Efficient Algorithms for Checking Fast Termination in VASS

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Abstract

Vector Addition Systems with States (VASS) consists of a finite state space equipped with d counters (d is called the dimension), where in each transition every counter is incremented, decremented, or left unchanged. VASS provide a fundamental model for analysis of concurrent processes, parametrized systems, and they are also used as abstract models for programs for bounds analysis. While termination is the basic liveness property that asks the qualitative question of whether a given model always terminates or not, the more general quantitative question asks for bounds on the number of steps to termination. In the realm of quantitative bounds a fundamental problem is to obtain asymptotic bounds on termination time. Large asymptotic bounds such as exponential or higher already suggest that either there is some error in modeling, or the model is not useful in practice. Hence we focus on polynomial asymptotic bounds for VASS. While some well-known approaches (e.g., lexicographic ranking functions) are neither sound nor complete with respect to polynomial bounds, other approaches only present sound methods for upper bounds. The existing approaches neither provide complete methods nor provide analysis of precise complexity bounds. In this work our main contributions are as follows: First, for linear asymptotic bounds we present a sound and complete method for VASS, and moreover, our algorithm runs in polynomial time. Second, we classify VASS according the normals of the vectors of the cycles. We show that singularities in the normal are the key reason for asymptotic bounds such as exponential (even in three dimensions) and non-elementary (even in four dimensions) for VASS. In absence of singularities, we show that the asymptotic complexity bound is always polynomial and of the form $\Theta(n^k)$, for some integer $k \leq d$. We present an algorithm, with time complexity polynomial in the size of the VASS and exponential in dimension d, to compute the optimal k. In other words, in absence of

singularities, we present an efficient sound and complete method to obtain precise (not only upper, but matching upper and lower) asymptotic complexity bounds for VASS.

1 Introduction

Static analysis for quantitative bounds. Static analysis of programs reasons about programs without running them. The most basic and important problem about *liveness* properties studied in program analysis is the *termination* problem that given a program asks whether it always terminates. The above problem seeks a *qualitative* or Boolean answer. However, given the recent interest in analysis of resource-constrained systems, such as embedded systems, as well as for performance analysis, it is vital to obtain quantitative performance characteristics. In contrast to the qualitative termination, the quantitative termination problem asks to obtain bounds on the number of steps to termination. The quantitative problem, which is more challenging than the qualitative one, is of great interest in program analysis in various domains, e.g., (a) in applications domains such as hard real-time systems, worst-case guarantees are required; and (b) the bounds are useful in early detection of egregious performance problems in large code bases [34].

Approaches for quantitative bounds. Given the importance of the quantitative termination problem significant research effort has been devoted, including important projects such as SPEED, COSTA [34, 35, 1]. Some prominent approaches are the following:

- The worst-case execution time (WCET) analysis is an active field of research on its own (with primary focus on sequential loop-free code and hardware aspects) [69].
- Advanced program-analysis techniques have also been developed for asymptotic bounds, such as resource analysis using abstract interpretation and type systems [35, 1, 45, 36, 37], e.g., linear invariant generation to obtain disjunctive and non-linear upper bounds [19], or potential-based methods [36, 37].
- Ranking functions based approach provides sound and complete approach for the qualitative termination problem, and for the quantitative problem it provides a sound approach to obtain asymptotic upper bounds [7, 9, 20, 59, 67, 21, 70, 63].

In summary, the WCET approach does not consider asymptotic bounds, while the other approaches consider asymptotic bounds, and present sound but not complete methods for upper bounds.

VASS and their modeling power. Vector Addition Systems (VASs) [50] or equivalently Petri Nets are fundamental models for analysis of parallel processes [25]. Enriching VASs with an underlying finite-state transition structure gives rise to Vector Addition Systems with States (VASS). Intuitively, a VASS consists of a finite set of control states and transitions between the control states, and and a set of d counters that hold non-negative integer values, where at every transition between the control states each counter is either incremented or decremented. VASS are a fundamental model for concurrent processes [25], and thus are often used for performing analysis of such processes [22, 30, 48, 49]. Besides that, VASS have been used as models of parametrized systems [6], as abstract models for programs for bounds and amortized analysis [66], as well as models of interactions between components of an API in component-based synthesis [27]. Thus VASS provide a rich modeling framework for a wide class of problems in program analysis.

Previous results for VASS. For a VASS, a *configuration* is a control state along with the values of counters. The termination problem for VASS can be defined as follows: (a) *counter termination* where the VASS terminates when one of the counters reaches value 0; (b) *control-state termination* where given a set of terminating control states the VASS terminates when one of the terminating states is reached. The termination question for VASS, given an initial configuration, asks whether all paths from the configuration terminate. The counter-termination problem is known to be EXPSPACE-complete: the EXPSPACE-hardness is shown in [56, 23] and the upper bound follows from [71, 5, 26].

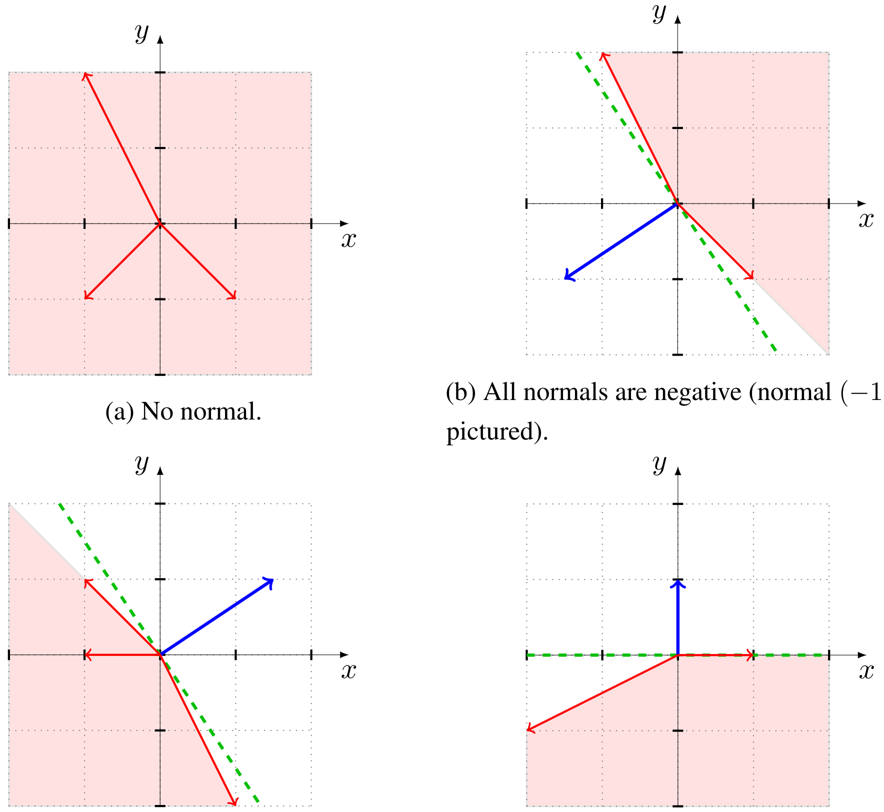
Asymptotic bounds analysis for VASS. While the qualitative termination problem has been studied extensively for VASS, the problem of quantitative bounds for the termination problem has received much less attention. In general, even for VASS whose termination can be guaranteed, the number of steps required to terminate can be non-elementary (tower of exponentials) in the magnitude of the initial configuration (i.e. in the maximal counter value appearing in the configuration). For practical purposes, bounds such as non-elementary or even exponential are too high as asymptotic complexity bounds, and the relevant complexity bounds are the polynomial ones. In this work we study the problem of computing asymptotic bounds for VASS, focusing on polynomial asymptotic bounds. Given a VASS and a configuration c, let n_c denote the maximum value of the counters in c. If for all configurations c all paths starting from c terminate, then let T_c denote the worst-case termination time from configuration c (i.e., the maximum number of steps till termination among all paths starting from c). The quantitative termination problem with *polynomial asymptotic bound* given a VASS and an integer k asks whether the asymptotic worst-case termination time is at most a polynomial of degree k, i.e., whether there exists a constant α such that for all c we have $T_c \leq \alpha \cdot n_c^k$. Note that with $k = 1$ (resp., $k = 2, 3$) the problem asks for asymptotic linear (resp., quadratic, cubic) bounds on the worst-case termination time. The asymptotic bound problem is rather different from the qualitative termination problem for VASS, and even the decidability of this problem is not obvious.

Limitations of the previous approaches for polynomial bounds for VASS. In the analysis of asymptotic bounds there are three key aspects, namely, (a) soundness, (b) completeness, and (c) precise (or tight complexity) bounds. For asymptotic bounds, previous approaches (such as ranking functions, potential-based methods etc) are sound (but not complete) for upper bounds. In other words, if the approaches obtain linear, or quadratic, or cubic bounds, then such bounds are guaranteed as asymptotic upper bounds (i.e., soundness is guaranteed), however, even if the asymptotic bound is linear or quadratic, the approaches may fail to obtain any asymptotic upper bound (i.e., completeness is not guaranteed). Another approach that has been considered for complexity analysis of programs are *lexicographic* ranking functions [4]. We show that with respect to polynomial bounds lexicographic ranking functions are not sound, i.e., there exists VASS for which lexicographic ranking function exists but the asymptotic complexity is exponential (see Example 4.11). Finally, none of the existing approaches are applicable for tight complexity bounds, i.e., the approaches consider $O(\cdot)$ bounds and are not applicable for $\Theta(\cdot)$ bounds. In summary, previous approaches do not provide sound and complete method for polynomial asymptotic complexity of VASS; and no approach provide techniques for precise complexity analysis.

Our contributions. Our main contributions are related to the complexity of the quantitative termination with polynomial asymptotic bounds for VASS and our results are applicable to counter termination.

1. We start with the important special case of linear asymptotic bounds. We present the first sound and complete algorithm that can decide linear asymptotic bounds for all VASS. Moreover, our algorithm is an efficient one that has polynomial time complexity. This contrast sharply with EXPSPACE-hardness of the qualitative termination problem and shows that deciding *fast (linear) termination*, which seems even more relevant for practical purposes, is computationally easier than deciding qualitative termination.
2. Next, we turn our attention to polynomial asymptotic bounds. For simplicity, we restrict ourselves to VASS where the underlying finite-state transition structure is strongly connected (see Section 7 for more comments). Given such a VASS $\mathcal{A}$, for every short[1] cycle C of the $\mathcal{A}$, the effect of executing the short cycle once can be represented as a d-dimensional vector, an analogue of loop summary (ignoring any nested sub-loops) for classical programs. Let Inc denote the set of all *increments*, i.e., short cycle effects in $\mathcal{A}$. We investigate the geometric properties of Inc to derive complexity bounds on $\mathcal{A}$. The property playing a key role is whether all cycle effects in Inc lie on one side of some hyperplane in $\mathbb{R}^d$. Formally, each hyperplane is uniquely determined by its normal vector $\mathbf{n}$ (a vector perpendicular to the hyperplane), and a hyperplane defined by $\mathbf{n}$ *covers* a vector effects $\mathbf{v}$ if $\mathbf{v} \cdot \mathbf{n} \leq 0$, where "$\cdot$" is the dot product of vectors. Geometrically, the hyperplane defined by $\mathbf{n}$ splits the whole d-dimensional space into two halves such that the normal $\mathbf{n}$ points into one of the halves, and its negative $-\mathbf{n}$ points into the other half. The hyperplane then "covers" vector $\mathbf{v}$ if $\mathbf{v}$ points into the same half as the vector $-\mathbf{n}$. We denote by $Normals(\mathcal{A})$ the set of all normals such that each $\mathbf{n} \in Normals(\mathcal{A})$ covers all cycle effects in $\mathcal{A}$. Depending on the properties of $Normals(\mathcal{A})$, we can distinguish the following cases:
   (A) *No normal:* if $Normals(\mathcal{A}) = \emptyset$ (Fig. 1a);
   (B) *Negative normal:* if all $\mathbf{n} \in Normals(\mathcal{A})$ have a negative component (Fig. 1b);
   (C) *Positive normal:* if there exists $\mathbf{n} \in Normals(\mathcal{A})$ whose all components are positive (Fig. 1c);

[1]A cycle C is short if its length is bounded by the number of control states of a given VASS.



(a) No normal.

(b) All normals are negative (normal $(-1.5, -1)$ pictured).

(c) Positive normal $(1.5, 1)$.

(d) Singular normal $(0, 1)$.

Figure 1: Classification of VASS into 4 sub-classes according to the geometric properties of vectors of cycle effects, pictured on 2D examples. Each figure pictures (as red arrows) vectors of simple cycle effects in some VASS (it is easy, for each figure, to construct a VASS whose simple cycle effects are exactly those pictured). The green dashed line, if present, represents the hyperplane (in 2D it is a line) covering the set of cycle effects. The thick blue arrow represents the normal defining the covering hyperplane. The pink shaded area represents the *cone* generated by cycle effects (see Section 2.3). Intuitively, we seek hyperplanes that do not intersect the interior of the cone (but can touch its boundary).

(D) *Singular normal:* if (C) does not hold, but there exists $\mathbf{n} \in \mathit{Normals}(\mathcal{A})$ such that all components of $\mathbf{n}$ are non-negative (in which some component of $\mathbf{n}$ is zero, Fig. 1d);

First, we observe that given a VASS, we can decide to which of the above category it belongs, in time which is polynomial in the number of control states of a given VASS for every fixed dimension (i.e., the algorithm is exponential only in the dimension d; see Section 2.2 for more comments). Second, we also show that if a VASS belongs to one of the first two categories, then there exist configurations with non-terminating runs from them (see Theorem 4.2). Hence asymptotic bounds are not applicable for the first two categories and we focus on the last two categories for polynomial asymptotic bounds.

3. For the positive normal category (C) we show that either there exist non-terminating runs or else the worst-case termination time is of the form $\Theta(n^k)$, where k is an integer and

$k \le d$. We show that given a VASS in this category, we can first decide whether all runs are terminating, and if yes, then we can compute the optimal asymptotic polynomial degree k such that the worst-case termination time is $\Theta(n^k)$ (see Theorem 4.8). Again, this is achievable in time polynomial in the number of control states of a given VASS for every fixed dimension. In other words, for this class of VASS we present an efficient approach that is sound, complete, and obtains precise polynomial complexity bounds. To the best of our knowledge, no previous work presents a complete approach for asymptotic complexity bounds for VASS, and the existing techniques only consider $O(\cdot)$ bounds, and not precise $\Theta(\cdot)$ bounds.

4. We show that singularities in the normal are the key reason for complex asymptotic bounds in VASS. More precisely, for VASS falling into the singular normal category (D), in general the asymptotic bounds are not polynomial, and we show that (a) by slightly adapting the results of [58], it follows that termination complexity of a VASS $\mathcal{A}$ in category (D) cannot be bounded by any primitive recursive function in the size of $\mathcal{A}$; (b) even with three dimensions, the asymptotic bound is exponential in general (see Example 4.9), (c) even with four dimensions, the asymptotic bound is non-elementary in general (see Example 4.10).

The main *technical contribution* of this paper is a novel geometric approach, based on hyperplane separation techniques, for asymptotic time complexity analysis of VASS. Our methods are sound for arbitrary VASS and complete for a non-trivial subclass.

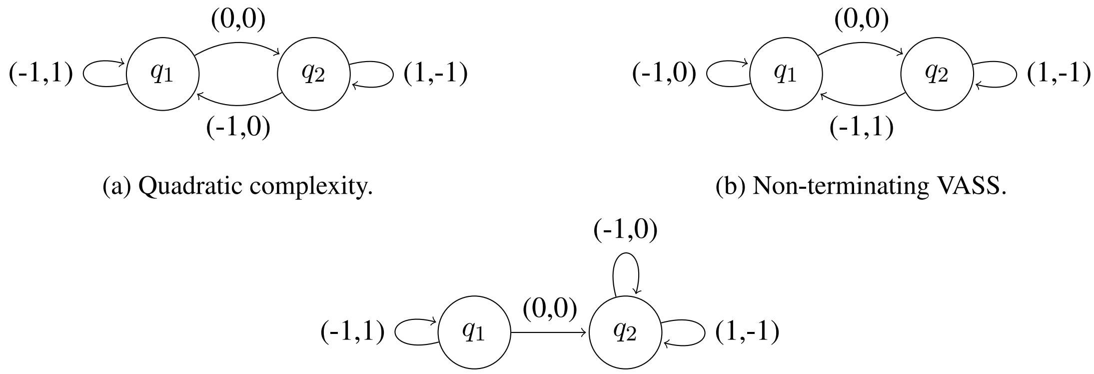
2 Preliminaries

2.1 Basic Notation

We use $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ to denote the sets of non-negative integers, rational numbers, and real numbers. The subsets of all *positive* elements of $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ are denoted by $\mathbb{N}^+$, $\mathbb{Q}^+$, and $\mathbb{R}^+$. Further, we use $\mathbb{N}_\infty$ to denote the set $\mathbb{N} \cup \{\infty\}$ where ∞ is treated according to the standard conventions. The cardinality of a given set M is denoted by $|M|$. When no confusion arises, we also use $|c|$ to denote the absolute value of a given $c \in \mathbb{R}$.

Given a function $f : \mathbb{N} \to \mathbb{N}$, we use $\mathcal{O}(f(n))$ and $\Omega(f(n))$ to denote the sets of all $g : \mathbb{N} \to \mathbb{N}$ such that $g(n) \le a \cdot f(n)$ and $g(n) \ge b \cdot f(n)$ for all sufficiently large $n \in \mathbb{N}$, where $a, b \in \mathbb{R}^+$ are some constants. If $h(n) \in \mathcal{O}(f(n))$ and $h(n) \in \Omega(f(n))$, we write $h(n) \in \Theta(f(n))$.

Let $d \ge 1$. The elements of $\mathbb{R}^d$ are denoted by bold letters such as $\mathbf{u}, \mathbf{v}, \mathbf{z}, \ldots$. The i-th component of $\mathbf{v}$ is denoted by $\mathbf{v}(i)$, i.e., $\mathbf{v} = (\mathbf{v}(1), \ldots, \mathbf{v}(d))$. For every $n \in \mathbb{N}$, we use $\vec{n}$ to denote the constant vector where all components are equal to n. The scalar product of $\mathbf{v}, \mathbf{u} \in \mathbb{R}^d$ is denoted by $\mathbf{v} \cdot \mathbf{u}$, i.e., $\mathbf{v} \cdot \mathbf{u} = \sum_{i=1}^{d} \mathbf{v}(i) \cdot \mathbf{u}(i)$. The other standard operations and relations on $\mathbb{R}$ such as $+$, $\le$, or $<$ are extended to $\mathbb{R}^d$ in the component-wise way. In particular, $\mathbf{v}$ is *positive* if $\mathbf{v} > \vec{0}$, i.e., all components of $\mathbf{v}$ are positive. The norm of $\mathbf{v}$ is defined by $norm(\mathbf{v}) = \sqrt{\mathbf{v}(1)^2 + \cdots + \mathbf{v}(d)^2}$.



(a) Quadratic complexity.

(b) Non-terminating VASS.

(c) Linear complexity.

Figure 2: An example of 2-dimensional VASS of varying complexity.

Half-spaces and Cones. An *open half-space* of $\mathbb{R}^d$ determined by a normal vector $\mathbf{n} \in \mathbb{R}^d$, where $\mathbf{n} \neq \vec{0}$, is the set $\mathcal{H}_{\mathbf{n}}$ of all $\mathbf{x} \in \mathbb{R}^d$ such that $\mathbf{x} \cdot \mathbf{n} < 0$. A *closed half-space* $\hat{\mathcal{H}}_{\mathbf{n}}$ is defined in the same way but the above inequality is non-strict. Given a finite set of vectors $U \subseteq \mathbb{R}^d$, we use $cone(U)$ to denote the set of all vectors of the form $\sum_{\mathbf{u} \in U} c_{\mathbf{u}} \mathbf{u}$, where $c_{\mathbf{u}}$ is a non-negative real constant for every $\mathbf{u} \in U$.

Example 2.1. *In Fig. 1, the cone, or more precisely its part that intersects the displayed area of $\mathbb{R}^2$, generated by the cycle effects (i.e., by the "red" vectors) is the pink-shaded area. As for the half spaces, e.g., in Fig. 1d, the closed half-space defined by the normal vector $(0,1)$ is the set $\{(x,y) \mid y \leq 0\}$, while the open half-space determined by the same normal is the set $\{(x,y) \mid y < 0\}$. Intuitively, each normal vector $\mathbf{n}$ determines a hyperplane (pictured by dashed lines in Fig. 1) that cuts $\mathbb{R}^d$ in two halves, and $\mathcal{H}_{\mathbf{n}}$ is the half which does not contain $\mathbf{n}$: depending on whether we are interested in closed or open half-space, we include the separating hyperplane into $\mathcal{H}_{\mathbf{n}}$ or not, respectively.*

2.2 Syntax and semantics of VASS

In this subsection we present a syntax of VASS, represented as finite state graphs with transitions labelled by vectors of counter changes.

Definition 2.2. *Let $d \in \mathbb{N}^+$. A d-*dimensional vector addition system with states (VASS) *is a pair $\mathcal{A} = (Q, T)$, where $Q \neq \emptyset$ is a finite set of* states *and $T \subseteq Q \times \{-1,0,1\}^d \times Q$ is a set of* transitions.

Example 2.3. *Fig. 2 shows examples of three small 2-dimensional VASS. The VASS in Fig. 2a has two states q_1, q_2 and four transitions $(q_1,(-1,1),q_2)$, $(q_1,(0,0),q_2)$, $(q_2,(-1,0),q_1)$, $(q_2,(1,-1),q_2)$.*

In some cases, we design algorithms where the time complexity is not polynomial in $\|\mathcal{A}\|$ (i.e., the size of $\mathcal{A}$), but polynomial in $|Q|$ and exponential just in d. Then, we say that the running time is polynomial in $|Q|$ for a fixed d.

We use simple operational semantics for VASS based on the view of VASS as finite-state machines augmented with non-negative integer-valued counters.

A *configuration* of $\mathcal{A}$ is a pair $p\mathbf{v}$, where $p \in Q$ and $\mathbf{v} \in \mathbb{N}^d$. The set of all configurations of $\mathcal{A}$ is denoted by $C(\mathcal{A})$. The *size* of $p\mathbf{v} \in C(\mathcal{A})$ is defined as $\|p\mathbf{v}\| = \max\{\mathbf{v}(i) \mid 1 \leq i \leq d\}$.

A *finite path* in $\mathcal{A}$ of length n is a finite sequence π of the form $p_0, \mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \ldots, \mathbf{u}_n, p_n$ where $n \geq 1$ and $(p_i, \mathbf{u}_{i+1}, p_{i+1}) \in T$ for all $0 \leq i < n$. If $p_0 = p_n$, then π is a *cycle*. A cycle is *short* if $n \leq |Q|$. The *effect* of π, denoted by $\mathit{eff}(\pi)$, is the sum $\mathbf{u}_1 + \cdots + \mathbf{u}_n$. Given two finite paths $\alpha = p_0, \mathbf{u}_1, \ldots, p_n$ and $\beta = q_0, \mathbf{v}_1, \ldots, q_m$ such that $p_n = q_0$, we use $\alpha \odot \beta$ to denote the finite path $p_0, \mathbf{u}_1, \ldots, p_n, \mathbf{v}_1, \ldots, q_m$.

Let π be a finite path in $\mathcal{A}$. A *decomposition of* π *into short*[2] *cycles*, denoted by $\mathit{Decomp}(\pi)$, is a finite list of short cycles (repetitions allowed) defined recursively as follows:

- If π does not contain any short cycle, then $\mathit{Decomp}(\pi) = []$, where $[]$ is the empty list.
- If $\pi = \alpha \odot \gamma \odot \beta$ where γ is the first short cycle occurring in π, then $\mathit{Decomp}(\pi) = \mathtt{Concat}([\gamma], \mathit{Decomp}(\alpha \odot \beta))$, where `Concat` is the list concatenation operator.

Observe that if $\mathit{Decomp}(\pi) = []$, then the length of π is at most $|Q| - 1$. Since the length of every short cycle is bounded by $|Q|$, the length of π is asymptotically the *same* as the number of elements in $\mathit{Decomp}(\pi)$, assuming a fixed VASS $\mathcal{A}$.

Given a path $\pi = p_0, \mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \ldots, \mathbf{u}_n, p_n$ and an initial configuration $p_0\mathbf{v}_0$, the *execution* of π in $p_0\mathbf{v}_0$ is a finite sequence $p_0\mathbf{v}_0, \ldots, p_n\mathbf{v}_n$ of configurations where $\mathbf{v}_i = \mathbf{v}_0 + \mathbf{u}_1 + \cdots + \mathbf{u}_i$ for all $0 \leq i \leq n$. If $\mathbf{v}_i \geq \vec{0}$ for all $0 \leq i \leq n$, we say that π is *executable* in $p_0\mathbf{v}_0$.

2.3 Termination Complexity of VASS

A *zero-avoiding computation* of length n initiated in a configuration $p\mathbf{v}$ is a finite sequence of configurations $\alpha = q_0\mathbf{z}_0, \ldots, q_n\mathbf{z}_n$ initiated in $p\mathbf{v}$ such that $\mathbf{z}_i > \vec{0}$ for all $0 \leq i \leq n$, and for each $0 \leq i < n$ there is a transition $(q_i, \mathbf{u}, q_{i+1}) \in T$ where $\mathbf{z}_{i+1} = \mathbf{z}_i + \mathbf{u}$. Every zero-avoiding computation α initiated in $q_0\mathbf{z}_0$ determines a unique finite path π_α in $\mathcal{A}$ such that α is the execution of π_α in $q_0\mathbf{z}_0$.

Definition 2.4. *Let $\mathcal{A} = (Q, T)$ be a d-dimensional VASS. For every configuration $p\mathbf{v}$ of $\mathcal{A}$, let $L(p\mathbf{v})$ be the least $\ell \in \mathbb{N}_\infty$ such that the length of every zero-avoiding finite computation initiated in $p\mathbf{v}$ is bounded by ℓ. The* termination complexity *of $\mathcal{A}$ is a function $\mathcal{L} : \mathbb{N} \to \mathbb{N}$ defined by*

$$\mathcal{L}(n) = \max\{L(p\mathbf{v}) \mid p\mathbf{v} \in C(\mathcal{A}) \textit{ where } \|p\mathbf{v}\| = n\}.$$

If $\mathcal{L}(n) = \infty$ for some $n \in \mathbb{N}$, we say that $\mathcal{A}$ is non-terminating*, otherwise it is* terminating.

[2] A standard technique for analysing paths in VASS are decompositions into *simple* cycles, where all states except for p_0 and p_n are pairwise different. The reason why we use short cycles instead of simple ones is clarified in Lemma 2.5.

Observe that if $\mathcal{A}$ is non-terminating, then $\mathcal{L}(n) = \infty$ for all sufficiently large $n \in \mathbb{N}$. Further, if $\mathcal{A}$ is terminating, then $\mathcal{L}(n) \in \Omega(n)$. In particular, if $\mathcal{L}(n) \in \mathcal{O}(n)$, we also have $\mathcal{L}(n) \in \Theta(n)$.

Given a path $\pi = p_0, \mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \ldots, \mathbf{u}_n, p_n$ and an initial configuration $p_0\mathbf{v}_0$, the *execution* of π in $p_0\mathbf{v}_0$ is a finite sequence $p_0\mathbf{v}_0, \ldots, p_n\mathbf{v}_n$ where $\mathbf{v}_i = \mathbf{v}_0 + \mathbf{u}_1 + \cdots + \mathbf{u}_i$ for all $0 \leq i \leq n$. If $\mathbf{v}_i \geq \vec{0}$ for all $0 \leq i \leq n$, we say that π is *executable* in $p_0\mathbf{v}_0$.

Let $\mathit{Inc} = \{\mathit{eff}(\pi) \mid \pi \text{ is a short cycle of } \mathcal{A}\}$. The elements of Inc are called *increments*. Note that if $\mathbf{u} \in \mathit{Inc}$, then $-|Q| \leq \mathbf{u}(i) \leq |Q|$ for all $1 \leq i \leq d$. Hence, $|\mathit{Inc}|$ is polynomial in $|Q|$, assuming d is a fixed constant. Although the total number of all short cycles can be exponential in $|Q|$, the set Inc is computable efficiently.[3]

Lemma 2.5. *Let $\mathcal{A} = (Q, T)$ be a d-dimensional VASS, and let $p \in Q$. The set Inc is computable in time $\mathcal{O}(\|\mathcal{A}\|^d)$, i.e., polynomial in $|Q|$ assuming d is a fixed constant.*

Proof. The set Inc is computable by the following standard algorithm: For all $q, q' \in Q$ and $1 \leq k \leq n$, let $E^k_{q,q'}$ be the set of all effects of paths from q to q' of length exactly k. Observe that

- $E^1_{q,q'} = \{\mathbf{u} \mid (q, \mathbf{u}, q') \in T\}$ for all $q, q' \in Q$;
- for every $1 < k \leq |Q|$, we have that $E^k_{q,q'} = \bigcup_{q'' \in Q}\{\mathbf{v}+\mathbf{u} \mid \mathbf{v} \in E^{k-1}_{q,q''} \text{ and } (q'', \mathbf{u}, q') \in T\}$.

Obviously, $\mathit{Inc} = \bigcup_{q \in Q} \bigcup_{k=1}^{n} E^k_{q,q}$, and the sets $E^k_{q,q'}$ for $k \leq |Q|$ are computable in time polynomial in $|Q|$, assuming d is a fixed constant. □

A *strongly connected component (SCC)* of $\mathcal{A}$ is maximal $R \subseteq Q$ such that for all $p, q \in R$ where $p \neq q$ there is a finite path from p to q. Given a SCC R of Q, we define the VASS $\mathcal{A}_R$ by restricting the set of control states to R and the set of transitions to $T \cap (R \times \{-1, 0, 1\}^d \times R)$. We say that $\mathcal{A}$ is *strongly connected* if Q is a SCC of $\mathcal{A}$.

3 Linear Termination Time

In this section, we give a complete and effective characterization of all VASS with *linear* termination complexity.

More precisely, we first provide a precise mathematical characterization of VASS with linear complexity: we show that if $\mathcal{A}$ is a d-dimensional VASS, then $\mathcal{L}(n) \in \mathcal{O}(n)$ iff there is an open half-space $\mathcal{H}_\mathbf{n}$ of $\mathbb{R}^d$ such that $\mathbf{n} > \vec{0}$ and $\mathit{Inc} \subseteq \mathcal{H}$.

Next we show that the mathematical characterization of VASS of linear complexity is equivalent to the existence of a *ranking function* of a special form for this VASS. We also show that existence of such a function for a given VASS $\mathcal{A}$ can be decided (and the function, if it exists, synthesized)

[3] Note that Lemma 2.5 would *not* hold if we used simple cycles instead of short cycles, because the problem whether a given vector $\mathbf{v}$ is an effect of a simple cycle is NP-complete, even if $d = 1$ (NP-hardness follows, e.g., by a straightforward reduction of the Hamiltonian path problem).

in time polynomial in the size of $\mathcal{A}$. Hence, we obtain a sound and complete polynomial-time procedure for deciding whether a given VASS has linear termination complexity.

We start with the mathematical characterization. Due to the next lemma, we can safely restrict ourselves to strongly connected VASS. A proof is trivial.

Lemma 3.1. *Let $d \in \mathbb{N}$, and let $\mathcal{A} = (Q, T)$ be a d-dimensional VASS. Then $\mathcal{L}(n) \in \mathcal{O}(n)$ iff $\mathcal{L}_R(n) \in \mathcal{O}(n)$ for every SCC R of Q, where $\mathcal{L}_R(n)$ is the termination complexity of $\mathcal{A}_R$.*

Now we show that if there is *no* open half-space $\mathcal{H}_{\mathbf{n}}$ such that $\mathbf{n} > \vec{0}$ and $Inc \subseteq \mathcal{H}_{\mathbf{n}}$, then there exist short cycles $\gamma_1, \ldots, \gamma_k$ and coefficients $b_1, \ldots, b_k \in \mathbb{N}^+$ such that the sum $\sum_{i=1}^{k} b_i \cdot \mathit{eff}(\gamma_i)$ is non-negative. Note that this does *not* yet mean that $\mathcal{A}$ is non-terminating—it may happen that the cycles $\pi_1, \ldots, \pi_k$ pass through disjoint subsets of control states and cannot be concatenated without including auxiliary finite paths decreasing the counters.

Lemma 3.2. *Let $\mathcal{A} = (Q, T)$ be a d-dimensional VASS. If there is no open half-space $\mathcal{H}_{\mathbf{n}}$ of $\mathbb{R}^d$ such that $\mathbf{n} > \vec{0}$ and $Inc \subseteq \mathcal{H}_{\mathbf{n}}$, then there exist $\mathbf{v}_1, \ldots, \mathbf{v}_k \in Inc$ and $b_1, \ldots, b_k \in \mathbb{N}^+$ such that $k \geq 1$ and $\sum_{i=1}^{k} b_i \mathbf{v}_i \geq \vec{0}$.*

Proof. We distinguish two possibilities.

(a) There exists a closed half-space $\hat{\mathcal{H}}_{\mathbf{n}}$ of $\mathbb{R}^d$ such that $\mathbf{n} > \vec{0}$ and $Inc \subseteq \hat{\mathcal{H}}_{\mathbf{n}}$.

(b) There is no closed half-space $\hat{\mathcal{H}}_{\mathbf{n}}$ of $\mathbb{R}^d$ such that $\mathbf{n} > \vec{0}$ and $Inc \subseteq \hat{\mathcal{H}}_{\mathbf{n}}$.

Case (a). We show that there exists $\mathbf{u} \in Inc$ such that $\mathbf{u} \neq \vec{0}$ and $-\mathbf{u} \in \mathit{cone}(Inc)$. Note that this immediately implies the claim of our lemma—since $-\mathbf{u} \in \mathit{cone}(Inc)$, there are $\mathbf{v}_1, \ldots, \mathbf{v}_k \in Inc$ and $c_1, \ldots, c_k \in \mathbb{R}^+$ such that $-\mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{v}_i$. Since all elements of Inc are vectors of non-negative integers, we can safely assume $c_i \in \mathbb{Q}^+$ for all $1 \leq i \leq k$. Let b be the least common multiple of $c_1, \ldots, c_k$. Then $b\mathbf{u} + (b \cdot c_1)\mathbf{v}_1 + \cdots + (b \cdot c_k)\mathbf{v}_k = \vec{0}$ and we are done.

It remains to prove the existence of $\mathbf{u}$. Let us fix a normal vector $\mathbf{n} > \vec{0}$ such that $Inc \subseteq \hat{\mathcal{H}}_{\mathbf{n}}$ and the set $Inc_{\mathbf{n}} = \{\mathbf{v} \in Inc \mid \mathbf{v} \cdot \mathbf{n} < 0\}$ is *maximal* (i.e., there is no $\mathbf{n}' > \vec{0}$ satisfying $Inc \subseteq \hat{\mathcal{H}}_{\mathbf{n}'}$ and $Inc_{\mathbf{n}} \subset Inc_{\mathbf{n}'}$). Further, we fix $\mathbf{u} \in Inc$ such that $\mathbf{u} \cdot \mathbf{n} = 0$. Note that such $\mathbf{u} \in Inc$ must exist, because otherwise $Inc_{\mathbf{n}} = Inc$ which contradicts the assumption of our lemma. We show $-\mathbf{u} \in \mathit{cone}(Inc)$. Suppose the converse. Then by Farkas' lemma there exists a separating hyperplane for $\mathit{cone}(Inc)$ and $-\mathbf{u}$ with normal vector $\mathbf{n}'$, i.e., $\mathbf{v} \cdot \mathbf{n}' \leq 0$ for all $\mathbf{v} \in Inc$ and $-\mathbf{u} \cdot \mathbf{n}' > 0$. Since $\mathbf{n} > \vec{0}$, we can fix a sufficiently small $\varepsilon > 0$ such that the following conditions are satisfied:

- $\mathbf{n} + \varepsilon \mathbf{n}' > \vec{0}$,
- for all $\mathbf{v} \in Inc$ such that $\mathbf{v} \cdot \mathbf{n} < 0$ we have that $\mathbf{v} \cdot (\mathbf{n} + \varepsilon \mathbf{n}') < \vec{0}$.

Let $\mathbf{w} = \mathbf{n} + \varepsilon \mathbf{n}'$. Then $\mathbf{w} > 0$, $\mathbf{v} \cdot \mathbf{w} < 0$ for all $\mathbf{v} \in Inc_{\mathbf{n}}$, and $\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{n} + \varepsilon(\mathbf{u} \cdot \mathbf{n}') = \varepsilon(\mathbf{u} \cdot \mathbf{n}') < 0$. This contradicts the maximality of $Inc_{\mathbf{n}}$.

Case (b). Let $B = \{\mathbf{v} \in \mathbb{R}^d \mid \mathbf{v} \geq \vec{0} \text{ and } 1 \leq \sum_{i=1}^{d} \mathbf{v}(i) \leq 2\}$. We prove $cone(Inc) \cap B \neq \emptyset$, which implies the claim of our lemma (there are $\mathbf{v}_1, \ldots, \mathbf{v}_k \in Inc$ and $c_1, \ldots, c_k \in \mathbb{Q}^+$ such that $\sum_{i=1}^{k} c_i \mathbf{v}_i \in B$). Suppose the converse, i.e., $cone(Inc) \cap B = \emptyset$. Since both $cone(Inc)$ and B are closed and convex and B is also compact, we can apply the "strict" variant of hyperplane separation theorem. Thus, we obtain a vector $\mathbf{n} \in \mathbb{R}^d$ and a constant $c \in \mathbb{R}$ such that $\mathbf{x} \cdot \mathbf{n} < c$ and $\mathbf{y} \cdot \mathbf{n} > c$ for all $\mathbf{x} \in cone(Inc)$ and $\mathbf{y} \in B$. Since $\vec{0} \in cone(Inc)$, we have that $c > 0$. Further, $\mathbf{n} \geq \vec{0}$ (to see this, realize that if $\mathbf{n}(i) < 0$ for some $1 \leq i \leq d$, then $\mathbf{y} \cdot \mathbf{n} < 0$ where $\mathbf{y}(i) = 1$ and $\mathbf{y}(j) = 0$ for all $j \neq i$; since $\mathbf{y} \in B$ and $c > 0$, we have a contradiction). Now we show $\mathbf{x} \cdot \mathbf{n} \leq 0$ for all $\mathbf{x} \in cone(Inc)$, which contradicts the assumption of Case (b). Suppose $\mathbf{x} \cdot \mathbf{n} > 0$ for some $\mathbf{x} \in cone(Inc)$. Then $(m \cdot \mathbf{x}) \cdot \mathbf{n} > c$ for a sufficiently large $m \in \mathbb{N}$. Since $m \cdot \mathbf{x} \in cone(Inc)$, we have a contradiction. □

Now we give the promised characterization of all VASS with linear termination complexity. Our theorem also reveals that the VASS termination complexity is either linear or at least quadratic (for example, it cannot be that $\mathcal{L}(n) \in \Theta(n \log n)$).

Theorem 3.3. *Let $\mathcal{A} = (Q, T)$ be a d-dimensional VASS. We have the following:*

(a) If there is an open half-space $\mathcal{H}_\mathbf{n}$ of $\mathbb{R}^d$ such that $\mathbf{n} > \vec{0}$ and $Inc \subseteq \mathcal{H}_\mathbf{n}$, then $\mathcal{L}(n) \in \mathcal{O}(n)$.

(b) If there is no open half-space $\mathcal{H}_\mathbf{n}$ of $\mathbb{R}^d$ such that $\mathbf{n} > \vec{0}$ and $Inc \subseteq \mathcal{H}_\mathbf{n}$, then $\mathcal{L}(n) \in \Omega(n^2)$.

Proof. We start with (a). Let $\mathcal{H}_\mathbf{n}$ be an open half-space of $\mathbb{R}^d$ such that $\mathbf{n} > \vec{0}$ and $Inc \subseteq \mathcal{H}_\mathbf{n}$, and let $q\mathbf{u}$ be a configuration of $\mathcal{A}$. Note that $\lceil \mathbf{n} \cdot \mathbf{u} \rceil \in \mathcal{O}(\|q\mathbf{u}\|)$ because $\mathbf{n}$ does not depend on $q\mathbf{u}$. Let $\delta = \min_{\mathbf{v} \in Inc} |\mathbf{v} \cdot \mathbf{n}|$. Each short cycle decreases the scalar product of the normal $\mathbf{n}$ and vector of counters by at least δ. Therefore, for every zero-avoiding computation α initiated in $q\mathbf{u}$ we have that $Decomp(\alpha)$ contains at most $\mathcal{O}(\|q\mathbf{u}\|)$ elements, so the length of α is $\mathcal{O}(\|q\mathbf{u}\|)$.

Now suppose there is no open half-space $\mathcal{H}_\mathbf{n}$ of $\mathbb{R}^d$ such that $\mathbf{n} > \vec{0}$ and $Inc \subseteq \mathcal{H}_\mathbf{n}$. We show that $\mathcal{L}(n) \in \Omega(n^2)$, i.e., there exist $p \in Q$ and a constant $a \in \mathbb{R}^+$ such that for all configurations $p\vec{n}$, where $n \in \mathbb{N}$ is sufficiently large, there is a zero-avoiding computation initiated in $p\vec{n}$ whose length is at least $a \cdot n^2$. Due to Lemma 3.1, we can safely assume that $\mathcal{A}$ is strongly connected. By Lemma 3.3, there are $\mathbf{v}_1, \ldots, \mathbf{v}_k \in Inc$ and $b_1, \ldots, b_k \in \mathbb{N}^+$ such that $k \geq 1$ and

$$\sum_{i=1}^{k} b_i \mathbf{v}_i \geq \vec{0}. \tag{1}$$

As the individual short cycles with effects $\mathbf{v}_1, \ldots, \mathbf{v}_k$ may proceed through disjoint sets of states, they *cannot* be trivially concatenated into one large cycle with non-negative effect. Instead, we fix a control state $p \in Q$ and a cycle π initiated in p visiting *all* states of Q (here we need that $\mathcal{A}$ is strongly connected). Further, for every $1 \leq i \leq k$ we fix a short cycle γ_i such that $\mathit{eff}(\gamma_i) = \mathbf{v}_i$. For every $t \in \mathbb{N}$, let π_t be a cycle obtained from π by inserting precisely $t \cdot b_i$ copies of every γ_i, where $1 \leq i \leq k$. Observe that the inequality (1) implies

$$\mathit{eff}(\pi_t) = \mathit{eff}(\pi) + t \cdot \sum_{i=1}^{k} b_i \mathbf{v}_i \geq \mathit{eff}(\pi) \quad \text{for every } t \in \mathbb{N}. \tag{2}$$

For every configuration $p\mathbf{u}$, let $t(\mathbf{u})$ be the largest $t \in \mathbb{N}$ such that π_t is executable in $p\mathbf{u}$ and results in a zero-avoiding computation. If such a $t(\mathbf{u})$ does not exist, i.e. π_t is executable in $p\mathbf{u}$ for all $t \in \mathbb{N}$, then $\mathcal{A}$ is non-terminating (since, e.g. $\mathbf{v}_1$ must be non-negative in such a case), and the proof is finished. Hence, we can assume that $t(\mathbf{u})$ is well-defined for each $\mathbf{u}$. Since the cycles π and $\gamma_1, \ldots, \gamma_k$ have fixed effects, there is $b \in \mathbb{R}^+$ such that for all configurations $p\mathbf{u}$ where all components of $\mathbf{u}$ (and thus also $\|p\mathbf{u}\|$) are above some sufficiently large threshold ξ we have that $t(\mathbf{u}) \geq b \cdot \|p\mathbf{u}\|$, i.e. $t(\mathbf{u})$ grows asymptotically at least linearly with the minimal component of $\mathbf{u}$. Now, for every $n \in \mathbb{N}$, consider a zero-avoiding computation $\alpha(n)$ initiated in $p\vec{n}$ defined inductively as follows: Initially, $\alpha(n)$ consists just of $p\mathbf{u}_0 = p\vec{n}$; if the prefix of $\alpha(n)$ constructed so far ends in a configuration $p\mathbf{u}_i$ such that $t(\mathbf{u}_i) \geq 1$ and $\mathbf{u}_i \geq \vec{\xi}$ (an event we call a *successful hit*), then the prefix is prolonged by executing the cycle $\pi_{t(\mathbf{u}_i)}$ (otherwise, the construction of $\alpha(n)$ stops). Thus, $\alpha(n)$ is obtained from $p\vec{n}$ by applying the inductive rule $I(n)$ times, where $I(n) \in \mathbb{N}_\infty$ is the number of successful hits before the construction of $\alpha(n)$ stops. Denote by $p\mathbf{u}_i$ the configuration visited by $\alpha(n)$ at i-th successful hit. Now the inequality (2) implies that $\mathbf{u}_i \geq \vec{n} + i \cdot \mathit{eff}(\pi)$, so there exists a constant e such that $\|p\mathbf{u}_i\| \geq n - i \cdot e$. In particular the decrease of all components of $\mathbf{u}_i$ is at most linear in i. This means that $I(n) \geq c \cdot n$ for all sufficiently large $n \in \mathbb{N}$, where $c \in \mathbb{R}^+$ is a suitable constant. But at the same time, upon each successful hit we have $\mathbf{u}_i \geq \vec{\xi}$, so length of the segment beginning with i-th successful hit and ending with the $(i+1)$-th hit or with the last configuration of $\alpha(n)$ is at least $b \cdot \|p\mathbf{u}_i\| \geq b \cdot (n - i \cdot e)$. Hence, the length of $\alpha(n)$ is at least $\sum_{i=1}^{c \cdot n} b \cdot (n - i \cdot e)$, i.e. quadratic. $\square$

Example 3.4. *Consider the VASS in Figure 2c. It consists of two strongly connected components,* $\{q_1\}$ *and* $\{q_2\}$*. In* $\mathcal{A}_{\{q_1\}}$ *we have Inc* $= \{(-1,1)\}$*. For* $\mathbf{n} = (1, \frac{1}{2})$ *the open half-space* $\hat{\mathcal{H}}_\mathbf{n}$ *contains Inc. Similarly, in* $\mathcal{A}_{\{q_2\}}$ *we have Inc* $= \{(-1,0),(1,-1)\}$*. For* $\mathbf{n} = (1,2)$ *we again have that Inc is contained in open half-space* $\hat{\mathcal{H}}_\mathbf{n}$*. Hence, the VASS has linear termination complexity.*

Now consider the VASS in Figure 2a. It is strongly connected and Inc $= \{(-1,1),(-2,2),(1,-1),(2,-2),(-1,0)\}$*. But there cannot be an open 2-dimensional half-space (i.e. an open half-plane) containing two opposite vectors, e.g.* $(-1,1)$ *and* $(1,-1)$*, because for any line going through the origin such that* $(-1,1)$ *does not lie on the line it holds that* $(1,-1)$ *lies on the "other side" of the line than* $(-1,1)$*. Hence, the VASS in Figure 2a has at least quadratic termination complexity. The same argument applies to VASS in Figure 2b.*

A straightforward way of checking the condition of Theorem 3.3 is to construct the corresponding linear constraints and check their feasibility by linear programming. This would yield an algorithm polynomial in $|\mathit{Inc}|$, i.e., polynomial in $|Q|$ for every fixed dimension d. Now we show that the condition can actually be checked in time *polynomial in the size of* $\mathcal{A}$. We do this by showing that the mathematical condition stated in Theorem 3.3 is equivalent to the existence of a ranking function of a special type for a given VASS. Formally, a *weighted linear map* for a VASS $\mathcal{A} = (Q,T)$ is defined by a vector of coefficients $\mathbf{c}$ and by a set of weights $\{h_q \mid q \in Q\}$, one constant for each state of $\mathcal{A}$. The weighted linear map $\mu = (\mathbf{c}, \{h_q \mid q \in Q\})$ defines a function (which we, slightly abusing the notation, also denote by μ) assigning numbers to configurations as follows:

$\mu(p\mathbf{v}) = \mathbf{c} \cdot \mathbf{v} + h_p$. A weighted linear map μ is a *weighted linear ranking (WLR) function* for $\mathcal{A}$ if $\mathbf{c} \geq \vec{0}$ and there exists $\epsilon > 0$ such that for each configuration $p\mathbf{v}$ and each transition $(p, \mathbf{u}, q)$ it holds $\mu(p\mathbf{v}) \geq \mu(q(\mathbf{v} + \mathbf{u})) + \epsilon$, which is equivalent, due to linearity, to

$$h_p - h_q \geq \mathbf{c} \cdot \mathbf{u} + \epsilon \tag{3}$$

We show that weighted linear ranking functions provide a sound and complete method for proving linear termination complexity of VASS.

Theorem 3.5. *Let $d \in \mathbb{N}$. The problem whether the termination complexity of a given d-dimensional VASS is linear is solvable in time polynomial in the size of $\mathcal{A}$. More precisely, the termination complexity of a VASS $\mathcal{A}$ is linear if and only if there exists a weighted linear ranking function for $\mathcal{A}$. Moreover, the existence of a weighted linear ranking function for $\mathcal{A}$ can be decided in time polynomial in $\|\mathcal{A}\|$.*

Proof Sketch. In the course of the proof we describe a polynomial time-algorithm for deciding whether given VASS has linear termination complexity. Once the algorithm is described, we will show that what it really does is checking the existence of a weighted linear ranking function for $\mathcal{A}$.

Let us start by sketching the underlying intuition. Our goal is to decide, in polynomial time, whether there is an open half-space $\mathcal{H}_\mathbf{n}$ of $\mathbb{R}^d$ such that $\mathbf{n} > \vec{0}$ and $Inc \subseteq \mathcal{H}_\mathbf{n}$. Note that this is equivalent to deciding whether there is an open half-space $\mathcal{H}_\mathbf{n}$ of $\mathbb{R}^d$ such that $\mathbf{n} \geq \vec{0}$ and $Inc \subseteq \mathcal{H}_\mathbf{n}$ (since we demand $\mathcal{H}_\mathbf{n}$ to be open and the scalar product is continuous, $\mathbf{n} \geq \vec{0}$ can be slightly tilted by adding a small $\vec{\delta} > 0$ to obtain a positive vector with the desired property).

Given a vector $\mathbf{n} \in \mathbb{R}^d$ and a configuration $q\mathbf{v}$, we say that $\mathbf{v} \cdot \mathbf{n}$ is the $\mathbf{n}$-*value of* $q\mathbf{v}$. Observe that if there is an open half-space $\mathcal{H}_\mathbf{n}$ such that $\mathbf{n} \geq \vec{0}$ and $Inc \subseteq \mathcal{H}_\mathbf{n}$, then there is $\varepsilon > 0$ such that the effect of every short cycle decreases the $\mathbf{n}$-value of a configuration by at least ε. As every path can be decomposed into short cycles, every path steadily decreases the $\mathbf{n}$-value of visited configurations. It follows that the mean change (per transition) of the $\mathbf{n}$-value along an infinite path is bounded from above by $-\varepsilon/|Q|$. On the other hand, if the maximum mean change in $\mathbf{n}$-values (over all infinite paths) is bounded from above by some negative constant, then every short cycle must decrease the $\mathbf{n}$-value by at least this constant. So, it suffices to decide whether there is $\mathbf{n} \geq \vec{0}$ such that for all infinite paths the mean change of the $\mathbf{n}$-value is negative. Thus, we reduce our problem to the classical problem of maximizing the mean payoff over a decision process with rewards. Using standard results (see, e.g., [60]), the latter problem polynomially reduces to the problem of solving a linear program that is (essentially) equivalent to the inequality (3). Finally, the linear program can be solved in polynomial time using e.g. [51]. $\square$

Remark 3.6. *The weighted linear ranking functions can be seen as a special case of well-known linear ranking functions for linear-arithmetic programs [20, 59], in particular state-based linear ranking functions, where a linear function of program variables is assigned to each state of the control flow graph. WLR ranking functions are indeed a special case, since the linear functions assigned to various state are almost identical, and they differ only in the constant coefficient h_q.*

Also, as the proof of the previous theorem shows, WLR functions in VASS can be computed directly by linear programming, without the need for any "supporting invariants," since effect of a transition in VASS is independent of the current values of the counters. Also, well-foundedness (i.e. the fact that the function is bounded from below) is guaranteed by the fact that $\mathbf{n} \geq 0$ *and counter values in VASS are always non-negative. It is a common knowledge that the existence of a state-based linear ranking function for a linear arithmetic program implies that the running time of the program is linear in the initial valuation of program variables. Hence, our main result can be interpreted as proving that for VASS, state-based linear ranking functions are both sound and* complete *for proving linear termination complexity.*

4 Polynomial termination time

In this section we concentrate on VASS with polynomial termination complexity. For simplicity, we restrict ourselves to *strongly connected VASS*. As we already indicated in Section 1, our analysis proceeds by considering properties of *normal vectors* perpendicular to hyperplanes covering the vectors of Inc.

Definition 4.1. *Let* $\mathcal{A} = (Q, T)$ *be a* d*-dimensional VASS. The set* $Normals(\mathcal{A})$ *consists of all* $\mathbf{n} \in \mathbb{R}^d$ *such that* $\mathbf{n} \neq \vec{0}$ *and* $Inc \subseteq \hat{\mathcal{H}}_{\mathbf{n}}$ *(i.e.,* $\mathbf{v} \cdot \mathbf{n} \leq 0$ *for all* $\mathbf{v} \in Inc$*).*

Let $\mathcal{A}$ be a strongly connected VASS. We distinguish four possibilities.

(A) $Normals(\mathcal{A}) = \emptyset$.

(B) $Normals(\mathcal{A}) \neq \emptyset$ and all $\mathbf{n} \in Normals(\mathcal{A})$ have a negative component.

(C) There exists $\mathbf{n} \in Normals(\mathcal{A})$ such that $\mathbf{n} > \vec{0}$.

(D) There exists $\mathbf{n} \in Normals(\mathcal{A})$ such that $\mathbf{n} \geq \vec{0}$ and (C) does not hold.

Note that one can easily decide which of the four conditions holds by linear programming. Due to Lemma 2.5, the decision algorithm is polynomial in the number of control states of $\mathcal{A}$ (assuming d is a fixed constant).

We start by showing that a VASS satisfying (A) or (B) is non-terminating. A proof is given in Section 5.2.

Theorem 4.2. *Let* $\mathcal{A} = (Q, T)$ *be a* d*-dimensional strongly connected VASS such that (A) or (B) holds. Then* $\mathcal{A}$ *is non-terminating.*

4.1 VASS satisfying condition (C)

Assume $\mathcal{A}$ is a d-dimensional VASS satisfying (C). We prove that if $\mathcal{A}$ is terminating, then $\mathcal{L}(n) \in \Theta(n^\ell)$ for some $\ell \in \{1, \ldots, d\}$. Further, there is a polynomial-time algorithm deciding whether $\mathcal{A}$ is terminating and computing the constant ℓ if it exists (assuming d is a fixed constant).

A crucial tool for our analysis is a *good normal*, introduced in the next definition.

Definition 4.3. *Let $\mathcal{A}$ be a VASS. We say that a normal $\mathbf{n} \in \mathit{Normals}(\mathcal{A})$ is* good *if $\mathbf{n} > \vec{0}$ and for every $\mathbf{v} \in \mathit{cone}(\mathit{Inc})$ we have that $-\mathbf{v} \in \mathit{cone}(\mathit{Inc})$ iff $\mathbf{v} \cdot \mathbf{n} = 0$.*

Example 4.4. *Consider the VASS of Fig. 2a. Here, a good normal is, e.g., the vector $\mathbf{n} = (1, 1)$. Observe that the effects of both self-loops (on q_1 and q_2) belong to the hyperplane defined by $(1, 1)$. Note that these loops compensate each other's effects so long as we stay in the hyperplane (this is the defining property of the good normal). This allows us to zig-zag in the plane without "paying" with decrements in the $\mathbf{n}$-value except when we need to switch between the loops (recall that the $\mathbf{n}$-value of a configuration $q\mathbf{v}$ is the product $\mathbf{v} \cdot \mathbf{n}$). This produces a path of quadratic length, which is asymptotically the worst case.*

The next lemma says that a good normal always exists and it is computable efficiently. A proof can be found in Section 5.3.

Lemma 4.5. *Let $\mathcal{A}$ be a d-dimensional VASS satisfying (C). Then there exists a good normal computable in time polynomial in $|Q|$, assuming d is a fixed constant.*

The next theorem is the key result of this section. It allows to reduce the analysis of termination complexity of a given VASS to the analysis of several smaller instances of the problem, which can be then solved recursively.

Theorem 4.6. *Let $\mathcal{A}$ be a VASS satisfying (C), and let $\mathbf{n} \in \mathit{Normals}(\mathcal{A})$ be a good normal. Consider a VASS $\mathcal{A}^{\mathbf{n}} = (Q, T_{\mathbf{n}})$ where*

$$T_{\mathbf{n}} = \{t \in T \mid \text{ there is a short cycle } \gamma \text{ of } \mathcal{A} \text{ containing } t \text{ such that } \mathit{eff}(\gamma) \cdot \mathbf{n} = 0\}.$$

Further, let $C_1, \ldots, C_k$ be all SCC of $\mathcal{A}^{\mathbf{n}}$ with at least one transition. We have the following:

(1) If $k = 0$ (i.e., if there is no SCC of $\mathcal{A}^{\mathbf{n}}$ with at least one transition), then $\mathcal{L}_{\mathcal{A}}(n) \in \Theta(n)$.

(2) If $k > 0$, all $\mathcal{A}^{\mathbf{n}}_{C_1}, \ldots, \mathcal{A}^{\mathbf{n}}_{C_k}$ are terminating, and the termination complexity of every $\mathcal{A}^{\mathbf{n}}_{C_i}$ is $\Theta(f_i(n))$, then $\mathcal{A}$ is terminating and $\mathcal{L}_{\mathcal{A}}(n) \in \Theta(n \cdot \max[f_1, \ldots, f_k](n))$, where $\max[f_1, \ldots, f_k] : \mathbb{N} \to \mathbb{N}$ is a function defined by $\max[f_1, \ldots, f_k](n) = \max\{f_1(n), \ldots, f_k(n)\}$.

To get some intuiting behind the proof of Theorem 4.6, consider the following example.

Example 4.7. *Consider the VASS of Fig. 2a. As mentioned in Example 4.4, there is a good normal $\mathbf{n} = (1, 1)$, which gives $T_{\mathbf{n}} = \{(-1, 1), (1, -1)\}$. Then Case (2) of Theorem 4.6 gives us two simpler VASS $\mathcal{A}^{\mathbf{n}}_{C_1}, \mathcal{A}^{\mathbf{n}}_{C_2}$ where $\mathcal{A}^{\mathbf{n}}_{C_1}$ has a single state q_1 and a single transition $(q_1, (-1, 1), q_1)$, and $\mathcal{A}^{\mathbf{n}}_{C_2}$ has a single state q_2 and a single transition $(q_2, (1, -1), q_2)$. Observe that both $\mathcal{A}^{\mathbf{n}}_{C_1}$ and $\mathcal{A}^{\mathbf{n}}_{C_2}$ can now be considered individually, and both of them have linear complexity. Also, as mentioned in Example 4.4, the good normal makes sure that the effect of the worst case behavior in $\mathcal{A}^{\mathbf{n}}_{C_1}$ can be compensated by a path in $\mathcal{A}^{\mathbf{n}}_{C_2}$, and vice versa. Moreover, following the worst case path in $\mathcal{A}^{\mathbf{n}}_{C_1}$ and its compensation in $\mathcal{A}^{\mathbf{n}}_{C_2}$ decreases the final $\mathbf{n}$-value of configurations only by a constant (caused by the switch between $\mathcal{A}^{\mathbf{n}}_{C_1}$ and $\mathcal{A}^{\mathbf{n}}_{C_2}$). So, we can follow such "almost compensating" loop $\Omega(n)$ times, and obtain a path of quadratic length.*

Note that in the general case the situation is more complicated since the compensating path may need to be composed using paths in several VASS of $\mathcal{A}^{\mathbf{n}}_{C_1}, \ldots, \mathcal{A}^{\mathbf{n}}_{C_k}$. So, we need to be careful about the number of switches and about geometry of the compensating path.

Proof sketch for Theorem 4.6. Claim (1) follows easily. It suffices to realize that if there is no SCC of $\mathcal{A}^{\mathbf{n}}$ with at least one transition, then there is no $\mathbf{v} \in \mathit{Inc}$ satisfying $\mathbf{v} \cdot \mathbf{n} = 0$. Hence, $\mathbf{v} \cdot \mathbf{n} < 0$ for all $\mathbf{v} \in \mathit{Inc}$, and we can apply Theorem 3.3.

Now we prove Claim (2). Let α be a zero-avoiding computation of $\mathcal{A}$ initiated in a configuration $q\mathbf{u}$. Since the last configuration $p\mathbf{v}$ of α satisfies $\mathbf{v} \geq \vec{0}$, we have that $\mathbf{v} \cdot \mathbf{n} \geq 0$. Hence,

$$\mathbf{v} \cdot \mathbf{n} \quad = \quad (\mathbf{u} + \mathit{eff}(\pi_\alpha)) \cdot \mathbf{n} \quad = \quad \mathbf{u} \cdot \mathbf{n} + \mathit{eff}(\pi_\alpha) \cdot \mathbf{n} \quad \geq \quad 0\,.$$

Let $\mathit{Decomp}(\pi_\alpha)$ by a decomposition of π_α into short cycles. For every short cycle γ of $\mathcal{A}$ we have that $\mathit{eff}(\gamma) \cdot \mathbf{n} \leq 0$. Since π_α can contain at most $|Q|$ transitions which are not contained in any cycle, we have that $\mathbf{u} \cdot \mathbf{n} \leq \mathbf{v} \cdot \mathbf{n} + c$, where $c \in \mathbb{N}$ is some fixed constant. This means that $\|p\mathbf{v}\|$ is $\mathcal{O}(\|q\mathbf{u}\|)$. Consequently, the same holds also for all *intermediate* configurations visited by α.

A short cycle γ of $\mathcal{A}$ such that $\mathit{eff}(\gamma) \cdot \mathbf{n} < 0$ is called $\mathbf{n}$-*decreasing*, otherwise it is $\mathbf{n}$-*neutral*. Clearly, the total number of $\mathbf{n}$-decreasing short cycles in $\mathit{Decomp}(\pi_\alpha)$ is $\mathcal{O}(\|q\mathbf{u}\|)$, because each of them decreases the scalar product with $\mathbf{n}$ by a fixed constant bounded away from zero, and $\mathbf{u} \cdot \mathbf{n}$ is $\mathcal{O}(\|q\mathbf{u}\|)$. This means that the total number of transitions in π_α which are *not* in $T_{\mathbf{n}}$ is $\mathcal{O}(\|q\mathbf{u}\|)$ (as we already noted, π_α can also contain transitions which are not contained in any short cycle, but their total number is bounded by $|Q|$). Let ϱ be a subpath of π_α with maximal length containing only transitions of $T_{\mathbf{n}}$. Note that ϱ is a concatenation of at most $|Q|$ subpaths which contain transitions of the same SCC C_i of $\mathcal{A}^{\mathbf{n}}$. Each of these subpaths is initiated in a configuration of size $\mathcal{O}(\|q\mathbf{u}\|)$, and hence its length is $\mathcal{O}(f_i(\|q\mathbf{u}\|))$. Hence, the length of ϱ is $\mathcal{O}(\|q\mathbf{u}\| \cdot \max[f_1, \ldots, f_k](\|q\mathbf{u}\|))$.

It remains to prove that $\mathcal{L}_{\mathcal{A}}(n) \in \Omega(n \cdot \max[f_1, \ldots, f_k](n))$. Let us fix some $i \leq k$. We prove that there exists a constant $a \in \mathbb{R}^+$ such that for all sufficiently large n there exists a zero-avoiding computation α_n of length at least $a \cdot n \cdot f_i(n)$ initiated in a configuration of size n. The construction of α_n is technically non-trivial, so we first explain the underlying idea informally. A formal proof is given in Section 5.4.

To achieve the length $\Omega(n \cdot f_i(n))$, the computation α_n needs to execute $\Omega(n)$ paths of length $\Theta(f_i(n))$ "borrowed" from $\mathcal{A}^{\mathbf{n}}_{C_i}$. The problem is that even after executing just one path π of length $\Theta(f_i(n))$, some counters can have very small values, which prevents the executing of another path of length $\Theta(f_i(n))$. Therefore, we need to "compensate" the effect of π and increase the counters. This is where we use the properties of a good normal. We can choose π so that it forms a cycle in $\mathcal{A}^{\mathbf{n}}_{C_i}$ (not necessarily a short one), and we prove that all cycles in $\mathcal{A}^{\mathbf{n}}_{C_i}$ are $\mathbf{n}$-neutral. From this we get $-\,\mathit{eff}(\pi) \in \mathit{cone}(\mathit{Inc})$, and hence the effect of π can be compensated by an appropriate combination of short cycles of $\mathcal{A}^{\mathbf{n}}$. So, after executing π, we execute the corresponding "compensating" path, and this is repeated $\Omega(n)$ times. Note that we need to ensure that the compensating paths do not decrease the counters too much in intermediate configurations, and the compensation ends in a configuration which is sufficiently close to the original configuration where we started executing π. $\square$

Now we can formulate and prove the main result of this section.

Theorem 4.8. *Let $\mathcal{A}$ be a d-dimensional VASS satisfying (C). The problem whether $\mathcal{A}$ is terminating is decidable in time polynomial in $|Q|$, assuming d is a fixed constant. Further, if $\mathcal{A}$ is terminating, then $\mathcal{L}(n) \in \Theta(n^k)$, where $k \in \{1, \ldots, d\}$ is a constant computable in time polynomial in $|Q|$, assuming d is a fixed constant.*

Proof. For a given a $\mathcal{A}$, the algorithm starts by computing a good normal $\mathbf{n}$ (see Lemma 4.5) and constructing the VASS $\mathcal{A}^{\mathbf{n}} = (Q, T_{\mathbf{n}})$ of Theorem 4.6. Here, the set $T_{\mathbf{n}}$ is computed as follows. Note that $\mathcal{A}$ can be seen as a directed multigraph where the nodes are the states and the edges correspond to transitions. To every transition $(q, \mathbf{u}, q')$ we assign its weight $-\mathbf{u} \cdot \mathbf{n}$. Note that the multigraph does not contain any negative cycles (a negative cycle in the multigraph would induce a cycle in $\mathcal{A}$ increasing the $\mathbf{n}$-value; however, such a cycle cannot exist with a good normal $\mathbf{n}$). To decide whether a given transition $(q, \mathbf{u}, q')$ belongs to $T_{\mathbf{n}}$, it suffices to find a path with the least accumulated weight from q' to q (which can be done using, e.g., Bellman-Ford algorithm [64]) and check whether the accumulated weight is equal to $\mathbf{u} \cdot \mathbf{n}$. Hence, $T_{\mathbf{n}}$ is computable in time polynomial in the size of $\mathcal{A}$ (for a given good normal $\mathbf{n}$).

Then, the algorithm proceeds by constructing the SCC $C_1, \ldots, C_k$ of $\mathcal{A}^{\mathbf{n}}$. If $k = 0$, then $\mathcal{L}(n) \in \Theta(n)$ (see Theorem 4.6 (1)). If $k = 1$ and $\mathcal{A}^{\mathbf{n}}_{C_1} = \mathcal{A}$, then $\mathcal{A}$ is non-terminating (this is a consequence of Theorem 4.6 (2); if $\mathcal{A}^{\mathbf{n}}_{C_1}$ was terminating with termination complexity $\Theta(f_1(n))$, then by Theorem 4.6 (2), the termination complexity of $\mathcal{A} = \mathcal{A}^{\mathbf{n}}_{C_1}$ is $\Theta(n \cdot f_1(n))$, which is impossible). Otherwise, the algorithm proceeds by analyzing $\mathcal{A}^{\mathbf{n}}_{C_1}, \ldots, \mathcal{A}^{\mathbf{n}}_{C_k}$ recursively. If some of them is non-terminating, then $\mathcal{A}$ is also non-terminating. Otherwise, the termination complexity of $\mathcal{A}$ is derived from the termination complexity of $\mathcal{A}^{\mathbf{n}}_{C_1}, \ldots, \mathcal{A}^{\mathbf{n}}_{C_k}$ as in Theorem 4.6 (2). Clearly, we obtain $\mathcal{L}(n) \in \Theta(n^k)$ for some $k \in \{1, \ldots, d\}$. It is easy to verify that the total the number of recursive calls is polynomial in the size of $\mathcal{A}$. $\square$

4.2 VASS satisfying condition (D)

Condition (D) is not sufficiently strong to guarantee polynomial termination time for terminating VASS. In fact, as d increases, the termination complexity can grow *very* fast. Even for $d = 3$, one can easily construct a terminating VASS satisfying (C) such that $\mathcal{L}(n) \in \Omega(2^n)$.

Example 4.9. *Consider the strongly connected 3-dimensional VASS $\mathcal{A}$ in Fig. 3. Let $n \in \mathbb{N}$ be arbitrary. We construct a zero-avoiding computation $\alpha(n)$ starting in $q_1\vec{n}$ whose length is exponential in n. For better readability, denote by x, y, and z the variables representing the first, second, and third counter, respectively.*

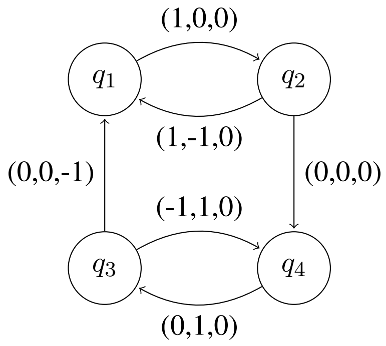


Figure 3: A 3-dimensional VASS satisfying condition (D) which has an exponential termination complexity.

The construction consist of iterating several phases. In Phase (a) we iterate the short cycle $q_1, (1,0,0), q_2, (1,-1,0), q_1$ *as long as* $y \geq 2$. *Then we perform the path* $q_1, (1,0,0), q_2, (0,0,0), q_4$ *to* q_4. *From there we continue with Phase (b), where we iterate the short cycle* $q_4, (0,1,0), q_3, (-1,1,0), q_4$ *as long as* $x \geq 2$. *After this we perform the path* $q_4, (0,1,0), q_3, (0,0,-1), q_1$ *to* q_1. *There we again switch to Phase (a), repeating the process until one of the counters hits zero.*

One can straightforwardly check that the total effect of performing Phase (a) once is setting y *to 1 while setting* x *to* x_a+2y_a, *where* x_a, y_a *are the values of* x, y *before the start of the phase. Similarly, The total effect of performing Phase (b) once is setting* x *to* 1 *while setting* y *to* $y_b + 2x_b$, *where* x_b, y_b *are the values of* x, y *before the start of the phase. Hence, the total effect of consecutively performing Phases (a) and (b) once can be bounded from below as follows: setting* x *to* 1 *and multiplying* y *by* 4. *Hence, the total effect of performing* N *consecutive iterations of Phases (a) and (b) is setting* x *to* 1, *multiplying* y *by* 4^N *and decreasing* z *by* N. *Since* z *decreases exactly during the witch from Phase (b) to Phase (a), we can perform exactly* n *consecutive iterations of (a) and (b). But increasing* y *from* n *to* 4^n *requires at least* $4^n - n$ *steps in VASS, hence the termination complexity of* $\mathcal{A}$ *is at least exponential. The matching asymptotic upper bound is easy to get.*

The key idea of the previous example can be used as building block for showing that higher-dimensional terminating VASS satisfying (D) can have even larger termination complexity than exponential. Already in dimension 4, the complexity can be non-elementary.

Example 4.10. *Consider the 4-dimensional strongly connected VASS in Fig. 4. As before, we denote by* x, y, z, w *the individual counters.*

For $n \in \mathbb{N}$ *we construct a zero-avoiding computation* $\alpha(n)$ *started in* $q_1\vec{n}$ *whose length in non-elementary. The construction again proceeds by switching between various phases and the phases we consider are the following: in Phase (a) we iterate cycle* $q_1, (-1,1,0,0), q_2, (0,1,0,0), q_1$ *from* q_1 *as long as* $x \geq 2$. *The effect of a single execution of (a) is setting* x *to* 1 *and* y *to* $y_a + 2x_a$ *(as before* v_p *denotes the value of counter* v *at the start of phase (p)). In Phase (b) we iterate the self-loop on* q_3 *as long as* $y \geq 2$, *the effect of the phase is setting* y *to* 1 *and* x *to* $x_b + y_b$.

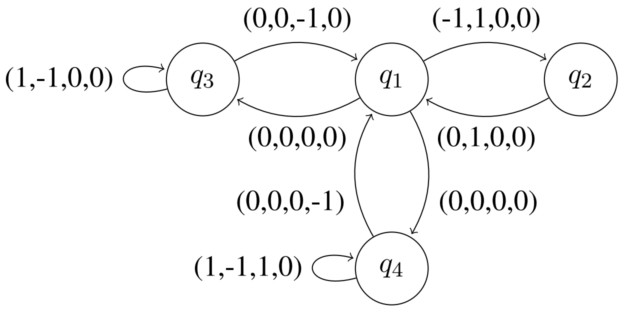


Figure 4: A 4-dimensional VASS satisfying condition (D) which has a non-elementary termination complexity.

Phase (c) consists of iterating the self-loop on q_4 as long as $y \geq 2$ and the effect is setting y to 1 and x and z to $x_c + y_c$ and $z_c + y_c$, respectively. Switching from (b) to (a) or (c) decreases z by 1, while switching from (c) to (a) or (b) decreases w by 1. Now the construction of $\alpha(n)$ proceeds as follows: we switch between Phases (a) and (b) as long as $z \geq 2$, after which we perform Phase (a) once more. We call this a Phase (d) and the total effect of (d) is setting x and z to 1, and y to a number at least $4^{z_d} \cdot x_d$. After Phase (d) we go to q_4 and execute Phase (c), after which we go to q_1 and start (d) again, repeating the process until a configuration with a zero counter is hit. The total effect of a single consecutive execution of (d) and (c) is setting y to 1 and x and z to a number at least $2^{z_d} \cdot x_d$. Since w is only decremented when switching from (d) to (c), we can repeat this consecutive execution at least n times. An easy induction shows that after i repeats of the consecutive executions of (d) and (c) the value of x is at least

$$\xi_n := n \cdot \underbrace{2^{2^{2^{\cdots 2^n}}}}_{n \text{ times}}.$$

Hence, the length of $\alpha(n)$ is at least ξ_n, i.e. non-elementary.

Figure 3 also provides an example showing that lexicographic ranking functions are not sound for polynomial bounds on termination complexity. We first define the notion of lexicographic ranking function for VASS: we specialize the standard definition of a lexicographic ranking functions for affine automata [4] (a generalization of VASS which models general linear arithmetic programs). Formally, an m-dimensional lexicographic map for a VASS $\mathcal{A} = (Q, T)$ is a collection $\{f_q^j \mid q \in Q, 1 \leq j \leq m\}$ of linear functions of counter values, one function per state and $1 \leq j \leq m$ (we allow m to be different from the dimension d of $\mathcal{A}$). A lexicographic map $\{f_q^j \mid q \in Q, 1 \leq j \leq m\}$ is a lexicographic ϵ-ranking function for $\mathcal{A}$ if each f_q^j is bounded from below on $\mathbb{N}$ and for each transition $(q, \mathbf{u}, q')$ of $\mathcal{A}$ there exists $1 \leq j \leq m$ such that $f_{q'}^j(\mathbf{u}) \leq f_q^j(\vec{0}) - \epsilon$ and for all $1 \leq j' < j$ it holds $f_{q'}^{j'}(\mathbf{u}) \leq f_q^{j'}(\vec{0})$. A standard argument shows that if $\mathcal{A}$ has a lexicographic ϵ-ranking function for, then it is terminating. However, lexicographic ranking functions are not sound for polynomial complexity bounds.

Example 4.11. *Consider the VASS $\mathcal{A}$ in Figure 3. Then, denoting the first, second, and third counter as x, y, z, respectively, there is the following 3-dimensional lexicographic $\frac{1}{2}$-ranking function for*

$\mathcal{A}$ *(we denote* $f_q = (f_q^1, \ldots, f_q^m)$*):* $f_{q_1} = (z, y, x)$*,* $f_{q_2} = (z, y - \frac{1}{2}, x)$*,* $f_{q_4} = (z - \frac{1}{2}, x, y)$*,* $f_{q_3} = (z - \frac{1}{2}, x - \frac{1}{2}, y)$*. But as shown in Example 4.9, the VASS has exponential termination complexity.*

5 Technical Proofs

5.1 Proof of Theorem 3.5

We describe a polynomial time-algorithm for deciding whether a given VASS has linear termination complexity. Recall from the proof sketch that it suffices to solve an equivalent problem whether there is an open half-space $\mathcal{H}_{\mathbf{n}}$ of $\mathbb{R}^d$ such that $\mathbf{n} \geq \vec{0}$ and $Inc \subseteq \mathcal{H}_{\mathbf{n}}$.

Let us formalize our intuition presented in the proof sketch. We need to introduce some additional notation: An *infinite path* π is an infinite sequence of the form $p_0, \mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \ldots$ where for each $n \geq 1$ the finite subsequence $p_0, \mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \ldots, \mathbf{u}_n, p_n$ is a finite path. We denote by $\pi_{\downarrow n}$ the finite prefix $p_0, \mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \ldots, \mathbf{u}_n, p_n$ of π. Given an infinite path π, we define the *mean change of* $\mathbf{n}$*-value* as

$$MC_{\mathbf{n}}(\pi) = \liminf_{n \to \infty} \frac{\mathit{eff}(\pi_{\downarrow n}) \cdot \mathbf{n}}{n}.$$

Consider the following linear program $\mathcal{L}$ obtained from [60], Section 8.8, by substituting the reward $r(s, a)$ with $\mathbf{u} \cdot \mathbf{n}$ where $\mathbf{u}$ is an effect of a transition:

Minimize *g with respect to the following constraints:*

For all $(q, \mathbf{u}, q') \in T$

$$g + h(q) - h(q') \geq \mathbf{u} \cdot \mathbf{n}$$

and

$$\mathbf{n} \geq \vec{0}.$$

Here, the variables are g, all $h(q)$, $q \in Q$, and all components of $\mathbf{n}$. By applying the results of [60], for every optimal solution $g, h, \mathbf{n}$ we have that

$$g = \sup_{\pi} MC_{\mathbf{n}}(\pi).$$

Moreover, there is at least one feasible solution.

We prove that there is $\mathbf{n} \geq \vec{0}$ such that the open half-space $\mathcal{H}_{\mathbf{n}}$ contains Inc iff an optimal solution $g, h, \mathbf{n}$ of the above program satisfies $g < 0$.

Consider an optimal solution $g, h, \mathbf{n}$ of the above program. Assume that $g < 0$. We show that $Inc \subseteq \mathcal{H}_{\mathbf{n}}$. For the sake of contradiction, assume that there is a short cycle π such that $\mathit{eff}(\pi) \cdot \mathbf{n} \geq 0$. Following the cycle π ad infinitum determines an infinite path π with $MC_{\mathbf{n}}(\pi) \geq 0$. However, this contradicts the fact that $0 > g = \sup_{\pi} MC_{\mathbf{n}}(\pi)$.

Now assume there is $\mathbf{n} \geq \vec{0}$ such that $Inc \subseteq \mathcal{H}_{\mathbf{n}}$. Let π be an infinite path. Let us fix $n \geq 1$ and consider $Decomp(\pi_{\downarrow n})$, the decomposition of $\pi_{\downarrow n}$ into short cycles. Let $Rest(\pi_{\downarrow n})$ be the remaining path obtained after removing all short cycles of $Decomp(\pi_{\downarrow n})$ from $\pi_{\downarrow n}$. Note that the length of $Rest(\pi_{\downarrow n})$ is at most $|Q|$, and hence $\mathit{eff}(Rest(\pi_{\downarrow n})) \cdot \mathbf{n} \leq |\vec{Q}| \cdot \mathbf{n}$.

Now let m be the length (i.e., the number of elements) of the list $Decomp(\pi_{\downarrow n})$. Note that $m \geq n/|Q| - 1$. Consider $\varepsilon > 0$ such that for all short cycles α we have that $\mathit{eff}(\alpha) \cdot \mathbf{n} \leq -\varepsilon$. Then

$$\mathit{eff}(\pi_{\downarrow n}) \quad\leq\quad m\cdot(-\varepsilon)+|\vec{Q}|\cdot\mathbf{n} \quad\leq\quad (n/|Q|-1)\cdot(-\varepsilon)+|\vec{Q}|\cdot\mathbf{n} \quad=\quad (n\cdot(-\varepsilon)/|Q|)+(|\vec{Q}|\cdot\mathbf{n}+\varepsilon)$$

and thus

$$\frac{\mathit{eff}(\pi_{\downarrow n})}{n} \quad\leq\quad \frac{(n\cdot(-\varepsilon)/|Q|)+(|\vec{Q}|\cdot\mathbf{n}+\varepsilon)}{n} \quad=\quad \frac{-\varepsilon}{|Q|}+\frac{(|\vec{Q}|\cdot\mathbf{n}+\varepsilon)}{n}.$$

Since $\lim_{n\to\infty}(|\vec{Q}| \cdot \mathbf{n} + \varepsilon)/n = 0$, we obtain that $MC_{\mathbf{n}}(\pi) \leq (-\varepsilon)/|Q|$. As π was chosen arbitrarily, we have that

$$\sup_{\pi} MC_{\mathbf{n}}(\pi) \quad\leq\quad \frac{-\varepsilon}{|Q|} \quad<\quad 0.$$

Hence, there is a solution $g, h, \mathbf{n}$ of the above linear program with $g < 0$.

In order to decide whether there is an open half-space $\mathcal{H}_{\mathbf{n}}$ of $\mathbb{R}^d$ such that $\mathbf{n} \geq \vec{0}$ and $Inc \subseteq \mathcal{H}_{\mathbf{n}}$, it suffices to compute an optimal solution $g, h, \mathbf{n}$ of the above linear program, which can be done in polynomial time (see, e.g., [51]), and check whether $g < 0$.

Now we get back to weighted linear ranking functions. Note that each solution $g, h, \mathbf{n}$ of the linear program $\mathcal{L}$ in which $g < 0$ yields a weighted linear ranking function $(\mathbf{c}, \{h_q \mid q \in Q\})$ by putting $\mathbf{c} := \mathbf{n}$ and $h_q := h(q)$ for each q. Conversely, each weighted linear ranking function yields a solution of $\mathcal{L}$ where $g < 0$, (we need to put $g := -\epsilon$, where ϵ is from the definition of a weighted lin. ranking function). Hence, a VASS $\mathcal{A}$ has linear termination complexity if and only if it has a weighted linear ranking function and this can be decided in polynomial time in size of $\mathcal{A}$. □

5.2 Proof of Theorem 4.2

If condition (A) or (B) holds, there is no $\mathbf{n} \geq \vec{0}$ such that $Inc \subseteq \hat{\mathcal{H}}_{\mathbf{n}}$. We show that then there exists $\mathbf{u} \in cone(Inc)$ such that $\mathbf{u} > \vec{0}$. Suppose there is no such $\mathbf{u}$. Let B be the set of all $\mathbf{v} > \vec{0}$. Since $cone(Inc)$ and B are convex and disjoint, there is a separating hyperplane with normal $\mathbf{n} \geq \vec{0}$ for $cone(Inc)$ and B. Since $cone(Inc) \subseteq \hat{\mathcal{H}}_{\mathbf{n}}$, we have a contradiction.

So, let $\mathbf{u} > \vec{0}$ such that $\mathbf{u} = \sum_{i=1}^{k} a_i \cdot \mathbf{v}_i$, where $a_i \in \mathbb{Q}^+$ and $\mathbf{v}_i \in Inc$ for all $1 \leq i \leq k$. Hence, there also exist $b_1, \ldots, b_k \in \mathbb{N}^+$ such that $\mathbf{w} = \sum_{i=1}^{k} b_i \cdot \mathbf{v}_i > \vec{0}$. Let us fix a cycle π in $\mathcal{A}$ visiting all control states (here we need that $\mathcal{A}$ is strongly connected). Clearly, there exists $c \in \mathbb{N}$ such that $\mathit{eff}(\pi) + c \cdot \mathbf{w} > 0$. Let ϱ be a cycle obtained from π by inserting $c \cdot b_i$ copies of a short cycle γ_i, where $\mathit{eff}(\gamma_i) = \mathbf{v}_i$. Then, $\mathit{eff}(\varrho) > \vec{0}$, and hence there exists an infinite computation initiated in $p\vec{n}$ for a sufficiently large $n \in \mathbb{N}$ (the control state p can be chosen arbitrarily).

5.3 Proof of Lemma 4.5

Due to condition (C), there exists at least one positive normal. Hence, we can fix a positive $\mathbf{n} \in \mathit{Normals}(\mathcal{A})$ such that the set $\{\mathbf{u} \in \mathit{Inc} \mid \mathbf{u} \cdot \mathbf{n} = 0\}$ is *minimal*. We show that for every $\mathbf{v} \in \mathit{cone}(\mathit{Inc})$ we have that $-\mathbf{v} \in \mathit{cone}(\mathit{Inc})$ iff $\mathbf{v} \cdot \mathbf{n} = 0$, i.e., $\mathbf{n}$ is a good normal. The "$\Rightarrow$" direction immediate—if $\mathbf{v}, -\mathbf{v} \in \mathit{cone}(\mathit{Inc})$, then $\mathbf{v} \cdot \mathbf{n} \leq 0$ and $-\mathbf{v} \cdot \mathbf{n} \leq 0$, which implies $\mathbf{v} \cdot \mathbf{n} = 0$. For the other direction, suppose there exists $\mathbf{v} \in \mathit{cone}(\mathit{Inc})$ such that $\mathbf{v} \cdot \mathbf{n} = 0$ and $-\mathbf{v} \notin \mathit{cone}(\mathit{Inc})$. Then there also exists $\mathbf{u} \in \mathit{Inc}$ such that $\mathbf{u} \cdot \mathbf{n} = 0$ and $-\mathbf{u} \notin \mathit{cone}(\mathit{Inc})$. For the rest of this proof, we fix such $\mathbf{u}$. By Farkas' lemma, there exists a separating hyperplane for $\mathit{cone}(\mathit{Inc})$ and $-\mathbf{u}$ with normal vector $\mathbf{n}'$, i.e., $-\mathbf{u} \cdot \mathbf{n}' > 0$ and $\mathbf{v} \cdot \mathbf{n}' \leq 0$ for every $\mathbf{v} \in \mathit{cone}(\mathit{Inc})$. Let us fix a sufficiently small $\varepsilon > 0$ such that $\mathbf{n} + \varepsilon\mathbf{n}' > \vec{0}$ and $\mathbf{v} \cdot (\mathbf{n} + \varepsilon\mathbf{n}') < 0$ for all $\mathbf{v} \in \mathit{Inc}$ where $\mathbf{v} \cdot \mathbf{n} < 0$. Clearly, $\mathbf{n} + \varepsilon\mathbf{n}'$ is a positive normal. Further, for all $\mathbf{v} \in \mathit{cone}(\mathit{Inc})$ such that $\mathbf{v} \cdot \mathbf{n} < 0$ we have that $\mathbf{v} \cdot (\mathbf{n} + \varepsilon\mathbf{n}') < 0$. Since $\mathbf{u} \cdot (\mathbf{n} + \varepsilon\mathbf{n}') < 0$, we obtain a contradiction with the minimality of $\mathbf{n}$.

To compute a good normal, first observe that the condition of Definition 4.3 can be safely relaxed just to the vectors of Inc, i.e., if $\mathbf{n} \in \mathit{Normals}(\mathcal{A})$ such that $\mathbf{n} > \vec{0}$ and $-\mathbf{v} \in \mathit{cone}(\mathit{Inc})$ iff $\mathbf{v} \cdot \mathbf{n} = 0$ for every $\mathbf{v} \in \mathit{Inc}$, then $\mathbf{n}$ is a good normal. To see this, fix some $\mathbf{n}$ with this property, and let $\mathbf{u} = \sum_{i=1}^{k} a_i \cdot \mathbf{v}_i$, where $a_i \in \mathbb{R}^+$ and $\mathbf{v}_i \in \mathit{Inc}$ for all $1 \leq i \leq k$. We need to show that $-\mathbf{u} \in \mathit{cone}(\mathit{Inc})$ iff $\mathbf{u} \cdot \mathbf{n} = 0$. If $-\mathbf{u} \in \mathit{cone}(\mathit{Inc})$, then $-\mathbf{u} = \sum_{i=1}^{k'} a'_j \cdot \mathbf{v}'_i$ where $a'_i \in \mathbb{R}^+$ and $\mathbf{v}'_i \in \mathit{Inc}$ for all $1 \leq i \leq k'$. Hence,

$$\vec{0} = \mathbf{u} + (-\mathbf{u}) = \sum_{i=1}^{k} a_i \cdot \mathbf{v}_i + \sum_{i=1}^{k'} a'_i \cdot \mathbf{v}'_i.$$

Hence,

$$0 = (\mathbf{u} + (-\mathbf{u})) \cdot \mathbf{n} = \sum_{i=1}^{k} a_i \cdot \mathbf{v}_i \cdot \mathbf{n} + \sum_{i=1}^{k'} a'_i \cdot \mathbf{v}'_i \cdot \mathbf{n}.$$

Since $\mathbf{v}_i \cdot \mathbf{n} \leq 0$ and $\mathbf{v}'_i \cdot \mathbf{n} \leq 0$ for all $1 \leq i \leq k$ and all $1 \leq i' \leq k'$, we obtain $\mathbf{v}_i \cdot \mathbf{n} = 0$ for all $1 \leq i \leq k$, hence $\mathbf{u} \cdot \mathbf{n} = 0$. On the other hand, if $\mathbf{u} \cdot \mathbf{n} = 0$, then $\sum_{i=1}^{k} a_i \cdot \mathbf{v}_i \cdot \mathbf{n} = 0$. Since $a_i > 0$ and $\mathbf{v}_i \cdot \mathbf{n} \leq 0$ for all $1 \leq i \leq k$, we have that $\mathbf{v}_i \cdot \mathbf{n} = 0$ for all $1 \leq i \leq k$. Hence $-\mathbf{v}_i \in \mathit{cone}(\mathit{Inc})$ for every $1 \leq i \leq k$ (by our assumption), and $-\mathbf{u} = \sum_{i=1}^{k} a_i \cdot (-\mathbf{v}_i) \in \mathit{cone}(\mathit{Inc})$.

Using the above observation, we can compute a good normal using linear programming as follows: First, compute the set $I = \{\mathbf{v} \in \mathit{Inc} \mid -\mathbf{v} \in \mathit{cone}(\mathit{Inc})\}$. Note that I can be computed easily by checking feasibility of the following linear constraints:

$$-\mathbf{v} = \sum_{\mathbf{u} \in \mathit{Inc}} a_{\mathbf{u}} \cdot \mathbf{u} \qquad \text{and} \qquad a_{\mathbf{u}} \geq 0.$$

Here, the variables are $a_{\mathbf{u}}$. A good normal can be computed using the following linear program:

Maximize *ε with respect to the following constraints:*

$$\mathbf{u} \cdot \mathbf{n} = 0 \textit{ for all } \mathbf{u} \in I$$
$$\mathbf{v} \cdot \mathbf{n} \leq -\varepsilon \textit{ for all } \mathbf{v} \in \textit{Inc} \setminus I$$
$$\mathbf{n} \geq \vec{\varepsilon}.$$

Here, the variables are ε and all components of $\mathbf{n}$.

Note that there is a good normal iff there is an optimal solution with $\varepsilon > 0$. Moreover, every optimal solution $\varepsilon, \mathbf{n}$ with $\varepsilon > 0$ gives a good normal $\mathbf{n}$.

5.4 Proof of Theorem 4.6

Now start by formulating an auxiliary technical lemma which is needed in the proof of Theorem 4.6.

Lemma 5.1. *Let $\mathcal{A}$ be a VASS satisfying (C), and let $\mathbf{n}$ be a good normal. Then there is a constant $\kappa \in \mathbb{R}^+$ such that for every $\mathbf{w} \in \mathit{cone}(\mathit{Inc})$, where $\mathbf{w} \cdot \mathbf{n} = 0$ and $\mathit{norm}(\mathbf{w}) = 1$, there exist $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathbb{R}^+$, and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathit{Inc}$ such that $\mathbf{w} = \sum_{j=1}^{k} a_j \cdot \mathbf{v}_j$, $\mathbf{v}_j \cdot \mathbf{n} = 0$ for all $1 \leq j \leq k$, and for all $k' \leq k$, the absolute values of all components of the vector $\sum_{j=1}^{k'} a_j \cdot \mathbf{v}_j$ are bounded by κ.*

Proof. Let $\mathbf{w} = \sum_{j=1}^{k} a_j \cdot \mathbf{v}_j$ where $a_j \in \mathbb{R}^+$, $\mathbf{v}_j \in \mathit{Inc}$ for all $1 \leq j \leq k$, and k is *minimal*. Clearly, $\mathbf{w} \cdot \mathbf{n} = \sum_{j=1}^{k} a_j \cdot (\mathbf{v}_j \cdot \mathbf{n}) = 0$, which implies $\mathbf{v}_j \cdot \mathbf{n} = 0$ for all $1 \leq j \leq k$ (recall that $\mathbf{v}_j \cdot \mathbf{n} \leq 0$ because $\mathbf{n} \in \mathit{Normals}(\mathcal{A})$). First, we show that for every $j \leq k$, the vector $-\mathbf{v}_j$ does not belong to $\mathit{cone}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_k\})$. Assume the converse, i.e., $-\mathbf{v}_1 \in \mathit{cone}(\{\mathbf{v}_2, \ldots, \mathbf{v}_k\})$. Then $-\mathbf{v}_1 = \sum_{j=2}^{k} b_j \cdot \mathbf{v}_j$, where $b_j \in \mathbb{R}^+$ for all $2 \leq j \leq k$. Further,

$$\mathbf{w} \quad = \quad (a_1 - c) \cdot \mathbf{v}_1 \;+\; (a_2 - cb_2) \cdot \mathbf{v}_2 \;+\; \cdots \;+\; (a_k - cb_k)\mathbf{v}_k$$

for every $c > 0$. Clearly, there exists $c > 0$ such that at least one of the coefficients $(a_1 - c)$, $(a_2 - cb_2), \ldots, (a_k - cb_k)$ is zero and the other remain positive, which contradicts the minimality of k. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \hat{\mathcal{H}}_{\mathbf{n}}$, there must exist $\mathbf{n}' > \vec{0}$ such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathcal{H}_{\mathbf{n}'}$ (otherwise, we can use the same argument as in the proof of Case (a) of Lemma 3.2 to show that $-\mathbf{v}_j \in \mathit{cone}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_k\})$ for some $1 \leq j \leq k$). Since $\mathbf{v}_j \cdot \mathbf{n}' < 0$ for all $1 \leq j \leq k$, each $\mathbf{v}_j$ moves in the direction of $-\mathbf{n}$ by some fixed positive distance. Since $\mathit{norm}(\mathbf{w}) = 1$, there is a bound $\delta_{\mathbf{v}_1,\ldots,\mathbf{v}_k} \in \mathbb{R}^+$ such that $a_j \leq \delta_{\mathbf{v}_1,\ldots,\mathbf{v}_k}$ for all $1 \leq j \leq k$, because no $a_j \cdot \mathbf{v}_j$ can go in the direction of $-\mathbf{n}$ by more than a unit distance.

The above claim applies to every $\mathbf{w} \in \mathit{cone}(\mathit{Inc})$ where $\mathbf{w} \cdot \mathbf{n} = 0$. Since Inc is finite, there are only finitely many candidates for the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ used to express $\mathbf{w}$, and hence there exists a fixed upper bound $\delta \in \mathbb{R}^+$ for all $\delta_{\mathbf{v}_1,\ldots,\mathbf{v}_k}$. This means that, for every $\mathbf{w} \in \mathit{cone}(\mathit{Inc})$ where $\mathbf{w} \cdot \mathbf{n} = 0$, there exist $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathbb{R}^+$, and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathit{Inc}$ such that $\mathbf{w} = \sum_{j=1}^{k} a_j \cdot \mathbf{v}_j$, $\mathbf{v}_j \cdot \mathbf{n} = 0$, and $a_j \leq \delta$ for all $1 \leq j \leq k$. This immediately implies the existence of κ. $\square$

Now can formalize the proof of Theorem 4.6.

All cycles of $\mathcal{A}^{\mathbf{n}}_{C_i}$ are $\mathbf{n}$*-neutral.* First, realize that for every cycle η of $\mathcal{A}$ (not necessarily short) we have that $\mathit{eff}(\eta)\cdot\mathbf{n} = \sum_{\gamma\in Decomp(\eta)} \mathit{eff}(\gamma)\cdot\mathbf{n} \le 0$. Now let $\beta = p_0, \mathbf{u}_1, p_1, \mathbf{u}_2, p_2, \ldots, \mathbf{u}_n, p_k$ be a cycle of $\mathcal{A}^{\mathbf{n}}_{C_i}$ (not necessarily short). Then each transition $(p_j, \mathbf{u}_{j+1}, p_{j+1})$ of β is contained in some $\mathbf{n}$-neutral short cycle γ_i of $\mathcal{A}$. Let ϱ_i be the (unique) path from p_{j+1} to p_j determined by γ_j, and let $\varrho = \varrho_{k-1}\odot\cdots\odot\varrho_0$. Then $\mathit{eff}(\beta)+\mathit{eff}(\varrho) = \sum_{j=0}^{k-1}\mathit{eff}(\gamma_j)$. Hence, $\mathit{eff}(\beta)\cdot\mathbf{n}+\mathit{eff}(\varrho)\cdot\mathbf{n} = \sum_{j=0}^{k-1}\mathit{eff}(\gamma_j)\cdot\mathbf{n} = 0$. Thus, we obtain $\mathit{eff}(\beta)\cdot\mathbf{n} = -\mathit{eff}(\varrho)\cdot\mathbf{n}$. Since both β and ϱ are cycles of $\mathcal{A}$, we have that $\mathit{eff}(\beta)\cdot\mathbf{n}\le 0$ and $\mathit{eff}(\varrho)\cdot\mathbf{n}\le 0$, which implies $\mathit{eff}(\beta)\cdot\mathbf{n} = 0$.

Constructing the paths of length $\Theta(f_i(n))$. Since the termination complexity of $\mathcal{A}^{\mathbf{n}}_{C_i}$ is $\Theta(f_i(n))$, there is $b\in\mathbb{R}^+$ such that for all sufficiently large $n\in\mathbb{N}$ there exist a configuration $p_n\vec{n}$ and a zero-avoiding computation β_n of length at least $b\cdot f_i(n)$ initiated in $p_n\vec{n}$. Since π_{β_n} inevitably contains a cycle whose length is at least $b'\cdot f_i(n)$ (for some fixed $b'\in\mathbb{R}^+$ independent of β_n), we can safely assume that π_{β_n} is actually a cycle, which implies $\mathit{eff}(\pi_{\beta_n})\in \mathit{cone}(\mathit{Inc})$.

Constructing the compensating path. Since π_{β_n} is $\mathbf{n}$-neutral and $\mathit{eff}(\pi_{\beta_n})\in\mathit{cone}(\mathit{Inc})$, we have that $-\mathit{eff}(\pi_{\beta_n})\in\mathit{cone}(\mathit{Inc})$. This is where we use the defining property of a good normal. Since $-\mathit{eff}(\pi_{\beta_n}) = \sum_{j=1}^{m} a_j\cdot\mathbf{v}_j$, where $m\in\mathbb{N}$, $a_j\in\mathbb{Q}^+$, and $\mathbf{v}_j\in\mathit{Inc}$ for all $1\le j\le m$, a straightforward idea is to define the compensating path by "concatenating" $\lfloor a_j\rfloor$ copies of γ_j, where $\mathit{eff}(\gamma_j) = \mathbf{v}_j$, for all $1\le j\le m$. This would produce the desired effect on the counters, but there is no bound on the counter decrease in intermediate configurations visited when executing this path. To overcome this problem, we construct the compensating path for π_{β_n} more carefully. Let $\mathbf{w}$ be the normalized $\mathit{eff}(\pi_{\beta_n})$, i.e., $\mathbf{w}$ has the same direction as $\mathit{eff}(\pi_{\beta_n})$ but its norm is equal to 1. By Lemma 5.1, $-\mathbf{w}$ is expressible as $-\mathbf{w} = \sum_{j=1}^{m} a_j\cdot\mathbf{v}_j$, where $m\in\mathbb{N}$, $a_j\in\mathbb{Q}^+$, and $\mathbf{v}_j\in\mathit{Inc}$, so that $\mathbf{v}_j\cdot\mathbf{n} = 0$ for all $1\le j\le m$, and for all $m'\le m$, the absolute values of all components of the vector $\sum_{j=1}^{m'} a_j\cdot\mathbf{v}_j$ are bounded by κ, where κ is a constant independent of $\mathbf{w}$. Let us fix some cycle η of $\mathcal{A}^{\mathbf{n}}_{C_i}$ visiting all of its states (recall that $\mathcal{A}^{\mathbf{n}}_{C_i}$ is strongly connected). The compensating path for π_{β_n} is obtained from η by inserting $\lfloor \mathit{norm}(\mathit{eff}(\pi_{\beta_n}))\cdot a_j\rfloor$ copies of a short cycle with effect $\mathbf{v}_j$, for every $1\le j\le m$. Observe that the difference between the effect of this compensating path and $-\mathit{eff}(\pi_{\beta_n})$ is bounded by a constant vector independent of n. Further, when executing the compensating path, the counters are never decreased by more that $\kappa\cdot\mathit{norm}(\mathit{eff}(\pi_{\beta_n}))$.

Constructing a zero-avoiding computation α_n of length $\Omega(n\cdot f_i(n))$. Now we are ready to put the above ingredients together, which still requires some effort. Let us fix a sufficiently large $n\in\mathbb{N}$ and a configuration $p\mathbf{v}$ where $\|p\mathbf{v}\| = n$ and p is a control state of $\mathcal{A}^{\mathbf{n}}_{C_i}$. Let q be the first state of π_{β_n}. If we started α_n in $p\mathbf{v}$ by executing a finite path which changes the control state from p to the first control state of π_{β_n} (which takes at most $|Q|$ transitions) and continued by executing π_{β_n}, the counters could potentially reach values arbitrarily close to zero (it might even happen that π_{β_n} is not executable). Instead, we fix a suitable $n'\le n$ satisfying $n-n'\ge\kappa\cdot\mathit{norm}(\mathit{eff}(\pi_{\beta_{n'}}))+|Q|$. Since $\mathit{norm}(\mathit{eff}(\pi_{\beta_{n'}}))\le\sqrt{d}\cdot n'$, we can safely put $n' = (n-|Q|)/(1+\kappa\sqrt{d})$. Now, we can initiate α_n by a short finite path which changes the control state from p to the first control state of $\pi_{\beta_{n'}}$, and continue by executing $\pi_{\beta_{n'}}$. Note that $(1+\kappa\sqrt{d})$ is a constant, so decreasing n to n' has no influence in the asymptotic length of the constructed computation. Then, we can safely execute the compensating path for $\pi_{\beta_{n'}}$, and thus reach a configuration $q\mathbf{u}$ where we continue in the same way as in $p\mathbf{v}$, i.e., execute another finite path of length $\Theta(f_i(n))$ and its corresponding compensating path. Since the $\mathbf{v}-\mathbf{u}$ is bounded by a constant vector, this can be repeated $\Omega(n)$ times before reaching a configuration where some counter value is not sufficiently large to perform another "round". Hence, the length of the resulting α_n is $\Omega(n \cdot f_i(n))$.

6 Related Work

In this section we discuss the related work.

Resource analysis. Our work is most closely related to automatic amortized analysis [38, 39, 40, 41, 42, 46, 45, 36, 31], as well as the SPEED project [34, 35, 33]. All these works focus on worst-case asymptotic bounds for programs, and present sound methods but not complete methods for upper bounds, i.e., even though the asymptotic bound is linear or quadratic, the approaches may still fail to provide any upper bound. However, all these works consider general programs rather than the model of VASS. In contrast, we consider VASS and present sound and complete method to derive tight (upper and matching lower) polynomial complexity bounds.

Recurrence relations. Other approaches for bounds analysis involve recurrence relations, such as [32, 29, 1, 2, 3]. Even for relatively simple programs the recurrence are quite complex, and cannot be obtained automatically. In contrast, we present a polynomial-time approach for optimal asymptotic bounds for VASS.

Ranking functions and extensions. Ranking functions for intraprocedural analysis have been widely studied [7, 9, 20, 59, 67, 21, 70, 63]. Most works have focussed on linear or polynomial ranking functions [20, 59, 67, 21, 70, 63], as well as non-polynomial bounds [14]. Again, these approaches are sound, but not complete even to derive upper bounds for VASS. The notion of ranking functions have been also extended to ranking supermartingales [10, 28, 15, 13, 16] for expected termination time of probabilistic programs, but such approaches do not present polynomial asymptotic bounds.

Results on VASS. The model of VASS [50] or equivalently Petri nets are a fundamental model for parallel programs [25, 50] as well as parameterized systems [6]. The termination problems (counter-termination, control-state termination) as well as the related problems of boundedness and coverability have been a rich source of theoretical problems that have been widely studied [56, 61, 23, 24, 8]. The complexity of the termination problem with fixed initial configuration is EXPSPACE-complete [56, 71, 5]. Recent work such as [66, 6] shows how VASS and subclass of VASS (such as lossy VASS) provide a natural model for abstraction and analysis of programs as well as parametrized systems. The work of [66] also considers lexicographic ranking functions to obtain sound asymptotic upper bounds for lossy VASS. However, this approach is not complete, and also do not consider tight complexity bounds (but only upper bounds). Besides the termination problem, the more general reachability problem where given a VASS, an initial and a final configuration, whether there exists a path between them has also been studied [57, 53, 55]. The reachability problem is decidable [57, 53, 55], and EXPSPACE-hard [56], and the current best-known upper bound is cubic Ackermannian [54], a complexity class belonging to the third level of a fast-growing complexity hierarchy introduced in [62].

Other related approaches are sized types [17, 43, 44], and polynomial resource bounds [65]. Again none of these approaches are complete for VASS nor they can yield tight asymptotic complexity bounds.

Hyperplane-separation technique and existence of infinite computation. The problem of existence of infinite computations in VASS has been studied in the literature. Polynomial-time algorithms have been presented in [11, 68] using results of [52]. In the more general context of games played on VASS, even deciding the existence of infinite computation is coNP-complete [11, 68], and various algorithmic approaches based on hyperplane-separation technique have been studied [12, 47, 18]. In this work we also consider normals of effects of cycles in VASS, which is related to hyperplane-separation technique. However all previous works consider hyperplane-based techniques to determine the existence of infinite computations on games played on VASS, and do not consider asymptotic time of termination. In contrast, we present the first approach to show that hyperplane-based techniques can be used to derive tight asymptotic complexity bounds on termination time for VASS.

7 Conclusion

In this paper, we studied the problem of obtaining precise polynomial asymptotic bounds for VASS. We obtained a full end efficient characterization of all VASS with linear termination complexity. Then we considered polynomial termination for strongly connected VASS, dividing them into four disjoint classes (A)–(D). For the first two classes, we proved that the VASS are non-terminating. For VASS in (C), we obtained a full and effective characterization of termination complexity. For the last class (D), we have shown that the termination complexity can be exponential even for dimension three. The results are applicable also to general (i.e., non-strongly connected VASS), by analyzing the individual SCCs. Some extra effort is needed in (C), because here a possible increase in the size of configurations accumulated in a given SCC before moving into another SCC must be taken into account. To keep our proofs reasonably simple, we considered just strongly connected VASS.

Our result gives rise to a number of interesting directions for future work. First, whether our precise complexity analysis or the complete method can be extended to other models in program analysis (such as affine programs with loops) is an interesting theoretical direction to pursue. Second, in the practical direction, using our result for developing a scalable tool for sound and complete analysis of asymptotic bounds for VASS and their applications in program analysis is also an interesting subject for future work.

References

[1] Elvira Albert, Puri Arenas, Samir Genaim, Miguel Gómez-Zamalloa, German Puebla, Diana V. Ramírez-Deantes, Guillermo Román-Díez, and Damiano Zanardini. Termination and cost analysis with COSTA and its user interfaces. *ENTCS*, 258(1):109–121, 2009.

[2] Elvira Albert, Puri Arenas, Samir Genaim, and Germán Puebla. Automatic inference of upper bounds for recurrence relations in cost analysis. In *SAS*, volume 5079 of *LNCS*, pages 221–237. Springer, 2008.

[3] Elvira Albert, Puri Arenas, Samir Genaim, Germán Puebla, and Damiano Zanardini. Cost analysis of java bytecode. In *ESOP*, volume 4421 of *LNCS*, pages 157–172. Springer, 2007.

[4] Christophe Alias, Alain Darte, Paul Feautrier, and Laure Gonnord. Multi-dimensional rankings, program termination, and complexity bounds of flowchart programs. In *SAS*, pages 117–133. Springer-Verlag, 2010.

[5] Mohamed Faouzi Atig and Peter Habermehl. On yen's path logic for petri nets. *IJFCS*, 22(04):783–799, 2011.

[6] Roderick Bloem, Swen Jacobs, Ayrat Khalimov, Igor Konnov, Sasha Rubin, Helmut Veith, and Josef Widder. Decidability in parameterized verification. *SIGACT News*, 47(2):53–64, 2016.

[7] Olivier Bournez and Florent Garnier. Proving positive almost-sure termination. In *RTA*, pages 323–337, 2005.

[8] Laura Bozzelli and Pierre Ganty. *Complexity Analysis of the Backward Coverability Algorithm for VASS*. In *RP*, pages 96–109. Springer, 2011.

[9] Aaron R. Bradley, Zohar Manna, and Henny B. Sipma. Linear ranking with reachability. In *CAV*, volume 3576 of *LNCS*, pages 491–504. Springer, 2005.

[10] Aleksandar Chakarov and Sriram Sankaranarayanan. Probabilistic program analysis with martingales. In *CAV*, volume 8044 of *LNCS*, pages 511–526. Springer, 2013.

[11] Krishnendu Chatterjee, Laurent Doyen, Thomas A. Henzinger, and Jean-Francois Raskin. Generalized mean-payoff and energy games. In *FSTTCS*, pages 505–516, 2010.

[12] Krishnendu Chatterjee and Yaron Velner. Hyperplane separation technique for multidimensional mean-payoff games. In *CONCUR*, pages 500–515. LNCS 8052, Springer, 2013.

[13] Krishnendu Chatterjee, Hongfei Fu, and Amir Kafshdar Goharshady. Termination analysis of probabilistic programs through positivstellensatz's. In *CAV (I)*, pages 3–22, 2016.

[14] Krishnendu Chatterjee, Hongfei Fu, and Amir Kafshdar Goharshady. Non-polynomial worst-case analysis of recursive programs. In *CAV*, pages 41–63, 2017.

[15] Krishnendu Chatterjee, Hongfei Fu, Petr Novotný, and Rouzbeh Hasheminezhad. Algorithmic analysis of qualitative and quantitative termination problems for affine probabilistic programs. In *POPL*, pages 327–342. ACM, 2016.

[16] Krishnendu Chatterjee, Petr Novotný, and Dorde Zikelic. Stochastic invariants for probabilistic termination. In *POPL*, pages 145–160, 2017.

[17] Wei-Ngan Chin and Siau-Cheng Khoo. Calculating sized types. In *Higher-Order and Symbolic Computation*, 14(2-3):261–300, 2001.

[18] Thomas Colcombet, Marcin Jurdzinski, Ranko Lazic, and Sylvain Schmitz. Perfect half space games. In *CoRR*, 2017.

[19] Michael Colón, Sriram Sankaranarayanan, and Henny Sipma. Linear invariant generation using non-linear constraint solving. In *CAV*, volume 2725 of *LNCS*, pages 420–432. Springer, 2003.

[20] Michael Colón and Henny Sipma. Synthesis of linear ranking functions. In *TACAS*, volume 2031 of *LNCS*, pages 67–81. Springer, 2001.

[21] Patrick Cousot. Proving program invariance and termination by parametric abstraction, Lagrangian relaxation and semidefinite programming. In *VMCAI*, volume 3385 of *LNCS*, pages 1–24. Springer, 2005.

[22] Emanuele D'Osualdo, Jonathan Kochems, and C. H. Luke Ong. Automatic Verification of Erlang-Style Concurrency. In *SAS*, pages 454–476. Springer 2013.

[23] Javier Esparza. Decidability and complexity of petri net problems—an introduction. *Petri nets*, pages 374–428, 1996.

[24] Javier Esparza, Ruslán Ledesma-Garza, Rupak Majumdar, Philipp Meyer, and Filip Niksic. An SMT-Based Approach to Coverability Analysis. In *CAV*, pages 603–619, Springer, 2014.

[25] Javier Esparza and Mogens Nielsen. Decidability issues for petri nets - a survey. In *Bulletin of the EATCS*, 52:245–262, 1994.

[26] Uli Fahrenberg, Line Juhl, Kim G. Larsen, Jiří Srba. Energy Games in Multiweighted Automata. In *Theoretical Aspects of Computing – ICTAC 2011*, volume 6916 of *LNCS*, pages 95–115. Springer, 2011.

[27] Yu Feng, Ruben Martins, Yuepeng Wang, Isil Dillig, and Thomas W. Reps. Component-based synthesis for complex apis. In *POPL*, pages 599–612, ACM, 2017.

[28] Luis María Ferrer Fioriti and Holger Hermanns. Probabilistic termination: Soundness, completeness, and compositionality. In *POPL*, pages 489–501. ACM, 2015.

[29] Philippe Flajolet, Bruno Salvy, and Paul Zimmermann. Automatic average-case analysis of algorithm. *TCS*, 79(1):37–109, 1991.

[30] Pierre Ganty and Rupak Majumdar. Algorithmic verification of asynchronous programs. *TOPLAS*, 34(1):6:1–6:48, May 2012.

[31] Stéphane Gimenez and Georg Moser. The complexity of interaction. In *POPL*, pages 243–255. ACM, 2016.

[32] Bernd Grobauer. Cost recurrences for DML programs. In *ICFP*, pages 253–264. ACM, 2001.

[33] Bhargav S. Gulavani and Sumit Gulwani. A numerical abstract domain based on expression abstraction and max operator with application in timing analysis. In *CAV*, volume 5123 of *LNCS*, pages 370–384. Springer, 2008.

[34] Sumit Gulwani. SPEED: symbolic complexity bound analysis. In *CAV*, volume 5643 of *LNCS*, pages 51–62. Springer, 2009.

[35] Sumit Gulwani, Krishna K. Mehra, and Trishul M. Chilimbi. SPEED: precise and efficient static estimation of program computational complexity. In *POPL*, pages 127–139. ACM, 2009.

[36] Jan Hoffmann, Klaus Aehlig, and Martin Hofmann. Multivariate amortized resource analysis. *TOPLAS*, 34(3):14, 2012.

[37] Jan Hoffmann, Klaus Aehlig, and Martin Hofmann. Resource aware ML. In *CAV*, volume 7358 of *LNCS*, pages 781–786. Springer, 2012.

[38] Jan Hoffmann and Martin Hofmann. Amortized resource analysis with polymorphic recursion and partial big-step operational semantics. In *APLAS*, volume 6461 of *LNCS*, pages 172–187. Springer, 2010.

[39] Jan Hoffmann and Martin Hofmann. Amortized resource analysis with polynomial potential. In *ESOP*, volume 6012 of *LNCS*, pages 287–306. Springer, 2010.

[40] Martin Hofmann and Steffen Jost. Static prediction of heap space usage for first-order functional programs. In *POPL*, pages 185–197. ACM, 2003.

[41] Martin Hofmann and Steffen Jost. Type-based amortised heap-space analysis. In *ESOP*, volume 3924 of *LNCS*, pages 22–37. Springer, 2006.

[42] Martin Hofmann and Dulma Rodriguez. Efficient type-checking for amortised heap-space analysis. In *CSL*, volume 5771 of *LNCS*, pages 317–331. Springer, 2009.

[43] John Hughes and Lars Pareto. Recursion and dynamic data-structures in bounded space: Towards embedded ML programming. In *ICFP*, pages 70–81. ACM, 1999.

[44] John Hughes, Lars Pareto, and Amr Sabry. Proving the correctness of reactive systems using sized types. In *POPL*, pages 410–423. ACM Press, 1996.

[45] Steffen Jost, Kevin Hammond, Hans-Wolfgang Loidl, and Martin Hofmann. Static determination of quantitative resource usage for higher-order programs. In *POPL*, pages 223–236. ACM, 2010.

[46] Steffen Jost, Hans-Wolfgang Loidl, Kevin Hammond, Norman Scaife, and Martin Hofmann. "carbon credits" for resource-bounded computations using amortised analysis. In *FM*, volume 5850 of *LNCS*, pages 354–369. Springer, 2009.

[47] Marcin Jurdzinski, Ranko Lazic, and Sylvain Schmitz. Fixed-dimensional energy games are in pseudo-polynomial time. In *ICALP* pages 260–272, 2015.

[48] Alexander Kaiser, Daniel Kroening, and Thomas Wahl. Dynamic Cutoff Detection in Parameterized Concurrent Programs. In *CAV*, volume 6174 of *LNCS*, pages 645–659. Springer, 2010.

[49] Alexander Kaiser, Daniel Kroening, and Thomas Wahl. Efficient Coverability Analysis by Proof Minimization. IN *CONCUR*, pages 500–515. Springer, 2012.

[50] Richard M. Karp and Raymond E. Miller. Parallel program schemata. *JCSS*, 3(2):147–195, 1969.

[51] Leonid Genrikhovich Khachiyan. A polynomial algorithm in linear programming. *Doklady Akademii Nauk SSSR*, 244:1093–1096, 1979.

[52] S. Rao Kosaraju and Gregory F. Sullivan. Detecting cycles in dynamic graphs in polynomial time. In *STOC*, pages 398–406, 1988.

[53] S. Rao Kosaraju. Decidability of reachability in vector addition systems (preliminary version). In *STOC*, pages 267–281. ACM, 1982.

[54] Jérôme Leroux and Sylvain Schmitz. Demystifying reachability in vector addition systems. In *LICS*, pages 56–67. ACM/IEEE 2015.

[55] Jérôme Leroux. Vector addition system reachability problem: A short self-contained proof. In *POPL*, pages 307–316. ACM, 2011.

[56] Richard J. Lipton. The reachability problem requires exponential space. Technical report 62, 1976.

[57] Ernst W. Mayr. An Algorithm for the General Petri Net Reachability Problem. In *SIAM*, 13:441–460, 1984.

[58] Ernst W. Mayr and Albert R. Meyer. The complexity of the finite conainment problem for Petri nets. *Journal of the ACM*, pages 561–576, 1981.

[59] Andreas Podelski and Andrey Rybalchenko. A complete method for the synthesis of linear ranking functions. In *VMCAI*, volume 2937 of *LNCS*, pages 239–251. Springer, 2004.

[60] Martin L. Puterman. *Markov Decision Processes*. Wiley, 1994.

[61] Charles Rackoff. The Covering and Boundedness Problems for Vector Addition Systems. *TCS*, 6:223–231, 1978.

[62] Sylvain Schmitz. Complexity hierarchies beyond elementary. *TOTC*, 8(1):3:1–3:36, February 2016.

[63] Liyong Shen, Min Wu, Zhengfeng Yang, and Zhenbing Zeng. Generating exact nonlinear ranking functions by symbolic-numeric hybrid method. *J. Systems Science & Complexity*, 26(2):291–301, 2013.

[64] Alfonso Shimbel. Applications of matrix algebra to communication nets. *The bull. math. biophysics*, 13(3):165–178, Sep 1951.

[65] Olha Shkaravska, Ron van Kesteren, and Marko C. J. D. van Eekelen. Polynomial size analysis of first-order functions. In *TLCA*, volume 4583 of *LNCS*, pages 351–365. Springer, 2007.

[66] Moritz Sinn, Florian Zuleger, and Helmut Veith. A simple and scalable static analysis for bound analysis and amortized complexity analysis. In *CAV*, pages 745–761, 2014.

[67] Kirack Sohn and Allen Van Gelder. Termination detection in logic programs using argument sizes. In *PODS*, pages 216–226. ACM Press, 1991.

[68] Yaron Velner, Krishnendu Chatterjee, Laurent Doyen, Thomas A. Henzinger, Alexander Rabinovich, and Jean-Francois Raskin. The complexity of multi-mean-payoff and multi-energy games. *Inf. Comput.*, 241:177–196, 2015.

[69] Reinhard Wilhelm. The worst-case execution-time problem - overview of methods and survey of tools. *TECS*, 7(3), 2008.

[70] Lu Yang, Chaochen Zhou, Naijun Zhan, and Bican Xia. Recent advances in program verification through computer algebra. *FCS*, 4(1):1–16, 2010.

[71] Hsu-Chun Yen. A unified approach for deciding the existence of certain petri net paths. *Information and Computation*, 96(1):119–137, 1992.