

Using Hoare Logic in a Process Algebra Setting

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Abstract. This paper concerns the relation between process algebra and Hoare logic. We introduce an extension of ACP (Algebra of Communicating Processes) with features that are relevant to processes in which data are involved, present a Hoare logic for the processes considered in this process algebra, and discuss the use of this Hoare logic as a complement to pure equational reasoning from the equational axioms of the process algebra.

Keywords: process algebra, data parameterized action, assignment action, guarded command, asserted process, Hoare logic.

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1 Introduction

ACP (Algebra of Communicating Processes) and its extensions provide a setting for equational reasoning about processes of some kind. The processes about which reasoning is in demand are often processes in which data are involved. It is quite common that certain data that are involved change in the course of such processes and that such processes proceed at certain stages in a way that depends on certain data. This means that reasoning about such a process involves reasoning about how certain data change in the course of certain subprocesses of that process. The question arises whether and how a Hoare logic can be used for the second kind of reasoning. After all, processes of the kind described above are reminiscent of the processes that arise from the execution of imperative programs.

This paper is concerned with the above-mentioned question. We investigate it using an extension of ACP [10] with features that are relevant to processes in which data are involved and a Hoare logic of asserted processes based on this extension of ACP. The extension concerned is called ACP_{ϵ}^* -D. Its additional features include assignment actions to deal with data that change in the course of a process and guarded commands to deal with processes that proceed at certain stages in a way that depends on certain data. In the Hoare logic concerned, an asserted process is a formula of the form $\{\phi\}p\{\psi\}$, where p is a term of ACP_{ϵ}^* -D that denotes a process and ϕ and ψ are terms of ACP_{ϵ}^* -D that denote conditions.

We define what it means that an asserted process is true in such a way that $\{\phi\}p\{\psi\}$ is true iff a set of equations that represents this judgment is derivable

from the axioms of ACP_ϵ^* -D. Such a definition is a prerequisite for an affirmative answer to the above-mentioned question. The set of equations that represents the judgment expresses that a certain equivalence relation holds between processes determined by the asserted process. The equivalence relation concerned is a useful equivalence relation when reasoning about processes in which data are involved. However, it is not a congruence relation, i.e. it is not preserved by all contexts. This complicates pure equational reasoning considerably. The presented Hoare logic can be considered to be a means to get partially round the complications concerned.

This paper is organized as follows. We begin with presenting ACP_ϵ^* , an extension of ACP with the empty process constant ϵ and the binary iteration operator $*$, and ACP_ϵ^* -D, an extension of ACP_ϵ^* with features that are relevant to processes in which data are involved (Sections 2 and 3). We also present a structural operational semantics of ACP_ϵ^* -D, define a notion of bisimulation equivalence based on this semantics, and show that the axioms of ACP_ϵ^* -D are sound with respect to this bisimulation equivalence (Section 4). After that, we present a Hoare logic of asserted processes based on ACP_ϵ^* -D, define what it means that an asserted process is true, and show that the axioms and rules of this Hoare logic are sound with respect to this meaning (Section 5). Following this, we go into the use of the presented Hoare logic as a complement to pure equational reasoning from the equational axioms of ACP_ϵ^* -D (Section 6). Finally, we discuss related work and make some concluding remarks (Sections 7 and 8).

2 ACP with the Empty Process and Iteration

In this section, we present ACP_ϵ^* , ACP [10] extended with the empty process constant ϵ as in [7, Section 4.4] and the binary iteration operator $*$ as in [8]. In ACP_ϵ^* , it is assumed that a fixed but arbitrary finite set A of *basic actions*, with $\delta, \epsilon \notin A$, and a fixed but arbitrary commutative and associative *communication* function $\gamma : (A \cup \delta) \times (A \cup \delta) \rightarrow (A \cup \delta)$, such that $\gamma(\delta, a) = \delta$ for all $a \in A \cup \delta$, have been given. Basic actions are taken as atomic processes. The function γ is regarded to give the result of synchronously performing any two basic actions for which this is possible, and to be δ otherwise. Henceforth, we write A_δ for $A \cup \delta$.

The algebraic theory ACP_ϵ^* has one sort: the sort \mathbf{P} of *processes*. We make this sort explicit to anticipate the need for many-sortedness later on. The algebraic theory ACP_ϵ^* has the following constants and operators to build terms of sort \mathbf{P} :

- the *inaction* constant $\delta : \rightarrow \mathbf{P}$;
- the *empty process* constant $\epsilon : \rightarrow \mathbf{P}$;
- for each $a \in A$, the *basic action* constant $a : \rightarrow \mathbf{P}$;
- the binary *alternative composition* operator $+: \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$;
- the binary *sequential composition* operator $\cdot : \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$;
- the binary *iteration* operator $* : \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$;
- the binary *parallel composition* operator $\parallel : \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$;
- the binary *left merge* operator $\ll : \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$;

- the binary *communication merge* operator $| : \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$;
- for each $H \subseteq \mathbf{A}$, the unary *encapsulation* operator $\partial_H : \mathbf{P} \rightarrow \mathbf{P}$.

We assume that there is a countably infinite set of variables of sort \mathbf{P} , which contains x, y and z . Terms are built as usual. We use infix notation for the binary operators. The following precedence conventions are used to reduce the need for parentheses: the operator \cdot binds stronger than all other binary operators and the operator $+$ binds weaker than all other binary operators.

The constants and operators of ACP_ϵ^* are the constants and operators of ACP_ϵ and additionally the iteration operator $*$. Let p and q be closed ACP_ϵ^* terms, $a \in \mathbf{A}$, and $H \subseteq \mathbf{A}$. Then the constants and operators of ACP_ϵ^* can be explained as follows:

- the constant δ denotes the process that is not capable of doing anything;
- the constant ϵ denotes the process that is only capable of terminating successfully;
- the constant a denotes the process that is only capable of first performing action a and next terminating successfully;
- a closed term of the form $p + q$ denotes the process that behaves either as the process denoted by p or as the process denoted by q , but not both;
- a closed term of the form $p \cdot q$ denotes the process that first behaves as the process denoted by p and on successful termination of that process next behaves as the process denoted by q ;
- a closed term of the form $p * q$ denotes the process that behaves either as the process denoted by q or as the process that first behaves as the process denoted by p and on successful termination of that process next behaves as $p * q$ again;
- a closed term of the form $p \parallel q$ denotes the process that behaves such that the processes denoted by p and q proceed in parallel;
- a closed term of the form $p \ll q$ denotes the process that behaves the same as the process denoted by $p \parallel q$, except that it starts with performing an action of the process denoted by p ;
- a closed term of the form $p | q$ denotes the process that behaves the same as the process denoted by $p \parallel q$, except that it starts with performing an action of the process denoted by p and an action of the process denoted by q synchronously;
- a closed term of the form $\partial_H(p)$ denotes the process that behaves the same as the process denoted by p , except that actions from H are blocked.

The axioms of ACP_ϵ^* are the equations given in Table 1. In these equations, a and b stand for arbitrary constants of ACP_ϵ^* that differ from ϵ and H stands for an arbitrary subset of \mathbf{A} . So, CM3, CM7, and D0–D4 are actually axiom schemas. Axioms A1–A9, CM1T, CM2T, CM3, CM4, CM5T, CM6T, CM7–CM9, and D0–D4 are the axioms of ACP_ϵ (cf. [7, Section 4.4]). Axioms BKS1 and RSP* have been taken from [9].

The iteration operator originates from [8], where it is called the binary Kleene star operator. The unary counterpart of this operator can be defined by the

Table 1. Axioms of ACP_ϵ^*

$x + y = y + x$	A1	$x \parallel y = x \parallel y + y \parallel x + x \mid y +$	
$(x + y) + z = x + (y + z)$	A2	$\partial_A(x) \cdot \partial_A(y)$	CM1T
$x + x = x$	A3	$\epsilon \parallel x = \delta$	CM2T
$(x + y) \cdot z = x \cdot z + y \cdot z$	A4	$a \cdot x \parallel y = a \cdot (x \parallel y)$	CM3
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	A5	$(x + y) \parallel z = x \parallel z + y \parallel z$	CM4
$x + \delta = x$	A6	$\epsilon \mid x = \delta$	CM5T
$\delta \cdot x = \delta$	A7	$x \mid \epsilon = \delta$	CM6T
$x \cdot \epsilon = x$	A8	$a \cdot x \mid b \cdot y = \gamma(a, b) \cdot (x \parallel y)$	CM7
$\epsilon \cdot x = x$	A9	$(x + y) \mid z = x \mid z + y \mid z$	CM8
		$x \mid (y + z) = x \mid y + x \mid z$	CM9
$x^* y = x \cdot (x^* y) + y$	BKS1	$\partial_H(\epsilon) = \epsilon$	D0
$z = x \cdot z + y \rightarrow z = x^* y$	RSP*	$\partial_H(a) = a$ if $a \notin H$	D1
		$\partial_H(a) = \delta$ if $a \in H$	D2
		$\partial_H(x + y) = \partial_H(x) + \partial_H(y)$	D3
		$\partial_H(x \cdot y) = \partial_H(x) \cdot \partial_H(y)$	D4

Table 2. Derivable equations for iteration

$x^* (y \cdot z) = (x^* y) \cdot z$	BKS2
$x^* (y \cdot ((x + y)^* z) + z) = (x + y)^* z$	BKS3
$\partial_H(x^* y) = \partial_H(x)^* \partial_H(y)$	BKS4
$\epsilon^* x = x$	BKS5

equation $x^* = x^* \epsilon$. From this defining equation, it follows that $x^* = x \cdot x^* + \epsilon$ and also that $x^* y = x^* \cdot y$.

Among the equations derivable from the axioms of ACP_ϵ^* are the equations concerning the iteration operator given in Table 2. In the axiom system of ACP^* given in [8], the axioms for the iteration operator are BKS1–BKS4 instead of BKS1 and RSP*. There exist equations derivable from the axioms of ACP_ϵ^* that are not derivable from the axioms of ACP_ϵ^* with BKS1 and RSP* replaced by BKS1–BKS4 (see [21]).

3 Data Enriched ACP_ϵ^*

In this section, we present ACP_ϵ^* -D, data enriched ACP_ϵ^* . This extension of ACP_ϵ^* has been inspired by [12]. It extends ACP_ϵ^* with features that are relevant to processes in which data are involved, such as guarded commands (to deal with processes that only take place if some data-dependent condition holds), data

parameterized actions (to deal with process interactions with data transfer), and assignment actions (to deal with data that change in the course of a process).

In ACP_ϵ^* -D, it is assumed that the following has been given with respect to data:

- a (single- or many-sorted) signature $\Sigma_{\mathfrak{D}}$ that includes a sort \mathbf{D} of *data* and constants and/or operators with result sort \mathbf{D} ;
- a minimal algebra \mathfrak{D} of the signature $\Sigma_{\mathfrak{D}}$.

Moreover, it is assumed that a countably infinite set \mathcal{V} of *flexible variables* has been given. A flexible variable is a variable whose value may change in the course of a process. Flexible variables are found under the name program variables in imperative programming. We write \mathbb{D} for the set of all closed terms of the first-order language with equality of \mathfrak{D} that are of sort \mathbf{D} . An *evaluation map* is a function σ from \mathcal{V} to $\mathbb{D} \cup \mathcal{V}$ where, for all $v \in \mathcal{V}$, $\sigma(v) = v$ if $\sigma(v) \in \mathcal{V}$. Let σ be an evaluation map and let V be a finite subset of \mathcal{V} . Then σ is a *V-evaluation map* if, for all $v \in \mathcal{V}$, $\sigma(v) \in \mathbb{D}$ iff $v \in V$.

Evaluation maps are intended to provide the data values assigned to flexible variables of sort \mathbf{D} when a term of sort \mathbf{D} is evaluated, but they provide closed terms of the first-order language with equality of \mathfrak{D} instead. This fits better in an algebraic setting. The requirement that \mathfrak{D} is a minimal algebra guarantees that each data value can be represented by a closed term. The possibility to map flexible variables to themselves allows for partial evaluation, i.e. evaluation where some flexible variables are not evaluated.

The algebraic theory ACP_ϵ^* -D has three sorts: the sort \mathbf{P} of *processes*, the sort \mathbf{C} of *conditions*, and the sort \mathbf{D} of *data*. ACP_ϵ^* -D has the constants and operators from $\Sigma_{\mathfrak{D}}$ and in addition the following constants to build terms of sort \mathbf{D} :

- for each $v \in \mathcal{V}$, the *flexible variable* constant $v : \rightarrow \mathbf{D}$.

ACP_ϵ^* -D has the following constants and operators to build terms of sort \mathbf{C} :

- the binary *equality* operator $= : \mathbf{D} \times \mathbf{D} \rightarrow \mathbf{C}$;
- the *truth* constant $\mathbf{t} : \rightarrow \mathbf{C}$;
- the *falsity* constant $\mathbf{f} : \rightarrow \mathbf{C}$;
- the unary *negation* operator $\neg : \mathbf{C} \rightarrow \mathbf{C}$;
- the binary *conjunction* operator $\wedge : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$;
- the binary *disjunction* operator $\vee : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$;
- the binary *implication* operator $\rightarrow : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$;
- the unary variable binding *universal quantification* operator $\forall : \mathbf{C} \rightarrow \mathbf{C}$ that binds a single variable of sort \mathbf{D} in its argument;
- the unary variable binding *existential quantification* operator $\exists : \mathbf{C} \rightarrow \mathbf{C}$ that binds a single variable of sort \mathbf{D} in its argument.

ACP_ϵ^* -D has the constants and operators of ACP_ϵ^* and in addition the following operators to build terms of sort \mathbf{P} :

- the binary *guarded command* operator $:\rightarrow : \mathbf{C} \times \mathbf{P} \rightarrow \mathbf{P}$;

- for each $n \in \mathbb{N}$ and each $a \in \mathbf{A}$, the n -ary *data parameterized action* operator $a : \underbrace{\mathbf{D} \times \dots \times \mathbf{D}}_{n \text{ times}} \rightarrow \mathbf{P}$;
- for each $v \in \mathcal{V}$, a unary *assignment action* operator $v := : \mathbf{D} \rightarrow \mathbf{P}$;
- for each evaluation map σ , a unary *evaluation* operator $V_\sigma : \mathbf{P} \rightarrow \mathbf{P}$.

We assume that there are countably infinite sets of variables of sort \mathbf{C} and \mathbf{D} and that the sets of variables of sort \mathbf{P} , \mathbf{C} , and \mathbf{D} are mutually disjoint and disjoint from \mathcal{V} . Terms are built as usual for a many-sorted signature (see e.g. [20,22]). We use the same notational conventions as before. We also use infix notation for the additional binary operators. Moreover, we use the notation $[v := e]$, where $v \in \mathcal{V}$ and e is a term of sort \mathbf{D} , for the term $v := (e)$.

We use the notation $\phi \leftrightarrow \psi$, where ϕ and ψ are terms of sort \mathbf{D} , for the term $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. Moreover, we use the notation $\bigvee \Phi$, where $\Phi = \{\phi_1, \dots, \phi_n\}$ and ϕ_1, \dots, ϕ_n are terms of sort \mathbf{D} , for the term $\phi_1 \vee \dots \vee \phi_n$.

We write \mathcal{P} for the set of all closed terms of sort \mathbf{P} , \mathcal{C} for the set of all closed terms of sort \mathbf{C} , and \mathcal{D} for the set of all closed terms of sort \mathbf{D} .

For each $\phi \in \mathcal{C}$, ϕ is a formula of the first-order language with equality of \mathfrak{D} if the flexible variables from \mathcal{V} are taken as additional variables of sort \mathbf{D} . We implicitly take the flexible variables from \mathcal{V} as additional variables of sort \mathbf{D} wherever the context asks for a formula. Two terms ϕ and ψ from \mathcal{C} are considered equal if the formula $\phi \leftrightarrow \psi$ holds in \mathfrak{D} .

Let p be a term from \mathcal{P} , ϕ be a term from \mathcal{C} , and e_1, \dots, e_n and e be terms from \mathcal{D} . Then the additional operators can be explained as follows:

- the term $\phi \rightarrow p$ denotes the process that behaves as the process denoted by p under condition ϕ ;
- the term $a(e_1, \dots, e_n)$ denotes the process that is only capable of first performing action $a(e_1, \dots, e_n)$ and next terminating successfully;
- the term $[v := e]$ denotes the process that is only capable of first performing action $[v := e]$, whose intended effect is the assignment of the result of evaluating e to flexible variable v , and next terminating successfully;
- the term $V_\sigma(p)$ denotes the process that behaves the same as the process denoted by p except that each subterm of p that belongs to \mathcal{D} is evaluated using the evaluation map σ updated according to the assignment actions that have taken place at the point where the subterm is encountered.

Evaluation operators are a variant of state operators (see e.g. [3]).

An evaluation map σ can be extended homomorphically from flexible variables to terms of sort \mathbf{D} and terms of sort \mathbf{C} . These extensions are denoted by σ as well. We write $\sigma\{e/v\}$ for the evaluation map σ' defined by $\sigma'(v') = \sigma(v')$ if $v' \neq v$ and $\sigma'(v) = e$.

The axioms of ACP_ϵ^* -D are the axioms of ACP_ϵ^* and in addition the equations given in Table 3. In these equations, ϕ and ψ stand for arbitrary terms from \mathcal{C} , e, e_1, e_2, \dots , and e'_1, e'_2, \dots stand for arbitrary terms from \mathcal{D} , v stands for an arbitrary flexible variable from \mathcal{V} , σ stands for an arbitrary evaluation map, a and b stand for arbitrary constants of ACP_ϵ^* -D that differ from ϵ , c stands for

Table 3. Axioms of ACP_ϵ^* -D

$\phi = \psi$	if $\mathfrak{D} \models \psi \leftrightarrow \phi$	IMP
$\mathbf{t} : \rightarrow x = x$		GC1
$\mathbf{f} : \rightarrow x = \delta$		GC2
$\phi : \rightarrow \delta = \delta$		GC3
$\phi : \rightarrow (x + y) = \phi : \rightarrow x + \phi : \rightarrow y$		GC4
$\phi : \rightarrow x \cdot y = (\phi : \rightarrow x) \cdot y$		GC5
$\phi : \rightarrow (\psi : \rightarrow x) = (\phi \wedge \psi) : \rightarrow x$		GC6
$(\phi \vee \psi) : \rightarrow x = \phi : \rightarrow x + \psi : \rightarrow x$		GC7
$(\phi : \rightarrow x) \parallel y = \phi : \rightarrow (x \parallel y)$		GC8
$(\phi : \rightarrow x) \mid y = \phi : \rightarrow (x \mid y)$		GC9
$x \mid (\phi : \rightarrow y) = \phi : \rightarrow (x \mid y)$		GC10
$\partial_H(\phi : \rightarrow x) = \phi : \rightarrow \partial_H(x)$		GC11
$\mathbf{V}_\sigma(\epsilon) = \epsilon$		V0
$\mathbf{V}_\sigma(a \cdot x) = a \cdot \mathbf{V}_\sigma(x)$		V1
$\mathbf{V}_\sigma(a(e_1, \dots, e_n) \cdot x) = a(\sigma(e_1), \dots, \sigma(e_n)) \cdot \mathbf{V}_\sigma(x)$		V2
$\mathbf{V}_\sigma([v := e] \cdot x) = [v := \sigma(e)] \cdot \mathbf{V}_{\sigma\{\sigma(e)/v\}}(x)$	if $\sigma(v) \in \mathbb{D}$	V3
$\mathbf{V}_\sigma([v := e] \cdot x) = [v := \sigma(e)] \cdot \mathbf{V}_\sigma(x)$	if $\sigma(v) \notin \mathbb{D}$	V4
$\mathbf{V}_\sigma(x + y) = \mathbf{V}_\sigma(x) + \mathbf{V}_\sigma(y)$		V5
$\mathbf{V}_\sigma(\phi : \rightarrow y) = \sigma(\phi) : \rightarrow \mathbf{V}_\sigma(x)$		V6
$a(e_1, \dots, e_n) \cdot x \parallel y = a(e_1, \dots, e_n) \cdot (x \parallel y)$		CM3D
$a(e_1, \dots, e_n) \cdot x \mid b(e'_1, \dots, e'_n) \cdot y =$ $(e_1 = e'_1 \wedge \dots \wedge e_n = e'_n) : \rightarrow c(e_1, \dots, e_n) \cdot (x \parallel y)$	if $\gamma(a, b) = c$	CM7Da
$a(e_1, \dots, e_n) \cdot x \mid b(e'_1, \dots, e'_m) \cdot y = \delta$	if $\gamma(a, b) = \delta$ or $n \neq m$	CM7Db
$a(e_1, \dots, e_n) \cdot x \mid b \cdot y = \delta$		CM7Dc
$a \cdot x \mid b(e_1, \dots, e_n) \cdot y = \delta$		CM7Dd
$\partial_H(a(e_1, \dots, e_n)) = a(e_1, \dots, e_n)$	if $a \notin H$	D1D
$\partial_H(a(e_1, \dots, e_n)) = \delta$	if $a \in H$	D2D
$[v := e] \cdot x \parallel y = [v := e] \cdot (x \parallel y)$		CM3A
$[v := e] \mid x = \delta$		CM5A
$x \mid [v := e] = \delta$		CM6A
$\partial_H([v := e]) = [v := e]$		D1A

an arbitrary constant of $\text{ACP}_\epsilon^*\text{-D}$ that differ from ϵ and δ , and H stands for an arbitrary subset of \mathbf{A} . Axioms GC1–GC11 have been taken from [4] (using a different numbering), but with the axioms with occurrences of conditional expressions of the form $p \triangleleft \phi \triangleright q$ replaced by simpler axioms. Axioms CM3D, CM7Da, CM7Db, D1D, and D2D have been inspired by [12].

The set \mathcal{A} of *actions* of $\text{ACP}_\epsilon^*\text{-D}$ is inductively defined by the following rules:

- if $a \in \mathbf{A}$, then $a \in \mathcal{A}$;
- if $a \in \mathbf{A}$ and $e_1, \dots, e_n \in \mathcal{D}$, then $a(e_1, \dots, e_n) \in \mathcal{A}$;
- if $v \in \mathcal{V}$ and $e \in \mathcal{D}$, then $[v := e] \in \mathcal{A}$.

The elements of \mathcal{A} are the processes that are considered to be atomic.

The set \mathcal{H} of *head normal forms* of $\text{ACP}_\epsilon^*\text{-D}$ is inductively defined by the following rules:

- $\delta \in \mathcal{H}$;
- if $\phi \in \mathcal{C}$, then $\phi : \rightarrow \epsilon \in \mathcal{H}$;
- if $\phi \in \mathcal{C}$, $\alpha \in \mathcal{A}$, and $p \in \mathcal{P}$, then $\phi : \rightarrow \alpha \cdot p \in \mathcal{H}$;
- if $p, p' \in \mathcal{H}$, then $p + p' \in \mathcal{H}$.

The following lemma and corollary about head normal forms are used in later sections.

Lemma 1. *For all terms $p \in \mathcal{P}$, there exists a term $q \in \mathcal{H}$ such that $p = q$ is derivable from the axioms of $\text{ACP}_\epsilon^*\text{-D}$.*

Proof. This is straightforwardly proved by induction on the structure of p . The cases where p is of the form δ , ϵ or α ($\alpha \in \mathcal{A}$) are trivial. The case where p is of the form $p_1 + p_2$ follows immediately from the induction hypothesis. The case where p is of the form $p_1 \parallel p_2$ follows immediately from the case that p is of the form $p_1 \parallel p_2$ and the case that p is of the form $p_1 | p_2$. Each of the other cases follow immediately from the induction hypothesis and a claim that is easily proved by structural induction. In the case where p is of the form $p_1 | p_2$, each of the cases to be considered in the inductive proof demands an additional proof by structural induction. \square

Some earlier extensions of ACP include Hoare’s ternary counterpart of the binary guarded command operator (see e.g. [4]). This operator can be defined by the equation $x \triangleleft u \triangleright y = u : \rightarrow x + (\neg u) : \rightarrow y$. From this defining equation, it follows that $u : \rightarrow x = x \triangleleft u \triangleright \delta$. In [13], a unary counterpart of the binary guarded command operator is used. This operator can be defined by the equation $\{u\} = u : \rightarrow \epsilon$. From this defining equation, it follows that $u : \rightarrow x = \{u\} \cdot x$ and also that $\{t\} = \epsilon$ and $\{f\} = \delta$. In [13], the processes denoted by closed terms of the form $\{\phi\}$ are called guards.

4 Structural Operational Semantics and Bisimulation Equivalence

In this section, we present a structural operational semantics of ACP_ϵ^* -D, define a notion of bisimulation equivalence based on this semantics, and show that the axioms of ACP_ϵ^* -D are sound with respect to this bisimulation equivalence.

We write \mathcal{C}^{sat} for the set of all terms $\phi \in \mathcal{C}$ for which $\mathfrak{D} \models \phi \leftrightarrow \text{f}$. As formulas of the first-order language with equality of \mathfrak{D} , the terms from \mathcal{C}^{sat} are exactly the formulas that are satisfiable in \mathfrak{D} .

We start with the presentation of the structural operational semantics of ACP_ϵ^* -D. The following transition relations on \mathcal{P} are used:

- for each $\phi \in \mathcal{C}^{sat}$, a unary relation $\{\phi\}\downarrow$;
- for each $\ell \in \mathcal{C}^{sat} \times \mathcal{A}$, a binary relation $\xrightarrow{\ell}$.

We write $p \{\phi\}\downarrow$ instead of $p \in \{\phi\}\downarrow$ and $p \xrightarrow{\{\phi\}a} q$ instead of $(p, q) \in \xrightarrow{(\phi, a)}$. The relations $\{\phi\}\downarrow$ and $\xrightarrow{\ell}$ can be explained as follows:

- $p \{\phi\}\downarrow$: p is capable of terminating successfully under condition ϕ ;
- $p \xrightarrow{\{\phi\}a} q$: p is capable of performing action a under condition ϕ and then proceeding as q .

The members of $\mathcal{C}^{sat} \times \mathcal{A}$ are sometimes called *guarded actions*.

The structural operational semantics of ACP_ϵ^* -D is described by the transition rules given in Table 4. In this table, a, b , and c stand for arbitrary basic actions from \mathbf{A} , v stands for an arbitrary flexible variable from \mathcal{V} , e and e_1, e_2, \dots stand for arbitrary terms from \mathcal{D} , ϕ and ψ stand for arbitrary terms from \mathcal{C}^{sat} , α stands for an arbitrary term from \mathcal{A} , H stands for arbitrary subset of \mathbf{A} , and σ stands for an arbitrary evaluation map.

A *bisimulation* is a binary relation R on \mathcal{P} such that, for all terms $p, q \in \mathcal{P}$ with $(p, q) \in R$, the following conditions hold:

- if $p \xrightarrow{\{\phi\}\alpha} p'$, then there exists a finite set $\Psi \subseteq \mathcal{C}^{sat}$ such that $\mathfrak{D} \models \phi \rightarrow \bigvee \Psi$ and, for all $\psi \in \Psi$, there exists a term $q' \in \mathcal{P}$ such that $q \xrightarrow{\{\psi\}\alpha} q'$ and $(p', q') \in R$;
- if $q \xrightarrow{\{\phi\}\alpha} q'$, then there exists a finite set $\Psi \subseteq \mathcal{C}^{sat}$ such that $\mathfrak{D} \models \phi \rightarrow \bigvee \Psi$ and, for all $\psi \in \Psi$, there exists a term $p' \in \mathcal{P}$ such that $p \xrightarrow{\{\psi\}\alpha} p'$ and $(p', q') \in R$;
- if $p \{\phi\}\downarrow$, then there exists a finite set $\Psi \subseteq \mathcal{C}^{sat}$ such that $\mathfrak{D} \models \phi \rightarrow \bigvee \Psi$ and, for all $\psi \in \Psi$, $q \{\psi\}\downarrow$;
- if $q \{\phi\}\downarrow$, then there exists a finite set $\Psi \subseteq \mathcal{C}^{sat}$ such that $\mathfrak{D} \models \phi \rightarrow \bigvee \Psi$ and, for all $\psi \in \Psi$, $p \{\psi\}\downarrow$.

Two terms $p, q \in \mathcal{P}$ are *bisimulation equivalent*, written $p \trianglelefteq q$, if there exists a bisimulation R such that $(p, q) \in R$. Let R be a bisimulation such that $(p, q) \in R$. Then we say that R is a bisimulation *witnessing* $p \trianglelefteq q$.

Table 4. Transition rules for ACP_ε^{*}-D

$\overline{\epsilon \{t\} \downarrow}$		
$\overline{a \{t\} a \rightarrow \epsilon}$	$\overline{a(e_1, \dots, e_n) \{t\} a(e_1, \dots, e_n) \rightarrow \epsilon}$	$\overline{[v := e] \{t\} [v := e] \rightarrow \epsilon}$
$\frac{x \{ \phi \} \downarrow}{x + y \{ \phi \} \downarrow}$	$\frac{y \{ \phi \} \downarrow}{x + y \{ \phi \} \downarrow}$	$\frac{x \{ \phi \} \alpha \rightarrow x'}{x + y \{ \phi \} \alpha \rightarrow x'} \quad \frac{y \{ \phi \} \alpha \rightarrow y'}{x + y \{ \phi \} \alpha \rightarrow y'}$
$\frac{x \{ \phi \} \downarrow, y \{ \psi \} \downarrow}{x \cdot y \{ \phi \wedge \psi \} \downarrow} \mathfrak{D} \not\models \phi \wedge \psi \leftrightarrow \mathbf{f}$	$\frac{x \{ \phi \} \downarrow, y \{ \psi \} \alpha \rightarrow y'}{x \cdot y \{ \phi \wedge \psi \} \alpha \rightarrow y'} \mathfrak{D} \not\models \phi \wedge \psi \leftrightarrow \mathbf{f}$	$\frac{x \{ \phi \} \alpha \rightarrow x'}{x \cdot y \{ \phi \} \alpha \rightarrow x' \cdot y}$
$\frac{y \{ \phi \} \downarrow}{x * y \{ \phi \} \downarrow}$	$\frac{y \{ \phi \} \alpha \rightarrow y'}{x * y \{ \phi \} \alpha \rightarrow y'}$	$\frac{x \{ \phi \} \alpha \rightarrow x'}{x * y \{ \phi \} \alpha \rightarrow x' \cdot (x * y)}$
$\frac{x \{ \phi \} \downarrow}{\psi \rightarrow x \{ \phi \wedge \psi \} \downarrow} \mathfrak{D} \not\models \phi \wedge \psi \leftrightarrow \mathbf{f}$	$\frac{x \{ \phi \} \alpha \rightarrow x'}{\psi \rightarrow x \{ \phi \wedge \psi \} \alpha \rightarrow x'} \mathfrak{D} \not\models \phi \wedge \psi \leftrightarrow \mathbf{f}$	
$\frac{x \{ \phi \} \downarrow, y \{ \psi \} \downarrow}{x \parallel y \{ \phi \wedge \psi \} \downarrow} \mathfrak{D} \not\models \phi \wedge \psi \leftrightarrow \mathbf{f}$	$\frac{x \{ \phi \} \alpha \rightarrow x'}{x \parallel y \{ \phi \} \alpha \rightarrow x' \parallel y}$	$\frac{y \{ \phi \} \alpha \rightarrow y'}{x \parallel y \{ \phi \} \alpha \rightarrow x \parallel y'}$
$\frac{x \{ \phi \} a \rightarrow x', y \{ \psi \} b \rightarrow y'}{x \parallel y \{ \phi \wedge \psi \} c \rightarrow x' \parallel y'} \gamma(a, b) = c, \mathfrak{D} \not\models \phi \wedge \psi \leftrightarrow \mathbf{f}$		
$\frac{x \{ \phi \} a(e_1, \dots, e_n) \rightarrow x', y \{ \psi \} b(e_1, \dots, e_n) \rightarrow y'}{x \parallel y \{ \phi \wedge \psi \} c(e_1, \dots, e_n) \rightarrow x' \parallel y'} \gamma(a, b) = c, \mathfrak{D} \not\models \phi \wedge \psi \leftrightarrow \mathbf{f}$		
$\frac{x \{ \phi \} \alpha \rightarrow x'}{x \parallel y \{ \phi \} \alpha \rightarrow x' \parallel y}$		
$\frac{x \{ \phi \} a \rightarrow x', y \{ \psi \} b \rightarrow y'}{x \mid y \{ \phi \wedge \psi \} c \rightarrow x' \mid y'} \gamma(a, b) = c, \mathfrak{D} \not\models \phi \wedge \psi \leftrightarrow \mathbf{f}$		
$\frac{x \{ \phi \} a(e_1, \dots, e_n) \rightarrow x', y \{ \psi \} b(e_1, \dots, e_n) \rightarrow y'}{x \mid y \{ \phi \wedge \psi \} c(e_1, \dots, e_n) \rightarrow x' \mid y'} \gamma(a, b) = c, \mathfrak{D} \not\models \phi \wedge \psi \leftrightarrow \mathbf{f}$		
$\frac{x \{ \phi \} \downarrow}{\partial_H(x) \{ \phi \} \downarrow}$	$\frac{x \{ \phi \} a \rightarrow x'}{\partial_H(x) \{ \phi \} a \rightarrow \partial_H(x')}$	$\frac{x \{ \phi \} a(e_1, \dots, e_n) \rightarrow x'}{\partial_H(x) \{ \phi \} a(e_1, \dots, e_n) \rightarrow \partial_H(x')}$
$\frac{x \{ \phi \} [v := e] \rightarrow x'}{\partial_H(x) \{ \phi \} [v := e] \rightarrow \partial_H(x')}$		
$\frac{x \{ \phi \} \downarrow}{V_\sigma(x) \{ \sigma(\phi) \} \downarrow}$	$\frac{x \{ \phi \} a \rightarrow x'}{V_\sigma(x) \{ \sigma(\phi) \} a \rightarrow V_\sigma(x')}$	$\frac{x \{ \phi \} a(e_1, \dots, e_n) \rightarrow x'}{V_\sigma(x) \{ \sigma(\phi) \} a(\sigma(e_1), \dots, \sigma(e_n)) \rightarrow V_\sigma(x')}$
$\frac{x \{ \phi \} [v := e] \rightarrow x'}{V_\sigma(x) \{ \sigma(\phi) \} [v := \sigma(e)] \rightarrow V_{\sigma\{\sigma(e)/v\}}(x')}$	$\sigma(v) \in \mathbb{D}$	$\frac{x \{ \phi \} [v := e] \rightarrow x'}{V_\sigma(x) \{ \sigma(\phi) \} [v := \sigma(e)] \rightarrow V_\sigma(x')} \sigma(v) \notin \mathbb{D}$

Because a transition on one side may be simulated by a set of transitions on the other side, a bisimulation as defined above is called a *splitting* bisimulation in [11].

Bisimulation equivalence is a congruence with respect to the operators of $\text{ACP}_\epsilon^*\text{-D}$ of which the result sort and at least one argument sort is \mathbf{P} .

Theorem 1 (Congruence). *For all terms $p, q, p', q' \in \mathcal{P}$ and all terms $\phi \in \mathcal{C}$, $p \dot{\sim} p'$ and $q \dot{\sim} q'$ only if $p + q \dot{\sim} p' + q'$, $p \cdot q \dot{\sim} p' \cdot q'$, $p * q \dot{\sim} p' * q'$, $\phi : \rightarrow p \dot{\sim} \phi : \rightarrow p'$, $p \parallel q \dot{\sim} p' \parallel q'$, $p \ll q \dot{\sim} p' \ll q'$, $p \mid q \dot{\sim} p' \mid q'$, $\partial_H(p) \dot{\sim} \partial_H(p')$, and $\mathbf{V}_\sigma(p) \dot{\sim} \mathbf{V}_\sigma(p')$.*

Proof. We can reformulate the transition rules such that:

- bisimulation equivalence based on the reformulated transition rules according to the standard definition of bisimulation equivalence coincides with bisimulation equivalence based on the original transition rules according to the definition of bisimulation equivalence given above;
- the reformulated transition rules make up a transition system specification in path format.

The reformulation is similar to the one for the transition rules for BPAs outlined in [5]. The proposition follows now immediately from the well-known result that bisimulation equivalence according to the standard definition of bisimulation equivalence is a congruence if the transition rules concerned make up a transition system specification in path format (see e.g. [6]). \square

The underlying idea of the reformulation referred to above is that we replace each transition $p \xrightarrow{\{\phi\}\alpha} p'$ by a transition $p \xrightarrow{\{\nu\}a} p'$ for each valuation of variables ν such that $\mathfrak{D} \models \phi[\nu]$, and likewise $p \xrightarrow{\{\phi\}\downarrow}$. Thus, in a bisimulation, a transition on one side must be simulated by a single transition on the other side. We did not present the reformulated structural operational semantics in this paper because it is, in our opinion, intuitively less appealing.

The axioms of $\text{ACP}_\epsilon^*\text{-D}$ are sound with respect to $\dot{\sim}$ for equations between terms from \mathcal{P} .

Theorem 2 (Soundness). *For all terms $p, q \in \mathcal{P}$, $p = q$ is derivable from the axioms of $\text{ACP}_\epsilon^*\text{-D}$ only if $p \dot{\sim} q$.*

Proof. Because $\dot{\sim}$ is a congruence, it is sufficient to prove the theorem for all substitution instances of each axiom of $\text{ACP}_\epsilon^*\text{-D}$. We will loosely say that a relation contains all closed substitution instances of an equation if it contains all pairs (p, q) such that $p = q$ is a closed substitution instance of the equation.

For each axiom, we can construct a bisimulation R witnessing $p \dot{\sim} q$ for all closed substitution instances $p = q$ of the axiom as follows:

- in the case of A1–A6, A8, A9, BKS1, CM3, CM4, CM7–CM9, D1, D3, D4, GC1, GC4–GC11, V1–V6, CM3D, CM7a, D1D, CM3A, and D1A, we take the relation R that consists of all closed substitution instances of the axiom concerned and the equation $x = x$;

- in the case of A7, CM2T, CM5T, CM6T, D0, D2, GC2, GC3, V0, CM7Db–CM7Dd, D2D, CM5A, and CM6A, we take the relation R that consists of all closed substitution instances of the axiom concerned;
- in the case of CM1, we take the relation R that consists of all closed substitution instances of CM1, the equation $x \parallel y = y \parallel x$, and the equation $x = x$;
- in the case of RSP*, we take the relation R that consists of all closed substitution instances $r = p * q$ of the consequent of RSP* for which $r \dot{=} p \cdot r + q$ and all closed substitution instances of the equation $x = x$. \square

5 A Hoare Logic of Asserted Processes

In this section, we present $\text{HL}_{\text{ACP}^*_\epsilon\text{-D}}$, a Hoare logic of asserted processes based on $\text{ACP}^*_\epsilon\text{-D}$, define what it means that an asserted process is true, and show that the axioms and rules of this logic are sound with respect to this meaning.

We write \mathcal{P}^{hl} for the set of all closed terms of sort \mathbf{P} in which the evaluation operators V_σ and the auxiliary operators \parallel and $|$ do not occur and we write \mathcal{C}^{hl} for the set of all terms of sort \mathbf{C} in which variables of sort \mathbf{C} do not occur. Clearly, $\mathcal{P}^{hl} \subset \mathcal{P}$ and $\mathcal{C} \subset \mathcal{C}^{hl}$.

An *asserted process* is a formula of the form $\{\phi\}p\{\psi\}$, where $p \in \mathcal{P}^{hl}$ and $\phi, \psi \in \mathcal{C}^{hl}$. Here, ϕ is called the *pre-condition* of the asserted process and ψ is called the *post-condition* of the asserted process.

The intuitive meaning of an asserted process $\{\phi\}p\{\psi\}$ is as follows: if ϕ holds at the start of p and p eventually terminates successfully, then ψ holds at the successful termination of p . The conditions ϕ and ψ concern the data values assigned to flexible variables at the start and at successful termination, respectively. Therefore, in general, one or more flexible variables occur in ϕ and ψ . Unlike in p , (logical) variables of sort \mathbf{D} may also occur in ϕ and ψ . This allows of referring in ψ to the data values assigned to flexible variables at the start, like in $\{v = u\}[v := v + 1]\{v = u + 1\}$.

Below, we use the notion of equivalence under V -evaluation to make the intuitive meaning of asserted processes more precise.

We write $FV(p)$, where $p \in \mathcal{P}$, for the set of all $v \in \mathcal{V}$ that occur in p and likewise $FV(\phi)$, where $\phi \in \mathcal{C}^{hl}$, for the set of all $v \in \mathcal{V}$ that occur in ϕ . We write $AFV(p)$, where $p \in \mathcal{P}$, for the set of all $v \in FV(p)$ that occur in subterms of p that are of the form $[v := e]$. Moreover, we write \mathcal{P}_V , where V is a finite subset of \mathcal{V} , for the set $\{p \in \mathcal{P} \mid FV(p) \subseteq V\}$.

Let V be a finite subset of \mathcal{V} and let $p, q \in \mathcal{P}_V$. Then p and q are *equivalent under V -evaluation*, written $p \stackrel{V}{\sim} q$, if, for all V -evaluation maps σ , $V_\sigma(p) = V_\sigma(q)$ is derivable from the axioms of $\text{ACP}^*_\epsilon\text{-D}$.

Notice that $\stackrel{V}{\sim}$, where V be a finite subset of \mathcal{V} , is an equivalence relation indeed. Notice further that, for all $p, q \in \mathcal{P}_W$, $W \subset V$ and $p \stackrel{W}{\sim} q$ only if $p \stackrel{V}{\sim} q$.

Let $\{\phi\}p\{\psi\}$ be an asserted process and let $V = FV(\phi) \cup FV(p) \cup FV(\psi)$. Then $\{\phi\}p\{\psi\}$ is *true* if, for all closed substitution instances $\{\phi'\}p\{\psi'\}$ of $\{\phi\}p\{\psi\}$, $\phi' \rightarrow p \stackrel{V}{\sim} (\phi' \rightarrow p) \cdot (\psi' \rightarrow \epsilon)$.

To justify the claim that the definition given above reflects the intuitive meaning given earlier, we mention that $\phi' \rightarrow p \stackrel{\vee}{\sim} (\phi' \rightarrow p) \cdot (\psi' \rightarrow \epsilon)$ only if, for all V -evaluation maps σ , there exists a V -evaluation maps σ' such that $V_\sigma(\phi' \rightarrow p) \stackrel{\vee}{\sim} V_\sigma(\phi' \rightarrow p) \cdot V_{\sigma'}(\psi' \rightarrow \epsilon)$.

Notice that, using the unary guard operator mentioned in Section 3, we can write $\{\phi'\} \cdot p \stackrel{\vee}{\sim} \{\phi'\} \cdot p \cdot \{\psi'\}$ instead of $\phi' \rightarrow p \stackrel{\vee}{\sim} (\phi' \rightarrow p) \cdot (\psi' \rightarrow \epsilon)$.

Below, we will present the axioms and rules of $\text{HL}_{\text{ACP}_\epsilon^*-\text{D}}$. In addition to axioms and rules that concern a particular constant or operator of $\text{ACP}_\epsilon^*-\text{D}$, there is a rule concerning auxiliary flexible variables and a rule for precondition strengthening and/or postcondition weakening.

We use some special terminology and notations with respect to auxiliary variables. Let $p \in \mathcal{P}^{hl}$, and let $A \subseteq FV(p)$. Then A is a *set of auxiliary variables of p* if each flexible variable in A occurs in p only in subterms of the form $[v := e]$ with $v \in A$. We write $AVS(p)$, where $p \in \mathcal{P}^{hl}$, for the set of all sets of auxiliary variables of p . Moreover, we write p_A , where $p \in \mathcal{P}^{hl}$ and $A \in AVS(p)$, for p with all occurrences of subterms of the form $[v := e]$ with $v \in A$ replaced by ϵ .

The axioms and rules of $\text{HL}_{\text{ACP}_\epsilon^*-\text{D}}$ are given in Table 5. In this table, p and q stand for arbitrary terms from \mathcal{P}^{hl} , ϕ, ψ, χ, ϕ' , and ψ' stand for arbitrary terms from \mathcal{C}^{hl} , a stands for an arbitrary basic action from \mathbf{A} , v stands for an arbitrary flexible variable from \mathcal{V} , and e and e_1, e_2, \dots stand for arbitrary terms from \mathcal{D} . The parallel composition rule may only be applied if the premises are disjoint. Premises $\{\phi\} p \{\psi\}$ and $\{\phi'\} q \{\psi'\}$ are *disjoint* if

- $AFV(p) \cap FV(q) = \emptyset$, $AFV(p) \cap FV(\phi') = \emptyset$, and $AFV(p) \cap FV(\psi') = \emptyset$;
- $AFV(q) \cap FV(p) = \emptyset$, $AFV(q) \cap FV(\phi) = \emptyset$, and $AFV(q) \cap FV(\psi) = \emptyset$.

In the consequence rule, the first premise and the last premise are not asserted processes. They assert that $\phi \rightarrow \phi' = \mathbf{t}$ and $\psi' \rightarrow \psi = \mathbf{t}$ are derivable from the axioms of $\text{ACP}_\epsilon^*-\text{D}$.

Before we move on to the soundness of the axioms and rules of $\text{HL}_{\text{ACP}_\epsilon^*-\text{D}}$, we consider two congruence related properties of the equivalences $\stackrel{\vee}{\sim}$ that are relevant to the soundness proof.

Theorem 3 (Congruence). *For all finite $V \subseteq \mathcal{V}$, for all terms $p, q, p', q' \in \mathcal{P}_V$, $p \stackrel{\vee}{\sim} p'$ and $q \stackrel{\vee}{\sim} q'$ only if $p+q \stackrel{\vee}{\sim} p'+q'$, $p \cdot q \stackrel{\vee}{\sim} p' \cdot q'$, and $p^* q \stackrel{\vee}{\sim} p'^* q'$. Moreover, for all finite $V \subseteq \mathcal{V}$, for all terms $p, p' \in \mathcal{P}_V$ and all terms $\phi \in \mathcal{C}^{hl}$ with $FV(\phi) \subseteq V$, $p \stackrel{\vee}{\sim} p'$ only if $\phi \rightarrow p \stackrel{\vee}{\sim} \phi \rightarrow p'$ and $\partial_H(p) \stackrel{\vee}{\sim} \partial_H(p')$.*

Proof. Assume $p \stackrel{\vee}{\sim} p'$ and $q \stackrel{\vee}{\sim} q'$. Then $p + q \stackrel{\vee}{\sim} p' + q'$ follows immediately and $p \cdot q \stackrel{\vee}{\sim} p' \cdot q'$ and $p^* q \stackrel{\vee}{\sim} p'^* q'$ follow easily by induction on the number of proper subprocesses of p , where use is made of Lemma 1. Assume $p \stackrel{\vee}{\sim} p'$. Then $\phi \rightarrow p \stackrel{\vee}{\sim} \phi \rightarrow p'$ follows immediately and $\partial_H(p) \stackrel{\vee}{\sim} \partial_H(p')$ follows easily by induction on the number of proper subprocesses of p , where use is made of Lemma 1. \square

Theorem 4 (Limited Congruence). *For all finite $V \subseteq \mathcal{V}$, for all terms $p, q, p', q' \in \mathcal{P}_V$ with $AFV(p) \cap FV(q) = \emptyset$, $AFV(q) \cap FV(p) = \emptyset$, $AFV(p') \cap FV(q') = \emptyset$, and $AFV(q') \cap FV(p') = \emptyset$, $p \stackrel{\vee}{\sim} p'$ and $q \stackrel{\vee}{\sim} q'$ only if $p \parallel q \stackrel{\vee}{\sim} p' \parallel q'$.*

Table 5. Axioms and rules of $\text{HL}_{\text{ACP}^*_\varepsilon\text{-D}}$

inaction axiom:	$\{\phi\} \delta \{\psi\}$
empty process axiom:	$\{\phi\} \epsilon \{\phi\}$
basic action axiom:	$\{\phi\} a \{\phi\}$
data parameterized action axiom:	$\{\phi\} a(e_1, \dots, e_n) \{\phi\}$
assignment axiom:	$\{\phi[e/v]\} [v := e] \{\phi\}$
alternative composition rule:	$\frac{\{\phi\} p \{\psi\}, \{\phi\} q \{\psi\}}{\{\phi\} p + q \{\psi\}}$
sequential composition rule:	$\frac{\{\phi\} p \{\psi\}, \{\psi\} q \{\chi\}}{\{\phi\} p \cdot q \{\chi\}}$
iteration rule:	$\frac{\{\phi\} p \{\phi\}, \{\phi\} q \{\psi\}}{\{\phi\} p^* q \{\psi\}}$
guarded command rule:	$\frac{\{\phi \wedge \psi\} p \{\chi\}}{\{\phi\} \psi \rightarrow p \{\chi\}}$
parallel composition rule:	$\frac{\{\phi\} p \{\psi\}, \{\phi'\} q \{\psi'\}}{\{\phi \wedge \phi'\} p \parallel q \{\psi \wedge \psi'\}} \text{ premises are disjoint}$
encapsulation rule:	$\frac{\{\phi\} p \{\psi\}}{\{\phi\} \partial_H(p) \{\psi\}}$
auxiliary variables rule:	$\frac{\{\phi\} p \{\psi\}}{\{\phi\} p_A \{\psi\}} \quad A \in \text{AVS}(p), \text{FV}(\psi) \cap A = \emptyset$
consequence rule:	$\frac{\vdash \phi \rightarrow \phi' = \mathbf{t}, \{\phi'\} p \{\psi'\}, \vdash \psi' \rightarrow \psi = \mathbf{t}}{\{\phi\} p \{\psi\}}$

Proof. Assume $\text{AFV}(p) \cap \text{FV}(q) = \emptyset$ and $\text{AFV}(q) \cap \text{FV}(p) = \emptyset$, $\text{AFV}(p') \cap \text{FV}(q') = \emptyset$ and $\text{AFV}(q') \cap \text{FV}(p') = \emptyset$, $p \sim^v p'$ and $q \sim^v q'$. Then $p \parallel q \sim^v p' \parallel q'$ follows easily by induction on the number of proper subprocesses of p , where use is made of Lemma 1. \square

Theorem 5 (Soundness). *For all terms $p \in \mathcal{P}^{hl}$, for all terms $\phi, \psi \in \mathcal{C}^{hl}$, the asserted process $\{\phi\} p \{\psi\}$ is derivable from the axioms and rules of $\text{HL}_{\text{ACP}^*_\varepsilon\text{-D}}$ only if $\{\phi\} p \{\psi\}$ is true.*

Proof. We will assume that $\phi, \psi \in \mathcal{C}$. We can do so without loss of generality because it is sufficient to consider arbitrary closed substitution instances of ϕ

and ψ if $\phi, \psi \notin \mathcal{C}$. We will prove the theorem by proving that each of the axioms is true and each of the rules is such that only true conclusions can be drawn from true premises. The theorem then follows by induction on the length of the proof.

The proofs for the axioms and the consequence rule are trivial. Theorems 3 and 4 facilitate the proofs of the other rules. By these theorems, the proofs for the alternative composition rule, the sequential composition rule, and the guarded command rule are also trivial and the proofs of the parallel composition rule, the encapsulation rule, and the auxiliary variables rule are straightforward proofs by induction on the number of proper subprocesses, in which use is made of Lemma 1. The parallel composition rule is proved simultaneously with similar rules for the left merge operator and the communication merge operator. The proof for the iteration rule goes in a less straightforward way.

In case of the iteration rule, we assume that

- (1) for all V -evaluation maps σ , $V_\sigma(\phi : \rightarrow p) = V_\sigma((\phi : \rightarrow p) \cdot (\phi : \rightarrow \epsilon))$ is derivable;
- (2) for all V -evaluation maps σ , $V_\sigma(\phi : \rightarrow q) = V_\sigma((\phi : \rightarrow q) \cdot (\psi : \rightarrow \epsilon))$ is derivable;

and we prove that

- (3) for all V -evaluation maps σ , $V_\sigma(\phi : \rightarrow (p * q)) = V_\sigma((\phi : \rightarrow (p * q)) \cdot (\psi : \rightarrow \epsilon))$ is derivable;

where $V = FV(\phi) \cup FV(p * q) \cup FV(\psi)$. We do so by induction on the number of proper subprocesses of $V_\sigma(\phi : \rightarrow (p * q))$.

The basis step is trivial. The inductive step is proved in the following way. It follows easily from assumption (1), where use is made of BKS1, that

- (4) for all V -evaluation maps σ , for some evaluation map σ' , $V_\sigma(\phi : \rightarrow (p * q)) = V_\sigma(\phi : \rightarrow p) \cdot V_{\sigma'}(\phi : \rightarrow (p * q)) + V_\sigma(\phi : \rightarrow q)$ is derivable.

We distinguish two cases: $\sigma \neq \sigma'$ and $\sigma = \sigma'$.

In the case where $\sigma \neq \sigma'$, (3) follows easily from (4), the induction hypothesis, and assumption (2), where use is made of BKS1.

In the case where $\sigma = \sigma'$, it follows easily from (4), where use is made of RSP*, that

- (5) for all V -evaluation maps σ , $V_\sigma(\phi : \rightarrow (p * q)) = V_\sigma((\phi : \rightarrow p) * (\phi : \rightarrow q))$ is derivable;

and (3) follows easily from (5) and assumption (2), making use of BKS1. \square

We do not see how Theorem 5 can be proved if RSP* is replaced by BKS2–BKS5.

6 Using the Hoare Logic for ACP_ϵ^* -D

In this section, we go into the use of $HL_{ACP_\epsilon^*D}$ as a complement to pure equational reasoning from the axioms of ACP_ϵ^* -D.

Let $\{\phi\}p\{\psi\}$ be an asserted process, and let $V = FV(\phi) \cup FV(p) \cup FV(\psi)$. Suppose that $\{\phi\}p\{\psi\}$ has been derived from the axioms and rules of $\text{HL}_{\text{ACP}_\epsilon^*-\text{D}}$. Then, by Theorem 5, $\{\phi\}p\{\psi\}$ is true. This means that, for all closed substitution instances $\{\phi'\}p\{\psi'\}$ of $\{\phi\}p\{\psi\}$, $\phi' \rightarrow p \stackrel{V}{\sim} (\phi' \rightarrow p) \cdot (\psi' \rightarrow \epsilon)$. In other words, for all closed substitution instances $\{\phi'\}p\{\psi'\}$ of $\{\phi\}p\{\psi\}$, for all V -evaluation maps σ , $V_\sigma(\phi' \rightarrow p) = V_\sigma(\phi' \rightarrow p) \cdot V_{\sigma'}(\psi' \rightarrow \epsilon)$ is derivable from the axioms of $\text{ACP}_\epsilon^*-\text{D}$. Thus, the derivation of $\{\phi\}p\{\psi\}$ from the axioms and rules of $\text{HL}_{\text{ACP}_\epsilon^*-\text{D}}$ has made a collection of equations available that can be considered to be derived by equational reasoning from the axioms of $\text{ACP}_\epsilon^*-\text{D}$.

Let us have a closer look at the equivalence relation $\stackrel{V}{\sim}$ on \mathcal{P}_V . Clearly, this equivalence relation is useful when reasoning about processes in which data are involved. However, it is plain from the proof of Theorem 4 that $\stackrel{V}{\sim}$ is not a congruence relation on \mathcal{P}_V . This complicates the use of equational reasoning to derive, among other things, the collection of equations referred to above considerably. The presented Hoare logic can be considered to be a means to get partially round the complications concerned.

Dissociated from its connection with $\text{HL}_{\text{ACP}_\epsilon^*-\text{D}}$, $\stackrel{V}{\sim}$ remains an interesting equivalence relation on \mathcal{P}_V when it comes to reasoning about processes in which data is involved. Therefore, we mention below a result on this equivalence relation which is a corollary of results from Section 5 used to prove the soundness of $\text{HL}_{\text{ACP}_\epsilon^*-\text{D}}$. The fact that $\stackrel{V}{\sim}$ is not a congruence relation on \mathcal{P}_V , and consequently that $\stackrel{V}{\sim}$ is not preserved by all contexts, makes this corollary to the point. In order to formulate the corollary, we first define a set of contexts.

For each finite $V \subseteq \mathcal{V}$, the set $\mathbb{C}_V^{\text{seq}}$ of *sequential evaluation supporting contexts for V* is the set $\bigcup_{W \subseteq V} \mathbb{C}_{V,W}^{\text{seq}}$, where the sets $\mathbb{C}_{V,W}^{\text{seq}}$, for finite $V, W \subseteq \mathcal{V}$ with $W \subseteq V$, are defined by simultaneous induction as follows:

- $\square \in \mathbb{C}_{V,W}^{\text{seq}}$;
- if $p \in \mathcal{P}^{\text{seq}}$, $C \in \mathbb{C}_{V,W}^{\text{seq}}$, $FV(p) \subseteq V$, and $AFV(p) \subseteq W$, then $p + C$, $C + p$, $p \cdot C$, $C \cdot p$, $p^* C$, $C^* p \in \mathbb{C}_{V,W}^{\text{seq}}$;
- if $\phi \in \mathcal{C}$ and $C \in \mathbb{C}_{V,W}^{\text{seq}}$, $FV(\phi) \subseteq V$, then $\phi : \rightarrow C \in \mathbb{C}_{V,W}^{\text{seq}}$;
- if $p \in \mathcal{P}^{\text{seq}}$, $C \in \mathbb{C}_{V,W}^{\text{seq}}$, $AFV(p) \cap V = \emptyset$, and $FV(p) \cap W = \emptyset$, then $p \parallel C$, $C \parallel p \in \mathbb{C}_{V \cup FV(p), W \cup AFV(p)}^{\text{seq}}$;
- if $H \subseteq \mathbf{A}$ and $C \in \mathbb{C}_{V,W}^{\text{seq}}$, then $\partial_H(C) \in \mathbb{C}_{V,W}^{\text{seq}}$.

We write $C[p]$, where $C \in \mathbb{C}_V^{\text{seq}}$ and $p \in \mathcal{P}$, for C with the occurrence of \square replaced by p .

The following is a corollary of Theorems 3 and 4.

Corollary 1. *Let V be a finite subset of \mathcal{V} . Then, for all $p, p' \in \mathcal{P}_V$, for all $C \in \mathbb{C}_V^{\text{seq}}$, $p \stackrel{V}{\sim} p'$ only if $C[p] \stackrel{V}{\sim} C[p']$.*

Of course, Corollary 1 can be applied to results from using $\text{HL}_{\text{ACP}_\epsilon^*-\text{D}}$. Let $\{\phi\}p\{\psi\}$ be an asserted process, let $V = FV(\phi) \cup FV(p) \cup FV(\psi)$, and let $C \in \mathbb{C}_V^{\text{seq}}$. Suppose that $\{\phi\}p\{\psi\}$ has been derived from the axioms and rules of $\text{HL}_{\text{ACP}_\epsilon^*-\text{D}}$. Then, for all closed substitution instances $\{\phi'\}p\{\psi'\}$ of $\{\phi\}p\{\psi\}$, we have that $C[\phi' \rightarrow p] \stackrel{V}{\sim} C[(\phi' \rightarrow p) \cdot (\psi' \rightarrow \epsilon)]$.

7 Related Work

The approach to the formal verification of programs that is now known as Hoare logic was proposed in [14]. The illustration of this approach was at the time confined to the very simple deterministic sequential programs that are mostly referred to as while programs (cf. [1]). The axioms, the sequential composition rule, the iteration rule, the guarded command rule, and the consequence rule from our Hoare logic savour strongly of the common rules for while programs. The alternative composition rule is the or rule due to [17], the parallel composition rule was proposed in [15], and the auxiliary variables rule was first introduced in [19]. The parallel composition rules proposed in [2,18,19] are more complicated than our parallel composition rule.

In the case of [2,18], the intention was to provide a Hoare logic for the first design of CSP [16]. In that design, one program may force another program to assign a data value sent by the former program to a program variable used by the latter program. This feature complicates the parallel composition rule considerably. Moreover, incorporating this feature in an ACP-like process algebra would lead to the situation that, in equational reasoning, certain axioms may not be applied in contexts of parallel processes (like in [13], see below). Because our concern is in the use of a Hoare logic as a complement to pure equational reasoning, we have not considered incorporating this feature.

In the case of [19], the rule is more complicated because, in the parallel programs covered, program variables may be shared variables, i.e. program variables that are assigned to in one program may be used in another program. Our process algebra also covers shared variables. However, covering shared variables in our Hoare logic as well would mean that the simple disjointness proof required by our parallel composition rule has to be replaced a sophisticated interference-freeness proof. We believe that this would diminish the usefulness of our Hoare logic as a complement to equational reasoning considerably. Therefore, we have not considered covering shared variables in the parallel composition rule.

In [13], an extension of ACP with the empty process constant and the unary counterpart of the binary guarded command operator is presented, the truth of an asserted sequential process is defined in terms of the transition relations from the given structural operational semantics of the presented extension of ACP, and it is shown that an asserted sequential process $\{\phi\}p\{\psi\}$ is true according to that definition iff $\{\phi\} \cdot p \xleftrightarrow{\quad} \{\phi\} \cdot p \cdot \{\psi\}$, where $\xleftrightarrow{\quad}$ is bisimulation equivalence as defined in [13] for sequential processes. Moreover, a Hoare logic of sequential asserted processes is presented and its soundness is shown. However, [13] does not go into the use of that Hoare logic as a complement to pure equational reasoning from the equational axioms.

Regarding the bisimulation equivalence $\xleftrightarrow{\quad}$ defined in [13] for sequential processes, we can mention that, if the data-states are evaluation maps, $p \xleftrightarrow{\quad} q$ iff $V_\sigma(p) \xleftrightarrow{\quad} V_\sigma(q)$ for all V -evaluation maps σ , where $V = FV(p) \cup FV(q)$. Due to the possibility of interference between parallel processes, a different bisimulation equivalence $\xleftrightarrow{\quad}'$, finer than $\xleftrightarrow{\quad}$, is needed in [13] for parallel processes. As a consequence, in equational reasoning, certain axioms may not be applied

in contexts of parallel processes. Moreover, $\stackrel{\vee}{\Rightarrow}$ together with the operators V_σ allows of dealing with local data-states, whereas the combination of $\stackrel{\vee}{\Rightarrow}'$ and $\stackrel{\vee}{\Rightarrow}''$ does not allow of dealing with local data-states.

8 Concluding Remarks

We have taken an extension of ACP with features that are relevant to processes in which data are involved, devised a Hoare logic of asserted processes based on this extension of ACP, and gone into the use of this Hoare logic as a complement to pure equational reasoning from the axioms of the extension of ACP.

We have defined what it means that an asserted process is true in terms of an equivalence relation ($\stackrel{\vee}{\sim}$) that had been found to be central to relating the extension of ACP and the Hoare logic. That this equivalence relation is not a congruence relation with respect to parallel composition is related to the fact that in the extension of ACP presented in [13] certain axioms may not be applied in contexts of parallel processes.

In this paper, we build on earlier work on ACP. The axioms of ACP_ϵ have been taken from [7, Section 4.4], the axioms for the iteration operator have been taken from [9], and the axioms for the guarded command operator have been taken from [4]. The evaluation operators have been inspired by [11] and the data parameterized action operator has been inspired by [12].

References

1. K. R. Apt, F. S. de Boer, and E.-R. Olderog. *Verification of Sequential and Concurrent Programs*. Texts in Computer Science. Springer-Verlag, Berlin, third edition, 2009.
2. K. R. Apt, N. Francez, and W. P. de Roever. A proof system for communicating sequential processes. *ACM Transactions on Programming Languages and Systems*, 2(3):359–385, 1980.
3. J. C. M. Baeten and J. A. Bergstra. Global renaming operators in concrete process algebra. *Information and Control*, 78(3):205–245, 1988.
4. J. C. M. Baeten and J. A. Bergstra. Process algebra with signals and conditions. In M. Broy, editor, *Programming and Mathematical Methods*, volume F88 of *NATO ASI Series*, pages 273–323. Springer-Verlag, 1992.
5. J. C. M. Baeten and J. A. Bergstra. Process algebra with propositional signals. *Theoretical Computer Science*, 177:381–405, 1997.
6. J. C. M. Baeten and C. Verhoef. A congruence theorem for structured operational semantics with predicates. In E. Best, editor, *CONCUR'93*, volume 715 of *Lecture Notes in Computer Science*, pages 477–492. Springer-Verlag, 1993.
7. J. C. M. Baeten and W. P. Weijland. *Process Algebra*, volume 18 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 1990.
8. J. A. Bergstra, I. Bethke, and A. Ponse. Process algebra with iteration and nesting. *Computer Journal*, 37:243–258, 1994.

9. J. A. Bergstra, W. J. Fokkink, and A. Ponse. Process algebra with recursive operations. In J. A. Bergstra, A. Ponse, and S. A. Smolka, editors, *Handbook of Process Algebra*, pages 333–389. Elsevier, Amsterdam, 2001.
10. J. A. Bergstra and J. W. Klop. Process algebra for synchronous communication. *Information and Control*, 60(1–3):109–137, 1984.
11. J. A. Bergstra and C. A. Middelburg. Splitting bisimulations and retrospective conditions. *Information and Computation*, 204(7):1083–1138, 2006.
12. J. A. Bergstra and C. A. Middelburg. A process calculus with finitary comprehended terms. *Theory of Computing Systems*, 53(4):645–668, 2013.
13. J. F. Groote and A. Ponse. Process algebra with guards: Combining Hoare logic with process algebra. *Formal Aspects of Computing*, 6(2):115–164, 1994.
14. C. A. R. Hoare. An axiomatic basis for computer programming. *Communications of the ACM*, 12(10):576–580, 583, 1969.
15. C. A. R. Hoare. Towards a theory of parallel programming. In C. A. R. Hoare and R. H. Perrott, editors, *Operating Systems Techniques*, pages 61–71. Academic Press, 1972.
16. C. A. R. Hoare. Communicating sequential processes. *Communications of the ACM*, 21(8):666–677, 1978.
17. P. E. Lauer. Consistent formal theories of the semantics of programming languages. Technical Report 25.121, IBM Laboratory Vienna, 1971.
18. G. M. Levin and D. Gries. A proof technique for communicating sequential processes. *Acta Informatica*, 15(3):281–302, 1981.
19. S. Owicki and D. Gries. An axiomatic proof technique for parallel programs I. *Acta Informatica*, 6(4):319–340, 1976.
20. D. Sannella and A. Tarlecki. Algebraic preliminaries. In E. Astesiano, H.-J. Krewski, and B. Krieg-Brückner, editors, *Algebraic Foundations of Systems Specification*, pages 13–30. Springer-Verlag, Berlin, 1999.
21. P. Sewell. Bisimulation is not finitely (first order) equationally axiomatisable. In *LICS'94*, pages 62–70. IEEE Computer Society Press, 1994.
22. M. Wirsing. Algebraic specification. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, pages 675–788. Elsevier, Amsterdam, 1990.