A New Type of Bases for Zero-dimensional Ideals

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Abstract

We formulate a substantial improvement on Buchberger's algorithm for Gröbner bases of zero-dimensional ideals. The improvement scales down the phenomenon of intermediate expression swell as well as the complexity of Buchberger's algorithm to a significant degree. The idea is to compute a new type of bases over principal ideal rings instead of over fields like Gröbner bases. The generalizations of Buchberger's algorithm from over fields to over rings are abundant in the literature. However they are limited to either computations of strong Gröbner bases or modular computations of the numeral coefficients of ideal bases with no essential improvement on the algorithmic complexity. In this paper we make pseudo-divisions with multipliers to enhance computational efficiency. In particular, we develop a new methodology in determining the authenticity of the factors of the pseudo-eliminant, i.e., we compare the factors with the multipliers of the pseudo-divisions instead of the leading coefficients of the basis elements. In order to find out the exact form of the eliminant, we contrive a modular algorithm of proper divisions over principal quotient rings with zero divisors. The pseudo-eliminant and proper eliminants and their corresponding bases constitute a decomposition of the original ideal. In order to address the ideal membership problem, we elaborate on various characterizations of the new type of bases. In the complexity analysis we devise a scenario linking the rampant intermediate coefficient swell to Bézout coefficients, partially unveiling the mystery of hight-level complexity associated with the computation of Gröbner bases. Finally we make exemplary computations to demonstrate the conspicuous difference between Gröbner bases and the new type of bases.

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1 Introduction

After Buchberger initiated his celebrated algorithm in his remarkable PhD thesis [Buc65], the theory of Gröbner bases has been established as a standard tool in algebraic geometry and computer algebra, yielding algorithmic solutions to many significant problems in mathematics, science and engineering [BW98]. As a result, there have been many excellent textbooks on the subject such as [AL94] [BW93] [CLO15] [KR00] [DL06] [GP08] [EH12] [GG13].

Nonetheless the computational complexity of Gröbner bases often demands an enormous amount of computing time and storage space even for problems of moderate sizes, which severely impedes its practicality and dissemination. A striking phenomenon is in the computation of Gröbner bases over the rational field with respect to the LEX ordering, when the coefficients of the final basis elements swell to extremely complicated rational numerals even though the coefficients of the original ideal generators are quite moderate. Example 9.2 in this paper should be impressive enough to illustrate such a phenomenon. An even more dramatic phenomenon is the "intermediate expression swell" referring to a generation of a huge number of intermediate polynomials with much more gigantic coefficients and larger exponents than those of the final basis elements during the implementation of the classical algorithm. In [CLO15, P116] and [GG13, P616, 21.7] there are some brief reviews on the complexity issues associated with the classical algorithm.

These challenges have stimulated decades of ardent endeavors in improving the efficiency of the classical algorithm. The methodologies such as the normal selection strategies and signatures effectively diminish the number of intermediate polynomials spawned during the process of algorithmic implementations [Buc85] [GMN91] [BW93, P222, 5.5] [Fau02] [EF17]. The modular and p-adic techniques based on "lucky primes" and Hensel lifting have been adopted to control the rampant growth of the intermediate coefficients albeit being limited to numeral coefficients only [Ebe83] [Win87] [ST89] [Tra89] [Pau92] [Gra93] [Arn03]. There are also Gröbner basis conversion methods such as the FGLM algorithm [FGL93] and Gröbner Walk [CKM97], a detailed description of which can be found in [CLO05, P49, §3; P436, §5] and [Stu95]. The idea of these methods is to compute another Gröbner basis with respect to a different but favorable monomial ordering before converting it to the desired Gröbner basis. Albeit with all these endeavors

over the decades, the high-level complexity associated with the Gröbner basis computations remains a conundrum.

Another train of thoughts over the past decades is to generalize Gröbner bases from over fields to over rings. Among the copious and disparate coefficient rings that we shall not enumerate here, the Gröbner bases over principal ideal rings are pertinent to the new type of bases in this paper. There is an excellent exposition on Gröbner bases over rings and especially over PIDs in [AL94, Chapter 4]. However the focal point of the exposition is on the strong Gröbner bases that resemble the Gröbner bases over fields and hence are still plagued with complexity problems.

In this paper we take a novel approach by defining a new type of bases over principal quotient rings instead of over numeral fields like Gröbner bases. It is a natural approach since for a zero-dimensional ideal, the final eliminant is always a univariate polynomial after eliminating all the other variables. With the principal ideals generated by the eliminant factors serving as moduli, we obtain an elegant decomposition of the original ideal into pairwise relatively prime ideals. We also use pseudo-divisions and multipliers to enhance computational efficiency. In the exemplary computations in Section 9, it is conspicuous that this new approach scales down both the high-level complexity and gigantic numeral coefficients of the Gröbner bases over rational fields.

In practice the Wu's method [Wu83] is more commonly used than the Gröbner basis method since it is based on pseudo-divisions and thus more efficient. However the pseudo-divisions adopted by Wu's method usually lose too much algebraic information of the original ideals. In Section 2 we recall some rudimentary facts on monomial orderings and then define pseudo-divisions over PIDs. The multipliers for the pseudo-divisions in this paper are always univariate polynomials so as to avoid losing too much algebraic information of the original ideals. The pseudo-divisions of this ilk also dispose of the solvability condition for the linear equations of leading coefficients imposed by the classical division algorithm over rings. Please refer to Remark 2.9 for details.

Algorithm 3.9 is one of the pivotal algorithms in the paper. It computes the pseudo-eliminant and pseudo-basis as per the elimination ordering in Definition 3.1. The purpose of Corollary 3.7 and Lemma 3.8 is to trim down the number of S-polynomials to be pseudo-reduced. They are highly effective in this respect as illustrated by the exemplary computations in Section 9.

The pseudo-eliminant might contain factors that are not the bona fide ones of the eliminant. The discrimination among these factors for authenticity is based on a crucial methodology, i.e., the pseudo-eliminant should be compared with the multipliers of the pseudo-divisions instead of the leading coefficients of the basis elements. Example 4.12 shows that the multipliers of the pseudo-divisions are more reliable than the leading coefficients of the basis elements. The multiplier methodology is incorporated into Theorem 4.10 establishing that the compatible part of the pseudo-eliminant constitutes a bona fide factor of the eliminant. This is one of the primary conclusions of the paper. The multiplier and its property in Lemma 4.8 generalize the syzygy theory over fields and PIDs for Gröbner bases and is another substantiation of the multiplier methodology. Please refer to Remark 4.9 for the comment. The compatible and incompatible parts of the pseudo-eliminant are defined in Definition 4.5 and computed via Algorithm 4.6. In particular, we obtain a squarefree decomposition of the incompatible part IP(χ_{ε}) by Algorithm 4.6

via a univariate squarefree factorization of the pseudo-eliminant χ_{ε} by Algorithm 4.2. We avoid a complete univariate factorization of the pseudo-eliminant χ_{ε} due to the concerns on computational complexity.

We conduct a complete analysis of the incompatible part $IP(\chi_{\varepsilon})$ of the pseudoeliminant χ_{ε} in Section 5 based on modular algorithms with the composite divisors obtained in Algorithm 4.6 as the moduli. The advantages are that we have one less variable than the classical algorithm and the composite divisors are usually small polynomial factors of the pseudo-eliminant χ_{ε} . However the disadvantage is that the computations are over the principal quotient rings (PQR) that might contain zero divisors. As a result, we redefine S-polynomials in Definition 5.11 carefully in order to obviate the zero multipliers incurred by the least common multiple of leading coefficients. Algorithm 5.20 is pivotal in procuring the proper eliminants and proper bases by proper divisions as in Theorem 5.10. We prove rigorously in Theorem 5.26 that the nontrivial proper eliminants obtained in Algorithm 5.20 are de facto the bone fide factors of the eliminant of the original ideal. The meticulous arguments in this primary conclusion are to ensure that our arguments are legitimate within the algebra $R_q[\tilde{\boldsymbol{x}}]$ that contains zero divisors. Similar to Lemma 4.8, we generalize the classical syzygy theory over fields and PIDs for Gröbner bases to the one in Lemma 5.25. Further, we also have Corollary 5.14 and Lemma 5.18 to trim down the number of S-polynomials for proper reductions.

We render two equivalent characterizations on the pseudo-bases B_{ε} obtained in Algorithm 3.9. The first characterization is the identity (6.15) in terms of leading terms whereas the second one is Theorem 6.13 via GCD-reductions as defined in Theorem 6.12. We have the same kind of characterizations in Theorem 6.15 for the proper bases B_q and B_p obtained in Algorithm 5.20. These bases as in (6.35) correspond to a decomposition of the original ideal in (6.2) whose modular version is in (6.36). We can define a unique normal form of a polynomial in $R_q[\tilde{x}]$ with respect to the original ideal by the Chinese Remainder Theorem as in Lemma 6.20 since the ideal decomposition in (6.2) is pairwise relatively prime. In the remaining part of this section we define irredundant, minimal and reduced bases that possess different levels of uniqueness.

In Section 7 we make some further improvements on the algorithms by Principle 7.1. The highlight of Section 8 is Lemma 8.3 in which we contrive a special scenario consisting of two basis elements, a detailed analysis of which reveals that the classical algorithm contains the Euclidean algorithm computing the greatest common divisor of the leading coefficients. Moreover, the results in (8.12) and (8.13) contain the Bézout coefficients u and v that might swell to an enormous size like in Example 8.1. By contrast the computation of our new type of S-polynomial as in (8.14) yields the above results in one step without the Bézout coefficients. This might help to unveil the mystery of intermediate coefficient swell as well as high-level complexity associated with the Gröbner basis computations.

We make two exemplary computations in Section 9 with the second one being more sophisticated than the first one. It contains a paradigmatic computation of proper eliminants and proper bases over principal quotient rings with zero divisors as in Algorithm 5.20. We provide a detailed explanation for each step of the computation to elucidate the ideas of this new type of bases.

As usual, we denote the sets of complex, real, rational, integral and natural numbers as \mathbb{C} , \mathbb{R} , \mathbb{Q} , \mathbb{Z} and \mathbb{N} respectively. In this paper, we use the following

notations for a ring $R: R^* := R \setminus \{0\}; R^*$ denotes the set of units in R. With $\mathbf{x} = (x_1, \dots, x_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, we denote a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ as \mathbf{x}^{α} and a term as $c\mathbf{x}^{\alpha}$ with the coefficient $c \in R^*$. We also use the boldface \mathbf{x} to abbreviate the algebra $R[x_1, \dots, x_n]$ over a ring R as $R[\mathbf{x}]$. The notation $\langle A \rangle$ denotes an ideal generated by a nonempty subset $A \subset R[\mathbf{x}]$. Further, K usually denotes a perfect field that is not necessarily algebraically closed unless specified. In most cases we treat the algebra $K[\mathbf{x}]$ as $(K[x_1])[\tilde{\mathbf{x}}]$ over the ring $R = K[x_1]$ with the variables $\tilde{\mathbf{x}} := (x_2, \dots, x_n)$.

2 A Pseudo-division Algorithm over Principal Ideal Domains

In this article we adopt a pseudo-divison of polynomials over a principal ideal domain as in Theorem 2.7. We shall abbreviate a principal ideal domain as a PID and denote it as R henceforth.

Let R be a PID and R[x] a polynomial algebra $R[x_1, \ldots, x_n]$ over R. Let us denote the set of monomials in $\mathbf{x} = (x_1, \ldots, x_n)$ as $[x] := \{x^{\alpha} : \alpha \in \mathbb{N}^n\}$. A nonzero ideal $I \subset R[x]$ is called a *monomial* ideal if I is generated by monomials in [x]. By Hilbert Basis Theorem we can infer that every monomial ideal in R[x] is finitely generated since a PID R is Noetherian.

Lemma 2.1. Let R be a PID. Consider a monomial ideal $I = \langle \mathbf{x}^{\alpha} : \alpha \in E \rangle$ in $R[\mathbf{x}]$ with $E \subset \mathbb{N}^n \setminus \{\mathbf{0}\}$. We have the following conclusions:

- (i) A term $c\mathbf{x}^{\beta} \in I$ for $c \in R^*$ if and only if there exists an $\alpha \in E$ such that \mathbf{x}^{β} is divisible by \mathbf{x}^{α} ;
- (ii) A polynomial $f \in I$ if and only if every term of f lies in I.

Proof. It suffices to prove the necessity of the two conclusions. Suppose that $g = \sum_{j=1}^{s} q_j \boldsymbol{x}^{\alpha_j}$ with g representing the term $c\boldsymbol{x}^{\beta} \in I$ as in (i), or $f \in I$ as in (ii). Here $q_j \in R[\boldsymbol{x}]$ and $\alpha_j \in E$ for $1 \leq j \leq s$. We expand each q_j into individual terms and compare those with the same multi-degrees on both sides of the equality. The conclusion readily follows since every term on the right hand side of the equality is divisible by some $\boldsymbol{x}^{\alpha_j}$ with $\alpha_j \in E$.

A total ordering \succ on the monomial set [x] satisfies that for every pair $\mathbf{x}^{\alpha}, \mathbf{x}^{\beta} \in [x]$, exactly one of the following relations holds: $\mathbf{x}^{\alpha} \succ \mathbf{x}^{\beta}, \mathbf{x}^{\alpha} = \mathbf{x}^{\beta}$, or $\mathbf{x}^{\alpha} \prec \mathbf{x}^{\beta}$. Moreover, $\mathbf{x}^{\alpha} \succeq \mathbf{x}^{\beta}$ means either $\mathbf{x}^{\alpha} \succ \mathbf{x}^{\beta}$ or $\mathbf{x}^{\alpha} = \mathbf{x}^{\beta}$. A well-ordering \succ on [x] satisfies that every nonempty subset $A \subset [x]$ has a minimal element. That is, there exists $\mathbf{x}^{\alpha} \in A$ such that $\mathbf{x}^{\beta} \succeq \mathbf{x}^{\alpha}$ for every $\mathbf{x}^{\beta} \in A$. A well-ordered set is always a totally ordered set since every subset consisting of two elements has a minimal element.

It is evident that under a well-ordering \succ on [x], there is no infinite strictly decreasing sequence $x^{\alpha_1} \succ x^{\alpha_2} \succ x^{\alpha_3} \succ \cdots$ in [x] (or every strictly decreasing sequence in [x] terminates). Nonetheless we have a much easier description as follows under the Noetherian condition.

Proposition 2.2. Let \succ be a total ordering on [x] such that $\mathbf{x}^{\alpha} \cdot \mathbf{x}^{\gamma} \succ \mathbf{x}^{\beta} \cdot \mathbf{x}^{\gamma}$ when $\mathbf{x}^{\alpha} \succ \mathbf{x}^{\beta}$ for all $\mathbf{x}^{\alpha}, \mathbf{x}^{\beta}, \mathbf{x}^{\gamma} \in [x]$. Then \succ is a well-ordering on [x] if and only if $\mathbf{x}^{\gamma} \succ 1$ for all $\gamma \in \mathbb{N}^n$.

Proof. Suppose that \succ is a well-ordering. Then [x] has the smallest element which we denot as x^{β_0} . If $1 \succ x^{\beta_0}$, then $x^{\beta_0} \succ x^{2\beta_0}$, contradicting the minimality of x^{β_0} . Hence follows the necessity of the conclusion.

Suppose that $A \subset [x]$ is nonempty. To prove the sufficiency, it suffices to prove that A has a minimal element in terms of the ordering \succ . Let K be a nontrivial field and $\langle A \rangle := \langle \{ \boldsymbol{x}^{\alpha} \colon \alpha \in A \} \rangle$ the monomial ideal generated by A in $K[\boldsymbol{x}]$. As per Hilbert Basis Theorem, $\langle A \rangle$ has a finite basis as $\langle A \rangle = \langle \boldsymbol{x}^{\alpha_1}, \boldsymbol{x}^{\alpha_2}, \dots, \boldsymbol{x}^{\alpha_s} \rangle$. Since \succ is a total ordering, we relabel the subscripts such that $\boldsymbol{x}^{\alpha_s} \succ \dots \succ \boldsymbol{x}^{\alpha_1}$. Now $\boldsymbol{x}^{\alpha_1}$ is the minimal element of A. In fact, for every $\boldsymbol{x}^{\alpha} \in \langle A \rangle$, according to Lemma 2.1, \boldsymbol{x}^{α} is divisible by one of $\{\boldsymbol{x}^{\alpha_j}\}$ for $1 \leq j \leq s$. Assume that \boldsymbol{x}^{α} is divisible by $\boldsymbol{x}^{\alpha_{j_0}}$, i.e., $\boldsymbol{x}^{\alpha} = \boldsymbol{x}^{\alpha_{j_0}} \cdot \boldsymbol{x}^{\gamma}$ for some $\gamma \in \mathbb{N}^n$. Then $\boldsymbol{x}^{\gamma} \succeq 1$ indicates that $\boldsymbol{x}^{\alpha} \succeq \boldsymbol{x}^{\alpha_{j_0}} \succeq \boldsymbol{x}^{\alpha_1}$.

Definition 2.3. A monomial ordering on [x] is a well-ordering on [x] such that $\mathbf{x}^{\alpha} \cdot \mathbf{x}^{\gamma} \succ \mathbf{x}^{\beta} \cdot \mathbf{x}^{\gamma}$ when $\mathbf{x}^{\alpha} \succ \mathbf{x}^{\beta}$ for all $\mathbf{x}^{\alpha}, \mathbf{x}^{\beta}, \mathbf{x}^{\gamma} \in [x]$. In particular, we have $\mathbf{x}^{\gamma} \succeq 1$ for all $\gamma \in \mathbb{N}^n$.

Notation 2.4. Let R be a PID and $f = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha}$ a polynomial in $R[\mathbf{x}]$. Let \succ be a monomial ordering. We denote the support of f as $\operatorname{supp}(f) := \{\mathbf{x}^{\alpha} \in [\mathbf{x}] : c_{\alpha} \neq 0\} \subset [\mathbf{x}]$. In particular, we define $\operatorname{supp}(f) := \{1\}$ when $f \in R^*$ and $\operatorname{supp}(f) := \emptyset$ when f = 0.

If f has a term $c_{\beta} \boldsymbol{x}^{\beta}$ that satisfies $\boldsymbol{x}^{\beta} := \max_{\succ} \{\boldsymbol{x}^{\alpha} \in \operatorname{supp}(f)\}$, then we use the following terminologies hereafter. The *leading term* of f is denoted as $\operatorname{LT}(f) := c_{\beta} \boldsymbol{x}^{\beta}$; The *leading monomial*¹ of f is denoted as $\operatorname{LM}(f) := \boldsymbol{x}^{\beta}$; The *leading coefficient* of f is denoted as $\operatorname{LC}(f) := c_{\beta} \in R^*$.

Let $B = \{b_j : 1 \leq j \leq s\}$ be a polynomial set in $R[x] \setminus \{0\}$. We denote the leading monomial set $\{LM(b_j) : 1 \leq j \leq s\}$ as LM(B). Let us also denote the monomial ideal generated by LM(B) in R[x] as $\langle LM(B) \rangle$.

In what follows we use gcd(a, b) and lcm(a, b) to denote the greatest common divisor and least common multiple of $a, b \in R^*$ respectively over a PID R.

Definition 2.5 (Term pseudo-reduction in R[x] over a PID R).

Let R be a PID and \succ a monomial ordering on [x]. For $f \in R[x] \setminus R$ and $g \in R[x] \setminus \{0\}$, suppose that f has a term $c_{\alpha}x^{\alpha}$ such that $x^{\alpha} \in \text{supp}(f) \cap \langle \text{LM}(g) \rangle$. Then we can make a *pseudo-reduction* of the term $c_{\alpha}x^{\alpha}$ of f by g as follows.

$$h = \mu f - \frac{m \mathbf{x}^{\alpha}}{\mathrm{LT}(q)} g \tag{2.1}$$

with the multipliers $m := \text{lcm}(c_{\alpha}, \text{LC}(g))$ and $\mu := m/c_{\alpha} \in \mathbb{R}^*$. We call h the remainder of the pseudo-reduction and μ the interim multiplier on f with respect to g.

¹It is also called the "leading power product" in the literature. Here we adopt the convention that is consistent with the terminology of "monomial ideals".

Definition 2.6 (Pseudo-reduced polynomial).

Let R be a PID and \succ a monomial ordering on [x]. A polynomial $r \in R[x]$ is pseudo-reduced with respect to a polynomial set $B = \{b_j : 1 \le j \le s\} \subset R[x] \setminus R$ if $\operatorname{supp}(r) \cap \langle \operatorname{LM}(B) \rangle = \emptyset$. In particular, this includes the special case when r = 0 and hence $\operatorname{supp}(r) = \emptyset$. We also say that r is pseudo-reducible with respect to B if it is not pseudo-reduced with respect to B, i.e., $\operatorname{supp}(r) \cap \langle \operatorname{LM}(B) \rangle \neq \emptyset$.

Theorem 2.7 (Pseudo-division in R[x] over a PID R).

Let R be a PID and \succ a monomial ordering on [x]. Suppose that $B = \{b_j : 1 \le j \le s\} \subset R[x] \setminus R$ is a polynomial set. For every $f \in R[x]$, there exist a multiplier $\lambda \in R^*$ as well as a remainder $r \in R[x]$ and quotients $q_j \in R[x]$ for $1 \le j \le s$ such that

$$\lambda f = \sum_{j=1}^{s} q_j b_j + r, \tag{2.2}$$

where r is pseudo-reduced with respect to G. Moreover, the polynomials in (2.2) satisfy the following condition:

$$LM(f) = \max\{\max_{1 \le j \le s} \{LM(q_j b_j)\}, LM(r)\}.$$

$$(2.3)$$

Proof. If f is already pseudo-reduced with respect to B, we just take r = f and $q_j = 0$ for $1 \le j \le s$. Otherwise we define $\boldsymbol{x}^{\alpha} := \max_{\succ} \{ \sup(f) \cap \langle \operatorname{LM}(B) \rangle \}$. There exists some j such that \boldsymbol{x}^{α} is divisible by $\operatorname{LM}(b_j)$ as per Lemma 2.1 (i). We make a pseudo-reduction of the term $c_{\alpha}\boldsymbol{x}^{\alpha}$ of f by b_j in the same way as the term pseudo-reduction in (2.1). We denote the remainder also as h and $\boldsymbol{x}^{\beta} := \max_{\succ} \{ \sup(h) \cap \langle \operatorname{LM}(B) \rangle \}$ if h is not pseudo-reduced with respect to B. It is easy to see that $\boldsymbol{x}^{\alpha} \succ \boldsymbol{x}^{\beta}$ after the above term pseudo-reduction. We repeat such term pseudo-reductions until the remainder h is pseudo-reduced with respect to B. Since the monomial ordering \succ is a well-ordering by Definition 2.3, the term pseudo-reductions terminate in finite steps. Hence follows the representation (2.2) in which the multiplier $\lambda \in R^*$ is a product of such interim multipliers μ as in (2.1).

To prove the equality in (2.3), it suffices to prove it for the term pseudo-reduction in (2.1). In fact, the pseudo-division in (2.2) is just a composition of the term pseudo-reductions in (2.1) and the remainder h in (2.1) shall eventually become the remainder r in (2.2). In (2.1) the leading monomial of $m\mathbf{x}^{\alpha}g/\text{LT}(g)$ is \mathbf{x}^{α} . Hence either $\text{LM}(f) = \mathbf{x}^{\alpha}$, or $\text{LM}(f) \succ \mathbf{x}^{\alpha}$ in which case LM(f) = LM(h) in (2.1). Thus follows the equality in (2.3).

Definition 2.8. Let R be a PID and $f \in R[x]$. Suppose that $B = \{b_j : 1 \le j \le s\} \subset R[x] \setminus R$ is a polynomial set over R. We call the expression in (2.2) a pseudo-division of f by B. More specifically, we name the polynomial r in (2.2) as a remainder of f on pseudo-division by B and $\lambda \in R^*$ in (2.2) a multiplier of the pseudo-division. We say that f pseudo-reduces to the remainder r via the multiplier $\lambda \in R^*$ modulo B. We also call it a pseudo-reduction of f by B.

The proof of Theorem 2.7 shows that the multiplier λ in (2.2) is a finite product of the interim multipliers μ as in (2.1). Based on the proof of Theorem 2.7 we can easily contrive a pseudo-division algorithm. We do not elaborate on it here since it is quite straightforward.

Remark 2.9. There is a difference between the above pseudo-division algorithm and the traditional division algorithm over a PID. In the traditional one as in [AL94, P207, Algorithm 4.1.1], it is required that the linear equation $LC(f) = \sum_{j=1}^{s} b_j \cdot LC(f_j)$ as in [AL94, P204, (4.1.1)] be solvable for b_j 's over R. The pseudo-division algorithm does not have this extra requirement. Their major difference is the multiplier $\lambda \in R^*$ in (2.2).

3 Pseudo-eliminants of Zero-dimensional Ideals

Let K be a field and \boldsymbol{x} denote variables (x_1, \ldots, x_n) as before. In this section let us consider the case when the PID R in Section 2 bears the particular form $R = K[x_1]$ with x_1 being the first variable of \boldsymbol{x} . In this case the polynomials in the algebra $K[\boldsymbol{x}]$ over K can be viewed as those in $(K[x_1])[\tilde{\boldsymbol{x}}]$ over $K[x_1]$ with the variables $\tilde{\boldsymbol{x}} = (x_2, \ldots, x_n)$ and coefficients in $K[x_1]$. It is evident that the pseudo-division of polynomials in Theorem 2.7 applies here without any essential change.

Unless specified, in what follows we shall always treat the algebra K[x] over K as the algebra $(K[x_1])[\tilde{x}]$ over $K[x_1]$. Hence for $f \in (K[x_1])[\tilde{x}]$ in this section, its leading coefficient LC(f) and leading monomial LM(f) in Notation 2.4 now satisfy $LC(f) \in (K[x_1])^*$ and $LM(f) \in [\tilde{x}]$ respectively. Here $[\tilde{x}]$ denotes the set of nonzero monomials in the variables $\tilde{x} = (x_2, \ldots, x_n)$. Moreover, we use (f) to denote a principal ideal in $K[x_1]$ generated by $f \in K[x_1]$. Recall that $\langle f \rangle$ denotes a principal ideal in $(K[x_1])[\tilde{x}] = K[x]$ generated by either $f \in K[x_1]$ or $f \in (K[x_1])[\tilde{x}]$.

Definition 3.1 (Elimination ordering² on $(K[x_1])[\tilde{\boldsymbol{x}}]$).

An elimination ordering on $(K[x_1])[\tilde{x}]$ is a monomial ordering on [x] such that the \tilde{x} variables are always larger than the x_1 variable. That is, $x_1^{\alpha}\tilde{x}^{\gamma} \succ x_1^{\beta}\tilde{x}^{\delta}$ if and only if $\tilde{x}^{\gamma} \succ \tilde{x}^{\delta}$ or, $\tilde{x}^{\gamma} = \tilde{x}^{\delta}$ and $\alpha > \beta$.

In what follows let us suppose that I is a zero-dimensional ideal³ of $K[\mathbf{x}] = (K[x_1])[\tilde{\mathbf{x}}]$. We have the following well-known conclusion.

Proposition 3.2. For a zero-dimensional ideal $I \subset (K[x_1])[\tilde{x}]$, we always have $I \cap K[x_1] \neq \{0\}$.

Proof. Please refer to [BW93, P272, Lemma 6.50] or [KR00, P243, Proposition 3.7.1(c)].

Definition 3.3 (Eliminant).

For a zero-dimensional ideal $I \subset (K[x_1])[\tilde{\boldsymbol{x}}]$, the principal ideal $I \cap K[x_1]$ in $K[x_1]$ is called the *elimination ideal* of I. Let us denote its generator as χ that satisfies $I \cap K[x_1] = (\chi)$ being a principal ideal in $K[x_1]$. We call χ the *eliminant* of the zero-dimensional ideal I henceforth.

In what follows let us elaborate on a revised version of Buchberger's algorithm. The purpose is to compute not only a pseudo-basis but also a pseudo-eliminant of

²Please also refer to [AL94, P69, Definition 2.3.1] and [EH12, P33, Definition 3.1].

³Please note that a zero-dimensional ideal I is always a proper ideal such that $I \neq K[x]$.

the elimination ideal $I \cap K[x_1]$. Let us first recall the S-polynomial over a PID as in the following definition⁴.

Definition 3.4 (S-polynomial).

Suppose that $f, g \in (K[x_1])[\tilde{\boldsymbol{x}}] \setminus K[x_1]$. Let us denote $m := \operatorname{lcm}(\operatorname{LC}(f), \operatorname{LC}(g)) \in K[x_1]$ and $\tilde{\boldsymbol{x}}^{\gamma} := \operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g)) \in [\tilde{\boldsymbol{x}}]$. Then the polynomial

$$S(f,g) := \frac{m\tilde{\boldsymbol{x}}^{\gamma}}{\operatorname{LT}(f)} f - \frac{m\tilde{\boldsymbol{x}}^{\gamma}}{\operatorname{LT}(g)} g \tag{3.1}$$

is called the S-polynomial of f and g.

It is easy to verify that the S-polynomial satisfies the following inequality due to the cancellation of leading terms in (3.1):

$$LM(S(f,g)) \prec \tilde{x}^{\gamma} = lcm(LM(f), LM(g)). \tag{3.2}$$

When $g \in (K[x_1])^*$ and $f \in (K[x_1])[\tilde{x}] \setminus K[x_1]$, we take LM(g) = 1 and m = lcm(LC(f), g). The S-polynomial in (3.1) becomes:

$$S(f,g) := \frac{m}{\operatorname{LC}(f)} f - m \cdot \operatorname{LM}(f). \tag{3.3}$$

Lemma 3.5. When $g \in (K[x_1])^*$ and $f \in (K[x_1])[\tilde{x}] \setminus K[x_1]$, the S-polynomial in (3.3) satisfies:

$$dS(f,g) = (f - \operatorname{LT}(f)) \cdot g := f_1 g \tag{3.4}$$

with $d := \gcd(LC(f), g) \in K[x_1]$ and $f_1 := f - LT(f)$.

Proof. Let us denote $l_f := LC(f)$. Based on the equality $m/l_f = g/d$, the S-polynomial in (3.3) becomes:

$$S(f,g) = \frac{gf}{d} - \frac{gl_f}{d} \cdot LM(f) = \frac{g}{d}(f - LT(f)) = \frac{f_1g}{d}.$$

Lemma 3.5 can be generalized to the following conclusion:

Lemma 3.6. For $f, g \in (K[x_1])[\tilde{x}] \setminus K[x_1]$, suppose that LM(f) and LM(g) are relatively prime. Let us denote d := gcd(LC(f), LC(g)). Then their S-polynomial in (3.1) satisfies:

$$dS(f,g) = f_1 g - g_1 f (3.5)$$

with $f_1 := f - LT(f)$ and $g_1 := g - LT(g)$. Moreover, we have:

$$LM(S(f,g)) = \max\{LM(f_1g), LM(g_1f)\}.$$
(3.6)

Proof. If LM(f) and LM(g) are relatively prime, then we have an identity $\tilde{\boldsymbol{x}}^{\gamma} = \text{LM}(f) \cdot \text{LM}(g)$ in (3.1). For convenience, let us denote $l_f := \text{LC}(f)$, $l_g := \text{LC}(g)$ and $d := \gcd(l_f, l_g)$. Then we have the identities $m/l_f = l_g/d$ and $m/l_g = l_f/d$ with $m = \text{lcm}(l_f, l_g)$ as in (3.1). We substitute these identities into (3.1) to obtain:

$$S(f,g) := \frac{l_g \cdot \text{LM}(g)}{d} f - \frac{l_f \cdot \text{LM}(f)}{d} g = \frac{1}{d} (\text{LT}(g) \cdot f - \text{LT}(f) \cdot g)$$

$$= \frac{1}{d} ((g - g_1)f - (f - f_1)g) = \frac{1}{d} (f_1g - g_1f)$$

$$= \frac{1}{d} (f_1(g_1 + \text{LT}(g)) - g_1(f_1 + \text{LT}(f))) = \frac{1}{d} (f_1 \cdot \text{LT}(g) - g_1 \cdot \text{LT}(f)). \quad (3.8)$$

⁴Please also refer to [AL94, P249, (4.5.1)] or [BW93, P457, Definition 10.9].

The identity (3.5) follows from (3.7). Now we show that the conclusion (3.6) is a consequence of the expression (3.8). In fact, for every term $c_{\alpha}\tilde{\boldsymbol{x}}^{\alpha}$ of f_1 and every term $c_{\beta}\tilde{\boldsymbol{x}}^{\beta}$ of g_1 , we have $\tilde{\boldsymbol{x}}^{\alpha} \cdot \text{LM}(g) \neq \tilde{\boldsymbol{x}}^{\beta} \cdot \text{LM}(f)$ since LM(g) and LM(f) are relatively prime and moreover, we have $\tilde{\boldsymbol{x}}^{\alpha} \prec \text{LM}(f)$ and $\tilde{\boldsymbol{x}}^{\beta} \prec \text{LM}(g)$. As a result, no term of $f_1 \cdot \text{LT}(g)$ cancels no term of $g_1 \cdot \text{LT}(f)$ in (3.8). Thus follows the equality (3.6).

Corollary 3.7. Suppose that LM(f) and LM(g) are relatively prime for $f, g \in (K[x_1])[\tilde{x}] \setminus K[x_1]$. Then their S-polynomial S(f,g) as in (3.1) can be pseudoreduced to 0 by f and g as in Theorem 2.7 with the multiplier $\lambda = d$ and quotients f_1 and g_1 as in (3.5).

In particular, for $g \in (K[x_1])^*$ and $f \in (K[x_1])[\tilde{x}] \setminus K[x_1]$, their S-polynomial S(f,g) in (3.3) can be pseudo-reduced to 0 by g with the multiplier $\lambda = d$ and quotient f_1 as in (3.4).

Proof. The conclusions readily follow from Lemma 3.6 and Lemma 3.5 based on Theorem 2.7.

For two terms $c_{\alpha}\tilde{\boldsymbol{x}}^{\alpha}, c_{\beta}\tilde{\boldsymbol{x}}^{\beta} \in (K[x_1])[\tilde{\boldsymbol{x}}]$ with $c_{\alpha}, c_{\beta} \in K[x_1]$, let us denote $\operatorname{lcm}(c_{\alpha}\tilde{\boldsymbol{x}}^{\alpha}, c_{\beta}\tilde{\boldsymbol{x}}^{\beta}) := \operatorname{lcm}(c_{\alpha}, c_{\beta}) \cdot \operatorname{lcm}(\tilde{\boldsymbol{x}}^{\alpha}, \tilde{\boldsymbol{x}}^{\beta})$. Then we have the following notation:

$$\operatorname{lcm}(\operatorname{LT}(f),\operatorname{LT}(g)) := \operatorname{lcm}(\operatorname{LC}(f),\operatorname{LC}(g)) \cdot \operatorname{lcm}(\operatorname{LM}(f),\operatorname{LM}(g)).$$

Lemma 3.8. For $f, g, h \in (K[x_1])[\tilde{x}] \setminus K[x_1]$, if $lcm(LM(f), LM(g)) \in \langle LM(h) \rangle$, then we have the following triangular relationship among their S-polynomials:

$$\lambda S(f,g) = \frac{\lambda \cdot \operatorname{lcm}(\operatorname{LT}(f), \operatorname{LT}(g))}{\operatorname{lcm}(\operatorname{LT}(f), \operatorname{LT}(h))} S(f,h) - \frac{\lambda \cdot \operatorname{lcm}(\operatorname{LT}(f), \operatorname{LT}(g))}{\operatorname{lcm}(\operatorname{LT}(h), \operatorname{LT}(g))} S(g,h), \quad (3.9)$$

where the multiplier $\lambda := LC(h)/d$ with $d := gcd(lcm(LC(f), LC(g)), LC(h)) \in K[x_1]$. Henceforth we also call the identity (3.9) the triangular identity of S(f,g) with respect to h.

Proof. From $lcm(LM(f), LM(g)) \in \langle LM(h) \rangle$ we can easily deduce that:

$$\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g)) \in \langle \operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(h)) \rangle \cap \langle \operatorname{lcm}(\operatorname{LM}(h), \operatorname{LM}(g)) \rangle.$$

In order to corroborate that the multiplier $\lambda = LC(h)/d$ suffices to make the two fractions in (3.9) terms in $(K[x_1])[\tilde{x}]$, we only need to consider the case when $\operatorname{mult}_p(LC(h)) > \gamma := \max\{\operatorname{mult}_p(LC(f)), \operatorname{mult}_p(LC(g))\}$ for every irreducible factor p of LC(h). In this case we have $\operatorname{mult}_p(\operatorname{lcm}(LC(f), LC(h))) = \operatorname{mult}_p(\operatorname{lcm}(LC(h), LC(g)))$ in the denominators of (3.9). Hence in the numerators of (3.9) we can take $\operatorname{mult}_p(\lambda) = \operatorname{mult}_p(LC(h)) - \gamma = \operatorname{mult}_p(LC(h)) - \operatorname{mult}_p(d)$. Now let us write the definition of S-polynomial in (3.1) into the following form:

$$S(f,g) = \frac{\operatorname{lcm}(\operatorname{LT}(f), \operatorname{LT}(g))}{\operatorname{LT}(f)} f - \frac{\operatorname{lcm}(\operatorname{LT}(f), \operatorname{LT}(g))}{\operatorname{LT}(g)} g. \tag{3.10}$$

Then the identity (3.9) readily follows if we also write S(f,h) and S(g,h) into the form of (3.10).

It is easy to verify that the identity (3.9) is consistent with the inequality (3.2) for S-polynomials since from (3.9) we can deduce that LM(S(f,g)) is dominated by one of the following leading monomials:

$$\frac{\operatorname{lcm}(\operatorname{LM}(f),\operatorname{LM}(g))}{\operatorname{lcm}(\operatorname{LM}(f),\operatorname{LM}(h))}\operatorname{LM}(S(f,h)), \text{ or } \frac{\operatorname{lcm}(\operatorname{LM}(f),\operatorname{LM}(g))}{\operatorname{lcm}(\operatorname{LM}(h),\operatorname{LM}(g))}\operatorname{LM}(S(g,h)).$$

For $f \in (K[x_1])[\tilde{x}] \setminus K[x_1]$ and $g \in (K[x_1])^*$, we shall use the following relation between (3.1) and (3.3):

$$S(f, g \cdot LM(f)) = S(f, g). \tag{3.11}$$

Moreover, the S-polynomial in (3.3) coincides with the term pseudo-reduction in (2.1) for $g \in R^* = (K[x_1])^*$.

Algorithm 3.9 (Pseudo-eliminant of a zero-dimensional ideal over a PID).

Input: A finite polynomial set $F \subset (K[x_1])[\tilde{x}] \setminus K$.

Output: A pseudo-eliminant $\chi_{\varepsilon} \in (K[x_1])^*$, pseudo-basis $B_{\varepsilon} \subset \langle F \rangle \setminus K[x_1]$ and multiplier set $\Lambda \subset K[x_1] \setminus K$.

Initialization: A temporary basis set $G := F \setminus K[x_1]$; a multiplier set $\Lambda := \emptyset$ in $K[x_1]$; a temporary set $\mathfrak{S} := \emptyset$ in $(K[x_1])[\tilde{x}] \setminus K$ for S-polynomials. If $F \cap K[x_1] \neq \emptyset$, we initialize $f_0 := \gcd(F \cap K[x_1])$; otherwise we initialize $f_0 := 0$.

For each pair $f, g \in G$ with $f \neq g$, we invoke Procedure Q as follows to compute their S-polynomial S(f, g).

Procedure Q:

Input: $f, g \in (K[x_1])[\tilde{\boldsymbol{x}}] \setminus K[x_1]$.

If LM(f) and LM(g) are relatively prime, we define d := gcd(LC(f), LC(g)) as in (3.5). If $d \in K[x_1] \setminus K$, we add d into the multiplier set Λ . Then we disregard the S-polynomial S(f,g).

If there exists an $h \in G \setminus \{f, g\}$ such that $\operatorname{lcm}(LM(f), LM(g)) \in \langle LM(h) \rangle$, and the triangular identity (3.9) has never been applied to the same triplet $\{f, g, h\}$, we compute the multiplier λ as in (3.9). If $\lambda \in K[x_1] \setminus K$, we add λ into the multiplier set Λ . Then we disregard the S-polynomial S(f, g).

If neither of the above two cases is true, we compute their S-polynomial S(f,g) as in (3.1). Then we add S(f,g) into the set \mathfrak{S} .

End of Q

We recursively repeat Procedure \mathcal{P} as follows for the pseudo-reductions of all the S-polynomials in the set \mathfrak{S} .

Procedure \mathcal{P} :

For an $S \in \mathfrak{S}$, we invoke Theorem 2.7 to make a pseudo-reduction of S by the temporary basis set G.

If the multiplier $\lambda \in K[x_1] \setminus K$ in (2.2), we add λ into the multiplier set Λ . If the remainder $r \in (K[x_1])[\tilde{x}] \setminus K[x_1]$, we add r into G. For every $f \in G \setminus \{r\}$, we invoke Procedure Q to compute the S-polynomial S(f,r).

If the remainder $r \in K[x_1] \setminus K$, we redefine $f_0 := \gcd(r, f_0)$.

If the remainder $r \in K^*$, we halt the algorithm and output $G = \{1\}$.

Then we delete S from the set \mathfrak{S} .

End of \mathcal{P}

Finally we define $\chi_{\varepsilon} := f_0$ and $B_{\varepsilon} := G$ respectively.

Procedure \mathcal{R} :

For every $f \in B_{\varepsilon}$, if $d := \gcd(LC(f), \chi_{\varepsilon}) \in K[x_1] \setminus K$, we add d into the multiplier set Λ .

End of \mathcal{R}

We output χ_{ε} , B_{ε} and Λ .

Remark 3.10. In Algorithm 3.9 we compute both $d := \gcd(LC(f), LC(g))$ when LM(f) and LM(g) are relatively prime in Procedure \mathcal{Q} and $d := \gcd(LC(f), \chi_{\varepsilon})$ for every $f \in B_{\varepsilon}$ in Procedure \mathcal{R} . The purpose of these computations is to procure the multipliers d in (3.5) of Lemma 3.6 and (3.4) of Lemma 3.5 respectively for the pseudo-reductions of the S-polynomials. It is the reason why we add d into the multiplier set Λ when $d \in K[x_1] \setminus K$.

Moreover, in Procedure \mathcal{Q} the condition that there exists an $h \in G \setminus \{f, g\}$ such that $\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g)) \in \langle \operatorname{LM}(h) \rangle$ is a condition for Lemma 3.8.

Definition 3.11 (Pseudo-eliminant; pseudo-basis; multiplier set).

We call the univariate polynomial χ_{ε} obtained via Algorithm 3.9 a pseudoeliminant of the zero-dimensional ideal I. We also call the polynomial set B_{ε} a pseudo-basis of I and Λ its multiplier set.

Please note that contrary to the convention, we do not include the pseudoeliminant χ_{ε} in the pseudo-basis B_{ε} since we shall contrive modular algorithms to compute modular bases with the factors of χ_{ε} as moduli in Section 5.

- **Lemma 3.12.** (i) A pseudo-eliminant χ_{ε} of a zero-dimensional ideal I is divisible by its eliminant χ . (ii) For each pair $f \neq g$ in the union set of pseudo-basis and pseudo-eliminant $B_{\varepsilon} \cup \{\chi_{\varepsilon}\}$, the pseudo-reduction of their S-polynomial S(f,g) by B_{ε} yields a remainder $r \in (\chi_{\varepsilon})$ in $K[x_1]$. In particular, this includes the case when r = 0. (iii) Algorithm 3.9 terminates in a finite number of steps.
- *Proof.* (i) The conclusion readily follows from the fact that $\chi_{\varepsilon} \in I \cap K[x_1] = (\chi)$.
- (ii) According to Procedure \mathcal{P} in Algorithm 3.9, if the remainder r of the pseudo-reduction by an intermediate polynomial set G satisfies $r \in (K[x_1])[\tilde{x}] \setminus K[x_1]$, we add it into G. That is, r eventually becomes an element of the pseudo-basis B_{ε} . It is evident that a pseudo-reduction of r by itself per se leads to the zero remainder. On the other hand, if $r \in K[x_1] \setminus K$, then as per $f_0 := \gcd(r, f_0)$ in Procedure \mathcal{P} of Algorithm 3.9, r is divisible by f_0 and hence by the pseudo-eliminant χ_{ε} , i.e., $r \in (\chi_{\varepsilon})$.
- (iii) The termination of the algorithm follows from the ring $(K[x_1])[\tilde{\boldsymbol{x}}] = K[\boldsymbol{x}]$ being Noetherian. In fact, whenever the remainder $r \in (K[x_1])[\tilde{\boldsymbol{x}}] \setminus K[x_1]$ in the Procedure \mathcal{P} of the algorithm, we add it to the intermediate polynomial set G. As a result, the monomial ideal $\langle LM(G) \rangle$ is strictly expanded since r is pseudoreduced with respect to $G \setminus \{r\}$. Hence the ascending chain condition imposed on the chain of ideals $\langle LM(G) \rangle$ ensures the termination of the algorithm.

Based on Lemma 3.12 (i), the following conclusion is immediate:

Corollary 3.13. If a pseudo-eliminant χ_{ε} of a zero-dimensional ideal I in K[x] satisfies $\chi_{\varepsilon} \in K^*$, then the reduced Gröbner basis⁵ of I is $\{1\}$.

In what follows we assume that the ideal I is a proper ideal of K[x], that is, $I \neq K[x]$. Thus let us exclude the trivial case of $\chi_{\varepsilon} \in K^*$ hereafter.

⁵Please refer to [AL94, P48, Definition 1.8.5] or [CLO15, P93, Definition 4] for a definition.

4 Pseudo-eliminant Divisors and Compatibility

Suppose that K is a perfect field whose characteristic is denoted as $\operatorname{char}(K)$. Recall that finite fields and fields of characteristic zero are perfect fields. In this section we begin to contrive an algorithm retrieving the eliminant χ of a zero-dimensional ideal I from its pseudo-eliminant χ_{ε} obtained in Algorithm 3.9. We first make a factorization of the pseudo-eliminant χ_{ε} into compatible and incompatible divisors. We prove that the compatible divisors of χ_{ε} are the authentic factors of χ . This shows that Algorithm 3.9 generates the eliminant χ when the pseudo-eliminant χ_{ε} is compatible. We compute the factors of χ that correspond to the incompatible divisors of χ_{ε} in Section 5.

Definition 4.1 (Squarefree factorization of univariate polynomials).

A univariate polynomial $f \in K[x] \setminus K$ is squarefree if it has no quadratic factors in $K[x] \setminus K$. That is, for every irreducible polynomial $g \in K[x] \setminus K$, f is not divisible by g^2 .

The squarefree factorization of a univariate polynomial $f \in K[x] \setminus K$ refers to a product $f = \prod_{i=1}^s g_i^i$ with $g_s \in K[x] \setminus K$ such that for those g_i 's that are not constants, they are both squarefree and pairwise relatively prime. Moreover, the squarefree part of f is defined as $\prod_{i=1}^s g_i$.

The squarefree factorization is unique up to multiplications by constants in K^* . Its existence and uniqueness follow from the fact that K[x] is a PID and hence a unique factorization integral domain. There are various algorithms for squarefree factorization depending on the field K being finite or of characteristic zero. Algorithm 4.2 as follows amalgamates these two cases of field characteristics. We improve the algorithm in [GP08, P539, Algorithm B.1.6] over a finite field and then apply it to the squarefree factorization over a filed of characteristic zero.

Consider the integer set $J := \mathbb{N}^*$ when $\operatorname{char}(K) = 0$, and $J := \mathbb{N} \setminus p\mathbb{N}$ when $\operatorname{char}(K) = p > 0$. That is, J stands for the set of positive integers that are not a multiple of p when $\operatorname{char}(K) = p > 0$. Let us enumerate the positive integers in J by the bijective enumeration map $\rho \colon \mathbb{N}^* \to J$ such that $\rho(i) < \rho(j)$ whenever i < j. We have the evident inequality $\rho(i) \geq i$ when $\operatorname{char}(K) = p > 0$. When $\operatorname{char}(K) = 0$, the enumeration map ρ is the identity map. In the algorithm below, for every $i \in \mathbb{N}^*$ we simply denote its image $\rho(i) \in J$ as [i], i.e., $\rho(i) = [i]$.

Algorithm 4.2 (Squarefree factorization of a univariate polynomial).

Input: A univariate polynomial $f \in K[x] \setminus K$.

Output: The squarefree decomposition $\{g_1, \ldots, g_s\}$ of f.

Procedure \mathcal{P} :

If $f' \neq 0$, we compute the greatest common divisor $f_{[1]} := \gcd(f, f')$ first. The squarefree part of f is defined as $h_{[1]} := f/f_{[1]}$.

We repeat the following procedure⁶ starting with i = 1 until i = s such that $f'_{[s]} = 0$:

$$\begin{array}{ll} h_{[i+1]} := \gcd(f_{[i]}, h_{[i]}); & \begin{cases} f_{[i+1]} := f_{[i]}/h_{[i+1]}^{[i+1]-[i]} \ if \ \mathrm{char}(K) > 0; \\ g_{[i]} := h_{[i]}/h_{[i+1]}. & if \ \mathrm{char}(K) = 0. \end{cases}$$

⁶If char(K) > 0, the exponent [i+1]-[i] of $h_{[i+1]}$ in the following definition of $f_{[i+1]}$ is the improvement on [GP08, P539, Algorithm B.1.6].

If $f_{[s]} \in K$, we define $g_{[s]} := h_{[s]}$ to obtain the squarefree factorization $f = \prod_{i=1}^s g_{[i]}^{[i]}$.

If $\operatorname{char}(K) = p > 0$ and $f_{[s]} \in K[x] \setminus K$, we invoke Procedure Q on $f_{[s]}$. End of P

Procedure Q:

If $\operatorname{char}(K) = p > 0$ and $f \in K[x] \setminus K$ satisfies f' = 0, we repeat the following procedure starting with $x_1 := x$ and $\psi_1 := f$ until i = t such that $\psi'_t \neq 0$:

$$x_{i+1} := x_i^p; \qquad \psi_{i+1}(x_{i+1}) := \psi_i(x_i)$$

We treat ψ_t as f and invoke Procedure \mathcal{P} on ψ_t . End of \mathcal{Q}

Procedure Q in Algorithm 4.2 is a composition of Frobenius automorphism.

Proposition 4.3. We can procure a squarefree factorization of f in finite steps via Algorithm 4.2.

Proof. The termination of the algorithm in finite steps readily follows from the fact that $\deg f_{[i+1]} < \deg f_{[i]}$ in Procedure \mathcal{P} as well as $\deg \psi_{i+1} < \deg \psi_i$ in Procedure \mathcal{Q} .

In the case of $\operatorname{char}(K)=0$, the bijective map $\rho\colon\mathbb{N}^*\to J$ is an identity map such that $\rho(i)=[i]=i$. To illustrate the procedure of the algorithm, suppose that $f=\prod_{i=1}^s g_i^i$ is a squarefree factorization of f. Then $h_1=\prod_{i=1}^s g_i$ is the squarefree part of f obtained in the beginning of Procedure \mathcal{P} . Moreover, the h_i in Procedure \mathcal{P} equals $\prod_{j=i}^s g_j$ for $1\leq i\leq s$. Hence we have $g_i=h_i/h_{i+1}$. Further, f_i equals $\prod_{j=i+1}^s g_j^{j-i}$ for $1\leq i< s$. Finally $f_s=1$ and Procedure \mathcal{P} terminates since $f_s'=0$. Thus follows the squarefree factorization.

In the case when $\operatorname{char}(K) = p > 0$, suppose that $f = \prod_{k \in p\mathbb{N}} g_k^k \cdot \prod_{i=1}^s g_{[i]}^{[i]}$. Let us denote $\varphi_p := \prod_{k \in p\mathbb{N}} g_k^k$ and $\varphi_q := \prod_{i=1}^s g_{[i]}^{[i]}$ such that $f = \varphi_p \varphi_q$. Then the $f_{[1]}$ in Procedure \mathcal{P} equals $\varphi_p \cdot \prod_{i=2}^s g_{[i]}^{[i]-1}$. Hence $h_{[1]} = \prod_{i=1}^s g_{[i]}$ is the squarefree part of φ_q . And the $h_{[i]}$ equals $\prod_{j=i}^s g_{[j]}$ for $1 \le i \le s$. Hence we have $g_{[i]} = h_{[i]}/h_{[i+1]}$. Moreover, $f_{[i]}$ equals $\varphi_p \cdot \prod_{j=i+1}^s g_{[j]}^{[j]-[i]}$ for $1 \le i < s$. Finally, $f_{[s]} = \varphi_p$ and thus $f'_{[s]} = 0$. Now we have a squarefree factorization of φ_q as $\prod_{i=1}^s g_{[i]}^{[i]}$.

Procedure \mathcal{Q} amounts to a variable substitution $x_t = x^{p^t}$ such that $f_{[s]}(x) = \psi_t(x_t)$. Since $\psi_t' \neq 0$, we treat ψ_t as f and assume that $\psi_t = \varphi_p \varphi_q$ with φ_p and φ_q being defined as above. We repeat Procedure \mathcal{P} on ψ_t to obtain a squarefree factorization $\varphi_q = \prod_{i=1}^s g_{[i]}^{[i]}$. Nonetheless here φ_q is in the variable x_t . Since the field K is perfect, we have $\varphi_q(x_t) = (\varphi_q(x))^{p^t} = \prod_{i=1}^s g_{[i]}^{[i]^t}$.

Remark 4.4. A remarkable thing about Algorithm 4.2 is that as long as the field K is perfect, it is independent of K and any of its field extensions. In fact, all the computations in Procedure \mathcal{P} are based on $f_{[1]} = \gcd(f, f')$ and $h_{[i+1]} := \gcd(f_{[i]}, h_{[i]})$ that are independent of the field extensions of K.

We do not attempt to make a complete factorization of a univariate polynomial due to the complexity of distinct-degree factorizations. There is a discussion on the various stages of univariate polynomial factorizations including both squarefree and distinct-degree factorizations on [GG13, P379].

Definition 4.5 (Compatible and incompatible divisors and parts).

For a zero-dimensional ideal I over a perfect field K, let χ_{ε} be a pseudoeliminant of I. Assume that Λ is the multiplier set for the pseudo-reductions of all the S-polynomials as in Algorithm 3.9. For an irreducible factor p of χ_{ε} with multiplicity i, if there exists a multiplier $\lambda \in \Lambda$ such that λ is divisible by p, then p^i is called an *incompatible divisor* of χ_{ε} . If p is relatively prime to every multiplier λ in Λ , then p^i is called a *compatible divisor* of χ_{ε} .

We name the product of all the compatible divisors of χ_{ε} as the *compatible* part of χ_{ε} and denote it as $CP(\chi_{\varepsilon})$. The *incompatible part* of χ_{ε} is defined as the product of all the incompatible divisors of χ_{ε} and denoted as $IP(\chi_{\varepsilon})$. In particular, we say that a pseudo-eliminant χ_{ε} per se is *compatible* if it has no incompatible divisors.

From the above Definition 4.5 it is evident that $\chi_{\varepsilon} = \text{CP}(\chi_{\varepsilon}) \cdot \text{IP}(\chi_{\varepsilon})$. In the following Algorithm 4.6, we compute the compatible part $\text{CP}(\chi_{\varepsilon})$ and make a squarefree decomposition of the incompatible part $\text{IP}(\chi_{\varepsilon})$ simultaneously.

Algorithm 4.6 (Compatible part $CP(\chi_{\varepsilon})$ of a pseudo-eliminant χ_{ε} and squarefree decomposition of its incompatible part $IP(\chi_{\varepsilon})$).

Input: A pseudo-eliminant $\chi_{\varepsilon} \in K[x_1]$ and multiplier set $\Lambda \subset K[x_1]$ that are obtained from Algorithm 3.9.

Output: Compatible part $CP(\chi_{\varepsilon})$ and squarefree decomposition $\{\Omega_i : 1 \leq i \leq s\}$ of the incompatible part $IP(\chi_{\varepsilon})$ with $\Omega_i \subset K[x_1]$.

We invoke Algorithm 4.2 to make a squarefree factorization $\chi_{\varepsilon} = \prod_{i=1}^{s} q_{i}^{i}$.

For each multiplicity i satisfying $1 \le i \le s$, we construct a polynomial set $\Omega_i \subset K[x_1]$ whose elements are pairwise relatively prime as follows:

For every $\lambda \in \Lambda$, we compute $d_{\lambda i} := \gcd(\lambda, q_i)$. If $d_{\lambda i} \in K[x_1] \setminus K$, we check whether $d_{\lambda i}$ is relatively prime to every element ω that is already in Ω_i . If not, we substitute $d_{\lambda i}$ by $d_{\lambda i}/\gcd(d_{\lambda i},\omega)$. And we substitute the ω in Ω_i by both $\gcd(d_{\lambda i},\omega)$ and $\omega/\gcd(d_{\lambda i},\omega)$. We repeat the process until either $d_{\lambda i} \in K$, or $d_{\lambda i}$ is relatively prime to every element in Ω_i . We add $d_{\lambda i}$ into Ω_i if $d_{\lambda i} \in K[x_1] \setminus K$.

Finally, for each multiplicity i satisfying $1 \le i \le s$, we compute the product $\omega_i := \prod_{\omega \in \Omega_i} \omega$. Then we output $\chi_{\varepsilon} / \prod_{i=1}^s \omega_i^i$ as the compatible part $\operatorname{CP}(\chi_{\varepsilon})$. We also output $\{\Omega_i \colon 1 \le i \le s\}$ as a squarefree decomposition of the incompatible part $\operatorname{IP}(\chi_{\varepsilon})$.

Definition 4.7 (Composite divisor set Ω_i ; composite divisor ω^i).

We call the univariate polynomial set Ω_i for $1 \leq i \leq s$ obtained in Algorithm 4.6 a composite divisor set of the incompatible part $IP(\chi_{\varepsilon})$ of the pseudo-eliminant χ_{ε} . For an element ω of Ω_i , we call its *i*-th power ω^i a composite divisor of the incompatible part $IP(\chi_{\varepsilon})$.

A composite divisor ω^i is a product of the incompatible divisors p^i in Definition 4.5. The incompatible part $IP(\chi_{\varepsilon})$ is the product of all the composite divisors ω^i according to the final step of Algorithm 4.6, that is:

$$IP(\chi_{\varepsilon}) = \prod_{i=1}^{s} \prod_{\omega \in \Omega_{i}} \omega^{i}. \tag{4.1}$$

The above composite divisors ω^i are pairwise relatively prime by the construction of the composite divisor set Ω_i in Algorithm 4.6.

It is evident that Algorithm 4.6 terminates in finite steps since the multiplier set Λ in Algorithm 4.6 is a finite set.

Lemma 4.8. Suppose that $F = \{f_j : 1 \leq j \leq s\} \subset (K[x_1])[\tilde{\boldsymbol{x}}] \setminus K[x_1]$ is a polynomial set. Moreover, each f_j has the same leading monomial $LM(f_j) = \tilde{\boldsymbol{x}}^{\alpha} \in [\tilde{\boldsymbol{x}}]$ for $1 \leq j \leq s$.

(i) If $f = \sum_{j=1}^{s} f_j$ satisfies $LM(f) \prec \tilde{x}^{\alpha}$, then there exist multipliers $b, b_j \in (K[x_1])^*$ for $1 \leq j < s$ such that

$$bf = \sum_{1 \le j < s} b_j S(f_j, f_s) \tag{4.2}$$

with the S-polynomial $S(f_i, f_s)$ being defined as in (3.1).

(ii) For each irreducible polynomial $p \in K[x_1] \setminus K$, we can always relabel the subscripts of the polynomial set $F = \{f_j : 1 \leq j \leq s\}$ such that the multiplier $b \in (K[x_1])^*$ of f in (4.2) is not divisible by p.

Proof. (i) Let us denote $l_j := LC(f_j)$ for $1 \le j \le s$ and the least common multiple $m_j := lcm(l_j, l_s)$ for $1 \le j < s$. From $LM(f) \prec \tilde{x}^{\alpha}$ we can deduce the following condition on the leading coefficients:

$$\sum_{j=1}^{s} l_j = 0. (4.3)$$

Now we define the multipliers in $K[x_1]$ as follows:

$$b := \lim_{1 \le j < s} \left(\frac{m_j}{l_j}\right); \qquad b_j := \frac{bl_j}{m_j} \quad (1 \le j < s)$$

$$\tag{4.4}$$

and prove that they satisfy the identity in (4.2). In fact, as per the definition of S-polynomials in (3.1) and the above definition of b_j for $1 \le j < s$, we have:

$$\sum_{1 \le j < s} b_j S(f_j, f_s) = \sum_{1 \le j < s} b_j \left(\frac{m_j f_j}{l_j} - \frac{m_j f_s}{l_s} \right) = b \sum_{1 \le j < s} f_j - f_s \sum_{1 \le j < s} \frac{b_j m_j}{l_s}$$
(4.5)

$$= b \sum_{1 \le j < s} f_j - f_s \sum_{1 \le j < s} \frac{bl_j}{l_s} = b \sum_{1 \le j \le s} f_j$$
 (4.6)

as per the condition (4.3). This proves the identity (4.2). Moreover, we should ensure that all our manipulations are over the PID $K[x_1]$. In fact, bl_j/l_s in (4.6) is in $K[x_1]$ because it equals $b_j m_j/l_s$ such that both b_j and m_j/l_s are in $K[x_1]$.

(ii) If none of $\{l_j\colon 1\leq j\leq s\}$ is a multiple of the irreducible polynomial p, the conclusion (ii) readily follows from the definition of the multiplier b in (4.4). For $1\leq j\leq s$, we denote the multiplicity of p in l_j as $\operatorname{mult}_p(l_j)\geq 0$. Let us relabel the subscripts of f_j and l_j for $1\leq j\leq s$ such that $\operatorname{mult}_p(l_s)=\min_{1\leq j\leq s}\{\operatorname{mult}_p(l_j)\}$. As a result, we have $\operatorname{mult}_p(\gcd(l_j,l_s))=\operatorname{mult}_p(l_s)$ for $1\leq j< s$. Hence $\operatorname{mult}_p(l_s/\gcd(l_j,l_s))=0$ for $1\leq j< s$. Then $\operatorname{mult}_p(m_j/l_j)=0$ since $m_j/l_j=l_s/\gcd(l_j,l_s)$ for $1\leq j< s$. Thus $b=\operatorname{lcm}_{1\leq j< s}(m_j/l_j)$ is not divisible by p.

Remark 4.9. The multiplier b in (4.2) and its associated property in Lemma 4.8 (ii) amount to a generalization of the syzygy theory for Gröbner bases over fields [AL94, P119, Prop. 3.2.3] and over PIDs [AL94, P247, Prop. 4.5.3]. In particular, Lemma 4.8 (ii) proves to be a vital property of the multiplier b for our subsequent line of reasoning in Theorem 4.10. For simplicity we do not use the language of syzygy modules in Lemma 4.8.

Theorem 4.10. Suppose that χ is the eliminant of a zero-dimensional ideal I in $(K[x_1])[\tilde{x}]$ over a perfect field K and χ_{ε} a pseudo-eliminant of I. Then χ is divisible by the compatible divisors of χ_{ε} , that is, for every compatible divisor p^i of χ_{ε} , we have $\operatorname{mult}_p(\chi_{\varepsilon}) = \operatorname{mult}_p(\chi) = i$. Hence χ is divisible by the compatible part $\operatorname{CP}(\chi_{\varepsilon})$ of χ_{ε} . In particular, $\chi = \chi_{\varepsilon}$ if χ_{ε} per se is compatible.

Proof. With $p \in K[x_1] \setminus K$ being an irreducible polynomial and $i \in \mathbb{N}^*$, let p^i be a compatible divisor of the pseudo-eliminant χ_{ε} as in Definition 4.5. We shall prove that the eliminant χ is also divisible by p^i . That is, $\operatorname{mult}_p(\operatorname{CP}(\chi_{\varepsilon})) = i \leq \operatorname{mult}_p(\chi)$. Thus as per Lemma 3.12 (i) we have:

$$\operatorname{mult}_p(\chi_{\varepsilon}) = \operatorname{mult}_p(\operatorname{CP}(\chi_{\varepsilon})) = \operatorname{mult}_p(\chi) = i.$$
 (4.7)

Let $G \cup \{f_0\} := \{f_j \colon 0 \leq j \leq s\} \subset (K[x_1])[\tilde{\boldsymbol{x}}] \setminus K$ be the basis of the ideal I after the Initialization in Algorithm 3.9 with $f_0 \in K[x_1] \setminus K^*$. In the generic case when $f_0 \neq 0$, i.e., $f_0 \in K[x_1] \setminus K$, the definition of pseudo-eliminant χ_{ε} in Algorithm 3.9 shows that $f_0 \in (\chi_{\varepsilon}) \subset K[x_1]$. That is, there exists $\rho \in (K[x_1])^*$ such that $f_0 = \rho \chi_{\varepsilon}$. The eliminant $\chi \in I \cap K[x_1]$ can be written as:

$$\chi = \sum_{j=0}^{s} h_j f_j = \sum_{j=1}^{s} h_j f_j + \rho h_0 \chi_{\varepsilon}$$

$$\tag{4.8}$$

with $h_j \in (K[x_1])[\tilde{\boldsymbol{x}}]$ for $0 \leq j \leq s$. Let us abuse the notation a bit and denote $F := G \cup \{f_0\} = \{f_j : 0 \leq j \leq s\}$. Suppose that $\max_{0 \leq j \leq s} \{\text{LM}(h_j f_j)\} = \tilde{\boldsymbol{x}}^{\beta}$. Let us collect and rename the elements in the set $\{f_j \in F : \text{LM}(h_j f_j) = \tilde{\boldsymbol{x}}^{\beta}, 0 \leq j \leq s\}$ into a new set $B_t := \{g_j : 1 \leq j \leq t\}$. And the subscripts of the functions $\{h_j\}$ are adjusted accordingly. In this way (4.8) can be written as follows:

$$\chi = \sum_{j=1}^{t} h_j g_j + \sum_{f_j \in F \setminus B_t} h_j f_j, \tag{4.9}$$

where the products $h_j f_j$ are those in (4.8) with $f_j \in F \setminus B_t$ for $0 \le j \le s$.

If we denote $LT(h_j) := c_j \tilde{x}^{\alpha_j}$ with $c_j \in (K[x_1])^*$ for $1 \leq j \leq t$ in (4.9), then it is evident that the following polynomial:

$$g := \sum_{j=1}^{t} \operatorname{LT}(h_j) \cdot g_j = \sum_{j=1}^{t} c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j$$
 (4.10)

is a summand of (4.9) and satisfies $LM(g) \prec \tilde{\boldsymbol{x}}^{\beta} = LM(\tilde{\boldsymbol{x}}^{\alpha_j}g_j)$ for $1 \leq j \leq t$ since the eliminant $\chi \prec \tilde{\boldsymbol{x}}^{\beta}$ in (4.9). According to Lemma 4.8 (i), there exist multipliers $b, b_j \in (K[x_1])^*$ for $1 \leq j < t$ that satisfy the following identity:

$$bg = \sum_{1 \le j < t} b_j S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c_t \tilde{\boldsymbol{x}}^{\alpha_t} g_t). \tag{4.11}$$

Moreover, by Lemma 4.8 (ii), we can relabel the subscript set $\{1 \leq j \leq t\}$ such that the multiplier b is not divisible by the irreducible polynomial $p \in K[x_1] \setminus K$ in (4.7), i.e., $\text{mult}_p(b) = 0$.

When $B_t \subset (K[x_1])[\tilde{\boldsymbol{x}}] \setminus K[x_1]$, if we define $\tilde{\boldsymbol{x}}^{\gamma_j} := \text{lcm}(LM(g_j), LM(g_t))$, we can simplify the S-polynomials in (4.11) based on (3.1):

$$S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c_t \tilde{\boldsymbol{x}}^{\alpha_t} g_t) = m_j \tilde{\boldsymbol{x}}^{\beta - \gamma_j} S(g_j, g_t)$$
(4.12)

with $m_j := \operatorname{lcm}(c_j \cdot \operatorname{LC}(g_j), c_t \cdot \operatorname{LC}(g_t)) / \operatorname{lcm}(\operatorname{LC}(g_j), \operatorname{LC}(g_t))$ for $1 \leq j < t$.

In particular, when $f_0 = \rho \chi_{\varepsilon}$ as in (4.8) satisfies $f_0 \in B_t$, we can deduce from $\mathrm{LM}(h_0 f_0) = \tilde{\boldsymbol{x}}^{\beta}$ that $\mathrm{LT}(h_0)$ bears the form $c\tilde{\boldsymbol{x}}^{\beta}$ with $c \in (K[x_1])^*$. Then the S-polynomials in (4.11) involving $c\tilde{\boldsymbol{x}}^{\beta} f_0$ bear the form $S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c \rho \tilde{\boldsymbol{x}}^{\beta} \chi_{\varepsilon})$ when $g_j \neq f_0$ for $1 \leq j < t$ and $g_t = f_0$, or $S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} \chi_{\varepsilon}, c_t \tilde{\boldsymbol{x}}^{\alpha_t} g_t)$, when $g_t \neq f_0$. Let us simply summarize these as $S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c \rho \tilde{\boldsymbol{x}}^{\beta} \chi_{\varepsilon})$ with $g_j \neq f_0$ and $1 \leq j \leq t$. Thus by (3.11) and (3.3), the simplification parallel to (4.12) now becomes:

$$S(c_i \tilde{\boldsymbol{x}}^{\alpha_j} g_i, c \rho \tilde{\boldsymbol{x}}^{\beta} \chi_{\varepsilon}) = S(c_i \tilde{\boldsymbol{x}}^{\alpha_j} g_i, c \rho \chi_{\varepsilon}) = n_i \tilde{\boldsymbol{x}}^{\alpha_j} S(g_i, \chi_{\varepsilon})$$
(4.13)

with $n_j := \operatorname{lcm}(c_j \cdot \operatorname{LC}(g_j), c\rho \chi_{\varepsilon}) / \operatorname{lcm}(\operatorname{LC}(g_j), \chi_{\varepsilon})$. From the definition of the set B_t it follows that $\tilde{\boldsymbol{x}}^{\alpha_j} \cdot \operatorname{LM}(g_j) = \tilde{\boldsymbol{x}}^{\beta}$.

Let $B_{\varepsilon} = \{g_k \colon 1 \leq k \leq \tau\} \subset (K[x_1])[\tilde{\boldsymbol{x}}] \setminus K[x_1]$ be the pseudo-basis of the ideal I obtained in Algorithm 3.9 such that the polynomial set B_t as in (4.9) is a subset of $B_{\varepsilon} \cup \{\chi_{\varepsilon}\}$. In Algorithm 3.9 we have pseudo-reduced every S-polynomial $S(g_j, g_t)$ in (4.12) by the pseudo-basis B_{ε} , either directly or indirectly like in Lemma 3.8. More specifically, according to Theorem 2.7, there exist a multiplier $\lambda_j \in (K[x_1])^*$ as well as a remainder $r_j \in K[x_1] \setminus K^*$ and $q_{jk} \in (K[x_1])[\tilde{\boldsymbol{x}}]$ for $1 \leq k \leq \tau$ such that the following pseudo-reduction of $S(g_j, g_t)$ by the pseudo-basis B_{ε} holds for $1 \leq j < t$:

$$\lambda_j S(g_j, g_t) = \sum_{k=1}^{\tau} q_{jk} g_k + r_j = \sum_{k=1}^{\tau} q_{jk} g_k + \rho_j \chi_{\varepsilon}.$$
 (4.14)

Please note that $S(g_j, g_t)$ is an S-polynomial between two elements g_j and g_t in B_t because we abuse the notation for the subscripts of the elements in B_ε and B_t in (4.14). The remainder r_j in (4.14) is a univariate polynomial in $(\chi_\varepsilon) \subset K[x_1] \setminus K^*$ according to Lemma 3.12 (ii). Hence in (4.14) we denote $r_j := \rho_j \chi_\varepsilon$ with $\rho_j \in K[x_1]$. Moreover, λ_j is relatively prime to the compatible divisor p^i of the pseudo-eliminant χ_ε as in (4.7), i.e., $\mathrm{mult}_p(\lambda_j) = 0$ for $1 \le j < t$. As per (2.3), we can deduce that $\mathrm{LM}(S(g_j, g_t)) = \max_{1 \le k \le \tau} \{\mathrm{LM}(q_{jk}g_k)\}$ holds in (4.14) for $1 \le j < t$. We further have $\mathrm{LM}(S(g_j, g_t)) \prec \tilde{x}^{\gamma_j}$ by (3.2) with $\tilde{x}^{\gamma_j} = \mathrm{lcm}(\mathrm{LM}(g_j), \mathrm{LM}(g_t))$ as in (4.12). Hence it readily follows that for $1 \le j < t$:

$$\max_{1 \le k \le \tau} \{ LM(q_{jk}g_k) \} = LM(S(g_j, g_t)) \prec \tilde{\boldsymbol{x}}^{\gamma_j}. \tag{4.15}$$

Based on (4.12) and (4.14), it is straightforward to obtain a pseudo-reduction of the S-polynomial $S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c_t \tilde{\boldsymbol{x}}^{\alpha_t} g_t)$ in (4.11) by the pseudo-basis B_{ε} as follows when $g_j, g_t \in (K[x_1])[\tilde{\boldsymbol{x}}] \setminus K[x_1]$ for $1 \leq j < t$.

$$\frac{\lambda_j}{d_j} S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c_t \tilde{\boldsymbol{x}}^{\alpha_t} g_t) = \frac{m_j}{d_j} \tilde{\boldsymbol{x}}^{\beta - \gamma_j} \left(\sum_{k=1}^{\tau} q_{jk} g_k + \rho_j \chi_{\varepsilon} \right). \tag{4.16}$$

In fact, with $d_j := \gcd(\lambda_j, m_j)$, it suffices to take λ_j/d_j as the multipliers for the above pseudo-reductions for $1 \le j < t$. It is evident that $\operatorname{mult}_p(\lambda_j/d_j) = 0$ if $\operatorname{mult}_p(\lambda_j) = 0$, i.e., the multipliers λ_j/d_j are still relatively prime to the compatible divisor p^i for $1 \le j < t$. Moreover, a combination of (4.15) and (4.16) leads to the inequalities:

$$LM(\tilde{\boldsymbol{x}}^{\beta-\gamma_j}q_{ik}q_k) \prec \tilde{\boldsymbol{x}}^{\beta}, \quad 1 \le j < t, \ 1 \le k \le \tau. \tag{4.17}$$

We can make a pseudo-reduction of the S-polynomial $S(g_j, \chi_{\varepsilon})$ in (4.13) by the pseudo-eliminant χ_{ε} as in (3.4) of Lemma 3.5. The multiplier $\lambda_j := \gcd(\operatorname{LC}(g_j), \chi_{\varepsilon})$ of the pseudo-reduction satisfies $\operatorname{mult}_p(\lambda_j) = 0$ since for every $f \in B_{\varepsilon}$, we added $\gcd(\operatorname{LC}(f), \chi_{\varepsilon}) \in K[x_1] \setminus K$ into the multiplier set Λ in Procedure \mathcal{R} of Algorithm 3.9. Then based on the relationship in (4.13), we can make pseudo-reductions of the S-polynomials $S(c_j\tilde{\boldsymbol{x}}^{\alpha_j}g_j, c\rho\chi_{\varepsilon})$ in (4.13) via multipliers λ_j/d_j with $d_j := \gcd(\lambda_j, n_j)$ as follows.

$$\frac{\lambda_j}{d_j} S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c \rho \chi_{\varepsilon}) = \frac{n_j}{d_j} \tilde{\boldsymbol{x}}^{\alpha_j} u_j \chi_{\varepsilon}$$
(4.18)

with $u_j := g_j - \operatorname{LT}(g_j)$ and $g_j \neq f_0$ for $1 \leq j \leq t$. Evidently we also have $\operatorname{mult}_p(\lambda_j/d_j) = 0$ as well as $\operatorname{LM}(\tilde{\boldsymbol{x}}^{\alpha_j}u_j) \prec \tilde{\boldsymbol{x}}^{\alpha_j} \cdot \operatorname{LM}(g_j) = \tilde{\boldsymbol{x}}^{\beta}$ by (4.13).

In (4.11) there appear two kinds of S-polynomials as $S(c_j\tilde{\boldsymbol{x}}^{\alpha_j}g_j, c_t\tilde{\boldsymbol{x}}^{\alpha_t}g_t)$ in (4.16) and $S(c_j\tilde{\boldsymbol{x}}^{\alpha_j}g_j, c_\rho\chi_{\varepsilon})$ in (4.18). Let λ denote the product of the multipliers λ_j/d_j in (4.16) and (4.18) for the pseudo-reductions of the S-polynomials in (4.11). It is evident that λ is still relatively prime to the compatible divisor p^i . Based on (4.11) and the pseudo-reductions of the S-polynomials in (4.16) and (4.18), we obtain the following representation:

$$b\lambda g = \sum_{k=1}^{\tau} q_k g_k + \eta \chi_{\varepsilon}. \tag{4.19}$$

With $1 \leq k \leq \tau$ here, $q_k \in (K[x_1])[\tilde{\boldsymbol{x}}]$ is a linear combination of the factors $\tilde{\boldsymbol{x}}^{\beta-\gamma_j}q_{jk}$ in (4.16) for $1 \leq j < t$ with coefficients $b_j \lambda m_j / \lambda_j \in K[x_1]$. And $\eta \in (K[x_1])[\tilde{\boldsymbol{x}}]$ is a linear combination of the factors $\tilde{\boldsymbol{x}}^{\beta-\gamma_j}\rho_j$ in (4.16) for $1 \leq j < t$ and the factors $\tilde{\boldsymbol{x}}^{\alpha_j}u_j$ in (4.18) for $1 \leq j \leq t$ with coefficients $b_j \lambda m_j / \lambda_j$ and $b_j \lambda n_j / \lambda_j$ in $K[x_1]$ respectively. Thus from (4.17) and $LM(\tilde{\boldsymbol{x}}^{\alpha_j}u_j) \prec \tilde{\boldsymbol{x}}^{\beta}$ in (4.18), we have the following inequality for (4.19):

$$\max\left\{\max_{1\leq k\leq \tau}\{q_k g_k\}, \eta \chi_{\varepsilon}\right\} \prec \tilde{\boldsymbol{x}}^{\beta}. \tag{4.20}$$

The multiplier $b\lambda$ as in (4.19) is relatively prime to the compatible divisor p^i since are both b and λ . We also know that the polynomial g in (4.10) is a summand of the representation of the eliminant χ in (4.9). We multiply both the polynomial g in (4.10) and the eliminant χ in (4.9) by the multiplier $b\lambda$. Then we substitute the summand $b\lambda g$ by its new representation in (4.19). In this way we obtain a new representation of $b\lambda \chi$ as follows.

$$b\lambda\chi = \sum_{k=1}^{\tau} q_k g_k + \eta \chi_{\varepsilon} + b\lambda \sum_{j=1}^{t} (h_j - \operatorname{LT}(h_j)) g_j + b\lambda \sum_{f_j \in F \setminus B_t} h_j f_j.$$
 (4.21)

The representation (4.21) also applies to the case when the univariate polynomial f_0 in (4.8) satisfies $f_0 \in F \setminus B_t$, that is, $LM(h_0 f_0) \prec \tilde{x}^{\beta}$ as per the definition of the set B_t in (4.9). In fact, in this case we can rewrite the last summation in (4.21) as follows.

$$b\lambda \sum_{f_j \in F \setminus B_t} h_j f_j = b\lambda h_0 f_0 + b\lambda \sum_{f_j \in G \setminus B_t} h_j f_j, \tag{4.22}$$

where $G = F \setminus K[x_1]$ is defined as in Algorithm 3.9. We can treat the above summand $b\lambda h_0 f_0$ as the summand $\eta \chi_{\varepsilon}$ in (4.21) since $f_0 = \rho \chi_{\varepsilon}$ as in (4.8).

Let $B_{\varepsilon} = \{g_k \colon 1 \leq k \leq \tau\}$ be the pseudo-basis obtained in Algorithm 3.9. Since $B_t \subset F = G \cup \{f_0\}$ and $G \subset B_{\varepsilon}$ in (4.21) and (4.22), we can rewrite the representation in (4.21), including the case of (4.22), into a new representation in terms of B_{ε} and χ_{ε} as follows.

$$b\lambda\chi = \sum_{k=1}^{\tau} \mu_k g_k + \mu_0 \chi_{\varepsilon} \tag{4.23}$$

with $\mu_k \in (K[x_1])[\tilde{x}]$ for $0 \le k \le \tau$. The leading monomials in (4.23) satisfy

$$\max\{\max_{1\leq k\leq\tau}\{LM(\mu_k g_k)\}, LM(\mu_0)\} \prec \tilde{\boldsymbol{x}}^{\beta}$$
(4.24)

according to (4.20) and the representations in (4.21) and (4.22). To summarize, the leading monomials in the representation (4.23) strictly decrease from those in the representations (4.8) and (4.9), up to the multiplier $b\lambda$ that satisfies $\text{mult}_p(b\lambda) = 0$.

Now we treat the representation in (4.23) as the one in (4.8) and repeat our discussions from (4.9) through (4.23). In this way we obtain a new representation of $b\lambda\chi$ with a new multiplier. The leading monomials of the new representation are strictly less than those in (4.23). This is similar to the strict inequality in (4.24) comparing the leading monomials of the representation in (4.23) with those in (4.8). Moreover, the new multiplier for the new representation is still relatively prime to the compatible divisor p^i of the pseudo-eliminant χ_{ε} .

We repeat the above procedure of rewriting the representations of the eliminant χ so as to strictly reduce the orderings of their leading monomials. Moreover, the multipliers for the representations are always relatively prime to the compatible divisor p^i of the pseudo-eliminant χ_{ε} . Since the elimination ordering on $(K[x_1])[\tilde{x}]$ as in Definition 3.1 and Definition 2.3 is a well-ordering, the above procedures halt after a finite number of repetitions. In this way we shall reach a representation bearing the following form:

$$\nu \chi = h \chi_{\varepsilon}. \tag{4.25}$$

The multiplier $\nu \in (K[x_1])^*$ in (4.25) is relatively prime to the compatible divisor p^i of the pseudo-eliminant χ_{ε} . Here the multiplier $h \in (K[x_1])^*$. Hence the eliminant χ is divisible by p^i since we assumed that χ_{ε} is divisible by its compatible divisor p^i according to the definition of compatible divisors in Definition 4.5. Thus follows the conclusion of Theorem 4.10.

In Definition 4.5 we defined the compatible part $CP(\chi_{\varepsilon})$ and incompatible part $IP(\chi_{\varepsilon})$ of a pseudo-eliminant χ_{ε} . The definition is based on the multiplier set Λ for the pseudo-reductions of S-polynomials by the pseudo-basis B_{ε} in obtaining χ_{ε} . As the following Corollary 4.11 shows, sometimes it is more convenient to determine

the compatibility by the leading coefficients of the pseudo-basis B_{ε} than by the multiplier set Λ as above.

Corollary 4.11. For a zero-dimensional ideal I over a perfect field K, let χ_{ε} be a pseudo-eliminant of I and B_{ε} its pseudo-basis. Suppose that p is an irreducible factor of χ_{ε} with multiplicity i. If p is relatively prime to every leading coefficient in $LC(B_{\varepsilon}) := \{LC(b) \in K[x_1] \setminus K : b \in B_{\varepsilon}\}$, then p^i is a compatible divisor of χ_{ε} . In particular, if χ_{ε} is relatively prime to every leading coefficient in $LC(B_{\varepsilon})$, then χ_{ε} is compatible and $\chi = \chi_{\varepsilon}$.

Proof. Let us prove that the divisor p^i of χ_{ε} satisfies Definition 4.5 and hence is compatible when p is relatively prime to every leading coefficient in $LC(B_{\varepsilon})$. That is, p^i is relatively prime to the multiplier set Λ for the pseudo-reductions of S-polynomials by the pseudo-basis B_{ε} in obtaining χ_{ε} . In what follows let us write an S-polynomial as S and the pseudo-basis as $B_{\varepsilon} = \{b_1, \ldots, b_s\}$. Then the multiplier $\lambda \in (K[x_1])^*$ in (2.2) for the pseudo-reduction of S by B_{ε} is just a finite product of the interim multipliers $\mu := m/c_{\alpha} \in (K[x_1])^*$ in (2.1) with $m = \text{lcm}(c_{\alpha}, \text{LC}(b_j))$. It is evident that if an irreducible factor p of χ_{ε} is relatively prime to $LC(b_j)$, then it is also relatively prime to the interim multiplier $\mu = m/c_{\alpha}$ since $m/c_{\alpha} = LC(b_j)/\gcd(c_{\alpha}, LC(b_j))$. As a result, p is relatively prime to the multiplier λ for the pseudo-reduction of S by B_{ε} . Thus p^i is a compatible divisor of χ_{ε} .

Definition 4.5 provides a criterion for the compatibility of a pseudo-eliminant's divisors based on the multiplier set Λ , whereas the criterion in Corollary 4.11 is based on the leading coefficients of the pseudo-basis B_{ε} . To distinguish we call them the multiplier criterion and coefficient criterion respectively. The following example shows that a divisor of a pseudo-eliminant χ_{ε} satisfying the multiplier criterion does not necessarily satisfy the coefficient criterion.

Example 4.12. In the algebra $(\mathbb{Q}[y])[x]$ with elimination ordering $x \succ y$, consider an ideal I generated by $f(x,y) = y(x^2+1)$ and g(x,y) = (y+1)(2x+1). As per (3.1), their S-polynomial is as follows with LC(f) = y and LC(g) = 2(y+1).

$$S(f,g) = 2(y+1)f - yxg = -y(y+1)(x-2).$$

The pseudo-reduction of S(f,g) is 2S(f,g)+yg=5y(y+1) with the multiplier $\lambda=2\in\mathbb{Q}$. The pseudo-eliminant $\chi_{\varepsilon}=5y(y+1)$ and pseudo-basis $F=\{f,g\}$. Now both the irreducible factors y and y+1 of χ_{ε} satisfy the multiplier criterion and hence are compatible divisors. Hence $\chi_{\varepsilon}=5y(y+1)$ is compatible and the eliminant $\chi=\chi_{\varepsilon}$ up to the unit coefficient $5\in\mathbb{Q}$. Nevertheless χ_{ε} does not satisfy the coefficient criterion.

5 Analysis of Incompatible Divisors via Modular Method

For a pseudo-eliminant χ_{ε} of a zero-dimensional ideal I in $(K[x_1])[\tilde{x}]$ over a perfect field K, we made a squarefree decomposition of its incompatible part $IP(\chi_{\varepsilon})$ in Algorithm 4.6. In order to determine the eliminant χ of I, we perform a complete

analysis of $IP(\chi_{\varepsilon})$ in this section. For the squarefree decomposition $\{\Omega_i\}$ of $IP(\chi_{\varepsilon})$ obtained in Algorithm 4.6, the elements in Ω_i are pairwise relatively prime and usually have small exponents due to the way they are constructed in Algorithm 4.6. Accordingly it is natural to contrive a modular algorithm modulo the elements in Ω_i so as to reduce the complexity of our algorithm. However this requires unorthodox computations in principal ideal rings with zero divisors.

Definition 5.1 (Principal ideal Quotient Ring: PQR).

Let R be a PID whose set of units is denoted as R^{\times} . With the principal ideal (q) generated by an element $q \in R^* \setminus R^{\times}$, the quotient ring $\bar{R} := R/(q)$ is called a principal ideal quotient ring and abbreviated as PQR henceforth.

Notation 5.2. Suppose that $q \in R^* \setminus R^\times$ in Definition 5.1 has a unique factorization $q = u \prod_{i=1}^s p_i^{\alpha_i}$ with $s, \alpha_i \in \mathbb{N}^*$ for $1 \le i \le s$ and $u \in R^\times$. Here the factors $\{p_i : 1 \le i \le s\} \subset R^* \setminus R^\times$ are irreducible and they are pairwise relatively prime when s > 1.

When $q \in R^* \setminus R^{\times}$ is irreducible in Definition 5.1, i.e., when $s = \alpha_1 = 1$ in Notation 5.2, the PQR \bar{R} becomes a field since q is prime in R. Nonetheless when $q \in R^* \setminus R^{\times}$ is not irreducible in Definition 5.1, i.e., in the case of either s > 1 or $\alpha_i > 1$ for an i satisfying $1 \le i \le s$, the PQR \bar{R} has zero divisors and is not an integral domain. In this case \bar{R} is no longer a factorial ring. Nonetheless \bar{R} still has nice properties to which we can resort in our computations.

Lemma 5.3. Let $\bar{R} = R/(q)$ be a PQR as in Definition 5.1 and $\varphi \colon R \to \bar{R}$ the canonical ring homomorphism. Suppose that $q \in R^* \setminus R^\times$ has a unique factorization $q = u \prod_{i=1}^s p_i^{\alpha_i}$ as in Notation 5.2. In what follows we also use the notation $\bar{a} := \varphi(a)$ for every $a \in R$.

- (i) An $r \in R$ is relatively prime to q, i.e., $\gcd(r,q) = 1$, if and only if $\varphi(r)$ is a unit in \bar{R} , that is, $\varphi(r) \in \bar{R}^{\times}$.
- (ii) For $1 \leq i \leq s$ and each $l \in \mathbb{N}$ satisfying $l \geq \alpha_i$, we have $\bar{p}_i^l \sim \bar{p}_i^{\alpha_i}$. Here the notation $\bar{a} \sim \bar{b}$ in \bar{R} means that \bar{a} is an associate of \bar{b} in \bar{R} , i.e., there is a unit $\bar{u} \in \bar{R}^{\times}$ such that $\bar{b} = \bar{u}\bar{a}$.
- (iii) For every $\bar{a} \in \bar{R}^*$, we have a unique representation $\bar{a} \sim \prod_{i=1}^s \bar{p}_i^{\beta_i}$ that satisfies $0 \le \beta_i \le \alpha_i$ for $1 \le i \le s$. We call such kind of representations a standard representation of \bar{a} in the PQR \bar{R} and denote it as \bar{a}_{st} . In particular, we define $\bar{a}_{st} := 1$ for $\bar{a} \in \bar{R}^{\times}$.

Proof. The conclusion (i) readily follows from the fact that R is a PID. In fact, gcd(r,q) = 1 means that there exist $u, v \in R$ such that ur + vq = 1, from which we can deduce that $\varphi(u)\varphi(r) = 1$. Conversely $\varphi(ur) = 1$ implies that there exists $v \in R$ such that ur - 1 = vq. Hence follows the conclusion.

When s=1 both the conclusions (ii) and (iii) are evident since $\bar{p}_1^l=0$ for $l\geq \alpha_1$. In particular, $\bar{R}^*=\bar{R}^\times$ when $\alpha_1=s=1$. In this case every $\bar{a}\in\bar{R}^*$ has a standard representation $\bar{a}\sim 1:=\bar{a}_{\rm st}$. So in what follows let us suppose that s>1.

Without loss of generality, let us prove (ii) in the case when i = s. For $l > \alpha_s$, we have $p_s^l \equiv p_s^l + q \mod q$. Moreover, we have the identity: $p_s^l + q = s$

 $\begin{array}{l} p_s^{\alpha_s}(p_s^{l-\alpha_s}+u\prod_{i=1}^{s-1}p_i^{\alpha_i}). \quad \text{Here } \varphi(p_s^{l-\alpha_s}+u\prod_{i=1}^{s-1}p_i^{\alpha_i}) \text{ is a unit in } \bar{R} \text{ by (i) since } \\ p_s^{l-\alpha_s}+u\prod_{i=1}^{s-1}p_i^{\alpha_i} \text{ is relatively prime to } q \text{ in } R. \text{ Thus } \bar{p}_s^l=\varphi(p_s^l+q)\sim \bar{p}_s^{\alpha_s}. \end{array}$

The existence of the standard representation in (iii) readily follows from the fact that R is factorial and the canonical homomorphism φ is an epimorphism. If p is irreducible and relatively prime to q, then $\bar{p}=\varphi(p)\in\bar{R}^{\times}$ is a unit. Hence for every $\bar{a}\in\bar{R}^{*}$, its standard representation can only bear the form $\bar{a}\sim\prod_{i=1}^{s}\bar{p}_{i}^{\beta_{i}}$ with $0\leq\beta_{i}\leq\alpha_{i}$ for $1\leq i\leq s$. Now suppose that \bar{a} has another standard representation $\bar{a}\sim\prod_{i=1}^{s}\bar{p}_{i}^{\gamma_{i}}$ with $0\leq\gamma_{i}\leq\alpha_{i}$ for $1\leq i\leq s$. Then there exists $h\in R$ such that $\prod_{i=1}^{s}p_{i}^{\beta_{i}}=u\cdot\prod_{i=1}^{s}p_{i}^{\gamma_{i}}+hq$ with $u\in R$ being relatively prime to q, from which we can easily deduce that $\beta_{i}=\gamma_{i}$ for $1\leq i\leq s$.

Let K be a perfect field and $q \in K[x_1] \setminus K$. It is easy to see that $K[x_1]/(q)$ is a PQR as defined in Definition 5.1. Hereafter we use R and \bar{R} to denote $K[x_1]$ and $K[x_1]/(q)$ respectively. Let us consider the following set:

$$R_q := \{ r \in K[x_1] \colon \deg(r) < \deg(q) \}$$
 (5.1)

with $\deg(r) = 0$ for every $r \in K$ including r = 0. Let us deem the canonical ring homomorphism $\varphi \colon R \to \bar{R}$ as a map. We restrict it on R_q and denote it as φ_q . It is evident that $\varphi_q \colon R_q \to \bar{R}$ is a bijective map with $\varphi_q(0) = 0$. We redefine the two binary operations on R_q , the addition and multiplication, as follows.

$$a+b:=\varphi_q^{-1}(\bar{a}+\bar{b}); \quad a\cdot b:=\varphi_q^{-1}(\bar{a}\cdot\bar{b}). \tag{5.2}$$

In this way the set R_q in (5.1) becomes a ring, which we still denote as R_q . It is easy to verify that φ_q is a ring isomorphism between R_q and \bar{R} . As a result, the conclusions in Lemma 5.3 apply to the ring R_q as well.

Definition 5.4 (Normal PQR R_q).

We call the ring R_q being constructed as in (5.1) and (5.2) a normal PQR henceforth.

Let a normal PQR R_q be defined as in Definition 5.4 for $q \in K[x_1] \setminus K$. For every $f \in R = K[x_1]$, there exist a quotient $h \in K[x_1]$ and unique remainder $r \in K[x_1]$ satisfying

$$f = hq + r; \qquad \deg(r) < \deg(q). \tag{5.3}$$

Hence by (5.1) we can define an epimorphism directly as follows.

$$\sigma_q \colon R \to R_q \colon \quad \sigma_q(f) := r.$$
 (5.4)

It is easy to verify that the epimorphism σ_q is a composition of the canonical ring homomorphism $\varphi \colon R \to \bar{R} = K[x_1]/(q)$ and the isomorphism $\varphi_q^{-1} \colon \bar{R} \to R_q$ in (5.2).

Since a normal PQR R_q is a subset of $R = K[x_1]$, for every $r \in R_q$, we can define an injection as follows.

$$\iota_q \colon R_q \hookrightarrow R \colon \quad \iota_q(r) := r.$$
(5.5)

Please note that ι_q is not a ring homomorphism since the binary operations on the ring R_q are different from those on R. Nonetheless $\sigma_q \circ \iota_q$ is the identity map

on R_q . For each pair $a, b \in R_q$, we define the binary operations between $\iota_q(a)$ and $\iota_q(b)$ as those defined on R.

Suppose that $\bar{a} \in \bar{R}^*$ has a standard representation $\bar{a} \sim \bar{a}_{\rm st} = \prod_{i=1}^s \bar{p}_i^{\beta_i}$ with $0 \leq \beta_i \leq \alpha_i$ as in Lemma 5.3 (iii). We can substitute $p_i = \varphi_q^{-1}(\bar{p}_i) \in R_q^* \setminus R_q^\times$ as in (5.2) for $\bar{p}_i \in \bar{R}^* \setminus \bar{R}^\times$ in this representation. In this way we obtain a standard representation of $a := \varphi_q^{-1}(\bar{a}) \in R_q^*$ in the normal PQR R_q as follows:

$$a \sim a_{\rm st} := \prod_{i=1}^{s} p_i^{\beta_i}, \quad 0 \le \beta_i \le \alpha_i; \qquad a = a^{\times} \cdot a_{\rm st},$$
 (5.6)

where $\{\alpha_i \colon 1 \leq i \leq s\}$ are the exponents for the unique factorization of the moduli q as in Lemma 5.3. For convenience we use $a^{\times} \in R_q^{\times}$ to denote the unit factor of a with respect to $a_{\rm st}$. We also call $a_{\rm st}$ the standard factor of a henceforth. In particular, we define $a_{\rm st} := 1$ for $a \in R_q^{\times}$. We can derive the existence and uniqueness of the standard representation $a_{\rm st}$ in (5.6) from Lemma 5.3 (iii) since the normal PQR R_q is isomorphic to the PQR \bar{R} under φ_q .

Remark 5.5. It is unnecessary to procure a complete factorization of $a_{\rm st}$ as in (5.6) in our computations. In fact, it suffices to make a factorization $a = a^{\times} \cdot a_{\rm st}$. This can be easily attained by a computation $a_{\rm st} = \gcd(\iota_q(a), q)$ with ι_q being defined as in (5.5). The soundness of the computation readily follows from Lemma 5.3 and (5.6).

An apparent difference between the PQR \bar{R} and normal PQR R_q is that the degree function deg is well defined on R_q , which is indispensable for polynomial divisions. More specifically, for all $a, b \in R_q^*$ with $\deg(b) > 0$, there exist a quotient $h \in R_q$ and unique remainder $r \in R_q$ satisfying the following equality:

$$a = hb + r$$
 such that $\deg(r) < \deg(b)$. (5.7)

Since all polynomials involved here including the product hb have degrees strictly less than deg(q) in (5.7), there is no multiplication of zero divisors leading to 0 for polynomial divisions. This includes the case when deg(a) < deg(b) and hence h = 0. That is, we make polynomial divisions on the normal PQR R_q in the same way as on R.

For $a, b \in R_q^*$ and their standard representations $a \sim a_{\rm st} = \prod_{i=1}^s p_i^{\beta_i}$ and $b \sim b_{\rm st} = \prod_{i=1}^s p_i^{\gamma_i}$ as in (5.6), let us define:

$$\gcd_{st}(a,b) := \gcd(a_{st},b_{st}) = \prod_{i=1}^{s} p_i^{n_i}; \ \operatorname{lcm}_{st}(a,b) := \operatorname{lcm}(a_{st},b_{st}) = \prod_{i=1}^{s} p_i^{m_i} \quad (5.8)$$

with $n_i := \min\{\beta_i, \gamma_i\}$ and $m_i := \max\{\beta_i, \gamma_i\}$. It is evident that we might have $\operatorname{lcm}_{\operatorname{st}}(a, b) = 0$ for $a, b \in R_q^*$ due to the possible existence of zero divisors in R_q^* .

Remark 5.6. The definition of $gcd_{st}(a, b)$ and $lcm_{st}(a, b)$ in (5.8) is based upon a complete factorization of a and b as in (5.6). In practice in order to minimize the complexity of our algorithm, we resort to Euclidean algorithm to compute gcd(a, b). The normal PQR R_q might have zero divisors and not be an Euclidean domain. However from our discussion on polynomial divisions in (5.7), we know that the polynomial division on R_q is the same as that on R. Moreover, for the

irreducible factor $p_i \in R_q^* \setminus R_q^{\times}$ in Notation 5.2 and $1 \leq e \leq \alpha_i$, if both a and b are divisible by p_i^e in (5.7), then so is the remainder r. Similarly if both b and r are divisible by p_i^e in (5.7), then so is a. Thus the computation of $\gcd(a,b)$ for $a,b \in R_q^*$ by Euclidean algorithm on R_q is sound and feasible. It differs from $\gcd_{\rm st}(a,b)$ only by a unit factor.

Let $lcm(\iota_q(a), \iota_q(b))$ be the least common multiple of $\iota_q(a)$ and $\iota_q(b)$ on R. The same for $gcd(\iota_q(a), \iota_q(b))$. For each pair $a, b \in R_q$, let us define:

$$\gcd_q(a,b) := \sigma_q(\gcd(\iota_q(a),\iota_q(b))); \quad \operatorname{lcm}_q(a,b) := \sigma_q(\operatorname{lcm}(\iota_q(a),\iota_q(b))) \tag{5.9}$$

with the epimorphism σ_q and injection ι_q defined as in (5.4) and (5.5) respectively. By Lemma 5.3 and (5.6), it is easy to verify the following relationship between the two definitions in (5.8) and (5.9):

$$\gcd_a(a,b) \sim \gcd_{\rm st}(a,b)$$
 and $\operatorname{lcm}_a(a,b) \sim \operatorname{lcm}_{\rm st}(a,b)$. (5.10)

In the identity (5.3), $\operatorname{mult}_p(r)$ is well-defined since $K[x_1]$ is a factorial domain and $r \in K[x_1]$. Therefore we can deduce that for every irreducible polynomial $p \in K[x_1]$, if $\max\{\operatorname{mult}_p(f), \operatorname{mult}_p(r)\} \leq \operatorname{mult}_p(q)$, then we have:

$$\operatorname{mult}_{p}(f) = \operatorname{mult}_{p}(r).$$
 (5.11)

Definition 5.7 (Elimination ordering on $R_q[\tilde{x}]$).

If the variable $x_1 \in R_q^*$, the *elimination ordering* on $R_q[\tilde{x}]$ is the monomial ordering such that the \tilde{x} variables are always larger than the variable $x_1 \in R_q^*$. That is, $x_1^{\alpha} \tilde{x}^{\gamma} \succ x_1^{\beta} \tilde{x}^{\delta}$ if and only if $\tilde{x}^{\gamma} \succ \tilde{x}^{\delta}$ or, $\tilde{x}^{\gamma} = \tilde{x}^{\delta}$ and $\alpha > \beta$.

We also say that the elimination ordering on $R_q[\tilde{x}]$ is induced from the one on $(K[x_1])[\tilde{x}]$ in Definition 3.1.

Definition 5.8 (Term reduction in $R_q[\tilde{\boldsymbol{x}}]$).

Let R_q be a normal PQR as in Definition 5.4 and \succ the elimination ordering on $R_q[\tilde{\boldsymbol{x}}]$ as in Definition 5.7. Let the epimorphism $\sigma_q \colon R \to R_q$ and injection $\iota_q \colon R_q \to R$ be defined as in (5.4) and (5.5) respectively. For $f \in R_q[\tilde{\boldsymbol{x}}] \setminus R_q$ and $g \in (R_q[\tilde{\boldsymbol{x}}])^* \setminus R_q^\times$ with $LC(g) \in R_q^*$, suppose that f has a term $c_\alpha \tilde{\boldsymbol{x}}^\alpha$ with $\tilde{\boldsymbol{x}}^\alpha \in \text{supp}(f) \cap \langle LM(g) \rangle$. We also define the multipliers $\mu := \sigma_q(\text{lcm}(l_\alpha, l_g)/l_\alpha)$ and $m := \sigma_q(\text{lcm}(l_\alpha, l_g)/l_g)$ with $l_\alpha := \iota_q(c_\alpha)$ and $l_g := \iota_q(LC(g))$. We can make a reduction of the term $c_\alpha \tilde{\boldsymbol{x}}^\alpha$ of f by g as follows.

$$h = \mu f - \frac{m\tilde{x}^{\alpha}}{\text{LM}(q)}g. \tag{5.12}$$

We call h the remainder of the reduction and μ the interim multiplier on f with respect to g.

In Definition 5.8 we might have $\operatorname{lcm}_{\operatorname{st}}(c_{\alpha},\operatorname{LC}(g))=0$ for $c_{\alpha},\operatorname{LC}(g)\in R_q^*$ due to the possible existence of zero divisors in R_q^* . We postpone to address this issue until Lemma 5.17 (ii) after the definition of S-polynomials over a normal PQR because in what follows we only consider a special kind of term reductions whose interim multipliers μ in (5.12) satisfy $\mu\in R_q^{\times}$.

Definition 5.9 (Properly reduced polynomial in $R_q[\tilde{x}]$).

Let R_q be a normal PQR as in Definition 5.4 and \succ the elimination ordering on $R_q[\tilde{\boldsymbol{x}}]$ as in Definition 5.7. A term $c_\alpha \tilde{\boldsymbol{x}}^\alpha \in R_q[\tilde{\boldsymbol{x}}]$ with the coefficient $c_\alpha \in R_q^*$ is said to be properly reducible with respect to $F = \{f_1, \ldots, f_s\} \subset R_q[\tilde{\boldsymbol{x}}] \setminus R_q$ if there exists an $f_j \in F$ such that $\tilde{\boldsymbol{x}}^\alpha \in \langle \text{LM}(f_j) \rangle$ and the interim multiplier μ with respect to f_j as in (5.12) satisfies $\mu \in R_q^\times$. We say that a polynomial $f \in R_q[\tilde{\boldsymbol{x}}]$ is properly reduced with respect to F if none of its terms is properly reducible with respect to F.

The condition $\mu \in R_q^{\times}$ for the properness in Definition 5.9 indicates that $\mu = \sigma_q(\operatorname{lcm}(l_{\alpha}, l_j)/l_{\alpha}) \in R_q^{\times}$. Here $l_{\alpha} := \iota_q(c_{\alpha})$ and $l_j := \iota_q(c_j)$ with $c_j := \operatorname{LC}(f_j)$. Hence we can deduce that $c_{\alpha} \in (c_j) \subset R_q$. Combined with the condition $\tilde{\boldsymbol{x}}^{\alpha} \in \langle \operatorname{LM}(f_j) \rangle$, the condition $\mu \in R_q^{\times}$ for the properness in Definition 5.9 is equivalent to the following term divisibility condition:

$$c_{\alpha}\tilde{\boldsymbol{x}}^{\alpha} \in \langle c_{i} \cdot \text{LM}(f_{i}) \rangle = \langle \text{LT}(f_{i}) \rangle \subset R_{q}[\tilde{\boldsymbol{x}}].$$
 (5.13)

Theorem 5.10 (Proper division in $R_q[\tilde{x}]$).

Let R_q be a normal PQR as in Definition 5.4 and \succ the elimination ordering on $R_q[\tilde{x}]$ as in Definition 5.7. Suppose that $F = \{f_1, \ldots, f_s\}$ are polynomials in $R_q[\tilde{x}] \setminus R_q$. For every $f \in R_q[\tilde{x}]$, there exist a multiplier $\lambda \in R_q^{\times}$ as well as a remainder $r \in R_q[\tilde{x}]$ and quotients $q_j \in R_q[\tilde{x}]$ for $1 \leq j \leq s$ such that:

$$\lambda f = \sum_{j=1}^{s} q_j f_j + r, \tag{5.14}$$

where r is properly reduced with respect to F. Moreover, the polynomials in (5.14) have to satisfy the following condition:

$$LM(f) = \max\{\max_{1 \le j \le s} \{LM(q_j) \cdot LM(f_j)\}, LM(r)\}.$$

$$(5.15)$$

Proof. The proof amounts to a verbatim repetition of that for Theorem 2.7 if we substitute the criterion of being properly reduced for that of being pseudoreduced. In fact, the polynomial division on a normal PQR R_q as in (5.7) is the same as that on $R = K[x_1]$. Moreover, a normal PQR R_q as in Definition 5.4 is also a Noetherian ring since it is isomorphic to the PQR \bar{R} in Definition 5.1 that is Noetherian.

Please note that the product $LM(q_j) \cdot LM(f_j)$ in (5.15) is based upon the term divisibility condition (5.13).

We also call the proper division in Theorem 5.10 a proper reduction henceforth. We can easily contrive a proper division algorithm based on Theorem 5.10.

For $f, g \in (R_q[\tilde{x}])^* \setminus R_q^{\times}$, suppose that $\operatorname{lcm}_{\operatorname{st}}(\operatorname{LC}(f), \operatorname{LC}(g)) = 0$ due to the existence of zero divisors in R_q^* . In this case if we employed Definition 3.4 for S-polynomials directly and in particular, the multiplier $m = \operatorname{lcm}_q(\operatorname{LC}(f), \operatorname{LC}(g))$, then their S-polynomial S(f,g) would equal 0 since $m = \operatorname{lcm}_q(\operatorname{LC}(f), \operatorname{LC}(g)) = 0$ in (3.1) as per (5.10). Hence we need to revise Definition 3.4 as follows.

Definition 5.11 (S-polynomial over a normal PQR R_q).

Let R_q be a normal PQR as in Defintion 5.4. Suppose that $f \in R_q[\tilde{x}] \setminus R_q$ and $g \in (R_q[\tilde{x}])^* \setminus R_q^{\times}$. Let us use c_f and c_g to denote LC(f) and LC(g) in R_q^* respectively. With the epimorphism $\sigma_q \colon R \to R_q$ and injection $\iota_q \colon R_q \to R$ defined as in (5.4) and (5.5) respectively, we denote $l_f := \iota_q(c_f)$ and $l_g := \iota_q(c_g)$. We also define multipliers $m_f := \sigma_q(\text{lcm}(l_f, l_g)/l_f)$ and $m_g := \sigma_q(\text{lcm}(l_f, l_g)/l_g)$ as well as the monomial $\tilde{x}^{\gamma} := \text{lcm}(\text{LM}(f), \text{LM}(g)) \in [\tilde{x}]$. Then the following polynomial:

$$S(f,g) := \frac{m_f \tilde{\boldsymbol{x}}^{\gamma}}{\text{LM}(f)} f - \frac{m_g \tilde{\boldsymbol{x}}^{\gamma}}{\text{LM}(g)} g$$
 (5.16)

is called the S-polynomial of f and g in $R_q[\tilde{x}]$.

In particular, when $f \in R_q[\tilde{x}] \setminus R_q$ and $g \in R_q^* \setminus R_q^*$, we can take LM(g) = 1 and $c_g = LC(g) = g$ in Definition 5.11. If we define $l_g := \iota_q(g)$, the definitions for m_f and m_g in (5.16) are unaltered. Now $\tilde{x}^{\gamma} = LM(f)$ and the S-polynomial in (5.16) becomes:

$$S(f,g) := m_f f - m_g g \cdot LM(f). \tag{5.17}$$

Lemma 5.12. For $f \in R_q[\tilde{x}] \setminus R_q$ and $g \in R_q^* \setminus R_q^\times$, and with the same notations as in (5.17), let us further denote $d := \gcd(l_f, l_g)$. Then the S-polynomial S(f, g) in (5.17) satisfies the following identity:

$$S(f,g) = \sigma_q \left(\frac{l_g}{d}\right) (f - LT(f)) = m_f(f - LT(f))$$
(5.18)

with $m_f = \sigma_q(\text{lcm}(l_f, l_g)/l_f)$ being defined as in (5.16).

Proof. It is evident that $m_f = \sigma_q(l_g/d)$ and $m_g = \sigma_q(l_f/d)$. Hence from (5.17) follows directly:

$$S(f,g) = \sigma_q \left(\frac{l_g}{d}\right) f - \sigma_q \left(\frac{l_f}{d}\right) \sigma_q(l_g) \cdot \text{LM}(f) = \sigma_q \left(\frac{l_g}{d}\right) (f - \text{LT}(f))$$

since $\sigma_q \circ \iota_q$ is the identity map on R_q .

There is a special kind of S-polynomials for $f \in R_q[\tilde{x}] \setminus R_q$ when $c_f = LC(f) \in R_q^* \setminus R_q^*$.

$$S(f,q) := n_f f = n_f (f - LT(f))$$
 (5.19)

with $n_f := \sigma_q(\operatorname{lcm}(l_f, q)/l_f) = \sigma_q(q/\operatorname{gcd}(l_f, q))$. Here $l_f := \iota_q(c_f)$ as in (5.16).

Lemma 5.13. For $f, g \in R_q[\tilde{x}] \setminus R_q$, suppose that LM(f) and LM(g) are relatively prime. With the same notations as in Definition 5.11, let us also denote $d := \gcd(l_f, l_g)$. Then their S-polynomial in (5.16) satisfies:

$$S(f,g) = \frac{f_1 \cdot \operatorname{LT}(g) - g_1 \cdot \operatorname{LT}(f)}{\sigma_q(d)} = \frac{f_1 g - g_1 f}{\gcd_q(\operatorname{LC}(f), \operatorname{LC}(g))}$$
(5.20)

with $f_1 := f - LT(f)$ and $g_1 := g - LT(g)$. Moreover, we have:

$$\max\{\operatorname{LM}(f_1) \cdot \operatorname{LM}(g), \operatorname{LM}(g_1) \cdot \operatorname{LM}(f)\} \prec \operatorname{LM}(f) \cdot \operatorname{LM}(g). \tag{5.21}$$

Proof. In the definition (5.16) we have $\tilde{\boldsymbol{x}}^{\gamma} = \text{LM}(f) \cdot \text{LM}(g)$. Furthermore, $m_f = \sigma_q(l_g/d)$ and $m_g = \sigma_q(l_f/d)$. Thus the first equality in (5.20) follows from (5.16) as follows.

$$S(f,g) = \sigma_q\left(\frac{l_g}{d}\right) \cdot LM(g)(f_1 + LT(f)) - \sigma_q\left(\frac{l_f}{d}\right) \cdot LM(f)(g_1 + LT(g)),$$

where we can write $\sigma_q(l_g/d)$ as $\sigma_q(l_g)/\sigma_q(d) = c_g/\sigma_q(d)$ and same for $\sigma_q(l_f/d)$.

The second equality in (5.20) follows from the first one by substituting $g - g_1$ and $f - f_1$ for LT(g) and LT(f) respectively. And the denominator $\sigma_q(d) = \gcd_q(LC(f), LC(g))$ is defined as in (5.9).

The inequality (5.21) is evident since $LM(f_1) \prec LM(f)$ and $LM(g_1) \prec LM(g)$. \square

From Lemma 5.12 and Lemma 5.13 and based on Theorem 5.10, we can easily deduce the following corollary for algorithmic simplifications.

Corollary 5.14. For $f, g \in R_q[\tilde{x}] \backslash R_q$, suppose that LM(f) and LM(g) are relatively prime. With the same notations as in Lemma 5.13, if $\sigma_q(d) \in R_q^{\times}$, then their S-polynomial S(f,g) as in (5.20) can be properly reduced to 0 by f and g as in Theorem 5.10 with the multiplier $\sigma_q(d)$ and quotients f_1 and $-g_1$.

In particular, for $f \in R_q[\tilde{x}] \setminus R_q$ and $g \in R_q^* \setminus R_q^\times$, with the same notations as in Lemma 5.12, if $\sigma_q(d) = \gcd_q(\operatorname{LC}(f), g) \in R_q^\times$, then their S-polynomial S(f, g) as in (5.18) can be properly reduced to 0 by g with the multiplier $\sigma_q(d) \in R_q^\times$ and quotient $f - \operatorname{LT}(f)$.

Definition 5.15 (LCM representation⁷).

For $F = \{f_1, \ldots, f_s\} \subset (R_q[\tilde{\boldsymbol{x}}])^* \setminus R_q^{\times}$, we say that an S-polynomial S(f, g) has an LCM representation with respect to F if there exist $\{h_1, \ldots, h_s\} \subset R_q[\tilde{\boldsymbol{x}}]$ satisfying:

$$S(f,g) = \sum_{j=1}^{s} h_j f_j$$

such that the following condition holds:

$$\max_{1 \le j \le s} \{ LM(h_j) \cdot LM(f_j) \} \prec lcm(LM(f), LM(g)).$$
 (5.22)

Remark 5.16. The LCM representation is especially suitable for the representation of S-polynomials over such rings with zero divisors as the PQR R_q . In particular, the condition (5.21) in Lemma 5.13 amounts to the condition (5.22) for the LCM representation with respect to $\{f,g\}$ when the multiplier $\sigma_q(d) \in R_q^{\times}$ in (5.20) as in Corollary 5.14. Similarly the identity (5.18) is also an LCM representation of S(f,g) with respect to $g \in R_q^{*} \setminus R_q^{\times}$ when the multiplier $\sigma_q(d) \in R_q^{\times}$.

For $g \in R_q^* \setminus R_q^{\times}$ and $f \in R_q[\tilde{\boldsymbol{x}}] \setminus R_q$, we shall also use the following relation between (5.16) and (5.17):

$$S(f, g \cdot LM(f)) = S(f, g). \tag{5.23}$$

Lemma 5.17. (i) The S-polynomial in (5.16) satisfies $LM(S(f,g)) \prec \tilde{x}^{\gamma}$. The S-polynomials in (5.17) and (5.19) satisfy $LM(S(f,g)) \prec LM(f)$ and $LM(S(f,q)) \prec LM(f)$ respectively. (ii) The two multipliers m_f and m_g in (5.16) and (5.17) are not zero, that is, we always have $m_f, m_g \in R_q^*$ even if $lcm_q(LC(f), LC(g)) = 0$ with lcm_q defined as in (5.9).

⁷Please refer to [CLO15, P107, Definition 5] for its definition over a field instead.

Proof. To prove the conclusion (i), it suffices to prove that $m_f c_f = m_g c_g$. In particular, we have $c_g = g$ in the case of (5.17). The composition $\sigma_q \circ \iota_q$ of the epimorphism σ_q in (5.4) and injection ι_q in (5.5) is the identity map on R_q . Hence $c_f = \sigma_q(\iota_q(c_f)) = \sigma_q(l_f)$ and we have the following verification:

$$m_f c_f = \sigma_q \left(\frac{\operatorname{lcm}(l_f, l_g)}{l_f} \right) \cdot \sigma_q(l_f) = \sigma_q(\operatorname{lcm}(l_f, l_g)).$$

Similarly we have $m_g c_g = \sigma_q(\text{lcm}(l_f, l_g))$ and thus the conclusion follows. The conclusion for (5.19) is easy to corroborate.

The conclusion (ii) follows from the identity:

$$m_f = \sigma_q \left(\frac{\operatorname{lcm}(l_f, l_g)}{l_f} \right) = \sigma_q \left(\frac{l_g}{\gcd(l_f, l_g)} \right).$$
 (5.24)

In fact, according to the definition of leading coefficients in Notation 2.4, $c_g = LC(g) \in R_q^*$. Hence $l_g = \iota_q(c_g) \in (K[x_1])^*$ and $deg(l_g) < deg(q)$ as in (5.1). Thus the multiplier $m_f \in R_q^*$ by (5.24). The same holds for m_g .

A conspicuous difference between the S-polynomials in Definition 5.11 over a normal PQR and those in Definition 3.4 over a PID is that the leading coefficients $m_f \cdot \text{LC}(f) = m_f c_f$ and $m_g \cdot \text{LC}(g) = m_g c_g$ in (5.16) and (5.17) might be zero due to the possible existence of zero divisors in R_q^* . We shall prove that this imposes no hindrance to the viability of our computations. For S-polynomials over a normal PQR R_q , Lemma 5.17 (i) is equivalent to the inequality (3.2).

For $f, g \in (R_q[\tilde{\boldsymbol{x}}])^* \setminus R_q^{\times}$, let us define:

$$\operatorname{lcm}_{q}(\operatorname{LT}(f), \operatorname{LT}(g)) := \operatorname{lcm}_{q}(\operatorname{LC}(f), \operatorname{LC}(g)) \cdot \operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g))$$
(5.25)

with lcm_q being defined as in (5.9).

Let us use the same notations as in Definition 5.11 for S-polynomials. For $f, g \in (R_q[\tilde{x}])^* \setminus R_q^{\times}$ without both of them in $R_q^* \setminus R_q^{\times}$, we define the LCM multiplier of f and g as:

$$\operatorname{cmr}(g|f) := \sigma_q \left(\frac{\operatorname{lcm}(l_f, l_g)}{l_f}\right) \frac{\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g))}{\operatorname{LM}(f)} = \frac{m_f \tilde{\boldsymbol{x}}^{\gamma}}{\operatorname{LM}(f)}.$$
 (5.26)

Then the definition for the S-polynomial S(f,g) in (5.16) can be written as:

$$S(f,g) = \operatorname{cmr}(g|f) \cdot f - \operatorname{cmr}(f|g) \cdot g. \tag{5.27}$$

Lemma 5.18. For $f, g, h \in (R_q[\tilde{x}])^* \setminus R_q^{\times}$ with at most one of them in $R_q^* \setminus R_q^{\times}$, if $lcm(LM(f), LM(g)) \in \langle LM(h) \rangle$, then we have the following relationship between their S-polynomials:

$$\lambda S(f,g) = \frac{\lambda \cdot \operatorname{cmr}(g|f)}{\operatorname{cmr}(h|f)} S(f,h) - \frac{\lambda \cdot \operatorname{cmr}(f|g)}{\operatorname{cmr}(h|g)} S(g,h)$$
 (5.28)

with the LCM multiplier cmr being defined as in (5.26). Here the multiplier $\lambda := \sigma_q(l_h/d) \in R_q^*$ with $l_h := \iota_q(\operatorname{LC}(h))$ and $d := \gcd(\operatorname{lcm}(l_f, l_g), l_h)$.

Proof. Same as the conclusion in Lemma 5.17 (ii), the denominators $\operatorname{cmr}(h|f)$ and $\operatorname{cmr}(h|g)$ in (5.28) are nonzero, which is the reason why we use the LCM multiplier cmr as in (5.26) instead of the lcm_q as in (5.25).

The multiplier $\lambda := \sigma_q(l_h/d) \in R_q^*$ can indeed render the two fractions in (5.28) terms in $R_q[\tilde{x}]$. The proof is totally similar to that for the multiplier λ in the identity (3.9) in Lemma 3.8.

We can corroborate the identity in (5.28) directly by the form of S-polynomials in (5.27) as well as the definition of LCM multipliers in (5.26).

Based on the above discussions we now analyze the incompatible part IP(χ_{ε}) of the pseudo-eliminant χ_{ε} . Our goal is to determine the corresponding factors of the eliminant χ of the original ideal I.

Let $\{\Omega_i \colon 1 \leq i \leq s\}$ be the composite divisor sets of the incompatible part $P(\chi_{\varepsilon})$ as in Definition 4.7. For a multiplicity i satisfying $1 \leq i \leq s$ and composite divisor ω^i with $\omega \in \Omega_i \subset K[x_1]$, let us denote ω^i as q and consider the normal PQR R_q that is isomorphic to the PQR $\bar{R} = K[x_1]/(q) = K[x_1]/(\omega^i)$ as in Definition 5.4. In short, from now on our discussions and computations are over the normal PQR R_q as follows.

$$R_q \cong K[x_1]/(q), \quad q = \omega^i, \quad \omega \in \Omega_i \subset K[x_1].$$
 (5.29)

We shall follow the pseudo-eliminant computation in Algorithm 3.9 to compute the eliminant of the ideal $I + (q) = I + (\omega^i)$ except that we shall compute it over the normal PQR R_q .

If we extend the ring epimorphism σ_q in (5.4) such that it is the identity map on the variables $\tilde{\boldsymbol{x}}$, then σ_q induces a ring epimorphism from $(K[x_1])[\tilde{\boldsymbol{x}}]$ to $R_q[\tilde{\boldsymbol{x}}]$ which we still denote as σ_q as follows.

$$\sigma_q : (K[x_1])[\tilde{\boldsymbol{x}}] \to R_q[\tilde{\boldsymbol{x}}]: \quad \sigma_q\left(\sum_{j=1}^s c_j \tilde{\boldsymbol{x}}^{\alpha_j}\right) := \sum_{j=1}^s \sigma_q(c_j) \tilde{\boldsymbol{x}}^{\alpha_j}.$$
 (5.30)

Please note that when the composite divisor q bears the form $x_1 - a$ with $a \in K$, the epimorphism σ_q in (5.4) becomes $\sigma_q(f) = f(a) \in K$ for $f \in K[x_1]$. In this case the coefficients $\sigma_q(c_j)$ in (5.30) become $c_j(a) \in K$. We call the induced epimorphism σ_q in (5.30) a specialization associated with $a \in K$.

Similarly we can extend the injection ι_q in (5.5) to an injection of $R_q[\tilde{x}]$ into $(K[x_1])[\tilde{x}]$ in the way that it is the identity map on the variables \tilde{x} as follows.

$$\iota_q \colon R_q[\tilde{\boldsymbol{x}}] \to (K[x_1])[\tilde{\boldsymbol{x}}] \colon \quad \iota_q\left(\sum_{j=1}^s c_j \tilde{\boldsymbol{x}}^{\alpha_j}\right) := \sum_{j=1}^s \iota_q(c_j) \tilde{\boldsymbol{x}}^{\alpha_j}.$$
(5.31)

Further, it is evident that $\sigma_q \circ \iota_q$ is the identity map on $R_q[\tilde{x}]$.

Lemma 5.19. Let \succ be an elimination ordering on [x] as in Definition 3.1 and $F := \{f_j : 0 \leq j \leq s\} \subset (K[x_1])[\tilde{x}] \setminus K$ a basis of a zero-dimensional ideal I. Suppose that $q = \omega^i$ is a composite divisor of the incompatible part $\operatorname{IP}(\chi_{\varepsilon})$ as in Definition 4.7 and R_q a normal PQR as in Definition 5.4. Then for $F \cap K[x_1]$ and $G := F \setminus K[x_1]$ as in the Initialization of Algorithm 3.9, we have $\sigma_q(F \cap K[x_1]) = \{0\}$ and $\sigma_q(G)$ is a basis of $I_q := \sigma_q(I)$ under the epimorphism σ_q in (5.30).

Proof. The construction of the composite divisor set Ω_i in Algorithm 4.6 indicates that the pseudo-eliminant χ_{ε} is divisible by the composite divisor $q = \omega^i$. The computation of χ_{ε} in Algorithm 3.9 shows that every element of $F \cap K[x_1]$ is divisible by f_0 in the Initialization of Algorithm 3.9 and thus by χ_{ε} . Hence $F \cap K[x_1] \subset (\omega^i) = (q) \subset K[x_1]$, which yields $\sigma_q(F \cap K[x_1]) = \{0\}$. Then readily follows the conclusion $I_q = \langle \sigma_q(G) \rangle$ with $\sigma_q(G) \subset R_q[\tilde{x}] \setminus R_q$.

In the following Algorithm 5.20 that is parallel to Algorithm 3.9, please note that all the binary operations over the ring R_q in (5.29), i.e., the additions and multiplications over the ring R_q in (5.29), are performed according to those defined in (5.2).

Based on Lemma 5.19, in what follows let us abuse the notations a bit and simply denote $\sigma_q(G)$ as $F = \{f_j : 1 \leq j \leq s\} \subset R_q[\tilde{x}] \setminus R_q$.

Algorithm 5.20 (Proper eliminant and proper basis over a normal PQR R_q).

Input: A finite polynomial set $F \subset R_q[\tilde{x}] \setminus R_q$.

Output: A proper eliminant $e_q \in R_q$ and proper basis $B_q \subset R_q[\tilde{x}] \setminus R_q$.

Initialization: A temporary set $\mathfrak{S} := \emptyset$ in $R_q[\tilde{\boldsymbol{x}}] \setminus R_q$ for S-polynomials; a temporary $e \in R_q$ as e := 0.

For each pair $f, g \in F$ with $f \neq g$, we invoke Procedure \mathcal{R} as follows to compute their S-polynomial S(f, g).

Procedure \mathcal{R} :

If LM(f) and LM(g) are relatively prime, we compute the multiplier $\sigma_q(d) := \gcd_q(\operatorname{LC}(f),\operatorname{LC}(g))$ as in (5.20). If $\sigma_q(d) \in R_q^* \setminus R_q^{\times}$, we compute and then add the S-polynomial S(f,g) into the set \mathfrak{S} . If $\sigma_q(d) \in R_q^{\times}$ as in Corollary 5.14, we disregard S(f,g).

If there exists an $h \in F \setminus \{f,g\}$ such that $\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g)) \in \langle \operatorname{LM}(h) \rangle$, and the triangular identity (5.28) has not been applied to the same triplet $\{f,g,h\}$ before, we compute the multiplier λ as in (5.28). If $\lambda \in R_q^* \setminus R_q^\times$, we compute and then add the S-polynomial S(f,g) into the set \mathfrak{S} . If $\lambda \in R_q^\times$, we disregard S(f,g).

If neither of the above two cases is true, we compute their S-polynomial S(f,g) as in (5.16). Then we add S(f,g) into the set \mathfrak{S} .

End of \mathcal{R}

We recursively repeat Procedure \mathcal{P} as follows for proper reductions of all the S-polynomials in \mathfrak{S} .

Procedure \mathcal{P} :

For an $S \in \mathfrak{S}$, we invoke Theorem 5.10 to make a proper reduction of S by F.

If the remainder $r \in R_q[\tilde{x}] \setminus R_q$, we add r into F. For every $f \in F \setminus \{r\}$, we invoke Procedure \mathcal{R} to compute the S-polynomial S(f,r).

If the remainder $r \in R_q^* \setminus R_q^*$ and e = 0, we redefine $e := \sigma_q(\gcd(\iota_q(r), q))$ with σ_q and ι_q as in (5.4) and (5.5) respectively.

If the remainder $r \in R_q^* \setminus R_q^*$ and $e \in R_q^*$, we compute $d = \gcd_q(r, e)$ as in (5.9). If d is not an associate of e, we redefine e := d.

If the remainder $r \in R_q^{\times}$, we halt the algorithm and output $e_q = 1$. Then we delete S from \mathfrak{S} .

End of \mathcal{P}

Next we recursively repeat Procedure Q as follows for proper reductions of the special kinds of S-polynomials in (5.17) and (5.19).

Procedure Q:

If $\mathfrak{S} = \emptyset$ and e = 0, then for every $f \in F$ with $LC(f) \in R_q^* \setminus R_q^*$, we compute the S-polynomial S(f,q) as in (5.19) and add it into \mathfrak{S} if this has not been done for f in a previous step.

Then we recursively repeat Procedure \mathcal{P} .

If $\mathfrak{S} = \emptyset$ and $e \in R_q^*$, then for every $f \in F$ with $LC(f) \in R_q^* \setminus R_q^\times$, if $\sigma_q(d) := \gcd_q(LC(f), e) \in R_q^* \setminus R_q^\times$ as in Corollary 5.14, we compute the S-polynomial S(f, e) as in (5.18) and add it into \mathfrak{S} unless an S-polynomial equal to uS(f, e) with $u \in R_q^\times$ had been added into \mathfrak{S} in a previous step.

Then we recursively repeat Procedure \mathcal{P} .

End of Q

Finally we define $e_q := e$ and $B_q := F$ respectively.

We output e_q and B_q .

Remark 5.21. In Procedure \mathcal{Q} of Algorithm 5.20, the condition $LC(f) \in R_q^* \setminus R_q^*$ in the case of $e \in R_q^*$ is necessary because if $LC(f) \in R_q^*$, we would have $d := \gcd_q(LC(f), e) \in R_q^*$ as in Corollary 5.14.

Definition 5.22 (Proper eliminant e_q ; proper basis B_q ; modular eliminant χ_q).

We call the standard representation e_q^{st} as in (5.6) of the univariate polynomial $e_q \in I_q \cap R_q$ obtained in Algorithm 5.20, whether it is zero or not, a *proper* eliminant of the ideal I_q . In what follows we shall simply denote $e_q := e_q^{\text{st}}$ except for a necessary discrimination in the context. We also call the final polynomial set B_q obtained in Algorithm 5.20 a *proper* basis of I_q .

Let χ be the eliminant of a zero-dimensional ideal I and q a composite divisor as in (5.29). Suppose that σ_q is the epimorphism as in (5.30). Then we define $\chi_q := \sigma_q(\chi)$ as the *modular* eliminant of $I_q = \sigma_q(I)$.

Lemma 5.23. Let χ_q and e_q be the modular and proper eliminants of the ideal I_q respectively as in Definition 5.22. Then the following conclusions hold.

- (a) If the modular eliminant $\chi_q \in R_q^*$, then the eliminant χ is divisible by $\iota_q(\chi_q^{\text{st}})$ with ι_q being the injection as in (5.5) and χ_q^{st} the standard representation of χ_q in R_q as in (5.6). And we have $\text{mult}_p(\chi) = \text{mult}_p(\chi_q^{\text{st}})$ for every irreducible factor p of the composite divisor $q = \omega^i$. If $\chi_q = 0$, then χ is divisible by q and we have $\text{mult}_p(\chi) = \text{mult}_p(q)$ for every irreducible factor p of q.
- (b) The epimorphism σ_q as in (5.30) is also an epimorphism from $I \cap K[x_1]$ to $I_q \cap R_q$. Moreover, the proper and modular eliminants e_q and χ_q of I_q satisfy $e_q \in (\chi_q) = I_q \cap R_q$. In particular, we have $e_q = 0$ if $\chi_q = 0$.
- (c) For each pair $f \neq g$ in the polynomial set $B_q \cup \{e_q\}$ with $e_q \in R_q^*$ and B_q being the proper basis of I_q , the proper reduction of their S-polynomial S(f,g) by B_q yields a remainder $r \in (e_q) \subset (\chi_q) \subset R_q$ including the special case of r = 0. The same holds for the polynomial set $B_q \cup \{q\}$ when $e_q = 0$.
- (d) Algorithm 5.20 terminates in finite steps.

Proof. Let us first prove (a). When $\chi_q \in R_q^*$, by Lemma 5.3 as well as the definition of the standard representation $\chi_q^{\rm st}$ of χ_q as in (5.6), we know that $\chi_q = u\chi_q^{\rm st}$ with $u \in R_q^{\times}$ being a unit. Moreover, for every irreducible factor p of q, we have $0 \leq \operatorname{mult}_p(\chi_q^{\rm st}) \leq \operatorname{mult}_p(q)$ as per Lemma 5.3 (iii) and (5.6). As per Lemma 3.12 (i), the pseudo-eliminant χ_{ε} is divisible by χ . By Definition 4.7, the composite divisor $q = \omega^i$ satisfies $\operatorname{mult}_p(q) = i = \operatorname{mult}_p(\chi_{\varepsilon})$ for every irreducible factor p of q. Hence follows the following inequality:

$$0 < \operatorname{mult}_{p}(\chi) \le \operatorname{mult}_{p}(q) = i \tag{5.32}$$

for every irreducible factor p of q. Based on the division identity $\chi = hq + \iota_q(\chi_q) = hq + \iota_q(u\chi_q^{\rm st})$ that is parallel to (5.3), we can deduce that $\operatorname{mult}_p(\chi_q^{\rm st}) = \operatorname{mult}_p(\iota_q(\chi_q^{\rm st})) = \operatorname{mult}_p(\chi)$ for every irreducible factor p of q, which is similar to the deduction of (5.11). Thus the eliminant χ is divisible by $\iota_q(\chi_q^{\rm st})$ as in the conclusion (a) due to the arbitrariness of the irreducible factor p of q.

When the modular eliminant $\chi_q = 0$, the divisibility of χ by q can be readily deduced from the definition of the epimorphism σ_q in (5.4). Then the equality $\operatorname{mult}_p(\chi) = \operatorname{mult}_p(q)$ for every irreducible factor p of q can be deduced from (5.32).

Next let us prove (b). For every $r \in I_q \cap R_q$, assume that there exists $f \in I \setminus K[x_1]$ such that $\sigma_q(f) = r$. Then f can be written into $f = gq + \iota_q(r)$ with $\iota_q(r) \in K[x_1]$. Let us denote $d := \gcd(\chi,q) \in K[x_1]$. It is evident that we have $gq\chi/d \in \langle \chi \rangle \subset I$ and hence $(f - gq)\chi/d = \chi \cdot \iota_q(r)/d \in I \cap K[x_1]$. Moreover, $\sigma_q(\chi \cdot \iota_q(r)/d) = r\sigma_q(\chi/d)$ such that $\sigma_q(\chi/d) \in R_q^{\times}$ by Lemma 5.3 (i) since χ/d is relatively prime to q. Thus $\sigma_q \colon I \cap K[x_1] \longrightarrow I_q \cap R_q$ is an epimorphism. As a result, we have $I_q \cap R_q = (\chi_q)$ based on $I \cap K[x_1] = (\chi)$ as per Definition 3.3. Then the conclusion (b) readily follows from the fact that $e_q \in I_q \cap R_q$.

The proofs for the conclusions (c) and (d) are almost verbatim repetitions of those for Lemma 3.12 (ii) and (iii). In particular, the argument for (d) is based on the fact that $R_q[\tilde{x}]$ is also a Noetherian ring. In fact, the normal PQR R_q in Definition 5.4 is isomorphic to the Noetherian PQR \bar{R} in Definition 5.1.

Corollary 5.24. If the proper eliminant $e_q \in R_q^*$, then the eliminant χ is not divisible by the composite divisor $q = \omega^i$ of the incompatible part $IP(\chi_{\varepsilon})$. Moreover, if the proper eliminant $e_q \in R_q^{\times}$, then the eliminant χ is relatively prime to the composite divisor $q = \omega^i$.

Proof. If the eliminant χ is divisible by the composite divisor q, then the modular eliminant $\chi_q = \sigma_q(\chi) = 0$. By Lemma 5.23 (b) we can deduce that $e_q = 0$, contradicting $e_q \in R_q^*$.

If the proper eliminant $e_q \in R_q^{\times}$, there exists $b \in R_q^{\times}$ such that $1 = be_q \in (\chi_q)$ since we have $e_q \in (\chi_q)$ by Lemma 5.23 (b). Hence the modular eliminant $\chi_q = \sigma_q(\chi) \in R_q^{\times}$, from which we can deduce that the eliminant χ is relatively prime to the composite divisor $q = \omega^i$ by Lemma 5.3 (i).

In what follows let us exclude the trivial case when the proper eliminant $e_q \in R_q^{\times}$. That is, let us assume that the eliminant χ is not relatively prime to the composite divisor $q = \omega^i$.

Lemma 5.25. Let $F = \{f_j : 1 \leq j \leq s\} \subset R_q[\tilde{x}] \setminus R_q$ be a polynomial set over a normal PQR R_q as in (5.29). Suppose that for $1 \leq j \leq s$, each f_j has the same leading monomial $LM(f_j) = \tilde{x}^{\alpha} \in [\tilde{x}]$.

(a) If $f = \sum_{j=1}^{s} f_j$ satisfies $LM(f) \prec \tilde{x}^{\alpha}$, then there exist multipliers $b, b_j \in R_q^*$ for $1 \leq j < s$ such that

$$bf = \sum_{1 \le j \le s} b_j S(f_j, f_s)$$
 (5.33)

with the S-polynomial $S(f_i, f_s)$ being defined as in (5.16).

(b) For every irreducible factor p of the composite divisor q, we can always relabel the subscripts of the polynomial set $F = \{f_j : 1 \leq j \leq s\}$ such that the multiplier $b \in R_q^*$ in (5.33) is not divisible by p.

Proof. The following meticulous discussions are to ensure that our computations and manipulations are in the algebra $R_q[\tilde{x}]$.

(a) Let us denote $c_j := LC(f_j) \in R_q^*$ for $1 \le j \le s$. And $l_j := \iota_q(c_j)$ for $1 \le j \le s$. We also define multipliers $m_j := lcm(l_j, l_s)/l_j$ and $n_j := lcm(l_s, l_j)/l_s$ in $R = K[x_1]$ for $1 \le j < s$. Same as the proof for Lemma 5.17 (ii), we can substitute m_j by $l_s/\gcd(l_j, l_s)$ for $1 \le j < s$, which leads to the following identities since σ_q is a homomorphism:

$$\sigma_q(m_j) \cdot \sigma_q(\gcd(l_j, l_s)) = \sigma_q(l_s / \gcd(l_j, l_s)) \cdot \sigma_q(\gcd(l_j, l_s)) = \sigma_q(l_s) = c_s \in R_q^*.$$

Hence we have $\sigma_q(m_j) \in R_q^*$ for $1 \le j < s$. Similarly we also have $\sigma_q(n_j) \in R_q^*$ for $1 \le j < s$.

Let us define a multiplier $a := \operatorname{lcm}_{1 \leq j < s}(m_j) \in K[x_1]$ and $b := \sigma_q(a) \in R_q$. Consider the following identities of univariate polynomials that are not difficult to verify with $r := \gcd_{1 \leq j \leq s}(l_j)$:

$$a = \lim_{1 \le j < s} (m_j) = \lim_{1 \le j < s} \left(\frac{l_s}{\gcd(l_j, l_s)} \right) = \frac{l_s}{\gcd_{1 \le j < s} (\gcd(l_j, l_s))} = \frac{l_s}{r}.$$
 (5.34)

Based on (5.34), we can infer that $\sigma_q(r)b = \sigma_q(r)\sigma_q(l_s/r) = \sigma_q(l_s) = c_s \in R_q^*$. Hence we have $b \in R_q^*$.

Similarly to the above, with $a_j := a/m_j \in K[x_1]$ and $b_j := \sigma_q(a_j)$ for $1 \le j < s$, we can deduce that $b_j \in R_q^*$ from $b_j \cdot \sigma_q(m_j) = b \in R_q^*$. With the S-polynomials $S(f_j, f_s)$ defined in (5.16), we have the following equalities by denoting $\mu_j := \sigma_q(m_j) \in R_q^*$ and $\nu_j := \sigma_q(n_j) \in R_q^*$ for $1 \le j < s$:

$$\sum_{1 \le j < s} b_j S(f_j, f_s) = \sum_{1 \le j < s} b_j (\mu_j f_j - \nu_j f_s) = b \sum_{1 \le j < s} f_j - f_s \sum_{1 \le j < s} \sigma_q \left(\frac{a n_j}{m_j}\right)$$

$$= b \sum_{1 \le j < s} f_j - f_s \sum_{1 \le j < s} \sigma_q \left(\frac{l_j}{r}\right), \tag{5.35}$$

where the final equality (5.35) is based on the identity $n_j/m_j = l_j/l_s$ for $1 \le j < s$ as well as the identity (5.34). Moreover, it is easy to verify that $\sigma_q(l_j/r) = \sigma_q(l_j)/\sigma_q(r) = c_j/\sigma_q(r) \in R_q^*$ in (5.35). Hence the equality (5.35) now becomes:

$$\sum_{1 \le j < s} b_j S(f_j, f_s) = b \sum_{1 \le j < s} f_j - \frac{f_s}{\sigma_q(r)} \sum_{1 \le j < s} c_j.$$
 (5.36)

The given condition $LM(f) \prec \tilde{x}^{\alpha}$ amounts to $0 = \sum_{j=1}^{s} LC(f_j) = \sum_{j=1}^{s} c_j$, from which we can deduce that $\sum_{1 \leq i \leq s} c_j = -c_s$. We plug this into (5.36) to obtain:

$$\sum_{1 \le j \le s} b_j S(f_j, f_s) = b \sum_{1 \le j \le s} f_j + \frac{c_s f_s}{\sigma_q(r)}.$$
 (5.37)

Finally as per (5.34) we can easily deduce that $\sigma_q(r) = \sigma_q(l_s)/\sigma_q(a) = c_s/b \in R_q^*$. This combined with (5.37) yield the conclusion (5.33).

(b) The proof depends on a substitution of $m_j = \operatorname{lcm}(l_j, l_s)/l_j$ by $l_s/\gcd(l_j, l_s)$ for $1 \leq j < s$. More specifically, given an irreducible factor p of the composite divisor q, we can always change the order of the elements in the polynomial set $F = \{f_j \colon 1 \leq j \leq s\}$ so that $\operatorname{mult}_p(l_s) = \min_{1 \leq j \leq s} \{\operatorname{mult}_p(l_j)\}$. Hence $\operatorname{mult}_p(m_j) = \operatorname{mult}_p(l_s/\gcd(l_j, l_s)) = 0$ for $1 \leq j < s$. And $\operatorname{mult}_p(a) = 0$ since $a := \operatorname{lcm}_{1 \leq j < s}(m_j)$. Similar to the deduction of (5.11), from the division of a by q as $a = hq + \sigma_q(a) = hq + b$ we can deduce that $\operatorname{mult}_p(b) = \operatorname{mult}_p(a) = 0$.

The proof of the following theorem is similar to that of Theorem 4.10. However in the proof there is an unprecedented phenomenon that the leading coefficients of the polynomials in $R_q[\tilde{x}]$ might be zero divisors in R_q^* . I believe that a meticulous elaboration on the proof constitutes a remedy for the situation, albeit at a price of being a bit repetitious.

Theorem 5.26. Suppose that χ is the eliminant of a zero-dimensional ideal I in $(K[x_1])[\tilde{\boldsymbol{x}}]$ over a perfect field K and χ_{ε} a pseudo-eliminant of I. Let $q = \omega^i$ be a composite divisor of the incompatible part $IP(\chi_{\varepsilon})$ and R_q the normal PQR as in (5.29). Let e_q and χ_q in R_q denote the proper and modular eliminants respectively as in Definition 5.22.

- (a) If the proper eliminant $e_q = 0$, then the eliminant χ is divisible by the composite divisor $q = \omega^i$ of the incompatible part $IP(\chi_{\varepsilon})$ and hence the modular eliminant $\chi_q = 0$. For every irreducible factor p of the composite divisor q, we have $mult_p(\chi) = i$.
- (b) If the proper eliminant $e_q \in R_q^*$, then the eliminant χ is divisible by $\iota_q(e_q)$ with ι_q being the injection as in (5.5) and $e_q = e_q^{\text{st}}$ as in Definition 5.22. Hence the modular eliminant $\chi_q^{\text{st}} = e_q$. For every irreducible factor p of the composite divisor q, we have $\text{mult}_p(\chi) = \text{mult}_p(e_q)$.

Proof. Let us fix an irreducible factor p of the composite divisor $q=\omega^i$. If F is the originally given basis of the ideal I in $(K[x_1])[\tilde{\boldsymbol{x}}]$, then $\sigma_q(F)$ is a basis of the ideal $I_q=\sigma_q(I)$ in R_q under the epimorphism σ_q in (5.30) according to Lemma 5.19. For simplicity let us abuse the notation a bit and still denote $\sigma_q(F)$ as $F=\{f_j\colon 1\leq j\leq s\}\subset R_q[\tilde{\boldsymbol{x}}]\setminus R_q$. Then there exist $h_j\in R_q[\tilde{\boldsymbol{x}}]$ for $1\leq j\leq s$ such that the modular eliminant $\chi_q=\sigma_q(\chi)\in R_q$ can be written as:

$$\chi_q = \sum_{j=1}^s h_j f_j. \tag{5.38}$$

Suppose that $\max_{1 \leq j \leq s} \{ \operatorname{LM}(h_j) \cdot \operatorname{LM}(f_j) \} = \tilde{\boldsymbol{x}}^{\beta}$. Similar to (4.9), we collect and rename the set $\{f_j : \operatorname{LM}(h_j f_j) = \tilde{\boldsymbol{x}}^{\beta}, 1 \leq j \leq s \}$ as a new set $B_t := \{g_j : 1 \leq j \leq t \}$. Let us first make an assumption that $B_t \neq \emptyset$. We shall address the special case

of $B_t = \emptyset$ shortly afterwards. Of course the subscripts of the functions $\{h_j\}$ are relabelled accordingly. It is easy to see that for $g_j \in B_t$ with $1 \le j \le t$, we have $LC(h_j) \cdot LC(g_j) \in R_q^*$. In this way (5.38) can be written as:

$$\chi_q = \sum_{j=1}^t h_j g_j + \sum_{f_j \in F \setminus B_t} h_j f_j,$$
 (5.39)

where the products $h_j f_j$ are those in (5.38) with $f_j \in F \setminus B_t$ for $1 \leq j \leq s$.

With $LT(h_j) := c_j \tilde{\boldsymbol{x}}^{\alpha_j}$ and $LC(h_j) = c_j \in R_q^*$ for $1 \le j \le t$, it suffices to study the following polynomial that is a summand of (5.39):

$$g := \sum_{j=1}^{t} \operatorname{LT}(h_j) \cdot g_j = \sum_{j=1}^{t} c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j.$$
 (5.40)

From $LM(\chi_q) = 1 \prec \tilde{x}^{\beta}$ in (5.39) we can deduce that $LM(g) \prec \tilde{x}^{\beta}$ in (5.40). As per Lemma 5.25 (a), there exist multipliers $b, b_j \in R_q^*$ for $1 \leq j < t$ such that:

$$bg = \sum_{1 \le j \le t} b_j S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c_t \tilde{\boldsymbol{x}}^{\alpha_t} g_t)$$
(5.41)

with $g_j \in B_t$ for $1 \leq j \leq t$. Moreover, let us fix an irreducible factor p of the composite divisor $q = \omega^i$. According to Lemma 5.25 (b), we can reorder the elements in B_t such that the multiplier b in (5.41) is not divisible by the irreducible factor p, i.e., $\text{mult}_p(b) = 0$.

In what follows let us elaborate on the simplifications of the S-polynomials in (5.41) that are similar to (4.12) and (4.13). The purpose of the following meticulous discussions is to ensure that all our manipulations in the proof are in the ring $R_q[\tilde{x}]$. For $1 \leq j \leq t$ we have $c_j \cdot \mathrm{LC}(g_j) = \mathrm{LC}(h_j) \cdot \mathrm{LC}(g_j) \in R_q^*$ by the definition of B_t in (5.39). Let us denote $C_j := c_j \cdot \mathrm{LC}(g_j) \in R_q^*$ for $1 \leq j \leq t$. And $L_j := \iota_q(C_j) \in (K[x_1])^*$ for $1 \leq j \leq t$. Let us also define multipliers $\mu_j := \sigma_q(\mathrm{lcm}(L_j, L_t)/L_j)$ and $\nu_j := \sigma_q(\mathrm{lcm}(L_t, L_j)/L_t)$ in R_q for $1 \leq j < t$. By the definition in (5.16), we have for $1 \leq j < t$:

$$S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c_t \tilde{\boldsymbol{x}}^{\alpha_t} g_t) = \frac{\mu_j c_j \tilde{\boldsymbol{x}}^{\beta}}{\operatorname{LM}(g_j)} g_j - \frac{\nu_j c_t \tilde{\boldsymbol{x}}^{\beta}}{\operatorname{LM}(g_t)} g_t.$$
 (5.42)

By Lemma 5.17 (ii), we have $\mu_j, \nu_j \in R_q^*$ for $1 \le j < t$.

For $1 \leq j \leq t$ let us denote $a_j := LC(g_j) \in R_q^*$ and $l_j := \iota_q(a_j) \in (K[x_1])^*$. The multipliers $m_j := \sigma_q(\operatorname{lcm}(l_j, l_t)/l_j)$ and $n_j := \sigma_q(\operatorname{lcm}(l_t, l_j)/l_t)$ are in R_q^* for $1 \leq j < t$ by Lemma 5.17 (ii). If we define $\tilde{\boldsymbol{x}}^{\gamma_j} := \operatorname{lcm}(LM(g_j), LM(g_t))$, we have for $1 \leq j < t$:

$$S(g_j, g_t) = \frac{m_j \tilde{\boldsymbol{x}}^{\gamma_j}}{\text{LM}(g_j)} g_j - \frac{n_j \tilde{\boldsymbol{x}}^{\gamma_j}}{\text{LM}(g_t)} g_t.$$
 (5.43)

by the definition in (5.16).

Now let us define the following multipliers for $1 \le j < t$:

$$w_j := \sigma_q \left(\frac{\operatorname{lcm}(L_j, L_t)}{\operatorname{lcm}(l_i, l_t)} \right). \tag{5.44}$$

From our assumption $C_j \in R_q^*$ we know that $\operatorname{mult}_{\varrho}(l_j) \leq \operatorname{mult}_{\varrho}(L_j)$ for every irreducible factor ϱ of the composite divisor q. Hence $\operatorname{lcm}(L_j, L_t)$ is divisible by $\operatorname{lcm}(l_j, l_t)$ and as a result, $w_j \in R_q$. We have the following representations of the S-polynomials in (5.42) by those in (5.43):

$$S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c_t \tilde{\boldsymbol{x}}^{\alpha_t} g_t) = w_j \tilde{\boldsymbol{x}}^{\beta - \gamma_j} S(g_j, g_t)$$
(5.45)

with $g_j \in B_t$ for $1 \le j \le t$ and B_t being the polynomial set as in (5.39). In fact, by $C_j = c_j \cdot LC(g_j) = c_j a_j \in R_q^*$ as above and $\sigma_q \circ \iota_q$ being the identity map on R_q , we have the following identities between the multipliers in (5.42) and those in (5.43):

$$\mu_{j}c_{j} = \frac{\mu_{j}C_{j}}{a_{j}} = \sigma_{q}\left(\frac{\operatorname{lcm}(L_{j}, L_{t})}{L_{j}}\right) \cdot \frac{\sigma_{q}(L_{j})}{a_{j}} = \frac{\sigma_{q}(\operatorname{lcm}(L_{j}, L_{t}))}{a_{j}};$$

$$w_{j}m_{j} = \sigma_{q}\left(\frac{\operatorname{lcm}(L_{j}, L_{t})}{\operatorname{lcm}(l_{j}, l_{t})}\right) \cdot \sigma_{q}\left(\frac{\operatorname{lcm}(l_{j}, l_{t})}{l_{j}}\right) \cdot \frac{\sigma_{q}(l_{j})}{a_{j}} = \frac{\sigma_{q}(\operatorname{lcm}(L_{j}, L_{t}))}{a_{j}}.$$

$$(5.46)$$

Thus by (5.46) we have $w_j m_j = \mu_j c_j$. Similarly we can prove that $w_j n_j = \nu_j c_t$. Hence follows the relationship (5.45) between the S-polynomials. Moreover, this shows that $w_j \in R_q^*$ whenever either $\mu_j c_j \in R_q^*$ or $\nu_j c_t \in R_q^*$ in (5.42).

Let $B_q = \{g_k \colon 1 \leq k \leq \tau\} \subset R_q[\tilde{x}] \setminus R_q$ be the proper basis of the ideal I_q obtained in Algorithm 5.20 such that the polynomial set B_t is a subset of B_q . In Algorithm 5.20 we have properly reduced every S-polynomial $S(g_j, g_t)$ in (5.43) by the proper basis B_q for $1 \leq j < t$, either directly or indirectly by the triangular identity (5.28) in Lemma 5.18. More specifically, according to Theorem 5.10, for $1 \leq j < t$, there exist a multiplier $\lambda_j \in R_q^{\times}$ as well as a remainder $r_j \in R_q$ and quotients $q_{jk} \in R_q[\tilde{x}]$ for $1 \leq k \leq \tau$ such that:

$$\lambda_j S(g_j, g_t) = \sum_{k=1}^{\tau} q_{jk} g_k + r_j = \sum_{k=1}^{\tau} q_{jk} g_k + \rho_j e_q.$$
 (5.47)

Here we abused the notations a bit and denote g_j as an element in B_t with $1 \le j \le t$ whereas g_k a proper basis element in $B_q \supset B_t$ with $1 \le k \le \tau$. The remainder r_j in (5.47) is a univariate polynomial in $(e_q) \subset R_q$ according to Lemma 5.23 (c). Hence in (5.47) we simply denote $r_j := \rho_j e_q$ with $\rho_j \in R_q$.

Based on (5.45) and (5.47), we obtain a pseudo-reduction of the S-polynomial $S(c_j \tilde{x}^{\alpha_j} g_j, c_t \tilde{x}^{\alpha_t} g_t)$ in (5.41) as follows for $1 \leq j < t$.

$$\lambda_j S(c_j \tilde{\boldsymbol{x}}^{\alpha_j} g_j, c_t \tilde{\boldsymbol{x}}^{\alpha_t} g_t) = w_j \tilde{\boldsymbol{x}}^{\beta - \gamma_j} \left(\sum_{k=1}^{\tau} q_{jk} g_k + \rho_j e_q \right).$$
 (5.48)

Here we still use the multiplier $\lambda_j \in R_q^{\times}$ in (5.47) for the pseudo-reduction in (5.48).

Let λ denote the product of all the multipliers $\lambda_j \in R_q^{\times}$ in (5.48) for $1 \leq j < t$. It is evident that we still have $\lambda \in R_q^{\times}$. Based on (5.41) and the pseudo-reductions of S-polynomials in (5.48), we obtain the following representation:

$$b\lambda g = \sum_{k=1}^{\tau} q_k g_k + \rho e_q \tag{5.49}$$

with $q_k := \sum_{1 \leq j < t} d_j \tilde{\boldsymbol{x}}^{\beta - \gamma_j} q_{jk}$ and $\rho := \sum_{1 \leq j < t} d_j \tilde{\boldsymbol{x}}^{\beta - \gamma_j} \rho_j$ if we denote $d_j := \lambda b_j w_j / \lambda_j \in R_q$ for $1 \leq j < t$. Moreover, the multiplier $b\lambda$ in (5.49) satisfies $\operatorname{mult}_p(b\lambda) = 0$ since $\operatorname{mult}_p(b) = 0$ in (5.41) for the fixed irreducible factor p of the composite divisor q and $\lambda \in R_q^{\times}$ in (5.49).

According to (5.15), we can deduce that $LM(S(g_j, g_t)) = \max_{1 \leq k \leq \tau} \{LM(q_{jk}) \cdot LM(g_k)\}$ holds for $1 \leq j < t$ in (5.47). The S-polynomial in (5.43) satisfies

$$LM(S(g_i, g_t)) \prec \tilde{\boldsymbol{x}}^{\gamma_j} \tag{5.50}$$

for $1 \le j < t$ according to Lemma 5.17 (i). Hence for $1 \le j < t$ and $1 \le k \le \tau$ we have the following estimates in (5.48):

$$\tilde{\boldsymbol{x}}^{\beta-\gamma_j} \cdot \text{LM}(q_{jk}) \cdot \text{LM}(g_k) \leq \tilde{\boldsymbol{x}}^{\beta-\gamma_j} \cdot \text{LM}(S(g_j, g_t)) \prec \tilde{\boldsymbol{x}}^{\beta}.$$
 (5.51)

From the above we can deduce the following inequality in (5.49):

$$\max\{\max_{1 \le k \le \tau} \{ LM(q_k) \cdot LM(g_k) \}, LM(\rho) \} \prec \tilde{\boldsymbol{x}}^{\beta}.$$
 (5.52)

There is a special kind of S-polynomials whose proper reductions as in (5.47) are performed in (5.20) of Lemma 5.13 instead of Algorithm 5.20. This is the case when $LM(g_j)$ and $LM(g_t)$ are relatively prime and $\sigma_q(d) \in R_q^{\times}$ as in Corollary 5.14. In this case (5.20) is an LCM representation, which is defined in Definition 5.15 and Remark 5.16. The condition (5.21) for the LCM representation amounts to the condition (5.50) so that the inequality (5.52) is still sound in this special case.

With an almost verbatim repetition of the arguments in (4.21) and (4.23), we can substitute the representation of $b\lambda g$ in (5.49) into that of the modular eliminant χ_q in (5.39) with the multiplier $b\lambda$ so as to obtain a new representation of $b\lambda \chi_q$ as follows.

$$b\lambda\chi_q = \sum_{k=1}^{\tau} v_k g_k + v_0 e_q \tag{5.53}$$

with $v_k \in R_q[\tilde{x}]$ for $0 \le k \le \tau$. Similar to (4.24) and according to (5.52) as well as a representation similar to (4.21), the leading monomials in (5.53) satisfy:

$$\max\left\{\max_{1\leq k\leq \tau}\left\{\mathrm{LM}(v_k)\cdot\mathrm{LM}(g_k)\right\},\mathrm{LM}(v_0)\right\} \prec \tilde{\boldsymbol{x}}^{\beta}. \tag{5.54}$$

In summary, the leading monomials in the representation (5.53) strictly decrease from those in the representation (5.38), up to the same multiplier $b\lambda$ as in (5.49) that satisfies $\operatorname{mult}_p(b\lambda) = 0$.

Now let us consider the case of $LM(h_jf_j) \prec LM(h_j) \cdot LM(f_j)$ in (5.38), i.e., $LC(h_j) \cdot LC(f_j) = 0$ for $1 \leq j \leq s$. In this case the set $B_t = \emptyset$ in (5.39) and the above discussions are of no avail any more. In this case we reorganize h_jf_j in the way of (5.40) with $LT(h_j) := c_j\tilde{\boldsymbol{x}}^{\alpha_j}$ and $c_j \in R_q^*$ as follows:

$$h_j f_j = \text{LT}(h_j) \cdot f_j + (h_j - \text{LT}(h_j)) \cdot f_j := c_j \tilde{\boldsymbol{x}}^{\alpha_j} f_j + (h_j - \text{LT}(h_j)) \cdot f_j$$
 (5.55)

such that $c_i \cdot LC(f_i) = 0$.

With $c_j \cdot LC(f_j) = 0$ as in (5.55), let us first consider the case when the proper eliminant $e_q \in R_q^*$. With $a_j := LC(f_j) \in R_q^* \setminus R_q^{\times}$ and ι_q the injection defined

in (5.5), let us denote $l_f := \iota_q(a_j)$ and $l_e := \iota_q(e_q)$. The S-polynomial $S(f_j, e_q)$ defined as in (5.17) satisfies the identity in (5.18):

$$S(f_j, e_q) = \sigma_q \left(\frac{l_e}{d}\right) (f_j - \text{LT}(f_j))$$
(5.56)

with $d := \gcd(l_f, l_e)$. Let us also denote $l_c := \iota_q(c_j)$ and $v := l_c d/l_e$. Then we have $v \in K[x_1]$. In fact, we have $v = l_c d/l_e = l_c l_f / \operatorname{lcm}(l_f, l_e)$. From $c_j \cdot \operatorname{LC}(f_j) = 0$ we can infer that $l_c l_f \in (q) \subset K[x_1]$. Since $e_q = e_q^{\operatorname{st}}$ as in Definition 5.22 and the composite divisor q is divisible by the proper eliminant $l_e = \iota_q(e_q)$ as per Lemma 5.3 (iii), $l_c l_f$ is divisible by l_e and hence divisible by $\operatorname{lcm}(l_f, l_e)$. Thus $v \in K[x_1]$. Moreover, $\sigma_q(v) \in R_q^*$ since $\sigma_q(v) \cdot \sigma_q(l_e/d) = \sigma_q(l_c) = c_j \in R_q^*$. A multiplication of $\sigma_q(v)\tilde{\boldsymbol{x}}^{\alpha_j}$ on both sides of (5.56) yields the following intriguing relationship between the polynomial $c_j\tilde{\boldsymbol{x}}^{\alpha_j}f_j$ in (5.55) and S-polynomial $S(f_j, e_q)$ in (5.56):

$$c_j \tilde{\boldsymbol{x}}^{\alpha_j} f_j = c_j \tilde{\boldsymbol{x}}^{\alpha_j} (f_j - \text{LT}(f_j)) = \sigma_q(v) \tilde{\boldsymbol{x}}^{\alpha_j} S(f_j, e_q). \tag{5.57}$$

With $c_j \cdot LC(f_j) = 0$ as in (5.55), if the proper eliminant $e_q = 0$, we computed the S-polynomial S(f,q) in Procedure \mathcal{Q} of Algorithm 5.20. In this case we have $a_j = LC(f_j) \in R_q^* \setminus R_q^*$, which is the condition for the definition of the S-polynomial $S(f_j,q)$ in (5.19). Thus we have the following relationship instead:

$$c_j \tilde{\boldsymbol{x}}^{\alpha_j} f_j = \sigma_q(\eta) \cdot \tilde{\boldsymbol{x}}^{\alpha_j} S(f_j, q), \tag{5.58}$$

where the S-polynomial $S(f_j,q) := n_f f_j$ with $n_f := \sigma_q(\operatorname{lcm}(l_f,q)/l_f)$ as in (5.19). Here $\eta := l_c l_f / \operatorname{lcm}(l_f,q)$ with the same notations $l_f = \iota_q(a_j)$ and $l_c := \iota_q(c_j)$ as in (5.56). From $c_j \cdot \operatorname{LC}(f_j) = 0$ we can infer that $l_c l_f \in (q) \subset K[x_1]$. Moreover, $\sigma_q(\eta) \in R_q^*$ since $\sigma_q(\eta) \cdot n_f = c_j \in R_q^*$.

If we have

$$\sigma_q(d) = \sigma_q(\gcd(l_f, l_e)) = \gcd_q(\operatorname{LC}(f), e_q) \in R_q^{\times}$$
(5.59)

in (5.56) as in Corollary 5.14, then we already have an LCM representation in terms of e_q in (5.56). Otherwise we made proper reductions of the S-polynomials $S(f_j, e_q)$ in (5.57) in Procedure $\mathcal Q$ of Algorithm 5.20 by the proper basis B_q . In Procedure $\mathcal Q$ we also made proper reductions of the S-polynomials $S(f_j, q)$ in (5.58) by B_q . They bear the same form as the one in (5.47) and also lead to strictly decreasing leading monomials as in (5.52). Moreover, it has a multiplier $\lambda_j \in R_q^{\times}$ like in (5.47). Thus these new kinds of S-polynomials in (5.57) and (5.58) have impact on the soundness of neither our arguments nor our conclusion.

If we define $g_0 := e_q$ in (5.53), then the representation of the modular eliminant $b\lambda\chi_q$ in (5.53) resembles the one in (5.38) except that we have the extra multiplier $b\lambda$ that is not divisible by the irreducible factor p of the composite divisor q. In particular, when the following holds in (5.53):

$$\operatorname{lm}(v_0 e_q) = \max_{1 \leq k \leq \tau} \{\operatorname{lm}(v_k) \cdot \operatorname{lm}(g_k)\} = \max_{1 \leq k \leq \tau} \{\operatorname{lm}(v_k g_k)\} := \tilde{\boldsymbol{x}}^{\gamma},$$

we have $g_0 = e_q \in B_t$ with B_t being defined similarly to that in (5.39) but with $\tilde{\boldsymbol{x}}^{\beta}$ substituted by $\tilde{\boldsymbol{x}}^{\gamma}$ here. Thus in this case there is a new kind of S-polynomials as follows besides the ilk in (5.41) and (5.45).

$$S(c_k \tilde{\boldsymbol{x}}^{\alpha_k} g_k, c \tilde{\boldsymbol{x}}^{\gamma} e_q) = S(c_k \tilde{\boldsymbol{x}}^{\alpha_k} g_k, c e_q) = w_k \tilde{\boldsymbol{x}}^{\alpha_k} S(g_k, e_q).$$
 (5.60)

Here $c_k \tilde{\boldsymbol{x}}^{\alpha_k} = \operatorname{LT}(v_k)$ and $c\tilde{\boldsymbol{x}}^{\gamma} = \operatorname{LT}(v_0)$. We have $\tilde{\boldsymbol{x}}^{\alpha_k} \cdot \operatorname{LM}(g_k) = \tilde{\boldsymbol{x}}^{\gamma}$. The S-polynomials $S(c_k \tilde{\boldsymbol{x}}^{\alpha_k} g_k, c_q)$ and $S(g_k, e_q)$ in (5.60) are defined according to (5.17) and (5.18). Similar to that in (5.44), we define the multiplier $w_k := \sigma_q(\operatorname{lcm}(L_k, L_e)/\operatorname{lcm}(l_k, l_e))$ with $L_k := \iota_q(C_k)$ and $L_e := \iota_q(C_e)$. Here $C_k := c_k \cdot \operatorname{LC}(g_k) \in R_q^*$ and $C_e := ce_q \in R_q^*$. And $l_k := \iota_q(a_k)$ and $l_e := \iota_q(e_q)$ with $a_k := \operatorname{LC}(g_k)$. The first equality in (5.60) is due to (5.23) and the second one is based on a deduction similar to (5.46). As we discussed in the paragraph of (5.59), this new kind of S-polynomials $S(g_k, e_q)$ in (5.60) have no impact on our conclusion.

Hence we can just treat the representation (5.53) as the one in (5.38) and repeat our discussions from (5.39) through (5.53). With a new multiplier not divisible by the irreducible factor p of the composite divisor q, we obtain a new representation of $b\lambda\chi_q$ whose leading monomials are strictly less than those in (5.53).

We continue to repeat the procedure in (5.53) of rewriting the representations of the modular eliminant $\chi_q = \sigma_q(\chi)$ so as to strictly reduce the orderings of their leading monomials like in (5.54). Moreover, the multipliers for the representations are never divisible by the irreducible factor p of the composite divisor q. Since the elimination ordering on $R_q[\tilde{x}]$ as in Definition 5.7 is a well-ordering, the above rewriting procedures halt after a finite number of repetitions. In this way we shall arrive at a representation of the modular eliminant χ_q bearing the following form:

$$\nu \chi_q = h e_q \tag{5.61}$$

with the multiplier ν not divisible by the irreducible factor p of the composite divisor q and hence $\nu \in R_q^*$. The other multiplier in (5.61) is $h \in R_q$. The following argument shows that it has no influence on our conclusion whether h = 0 or not.

In fact, if the proper eliminant $e_q = 0$ in (5.61), then $\nu \chi_q = 0$ in R_q . The modular eliminant χ_q is divisible by the incompatible divisor p^i since the multiplier ν is not divisible by p but the composite divisor $q = \omega^i$ satisfies $\operatorname{mult}_p(q) = i$. Since the incompatible divisor p^i of the composite divisor $q = \omega^i$ is arbitrary, the modular eliminant χ_q is divisible by $q = \omega^i$ and hence $\chi_q = 0$. Then by Lemma 5.23 (a) follows Theorem 5.26 (a) that the eliminant χ is divisible by the composite divisor $q = \omega^i$. Hence for every incompatible divisor p^i of $q = \omega^i$, we have $\operatorname{mult}_p(q) = i \leq \operatorname{mult}_p(\chi)$. Finally by (5.32) we can infer that $\operatorname{mult}_p(\chi) = i$.

Let us consider the case when the proper eliminant $e_q \in R_q^*$ in Theorem 5.26 (b). In the case of $he_q \in R_q^*$ in (5.61), from $\operatorname{mult}_p(\nu) = 0$ we can deduce that $\operatorname{mult}_p(e_q) + \operatorname{mult}_p(h) = \operatorname{mult}_p(\chi_q)$. Hence $\operatorname{mult}_p(e_q) \leq \operatorname{mult}_p(\chi_q)$. In the case of $he_q = 0$ in (5.61), which includes the case when h = 0, we have $\operatorname{mult}_p(\chi_q) = \operatorname{mult}_p(q)$ since $\operatorname{mult}_p(\nu) = 0$. Thus $\operatorname{mult}_p(e_q) \leq \operatorname{mult}_p(q) = \operatorname{mult}_p(\chi_q)$ as per Lemma 5.3 (ii). So from (5.61) we can infer that $\operatorname{mult}_p(e_q) \leq \operatorname{mult}_p(\chi_q)$ always holds. On the other hand, by Lemma 5.23 (b) we have $\operatorname{mult}_p(e_q) \geq \operatorname{mult}_p(\chi_q)$. Thus follows the equality $\operatorname{mult}_p(e_q) = \operatorname{mult}_p(\chi_q)$ for every irreducible factor p of the composite divisor q. We can infer that they have equal standard representations $\chi_q^{st} = e_q$ based on the definition in (5.6) and $e_q = e_q^{st}$ in Definition 5.22. Hence the eliminant χ is divisible by $\iota_q(\chi_q^{st}) = \iota_q(e_q)$ according to Lemma 5.23 (a). Moreover, Lemma 5.23 (a) also indicates that $\operatorname{mult}_p(\chi) = \operatorname{mult}_p(\chi_q^{st}) = \operatorname{mult}_p(e_q)$ for every irreducible factor p of q. This completes the proof of Theorem 5.26 (b).

6 A New Type of Bases for Zero-dimensional Ideals

In this section we procure the exact form of the eliminant χ of a zero-dimensional ideal I. This is based on our former analyses of the pseudo-eliminant χ_{ε} of I, i.e., of its compatible part $CP(\chi_{\varepsilon})$ in Theorem 4.10 and incompatible part $IP(\chi_{\varepsilon})$ in Theorem 5.26 respectively. We also formulate a decomposition of I according to $CP(\chi_{\varepsilon})$ and the composite divisors of $IP(\chi_{\varepsilon})$. In this way we acquire a new type of bases for I based on the exact form of χ , pseudo-basis B_{ε} obtained in Algorithm 3.9 and proper bases B_q obtained in Algorithm 5.20. Moreover, we address the ideal membership problem for this new type of bases and characterize the new type of bases in terms of their leading terms.

Definition 6.1 (Proper divisors θ_q and proper factor χ_{IP}).

For every composite divisor q of the incompatible part $IP(\chi_{\varepsilon})$ as in Definition 4.7, there corresponds to a proper eliminant e_q as in Definition 5.22. We define a proper divisor $\theta_q \in K[x_1]$ in accordance with e_q as follows.

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If e_q \in R_q^{\times}, we define \theta_q := 1;
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If $e_q = 0$, we define $\theta_q := q$;

If $e_q \in R_q^* \setminus R_q^*$, we define $\theta_q := \iota_q(e_q)$ with ι_q being the injection as in (5.5) and $e_q = e_q^{\text{st}}$ as in Definition 5.22.

We say that a proper divisor $\theta_q \in K[x_1]$ is nontrivial if $\theta_q \neq 1$.

We define the proper factor χ_{IP} as the product of all the proper divisors θ_q .

Theorem 6.2. The eliminant χ of a zero-dimensional ideal I is the product of the compatible part $CP(\chi_{\varepsilon})$ of the pseudo-eliminant χ_{ε} and proper factor χ_{IP} as in Definition 6.1. That is, $\chi = CP(\chi_{\varepsilon}) \cdot \chi_{IP}$. Moreover, the compatible part $CP(\chi_{\varepsilon})$ is relatively prime to the proper factor χ_{IP} .

Proof. According to Lemma 3.12 (i), every irreducible factor p of the eliminant χ is also a factor of either the compatible part $CP(\chi_{\varepsilon})$ or a composite divisor $q = \omega^i$ of the incompatible part $IP(\chi_{\varepsilon})$ as per (4.1). Moreover, the proper factor χ_{IP} is defined as the product of the proper divisors θ_q for all the composite divisors q of the incompatible part $IP(\chi_{\varepsilon})$. Hence we can easily deduce the conclusion from Theorem 4.10 and Theorem 5.26 as well as the definition of proper divisors θ_q in Definition 6.1. Finally, $CP(\chi_{\varepsilon})$ and χ_{IP} are relatively prime since $CP(\chi_{\varepsilon})$ and $IP(\chi_{\varepsilon})$ are relatively prime.

The nontrivial proper divisors θ_q in Definition 6.1 as well as the compatible part $CP(\chi_{\varepsilon})$ of the pseudo-eliminant χ_{ε} are pairwise relatively prime. In fact, the composite divisor q is divisible by the proper divisor $\theta_q = \iota_q(e_q^{\text{st}})$ when the proper eliminant $e_q \in R_q^* \setminus R_q^{\times}$ in Definition 6.1. And $\theta_q = q$ when $e_q = 0$. The composite divisors q are pairwise relatively prime as in (4.1). In this way the nontrivial proper divisors θ_q and the compatible part $CP(\chi_{\varepsilon})$ constitute a factorization of the eliminant χ according to Theorem 6.2. In the following lemma we make a decomposition of the zero-dimensional ideal I in accordance with the factorization of the eliminant χ .⁸

⁸Please also refer to [BW93, P337, Lemma 8.5] for a similar decomposition.

Lemma 6.3. Let $\{b_j : 1 \leq j \leq s\} \subset K[x_1] \setminus K$ be pairwise relatively prime and $b := \prod_{j=1}^s b_j$. Then for an arbitrary ideal $J \subset (K[x_1])[\tilde{x}]$, we have:

$$J + \langle b \rangle = \bigcap_{j=1}^{s} (J + \langle b_j \rangle), \tag{6.1}$$

where $\langle b \rangle$ and $\langle b_j \rangle$ denote the principal ideals in $(K[x_1])[\tilde{\boldsymbol{x}}]$ generated by b and b_j respectively for $1 \leq j \leq s$.

Proof. It is evident that the inclusion " \subset " holds. The proof of the reverse inclusion is as follows. For every $f \in \bigcap_{j=1}^s (J + \langle b_j \rangle)$, there exist $g_j \in J$ and $h_j \in (K[x_1])[\tilde{\boldsymbol{x}}]$ such that $f = g_j + h_j b_j$ for $1 \le j \le s$. Moreover, $\{b/b_j \colon 1 \le j \le s\}$ have no nontrivial common factors. Hence there exist $a_j \in K[x_1]$ for $1 \le j \le s$ such that $\sum_{j=1}^s a_j b/b_j = 1$. Now we have:

$$f = \sum_{j=1}^{s} f a_j b / b_j = \sum_{j=1}^{s} (g_j + h_j b_j) a_j b / b_j = \sum_{j=1}^{s} \frac{a_j b}{b_j} g_j + b \sum_{j=1}^{s} h_j a_j \in J + \langle b \rangle. \quad \Box$$

Let $\Theta:=\{q_j\colon 1\leq j\leq t\}$ be the set of composite divisors of the incompatible part $\mathrm{IP}(\chi_\varepsilon)$ such that their corresponding proper divisors θ_{q_j} as in Definition 6.1 are not trivial, i.e., $\theta_{q_j}\neq 1$ for $1\leq j\leq t$. We have the following decomposition of the zero-dimensional ideal $I=I+\langle\chi\rangle$ according to Theorem 6.2 and Lemma 6.3.

$$I = (I + \langle CP(\chi_{\varepsilon}) \rangle) \cap \bigcap_{q \in \Theta} (I + \langle \theta_q \rangle).$$
 (6.2)

Lemma 6.4. Suppose that $I \subset (K[x_1])[\tilde{x}]$ is a zero-dimensional ideal with an elimination ordering on [x] as in Definition 3.1. Let $B_{\varepsilon} = \{g_k : 1 \leq k \leq s\}$ be a pseudo-basis of I and $d = CP(\chi_{\varepsilon})$ the compatible part of the pseudo-eliminant χ_{ε} associated with B_{ε} . For every $f \in I$, there exist $\{v_k : 0 \leq k \leq \tau\} \subset (K[x_1])[\tilde{x}]$ and a multiplier λ relatively prime to d such that:

$$\lambda f = \sum_{k=1}^{\tau} v_k g_k + v_0 \chi_{\varepsilon}. \tag{6.3}$$

Moreover, the polynomials in (6.3) satisfy the following condition:

$$LM(f) = \max \{ \max_{1 \le k \le \tau} \{ LM(v_k g_k) \}, LM(v_0) \}.$$
 (6.4)

Proof. The proof for the conclusion is almost a verbatim repetition of that for Theorem 4.10. More specifically, suppose that $f \in I$ can be written as

$$f = \sum_{j=0}^{s} h_j f_j \tag{6.5}$$

with $\{f_j\colon 0\leq j\leq s\}\subset (K[x_1])[\tilde{\boldsymbol{x}}]\setminus K$ being the basis $G\cup\{f_0\}$ of the ideal I after the Initialization in Algorithm 3.9 and $\{h_j\colon 0\leq j\leq s\}\subset (K[x_1])[\tilde{\boldsymbol{x}}]$. In particular, $f_0\in (\chi_\varepsilon)\subset (d)\subset K[x_1]\setminus K^*$. It is evident that the conclusion holds when $\mathrm{LM}(f)=\max_{0\leq j\leq s}\{\mathrm{LM}(h_jf_j)\}$.

So in what follows let us suppose that $LM(f) \prec \max_{0 \leq j \leq s} \{LM(h_j f_j)\}$. In this case we treat f as the eliminant χ in (4.8). Let us fix an irreducible factor p of the compatible part $d = CP(\chi_{\varepsilon})$. We repeat the arguments verbatim from (4.9) through (4.23) to obtain a new representation like (4.23) as follows.

$$b\lambda f = \sum_{k=1}^{\tau} \mu_k g_k + \mu_0 \chi_{\varepsilon} \tag{6.6}$$

such that the multiplier $b\lambda$ is relatively prime to p, i.e., $\operatorname{mult}_p(b\lambda) = 0$. Here $\{g_k \colon 1 \leq k \leq \tau\} = B_{\varepsilon}$ is the pseudo-basis obtained in Algorithm 3.9 and $\mu_k \in (K[x_1])[\tilde{x}]$ for $0 \leq k \leq \tau$. The leading monomials of the representation in (6.6) are strictly less than those in (6.5), which is similar to (4.24). We repeat this procedure of rewriting the representations of $b\lambda f$ so that their leading monomials strictly decrease. Moreover, the multipliers for the representations are always relatively prime to the irreducible factor p of the compatible part d. Since the elimination ordering on $(K[x_1])[\tilde{x}]$ is a well-ordering and the leading monomials of a representation of f cannot be strictly less than $\operatorname{LM}(f)$, after a finite number of repetitions we obtain a representation bearing the following form:

$$\nu f = \sum_{k=1}^{\tau} w_k g_k + w_0 \chi_{\varepsilon} \tag{6.7}$$

with $w_k \in (K[x_1])[\tilde{\boldsymbol{x}}]$ for $0 \le k \le \tau$ such that

$$\max\left\{\max_{1\leq k\leq \tau}\left\{\mathrm{LM}(w_k g_k)\right\}, \mathrm{LM}(w_0 \chi_{\varepsilon})\right\} = \mathrm{LM}(f). \tag{6.8}$$

Moreover, the multiplier $\nu \in (K[x_1])^*$ in (6.7) is relatively prime to the irreducible factor p of the compatible part $d = CP(\chi_{\varepsilon})$.

Suppose that the compatible part d has a factorization into a product of compatible divisors that are pairwise relatively prime as in Definition 4.5, i.e., $d = \prod_{l=1}^t p_l^{n_l}$ with $n_l \in \mathbb{N}^*$. For each irreducible factor p_l of d, there corresponds to a representation of f in (6.7) that can be indexed by the subscript l of p_l with $1 \le l \le t$ as follows.

$$\nu_l f = \sum_{k=1}^{\tau} w_k^{(l)} g_k + w_0^{(l)} \chi_{\varepsilon}, \tag{6.9}$$

where the multiplier $\nu_l \in (K[x_1])^*$ in (6.9) is relatively prime to the irreducible factor p_l of the compatible part d. Moreover, the leading monomial identity (6.8) still holds for $w_k^{(l)}$ and $w_0^{(l)}$ with $1 \leq l \leq t$ in (6.9), i.e.,

$$\max\{\max_{1 \le k \le \tau} \{LM(w_k^{(l)}g_k)\}, LM(w_0^{(l)}\chi_{\varepsilon})\} = LM(f).$$
(6.10)

Let us denote $\lambda := \gcd_{1 \leq l \leq t} \{\nu_l\} \in (K[x_1])^*$. Then λ is relatively prime to the compatible part d. There exist $\{u_l \in K[x_1]: 1 \leq l \leq t\}$ such that $\lambda = \sum_{l=1}^t u_l \nu_l$. Hence we can obtain a representation of f as follows.

$$\lambda f = \sum_{l=1}^{t} u_l \nu_l f = \sum_{k=1}^{\tau} g_k \sum_{l=1}^{t} u_l w_k^{(l)} + \chi_{\varepsilon} \sum_{l=1}^{t} u_l w_0^{(l)} := \sum_{k=1}^{\tau} v_k g_k + v_0 \chi_{\varepsilon}$$
 (6.11)

with $v_k := \sum_{l=1}^t u_l w_k^{(l)}$ for $0 \le k \le \tau$ in $(K[x_1])[\tilde{\boldsymbol{x}}]$. This is (6.3) proved. Based on the identities $v_k = \sum_{l=1}^t u_l w_k^{(l)}$ in (6.11) for $0 \le k \le \tau$, we can infer the following inequalities between their leading monomials:

$$LM(v_0\chi_{\varepsilon}) \leq \max_{1\leq l\leq t} \{LM(w_0^{(l)}\chi_{\varepsilon})\}; \quad LM(v_kg_k) \leq \max_{1\leq l\leq t} \{LM(w_k^{(l)}g_k)\}$$
(6.12)

for $1 \le k \le \tau$ since $u_l \in K[x_1]$ for $1 \le l \le t$. A combination of (6.12) and (6.10) leads to:

$$\max\{\max_{1\leq k\leq \tau}\{\operatorname{LM}(v_kg_k)\}, \operatorname{LM}(v_0\chi_{\varepsilon})\} \leq \operatorname{LM}(f). \tag{6.13}$$

We can also infer the reverse inequality of (6.13) from (6.11). Thus follows the equality (6.4).

The following intriguing observation is crucial for its ensuing conclusions.

Lemma 6.5. Suppose that $I \subset (K[x_1])[\tilde{x}]$ is a zero-dimensional ideal with an elimination ordering on [x] as in Definition 3.1. Let $d = CP(\chi_{\varepsilon})$ be the compatible part of a pseudo-eliminant χ_{ε} of I. Consider the epimorphism $\sigma_d: (K[x_1])[\tilde{\boldsymbol{x}}] \to$ $R_d[\tilde{x}]$ as in (5.30) such that $I_d := \sigma_d(I)$. If we define $I_* := \{f \in I : \sigma_d(LC(f)) \in I\}$ R_d^* , then $\sigma_d(I_*) = I_d$.

Proof. Let χ_{IP} be the proper factor as in Definition 6.1 such that $d \cdot \chi_{\text{IP}} = \chi$ with χ being the eliminant of I as per Theorem 6.2. For every $h \in I_d$ with $LC(h) \in R_d^*$, suppose that $\sigma_d(g) = h$ with $g \in I$. Then $g - \iota_d(h) \in \langle d \rangle$ with the injection ι_d defined like in (5.31). Hence $\chi_{\text{IP}}g - \chi_{\text{IP}} \cdot \iota_d(h) \in \langle \chi \rangle \subset I$. Thus $\chi_{\text{IP}} \cdot \iota_d(h) \in I$ and $\sigma_d(\chi_{\text{IP}} \cdot \iota_d(h)) = \sigma_d(\chi_{\text{IP}}) \cdot \sigma_d(\iota_d(h)) = \sigma_d(\chi_{\text{IP}}) \cdot h \text{ since } \sigma_d \circ \iota_d \text{ is the identity map. We}$ can further infer that $\sigma_d(\chi_{\text{IP}})$ is a unit in R_d by Lemma 5.3 (i) since χ_{IP} is relatively prime to d as per Theorem 6.2. Hence $\sigma_d(\chi_{\text{IP}}) \cdot \text{LC}(h) \in R_d^*$, from which we can deduce that $\chi_{\text{IP}} \cdot \iota_d(h) \in I_*$. From $\sigma_d(\chi_{\text{IP}}) \in R_d^{\times}$ and $\sigma_d(\chi_{\text{IP}}) \cdot h = \sigma_d(\chi_{\text{IP}} \cdot \iota_d(h))$, we can also deduce that $I_d = \sigma_d(\chi_{\text{\tiny IP}}) \cdot I_d := \{\sigma_d(\chi_{\text{\tiny IP}}) \cdot h \colon h \in I_d\} \subset \sigma_d(I_*)$. The inclusion $I_d = \sigma_d(I) \supset \sigma_d(I_*)$ is evident.

For the ideal $I + \langle d \rangle = I + \langle \text{CP}(\chi_{\varepsilon}) \rangle$ in (6.2), we provide a characterization of the basis $B_{\varepsilon} \cup \{d\}$ in the following conclusions with B_{ε} being a pseudo-basis of I as in Definition 3.11. In particular, we address the ideal membership problem for the ideal $I + \langle d \rangle$.

Lemma 6.6. Suppose that $I \subset (K[x_1])[\tilde{x}]$ is a zero-dimensional ideal with an elimination ordering on [x] as in Definition 3.1. Let $B_{\varepsilon} = \{g_k : 1 \leq k \leq s\}$ be a pseudo-basis of I and $d = CP(\chi_{\varepsilon})$ the compatible part of the pseudo-eliminant χ_{ε} associated with B_{ε} . Consider the epimorphism $\sigma_{\rm d}$ as follows.

$$\sigma_d \colon (K[x_1])[\tilde{\mathbf{x}}] \longrightarrow R_d[\tilde{\mathbf{x}}]$$
 (6.14)

is defined like in (5.30) such that $I_d := \sigma_d(I)$ and $B_d := \sigma_d(B_{\varepsilon})$. Then with an elimination ordering on $R_d[\tilde{x}]$ like in Definition 5.7, we have the following ideal identity in $R_d[\tilde{\boldsymbol{x}}]$:

$$\langle \text{LT}(I_d) \rangle = \langle \text{LT}(B_d) \rangle.$$
 (6.15)

Proof. For every $g \in I_d$, let $f \in I_*$ as in Lemma 6.5 such that $\sigma_d(f) = g$ and in particular, $\sigma_d(LC(f)) = LC(g) \in R_d^*$. According to Lemma 6.4, there exist $\{v_k : 0 \le k \le s\} \subset (K[x_1])[\tilde{x}]$ as well as a multiplier $\lambda \in K[x_1]$ that is relatively prime to $d = CP(\chi_{\varepsilon})$ such that both (6.3) and (6.4) hold for this $f \in I_*$. We apply the epimorphism σ_d like in (5.30) to the identity (6.3). Then $\sigma_d(\lambda) \in R_d^*$ by Lemma 5.3 (i) since λ is relatively prime to d. We have $\sigma_d(LT(\lambda f)) = \sigma_d(\lambda) \cdot LT(g) \ne 0$ since $\sigma_d(LC(f)) = LC(g) \in R_d^*$ whereas $\sigma_d(LT(v_0\chi_{\varepsilon})) = \sigma_d(\chi_{\varepsilon} \cdot LT(v_0)) = 0$ since $\chi_{\varepsilon} \in (d) \subset K[x_1]$. For $1 \le k \le s$, we collect the subscript k into a set λ if $LM(v_k) \cdot LM(g_k) = LM(g) = LM(f)$ and $\sigma_d(LC(v_kg_k)) = \sigma_d(LC(v_k) \cdot LC(g_k)) \in R_d^*$. Then based on (6.4) we have $\lambda \ne \emptyset$ since $\sigma_d(LT(\lambda f)) \ne 0$ on the left hand side of (6.3). In the case of $\sigma_d(LC(g_k)) \in R_d^*$ for $k \in \lambda$, we also have $\sigma_d(LT(g_k)) = LT(\sigma_d(g_k))$. Hence the following identity:

$$LT(g) = \sigma_d(\lambda)^{-1} \cdot \sum_{k \in \Lambda} \sigma_d(LT(v_k)) \cdot LT(\sigma_d(g_k)) \in \langle LT(B_d) \rangle$$
 (6.16)

indicates the ideal identity (6.15).

Lemma 6.7. Let R_q be a normal PQR as in Definition 5.4. Suppose that $\{c_j : 0 \le j \le s\} \subset R_q^*$. There exist $\{b_j : 1 \le j \le s\} \subset R_q$ such that $c_0 = \sum_{j=1}^s b_j c_j$ if and only if $c_0 \in (c) \subset R_q$ with $c := \gcd(\{c_j : 1 \le j \le s\})$ as in (5.8) or (5.9). In particular, for every proper subset $\Lambda \subset \{1 \le j \le s\}$, the identity $c_0 = \sum_{j \in \Lambda} d_j c_j$ holds with $\{d_j : j \in \Lambda\} \subset R_q$ only if $c_0 \in (c)$.

Proof. First of all, there exist $\{a_j: 1 \leq j \leq s\} \subset R_q$ such that

$$c = \sum_{j=1}^{s} a_j c_j. (6.17)$$

In fact, let ι_q be the injection defined in (5.5). If $d := \gcd(\{\iota_q(c_j) : 1 \le j \le s\})$, then there exist $\{d_j : 1 \le j \le s\} \subset K[x_1]$ such that $d = \sum_{j=1}^s d_j \cdot \iota_q(c_j)$. Applying σ_q to this identity, we have

$$\sigma_q(d) = \sum_{j=1}^s \sigma_q(d_j) \cdot c_j \tag{6.18}$$

since $\sigma_q \circ \iota_q$ is the identity map. Hence $\sigma_q(d) \in (c)$ since $c_j \in (c)$ for $1 \leq j \leq s$. On the other hand, $\iota_q(c_j) \in (d)$ for $1 \leq j \leq s$. Thus $c_j \in (\sigma_q(d))$ for $1 \leq j \leq s$ and hence their common divisor $c \in (\sigma_q(d))$. By the standard representations of c and $\sigma_q(d)$ as in (5.6), we have $c = u \cdot \sigma_q(d)$ with $u \in R_q^{\times}$. As a result, it suffices to take $a_j := u \cdot \sigma_q(d_j)$ for $1 \leq j \leq s$ in (6.18) in order to deduce (6.17).

Now the necessity of the conclusion is easy to verify. And the sufficiency of the conclusion readily follows from (6.17).

Notation 6.8. Suppose that $B \subset R_q[\tilde{\boldsymbol{x}}] \setminus R_q$ is a finite polynomial set and $\tilde{\boldsymbol{x}}^{\alpha} \in [\tilde{\boldsymbol{x}}]$. We denote $B|_{\tilde{\boldsymbol{x}}^{\alpha}} := \{b \in B : \tilde{\boldsymbol{x}}^{\alpha} \in \langle LM(b) \rangle \}$.

Hereafter we shall simply use gcd to denote the gcd_{st} in (5.8) or gcd_q in (5.9). The two definitions only differ by a unit as in (5.10), which has no impact on our conclusions.

It is evident that $B|_{\tilde{x}^{\alpha}} \neq \emptyset$ is equivalent to $\tilde{x}^{\alpha} \in \langle LM(B) \rangle$ since Lemma 2.1 applies to PQR as well. Please note that this is also the condition for a nonzero term $c_{\alpha}\tilde{x}^{\alpha}$ to be pseudo-reducible with respect to B in Definition 2.6.

Definition 6.9 (GCD-reducible terms and polynomials in $R_q[\tilde{x}]$).

Let $\rho \in R_q^*$ and $B \subset R_q[\tilde{\boldsymbol{x}}] \backslash R_q$ be a finite polynomial set. We say that a nonzero term $c_{\alpha}\tilde{\boldsymbol{x}}^{\alpha}$ is GCD-reducible with respect to B if $B|_{\tilde{\boldsymbol{x}}^{\alpha}} \neq \emptyset$ and $c_{\alpha} \in (d) \subset R_q$ with $d := \gcd(\{\operatorname{LC}(b) \colon b \in B|_{\tilde{\boldsymbol{x}}^{\alpha}}\})$ as in Notation 6.8. We also say that $c_{\alpha}\tilde{\boldsymbol{x}}^{\alpha}$ is GCD-reduced with respect to B if it is not GCD-reducible with respect to B.

A polynomial $f \in R_q[\tilde{x}] \setminus R_q$ is said to be GCD-reducible with respect to B if f has a GCD-reducible term. Otherwise f is said to be GCD-reduced with respect to B.

Lemma 6.10. Let $B \subset R_q[\tilde{x}] \setminus R_q$ be a finite polynomial set. Then a nonzero term $c_{\alpha}\tilde{x}^{\alpha} \in \langle LT(B) \rangle$ if and only if $c_{\alpha}\tilde{x}^{\alpha}$ is GCD-reducible with respect to B.

Proof. We only prove the necessity condition since the sufficiency condition is evident by Lemma 6.7. Suppose that $\operatorname{LT}(B) = \{a_j \tilde{\boldsymbol{x}}^{\alpha_j} : 1 \leq j \leq s\}$ and $c_\alpha \tilde{\boldsymbol{x}}^\alpha = \sum_{j=1}^s a_j h_j \tilde{\boldsymbol{x}}^{\alpha_j}$ with $h_j \in R_q[\tilde{\boldsymbol{x}}]$ for $1 \leq j \leq s$. We expand each h_j into individual terms and compare the terms with the same monomial $\tilde{\boldsymbol{x}}^\alpha$ on both sides of the equality. In this way we obtain an equality $c_\alpha \tilde{\boldsymbol{x}}^\alpha = \sum_{j=1}^s c_{\beta_j} a_j \tilde{\boldsymbol{x}}^{\alpha_j} \tilde{\boldsymbol{x}}^{\beta_j}$ instead with $c_{\beta_j} \tilde{\boldsymbol{x}}^{\beta_j}$ being a term of h_j such that $\alpha_j + \beta_j = \alpha$ for $1 \leq j \leq s$. Now it is evident that $B|_{\tilde{\boldsymbol{x}}^\alpha} \neq \emptyset$ and $c_\alpha = \sum_{j=1}^s a_j c_{\beta_j}$. Then the conclusion readily follows from Lemma 6.7.

Definition 6.11 (GCD-term reduction in $R_q[\tilde{x}]$).

Suppose that $f \in R_q[\tilde{\boldsymbol{x}}] \setminus R_q$ has a term $c_{\alpha}\tilde{\boldsymbol{x}}^{\alpha}$ that is GCD-reducible with respect to a finite set $B \subset R_q[\tilde{\boldsymbol{x}}] \setminus R_q$. Let us denote $d := \gcd(\{\operatorname{LC}(b) : b \in B|_{\tilde{\boldsymbol{x}}^{\alpha}}\})$ as in Notation 6.8. With $l_{\alpha} := \iota_q(c_{\alpha})$ and $l_d := \iota_q(d)$, let us define the multipliers $\mu := \sigma_q(\operatorname{lcm}(l_{\alpha}, l_d)/l_{\alpha})$ and $m := \sigma_q(\operatorname{lcm}(l_{\alpha}, l_d)/l_d)$. Then we can make a GCD-term reduction of f by B as follows.

$$h = \mu f - \sum_{b \in B|_{\Xi^{\alpha}}} \frac{mc_b \boldsymbol{x}^{\alpha}}{\mathrm{LM}(b)} b, \tag{6.19}$$

where $d = \sum_{b \in B|_{\bar{x}^{\alpha}}} c_b \cdot LC(b)$ with $c_b \in R_q$. We call h the remainder of the reduction and μ the interim multiplier on f with respect to B.

Theorem 6.12 (GCD-division in $R_q[\tilde{x}]$).

With the elimination ordering on $R_q[\tilde{\mathbf{x}}]$ as in Definition 5.7, suppose that $B = \{g_j \colon 1 \leq j \leq s\} \subset R_q[\tilde{\mathbf{x}}] \setminus R_q$ is a polynomial set. For every $f \in R_q[\tilde{\mathbf{x}}]$, there exist a multiplier $\lambda \in R_q^{\times}$ as well as a remainder $r \in R_q[\tilde{\mathbf{x}}]$ and quotients $h_j \in R_q[\tilde{\mathbf{x}}]$ for $1 \leq j \leq s$ such that

$$\lambda f = \sum_{j=1}^{s} h_j g_j + r, \tag{6.20}$$

where r is GCD-reduced with respect to B. Moreover, the polynomials in (6.20) satisfy the following condition:

$$LM(f) = \max\{\max_{1 \le j \le s} \{LM(h_j) \cdot LM(g_j)\}, LM(r)\}.$$

$$(6.21)$$

Proof. Similar to the proof of Theorem 5.10 for proper division, the proof is almost a verbatim repetition of that for Theorem 2.7 if we substitute $R_q[\tilde{x}]$ for $(K[x_1])[\tilde{x}]$. In fact, the maximal term of h that is GCD-reducible with respect to B is strictly less than that of f after we make a GCD-term reduction as in (6.19). Moreover, it suffices to prove that the condition (6.21) applies to the GCD-term reduction in (6.19), same as in the proof of Theorem 2.7.

We also call the GCD-division with respect to B a GCD-reduction with respect to B henceforth.

Theorem 6.13. Suppose that $I \subset (K[x_1])[\tilde{x}]$ is a zero-dimensional ideal with an elimination ordering on [x] as in Definition 3.1. Let $B_{\varepsilon} = \{g_k : 1 \leq k \leq s\}$ be a pseudo-basis of I and $d = CP(\chi_{\varepsilon})$ the compatible part of the pseudo-eliminant χ_{ε} associated with B_{ε} . Let $\sigma_d : (K[x_1])[\tilde{x}] \to R_d[\tilde{x}]$ be the epimorphism as in (5.30) such that $I_d := \sigma_d(I)$ and $B_d := \sigma_d(B_{\varepsilon})$. Then the identity (6.15) in Lemma 6.6 is equivalent to a characterization of B_d as following:

For every $f \in R_d[\tilde{x}]$, we have $f \in I_d$ if and only if we can make a GCD-reduction of f with respect to B_d as in (6.20) and (6.21) such that the remainder r = 0.

Proof. Suppose that the identity (6.15) holds. For every $f \in I_d$, we make a GCD-reduction of f with respect to B_d as in (6.20) and (6.21) such that the remainder r is GCD-reduced with respect to B_d as in Definition 6.9. On the other hand, the remainder $r \in I_d$ as per the expression in (6.20). If $r \neq 0$, there exist $\{h_k : 1 \leq k \leq s\} \subset R_d[\tilde{x}]$ such that $\operatorname{LT}(r) = \sum_{k=1}^s h_k \cdot \operatorname{LT}(\sigma_d(g_k))$ according to (6.15). For $1 \leq k \leq s$, we collect the subscript k into a set Λ if h_k has a term denoted as $c_k \tilde{x}^{\alpha_k}$ that satisfies $\tilde{x}^{\alpha_k} \cdot \operatorname{LM}(\sigma_d(g_k)) = \operatorname{LM}(r)$ and $c_k \cdot \operatorname{LC}(\sigma_d(g_k)) \in R_d^*$. Then we have $\Lambda \neq \emptyset$ since $\operatorname{LC}(r) \in R_d^*$. Moreover, $\operatorname{LC}(r) = \sum_{k \in \Lambda} c_k \cdot \operatorname{LC}(\sigma_d(g_k))$ and hence $\operatorname{LC}(r) \in (a)$ by Lemma 6.7 with $a := \gcd(\{\operatorname{LC}(\sigma_d(g_k)) : k \in \Lambda\})$. According to Definition 6.9, r is GCD-reducible with respect to B_d , which constitutes a contradiction. This proves the necessity of the conclusion. The sufficiency of the conclusion is evident since $B_d \subset I_d$.

Next let us prove the identity (6.15) under the assumption that every $f \in I_d$ can be GCD-reduced to r=0 by B_d . In fact, $f=\sum_{k=1}^s q_k \cdot \sigma_d(g_k)$ with $q_k \in R_d[\tilde{x}]$ such that $\mathrm{LM}(f)=\max_{1\leq k\leq s}\{\mathrm{LM}(q_k)\cdot \mathrm{LM}(\sigma_d(g_k))\}$ according to (6.21). For $1\leq k\leq s$, we collect the subscript k into a set Λ if $\mathrm{LM}(q_k)\cdot \mathrm{LM}(\sigma_d(g_k))=\mathrm{LM}(f)$. Then $\mathrm{LT}(f)=\sum_{k\in\Lambda}\mathrm{LT}(q_k)\cdot \mathrm{LT}(\sigma_d(g_k))$. Thus $\mathrm{LT}(I_d)\subset \langle \mathrm{LT}(B_d)\rangle$. The other direction of (6.15) is trivial to prove.

In what follows we provide a modular characterization of the ideal $I + \langle \theta_q \rangle$ in (6.2) over the normal PQR R_q . We shall use a modular argument for the characterization resorting to the proper basis B_q and proper eliminant e_q obtained in Algorithm 5.20.

Lemma 6.14. Suppose that χ_{ε} is a pseudo-eliminant of a zero-dimensional ideal I with q being a composite divisor of its incompatible part $IP(\chi_{\varepsilon})$. Let e_q and $B_q = \{g_k : 1 \leq k \leq \tau\}$ be the proper eliminant and proper basis of $I_q = \sigma_q(I)$ respectively with σ_q being the epimorphism as in (5.30). For every $f \in I_q$, there

exist a multiplier $\lambda \in R_q^{\times}$ and $\{v_k : 0 \le k \le \tau\} \subset R_q[\tilde{x}]$ such that:

$$\lambda f = \sum_{k=1}^{\tau} v_k g_k + v_0 e_q. \tag{6.22}$$

Moreover, the polynomials in (6.22) satisfy the following condition:

$$LM(f) = \max \left\{ \max_{1 \le k \le \tau} \left\{ LM(v_k) \cdot LM(g_k) \right\}, LM(v_0) \right\}. \tag{6.23}$$

In particular, the above conclusions are still sound when the proper eliminant $e_q = 0$.

Proof. The proof is an almost verbatim repetition of that for Theorem 5.26, which is similar to the proof for Lemma 6.4. In fact, suppose that F is the originally given basis of the ideal I in $(K[x_1])[\tilde{x}]$ such that $\sigma_q(F) = \{f_j : 1 \leq j \leq s\} \subset R_q[\tilde{x}] \setminus R_q$ is a basis of the ideal $I_q = \sigma_q(I)$. Then for every $f \in I_q$, there exist $h_j \in R_q[\tilde{x}]$ for $1 \leq j \leq s$ such that f can be written as:

$$f = \sum_{j=1}^{s} h_j f_j. (6.24)$$

Thus the conclusion readily follows when $LM(f) = \max_{1 \leq j \leq s} \{LM(h_j) \cdot LM(f_j)\}$ since $\sigma_q(F) \subset B_q$.

Now let us suppose that $LM(f) \prec \max_{1 \leq j \leq s} \{LM(h_j) \cdot LM(f_j)\}$. In this case we treat f as the modular eliminant χ_q in (5.38). Let us fix an irreducible factor p of the composite divisor q. We repeat the arguments verbatim from (5.39) through (5.53) to obtain a new representation like in (5.53) as follows.

$$b\lambda f = \sum_{k=1}^{\tau} \mu_k g_k + \mu_0 e_q \tag{6.25}$$

with $\mu_k \in R_q[\tilde{x}]$ for $0 \le k \le \tau$. The leading monomials of the representation in (6.25) are strictly less than those in (6.24), which resembles (5.54) closely. Moreover, the multiplier $b\lambda$ in (6.25) is relatively prime to the irreducible factor p of the composite divisor q, i.e., $\operatorname{mult}_p(b\lambda) = 0$. We repeat this procedure of rewriting the representations of $b\lambda f$ so that their leading monomials strictly decrease. And the multipliers for the representations are always relatively prime to the irreducible factor p of the composite divisor q. After a finite number of repetitions we shall obtain a representation in the following form:

$$\nu f = \sum_{k=1}^{\tau} w_k g_k + w_0 e_q \tag{6.26}$$

with $w_k \in R_q[\tilde{\boldsymbol{x}}]$ for $0 \le k \le \tau$ such that

$$\max\{\max_{1 \le k \le \tau} \{ LM(w_k) \cdot LM(g_k) \}, LM(w_0) \} = LM(f).$$
(6.27)

The multiplier $\nu \in R_q^*$ in (6.26) is relatively prime to the irreducible factor p of the composite divisor q.

Since for every irreducible factor p of the composite divisor q, we have (6.26) and (6.27), we can repeat the arguments almost verbatim in (6.9) and (6.11) to show that there exist a multiplier $\lambda \in R_q^{\times}$ and $\{v_k \colon 0 \le k \le \tau\} \subset R_q[\tilde{x}]$ such that (6.22) holds. In the arguments we substitute the proper eliminant e_q for the pseudo-eliminant χ_{ε} and use (6.17) for the representation of a greatest common divisor in R_q . Moreover, we can corroborate (6.23) by repeating almost verbatim the arguments in (6.10), (6.12) and (6.13). In fact, it suffices to substitute $LM(w_k^{(l)}) \cdot LM(g_k)$ for $LM(w_k^{(l)}g_k)$ in (6.10) and (6.12), as well as to substitute $LM(v_k) \cdot LM(g_k)$ for $LM(v_kg_k)$ in (6.12) and (6.13), as regards the existence of zero divisors in R_q . \square

Let $I \subset (K[x_1])[\tilde{x}]$ be a zero-dimensional ideal. For a composite divisor q of the incompatible part $IP(\chi_{\varepsilon})$ of a pseudo-eliminant χ_{ε} of I, let R_q be the normal PQR defined as in (5.29). Suppose that $e_q \in R_q^* \setminus R_q^{\times}$ is the proper eliminant obtained in Algorithm 5.20. In particular, let e_q stand for the standard representation e_q^{st} as in Definition 5.22. As per (5.6), we have $q \in (\iota_q(e_q)) \subset K[x_1]$ with ι_q being the injection defined as in (5.5). Hence similar to (5.1), let us define:

$$R_p := \{ r \in R_q \colon \deg(r) < \deg(e_q) \}.$$
 (6.28)

Similar to (5.2), we can redefine the binary operations on R_p such that R_p is a normal PQR satisfying $R_p \cong R_q/(e_q)$. For every $f \in R_q$, there exist a quotient $h \in R_q$ and unique remainder $r \in R_q$ satisfying $f = he_q + r$ such that $\deg(r) < \deg(e_q)$. Like in (5.4) we can define an epimorphism $\sigma_p \colon R_q \to R_p$ as $\sigma_p(f) := r$. This combined with (5.3) and (5.4) lead to an epimorphism $\sigma_p \circ \sigma_q \colon K[x_1] \to R_p$. For every $f \in K[x_1]$, there exist a quotient $h \in K[x_1]$ and unique remainder $r \in K[x_1]$ such that $f = h \cdot \iota_q(e_q) + r$. Hence we can also define an epimorphism $\tilde{\sigma}_p \colon K[x_1] \to R_p$ as $\tilde{\sigma}_p(f) := r$. Since $q \in (\iota_q(e_q))$, it is evident that $\tilde{\sigma}_p = \sigma_p \circ \sigma_q$. For simplicity we still denote $\tilde{\sigma}_p$ as σ_p henceforth.

Similar to (5.30), σ_p can be extended to a ring epimorphism from $(K[x_1])[\tilde{x}]$ or $R_q[\tilde{x}]$ to $R_p[\tilde{x}]$ which we still denote as σ_p as follows.

$$\sigma_p \colon (K[x_1])[\tilde{\boldsymbol{x}}] \text{ or } R_q[\tilde{\boldsymbol{x}}] \to R_p[\tilde{\boldsymbol{x}}] \colon \quad \sigma_p\left(\sum_{j=1}^s c_j \tilde{\boldsymbol{x}}^{\alpha_j}\right) := \sum_{j=1}^s \sigma_p(c_j) \tilde{\boldsymbol{x}}^{\alpha_j}.$$
 (6.29)

Similar to (5.31), we also have an injection ι_p as follows.

$$\iota_p \colon R_p[\tilde{\boldsymbol{x}}] \to (K[x_1])[\tilde{\boldsymbol{x}}] \text{ or } R_q[\tilde{\boldsymbol{x}}] \colon \quad \iota_p\left(\sum_{j=1}^s c_j \tilde{\boldsymbol{x}}^{\alpha_j}\right) := \sum_{j=1}^s \iota_p(c_j) \tilde{\boldsymbol{x}}^{\alpha_j}.$$
 (6.30)

Theorem 6.15. Suppose that $I \subset (K[x_1])[\tilde{x}]$ is a zero-dimensional ideal over a perfect field K and χ_{ε} a pseudo-eliminant of I. Let q be a composite divisor of the incompatible part $\operatorname{IP}(\chi_{\varepsilon})$ and R_q the normal PQR as in (5.29). Let $e_q \in R_q \setminus R_q^{\times}$ and B_q denote the proper eliminant and proper basis obtained in Algorithm 5.20 respectively.

If $e_q = 0$, then we have the following two equivalent characterizations of B_q :

(1) A characterization of B_q through its leading terms via an ideal identity as follows.

$$\langle \text{LT}(I_q) \rangle = \langle \text{LT}(B_q) \rangle.$$
 (6.31)

(2) A characterization of B_q through GCD-reductions:

For every $f \in R_q[\tilde{x}]$, we have $f \in I_q$ if and only if we can make a GCD-reduction of f with respect to B_q as in (6.20) and (6.21) with the remainder r = 0.

If $e_q \in R_q^* \setminus R_q^*$, we define $I_p := \sigma_p(I_q) = \sigma_p(I)$ and $B_p := \sigma_p(B_q) = \sigma_p(B_\varepsilon)$ with σ_p being the epimorphism as in (6.29). Then we have the following two equivalent characterizations of B_p :

(3) A characterization of B_p through its leading terms via an ideal identity as follows.

$$\langle LT(I_p) \rangle = \langle LT(B_p) \rangle.$$
 (6.32)

(4) A characterization of B_p through GCD-reductions:

For every $f \in R_p[\tilde{x}]$, we have $f \in I_p$ if and only if we can make a GCD-reduction of f with respect to B_p as in (6.20) and (6.21) with the remainder r = 0.

Proof. The identities (6.31) and (6.32) follow directly from Lemma 6.14. The proofs are similar to and even simpler than that for the identity (6.15) in Lemma 6.6 since we need a conclusion like Lemma 6.5 in neither cases. In fact, we can obtain (6.31) as an identity of leading terms from the identity (6.22). We first define a subscript set Λ for (6.22) as $\Lambda := \{1 \leq k \leq \tau \colon LM(v_k) \cdot LM(g_k) = LM(f), LC(v_k) \cdot LC(g_k) \in R_q^* \}$. Then we can obtain an identity of leading terms as follows.

$$LT(f) = \lambda^{-1} \sum_{k \in \Lambda} LT(v_k) \cdot LT(g_k) \in \langle LT(B_q) \rangle$$
(6.33)

since we have $LC(f) \in \mathbb{R}_q^*$ as well as $e_q = 0$ in (6.22) in this case.

To obtain (6.32), for every $f \in I_p$, let ι_p be the injection defined as in (6.30) such that $\iota_p(f) \in I_q$. Now consider the identity (6.22) that holds for $\iota_p(f)$. We can apply the epimorphism σ_p in (6.29) to the identity (6.22) for $\iota_p(f)$ to obtain an identity of leading terms that is similar to (6.33) since $LC(f) \in R_p^*$ and $\sigma_p(e_q) = 0$.

The proofs for the equivalence between the characterizations in (1) and (2), as well as between those in (3) and (4), are verbatim repetitions of that for Theorem 6.13. In fact, it suffices to substitute I_q , B_q and σ_q , as well as I_p , B_p and σ_p , for I_d , B_d and σ_d respectively.

Remark 6.16. We used GCD-reductions in Theorem 6.13 and Theorem 6.15 (2) and (4) to address the ideal membership problem for the new type of bases. However we avoided making GCD-reductions of the S-polynomials in the computations of the pseudo-bases and pseudo-eliminants in Algorithm 3.9, as well as the proper bases and proper eliminants in Algorithm 5.20. The reason becomes clear in Section 8 when we show that the GCD-computations not only incur complexity issues but also contain Bézout coefficients that tend to swell to an excruciating magnitude over the rational field $K = \mathbb{Q}$. This is also the reason why we do not adopt the so-called strong Gröbner bases for polynomial rings over principal ideal rings. And the PQR is a special kind of principal ideal rings. Please refer to [AL94, P251, Definition 4.5.6] for the definition of strong Gröbner bases over PIDs.

Notation 6.17 (Unified notations for modular bases).

Let us use B_q to denote the various modular bases that were defined previously as follows: (i) The proper basis B_q with the moduli being a composite divisor q as in Definition 5.22; (ii) The modular basis $B_d = \sigma_d(B_\varepsilon)$ with the moduli being the compatible part $d = \text{CP}(\chi_\varepsilon)$ as in Lemma 6.6; (iii) The modular basis $B_p = \sigma_p(B_q)$ with the moduli being a proper eliminant e_q as in Theorem 6.15 (3).

In accordance with the above notation of B_q , we also use σ_q as a unified notation for the epimorphisms σ_q in (5.30), σ_d in (6.14) and σ_p in (6.29). Moreover, we denote the coefficient ring R_q , R_d or R_p simply as R_q and ideal I_q , I_d or I_p as I_q respectively such that $B_q \subset I_q \subset R_q[\tilde{x}]$ henceforth.

The unified modular basis B_q satisfies the similar identities (6.15), (6.31) and (6.32) that can be assimilated into a unified identity as follows.

$$\langle \operatorname{LT}(I_a) \rangle = \langle \operatorname{LT}(B_a) \rangle.$$
 (6.34)

Now we furnish the unified identity (6.34) with an interpretation via the ideal $I + \langle q \rangle$. Let $B_{\varepsilon} = \{g_k \colon 1 \le k \le s\}$ be a pseudo-basis of I and $B_q = \sigma_q(B_{\varepsilon}) = \{b_j \colon 1 \le j \le t\} \subset I_q$ the unified notation for the modular bases as in Notation 6.17. We can also define a unified notation for the injection $\iota_q \colon R_q[\tilde{x}] \to (K[x_1])[\tilde{x}]$ similar to (5.31) and (6.30) such that $\sigma_q \circ \iota_q$ is the identity map. By the canonical isomorphism as follows:

$$(I + \langle q \rangle)/\langle q \rangle \simeq I/(I \cap \langle q \rangle) \simeq \sigma_q(I) = I_q,$$

it is easy to deduce that $\iota_q(B_q) \subset B_{\varepsilon} + \langle q \rangle := \{g_k + fq : 1 \leq k \leq s, f \in (K[x_1])[\tilde{x}]\}$. Then it readily follows that $(\iota_q(B_q) \cup \{0\}) + \langle q \rangle = B_{\varepsilon} + \langle q \rangle$. Here $\iota_q(B_q) \cup \{0\}$ is for the possibility that $0 \in \sigma_q(B_{\varepsilon})$.

We can deduce that for every $f \in I + \langle q \rangle$, there exists $g \in (K[x_1])[\tilde{x}]$ such that $f - gq \in \langle \iota_q(B_q) \rangle$. In fact, we can invoke Theorem 6.13 and Theorem 6.15 (2) and (4) on $\sigma_q(f) \in I_q$. According to the GCD-reduction by the modular basis $B_q = \{b_j \colon 1 \leq j \leq t\}$ as in (6.20), there exist a multiplier $\lambda \in R_q^{\times}$ and quotients $h_j \in R_q[\tilde{x}]$ with $1 \leq j \leq t$ such that $\lambda \cdot \sigma_q(f) = \sum_{j=1}^t h_j b_j$. Since $h_j b_j = \sigma_q(\iota_q(h_j) \cdot \iota_q(b_j))$ for $1 \leq j \leq t$, there exists $h \in (K[x_1])[\tilde{x}]$ such that $\lambda \cdot \iota_q(\sigma_q(f)) - hq = \sum_{j=1}^t \iota_q(h_j) \cdot \iota_q(b_j) \in \langle \iota_q(B_q) \rangle$. Moreover, we also have $f - \iota_q(\sigma_q(f)) \in \langle q \rangle$. Hence $f - gq \in \langle \iota_q(B_q) \rangle$ for some $g \in (K[x_1])[\tilde{x}]$.

Thus we have proved the following interpretation for the identity (6.34) since $(\iota_q(B_q) \cup \{0\}) + \langle q \rangle = B_{\varepsilon} + \langle q \rangle$.

Lemma 6.18. If a modular basis B_q satisfies the identity (6.34), then $B_{\varepsilon} \cup \{q\}$ or $\iota_q(B_q) \cup \{q\}$ constitute a basis for $I + \langle q \rangle$.

Let B_{ε} be a pseudo-basis and $d = \operatorname{CP}(\chi_{\varepsilon})$ the compatible part of a pseudoeliminant χ_{ε} of I. We still use θ_q to denote the nontrivial proper divisors for $q \in \Theta$ with Θ being the corresponding set of composite divisors as in (6.2). Then we have the following new type of bases in accordance with the ideal decomposition of Iin (6.2):

$$(B_{\varepsilon} \cup \{d\}) \cup \bigcup_{q \in \Theta} (B \cup \{\theta_q\}),$$
 (6.35)

where the basis B stands for either B_{ε} or $\iota_q(B_q)$.

The modular version of the new type of bases specified in (6.35) correspond to the modular bases B_q , B_d and B_p as in Notation 6.17. These modular bases are especially suited for the Chinese Remainder Theorem.

Lemma 6.19. Let Θ be the set of composite divisors whose proper divisors are nontrivial as in (6.2) and (6.35). We use the unified notation I_q as in Notation 6.17 for the modular ideals I_q in (6.31) and I_p in (6.32) but we keep the notation for I_d as in (6.15) unaltered. If we denote $I_{\chi} := I/I \cap \langle \chi \rangle$, then we have a decomposition as follows.

$$I_{\chi} \simeq I_d \times \prod_{q \in \Theta} I_q.$$
 (6.36)

Proof. The identity (6.36) amounts to proving that the canonical homomorphism φ as follows is an isomorphism:

$$\varphi \colon I/I \cap \langle \chi \rangle \longrightarrow (I/I \cap \langle d \rangle) \times \prod_{q \in \Theta} I/I \cap \langle q \rangle.$$
 (6.37)

The proof is similar to that for the Chinese Remainder Theorem. In fact, it is obvious that φ is an injection since we have the decomposition:

$$(I \cap \langle d \rangle) \cap \bigcap_{q \in \Theta} (I \cap \langle q \rangle) = I \cap \langle \chi \rangle \tag{6.38}$$

by the factorization of the eliminant χ in Theorem 6.2.

For every $q \in \Theta \cup \{d\} := \widehat{\Theta}$, let us define $J_q := \bigcap_{q' \in \widehat{\Theta} \setminus \{q\}} (I \cap \langle q' \rangle)$. Fix a $q \in \widehat{\Theta}$. For every $q' \in \widehat{\Theta} \setminus \{q\}$, there exist $c_q, c_{q'} \in K[x_1]$ such that $c_q q + c_{q'} q' = 1$. Hence for every $f \in I$, we have the following identity:

$$f = f \prod_{q' \in \widehat{\Theta} \setminus \{q\}} (c_q q + c_{q'} q') \in (I \cap \langle q \rangle) + J_q,$$

from which we can deduce the following ideal identity for every $q \in \widehat{\Theta}$:

$$I = (I \cap \langle q \rangle) + J_q. \tag{6.39}$$

We can substitute (6.39) into (6.37) to obtain the following equivalence for every $q \in \widehat{\Theta}$:

$$I/I \cap \langle q \rangle = ((I \cap \langle q \rangle) + J_q)/(I \cap \langle q \rangle) \simeq J_q/((I \cap \langle q \rangle) \cap J_q) = J_q/(I \cap \langle \chi \rangle), (6.40)$$

where the last equality is based on (6.38). The identity (6.40) simplifies the canonical monomorphism φ in (6.37) into the following form:

$$\varphi \colon I/I \cap \langle \chi \rangle \longrightarrow \prod_{q \in \widehat{\Theta}} J_q/(I \cap \langle \chi \rangle) \simeq \prod_{q \in \widehat{\Theta}} I/I \cap \langle q \rangle \xrightarrow{\pi_q} I/I \cap \langle q \rangle, \tag{6.41}$$

where π_q is the canonical projection onto the components. The surjectivity of φ follows from the fact that for $s_q \in J_q$ with $q \in \widehat{\Theta}$, we have $\pi_q \circ \varphi(s) = s_q$ with $s := \sum_{q \in \widehat{\Theta}} s_q$.

Lemma 6.20. For the eliminant χ of an ideal I, let $R_{\chi} := (K[x_1])[\tilde{x}]/\langle \chi \rangle$ and I_{χ} be defined as in (6.36) respectively. Then we have the following isomorphism in accordance with the ideal decompositions in (6.2) and (6.38):

$$R_{\chi}/I_{\chi} \simeq (R_d/I_d) \times \prod_{q \in \Theta} (R_q/I_q),$$
 (6.42)

where Θ is defined as in (6.2) and (6.36) and I_d and I_q are defined as in (6.36).

Proof. By Chinese Remainder Theorem we have the algebra decomposition:

$$R/(I + \langle \chi \rangle) \simeq R/(I + \langle d \rangle) \times \prod_{q \in \Theta} R/(I + \langle q \rangle)$$
 (6.43)

with $R = (K[x_1])[\tilde{x}]$. Then the decomposition (6.42) follows from (6.43) by the following observation:

$$R/(I+\langle q\rangle) \simeq (R/\langle q\rangle)/((I+\langle q\rangle)/\langle q\rangle) \simeq R_q/I_q$$
.

Similarly we have
$$R/(I+\langle \chi \rangle) \simeq R_{\chi}/I_{\chi}$$
 and $R/(I+\langle d \rangle) \simeq R_d/I_d$.

It is easy to show that the GCD-reduced remainder r obtained in the GCD-reduction (6.20) of Theorem 6.12 is unique. In this way we can define a unique normal form r for every $f \in R_q[\tilde{x}]$ with respect to B. This combined with the identity (6.42) yield a normal form for every $f \in R_\chi$ with respect to I_χ .

It is evident that Definition 6.9 and Lemma 6.10 apply to all these modular bases denoted as B_q . In the remaining part of this section, let us address the uniqueness of this new type of bases. The first step is to minimize the number of elements in a basis.

Definition 6.21 (Irredundant basis).

A basis B_q as in Notation 6.17 satisfying (6.34) is said to be *irredundant* if $\langle LT(B_q \setminus \{b\}) \rangle$ is a proper subset of $\langle LT(B_q) \rangle$ for every element $b \in B_q$. That is, LT(b) is GCD-reduced with respect to $B_q \setminus \{b\}$ for every $b \in B_q$ by Lemma 6.10.

Lemma 6.22. If B_q and B'_q are irredundant bases as in Definition 6.21 of the same ideal I_q , then we have two equal sets of leading monomials $LM(B_q) = LM(B'_q)$.

Proof. From $\operatorname{LT}(b) \in \langle \operatorname{LT}(B'_q) \rangle$ for every $b \in B_q$, we know that $\operatorname{LT}(b)$ is GCD-reducible with respect to B'_q as per Lemma 6.10. Further, for every $b' \in B'_q|_{\operatorname{LM}(b)}$ as in Notation 6.8, $\operatorname{LT}(b')$ is also GCD-reducible with respect to B_q . Nonetheless we know for sure that not every element of $\operatorname{LT}(B'_q|_{\operatorname{LM}(b)})$ be GCD-reducible with respect to $B_q \setminus \{b\}$. Otherwise $\operatorname{LT}(b)$ would be GCD-reducible with respect to $B_q \setminus \{b\}$, contradicting the assumption that B_q is irredundant. As a result, there exists a $b' \in B'_q|_{\operatorname{LM}(b)}$ satisfying $b \in B_q|_{\operatorname{LM}(b')}$. These two conditions indicate that $\operatorname{LM}(b)$ is divisible by $\operatorname{LM}(b')$ and vice versa. Hence we have $\operatorname{LM}(b') = \operatorname{LM}(b)$, from which we can deduce that $\operatorname{LM}(B_q) \subset \operatorname{LM}(B'_q)$ since $b \in B_q$ is arbitrary. Similarly we have $\operatorname{LM}(B'_q) \subset \operatorname{LM}(B_q)$.

Remark 6.23. The two irredundant bases B_q and B'_q in Lemma 6.22 do not have to contain the same number of elements. In fact, consider the scenario of $b_j \in B_q$ with j = 1, 2 such that $LM(b_1) = LM(b_2) = \min\{LM(B_q)\}$. Suppose that $LC(b_j) \in$

 $R_q^* \setminus R_q^{\times}$ and $LC(b_j)/d \in R_q^* \setminus R_q^{\times}$ for j = 1, 2 with $d := \gcd(LC(b_1), LC(b_2))$. By Lemma 6.7 there exist $c_j \in R_q$ for j = 1, 2 such that $d = c_1 \cdot LC(b_1) + c_2 \cdot LC(b_2)$. Now we construct a new irredundant basis B_q' by substituting $b := c_1b_1 + c_2b_2$ for both b_1 and b_2 in B_q . Then we still have $\langle LT(B_q) \rangle = \langle LT(B_q') \rangle$ as in (6.34) and moreover, $LM(B_q) = LM(B_q')$.

Definition 6.24 (Minimal basis).

An irredundant basis B_q as in Definition 6.21 is called a *minimal* basis if for every $f \in B_q$, its leading coefficient $LC(f) = \lambda \cdot \gcd(\{LC(b): b \in B_q|_{LM(f)}\})$ with $\lambda \in B_q^{\times}$ being a unit. Here $B_q|_{LM(f)}$ and gcd are defined as in Notation 6.8.

In the example in Remark 6.23, the basis B_q is irredundant but not minimal.

Lemma 6.25. Let $B_q \subset R_q[\tilde{x}]$ be an irredundant basis as in Definition 6.21. For every $f \in B_q$, let us denote $d := \gcd(\{LC(b): b \in B_q|_{LM(f)}\})$ as in Notation 6.8. Assume that $d = \sum_{b \in B_q|_{LM(f)}} c_b \cdot LC(b)$ with $c_b \in R_q$ as in Lemma 6.7. For every $f \in B_q$, if we substitute f by

$$g := \sum_{b \in B_q|_{LM(f)}} \frac{c_b \cdot LM(f)}{LM(b)} b, \tag{6.44}$$

we shall obtain a new basis denoted as B'_q . We delete every $g \in B'_q$ for which there exists $g' \in B'_q$ with $g' \neq g$ such that $\mathrm{LT}(g') = \lambda \cdot \mathrm{LT}(g)$ with $\lambda \in R^\times_q$. If we still denote the new basis set as B'_q , then B'_q is a minimal basis.

Proof. We prove the irredundancy of B'_q by contradiction. Assume that there exists $g \in B'_q$ such that g is GCD-reducible with respect to $B'_q \setminus \{g\}$. Let us denote $\Omega := (B'_q \setminus \{g\})|_{\mathrm{LM}(g)} \neq \emptyset$ as in Notation 6.8. That is, for every $b \in \Omega$ there is $c_b \in R_q$ such that the following identity holds.

$$LC(g) = \sum_{b \in \Omega} c_b \cdot LC(b). \tag{6.45}$$

Moreover, for every $b \in \Omega$, we have $LM(b) \prec LM(g)$. The reason is that if LM(b) = LM(g), we would have $LT(b) = \lambda \cdot LT(g)$ with $\lambda \in R_q^{\times}$ by the definition of B_q' based on (6.44). Then one of b and b would have been deleted from B_q' .

Now every $b \in \Omega$ satisfies $\operatorname{LT}(b) \in \langle \operatorname{LT}(B_q) \rangle$ since $\Omega \subset B'_q \subset I_q$ and B_q satisfies the identity $\langle \operatorname{LT}(I_q) \rangle = \langle \operatorname{LT}(B_q) \rangle$. Hence $\operatorname{LT}(b)$ is GCD-reducible with respect to B_q as per Lemma 6.10, i.e., for every $a \in B_q|_{\operatorname{LM}(b)} := \Gamma_b \neq \emptyset$, there exists $h_a \in R_q$ such that $\operatorname{LC}(b) = \sum_{a \in \Gamma_b} h_a \cdot \operatorname{LC}(a)$. This combined with (6.45) lead to:

$$LC(g) = \sum_{b \in \Omega} \sum_{a \in \Gamma_b} c_b h_a \cdot LC(a). \tag{6.46}$$

From (6.46) we can infer that LT(g) is GCD-reducible with respect to $\bigcup_{b\in\Omega}\Gamma_b$. Moreover, the construction of $g\in B'_q$ in (6.44) is based on an $f\in B_q$ such that LM(f)=LM(g) and LC(f) is divisible by LC(g). Hence LT(f) is GCD-reducible with respect to $\bigcup_{b\in\Omega}\Gamma_b$. But we already proved that $LM(b)\prec LM(g)=LM(f)$ for every $b\in\Omega$. This indicates that $f\notin\bigcup_{b\in\Omega}\Gamma_b$ and thus we have $\bigcup_{b\in\Omega}\Gamma_b\subset B_q|_{LM(g)}\setminus\{f\}\subset B_q\setminus\{f\}$. Hence LT(f) is GCD-reducible with respect to $B_q\setminus\{f\}$. This contradicts the assumption that B_q is irredundant.

The minimality of B'_q readily follows from Definition 6.24.

Lemma 6.26. If B_q and B'_q are minimal bases of the same ideal I_q as in Definition 6.24, then we have two equal sets of leading terms $LT(B_q) = LT(B'_q)$ and moreover, B_q and B'_q have the same number of basis elements.

Proof. Since minimal bases are irredundant, we already have $LM(B_q) = LM(B'_q)$ as per Lemma 6.22. For every $b \in B_q$, there exists $b' \in B'_q$ such that LM(b) = LM(b'). Moreover, LT(b') is GCD-reducible with respect to B_q and hence LC(a) is divisible by LC(b') for every $a \in B_q|_{LM(b')}$ due to the minimality of $B'_q \ni b'$. Now $b \in B_q|_{LM(b')}$ and hence LC(b) is divisible by LC(b'). Similarly LC(b') is divisible by LC(b) as well. Thus we also have $LC(b) = \lambda \cdot LC(b')$ with $\lambda \in R_q^{\times}$.

Definition 6.27 (Reduced basis).

A minimal basis B_q as in Definition 6.24 is said to be a *reduced* basis if every $b \in B_q$ is GCD-reduced with respect to $B_q \setminus \{b\}$. That is, every nonzero term of b is GCD-reduced with respect to $B_q \setminus \{b\}$.

Lemma 6.28. If both B_q and B'_q are reduced bases of the same ideal I_q as in Definition 6.27, then we have two equal bases $B_q = B'_q$.

Proof. According to Lemma 6.26, we have $LT(B_q) = LT(B'_q)$ since both of them are minimal bases. Hence a term $c_{\alpha}\tilde{x}^{\alpha}$ is GCD-reduced with respect to B_q if and only if it is GCD-reduced with respect to B'_q .

Now for every $b \in B_q$, there is $b' \in B'_q$ such that LT(b) = LT(b'). Let us assume that $b - b' \neq 0$. By $b - b' \in I_q$, we have $LT(b - b') \in \langle LT(B_q) \rangle$ as per the identity (6.34). Then we have $LT(b - b') \in \langle LT(B_q \setminus \{b\}) \rangle$ since $LT(b - b') \prec LT(b)$. Hence LT(b - b') is GCD-reducible with respect to $B_q \setminus \{b\}$ by Lemma 6.10.

On the other hand, every nonzero term of b and b' is GCD-reduced with respect to $B_q \setminus \{b\}$ and $B'_q \setminus \{b'\}$ respectively by Definition 6.27. By $\operatorname{LT}(B_q \setminus \{b\}) = \operatorname{LT}(B'_q \setminus \{b'\})$, we can infer that every nonzero term of b' is GCD-reduced with respect to $B_q \setminus \{b\}$ as well. Thus we can conclude that every nonzero term of b-b' is GCD-reduced with respect to $B_q \setminus \{b\}$. In particular, $\operatorname{LT}(b-b')$ is GCD-reduced with respect to $B_q \setminus \{b\}$. This constitutes a contradiction.

Remark 6.29. Please note that a reduced basis is not necessarily a strong basis like the strong Gröbner basis. A strong Gröbner basis is defined as a finite set $G := \{g_j : 1 \leq j \leq s\}$ such that for every $f \in \langle G \rangle$, there exists a $g_j \in G$ such that $\operatorname{LT}(f)$ is divisible by $\operatorname{LT}(g_j)$. Consider the following counterexample. Let $I = \langle f, g \rangle \subset R_q[x, y]$ be an ideal with basis $f = (z+1)^2x + r(z)$ and $g = (z^2-1)y+s(z)$ such that $r,s \in R_q$. An invocation of Algorithm 5.20 shows that their S-polynomial as in (5.16) can be properly reduced to 0. Hence $\{f,g\}$ constitutes a proper basis of I by Definition 5.22 and satisfies the identity (6.31) by Theorem 6.15. Then it is easy to verify that $\{f,g\}$ constitutes a reduced basis by Definition 6.27. Nevertheless it does not constitute a strong basis of I. In fact, consider $h := yf - xg \in I$. Then $\operatorname{LT}(h) = 2(z+1)xy$ is divisible by neither $\operatorname{LT}(f) = (z+1)^2x$ nor $\operatorname{LT}(g) = (z^2-1)y$.

⁹Please also refer to [AL94, P251, Definition 4.5.6].

Remark 6.30. If you are enticed by the uniqueness of the reduced ideal bases as in Lemma 6.28, you should be prepared to embrace the phenomenal sizes of the intermediate Bézout coefficients such as the c_b 's in (6.44). We shall elaborate on this issue when we make a complexity comparison between the new type of bases and Gröbner bases in Section 8. The exemplary wild growth of Bézout coefficients can be found especially in Example 8.1.

7 Further Improvements on the Algorithm

We already strove to simplify our algorithms via Corollary 3.7 and Corollary 5.14 as well as the triangular identities in Lemma 3.8 and Lemma 5.18. In this section we make further improvements on the efficiency and complexity of Algorithm 3.9 and Algorithm 5.20.

Recall that in Algorithm 3.9 and Algorithm 5.20 we have temporary sets G and F respectively containing the basis elements, as well as the temporary sets \mathfrak{S} containing the S-polynomials. We also have Procedure \mathcal{P} for the pseudo-reduction and proper reduction of the S-polynomials in \mathfrak{S} respectively. Let us supplement the following principles to improve the efficiency of Algorithm 3.9 and Algorithm 5.20.

Principle 7.1 (Minimal principle).

- (i) We always list the elements in the temporary set G in Algorithm 3.9 and temporary set F in Algorithm 5.20 such that their leading terms are in increasing order with respect to the monomial ordering.
- (ii) When we make a pseudo-reduction or proper reduction of an S-polynomial in \mathfrak{S} in Procedure \mathcal{P} , we always use the basis elements in the temporary set G in Algorithm 3.9 or F in Algorithm 5.20 first whose leading terms are as small as possible with respect to the monomial ordering.
- (iii) When we choose an S-polynomial in \mathfrak{S} for pseudo-reduction or proper reduction by Procedure \mathcal{P} , we always choose the one whose leading term is as small as possible with respect to the monomial ordering.
- (iv) For each triplet we can invoke a triangular identity as in (3.9) or (5.28) at most once. Moreover, we choose the triangular identity such that the multiplier λ in (3.9) or (5.28) is as simple as possible. More specifically, in the case of (3.9) we require that the degree of the squarefree part of λ as in Definition 4.1 be as small as possible. In the case of (5.28), we require that the degrees of both the unit factor λ^{\times} and standard factor $\lambda_{\rm st}$ as in (5.6) be as small as possible. And the degree of $\lambda_{\rm st}$ has the priority for the comparison.
- (v) In Algorithm 5.20 when we initialize the temporary set $F := \sigma_q(G)$ by applying the epimorphism σ_q in (5.30) to the original basis G, we should choose the representations in R_q of the coefficients of the basis elements in F such that their squarefree parts are as simple as possible in terms of both degrees and coefficients. The priority of the comparison is given to the degrees of the squarefree parts of the leading coefficients. This is especially true for the case when we already have the factorizations of the coefficients.

Principle 7.1 enhances efficiency by imposing a preference or direction for the implementation of Algorithm 3.9 to follow. In effect whenever we make a pseudo-reduction of an S-polynomial in \mathfrak{S} , we usually obtain a remainder r with the least

leading term in G. Then the remainder r generates a new S-polynomial with the least leading term in \mathfrak{S} . We choose to make a pseudo-reduction of this new S-polynomial in \mathfrak{S} according to Principle 7.1 (iii). A repetition of this process with strictly decreasing leading terms inevitably leads to a temporary pseudo-eliminant in $K[x_1]$ before we make a pseudo-reduction of all the S-polynomials in \mathfrak{S} . Let us denote the temporary pseudo-eliminant as χ_{δ} . It is easy to see that the pseudo-eliminant χ_{ε} , the final output of Algorithm 3.9, satisfies $\chi_{\delta} \in (\chi_{\varepsilon}) \subset K[x_1]$.

In what follows let us use the temporary pseudo-eliminant χ_{δ} to further improve Lemma 3.8 and Corollary 3.7.

Lemma 7.2. Let χ_{δ} be a temporary pseudo-eliminant in Algorithm 3.9 such that $\chi_{\delta} \in (\chi_{\varepsilon}) \subset K[x_1]$ with χ_{ε} being the pseudo-eliminant. For $f, g, h \in (K[x_1])[\tilde{\boldsymbol{x}}] \setminus K[x_1]$, suppose that $\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g)) \in \langle \operatorname{LM}(h) \rangle$. If the multiplier $\lambda = \operatorname{LC}(h)/d$ as in (3.9) is relatively prime to χ_{δ} , then it is unnecessary to add λ into the multiplier set Λ in Procedure \mathcal{Q} of Algorithm 3.9. We simply disregard the S-polynomial S(f,g).

Proof. The multiplier λ in (3.9) is relatively prime to the temporary pseudo-eliminant χ_{δ} and hence the pseudo-eliminant χ_{ε} as well as the eliminant χ . The proof of Theorem 4.10 demonstrates that such kind of multipliers have impact on the soundness of neither our arguments nor our conclusions since they would appear as factors of the multiplier ν in (4.25).

Lemma 7.3. Let χ_{δ} be a temporary pseudo-eliminant in Algorithm 3.9 such that $\chi_{\delta} \in (\chi_{\varepsilon}) \subset K[x_1]$ with χ_{ε} being the pseudo-eliminant. Suppose that LM(f) and LM(g) are relatively prime for $f, g \in (K[x_1])[\tilde{x}] \setminus K[x_1]$. If $d = \gcd(LC(f), LC(g))$ as in (3.5) is relatively prime to χ_{δ} , then it is unnecessary to add $\lambda = d$ into the multiplier set Λ in Procedure Q of Algorithm 3.9. We simply disregard the S-polynomial S(f,g).

The reason for disregarding the multiplier d in Lemma 7.3 is the same as that for disregarding the multiplier λ in Lemma 7.2.

Remark 7.4. Please note that after we obtain the temporary pseudo-eliminant χ_{δ} in Algorithm 3.9, we refrain from simplifying the leading coefficients of the basis elements by substituting $gcd(LC(f), \chi_{\delta})$ for LC(f) notwithstanding the temptation of converting f into a monic polynomial. The reason is that such simplifications involve the Bézout coefficients that most probably have gigantic sizes over \mathbb{Q} . Please refer to Example 8.1 for an example on this.

The conformity to Principle 7.1 during the implementation of Algorithm 5.20 also yields a temporary proper eliminant denoted as e_{δ} in most cases before the output of the proper eliminant e_q such that $e_{\delta} \in (e_q) \subset R_q$. When $e_{\delta} \in R_q^*$, similar to Lemma 7.2 and Lemma 7.3, we can make improvements on Lemma 5.18 and Corollary 5.14 as follows.

Lemma 7.5. Let $e_{\delta} \in R_q^*$ be a temporary proper eliminant in Algorithm 5.20 such that $e_{\delta} \in (e_q) \subset R_q$ with e_q being the proper eliminant. For $f, g, h \in (R_q[\tilde{x}])^* \setminus R_q^{\times}$ with at most one of them in $R_q^* \setminus R_q^{\times}$, suppose that $\operatorname{lcm}(\operatorname{LM}(f), \operatorname{LM}(g)) \in \langle \operatorname{LM}(h) \rangle$. If the multiplier $\lambda = \sigma_q(l_h/d)$ as in (5.28) is relatively prime to the temporary proper eliminant e_{δ} , then we simply disregard the S-polynomial S(f,g) in Procedure \mathcal{R} of Algorithm 5.20.

We omit the proof of Lemma 7.5 since it is almost a verbatim repetition of that for Lemma 7.2. In fact, if the multiplier λ is relatively prime to the temporary proper eliminant e_{δ} , then so is it to the proper eliminant e_{q} . Such kind of multipliers have impact on neither our arguments nor our conclusions in Theorem 5.26.

Lemma 7.6. Let $e_{\delta} \in R_q^*$ be a temporary proper eliminant in Algorithm 5.20 such that $e_{\delta} \in (e_q) \subset R_q$ with e_q being the proper eliminant. Suppose that LM(f) and LM(g) are relatively prime for $f, g \in R_q[\tilde{x}] \setminus R_q$. If $d := \gcd_q(LC(f), LC(g))$ as in (5.20) is relatively prime to the temporary proper eliminant e_{δ} , then we simply disregard their S-polynomial S(f, g) in Procedure \mathcal{R} of Algorithm 5.20.

Remark 7.7. After we obtain a temporary proper eliminant e_{δ} in Algorithm 5.20, we can simplify computations by implementing the remaining part of the algorithm in $R_p[\tilde{x}]$ over the normal PQR $R_p \simeq R_q/(e_{\delta})$, which are defined similar to (6.28) and (6.29).

In Lemma 7.6 we can deduce that the multiplier d for the proper reduction is relatively prime to the proper eliminant e_q since $e_{\delta} \in (e_q) \subset R_q$. By Lemma 5.3 (i) we have a unit multiplier $\sigma_p(d) \in R_p^{\times}$ with σ_p defined as in (6.29). Hence the S-polynomial S(f,g) can be disregarded for our conclusions.

8 Complexity Comparison with Gröbner Bases

A conspicuous phenomenon in the computation of Gröbner bases over \mathbb{Q} in Lex ordering is the explosion of intermediate coefficients. The consumption of time and memory in the computation of S-polynomials constitutes another burdensome computational complexity. In this section we prove two lemmas as exemplary illustrations showing that our new type of bases minimize these two problems to a substantial extent.

Let K be a field and $f, g \in (K[x])^*$. The polynomials $u, v \in K[x]$ satisfying $uf + vg = \gcd(f, g)$ are called the *Bézout coefficients* of f and g.

The following Example 8.1 shows that albeit $f, g \in \mathbb{Q}[x]$ have moderate integral coefficients, their Bézout coefficients can swell to quite unpalatable sizes.

Example 8.1.

$$f(x) := (7x^{10} - 9x^8 - 21x^7 + 13x^6 + 29x^5 - 34x^4 - 56x^3 - 14x^2 + 3x + 1)^2;$$

$$g(x) := (6x^{10} + 15x^9 + x^8 - 16x^7 - 37x^6 + 64x^5 + 18x^4 + 5x^3 - 3x^2 - 4x - 1)^2.$$

I refrain from printing out the Bézout coefficients in Example 8.1 since they can trigger a bit discomfort and be calculated for private appreciations via any popular software for symbolic computations.

For a field K and polynomial $f \in K[x]$, let us use lt(f), $lm(f) \in [x]$ and $lc(f) \in K^*$ to denote the leading term, leading monomial and leading coefficient of f over the field K respectively. This is to discriminate from our previous notations for the leading term LT(f), leading monomial $LM(f) \in [\tilde{x}]$ and leading coefficient $LC(f) \in K[x_1]$ of f over the PID $K[x_1]$ when we treat K[x] as $(K[x_1])[\tilde{x}]$.

For $f, g \in K[x] \setminus \{0\}$ with $lm(f) \in \langle lm(g) \rangle$, recall that after the first step of polynomial division of f by g, we have:

$$h = f - \frac{\operatorname{lt}(f)}{\operatorname{lt}(g)}g. \tag{8.1}$$

A continuation of the polynomial division in (8.1) yields a representation:

$$f = qg + r \tag{8.2}$$

with the quotient q and remainder r in K[x] such that r is reduced with respect to g, that is, $r \notin \langle \operatorname{lm}(g) \rangle \setminus \{0\}$.

Also recall that in the computation of Gröbner bases over the field K, the S-polynomial of $f, g \in K[x] \setminus \{0\}$ over K is defined as:

$$S(f,g) := \frac{\boldsymbol{x}^{\gamma}}{\operatorname{lt}(f)} f - \frac{\boldsymbol{x}^{\gamma}}{\operatorname{lt}(g)} g \tag{8.3}$$

with $\boldsymbol{x}^{\gamma} := \operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g)) \in [\boldsymbol{x}].$

Please note that when $\operatorname{lm}(f) \in \langle \operatorname{lm}(g) \rangle$ in (8.3), then $\boldsymbol{x}^{\gamma} = \operatorname{lm}(f)$ and we have the following relationship between the S-polynomial S(f,g) in (8.3) and polynomial division in (8.1):

$$lc(f) \cdot S(f,g) = h. \tag{8.4}$$

Lemma 8.2. For a field K and elimination ordering $z \prec x$ on K[x, z], consider the ideal $I := \langle ax + c, bx + d \rangle$ with $a, b, c, d \in (K[z])^*$. Suppose that I is a zero-dimensional ideal such that $ad - bc \neq 0$. Then the computation of Gröbner basis of I contains Euclidean algorithm for the computation of $\gcd(a, b)$. In particular, the Bézout coefficients of a and b appear in the intermediate coefficients of the computation.

Proof. Without loss of generality, suppose that $lt(a) = c_{\alpha}z^{\alpha}$ and $lt(b) = c_{\beta}z^{\beta}$ with $c_{\alpha}, c_{\beta} \in K^*$ and $\alpha \geq \beta$.

Suppose that the first step of polynomial division of ax + c by bx + d as in (8.1) is $a_1x + c_1 := ax + c - (c_{\alpha}/c_{\beta})z^{\alpha-\beta}(bx+d)$. According to the identity in (8.4), the classical S-polynomial as in (8.3) is essentially $a_1x + c_1$ up to a unit multiplier c_{α} , that is, $c_{\alpha}S(ax+c,bx+d) = a_1x+c_1$. If we have $\deg(a_1) \geq \deg(b)$, a continuation of the division of $a_1x + c_1$ by bx + d as in (8.2) coincides with the reduction of the S-polynomial $c_{\alpha}S(ax+c,bx+d)$ by bx+d in the computation of Gröbner basis for I:

$$c_{\alpha}S(ax+c,bx+d) = a_1x + c_1 = q(bx+d) + b_1x + d_1 \tag{8.5}$$

with $q, b_1, d_1 \in K[z]$ such that $\deg(b_1) < \deg(b)$.

Let us assume that $b_1d_1 \neq 0$ in (8.5). In the computation of Gröbner basis, we add $b_1x + d_1$ into the temporary set of basis for I in this case. Then we compute the S-polynomial $S(bx + d, b_1x + d_1)$ and further reduce it by $b_1x + d_1$. Similar to the above discussion in (8.5) based on the identity (8.4), this exactly coincides with the step of Euclidean algorithm in which we make a polynomial division of bx + d by $b_1x + d_1$.

A repetition of the above process shows that the computation of Gröbner basis for I amounts to an implementation of Euclidean algorithm for the computation of

 $\gcd(a,b)$, i.e., the greatest common divisor of the leading coefficients $a,b \in (K[z])^*$, albeit with unit multipliers like $c_{\alpha} = \operatorname{lc}(a)$ in (8.5). Let us denote $\rho := \gcd(a,b)$ with Bézout coefficients $u,v \in K[z]$ such that $\rho = ua + vb$. According to Euclidean algorithm, after the above computations we shall obtain $u(ax+c) + v(bx+d) = \rho x + uc + vd$. We can obtain the same result via the computation and reduction of S-polynomials if we assimilate the unit multipliers like c_{α} in (8.5) into the Bézout coefficients u,v. We add $\rho x + uc + vd$ into the temporary set of basis for I and use it to eliminate the variable x so as to obtain the eliminant χ of I. Let us denote $a = m\rho$ and $b = n\rho$ with $m, n \in K[z]$. Similar to the above discussions, the process of reducing the S-polynomial $c_{\alpha}S(ax+c, \rho x + uc + vd)$ by $\rho x + uc + vd$ amounts to the elimination of the variable x as follows.

$$ax + c - m(\rho x + uc + vd) = c - m(uc + vd) = \frac{c\rho - a(uc + vd)}{\rho} = \frac{v(bc - ad)}{\rho}$$
 (8.6)

since $\rho = ua + vb$. Similarly we have:

$$bx + d - n(\rho x + uc + vd) = -\frac{u(bc - ad)}{\rho}.$$
 (8.7)

From $u(a/\rho) + v(b/\rho) = 1$ we can infer that the Bézout coefficients u and v are relatively prime to each other. Hence the eliminant χ can be obtained from (8.6) and (8.7) as following:

$$\chi = \gcd\left(\frac{v(bc - ad)}{\rho}, -\frac{u(bc - ad)}{\rho}\right) = \frac{bc - ad}{\rho}.$$
 (8.8)

The Bézout coefficients u and v appear in (8.6) and (8.7) but not in the eliminant χ in (8.8).

Lemma 8.2 represents a more generic scenario than the ideal $I = \langle ax+c, bx+d \rangle$ appears to be. In fact, in the final steps in the computation of the eliminant of a zero-dimensional ideal, we often run into the situation in the lemma. In the case of Lemma 8.2 over the PID K[z], the intermediate coefficients in (8.6) and (8.7) contain the Bézout coefficients u and v of the leading coefficients a and b as their factors that tend to swell over the rational field $\mathbb Q$ like in Example 8.1. Nonetheless in terms of our new type of S-polynomials as in (3.1) over the PID K[z], we have a straightforward computation as follows.

$$S(ax + c, bx + d) = \lambda(ax + c) - \mu(bx + d) = \lambda c - \mu d = \frac{bc - ad}{\rho} = \chi,$$
 (8.9)

where the two multipliers $\lambda := l/a = b/\rho = n$ and $\mu := l/b = a/\rho = m$ with l := lcm(a, b) and $\rho = \text{gcd}(a, b)$ such that $a = m\rho$ and $b = n\rho$.

The eliminant χ in (8.8) is obtained in one step in (8.9) without resorting to the Bézout coefficients u and v of the leading coefficients a and b.

Now let us generalize Lemma 8.2 to contrive a generic scenario as follows.

Lemma 8.3. With a field K and elimination ordering on [x] as in Definition 3.1, suppose that the generators f and g of the ideal $I = \langle f, g \rangle \subset (K[x_1])[\tilde{x}]$ satisfy $\operatorname{LT}(f) = a\tilde{x}^{\alpha}$ and $\operatorname{LT}(g) = b\tilde{x}^{\beta}$ with the leading coefficients $a, b \in (K[x_1])^*$. Then the computation of Gröbner basis of I contains Euclidean algorithm for the computation of $\operatorname{gcd}(a,b)$. In particular, the Bézout coefficients of a and b appear in the intermediate coefficients of the computation.

Proof. Without loss of generality, suppose that $s = \deg(a) \ge \deg(b) = t$. Let us denote $\operatorname{lc}(f) = c$ and $\operatorname{lc}(g) = d$ in K^* respectively. Then $\operatorname{lt}(f) = cx_1^s \tilde{\boldsymbol{x}}^{\alpha}$ and $\operatorname{lt}(g) = dx_1^t \tilde{\boldsymbol{x}}^{\beta}$. With $\tilde{\boldsymbol{x}}^{\gamma} := \operatorname{lcm}(\tilde{\boldsymbol{x}}^{\alpha}, \tilde{\boldsymbol{x}}^{\beta})$, we have $\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g)) = x_1^s \tilde{\boldsymbol{x}}^{\gamma}$. The S-polynomial in (8.3) now bears the following form:

$$cS(f,g) = \tilde{\boldsymbol{x}}^{\gamma-\alpha}f - \frac{cx_1^{s-t}}{d}\tilde{\boldsymbol{x}}^{\gamma-\beta}g = \left(a - \frac{cx_1^{s-t}}{d}b\right)\tilde{\boldsymbol{x}}^{\gamma} + \tilde{\boldsymbol{x}}^{\gamma-\alpha}\left(f_1 - \frac{\operatorname{lt}(f)}{\operatorname{lt}(g)}g_1\right) \tag{8.10}$$

with $f_1 := f - LT(f) = f - a\tilde{\boldsymbol{x}}^{\alpha}$ and $g_1 := g - LT(g) = g - b\tilde{\boldsymbol{x}}^{\beta}$.

Since we also have c = lc(a) and d = lc(b) respectively, the leading coefficient of cS(f,g) in $(K[x_1])[\tilde{x}]$ in (8.10), i.e., $LC(cS(f,g)) = a - (c/d)x_1^{s-t}b := a_1 \in K[x_1]$, is exactly the first step of the polynomial division of a by b in $K[x_1]$.

If $\deg(a_1) \geq \deg(b)$, then a further reduction of the S-polynomial cS(f,g) in (8.10) by g amounts to a polynomial division of a_1 by b as in (8.2). Suppose that we have $a_1 = qb + r$ with $q, r \in K[x_1]$ such that $\deg(r) < \deg(b)$. Then cS(f,g) is reduced by g to a polynomial $h \in (K[x_1])[\tilde{x}]$ bearing the form $h = r\tilde{x}^{\gamma} + h_1$ such that $LT(h) = r\tilde{x}^{\gamma}$. For simplicity, let us assume that h_1 is already reduced with respect to g, that is, no term of h_1 is in $\langle \operatorname{Im}(g) \rangle \subset K[x]$.

Next in the computation of Gröbner basis, we add h into the basis $\{f, g\}$ of I and compute the S-polynomial S(g, h). Similar to (8.10), the leading coefficient LC(dS(g, h)) of \tilde{x}^{γ} amounts to the first step of the polynomial division of b by r in $K[x_1]$. This together with a further reduction of dS(g, h) by h exactly coincide with the step of Euclidean algorithm in which we make a polynomial division of b by r in $K[x_1]$.

A repetition of the above process shows that the computation of Gröbner basis for I contains an implementation of Euclidean algorithm for the computation of gcd(a,b), i.e., the greatest common divisor of the leading coefficients a = LC(f) and b = LC(g) in $K[x_1]$, albeit with unit multipliers like c in (8.10). Let us denote $\rho := gcd(a,b)$ with Bézout coefficients $u,v \in K[x_1]$ such that $\rho = ua + vb$. In essence the computation of Gröbner basis amounts to a computation of the greatest common divisor of the leading coefficients. Hence based on Euclidean algorithm we have:

$$u\tilde{\boldsymbol{x}}^{\gamma-\alpha}(a\tilde{\boldsymbol{x}}^{\alpha}+f_1)+v\tilde{\boldsymbol{x}}^{\gamma-\beta}(b\tilde{\boldsymbol{x}}^{\beta}+g_1)=\rho\tilde{\boldsymbol{x}}^{\gamma}+u\tilde{\boldsymbol{x}}^{\gamma-\alpha}f_1+v\tilde{\boldsymbol{x}}^{\gamma-\beta}g_1:=w. \quad (8.11)$$

We add w in (8.11) into the basis of I and then compute the S-polynomial S(f, w). We make a reduction of the S-polynomial S(f, w) by w. Let us denote $a = m\rho$ and $b = n\rho$ with $m, n \in K[z]$. From the perspective of $(K[x_1])[\tilde{x}]$, this reduction process can be summarized as being equivalent to the following elimination of the leading term $a\tilde{x}^{\alpha}$ of $f = a\tilde{x}^{\alpha} + f_1$:

$$\tilde{\boldsymbol{x}}^{\gamma-\alpha}f - mw = \rho[(1-mu)\tilde{\boldsymbol{x}}^{\gamma-\alpha}f_1 - mv\tilde{\boldsymbol{x}}^{\gamma-\beta}g_1]/\rho
= \frac{v(b\tilde{\boldsymbol{x}}^{\gamma-\alpha}f_1 - a\tilde{\boldsymbol{x}}^{\gamma-\beta}g_1)}{\rho}.$$
(8.12)

Similarly we have:

$$\tilde{\boldsymbol{x}}^{\gamma-\beta}g - nw = -\frac{u(b\tilde{\boldsymbol{x}}^{\gamma-\alpha}f_1 - a\tilde{\boldsymbol{x}}^{\gamma-\beta}g_1)}{\rho}.$$
(8.13)

Thus the Bézout coefficients u and v appear in the computation of Gröbner basis for I. And they might swell over the rational field $K = \mathbb{Q}$.

With l := lcm(a, b), let us denote two multipliers $\lambda := l/a = b/\rho = n$ and $\mu := l/b = a/\rho = m$. In terms of the S-polynomials as in (3.1) over the PID $K[x_1]$, we have a straightforward computation of the S-polynomial S(f, g) as follows.

$$S(f,g) = \lambda \tilde{\boldsymbol{x}}^{\gamma-\alpha} (a\tilde{\boldsymbol{x}}^{\alpha} + f_1) - \mu \tilde{\boldsymbol{x}}^{\gamma-\beta} (b\tilde{\boldsymbol{x}}^{\beta} + g_1)$$

$$= \lambda \tilde{\boldsymbol{x}}^{\gamma-\alpha} f_1 - \mu \tilde{\boldsymbol{x}}^{\gamma-\beta} g_1 = (b\tilde{\boldsymbol{x}}^{\gamma-\alpha} f_1 - a\tilde{\boldsymbol{x}}^{\gamma-\beta} g_1)/\rho.$$
(8.14)

We obtained a simpler result in (8.14) in one step than those in (8.12) and (8.13) without the Bézout coefficients u and v of the leading coefficients a = LC(f) and b = LC(g) that might swell to an unexpected size over the rational field $K = \mathbb{Q}$ like in Example 8.1.

9 Examples and Paradigmatic Computations

I furnish this section with two examples to demonstrate the computations of the new type of bases for zero-dimensional ideals. In theses examples it is conspicuous that the intermediate coefficients as well as the coefficients of the basis elements are restrained to moderate sizes and do not swell like in the case of Gröbner bases over \mathbb{Q} .

Example 9.1 is an excerpt from the textbook [CLO05, Chapter 8, P426, §4] with minor modifications. In this simple example the multiplier set Λ in Algorithm 3.9 is empty except in the final step. The pseudo-eliminant χ_{ε} procured in this way is exactly the eliminant χ .

Example 9.1. Suppose that the ideal $I = \langle f, g, h \rangle \subset \mathbb{Q}[x, y, z]$ with

$$f = -x + y + z^2 - 1;$$
 $g = -zx + y^3 + 2;$ $h = x^2 + x - zy.$ (9.1)

For the purpose of comparison, we list its classical Gröbner basis with respect to the LEX ordering $z \prec y \prec x$ as $\{p, g_1, g_2\}$ such that:

$$p = z^{12} - 3z^{10} - 2z^8 + 4z^7 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z + 6;$$

$$g_1 = 38977y + 1055z^{11} + 515z^{10} + 42z^9 - 3674z^8 - 12955z^7 + 5285z^6 - 1250z^5 + 36881z^4 + 7905z^3 + 42265z^2 - 63841z - 37186;$$

$$g_2 = 38977x + 1055z^{11} + 515z^{10} + 42z^9 - 3674z^8 - 12955z^7 + 5285z^6 - 1250z^5 + 36881z^4 + 7905z^3 + 3288z^2 - 63841z + 1791.$$

$$(9.2)$$

Let us denote the temporary pseudo-basis set $G := \{f, g, h\}$. As per Principle 7.1 (i), we list the elements in G in increasing order of their leading terms with respect to the LEX ordering as in (9.1).

As in Procedure \mathcal{Q} of Algorithm 3.9, we can disregard the S-polynomial S(g,h) as per the triangular identity (3.9). In fact, $\operatorname{lcm}(\operatorname{LT}(g),\operatorname{LT}(h)) = -zx^2$ is divisible by $\operatorname{LT}(f) = -x$ and hence the multiplier $\lambda = 1$ in (3.9) in this case. The temporary S-polynomial set is $\mathfrak{S} = \{S(f,g),S(f,h)\}$.

According to Principle 7.1 (iii), we first compute the S-polynomial:

$$S(f,g) = zf - g = -y^3 + zy + z^3 - z - 2 := e.$$
(9.3)

We add it into G such that $G = \{e, f, g, h\}$. We name it as the first element e according to Principle 7.1 (i) because LT(e) is less than every element in $LT(G \setminus \{e\})$

and hence cannot be pseudo-reduced by $G \setminus \{e\}$ as in Theorem 2.7. Then we delete S(f,g) from \mathfrak{S} .

We can disregard the S-polynomials S(e, f), S(e, g) and S(e, h) as in Procedure Q of Algorithm 3.9. This is based on Corollary 3.7 since LT(e) is relatively prime to every element in $LT(G \setminus \{e\})$.

We compute the S-polynomial

$$S(f,h) = xf + h = xy + z^2x - zy$$

and pseudo-reduce its leading term xy by f like in (2.1) as per Principle 7.1 (ii). The remainder of the term pseudo-reduction is as follows:

$$r = S(f,h) + yf = z^2x + y^2 + (z^2 - z - 1)y.$$

We make a pseudo-reduction of the leading term $LT(r) = z^2x$ by f like in (2.1) for another time. The final remainder is as follows.

$$d := r + z^{2} f = y^{2} + (2z^{2} - z - 1)y + z^{2}(z^{2} - 1)$$

$$(9.4)$$

that cannot be further pseudo-reduced by G. We add it into G such that $G = \{d, e, f, g, h\}$. The reason for naming it as the first element d of G is still Principle 7.1 (i). Then we delete S(f, h) from \mathfrak{S} .

We disregard the S-polynomials S(d, f), S(d, g) and S(d, h) as in Procedure Q of Algorithm 3.9 based on Corollary 3.7. We add S(d, e) into \mathfrak{S} such that $\mathfrak{S} = \{S(d, e)\}.$

We compute the S-polynomial S(d, e) as following:

$$S(d,e) = yd + e = (2z^2 - z - 1)y^2 + (z^3 - z + 1)zy + z^3 - z - 2$$

and pseudo-reduce its leading term $(2z^2 - z - 1)y^2$ by d as in (2.1). For the same reason as above, we name the remainder of the term pseudo-reduction as the first element in G such that:

$$c := S(d, e) - (2z^{2} - z - 1)d$$

= $(3z^{4} - 4z^{3} - 2z^{2} + z + 1)y + 2z^{6} - z^{5} - 3z^{4} + z^{2} + z + 2.$ (9.5)

Now $G = \{c, d, e, f, g, h\}$. Then we delete S(d, e) from \mathfrak{S} .

We disregard the S-polynomials S(c, f), S(c, g) and S(c, h) as in Procedure \mathcal{Q} of Algorithm 3.9 based on Corollary 3.7 due to their relatively prime leading terms. By the triangular identity (3.9), we also disregard the S-polynomial S(c, e) as in Procedure \mathcal{Q} of Algorithm 3.9 since $\operatorname{lcm}(\operatorname{LT}(c), \operatorname{LT}(e)) = -(3z^4 - 4z^3 - 2z^2 + z + 1)y^3$ is divisible by $\operatorname{LT}(d) = y^2$. Here the multiplier λ as in (3.9) satisfies $\lambda = 1$. We add S(c, d) into \mathfrak{S} such that $\mathfrak{S} = \{S(c, d)\}$.

We compute the S-polynomial

$$S(c,d) = -yc + (3z^4 - 4z^3 - 2z^2 + z + 1)d = y(4z^6 - 10z^5 + 8z^3 + 2z^2 - 3z - 3) + (3z^6 - 4z^5 - 5z^4 + 5z^3 + 3z^2 - z - 1)z^2$$

and then pseudo-reduce it by c. The multiplier for the pseudo-reduction is $3z^4 - 4z^3 - 2z^2 + z + 1$ and we add it into a multiplier set $\Lambda = \{3z^4 - 4z^3 - 2z^2 + z + 1\}$. The remainder of the pseudo-reduction is a temporary pseudo-eliminant

$$\chi_{\delta} := z^{12} - 3z^{10} - 2z^8 + 4z^7 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + 2z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z^2 + 6z^6 + 12z^6 +$$

Then we delete S(c,d) from \mathfrak{S} such that $\mathfrak{S} = \emptyset$ and we have completed the procedure of Algorithm 3.9. Hence the pseudo-eliminant $\chi_{\varepsilon} = \chi_{\delta}$. Moreover, the pseudo-eliminant χ_{ε} is relatively prime to the multiplier λ in Λ and hence is compatible. Hence the eliminant $\chi = \chi_{\varepsilon}$ according to Theorem 4.10.

Now the pseudo-basis of I as in Definition 3.11 is $B_{\varepsilon} = G = \{c,d,e,f,g,h\}$ as in (9.1), (9.3), (9.4) and (9.5) respectively. Let us define $q := \chi$ and the epimorphism $\sigma_q \colon (K[z])[x,y] \longrightarrow R_q[x,y]$ as in (6.14) over the normal PQR $R_q \simeq K[z]/(q)$. Now both $\operatorname{LT}(\sigma_q(g)) = -zx$ and $\operatorname{LT}(\sigma_q(h)) = x^2$ are divisible and hence GCD-reducible with respect to $\sigma_q(f)$ as in Definition 6.9. Thus in order to obtain an irredundant modular basis as in Definition 6.21, we can delete $\sigma_q(g)$ and $\sigma_q(h)$ from $B_q = \sigma_q(B_{\varepsilon})$. Moreover, we also delete $\sigma_q(d)$ and $\sigma_q(e)$ from B_q since both $\operatorname{LT}(\sigma_q(d)) = y^2$ and $\operatorname{LT}(\sigma_q(e)) = -y^3$ are GCD-reducible with respect to $\sigma_q(c)$ as per Definition 6.9 for GCD-reducibility. In fact, we have $\operatorname{LC}(\sigma_q(c)) = 3z^4 - 4z^3 - 2z^2 + z + 1 \in R_q^{\times}$ is a unit. Hence the following basis constitutes an irredundant basis of $I_q = \sigma_q(I)$ under the above epimorphism σ_q :

$$\sigma_q(c) = (3z^4 - 4z^3 - 2z^2 + z + 1)y + 2z^6 - z^5 - 3z^4 + z^2 + z + 2;$$

$$\sigma_q(f) = -x + y + z^2 - 1.$$
(9.6)

For simplicity we also call $\{c, f\}$ an irredundant basis of I.

The irredundant basis in (9.6) is automatically a minimal basis as in Definition 6.24 but not a reduced basis as in Definition 6.27. We can make a GCD-term reduction of the term y in $\sigma_q(f)$ by $\sigma_q(c)$ with multiplier $\mu = 3z^4 - 4z^3 - 2z^2 + z + 1 \in R_q^{\times}$ as in Definition 6.11 to obtain a remainder denoted as b. Now $\{\sigma_q(c), b\}$ constitutes a reduced basis for I_q . For simplicity we still denote $\sigma_q(c)$ as c and they are our new type of basis for I_q that we still denote as B_q as follows.

$$\begin{cases}
c = (3z^4 - 4z^3 - 2z^2 + z + 1)y + 2z^6 - z^5 - 3z^4 + z^2 + z + 2; \\
b = (3z^4 - 4z^3 - 2z^2 + z + 1)x - z^6 + 3z^5 + 2z^4 - 5z^3 - 2z^2 + 2z + 3.
\end{cases} (9.7)$$

The reduced basis B_q of I_q in (9.7) is unique by Lemma 6.28. It satisfies the identity (6.15) in Lemma 6.6 and hence the unified identity (6.34). Since $b = \iota_q(b)$ and $c = \iota_q(c)$, the set $\{b, c, \chi\}$ constitutes another form of our new type of basis for $I + \langle \chi \rangle = I$. It is in a simpler form than the one in (9.2) with moderate coefficients and exponents for the variable z.

The following Example 9.2 is a slight complication of Example 9.1 in that the multiplier set Λ is not empty during the implementation of Algorithm 3.9. Hence we have to invoke Algorithm 5.20 over a PQR with zero divisors to procure the exact form of the eliminant and new type of bases. The coefficients of the classical Gröbner basis in Example 9.2 also cause a bit more psychological disturbances than those in Example 9.1.

Example 9.2. Suppose that the ideal $I = \langle f, g, h \rangle \subset \mathbb{Q}[x, y, z]$ with

$$f = -z^2(z+1)^3x + y; \ g = z^4(z+1)^6x - y^2; \ h = -x^2y + y^3 + z^4(z-1)^5.$$
 (9.8)

For the purpose of comparison, in the following we list its classical Gröbner basis $G = \{p, g_1, g_2, g_3, g_4\}$ with respect to the LEX ordering $z \prec y \prec x$:

$$p = (z-1)^{5}z^{6}(z^{13} + 9z^{12} + 36z^{11} + 84z^{10} + 126z^{9} + 126z^{8} + 85z^{7} + 31z^{6} + 19z^{5} - 9z^{4} + 4z^{3} - 4z^{2} - 3z - 1).$$

$$(9.9)$$

```
g_1 = 20253807z^2y + 264174124z^{23} + 1185923612z^{22} + 850814520z^{21} -
     -3776379304z^{20} -6824277548z^{19} +1862876196z^{18} +12815317453z^{17} +
     +\,3550475421z^{16} + 2124010584z^{15} - 35582561480z^{14} + 42918431554z^{13} -
     -41728834070z^{12} + 35649844325z^{11} - 17049238505z^{10} + 3388659963z^9 +
     +930240431z^8 - 61146095z^7 - 518331181z^6:
q_2 = 20253807y^2 + 903303104z^{23} + 4102316224z^{22} + 3140448384z^{21} -
     -12683487983z^{20} - 23996669428z^{19} + 4804720290z^{18} + 43739947868z^{17} +
     + 14906482335z^{16} + 9051639768z^{15} - 121400613331z^{14} +
     \phantom{z_5}+139970660534z^{13}-138071007235z^{12}+118589702914z^{11}-
     -55199680030z^{10} + 11927452134z^9 + 2021069107z^8 - 38017822z^7 -
     -1768266833z^6;
g_3 = 2592487296z^2x + (7777461888z - 2592487296)y + 108083949263z^{23} +
     -2820179010211z^{19} + 788268739077z^{18} + 5350420983851z^{17} +
     +1476923019345z^{16}+689330555757z^{15}-14602936038043z^{14}+
     + 17386123487861z^{13} - 16350039201517z^{12} + 13787524468420z^{11} -
     -6235683207154z^{10} + 786997920594z^9 + 628350552934z^8 -
     -64382649769z^7 - 206531133875z^6:
g_4 = 20253807x^2y + 1037047036z^{23} + 4686773132z^{22} + 3455561112z^{21} -
     -14868243976z^{20} - 27470438972z^{19} + 6731446644z^{18} + 51651585868z^{17} +
     +16267315284z^{16} + 7429467573z^{15} - 141636109619z^{14} +
     + 163168836472z^{13} - 155454190640z^{12} + 135706468958z^{11} -
     -62903516282z^{10} + 11263865469z^9 + 2500312823z^8 + 197272975z^7 -
     -1682438629z^6 - 101269035z^5 + 20253807z^4.
```

We define the temporary pseudo-basis set $G := \{f, g, h\}$ as in (9.8). As per Principle 7.1 (i), we list the elements in G in increasing order of their leading terms with respect to the LEX ordering as in (9.8).

We can disregard the S-polynomial S(g,h) as in Procedure $\mathcal Q$ of Algorithm 3.9 based on the triangular identity in Lemma 3.8. In fact, $\operatorname{lcm}(\operatorname{LT}(g),\operatorname{LT}(h)) = -z^4(z+1)^6x^2y$ is divisible by $\operatorname{LT}(f) = -z^2(z+1)^3x$ and hence in this case the multiplier $\lambda = 1$ in (3.9). We shall not take into account the triangular identity of S(f,h) with respect to g since we already invoked a triangular identity in the above on the same triplet $\{f,g,h\}$ according to Principle 7.1 (iv). Hence the temporary S-polynomial set is $\mathfrak S = \{S(f,g),S(f,h)\}$.

According to Principle 7.1 (iii), we first compute the S-polynomial

$$S(f,g) = z^{2}(z+1)^{3}f + g = -y^{2} + z^{2}(z+1)^{3}y := e$$
(9.10)

that cannot be further pseudo-reduced by G as in Theorem 2.7. We add e into G such that $G = \{e, f, g, h\}$. We name it as the first element e in G because LT(e) is less than every element in $LT(G \setminus \{e\})$. Then we delete S(f, g) from \mathfrak{S} such that $\mathfrak{S} = \{S(f, h)\}$.

We can disregard the S-polynomials S(e, f) and S(e, g) like in Procedure \mathcal{Q} of Algorithm 3.9 based on Corollary 3.7 since LT(e) is relatively prime to LT(f)

and LT(g). We can also disregard the S-polynomial S(e,h) as in Procedure \mathcal{Q} of Algorithm 3.9 based on the triangular identity (3.9) with respect to f. In fact, the leading monomials of the triplet satisfy $\operatorname{lcm}(\operatorname{LM}(e),\operatorname{LM}(h))=x^2y^2$ being divisible by $\operatorname{LM}(f)=x$. The multiplier λ in the identity (3.9) equals $\lambda=\operatorname{LC}(f)=-z^2(z+1)^3$. We add λ into the multiplier set Λ such that $\Lambda=\{z^2(z+1)^3\}$.

We compute the S-polynomial

$$S(f,h) = z^{2}(z+1)^{3}h - xyf = -xy^{2} + z^{2}(z+1)^{3}y^{3} + z^{6}(z+1)^{3}(z-1)^{5}$$

and pseudo-reduce it as in Theorem 2.7 by e and f according to Principle 7.1 (ii). More specifically, we first pseudo-reduce $LT(S(f,h)) = -xy^2$ by e in (9.10) with interim multiplier $\mu = 1$ as in (2.1) to obtain the following remainder:

$$r_1 = S(f,h) - xe = -z^2(z+1)^3xy + z^2(z+1)^3y^3 + z^6(z+1)^3(z-1)^5.$$

Then we make a further pseudo-reduction of $LT(r_1) = -z^2(z+1)^3xy$ by f in (9.8) as in (2.1) also with interim multiplier $\mu = 1$. The new remainder is as follows.

$$r_2 = r_1 - yf = z^2(z+1)^3y^3 - y^2 + z^6(z+1)^3(z-1)^5.$$

An ensuing pseudo-reduction of $LT(r_2) = z^2(z+1)^3y^3$ by e in (9.10) with interim multiplier $\mu = 1$ yields the following remainder:

$$r_3 = r_2 + z^2(z+1)^3 ye = (z^4(z+1)^6 - 1)y^2 + z^6(z+1)^3(z-1)^5.$$

We obtain the following final remainder d after a repetition of the above pseudoreduction of $LT(r_3) = (z^4(z+1)^6 - 1)y^2$ by e in (9.10) with interim multiplier $\mu = 1$:

$$d := r_3 + (z^4(z+1)^6 - 1)e = z^2(z+1)^3[(z^4(z+1)^6 - 1)y + z^4(z-1)^5].$$
 (9.11)

For the same reason as above, we name the remainder d as the first element in G such that $G = \{d, e, f, g, h\}$. Then we delete S(f, h) from \mathfrak{S} such that $\mathfrak{S} = \emptyset$.

The leading monomial LM(d) = y is relatively prime to LM(f) = LM(g) = x. But their leading coefficients satisfy $\gcd(LC(d), LC(f)) = \gcd(LC(d), LC(g)) = z^2(z+1)^3$, which is already in the multiplier set Λ . Hence we can just disregard the S-polynomials S(d,f) and S(d,g) as in Procedure Q of Algorithm 3.9. Moreover, we also disregard the S-polynomial S(d,h) by the triangular identity with respect to f as in (3.9) since $\operatorname{lcm}(LT(d), LT(h)) = z^2(z+1)^3(z^4(z+1)^6-1)x^2y$ is divisible by $\operatorname{LT}(f) = -z^2(z+1)^3x$. We add the S-polynomial S(d,e) into $\mathfrak S$ such that $\mathfrak S = \{S(d,e)\}$.

We compute the S-polynomial S(d, e) as following:

$$S(d,e) = yd + z^{2}(z+1)^{3}(z^{4}(z+1)^{6} - 1)e$$

= $z^{4}(z+1)^{3}(z^{13} + 9z^{12} + 36z^{11} + 84z^{10} + 126z^{9} + 126z^{8} + 85z^{7} + 31z^{6} + 19z^{5} - 9z^{4} + 4z^{3} - 4z^{2} - 3z - 1)y.$

Then we make a pseudo-reduction of LT(S(d,e)) = S(d,e) by d in (9.11) as in (2.1) with the interim multiplier $\mu = z^4(z+1)^6 - 1$. The remainder of the term pseudo-reduction is:

$$r_4 = z^4(z+1)^3(z^{13} + 9z^{12} + 36z^{11} + 84z^{10} + 126z^9 + 126z^8 + 85z^7 + 31z^6 + 19z^5 - 9z^4 + 4z^3 - 4z^2 - 3z - 1)[2(z^4(z+1)^6 - 1)y + z^4(z-1)^5].$$

We add the multiplier μ into the multiplier set Λ such that

$$\Lambda = \{z^2(z+1)^3, \ z^4(z+1)^6 - 1\}. \tag{9.12}$$

After a further pseudo-reduction of the above remainder r_4 by d in (9.11) with interim multiplier $\mu = 1$, we can obtain a temporary pseudo-eliminant as follows.

$$\chi_{\delta} = (z-1)^{5} z^{8} (z+1)^{3} (z^{13} + 9z^{12} + 36z^{11} + 84z^{10} + 126z^{9} + 126z^{8} + 85z^{7} + 31z^{6} + 19z^{5} - 9z^{4} + 4z^{3} - 4z^{2} - 3z - 1).$$

$$(9.13)$$

Then we delete S(d, e) from \mathfrak{S} such that $\mathfrak{S} = \emptyset$. Since we have exhausted all the S-polynomials in \mathfrak{S} , the pseudo-eliminant $\chi_{\varepsilon} = \chi_{\delta}$.

By a comparison as in Algorithm 4.6 between the pseudo-eliminant χ_{ε} in (9.13) and multiplier set Λ in (9.12), we can compute the compatible part $CP(\chi_{\varepsilon})$ of the pseudo-eliminant χ , which is defined in Definition 4.5, as following:

$$CP(\chi_{\varepsilon}) = (z-1)^{5}(z^{13} + 9z^{12} + 36z^{11} + 84z^{10} + 126z^{9} + 126z^{8} + 85z^{7} + 31z^{6} + 19z^{5} - 9z^{4} + 4z^{3} - 4z^{2} - 3z - 1).$$

$$(9.14)$$

Moreover, the composite divisors of the incompatible part IP(χ_{ε}) of the pseudo-eliminant χ_{ε} in (9.13) are z^{8} and $(z+1)^{3}$ as per Definition 4.7.

For the composite divisor $q=z^8$, in what follows let us invoke Algorithm 5.20 to compute its corresponding proper eliminant e_q . The computations are based on Theorem 5.26 over the normal PQR $R_q \simeq K[z]/(z^8)$.

The epimorphism σ_q as in (5.30) transforms the basis elements g and h into:

$$\sigma_q(g) = (20z^3 + 15z^2 + 6z + 1)z^4x - y^2;$$

$$\sigma_q(h) = -x^2y + y^3 + (10z^3 - 10z^2 + 5z - 1)z^4.$$

The squarefree part of LC(g) in (9.8) equals z(z+1) whereas it equals $z(20z^3+15z^2+6z+1)$ as above. Hence we should use the representation of g in (9.8) as per Principle 7.1 (v). The same reason for using the representation of h in (9.8). Thus although we start with the basis elements $F = \{\sigma_q(f), \sigma_q(g), \sigma_q(h)\}$, we still denote it as $\{f, g, h\}$ in (9.8) henceforth with the understanding that $F \subset R_q[x, y]$ and $R_q \simeq K[z]/(z^8)$.

We can disregard the S-polynomial S(g,h) as in Procedure \mathcal{R} of Algorithm 5.20 by the triangular identity with respect to f as in Lemma 5.18. In fact, by the representations in (9.8), we have $\operatorname{lcm}(\operatorname{LT}(g),\operatorname{LT}(h)) = -z^4(z+1)^6x^2y$ being divisible by $\operatorname{LT}(f) = -z^2(z+1)^3x$. Hence it is easy to deduce that the multiplier λ in (5.28) now becomes $\lambda = 1$ and thus it is redundant to compute the S-polynomials S(g,h). The temporary S-polynomial set is $\mathfrak{S} = \{S(f,g),S(f,h)\}$.

According to Principle 7.1 (iii), we first compute the S-polynomial S(f, g):

$$S(f,g) = z^{2}(z+1)^{3}f + g \mod (q=z^{8})$$

= $-y^{2} + z^{2}(z+1)^{3}y := e.$ (9.15)

The S-polynomial S(f,g) cannot be properly reduced by $F = \{f,g,h\}$. We add S(f,g) into F and name it as the first element e in F according to Principle 7.1 (i) such that $F = \{e, f, g, h\}$. Then we delete S(f,g) from \mathfrak{S} .

We can disregard the S-polynomials S(e, f) and S(e, g) as in Procedure \mathcal{R} of Algorithm 5.20 based on Corollary 5.14 because LT(e) is relatively prime to both LT(f) and LT(g). We add the S-polynomial S(e, h) into \mathfrak{S} such that $\mathfrak{S} = \{S(e, h), S(f, h)\}$.

By Principle 7.1 (iii) we first compute the S-polynomial S(f, h):

$$S(f,h) = xyf - z^2(z+1)^3h \mod (q=z^8)$$

= $xy^2 - z^2(z+1)^3y^3 - (2z-1)z^6$.

We make a proper reduction of S(f,h) as in Theorem 5.10 by e and f according to Principle 7.1 (ii). More specifically, we first properly reduce $LT(S(f,h)) = xy^2$ by e in (9.15) with interim multiplier $\mu = 1$ as in (5.12). The remainder r_1 is as follows.

$$r_1 = S(f,h) + xe = z^2[(z+1)^3xy - (z+1)^3y^3 + z^4 - 2z^5].$$

Then we make a further proper term reduction of $LT(r_1) = z^2(z+1)^3xy$ by f also with interim multiplier $\mu = 1$. The remainder of the reduction is as follows.

$$r_2 = r_1 + yf = -z^2(z+1)^3y^3 + y^2 - 2z^7 + z^6.$$

An ensuing proper term reduction of $LT(r_2) = -z^2(z+1)^3y^3$ by e with interim multiplier $\mu = 1$ yields the following remainder:

$$r_3 = r_2 - z^2(z+1)^3 ye = (-20z^7 - 15z^6 - 6z^5 - z^4 + 1)y^2 - 2z^7 + z^6.$$

We obtain the following final remainder after a repetition of the above proper term reduction of $LT(r_3)$ by e with interim multiplier $\mu = 1$:

$$d := z^{2}[(-9z^{5} - z^{4} + z^{3} + 3z^{2} + 3z + 1)y - 2z^{5} + z^{4}].$$
 (9.16)

According to Principle 7.1 (i), we name the remainder d as the first element in F such that $F = \{d, e, f, g, h\}$. Then we delete S(f, h) from \mathfrak{S} .

We disregard the S-polynomials S(d,g) and S(d,h) like in Procedure \mathcal{R} of Algorithm 5.20 by the triangular identities with respect to f as in Lemma 5.18. In fact, in the case of S(d,g) the multiplier λ as in (5.28) satisfies $\lambda=1$ whereas in the case of S(d,h) the multiplier $\lambda=(z+1)^3\in R_q^{\times}$. We add the S-polynomials S(d,e) and S(d,f) into \mathfrak{S} such that $\mathfrak{S}=\{S(d,e),S(d,f),S(e,h)\}$.

By Principle 7.1 (iii) we compute the S-polynomial S(d, e) first:

$$S(d,e) = yd + (-9z^5 - z^4 + z^3 + 3z^2 + 3z + 1)z^2e \mod (q = z^8)$$

= $z^4(18z^3 + 16z^2 + 6z + 1)y$.

We make a proper term reduction of LT(S(d, e)) by d as in (5.12) with interim multiplier $\mu = -9z^5 - z^4 + z^3 + 3z^2 + 3z + 1 \in R_q^{\times}$. The remainder of the reduction is 0 over R_q . We delete S(d, e) from \mathfrak{S} .

By Principle 7.1 (iii) we then compute the S-polynomial S(d, f):

$$S(d, f) = (z+1)^3 x d + (-9z^5 - z^4 + z^3 + 3z^2 + 3z + 1) y f \mod (q = z^8)$$

= $(z+1)z^6 x + (-9z^5 - z^4 + z^3 + 3z^2 + 3z + 1) y^2$.

We make a proper term reduction of $LT(S(d, f)) = (z + 1)z^6x$ by f in (9.8) with interim multiplier $\mu = (z + 1)^2 \in R_q^{\times}$ as in (5.12). The remainder of the term reduction is:

$$r_1 = -(z+1)^2(9z^5 + z^4 - z^3 - 3z^2 - 3z - 1)y^2 + z^4y.$$

Next we make a proper term reduction of $LT(r_1)$ by e in (9.15) with interim multiplier $\mu = 1$ to obtain a new remainder:

$$r_2 = z^2 (42z^5 + 69z^4 + 56z^3 + 29z^2 + 8z + 1)y.$$

A further proper term reduction of $LT(r_2)$ by d in (9.16) with interim multiplier $\mu = -9z^5 - z^4 + z^3 + 3z^2 + 3z + 1 \in R_q^{\times}$ as in (5.12) yields a temporary proper eliminant:

$$e_{\delta} = -z^6 (6z + 1). \tag{9.17}$$

Then we delete S(d, f) from \mathfrak{S} .

Since e_{δ} has a standard representation $e_{\delta}^{\text{st}} = z^6$, according to Remark 7.7, we can simplify computations by implementing the algorithm over the normal PQR $R_p \simeq R_q/(z^6)$, which is similar to (6.29).

Let us now compute the final S-polynomial $S(e,h) \in \mathfrak{S}$ in $R_p[x,y]$ as follows.

$$S(e,h) = x^{2}e - yh \mod (q = z^{6})$$

= $z^{2}(z+1)^{3}x^{2}y - y^{4} + (z^{4} - 5z^{5})y$.

This is followed by a proper term reduction of $LT(S(e, h)) = z^2(z+1)^3x^2y$ by h as in (5.12) with interim multiplier $\mu = 1$. The remainder of the term reduction is as follows.

$$r_1 = S(e,h) + z^2(z+1)^3h = -y^4 + z^2(z+1)^3y^3 + (z^4 - 5z^5)y.$$

A further proper term reduction of $LT(r_1) = -y^4$ by e also with interim multiplier $\mu = 1$ leads to the remainder $r_2 = r_1 - y^2 e = z^4 (1 - 5z)y$. We make a proper term reduction of this remainder r_2 by d as in (5.12) with interim multiplier $\mu = -9z^5 - z^4 + z^3 + 3z^2 + 3z + 1 \in R_p^{\times}$. The remainder of the reduction is 0 over R_p . We delete S(e, h) from \mathfrak{S} such that $\mathfrak{S} = \emptyset$.

For the Procedure \mathcal{Q} in Algorithm 5.20, now we have $e_{\delta}^{\mathrm{st}} = z^6$. Hence in (5.19) the multiplier $n_f = z^4$ and the S-polynomial $S(f, e_{\delta}^{\mathrm{st}}) = z^4 y$ over R_p . A proper term reduction of $S(f, e_{\delta}^{\mathrm{st}})$ by $d \in F$ in (9.16) with the multiplier $-9z^5 - z^4 + z^3 + 3z^2 + 3z + 1 \in R_p^{\times}$ yields the remainder 0 over R_p . We also disregard the S-polynomial $S(g, e_{\delta}^{\mathrm{st}})$. In fact, over the normal PQR $R_q = K[z]/(z^8)$ as above, we have $\mathrm{lcm}(l_g, l_e) = z^6(z+1)^6$ is divisible by $l_f = -z^2(z+1)^3$ with $l_g := \iota_q(\mathrm{LC}(g)) = z^4(z+1)^6$, $l_e := \iota_q(e_{\delta}^{\mathrm{st}}) = z^6$ and $l_f := \iota_q(\mathrm{LC}(f)) = -z^2(z+1)^3$. Hence we can invoke the triangular identity (5.28) on $S(g, e_{\delta}^{\mathrm{st}})$ with respect to f to show that the S-polynomial $S(g, e_{\delta}^{\mathrm{st}})$ can be disregarded. Moreover, back to the normal PQR $R_p = K[z]/(z^6)$, we can prove that the S-polynomial $S(d, e_{\delta}^{\mathrm{st}}) = 0$ as in (5.19).

For the composite divisor $q = (z+1)^3$ as in (9.13), our computations are over the normal PQR $R_q \simeq K[z]/((z+1)^3)$. Under the epimorphism σ_q as in (5.30), the ideal $I_q := \sigma_q(I)$ is generated by $F := \sigma_q(G) \subset R_q[x,y]$ with G as in (9.8):

$$f = y;$$
 $g = -y^2;$ $h = -x^2y + y^3 + z^4(z-1)^5.$

Please note that here we choose the representation of h in (9.8) by Principle 7.1 (v). And we abuse the notations a bit and still use $\{f, g, h\}$ to denote the elements in F. It is easy to corroborate that F is not consistent such that the ideal $I_q = \{1\}$ and should be disregarded.

Now the pseudo-basis of I as in Definition 3.11 is $B_{\varepsilon} := G = \{d, e, f, g, h\}$ as in (9.8), (9.10) and (9.11). To obtain an irredundant basis of I_q as in Definition 6.21 over the normal PQR $R_q \simeq K[z]/(q)$ with $q := \mathrm{CP}(\chi_{\varepsilon})$ as in (9.14), we delete g from B_{ε} since $\mathrm{LT}(g) = z^4(z+1)^6x$ is divisible by $\mathrm{LT}(f) = -z^2(z+1)^3x$. In fact, under the epimorphism $\sigma_q \colon (K[z])[x,y] \to R_q[x,y]$ as in (6.14), $\mathrm{LT}(\sigma_q(g))$ is GCD-reducible with respect to $\sigma_q(f)$ as per Definition 6.9 for GCD-reduciblity. Similarly we also delete h from B_{ε} since $\mathrm{LT}(\sigma_q(h)) = -x^2y$ is GCD-reducible with respect to $\sigma_q(f)$ based on $\mathrm{LC}(\sigma_q(f)) = -z^2(z+1)^3 \in R_q^{\times}$. Further, we delete e in (9.10) from B_{ε} since $\mathrm{LT}(\sigma_q(e)) = -y^2$ is GCD-reducible with respect to $\sigma_q(d)$ as in (9.11) due to the fact that $\mathrm{LC}(\sigma_q(d)) = (z+1)^3 z^2 (z^4(z+1)^6 - 1) \in R_q^{\times}$. Hence an irredundant basis of I_q , which we denote as B_q , is $B_q = \{\sigma_q(d), \sigma_q(f)\}$ with d and f being defined in (9.11) and (9.8) respectively.

This irredundant basis B_q is automatically a minimal basis as in Definition 6.24 but not a reduced basis as in Definition 6.27. We can make a proper term reduction of the term y in f by d as in (5.12) to obtain a remainder denoted as c. In this way we obtain a reduced basis that we still denote as B_q . That is, $B_q = \{c, \sigma_q(d)\}$ with d defined in (9.11) and

$$c := -z^{4}(z+1)^{6}(z^{4}(z+1)^{6} - 1)x - z^{6}(z+1)^{3}(z-1)^{5}.$$
 (9.18)

The reduced basis $B_q = \{c, \sigma_q(d)\}$ is unique by Lemma 6.28 and satisfies the identity (6.15) in Lemma 6.6 and hence the unified identity (6.34).

For the composite divisor $q=z^8$, our computations over the normal PQR $R_q \simeq K[z]/(z^8)$ started with the basis F of I_q that bears the same form as the one in (9.8) and ended with the proper eliminant $e_q=z^6$ as the standard factor of e_δ in (9.17). Hence let us consider the modular basis of $I_p=\sigma_p(I)$ over the normal PQR $R_p \simeq R_q/(z^6)$. What we already have is a modular basis $F=\{d,e,f,g,h\}$ of $I_q=\sigma_q(I)$ over $R_q\simeq K[z]/(z^8)$ that are defined in (9.8), (9.15) and (9.16) respectively.

The basis elements g and h of I_q in (9.8) would bear the following form over R_p :

$$g = z^4(6z+1)x - y^2;$$
 $h = -x^2y + y^3 + (5z-1)z^4.$

Nonetheless by Principle 7.1 (v), we still use the representations of g and h in (9.8). In order to have an irredundant proper basis of I_p , we delete $\sigma_p(g)$ from $\sigma_p(F)$ since $\operatorname{LT}(\sigma_p(g)) = z^4(z+1)^6x$ is divisible by $\operatorname{LT}(\sigma_p(f)) = -z^2(z+1)^3x$. The supplemented basis element e in (9.15) is invariant under the epimorphism σ_p whereas the supplemented basis element d in (9.16) bears the form $\sigma_p(d) = z^2(z+1)^3y$ over R_p . Now it is evident that we can use $\sigma_p(d)$ to make a term reduction of $\sigma_p(e)$ such that it bears a reduced form denoted as $b_2 := y^2$. Moreover, we can render $\sigma_p(h)$ reduced by b_2 such that $\sigma_p(h) = -x^2y + z^4(z-1)^5$.

Altogether we obtain a reduced basis of I_p denoted as B_p over $R_p \simeq K[z]/(z^6)$ as follows.

$$B_p \begin{cases} b_1 := \sigma_p(d) = z^2(z+1)^3 y; & b_2 = y^2; \\ b_3 := -\sigma_p(f) = z^2(z+1)^3 x - y; & b_4 := -\sigma_p(h) = x^2 y - z^4(z-1)^5. \end{cases}$$
(9.19)

With the compatible part $q = CP(\chi_{\varepsilon})$ of the pseudo-eliminant χ_{ε} defined in (9.14), a reduced basis B_q of I_q is defined in (9.18) and (9.11) respectively as follows.

$$B_q \begin{cases} a_1 := \sigma_q(d) = z^2(z+1)^3 [(z^4(z+1)^6 - 1)y + z^4(z-1)^5]; \\ a_2 := c = z^4(z+1)^3 [(z+1)^3(z^4(z+1)^6 - 1)x + z^2(z-1)^5]. \end{cases}$$
(9.20)

Let ι_q be the injection as in (5.31) associated with $R_q \simeq K[z]/(z^8)$. In this example we have a unique proper divisor $\theta_q = \iota_q(e_\delta^{\rm st}) = z^6$ and hence the proper factor $\chi_{\rm IP} = \theta_q = z^6$ as per Definition 6.1. By Theorem 6.2 the eliminant $\chi = {\rm CP}(\chi_\varepsilon) \cdot \chi_{\rm IP}$ with ${\rm CP}(\chi_\varepsilon)$ as in (9.14). This coincides with the eliminant p obtained in the classical Gröbner basis as in (9.9). Nonetheless our new type of bases in (9.19) and (9.20) not only have much more moderate coefficients than those of the classical Gröbner basis under (9.9) but also completely obviate the intermediate coefficient swell problem.

Moreover, based on the modular bases B_q in (9.20) and B_p in (9.19), we can use Lemma 6.18 to obtain the new type of bases $\iota_q(B_q) \cup \{q\}$ and $\iota_p(B_p) \cup \{z^6\}$ for $I + \langle q \rangle$ and $I + \langle z^6 \rangle$ respectively. Here $q = \text{CP}(\chi_{\varepsilon})$ is as in (9.14) and ι_p as in (6.30). According to Lemma 6.3, we have $I = (I + \langle q \rangle) \cap (I + \langle z^6 \rangle)$.

10 Conclusion and Remarks

In this paper we defined a new type of bases as in (6.34) and (6.35) in accordance with a decomposition of the original ideal in (6.36) and (6.2) respectively. The characterizations of the new type of bases in Theorem 6.13 and Theorem 6.15 are in effect solutions to the ideal membership problem. The computations and logical deductions in this paper suggest that it is much easier to study a zero-dimensional ideal over principal quotient rings (PQR) modulo the factors of its eliminant or even pseudo-eliminant than over fields.

An obvious direction for future research is to generalize this new type of bases to ideals of positive dimensions. The new type of bases and their algorithms can be easily generalized to such kind of ideals $I \subset \mathbb{Z}[x]$ as $I \cap \mathbb{Z} \neq \{0\}$. We shall address the generic case of $I \cap \mathbb{Z} = \{0\}$ in forthcoming papers. It is meaningful to enhance the computational efficiency of the new type of bases by the normal selection strategies and signatures as well as the conversions between different monomial orderings as aforementioned in the Introduction. A complexity analysis on the new type of bases that is similar to those in [MM82] [MM84] [May89] [Dub90] [KM96] [MR13] on Gröbner bases should be interesting.

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