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Presburger Vector Addition Systems

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***Abstract*—The reachability problem for Vector Addition Systems (VAS) is a central problem of net theory. The problem is known to be decidable by inductive invariants definable in the Presburger arithmetic. When the reachability set is definable in the Presburger arithmetic, the existence of such an inductive invariant is immediate. However, in this case, the computation of a Presburger formula denoting the reachability set is an open problem. In this paper we close this problem by proving that if the reachability set of a VAS is definable in the Presburger arithmetic, then the VAS is flatable, i.e. its reachability set can be obtained by runs labeled by words in a bounded language. As a direct consequence, classical algorithms based on acceleration techniques effectively compute a formula in the Presburger arithmetic denoting the reachability set.**

***Keywords*-Infinite State Systems, Acceleration, flatability, Presburger, Vector Addition Systems, Petri nets, Reachability**

I. Introduction

Vector Addition Systems (VAS) or equivalently Petri Nets are one of the most popular formal methods for the representation and the analysis of parallel processes [1]. The reachability problem is central since many computational problems (even outside the realm of parallel processes) reduce to this problem. Sacerdote and Tenney provided in [2] a partial proof of decidability of this problem. The proof was completed in 1981 by Mayr [3] and simplified by Kosaraju [4] from [2], [3]. Ten years later [5], Lambert provided a further simplified version based on [4]. This last proof still remains difficult and the upper-bound complexity of the corresponding algorithm is just known to be non-primitive recursive. Nowadays, the exact complexity of the reachability problem for VAS is still an open-question. Even an Ackermannian upper bound is open (this bound holds for VAS with finite reachability sets [6]).

Recently, in [7], the reachability sets of VAS are proved to be *almost semilinear*, a class of sets that extends the class of Presburger sets (the sets definable in $\mathrm{FO}(\mathbb{Z},+,\leq)$) inspired by the *semilinear sets* [8]. Note that in general reachability sets are not definable in the Presburger arithmetic [9]. An application of the almost semilinear sets was provided; a final configuration is not reachable from an initial one if and only if there exists a forward inductive invariant definable in the Presburger arithmetic that contains the initial configuration but not the final one. Since we can decide if a Presburger formula denotes a forward inductive invariant, we deduce that there exist checkable certificates of non-reachability in the Presburger arithmetic. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by enumerating finite sequences of actions and a second one that tries to prove the non-reachability by enumerating Presburger formulas. Such an algorithm always terminates in theory but in practice an enumeration does not provide an efficient way for deciding the reachability problem. In particular the problem of deciding *efficiently* the reachability problem is still an *open question*.

When the reachability set is definable in the Presburger arithmetic, the existence of checkable certificates of non-reachability in the Presburger arithmetic is immediate since the reachability set is a forward inductive invariant (in fact the most precise one). The problem of deciding if the reachability set of a VAS is definable in the Presburger arithmetic was studied twenty years ago independently by Dirk Hauschildt during his PhD [10] and Jean-Luc Lambert. Unfortunately, these two works were never published. Moreover, from these works, it is difficult to deduce a simple algorithm for computing a Presburger formula denoting the reachability set when such a formula exists.

For the class of *flatable* vector addition systems [11], [12], such a computation can be performed with *accelerations techniques*. Let us recall that a VAS is said to be *flatable* if there exists a language included in $w_1^* \ldots w_m^*$ for some words $w_1, \ldots, w_m$ such that every reachable configuration is reachable by a run labeled by a word in this language (such a language is said to be *bounded* [13]). *Acceleration techniques* provide a framework for deciding reachability properties that works well in practice but without termination guaranty in theory. Intuitively, acceleration techniques consist in computing with some symbolic representations transitive closures of sequences of actions. For vector addition systems, the Presburger arithmetic is known to be expressive enough for this computation. As a direct consequence, when the reachability set of a vector addition system is computable with acceleration techniques, this set is necessarily definable in the Presburger arithmetic. In [12], we proved that a VAS is flatable if, and only if, its reachability set is computable by acceleration.

Recently, we proved that many classes of VAS with known Presburger reachability sets are flatable [12] and we conjectured that VAS with reachability sets definable in the Presburger arithmetic are flatable. In this paper, we prove this conjecture. As a direct consequence, classical acceleration techniques always terminate on the computation of Presburger formulas denoting reachability sets of VAS when such a formula exists.

Outline In section III we introduce the acceleration framework and the notion of flatable subreachability sets and flat-

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able subreachability relations. We also recall why Presburger formulas denoting reachability sets of flatable VAS are computable with acceleration techniques. In section IV we recall the definition of well-preorders, the Dickson's lemma and the Higman's lemma. In Section V we provide some classical elements of linear algebra. We recall the characterization of Presburger sets as finite union of linear sets. We also introduce in this section the central notion of *smooth periodic sets*. Intuitively smooth periodic sets are sets of vectors of rational numbers stable by finite sums, and such that from any infinite sequence of elements, a so-called limit vector can be extracted. The definition of smooth periodic sets also requires that the possible limits forms a set definable in the first order logic $\mathrm{FO}(\mathbb{Q},+,\leq)$. In Section VI we recall the well-order over the runs first introduced in [14] central in the analysis of vector addition systems. Sections VII and VIII provide independent results that are used in Section IX to prove that reachability sets of vector additions systems intersected with Presburger sets are finite unions of sets $\mathbf{b}+\mathbf{P}$ where $\mathbf{b}$ is a vector and $\mathbf{P}$ is a smooth periodic set such that for every linear set $\mathbf{Y}\subseteq\mathbf{b}+\mathbf{P}$ there exists $\mathbf{p}\in\mathbf{P}$ such that $\mathbf{p}+\mathbf{Y}$ is a flatable subreachability set (intuitively a subset of the reachability set computable by acceleration). The last Sections X and XI show that this decomposition of the reachability set is sufficient for proving that if the reachability set of a VAS is definable in the Presburger arithmetic then it is flatable. Due to space limitation, most mathematical results are only proved in a technical report [15].

II. Vectors And Numbers

We denote by $\mathbb{N},\mathbb{N}_{>0},\mathbb{Z},\mathbb{Q},\mathbb{Q}_{\geq 0},\mathbb{Q}_{>0}$ the set of *natural numbers*, *positive integers*, *integers*, *rational numbers*, *non negative rational numbers*, and *positive rational numbers*. *Vectors* and *sets of vectors* are denoted in bold face. The ith *component* of a vector $\mathbf{v}\in\mathbb{Q}^d$ is denoted by $\mathbf{v}(i)$. We introduce $||\mathbf{v}||_\infty=\max_{1\leq i\leq d}|\mathbf{v}(i)|$ where $|\mathbf{v}(i)|$ is the *absolute value* of $\mathbf{v}(i)$. A set $\mathbf{B}\subseteq\mathbb{Q}^d$ is said to be *bounded* if there exists $m\in\mathbb{Q}_{\geq 0}$ such that $||\mathbf{b}||_\infty\leq m$ for every $\mathbf{b}\in\mathbf{B}$. The addition function $+$ is extended component-wise over $\mathbb{Q}^d$.

The *dot product* of two vectors $\mathbf{x},\mathbf{y}\in\mathbb{Q}^d$ is the rational number $\sum_{i=1}^d\mathbf{x}(i)\mathbf{y}(i)$ denoted by $\mathbf{x}\cdot\mathbf{y}$.

Given two sets $\mathbf{V}_1,\mathbf{V}_2\subseteq\mathbb{Q}^d$ we denote by $\mathbf{V}_1+\mathbf{V}_2$ the set $\{\mathbf{v}_1+\mathbf{v}_2 \mid (\mathbf{v}_1,\mathbf{v}_2)\in\mathbf{V}_1\times\mathbf{V}_2\}$, and we denote by $\mathbf{V}_1-\mathbf{V}_2$ the set $\{\mathbf{v}_1-\mathbf{v}_2 \mid (\mathbf{v}_1,\mathbf{v}_2)\in\mathbf{V}_1\times\mathbf{V}_2\}$. In the same way given $T\subseteq\mathbb{Q}$ and $\mathbf{V}\subseteq\mathbb{Q}^d$ we let $T\mathbf{V}=\{t\mathbf{v}\mid(t,\mathbf{v})\in T\times\mathbf{V}\}$. We also denote by $\mathbf{v}_1+\mathbf{V}_2$ and $\mathbf{V}_1+\mathbf{v}_2$ the sets $\{\mathbf{v}_1\}+\mathbf{V}_2$ and $\mathbf{V}_1+\{\mathbf{v}_2\}$, and we denote by $t\mathbf{V}$ and $T\mathbf{v}$ the sets $\{t\}\mathbf{V}$ and $T\{\mathbf{v}\}$. In the sequel, an empty sum of sets included in $\mathbb{Q}^d$ denotes the set reduced to the zero vector $\{\mathbf{0}\}$.

III. Flatable Vector Addition Systems

A *Vector Addition System (VAS)* is a pair $(\mathbf{c}_{\text{init}},\mathbf{A})$ where $\mathbf{c}_{\text{init}}\in\mathbb{N}^d$ is an *initial configuration* and $\mathbf{A}\subseteq\mathbb{Z}^d$ is a finite set of *actions*.

The semantics of vector addition systems is obtained as follows. A vector $\mathbf{c}\in\mathbb{N}^d$ is called a *configuration*. We introduce the labeled relation $\rightarrow$ defined by $\mathbf{x}\xrightarrow{\mathbf{a}}\mathbf{y}$ if $\mathbf{x},\mathbf{y}\in\mathbb{N}^d$ are configurations, $\mathbf{a}\in\mathbf{A}$ is an action, and $\mathbf{y}=\mathbf{x}+\mathbf{a}$. As expected, a *run* is a non-empty word $\rho=\mathbf{c}_0\ldots\mathbf{c}_k$ of configurations $\mathbf{c}_j\in\mathbb{N}^d$ such that $\mathbf{a}_j=\mathbf{c}_j-\mathbf{c}_{j-1}$ is a vector in $\mathbf{A}$ (see e.g., Figure 1). The word $w=\mathbf{a}_1\ldots\mathbf{a}_k$ is called the *label* of ρ. The configurations $\mathbf{c}_0$ and $\mathbf{c}_k$ are respectively called the *source* and the *target* and they are denoted by $\mathrm{src}(\rho)$ and $\mathrm{tgt}(\rho)$. We also denote by $\mathrm{dir}(\rho)$ the pair $(\mathrm{src}(\rho),\mathrm{tgt}(\rho))$ called the *direction* of ρ. The relation $\rightarrow$ is extended over the words $w=\mathbf{a}_1\ldots\mathbf{a}_k$ of actions $\mathbf{a}_j\in\mathbf{A}$ by $\mathbf{x}\xrightarrow{w}\mathbf{y}$ if there exists a run from $\mathbf{x}$ to $\mathbf{y}$ labeled by w. Given a language $W\subseteq\mathbf{A}^*$, we denote by $\xrightarrow{W}$ the relation $\bigcup_{w\in W}\xrightarrow{w}$. The relation $\xrightarrow{\mathbf{A}^*}$ is called the reachability relation and it is denoted by $\xrightarrow{*}$. A *subreachability relation* is a relation included in $\xrightarrow{*}$.

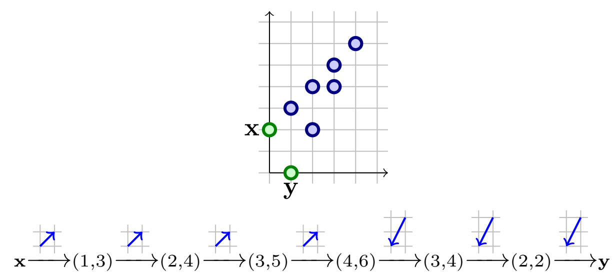

Figure 1. The run labeled by $(1,1)^4(-1,-2)^3$ with $\mathrm{dir}(\rho)=(\mathbf{x},\mathbf{y})$.

Given a configuration $\mathbf{c}\in\mathbb{N}^d$ and a language $W\subseteq\mathbf{A}^*$ we denote by $\mathrm{post}(\mathbf{c},W)$ the set of configurations $\mathbf{y}\in\mathbb{N}^d$ such that $\mathbf{c}\xrightarrow{W}\mathbf{y}$. Given a set of configurations $\mathbf{C}\subseteq\mathbb{N}^d$ and a language $W\subseteq(\mathbb{Z}^d)^*$ we denote by $\mathrm{post}(\mathbf{C},W)$ the set of configurations $\bigcup_{\mathbf{c}\in\mathbf{C}}\mathrm{post}(\mathbf{c},W)$. The set $\mathrm{post}(\mathbf{c}_{\text{init}},\mathbf{A}^*)$ is called the *reachability set*. A subset of this set if called a *subreachability set*.

Flatability properties [11], [12] are defined thanks to *bounded languages* [13]. A language $W\subseteq\mathbf{A}^*$ is said to be *bounded* if there exists a finite sequence $w_1,\ldots w_m$ of words $w_j\in\mathbf{A}^*$ such that $W\subseteq w_1^*\ldots w_m^*$. Let us recall that bounded languages are stable by concatenation, union, intersection, and subset. A subreachability relation is said to be *flatable* if it is included in $\xrightarrow{W}$ where $W\subseteq\mathbf{A}^*$ is a bounded language. A subreachability set is said to be *flatable* if it is included in $\mathrm{post}(\mathbf{c}_{\text{init}},W)$ where $W\subseteq\mathbf{A}^*$ is a bounded language.

Definition III.1. *A VAS is said to be* flatable *if its reachability set is flatable. A VAS is said to be* Presburger *if its reachability set is definable in the Presburger arithmetic.*

In this paper we show that the class of Presburger VAS coincides with the class of flatable VAS. In the remainder of this section we recall elements of acceleration techniques that explain why flatable VAS are Presburger. We also explain why a Presburger formula denoting the reachability set is effectively computable in this case.

The *displacement* of a word $w = \mathbf{a}_1 \dots \mathbf{a}_k$ of actions $\mathbf{a}_j \in \mathbf{A}$ is the vector $\Delta(w) = \sum_{j=1}^{k} \mathbf{a}_j$. Observe that $\mathbf{x} \xrightarrow{w} \mathbf{y}$ implies $\mathbf{x} + \Delta(w) = \mathbf{y}$ but the converse is not true in general. The converse property can be obtained by associating to every word $w = \mathbf{a}_1 \dots \mathbf{a}_k$ the configuration $\mathbf{c}_w$ defined for every $i \in \{1, \dots, d\}$ by:

$$\mathbf{c}_w(i) = \max\{-(\mathbf{a}_1 + \dots + \mathbf{a}_j)(i) \mid 0 \leq j \leq k\}$$

The following lemma shows that $\mathbf{c}_w$ is the minimal for $\leq$ configuration from which there exists a run labeled by w.

Lemma III.2. *There exists a run from a configuration $\mathbf{x} \in \mathbb{N}^d$ labeled by a word $w \in \mathbf{A}^*$ if, and only if, $\mathbf{x} \geq \mathbf{c}_w$.*

Proof: We assume that $w = \mathbf{a}_1 \dots \mathbf{a}_k$ where $\mathbf{a}_j \in \mathbf{A}$. Assume first that there exists a run $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ labeled by w from $\mathbf{c}_0 = \mathbf{x}$. Since $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$ we deduce that $\mathbf{c}_j = \mathbf{x} + \mathbf{a}_1 + \dots + \mathbf{a}_j$. Since $\mathbf{c}_j \geq \mathbf{0}$ we get $\mathbf{x} \geq -(\mathbf{a}_1 + \dots + \mathbf{a}_j)$. We have proved that $\mathbf{x} \geq \mathbf{c}_w$. Conversely, let us assume that $\mathbf{x} \geq \mathbf{c}_w$ and let us prove that there exists a run from $\mathbf{x}$ labeled by w. We introduce the vectors $\mathbf{c}_j = \mathbf{x} + \mathbf{a}_1 + \dots + \mathbf{a}_j$. Since $\mathbf{x} \geq \mathbf{c}_w$ we deduce that $\mathbf{c}_j \in \mathbb{N}^d$. Therefore $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ is a run. Just observe that $\mathbf{c}_0 = \mathbf{x}$ and ρ is labeled by w. ∎

The following lemma shows that the set of triples $(\mathbf{x}, n, \mathbf{y}) \in \mathbb{N}^d \times \mathbb{N} \times \mathbb{N}^d$ such that $\mathbf{x} \xrightarrow{w^n} \mathbf{y}$ is effectively definable in the Presburger arithmetic. In particular with an existential quantification of the variable n, we deduce that the relation $\xrightarrow{w^*}$ is effectively definable in the Presburger arithmetic. Hence if a set of configurations $\mathbf{C} \subseteq \mathbb{N}^d$ is denoted by a Presburger formula then for every word $w \in \mathbf{A}^*$ we can effectively compute a Presburger formula denoting $\mathrm{post}(\mathbf{C}, w^*)$.

Lemma III.3. *A pair $(\mathbf{x}, \mathbf{y}) \in \mathbb{N}^d \times \mathbb{N}^d$ of configurations satisfies $\mathbf{x} \xrightarrow{w^n} \mathbf{y}$ where $w \in \mathbf{A}^*$ and $n \in \mathbb{N}_{>0}$ if and only if:*

$$\mathbf{x} \geq \mathbf{c}_w \quad \wedge \quad \mathbf{x} + n\Delta(w) = \mathbf{y} \quad \wedge \quad \mathbf{y} - \Delta(w) \geq \mathbf{c}_w$$

Proof: Assume first that we have a run $\mathbf{x} \xrightarrow{w^n} \mathbf{y}$. Since $n \geq 1$, a prefix and a suffix of this run show that $\mathbf{x} \xrightarrow{w} \mathbf{x} + \Delta(w)$ and $\mathbf{y} - \Delta(w) \xrightarrow{w} \mathbf{y}$. Lemma III.2 shows that $\mathbf{x} \geq \mathbf{c}_w$ and $\mathbf{y} - \Delta(w) \geq \mathbf{c}_w$. Moreover, since $\mathbf{x} + n\Delta(w) = \mathbf{y}$ we have proved one way of the lemma. For the other way, let us assume that $\mathbf{x} \geq \mathbf{c}_w$, $\mathbf{x} + n\Delta(w) = \mathbf{y}$, and $\mathbf{y} - \Delta(w) \geq \mathbf{c}_w$. We introduce the sequence $\mathbf{c}_0, \dots, \mathbf{c}_n$ defined by $\mathbf{c}_j = \mathbf{x} + j\Delta(w)$. Let us prove that $\mathbf{c}_{j-1} \geq \mathbf{c}_w$ for every $1 \leq j \leq n$. Let $i \in \{1, \dots, d\}$. If $\Delta(w)(i) \geq 0$ then $\mathbf{c}_{j-1}(i) \geq \mathbf{x}(i) \geq \mathbf{c}_w(i)$. Next, assume that $\Delta(w)(i) < 0$. In this case, since $\mathbf{x} + n\Delta(w) = \mathbf{y}$ we deduce that $\mathbf{c}_{j-1} = \mathbf{y} - \Delta(w) + (n - j)(-\Delta(w))$. Thus $\mathbf{c}_{j-1}(i) \geq \mathbf{y}(i) - \Delta(w)(i) \geq \mathbf{c}_w(i)$. We have proved that $\mathbf{c}_{j-1} \geq \mathbf{c}_w$. Lemma III.2 shows that $\mathbf{c}_{j-1} \xrightarrow{w} \mathbf{c}_j$. We have proved that $\mathbf{c}_0 \xrightarrow{w^n} \mathbf{c}_n$. Since $\mathbf{c}_0 = \mathbf{x}$ and $\mathbf{c}_n = \mathbf{y}$ we have proved the other way. ∎

We deduce the following theorem also proved in [16] in a more general context. This theorem shows that we can effectively compute a Presburger formula denoting the reachability set of flatable VAS.

Theorem III.4 ([16]). *There exists an algorithm computing for any flatable VAS $(\mathbf{c}_{init}, \mathbf{A})$ a sequence $w_1, \dots, w_m \in \mathbf{A}^*$ such that:*

$$\mathrm{post}(\mathbf{c}_{init}, \mathbf{A}^*) = \mathrm{post}(\mathbf{c}_{init}, w_1^* \dots w_m^*)$$

Proof: Let us consider an algorithm that takes as input a VAS $(\mathbf{c}_{\mathrm{init}}, \mathbf{A})$ and it computes inductively a sequence $(w_m)_{m \geq 1}$ of words $w_m \in \mathbf{A}^*$ such that every finite sequence $(\sigma_j)_{1 \leq j \leq n}$ of words $\sigma_j \in \mathbf{A}^*$ is a sub-sequence. Note that such an algorithm exists. From this sequence, another algorithm computes inductively Presburger formulas denoting sets of configurations $\mathbf{C}_m \subseteq \mathbb{N}^d$ satisfying $\mathbf{C}_0 = \{\mathbf{c}_{\mathrm{init}}\}$ and $\mathbf{C}_m = \mathrm{post}(\mathbf{C}_{m-1}, w_m^*)$ for every $m \in \mathbb{N}_{>0}$. The algorithm stops and it returns $w_1, \dots, w_m$ when $\mathrm{post}(\mathbf{C}_m, \mathbf{A}) \subseteq \mathbf{C}_m$. Note that such a test is implementable since $\mathbf{C}_m$ is denoted by a Presburger formula and the Presburger arithmetic is a decidable logic. When the algorithm stops the set $\mathbf{C}_m$ is included in the reachability set and it satisfies $\mathrm{post}(\mathbf{C}_m, \mathbf{A}) \subseteq \mathbf{C}_m$. We deduce that $\mathbf{C}_m$ is equal to the reachability set. In particular the reachability set if equal to $\mathrm{post}(\mathbf{c}_{\mathrm{init}}, w_1^* \dots w_m^*)$ and the algorithm is correct.

For the termination, since the VAS is flatable, there exists a bounded language $W \subseteq \mathbf{A}^*$ such that the reachability set is included in $\mathrm{post}(\mathbf{c}_{\mathrm{init}}, W)$. As W is bounded, there exists a finite sequence $\sigma_1, \dots, \sigma_n \in \mathbf{A}^*$ such that $W \subseteq \sigma_1^* \dots \sigma_n^*$. There exists $m \in \mathbb{N}$ such that this sequence is a sub-sequence of $w_1, \dots, w_m$. Let us observe that $W \subseteq \sigma_1^* \dots \sigma_n^* \subseteq w_1^* \dots w_m^*$. From the following inclusions we deduce that $\mathbf{C}_m$ is equal to the reachability set:

$$\begin{aligned}
\mathrm{post}(\mathbf{c}_{\mathrm{init}}, \mathbf{A}^*) &\subseteq \mathrm{post}(\mathbf{c}_{\mathrm{init}}, W) \\
&\subseteq \mathrm{post}(\mathbf{c}_{\mathrm{init}}, w_1^* \dots w_m^*) \\
&= \mathbf{C}_m \\
&\subseteq \mathrm{post}(\mathbf{c}_{\mathrm{init}}, \mathbf{A}^*)
\end{aligned}$$

In particular $\mathrm{post}(\mathbf{C}_m, \mathbf{A}) \subseteq \mathbf{C}_m$ and the algorithm terminates before the mth iteration. ∎

Corollary III.5. *Reachability sets of flatable VAS are effectively definable in the Presburger arithmetic.*

In the remainder of this paper, we proved that Presburger VAS are flatable. As a direct consequence a Presburger formula denoting the reachability set of a Presburger VAS is effectively computable using classical acceleration techniques.

IV. Well-Preorders

A relation R over a set S is a subset $R \subseteq S \times S$. The composition of two relations R_1, R_2 over S is the relation over S denoted by $R_1 \circ R_2$ and defined as the set $\bigcup_{i \in S} \{(s,t) \in S \times S \mid (s,i) \in R_1 \ \wedge \ (i,t) \in R_2\}$. A relation R over S is said to be *reflexive* if $(s,s) \in R$ for every $s \in S$, *transitive* if

$R \circ R \subseteq R$, *antisymmetric* if $(s,t),(t,s) \in R$ implies $s = t$, a *preorder* if R is reflexive and transitive, and an *order* if R is an antisymmetric preorder. The composition of R by itself n times where $n \in \mathbb{N}_{>0}$ is denoted by R^n. The transitive closure of a relation R is the relation $\bigcup_{n \geq 1} R^n$ denoted by R^+.

A preorder $\sqsubseteq$ over a set S is said to be *well* if for every sequence $(s_n)_{n \in \mathbb{N}}$ of elements $s_n \in S$ there exists an infinite set $N \subseteq \mathbb{N}$ such that $s_n \sqsubseteq s_m$ for every $n \leq m$ in N. Observe that $(\mathbb{N}, \leq)$ is a well-ordered set whereas $(\mathbb{Z}, \leq)$ is not well-ordered. As another example, the pigeonhole principle shows that a set S is well-ordered by the equality relation if, and only if, S is finite. Well-preorders can be easily defined thanks to *Dickson's lemma* and *Higman's lemma* as follows.

Dickson's lemma: Dickson's lemma shows that the Cartesian product of two well-preordered sets is well-preordered. More formally, given two preordered sets $(S_1, \sqsubseteq_1)$ and $(S_2, \sqsubseteq_2)$ we denote by $\sqsubseteq_1 \times \sqsubseteq_2$ the preorder defined component-wise over the Cartesian product $S_1 \times S_2$ by $(s_1, s_2) \sqsubseteq_1 \times \sqsubseteq_2 (s_1', s_2')$ if $s_1 \sqsubseteq_1 s_1'$ and $s_2 \sqsubseteq_2 s_2'$. *Dickson's lemma* says that $(S_1 \times S_2, \sqsubseteq_1 \times \sqsubseteq_2)$ is well-preordered for every well-preordered sets $(S_1, \sqsubseteq_1)$ and $(S_2, \sqsubseteq_2)$. As a direct application, the set $\mathbb{N}^d$ equipped with the component-wise extension of $\leq$ is well-ordered.

Higman's lemma: Higman's lemma shows that words over well-preordered alphabets can be well-preordered. More formally, given a preordered set $(S, \sqsubseteq)$, we introduce the set S^* of words over S equipped with the preorder $\sqsubseteq^*$ defined by $w \sqsubseteq^* w'$ if w and w' can be decomposed into $w = s_1 \dots s_k$ and $w' \in S^* s_1' S^* \dots s_k' S^*$ where $s_j \sqsubseteq s_j'$ are in S for every $j \in \{1, \dots, k\}$. *Higman's lemma* says that $(S^*, \sqsubseteq^*)$ is well-preordered for every well-preordered set $(S, \sqsubseteq)$. As a classical application, the set of words over a finite alphabet is well-ordered by the sub-word relation.

V. Vector Spaces, Conic Sets, Periodic Sets, and Lattices

In this section we recall some elements of linear algebra. We also introduce the central notions of *definable conic sets* and *smooth periodic sets*.

A *vector space* is a set $\mathbf{V} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{V}$, $\mathbf{V} + \mathbf{V} \subseteq \mathbf{V}$, and $\mathbb{Q}\mathbf{V} \subseteq \mathbf{V}$. Any set $\mathbf{X} \subseteq \mathbb{Q}^d$ is included in a unique minimal under set inclusion vector space. This vector space called the *vector space generated* by $\mathbf{X} \subseteq \mathbb{Q}^d$. Let us recall that every vector space $\mathbf{V}$ is generated by a finite set. The *rank* $\operatorname{rank}(\mathbf{V})$ of a vector space $\mathbf{V}$ is the minimal natural number $r \in \mathbb{N}$ such that there exists a finite set $\mathbf{B}$ with r vectors that generates $\mathbf{V}$. Let us recall that $\operatorname{rank}(\mathbf{V}) \leq \operatorname{rank}(\mathbf{W})$ for every pair of vector spaces $\mathbf{V} \subseteq \mathbf{W}$. Moreover, if $\mathbf{V}$ is strictly included in $\mathbf{W}$ then $\operatorname{rank}(\mathbf{V}) < \operatorname{rank}(\mathbf{W})$. Vectors spaces are geometrically characterized as follows:

Lemma V.1 ([17]). *A set $\mathbf{V} \subseteq \mathbb{Q}^d$ is a vector space if and only if there exists a finite set $\mathbf{H} \subseteq \mathbb{Q}^d$ such that:*

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbb{Q}^d \;\middle|\; \bigwedge_{\mathbf{h} \in \mathbf{H}} \mathbf{h} \cdot \mathbf{v} = 0 \right\}$$

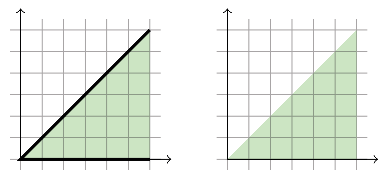

Figure 2. The finitely generated conic set $\mathbb{Q}_{\geq 0}(1,1) + \mathbb{Q}_{\geq 0}(1,0)$ and the definable conic set $\{(0,0)\} \cup \{(c_1,c_2) \in \mathbb{Q}^2_{>0} \mid c_2 < c_1\}$

A *conic set* is a set $\mathbf{C} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{C}$, $\mathbf{C} + \mathbf{C} \subseteq \mathbf{C}$ and $\mathbb{Q}_{\geq 0}\mathbf{C} \subseteq \mathbf{C}$. Any set $\mathbf{X} \subseteq \mathbb{Q}^d$ is included in a unique minimal under set inclusion conic set. This conic set is called the *conic set generated* by $\mathbf{X} \subseteq \mathbb{Q}^d$. Contrary to the vector spaces, some conic sets are not finitely generated. Finitely generated conic sets are geometrically characterized by the following lemma.

Lemma V.2 ([17]). *A set $\mathbf{C} \subseteq \mathbb{Q}^d$ is a finitely generated conic set if and only if there exists a finite set $\mathbf{H} \subseteq \mathbb{Q}^d$ such that:*

$$\mathbf{C} = \left\{ \mathbf{c} \in \mathbb{Q}^d \;\middle|\; \bigwedge_{\mathbf{h} \in \mathbf{H}} \mathbf{h} \cdot \mathbf{c} \geq 0 \right\}$$

Definition V.3. *A conic set is said to be* definable *(*polytope *in [18]) if it can be defined by a formula in* $\mathrm{FO}(\mathbb{Q}, +, \leq)$.

Example V.4. *The conic set $\mathbf{C} = \{(c_1, c_2) \in \mathbb{Q} \times \mathbb{Q} \mid c_1 \leq \sqrt{2} c_2\}$ is not definable. Fig. 2 depicts a finitely generated conic set and a definable conic set which is not finitely generated.*

A *periodic set* is a set $\mathbf{P} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{P}$, and $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$. Any set $\mathbf{X} \subseteq \mathbb{Q}^d$ is included in a unique minimal under set inclusion periodic set. This periodic set is called the *periodic set generated* by $\mathbf{X}$. Observe that the conic set $\mathbf{C}$ generated by a periodic set $\mathbf{P}$ is $\mathbf{C} = \mathbb{Q}_{\geq 0}\mathbf{P}$. The finitely generated periodic sets are characterized as follows. Given a periodic set $\mathbf{P}$ we denote by $\leq_{\mathbf{P}}$ the preorder over $\mathbf{P}$ defined by $\mathbf{p} \leq_{\mathbf{P}} \mathbf{q}$ if $\mathbf{q} \in \mathbf{p} + \mathbf{P}$. A periodic set $\mathbf{P} \subseteq \mathbb{Q}^d$ is said to be *discrete* if there exists $n \in \mathbb{N}_{>0}$ such that $\mathbf{P} \subseteq \frac{1}{n}\mathbb{Z}^d$. Observe that finitely generated periodic sets are discrete. The following lemma characterizes the discrete periodic sets that are finitely generated. The proof is given in [15].

Lemma V.5. *Let $\mathbf{P}$ be a discrete periodic set. The following conditions are equivalent:*

- $\mathbf{P}$ *is finitely generated as a periodic set.*
- $(\mathbf{P}, \leq_{\mathbf{P}})$ *is well-preordered.*
- $\mathbb{Q}_{\geq 0}\mathbf{P}$ *is finitely generated as a conic set.*

Remark V.6. *A set $\mathbf{X} \subseteq \mathbb{Z}^d$ is definable in the Presburger arithmetic* $\mathrm{FO}(\mathbb{Z}, +, \leq)$ *if, and only if, it is a finite union of* linear sets $\mathbf{b} + \mathbf{P}$ *where $\mathbf{b} \in \mathbb{Z}^d$ and $\mathbf{P} \subseteq \mathbb{Z}^d$ is a finitely generated periodic set [8].*

A *limit* of a periodic set $\mathbf{P} \subseteq \mathbb{Q}^d$ is a vector $\mathbf{v} \in \mathbb{Q}^d$ such that there exists $\mathbf{p} \in \mathbf{P}$ and $n \in \mathbb{N}_{>0}$ satisfying $\mathbf{p} + n\mathbb{N}\mathbf{v} \subseteq \mathbf{P}$. The set of limits of $\mathbf{P}$ is denoted by $\lim(\mathbf{P})$.

Lemma V.7. $\lim(\mathbf{P})$ *is a conic set.*

Proof: Let $\mathbf{C} = \lim(\mathbf{P})$. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{C}$. There exist $\mathbf{p}_1, \mathbf{p}_2 \in \mathbf{P}$ and $n_1, n_2 \in \mathbb{N}_{>0}$ such that $\mathbf{p}_1 + n_1\mathbb{N}\mathbf{v}_1$ and $\mathbf{p}_2 + n_2\mathbb{N}\mathbf{v}_2$ are included in $\mathbf{P}$. Let $n = n_1 n_2$. Since $n\mathbb{N}$ is included in $n_1\mathbb{N}$ and $n_2\mathbb{N}$ we deduce that $\mathbf{p}_1 + n\mathbb{N}\mathbf{v}_1$ and $\mathbf{p}_2 + n\mathbb{N}\mathbf{v}_2$ are included in $\mathbf{P}$. As $\mathbf{P}$ is periodic we deduce that $\mathbf{p} + n\mathbb{N}\mathbf{v} \subseteq \mathbf{P}$ where $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. As $\mathbf{p} \in \mathbf{P}$ we get $\mathbf{v} \in \mathbf{C}$. We deduce that $\mathbf{C} + \mathbf{C} \subseteq \mathbf{C}$. Since $\mathbf{0} \in \mathbf{C}$ and $\mathbb{Q}_{\geq 0}\mathbf{C} \subseteq \mathbf{C}$ are immediate, we have proved that $\mathbf{C}$ is a conic set. ■

A *periodic set* $\mathbf{P}$ is said to be *well-limit* if for every sequence $(\mathbf{p}_n)_{n\in\mathbb{N}}$ of vectors $\mathbf{p}_n \in \mathbf{P}$ there exists an infinite set $N \subseteq \mathbb{N}$ such that $\mathbf{p}_m - \mathbf{p}_n \in \lim(\mathbf{P})$ for every $n \leq m$ in N. The periodic set $\mathbf{P}$ is said to be *smooth* if $\lim(\mathbf{P})$ is a definable conic set and $\mathbf{P}$ is well-limit.

Example V.8. *Let us consider the periodic set* $\mathbf{P} \subseteq \mathbb{N}^2$ *generated by* $(0,1)$ *and the pairs* $(2^m, 1)$ *where* $m \in \mathbb{N}$. *The limit of* $\mathbf{P}$ *is the definable conic set* $\mathbf{C} = \{(0,0)\} \cup (\mathbb{Q}_{\geq 0} \times \mathbb{Q}_{>0})$. *Note that* $\mathbf{P}$ *is not well-limit since the sequence* $(\mathbf{p}_n)_{n\in\mathbb{N}}$ *defined by* $\mathbf{p}_n = (2^n, 1)$ *is such that* $\mathbf{p}_m - \mathbf{p}_n = (2^m - 2^n, 0) \notin \mathbf{C}$ *for every* $n < m$.

Example V.9. *The periodic set* $\mathbf{P} = \{(0,0)\} \cup (\mathbb{N}_{>0} \times \mathbb{N}_{>0})$ *is smooth and* $\lim(\mathbf{P}) = \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0}$.

A *lattice* is a set $\mathbf{L} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{L}$, $\mathbf{L} + \mathbf{L} \subseteq \mathbf{L}$ and $-\mathbf{L} \subseteq \mathbf{L}$. Any set $\mathbf{X} \subseteq \mathbb{Q}^d$ is included in a unique minimal under set inclusion lattice. This lattice is called the *lattice generated* by $\mathbf{X}$. Observe that the conic set generated by a lattice $\mathbf{L}$ is equal to the vector space $\mathbf{V} = \mathbb{Q}_{\geq 0}\mathbf{L}$. Since vector spaces are finitely generated, the previous Lemma V.5 shows that discrete lattices are finitely generated (as periodic sets and in particular as lattices).

Remark V.10. *The following relations hold:*

conic sets	$\subset$	*periodic sets*	$\supset$	*discrete periodic sets*	$\supset$	*finitely gen. periodic sets*
$\cup$		$\cup$		$\cup$		$\cup$
vector spaces	$\subset$	*lattices*	$\supset$	*discrete lattices*	$=$	*finitely gen. lattices*

Example V.11. *The following periodic sets provide the strictness of the previous inclusion relations :* $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_{\geq 0}$, $\{(x,y) \in \mathbb{Q} \times \mathbb{Q} \mid x \leq \sqrt{2}y\}$.

VI. Well-Order Over The Runs

We define a well-order over the runs as follows. We introduce the relation $\trianglelefteq$ over the runs defined by $\rho \trianglelefteq \rho'$ if ρ is a run of the form $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ where $\mathbf{c}_j \in \mathbb{N}^d$ and if there exists a sequence $(\mathbf{v}_j)_{0\leq j\leq k+1}$ of vectors $\mathbf{v}_j \in \mathbb{N}^d$ such that ρ' is a run of the form $\rho' = \rho_0 \dots \rho_k$ where ρ_j is a run from $\mathbf{c}_j + \mathbf{v}_j$ to $\mathbf{c}_j + \mathbf{v}_{j+1}$.

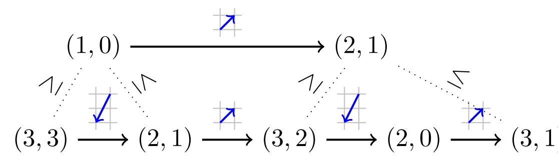

Figure 3. $(1,0)(2,1) \quad \trianglelefteq \quad (3,3)(2,1)(3,2)(2,0)(3,1)$

Example VI.1. *This example is depicted on Figure 3. Let* $\rho = (1,0)(2,1)$ *and observe that* $\rho \trianglelefteq \rho_1\rho_2$ *where* $\rho_1 = (3,3)(2,1)$ *and* $\rho_2 = (3,2)(2,0)(3,1)$.

Let us recall the following lemma based on the Higman's Lemma.

Lemma VI.2 ([14], [19]). *The relation* $\trianglelefteq$ *is a well-order.*

Lemma VI.3. *For every pair of runs* $\rho \trianglelefteq \rho'$*, the pair* $(\mathbf{e}, \mathbf{f}) = \mathrm{dir}(\rho') - \mathrm{dir}(\rho)$ *satisfies* $\mathrm{dir}(\rho) + \mathbb{N}(\mathbf{e}, \mathbf{f})$ *is a flatable subreachability relation.*

Proof: Assume that $\rho \trianglelefteq \rho'$. In this case $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ where $\mathbf{c}_j \in \mathbb{N}^d$ and there exists a sequence $\mathbf{v}_0, \dots, \mathbf{v}_{k+1} \in \mathbb{N}^d$ such that $\rho' = \rho_0 \dots \rho_k$ where ρ_j is a run from $\mathbf{c}_j + \mathbf{v}_j$ to $\mathbf{c}_j + \mathbf{v}_{j+1}$ labeled by a word σ_j. We introduce the actions $\mathbf{a}_1, \dots, \mathbf{a}_k$ defined by $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$. By monotony we deduce that for every $r \in \mathbb{N}$ we have a run from $\mathbf{c}_j + r\mathbf{v}_j$ to $\mathbf{c}_j + r\mathbf{v}_{j+1}$ labeled by σ_j^r. We also have $\mathbf{c}_j + r\mathbf{v}_{j+1} \xrightarrow{\mathbf{a}_j} \mathbf{c}_{j+1} + r\mathbf{v}_{j+1}$. We obtain from these runs, a run ρ_r from $\mathbf{c}_0 + r\mathbf{v}_0$ to $\mathbf{c}_k + r\mathbf{v}_{k+1}$ labeled by $\sigma_0^r\mathbf{a}_1\sigma_1^r \dots \mathbf{a}_k\sigma_k^r$. Since $(\mathbf{e}, \mathbf{f}) = \mathrm{dir}(\rho') - \mathrm{dir}(\rho)$ is the pair $(\mathbf{v}_0, \mathbf{v}_{k+1})$ we deduce that $\mathrm{dir}(\rho) + \mathbb{N}(\mathbf{e}, \mathbf{f})$ is included in $\xrightarrow{W}$ where $W = \sigma_0^*\mathbf{a}_1\sigma_1^* \dots \mathbf{a}_k\sigma_k^*$. ■

Based on the definition of the well-order $\trianglelefteq$, we introduce the *transformer relation with capacity* $\mathbf{c} \in \mathbb{N}^d$ as the relation $\overset{\mathbf{c}}{\curvearrowright}$ over $\mathbb{N}^d$ defined by $\mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}$ if there exists a run from $\mathbf{c} + \mathbf{x}$ to $\mathbf{c} + \mathbf{y}$. By monotony, let us observe that $\overset{\mathbf{c}}{\curvearrowright}$ is a periodic relation.

Remark VI.4. *In [19], the conic relation* $\mathbb{Q}_{\geq 0} \overset{\mathbf{c}}{\curvearrowright}$ *is shown to be definable.*

VII. Reflexive Definable Conic Relations

The class of finite unions of reflexive definable conic relations over $\mathbb{Q}_{\geq 0}^d$ are clearly stable by composition, sum, intersection, and union. In the technical report [15], the following theorem is proved:

Theorem VII.1. *Transitive closures of finite unions of reflexive definable conic relations over* $\mathbb{Q}_{\geq 0}^d$ *are reflexive definable conic relations.*

Example VII.2. *Let us consider the reflexive definable conic relation* $R = \{(x, x') \in \mathbb{Q}_{\geq 0}^2 \mid x \leq x' \leq 2x\}$. *Observe that* R^n *where* $n \geq 1$ *is the reflexive definable conic relation* $\{(x, x') \in \mathbb{Q}_{\geq 0}^2 \mid x \leq x' \leq 2^n x\}$. *Thus* $R^+ = \{(0,0)\} \cup \{(x, x') \mid 0 < x \leq x'\}$. *Observe that* R^n *is strictly included* *in R^+ for every $n \geq 1$. Hence R^+ cannot be computed with a finite Kleene iteration $R^1 \cup \ldots \cup R^n$.*

VIII. Transformer Relations

In this section, we prove the following theorem. All other results are not used in the sequel.

Theorem VIII.1. *For every capacity $\mathbf{c} \in \mathbb{N}^d$ and for every periodic relation P included in $\overset{\mathbf{c}}{\curvearrowright}$, there exists a definable conic relation $R \subseteq \mathbb{Q}^d_{\geq 0} \times \mathbb{Q}^d_{\geq 0}$ such that $\lim(P) \subseteq R$ and such that for every $(\mathbf{e}, \mathbf{f}) \in \bar{R}$ there exists $(\mathbf{x}, \mathbf{y}) \in P$ and $n \in \mathbb{N}_{>0}$ such that*

$$(\mathbf{c}, \mathbf{c}) + (\mathbf{x}, \mathbf{y}) + n\mathbb{N}(\mathbf{e}, \mathbf{f})$$

is a flatable subreachability relation.

Theorem VIII.1 is obtained by following the approach introduced in [19]. Note that even if some lemmas are very similar to the ones given in that paper, proofs must be adapted to our context. In this new context, Theorem VII.1 is central for proving the existence of the relation R introduced by Theorem VIII.1 (introduced as R_γ in the sequel).

In the remainder of this section, γ denotes a pair $(\mathbf{c}, P)$ where $\mathbf{c} \in \mathbb{N}^d$ is a capacity, and $P \subseteq \overset{\mathbf{c}}{\curvearrowright}$ is a periodic relation. We introduce the set Ω_γ of runs ρ such that $\mathrm{dir}(\rho) \in (\mathbf{c}, \mathbf{c}) + P$. Note that Ω_γ is non empty since it contains the run reduced to the single configuration $\mathbf{c}$. We denote by $\mathbf{Q}_\gamma$ the set of configurations $\mathbf{q} \in \mathbb{N}^d$ such that there exists a run $\rho \in \Omega_\gamma$ in which $\mathbf{q}$ occurs. We denote by I_γ the set of indexes $i \in \{1, \ldots, d\}$ such that $\{\mathbf{q}(i) \mid \mathbf{q} \in \mathbf{Q}_\gamma\}$ is finite. We consider the projection function $\pi_\gamma : \mathbf{Q}_\gamma \to \mathbb{N}^{I_\gamma}$ defined by $\pi_\gamma(\mathbf{q})(i) = \mathbf{q}(i)$. We introduce the finite set of *states* $S_\gamma = \pi_\gamma(\mathbf{Q}_\gamma)$ and the set T_γ of *transitions* $(\pi_\gamma(\mathbf{q}), \mathbf{q}' - \mathbf{q}, \pi_\gamma(\mathbf{q}'))$ where $\mathbf{q}\mathbf{q}'$ is a factor of a run in Ω_γ. We introduce $s_\gamma = \pi_\gamma(\mathbf{c})$. Since $T_\gamma \subseteq S_\gamma \times \mathbf{A} \times S_\gamma$ we deduce that T_γ is finite. We introduce the graph $G_\gamma = (S_\gamma, T_\gamma)$.

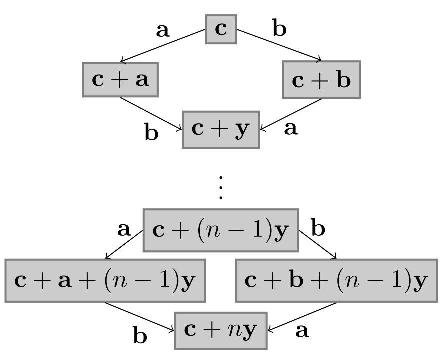

Figure 4. Set of runs Ω_γ.

Example VIII.2. *Let us consider the VAS $\mathbf{A} = \{\mathbf{a}, \mathbf{b}\}$ where $\mathbf{a} = (1, 1, -1)$ and $\mathbf{b} = (-1, 0, 1)$, and let us consider the pair $(\mathbf{c}, P)$ where $\mathbf{c} = (1, 0, 1)$, and $P = \mathbb{N}(\mathbf{0}, \mathbf{y})$ with $\mathbf{y} =$*

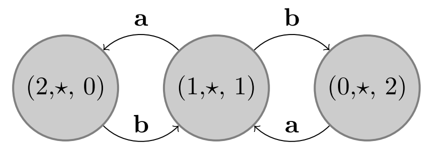

Figure 5. Graph G_γ.

$(0, 1, 0)$. Note that P is included in $\overset{\mathbf{c}}{\curvearrowright}$ since there exists a run $\mathbf{c} \xrightarrow{(\mathbf{ab})^n} \mathbf{c} + n\mathbf{y}$ for every $n \in \mathbb{N}$. The set Ω_γ is depicted in Figure 4. This set is equal to $\{\mathbf{c} \xrightarrow{w_1 \ldots w_n} \mathbf{c} + n\mathbf{y} \mid n \in \mathbb{N}\ w_j \in \{\mathbf{ab}, \mathbf{ba}\}\}$. Observe that $\mathbf{Q}_\gamma = (\mathbf{c} + \mathbf{a} + \mathbb{N}\mathbf{y}) \cup (\mathbf{c} + \mathbb{N}\mathbf{y}) \cup (\mathbf{c} + \mathbf{b} + \mathbb{N}\mathbf{y})$. Hence the set of bounded components is $I_\gamma = \{1, 3\}$. Observe that $\pi_\gamma(\mathbf{c} + \mathbf{a} + n\mathbf{y}) = (2,\star, 0)$, $\pi_\gamma(\mathbf{c} + n\mathbf{y}) = (1,\star, 1)$, and $\pi_\gamma(\mathbf{c} + \mathbf{b} + n\mathbf{y}) = (0,\star, 2)$ where $\star$ denotes a projected component. Hence $s_\gamma = (1,\star, 1)$ and $S_\gamma = \{(2,\star, 0), (1,\star, 1), (0,\star, 2)\}$. The graph G_γ is depicted on Figure 5.

An *intraproduction* for γ is a vector $\mathbf{h} \in \mathbb{N}^d$ such that $\mathbf{c} + \mathbf{h} \in \mathbf{Q}_\gamma$. We denote by $\mathbf{H}_\gamma$ the set of intraproduction for γ. The following Lemma VIII.3 shows that this set is periodic. In particular for every $\mathbf{h} \in \mathbf{H}_\gamma$, from $\mathbf{c} + \mathbb{N}\mathbf{h} \subseteq \mathbf{Q}_\gamma$ we deduce that $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$.

Lemma VIII.3. *We have $\mathbf{Q}_\gamma + \mathbf{H}_\gamma \subseteq \mathbf{Q}_\gamma$.*

Proof: Let $\mathbf{q} \in \mathbf{Q}_\gamma$ and $\mathbf{h} \in \mathbf{H}_\gamma$. As $\mathbf{q} \in \mathbf{Q}_\gamma$, there exist $(\mathbf{x}, \mathbf{y}) \in P$ and words $u, v \in \mathbf{A}^*$ such that $\mathbf{c} + \mathbf{x} \xrightarrow{u} \mathbf{q} \xrightarrow{v} \mathbf{c} + \mathbf{y}$. Since $\mathbf{h} \in \mathbf{H}_\gamma$ there exist $(\mathbf{x}', \mathbf{y}') \in P$ and words $u', v' \in \mathbf{A}^*$ such that $\mathbf{c} + \mathbf{x}' \xrightarrow{u'} \mathbf{c} + \mathbf{h} \xrightarrow{v'} \mathbf{c} + \mathbf{y}'$. By monotony, we have $\mathbf{c} + (\mathbf{x} + \mathbf{x}') \xrightarrow{u'u} \mathbf{q} + \mathbf{h} \xrightarrow{vv'} \mathbf{c} + (\mathbf{y} + \mathbf{y}')$. As P is periodic, we deduce that $\mathbf{q} + \mathbf{h} \in \mathbf{Q}_\gamma$. ■

Example VIII.4. *Let us come back to Example VIII.2. Note that $\mathbf{H}_\gamma = \mathbb{N}\mathbf{y}$. Observe that $\mathbf{Q}_\gamma + \mathbf{H}_\gamma = \mathbf{Q}_\gamma$.*

Corollary VIII.5. *We have $\pi_\gamma(\mathrm{src}(\rho)) = s_\gamma = \pi_\gamma(\mathrm{tgt}(\rho))$ for every run $\rho \in \Omega_\gamma$.*

Proof: Since $\rho \in \Omega_\gamma$ there exists $(\mathbf{x}, \mathbf{y}) \in P$ such that ρ is a run from $\mathbf{c} + \mathbf{x}$ to $\mathbf{c} + \mathbf{y}$. In particular $\mathbf{x}$ and $\mathbf{y}$ are two intraproductions for γ. We deduce that $\mathbf{x}(i) = 0 = \mathbf{y}(i)$ for every $i \in I_\gamma$. Hence $\pi_\gamma(\mathrm{src}(\rho)) = \pi_\gamma(\mathbf{c}) = \pi_\gamma(\mathrm{tgt}(\rho))$. ■

A *path* in G_γ is a word $p = (s_0, \mathbf{a}_1, s_1) \ldots (s_{k-1}, \mathbf{a}_k, s_k)$ of transitions $(s_{j-1}, \mathbf{a}_j, s_j)$ in T_γ. Such a path is called a path from s_0 to s_k labeled by $w = \mathbf{a}_1 \ldots \mathbf{a}_k$. When $s_0 = s_k$ the path is called a *cycle*. The previous corollary shows that every run $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$ in Ω_γ labeled by a word $w = \mathbf{a}_1 \ldots \mathbf{a}_k$ provides the cycle $t_1 \ldots t_k$ in G_γ on s_γ labeled by w where $t_j = (\pi_\gamma(\mathbf{c}_{j-1}), \mathbf{a}_j, \pi_\gamma(\mathbf{c}_j))$. We deduce that G_γ is strongly connected.

Lemma VIII.6. *For every $\mathbf{q} \leq \mathbf{q}'$ in $\mathbf{Q}_\gamma$ there exists an intraproduction $\mathbf{h} \in \mathbf{H}_\gamma$ such that $\mathbf{q}' \leq \mathbf{q} + \mathbf{h}$.*

Proof: As $\mathbf{q}, \mathbf{q}' \in \mathbf{Q}_\gamma$ there exist $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in P$, and there exist $u, v, u', v' \in \mathbf{A}^*$ such that:

$$\mathbf{c}+\mathbf{x} \xrightarrow{u} \mathbf{q} \xrightarrow{v} \mathbf{c}+\mathbf{y} \qquad \text{and} \qquad \mathbf{c}+\mathbf{x}' \xrightarrow{u'} \mathbf{q}' \xrightarrow{v'} \mathbf{c}+\mathbf{y}'$$

Let us introduce $\mathbf{z} = \mathbf{q}' - \mathbf{q}$. By monotony:

$$\begin{gathered}
\mathbf{c}+\mathbf{x}+\mathbf{x}' \xrightarrow{u'} \mathbf{q}'+\mathbf{x} \\
\mathbf{q}+\mathbf{z}+\mathbf{x} \xrightarrow{v} \mathbf{c}+\mathbf{y}+\mathbf{z}+\mathbf{x} \\
\mathbf{c}+\mathbf{x}+\mathbf{z}+\mathbf{y} \xrightarrow{u} \mathbf{q}+\mathbf{z}+\mathbf{y} \\
\mathbf{q}'+\mathbf{y} \xrightarrow{v'} \mathbf{c}+\mathbf{y}+\mathbf{y}'
\end{gathered}$$

Since $\mathbf{q}'+\mathbf{x} = \mathbf{q}+\mathbf{z}+\mathbf{x}$ and $\mathbf{q}+\mathbf{z}+\mathbf{y} = \mathbf{q}'+\mathbf{y}$, we have proved that $\mathbf{c}+\mathbf{x}+\mathbf{x}' \xrightarrow{u'v} \mathbf{c}+\mathbf{h} \xrightarrow{uv'} \mathbf{c}+\mathbf{y}+\mathbf{y}'$ with $\mathbf{h} = \mathbf{x}+\mathbf{z}+\mathbf{y}$. Thus $\mathbf{h}$ is an intraproduction. Observe that $\mathbf{q}+\mathbf{h} = \mathbf{q}'+\mathbf{x}+\mathbf{y} \geq \mathbf{q}'$. ■

Lemma VIII.7. *There exist intraproductions $\mathbf{h} \in \mathbf{H}_\gamma$ such that $I_\gamma = \{i \mid \mathbf{h}(i) = 0\}$.*

Proof: Let $i \notin I_\gamma$. There exists a sequence $(\mathbf{q}_k)_{k\in\mathbb{N}}$ of configurations $\mathbf{q}_k \in \mathbf{Q}_\gamma$ such that $(\mathbf{q}_k(i))_{k\in\mathbb{N}}$ is strictly increasing. Since $(\mathbb{N}^d, \leq)$ is well-ordered there exists $k < k'$ such that $\mathbf{q}_k \leq \mathbf{q}_{k'}$. Lemma VIII.6 shows that there exists an intraproduction $\mathbf{h}_i$ for γ such that $\mathbf{q}_{k'} \leq \mathbf{q}_k + \mathbf{h}_i$. In particular $\mathbf{h}_i(i) > 0$. As the set of intraproductions $\mathbf{H}_\gamma$ is periodic we deduce that $\mathbf{h} = \sum_{i \notin I} \mathbf{h}_i$ is an intraproduction for γ. By construction we have $\mathbf{h}(i) > 0$ for every $i \notin I_\gamma$. Since $\mathbf{h} \in \mathbf{H}_\gamma$ we deduce that $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$. Therefore $I_\gamma = \{i \mid \mathbf{h}(i) = 0\}$. ■

Given $s \in S_\gamma$ we introduce the relation $R_{\gamma,s}$ of pairs $(\mathbf{e}, \mathbf{f}) \in \mathbb{Q}_{\geq 0}^d \times \mathbb{Q}_{\geq 0}^d$ such that $\mathbf{f} - \mathbf{e} \in \mathbb{Q}_{\geq 0}\Delta(\sigma)$ where σ is the label of a cycle on s in G_γ. Observe that $R_{\gamma,s}$ is a reflexive definable conic relation. From Theorem VII.1 we deduce that the transitive closure $R_\gamma = (\bigcup_{s\in S_\gamma} R_{\gamma,s})^+$ is a reflexive definable conic relation.

Example VIII.8. *Let us come back to Example VIII.2. Cycles in the graph G_γ (depicted in Figure 5) are labeled by words such that the number of occurrences of* $\mathbf{a}$ *is equal to the number of occurrences of* $\mathbf{b}$. *We deduce that $R_{\gamma,s}$ is equal $\{(\mathbf{e}, \mathbf{f}) \in \mathbb{Q}_{\geq 0}^3 \times \mathbb{Q}_{\geq 0}^3 \mid \mathbf{f} - \mathbf{e} \in \mathbb{Q}_{\geq 0}\mathbf{y}\}$ whatever the state $s \in S_\gamma$. We derive that R_γ is also equal to this relation.*

Lemma VIII.9. *For every $s_1, \ldots, s_k \in S_\gamma$ there exists $(\mathbf{x}, \mathbf{y}) \in P$ and $\mathbf{q}_1, \ldots, \mathbf{q}_k \in \mathbf{Q}_\gamma$ such that $s_j = \pi_\gamma(\mathbf{q}_j)$ for every $1 \leq j \leq k$ and such that:*

$$\mathbf{c}+\mathbf{x} \xrightarrow{*} \mathbf{q}_1 \cdots \xrightarrow{*} \mathbf{q}_k \xrightarrow{*} \mathbf{c}+\mathbf{y}$$

Proof: Since $s_j \in S_\gamma$ there exists $\mathbf{p}_j \in \mathbf{Q}_\gamma$ and $(\mathbf{x}_j, \mathbf{y}_j) \in P$ such that $\mathbf{c}+\mathbf{x}_j \xrightarrow{*} \mathbf{p}_j \xrightarrow{*} \mathbf{c}+\mathbf{y}_j$. Let us introduce $(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^k (\mathbf{x}_j, \mathbf{y}_j)$. Since P is periodic, this pair is in P. Let us introduce $\mathbf{h}_j = \mathbf{y}_1 + \cdots + \mathbf{y}_{j-1} + \mathbf{x}_j + \cdots + \mathbf{x}_k$. By monotony, since $\mathbf{c}+\mathbf{x}_j \xrightarrow{*} \mathbf{p}_j \xrightarrow{*} \mathbf{c}+\mathbf{y}_j$, we deduce that $\mathbf{c}+\mathbf{h}_j \xrightarrow{*} \mathbf{q}_j \xrightarrow{*} \mathbf{c}+\mathbf{h}_{j+1}$ where $\mathbf{q}_j = \mathbf{p}_j + (\mathbf{h}_j - \mathbf{x}_j)$. Since $\mathbf{h}_j - \mathbf{x}_j$ is a sum of intraproductions, we deduce that $\mathbf{h}_j - \mathbf{x}_j$ is an intraproduction. In particular $\pi_\gamma(\mathbf{q}_j) = \pi_\gamma(\mathbf{p}_j) = s_j$. We have proved the lemma. ■

Lemma VIII.10. *For every $(\mathbf{e}, \mathbf{f}) \in R_\gamma$ there exists $(\mathbf{x}, \mathbf{y}) \in P$ and $n \in \mathbb{N}_{>0}$ such that:*

$$(\mathbf{c}, \mathbf{c}) + (\mathbf{x}, \mathbf{y}) + n\mathbb{N}(\mathbf{e}, \mathbf{f})$$

is a flatable subreachability relation.

Proof: Let us consider $(\mathbf{e}, \mathbf{f}) \in R_\gamma$. There exists a non-empty sequence $s_1, \ldots, s_k$ of states $s_j \in S_\gamma$ such that $(\mathbf{e}, \mathbf{f}) \in R_{\gamma,s_1} \circ \cdots \circ R_{\gamma,s_k}$. We introduce s_0, s_{k+1} equal to s_γ. Let us consider the sequence $(\mathbf{v}_j)_{0 \leq j \leq k}$ such that $\mathbf{v}_0 = \mathbf{e}$, $\mathbf{v}_k = \mathbf{f}$ and such that $(\mathbf{v}_{j-1}, \mathbf{v}_j) \in R_{\gamma,s_j}$ for every $j \in \{1, \ldots, k\}$. By definition of R_{γ,s_j}, there exists $\lambda_j \in \mathbb{Q}_{\geq 0}$ and a cycle in G_γ on s_j labeled by a word σ_j such that $\mathbf{v}_j - \mathbf{v}_{j-1} = \lambda_j \Delta(\sigma_j)$. By multiplying $(\mathbf{e}, \mathbf{f})$ by a positive natural number, we can assume without loss of generality that $\lambda_j \in \mathbb{N}$ for every $j \in \{1, \ldots, k\}$, and $\mathbf{v}_j \in \mathbb{N}^d$ for every $j \in \{0, \ldots, k\}$. Moreover, by replacing σ_j by $\sigma_j^{\lambda_j}$ we can assume that $\mathbf{v}_j - \mathbf{v}_{j-1} = \Delta(\sigma_j)$.

Lemma VIII.9 shows that there exist $(\mathbf{x}, \mathbf{y}) \in P$, words $w_0, \ldots, w_k \in \mathbf{A}^*$, and configurations $\mathbf{q}_1, \ldots, \mathbf{q}_k \in \mathbf{Q}_\gamma$ such that $s_j = \pi_\gamma(\mathbf{q}_j)$ for every $1 \leq j \leq k$ and such that:

$$\mathbf{c}+\mathbf{x} \xrightarrow{w_0} \mathbf{q}_1 \cdots \xrightarrow{w_{k-1}} \mathbf{q}_k \xrightarrow{w_k} \mathbf{c}+\mathbf{y}$$

Note that $w = w_0\sigma_1 w_1 \ldots \sigma_k w_k$ is the label of a cycle on s_γ. Lemma VIII.7 shows that there exist intraproductions $\mathbf{h} \in \mathbf{H}_\gamma$ such that $I_\gamma = \{i \mid \mathbf{h}(i) = 0\}$. Since the set of intraproductions is periodic, by multiplying $\mathbf{h}$ by a large positive natural number we can assume without loss of generality that there exists a run from $\mathbf{c}+\mathbf{h}$ labeled by w. As $\mathbf{h}$ is an intraproduction there exist $(\mathbf{x}', \mathbf{y}') \in P$ and $u, v \in \mathbf{A}^*$ such that $\mathbf{c}+\mathbf{x}' \xrightarrow{u} \mathbf{c}+\mathbf{h} \xrightarrow{v} \mathbf{c}+\mathbf{y}'$. By monotony, we deduce that for every $r \in \mathbb{N}$ we have:

$$\mathbf{c}+\mathbf{x}+\mathbf{x}'+r\mathbf{e} \xrightarrow{uw_0\sigma_1^r w_1 \ldots \sigma_k^r w_k v} \mathbf{c}+\mathbf{y}+\mathbf{y}'+r\mathbf{f}$$

Since P is periodic we deduce that $(\mathbf{x}+\mathbf{x}', \mathbf{y}+\mathbf{y}') \in P$. We have proved the lemma with the bounded language $W = uw_0\sigma_1^* w_1 \ldots \sigma_k^* w_k v$. ■

Lemma VIII.11. *States in S_γ are incomparable.*

Proof: Let us consider $s \leq s'$ in S_γ. There exists $\mathbf{q}, \mathbf{q}' \in \mathbf{Q}_\gamma$ such that $s = \pi_\gamma(\mathbf{q})$ and $s' = \pi_\gamma(\mathbf{q}')$. Lemma VIII.7 shows that there exists an intraproduction $\mathbf{h}' \in \mathbf{H}_\gamma$ such that $I_\gamma = \{i \mid \mathbf{h}'(i) = 0\}$. By replacing $\mathbf{h}'$ by a vector in $\mathbb{N}_{>0}\mathbf{h}'$ we can assume without loss of generality that $\mathbf{q}(i) \leq \mathbf{q}'(i) + \mathbf{h}'(i)$ for every $i \notin I_\gamma$. As $\mathbf{q}(i) = s(i) \leq s'(i) = \mathbf{q}'(i) = \mathbf{q}'(i) + \mathbf{h}'(i)$ for every $i \in I_\gamma$ we deduce that $\mathbf{q} \leq \mathbf{q}' + \mathbf{h}'$. Lemma VIII.3 shows that $\mathbf{q}'+\mathbf{h}' \in \mathbf{Q}_\gamma$. Lemma VIII.6 shows that there exists an intraproduction $\mathbf{h} \in \mathbf{H}_\gamma$ such that $\mathbf{q}'+\mathbf{h}' \leq \mathbf{q}+\mathbf{h}$. As $\mathbf{h} \in \mathbf{H}_\gamma$ we deduce that $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$. In particular $\mathbf{q}'(i) \leq \mathbf{q}(i)$ for every $i \in I_\gamma$. Hence $s' \leq s$, and we get $s = s'$. ■

Lemma VIII.12. *We have* $\lim(P) \subseteq R_\gamma$.

Proof: Let $(\mathbf{e}, \mathbf{f}) \in \lim(P)$. By multiplying this pair by a positive integer, we can assume that there exists $(\mathbf{x}, \mathbf{y}) \in P$ such that $(\mathbf{x}, \mathbf{y}) + \mathbb{N}(\mathbf{e}, \mathbf{f}) \subseteq \mathbf{P}$. Thus for every $n \in \mathbb{N}$ there exists a run ρ_n labeled by a word in $\mathbf{A}^*$ such that $\mathrm{dir}(\rho_n) =$

$(\mathbf{c},\mathbf{c})+(\mathbf{x},\mathbf{y})+n(\mathbf{e},\mathbf{f})$. Lemma VI.2 shows that there exists $n<m$ such that $\rho_n \trianglelefteq \rho_m$. Assume that ρ_n is the run $\mathbf{c}_0 \dots \mathbf{c}_k$ where $\mathbf{c}_j \in \mathbb{N}^d$. There exists a sequence $\mathbf{v}_0,\dots,\mathbf{v}_{k+1} \in \mathbb{N}^d$ such that $\rho_m = \rho_0' \dots \rho_k'$ where ρ_j' is a run from $\mathbf{c}_j+\mathbf{v}_j$ to $\mathbf{c}_j+\mathbf{v}_{j+1}$ labeled by a word σ_j. Observe that $s_j = \pi_\gamma(\mathbf{c}_j)$ is in S_γ. Since $s_j \leq \pi_\gamma(\mathbf{c}_j+\mathbf{v}_j)$, Lemma VIII.11 shows that $s_j = \pi_\gamma(\mathbf{c}_j+\mathbf{v}_j)$. Since $s_j \leq \pi_\gamma(\mathbf{c}_j+\mathbf{v}_{j+1})$, we also deduce that $s_j = \pi_\gamma(\mathbf{c}_j+\mathbf{v}_{j+1})$. Thus σ_j is the label of a cycle on s_j in G_γ. We deduce that $(\mathbf{v}_j,\mathbf{v}_{j+1}) \in R_{\gamma,s_j}$. Thus $(\mathbf{v}_0,\mathbf{v}_{k+1}) \in R_\gamma$. Since this pair is equal to $(\mathbf{e},\mathbf{f})$, we are done. ■

We have proved Theorem VIII.1 thanks to the relation R_γ, denoted as R in that theorem.

IX. Reachability Decomposition

In this section, we prove the following theorem. All other results are not used in the sequel.

Theorem IX.1. *For every Presburger set $\mathbf{X} \subseteq \mathbb{N}^d$, the set $\mathrm{post}(\mathbf{c}_{init},\mathbf{A}^*) \cap \mathbf{X}$ is a finite union of sets $\mathbf{b}+\mathbf{P}$ where $\mathbf{b} \in \mathbb{N}^d$ and $\mathbf{P} \subseteq \mathbb{N}^d$ is a smooth periodic set such that for every linear set $\mathbf{Y} \subseteq \mathbf{b}+\mathbf{P}$ there exists $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{p}+\mathbf{Y}$ is flatable.*

The proof of the previous theorem is based on the following simple lemma.

Lemma IX.2. *For every relations $R_1, R_2 \subseteq \mathbb{N}^d \times \mathbb{N}^d$ and for every capacity $\mathbf{c} \in \mathbb{N}^d$ such that $(\mathbf{c},\mathbf{c})+R_1$ and $(\mathbf{c},\mathbf{c})+R_2$ are flatable subreachability relations, then $(\mathbf{c},\mathbf{c})+R_1+R_2$ is a flatable subreachability relation.*

Proof: There exist bounded languages $W_1, W_2 \subseteq \mathbf{A}^*$ such that $(\mathbf{c},\mathbf{c})+R_1$ and $(\mathbf{c},\mathbf{c})+R_2$ are included respectively in $\xrightarrow{W_1}$ and $\xrightarrow{W_2}$. By monotony, we deduce that $(\mathbf{c},\mathbf{c})+R_1+R_2$ is included in $\xrightarrow{W_1 W_2}$. ■

Since Presburger sets are finite unions of linear sets, we can assume that $\mathbf{X}$ is a linear set in the previous Theorem IX.1. Hence, we can assume that there exists a configuration $\mathbf{x} \in \mathbb{N}^d$ and a finitely generated periodic set $\mathbf{M} \subseteq \mathbb{N}^d$ such that $\mathbf{X} = \mathbf{x}+\mathbf{M}$. We introduce the set Ω of runs ρ from the initial configuration $\mathbf{c}_{\text{init}}$ to a configuration in $\mathbf{X}$. Lemma VI.2 shows that $\trianglelefteq$ is a well-order over Ω and Lemma V.5 shows that $\leq_{\mathbf{M}}$ is a well-order over $\mathbf{M}$. We deduce that Ω is well-ordered by the relation $\sqsubseteq$ defined by $\rho \sqsubseteq \rho'$ if $\rho \trianglelefteq \rho'$ and $\mathrm{tgt}(\rho)-\mathbf{x} \leq_{\mathbf{M}} \mathrm{tgt}(\rho')-\mathbf{x}$. In particular $\Omega_0 = \min_{\sqsubseteq}(\Omega)$ is a finite set. Let us observe that we have the following equality:

$$\mathbf{X} = \bigcup_{\rho \in \Omega_0} \mathrm{tgt}(\rho)+\mathbf{M}_\rho$$

Where $\mathbf{M}_\rho$ is the following periodic set:

$$\mathbf{M}_\rho = \{\mathbf{m} \in \mathbf{M} \mid \mathbf{0} \overset{\mathbf{c}_0}{\curvearrowright} \circ \dots \circ \overset{\mathbf{c}_k}{\curvearrowright} \mathbf{m}\}$$

So, the proof of Theorem IX.1 reduces to show that $\mathbf{M}_\rho$ is a smooth periodic set such that for every $\mathbf{y} \in \mathbb{N}^d$ and for every finitely generated periodic set $\mathbf{Q} \subseteq \mathbb{N}^d$ such that $\mathbf{y}+\mathbf{Q} \subseteq \mathrm{tgt}(\rho)+\mathbf{M}_\rho$, there exists $\mathbf{m} \in \mathbf{M}_\rho$ such that $\mathbf{y}+\mathbf{m}+\mathbf{Q}$ is flatable.

In the sequel ρ is a run in Ω of the form $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$. We introduce the periodic set P of tuples $(\mathbf{x}_0,\dots,\mathbf{x}_{k+1}) \in (\mathbb{N}^d)^{k+2}$ such that $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{x}_{k+1} \in \mathbf{M}$ and $\mathbf{x}_j \overset{\mathbf{c}_j}{\curvearrowright} \mathbf{x}_{j+1}$ for every j. We consider the projection function $\pi_j : (\mathbb{N}^d)^{k+2} \rightarrow \mathbb{N}^d \times \mathbb{N}^d$ defined by $\pi_j(\mathbf{x}_0,\dots,\mathbf{x}_{k+1}) = (\mathbf{x}_j,\mathbf{x}_{j+1})$. We also introduce the periodic set $P_j = \pi_j(P)$. Theorem VIII.1 shows that there exists a definable conic relation $R_j \subseteq \mathbb{Q}_{\geq 0}^d \times \mathbb{Q}_{\geq 0}^d$ such that $\lim(P_j) \subseteq R_j$ and such that for every $r_j \in R_j$, there exists $p_j \in P$ and $n_j \in \mathbb{N}_{>0}$ such that $(\mathbf{c}_j,\mathbf{c}_j)+\pi_j(p_j)+n_j\mathbb{N}r_j$ is a flatable subreachability relation.

We introduce the following definable conic set:

$$\mathbf{C} = \{\mathbf{c} \in \mathbb{Q}_{\geq 0}^d \mid \mathbf{0} \; R_0 \circ \dots \circ R_k \; \mathbf{c}\}$$

Lemma IX.3. *The periodic set $\mathbf{M}_\rho$ is well-limit and its limit is included in $\mathbf{C} \cap \mathbb{Q}_{\geq 0}M$.*

Proof: Let us consider a sequence $(\mathbf{m}_n)_{n \in \mathbb{N}}$ of vectors $\mathbf{m}_n \in \mathbf{M}_\rho$. For every n, there exists a sequence $(\mathbf{x}_{0,n},\dots,\mathbf{x}_{k+1,n})$ in P such that $\mathbf{x}_{k+1,n} = \mathbf{m}_n$. So, there exists a run $\rho_{j,n}$ from $\mathbf{c}_j+\mathbf{x}_{j,n}$ to $\mathbf{c}_j+\mathbf{x}_{j+1,n}$ labeled by a word in $\mathbf{A}^*$. Lemma VI.2 shows that $\trianglelefteq$ is a well-order over the runs and Lemma V.5 shows that $\leq_{\mathbf{M}}$ is a well-order over $\mathbf{M}$. We deduce that there exists an infinite set $N \subseteq \mathbb{N}$ such that $\rho_{j,n} \trianglelefteq \rho_{j,m}$ and $\mathbf{m}_n \leq_{\mathbf{M}} \mathbf{m}_m$ for every $n \leq m$ in N and for every $0 \leq j \leq k$. Lemma VI.3 shows that for every $r \in \mathbb{N}$ there exists a run labeled by a word in $\mathbf{A}^*$ with a direction equals to $\mathrm{dir}(\rho_{j,n})+r(\mathrm{dir}(\rho_{j,m})-\mathrm{dir}(\rho_{j,n}))$. Let us introduce $\mathbf{z}_{j,r} = \mathbf{x}_{j,n}+r(\mathbf{x}_{j,m}-\mathbf{x}_{j,n})$ and observe that the previous direction is equal to $(\mathbf{c}_j,\mathbf{c}_j)+(\mathbf{z}_{j,r},\mathbf{z}_{j+1,r})$. Thus $\mathbf{z}_{j,r} \overset{\mathbf{c}_j}{\curvearrowright} \mathbf{z}_{j+1,r}$. Since $\mathbf{z}_{0,r} = \mathbf{0}$ and $\mathbf{z}_{k+1,r} = \mathbf{m}_n + r(\mathbf{m}_m-\mathbf{m}_n) \in \mathbf{M}$ from $\mathbf{m}_n \leq_{\mathbf{M}} \mathbf{m}_m$, we deduce that $(\mathbf{z}_{0,r},\dots,\mathbf{z}_{k+1,r}) \in P$. Thus $\mathbf{m}_n+r(\mathbf{m}_m-\mathbf{m}_n) \in \mathbf{M}_\rho$. We deduce that $\mathbf{m}_m-\mathbf{m}_n \in \lim(\mathbf{M}_\rho)$. Therefore $\mathbf{M}_\rho$ is well-limit periodic.

Now, let us consider $\mathbf{v} \in \lim(\mathbf{M}_\rho)$. By multiplying this vector by a positive integer, we can assume that there exists $\mathbf{m} \in \mathbf{M}$ such that $\mathbf{m}_n = \mathbf{m}+n\mathbf{v}$ is in $\mathbf{M}_\rho$ for every $n \in \mathbb{N}$. We can then apply the previous paragraph on this sequence. Let $n<m$ in N. Since $(\mathbf{z}_{0,r},\dots,\mathbf{z}_{k+1,r}) \in P$ we deduce that $(\mathbf{z}_{j,r},\mathbf{z}_{j,r+1}) \in P_j$. Thus $(\mathbf{x}_{j,n},\mathbf{x}_{j+1,n})+\mathbb{N}((\mathbf{x}_{j,m},\mathbf{x}_{j+1,m})-(\mathbf{x}_{j,n},\mathbf{x}_{j+1,n}))$ is included in $\mathbf{P}_j$ and we deduce that $(\mathbf{x}_{j,m},\mathbf{x}_{j+1,m})-(\mathbf{x}_{j,n},\mathbf{x}_{j+1,n}) \in \lim(P_j)$. Hence $(\mathbf{x}_{j,m},\mathbf{x}_{j+1,m})-(\mathbf{x}_{j,n},\mathbf{x}_{j+1,n}) \in R_j$. We deduce that $(\mathbf{x}_{0,m}-\mathbf{x}_{0,n},\mathbf{x}_{k+1,m}-\mathbf{x}_{k+1,n}) \in R_0 \circ \dots \circ R_k$. From $\mathbf{x}_{0,m}-\mathbf{x}_{0,n} = \mathbf{0}$ and $\mathbf{x}_{k+1,m}-\mathbf{x}_{k+1,n} = \mathbf{m}_m-\mathbf{m}_n = (m-n)\mathbf{v}$, we deduce that $\mathbf{v} \in \mathbf{C}$. Moreover, from $\mathbf{m}_n \leq_{\mathbf{M}} \mathbf{m}_n$ we get $(m-n)\mathbf{v} \in \mathbf{M}$. We have proved that $\mathbf{v} \in \mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{M}$. ■

Lemma IX.4. *For every $\mathbf{v} \in \mathbf{C}$, there exist relations $\tilde{R}_0,\dots,\tilde{R}_k \subseteq \mathbb{N}^d \times \mathbb{N}^d$ such that $(\mathbf{c}_j,\mathbf{c}_j)+\tilde{R}_j$ is a flatable subreachability relation, $\mathbf{m} \in \mathbf{M}$, and $n \in \mathbb{N}_{>0}$ such that for every $r \in \mathbb{N}$:*

$$\mathbf{0} \; \tilde{R}_0 \circ \dots \circ \tilde{R}_k \; \mathbf{m}+rn\mathbf{v}$$

Proof: Let us consider $\mathbf{v} \in \mathbf{C}$. There exists a sequence $(\mathbf{v}_0, \ldots, \mathbf{v}_{k+1}) \in (\mathbb{Q}^d_{\geq 0})^{k+1}$ such that $\mathbf{v}_0 = \mathbf{0}$, $\mathbf{v}_{k+1} = \mathbf{v}$ and $(\mathbf{v}_j, \mathbf{v}_{j+1}) \in R_j$ for every j. There exist $n_j \in \mathbb{N}_{>0}$, $p_j \in P$, such that $(\mathbf{c}_j, \mathbf{c}_j) + \pi_j(p_j) + n_j\mathbb{N}(\mathbf{v}_j, \mathbf{v}_{j+1})$ is a flatable subreachability relation. Let $n = \prod_{j=0}^k n_j$. Since $n\mathbb{N} \subseteq n_j\mathbb{N}$ we deduce that $(\mathbf{c}_j, \mathbf{c}_j) + \pi_j(p_j) + n\mathbb{N}(\mathbf{v}_j, \mathbf{v}_{j+1})$ is a flatable subreachability relation. Let us consider $p = \sum_{j=1}^k p_j$. Note that $p - p_j \in P$ and in particular $(c_j, c_j) + \pi_j(p - p_j)$ is in the reachability relation. Lemma IX.2 shows that $(c_j, c_j) + \tilde{R}_j$ is a flatable subreachability relation where $\tilde{R}_j = \pi_j(p) + n\mathbb{N}(\mathbf{v}_j, \mathbf{v}_{j+1})$. Assume that $p = (\mathbf{x}_0, \ldots, \mathbf{x}_{k+1})$. We have proved that for every $r \in \mathbb{N}$ we have $\mathbf{x}_j + nr\mathbf{v}_j \ \tilde{R}_j \ \mathbf{x}_{j+1} + nr\mathbf{v}_{j+1}$. Since $p \in P$ we deduce that $\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{m} = \mathbf{x}_{k+1}$ is a vector in $\mathbf{M}$. Since $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{v}_{k+1} = \mathbf{v}$, we have proved the lemma. ■

The previous Lemma IX.4 shows that $\mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{M}$ is included in $\lim(\mathbf{M}_\rho)$. Hence, with Lemma IX.3 we deduce that $\lim(\mathbf{M}_\rho)$ is equal to the definable conic set $\mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{M}$.

Lemma IX.5. *For every $\mathbf{y} \in \mathbb{N}^d$ and for every finitely generated periodic set $\mathbf{Q} \subseteq \mathbb{N}^d$ such that $\mathbf{y} + \mathbf{Q} \subseteq \mathrm{tgt}(\rho) + \mathbf{M}_\rho$, there exists $\mathbf{m} \in \mathbf{M}_\rho$ such that $\mathbf{y} + \mathbf{m} + \mathbf{Q}$ is flatable.*

Proof: Since $\mathbf{Q}$ is finitely generated, there exists a finite set $\mathbf{V} \subseteq \mathbf{Q}$ that generates $\mathbf{Q}$. Observe that $\mathbf{x} - \mathrm{tgt}(\rho) + \mathbb{N}\mathbf{v} \subseteq \mathbf{M}_\rho$ for every $\mathbf{v} \in \mathbf{V}$. Thus $\mathbf{v} \in \lim(\mathbf{M}_\rho)$. As $\lim(\mathbf{M}_\rho) \subseteq \mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{M}$, we deduce that there exist relations $\tilde{R}_{0,\mathbf{v}}, \ldots, \tilde{R}_{k,\mathbf{v}} \subseteq \mathbb{N}^d \times \mathbb{N}^d$ such that $(\mathbf{c}_j, \mathbf{c}_j) + \tilde{R}_{j,\mathbf{v}}$ is a flatable subreachability relation, $\mathbf{m}_\mathbf{v} \in \mathbf{M}$, and $n_\mathbf{v} \in \mathbb{N}_{>0}$ such that for every $r \in \mathbb{N}$:

$$\mathbf{0} \ \tilde{R}_{0,\mathbf{v}} \circ \cdots \circ \tilde{R}_{k,\mathbf{v}} \ \mathbf{m}_\mathbf{v} + rn_\mathbf{v}\mathbf{v}$$

Let us consider $n = \prod_{\mathbf{v} \in \mathbf{V}} n_\mathbf{v}$, $\mathbf{m} = \sum_{\mathbf{v} \in \mathbf{V}} \mathbf{m}_\mathbf{v}$ and $\tilde{R}_j = \sum_{\mathbf{v} \in \mathbf{V}} \tilde{R}_{j,\mathbf{v}}$. Lemma IX.2 shows $(\mathbf{c}_j, \mathbf{c}_j) + \tilde{R}_j$ is a flatable subreachability relation. Moreover, since $\mathbf{Q}$ is generated by $\mathbf{V}$ we deduce that for every $\mathbf{q} \in \mathbf{Q}$ we have:

$$\mathbf{0} \ \tilde{R}_0 \circ \cdots \circ \tilde{R}_k \ \mathbf{m} + n\mathbf{q}$$

Now, let us consider the set $\mathbf{Z} = \sum_{\mathbf{v} \in \mathbf{V}} \{0, \ldots, n-1\}\mathbf{v}$. Observe that $\mathbf{Z}$ is finite and since $\mathbf{Z} \subseteq \mathbf{M}_\rho$, we deduce that for every $\mathbf{z} \in \mathbf{M}_\rho$, there exists $p_\mathbf{z} = (\mathbf{x}_{0,\mathbf{z}}, \ldots, \mathbf{x}_{k+1,\mathbf{z}}) \in P$ such that $\mathbf{x}_{k+1,\mathbf{z}} = \mathbf{z}$. Let us consider the relation $\tilde{R}'_j = \bigcup_{\mathbf{z} \in \mathbf{Z}} (\tilde{R}_j + \pi_j(p_\mathbf{z}))$. Lemma IX.2 shows that $(\mathbf{c}_j, \mathbf{c}_j) + \tilde{R}'_j$ is flatable. Since $\mathbf{Q} = \mathbf{Z} + n\mathbf{Q}$ we deduce that for every $\mathbf{q} \in \mathbf{Q}$ we have:

$$\mathbf{0} \ \tilde{R}'_0 \circ \cdots \circ \tilde{R}'_k \ \mathbf{m} + \mathbf{q}$$

Finally, since $\mathbf{y} - \mathrm{tgt}(\rho) \in \mathbf{M}_\rho$ we deduce that there exists $p = (\mathbf{x}_0, \ldots, \mathbf{x}_{k+1})$ in P such that $\mathbf{x}_{k+1} = \mathbf{y} - \mathrm{tgt}(\rho)$. Lemma IX.2 shows that $\tilde{R}''_j = \tilde{R}'_j + \pi_j(p)$ is such that $(\mathbf{c}_j, \mathbf{c}_j) + \tilde{R}''_j$ is flatable. Hence, this relation is included in $\xrightarrow{W_j}$ where $W_j \subseteq \mathbf{A}^*$ is a bounded language.

Let us introduce the actions $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$ and the bounded language $W = W_0\mathbf{a}_1W_1 \ldots \mathbf{a}_kW_k$. We have proved that $\mathrm{post}(\mathbf{c}_{\text{init}}, W)$ contains $\mathbf{y} + \mathbf{m} + \mathbf{Q}$. Thus this set is flatable. ■

We have proved Theorem IX.1.

X. Equivalent Presburger Sets

In this section, we first extend the notion of dimension introduced in [18] for sets included in $\mathbb{Z}^d$ to sets included in $\mathbb{Q}^d$. This definition provides a simple way for defining an equivalence relation over the subsets of $\mathbb{Q}^d$. Finally, we provide a characterization of sets equivalent to Presburger sets and that can be decomposed as finite unions of sets of the form $\mathbf{b} + \mathbf{P}$ where $\mathbf{b} \in \mathbb{Z}^d$ and $\mathbf{P} \subseteq \mathbb{Z}^d$ is a smooth periodic set.

The *dimension* of a set $\mathbf{X} \subseteq \mathbb{Q}^d$ is the minimal integer $r \in \{-1, \ldots, d\}$ such that $\mathbf{X} \subseteq \bigcup_{j=1}^k (\mathbf{B}_j + \mathbf{V}_j)$ where $\mathbf{B}_j$ is a bounded subset of $\mathbb{Q}^d$ and $\mathbf{V}_j \subseteq \mathbb{Q}^d$ is a vector space satisfying $\mathrm{rank}(\mathbf{V}_j) \leq r$ for every j. We denote by $\dim(\mathbf{X})$ the dimension of $\mathbf{X}$. Observe that $\dim(\mathbf{v} + \mathbf{X}) = \dim(\mathbf{X})$ for every $\mathbf{X} \subseteq \mathbb{Q}^d$ and for every $\mathbf{v} \in \mathbb{Q}^d$. Observe that $\dim(\mathbf{X}) = -1$ if and only if $\mathbf{X}$ is empty. Note that $\dim(\mathbf{X} \cup \mathbf{Y}) = \max\{\dim(\mathbf{X}), \dim(\mathbf{Y})\}$ for every subsets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{Q}^d$.

Example X.1. $\dim(\mathbb{N}) = 1$, $\dim(\mathbb{Q}) = 1$, $\dim(\mathbb{N}(1,0) + \mathbb{N}(1,1)) = 2$, $\dim(\mathbb{N}(1,0) \cup \mathbb{N}(1,1)) = 1$.

The dimension of a periodic set is obtained as follows.

Lemma X.2. *We have $\dim(\mathbf{P}) = \mathrm{rank}(\mathbf{V})$ for every periodic set $\mathbf{P}$ where $\mathbf{V}$ is the vector space generated by $\mathbf{P}$.*

Given a natural number $r \in \{0, \ldots, d\}$, we introduce the equivalence relation $\equiv_r$ over the subsets of $\mathbb{Q}^d$ by $\mathbf{X} \equiv_r \mathbf{Y}$ if $\dim(\mathbf{X}\Delta\mathbf{Y}) < r$. Note that $\equiv_r$ is distributive over $\cup$ and $\cap$. In the technical report [15], the following Theorem X.3 is proved.

Theorem X.3. *Let $\mathbf{X} = \bigcup_{j=1}^k (\mathbf{b}_j + \mathbf{P}_j)$ where $\mathbf{b}_j \in \mathbb{Z}^d$ and $\mathbf{P}_j \subseteq \mathbb{Z}^d$ is a smooth periodic set. We assume that $\mathbf{X}$ is non empty and we introduce $r = \dim(\mathbf{X})$. If $\mathbf{X}$ is equivalent for $\equiv_r$ to a Presburger set then there exists a sequence $(\mathbf{Y}_j)_{1 \leq j \leq k}$ of linear sets $\mathbf{Y}_j \subseteq \mathbf{b}_j + \mathbf{P}_j$ such that $\mathbf{X} \equiv_r \bigcup_{j=1}^k \mathbf{p}_j + \mathbf{Y}_j$ for every sequence $(\mathbf{p}_j)_{1 \leq j \leq k}$ of vectors $\mathbf{p}_j \in \mathbf{P}_j$.*

XI. Presburger Reachability Sets

In this section we prove that Presburger subreachability sets are flatable. As a direct consequence, we deduce that Presburger VAS are flatable.

Lemma XI.1. *Presburger subreachability sets are flatable.*

Proof: We prove by induction over $r \in \{-1, \ldots, d\}$ that Presburger subreachability sets $\mathbf{X}$ with $\dim(\mathbf{X}) \leq r$ are flatable. Note that if $\dim(\mathbf{X}) = -1$ then $\mathbf{X}$ is empty and the proof is immediate. Let us assume that the lemma is proved in dimension $r \in \{-1, \ldots, d\}$ and let us consider a Presburger subreachability set $\mathbf{X} \subseteq \mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*)$ such that $\dim(\mathbf{X}) = r + 1$. In particular $\mathbf{X}$ is non empty. Theorem IX.1 shows $\mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*) \cap \mathbf{X}$ is a finite union of sets $\bigcup_{j=1}^k (\mathbf{b}_j + \mathbf{P}_j)$ where $\mathbf{b}_j \in \mathbb{N}^d$ and $\mathbf{P}_j \subseteq \mathbb{N}^d$ is a smooth periodic set such that for every linear set $\mathbf{Y}_j \subseteq \mathbf{b}_j + \mathbf{P}_j$ there exists $\mathbf{p}_j \in \mathbf{P}_j$ such that $\mathbf{p}_j + \mathbf{Y}_j$ is flatable.

Since $\mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*) \cap \mathbf{X}$ is equal to $\mathbf{X}$ which is a Presburger set, Theorem X.3 shows that there exists a sequence $(\mathbf{Y}_j)_{1 \leq j \leq k}$ of linear sets $\mathbf{Y}_j \subseteq \mathbf{b}_j + \mathbf{P}_j$ such that $\mathbf{X} \equiv_r \bigcup_{j=1}^{k} \mathbf{p}_j + \mathbf{Y}_j$ for every sequence $(\mathbf{p}_j)_{1 \leq j \leq k}$ of vectors $\mathbf{p}_j \in \mathbf{P}_j$.

Let us consider a sequence $(\mathbf{p}_j)_{1 \leq j \leq k}$ of vectors $\mathbf{p}_j \in \mathbf{P}_j$ such that $\mathbf{p}_j + \mathbf{Y}_j$ is flatable. We deduce that $\mathbf{Y} = \bigcup_{j=1}^{k} \mathbf{p}_j + \mathbf{Y}_j$ is flatable. Since $\mathbf{X} \equiv_r \mathbf{Y}$ we deduce that $\dim(\mathbf{X} \backslash \mathbf{Y}) < r$. Since $\mathbf{X} \backslash \mathbf{Y}$ is a Presburger subreachability set, by induction, this set is flatable. From $\mathbf{X} \subseteq (\mathbf{X} \backslash \mathbf{Y}) \cup \mathbf{Y}$, we deduce that $\mathbf{X}$ is flatable. We have proved the rank $r + 1$. ■

Theorem XI.2. *The class of flatable VAS coincides with the class of Presburger VAS.*

Proof: Assume first that the VAS is Presburger. Then $\mathbf{X} = \mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*)$ is a Presburger set. The previous lemma shows that $\mathbf{X}$ is flatable. Hence the VAS is flatable. Conversely, if the VAS is flatable, Theorem III.4 shows that the VAS is Presburger. ■

Corollary XI.3. *Presburger subreachability relations are flatable.*

Proof: Let $\mathbf{A} \subseteq \mathbb{Z}^d$ be a finite set of actions. We consider the VAS $((\mathbf{0}, \mathbf{0}), A')$ in dimension $2d$ where A' is the set $\{\mathbf{0}\} \times \mathbf{A}$ and the vectors $(\mathbf{u}_i, \mathbf{u}_i)$ where $\mathbf{u}_i \in \mathbb{Z}^d$ satisfies $\mathbf{u}_i(j) = 0$ if $j \neq i$ and $\mathbf{u}_i(i) = 1$. Observe that the reachability set of this VAS is $\xrightarrow{\mathbf{A}^*}$. Hence, if a subreachability relation R of $\xrightarrow{\mathbf{A}^*}$ is Presburger, we deduce that there exists a bounded language $W' \subseteq (A')^*$ such that $R \subseteq \mathrm{post}((\mathbf{0}, \mathbf{0}), W')$. Let us consider the word morphism $\phi : (A')^* \to A^*$ defined by $\phi(\mathbf{0}, \mathbf{a}) = \mathbf{a}$ and $\phi(\mathbf{u}_i, \mathbf{u}_i) = \varepsilon$. Observe that $W = \phi(W')$ is a bounded language and $\mathrm{post}((\mathbf{0}, \mathbf{0}), W')$ is included in $\xrightarrow{W}$. We deduce that R is flatable. ■

Example XI.4. *Let* $\mathbf{A} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *with* $\mathbf{a} = (1, 1, 1)$, $\mathbf{b} = (-1, 2, 1)$ *and* $\mathbf{c} = (2, -1, 1)$. *The reachability relation* $\xrightarrow{*}$ *is equal to* $\xrightarrow{W}$ *with* $W = \mathbf{a}^*\mathbf{b}^*\mathbf{c}^*\mathbf{b}^*(\mathbf{bc})^*\mathbf{b}^*\mathbf{c}^*$.

XII. Conclusion

This paper proved that acceleration techniques are complete for computing Presburger formulas denoting reachability sets of Presburger VAS. Since there exist VAS with finite reachability sets of Ackermann cardinals [20], acceleration-based algorithms have an Ackermann lower bound of complexity. Note that deciding reachability problems for Presburger VAS in Ackermannian complexity is an open problem. Moreover, an Ackermannian worst case complexity does not prevent algorithms like the Karp and Miller one [21] to decide some reachability problems (so-called coverability problems) efficiently in practice.

In the future, we are interested in improving acceleration techniques to avoid the Presburger definability condition of the reachability sets. As a first step, we are interested in characterizing vector addition systems with reachability sets not definable in the Presburger arithmetic. These vector addition systems are interesting since we know that there exist inductive invariants definable in the Presburger arithmetic obtained by over-approximating reachability sets. The main objective is an algorithm for deciding the general reachability problem for vector addition systems based on accelerations and on-demand over-approximations that works well in practice.

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