ON THE VALUE GROUP OF A DIFFERENTIAL VALUATION II

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In much of classical asymptotic theory one works with a field of functions of a real or complex variable that is closed under differentiation and in which each function approaches a limit, possibly ∞, as the variable approaches some ideal value, all in such a way that some version of L'Hospital's rule holds. This gives rise to the notion of a differential valuation of a differential field of characteristic zero, which has been discussed by the author in [3] and [4], and which leads to a certain amount of algebraic gadgetry that in effect generalizes important parts of the classical theory. The algebraic aspects of this theory are further developed in the present paper, two main results of which (Theorems 2 and 3) say roughly that solutions of algebraic differential equations with elementary coefficients exhibit no worse singularities than do elementary functions.

1. Generalities on asymptotic couples. By an *asymptotic couple* we mean a pair (Γ, ψ), where Γ is an ordered abelian group and ψ is a map from $\Gamma^* = \Gamma - \{0\}$ into Γ such that

(a) if $\alpha \in \Gamma^*$ and $n \in \mathbf{Z}$, $n \neq 0$, then $\psi(n\alpha) = \psi(\alpha)$
(b) if $\alpha, \beta \in \Gamma^*$, $\beta \neq -\alpha$, then $\psi(\alpha + \beta) \geq \min\{\psi(\alpha), \psi(\beta)\}$
(c) for any $\alpha, \beta \in \Gamma^*$, $\psi(\beta) < \psi(\alpha) + |\alpha|$.

It is known [3, Theorem 4] that if k is a differential field and ν a differential valuation of k with value group Γ, then there is a map $\psi: \Gamma^* \to \Gamma$ such that for all $a \in k^*$ with $\nu(a) \neq 0$ we have $\psi(\nu(a)) = \nu(a'/a)$ and (Γ, ψ) is an asymptotic couple. Conversely [4, Theorem 1] any asymptotic couple (Γ, ψ) arises in this manner from a differential valuation of a differential field, at least if the ordered subset $\psi(\Gamma^*)$ of Γ, in the opposite ordering, is well ordered (in particular if $\psi(\Gamma^*)$ is a finite set).

An immediate consequence of (a) and (b) above is that if $\alpha, \beta \in \Gamma^*$ and $\psi(\alpha) \neq \psi(\beta)$, then $\psi(\alpha + \beta) = \min\{\psi(\alpha), \psi(\beta)\}$.

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Our first two results are in effect generalizations of the two corollaries of [3, Theorem 1].

PROPOSITION 1. *Let* (Γ, ψ) *be an asymptotic couple. Then if* $\alpha, \beta \in \Gamma^*$ *and* $\alpha > \beta$ *we have*

$$\psi(\alpha) + \alpha > \psi(\beta) + \beta.$$

This is clear if $\psi(\alpha) \geq \psi(\beta)$, so suppose that $\psi(\alpha) < \psi(\beta)$. Then by property (b) above we have $\psi(\alpha - \beta) = \psi(\alpha)$. By property (c), $\psi(\beta) < \psi(\alpha - \beta) + |\alpha - \beta| = \psi(\alpha) + \alpha - \beta$.

PROPOSITION 2. *Let* (Γ, ψ) *be an asymptotic couple, with* $\Gamma \neq \{0\}$. *Then for any* $\alpha \in \Gamma$ *there are* $\xi, \eta \in \Gamma^*$ *such that*

$$\psi(\xi) + \xi > \alpha > \psi(\eta) + \eta.$$

Fix some $\beta \in \Gamma^*$ and take $\eta \in \Gamma^*$, $\eta \leq \alpha - \psi(\beta) - |\beta|$. Then $\psi(\eta) < \psi(\beta) + |\beta|$, by property (c), so that $\psi(\eta) + \eta < \psi(\beta) + |\beta| + \eta \leq \alpha$. Next, choose $\gamma \in \Gamma^*$ such that $\gamma > \max\{\beta, \alpha - \psi(\beta) - \beta\}$, and set $\xi = 2\gamma$. Then $\psi(\xi) + \xi = \psi(\gamma) + 2\gamma > \psi(\beta) + \beta + \gamma > \alpha$.

The next lemma and proposition will not be used in the rest of this paper.

LEMMA. *Let* (Γ, ψ) *be an asymptotic couple, let* $\gamma_0 \in \Gamma$, *and let* $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma^*$ *be such that* $\psi(\gamma_1) < \psi(\gamma_2) < \cdots < \psi(\gamma_n)$. *Then*

(1) (Γ, ψ_1) *is an asymptotic couple, if* $\psi_1(\gamma) = \gamma_0$ *for each* $\gamma \in \Gamma^*$
(2) (Γ, ψ_2) *is an asymptotic couple, if* $\psi_2(\gamma) = \psi(\gamma) - \gamma_0$ *for each* $\gamma \in \Gamma^*$
(3) (Γ, ψ_3) *is an asymptotic couple, if* $\psi_3(\gamma) = \max\{\psi(\gamma), \gamma_0\}$ *for each* $\gamma \in \Gamma^*$
(4) (Γ, ψ_4) *is an asymptotic couple if, for each* $\gamma \in \Gamma^*$,

$$\psi_4(\gamma) = \psi(\gamma_1) \text{ if } \psi(\gamma) \leq \psi(\gamma_1)$$

$$\psi_4(\gamma) = \psi(\gamma_i) \text{ if } i = 2, 3, \ldots, n \text{ and } \psi(\gamma_{i-1}) < \psi(\gamma) \leq \psi(\gamma_i)$$

$$\psi_4(\gamma) = \psi(\gamma_n) \text{ if } \psi(\gamma) > \psi(\gamma_n)$$

Furthermore, if (Γ, ψ_5) *and* (Γ, ψ_6) *are asymptotic couples, then* (Γ, ψ_7) *is an asymptotic couple, where* $\psi_7 : \Gamma^* \to \Gamma$ *is given by* $\psi_7(\gamma) = \min\{\psi_5(\gamma), \psi_6(\gamma)\}$ *for each* $\gamma \in \Gamma^*$.

The various verifications are tedious, but completely straightforward. Note that the general case of (4) follows from repeated application of the special case $n = 2$ and the final contention.

PROPOSITION 3. *Let* (Γ, ψ) *be an asymptotic couple, and* $\alpha, \beta \in \Gamma^*$. *Then if* $\psi(\alpha) \neq \psi(\beta)$, *we have*

$$\psi(\psi(\alpha) - \psi(\beta)) > \min\{\psi(\alpha), \psi(\beta)\}.$$

Suppose that, on the contrary, we have $\alpha, \beta \in \Gamma^*$, with $\psi(\alpha) < \psi(\beta)$ and $\psi(\psi(\alpha) - \psi(\beta)) \leq \psi(\alpha)$. Replacing ψ by ψ_2 of part (2) of the lemma, with $\gamma_0 = \psi(\alpha)$, we get the same situation for ψ, except that now $\psi(\alpha) = 0$. Hence we have $\psi(\alpha) = 0$, $\psi(\beta) > 0$, $\psi(\psi(\beta)) \leq 0$. Now use part (4) of the lemma, with the present ψ and with $\gamma_1 = \alpha$ and $\gamma_2 = \beta$ to obtain the appropriate ψ_4. We have $\psi_4(\alpha) = \psi(\alpha) = 0$, $\psi_4(\beta) = \psi(\beta) > 0$, $\psi_4(\psi(\beta)) = 0$. Since (Γ, ψ_4) is an asymptotic couple, we have $\psi_4(\beta) < \psi_4(\psi(\beta)) + |\psi(\beta)|$, contradicting $\psi_4(\psi(\beta)) = 0$ and $\psi(\beta) = \psi_4(\beta) > 0$. This proves the proposition.

PROPOSITION 4. *Let* (Γ, ψ) *be an asymptotic couple. If* Δ *is a minimal nonzero convex subgroup of* Γ, *then* ψ *is constant on* Δ^* *and at most this constant on the rest of* Γ^*. *If* $\psi(\Gamma^*)$ *contains a greatest element* γ_0 *and a next greatest element* γ_1, *then for each nonzero element* δ *of the smallest convex subgroup of* Γ *that contains* $\gamma_0 - \gamma_1$ *we have* $\psi(\delta) = \gamma_0$.

An ordered group Δ as given is of rank one, therefore order-isomorphic to a subgroup of $\mathbf{R}$. Suppose there exist $\alpha, \beta \in \Delta^*$ with $\psi(\alpha) < \psi(\beta)$. The group $\mathbf{Z}\alpha + \mathbf{Z}\beta$ cannot be discrete, for then it would be cyclic, contradicting property (a). Therefore $\mathbf{Z}\alpha + \mathbf{Z}\beta$ contains arbitrarily small positive elements of Γ. Thus we can find $\delta \in \Delta^*$ such that $|\delta| < \min\{\psi(\beta) - \psi(\alpha), |\beta|\}$. Since $|\delta| < |\beta|$ we have $\delta \notin \mathbf{Z}\beta$, so by properties (a) and (b) we have $\psi(\delta) = \psi(\alpha)$. By property (c), we get $\psi(\beta) < \psi(\delta) + |\delta| < \psi(\alpha) + \psi(\beta) - \psi(\alpha) = \psi(\beta)$. This contradiction shows that ψ is constant on Δ^*. Thus if $\gamma \in \Gamma^*$, we have $\psi(\gamma) < \psi(\delta) + |\delta|$, for any $\delta < \Delta^*$. Since Δ contains arbitrarily small positive elements of Γ, this implies that $\psi(\gamma) \leq \psi(\delta)$, completing the proof of the first statement. For the second statement, note first that the elements δ of the smallest convex subgroup of Γ containing $\gamma_0 - \gamma_1$ are precisely those $\delta \in \Gamma$ such that $|\delta| < n(\gamma_0 - \gamma_1)$ for some positive integer n. For such a δ, if $\delta \neq 0$, and for any $\gamma \in \Gamma^*$, we have $n(\psi(\gamma) - \psi(\delta)) < |\delta|$ by [3, Theorem 5], so that $n(\gamma_0 - \psi(\delta)) < |\delta| < n(\gamma_0 - \gamma_1)$. This implies $\psi(\delta) > \gamma_1$, so that $\psi(\delta) = \gamma_0$.

If (Γ_1, ψ_1), (Γ_2, ψ_2) are asymptotic couples, define a *morphism from* (Γ_1, ψ_1) *into* (Γ_2, ψ_2) to be an order-preserving (that is, $\geq$-preserving) homomorphism $\tau: \Gamma_1 \to \Gamma_2$ such that $\psi_2 \tau \alpha = \tau \psi_1 \alpha$ for all $\alpha \in \Gamma_1 - \ker \tau$. These morphisms make the set of all asymptotic couples the objects of a category, the *category of asymptotic couples.* The couple $(\{0\}, \phi)$ is both an initial and a terminal object of this category.

PROPOSITION 5. *Let* $\tau:(\Gamma_1, \psi_1) \to (\Gamma_2, \psi_2)$ *be a morphism of asymptotic couples. Then* $\psi_2((\tau\Gamma_1)^*) \subset \tau\Gamma_1$. *Furthermore, if* $\Delta = \ker \tau$, *then* Δ *is a convex subgroup of* Γ_1 *and if* $\delta \in \Delta^*$ *and* $\Delta \neq \Gamma_1$ *then* $\tau\psi_1\delta =$ l.u.b. $\psi_2((\tau\Gamma_1)^*)$, *the* l.u.b. *being taken in* $\tau\Gamma_1$.

The first assertion holds since $\psi_2\tau\alpha = \tau\psi_1\alpha$ for all $\alpha \in \Gamma_1 - \Delta$. The subgroup Δ of Γ_1 is convex since τ is order-preserving. Now fix some $\delta \in \Delta^*$. Then for any $\alpha \in \Gamma_1 - \Delta$ we have $\psi_1(\alpha) < \psi_1(\delta) + |\delta|$, so that $\tau\psi_1\alpha \leq \tau\psi_1\delta$, or $\psi_2\tau\alpha \leq \tau\psi_1\delta$. Also, by [3, Theorem 5], $2(\psi_1(\delta) - \psi_1(\alpha)) < |\alpha|$, so that $2(\tau\psi_1\delta - \tau\psi_1\alpha) \leq |\tau\alpha|$, or $2(\tau\psi_1\delta - \psi_2\tau\alpha) \leq |\tau\alpha|$, showing that $\tau\psi_1\delta - \psi_2\tau\alpha < |\tau\alpha|$ and proving the l.u.b. statement.

COROLLARY. *If* $\delta_1, \delta_2 \in \Delta^*$, *then* $\psi_1(\delta_1) - \psi_1(\delta_2) \in \Delta$.

PROPOSITION 6. *Let* (Γ, ψ) *be an asymptotic couple and let* Δ *be a convex subgroup of* Γ. *Then* Γ/Δ *has a unique structure of ordered abelian group such that the natural surjection* $\Gamma \to \Gamma/\Delta$ *is order-preserving. If* $\alpha, \beta \in \Gamma^*$ *and* $\alpha - \beta \in \Delta$, *then* $\psi(\alpha) - \psi(\beta) \in \Delta$. *Let the map* $\psi':(\Gamma/\Delta)^* \to \Gamma/\Delta$ *be defined by* $\psi'(\alpha + \Delta) = \psi(\alpha) + \Delta$ *for all* $\alpha \in \Gamma - \Delta$. *Then* $(\Gamma/\Delta, \psi')$ *is an asymptotic couple and* $\gamma \mapsto \gamma + \Delta$ *a morphism from* (Γ, ψ) *to* $(\Gamma/\Delta, \psi')$.

The first contention, that Γ/Δ is an ordered abelian group, is well known [5, p. 43]. Now let $\alpha, \beta \in \Gamma^*$ and $\alpha - \beta \in \Delta$. Let $\alpha - \beta = \delta \neq 0$. Then if $\psi(\delta) > \psi(\beta)$, property (b) implies that $\psi(\alpha) = \psi(\beta)$, while otherwise we have $\psi(\delta) \leq \psi(\beta) < \psi(\delta) + |\delta|$. By symmetry, since $\psi(-\delta) = \psi(\delta)$, either $\psi(\alpha) = \psi(\beta)$ or $\psi(\delta) \leq \psi(\alpha) < \psi(\delta) + |\delta|$. In any case we have $|\psi(\beta) - \psi(\alpha)| < |\delta|$, so that $\psi(\beta) - \psi(\alpha) \in \Delta$. Hence the indicated map $\psi':(\Gamma/\Delta)^* \to \Gamma/\Delta$ is well defined. To complete the proof it remains only to show that $(\Gamma/\Delta, \psi')$ is an asymptotic couple, that is, that properties (a), (b), (c) hold for $(\Gamma/\Delta, \psi')$. Of these, (a) is a trivial consequence of the same property for (Γ, ψ), and the proof of (b) is straightforward. To prove (c), we must show that for all $\alpha, \beta \in \Gamma - \Delta$ we have $\psi'(\beta + \Delta) < \psi'(\alpha + \Delta) + |\alpha + \Delta|$. Assume, as we may, that $\alpha > 0$. Then since $\psi(\beta) < \psi(\alpha) + \alpha$, we have $\psi(\beta) + \Delta \leq \psi(\alpha) + \alpha + \Delta$, or $\psi'(\beta + \Delta) \leq \psi'(\alpha + \Delta) + |\alpha + \Delta|$. By [3, Theorem 5], we have $2(\psi(\beta) - \psi(\alpha)) < \alpha$, so that $2(\psi'(\beta + \Delta) - \psi'(\alpha + \Delta)) \leq \alpha + \Delta$. If we had $\psi'(\beta + \Delta) = \psi'(\alpha + \Delta) + |\alpha + \Delta|$ we would have $2\alpha + \Delta = \alpha + \Delta$, or $\alpha \in \Delta$, which is false. This completes the proof.

If $\tau:(\Gamma_1, \psi_1) \to (\Gamma_2, \psi_2)$ is a morphism of asymptotic couples and $\Delta = \ker \tau$, there is a unique morphism from the couple $(\Gamma_1/\Delta, \psi_1')$ of Proposition 6 into (Γ_2, ψ_2) such that τ is the composite of two morphisms

$$(\Gamma_1, \psi_1) \to (\Gamma_1/\Delta, \psi_1') \to (\Gamma_2, \psi_2).$$

In the morphism on the left, the underlying group homomorphism is surjective, in the morphism on the right it is injective.

COROLLARY. *Let Γ, ψ, Δ and ψ' be as above, let τ be the natural homomorphism from Γ to Γ/Δ, and let Γ_1 be a subgroup of Γ that contains Δ. Then $\psi(\Gamma_1^*) \subset \Gamma_1$ if and only if $\psi(\Delta^*) \subset \Gamma_1$ and $\psi'((\tau\Gamma_1)^*) \subset \tau\Gamma_1$, and in this case the restriction of τ to Γ_1 gives a morphism from $(\Gamma_1, \psi|\Gamma_1^*)$ to $(\tau\Gamma_1, \psi'|(\tau\Gamma_1)^*)$.*

PROPOSITION 7. *Let (Γ, ψ) be an asymptotic couple, with $\Gamma \neq \{0\}$ and $\psi(\Gamma^*)$ a finite set. Then for any $\alpha \in \Gamma$ there is an element $\beta \in \Gamma^*$ such that $\psi(\beta) + \beta = \alpha$, except when $\alpha = \max \psi(\Gamma^*)$.*

This is simply [4, Theorem 2], in the special case that $\psi(\Gamma^*)$ is finite, restated here for the convenience of the reader.

LEMMA. *Let (Γ, ψ) be an asymptotic couple, let Γ_0 be a subgroup of Γ and let $\gamma \in \Gamma$. Then the set $\psi((\Gamma_0 + \mathbf{Z}\gamma)^*) - \psi(\Gamma_0^*)$ consists of at most one element.*

For otherwise there exist $\alpha, \beta \in \Gamma_0$ and $m, n \in \mathbf{Z}$ such that $\alpha + m\gamma$, $\beta + n\gamma$ are nonzero and $\psi(\alpha + m\gamma)$, $\psi(\beta + n\gamma)$ are distinct and not in $\psi(\Gamma_0^*)$. Then $m, n \neq 0$, $n\alpha \neq m\beta$, and $\psi(n\alpha - m\beta) = \psi(n(\alpha + m\gamma) - m(\beta + n\gamma)) = \min\{\psi(\alpha + m\gamma), \psi(\beta + n\gamma)\} \notin \psi(\Gamma_0^*)$, a contradiction.

THEOREM 1. *Let (Γ, ψ) be an asymptotic couple, with $\psi(\Gamma^*)$ finite, and suppose that Γ_0 is a subgroup of Γ such that $\psi(\Gamma_0^*) \subset \Gamma_0$ and $\psi(\Gamma_0^*) \neq \psi(\Gamma^*)$. Then there is a subgroup Γ_1 of Γ such that $\Gamma_0 \subset \Gamma_1$, $\psi(\Gamma_1^*) \subset \Gamma_1$, and $\psi(\Gamma_1^*) - \psi(\Gamma_0^*)$ consists of one element.*

We prove the theorem by induction on card $\psi(\Gamma^*) \geq 1$, assuming that it is true for all lesser values. By Proposition 7, there exists $\beta \in \Gamma^*$ such that either $\psi(\beta) = -\beta$ or $\psi(\beta) = 0$, so that if $\Gamma_0 = \{0\}$ we may take $\Gamma_1 = \mathbf{Z}\beta$. We may therefore suppose $\Gamma_0 \neq \{0\}$. In particular, card $\psi(\Gamma^*) \geq 2$. By the Lemma, if $\gamma \in \Gamma^*$ and $\psi(\gamma) \in \Gamma_0 - \psi(\Gamma_0^*)$, we can take $\Gamma_1 =$

$\Gamma_0 + \mathbf{Z}\gamma$. We may therefore assume that for each $\gamma \in \Gamma^*$ such that $\psi(\gamma) \notin \psi(\Gamma_0^*)$ we also have $\psi(\gamma) \notin \Gamma_0$. If $\max \psi(\Gamma_0^*) \neq \max \psi(\Gamma^*)$, Proposition 7 guarantees the existence of $\beta \in \Gamma - \Gamma_0$ such that $\psi(\beta) + \beta = \max \psi(\Gamma_0^*)$. Then $\psi(\beta) \notin \Gamma_0$, $\psi((\Gamma_0 + \mathbf{Z}\beta)^*) = \psi(\Gamma_0^*) \cup \{\psi(\beta)\} \subset \Gamma_0 + \mathbf{Z}\beta$ and we may take $\Gamma_1 = \Gamma_0 + \mathbf{Z}\beta$. We are thus reduced to the case where $\max \psi(\Gamma^*) = \max \psi(\Gamma_0^*)$. Let Δ be the nonzero convex subgroup of Γ of the second part of Proposition 4 which contains the difference of the two greatest elements of $\psi(\Gamma^*)$. Then $\max \psi(\Gamma^*) = \psi(\delta)$ for any $\delta \in \Delta^*$. Therefore $\psi((\Gamma_0 + \Delta)^*) = \psi(\Gamma_0^*) \cup \{\psi(\delta)\} = \psi(\Gamma_0^*)$, so that we may if necessary replace Γ_0 by $\Gamma_0 + \Delta$ to reduce ourselves to the case $\Delta \subset \Gamma_0$. Now consider the natural homomorphism $\Gamma \to \Gamma/\Delta$ and the function $\psi' : (\Gamma/\Delta)^* \to \Gamma/\Delta$ of Proposition 6. For any $\gamma \in \Gamma - \Delta$ we have $\psi'(\gamma + \Delta) = \psi(\gamma) + \Delta$, so that $\psi'(\gamma + \Delta)$ depends only on $\psi(\gamma)$. If $\alpha \in \Gamma^*$ is such that $\psi(\alpha)$ is the second largest element of $\psi(\Gamma^*)$ then $\psi(\delta) - \psi(\alpha) \in \Delta$, so that $\psi(\alpha) + \Delta = \psi(\delta) + \Delta$. Thus card $\psi'((\Gamma/\Delta)^*) <$ card $\psi(\Gamma^*)$. Also, there exists $\gamma \in \Gamma^*$ such that $\psi(\gamma) \notin \psi(\Gamma_0^*)$, and then $\psi(\gamma) \notin \Gamma_0$, so that $\psi(\gamma) + \Delta \not\subset \Gamma_0$ and $\psi'(\gamma + \Delta) \notin \psi'((\Gamma_0/\Delta)^*)$. Therefore $\psi'((\Gamma/\Delta)^*) \neq \psi'((\Gamma_0/\Delta)^*)$. We may therefore apply our induction hypothesis to $\Gamma_0/\Delta \subset \Gamma/\Delta$ to get a subgroup Γ_2 of Γ that contains Γ_0 and is such that $\psi'((\Gamma_2/\Delta)^*) \subset \Gamma_2/\Delta$ and $\psi'((\Gamma_2/\Delta)^*)$ contains one more element than $\psi'((\Gamma_0/\Delta)^*)$. Since $\psi(\Delta^*) \subset \psi(\Gamma_0^*) \subset \Gamma_0 \subset \Gamma_2$, we deduce from the Corollary of Proposition 6 that $\psi(\Gamma_2^*) \subset \Gamma_2$. Since $\psi'((\Gamma_2/\Delta)^*) \not\subset \psi'((\Gamma_0/\Delta)^*)$, we have $\psi(\Gamma_2^*) \not\subset \psi(\Gamma_0^*)$. Also, for any $\gamma \in \Gamma_2^*$ such that $\psi(\gamma) \notin \psi(\Gamma_0^*)$ we have $\psi(\gamma) \notin \Gamma_0$ and $\psi(\gamma) + \Delta \notin \psi'((\Gamma_0/\Delta)^*)$, so that the element $\psi(\gamma) + \Delta$ of $\psi'((\Gamma_2/\Delta)^*)$ is independent of γ, showing that the subgroup $\Gamma_3 = \Gamma_0 + \mathbf{Z}\psi(\gamma)$ of Γ_2 is independent of γ. Clearly $\psi(\Gamma_2^*) \subset \Gamma_3$, so $\psi(\Gamma_3^*) \subset \Gamma_3$. By the Lemma, $\psi(\Gamma_3^*)$ contains at most one more element than $\psi(\Gamma_0^*)$. If it contains one more element, we set $\Gamma_1 = \Gamma_3$, otherwise $\psi(\Gamma_3^*) = \psi(\Gamma_0^*)$ and we take $\Gamma_1 = \Gamma_3 + \mathbf{Z}\gamma$, for some $\gamma \in \Gamma_2$ such that $\psi(\gamma) \notin \psi(\Gamma_0^*)$. This completes the proof.

PROPOSITION 8. *Let (Γ, ψ) be an asymptotic couple, with $\psi(\Gamma^*)$ finite, and let $\Gamma_0 \neq \{0\}$ be a subgroup of Γ such that $\psi(\Gamma_0^*) \subset \Gamma_0$ and $\psi(\Gamma^*) - \psi(\Gamma_0^*)$ consists of one element α. If $\alpha > \max \psi(\Gamma_0^*)$ then there is a unique $\theta \in \Gamma - \Gamma_0$ such that $\psi(\theta) + \theta \in \Gamma_0$ and for this θ we have $\theta < 0$, $\psi(\theta) + \theta = \max \psi(\Gamma_0^*)$, $\alpha = \psi(\theta)$, and $\psi((\Gamma_0 + \mathbf{Z}\theta)^*) = \psi(\Gamma^*) \subset \Gamma_0 + \mathbf{Z}\theta$. If $\alpha < \max \psi(\Gamma_0^*)$ then there is a unique $\beta \in \Gamma^*$ such that $\alpha = \psi(\beta) + \beta$, and for this β we have $\beta < 0$, $\alpha \in \Gamma_0 + \mathbf{Z}\beta$, and $\psi((\Gamma_0 + \mathbf{Z}\beta)^*) = \psi(\Gamma_0^*) \subset \Gamma_0$.*

First suppose $\alpha > \max \psi(\Gamma_0^*)$. By Proposition 1, any element of Γ of the form $\psi(\gamma) + \gamma$, for some $\gamma \in \Gamma^*$, is of this form for a unique γ. By Proposition 7, any element of Γ_0 except max $\psi(\Gamma_0^*)$ is of this form for some $\gamma \in \Gamma_0^*$. Therefore if $\theta \in \Gamma - \Gamma_0$ is such that $\psi(\theta) + \theta \in \Gamma_0$, we must have $\psi(\theta) + \theta = \max \psi(\Gamma_0^*)$; furthermore such a θ exists. By asymptotic couple property (c), $\theta < 0$. Thus $\psi(\theta) > \max \psi(\Gamma_0^*)$, so $\psi(\theta) = \alpha$. Clearly $\psi((\Gamma_0 + \mathbf{Z}\theta)^*) = \psi(\Gamma^*) \subset \Gamma_0 + \mathbf{Z}\theta$. Now suppose $\alpha < \max \psi(\Gamma_0^*)$. As before, there is a unique $\beta \in \Gamma^*$ such that $\alpha = \psi(\beta) + \beta$, and we have $\beta < 0$. Therefore $\psi(\beta) > \alpha$, so that $\psi(\beta) \in \psi(\Gamma_0^*) \subset \Gamma_0$. Finally, if we had $\psi((\Gamma_0 + \mathbf{Z}\beta)^*) \not\subset \psi(\Gamma_0^*)$, then for some $\gamma_0 \in \Gamma_0$ and $n \in \mathbf{Z}$ we would have $\gamma_0 + n\beta \neq 0$ and $\psi(\gamma_0 + n\beta) = \alpha$. Since $\alpha \notin \psi(\Gamma_0^*)$, we have $n \neq 0$, and since $\psi(n\beta) = \psi(\beta) > \alpha$ we deduce from asymptotic couple property (b) that $\psi(\gamma_0) = \alpha$, which is impossible. Thus $\psi((\Gamma_0 + \mathbf{Z}\beta)^*) \subset \psi(\Gamma_0^*)$.

For an example of the second case of Proposition 8, take $\Gamma = \{(m, n + pa) : m, n, p \in \mathbf{Z}\}$ for some real number $a < 1$, a subgroup of the lexicographically ordered group $\mathbf{R}^2$, with $\psi((m, n + pa)) = (0, a)$ if $m \neq 0$, $\psi((0, n + pa)) = (0, 1)$ if $n + pa \neq 0$, and $\Gamma_0 = \{(0, n) : n \in \mathbf{Z}\}$. Here $\psi(\Gamma_0^*) = \{(0, 1)\}$, $\alpha = (0, a)$ and $\beta = (0, a - 1)$. The smallest subgroup of Γ containing $\psi(\Gamma^*)$ is $\Gamma_0 + \mathbf{Z}\beta = \{(0, n + pa) : n, p \in \mathbf{Z}\}$.

Recall that if (Γ, ψ) is an asymptotic couple, then the map $\gamma \mapsto \gamma \otimes 1$ is an embedding of Γ in the $\mathbf{Q}$-vector space $\mathbf{Q}\Gamma = \Gamma \otimes_{\mathbf{Z}} \mathbf{Q}$, so we can identify Γ with a subset of $\mathbf{Q}\Gamma$ [4, p. 259]. The group $\mathbf{Q}\Gamma$ has a unique ordering that induces the given ordering of Γ and there is a unique extension of the map ψ to $(\mathbf{Q}\Gamma)^*$ which satisfies property (a). If we denote the extension of ψ to $(\mathbf{Q}\Gamma)^*$ by the same symbol ψ, then $(\mathbf{Q}\Gamma, \psi)$ is an asymptotic couple and $\psi((\mathbf{Q}\Gamma)^*) = \psi(\Gamma^*)$. The groups Γ and $\mathbf{Q}\Gamma$ have the same rank and the same rational rank, and the latter is just the vector space dimension $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma$.

COROLLARY. *Let (Γ, ψ) be an asymptotic couple, with $\psi(\Gamma^*)$ finite, and let $\Gamma_0 \neq \{0\}$ be a subgroup of Γ such that $\psi(\Gamma_0^*) \subset \Gamma_0$ and $\Gamma \not\subset \mathbf{Q}\Gamma_0$. Then there is a subgroup Γ_1 of Γ such that $\Gamma_0 \subset \Gamma_1$, $\psi(\Gamma_1^*) \subset \Gamma_1$, and* $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma_1/\mathbf{Q}\Gamma_0 = 1$. *Furthermore, for any such Γ_1, $\psi(\Gamma_1^*)$ is contained in the $\mathbf{Q}$-vector space spanned by $\psi(\Gamma_0^*)$ if and only if* $\max \psi(\Gamma_1^*) \in \psi(\Gamma_0^*)$.

We first show that a Γ_1 with the desired properties exists. By Theorem 1, we may assume that $\psi(\Gamma^*) - \psi(\Gamma_0^*)$ consists of at most one element. If $\psi(\Gamma^*) = \psi(\Gamma_0^*)$, we may take $\Gamma_1 = \Gamma_0 + \mathbf{Z}\beta$, for any $\beta \in \Gamma - \mathbf{Q}\Gamma_0$. Next suppose that $\psi(\Gamma^*) - \psi(\Gamma_0^*) = \{\alpha\}$. If $\alpha > \max \psi(\Gamma_0^*)$, we obtain $\theta \in \Gamma - \Gamma_0$ such that $\psi(\theta) + \theta \in \Gamma_0$ and we take $\Gamma_1 = \Gamma_0 + \mathbf{Z}\theta$. If $\alpha < \max \psi(\Gamma_0^*)$, we find $\beta \in \Gamma^*$ such that $\alpha = \psi(\beta) + \beta$ and then take

$\Gamma_1 = \Gamma_0 + \mathbf{Z}\beta$ if $\beta \notin \mathbf{Q}\Gamma_0$, otherwise take $\Gamma_1 = \Gamma_0 + \mathbf{Z}\beta + \mathbf{Z}\gamma$, for any $\gamma \in \Gamma^*$ such that $\psi(\gamma) = \alpha$. This proves the existence of Γ_1. To prove the last assertion, note that it suffices to assume that $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma/\mathbf{Q}\Gamma_0 = 1$. Then for any subgroup Γ_1 of Γ such that $\Gamma_0 \subset \Gamma_1$, $\psi(\Gamma_1{}^*) \subset \Gamma_1$ and $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma_1/\mathbf{Q}\Gamma_0 = 1$ we have $\Gamma \subset \mathbf{Q}\Gamma_1$ and so $\psi(\Gamma^*) = \psi(\Gamma_1{}^*)$. Thus it suffices to verify the last assertion for the Γ_1 we have constructed above, which is easy.

2. Elementary field extensions with given value group extension. The following restatement of [3, Theorem 2] is included for the convenience of the reader. It is somewhat stronger than the original, except that we here assume that $\mathcal{C} = C$; we remark however that the reader wishing to consider the case $\mathcal{C} \neq C$ can easily modify the statement and proof to include this case, at the expense of rather more complicated wording.

PROPOSITION 9. *Let K be a differential field of characteristic zero, k a differential subfield of K, and C the field of constants of k. Let T be a subgroup of the multiplicative group K^* such that $k^* \subset T$, $K = C(T)$, and such that any constant in T is in C. Let $v: T \to \Gamma$ be a surjective homomorphism from T to an ordered abelian group Γ whose kernel is contained in k^*, such that v induces a differential valuation of k, and such that elements of T with distinct v-images in Γ are linearly independent over C. Then v can be extended in one and only one way to a valuation of K, also denoted v, and this valuation v has residue class field C and value group Γ. If for all $a, b \in T$ such that $v(a), v(b) > 0$ we have $v(a'b/b') > 0$, then v is a differential valuation of K.*

Any nonzero element x of the ring $C[T]$ can be written $x = \Sigma_{i=1}^{n} t_i$, with each $t_i \in T$ and $t_i/t_j \notin k$ for $i \neq j$. By our assumptions on the kernel of $v: T \to \Gamma$ and linear independence over C, this representation is unique, except for the order of the summands. If we define $v(x) = \min\{v(t_1), \ldots, v(t_n)\}$, we get a well-defined extension of v from T to a map $v: (C[T])^* \to \Gamma$. For $x_1, x_2 \in C[T]$ with $x_1 x_2 \neq 0$, $x_1 + x_2 \neq 0$ we get $v(x_1 x_2) = v(x_1) + v(x_2)$, $v(x_1 + x_2) \geq \min\{v(x_1), v(x_2)\}$, so the map v on $(C[T])^*$ extends to a valuation of $C(T) = K$. This valuation is clearly the only valuation on K that extends the given map on T, since this map is trivial on C. The extended valuation, which we also denote v, clearly has value group Γ. At this point, we can apply the original [3, Theorem 2] to get the present result. Alternately, we can quickly complete the proof by referring to the original proof and bypassing the lemma given therein.

PROPOSITION 10. *Let* (Γ, ψ) *be an asymptotic couple, and let* Γ_0 *be a subgroup of* Γ *such that* $\psi(\Gamma_0^*) \subset \Gamma_0$ *and* $\Gamma \subset \mathbf{Q}\Gamma_0$. *Let* ν *be a differential valuation of a differential field* k *whose corresponding asymptotic couple is* $(\Gamma_0, \psi | \Gamma_0)$, *and let* S *be a subgroup of the multiplicative group of* k *such that* $\nu(S) = \Gamma_0$. *Then there exists a differential extension field* K *of* k *having the same constants as* k *and an extension of* ν *to a differential valuation of* K, *which we also denote* ν, *such that if* S' *is the set of all elements of* K *of nonzero value some nonzero power of which is in* S, *then* $K = k(S')$ *and* $\nu(K^*) = \nu(S') = \Gamma$.

Extend ν to a differential valuation of the algebraic closure $\bar{k}$ of k. For simplicity, denote this differential valuation of $\bar{k}$ by the same letter ν, so that $\nu(\bar{k}^*) = \mathbf{Q}\nu(k^*) = \mathbf{Q}\Gamma_0$. Consider differential fields K_1 such that $k \subset K_1 \subset \bar{k}$, such that K_1 has the same constants as k, and such that if S_1 is the subset of K_1 consisting of all elements of nonzero value which have some nonzero power in S, then $K_1 = k(S_1)$ and $\nu(K_1^*) = \nu(S_1) \subset \Gamma$. Applying Zorn's lemma to the inclusion-ordered set $\{K_1\}$, we get a maximal $K \in \{K_1\}$. We shall show that this K satisfies our demands, and for this purpose we may assume $K = k$, which means that $\{K_1\} = \{k\}$. We have only to prove that $\Gamma_0 = \Gamma$. If not, there is an element $u \in S$ and a prime number p such that $(1/p)\nu(u) \in \Gamma - \Gamma_0$. By [3, Lemma to Theorem 6], the field $k_1 = k(u^{1/p})$ has the same constants as k. Since $p \leq [\nu(k_1^*):\nu(k^*)] \leq [k_1^*:k] \leq p$, we deduce that k_1 is one of the fields K_1 and $k_1 \neq k$, a contradiction.

LEMMA 1. *Let* k *be a differential field of characteristic zero of finite transcendence degree over its subfield of constants* C. *Then any finite subset of* k *generates with* C *a differential subfield of* k *that is a finite field extension of* C.

Let $\{x_1, \ldots, x_n\}$ be a finite subset of k, let k_1 be the differential field $C\langle x_1, \ldots, x_n\rangle = C(x_1, x_1', x_1'', \ldots, x_n, x_n', x_n'', \ldots)$ and let $y_1, \ldots, y_r$ be a transcendence basis of k_1 over C. Let $U, Y_1, \ldots, Y_r$ be indeterminates over k. Then for any $u \in k_1$ there exists an irreducible $F \in C[U, Y_1, \ldots, Y_r]$ such that $F(u, y_1, \ldots, y_r) = 0$. Then

$$\frac{\partial F}{\partial U}(u, y_1, \ldots, y_r)u' + \sum_{i=1}^{r} \frac{\partial F}{\partial Y_i}(u, y_1, \ldots, y_r)y_i' = 0.$$

Since $(\partial F/\partial U)(u, y_1, \ldots, y_r) \neq 0$, we get $u' \in C(u, y_1, \ldots, y_r, y_1', \ldots, y_r')$. This shows, in particular, that $k_2 = C(y_1, \ldots, y_r, y_1', \ldots, y_r')$ is a differential field. Since $x_1, \ldots, x_n$ are algebraic over k_2, the field $k_2(x_1, \ldots, x_n)$ is also a differential field. That is, $k_1 = k_2(x_1, \ldots, x_n)$, which is a finite extension of C.

LEMMA 2. *Let k be a differential field of characteristic zero that is of finite transcendence degree over its subfield of constants C and let $b \in k^*$. Then the set of all $c \in C$ such that there exist $a \in k^*$ and a nonzero $n \in \mathbf{Z}$ such that $a'/a = ncb$ form an additive subgroup of C of finite rank.*

Let S be the set of all such $c \in C$. For each $c \in S$, choose $a_c \in k^*$ and $n_c \in \mathbf{Z}^*$ such that $a_c'/a_c = n_c cb$. Clearly $0 \in S$ and if $c, e \in S$ then $n_c n_e(c - e)b = (a_c^{n_e}/a_e^{n_c})'/(a_c^{n_e}/a_e^{n_c})$, so that S is a group. Now assume, provisionally, that k is a finite extension of C. Consider the k-vector space $\Omega_{k/C}$ of C-differentials of k and the operation D^1 on $\Omega_{k/C}$ induced by the given derivation $'$ of k [2]. We compute, for $c, e \in S$,

$$D^1\left(cn_c\frac{da_e}{a_e} - en_e\frac{da_c}{a_c}\right) = cn_cD^1\left(\frac{da_e}{a_e}\right) - en_eD^1\left(\frac{da_c}{a_c}\right)$$

$$= cn_cd(a_e'/a_e) - en_ed(a_c'/a_c) = d(cn_ca_e'/a_e - en_ea_c'/a_c)$$

$$= d0 = 0.$$

Any set of differentials of the type $cn_cda_e/a_e - en_eda_c/a_c$ in number greater than deg tr k/C will be linearly dependent over k, therefore linearly dependent over C [2, Proposition 6]. Therefore all such differentials lie in a finite dimensional C-subspace of $\Omega_{k/c}$. Therefore the differentials $\{da_c/a_c\}_{c\in S}$ lie in a finite dimensional C-subspace of $\Omega_{k/C}$. Since C is algebraically closed in k, we can consider a normal projective algebraic variety V defined over C whose function field $C(V)$ is C-isomorphic to K. By means of this isomorphism we identify elements of k with elements of $C(V)$ and elements of $\Omega_{k/C}$ with differentials on V. There are distinct subvarieties $W_1, \ldots, W_N$ of V of codimension one such that the differentials $\{da_c/a_c\}_{c\in S}$ are regular on $V - W_1 - \cdots - W_N$. Thus for each $c \in S$, the divisor (a_c) of a_c on V is a linear combination with integral coefficients of $W_1, \ldots, W_N$. Thus if $c_1, \ldots, c_M \in S$ and $M > N$, there are integers $r_1, \ldots, r_M$, not all zero, such that $(a_{c_1}{}^{r_1} \cdots a_{c_M}{}^{r_M}) = 0$. Thus $a_{c_1}{}^{r_1} \cdots a_{c_M}{}^{r_M} \in C^*$, so that $r_1a_{c_1}'/a_{c_1} + \cdots + r_Ma_{c_M}'/a_{c_M} = 0$, or $r_1n_{c_1}c_1b + \cdots + r_Mn_{c_M}c_Mb = 0$, showing that $c_1, \ldots, c_M$ are linearly dependent over $\mathbf{Q}$. This concludes the proof if k is a finite extension of C. In the general case, use the preceding lemma to find a differential subfield k_1 of k that contains C, contains a transcendence basis of k over C, and is a finite field extension of C. We shall complete the proof by showing that if $c \in C^*$ is such that there exists an $a \in k^*$ such that a'/acb is a nonzero integer, then such an a can be found in k_1. For this, let k_2 be a normal algebraic extension field of k_1 that contains a and is of finite degree over k_1. Then the derivation on k_1 extends uniquely to a derivation on k_2 and for each $\sigma \in \operatorname{Aut}(k_2/k_1)$ we have $\sigma(a)'/\sigma(a)cb = a'/acb = n \in \mathbf{Z}^*$. Taking the product over all $\sigma \in \operatorname{Aut}(k_2/k_1)$, we get $\Pi\sigma(a) \in k_1{}^*$ and $(\Pi\sigma(a))'/\Pi\sigma(a)cb = [k_2:k_1]n$. This ends the proof.

THEOREM 2. *Let (Γ, ψ) be an asymptotic couple, with Γ of finite rational rank, and let Γ_0 be a nonzero subgroup of Γ such that $\psi(\Gamma_0{}^*) \subset \Gamma_0$. Let k be a differential field of finite transcendence degree over its field of constants C, which is an infinite dimensional vector space over $\mathbf{Q}$, and let v be a differential valuation of k whose corresponding asymptotic couple is $(\Gamma_0, \psi|\Gamma_0)$. Then there exists a differential extension field K of k with field of constants C and a differential valuation of K extending v whose corresponding asymptotic couple is (Γ, ψ). Furthermore, K can be chosen to be an extension of k by repeated adjunctions of n^{th} roots, for various n, and logarithms, exponentials, and irrational powers of single elements of negative value, the number of the last three types of extension being* $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma/\mathbf{Q}\Gamma_0$.

In the case where $\Gamma \subset \mathbf{Q}\Gamma_0$, this is a direct consequence of Proposition 10, using $S = k^*$; note also that if Γ/Γ_0 is finite, we may take K to be of finite degree over k. Now try to prove the theorem by induction on $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma/\mathbf{Q}\Gamma_0$. We have just shown that the theorem holds if $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma/\mathbf{Q}\Gamma_0 = 0$. Suppose for a moment that the theorem holds if $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma/\mathbf{Q}\Gamma_0 = 1$. If $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma/\mathbf{Q}\Gamma_0 > 1$, then the Corollary to Proposition 8 shows the existence of a subgroup Γ_1 of Γ such that $\Gamma_0 \subset \Gamma_1$, $\psi(\Gamma_1{}^*) \subset \Gamma_1$, and $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma_1/\mathbf{Q}\Gamma_0 = 1$. Then $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma/\mathbf{Q}\Gamma_1 = \dim_{\mathbf{Q}} \mathbf{Q}\Gamma/\mathbf{Q}\Gamma_0 - 1$ and the induction proof will go through in two steps, Γ_0 to Γ_1 then Γ_1 to Γ. It therefore suffices to assume $\dim_{\mathbf{Q}} \mathbf{Q}\Gamma/\mathbf{Q}\Gamma_0 = 1$. Suppose for a moment that $\max \psi(\Gamma^*) > \max \psi(\Gamma_0{}^*)$. Then for $\theta \in \Gamma$ as in Proposition 8 and $\Gamma_1 = \Gamma_0 + \mathbf{Z}\theta$, we have $\Gamma_0 \subset \Gamma_1 \subset \Gamma$, $\psi(\Gamma_1{}^*) \subset \Gamma_1$, $\theta \notin \mathbf{Q}\Gamma_0$, and $\Gamma \subset \mathbf{Q}\Gamma_1$. To prove the theorem for Γ_0 and Γ it suffices to prove it for Γ_0 and Γ_1, using the case already proved for the passage from Γ_1 to Γ. In the remaining case $\max \psi(\Gamma^*) = \max \psi(\Gamma_0{}^*)$, the Corollary to Proposition 8 implies $\psi(\Gamma^*) \subset \mathbf{Q}\Gamma_0$, so that if Γ_2 is the group generated by Γ_0 and $\psi(\Gamma^*)$ then $\Gamma_0 \subset \Gamma_2 \subset \Gamma$ and $\psi(\Gamma_2{}^*) \subset \Gamma_0$. Since $\Gamma_2 \subset \mathbf{Q}\Gamma_0$, it suffices to prove the theorem for Γ_2 and Γ. Thus we may assume that $\psi(\Gamma^*) \subset \Gamma_0$. Now let $\theta \in$

$\Gamma - \mathbf{Q}\Gamma_0$ be such that $\psi(\theta) = \max \psi(\Gamma - \mathbf{Q}\Gamma_0)$. Then $\Gamma_0 \subset \Gamma_0 + \mathbf{Z}\theta \subset \Gamma$, $\psi(\Gamma_0 + \mathbf{Z}\theta) \subset \Gamma_0$, and $\Gamma \subset \mathbf{Q}(\Gamma_0 + \mathbf{Z}\theta)$. In view of the case already proved, it suffices to prove the theorem for Γ_0 and $\Gamma_0 + \mathbf{Z}\theta$. Therefore in all cases we may assume that $\Gamma = \Gamma_0 + \mathbf{Z}\theta$, where $\theta \in \Gamma - \mathbf{Q}\Gamma_0$ is such that $\psi(\Gamma) = \max \psi(\Gamma - \mathbf{Q}\Gamma_0)$. Clearly $\psi(\gamma_0 + n\theta) = \min\{\psi(\gamma_0), \psi(\theta)\}$ whenever $\gamma_0 \in \Gamma_0{}^*$ and $n \in \mathbf{Z}^*$. Let y be transcendental over k. We shall apply Proposition 9 to the fields $k \subset K = k(y)$ (the latter with appropriate extra structure, to be defined) and the subgroup $T = \bigcup_{i \in \mathbf{Z}} k^* y^i$ of K^*. Clearly $k^* \subset T$ and $K = C(T)$. Let $v: T \to \Gamma$ be the surjective homomorphism given by $v(ay^i) = v(a) + i\theta$ if $a \in k^*$, $i \in \mathbf{Z}$, which extends the v given on k^*. The kernel of v on T is clearly contained in k^* and elements of T with distinct v-images are linearly independent over C. We now make K into a differential extension field of k by setting $y' = cby$, where $c \in C^*$ is to be determined and $b \in T$ is chosen as follows. If $\psi(\theta) > \max \psi(\Gamma_0{}^*)$, we take θ as in Proposition 8, where $\psi(\theta) + \theta = \max \psi(\Gamma_0{}^*)$. Choose $u \in k^*$ such that $v(u) < 0$ and $\psi(v(u)) = \max \psi(\Gamma_0{}^*)$ and take $c = 1$ and $b = u'/uy$. Then $y' = u'/u$, so that y is a logarithm of an element of k of negative value and $v(y'/y) = v(u'/uy) = \psi(v(u)) - v(y) = \max \psi(\Gamma_0{}^*) - \theta = \psi(\theta)$. In the cases $\psi(\theta) \leq \max \psi(\Gamma_0{}^*)$, we have $\psi(\theta) \in \Gamma_0$. If $\psi(\theta) < \max \psi(\Gamma_0{}^*)$, there exists $\beta \in \Gamma_0$, $\beta < 0$, such that $\psi(\theta) = \psi(\beta) + \beta$. Choose $b = u'$, where $u \in k^*$ is chosen such that $v(u) = \beta$. Then $y' = (cu)'y$, so that y is an exponential of an element of k of negative value, and $v(y'/y) = v(u') = v(u'/u) + v(u) = \psi(\beta) + \beta = \psi(\theta)$. Finally, if $\psi(\theta) = \max \psi(\Gamma_0{}^*)$, choose $u \in k^*$ such that $v(u'/u) = \max \psi(\Gamma_0{}^*)$ and $v(u) < 0$, choose $b = u'/u$ and let $c \in C^*$ remain to be determined. Then $y'/y = cu'/u$, so that if $c \notin \mathbf{Q}$ then y is an irrational power of an element of k of negative value and $v(y'/y) = v(u'/u) = \psi(\theta)$. Thus in all cases we have $v(y'/y) = \psi(\theta)$. We now show that any constant in T is in C, at least if c is properly chosen. For suppose that $a \in k^*$, $n \subset \mathbf{Z}$ and ay^n is constant. Then $(ay^n)'/(ay^n) = 0 = a'/a + ny'/y = a'/a + ncb$. If $n = 0$, clearly $a \in C$. If $n \neq 0$, then $b \in k$ and we are in one of the cases $\psi(\alpha) \in \Gamma_0$. Lemma 2 shows that $a'/a + ncb = 0$ is possible only if c lies in a certain finite-dimensional $\mathbf{Q}$-subspace of C; fixing c outside this subspace guarantees that all constants of T are in C. Note also that c can be chosen so that any nonzero rational multiple of c satisfies the condition on constants. We now proceed to verify that for any $t \in T$ such that $v(t) \neq 0$ we have $v(t'/t) = \psi(v(t))$. Let $t = ay^n$, with $a \in k^*$, $n \in \mathbf{Z}$. Our assertion is true for $n = 0$, so suppose $n \neq 0$. Then $v(t'/t) = v(a'/a + ny'/y)$. If $v(a) \neq 0$, then $v(a'/a) = \psi(v(a)) \in \psi(\Gamma_0{}^*)$ while $v(y'/y) = \psi(\theta)$. Since $\psi(\theta) = \max \psi(\Gamma$

$- \mathbf{Q}\Gamma_0$), we have $\nu(t'/t) = \min\{\nu(a'/a), \psi(\theta)\} = \min\{\psi(\nu(a)), \psi(\theta)\} = \psi(\nu(a) + n\theta) = \psi(\nu(t))$, as was to be shown. If $\nu(a) = 0$, there is nothing to prove if $a \in C$, while otherwise we can write $a = c_1 + a_1$, with $c_1 \in C^*$, $a_1 \in k^*$ and $\nu(a_1) > 0$, so that $\nu(a'/a) = \nu(a_1') = \nu(a_1'/a_1) + |\nu(a_1)| > \psi(\theta)$. Thus $\nu(t'/t) = \nu(a'/a + ny'/y) = \nu(y'/y) = \psi(\theta) = \psi(\nu(t))$. Therefore $\nu(t'/t) = \psi(\nu(t))$ for all $t \in T$ such that $\nu(t) \neq 0$. Now suppose that $a, b \in T$, with $\nu(a), \nu(b) > 0$. Then $\nu(a'b/b') = \nu(a'/a) - \nu(b'/b) + |\nu(a)| > 0$. By Proposition 9, ν extends to a differential valuation of K with constants C and value group Γ. It remains only to show that $\nu(a'/a) = \psi(\nu(a))$ for each $a \in K^*$ such that $\nu(a) \neq 0$. But given such an a, there exists $t \in T$ such that $\nu(a) = \nu(t)$, and then $\nu(a'/a) = \nu(t'/t) = \psi(\nu(t)) = \psi(\nu(a))$. This completes the proof of the theorem.

Remark 1. If the group Γ/Γ_0 is finitely generated then the field K may be chosen so that K is a finite extension of k; this follows directly from the proof. Furthermore, if k is a field of meromorphic functions on an open subset of $\mathbf{C}$, the field K can be taken to be a field of meromorphic functions on some smaller open set: To verify this assertion it suffices to show that in the part of the proof where $\Gamma = \Gamma_0 + \mathbf{Z}\theta$, with θ of special type, where we take $K = k(y)$, with y transcendental over k and $y' = cby$, for suitable $c \in C^*$ and $b \in T$, we may take y to be a meromorphic function on some nonempty open subset of our given open set. But the equation $y' = cby$ can always be solved for a meromorphic function on an open subset, so we have merely to show that such a solution is transcendental over k. This reduces to several cases. In the first case, $\psi(\theta) > \max \psi(\Gamma_0^*)$, we have the differential equation $y' = u'/u$, where $u \in k^*$, $\nu(u) < 0$ and $\psi(\nu(u)) = \max \psi(\Gamma_0^*)$. If y is algebraic over k then, since $y' \in k$, we must have $y \in k$. By subtracting a constant from y if necessary, we may suppose $\nu(y) \neq 0$. If $\nu(y) < 0$, then $\psi(\nu(y)) = \nu(y'/y) > \nu(y') = \nu(u'/u) = \max \psi(\Gamma_0^*)$, a contradiction. If $\nu(y) > 0$, then $\psi(\nu(u)) = \nu(u'/u) = \nu(y') = \psi(\nu(y)) + |\nu(y)|$, a contradiction to asymptotic couple property (c). In the second case $\psi(\theta) < \max \psi(\Gamma_0^*)$, and also in the last case $\psi(\theta) = \max \psi(\Gamma_0^*)$, the differential equation is of the form $y' = cby$, with $c \in C^*$ undetermined and b a specific element of k^*. If y is algebraic over k, then for some positive integer n we have $y^n \in k$, so that $(y^n)'/y^n = ncb$, and by Lemma 2 this last equation can be avoided by a proper choice of c. In each case y is transcendental over k, as desired.

Remark 2. Under the general hypotheses of Theorem 2, we may choose $\alpha_1, \ldots, \alpha_n \in \Gamma$ such that $\Sigma_{i=1}^n \mathbf{Q}\alpha_i = \mathbf{Q}\Gamma$ and $\psi(\Gamma^*) \subset \{\alpha_1, \ldots,$

$\alpha_n\}$. If $\Gamma_1 = \Sigma_{n=1}^{n} \mathbf{Z}\alpha_i$, then $\psi(\Gamma_1^*) \subset \Gamma_1$. The theorem can be applied to the two cases $\Gamma_0 \subset \Gamma_1$ and $\Gamma_1 \subset \Gamma$, giving a field K_1 corresponding to Γ_1 which is a finite extension of k and our field K for Γ which is an extension of K_1 by repeated adjunctions, possibly infinite in number, of n^{th} roots.

Remark 3. A slightly weaker version of Theorem 2 holds if the hypothesis $\Gamma_0 \neq \{0\}$ is omitted. To see this, suppose that $\Gamma_0 = \{0\}$, $\Gamma \neq \{0\}$. Then Proposition 7 implies the existence of $\alpha \in \Gamma^*$ such that $\psi(\alpha) = -\alpha$ or $\psi(\alpha) = 0$. Then $\mathbf{Z}\alpha$ is a nonzero subgroup of Γ such that $\psi((\mathbf{Z}\alpha)^*) \subset \mathbf{Z}\alpha$, so that we can apply the stated version of Theorem 2 to (Γ, ψ) and the subgroup $\mathbf{Z}\alpha$ of Γ, provided only that we can find a differential field k, a finite extension of some suitable prescribed field of constants C, and a differential valuation v of k whose corresponding asymptotic couple is $(\mathbf{Z}\alpha, \psi | \mathbf{Z}\alpha)$. We set $k = C(y)$, where y is transcendental over C and $v(y) = \alpha$ (which determines the valuation v on k) and we fix the differential field structure of k as follows. If $\psi(\alpha) = -\alpha$, we take $y' = 1$, while if $\psi(\alpha) = 0$, we assume, as we may, that $\alpha > 0$ and take $y' = -y$. Thus, if $C = \mathbf{R}$, k may be taken to be a Hardy field of germs of differentiable real-valued functions on $\mathbf{R}$, with $k = \mathbf{R}(x)$ (here $y = x$) in the case $\psi(\alpha) = -\alpha$, the germs being germs at 0^+ if $\alpha > 0$ and at $+\infty$ if $\alpha < 0$, while in the case $\psi(\alpha) = 0$ the germs are at $+\infty$ and $k = \mathbf{R}(e^{-x})$ (and here $y = e^{-x}$).

3. The case of Hardy fields. We first recall that an ordered abelian group Γ of finite rank n can be embedded in the lexicographically ordered group $\mathbf{R}^n$. To see this, we may assume that $\Gamma = \mathbf{Q}\Gamma$. Let the maximal chain of convex subgroups of Γ be $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n = \{0\}$, so that for $i = 0, \ldots, n-1$ we have an order-preserving homomorphism τ_i from Γ_i into $\mathbf{R}$ with kernel Γ_{i+1}. For $i = 1, \ldots, n$ let S_i be a $\mathbf{Q}$-vector subspace of Γ_{i-1} that is complementary to the subspace Γ_i, so that $\Gamma = S_1 \oplus \cdots \oplus S_n$. Then any $s \in \Gamma$ can be uniquely written as $s = s_1 + \cdots + s_n$, with each $s_i \in S_i$, and we have a well-defined map $\Gamma \to \mathbf{R}^n$ given by $s \mapsto (\tau_1(s_1), \ldots, \tau_n(s_n))$. This map is an order-preserving embedding of Γ in $\mathbf{R}^n$.

Nonzero elements α, β of an ordered abelian group Γ are called *comparable* if there are positive integers m, n such that $m|\alpha| > |\beta|$, $n|\beta| > |\alpha|$. Comparability is clearly an equivalence relation on Γ^*. If $\Delta_1 \subset \Delta_2$ are convex subgroups of Γ, with Δ_1 a maximal proper convex subgroup of Δ_2, then any two elements of $\Delta_2 - \Delta_1$ are comparable and these are comparable to no other elements of Γ. Thus the number of comparability classes on Γ^* is rank Γ. If (Γ, ψ) is an asymptotic couple, we say that ψ *respects rank* if $\psi(\alpha) = \psi(\beta)$ whenever $\alpha, \beta \in \Gamma^*$ are comparable. In this case, if $\alpha, \beta \in \Gamma^*$ and $|\alpha| > n|\beta|$ for all $n \in \mathbf{Z}$, then $\alpha + \beta$ and α are comparable, implying $\psi(\alpha + \beta) = \psi(\alpha)$, so that $\psi(\beta) \geq \psi(\alpha)$. An asymptotic couple where this last property fails is [3, Example 13]; another example of an asymptotic couple where rank is not respected is on p. 262 of [4].

Let Γ be a subgroup of rank n of the lexicographically ordered group $\mathbf{R}^n$, and let (Γ, ψ) be an asymptotic couple where rank is respected. We extend ψ to a map $(\mathbf{R}^n)^* \to \mathbf{R}^n$ denoting the extension by the same letter ψ, in such a way that ψ is constant on comparability classes of $(\mathbf{R}^n)^*$. Then $(\mathbf{R}^n, \psi)$ is an asymptotic couple. (The verification of property (c) uses [3, Theorem 5].) This asymptotic couple $(\mathbf{R}^n, \psi)$ is completely determined by the real $n \times n$ matrix (c_{ij}), where $\psi(x_1, \ldots, x_n) = (c_{i1}, \ldots, c_{in})$ if $x_1 = \cdots = x_{i-1} = 0$, $x_i \neq 0$. We know that $(c_{i1}, \ldots, c_{in}) \geq (c_{j1}, \ldots, c_{jn})$ if $1 \leq j < i \leq n$. Also, if $1 \leq j < i \leq n$ and $\epsilon > 0$, then asymptotic couple property (c) implies that $(c_{i1}, \ldots, c_{in}) - (c_{j1}, \ldots, c_{jn}) < (0, \ldots, 0, \epsilon, 0, \ldots, 0)$ (where ϵ is in the j^{th} place), so that $c_{i1} \leq c_{j1}, \ldots, c_{ij} \leq c_{jj}$. Thus $c_{i1} = c_{j1}, \ldots, c_{ij} = c_{jj}$, that is, all the elements in any column of the matrix (c_{ij}) from the diagonal element down are equal. Conversely, if (c_{ij}) is a real $n \times n$ matrix whose downward sequence of rows increases monotonically (but not necessarily strictly) and such that any two elements in the same column and on or below the main diagonal are equal, then there is a unique rank-respecting asymptotic couple $(\mathbf{R}^n, \psi)$ giving rise to (c_{ij}).

If (a_{ij}) is a real $n \times n$ upper triangular matrix with positive diagonal elements, then the map $(x_1, \ldots, x_n) \mapsto (x_1a_{11} + \cdots + x_na_{n1}, \ldots, x_1a_{1n} + \cdots + x_na_{nn})$ (that is, the matrix product $(x)(a_{ij})$) is an order-preserving automorphism of $\mathbf{R}^n$. Under an automorphism of this type, any given subgroup of $\mathbf{R}^n$ that is isomorphic to the lexicographically ordered $\mathbf{Z}^n$ can be mapped onto $\mathbf{Z}^n$, and in such a way that a given set of mutually incomparable positive generators of the subgroup goes into $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1)$. Furthermore, any automorphism of $\mathbf{R}^n$ of this type sends any $n \times n$ matrix of the type (c_{ij}) of the preceding paragraph (such a matrix is just an ordered set of elements of $\mathbf{R}^n$) into a matrix of the same type. Also, by means of a suitable matrix (a_{ij}) the matrix (c_{ij}) can be put into a fairly simple form. If the element $c_{11} \neq 0$, we can take (a_{ij}) to consist entirely of zeros below the first row, except for ones on the main diagonal, and choose the first row of (a_{ij}) in such a way that the first row of $(c_{ij})\cdot(a_{ij})$ is $(\pm 1, 0, \ldots, 0)$. If $c_{11} = 0$, we start with c_{22}.

Continuing in this way, we put (c_{ij}) into the special form in which each nonzero diagonal element is either 1 or -1 and all elements to its right are zero. As an application of this special form, note that if α is the first row of (c_{ij}) with a nonzero diagonal element, we have $\psi(\alpha) = \pm\alpha$, and if there is no such α then (c_{ij}) has only zeros in its last row, so that $\max \psi((\mathbf{R}^n)^*) = 0$, a special case of Proposition 7.

We call an asymptotic couple (Γ, ψ) *of Hardy type* if the following principle (*) of [3] holds:

(*) Let $\alpha > \beta$ be positive elements of Γ. Then $\psi(\alpha) \leq \psi(\beta)$. Indeed if $\alpha > n\beta$ for all positive integers n then $\psi(\alpha) < \psi(\beta)$, and in the contrary case $\psi(\alpha) = \psi(\beta)$.

An asymptotic couple (Γ, ψ) of Hardy type is clearly rank-respecting, and furthermore rank Γ = card $\psi(\Gamma^*)$. If Γ is of finite rank, the previous paragraphs are applicable. In their context, we have an asymptotic couple of Hardy type if and only if the sequence of rows of the matrix (c_{ij}) is strictly increasing.

We now modify the definition of [3, Example 1] to define a *Hardy field* to be a set of germs of differentiable real-valued functions on neighborhoods of $+\infty$, or of $-\infty$, in $\mathbf{R}$, or on deleted right or left sided neighborhoods in $\mathbf{R}$ of any point of $\mathbf{R}$ that is closed under differentiation and form a field with respect to the usual addition and multiplication of germs. The cases of $+\infty$ and $-\infty$ are clearly isomorphic, and if the Hardy field includes $\mathbf{R}$ then all the other cases are isomorphic to that of neighborhoods of 0^+ in $\mathbf{R}$. By [3, (*)], the asymptotic couple corresponding to a Hardy field that contains $\mathbf{R}$ is of Hardy type.

PROPOSITION 11. *Let k be a Hardy field that contains $\mathbf{R}$ and let (Γ, ψ) be its associated asymptotic couple. If k consists of germs of functions on $\mathbf{R}$ near $+\infty$, then for each positive $\alpha \in \Gamma$ and positive integer n we have $\psi(\alpha) + \alpha/n > 0$. If k consists of germs of functions on $\mathbf{R}$ near 0^+, then for each $\alpha \in \Gamma^*$ we have $\psi(\alpha) < 0$. If k is a Hardy field of both types and $\psi(\Gamma^*)$ is a finite set, then $k = \mathbf{R}$.*

Recall that the adjunction to a Hardy field of a primitive, or the exponential, of one of its elements gives another Hardy field [1, p. 109]; in particular, if x is the coordinate function on $\mathbf{R}$, then in each case $k(x)$ is a Hardy field of the same type as k. First let k consist of germs near $+\infty$ and let α, n be as above. Choose $y \in k^*$ such that $\nu(y) = \alpha$, where ν is the natural differential valuation of a Hardy field. Then $k(|y|^{1/n})$ is also a Hardy field of germs near $+\infty$ and $\nu(|y|^{1/n}) > 0 > \nu(x)$. Therefore $\nu((|y|^{1/n})') > \nu(x') = 0$, so that $\psi(\nu(|y|^{1/n})) + \nu(|y|^{1/n}) > 0$, or $\psi(\alpha) + \alpha/n > 0$. If now k consists of germs near 0^+ and α is as above, again choose $y \in k^*$ such that $\nu(y) = \alpha$. Then $k(\log|y|)$ is a Hardy field. Since $\log|y|$ has large absolute value near 0^+, we have $\nu(\log|y|) < 0 < \nu(x)$, so that $\nu((\log|y|)') < \nu(x') = 0$. Hence $\nu(y'/y) < 0$, or $\psi(\alpha) < 0$. Finally suppose that k is a Hardy field of both types and that $\psi(\Gamma^*)$ is finite. Then for $\alpha \in \Gamma^*$ we have $\psi(\alpha) + \alpha$ positive or negative according as α is positive or negative, hence never zero. By Proposition 7, if $\Gamma \neq \{0\}$ then $\max \psi(\Gamma^*) = 0$, a contradiction. Thus $\Gamma = \{0\}$ and $k = \mathbf{R}$.

LEMMA 1. *Let* (Γ, ψ) *be an asymptotic couple of Hardy type, with* Γ *of finite rank, and let* Γ_0 *be a subgroup of* Γ *of the same rank that contains* $\psi(\Gamma^*)$. *Let* k *be a Hardy field containing* $\mathbf{R}$ *whose corresponding asymptotic couple is* $(\Gamma_0, \psi|\Gamma_0)$. *Then* k *is contained in a Hardy field* K *whose asymptotic couple is* $(\Gamma_1, \psi|\Gamma_1)$, *where* Γ_1 *is a subgroup of* Γ *that contains* Γ_0 *and is such that* $\Gamma \subset \mathbf{Q}\Gamma_1$. *Furthermore, letting* ν *denote the valuation associated with any Hardy field, letting* rank $\Gamma = n$, *and letting* $y_1, \ldots, y_n$ *be fixed positive elements of* k *such that* $\nu(y_n) > 0$ *and* $\nu(y_i) > \nu(y_{i+1}^N)$ *for each* $i = 1, \ldots, n-1$ *and all* $N \in \mathbf{Z}$, *there exists a subset* $A \subset \{1, \ldots, n\} \times \mathbf{R}$ *such that the field* K *may be taken to be* $k(\{y_i^s\}_{(i,s)\in A})$, *with* $\nu(K^*)$ *generated by* $\nu(k^*)$ *and* $\{\nu(y_i^s)\}_{(i,s)\in A}$.

Let k, ν, n be as given. Since $\nu(k^*) = \Gamma_0$ has rank n, we can find $y_1, \ldots, y_n \in k^*$ whose values are decreasing, incomparable, and positive, as indicated. Recall that adjoining to a Hardy field a real power of one of its positive elements gives a bigger Hardy field. Now consider subsets $A \subset \{1, \ldots, n\} \times \mathbf{R}$ such that the Hardy field $k_A = k(\{y_i^s\}_{(i,s)\in A})$ has value group Γ_A generated by $\nu(k^*)$ and $\{\nu(y_i^s)\}_{(i,s)\in A}$, together with order-preserving embeddings $\Gamma_A \to \Gamma$ which extend the identity map on Γ_0. Since rank is respected, such embeddings $\Gamma_A \to \Gamma$ correspond to embeddings in the category of asymptotic couples. Applying Zorn's lemma to the set of pairs of A's and embeddings $\Gamma_A \to \Gamma$ gives a maximal A, and we claim that for this A, the field $K = k_A$ meets our demands. To prove this, it suffices to assume that $k_A = k$. Then for any $i = 1, \ldots, n$ and $s \in \mathbf{R}$ we have either $y_i^s \in k$, or $\nu((k(y_i^s))^*) \neq \nu(k^*) + \mathbf{Z}\nu(y_i^s)$, or there is no order-embedding of $\nu((k(y_i^s))^*)$ into Γ which extends the given isomorphism between $\nu(k^*)$ and Γ_0, and under these conditions we must show that $\Gamma \subset \mathbf{Q}\Gamma_0$. We shall assume that $\Gamma \not\subset \mathbf{Q}\Gamma_0$ and derive a contradiction. Let $\Gamma =$

$\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_{n+1} = \{0\}$ be the chain of convex subgroups of Γ. For each $i = 1, \ldots, n$ we have $\nu(y_i) \in \Delta_i - \Delta_{i+1}$. Let $d \in \{1, \ldots, n\}$ be the largest integer such that $\Delta_d \not\subset \mathbf{Q}\Gamma_0$. Then $\Delta_{d+1} \subset \mathbf{Q}\Gamma_0$. Let $\tau: \Delta_d \to \mathbf{R}$ be the order-preserving homomorphism with kernel Δ_{d+1}, normalized so that $\tau(\nu(y_d)) = 1$. Fix $\alpha \in \Delta_d - \mathbf{Q}\Gamma_0$ and let $c = \tau(\alpha)$. If $c \in \mathbf{Q}\tau(\Delta_d \cap \Gamma_0)$, then there is a positive integer N and an element $\delta \in \Delta_d \cap \Gamma_0$ such that $Nc = \tau(\delta)$, so that $N\alpha - \delta \in \Delta_{d+1} \subset \mathbf{Q}\Gamma_0$, so $\alpha \in \mathbf{Q}\Gamma_0$, which is false. Therefore $c \notin \mathbf{Q}\tau(\Delta_d \cap \Gamma_0)$. In particular, $c \notin \mathbf{Q}\tau(\nu(y_d)) = \mathbf{Q}$. Now consider the Hardy field $k(y_d^c)$. Suppose we have $\nu(ay_d^{mc}) = 0$ for some $a \in k^*$ and some nonzero $m \in \mathbf{Z}$. Then for $N \in \mathbf{Z}$, $N > |mc|$, the images under the place of the Hardy field $k(y_d^c)$ of y_d^{mc-N} and y_d^{mc+N} are respectively ∞ and 0, so that $\nu(y_d^{-N}) < \nu(y_d^{mc}) < \nu(y_d^N)$, which gives $|\nu(a)| < \nu(y_d^N) = N\nu(y_d)$, so that $\nu(a) \in \Delta_d$. If $p, q \in \mathbf{Z}$, $q > 0$ and $p/q < mc$, then $\nu(y_d^p) < \nu(y_d^{qmc})$, so that $\nu(a^q y_d^p) < 0$, so that $q\tau(\nu(a)) + p \leq 0$, so $\tau(\nu(a)) + p/q \leq 0$. Similarly, if $p, q \in \mathbf{Z}$, $q > 0$ and $p/q > mc$, then $\tau(\nu(a)) + p/q \geq 0$. Therefore $\tau(\nu(a)) + mc = 0$, contradicting $c \notin \mathbf{Q}\tau(\Delta_d \cap \Gamma_0)$. Thus for any $a \in k^*$ and any nonzero $m \in \mathbf{Z}$, we have $\nu(ay_d^{mc}) \neq 0$. Therefore the Hardy field $k(y_d^c)$, which is the field of quotients of the ring $k[y_d^c]$, each element of which can be uniquely written in the form $\Sigma_{m\in I} a_m y_d^{mc}$, where I is a finite set of nonnegative integers and each $a_m \in k^*$, has value group $\nu(k^*) \oplus \mathbf{Z}\nu(y_d^c) = \Gamma_0 \oplus \mathbf{Z}\nu(y_d^c)$, which is isomorphic to the subgroup $\Gamma_0 + \mathbf{Z}\alpha$ of Γ under the mapping which is the identity on Γ_0 and which sends $\nu(y_d^c)$ into α. We now wish to show that this isomorphism is order-preserving; this will contradict our assumptions on k and Γ and so complete the proof. It remains therefore only to show that if $a \in k^*$ and $m \in \mathbf{Z}$, then the statement $\nu(ay_d^{mc}) > 0$ implies $\nu(a) + m\alpha > 0$. This is clear if $m = 0$, so suppose $m \neq 0$. If $N \in \mathbf{Z}$ and $N > mc$ then $\nu(ay_d^N) > 0$, or $\nu(a) + N\nu(y_d) > 0$. Therefore the image of $\nu(a)$ in the homomorphism $\Gamma \to \Gamma/\Delta_d$ is nonnegative. If $\nu(a) \notin \Delta_d$, then $\nu(a) > N\nu(y_d)$ for all $N \in \mathbf{Z}$ and also $\nu(a) > N\alpha$ for all $N \in \mathbf{Z}$, so that $\nu(a) + m\alpha > 0$. If $\nu(a) \in \Delta_d$, then for any $p, q \in \mathbf{Z}$, $q > 0$, $p/q > mc$, we have $\nu(a^q y_d^p) > 0$, so that $q\nu(a) + p\nu(y_d) > 0$, so $q\tau(\nu(a)) + p > 0$, or $\tau(\nu(a)) + p/q > 0$; this implies that $\tau(\nu(a)) + mc \geq 0$, so that $\nu(a) + m\alpha > 0$. This completes the proof.

LEMMA 2. *Let (Γ, ψ) be an asymptotic couple of Hardy type, and let Γ_0 be a subgroup of Γ such that $\psi(\Gamma_0^*) \subset \Gamma_0$ and $\Gamma \subset \mathbf{Q}\Gamma_0$. Let k be a Hardy field containing* $\mathbf{R}$ *whose corresponding asymptotic couple is $(\Gamma_0, \psi | \Gamma_0)$, let ν denote the valuation associated with any Hardy field, and let S*

be a subgroup of the multiplicative group of positive elements of k such that $\nu(S) = \Gamma_0$. Then there exists a Hardy field K containing k such that if S' is the set of all positive elements of nonzero value of K some nonzero integral power of which is in S, then $K = k(S')$ and $\nu(K^*) = \nu(S') = \Gamma$.

This result is similar to Proposition 10, whose proof obtains here if we replace $\bar{k}$ by any Hardy field containing k which is closed under the process of taking positive n^{th} roots of positive elements, for any n, and we replace S_1 by its subset of positive elements of nonzero value.

THEOREM 3. *Let (Γ, ψ) be an asymptotic couple of Hardy type, with Γ of finite rank, and let Γ_0 be a nonzero subgroup of Γ such that $\psi(\Gamma_0) \subset \Gamma_0$. Let k be a Hardy field containing* **R** *whose corresponding asymptotic couple is $(\Gamma_0, \psi | \Gamma_0^*)$. Then there exists a Hardy field K containing k whose corresponding asymptotic couple is (Γ, ψ). Furthermore, K can be chosen to be an extension of k by repeated adjunctions of real powers of positive elements of nonzero value and logarithms and exponentials of positive elements of negative value, with the number of the last two types of extension being* rank Γ_1 − rank Γ_0.

If Γ_1 is a subgroup of Γ that contains Γ_0 and satisfies $\psi(\Gamma_1^*) \subset \Gamma_1$, then it suffices to prove the theorem for the passage from Γ_0 to Γ_1, and then for the passage from Γ_1 to Γ. Applying Theorem 1, it suffices to prove the theorem when $\psi(\Gamma^*) - \psi(\Gamma_0^*)$ consists of at most one element, that is, rank $\Gamma \leq$ rank $\Gamma_0 + 1$. The case rank $\Gamma =$ rank Γ_0 is handled by Lemmas 1 and 2 together. For the case where the ranks differ by one, we apply Proposition 8. Noting that the theorem has already been proved when the ranks are equal, we are reduced to the following two cases:

Case (1). $\Gamma = \Gamma_0 + \mathbf{Z}\theta$, where $\theta < 0$ and
$\psi(\theta) + \theta = \max \psi(\Gamma_0^*)$

Case (2). $\Gamma = \Gamma_0 + \mathbf{Z}\theta$, where $\theta < 0$ and
$\psi(\theta) \in \Gamma_0 - \psi(\Gamma_0^*)$.

First we remark that in each of these two cases, the asymptotic couple (Γ, ψ) is determined by the knowledge that it is of Hardy type and the given data for $(\Gamma_0, \psi | \Gamma_0)$ and θ. For if $\alpha \in \Gamma_0^*$ and $N \in \mathbf{Z}^*$, then $\psi(\alpha + N\theta) = \min\{\psi(\alpha), \psi(\theta)\}$, and this determines ψ on Γ; also, since $\psi(\alpha) \neq \psi(\theta)$, we have $|\alpha|$ and $|\theta|$ incomparable, and we know which is larger, and this determines whether $\alpha + N\theta$ is positive or negative, which determines the order on the group Γ. In case (1), choose $u \in k^*$ such that $\nu(u'/u) = \max \psi(\Gamma_0^*)$. We may assume u positive and $\nu(u) < 0$. Then K

$= k(\log u)$ is a Hardy field. For any $N \in \mathbf{Z}$, $(\log u)^N/u$ maps into 0 in the place of K, so that $\nu((\log u)^N/u) > 0$, so $\nu(1/u) > N\nu(1/\log u)$, and therefore $\nu(K^*) = \Gamma_0 + \nu(\log u)$. Since $\nu(\log u) < 0$ and $\psi(\nu(\log u)) + \nu(\log u) = \nu((\log u)') = \nu(u'/u) = \max \psi(\Gamma_0{}^*)$, the field K works for case (1). In case (2), by Proposition 7 there is a $\beta \in \Gamma_0{}^*$ such that $\psi(\theta) = \psi(\beta) + \beta$, and we must have $\beta < 0$. Here we choose $u \in k^*$ such that $\nu(u) = \beta$, with u positive and $\nu(u) < 0$. Here we take $K = k(e^u)$. Then if $a \in k^*$, $N \in \mathbf{Z}^*$ and $\nu(ae^{Nu}) = 0$, we deduce $\psi(\nu(a)) = \psi(\nu(e^u)) = \nu(u') = \psi(\nu(u)) + \nu(u) = \psi(\beta) + \beta = \psi(\theta) \notin \psi(\Gamma_0{}^*)$, a contradiction. Therefore $\nu(K^*) = \Gamma_0 + \mathbf{Z}\nu(e^u)$, with $\nu(e^u) < 0$ and $\psi(\nu(e^u)) = \psi(\theta)$. Hence the present field K works for case (2), and we are done.

Remark 1. If the group Γ/Γ_0 is finitely generated, the field K may be chosen a finite field extension of k, got by adjunction of elements of the appropriate type.

Remark 2. As in the corresponding Remark following Theorem 2, we may choose K so that it is obtained by just taking a finite extension of k of the appropriate type, then taking an extension of the type of Lemma 1, then an extension of the type of Lemma 2.

Remark 3. A slightly weaker version of Theorem 3 holds if the hypothesis $\Gamma_0 \neq \{0\}$ is omitted, exactly as in the corresponding Remark following Theorem 2.

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