Intrinsic Co-Heyting Boundaries and the Leibniz Rule in Certain Toposes

F. William Lawvere
Department of Mathematics, S.U.N.Y. at Buffalo
Buffalo, NY 14214

Certain lattices, such as that of all closed subsets of a topological space or that of all subtoposes of a given topos, may be called "co-Heyting algebras" in that they enjoy a "subtraction" operator left adjoint to join, dually to the "implication" operator right adjoint to meet which Heyting algebras have. The cited examples illustrate that such lattices may occur in practice directly, not only as formal opposites of Heyting algebras. In particular, there is for each element A of a co-Heyting algebra a smallest element *non A* whose join with A is the top element, and the meet *A and non A* is a further element which deserves to be called the *boundary* of A.

For example, given a retract-closed locally finite category, the subtoposes of the topos of presheaves on it correspond to the retract-closed subcategories A of the locally finite category itself, and the boundary of A is the category of those objects which are simultaneously objects in A and retracts of objects not in A. The subtoposes classify the models of positive (or "geometric") extensions of the theory for which the full presheaf category is classifying. For example, any positive extension of the theory of distributive lattices (such as the theory of Boolean algebras or the theory of totally-ordered sets with end points) corresponds to a subcategory A of the category of finite posets; the boundary of each such theory is another theory which in principle can be calculated by the method just described. It would be interesting to compute the boundary classes in this sense of various known classes of abelian groups. However, in this note I concentrate on another issue and on a different kind of example, the algebra of subobjects of a given object in a special kind of topos.

In any co-Heyting algebra, the boundary operator satisfies a Leibniz product rule: the boundary of any meet of two elements is the join of two meets, each involving one of the elements and the boundary of the other. (This relationship is evident in the usual diagram showing a meet as the intersection of overlapping ovals, if one takes seriously the bounding curves of the ovals). There are several other universally-valid relationships between the co-Heyting operation "non" and the lattice operations, for example any element A is the join of its boundary with its "core" non non A.

In any presheaf topos (and more generally any essential subtopos of a presheaf topos), the lattice of all subobjects of any given object is another example of a co-Heyting algebra (as well as a Heyting algebra). The co-Heyting operations are in general not preserved by substitution (inverse image) along maps, unlike the Heyting "not" and the "possibility" operators provided by Grothendieck topologies. More like those "necessity" operators for which *de dicto/de re* is a genuine distinction, we have for non in such a topos in general only the "lax" preservation

$$\text{non}(Af) \subseteq (\text{non } A)f$$

for subobjects A of the codomain of a map f. Thus in particular, the non operator is not induced by an endomap of the truth-value object. Nonetheless, it is subject to some control because of the adjointness to join, which *is* preserved by substitution.

In a presheaf topos, the elements of kind C in non A (where A is a subobject of X) are easily seen to be just those elements x of kind C in X for which there exists some map u from C to another representable D and some z in X of kind D for which x = zu and z is not in A(D). For example, in the topos of directed graphs (where u can be either "source" or "target"), the boundary of a subgraph A consists of all its nodes which are either sources or targets of arrows (in the ambient graph) which are not in the subgraph A.

Proposition: In the topos of presheaves on a small category, the co-Heyting non is preserved by substitution along all maps iff the small category is a groupoid (so that the Heyting and co-Heyting structures collapse to a Boolean one).

Some toposes support the intuition that their objects are arbitrary "spaces" of a specified kind, and this suggests another Leibniz rule for the boundary of a *cartesian* product: the boundary of a cylinder (such as the tin enclosing a tin can) is the union of two cartesian products, each involving the boundary of one of the two factors. This turns out to be true in *some* presheaf toposes; in the others, *de dicto/de re* is a real distinction even for *projection* maps, starkly underlining the need for "declaration of variables " in order to have meaningful formulas. The connection between the two issues comes from the fact that a sub-cartesian product *is* a meet, namely of inverse images along projections.

Theorem: If a small category has the property that every map factors as a split mono followed by a split epi, then, in the topos of presheaves on it, the co-Heyting non is preserved by substitution along any product projection, and in particular the Leibniz product rule for the boundary of a cartesian product (within a larger cartesian product) holds.

Since one way to obtain such a factorization of a map is to use its "graph" together with the graph of any map in the reverse direction, we have

Corollary: If a small category has all hom-sets nonempty and has binary products (*or* coproducts), then in the topos of presheaves on it, the Leibniz product rule for the co-Heyting boundary of cartesian products is valid.

There is at least one important example in which the models have neither products nor coproducts but yet have the factorization property of the theorem:

Corollary: In the topos of simplicial sets, if $A \subseteq X$, $B \subseteq Y$ then

$$\partial(A\times B) = (\partial A)\times B \cup A\times\partial B$$

for the co-Heyting boundaries ∂ in $X\times Y$, X, Y respectively.

Reference: F. W. Lawvere, Introduction to *Categories in Continuum Physics*, Springer Lecture Notes in Mathematics **1174**, 1986.

This paper is in final form and will not be published elswhere.