On Transfinite Hybrid Automata

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Abstract. In this paper, we propose a new method to deal with hybrid systems based on the concept of the nonstandard analysis and the Büchi's transfinite automata. An essential point of the method is a generalization of hybrid automata with hyperfinite iteration of an infinitesimal transition in ${}^*\mathbb{R}$. This nonstandard model of hybrid automata allows discrete but hyperfinite state transition, so that we can describe and reason about the interaction of the continuous and discrete dynamics in the algebraic framework. In this enlarged perspective of the hybrid automata, we discuss about the asymptotic orbit of the dynamics that is peculiar to the hybrid systems such as Zeno.

1 Introduction

A hybrid automaton is a well known as a standard framework in the formalization of the hybrid system in which continuous and discrete dynamics are interacting each other [1, 13]. Especially, it has been providing a useful basis for the analysis or verification for the dynamical property of the hybrid systems such as the automotive engine [2] or the robotics [9]. However, the hybrid automaton has several limitations due to its simplicity and it is sometimes insufficient for the fully treatment of hybrid dynamics. Among them, we focus on two problems: the description of continuous dynamics and Zeno problem. The continuous dynamics is described in the form of differential equations in the hybrid automata. This is one of advantages of the hybrid automata because many mathematical methods and techniques are available in the fully established fields of control theory. However, it is also one of reasons for the difficulty in the logical or mechanical treatment of hybrid dynamics. In the automatic synthesis or automatic verification for the engine controllers, for example, the axiomatic framework is necessary for the differential equation of the combustion dynamics in cylinder. But we have no efficient reasoning system till now for the analysis and the real number theory which is actually available for the complex differential equations. On the other hands, so-called Zeno problem is related to the state transition of automata. Namely, it is incomplete in the meaning that the limiting state cannot be defined for the infinite sequence of discrete state transitions. The Zeno is a phenomenon in which the infinite iteration of a discrete value change occurs in the finite time that is familiar not only in the industrial hybrid system but

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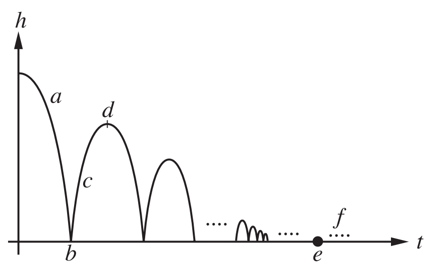
also in our daily life [8, 12]. For example, a bouncing ball shows the Zeno orbit in which a ball becomes to be rest in the finite time after the infinite iteration of bouncing (discontinuous change of velocity). It is often pointed out that the Zeno is a significant but hard problem from the standard treatment of the hybrid system, so that the non-Zeno condition is prerequisite assumption at many studies of hybrid system. There have been many studies to deal with Zeno in the hybrid automata [14, 17, 6], and the timed automata [3].

In this paper, we propose a new type of hybrid automata called a transfinite hybrid automaton, which is a generalization of the standard one at two points. First, we use the infinite recurrence equation instead of the differential equation based on the nonstandard analysis so that the whole behaviors of systems including the continuous dynamics are represented by the discrete transition of states. Secondly, the construction of Büchi's transfinite automaton is introduced in the frame of hybrid automata, which allows the infinite transitions among the set of states [5]. In comparison with other methods, the transfinite hybrid automaton has the following advantages:

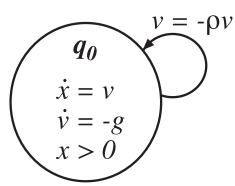
(1) Since the continuous change is defined by the sequence of the actions with the infinitesimal effect in the very small duration, we can deal with both the continuous and discrete dynamics uniformly in the discrete but hyperfinite state transition paradigm. Therefore, we need not deal with the derivative and integration. Instead, we use only arithmetic and simple algebra for $\mathbb{R}$ and $^*\mathbb{R}$ so that it is appropriate for the logical treatment of the hybrid system.
(2) The completeness in the space of discrete and continuous dynamics is naturally introduced so that it allows to deal with the fixed point, for example, an asymptotic behavior toward limits such as Zeno.

1.1 A Bouncing Ball

Let consider the motion of a bouncing ball. It is described by two variables x, v which denote the height and velocity of a ball.



(a) Bouncing actions $a - f$



(b) Model of bouncing ball as Hybrid Automaton

Fig. 1. Bouncing ball system

Figure 1 gives the motion of the bouncing ball and its corresponding hybrid automaton. In this paper, we keep the word "state" for the automaton, and use

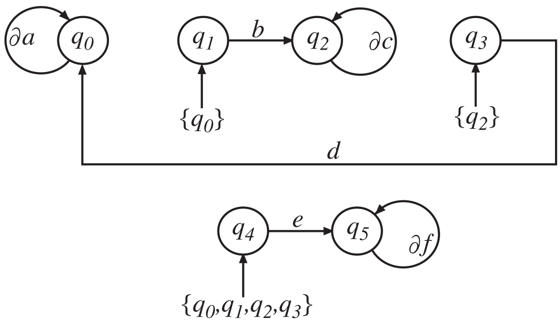


Fig. 2. Model of bouncing ball as Transfinite Automaton

the word "situation" to denote the physical status of the system. In Figure 1(b), the system continuously evolves within the state q_0 according to the differential equation while it satisfies the invariant condition in q_0. The state transition occurs whenever the condition is violated. Although this model gives a simple and elegant formulation, it is sometimes weak. For example, the state transition eventually halts after the infinite repetition of bouncing but we cannot find out this property from the visual representation. Figure 2 gives the automaton proposed here for the bouncing ball. We use two types of transition. The first is the usual transition represented by the arc from one state to another. The second type is represented by the arc from a set of states to another state, which means that after infinite visits to all states of the set, the machine transfers to the designated state. Every transition is instantaneous. We denote by ∂a the infinitesimal segment of the action a. Therefore, the ball moves to the situation q_1 after iterating the small falling ∂a infinitely at q_0. The action b represents the instantaneous motion of the ball bouncing. In the situation q_2, it ascends toward the highest point by repeating ∂c and reaches to the point at q_3. Then it moves to falling again. After the infinite iteration of these behaviors, the ball is at rest in q_4 and repeats the halting action forever in q_5. The rational expression $((\partial a)^\omega b(\partial c)^\omega d)^\omega e(\partial f)^\omega$ gives the action schema for the behavior of the ball.

1.2 Outline of the Methodology

The proposed method is essentially to build the transfinite automata which simulate the standard hybrid automata. It is schematically illustrated in Figure 3. We have two worlds, $\mathbb{Z}$ (a world of a hybrid automata $\mathcal{A}$ moving in $\mathbb{R}$) and ${}^*\mathbb{Z}$(a world of a transfinite hybrid automata ${}^*\mathcal{A}$ moving in ${}^*\mathbb{R}$). The two mappings α and β gives the simulation condition. Namely, we can use $\beta({}^*\mathcal{A}(\alpha(s_1)))$ in ${}^*\mathbb{Z}$ instead of $s_2 = \mathcal{A}(s_1)$ in $\mathbb{Z}$.

From a logical point of view, this method corresponds to the nonstandard model of hybrid automata. Let's assume that an action is decomposable into a sequence of n action-steps via the equi-distant discretization. We describe the dynamics of each action-step in the standard world $\mathbb{Z}$. We denote by $\mathbf{L}$ the logical closure of the axioms and theorems in $\mathbb{Z}$. We re-interpret the all formulas

$$\begin{tikzcd} \text{World } {}^*\mathbb{Z} & {}^*s_1 \arrow[r, "{}^*\mathcal{A}"] & {}^*s' \arrow[d, "\beta"] \\ \text{World } \mathbb{Z} & s_1 \arrow[u, "\alpha"] \arrow[r, "\mathcal{A}"'] & s_2 = {}^\circ({}^*s') \end{tikzcd}$$

Fig. 3. Simulation diagram

of **L** in ${}^*\mathbb{R}$ by the mapping α. By this transfer, all formulas in **L** are transformed to the formulas in ${}^*\mathbb{R}$. We call it ${}^*\mathbf{L}$. For example n is transfer to the infinite integer ${}^*n$. Since ${}^*\mathbb{R}$ allows a discrete but hyperfinite state transition, ${}^*\mathcal{A}$ gives a new situation as the result of performing the infinitesimal action or the infinite iteration of action. Since the situations in ${}^*\mathbb{Z}$ contains infinitesimal or infinite values of system variables and these values have no actual meaning from the physical point of view, the nonstandard situation must be pulled back to the standard one by the mapping β. The mapping β must designate a standard situation which is close to the given nonstandard situation.

Note that ${}^*\mathbf{L}$ is syntactically equivalent to **L** because we alter only its interpretation. So that all theorems in **L** hold even in ${}^*\mathbf{L}$(Transfer principle [16]). This is an essential point of this method.

2 Nonstandard Discretization

In this paper, we treat the continuous dynamics in terms of hyperfinite recurrence equations. Therefore, we introduce the infinite division of an action with continuous motion based on the nonstandard analysis. The infinitesimal segment of action is called an infinitesimal action. We prove that any standard action is equivalent to an infinite iteration of the infinitesimal action.

2.1 Nonstandard Real ${}^*\mathbb{R}$

The nonstandard real number ${}^*\mathbb{R}$ is constructed from the real number $\mathbb{R}$ via the ultra product formation [16].

Definition 1 (Ultra filter). *Let $\mathcal{F}$ be a family of the subsets of $\mathbb{N}$ which satisfy the following conditions, where $\mathbb{N}$ is the set of all natural numbers.*

(1) $\mathbb{N} \in \mathcal{F}$, $\phi \notin \mathcal{F}$
(2) if $A \in \mathcal{F}$ *and* $A \subset B$ *then* $B \in \mathcal{F}$
(3) if $A \in \mathcal{F}$ *and* $B \in \mathcal{F}$ *then* $A \cap B \in \mathcal{F}$
(4) for any $A \subset \mathbb{N}$, $A \in \mathcal{F}$ *or* $\mathbb{N} - A \in \mathcal{F}$
(5) if $A \subset \mathbb{N}$ *is finite then* $\mathbb{N} - A \in \mathcal{F}$

$\mathcal{F}$ is called an ultra filter. For example, the set of all co-finite sets $\{\mathbb{N} - A \mid A$ is finite $\}$ is an ultra filter.

Definition 2 (Hyperreal number). *We fix an ultra filter $\mathcal{F}$.*

(1) Let W denote a set of sequences of real numbers $(a_1, a_2, \cdots)$. The nonstandard real number ${}^*\mathbb{R}$ is defined by introducing the following equivalence relation into W

$$(a_1, a_2, \cdots) \sim (b_1, b_2, \cdots) \Leftrightarrow \{k \mid a_k = b_k\} \in \mathcal{F}$$

Namely, ${}^*\mathbb{R} = W/\sim$. We denote the equivalence class of $(a_1, a_2, \cdots)$ by $[(a_1, a_2, \cdots)]$.
This means that the hyperreal number is regarded as the infinite sequence of real numbers ignoring the difference of finite part.

(2) the nonstandard integer ${}^*\mathbb{N} \subseteq {}^*\mathbb{R}$ is also introduced via the ultra product formation. A set $A \subseteq {}^*\mathbb{N}$ is hyperfinite if there exists $N \in {}^*\mathbb{N}$ such that for every $x \in A, x < N$.

We distinguish the nonstandard variables and function symbols from the standard one by attaching * to them, although it is omitted in the clear cases. The usual real number a is treated by $[(a, a, a, ...)]$, so that ${}^*\mathbb{R}$ contains $\mathbb{R}$ itself. An element of $\mathbb{R}$ in ${}^*\mathbb{R}$ is called a standard number. For the elements of ${}^*\mathbb{R}$, we can define arithmetic operations and relations in the following way [10].

The usual arithmetic operations such as $+, -, \times, /$ are given by

$$[(a_1, a_2, ...)] + [(b_1, b_2, ...)] = [(a_1 + b_1, a_2 + b_2, ...)]$$

The usual relations such as $<$ are given by

$$[(a_1, a_2, ...)] < [(b_1, b_2, ...)] \equiv \{n \mid a_n < b_n\} \in \mathcal{F}$$

Note that an element $[(1, \frac{1}{2}, \frac{1}{3}, ...)]$ is smaller than any real number $[(a, a, ...)]$, namely it is an infinitesimal. Also, a number $\omega = [(1, 2, 3, ...)]$ is infinite.

Definition 3 (Closeness). *We define a relation $a \approx b$ if the distance from a to b is infinitesimal, that is*

$$(\forall a, b \in {}^*\mathbf{R})[a \approx b \text{ iff } a - b \text{ is infinitesimal}]$$

A relation $a \approx b$ is an equivalence relation.

From the construction of ${}^*\mathbb{R}$, every standard function $f(x)$ is extended to the nonstandard function ${}^*f(x)$, which holds any properties of the original function. This is called the transfer principle. Dual to the transfer principle, the finite hyperreal number satisfying a certain property has the corresponding real number with this property [10].

Theorem 1 (Transfer Principle). *If a property holds for all real numbers, then it holds for all hyperreal numbers.*

Theorem 2 (monad, shadow). *For every finite number $a \in {}^*\mathbb{R}$, there is only one standard number $b \in \mathbb{R}$ such that $a \approx b$. b is called a monad of a and denoted by $b = {}^\circ a$.*

The continuity, differentiation and Riemann integral of the $\mathbb{R}$-valued function is defined in the $^*\mathbb{R}$ in the following way.

Definition 4 (Continuity, Differentiation and Riemann integral).

(1) A standard function $f(x)$ is continuous at a standard number x if and only if for all $y \in {}^*\mathbb{R}$, $^*f(y) \approx {}^*f(x)$ if $y \approx x$.
(2) A standard function $f(x)$ is differentiable at a standard number x if and only if there exists some $d \in \mathbb{R}$ such that for every nonzero infinitesimal ε,

$$\frac{1}{\varepsilon}[{}^*f(x+\varepsilon) - f(x)] \approx d$$

(3) Let A standard function $f(x)$ be continuous over $[a,b]$. For the nonstandard sequence, $x_0 = a, x_1, x_2 \cdots, x_\omega = b$ such that $x_{i+1} - xi = \frac{b-a}{\omega}$ for every i,

$$\int_a^b f(x)dx = {}^\circ\left(\sum_{k=0}^{\omega-1} {}^*f(x_k)\frac{b-a}{\omega}\right)$$

2.2 Description of the Dynamics

Situation: The behavior of the hybrid system is represented in the terms of the situation and the action. The situation is the physical status of the system at each time instance and the action causes the time evolution via changing the situation. The situation is described by the vector of $\mathbb{R}$-valued variables $(x_1, x_2, \cdots, x_n)$. We denote the set of situations by **Sit**. We assume that the situation contains a special variable *time*. The set of action is $\mathbf{Act} \subseteq \mathbf{Sit} \times \mathbf{Sit}$. We define the duration of action a at the situation s as $\tau(a,s) = time(a(s)) - time(s)$.

The nonstandard extension of situation is introduced via the similar way to $^*\mathbb{R}$.

Definition 5 (Nonstandard situation). *We fix an ultra filter $\mathcal{F}$. Let $s_1 = (x_1^1, x_2^1, ..., x_m^1), s_2 = (x_1^2, x_2^2, ..., x_m^2), \cdots$ be a sequence of situations. Then the nonstandard situation is defined by*

$$\begin{aligned}{}^*\mathbf{Sit} = \{([(x_1^1, x_1^2, \cdots)], [(x_2^1, x_2^2, \cdots)], \cdots, [(x_m^1, x_m^2, \cdots)] > \\ \mid \{n \mid (x_1^n, x_2^n, \cdots, x_m^n) \in \mathbf{Sit}\} \in \mathcal{F}\}\end{aligned}$$

From this definition, we can always find the limiting situation $[(s_1, s_2, ...)] \in {}^*\mathbf{Sit}$ of any sequence of standard situations $s_1, s_2, ... \in \mathbf{Sit}$. Namely,

$$\begin{aligned}[(s_1, s_2, ...)] = ([(x_1^1, x_1^2, ...)], [(x_2^1, x_2^2, ...)], ..., [(x_m^1, x_m^2, ...)]) \\ \text{where } s_n = (x_1^n, x_2^n, ..., x_m^n) \text{ for each } n\end{aligned}$$

Action: A dynamical system can be characterized generally as an infinite iteration of an action a, namely $(s, a(s), a(a(s)), ...)$. In the following theorem, we prove that there always exists the nonstandard situation which is the result of infinite iteration of action.

Theorem 3 (Infinite iteration of action). *For any standard action a and any situation $s \in \mathbf{Sit}$, $a^\omega(s) \in {}^*\mathbf{Sit}$ for $\omega = [(1, 2, ...)]$.*

Proof. Clearly, $a^\omega(s) = [(a^1(s), a^2(s), ...)]$ by the definition. From Definition 5, $a^\omega(s) \in {}^*\mathbf{Sit}$.

Definition 6 (Fixed point). *An action a has a fixed point if and only if*

$$\forall n \in \mathbb{N}[a^\omega(s) \approx a^{\omega+n}(s)]$$

A fixed point s is **attractive** *if and only if there exists $Z \subseteq {}^*\mathbf{Sit}$ such that*

$$s \in Z \text{ and } \forall s' \in Z[s' \neq s \rightarrow (\forall n \in \mathbb{N}[a^n(s') \in Z]) \wedge a^\omega(s') \approx s]$$

The fixed point s is **repelling** *if and only if there exists $Z \subseteq {}^*\mathbf{Sit}$ such that*

$$s \in Z \text{ and } \exists n \in \mathbb{N}\forall s' \in Z[s' \neq s \wedge a^n(s') \notin Z]$$

A fixed point $s = a^\omega(s_0)$ is **Zeno** *related to a, x if and only if*

$$\exists r \in \mathbb{N}[x(s) \not\approx x(a^r(s)) \wedge time(s) \text{ is finite}$$

Definition 7 (Equi-time division of action). *A standard action a is $n-$divisible for any $n \in N$ if there exists an action a_n such that ${a_n}^n(s) = a(s)$ and $\tau(a_n, s_k)$ is constant for every $s_k = {a_n}^k(s)$ for $k < n$.*

Theorem 4 (Infinite division of action). *Assume that the action a is n-divisible for every n. There exists $\partial a \in {}^*Act$ such that $\partial a^k(s) \in {}^*\mathbf{Sit}$ for each $k = 1, 2, ..., \omega$.*

Proof. Let $\xi(n, k) = {a_n}^k(s)$ for finite n, k. We define nonstandard situations $\xi(\omega, k) = [(\xi(1, k_1), \xi(2, k_2), ..., \xi(n, k_n), ...)]$, where $k_\omega = k$ and if $k_m < \frac{m}{2}$ then $k_{m-1} = k_m$ else $k_{m-1} = k_m - 1$.

By the definition 5, $\xi(\omega, k) \in {}^*\mathbf{Sit}$. And also,
$time(\xi(\omega, k)) = [(time(\xi(1, k_1), time(\xi(2, k_2), ...,)] = time(s) + \frac{[(k_1, k_2, ...)]}{\omega}\tau$.
Therefore, we can conclude $\xi(\omega, k) = \partial a^k(s)$.

In the following, we denote the duration of ∂a by ν, that is $\nu = \frac{\tau}{\omega}$ for the duration τ of action a.

Clearly, $a(s) = \partial a^\omega(s)$. Namely, we can regard any standard action as an infinite iteration of the infinitesimal action.

2.3 Nonstandard Recurrence Equation

The continuous parts of the hybrid dynamics can be easily transformed to the hyperfinite recurrence equations via the infinite iteration of the infinitesimal action if their differential equations satisfy the Lipschitz condition.

Theorem 5 (Discretization). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a standard function which satisfies the Lipschitz condition:* $|f(x) - f(y)| < c|x - y|$
Let ∂a be an infinitesimal action with the duration $\nu = \frac{T}{\omega}$ such that $\partial a : x' = x + f(x)\nu$
Then the function $g(t) = {}^\circ z(t)$ where $z(t) = (\partial a)^i(x_0)$ for $(i+1)\nu > t \geq i\nu$ gives a solution at $t \in [0, T]$ for

$$\frac{dx}{dt} = f(x); \quad x(t_0) = x_0$$

Proof. Let $\xi_i = (\partial a)^i(x_0)$, namely $\xi_{i+1} = \xi_i + f(\xi_i)\nu$.
$|\xi_{i+1} - \xi_i| = |\xi_i + f(\xi_i)\nu - (\xi_{i-1} + f(\xi_{i-1})\nu)| < |\xi_i - \xi_{i-1}| + |f(\xi_i)\nu - f(\xi_{i-1})\nu)| < (1 + c\nu)|\xi_i - \xi_{i-1}|$. Therefore,
$|\xi_{i+1} - \xi_i| < (1 + c\nu)^i|\xi_1 - \xi_0| = (1 + c\frac{T}{\omega})^{\frac{t}{T}\omega}|\xi_1 - \xi_0| < e^{cT}|f(\xi_0)|\nu \approx 0$. Namely, $\xi_{i+1} \approx \xi_i$, so that $z(t) \approx z(t')$ if $t \approx t'$. This means that $z(t)$ is an S-continuous function. It is known that every S-continuous function h(t) has a continuous function ${}^\circ h(t)$ as a shadow [10]. Therefore, there exists a standard continuous function $g(t) = {}^\circ z(t)$. Clearly, $g(t_0) = x_0$ and

$$\int_{t_0}^{t} f(x)dt \approx \sum_{k=0}^{i} f(\xi_k)\nu = \xi_i - \xi_0 \approx {}^\circ z(t) - {}^\circ z(t_0) = g(t) - g(t_0)$$

Example 1. Consider the differential equation of the action a for the situation $(x, time)$

$$\frac{dx}{dt} + kx = p, \quad x(s_0) = x_0, time(s_0) = 0$$

Let assume that we want to predict the situation s' after t seconds. We deal with this differential equation by using the piecewise constant model. The dynamics for the variable x is described by an action ∂a with duration ν.

For every situation s_i,
$x(s_{i+1})) = (1 - k\nu)x(s_i) + p\nu$, $time(s_i) = time(s_i) + \nu$ for $i = 1, 2, ..., \omega$ where $\nu = \frac{t}{\omega}$.

The desired situation s' is given by s_ω. By mathematical Induction rule (the mathematical induction is also available in ${}^*\mathbb{N}$ [16]), we have

$$x(s_\omega) = x_0(1 - k\frac{t}{\omega})^\omega$$

We must find out the standard value near $x(s_\omega)$. We use a knowledge related to infinitesimal arithmetic: $(1 - \frac{1}{n})^n \approx e^{-1}$ if n is infinite. From this equation, we have $(1 - k\frac{t}{\omega})^\omega \approx e^{-kt}$

Finally, we have the desired situation $s' = (x_0 e^{-kt}, t)$

Note that we use only the simple arithmetic rules for real and hyperreal numbers in the above argument.

3 Transfinite Hybrid Automata

Büchi introduces the automaton on the infinite words indexed by ordinals [5], which is a generalization of the usual automaton. Namely, Büchi's transfinite automaton contains the transition rules for the limit ordinals in addition to the usual successor transitions. We use this construction to define the automata which recognize the hyperfinite words whose letters are indexed by the nonstandard integer. Since the hyperfinite integer has a similar linear and discrete order structure to the ordinals, the idea of limit transitions can be used also to the automata on hyperfinite words. Although the hybrid automaton defined here is based on the nonstandard integers rather than ordinals, we call it "transfinite" hybrid automata because it inherits the limit transition rules from Büchi's transfinite automaton.

In the following, we use n instead of $*n$ to represent the hyper integer, that is $n \in {}^*\mathbb{N}$).

Definition 8 (word). *Let Σ is a finite set of letters (called alphabet).*

(1) For a (nonstandard) integer n, the n-sequence of letters $a_1 a_2, \cdots a_n$ is called a word on Σ with the length n. ϵ is the word with the length 0.
(2) For the sets U, V of words of Σ, we define the rational operation
$U + V = \{x \mid x \in U \cup V\}$
$U \cdot V = \{x_1 x_2 \mid x_1 \in U \wedge x_2 \in V\}$
$U^\omega = \{x^\omega \mid x \in U\}$
$U^\dagger = \{\epsilon\} \cup U \cup U \cdot U \cdots \cup U^\omega \cdots$ *, $U^\dagger$ is the nonstandard extension of closure U^**
(3) $S \subseteq \Sigma^\dagger$ is called rational if and only if it can be obtained from finite subsets using a hyperfine number of rational operations.

Definition 9 (Transfinite Hybrid Automaton).
A transfinite hybrid automaton $\mathcal{A}$ is an 8-tuple $(Q, A, E, I, F, X, X_0, D)$ where

1. *Q is the finite set of states.*
2. *A is the finite set of infinitesimal actions.*
3. *E is the finite set of transition rules ; $E \subseteq (Q \times A \times Q) \cup (\mathcal{P}(Q) \times Q)$ where $(P, q) \notin E$ if $q \in P$*
4. *$I \in Q$ is the initial state. $X_0 \in X$ is the initial situation.*
5. *$F \subset Q$ is the set of the final states.*
6. *For the sequence of states $q_1, q_2, \cdots, q_n$, $Inf(\{q_1, q_2, \cdots, q_n\}) = \{q \mid \{k \mid q_k = q\}$ is infinite $\}$*
7. *$X \subseteq \mathbb{R}^m$ where $m \in {}^*\mathbb{N}$ is the set of situations. Namely, $x = (x_1, x_2, \cdots, x_m) \in X$ is a vector of the continuous or discrete values of physical entities. We use $x' = (x'_1, x'_2, \cdots, x'_m) \in X'$ for the result situation of the action. The situation contains the special variable time or t.*
8. *D is the set of dynamics for each action of A. The dynamics of each action is given by specifying its precondition and effect in the form of $p(x) \rightarrow x' = f(x)$ where $f(x)$ is given by the algebraic formula formed from the arithmetic operations of ${}^*\mathbb{R}$.*

Example 2 (A bouncing ball automaton). (Figure 2)
$\mathcal{A} = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, E, q_0, \{q_5\}, \{(x, v, t)\}, D)$ where
$E = \{(q_0, \partial a, q_0), (q_1, b, q_2), (q_2, \partial c, q_2), (q_3, d, q_0), (q_4, e, q_5), (q_5, \partial f, q_5),$
$(\{q_0\}, q_1), (\{q_2\}, q_3), (\{q_0, q_1, q_2, q_3\}, q_4)\}$
The dynamics D for each action is
$a : x \not\approx 0 \to (x' = x - v\tau(\partial a); v' = v - g\tau(\partial a); t' = t + \tau(\partial a);)$
$b : x \approx 0 \to (x' = x; v' = -v\rho; t' = t + \tau(b);)$, where $0 < \rho < 1$
$c : v \not\approx 0 \to (x' = x - v\tau(\partial c); v' = v - g\tau(\partial c); t' = t + \tau(\partial c);)$
$d : v \not\approx 0 \to (x' = x; v' = -v; t' = t + \tau(d);)$
$e : x \approx 0 \to (x' = 0; v' = 0; t' = t + \tau(e);)$
$f : x = 0 \to (x' = 0; v' = 0; t' = t + \tau(\partial f);)$

Definition 10 (Transition).

(1) *A word $\alpha = a_1 a_2 \cdots a_n \in A^\dagger$ has a transition $(q_0, q_1, \cdots q_n)$ of the automaton $\mathcal{A} = (Q, A, E, q_0, F, X, X_0, D)$ if and only if there exists either $(q_i, a_i, q_{i+1}) \in E$ or $\{q_{i1}, q_{i2}, \cdots, q_{ir}\} \subseteq Inf(\{q_0, q_1, \cdots q_i\})$ such that $(\{q_{i1}, q_{i2}, \cdots, q_{ir}\}, q_{i+1}) \in E$*

(2) *$\alpha = a_1 a_2 \cdots a_n \in A^\dagger$ is recognizable if its transition $(q_0, q_1, \cdots q_n)$ ends at the final state $q_n \in F$. The set of all recognizable words of $\mathcal{A}$ is called the language of $\mathcal{A}$ which we denote by $L(\mathcal{A})$.*

Kleene's theorem gives the fundamental result that the language $L(\mathcal{A})$ of a finite automaton $\mathcal{A}$ is equivalent to some rational expression. This corresponding theorem is extended to the case of the transfinite automata [7, 18]. For the transfinite automata defined here, the theorem also holds because of the straightforward application of the transfer principle in the nonstandard analysis.

Theorem 6 (Action schema). *For given transfinite hybrid automata $\mathcal{A} = (Q, E, q_0, F, X, X_0, D)$, there exists the rational expression $\mathcal{E}$, $\mathcal{E} = L(\mathcal{A})$. A set $AS \subseteq A^\dagger$ is called a set of an action schema if it is rational.*

Theorem 7 (Boolean operation of automata). *Let $\mathcal{A}, \mathcal{B}$ be two transfinite hybrid automata over A. There exist transfinite hybrid automata which recognize $L(\mathcal{A} \cup \mathcal{B}), L(\mathcal{A} \cap \mathcal{B}), A^\dagger - \mathcal{A} \cup L(\mathcal{B})$.*

Definition 11 (Execution).

(1) **nonstandard execution.** *Let $\alpha = a_1 a_2, \cdots a_n \in AS$ be an action schema of $\mathcal{A} = (Q, E, q_0, F, X, X_0, D)$ with the transition $q_0, q_1, \cdots q_n$. α is $*-$executable if and only if there exists a sequence of situations $w_0, w_1, \cdots, w_n \subseteq X$ such that*

- *(a)* $w_0 = X_0$
- *(b)* *for all $i = 1, 2, \cdots, n$*

  case 1: If $(q_i, a_i, q_{i+1}) \in E$ and $p(x(w_i))$ holds where $p(x)$ is the precondition of action a_i, then $x(w_{i+1}) = x'(w_i)$.

  case 2: if $(\{q_{i1}, q_{i2}, \cdots, q_{ir}\}, q_{i+1}) \in E$ and $x(w_{ij}) \approx x(w_{ik})$ for any $r \geq i, j > 0$, then $x(w_{i+1}) = x(w_{ij})$ for some $j = 1, 2, \cdots, r$.

(2) **execution in the standard sense.** *Note that each situation w_i of α is always defined over ${}^*\mathbb{R}$ if it is $*-$executable. However, it is feasible only when all values in w_i have their shadows in the standard world. Namely, α is executable in the standard sense if and only if it is $*-$executable and the all situations $w_0, w_1, \cdots, w_n \subseteq X$ have finite values. Especially, time(w_{ij}) must be finite for each ij in every limit transition $(\{q_{i1}, q_{i2}, \cdots, q_{ir}\}, q_{i+1}) \in E$. This means that a state is actually reachable after the limit transition only when the machine spent finite time for the previous ω-iteration).*

4 The Description of Hybrid System

4.1 Alternative Water Tank

Consider a coupling of two water tanks [14]. Let x, y denotes the level of water in Tank A and Tank B. We assume that the tap in the bottom of each tank discharges the water at a rate proportional to the level of each tank. Also the constant flow denoted by p of the water is poured exclusively to either Tank A (we call the state A) or Tank B (the state B) at each time (Figure 4).We use the control strategy.

$$\text{if } st = A \wedge x \geq h \wedge y < h \text{ then switch to } B$$
$$\text{if } st = B \wedge y \geq h \wedge x < h \text{ then switch to } A$$

The levels of water x, y are described below: (where st is the state of Tank)

$$\text{if } st = A \text{ then } \frac{dx}{dt} + kx - p = 0 \wedge \frac{dy}{dt} + ky = 0$$
$$\text{if } st = B \text{ then } \frac{dx}{dt} + kx = 0 \wedge \frac{dy}{dt} + ky - p = 0$$

The corresponding hybrid automaton is given in Figure 5).

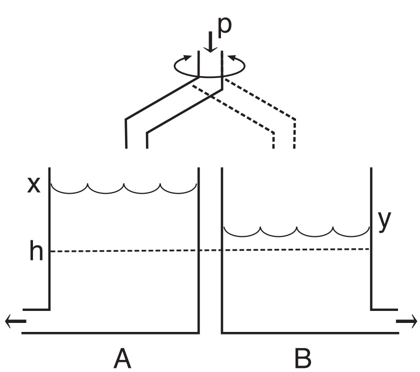


Fig. 4. Alternative water tank

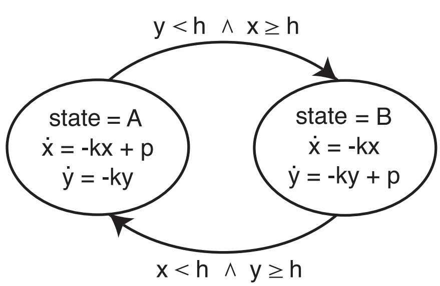


Fig. 5. Standard hybrid automaton

4.2 The System Description in a Transfinite Hybrid Automaton

We can describe this system in a transfinite hybrid automaton as below.

$\mathbf{Q} = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}$
$\mathbf{A} = \{\partial a, a, \partial b, b, c, d, e, f, g\}$
$\mathbf{E} = \{\{q_0, \partial a, q_0\}, \{q_2, \partial b, q_2\}, \{q_1, c, q_2\}, \{q_3, d, q_0\}, \{q_4, e, q_5\}, \{q_5, f, q_0\}, \{q_5, g, q_2\},$
$\{\{q_0\}, q_1\}, \{\{q_2\}, q_3\}, \{\{q_0, q_1, q_2, q_3\}, q_4\}, \{\{q_0, q_1, q_2, q_3\}, q_6\}\}$
$\mathbf{I} = \{Q = q_0, x = h_A, y = h_B, t = 0\}$
$\mathbf{F} = \{q_6\}$
$\mathbf{X} = \{x, y, t\}$
$\mathbf{D} = \{$
$\partial a : y \not\approx h \rightarrow (x' = (1 - k\tau(\partial a))x + p\tau(\partial a)), y' = (1 - k\tau(\partial a))y, t' = t + \tau(\partial a))$
$\partial b : x \not\approx h \rightarrow (x' = (1 - k\tau(\partial b))x, y' = (1 - k\tau(\partial b))y + p\tau(\partial b), t' = t + \tau(\partial b))$
$c : y \approx h \rightarrow (x' = x, y' = h, t' = t + \Delta)$
$d : x \approx h \rightarrow (x' = h, y' = y, t' = t + \Delta)$
$e : x \approx h \wedge y \approx h \rightarrow (x' = h, y' = h, t' = t + \Delta)$
$f : x' = (1 - k\Delta)x + p\Delta, y' = (1 - k\Delta)y, t' = t + \Delta$
$g : \ x' = (1 - k\Delta)x, y' = (1 - k\Delta)y + p\Delta, t' = t + \Delta$
$\}$

where Δ is the infinitesimal which denotes the minimal unit time of the system (the minimal clock or sampling time).

We present this automaton in Figure 6. The action schema of this automaton is

$$((\partial a)^\omega c(\partial b)^\omega d)^\omega e(f((\partial a)^\omega c(\partial b)^\omega d)^\omega + g((\partial b)^\omega d(\partial a)^\omega c)^\omega)$$

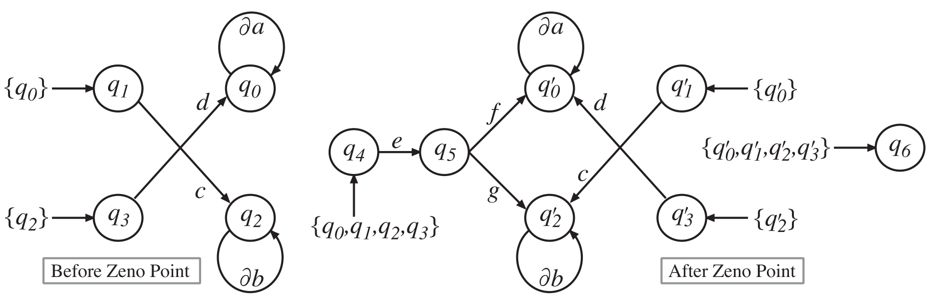


Fig. 6. A transfinite hybrid automaton for the alternative water tanks

4.3 Simulation of System Behavior

By using this example, we present the simulation of system dynamics. This system contains Zeno. We prove its existence and localize the Zeno point, and discuss about how we can escape from it.

[The existence of Zeno point]

The level of water x, y at the end of the action $(\partial a)^\omega, (\partial b)^\omega$ was shown below.

$$x_1 = \frac{p}{k} + \frac{h(h_A - \frac{p}{k})}{h_A}, y_1 = h \text{ when } (\partial a)^\omega \text{ was executed}$$
$$x_1 = h, y_1 = \frac{p}{k} + \frac{h(h_B - \frac{p}{k})}{h_B}, \text{ when } (\partial b)^\omega \text{ was executed}$$

If we assume that the index $2n$ of values means n times iteration of $(\partial a)^\omega \cdot (\partial b)^\omega$ and $2n+1$ means the execution of $(\partial a)^\omega$ after n times iteration of $(\partial a)^\omega \cdot (\partial b)^\omega$, then

$$x_{2n+1} = \tfrac{p}{k} + \tfrac{h(h-\frac{p}{k})}{y_{2n}}, \qquad y_{2n+1} = h, \qquad \tau_{2n+1} = \tfrac{1}{k}log(\tfrac{y_{2n}}{h})$$
$$x_{2n} = h, \qquad y_{2n} = \tfrac{p}{k} + \tfrac{h(h-\frac{p}{k})}{x_{2n-1}}, \tau_{2n} = \tfrac{1}{k}log(\tfrac{x_{2n-1}}{h}, t_n) = \textstyle\sum_{i=1}^{n} \tau_i$$

The symbolic simulator gives the values x_{2n+1}, y_{2n} in the form of infinite continued fraction.

$$x_{2n+1} = y_{2n} = \frac{p}{k} + \cfrac{h\left(h - \frac{p}{k}\right)}{\frac{p}{k} + \cfrac{h\left(h-\frac{p}{k}\right)}{\frac{p}{k} + \cfrac{h\left(h-\frac{p}{k}\right)}{\frac{p}{k} + \cfrac{h\left(h-\frac{p}{k}\right)}{\vdots}}}}$$

We can prove the relations $h < x_{2n+1} < x_{2n-1}, h < y_{2n} < y_{2n-2}$ from the mathematical induction. Namely, the sequences of x, y are monotonously decreasing but bounded so that x, y have a limit point by the Weierstrass's theorem. Namely, we have a standard $\hat{x}, \hat{y}, \hat{t}$ such that $x_{2n+1} \approx \hat{x}, y_{2n} \approx \hat{y}, t_n \approx \hat{t}$. From $x_{2n+1} \approx x_{2n-1} \approx \hat{x}$ and a recurrence equation $x_{2n+1} = \frac{p}{k} + \frac{h(h-\frac{p}{k})}{\frac{p}{k}+\frac{h(h-\frac{p}{k})}{x_{2n-1}}}$, we have $\hat{x} = h$. Similarly, we can get $\hat{y} = h$. On the other hand, we have $(x_{n+1} + y_{n+1}) = (1 - k\Delta)(x_n + y_n) + p\tau$ from the system description. By solving this recurrence equation, we have $\hat{t} = \frac{1}{k}log(\frac{(h_A+h_B)k-p}{2hk-p})$. $\hat{t}$ is finite so that we can prove that the situation $\hat{s} =< h, h, \frac{1}{k}log(\frac{(h_A+h_B)k-p}{2hk-p}) >$ is Zeno point.

[Escape from Zeno]

This Zeno point is repelling for $t > \hat{t}$ so that the behavior of the system at $t > \hat{t}$ depends on the situation immediately after the Zeno point. Assume that ∂a is currently performed. Since the duration of any action should not be smaller than the minimal sampling time Δ, the action ∂a continues during Δ even $\partial a(\tau)$ becomes smaller than Δ. After passing Δ, the next value of y becomes lower than h and the next system time is over $\hat{t}$ (See Figure 7). So we can jump out from the fixed point.

[Uncertainty]

It is uncertain which action f or g will be selected at the state q_5 after Zeno. If ∂a is executed immediately before q_5, f is selected, and otherwise g is taken.

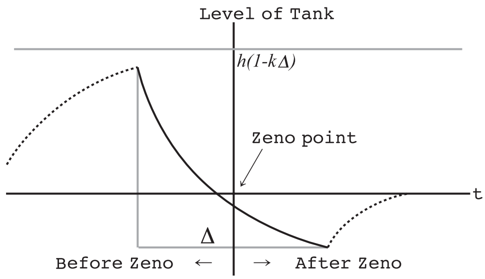


Fig. 7. Monad around Zeno point

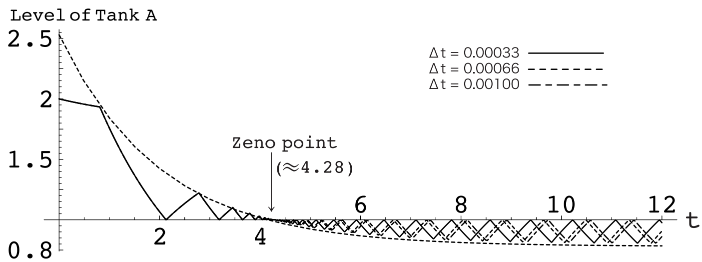


Fig. 8. The orbit of the water level of Tank A

However, ∂a and ∂b are exchanged intensively at the state q_4, so that the next state is undecidable. This is the inherent uncertainty in after-Zeno due to the infinite discrete value change within a finite time.

The Figure 8 shows a result of the numerical simulation. We use some very small numbers for Δ, and the other parameters are $x_0 = 2.0 \wedge y_0 = 1.5 \wedge k = 0.5 \wedge h = 1.0 \wedge p = 0.9$. The excursion of the level of tank is contained in the envelope $u = (h_A + h_B - \frac{p}{k})e^{-kt} + \frac{p}{k}$

5 Concluding Remarks

We propose a new type of hybrid automata which aims at the formal analysis and synthesis of the control plan for hybrid systems. In this system, executions can be represented by transfinite sequences of infinitesimal actions over $^*\mathbb{R}$ for the both continuous and discrete dynamics. We can deal with the convergent (even also divergent) sequences such as Zeno in this framework. Although Büchi's transfinite automaton is formed over the countable ordinal, we use his idea for the hypernumber $^*\mathbb{N}$ and $^*\mathbb{R}$. The formulation based on the nonstandard analysis is required because the state and its physical situation is defined after the limit transition depending on the times of iteration $(\omega, 2\omega, \omega^2, \cdots)$. Namely, we need to incorporate the limit of functions over $\mathbb{R}$ with the limit transition. The transfinite transition on ordinal is possibly introduced to the standard hybrid automata directly, but it may be weak in this sense. The nonstandard formulation always allows the transfinite state transition to be complete.

In order to deal with the hybrid dynamics in the transfinite hybrid automata, the following inferential devices are required:

(1) a reasoning system for the extended arithmetic of the hyperfinite integer and $^*\mathbb{R}$
(2) a set of transfer rules between $^*\mathbb{R}$ and $\mathbb{R}$ such as

$$^\circ(x+y) = {}^\circ x + {}^\circ y, {}^\circ(xy) = {}^\circ x {}^\circ y,$$
$$^\circ x \neq 0 \rightarrow {}^\circ(1/x) = 1/{}^\circ x, {}^\circ(1+\frac{x}{\omega})^\omega = e^x$$

A symbolic and numerical simulator is partially developed on $Mathematica^{TM}$ of Wolfram Research Inc. The symbolic simulation of the recurrence equation is troublesome because its solution is usually given in the form of very long arithmetic formula.

Many problems remains for the future study. Especially, the synthesis or verification of the transfinite hybrid automata may not necessarily be easy. Büchi proved the equivalence of the transfinite automata to the formulas of some second order language for the countable ordinals [5]. This suggests that it may be possibly used as the specification language for the transfinite behaviors of dynamics.

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