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Journal of Combinatorial Theory, Series B 96 (2006) 754–757
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

Note

Proper minor-closed families are small

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Received 28 September 2005

Available online 10 March 2006

Abstract

We prove that for every proper minor-closed class $\mathcal{I}$ of graphs there exists a constant c such that for every integer n the class $\mathcal{I}$ includes at most $n!c^n$ graphs with vertex-set $\{1, 2, \dots, n\}$. This answers a question of Welsh.

Keywords: Graph; Minor; Minor-closed family

1. Introduction

All graphs in this paper are finite and simple. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges and deleting resulting loops and parallel edges. A *lower ideal* is a class of graphs closed under isomorphism and taking minors, and it is called *proper* if it is not the class of all graphs. We say that a class $\mathcal{C}$ of graphs is *small* if there exists a constant c such that the number of graphs in $\mathcal{C}$ with vertex-set $[n] := \{1, 2, \dots, n\}$ is at most $n!c^n$ for all integers $n \geqslant 1$. Our goal is to answer a question of Welsh [4] by proving the following theorem.

Theorem 1.1. *Every proper lower ideal of graphs is small.*

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[1] Partially supported by ONR Grant N00014-01-1-0608 and NSF Grant DMS-0070912.
[2] Partially supported by NSF Grant DMS-0200595.

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2006.01.006

The question was motivated by the work of McDiarmid, Steger and Welsh [2] on random planar graphs. It turns out that their results hold for more general classes than planar graphs, namely those that are small and "addable" (but not necessarily closed under taking minors). McDiarmid, Steger and Welsh define a class $\mathcal{C}$ of graphs to be *addable* if (a) a graph G is in $\mathcal{C}$ if and only if each component of G is in $\mathcal{C}$; and (b) for each graph $G \in \mathcal{C}$, if u and v are vertices in distinct components of G then the graph obtained from G by adding an edge joining u and v is also in $\mathcal{C}$. For instance, if H is a 2-connected graph, then the class $\mathcal{F}$ of all graphs with no minor isomorphic to H is addable. By Theorem 1.1 the class $\mathcal{F}$ is small, and hence the results of McDiarmid, Steger and Welsh apply to $\mathcal{F}$.

Let us give a brief overview of the results of [2]. If $\mathcal{C}$ is a non-empty class of graphs and n is a positive integer, then we denote by $\mathcal{C}_n$ the set of graphs in $\mathcal{C}$ with vertex-set $[n]$. Theorem 3.3 of [2] states that if $\mathcal{C}$ is a non-empty class of graphs which is small and addable, then there is a finite constant $c > 0$ such that $(|\mathcal{C}_n|/n!)^{1/n} \to c$ as $n \to \infty$. Theorem 4.2 of [2] says that if $\mathcal{C}$ includes the star on $k+1$ vertices, then there exists a constant d such that, letting $a_k = d/(c^k(k+2)!)$, for all sufficiently large integers n the probability that a graph in $\mathcal{C}_n$, selected uniformly at random, has fewer than $a_k n$ vertices of degree k is at most $e^{-a_k n}$. This is, in fact, an immediate corollary of the more general Theorem 4.1, where the same conclusion is established for the number of "appearances" of arbitrary connected graphs in graphs from $\mathcal{C}_n$. As a final example of the many results obtained in [2] let us mention that Theorem 5.1, even though stated for planar graphs only, actually shows that the probability that a graph in $\mathcal{C}_n$ has an isolated vertex is at least $a_1 e^{-1} + o(1)$.

2. Proof of Theorem 1.1

By a result of Kostochka [1] and Thomason [3] for every integer $t \geqslant 2$ there exists a least constant α_t such that every graph with no K_t minor has average degree at most α_t.

Lemma 2.1. *For every integer $t \geqslant 3$, every graph on $n \geqslant 1$ vertices with no K_t minor has at most $\alpha_t^t n$ complete subgraphs.*

Proof. The lemma clearly holds for $t = 3$, because every graph with no K_3 minor is a forest and $\alpha_3 = 2$. We proceed by induction on $n + t$. Assume that the lemma holds for $n + t - 1$, and let G be a graph on n vertices with no K_t minor. We may assume that G has no isolated vertices, for otherwise the lemma follows by induction. Let $v \in V(G)$. By the induction hypothesis the neighborhood of v has at most $\alpha_{t-1}^{t-1} \deg(v)$ complete subgraphs, including the null graph. Thus v is in at most $\alpha_{t-1}^{t-1} \deg(v)$ complete subgraphs, and hence G has at most

$$\sum_{v \in V(G)} \alpha_{t-1}^{t-1} \deg(v) \leqslant \alpha_{t-1}^{t-1} \alpha_t n \leqslant \alpha_t^t n$$

non-null complete subgraphs. Since at least one complete subgraph was counted twice, the lemma follows. □

A vertex v of a graph G is a *clone* if G has a vertex $u \neq v$ with the same neighborhood as v. In that case we say that v is a *clone* of u.

Lemma 2.2. *For every integer $t \geqslant 3$ there exists a constant d such that every graph G with no K_t minor has a set $S \subseteq V(G)$ of size at least $|V(G)|/d$ such that every vertex in S has degree at most d and either is a clone or is adjacent to a vertex of degree at most d.*

Proof. Let $t \geqslant 3$ be fixed, let $\alpha := \alpha_t$, and let d satisfy $d \geqslant (\alpha^t + \alpha + 1)(\alpha + 1) + 1$. We claim that d satisfies the conclusion of the theorem. To see that, let G be a graph on n vertices with no K_t minor. Let X be the set of all vertices of G that have degree at least d. Since the sum of the degrees of vertices of G is at most αn, we see that $|X| \leqslant \alpha n/d$. Let Y be the set of all vertices of $V(G) - X$ that are adjacent to another vertex of $V(G) - X$. If $|Y| \geqslant n/d$, then Y satisfies the conclusion of the lemma, and so we may assume that $|Y| < n/d$. Let $Z = V(G) - X - Y$; then every neighbor of a vertex in Z belongs to X. In particular, no two vertices in Z are adjacent. Let Z' be a maximal subset of Z such that for every vertex $z \in Z'$ there exists a pair of distinct non-adjacent neighbors $a(z), b(z)$ of z such that $\{a(z), b(z)\} \neq \{a(z'), b(z')\}$ whenever $z, z' \in Z'$ are distinct. Let H be the graph obtained from $G[X \cup Y]$ by adding the edge $a(z)b(z)$ for all $z \in Z'$. Since G, and hence H, has no K_t minor, we deduce that $|Z'| \leqslant \alpha|X \cup Y|$. The choice of Z' implies that the neighborhood of every vertex of $Z - Z'$ is a complete subgraph of H. By Lemma 2.1 there are at most $\alpha^t|X \cup Y|$ distinct such neighborhoods, and so all but possibly $\alpha^t|X \cup Y|$ vertices of $Z - Z'$ have degree at most d and are clones. But

$$\begin{aligned} |Z| - |Z'| - \alpha^t|X \cup Y| &\geqslant n - |X \cup Y| - \alpha|X \cup Y| - \alpha^t|X \cup Y| \\ &\geqslant \big(1 - \big(\alpha^t + \alpha + 1\big)(\alpha + 1)/d\big)n \geqslant n/d \end{aligned}$$

by the choice of d, as desired. □

We are now ready to prove Theorem 1.1. Let $\mathcal{I}$ be a proper lower ideal. Since $\mathcal{I}$ is proper there exists an integer t such that $K_t \notin \mathcal{I}$. Let d be as in Lemma 2.2. We say that a vertex v of a graph G is *good* if it has one of the properties of Lemma 2.2; that is, if it has degree at most d and either it is a clone or is adjacent to a vertex of degree at most d. Then every graph in $\mathcal{I}$ has a set of good vertices of size at least $|V(G)|/d$. Let $\mathcal{K}$ be the set of all graphs $G \in \mathcal{I}$ with vertex-set $[n]$ for some n such that vertex n is good. Let us recall that $\mathcal{I}_n$ denotes the set of all graphs in $\mathcal{I}$ with vertex-set $[n]$.

Let $c = d(3^{2d} + 1)$; we will prove by induction on n that $|\mathcal{I}_n| \leqslant n!c^n$. This is clearly true for $n = 0$, and so we may assume that the assertion holds for $n - 1$. By counting pairs (G, i) in two different ways, where $G \in \mathcal{I}_n$ and $i \in [n]$ is good, and using Lemma 2.2 we deduce that $|\mathcal{I}_n| \leqslant d|\mathcal{K}_n|$. We wish to define a mapping $\phi : \mathcal{K}_n \to [n-1] \times \mathcal{I}_{n-1}$. To that end let $G \in \mathcal{K}_n$. Then there exists a vertex $i \in [n-1]$ of degree at most d such that either n is a clone of i, or n is adjacent to i. In the former case we define $\phi(G) = (i, G\backslash n)$, and in the latter case we put $\phi(G) = (i, G')$, where G' is the graph obtained from G by contracting the edge with ends i and n and deleting the resulting parallel edges. Then the inverse image of every element of $[n-1] \times \mathcal{I}_{n-1}$ has at most $1 + 3^{2d}$ elements. (More precisely, one plus the number of ways to split a vertex of degree at most $2d$ into two vertices.) It follows that

$$|I_n| \leqslant d|K_n| \leqslant d\big(1 + 3^{2d}\big)(n-1)(n-1)!c^{n-1} \leqslant n!c^n,$$

as desired. This completes the proof of Theorem 1.1.

References

[1] A.V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, Metody Diskret. Anal. 38 (1982) 37–58 (in Russian).
[2] C. McDiarmid, A. Steger, D. Welsh, Random planar graphs, J. Combin. Theory Ser. B 93 (2005) 187–206.
[3] A. Thomason, An extremal function for contractions of graphs, Math. Proc. Cambridge Philos. Soc. 95 (1984) 261–265.
[4] D. Welsh, Lecture at the NZIMA Conference on Combinatorics and Its Applications, Lake Taupo, New Zealand, December 2004.