Provability in BI's Sequent Calculus is Decidable

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Abstract. The logic of Bunched Implications (BI) combines both additive and multiplicative connectives, which include two primitive intuitionistic implications. As a consequence, contexts in the sequent presentation are not lists, nor multisets, but rather tree-like structures called bunches. This additional complexity notwithstanding, the logic has a well-behaved metatheory admitting all the familiar forms of semantics and proof systems. However, the presentation of an effective proof-search procedure has been elusive since the logic's debut. We show that one can reduce the proof-search space for any given sequent to a primitive recursive set, the argument generalizing Gentzen's decidability argument for classical propositional logic and combining key features of Dyckhoff's contraction-elimination argument for intuitionistic logic. An effective proof-search procedure, and hence decidability of provability, follows as a corollary.

Keywords: Logic · Proof-search · Sequent Calculus · BI · Decidability

1 Introduction

The logic of Bunched Implication (BI) [15] is a logic in which multiplicative and additive intuitionistic implications live side-by-side. Consequently, there are two different types of data (additive and multiplicative), which are distinguished by introducing two different context-formers: multiplicative composition (Δ , Γ) denies the structural rules of weakening and contraction, whereas additive composition (Δ , Γ) admits them. It follows that contexts in BI's sequents are not a flat data structure, such as lists or multisets, but instead are tree structures called bunches, a term that derives from the relevance logic literature (see, for example, [18]). The uniqueness of BI lies in the presence of two primitive implications (additive \rightarrow and multiplicative \rightarrow): the logic arises naturally from investigating the proof-theoretic relationship between conjunction and implication [15].

A natural question to ask about a logic is whether or not logical consequence is decidable. The approaches to the problem can be partitioned into two broad categories, the syntactic and the semantic, which are characterized by making use of the proof theory and model theory of the logic respectively. Here, we propose a purely syntactic approach, based on proof-search in BI's sequent calculus LBI [16]. Although BI enjoys the full gamut of Hilbert, natural deduction, tableaux, and display calculi [16,8,2], we choose the sequent calculus because it has local

correctness and analytic proofs are complete, meaning that one can structure the proof-search space for any given sequent through combinatorics on the subformulas appearing in it.

Consider the following naive construction:

- combine multisets of formulas and sub-formulas using additive context-formers to form a zeroth generation of bunches;
- combine multisets of previously constructed bunches using multiplicative context-formers to form a next generation of bunches; and
- combine multisets of previously constructed bunches using additive contextformers to complete the next generation of bunches.

This process is infinite, yielding an infinite space, and is therefore unusable for proof-search. However, if one can restrict to a primitive recursive subspace (i.e., a subset constructed using only primitive recursive functions) that contains all the sequents of some proof whenever one exists, then a procedure follows immediately by arranging the constructed sequents into proof candidates and checking correctness. The proposed approach is to establish such a space by generalizing Gentzen's decidability argument for classical propositional logic [9].

Gentzen's method relies on the completeness of a class of proofs where one can prove that every formula occurs at most three times in any sequent. There are two essential requirements for this method to work: first, that there is a uniform bound for the increase in the multiplicity of formulas in the conclusion with respect to a single inference step in the proof calculus; second, that after each deductive inference step one can use the structural rules to reduce the resulting sequent by removing extraneous occurrences. Call the first insight the steady growth property (SGP), then the second describes a control régime establishing the class of regimented proofs where the number of formula occurrences can be effectively and uniform bounded.

The bunched structure of BI means that the situation is more complex than for classical logics since there are now three measures to control, corresponding to the above construction: additive width (maximal size of additive combinations), multiplicative width (maximal size of multiplicative combinations), and depth (number of generations). Moreover, the extra complexity afforded by, and interactions of, the additive and multiplicative structures renders the notions of extraneous and reduction quite delicate.

The structure of the paper is as follows. First, the logic of Bunched Implications is introduced in Section 2.

The study of control begins in Section 3 with a formal characterization of reduction, as well as the class of regimented proofs, which follows Gentzen's method. However, we work in a restricted sequent calculus dLBI that arises from LBI by requiring that the structural rules are *small-step*, in the sense that they can only introduce a bounded amount of extraneous data, and that the unit laws for each context-former are no longer present. The first transformation means the system has the SGP, and the second means that multiplicative width and depth of sequents can be bounded. The elimination of the unit laws is based on Dyckhoff's contraction-elimination argument for intuitionistic propositional

logic [7], wherein one embeds a sufficient amount of a structural rule into the logical rules so as to make it redundant.

Formal measures for additive width, multiplicative width, and depth are given in Section 4. Moreover, we give uniform bounds for these measures with respect to regimented proofs in dLBI. An effective proof-search procedure, and decidability, follows immediately.

While the body of the paper contains the decidability argument as a whole, full details of proofs can be found in the appendices.

2 The Logic of Bunched Implications

As a logic, BI can be regarded as the free combination of intuitionistic logic (IL) and the multiplicative fragment of intuitionistic linear logic (MILL), arising from the presence of two distinct context-formers in its sequent presentation. The two conjunctions \land and * are represented at the meta-level by context-formers \mathsection and \mathsection , in place of the usual commas for IL and MILL, respectively.

Definition 1 (Formula). Let \mathbb{A} be a denumerable set of propositional letters. The formulas of BI are defined by the following grammar:

$$\phi ::= \top \mid \bot \mid 1 \mid A \in \mathbb{A} \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \to \phi) \mid (\phi * \phi) \mid (\phi \to \phi)$$

If $\circ \in \{\land, \lor, \rightarrow, \top, \bot\}$, then it is an additive connective, and if $\circ \in \{*, -*, 1\}$ then it is a multiplicative connective. The set of all formulas is denoted \mathbb{F} .

Definition 2 (Bunch). A bunch is constructed from the following grammar:

$$\Delta ::= \phi \in \mathbb{F} \mid \varnothing_{+} \mid \varnothing_{\times} \mid (\Delta ; \Delta) \mid (\Delta , \Delta)$$

The symbols \varnothing_+ and \varnothing_\times are the additive and multiplicative units, respectively, and the symbols \S and \S are the additive and multiplicative context-formers, respectively. A bunch is basic if it is a formula, \varnothing_+ , or \varnothing_\times , and complex otherwise. The set of all bunches is denoted $\mathbb B$; the set of complex bunches with additive and multiplicative principal context-formers are denoted $\mathbb B^+$ and $\mathbb B^\times$, respectively.

Definition 3 (Sequent). A sequent is a pair of a bunch Γ , called the context, and a formula ϕ , and is denoted $\Gamma \Longrightarrow \phi$.

Since each of the context-formers of a bunch represents a comma in IL or MILL, they have the same properties; in particular, they may each be read as a multiset constructor instead, witnessed in the proof theory by the presence of an exchange rule.

Definition 4 (Permutation, Coherent Equivalence). Two bunches are permutations when $\Gamma \cong \Gamma'$, where \cong is the least equivalence relation on bunches satisfying: commutative semi-group equations for the additive context-former; commutative semi-group equations for the multiplicative context-former; and coherence; that is, if $\Delta \cong \Delta'$, then $\Gamma(\Delta) \cong \Gamma(\Delta')$.

Permutation is extended to coherent equivalence \equiv , when \varnothing_+ is a unit for the additive context-former and \varnothing_\times is a unit for the multiplicative context-former.

$$\overline{A} \Longrightarrow A \text{ Ax } \overline{\Gamma(\bot)} \Longrightarrow \phi^{\bot} \bot \overline{\varnothing_{\times}} \Longrightarrow 1^{\bot} R \overline{\varnothing_{+}} \Longrightarrow \overline{\top} \top_{R}$$

$$\frac{\Delta' \Longrightarrow \phi \Gamma(\Delta'', \psi) \Longrightarrow \chi}{\Gamma(\Delta', \Delta'', \phi \twoheadrightarrow \psi) \Longrightarrow \chi} \twoheadrightarrow_{L} \frac{\Delta, \phi \Longrightarrow \psi}{\Delta \Longrightarrow \phi \twoheadrightarrow \psi} \twoheadrightarrow_{R}$$

$$\frac{\Gamma(\phi, \psi) \Longrightarrow \chi}{\Gamma(\phi \ast \psi) \Longrightarrow \chi} \twoheadrightarrow_{L} \frac{\Delta \Longrightarrow \phi \Delta' \Longrightarrow \psi}{\Delta, \Delta' \Longrightarrow \phi \ast \psi} \twoheadrightarrow_{R} \frac{\Gamma(\varnothing_{\times}) \Longrightarrow \chi}{\Gamma(\top^{*}) \Longrightarrow \chi} \Longrightarrow 1$$

$$\frac{\Gamma(\phi, \psi) \Longrightarrow \chi}{\Gamma(\phi \land \psi) \Longrightarrow \chi} \land_{L} \frac{\Delta \Longrightarrow \phi \Delta' \Longrightarrow \psi}{\Delta, \Delta' \Longrightarrow \phi \land \psi} \land_{R} \frac{\Gamma(\varnothing_{+}) \Longrightarrow \chi}{\Gamma(\top) \Longrightarrow \chi} \top_{L}$$

$$\frac{\Gamma(\phi) \Longrightarrow \chi}{\Gamma(\phi \lor \psi) \Longrightarrow \chi} \lor_{L} \frac{\Delta \Longrightarrow \phi}{\Delta, \Delta' \Longrightarrow \phi \lor \psi} \lor_{R1} \frac{\Delta \Longrightarrow \psi}{\Delta \Longrightarrow \phi \lor \psi} \lor_{R2}$$

$$\frac{\Delta' \Longrightarrow \phi \Gamma(\Delta'', \psi) \Longrightarrow \chi}{\Gamma(\Delta', \psi) \Longrightarrow \chi} \to_{L} \frac{\Delta, \phi \Longrightarrow \psi}{\Delta, \omega} \to_{R} \frac{\Gamma(\Delta', \psi) \Longrightarrow \chi}{\Gamma(\Delta', \psi) \Longrightarrow \chi} \subset C$$

$$\frac{\Gamma(\Delta', \psi) \Longrightarrow \chi}{\Gamma(\Delta', \psi) \Longrightarrow \chi} \lor_{L} \frac{\Delta, \psi}{\Delta, \omega} \to \psi} \to_{R} \frac{\Gamma(\Delta', \psi) \Longrightarrow \chi}{\Gamma(\Delta', \psi) \Longrightarrow \chi} \subset C$$

$$\frac{\Gamma(\Delta', \psi) \Longrightarrow \chi}{\Gamma(\Delta', \psi) \Longrightarrow \chi} \lor_{L} \frac{\Delta, \psi}{\Delta, \omega} \to \psi} \to_{R} \frac{\Gamma(\Delta', \psi) \Longrightarrow \chi}{\Gamma(\Delta', \psi) \Longrightarrow \chi} \subset C$$

Fig. 1. Sequent Calculus LBI

Let $\Delta \triangleleft \Gamma$ denote that Δ is a proper sub-tree of Γ , and let $\Delta \unlhd \Gamma$ denote $\Delta \triangleleft \Gamma$ or $\Delta = \Gamma$, in which case Δ is called a sub-bunch of Γ . We may write $\Gamma(\Delta)$ to mean that Δ is a sub-bunch, and let $\Gamma[\Delta \mapsto \Delta']$ — abbreviated to $\Gamma(\Delta')$ where no confusion arises — be the result of replacing the occurrence of Δ by Δ' .

Definition 5 (Sequent Calculus LBI). LBI is given in Figure 1.

We will restrict attention to proofs constructed without the use of cut, which is adequate [2]. Furthermore, though the usual inductive definition of proofs suffices, some extra flexibility in the meta-language is useful. For example, we may regard a rule R as a relation $\mathbf{R}(P_0,...,P_n,C)$ which holds if and only if the conclusion C may be inferred from the premisses $P_0,...,P_n$ by the rule.

Definition 6 (Derivation). Let L be a sequent calculus, let Σ be a set of sequents, and let S be some sequent. A rooted finite tree \mathcal{D} of sequents is a L-derivation of S from Σ if, for any node ζ ,

- if ζ is a leaf, then either $\zeta \in \Sigma$ or $\mathbf{A}(\zeta)$ holds for some axiom $\mathbf{A} \in \mathsf{L}$;
- if ζ has children $P_0, ..., P_n$ in \mathcal{D} , then there is a rule s.t. $\mathbf{R}(P_0, ..., P_n, \zeta)$; and
- if ζ is the root, then $\zeta = S$.

We may write $\mathcal{D}: \mathcal{L} \vdash_{\mathsf{L}} S$ to denote that \mathcal{D} is a L-derivation of S from \mathcal{L} , or simply a proof of S when $\mathcal{L} = \emptyset$. We will suppress the set constructors when there is no confusion. Derivation allows us to talk about *sections* of proofs which may then be replaced by other sections with the same leaves and root, but which have a particular desirable structure. If \mathcal{D} is a derivation, we write $S \in \mathcal{D}$ if S is a sequent occurring in the tree.

3 The Control Régime

We begin by considering how Gentzen's method of reduction, handled by the structural rules, generalizes to BI. The more complex structure of contexts results in a more complex definition; for example, it is not only occurrences of formulas that need to be controlled, but entire sub-bunches.

Unfortunately, Gentzen's control alone is insufficient to yield a primitive recursive subset of the search space for BI. Firstly, because LBI lacks the SGP. Secondly, because the unit law of \varnothing_{\times} within the exchange rule E means that there is no bound on the number of generations in the naive construction given in Section 1. Thus we introduce dLBI which has *small-step* structural rules, and no unit laws. This last step is achieved by using Dyckhoff's method of embedding the content of the unit laws into the logical rules so that they are no longer required.

3.1 Reduction and Normality

We establish here some computational and proof-theoretic requirements that the reduction of contexts must satisfy. However, since the structural rules are subsequently restricted to form dLBI, the *modus operandi* for the control régime, we must similarly restrict the reduction to be small-step.

Definition 7 (Reduction and Normality). Let $\Gamma, \Gamma', \Gamma'' \in \mathbb{B}$ be arbitrary. Big-step reductions are defined by,

$$\Gamma \geq \Gamma' \iff \begin{cases} \Gamma \cong \Gamma' \\ \Gamma = \Gamma(\Delta , \varnothing_{\times}) \text{ and } \Gamma' = \Gamma(\Delta) \\ \Gamma = \Gamma(\Delta , \varnothing_{+}) \text{ and } \Gamma' = \Gamma(\Delta) \\ \Gamma = \Gamma(\Sigma , \Sigma) \text{ and } \Gamma' = \Gamma(\Sigma) \end{cases}$$

A bunch Γ is normal if and only if $\Gamma \geq \Gamma'' \geq \Gamma' \implies \Gamma \cong \Gamma'$. A big-step reduction is a small-step reduction, denoted $\Gamma \succcurlyeq \Gamma'$, when Σ is normal.

Example 8. Let $\Gamma = (\phi, (\chi; \varnothing_+)); (\psi; (\varnothing_\times; \psi))$, one can normalize it by first permuting ψ and \varnothing_\times , and then removing one of the ψ ,

$$\Gamma \succcurlyeq ((\phi, (\chi, \mathcal{S}, \mathcal{O}_{+})), \mathcal{S}(\mathcal{O}_{\times}, \mathcal{S}(\psi, \mathcal{S}, \psi))) \succcurlyeq ((\phi, (\chi, \mathcal{S}, \mathcal{O}_{+})), \mathcal{S}(\mathcal{O}_{\times}, \mathcal{S}, \psi)) \qquad \Box$$

Let $\bar{\mathbb{B}}$ denote the set of normal bunches and, given a bunch Γ , denote $\bar{\Gamma}$ for a normal form of it.

Lemma 9. Reduction is normalizing; that is, $\forall \Gamma \in \mathbb{B} \exists \bar{\Gamma} \in \bar{B} : \Gamma \succcurlyeq^* \bar{\Gamma}$

Proof (sketch). Follows by induction on the size of Γ .

In general, we cannot adhere to a particular reduction pattern, but it is crucial that one can freely choose the eventual normal form, thus it is required that reduction is confluent.

Lemma 10. Reduction is confluent; that is, $\forall \Gamma, \Gamma', \Gamma'' \in \mathbb{B} \exists \Gamma''' \in \mathbb{B}$ such that if $\Gamma \succcurlyeq^* \Gamma'$ and $\Gamma \succcurlyeq^* \Gamma''$, then $\Gamma' \succcurlyeq^* \Gamma'''$ and $\Gamma'' \succcurlyeq^* \Gamma'''$.

Proof (sketch). This follows from lengthy induction on the size of bunches, considering the effect of permutation, and employing Newman's lemma [14]. See Appendix A for details. \Box

Reduction is instantiated in the sequent calculus by the structural rules, sequences of reduction are represented by *strategies*.

Definition 11 (Strategy). A derivation $\mathcal{D}: S \vdash S'$ is said to be a positive (resp. negative) strategy if every inference is either an instance of a contraction (resp. weakening) rule or exchange. A positive strategy is normalizing if and only if S' is normal.

Lemma 12. There is a reduction sequence σ for $S = S_0 \succeq ... \succeq S_n = S'$ if and only if there is a positive strategy $\mathcal{D}_{\sigma} : S \vdash S'$ where the (i,j) parent-child pair is (S_i, S_j) . Furthermore, if there is a positive strategy $\mathcal{D} : S \vdash S'$, then there is a negative strategy $\mathcal{D} : S' \vdash S$ containing the same sequents.

Proof. The first claim follows from observing that the conditions defining reduction describe instances of the structural rules. The second claim is immediate since a reverse reading an instance of contraction is an instance of weakening. \Box

The invertibility of reduction allows the proof-search problem to be reduced to considering normal forms, which are, generally speaking, better behaved.

Lemma 13. A sequent is provable if and only if any normal form is provable; that is, $\vdash_{\mathsf{LBI}} S \iff \vdash_{\mathsf{LBI}} \bar{S}$.

Proof. Each direction follows immediately from Lemma 12 and Lemma 9. \Box

3.2 Regimented Proofs

Gentzen's control régime is not implemented in LBI, but rather in a variant that has the SGP and in which depth can be controlled. The system arises from LBI by restricting all the rules to introduces a bounded number of sub-bunches, and by embedding the unit laws \varnothing_+ and \varnothing_\times into the other rules.

Definition 14 (System dLBI). System dLBI is composed of the rules in Figure 2, where Σ is a normal bunch.

Since the unit laws for \varnothing_{\times} and \varnothing_{+} are no longer available in dLBI, a positive strategy in dLBI does not include reductions of the form $\Gamma(\Delta, \varnothing_{\times}) \succcurlyeq \Gamma(\Delta)$ or $\Gamma(\Delta, \varnothing_{+}) \succcurlyeq \Gamma(\Delta)$. Though seemingly limiting, the additional rules in dLBI are designed to make the restriction adequate.

The class of regimented proofs is characterized by having proofs repeating the following three phases in succession:

$$\overline{A \Rightarrow A} \stackrel{\mathsf{Ax}}{} \overline{E(\bot)} \Rightarrow \phi \stackrel{\mathsf{L}'}{} \overline{\varnothing_{\mathsf{X}}} \Rightarrow 1 \stackrel{\mathsf{IR}}{} \overline{\varnothing_{+}} \Rightarrow \mathsf{T} \stackrel{\mathsf{TR}}{} \overline{\mathsf{R}}$$

$$\frac{\Delta' \Rightarrow \phi \quad \Gamma(\Delta'', \psi) \Rightarrow \chi}{\Gamma(\Delta', \Delta'', \phi * \psi) \Rightarrow \chi} \stackrel{\mathsf{*L}}{} \frac{\Delta, \phi \Rightarrow \psi}{\Delta \Rightarrow \phi * \psi} \stackrel{\mathsf{*R}}{} \overline{\mathsf{R}}$$

$$\frac{\Delta \Rightarrow \phi \quad \Gamma(\varnothing_{\mathsf{X}}, \psi) \Rightarrow \chi}{\Gamma(\Delta, \phi * \psi) \Rightarrow \chi} \stackrel{\mathsf{*L}_{1}}{} \frac{\varnothing_{\mathsf{X}}}{} \Rightarrow \phi \quad \Gamma(\Delta, \psi) \Rightarrow \chi}{\Gamma(\Delta, \phi * \psi) \Rightarrow \chi} \stackrel{\mathsf{*L}_{2}}{} \overline{\mathsf{R}}$$

$$\frac{\varnothing_{\mathsf{X}}}{} \Rightarrow \phi \quad \Gamma(\varnothing_{\mathsf{X}}, \psi) \Rightarrow \chi}{\Gamma(\phi * \psi) \Rightarrow \chi} \stackrel{\mathsf{*R}_{1}}{} \frac{\Gamma \Rightarrow \phi \quad \varnothing_{\mathsf{X}}}{} \Rightarrow \psi} \stackrel{\mathsf{*R}_{2}}{} \overline{\mathsf{R}}$$

$$\frac{\varphi_{\mathsf{X}}}{} \Rightarrow \phi \quad \Gamma \Rightarrow \psi}{} \stackrel{\mathsf{*R}_{1}}{} \frac{\Gamma \Rightarrow \phi \quad \varnothing_{\mathsf{X}}}{} \Rightarrow \psi} \stackrel{\mathsf{*R}_{2}}{} \overline{\mathsf{R}}$$

$$\frac{\Gamma(\phi, \psi) \Rightarrow \chi}{\Gamma(\phi * \psi) \Rightarrow \chi} \stackrel{\mathsf{*L}}{} \frac{\Delta \Rightarrow \phi \quad \Delta' \Rightarrow \psi}{} \Rightarrow \chi} \stackrel{\mathsf{*R}_{2}}{} \frac{\Gamma(\varnothing_{\mathsf{X}}) \Rightarrow \chi}{\Gamma(\mathsf{T}^{*}) \Rightarrow \chi} \stackrel{\mathsf{*L}}{} 1$$

$$\frac{\Gamma(\phi, \psi) \Rightarrow \chi}{\Gamma(\phi \land \psi) \Rightarrow \chi} \wedge_{\mathsf{L}} \frac{\Delta \Rightarrow \phi \quad \Delta' \Rightarrow \psi}{\Delta, \chi} \stackrel{\mathsf{*R}}{} \frac{\Gamma(\varnothing_{\mathsf{X}}) \Rightarrow \chi}{\Gamma(\mathsf{T}^{*}) \Rightarrow \chi} \stackrel{\mathsf{*L}}{} 1$$

$$\frac{\Gamma(\phi, \psi) \Rightarrow \chi}{\Gamma(\phi \land \psi) \Rightarrow \chi} \wedge_{\mathsf{L}} \frac{\Delta \Rightarrow \phi \quad \Delta' \Rightarrow \psi}{\Delta, \chi} \wedge_{\mathsf{R}} \frac{\Gamma(\varnothing_{\mathsf{X}}) \Rightarrow \chi}{\Gamma(\mathsf{T}^{*}) \Rightarrow \chi} \stackrel{\mathsf{*L}}{} 1$$

$$\frac{\Gamma(\phi) \Rightarrow \chi}{\Gamma(\phi \land \psi) \Rightarrow \chi} \vee_{\mathsf{L}} \frac{\Delta \Rightarrow \phi}{\Delta, \chi} \stackrel{\mathsf{*L}}{} 2 \Rightarrow \phi \vee_{\mathsf{V}} \vee_{\mathsf{R}_{2}} \stackrel{\mathsf{*L}}{} 2 \Rightarrow \phi \vee_{\mathsf{V}} 2 \Rightarrow \phi \vee_{\mathsf{V}} 2 \Rightarrow \phi \vee_{\mathsf{V}} \stackrel{\mathsf{*L}}{} 2 \Rightarrow \phi \vee_{\mathsf{V}} 2 \Rightarrow \phi \vee_$$

Fig. 2. Sequent Calculus dLBI

- Action: Logical deductions are made; that is, information is introduced, combined, or modified for the sequents so far constructed;
- Normalization: Any computational effects of the action phase, such as the introduction of extraneous data, are removed; and
- Loading: The extraneous data required for the next action are added.

It is important to note that actions are not merely logical rules. Firstly, because action refers to an instance of a rule, not its shape. Secondly, because some instances of weakening add information to the sequent. Call a sub-bunch duplicit if it is additively combined with a permutation of itself. Suppose the bunch introduced by weakening is duplicit, then no actual information has been added, but if it is not duplicit, then the overall information of the context has increased. The first use is a *loading* inference, and the second is an *action*.

Definition 15 (Action). An inference $\mathbf{R}(P_0,...,P_n,C)$ is an action if the rule \mathbf{R} is an logical rule, an axiom; or, \mathbf{R} is one of \mathbf{W}' or conv , with the constraint that the introduced bunch Σ is not duplicit in the context of C. An action is regimented when all duplicit sub-bunches of the premisses are active in the rule application.

Example 16. In the following instance of \rightarrow_L , occurrences of formulas are labelled by superscripts that are assigned using the smallest fresh natural number available when reading the sequent from the left:

$$\frac{\phi^0 \, ; \, \phi^1 \implies \phi^2 \qquad \psi^0 \, ; \, \psi^1 \implies \chi^0}{\phi^0 \, ; \, \phi^1 \, ; \, \psi^0 \, ; \, (\phi \to \psi)^0 \implies \chi^0} \qquad \frac{\phi^0 \implies \phi^1 \qquad \psi^0 \, ; \, \psi^1 \implies \chi^0}{\phi^0 \, ; \, \psi^0 \, ; \, (\phi \to \psi)^0 \implies \chi^0}$$

The first inference is unregimented since in the left antecedent ϕ^0 and ϕ^1 are duplicit but not active. Meanwhile, the second inference is regimented since, though the right antecedent contains ψ^0 and ψ^1 , they are both active.

Let $\mathbf{R}(P_0, ..., P_n, C)$ be a regimented action, let $\hat{P}_0,, \hat{P}_n$ denote the sequents resulting from P_i after removing (through reduction) any bunches not active in the inference, and denote \hat{C} for the result of applying \mathbf{R} to the same active bunches of $\hat{P}_0, ..., \hat{P}_n$.

Lemma 17. For any action, \hat{C} reduces to a normal form of C.

Proof. Let σ_i be sequences witnessing $P_i \succcurlyeq^* \hat{P}_i$ and respectively. The applicable reductions in the composition $\sigma_0, ..., \sigma_n$ is a reduction sequence witnessing $C \succcurlyeq^* \hat{C}$. It follows from Lemma 10 that $\hat{C} \succcurlyeq^* \bar{C}$, as required.

What remains to define is the *loading* phase, which is precisely the introduction of data required to make the next action.

Definition 18 (Loading Strategy). Let $\mathbf{R}(P_0,...,P_n,C)$ be an action, any negative strategy witnessing $\bar{P}_i \geq^* \hat{P}_i$ is a loading strategy.

Definition 19 (Regimented Derivations). A derivation is regimented if every action is regimented, is preceded by a negative strategy, and is succeeded by a loading strategy.

If $\mathcal{D}: \Sigma \vdash_{\mathsf{L}} S$ is a regimented L-derivation, then it may be denoted $\mathcal{D}: \Sigma \Vdash_{\mathsf{L}} S$. The normalizing and loading phases may be empty if the conclusion of the action is normal or nothing needs to be loaded for the next action.

3.3 Completeness of dLBI

To show that dLBI is complete, we partially re-introduce a restricted form of the unit laws of coherent equivalence, which are otherwise absent, as the rad rule.

Definition 20 (rad). The rad rule allows a multiplicatively combined \varnothing_+ in a context to be removed along side the context-former,

$$\frac{\Gamma(\Delta,\varnothing_+) \implies \chi}{\Gamma(\Delta) \implies \chi} \text{ rad}$$

When present, an instance of the rad rule is an action.

Lemma 21. A normal sequent is LBI-provable if and only if it has a regimented dLBI + rad-proof; that is, $\vdash_{LBI} \bar{S} \iff \vdash_{dLBI+rad} \bar{S}$.

Proof (sketch). (\iff) Immediate, since the rules of dLBI + rad are derivable in LBI, thus a proof in the former system can be transformed into a proof in the latter system.

(\Longrightarrow) Consider system sLBI, which is LBI replacing W, C, and E by W', $W_{\varnothing_+}, W_{\varnothing_+}, C', E'$, together with

$$\frac{\Gamma(\Delta,\varnothing_{\times}) \Longrightarrow \chi}{\Gamma(\Delta) \Longrightarrow \chi} \mathsf{C}_{\varnothing_{\times}} \qquad \frac{\Gamma(\Delta,\varnothing_{+}) \Longrightarrow \chi}{\Gamma(\Delta) \Longrightarrow \chi} \mathsf{C}_{\varnothing_{\times}}$$

Any LBI-proof of a normal sequent can be transformed into a sLBI-proof by applying Gentzen's method of normalizing and loading, thus sLBI-proofs are complete for normal forms. Any instance of $C_{\varnothing_{\times}}$ and $C_{\varnothing_{+}}$ can be permuted upward and eliminated by adding $-*_{L1}$, $-*_{L2}$, $-*_{L3}$, $*_{R1}$, $*_{R2}$, \rightarrow_{L1} , \rightarrow_{L2} , \rightarrow_{L3} , \land_{R1} , \land_{R2} , and conv to the system. The result is a regimented dLBI-proof.

See Appendix B for details of both directions.

The rad rule is an intermediate step in the eliminations of the unit-law (explicitly given as $C_{\varnothing_{\times}}$ and $C_{\varnothing_{+}}$), but has the same undesired effect as $C_{\varnothing_{\times}}$.

Lemma 22. A normal sequent has a regimented dLBI+rad-proof if and only if it has a regimented dLBI-proof; that is, $\Vdash_{\mathsf{dLBI+rad}} \bar{S} \iff \Vdash_{\mathsf{dLBI}} \bar{S}$.

Proof (sketch). Let \mathcal{D} be a dLBI + rad-proof, and consider an arbitrary instance of rad in the proof. If the unit on which the rule is applied is not active in the inference preceding it, then the two rules may trivially be permuted. Suppose this has been done, the loading strategy preceding the rad is empty as it is not possible for the unit to be active in both inferences, thus the rad follows either from another regimented action or from a normalizing strategy. In the

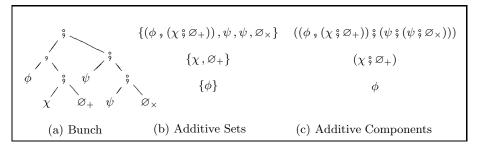


Fig. 3. Bunch with Additive Sets and Components.

first case, the inferences may be replaced by either one or two other regimented actions neither of which is rad, and in the second case the rad may be permuted upwards. Permuting and rewriting in this way, beginning with the topmost rad in \mathcal{D} produces a regimented proof \mathcal{D}' , which is rad-free, as required. See Appendix C for details.

4 The Proof-search Space

It remains to show that restricting attention to regimented dLBI-proof is a sufficient control régime. To do this, we formally introduce three measures, μ , ω , and δ , and show that bounding them is sufficient to yield a primitive recursive search space. We then establish some effective bounds.

4.1 Bunches and Data

Defining the functions that measure data content, especially extraneous data content, first requires an understanding of how data in bunches is structured.

Definition 23 (Types of Data). Let $\Delta \subseteq \Gamma$, then Δ is said to be additive (resp. multiplicative) data if it complex with principal context-former multiplicative (resp. additive), or it is basic.

Definition 24 (Additive Sets and Components). Let $\Delta_1, \Delta_2 \leq \Gamma \in \mathbb{B}$ both be additive data. The relation $\Delta_1 \sim \Delta_2$ holds if and only if there is a path from the principal context-former of Δ_1 to the principal context-former of Δ_2 in the parse-tree of Γ passing through only additive context-formers. The equivalence classes $\{\Delta \in \mathbb{B} \mid \Delta \leq \Gamma\}/\sim$ are the additive sets of the bunch. The least subbunch of Γ (with respect to the sub-bunch relation) containing every member of an additive set is its additive component.

Example 25. In Figure 3, a bunch together with its additive sets and components is presented. The additive sets are determined by choosing some additive data and finding all other additive data that can be reached when passing though only additive context-formers.

4.2 The Search Space

There are two *widths* of bunches that must be studied, additive and multiplicative, but both are simply the number of times a certain construction is allowed to appear.

Additive width is captured by *multiplicity*: the number of occurrences of additive data in the same environment. The number of occurrences of a *given* sub-bunch is captured by *duplicity*.

Definition 26 (Duplicity). Let $\Delta \subseteq \Gamma$. Then its duplicity set is the collection of occurrences of permutation in its neighbourhood,

$$d\Delta = \{ \Delta' \triangleleft \Gamma \mid \Delta \cong \Delta' \text{ and } \Delta \sim \Delta' \}$$

The duplicity of a sub-bunch Δ in Γ is $|d\Delta|-1$, and the duplicity of an additive set A is $\partial A := \sup_{\Delta \in A} |d\Delta| - 1$. Duplicity sets and additive sets are trivial if they have zero duplicity, and a bunch is duplicit if its duplicity set is non-trivial.

Gentzen's bound for intuitionistic logic can be stated formally as $\partial A \leq 2$ for every sequent in a certain class of proofs. However, the requirement is actually a simultaneous bound over all additive sets.

Definition 27 (Multiplicity). Let $\Gamma \in \mathbb{B}$, and let \mathcal{A} be the set of all additive sets in the bunch. Then the multiplicity of Γ is defined by

$$\mu(\Gamma) := \sum_{A \in \mathcal{A}} \partial A = \sum_{A \in \mathcal{A}} \left(\sup_{\Delta \in A} |d\Delta| - 1 \right)$$

The subtraction is essential as it allows the sum to characterize extraneous data, in the sense of data occurrences that are repeated without adding information. In the case of a single additive set (e.g., when working in the additive fragment of BI), the measure recovers Gentzen's bound of $\mu(\Gamma) < 2$.

Example 28. Let Γ be the bunch in Figure 3, then ψ is duplicit as $|d\psi| = 2$. This is the only non-trivial duplicity set, and $\mu(\Gamma) = 1$.

The steady growth property (SGP) is precisely the statement that there is a $k \in \mathbb{N}$ such that, for every rule R, and any sequents $P_0, ..., P_1, C$, in the sequent calculus,

$$\mathbf{R}(P_0, ...P_n, C) \implies \mu(C) \le \max\{\mu(P_0), ..., \mu(P_n)\} + k$$

The inference, $\mathbf{W}(\varnothing_+ \Longrightarrow A, \varnothing_+ \, \S(\varnothing_+)^n \Longrightarrow A)$, where $(\varnothing_+)^n = \varnothing_+ \, \S \dots \, \S \, \varnothing_+$ with *n*-occurrences, witnesses that LBI does not have the SGP, and justifies the restriction of the structural rules when forming dLBI.

Multiplicative width is less complex, since multiplicative data is characterized by not proliferating during proof-search. This holds for BI, with the exception of the multiplicative unit \varnothing_{\times} , because of the unit laws of coherent equivalence.

Definition 29 (Multiplicative Width). The multiplicative width of a bunch is defined by the following:

$$\omega(\Gamma) = \begin{cases} 0 & \text{if } \Gamma \in \mathbb{A} \cup \{\bot, \top, 1, \varnothing_+, \varnothing_\times\} \\ \max\{\omega(\psi_1), \omega(\psi_2)\} & \text{if } \Gamma = \psi_1 \circ \psi_2 \text{ for } \circ \in \{\land, \to, \lor\} \\ \omega(\psi_1) + \omega(\psi_2) + 1 & \text{if } \Gamma = \psi_1 \circ \psi_2 \text{ for } \circ \in \{*, -*\} \\ \max\{\omega(\Delta), \omega(\Delta')\} & \text{if } \Gamma = (\Delta; \Delta') \\ \omega(\Delta) + \omega(\Delta') + 1 & \text{if } \Gamma = (\Delta, \Delta') \end{cases}$$

The depth of a bunch is, heuristically, the maximal number of layers of multiplicative context-formers between leaf and root. The presence of permutation as an equivalence relation on bunches means that the *layers* are characterized by a change in context-former, as opposed to merely distance from the root.

Definition 30 (Topset). Let Γ be a complex bunch. Denote by $\Delta \in \Gamma$ that $\Delta \triangleleft \Gamma$, with the additional constraint that Δ is of a different type to Γ and that there are no context-former alternations between the principal context-former of Δ and the root of Γ

Example 31. Let Γ be the bunch in Figure 3, then $(\phi, (\chi; \varnothing_+)), \psi, \psi, \varnothing_\times \in \Gamma$, since in each case there are no context-former alternations between their principal context-formers and the root of the bunch. However, we do not have $\psi : \varnothing_\times \in \Gamma$, since, although there are no alternations, its principal context-former is the same as the root of the bunch.

Definition 32 (Depth). The depth of a bunch is defined as follows:

$$\delta(\Gamma) = \begin{cases} \omega(\Gamma) & \text{if } \Gamma \in \mathbb{F} \cup \{\varnothing_+, \varnothing_\times\} \\ \max\{\delta(\Pi) \mid \Pi \in \Gamma\} & \text{if } \Gamma \in \mathbb{B}^+ \\ \max\{\delta(\Sigma) \mid \Sigma \in \Gamma\} + 1 & \text{if } \Gamma \in \mathbb{B}^\times \end{cases}$$

Example 33. Let Γ be the bunch in Figure 3, and suppose the formulas contain no multiplicative connectives, then since there is only one context-former alternation downward, we see $\delta(\Gamma) = 1$.

The measures of width and depth extend to sequents by distributing over the projections, and to proofs \mathcal{D} by taking the least uniform bound; that is, for $f \in \{\mu, \omega, \delta\}$,

$$f(\Gamma \implies \phi) := f(\Gamma) + f(\phi)$$
 and $f(\mathcal{D}) := \sup_{S \in \mathcal{D}} f(S)$

Bounds in these measures suffice to limit the space to a primitive recursive set. Let Σ be a set of formula and let $\int(\Sigma)$ denote the set of sequents that are constructable from those formulas.

Lemma 34. Let $a, m, d \in \mathbb{N}$ be arbitrary and Σ be a finite set of formula, then the following set is primitive recursive:

$$\mathbb{S}^{\varSigma}_{a,m,d} := \{S \in \digamma(\varSigma) \mid \mu(S) \leq a \text{ and } \omega(S) \leq m \text{ and } \delta(S) \leq d\}$$

Proof (sketch). We effectively generate a set of bunches Γ out of formulas of Σ by iteratively taking additive and multiplicative combinations of size a and m, respectively, from previously constructed bunches, beginning with the base case of Σ . At the nth iteration, all the bunches of depth n have been constructed, so we stop the inductive process after d time, collecting all the bunches in a set $\mathcal{V}^{a,m,d}$, which is primitive recursive. The set $\mathcal{V}^{a,m,d}$ contains all bunches Γ satisfying the bounds $\mu(\Gamma) \leq a$, $\omega(\Gamma) \leq m$, and $\delta(\Gamma) \leq d$, thus $\mathbb{S}^{\Sigma}_{a,m,d} \subseteq \mathcal{V}_{a,m,d} \times \Sigma$. Since the separating predicate decidable and the bounding set is primitive recursive, so is $\mathbb{S}^{\Sigma}_{a,m,d}$. See Appendix Γ for details.

4.3 Bounds

We now show that appealing to regimented dLBI-proofs suffices to have computable bounds for multiplicity, multiplicative width, and depth.

First, in the absence of the unit-law for \varnothing_{\times} , control over multiplicative width is automatic.

Lemma 35. The multiplicative width of sequents in a regimented proof is uniformly bounded by the multiplicity of the end-sequent; that is, $\mathcal{D}: \varnothing \Vdash \bar{S} \Longrightarrow \omega(\mathcal{D}) \leq \omega(\bar{S})$.

Proof. Suppose there was a sequent S' in the proof such that $\omega(S') > \omega(\bar{S})$. Let S'' be an immediate successor of S', then since for every rule in dLBI the conclusion is at least as multiplicatively wide as the premiss, we have $\omega(S'') > \omega(\bar{S})$ too. Taking such successors can be repeated *ad infinitum*, but this is impossible since proofs are finite trees. Hence it must be the case that $\omega(\mathcal{D}) \leq \omega(\bar{S})$.

Second, multiplicity may be bounded by witnessing the effect of Gentzen's control régime.

Lemma 36. The multiplicity of sequents in a regimented proof is uniformly bounded above by three; that is, $\mathcal{D}: \varnothing \Vdash \bar{S} \implies \mu(\mathcal{D}) \leq 3$.

Proof (sketch). The following claims can be proved by contradiction:

- Suppose all duplicit sub-bunches of $\Gamma(\Delta)$ are contained in Δ , then for any Δ' , the bound $\mu(\Gamma(\Delta')) \leq 1 + \mu(\Delta')$ holds;
- Let $\Gamma = \Pi_1 \circ \Pi_2$, where $\circ \in \{9, 9\}$ and Π_1, Π_2 are normal, then $\mu(\Gamma) \leq 1$

It follows by inspection that, if $\mathbf{R}(P_0,...,P_n,C)$ is a regimented action, then $\mu(P_0),...,\mu(P_n), \ \mu(C) \leq 3$, and, by induction on the length, that if \mathcal{D} is a strategy with root S, then $\mu(D) \leq \mu(S)$. Hence, by induction on the number of steps (i.e., use of a loading strategy, action, and normalizing strategy) that the result holds. See Appendix D for details.

Finally, we attend to depth. However, even in the case of regimented proofs, due to the interaction between the resolution rules (i.e., the implication left rules) and contraction, it can increase during proof-search.

Example 37. Consider the following proof where neither ϕ nor ψ contains a multiplicative connective:

$$\begin{array}{c|c} & \varnothing_{+} \Longrightarrow \top \\ \hline (\phi \twoheadrightarrow \top) \rightarrow \psi \text{, } \phi \Longrightarrow \top \\ \hline (\phi \twoheadrightarrow \top) \rightarrow \psi \Longrightarrow (\phi \twoheadrightarrow \top) \\ \hline (\phi \twoheadrightarrow \top) \rightarrow \psi \Longrightarrow (\phi \twoheadrightarrow \top) \rightarrow \psi \Longrightarrow \top \\ \hline (\phi \twoheadrightarrow \top) \rightarrow \psi \text{, } (\phi \twoheadrightarrow \top) \rightarrow \psi \Longrightarrow \top \\ \hline (\phi \twoheadrightarrow \top) \rightarrow \psi \Longrightarrow \top \\ \hline \end{array} \text{C'}$$

Let A be the left premiss of the $-*_L$ rule, and B be the premiss of the C rule, then observe that depth has indeed increased, since $\delta(A) = 2$ and $\delta(B) = 1$. \square

This example witnesses the only way in which depth can increase. Fortunately the phenomenon does not compound and can be bounded *a priori*.

Lemma 38. The depth of sequents in a regimented proof is uniformly bounded by twice the depth of the end-sequent; that is, $\mathcal{D}: \varnothing \Vdash \bar{S} \implies \delta(\mathcal{D}) \leq 2\delta(\bar{S})$.

Proof (sketch). By properly implementing a formal track-and-trace scheme, one can effectively analyse where and how the undesirable increases in depth happen. This is based on the inductive labelling of proofs, which annotates each multiplicative context-former and connective with a label that is copied only in uses of contraction. Thus we see that it is only through a combination of contraction and the resolution rules that depth increases, which happens when the same label occurs both on a context-former and a connective. However, since the formula is necessarily below the context-former, the value can at most double. See Appendix E for details.

Since the measures have all been bounded, a proof-search procedure follows immediately.

Theorem 39. There is an effective proof-search procedure for LBI.

Proof. For any putative goal S, we reduce the proof-search problem as follows:

$$\vdash_{\mathsf{LBI}} S \quad \xleftarrow{\mathsf{Lemma} \ 13} \quad \vdash_{\mathsf{LBI}} \bar{S} \xleftarrow{\mathsf{Lemma} \ 21} \quad \Vdash_{\mathsf{dLBI}+\mathsf{rad}} \bar{S} \xleftarrow{\mathsf{Lemma} \ 22} \quad \Vdash_{\mathsf{dLBI}} \bar{S}$$

Call a proof *concise* if no sequent appears twice in the same branch; we may restrict attention to concise regimented proofs by removing any section of the branch of a given proof where a repetition occurs.

Now consider a concise regimented dLBI-proof \mathcal{D} for \bar{S} , then by Lemma 35, Lemma 36, and Lemma 38, we have $\omega(\mathcal{D}) \leq \omega(\bar{S}) = m$, $\mu(\mathcal{D}) \leq 3 = a$, and $\delta(\mathcal{D}) \leq 2\delta(\bar{S}) = d$. Let Σ be the set of formulas in \bar{S} closed under subformula relation, and note that this finite since S has finitely many formulas and the subformula relation is a well-order, thus it follows from Lemma 34 that $S = \mathbb{S}^{\Sigma}_{a,m,d}$, is primitive recursive. Observe that all the sequents of \mathcal{D} must appear in this set.

The rules of dLBI require at most two premisses, and there are no more than $2^{|S|}$ concise binary tree with labels from S ending with \bar{S} . Denote the set of such trees by \mathfrak{D} and note that it is also primitive recursive. If \mathcal{D} exists then $\mathcal{D} \in \mathfrak{D}$, thus checking correctness of the proof candidates completes the proofsearch procedure.

5 Conclusion

We have shown, using traditional and exclusively syntactic techniques from proof theory, that one can restrict the proof-search space for BI, with respect to its sequent calculus, to a primitive recursive set. Analysing Gentzen's decidability argument for classical (and intuitionistic logic) gave a partial solution, but BI combines additional multiplicative structures, and the latter are poorly behaved in the sense that they admit a limited form of contraction in the unit laws. The problem is solved using the ideas of Dyckhoff's contraction-elimination argument for intuitionistic logic, where a sufficient amount of the offending rule is embedded into the remaining rules so that it becomes impotent. Effective proof-search is a corollary.

The analysis of Gentzen's methodology yields the notion of a regimented proof, and future work includes exploring the significance of these proofs within the study of proof-search in particular, and proof theory in general. Moreover, since the bounding of regimented proofs largely entails an analysis of data and information of structures (formulas and bunches) within a sequent, one might also consider the semantic significance of such proofs.

Curiously, the neighbourhood of BI is largely undecidable; that is, there cannot be a decision procedure for Boolean BI, Separation Logic, or Classical BI [3,4,11,12]. The analogous treatment of regimented proofs in their respective sequent calculi (of the same general form) fails because cut-free proofs are not complete [2]. However, there is a uniform proof theory using hypersequent calculi for a class of logics related to BI for which the problem remains open [6]. A parallel treatment of substructural logics has been completed successfully [5,17].

Finally, one may consider the relationship between the combinatorial and proof theoretic techniques presented here with Kripke's tour de force decidability argument for the full Lambek calculus with contraction [10,19], which has also seen successful generalization [1,13,17]. In particular, one might consider to what extent the combinatorics are related.

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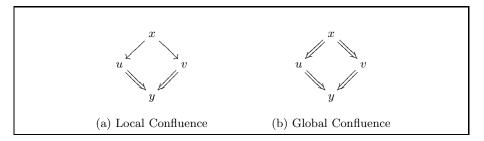


Fig. 4. Types of Confluence.

A Reduction is Confluent

A reduction $\Gamma \geq \Gamma'$ is proper, denoted $\Gamma > \Gamma'$ and $\Gamma \succ \Gamma'$ for big-step and small-step reductions, respectively, if $\Gamma \ncong \Gamma'$.

Definition 40 (Local and Global Confluence). A relation \downarrow with reflexive and transitive closure \Downarrow is said to be locally confluent (resp. globally confluent) if, whenever $x \downarrow u$ and $x \downarrow v$ (resp. $x \Downarrow u$ and $x \Downarrow v$) there is y such that $u \Downarrow y$ and $v \Downarrow y$.

Global confluence (henceforth: confluence) follows immediately from local confluence for terminating relations. However, reduction is only terminating upto permutation, thus we work instead with equivalence classes. Reduction is then replaced by a well-founded relation called *class reduction*.

For $\Gamma \in \mathbb{B}$ denote its equivalence class under permutation by,

$$[\Gamma] := \{ \Gamma' \in \mathbb{B} \mid \Gamma' \cong \Gamma \}$$

Let $\mathcal{G}, \mathcal{G}' \in \mathbb{B}/_{\cong}$ then class reduction is defined by,

$$\mathcal{G} \hookrightarrow \mathcal{G}' \iff \exists \Gamma', \Gamma'' \in \mathbb{B} \text{ such that } (\Gamma \in \mathcal{G}) \text{ and } (\Gamma' \in \mathcal{G}') \text{ and } (\Gamma \succ \Gamma')$$

To establish a correspondence between the results on bunches and their equivalence classes, we use the following measure which can be regarded as a extended quasi-metric on \mathbb{B} :

$$d(\Gamma, \Gamma') := \min\{n \mid \Gamma = \Gamma_0 \succcurlyeq \ldots \succcurlyeq \Gamma_n = \Gamma'\}$$

Lemma 41.
$$[\Gamma] \hookrightarrow^* [\Gamma'] \iff \Gamma \succcurlyeq^* \Gamma'$$

Proof. (\Rightarrow) Let Γ'' be such that $[\Gamma] \hookrightarrow^* [\Gamma''] \hookrightarrow^* [\Gamma']$. We proceed by induction on the number of context-formers in Γ .

Base Case. If Γ is basic, then $\Gamma = \Gamma'' = \Gamma'$ as proper reduction is not possible. Hence $\Gamma \succcurlyeq^* \Gamma'$ by reflexivity.

Inductive step.If $\Gamma \cong \Gamma'$ then $\Gamma \succcurlyeq^* \Gamma'$ and we are done, otherwise, without loss of generality, $[\Gamma] \hookrightarrow [\Gamma'']$; that is, there are $\tilde{\Gamma} \in [\Gamma]$ and $\tilde{\Gamma}'' \in [\Gamma'']$ such

that $\Gamma \cong \tilde{\Gamma} \succ \tilde{\Gamma}'' \cong \Gamma''$. However, as the reduction is proper, Γ'' contains strictly fewer context-formers than Γ , thus it follows by inductive hypothesis that $\Gamma'' \succcurlyeq^* \Gamma'$. The claim follows from transitivity.

 (\Leftarrow) We proceed by induction on $d(\Gamma, \Gamma')$, the length of the minimal sequence of reductions.

Base Case. When $d(\Gamma, \Gamma') = 0$ we have $\Gamma = \Gamma'$ so the result holds immediately by reflexivity.

Inductive Step. Without loss of generality, $d(\Gamma, \Gamma') \geq 1$ so there is $\Gamma'' \in \mathbb{B}$ satisfying $\Gamma \succcurlyeq \Gamma'' \succcurlyeq^* \Gamma'$ in a sequence of reduction of minimal length. By the induction hypothesis, $[\Gamma] \hookrightarrow^* [\Gamma''] \hookrightarrow^* [\Gamma']$ and the claim follows by transitivity.

Lemma 42. Class reduction is locally confluent.

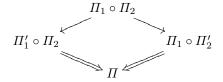
Proof. Suppose $[\Gamma] \hookrightarrow [\Gamma']$ and $[\Gamma] \hookrightarrow [\Gamma'']$, then denote the removed sub-bunch in each case $\hat{\Delta}_1$ and $\hat{\Delta}_2$ respectively. We proceed by induction on the number of context-formers in Γ .

Base Case. Since the reductions are proper Γ must have at least one context-former. It follows that either $\Gamma \cong (\Delta \, ; \, \Delta)$, or $\Gamma \cong (\Delta \, ; \, \varnothing_+)$, or $\Gamma \cong (\Delta \, , \, \varnothing_\times)$, where Δ is basic. In any case we have $\Gamma' = \Gamma'' = \Delta$ so the result holds by reflexivity as then $[\Gamma'] = [\Gamma'']$.

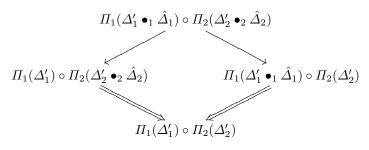
Inductive Step. We proceed by case analysis on the possible relationships of the removed sub-bunches. In the case where $\hat{\Delta}_1 \leq \hat{\Delta}_2$, then it cannot be that $\hat{\Delta}_1 \lhd \hat{\Delta}_2$ by normality of $\hat{\Delta}_2$, thus it must be that $\hat{\Delta}_1$ and $\hat{\Delta}_1$ are the same subbunch, but then confluence follows trivially; mutatis mutandis for $\hat{\Delta}_1 \leq \hat{\Delta}_2$. Therefore, without loss of generality, let $\Gamma = \Gamma(\Pi_1 \circ \Pi_2)$, where \circ is a context-former, and let $\hat{\Delta}_1 \leq \Pi_1$ and $\hat{\Delta}_2 \leq \Pi_2$. There are four cases to consider, and in each case local confluence follows from Lemma 41:

- $-\hat{\Delta}_1 = \Pi_1$ and $\hat{\Delta}_2 = \Pi_2$. By definition of reduction it must be that $\Pi_1 \circ \Pi_2 = \hat{\Delta}_2 \circ \Delta$ and $\Delta = \hat{\Delta}_1 = \hat{\Delta}_2$. Hence, $\Gamma' = \Gamma''$, as required.
- $-\hat{\Delta}_1 = \Pi_1$ and $\hat{\Delta}_2 \triangleleft \Pi_2$. It follows that there is $\Pi'_2 \in \mathbb{B}$ such that $\Gamma'' \cong \Gamma(\Pi_1 \circ \Pi'_2)$, where $\Pi_2 \succ \Pi'_2$. Thus $\Gamma' = \Gamma(\Pi_2) \succcurlyeq \Gamma(\Pi'_2)$. It cannot be that $\Pi_1 \equiv \Pi_2$, since then Π_1 is not normal contradiction the reduction hypothesis, hence Π_1 is still removable in Γ'' , whence $\Gamma'' \succcurlyeq^* \Gamma(\Pi'_2)$ as required.
- $-\hat{\Delta}_1 \triangleleft \Pi_1$ and $\hat{\Delta}_2 = \Pi_2$. Similar to preceding.
- $-\hat{\Delta}_1 \triangleleft \Pi_1$ and $\hat{\Delta}_2 \triangleleft \Pi_2$. There must be $\Pi'_1, \Pi'_2 \in \mathbb{B}$ such that $\Gamma' \cong \Gamma(\Pi'_1 \circ \Pi_2)$ and $\Gamma'' \cong \Gamma(\Pi_1 \circ \Pi'_2)$, where $\Pi_1 \succ \Pi'_1$ and $\Pi_2 \succ \Pi'_2$. There are two sub-cases to consider.

First, if $(\Pi_1 \circ \Pi_2) \triangleleft \Gamma$, then by inductive hypothesis there is $\Pi \in \mathbb{B}$ satisfying the following diagram,



Second, if $(\Pi_1 \circ \Pi_2) = \Gamma$, then $\Pi_1 = \Pi_1(\Delta'_1 \bullet_1 \hat{\Delta}_1)$ and $\Pi_2 = \Pi_2(\Delta'_2 \bullet_2 \hat{\Delta}_2)$, where \bullet_1 and \bullet_2 are context formers. The result follows from the following diagram:



Lemma 43. Class reduction is confluent.

Proof. Complexity of bunches is invariant under permutation; so, if $\mathcal{G} \hookrightarrow \mathcal{G}'$, then all members of \mathcal{G}' have strictly fewer context-formers than the bunches in \mathcal{G} . It follows that class reduction is terminating. Confluence follows from Newman's Lemma [14] and Lemma 42.

Lemma 44. Reduction is confluent.

Proof. Suppose $\Gamma \succcurlyeq^* \Gamma'$ and $\Gamma \succcurlyeq^* \Gamma''$, then, by Lemma 41, we have $[\Gamma] \hookrightarrow^* [\Gamma']$ and $[\Gamma] \hookrightarrow^* [\Gamma'']$. By Lemma 43, we have $\exists \Gamma'''$ satisfying $[\Gamma'] \hookrightarrow^* [\Gamma'''] * \hookleftarrow [\Gamma'']$. Confluence follows by a final use of Lemma 41.

B Completeness of dLBI

We denote $\mathcal{D}_{\sigma}: S \succcurlyeq^* S'$ for the positive strategy corresponding to the sequence σ , and $\bar{\mathcal{D}}_{\sigma}: S \ ^* \preccurlyeq S'$ for the associated negative strategy. Moreover, given a reduction sequence σ for $\Delta \succcurlyeq^* \Delta'$ we may treat it as a reduction sequence for $\Gamma(\Delta) \succcurlyeq^* \Gamma(\Delta')$ since the relation applies on any sub-bunch. Finally, we denote $\mathcal{D}: S \succcurlyeq^* S'$ if there is σ such that $\mathcal{D} = \mathcal{D}_{\sigma}$.

Definition 45 (System sLBI). System **sLBI** is composed of the rules in Figure 5, where Σ is a normal bunch.

Since sLBI is a restriction of LBI , soundness is immediate. Meanwhile, for completeness it is sufficient to show admissibility for each rule R in LBI ; that is

$$\mathbf{R}(P_0,...,P_n,C) \implies P_0,...,P_n \vdash_{\mathsf{sLBI}} C$$

Lemma 46 (Soundness and Completeness of sLBI). $\vdash_{LBI} S \iff \vdash_{sLBI} S$.

Proof. (\iff) Soundness is immediate as any instance of a sLBI rule is an instance of a LBI rule. (\implies) For completeness, let strategy refer to sLBI-strategy which is necessarily a sequence of small-step reductions. The operational rules of LBI are already rules of sLBI so admissibility is trivial, the same holds for all the axioms except \bot which may be simulated as follows:

$$\overline{A} \Longrightarrow \overline{A} \xrightarrow{A \times} \overline{\Sigma(\bot)} \Longrightarrow \phi \xrightarrow{\bot \bot'} \overline{\varnothing_{\times}} \Longrightarrow 1 \xrightarrow{1_{R}} \overline{\varnothing_{+}} \Longrightarrow \overline{\top} \xrightarrow{\top_{R}}$$

$$\frac{\Delta' \Longrightarrow \phi \quad \Gamma(\Delta'', \psi) \Longrightarrow \chi}{\Gamma(\Delta', \Delta'', \phi \twoheadrightarrow \psi)} \Longrightarrow \chi \xrightarrow{*_{L}} \frac{\Delta, \phi \Longrightarrow \psi}{\Delta \Longrightarrow \phi \twoheadrightarrow \psi} \xrightarrow{*_{R}}$$

$$\frac{\Gamma(\phi, \psi) \Longrightarrow \chi}{\Gamma(\phi \ast \psi) \Longrightarrow \chi} \xrightarrow{*_{L}} \frac{\Delta \Longrightarrow \phi \quad \Delta' \Longrightarrow \psi}{\Delta, \Delta' \Longrightarrow \phi \ast \psi} \xrightarrow{*_{R}} \frac{\Gamma(\varnothing_{\times}) \Longrightarrow \chi}{\Gamma(\top^{*}) \Longrightarrow \chi} \xrightarrow{1_{L}}$$

$$\frac{\Gamma(\phi, \psi) \Longrightarrow \chi}{\Gamma(\phi \land \psi) \Longrightarrow \chi} \xrightarrow{\Lambda_{L}} \frac{\Delta \Longrightarrow \phi \quad \Delta' \Longrightarrow \psi}{\Delta, \Delta' \Longrightarrow \phi \land \psi} \xrightarrow{\Lambda_{R}} \frac{\Gamma(\varnothing_{+}) \Longrightarrow \chi}{\Gamma(\top) \Longrightarrow \chi} \xrightarrow{T_{L}}$$

$$\frac{\Gamma(\phi) \Longrightarrow \chi}{\Gamma(\phi \lor \psi) \Longrightarrow \chi} \xrightarrow{V_{L}} \frac{\Delta \Longrightarrow \phi}{\Delta \Longrightarrow \phi \lor \psi} \xrightarrow{\Lambda_{R}} \frac{\Delta \Longrightarrow \psi}{\Lambda} \xrightarrow{\Lambda_{R}} \xrightarrow{\Gamma(\Delta') \Longrightarrow \chi} \xrightarrow{\Gamma(\Delta')$$

Fig. 5. Sequent Calculus sLBI

$$\frac{\bar{\Gamma}(\bot) \Longrightarrow \phi}{\vdots} \\
\mathcal{D}: (\bar{\Gamma}(\bot) \Longrightarrow \phi) *_{\preccurlyeq} (\Gamma(\bot) \Longrightarrow \phi)$$

$$\vdots \\
\Gamma(\bot) \Longrightarrow \phi$$

It remains to analyse the structural rules. Let $R \in \{E, WC\}$, then for any instance $\mathbf{R}(P,C)$ let A be the context of P and let B the context of C. We show admissibility of each rule independently.

- E: Observe that $A \equiv B$ if and only if there is a transformation of the premise to the conclusion using the commutative monoid equations, but these equations are given by $\mathsf{C}_{\varnothing_{\times}}$, $\mathsf{C}_{\varnothing_{+}}$, $\mathsf{W}_{\varnothing_{+}}$, $\mathsf{W}_{\varnothing_{\times}}$, and E'. Thus, the proof starting with P and applying these rules in sequence and concluding C gives the desired simulation.
- W: We have $A = \Gamma(\Delta)$ and $B = \Gamma(\Delta; \Delta')$, where Δ' is some bunch. By Lemma 9, there is a normal form $\bar{\Delta}'$ of Δ' , and there is a sequence σ witnessing the reduction $\Delta' \succcurlyeq^* \bar{\Delta}'$. Consider the following proof:

$$\frac{\Gamma(\Delta) \Longrightarrow \phi}{\Gamma(\Delta \, ; \bar{\Delta}') \Longrightarrow \phi}$$

$$\vdots$$

$$\mathcal{D}_{\sigma} : (\Gamma(\Delta \, ; \bar{\Delta}') \Longrightarrow \phi) * \preccurlyeq (\Gamma(\Delta \, ; \Delta') \Longrightarrow \phi)$$

$$\vdots$$

$$\Gamma(\Delta \, ; \Delta') \Longrightarrow \phi$$

The proof is the desired simulation; it is sLBI-sound since negative strategies are sound and the first step is an instance of W'.

C': We have $A = \Gamma(\Delta \circ \Delta)$ and $B = \Gamma(\Delta)$. By Lemma ?? there is a normal form $\bar{\Delta}$ of Δ , and a sequence σ witnessing the reduction $\Delta \succeq^* \bar{\Delta}$. Consider the following proof,

$$\frac{\Gamma(\Delta;\Delta) \Longrightarrow \phi}{\vdots}$$

$$\mathcal{D}_{\sigma}: (\Gamma(\Delta;\Delta) \Longrightarrow \phi) \succcurlyeq^{*} (\Gamma(\Delta;\bar{\Delta}) \Longrightarrow \phi)$$

$$\vdots$$

$$\underline{\Gamma(\Delta;\bar{\Delta}) \Longrightarrow \phi}$$

$$\vdots$$

$$\mathcal{D}'_{\sigma}: (\Gamma(\Delta;\bar{\Delta}) \Longrightarrow \phi) \succcurlyeq^{*} (\Gamma(\bar{\Delta};\bar{\Delta}) \Longrightarrow \phi)$$

$$\vdots$$

$$\underline{\Gamma(\bar{\Delta};\bar{\Delta}) \Longrightarrow \phi}$$

$$\underline{\Gamma(\bar{\Delta}) \Longrightarrow \phi}$$

$$\vdots$$

$$\underline{\Gamma(\bar{\Delta}) \Longrightarrow \phi}$$

$$\vdots$$

$$\mathcal{D}''_{\sigma}: (\Gamma(\bar{\Delta}) \Longrightarrow \phi) * \preccurlyeq (\Gamma(\Delta) \Longrightarrow \phi)$$

$$\vdots$$

$$\underline{\Gamma(\Delta) \Longrightarrow \phi}$$

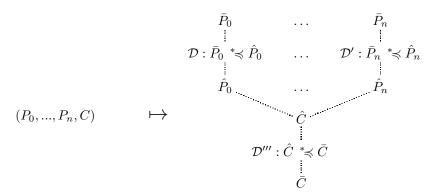
Since the positive and negative strategies are sLBI -sound and the only other inference is an instance of the C' rule, the whole proof is sLBI -sound. Since it satisfies the appropriate start and end criterion, it gives the desired simulation.

We now proceed to showing the completeness of regimented sLBI-proofs, as they can be transformed into regimented dLBI proofs.

Lemma 47. $\vdash_{\mathsf{sLBI}} \bar{S} \iff \Vdash_{\mathsf{sLBI}} \bar{S}$

Proof. (\Leftarrow) Immediate since regimented sLBI-proofs are sLBI-proofs. (\Rightarrow) Given an sLBI-proof of a sequent, denote \mathcal{T} for the tree whose nodes are tuples of actions in the given proof, and call it the *action tree*. The nodes in the action

tree may be expanded into regimented actions preceded by loading strategies and succeeded by normalising strategies as follows:



Hence, we may recursively rewrite \mathcal{T} into a regimented dLBI-proof. The inductive procedure is defined on the maximal number of inferences in a branch in the action tree:

Base Case. If \mathcal{T} is a single node then the original proof contained only one action, and it follows from the definition that this action was an axiom. By Lemma 10 and Lemma 12, the sequent may be extended with a normalising strategy with endsequent \bar{S} , which gives the desired regimented proof.

Inductive Step. Let $(P_0, ..., P_n, C)$ be the last node of the action tree \mathcal{T} , and let R be a rule witnessing the inference. Recur on the immediate sub-trees with end-sequents $P_0, ..., P_n$ to obtain regimented proofs \mathcal{D}_i for \bar{P}_i respectively. By Lemma 10 and Lemma 12, the proofs \mathcal{D}_i may be extended with loading strategies to yield proofs $\hat{\mathcal{D}}_i$ for \hat{P}_i , respectively. These proofs may be combined into a proof of \hat{C} , and it follows from Lemma 17 and Lemma 12 that this proofs can be extended with a normalising strategy to obtain a proof of \bar{C} . It follows from Lemma 10 that we can take, without loss of generality, $\bar{C} = \bar{S}$ as, by definition of \mathcal{T} , there are no actions between C and \bar{S} in the input proof.

The procedure is terminating since each recursion is on a strictly smaller action tree. By construction, \mathcal{D} is a regimented sLBI-proof with endsequent \bar{S} as required.

We turn now back to dLBI, and relate the two systems through the regimented proofs. Soundness is immediate.

Lemma 48. $\Vdash_{\mathsf{sLBI}} \bar{S} \iff \Vdash_{\mathsf{dLBI}} \bar{S}$.

Proof. All the inferences of dLBI are derivable in sLBI: the rules not already present (i.e., $-*_{L1}$, $-*_{L2}$, $-*_{L3}$, $*_{R1}$, $*_{R2}$, $-*_{L1}$, $-*_{L2}$, $-*_{L3}$, \wedge_{R1} , \wedge_{R2} , conv, rad) are shown below.

$$\frac{\Delta \Longrightarrow \phi \quad \Gamma(\varnothing_{\times}, \psi) \Longrightarrow \chi}{\Gamma(\Delta, \varnothing_{\times}, \phi * \psi) \Longrightarrow \chi} \mathsf{C}_{\varnothing_{\times}}$$

$$\frac{\Gamma(\Delta, \varphi * \psi) \Longrightarrow \chi}{\Gamma(\Delta, \varphi * \psi) \Longrightarrow \chi} \mathsf{C}_{\varnothing_{\times}}$$

$$\frac{\Gamma(\varnothing_{\times}, \Delta, \varphi * \psi) \Longrightarrow \chi}{\Gamma(\Delta, \varphi * \psi) \Longrightarrow \chi} \mathsf{C}_{\varnothing_{\times}}$$

$$\begin{array}{c} \underbrace{\frac{\varnothing_{\times} \Rightarrow \phi}{\Gamma(\varnothing_{\times},\varnothing_{\times},\phi * * \psi) \Rightarrow \chi}_{\Gamma(\varnothing_{\times},\varphi * \psi) \Rightarrow \chi}_{\Gamma(\varphi * \psi) \Rightarrow \chi}_{\Gamma(\varphi$$

For the completeness proof we must eliminate all instances of the C_{\varnothing_+} and C_{\varnothing_\times} rules in an arbitrary regimented sLBI proof, possibly by introducing one of the new rules of dLBI such as rad and conv. To do this, it is first helpful to know where the instances are.

Definition 49 (d-property). A sLBI \cup dLBI \cup {rad}-proof has the d-property if all instances of C_{\varnothing_+} and C_{\varnothing_\times} either follow an action or follow an instance of the same rule following an action.

We may write $\mathcal{D}: \varnothing \Vdash^{\mathsf{d}}_{\mathsf{L}} \bar{S}$ to denote that \mathcal{D} is a regimented L-proof which has the d property.

Lemma 50.
$$\Vdash_{\mathsf{sLBI}} \bar{S} \implies \Vdash_{\mathsf{sLBI}+\mathsf{rad}}^d \bar{S}$$

Proof. Call a unit a zero when it the child of a matching context-former, then $C_{\varnothing_{\times}}$ and $C_{\varnothing_{+}}$ are applicable if and only if a zero has been introduced. Without loss of generality, by Lemma 10, assume they are applied eagerly; that is, whenever a zero appears after an action or in a normalising phase the unit-law is immediately applied.

Consider the first instance of $C_{\varnothing_{\times}}$ or $C_{\varnothing_{+}}$ in the regimented sLBI-proof. It can follow C only if the following happens:

$$\frac{\Gamma(\Delta, (\varnothing_{\times}, \varnothing_{\times})) \Longrightarrow \chi}{\Gamma(\Delta, \varnothing_{\times}) \Longrightarrow \chi} C_{\varnothing_{\times}}$$

However, then it must be that the first sequent in the strategy contained an additive set of multiplicative units, and therefore that the action preceding the normalising phases can additively combine data not combined in the premisses. Hence, the action must be one of \land_R , \multimap_L , and W', but for none of these rules can the configuration occur:

- For \wedge_R the additive context-former would be principal in the bunch, but there is a multiplicative context-former above it;
- For $-*_L$ the additive set would contain an implication formula, but it does not;
- For W' the inference would not be an action since it introduces duplicity.

Hence we conclude that unit contractions follow either actions or other unit contractions. However, the case of C_{\varnothing_+} following C_{\varnothing_\times} ,

$$\frac{\Gamma(\Delta\,;(\varnothing_{\times}\,,\varnothing_{+})) \Longrightarrow \chi}{\Gamma(\Delta\,;\varnothing_{+}) \Longrightarrow \chi} \mathsf{C}_{\varnothing_{+}}$$

$$\frac{\Gamma(\Delta\,;\varnothing_{+}) \Longrightarrow \chi}{\Gamma(\Delta) \Longrightarrow \chi} \mathsf{C}_{\varnothing_{+}}$$

is also impossible as no action has the requisite conclusion. Meanwhile, the case of $C_{\varnothing_{\times}}$ following $C_{\varnothing_{+}}$ can only happen when the action was a weakening, in which case the inferences may be collectively replaced by rad.

$$\frac{\Gamma(\Delta\,,\varnothing_{+}) \implies \chi}{\frac{\Gamma(\Delta\,,(\varnothing_{+}\,;\varnothing_{\times})) \implies \chi}{\Gamma(\Delta\,,\varnothing_{\times}) \implies \chi}} \underset{\Gamma}{\mathsf{C}_{\varnothing_{+}}} \qquad \longmapsto \qquad \frac{\Gamma(\Delta\,,\varnothing_{+}) \implies \chi}{\Gamma(\Delta) \implies \chi} _{\mathsf{rad}}$$

We proceed by recursively rewriting the regimented sLBI-proof $\mathcal D$ using the following inductive procedure:

Base Case. If \mathcal{D} has only one action, then the action must be an axiom which all introduce normal bunches, hence it already satisfies the criterion.

Inductive Step. Consider the last action $\mathbf{R}(P_0,...,P_n,C)$, and recur on the sub-proofs of $P_0,...,P_n$ to achieve proofs satisfying the criterion. Thus all instance of $\mathsf{C}_{\varnothing_{\times}}$ and $\mathsf{C}_{\varnothing_{+}}$, remaining are below the action, and consider the first instance contradicting the d-property. From the above we see that the inferences must take the following form, which may be replaced by a single instance of rad.

The procedure halts because each recursion occurs on a strictly smaller sub-tree, and after each inductive step the number of instances not satisfying the criterion decreases. \Box

Lemma 51.
$$\Vdash^d_{\mathsf{sLBI+rad}} \bar{S} \implies \Vdash_{\mathsf{dLBI+rad}} \bar{S}$$

Proof. We can transform any regimented sLBI-proof with the d-property into a regimented dLBI-proof by eliminating all the instances of $C_{\varnothing_{\times}}$ or $C_{\varnothing_{+}}$. The rules are only applicable after certain instances of $*_R$, $-*_L$, \wedge_R , \rightarrow_L , and W', the transformations are as follows:

Cases of ∗_L:

$$\frac{\Delta \Longrightarrow \phi \quad \Gamma(\varnothing_{\times}, \psi) \Longrightarrow \chi}{\Gamma(\Delta, \varnothing_{\times}, \phi \twoheadrightarrow \psi) \Longrightarrow \chi} C_{\varnothing_{\times}} \qquad \longmapsto \qquad \frac{\Delta \Longrightarrow \phi \quad \Gamma(\varnothing_{\times}, \psi) \Longrightarrow \chi}{\Gamma(\Delta, \phi \twoheadrightarrow \psi) \Longrightarrow \chi} *_{\text{L1}}$$

$$\frac{\varnothing_{\times} \Longrightarrow \phi \quad \Gamma(\Delta, \psi) \Longrightarrow \chi}{\Gamma(\varnothing_{\times}, \Delta, \phi \twoheadrightarrow \psi) \Longrightarrow \chi} C_{\varnothing_{\times}} \qquad \longmapsto \qquad \frac{\varnothing_{\times} \Longrightarrow \phi \quad \Gamma(\Delta, \psi) \Longrightarrow \chi}{\Gamma(\Delta, \phi \twoheadrightarrow \psi) \Longrightarrow \chi} *_{\text{L2}}$$

$$\frac{\Gamma(\varnothing_{\times}, \Delta, \phi \twoheadrightarrow \psi) \Longrightarrow \chi}{\Gamma(\Delta, \phi \twoheadrightarrow \psi) \Longrightarrow \chi} C_{\varnothing_{\times}} \qquad \mapsto \qquad \frac{\varphi_{\times} \Longrightarrow \phi \quad \Gamma(\Delta, \psi) \Longrightarrow \chi}{\Gamma(\Delta, \phi \twoheadrightarrow \psi) \Longrightarrow \chi} *_{\text{L2}}$$

$$\frac{\varnothing_{\times} \Longrightarrow \phi \qquad \Gamma(\varnothing_{\times}, \psi) \Longrightarrow \chi}{\frac{\Gamma(\varnothing_{\times}, \varphi \to \psi) \Longrightarrow \chi}{\Gamma(\varphi \to \psi) \Longrightarrow \chi} C_{\varnothing_{\times}}} \mapsto \frac{\varnothing_{\times} \Longrightarrow \phi \qquad \Gamma(\varnothing_{\times}, \psi) \Longrightarrow \chi}{\Gamma(\varphi \to \psi) \Longrightarrow \chi} {}_{*L_{3}}$$

Cases of $*_R$:

$$\frac{\varnothing_{\times} \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Delta \Rightarrow \phi * \psi} {}^{*} C_{\varnothing_{\times}} \qquad \mapsto \qquad \frac{\varnothing_{\times} \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Delta \Rightarrow \phi * \psi} {}^{*} R_{1}$$

$$\frac{\varnothing_{\times} \Rightarrow \phi \qquad \varnothing_{\times} \Rightarrow \psi}{\Delta \Rightarrow \phi * \psi} {}^{*} C_{\varnothing_{\times}} \qquad \mapsto \qquad \frac{\varnothing_{\times} \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Delta \Rightarrow \phi * \psi} {}^{*} R_{1}$$

$$\frac{\varnothing_{\times} \Rightarrow \phi \qquad \varnothing_{\times} \Rightarrow \psi}{\varnothing_{\times} \Rightarrow \phi * \psi} {}^{*} C_{\varnothing_{\times}} \qquad \mapsto \qquad \frac{\varnothing_{\times} \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\varnothing_{\times} \Rightarrow \phi * \psi} {}^{*} R_{1}$$

$$\frac{\Delta \Rightarrow \phi \qquad \varnothing_{\times} \Rightarrow \psi}{\varnothing_{\times} \Rightarrow \phi * \psi} {}^{*} C_{\varnothing_{\times}} \qquad \mapsto \qquad \frac{\Delta \Rightarrow \phi \qquad \varnothing_{\times} \Rightarrow \psi}{\Delta \Rightarrow \phi * \psi} {}^{*} R_{2}$$

$$\frac{\Delta \Rightarrow \phi \qquad \varnothing_{\times} \Rightarrow \psi}{\Delta \Rightarrow \phi * \psi} {}^{*} C_{\varnothing_{\times}} \qquad \mapsto \qquad \frac{\Delta \Rightarrow \phi \qquad \varnothing_{\times} \Rightarrow \psi}{\Delta \Rightarrow \phi * \psi} {}^{*} R_{2}$$

$$\text{ses of } \rightarrow_{\mathsf{L}}:$$

$$\frac{\Delta \Longrightarrow \phi \qquad \Gamma(\varnothing_{\times}\, \mathring{\varsigma}\, \psi) \Longrightarrow \chi}{\Gamma(\Delta\, \mathring{\varsigma}\, \varphi \to \psi) \Longrightarrow \chi} \xrightarrow{\mathsf{C}_{\varnothing_{+}}} \mathsf{L} \qquad \longmapsto \qquad \underline{\Delta \Longrightarrow \phi \qquad \Gamma(\varnothing_{+}\, \mathring{\varsigma}\, \psi) \Longrightarrow \chi} \xrightarrow{\mathsf{L}_{1}} \mathsf{L}$$

$$\frac{\varnothing_{+} \Longrightarrow \phi \qquad \Gamma(\Delta \, \mathring{\varsigma} \, \psi) \implies \chi}{\frac{\Gamma(\varnothing_{+} \, \mathring{\varsigma} \, \Delta \, \mathring{\varsigma} \, \phi \to \psi) \implies \chi}{\Gamma(\Delta \, \mathring{\varsigma} \, \phi \to \psi) \implies \chi}} \, \mathsf{C}_{\varnothing_{+}} \qquad \longmapsto \qquad \frac{\varnothing_{+} \implies \phi \qquad \Gamma(\Delta \, \mathring{\varsigma} \, \psi) \implies \chi}{\Gamma(\Delta \, \mathring{\varsigma} \, \phi \to \psi) \implies \chi} \, \to_{\mathsf{L}_{2}}$$

$$\frac{\varnothing_{+} \Longrightarrow \phi \qquad \Gamma(\varnothing_{+};\psi) \Longrightarrow \chi}{\Gamma(\varnothing_{+};\varphi \to \psi) \Longrightarrow \chi} \xrightarrow{\mathsf{C}_{\varnothing_{+}}} \qquad \longmapsto \frac{\varnothing_{+} \Longrightarrow \phi \qquad \Gamma(\varnothing_{\times};\psi) \Longrightarrow \chi}{\Gamma(\varphi \to \psi) \Longrightarrow \chi} \xrightarrow{\mathsf{C}_{\varnothing_{\times}}} \qquad \longmapsto \frac{\varphi_{+} \Longrightarrow \varphi \qquad \Gamma(\varnothing_{\times};\psi) \Longrightarrow \chi}{\Gamma(\varphi \to \psi) \Longrightarrow \chi} \xrightarrow{\mathsf{L}_{3}}$$

Cases of \wedge_R :

$$\frac{\varnothing_{+} \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Delta \Rightarrow \phi \land \psi} \land_{R} \qquad \mapsto \frac{\varnothing_{+} \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Delta \Rightarrow \phi \land \psi} \land_{R1}$$

$$\frac{\varnothing_{+} \Rightarrow \phi \qquad \varnothing_{+} \Rightarrow \phi \qquad \Delta \Rightarrow \psi}{\Delta \Rightarrow \phi \land \psi} \land_{R1}$$

$$\frac{\varnothing_{+} \Rightarrow \phi \qquad \varnothing_{+} \Rightarrow \psi}{\varnothing_{+} \Rightarrow \phi \land \psi} \land_{R}$$

$$\frac{\varnothing_{+} \Rightarrow \phi \qquad \varnothing_{+} \Rightarrow \psi}{\varnothing_{+} \Rightarrow \phi \land \psi} \land_{R1}$$

$$\frac{\Delta \Rightarrow \phi \qquad \varnothing_{+} \Rightarrow \psi}{\Delta \stackrel{?}{?} \varnothing_{+} \Rightarrow \phi \land \psi} \land_{R}$$

$$\frac{\Delta \Rightarrow \phi \qquad \varnothing_{+} \Rightarrow \psi}{\Delta \Rightarrow \phi \land \psi} \land_{R2}$$

$$\frac{\Delta \Rightarrow \phi \qquad \varnothing_{+} \Rightarrow \psi}{\Delta \Rightarrow \phi \land \psi} \land_{R2}$$

Case of W:

$$\frac{\Gamma(\varnothing_+) \Longrightarrow \chi}{\Gamma(\varnothing_+; \Sigma) \Longrightarrow \chi} \mathop{\rm V}_{{\rm C}_{\varnothing_+}} \qquad \longmapsto \qquad \frac{\Gamma(\varnothing_+) \Longrightarrow \chi}{\Gamma(\Sigma) \Longrightarrow \chi} \mathop{\rm conv}$$

After transformation, the proof remains regimented since the phase structure is preserved, but there are no instances of C_{\varnothing_+} or C_{\varnothing_\times} remaining, so that the proof is a regimented dLBI-proof.

C Admissibility of rad in dLBI

We call any *additive* unit that is the child of a multiplicative context-former a radical.

Lemma 22.
$$\Vdash_{\mathsf{dLBI}+\mathsf{rad}} \bar{S} \Longrightarrow \Vdash_{\mathsf{dLBI}} \bar{S}$$
.

Proof. Consider an arbitrary instance of rad in a regimented dLBI+rad proof. If the unit on which the rule is applied is not active in the inference preceding it, then the two rules may be trivially permuted. Suppose this has been done, then the loading strategy preceding the rad is empty as it is not possible for the unit be active in the loading inference. Therefore, the rad follows either from another regimented action or from a normalising strategy.

If rad follows from a normalising strategy, then the preceding rule was an instance of C', and there are two possibilities to consider: either the radical was active in the C', or it was not. The first case is impossible since it would require the conclusion of the action preceding the normalising strategy to contain an additive set containing only additive units. The second case can be handled by the following transformation:

$$\frac{\Gamma(\varSigma(\Delta\,,\varnothing_+)\,;\varSigma(\Delta\,,\varnothing_+))\,\Longrightarrow\,\chi}{\Gamma(\varSigma(\Delta))\,\Longrightarrow\,\chi}\,\operatorname{rad}\,\operatorname{C}'\,\mapsto\,\frac{\frac{\Gamma(\varSigma(\Delta\,,\varnothing_+)\,;\varSigma(\Delta\,,\varnothing_+))\,\Longrightarrow\,\chi}{\Gamma(\varSigma(\Delta))\,\Longrightarrow\,\chi}\,\operatorname{rad}}{\frac{\Gamma(\varSigma(\Delta)\,;\varSigma(\Delta))\,\Longrightarrow\,\chi}{\Gamma(\varSigma(\Delta))\,\Longrightarrow\,\chi}\,\operatorname{C}'}$$

It remains to consider the case where the instance of rad follows from a regimented action. The only axiom that can introduce a radical is \perp' ; however, the

instance may simply be replaced by another, which does not have the offending sub-bunch. Similarly, we may disregard ι_L , $-*_{R_i}$, $*_L$, W', and conv, as the conclusion cannot contain a radical not present in the premiss. What remains are $-*_{L_i}$, $*_{R_i}$, W' and conv, which are transformed as follows:

D **Bounding Multiplicity**

First, we find bounds for the normalization and loading phases of regimented proofs, which is achieved by finding bounds for the kind of reductions that appear in them. That is, we observe that the conclusion of an action in a regimented proof contains at most one non-trivial additive set, namely the one containing the active bunches, as otherwise the action was not regimented.

Lemma 52. Let $\Gamma \succcurlyeq^* \Gamma'$, and suppose there is a unique non-trivial additive set in Γ , then any non-trivial additive set in Γ' is also unique.

Proof. If the reduction is a permutation, then we are done. If all the additive sets in Γ' are trivial, then we are done. Otherwise, let A be the unique nontrivial additive set of Γ , and note that it contains the bunch that is removed

by the reduction. If no bunch has become duplicit after the reduction, then A is still unique in Γ' and we are done. Otherwise, any bunch that becomes duplicit after reduction must be in the duplicity set of a bunch containing the additive component corresponding to A as a sub-bunch, as otherwise it would not have been affected by reduction. Furthermore, the additive set A must have become trivial as otherwise Γ contained two non-trivial and distinct additive sets contradicting the hypothesis. The new non-trivial additive set must be unique for the same reason: otherwise there would be a non-trivial additive set containing it contradicting the hypothesis.

Lemma 53. If \mathcal{D} is a normalizing strategy witnessing $S \succcurlyeq^* \bar{S}$, and there is unique non-trivial additive set in S, then $\mu(\mathcal{D}) \leq \mu(S)$

Proof. It follows from Lemma 52 that every sequent in the normalizing strategy has at most one non-trivial additive set. Let Γ and Γ' be the contexts of a redex and reduct respectively, and suppose Γ has a non-trivial additive set A. If the reduction does not increase the duplicity of any sub-bunch, then trivially $\mu(\Gamma') \leq \mu(\Gamma)$ as required. Otherwise there is $\Delta \leq \Gamma$ such that after reduction its duplicity increases, but then there must be a bunch $\tilde{\Delta}$ in its duplicity set after reduction, not there before the reduction, containing the additive component corresponding to A as a sub-bunch. However, by uniqueness of A, there can be at most one such-bunch, hence the duplicity set has increased by at most one. At the same time, it must be that the duplicity of the additive set decreases, as otherwise A was not unique in Γ . Hence, at most $\mu(\Gamma') = \mu(\Gamma)$, as required. \square

Since loading strategies are negative strategies whose corresponding positive strategy satisfies the hypothesis of the above lemma, the same result applies; that is, one can bound the multiplicity of the phase by the multiplicity of the end-sequent.

The following two technical results will establish the bounds on the actions—the steady growth property.

Lemma 54. Suppose all duplicit sub-bunches of $\Gamma(\Delta)$ are contained in Δ , then for any Δ' ,

$$\mu(\Gamma(\Delta')) \le 1 + \mu(\Delta')$$

Proof. In $\Gamma(\Delta')$ there is at most one duplicity set outside Δ' containing more than one member, as otherwise there were duplicit sub-bunches outside Δ . Furthermore, for the same reason, this duplicity set has at most two members. Hence, $\mu(\Gamma(\Delta')) \leq 1 + \mu(\Delta')$ as required.

Lemma 55. Let $\Gamma = \Pi_1 \circ \Pi_2$ where \circ is a context-former and Π_1, Π_2 are normal, then $\mu(\Gamma) \leq 1$

Proof. For any additive set A in Γ of the following hold:

- A is an additive set of Π_1 ,
- A is an additive set of Π_2 ,

-A is the topset of Γ .

By normality hypothesis for Π_1 and Π_2 , in the first two cases $\partial A = 0$, and in the last $\partial A \leq 1$ since Π_1 and Π_2 may contain some permutation equivalent bunches. Hence $\mu(\Gamma) \leq 1$ as required.

Lemma 56 (Steady Growth Property). Let $R(P_0, ..., P_n, C)$ be a regimented action. Then

$$\mu(P_0), ..., \mu(P_n), \mu(C) \le 3$$

Proof. Let Γ_i be the contexts of P_i , and let Δ_i be a minimal sub-bunch of Γ_i containing all the active sub-bunches in the inference. If Σ is any normal bunch not appearing in Γ_i then, by hypothesis, $\Gamma_i(\Sigma_i)$ is normal. Hence, by Lemma 54,

$$\mu(\Gamma_i(\Delta_i)) \le \mu(\Delta_i) + 1$$

By case analysis $\mu(\Delta) \leq 1$, since in every rule there are at most two sub-bunches that are active in the same additive set. Hence, $\mu(\Gamma_i) \leq 2$, whence $\mu(P_i) \leq 2$, as required.

It remains to establish the bound for \hat{C} , which we do by cases analysis on R with the following partition:

$$\begin{split} \mathcal{R}_1 &:= \{ \vee_R \} \\ \mathcal{R}_2 &:= \{ \rightarrow_R \,, \, \twoheadrightarrow_R \,, \wedge_R \,, \ast_R \} \\ \mathcal{R}_3 &:= \{ \iota_L \,, \, \top_L \,, \wedge_L \,, \ast_L \,, \mathsf{conv} \} \\ \mathcal{R}_4 &:= \{ \rightarrow_L \,, \, \twoheadrightarrow_L \,, \mathsf{W}' \,, \mathsf{E}' \,, \, \vee_L \} \end{split}$$

The variants on $-*_L$, \rightarrow_L , $*_R$, \land_R are identified with the rule itself. The cases are handled as follows:

 \mathcal{R}_1 : There are no active sub-bunches, so $\mu(\hat{C}) = 0$, by hypothesis.

 \mathcal{R}_2 : The conclusion of the rules takes the form $\Pi_1 \circ \Pi_2$, where \circ is a context-former, and Π_1 and Π_2 are normal, by hypothesis. Thus, by Lemma 55,

$$\mu(\hat{C}) \le 1$$

 \mathcal{R}_3 : All the rules are substitutions in a bunch $\Gamma(\Delta)$ by a bunch Δ' , hence, by Lemma 54,

$$\mu(\hat{C}) \le \mu(\Delta') + 1$$

However, $\mu(\Delta') = 0$ since Δ' is normal, thus $\mu(\hat{C}) \leq 1$

 \mathcal{R}_4 : All the rules are substitution in $\Gamma(\Delta)$ by a bunch Δ' , hence by Lemma 54,

$$\mu(\hat{C}) \le \mu(\Delta') + 1$$

For the W' and E' observe $\Delta' = \Sigma \circ \Sigma'$ where Σ and Σ' are normal by hypothesis, so by Lemma 55 we have $\mu(\Delta') \leq 1$. Similarly, for $\to L$ and $\to R$ we have $\Delta' = \Sigma \circ \Sigma' \circ \Sigma''$ and each of Σ, Σ' and Σ is normal. Thus by Lemma 55 twice $\mu(\Delta') \leq 2$. In both cases we have the bound $\mu(\hat{C}) \leq 3$ as required.

he bound for regimented proofs follows immediately.

Lemma 57. $\mathcal{D}: \varnothing \vdash \bar{S} \implies \mu(\mathcal{D}) \leq 3$.

Proof. We proceed by induction on the number of actions in \mathcal{D} ; that is, the maximum number of regimented actions in any branch from root to leaf.

Base Case. If the number of actions is one, then \mathcal{D} consists of one node which is an axiom, and therefore normal. Thus $\mu(\mathcal{D}) = \mu(\bar{S}) = 0$.

Inductive Step. Assume the property holds for proofs with at most n actions, and suppose \mathcal{D} has n+1. Let $\mathbf{R}(\hat{P}_0,\hat{P}_n,\hat{C})$ be the last action, then there are sub-trees \mathcal{D}_i that are proofs of \bar{P}_i respectively. The proofs \mathcal{D}_i have at most n actions, thus by inductive hypothesis $\mu(\mathcal{D}_i) \leq 3$. By Lemma 53 and Lemma 12, extending the \mathcal{D}_i with loading strategies for \hat{P}_i proofs \mathcal{D}'_i satisfying $\mu(\mathcal{D}'_i) \leq 3$. Moreover, applying R gives a proof \mathcal{D}_C of \hat{C} , which by Lemma 56 satisfies $\mu(\mathcal{D}_C) \leq 3$. Observe that \hat{C} has all duplicit bunches in the same additive set, as otherwise there were duplicit bunches not active in the premises. Hence, extending with the normalising strategy gives \mathcal{D} , which by Lemma 53 satisfies $\mu(\mathcal{D}) \leq 3$ as required.

E Bounding Depth

In Example 37, we see that that reason depth increases is that the same multiplicative connective can, in a sense, be counted twice in the computation of δ —once as a connective, and once as a context-former. This sense can be made rigorous by introducing a labelling scheme to track the relationship between connectives and context-formers.

Definition 58 (Labelling). A label is an integer; a formula is labelled if every multiplicative connective has a label associated to it, and a bunch is labelled if every formula in it is labelled and every multiplicative context-former carries a label. A sequent is labelled when both the context and the formula are labelled.

A proof is well-labelled if every sequent in it is labelled according to the following scheme:

Base Case. In the end-sequent every multiplicative connective and context-former is given a unique integer label.

Inductive Step. For each rule, the connective and context-formers inherit their labels if still present and unchanged in the reductive reading. New context-formers inherit the labels of their justifying connectives; that is, the label on the main connective in the principal formula of the conclusion.

Well-labellings are identical up-to bijection of the set of labels used.

Example 59. A well-labelling of the above proof in Example 37 is as follows:

$$\frac{\varnothing_{+} \Rightarrow \top}{((\phi - *^{0} \top) \rightarrow \psi), {}_{\circ} \phi \Rightarrow \top} \frac{\text{conv}}{*_{R}} \frac{\varnothing_{+} \Rightarrow \top}{\psi \Rightarrow \top} \frac{C}{\psi \Rightarrow \top} \frac{(\phi - *^{0} \top) \rightarrow \psi \Rightarrow (\phi - *^{0} \top) \rightarrow \psi)}{((\phi - *^{0} \top) \rightarrow \psi) \Rightarrow \top} C'$$

The label on the multiplicative context-former in the antecedent of the \rightarrow_R rule witnesses how the depth came to be increased, justifying the claim that the \rightarrow is counted twice.

Lemma 60. In a well-labelled proof every sequent is labelled.

Proof. Let \mathcal{D} be a well-labelled proof, and let $S \in \mathcal{D}$. We proceed by induction on the height of S; that is, the number of nodes between it and the end-sequent.

Base Case. Since S is the end-sequent of \mathcal{D} , it follows from the definition of the well-labelling scheme that it labelled.

Inductive Step. Let S' be the conclusion of the inference whose antecedent is S. By inductive hypothesis S' is labelled, since for any rule every context-former or connective in S is justified by a connective or context-former in S' it is also well-labelled.

Using the well-labelling scheme we can track the relationship between connectives and context-formers. First, however, we establish a connection between the connectives, the labels, and the computation of depth. Let $\sigma = (h, \tau)$ denote a (finite) sequence σ with head h and tail τ . Moreover, we use the term neighbourhood to denote either additive or multiplicative set.

Definition 61 (Line, Critical Line). Let $\Gamma \in \mathbb{B}$. A sequence of bunches $\sigma = (h, \tau)$ is a line in Γ if there is a sub-bunch $\Delta \in \Gamma$ such that $h = \Delta$ and τ is a line of Δ . The line is empty in the case that Δ is a unit or a formula.

A line in Γ is critical when every bunch in it is a bunch of maximal depth in its neighbourhood.

Definition 62 (Label Set, Critical Label Set). Let $\phi \in B$ be a labelled formula, let $\lambda(\phi)$ be the set of labels appearing in it. The function extends to bunches as follows, where $\Delta \in \mathbb{B}$ is labelled:

$$\lambda(\Delta) := \begin{cases} \lambda(\phi) & \text{if } \Delta = \phi \in \mathbb{F} \\ \varnothing & \text{if } \Delta \in \{\varnothing_+, \varnothing_\times\} \\ \varnothing & \text{if } \Delta = (\varSigma \, \mathring{\varsigma} \, \varSigma) \in \mathbb{B}^+ \\ \{n\} & \text{if } \Delta = (\varSigma \, \mathring{\varsigma}^n \, \varSigma) \in \mathbb{B}^\times \end{cases}$$

Let $\Gamma \in \mathbb{B}$ be labelled, let $\sigma = (h, \tau)$ be a line in it, and let \sqcup denote multiset union. Then define,

$$\mathcal{L}(\sigma) := \lambda(h) \sqcup \mathcal{L}(\tau)$$

A a multiset of integers ℓ is a label set in Γ if there is a line σ in Γ such that $\ell = \mathcal{L}(\sigma)$. It is a critical label set if σ is a critical line.

Lemma 63. Let \mathcal{D} be a well-labelled proof, let $(\Gamma \implies \phi) \in \mathcal{D}$, and let ℓ be a critical label set for Γ , then $|\ell| = \delta(\Gamma)$.

Proof. Since ℓ is a critical label set there is a critical line σ in Γ such that $\ell = \mathcal{L}(\sigma)$. We proceed by induction on the construction of σ .

Base Case. It must be that σ contains only one member which must be Γ , but then Γ is either a formula or, a unit. In either case $|\ell| = \mu \delta(\Gamma)$ since every multiplicative connective is labelled and is counted toward the depth.

Inductive Step. The selection during the recursion of a critical label set is precisely the selection during depth, and a label is added to the set if and only if depth increases, so the result follows by inductive hypothesis.

Since depth increased only because of the resolution rules where the head and body of the left antecedent must have belonged to the same additive or multiplicative set, it follows that depth can at most double in any such instance. However, further increase is not possible, thus an upper bound for depth is doubling.

Lemma 64. Let \mathcal{D} be a well-labelled proof, let $(\Gamma \implies \phi) \in \mathcal{D}$, and let $\Delta, \Delta' \in \mathbb{B}$ satisfy either

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-\Delta \sim \Delta', and \Delta' is a formula; or -\Delta = \Gamma, and \Delta' = \phi.
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Let ℓ and ℓ' be label sets for Δ and Δ' respective, then a label in $\ell \sqcup \ell'$ occurs at most twice.

Proof. We proceed by induction on the height of the occurrence of $(\Gamma \implies \phi)$ —the number of nodes between it and the root of the proof.

Base Case. By definition of the well-labelling scheme, in the end-sequent every context-former and connective has a unique label, thus ℓ and ℓ' each contains a label at most once. It follows that $\ell \sqcup \ell'$ contains a label at most twice as required.

Inductive Step. In each rule the property is preserved since the only case where the label sets of the premises are not sub-multisets of those in the conclusion is in the resolution rules, but then it follows by inductive hypothesis. \Box

The bound follows as a corollary.

Lemma 65.
$$\mathcal{D}: \varnothing \vdash \bar{S} \implies \delta(\mathcal{D}) < 2\delta(\bar{S})$$
.

Proof. Let $s = (\Gamma \implies \phi) \in \mathcal{D}$, then by definition $\delta(s) = \delta(\Gamma) + \delta(\phi)$. It follows from Lemma 64 and Lemma 63 that $\delta(s) \leq 2n$ where n is the number of possible labels, but by definition of the labelling scheme $n = \delta(\bar{S})$, as required. \square

F Bounded Search Space

Let $A \sqsubseteq B$ denote that A is a multiset of elements of B. We may regard a multiset as a set for which each occurrence of a member is counted as a separate element, in which case we use the notation |A| unambiguously to denote both the cardinality of a set and the number of elements in a multiset.

Definition 66 (Additive and Multiplicative Combinations). Let B be a set of bunches. The additive and multiplicative combinations of elements are given by $\sum B$ and $\prod B$ respectively.

Additive and multiplicative combinations allow us to define the construction of the proof-search space. First we have following operators,

The iterative application of taking additive and multiplicative combinations is defined as follows:

$$(\otimes^m \oplus^a)^d \Sigma := \begin{cases} \Sigma & \text{if } d = 0 \\ (\otimes^m \oplus^a)^{d-1} \otimes^m \oplus^a \Sigma & \text{otherwise} \end{cases}$$

Let Σ be a set of bunches, denote $\mathcal{G}(\Sigma)$ for the set of all bunches that can be constructed out of the elements of Σ .

Lemma 67. If $\Delta \in \mathcal{G}(\Sigma)$ satisfies $\mu(\Delta) \leq a$, $\omega(\Delta) \leq m$, and $\delta(\Delta) \leq d$, then

$$\varDelta \in \mathcal{V}^d := \bigcup_{i=0}^d \left((\otimes^m \oplus^a)^i \varSigma \right) \cup \bigcup_{i=0}^d \left((\oplus^a \otimes^m)^i \varSigma \right)$$

Proof. We proceed by induction on the d parameter. In the base case d=0, but then m=0, and we may conclude that $\Delta\in \oplus^a \Sigma$, but $\Sigma\subseteq \mathcal{V}$, so we are done. For the inductive step, observe $\delta(\Delta)\leq d+1$ if and only if $\delta(\Delta)< d$ or $\delta(\Delta)=d+1$. In the first case, we immediately have $\Delta\in \mathcal{V}^{d+1}$ by the induction hypothesis (since $\mathcal{V}^n\subseteq \mathcal{V}^{n+1}$). In the second case, we note that from the definition of depth, either Δ is a multiplicative composition of (additive) bunches of depth at most d or it is an additive composition of multiplicative bunches which are combinations of (additive) bunches of depth at most d; that is, either $\Delta\in\sum\prod_{i=0}^{m}M$ or $\Delta\in\prod M$ where $M\subset\mathcal{V}^d$. Hence $\Delta\in\oplus^a\bigcup_{i=0}^d\left((\otimes^m\oplus^a)^i\Sigma\right)$ or $\Delta\in\otimes^m\bigcup_{i=0}^d\left((\oplus^a\otimes^m)^i\Sigma\right)$, whence $\Delta\in\mathcal{V}^{d+1}$.

Lemma 34. Let $a, m, d \in \mathbb{N}$ be arbitrary and Σ be a finite set of formula, then the following set is finite and primitive recursive:

$$\mathbb{S}^{\varSigma}_{a,m,d} := \{S \in \digamma(\varSigma) \mid \mu(S) \leq a \text{ and } \omega(S) \leq m \text{ and } \delta(S) \leq d\}$$

Proof. The contexts of the elements in $\mathbb{S}^{\Sigma}_{a,m,d}$ are members of $\mathcal{G}(\Sigma)$ satisfying $\mu(\Delta) \leq a, \ \omega(\Delta) \leq m$, and $\delta(\Delta) \leq d$, thus by Lemma 67 they are member of \mathcal{V}^d . It follows that $\mathbb{S}^{\Sigma}_{a,m,d} \subseteq \mathcal{V}^d \times \Sigma$, and since the right hand side is primitive recursive and finite and the separating condition is decidable, the left hand side is primitive recursive and finite.