Existential Least Fixed-Point Logic and Its Relatives

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Abstract

The main objects of our interest are the existential fragment ∃LFP of least fixed-point logic, stratified fixed-point logic SFP, which is the smallest regular logic containing ∃LFP, and transitive closure logic TC. The main result of the first part of this paper is a normal form for ∃LFP, which transfers to SFP to a certain extent. We study some of the consequences of this normal form and show that TC can be seen as a natural fragment of SFP. The second part of the paper is concerned with separating the logics under consideration. Furthermore, it shows that the existential preservation theorem fails for TC and SFP (both on finite and arbitrary structures). The method used to show this also yields a negative answer to a question posed by Rosen and Weinstein concerning first-order sentences preserved under extensions of finite structures.

Keywords: Inductive definitions, finite model theory, Datalog.

1 Introduction

Inductive definitions by positive existential formulae have first been studied in generalized recursion theory [1]. Chandra and Harel [4] introduced queries defined by positive existential inductions in the context of database theory and finite model theory. They observed that these are exactly the queries definable in the database language DATALOG. Blass and Gurevich [2] studied *existential least fixed-point logic* ∃LFP, which is the closure of existential first-order logic under positive existential inductions. They showed that this logic has some nice properties making it interesting both from the computer scientist's and the model theorist's point of view. For example, on finite structures with a successor relation ∃LFP is equally expressive as least fixed-point logic LFP and hence captures polynomial time [2]. On the integers, ∃LFP defines precisely the recursively enumerable sets (see [1]). Compared to LFP on arbitrary structures, its existential fragment is much easier to handle since the closure ordinal of an ∃LFP-formula is at most ω. As a result, ∃LFP is contained in the infinitary logic $L_{\omega_1\omega}$.

The main contribution of this paper to the investigation of ∃LFP is a strong normal form, which has several results of Blass and Gurevich as immediate consequences. Furthermore, it yields an Ehrenfeucht–Fraïssé-like characterization of ∃LFP and gives a complete problem for ∃LFP under quantifier-free reductions. In fact, this complete problem turns out to be the familiar PATH SYSTEMS problem, which was the first problem shown to be complete for polynomial time [6].

Database theory gives one of the main motivations to study ∃LFP particularly on finite relational structures, since the logic corresponds to the database language DATALOG (with negations allowed in front of extensional relation symbols). If we allow function symbols (as we do in this paper), this correspondence extends to arbitrary logic programs. As a reasonable concept of negation in logic programs, so-called stratification has received some interest. This led Kolaitis [18] to the introduction of *stratified fixed-point logic* SFP as the extension of $\exists$LFP corresponding to stratified logic programs.

Model-theoretically speaking, stratification is nothing else but substitution of relation symbols by formulae already defined. Basically, this shows that SFP is the smallest regular logic containing $\exists$LFP.[1] This observation implies that SFP inherits many of the nice qualities of $\exists$LFP. Furthermore, since SFP is regular it often behaves even more favourably. For example, SFP-formulas still have closure ordinal ω. Considering its closure properties, this implies that SFP is a *fragment* of $L_{\omega_1\omega}$, providing a completeness theorem for SFP (see [17]; also cf [5] for an explicit proof of a completeness theorem for $\exists$LFP).

We can transfer our normal form result to SFP; as a consequence, we get an Ehrenfeucht–Fraïssé game for this logic and also a representation by a vectorized Lindström-quantifier.

SFP can also be seen as a natural generalization of the well-known transitive closure logic TC, or the other way around, TC is a natural fragment of SFP. Comparing TC with its existential fragment $\exists$TC (thereby referring to known results due to Immerman [16], Grädel and McColm [10, 9]) will give further evidence that this is a reasonable point of view; the pairs SFP–$\exists$LFP and TC–$\exists$TC behave quite similarly.

We then turn to questions concerning the expressive power of the logics we have considered so far. We give a short(er) proof of Immerman's [15] result that $\exists$LFP is not contained in TC. Using a similar idea, we will show that $\exists$TC is strictly contained in the intersection of $\exists$LFP with TC, implying that there is no existential preservation theorem (or substructure preservation theorem) for TC (neither on finite nor on arbitrary structures).

In the final section of the paper we show that existential preservation also fails for SFP and thus for LFP. Furthermore, by the same methods we show that there exists a first-order sentence which is preserved under extensions of finite structures, but is not equivalent to an existential sentence of the infinitary logic $L^{\omega}_{\infty\omega}$. This answers negatively a question posed by Rosen and Weinstein [21].

The logics examined in this paper are usually studied in the context of finite model theory, especially when considering the connections with database theory. Nevertheless, most of our results also hold on infinite structures, so there is no need to restrict ourselves to the finite. But it should be emphasized that in particular the results separating logics already hold in the finite.

Notation that deviates from the standard will usually be explained in the footnotes.

2 Some basic definitions and facts

Least fixed-point logic LFP is the extension of first-order logic by a *(least) fixed-point operator* that allows, given a formula $\varphi(\bar{x}, X)$ which is positive in the relation variable X[2], to build a new formula $[\mathrm{LFP}_{\bar{x},X}\varphi(\bar{x}, X)]\bar{t}$ (where $\bar{x}$ is a tuple of distinct individual variables, $\bar{t}$ is a tuple of terms, and the length of $\bar{x}$ and $\bar{t}$ matches the arity of X, say k).

To define the semantics of the new formula, in each structure $\mathfrak{A}$ we define an operator

[1] A logic is called *regular* if it satisfies some basic closure properties (being closed under Boolean combinations, existential quantification, relativization, and substitution of predicates). We refer the reader to [8] for a precise definition and discussion of regular logics.

[2] We call φ *positive in* X if X only occurs in the scope of an even number of negation symbols in φ.

$F_{\varphi(\bar{x},X)} : \mathrm{Pow}(A^k) \to \mathrm{Pow}(A^k)$ by

$$F_{\varphi(\bar{x},X)}(B) = \{\bar{a} \mid \mathfrak{A} \models \varphi(\bar{a}, B)\}.^3$$

Since $\varphi(\bar{x}, X)$ is positive in X, the operator $F_{\varphi(\bar{x},X)}$ is monotone, and hence it has a least fixed-point $F^{\infty}_{\varphi(\bar{x},X)}$. Now the meaning of the new formula is defined by

$$\mathfrak{A} \models [\mathrm{LFP}_{\bar{x},X}\varphi(\bar{x}, X)]\bar{a} \iff \bar{a} \in F^{\infty}_{\varphi(\bar{x},X)}.$$

Note that $F^{\infty}_{\varphi(\bar{x},X)}$ can also be obtained in the following way: For each $B \subseteq A^k$ we let $F^{0}_{\varphi(\bar{x},X)}(B) = B$, $F^{\alpha+1}_{\varphi(\bar{x},X)}(B) = F_{\varphi(\bar{x},X)}(F^{\alpha}_{\varphi(\bar{x},X)}(B))$ (for each ordinal $\alpha \geq 0$), and $F^{\beta}_{\varphi(\bar{x},X)} = \bigcup_{\alpha<\beta} F^{\alpha}_{\varphi(\bar{x},X)}(B)$ (for each limit ordinal β). Then we have $F^{\alpha}_{\varphi(\bar{x},X)}(B) \subseteq F^{\beta}_{\varphi(\bar{x},X)}(B)$ for all ordinals $\alpha < \beta$ and $B \subseteq A^k$ such that $B \subseteq F_{\varphi(\bar{x},X)}(B)$. Thus $F^{\alpha}_{\varphi(\bar{x},X)}(B) = F^{\alpha+1}_{\varphi(\bar{x},X)}(B)$ for all such B and $\alpha \geq |A^k|^+$. This implies

$$F^{\infty}_{\varphi(\bar{x},X)} = \bigcup_{\alpha \leq |A^k|^+} F^{\alpha}_{\varphi(\bar{x},X)}(\emptyset).$$

An LFP-formula χ is called *existential*, if it contains no universal quantifiers and if negation symbols only occur in front of atomic subformulas.

DEFINITION 2.1
Existential least fixed-point logic ∃LFP is the fragment of LFP whose formulas are the existential LFP-formulas.

The following well-known lemma is due to Moschovakis [20], and the existential case is explicitly stated in [1].

THEOREM 2.2 (Transitivity Lemma)
Let $\varphi(\bar{x}, X, Y), \psi(\bar{y}, X, Y)$ be first-order formulas positive in X and Y. Then there exists a first-order formula $\varphi'(\bar{z}, Z)$ (positive in Z) such that

$$\models [\mathrm{LFP}_{\bar{x},X}\varphi(\bar{x}, X, [\mathrm{LFP}_{\bar{y},Y}\psi(\bar{y}, X, Y)]\_\!\_)]\bar{t} \longleftrightarrow [\mathrm{LFP}_{\bar{z},Y}\varphi'(\bar{z}, Z)]\bar{t}t_1 \ldots t_1.^4$$

Furthermore, if φ, ψ are existential then φ' can also be chosen to be existential.

As a corollary we obtain a normal form result for ∃LFP. It is convenient to use the following notational convention:

NOTATION 2.3
Let $[\mathrm{LFP}_{\bar{x},X}\varphi]\bar{t}$ be an LFP-formula and v a variable not occurring in φ. Then we abbreviate $\exists v[\mathrm{LFP}_{\bar{x},X}\varphi]v \ldots v$ by $[\mathrm{LFP}_{\bar{x},X}\varphi]$.

COROLLARY 2.4
Every ∃LFP-formula is equivalent to a formula of the form

$$[\mathrm{LFP}_{\bar{x},X}\varphi]$$

where φ is an existential first-order formula (positive in X).

Immerman [16] proved that on finite structures the analogous result holds for LFP, but this is deeper since it has to deal with negated fixed-point operators.

[3] For brevity we usually suppress parameters.

[4] For any two formulas $\varphi(X)$ and $\psi(\bar{x})$, where the length of $\bar{x}$ matches the arity of X, $\varphi(\psi(\_\_))$ denotes the formula obtained from φ by replacing each subformula of the form $X\bar{t}$ by $\psi(\bar{t})$. Similar notation will be used throughout this paper without further explanation.

3 The bounded normal form

A formula $\varphi \in$ LFP is called *positive existential in a relation variable* X if X does not occur in the scope of any universal quantifiers or negation symbols in φ.

LEMMA 3.1
Let $\varphi(X) \in$ FO be positive existential in X, say with k occurrences of X. Then there exists a formula $\psi(\bar{x}_1, \dots, \bar{x}_k)$ in which X does not occur such that for any structure $\mathfrak{A}$ and non-empty set $B \subseteq A^l$ (where l is the arity of X) we have

$$\mathfrak{A} \models \varphi(B) \quad \Longleftrightarrow \quad \exists \bar{a}_1, \dots, \bar{a}_k \in B : \mathfrak{A} \models \psi(\bar{a}_1, \dots, \bar{a}_k).$$

Furthermore, if φ is existential then ψ can also be chosen to be existential.

PROOF. The proof is an easy induction on φ. ∎

The next theorem looks quite technical, but it has many nice consequences.

THEOREM 3.2
Every $\exists$LFP-formula χ is equivalent to a formula χ' of the form

$$\left[\mathrm{LFP}_{\bar{x},X}\ \theta_0(\bar{x}) \vee \left(\exists \bar{y} \in X \theta_1(\bar{x}, \bar{y}) \wedge \exists \bar{z} \in X \theta_2(\bar{x}, \bar{z})\right)\right]$$

where $\theta_0, \theta_1, \theta_2$ are quantifier-free formulas in which X does not occur. We call χ' an $\exists$LFP-formula in *bounded normal form.*

PROOF. By Corollary 2.4 it suffices to prove the statement for formulas $\chi = [\mathrm{LFP}_{\bar{x},X}\varphi]$ where φ is an existential first-order formula (positive in X).

STEP 1: An application of Lemma 3.1 to φ gives us an existential first-order formula $\psi(\bar{x}, \bar{y}_1, \dots, \bar{y}_k)$ in which X does not occur such that

$$\models X \neq \emptyset \rightarrow \left(\varphi(\bar{x}, X) \leftrightarrow \exists \bar{y}_1 \in X \dots \exists \bar{y}_k \in X\ \psi(\bar{x}, \bar{y}_1, \dots, \bar{y}_k)\right).$$

Denoting by $\varphi(\bar{x}, \emptyset)$ the (existential first-order) formula obtained from φ by replacing each subformula of the form $X\bar{t}$ by $\neg t_1 = t_1$ this implies (using the monotonicity of $F_{\varphi(\bar{x},X)}$)

$$\models \varphi(\bar{x}, X) \leftrightarrow \left(\varphi(\bar{x}, \emptyset) \vee \exists \bar{y}_1 \in X \dots \exists \bar{y}_k \in X\ \psi(\bar{x}, \bar{y}_1, \dots, \bar{y}_k)\right). \qquad (*)$$

STEP 2: Say, X is l-ary, and let $\tilde{X}$ be a new kl-ary relation variable. Let $\xi_0(\bar{x}_1 \dots \bar{x}_k) = \bigwedge_{i=1}^{k} \varphi(\bar{x}_i, \emptyset)$,

$$\xi_1(\bar{x}_1 \dots \bar{x}_k, \bar{y}_1 \dots \bar{y}_k, \bar{z}_1 \dots \bar{z}_k) = \bigwedge_{i=1}^{k} \left(\psi(\bar{x}_i, \bar{y}_1, \dots, \bar{y}_k) \vee \bar{x}_i = \bar{y}_i \vee \bar{x}_i = \bar{z}_i\right),$$

and

$$\xi(\bar{x}_1 \dots \bar{x}_k, \tilde{X}) = \xi_0(\bar{x}_1 \dots \bar{x}_k) \vee \exists \bar{y}_1 \dots \bar{y}_k \in \tilde{X}\ \exists \bar{z}_1 \dots \bar{z}_k \in \tilde{X}\ \xi_1(\bar{x}_1 \dots \bar{x}_k, \bar{y}_1 \dots \bar{y}_k, \bar{z}_1 \dots \bar{z}_k).$$

Consider the operator $F_{\xi(\bar{x}_1\ldots\bar{x}_k,\tilde{X})} : \mathrm{Pow}(A^{kl}) \to \mathrm{Pow}(A^{kl})$ in some structure $\mathfrak{A}$. It is quite obvious (using (*) and the monotonicity of the operators) that

$$F_{\xi(\bar{x}_1\ldots\bar{x}_k,\tilde{X})}(B \times \ldots \times B) \subseteq F_{\varphi(\bar{x},X)}(B) \times \ldots \times F_{\varphi(\bar{x},X)}(B)$$

for all $B \subseteq A^l$ with $B \subseteq F_{\varphi(\bar{x},X)}(B)$ (in particular for all $B = F^\alpha(\emptyset)$). Hence $F^\infty_{\xi(\bar{x}_1\ldots\bar{x}_k,\tilde{X})}(\emptyset) \subseteq F^\infty_{\varphi(\bar{x},X)}(\emptyset) \times \ldots \times F^\infty_{\varphi(\bar{x},X)}(\emptyset)$.

On the other hand we have

$$F_{\varphi(\bar{x},X)}(B) \times \ldots \times F_{\varphi(\bar{x},X)}(B) \subseteq F^k_{\xi(\bar{x}_1\ldots\bar{x}_k,\tilde{X})}(B \times \ldots \times B)$$

for all $B \subseteq A^l$, implying $F^\infty_{\varphi(\bar{x},X)}(\emptyset) \times \ldots \times F^\infty_{\varphi(\bar{x},X)}(\emptyset) \subseteq F^\infty_{\xi(\bar{x}_1\ldots\bar{x}_k,\tilde{X})}(\emptyset)$. Thus

$$F^\infty_{\xi(\bar{x}_1\ldots\bar{x}_k,\tilde{X})} = F^\infty_{\varphi(\bar{x},X)} \times \ldots \times F^\infty_{\varphi(\bar{x},X)},$$

which shows that

$$\models \forall v\big([\mathrm{LFP}_{\bar{x},X}\varphi(\bar{x},X)]v\ldots v \longleftrightarrow [\mathrm{LFP}_{\bar{\bar{x}},\tilde{X}}\xi_0(\bar{\bar{x}}) \vee \exists\bar{\bar{y}} \in \tilde{X}\exists\bar{\bar{z}} \in \tilde{X}\xi(\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{z}})]v \ldots. v\big).$$

STEP 3: Simplifying the notation, up to now we hâve obtained a formula χ' equivalent to χ of the form

$$[\mathrm{LFP}_{\bar{x},X}\xi_0(\bar{x}) \vee \exists\bar{y} \in X\exists\bar{z} \in X\ \xi_1(\bar{x},\bar{y},\bar{z})]$$

where ξ_0 and ξ_1 are existential first-order formulas in which X does not occur.

Let m be the arity of X and X' be $(m+1)$-ary. We define a quantifier-free formula $\eta_0(x\bar{x})$ to be $\neg x = x_1$ and let

$$\xi_1'(x\bar{x},y\bar{y},z\bar{z}) = \xi_0(\bar{x}) \vee (\xi_1(\bar{x},\bar{y},\bar{z}) \wedge y = y_1 \wedge z = z_1).$$

It is not hard to see that we have

$$\models \chi' \longleftrightarrow [\mathrm{LFP}_{x\bar{x},X'}\ \eta_0(x\bar{x}) \vee \exists y\bar{y} \in X'\exists z\bar{z} \in X'\xi_1'(x\bar{x},y\bar{y},z\bar{z})].$$

Now we choose a quantifier-free formula $\eta_1(x\bar{x},y\bar{y},z\bar{z},w_1,\ldots,w_r)$ such that

$$\models \xi_1'(x\bar{x},y\bar{y},z\bar{z}) \longleftrightarrow \exists w_1\ldots\exists w_r\eta_1(x\bar{x},y\bar{y},z\bar{z},w_1,\ldots,w_r).$$

Letting X'' be $(m+1+r)$-ary we obtain a formula

$$\begin{aligned}&[\mathrm{LFP}_{x\bar{x},X''}\eta_0(x\bar{x})\vee\\ &\qquad \exists y\bar{y} \in X''\,\exists z\bar{z} \in X''\eta_1(xx_1\ldots x_m,yy_1\ldots y_m,zz_1\ldots z_m,y_{m+1},\ldots,y_{m+r})]\end{aligned}$$ [5]

which is still equivalent to χ.

STEP 4: Abusing notation again, after Step 3 we have arrived at a formula equivalent to χ of the form

$$[\mathrm{LFP}_{\bar{x},X}\ \underbrace{\eta_0(\bar{x}) \vee \exists\bar{y} \in X\exists\bar{z} \in X\ \eta_1(\bar{x},\bar{y},\bar{z})}_{=\eta(\bar{x},X)}]$$

where η_0 and η_1 are quantifier-free formulas in which X does not occur.

[5]Note that in this formula the tuples $\bar{x},\bar{y},\bar{z}$ are of length $(m+r)$.

Let s be the arity of X, X^* be $2s$-ary, and

$$\theta(\bar{x}_1\bar{x}_2, X^*) = \underbrace{\big(\eta_0(\bar{x}_1) \wedge \eta_0(\bar{x}_2)\big)}_{=\theta_0(\bar{x}_1\bar{x}_2)} \vee \big(\exists \bar{y}_1\bar{y}_2 \in X^* \underbrace{\eta_1(\bar{x}_1, \bar{y}_1, \bar{y}_2)}_{=\theta_1(\bar{x}_1\bar{x}_2, \bar{y}_1\bar{y}_2)} \wedge \exists \bar{z}_1\bar{z}_2 \in X^* \underbrace{\eta_1(\bar{x}_2, \bar{z}_1, \bar{z}_2)}_{=\theta_2(\bar{x}_1\bar{x}_2, \bar{z}_1\bar{z}_2)}\big).$$

Then for each structure $\mathfrak{A}$ and $B \subseteq A^s$ we have

$$F_{\theta(\bar{x}_1\bar{x}_2, X^*)}(B \times B) = F_{\eta(\bar{x}, X)} \times F_{\eta(\bar{x}, X)}.$$

Hence

$$\models \forall v\big([\mathrm{LFP}_{\bar{x},X}\varphi(\bar{x}, X)]v \dots v \longleftrightarrow [\mathrm{LFP}_{\bar{x}_1\bar{x}_2,X^*}\theta(\bar{x}_1\bar{x}_2, X^*)]v \dots vv \dots v\big).$$

∎

One consequence of this theorem is a method to check whether two classes of structures can be separated in ∃LFP. Basically, ∃LFP-formulas of the form

$$\Big[\mathrm{LFP}_{\bar{x},X}\ \theta_0(\bar{x}) \vee \big(\exists \bar{y} \in X\theta_1(\bar{x}, \bar{y}) \wedge \exists \bar{z} \in X\theta_2(\bar{x}, \bar{z})\big)\Big]\bar{t}$$

state the existence of certain binary trees.

In this paper, a *forest* is an acyclic directed graph where every node has in-degree at most 1; a *tree* is a connected forest. The *roots* of a forest are all nodes with in-degree 0, and the *leaves* are all nodes with out-degree 0. The *height* of a node v in a forest is the length of the (unique) path from a root to v. The *height* of a forest is the maximal height of its leaves.

DEFINITION 3.3
(1) A *2-tree* is an $\{E_1, E_2, L, \underline{r}\}$-structure $\mathfrak{T}$ (where E_1 and E_2 are binary relation symbols, L is a unary relation symbol, and $\underline{r}$ is a constant symbol) such that:

- $(T, E_1^{\mathfrak{T}} \cup E_2^{\mathfrak{T}})$ is a tree with root $\underline{r}^{\mathfrak{T}}$.
- $L^{\mathfrak{T}}$ is the set of leaves of this tree.
- Each $t \in T \setminus L^{\mathfrak{T}}$ has exactly one E_1-successor and one E_2-successor.

(2) A *k-ary 2-tree on a structure* $\mathfrak{A}$ is a pair $(\mathfrak{T}, f)$ consisting of a 2-tree $\mathfrak{T}$ and a mapping $f : T \to A^k$ such that $f(\underline{r}^{\mathfrak{T}}) = a \dots a$ for some $a \in A$.

LEMMA 3.4 (Tree Lemma)
Let $\chi = \big[\mathrm{LFP}_{\bar{x},X}\ \theta_0(\bar{x}) \vee \big(\exists \bar{y} \in X\theta_1(\bar{x}, \bar{y}) \wedge \exists \bar{z} \in X\theta_2(\bar{x}, \bar{z})\big)\big]$ be an ∃LFP-sentence in bounded normal form. Then for any structure we have $\mathfrak{A} \models \chi$ if and only if there exists a finite k-ary 2-tree $(\mathfrak{T}, f)$ on $\mathfrak{A}$ (where k is the arity of X) such that

(i) $L^{\mathfrak{T}}t \implies \mathfrak{A} \models \theta_0\big(f(t)\big)$.
(ii) $E_i^{\mathfrak{T}}tu \implies \mathfrak{A} \models \theta_i\big(f(t), f(u)\big)$ (for $i = 1, 2$).

PROOF. The direction from the right to the left is easy. So suppose that $\mathfrak{A} \models \big[\mathrm{LFP}_{\bar{x},X}\ \theta_0(\bar{x}) \vee \big(\exists \bar{y} \in X\theta_1(\bar{x}, \bar{y}) \wedge \exists \bar{z} \in X\theta_2(\bar{x}, \bar{z})\big)\big]a \dots a$ for some $a \in A$. Let us denote the operator $F_{\theta_0(\bar{x}) \vee \left(\exists \bar{y} \in X\theta_1(\bar{x},\bar{y}) \wedge \exists \bar{z} \in X\theta_2(\bar{x},\bar{z})\right)}$ simply by F for the course of this proof. Then there exists an ordinal α such that $a \dots a \in F^{\alpha}(\emptyset)$.

Inductively we define 2-trees $\mathfrak{T}_i$ and a mapping $f_i : T_i \to A^k$ for each $i \geq 0$ such that

(1) $f(r^{\mathfrak{T}_i}) = a \dots a$.

(2) $E_j^{\mathfrak{T}_i} tu \implies \mathfrak{A} \models \theta_j(f_i(t), f_i(u))$ (for $j = 1, 2$).

(3) The height of $\mathfrak{T}_i$ is $\leq i$.

(4) There exists an ordinal $\alpha(i)$ such that for each $t \in L^{\mathfrak{T}_i}$ we have $f_i(t) \in F^{\alpha(i)}(\emptyset)$; and that for $j < i$ we either have $\alpha(i) < \alpha(j)$ or $\alpha(i) = \alpha(j) = 1$.

(5) If the height of a leaf $t \in L^{\mathfrak{T}}$ is smaller than i, then $f_i(t) \in F^1(\emptyset) = F(\emptyset)$.

This implies our statement, since there must exist a finite $i \geq 0$ such that $\alpha(i) = 1$ and hence

$$\forall t \in L^{\mathfrak{T}_i} : f_i(t) \in F(\emptyset) = \{\bar{a} \in A \mid \mathfrak{A} \models \theta_0(\bar{a})\}.$$

$\mathfrak{T}_i$ and f_i satisfy (i) and (ii).

But the definition of the $\mathfrak{T}_i$ is quite easy: We let $\mathfrak{T}_0$ be the 2-tree consisting only of one node t, and $f_0(t) = a \dots a$. (4) is satisfied with $\alpha(0) = \alpha$.

$\mathfrak{T}_{i+1}$ and f_{i+1} are constructed from $\mathfrak{T}_i$ and f_i as follows: for each $t \in L^{\mathfrak{T}_i}$ with $f_i(t) \notin F(\emptyset)$ there exist some $\beta_t < \alpha(i)$ and $\bar{b}, \bar{c} \in F^{\beta_t}(\emptyset)$ such that $\mathfrak{A} \models \theta_1(f_i(t), \bar{b}) \wedge \theta_2(f_i(t), \bar{c})$. We add two new nodes t_1, t_2 to T_i and an E_j-edge from t to t_j (for $j = 1, 2$). t is removed from $L^{\mathfrak{T}_{i+1}}$ and t_1, t_2 are added. f_{i+1} maps t_1 to $\bar{b}$ and t_2 to $\bar{c}$. The rest is unchanged. Note that the maximum of the finitely many β_t is smaller than $\alpha(i)$ and can be taken as $\alpha(i + 1)$. ∎

A common way to avoid some of the difficulties that usually come along with function symbols is to work with so-called term-reduced formulas: $\varphi \in$ LFP is called *term-reduced* if all of its atomic subformulas are of the form $Rx_1 \dots x_m$ (for an m-ary relation symbol or relation variable R and (not necessarily distinct) individual variables $x_1, \dots, x_m$), $fx_1 \dots x_m = y$ (for an m-ary function symbol f and (not necessarily distinct) individual variables $x_1, \dots, x_m, y$), or $c = x$ (for a constant symbol c and an individual variable x).

Observe that for any two structure $\mathfrak{A}$ and $\mathfrak{B}$ and tuples $a_1 \dots a_k \in A$ and $b_1 \dots b_k \in B$, the mapping defined by $a_i \mapsto b_i$ (for $1 \leq i \leq k$) is a partial isomorphism if and only if for all term-reduced atomic formulas $\theta(x_1, \dots, x_k)$ with variables among $\{x_1, \dots, x_k\}$ we have

$$\mathfrak{A} \models \theta(a_1 \dots a_k) \iff \mathfrak{B} \models \theta(b_1 \dots b_k).$$

Also note that each $\exists$LFP-sentence is actually equivalent to a term-reduced sentence in bounded normal form.

A formula $\varphi \in$ LFP is *k-ary*, if it contains at most k-ary relation variables.

COROLLARY 3.5

Let σ be a finite signature. Then for all σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ the following are equivalent:

(1) Each k-ary, term-reduced $\exists$LFP-sentence in bounded normal form that holds in $\mathfrak{A}$ also holds in $\mathfrak{B}$.

(2) For any finite k-ary 2-tree $(\mathfrak{T}, f)$ on $\mathfrak{A}$ there exists a finite k-ary 2-tree $(\mathfrak{U}, g)$ on $\mathfrak{B}$ such that
   - For each $u \in L^{\mathfrak{U}}$ there exists a $t \in L^{\mathfrak{T}}$ such that the canonical mapping π between (the k-tuples) $g(u)$ and $f(t)$ is a partial isomorphism between $\mathfrak{B}$ and $\mathfrak{A}$.
   - For each $i = 1, 2$, $uu' \in E_i^{\mathfrak{U}}$ there exist $tt' \in E_i^{\mathfrak{T}}$ such that the canonical mapping between (the $2k$-tuples) $g(u)g(u')$ and $f(t)f(t')$ is a partial isomorphism between $\mathfrak{B}$ and $\mathfrak{A}$.

PROOF. Having the Tree Lemma available, a proof of this corollary only requires standard techniques which are used to prove logics being captured by Ehrenfeucht–Fraïssé games. We refer the reader to [10] as an example where such a proof is carried out. ∎

We also get some results of Blass and Gurevich [2] as immediate consequences of the Tree Lemma.

The *closure ordinal* of an LFP-formula χ is the least ordinal α such that for each subformula $[\mathrm{LFP}_{\bar{x},X}\varphi]\bar{t}$ of χ and each structure $\mathfrak{A}$ we have $F_\varphi^\alpha(\emptyset) = F_\varphi^{\alpha+1}(\emptyset)$, if such an α exists, and ∞ otherwise.

COROLLARY 3.6 ([2])

(1) For each sentence $\varphi \in \exists$LFP and structure $\mathfrak{A}$ we have $\mathfrak{A} \models \varphi$ if and only if there exists a finitely generated substructure of $\mathfrak{A}$ which is a model of φ.

(2) $\exists$LFP-formulas are preserved under extensions.

(3) The closure ordinal of an $\exists$LFP-formula is at most ω.

PROOF. (1) and (2) are immediate, and so is (3) for formulas in bounded normal form. Thus it only has to be checked that the proofs of the normal form results (including the Transitivity Lemma) do not change the closure ordinal by more than a finite factor, which is immediately done. ∎

REMARK 3.7

Without giving any definitions of logical reducibilities, let me remark another consequence of the Tree Lemma: let $\mathcal{E}$ be the class of all $\{E_1, E_2, L, R\}$-structures $\mathfrak{A}$ (where E_1, E_2, L are as before and R is another unary relation symbol) such that there exists a finite unary 2-tree $(\mathfrak{T}, f)$ on $\mathfrak{A}$ where f is a homomorphism from $\mathfrak{T}\restriction_{\{E_1,E_2,L\}}$ to $\mathfrak{A}\restriction_{\{E_1,E_2,L\}}$ and $f(\underline{r}^{\mathfrak{T}}) \in R^{\mathfrak{A}}$.

Then the class $\mathcal{E}$ is a complete problem for $\exists$LFP *under quantifier-free reductions.*

Since on finite structures with a successor relation $\exists$LFP captures polynomial time, as a corollary we see that the class $\mathcal{E}$ is complete for polynomial time under quantifier-free reductions. Actually $\mathcal{E}$ is only a slight modification of the well-known *path systems* problem which was the first problem proved to be complete for polynomial time [6]. Replacing the 2 binary by a ternary relation it is also easy to prove completeness of path systems without any modifications from our results.

4 Stratified fixed-point logic

In the previous section we have seen some nice properties of the logic $\exists$LFP. But it also has certain shortcomings. In particular, it does not even contain first-order logic (since $\exists$LFP-formulas are preserved under extensions), which is often considered as being a basic property of a logic.[6] In this section we are going to study an extension of $\exists$LFP which overcomes these shortcomings, but still preserves many nice features of $\exists$LFP.

Recall that a formula φ is positive existential in a relation variable X if X does not occur in the scope of any universal quantifiers or negation symbols in φ.

DEFINITION 4.1

Stratified fixed-point logic SFP is the sublogic of LFP where a fixed-point operator $[\mathrm{LFP}_{\bar{x},X}\ \_\ ]\bar{t}$ is only allowed to be applied to formulas which are positive existential in X.

[6] Being closed under the first-order operations (Boolean combinations and existential quantification) is a property required of regular logics. See also footnote [1] on page 206

We say that a formula φ is obtained from a formula ψ by *substitution of a k-ary relation symbol R by a formula* $\chi(x_1, \ldots, x_k)$ if φ is obtained from ψ by replacing each subformula of the form $Rt_1 \ldots t_k$ by $\chi(t_1, \ldots, t_k)$.

LEMMA 4.2
Up to logical equivalence, SFP-sentences are precisely those LFP-sentences that can be obtained from an atomic formula by repeatedly substituting relation symbols by ∃LFP-formulas without free relation variables.

In particular, this shows that SFP is the smallest regular logic containing ∃LFP.

PROOF. An easy induction on χ which may be illustrated by the following example: The formula

$$\chi = \forall x \forall y \big[\mathrm{LFP}_{z,Z} z = x \vee \exists z'(Zz' \wedge Ezz')\big] y$$

can be obtained by the following sequence of substitions:

$$\begin{array}{rl} P \rightsquigarrow \neg Q \rightsquigarrow \neg\exists x \exists y Rxy \rightsquigarrow & \neg\exists x \exists y \neg Sxy \\ & \quad\updownarrow \\ & \forall x \forall y Sxy \quad \rightsquigarrow \chi \end{array}$$

(where P, Q are 0-ary and R, S are binary relation symbols). ∎

REMARK 4.3
Refering to Remark 3.7, we let $Q_{\mathcal{E}}$ be the *Lindström quantifier* associated with the class $\mathcal{E}$. The facts that SFP is the smallest regular logic containing ∃LFP and that $\mathcal{E}$ is complete for ∃LFP under quantifier-free reductions imply that SFP *has the same expressive power as the logic* $\mathrm{FO}(Q_{\mathcal{E}}^{\leq\omega})$ (obtained by augmenting first-order logic by the vectorization of the quantifier $Q_{\mathcal{E}}$).

Although we cannot prove a normal-form result showing that one fixed-point operator suffices for SFP, we can simplify the structure of SFP-formulas considerably using our bounded normal form for ∃LFP.

We define a hierarchy of SFP-formulas as follows:

- SFP_0 is the class of quantifier-free formulas.
- SFP_{i+1} is the class of all boolean combinations of formulas of the form

$$[\mathrm{LFP}_{\bar{x},X}\varphi_0 \vee \exists \bar{y} \in X \exists \bar{z} \in X\ \varphi]$$

  where $\varphi_0, \varphi \in \mathrm{SFP}_i$ do not contain X freely.

A result of Grädel and McColm [9] shows that this induces a strict hierachy, i.e. that for each $i \geq 0$ there exists a formula $\chi \in \mathrm{SFP}_{i+1}$ which is not equivalent to any $\chi' \in \mathrm{SFP}_i$.

THEOREM 4.4
Every SFP-formula is equivalent to a formula in $\bigcup_{i\geq 0} \mathrm{SFP}_i$.

PROOF. This follows immediately from Theorem 3.2 and Lemma 4.2. ∎

Together with Corollary 3.5, our theorem delivers an Ehrenfeucht–Fraïssé game for stratified fixed-point logic:

DEFINITION 4.5
The k-ary r-move SFP-game is played by two players, the challenger and the duplicator, on a pair $\mathfrak{A}, \mathfrak{B}$ of structures of the same signature using pairs (P, Q) of corresponding pebbles.

The game lasts r rounds, and in each round the challenger selects one of the following moves:

SFP-MOVE The challenger selects an $l \leq k$ and a finite l-ary 2-tree $(\mathfrak{T}, f)$ on $\mathfrak{A}$. The duplicator answers by giving a finite l-ary 2-tree $(\mathfrak{U}, g)$ on $\mathfrak{B}$.
Then the challenger either (1) picks a leaf $u \in L^{\mathfrak{U}}$ and places pebbles $Q_1, \ldots, Q_l$ on $g(u)$, or (2) he picks nodes $u, u' \in U$ such that $E_1^{\mathfrak{U}} uu'$ and places pebbles $Q_1, \ldots, Q_{2l}$ on $g(u)g(u')$, or (3) he picks nodes $u, u' \in U$ such that $E_2^{\mathfrak{U}} uu'$ and places pebbles $Q_1, \ldots, Q_{2l}$ on $g(u)g(u')$.
The duplicator answers in case (1) by picking a leaf $t \in L^{\mathfrak{T}}$ and placing the corresponding pebbles $P_1, \ldots, P_l$ on $f(t)$. In case (2) (and (3)) she picks nodes $t, t' \in T$ such that $E_1^{\mathfrak{T}} tt'$ ($E_2^{\mathfrak{T}} tt'$, respectively) and places the corresponding pebbles $P_1, \ldots, P_{2l}$ on $f(t)f(t')$.

$\neg$SFP-MOVE Analogous to the SFP-move with reversed boards.

The duplicator wins the game if after r moves the canonical mapping between the pebbled elements in $\mathfrak{A}$ and the corresponding pebbled elements in $\mathfrak{B}$ is a partial isomorphism.

THEOREM 4.6
Let σ be a finite signature and $k, r \geq 0$. Then for all σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ the following are equivalent:

(1) The same k-ary term-reduced SFP_r-sentences hold in $\mathfrak{A}$ and $\mathfrak{B}$.

(2) The duplicator has a winning strategy for the k-ary r-move SFP-game on $\mathfrak{A}$ and $\mathfrak{B}$.

PROOF. An induction on r using Corollary 3.5 and the fact that there are only finitely many in-equivalent quantifier-free σ-sentences and thus only finitely many k-ary SFP_i-sentences (for each $i \geq 0$). ∎

5 Transitive closure logic

This section contains no new results, but shows how to embed *transitive closure logic* into our framwork.

The *transitive closure* $\mathrm{TC}(R)$ of a $2k$-ary relation R (for some $k \geq 1$) on a set A is defined to be

$$\{\bar{a}\bar{b} \mid \exists m \geq 1, \bar{a}_0 = \bar{a}, \bar{a}_1, \ldots, \bar{a}_m = \bar{b}\, \forall i < m: \; R\bar{a}_i\bar{a}_{i+1}\}.$$

Transitive closure logic TC is the extension of first-order logic by an operator that allows one, given a formula $\varphi(\bar{x}, \bar{y})$, to build a new formula $[\mathrm{TC}_{\bar{x},\bar{y}}\varphi(\bar{x}, \bar{y})]\bar{t}, \bar{u}$ (where $\bar{x}, \bar{y}$ are tuples of individual variables and $\bar{t}, \bar{u}$ are tuples of terms, all of the same length).[7]

This new formula is intended to define the transitive closure of the relation defined by φ, i.e. for any structure $\mathfrak{A}$ and $\bar{a}, \bar{b} \in A$ we let

$$\mathfrak{A} \models [\mathrm{TC}_{\bar{x},\bar{y}}\varphi(\bar{x}, \bar{y})]\bar{a}, \bar{b} \iff \bar{a}\bar{b} \in \mathrm{TC}\big(\{\bar{c}\bar{d} \mid \mathfrak{A} \models \varphi(\bar{c}, \bar{d})\}\big).$$

[7]Let me remark that we do *not* define TC in terms of the *reflexive* transitive closure, as is often done.

Let us first observe that the transitive closure of a formula can be defined in SFP: $[\mathrm{TC}_{\bar{x},\bar{y}} \varphi(\bar{x},\bar{y})]\bar{t},\bar{u}$ is equivalent to

$$[\mathrm{LFP}_{\bar{y},Y}\varphi(\bar{t},\bar{y}) \vee \exists\bar{x} \in Y\varphi(\bar{x},\bar{y})]\bar{u}.$$

But this equivalence reveals more information than just TC being contained in SFP: it shows that a TC-operator can be simulated by a fixed-point operator applied to a formula $\xi(\bar{y}, Y)$ of the form $\psi_0(\bar{y}) \vee \exists\bar{x} \in Y\psi(\bar{x},\bar{y})$ (where Y does not occur in ψ_0 or ψ). The special quality of this formula is that each tuple entering the fixed-point process at some stage only needs one tuple from the previous stage to witness this. Hence we are dealing with a very simple type of recursion here.

On the other hand, each SFP-formula of the form $[\mathrm{LFP}_{\bar{y},Y}\psi_0(\bar{y}) \vee \exists\bar{x} \in Y\psi(\bar{x},\bar{y})]\bar{t}$ (where Y does not occur in ψ_0 or ψ) is equivalent to the TC-formula

$$\psi_0(\bar{t}) \vee \exists\bar{z}\big(\psi_0(\bar{z}) \wedge [\mathrm{TC}_{\bar{x},\bar{y}}\psi(\bar{x},\bar{y})]\bar{z},\bar{t}\big).$$

Thus TC corresponds to a natural, syntactically defined sublogic of SFP.

Slightly generalizing the simple type of recursion occurring in TC-formulas, we come to the class of *l-bounded fixed-point formulas*, which are defined to be those LFP-formulas whose fixed-point operators are only applied to formulas $\xi(\bar{x}, X)$ of the form

$$\psi_0(\bar{x}) \vee \exists\bar{y}_1 \in Y \ldots \exists\bar{y}_l \in Y\psi(\bar{x},\bar{y}_1,\ldots,\bar{y}_l).$$

Using Lemma 3.1 it can be easily seen that each SFP-formula is equivalent to an l-bounded formula for some $l \geq 0$.

Actually, by results of the previous section, every SFP-formula is equivalent to a 2-bounded formula. This is the only non-trivial part of the following theorem. Henrik Imhof [14] proved this result independently. Let me also remark that the essence of (2) has been observed by Grädel [10] when he showed that TC corresponds to the linear fragment of STRATIFIED DATALOG.

THEOREM 5.1

(1) FO = 0-bounded LFP.[8]

(2) TC = 1-bounded LFP.

(3) SFP = $\bigcup_{l\geq 0}$ l-bounded LFP = 2-bounded LFP.

Let me remark here that boundedness has also turned out to be quite useful in the more general context of *partial* and *inductive fixed-point logic* as a concept to classify different fixed-point logics in a natural way (see [11, 12, 19]).

DEFINITION 5.2

Existential transitive closure logic $\exists$TC is the sublogic of TC whose formulas are those TC-formulas that do not contain any universal quantifiers and where negation symbols only occur in front of atomic subformulas.

It is quite obvious that TC is the smallest regular logic containing $\exists$TC (just as SFP is the smallest regular logic containing $\exists$LFP). The pair $\exists$TC—TC also behaves very similarly to the pair $\exists$LFP—SFP with respect to normal forms.

[8] Using the obvious notation, i.e. l-bounded LFP is the sublogic of LFP whose formulas are the l-bounded fixed-point formulas.

Analogously to fixed-point formulas, let us write $[\mathrm{TC}_{\bar{x},\bar{y}}\varphi]$ for a formula

$$\exists v[\mathrm{TC}_{\bar{x},\bar{y}}\varphi]v\dots v, v\dots v$$

(where v does not occur in φ).

Using methods Immerman developed to prove a normal form for TC on ordered finite structures one obtains the following theorem. More explicitly, the results can be found in [9].

THEOREM 5.3 ([16])
(1) Each $\exists$TC-formula is equivalent to a formula of the form $[\mathrm{TC}_{\bar{x},\bar{y}}\theta(\bar{x},\bar{y})]$ where θ is quantifier free.

(2) Each TC-formula is equivalent to a formula of the form

$$\Big[\mathrm{TC}_{\bar{x}_1,\bar{y}_1}\neg[\mathrm{TC}_{\bar{x}_2,\bar{y}_2}\neg[\mathrm{TC}_{\bar{x}_3,\bar{y}_3}\dots\neg[\mathrm{TC}_{\bar{x}_m,\bar{y}_m}\theta(\bar{x}_1,\bar{y}_1,\dots,\bar{x}_m,\bar{y}_m]\dots]]\Big] \quad (*)$$

where θ is quantifier-free.

On the other hand, Grädel and McColm [9] showed that the hierarchy inside of TC induced by the index m in $(*)$ is strict.

6 Separating the logics

THEOREM 6.1
(1) [7, 18] SFP is strictly contained in LFP.
(2) [15] $\exists$LFP is not contained in TC. Hence TC is strictly contained in SFP.

The first statement was shown by Dahlhaus and Kolaitis. The basic idea of their proofs is to use certain structures, so called *game trees*[9], which can be distinguished in LFP, but not in SFP.[10]

Immerman proved the second statement in 1981. At this time the game tree technique was not known, and Immerman's proof is quite involved. So it seems reasonable to sketch a simpler proof using game trees.

In the following, we let E be a binary and BLACK, WHITE be unary relation symbols. Furthermore, we let $\underline{r}$ be a constant symbol. For each $w \geq 1$ we inductively define two classes $\mathcal{B}_w = \{\mathfrak{B}_w^h \mid h \geq 1\}$ and $\mathcal{W}_w = \{\mathfrak{W}_w^h \mid h \geq 1\}$ of $\{E, \text{BLACK}, \text{WHITE}, \underline{r}\}$-structures. $\mathfrak{B}_w^h$ ($\mathfrak{W}_w^h$) is called the ***black (white, respectively) game tree of width w and height h.***

$h = 0$: The only node of $\mathfrak{B}_w^0$ is the interpretation of $\underline{r}$. $\text{BLACK}^{\mathfrak{B}_w^0}$ holds for this node, whereas $\text{WHITE}^{\mathfrak{B}_w^0}$ and $E^{\mathfrak{B}_w^0}$ are empty.

$\mathfrak{W}_w^0$ is the same, except $\text{WHITE}^{\mathfrak{W}_w^0}$ holding for $\underline{r}^{\mathfrak{W}_w^0}$ and $\text{BLACK}^{\mathfrak{W}_w^0}$ being empty.

$h \to h+1$: $\mathfrak{B}_w^{h+1}$ consists of w disjoint copies of $\mathfrak{B}_w^h\restriction_{\{E,\,\text{BLACK},\,\text{WHITE}\}}$ together with a copy of $\mathfrak{W}_w^h\restriction_{\{E,\,\text{BLACK},\,\text{WHITE}\}}$ (which is disjoint to all of the $\mathfrak{B}_w^h$s) and a new node interpreting $\underline{r}$. In addition, there are edges from $\underline{r}^{\mathfrak{B}_w^{h+1}}$ to the roots of the $\mathfrak{B}_w^h$s and the $\mathfrak{W}_w^h$.

$\mathfrak{W}_w^{h+1}$ consists of $w+1$ disjoint copies of $\mathfrak{B}_w^h\restriction_{\{E,\,\text{BLACK},\,\text{WHITE}\}}$ and a new node $\underline{r}^{\mathfrak{W}_w^{h+1}}$ having an edge to the roots of the $\mathfrak{B}_w^h$s.

---

[9] These structures have been introduced by Chandra and Harel [3] to establish a hierarchy in first-order logic.

[10] After having seen the following proof, it should be no difficulty for the reader to prove this, using game trees and the Ehrenfeucht–Fraïssé game for SFP we have introduced in Section 4.

The first observation is that for each $w \geq 1$ there exists an $\exists$LFP-formula separating $\mathcal{B}_w$ from $\mathcal{W}_w$. Thus to show that $\exists$LFP is not contained in TC it suffices to prove:

PROPOSITION 6.2
$\mathcal{B}_2$ cannot be separated from $\mathcal{W}_2$ by a TC-formula.

Since from now on we are only going to consider game trees of width 2, we are going to suppress the subscript 2 indicating this width.

I guess it is no surprise for the reader that we are proving this with an Ehrenfeucht–Fraïssé game. A game that characterizes TC has been introduced by Grädel [10]. However, we are not going to use exactly Grädel's game, but a variant which makes inductions easier to handle. (From a theoretical point of view this game is quite unsatisfying because it is too strong to characterize TC exactly.)

DEFINITION 6.3
The *k-ary r-move path-game* is played by two players, the challenger and the duplicator, on a pair $\mathfrak{A}$, $\mathfrak{B}$ of structures of the same signature using pairs of corresponding pebbles.

The game lasts r rounds, and in each round the challenger selects one of the following moves:

P-MOVE The challenger selects a path $\bar{a}_1, \ldots, \bar{a}_m$ of tuples of length $\leq k$ in $\mathfrak{A}$. The duplicator answers by selecting a path $\bar{b}_1, \ldots, \bar{b}_m$ in $\mathfrak{B}$ such that the length of $\bar{b}_i$ equals the length of $\bar{a}_i$ (for each $1 \leq i \leq m$).[11]
Then the challenger chooses an $i < m$ and places pebbles on $\bar{b}_i\bar{b}_{i+1}$. The duplicator places the corresponding pebbles on $\bar{a}_i\bar{a}_{i+1}$.[12]

$\neg$P-MOVE Analogously with reversed structures.

The duplicator wins the game if after r moves the canonical mapping between the pebbled elements in $\mathfrak{A}$ and the corresponding pebbled elements in $\mathfrak{B}$ is a partial isomorphism.

The following lemma is an easy consequence of Grädel's result on his TC-game:

LEMMA 6.4
Let σ be a finite signature and φ a TC-sentence of signature σ. Then there exist $k, r \geq 1$ such that for all σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ on which the duplicator wins the k-ary r-move path-game we have:

$$\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi.$$

Thus to prove Propostion 6.2 it suffices to show:

PROPOSITION 6.5
For each $k, r \geq 1$ the duplicator wins the k-ary r-move path-game on $\mathfrak{B}^{2kr}$ and $\mathfrak{W}^{2kr}$.

LEMMA 6.6
For each $k \geq 1$ the duplicator wins the k-ary 1-move path-game on $\mathfrak{B}^{2k}$ and $\mathfrak{W}^{2k}$.

PROOF. Suppose first that the duplicator can answer a k-ary P-move on $\mathfrak{W}^{2k-1}$, $\mathfrak{B}^{2k-1}$ for some $k \geq 1$.

---

[11] Note that both paths have the same length. This is a crucial difference to Grädel's TC-game.

[12] Here is the other important difference: The duplicator has to place her pebbles on the tuples with the same indices as the challenger.

- Then she can answer a $\neg$P-move on $\mathfrak{B}^{2k}$, $\mathfrak{W}^{2k}$, because $\mathfrak{B}^{2k}$ ($\mathfrak{W}^{2k}$) can be obtained from $\mathfrak{B}^{2k-1}$ ($\mathfrak{W}^{2k-1}$, respectively) by replacing the black leaves, i.e. all nodes on which BLACK holds, by $\mathfrak{B}^1$ and the white leaves by $\mathfrak{W}^1$.
- She can also answer a P-move on $\mathfrak{B}^{2k}$, $\mathfrak{W}^{2k}$, because $\mathfrak{W}^{2k}$ can be obtained from $\mathfrak{B}^{2k}$ by replacing one subtree of the form $\mathfrak{W}^{2k-1}$ by a tree of the form $\mathfrak{B}^{2k-1}$.

Thus it suffices consider P-moves on pairs $\mathfrak{W}^{2k-1}, \mathfrak{B}^{2k-1}$. We proceed by induction on k.

$k = 1$: We are dealing with the following structures:

The challenger picks a path $w_1, \ldots, w_m \in W^1$. Let $w_0 = \underline{r}^{\mathfrak{W}^1}$ and $b_0 = \underline{r}^{\mathfrak{B}^1}$. The duplicator defines her path $b_1, \ldots, b_m$ inductively as follows:

- If $w_i = \underline{r}^{\mathfrak{W}^1}$ then $b_i = \underline{r}^{\mathfrak{B}^1}$.
- If $w_i \in \{w, w', w''\}$ then she selects $b_i \in \{b', b''\}$ such that $b_{i-1} = b_i$ if and only if $w_{i-1} = w_i$.

Clearly, this is a winning move.

$k \to k+1$: Here the structures look as follows:

where $\mathfrak{W}, \mathfrak{W}', \mathfrak{W}'', \mathfrak{B}'$, and $\mathfrak{B}''$ are isomorphic copies of $\mathfrak{B}^{2k}$ and $\mathfrak{B}$ is isomorphic to $\mathfrak{W}^{2k}$.
The challenger chooses the path $\bar{w}_1, \ldots, \bar{w}_m \in W^{2k+1}$ of tuples of length $\leq k+1$.
To define a path $\bar{b}_1, \ldots, \bar{b}_m \in B^{k+1}$ the duplicator first lets $b_{ij} := \underline{r}^{\mathfrak{B}^{2k+1}}$ whenever $w_{ij} = \underline{r}^{\mathfrak{W}^{2k+1}}$.

Next she considers the subsequence $\bar{w}_{i_1}, \ldots, \bar{w}_{i_n}$ of those tuples that are completely contained in one of the subtrees $\mathfrak{W}$, $\mathfrak{W}'$, or $\mathfrak{W}''$ of $\mathfrak{W}^{2k+1}$. Similarly to the case $k = 1$ she can copy them into the subtrees $\mathfrak{B}'$ and $\mathfrak{B}''$ of $\mathfrak{B}^{2k+1}$ (recall that all these trees are isomorphic) such that whenever two succeeding tuples $\bar{w}_{i_j}, \bar{w}_{i_{j+1}}$ are in distinct subtrees so are the corresponding tuples $\bar{b}_{i_j}, \bar{b}_{i_{j+1}}$ (and vice versa).

For the remaining w_{ij} consider two succeeding tuples $\bar{w}_{i_j}, \bar{w}_{i_{j+1}}$ that are completely contained, say, in $\mathfrak{W}$ and $\mathfrak{W}'$, respectively. Then $\bar{b}_{i_j}$ and $\bar{b}_{i_{j+1}}$ will be in $\mathfrak{B}'$ and $\mathfrak{B}''$, respectively, or vice versa. Let us assume that $\bar{b}_{i_j} \in \mathfrak{B}'$ and $\bar{b}_{i_{j+1}} \in \mathfrak{B}''$. This fixes a correspondence $\mathfrak{W} \mapsto \mathfrak{B}'$, $\mathfrak{W}' \mapsto \mathfrak{B}''$ and $\mathfrak{W}'' \mapsto \mathfrak{B}$. Now for every $w_{ih} \in \mathfrak{W}$ with $i_j < i < i_{j+1}$ the duplicator picks the corresponding b_{ih} in $\mathfrak{B}'$. For every $w_{ih} \in \mathfrak{W}'$ with $i_j < i < i_{j+1}$ she picks the corresponding b_{ih} in $\mathfrak{B}''$. The remaining w_{ij} (with $i_j < i < i_{j+1}$) form a path of at most k-tuples in the subtrees $\mathfrak{W}''$. Applying the induction hypothesis the duplicator can find a corresponding path in $\mathfrak{B}$.

Obviously, all intervals $\bar{w}_{i_j}, \bar{w}_{i_j+1}$ can be treated independently in this way. ∎

Using this lemma, it is not too difficult to prove Proposition 6.5 by induction on r. This completes our proof of Theorem 6.1(2). ∎

Observe that the previous theorem also shows that ∃TC is strictly contained in ∃LFP. We can strengthen this slightly, generalizing an argument used by Gurevich [13] to prove that the existential preservation theorem for first-order logic fails in the finite. But note that finiteness is not relevant in the following theorem: we define an ∃LFP-sentence which is equivalent to a TC-sentence on arbitrary structures, but not equivalent to an ∃TC-sentence even on finite structures.

THEOREM 6.7
∃TC is strictly contained in ∃LFP ∩ TC.

PROOF. Let $\sigma = \{E, \text{BLACK}, \underline{r}\}$ where E is a binary relation symbol, BLACK is a unary relation symbol, and $\underline{r}$ is a constant symbol.

We say that a forest has *width at most w* (for some $w \geq 1$), if the nodes have out-degree at most w. A forest has *width exactly w*, if all nodes have out-degree w or 0.

Consider the σ-query ν which says

If E is the edge relation of a forest of width at most 3 *and* BLACK *only contains leaves then the subtree with root* $\underline{r}$ *is a finite tree of width exactly* 3 *whose leaves are all* BLACK.

It can be defined by the ∃LFP-formula

$$\left.\begin{array}{ll} (& \forall x\, \neg Exx \\ \wedge & \forall x \forall y \forall z\, \neg(y \neq z \wedge Eyx \wedge Ezx) \\ \wedge & \forall x\, \neg[\mathrm{LFP}_{y,Y} Exy \vee \exists z \in Y\, Ezy]x \\ \wedge & \forall x \forall y_1 \ldots \forall y_4\, \neg \bigwedge_{1 \leq i < j \leq 4} (y_i \neq y_j \wedge Exy_i) \\ \wedge & \forall x \forall y\, (\text{BLACK}x \rightarrow \neg Exy) \quad) \end{array}\right\} =: \varphi$$

$$\longrightarrow \quad [\mathrm{LFP}_{x,X} \text{BLACK}x \vee \exists y_1, y_2, y_3 \in X \bigwedge_{1 \leq i < j \leq 3} (y_i \neq y_j \wedge Exy_i)]\underline{r}\,.$$

Note that φ is equivalent to a TC-formula φ'. The following TC-formula ψ says (in models of φ) that the subtree with root $\underline{r}$ is finite (by stating the existence of an element with maximal (finite) distance to $\underline{r}$):

$$\psi = L\underline{r} \vee \exists x \Big([\mathrm{TC}_{u,v}\ Euv]\underline{r}, x$$
$$\wedge \forall y \big([\mathrm{TC}_{u,v}\ Euv]\underline{r}, y \rightarrow \exists z [\mathrm{TC}_{u_1 u_2, v_1 v_2} Eu_1 v_1 \wedge \neg u_1 = \underline{r} \wedge Eu_2 v_2] xy, z\underline{r} \big) \Big).$$

Hence we can also define ν by the TC-formula

$$\varphi' \longrightarrow \Big(\psi \wedge \forall x \big([\mathrm{TC}_{u,v}\ u = v \vee Euv]\underline{r}, x$$
$$\rightarrow (\text{BLACK}x \vee \exists y_1, y_2, y_3 \bigwedge_{1 \leq i < j \leq 3} (y_i \neq y_j \wedge Exy_i)) \big) \Big).$$

However, ν is not $\exists$TC-definable. To see this, for each $h \geq 0$ we let $\mathfrak{A}_h$ be the complete ternary tree of height h with all leaves coloured BLACK and $\mathfrak{B}_h$ the complete ternary tree of height h with all but one leaf coloured BLACK.

Clearly we have $\mathfrak{A}_h \models \nu$ and $\mathfrak{B}_h \not\models \nu$ (for all $h \geq 0$). An argument analogous to Lemma 6.6 shows that the duplicator can answer one h-ary P-move on $\mathfrak{A}_h$, $\mathfrak{B}_h$, i.e. for each path $\bar{a}_1, \ldots, \bar{a}_m$ of at most h-tuples in $\mathfrak{A}_h$ she can define a path $\bar{b}_1, \ldots, \bar{b}_m$ in $\mathfrak{B}_h$ such that the tuples $\bar{a}_i\bar{a}_{i+1}$ and $\bar{b}_i\bar{b}_{i+1}$ induce a partial isomorphism. Considering Theorem 5.3(1), this shows that for each $\exists$TC-sentence φ there exists an $h \geq 1$ such that $\mathfrak{A}_h \models \varphi$ implies $\mathfrak{B}_h \models \varphi$. Hence ν cannot be defined in $\exists$TC. ∎

COROLLARY 6.8
There is no existential preservation theorem for transitive closure logic (neither on finite nor on infinite structures).

7 The failure of existential preservation for fixed-point logics

We have seen in the previous section that the failure of existential preservation for TC can be proved relatively easily by known techniques which have also been applied to separate logics. To obtain the analogous result for the logic SFP (and thus LFP) some new ideas are needed.

Although the basic idea of the following proof is quite simple (I learned it from Rosen and Weinstein's [21] proof of the failure of a substructure preservation theorem for the logic $L^{\omega}_{\infty\omega}$[13]), it requires many technical details. However, it yields an even stronger result which also implies Rosen and Weinstein's theorem concerning $L^{\omega}_{\infty\omega}$. Furthermore, the same method can be applied to answer a question by Rosen and Weinstein whether first-order sentences which are preserved under extensions of finite structures are equivalent to existential $L^{\omega}_{\infty\omega}$-sentences on finite structures. (As usual, an $L^{\omega}_{\infty\omega}$-sentence is called *existential* if it contains no universal quantifiers and negation symbols only occur in front of atomic subformulas. In accordance with our previous notation, existential $L^{\omega}_{\infty\omega}$ is denoted by $\exists L^{\omega}_{\infty\omega}$. Observe that $\exists\text{LFP} \subseteq \exists L^{\omega}_{\infty\omega}$.)

THEOREM 7.1
There exists a TC-sentence which is preserved under extensions, but not equivalent to any sentence of the logic $\exists L^{\omega}_{\infty\omega}$ (even on finite structures).

PROOF. We start by defining, for each $k \geq 1$, two structures $\mathfrak{A}_k$ and $\mathfrak{B}_k$. Essentially, they are the same structures as those used by Rosen and Weinstein to prove the failure of an existential preservation theorem for $L^{\omega}_{\infty\omega}$. However, to make the proof more transparent, we allow our signature to contain more than only one binary relation symbol. Let E be a binary relation symbol and R, L, P unary relation symbols.

- The *(k,p)-flag* $\mathfrak{F}_{k,p}$ is the $\{E, R, L, P\}$-structure with universe $F_{k,p} = \{0, \ldots, k-1\}$, $E^{\mathfrak{F}_{k,p}}$ being the natural successor relation on $F_{k,p}$ (i.e. $E = \{(i, i+1) \mid 0 \leq i < k-1\}$), $R^{\mathfrak{F}_{k,p}} = \{0\}$, $L^{\mathfrak{F}_{k,p}} = \{k-1\}$, and $P^{\mathfrak{F}_{k,p}} = \{p-1\}$.
- $\mathfrak{A}_k$ is the disjoint union of the flags $\mathfrak{F}_{2k+1,k+1}, \ldots \mathfrak{F}_{2k+1,2k+1}$.
- The *(k,j)-tree* $\mathfrak{T}_{k,j}$ is the tree obtained from $\mathfrak{A}_k$ by fusing the ith node of each flag, for $0 \leq i < j$.
- $\mathfrak{B}_k$ is the disjoint union of the trees $\mathfrak{T}_{k,1}, \ldots \mathfrak{T}_{k,k}$.

[13] $L^{\omega}_{\infty\omega}$ is the fragment of the infinitary logic $L_{\infty\omega}$ in which each formula only contains finitely many variables.

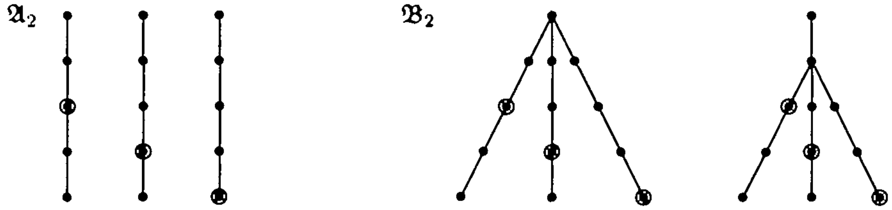

FIG. 1. The structures $\mathfrak{A}_2$ and $\mathfrak{B}_2$

The crucial observation Rosen and Weinstein made is that although $\mathfrak{A}_k$ is *not* a substructure of $\mathfrak{B}_k$, the following holds:

PROPOSITION 7.2 ([21])
Each $\exists L^k_{\infty\omega}$-sentence holding in $\mathfrak{A}_k$ also holds in $\mathfrak{B}_k$.

Intuitively, the reason for this is that each of the $(k+1)$ connected components of $\mathfrak{A}_k$ can be embedded into any of the k connected compenents of $\mathfrak{B}_k$; and that k variables can speak about at most k components at the same time. ∎

Suppose next we could find two TC-sentences φ and ψ such that:

- φ is preserved under substructures.
- $\mathfrak{A}_k \models \varphi$ and $\mathfrak{B}_k \models \varphi$ for each $k \geq 1$.
- For all $\{E, R, L, P\}$-structures $\mathfrak{C}$, if $\mathfrak{C} \models \varphi$ then

$$\mathfrak{C} \models \psi \iff \mathfrak{C} \text{ has a substructure isomorphic to } \mathfrak{A}_k \text{ for some } k \geq 1.$$

Then we could complete the proof of our theorem as follows: Let $\chi = \varphi \rightarrow \psi$. χ is preserved under extensions, because suppose that $\mathfrak{C} \models \chi$ and $\mathfrak{D} \supseteq \mathfrak{C}$. If $\mathfrak{D} \not\models \varphi$ then $\mathfrak{D} \models \chi$, as desired. If $\mathfrak{D} \models \varphi$, then $\mathfrak{C} \models \varphi$ (since φ is preserved under substructures) and thus $\mathfrak{C} \models \psi$. Then there exists a $k \geq 1$ such that $\mathfrak{A}_k \subseteq \mathfrak{C}$. Hence $\mathfrak{A}_k \subseteq \mathfrak{D}$. Since $\mathfrak{D} \models \varphi$, this implies $\mathfrak{D} \models \psi$. Thus $\mathfrak{D} \models \chi$.

Observe next that $\mathfrak{A}_k \models \chi$ for each $k \geq 1$, whereas $\mathfrak{B}_k \not\models \chi$ (because $\mathfrak{B}_k$ does not contain a substructure isomorphic to any $\mathfrak{A}_l$).

Then Proposition 7.2 would imply that χ is not equivalent to a sentence $\chi' \in \exists L^{\omega}_{\infty\omega}$.

Unfortunately we cannot find such sentences φ and ψ (at least I cannot). But we can slightly extend the idea to make it work. Let therefore $<$ be another binary and O a unary relation symbol, and let $\underline{0}$ be a constant symbol. The signature we will work with from now on is $\tau = \{O, <, \underline{0}, E, R, L, P\}$. Let $\mathfrak{O}_k$ be the $\{O, <, \underline{0}\}$-structure with universe $O_k = \{0, \dots, 2^{2^{2k+3}} - 1\}$, $O^{\mathfrak{O}_k} = O_k$, $<^{\mathfrak{O}_k}$ being the natural ordering and $\underline{0}^{\mathfrak{O}_k} = 0$. Let $\mathfrak{A}^*_k$ ($\mathfrak{B}^*_k$) be the disjoint union of $\mathfrak{O}_k$ and $\mathfrak{A}_k$ ($\mathfrak{B}_k$, respectively).[14] Clearly, we still have:

PROPOSITION 7.3
Each $\exists L^k_{\infty\omega}$-sentence holding in $\mathfrak{A}^*_k$ also holds in $\mathfrak{B}^*_k$.

But now we can prove the following:

[14]More formally, $\mathfrak{A}_k$ is the structure with universe $O_k \dot{\cup} A_k$, relations $O^{\mathfrak{A}^*_k} = O^{\mathfrak{O}_k}$, $<^{\mathfrak{A}^*_k} = <^{\mathfrak{O}_k}$, $E^{\mathfrak{A}^*_k} = E^{\mathfrak{A}_k}$, $R^{\mathfrak{A}^*_k} = R^{\mathfrak{A}_k}$, $L^{\mathfrak{A}^*_k} = L^{\mathfrak{A}_k}$, $P^{\mathfrak{A}^*_k} = P^{\mathfrak{A}_k}$ and the constant $\underline{0}^{\mathfrak{A}^*_k} = \underline{0}^{\mathfrak{O}_k}$. $\mathfrak{B}^*_k$ is defined analogously.

PROPOSITION 7.4
There are TC-sentences φ and ψ such that:

(i) φ is preserved under substructures.
(ii) $\mathfrak{A}_k^* \models \varphi$ and $\mathfrak{B}_k^* \models \varphi$ for each $k \geq 1$.
(iii) For all τ-structures $\mathfrak{C}$, if $\mathfrak{C} \models \varphi$ then

$$\mathfrak{C} \models \psi \quad \Longleftrightarrow \quad \mathfrak{C} \text{ has a substructure isomorphic to } \mathfrak{A}_k^* \text{ for some } k \geq 1.$$

Having seen this, we can complete the proof of Theorem 7.1 just as we showed above.

PROOF. [of Proposition 7.4] Let us start by defining some auxiliary $\exists$TC-formulas. The intended meaning described below the formulas holds whenever E is the edge relation of a forest.

$$\begin{aligned}
\text{BRANCH}(x) &= \exists y \exists z\, (Exy \wedge Exz \wedge y \neq z). \\
\text{PATH}_2(x,y) &= [\text{TC}_{u,v} Euv \vee u = v]x, y. \\
\text{PATH}_3(x,y,z) &= \text{PATH}_2(x,y) \wedge \text{PATH}_2(y,z).
\end{aligned}$$

y lies on the (in forests: unique (!)) path from x to z.

$$\text{DIST}_=(w,x,y,z) = (w = x \wedge y = z) \vee [\text{TC}_{u_1 u_2, v_1 v_2} Eu_1 v_1 \wedge Eu_2 v_2]wy, xz.$$

The length of the (unique) E-path from w to x is the same as the length of the E-path from y to z.

$$\begin{aligned}
\text{DIST}_\leq(w,x,y,z) &= \exists z' \big(\text{PATH}_3(y,z',z) \wedge \text{DIST}_=(w,x,y,z')\big). \\
\text{DIST}_<(w,x,y,z) &= \exists z' \big(\text{PATH}_3(y,z',z) \wedge z \neq z' \wedge \text{DIST}_=(w,x,y,z')\big).
\end{aligned}$$

We now define a formula φ' which will almost be the φ we are heading for. It is the conjunction over the following formulas (cf below for their intuitive meaning):

(1) $\forall x \forall y \forall z\, \big((Eyx \wedge Ezx) \rightarrow y = z\big)$
(2) $\forall x\, \neg[\text{TC}_{u,v} Euv]x, x$
(3) $\forall x \forall y\, (Rx \rightarrow \neg Eyx)$
(4) $\forall x \forall y\, (Lx \rightarrow \neg Exy)$
(5) $\forall x_1 \forall y_1 \forall x_2 \forall y_2\, \neg\big(Rx_1 \wedge Ly_1 \wedge Rx_2 \wedge Ly_2 \wedge \text{DIST}_<(x_1,y_1,x_2,y_2)\big)$
(6) $\forall x\, \neg(Rx \wedge Lx)$
(7) $\forall w \forall x \forall y \forall z\, \neg\big(Rw \wedge Lz \wedge \text{PATH}_2(w,x) \wedge Exy \wedge \text{PATH}_2(y,z) \wedge \text{DIST}_=(w,x,y,z)\big)$
(8) $\forall x \forall y \forall z\, \big((\text{BRANCH}(x) \wedge \text{BRANCH}(y) \wedge \text{PATH}_2(z,x) \wedge \text{PATH}_2(z,y)) \rightarrow x = y\big)$
(9) $\forall x_1 \forall y_1 \forall x_2 \forall y_2\, \Big(\big(Rx_1 \wedge \text{BRANCH}(y_1) \wedge Rx_2 \wedge \text{BRANCH}(y_2) \wedge \text{DIST}_=(x_1,y_1,x_2,y_2)\big) \rightarrow x_1 = x_2\Big)$
(10) $\forall x \forall y\, \big((Px \wedge Py \wedge \text{PATH}_2(x,y)) \rightarrow x = y\big)$
(11) $\forall x \forall y_1 \forall y_2\, \Big(\big(Rx \wedge Py_1 \wedge Py_2 \wedge \text{DIST}_=(x,y_1,x,y_2)\big) \rightarrow y_1 = y_2\Big)$.

Let us first try to understand what φ' says: (1) and (2) cause E to be the edge relation of a forest, and by (3) and (4), R only contains roots and L only contains leaves. By (5), all paths from R to L have the same length. (6) and (7) guarantee that this length is > 0 and even[15] (just like in $\mathfrak{A}_k$ and $\mathfrak{B}_k$).

(8) says that each tree in the forest contains at most one vertex with more than one successor. Let us call such vertices *branching vertices*. The *branching height* of a tree with a branching vertex is one plus the height of this vertex.[16] Trees without any branching vertices have *branching height* 0. (9) guarantees that for each (strictly) positive branching height there is at most one tree.

Now we come to the relation P. (10) says that each path contains at most one element in P, and finally by (11) there are no two elements of P of the same height in one tree.

It is now obvious that $\mathfrak{A}_k^*$ and $\mathfrak{B}_k^*$ are models of φ' for each $k \geq 1$. Furthermore, observe that φ' is a universal TC-sentence. Hence it is preserved under substructures. Thus (i) and (ii) hold for φ'.

Before passing from φ' to φ, let us turn to defining ψ. It has to say, in models $\mathfrak{C}$ of φ', that they contain a substructure isomorphic to $\mathfrak{A}_k^*$, where $2k$ is the length of (all) the paths from R to L.

In other words, ψ has to say that there are elements $p_1, \ldots, p_{k+1}$ of P all lying on disjoint paths from R to L such that the height of p_i is $i + k$. Since the $\{E\}$-reducts of models of φ' are forests, this is the case if and only if

There exist triples $(r_1, p_1, l_1), \ldots, (r_{k+1}, p_{k+1}, l_{k+1})$ such that $r_i \in R$, $l_i \in L$, p_i lies on a path from r_i to l_i and the height of p_i is $i + k$, and all the r_i are distinct. $(*)$

We can express this by a second-order formula, but not in TC (recalling that k depends on the structure).

However, we have not used too many of the properties fixed by φ' yet. Recall that each tree (in a model of φ') has a uniquely determined branching height. So two elements of P are on disjoint paths if and only if their trees have a distinct branching height or branching height 0. Considering this, we can restate $(*)$ as

There exist tuples $(r_1, p_1, l_1, h_1), \ldots, (r_{k+1}, p_{k+1}, l_{k+1}, h_{k+1})$ such that $r_i \in R$, $l_i \in L$, p_i lies on a path from r_i to l_i and the height of p_i is $i + k$, h_i is the branching height of the tree with root r_i, and the h_i are either 0 or distinct (i.e. $\forall i \neq j \leq k + 1 (h_i \neq \underline{0} \rightarrow h_i \neq h_j)$). $(**)$

Instead of $(**)$ we can also say:

There exists a set β of pairs of natural numbers such that

- *For all x such that $k+1 \leq x \leq 2k+1$ there exists a y such that $(x, y) \in \beta$.*
- *For all $(x, y) \in \beta$ there exists a $p \in P$ occurring on a path from R to L such that x is the height of p and y is the branching height of the tree containing p.* $(* * *)$
- *The second components of pairs in β (the branching heights) are either* 0 *or distinct.*

The achievement of these transformations is that $(* * *)$ is a statement which basically speaks about a set of pairs of natural numbers. This is where the ordering (of our structures $\mathfrak{A}_k^*$ and $\mathfrak{B}_k^*$) comes in: in a sufficiently large (compared to k) ordering (considered as an ini-

[15] By definition, a path consisting of $(2k + 1)$ nodes has (the even) length $2k$.

[16] Recall that the height of a vertex in a tree is the length of the (unique) path from the root to this vertex.

tial segment of the natural numbers) we can code sets of $(k+1)$ pairs of numbers by a single number, and thus quantify over such sets.

We consider τ-structures where the O-part is a linear order and the $\neg O$-part carries an $\{E, R, L, P\}$-structure. We use variable symbols $\alpha, \beta, \ldots$ for those variables that are intended to speak about the ordering (although formally we are not dealing with a two sorted structure, so the range of the variables is not restricted).

Let φ'' be a universal first-order sentence saying that $<$ is an ordering of O with minimum $\underline{0}$, and E, R, L, P only hold for elements not contained in O. Now, the desired φ is the conjunction of φ' and φ''. Obviously, (i) and (ii) still hold for φ.

By $\mathfrak{n}$ we denote the structure $(\{0, \ldots, n-1\}, <, 0)$ where $<$ is the natural ordering. Using standard techniques to define arithmetical predicates in TC, we can find a formula $\textsc{El}(\alpha, \beta, \gamma, \delta)$ such that for each $l \geq 0, n \geq 2^{2^{l+2}}$ we have:

$\{(a, b, c) \mid \mathfrak{n} \models \textsc{El}(l, a, b, c)\}$ is the graph of an injective function

$$f : \mathrm{Pow}(\{0, \ldots, l\} \times \{0, \ldots, l\}) \longrightarrow \{0, \ldots, 2^{2^{l+2}} - 1\}.$$

Furthermore, we can find TC-formulas $\textsc{LaO}(\alpha)$ and $\textsc{SecHalf}(\alpha, \beta)$ such that for all $l \geq 0, n > l, m < n$ we have

$$\mathfrak{n} \models \textsc{LaO}(l) \iff n \geq 2^{2^{l+2}}$$

and

$$\mathfrak{n} \models \textsc{SecHalf}(l, m) \iff (m \leq l \wedge 2m \geq l).$$

For brevity, we also introduce the formula $\textsc{Succ}(\alpha, \beta) = \alpha < \beta \wedge \forall \gamma \neg(\alpha < \gamma \wedge \gamma < \beta)$ whose meaning is obvious.

Next, we have to relate the order of our τ-structures with the $\{E, R, L, P\}$-part. Let

$$\textsc{Height}(x, \alpha) = (Rx \wedge \alpha = \underline{0}) \vee \exists y (Ry \wedge [\mathrm{TC}_{u\gamma, v\delta} Euv \wedge \textsc{Succ}(\gamma, \delta)] y\underline{0}, x\alpha.$$

Observe that $\textsc{Height}(x, \alpha)$ holds in a model of φ if the height of x in its tree equals the size of α in the ordering.

To express the condition $(***)$, we need to define the branching height of the tree containing a vertex x. This can be done by

$$\begin{aligned} \textsc{BH}(x, \alpha) = \exists y \Big(& Ry \wedge \textsc{Path}_2(y, x) \wedge \\ & (\exists z (\textsc{Path}_2(y, z) \wedge \textsc{Branch}(z) \wedge \exists \beta (\textsc{Succ}(\beta, \alpha) \wedge \textsc{Height}(z, \beta))) \\ & \vee (\neg \exists z (\textsc{Path}_2(y, z) \wedge \textsc{Branch}(z)) \wedge \alpha = \underline{0})) \Big). \end{aligned}$$

Out of all these parts we can assemble the desired formula ψ to be

$$\begin{aligned} & \exists x \exists \alpha (Lx \wedge \textsc{Height}(x, \alpha) \wedge \textsc{LaO}(\alpha) \wedge \\ & \quad \exists \beta \Big(\forall \gamma (\textsc{SecHalf}(\alpha, \gamma) \\ & \qquad \rightarrow \exists w \exists y \exists z \exists \delta (Rw \wedge Py \wedge Lz \wedge \textsc{Path}_3(w, y, z) \\ & \qquad\qquad \wedge \textsc{Height}(y, \gamma) \wedge \textsc{BH}(y, \delta) \wedge \textsc{El}(\alpha, \gamma, \delta, \beta))) \\ & \quad \wedge \forall \varepsilon \forall \zeta \forall \varepsilon' \forall \zeta' ((\varepsilon \neq \varepsilon' \wedge \textsc{El}(\alpha, \varepsilon, \zeta, \beta) \wedge \textsc{El}(\alpha, \varepsilon', \zeta', \beta)) \rightarrow (\zeta \neq \zeta' \vee \zeta = \underline{0})) \Big). \end{aligned}$$

Suppose now, $\mathfrak{C} \models \varphi$. We have to prove that (iii) holds. $\mathfrak{C}$ has a substructure isomorphic to

$\mathfrak{A}_k^*$ if and only if its order contains $\mathfrak{O}_k$, i.e. is longer than $2^{2^{2k+3}}$ (where $2k+1$ is the length of the paths from $R^{\mathfrak{C}}$ to $L^{\mathfrak{C}}$) and if it satisfies $(***)$. The first line of ψ takes care of the order, and the rest ensures $(***)$. (β encodes a set of pairs of natural numbers such that for each $i \leq k+1$, there exists an element of P of height $i+k$ on a path from R to L in a tree of branching height h, such that $(i+k, h) \in \beta$, and the second components (the branching heights) of elements of β are either distinct or $\underline{0}$. This yields $(***)$.) ∎

COROLLARY 7.5

There is no substructure preservation theorem for SFP or LFP (neither on finite nor on arbitrary structures).

THEOREM 7.6

There exists a first-order sentence which is preserved under extensions of finite structures, but is not even on finite structures equivalent to a sentence of the logic $\exists\mathrm{L}^{\omega}_{\infty\omega}$.

PROOF. *In this proof we only consider finite structures, and statements like two formulas being equivalent always refer to finite structures.*

We use the same idea as in the previous proof. The lower expressive power of first-order logic can be compensated by a larger signature, for the price of restricting to finite structures.

Let us first take care of the arithmetical part. Let $\sigma_{Ar} = \{+, \times, <, S, \underline{0}\}$ be a relational signature of arithmetic, i.e. $+$ and $\times$ are ternary relation symbols, $<$ and S are binary relation symbols and $\underline{0}$ is a constant symbol.

By n^+ we denote the σ_{Ar}-structure with universe $\{0, \dots, n-1\}$ where $+$, $\times$, $<$, S (the successor relation), and $\underline{0}$ are interpreted in the natural way.

Let φ_{Ar} be the conjunction of the following universal first-order sentences:

(A1) $\forall x\, \neg x < x$

(A2) $\forall x \forall y\, (x < y \vee x = y \vee y < x)$

(A3) $\forall x \forall y \forall z\, ((x < y \wedge y < z) \to x < z)$

(A4) $\forall x\, \neg x < \underline{0}$

(A5) $\forall x \forall y\, (Sxy \to (x < y \wedge \forall z \neg(x < z \wedge z < y)))$

(A6) $\forall x \forall y \forall z \forall z'\, ((x + y = z \wedge x + y = z') \to z = z')$

(A7) $\forall x\, x + \underline{0} = x$

(A8) $\forall x \forall y \forall y' \forall z \forall z'\, ((x + y = z \wedge Syy' \wedge Szz') \to x + y' = z')$

(A9) $\forall x \forall y \forall z \forall z'\, ((x \times y = z \wedge x \times y = z') \to z = z')$

(A10) $\forall x\, x \times \underline{0} = \underline{0}$

(A11) $\forall x \forall y \forall y' \forall z \forall z'\, ((x \times y = z \wedge Syy' \wedge z + x = z') \to x \times y' = z')$.

Observe that if $\mathfrak{A} \models \varphi_{Ar}$ then the initial segment of $\mathfrak{A}$ on which S is completely defined is isomorphic to a structure n^+ for some $n \geq 0$.

Since on structures n^+ we have enough arithmetic available, here we can find a first-order formula $\mathrm{EL}^+(w, x, y, z)$ such that for all $l \geq 0$, $n \geq 2^{2^{l+2}}$, and σ_{Ar}-structures $\mathfrak{A} \models \varphi_{Ar}$ with $\mathrm{n}^+ \subseteq \mathfrak{A}$

$$\{(a, b, c) \mid \mathfrak{A} \models \mathrm{EL}^+(l, a, b, c)\} \text{ is the graph of an injective function}$$

$$f : \mathrm{Pow}(\{0, \dots, l\} \times \{0, \dots, l\}) \longrightarrow \{0, \dots, 2^{2^{l+2}} - 1\}$$

and a first-order formula $\textsc{SecHalf}^+(x,y)$ such that for all $l \geq 0$, $n \geq l$, σ_{Ar}-structures $\mathfrak{A} \models \varphi_{Ar}$ with $\mathrm{n}^+ \subseteq \mathfrak{A}$ and $a \in A$

$$\mathfrak{A} \models \textsc{SecHalf}^+(l,a) \iff (a \leq l \wedge 2a \geq l).$$

Moreover, we can define a formula $\textsc{LaO}^+(x)$ such that for all $l \geq 0$, $n = 2^{2^{l+2}}$ and for each σ_{Ar}-structure $\mathfrak{A}$ and $a \in A$ such that a is the l^{th} element in $<^{\mathfrak{A}}$ we have

$$\mathfrak{A} \models \textsc{LaO}^+(a) \iff \mathrm{n}^+ \text{ is isomorphic to an initial segment of } \mathfrak{A}.$$

Let $\tau^+ = \sigma_{Ar} \cup \{E, R, L, P, \prec, H, O\}$ where E, R, L, P, O are as before and H, $\prec$ are new binary relation symbols. For $k \geq 1$, we define τ^+-structures $\mathfrak{A}_k^+$ and $\mathfrak{B}_k^+$ (playing the role of $\mathfrak{A}_k^*$ and $\mathfrak{B}_k^*$, respectively) as follows:

- $\mathfrak{A}_k^+ \restriction_\tau = \mathfrak{A}_k^*$ and $\mathfrak{B}_k^+ \restriction_\tau = \mathfrak{B}_k^*$.
- $+, \times, S$ are interpreted on the O-part in the natural way.
- $\prec$ is the partial ordering whose successor-relation is E.
- H relates each element of the $\neg O$-part to its height (considered as an element of the ordering).

All these additional relations do not change the basic fact that

PROPOSITION 7.7
Each $\exists\mathrm{L}^k_{\infty\omega}$-sentence holding in $\mathfrak{A}_k^+$ also holds in $\mathfrak{B}_k^+$.

We also have

PROPOSITION 7.8
There are first-order sentences φ^+ and ψ^+ such that:

(i) φ^+ is preserved under substructures.
(ii) $\mathfrak{A}_k^+ \models \varphi^+$ and $\mathfrak{B}_k^+ \models \varphi^+$ for each $k \geq 1$.
(iii) For all τ-structures $\mathfrak{C}$, if $\mathfrak{C} \models \varphi^+$ then

$$\mathfrak{C} \models \psi^+ \iff \mathfrak{C} \text{ has a substructure isomorphic to } \mathfrak{A}_k^+ \text{ for some } k \geq 1.$$

Again, this yields the theorem.

PROOF. [of Proposition 7.8] Let φ^O_{Ar} denote the relativizations of φ_{Ar} to O and φ^+ be the conjunction of φ^O_{Ar} with the following formulae:

(1$^+$) $\forall x \forall y \big((Exy \vee Eyx \vee x \prec y \vee y \prec x \vee Rx \vee Lx \vee Px) \rightarrow \neg Ox\big)$
(2$^+$) $\forall \alpha \forall \beta \forall \gamma \big((\alpha + \beta = \gamma \vee \beta + \alpha = \gamma \vee \ldots \vee \alpha = \underline{0}) \rightarrow O\alpha\big)$
(3$^+$) $\forall x\, \neg x \prec x$
(4$^+$) $\forall x \forall y \forall z \ \big((x \prec y \wedge y \prec z) \rightarrow x \prec z\big)$
(5$^+$) $\forall x \forall y \ \big(Exy \rightarrow (x \prec y \wedge \forall z \neg(x \prec z \wedge z \prec y))\big)$
(6$^+$) $\forall x \forall y \forall z \ \big((Eyx \wedge Ezx) \rightarrow y = z\big)$
(7$^+$) $\forall x \forall y \ (Rx \rightarrow \neg Eyx)$
(8$^+$) $\forall x \forall y \ (Lx \rightarrow \neg Exy)$
(9$^+$) $\forall x \forall \alpha \ \big(Hx\alpha \rightarrow (\neg Ox \wedge O\alpha)\big)$

(10^+) $\forall x \forall \alpha \forall \beta\ \big((Hx\alpha \wedge Hx\beta) \rightarrow \alpha = \beta\big)$

(11^+) $\forall x\ (Rx \rightarrow Hx\underline{0})$

(12^+) $\forall x \forall y \forall \alpha \forall \beta\ \big((Hx\alpha \wedge S\alpha\beta \wedge Exy) \rightarrow Hy\beta\big)$

(13^+) $\forall x \forall y \forall \alpha \forall \beta\ \big((Lx \wedge Ly \wedge Hx\alpha \wedge Hy\beta) \rightarrow \alpha = \beta\big)$

(14^+) $\forall x\ \neg(Rx \wedge Lx)$

(15^+) $\forall x \forall \alpha \forall \beta \forall \gamma\ \neg\big(Lx \wedge Hx\alpha \wedge S\beta\gamma \wedge \beta + \gamma = \alpha\big)$

(16^+) $\forall x \forall y \forall z\ \big((\mathrm{BRANCH}(x) \wedge \mathrm{BRANCH}(y) \wedge z \prec x \wedge z \prec y) \rightarrow x = y\big)$

(17^+) $\forall x \forall y \forall \alpha\ \neg\big(\mathrm{BRANCH}(x) \wedge \mathrm{BRANCH}(y) \wedge Hx\alpha \wedge Hy\alpha\big)$

(18^+) $\forall x \forall y\ \big((Px \wedge Py \wedge x \prec y) \rightarrow x = y\big)$

(19^+) $\forall x \forall y \forall z \forall \alpha\ \neg\big(Rx \wedge Py \wedge Pz \wedge x \prec y \wedge x \prec z \wedge Hx\alpha \wedge Hx\beta)\big)$.

(1^+) and (2^+) guarantee that the two parts of the structure are properly separated. By (3^+)–(5^+), $\prec$ is a partial order whose successor relation is compatible with E, hence E is anti-cyclic, and together with (6^+) this means that it is the edge relation of a forest. (7^+) and (8^+) say that R only contains roots and L only contains leaves of this forest.

(9^+)–(12^+) guarantee that the height function H is reasonably defined. In particular, if there is an E-path from a root to an x of length, say l, and S is defined for the first $l + 1$ elements of the ordering, then x is related (via H) to a unique α. Thus (13^+) guarantees that, provided S is defined on an initial segment of $<$ which is longer than any path from R to L, all paths from R to L have the same length. Furthermore, by (14^+) this length is > 0 and by (15^+) it is even.

Again under the assumption that S is defined on a sufficiently large initial segment of $<$, (16^+)–(19^+) have about the same meaning as (8)–(11) of the formula φ' in the proof of Proposition 7.4.

Again, it can be easily checked that (i) and (ii) hold.

It should be no difficulty for the reader to complete the proof now (analogously to the proof of Proposition 7.4). ∎

REMARK 7.9

Observe that the main reason this last proof fails for infinite structures is that we cannot guarantee in first-order logic that the distance from R to L is finite (i.e. that there is any path from R to L).

However, we can express this in transitive closure logic. Consequently, it is not hard to see that we can also obtain Theorem 7.1 as a corollary of the proof of Theorem 7.6.

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