SUBRECURSIVE HIERARCHIES VIA DIRECT LIMITS

E. C. Dennis-Jones and S.S. Wainer
Manchester University Leeds University

Introduction

The subrecursive classification problem is to find 'natural' ordinal assignments for (classes of) recursive functions, which reflect as accurately as possible their computational complexity.

The most common approach is to first view the problem in reverse, associating a hierarchy of increasing functions f_α with ordinals α so that f_α dominates f_β whenever $\beta < \alpha$. Then the complexity of a function h is measured by the least α such that h is computable within f_α-bounded time or space. The naturalness of such an assignment depends upon the acceptability of f_α as a 'functional representation' of ordinal α.

A typical example - and one which provides useful classifications of many subrecursive classes - is the transfinitely extended Grzegorczyk Hierarchy, first developed to level ω^ω by Robbin and then more generally to level ε_0 and beyond by Löb and Wainer [4, 7] and Schwichtenberg [6]. Details vary inessentially from author to author but the basic scheme is

$$F_0 = \text{initial function, e.g. exponential}$$

$$F_{\alpha+1} = It(F_\alpha)$$

$$F_\lambda = \text{Diagonal } (F_{\lambda_x})_{x<\omega}$$

where It is some suitable iteration functional and $(\lambda_x)_{x<\omega}$ is a chosen fundamental sequence to limit λ. This is a standard kind of recursion-theoretic construction, motivated by the fact that the iteration functional plays an analogous role (w.r.t elementary recursiveness) to that of the jump operator in the generation of the hyperarithmetic hierarchy. However, the F_α-hierarchy is unsatisfactory on two counts:

(i) it is not sufficiently refined - $h_1(x) = F_\alpha(x^x)$ is clearly more complex than $h_2(x) = F_\alpha(x^3)$ and yet they will both appear at level $\alpha + 1$.

(ii) There seems to be no good mathematical reason for accepting F_α as a natural 'representation' of ordinal α.

If, on the other hand, one merely demands that a hierarchy should be as *refined* as possible - ignoring for the moment any consideration of its potential usefulness - then there is another, very obvious candidate: namely, the Slow Growing Hierarchy:

$$G_0 = \text{constant } 0$$
$$G_{\alpha+1} = G_\alpha + 1$$
$$G_\lambda = \text{Diagonal } (G_{\lambda_x})_{x<\omega}, \quad \text{i.e. } G_\lambda(x) = G_{\lambda_x}(x).$$

Furthermore, this seems to give a quite natural assignment of functions to ordinals, since diagonalisation is the functional counterpart to ordinal supremum in the following sense: if the fundamental sequence $(\lambda_x)_{x<\omega}$ is chosen so that $x \le y \Rightarrow G_{\lambda_x}(y) \le G_{\lambda_y}(y)$, as will normally be the case (see later), then G_λ is the *least* function h (w.r.t. domination) such that $x \le y \Rightarrow G_{\lambda_x}(y) \le h(y)$.

Our aims here are in §1 to give a mathematically convincing reason why the G_α's can genuinely be regarded as natural representations of ordinals α, and in §2 as illustration, to show that the Slow Growing Hierarchy eventually produces an extremely refined form of the Grzegorczyk Hierarchy in such a way that the F_α's themselves turn out to represent certain 'large' Bachmann ordinals. In fact, G collapses the Bachmann Hierarchy of ordinal functions onto a version of the Grzegorczyk Hierarchy, thus providing a more delicate measure of the complexity of the F_α's.

The underlying ideas in §1 and indeed many of the results in §2, are already to be found - either explicit or implicit - in Girard [2] though in a different form and within a more elaborate framework. Our more down-to-earth approach, following on from [1], [8], is perhaps more closely related to Jervell [3].

§1. Direct Limit Representation of Ordinals

Girard's theory of dilators and their associated denotation systems is based on the simple observation that any ordinal can be represented as the direct limit of the partially ordered system of inclusions between all its finite subsets. For a countable ordinal one can go a step further and extract direct systems of finite embeddings which are ordered by the integers N and can therefore be coded in a canonical fashion by number-theoretic functions! It is this representation of *countable* ordinals which we shall develop.

The direct system one obtains for a given countable limit ordinal will depend upon the choice of a fundamental sequence for it, and will be built up inductively. So we need to consider ordinals not simply as set-theoretic objects, but rather as well-founded tree-structures which specify the particular choices of fundamental sequences used in their generation. This means that there will be many such structures corresponding to the same set-theoretic ordinal and that when thinking of an ordinal, one must have in mind a particular way in which it is built up.

Definition 1. The set Ω of countable *ordinal structures* or *tree-ordinals* consists of the infinitary terms generated inductively by:

(i) $0 \in \Omega$

(ii) $\alpha \in \Omega \Rightarrow \alpha +_0 1 \in \Omega$

(iii) $\forall x \in N(\alpha_x \in \Omega) \Rightarrow (\alpha_x)_{x \in N} \in \Omega.$

Lower-case Greek letters $\alpha, \beta, \gamma, \delta, \lambda$ henceforth denote tree-ordinals and λ will always denote a 'limit' i.e. $\lambda = (\lambda_x)_{x \in N}$. Each tree-ordinal α is a 'notation' for a set-theoretic ordinal $|\alpha|$ defined by $|0| = 0, |\alpha +_0 1| = |\alpha| + 1$ and $|(\alpha_x)_{x \in N}| = \sup_{x \in N} |\alpha_x|$. Thus, we will often write $\sup \alpha_x$ to denote a sequence $(\alpha_x)_{x \in N} \in \Omega$. Ω is partially-ordered by the obvious tree-ordering $\prec$ which is just the transitive closure of $0 \preccurlyeq \alpha$, $\alpha \prec \alpha +_0 1$ and $\alpha_n \prec \sup \alpha_x$ for each n.

Definition 2. Make Ω into a category by taking as morphisms the *embeddings* $g : \alpha \to \beta$, i.e. $\gamma \preccurlyeq \delta \prec \alpha \Leftrightarrow g(\gamma) \preccurlyeq g(\delta) \prec \beta$.

Definition 3. (a) A system $(\alpha_x, g_{xy})_{x<y \in N}$ of morphisms $g_{xy} : \alpha_x \to \alpha_y$ is *directed* if

$$x < y < z \Rightarrow g_{xz} = g_{yz} \circ g_{xy}.$$

(b) A direct system $(\alpha_x, g_{xy})_{x<y \in N}$ has $(\alpha, g_x)_{x \in N}$ as a *direct limit*, written

$$(\alpha, g_x)_N = \underset{\rightarrow}{\lim} (\alpha_x, g_{xy})_N$$

if (i) $g_x : \alpha_x \to \alpha$

(ii) $x < y \Rightarrow g_x = g_y \circ g_{xy}$

(iii) given any other $(\alpha', g'_x)_{x \in N}$ satisfying (i) and (ii) w.r.t.

$(\alpha_x, g_{xy})_N$ there is a unique $h:\alpha \to \alpha'$ such that for all $x \in N$, $g'_x = h \circ g_x$.

Note 1. Although a direct system $(\alpha_x, g_{xy})_N$ may in general have many different (though isomorphic) direct limits $(\alpha, g_x)_N$ in Ω, the ordinal $|\alpha|$ is *uniquely* determined (since the existence of a morphism $h:\alpha \to \alpha'$ implies that $|\alpha| \le |\alpha'|$).

Definition 4. For each $\alpha \in \Omega$ and each $x < y \in N$, define the integer n_x^α and the maps $g_{xy}^\alpha, g_x^\alpha$ with domain n_x^α by the recursions:

$$
\begin{array}{lll}
n_x^0 = 0 & g_{xy}^0 = \emptyset & g_x^0 = \emptyset \\
n_x^{\alpha+1} = n_x^\alpha + 1 & g_{xy}^{\alpha+1} = g_{xy}^\alpha \cup \{<n_x^\alpha, n_y^\alpha>\} & g_x^{\alpha+1} = g_x^\alpha \cup \{<n_x^\alpha, \alpha>\} \\
n_x^\lambda = n_x^{\lambda x} & g_{xy}^\lambda = g_{xy}^{\lambda_x} & g_x^\lambda = g_x^{\lambda_x} .
\end{array}
$$

Definition 5.(a) For each fixed x the "x-ordering" $\prec_x$ is the transitive closure of $\alpha \prec_x \alpha +_0 1$ and $\lambda_x \prec_x \lambda$.

(b) $\alpha[x] = \{\beta \mid \beta +_0 1 \preccurlyeq_x \alpha\}$.

In Ω we shall identify the integers with the tree-ordinals $0 +_0 1 +_0 1 +_0 \ldots +_0 1$. It is easy to see that the cardinality of $\alpha[x]$ is n_x^α and that $\alpha[x]$ is the range of g_x^α. Thus g_x^α can be considered as a morphism: $n_x^\alpha \to \alpha$ and g_{xy}^α can be considered as a morphism from n_x^α to n_y^α provided $g_{xy}^{\lambda_x}(\ ) < n_y^\lambda$ at limits λ.

Lemma 1. The following are equivalent for each $\alpha \in \Omega$:

(a) $(n_x^\alpha, g_{xy}^\alpha)_{x<y\in N}$ is a direct system and

$$x < y \Rightarrow g_x^\alpha = g_y^\alpha \circ g_{xy}^\alpha \ ,$$

(b) $x < y \Rightarrow \alpha[x] \subseteq \alpha[y]$.

Proof. The second part of (a) implies that range $\left(g_x^\alpha\right) \subseteq$ range $\left(g_y^\alpha\right)$ whenever $x < y$. But this is just (b) because range $g_x^\alpha = \alpha[x]$ for each x. Now assume (b). From the above definitions, we note that

$$g_{xy}^\alpha(n) = m \iff \exists \beta \in \alpha[x](n = n_x^\beta \ \&\ m = n_y^\beta)$$
$$g_x^\alpha(n) = \beta \iff \beta \in \alpha[x] \ \&\ n = n_x^\beta \ .$$

Then from (b) we have (i) $n_y^\beta < n_y^\alpha$ when $\beta \in \alpha[x]$ and so $g_{xy}^\alpha : n_x^\alpha \to n_y^\alpha$; (ii) if $x < y < z$ then for each $\beta \in \alpha[x] \subseteq \alpha[y]$,

$$g_{yz}^\alpha(g_{xy}^\alpha(n_x^\beta)) = g_{yz}^\alpha(n_y^\beta) = n_z^\beta = g_{xz}^\alpha(n_x^\beta),$$

hence $(n_x^\alpha, g_{xy}^\alpha)_{x<y\in N}$ is a direct system; and (iii) if $x < y$ then for each $\beta \in \alpha[x]$,

$$g_y^\alpha(g_{xy}^\alpha(n_x^\beta)) = g_y^\alpha(n_y^\beta) = \beta = g_x^\alpha(n_x^\beta),$$

so $g_x^\alpha = g_y^\alpha \circ g_{xy}^\alpha$.

□

Lemma 2. For each $\alpha \in \Omega$,

if $x < y \Rightarrow \alpha[x] \subseteq \alpha[y]$

and $\beta \prec \alpha \Rightarrow \exists x(\beta \in \alpha[x])$

then $(\alpha, g_x^\alpha)_N = \varinjlim (n_x^\alpha, g_{xy}^\alpha)_N$.

Proof. It only remains, from Lemma 1, to check that given any system of morphisms $g_x' : n_x^\alpha \to \alpha'$ with $g_x' = g_y' \circ g_{xy}^\alpha$ there is a unique $h : \alpha \to \alpha'$ satisfying $g_x' = h \circ g_x^\alpha$ for each x. Given any β α we have $\beta \in \alpha[x]$ for all but finitely many x. Choose any such x, so $\beta = g_x^\alpha(n_x^\beta)$, and then define $h(\beta) = g_x'(n_x^\beta)$. If $x < y$ and $\beta = g_x^\alpha(n_x^\beta) = g_y^\alpha(n_y^\beta) = g_y^\alpha(g_{xy}^\alpha(n_x^\beta))$ then $g_x'(n_x^\beta) = g_y'(g_{xy}^\alpha(n_x^\beta)) = g_y'(n_y^\beta)$ so h is well-defined and it must be unique .

□

Towards the converse of Lemma 2, suppose $(\alpha, g_x^\alpha)_N$ is a direct limit for the system $(n_x^\alpha, g_{xy}^\alpha)_N$. Then by Lemma 1 we certainly have $x < y \Rightarrow \alpha[x] \subseteq \alpha[y]$. Let $\alpha' = \sup \alpha_x'$ where α_x' is the top element of $\alpha[x]$ and define $g_x' : n_x^\alpha \to \alpha'$ by $g_x'(n_x^\beta) = g_x^\alpha(n_x^\beta)$ for each $\beta \in \alpha[x]$. Then there must be a unique $h : \alpha \to \alpha'$ with $g_x' = h \circ g_x^\alpha$ for each x. Clearly, h must be the identity on $\bigcup_x \alpha[x]$. So for each $\beta \prec \alpha$, $h(\beta) \prec \alpha'$ and therefore $h(\beta) \preccurlyeq \alpha_x' = h(\alpha_x')$ for some x. Since h is an embedding, we then have $\beta \preccurlyeq \alpha_x' \prec \alpha$. Now if we were to assume, inductively, the direct limit representation of α_x' and the converse of Lemma 2 for α_x', then in the case of $\beta \prec \alpha_x'$ there would be a $y > x$ such that $\beta \in \alpha_x'[y]$. But since $\alpha_x' \in \alpha[x] \subseteq \alpha[y], \beta +_0 1 \preccurlyeq_y \alpha_x' \prec_y \alpha$ and so $\beta \in \alpha[y]$. We have now proved:

Theorem 1. The following are equivalent for each $\alpha \in \Omega$:

(a) For every $\gamma \preccurlyeq \alpha$, $(\gamma, g_x^\gamma)_N = \varinjlim (n_x^\gamma, g_{xy}^\gamma)_N$.

(b) For every $\gamma \preccurlyeq \alpha$,

$$x < y \Rightarrow \gamma[x] \subseteq \gamma[y]$$

and $$\beta \prec \gamma \Rightarrow \exists x(\beta \in \gamma[x]).$$

Note 2. Looking back at Definition 4, one sees that $(g^{\alpha}_{xy})_{x<y\in N}$ and $(g^{\alpha}_{x})_{x\in N}$ are completely determined by α and the sequence $(n^{\alpha}_{x})_{x\in N}$. Furthermore, it is obvious that $(n^{\alpha}_{x})_{x\in N}$ *is just the Slow-Growing function* G_{α}. In order to emphasise the 'pointwise-at-x' definition of G *we shall henceforth denote* G_{α} *instead by* $G(\alpha)$ *and its value at* x *by* $G_x(\alpha)$. Thus $G_x(\alpha) = n^{\alpha}_x$ and instead of writing

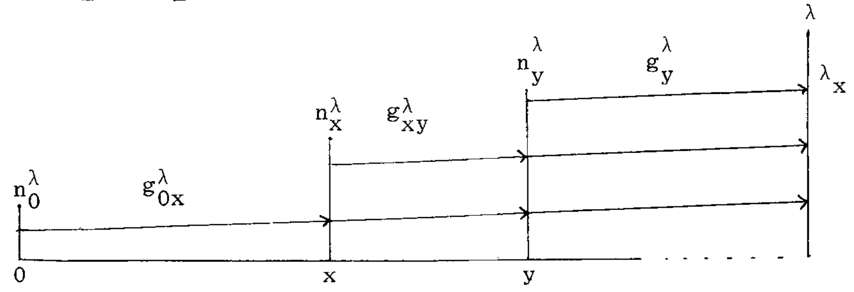
$$(\alpha, g^{\alpha}_{x})_N = \lim_{\rightarrow}(n^{\alpha}_{x}, g^{\alpha}_{xy})_N$$

we may simply write

$$\boxed{\alpha = \lim_{\rightarrow} G(\alpha)}$$

Note 3. The inductive generation of Ω imposes little structure on tree-ordinals other than that $\prec$ is a well-founded partial ordering. Condition (b) of Theorem 1 gives α a more ordinal-like structure - if $\beta_1 \prec \alpha$ and $\beta_2 \prec \alpha$ then $\beta_1, \beta_2 \in \alpha[x]$ for all but finitely many x, so $\beta_1 \preccurlyeq_x \beta_2$ or $\beta_2 \preccurlyeq_x \beta_1$. Therefore, α is well-ordered by $\prec$ and $\beta \prec \alpha \Rightarrow \beta +_0 1 \preccurlyeq \alpha$. However, if $\alpha = \sup \alpha_x$ it still need not be the case that $(\alpha_x)_{x\in N}$ is a *fundamental* sequence, i.e. $\alpha_0 \prec \alpha_1 \prec \alpha_2 \prec \ldots$. Related to this is a further problem - it still may not be the case that $G(\alpha)$ dominates each $G(\alpha_x)$, since we may have $n^{\alpha_x}_y \geq n^{\alpha}_y$ for infinitely many $y > x$. Clearly, we need to ensure that the sequences $(\lambda_x)_{x\in N} \preccurlyeq \alpha$ mesh together in some suitable way.

The direct-limit picture suggests a natural additional requirement on the way in which $\lambda = \sup \lambda_x$ is built up by the enumeration $\lambda[0] \subseteq \lambda[1] \subseteq \lambda[2] \subseteq \ldots$.



We shall demand that λ_x appears at the earliest possible stage, namely $y = x + 1$.

Definition 6. Write $\alpha = \underset{\rightarrow}{\mathrm{Lim}}\, G(\alpha)$ with a capital 'L' to mean that

(i) for every $\gamma \preccurlyeq \alpha$, $(\gamma, g_x^\gamma)_N = \underset{\rightarrow}{\lim}(n_x^\gamma, g_{xy}^\gamma)_N$ and

(ii) for every $\lambda = \sup \lambda_x \preccurlyeq \alpha$, $x < y \Rightarrow \lambda_x \in \lambda[y]$.

Definition 7. Call α a *nice* tree-ordinal if it satisfies condition (ii) above, i.e. if $\lambda = \sup \lambda_x \preccurlyeq \alpha$ then $x < y \Rightarrow \lambda_x +_0 1 \preccurlyeq_y \lambda_y$. (See Schmidt [5] for other conditions on fundamental sequences which are similar to, though not the same as, niceness.)

Lemma 3. If α is nice, then for every $\gamma \preccurlyeq \alpha$

$$x < y \Rightarrow \gamma[x] \subseteq \gamma[y]$$

and

$$\beta \prec \gamma \Rightarrow \exists\, x(\beta \in \gamma[x]).$$

Proof. By induction on $\gamma \preccurlyeq \alpha$ noting that if $\gamma = \sup \gamma_x \preccurlyeq \alpha$ then $x < y \Rightarrow \gamma_x \in \gamma[y] \Rightarrow \gamma_x[y] \subset \gamma[y]$.

□

Theorem 2. For each $\alpha \in \Omega$,

$\alpha = \underset{\rightarrow}{\mathrm{Lim}}\, G(\alpha)$ if and only if α is nice.

Proof. Immediate from Theorem 1, Lemma 3, and the Definitions.

□

Theorem 3. Each nice α is well-ordered by $\prec$ in such a way that

$$\beta \prec \gamma \preccurlyeq \alpha \Rightarrow G(\beta) \text{ is dominated by } G(\gamma).$$

Proof. We already know from Note 3 and Lemma 3, that $\prec$ well-orders α if it's nice. The rest follows by induction on γ for if $\gamma = \sup \gamma_x \preccurlyeq \alpha$ then since α is nice

$$x < y \Rightarrow \gamma_x[y] \subset \gamma[y] \Rightarrow n_y^{\gamma_x} < n_y^\gamma$$

and so $G(\gamma)$ dominates each $G(\gamma_x)$.

□

Thus, not only does a nice $\alpha \in \Omega$ provide a system of unique notations $\beta \preccurlyeq \alpha$ for the ordinals $\leq |\alpha|$, but in addition each such β naturally induces a direct-limit representation $G(\beta)$ of the ordinal $|\beta|$.

An obvious question at this point is whether *every* proper initial segment of the countable ordinals can be so represented? The answer is easily seen to be 'yes' .... .

First define addition $+_0$ on Ω by

$$\alpha +_0 0 = \alpha$$
$$\alpha +_0 (\beta +_0 1) = (\alpha +_0 \beta) +_0 1$$
$$\alpha +_0 \sup \lambda_x = \sup(\alpha +_0 \lambda_x)$$

Lemma 4.(i) $+_0$ is associative

(ii) $\gamma \preccurlyeq \alpha +_0 \beta \Rightarrow \gamma \preccurlyeq \alpha$ or $\exists \delta \preccurlyeq \beta(\gamma = \alpha +_0 \delta)$

(iii) $\delta \prec_y \beta \Rightarrow \alpha +_0 \delta \prec_y \alpha +_0 \beta$.

(iv) If α and β are nice, then $\alpha +_0 \beta$ is nice.

Now given any sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ of tree-ordinals, define

$$\Sigma\alpha_x = \sup_x(\alpha_0 +_0 \alpha_1 +_0 \cdots +_0 \alpha_x)$$

Lemma 5. If $\alpha_0, \alpha_1, \alpha_2, \ldots$ are all nice and $\neq 0$, then $\Sigma\alpha_x$ is nice.

Proof. By Lemma 4, since if $\alpha \neq 0$ is nice and $y > 0$, then $1 \preccurlyeq_y \alpha$.

□

Theorem 4. For every countable ordinal τ there is a nice $\alpha \in \Omega$ such that $|\alpha| = \tau$.

Proof. Clearly, 0 is nice and $\alpha +_0 1$ is nice if α is. If τ is a limit we can choose, inductively, a sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ of non-zero nice tree-ordinals such that $\tau = \sup|\alpha_x|$. But then, $\tau \leq |\Sigma\alpha_x|$ and Σ_x is nice by Lemma 5, so there is a nice $\beta \preccurlyeq \Sigma\alpha_x$ such that $\tau = |\beta|$.

□

Remark. Since tree ordinals are just countable well-founded trees with a certain structure, they can be coded in a standard way as reals. Thus it makes perfectly good sense to talk about 'recursive tree ordinals', 'Δ^1_2 tree ordinals', etc. It should be clear that if $\alpha_0, \alpha_1, \alpha_2, \ldots$ is a recursively-given sequence of recursive tree ordinals, then $\Sigma\alpha_x$ is also recursive. So a special case of Theorem 4 is that τ is a recursive ordinal if and only if there is a nice recursive $\alpha \in \Omega$ such that $|\alpha| = \tau$.

§2. The Bachmann and Grzegorczyk Hierarchies

Here we compute out some examples of

$$\alpha = \underset{\to}{\mathrm{Lim}}\, G(\alpha)$$

for certain well-known recursive ordinals α.

First some very simple examples:

Define multiplication and exponentiation on Ω by

$$\begin{array}{lll} \alpha.0 = 0 & \qquad & \alpha^0 = 1 \\ \alpha.(\beta+_0 1) = \alpha.\beta +_0 \alpha & & \alpha^{\beta+1} = \alpha^\beta.\alpha \\ \alpha.\lambda = \sup(\alpha.\lambda_x) & & \alpha^\lambda = \sup(\alpha^{\lambda_x}) \end{array}$$

Lemma 6. (i) If $\alpha \neq 0$ is nice then for each $y > 0$,

$$\delta \prec_y \gamma \Rightarrow \alpha.\delta \prec_y \alpha.\gamma.$$

Hence, if $\alpha \neq 0$ is nice and β is nice, then $\alpha.\beta$ is nice.

(ii) If α is nice and $\forall y > 0(2 \preccurlyeq_y \alpha)$ then for each $y > 0$

$$\delta \prec_y \gamma \Rightarrow \alpha^\delta \prec_y \alpha^\gamma.$$

Hence if α and β are nice and $\forall y > 0(2 \preccurlyeq_y \alpha)$ then α^β is nice.

Proof. By quite straightforward inductions over Ω. The additional conditions on α are needed to ensure that (i) $\alpha.\gamma +_0 1 \preccurlyeq_y \alpha(\gamma+_0 1)$, (ii) $\alpha^\gamma +_0 1 \preccurlyeq_y \alpha^{\gamma+1}$.

□

The following is also easily proved by inductions on $\beta \in \Omega$.

Lemma 7. (i) $G_x(\alpha+_0\beta) = G_x(\alpha) + G_x(\beta)$

(ii) $G_x(\alpha.\beta) = G_x(\alpha).G_x(\beta)$

(iii) $G_x(\alpha^\beta) = G_x(\alpha)^{G_x(\beta)}$.

If we now take, as our standard representation of the first limit ordinal,

$$\omega = \sup(x+1) \in \Omega$$

then ω is nice, $\forall y > 0(2 \preccurlyeq_y \omega)$ and for each x, $G_x(\omega) = x + 1$. Thus, from Lemmas 6,7 and Theorem 2, we see immediately that *if* α *is in Cantor Normal Form to base* ω *then* $\alpha = \underset{\to}{\mathrm{Lim}}\, G(\alpha)$ *where* $G_x(\alpha)$ *is the result of replacing* ω *by* $x + 1$ *throughout the normal form.*

To see how the direct limit representation extends to higher ordinals, define the finite levels $\phi_n : \Omega \to \Omega$ of the Bachmann-Veblen Hierarchy on Ω by:

$$\begin{aligned}
\phi_0(\alpha) &= \omega^{\alpha} \\
\phi_{n+1}(0) &= \sup_x \phi_n^x(1) \\
\phi_{n+1}(\beta +_0 1) &= \sum_x \phi_n^x(\phi_{n+1}(\beta) +_0 1) \\
\phi_{n+1}(\lambda) &= \sup_x \phi_{n+1}(\lambda_x)
\end{aligned}$$

where $\phi^0(\alpha) = \alpha$ and $\phi^{x+1}(\alpha) = \phi(\phi^x(\alpha))$.

Define also $\phi_\omega(0) = \sup_x \phi_{x+1}(0)$.

Then, $|\phi_n(\alpha)|$ = Bachmann-Veblen $\emptyset_n(|\alpha|)$
= $|\alpha|$-th fixed point of $\emptyset_{n-1}$.

So, for example, $|\phi_1(\alpha)| = \varepsilon_{|\alpha|}, |\phi_2(\alpha)|$ is the $|\alpha|$-th critical epsilon-number, and $|\phi_\omega(0)|$ is the first 'prim-closed' ordinal.

Lemma 8. (i) $\phi_n(\beta) +_0 1 \preccurlyeq_y \phi_n(\beta +_0 1)$.

(ii) $\gamma \prec_y \beta \Rightarrow \phi_n(\gamma) \prec_y \phi_n(\beta)$.

(iii) $x < y \le z \Rightarrow \phi_n^x(1) +_0 1 \preccurlyeq_z \phi_n^y(1)$.

Lemma 9. (i) For each n, if α is nice, then $\phi_n(\alpha)$ is nice.

(ii) $\phi_\omega(0)$ is nice.

Lemma 10. For each n,x and each $\alpha \in \Omega$,

$$G_x(\phi_n(\alpha)) = f_{n,x}(G_x(\alpha))$$

where the functions $f_{n,x}$ are given by:

$$\begin{aligned}
f_{0,x}(a) &= (x+1)^a \\
f_{n+1,x}(0) &= f_{n,x}^x(1) \\
f_{n+1,x}(b+1) &= \sum_{i \le x} f_{n,x}^i(f_{n+1,x}(b)+1).
\end{aligned}$$

(again, f^x denotes the x-th iterate of f).

Notice that $f_{n+1,x}$ is defined from $f_{n,x}$ by what is essentially an *iteration*, i.e. $f_{n+1,x} = It_x(f_{n,x})$ where

$It_x(f)(0) = f^x(1)$, $It_x(f)(b+1) = \sum_{i \le x} f^i(It_x(f)(b)+1)$.

For each $x > 0$, $\{f_{n,x}\}_{n<\omega}$ is a version of the Grzegorczyk Hierarchy - every $f_{n,x}$ is primitive recursive and each primitive recursive function is dominated by some $f_{n,x}$.

From this family of hierarchies we can extract a 1-variable version $\{F_n\}_{n<\omega}$ by setting

$$F_n(x) = f_{n,x}(0).$$

Because of the uniformity in the definition of $f_{n,x}$ it is clear that each F_n is primitive recursive. Furthermore, it is not difficult to check that for each n and x, and all $a > x$

$$f_{n,x}(a) \le f_{n,a}(a) \le f_{n,a}(f_{n,a}^{a-1}(1)) = f_{n+1,a}(0) = F_{n+1}(a).$$

Thus, if we place 'on top' of the F_n-hierarchy the function

$$F_\omega(x) = F_{x+1}(x) = f_{x+1,x}(0)$$

then F_ω is our version of the Ackermann function, and

$$G_x(\phi_\omega(0)) = G_x(\phi_{x+1}(0)) = F_\omega(x).$$

By Lemmas 9,10 and Theorem 2, we then get

<u>Theorem 5</u>. (i) For each n, $\phi_n(0) = \operatorname{Lim}_{\rightarrow} F_n$.

(ii) $\phi_\omega(0) = \operatorname{Lim}_{\rightarrow} F_\omega$.

Therefore, in terms of the Slow-Growing hierarchy G, we see that the ordinal complexity of the 'first' non-elementary function

$$F_1(x) = \left.(x+1)^{(x+1)^{\cdot^{\cdot^{(x+1)}}}}\right\} x \text{ times}$$

is its direct limit ε_0, and the complexity of the 'first non-primitive recursive function F_ω is the first primitive recursively closed ordinal $\phi_\omega(0)$.

The results of [1], [8] suggest how Theorem 5 extends to higher transfinite levels of the Bachmann and Grzegorczyk hierarchies, so that, for example, $\operatorname{Lim}_{\rightarrow} F_{\varepsilon_0}$ = Howard Ordinal. However, the hierarchies originally developed paid no attention to 'niceness' and need to be modified slightly in order to incorporate this property. The second author plans a more detailed treatment of these higher-level results.

References

[1] E.A. Cichon and S.S. Wainer 'The Slow-Growing and the Grzegorczyk Hierarchies', J. Symb. Logic 48 (1983) 399-408.

[2] J-Y. Girard, 'Π_2^1-Logic Part I', Annals. Math. Logic 21 (1981) 75-219.

[3] H.R. Jervell, 'Introducing Homogeneous Trees', Proc. Herbrand Symposium, Ed. J. Stern, North-Holland (1982).

[4] M.H. Löb and S.S. Wainer, 'Hierarchies of Number-Theoretic Functions I, II', Archiv. für math. Logik 13 (1970) 39-51, and 97-113.

[5] D. Schmidt, 'Built-Up Systems of Fundamental Sequences and Hierarchies of Number-Theoretic Functions', Archiv. für math. Logic 18 (1976) 47-53. Postcript 145-146.

[6] H. Schwichtenberg, 'Eine Klassifikation der ε_0-rekursiven Funktionen', Zeitschrift für math. Logik 17 (1971) 61-74.

[7] S.S. Wainer, 'A Classification of the Ordinal Recursive Functions' Archiv. für math. Logik 13 (1970) 136-153.

[8] S.S. Wainer, 'The "Slow-Growing" Π_2^1 Approach to Hierarchies', to appear in Proc. AMS Summer Research Inst. on Recursion Theory, 1982, Eds. A. Nerode and R.A. Shore.