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Isomorphism Testing for Graphs of Bounded Rank Width

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Abstract

We give an algorithm that, for every fixed k, decides isomorphism of graphs of rank width at most k in polynomial time. As the rank width of a graph is bounded in terms of its clique width, we also obtain a polynomial time isomorphism test for graph classes of bounded clique width.

I. Introduction

Rank width, introduced by Oum and Seymour [27], is a graph invariant that measures how well a graph can be recursively decomposed along "simple separations". In this sense, it resembles tree width, but it fundamentally differs from tree width in how the "simplicity" of a separation is measured: for rank width, the idea is to take the row rank (over the field $\mathbb{F}_2$) of the matrix that records the adjacencies between the two parts of a separation, whereas for tree width one simply counts how many vertices the two parts have in common. Rank width is bounded in terms of tree width, but not vice versa. For example, the complete graph K_n has rank width 1 and tree width $n-1$. This also shows that graphs of bounded rank width are not necessarily sparse (as opposed to graphs of bounded tree width). An interesting aspect of rank width when dealing with problems like graph isomorphism testing (or various problems related to logical definability) that make no real distinction between the edge relation and the "non-edge relation" of a graph is that the rank width of a graph and its complement differ by at most one. Another well-known graph invariant is clique width [6]; it measures how many labels are needed to generate a graph in a certain grammar. Rank width is equivalent to clique width, in the sense that each of the two invariants is bounded in terms of the other [27]. As for bounded tree width, many hard algorithmic problems can be solved in polynomial time (often cubic time) on graph classes of bounded rank width, or equivalently, bounded clique width (e.g. [5], [8], [10], [19]). However, until now it was open whether the isomorphism problem is among them.

We give an algorithm that, for every fixed k, decides isomorphism of graphs of rank width at most k in polynomial time. Many of the best known graph classes where the isomorphism problem is known to be in polynomial time are classes of sparse graphs [17], [9], [26], [25], [29], [2], [14], among them planar graphs, graphs of bounded degree, and graphs of bounded tree width. Less is known for dense graphs; among the known results are polynomial time isomorphism tests for classes with bounded eigenvalue multiplicities [1] and various hereditary graph classes, specifically classes intersection graphs [7], [20], among them interval graphs [24], and classes defined by excluding specific induced subgraphs [3], [21], [33]. Our result substantially extends the realm of hereditary graph classes with a tractable isomorphism problem. While it subsumes several known results [2], [4], [22], [33], for the classes of clique width at most k a polynomial time isomorphism algorithm was only known for the case $k \leq 2$ ([22]).

Technically, we found the isomorphism problem for bounded rank width graphs much harder than anticipated. The overall proof strategy is generic: first compute a canonical decomposition of a graph, or if that is impossible, a canonical family of decompositions with a compact representation, and then use dynamic programming to solve the isomorphism problem. Indeed, this is the strategy taken for bounded tree width graphs in [2], [23]. However, for graphs of bounded rank width, both steps of this general strategy turned out to be difficult to implement. To compute canonical decompositions, we heavily rely on the general theory of connectivity functions, branch decompositions, and tangles [30], [11], and in particular on computational aspects of the theory recently developed in [15]. Our starting point is an algorithm for canonically decomposing a connectivity function into highly connected regions described by maximal tangles [15]. The technical core of the first part of this paper is a decomposition of these highly connected regions into pieces of bounded width (Lemma VII.1). It has been slightly disturbing to find that even with a canonical decomposition given, the isomorphism problem is still nontrivial and requires a complicated (though elementary) group theoretic machinery. The intuitive reason for this can be explained by a comparison with bounded tree width. In a bounded-width tree decomposition of a graph, we have low order vertex separations of the graph, and after removing the separating vertices (a bounded number) we can deal with the two parts of a separation independently. In a bounded-rank-width decomposition, we have partitions of the graph into two parts such that

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the adjacency matrix between these parts has low rank. For such a partition, removing a bounded number of vertices shows no effect. Instead, we need to fix a bounded number of rows and columns in the matrix, but even then there is a nontrivial interaction between the two parts, which fortunately we can capture group theoretically.

The paper is organised as follows: after reviewing the necessary background in Section II, in the short Section III, we show that all tangles of a connectivity function have "triple covers" of bounded size, providing another technical tool for dealing with tangles (which may be of independent interest). In Section IV, we introduce treelike decompositions of connectivity functions, which may be viewed as compact representations of families of tree decompositions. Sections V–VII are devoted to a proof of the canonical decomposition theorem (Theorem VII.2). In Section VIII, we describe the situation at a single node of our decomposition and its children in matrix form and introduce the notion of partition rank of a matrix to capture the width of the decomposition at this node. Finally, in Sections IX we develop the group theoretic machinery and give the actual isomorphism algorithm. In the paper, some proofs have been omitted, they can be found in the full version [16].

Throughout this paper, we often speak of "canonical" constructions. The precise technical meaning depends on the context, but in general a construction (or algorithm) is *canonical* if every isomorphism between its input objects commutes with an isomorphism between the output objects.

II. Connectivity Functions, Tangles, and Branch Decompositions

A *connectivity function* on a finite set A is a symmetric and submodular function $\kappa\colon 2^A \to \mathbb{N}$ with $\kappa(\emptyset) = 0$. *Symmetric* means that $\kappa(X) = \kappa(\overline{X})$ for all $X \subseteq A$; here and whenever the ground set A is clear from the context we write $\overline{X}$ to denote $A \setminus X$, the complement of X. *Submodular* means that $\kappa(X) + \kappa(Y) \geq \kappa(X \cap Y) + \kappa(X \cup Y)$ for all $X, Y \subseteq A$. Observe that a symmetric and submodular set function is also *posimodular*, that is, it satisfies $\kappa(X) + \kappa(Y) \geq \kappa(X \setminus Y) + \kappa(Y \setminus X)$ (apply submodularity to X and $\overline{Y}$).

The only connectivity function that we consider in this paper is the *cut rank* function ρ_G of a graph G. For all subsets $X, Y \subseteq V(G)$, we let $M_{X,Y}$ be the $X \times Y$-matrix over the 2-element field $\mathbb{F}_2$ with entries $m_{xy} = 1 \iff xy \in E(G)$. We define $\rho_G : 2^{V(G)} \to \mathbb{N}$ by letting $\rho_G(X)$ be the row rank of the matrix $M_{X,Y}$ over $\mathbb{F}_2$. It is not hard to prove that ρ_G is indeed a connectivity function.

For the rest of this section, let κ be a connectivity function on a finite set A. We often think of a subset $Z \subseteq A$ as a *separation* of A into Z and $\overline{Z}$ and of $\kappa(Z)$ as the *order* of this separation; consequently, we also refer to $\kappa(Z)$ as the *order of* Z. For disjoint sets $X, Y \subseteq A$, an (X,Y)*-separation* is a set $Z \subseteq A$ such that $X \subseteq Z \subseteq \overline{Y}$. Such a separation Z is minimum if its order is minimum, that is, if $\kappa(Z) \leq \kappa(Z')$ for all (X,Y)-separations Z'. It is an easy consequence of the submodularity of κ that there is a unique minimum (X,Y)-separation Z such that $Z \subseteq Z'$ for all other minimum (X,Y)-separations Z'. We call Z the *leftmost minimum* (X,Y)*-separation*. There is also a unique *rightmost minimum* (X,Y)*-separation*, which is easily seen to be the complement of the leftmost minimum (Y,X)-separation.

A κ*-tangle* of order $k \geq 0$ is a set $\mathcal{T} \subseteq 2^A$ satisfying the following conditions.

(T.0) $\kappa(X) < k$ for all $X \in \mathcal{T}$,

(T.1) For all $X \subseteq A$ with $\kappa(X) < k$, either $X \in \mathcal{T}$ or $\overline{X} \in \mathcal{T}$.

(T.2) $X_1 \cap X_2 \cap X_3 \neq \emptyset$ for all $X_1, X_2, X_3 \in \mathcal{T}$.

(T.3) $\mathcal{T}$ does not contain any singletons, that is, $\{a\} \notin \mathcal{T}$ for all $a \in A$.

We denote the order of a κ-tangle $\mathcal{T}$ by $\operatorname{ord}(\mathcal{T})$.[1] It is known [30] that for each k there is at most a linear number of κ-tangles of order k.

Let $\mathcal{T}, \mathcal{T}'$ be κ-tangles. If $\mathcal{T}' \subseteq \mathcal{T}$, we say that $\mathcal{T}$ is an *extension* of $\mathcal{T}'$. The tangles $\mathcal{T}$ and $\mathcal{T}'$ are *incomparable* (we write $\mathcal{T} \bot \mathcal{T}'$) if neither is an extension of the other. The *truncation* of $\mathcal{T}$ to order $k \leq \operatorname{ord}(\mathcal{T})$ is the set $\{X \in \mathcal{T} \mid \kappa(X) < k\}$, which is obviously a tangle of order k. Observe that if $\mathcal{T}$ is an extension of $\mathcal{T}'$, then $\operatorname{ord}(\mathcal{T}') \leq \operatorname{ord}(\mathcal{T})$, and $\mathcal{T}'$ is the truncation of $\mathcal{T}$ to order $\operatorname{ord}(\mathcal{T}')$.

A κ-tangle $\mathcal{T}$ is *maximal* if there is no κ-tangle $\mathcal{T}' \subseteq \mathcal{T}$ with $\operatorname{ord}(\mathcal{T}') > \operatorname{ord}(\mathcal{T})$. A κ-tangle $\mathcal{T}$ is ℓ*-maximal*, for some $\ell \geq 0$, if either $\operatorname{ord}(\mathcal{T}) = \ell$ or $\mathcal{T}$ is maximal.

A $(\mathcal{T}, \mathcal{T}')$*-separation* is a set $Z \subseteq A$ such that $Z \in \mathcal{T}$ and $\overline{Z} \in \mathcal{T}'$. Obviously, if Z is a $(\mathcal{T}, \mathcal{T}')$-separation then $\overline{Z}$ is a $(\mathcal{T}', \mathcal{T})$-separation. Observe that there is a $(\mathcal{T}, \mathcal{T}')$-separation if and only if $\mathcal{T}$ and $\mathcal{T}'$ are incomparable. The *order* of a $(\mathcal{T}, \mathcal{T}')$-separation Z is $\kappa(Z)$. A $(\mathcal{T}, \mathcal{T}')$-separation Z is *minimum* if its order is minimum. It can be shown [15] that if

[1] There is a small technical issue that one needs to be aware of, but that never causes any real problems: if we view tangles as families of sets, then their order is not always well-defined. Indeed, if there is no set X of order $\kappa(X) = k-1$, then every tangle of order k is equal to its truncation to order $k-1$. In such a situation, we have to explicitly annotate a tangle with its order, formally viewing a tangle as a pair $(\mathcal{T}, k)$ where $\mathcal{T} \subseteq 2^A$ and $k \geq 0$.

$\mathcal{T}\perp\mathcal{T}'$ then there is a unique minimum $(\mathcal{T},\mathcal{T}')$-separation Z such that $Z \subseteq Z'$ for all minimum $(\mathcal{T},\mathcal{T}')$-separations Z'. We call Z the *leftmost minimum* $(\mathcal{T},\mathcal{T}')$*-separation*. Of course there is also a *rightmost minimum* $(\mathcal{T},\mathcal{T}')$*-separation*, which is the complement of the leftmost minimum $(\mathcal{T}',\mathcal{T})$-separation.

Now that we have defined (X,Y)-separations for sets X,Y and $(\mathcal{T},\mathcal{T}')$-separations for tangles $\mathcal{T},\mathcal{T}'$, we also need to define combinations of both. For a κ-tangle $\mathcal{T}$ and a set $X \subseteq A$ such that $X \notin \mathcal{T}$, a $(\mathcal{T},X)$*-separation* is a set $Z \in \mathcal{T}$ such that $Z \subseteq \overline{X}$. A $(\mathcal{T},X)$-separation is *minimum* if its order is minimum, and again it can be proved that if there is a $(\mathcal{T},X)$-separation, then there is a unique *leftmost minimum* $(\mathcal{T},X)$*-separation* and a *rightmost minimum* $(\mathcal{T},X)$*-separation*. Analogously, we define (leftmost, rightmost minimum) $(X,\mathcal{T})$-separations.

Lemma II.1. *Let $\mathcal{T},\mathcal{T}'$ be κ-tangles and $X,Y \subseteq A$ such that neither $Y \subseteq X$ nor $\overline{Y} \subseteq X$.*
(1) If X is a minimum $(\mathcal{T},\mathcal{T}')$-separation, then $\kappa(X\cap Y) \le \kappa(Y)$ or $\kappa(X\cap\overline{Y}) \le \kappa(Y)$.
(2) If X is a rightmost minimum $(\mathcal{T},\mathcal{T}')$-separation, then $\kappa(X\cap Y) < \kappa(Y)$ or $\kappa(X\cap\overline{Y}) < \kappa(Y)$.

The last concept we need to define is that of branch decompositions and branch width of a connectivity function. A *cubic tree* is a tree where every node that is not a leaf has degree 3. An *oriented edge* of a tree T is a pair (s,t), where $st \in E(T)$. We denote the set of all oriented edges of T by $\overrightarrow{E}(T)$ and the set of leaves of T by $L(T)$. A *branch decomposition* of κ is a pair (T,ξ), where T is a cubic tree and $\xi\colon L(T) \to A$ is a bijective mapping. For every oriented edge $(s,t) \in \overrightarrow{E}(T)$, we let $\widetilde{\xi}(s,t) \subseteq A$ be the set of all $\xi(u)$ where u is a leaf in the component of $T-\{st\}$ that contains t (so the oriented edge (s,t) points towards u). Observe that $\widetilde{\xi}(s,t) = \overline{\widetilde{\xi}(t,s)}$. We define the *width* of the branch decomposition (T,ξ) to be

$$\mathrm{wd}(T,\xi) := \max\{\kappa(\widetilde{\xi}(s,t)) \mid (s,t) \in \overrightarrow{E}(s,t)\}.$$

The *branch width* $\mathrm{bw}(\kappa)$ of κ is the minimum of the width of all branch decompositions of κ. The *rank width* of a graph G is defined to be the branch width of the cut rank function ρ_G.

Theorem II.2 (Duality Theorem [30]). *The branch width of κ is exactly the maximum order of a κ-tangle.*

For disjoint sets $X,Y \subseteq A$ we define $\kappa_{\min}(X,Y) := \min\{\kappa(Z) \mid X \subseteq Z \subseteq \overline{Y}\}$. Note that for all X,Y the two functions $X' \mapsto \kappa_{\min}(X',Y)$ and $Y' \mapsto \kappa_{\min}(X,Y')$ are monotone and submodular.

For sets $Y \subseteq X$, we say that a set Y is *free* in X if $\kappa_{\min}(Y,\overline{X}) = \kappa(X)$ and $|Y| \le \kappa(X)$. It can be shown that for every $X \subseteq A$ there is a set Y that is free in X [31], [15].

A. Computing with Tangles

Algorithms expecting a set function $\kappa : 2^A \to \mathbb{N}$ as input are given the ground set A as actual input (say, as a list of objects), and they are given an oracle that returns for $X \subseteq A$ the value of $\kappa(X)$. The running time of such algorithms is measured in terms of the size $|A|$ of the ground set. We assume this computation model for all algorithms dealing with abstract connectivity functions κ. Of course, if $\kappa = \rho_G$ is the cut rank function of a graph G, then we assume a standard computation model (without oracles), where the graph G is given as input; we can use G to simulate oracle access to ρ_G.

An important fact underlying most of our algorithms is that, under this model of computation, submodular functions can be efficiently minimised [18], [32].

In [15], we introduced a data structure for representing all tangles of a graph up to a certain order. A *comprehensive tangle data structure* of order k for a connectivity function κ over a set A is a data structure $\mathcal{D}$ with functions $\textsc{Order}_{\mathcal{D}}$, $\textsc{Size}_{\mathcal{D}}$, $\mathcal{T}_{\mathcal{D}}$, $\textsc{TangOrd}_{\mathcal{D}}$, $\textsc{Trunc}_{\mathcal{D}}$, $\textsc{Sep}_{\mathcal{D}}$, and $\textsc{Find}_{\mathcal{D}}$ that provide the following functionalities.

(1) The function $\textsc{Order}_{\mathcal{D}}()$ returns the fixed integer k.
(2) For $\ell \in [k]$ the function $\textsc{Size}_{\mathcal{D}}(\ell)$ returns the number of κ-tangles of order at most ℓ. We denote the number of κ-tangles of order at most k by $|\mathcal{D}|$.
(3) For each $i \in \big[|\mathcal{D}|\big]$ the function $\mathcal{T}_{\mathcal{D}}(i,\cdot)\colon 2^A \to \{0,1\}$ is a tangle $\mathcal{T}_i$ of order at most k, (i.e., the function call $\mathcal{T}_{\mathcal{D}}(i,X)$ determines whether $X \in \mathcal{T}_i$).
We call i the *index* of the tangle $\mathcal{T}_i$ within the data structure.
(4) For $i \in \big[|\mathcal{D}|\big]$ the call $\textsc{TangOrd}_{\mathcal{D}}(i)$ returns $\mathrm{ord}(\mathcal{T}_i)$.
(5) For $i \in \big[|\mathcal{D}|\big]$ and $\ell \le \mathrm{ord}(\mathcal{T}_i)$ the call $\textsc{Trunc}_{\mathcal{D}}(i,\ell)$ returns an integer j such that $\mathcal{T}_j$ is the truncation of $\mathcal{T}_i$ to order ℓ. If $\ell > \mathrm{ord}(\mathcal{T}_i)$ the function returns i.
(6) For distinct $i,j \in \big[|\mathcal{D}|\big]$ the call $\textsc{Sep}_{\mathcal{D}}(i,j)$ outputs a set $X \subseteq A$ such that X is the leftmost minimum $(\mathcal{T}_i,\mathcal{T}_j)$-separation or states that no such set exists (in which case one of the tangles is a truncation of the other).
(7) Given $\ell \in \{0,\dots,k\}$ and a tangle $\mathcal{T}'$ of order ℓ (via a membership oracle) the function $\textsc{Find}_{\mathcal{D}}(\ell,\mathcal{T}')$, returns the index of $\mathcal{T}'$, that is, the unique integer $i \in \big[|\mathcal{D}|\big]$ such that $\mathrm{ord}(\mathcal{T}_i) = \ell$ and $\mathcal{T}' = \mathcal{T}_i$.

We say a comprehensive tangle data structure $\mathcal{D}$ is *efficient* if all functions $\textsc{Order}_{\mathcal{D}}$, $\textsc{Size}_{\mathcal{D}}$, $\mathcal{T}_{\mathcal{D}}$, $\textsc{TangOrd}_{\mathcal{D}}$, $\textsc{Trunc}_{\mathcal{D}}$, $\textsc{Sep}_{\mathcal{D}}$, and $\textsc{Find}_{\mathcal{D}}$ can be evaluated in polynomial time.

Theorem II.3 ([15]). *For every constant k there is a polynomial time algorithm that, given oracle access to a connectivity function κ, computes an efficient comprehensive tangle data structure of order k.*

Using a comprehensive tangle data structure, we can design polynomial time algorithms for other computational problems related to tangles.

Lemma II.4. *Let $k \geq 0$.*

(1) There is a polynomial time algorithm that, given a set $X \subseteq A$ and a tangle $\mathcal{T}$ of order k (via its index in a comprehensive tangle data structure), computes the leftmost minimum $(\mathcal{T}, X)$-separation if it exists or reports that there is no $(\mathcal{T}, X)$-separation.

(2) There is a polynomial time algorithm that, given a tangle $\mathcal{T}$ of order k (via its index in a comprehensive tangle data structure), computes a list of all inclusionwise minimal elements of $\mathcal{T}$.

Proof: Assertion (1) follows from Lemma 2.20 of [15].

To prove (2), we claim that a set $X \in A$ is an inclusionwise minimal element of a tangle $\mathcal{T}$ of order k if and only if the following two conditions are satisfied.

(i) There is a set $Y \subseteq \overline{X}$ of size $|Y| \leq k$ such that X is the leftmost minimum $(\mathcal{T}, Y)$-separation.

(ii) There is a no set $Z \subseteq A$ of size $|Z| \leq k$ such that the leftmost minimum $(\mathcal{T}, Z)$-separation is a proper subset of X.

To see this, we simply observe that if X is an inclusionwise minimal element of $\mathcal{T}$, then it trivially satisfies (ii), and it satisfies (i), because we can let Y be a set that is free in $\overline{X}$. Conversely, if X satisfies (i) then it is an element of $\mathcal{T}$, and (ii) makes sure that it is inclusionwise minimal.

There are at most $\binom{|A|}{k}$ sets X satisfying (i), and using (1) we can list these in polynomial time. Then, using (1) again, for each of these sets we can check whether they satisfy (ii). ∎

B. Contractions

Contractions give a way to construct new connectivity functions from given ones. To define a *contraction*, we take one or several disjoint subsets of the ground set and "contract" these sets to single points. In the new decomposition, these new points represent the sets of the original decomposition.

For the formal treatment, let κ be a connectivity function on a set A.

Let $C_1, \ldots, C_m \subseteq A$ be mutually disjoint subsets of A. Let $B := A \setminus (C_1 \cup \ldots \cup C_m)$, and let $c_1, \ldots, c_m$ be fresh elements (mutually distinct, and distinct from all elements of B). We define $A{\downarrow}_{C_1,\ldots,C_m} := B \cup \{c_1, \ldots, c_m\}$. To simplify the notation, here and in the following we omit the index ${}_{C_1,\ldots,C_m}$ if the sets C_i are clear from the context. For every subset $X \subseteq A{\downarrow}$, we define its *expansion* to be the set

$$X{\uparrow} := X{\uparrow}_{C_1,\ldots,C_m} := (X \cap B) \cup \bigcup_{\substack{i \in [m] \\ c_i \in X}} C_i.$$

The $C_1, \ldots, C_m$*-contraction* of κ is the function $\kappa{\downarrow}$, or $\kappa{\downarrow}_{C_1,\ldots,C_m}$, on $2^{A\downarrow}$ defined by $\kappa{\downarrow}(X) := \kappa(X{\uparrow})$. It is easy to verify that $\kappa{\downarrow}$ is indeed a connectivity function.

Remark II.5. A different view on contractions is to maintain the ground set, but define the connectivity function on a sublattice of the power set lattice. That is, not all separations of the ground set get an order, but only some of them.

Formally, we let $\mathcal{L} := \mathcal{L}(A{\downarrow}C_1, \ldots, C_m)$ be the sublattice of $\mathcal{P}(A) := (2^A, \cap, \cup)$ consisting of all sets $X \subseteq A$ such that for all $i \in [m]$ either $C_i \subseteq X$ or $C_i \subseteq \overline{X}$. Obviously, $\mathcal{L}$ is closed under intersection and union and thus indeed a sublattice. Observe that every $X \in \mathcal{L}$ has a natural contraction

$$X{\downarrow} := (X \cap B) \cup \{c_i \mid i \in [m] \text{ with } C_i \subseteq X\},$$

and we have $X{\downarrow}{\uparrow} = X$. As we also have $X'{\uparrow} \in \mathcal{L}$ for all $X' \subseteq A{\downarrow}$, the contraction mapping is a bijection between $\mathcal{L}(A{\downarrow}C_1, \ldots, C_m)$ and $A{\downarrow}$. It follows immediately from the definition of $\kappa{\downarrow}$ that for all $X \in \mathcal{L}(A{\downarrow}C_1, \ldots, C_m)$ we have

$$\kappa(X) = \kappa{\downarrow}(X{\downarrow}).$$

Thus the contraction mapping is an isomorphism from the *connectivity system* $\big(\mathcal{L}, \kappa|_{\mathcal{L}}\big)$, where $\kappa|_{\mathcal{L}}$ denotes the restriction of κ to $\mathcal{L}$, and the connectivity system $(2^{A\downarrow}, \kappa{\downarrow})$.

The view of a contraction of κ as a restriction to a sublattice will be useful when dealing with contractions of the cut-rank function of a graph in Section VI-B. ⌟

Let $\mathcal{T}$ be a κ-tangle of order k. We define $\mathcal{T}{\downarrow} := \mathcal{T}{\downarrow}_{C_1,\ldots,C_m} := \{X \subseteq A{\downarrow} \mid X{\uparrow} \in \mathcal{T}\}$. Note that $\mathcal{T}{\downarrow}$ is not necessarily a $\kappa{\downarrow}$-tangle: if $C_i \in \mathcal{T}$ for some $i \in [m]$, then $\{c_i\} \in \mathcal{T}{\downarrow}$, and thus $\mathcal{T}{\downarrow}$ violates (T.3). However, it is straightforward to verify that $\mathcal{T}{\downarrow}$ is a $\kappa{\downarrow}$-tangle (of the same order k) if and only if $C_1, \ldots, C_m \notin \mathcal{T}$.

III. Triple Covers

A *cover* of a κ-tangle $\mathcal{T}$ is a set $C \subseteq A$ such that $C \cap Y \neq \emptyset$ for all $Y \in \mathcal{T}$. It is not hard to prove that every κ-tangle of order k has a cover of size at most k.

A *triple cover* of a $\mathcal{T}$ is a set $Q \subseteq A$ such that $Q \cap Y_1 \cap Y_2 \cap Y_3 \neq \emptyset$ for all $Y_1, Y_2, Y_3 \in \mathcal{T}$. We shall prove that every tangle of order k has a triple cover of size bounded in terms of k.

Observe that we can test in polynomial time whether a given set Q is a triple cover for a κ-tangle $\mathcal{T}$, given by its index in a comprehensive tangle data structure: using the data structure, we produce a list of all inclusionwise minimal elements of $\mathcal{T}$, and then we check if any three of them have a nonempty intersection with Q.

Let $\theta : \mathbb{N} \to \mathbb{N}$ be defined by $\theta(0) := 0$ and $\theta(i+1) := \theta(i) + 3^{\theta(i)}$.

Lemma III.1. *Let $\mathcal{T}$ be a κ-tangle of order k. Then $\mathcal{T}$ has a triple cover of size at most $\theta(3k-2)$.*

IV. Treelike Decompositions

In a directed graph D, by $N_+^D(t)$ or just $N_+(t)$ if D is clear from the context, we denote the set out-neighbours of a node t. By $\trianglelefteq^D$ or just $\trianglelefteq$ we denote the reflexive transitive closure of $E(D)$, which is a partial order if D is acyclic. A *directed tree* is a directed graph T where for all nodes t the set $\{s \mid s \trianglelefteq t\}$ is linearly ordered by $\trianglelefteq$.

Let A be a set. A *directed decomposition* of A is a pair (D, γ), where D is a directed graph and $\gamma : V(D) \to 2^A$. If κ is a connectivity function on A, we also say that (D, γ) is a directed decomposition of κ. For every node $t \in V(D)$, we let

$$\beta(t) := \gamma(t) \setminus \bigcup_{u \in N_+^D(t)} \gamma(u). \tag{4.A}$$

We call $\beta(t)$ the *bag* and $\gamma(t)$ the *cone* at t. We always denote the bag function of a directed decomposition (D, γ) by β, and we use implicit naming conventions by which, for example, we denote the bag function of (D', γ') by β'.

A directed decomposition (D, γ) of A is *treelike*, or a *treelike decomposition*, if it satisfies the following axioms.

(TL.1) D is a acyclic.

(TL.2) For all $(t, u) \in E(D)$,

$$\gamma(t) \supseteq \gamma(u).$$

(TL.3) For all $t \in V(D)$ and $u_1, u_2 \in N_+^D(t)$,

$$\gamma(u_1) = \gamma(u_2) \quad \text{or} \quad \gamma(u_1) \cap \gamma(u_2) = \emptyset.$$

(TL.4) There is a $t \in V(D)$ such that $\gamma(t) = A$.

If (D, γ) only satisfies (TL.1)–(TL.3), we call it a *partial treelike decomposition*. The treelike decompositions of connectivity functions introduced here are adaptations of treelike decompositions of graphs introduced in [12], [13].

In the following, let (D, γ) be a partial treelike decomposition of A. Observe that for all $t \in V(D)$,

$$\gamma(t) = \bigcup_{u \trianglerighteq t} \beta(u). \tag{4.B}$$

(D, γ) is a *(partial) directed tree decomposition*[2] if D is a directed tree and for all $t \in V(D)$ and all distinct $u_1, u_2 \in N_+^D(t)$,

$$\gamma(u_1) \cap \gamma(u_2) = \emptyset.$$

Observe that (D, γ) is a directed tree decomposition if and only if D is a directed tree and the bags $\beta(t)$ for $t \in V(D)$ are mutually disjoint and have union A (that is, they form a partition of A with possibly empty parts).

[2]Deviating from previous work [11], [15], we view the trees in tree decompositions as being directed.

Now assume that κ is a connectivity function on A and (D, γ) a (partial) treelike decomposition of κ. The *width* of a node $t \in V(D)$ in (D, γ) is

$$\operatorname{wd}(D, \gamma, t) := \max_{\substack{X \subseteq \beta(t) \\ U \subseteq N_+^D(t)}} \kappa\left(X \cup \bigcup_{u \in U} \gamma(u)\right).$$

The *width* $\operatorname{wd}(D, \gamma)$ of the decomposition is the maximum of the widths of its nodes. This width measure is justified by the fact that the minimum width of a treelike decomposition of κ is equal to the branch width of κ (see the full version of this paper [16] for details).

It is sometimes convenient to normalise treelike decompositions. We call a treelike decomposition (D, γ) *normal* if all its bags at leaves (i.e., nodes of out-degree 0) have size exactly one and all bags at inner nodes (i.e., nodes of out degree > 0) are empty, and if furthermore for every node t either all children of t have the same cone or all children of t have mutually disjoint cones. It can be shown that every treelike decomposition can be transformed into a normal treelike decomposition of the same width by a canonical polynomial time algorithm.

Now let $\mathfrak{T}$ be a family of mutually incomparable κ-tangles. A *directed tree decomposition for* $\mathfrak{T}$ is a triple (T, γ, τ), where (T, γ) is a directed tree decomposition of κ and $\tau : \mathfrak{T} \to V(T)$ a bijective mapping such that the following two conditions are satisfied.

(DTD.1) For all nodes $t, u \in V(T)$ with $u \not\trianglelefteq t$ there is a minimum $(\tau^{-1}(u), \tau^{-1}(t))$-separation Y such that $\gamma(u) \subseteq Y$.

(DTD.2) For all nodes $t \in V(T)$ except the root, there is a node $u \in V(T)$ such that $t \not\trianglelefteq u$ and $\gamma(t)$ is a leftmost minimum $(\tau^{-1}(t), \tau^{-1}(u))$-separation.

Observe that (DTD.1) implies that for all nodes $t \in V(T)$ and children $u \in N_+(t)$ we have $\gamma(u) \notin \tau^{-1}(t)$. Furthermore, (DTD.2) implies that $\gamma(t) \in \tau^{-1}(t)$.

Recall that a κ-tangle $\mathcal{T}$ is *k-maximal*, for some $k \geq 0$, if either $\operatorname{ord}(\mathcal{T}) = k$ or $\operatorname{ord}(\mathcal{T}) < k$ and $\mathcal{T}$ is an (inclusionwise) maximal tangle. We denote the family of all k-maximal κ-tangles by $\mathfrak{T}^{\leq k}_{\max}$. Observe that for $k = \operatorname{bw}(\kappa)$ the k-maximal κ-tangles are precisely the maximal κ-tangles.

Theorem IV.1 ([15]). *Let $\ell \geq 0$. Then there is a polynomial time algorithm that, given oracle access to a connectivity function κ and a κ-tangle $\mathcal{T}_{\text{root}} \in \mathfrak{T}^{\leq \ell}_{\max}$ (via a membership oracle or its index in a comprehensive tangle data structure for κ), computes a canonical directed tree decomposition (T, γ, τ) for the set $\mathfrak{T}^{\leq \ell}_{\max}$ such that $\tau^{-1}(r) = \mathcal{T}_{\text{root}}$ for the root r of T.*

Here canonical means that if $\kappa' : 2^{A'} \to \mathbb{N}$ is another connectivity function and $\mathcal{T}'_{\text{root}}$ an ℓ-maximal κ'-tangle, and (T', γ', τ') is the decomposition computed by our algorithm on input $(\kappa', \mathcal{T}'_{\text{root}})$, then for every isomorphism f from $(\kappa, \mathcal{T}_{\text{root}})$ to $(\kappa', \mathcal{T}'_{\text{root}})$, that is, bijective mapping $f : A \to A'$ with $\kappa(X) = \kappa'(f(X))$ and $X \in \mathcal{T}_{\text{root}} \iff f(X) \in \mathcal{T}'_{\text{root}}$ for all $X \subseteq A$, there is an isomorphism g from T to T' such that that $f(\gamma(t)) = \gamma'(g(t))$ for all $t \in V(T)$ and $X \in \tau^{-1}(t) \iff f(X) \in (\tau')^{-1}(g(t))$ for all $X \subseteq A, t \in V(T)$.

V. Partitioning with Respect to a Maximal Tangle

Let G be a graph of rank width at most k. In this and the following two sections, we describe our construction of a canonical treelike decomposition of ρ_G of width at most $a(k)$ (for some function a). Since large parts of the construction go through for arbitrary connectivity functions, we find it convenient to let $\kappa := \rho_G$ and $A := V(G)$.

We start from a directed tree decomposition (T, γ, τ) for $\mathfrak{T}^{\leq k}_{\max}$. The idea is to decompose the "pieces" of this decomposition, corresponding to the nodes of T, further into decompositions of bounded width and then merge all these bounded-width decompositions into one big decomposition. The largest part of the construction, resulting in Lemma VII.1, deals with a single node of T.

So we fix a node $t \in V(T)$. We let $\mathcal{T}_0 := \tau^{-1}(t)$ be the maximal tangle associated with t and $k_0 := \operatorname{ord}(\mathcal{T}_0)$. Let $B := \beta(t)$ and $C_0 := \overline{\gamma(t)}$. Assuming that the children of t in T are $u_1, \ldots, u_m$, we let $C_i := \gamma(u_i)$ for $i \in [m]$ Observe that the sets $B, C_0, \ldots, C_m$ form a partition of A (the set C_0 may be empty). Now we contract the sets $C_0, \ldots, C_m$. We shall construct a bounded width decomposition of the resulting connectivity function $\kappa{\downarrow}$ on the contracted set $A{\downarrow}$.

We construct the decomposition recursively. At any time, we have a set $X \subseteq A{\downarrow}$ that still needs to be decomposed, and we will show how to partition X in a canonical way, at any time keeping control of the width of the resulting decomposition.

We initialise the construction by taking a triple cover Q of the tangle $\mathcal{T}_0$ of size $|Q| \leq \theta(3k_0 - 2)$. We let $Q^{\vee}$ be the "projection" of Q into $A{\downarrow}$ (precise definitions follow). The set $Q^{\vee} \cup \{c_0\}$ will be the bag at the root of our decomposition, and the first set X to be decomposed further is $A{\downarrow} \setminus (Q^{\vee} \cup \{c_0\})$.

Now suppose we are in some decomposition step where we need to decompose $X \subseteq A\!\downarrow \setminus (Q^\vee \cup \{c_0\})$. Depending on $\kappa\!\downarrow\!(X)$, we do this in two completely different ways. In this section (Section V), we consider the case $\kappa\!\downarrow\!(X) < (3k+2)\cdot k$, and in Section VI we shall consider the case $\kappa\!\downarrow\!(X) \geq (3k+2)\cdot k$.

A. Assumptions

Before we start the technical construction, we step back and collect the assumptions we make in a slightly more abstract setting, which we fix for the rest of the section.

Assumptions V.1. (1) $\kappa : 2^A \to \mathbb{N}$ is a connectivity function on a set A.
(2) $k := \mathrm{bw}(\kappa) \geq 1$.
(3) $C_0, \dots, C_m \subseteq A$ are mutually disjoint sets with $\kappa(C_i) < k$, and $C_1, \dots, C_m$ are nonempty.
(4) For all $i \in [m]$ there are tangles $\mathcal{T}_i, \mathcal{T}_i'$ such that C_i is a leftmost minimum $(\mathcal{T}_i, \mathcal{T}_i')$-separation. Furthermore, if $C_0 \neq \emptyset$ then there are tangles $\mathcal{T}_0, \mathcal{T}_0'$ such that such that $\overline{C}_0$ is a leftmost minimum $(\mathcal{T}_0, \mathcal{T}_0')$-separation.
(5) $B := A \setminus (C_0 \cup \ldots \cup C_m)$.
(6) $A\!\downarrow := A\!\downarrow_{C_0,\dots,C_m}$, and c_i is the element of $A\!\downarrow$ corresponding to the contracted set C_i, for $i = 0, \dots, m$.
(7) $\kappa\!\downarrow := \kappa\!\downarrow_{C_0,\dots,C_m}$.

The assumption $\mathrm{bw}(\kappa) \geq 1$ is without loss of generality, because if $\mathrm{bw}(\kappa) = 0$ then $\kappa(\{x\}) = 0$ for all $x \in A$ and thus $\kappa(X) = 0$ for all $X \subseteq A$.

Assumptions V.2. (1) There is a maximal κ-tangle $\mathcal{T}_0$ such that $\overline{C}_0 \in \mathcal{T}_0$ and $C_i \notin \mathcal{T}_0$ for $i = 1, \dots, m$.[3]
(2) For every κ-tangle $\mathcal{T} \perp \mathcal{T}_0$, there is an $i \in \{0, \dots, m\}$ and a set $Y \subseteq C_i$ such that $Y \in \mathcal{T}$.
(3) $k_0 := \mathrm{ord}(\mathcal{T}_0)$. (Note that $1 \leq k_0 \leq k$.)

Observe that

$$\mathcal{T}_0\!\downarrow := \mathcal{T}_0\!\downarrow_{C_0,\dots,C_m}.$$

is a $\kappa\!\downarrow$-tangle, because $C_i \notin \mathcal{T}_0$ for $i = 0, \dots, m$ by Assumption V.2(1).

Assumptions V.3. (1) $Q \subseteq A$ is a triple cover of the tangle $\mathcal{T}_0$ of size $|Q| \leq \theta(3k_0 - 2)$.
(2) $Q^\vee := (B \cap Q) \cup \{c_i \mid 0 \leq i \leq m, C_i \cap Q \neq \emptyset\}$.

Observe that $Q^\vee$ is a triple cover for the $\kappa\!\downarrow$-tangle $\mathcal{T}_0\!\downarrow$.

All algorithms we devise in this section will get κ and $C_0, \dots, C_m$ and Q as input, and possibly other objects. We assume that we have constructed a comprehensive tangle data structure for κ and have determined the index of $\mathcal{T}_0$ in this data structure. Thus our algorithms also have access to $\mathcal{T}_0$.

Whenever we refer to a construction in this section as being *canonical*, what we mean is that it is canonical given κ and $C_0, \dots, C_m$ and Q. Note that $\mathcal{T}_0$ is canonical given κ and $C_0, \dots, C_m$, because $\mathcal{T}_0$ is the unique maximal κ-tangle with $\overline{C}_0 \in \mathcal{T}_0$ and $C_i \notin \mathcal{T}_0$ for $i = 1, \dots, m$. Thus we may depend on $\mathcal{T}_0$ in canonical constructions.

Our goal is to prove the following lemma.

Lemma V.4. *For every $k_1 \in \mathbb{N}$ there are $a_1 = a_1(k, k_1)$, $b_1 = b_1(k, k_1)$, and $f_1 = f_1(k, k_1) > 0$ such that for every $X \subseteq A\!\downarrow \setminus (Q^\vee \cup \{c_0\})$ of order $\kappa\!\downarrow\!(X) = k_1$ and size $|X| \geq 2$, one of the following two conditions is satisfied.*
(i) There is a canonical partition of X into $b \leq b_1$ sets $X_1, \dots, X_b$ such that $\kappa\!\downarrow\!(X_i) \leq a_1$ and $|X_i| \leq (1 - 1/f_1)|X|$ for all $i \in [b]$.
(ii) There is a canonical partition of X into sets $X_0, X_1, \dots, X_n$ such that
a. $\kappa\!\downarrow\!(X_0) \leq k_1$,
b. $\kappa\!\downarrow\!\left(\bigcup_{i\in I} X_i\right) \leq 2k_1$ for every set $I \subseteq [n]$,
c. $|X_i| \leq (1 - 1/f_1)|X|$ for every $i \in [n]$.
Furthermore, given X (in addition to $\kappa\!\downarrow$ and $C_0, \dots, C_m$ and Q), the partition in (i) or (ii) can be computed in polynomial time (for fixed k, k_1).

The lemma will be proved in Section V-D.

For the rest of Section V, we fix a set $X \subseteq A\!\downarrow \setminus (Q^\vee \cup \{c_0\})$. Let $k_1 := \kappa\!\downarrow\!(X)$ and $k_2 := k_0 + k_1$.

We assume that

$$|X| \geq 6k_2. \tag{5.A}$$

[3]If $C_0 \neq \emptyset$, then the tangles $\mathcal{T}_0$ in Assumption V.1(4) and Assumption V.2(1) are the same, but this is irrelevant.

Note that this implies $|X| \geq 6$, because $k_2 \geq k_0 \geq 1$.

B. Existence of a Balanced Separations

We call a set $Z \subseteq X$ a *balanced X-separation* if $\kappa{\downarrow}(Z) \leq k_1 = \kappa{\downarrow}(X)$ and

$$\frac{1}{3}|X| - k_2 + \kappa{\downarrow}(Z) \leq |Z| \leq \frac{2}{3}|X| + k_2 - \kappa{\downarrow}(Z)$$

Note that this notion does not only depend on X, but via k_2 also on k_0, the order of the tangle $\mathcal{T}_0$.

Lemma V.5. *There is a balanced X-separation.*

The proof of this lemma crucially depends on the fact that $\mathcal{T}_0$ is the unique maximal tangle associated with the tree node t and there is a triple cover Q of $\mathcal{T}_0$ that has an empty intersection with $X{\uparrow}$, which intuitively means that the "essential" part of $\mathcal{T}_0$ is outside of X. Together, these two facts imply that X cannot be very highly connected, and this allows us to find Z.

C. A Canonical Family of Separations

For $0 \leq \ell \leq k_1$, let $p(\ell) := 2^{-3^{k_2-\ell}}$. Note that

$$\frac{1}{8} \geq p(k_1) \geq p(k_1 - 1) \geq \ldots \geq p(0) \tag{5.B}$$

and

$$p(\ell - 1) = p(\ell)^3 \tag{5.C}$$

for all $\ell \geq 1$.

Let us call a set $Z \subseteq X$ of order $\ell := \kappa{\downarrow}(Z)$ *good* (or a *good separation*) if

$$p(\ell) \cdot |X| \leq |Z| < |X|. \tag{5.D}$$

Recall that $\frac{1}{6}|X| \geq k_2$ by (5.A). Thus for $0 \leq \ell \leq k_1$ we have

$$\frac{1}{3}|X| - k_2 + \ell \geq \frac{1}{3}|X| - k_2 \geq \frac{1}{6}|X| \geq \max\bigl\{1, p(\ell) \cdot |X|\bigr\}$$

It follows that every balanced X-separation is good. Hence by Lemma V.5, there is a good separation Z of order $\kappa{\downarrow}(Z) \leq k_1$.

Let ℓ be minimum such that there is a good separation Z of order $\kappa{\downarrow}(Z) = \ell$.

Let $\mathcal{Z}$ be the set of all $Z \subseteq X$ such that

(i) Z is good;
(ii) $\kappa{\downarrow}(Z) = \ell$;
(iii) $|Z|$ is maximum subject to (i) and (ii).

Observe that $|Z| = |Z'| \geq p(\ell) \cdot |X|$ for all $Z, Z' \in \mathcal{Z}$. Let

$$\mathcal{Y} := \{Y \subseteq A{\downarrow} \mid \overline{Y} \in \mathcal{Z}\}.$$

Note that $\overline{X} \subset Y$ and $\kappa{\downarrow}(Y) = \ell$ and $|Y| = |Y'|$ for all $Y, Y' \in \mathcal{Y}$. Let us call two sets Y, Y' *X-disjoint* if $Y \cap Y' \subseteq \overline{X}$. Observe that for X-disjoint sets $Y, Y' \in \mathcal{Y}$ we have $Y \cap Y' = \overline{X}$.

Our next goal is to prove the following lemma.

Lemma V.6. *There is a $b_2 = b_2(k_0, k_1)$ such that if $|\mathcal{Y}| > b_2$ then the elements of $\mathcal{Y}$ are mutually X-disjoint.*

The idea of the proof is as follows. Assume that there are $Y, Y' \in \mathcal{Y}$ that are not X-disjoint. Then there are $Z, Z' \in \mathcal{Z}$ whose union is a proper subset of X. The choice of the function p and a submodularity argument guarantee that these sets Z, Z' have a small intersection. Thus $|Y \setminus Y'| = |Z' \setminus Z|$ is relatively large (close to $p(\ell)|X|$, i.e., a constant fraction of $|X|$) and thus $|Y \setminus \overline{X}| \geq |Y \setminus Y'|$ is relatively large. As all elements of $\mathcal{Y}$ have the same size, this holds for all $Y \in \mathcal{Y}$. Now we apply Ramsey's Theorem and find that if $\mathcal{Y}$ is very large either (i) there is a large family $Z_1, \ldots, Z_n \in \mathcal{Z}$ such that all pairwise unions $Z_i \cup Z_j$ are proper subsets of X, or (ii) there is a large family $Z_1, \ldots, Z_n \in \mathcal{Z}$ such that all pairwise unions $Z_i \cup Z_j$ are equal to X. In case (i), we argue that the Z_i are relatively large, but have a small intersection, and thus for large n their union becomes larger than $|X|$, which is impossible. In case (ii) we argue that the $Y_i := \overline{Z}_i$ are mutually X-disjoint, and as the sets $Y_i \setminus \overline{X}$ are relatively large, for large n their union becomes larger than $|X|$. Again, this is impossible. Thus the size of $\mathcal{Y}$ must be bounded.

More concretely, we obtain the following four lemmas.

Lemma V.7. *Let $Z, Z' \in \mathcal{Z}$ with $Z \neq Z'$ and $Z \cup Z' \neq X$. Then*

$$|Z \cap Z'| < p(\ell)^3 \cdot |X|$$

Lemma V.8. *Let $Z_1, \ldots, Z_m \in \mathcal{Z}$ such that $Z_i \cup Z_j \neq X$ for all distinct $i, j \in [m]$. Then*

$$m < \frac{2}{p(\ell)}.$$

Lemma V.9. *For all $n \geq 1$ there is an $m = m(\ell, n)$ such that if $|\mathcal{Z}| > m$ then there are $Z_1, \ldots, Z_n \in \mathcal{Z}$ such that $Z_i \cup Z_j = X$ for all distinct $i, j \in [n]$.*

Lemma V.10. *If there are distinct sets in $\mathcal{Y}$ that are not X-disjoint, then for all $Y \in \mathcal{Y}$,*

$$|Y \setminus \overline{X}| \geq \big(p(\ell) - p(\ell)^3\big) \cdot |X|.$$

These four lemmas provide all the required properties of sets in $\mathcal{Y}$ and $\mathcal{Z}$ to obtain Lemma V.6 as follows.

Proof of Lemma V.6: We let

$$n(\ell) := \left\lfloor \frac{1}{\big(p(\ell) - p(\ell)^3\big)} \right\rfloor + 1,$$

and choose $m = m(\ell, n(\ell))$ according to Lemma V.9. Suppose that $|\mathcal{Z}| = |\mathcal{Y}| > m$. Then there are $Z_1, \ldots, Z_{n(\ell)}$ such that $Z_i \cup Z_j = X$ for all distinct $i, j \in [n(\ell)]$. For all $i \in [n(\ell)]$, let $Y_i = \overline{Z}_i$. Then $Y_1, \ldots, Y_{n(\ell)}$ are mutually X-disjoint. Thus

$$\sum_{i=1}^{n(\ell)} |Y_i \setminus \overline{X}| \leq |X|.$$

By the choice of $n(\ell)$ and Lemma V.10, it follows that all sets in $\mathcal{Y}$ are mutually X-disjoint.

To complete the proof, we let

$$b_2(k_0, k_1) := \max_{0 \leq \ell \leq k_1} m\left(\ell, n(\ell)\right).$$

■

Lemma V.11. *Suppose that the elements of $\mathcal{Y}$ are mutually X-disjoint. Then for all $\mathcal{Y}_0 \subseteq \mathcal{Y}$,*

$$\kappa{\downarrow}\Big(\bigcup_{Y \in \mathcal{Y}_0} Y\Big) \leq \ell,$$

with equality for all $\mathcal{Y}_0 \subset \mathcal{Y}$.

Proof: We prove by induction on $i \leq |\mathcal{Y}|$ that for all $Y_1, \ldots, Y_i \in \mathcal{Y}$,

$$\kappa{\downarrow}(Y_1 \cup \ldots \cup Y_i) \leq \ell, \tag{5.E}$$

with equality if $i < |\mathcal{Y}|$.

The base step $i = 1$ is trivial. For $i = 2$, let $Y_1, Y_2 \in \mathcal{Y}$. Then $Y_1 \cap Y_2 = \overline{X}$ and thus $\kappa{\downarrow}(Y_1 \cap Y_2) = \kappa{\downarrow}(X) = k_1 \geq \ell$. By submodularity, $\kappa{\downarrow}(Y_1 \cup Y_2) \leq \ell$.

Now let $2 \leq i < |\mathcal{Y}|$, and suppose that

$$\kappa{\downarrow}(Y_1 \cup \ldots \cup Y_{i-1}) = \ell \tag{5.F}$$

for all $Y_1, \ldots, Y_{i-1} \in \mathcal{Y}$ and (5.E) for all $Y_1, \ldots, Y_i \in \mathcal{Y}$.

Let $Y_1, \ldots, Y_{i+1} \in \mathcal{Y}$ be mutually distinct. By posimodularity

$$\kappa{\downarrow}(Y_1 \cup \ldots \cup Y_i) + \kappa{\downarrow}(Y_i \cup Y_{i+1}) \geq \kappa{\downarrow}(Y_1 \cup \ldots \cup Y_{i-1}) + \kappa{\downarrow}(Y_{i+1}).$$

As $\kappa{\downarrow}(Y_1 \cup \ldots \cup Y_{i-1}) = \kappa{\downarrow}(Y_{i+1}) = \ell$ by (5.F) and $\kappa{\downarrow}(Y_i \cup Y_{i+1}) \leq \ell$ by (5.E), it follows that $\kappa{\downarrow}(Y_1 \cup \ldots \cup Y_i) \geq \ell$, which combined with (5.E) implies equality.

Furthermore, by submodularity,

$$\kappa{\downarrow}(Y_1 \cup \ldots \cup Y_i) + \kappa{\downarrow}(Y_i \cup Y_{i+1}) \geq \kappa{\downarrow}(Y_i) + \kappa{\downarrow}(Y_1 \cup \ldots \cup Y_{i+1}).$$

As $\kappa{\downarrow}(Y_1 \cup \ldots \cup Y_i) = \kappa{\downarrow}(Y_i \cup Y_{i+1}) = \kappa{\downarrow}(Y_1) = \ell$, this implies $\kappa{\downarrow}(Y_1 \cup \ldots \cup Y_{i+1}) \leq \ell$. ■

D. *Proof of Lemma V.4*

We continue to use the notation of Section V-C. Essentially, the lemmas proved there show how to use the family $\mathcal{Y}$ to obtain the desired partition of X. The main question that remains to be solved is how to compute $\mathcal{Y}$.

Lemma V.12. *There is a polynomial time algorithm that, given X and oracle access to $\kappa{\downarrow}$, computes $\mathcal{Y}$.*

Proof: Let $\mathcal{Z}^*$ be the family of all $Z \subseteq A$ satisfying the following three conditions:

(i) Z satisfies (5.D), that is, $p(\ell) \cdot |X| \leq |Z| < |X|$.

(ii') $\kappa{\downarrow}_{\min}(Z, \overline{X}) = \ell$;

(iii') there are a set $Z_0 \subseteq X$ of size $|Z_0| \leq \ell$ and an element $x \in X$ such that Z is a rightmost minimum $(Z_0, \overline{X} \cup \{x\})$-separation.

Let $m := \max\big\{|Z| \;\big|\; Z \in \mathcal{Z}^*\big\}$.

Claim 1.

$$\mathcal{Z} = \big\{Z \in \mathcal{Z}^* \;\big|\; |Z| = m\big\}.$$

Proof. We first prove that $\mathcal{Z} \subseteq \mathcal{Z}^*$. Let $Z \in \mathcal{Z}$. Clause (i) in the definition of $\mathcal{Z}$ is the same as clause (i) above.

If there was some Z' such that $Z \subseteq Z' \subseteq X$ and $\kappa{\downarrow}(Z') < \ell$, then $Z' \subset X$, because $\kappa{\downarrow}(X) = k_1 \geq \ell$, and Z' would also satisfy (5.D), because $|Z'| \geq |Z|$. Thus (ii) would be violated. This proves (ii').

To see that Z satisfies (iii'), let $Z_0 \subseteq Z$ be inclusionwise minimal such that $\kappa{\downarrow}_{\min}(Z_0, \overline{X}) = \ell$. Suppose for contradiction that $|Z_0| = n > \ell$, and let $z_1, \ldots, z_n$ be an enumeration of Z_0. For every $i \in [n]$, let $Z^i = \{z_1, \ldots, z_i\}$. Then $\kappa{\downarrow}_{\min}(Z^i, \overline{X}) \leq \kappa{\downarrow}_{\min}(Z^{i+1}, \overline{X})$ for all $i < \ell$, because $\kappa{\downarrow}_{\min}$ is monotone in the first argument. As $\kappa{\downarrow}_{\min}(Z_0, \overline{X}) = \ell$, there is an $i < \ell$ such that $\kappa{\downarrow}_{\min}(Z^i, \overline{X}) = \kappa{\downarrow}_{\min}(Z^{i+1}, \overline{X})$. By the submodularity of $\kappa{\downarrow}_{\min}$ in the first argument,

$$\kappa{\downarrow}_{\min}(Z_0 \setminus \{z_{i+1}\}, \overline{X}) + \kappa{\downarrow}_{\min}(Z^{i+1}, \overline{X}) \geq \kappa{\downarrow}_{\min}(Z^i, \overline{X}) + \kappa{\downarrow}_{\min}(Z_0, \overline{X}).$$

It follows that $\kappa{\downarrow}_{\min}(Z_0 \setminus \{z_{i+1}\}, \overline{X}) = \kappa{\downarrow}_{\min}(Z_0, \overline{X})$, contradicting the minimality of Z_0. This proves that $|Z_0| \leq \ell$.

Let $x \in X \setminus Z$. Then $Z_0 \subseteq Z \subseteq X \setminus \{x\} \subseteq X$, and $\kappa{\downarrow}_{\min}(Z_0, \overline{X}) = \ell$ and $\kappa(Z) = \ell$ imply $\kappa{\downarrow}_{\min}(Z_0, \overline{X} \cup \{x\}) = \ell$. Thus Z is a minimum $(Z_0, \overline{X} \cup \{x\})$-separation, and now clause (iii) in the definition of $\mathcal{Z}$ (the maximality of $|Z|$) implies that Z is rightmost. This completes the proof of (iii') and thus of the inclusion $\mathcal{Z} \subseteq \mathcal{Z}^*$. The maximality of the elements of $\mathcal{Z}$ (clause (iii) in the definition) then implies that

$$\mathcal{Z} \subseteq \big\{Z \in \mathcal{Z}^* \;\big|\; |Z| = m\big\}.$$

To prove the converse inclusion, let $Z \in \mathcal{Z}^*$ with $|Z| = m$. Then by (iii'), $Z \subset \overline{X}$. Clauses (i) and (ii') above imply clauses (i) and (ii) in the definition of $\mathcal{Z}$.

Suppose that there is some $Z' \subset X$ satisfying (i) and (ii) such that $|Z'| > |Z|$. Choose such a Z' of maximum size. Then $Z' \in \mathcal{Z}$, and thus $|Z'| = m = |Z|$. This is a contradiction. ⌟

It is easy to see that $\mathcal{Z}^*$ can be computed in polynomial time, and this implies that $\mathcal{Z}$ and thus $\mathcal{Y}$ can be computed in polynomial time. ∎

Proof of Lemma V.4: Recall that $k_0 \leq k$. We let

$$f_1 := f_1(k, k_1) := \frac{1}{p(0) - p(0)^3} \tag{5.G}$$

and

$$a_1 := a_1(k, k_1) := \max\{k, 2k_1 \cdot b_2(k, k_1)\}, \tag{5.H}$$

$$b_1 := b_1(k.k_1) := \max\{6(k + k_1), 2^{b_2(k,k_1)}\}, \tag{5.I}$$

where $b_2 := b_2(k, k_1)$ is chosen according to Lemma V.6.

If $|X| < 6(k + k_1)$, we simply partition X into 1-element sets. Note that $\kappa{\downarrow}(\{b\}) = \kappa(\{b\}) \leq k$ for all $b \in B$, because $\mathrm{bw}(\kappa) \leq k \leq a_1$, and $\kappa(\{c_i\}) = \kappa(C_i) < k \leq a_1$ for $0 \leq i \leq m$ by Assumption V.1(3). Thus (i) is satisfied.

In the following, we assume that $|X| \geq 6(k + k_1) \geq 6k_2$. This is the assumption needed for the previous results.

Case 1: There are distinct $Y, Y' \in \mathcal{Y}$ that are not X-disjoint.

Then $|\mathcal{Y}| \le b_2(k, k_2)$ by Lemma V.6 and

$$|Y \cap X| > (p(\ell) - p(\ell)^3)|X| \ge \frac{1}{f_1} \cdot |X| \tag{5.J}$$

by Lemma V.10. Moreover, $\overline{Y} \in \mathcal{Z}$ for all $Y \in \mathcal{Y}$, which implies

$$|\overline{Y}| \ge p(\ell) \cdot |X| \ge \frac{1}{f_1} \cdot |X|. \tag{5.K}$$

Let $Y_1^+, \dots, Y_n^+$ be an enumeration of all sets $|Y \cap X|$ for $Y \in \mathcal{Y}$. Note that $n \le b_2(k, k_1)$. For every $i \in [n]$, let $Y_i^- := X \setminus Y_i^+$. For every $i \in [n]$, we have $\kappa{\downarrow}(Y_i^-) = \ell$ and, by submodularity, $\kappa{\downarrow}(Y_i^+) \le k_1 + \ell \le 2k_1$.

Let $X_1, \dots, X_b$ be a list of all nonempty sets of the form

$$\bigcap_{i=1}^{n} Y_i^{\sigma(i)},$$

for some function $\sigma : [n] \to \{+, -\}$. Then $b \le 2^n \le b_1$. Submodularity implies that

$$\kappa{\downarrow}(X_i) \le 2k_1 n \le a_1.$$

It follows from (5.J) and (5.K) that $|Y_i^\sigma| \le (1 - 1/f_1)|X|$ for all $i \in [n]$ and $\sigma \in \{+, -\}$. Thus the partition $X_1, \dots, X_b$ satisfies assertion (i) of the lemma.

Case 2: The elements of $\mathcal{Y}$ are mutually X-disjoint.

We let

$$X_0 := \bigcap_{Y \in \mathcal{Y}} \overline{Y},$$

and we let $X_1, \dots, X_n$ be an enumeration of the sets $Y \cap X$ for $Y \in \mathcal{Y}$. Note that the sets $X_0, \dots, X_n$ form a partition of X. It follows from Lemma V.11 that $\kappa{\downarrow}(X_0) \le \ell \le k$ and $\kappa{\downarrow}\left(\bigcup_{i \in I} X_i\right) \le k + \ell \le 2k$ for every set $I \subseteq [n]$. It follows from (5.D) that $|X_i| \le (1 - 1/f_1)|X|$ for every $i \in [n]$.

Thus the partition $X_0, \dots, X_n$ satisfies assertion (ii) of the lemma.

It follows from Lemma V.12 that in both cases the partition can be computed in polynomial time. ∎

VI. The Non-Well-Linked Case

A. *Partitioning with Respect to an Independent Set*

In this section, we make Assumptions V.1 again (but not Assumptions V.2 and V.3).

Let $X \subseteq A{\downarrow}$ such that

$$k_1 := \kappa{\downarrow}(X) \ge (3k + 2) \cdot k. \tag{6.A}$$

We define a function $\lambda : 2^{\overline{X}} \to \mathbb{N}$ by letting

$$\lambda(Y) := \kappa{\downarrow}_{\min}(Y, X)$$

for all $Y \subseteq \overline{X}$. Then λ is submodular and monotone, and we have $\lambda(\emptyset) = 0$. Such a function is known as an *integer polymatroid*. It induces a matroid $\mathcal{M}(\lambda)$ on $\overline{X}$ whose independent sets are all $Y \subseteq \overline{X}$ satisfying

$$|Z| \le \lambda(Z) \text{ for all } Z \subseteq Y \tag{6.B}$$

(see [28], Proposition 12.1.2). The rank function r_λ of $\mathcal{M}(\lambda)$ is defined by

$$r_\lambda(Y) := \min\left\{\lambda(Z) + |Y \setminus Z| \;\middle|\; Z \subseteq Y\right\}.$$

(see [28], Proposition 12.1.7). Observe that for all $y \in \overline{X}$ we have $\lambda(\{y\}) \le \kappa{\downarrow}(\{y\}) \le k$. If $y \in B$ then this holds because $\kappa{\downarrow}(\{y\}) = \kappa(\{y\}) \le \mathrm{bw}(\kappa)$, and if $y = c_i$ it follows from Assumption V.1(3). A straightforward induction based on the submodularity of λ then implies that $\lambda(Y) \le k|Y|$ for all $Y \subseteq \overline{X}$. Thus for all $Z \subseteq \overline{X}$,

$$\lambda(Z) \ge \lambda(\overline{X}) - \lambda(\overline{X} \setminus Z) \ge k_1 - k \cdot |\overline{X} \setminus Z|,$$

which implies $\lambda(Z) + k|\overline{X} \setminus Z| \geq k_1$ and hence $\lambda(Z) + |\overline{X} \setminus Z| \geq k_1/k$. By the definition of r_λ, we get

$$r_\lambda(\overline{X}) \geq \frac{k_1}{k} \geq (3k+2).$$

Thus there is a set $Y' \subseteq \overline{X}$ of size $|Y'| = 3k+2$ that is an independent set of $\mathcal{M}(\lambda)$. As all subsets of an independent set are independent as well, there is an independent set $Y \subseteq \overline{X} \setminus \{c_0\}$ of size $|Y| = 3k+1$. We keep such a set Y fixed in the following.

Lemma VI.1. *Let $Z \subseteq A{\downarrow}$ such that $\kappa{\downarrow}(Z) < |Y \setminus Z|$.*
Then

$$\kappa{\downarrow}(X \cap Z) < \kappa{\downarrow}(X).$$

Proof: As Y is independent, we have $|Y \setminus Z| \leq \lambda(Y \setminus Z)$. As $Y \setminus Z \subseteq \overline{X} \setminus Z \subseteq \overline{X}$, we have $\lambda(Y \setminus Z) \leq \kappa{\downarrow}(\overline{X} \setminus Z)$. Thus $\kappa{\downarrow}(\overline{X} \setminus Z) - |Y \setminus Z| \geq 0$ and therefore

$$\begin{aligned}
\kappa{\downarrow}(X \cap Z) &\leq \kappa{\downarrow}(X \cap Z) + \kappa{\downarrow}(\overline{X} \setminus Z) - |Y \setminus Z| \\
&= \kappa{\downarrow}(X \cap Z) + \kappa{\downarrow}(X \cup Z) - |Y \setminus Z| && \text{(symmetry)} \\
&\leq \kappa{\downarrow}(X) + \kappa{\downarrow}(Z) - |Y \setminus Z| && \text{(submodularity)} \\
&< \kappa{\downarrow}(X) + |Y \setminus Z| - |Y \setminus Z| && \text{(assumption of the lemma)} \\
&= \kappa{\downarrow}(X).
\end{aligned}$$

■

Lemma VI.2. *There is a set $Z \subseteq A{\downarrow}$ such that $\kappa{\downarrow}(Z) \leq k$ and*

$$\kappa{\downarrow}(Z) < \min\{|Y \cap Z|, |Y \setminus Z|\}. \tag{6.C}$$

Furthermore, we can compute such a set Z in polynomial time (for fixed k).

Proof: We define a weight function $\varphi : A \to \mathbb{R}$ as follows:

$$\varphi(x) := \begin{cases} \frac{1}{|C_i|} & \text{if } x \in C_i \text{ for some } i \in \{0, \dots, m\} \text{ such that } c_i \in Y, \\ 1 & \text{if } x \in Y \cap B, \\ 0 & \text{otherwise.} \end{cases}$$

Claim 1. There is a $Z \subseteq A$ such that $\kappa(Z) \leq k$ and

$$\min\{\varphi(Y{\uparrow} \cap Z),\ \varphi(Y{\uparrow} \setminus Z)\} > k.$$

Proof. Suppose for contradiction that for all $Z \subseteq A$ such that $\kappa(Z) \leq k$, either $\varphi(Y{\uparrow} \cap Z) \leq k$ or $\varphi(Y{\uparrow} \setminus Z) \leq k$. Let

$$\mathcal{T} := \{Z \subseteq A \mid \kappa(Z) \leq k, \varphi(Y{\uparrow} \cap Z) \geq 2k+1\}.$$

Then $\mathcal{T}$ is a κ-tangle of order $k+1$. Indeed, it obviously satisfies (T.0). It satisfies (T.1), because

$$\varphi(Y{\uparrow} \cap Z) + \varphi(Y{\uparrow} \setminus Z) = \varphi(Y{\uparrow}) = |Y| = 3k+1.$$

It satisfies (T.2), because $2k+1 > (2/3)|Y|$, and it satisfies (T.3) because $\varphi(\{x\}) \leq 1 < 2k+1$ for all $x \in A$.
However, as $\mathrm{bw}(\kappa) = k$ no tangle of order $k+1$ exists. ⌟

Let ℓ be minimum such that there is a $Z \subseteq A$ such that $\kappa(Z) \leq \ell$ and

$$\min\{\varphi(Y{\uparrow} \cap Z),\ \varphi(Y{\uparrow} \setminus Z)\} > \ell. \tag{6.D}$$

Let $Z \subseteq A$ such that $\kappa(Z) \leq \ell$ and (6.D).

Without loss of generality we may assume that either $Z \cap C_0 = \emptyset$ or $C_0 \subseteq Z$. To see this, suppose that neither $Z \cap C_0 = \emptyset$ nor $C_0 \subseteq Z$, or equivalently, neither $Z \subseteq \overline{C}_0$ nor $\overline{Z} \subseteq \overline{C}_0$. By Assumption V.1(4), $\overline{C}_0$ is a minimum $(\mathcal{T}_0, \mathcal{T}_0')$-separation. Thus by Lemma II.1(1) (applied to $X = \overline{C}_0$), either $\kappa(Z \cap \overline{C}_0) \leq \kappa(Z)$ or $\kappa(\overline{Z} \cap \overline{C}_0) \leq \kappa(Z)$. As $c_0 \notin Y$, we have $\varphi(Z \cap \overline{C}_0) = \varphi(Z \cup C_0) = \varphi(Z)$. Thus if $\ell' := \kappa(Z \cap \overline{C}_0) \leq \kappa(Z)$, then $Z' := Z \cap \overline{C}_0$ satisfies (6.D) with Z', ℓ' instead of Z, ℓ, and if $\ell' := \kappa(\overline{Z} \cap \overline{C}_0) \leq \kappa(Z)$, then $Z' := \overline{Z} \cap \overline{C}_0$ satisfies (6.D) with Z', ℓ' instead of Z, ℓ. In both cases, we have $Z' \cap C_0 = \emptyset$. This justifies the assumption that either $Z \cap C_0 = \emptyset$ or $C_0 \subseteq Z$.

Claim 2. For all $i \in [m]$, either $C_i \cap Z = \emptyset$ or $C_i \subseteq Z$.

Proof. Suppose for contradiction that neither $C_i \cap Z = \emptyset$ nor $C_i \subseteq Z$. Then neither $Z \subseteq \overline{C}_i$ nor $\overline{Z} \subseteq \overline{C}_i$. By Assumption V.1(4) and Lemma II.1, either $\kappa(Z \cap \overline{C}_i) < \kappa(Y)$ or $\kappa(\overline{Z} \cap \overline{C}_i) < \kappa(Y)$.

Without loss of generality we assume that $\ell' := \kappa(Z \cap \overline{C}_i) < \kappa(Y) = \ell$. Let $Z' := Z \cap \overline{C}_i$. Then

$$\varphi(Y{\uparrow} \cap Z') \geq \varphi(Y{\uparrow} \cap Z) - 1 \geq \ell - 1 \geq \ell'$$

and

$$\varphi(Y{\uparrow} \setminus Z') \geq \varphi(Y{\uparrow} \setminus Z) \geq \ell \geq \ell'.$$

This contradicts the minimality of ℓ. ⌟

It follows that there is a $Z^\vee \subseteq A{\downarrow}$ such that $Z = Z^\vee{\uparrow}$. Then

$$|Y \cap Z^\vee| = \varphi(Y{\uparrow} \cap Z) > \ell = \kappa(Z) = \kappa{\downarrow}(Z^\vee)$$

and, similarly, $|Y \setminus Z^\vee| > \kappa{\downarrow}(Z^\vee)$.

We can compute a set Z satisfying (6.C) in polynomial time as follows: for every $Z_0 \subseteq Y$ we compute a leftmost minimum $(Z_0, Y \setminus Z_0)$-separation Z until we find one with $\kappa{\downarrow}(Z) < |Z_0| = |Y \cap Z|$ and $\kappa{\downarrow}(Z) < |Y \setminus Z_0| = |Y \setminus Z|$. ■

Lemma VI.3. *Let $Z \subseteq A{\downarrow}$ such that $\kappa{\downarrow}(Z) < \min\{|Y \cap Z|, |Y \setminus X|\}$. Then $X \cap Z, X \setminus Z$ is a partition of X into two nonempty sets with $\kappa{\downarrow}(X \cap Z), \kappa{\downarrow}(X \setminus Z) < \kappa{\downarrow}(X)$.*

Proof: As Y is independent in $\mathcal{M}(\lambda)$, for each Z with $\kappa{\downarrow}(Z) < \min\{|Y \cap Z|, |Y \setminus X|\}$ we have $X \cap Z \neq \emptyset$, because for $Z' \subseteq \overline{X}$ we have $\kappa{\downarrow}(Z') \geq \lambda(Y \cap Z') \geq |Y \cap Z'|$ by (6.B). By Lemma VI.1, we have $\kappa{\downarrow}(X \cap Z) < \kappa{\downarrow}(X)$. By symmetry, we also have $X \setminus Z = X \cap \overline{Z} \neq \emptyset$ and $\kappa{\downarrow}(X \setminus Z) < \kappa{\downarrow}(X)$. ■

B. A Canonical Family of Partitions

While so far, all our constructions work for general connectivity functions, in this section we need to restrict our attention to the cut rank function of a graph. In addition to Assumptions V.1, which we still maintain, we make the following assumption.

Assumption VI.4. There is a graph G such that $\kappa = \rho_G$.

As in the previous subsection, let $X \subseteq A{\downarrow}$ such that $k_1 := \kappa{\downarrow}(X) \geq (3k+2)k$. Then

$$\mathrm{rk}(M_{X{\uparrow}, \overline{X}{\uparrow}}) = k_1.$$

For every $W \subseteq A{\downarrow} = V(G){\downarrow}$, we let $G[W{\uparrow}]$ be the induced subgraph of G with vertex set $W{\uparrow}$, and we let $\kappa{\downarrow}_W := \rho_{G[W{\uparrow}]}{\downarrow}$, where of course we contract only those C_i that are contained in $W{\uparrow}$. Observe that for every $Z \subseteq W$ we have

$$\kappa{\downarrow}_W(Z) = \mathrm{rk}\left(M_{Z{\uparrow}, (W \setminus Z){\uparrow}}\right).$$

By $\overline{X}^{\underline{\ell}}$ we denote the set of all ℓ-tuples of elements of $\overline{X}$ with mutually distinct entries. For every $\ell \geq 1$, we shall define an equivalence relation $\equiv_X^\ell$ on $\overline{X}^{\underline{\ell}}$ with index (that is, number of equivalence classes) bounded in terms of k_1 and ℓ such that the following holds.

Lemma VI.5. *Let $\boldsymbol{w} = (w_1, \ldots, w_\ell), \boldsymbol{w}' = (w'_1, \ldots, w'_\ell) \in \overline{X}^{\underline{\ell}}$ such that $\boldsymbol{w} \equiv_X^\ell \boldsymbol{w}'$, and let $W := \{w_1, \ldots, w_\ell\}$, $W' := \{w'_1, \ldots, w'_\ell\}$.*

Let $Z \subseteq X \cup W$ and $Z' := (X \cap Z) \cup \{w'_i \mid i \in [\ell]$ such that $w_i \in Z\}$. Then

$$\kappa{\downarrow}_{X \cup W}(Z) = \kappa{\downarrow}_{X \cup W'}(Z').$$

Let us first consider the special case that no sets are contracted, that is, $m = -1$ (this is not a case that we actually need to consider, but it is helpful to explain the ideas). Then $A{\downarrow} = A = V(G)$ and $\kappa{\downarrow} = \kappa = \rho_G$. For $\boldsymbol{w} = (w_1, \ldots, w_\ell), \boldsymbol{w}' = (w'_1, \ldots, w'_\ell) \in \overline{X}^{\underline{\ell}}$, we let $\boldsymbol{w} \equiv_X^\ell \boldsymbol{w}'$ if for all $i \in [\ell]$ the columns of the matrix $M_{X, \overline{X}}$ indexed by w_i and w'_i are equal and for all $i, j \in [\ell]$ we have $w_i w_j \in E(G) \iff w'_i w'_j \in E(G)$. We can rephrase these two conditions as follows:

(i) For all $i \in [\ell]$ the matrices $M_{X, \{w_i\}}$ and $M_{X, \{w'_i\}}$ are equal.
(ii) For all $i \in [\ell]$ the matrices $M_{\{w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_\ell\}, \{w_i\}}$ and $M_{\{w'_1, \ldots, w'_{i-1}, w'_{i+1}, \ldots, w'_\ell\}, \{w'_i\}}$ are equal.

Let $W := \{w_1, \dots, w_\ell\}$ and $W' := \{w'_1, \dots, w'_\ell\}$. For all $Z \subseteq X \cup W$ and $Z' := (X \cap Z) \cup \{w'_i \mid w_i \in Z\} \subseteq X \cup W'$, condition (i) implies that

$$M_{X \cap Z, W \setminus Z} = M_{X \cap Z', W' \setminus Z'}, \tag{6.E}$$
$$M_{X \setminus Z, W \cap Z} = M_{X \setminus Z', W' \cap Z'}. \tag{6.F}$$

Condition (ii) implies

$$M_{W \cap Z, W \setminus Z} = M_{W' \cap Z', W' \setminus Z'}. \tag{6.G}$$

Lemma VI.5 (in the special case) follows easily, because

$$\kappa{\downarrow}_{X \cup W}(Z) = \rho_{G[X \cup W]}(Z) = \mathrm{rk}(M_{Z, (X \cup W) \setminus Z}) = \mathrm{rk}\left(\begin{pmatrix} M_{X \cap Z, X \setminus Z} & M_{X \cap Z, W \setminus Z} \\ M_{W \cap Z, X \setminus Z} & M_{W \cap Z, W \setminus Z} \end{pmatrix}\right) \tag{6.H}$$

and

$$\kappa{\downarrow}_{X \cup W'}(Z') = \mathrm{rk}\left(\begin{pmatrix} M_{X \cap Z', X \setminus Z'} & M_{X \cap Z', W' \setminus Z'} \\ M_{W' \cap Z', X \setminus Z'} & M_{W' \cap Z', W' \setminus Z'} \end{pmatrix}\right). \tag{6.I}$$

Since $X \cap Z' = X \cap Z$ and $X \setminus Z' = X \setminus Z$, equations (6.E), (6.F), and (6.G) imply that the matrices in the rightmost terms in (6.H) and (6.I) are equal.

Let us now turn to the general case. The situation is more difficult here because the sets C_i and hence the matrices involved in our argument above, in particular the matrices $M_{Z\uparrow, W \setminus Z\uparrow}$ may have unbounded size (in terms of k and ℓ). The crucial observation is that we can bound the size of the C_i in terms of $k \le k_1$, exploiting the fact that $\rho_G(C_i) = \kappa{\downarrow}(\{c_i\}) < k$. To simplify the notation, we assume that $B = \{c_{m+1}, \dots, c_n\}$ for some $n \ge m$, and for $i = m+1, \dots, n$, we let $C_i := \{c_i\}$, so that actually $A = \bigcup_{i=0}^{n} C_i$ and $A{\downarrow} = \{c_0, \dots, c_n\}$.

Suppose that for some $i \in [m]$ there are distinct vertices $v, v' \in C_i$ such that for all $w \in V(G) \setminus C_i$ we have $vw \in E(G) \iff v'w \in E(G)$. Let $G' := G \setminus \{v'\}$ and $C'_i := C_i \setminus \{v'\}$ and $C'_j := C_j$ for $j \ne i$. Then contracting $C'_1, \dots, C'_m$ in G' has the same effect as contracting $C_1, \dots, C_m$ in G, that is,

$$A{\downarrow} = V(G){\downarrow}_{C_1, \dots, C_m} = V(G'){\downarrow}_{C'_1, \dots, C'_m},$$
$$\kappa{\downarrow} = \rho_G{\downarrow}_{C_1, \dots, C_m} = \rho_{G'}{\downarrow}_{C'_1, \dots, C'_m}.$$

By repeating this construction we arrive at an induced subgraph $G'' \subseteq G$ and a partition $C''_1, \dots, C''_n \subseteq V(G'')$, where $C''_i \subseteq C_i$ for $0 \le i \le m$ and $C''_i = C_i = \{c_i\}$ for $m+1 \le i \le n$, such that

$$A{\downarrow} = V(G''){\downarrow}_{C''_1, \dots, C''_m},$$
$$\kappa{\downarrow} = \rho_{G''}{\downarrow}_{C''_1, \dots, C''_m},$$

and for all $i \in [n]$ and distinct $v, v' \in C''_i$ there is a $w \in V(G'') \setminus C''_i$ such that $vw \in E(G) \not\Leftrightarrow v'w \in E(G)$. Observe that the construction of G'' and $C''_1, \dots, C''_m$ from G and $C_1, \dots, C_m$ can be carried out in polynomial time and that the connectivity function $\kappa{\downarrow} = \rho_{G''}{\downarrow}_{C''_1, \dots, C''_m}$ obtained from G'' by contracting the sets $C_1, \dots, C_m$ is canonical. To simplify the notation, in the following we assume that $G'' = G$ and $C''_i = C_i$ for all $i \in [n]$.

Let $i \in [n]$. As for all distinct $v, v' \in C_i$ there is a $w \in \overline{C}_i$ such that $vw \in E(G) \not\Leftrightarrow v'w \in E(G)$, the rows of the matrix $M_{C_i, \overline{C}_i}$ are mutually distinct. As $k > \kappa{\downarrow}(\{c_i\}) = \rho_G(C_i) = \mathrm{rk}(M_{C_i, \overline{C}_i})$, the matrix $M_{C_i, \overline{C}_i}$, being a matrix over $\mathbb{F}_2$, has at most 2^{k-1} distinct rows. This implies that

$$|C_i| \le 2^{k-1}.$$

Now we are ready to define the equivalence relation $\equiv^\ell_X$. For this we let $\boldsymbol{w} = (w_1, \dots, w_\ell), \boldsymbol{w}' = (w'_1, \dots, w'_\ell)' \in \overline{X}^\ell$. To simplify the notation, for every $i \in [\ell]$ we let $w_i{\uparrow} := \{w_i\}{\uparrow}$, that is, if $w_i = c_j$ then $w_i{\uparrow} = C_j$. Similarly, we let $w'_i{\uparrow} := \{w'_i\}{\uparrow}$. We let $\boldsymbol{w} \equiv^\ell_X \boldsymbol{w}'$ if for every $i \in [\ell]$ there are linear orders $\le_i$ of $w_i{\uparrow}$ and $\le'_i$ of $w'_i{\uparrow}$ such that the following two conditions are satisfied.

(i) For $i \in [\ell]$ the matrices $M_{X, w_i\uparrow}$ and $M_{X, w'_i\uparrow}$ are equal if the columns of the matrices are ordered according to the linear orders $\le_i$ and $\le'_i$, respectively.

(ii) For $i \in [\ell]$ the matrices $M_{w_1\uparrow \cup \dots \cup w_{i-1}\uparrow \cup w_{i+1}\uparrow \cup \dots \cup w_\ell\uparrow, w_i\uparrow}$ and $M_{w'_1\uparrow \cup \dots \cup w'_{i-1}\uparrow \cup w'_{i+1}\uparrow \cup \dots \cup w'_\ell\uparrow, w'_i\uparrow}$ are equal if

(ii-a) the rows of the matrices are ordered lexicographically according to the natural order on the indices j of the w_j, w'_j and, within the sets $w_j{\uparrow}, w'_j{\uparrow}$, according to the linear orders $\le_j, \le'_j$, respectively;

(ii-b) columns of the matrices are ordered according to the linear orders $\leq_i$, $\leq'_i$, respectively.

Proof of Lemma VI.5: We have

$$\kappa{\downarrow}_{X\cup W}(Z) = \rho_{G[(X\cup W)\uparrow]}(Z\uparrow) = \mathrm{rk}(M_{Z\uparrow,((X\cup W)\setminus Z)\uparrow}) = \mathrm{rk}\left(\begin{pmatrix} M_{(X\cap Z)\uparrow,(X\setminus Z)\uparrow} & M_{(X\cap Z)\uparrow,(W\setminus Z)\uparrow} \\ M_{(W\cap Z)\uparrow,(X\setminus Z)\uparrow} & M_{(W\cap Z)\uparrow,(W\setminus Z)\uparrow} \end{pmatrix}\right). \tag{6.J}$$

Similarly, for $Z' := (X\cap Z)\cup\{w'_i \mid w_i\in Z\}$,

$$\kappa{\downarrow}_{X\cup W}(Z') = \mathrm{rk}\left(\begin{pmatrix} M_{(X\cap Z)\uparrow,(X\setminus Z)\uparrow} & M_{(X\cap Z)\uparrow,(W'\setminus Z')\uparrow} \\ M_{(W'\cap Z')\uparrow,(X\setminus Z)\uparrow} & M_{(W'\cap Z')\uparrow,(W'\setminus Z')\uparrow} \end{pmatrix}\right), \tag{6.K}$$

where we use the fact that $X\cap Z = X\cap Z'$ and $X\setminus Z = X\setminus Z'$. We may assume that in all these matrices the rows and columns indexed by entries of W, W' are ordered lexicographically according to the indices of the i and the orders $\leq_i, \leq'_i$ as in (ii-a) above.

Observe that (i) implies

$$M_{(X\cap Z)\uparrow,(W\setminus Z)\uparrow} = M_{(X\cap Z)\uparrow,(W'\setminus Z')\uparrow},$$
$$M_{(W\cap Z)\uparrow,(X\setminus Z)\uparrow} = M_{(W'\cap Z')\uparrow,(X\setminus Z)\uparrow}.$$

Furthermore, (ii) implies that

$$M_{(W\cap Z)\uparrow,(W\setminus Z)\uparrow} = M_{(W'\cap Z')\uparrow,(W'\setminus Z')\uparrow}.$$

Thus the matrices in the rightmost terms in (6.J) and (6.K) are equal. ■

The following lemma collects further useful properties of the equivalence relation $\equiv^\ell_X$.

Lemma VI.6. *Let $\ell\geq 1$*

(1) Given X, the equivalence relation $\equiv^\ell_X$ is canonical.
(2) There is an $e_1 = e_1(k_1,\ell)$ (independent of $\kappa{\downarrow}$) such that the index of $\equiv^\ell_X$ is at most $e_1(k_1,\ell)$.
(3) Given the graph G, the sets $C_0,\dots,C_m$, and the set X, the equivalence relation $\equiv^\ell_X$ can be computed in polynomial time (for fixed k_1 and ℓ).

Proof: (1) and (3) are obvious from the construction.

(2) follows easily from the following two observations.

- For every $\boldsymbol{w} = (w_1,\dots,w_\ell)\in\overline{X}^\ell$ the set $\bigcup_{i=1}^\ell w_i\uparrow$ has at most $\ell\cdot 2^{k-1}\leq\ell\cdot 2^{k_1-1}$ elements.
- The matrix $M_{X\uparrow,\overline{X}\uparrow}$ has rank k_1 and thus at most 2^{k_1} different columns.

■

We are now ready to prove the main result of this section. Let

$$e_2(k_1) := \max\{e_1(k_1,\ell) \mid 1\leq\ell\leq 2^{k_1}\}, \tag{6.L}$$

where e_1 is chosen according to Lemma VI.6.

Lemma VI.7. *Let $k_1\geq(3k+2)k$ and $e_2 = e_2(k_1)$. Then for every $X\subseteq A{\downarrow}$ of order $\kappa{\downarrow}(X) = k_1$ there is a canonical family of $e\leq e_2$ partitions $X^{(i)}_1, X^{(i)}_2$ of X, for $1\leq i\leq e$, such that $\kappa{\downarrow}(X^{(i)}_j) < k_1$ for $1\leq i\leq e$, $j = 1,2$.*

Furthermore, given X and oracle access to $\kappa{\downarrow}$, the family of partitions can be computed in polynomial time (for fixed k, k_1).

Proof: Let ℓ be the number of distinct columns of $M_{X\uparrow,\overline{X}\uparrow}$. Observe that $k\leq\ell\leq 2^{k_1}$. We call a set $W\subseteq\overline{X}$ *complete* if all columns of the matrix $M_{X\uparrow,\overline{X}\uparrow}$ already appear in the matrix $M_{X\uparrow,W\uparrow}$. A tuple $\boldsymbol{w}\in\overline{X}^{\ell}$ is *complete* if the set of its entries is complete. Observe that if $\boldsymbol{w}$ is complete and $\boldsymbol{w}'\equiv^\ell_X\boldsymbol{w}$, then $\boldsymbol{w}'$ is complete as well.

Let $\boldsymbol{w}^{(1)},\dots,\boldsymbol{w}^{(e)}$ be a system of representatives of the $\equiv^\ell_X$-equivalence classes consisting of complete tuples. Note that $e\leq e_2(k_1)$. For $1\leq i\leq e$, let $W^{(i)}$ be the set of entries of $\boldsymbol{w}^{(i)}$ and $A{\downarrow}^{(i)} := X\cup W^{(i)}$ and $\kappa{\downarrow}^{(i)} := \kappa{\downarrow}_{A{\downarrow}^{(i)}}$. By Lemma VI.5, up to renaming of the elements, the connectivity function $\kappa{\downarrow}^{(i)}$ only depends on the equivalence class of $\boldsymbol{w}^{(i)}$ and not on the choice of the specific tuple. Thus, up to renaming, the family of connectivity function $\kappa{\downarrow}^{(i)}$ is canonical.

Claim 1. Let $i\in[e]$. For each $Z\subseteq X$, we have $\kappa{\downarrow}^{(i)}(Z) = \kappa{\downarrow}(Z)$.

Proof. This follows from the completeness of $W^{(i)}$. ⌟

In particular, we have $\kappa{\downarrow}^{(i)}(X) = \kappa{\downarrow}(X) = k_1$. Now we apply the construction of Section VI-A to $\kappa{\downarrow}^{(i)}$. We define $\lambda^{(i)} : 2^{W^{(i)}} \to \mathbb{N}$ by $\lambda^{(i)}(Y) := \kappa{\downarrow}^{(i)}_{\min}(Y, X)$ and let $\mathcal{M}(\lambda^{(i)})$ be the matroid induced by $\lambda^{(i)}$. Note that the order of the entries of the tuple $\boldsymbol{w}^{(i)}$ gives us a linear order $\leq^{(i)}$ on $W^{(i)}$. We let $Y^{(i)}$ be the lexicographically first subset of $W^{(i)} \setminus \{c_0\}$ of size $3k+1$ that is independent in $\mathcal{M}(\lambda^{(i)})$. We let $Z_0^{(i)}$ be the lexicographically first subset of $Y^{(i)}$ such that for the leftmost minimum $(Z_0^{(i)}, Y^{(i)} \setminus Z_0^{(i)})$-separation $Z^{(i)}$ we have

$$\kappa{\downarrow}(Z^{(i)}) < \min\{|Y^{(i)} \cap Z^{(i)}|, |Y^{(i)} \setminus Z^{(i)}|\}, \tag{6.M}$$

and we let $X_1^{(i)} := X \cap Z^{(i)}$ and $X_2^{(i)} := X \setminus Z^{(i)}$. By Lemma VI.3 we have $\kappa{\downarrow}^{(i)}(X_j^{(i)}) < \kappa{\downarrow}^{(i)}(X)$, and by Claim 1 this implies $\kappa{\downarrow}(X_j) < \kappa{\downarrow}(X)$.

Clearly, the construction is canonical and can be carried out in polynomial time. ■

Corollary VI.8. *For every k_1 there is a $c_1 = c_1(k, k_1)$ such that for every $X \subseteq A{\downarrow}$ of order $\kappa{\downarrow}(X) = k_1$ there is a canonical partial treelike decomposition (T_X, γ_X) with the following properties.*

(i) T_X is a directed tree.
(ii) $\gamma_X(r) = X$ for the root r of T_X.
(iii) $\bigcup_{t \in L(T_X)} \gamma_X(t) = X$ (but the sets $\gamma_X(t)$ for the leaves $t \in L(T_X)$ are not necessarily disjoint).
(iv) $\kappa{\downarrow}(\gamma_X(t)) \leq k_1$ for all $t \in V(T_X)$.
(v) $\kappa{\downarrow}(\gamma_X(t)) < (3k+2)k$ for all leaves $t \in L(T_X)$.
(vi) T_X has at most $c_1(k, k_1)$ nodes.

Furthermore, given X and oracle access to $\kappa{\downarrow}$, the decomposition (T_X, γ_X) can be computed in polynomial time (for fixed k, k_1).

Proof: We define c_1 inductively by letting $c_1(k, k_1) = 1$ for all $k_1 < (3k+2)k$ and

$$c_1(k, k_1) = 4e_2(k_1) \cdot c_1(k, k_1 - 1)$$

for every $k_1 \geq (3k+2)k$.

We construct the decomposition T_X recursively as follows: we start with a root r and let $\gamma_X(r) := X$. If $\kappa{\downarrow}(X) < (3k+2)k$, this is the whole decomposition. So suppose that $\kappa{\downarrow}(X) \geq (3k+2)k$. Then we choose a canonical family of $e \leq e_1(k_1)$ partitions $X_1^{(i)}, X_2^{(i)}$ of X according to Lemma VI.7. For every $i \in [e]$ and $j = 1, 2$, let $(T_j^{(i)}, \gamma_j^{(i)})$ be the recursively constructed decomposition for $X_j^{(i)}$ (note that $\kappa{\downarrow}(X_j^{(i)}) < \kappa{\downarrow}(X) = k_1$).

To construct (T_X, γ_X) attach children $t^{(1)}, \ldots, t^{(e)}$ to r and let $\gamma_X(t^{(i)}) := X$. For every i, we attach the trees $T_1^{(i)}, T_2^{(i)}$ to $t^{(i)}$ such that the root of $T_j^{(i)}$ becomes a child of $t^{(i)}$ in T_X. For all $t \in V(T_j^{(i)})$, we let $\gamma_X(t) := \gamma_j^{(i)}(t)$.

It is easy to see that this construction has the desired properties. ■

VII. Constructing Canonical Treelike Decompositions

For the following lemma, we make Assumptions V.1, V.2, and VI.4.

Lemma VII.1. *There are $a_2 = a_2(k)$ and $g_1 = g_1(k)$ such that there is a treelike decomposition (T, γ) of $\kappa{\downarrow}$ with the following properties.*

(i) T is a directed tree.
(ii) T has at most n^{g_1} nodes, where $n := |A| = |V(G)|$.
(iii) (T, γ) has width at most a_2.
(iv) $\beta(t) = \{c_0\}$ for the root r of T.
(v) $|\gamma(t)| = 1$ for all leaves $t \in L(T)$.

Furthermore, given κ and $C_0, \ldots, C_m$ the construction of (T, γ) is canonical and can be carried out by a polynomial time algorithm (for fixed k).

As said earlier, the idea is to built the decomposition recursively, starting with the complement of a triple cover Q. As we cannot choose this triple cover canonically, we build a separate decomposition for each triple cover and then take the union of all these decomposition (joined at a common root). In a treelike decomposition, this is allowed.

The difficulty is to control the size of the decomposition, because whenever we are in the situation $\kappa{\downarrow}(X) \geq (3k+2)k$ (Section VI), we do not obtain a single partition, but a family of $e(k)$ partitions. This is potentially dangerous, because if we repeatedly decompose the same set in different ways, then we potentially end up with an exponentially large tree. However, whenever we apply the construction of Section VI, we strictly decrease the $\kappa\downarrow$-value, and careful analysis of the recurrences involved shows that this way we can control the size of the decomposition.

By merging the decompositions Lemma VII.1 gives us for all nodes of the decomposition (T, γ) from Theorem IV.1, we finally obtain the desired treelike decomposition, as stated in the following theorem.

Theorem VII.2 (Canonical Decomposition Theorem). *Let $k \in \mathbb{N}$. Then there is an $a = a(k) \in \mathbb{N}$ and a polynomial time algorithm that, given a graph G of rank width at most k, computes a canonical treelike decomposition of ρ_G of width at most a.*

VIII. Matrices of Bounded Partition Rank

In this section we consider symmetric matrices $P \in \{0, 1, ?\}^{V \times V}$ with entries $0, 1, ?$ and row and column indices from a set V. We usually denote the entries of such a matrix P by p_{vw}, for $v, w \in V$, and we denote the row $(p_{vw} \mid w \in V)$ with index v by $\boldsymbol{p}_v$. We need no special notation for the columns and just refer to them via their indices $w \in V$.

If the ?-entries of such a matrix P form a block diagonal matrix, we call P a *?-block matrix*. That is, $P \in \{0, 1, ?\}^{V \times V}$ is a ?-block matrix if it is symmetric and there are mutually disjoint subsets $I_1, \ldots, I_m \subseteq V$ such that $p_{vw} =?$ if and only if there is a $j \in [m]$ such that $v, w \in I_j$. We call the sets $I_1, \ldots, I_m$ the *?-indices* of P, and we say that row $\boldsymbol{p}_v$ *has ?-index* I_j if $v \in I_j$ (similarly for columns). For disjoint subsets $B, C \subseteq \{I_1, \ldots, I_m\}$, we let $P_{B,C}$ be the submatrix of P obtained by deleting all rows corresponding to indices that are not in B and deleting all columns corresponding to indices that are not in C. Note that $P_{B,C}$ is a $\{0, 1\}$-matrix. We denote by $P_{B,\overline{B}}$ the matrix $P_{B,\{I_1,\ldots,I_m\}\setminus B}$.

We say that the matrix P has *partition rank at most* k if for each partition of the family of ?-indices into two parts B and $\overline{B}$, the submatrix $P_{B,\overline{B}}$ has rank at most k over $\mathbb{F}_2$.

We are interested in ?-block matrices and their partition rank because we can use them to describe the width of treelike decompositions of cut-rank functions. Let (D, γ) be a normal treelike decomposition of the cut rank function ρ_G of a graph G, and let $t \in V(D)$ be a node with children $u_1, \ldots, u_\ell$ such that the children have pairwise disjoint cones. We define an *associated ?-block matrix* $P \in \{0, 1, ?\}^{V(G) \times V(G)}$ with entries p_{vw} defined as follows:

- if there is an $i \in [\ell]$ such that $v, w \in \gamma(u_i)$ then $p_{uw} =?$;
- if $v, w \in \overline{\gamma(t)}$ then $p_{uw} =?$;
- otherwise, if $vw \in E(G)$ then $p_{vw} = 1$ and if $vw \notin E(G)$ then $p_{vw} = 0$.

Note that the ?-indices of P are the sets $\gamma(u_1), \ldots, \gamma(u_\ell), \overline{\gamma(t)}$.

Lemma VIII.1. *Let (D, γ) be a normal treelike decomposition of the cut rank function ρ_G of a graph G. Let $t \in V(D)$ be a node whose children have mutually disjoint cones, and let P be the ?-block matrix associated with t. Then the partition rank of P is equal to the width of (D, γ) at t.*

An *extension* of a $\{0, 1, ?\}$-vector is a $\{0, 1\}$-vector obtained by replacing each '?'-entry by a 0 or a 1. That is, $\boldsymbol{x} = (x_v \mid v \in V) \in \{0, 1\}^V$ is an extension of $\boldsymbol{p} = (p_v \mid v \in V) \in \{0, 1, ?\}^V$ if $p_v \in \{0, 1\}$ implies $x_v = p_v$, for all $v \in V$. We say that two $\{0, 1, ?\}$-vectors are *compatible* if they have a common extension. An *isomorphism* from a matrix $P \in \{0, 1, ?\}^{V \times V}$ to a matrix $P' \in \{0, 1, ?\}^{V' \times V'}$ is a bijective mapping $\varphi : V \to V'$ such that $p_{vw} = p'_{\varphi(v)\varphi(v)}$ for all $v, w \in V$, where as usual we denote the entries of P by p_{vw} and the entries of P' by $p'_{v'w'}$.

Let $P \in \{0, 1, ?\}^{V \times V}$ be a ?-block matrix. An *extension set* for P is a set of vectors $\mathrm{Ext} \subseteq \{0, 1\}^V$ such that every row in $\boldsymbol{p}_v$ of P has an extension in Ext. If Ext is an extension set for P, then for every $v \in V$ we denote the set of all extensions of $\boldsymbol{p}_v$ in Ext by $\mathrm{Ext}(v)$. We call a construction that assigns an extension set to every ?-block matrix *canonical* if for every two isomorphic ?-block matrices $P \in \{0, 1, ?\}^{V \times V}$ and $P' \in \{0, 1, ?\}^{V' \times V'}$ and every isomorphism ψ from P to P' the following two conditions are satisfied.

(i) There is a bijection χ from Ext to Ext$'$ such that $\chi(\mathrm{Ext}(v)) = \mathrm{Ext}'(\psi(v))$ for all $v \in V$.
(ii) For every $\boldsymbol{x} = (x_v \mid v \in V) \in \mathrm{Ext}$ with $\chi(\boldsymbol{x}) =: \boldsymbol{x}' = (x'_{v'} \mid v' \in V') \in \mathrm{Ext}'$ and every $v \in V$ we have $x_v = x'_{\psi(v)}$.

Theorem VIII.2. *Let $k \in \mathbb{N}$. Then there is an $e = e(k) \in \mathbb{N}$ and a polynomial time algorithm that, given a ?-block matrix $P \in \{0, 1, ?\}^{V \times V}$ of partition rank at most k, computes a canonical extension set* $\mathrm{Ext} \subseteq \{0, 1\}^V$ *for P of size* $|\mathrm{Ext}| \leq e$.

IX. Computing the Automorphism Groups

We use various standard algorithms for permutation groups. Recall that a permutation group Γ that permutes elements in some set V can be succinctly represented by a generating set of polynomial size. For a set $\{g_1, \ldots, g_t\}$ of permutations on V the group generated by the set is denoted by $\langle g_1, \ldots, g_t \rangle$ or by $\langle \{g_1, \ldots, g_t\} \rangle$. For sets V' and V, slightly abusing terminology, we call a set of bijections Λ from V' to V a (V, V')-*coset*, or just a *coset* if V and V' are clear from the context, if there is a bijection σ in Λ and a permutation group Γ on V' such that $\sigma\Gamma = \Lambda$.[4] We also regard the empty set as a coset. We will always assume that cosets are succinctly represented by one explicit bijection σ in Λ and a generating set for the permutation group Γ. For more details on the algorithmic theory of permutation groups we refer to [34].

Let ρ_G and $\rho_{G'}$ be the cut rank function of graphs G and G' with vertex sets V, V', respectively. Let (D, γ) and (D', γ') be directed decompositions of V and V', respectively. An isomorphism φ from G to G' is said to *respect* (D, γ) *and* (D', γ') if there is an isomorphism $\widehat{\varphi}$ from D to D' such that $\gamma'(\widehat{\varphi}(t)) = \varphi(\gamma(t))$ for all $t \in V(D)$. We sometimes say that $\widehat{\varphi}$ is an *isomorphism from* (D, γ) *to* (D', γ') *extending* φ. We denote the coset of all isomorphism from G to G' that respect (D, γ) and (D', γ') by $\mathrm{Iso}(G_{D,\gamma}, G'_{D',\gamma'})$. Note that $\mathrm{Iso}(G_{D,\gamma}, G'_{D',\gamma'}) \leq \mathrm{Iso}(G, G')$.

In the following we describe an algorithm computing a coset Λ from $V(G)$ to $V(G')$ that satisfies $\mathrm{Iso}(G_{D,\gamma}, G'_{D',\gamma'}) \leq \Lambda \leq \mathrm{Iso}(G, G')$. We do this by dynamic programming over the directed acyclic graphs D and D'. We will at several points work with coloured graphs. Isomorphisms of coloured graphs are always required to map every vertex to a vertex of the same colour.

Let us fix k and G, G' and (D, γ), (D', γ') and let $V := V(G)$ and $V' := V(G')$. Furthermore, let $t \in V(D)$. We shall define a graph G_t that represents the induced subgraph $G[\gamma(t)]$ as well as an "abstraction" of the edges from $\gamma(t)$ to $\overline{\gamma(t)} = V \setminus \gamma(t)$. Let $W_t \subseteq \{0,1\}^{\gamma(t)}$ be the set of rows that appear in the matrix $M_{\overline{\gamma(t)},\gamma(t)}$. Since the width of $(D, \gamma(t))$ is at most k, the rank of the matrix $M_{\overline{\gamma(t)},\gamma(t)}$ is at most k, and thus the set W_t has size at most 2^k. We may view the elements $\boldsymbol{w} = (w_v \mid v \in \gamma(t))$ in W_t as "types", or equivalence classes of vertices $w \in \overline{\gamma(t)}$, where two vertices w, w' have the same type, or are equivalent, if they have the same adjacencies with the vertices in $\gamma(t)$. The entries of the vector $\boldsymbol{w}$ are these adjacencies; $w_v = 1$ means that all vectors of this type are adjacent to v and $w_v = 1$ means that they are not adjacent.

Now we are ready to define the graph G_t. The vertex set is $V(G_t) := \gamma(t) \cup W_t$, and the edge set is

$$\begin{aligned} E(G_t) := &\big\{vv' \;\big|\; v, v' \in \gamma(t) \text{ such that } vv' \in E(G)\big\} \\ &\cup \big\{v\boldsymbol{w} \;\big|\; v \in \gamma(t) \text{ and } \boldsymbol{w} = (w_{v'} \mid v' \in \gamma(t)) \in W_t \text{ such that } w_v = 1\big\}. \end{aligned}$$

Thus W_t is an independent set in G_t, and $G_t[\gamma(t)] = G[\gamma(t)]$. We colour the graph G_t so that the vertices in W_t are coloured red and all other vertices are coloured blue. We let D_t be the induced subgraph of D whose vertex set consist of all vertices that are reachable from t in D, and we let γ_t be the restriction of γ to $V(D_t)$. Then (D_t, γ_t) is a normal treelike decomposition of $V(G_t) \setminus W_t$. (We may also view it as a partial treelike decomposition of $V(G_t)$ or even $V(G)$; this does not matter, as we are not interested in the width of this decomposition.) Note that if r is the unique root of D then we have $G_r = G$ and $(D_r, \gamma_r) = (D, \gamma)$.

We define sets $W'_t \subseteq \{0,1\}^{\gamma'(t)}$, graphs G'_t, and decompositions (D'_t, γ'_t) analogously for all nodes $t \in V(D')$.

Recall that our goal is to compute a coset Λ such that $\mathrm{Iso}(G_{D,\gamma}, G'_{D',\gamma'}) \leq \Lambda \leq \mathrm{Iso}(G, G')$. We do this by a dynamic programming algorithm that processes the nodes of D in a bottom-up manner (starting from the leaves). The next lemma describes the inductive step.

Lemma IX.1. *There is a polynomial time algorithm that, given* $G, G', (D, \gamma), (D', \gamma')$ *as above and in addition*

- *nodes* $t \in V(D)$ *and* $t' \in V(D')$*;*
- *for all* $u \in N_+^D(t)$ *and* $u' \in N_+^{D'}(t')$ *a coset* $\Lambda(u, u')$ *satisfying*

$$\mathrm{Iso}((G_u)_{D_u,\gamma_u}, (G'_{u'})_{D'_{u'},\gamma'_{u'}}) \leq \Lambda(u, u') \leq \mathrm{Iso}(G_u, G'_{u'}), \tag{9.A}$$

computes a coset Λ *such that*

$$\mathrm{Iso}((G_t)_{D_t,\gamma_t}, (G'_{t'})_{D'_{t'},\gamma'_{t'}}) \leq \Lambda \leq \mathrm{Iso}(G_t, G'_{t'}).$$

Via dynamic programming we obtain the following corollary.

Corollary IX.2. *For every* $k \in \mathbb{N}$ *there is a polynomial time algorithm that, given graphs* G, G' *and normal treelike decompositions* (D, γ), (D', γ') *of* ρ_G, $\rho_{G'}$*, respectively, of width at most* k*, computes a coset* Λ *such that*

$$\mathrm{Iso}(G_{D,\gamma}, G'_{D',\gamma'}) \leq \Lambda \leq \mathrm{Iso}(G, G').$$

Combined with the Canonical Decomposition Theorem (Theorem VII.2), the corollary yields a polynomial time isomorphism test for graphs of bounded rank width (Theorem IX.3).

Theorem IX.3. *For every* $k \in \mathbb{N}$ *there is a polynomial time algorithm that, given graphs* G *and* G' *of rank width at most* k*, computes the set* $\mathrm{Iso}(G, G')$ *of all isomorphisms from* G *to* G'.

[4]To relate this to the standard group theoretic notion of coset, note that if $V = V'$, then a (V, V')-coset is a left coset of a subgroup of the symmetric group on V.

X. Conclusions

For every fixed k we obtain a polynomial time isomorphism test for graph classes of bounded rank width, unfortunately with a horrible running time: we only have a non-elementary upper bound (in terms of k) for the degree of the polynomial bounding the running time. Thus before even asking whether the isomorphism problem is fixed-parameter tractable if parameterized by rank-width, we ask for an algorithm with a running time $n^{O(k)}$. The bottleneck is the bound we obtain for the size of a triple cover of a tangle (see Lemma III.1); our algorithm has to enumerate all triple covers of all maximal tangles. But maybe there is a way to avoid this.

Our algorithm uses the group theoretic machinery, but the group theory involved is fairly elementary. It seems conceivable that it can be avoided altogether and there is a combinatorial algorithm deciding isomorphism of rank width at most k. Specifically, we ask whether for any k there is an ℓ such that the ℓ-dimensional Weisfeiler-Lehman algorithm decides isomorphism of graphs of rank width at most k.

Most of the arguments that we use in the construction of canonical bounded width decompositions apply to arbitrary connectivity functions and not just the cut rank function. (Only from Section VI onwards we use specific properties of the cut rank function.) It is an interesting question whether there is a polynomial time isomorphism test for arbitrary connectivity functions of bounded branch width. Even if this is not the case, it would be interesting to understand for which connectivity functions beyond the cut rank function such an isomorphism test exists.

In the end, the main question is whether our results help to solve the isomorphism problem for general graphs. The immediate answer is 'no'. However, we do believe that structural techniques such as those developed here (and also in [12], [14]), in combination with group theoretic techniques, may help to design graph isomorphism test with an improved worst-case running time.

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