

# DENSITY OF ALGEBRAIC POINTS ON NOETHERIAN VARIETIES

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**Abstract.** Let  $\Omega \subset \mathbb{R}^n$  be a relatively compact domain. A finite collection of real-valued functions on  $\Omega$  is called a *Noetherian chain* if the partial derivatives of each function are expressible as polynomials in the functions. A *Noetherian function* is a polynomial combination of elements of a Noetherian chain. We introduce *Noetherian parameters* (degrees, size of the coefficients) which measure the complexity of a Noetherian chain. Our main result is an explicit form of the Pila–Wilkie theorem for sets defined using Noetherian equalities and inequalities: for any  $\varepsilon > 0$ , the number of points of height  $H$  in the transcendental part of the set is at most  $C \cdot H^\varepsilon$  where  $C$  can be *explicitly* estimated from the Noetherian parameters and  $\varepsilon$ . We show that *many functions of interest in arithmetic geometry fall within the Noetherian class, including elliptic and abelian functions, modular functions and universal covers of compact Riemann surfaces, Jacobi theta functions, periods of algebraic integrals, and the uniformizing map of the Siegel modular variety  $\mathcal{A}_g$ .* We thus effectivize the (geometric side of) Pila–Zannier strategy for unlikely intersections in various contexts.

## 1 Introduction

**1.1 The (real) Noetherian class.** Let  $\Omega_{\mathbb{R}} \subset \mathbb{R}^n$  be a bounded domain, and denote by  $\mathbf{x} := (x_1, \dots, x_n)$  a system of coordinates on  $\mathbb{R}^n$ . A collection of analytic functions  $\phi := (\phi_1, \dots, \phi_\ell) : \Omega_{\mathbb{R}} \rightarrow \mathbb{R}^\ell$  is called a (complex) *real Noetherian chain* if it satisfies an overdetermined system of algebraic partial differential equations,

$$\frac{\partial \phi_i}{\partial x_j} = P_{i,j}(\mathbf{x}, \phi), \quad \begin{array}{l} i = 1, \dots, \ell, \\ j = 1, \dots, n, \end{array} \quad (1)$$

where  $P_{i,j}$  are polynomials. We call  $\ell$  the order and  $\alpha := \max_{i,j} \deg P_{i,j}$  the degree of the chain. If  $P \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$  is a polynomial of degree  $\beta$  then  $P(\mathbf{x}, \phi) : \Omega_{\mathbb{R}} \rightarrow \mathbb{R}$  is called a *real Noetherian function* of degree  $\beta$ .

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multivariate

We call the set of common zeros of a collection of real Noetherian functions of degree at most  $\beta$  a real Noetherian variety of degree  $\beta$ . We call a set defined by a finite sequence of Noetherian equations or inequalities a basic semi-Noetherian set, and a finite union of such sets a semi-Noetherian set. We define the *complexity*  $\beta$  of a semi-Noetherian set (more precisely the formula defining it) to be the maximum of the degrees of the Noetherian functions appearing in the definition, plus the total number of relations. We use an analogous definition for the complexity of a semialgebraic set.

We define the *Noetherian size* of  $\phi$ , denoted  $\mathcal{S}(\phi)$ , to be

$$\mathcal{S}(\phi) := \max_{x \in \Omega_{\mathbb{R}}} \max_{\substack{i=1, \dots, \ell \\ j=1, \dots, n}} \{|x_j|, |\phi_i(\mathbf{x})|, \|P_{i,j}\|_{\infty}\}. \quad (2)$$

Here and below  $\|P\|_{\infty}$  denotes the maximum norm on the coefficients of  $P$ . For simplicity of the notation we always assume  $\mathcal{S}(\phi) \geq 2$ . In this paper we will be concerned with the problem of producing explicit estimates for some quantities associated to Noetherian varieties and semi-Noetherian sets. When we say that a quantity can be *explicitly estimated* in terms of the Noetherian parameters, we mean that it admits an explicit upper bound in terms of the parameters  $n, \ell, \alpha, \mathcal{S}(\phi), \beta$ .

**1.2 Main statement.** For a set  $A \subset \mathbb{R}^n$  we define the *algebraic part*  $A^{\text{alg}}$  of  $A$  to be the union of all connected semialgebraic subsets of  $A$  of positive dimension. We define the *transcendental part*  $A^{\text{trans}}$  of  $A$  to be  $A \setminus A^{\text{alg}}$ . Recall that the *height* of a (reduced) rational number  $\frac{a}{b} \in \mathbb{Q}$  is defined to be  $\max(|a|, |b|)$ . More generally, for  $\alpha \in \mathbb{Q}^{\text{alg}}$  we denote by  $H(\alpha)$  its absolute multiplicative height as defined in [BG06]. For a vector  $\alpha$  of algebraic numbers we denote by  $H(\alpha)$  the maximum among the heights of the coordinates. For a set  $A \subset \Omega_{\mathbb{R}}$  we denote the set of  $\mathbb{Q}$ -points of  $A$  by  $A(\mathbb{Q}) := A \cap \mathbb{Q}^n$  and denote

$$A(\mathbb{Q}, H) := \{\mathbf{x} \in A(\mathbb{Q}) : H(\mathbf{x}) \leq H\}. \quad (3)$$

We denote by  $f^{\circ j}$  the  $j$ -th compositional iterate of  $f$ . Throughout the paper we let  $C_j(\phi, d)$  denote the asymptotic class

$$\bullet \quad C_j(\phi, d) := (\mathcal{S}(\phi))^{\exp^{O(j)}(O(d^2))}, \quad C(\phi, d) := C_1(\phi, d), \quad (4)$$

where it is understood that each occurrence may represent a different function from the class. In all instances of asymptotic notation in this paper, it is understood that the implied constants *can be explicitly and straightforwardly estimated in terms of the Noetherian parameters*, even though we do not always produce explicit expressions for the constants. The following is a basic form of our main theorem.

**Theorem 1.** *Let  $X \subset \Omega_{\mathbb{R}}$  be a semi-Noetherian set of complexity  $\beta$  and  $\varepsilon > 0$ . There exists a constant*

$$N = C_n(\phi, \beta \varepsilon^{1-n}) \quad (5)$$

*such that for any  $H \in \mathbb{N}$  we have*

$$\#X^{\text{trans}}(\mathbb{Q}, H) \leq N \cdot H^{\varepsilon}. \quad (6)$$

Theorem 1 is a direct corollary of the following more general statement. First, we consider *algebraic* points of a fixed degree  $k \in \mathbb{N}$  instead of rational points. Toward this end we introduce the notation

$$A(k) := \{\mathbf{x} \in A : [\mathbb{Q}(x_1) : \mathbb{Q}], \dots, [\mathbb{Q}(x_n) : \mathbb{Q}] \leq k\}, \quad (7)$$

$$A(k, H) := \{\mathbf{x} \in A(k) : H(\mathbf{x}) \leq H\}. \quad (8)$$

Second, we obtain a more accurate description of the part of  $X^{\text{alg}}$  where algebraic points of a given height may lie. Toward this end we introduce the following notation.

**DEFINITION 1.** *Let  $A, W$  be two subsets of a topological space. We denote by*

$$A(W) := \{w \in W : W_w \subset A\} \quad (9)$$

*the set of points of  $W$  such that  $A$  contains the germ of  $W$  around  $w$ , i.e. such that  $w$  has a neighborhood  $U_w$  with  $U_w \cap W \subset A$ .*

In particular, when  $W \subset \mathbb{R}^n$  is a connected positive dimensional semialgebraic set then we have  $X(W) \subset X^{\text{alg}}$ . With these notations, the general form of our main theorem is as follows.

**Theorem 2.** *Let  $X \subset \Omega_{\mathbb{R}}$  be a semi-Noetherian set of complexity  $\beta$  and  $\varepsilon > 0$ . There exist constants*

$$d, N = C_{n(k+1)}(\phi, \beta\varepsilon^{1-n}) \quad (10)$$

*with the following property. For every  $H \in \mathbb{N}$  there exist at most  $NH^\varepsilon$  smooth connected semialgebraic sets  $\{S_\alpha\}$  of complexity at most  $d$  such that*

$$X(k, H) \subset \bigcup_{\alpha} X(S_\alpha). \quad (11)$$

We remark that in Theorem 2 we allow the asymptotic constants to depend on the degree  $k$  as well.

### 1.3 Motivation.

**1.3.1 The Pila–Wilkie theorem and the Pila–Zannier strategy for problems of unlikely intersections.** Following the fundamental work of Bombieri and Pila [BP89], Pila and Wilkie proved in [PW06] that for any set definable in an o-minimal structure, the number of rational points of height  $H$  in the transcendental part grows sub-polynomially with  $H$  in the sense of Theorem 1. In other words, without the added condition of effectivity, Theorem 1 is already known in vast generality. Similarly, a non-effective result similar in spirit to Theorem 2, valid for arbitrary definable sets, has been established in [Pil09].

Beyond the intrinsic interest in the study of density of rational points on transcendental sets, this direction of research has attracted considerable attention following the discovery of a surprising link to various problems of unlikely intersections in

arithmetic geometry. The first and prototypical example of this link was produced in Pila–Zannier’s [PZ08] proof of the Manin–Mumford conjecture (first proved by Raynaud [Ray83]). We briefly recall the statement and strategy of proof to motivate the following discussion.

Let  $A$  be an abelian variety and  $V \subset A$  an algebraic subvariety, both defined over a number field. Suppose that  $V$  does not contain a coset of an infinite abelian subvariety. Then (a particular case of) the Manin–Mumford conjecture asserts that the number of torsion points is finite. The strategy of [PZ08] proceeds as follows. Let  $\pi : \mathbb{C}^g \rightarrow A$  denote the universal cover of  $A$  and  $\Omega \subset \mathbb{C}^g$  denote the standard fundamental domain. Identify  $\mathbb{C}^g$  with  $\mathbb{R}^{2g}$  in such a way that  $\Omega$  corresponds to the unit cube, and observe that under this identification the torsion points of order  $H$  in  $V$  correspond to rational points of height  $H$  in  $X := \pi^{-1}(V) \cap \Omega$ . One now obtains two competing estimates for  $\#X(\mathbb{Q}, H)$ :

- (1) One checks that under the assumptions on  $V$  one has  $X^{\text{trans}} = X$ . Thus by the Pila–Wilkie theorem  $\#X(\mathbb{Q}, H)$  grows sub-polynomially with  $H$ .
- (2) By a result of Masser [Mas84], if  $p \in A$  is torsion of order  $H$  then the number of its Galois conjugates is at least  $cH^\delta$  for some  $c, \delta > 0$ . Since  $V$  is defined over a number field, a constant fraction of these conjugates belong to  $V$ , and we conclude that  $\#X(\mathbb{Q}, H) > c'H^\delta$  for some  $c' > 0$ .

The inconsistency of these two estimates implies that for  $H$  sufficiently large,  $V$  contains no torsion points of order  $H$ . In particular the number of torsion points is finite.

The Manin–Mumford conjecture has a prototypical form: given an arithmetic condition on  $p \in A$  (being torsion) and a geometric condition ( $p \in V$ ), the number of solutions is finite unless for some “obvious” reasons (e.g.  $V$  contains an abelian subvariety). Various other problems of a similar prototype have been solved by using the same basic Pila–Zannier strategy. We list two prominent examples:

**The <sup>Yves</sup>André–Oort conjecture** We describe the special case considered in [And98] for simplicity. Consider the product  $Y(N_1) \times Y(N_2)$  of two modular curves, and its irreducible algebraic subvariety  $V$ , and suppose  $V$  is not defined by a *modular polynomial*. Then the number of points  $(p_1, p_2) \in V$  where both  $p_1, p_2$  are CM-points (i.e. correspond to elliptic curves with complex multiplication) is finite [And98]. A more general case of this statement, the *André–Oort conjecture* for modular curves (involving the products of an arbitrary number of modular curves as well as abelian varieties and complex tori) was proved using the Pila–Zannier strategy in [Pil11]. The uniformization maps of modular curves play a key role in this proof. Note that since the fundamental domains of modular curves are never compact, the definable sets appearing in this proof are not subanalytic and the full strength of the Pila–Wilkie theorem in the o-minimal setting is required to study their behavior near the cusps. This proof was later extended, with significant effort on the Galois-theoretic side, to various other contexts involving Shimura varieties. We refer the reader to [ET14]

for a survey of various developments in this area, and to [Tsi18] for the more recent unconditional proof of the André–Oort conjecture for  $\mathcal{A}_g$ .

**Torsion anomalous points** Consider the two points

$$P(\lambda) = (2, \sqrt{2(2-\lambda)}), \quad Q(\lambda) = (3, \sqrt{6(3-\lambda)}) \quad (12)$$

on the Legendre elliptic curve  $E_\lambda$  defined by  $y^2 = x(x-1)(x-\lambda)$ . What can be said about the set of points  $\lambda$  where both  $P(\lambda)$  and  $Q(\lambda)$  are torsion on  $E_\lambda$ ? In [MZ10] Masser and Zannier use the Pila–Zannier strategy to show that this set is finite. Here the analytic uniformization  $\wp_\lambda(z)$  as a function of both variables plays the role of the uniformization, and it suffices to consider this function restricted to a certain compact set. Many other results in a similar direction have been derived using a similar strategy, see e.g. [MZ15, MZ14, MZ12, BMPZ16]. The original question for  $P, Q$  above has been resolved effectively by Stoll [Sto14] using different methods. However for many of the subsequent developments the strategy of Masser and Zannier gives the only currently known proofs, whose principal source of ineffectivity stems from the use of the Pila–Wilkie theorem.

*1.3.2 Questions of effectivity.* It is natural to ask to which extent, and in what instances, can an effective form of the Pila–Zannier strategy be established. This question is split into two parts. One problem is to effectivize the lower bounds on sizes of Galois orbits, and the other is to effectivize the Pila–Wilkie upper bound. Of course, in order to expect some type of effectivity in the Pila–Wilkie theorem one must restrict to a structure where the definable sets admit some form of effective description.

The proof of the Manin–Mumford conjecture given in [PZ08] relies on the orbit lower bounds of [Mas84], which are effective. For the upper bound, the Pila–Wilkie theorem is applied to sets defined using the uniformizing maps of abelian varieties. In the final section of [PZ08] Pila and Zannier hypothesize that an estimate may be derived from an explicit description of the abelian variety and its algebraic subvariety in terms of theta functions. Effective proofs of the Manin–Mumford conjecture have been obtained using entirely different methods in [Rém00, Hru01]. We show in Sect. 3.3–3.4 that elliptic and abelian functions belong to the Noetherian category, thus effectivizing the upper bound and allowing an effective version of the Manin–Mumford conjecture to be derived using the Pila–Zannier strategy.

The proof of the André–Oort conjecture for modular curves given in [Pil11] and in subsequent works relies on lower bounds that are generally not known to be effective (but can be made effective assuming the Generalized Riemann Hypothesis). For the upper bounds, the Pila–Wilkie theorem is applied to sets defined using uniformizing maps of modular curves (and in subsequent work of Shimura varieties). Even without an effective lower bound, an effective upper bound could lead for example to asymptotic estimates (with an undetermined constant) in terms of the data involved. For discussion in this direction see [Pil11, Section 13]. An effective

version of André’s original theorem (for a product of two modular curves) without the assumption of GRH was obtained in [Küh12] (and also in [BMZ13]). We are not aware of effective results in higher dimensions or for other Shimura curves. We show in Sect. 3.7 that the uniformizing map of the moduli space of principally polarized abelian varieties  $\mathcal{A}_g$  belongs to the Noetherian category, thus effectivizing the upper-bound for compact subvarieties of  $\mathcal{A}_g$ .

The proof of [MZ10] concerning torsion anomalous points relies on lower bounds which are effective [Zan12, Section 3.4.3]. For the upper bound, the Pila–Wilkie theorem is applied to sets defined using the uniformizing maps of elliptic families, which can be described for instance using the Weierstrass function  $\wp(z; \tau)$  as a function of both parameters (or equivalently in terms of theta function of both parameters). We show in 3.5 that  $\wp(z; \tau)$  is a Noetherian function of both its variables, implying that questions in the Masser–Zannier direction can also be treated within the Noetherian category. We note however that unlike the applications around André–Oort, some (though not all) of the applications in the Masser–Zannier direction require sub-Noetherian rather than semi-Noetherian sets (as some variables need to be eliminated). It seems likely that such cases can also be handled either by application of the implicit function theorem for Noetherian function (Corollary 9) or by a variation of the methods developed in this paper, but we do not address these issues presently. We also remark that some effective results in this direction have been obtained in [HJM16], including through the effectivization of the Pila–Wilkie theorem for some specific curves, thus effectivizing the upper bound in this context.

*1.3.3 Effectivity through differential equations.* The arithmetic-geometry applications of the Pila–Wilkie theorem involve the use of classical functions such as exponential, elliptic and abelian functions (for Manin–Mumford); modular functions and universal covers of more general Shimura varieties (André–Oort); and theta functions (torsion anomaly in families). An effective version of the Pila–Wilkie theorem that unifies the treatment of these various applications would have to start with a framework allowing a uniform and effective description of each of these functions. It is natural to look at differential equations as a possible way of describing such functions explicitly.

The effective study of the quantitative geometry of transcendental functions through differential equations is of course not new. Khovanskii’s theory of Pfaffian functions [Kho91] provides a very successful example of this sort. Pfaffian functions are defined in a manner similar to the Noetherian functions, but with an extra assumption of triangularity in the system (1). With this extra assumption, general estimates on the geometric complexity of “Pfaffian sets” have been established in [Kho91] (see also [GN09, GV04] for additional developments). This theory has been utilized for deriving effective versions of the Pila–Wilkie theorem for Pfaffian curves [Pil07] and for certain Pfaffian surfaces [JT12]. These works also prove a stronger form of the Pila–Wilkie theorem, improving the asymptotic from sub-polynomial to polylogarithmic.

In [BN17b] an effective form of the Pila–Wilkie theorem (and its strengthening to polylogarithmic asymptotics) was established for definable sets of arbitrary dimension in  $\mathbb{R}^{\text{RE}}$ , the structure generated by the restricted exponential and sine functions. The proof relies on a combination of Pfaffian methods and complex geometry, and it appears likely that the method of proof would extend to allow (compact restrictions of) elliptic and abelian functions. This puts the Manin–Mumford conjecture in arbitrary dimension within the scope of this method. On the other hand, modular functions, theta functions and other functions required in the applications of the Pila–Wilkie theorem to arithmetic geometry do not appear to be Pfaffian (at least not to our knowledge), and a fundamentally different approach seems to be required for handling them.

It seems a-priori likely (and in fact this is verified for many cases in Sect. 3) that the functions needed in applications to arithmetic geometry would fall within the framework of Noetherian functions. However, the quantitative geometric theory of Noetherian functions is far less developed than that of the Pfaffian functions. Khovanskii has conjectured that some “local” form of his theory of Pfaffian functions should hold for Noetherian functions, and most work has been focused on problems of a local nature [GK98, BN12, BN15] (and even in this local setting the conjecture is not yet fully settled). Barring very significant progress on the general theory of Noetherian functions it seems unlikely that the proof strategy of [BN17b] could be carried over to the Noetherian category. For instance, we currently do not know how to effectively bound the number of solutions of a system of two Noetherian functions in two variables in terms of the Noetherian parameters. In the following section we explain how it is still possible to obtain effective estimates for the seemingly much more complicated Pila–Wilkie theorem in arbitrary dimension without ever addressing this very basic question.

**1.4 Sketch of the proof of Theorem 1.** Let  $\{F_j\}$  be the set of Noetherian functions defining our set  $X$  (we suppose for simplicity that only equalities are used). Since real Noetherian functions remain Noetherian in the complex domain as explained in Sect. 2.1, there is no harm in viewing  $\{F_j\}$  as complex analytic functions and replacing  $X$  by their common zero locus in the complex domain. We note that this simple step which is almost automatic in the Noetherian category is unavailable in the essentially real Pfaffian category. Even if one only wishes to prove the effective Pila–Wilkie theorem for restricted Pfaffian functions, our proof goes through their complex continuations which are Noetherian but no longer Pfaffian. We do not know of any simpler method for treating the Pfaffian case in arbitrary dimension.

*1.4.1 The basic inductive step.* The proof follows an induction over dimension which is similar to the one used in [PW06]. However, for our purposes the book-keeping needs to be done a little differently. To simplify the presentation we assume that  $X^{\text{alg}} = \emptyset$ . Our inductive step can then be formulated as follows.



PROPOSITION (cf. Proposition 28). *Let  $W \subset \mathbb{C}^n$  be an irreducible algebraic variety and suppose that  $X \subset W$ . Then there exist  $NH^\varepsilon$  hypersurfaces  $\mathcal{H}_\alpha$  of degree  $d$  (both  $N$  and  $d$  explicitly estimated in terms of the Noetherian parameters and  $\deg W$ ) such that none of the  $\mathcal{H}_\alpha$  contain  $W$  and  $X(\mathbb{Q}, H)$  is contained in their union.*

Having established this proposition, one can start with  $W = \mathbb{C}^n$  and at the inductive step replace  $W$  by its intersection with each of the hypersurfaces  $\mathcal{H}_\alpha$  (more precisely each irreducible component), and replace  $X$  by  $X \cap W$ . In this way one eventually obtains (when  $\dim W = 0$ ) a collection of  $O(H^\varepsilon)$  points containing  $X(\mathbb{Q}, H)$ .

*1.4.2 The construction of the hypersurfaces.* The construction of the hypersurfaces  $\mathcal{H}_\alpha$  follows a complex-analytic strategy based on the notion of a *Weierstrass polydisc* developed in [BN17a, BN17b]. Recall that a Weierstrass polydisc for a pure-dimensional analytic set  $Y \subset \Omega$  is a polydisc  $\Delta := \Delta_z \times \Delta_w$  such that  $\dim \Delta_z = \dim Y$  and  $Y \cap (\Delta_z \times \partial \Delta_w) = \emptyset$ . Under these assumptions  $Y \cap \Delta$  is a ramified cover of  $\Delta_z$  of some degree  $e(Y, \Delta)$ . We denote by  $\Delta^\rho$  the shrinking of  $\Delta$  about its center by a factor of  $\rho$  (see Definition 14). By the method of [BN17a] (analogous to [BP89]) one can construct a hypersurface  $\mathcal{H}$  of degree  $O(e(Y, \Delta))$  containing  $(Y \cap \Delta^{H^\varepsilon})(\mathbb{Q}, H)$ .

A straightforward strategy would be to cover  $X$  by sets of the form  $\Delta_\alpha^{H^\varepsilon}$  where  $\Delta_\alpha$  is a Weierstrass polydisc for  $X$ , with their number and degrees  $e(X, \Delta_\alpha)$  explicitly estimated in terms of the Noetherian parameters. This is similar to the strategy of [BN17b]. However, the proof in [BN17b] relies on essentially real ideas for the construction of the Weierstrass polydiscs (entropy estimates, Vitushkin's formula) and requires estimates which are not available in the Noetherian category. We proceed to explain how this is replaced in the present paper by complex-analytic considerations.

*1.4.3 The  $W = \mathbb{C}^n$  case.* Consider the first step where  $W = \mathbb{C}^n$ . Choose one of the Noetherian functions defining  $X$  which is not identically zero, say  $F$ . A key observation is that to prove the inductive step, it suffices to replace  $X$  in the statement of the basic inductive step by the set  $Y := \{F = 0\}$  (note that  $X \subset Y$  and possibly  $\dim X < \dim Y$ ). Rather than covering  $X$  by Weierstrass polydiscs, we will cover  $Y$ . More specifically, we will show that for every point  $p \in \Omega$  one can construct a Weierstrass polydisc for  $Y$  centered at  $p$  whose size and degree is bounded from below in terms of the Noetherian parameters. Suppose for simplicity that  $p$  is the origin.

We study the problem of constructing Weierstrass polydiscs for a holomorphic hypersurface  $\{F = 0\}$ . After rescaling we may suppose that  $F$  is defined on the unit disc and has maximum norm 1 there. Suppose that we can find a complex line  $L_w$  through the origin, and a circle  $S$  of radius  $r$  around the origin in  $L$  such that,

- (1)  $r$  is bounded from below in terms of the Noetherian parameters,
- (2)  $|F(w)|$  for  $w \in S$  is bounded from below by some quantity  $\delta$  depending on the Noetherian parameters.



Then a simple argument shows that  $\Delta_z \times \{|w| \leq r\}$  is a Weierstrass polydisc for  $\{F = 0\}$ , where  $\Delta_z$  is a polydisc of radius  $\sim \delta$  orthogonal to  $L$ . Moreover, the degree of  $Y$  in this Weierstrass polydisc is given by the number of zeros of  $F$  in  $\{0\} \times \{|w| \leq r\}$ .

To construct  $S$  as above we use the notion of the *Bernstein index* (see Definitions 15 and 20). For the purposes of this introduction, if  $F$  is a function of one complex variable in a disc  $D$  then one may consider its Bernstein index given by  $\log(M/m)$  where  $M$  is the maximum of  $F$  on  $D$  and  $m$  is the maximum on the 2-shrinking  $D^2$ . For functions of several variables we take the maximum over all complex lines through the origin of the Bernstein index of the restriction. In Proposition 21 we reduce the problem of finding the circle  $S$  above to the estimation of the Bernstein index. Moreover, the Bernstein index is also known to give an upper bound for the number of zeros of  $F$  in  $D$  (see Theorem 3), thus allowing us to estimate the degree of the Weierstrass polydisc.

Much is known about the estimation of Bernstein indices for solutions of scalar linear differential equations (of any order) due to work of [IS96, NY97]. Moreover, these results can be extended to solutions of polynomial (non-linear) differential equations by using the methods of [NY99]. A combination of these tools suffices for producing an estimate for the Bernstein index of a Noetherian function in terms of the Noetherian parameters (see Theorem 6), and allows us to finish the proof for this case.

*1.4.4 Higher codimensions.* We now discuss the inductive step of the proof with  $W$  of arbitrary dimension. We again note that we may as well replace  $X$  by the larger set  $Y := W \cap \{F = 0\}$ , where  $F$  is one of the Noetherian functions defining  $X$  which is not identically vanishing on  $W$ . We will again seek to construct a Weierstrass polydisc for  $Y$  around the origin.

We start by constructing a Weierstrass polydisc  $\Delta = \Delta_z \times \Delta_w$  for  $W$  (see Theorem 8): since  $W$  is algebraic  $\Delta$  can be constructed using the methods of [BN17b]. Then  $W$  is a ramified cover of  $\Delta_z$ , and the projection  $Y_F$  of  $Y$  to  $\Delta_z$  is given by the zero locus of an analytic function  $\mathcal{R}_F$  which we call an *analytic resultant* (77): for  $z \in \Delta_z$  it is given by the product of  $F(z, w)$  over the different branches of  $W$  over  $z$ .

If we can construct a Weierstrass polydisc  $\Delta'$  for  $Y_F$  in  $\Delta_z$  then a simple topological argument shows that  $\Delta' \times \Delta_w$  is a Weierstrass polydisc for  $Y$  (see Lemma 22), thus concluding our construction. Since  $Y_F$  is a hypersurface in  $\Delta_z$  we are essentially reduced back to the situation already considered, except that now we must estimate the Bernstein index of the analytic resultant  $\mathcal{R}_F$  instead of  $F$  itself. This translates to choosing a complex line  $L$  in  $\Delta_z$ , and then studying the restriction of  $F$  to the algebraic curve  $C$  obtained by lifting  $L$  through the ramified cover back to  $W$ .

A natural approach for studying the restriction of  $F$  to an algebraic curve  $C$  is to parameterize  $C$  using a map  $\gamma : \mathbb{C} \rightarrow C$  which itself satisfies a differential equation, and then replace  $F|_C$  by  $F \circ \gamma$ , now defined on  $\mathbb{C}$  and satisfying a system of auxiliary differential equations obtained by composing the equations of  $F$  and  $\gamma$ . There are two

primary obstacles to this idea: first, the curve  $C$  need not be smooth, and one must somehow handle the singular points; and second, even if  $C$  is smooth, it is not clear how to write a differential equation for the parameterization of  $C$  whose Noetherian parameters depend only of the degree of  $C$ . One natural option, for instance for a plane curve  $C = \{P(x, y) = 0\}$ , would be to parameterize  $C$  as a trajectory of the Hamiltonian field  $P_y \frac{\partial}{\partial x} - P_x \frac{\partial}{\partial y}$ . However, this produces Noetherian sizes tending to infinity along degenerating families of algebraic curves such as  $C_\varepsilon := \{y^2 + \varepsilon x = 0\}$ .

To overcome this problem we appeal to the theory of linear scalar differential equations. More specifically, for every algebraic function  $y(x)$  of degree  $d$  one can construct a scalar differential operator

$$L = a_0(t)\partial_t^k + \cdots + a_k(t)y, \quad a_0, \dots, a_k \in \mathbb{C}[t], \quad a_0 \neq 0 \quad (13)$$

satisfying  $Ly = 0$ . Moreover we show that the *slope*

$$\angle L := \max_{i=1, \dots, k} \frac{\|a_i\|_\infty}{\|a_0\|_\infty} \quad (14)$$

can be uniformly bounded in terms of  $d$ . This is a consequence of a much more general phenomenon of “uniform boundedness of slopes in regular families” which was discovered in the work of Grigoriev [Gri01, Yak06]. For instance, for the family  $C_\varepsilon$  above the operator  $Ly = x\partial_x y - \frac{1}{2}y$  provides a differential equation for  $y(x)$ , uniform in  $\varepsilon$ . The boundedness of the slope translates into a bound on the Noetherian size of the auxiliary system constructed from  $F \circ \gamma$ . We then use various analytic properties of the Bernstein index (notably subadditivity under products, see Lemma 19) to deduce estimates for the Bernstein indices of  $\mathcal{R}_F$  and finish the proof.

**REMARK 2.** We remark that the non-degenerating differential equations for parameterizations of algebraic curves are in some abstract sense similar to the  $C^r$ -parameterizations of Yomdin-Gromov [Yom87a, Yom87b, Gro87] that are used (in a form generalized to o-minimal structures) in the original work of Pila–Wilkie [PW06]. They are used in our proof to avoid the same type of problem.

**1.5 Contents of this paper.** In Sect. 2 we introduce the notion of complex Noetherian functions; prove closure of the Noetherian functions under various operations including multiplicative inverse, composition and compositional inverse, and forming of implicit functions; discuss Noetherian systems with poles and Noetherian systems over smooth varieties in place of  $\mathbb{C}^n$ ; and prove a statement about the behavior of a Noetherian function near the boundary of  $\Omega$ .

In Sect. 3 we develop a large number of examples of Noetherian functions: Klein’s  $j$ -invariant and other modular functions; universal covers of compact Riemann surfaces; elliptic and abelian functions, Jacobi theta functions and  $\wp(z; \tau)$  (with respect to both variables); periods over algebraic integrals over smooth families; and for the uniformizing map of the Siegel modular variety  $\mathcal{A}_g$ .

In Sect. 4 we develop the general analytic theory of Weierstrass polydiscs: we define the Bernstein index and recall its basic properties; show how estimates on the Bernstein index of a function  $F$  can be used to effectively construct a Weierstrass polydisc for its zero locus; introduce the notion of an analytic resultant, and show how it can be used to inductively construct a Weierstrass polydisc for a set  $X \cap \{F = 0\}$  from the Weierstrass polydisc of  $X$ .

In Sect. 5 we introduce the relevant background information on linear differential equations for algebraic functions and the boundedness of their slope; state an estimate for the Bernstein index of a function satisfying a polynomial (non-linear) differential equation in terms of the Noetherian parameters; study the restriction of a Noetherian function to an algebraic curve by parameterization using linear differential equations; and prove the key estimates on the Bernstein indices of analytic resultants of Noetherian functions with respect to algebraic curves.

Finally in Sect. 6 we recall the relation between Weierstrass polydiscs and the study of rational (and more generally algebraic) points on an analytic set; prove a complex-analytic analog of Theorem 2 for complex Noetherian varieties; and reduce the general case of Theorem 2 to its complex version.

## 2 The Noetherian Class

In this section we develop some elementary properties of the class of Noetherian functions. We begin by introducing the complex analog of the real Noetherian functions, which will be the main class considered throughout the paper.

**2.1 Complex Noetherian functions.** Let  $\Omega \subset \mathbb{C}^n$  be a bounded domain. Then a system of the form (1) where  $P_{ij}$  are now allowed to be complex polynomials is called a (complex) Noetherian system; its holomorphic solution  $\phi : \Omega \rightarrow \mathbb{C}^\ell$  is called a (complex) Noetherian chain; a function of the form  $P(\mathbf{x}, \phi)$  where  $P \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$  is called a (complex) Noetherian function; and the common zero locus in  $\Omega$  of a collection of Noetherian functions is called a (complex) Noetherian variety. We use the convention that, unless the prefix real- is explicitly used, all Noetherian constructs are assumed to be complex.

A real Noetherian system (1) may be viewed as a complex Noetherian system. Any Noetherian chain  $\phi : \Omega_{\mathbb{R}} \rightarrow \mathbb{R}^\ell$  extends as a holomorphic function to some complex domain  $\Omega_{\mathbb{R}} \subset \Omega \subset \mathbb{C}^n$ . In fact, the size of the domain to which this complex continuation is possible can be explicitly estimated from below in terms of the Noetherian parameters, see Lemma 11. There is therefore little harm in considering a real Noetherian chain as the restriction to the reals of a complex Noetherian chain.

Conversely, under the identification of  $\mathbb{C}^n$  with  $\mathbb{R}^{2n}$  every complex Noetherian  $\phi$  chain of dimension  $n$  and length  $\ell$  becomes a real Noetherian chain  $\phi_{\mathbb{R}} = (\operatorname{Re} \phi, \operatorname{Im} \phi)$  of dimension  $2n$  and length  $2\ell$ . Indeed, the system (1) provides derivation rules for  $\phi_{\mathbb{R}}$  with respect to  $\frac{\partial}{\partial z_j}$ , whereas the Cauchy-Riemann equations provides derivations rules with respect to  $\frac{\partial}{\partial \bar{z}_j}$ . We leave the detailed derivation to the reader.

In conclusion, we see that in the Noetherian category the real and complex settings are in some sense mutually-reducible. In the present paper we will employ essentially complex arguments to the study of Noetherian functions and assume unless otherwise stated that all Noetherian functions are complex. The equivalence above will imply that this causes no loss of generality. We remark that this situation stands in stark contrast to the theory of Pfaffian functions, which is an essentially real theory: the holomorphic continuation of a Pfaffian function defined on  $\mathbb{R}^n$  need not itself be Pfaffian when considered as function on  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ . The key difference is that the Cauchy-Riemann equations, while algebraic, do not satisfy the triangularity condition required of Pfaffian chains. It is this added generality of the Noetherian class that allows our complex-analytic treatment to go through in full generality.

not really  
a proof that  
this cannot  
be done

**2.2 Closure properties.** We begin by noting that a union of Noetherian chains is itself Noetherian.

LEMMA 3. *Let  $\phi, \tilde{\phi} : \Omega \rightarrow \mathbb{C}$  be two Noetherian chains of complexity  $(n, \ell, \alpha)$  and  $(n, \tilde{\ell}, \tilde{\alpha})$  respectively. Then  $(\phi, \tilde{\phi})$  is a Noetherian chain of complexity  $(n, \ell + \tilde{\ell}, \max(\alpha, \tilde{\alpha}))$ . Moreover,  $\mathcal{S}(\phi, \tilde{\phi})$  is the maximum of  $\mathcal{S}(\phi)$  and  $\mathcal{S}(\tilde{\phi})$ .*

The Noetherian class is clearly closed under differentiation.

LEMMA 4. *Let  $\phi : \Omega \rightarrow \mathbb{C}$  be a Noetherian chain with complexity  $(n, \ell, \alpha)$  and let  $F : \Omega \rightarrow \mathbb{C}$  be a Noetherian function of degree  $\beta$  over  $\phi$ . Then for  $j = 1, \dots, n$  the derivative  $\frac{\partial F}{\partial x_j}$  is a Noetherian function of degree  $\beta + \alpha - 1$ .*

**2.2.1 Closure under arithmetic operations.** Next we summarize closure properties under the basic arithmetic operations. In light of Lemma 3 there is no harm in assuming all functions involved share one Noetherian chain  $\phi$  of complexity  $(n, \ell, \alpha)$ .

LEMMA 5. *Let  $F_1, F_2$  be two Noetherian functions over the chain  $\phi$  with degrees  $\beta_1, \beta_2$ . Then  $F + G$  (resp.  $F \cdot G$ ) is a Noetherian function over  $\phi$  with degree  $\max(\beta_1, \beta_2)$  (resp.  $\beta_1 + \beta_2$ ).*

LEMMA 6. *Let  $F = P(\mathbf{x}, \phi) : \Omega \rightarrow \mathbb{C}$  be a Noetherian function of degree  $\beta$ , and suppose that  $|F(\mathbf{x})| \geq \varepsilon > 0$  for all  $\mathbf{x} \in \Omega$ . Then  $1/F$  is a Noetherian function of degree 1 with respect to a Noetherian chain  $\tilde{\phi}$  with complexity  $(n, \ell + 1, \alpha + \beta + 1)$  and  $\mathcal{S}(\tilde{\phi})$  is explicitly computable from  $1/\varepsilon$  and the Noetherian parameters.*

A similar statement holds for the inverse of a matrix of Noetherian functions, where  $\varepsilon$  is now a lower bound for the modulus of the determinant.

*Proof.* We let  $\tilde{\phi} = (\phi, 1/F)$ . To make this into a Noetherian chain we introduce differential equations for  $1/F$ ,

$$\frac{\partial(1/F)}{\partial x_j} = -(1/F)^2 \frac{\partial F}{\partial x_j} = -(1/F)^2 \left( \frac{\partial P}{\partial x_j}(\mathbf{x}, \phi) + \sum_{k=1}^{\ell} \frac{\partial P}{\partial \phi_k}(\mathbf{x}, \phi) \frac{\partial \phi_k}{\partial x_j} \right) \quad (15)$$

and note that the right hand side is a polynomial in  $\mathbf{x}, \tilde{\phi}$  of degree  $\alpha + \beta + 1$ . An upper bound for  $\mathcal{S}(\tilde{\phi})$  follows by a simple estimate.

For the second statement it suffices to write  $A^{-1} = (\det A)^{-1} \operatorname{adj} A$  which reduces the claim to the first statement.  $\square$

2.2.2 Closure under compositions and compositional inverse. Next we consider closure under composition and compositional inverse.

LEMMA 7. *For  $i = 1, 2$  let  $\phi_i : \Omega_i \rightarrow \mathbb{C}$  be a Noetherian chain with complexity  $(n_i, \ell_i, \alpha_i)$ . Let  $\mathbf{F} = (F_1, \dots, F_{n_2}) : \Omega_1 \rightarrow \Omega_2$  be a tuple of Noetherian functions of degree at most  $\beta_1$  over  $\phi_1$ , and  $G : \Omega_2 \rightarrow \mathbb{C}$  be a Noetherian function of degree  $\beta_2$  over  $\phi_2$ . Then  $G \circ F : \Omega_1 \rightarrow \mathbb{C}$  is a Noetherian function of degree  $\beta_2$  over a chain  $\tilde{\phi}$  with complexity  $(n_1, \ell_1 + \ell_2, \max(\alpha_1 + \beta_1, \alpha_2))$  and  $\mathcal{S}(\tilde{\phi})$  explicitly computable from the Noetherian parameters.*

*Proof.* We let  $\tilde{\phi} = (\phi_1, \phi_2 \circ \mathbf{F})$ . To make this into a Noetherian chain we use the given equations for  $\phi_1$ , and the chain rule for the derivatives of  $\phi_2 \circ \mathbf{F}$ , which gives a polynomial combination of the components of  $\phi_2 \circ F$  and the derivatives of  $\mathbf{F}$ , both of which are expressible as polynomials in  $\tilde{\phi}$ . We leave the details for the reader.  $\square$

LEMMA 8. *Let  $\phi : \Omega \rightarrow \mathbb{C}$  be a Noetherian chain and let  $\mathbf{F} = (F_1, \dots, F_n) : \Omega \rightarrow \tilde{\Omega}$  be a tuple of Noetherian functions. Suppose that  $F$  is bijective and that  $|\det \frac{\partial \mathbf{F}}{\partial \mathbf{x}}| \geq \varepsilon > 0$  for all  $\mathbf{x} \in \Omega$ . Then the compositional inverse  $F^{-1}$  is a tuple of Noetherian functions over a Noetherian chain  $\tilde{\phi} : \tilde{\Omega} \rightarrow \mathbb{C}$ , with the Noetherian parameters explicitly computable in terms of the Noetherian parameters of  $\phi, \mathbf{F}$  and  $\varepsilon^{-1}$ .*

*Proof.* For simplicity of the notation we assume that the tuple  $\phi$  contains all the coordinate functions  $x_j$ . Denote the variables on  $\tilde{\Omega}$  by  $\mathbf{y}$ . Then by the chain rule,

$$\frac{\partial \mathbf{F}^{-1}}{\partial \mathbf{y}} = \left( \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \circ \mathbf{F}^{-1} \right)^{-1} = \left[ \det^{-1} \left( \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right) \operatorname{adj} \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right] \circ \mathbf{F}^{-1} \quad (16)$$

where  $\operatorname{adj}$  denotes the adjugate matrix. Note that by Noetherianity of  $\mathbf{F}$ , the right hand side of (16) is a polynomial in  $(\phi, \det^{-1} \frac{\partial \mathbf{F}}{\partial \mathbf{x}}) \circ \mathbf{F}^{-1}$ .

We will show that  $\tilde{\phi} := (\phi, \det^{-1} \frac{\partial \mathbf{F}}{\partial \mathbf{x}}) \circ \mathbf{F}^{-1} : \tilde{\Omega} \rightarrow \mathbb{C}^{\ell+1}$  forms a Noetherian chain. By our assumption on  $\phi$  it will follow in particular that each component of  $\mathbf{F}^{-1}$  is a Noetherian function of degree 1 with respect to this chain. To write equations for  $\phi \circ \mathbf{F}^{-1}$  we write

$$\frac{\partial(\phi \circ \mathbf{F}^{-1})}{\partial \mathbf{y}} = \left( \frac{\partial \phi}{\partial \mathbf{x}} \circ \mathbf{F}^{-1} \right) \cdot \frac{\partial \mathbf{F}^{-1}}{\partial \mathbf{y}} \quad (17)$$

and note that the first factor is a polynomial in  $\tilde{\phi}$  by Noetherianity of  $\phi$  while the second factor is a polynomial in  $\tilde{\phi}$  by the note following (16). Finally, to write equations for  $(\det^{-1} \frac{\partial \mathbf{F}}{\partial \mathbf{x}}) \circ \mathbf{F}^{-1}$  we proceed as in Lemma 6, noting that the derivatives of  $(\det \frac{\partial \mathbf{F}}{\partial \mathbf{x}}) \circ \mathbf{F}^{-1}$  with respect to  $\mathbf{y}$  are expressible as polynomials in  $\tilde{\phi}$  by what was already shown since it is a polynomial in  $\phi \circ \mathbf{F}^{-1}$ .

It is clear that the construction is entirely explicit and the noetherian parameters of  $\tilde{\phi}$  can be explicitly estimated from those of  $\phi$  and  $\mathbf{F}$  and  $\varepsilon^{-1}$  (which enters into the size  $\mathcal{S}(\tilde{\phi})$  as in Lemma 6).  $\square$

As a simple corollary we deduce that the Noetherian class is closed under forming implicit functions.

*implicit function theorem*

- COROLLARY 9. Let  $\Omega_x \subset \mathbb{C}_x^n, \Omega_y \subset \mathbb{C}_y^m$  be relatively compact domains and let  $\underline{F} : \Omega_x \times \Omega_y \rightarrow \mathbb{C}^m$  be a tuple of Noetherian functions such that  $\left| \det \frac{\partial \underline{F}}{\partial \mathbf{y}} \right| > \varepsilon > 0$  on the set  $\mathcal{G} = \underline{F}^{-1}(0)$ . If  $\mathcal{G}$  is the graph of a function  $G : \Omega_x \rightarrow \Omega_y$  then  $G$  is Noetherian with parameters explicitly computable from the Noetherian parameters of  $\underline{F}$  and  $\varepsilon$ .

$$F(x, y) = 0$$

*Proof.* By assumption the map  $H : (\mathbf{x}, \mathbf{y}) \rightarrow (\mathbf{x}, F)$  satisfies  $\left| \det \frac{\partial H}{\partial (\mathbf{x}, \mathbf{y})} \right| > \varepsilon$  on  $\mathcal{G}$ . We choose a neighborhood  $\Omega \subset \Omega_x \times \Omega_y$  of  $\mathcal{G}$  such that  $H$  is injective on  $\Omega$ : it is enough to ensure that  $H(\mathbf{x}, \mathbf{y})$  is injective in  $\mathbf{y}$  for fixed  $\mathbf{x}$ , and  $\Omega$  with this property can be chosen by the inverse mapping theorem (here we use that  $\mathcal{G}$  has only one point with a given  $\mathbf{x}$  coordinate). By Lemma 8 the inverse  $H^{-1} : F(\Omega) \rightarrow \Omega$  is Noetherian with computable parameters as above. Finally note that  $\Omega_x \times \{0\} = H(\mathcal{G}) \subset H(\Omega)$  and  $H^{-1}(\mathbf{x}, 0) : \Omega_x \rightarrow \Omega$  is a Noetherian function of the form  $H^{-1}(\mathbf{x}, 0) = (\mathbf{x}, G(\mathbf{x}))$ , with  $G$  satisfying the conditions of the statement.  $\square$

**2.3 Systems with poles.** In our definition of a Noetherian system (1) we assume that the derivatives of  $\phi$  are expressible as *polynomials* in  $\mathbf{x}, \phi$ . In general one may also consider rational systems

$$\frac{\partial \phi_i}{\partial x_j} = \frac{Q_{i,j}(\mathbf{x}, \phi)}{R_{i,j}(\mathbf{x}, \phi)}, \quad \begin{array}{l} i = 1, \dots, \ell, \\ j = 1, \dots, n, \end{array} \quad (18)$$

but in this case a more careful notion of “Noetherian size” is required to make our main results hold.

EXAMPLE 10. Consider the following Noetherian system in independent variable  $x$  and dependent variables  $(e, f, g)$ ,

$$\frac{\partial e}{\partial x} = 0, \quad \frac{\partial f}{\partial x} = g/e, \quad \frac{\partial g}{\partial x} = -f/e. \quad (19)$$

For every fixed value of  $0 < \varepsilon < 1$ , the tuple  $(\varepsilon, \sin(x/\varepsilon), \cos(x/\varepsilon))$  in the domain  $\Omega = \{0 < x < 1\}$  forms a Noetherian chain for this system. However, the number of rational points of height  $H$  in the set  $\Omega \cap \{\sin(x/\varepsilon) = 0\}$  for  $\varepsilon = 1/(\pi H)$  is  $H$ , so clearly our notion of Noetherian size for such systems must tend to infinity as  $\varepsilon \rightarrow 0$  if we are to expect a sub-polynomial asymptotic for the number of rational points, with the constant depending only on the Noetherian size.

In this paper we will only consider the case where the Noetherian chain  $\phi$  remain bounded away from the polar locus. In this case one may, by a simple reduction, translate (18) back into polynomial form as follows. We introduce additional dependent variables  $\rho_{i,j}$  for  $\frac{1}{R_{i,j}}$  and recast (18) in the form

$$\begin{array}{l} \frac{\partial \phi_i}{\partial x_j} = \rho_{i,j} Q_{i,j}(\mathbf{x}, \phi) \\ \frac{\partial \rho_{i,j}}{\partial x_k} = -\rho_{i,j}^2 \frac{\partial R_{i,j}(\mathbf{x}, \phi)}{\partial x_k} \end{array} \quad \text{for} \quad \begin{array}{l} i = 1, \dots, \ell, \\ j, k = 1, \dots, n, \end{array} \quad (20)$$

*invertible as a power series*



where we express  $\frac{\partial R_{i,j}(\mathbf{x}, \phi)}{\partial x_k}$  as a polynomial in  $\mathbf{x}, \phi, \rho_{i,j}$  using the derivation rule (18), replacing all occurrences of  $\frac{1}{R_{i,j}}$  by  $\rho_{i,j}$ . It is then simple to see that for any solution  $\phi$  of (18),  $(\phi, \rho)$  forms a solution of the system above, and moreover that the Noetherian size of this solution can be explicitly estimated in terms of the Noetherian size of  $\phi$  and the minimum value attained by  $|R_{i,j}(\mathbf{x}, \phi)|$ , i.e. the “distance” to the polar locus of (18).

**2.4 Systems over smooth algebraic varieties.** Let  $S \subset \mathbb{C}^N$  be an irreducible smooth algebraic variety of dimension  $n$ . One can generalize the notion of a Noetherian system on  $\mathbb{C}^n$  to Noetherian systems over  $S$ . For this purpose it is more convenient to use differential form notation. Let  $T^*S$  denote the cotangent bundle of  $S$ . We denote by  $\Omega_S^1$  the (algebraic) sheaf of sections of this bundle. If  $U \subset S$  is a (possibly non-algebraic) domain then we will denote by  $\Omega_S^1(U)$  the sections admitting a regular extension to some Zariski open neighborhood of  $U$ . We denote by

$$\Omega_S^1[z_1, \dots, z_k] := \Omega_S^1 \otimes \mathbb{C}[z_1, \dots, z_k] \quad (21)$$

the sheaf of sections of  $T^*S$  depending polynomially on additional variables  $z_1, \dots, z_k$ . We will use similar notations with  $\mathcal{O}_S$  for the structure sheaf of  $S$ .

Let  $U \subset S$  be a relatively compact domain. A collection of holomorphic functions  $\phi := (\phi_1, \dots, \phi_\ell) : U \rightarrow \mathbb{C}^\ell$  is called a Noetherian chain if it satisfies a *Noetherian system over  $S$* ,

$$d\phi = \omega, \quad \omega \in \mathbb{C}^\ell \otimes \Omega_S^1[\phi](U) \quad (22)$$

where  $\omega$  is a vector of one-forms depending polynomially on  $\phi$ . A function from the ring  $\mathcal{O}_S[\phi](U)$  is called a Noetherian function over  $S$ .

Much of the material developed in this paper could be generalized to Noetherian systems over a smooth variety. However, this would introduce additional (mostly notational) difficulties that we prefer to avoid in the interest of readability. Instead we show that one can form local charts on  $S$  where the system (22) pulls back to a standard Noetherian system on a subset of  $\mathbb{C}^n$ . More formally let  $s_0 \in S$ . Since  $S \subset \mathbb{C}^N$  is smooth, one can choose  $N - n$  polynomials vanishing on  $S$  with linearly independent differentials in a neighborhood of  $s_0$ . Let  $\pi : S \rightarrow \mathbb{C}^n$  be a linear projection which is submersive at  $s_0$ . By Corollary 9 applied to the collection of polynomials above, one may construct a biholomorphic Noetherian map  $\Psi = (\psi_1, \dots, \psi_N) : U_0 \rightarrow S$  for some domain  $U_0 \subset \mathbb{C}^n$  whose image contains  $s_0$ , and whose inverse is given by  $\pi$ . Since  $\pi$  is a linear projection, we see that  $\Psi^{-1}$  is Noetherian over  $S$ .

Represent the Noetherian system for  $\Psi$  in differential form

$$d\Psi = \eta(\Psi), \quad \eta \in \mathbb{C}^N \otimes \Omega_{\mathbb{C}^n}^1[\Psi](U_0). \quad (23)$$

Since  $S$  is smooth, among the one-forms  $dx_1, \dots, dx_N \in (\mathbb{C}^N)^*$  there are  $n$  forms whose restrictions to  $S$  form a basis for  $T_S^*$  over  $s_0$ . Expressing  $\omega$  in terms of this basis, we see that after possibly restricting to a Zariski open neighborhood  $\tilde{U}$  of  $s_0$



we may write  $\omega$  as the restriction to  $\tilde{U}$  of an element of  $\mathbb{C}^\ell \otimes (\mathbb{C}^N)^* \otimes \mathcal{O}_S[\phi](\tilde{U})$ . Similarly, since any function in  $\mathcal{O}_S(\tilde{U})$  is the restriction of some rational function on  $\mathbb{C}^N$  with polar locus disjoint from  $\tilde{U}$  we may take  $\omega$  to be the restriction to  $\tilde{U}$  of  $\tilde{\omega} \in \mathbb{C}^\ell \otimes (\mathbb{C}^N)^* \otimes \mathbb{C}[\mathbf{x}, \phi]_f$  where the localization is by some polynomial  $f \in \mathbb{C}[\mathbf{x}]$  non-vanishing in  $\tilde{U}$ . Shrinking  $U_0$  we may also assume that  $\Psi(U_0) \subset \tilde{U}$ .

We claim that the tuple  $(\Psi, \phi \circ \Psi) : U_0 \rightarrow \mathbb{C}^{N+\ell}$  containing the pullbacks of  $\phi$  by  $\Psi$  is a Noetherian chain. The Eq. (23) is already in Noetherian form, and we proceed to derive Noetherian equations for  $\phi \circ \Psi$ . From (22) and (23) we have

$$\begin{aligned} d(\phi \circ \Psi) &= (d\phi)_\Psi \cdot d\Psi = \tilde{\omega}(\phi \circ \Psi) \cdot \eta(\Psi) \\ &\in \mathbb{C}^\ell \otimes \Omega_{\mathbb{C}^n}^1(U_0) \otimes \mathbb{C}[\phi \circ \Psi, \Psi]_{f(\Psi)} \end{aligned} \quad (24)$$

where we used the pairing of  $\mathbb{C}^N$  and  $(\mathbb{C}^N)^*$ . Since  $f$  is non-vanishing on  $\Psi(U_0)$  by assumption we may view this as a Noetherian system with poles by the construction of Sect. 2.3. Finally, any Noetherian function  $F \in \mathcal{O}_S[\phi](U)$  may be expressed as an element of  $\mathbb{C}[\mathbf{x}, \phi]_g$  as explained above, and its pullback  $F \circ \Psi \in \mathbb{C}[\Psi, \phi \circ \Psi]_{g(\Psi)}$  is Noetherian over  $\Psi, \phi \circ \Psi$  (with poles at the zeros of  $g(\Psi)$ ).

**2.5 Boundary behavior.** We finish this section with a simple lemma showing that the behavior of the Noetherian chain in a neighborhood of the boundary of a domain can be controlled in terms of the Noetherian size.

**LEMMA 11.** *Let  $\phi : \Omega \rightarrow \mathbb{C}^n$  be a (possibly real) Noetherian chain (so that we allow  $\Omega$  to be a bounded domain in  $\mathbb{R}^n$  as well) and set  $S := \mathcal{S}(\phi)$ . Then  $\phi$  extends as a complex Noetherian chain on the complex  $\rho$ -neighborhood of  $\Omega$ , where  $\rho = S^{-O(1)}$  and the Noetherian size of  $\phi$  in this larger domain is bounded by  $2S$ .*

*Proof.* It is a simple exercise to verify using (1) that for  $t_0 \in \Omega$  we have

$$\left| \frac{\partial^{\mathbf{j}} \phi_i}{\partial \mathbf{x}^{\mathbf{j}}} (t_0) \right| \leq \mathbf{j}! S^{O(|\mathbf{j}|)} \quad (25)$$

where the asymptotic constants depend on  $n, l, \alpha$ . From this it follows that  $\phi$  extends holomorphically to a  $\rho$ -neighborhood of  $\Omega$  with  $\rho = S^{-O(1)}$ , and it obviously continues to satisfy (1) in this larger domain. It is also clear that one may choose  $\rho$  as above such that the Noetherian size of  $\phi$  in this larger domain is bounded by  $2S$ .  $\square$

### 3 Examples of Noetherian functions.

The elementary functions  $\underline{e^z}$ ,  $\underline{\sin z}$  and  $\underline{\cos z}$  (restricted to any relatively compact domain in  $\mathbb{C}$ ) form classical examples of Noetherian functions. We proceed with some less trivial examples. In this section we shall freely use Noetherian systems with poles as developed in Sect. 2.3.

**3.1 Klein's  $j$ -invariant and other modular functions.** The Klein  $j$ -invariant is the unique function  $j : \mathbb{H} \rightarrow \mathbb{C}$  which is  $\mathrm{SL}(2, \mathbb{Z})$ -invariant, holomorphic in  $\mathbb{H}$ , has a simple pole at the cusp, and satisfies  $j(e^{2\pi i/3}) = 0$  and  $j(i) = 1728$ . We will denote the coordinate on  $\mathbb{H}$  by  $\tau$  and on  $\mathbb{C}$  by  $z$ .

The  $j$  function realizes the identification  $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H} \simeq \mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$ , but note that it is not a covering map: it is ramified over the points  $0, 1728$  corresponding to the orbits of  $e^{2\pi i/3}, i$  (which have a non-trivial stabilizer in  $\mathrm{SL}(2, \mathbb{Z})$ ). The  $j$ -function is known to satisfy a differential equation of order 3. To define it, recall that the Schwarzian derivative  $S_z(f)$  and automorphic derivative  $D_z(f)$  of a function  $f$  are defined by

$$\bullet \quad S_z(f) := \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2, \quad \bullet \quad D_z(f) := S_z(f)/(f')^2. \quad (26)$$

If  $g \in \mathrm{Aut} \mathbb{CP}^1$  is a Möbius transformation and  $f$  is a holomorphic function on some domain in  $\mathbb{CP}^1$  then

$$S_z(g \circ f) = S_z(f), \quad D_z(f \circ g) = D_z(f) \circ g, \quad (27)$$

wherever both sides are defined. In particular, since  $j$  is automorphic with respect to  $\mathrm{SL}(2, \mathbb{Z})$  it follows that  $D_\tau j$  is as well. Since  $j$  is a Hauptmodul for  $\mathrm{SL}(2, \mathbb{Z})$  it follows that  $D_\tau j$  is a rational function of  $j$ , and more explicitly [Mas03, page 20]

$$D_\tau j = \frac{j^2 - 1968j + 2645208}{2j^2(j - 1728)^2}. \quad (28)$$

We now cast (28) as a Noetherian system in the free variable  $\tau$  and dependent variables  $j, j_1, j_2$ ,

$$\begin{aligned} \frac{\partial j}{\partial \tau} &= j_1, \quad \frac{\partial j_1}{\partial \tau} = j_2, \\ \frac{\partial j_2}{\partial \tau} &= \frac{3j_2^2}{2j_1} + \frac{j^2 - 1968j + 2645208}{2j^2(j - 1728)^2} j_1^3 \end{aligned} \quad (29)$$

does not have a power series expansion at 0.

It is straightforward to verify using (28) that  $\phi_j$  given by

$$j = j(\tau), \quad j_1 = j'(\tau), \quad j_2 = j''(\tau), \quad (30)$$

forms a solution of (29). If  $\Omega \subset \mathbb{H} \setminus \mathrm{SL}(2, \mathbb{Z}) \cdot \{e^{2\pi i/3}, i\}$  is relatively compact then (29) is defined in  $\Omega$  since  $j'$  and  $j(j - 1728)$  only vanish on the orbits of  $e^{2\pi i/3}, i$ . The singularities of our equations over the orbits of  $e^{2\pi i/3}, i$  correspond to the fact that  $j$  is ramified at these points. In fact the methods developed in this paper may be extended to cover such “mild” singularities, but in the interest of clarity we avoid this extra generality. For “reasonable” domains  $\Omega$  it should also be possible to estimate  $\mathcal{S}(\phi_j)$  explicitly, but we do not pursue this direction.

If instead of  $\mathrm{SL}(2, \mathbb{Z})$  one considers for instance its modular subgroup  $\Gamma(2)$  then the resulting map  $\lambda : \mathbb{H} \rightarrow \Gamma(2) \backslash \mathbb{H} \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$  is a covering map. By the same arguments as above, the  $\lambda$  function satisfies a differential equation which can

be transformed into a Noetherian system. Note that in this case, since the  $\lambda$  function is a non-ramified cover, the resulting solution  $\phi_\lambda$  is defined for any relatively compact domain  $\Omega \subset \mathbb{H}$ .

**3.2 Universal covers of compact Riemann surfaces.** Let  $C$  be a compact Riemann surface of genus greater than one. Then by the uniformization theorem there is a (uniquely defined up to a Möbius transformation) universal covering map  $f : \mathbb{H} \rightarrow \Gamma \backslash \mathbb{H} \simeq C$  where  $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$  is a discrete subgroup acting freely on  $\mathbb{H}$ . We fix an embedding  $C \rightarrow \mathbb{C}P^k$  of  $C$  as a smooth algebraic curve in projective space.

Let  $p \in C$ , and let  $\mathbf{x} = (x_1, \dots, x_k) : \mathbb{C}P^k \rightarrow \mathbb{C}^k$  denote an affine chart on  $\mathbb{C}P^k$ . We may and do assume that  $p$  is in the finite part of  $\mathbf{x}$ , and moreover that  $dx_j|_C$  does not vanish at  $p$  for  $j = 1, \dots, k$ . Set  $f_j := x_j \circ f : \mathbb{H} \rightarrow \mathbb{C}$ . We will show that there exists a neighborhood  $p \in U_p \subset C$  such that  $f_j$  form Noetherian functions in any relatively compact domain  $\Omega_p \subset f^{-1}(U_p)$ . By compactness of  $C$  we can then study the universal cover  $f$  using finitely many such charts.

Since  $f_j$  is meromorphic and  $\Gamma$ -invariant, the same is true for  $D_\tau f_j$  which therefore defines a meromorphic function on  $\Gamma \backslash \mathbb{H} \simeq C$ . By the GAGA principle (or Chow's theorem) it follows that

$$D_\tau f_j = R_j \circ f \quad (31)$$

where  $R_j$  is a rational function on  $C$ . Moreover, using (26) it is easy to verify that  $R_j$  is regular at  $p$ : the poles of  $D_\tau f_j$  only occur at poles or zeros of  $f'_j(\tau) = dx_j(f'(\tau))$ , neither of which contains  $p$  by our assumption on  $\mathbf{x}$ . Therefore one can write  $R_j = P_j/Q_j$  where  $P_j, Q_j$  are polynomials in the coordinates  $\mathbf{x}$  and  $Q_j(p) \neq 0$ .

Fix a neighborhood  $p \in U_p \subset C$  where  $\mathbf{x}$  is finite,  $f'_1, \dots, f'_k$  are finite, and  $Q_1, \dots, Q_k$  are non-zero. Then we have a system

$$D_\tau f_j = \frac{P_j(f_1, \dots, f_k)}{Q_j(f_1, \dots, f_k)} \quad (32)$$

and moreover the  $f_1, \dots, f_k$  are bounded and the denominators  $Q_j(f_1, \dots, f_k)$  and the derivatives  $f'_j$  are bounded away from zero in any relatively compact domain  $\Omega_p \subset f^{-1}(U_p)$ . Then (32) may be viewed as a Noetherian system (with poles) and the conditions above guarantee that  $f_1, \dots, f_k$  correspond to a holomorphic solution of this Noetherian system in  $\Omega_p$ .

**3.3 Elliptic functions.** In Sect. 3.4 we show that all abelian functions satisfy Noetherian systems. However, since the case of dimension one (elliptic functions) is of classical importance we begin with a more explicit description of this case. Recall that the field of doubly-periodic meromorphic functions with periods  $\omega_1, \omega_2$  is generated by  $\wp, \wp'$  where

$$\wp(z) = \wp(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{n^2+m^2 \neq 0} \left[ \frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right]. \quad (33)$$

$$2\wp'' = 12\wp^2 - g_2 \quad 2\wp''' = 24\wp\wp' - g_3 \quad !$$

The  $\wp$ -function satisfies the differential equation

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3 \quad (34)$$

where  $g_2, g_3$  are certain invariants depending on  $\omega_1, \omega_2$ . Taking derivative, dividing by  $\wp'$  and taking another derivative we see that the  $\wp$  function satisfies the stationary Korteweg-de Vries equation

$$\bullet \quad \wp'''(z) = 12\wp\wp', \quad (35)$$

which is expressible as a Noetherian system in the dependent variables  $\wp, \wp_1, \wp_2$  as follows

$$\wp' = \wp_1, \quad \wp'_1 = \wp_2, \quad \wp'_2 = 12\wp\wp_1. \quad (36)$$

We remark that the Eqs. (36) do not depend on the invariants  $g_2, g_3$ , meaning that the Noetherian complexity does not diverge as the periods  $\omega_1, \omega_2$  degenerate.

**3.4 Abelian functions.** Recall that a *complex torus*  $X$  of dimension  $g$  is a quotient  $\mathbb{C}^g/\Lambda$  where  $\Lambda \subset \mathbb{C}^g$  is a lattice of rank  $2g$ . A complex torus  $A$  which is also a projective variety over  $\mathbb{C}$  is said to be an *abelian variety*. A meromorphic function on an abelian variety is called an *abelian function*. We fix an abelian variety  $A$  and an embedding  $A \rightarrow \mathbb{C}P^k$  of  $A$  as a smooth projective variety.

Let  $\pi : \mathbb{C}^g \rightarrow \mathbb{C}^g/\Lambda \simeq A$  denote the quotient map, and denote by  $\mathbf{z} = (z_1, \dots, z_g)$  the coordinates on  $\mathbb{C}^g$ . Let  $\mathbf{x} = (x_1, \dots, x_k) : \mathbb{C}P^k \rightarrow \mathbb{C}^k$  denote an affine chart on  $\mathbb{C}P^k$  and set  $f_j := x_j \circ \pi$ . We will show that  $f_1, \dots, f_k$  are Noetherian functions of the variables  $\mathbf{z}$  away from the polar locus. Since  $A$  is compact, one can then study the map  $\pi$  using finitely many affine charts  $\mathbf{x}$ . The proof is similar to the one given in Sect. 3.2.

For  $i = 1, \dots, k$  and  $j = 1, \dots, g$ , the derivative  $\frac{\partial f_i}{\partial z_j}$  is  $\Lambda$ -invariant (since  $f_j$  is), and it therefore defines a meromorphic function on  $\mathbb{C}^g/\Lambda \simeq A$ . By the GAGA principle (or Chow's theorem) it follows that

$$\frac{\partial f_i}{\partial z_j} = R_i \circ \pi \quad (37)$$

where  $R_i$  is a rational function on  $A$ . Moreover,  $R_i$  is regular on the finite part of the chart  $\mathbf{x}$ , and can therefore be expressed as a polynomial  $P$  in the  $\mathbf{x}$  variables. Thus we have a system of Noetherian equations

$$\bullet \quad \frac{\partial f_i}{\partial z_j} = P_i(f_1, \dots, f_k). \quad (38)$$

**3.5 Jacobi theta functions, thetanulls and  $\wp(z; \tau)$ .** The theta functions are a subject of a large number of inconsistent notational variations. We stick here to the conventions employed by [WW96, AS64]. We remark that in these sources the notation for the elliptic function  $\wp(z; \omega_1, \omega_2)$  varies slightly from the one used previously in Sect. 3.3: it denotes a function with periods  $2\omega_1, 2\omega_2$  rather than  $\omega_1, \omega_2$ . To avoid further complicating the references, we use this alternative normalization for this section. We write  $\wp(z; \tau) := \wp(z; 1, \tau)$ .

The Jacobi theta function is the holomorphic function given by [WW96, 21.1]

$$\vartheta : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}, \quad \vartheta(z; \tau) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2 \tau + 2i n z}. \quad (39)$$

We also write  $\vartheta_4$  for  $\vartheta$ . Three additional variants are [WW96, 21.11]

$$\begin{aligned} \vartheta_3(z; \tau) &:= \vartheta_4(z + \tfrac{1}{2}\pi; \tau), \\ \vartheta_1(z; \tau) &:= -ie^{iz + \frac{1}{4}\pi i \tau} \vartheta_4(z + \tfrac{1}{2}\pi\tau; \tau), \\ \vartheta_2(z; \tau) &:= \vartheta_1(z + \tfrac{1}{2}\pi; \tau). \end{aligned} \quad (40)$$

It is common to omit  $\tau$  from the notation, writing  $\vartheta(z)$  for  $\vartheta(z; \tau)$ . The theta functions  $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$  vanish at  $(z; \tau)$  if and only if  $z$  is congruent modulo  $\pi, \pi\tau$  to  $0, \frac{\pi}{2}, \frac{\pi}{2}(1 + \tau), \frac{\pi}{2}\tau$  respectively [WW96, 21.12]. Each of the theta functions satisfies a variant of the heat equation [WW96, 21.4]

$$\bullet \quad \delta \vartheta_j = -\tfrac{1}{4} \partial_z^2 \vartheta_j \quad (41)$$

where  $\delta := \frac{1}{\pi i} \frac{\partial}{\partial \tau}$  and  $\partial_z := \frac{\partial}{\partial z}$ .

We now restrict attention to the *thetanulls*, i.e. the functions  $\vartheta_j(0) := \vartheta_j(0; \tau)$  for  $j = 2, 3, 4$  (note  $\vartheta_1(0) \equiv 0$ ). From the description of the zeros of the theta functions above, we see that the thetanulls are nowhere vanishing on  $\mathbb{H}$ . The logarithmic derivatives  $\psi_j := \delta \vartheta_j(0) / \vartheta_j(0)$  satisfy a system of non-linear differential equations due to Halphen [Hal81],

$$\begin{aligned} \delta \psi_2 &= 2(\psi_2 \psi_3 + \psi_2 \psi_4 - \psi_3 \psi_4), \\ \delta \psi_3 &= 2(\psi_2 \psi_3 + \psi_3 \psi_4 - \psi_2 \psi_4), \\ \delta \psi_4 &= 2(\psi_2 \psi_4 + \psi_3 \psi_4 - \psi_2 \psi_3). \end{aligned} \quad (42)$$

It is easy to recast (42) as a rational Noetherian system for the thetanulls  $\vartheta_j(0)$  and their first derivatives  $\delta \vartheta_j(0)$ , for example

$$\delta^2 \vartheta_2(0) = 2\vartheta_2(0)(\psi_2 \psi_3 + \psi_2 \psi_4 - \psi_3 \psi_4) + (\delta \vartheta_2)^2 / \vartheta_2(0) \quad (43)$$

and similarly for  $j = 3, 4$ . Since the denominators of these system are given by the thetanulls which are nowhere vanishing in  $\mathbb{H}$ , we conclude that the thetanulls are Noetherian in every relatively compact subdomain of  $\mathbb{H}$ .

We now return to the general study of the theta functions and their relation to the Weierstrass elliptic functions. Recall that the Weierstrass  $\zeta$  function is defined by the conditions

$$\partial_z \zeta(z; \tau) = -\wp(z; \tau), \quad \lim_{z \rightarrow 0} \zeta(z; \tau) - z^{-1} = 0. \quad (44)$$

From (35) it follows that  $\zeta$  satisfies

$$\partial_z^4 \zeta = -12(\partial_z \zeta)(\partial_z^2 \zeta). \quad (45)$$

The  $\zeta$ -function can be expressed in terms of theta functions as follows [AS64, 18.10.7, 18.10.18]

$$\zeta(z; \tau) = \eta z + \partial_z(\log \vartheta_1(\frac{\pi z}{2}; \tau)), \quad \eta := -\frac{\pi^2}{12} \frac{\vartheta_1'''(0)}{\vartheta_1'(0)}. \quad (46)$$

We claim that  $\eta$  is a Noetherian function with respect to the system constructed above for the thetanulls. To see this recall that the thetanulls satisfy the fundamental relation [WW96, 21.41]

$$\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0). \quad (47)$$

Combining with (41) this gives

$$\eta = \frac{\pi^2}{3} \frac{\delta(\vartheta_2(0)\vartheta_3(0)\vartheta_4(0))}{\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)} \quad (48)$$

and the  $\delta$ -derivative can be expressed in terms of the Noetherian derivation rules.

We now construct a Noetherian system in the independent variables  $z, \tau$  including the functions  $\zeta(z; \tau)$  and  $\tilde{\vartheta}_1(z, \tau) := \vartheta_1(\pi z/2; \tau)$ . We start with the system constructed above for the thetanulls (by definition independent of the variable  $z$ ), and add five additional dependent variables  $\vartheta_1$  and  $\zeta = \zeta_0, \zeta_1, \zeta_2, \zeta_3$  for  $\zeta$  and its first three  $\partial_z$ -derivatives. For the  $z$  derivatives we have the following derivation rules

$$\begin{aligned} \partial_z \zeta_j &= \zeta_{j+1}, & j &= 0, 1, 2, \\ \partial_z \zeta_3 &= -12\zeta_1\zeta_2, \\ \partial_z \tilde{\vartheta}_1 &= (\zeta - \eta z)\tilde{\vartheta}_1 \end{aligned} \quad (49)$$

where the last two equations follow from (45) and (46) respectively. For the  $\tau$  derivatives we have

$$\begin{aligned} \partial_\tau \tilde{\vartheta}_1 &= -\frac{i}{\pi} \partial_z^2 \tilde{\vartheta}_1, \\ \partial_\tau \zeta &= \partial_\tau(\eta z) + \partial_\tau \frac{\partial_z \tilde{\vartheta}_1}{\tilde{\vartheta}_1}, \\ \partial_\tau \zeta_j &= \partial_z^j(\partial_\tau \zeta), & j &= 1, 2, 3, \end{aligned} \quad (50)$$

where the first two equations follow from (41) and (46) respectively. The system (50) can be rewritten as a Noetherian system: for the first equation, one can rewrite  $\partial_z^2 \tilde{\vartheta}_1$  as a polynomial using the derivation rules of (49); for the second equation one can proceed similarly using additionally the first equation for  $\partial_\tau \tilde{\vartheta}_1$ ; and for the final three equations one proceeds similarly using the second equation for  $\partial_\tau \zeta$ . We leave the detailed derivation for the reader.

In conclusion, we have obtained a rational Noetherian system for  $\tilde{\vartheta}_1(z; \tau)$  and  $\zeta(z; \tau)$  with the polar locus given by  $\{\tilde{\vartheta}_1 = 0\}$ . This polar locus is given by all pairs  $(z; \tau)$  such that  $z$  is congruent to 0 modulo  $2, 2\tau$ , which is the same as the polar locus of  $\zeta(z; \tau)$ . Since  $\wp(z; \tau) = -\partial_z \zeta(z; \tau)$ , we see that  $\wp(z; \tau)$  is also Noetherian outside its polar locus.

**3.6 Periods of algebraic integrals, Gauss–Manin connection.** We follow [Gri70] for basic facts concerning the Gauss–Manin connection. Let  $X, S$  be smooth algebraic varieties over  $\mathbb{C}$ , and  $f : X \rightarrow S$  be a rational proper holomorphic map such that  $df$  is everywhere of maximal rank. For  $s \in S$  we denote  $V_s := f^{-1}(s)$ , and think of  $\{V_s\}_{s \in S}$  as a family of smooth, complete varieties depending on the parameter  $s$ . There is an algebraic vector bundle  $H_{\text{dR}}^1(X/S) \rightarrow S$ , called *relative cohomology of  $X$  over  $S$* , whose fiber over a point  $s \in S$  is given by  $H^1(X_s, \mathbb{C})$ . This vector bundle is equipped with a canonical flat connection  $\nabla$  called the *Gauss–Manin connection*,

$$\nabla : H_{\text{dR}}^1(X/S) \rightarrow \Omega_S^1 \otimes H_{\text{dR}}^1(X/S) \quad (51)$$

where  $\Omega_S^1$  denotes regular 1-forms on  $S$ . Geometrically, if  $\delta \in H_1(X_{s_0}, \mathbb{Z})$  and  $\delta(s)$  denotes the continuation of  $\delta$  for  $s$  near  $s_0$  using Ehreshmann’s lemma, then  $\delta(s)$  are flat sections of the dual connection  $\nabla^*$  on the relative homology bundle  $H_1(X/S)$ .

Fix some point  $s_0 \in S$  and let  $\Phi_1, \dots, \Phi_m : S \rightarrow H_{\text{dR}}^1(X/S)$  be rational sections which form a basis at  $s_0$ . Choose any basis  $\delta_1(s_0), \dots, \delta_m(s_0)$  of  $H_1(X_{s_0}, \mathbb{Q})$  and extend it to horizontal sections  $\delta_1, \dots, \delta_m : S \rightarrow H_1(X/S)$  as above. If we denote by  $(\cdot, \cdot)$  the pairing between homology and cohomology then we have the *period matrix*,

$$X(s) = \begin{pmatrix} (\Phi_1(s), \delta_1(s)) & \cdots & (\Phi_1(s), \delta_m(s)) \\ & \ddots & \\ (\Phi_m(s), \delta_1(s)) & \cdots & (\Phi_m(s), \delta_m(s)) \end{pmatrix}. \quad (52)$$

Since  $\delta_j$  are horizontal with respect to  $\nabla^*$  we have  $d(\Phi_i, \delta_j) = (\nabla(\Phi_i), \delta_j)$  and in matrix form we have a linear differential equation

$$dX = A \cdot X \quad (53)$$

where  $A$  is an  $m \times m$  matrix whose coefficients are rational one-forms in  $S$  and regular whenever  $\Phi_1, \dots, \Phi_m$  form a basis.

If for instance  $S = \mathbb{C}^n$  then (53) restricted to each of the columns of  $X(s)$  is a (rational) Noetherian system which is regular wherever  $A$  is (and in particular at  $s_0$ ). For a general variety  $S$ , the system (53) is a Noetherian system over  $S$  in the sense of Sect. 2.4. In conclusion, we see that the entries of the period matrix are Noetherian functions in a neighborhood of any point  $s_0 \in S$ .

**3.7 The moduli space of principally polarized abelian varieties.** Periods play a central role in the construction of moduli spaces of principally polarized abelian varieties (with additional structure). The first classical example is that of elliptic curves. If  $E$  is an elliptic curve,  $\omega$  a holomorphic one-form on  $E$  and  $\delta_1, \delta_2 \in H_1(E, \mathbb{Z})$  two cycles with intersection number 1, then the upper half of the period matrix (52) consists of the two elliptic integrals

$$I_1 := \oint_{\delta_1} \omega, \quad I_2 := \oint_{\delta_2} \omega. \quad (54)$$



In the universal cover  $\mathbb{C} \rightarrow \mathbb{C}/\Lambda \simeq E$  the form  $\omega$  corresponds (up to scalar) to  $dz$  and the two periods correspond to two generators of the lattice  $\Lambda$ . The ratio of the two periods is one of the points in the upper half-space  $\mathbb{H}$  representing  $E$ .

If we consider a smooth family of elliptic curves  $\{E_s\}_{s \in S}$  then the periods  $I_1, I_2$  continue as maps  $I_1(s), I_2(s)$  as in Sect. 3.6. In a neighborhood of any point  $s_0 \in S$  these two functions are Noetherian, and we may assume without loss of generality that  $I_2(s) \neq 0$  in the neighborhood so that the function  $I_1(s)/I_2(s)$  mapping each  $s$  to its representative in  $\mathbb{H}$  is also Noetherian. For instance if we let

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \equiv \pm 1 \pmod{N}, \quad b \equiv c \equiv 0 \pmod{N} \right\} \quad (55)$$

for  $N \geq 3$  and  $Y(N) = \Gamma(N) \backslash \mathbb{H}$  then  $Y(N)$  admits the structure of a smooth quasi-projective variety and there exists a canonical family  $E \rightarrow Y(N)$  of elliptic curves (with a level  $N$  structure) over  $Y(N)$  [MF82, Chapter 7]. In this case the projection  $\pi : \mathbb{H} \rightarrow Y(N)$  is locally inverse to the ratio  $I_1(s)/I_2(s)$ . Using Lemma 8 one can then show that  $\pi$  is Noetherian as well, when we identify  $Y(N)$  with its projective embedding. See the end of this section for a formal treatment of this implication using the charts of Sect. 2.4 in a more general setting. Of course, the Noetherianity of  $\pi$  also essentially follows from the construction of Sect. 3.1. However, the present construction has the advantage of generalizing to higher dimensions as we illustrate below.

Let  $\mathbb{H}_g$  denote the Siegel half-space of genus  $g$ , consisting of  $g \times g$  symmetric matrices over  $\mathbb{C}$  with positive-definite imaginary part. The symplectic group  $\mathrm{Sp}_{2g}(\mathbb{R})$  acts on  $\mathbb{H}_g$  (see e.g. [BZ03]) and the quotient  $\mathcal{A}_g := \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$  is called the Siegel modular variety. Then  $\mathcal{A}_g$  is the moduli space of principally polarized abelian varieties of genus  $g$ , and it can be shown to have the structure of a quasi-projective algebraic variety. We will show that the uniformization of  $\mathcal{A}_g$  by  $\mathbb{H}_g$  (given by the quotient map) is a Noetherian map.

It will be convenient for our purposes to pass from  $\mathrm{Sp}_{2g}(\mathbb{Z})$  to a finite-index subgroup such that the quotient becomes a smooth manifold. Following [BZ03] we take  $\Gamma = \Gamma_{4,8}$  to be the theta-group of level  $(4, 8)$ ,

$$\Gamma_{4,8} := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}) : \gamma = \mathbf{1}_g \pmod{4}, \right. \\ \left. \mathrm{diag}(a^t b) \equiv \mathrm{diag}(c^t d) = 0 \pmod{8} \right\}. \quad (56)$$

The quotient  $S := \Gamma \backslash \mathbb{H}_g$  is a complex manifold which has a natural structure of a smooth quasi-projective variety [MF82, p.190]. The variety  $S$  is equipped with a canonical family  $A \rightarrow S$  of principally-polarized abelian varieties with a level  $(4, 8)$ -structure [MF82, Appendix 7, A–B], and is the moduli space for such varieties. We denote by  $\pi : \mathbb{H}_g \rightarrow S$  the canonical projection. Since  $\Gamma$  is a finite index subgroup of  $\mathrm{Sp}_{2g}(\mathbb{Z})$  there is a regular, finite algebraic map  $\kappa : S \rightarrow \mathcal{A}_g$ . By the results of Sect. 2.4

if we show that  $\pi$  is Noetherian then  $\kappa \circ \pi$ , which is the uniformization  $\mathbb{H}_g \rightarrow \mathcal{A}_g$ , is Noetherian as well. We proceed to prove that  $\pi$  is indeed a Noetherian map.

Fix  $s_0 \in S$  and let  $\tau_0 \in \mathbb{H}_g$  with  $\pi(\tau_0) = s_0$ . According to [Shi98, Theorem 30.3] we may choose  $g$  rational sections  $\omega_1, \dots, \omega_g : S \rightarrow H_{\text{dR}}^1(A/S)$  which define a basis of the holomorphic differentials on  $A$  over a generic point of  $S$ , are regular at  $\tau_0$ , and which admit  $2\pi i P(\tau)(\mathbf{1}_g \ \tau)$  as a period matrix<sup>1</sup> (where we think of  $\tau$  as a function of  $s$ ) and  $P(\tau_0)$  is an invertible matrix. As periods, each of the entries of this matrix are Noetherian functions in a neighborhood of  $s_0$ , and from Lemma 6 it follows that  $\tau(s)$ , being the product of the second block by the inverse of the first, is also a Noetherian function in a neighborhood of  $s_0$ .

Having shown that the local inverse of  $\pi$  in a neighborhood of  $s_0$  is Noetherian, we may also conclude that  $\pi$  is Noetherian in a neighborhood of  $\tau_0$ . We do this in detail to illustrate the technique involving Noetherian charts. Recall that by Sect. 2.4 there exists a Noetherian chart  $\Psi : U_0 \rightarrow S$  with  $U_0 \subset \mathbb{C}^{\dim S}$ , say mapping 0 to  $s_0$ , such that  $\tau \circ \Psi : U_0 \rightarrow \mathbb{H}_g$  is Noetherian. By construction  $\tau \circ \Psi$  is also invertible in a neighborhood of 0, and by Lemma 8 the inverse map  $\tilde{\pi} : \mathbb{H}_g \rightarrow U_0$  mapping  $\tau_0$  to 0 is also Noetherian around  $\tau_0$ . But then  $\pi = \Psi \circ \tilde{\pi}$  is Noetherian in a neighborhood of  $\tau_0$  by Lemma 7, as claimed.

We remark that for explicit computation of the Noetherian parameters one would need an explicit description of the universal family and associated Gauss–Manin connection, which may be a non-trivial problem. An alternative approach would be to derive Noetherian systems for higher dimensional thetanulls analogous to the Halphen system (42) directly, using the Riemann theta functions and their various identities. Such an approach is pursued for genus two in [Ohy96] and for general genus in [Zud00], where the thetanulls of any genus are shown to satisfy a nonlinear system of differential equations which may indeed be regarded as a Noetherian system. Unfortunately this system admits singularities at the zeros of the thetanulls and therefore does not quite establish their Noetherianity in the entire Siegel half-space – but it does seem to indicate that this direct approach is feasible.

## 4 Analytic theory of Weierstrass Polydiscs

We begin by recalling the notion of a Weierstrass polydisc from [BN17b]. We call a system  $\mathbf{x}$  of coordinates on  $\mathbb{C}^n$  *standard* if it is obtained from the standard coordinates by an affine unitary transformation.

**DEFINITION 12.** *Let  $X \subset \Omega$  be an analytic subset of pure dimension  $m$ . We say that a polydisc  $\Delta = \Delta_z \times \Delta_w$  in a standard coordinate system  $\mathbf{x} = \mathbf{z} \times \mathbf{w}$  coordinates is a Weierstrass polydisc for  $X$  if  $\dim \mathbf{z} = m$ ,  $\bar{\Delta} \subset \Omega$  and  $(\bar{\Delta}_z \times \partial \Delta_w) \cap X = \emptyset$ . We call  $\Delta_z$  the base and  $\Delta_w$  the fiber of  $\Delta$ .*

We will denote by  $\pi_z : \mathbb{C}^n \rightarrow \mathbb{C}^m$  the projection map, and by  $\pi_z^X := \pi_z|_{X \cap \Delta}$ .

---

<sup>1</sup> For an appropriate choice of basis for  $H_1(A_{s_0}, \mathbb{C})$  with respect to the principal polarization.

**Fact 13.** ([BN17b, Fact 5]) *Let  $\Delta$  be a Weierstrass polydisc for  $X$ . Then  $\pi_z^X : X \cap \Delta \rightarrow \mathbb{C}^m$  is a proper  $e(X, \Delta)$ -to-1 map for some number  $e(X, \Delta) \in \mathbb{N}$  called the degree.*

In [BN17a, BN17b] Weierstrass polydiscs play a key role in the study of rational points on analytic varieties. Proving the existence of Weierstrass polydiscs with effective estimates on the size and degree is the main step in establishing an effective Pila–Wilkie result using this method. In this section we develop some analytic tools to approach this problem. Namely, we recall the notion of *Bernstein index* of an analytic function, and show that estimates on the Bernstein indices of appropriately constructed functions imply the existence of Weierstrass polydiscs. Later, in Sect. 5 we show that in the Noetherian category one can indeed estimate the relevant Bernstein indices in terms of the Noetherian parameters.

**4.1 Bernstein indices.** We recall the notion of  $\delta$ -extension from [BN17a].

**DEFINITION 14** ( $\delta$ -extension). *Let  $A \subset \mathbb{C}^n$  be a ball or a polydisc centered at a point  $p \in \mathbb{C}^n$ , and let  $\delta > 0$ . The  $\delta$ -extension  $A^\delta$  of  $A$  is defined to be  $p + \delta^{-1}(A - p)$ , i.e., the stretching of  $A$  by a factor of  $\delta^{-1}$  around its center.*

We recall the following definition from [NY97].

**DEFINITION 15.** *Let  $U \subset \mathbb{C}$  be a domain with a connected piecewise smooth boundary and  $K \subset U$  a compact subset. The Bernstein index of a holomorphic function  $f : \bar{U} \rightarrow \mathbb{C}$  with respect to the pair  $K \subset U$  is the number*

$$\mathfrak{B}_{K,U}(f) := \ln \frac{M_U(f)}{M_K(f)}, \quad M_A(f) := \max_{z \in A} |f(z)|. \quad (57)$$

For  $U$  a disc and  $\eta > 1$  called the gap, we denote

$$\mathfrak{B}_U^\eta(f) := \mathfrak{B}_{\bar{U}^\eta, U}(f). \quad (58)$$

The following Theorem 3, Lemma 16 of [IS96] and Lemma 19 of [NY97] hold, with suitable constants, for arbitrary pairs  $K \subset U$  as in Definition 15. We will only require them in the case of concentric discs and state them for this case in order to give explicit asymptotics for the constants (the constants as given in [IS96] are fully explicit and we use asymptotic notation only to simplify the presentation).

**Theorem 3.** ([IS96, Lemma 1, Example 1]) *Let  $U$  be a disc and  $\varepsilon > 0$ . For any  $f \in \mathcal{O}(\bar{U})$  we have*

$$\#\{z \in \bar{U}^{1+\varepsilon} : f(z) = 0\} \leq \gamma_\varepsilon \cdot \mathfrak{B}_U^{1+\varepsilon}(f), \quad \gamma_\varepsilon = \frac{2}{\varepsilon^2} + O(\varepsilon). \quad (59)$$

Here each root of  $f$  is counted with its multiplicity.

The following lemma of [IS95] gives a lower bound for the values of a holomorphic function in terms of its Bernstein index.

LEMMA 16. ([IS96, Lemma 3]) *Let  $U$  be a disc of radius 1 and  $\varepsilon > 0$ . For any  $f \in \mathcal{O}(\bar{U})$  and any  $h > 0$  one can find a finite union  $D_h$  of discs around roots of  $f$  in  $U$ , with the sum of the diameters less than  $h$ , such that*

$$\min_{z \in \bar{U}^{1+\varepsilon} \setminus D_h} |f(z)| \geq M \left( \frac{m}{M} \right)^{\chi_\varepsilon - \tau_\varepsilon \ln h} \quad (60)$$

where  $M = M_U(f)$ ,  $m = M_{U^{1+\varepsilon}}(f)$  and

$$\chi_\varepsilon = \frac{8}{\varepsilon^4} \ln \frac{1}{\varepsilon} + O(\varepsilon^{-4}), \quad \tau_\varepsilon = \frac{2}{\varepsilon^2} + O(\varepsilon). \quad (61)$$

*Proof.* For the computation of the constants we note that (in the notation of [IS96, Lemma 3]) we choose  $V$  to be a disc of radius  $1 - \frac{\varepsilon}{2}$  and by [IS96, Example 1] we have

$$\rho = 1 - \frac{1}{2}\varepsilon^2 + O(\varepsilon^3), \quad \gamma = \frac{2}{\varepsilon^2} + O(\varepsilon), \quad \sigma = \frac{4}{\varepsilon^2} + O(\varepsilon) \quad (62)$$

and then

$$\lambda_1 = \frac{2}{\varepsilon} + O(1), \quad \lambda_2 = 1 - \frac{1}{2}\varepsilon + O(\varepsilon^2), \quad \theta = \frac{8}{\varepsilon^4} \ln \frac{1}{\varepsilon} + O(\varepsilon^{-4}) \quad (63)$$

from which the claim follows.

We also note that the fact that the centers of the discs can be chosen to be roots of  $f$  is not part of the original statement of [IS96, Lemma 3] but it is evident from the proof.  $\square$

As an easy consequence we have the following.

LEMMA 17. *Let  $U$  be a disc and  $f \in \mathcal{O}(\bar{U})$  with  $M_U(f) = 1$ . Then there exists a disc  $U^4 \subset D \subset \bar{U}^2$  concentric with  $U$  such that*

$$\min_{z \in \partial D} |f(z)| \geq e^{-O(\mathfrak{B}_U^2(f))}. \quad (64)$$

*Proof.* Since the claim is invariant under rescaling we may suppose that  $U$  is a disc of radius 1. Apply Lemma 16 with  $h = 1/4$ . Since the sum of the diameters of the discs in  $D_h$  does not exceed  $1/4$  we see that  $D_h$  must be disjoint from  $\partial D$  where  $D$  is a disc concentric with  $U$  with radius  $1/4 < r \leq 1/2$ .  $\square$

We record a simple corollary of Lemma 16 which allows one to control the Bernstein index with the standard gap 2 in terms of the Bernstein index with a smaller gap.

COROLLARY 18. *Let  $U$  be a disc and  $f \in \mathcal{O}(\bar{U})$ . Then for any  $0 < \varepsilon < 1$  we have*

$$\mathfrak{B}_U^2(f) \leq (\chi_\varepsilon + \tau_\varepsilon \ln 2) \cdot \mathfrak{B}_U^{1+\varepsilon}(f). \quad (65)$$

*Proof.* Since the claim is invariant by rescaling we may assume that  $U$  is the unit disc. Apply Lemma 16 to obtain a finite union  $D$  of discs with the sum of the diameters less than  $1/2$ , such that

$$\min_{z \in \bar{U}^{1+\varepsilon} \setminus D} |f(z)| \geq M \left( \frac{m}{M} \right)^{\chi_\varepsilon + \tau_\varepsilon \ln 2} \quad (66)$$

where  $M = M_U(f)$ ,  $m = M_{U^{1+\varepsilon}}(f)$ . In particular, since  $D$  cannot cover  $U^2$ , we have

$$\frac{M_{U^2}(f)}{M_U(f)} \geq \left( \frac{m}{M} \right)^{\chi_\varepsilon + \tau_\varepsilon \ln 2}. \quad (67)$$

The claim now follows by taking inverse and log.  $\square$

We will also require the following subadditivity property of [NY97, Lemma 3]. We will not need to use the explicit form of the constants, but we remark that they can be easily explicitly recovered from the proof.

LEMMA 19. *Let  $U$  be a disc and  $f_1, \dots, f_p \in \mathcal{O}(\bar{U})$ . Then*

$$\mathfrak{B}_U^2(f_1 \cdots f_p) \leq O(\ln(p+1)) \sum_{j=1}^p \mathfrak{B}_U^2(f_j). \quad (68)$$

For dimension greater than one we introduce the following version of the Bernstein index.

DEFINITION 20. *Let  $B \subset \mathbb{C}^n$  be a Euclidean ball centered at  $p$  and  $K \subset U$  a compact subset. Let  $F : \bar{B} \rightarrow \mathbb{C}$  be holomorphic. We define*

$$\mathfrak{B}_{K,B}(F) := \max_{L \ni p} \mathfrak{B}_{K \cap L, B \cap L}(f) \quad (69)$$

where  $L$  ranges over the complex lines containing  $p$ . For  $\eta > 1$  called the gap, we denote

$$\mathfrak{B}_B^\eta(f) := \mathfrak{B}_{\bar{B}^\eta, B}(f). \quad (70)$$

**4.2 Weierstrass polydiscs for holomorphic hypersurfaces.** Let  $B \subset \mathbb{C}^m$  be a Euclidean ball and  $R : \bar{B} \rightarrow \mathbb{C}$  a holomorphic function. Our goal in this section is to construct a Weierstrass polydisc  $\Delta$  for the hypersurface  $X_R := \{R = 0\}$ . More specifically, we would like to control the size of  $\Delta$  and the degree  $e(X_R, \Delta)$  in terms of the Bernstein index  $\mathfrak{B}_B^2(R)$ .

PROPOSITION 21. *Let  $R : \bar{B} \rightarrow \mathbb{C}$  be a holomorphic function and set  $\mathfrak{B} := \mathfrak{B}_B^2(R)$ . Then there exists a Weierstrass polydisc  $\Delta$  for  $X_R := B \cap \{R = 0\}$  such that*

$$B^\eta \subset \Delta \subset B \text{ where } \eta = e^{O(\mathfrak{B})}, \quad (71)$$

$$e(X_R, \Delta) = O(\mathfrak{B}). \quad (72)$$

Here (72) holds even if we consider  $X_R$  with its cycle structure, i.e. count each component of  $X_R$  with its associated multiplicity as a zero of  $R$  (see [Ful98] for the notion of cycles and multiplicities).

*Proof.* Since the claim is invariant under rescaling of  $R$  and  $B$  we may assume that  $B$  is the unit ball at the origin and that the maximum of  $R$  on  $B$  is achieved at some point  $p \in B$  with  $|R(p)| = 1$ . Let  $L$  denote the complex line passing through the center of  $B$  and the point  $p$  and write  $U = B \cap L$ . Let  $\mathbf{z} = z_1 \times \mathbf{z}'$  be a system of Euclidean coordinates on  $B$  where the origin corresponds to the center of  $B$  and  $\mathbf{z}' = 0$  corresponds to  $L$ .

According to Lemma 17 there exists a disc  $U^4 \subset D \subset \bar{U}^2$  such that

$$\min_{z_1 \in \partial D} |R(z_1, 0)| \geq e^{-O(\mathfrak{B})}. \quad (73)$$

Since  $R$  is assumed to have maximum norm 1 on  $B$ , it follows from the Cauchy estimates that

$$\left\| \frac{\partial R}{\partial \mathbf{z}'}(z_1, \mathbf{z}') \right\| = O(1), \quad \forall z_1 \in \partial D, \|\mathbf{z}'\| < 1/2. \quad (74)$$

Combining (73) and (74) we see that we may choose a polydisc  $\Delta_w$  of polyradius  $e^{-O(\mathfrak{B})}$  around the origin such that  $R$  does not vanish on  $\Delta_w \times \partial D$ . Then  $\Delta := \Delta_{z'} \times D$  is a Weierstrass polydisc for  $X_R$ . To estimate  $e(X_R, \Delta)$  it will suffice to count the number of zeros of  $R$  in the fiber  $\mathbf{z}' = 0$ , i.e. the number of zeros of  $R$  restricted to  $D \subset L$ . This follows directly from Theorem 3.  $\square$

**4.3 Weierstrass polydisc for an intersection with a hypersurface.** Let  $X \subset \Omega$  be an analytic subset of pure dimension  $m$  and  $\Delta = \Delta_z \times \Delta_w$  a Weierstrass polydisc for  $X$ . Let  $F : \Omega \rightarrow \mathbb{C}$  be a holomorphic function, and set

$$X_F := X \cap \Delta \cap \{F = 0\}, \quad Y_F := \pi_z(X_F). \quad (75)$$

LEMMA 22. *Suppose  $\Delta'_z = \Delta_z \times \Delta_{w'} \subset \Delta_z$  is a Weierstrass polydisc for  $Y_F$ . Then  $\Delta' := \Delta'_z \times \Delta_w$  is a Weierstrass polydisc for  $X_F$ .*

*Proof.* Recall that

$$\Delta_{z'} \times \partial(\Delta_{w'} \times \Delta_w) = [\Delta_{z'} \times \partial\Delta_{w'} \times \Delta_w] \cup [\Delta'_z \times \partial\Delta_w]. \quad (76)$$

It thus suffices to note that  $X_F$  does not meet  $\Delta'_z \times \partial\Delta_w$  since  $X$  does not meet  $\Delta_z \times \partial\Delta_w$ ; and  $X_F$  does not meet  $\Delta_{z'} \times \partial\Delta_{w'} \times \Delta_w$  since its  $\pi_z$  projection  $Y_F$  does not meet  $\Delta_{z'} \times \partial\Delta_{w'}$ .  $\square$

We define the *analytic resultant* of  $F$  with respect to  $X, \Delta$ ,

$$\mathcal{R}_F = \mathcal{R}(X, \Delta, F) : \Delta_z \rightarrow \mathbb{C}, \quad \mathcal{R}_F(z) = \prod_{w: (z, w) \in X \cap \Delta} F(z, w). \quad (77)$$

By Fact 13 we see that  $\mathcal{R}_F$  is a holomorphic function, and by definition

$$Y_F = \{z \in \Delta_z : \mathcal{R}_F(z) = 0\}. \quad (78)$$

Here equality holds even if we consider  $Y_F$  and  $\{\mathcal{R}_F = 0\}$  with their natural cycle structures.

PROPOSITION 23. *Let  $B \subset \Delta_z$  be a Euclidean ball with the same center as  $\Delta_z$ , and set  $\mathfrak{B} := \mathfrak{B}_B^2(\mathcal{R}_F)$ . Then there exists a Weierstrass polydisc  $\Delta' := \Delta'_z \times \Delta_w$  for  $X_F$  such that*

$$B^\eta \subset \Delta'_z \subset B \text{ where } \eta = e^{O(\mathfrak{B})}, \quad (79)$$

$$e(X_F, \Delta') = O(\mathfrak{B}). \quad (80)$$

*Proof.* By Proposition 21 applied to  $\mathcal{R}_F$ , there exists a Weierstrass polydisc  $\Delta'_z \subset B$  for  $Y_F$  satisfying (79). Then by Lemma 22,  $\Delta'$  is a Weierstrass polydisc for  $X_F$ . Finally  $e(X_F, \Delta')$  is equal by definition to  $e(Y_F, \Delta'_z)$  if we count each component of  $Y_F$  with its associated multiplicity, and (80) follows from (72) and the remark following it.  $\square$

Proposition 23 can be used to inductively construct a Weierstrass polydisc of controllable size and degree for the zero locus of a collection of functions, assuming one can explicitly estimate the Bernstein indices  $\mathfrak{B}$  involved. Unfortunately it appears that the techniques presently at our disposal do not suffice to produce such estimates for arbitrary collections of Noetherian functions. Instead, we will focus on the case where  $X$  is an algebraic variety and  $F$  is a Noetherian function, where a wider range of techniques is available. As it turns out, this more restrictive case will be sufficient for our purposes.

## 5 Weierstrass polydiscs and Bernstein Indices of Noetherian Functions

In this section we produce estimates for the Bernstein indices of Noetherian functions and, more generally, their analytic resultants (77) with respect to an algebraic variety. In combination with the results of Sect. 4 this allows us to construct Weierstrass polydiscs for the intersection between an algebraic variety and a Noetherian hypersurface. <sup>algebraic</sup> The main statements are given in Sect. 5.1. Two principal ingredients from the qualitative theory of differential equations are used in producing these estimates. First, in Sect. 5.2 we use some results from the theory of linear differential equations to obtain parametrizations of an algebraic curve which are well-behaved (in terms of the differential equations involved). Consequently in Sect. 5.4 we produce a well-behaved non-linear differential equation for the restriction of a Noetherian function to an algebraic curve. In Sect. 5.3 we introduce a result of [NY99] on the oscillation of trajectories of (non-linear) polynomial vector fields. Finally in Sect. 5.5 we use this result to estimate the Bernstein index of a Noetherian function restricted to an algebraic curve, and consequently finish the proof of the main statement of this section.

**5.1 Main statement.** If  $X \subset \Omega$  is an analytic set and  $m \in \mathbb{N}$ , we denote by  $W^m$  the union of the  $m$ -dimensional irreducible components of  $X$ . Our goal in this section is to prove the following.



**Theorem 4.** *Let  $V \subset \mathbb{C}^n$  be an algebraic variety of pure dimension  $m$  and degree at most  $\beta$ . Let  $F : \Omega \rightarrow \mathbb{C}$  be a Noetherian function of degree at most  $\beta$ . Let  $B \subset \Omega$  be a Euclidean ball. Write*

$$V_F := (\Omega \cap V \cap \{F = 0\})^{m-1}. \quad (81)$$

*Then there exists a Weierstrass polydisc  $\Delta$  for  $V_F$  and  $\eta > 0$  such that*

- (1)  $B^\eta \subset \Delta \subset B$  where  $\eta = e^{C(\phi, \beta)}$ ,
- (2)  $e(V_F, \Delta) = C(\phi, \beta)$ .

Theorem 4 implies that one can cover any compact piece of  $V_F$  by an explicitly estimated number of Weierstrass polydiscs with explicitly estimated degrees. The special case  $\dim V = 1$  is of some independent interest and we record it separately.

**COROLLARY 24.** *Let  $C \subset \mathbb{C}^n$  be an algebraic curve and  $F : \Omega \rightarrow \mathbb{C}$  a Noetherian function, both of degree at most  $\beta$ . Let  $B \subset \Omega$  be a Euclidean ball. Then for  $\eta = e^{C(\phi, \beta)}$  the number of isolated zeros of  $F$  in the set  $V \cap B^\eta$  is bounded by  $C(\phi, \beta)$ .*

*Proof.* We observe that in this case  $C_F$  of (81) has dimension zero and consists of the isolated zeros of  $F$  on  $C \cap \Omega$ . In this context a Weierstrass polydisc  $\Delta$  is just a disc with  $\partial\Delta \cap C_F = \emptyset$  and  $e(C_F, \Delta)$  is the number of points in  $C_F \cap \Delta$ . The claim follows since  $B^\eta \subset \Delta$ .  $\square$

By a covering argument, Corollary 24 allows one to explicitly estimate the number of zeros of a Noetherian function restricted to a compact piece of an algebraic curve. Crucially, the estimate depends only on the degree of the curve and the Noetherian parameters. We conjecture that a similar statement, with the curve  $V$  replaced by the zero locus of arbitrary additional Noetherian functions, is likely to be true. A conjecture in this spirit (in a more local setting) is due to Gabrielov and Khovanskii [GK98], with some partial results established in [GK98, BN12, BN15].

**5.2 Linear ODEs and algebraic functions.** Let  $y(t)$  be an algebraic function defined by the polynomial  $P(y, t) = 0$  of degree  $d$ . It is classically known that  $y$  satisfies a scalar linear differential equation  $L(y) = 0$  of order  $k \leq d$ ,

$$L = a_0(t)\partial_t^k + \cdots + a_k(t)y, \quad a_0, \dots, a_k \in \mathbb{C}[t], \quad a_0 \not\equiv 0. \quad (82)$$

Following [BN10] we define the slope<sup>2</sup> of  $L$  to be

$$\angle L := \max_{i=1, \dots, k} \frac{\|a_i\|_\infty}{\|a_0\|_\infty}.$$

It is not difficult to see that the degrees of  $a_0, \dots, a_k$  can be estimated in terms of  $d$ . It is less trivial, but still true, that the same is true for the slope  $\angle L$ . More explicitly, we have the following.

<sup>2</sup> We use  $\ell_\infty$  rather than  $\ell_2$ -norms for convenience but this is of no significance.

*D-finite*

*can  $Q_0(t) = 1$   
wlog?*

*or invertible  
(83) as a power*

*series?*

**Theorem 5.** *The operator  $L$  can be chosen such that*

$$\deg a_0, \dots, \deg a_k = d^{O(1)}, \quad (84)$$

$$\angle L = 2^{2^{\text{poly}(d)}}. \quad (85)$$

*Proof.* Let  $\mathcal{P}$  denote the space of polynomials of degree  $d$  in the variables  $y, t$ , which we identify with their tuple of coefficients. Consider  $P(y, t)$  as a general polynomial of degree  $d$  with indeterminate coefficients  $\mathbf{p} \in \mathcal{P}$ . According to [BNY11, Corollary 3.3] the vector function  $\mathbf{y} = (y, y^2, \dots, y^d)$  satisfies an integrable regular system of the form

$$d\mathbf{y} = \Omega \cdot \mathbf{y} \quad (86)$$

where  $\Omega$  is a rational matrix one-form in the variables  $\mathbf{p}, t$  with coefficients from  $\mathbb{Q}(\mathbf{p}, t)$ . Moreover, the degree of the entries is bounded by  $d^{O(1)}$  and the complexity (i.e. maximal height of any of the coefficients) is bounded by  $2^{d^{O(1)}}$  (this is similar, but much easier, than [BN10, Theorem 9]).

By a standard reduction from linear first-order systems to high-order scalar equations (e.g. [BY09, Lemma 5]) one can then derive a family of linear operators

$$L_{\mathbf{p}} = a_0(\mathbf{p}, t) \partial_t^k + \dots + a_k(\mathbf{p}, t) y, \quad a_0, \dots, a_k \in \mathbb{C}[t], \quad a_0 \neq 0. \quad (87)$$

of order  $k \leq d$ , degree  $d^{O(1)}$  and complexity  $2^{d^{O(1)}}$  such that  $L_{\mathbf{p}}(y(\mathbf{p}, t)) = 0$ . By [BN10, Principal Lemma 33] there exists a proper algebraic subset  $\Sigma \subset \mathcal{P}$  such that for  $\mathbf{p} \notin \Sigma$  the slope  $\angle L_{\mathbf{p}}$  is bounded by some explicit constant of the form  $2^{2^{\text{poly}(d)}}$ .

It remains to consider the case  $\mathbf{p} \in \Sigma$ . Note that in this case it is possible that  $a_0(\mathbf{p}, t) \equiv 0$  so that the expression defining the slope of  $L_{\mathbf{p}}$  is not well-defined. However, this problem is only apparent. Let  $\gamma : (\mathbb{C}, 0) \rightarrow \mathcal{P}$  be a one-parametric family that meets  $\Sigma$  only at  $\gamma(0) = \mathbf{p}$  and consider the family  $L_s := L_{\gamma(s)}$ . Then  $L_s \neq 0$  for  $s \neq 0$ , but  $L_0$  may vanish identically. Let  $\nu$  denote the order of vanishing in  $s$ , so that

$$L_s = s^\nu \tilde{L}_s, \quad \tilde{L}_0 \neq 0. \quad (88)$$

For  $s \neq 0$  we have

$$\tilde{L}_s(y(\gamma(s), t)) = s^{-\nu} L_s(y(\gamma(s), t)) = 0 \quad (89)$$

and since both  $\tilde{L}_s$  and  $y(\gamma(s), t)$  are continuous (even analytic) in  $s$  it follows that  $\tilde{L}_0(y(\mathbf{p}, t)) = 0$ . We now note that since the slope is invariant under scalar multiplication,  $\angle \tilde{L}_s = \angle L_s$  for  $s \neq 0$  and is therefore bounded by the uniform constant  $2^{2^{\text{poly}(d)}}$  as above. The slope of the limit  $\tilde{L}_0$  is therefore bounded by the same constant, and  $\tilde{L}_0$  thus satisfies the conditions of the lemma.  $\square$

In [BN10] a result of the type of Theorem 5 is proved for a much more general class of functions known as Q-functions, which includes the algebraic functions as well as abelian integrals. The qualitative theory of linear ODEs is then used to estimate the number of zeros of Q-functions, and similar ideas could be used to give estimates for Bernstein indices as well. In this approach the boundedness of the slope plays a

key role. However, in the context of the present paper we must consider *Noetherian* functions which satisfy *non-linear* differential equations, making the class of Q-functions inadequate for our purposes. As we shall see, the boundedness of the slope for algebraic functions will play a key role none the less.

**5.3 Bernstein indices for non-linear polynomial ODEs.** We consider a polynomial non-linear system of ODEs,

$x: \mathbb{C} \rightarrow \mathbb{C}$  *univariate*

$$\partial_t x = \xi(t, x) \quad (90)$$

*die derivative*  
where  $\xi$  is a polynomial vector field on  $\mathbb{C}_t \times \mathbb{C}^N$ ,

$$\xi = \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x_1} + \cdots + \xi_N \frac{\partial}{\partial x_N}, \quad \xi_1, \dots, \xi_N \in \mathbb{C}[t, x_1, \dots, x_N]. \quad (91)$$

Let

$$d = \deg \xi := \max_{i=1, \dots, N} \deg \xi_i, \quad (92)$$

$$\|\xi\|_\infty := \max_{i=1, \dots, N} \|\xi_i\|_\infty. \quad (93)$$

We will require the following result of [NY99].

**Theorem 6.** ([NY99, Theorem 2]) *Let  $S > 2$  and  $p = (t_0, x_0) \in \mathbb{C}_t \times \mathbb{C}^N$ . Suppose that  $\|\xi\|_\infty \leq S$  and  $\|p\|_\infty \leq S$ . Denote by  $x = x(t)$  the solution of (90) passing through  $p$ .*

*Let  $P \in \mathbb{C}[t, x_1, \dots, x_N]$  have degree bounded by  $d$  and suppose  $P(t, x(t)) \not\equiv 0$ . Then  $x(t)$  can be extended to a disc  $D = D_\rho(t_0)$  where*

$$\rho = S^{-\exp^{o4}(O(Nd))}, \quad (94)$$

$$\mathfrak{B}_D^2(P(t, x(t))) \leq d^{N^{O(N^2)}}. \quad (95)$$

*More generally, if  $x(t)$  can be extended to a disc  $D = D_\rho(t_0)$  for some  $\rho > 0$  and remains bounded by  $2S$  in  $D$  then*

$$\mathfrak{B}_D^2(P(t, x(t))) \leq d^{N^{O(N^2)}} + \rho \cdot S^{\exp^{o4}(O(Nd))}. \quad (96)$$

*Proof sketch.* We let  $R$  denote ring of polynomials in the variables  $t, x_1, \dots, x_N$  and  $\{c_\alpha\}_{|\alpha| \leq d}$ . We denote by  $P$  the universal polynomial  $\sum_\alpha c_\alpha x^\alpha$  (so that  $P$  in the statement of the theorem is obtained by specializing the  $c_\alpha$  variables). Consider the chain of  $R$ -ideals defined recursively as follows

$$I_0 = \langle P \rangle, \quad I_{k+1} = \langle I_k, \xi^k P \rangle. \quad (97)$$

Since  $R$  is a noetherian ring, the chain  $I_k$  must stabilize. Moreover, it follows from the Leibnitz rule that if  $I_k = I_{k+1}$  then already  $I_k = I_\infty$ . Using methods of effective

commutative algebra it is possible to give an effective upper bound for the first index of stabilization  $k$  in terms of  $N$  and  $d$ , namely [NY99, Theorem 4] gives

$$k = d^{N^{O(N^2)}}. \quad (98)$$

With this  $k$  stabilization implies that we have an equation

$$\xi^{k+1}P = \sum_{j=0}^k c_j \cdot \xi^{k-j}P, \quad c_j \in R \quad (99)$$

and restricting to the solution  $(t, x(t))$  we have

More!

$$\bullet \quad \partial_t^{k+1}P(t, x(t)) = \sum_{j=0}^k c_j(t, x(t)) \cdot \partial_t^{k-j}P(t, x(t)). \quad (100)$$

Moreover, by methods of effective commutative algebra it is possible to give an effective upper bound for the  $\ell_\infty$ -norms of the polynomials  $c_j$  in terms of  $N, d, S$ . By simple growth estimates we can then choose  $\rho$  satisfying (94) such that  $(t, x(t))$  remains in the ball of radius  $2S$  for  $t \in D = D_\rho(t_0)$ . For any  $\rho$  that satisfies this condition, the coefficients of (100) are explicitly bounded by  $S^{\exp^{O(4)}(O(Nd))}$  in  $D$ .

We view (100) as a linear differential operator  $L$  with holomorphic coefficients in  $D$  satisfying  $L(P(t, x(t))) = 0$ . We pass to the coordinate  $s = (t - t_0)/\rho$  so that  $D$  corresponds to the unit disc in  $s$ . We express  $L$  in the  $s$ -coordinate and multiply by  $\rho^{k+1}$  to obtain a monic operator  $\tilde{L}$ . Then the coefficients of  $\tilde{L}$  are bounded by  $\rho \cdot S^{\exp^{O(4)}(O(Nd))}$  in the unit disc, and the proof is then concluded by applying Theorem 7 below. We remark that at the final step [NY99] uses a lemma of Kim to estimate the number of zeros in  $D$  and state the conclusion concerning this number rather than the Bernstein index, but this is of course a minor technical difference.  $\square$

To finish the proof of Theorem 6 we recall the following theorem of [NY97].

**Theorem 7.** ([NY97, Theorem 1]) *Let  $K, U$  be as in Definition 15 and let  $L$  be a linear differential operator*

$$L = \partial_t^{k+1} + \sum_{j=1}^k a_j(t) \partial_t^{n-j}, \quad a_0, \dots, a_k : \bar{U} \rightarrow \mathbb{C} \quad (101)$$

*with holomorphic coefficients of absolute value bounded by  $M$  in  $U$ . Then for any solution  $Lf = 0$  we have*

$$\mathfrak{B}_{K,U}(f) \leq O(M + k \ln k) \quad (102)$$

*where the asymptotic constant depends only on  $K, U$ .*

We remark that in [NY97], Theorem 7 is stated as a bound on the number of zeros of  $f$  rather than the Bernstein index. However, the proof goes through an estimate for the Bernstein index, which is given in the second displayed equation in [NY97, p. 317].

**5.4 Restriction of a Noetherian function to an algebraic curve.** Let  $C \subset \mathbb{C}^n$  be an algebraic curve of degree  $d$ , and let  $\pi_t : C \rightarrow \mathbb{C}_t$  be the restriction to  $C$  of some affine coordinate on  $\mathbb{C}^n$  which is not constant on any component of  $C$ . Then there is a (ramified) inverse map  $t \rightarrow (x_1(t), \dots, x_n(t))$  from  $\mathbb{C}_t$  to  $C$  where each  $x_j(t)$  is an algebraic function of degree at most  $d$ . For  $j = 1, \dots, n$  we choose a linear differential equation  $L^j(x_j(t)) = 0$  of order  $k_j \leq d$ ,

$$L^j = a_0^j(t) \partial_t^{k_j} + \dots + a_{k_j}^j(t) y, \quad a_0^j, \dots, a_{k_j}^j \in \mathbb{C}[t], \quad \left\| a_0^j \right\|_\infty = 1. \quad (103)$$

where the operators  $L_j$  satisfy the estimates of Theorem 5.

Recall that  $\phi : \Omega \rightarrow \mathbb{C}$  is a collection of Noetherian functions satisfying (1). We will study the restriction  $\phi|_C$  by writing a non-linear system of the form (90) for the map  $t \rightarrow (\mathbf{x}(t), \phi(x(t)))$ . Toward this end we write  $N = k_1 + \dots + k_n + \ell$  and work in the ambient space  $\mathbb{C}^N$  with the coordinates  $x_j^{(k)}$  and  $Q_j$  for  $j = 1, \dots, n$  and  $k = 1, \dots, k_j - 1$  and  $\phi_1, \dots, \phi_\ell$ . In this space we introduce the following system,

$$\begin{aligned} \partial_t x_j^{(k)} &= x_j^{(k+1)}, & k &= 1, \dots, k_j - 2, \\ \partial_t x_j^{(k_j-1)} &= -Q_j (a_1^{k_j-1} x_j^{(k_j-1)} + \dots + a_{k_j} x_j^{(0)}), \\ \partial_t Q_j &= -\partial_t(a_0^j) \cdot (Q_j)^2, \\ \partial_t \phi_l &= P_{l,1}(\mathbf{x}^{(0)}, \phi) \cdot x_1^{(1)} + \dots + P_{l,n}(\mathbf{x}^{(0)}, \phi) \cdot x_n^{(1)}. \end{aligned} \quad (104)$$

Let  $\xi_{\phi,C}$  denote the vector field corresponding to (104) in the sense of Sect. 5.3. Then it is straightforward to verify using the estimates of Theorem 5 that

$$\|\xi_{\phi,C}\|_\infty = 2^{2^{\text{poly}(d)}} \cdot \mathcal{S}(\phi). \quad (105)$$

Let  $t \in \mathbb{C}_t$  be a point where  $\pi_t|_C$  is unramified and the polynomials  $a_0^1, \dots, a_0^n$  are non-vanishing, and let  $\tilde{\mathbf{x}}(t)$  be an analytic branch of the algebraic map  $\mathbf{x}(t)$ . Then it is straightforward to check that the map  $\Phi : \mathbb{C}_t \rightarrow \mathbb{C}^N$  defined by

$$x_j^{(k)} = \partial_t^k \tilde{x}_j(t), \quad Q_j = 1/a_0^j(t), \quad \phi_l = \phi_l(\tilde{\mathbf{x}}(t)) \quad (106)$$

is a solution of (104).

**5.5 Proof of Theorem 4.** Let  $V \subset \mathbb{C}^n$  be an algebraic variety of pure dimension  $m$  and degree  $d$ . Let  $F : \Omega \rightarrow \mathbb{C}$  be a Noetherian function of degree  $d$ . Finally let  $B \subset \Omega$  be a Euclidean ball. We recall the following result of [BN17b].

**Theorem 8** [BN17b, Theorem 7]. *Let  $B \subset \mathbb{C}^n$  be a Euclidean ball and  $V \subset \mathbb{C}^n$  be an algebraic variety of pure dimension  $m$  and degree  $d$ . Then there exists a Weierstrass polydisc  $\Delta := \Delta_z \times \Delta_w$  for  $V$  with the same center as  $B$  such that  $B^{\tilde{\eta}} \subset \Delta \subset B$  and  $\tilde{\eta} = d^{O(1)}$ .*

Note that Theorem 8 is originally stated for general sub-Pfaffian sets, and above we give only the algebraic case which will suffice for our purposes. We also note that the theorem is originally stated for a ball around the origin, but this is clearly of no significance in the formulation above. Finally, in our formulation we implicitly used the fact the algebraic varieties of degree  $d$  are set-theoretically cut out by polynomials of degree at most  $d$ , see e.g. [BN17b, Lemma 29].

We remark that Theorem 8 could also be established inductively by the methods used in this paper, but the estimates obtained in this way would be significantly weaker.

Let  $\Delta \subset B$  be a Weierstrass polydisc for  $V$  as in Theorem 8. Then  $\Delta \cap V$  decomposes into a union of irreducible analytic components. We denote by  $\tilde{V} \subset \Delta$  the union of these components where  $F$  does not vanish identically, so that

$$V_F \cap \Delta = \tilde{V} \cap \{F = 0\}. \quad (107)$$

By definition  $\Delta$  is also a Weierstrass polydisc for  $\tilde{V}$ . We let  $\mathcal{R}_F := \mathcal{R}(\tilde{V}, \Delta, F)$  denote the analytic resultant (77) of  $F$  with respect to  $\tilde{V}, \Delta$ . Denote by  $B_z \subset \Delta_z$  the largest ball in  $\Delta_z$  with the same center. Evidently

$$\Delta_z^{O(1)} \subset B_z. \quad (108)$$

We will study the restriction of  $\mathcal{R}_F$  to  $B_z$ .

Let  $L \in \mathbb{C}^m$  be a complex line through the center of  $B_z$  and  $C \subset \mathbb{C}^n$  be the complex curve  $C = V \cap \pi_z^{-1}(L)$ . Let  $\pi_t : \mathbb{C}^n \rightarrow \mathbb{C}$  be an affine combination of the  $\mathbf{z}$  coordinates which maps  $L \cap B_z$  onto the unit disc  $D$ . Denote by  $\Sigma$  the ramification locus of  $\pi_t|_C$ . Consider the system (104) for the pair  $C, \pi_t$ . Then for any  $t_0 \in D \setminus \Sigma$  there exist exactly  $\nu \leq d$  points such that

$$p_1(t_0), \dots, p_\nu(t_0) \in \tilde{V} \cap \pi_t^{-1}(t_0). \quad (109)$$

Moreover, these points extend as ramified algebraic functions for  $t \notin \Sigma$ .

Suppose that  $t_0$  is also not a root of the polynomials  $a_0^1, \dots, a_0^n$ . Then for  $i = 1, \dots, \nu$ , we have a map  $\Phi_i : \mathbb{C}_t \rightarrow \mathbb{C}^N$  defined as in (106) with  $\tilde{x}(t) = p_i(t)$ . In general,  $\|\Phi_i(t_0)\|_\infty$  cannot be bounded in terms of the Noetherian parameters alone: it tends to infinity as  $t_0$  tends to  $\Sigma$  or to a zero of a polynomial  $a_0^j$ . However, we will show that on a suitably chosen annulus around the origin one can indeed control the norms. The key estimate is contained in the following lemma.

**LEMMA 25.** *One can choose  $1/2 < r < 3/4$  and  $\rho = d^{-O(1)}$  such that  $A_{r,\rho} := \{r - \rho < |t| < r + \rho\}$  satisfies*

$$\text{dist}(A_r, \Sigma) = d^{-O(1)}, \quad \min_{\substack{t \in A_{r,\rho} \\ j=1, \dots, n}} |a_0^j(t)| = e^{-d^{O(1)}}. \quad (110)$$

*Proof.* Recall that we have  $\|a_0^j\|_\infty = 1$  and  $\deg a_0^j = d^{O(1)}$ . Then

$$M_D(a_0^j) \geq 1, \quad M_{D^{1/2}}(a_0^j) \leq 2^{d^{O(1)}}, \quad (111)$$

where the lower bound follows from the Cauchy formula on the unit disc and the upper bound is straightforward. Therefore  $\mathfrak{B}_D^2(a_0^j) \leq d^{-O(1)}$ . Then by Lemma 16 we can choose a union  $D_j$  of discs with the sum of the diameters less than  $1/(9n)$  such that

$$\min_{t \in D \setminus D_j} |a_0^j(t)| \geq e^{-d^{O(1)}}. \quad (112)$$

Moreover the center of each disc is a root of  $a_0^j$  and in particular the number of discs does not exceed  $d^{O(1)}$ .

We also have  $\#\Sigma = d^{O(1)}$ , for instance since every point of  $\Sigma$  is a root of some  $a_0^j$ . We let  $D'$  denote the union of discs of radius  $1/(9 \cdot \#\Sigma)$  around each point of  $\Sigma$ , so that the sum of diameters is at most  $1/9$  and we have

$$\text{dist}(\Sigma, D \setminus D') = d^{-O(1)}. \quad (113)$$

In the collection  $D', D_1, \dots, D_n$  we have  $N = d^{O(1)}$  discs with the sum of the diameters at most  $2/9$ . It is then a simple geometric exercise to show that one can choose an annulus  $A_{r,\rho}$  with  $1/4 < r < 1/2$  and  $\rho = d^{-O(1)}$  which is disjoint from the union, which concludes the proof.  $\square$

We are now ready to establish an upper bound for  $\|\Phi_i(t)\|_\infty$ . We choose and fix  $A := A_{r,\rho}$  as in Lemma 25.

LEMMA 26. *For every  $i = 1, \dots, \nu$  and  $t_0 \in A$  we have  $\|\Phi_i(t_0)\|_\infty = e^{d^{O(1)}} \cdot \mathcal{S}(\phi)$ .*

*Proof.* The  $x_j = x_j^{(0)}$  and  $\phi_l$  coordinates are bounded by  $\mathcal{S}(\phi)$  by definition. The  $Q_j$  coordinates are bounded by  $e^{d^{O(1)}}$  by Lemma 25. The  $x_j^{(k)}$  coordinates (for  $0 < k < k_j$ ) are given by  $\partial_t^k(p_i)_j(t_0)$ , where  $(p_i)_j$  denotes the  $x_j$  coordinate of the branch  $p_i$ . Recall that  $p_i$  extends holomorphically as long as  $t \notin \Sigma$ , i.e. by Lemma 25 to a disc of radius  $d^{-O(1)}$ . Moreover in this disc the image of  $p_i$  remains in  $C \cap \Delta \subset \Omega$  and in particular  $|(p_i)_j(t)|$  is bounded by  $\mathcal{S}(\phi)$  in this disc. Applying the Cauchy estimate for  $\partial_t^k(p_i)_j(t_0)$  in the disc we obtain the bound  $\mathcal{S}(\phi) \cdot d^{-O(k)}$  for  $x_j^{(k)}$ , and since we have  $k < k_j \leq d$  this bound is of the required form.  $\square$

Let  $|t_0| = r$  and  $i = 1, \dots, \nu$ . By the choice of  $A$  the trajectory  $\Phi_i(t)$  can be extended to a disc  $D_\rho(t_0)$  and by Lemma 26 we have

$$\|\Phi_i(t)\|_\infty = e^{-d^{O(1)}} \cdot \mathcal{S}(\phi), \quad \forall t \in D_\rho(t_0). \quad (114)$$

We recall also that by (105) we have

$$\|\xi_{\phi,C}\|_\infty = 2^{2^{\text{poly}(d)}} \cdot \mathcal{S}(\phi). \quad (115)$$



We now return to the analysis of the analytic resultant  $\mathcal{R}_F$ , and more specifically its restriction to the complex line  $L$ . Working in the  $t$  coordinate for  $t \in D$ , we have by definition

$$\mathcal{R}_F(t) = F(p_1(t)) \cdots F(p_\nu(t)). \quad (116)$$

Since  $F$  is a Noetherian function of degree  $d$ , the function  $F(p_i(t))$  can be written in the form  $P(\Phi_i(t))$  for  $P$  a polynomial of degree  $d$  on  $\mathbb{C}^N$ . Then by Theorem 6, using (114), (115) and  $N \leq nd + \ell = O(d)$ , we have

$$\mathfrak{B}_{D_\rho(t_0)}^2(P(\Phi_i(t))) = C(\phi, d). \quad (117)$$

Since (117) is true for  $i = 1, \dots, \nu$  we have by Lemma 19 also

$$\mathfrak{B}_{D_\rho(t_0)}^2(\mathcal{R}_F(t)) = C(\phi, d). \quad (118)$$

Recall that  $\mathcal{R}_F$  is in fact a holomorphic function in  $D$ . In particular, its maximum on the disc  $\bar{D}_{r+\rho}(0)$  is obtained somewhere on the boundary. For a suitable choice of  $t_0$ , this same maximum is obtained on  $D_\rho(t_0)$ . On the other hand, the maximum of  $\mathcal{R}_F$  on the disc  $\bar{D}_{r+\rho/2}(0)$  is certainly not smaller than the maximum on the disc  $D_{\rho/2}(t_0)$ . Thus from (118) we see that

$$\mathfrak{B}_{\bar{D}_{r+\rho/2}(0), D_{r+\rho}(0)}(\mathcal{R}_F(t)) = C(\phi, d). \quad (119)$$

The gap in the Bernstein index above is  $\sim \rho$ , and by Corollary 18 we have

$$\mathfrak{B}_{D_{r+\rho}(0)}^2(\mathcal{R}_F(t)) = C(\phi, d). \quad (120)$$

Since  $r > 1/2$  and  $(r + \rho)/2 < 3/8 + o(1)$  the extension  $D_{(r+\rho)/2}(0) \subset D_{1/2}(0)$  has gap uniformly bounded from zero, and we have again by Corollary 18,

$$\mathfrak{B}_{D^2}^2(\mathcal{R}_F(t)) = O(\mathfrak{B}_{\bar{D}_{(r+\rho)/2}(0), D_{1/2}(0)}(\mathcal{R}_F(t))) = C(\phi, d). \quad (121)$$

Since (121) holds for any complex line  $L$  through the center of  $B_z$ , and since  $D^2$  corresponds in the  $t$ -chart to  $B_z^2 \cap L$ , we finally have

$$\mathfrak{B}_{B_z^2}^2(\mathcal{R}_F) = C(\phi, d). \quad (122)$$

By (107) and Proposition 23 applied to the ball  $B_z^2$ , there exists a Weierstrass polydisc  $\Delta' := \Delta'_z \times \Delta'_w$  for  $V_F$  such that

$$B_z^{2\eta'} \subset \Delta'_z \subset B_z^2 \text{ where } \eta' = e^{C(\phi, d)}, \quad (123)$$

$$e(V_F, \Delta') = C(\phi, d). \quad (124)$$

Finally, we deduce from (108), (123) and Theorem 8 that

$$B^\eta \subset \Delta' \subset B, \quad \eta = e^{C(\phi, d)}. \quad (125)$$

which concludes the proof of Theorem 4.

## 6 Rational and Algebraic Points on Noetherian Varieties.

In this section we study rational (and more generally algebraic) points on Noetherian varieties and prove Theorems 1 and 2.

**6.1 Rational points in a Weierstrass polydisc.** We begin by recalling the relation, established in [BN17a, BN17b], between Weierstrass polydiscs and the study of rational points on analytic sets. Let  $X \subset \Omega$  be an analytic set of pure dimension  $m$ . Let  $\Delta := \Delta_z \times \Delta_w$  be a Weierstrass polydisc for  $X$  and set  $\Delta' := \Delta_z \times \Delta_w^{1/3}$  and  $\nu := e(X, \Delta)$ .

**PROPOSITION 27.** *Let  $M, H \geq 3$  and suppose  $\mathbf{f} := (f_1, \dots, f_{m+1}) \in \mathcal{O}(\bar{\Delta}')$  satisfy  $M_{\Delta'}(f_i) \leq M$ . Let*

$$Y := \mathbf{f}(X \cap \Delta^{H^\varepsilon}) \subset \mathbb{C}^{m+1}. \quad (126)$$

*For every  $\varepsilon > 0$  there exists a number*

$$d = O(\nu^{n-m} \varepsilon^{-m} (\log M)^m) \quad (127)$$

*such that  $Y(\mathbb{Q}, H)$  is contained in an algebraic hypersurface of degree at most  $d$  in  $\mathbb{C}^{m+1}$ .*

*Proof.* According to [BN17a, Proposition 11], and plugging in the values of  $\|\mathcal{D}\|$ ,  $e(\mathcal{D})$  from [BN17b, Theorem 3], we see that it suffices to choose  $d$  such that

$$\varepsilon \log H > C_1 \frac{d^{-1} \log(\nu^{n-m}) + \log M + \log H}{(d/\nu^{n-m})^{1/m}}. \quad (128)$$

In particular it is enough to have

$$d > (n - m) \log \nu \quad \text{and} \quad \varepsilon > C_1 \frac{\log M + 2}{(d/\nu^{n-m})^{1/m}}, \quad (129)$$

which is compatible with (127).  $\square$

**6.2 Exploring rational points in complex Noetherian varieties.** The following is our main result for this section.

**Theorem 9.** *Let  $X \subset \Omega$  be a Noetherian variety of degree  $\beta$  and  $\varepsilon > 0$ . There exist constants*

$$d, N = C_n(\phi, \beta \varepsilon^{1-n}) \quad (130)$$

*with the following property. For every  $H \in \mathbb{N}$  there exist at most  $NH^\varepsilon$  irreducible algebraic varieties  $V_\alpha \subset \mathbb{C}^n$  with  $\deg V_\alpha \leq d$  such that*

$$X(\mathbb{Q}, H) \subset \bigcup_{\alpha} X(V_\alpha). \quad (131)$$

The following proposition provides the key inductive step in the proof of Theorem 9.

PROPOSITION 28. *Let  $W \subset \mathbb{C}^n$  be an irreducible algebraic variety of dimension  $m+1$  of degree at most  $\beta$  and let  $X \subset \Omega \cap W$  be a Noetherian variety of degree at most  $\beta$ . Let  $\varepsilon > 0$ . There exist constants*

$$d = C(\phi, \beta)\varepsilon^{-m}, \quad (132)$$

$$N = e^{C(\phi, \beta)}, \quad (133)$$

*with the following property. For every  $H \in \mathbb{N}$  there exist at most  $NH^\varepsilon$  hypersurfaces  $\mathcal{H}_\alpha \subset \mathbb{C}^n$  with  $\deg \mathcal{H}_\alpha \leq d$  such that  $W \not\subset \mathcal{H}_\alpha$  and*

$$X(\mathbb{Q}, H) \subset X(W) \cup \bigcup_{\alpha} \mathcal{H}_\alpha. \quad (134)$$

*Proof.* Let  $\{F_i\}$  denote the finite collection of Noetherian functions of degrees bounded by  $\beta$  such that  $X$  is their common zero locus. As an analytic set,  $W$  may contain several irreducible components  $\{W_h\}$ . We let  $F$  denote a generic linear combination of the  $F_i$  such that for every  $h$ ,  $F$  vanishes identically on  $W_h$  if and only if every  $F_i$  does. Set

$$W_F := (W \cap \{F = 0\})^m. \quad (135)$$

By [BN17b, Lemma 29] there exists a hypersurface  $\mathcal{H}_0 \subset \mathbb{C}^n$  containing  $\text{Sing } W$  and not  $W$  with  $\deg \mathcal{H}_0 \leq \beta$ . If  $p \in X \setminus \mathcal{H}_0$  then  $W$  is smooth at  $p$  and in particular the germ  $W_p$  consists of a single analytic component. If  $F$  vanishes identically on this component then by construction  $W_p \subset X$ , so that  $p \in X(W)$ . Otherwise  $p \in W_F$ , and it remains to construct a collection of hypersurfaces  $\mathcal{H}_\alpha$  as in the statement with

$$W_F(\mathbb{Q}, H) \subset \bigcup_{\alpha} \mathcal{H}_\alpha. \quad (136)$$

Set  $S := \mathcal{S}(\phi)$ . Recall from Lemma 11 that our Noetherian system extends to a  $\rho$ -neighborhood of  $\Omega$  with  $\rho = O(S^{-O(1)})$ , and the Noetherian size of our system in this larger domain is at most  $O(S)$ . Let  $p \in \Omega$  and let  $B$  denote the ball of radius  $\rho$  around  $p$ . According to Theorem 4 there exists a Weierstrass polydisc  $\Delta$  for  $W_F$  and  $\eta > 0$  such that

- (1)  $B^\eta \subset \Delta \subset B$  where  $\eta = e^{C(\phi, \beta)}$ ,
- (2)  $e(W_F, \Delta) = C(\phi, \beta)$ .

We choose  $m+1$  coordinates  $\mathbf{f} := (f_1, \dots, f_{m+1})$  among the standard coordinates on  $\mathbb{C}^n$  such that the projection  $\mathbf{f} : W \rightarrow \mathbb{C}^{m+1}$  is dominant. We apply Proposition 27 to  $W_F, \Delta, \mathbf{f}$  and  $\varepsilon/n$  and note that  $M = O(S)$  to conclude that

$$(W_F \cap B^{H^{\varepsilon/n}\eta})(\mathbb{Q}, H) \subset \mathbf{f}^{-1}[(\mathbf{f}(W_F \cap \Delta^{H^{\varepsilon/n}}))(\mathbb{Q}, H)] \quad (137)$$

is contained in an algebraic hypersurface of degree  $d$  as in the statement, which does not contain  $W$  since  $\mathbf{f}$  is dominant.

Finally it remains to cover  $\Omega$  by balls of the form  $B^{H^{\varepsilon/n}\eta}$ , i.e. balls of radius  $H^{\varepsilon/n} \cdot \eta \cdot \rho = H^{\varepsilon/n} \cdot e^{C(\phi, \beta)}$  with centers  $p \in \Omega$ , and take the collection of corresponding hypersurfaces. The domain  $\Omega$  is contained in a ball of radius  $S$ , and a simple subdivision argument shows that this can be done with  $NH^\varepsilon$  balls as above.  $\square$

The following lemma gives an inductive proof of Theorem 9, which is obtained for the case  $W = \mathbb{C}^n$ .

**LEMMA 29.** *Let  $W \subset \mathbb{C}^n$  be an irreducible algebraic variety of dimension  $m+1$  and degree at most  $\beta$  and let  $X \subset \Omega \cap W$  be a Noetherian variety of degree at most  $\beta$ . There exist constants*

$$d, N = C_{m+1}(\phi, \beta \varepsilon^{-m}) \quad (138)$$

*with the following property. For every  $H \in \mathbb{N}$  there exist at most  $NH^\varepsilon$  irreducible algebraic varieties  $V_\alpha \subset W$  with  $\deg V_\alpha \leq d$  such that*

$$X(\mathbb{Q}, H) \subset \bigcup_{\alpha} X(V_\alpha). \quad (139)$$

*Proof.* We proceed by induction on  $m$ , where the case  $m = -1$  is trivial. By Proposition 28 applied to  $X, W$  we have a collection of at most  $e^{C(\phi, \beta)} H^{\varepsilon/2}$  hypersurfaces  $\mathcal{H}_{\alpha'}$  of degrees  $C(\phi, \beta) \varepsilon^{-m}$  such that

$$X(\mathbb{Q}, H) \subset X(W) \cup \bigcup_{\alpha'} \mathcal{H}_{\alpha'}. \quad (140)$$

Let  $\{W_\alpha\}$  denote the union over  $\alpha'$  of the sets of irreducible components of  $W \cap \mathcal{H}_{\alpha'}$ . The degree of the intersection is bounded by the product of the degrees, and since the number of irreducible components of a variety is bounded by its degree we have

$$\deg W_\alpha = C(\phi, \beta) \varepsilon^{-m}, \quad \dim W_\alpha = m, \quad \#\{W_\alpha\} = e^{C(\phi, \beta)} \varepsilon^{-m} H^{\varepsilon/2}, \quad (141)$$

and

$$X(\mathbb{Q}, H) \subset X(W) \cup \bigcup_{\alpha} (X \cap W_\alpha). \quad (142)$$

We now apply the inductive hypothesis to each pair  $W_\alpha, X \cap W_\alpha$  with the exponent  $\varepsilon/2$  to obtain collections  $W_{\alpha, \beta}$  with

$$\deg W_{\alpha, \beta} = C_m(\phi, C(\phi, \beta) \varepsilon^{-2m}), \quad (143)$$

$$\#\{W_{\alpha, \beta}\} = C_m(\phi, C(\phi, \beta) \varepsilon^{-2m}) \cdot H^{\varepsilon/2}, \quad (144)$$

such that

$$(X \cap W_\alpha)(\mathbb{Q}, H) \subset X(W_{\alpha, \beta}). \quad (145)$$

Finally we take  $\{V_\alpha\}$  to be the union of the sets  $\{W\}$  and  $\{W_{\alpha, \beta}\}$ .  $\square$

**6.3 Exploring algebraic points in complex Noetherian varieties.** Our goal in the section is to establish the following generalization of Theorem 9.

**Theorem 10.** *Let  $X \subset \Omega$  be a Noetherian variety of degree  $\beta$  and  $\varepsilon > 0$ . There exist constants*

$$d, N = C_{n(k+1)}(\phi, \beta \varepsilon^{1-n}) \quad (146)$$

*with the following property. For every  $H \in \mathbb{N}$  there exist at most  $NH^\varepsilon$  irreducible algebraic varieties  $V_\alpha \subset \mathbb{C}^n$  with  $\deg V_\alpha \leq d$  such that*

$$X(k, H) \subset \bigcup_{\alpha} X(V_\alpha). \quad (147)$$

The proof of Theorem 10, adapted from [Pil09], is given in the remainder of this section. We begin by setting up some notation. Let  $\mathcal{P}_{\leq k} := \mathbb{R}^{k+1} \setminus \{\mathbf{0}\}$ . For  $\mathbf{c} \in \mathcal{P}_{\leq k}$  let  $P_{\mathbf{c}} \in \mathbb{R}[x]$  denote the polynomial

$$P_{\mathbf{c}}(X) := \sum_{j=0}^k c_j X^j. \quad (148)$$

Following [Pil09] we introduce the following height function. For an algebraic number  $\alpha \in \mathbb{Q}^{\text{alg}}$  we define

$$H_k^{\text{poly}}(\alpha) = \min\{H(\mathbf{c}) : \mathbf{c} \in \mathcal{P}_{\leq k}(\mathbb{Q}), \quad P_{\mathbf{c}}(\alpha) = 0\} \quad (149)$$

and  $H_k^{\text{poly}}(\alpha) = \infty$  if  $[\mathbb{Q}(\alpha) : \mathbb{Q}] > k$ . Then whenever  $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq k$  we have [Pil09, 5.1]

$$H_k^{\text{poly}}(\alpha) \leq 2^k H(\alpha)^k. \quad (150)$$

We define  $X^{\text{poly}}(k, H)$  in analogy with  $X(k, H)$  replacing  $H(\cdot)$  by  $H^{\text{poly}}(\cdot)$ . In light of (150), it will suffice to prove the claim for  $X^{\text{poly}}(k, H)$ .

Let  $\Sigma \subset \mathbb{C}^n \times \mathcal{P}_{\leq k}^n$  be the algebraic variety given by

$$\Sigma := \{(\mathbf{x}, \mathbf{c}_1, \dots, \mathbf{c}_n) : P_{\mathbf{c}_1}(x_1) = \dots = P_{\mathbf{c}_n}(x_n) = 0\}. \quad (151)$$

and denote by  $\pi_1, \pi_2$  the projections to  $\mathbb{C}^n, \mathcal{P}_{\leq k}^n$  respectively. Let  $Y := \pi_1^{-1}(X) \cap \Sigma$ . Note that  $Y$  is a Noetherian variety of degree  $O(\beta)$  and Noetherian size  $O(\mathcal{S}(\phi))$ .

Set  $\tilde{\Omega} = \Omega \times U_{\leq k}^n$  where  $U_{\leq k} \subset \mathcal{P}_{\leq k}$  is given by

$$U_{\leq k} = \{\mathbf{c} : 1/2 < \max_{j=0, \dots, k} |c_j| < 2\}. \quad (152)$$

Denote  $\tilde{Y} := \tilde{\Omega} \cap Y$  and

$$\tilde{Y}(\mathbb{Q}, H; \pi_2) := \{\mathbf{y} \in \tilde{Y} : H(\pi_2(\mathbf{y})) \leq H\}. \quad (153)$$

We claim that

$$X^{\text{poly}}(k, H) \subset \pi_1[\tilde{Y}(\mathbb{Q}, H^2; \pi_2)]. \quad (154)$$

Indeed, let  $\mathbf{x} \in X^{\text{poly}}(k, H)$ , and for every coordinate  $x_i$  choose the corresponding polynomials  $P_{\mathbf{c}_i}$  as in (149). Then the coefficient of each  $\mathbf{c}_i$  are bounded by  $H$ , and we let  $\mathbf{c}'_i$  be the vector obtained by normalizing them to have maximum 1 so that  $\mathbf{c}_i \in U_{\leq k}$ . Then we have  $H(\mathbf{c}'_i) \leq H^2$  so  $(\mathbf{x}, \mathbf{c}'_1, \dots, \mathbf{c}'_n) \in \tilde{Y}(\mathbb{Q}, H^2; \pi_2)$ . We now turn to the description of  $\tilde{Y}(\mathbb{Q}, H; \pi_2)$ .

LEMMA 30. *There exist constants*

$$d, N = C_{n(k+1)}(\phi, \beta \varepsilon^{-m}) \quad (155)$$

*with the following property. For every  $H \in \mathbb{N}$  there exist at most  $NH^\varepsilon$  irreducible algebraic varieties  $\tilde{V}_\alpha \subset \mathbb{C}^n \times \mathcal{P}_{\leq k}^n$  with  $\deg \tilde{V}_\alpha \leq d$  such that*

$$\tilde{Y}(\mathbb{Q}, H; \pi_2) \subset \bigcup_{\alpha} \tilde{Y}(\tilde{V}_\alpha). \quad (156)$$

*Proof.* The claim follows essentially by repetition of the proof of Theorem 9 with the following modification. Since  $Y$  is a subset of the algebraic variety  $\Sigma$  we begin our induction in Lemma 29 with  $W = \Sigma$  rather than  $W = \mathbb{C}^n \times \mathcal{P}_{\leq k}^n$ . Note that  $\dim \Sigma = \dim \mathcal{P}_{\leq k}^n = n(k+1)$ .

In the notations of the proof of Proposition 28, rather than choosing the coordinates  $\mathbf{f}$  from all coordinates on  $\mathbb{C}^n \times \mathcal{P}_{\leq k}^n$ , we claim that it suffices to consider only coordinates on  $\mathcal{P}_{\leq k}^n$  (i.e. coordinates of  $\pi_2$ ), thereby obtaining a description of  $\tilde{Y}(\mathbb{Q}, H; \pi_2)$  instead of  $\tilde{Y}(\mathbb{Q}, H)$ . This is permissible since the projection  $\pi_2$  has finite fibers when restricted to  $\Sigma \cap \tilde{\Omega}$ , and  $\pi_2$  is therefore dominant on it as required. When we continue the induction  $\Sigma$  is replaced by a collection of its irreducible subvarieties (and we may as well consider only those that meet  $\tilde{\Omega}$ ), and the same argument applies. The rest of the inductive proof proceeds as in Lemma 29.  $\square$

The following lemma, in combination with (154) and Lemma 30, completes the proof of Theorem 10.

LEMMA 31. *Let  $\tilde{V} \subset \mathbb{C}^n \times \mathcal{P}_{\leq k}^n$  be an irreducible variety of degree  $d$ . Then there exists a collection of  $d^{O(1)}$  irreducible varieties  $V_j \subset \mathbb{C}^n$  of degree  $d^{O(1)}$  such that*

$$\pi_1(\tilde{Y}(\tilde{V}_\alpha)) \subset \bigcup_j X(V_j). \quad (157)$$

*Proof.* Write  $V := \pi_1(\tilde{V})$ . By [BN17b, Lemma 29] there exist a hypersurface  $\tilde{\mathcal{H}} \subset \mathbb{C}^n \times \mathcal{P}_{\leq k}^n$  (resp.  $\mathcal{H} \subset \mathbb{C}^n$ ) containing  $\text{Sing } \tilde{V}$  (resp.  $\text{Sing } V$ ) and not containing  $\tilde{V}$  (resp.  $V$ ). Proceeding by induction over dimension for the irreducible components of  $\tilde{V} \cap \tilde{\mathcal{H}}$  and  $\tilde{V} \cap \pi_1^{-1}(\mathcal{H})$ , we obtain a collection of varieties  $V'_j$  such that

$$\pi_1(\tilde{Y}(\text{Sing } \tilde{V})) \cup \pi_1(\tilde{Y}(\tilde{V} \cap \pi_1^{-1}(\text{Sing } \tilde{V}))) \subset \bigcup_j X(V'_j). \quad (158)$$

It is easy to verify inductively that size and degrees of the collection  $V'_j$  satisfy the required asymptotic estimates. To complete the construction, we let  $\tilde{p} \in \tilde{Y}(\tilde{V})$  and suppose that  $\tilde{p} \notin \text{Sing } \tilde{V}$  and  $p := \pi_1(\tilde{p}) \notin \text{Sing } V$ . We claim that in this case  $p \in X(V)$  (so taking the collection  $V'_j$  in addition to  $V$  completes the proof).

Fix a small ball  $\tilde{B} \subset \mathbb{C}^n \times \mathcal{P}_{\leq k}^n$  around  $\tilde{p}$  such that  $\tilde{V}$  is smooth in  $\tilde{B}$  and  $V$  is smooth in  $B := \pi_1(\tilde{B})$ . By the Sard theorem applied to  $\pi_1|_{\tilde{B} \cap \tilde{V}}$ , we may find a point  $\tilde{q} \in \tilde{B} \cap \tilde{V}$  arbitrarily close to  $\tilde{p}$  which is a non-critical point of  $\pi_1|_{\tilde{B} \cap \tilde{V}}$ . In particular  $\pi_1|_{\tilde{B} \cap \tilde{V}}$  is submersive at  $\tilde{q}$ , so there exists a neighborhood  $U_{\tilde{q}} \subset \tilde{B} \cap \tilde{V}$  of  $\tilde{q}$  such that  $U_q := \pi_1(U_{\tilde{q}}) \subset B \cap V$  is a neighborhood of  $q := \pi_1(\tilde{q})$  in  $B \cap V$ . Now since  $\tilde{p} \in \tilde{Y}(\tilde{V})$ , we may assume (for an appropriate choice of  $\tilde{q}$ ) that  $U_{\tilde{q}} \subset \tilde{Y}$ . Then by definition of  $\tilde{Y}$  it follows that  $U_q = \pi_1(U_{\tilde{q}}) \subset X$ .

In conclusion, we see that  $X$  contains the germ of  $V$  at points  $q$  arbitrarily close to  $p$ . Since the germ of  $V$  at  $p$  is irreducible (in fact smooth) and  $X$  is analytic, it follows that  $X$  contains the germ of  $V$  at  $p$ , i.e.  $p \in X(V)$  as claimed.  $\square$

**6.4 The main result in the real setting.** Finally we are ready to conclude the proof of Theorem 2 by reduction to the case of (complex) Noetherian varieties.

*Proof of Theorem 2.* It clearly suffices to consider the case of a single basic semi-Noetherian set. Next, one can easily reduce to the case of Noetherian varieties by dropping all inequalities. Indeed suppose that  $X = Y \cap U$  where  $Y$  is a Noetherian variety and  $U$  is defined by a collection of strict Noetherian inequalities. Then if  $\{S_\alpha\}$  is a collection constructed for  $Y$  as in the conclusion of Theorem 2, we have

$$X(k, H) \subset X \cap Y(k, H) \subset X \cap \left[ \bigcup_{\alpha} Y(S_\alpha) \right] \subset \bigcup_{\alpha} X(S_\alpha) \quad (159)$$

where the final inclusion follows since  $X$  is locally open in  $Y$ , see e.g. [BN17b, Lemma 26]. Henceforth we assume that  $X$  is a real Noetherian variety of degree  $\beta$ .

Recall from Sect. 2.1 that the real Noetherian chain used in the definition of  $X$  admits holomorphic continuation to a complex Noetherian chain in a domain  $\Omega \supset \Omega_{\mathbb{R}}$  with  $\Omega \cap \mathbb{R}^n = \Omega_{\mathbb{R}}$ , and that the Noetherian size of this chain is at most twice the Noetherian size of the real chain. We let  $\tilde{X} \subset \Omega$  denote the complex Noetherian variety defined by the (holomorphic continuations of) the real Noetherian functions defining  $X$ , so that  $X = \tilde{X} \cap \mathbb{R}^n$ .

Now let  $\{\tilde{V}_\alpha\}$  denote the collection constructed for  $\tilde{X}$  as in Theorem 10 and set  $V_\alpha := \mathbb{R}^n \cap \tilde{V}_\alpha$ . Then  $V_\alpha$  is cut out by the equations of  $\tilde{V}_\alpha$  in addition to linear equations for the vanishing of the imaginary parts, and in particular has complexity bounded by  $O(\beta)$ . By [GV95, Theorem 2] one can decompose  $V_\alpha$  into a union of  $\beta_2 := \beta_1^{2^{O(m)}}$  smooth (but not necessarily connected) semialgebraic sets of complexity  $\beta_2$ . Finally, by [BPR06, Theorem 16.13] each such semialgebraic set can be decomposed into its connected components, with the number of connected components bounded by  $\beta_3 = \beta_2^{O(m^4)}$  and their complexity bounded by  $\beta_3$ . We let  $\{S_\eta\}$  denote the union of the collections of these components for every  $V_\alpha$ . One then easily verifies that  $\{S_\eta\}$



satisfies the stated asymptotic estimates for the size and complexity, and finally we have

$$X(k, H) \subset \mathbb{R}^n \cap \tilde{X}(k, H) \subset \mathbb{R}^n \cap \left[ \bigcup_{\alpha} \tilde{X}(\tilde{V}_{\alpha}) \right] \subset \bigcup_{\alpha} X(V_{\alpha}) \subset \bigcup_{\eta} X(S_{\eta}). \quad (160)$$

□

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