Then the* $\mathcal{L}_1$ *definability problem is decidable for* $\mathrm{GN}^l[\sigma]$*-invariant* $\mathrm{GSO}[\sigma]$*.*

Above, we mean that there is an algorithm that decides $\mathcal{L}_1$ definability for any input $\mathrm{GSO}[\sigma]$ sentence that is $\mathrm{GN}^l[\sigma]$-invariant, with the output being arbitrary otherwise. The decidability is relying on the fact that logical equivalence is decidable for $\mathrm{GN}^l[\sigma]$-invariant $\mathrm{GSO}[\sigma]$: this follows from performing the forward mapping and checking equivalence of the corresponding sentences on the encodings. The approach above gives a definability test in the usual sense for inputs in $\mathrm{GNFP}[\sigma]$, since these are all $\mathrm{GN}^l[\sigma]$-invariant for some l. In particular we will see that we can test whether a $\mathrm{GNFP}^l[\sigma]$ sentence is in $\mathrm{GFP}[\sigma']$ or in $\mathrm{GNFP}^k[\sigma']$ for some subsignature σ' of σ. But there are larger GN^l-invariant logics (e.g. the GNFP-UP logic in [10]), so we can actually apply Proposition 3.4 to decide GFP or GNFP definability starting with inputs in these more expressive logics as well.

4. Identifying GFP definable sentences

4.1. **Decidability of GFP-definability.** For GFP, we can instantiate the high-level algorithm by giving a backward mapping. The backward mapping starts with a μ-calculus formula describing tree codes with some bag size m, and produces a GFP-formula describing relational structures. This backward mapping is tuned to a particular subsignature σ' of the original signature σ, with guards taken from $\sigma_g \subseteq \sigma'$. Throughout this section, we assume that $\sigma_g \subseteq \sigma' \subseteq \sigma$.

Lemma 4.1 (GFP-Bwd, adapted from [25])**.** *Given* $\phi^\mu \in \mathrm{L}_\mu[\Sigma_{\sigma,m}^{code}]$ *and* $\sigma_g \subseteq \sigma' \subseteq \sigma$, ϕ^μ *can be translated into* $\psi \in \mathrm{GFP}[\sigma', \sigma_g]$ *such that for all* σ*-structures* $\mathfrak{B}$, $\mathfrak{B} \models \psi$ *iff* $\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{B}) \models \phi^\mu$.

As with the forward mapping, the translation proceeds by induction on the structure of the formula ϕ^μ. Since each node in $\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{B})$ is strictly σ_g-guarded, each node is based on at most width(σ_g) elements from $\mathfrak{B}$. To deal with this, the translation of some formula θ

actually generates a family of formulas: for each $0 \leq k \leq \mathrm{width}(\sigma_g)$, a formula $\theta_k^{\leftarrow}$ with k free first-order variables is produced such that it correctly captures the meaning of the μ-calculus formula from a node of $\mathcal{U}_{\mathrm{G}[\sigma',\sigma_g]}(\mathfrak{B})$ that represents exactly k elements from $\mathfrak{B}$. The desired sentence ψ for Lemma 4.1 is $(\phi^\mu)_0^{\leftarrow}$, since the root of $\mathcal{U}_{\mathrm{G}[\sigma',\sigma_g]}(\mathfrak{B})$ has an empty domain.

For the purposes of the induction, we must also deal with formulas with free second-order variables. For each fixpoint variable X, each $1 \leq j \leq \mathrm{width}(\sigma_g)$, and each $P \in \Sigma_{\sigma_g,\mathrm{width}(\sigma_g)}^{\mathrm{code}}$ with $\mathrm{names}(P) = \{1,\ldots,j\}$, we introduce a second-order variable $X_{j,P}$ to represent nodes of size j whose indices are strictly σ_g-guarded by P (please refer to the definitions on page 10). The relation $X_{j,P}$ is a j-ary relation. In order to handle nodes with empty domain or domain of size 1 that are trivially σ_g-guarded, we also introduce $X_{0,\top}$ and $X_{1,\top}$. We define $X^{\leftarrow}$ to be the set of these second-order variables based on X.

Fix some σ-structure $\mathfrak{B}$ and $\mathcal{U}_{\mathrm{G}[\sigma',\sigma_g]}(\mathfrak{B})$. We write $\mathrm{elem}(v)$ to denote the ordered tuple of elements from $\mathfrak{B}$ represented at v in $\mathcal{U}_{\mathrm{G}[\sigma',\sigma_g]}(\mathfrak{B})$. A set V of nodes in $\mathcal{U}_{\mathrm{G}[\sigma',\sigma_g]}(\mathfrak{B})$ is a *bisimulation-invariant valuation for a free variable* X if it satisfies the following property: if it contains a node then it contains every node that is the root of a bisimilar subtree. We write $V^{\leftarrow}$ for its representation in $\mathfrak{B}$. Specifically, $V^{\leftarrow}$ consists of valuations $V_{j,P}$ for each $X_{j,P}$ in $X^{\leftarrow}$, where

$$V_{j,P} = \{\mathrm{elem}(v) : v \in V, |\mathrm{elem}(v)| = j, \text{ and the label } \tau \text{ at } v \text{ is strictly } \sigma_g\text{-guarded by } P\}.$$

We also set $V_{0,\top}$ to $\top$ (respectively, $\bot$) if J contains all nodes with empty domain (respectively, if J contains no nodes with empty domain), and $V_{1,\top} = \{\mathrm{elem}(v) : v \in V, |\mathrm{elem}(v)| = 1\}$.

Lemma 4.2. *Let $\phi \in \mathrm{L}_\mu[\Sigma_{\sigma,m}^{code}]$ with free second-order variables $\boldsymbol{X}$, and let $\sigma_g \subseteq \sigma' \subseteq \sigma$. For each $0 \leq k \leq \mathrm{width}(\sigma_g)$, we can construct a $\mathrm{GFP}[\sigma',\sigma_g]$-formula $\phi_k^{\leftarrow}(x_1,\ldots,x_k,\boldsymbol{X}^{\leftarrow})$ such that for all σ-structures $\mathfrak{B}$, for all bisimulation-invariant valuations $\boldsymbol{V}$ of $\boldsymbol{X}$, and for all nodes v in $\mathcal{U}_{\mathrm{G}[\sigma',\sigma_g]}(\mathfrak{B})$ with $|\mathrm{elem}(v)| = k$,*

$$\mathfrak{B}, \mathrm{elem}(v), \boldsymbol{V}^{\leftarrow} \models \phi_k^{\leftarrow} \quad \textit{iff} \quad \mathcal{U}_{\mathrm{G}[\sigma',\sigma_g]}(\mathfrak{B}), v, \boldsymbol{V} \models \phi.$$

Proof sketch. We proceed by induction on the structure of ϕ.

- If $\phi = D_n$, then $\phi_k^{\leftarrow}$ is $\top$ if $k = n$ and $\bot$ otherwise.
- If $\phi = R_{i_1 \ldots i_l}$ such that $R \notin \sigma'$ or $\{i_1,\ldots,i_l\} \not\subseteq \{1,\ldots,k\}$, then $\phi_k^{\leftarrow} := \bot$. Otherwise $\phi_k^{\leftarrow} := R\, x_{i_1} \ldots x_{i_l}$.
- If $\phi = X$, then $\phi_k^{\leftarrow} := \bigvee_\alpha (\alpha(x_1,\ldots,x_k) \wedge X_{k,P}\, x_1 \ldots x_k)$ where α ranges over atomic formulas that are strict σ_g-guards for $\{x_1,\ldots,x_k\}$, and P is the encoding of α.
- The translation commutes with $\vee$ and $\wedge$ and $\neg$ for each k.
- If $\phi = \langle \rho \rangle \chi$ with $\mathrm{dom}(\rho) = \{i_1,\ldots,i_l\} \not\subseteq \{1,\ldots,k\}$, then $\phi_k^{\leftarrow} := \bot$. Otherwise $\phi_k^{\leftarrow}$ is

$$\bigvee_{l \leq j \leq \mathrm{width}(\sigma_g)} \bigvee_\alpha \exists y_1 \ldots y_j. \left(\alpha(y_1,\ldots,y_j) \wedge \chi_j^{\leftarrow}(y_1,\ldots,y_j) \wedge \bigwedge_{i \in \mathrm{dom}(\rho)} x_i = y_{\rho(i)} \right)$$

  where α ranges over atomic formulas that are strict σ_g-guards for $y_1,\ldots,y_j$.
- Finally, if $\phi = \mu Y. \chi$ then $\phi_k^{\leftarrow}$ is

$$\bigvee_\alpha \left(\alpha(x_1,\ldots,x_k) \wedge [\mathbf{lfp}_{Y_{k,P},y_1,\ldots,y_k}.S_{\mu Y.\chi}](x_1,\ldots,x_k) \right)$$

where α ranges over atomic formulas that are strict σ_g-guards for $y_1, \ldots, y_j$, the relation P is the encoding of α, and $S_{\mu Y.\chi}$ is a system consisting of equations

$$Y_{j,P}, y_1 \ldots, y_j := P_j^{\leftarrow}(y_1, \ldots, y_j) \wedge \chi_j^{\leftarrow}(y_1, \ldots, y_j)$$

for each $Y_{j,P}$ in $Y^{\leftarrow}$.

The formulas produced by this translation are in $\mathrm{GFP}[\sigma', \sigma_g]$; in particular, note the σ_g-guarded existential quantification in the diamond modality translation, and the σ_g-guarded fixpoints in the fixpoint translation (we use simultaneous fixpoints here, but these can be eliminated if required). The correctness of this translation comes from the fact that every node in $\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{B})$ represents elements that are strictly σ_g-guarded. Hence, the translation of a diamond modality that expresses the existence of a neighboring node in the tree translates into a σ_g-guarded existential quantification. Likewise, the fixpoint formulas that are defining a set of nodes in the tree can be translated into fixpoint formulas defining sets of tuples that are all σ_g-guarded. We omit the formal proof of correctness, since it is similar to the more complicated proof of correctness for Lemma 5.4 that we will give later. $\quad\square$

As mentioned earlier, the desired formula ψ for Lemma 4.1 is $(\phi^\mu)_0^{\leftarrow}$ obtained using Lemma 4.2.

Plugging Lemma 4.1 into our high-level algorithm, with $\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{B})$ as $\mathcal{U}_{\mathcal{L}_1}(\mathfrak{B})$, we get decidability of the GFP-definability problem:

Theorem 4.3. *The $\mathrm{GFP}[\sigma', \sigma_g]$ definability problem is decidable for $\mathrm{GN}^l[\sigma]$-invariant $\mathrm{GSO}[\sigma]$ where $l \geq \mathrm{width}(\sigma)$ and $\sigma_g \subseteq \sigma' \subseteq \sigma$.*

4.2. **Isolating the complexity of GFP-definability.** We now see if we can get a more efficient GFP-definability test, with the goal of obtaining a tight bound on the complexity of this problem.

There are two sources of inefficiency in the high-level algorithm. First, the forward mapping is non-elementary since we pass through MSO on the way to a μ-calculus formula. Second, testing equivalence of the original sentence with the sentence produced by the forward and backward mappings naïvely would cause an additional blow-up: we would apply a forward mapping again in order to produce tree automata, and then check their equivalence using an EXPTIME algorithm.

For the special case of input in GNFP, we can avoid these inefficiencies and obtain an optimal complexity bound.

Theorem 4.4. *The $\mathrm{GFP}[\sigma', \sigma_g]$ definability problem is 2-EXPTIME-complete for input in $\mathrm{GNFP}[\sigma]$ where $\sigma_g \subseteq \sigma' \subseteq \sigma$.*

The proof of Theorem 4.4 will require some extra machinery. The main idea behind the optimized procedure is to directly use automata throughout the process. First, for input ϕ in GNFP it is known from [12] that there is a forward mapping directly producing a tree automaton $\mathcal{A}_\phi$ with exponentially-many states that accepts a consistent tree $\mathcal{T}$ iff $\mathfrak{D}(\mathcal{T}) \models \phi$; $\mathcal{A}_\phi$ accepts exactly the consistent trees that satisfy the formula ϕ^μ from Lemma 3.1. This direct construction avoids passing through MSO, and can be done in 2-EXPTIME. We can then construct an automaton $\mathcal{A}'_\phi$ from $\mathcal{A}_\phi$ that accepts a tree $\mathcal{T}$ iff $\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{D}(\mathcal{T}))$ is accepted by $\mathcal{A}_\phi$; we call this the $\mathrm{G}[\sigma', \sigma_g]$-*view automaton*, since it mimics the view of $\mathcal{A}_\phi$ running on the guarded unravelling of $\mathfrak{D}(\mathcal{T})$. This can be seen as an automaton that

represents the composition of the backward mapping with the forward mapping. With these constructions in place, we have the following improved algorithm to test definability of ϕ in GFP: construct $\mathcal{A}_\phi$ from ϕ, construct $\mathcal{A}'_\phi$ from $\mathcal{A}_\phi$, and test equivalence of $\mathcal{A}_\phi$ and $\mathcal{A}'_\phi$ over consistent trees. Note that with this improved procedure it is not necessary to actually construct the backward mapping, or to pass forward to trees for a second time in order to test equivalence. Overall, the procedure can be shown to run in 2-EXPTIME. A reduction from GFP-satisfiability testing, which is known to be 2-EXPTIME-hard, yields the lower bound.

Upper bound. We now give more details of the upper bound in Theorem 4.4. As mentioned earlier, there is an improved forward mapping from formulas in GNFP$[\sigma]$ directly to automata, without passing through MSO. It is known from prior work how to do this in 2-EXPTIME:

Lemma 4.5 (GNFP-Fwd Automaton, [12])**.** *Given $\phi \in$ GNFP$^l[\sigma]$ and given some $m \geq \max\{l, \mathrm{width}(\sigma)\}$, we can construct in 2-EXPTIME a 2-way alternating μ-automaton $\mathcal{A}_\phi$ such that for all consistent $\Sigma_{\sigma,m}^{code}$-trees $\mathcal{T}$, $\mathcal{T} \in L(\mathcal{A}_\phi)$ iff $\mathfrak{D}(\mathcal{T}) \models \phi$.*
    The number of states of $\mathcal{A}_\phi$ is exponential in $|\phi|$, and the number of priorities is linear in $|\phi|$.

It is straightforward to construct a 2-way alternating μ-automaton that checks whether a $\Sigma_{\sigma',m}^{code}$-tree is consistent. This is also known from prior work, e.g. [12].

Lemma 4.6 (Consistency Automaton)**.** *We can construct in 2-EXPTIME a 2-way alternating μ-automaton $\mathcal{C}$ such that for all $\Sigma_{\sigma,m}^{code}$-trees $\mathcal{T}$, $\mathcal{T} \in L(\mathcal{A}_\phi)$ iff $\mathcal{T}$ is consistent.*
    The number of states of $\mathcal{A}_\phi$ is exponential in $|\phi|$, and the number of priorities is linear in $|\phi|$.

As mentioned above, we can then construct a G$[\sigma', \sigma_g]$-view automaton, which can be seen as the composition of the backward mapping with the forward mapping. This results in an additional blow-up of the state set by a factor of 2^{k+1} (for $k = \mathrm{width}(\sigma')$) but no further increase in size.

Lemma 4.7 (GFP-View Automaton)**.** *Let $\sigma_g \subseteq \sigma' \subseteq \sigma$. Given a 2-way alternating μ-automaton $\mathcal{A}$ over $\Sigma_{\sigma,m}^{code}$-trees with $m \geq \mathrm{width}(\sigma')$, we can construct a G$[\sigma', \sigma_g]$-view automaton $\mathcal{A}'$ such that $\mathcal{T} \in L(\mathcal{A}')$ iff $\mathcal{U}_{G[\sigma',\sigma_g]}(\mathfrak{D}(\mathcal{T})) \in L(\mathcal{A})$. The view automaton can be constructed in time polynomial in the size of $\mathcal{A}$ and exponential in $k = \mathrm{width}(\sigma')$. The number of states increases by a factor of 2^{k+1} and the number of priorities remains the same.*

Proof. We need to design $\mathcal{A}'$ so that when it is run on a consistent $\Sigma_{\sigma,m}^{code}$-tree $\mathcal{T}$, it mimics the run of $\mathcal{A}$ on $\mathcal{U}_{G[\sigma',\sigma_g]}(\mathfrak{D}(\mathcal{T}))$. Before we do this, it is helpful to recall what these tree codes look like, and what their relationship is.

In $\mathcal{T}$, each node represents at most m elements of $\mathfrak{D}(\mathcal{T})$, and these elements are not necessarily guarded. On the other hand, each node in $\mathcal{U}_{G[\sigma',\sigma_g]}(\mathfrak{D}(\mathcal{T}))$ represents a strictly σ_g-guarded set of elements of size at most $k = \mathrm{width}(\sigma')$. But there is a strong relationship between $\mathcal{T}$ and $\mathcal{U}_{G[\sigma',\sigma_g]}(\mathfrak{D}(\mathcal{T}))$: each node in $\mathcal{U}_{G[\sigma',\sigma_g]}(\mathfrak{D}(\mathcal{T}))$ is a copy of a strictly σ_g-guarded subset of $\mathfrak{D}(\mathcal{T})$, and hence can be identified with a strictly σ_g-guarded subset of elements. Since every atom must be represented in at least one node of the tree decomposition, these elements must occur together in a *single node* of $\mathcal{T}$.

The construction of $\mathcal{A}'$ from $\mathcal{A}$ reflects this. We augment each state of $\mathcal{A}$ to also include the current *strictly σ_g-guarded view*, which is just some strictly σ_g-guarded subset of the at most m elements represented in the current node. Each strictly σ_g-guarded subset is of size at most k. The view automaton $\mathcal{A}'$ simulates $\mathcal{A}$ as if it could only see the strictly σ_g-guarded view of the label.

For each single move of the original automaton, we allow the view automaton to make a finite (but unbounded) series of moves before selecting the next strictly σ_g-guarded view. Why is this? Observe that each single move of $\mathcal{A}$ on some guarded unravelling $\mathcal{U}_{\mathrm{G}[\sigma',\sigma_g]}(\mathfrak{D}(\mathcal{T}))$ leads to a neighboring node that is based on a new strictly σ_g-guarded set of elements from $\mathfrak{D}(\mathcal{T})$. Although these elements must be represented in a single node in $\mathcal{T}$ (since they are guarded), the node in $\mathcal{T}$ that represents this new guarded set could be far away from node that represents the current guarded view. Hence, we allow the view automaton to navigate to a node in $\mathcal{T}$ representing this next strictly σ_g-guarded view before continuing the simulation.

We give more details on the construction. It may be helpful at this stage to refer to Section 2.6 for a brief introduction to the μ-automata used here.

Let $\mathcal{A} = \langle \Sigma_{\sigma,m}^{\mathrm{code}}, Q_E, Q_A, q_0, \delta, \Omega \rangle$, where Q_E and Q_A represent existential and universal states, respectively, q_0 and δ are the initial state and transition function, while Ω is a priority function used to define the acceptance condition. We construct the $\mathrm{G}[\sigma', \sigma_g]$-view automaton $\mathcal{A}' = \langle \Sigma_{\sigma,m}^{\mathrm{code}}, Q'_E, Q'_A, q'_0, \delta', \Omega' \rangle$ as follows.

Let VIEWS consist of subsets of $\{1, \ldots, m\}$ of size at most $k = \mathrm{width}(\sigma')$. Then let $Q'_E := Q_E \times \mathrm{VIEWS} \times \{\mathrm{select}, \mathrm{move}\}$ and $Q'_A := Q_A \times \mathrm{VIEWS} \times \{\mathrm{select}, \mathrm{move}\}$, with initial state $q'_0 := (q_0, \emptyset, \mathrm{select})$.

Let τ denote a node label in a $\Sigma_{\sigma,m}^{\mathrm{code}}$-tree. For $I \in \mathrm{VIEWS}$, let $\tau \restriction_{\sigma',I}$ denote the restriction of the label τ to the indices in I and $\Sigma_{\sigma',m}^{\mathrm{code}}$.

In select mode: $\delta'((q, I, \mathrm{select}), \tau)$ is the set of moves of the form $(0, (q', I', \mathrm{move}))$ such that $(d, \rho', q') \in \delta(q, \tau \restriction_{\sigma',I})$ and $I' = \mathrm{dom}(\rho')$. In other words, the automaton selects the next state in the simulation of $\mathcal{A}$ based on its view and then switches to move mode.

In move mode: $\delta'((q, I, \mathrm{move}), \tau)$ is the set consisting of

- $(\uparrow, \rho, (q, \rho(I), \mathrm{move}))$ for each $\rho \in \mathrm{EDGES}$ with $\mathrm{dom}(\rho) \supseteq I$,
- $(\downarrow, \rho, (q, \rho(I), \mathrm{move}))$ for each $\rho \in \mathrm{EDGES}$ with $\mathrm{dom}(\rho) \supseteq I$, and
- $(0, (q, I', \mathrm{select}))$ for each $I' \supseteq I$ that is strictly σ_g-guarded in τ.

Thus, in move mode, the automaton can either move to a neighboring node that contains I (renamed according to some ρ) and stay in move mode, or it can expand to a new strictly σ_g-guarded view and switch to select mode in order to continue the simulation.

The priority function Ω' is defined such that states in select mode inherit the priority of the underlying state from $\mathcal{A}$. In move mode, $\Omega'((q, I, \mathrm{move}))$ is 1 if $q \in Q_E$, and 0 if $q \in Q_A$; this ensures that the controlling player cannot cheat by forever delaying the next step in the simulation. $\qquad \square$

We can use these automata for an improved 2-ExpTime decision procedure. Suppose the input is $\phi \in \mathrm{GNFP}^l[\sigma]$. Let $m = \max\{l, \mathrm{width}(\sigma')\}$. We start by constructing $\mathcal{A}_\phi$ and $\mathcal{C}$ using Lemmas 4.5 and 4.6, and then construct the view automaton $\mathcal{A}'_\phi$ from $\mathcal{A}_\phi$ using Lemma 4.7. This can all be done in 2-ExpTime.

We claim that $\mathcal{A}_\phi$ is equivalent to $\mathcal{A}'_\phi$ for $\Sigma_{\sigma,m}^{\mathrm{code}}$-trees in $L(\mathcal{C})$ iff ϕ is $\mathrm{GFP}[\sigma', \sigma_g]$-definable. First, suppose $\mathcal{A}_\phi$ and $\mathcal{A}'_\phi$ are equivalent with respect to tree codes in $L(\mathcal{C})$. Let $\psi \in$

$\mathrm{GFP}[\sigma', \sigma_g]$ be the formula obtained by converting $\mathcal{A}_\phi$ to an equivalent μ-calculus formula using Theorem 2.8, and then applying Lemma GFP-Bwd. We show that ϕ is $\mathrm{GFP}[\sigma', \sigma_g]$-definable using this sentence ψ. For all σ-structures $\mathfrak{B}$:

$$
\begin{array}{rll}
& \mathfrak{B} \models \phi & \\
\Leftrightarrow & \mathfrak{D}(\mathcal{U}_{\mathrm{GN}^m[\sigma]}(\mathfrak{B})) \models \phi & \text{by } \mathrm{GN}^l[\sigma]\text{-invariance of } \phi \\
\Leftrightarrow & \mathcal{U}_{\mathrm{GN}^m[\sigma]}(\mathfrak{B}) \in L(\mathcal{A}_\phi) & \text{by Lemma GNFP-Fwd Automaton} \\
\Leftrightarrow & \mathcal{U}_{\mathrm{GN}^m[\sigma]}(\mathfrak{B}) \in L(\mathcal{A}'_\phi) & \text{by language equivalence} \\
\Leftrightarrow & \mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{D}(\mathcal{U}_{\mathrm{GN}^m[\sigma]}(\mathfrak{B}))) \in L(\mathcal{A}_\phi) & \text{by Lemma GFP-View Automaton} \\
\Leftrightarrow & \mathfrak{D}(\mathcal{U}_{\mathrm{GN}^m[\sigma]}(\mathfrak{B})) \models \psi & \text{by Lemma GFP-Bwd} \\
\Leftrightarrow & \mathfrak{B} \models \psi & \text{by } \mathrm{GN}^l[\sigma]\text{-invariance of } \psi.
\end{array}
$$

Hence, ϕ and ψ are logically equivalent, so ψ witnesses the $\mathrm{GFP}[\sigma', \sigma_g]$-definability of ϕ.

In the other direction, suppose that ϕ is $\mathrm{GFP}[\sigma', \sigma_g]$-definable. We must show that $\mathcal{A}_\phi$ and $\mathcal{A}'_\phi$ are equivalent with respect to $\Sigma_{\sigma,m}^{\mathrm{code}}$-trees in $L(\mathcal{C})$. For all $\Sigma_{\sigma,m}^{\mathrm{code}}$-trees $\mathcal{T}$ in $L(\mathcal{C})$:

$$
\begin{array}{rll}
& \mathcal{T} \in L(\mathcal{A}_\phi) & \\
\Leftrightarrow & \mathfrak{D}(\mathcal{T}) \models \phi & \text{by Lemma GNFP-Fwd Automaton} \\
\Leftrightarrow & \mathfrak{D}(\mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{D}(\mathcal{T}))) \models \phi & \text{by GFP-definability of } \phi \\
\Leftrightarrow & \mathcal{U}_{\mathrm{G}[\sigma', \sigma_g]}(\mathfrak{D}(\mathcal{T})) \in L(\mathcal{A}_\phi) & \text{by Lemma GNFP-Fwd Automaton} \\
\Leftrightarrow & \mathcal{T} \in L(\mathcal{A}'_\phi) & \text{by Lemma GFP-View Automaton.}
\end{array}
$$

Hence, we have shown the equivalence of $\mathcal{A}_\phi$ and $\mathcal{A}'_\phi$ with respect to $\Sigma_{\sigma',m}^{\mathrm{code}}$-trees in $L(\mathcal{C})$, which concludes the proof of correctness.

It remains to show that testing equivalence of $\mathcal{A}_\phi$ and $\mathcal{A}'_\phi$ with respect to trees in $L(\mathcal{C})$ can be done in 2-ExpTime. Using standard constructions from automata theory, we can construct 2-way alternating μ-automata recognizing $L(\mathcal{C}) \cap L(\mathcal{A}_\phi) \cap \overline{L(\mathcal{A}'_\phi)}$ and $L(\mathcal{C}) \cap L(\mathcal{A}'_\phi) \cap \overline{L(\mathcal{A}_\phi)}$, with only a constant blow-up in size thanks to the use of alternating automata. It then suffices to test emptiness of the language accepted by the automaton for $L(\mathcal{C}) \cap L(\mathcal{A}_\phi) \cap \overline{L(\mathcal{A}'_\phi)}$ and the automaton for $L(\mathcal{C}) \cap L(\mathcal{A}'_\phi) \cap \overline{L(\mathcal{A}_\phi)}$. This can be done in time exponential in the number of states and number of priorities (see [25, 40]). Overall, this means that the decision procedure is in 2-ExpTime as claimed. This completes the upper bound portion of Theorem 4.4.

Lower bound. Theorem 4.3 also states 2-ExpTime hardness of the GFP-definability problem. This is a straightforward reduction from satisfiability of $\mathrm{GFP}[\sigma]$ sentences, which is known to be 2-ExpTime-hard [27].

Fix a sentence ϕ_0 over a signature σ_0 that is in GNFP^k over its signature but not in GFP. Given $\phi \in \mathrm{GFP}[\sigma]$ our reduction produces $\phi \wedge \phi_0$, where we first modify the signatures so that σ is disjoint from σ_0.

We claim that $\phi \wedge \phi_0$ is definable in $\mathrm{GFP}[\sigma]$ iff ϕ is unsatisfiable.

Clearly if ϕ is unsatisfiable $\phi \wedge \phi_0$ is definable in $\mathrm{GFP}[\sigma]$. In the other direction, suppose for the sake of contradiction that $\phi \wedge \phi_0$ is in $\mathrm{GFP}[\sigma]$ but ϕ holds in a model $\mathfrak{A}$. Then $\phi \wedge \phi_0$ is $\mathrm{G}[\sigma]$-invariant.

We claim that ϕ_0 is G$[\sigma_0]$-invariant. Consider σ_0 structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ that are G$[\sigma_0]$-bisimilar, and where $\mathfrak{A}_1$ satisfies ϕ_0. We can assume $\mathfrak{A}_1 \cup \mathfrak{A}_2$ has a domain that is disjoint from that of $\mathfrak{A}$, since taking an isomorphic copy does not change either the guarded bisimilarity or the truth of ϕ_0. Form the $\sigma \cup \sigma_0$ structures $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$ by interpreting the σ relations as in $\mathfrak{A}$. We can extend the guarded bisimulation of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over σ_0 by the identity mapping for elements in $\mathfrak{A}$, and this clearly gives a guarded bisimulation of $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$ over $\sigma \cup \sigma_0$. $\mathfrak{A}_1'$ satisfies $\phi_0 \wedge \phi$, so $\mathfrak{A}_2'$ satisfies $\phi_0 \wedge \phi$ as well, hence $\mathfrak{A}_2$ satisfies ϕ_0, completing the proof that ϕ_0 is G$[\sigma_0]$-invariant.

Since ϕ_0 is G$[\sigma_0]$-invariant, ϕ_0 is definable in GFP$[\sigma]$ by [25], a contradiction since ϕ_0 was chosen to be outside of GFP.

4.3. **Further applications of the machinery.** Our decidability results give us a corollary on definability in fragments of FO when the input is in FO:

Corollary 4.8. *The* GF$[\sigma', \sigma_g]$ *definability problem is decidable for* GN$^l[\sigma]$*-invariant* FO$[\sigma]$ *where* $l \geq \mathrm{width}(\sigma)$ *and* $\sigma_g \subseteq \sigma' \subseteq \sigma$.

In the special case that the input is in GNF$[\sigma]$*, the* GF$[\sigma', \sigma_g]$ *definability problem is* 2-ExpTime-*complete.*

Proof. It was known from [1] that if ϕ is in FO$[\sigma]$ and is guarded bisimulation invariant with respect to σ, then it is in GF$[\sigma]$. By a straightforward refinement of the argument in [1], we see that if ϕ is in FO$[\sigma]$ and is G$[\sigma', \sigma_g]$-invariant, then it is in GF$[\sigma', \sigma_g]$.

Hence, given an input formula ϕ, we just use the algorithm of Theorem 4.4 to see if ϕ is in GFP$[\sigma', \sigma_g]$. If it is, then we conclude that ϕ is actually in GF$[\sigma', \sigma_g]$.

In the special case that the input is in GNF$[\sigma]$, we can use Theorem 4.4 to get the 2-ExpTime upper bound. The lower bound follows from a standard reduction from satisfiability (see the proof of the lower bound in Theorem 4.4). $\qquad\square$

Note that in this work we are characterizing sublogics within fragments of fixpoint logics and within fragments of first-order logic. We do not deal with identifying first-order definable formulas within a fixpoint logic, as in [15, 6, 12].

We can also apply our theorem to answer some questions about *conjunctive queries (CQs)*: formulas built up from relational atoms via $\wedge$ and $\exists$. When the input ϕ to our definability algorithm is a CQ, ϕ can be written as a GF sentence exactly when it is *acyclic*: roughly speaking, this means it can be built up from guarded existential quantification (see [24]). Transforming a query to an acyclic one could be quite relevant in practice, since acyclic queries can be evaluated in linear time [42]. There are well-known methods for deciding whether a CQ ϕ is acyclic, and recently these have been extended to the problem of determining whether ϕ is acyclic for all structures satisfying a set of constraints (e.g., Guarded Universal Horn constraints [9] or Functional Dependencies [22]). Using Corollary 4.8 above we can get an analogous result for arbitrary constraints in the guarded fragment:

Theorem 4.9. *Given a finite set of* GF *sentences* Σ *and a CQ sentence* Q*, we can decide whether there is a union of acyclic CQs* Q' *equivalent to* Q *for all structures satisfying* Σ*. The problem is* 2-ExpTime-*complete.*

For the purposes of this proof, an acyclic CQ is one built up from atomic relations by conjunction and guarded existential quantification alone. [24] showed that this is equivalent

to the more usual definitions, via the associated graph being chordal and conformal, or the associated graph being tree decomposable. First, we need the following basic result:

Claim 4.10. Let Σ be a finite set of GF sentences, Q a CQ, and suppose there is a GF sentence ϕ such that Q is equivalent to ϕ for all structures satisfying Σ. Then there is a union of acyclic CQs Q' such that Q is equivalent to Q' for all structures satisfying Σ.

Note that in the case Σ is empty, this states that a CQ is in GF iff it is a union of acyclic queries. If a CQ is equivalent to a disjunction of CQs, then it is equivalent to one of its disjuncts [35], thus in the case that Σ is empty (or more generally, when Σ is universal Horn) we can strengthen the conclusion to be that Q' is a single acyclic CQ. Although the characterization in the claim is probably well-known, we provide a proof:

Proof of Claim. We apply the "treeification lemma" of [5], which states that for every CQ sentence Q we have a union of acyclic queries Q' such that:

- Q' implies Q
- for every χ in GF: χ implies Q' if and only if χ implies Q

Suppose there is a formula ϕ in GF such that Q is equivalent to ϕ for all structures satisfying Σ. Then clearly $\Sigma \wedge \phi$ implies Q, and hence by the second item above, $\Sigma \wedge \phi$ implies Q' and thus for structures satisfying Σ, Q implies Q'.

Therefore for structures satisfying Σ, Q is equivalent to Q' as required. $\qquad\square$

Returning to the proof of Theorem 4.9, the above claim tells us that Q is equivalent to an acyclic Q' for structures satisfying Σ exactly when $\Sigma \wedge Q$ is equivalent to a sentence in GF. Since $\Sigma \wedge Q \in$ GNF when Σ is a set of GF sentences and Q is a CQ, Corollary 4.8 implies that we can decide this in 2-EXPTIME time. This gives the desired upper bound. As before, the lower bound follows from a standard reduction from satisfiability.

Note that if Σ consists of universal Horn constraints ("TGDs"), then a CQ Q is equivalent to a union of CQs Q' relative to Σ implies that it is equivalent to one of the disjuncts of Q'. Thus the result above implies decidability of acyclicity relative to universal Horn GF sentences, one of the main results of [9]. Also note that the results of [5] imply that in Theorem 4.9 the quantification "for all structures" can equivalently be replaced by "for all finite structures".

5. IDENTIFYING GNFP^k AND UNFP^k SENTENCES

We now turn to extending the prior results to GNFP and UNFP. In order to make use of the back-and-forth approach described in the previous section, we need to know that it suffices to check for definability on structures of some bounded tree-width. If we can focus on structures of bounded tree-width, we can make use of tree automata and other results about regular tree languages in solving the definability problem, since there is a fixed finite alphabet for the encodings of such structures.

This was true for definability within $\text{GFP}[\sigma', \sigma_g]$, where the tree-width depended only on the signature σ'. For definability in GNFP and UNFP, it does not suffice to check structures of some fixed tree-width. However, it does suffice for GNFP^k and UNFP^k. Hence, in this section, we will consider characterizing and deciding definability within GNFP^k and UNFP^k.

The overall approach remains the same: we apply the high-level algorithm of Proposition 3.4, using the forward mapping of Lemma 3.1. However, the unravelling and backward mapping for GNFP^k is more technically challenging than the corresponding constructions

for GFP. The naïve backward mapping from L_μ into LFP is a straightforward structural induction, but it fails to be in GNFP^k for two reasons. First, the inductive step for negation in the naïve algorithm simply applies negation to the recursively-produced formula. Clearly this can produce unguarded negation. Similarly, the recursive step for fixpoints may use unguarded fixpoints.

This section focuses on these issues. Section 5.1 introduces a special unravelling construction called a plump unravelling that is more complicated than the block k-width guarded negation unravelling defined earlier, but still preserves all GNFP^k-sentences. Section 5.2 then provides the backward mapping for GNFP^k by showing that problematic subformulas in the original L_μ-formula can be eliminated, with the correctness of this simplification holding over tree codes of these plump unravellings. Section 5.3 then summarizes the definability results that come from this.

As in the previous section, we will be working with signatures $\sigma_g \subseteq \sigma' \subseteq \sigma$, where σ' is the subsignature of σ targeted by the backward mapping, with guards taken from σ_g.

5.1. **Plump unravellings.** We first need an appropriate notion of unravelling. We use a variant of the block k-width guarded negation unravelling discussed in Section 2, but we will need to assume that we have a certain copies of pieces of the structure, over and above the usual copies present in every unravelling construction. A property like this was defined in [13] for UNFP^k, called "shrewdness", but we will need a more subtle property for GNFP^k, which we call "plumpness".

In order to define the property that this special unravelling has, we need to define how we can modify copies of certain parts of the structure in a way that is not distinguishable by GNFP^k. Let τ and τ' be sets of σ'-atoms over some set A of elements. Let $I, J \subseteq A$. We say τ and τ' agree on J if for all σ'-atoms $\alpha(a_1, \ldots, a_l)$ with $\{a_1, \ldots, a_l\} \subseteq J$, $\alpha(a_1, \ldots, a_l) \in \tau$ iff $\alpha(a_1, \ldots, a_l) \in \tau'$. We say τ' is an (σ_g, I)-*safe restriction* of τ if

(1) $\tau' \subseteq \tau$;
(2) τ' agrees with τ on I;
(3) τ' agrees with τ on every $J \subseteq A$ that is σ_g-guarded in τ'.

We will use the same terminology and notation when τ and τ' are encoded sets of atoms (rather than sets of atoms) and I, J are sets of indices encoding elements (rather than elements themselves).

Note that τ itself is considered a trivial (σ_g, I)-safe restriction of τ. Here is another example:

Example 5.1. Consider signatures $\sigma' = \{U, R, T\}$ and $\sigma_g = \{R\}$, where U is a unary relation, R is a binary relation, and T is a ternary relation. For readability in this example, we will write, e.g., $R(u, v)$ instead of Ruv. Consider $I = \{1, 2\}$ and

$$\tau = \left\{ \begin{array}{c} U(1),\ U(3) \\ R(1,2),\ R(2,3),\ R(3,1) \\ T(3,2,2) \end{array} \right\}.$$

Then the possible (σ_g, I)-safe restrictions of τ are τ itself and

$$\tau_1' = \left\{ \begin{array}{c} U(1),\, U(3) \\ R(1,2),\, R(2,3) \\ T(3,2,2) \end{array} \right\} \qquad \tau_2' = \left\{ \begin{array}{c} U(1),\, U(3) \\ R(1,2),\, R(3,1) \\ T(3,2,2) \end{array} \right\} \qquad \tau_4' = \left\{ \begin{array}{c} U(1),\, U(3) \\ R(1,2) \\ T(3,2,2) \end{array} \right\}$$

$$\tau_3' = \left\{ \begin{array}{c} U(1),\, U(3) \\ R(1,2),\, R(3,1) \end{array} \right\} \qquad \tau_5' = \left\{ \begin{array}{c} U(1),\, U(3) \\ R(1,2) \end{array} \right\}.$$

Note that in some of the restrictions, we drop some atoms that use relations from σ' or even σ_g. However, we cannot drop atoms over unary relations (since these are always trivially guarded), and we can never drop atoms using elements from the set I. Furthermore, the σ_g-atoms that we keep restrict what other atoms that we can drop, since for any σ_g-guarded set that remains we must preserve atoms over that set.

We are almost ready to define the plumpness property that a plump unravelling will exhibit. The idea is that a plump tree is a special type of σ_g-guarded interface tree. Recall that a σ_g-guarded interface tree alternates between interface nodes that are σ_g-guarded and bag nodes of some bounded width (see the description on page 13). The additional property that a plump tree must satisfy is that that for every interface node u, all safe restrictions of the atoms represented at u are realized by siblings of u.

By a (σ_g, I)-*safe restriction of a node* v in a tree code, we mean a (σ_g, I)-safe restriction of the atoms represented by the node v.

A $\Sigma_{\sigma',k}^{\text{code}}$-tree has the σ_g-*plumpness property* if for all interface nodes v: if w is a ρ_0-child of v over names J with $I = \text{rng}(\rho_0)$ and τ is the encoded set of σ'-atoms that hold at w, then for any (σ_g, I)-safe restriction τ' of τ, there is a ρ_0-child w' of v such that

(1) τ' is the encoded set of σ'-atoms that hold at w';
(2) for each ρ-child u' of w', there is a ρ-child u of w such that the subtrees rooted at u and u' are bisimilar; and
(3) for each ρ-child u of w such that $\text{dom}(\rho)$ is strictly σ_g-guarded in τ', there is a ρ-child u' of w' such that the subtrees rooted at u' and u are bisimilar.

We say a tree code is σ_g-*plump* if it satisfies this property.

Example 5.2. Let σ', σ_g be as in Example 5.1. Let $\mathcal{T}$ be a σ_g-plump tree. Suppose there is an interface node v in $\mathcal{T}$ with label encoding $\tau_0 = \{U(1), R(1,2)\}$, and there is a ρ_0-child w of v such that the label of w encodes $\tau = \{U(1), U(3), R(1,2), R(2,3), R(3,1), T(3,2,2)\}$, and ρ_0 is the identity function with domain $\{1,2\}$. Then by plumpness there must also be ρ_0-children $w_1, \ldots, w_5$ of v with labels encoding $\tau_1', \ldots, \tau_5'$ from Example 5.1.

The following proposition shows that one can obtain unravellings that are plump:

Proposition 5.3. *Let* $\mathfrak{B}$ *be a* σ-*structure,* $k \in \mathbb{N}$, *and* $\sigma_g \subseteq \sigma' \subseteq \sigma$. *There is a consistent, plump,* σ_g-*guarded-interface tree* $\mathcal{U}_{\text{BGN}^k[\sigma', \sigma_g]}^{\text{plump}}(\mathfrak{B})$ *such that* $\mathfrak{B}$ *is* $\text{BGN}^k[\sigma', \sigma_g]$-*bisimilar to* $\mathfrak{D}(\mathcal{U}_{\text{BGN}^k[\sigma', \sigma_g]}^{\text{plump}}(\mathfrak{B}))$. *We call* $\mathfrak{D}(\mathcal{U}_{\text{BGN}^k[\sigma', \sigma_g]}^{\text{plump}}(\mathfrak{B}))$ *the* plump unravelling *of* $\mathfrak{B}$.

The proof of the proposition will take up the remainder of this section. Before proceeding with the proof, we would recommend reviewing the notation and definitions on page 12.

Construction of plump unravelling. Let $\mathfrak{A}$ be a σ-structure and let $k \in \mathbb{N}$. Consider the set Π'_k of finite sequences of the form $X_0(Y_1, \tau_1)X_1 \ldots (Y_m, \tau_m)$ or $X_0(Y_1, \tau_1)X_1 \ldots (Y_m, \tau_m)X_m$, where $X_0 = \emptyset$ and for all $1 \leq i \leq m$,

- X_i is a set of elements of $\mathfrak{A}$ that is strictly σ_g-guarded by an atom in τ_i;
- Y_i is a set of elements of $\mathfrak{A}$ of size at most k;
- Y_i contains both X_{i-1} and X_i;
- τ_i is a (σ_g, X_{i-1})-safe restriction of $\mathrm{AT}_{\mathfrak{A}, \sigma'}(Y_i)$.

Let $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma', \sigma_g]}(\mathfrak{A})$ denote the $\Sigma^{\mathrm{code}}_{\sigma', k}$-tree of sequences of Π'_k, arranged based on prefix order. Roughly speaking, the node labels indicate the encoding of the atomic formulas holding at each position—this is based on $\mathrm{AT}_{\mathfrak{A}, \sigma'}(X_i)$ if the node corresponds to a sequence ending in X_i, and τ_i if the node corresponds to a sequence ending in (Y_i, τ_i). The edge labels ρ indicate the shared elements between X_i and Y_{i+1} or Y_{i+1} and X_{i+1}. This is similar to $\mathcal{U}_{\mathrm{BGN}^k[\sigma', \sigma_g]}(\mathfrak{A})$, except it includes all of the variations to the labels coming from safe restrictions of the bag nodes that are needed to make it a plump tree.

Formally, we build up the labels inductively based on the depth of the tree. As we go, we also define for each $v \in \Pi'_k$ ending in Z_i or (Z_i, τ_i), a bijective function ν_v from Z_i to $\{1, \ldots, |Z_i|\}$ that defines the element index assigned at that node to each element in Z_i in the tree encoding.

- The label at the root v_0 consists only of D_0, and we have the empty map ν_{v_0} (since $X_0 = \emptyset$).
- Consider the node $v = X_0(Y_1, \tau_1)X_1 \ldots X_{m-1}(Y_m, \tau_m)$ and its parent of the form $u = X_0(Y_1, \tau_1)X_1 \ldots X_{m-1}$ with inductively defined ν_u. Fix some bijective ν_v from Y_m to $\{1, \ldots, |Y_m|\}$ that agrees with ν_u on X_{m-1}. For each $R(a_1, \ldots, a_l) \in \tau_m$, add $R_{\nu_v(a_1), \ldots, \nu_v(a_l)}$ to the label of v. Add $D_{|Y_m|}$ to the label at v. The edge label ρ between u and v is defined to be the identity map from $\{1, \ldots, |X_{m-1}|\}$ to $\{1, \ldots, |X_{m-1}|\}$.
- Consider the node $v = X_0(Y_1, \tau_1)X_1 \ldots (Y_m, \tau_m)X_m$ and its parent of the form $u = X_0(Y_1, \tau_1)X_1 \ldots (Y_m, \tau_m)$ with inductively defined ν_u. Fix some bijective ν_v from X_m to $\{1, \ldots, |X_m|\}$. Then for each $R(a_1, \ldots, a_l) \in \mathrm{AT}_{\mathfrak{A}, \sigma'}(X_m)$, add $R_{\nu_v(a_1), \ldots, \nu_v(a_l)}$ to the label at v. Add $D_{|X_m|}$ to the label at v. The edge label between u and v is the function ρ with domain $\ell_u(Y_m \cap X_m)$ such that for each $a \in Y_m \cap X_m$, $\rho(\ell_u(a)) := \ell_v(a)$.

This completes the definition of $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma', \sigma_g]}(\mathfrak{A})$.

One worry with a definition like this is that the resulting tree is not consistent; in particular, one could worry that information about shared elements is not propagated correctly between neighboring nodes because the labels come from encoding safe restrictions of the atomic type (rather than just always encoding the exact atomic type). Suppose that we have some encoded atom $R_{i_1, \ldots, i_n}$ in some node. We can show that this information is propagated to all appropriate nodes, by induction on the length of the propagation path. The base case (for a path of length 0) is trivial. Now suppose that $R_{i_1, \ldots, i_n}$ is in an interface node v and there is some ρ-child w such that $\mathrm{names}(R) \subseteq \mathrm{dom}(\rho) = \mathrm{rng}(\rho)$. The label at w could correspond to a $(\sigma_g, \mathrm{rng}(\rho))$-safe restriction of the atomic type. But because such a restriction must agree on $\mathrm{rng}(\rho)$, this means that $R_{\rho(i_1), \ldots, \rho(i_n)}$ must appear in w as required. If $R_{i_1, \ldots, i_n}$ is in a bag node w and there is some ρ-child v such that $\mathrm{names}(R) \subseteq \mathrm{dom}(\rho) = \mathrm{rng}(\rho)$, then $\mathrm{dom}(\rho)$ must be strictly σ_g-guarded in w (by the definition of the plump unravelling). Because w must agree exactly with the atomic type on any

σ_g-guarded set in w, this means that $R_{\rho(i_1),...,\rho(i_n)}$ will also appear in v as required. Similar reasoning can be used for propagation to a parent node as well.

Thus, $\mathcal{U}^{\text{plump}}_{\text{BGN}^k[\sigma',\sigma_g]}(\mathfrak{A})$ is consistent. It is also straightforward to check that it is a plump, σ_g-guarded-interface tree.

Proof of bisimilarity. The proof that this unravelling is $\text{BGN}^k[\sigma',\sigma_g]$-bisimilar to the original structure is fairly standard. It suffices to show that Duplicator has a winning strategy in the $\text{BGN}^k[\sigma',\sigma_g]$-game between $\mathfrak{U} := \mathfrak{D}(\mathcal{U}^{\text{plump}}_{\text{BGN}^k[\sigma',\sigma_g]}(\mathfrak{A}))$ and $\mathfrak{A}$.

We build up the strategy inductively, ensuring that every partial play of even length in the strategy ends in a position f that is *good*. We say a position f with active structure $\mathfrak{A}$ is good if it is a partial σ'-isomorphism and there is an interface node v in $\mathcal{U} := \mathcal{U}^{\text{plump}}_{\text{BGN}^k[\sigma',\sigma_g]}(\mathfrak{A})$ where $\text{rng}(f)$ is represented, and this node is based on $\text{dom}(f)$ from $\mathfrak{A}$. Likewise, we say a position f with active structure $\mathfrak{U}$ is good if it is a partial σ'-isomorphism and there is an interface node v in $\mathcal{U}$ where $\text{dom}(f)$ is represented, and this node is based on $\text{rng}(f)$ from $\mathfrak{A}$.

The empty play is trivially good, since the starting position has empty domain, and is represented at the root of $\mathcal{U}$.

Assume that a partial play consistent with the strategy ends in a good position $f : A \to U$ with active structure $\mathfrak{A}$, with $\text{rng}(f)$ represented in node v in $\mathcal{U}$. We show how to extend the strategy in the block k-width game.

- If Spoiler switches structures and then collapses to some strictly σ_g-guarded set U' of U, then the resulting position f' is clearly still a partial σ'-isomorphism. In the plump unravelling, we know that v is a node based on a σ_g-guarded set A (i.e. a sequence ending in A) and there is a successor w of v such that w is also based on A (i.e. a sequence ending in some (A, τ)). Moreover, both v and w must represent exactly the atomic σ'-type of $\text{dom}(f)$, since interface nodes in $\mathcal{U}$ represent the exact σ_g-type of the underlying elements of $\mathfrak{A}$, and this is propagated to a successor that shares the same elements. Hence, w has a successor v' based on the restriction of A to $\text{rng}(f')$, which represents the strictly σ_g-guarded subset U' of U. Thus, we have extended the play to another good position f'.
- Now consider the case when Spoiler extends to $A' \supseteq A$ with $|A'| \leq k$. Consider the successor w of v that is based on A' and the exact σ_g-type of A'; this exists in $\mathcal{U}$ since v was based on A, $A' \subseteq A$ and $|A'| \leq k$. Let U' be the elements from $\mathfrak{U}$ that come from this node w. Then $f' : A' \to U'$ is a partial σ'-isomorphism based on how we selected w. Spoiler must then select some strictly σ_g-guarded A'' that is a subset of A' to collapse to. The resulting position $f'' : A'' \to U''$ is also a partial σ'-isomorphism. Moreover, by the definition of the unravelling, $\text{rng}(f'')$ must be represented at some interface node v' that is a successor of w and is based on A''. This means we have extended the strategy so that the partial play ends in a good position.

Now assume that the play ends in a good position $f : U \to A$ with active structure $\mathfrak{U}$, with $\text{dom}(f)$ represented in node v in $\mathcal{U}$. We show how to extend the strategy.

- If Spoiler switches structures and then collapses to some strictly σ_g-guarded set, then we can use similar reasoning as above to argue that the resulting position is good.
- So assume Spoiler extends to $U' \supseteq U$ with $|U'| \leq k$. Let Duplicator select the elements A' which were the basis for U'. The resulting $f' : U' \to A'$ is guaranteed to be a partial σ'-homomorphism since the atomic information about U' must be a subset of the atomic type of the underlying elements A' from $\mathfrak{A}$.

We cannot guarantee at this stage that f' is a partial σ'-isomorphism. First, multiple elements from U' might be derived from a single element in A'. This is the case in any unravelling construction, since the unravelling creates 'copies' of pieces of the structure. Second, elements from U' might come from a node that is a restriction of the atomic type of the underlying elements A' from $\mathfrak{A}$. This is a particular feature of the plump unravellings, since we include variations of the atomic type of the elements A' based on safe restrictions of the atomic type of those elements. We also cannot guarantee that these elements U are all represented at a single node in $\mathcal{U}$.

However, when Spoiler collapses to a strictly σ_g-guarded set $U'' \subseteq U'$, this strictly σ_g-guarded set must be represented in at least one node w in $\mathcal{U}$; suppose this is a bag node. This node w could represent a restriction of the type. But the key property of the plump unravelling is that we only allow safe restrictions of the type, which must agree exactly with the underlying elements from $\mathfrak{A}$ on any σ_g-guarded set. Hence, this restriction to a strictly σ_g-guarded set must be a partial σ'-isomorphism, and the domain of this partial σ'-isomorphism is represented in a successor v' of w. The reasoning is easier if w is an interface node, since an interface node must be based on the exact type of the underlying elements from $\mathfrak{A}$. In any case, the resulting partial plays end in good positions, as desired.

This means that we can build up a winning strategy for Duplicator in the $\mathrm{BGN}^k[\sigma', \sigma_g]$-game between $\mathfrak{A}$ and its plump unravelling, which concludes the proof of Proposition 5.3.

5.2. **Backward mapping.** Returning to the components required for the application of Proposition 3.4, we see that Proposition 5.3 says that the structure $\mathfrak{B}$ is $\mathrm{BGN}^k[\sigma', \sigma_g]$-bisimilar to $\mathfrak{D}(\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma', \sigma_g]}(\mathfrak{B}))$ as required for an application of Proposition 3.4. Plumpness will come into play in the backward mapping, which is given by the following lemma:

Lemma 5.4 (GNFP^k-Bwd). *Given $\phi^\mu \in \mathrm{L}_\mu[\Sigma^{code}_{\sigma, m}]$, relational signatures σ_g and σ' with $\sigma_g \subseteq \sigma' \subseteq \sigma$, and $k \leq m$, we can construct $\psi \in \mathrm{GNFP}^k[\sigma', \sigma_g]$ such that for all σ-structures $\mathfrak{B}$, $\mathfrak{B} \models \psi$ iff $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma', \sigma_g]}(\mathfrak{B}) \models \phi^\mu$.*

There are two main challenges for this backward mapping, compared to the backward mapping for GFP described earlier. First, we must understand where negations occur in the μ-calculus formula, and ensure that these negations will translate into σ_g-guarded negations. Second, we must ensure that fixpoints reference only interface positions, so they can be translated into σ_g-guarded fixpoints.

For example, the original μ-calculus formula ϕ^μ can include subformulas of the form $\langle \rho \rangle \mathrm{EXACTLABEL}(\tau)$ where τ is a set of unary relations from $\Sigma^{code}_{\sigma', k}$, and $\mathrm{EXACTLABEL}(\tau)$ asserts P for all $P \in \tau$ and $\neg P$ for all unary relations P not in τ. This would be problematic in a naïve backward mapping, since the backward translation of some $\neg R_{i_1, \ldots, i_n}$ would be converted into an unguarded negation $\neg R(x_{i_1}, \ldots, x_{i_n})$. On the other hand the formula $\langle \rho \rangle \mathrm{GNLABEL}(\tau)$ where $\mathrm{GNLABEL}(\tau)$ asserts P for all $P \in \tau$ but only asserts $\neg P$ for unary relations P that are not in τ but whose indices are σ_g-guarded by some $P' \in \tau$ would be unproblematic, since this could be translated to a formula with σ_g-guarded negation. The key observation is that from an interface node in a plump tree, these two formulas are equivalent: if $\mathcal{T}, v \models \langle \rho \rangle \mathrm{GNLABEL}(\tau)$ at any interface node v, then plumpness ensures that if there is some ρ-child w' of v with label τ' satisfying $\mathrm{GNLABEL}(\tau)$, then there is a ρ-child w of v with label τ satisfying $\mathrm{EXACTLABEL}(\tau)$—it can be checked that τ is a $(\sigma_g, \mathrm{rng}(\rho))$-safe restriction of τ'. Thus the main technical work in this section is to show

that problematic subformulas like $\text{EXACTLABEL}(\tau)$ can be eliminated, with the correctness of the simplification holding at least over plump trees.

Hence, the structure of the proof is as follows. We first define a fragment of $\mathrm{L}_\mu[\Sigma^{\mathrm{code}}_{\sigma',k}]$, which we call "$\text{GNFP}^k[\sigma',\sigma_g]$-safe formulas", and show that these formulas can be converted into $\text{GNFP}^k[\sigma',\sigma_g]$. After this, we will show that any μ-calculus formula over $\Sigma^{\mathrm{code}}_{\sigma,m}$-trees can be converted to a $\text{GNFP}^k[\sigma',\sigma_g]$-safe form that is equivalent on plump, σ_g-guarded-interface $\Sigma^{\mathrm{code}}_{\sigma',k}$-trees like $\mathcal{U}^{\mathrm{plump}}_{\text{BGN}^k[\sigma',\sigma_g]}(\mathfrak{B})$.

We say an $\mathrm{L}_\mu[\Sigma^{\mathrm{code}}_{\sigma',k}]$ formula is $\text{GNFP}^k[\sigma',\sigma_g]$-*safe for interface nodes* (respectively, $\text{GNFP}^k[\sigma',\sigma_g]$-*safe for bag nodes*) if

- every occurrence of a fixpoint $\lambda X.\chi$ or fixpoint variable X is in the scope of an even (respectively, odd) number of modalities and has one of the following forms:

$$P \wedge D_m \wedge \lambda X.\chi \qquad\qquad P \wedge D_m \wedge \neg\lambda X.\chi$$
$$P \wedge D_m \wedge X \qquad\qquad P \wedge D_m \wedge \neg X$$

  for some P encoding a relation in σ_g with $\text{names}(P) = \{1,\ldots,m\}$;
- every negation has one of the following forms:

$$P' \wedge \neg R \qquad\qquad P \wedge D_m \wedge \neg X$$
$$P'' \wedge \neg\langle\rho\rangle\chi \qquad\qquad P \wedge D_m \wedge \neg\lambda X.\chi$$

  for some P, P', P'' encoding relations in σ_g and with $\text{names}(P) = \{1,\ldots,m\}$, $\text{names}(P') \supseteq \text{names}(R)$, $\text{names}(P'') = \text{dom}(\rho)$;
- every modality has one of the following forms:

$$P \wedge \langle\rho\rangle\chi$$
$$P \wedge \neg\langle\rho\rangle\chi$$

  for some P encoding a relation in σ_g with $\text{names}(P) = \text{dom}(\rho)$.

Note that these safe formulas impose σ_g-guardedness conditions on fixpoints, negations, and modalities.

We already defined $\text{names}(\chi)$ for χ a propositional variable. We generalize this to all $\text{GNFP}^k[\sigma',\sigma_g]$-safe formulas χ as follows:

- $\text{names}(\top) = \text{names}(\bot) := \emptyset$;
- $\text{names}(D_m) = \{1,\ldots,m\}$;
- $\text{names}(\chi_1 \wedge \chi_2) = \text{names}(\chi_1 \vee \chi_2) := \text{names}(\chi_1) \cup \text{names}(\chi_2)$;
- $\text{names}(P \wedge \neg R) := \text{names}(P)$;
- $\text{names}(P \wedge \langle\rho\rangle\chi) = \text{names}(P \wedge \neg\langle\rho\rangle\chi) := \text{dom}(\rho)$;
- $\text{names}(P \wedge D_m \wedge X) = \text{names}(P \wedge D_m \wedge \neg X) = \{1,\ldots,m\}$;
- $\text{names}(P \wedge D_m \wedge \lambda X.\chi) = \text{names}(P \wedge D_m \wedge \neg\lambda X.\chi) := \{1,\ldots,m\}$.

These names determine the free first-order variables in the backwards translation.

For each fixpoint variable X we introduce multiple second-order variables of the form $X_{j,P}$ for $0 \leq j \leq k$ and $P \in \Sigma^{\mathrm{code}}_{\sigma',k}$ with $\text{names}(P) = \{1,\ldots,j\}$, as we did for the GFP backward mapping. Let $X^\leftarrow$ denote the set of these new second-order variables based on X. A set V of nodes in $\mathcal{U}^{\mathrm{plump}}_{\text{BGN}^k[\sigma',\sigma_g]}(\mathfrak{B})$ is a *safe valuation for a free variable X* if it

(1) contains only interface nodes, and

(2) if it contains an interface node then it contains every interface node that is the root of a bisimilar subtree.

We write $V^{\leftarrow}$ for its representation in $\mathfrak{B}$ (as we did for GFP).

We will now show that from a $\mathrm{GNFP}^k[\sigma', \sigma_g]$-safe formula in $\mathrm{L}_\mu[\Sigma^{code}_{\sigma',k}]$ we can produce $\mathrm{GNFP}^k[\sigma', \sigma_g]$ formulas described below. As in the backward mapping for GFP, we must generate a family of formulas during the inductive translation. This time we must not only deal with different domain sizes in the nodes of $\mathcal{U}^{plump}_{\mathrm{BGN}^k[\sigma',\sigma_g]}(\mathfrak{B})$, but we must also distinguish between bag nodes and interface nodes. We use $\phi^{\leftarrow}_m$ to indicate a formula related to a bag node of size m, and $\phi^{\leftarrow}_{m,I}$ to indicate a formula related to an interface node with size m.

Lemma 5.5. *Let* $\phi \in \mathrm{L}_\mu[\Sigma^{code}_{\sigma',k}]$ *be* $\mathrm{GNFP}^k[\sigma', \sigma_g]$-*safe for interface nodes (respectively, bag nodes) with free second-order variables* $\boldsymbol{X}$. *For each* $m \leq k$, *we can construct a* $\mathrm{GNFP}^k[\sigma', \sigma_g]$-*formula* $\phi^{\leftarrow}_{m,I}(x_1, \ldots, x_m, \boldsymbol{X}^{\leftarrow})$ *(respectively,* $\phi^{\leftarrow}_m(x_1, \ldots, x_m, \boldsymbol{X}^{\leftarrow})$*) such that for all* σ-*structures* $\mathfrak{B}$, *for all safe valuations* $\boldsymbol{V}$ *of* $\boldsymbol{X}$, *and for all nodes* v *in* $\mathcal{U}(\mathfrak{B}) = \mathcal{U}^{plump}_{\mathrm{BGN}^k[\sigma',\sigma_g]}(\mathfrak{B})$ *with* $|\mathrm{elem}(v)| = m$ *and label* τ:

if v *is an interface node:*	$\mathcal{U}(\mathfrak{B}), v, \boldsymbol{V} \models \phi$	$\Rightarrow$	$\mathfrak{B}, \mathrm{elem}(v), \boldsymbol{V}^{\leftarrow} \models \phi^{\leftarrow}_{m,I}$ (1)
if v *is an interface node:*	$\mathfrak{B}, \mathrm{elem}(v), \boldsymbol{V}^{\leftarrow} \models \phi^{\leftarrow}_{m,I}$	$\Rightarrow$	$\mathcal{U}(\mathfrak{B}), v, \boldsymbol{V} \models \phi$ (2)
if v *is a bag node:*	$\mathcal{U}(\mathfrak{B}), v, \boldsymbol{V} \models \phi$	$\Rightarrow$	$\mathfrak{B}, \mathrm{elem}(v), \boldsymbol{V}^{\leftarrow} \models \phi^{\leftarrow}_m$ (3)
if v *is a bag node and* τ *encodes* $\mathrm{AT}_{\mathfrak{B},\sigma'}(\mathrm{elem}(v))$:	$\mathfrak{B}, \mathrm{elem}(v), \boldsymbol{V}^{\leftarrow} \models \phi^{\leftarrow}_m$	$\Rightarrow$	$\mathcal{U}(\mathfrak{B}), v, \boldsymbol{V} \models \phi$. (4)

Moreover, if $\mathrm{names}(\phi) = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, m\}$ *then we have* $\mathrm{free}(\phi^{\leftarrow}_{m,I}) = \mathrm{free}(\phi^{\leftarrow}_m) = \{x_{i_1}, \ldots, x_{i_n}\}$, *and any subformula of* $\phi^{\leftarrow}_{m,I}$ *or* $\phi^{\leftarrow}_m$ *that begins with an existential quantifier and is not directly below another existential quantifier is strictly* σ_g-*answer-guarded, and any negation is strictly* σ_g-*guarded.*

It is helpful to compare the statement of this lemma to the analogous one for GFP in Lemma 4.1. Note that we cannot replace conditions (1)–(4) by, say, the stronger and simpler statement $\mathcal{U}^{plump}_{\mathrm{BGN}^k[\sigma',\sigma_g]}(\mathfrak{B}), v, \boldsymbol{V} \models \phi$ iff $\mathfrak{B}, \mathrm{elem}(v), \boldsymbol{V}^{\leftarrow} \models \phi^{\leftarrow}_m$. In particular, if v is a bag node, we cannot guarantee that $\mathcal{U}^{plump}_{\mathrm{BGN}^k[\sigma',\sigma_g]}(\mathfrak{B}), v, \boldsymbol{V} \models \phi$ implies $\mathfrak{B}, \mathrm{elem}(v), \boldsymbol{V}^{\leftarrow} \models \phi^{\leftarrow}_m$, since bag nodes in a plump unravelling are based on safe restrictions of the atomic information, rather than an exact copy. The weaker condition in (4) is sufficient for our purposes.

Although the following inductive proof is long, it does not use any complicated machinery. We are translating L_μ-formulas that talk about the tree codes of plump unravellings into formulas talking about relational structures (e.g. diamond modalities in the μ-calculus formula become existentially quantified formulas in the translation). The conditions for GNFP^k-safe L_μ-formulas ensure that this translation always stays within GNFP^k.

Proof of Lemma 5.5. We start with the inductive translation, and then give the proof of correctness for some illustrative cases.

Translation. If $\mathrm{names}(\phi) \not\subseteq \{1, \ldots, m\}$ then $\phi^{\leftarrow}_{m,I} = \phi^{\leftarrow}_m = \bot$. Otherwise, we proceed by induction on the structure of the $\mathrm{GNFP}^k[\sigma', \sigma_g]$-safe formula ϕ to define $\phi^{\leftarrow}_{m,I}$ and $\phi^{\leftarrow}_m$.

- If $\phi = D_j$, then $\phi^{\leftarrow}_{m,I} = \phi^{\leftarrow}_m$ is $\top$ if $m = j$ and $\bot$ otherwise.

- If $\phi = R_{i_1 \ldots i_n}$, then

$$\phi^{\leftarrow}_{m,I} = \phi^{\leftarrow}_m := R\, x_{i_1} \ldots x_{i_n}.$$

- If $\phi = P \wedge \neg R$, then $\phi^{\leftarrow}_{m,I} := (P)^{\leftarrow}_{m,I} \wedge \neg((P)^{\leftarrow}_{m,I} \wedge (R)^{\leftarrow}_{m,I})$ and $\phi^{\leftarrow}_m := (P)^{\leftarrow}_m \wedge \neg((P)^{\leftarrow}_m \wedge (R)^{\leftarrow}_m)$.
- If $\phi = P \wedge D_j \wedge X$ and $m = j$, then $\phi^{\leftarrow}_{m,I} := (P)^{\leftarrow}_{j,I} \wedge X_{j,P}(x_1, \ldots, x_j)$; otherwise, if $m \neq j$, then $\phi^{\leftarrow}_{m,I} := \bot$. Similarly for $\phi = P \wedge D_j \wedge \neg X$.
- The translation commutes with $\vee$ and $\wedge$ for each m.
- If $\phi = P \wedge \langle \rho \rangle \chi$ where $\mathrm{dom}(\rho) = \{i_1, \ldots, i_n\}$, then

$$\phi^{\leftarrow}_{m,I} := (P)^{\leftarrow}_{m,I} \wedge \bigvee_{n \leq j \leq k} \exists y_1 \ldots y_j. \left((P)^{\leftarrow}_{m,I} \wedge \chi^{\leftarrow}_j(y_1, \ldots, y_j)[x_i/y_{\rho(i)} : i \in \mathrm{dom}(\rho)] \right)$$

$$\phi^{\leftarrow}_m := (P)^{\leftarrow}_m \wedge \chi^{\leftarrow}_{n,I}(y_1, \ldots, y_n)[x_i/y_{\rho(i)} : i \in \mathrm{dom}(\rho)].$$

Similarly for $\phi = P \wedge \neg \langle \rho \rangle \chi$.

- If $\phi = P \wedge D_j \wedge \mu Y.\chi$, then $\phi^{\leftarrow}_{m,I} := (P)^{\leftarrow}_{m,I} \wedge (D_j)^{\leftarrow}_{m,I} \wedge [\mathbf{lfp}_{Y_{j,P}, y_1, \ldots, y_j}.S_{\mu Y.\chi}](x_1, \ldots, x_j)$ where $S_{\mu Y.\chi}$ is a system consisting of equations

$$Y_{n,P'}, y_1 \ldots, y_n := (P')^{\leftarrow}_{n,I}(y_1, \ldots, y_n) \wedge \chi^{\leftarrow}_{n,I}(y_1, \ldots, y_n)$$

for each $Y_{n,P'}$ in $Y^{\leftarrow}$. Similarly for $\phi = P \wedge D_j \wedge \neg \mu Y.\chi$.

- If $\phi = P \wedge D_j \wedge \nu Y.\chi$, then $\phi^{\leftarrow}_{m,I} := (P)^{\leftarrow}_{m,I} \wedge (D_j)^{\leftarrow}_{m,I} \wedge \neg[\mathbf{lfp}_{Y_j, y_1, \ldots, y_j}.S_{\nu Y.\chi}](x_1, \ldots, x_j)$ where $S_{\nu Y.\chi}$ is a system consisting of equations

$$Y_{n,P'}, y_1 \ldots, y_n := (P')^{\leftarrow}_{n,I}(y_1, \ldots, y_n) \wedge \neg\chi^{\leftarrow}_{n,I}(y_1, \ldots, y_n)[\neg Y_{n'',P''}/Y_{n'',P''} : Y_{n'',P''} \in Y^{\leftarrow}]$$

for each $Y_{n,P'}$ in $Y^{\leftarrow}$. Similarly for $\phi = P \wedge D_j \wedge \neg \nu Y.\chi$.

The formulas $\phi^{\leftarrow}_m$ and $\phi^{\leftarrow}_{m,I}$ are in $\mathrm{GNFP}[\sigma', \sigma_g]$ of width k, but are not necessarily in strict normal form. However, it can be checked that the negations are all strictly σ_g-guarded; for instance, in the case of $\phi = P \wedge \neg R$, then $\phi^{\leftarrow} = (P)^{\leftarrow}_m \wedge \neg((P)^{\leftarrow}_m \wedge (R)^{\leftarrow}_m)$, which is a strictly σ_g-guarded negation since $\mathrm{names}(P) \supseteq \mathrm{names}(R)$ and P encodes a σ_g-relation (by definition of $\mathrm{GNFP}^k[\sigma', \sigma_g]$-safety). Also, every subformula starting with an existential quantifier and not immediately below another existential quantifier is strictly σ_g-answer-guarded. Thus, we can convert to strict normal form using Proposition 2.2 without increasing the width, so the formulas are in $\mathrm{GNFP}^k[\sigma', \sigma_g]$ as desired.

Proof of correctness. We now give the proof of correctness for some illustrative cases. We write IH1–IH4 to denote the application of the inductive hypothesis based on properties (1)–(4). We will assume that $\mathrm{names}(\phi) \subseteq \{1, \ldots, m\}$.

Atom. Consider the base case when $\phi = R_{i_1 \ldots i_n}$.

Recall that in the plump unravelling, the label at a node v is based on $\mathrm{AT}_{\mathrm{elem}(v), \sigma'}(\mathfrak{B})$. For an interface node, the label is *exactly* the encoding of $\mathrm{AT}_{\mathrm{elem}(v), \sigma'}(\mathfrak{B})$, which is enough to ensure that properties (1) and (2) hold. For a bag node, the label is based on a safe *restriction* of $\mathrm{AT}_{\mathrm{elem}(v), \sigma'}(\mathfrak{B})$, which can result in the label at v including only a subset of the encoding of $\mathrm{AT}_{\mathrm{elem}(v), \sigma'}(\mathfrak{B})$. Property (3) follows from this. Not all bag nodes v such that $\mathfrak{B}, \mathrm{elem}(v) \models \phi^{\leftarrow}_m$ would satisfy $\mathcal{U}(\mathfrak{B}), v \models \phi$. However, for bag nodes v that do represent $\mathrm{AT}_{\mathrm{elem}(v), \sigma'}(\mathfrak{B})$ in its entirety, $\mathfrak{B}, \mathrm{elem}(v) \models \phi^{\leftarrow}_m$ implies $\mathcal{U}(\mathfrak{B}), v \models \phi$. Hence, property (4) holds.

Negated atom. Consider the case when $\phi = P \wedge \neg R$ with $\mathrm{names}(P) \supseteq \mathrm{names}(R) = \{i_1, \ldots, i_n\}$ and $\phi_{m,I}^{\leftarrow} := (P)_{m,I}^{\leftarrow} \wedge \neg((P)_{m,I}^{\leftarrow} \wedge (R)_{m,I}^{\leftarrow})$ and $\phi_m^{\leftarrow} := (P)_m^{\leftarrow} \wedge \neg((P)_m^{\leftarrow} \wedge (R)_m^{\leftarrow})$.

For (1), (2), and (4) correctness essentially follows from the inductive hypothesis. Consider (3). Suppose $\mathcal{U}(\mathfrak{B}), v \models \phi$ for a bag node v. By the properties of the plump unravelling, the label at a bag node v is the encoding of a safe restriction of $\mathrm{AT}_{\mathrm{elem}(v),\sigma'}(\mathfrak{B})$—this means that for any set of elements that is still σ_g-guarded after the restriction, the label must encode exactly the atoms in $\mathrm{AT}_{\mathrm{elem}(v),\sigma'}(\mathfrak{B})$ about these elements. Because we know that $i_1, \ldots, i_n$ is σ_g-guarded by P in the label at v, this means that the label at v must reflect $\mathrm{AT}_{\mathrm{elem}(v),\sigma'}(\mathfrak{B})$ exactly over the elements corresponding to $i_1, \ldots, i_n$. Hence, $\mathfrak{B}, \mathrm{elem}(v) \models (R)_m^{\leftarrow}$ iff $\mathcal{U}(\mathfrak{B}), v \models R$, so $\mathfrak{B}, \mathrm{elem}(v) \models \neg(R)_m^{\leftarrow}$. By IH3 applied to P, we also have $\mathfrak{B}, \mathrm{elem}(v) \models (P)_m^{\leftarrow}$, so $\mathfrak{B}, \mathrm{elem}(v) \models \phi_m^{\leftarrow}$ as desired.

Modality. Consider the case when $\phi = P \wedge \neg\langle\rho\rangle\chi$ with $\mathrm{names}(P) = \mathrm{dom}(\rho) = \{i_1, \ldots, i_n\}$ and

$$\phi_{m,I}^{\leftarrow} := (P)_{m,I}^{\leftarrow} \wedge \neg \bigvee_{n \leq j \leq k} \exists y_1 \ldots y_j. \left((P)_{m,I}^{\leftarrow} \wedge \chi_j^{\leftarrow}(y_1, \ldots, y_j)[x_i/y_{\rho(i)} : i \in \mathrm{dom}(\rho)] \right)$$

$$\phi_m^{\leftarrow} := (P)_m^{\leftarrow} \wedge \neg\chi_{n,I}^{\leftarrow}(y_1, \ldots, y_n)[x_i/y_{\rho(i)} : i \in \mathrm{dom}(\rho)].$$

For (1), suppose v is an interface node and $\mathcal{U}(\mathfrak{B}), v \models \phi$. Let $\mathrm{elem}(v) = a_1 \ldots a_m$. Then by IH1, $\mathfrak{B}, \mathrm{elem}(v) \models (P)_{m,I}^{\leftarrow}$. Suppose for the sake of contradiction that

$$\exists y_1 \ldots y_j. \left(\chi_j^{\leftarrow}(y_1, \ldots, y_j)[x_i/y_{\rho(i)} : i \in \mathrm{dom}(\rho)] \right)$$

for some $n \leq j \leq k$. Then there are elements $b_1, \ldots, b_j$ in $\mathfrak{B}$ such that $\mathfrak{B}, b_1, \ldots, b_j \models \chi_j^{\leftarrow}(y_1, \ldots, y_j)$, where $a_i = b_{\rho(i)}$. By the properties of the unravelling, there is a ρ-child w of v such that $\mathrm{elem}(w) = b_1, \ldots, b_j$ and such that the label at w is the encoding of $\mathrm{AT}_{\mathrm{elem}(w),\sigma'}(\mathfrak{B})$. Hence, by IH4, $\mathcal{U}(\mathfrak{B}), w \models \chi$ (we can apply IH4 since we have chosen w such that it represents the full atomic type of $\mathrm{elem}(w)$ in $\mathfrak{B}$). This means $\mathcal{U}(\mathfrak{B}), v \models \langle\rho\rangle\chi$, a contradiction. Therefore, it must be the case that $\mathfrak{B}, \mathrm{elem}(v) \models \phi_{m,I}^{\leftarrow}$.

For (2), suppose v is an interface node and $\mathfrak{B}, \mathrm{elem}(v) \models \phi_{m,I}^{\leftarrow}$. Let $\mathrm{elem}(v) = a_1 \ldots a_m$. By IH2, $\mathcal{U}(\mathfrak{B}), v \models P$. Suppose for the sake of contradiction that $\mathcal{U}(\mathfrak{B}), v \models \langle\rho\rangle\chi$. Then there is some ρ-child w of v such that $\mathcal{U}(\mathfrak{B}), w \models \chi$. The node w must be a bag node and must have some domain predicate D_j. Let $b_1 \ldots b_j$ be the elements from $\mathfrak{B}$ represented there, with $a_i = b_{\rho(i)}$. By IH3, this means that $\mathfrak{B}, \mathrm{elem}(w) \models \chi_j^{\leftarrow}(y_1, \ldots, y_j)$, and hence $\mathfrak{B}, \mathrm{elem}(v) \models \exists y_1 \ldots y_j. \left(\chi_j^{\leftarrow}(y_1, \ldots, y_j)[x_i/y_{\rho(i)} : i \in \mathrm{dom}(\rho)] \right)$, a contradiction of the fact that $\mathfrak{B}, \mathrm{elem}(v) \models \phi_{m,I}^{\leftarrow}$. Hence, $\mathcal{U}(\mathfrak{B}), v \models \phi$ as desired.

For (3), suppose that v is a bag node and $\mathcal{U}(\mathfrak{B}), v \models \phi$, with $\mathrm{elem}(v) = a_1 \ldots a_m$. By IH3, $\mathfrak{B}, \mathrm{elem}(v) \models (P)_m^{\leftarrow}$. Suppose for the sake of contradiction that $\mathfrak{B}, \mathrm{elem}(v) \models \chi_{n,I}^{\leftarrow}[x_i/y_{\rho(i)} : i \in \mathrm{dom}(\rho)]$. Let w be a ρ-child of v with domain of size $|\mathrm{rng}(\rho)| = n$, and satisfying $a_i = \rho(i)$ (this must exist by properties of unravelling). Hence, $\mathfrak{B}, \mathrm{elem}(w) \models \chi_{n,I}^{\leftarrow}(y_1, \ldots, y_n)$. By IH2, this means that $\mathcal{U}(\mathfrak{B}), w \models \chi$, and hence $\mathcal{U}(\mathfrak{B}), v \models \langle\rho\rangle\chi$, a contradiction. Therefore $\mathfrak{B}, \mathrm{elem}(v) \models \phi_m^{\leftarrow}$ as desired.

For (4), suppose v is a bag node representing $\mathrm{AT}_{\mathrm{elem}(v),\sigma'}(\mathfrak{B})$ and $\mathfrak{B}, \mathrm{elem}(v) \models \phi_m^{\leftarrow}$. Let $\mathrm{elem}(v) = a_1 \ldots a_m$. By IH4, $\mathcal{U}(\mathfrak{B}), v \models P$. Suppose for the sake of contradiction that $\mathcal{U}(\mathfrak{B}), v \models \langle\rho\rangle\chi$. Then there is some ρ-child w of v such that $\mathcal{U}(\mathfrak{B}), w \models \chi$. The node w must be an interface node and must have domain D_n (where $n = |\mathrm{rng}(\rho)|$). Let $b_1 \ldots b_n$ be the

elements from $\mathfrak{B}$ represented at w, with $a_i = b_{\rho(i)}$. By IH1, $\mathfrak{B}, \mathrm{elem}(w) \models \chi_{n,I}^{\leftarrow}(y_1, \ldots, y_n)$, so $\mathfrak{B}, \mathrm{elem}(v) \models \chi_{n,I}^{\leftarrow}(y_1, \ldots, y_n)[x_i/y_{\rho(i)} : i \in \mathrm{dom}(\rho)]$, a contradiction. Thus, $\mathcal{U}(\mathfrak{B}), v \models \phi$.

Fixpoint. Consider the case $\phi = P \wedge D_j \wedge \mu Y.\chi$, where

$$\phi_{m,I}^{\leftarrow} := (P)_{m,I}^{\leftarrow} \wedge (D_j)_{m,I}^{\leftarrow} \wedge [\mathbf{lfp}_{Y_j, P, y_1, \ldots, y_j} . S_{\mu Y.\chi}](x_1, \ldots, x_j)$$

and $S_{\mu Y.\chi}$ is a system consisting of equations

$$Y_{n,P'}, y_1 \ldots, y_n := (P')_{j,I}^{\leftarrow}(y_1, \ldots, y_n) \wedge \chi_{n,I}^{\leftarrow}(y_1, \ldots, y_n)$$

for each $Y_{n,P'}$ in $Y^{\leftarrow}$.

We must prove properties (1) and (2). For ordinals β, we write χ^{β} for the β-approximant of the fixpoint $\mu Y.\chi$ and $(\chi_{j,I}^{\leftarrow})^{\beta}$ for the β-approximant of $[\mathbf{lfp}_{Y_j, y_1, \ldots, y_j} . S_{\mu Y.\chi}](x_1, \ldots, x_j)$. We first show the result for the fixpoint approximants. That is, for all interface nodes v with $|\mathrm{elem}(v)| = j$, $\mathfrak{B}, \mathrm{elem}(v) \models (\chi_{j,I}^{\leftarrow})^{\beta}$ iff $\mathcal{U}(\mathfrak{B}), v \models \chi^{\beta}$. We proceed by induction on the fixpoint approximant β; we will refer to this as the inner induction to distinguish it from the outer induction on the structure of the formula.

For $\beta = 0$, the result follows by the outer inductive hypothesis IH1 and IH2 applied to the formulas that result from substituting $\perp$ for Y in χ, and $\perp$ for $Y_0, \ldots, Y_k$ in $\chi_j^{\leftarrow}$.

Now assume $\beta > 0$ is a successor ordinal $\beta = \delta + 1$.

For (1), assume $\mathcal{U}(\mathfrak{B}), v \models \chi^{\beta}$ for v an interface node with $|\mathrm{elem}(v)| = j$. Then $\mathcal{U}(\mathfrak{B}), v, V_{\delta} \models \chi$ where $V_{\delta} := \{w : w \text{ is an interface node and } \mathcal{U}(\mathfrak{B}), w \models \chi^{\delta}\}$ is the valuation for Y. Note that this is a safe valuation: it clearly contains only interface nodes, and if it contains interface node w, then it contains all w' such that the subtrees rooted at w and w' are bisimilar because χ is in the μ-calculus, and the μ-calculus is bisimulation invariant. By the outer inductive hypothesis, this implies that $\mathfrak{B}, \mathrm{elem}(v), V_{\delta}^{\leftarrow} \models \chi_j^{\leftarrow}$ for $V_{\delta}^{\leftarrow} = (V_0^{\delta}, \ldots, V_k^{\delta})$ and $V_i^{\delta} = \{\mathrm{elem}(w) : \mathrm{elem}(w) = i \text{ and } w \in V\}$. However, by the inner inductive hypothesis, $V_i^{\delta} = \{\mathrm{elem}(w) : \mathrm{elem}(w) = i \text{ and } \mathfrak{B}, \mathrm{elem}(w) \models (\chi_{i,I}^{\leftarrow})^{\delta}\}$. This means that $\mathfrak{B}, \mathrm{elem}(v) \models (\chi_{j,I}^{\leftarrow})^{\beta}$ as desired.

Next, assume $\mathfrak{B}, \mathrm{elem}(v) \models (\chi_{j,I}^{\leftarrow})^{\beta}$ for v an interface node with $|\mathrm{elem}(v)| = j$. Then $\mathfrak{B}, \mathrm{elem}(v), V_{\delta}^{\leftarrow} \models \chi_j^{\leftarrow}$ where $V_i^{\delta} = \{\mathrm{elem}(w) : \mathrm{elem}(w) = i \text{ and } \mathfrak{B}, \mathrm{elem}(w) \models (\chi_{i,I}^{\leftarrow})^{\delta}\}$. Define $V_{\delta} = \{w : w \in V_i^{\delta} \text{ for some } i\}$. By the inner inductive hypothesis, V_{δ} is equivalent to the valuation $\{w : \mathcal{U}(\mathfrak{B}), w \models \chi^{\delta}\}$. By the bisimulation-invariance of μ-calculus, this is a safe valuation for Y. Hence, the outer inductive hypothesis implies that $\mathcal{U}(\mathfrak{B}), v, V_{\delta} \models \chi$, and hence $\mathcal{U}(\mathfrak{B}), v \models \chi^{\beta}$ as desired.

The proof is similar when β is a limit ordinal.

The overall result for this case follows by appealing to the fact that the least fixpoint corresponds to some β-approximant, and noting that because the fixpoint references only interface nodes, it is correct to add a σ_g-guardedness requirement to each formula in the simultaneous fixpoint.

Note that the greatest fixpoint cases rely on the fact that a greatest fixpoint can be expressed using negation and least fixpoint, e.g. $\nu Y.\chi = \neg \mu Y.\neg \chi[\neg Y/Y]$. This is reflected in the translation. The only additional technicality in the translation is that we add an extra σ_g-guard at the beginning of each formula in the simultaneous fixpoint. $\square$

Lemma 5.4 follows by converting ϕ^μ to a $\mathrm{GNFP}^k[\sigma', \sigma_g]$-safe form ψ (see Lemma 5.6 below) that is equivalent over plump, σ_g-guarded-interface $\Sigma^{code}_{\sigma',k}$-trees, and then using Lemma 5.5, taking $\psi^\leftarrow := \phi^\leftarrow_{0,I}$ as the desired sentence in $\mathrm{GNFP}^k[\sigma', \sigma_g]$.

Conversion to $\mathrm{GNFP}^k[\sigma', \sigma_g]$*-safe formula.* It remains to show that we can actually convert a μ-calculus formula ϕ^μ into a GNFP^k-safe formula.

Lemma 5.6. *Given* $\phi^\mu \in \mathrm{L}_\mu[\Sigma^{code}_{\sigma,m}]$*, we can construct* $\phi \in \mathrm{L}_\mu[\Sigma^{code}_{\sigma',k}]$ *that is* $\mathrm{GNFP}^k[\sigma', \sigma_g]$*-safe for interface nodes and such that for all plump,* σ_g*-guarded-interface* $\Sigma^{code}_{\sigma',k}$*-trees* $\mathcal{T}$*, we have* $\mathcal{T} \models \phi$ *iff* $\mathcal{T} \models \psi$.

This requires a series of transformations, described below. At each stage, we ensure equivalence with ϕ^μ, at least over plump trees.

Before we give these transformations, we review and introduce some additional notation.

We utilize vectorial (simultaneous) fixpoints, using the standard notation. We also use the standard notation for the box modality and greatest fixpoint operator, writing $[\rho]\psi$ as an abbreviation for $\neg\langle\rho\rangle\neg\psi$, and $\nu Y.\psi$ for $\neg\mu Y.\neg\psi[\neg Y/Y]$.

Recall that each node label is a set of propositions τ, and each edge label is a mapping ρ describing the relationship between names in neighboring nodes. As before, we write EDGES to denote the set of functions ρ such that the binary predicate E_ρ is in $\Sigma^{code}_{\sigma',k}$, and we write NODELABELS for the set of internally consistent node labels. We write BAGLABELS and INTERFACELABELS for the subset of NODELABELS allowed in a bag node and interface node, respectively, in a σ_g-guarded-interface tree of width k. We write BAGLABELS(τ_0) for the subset of labels from BAGLABELS that contain the encoded atoms of τ_0, i.e. labels that extend τ_0 with additional encoded atoms (but may differ in domain size). Finally, for $S \subseteq \mathrm{EDGES} \times Q$, we write EDGES($S$) to denote the set of edge labels appearing in S.

Given $\tau \in \mathrm{NODELABELS}$ and a set of names I, we let $\mathrm{guard}^{\sigma_g}_I(\tau)$ denote some $P \in \tau$ that is in σ_g and satisfies $\mathrm{names}(P) = I$ ($\bot$ if no such P exists, and $\top$ if I is of size at most 1). We let $\mathrm{guard\text{-}dom}(\rho, \tau)$ denote $\mathrm{guard}^{\sigma_g}_{\mathrm{dom}(\rho)}(\tau)$, and let $\mathrm{guard\text{-}rng}(\rho, \tau)$ be the result of replacing the names appearing in $\mathrm{guard}^{\sigma_g}_{\mathrm{dom}(\rho)}(\tau)$ according to ρ. The idea is that these are macros that give a σ_g-guard related to τ.

We also define some auxiliary formulas to improve readability in the formulas in this section. For $\tau \in \mathrm{NODELABELS}$, we define

$$\mathrm{EXACTLABEL}(\tau) := \bigwedge_{P \in \tau} P \wedge \bigwedge_{P \in \mathrm{NPROPS}(\tau)} \neg P$$

$$\mathrm{GNLABEL}(\tau) := \bigwedge_{P \in \tau} P \wedge \bigwedge_{P \in \mathrm{GNPROPS}(\tau)} \bigwedge_{\substack{\mathrm{names}(P) \subseteq I \subseteq \{1,\ldots,k\} \\ \text{s.t. } \mathrm{guard}^{\sigma_g}_I(\tau) \neq \bot}} \left(\mathrm{guard}^{\sigma_g}_I(\tau) \wedge \neg P\right).$$

where $\mathrm{NPROPS}(\tau)$ consists of unary propositions from $\Sigma^{code}_{\sigma',k}$ that do not appear in τ, and $\mathrm{GNPROPS}(\tau)$ consists of unary propositions P from $\Sigma^{code}_{\sigma',k}$ that do not appear in τ but use names that are σ_g-guarded by some $P' \in \tau$. Both $\mathrm{EXACTLABEL}(\tau)$ and $\mathrm{GNLABEL}(\tau)$ assert all of the positive information about the propositions in τ. However, $\mathrm{GNLABEL}(\tau)$ only asserts some of the negative information, namely it only asserts that some proposition is missing from τ if the indices used by that proposition are σ_g-guarded in τ. Hence, $\mathrm{GNLABEL}(\tau)$ can be seen as an approximation of $\mathrm{EXACTLABEL}(\tau)$: if $\mathcal{T}, v \models \mathrm{EXACTLABEL}(\tau)$ then

$\mathcal{T}, v \models \text{GNLABEL}(\tau)$, but the converse does not always hold. Note that $\text{GNLABEL}(\tau)$ is $\text{GNFP}^k[\sigma', \sigma_g]$-safe but $\text{EXACTLABEL}(\tau)$ is not.

We now proceed with the series of transformations taking ψ to a $\text{GNFP}^k[\sigma', \sigma_g]$-safe version that is equivalent over plump trees.

Step 1: Refinement to σ_g-guarded-interface $\Sigma^{\text{code}}_{\sigma', k}$-trees. By Theorem 2.8, there is some μ-automaton $\mathcal{A}'$ that is equivalent to ϕ^μ. This automaton runs on $\Sigma^{\text{code}}_{\sigma, m}$-trees. However, for the purposes of the backward mapping, we are only interested in it running on $\Sigma^{\text{code}}_{\sigma', k}$-trees—and more specifically on plump σ_g-guarded-interface trees that encode plump $\text{BGN}^k[\sigma', \sigma_g]$-unravellings of σ-structures.

Therefore, in this first step, we make some straightforward modifications to this automaton that are correct on σ_g-guarded-interface $\Sigma^{\text{code}}_{\sigma', k}$ trees, and then convert it back into an $\text{L}_\mu[\Sigma^{\text{code}}_{\sigma', k}]$-formula. The shape of the resulting formula is described in Claim 5.7, but we defer the formal statement until after we have described these modifications. Along the way, we also introduce some notation and terminology (e.g. bag states, interface states, etc.) that we will use in the later steps.

Let $\mathcal{A}'$ be the μ-automaton with state set Q', transition function δ', and priority function Ω' that runs on $\Sigma^{\text{code}}_{\sigma, m}$-trees and is equivalent to ϕ^μ (see Theorem 2.8).

We start by refining this automaton to run on $\Sigma^{\text{code}}_{\sigma', k}$-trees. This means that we can limit the alphabet for the automaton to just node labels in NODELABELS and edges in EDGES, and eliminate all of the transition function information related to labels outside of this.

We then refine it to run on σ_g-guarded-interface trees. In particular, we can modify the automaton so that states in interface nodes are disjoint from the states of the automaton in bag nodes. We will refer to *interface states* and *bag states* as appropriate. Because a guarded-interface tree alternates between interface nodes and bag nodes, it is possible to enforce that

(1) bag states are assigned priority 0,
(2) the initial state is assigned the maximum priority,
(3) for all $S \in \delta(s, \tau)$ and $(\rho, s') \in S$, $\text{dom}(\rho)$ is strictly σ_g-guarded in τ, and
(4) for all bag states r, $S \in \delta(r, \tau)$, and $\rho \in$ EDGES such that $\text{dom}(\rho)$ is strictly σ_g-guarded in τ, there is some $(\rho, q) \in S$.

Making these changes to the automaton results in only a polynomial blow-up in the size of the automaton. Let $\mathcal{A}$ be the resulting automaton with state set Q, transition function δ, and priority function Ω.

We can then write in the usual way (see, e.g., [41]) a vectorial $\text{L}_\mu[\Sigma^{\text{code}}_{\sigma', k}]$-formula ψ describing the operation of $\mathcal{A}$:

$$\lambda_{n'} X_{s_{n'}} \ldots \lambda_1 X_{s_1} . \begin{pmatrix} \delta_{s_1} \\ \vdots \\ \delta_{s_{n'}} \end{pmatrix}$$

where $s_1, \ldots, s_{n'}$ is an ordering of the states based on the priority (from least to greatest priority), λ_i is μ (respectively, ν) if s_i has an odd (respectively, even) priority, and δ_s are transition formulas defined below. We can assume that the ordering is chosen so that (for some i) $s_1, \ldots, s_i$ consists of bag states, no bag states are present in $s_{i+1}, \ldots, s_{n'}$, and $s_{n'}$ is the initial state. We will refer to X_{s_i} as an *interface predicate* (respectively, *bag predicate*) if s_i is an interface state (respectively, bag state).

The formulas δ_s describing the transitions from state s are defined as follows:

$$\delta_s := \bigvee_{\tau \in \text{NodeLabels}} (\text{ExactLabel}(\tau) \wedge \delta_{s,\tau})$$

$$\delta_{s,\tau} := \bigvee_{S \in \delta(s,\tau)} \left(\bigwedge_{(\rho,s') \in S} \langle \rho \rangle X_{s'} \wedge \bigwedge_{\rho \in \text{Edges}} [\rho] \bigvee_{(\rho,s') \in S} X_{s'} \right)$$

This captures precisely the meaning of the transition function in a μ-automaton. The idea is that it picks out exactly the label τ at the current node, and then ensures that the successors of this node satisfy the requirements specified by the transition function when in state s and at a position with label τ.

In order to improve readability, from now on we will use q to range over the interface states, r to range over the bag states, and s to range over both of these. Because the priority of bag states is 0 and the automaton alternates between interface and bag states, we can eliminate all bag predicates by inlining the formulas δ_r any time a bag predicate X_r appears. While we are doing this, we can further refine the transition formulas based on whether it is a bag state or an interface state.

For all bag states r, we construct $\delta_{r,\tau}^2$ from $\delta_{r,\tau}$ by performing the following operations:

- substitute $\bigwedge_{(\rho,q) \in S} \langle \rho \rangle (\text{guard-rng}(\rho, \tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q)$ for $\bigwedge_{(\rho,q) \in S} \langle \rho \rangle X_q$;
- substitute $\bigwedge_{\rho \in \text{Edges}(S)} [\rho] (\text{guard-rng}(\rho, \tau) \wedge D_{|\text{rng}(\rho)|} \rightarrow \bigvee_{(\rho,q) \in S} X_q)$ for
  $\bigwedge_{\rho \in \text{Edges}} [\rho] \bigvee_{(\rho,q) \in S} X_q$.

Guarding the range of ρ-successors is correct, since in a σ_g-guarded-interface tree, every successor of a bag node is an interface node with a strictly σ_g-guarded domain that is a subset of the bag domain. It is correct to restrict the conjunction over $\rho \in \text{Edges}$ to just $\rho \in \text{Edges}(S)$ since we have enforced that $S \in \delta(r, \tau)$ satisfies the property that $\text{Edges}(S)$ includes every possible outgoing edge label when the node label is τ.

Likewise, for all interface states q, we construct formulas δ_{q,τ_0}^2 from δ_{q,τ_0} using the following operations:

- substitute $\bigwedge_{(\rho_0,r) \in S} \text{guard-dom}(\rho_0, \tau_0) \wedge \langle \rho_0 \rangle \delta_r^{2,\tau_0}$ for $\bigwedge_{(\rho_0,r) \in S} \langle \rho_0 \rangle X_r$;
- substitute $\bigwedge_{\rho_0 \in \text{Edges}} \text{guard-dom}(\rho_0, \tau_0) \wedge [\rho_0] \bigvee_{(\rho_0,r) \in S} \delta_r^{2,\tau_0}$ for
  $\bigwedge_{\rho_0 \in \text{Edges}} [\rho] \bigvee_{(\rho_0,r) \in S} X_r$;

where δ_r^{2,τ_0} is obtained from δ_r by

- substituting $\bigvee_{\tau \in \text{BagLabels}(\tau_0)} \left(\text{ExactLabel}(\tau) \wedge \delta_{r,\tau}^2 \right)$ for
  $\bigvee_{\tau \in \text{NodeLabels}} \left(\text{ExactLabel}(\tau) \wedge \delta_{r,\tau} \right)$.

Guarding the domain of ρ-successors is correct, since in a σ_g-guarded-interface tree, every interface node has a strictly σ_g-guarded domain, even though the domain of the successor need not be guarded. It is correct to replace the disjunction over $\tau \in \text{NodeLabels}$ with $\tau \in \text{BagLabels}(\tau_0)$, since in a σ_g-guarded-interface tree, any bag node must extend the label τ_0 of its parent.

Finally, for all interface states q, we construct δ_q^2 from δ_q by

- substituting $\bigvee_{\tau_0 \in \text{InterfaceLabels}} \left(\text{GNLabel}(\tau_0) \wedge \delta_{q,\tau_0}^2 \right)$ for
  $\bigvee_{\tau_0 \in \text{NodeLabels}} \left(\text{ExactLabel}(\tau_0) \wedge \delta_{q,\tau_0} \right)$.

It is correct to replace the disjunction over NodeLabels with InterfaceLabels since this is an interface state formula. Replacing $\text{ExactLabel}(\tau_0)$ with $\text{GNLabel}(\tau_0)$ is correct,

since EXACTLABEL(τ_0) is equivalent to GNLABEL(τ_0) for all $\tau_0 \in$ INTERFACELABELS (since the domain of τ_0 must be strictly σ_g-guarded).

Note that after these substitutions, there are no occurrences of bag predicates X_r in the vectorial components, so these fixpoint variables can be eliminated. This leaves the interface predicates X_q and vectorial components δ_q^2.

The resulting formula now satisfies the conditions described in the following claim.

Claim 5.7. There is a formula $\psi^2 \in \mathrm{L}_\mu[\Sigma_{\sigma',k}^{\mathrm{code}}]$ obtained effectively from $\phi^\mu \in \mathrm{L}_\mu[\Sigma_{\sigma,m}^{\mathrm{code}}]$ of the form

$$\lambda_n X_{q_n} \ldots \lambda_1 X_{q_1} . \begin{pmatrix} \delta_{q_1}^2 \\ \vdots \\ \delta_{q_n}^2 \end{pmatrix}$$

such that for all i, $\lambda_i \in \{\mu, \nu\}$ and for all $q \in \{q_1, \ldots, q_n\}$, δ_q^2 is of the form

$$\delta_q^2 := \bigvee_{\tau_0 \in \text{INTERFACELABELS}} \left(\text{GNLABEL}(\tau_0) \wedge \delta_{q,\tau_0}^2 \right)$$

$$\delta_{q,\tau_0}^2 := \bigvee_{S \in \delta(q,\tau_0)} \left(\bigwedge_{(\rho_0,r) \in S} \text{guard-dom}(\rho_0, \tau_0) \wedge \langle \rho_0 \rangle \delta_r^{2,\tau_0} \wedge \right.$$

$$\left. \bigwedge_{\rho_0 \in \text{EDGES}} \text{guard-dom}(\rho_0, \tau_0) \wedge [\rho_0] \bigvee_{(\rho_0,r) \in S} \delta_r^{2,\tau_0} \right)$$

$$\delta_r^{2,\tau_0} := \bigvee_{\tau \in \text{BAGLABELS}(\tau_0)} \left(\text{EXACTLABEL}(\tau) \wedge \delta_{r,\tau}^2 \right)$$

$$\delta_{r,\tau}^2 := \bigvee_{S \in \delta(r,\tau)} \left(\bigwedge_{(\rho,q) \in S} \langle \rho \rangle (\text{guard-rng}(\rho, \tau) \wedge D_{|\mathrm{rng}(\rho)|} \wedge X_q) \wedge \right.$$

$$\left. \bigwedge_{\rho \in \text{EDGES}(S)} [\rho] (\text{guard-rng}(\rho, \tau) \wedge D_{|\mathrm{rng}(\rho)|} \rightarrow \bigvee_{(\rho,q) \in S} X_q) \right).$$

Moreover, for all σ_g-guarded-interface $\Sigma_{\sigma',k}^{\mathrm{code}}$-trees $\mathcal{T}$, and for all interface nodes v, we have $\mathcal{T}, v \models \psi^2$ iff $\mathcal{T}, v \models \phi^\mu$.

Step 2: Refinement to plump trees. The subformulas $\langle \rho \rangle \delta_r^2$ and $[\rho] \bigvee_{(\rho,r) \in S} \delta_r^2$ both may have negations that are not allowed in GNFP$^k[\sigma', \sigma_g]$-safe formulas since the transition function formula δ_r^2 depends on knowing the exact label τ at the current node: all of the positive information about which propositions hold and all negative information about which propositions do not. In a plump tree, however, we will see that it is not necessary to know the exact label; instead, we will make a number of modifications to the formulas, which will allow us to replace EXACTLABEL(τ) with GNLABEL(τ).

We first prove some auxiliary claims that will help with this. Recall that we say $\boldsymbol{V}$ is a *safe valuation* for predicates $\boldsymbol{X}$ if it satisfies the following property: if w and w' are interface nodes that are roots of bisimilar subtrees, then w is in the valuation for $X \in \boldsymbol{X}$ iff w' is in the valuation for $X \in \boldsymbol{X}$.

Claim 5.8. Let v be an interface node with label τ_0 in a plump σ_g-guarded-interface $\Sigma_{\sigma',k}^{\mathrm{code}}$-tree $\mathcal{T}$.

For each ρ_0-child w of v and each $\tau \in \text{BagLabels}(\tau_0)$ such that $\mathcal{T}, w \models \text{GNLabel}(\tau)$, there is some ρ_0-child w' of v with label τ such that $\mathcal{T}, w' \models \text{ExactLabel}(\tau)$.

Moreover, if $\boldsymbol{V}$ is a safe valuation for the interface predicates $\boldsymbol{X}$, then $\mathcal{T}, w, \boldsymbol{V} \models \delta^2_{r,\tau}$ iff $\mathcal{T}, w', \boldsymbol{V} \models \delta^2_{r,\tau}$.

Proof. Let w be a ρ_0-child of v with label τ_1. It must be the case that $\tau_1 \in \text{BagLabels}(\tau_0)$ by the properties of σ_g-guarded-interface trees. Let $\tau \in \text{BagLabels}(\tau_0)$ such that $w \models \text{GNLabel}(\tau)$ (for notational simplicity, we write $w \models \dots$ rather than $\mathcal{T}, w, \boldsymbol{V} \models \dots$).

We first show that τ is a $(\sigma_g, \text{rng}(\rho_0))$-safe restriction of τ_1.

It is clear that $\tau_0 \subseteq \tau \subseteq \tau_1$ since both τ and τ_1 are in $\text{BagLabels}(\tau_0)$, and if $w \models \text{GNLabel}(\tau)$ then all $P \in \tau$ must appear in the label τ_1 of w.

Consider some proposition P such that $\text{names}(P)$ is σ_g-guarded in τ. We must show that $P \in \tau$ iff $P \in \tau_1$. If $P \in \tau$, then $P \in \tau_1$ since $\tau \subseteq \tau_1$. If $P \notin \tau$, then $\text{GNLabel}(\tau)$ asserts that $\neg P$ holds (since $\text{names}(P)$ is σ_g-guarded), so it must be the case that $P \notin \tau_1$.

Now consider some proposition P using only names in $\text{rng}(\rho_0)$. Note that $\text{dom}(\rho_0)$ is strictly σ_g-guarded in τ_0, and since $\tau \supseteq \tau_0$, $\text{rng}(\tau_0)$ is strictly σ_g-guarded in τ. Hence, $\text{names}(P)$ is σ_g-guarded in τ, and similar reasoning as above implies that $P \in \tau$ iff $P \in \tau_1$.

This is enough to conclude that τ is a $(\sigma_g, \text{rng}(\rho_0))$-safe restriction of τ_1. Therefore, by plumpness, there is some w' such that $w' \models \text{ExactLabel}(\tau)$ as desired.

Now make the further assumption that $w \models \delta^2_{r,\tau}$; we must show that $w' \models \delta^2_{r,\tau}$. If $w \models \delta^2_{r,\tau}$ then there is some $S \in \delta(r, \tau)$ such that

- (existential requirement) for every $(\rho, q) \in S$, there is some child u of w where X_q holds, and
- (universal requirement) for every ρ-child u of w such that $\rho \in \text{Edges}(S)$, there is some q such that $(\rho, q) \in S$ and X_q holds at u.

We claim that the same property holds at w', using the same $S \in \delta(r, \tau)$. For each $\rho \in \text{Edges}(S)$, $\text{dom}(\rho)$ must be strictly σ_g-guarded in τ (otherwise, the existential requirement would not be fulfilled). Hence, by the definition of a plump tree, for each $\rho \in \text{Edges}(S)$ and each ρ-child u of w, there is a corresponding ρ-child u' in w' such that the subtrees rooted at u and u' are bisimilar. If X_q holds at u, and this is used to satisfy some existential requirement with $(\rho, q) \in S$, then X_q also holds at u' so this existential requirement is also satisfied for w' (this relies on the fact that the valuations for $\boldsymbol{X}$ are safe). For the universal requirement at some ρ-child u' of w' for $\rho \in \text{Edges}(S)$, consider the corresponding child u of w such that the subtrees rooted at u and u' are bisimilar, which is guaranteed by plumpness. Since the universal requirement is satisfied at u, there is some X_q holding at u with $(\rho, q) \in S$. Since the valuations for $\boldsymbol{X}$ are safe, this means that X_q also holds at u', so the universal requirement at u' holds. Using this reasoning, we can conclude that $w' \models \delta^2_{r,\tau}$.

The reasoning is similar in the other direction, assuming $w' \models \delta^2_{r,\tau}$.

Thus, we can conclude that $w' \models \text{ExactLabel}(\tau)$, and $w' \models \delta^2_{r,\tau}$ iff $w \models \delta^2_{r,\tau}$. $\qquad\square$

Using the previous claim, we will prove that we can replace $\text{ExactLabel}(\tau)$ with $\text{GNLabel}(\tau)$ when considering equivalence only over plump trees. The exact way we do this replacement, however, will depend on whether $\text{ExactLabel}(\tau)$ is under a diamond modality or a box modality. We define the following auxiliary formulas to handle these

cases

$$\delta_r^{\Diamond,\tau_0} := \bigvee_{\tau \in \text{BagLabels}(\tau_0)} \left(\text{GNLabel}(\tau) \wedge \delta_{r,\tau}^2 \right)$$

$$\delta_S^{\Box,\tau_0} := \bigwedge_{\tau \in \text{BagLabels}(\tau_0)} \left(\text{GNLabel}(\tau) \rightarrow \bigvee_{(\rho,r) \in S} \delta_{r,\tau}^2 \right)$$

and prove the correctness of the following transformation:

Claim 5.9. Let v be an interface node with label τ_0 in a plump σ_g-guarded-interface $\Sigma_{\sigma',k}^{\text{code}}$-tree $\mathcal{T}$, and let $\boldsymbol{V}$ be a safe valuation for the interface predicates $\boldsymbol{X}$. Then

$$\mathcal{T}, v, \boldsymbol{V} \models \langle \rho \rangle \delta_r^2 \quad \text{iff} \quad \mathcal{T}, v, \boldsymbol{V} \models \langle \rho \rangle \delta_r^{\Diamond,\tau_0}, \tag{1}$$

$$\mathcal{T}, v, \boldsymbol{V} \models [\rho] \bigvee_{(\rho,r) \in S} \delta_r^2 \quad \text{iff} \quad \mathcal{T}, v, \boldsymbol{V} \models [\rho] \delta_S^{\Box,\tau_0}. \tag{2}$$

Proof of claim. Fix a plump σ_g-guarded-interface $\Sigma_{\sigma',k}^{\text{code}}$-tree $\mathcal{T}$, an interface node v with label τ_0, and a safe valuation $\boldsymbol{V}$ for the interface predicates $\boldsymbol{X}$. We write $v \models \psi$ for $\mathcal{T}, v, \boldsymbol{V} \models \psi$.

(1) We start with the easier left-to-right direction. Assume $v \models \langle \rho \rangle \delta_r^2$. Then there is some ρ-child w of v such that $w \models \delta_r^2$. Let τ be the exact label at w; note that $\tau \in \text{BagLabels}(\tau_0)$ by properties of σ_g-guarded-interface trees. Then $w \models \text{ExactLabel}(\tau) \wedge \delta_{r,\tau}^2$. This implies that $w \models \text{GNLabel}(\tau) \wedge \delta_{r,\tau}^2$. This is enough to conclude that $w \models \delta_r^{\Diamond,\tau_0}$ and $v \models \langle \rho \rangle \delta_r^{\Diamond,\tau_0}$.

Next, we prove the right-to-left direction, which makes use of Claim 5.8 (and hence makes use of plumpness). Assume that $v \models \langle \rho \rangle \delta_r^{\Diamond,\tau_0}$. Then there is some ρ-child w and some $\tau \in \text{BagLabels}(\tau_0)$ such that $w \models \text{GNLabel}(\tau) \wedge \delta_{r,\tau}^2$. By Claim 5.8, there is some ρ-child w' such that $w' \models \text{ExactLabel}(\tau) \wedge \delta_{r,\tau}^2$, so $v \models \langle \rho \rangle \delta_r^2$ as desired.

(2) The easier direction is the right-to-left direction: assume $v \models [\rho] \delta_S^{\Box,\tau_0}$. Let w be a ρ-child of v, and let τ be the label at w. It must be the case that $\tau \in \text{BagLabels}(\tau_0)$ by the properties of a σ_g-guarded-interface tree. Hence, by the definition of $\delta_S^{\Box,\tau_0}$, there is some $(\rho, r) \in S$, such that $w \models \delta_{r,\tau}^2$. This means $w \models \text{GNLabel}(\tau) \wedge \delta_{r,\tau}^2$, so $w \models \bigvee_{(\rho,r) \in S} \delta_r^2$. Overall, this means $v \models [\rho] \bigvee_{(\rho,r) \in S} \delta_r^2$ as desired.

Now assume $v \models [\rho] \bigvee_{(\rho,r) \in S} \delta_r^2$ for the left-to-right direction. Let w be a ρ-child of v. Consider $\tau \in \text{BagLabels}(\tau_0)$ such that $w \models \text{GNLabel}(\tau)$. It suffices to show that $w \models \bigvee_{(\rho,r) \in S} \delta_{r,\tau}^2$. By Claim 5.8, there is a ρ-child w' of v such that $w' \models \text{ExactLabel}(\tau)$. Since $v \models [\rho] \bigvee_{(\rho,r) \in S} \delta_r^2$ and the label at w' is τ, this means that $w' \models \delta_{r,\tau}^2$ for some $(\rho, r) \in S$. By Claim 5.8, $w \models \delta_{r,\tau}^2$ as well, so $w \models \bigvee_{(\rho,r) \in S} \delta_{r,\tau}^2$ as required. $\square$

This allows us to take ψ^2 from the previous step and refine it further based on the assumption that we are only interested in plump σ_g-guarded-interface $\Sigma_{\sigma',k}^{\text{code}}$-trees. The shape of the resulting formula is stated in the following claim:

Claim 5.10. There is a formula ψ^3 obtained effectively from ψ^2 such that the vectorial component δ_q^3 for each q is of the form

$$\delta_q^3 := \bigvee_{\tau_0 \in \text{INTERFACELABELS}} \left(\text{GNLABEL}(\tau_0) \wedge \delta_{q,\tau_0}^3 \right)$$

$$\delta_{q,\tau_0}^3 := \bigvee_{S \in \delta(q,\tau_0)} \left(\bigwedge_{(\rho_0,r) \in S} \text{guard-dom}(\rho_0, \tau_0) \wedge \langle \rho_0 \rangle \delta_r^{\Diamond,\tau_0} \wedge \right.$$

$$\left. \bigwedge_{\rho_0 \in \text{EDGES}} \text{guard-dom}(\rho_0, \tau_0) \wedge [\rho_0] \delta_S^{\Box,\tau_0} \right)$$

$$\delta_r^{\Diamond,\tau_0} := \bigvee_{\tau \in \text{BAGLABELS}(\tau_0)} \left(\text{GNLABEL}(\tau) \wedge \delta_{r,\tau}^3 \right)$$

$$\delta_S^{\Box,\tau_0} := \bigwedge_{\tau \in \text{BAGLABELS}(\tau_0)} \left(\text{GNLABEL}(\tau) \rightarrow \bigvee_{(\rho,r) \in S} \delta_{r,\tau}^3 \right)$$

$$\delta_{r,\tau}^3 := \bigvee_{S \in \delta(r,\tau)} \left(\bigwedge_{(\rho,q) \in S} \langle \rho \rangle (\text{guard-rng}(\rho, \tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q) \wedge \right.$$

$$\left. \bigwedge_{\rho \in \text{EDGES}(S)} [\rho](\text{guard-rng}(\rho, \tau) \wedge D_{|\text{rng}(\rho)|} \rightarrow \bigvee_{(\rho,q) \in S} X_q) \right).$$

Moreover, for all plump $\Sigma_{\sigma',k}^{\text{code}}$-trees $\mathcal{T}$ and for all interface nodes v, we have $\mathcal{T}, v \models \psi^3$ iff $\mathcal{T}, v \models \psi^2$.

Proof. For all interface states q and bag states r:

- substitute $\langle \rho \rangle \delta_r^{\Diamond,\tau_0}$ for $\langle \rho \rangle \delta_r^2$ in δ_{q,τ_0}^2;
- substitute $[\rho] \delta_S^{\Box,\tau_0}$ for $[\rho] \bigvee_{(\rho,r) \in S} \delta_r^2$ in δ_{q,τ_0}^2.

Let ψ^3 be the resulting formula, with vectorial components δ_q^3.

Equivalence over plump σ_g-guarded-interface trees essentially follows from Claim 5.9. Technically, one would show that the result is correct by induction on the number of fixpoint operators, and a transfinite induction on the fixpoint approximant required for each fixpoint. Since all of the fixpoint approximations give safe valuations for the fixpoint predicates, Claim 5.9 can be applied at each step in the induction. $\square$

Step 3: Clean up to obtain $\text{GNFP}^k[\sigma', \sigma_g]$-safe formula. The formula ψ^3 obtained in the previous step is almost $\text{GNFP}^k[\sigma', \sigma_g]$-safe. We now perform some clean-up operations in order to get the required $\text{GNFP}^k[\sigma', \sigma_g]$-safe formula.

The first clean-up operation deals with the negations that are implicit in the box modalities. The following claim shows the shape of the formula after we have eliminated box modalities, and pushed some of these negations inside.

Claim 5.11. There is a formula ψ^4 obtained effectively from ψ^3 such that the vectorial component δ_q^4 for each q is of the form:

$$\delta_q^4 := \bigvee_{\tau_0 \in \text{INTERFACELABELS}} \left(\text{GNLABEL}(\tau_0) \wedge \delta_{q,\tau_0}^4 \right)$$

$$\delta_{q,\tau_0}^4 := \bigvee_{S \in \delta(q,\tau_0)} \Big(\bigwedge_{(\rho_0,r) \in S} \text{guard-dom}(\rho_0,\tau_0) \wedge \langle \rho_0 \rangle \delta_r^{\Diamond,\tau_0} \wedge$$
$$\bigwedge_{\rho_0 \in \text{EDGES}} \text{guard-dom}(\rho_0,\tau_0) \wedge \neg \langle \rho_0 \rangle \delta_S^{\neg \Diamond,\tau_0} \Big)$$

$$\delta_r^{\Diamond,\tau_0} := \bigvee_{\tau \in \text{BAGLABELS}(\tau_0)} \left(\text{GNLABEL}(\tau) \wedge \delta_{r,\tau}^4 \right)$$

$$\delta_{r,\tau}^4 := \bigvee_{S \in \delta(r,\tau)} \Big(\bigwedge_{(\rho,q) \in S} \text{guard-dom}(\rho,\tau) \wedge \langle \rho \rangle (\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q) \wedge$$
$$\bigwedge_{\rho \in \text{EDGES}(S)} \text{guard-dom}(\rho,\tau) \wedge \neg \langle \rho \rangle (\bigwedge_{(\rho,q) \in S} \text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge \neg X_q) \Big)$$

$$\delta_S^{\neg \Diamond,\tau_0} := \bigvee_{\tau \in \text{BAGLABELS}(\tau_0)} \Big(\text{GNLABEL}(\tau) \wedge \bigwedge_{(\rho,r) \in S} \overline{\delta}_{r,\tau}^4 \Big)$$

$$\overline{\delta}_{r,\tau}^4 := \bigwedge_{S \in \delta(r,\tau)} \Big(\bigvee_{(\rho,q) \in S} \text{guard-dom}(\rho,\tau) \wedge \neg \langle \rho \rangle (\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q) \vee$$
$$\bigvee_{\rho \in \text{EDGES}(S)} \text{guard-dom}(\rho,\tau) \wedge \langle \rho \rangle (\bigwedge_{(\rho,q) \in S} \text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge \neg X_q) \Big).$$

Moreover, for all σ_g-guarded-interface $\Sigma_{\sigma',k}^{\text{code}}$-trees $\mathcal{T}$ and for all interface nodes v, we have $\mathcal{T}, v \models \psi^4$ iff $\mathcal{T}, v \models \psi^3$.

Proof. Recall that $[\rho]\chi$ is equivalent to $\neg\langle\rho\rangle\neg\chi$. Simply by using this equivalence, and pushing negations inside, we can rewrite the transition formulas to:

$$\delta_q^4 := \bigvee_{\tau_0 \in \textsc{InterfaceLabels}} \left(\textsc{GNLabel}(\tau_0) \wedge \delta_{q,\tau_0}^4\right)$$

$$\delta_{q,\tau_0}^4 := \bigvee_{S \in \delta(q,\tau_0)} \left(\bigwedge_{(\rho_0,r) \in S} \text{guard-dom}(\rho_0,\tau_0) \wedge \langle\rho_0\rangle\delta_r^{\Diamond,\tau_0} \wedge \right.$$
$$\left. \bigwedge_{\rho_0 \in \textsc{Edges}} \text{guard-dom}(\rho_0,\tau_0) \wedge \neg\langle\rho_0\rangle\delta_S^{\neg\Diamond,\tau_0} \right)$$

$$\delta_r^{\Diamond,\tau_0} := \bigvee_{\tau \in \textsc{BagLabels}(\tau_0)} \left(\textsc{GNLabel}(\tau) \wedge \delta_{r,\tau}^4\right)$$

$$\delta_{r,\tau}^4 := \bigvee_{S \in \delta(r,\tau)} \left(\bigwedge_{(\rho,q) \in S} \langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q) \wedge \right.$$
$$\left. \bigwedge_{\rho \in \textsc{Edges}(S)} \neg\langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge \bigwedge_{(\rho,q) \in S} \neg X_q) \right)$$

$$\delta_S^{\neg\Diamond,\tau_0} := \bigvee_{\tau \in \textsc{BagLabels}(\tau_0)} \left(\textsc{GNLabel}(\tau) \wedge \bigwedge_{(\rho,r) \in S} \overline{\delta}_{r,\tau}^4\right)$$

$$\overline{\delta}_{r,\tau}^4 := \bigwedge_{S \in \delta(r,\tau)} \left(\bigvee_{(\rho,q) \in S} \neg\langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q) \vee \right.$$
$$\left. \bigvee_{\rho \in \textsc{Edges}(S)} \langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge \bigwedge_{(\rho,q) \in S} \neg X_q) \right).$$

Finally, we perform the following substitutions:

- substitute $\text{guard-dom}(\rho,\tau) \wedge \langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q)$ for $\langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q)$ in $\delta_{r,\tau}^4$;
- substitute $\text{guard-dom}(\rho,\tau) \wedge \neg\langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q)$ for $\neg\langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge X_q)$ in $\overline{\delta}_{r,\tau}^4$;
- substitute $\text{guard-dom}(\rho,\tau) \wedge \neg\langle\rho\rangle(\bigwedge_{(\rho,q) \in S} \text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge \neg X_q)$ for $\neg\langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge \bigwedge_{(\rho,q) \in S} \neg X_q)$ in $\delta_{r,\tau}^4$;
- substitute $\text{guard-dom}(\rho,\tau) \wedge \langle\rho\rangle(\bigwedge_{(\rho,q) \in S} \text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge \neg X_q)$ for $\langle\rho\rangle(\text{guard-rng}(\rho,\tau) \wedge D_{|\text{rng}(\rho)|} \wedge \bigwedge_{(\rho,q) \in S} \neg X_q)$ in $\overline{\delta}_{r,\tau}^4$;

This is correct since the domain of any edge label exiting a bag node must be strictly σ_g-guarded in a σ_g-guarded-interface tree. This results in a formula of the desired shape. $\square$

The formula resulting from Claim 5.11 is $\text{GNFP}^k[\sigma',\sigma_g]$-safe except for the fact that it uses a simultaneous fixpoint. As a final clean-up step, we convert the vectorial fixpoint formula ψ^4 to a standard L_μ-formula using the Bekič principle, with the outermost fixpoint testing membership in the interface state q_n component (recall that the outermost fixpoint operator based on X_{q_n} corresponded to the initial state q_n of the automaton that was equivalent to the original L_μ-formula). This yields an equivalent formula $\lambda_n X_{q_n}.\chi$, where χ uses no vectorial fixpoints. The $\text{GNFP}^k[\sigma',\sigma_g]$-safe formula required for Lemma 5.6 is just

$\top \wedge D_0 \wedge \lambda_n X_{q_n}.\chi$. This is correct since we are interested in evaluating this starting at the root of $\mathcal{U}^{\text{plump}}_{\text{BGN}^k[\sigma',\sigma_g]}(\mathfrak{B})$, which has an empty set of names.

This concludes the proof of Lemma 5.6 and Lemma 5.4.

5.3. **Decidability of definability.** Using the above lemma and Proposition 3.4, we obtain the following analog of Theorem 4.3.

Theorem 5.12. *The* $\text{GNFP}^k[\sigma',\sigma_g]$ *definability problem is decidable for* $\text{GN}^l[\sigma]$-*invariant* $\text{GSO}[\sigma]$ *and* $k,l \geq \text{width}(\sigma)$.

Since $\text{UNFP}^k[\sigma']$ is just $\text{GNFP}^k[\sigma',\emptyset]$, we obtain the following corollary:

Corollary 5.13. *The* $\text{UNFP}^k[\sigma']$ *definability problem is decidable for* $\text{GN}^l[\sigma]$-*invariant* $\text{GSO}[\sigma]$ *and* $k,l \geq \text{width}(\sigma)$.

We get corollaries for fragments of FO, analogous to Corollary 4.8:

Corollary 5.14. *The* $\text{GNF}^k[\sigma',\sigma_g]$ *and* $\text{UNF}^k[\sigma']$ *definability problems are decidable for* $\text{GN}^l[\sigma]$-*invariant* $\text{FO}[\sigma]$ *and* $k,l \geq \text{width}(\sigma)$.

We can also apply the backward and forward mappings to get a semantic characterization for GNFP^k, analogous to the Janin-Walukiewicz theorem. The following extends a result of [8] characterizing GNF^k formulas as the BGN^k-invariant fragment of FO.

Theorem 5.15. $\text{GNFP}^k[\sigma',\sigma_g]$ *is the* $\text{BGN}^k[\sigma',\sigma_g]$-*invariant fragment of* $\text{GSO}[\sigma']$.

The proof is similar to the characterizations of Janin-Walukiewicz and [25], and can also be seen as a variant of Proposition 3.4, where we use $\text{BGN}^k[\sigma',\sigma_g]$-invariance rather than equivalence to a $\text{GNFP}^k[\sigma',\sigma_g]$ sentence in justifying that the input formula is equivalent to the result of the composition of backward and forward mappings.

6. INTERPOLATION

6.1. **Positive results.** The forward and backward mappings utilized for the definability questions can also be used to prove that GFP and GNFP^k have a form of interpolation.

Let ϕ_L and ϕ_R be sentences over signatures σ_L and σ_R such that $\phi_L \models \phi_R$ (ϕ_L entails ϕ_R). An *interpolant* for such a validity is a formula θ for which $\phi_L \models \theta$ and $\theta \models \phi_R$, and θ mentions only relations appearing in both ϕ_L and ϕ_R (their *common signature*). We say a logic $\mathcal{L}$ has *Craig interpolation* if for all $\phi_L, \phi_R \in \mathcal{L}$ with $\phi_L \models \phi_R$, there is an interpolant $\theta \in \mathcal{L}$ for it. We say a logic $\mathcal{L}$ has the stronger *uniform interpolation* property if one can obtain θ from ϕ_L and a signature σ', and θ can serve as an interpolant for any ϕ_R entailed by ϕ_L and such that the common signature of ϕ_R and ϕ_L is contained in σ'. A uniform interpolant can be thought of as the best approximation from above of ϕ_L over σ'.

Uniform interpolation holds for the μ-calculus [19], and it also holds for UNFP^k [13]. Unfortunately, $\text{GFP}[\sigma]$ and $\text{GNFP}^k[\sigma]$ both fail to have uniform interpolation and Craig interpolation [29, 13]. However, if we disallow subsignature restrictions that change the guard signature, then we can regain this interpolation property. This "preservation of guard" variant was investigated first by Hoogland, Marx, and Otto in the context of Craig interpolation [29]. The uniform interpolation variant was introduced by D'Agostino and Lenzi [20], who called it *uniform modal interpolation*. Formally, we say a guarded logic

$\mathcal{L}[\sigma, \sigma_g]$ with guard signature $\sigma_g \subseteq \sigma$ has *uniform modal interpolation* if for any $\phi_L \in \mathcal{L}[\sigma, \sigma_g]$ and any subsignature $\sigma' \subseteq \sigma$ containing σ_g, there exists a formula $\theta \in \mathcal{L}[\sigma', \sigma_g]$ such that ϕ_L entails θ and for any σ'' containing σ_g with $\sigma'' \cap \sigma \subseteq \sigma'$ and any $\phi_R \in \mathcal{L}[\sigma'', \sigma_g]$ entailed by ϕ_L, θ entails ϕ_R. It was shown in [20] that GF has uniform modal interpolation. We strengthen this to GFP and GNFP^k.

Theorem 6.1. *For σ a relational signature, $\sigma_g \subseteq \sigma$, and $k \in \mathbb{N}$: $\text{GFP}[\sigma, \sigma_g]$ and $\text{GNFP}^k[\sigma, \sigma_g]$ sentences have uniform modal interpolation, and the interpolants can be found effectively.*

Theorem 6.1 also implies that UNFP^k has the traditional uniform interpolation property: since the guard signature is empty for UNFP^k, uniform modal interpolation and uniform interpolation coincide. Another corollary is that UNFP (not just UNFP^k) has Craig interpolation. Consider sentences ϕ_L and ϕ_R in UNFP such that $\phi_L \models \phi_R$ and with k the maximum width of ϕ_L and ϕ_R. Then the UNFP^k uniform interpolant for ϕ_L with respect to $\sigma_L \cap \sigma_R$ can serve as a Craig interpolant for $\phi_L \models \phi_R$. Note that uniform interpolation for UNFP^k and Craig interpolation for UNFP were known already from [13].

Corollary 6.2. *For σ a relational signature and $k \in \mathbb{N}$: $\text{UNFP}^k[\sigma]$ has uniform interpolation and $\text{UNFP}[\sigma]$ has Craig interpolation. In both cases, the interpolants can be found effectively.*

The idea for the proof of Theorem 6.1 is to use the back-and-forth method from before, together with the uniform interpolation property of the μ-calculus. To illustrate this, we sketch the argument for $\text{GFP}[\sigma, \sigma_g]$, before giving the formal proof for GNFP^k below.

Consider $\phi_L \in \text{GFP}[\sigma, \sigma_g]$ of width k and subsignature $\sigma' \subseteq \sigma$ containing σ_g. We apply Lemma Fwd to get a formula $\phi_L^\mu \in \text{L}_\mu[\Sigma_{\sigma,k}^{\text{code}}]$ that captures codes of tree-like models of ϕ_L.

We want to go backward now, to get a formula over the subsignature σ'. We saw that the backward mapping from earlier can do this: it can start with a μ-calculus formula over $\Sigma_{\sigma,k}^{\text{code}}$, and produce a formula in $\text{GFP}[\sigma', \sigma_g]$. The formula produced by this backward mapping has a nice property related to definability: it is equivalent to ϕ_L exactly when ϕ_L is definable in $\text{GFP}[\sigma', \sigma_g]$.

In general, however, we do not expect ϕ_L to be equivalent to a formula over the subsignature—for uniform interpolation we just want to *approximate* the formula over this subsignature. The backward mapping of ϕ_L^μ (e.g. using Lemma $\text{GFP}[\sigma', \sigma_g]$-Bwd from Section 4 or Lemma $\text{GNFP}^k[\sigma', \sigma_g]$-Bwd from Section 5), does not always do this. Hence, it is necessary to add one additional step before taking the backward mapping: we apply uniform interpolation for the μ-calculus [19], obtaining $\theta^\mu \in \Sigma_{\sigma',k}^{\text{code}}$ which is entailed by ϕ_L^μ and entails each $\text{L}_\mu[\Sigma_{\sigma',k}^{\text{code}}]$-formula implied by ϕ_L^μ. Finally, we apply Lemma $\text{GFP}[\sigma', \sigma_g]$-Bwd to θ^μ to get $\theta \in \text{GFP}[\sigma', \sigma_g]$. We can check that $\theta \in \text{GNFP}^k[\sigma', \sigma_g]$ is the required uniform modal interpolant for ϕ_L over subsignature σ'.

We emphasize that although our uniform interpolation results and definability decision procedures both use this back-and-forth approach, the definability questions are easier in the sense that they do not require interpolation for the μ-calculus.

We now give the proof of Theorem 6.1.

Proof of Theorem 6.1. We prove this for $\text{GNFP}^k[\sigma, \sigma_g]$, but the proof is similar for $\text{GFP}[\sigma, \sigma_g]$ (using the guarded unravelling $\mathcal{U}_{\text{G}[\sigma,\sigma_g]}(\mathfrak{B})$ instead of the plump unravelling $\mathcal{U}_{\text{BGN}^k[\sigma,\sigma_g]}^{\text{plump}}(\mathfrak{B})$, and Lemma $\text{GFP}[\sigma', \sigma_g]$-Bwd instead of Lemma $\text{GNFP}^k[\sigma', \sigma_g]$-Bwd).

We construct the interpolant for ϕ_L as follows:

(1) apply Lemma Fwd to get $\phi_L^\mu \in \text{L}_\mu[\Sigma_{\sigma,k}^{\text{code}}]$;

(2) let consistent$_{\sigma,k}$ be the L$_\mu$[$\Sigma^{\mathrm{code}}_{\sigma,k}$]-formula that expresses that a tree is consistent with respect to $\Sigma^{\mathrm{code}}_{\sigma,k}$;

(3) get the uniform interpolant $\chi \in$ L$_\mu$[$\Sigma^{\mathrm{code}}_{\sigma',k}$] for $\phi^\mu_{\mathrm{L}} \wedge$ consistent$_{\sigma,k}$ and subsignature $\Sigma^{\mathrm{code}}_{\sigma',k}$ (using [19]);

(4) apply Lemma GNFP^k[σ', σ_g]-Bwd to χ to get $\theta \in$ GNFP^k[σ', σ_g].

We can see that θ is a formula over the subsignature σ' by the properties of the backward mapping. We must show that θ satisfies the other properties required of a uniform interpolant.

Original sentence entails interpolant. First, we prove that $\phi_{\mathrm{L}} \models \theta$. Let $\mathfrak{B}$ be a σ-structure and assume $\mathfrak{B} \models \phi_{\mathrm{L}}$. Then $\mathfrak{D}(\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma,\sigma_g]}(\mathfrak{B})) \models \phi_{\mathrm{L}}$ since ϕ_{L} is BGN^k[σ, σ_g]-invariant. Hence, by Lemma Fwd, we have $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma,\sigma_g]}(\mathfrak{B}) \models \phi^\mu_{\mathrm{L}}$. Since $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma,\sigma_g]}(\mathfrak{B})$ is a consistent $\Sigma^{\mathrm{code}}_{\sigma,k}$-tree, this means that $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma,\sigma_g]}(\mathfrak{B}) \models \phi^\mu_{\mathrm{L}} \wedge$ consistent$_{\sigma,k}$. We can now use the fact that χ is a uniform interpolant for $\phi^\mu_{\mathrm{L}} \wedge$ consistent$_{\sigma,k}$, to conclude that $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma,\sigma_g]}(\mathfrak{B}) \models \chi$. But χ is in L$_\mu$[$\Sigma^{\mathrm{code}}_{\sigma',k}$], so the restriction of $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma,\sigma_g]}(\mathfrak{B})$ to the subsignature $\Sigma^{\mathrm{code}}_{\sigma',k}$ also satisfies χ. Moreover, the restriction of $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma,\sigma_g]}(\mathfrak{B})$ to the subsignature is $\Sigma^{\mathrm{code}}_{\sigma',k}$-bisimilar to the unravelling with respect to this subsignature $\Sigma^{\mathrm{code}}_{\sigma',k}$ (this relies on the fact that the guard signature σ_g is the same in both cases). Hence, $\mathcal{U}^{\mathrm{plump}}_{\mathrm{BGN}^k[\sigma',\sigma_g]}(\mathfrak{B}) \models \chi$, which by the backward mapping means that $\mathfrak{B} \models \theta$.

Interpolant entails appropriate sentences in subsignature. Next, assume that $\phi_{\mathrm{L}} \models \phi_{\mathrm{R}}$ for some $\phi_{\mathrm{R}} \in$ GNFP^k[$\sigma_{\mathrm{R}}, \sigma_g$] with $\sigma_{\mathrm{R}} \cap \sigma \subseteq \sigma'$. Let $\sigma'' = \sigma \cup \sigma_{\mathrm{R}}$.

We need to show that $\theta \models \phi_{\mathrm{R}}$. In order to prove this, our reasoning will go back and forth between relational and tree structures.

We start by applying Lemma Fwd to ϕ_{R} to get ϕ^μ_{R}. Let consistent$_{\sigma_{\mathrm{R}},k}$ be the L$_\mu$[$\Sigma^{\mathrm{code}}_{\sigma_{\mathrm{R}},k}$]-formula that expresses that a tree is consistent with respect to $\Sigma^{\mathrm{code}}_{\sigma_{\mathrm{R}},k}$. We now want to show that consistent$_{\sigma,k} \wedge \phi^\mu_{\mathrm{L}}$ entails consistent$_{\sigma_{\mathrm{R}},k} \to \phi^\mu_{\mathrm{R}}$.

Claim 6.3. consistent$_{\sigma,k} \wedge \phi^\mu_{\mathrm{L}} \models$ consistent$_{\sigma_{\mathrm{R}},k} \to \phi^\mu_{\mathrm{R}}$ over all $\Sigma^{\mathrm{code}}_{\sigma'',k}$-structures.

Proof of claim. We first show that consistent$_{\sigma,k} \wedge \phi^\mu_{\mathrm{L}} \models$ consistent$_{\sigma_{\mathrm{R}},k} \to \phi^\mu_{\mathrm{R}}$ over all $\Sigma^{\mathrm{code}}_{\sigma'',k}$-trees. Suppose that $\mathcal{T}$ is a $\Sigma^{\mathrm{code}}_{\sigma'',k}$-tree and $\mathcal{T} \models$ consistent$_{\sigma,k} \wedge \phi^\mu_{\mathrm{L}}$. Then $\mathcal{T}$ must be consistent with respect to the subsignature $\Sigma^{\mathrm{code}}_{\sigma,k}$. If $\mathcal{T}$ is not consistent with respect to $\Sigma^{\mathrm{code}}_{\sigma_{\mathrm{R}},k}$, then $\mathcal{T}$ trivially satisfies consistent$_{\sigma_{\mathrm{R}},k} \to \phi^\mu_{\mathrm{R}}$ and we are done. Otherwise, $\mathcal{T}$ is consistent with respect to both $\Sigma^{\mathrm{code}}_{\sigma,k}$ and $\Sigma^{\mathrm{code}}_{\sigma_{\mathrm{R}},k}$, which is enough to conclude that it is a consistent $\Sigma^{\mathrm{code}}_{\sigma'',k}$-tree. Hence, by Lemma Fwd, we have $\mathfrak{D}(\mathcal{T}) \models \phi_{\mathrm{L}}$. Since $\phi_{\mathrm{L}} \models \phi_{\mathrm{R}}$, this means that $\mathfrak{D}(\mathcal{T}) \models \phi_{\mathrm{R}}$. Another application of Lemma Fwd allows us to conclude that $\mathcal{T} \models \phi^\mu_{\mathrm{R}}$, and hence by weakening, $\mathcal{T} \models$ consistent$_{\sigma_{\mathrm{R}},k} \to \phi^\mu_{\mathrm{R}}$ as desired.

We can use this to prove that consistent$_{\sigma,k} \wedge \phi^\mu_{\mathrm{L}} \models$ consistent$_{\sigma_{\mathrm{R}},k} \to \phi^\mu_{\mathrm{R}}$ over all $\Sigma^{\mathrm{code}}_{\sigma'',k}$-structures. Assume not. Then there is some $\Sigma^{\mathrm{code}}_{\sigma'',k}$-structure $\mathfrak{G}$ such that $\mathfrak{G} \models$ consistent$_{\sigma,k} \wedge \phi^\mu_{\mathrm{L}} \wedge \neg($consistent$_{\sigma_{\mathrm{R}},k} \to \phi^\mu_{\mathrm{R}})$. By the tree-model property of L$_\mu$ [17], this means there is

some $\Sigma_{\sigma'',k}^{\text{code}}$-tree $\mathcal{T}$ that witnesses this, which contradicts the previous paragraph. Therefore consistent$_{\sigma,k} \wedge \phi_{\text{L}}^{\mu} \models$ consistent$_{\sigma_{\text{R}},k} \rightarrow \phi_{\text{R}}^{\mu}$ over all $\Sigma_{\sigma'',k}^{\text{code}}$-structures as required. $\qquad\square$

Since consistent$_{\sigma,k} \wedge \phi_{\text{L}}^{\mu}$ entails consistent$_{\sigma_{\text{R}},k} \rightarrow \phi_{\text{R}}^{\mu}$ by the previous claim and χ is a uniform interpolant for consistent$_{\sigma,k} \wedge \phi_{\text{L}}^{\mu}$ over the subsignature $\Sigma_{\sigma',k}^{\text{code}}$, we know that $\chi \models$ consistent$_{\sigma_{\text{R}},k} \rightarrow \phi_{\text{R}}^{\mu}$ over all $\Sigma_{\sigma'',k}^{\text{code}}$-structures. We can use this to show that $\theta \models \phi_{\text{R}}$.

Let $\mathfrak{B}$ be a σ''-structure such that $\mathfrak{B} \models \theta$. Then $\mathfrak{D}(\mathcal{U}_{\text{BGN}^k[\sigma',\sigma_g]}^{\text{plump}}(\mathfrak{B})) \models \theta$ since θ is BGN$^k[\sigma',\sigma_g]$-invariant (since it is in GNFP$^k[\sigma',\sigma_g]$). Hence, $\mathcal{U}_{\text{BGN}^k[\sigma',\sigma_g]}^{\text{plump}}(\mathfrak{B}) \models \chi$ by properties of the backward mapping. But $\mathcal{U}_{\text{BGN}^k[\sigma'',\sigma_g]}^{\text{plump}}(\mathfrak{B}) \models \chi$ as well, since $\mathcal{U}_{\text{BGN}^k[\sigma'',\sigma_g]}^{\text{plump}}(\mathfrak{B})$ is $\Sigma_{\sigma',k}^{\text{code}}$-bisimilar to $\mathcal{U}_{\text{BGN}^k[\sigma',\sigma_g]}^{\text{plump}}(\mathfrak{B})$. Hence, by the previous paragraph, we must have $\mathcal{U}_{\text{BGN}^k[\sigma'',\sigma_g]}^{\text{plump}}(\mathfrak{B}) \models$ consistent$_{\sigma_{\text{R}},k} \rightarrow \phi_{\text{R}}^{\mu}$. Since $\mathcal{U}_{\text{BGN}^k[\sigma'',\sigma_g]}^{\text{plump}}(\mathfrak{B})$ is a consistent $\Sigma_{\sigma'',k}^{\text{code}}$-tree, it is also $\Sigma_{\sigma_{\text{R}},k}^{\text{code}}$-consistent. Hence, $\mathcal{U}_{\text{BGN}^k[\sigma'',\sigma_g]}^{\text{plump}}(\mathfrak{B}) \models \phi_{\text{R}}^{\mu}$ and $\mathfrak{D}(\mathcal{U}_{\text{BGN}^k[\sigma'',\sigma_g]}^{\text{plump}}(\mathfrak{B})) \models \phi_{\text{R}}$. Since $\mathfrak{B}$ and $\mathfrak{D}(\mathcal{U}_{\text{BGN}^k[\sigma'',\sigma_g]}^{\text{plump}}(\mathfrak{B}))$ are BGN$^k[\sigma'',\sigma_g]$-bisimilar, and $\phi_{\text{R}} \in$ GNFP$^k[\sigma_{\text{R}},\sigma_g]$ with $\sigma_{\text{R}} \subseteq \sigma''$, this means that $\mathfrak{B} \models \phi_{\text{R}}$ as desired.

This completes the proof that θ entails ϕ_{R}, and hence completes the proof that θ is a uniform modal interpolant. $\qquad\square$

6.2. **Failure of uniform interpolation.** In this section we will see that some natural extensions and variants of our main interpolation theorems fail.

Although we have shown that UNFP has Craig interpolation, it fails to have uniform interpolation.

Proposition 6.4. *Uniform interpolation fails for* UNFP*. In particular, there is a* UNF *antecedent with no uniform interpolant in* LFP*, even when the consequents are restricted to sentences in* UNF*. The variant of uniform interpolation where entailment is considered only over finite structures also fails for* UNFP*.*

Proof. There is a UNF sentence ϕ that expresses that unary relations R,G,B form a 3-coloring of a graph with edge relation E. This is because we only need unary negation to say:

- every node has a color: $\neg \exists x.(\neg Rx \wedge \neg Gx \wedge \neg Bx)$,
- neighbouring nodes do not share a color: $\neg \exists xy. \big(Exy \wedge ((Rx \wedge Ry) \vee \dots)\big)$.

For readability, we have omitted the trivial guards $x = x$ in the unary negations above.

Consider a uniform interpolant θ (in any logic) for the UNF sentence ϕ with respect to its UNF-consequences in the signature $\{E\}$. We claim that there cannot be an LFP formula equivalent to θ.

For all finite graphs G that are not 3-colorable, let ψ_G be the UNF sentence corresponding to the canonical conjunctive query of G over relation E—that is, if G consists of edges E mentioning vertices $v_1 \dots v_n$, ψ_G is $\exists v_1 \dots v_n. \big(\bigwedge_{e \in G, e=(v_i,v_j)} Ev_iv_j\big)$. Then ϕ must entail $\neg \psi_G$, since the 3-coloring of a graph G' satisfying ψ_G is easily seen to induce a 3-coloring on G.

Now consider a finite graph G. If G is 3-colorable, then $G \models \phi$, and hence $G \models \theta$. On the other hand, if G is not 3-colorable, then $G \models \psi_G$, so $G \models \neg\theta$ because ϕ entails $\neg\psi_G$ and thus, by the assumption on θ, θ entails $\neg\psi_G$. Therefore, θ holds in G iff G is 3-colorable.

Dawar [21] showed that 3-colorability is not expressible in the infinitary logic $\mathcal{L}^\omega_{\infty\omega}$ over finite structures. Since LFP can be translated into $\mathcal{L}^\omega_{\infty\omega}$ over finite structures, this implies that θ cannot be in LFP.

The above argument only makes use of the properties of θ over finite structures, and thus demonstrates the failure of the variant of uniform interpolation in the finite. ▢

We have trivial uniform interpolants in existential second-order logic, i.e. in NP. The previous arguments shows that interpolants for UNFP express NP-hard problems, and thus cannot be in any PTime language if PTime is not equal to NP. We remark that one could still hope to find uniform interpolants for UNFP by allowing the interpolants to live in a larger fragment that is still "tame", but we leave this as an open question.

Uniform interpolation also fails for GSO.

Proposition 6.5. *Uniform interpolation fails for* GSO. *In particular, there is a* GF *antecedent with no uniform interpolant in* GSO, *even when the consequents are restricted to sentences of* GF *(or* UNF*) of width 2.*

Proof. Let $\phi \in \text{GF}[\sigma]$ for $\sigma = \{G, P, Q, R_1, R_2, S\}$ be

$$\forall z.[Qz \rightarrow \exists xy.(Gzzxy \wedge Sxy \wedge R_1\, zx \wedge R_2\, zy)] \wedge$$

$$\forall xy.[Sxy \rightarrow \exists x'y'.\Big(Gxyx'y' \wedge Sx'y' \wedge R_1\, xx' \wedge R_2\, yy' \wedge$$

$$\big((Px' \wedge Py') \vee (\neg Px' \wedge \neg Py')\big)\Big)\Big]$$

which implies that there is a "ladder" starting at every Q-node (where S connects pairs of elements on the same rung, and R_i connects corresponding elements on different rungs) and the pair of elements on each rung agree on P. The relation G is used as a dummy guard to ensure that the formula is in GF.

Then for each n, we can define over $\sigma' = \{P, Q, R_1, R_2\}$ a formula ψ_n

$$\Big(\exists x.\Big(Qx \wedge \forall x_1 \ldots x_n.\big((\textstyle\bigwedge_i R_1\, x_i x_{i+1} \wedge x_1 = x) \rightarrow Px_n\big)\Big)\Big) \rightarrow$$

$$\Big(\exists y.\big(Qy \wedge \exists y_1 \ldots y_n.(\textstyle\bigwedge_i R_2\, y_i y_{i+1} \wedge y_1 = y \wedge Py_n)\big)\Big)$$

which expresses that if there is some Q-position x such that every R_1-path of length n from x ends in a position satisfying P, then there is an R_2-path of length n from some Q-position y that ends in a position satisfying P. Note that for all n, ψ_n can be written in either GF or UNF of width 2, and $\phi \models \psi_n$.

Assume for the sake of contradiction that there is some uniform interpolant θ in GSO.

Over trees, GSO coincides with MSO ([18], as cited in [25]). Hence, there is an equivalent θ' in MSO over tree structures. This means we can construct from θ' a nondeterministic parity tree automaton $\mathcal{A}$ that recognizes precisely the language of trees (with branching degree at most 2, say) where θ' holds.

Let m be the number of states in $\mathcal{A}$. Consider the ladder structure $\mathfrak{A}_m$ consisting of a single element a from which there is an infinite R_1-chain of distinct elements and an infinite R_2-chain of distinct elements, where the i-th elements on each chain are connected by S,

elements on level i and $i+1$ are guarded by G, P holds only at the $(m+1)$-st element in each chain, and Q holds only at a.

Because $\mathfrak{A}_m \models \phi$, we have $\mathfrak{A}_m \models \theta$. But over σ', $\mathfrak{A}_m$ is a tree with branching degree at most 2, so $\mathfrak{A}_m \models \theta'$. Hence, there is an accepting run of $\mathcal{A}$ on $\mathfrak{A}_m$. Using a pumping argument, we can pump a section of the R_2 branch before the P-labelled element in order to generate an accepting run of $\mathcal{A}$ on a new tree $\mathfrak{A}'_m$ where P holds at the $(m+1)$-st element in the R_1-chain and P does not hold at that position in the R_2 chain. Hence, this new tree $\mathfrak{A}'_m$ is a model for both θ' and θ. But $\mathfrak{A}'_m \not\models \psi_m$, contradicting the fact that θ is a uniform interpolant. $\qquad\square$

It has been known for some time that GF fails to have even ordinary Craig interpolation [28], and hence fails to have uniform interpolation. The previous proposition shows that we cannot get uniform interpolants for GF even when we allow the uniform interpolants to come from GSO.

6.3. **Failure of Craig interpolation for GNFP.** It is natural to try to extend our results about GNFP^k to the logic GNFP. Unfortunately, Craig interpolation fails for GNFP.

Proposition 6.6. *Craig interpolation fails for* GNFP. *In particular, there is an entailment of* GFP *sentences with no* GNFP *interpolant, even over finite structures.*

Proof. Define the GFP$[\sigma]$ sentence ϕ over signature $\sigma = \{G, Q, R\}$ to be $\forall x.(Qx \to \phi'(x))$ where $\phi'(x)$ is

$$[\mathbf{lfp}_{X,xy} . Gxyy \wedge (Rxy \vee \exists y'.(Gxy'y \wedge Rxy' \wedge Xy'y))](xx)$$

Note that $\phi'(x)$ implies that x has an R-path to itself. Thus ϕ implies that every element where Q holds has an R-path to itself.

Define the GFP$[\sigma']$ sentence ψ over signature $\sigma' = \{P, Q, R\}$ to be

$$\forall x.\Big((Qx \wedge Px) \to [\mathbf{lfp}_{X,x} . \exists y.\big(Rxy \wedge (Py \vee Xy)\big)](x)\Big).$$

The sentence ψ expresses that for all Q and P labelled elements x, there is an R-path from x leading to some node y with Py.

We first argue that $\phi \models \psi$. Assume ϕ holds in some $(\sigma \cup \sigma')$-structure, and consider some element x_0; we must show that x_0 satisfies $(Qx_0 \wedge Px_0) \to [\mathbf{lfp}_{X,x} . \exists y.\big(Rxy \wedge (Py \vee Xy)\big)](x_0)$. If Q and P do not hold at x_0, then the condition is trivially satisfied here. Otherwise, if Q and P do hold at x_0, then ϕ ensures that there is an R-path from x_0 to itself, and hence there is an R-path from x_0 to a node where P holds as required by ψ.

Now suppose for the sake of contradiction that there is a GNFP$[\sigma \cap \sigma']$-interpolant χ for $\phi \models \psi$. Note that χ only uses relations Q and R. Let k be the width of χ in strict normal form.

We now define two $(\sigma \cup \sigma')$-structures, $\mathfrak{A}$ and $\mathfrak{B}$, that we will use to obtain a contradiction. The graph structures for $\mathfrak{A}$ and $\mathfrak{B}$ are pictured in Figure 2 (i.e. this shows the structure in terms of relation R only).

Let $\mathfrak{A}$ be the structure consisting of elements $\{a_1, \ldots, a_{k+1}\}$ arranged in an R-cycle (i.e. $Ra_1a_2, Ra_2a_3, \ldots, Ra_{k+1}a_1$). Q holds of all elements, and G holds of all triples of elements (in particular, $Ga_1a_2a_2$, $Ga_2a_3a_3$, etc.). The only element satisfying P is a_1.

Let $\mathfrak{B}$ be the structure with elements $\{b_0, b_1, \ldots, b_k, b_{k+1}\} \cup \{c_1, \ldots, c_{k+1}\}$ where the elements $c_1, \ldots, c_{k+1}$ are arranged in an R-cycle, $b_1, \ldots, b_{k+1}$ are arranged in an R-cycle,

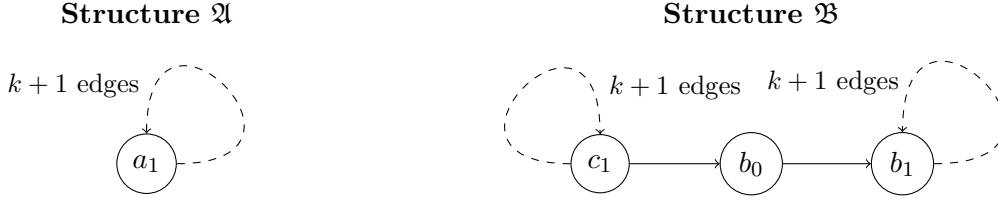**Structure $\mathfrak{A}$** **Structure $\mathfrak{B}$**

FIGURE 2. Structures used in the proof of Proposition 6.6.

and Rc_1b_0 and Rb_0b_1. As in $\mathfrak{A}$, Q holds of all elements and G holds of all triples of elements. P holds only at b_0.

Notice that $\mathfrak{A}$ satisfies ϕ, and hence satisfies ψ. But $\mathfrak{B}$ does not satisfy ψ, because b_0 does not have an R-path to a node labelled with P.

We claim that $\mathfrak{A}$ and $\mathfrak{B}$ are indistinguishable by $\mathrm{GNFP}[\sigma \cap \sigma']$ sentences of width k. We must define a winning strategy for Duplicator in the $\mathrm{BGN}^k[\sigma \cap \sigma']$-bisimulation game between $\mathfrak{A}$ and $\mathfrak{B}$. Because the structures agree on Q (since Q holds of every element in both structures), it suffices to show that they are indistinguishable with respect to relation R.

Consider a position that is strictly guarded by R or Q. Such a position consists of at most two elements (and if there are two elements, u and v must satisfy Ruv). The initial position in the game (consisting of the empty partial homomorphism) is like this, so we must show that Duplicator has a strategy to ensure that she is always gets back to a position like this.

Suppose the active structure is $\mathfrak{A}$. Without loss of generality, we can assume the partial isomorphism f in the position has domain a_1, a_2. It must be the case that $Rf(a_1)f(a_2)$ holds in $\mathfrak{B}$. Because Spoiler can only extend his selection to at most k elements, it is not possible for him to select all of the elements in the R-cycle in $\mathfrak{A}$. We can assume, again without loss of generality, that he extends to a partial homomorphism that includes all but 1 element, a_{i+1}, in the R-cycle of $\mathfrak{A}$. That is, we suppose that Spoiler extends his selection to elements $a_1, a_2, \ldots, a_i$ and $a_{i+2}, \ldots, a_{k+1}$. The sequence $a_1, a_2, \ldots, a_i$ forms an R-successor chain, so Duplicator responds by mapping $a_3 \ldots a_i$ so that the images $f(a_2), \ldots, f(a_i)$ form a chain of R-successors starting at $f(a_2)$. The chain is unique unless $f(a_2)$ is c_1; if $f(a_2) = c_1$, she can choose to obtain either the successor chain $c_1, c_2, c_3, \ldots$ or the chain $c_1, b_0, b_1, \ldots$. Likewise $a_{i+2}, \ldots, a_{k+1}a_1$ forms a successor chain leading to a_1. Duplicator maps $a_{i+2}, \ldots, a_{k+1}$ so that $f(a_{i+2}), \ldots, f(a_{k+1}), f(a_1)$ forms an R-successor chain leading to $f(a_1)$. The chain is unique unless $f(a_1)$ is b_1, and in this case she can choose either of the two candidate chains. This is a new partial homomorphism with respect to R, and when Spoiler collapses to a single element or pair of elements satisfying Ruv, we have a partial isomorphism as required.

Now consider the case where the active structure is $\mathfrak{B}$, with elements u' and v' with $Ru'v'$, and Spoiler extends to a set of elements E. Note that the subgraph on E induced by R in $\mathfrak{B}$ is acyclic, due to the size of E. Let V^+ be the maximal subset of E that contains v' and is closed under R. The set V^+ could be the union of two chains of R-successors, or a single chain of R-successors. Similarly let U^- be the maximal subset of E that contains u' and is closed under R-predecessors. U^- can consist of two chains, or it can be a single R-chain. Note that an element of V^+ is the i-th successor of u' for some i, and thus Duplicator has no choice but to play the unique i-th R-successor element of $f(v')$

in her response. Similarly on U^- Duplicator must play the corresponding R-predecessor of $f(u')$. On the remaining elements O of E, Duplicator can choose any homomorphism into $\mathfrak{A}$. Such a homomorphism can be found by breaking the subgraph induced on O into connected components: for each component C choose an element $e_0 \in C$ and map it to an $f(e_0)$ arbitrarily; each other element in C is the i-th predecessor or i-th successor of e_0, so we can map it to the unique i-th predecessor or successor of $f(e_0)$. The acylicity of E guarantees that this mapping is a homomorphism. Although it is not injective, two elements e_1 and e_2 of E map to the same element in $\mathfrak{A}$ only if there is some i such that either the i-th successor of e_1 and e_2 are equal or the i-th predecessor of e_1 and e_2 are equal. Thus acyclicity of E guarantees that we cannot have $Re_1 e_2$ for such an e_1 and e_2. Hence, when Spoiler collapses to a strictly guarded position in his next move, the resulting position is a partial isomorphism as required.

Playing like this, Duplicator can continue to play indefinitely, so we she wins the bisimulation game. This shows that $\mathfrak{A}$ and $\mathfrak{B}$ are indistinguishable by strict normal form $\mathrm{GNFP}^k[\sigma \cap \sigma']$-sentences, so they must agree on χ.

Since $\mathfrak{A} \models \phi$, we have $\mathfrak{A} \models \chi$. Hence, $\mathfrak{B} \models \chi$. But this implies that $\mathfrak{B} \models \psi$, which is a contradiction. $\qquad\square$

7. Conclusions

In this paper we explored effective characterizations of definability in expressive fixpoint logics. In the process, we have extended and refined the approach of going back and forth between relational structures and trees. Boot-strapping from results about trees also allowed us to obtain results about interpolation for these logics. We did not allow constants in the formulas in this paper, but we believe that similar effective characterization and interpolation results hold for guarded fixpoint logics with constants.

There are a number of open questions related to this work. For GNFP^k-definability, we proved only decidability results in this paper. It would be interesting to determine the exact complexity of this problem, perhaps using a direct automaton construction in the spirit of the construction given for GFP. We also leave open the question of deciding definability in GNFP and UNFP, without any width restriction. For this question, one natural way to proceed is to try to bound the width of a defining sentence in terms of some parameter of the input (e.g., its length). For example, if we could show that a sentence of length n in some larger logic $\mathcal{L}$ is definable in GNFP iff it is definable in $\mathrm{GNFP}^{f(n)}$ for some fixed function f, then we could test for membership in GNFP using the results of this paper. We also note that our results on fixpoint logics hold only when equivalence is considered over all structures, leaving open the corresponding questions over finite structures.

In Corollary 7 of the conference version of this paper ([11]), we claimed to have proven that it was possible to decide membership in alternation-free GFP, a restriction of GFP to formulas with no nesting of both least and greatest fixpoints. However, the proof of this claim was incorrect, and hence this question is open. It is desirable to know if a sentence is in this alternation-free fragment of GFP since it has better computational properties: for instance, model checking for this alternation-free fragment can be done in linear time [23]. This alternation-free fragment also corresponds to another previously studied logic called DATALOG-LITE [23]. Hence, deciding membership in alternation-free GFP (equivalently, DATALOG-LITE) remains an interesting open problem.

Finally, we showed that GNFP fails to have Craig interpolation. This leaves open the question of whether there is a decidable fixpoint logic that contains GNFP and has interpolation. One candidate for this larger logic is called GNFP-UP [10], but it is not clear whether the methods in this paper could be adapted to prove such a result.

Acknowledgment

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