ON THE ESTIMATION OF $N(\sigma, T)$

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LET $N(\sigma, T)$ be the number of zeros $\rho = \beta+\gamma i$ of the Riemann zeta-function $\zeta(s)$ for which $\beta \geqslant \sigma$, $0 < \gamma \leqslant T$. It is proved in my paper [2] that the estimate

$$N(\sigma, T) = O(T^{\lambda(\sigma)(1-\sigma)}\log^5 T) \tag{1}$$

holds uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ as $T \to \infty$, when

$$\lambda(\sigma) = 1+2\sigma, \tag{2}$$

and also when

$$\lambda(\sigma) = 2+4c, \tag{3}$$

where c is a constant for which

$$\zeta(\tfrac{1}{2}+ti) = O(t^c) \quad \text{as } t \to \infty.$$

In this note we sketch a proof of (1) with

$$\lambda(\sigma) = \frac{3}{2-\sigma}.$$

This is an improvement on (2), since

$$(1+2\sigma)-\frac{3}{2-\sigma} = \frac{(2\sigma-1)(1-\sigma)}{2-\sigma} > 0 \quad (\tfrac{1}{2} < \sigma < 1),$$

but does not help in the application to the problem of the order of magnitude of the difference between consecutive primes, because it is the maximum of $\lambda(\sigma)$ for $\frac{1}{2} \leqslant \sigma \leqslant 1$ that is important in this problem.

We assume that the reader is familiar with § 3 and § 5 of [2], and merely indicate the essential changes in the argument.

As in [2], let

$$f_X(s) = \zeta(s) \sum_{n<X} \mu(n)n^{-s}-1 = \zeta(s)M_X(s)-1.$$

For $\sigma = \sigma_1 = \frac{1}{2}$, $\mu = \mu_1 = \frac{4}{3}$, we have, using (26) and a formula near the bottom of page 260 of [2],

$$\begin{aligned}
\int_0^T |f_X(\sigma+ti)|^\mu \, dt &\leqslant \int_0^T A_1(|\zeta|^{\frac{4}{3}}|M_X|^{\frac{4}{3}}+1)\, dt \\
&\leqslant A_1\left(\int_0^T |\zeta|^4 \, dt\right)^{\frac{1}{3}}\left(\int_0^T |M_X|^2 \, dt\right)^{\frac{2}{3}}+A_1 T \\
&< A_2\{T\log^4(T+2)\}^{\frac{1}{3}}\{(T+X)\log X\}^{\frac{2}{3}}+A_1 T \\
&< A_3(T+X)\log^2(T+X) \\
&< A_4(T+X)^{1+\delta}\delta^{-2} \quad (T > 0,\ X \geqslant 3,\ \delta > 0),
\end{aligned}$$

where $A_1, A_2, \ldots$ are positive absolute constants.

For $\sigma = \sigma_2 = 1+\delta$ $(0 < \delta < 1)$, $\mu = \mu_2 = 2$, we have, by [2], (17),

$$\int_0^T |f_X(\sigma+ti)|^\mu\, dt < A_5(T+X)X^{-1}\delta^{-4} \quad (T > 0,\ X \geqslant 3).$$

Arguing substantially as on page 261 of [2], but using a two-variable convexity theorem of Gabriel [1, Theorem 2]*, we deduce that, if $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$ $(0 < \delta < 1)$, and μ, ρ_1, ρ_2 are defined by

$$\frac{1}{\mu} = \frac{\vartheta_1}{\mu_1}+\frac{\vartheta_2}{\mu_2}, \qquad \rho_1 = \frac{\vartheta_1 \mu}{\mu_1}, \qquad \rho_2 = \frac{\vartheta_2 \mu}{\mu_2},$$

where
$$\vartheta_1 = \frac{1+\delta-\sigma}{\frac{1}{2}+\delta}, \qquad \vartheta_2 = \frac{\sigma-\frac{1}{2}}{\frac{1}{2}+\delta},$$

then, for $T > 1$, $X \geqslant 3$,

$$\int_1^T |f_X(\sigma+ti)|^\mu\, dt < A_6\{(T+X)^{1+\delta}\delta^{-2}\}^{\rho_1}\{(T+X)X^{-1}\delta^{-4}\}^{\rho_2}. \tag{4}$$

Now $\rho_1+\rho_2 = 1$,

$$\therefore\ \frac{1}{\mu} = \frac{\frac{3}{4}(1+\delta-\sigma)}{\frac{1}{2}+\delta}+\frac{\frac{1}{2}(\sigma-\frac{1}{2})}{\frac{1}{2}+\delta} = \frac{1+\frac{3}{2}\delta-\frac{1}{2}\sigma}{1+2\delta},$$

$$\rho_2 = \frac{(\sigma-\frac{1}{2})(1+2\delta)}{(\frac{1}{2}+\delta)(2+3\delta-\sigma)} = \frac{2\sigma-1}{2+3\delta-\sigma} > \frac{2\sigma-1}{2-\sigma}-A_7\delta.$$

Substituting in (4) and taking $\delta = A_8/\log(T+X)$, we obtain

$$\int_1^T |f_X(\sigma+ti)|^\mu\, dt < A_9(T+X)X^{-(2\sigma-1)/(2-\sigma)}\log^4(T+X),$$

for $T > 1$, $X \geqslant 3$, $\frac{1}{2} \leqslant \sigma \leqslant 1$, and a certain $\mu = \mu(\sigma, T, X)$ in the range $\frac{4}{3} \leqslant \mu \leqslant 2$.

Arguing now as on pages 262–3 of [2], using the fact that

$$\log|1-f_X^2| \leqslant \log(1+|f_X|^2) \leqslant A_{10}|f_X|^\mu \quad (\tfrac{4}{3} \leqslant \mu \leqslant 2),$$

and taking $X = T$, we obtain the stated result.

A similar argument with $\mu_1 = 1$, and with the mean value of $|\zeta(\frac{1}{2}+ti)|^2$ in place of that of $|\zeta(\frac{1}{2}+ti)|^4$, would give Titchmarsh's value $\lambda(\sigma) = 4/(3-2\sigma)$.

* The theorem is applied, with $\alpha = \sigma_1$ $(= \frac{1}{2})$, $\beta = \sigma_2$ $(= 1+\delta)$, $a = 1/\mu_1$, $b = 1/\mu_2$, $\lambda = \vartheta_1$, $\lambda' = \vartheta_2$, to the auxiliary function $\phi(s) = \phi_{X,\tau}(s)$ constructed in [2], 261.

REFERENCES

1. R. M. Gabriel, 'Some results concerning the integrals of moduli of regular functions along certain curves': *J. of London Math. Soc.* 2 (1927), 112–17.
2. A. E. Ingham, 'On the difference between consecutive primes': *Quart. J. of Math.* (Oxford), 8 (1937), 255–66.