Some Aspects of Model Theory and Finite Structures

Eric Rosen

Bulletin of Symbolic Logic / Volume 8 / Issue 03 / September 2002, pp 380 - 403
DOI: 10.2178/bsl/1182353894, Published online: 15 January 2014

The Bulletin of Symbolic Logic
Volume 8, Number 3, Sept. 2002

SOME ASPECTS OF MODEL THEORY AND FINITE STRUCTURES

ERIC ROSEN

Model theory is concerned mainly, although not exclusively, with infinite structures. In recent years, finite structures have risen to greater prominence, both within the context of mainstream model theory, e.g., in work of Lachlan, Cherlin, Hrushovski, and others, and with the advent of *finite model theory*, which incorporates elements of classical model theory, combinatorics, and complexity theory. The purpose of this survey is to provide an overview of what might be called the *model theory of finite structures*. Some topics in finite model theory have strong connections to theoretical computer science, especially descriptive complexity theory (see [26, 46]). In fact, it has been suggested that finite model theory really is, or should be, logic for computer science. These connections with computer science will, however, not be treated here.

It is well-known that many classical results of 'infinite model theory' fail over the class of finite structures, including the compactness and completeness theorems, as well as many preservation and interpolation theorems (see [35, 26]). The failure of compactness in the finite, in particular, means that the standard proofs of many theorems are no longer valid in this context. At present, there is no known example of a classical theorem that remains true over finite structures, yet must be proved by substantially different methods. It is generally concluded that first-order logic is 'badly behaved' over finite structures.

From the perspective of expressive power, first-order logic also behaves badly: it is both too weak and too strong. Too weak because many natural properties, such as the size of a structure being even or a graph being connected, cannot be defined by a single sentence. Too strong, because every class of finite structures with a finite signature can be defined by an infinite set of sentences. Even worse, every finite structure is defined up to isomorphism by a single sentence. In fact, it is perhaps because of this last point more than anything else that model theorists have not been very interested in finite structures. Modern model theory is concerned largely with complete first-order theories, which are completely trivial here.

Received May 23, 2000; revised form May 20, 2002.

1079-8986/02/0803-0003/$3.40

Nevertheless, finite structures do arise naturally in 'infinite model theory'. For example, Zilber proved that any sentence in a totally categorical theory has a finite model. Around the same time, Lachlan developed a general classification theory for finite homogeneous structures which is intimately connected to his theory of stable finitely homogeneous structures. Some more ways that finite structures figure in infinite model theory are discussed below. One aim of this paper is to highlight some of these aspects of the model theory of finite structures, where the finite and infinite interact fruitfully, in order to dispel the perhaps too common perception that (first-order) model theory has little to say about finite structures.

On the other hand, what seems to be lacking in the model theory of finite structures is some kind of general or unifying program, calling attention to interesting problems and projects. Shelah's program for classifying first-order theories and his invention of stability theory provided a unifying framework for model theory — as well as many ideas and tools for further developments, for example, in geometric model theory and in o-minimal and simple theories. Is there perhaps some kind of general classification theory for finite structures? One obstacle, mentioned above, is that first-order logic does not seem to be the right logic to use for investigating finite structures. Recent work by Baldwin, Djordjević, Hyttinen, Lessmann, and others, developing stability theory for finite variable logic, may perhaps provide the appropriate framework.

A large portion of this paper, including most of Sections 2 through 5, considers the following general question, which has been a central aspect of finite model theory. What happens to various 'metamathematical' theorems, such as preservation and interpolation theorems, over the class of finite structures? (See [35, 26].) One point that I wish to make is that there are a number of possibilities, beyond discovering that a theorem becomes false, or remains true. For example, although the completeness theorem fails, Trakhtenbrot's theorem says that the set of sentences valid over the class of finite structures is co-r.e. complete. In Section 1.2, I suggest a taxonomy of phenomena of this kind.

In Section 6, as well as in parts of Section 2, we briefly survey some work on finite structures that is part of, or arises out of, modern model theory, from which a more systematic theory of finite structures might emerge.

Most of our notation is standard. An $\exists_n$, resp. $\forall_n$ sentence is a first-order sentence in prenex normal form, whose quantifer prefix consists of n alternating blocks of $\exists$s and $\forall$s, beginning with $\exists$, resp. $\forall$. $\mathrm{Mod}_f(\varphi)$ is the class of finite models of φ.

Standard references for model theory are [13, 40]; for finite model theory and descriptive complexity theory, see [26, 46].

Acknowledgements. I would like to thank Alexei Stolboushkin for some helpful comments and for allowing me to include an example due to him in

Section 3. I am also extremely grateful to John Baldwin for many helpful discussions on the subject of this paper, as well as for carefully reading an earlier version.

§1. Finite structures.

1.1. First-order logic over finite structures. We provide a very brief overview of the classical model theory of first-order logic over finite structures. That is, all semantic notions are redefined by restricting all structures to be finite. We also assume throughout that each signature is finite.

The failure of classical theorems. Most classical theorems about first-order logic (the compactness theorem, interpolation theorems, and preservation theorems) fail in the finite. For example, the failure of the compactness theorem is witnessed by the theory $T = \{\varphi_n \mid n \in \omega \setminus \{0\}\}$, where φ_n says that there are at least n elements. Clearly, every finite subset of T has a finite model, although T itself does not. Counterexamples to other classical theorems are more complicated, but few such results involve a difficult argument. Preservation theorems are discussed at length in Section 2.

Definability and Ehrenfeucht-Fraïssé games. An important concern in model theory is the notion of definability. The compactness theorem is a simple and effective tool for proving that certain classes of structures are not definable, but it is no longer available in the finite context. Nevertheless, questions of definability play a central role in finite model theory. As it is clear that *every* class of finite structures (over a finite signature) can be defined by an infinite set of first-order sentences, interest is restricted to the notion of definability by a single sentence. (On the other hand, in Section 5.1 I briefly discuss the study of finite models of complete L^k-theories.) By far the most important method available for proving non-definability results is the technique of Ehrenfeucht-Fraïssé which gives an algebraic, or game-theoretic, characterization of definability. This is one of the few results from classical model theory that remain true over finite structures. Ehrenfeucht-Fraïssé games provide a simple way to show that many classes of structures, for example the finite linear orders of even length, are not definable.

Decidability; questions of effectiveness. Model theorists are occasionally interested in questions involving decidability or effectiveness. Is a particular theory decidable? Does it have a recursive model? Of course every finite structure is recursive, so it is natural to consider somewhat different kinds of questions in finite model theory, often connected to computational complexity theory. For example, Fagin proved that a class of finite structures is defined by a Σ^1_1 sentence if and only if it is in the complexity class NP.

A rather old subject is the *classical decision problem*. Which classes of first-order formulas have a decidable satisfiability problem? For prefix classes this has been completely answered (see [12]). In fact, the analagous problem for finite structures was solved simultaneously, using essentially the same proofs. It turns out that, for any prefix class P, the class of first-order sentences whose prefix is in P has a decidable satisfiability problem iff it has a decidable finite satisfiability problem (though there is no 'meta-theorem' explaining why it must be so).

On the other hand, there is an important difference between the finite and the infinite. By the completeness theorem, the satisfiability problem for any class of sentences is always co-r.e., though not necessarily r.e. For finite structures, the situation is dual. Satisfiability is trivially r.e., though not necessarily co-r.e. In fact, for every prefix class P whose satisfiability problem is not decidable, the set of satisfiable sentences is co-r.e. complete, whereas the set of finitely satisfiable sentences is r.e. complete.

One explanation for the difference is that being a well-ordering is definable over the class of finite structures, as every finite order is well-ordered. This makes it possible, for example, to code Turing machine computations over finite structures, so that for every Turing machine M one can effectively find a sentence φ_M that has a finite model iff M halts. (This essentially proves Trakhtenbrot's theorem that the set of finitely satisfiable sentences is r.e.-complete).

The definability of well-orderings also explains why *in some sense* first-order logic over finite structures is more expressive than over the class of all structures. It follows rather easily that Beth's definability theorem fails over finite structures and that every class of finite structures defined by a sentence in least-fixed point logic is implicitly definable in first-order logic [50].

Stability and classification theory. Stability theory has had very little to say about finite structures, though recently there has been an attempt to develop stability theory for finite structures (see [5, 25, 45, 44, 6, 54, 24]).

On the other hand, Lachlan has developed a very general theory of finite homogeneous relational structures, which is intimately connected to the investigation of stable finitely homogeneous structures and $\aleph_0$-categorical, $\aleph_0$-stable structures (see the survey articles [51, 52, 18]). In this paper, a structure A is called *homogeneous* if every finite partial isomorphism from A to itself extends to an automorphism. Lachlan shows that for any fixed finite relational signature σ, the class of finite homogeneous σ-structures contains finitely many infinite families of structures as well as finitely many 'sporadics'. (For example, with finitely many exceptions every finite homogeneous graph is either a complete k-partite graph or the complement of such a graph.) Cherlin's paper [18] contains a nice discussion of the kind of classification problems that arise in the context of Lachlan's work. Cherlin and Hrushovski [17] have extended this theory in their work on large finite structures with few 4-types and smoothly approximated structures.

In a few particular cases, the finite homogeneous structures have been completely classified, for example, for graphs [29], 3-graphs [53], groups [19], and rings (see [65]).

Extending partial isomorphisms. Hrushovski proved that every finite graph G can be extended to another finite graph H such that every partial isomorphism of G extends to an automorphism of H [42]. Hodges, Hodkinson, Lascar, and Shelah used this result to establish that the random graph has the small index property [41]. Hrushovski's theorem has been generalized by Herwig and Lascar to other classes of finite structures [36, 37, 38].

1.2. Relativization to finite structures. A natural place to start when considering model theory over finite structures is to ask what happens when we try to relativize well-known classical results from model theory to the class of finite structures. For example, it was mentioned above that the compactness theorem fails in the finite. Below is a list of other kinds of possibilities beyond simply 'failure' or 'success'.

Results that become meaningless, e.g., the Löwenheim-Skolem theorem. Certain results, such as the Löwenheim-Skolem theorem, are only about infinite structures. As such, they have nothing to say about finite structures and there is nothing more to be done.

Purely negative results, e.g., compactness. Other theorems, such as the compactness theorem, can be rephrased or 'relativized' to finite structures, though the new statement may be, and usually is, false. In certain cases, there may be some possibility that a similar or related version of the theorem does hold. The compactness theorem, though, seems to be an example of a theorem which fails so badly that nothing of it can be salvaged.

New results, e.g., Trakhtenbrot's theorem. The completeness theorem too fails over finite structures. But in contrast to the compactness theorem, this is not the end of the story. Whereas the set of valid sentences is r.e.-complete, Trakhtenbrot proved that the set of sentences valid over finite structures is rather co-r.e.-complete. Here is an example, probably the best one, where the failure of a classical result led to something new that perhaps even sheds some light on the original theorem.

In Section 4.1, we will mention a number of open questions with a similar motivation: they ask whether suitably modified versions hold of theorems that are known to fail on finite structures.

Results that remain true. Certain results do remain true over the class of finite structures, for example, the theorem of Ehrenfeucht-Fraïssé. In this case, the proof simply works equally well in the finite, so there was nothing additional to verify. Likewise, all results about the decidability of the satisfaction problem for first-order prefix classes remain true over finite structures. A slight modification of the argument is required, though the proof strategy remains similar.

In contrast, there seems to be no example of a theorem that remains true when relativized to finite structures but for which there are entirely different proofs for the two cases. It would be interesting to find a theorem proved using the compactness theorem that can be established using a new method over finite structures. It is known that many of the candidates for such a theorem, such as preservation and interpolation theorems, fail in the finite.

1.3. The difficulty with finite structures. We have already mentioned that the failure of the compactness theorem over finite structures deprives us of the most important tool of model theory. But there are also a number of other reasons why the situation in the finite is so different, and often more difficult, than in the more general setting. For example, over finite structures there is no adequate notion of a free structure. This, rather than anything to do with compactness, seems to be the reason why Birkhoff's preservation theorem (see Section 2.4 below) fails in the finite.

More importantly, it can be extremely difficult to construct finite structures. To a certain extent, this is connected to the failure of compactness, which provides a powerful way to build structures, but there are also other reasons. For example, certain natural constructions involve taking limits of chains of structures, but these limits can be infinite, even when each structure in the chain is finite. In other words, the class of finite structures is not closed under certain natural operations. We illustrate this point below.

Take the set of 'extension axioms' that axiomatize the random graph, consisting of all those sentences that say, for each m and n in $\omega, 0 \leq m \leq n$, for any sequence of n distinct vertices, there is another vertex adjacent to the first m vertices, but none of the remaining ones. It is straightforward to construct a model of these sentences 'by hand' by building a countable structure M as the union of an infinite chain of finite graphs $M_i, i \in \omega$, such that all the witnesses needed to satisfy the extension axioms for tuples in M_i can be found in M_{i+1}. On the other hand, it is much more difficult to prove that the theory T of the random graph has the finite model property, i.e., every sentence in T has a finite model. (And it remains an open question whether the universal homogeneous triangle-free graph shares this property.)

Likewise, it is easy to show that every graph G can be extended to a graph H such that every partial isomorphism of G extends to an automorphism of H. Hrushovski [42] proved that holds also for finite structures, but the construction is considerably harder.

Lastly, a more conceptual point. It is not yet clear what are the interesting questions and research programs in the model theory of finite structures. One hopes that some kind of Shelah style classification theory could be developed (and perhaps the work of Lachlan et al. is a start). But then first-order logic cannot be the right language to consider, as every complete first-order theory has at most one finite model. An alternative is *finite variable logic*, discussed below in Section 5.1.

§2. Preservation theorems. This section contains a rather complete discussion of what is known about preservation theorems over finite structures.

2.1. Preservation under extensions. Łoś and Tarski proved the existential preservation theorem, which says that a first-order sentence φ defines a class that is closed under extensions iff φ is equivalent to an existential first-order sentence. Tait [69] showed that this does not remain true over the class of finite structures.

THEOREM 2.1.1. *There is a sentence φ such that the class of finite models of φ is closed under extensions and φ is not equivalent over finite structures to any existential sentence.*

Below we give such an example, due to Gurevich and Shelah, that we will also use later (Section 2.1.2). Given a class $\mathcal{K}$ of finite structures that is closed under extensions, say that $A \in \mathcal{K}$ is *minimal* in $\mathcal{K}$ if for all proper substructures $B \subset A$, $B \notin \mathcal{K}$. Clearly, such a $\mathcal{K}$ is defined by an existential first-order sentence iff it has finitely many minimal structures, up to isomorphism.

PROOF. Let $\sigma = \{S, <, c, d\}$, $S, <$, binary relations, and c, d, constants. Define

$\psi := \forall xyz(\neg x < x \wedge (x < y \vee y < x \vee x = y) \wedge (x < y \wedge y < z \longrightarrow x < z))$,

the axioms for linear order. Then let

$$\theta := \psi \wedge \forall xyz((c \leq x \wedge x \leq d) \wedge (Sxy \longrightarrow x < y) \\ \wedge \neg(Sxy \wedge x < z \wedge z < y)) \wedge \exists x \forall y(x \neq d \wedge \neg Sxy)$$

Clearly, $A \models \theta$ iff $<$ is a linear order with minimal, resp. maximal, element c, resp. d, such that S is a (properly) partial successor relation. It is easy to verify that $\mathrm{Mod}_f(\theta)$ is closed under substructures.

We now claim that $\varphi := \neg\theta$ is the required sentence. By the preceding remark, $\mathrm{Mod}_f(\varphi)$ is closed under extensions. For all $n > 0$, let A_n be the structure with universe $\{0, 1, \ldots, n\}$, $c^{A_n} = 0$, $d^{A_n} = n$, and such that $<$ and S have standard interpretation. It is again easy to verify that for all n, A_n is a minimal model of $\mathrm{Mod}_f(\varphi)$ as desired. ⊣

Observe that the sentence φ above is $\forall_2$. We prove the following known fact that we use below.

FACT 2.1.2. *Let φ be an $\exists_2$ sentence, such that the class of finite models of φ is closed under extensions. Then φ is equivalent over finite structures to an existential sentence.*

Given such a φ, let m be the number of existential quantifiers in φ. It is clear that for all A, such that $A \models \varphi$, there is a $B \subseteq A$, of size $\leq m$, such that $B \models \varphi$. Thus every minimal structure in $\mathrm{Mod}_f(\varphi)$ has cardinality $\leq m$.

2.1.1. *Preservation under intersection of submodels.* We observe that the following preservation theorem for $\forall_2$ sentences, due to Abraham Robinson [59] (see also [40], p. 299), also fails in the finite. This is essentially an immediate consequence of the preceding result. Of course, the better known characterization of $\forall_2$ sentences, due to Chang, Łoś, and Suszko, as those sentences which define classes of models that are closed under unions of chains, is not very meaningful over finite structures.

THEOREM 2.1.3. *Let φ be such that for all C, if $C \models \varphi$, then for all substructures $A, B \subseteq C$, with non-empty intersection, that are also models of φ, the substructure with universe $A \cap B$ is also a model of φ. Then φ is equivalent to an $\forall_2$ sentence.*

THEOREM 2.1.4. *The preceding theorem fails over the class of finite structures.*

PROOF. Let φ be any sentence such that $\mathrm{Mod}_f(\varphi)$ is closed under substructures, and φ is not equivalent to a universal sentence. Clearly, φ fulfills the conditions of Robinson's theorem. By the preceding Fact, if φ is equivalent to an $\forall_2$ sentence, then it is equivalent to a universal sentence, contradicting our assumption. ⊣

2.1.2. *Preservation under images of endomorphisms.* In this section, we observe that a preservation theorem for classes closed under images of endomorphisms, due to Scott and Węglorz (see [40], p. 505), fails for finite structures. Conveniently, this is witnessed by the sentence θ from from the proof of Theorem 2.1.1.

Let σ be a signature, A and B σ-structures, and π a map from A to B. Recall that π is a homomorphism if for each n-ary relation $R \in \sigma$ and each n-tuple $a_1, \dots, a_n$ in A, if $A \models R[a_1, \dots, a_n]$, then $B \models R[\pi(a_1), \dots, \pi(a_n)]$. (Note that homomorphisms are not required to be surjective.) An *endomorphism* is a homomorphism from a structure to itself. We say that a class $\mathcal{K}$ is closed under images of endomorphisms if for every $A \in \mathcal{K}$ and endomorphism $\pi : A \longrightarrow A$, the image $\pi(A)$ is also in $\mathcal{K}$.

THEOREM 2.1.5. *Let φ be a sentence such that* $\mathrm{Mod}(\varphi)$ *is closed under images of endomorphisms. Then φ is equivalent to a positive boolean combination of positive and universal sentences.*

THEOREM 2.1.6. *The preceding theorem fails over the class of finite structures.*

PROOF. We choose our sentence to be θ from the proof of Theorem 2.1.1. First, it is easy to observe that for all $A \in \mathrm{Mod}_f(\theta)$, the only endomorphism of A is the identity map (because of the linear order). To show that $\mathrm{Mod}_f(\theta)$ is not defined by a positive boolean combination of positive and universal sentences, it suffices to show that for each sentence $\psi := \psi_1 \wedge \psi_2$, ψ_1 positive and ψ_2 universal, there are structures $A \in \mathrm{Mod}_f(\theta)$, $B \notin \mathrm{Mod}_f(\theta)$, such that if $A \models \psi$, then $B \models \psi$. We assume that the ψ_i are in prenex normal form.

Let n be the number of universal quantifiers in ψ_2. We choose B to have universe $\{0, \dots, 2n-1\}$, with $<$ interpreted as the standard linear order, S as the successor relation, and $c^B = 0, d^B = 2n-1$. Let A be the structure obtained from B by removing the S-edge from $n-1$ to n. Observe that $A \models \theta$ and $B \not\models \theta$. By the monotonicity of positive formulas, every positive formula true in A is also true in B. One can easily check that every substructure of B of cardinality n is isomorphic to a substructure of A of the same cardinality; therefore if $A \models \psi_2$, then $B \models \psi_2$. This shows that A and B are as desired. $\dashv$

2.2. Monotone classes. We discuss two closely related preservation theorems due to Lyndon [55, 56] which fail on finite structures. Let σ be a signature and let τ be a set of relation symbols in σ. A class $\mathcal{K}$ of σ-structures is *monotone* in τ if for any pair of σ-structures A and B with the same universe, if A is in $\mathcal{K}$ and B is obtained from A by adding tuples to relations P^A, $P \in \tau$, then B is also in $\mathcal{K}$. Observe that if $\mathcal{K}$ is closed under surjective homomorphisms, then it is monotone in σ itself.

It is easily verified that, if all the relation symbols in τ occur positively in a sentence φ, then $\mathrm{Mod}(\varphi)$ is monotone in τ and, if φ is positive, then $\mathrm{Mod}(\varphi)$ is closed under surjective homomorphisms. Lyndon established converses of these statements.

THEOREM 2.2.1. *Let φ be a sentence that defines a class that is monotone in τ. Then φ is equivalent to a sentence in which every relation symbol in τ occurs positively.*

COROLLARY 2.2.2. *Let φ be a sentence that defines a class that is closed under surjective homomorphisms. Then φ is equivalent to a positive sentence.*

Ajtai and Gurevich [1] showed that the first statement fails over the class of finite structures. Stolboushkin [68] gave a simpler counterexample, which can be modified to show that the second statement also fails (see [60]).

2.3. Preservation under homomorphisms. In this section we discuss what is certainly the most well-known open question regarding preservation theorems over finite structures. A class $\mathcal{K}$ is closed under homomorphisms if for all A and B, if A is in $\mathcal{K}$, and there is a homomorphism from A into B, then $B \in \mathcal{K}$. Such a class is always closed under extensions. An easy consequence of the existential preservation theorem is the homomorphism preservation theorem. A sentence φ defines a class that is closed under homomorphisms iff φ is equivalent to a positive existential sentence.

> Does the homomorphism preservation theorem remain true over the class of finite stuctures?

We observe a few facts related to this question. Ajtai and Gurevich ([2], Theorem 10.2) show that the injective homomorphism preservation theorem fails over the class of finite structures. Their counterexample is based on the Gurevich-Shelah sentence above. A positive existential sentence with inequalities is an existential sentence in which $=$ is the only relation which may occur in the scope of a negation.

THEOREM 2.3.1. *There is a sentence φ such that $\mathrm{Mod}_f(\varphi)$ is closed under injective homomorphisms, and φ is not equivalent in the finite to any positive existential sentence with inequalities.*

We also discuss some partial positive solutions to the preceding open question. It is almost immediate that any existential sentence that defines a class that is closed under homomorphisms is equivalent to a positive existential sentence. It has also been observed [60] that the following 'dual' result holds. Every positive sentence that defines a class that is closed under homomorphisms is equivalent to a positive existential sentence. Below, we will give a proof, also from [60], that the homomorphism preservation theorem holds for $\forall_3$ sentences. (In contrast, there are $\forall_2$ counterexamples to the existential preservation theorem, containing a single $\forall$.)

We introduce the following notation. For all $l, m, n \in \omega \setminus \{0\}$, $\mathrm{FO}[\forall^l \exists^m \forall^n]$ denotes the set of $\forall_3$ sentences whose quantifier prefix is a subword of $\forall^l \exists^m \forall^n$. Given structures A and B, $A \oplus B$ denotes the structure which is the disjoint union of A and B, and $p \cdot A$ denotes the disjoint union of p copies of A, $p \in \omega \setminus \{0\}$.

We also need the following version of the Ehrenfeucht-Fraïssé game. The $\forall^l \exists^m \forall^n$-game on A and B is a 3 round game played with $l + m + n$ labeled pebble pairs such that:

1. In round 1, the Spoiler plays l pebbles $\overline{\beta}^1 = (\beta_1, \dots, \beta_l)$ on B.
   The Duplicator then puts l pebbles $\overline{\alpha}^1 = (\alpha_1, \dots, \alpha_l)$ on A.
2. In round 2, the Spoiler plays m pebbles $\overline{\alpha}^2 = (\alpha_{l+1}, \dots, \alpha_{l+m})$ on A.
   The Duplicator then puts m pebbles $\overline{\beta}^2 = (\beta_{l+1}, \dots, \beta_{l+m})$ on B.
3. In round 3, the Spoiler plays n pebbles $\overline{\beta}^3 = (\beta_{l+m+1}, \dots, \beta_{l+m+n})$ on B.
   The Duplicator then puts n pebbles $\overline{\alpha}^3 = (\alpha_{l+m+1}, \dots, \alpha_{l+m+n})$ on A.

As usual, the Duplicator wins just in case the pebbles determine a partial isomorphism from A to B. The following lemma is easy to verify.

LEMMA 2.3.2. *Suppose that the Duplicator has a winning strategy in the $\forall^l \exists^m \forall^n$-game on A and B. Then for all $\varphi \in \mathrm{FO}[\forall^l \exists^m \forall^n]$, if $A \models \varphi$, then $B \models \varphi$.*

(The converse to this lemma is not true.)

THEOREM 2.3.3. *Let $\varphi \in \forall_3$ be such that $\mathrm{Mod}_f(\varphi)$ is closed under homomorphisms. Then φ is equivalent to a positive existential sentence* (*which can even be found effectively*).

PROOF. Let φ in FO[$\forall^l\exists^m\forall^n$] define a class that is closed under homomorphisms and let σ be the signature of φ. We show that there is an $s \in \omega$ that bounds the size of every minimal model of $\mathcal{K} = \mathrm{Mod}_f(\varphi)$. This implies that $\mathcal{K}$ is defined by a sentence in FO($\exists$) and thus that it is actually defined by a positive existential sentence. We can calculate s as a function of m and σ, which implies that there is an effective procedure for finding a positive existential sentence equivalent to φ. Let r be the number of structures, up to isomorphism, of signature σ and cardinality m; choose $s := r \cdot m$.

Let A be a minimal model of $\mathcal{K}$. We want to show that there is a $B \in \mathcal{K}$, of cardinality $\leq s$, such that there is a homomorphism from B into A. By the minimality of A, the homomorphism must be onto, implying that the cardinality of A is also $\leq s$. Assume for contradiction that the cardinality of A is $> s$ and let $\{M_1, \dots, M_q\}$ be the set of substructures of A of cardinality $= m$, again up to isomorphism. Define $t := l + m + n$. Let $G = (t \cdot M_1) \oplus \cdots \oplus (t \cdot M_q)$, and let $B = M_1 \oplus \cdots \oplus M_q$. It is obvious that there are homomorphisms from G onto B, and from B into A. Observe also that the cardinality of B is $q \cdot m \leq s$. Since $\mathcal{K}$ is closed under homomorphisms, it suffices to show that $G \in \mathcal{K}$.

To establish this fact, we define an extension A' of A, and describe the Duplicator's winning strategy for the $\forall^l\exists^m\forall^n$-game on A' and G. Since $A' \models \varphi$, this implies that $G \models \varphi$. Let $A' = A \oplus G$, and let f be the canonical injection from G into A'. In round 1, the Spoiler plays l pebbles $\overline{\beta}^1$ on some l-tuple in G. The Duplicator then plays on the l-tuple $f(\overline{\beta}^1)$ in A'. In round 2, the Spoiler plays some pebbles $\overline{\alpha}^{2,0}$ on $A \subseteq A'$, and plays his other pebbles $\overline{\alpha}^{2,1}$ on $G \subseteq A'$. Conceptually, the Duplicator makes her move in two stages. She first plays her pebbles $\overline{\beta}^{2,1}$ on $f^{-1}(\overline{\alpha}^{2,1})$. She then chooses an unpebbled component M'_p of G, one of the copies of M_p, such that there is an embedding, h, from M'_p into G that contains the tuple $\overline{\alpha}^{2,0}$ in its range. There must be such a component since G contains t copies of each M_p. The Duplicator then plays her pebbles $\overline{\beta}^{2,0}$ on the preimage of $\overline{\alpha}^{2,0}$ under h. It is clear that the Duplicator succeeds in maintaining a partial isomorphism. Now let f' be the embedding of G into A' that equals h on M'_p, and equals f on $G \setminus M'_p$. In round 3, after the Spolier has moved, the Duplicator plays her pebbles $\overline{\alpha}^3$ on the image $f'(\overline{\beta}^3) \subseteq A'$ of the pebbles played by the Spoiler. It is easy to see that this is indeed a winning strategy for the Duplicator. $\dashv$

2.4. Algebras, Horn formulas, and preservation under products. We discuss briefly some preservation theorems involving classes closed under direct products. (For more information about other kinds of products important in model theory, especially ultraproducts, see Chang and Keisler [13] or Hodges [40].) The material presented below originated in the context of universal algebra, in which signatures are purely functional, i.e., contain no relation symbols, but can be generalized to allow relations.

2.4.1. *Varieties.* The following theorem of Birkhoff [11] predates by about twenty years work in model theory on preservation theorems. (Below is actually a generalization allowing relation symbols in the signature (see Hodges [40], Corollary 9.2.8).)

THEOREM 2.4.1. *Let σ be a signature and $\mathcal{K}$ a class of σ-structures. Then the following are equivalent.*

1. *$\mathcal{K}$ is closed under direct products, substructures, and homomorphic images (that is, $\mathcal{K}$ is a* variety).
2. *$\mathcal{K}$ is axiomatized by a set of sentences of the form $\forall\overline{x}\varphi$, where φ is atomic.*

(Observe that if σ contains only function symbols, then the second condition above says that $\mathcal{K}$ is defined by equations.) It is interesting to note that $\mathcal{K}$ is not required to be first-order definable. In fact, the proof makes no use of the compactness theorem, but uses rather the existence of free structures in varieties. Nevertheless, it has long been known that Birkhoff's theorem fails over finite structures. (The result seems to be folklore.)

PROPOSITION 2.4.2. *There exists a class of finite structures that is closed under (finite) direct products, substructures, and homomorphic images but is not definable by a set of sentences of the form $\forall\overline{x}\varphi$, where φ is atomic.*

We mention two such examples. First, let $\mathcal{K}$ be the class of finite structures containing a single bijective unary function. Second, let $\sigma = \{\cdot\}$ contain a single binary function and let $\mathcal{G}$ be the set of finite groups.

2.4.2. *Preservation under direct products and substructures.* A first-order definable class of structures is closed under taking direct products and substructures iff it is defined by a set of universal Horn sentences. Alechina and Gurevich [3] ask whether a version of this theorem for single sentences holds over finite structures.

> Is every class of finite structures that is defined by a single sentence and closed under finite direct products and substructures defined by a universal Horn sentence?

We observe that a positive answer to this question would imply that the homomorphism preservation theorem holds over finite structures.

PROPOSITION 2.4.3. *Suppose that every sentence φ that defines a class of finite structures closed under finite direct products and substructures is equivalent to a universal Horn sentence. Then the homomorphism preservation theorem holds for finite structures.*

PROOF. The proposition follows easily from the next simple observation.

LEMMA 2.4.4. *Let $\mathcal{K}$ be a class of finite structures that is closed under homomorphisms. The complement $\overline{\mathcal{K}}$ is closed under finite direct products and substructures.*

Proof of Lemma. By contradiction. Suppose that $\mathcal{K}$ is closed under homomorphisms, but $\overline{\mathcal{K}}$ is not closed under direct products and substructures. Clearly $\overline{\mathcal{K}}$ is closed under substructures, so there must be A and B in $\mathcal{K}$ such that $D := A \times B$ is in $\mathcal{K}$. It is then easy to verify that the natural projection function $\pi : D \longrightarrow A$ is a homomorphism, contradicting the closure condition on $\mathcal{K}$. End of Proof.

Let φ define a class of finite structures that is closed under homomorphisms. By the preceding lemma, $\neg\varphi$ defines a class that is closed under finite direct products and substructures so, by hypothesis, it is equivalent to a universal Horn sentence. Thus φ is equivalent to an existential sentence. But we have already remarked that every existential sentence that defines a class that is closed under homomorphisms is equivalent to a positive existential sentence, as desired. $\dashv$

Observe that each of the two examples from the previous section, the class of bijective unary functions and the class of groups, is defined in the finite by a universal Horn sentence.

We mention that there are first-order definable classes of structures that are closed under extensions and direct products but not defined by an existential Horn sentence. One such example is the class of structures of cardinality ≥ 3, over the empty signature.

2.5. A preservation theorem for finite variable logic. We briefly discuss a preservation theorem involving finite variable logic, a fragment of first-order logic which has received considerable attention in finite model theory. This logic is discussed further in Section 5.1. For each k, let L^k be the set of first-order formulas which contain only the (reusable) variables $x_1, \dots, x_k$. Two structures are L^k-equivalent if they satisfy the same L^k sentences.

The next theorem was proved by Immerman and Kozen [47].

Theorem 2.5.1. *Let φ define a class that is closed under L^k-equivalence. Then φ is equivalent to a sentence in L^k.*

Proof. Let φ be such that $\mathrm{Mod}(\varphi)$ is closed under L^k-equivalence. Define $\Gamma := \{\theta \in L^k | \varphi \models \theta\}$. We show that φ is equivalent to Γ; the result then follows immediately by compactness. Clearly, $\mathrm{Mod}(\varphi) \subseteq \mathrm{Mod}(\Gamma)$. To prove the reverse inclusion, choose $A \notin \mathrm{Mod}(\varphi)$, and let $\Delta = \{\, \theta \in L^k | A \models \theta\}$, the L^k-theory of A. Because $\mathrm{Mod}(\varphi)$ is closed under L^k-equivalence, $\Delta \models \neg\varphi$, so there is a $\theta \in \Delta$ such that $\theta \models \neg\varphi$. Therefore $\theta \notin \Gamma$ and $A \notin \mathrm{Mod}(\Gamma)$. $\dashv$

Observe that the only fact about L^k that is used is that it is closed under boolean operations. Immerman and Kozen indeed prove the corresponding more general result.

The following observation showing that the previous theorem does not remain true over finite structures is folklore.

PROPOSITION 2.5.2. *There is a class $\mathcal{K}$ of finite structures that is first-order definable and closed under L^2-equivalence, that is not defined by any L^2 sentence.*

PROOF. Let $\mathcal{K}$ be the class of (finite) linear orders. It is well-known and easy to show that every finite linear order is characterized up to isomorphism by a single L^2 sentence, so $\mathcal{K}$ is clearly closed under L^2-equivalence.

To prove that $\mathcal{K}$ is not defined by any L^2 sentence, it suffices to show that for all n, there are structures $A_n \in \mathcal{K}$, $B_n \notin \mathcal{K}$, that are not distinguished by any L^2 sentence of quantifier rank $\leq n$. Choose A_n to be the linear order on $\{0, \dots, 2n\}$, with the standard ordering, and let B_n be the identical structure, except that $B_n \models (n+1) < (n-1) \wedge \neg((n-1) < (n+1))$. A standard pebble game argument can be used to show that A_n and B_n are as desired (see, e.g., [26]). ⊣

There are a number of further related questions that one may ask. For example, for what numbers $k, n, 1 < k < n$, is it the case that for every sentence φ, if $\mathrm{Mod}_f(\varphi)$ is closed under L^k-equivalence, then φ is equivalent to a sentence in L^n? Another possible refinement to the original theorem is known to be false. That is, for all $n \geq 2$, there is an existential first-order sentence φ such that $\mathrm{Mod}(\varphi)$ is closed under L^n-equivalence, but φ is not equivalent to any existential L^n sentence (see [4, 64, 32]).

Propositional modal logic. Propositional modal logic can be viewed in a natural way as a fragment of first-order logic via the use of Kripke structures. Each modal sentence corresponds to a first-order formula with one free variable over a signature containing a single binary relation and any number of unary relations. The image of this mapping is the *modal fragment of first-order logic*.

Van Benthem proved a version of the theorem of Immerman and Kozen for the modal fragment of first-order logic. It says that a class of structures with a distinguished element that is defined by a first-order formula with one free variable, is 'closed under bisimulations' iff it is definable by a formula in the modal fragment of first-order logic [10]. In this case, though, the theorem does remain true over the class of finite structures [61].

Further questions. There are of course other preservation theorems about which one could ask whether they remain true over finite structures. For example, Compton [21] established a number of related preservation theorems for classes closed, for example, under closed extensions and under components and disjoint unions, involving sentences containing 'connecting quantifiers'. It would be interesting to know whether these theorems hold over finite structures, especially since Compton was led to them by questions involving finite combinatorics.

In another direction, one can ask whether a particular preservation theorem, which is known to fail over finite structures, holds for a restricted class of structures. For example, Hodges [39] asked whether over the class of finite groups, every sentence that is preserved under extensions is equivalent to an existential sentence.

§3. Other results.

3.1. Definability. Little is known about the expressive power of single first-order formulas over finite structures. There are however a few interesting and non-trivial results. For example, Felgner proved that the class of finite non-abelian simple groups is defined by a single sentence. Chatzidakis, van den Dries, and Macintyre [15] investigated definable sets in finite fields and showed that there is no formula $\psi(y)$ that defines, in each finite field $\boldsymbol{F}_{q^2}$, the subfield $\boldsymbol{F}_q$.

3.2. Δ_n-definability. A class $\mathcal{K}$ is Δ_n-definable if there are sentences $\varphi_1 \in \exists_n$ and $\varphi_2 \in \forall_n$ such that $\mathcal{K} = \mathrm{Mod}(\varphi_1) = \mathrm{Mod}(\varphi_2)$. The following theorem appears in Shoenfield's textbook [67].

THEOREM 3.2.1. *For all $n \geq 1$, if a class $\mathcal{K}$ of structures is Δ_n-definable, then it is also defined by a boolean combination of $\exists_{n-1}$ sentences.*

We observe that for $n = 2$ this does not remain true over the class of finite structures. The counterexample is well-known in formal language and automata theory, though the connection with classical model theory had perhaps not been noticed.

THEOREM 3.2.2. *There is a Δ_2-definable class of finite structures that is not defined by a boolean combination of $\exists_1$ sentences.*

PROOF. Let ψ be a universal sentence that axiomatizes the class of linear orders. We define $\varphi = \psi \wedge \exists x \forall y (y \leq x \wedge Px)$, and $\theta = \psi \wedge \forall x \exists y (x \leq y \wedge Py)$. It is easy to see that $\mathrm{Mod}_f(\varphi) = \mathrm{Mod}_f(\theta)$ is the class of finite linear orders with maximal element in the relation P. Obviously φ, resp. θ, is equivalent to an $\exists_2$ sentence, resp. an $\forall_2$ sentence.

Let $\mathcal{B}_1^n$ be the set of sentences which are boolean combinations of prenex existential sentences, each of which contains at most n quantifiers. Observe that two structures satisfy the same $\mathcal{B}_1^n$ sentences just in case they have, up to isomorphism, the same set of substructures of cardinality $\leq n$.

It suffices to show that for all n, there are finite structures $A_n \in \mathrm{Mod}_f(\varphi)$ and $B_n \notin \mathrm{Mod}_f(\varphi)$ that are not distinguished by any sentence in $\mathcal{B}_1^n$. Let A_n be the structure with universe $\{0, \dots, 2n-1\}$ such that $<^A$ is the natural order relation and $P^A = \{1, 3, \dots, 2n-1\}$. Let B_n be the structure with the same universe and same order relation, and $P^B = \{0, 2, \dots, 2n-2\}$. Clearly, $A_n \in \mathrm{Mod}_f(\varphi)$ and $B_n \notin \mathrm{Mod}_f(\varphi)$. It is easy to see that the substructures of A_n, resp. B_n, of cardinality $\leq n$ are exactly those structures over the signature $\{<, P\}$ which are linearly ordered. Therefore, by the above observation, A_n and B_n are not distinguished by any sentence in $\mathcal{B}_1^n$. $\dashv$

To the best of our knowledge, it is an open question whether Theorem 3.2.1 fails over finite structures for all n.

3.3. Connectedness. We give another example witnessing the failure of compactness type phenomena over finite structures. One can define a natural metric for relational structures that generalizes the ordinary metric on graphs. Let A be a relational structure with signature σ. The *Gaifman graph* of A, $\mathcal{G}(A)$, is a simple graph, with vertex set A, such that for all $a, b \in A$, there is an edge between a and b iff there is a k-ary relation $R \in \sigma$ and a k-tuple $\overline{c}$ such that a, b are in $\overline{c}$ and $A \models R\overline{c}$. The distance between two points $a, b \in A$ is the distance between them in $\mathcal{G}(A)$, which is either a finite number or ∞. The diameter of A is $\geq n$ if there are elements of A of distance n; A is *connected* if the distance between any two elements is finite.

An immediate consequence of the compactness theorem is the following. Let $\mathcal{K}$ be the class of models of a sentence such that for all n, there is an $A \in \mathcal{K}$ with diameter $\geq n$. Then there is a $B \in \mathcal{K}$ such that B is not connected. Stolboushkin asked whether this remains true over the class of finite structures. We answered this question negatively; below we give a simpler example due to Stolboushkin.

THEOREM 3.3.1. *There is a sentence φ such that* (i) *every finite model of φ is connected, and* (ii) *there are finite models of φ of arbitrarily large diameter.*

PROOF. Let $\sigma = \{S, <\}$, such that S and $<$ are both binary relations. For each $n \geq 2$, we describe a connected structure A_n of diameter n.

For $n \geq 2$, the universe of A_n is the set of ordered pairs $\{(i, j) \mid 0 \leq i \leq j < n\}$, which we view as a triangular region of the plane. We define $S^{A_n} := \{((i, j), (k, l)) \mid i = k \text{ and } j + 1 = l, \text{ or } i + 1 = k \text{ and } j = l\}$; and $<^{A_n} = \{(i, j), (k, l) \mid i = k \text{ and } j < l\}$. It is clear that A_n has the desired properties; e.g., the distance between $(0, 0)$ and $(n - 1, n - 1)$ equals n.

It is straightforward to show that $\mathcal{K} := \{A_n \mid n \geq 2\}$ is defined by a single sentence. ⊣

§4. Other questions.

4.1. Generalized preservation theorems. Another avenue is to look for 'generalized preservation theorems' in instances where it is known a classical preservation theorem fails. For example, given the failure of the existential preservation theorem over finite structures, one can ask whether there is a natural logic L, extending first-order logic, such that every extension-closed class of finite structures that is defined by a first-order sentence is also defined by an existential L sentence (see [64]). This holds trivially for $L := L_{\infty\omega}$, as every class of finite structures that is closed under extensions can be defined by a single existential $L_{\infty\omega}$ sentence, so only weaker logics are of any interest. Unfortunately Grohe [33] has proved a strong negative result in this direction: there is a first-order sentence that defines a class of finite structures closed under extensions but that is not equivalent to any existential $L^{\omega}_{\infty\omega}$ sentence. It is perhaps unlikely, then, that one will be able to find generalized preservation theorems of this sort.

Alechina and Gurevich [3] and others have suggested another way to look for preservation theorems over finite structures. Viewed abstractly, a preservation theorem says that for some semantic property of classes of structures (for example, classes that are closed under extensions), there is a *recursive* set S of first-order sentences such that (i) any set of sentences from S defines a class with this property and, in the other direction, (ii) every set of first-order sentences that defines such a class is equivalent to a set of sentences in S. Thus S is recursive and 'semantically complete' for this property. For example, we can ask whether there is a recursive set S of first-order sentences such that every sentence in S defines a class of finite structures that is closed under extensions and every first-order sentence that defines a class of finite structures that is closed under extensions is equivalent to a sentence in S.

4.2. First-order prefix classes. We briefly discuss a different kind of question about the expressive power of first-order logic. We will also have the opportunity to illustrate one reason why model theoretic questions about finite structures can be more difficult than in the general case: most powerful model theoretic constructions produce only infinite structures. In particular, the problem here is not simply the failure of the compactness theorem.

Let p and q be two first-order prefixes, that is, finite strings of $\forall$s and $\exists$s. We are interested in determining when it is the case that there is a sentence φ, in prenex normal form, with prefix p, such that φ is not equivalent to any sentence with prefix q. Walkoe [71] proved that if p and q are different prefixes of the same length n, then there is such a sentence, containing a single n-ary relation. Keisler and Walkoe [48] improved this result by showing that it remains true over the class of finite structures. Both arguments involve applications of Ramsey theory, so it is perhaps not too surprising that they were able to establish the result in the finite case.

Grädel and McColm [31] asked whether it was possible to strengthen these results by considering prefixes of different lengths, or by placing restrictions on the signature. Observe that if p is a subword of q, then every sentence with prefix p is equivalent to one with prefix q. We proved a converse [62], though only for infinite structures.

THEOREM 4.2.1. *For every first-order prefix p, there is a sentence φ_p containing a single binary relation, such that for all prefixes q which do not embed p, φ_p is not equivalent to any sentence with prefix q.*

It is natural to ask, then, whether this result also holds over finite structures. We believe that it does, and that the same sentences φ_p witness this. But the proof of Theorem 4.2.1 involves the use of well-behaved and highly symmetric ultrahomogeneous structures, built using the Fraïssé construction, which produces structures of cardinality $\aleph_0$. If it were shown that these structures are pseudofinite (see Section 5.2), the proof would transfer directly to the finite case, but this appears to be a difficult problem. Failing that, it seems that one needs either a powerful new way of constructing finite structures with certain logical properties, or a rather different proof altogether.

§5. New directions.

5.1. Other logics. A number of logics other than first-order logic have been studied in the context of finite model theory, among them second-order logic, fixed-point logics, and finite variable logics. From the point of view of *descriptive complexity theory*, first-order logic is rather weak, in the sense that many computationally simple properties of finite structures, such as parity, are not first-order definable. This has motivated the investigation of stronger logics that capture complexity classes. (See [46] for a full account of descriptive complexity theory.)

From a model theoretic perspective, though, first-order logic is rather too strong over finite structures. Every finite structure is defined up to isomorphism by a single sentence, and every class of finite structures is definable by a set of sentences. (The first point obviously renders the study of complete first-order theories with finite models trivial. Likewise, the relation of elementary equivalence in this context becomes nothing other than isomorphism.)

Modern model theory concerns itself largely with complete first-order theories or, put another way, the elementary equivalence relation. In order to develop an analagous theory for finite structures, one must consider logics weaker than first-order logic. The most interesting candidate so far seems to be finite variable logic, introduced above.

In an early paper on the subject, Poizat [58] began investigating finite variable logic over finite structures and proved (or mentioned) a number of fundamental results, including the fact that the L^k-theory of a finite structure is finitely axiomatizable (see also [23]). He also studied small models of L^k-theories, i.e., those of cardinality $\leq k + 1$. (See also [16].)

Further work has revealed that complete L^k-theories can be rather complex, as well as badly behaved. Thomas [70], answering a question of Poizat, showed that there are complete L^k-theories with more than one but only finitely many models. Grohe [34] showed how to encode diophantine equations in complete L^k-theories so that positive integer solutions to an equation correspond to models of the theory. This provided a negative answer to the following question of Dawar [22], which asked for a kind of effective downward Löwenheim-Skolem theorem for L^k-theories over finite structures. For each k, is there a recursive function f_k such that every complete L^k-theory with n types that has a finite model has a model of size $\leq f_k(n)$? Building on this work, Barker [9] gave a negative answer to a second, related question of Dawar about the existence of an effective upward Löwenheim-Skolem theorem for L^k. For $k \geq 4$, there is no recursive function g_k such that if a complete L^k-theory with n types has a finite model of size $\geq g_k(n)$, then it has arbitrarily large finite models. Grohe's work also implies that the question whether, given a finite structure M and a number k, M is characterized up to isomorphism by its L^k-theory, is undecidable.

Recently, a number of authors have begun developing stability theory for finite variable logics [5, 25, 45, 44, 6, 54], which promises to offer exciting new developments in 'pure finite model theory'. For example, Djordjević [25] has shown that in many ways stable L^k-theories that satisfy a certain amalgamation property are much better behaved than arbitrary theories. In particular, he proves that every stable L^k-theory with the set amalgamation property and finitely many L^k types has infinitely many finite models. Further, using results of Lachlan on finite homogeneous structures, he shows that such theories do possess an effective downward Löwenheim-Skolem theorem. More generally, Lessmann [54] has developed simplicity theory for L^k, establishing finite variable versions of many results for first-order logic. Still, many interesting classification type questions about classes of finite models of complete L^k-theories remain.

From a different point of view, there has been some work on the existential fragment of finite variable logic. Finite k-universal graphs, i.e., graphs that satisfy every consistent existential L^k sentence, are investigated in [63]. It had also been observed that the existential L^k-theory of such (finite) structures is not finitely axiomatizable [64]. One can also find finite structures whose universal L^k-theory is not finitely axiomatizable. For example, let $\sigma = \{E, s, t\}$ and let A be a finite σ-structure with two connected components, each satisfying the $k+1$-extension axioms (that is, the L^{k+1} fragment of the theory of the random graph), so that s^A and t^A are on different components. This suggests various questions involving finite axiomatizability of finite structures. For example, is it decidable whether, given a finite structure A and a number k, the existential (universal) L^k-theory of A is finitely axiomatizable?

Finally, we mention a well-known open question about L^k-definability. Over arbitrary finite structures, it is easy to see that the languages $L^k, k \in \omega$ form a strict hierarchy in terms of expressive power, as the property that there are at least $k+1$ elements can be defined by an L^{k+1} sentence but not by an L^k sentence. Does the L^k-hierarchy remain strict over the class of finite ordered graphs, in the signature $\{<, E\}$?

5.2. Connections with the infinite.

Pseudofiniteness. An infinite structure M is *pseudofinite* if every sentence true in M has a finite model. (Equivalently, M is a model of the theory of the class of finite structures, *or* $\mathrm{Th}(M)$ contains no infinity axioms.) This notion arises naturally in model theory though the class of pseudofinite structures as such has not been extensively investigated. Unlike the class of finite structures, this class is first-order definable and thus classical theorems remain true in this context.

An interesting and often difficult problem is to determine whether a particular structure is pseudofinite. An important result of Zilber [73] says that totally categorical structures are pseudofinite. Clearly, the theory of such a structure is not finitely axiomatizable. Nevertheless, Hrushovski proved that the theory of a totally categorical structure is *pseudo-finitely axiomatized*, that is, axiomatized by a finite set of sentences and a single axiom schema saying that there are infinitely many elements. Taken together, these results say that a totally categorical structure, such as an infinite dimensional vector space over a finite field, strongly resembles certain finite structures (in this case, finite dimensional vector spaces over the same field).

Cherlin, Harrington, and Lachlan [20] generalized Zilber's theorem to the class of $\aleph_0$-stable, $\aleph_0$-categorical structures. (They proved further that such structures are actually *smoothly approximable* by finite structures.) Macpherson [57] has conjectured that a finitely axiomatized $\aleph_0$-categorical theory has the strict order property, which would further generalize part of the result of Cherlin, Harrington, and Lachlan.

More generally, we do not know of any example of a non-pseudofinite $\aleph_0$-categorical structure whose theory does not have the strict order property. Among unstable structures, it follows immediately from the 0-1 law [27, 30], discussed below, that the random graph is pseudofinite. Pillay also has an example of a non-simple $\aleph_0$-categorical theory that is pseudofinite, answering a question from [49]. Cherlin has asked whether this also holds for the countable universal homogeneous triangle-free graph.

Moving in the other direction, one can start with an interesting class $\mathcal{K}$ of finite structures and consider the theory $\mathrm{Th}(\mathcal{K})$, whose infinite models will be pseudofinite non-standard structures in $\mathcal{K}$. Much work has been done on pseudofinite fields (see [14]). Pseudofinite groups have been studied by Felgner [28] and by Wilson [72], and the simple ones have been classified: a simple pseudofinite group is elementarily equivalent to a Chevalley group over a pseudofinite field.

0-1 *laws and the Hrushovski construction.* A 0-1 law for a logic over a class $\mathcal{K}$ of finite structures (with a fixed probability measure) says that every sentence is either *almost surely true* or almost surely false over $\mathcal{K}$. The existence of a 0-1 law means that the *almost sure theory* (consisting of those sentences that are almost surely true) is complete; thus 0-1 laws can be sources of new complete theories. Furthermore, any theory T obtained in this way has the finite model property: every sentence in T is satisfied by almost every finite structure in $\mathcal{K}$. Obtaining a (previously known) theory T as an almost sure theory for some class $\mathcal{K}$ provides one way to show that the T has the finite model property (equivalently, that models of T are pseudofinite).

Baldwin and Shelah revealed deep connections between some 0-1 laws and work in pure stability theory. Shelah and Spencer [66] established a 0-1 law for graphs with edge probability $n^{-\alpha}$, for all irrational α between 0 and 1. Around the same time, Hrushovski [43], refuting a conjecture of Lachlan, proved the existence of (uncountably many) $\aleph_0$-categorical stable but not superstable theories. In order to do so, he devised a new and flexible technique for constructing infinite structures as limits of finite structures with a variety of stability theoretic properties. Baldwin and Shelah [7] noticed the striking similarity between the Shelah and Spencer almost sure theories and Hrushovski's examples, which enabled them to prove (among other things) the stability of the former, using work of Baldwin and Shi [8].

REFERENCES

[1] M. Ajtai and Y. Gurevich, *Monotone versus positive*, ***Journal of the ACM***, vol. 34 (1987), pp. 1004–1015.

[2] ———, *Datalog vs first-order logic*, ***Journal of Computer and System Sciences***, vol. 49 (1994), pp. 562–58.

[3] N. Alechina and Y. Gurevich, *Syntax vs. semantics on finite structures*, ***Structures in logic and computer science. A selection of essays in honor of A. Ehrenfeucht*** (J. Mycielski, G. Rozenberg, and A. Salomaa, editors), LNCS, vol. 1261, Springer-Verlag, 1997, pp. 14–33.

[4] H. Andréka, J. van Benthem, and I. Németi, *Modal languages and bounded fragments of predicate logic*, 1996, ILLC Research Report ML–96–03.

[5] J. T. Baldwin, *Theories in finite model theory*, 1999, preprint.

[6] J. T. Baldwin and O. Lessmann, *Amalgamation properties and finite models in L^n-theories*, ***Archive for Mathematical Logic***, vol. 41 (2002), pp. 155–167.

[7] J. T. Baldwin and S. Shelah, *Randomness and semigenericity*, ***Transactions of the American Mathematical Society***, vol. 349 (1997), pp. 1359–1376.

[8] J. T. Baldwin and N. Shi, *Stable generic structures*, ***Annals of Pure and Applied Logic***, vol. 79 (1996), pp. 1–35.

[9] R. Barker, *A proof that there is no recursive link between the k-size of a model and its cardinality*, 2000, preprint.

[10] J. van Benthem, ***Modal logic and classical logic***, Bibliopolis, 1983.

[11] G. Birkhoff, *On the structure of abstract algebras*, ***Proc. Camb. Phil. Soc.***, vol. 31 (1935), pp. 433–454.

[12] E. Börger, E. Grädel, and Y. Gurevich, ***The classical decision problem***, Springer-Verlag, 1997.

[13] C. C. Chang and H. J. Keisler, ***Model theory***, 3rd ed., North-Holland, 1990.

[14] Z. Chatzidakis, *Model theory of finite fields and pseudo-finite fields*, ***Annals of Pure and Applied Logic***, vol. 88 (1997), pp. 95–108.

[15] Z. Chatzidakis, L. van den Dries, and A. Macintyre, *Definable sets over finite fields*, ***Journal für die Reine und Angewandte Mathematik***, vol. 427 (1992), pp. 107–135.

[16] G. Cherlin, *Theories categorical in power $n+2$*, ***Discrete Mathematics***, to appear.

[17] ———, *Large finite structures with few types*, ***Algebraic model theory*** (B. T. Hart, A. H. Lachlan, and M. A. Valeriote, editors), Kluwer Academic Publishers, 1997, pp. 53–105.

[18] ———, *Sporadic homogeneous structures*, ***The Gelfand Mathematical Seminars***, Birkhäuser, 2000, pp. 15–48.

[19] G. CHERLIN and U. FELGNER, *Homogeneous finite groups*, ***Journal of the London Mathematical Society (2)***, vol. 62 (2000), pp. 784–794.

[20] G. CHERLIN, L. HARRINGTON, and A. H. LACHLAN, *$\aleph_0$-categorical, $\aleph_0$-stable structures*, ***Annals of Pure and Applied Logic***, vol. 28 (1985), pp. 103–135.

[21] K. COMPTON, *Some useful preservation theorems*, ***The Journal of Symbolic Logic***, vol. 48 (1983), pp. 427–440.

[22] A. DAWAR, *Types and indiscernibles in finite models*, ***Logic Colloquium '95*** (J. Makowsky and E. Ravve, editors), Springer-Verlag, 1998, pp. 51–65.

[23] A. DAWAR, S. LINDELL, and S. WEINSTEIN, *Infinitary logic and inductive definability over finite structures*, ***Information and Computation***, vol. 119 (1994), pp. 160–175.

[24] M. DJORDJEVIĆ, *Finite variable logic and homogeneous structures*, 2000, preprint.

[25] ———, *Finite variable logic, stability and finite models*, ***The Journal of Symbolic Logic***, vol. 66 (2001), pp. 837–858.

[26] H.-D. EBBINGHAUS and J. FLUM, ***Finite model theory***, Springer-Verlag, 1995.

[27] R. FAGIN, *Probabilities on finite models*, ***The Journal of Symbolic Logic***, vol. 41 (1976), pp. 50–58.

[28] U. FELGNER, *Pseudo-endliche Gruppen*, ***Seminarber., Humboldt-Univ. Berlin, Sekt. Math.***, vol. 110 (1990), pp. 82–96.

[29] A. GARDINER, *Homogeneous graphs*, ***Journal of Combinatorial Theory. Series B.***, vol. 20 (1976), pp. 94–102.

[30] JU. V. GLEBSKIĬ, D. I. KOGAN, M. I. LIOGONKIĬ, and V. A. TALANOV, *Volume and fraction of satisfiability of formulas of the lower predicate calculus*, ***Kibernetika (Kiev)***, vol. 2 (1969), pp. 17–27, (Russian).

[31] E. GRÄDEL and G. MCCOLM, *Hierarchies in transitive closure logic, stratified Datalog and infinitary logic*, ***Annals of Pure and Applied Logic***, vol. 77 (1996), pp. 166–199.

[32] E. GRÄDEL and E. ROSEN, *On preservation theorems for two-variable logic*, ***Mathematical Logic Quarterly***, vol. 45 (1999), pp. 315–325.

[33] M. GROHE, *Existential least fixed-point logic and its relatives*, ***Journal of Logic and Computation***, vol. 7 (1997), pp. 205–228.

[34] ———, *Finite variable logics in descriptive complexity theory*, this BULLETIN, vol. 4 (1998), pp. 345–398.

[35] Y. GUREVICH, *Toward logic tailored for computational complexity*, ***Computation and proof theory*** (M. M. Richter et al., editors), Lecture Notes in Mathematics, vol. 1104, Springer-Verlag, 1984, pp. 175–216.

[36] B. HERWIG, *Extending partial isomorphisms on finite structures*, ***Combinatorica***, vol. 115 (1995), pp. 365–371.

[37] ———, *Extending partial isomorphisms for the small index property of many ω-categorical structures*, ***Israel Journal of Mathematics***, vol. 107 (1997), pp. 93–123.

[38] B. HERWIG and D. LASCAR, *Extending partial isomorphisms and the profinite topology on free groups*, ***Transactions of the American Mathematical Society***, vol. 352 (2000), pp. 1985–2021.

[39] W. HODGES, *Finite extensions and finite groups*, ***Models and sets (Aachen, 1983)*** (G. H. Müller and M. M. Richter, editors), Lecture Notes in Mathematics, vol. 1103, Springer-Verlag, 1984, pp. 193–206.

[40] ———, ***Model theory***, Cambridge University Press, 1993.

[41] W. HODGES, I. HODKINSON, D. LASCAR, and S. SHELAH, *The small index property for ω-categorical ω-stable structures and for the random graph*, ***Journal of the London Mathematical Society (2)***, vol. 48 (1993), pp. 204–218.

[42] E. HRUSHOVSKI, *Extending partial isomorphisms of graphs*, ***Combinatorica***, vol. 12 (1992), pp. 411–416.

[43] ———, *A stable $\aleph_0$-categorical pseudoplane*, 1998, preprint.

[44] T. Hyttinen, *Forking in finite models*, 1999, preprint.

[45] ———, *On stability in finite models*, ***Archive for Mathematical Logic***, vol. 39 (2000), pp. 89–102.

[46] N. Immerman, ***Descriptive complexity***, Springer-Verlag, 1998.

[47] N. Immerman and D. Kozen, *Definability with bounded number of bound variables*, ***Information and Computation***, vol. 83 (1989), pp. 121–139.

[48] H. J. Keisler and W. Walkoe, Jr., *The diversity of quantifier prefixes*, ***The Journal of Symbolic Logic***, vol. 38 (1973), pp. 79–85.

[49] B. Kim and A. Pillay, *From stability to simplicity*, this Bulletin, vol. 4 (1998), pp. 17–36.

[50] Ph. Kolaitis, *Implicit definability on finite structures and unambiguous computations*, ***Proceedings of 5th IEEE Symposium on Logic in Computer Science***, 1990, pp. 168–180.

[51] A. H. Lachlan, *Homogeneous structures*, ***Proceedings of the ICM 1986*** (A. Gleason, editor), American Mathematical Society, 1987, pp. 314–321.

[52] ———, *Stable finitely homogeneous structures a survey*, ***Algebraic model theory*** (B. T. Hart, A. H. Lachlan, and M. A. Valeriote, editors), Kluwer Academic Publishers, 1997, pp. 145–159.

[53] A. H. Lachlan and A. Tripp, *Finite homogeneous* 3*-graphs*, ***Mathematical Logic Quarterly***, vol. 41 (1995), pp. 287–306.

[54] O. Lessmann, *Simple theories in finite variable logic*, 2000, preprint.

[55] R. C. Lyndon, *An interpolation theorem in the predicate calculus*, ***Pacific Journal of Mathematics***, vol. 9 (1959), pp. 129–142.

[56] ———, *Properties preserved under homomorphism*, ***Pacific Journal of Mathematics***, vol. 9 (1959), pp. 143–154.

[57] D. Macpherson, *Finite axiomatizability and theories with trivial algebraic closure*, ***Notre Dame Journal of Formal Logic***, vol. 32 (1991), pp. 188–192.

[58] B. Poizat, *Deux ou trois choses que je sais de* L_n, ***The Journal of Symbolic Logic***, vol. 47 (1982), pp. 641–658.

[59] A. Robinson, ***Introduction to model theory and to the metamathematics of algebra***, North-Holland, 1963.

[60] E. Rosen, *Finite model theory and finite variable logic*, ***Ph.D. thesis***, University of Pennsylvania, 1995.

[61] ———, *Modal logic over finite structures*, ***Journal of Logic, Language and Information***, vol. 6 (1997), pp. 427–439.

[62] ———, *On the first-order prefix hierarchy*, 1998, preprint.

[63] E. Rosen, S. Shelah, and S. Weinstein, k*-universal graphs*, ***Logic and random structures***, American Mathematical Society, 1997, pp. 65–77.

[64] E. Rosen and S. Weinstein, *Preservation theorems in finite model theory*, ***Logic and computational complexity*** (D. Leivant, editor), LNCS, vol. 960, Springer-Verlag, 1995, pp. 480–502.

[65] D. Saracino and C. Wood, *Homogeneous finite rings in characteristic* 2^n, ***Annals of Pure and Applied Logic***, vol. 40 (1988), pp. 11–28.

[66] S. Shelah and J. Spencer, *Zero-one laws for sparse random graphs*, ***Journal of the American Mathematical Society***, vol. 1 (1988), pp. 97–115.

[67] J. Shoenfield, ***Mathematical logic***, Addison-Wesley, 1967.

[68] A. Stolboushkin, *Finite monotone properties*, ***Proceedings of 10th IEEE Symposium on Logic in Computer Science***, 1995, pp. 324–330.

[69] W. W. Tait, *A counterexample to a conjecture of Scott and Suppes*, ***The Journal of Symbolic Logic***, vol. 24 (1959), pp. 15–16.

[70] S. Thomas, *Theories with finitely many models*, ***The Journal of Symbolic Logic***, vol. 51 (1986), pp. 374–376.

[71] W. WALKOE, JR., *Finite partially-ordered quantification*, ***The Journal of Symbolic Logic***, vol. 35 (1970), pp. 535–555.

[72] J. S. WILSON, *On simple pseudofinite groups*, ***Journal of the London Mathematical Society (2)***, vol. 51 (1995), pp. 471–490.

[73] B. ZILBER, *Totally categorical theories; structural properties and non-finite axiomatizability*, ***Proceedings: Model Theory of Algebra and Arithmetic*** (L. Pacholski et al., editors), 1979, pp. 381–410.

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO, IL 60607, USA
E-mail: erosen@math.uic.edu