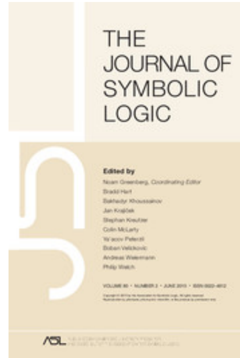

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J. Denef and L. Lipshitz

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DECISION PROBLEMS FOR DIFFERENTIAL EQUATIONS

J. DENEF AND L. LIPSHITZ

Dedicated to the memory of Julia Robinson

§1. Introduction. In this paper we shall consider some decision problems for ordinary differential equations. All differential equations will be algebraic differential equations, i.e. equations of the form $P(x, y, y', \dots, y^{(n)}) = 0$ (or $P(x, y_1, \dots, y_m, y'_1, \dots, y'_m, \dots) = 0$ in the case of several dependent variables), where P is a polynomial in all its variables with rational coefficients. (We call P a differential polynomial.) 1954

Jaśkowski [6] showed that there is no algorithm to determine if a system of algebraic differential equations (in several dependent variables) has a solution on [0, 1]. An easier proof of this using results on Hilbert's tenth problem is given in [2]. Adler'69 It is natural to restrict the problem in the hope of finding something which is decidable. Two ways to do this are the following: (a) One can ask only for the existence of solutions locally, say around $x = 0$. Here solution may mean (i) analytic functions, (ii) germs of C^∞ functions, or (iii) formal power series. (b) One can consider only one equation in one dependent variable, but ask for the existence of a solution on an interval. In [4] we gave an algorithm for deciding (a)(ii) and (a)(iii) and showed that (a)(i) is undecidable. In §4 (Theorem 4.1) we shall show that (b) (for real analytic solutions) is undecidable. This partially answers Problem 9 of Rubel [13], which stimulated this investigation. We shall also prove (Theorem 4.2) that, given an analytic function $f(x)$ which is the solution of an algebraic differential equation, determined by some initial conditions at $x = 0$, one cannot in general determine if the radius of convergence of f is < 1 or ≥ 1.

1968 Richardson [11] has shown that there is no algorithm to determine whether a function $f(x)$ built up from $+, -, \cdot, 0, 1, \pi$, sin and the variable x, by composition, has a zero on $\mathbb{R}^+ = [0, \infty)$. (Actually he shows something slightly weaker than this, but this follows from his proof and the undecidability of Hilbert's tenth problem.) In §3, we shall slightly modify his proof to obtain a family of functions $f_\nu(x)$ which all satisfy algebraic differential equations over $\mathbb{Q}$ and such that there is no algorithm

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which determines for which v $\exists x(f_v(x) = 0)$. In §2 we mention some facts about differential equations which we need in the rest of the paper. The undecidability results are proved in §4. Noncomputability results for (nonalgebraic) differential equations are given in [1], [9] and [10].

In §5 we shall give some applications to word problems. These results extend some of the results for addition, multiplication and exponentiation proved in [5], [7], [12] and [15] to a more general setting. We would like to thank Fred Rickey for bringing reference [6] to our attention.

§2. Some facts about differential equations. (i) Let $f(x) = \sum_{i=0}^{\infty} c_i x^i$ and suppose that $f(x)$ satisfies the algebraic differential equation $P(x, y, y', \ldots, y^{(m)}) = 0$ and also that $S(x, f, f', \ldots, f^{(m)}) = d_k x^k + d_{k+1} x^{k+1} + \cdots$ with $d_k \neq 0$, where $S(x, y, y', \ldots y^{(m)}) = \partial P/\partial y^{(m)}$ is the separant of P. Then the c_i satisfy a recursion of the form

order m
valuation K

$$p(i)c_i = q_i(c_0, \ldots, c_{i-1}) \tag{2.1}$$

for $i > 2k$, where $p(i)$ is a fixed polynomial (depending on $c_0, \ldots, c_{2k}$) and the q_i are polynomials. Hence, if v is chosen larger than $2k$ and all the integer zeros of $p(i)$, then the series $f(x) = \sum c_i x^i$ is completely determined by the conditions that it satisfy the above differential equation and the initial conditions $(\partial^i f/\partial x^i)(0) = c_i$ for $i < v$. The exact form of the recursion formula (2.1) (which we shall not need) is given in [8], where the above facts are also proved. We note that if the $c_i \in \mathbf{Q}$ are effectively given and the differential equation $P(x, y, y', \ldots, y^{(n)}) = 0$ has coefficients in $\mathbf{Q}$ then $p(i)$ and hence also v can be effectively found. We note further that if $f(x)$ and P are as above and we make the change of (dependent) variables $y = \sum_{i=0}^{v-1} c_i x^i + x^v u$ then $P(x, y, y', \ldots y^{(n)}) = Q(x, u, u', \ldots, u^{(n)})$ and $Q = 0$ has a unique power series solution $u(x) = \sum_{i=0}^{\infty} c_{i+v} x^i$. This shows that by making the above type of change of variables we can absorb the initial conditions into the differential equation, in the case that we are looking for analytic solutions (or formal power series solutions).

DL'84 Theorem 3.1 of [4] gives an algorithm for deciding when a system of algebraic differential equations in several unknowns has a power series solution. By using the above device of absorbing the initial conditions into the equations this also gives an algorithm for deciding when a system of algebraic differential equations together with initial conditions (in $\mathbf{Q}$) has a power series solution.

no complexity was given

(ii) Now let K be a computable field of characteristic zero (i.e. a field in which the operations $+$ and $\cdot$ are computable) and suppose further that we have an algorithm for deciding when a polynomial in $K[x]$ is irreducible, and that $\mathbf{N}$ is a computable subset of K (i.e. there is an algorithm for deciding when an element of K is a positive integer). In §4 we shall only need the case $K = \mathbf{Q}$; in §5 we shall need the more general case. (To put our results in an even more general setting we may assume that K is a countable field of characteristic zero and that we are given oracles for these two problems—irreducibility in $K[x]$ and membership in $\mathbf{N}$.) Then all the above statements remain true with K in place of $\mathbf{Q}$. Theorem 3.1 of [4] also remains true in this case (cf. Remark 3.2 of [4]). Let $f_i(x) = \sum a_{ij} x^j$ be power series with the $a_{ij} \in K$ and suppose that each f_i is determined by an algebraic differential equation and initial conditions—i.e. we are given equations $P_i(x, y, y', \ldots, y^{(m_i)}) = 0$ satisfied by $f_i(x)$ such that $S_i(x, f_i, \ldots, f_i^{(m_i)}) \neq 0$, where $S_i = \partial P_i/\partial y^{(m_i)}$, and sufficient initial conditions for the f_i (i.e. $f_i^{(j)}(0) = a_{ij}$, $j \leq I_i$) to determine the f_i as described above. Notice that this implies that each $f_i(x)$ is a computable power series—i.e. we can compute the Taylor coefficients a_{ij}. Let $\mathscr{T}$ be the set of terms built up from constants in K, the variable x, by composition starting with the functions $+, -, \cdot, ^{-1}$, and the f_i, where compositions $f_i(t(x))$ are only allowed when $t(0) = 0$, and t^{-1} is only allowed when $t(0) \neq 0$. Each term then corresponds to a power series in $K[[x]]$. We shall show that to each term $t(x)$ in $\mathscr{T}$ one can effectively associate a system of equations Σ_t, in several dependent variables $y_1, \ldots, y_r$, with some initial conditions, so that Σ_t has a unique solution in $K[[x]]$ and in fact in $K'[[x]]$ for any field $K' \supseteq K$, and in this unique solution $y_r(x) = t(x)$ (identifying the terms in $\mathscr{T}$ with their Taylor series). We do this by induction on the complexity of terms. For example if $t_3 = t_1 + t_2$ then

$$\Sigma_{t_3} = \Sigma_{t_1}(y_1, \ldots, y_{r_1}) \cup \Sigma_{t_2}(y_{r_1+1}, \ldots, y_{r_1+r_2}) \cup \{y_{r_1+r_2+1} = y_{r_1} + y_{r_1+r_2}\}.$$

Multiplication, subtraction and division are handled in a similar way. The only nontrivial step is composition $t_2(x) = f(t_1(x))$, where f is one of the f_i. Suppose we have such a system Σ_1 for $t_1(x)$. We may check if $t_1'(x) \equiv 0$ by applying the above-mentioned algorithm to the system $\Sigma_1(y_1, \ldots, y_r) \cup \{y_r' = 0\}$. If $t_1'(x) \not\equiv 0$, let $t_2(x) = f(t_1(x))$. Then

$$t_2'(x) = f'(t_1(x))t_1'(x), \qquad t_2''(x) = f''(t_1(x))(t_1'(x))^2 + f'(t_1(x))t_1''(x),$$

etc. Solving these equations for the $f^{(i)}(t_1(x))$ gives

$$f'(t_1(x)) = t_2'/t_1, \quad f''(t_1(x)) = t_2''/(t_1)^2 - t_2't_1''/(t_1')^3, \quad \text{etc.}$$

From $P_i(x, f) = 0$ and $S_i(x, f) \neq 0$ we have, by substituting $t_1(x)$ for x, that

$$P_i(t_1(x), f(t_1(x)), f'(t_1(x)), f''(t_1(x)), \ldots) = 0$$

and

$$P_i(t_1(x), t_2(x), t_2'/t_1', t_2''/(t_1')^2 - t_2't_1''/(t_1')^3, \ldots) = 0.$$

Multiplying by a suitable power (say $(t_1)^\alpha$) of t_1 to clear denominators gives an equation

$$R(t_1, t_1', \ldots, t_1^{(m_i)}, t_2, t_2', \ldots, t_1^{(m_i)}) = 0,$$

with

$$\frac{\partial R}{\partial t_2^{(m_i)}}(t_1, t_2) = (t_1')^{\alpha - m_i} \frac{\partial P_i}{\partial y_i^{(m_i)}}(t_1(x),\ f(t_1(x))) \neq 0.$$

We can use these to find a recursion formula for the Taylor coefficients of $t_2(x)$ and then determine how many initial conditions are needed to determine $t_2(x)$ from this equation, as indicated in (i) above. Then $\Sigma_{t_2} = \Sigma_1(y_1, \ldots, y_r) \cup \{R(y_r, y_{r+1}) = 0\}$ plus this set of initial conditions.

Finally we remark that we can use the system $\Sigma_t(y_1, \ldots, y_r)$ to find an equation $P_t(x, y_r, y_r', \ldots, y_r^{(m)}) = 0$ satisfied by $t(x)$ and with $(\partial P_t/\partial y^{(m)})(x, t(x)) \neq 0$. To do this apply the algorithm of Theorem 3.1 of [4] successively to the systems

$$\Sigma_1 = \Sigma_t \cup \{H(x, y_r, y_r', \ldots, y_r^{(k)}) = 0\} \quad \text{and} \quad \Sigma_2 = \Sigma_1 \cup \{\partial H/\partial y_r^{(k)} = 0\}$$

as H varies over all differential polynomials in y_r over $K[x]$, until we find an H such that Σ_1 has a solution and Σ_2 does not. (Under the above conditions there always is an algebraic differential equation $R(x, y, y', \ldots) = 0$ satisfied by $t(x)$, and if we take R of lowest possible order, m say, and lowest possible degree in $y^{(m)}$, then $S(x, t) = (\partial R/\partial y^{(m)})(x, t) \neq 0$.) Once we know that $S(x, t) \neq 0$ we can find k such that $S(x, t) = d_k x^k$ + higher order forms with $d_k \neq 0$ by computing the Taylor coefficients of $t(x)$ and plugging into S.

§3. **Richardson's functions.** In this section we modify the construction of [11] to obtain a family $f_v(x)$ of functions on $(-\infty, \infty)$ with the following properties:

(i) There is no algorithm for deciding whether $\exists x \in \mathbb{R}^+(f_v(x) < 1/4)$.

(ii) The f_v are real analytic and the Taylor series for the f_v at $x = 0$ have rational coefficients.

(iii) (see §4) There is an algorithm which will produce an algebraic differential equation $P_v(x, y, y', \ldots, y^{(n)}) = 0$ such that $P_v(x, f_v(x)) = 0$ and $(\partial P_v/\partial y^{(n)})(x, f_v(x)) \neq 0$.

From the undecidability of Hilbert's tenth problem (see for example [3]) there is a polynomial $P(y, x_1, \ldots, x_n) \in \mathbb{Z}[y, x_1, \ldots, x_n]$ such that $\{v \in \mathbb{N}: \exists x_1, \ldots, x_n \in \mathbb{N}\ (P(v, x_1, \ldots, x_n) = 0)\}$ is nonrecursive. Fix such a P. We may take $n \leq 9$. Choose polynomials $k_i(v, x_1, \ldots, x_n)$ such that if $|x_i - \tilde{x}_i| < 1$ for $i = 1, \ldots, n$ then

$$k_i(v, x_1, \ldots, x_n) > |D_{x_i} P^2(v, \tilde{x}_1, \ldots, \tilde{x}_n)|,$$

where $D_{x_i} = d/dx_i$. Given P, this is easy to do effectively (see [11, p. 516]). Let

$$f(v, v, x_1, \ldots, x_n) = (n + 1)^4 \left[P^2(v, x_1, \ldots, x_n) + \sum_{i=1}^{n} \sin^2(v x_i) k_i^4(v, x_1, \ldots, x_n) \right],$$

where the k_i are as above.

LEMMA 3.1 [11, Theorem 1, p. 516].

(a) $\qquad \exists x_1, \ldots, x_n \in \mathbb{N}(P(v, x_1, \ldots, x_n) = 0)$

(b) $\qquad \Leftrightarrow \exists x_1, \ldots, x_n \in \mathbb{R}^+(f(v, \pi, x_1, \ldots, x_n) = 0)$

(c) $\qquad \Leftrightarrow \exists x_1, \ldots, x_n \in \mathbb{R}^+(f(v, \pi, x_1, \ldots, x_n) \leq 1)$.

PROOF. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are immediate. The idea of (c) $\Rightarrow$ (a) is to let $\tilde{x}_i$ be the nearest integer to x_i and to use the definition of f and the construction of the k_i to see that $P(v, \tilde{x}_1, \ldots, \tilde{x}_n) < 1$. Since P has integer coefficients this forces $P(v, \tilde{x}_1, \ldots, \tilde{x}_n) = 0$. The details can be found in [11]. ■

Let $g_i(w) = w \sin w^{2i-1}$.

LEMMA 3.2 (cf. [11, Theorem 2, p. 518]). *For all $x_1, \ldots, x_n \in \mathbb{R}$ and all $\delta > 0$ there exist arbitrarily large w such that $|g_i(w) - x_i| < \delta$ for $i = 1, \ldots, n$.*

PROOF. By induction on n, let $\bar{w}$ be greater than $|x_1|, \ldots, |x_n|, (6\pi/\delta)^2$ and 100, and satisfy $|g_i(\bar{w}) - x_i| < \delta/2$ for $i = 1, \ldots, n - 1$. Let $w = \bar{w} + a$. Then

$$\begin{aligned} g_n(w) &= (\bar{w} + a)\sin(\bar{w} + a)^{2n-1} \\ &= (\bar{w} + a)\sin\left(\bar{w}^{2n-1} + (2n-1)\bar{w}^{2n-2}a + \binom{2n-1}{2}\bar{w}^{2n-3}a^2 + \cdots\right). \end{aligned}$$

As a varies from 0 to $3\pi/(2n-1)\bar{w}^{2n-2}$ the argument of the sine increases by at least 2π, and hence $g_n(w)$ takes on all values between $\pm\bar{\omega}$. Hence there is an a in this range such that $g_n(w) = x_n$. Now for $i < n$, by the intermediate value theorem,

$$g_i(\bar{w}+a) = g_i(\bar{w}) + a[\sin(\bar{w}+c)^{2i-1} + (\bar{w}+c)\cos(\bar{w}+c)^{2i-1}(2i-1)(\bar{w}+c)^{2i-2}]$$

for some c with $0 < c < a$. Hence

$$|g_i(\bar{w}+a) - g_i(\bar{w})| \leq a[1 + (2i-1)(\bar{w}+c)^{2i-1}] < a2i(\bar{w}+a)^{2i-1}$$

$$< \frac{3\pi}{(2n-1)\bar{w}^{2n-2}} 2i(\bar{w}+1)^{2i-1} < \frac{3\pi}{\bar{w}^{1/2}} < \frac{\delta}{2}. \quad \blacksquare$$

Now let

$$F(v, v, x) = f(v, g_1^2(x), \ldots, g_n^2(x)).$$

Choose $k \in \mathbb{N}$ such that for $|x| \geq 2$

$$|x|^{2k-2} > 8(n+1)^4 \sum_{i=1}^{n} k_i^4(v, g_1^2(x), \ldots, g_n^2(x)).$$

Since the $|g_i(x)| \leq |x|$, this is easy to do effectively. Let

$$G(x) = F(v, 2\tan^{-1}(x^{2k}), x) + A/(x^2+1),$$

where $A \in \mathbb{N}$ is chosen so that $G(x) > 1$ for $|x| \leq 2$. Again this is easy to do effectively.

LEMMA 3.3. $\exists x_1, \ldots, x_n \in \mathbb{N}(P(v, x_1, \ldots, x_n) = 0) \Leftrightarrow \exists x \in \mathbb{R}^+(|G(x)| \leq 1/4)$.

PROOF. The direction $\Rightarrow$ is immediate from Lemmas 3.1 and 3.2 and the fact that

$$\lim_{x\to\infty} 2\tan^{-1}(x^{2k}) = \pi.$$

For the other direction suppose that $|G(x)| \leq 1/4$. Let $\tilde{G}(x) = F(v, \pi, x)$. From the definitions and Lemma 3.1 it is sufficient to show that $|\tilde{G}(x)| \leq 1$. Since $|G(x)| \leq 1/4$ and F is nonnegative it follows that $A/(x^2+1) \leq 1/4$ and $F(v, 2\tan^{-1}(x^{2k}), x) \leq 1/4$, and $|x| \geq 2$ (from the choice of A). Now

$$\tilde{G}(x) - G(x) - \frac{A}{x^2+1} = -F(v, 2\tan^{-1}(x^{2k}), x) + F(v, \pi, x)$$

$$= \int_{2\tan^{-1}(x^{2k})}^{\pi} F_v(v, v, x)\, dv,$$

where

$$F_v(v, v, x) = \partial F(v, v, x)/\partial v$$

$$= (n+1)^4 \left[\sum_{i=1}^{n} 2\sin(vg_i^2(x))\cos(vg_i^2(x))g_i^2(x)k_i^4(v, g_1^2(x), \ldots, g_n^2(x))\right]$$

$$\leq 2(n+1)^4 \sum_{i=1}^{n} g_i^2(x)k_i^4(v, g_1^2(x), \ldots, g_n^2(x))$$

$$\leq 2(n+1)^4 \sum_{i=1}^{n} x^2 k_i^4(v, g_1^2(x), \ldots, g_n^2(x)).$$

So

$$\left|\tilde{G}(x) - G(x) - \frac{A}{x^2+1}\right| \le 2(n+1)^4 x^2 \sum_{i=1}^{n} k_i^2(v, g_1^2(x), \ldots, g_n^2(x)) |\pi - 2\tan^{-1}(x^{2k})|.$$

Now $|\pi - 2\tan^{-1}(x^{2k})| < 2/x^{2k}$, so

$$\left|\tilde{G}(x) - G(x) - \frac{A}{x^{2+1}}\right| < \frac{4(n+1)^4}{x^{2k-2}} \sum_{i=1}^{n} k_i^4(v, g_1^2(x), \ldots, g_n^2(x)) \le 1/2$$

by our choice of k, since $|x| \ge 2$. Hence $|\tilde{G}(x)| \le 1$. ■

REMARK. If $\exists x \in \mathbb{R}^+(|G(x)| \le 1/4)$ then in fact for every $\varepsilon > 0$ there are arbitrarily large values of x for which $|G(x)| < \varepsilon$.

§4. Undecidability results. Let $G(x)$ (actually depending on v) be as above, and let $\bar{u}(x) = [\frac{1}{4} - G(x)]^{-1}$. The function $G(x)$ is analytic near $x = 0$, $G(0) > 1$, and all the Taylor coefficients of G are rational. Notice that $\bar{u}(x)$ is built up from x, $\sin x$, $\tan^{-1}(x)$, $+, \cdot, {}^{-1}$ and rational constants by composition and that in this building up sin and $\tan^{-1}$ are only applied to terms which vanish at 0. Hence as described in §2(ii) (with $K = \mathbb{Q}$) we can effectively find an algebraic differential equation $A(x, w, w', \ldots, w^{(m)}) = 0$ satisfied by $\bar{u}$ and such that $(\partial A/\partial w^{(m)})(x, \bar{u}(x)) \ne 0$. Then also as described in 2(ii) we can find a k and initial conditions $\bar{u}^{(i)}(0) = a_i$, $i = 0, \ldots, k-1$, which together with $A = 0$ completely determine $\bar{u}(x)$ as a power series. If we make the change of variables $\bar{u}(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + x^k u(x)$ and substitute into $A(x, \bar{u}(x)) = 0$ we get a differential equation $B(x, u, u', \ldots, u^{(m)}) = 0$ which has a unique analytic solution at $x = 0$. This solution extends to a real analytic solution on $[0, \infty)$ if and only if $G(x) > 1/4$ for all $x \in \mathbb{R}^+$, which is the case if and only if $P(v, x_1, \ldots, x_u) = 0$ has no solution for $x_1, \ldots, x_n \in \mathbb{N}$. Thus we have proved

THEOREM 4.1. *Given an algebraic differential equation in one variable, with coefficients from $\mathbb{Z}$, it is not in general possible to test if it has an analytic solution on $[0, \infty)$. The same remains true even if we know that the equation has a unique analytic solution at $x = 0$.*

Next we will extend the above construction to prove

$\Rightarrow$ DA

THEOREM 4.2. *Given a power series $v(z) = \sum a_i z^i$, with the $a_i \in \mathbb{Q}$, which is determined by an algebraic differential equation and initial conditions, one cannot in general tell if the radius of convergence of $v(z)$ is < 1 or ≥ 1.*

PROOF. The idea of the proof is very simple. Let $G(x)$ be as above (actually depending on v). If $G(x) > 1/4$ on all of $\mathbb{R}^+$ then there is an open region Ω in $\mathbb{C}$ with $\mathbb{R}^+ \subset \Omega$ such that $G(x) \ne 1/8$ on Ω. On the other hand if $\exists x_{\in \mathbb{R}^+}(G(x) < 1/4)$ then, by the remark following Lemma 3.3, $\exists x_{\in \mathbb{R}^+}(G(x) = 1/8)$. Hence $\bar{\bar{u}}(x) = [1/8 - G(x)]^{-1}$ is analytic on Ω if and only if $P(v, x_1, \ldots, x_n)$ has no zero in $\mathbb{N}^n$. Next we will find an analytic function $x = \varphi(z)$ which maps the open unit disc D in $\mathbb{C}$ into a region U with $\mathbb{R}^+ \subseteq U \subseteq \Omega$. Then $v(z) = \bar{\bar{u}}(\varphi(z))$ is analytic on D if and only if $P(v, x_1, \ldots, x_n)$ has no zero on $\mathbb{N}^n$. Of course, φ must be chosen so that $v(z)$ satisfies an algebraic differential equation, and we must be able to effectively find these equations (depending on v). Notice that for $0 \ll |z|$ we have $|\sin z| < 2e^{|z|}$ $(z \in \mathbb{C})$. Hence it is easy to see, from the definition of $G(x)$, that for large values of $|z|$ we have $|G'(z)| < e^{e^{e^{|z|}}}$. Using this and the fact that $G(z)$ is computable we can effectively find $N, M \in \mathbf{N}$ such that on the strip $z = x + iy$, $x \geq -1/N$, $|y| \leq 1/N$, we have $|G'(z)| < Me^{e^{e^{|z|}}}$. ($N$ must be chosen large enough so that we avoid the singularities of $\tan^{-1}(z^{4k})$ and $1/(1+z^2)$. These occur at $z = \pm i$ and $z = e^{ij\pi/k}$ for $j = 1, 2, \ldots, 2k-1$.) Hence what we need is a φ, mapping the unit disc D into a region as shown in Figure 1, below.

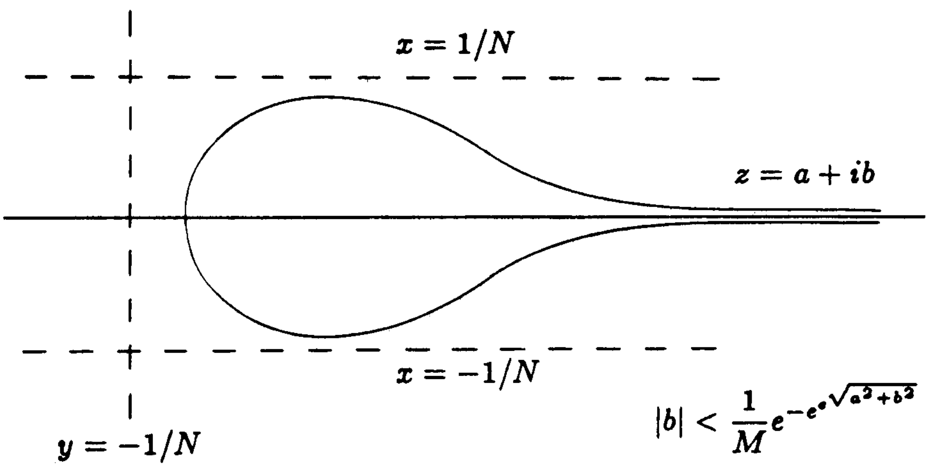

FIGURE 1

If we let $\psi(w) = \log(1 + w)$, then for K large enough $\psi^4(-\log\sqrt{1-z})/K$ works. We leave the verification of this to the reader—it is a routine calculation. What this function does to the unit disc is indicated in Figure 2, below.

Hence one can effectively find a $K \in \mathbf{N}$ such that if we take

$$\varphi(z) = \psi^4(-\log\sqrt{1-z})/K,$$

then

$$|\operatorname{Im} \varphi(z)| < \frac{1}{8M} e^{-e^{e^{|\varphi(z)|}}} \quad \text{for } z \in D.$$

($\operatorname{Im} \varphi(z)$ is the imaginary part of $\varphi(z)$.) It follows that if $G(x) > 1/4$ on $\mathbf{R}^+$ then $G(\varphi(z)) \neq 1/8$ on D and hence $v(z) = \bar{\bar{u}}(\varphi(z))$ is analytic on D. Exactly the same argument as in the proof of Theorem 4.1 (using the functions $\tan^{-1}(x)$, $\sin x$, $\log\sqrt{1-x}$ and $\log(1+x)$) shows that we can find an algebraic differential equation $E(z, v, v', \ldots, v^{(m)}) = 0$ and initial conditions which completely determine the power series $v(z)$. ■

REMARK. It would be interesting to know if theorems analogous to 4.1 and 4.2 hold for linear differential equations.

probably not ?

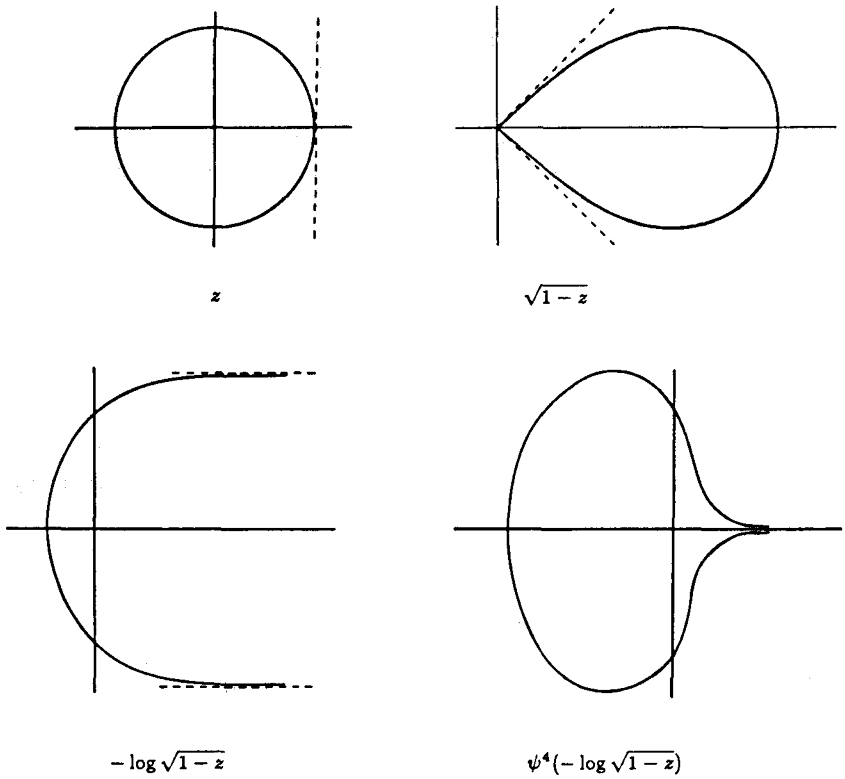

FIGURE 2

§5. **Identity problems.** Several proofs have been given (cf. [5], [7], [12] and [15]) showing that if one has an algorithm to decide identities among constants built up from $\mathbb{Q}$ by $+$, $-$, $\cdot$, $^{-1}$ and e^x (or x^y) then one also has an algorithm to decide identities among terms built up in the same way but also involving variables. In this section we shall show that the same is true in a much broader context.

THEOREM 5.1. *Let K be a computable field of characteristic* 0 *and assume that $\mathbb{N}$ is a computable subset of K and that we have an algorithm to check polynomials in $K[x]$ for irreducibility.* (*Or let K be any countable field of characteristic zero and assume that we are given oracles for determining membership in $\mathbb{N}$ and irreducibility of polynomials in $K[x]$.*) *Let $f_i(x) = \sum a_{ij}x^j \in K[[x]]$ be power series for $i = 1, \ldots, I$,* DA *and suppose that each $f_i(x)$ is determined by an algebraic differential equation $P_i(x, w, w', \ldots, w^{(m_i)}) = 0$ and initial conditions $f_i^{(j)}(0) = a_{ij}$, $j = 0, \ldots, k_i$. Let $\mathscr{T}$ be the set of all terms built up from variables $x_1, x_2, \ldots$, elements of K, $+$, $-$, $\cdot$, $^{-1}$ and the f_i by composition, where $f_i(t(x))$ is only allowed when $t(0) = 0$ and $t^{-1}(x)$ is only allowed when $t(0) \neq 0$. Then there is an algorithm for determining when a term in $\mathscr{T}$ is identically zero* (*as a power series*).

PROOF. Let $t(x_1, \ldots, x_n)$ be given. Let $\bar{K} = K(\tau_1, \ldots, \tau_n)$, where the τ_i are algebraically independent over K. Observe that $\bar{K}$ also has the properties which we assumed for K, and that

$$t(x_1, \ldots, x_n) = 0 \Leftrightarrow t(\tau_1 x, \ldots, \tau_n x) = 0.$$

As described in §2(ii) we can find a system $\Sigma(y_1, \ldots, y_r)$ such that Σ has a unique solution $\bar{y}_1, \ldots, \bar{y}_r$ (in $\mathbf{C}[[x]]$) and $\bar{y}_r = t(\tau_1 x, \ldots, \tau_n x)$. We can now apply the algorithm of Theorem 3.1 of [~~2~~4] to the system $\Sigma' = \Sigma \cup \{y_r = 0\}$ to determine if $t \equiv 0$. (It is worth remarking that we are applying this algorithm to quite special systems—Σ' has at most one solution, and we can generate as much of this solution as we wish—several of the complications in the algorithm fall away.) ■

Solvability in power series

REMARKS. (1) The restriction on composition, that $f(t(x))$ is only allowed when $t(0) = 0$, is not crucial. If we want a term of the form $f(t(x))$ where $t(0) = a \neq 0$, and f is analytic and a lies in the circle of convergence of f, we can take $g(x) = f(a + x)$ as a new initial function and consider $g(t_1(x))$ where $t_1(x) = t(x) - a$. One can easily devise a differential equation for $g(x)$. Of course K must contain all the Taylor coefficients of g—that might force one to work over a larger field.

(2) The algorithm in Theorem 3.1 of [4] actually gives a little more information. Given a term $t(x)$, the above procedure computes $\beta \in \mathbb{N}$ such that if $t(x) \equiv 0 \bmod x^\beta$ (i.e. the first β Taylor coefficients of t are 0) then $t(x) \equiv 0$.

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DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY OF LEUVEN
B-3030 LEUVEN, BELGIUM

DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907