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Apparent singularities of linear difference equations with polynomial coefficients

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Abstract Let L be a linear difference operator with polynomial coefficients. We consider singularities of L that correspond to roots of the trailing (resp. leading) coefficient of L . We prove that one can effectively construct a left multiple with polynomial coefficients \tilde{L} of L such that every singularity of \tilde{L} is a singularity of L that is not apparent. As a consequence, if all singularities of L are apparent, then L has a left multiple whose trailing and leading coefficients equal 1.

Keywords Linear Differences Equations · Linear Differential Equations · Apparent Singularities · Desingularization · Computer Algebra

1 Introduction

Investigation of singular points (also called singularities) of a linear differential or difference operator L gives an opportunity to study singularities of solutions of the equation $L(y) = 0$ without solving this equation.

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In the differential case, take $L = \sum_{i=0}^d a_i(z) \partial^i \in \mathbb{C}[z, \partial]$ where d is the order of L (so $a_d(z) \neq 0$) and $\partial = d/dz$. Let $S(L) \subset \mathbb{C}$ be the set of roots of $a_d(z)$. The finite singularities of L are the elements of $S(L)$. The singularity at ∞ will not be considered in this paper. If $p \in S(L)$ and if there exist d linearly independent holomorphic solutions at $z = p$ then p is called an *apparent* singularity. Suppose L has apparent singularities. The question is if it is possible to construct another operator $\tilde{L} \in \mathbb{C}[z, \partial]$ of higher order such that any solution of $L(y) = 0$ is a solution of $\tilde{L}(y) = 0$, and $S(\tilde{L}) = \{p \in S(L) \mid p \text{ not apparent}\}$. In the differential case the answer is affirmative, see [13]. In this paper we give the affirmative answer to the corresponding question for the difference case.

In the remainder of this paper (except the appendix) only the difference case will be considered. The shift operator E acts on functions of the complex variable z as $Ey(z) = y(z + 1)$. We consider non-commutative operator rings $\mathbb{C}[z, E]$ and $\mathbb{C}(z)[E]$ (the rings of linear difference operators with polynomial and, resp., rational function coefficients over \mathbb{C}). Let

$$L = a_d(z)E^d + \cdots + a_1(z)E + a_0(z) \in \mathbb{C}[z, E]. \quad (1)$$

Assume that the leading coefficient $a_d(z)$ and the trailing coefficient $a_0(z)$ are both non-zero, and that $a_0(z), \dots, a_d(z)$ do not have a non-constant common factor. Set $\text{ord } L = d$.

Definition 1 A root p of $a_0(z)$ is called a *t-singularity* (a trailing singularity). A root p of $a_d(z - d)$ is called an *l-singularity* (a leading singularity).

Definition 2 A *right-holomorphic* (resp. *left-holomorphic*) function is a meromorphic function on \mathbb{C} that is holomorphic on some right (resp. left) half plane. In other words, holomorphic when $\text{Re } z$ (resp. $-\text{Re } z$) is sufficiently large. A *half-holomorphic* function is a function that is right- or left-holomorphic.

Definition 3 A root p of $a_0(z)$ (resp. of $a_d(z - d)$) is called an *apparent t-*(resp. *l-*)singularity if no right-(resp. left)-holomorphic solution has a pole at p . An operator \tilde{L} is a *t-*(resp. *l-*)desingularization of L if every meromorphic solution of L is a solution of \tilde{L} , and every *t-*(resp. *l-*)singularity of \tilde{L} is a *t-*(resp. *l-*)singularity of L that is not apparent.

We show that both *t-* and *l-*desingularizations exist. We give algorithms `t-desing` and `l-desing` for constructing a *t-*(resp. *l-*)desingularization and algorithm `desingboth` for constructing a desingularization related to both trailing and leading coefficients.

The above definition of a desingularization is not the same as in [2] (see the summary of [2] given in Section 4.2).

Our approach is based on some specific properties of apparent singularities that are proved in this paper (Propositions 3,5).

Besides of theoretical interest, it is useful to have a desingularization \tilde{L} of L for solving the continuation problem. The equation $L(y) = 0$ can be used as a tool to define a sequence or a function. If we know the value $y(z)$ at every point z of a given strip $\lambda \leq \text{Re } z < \lambda + \delta$, where δ is larger or equal to the order of L then we can find the value of $y(z)$ in the strip $\lambda - 1 \leq \text{Re } z < \lambda$, and then in the strip $\lambda - 2 \leq \text{Re } z < \lambda - 1$, and so on. We can keep continuing $y(z)$ to the left

in this way except when we encounter t -singularities. Similarly, we can continue $y(z)$ to the right except at l -singularities. Thus, the singularities of L may present obstacles to continuing solutions of $L(y) = 0$. If the singularities are apparent, one can overcome those obstacles using \tilde{L} instead of L .

Some computer algebra algorithms for computing closed-form solutions of equations of the form $Ly = 0$, where L is as in (1), begin by constructing an “ansatz” or template that contains undetermined components. Substituting such a template into the original equation one gets some relations which can be used to find those components if they exist. In some cases the initial form of the template can be constructed using only the leading and trailing coefficients $a_d(z)$, $a_0(z)$, and the order d of L . For example, the algorithm `Hyper` for constructing hypergeometric solutions [11] looks for solutions $T(z)$ such that

$$\frac{T(z+1)}{T(z)} = \zeta \frac{a(z)}{b(z)} \frac{c(z+1)}{c(z)},$$

where a, b, c are monic polynomials, ζ is a constant and $a(z)|a_0(z)$, $b(z)|a_d(z-d+1)$. This gives a finite set of such templates, and for each of them the algorithm tries to find the corresponding ζ and c . The algorithm from [1] looks for rational solutions of the form $\frac{P(z)}{U(z)}$, where $U(z)$ is a “universal denominator” which can be constructed using $a_d(z-d)$ and $a_0(z)$. The template is substituted into the original equation and then the algorithm tries to find all the corresponding polynomials $P(z)$. Since any solution of the original equation is a solution of a desingularization of this equation, such algorithms could use a desingularization (whose leading and trailing coefficients can be much simpler than those of the original equation) for constructing the templates, while the templates themselves are substituted into the original equation (whose order may be less than the order of a desingularization).

Another use of desingularization is the following: In the process of continuing sequences to the left (resp. to the right) one must always divide by the trailing (resp. leading) coefficient. If all t -(resp. l)-singularities are apparent, then one can avoid such divisions by computing a desingularization, see Sect. 4.1 for an application. However, there is a price to pay, namely that the order increases.

In the appendix we give algorithm `∂ -desing` for desingularization in the differential case. More general results for the differential case can be found in [13], where $\mathbb{C}[z, \partial] \cap \mathbb{C}(z)[\partial]L$ is computed. We include this appendix for completeness and because the proof is short.

2 The sets $C_{q,\sigma}(L)$ and $R_{q,\sigma}(L)$

2.1 The set of singularities

We consider a linear difference operator (1). The set of trailing (resp. leading) singularities is $S_t(L) = \{p \in \mathbb{C} \mid a_0(p) = 0\}$ (resp. $S_l(L) = \{p \in \mathbb{C} \mid a_d(p-d) = 0\}$). The set of singularities is $S(L) = S_t(L) \cup S_l(L)$.

A point $p \in \mathbb{C}$ is said to be *congruent* to a t -(resp. l)-singularity of L if $a_0(p+v) = 0$ (resp. $a_d(p-d-v) = 0$) for some $v \in \mathbb{N}$. Let L be of the form (1) and $\sigma_1, \dots, \sigma_m$ be all singularities of L . Set

$$\iota(L) = \min\{\infty, \operatorname{Re} \sigma_1, \dots, \operatorname{Re} \sigma_m\}, \quad \kappa(L) = \max\{-\infty, \operatorname{Re} \sigma_1, \dots, \operatorname{Re} \sigma_m\}. \quad (2)$$

So L has no singularity in the half-planes $\operatorname{Re} z < \iota(L)$ and $\operatorname{Re} z > \kappa(L)$.

Note that any solution $F(z)$ of $L(y) = 0$ which is defined and holomorphic on a half-plane $\operatorname{Re} z > \kappa(L)$ (or, resp., $\operatorname{Re} z < \iota(L)$), can be continued to a meromorphic solution defined on \mathbb{C} whose poles are congruent to t -(resp. l)-singularities.

Starting from this point until the end of Section 3.2 we will consider only the t -singularities (the l -singularities can be handled similarly).

2.2 Systems of linear relations

Let Ω be a non-empty open subset of \mathbb{C} which is stable under E , i.e., $z \in \Omega$ implies $z + 1 \in \Omega$. For example let Ω be the right half-plane $\operatorname{Re} z > \kappa(L)$. Let $\varphi : \Omega \rightarrow \mathbb{C}$ be an arbitrary holomorphic function. We associate to φ a new function $\hat{\varphi}$ whose values are formal series in ε :

$$\hat{\varphi} : \Omega \longrightarrow \mathbb{C}[[\varepsilon]], \quad \hat{\varphi}(z) = \sum_{v=0}^{\infty} \frac{\varphi^{(v)}(z)}{v!} \varepsilon^v.$$

(As usual, $\mathbb{C}[[\varepsilon]]$ is the ring of formal power series in ε and $\mathbb{C}((\varepsilon)) = \mathbb{C}[[\varepsilon]][\varepsilon^{-1}]$ is its field of fractions.) Here ε is a new variable, rather than a “small number”. Of course, when $\varepsilon \in \mathbb{C}$ with $|\varepsilon|$ small enough the formal series $\hat{\varphi}(z)$ converges and its sum is equal to $\varphi(z + \varepsilon)$.

If φ is a polynomial then we can identify $\hat{\varphi}(z) = \varphi(z + \varepsilon)$. The operator $L \in \mathbb{C}[z, E]$ has polynomial coefficients. We associate to L the operator

$$\hat{L} = \hat{a}_d(z)E^d + \cdots + \hat{a}_0(z) = a_d(z + \varepsilon)E^d + \cdots + a_0(z + \varepsilon)$$

which acts on functions $\Phi(z)$ whose values are elements of the field $\mathbb{C}((\varepsilon))$.

If $\Phi(q)$, $\Phi(q + 1)$, \dots , $\Phi(q + d - 1)$ are given elements of $\mathbb{C}((\varepsilon))$ for some $q \in \mathbb{C}$, then, by using the operator \hat{L} , one can compute series

$$\Phi(q - 1), \Phi(q - 2), \dots \quad (3)$$

An advantage of \hat{L} in comparison with L is that neither the leading nor the trailing coefficient of \hat{L} vanishes when z is any complex number. However, it may happen that for some positive integer m , the series $\Phi(q - m)$ contains terms with negative exponents of ε even when $\Phi(q)$, \dots , $\Phi(q + d - 1)$ are in $\mathbb{C}[[\varepsilon]]$. Indeed, if $a_0(q - m) = 0$ then the series $\hat{a}_0(q - m)$ is not invertible in the ring $\mathbb{C}[[\varepsilon]]$. The problem of defining complex values of solutions at singularities of L is transformed into the problem of obtaining an element of $\mathbb{C}[[\varepsilon]]$ (truncated in computations).

Let σ be some t -singularity of L . Now choose $q \in \sigma + \mathbb{N}$ for which $\operatorname{Re} q > \kappa(L)$. Let $\Phi(z)$ be a function whose values are elements of $\mathbb{C}((\varepsilon))$ and suppose that $\hat{L}(\Phi) = 0$. If the values of $\Phi(z)$ at points $q, q + 1, \dots, q + d - 1$ are in the ring $\mathbb{C}[[\varepsilon]]$

$$\begin{array}{ccccccc} \Phi(q) & = & F_{q,0} & + & F_{q,1}\varepsilon & + & F_{q,2}\varepsilon^2 & + \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & \\ \Phi(q + d - 1) & = & F_{q+d-1,0} & + & F_{q+d-1,1}\varepsilon & + & F_{q+d-1,2}\varepsilon^2 & + \cdots \end{array} \quad (4)$$

with $F_{i,j} \in \mathbb{C}$ then using the equality $\hat{L}(\Phi) = 0$ we can compute $\Phi(\sigma) \in \mathbb{C}((\varepsilon))$, and each coefficient of this series will be a linear form in a finite set of $F_{i,j}$. The series $\Phi(\sigma)$ may contain negative exponents of ε . We can find conditions on the coefficients $F_{i,j}$ in (4) that guarantee that $\Phi(\sigma)$ belongs also to the ring $\mathbb{C}[[\varepsilon]]$. Indeed, if we use the generic power series (4) for this computation and if we get some terms with negative exponents of ε , then after equating their coefficients to zero, we get a system of linear relations denoted as $C_{q,\sigma}(L)$. This system forms a necessary and sufficient condition that the initial conditions (4) lead to $\Phi(\sigma) \in \mathbb{C}[[\varepsilon]]$ when Φ satisfies $\hat{L}(\Phi) = 0$. Constructing $C_{q,\sigma}(L)$ can be carried out algorithmically, see also [3] and [6].

Example 1 For the operator

$$(z-1)zE^2 - (3z+7)(z-3)E + (z+2)(z+1)$$

we can take generic formal series

$$\begin{aligned}\Phi(4) &= F_{4,0} + F_{4,1}\varepsilon + F_{4,2}\varepsilon^2 + \cdots \\ \Phi(5) &= F_{5,0} + F_{5,1}\varepsilon + F_{5,2}\varepsilon^2 + \cdots\end{aligned}$$

Then the system $C_{4,-1}(L)$ is $\{20F_{4,0} - 39F_{5,0} = 0\}$. This means that if $\Phi(4)$ and $\Phi(5)$ are as above, and if Φ is a solution of \hat{L} , then $\Phi(-1) \in \mathbb{C}[[\varepsilon]]$ if and only if $20F_{4,0} - 39F_{5,0} = 0$.

Proposition 1 *With notations as above, if $C_{q,\sigma}(L) \neq \emptyset$ then $C_{q,\sigma}(L)$ contains a relation $f(F_{q,0}, F_{q+1,0}, \dots, F_{q+d-1,0}) = 0$, where f is a non-zero linear form in the variables $F_{q,0}, F_{q+1,0}, \dots, F_{q+d-1,0}$.*

Proof Write $\Phi(\sigma) = \sum_{i=-N}^{\infty} F_{\sigma,i} \varepsilon^i$ with $F_{\sigma,-N}$ not zero. Now $N > 0$ since $C_{q,\sigma}(L) \neq \emptyset$. Furthermore, N is the highest pole order of any solution of \hat{L} that has formal series as initial values at the points $q, q+1, \dots, q+d-1$.

Now $F_{\sigma,-N}$ can be written as a linear combination of the $F_{q+i,j}$ with $0 \leq i < d$ and $0 \leq j$. Suppose that some $F_{q+i,j}$ with $j > 0$ appears with non-zero coefficient. Now assign the value 1 for this $F_{q+i,j}$ and the value 0 for all $F_{q+i',j'}$ with $(i', j') \neq (i, j)$. Then we have obtained a function Φ with initial values in $\{0, \varepsilon^j\}$ at $q, q+1, \dots, q+d-1$ and a pole of order N at σ . Dividing by ε^j , we get initial values in $\{0, 1\} \subset \mathbb{C}[[\varepsilon]]$ and a pole of order $N+j$ at σ which is a contradiction since $N+j > N$. \square

2.3 The set $R_{q,\sigma}(L)$

Let σ be some t -singularity of L and $q \in \sigma + \mathbb{Z}$. Let $\Phi(z)$ be a function whose values are rational functions of ε (we can expand them into power series if necessary), and suppose that $\hat{L}(\Phi) = 0$. If the values of $\Phi(z)$ at points $q, q+1, \dots, q+d-1$ are known rational functions of ε

$$\Phi(q), \Phi(q+1), \dots, \Phi(q+d-1), \quad (5)$$

then we can compute step-by-step the rational functions

$$\dots, \Phi(q-2), \Phi(q-1), \dots, \Phi(q+d), \Phi(q+d+1), \dots \quad (6)$$

(in particular $\Phi(\sigma)$) using \hat{L} . Consider the following d -tuples

$$(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1) \quad (7)$$

as d sets of initial values (5). For each set of initial values one obtains a sequence of the form (6) so we get rational functions $\Phi_i(\sigma)$, $i = 1, \dots, d$. Let

$$R_{q,\sigma}(L) = \{\Phi_1(\sigma), \dots, \Phi_d(\sigma)\};$$

so $R_{q,\sigma}(L)$ consists of d rational functions of ε .

Proposition 2 *With notations as above, if $\operatorname{Re} q > \kappa(L)$ and $C_{q,\sigma}(L) = \emptyset$ then no rational function from $R_{q,\sigma}(L)$ has a pole at $\varepsilon = 0$.*

Proof Each of the d -tuples (7) is a d -tuple of formal series (whose non-constant terms are equal to 0). Any rational function from $R_{q,\sigma}(L)$ can be represented as an element of $\mathbb{C}((\varepsilon))$. If $C_{q,\sigma}(L) = \emptyset$, then every such series has to be in $\mathbb{C}[[\varepsilon]]$. \square

Remark 1 Let $q \in \sigma + \mathbb{N}$. Set $q' = \sigma + n + 1$ where $n \in \mathbb{Z}$ is some integer for which there exists no integer $i > n$ with $a_d(\sigma + i) = 0$. Suppose that $q - q' \geq 0$. If Φ is a solution of \hat{L} , then $\Phi(q), \dots, \Phi(q + d - 1)$ can be computed from $\Phi(q'), \dots, \Phi(q' + d - 1)$ without dividing by ε . This implies that no rational function from $R_{q',\sigma}(L)$ has a pole at $\varepsilon = 0$.

3 Existence of a desingularization; algorithms

3.1 The sets $C_{q,\sigma}(L)$ and $R_{q,\sigma}(L)$ in the case of apparent singularities

We will use the following known result (see [7, Proposition 4.4 and Theorem 4.5]).

Theorem 1 (Ramis [12]; Barkatou [4]; Immink [7]) *The difference equation $L(y) = 0$ admits linearly independent meromorphic solutions F_1, \dots, F_d that are holomorphic in the right half-plane $\operatorname{Re} z > \kappa(L)$. Moreover, for some sufficiently large integer N , the sequences $\{F_j(n)\}_{n>N}$, $j = 1, \dots, d$, are linearly independent, i.e.,*

$$\begin{vmatrix} F_1(n) & \dots & F_1(n+d-1) \\ \vdots & & \vdots \\ F_d(n) & \dots & F_d(n+d-1) \end{vmatrix} \neq 0 \quad (8)$$

for all $n > N$.

As a consequence of Theorem 1 we get that if L is of the form (1), then for any complex number q with $\operatorname{Re} q$ large enough, there exist meromorphic solutions F_1, \dots, F_d that are holomorphic in the half plane $\operatorname{Re} z > \kappa(L)$, such that

$$\begin{vmatrix} F_1(q) & \dots & F_1(q+d-1) \\ \vdots & & \vdots \\ F_d(q) & \dots & F_d(q+d-1) \end{vmatrix} \neq 0. \quad (9)$$

Proposition 3 *Let σ be an apparent t -singularity of $L \in \mathbb{C}[z, E]$. Then $C_{q,\sigma}(L) = \emptyset$ for all $q \in \sigma + \mathbb{N}$ with $\operatorname{Re} q > \kappa(L)$.*

Proof Suppose that $q \in \sigma + \mathbb{N}$, $\operatorname{Re} q > \kappa(L)$ and $C_{q,\sigma}(L) \neq \emptyset$. Then for any q' such that $q' - q \in \mathbb{N}$ we have $C_{q',\sigma}(L) \neq \emptyset$, since if Φ is a solution of \hat{L} , then $\Phi(q'), \dots, \Phi(q' + d - 1)$ can be computed from $\Phi(q), \dots, \Phi(q + d - 1)$ without dividing by ε (see Remark 1 with q, q' interchanged). So we can assume that $\operatorname{Re} q$ is larger than any given; real number. By Definition 3 the functions F_1, \dots, F_d mentioned in the consequence of Theorem 1 have no pole at σ . If $C_{q,\sigma}(L) \neq \emptyset$ then by Proposition 1 there exists a non-trivial relation of the form $u_0 F_{q,0} + u_1 F_{q+1,0} + \dots + u_{d-1} F_{q+d-1,0} = 0$ in $C_{q,\sigma}(L)$ and so the columns of (9) are linearly dependent which is a contradiction, because inequality (9) has to be valid for all q large enough. \square

Proposition 4 *Let σ be a t -singularity of $L \in \mathbb{C}[z, E]$, then $C_{q,\sigma}(L) = \emptyset$ for all $q \in \sigma + \mathbb{N}$ with $\operatorname{Re} q > \kappa(L)$ if and only if σ is apparent.*

Proof It is obvious that if $C_{q,\sigma}(L) = \emptyset$ then σ is apparent. The converse is true by Proposition 3. \square

As a consequence of Propositions 2, 3 we get

Proposition 5 *Let σ be an apparent t -singularity of $L \in \mathbb{C}[z, E]$, $q \in \sigma + \mathbb{N}$, $\operatorname{Re} q > \kappa(L)$. Then no rational function from $R_{q,\sigma}(L)$ has a pole at $\varepsilon = 0$.*

3.2 t - and l -desingularizations

Algorithm `t-desing`.

Input: $L \in \mathbb{C}[z, E]$ with non-zero E^0 coefficient.

Output: A t -desingularization of L .

1. Let $a_0, a_d \in \mathbb{C}[z]$ be the trailing and leading coefficient of L .
2. Let $n \in \mathbb{N}$ be the *dispersion* of a_d, a_0 , which is the largest integer such that a_d has some root that equals n plus some root of a_0 . If such $n \in \mathbb{N}$ does not exist then set n equal 0.
3. Set $L_1 := \frac{1}{a_0} L \in \mathbb{C}(z)[E]$.
4. For i from 1 to n , clear the E^i coefficient of L_1 by subtracting $CE^i \frac{1}{a_0}$ from L_1 where c is the E^i coefficient of L_1 .
5. Let $b_0 \in \mathbb{C}[z]$ be the least common multiple of the denominators of the coefficients of L_1 , and set $L_2 := b_0 L_1$.
6. Compute $s, t \in \mathbb{C}[z]$ for which $sa_0 + tb_0 = \gcd(a_0, b_0)$.
7. Output: $sL + tL_2$.

The dispersion of a_d, a_0 (Step 2) can be computed as the largest integer root of the resultant $r(h) = \operatorname{Res}_z(a_0(z+h), a_d(z))$, or using the full factorization of a_d, a_0 (which makes sense when, e.g., the polynomials a_d, a_0 have rational-number coefficients) [9].

Theorem 2 *The algorithm `t-desing` produces a t -desingularization of L .*

Proof Let σ be an apparent t -singularity. If n is a non-negative integer then there exists precisely one operator of the form

$$L_1 = r_{n+d}(z)E^{n+d} + \cdots + r_{n+1}(z)E^{n+1} + 1 \in \mathbb{C}(z)[E]$$

that is right-divisible by L (recall that $d = \text{ord } L$). Algorithm `t-desing` computes this operator L_1 where n is the dispersion of a_d and a_0 .

Let $\Phi(z)$ be a function whose values are rational functions of ε . If $\hat{L}(\Phi) = 0$, then we have also $\hat{L}_1(\Phi) = 0$ and so

$$\Phi(\sigma) = - \sum_{i=1}^d \hat{r}_{n+i}(\sigma) \Phi(\sigma + n + i).$$

By taking the initial values $(1, 0, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, etc., for $\Phi(\sigma + n + i)$, $i = 1, \dots, d$, we obtain the rational functions in the set $R_{\sigma+n+1,\sigma}(L)$ on the left-hand side of this equation, and the rational functions $-\hat{r}_{n+i}(\sigma)$ on the right-hand side. Since σ is apparent, the rational functions in the set $R_{\sigma+n+1,\sigma}(L)$ have no pole at $\varepsilon = 0$ by Proposition 5 when n is sufficiently large (the dispersion is large enough by Remark 1). Hence, the $-\hat{r}_{n+i}$ have no pole at σ . Thus, σ is not a root of the denominators of the coefficients of $L_1 \in \mathbb{C}(z)[E]$. The least common multiple of these denominators equals the trailing coefficient b_0 of L_2 , so σ is not a root of b_0 . The trailing coefficient of the output is $sa_0 + tb_0 = \gcd(a_0, b_0)$. Since this is a factor of a_0 , the t -singularities of the output form a subset of the t -singularities of L . And since this gcd divides b_0 , it follows that σ is not a t -singularity of the output. The same argument applies to every apparent t -singularity σ , and hence the output is a t -desingularization of L . \square

Therefore the following theorem (the main theorem for t -singularities) is proven:

Theorem 3 *Every $L \in \mathbb{C}[z, E]$ is t -desingularizable (in other words, there exists a t -desingularization \tilde{L} of L).*

Example 2 For the operator

$$L = (2z - 1)(z - 1)E^2 + (5z - 1 - 9z^2 + 2z^3)E + z(1 + 2z),$$

algorithm `t-desing` returns: $\frac{1}{3}(-1 + 4z)(5z - 1 - 9z^2 + 2z^3)E^3 + (\frac{26}{3}z^2 - \frac{43}{3}z + 11/3 + \frac{85}{3}z^3 - 18z^4 + 8/3z^5)E^2 + \frac{1}{3}(7 + 4z)(5z - 1 - 9z^2 + 2z^3)E + 1$.

If the dispersion, the number n in the algorithm, is not positive then it follows from Theorem 2 that no t -singularity is apparent. This fact already followed from the approach in [2], see the summary of [2] given in Section 4.2.

Definition 4 We call \tilde{L} a *complete t -desingularization* of L if $\tilde{L} \in \mathbb{C}[z, E]$ is right-divisible by L and its trailing coefficient is a non-zero constant. If a complete t -desingularization of L exists, then we say that L is *completely t -desingularizable*.

Theorem 4 *A complete t -desingularization of L exists if and only if all t -singularities of L are apparent.*

Proof Suppose a complete t -desingularization \tilde{L} exists. If $F(z)$ is a right-holomorphic solution of L , then it is also a right-holomorphic solution of \tilde{L} . Then $F(z)$ must be holomorphic since the trailing coefficient of \tilde{L} is a non-zero constant. Hence all t -singularities of L are apparent. Conversely, if all t -singularities of L are apparent then a complete t -desingularization exists by Theorem 2. \square

Algorithm `t-desing` removes at least all apparent t -singularities by Theorem 2. If a complete t -desingularization exists, in other words, if all t -singularities can be removed, then the above theorem shows that algorithm `t-desing` will do so. Besides removing all apparent singularities our algorithm may also remove some non-apparent singularities. However, this does not imply that `t-desing` always removes as many t -singularities as possible. This is illustrated in the next example.

Example 3 Let $L = (z + 2)^2(z - 1)^2E - (z + 1)z(z - 2)^2$. The t -singularities of L are $-1, 0, 2$, none of which are apparent. The application of `t-desing` to L gives

$$-(z + 3)(z + 4)^2E^3 + z(z - 2)^2.$$

So one t -singularity disappeared even though no t -singularity was apparent. Note that the t -singularity 0 can be removed as well: the operator

$$4(z + 4)^2E^3 - 3z(z + 3)(z + 4)E^2 + 3(z + 2)(z - 1)^2E + 2(z - 2)^2$$

is right-divisible by L . The t -singularity $z - 2$ cannot be removed, because if all t -singularities could be removed then all t -singularities would have to be apparent, and this is not the case.

We also implemented an algorithm that removes all singularities that can be removed, by reducing this problem to a linear algebra problem over the constants. This implementation is available at:

<http://www.math.fsu.edu/~hoeij/papers/desing/> and tends to produce nicer desingularizations than `t-desing`. We used `t-desing` in this paper because it is shorter than the linear algebra based desingularization algorithm, and because the proof that all apparent singularities can be removed is easier with `t-desing`.

Example 4 For the operator

$$L = (z - 3)(z - 2)E + z(z - 1) \quad (10)$$

we get $C_{4,1}(L) = C_{4,0}(L) = \emptyset$. So L must be completely t -desingularizable; algorithm `t-desing` returns: $\frac{1}{72}(5z - 6)(z - 3)(z - 2)^2(z - 1)E^4 + \frac{1}{72}(108 + 106z + 5z^3 + 39z^2)(z - 3)(z - 2)E + 1$. Note that nicer desingularizations are possible, for example $(E + 1)^5$ of order 5, or $-(z + 1)E^4 + (17z - 29)E^3 + (17z + 56)E^2 + (-z + 5)E + 1$ of order 4.

The operator L from Example 1 is not completely desingularizable because $C_{4,-1}(L) \neq \emptyset$ there. The output of our algorithm applied to the operator L from Example 1 is L itself.

For definiteness, we considered the trailing singularities. If one already has an implementation of a t -desingularization algorithm, then one can obtain an l -desingularization algorithm by changing a small number of lines. However, one can avoid this duplication of code because one can reduce l -desingularization to t -desingularization (and get the algorithm `l-desing`) with the following trick: interchange the roles of the leading and trailing coefficient by using the automorphism of $\mathbb{C}[z, E, E^{-1}]$ given by $z \mapsto -z, E \mapsto E^{-1}$. This trick was used in an implementation [10] of `ds`, an old naive desingularization algorithm [2].

3.3 An operator which is a desingularization of L related to both trailing and leading coefficients

So far we considered mainly trailing apparent singularities. Note that Theorem 3 is valid, *mutatis mutandis*, for leading apparent singularities and a desingularization related to the leading coefficient. The prefix “ lt –” indicates that we consider both leading and trailing singularities, just like l - and t - indicate leading and trailing. The following theorem generalizes Theorem 3.

Theorem 5 (*The main theorem.*) Any $L \in \mathbb{C}[z, E]$ is lt -desingularizable.

Proof Let $R_t, R_l \in \mathbb{C}(z)[E]$, $L_t, L_l \in \mathbb{C}[z, E]$ be such that $L_t = R_t L$ (resp. $L_l = R_l L$) is a t -(resp. l)-desingularization of L . Consider the operator

$$\tilde{L} = L_t + E^m L_l, \quad (11)$$

where $m = \max\{1, \text{ord } L_t - \text{ord } L_l + 1\}$. It is clear that \tilde{L} belongs to $\mathbb{C}[z, E]$, and \tilde{L} is right-divisible by L since $\tilde{L} = (R_t + E^m R_l)L$. The operator \tilde{L} is simultaneously a t - and an l -desingularization of L because its t -singularities are t -singularities of L_t and its l -singularities are l -singularities of L_l . \square

Analogously to the t -case an operator L is completely lt -desingularizable if and only if any half-holomorphic solution of $L(y) = 0$ is holomorphic on \mathbb{C} . In other words, L is completely lt -desingularizable if and only if all singularities (leading and trailing) are apparent.

Example 5 The operator $(z - 2)E - z$ has a complete lt -desingularization $E^3 - 3E^2 + 3E - 1$.

We name `desingboth` the algorithm given above. Thus, the algorithm is given in the following manner:

- Use algorithms `l-desing` and `t-desing` for constructing L_l and L_t .
- Return $\tilde{L} = L_t + E^m L_l$, where $m = \max\{1, \text{ord } L_t - \text{ord } L_l + 1\}$.

Example 6 For the operator L from Exp. 2, algorithm `l-desing` returns:

$$\begin{aligned} L_l = & E^3 - 1/3 (4z - 5) (-3 - 7z - 3z^2 + 2z^3) E^2 \\ & + \left(-8/3 z^5 + \frac{46}{3} z^4 - 11z^3 - \frac{98}{3} z^2 - 1/3 z + 9 \right) E \\ & - 1/3 (4z + 3) (-3 - 7z - 3z^2 + 2z^3) \end{aligned}$$

Algorithm `desingboth` returns:

$$\begin{aligned}\tilde{L} = & E^4 + \left(-16z^3 + 20z^2 + \frac{28}{3}z - 10/3\right)E^3 \\ & + \left(-16z^4 + 52z^3 + \frac{25}{3}z^2 - 65z - \frac{56}{3}\right)E^2 \\ & + \left(\frac{70}{3} + \frac{124}{3}z - 12z^2 - 16z^3\right)E + 1.\end{aligned}$$

4 An application, previous work, and a conjecture

4.1 An application of desingularization

As we have mentioned, it is useful to have a desingularization \tilde{L} of L for solving the continuation problem. Below we consider another application of the desingularization.

Suppose that the sequence $u(0), u(1), \dots$ satisfies the following relation

$$(1 + 16n)^2 u(n + 2) - (224 + 512n)u(n + 1) - (n + 1)(17 + 16n)^2 u(n) = 0$$

which corresponds to $L = (1 + 16z)^2 E^2 - (224 + 512z)E - (z + 1)(17 + 16z)^2$. Assume that $u(0), u(1) \in \mathbb{Z}$. By substituting $z = 0, 1, \dots$ in the relation we find:

$$u(2) = 289u(0) + 224u(1), \quad u(3) = 736u(0) + 578u(1), \dots$$

One sees that $u(2), u(3) \in \mathbb{Z}$ since we assumed that $u(0), u(1) \in \mathbb{Z}$. The question is now the following:

Prove that $u(n) \in \mathbb{Z}$ for every nonnegative integer n .

Each time we use L to compute the next term $u(n + 2)$ from the two previous terms $u(n), u(n + 1)$ we perform additions, multiplications, and one division, namely by the *leading coefficient* of L which is $(1 + 16n)^2$. How to prove that this division does not cause $u(n + 2)$ to become a fraction? If we use the algorithm `l-desing` (Sect. 3.2), then it produces the following operator:

$$\begin{aligned}L_l = & E^3 + \frac{1}{64} (16n + 23) (16n - 7) E^2 + \left(-9n - \frac{29}{2}\right) E \\ & - \frac{1}{64} (16n + 23) (16n + 25) (n + 1).\end{aligned}$$

Now $u(n + 3)$ can be computed from $u(n), u(n + 1), u(n + 2)$ using L_l . This can only introduce powers of 2 in the denominator of $u(n + 3)$ because L_l has only powers of 2 in the denominator and has leading coefficient 1. So the denominators in the sequence must be powers of 2, but must simultaneously be odd (and hence equal to 1) because the leading coefficient $(1 + 16n)^2$ of L is always odd. Hence $u(n) \in \mathbb{Z}$ for all nonnegative integers n .

4.2 Some remarks to algorithm `ds` in [2]

In this section we will review algorithm `ds` from [2] and compare it with algorithm `t-desing`. As it was mentioned above, the definition of a desingularization that we use in the present paper is not the same as in [2]. To avoid confusion, we will use prefixes t' - and l' - instead of t - , l - , when we discuss a desingularization in the sense of [2]. For self-containedness we recall the definition of a t' -desingularization. Consider an operator L of the form (1). Set $\mu_t(L) = \max(\{p \in \mathbb{Z} \mid a_0(p) = 0\} \cup \{-\infty\})$. An operator $M \in \mathbb{C}[z, E]$ is a t' -desingularization of L if

- (a) M is right divisible by L (i.e., $M = R_t L$ for some $R_t \in \mathbb{C}(z)[E]$) and the trailing coefficient of L is divisible in $\mathbb{C}[z]$ by the trailing coefficient of M ,
- (b) there is no operator $M' \in \mathbb{C}[z, E]$ such that M' is right-divisible by L , the trailing coefficient of L is divisible in $\mathbb{C}[z]$ by the trailing coefficient of M' , and $\mu_t(M') < \mu_t(M)$.

Consider the operator

$$L = (z - 1)(z - 2)(z + 1)E^2 + (z^5 - 3z^3 + 3z + 2)E + z^2(z + 2). \quad (12)$$

The trailing coefficient has integer roots $z = 0$ and $z = -2$. We would like to construct a t' -desingularization of L . We will start with the largest integer root first, so we first only consider the factor z^2 . The question is if there exists an operator $R_t \in \mathbb{C}(z)[E]$ for which $R_t L$ is in $\mathbb{C}[z, E]$ and has a trailing coefficient that is not divisible by z^2 , or even not divisible by z . If such $R_t \in \mathbb{C}(z)[E]$ exists, then there must exist an R_t with the same property in $z^{-1}\mathbb{C}[z^{-1}][E]$. One sees this by first replacing each coefficient of R_t in $\mathbb{C}(z)$ by its series expansion in $\mathbb{C}((z)) = \mathbb{C}[[z]][z^{-1}]$, and then throwing away all non-negative powers of z to end up with an element of $z^{-1}\mathbb{C}[z^{-1}]$. So, if an operator R_t with the desired property (that $R_t L$ is in $\mathbb{C}[z, E]$ and that the factor z^2 of the trailing coefficient has disappeared) exists in $\mathbb{C}(z)[E]$, then there exists an operator R_t in $z^{-1}\mathbb{C}[z^{-1}][E]$ with the same property. We can write this R_t as

$$R_t = \sum_{i=0}^N r_i E^i, \quad r_i = \sum_{j=-K}^{-1} c_{ij} z^j \quad (13)$$

for some non-negative integers N, K and some $c_{ij} \in \mathbb{C}$.

Suppose that there is a $c_{ij} \neq 0$ for some $j < -2$. Then take i minimal with this property. Now i can not be 0 because then the trailing coefficient of $R_t L$ can not be in $\mathbb{C}[z]$. The E^i coefficient of $r_i E^i L$ is $r_i(z + i)^2(z + 2 + i)$ which has a pole of order > 2 at $z = 0$ since $(z + i)^2(z + 2 + i)$ is not divisible by z (this is the reason we started with z^2 and not with $z + 2$). Then the E^i coefficient of $R_t L$ has a pole of order > 2 as well (the pole in the E^i term of $r_i E^i L$ can not cancel against the E^i terms of $r_0 E^0 L, \dots, r_{i-1} E^{i-1} L$ since those terms have pole orders ≤ 2 by the minimality of i). This means that $R_t L$ is not in $\mathbb{C}[z, E]$ which is a contradiction. It follows that there can be no $j < -2$ for which $c_{i,j} \neq 0$ for some i , thus we can take $K = 2$ without loss of generality.

Now take N minimal with $r_N \neq 0$. Then the leading coefficient of $R_t L$ is $r_N(z - 1 + N)(z - 2 + N)(z + 1 + N)$. This must be in $\mathbb{C}[z]$, however, r_N only has negative powers of z . It follows that $(z - 1 + N)(z - 2 + N)(z + 1 + N)$

must be divisible by z . Hence N can be no greater than the largest integer root of the leading coefficient, which is 2. So we can take $N = 2$. In the more general situation where we want to eliminate several roots of the trailing coefficient at the same time one can use the same argument to show that one may assume without loss of generality that the order of R_t is bounded by the dispersion (the largest root difference in \mathbb{Z}) of the leading and trailing coefficient of L .

We now see that if R_t exists, then we may assume it to be of the form in equation (13), and from the preceding we see that we may also assume $N = 2$ and $K = 2$. This turns the problem into a finite dimensional system of linear equations for the c_{ij} . Solving this system decides whether or not the factor z^2 can be removed, and if so, how to do this. We find the following solution

$$R_t = \frac{-1}{12z}E^2 + \left(\frac{5}{9z} - \frac{2}{3z^2}\right)E + \frac{1}{z^2}.$$

Then $R_t L$ is in $\mathbb{C}[z, E]$ and has trailing coefficient $z + 2$. Algorithm `ds` in [2] finds this R_t in a slightly different way. Write $R_t = \sum r_i E^i = z^{-2} \sum \tilde{r}_i E^i$ where $\tilde{r}_i = z^2 r_i \in \mathbb{C}[z]$. We can now work modulo z^2 , so we can view \tilde{r}_i as an element of $\mathbb{C}[z]/(z^2)$. Now equate $(\sum \tilde{r}_i E^i)L$ to zero modulo z^2 . This leads to a system of linear equations for the \tilde{r}_i . This system is already in a triangular form so it can be solved quickly by applying Gaussian elimination over $\mathbb{C}[z]/(z^2)$. If we set $\tilde{r}_0 := 1$ then one first finds $\tilde{r}_1 = (5/9)z - 2/3$. This involves a division in $\mathbb{C}[z]/(z^2)$, namely one divides by $a_0(z + 1)$ where $a_0(z) = z^2(z + 2)$ is the trailing coefficient of L . After that one computes $\tilde{r}_2 = -z/12$, which involves a division by $a_0(z + 2)$.

After eliminating z^2 we can apply a similar process to the factor $z + 2$ in the trailing coefficient. Then one obtains linear equations over $\mathbb{C}[z]/(z + 2)$ instead of over $\mathbb{C}[z]/(z^2)$ and one can proceed along the same lines, see [2] for details. It turns out that $z + 2$ can be removed as well.

Algorithm `ds` tries to remove one root from the trailing coefficient at a time. If a root can not be removed, that is, if we encounter a non-apparent t -singularity, then algorithm `ds` stops, and later roots will not be removed even if some of them correspond to apparent t -singularities. The algorithm `t-desing` presented in this paper has the advantage that it removes all apparent t -singularities, even if there are non-apparent t -singularities between them. Furthermore, it is shorter than algorithm `ds` and does not need to compute with roots of a_0 . Algorithm `t-desing` is also simpler than differential desingularization (see Appendix) since it does not need to know which singularities are apparent.

The paper [2] gives a proof that the so-called ε -criterion is an alternative way to decide if a desingularization exists (from the viewpoint of [2]). The ε -criterion is based on a construction similar to $R_{q,\sigma}(L)$.

We return to the notion of a desingularization in terms of Definition 3.

4.3 A conjecture

Let $L \in \mathbb{Q}[z, E]$ with leading and trailing coefficient $a_d(z)$ and $a_0(d)$. Let N_1 be an integer larger than any integer root of $a_d(z - d)a_0(z)$ and let N_2 be an integer smaller than any integer root of $a_d(z - d)a_0(z)$. Then there exist d linearly independent sequences $u_1, \dots, u_d : N_1 + \mathbb{N} \rightarrow \mathbb{Q}$ and d linearly independent

sequences $v_1, \dots, v_d : N_2 - \mathbb{N} \rightarrow \mathbb{Q}$ that satisfy the recurrence given by L . Let P denote the set of all prime numbers p that occur as a factor in a denominator of a number that appears in a sequence u_i or v_i . So p is not in P if and only if every entry of every u_i and v_i does not have p in the denominator. We call L *smooth* if the set P is finite. Note that the set P may depend on the choices of the u_i and v_i , but whether P is finite or not only depends on L .

Conjecture L is smooth if and only if L is completely lt -desingularizable.

The conjecture relates analytic properties to number theoretic properties, namely it states that sequence solutions (where we consider sequences that extend to the right as well as sequences that extend to the left) have only finitely many primes in the denominators iff all half-holomorphic solutions are holomorphic.

If L has a complete lt -desingularization \tilde{L} then we can extend sequences u_i to the right and sequences v_i to the left with \tilde{L} . This will only introduce finitely many primes in denominators (so L is smooth) since \tilde{L} has constant leading and trailing coefficient. This shows that the conjecture is true in one direction.

Appendix: The differential case

In this appendix we prove that the statement in Theorem 3 is also true in the differential case. The proof is essentially a desingularization algorithm (we name it ∂ -desing). An implementation can be found at:

<http://www.math.fsu.edu/~hoeij/papers/desing/> However, the results in this appendix are not new; more general results were given by Tsai in [13] (a desingularization is one of the elements in the output of Tsai's Weyl Closure algorithm). We include this appendix for completeness and because the proof is short since the result is less general than [13].

Let ∂ be the differentiation d/dz . If $L \in \mathbb{C}[z, \partial]$ and the coefficients of L do not have a non-constant common factor, then a *singular point* or *singularity* of L is a zero of the leading coefficient of L . If $L \in \mathbb{C}(z)[\partial]$ and L is monic

$$L = \partial^n + a_{n-1}(z)\partial^{n-1} + \dots + a_0(z), \quad (14)$$

then singularities are poles of a_i 's. We will consider differential operators in the form (14). Definitions of *regular singularity* and, resp., *irregular singularity* of a given operator L of the form (14) can be found, e.g., in [8]. A point that is not a singularity (neither regular nor irregular) is *ordinary*.

Definition 5 Let $L \in \mathbb{C}(z)[\partial]$. A singularity p of L is *apparent* if every solution of L is holomorphic at $z = p$ (in other words, p is an apparent singularity of L if the equation $L(y) = 0$ admits a fundamental set of solutions that are holomorphic in a neighborhood of p).

Definition 6 Let $L \in \mathbb{C}(z)[\partial]$ be monic. Then \tilde{L} is called a *desingularization* of L if it has L as a right-hand factor, and if for every $p \in \mathbb{C}$, if p is either an ordinary point of L or an apparent singularity of L , then p is an ordinary point of \tilde{L} .

Proposition 6 Let $L \in \mathbb{C}(z)[\partial]$ be monic and have order n , and let $p \in \mathbb{C}$. The following statements are equivalent.

- a) *There exist an open set U with $p \in U$ and a basis of solutions y_1, \dots, y_n of L , holomorphic on U , for which y_i vanishes at p with order $i - 1$.*
- b) *p is not an irregular singularity of L , the local exponents at p are $0, 1, \dots, n-1$, and the formal solutions of L at p are in $\mathbb{C}[[z - p]]$.*
- c) *p is not a pole of any of the coefficients in $\mathbb{C}(z)$ of L (i.e., p is an ordinary point of L).*

Proof a) \Rightarrow b) is clear; c) \Rightarrow a) is Cauchy's theorem; a proof for b) \Rightarrow c) can be found in [5, Lemma 9.2]. \square

Proposition 7 *Let $L \in \mathbb{C}(z)[\partial]$ be monic, have order n , and let $p \in \mathbb{C}$. The following statements are equivalent.*

- a) *p is either an ordinary point or an apparent singularity of L .*
- b) *L has n linearly independent solutions in $\mathbb{C}[[z - p]]$ (this is equivalent to: p is not an irregular singular point of L , the local exponents are non-negative integers and the formal solutions at p do not involve logarithms, see [5] for more details).*
- c) *There exists a monic operator $\tilde{L} \in \mathbb{C}(z)[\partial]$ that has L as a right-hand factor such that p is an ordinary point of \tilde{L} .*

Proof a) \Rightarrow b) is clear; c) \Rightarrow a) follows from Cauchy's theorem. For b) \Rightarrow c), let m be the highest local exponent. Consider the set V of all integers $0 \leq i \leq m$ for which i is not an exponent of L at p . Let L_1 be an operator with the following as basis of solutions: $L((z - p)^i)$, $i \in V$. Let $\tilde{L} = L_1 L$. Then \tilde{L} satisfies part a) of Proposition 6. \square

Remark 2 In general, solutions in $\mathbb{C}[[z - p]]$ need not be convergent, but b) \Rightarrow a) of Proposition 7 says that if *all* formal solutions of L are in $\mathbb{C}[[z - p]]$ then they are automatically convergent.

Theorem 6 *Every monic operator $L \in \mathbb{C}(z)[\partial]$ is desingularizable (in other words, there exists a desingularization \tilde{L} of L).*

Proof Let A be the set of all apparent singularities $p \in \mathbb{C}$. For $p \in A$, let $m(p)$ be the highest exponent at p . Let m be the maximum of all $m(p)$, and let $e(p) \subset \{0, 1, \dots, m\}$ be the set of exponents of L at p .

Take $p \in A$ for which $m = m(p)$. Suppose a desingularization \tilde{L} exists. The exponents of L at p must be a subset of the exponents of \tilde{L} at p because L is a right-hand factor of \tilde{L} . Hence m is an exponent of \tilde{L} at p , but since p is an ordinary point of \tilde{L} it follows that $0, 1, \dots, m$ are exponents of \tilde{L} at p as well, so the order of \tilde{L} must be at least $m + 1$.

We will now show that a desingularization \tilde{L} of order $m + 1$ exists. First, construct polynomials $y_1, \dots, y_{m+1-n} \in \mathbb{C}[z]$ such that for every $p \in A$ and every $i \in \{0, 1, \dots, m\} \setminus e(p)$ there is precisely one y_j that vanishes at p with order i . Let L_1 be the monic operator whose solutions are spanned by $L(y_1), \dots, L(y_{m+1-n})$. Then $L_1 L$ satisfies condition a) of Proposition 6 at every $p \in A$. Thus, p is an ordinary point of $L_1 L$ for every $p \in A$. However, $L_1 L$ need not satisfy the definition of a desingularization because we may have created new apparent singularities. Write $L_1 = \partial^{m+1-n} + \sum a_i \partial^i$ with $a_i \in \mathbb{C}(z)$. Take $b_i \in \mathbb{C}(z)$ as follows: b_i has

no poles in $\mathbb{C} \setminus A$, and $b_i - a_i$ must vanish at every $p \in A$ with order at least $M(p)$ for some integers $M(p)$. We take these integers $M(p)$ high enough to make sure that L_2L is an element of $\mathbb{C}(z)[\partial]$ whose coefficients in $\mathbb{C}(z)$ have no pole at any $p \in A$, where $L_2 = \sum (b_i - a_i)\partial^i$. Now let $L_3 = \partial^{m+1-n} + \sum b_i\partial^i$ and $\tilde{L} = L_3L$. Since the b_i have no poles in $\mathbb{C} \setminus A$, we see that every ordinary point in \mathbb{C} of L is an ordinary point of L_3 and hence an ordinary point of \tilde{L} . The operators L_1L and L_2L have coefficients that do not have poles at any $p \in A$, hence the same is true for $\tilde{L} = L_1L + L_2L$, and since \tilde{L} is monic it follows that every $p \in A$ is an ordinary point of \tilde{L} . \square

Example 7 Let L be the monic operator with $z \cos(z)$ and $z \sin(z)$ as basis of solutions. Then L has one singularity in \mathbb{C} , namely at $z = 0$, which is an apparent singularity with local exponents 1 and 2. Thus, to desingularize L we must add a solution with exponent 0. Take $y_1 = z^0 = 1$ and compute $L(y_1)$. The explicit form of L is

$$L = \partial^2 - \frac{2}{z}\partial + 1 + \frac{2}{z^2}$$

so we find $L(y_1) = 1 + 2/z^2$. Now let L_1 be the monic operator with $1 + 2/z^2$ as a basis of solutions, then

$$L_1 = \partial + \frac{4}{z(z^2 + 2)}.$$

Multiplying L on the left by L_1 adds a solution (namely $y_1 = z^0$) to L with the missing exponent 0. Then L_1L satisfies part b) of Proposition 6. Hence $z = 0$ is a regular point of L_1L . Unfortunately, L_1 introduces new singularities, namely at $z^2 + 2 = 0$. We will illustrate how to remedy this with a truncated Laurent series at $z = 0$. We have

$$L_1 = \partial + \frac{4}{z(z^2 + 2)} = \partial + 2z^{-1} - z + \frac{1}{2}z^3 - \frac{1}{4}z^5 + \dots$$

Since the highest power of z in the denominator of L is z^2 , we see that if we change any z^i -term in L_1 with $i \geq 2$ then this cannot introduce poles at $z = 0$ in the coefficients of L_1L . If we remove these terms from L_1 we get an operator $L_3 := \partial + 2z^{-1} - z$ for which the coefficients of L_3L still have no pole at $z = 0$. Since L and L_3 have no other singularities in \mathbb{C} apart from $z = 0$, we see that $\tilde{L} := L_3L$ has no other singularities in \mathbb{C} either, so \tilde{L} must be in $\mathbb{C}[z, \partial]$ and indeed:

$$\tilde{L} = \partial^3 - z\partial^2 + 3\partial - z.$$

Note that this desingularization process generally makes the singularity at $z = \infty$ worse in the sense that the highest slope in the Newton polygon at $z = \infty$ is higher for L_3 (and hence for \tilde{L}) than it is for L . In Example 7 this is unavoidable if we want \tilde{L} to have minimal order, although for this particular example a desingularization $\tilde{L} = (\partial^2 + 1)^2$ of non-minimal order exists whose singularity at $z = \infty$ is not worse than that of L . Our implementation always computes a desingularization of minimal order but it is not optimal in the sense that it makes no effort to avoid making the singularity at $z = \infty$ worse even in examples where this could be done.

Definition 7 We call \tilde{L} a *complete desingularization* of L if $\tilde{L} \in \mathbb{C}[z, \partial]$ has L as right-hand factor and is monic with respect to ∂ .

Theorem 6 implies that a complete desingularization of L exists if and only if all singularities of L in \mathbb{C} are apparent singularities.

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