FIXED POINT THEOREMS AND SEMANTICS: A FOLK TALE *

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Folk theorems, least fixed point theorems

1. Introduction

In this paper we search for the theorem on which the fixed point theory of recursion is based. This leads us to a variety of sources, and the difficulties encountered suggest that we are in the presence of a folk theorem as defined by David Harel [18].

In Section 2 we examine eight textbooks which are likely to be primary source information for people interested in the so called fixed point theory of recursion. In these books a more or less formalized fixed point theorem is mentioned as the basis for the semantics of recursive definitions. However there is substantial disagreement on what exactly this theorem is, when it was written and by whom.

In Section 3 we look for the original works referred to in the textbooks. We expose the factors creating these problems and finally explain most of the discrepancies.

In Section 4 we discuss an extension of the folk theorem, for which we find similar problems. The further theorems found are also relevant to Section 3.

In this paper we only talk about various theorems, so only an intuitive understanding of mathematical notions such as continuity and partial ordering is necessary. The interested reader will find definitions in the references.

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2. The folk statement of the fixed point theory of recursive definitions

We refer to David Harel's paper [18] for a full explanation of what are folk theorems. We just give here some criteria that he established, making more precise the difference between a folk statement (informal) and a folk theorem (formal):

(i) A folk statement is the informal version of a theorem (or theorems) which tends to be remembered by people not working in the actual area of research relevant to this or those theorems.

(ii) The effort involved in tracing back the actual theorem(s) behind the folk statement is far greater than that involved in reproving them.

(iii) A real folk theorem will be reproved over the years again and again, an activity which is completely justified and which should be regarded as part of the culture. Therefore a folk theorem will often have many versions.

(iv) The roots of a folk statement or theorem might be buried in some obscure lecture notes, in a letter to an editor, or worse in a private communication.

David Harel suggested that a compilation of folk theorems could be made. We present here a candidate. We first give the statement, then try to find the theorem(s) on which it is based.

The fundamental result providing the fixed point semantics of a recursive construct is frequently loosely worded as:

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The function defined by a recursive program is the least fixed point of some functional operator.

This statement will be referred to as the *folk statement.*

Suppose we take the naive approach of a reader, who has only heard the folk statement, and who looks in various textbooks for a mention of the theorem(s) to which the author relates the folk statement.

2.1. Primary sources (in chronological order)

Manna [26, p. 370] bases the justification for the folk statement on Kleene's first recursion theorem given as follows.

Theorem 1. Every continuous functional F has a least fixed point which is lub$\{F^n(\perp)\}_{n\in N}$.

Bird [4, p. 158] gives the same result but calls it Kleene's fixed point theorem. Milne and Strachey [29, p. 327–328] refer to a version of Theorem 1 which they call the "Knaster fixed point theorem".

Theorem 2. Every continuous function F over a complete lattice has a least fixed point which is lub$\{F^n(\perp)\}$.

A proof is given as part of a more general result [29, p. 330]. In discussions leading up to this they mention a theorem of Tarski [38] about fixed points of a monotonic function over a complete lattice, but do not consider it further as they are mainly interested in continuous functions.

Stoy [36, p. 80] explains that least fixed points of certain operators are required for the semantics of recursive definitions and observes that, by Tarski's theorem, monotonicity of the operators would be enough to guarantee the existence of least fixed points [36, p. 97]. But he goes on to say that monotonicity is not a sufficient restriction to enable the solution of another (related) problem and finally [36, p. 112] gives, without credit, a version of Theorem 2 for countably based continuous complete lattices.

Brady [8, p. 229] indicates that to guarantee the existence of a solution to recursive equations Scott [33] and Scott and Strachey [34] exploit "a result due to Tarski and Knaster which guarantees that a continuous function over a complete lattice has a fixed point". Brady's reference for this result is to a paper by Scott [33] which gives a theorem with credit to Tarski only. Later [8, p. 244] Brady mentions the use of *least* fixed points and refers the reader to Stoy for details.

Livercy [25, p. 18] gives a version of Theorem 2 and an extension with a very indirect reference to Kleene. Cutland [14, p. 192] cites Kleene's first recursion theorem given as follows: every recursive operator has a least fixed point which is a computable function. De Bakker [3, p. 154] bases the semantics of recursive procedures as least fixed points on a version of Theorem 2 for complete partial orderings (c.p.o.). He gives Scott credit for much of the mathematics of this chapter, particularly the results on least fixed points [3, p. 472]. Also he gives Tarski's theorem but with the mention that it has interest for future applications [3, p. 142]. He describes Tarski's theorem as a lattice theoretic generalization of an old result in set theory [24, p. 473]. Scott, in his foreword to Stoy, includes the following quote from the draft of Milne and Strachey: "The relevant theorems about fixed points, due to Knaster and Tarski . . .".

After examination of the various theorems cited by the textbooks it becomes apparent that the folk statement is based on the following theorem (whose origins we examine in the next section).

Theorem 3. Every continuous function F over a c.p.o. has a least fixed point which is lub$\{F^n(\perp)\}_{n\in N}$.

Observe that, strictly speaking, particular cases of this theorem restricted to flat lattices or complete lattices are appropriate according to the particular setting in which different authors develop the theory. For instance, Manna is dealing mainly with flat lattices, Milne and Strachey with complete lattices and de Bakker with c.p.o.'s.

3. Searching for the origins

To solve the apparent discrepancies regarding the origins of Theorem 3 we look first at the work of Kleene, Knaster and Tarski.

The first recursion theorem of Kleene [22, p. 348] was known to him before 1938 (see for example his comments on [23, p. 376] and also [13]). It is correctly given by Cutland. Manna and Bird give what in fact is a partial version of this theorem which in their setting is more appropriate than the original full version. The statement of the theorem in [22] mentions only the existence of a least fixed point but the proof uses an iterative method leading to its characterization as lub$\{(F^n(\perp))\}$.

The result in [24] which relates to this work is in fact given as a lemma. We translate it as follows.

Theorem 4. Every monotonic function F over sets such that there exists a set A with $F(A) \subseteq A$ has a fixed point, $F(D) = D$, $D \subseteq A$.

This result was obtained jointly with Tarski as is stated at the beginning of [24] (see also [38, footnote 2]). But the paper contains no proof, so we do not know if this fixed point, guaranteed to exist, is the least one. And even if it is the least one the proof may not have given a lub characterisation to it. Also monotonic functions are not necessarily continuous. Therefore Theorem 2 is not due to Knaster.

The result in [38] which has been mentioned previously is the following:

Theorem 5. Every monotonic function over a complete lattice has a complete lattice of fixed points (and hence a least fixed point).

This theorem was obtained by Tarski in 1939 (and discussed by him in a few public lectures in 1939–1942 (see [38, footnote 2])). An abstract appeared in 1949 [37]. Although not of joint authorship Theorem 5 improves the earlier result of Knaster and Tarski (Theorem 4) and is sometimes referred to as the Knaster–Tarski theorem (e.g. see [31, p. 193]).

Theorem 5 was given in the second edition of [5, p. 54]. Birkhoff also pointed out the relationship between this result and earlier work of Kantorovitch [21] and Knaster [24] but omitted reference to Tarski. The third edition of [5, p. 115] gives credit to Tarski and also mentions [20] where a version of Theorem 3 is proved in the setting of ordered linear spaces.

Discussion. The investigations reported above lead us to make the claim that Theorem 3 is not only the theorem on which the folk statement is based but indeed qualifies as a folk theorem.

An important ingredient of this folk theorem is the lub characterisation it provides of the least fixed point of a continuous function. This may be contrasted with Tarski's result (Theorem 5) in which he proves the existence of a least fixed point but not in a constructive way. From a denotational point of view this may be considered as sufficient, as we may unambiguously associate to each recursive construct a mathematical function (its meaning) independently of any implementation. The folk theorem is restricted to continuous functions but provides a constructive proof of the existence of a least fixed point which is closely related to a particular implementation. Therefore from an operational point of view it is more satisfactory than Tarski's theorem. Most authors who cite both theorems invariably attach more importance to the folk theorem but sometimes Tarski's theorem is cited in a context where the folk theorem is intended.

Now Theorem 5 appears at the beginning of [38] but 20 pages further on, in an easily overlooked discussion of generalizations of later results, Tarski mentions that under certain conditions the formula lub$\{F^n(\perp)\}$ yields a fixed point.

So even though neither Tarski nor Kleene explicitly stated the folk theorem as a theorem it is clear that they knew the result, as did Kantorovitch and others.

Perhaps this part of the tale is best concluded with the name that Scott gave in an important paper [32] to his version of the folk theorem viz. "a well-known theorem".

4. Extensions of the folk theorem

The interest which we have seen above in general ordered spaces such as complete lattices and in continuous functions comes from developments in programming language semantics which enable the modelling of features other than recursion. However, there are instances for example in the study of unbounded nondeterminism, and in considering fair parallelism where one may need monotonic non-continuous functions and a constructive version of Tarski's theorem, that is the generalization of the folk theorem to monotonic functions (see, for example, [2,9]). Sonenberg [35] has a list of other references.

This extension is obtained by replacing the ω iteration by a transfinite iteration.

Extended Folk Theorem 6 (with ordinals). If f is a monotonic function from a complete lattice D into itself, there is an ordinal α of cardinality less than or equal to that of D such that $\beta \geqslant \alpha \Rightarrow$ the least fixed point of f is $f^{\beta}(\perp)$ where $\perp$ is the bottom element of the lattice and $f^{\lambda}(\perp) = f(\bigcup_{\mu<\lambda} f^{\mu}(\perp))$ for all $\lambda \geqslant 1$, $f^{0}(\perp) = \perp$.

This version was presented by Hitchcock and Park [19] in relation with a problem of termination and independently by Cadiou [10]. Gallier, finding Hitchcock and Park's proof too sketchy, reproves it in [16]. Cousot and Cousot [12] give a full treatment of this theorem and its corollaries and cite Devide [15] and Pasini [30] as other authors proving this result. The following version of the extended folk theorem makes it very similar to a theorem of Bourbaki [6]. The difference being increasing functions are used instead of monotonic and the fixed point is not necessarily the least, but is still canonical.

Extended Folk Theorem 7 (without ordinals). Let f be a monotonic function from a chain completed poset D into itself. f has a least fixed point equal to the lub of a well ordered complete lattice S defined recursively as the smallest set satisfying:

(i) $\perp \in S$.
(ii) If $t \in S$, then $f(t) \in S$.
(iii) If $T \subset S$ and lub T exists, then lub $T \in S$.

In fact one can argue that Bourbaki's theorem is on equal footing with Kantorovitch's, Kleene's and Tarski's as a foster parent to the folk theorem, at least in the elementary setting of flat lattices. Indeed we want to approximate the function recursively defined by a sequence of better and better approximations, which means that we are interested in the fact that the operator is increasing. For this we need Bourbaki's theorem: it provides a canonical fixed point, which is necessary for denotational purposes and as discussed by Manna and Shamir [27]; it does not need to be the least. And as Markowsky [28] has shown, the extended folk theorem is a simple corollary to Bourbaki's.

Now the proof of Bourbaki's theorem appeared in 1950 with the following mention: This theorem was given, without proof, eleven years before in "Eléments de la Théorie des Ensembles" and as readers of this book kept asking for its proof we finally decided to publish it even though it is to be found in essence in Zermelo's paper. The English edition of Bourbaki [7] includes an extensive historical note. In [7, p. 326] it is noted that Dedekind introduced the notion of the 'chain' of an element a of a set E relative to a function $f: E \rightarrow E$, namely $\bigcap \{K \subseteq E: a \in K, f(K) \subseteq K\}$, and that a notion closely related to this forms the basis of Zermelo's second proof of his theorem and also the proof of Theorem 7. The question of the origins of Bourbaki's theorem is intimately related to that for many maximal principles and for the axiom of choice. Campbell [11] gives a comprehensive account.

The interested reader will find many more related fixed point theorems in the literature, dating back at least to Cantor (1897). For a history of fixed point theorems in mathematical analysis, see [17].

Let us conclude with the remark that the Abian–Brown theorem [1] seems to be the most general fixed point theorem from which one can easily derive Bourbaki's theorem, the folk theorem and its extension.

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References

[1] S. Abian and A.B. Brown, A theorem on partially ordered sets with application to fixed point theorems, Canad. J. Math. 13 (1961) 78–82.
[2] K.R. Apt and G.D. Plotkin, A Cook's tour of countable non-determinism, ICALP (1981).
[3] J. de Bakker, Mathematical Theory of Program Correctness (Prentice-Hall, Englewood Cliffs, 1980).
[4] R. Bird, Programs and Machines (Wiley, New York, 1976).
[5] G. Birkhoff, Lattice Theory, Vol. 25 (American Mathe-

matical Society, Providence, RI, 1st ed. 1940; 2nd ed., 1948; 3rd ed., 1967).
[6] N. Bourbaki, Sur le théorème de Zorn, Arch. Math. (1950) 434–437.
[7] N. Bourbaki, Elements of mathematics, Theory of Sets (1968).
[8] J.M. Brady, The Theory of Computer Science, A Programming Approach (Chapman and Hall, London, 1977).
[9] M. Broy, Are fairness assumptions fair? 2nd Internat. Conf. Distributed Computing Systems, April, 1981.
[10] J.M. Cadiou, Recursive definitions of partial functions and their computations, Ph.D. Thesis, Computer Science Department, Stanford University, 1972.
[11] P.J. Campbell, The origin of Zorn's lemma, Historia Math. 5 (1978) 77–89.
[12] P. Cousot and R. Cousot, Constructive versions of Tarski's fixed point theorem, Pacific J. Math. 82(1) (1979).
[13] J.N. Crossley, ed., Reminiscences of Logicians, in Algebra and Logic, Springer Lecture Notes in Math. 450 (1975) 1–63.
[14] N.J. Cutland, Computability, An Introduction to Recursive Function Theory (Cambridge University Press, London, 1980).
[15] V. Devide, On monotonous mappings of complete lattices, Fund. Math. 53 (1964) 147–154.
[16] J.H. Gallier, On the existence of optimal fixpoints, Math. Systems Theory 13 (1980) 209–217.
[17] H. Goldstine, A History of Numerical Analysis from the 16th through the 19th Century (Springer, Berlin, 1979).
[18] D. Harel, On folk theorems, Comm. ACM 23(7) (1980) 379–389.
[19] P. Hitchcock and D. Park, Induction rules and proofs of termination, in: Nivat, ed., Automata, Languages and Programming (North-Holland, Amsterdam, 1972).
[20] L. Kantorovitch, The method of successive approximations for functional equations, Acta Math. 71 (1939) 63–97.
[21] L. Kantorovitch, On a class of functional equations, Doklady URSS 4 (1946) 219–224.
[22] S.C. Kleene, Introduction to Meta Mathematics (Van Nostrand, New York, 1962).
[23] S.C. Kleene, The origins of recursive function theory, in 20th century, IEEE Foundations Comput. Sci. (1979).
[24] B. Knaster, Un théorème sur les fonctions d'ensembles, Ann. Soc. Pol. Math. 6 (1928) 133–134.
[25] C. Livercy, Théorie des Programmes. Schemas, Preuves, Semantique (Dunod, Paris, 1978).
[26] Z. Manna, Mathematical Theory of Computation (McGraw-Hill, New York, 1974).
[27] Z. Manna and A. Shamir, The convergence of functions to fixed points of recursive definitions, Theoretical Comput. Sci. 6 (1978) 109–141.
[28] G. Markowsky, Chain-complete posets and directed sets with applications, Algebra Universalis 6 (1976) 53–68.
[29] R. Milne and C.A. Strachey, Theory of Programming Language Semantics (Chapman and Hall, London, 1976).
[30] A. Pasini, Some fixed point theorems of the mappings of partially ordered sets, Rend. Sem. Math. Univ. Padova 51 (1974) 167–177.
[31] H. Rogers, Theory of Recursive Functions and Effective Computability (McGraw-Hill, New York, 1967).
[32] D.S. Scott, Data types as lattices, SIAM J. Comput. 5 (1976) 522–587.
[33] D.S. Scott, The lattice of flow diagrams, in: E. Engeler, ed., Springer Lecture Notes in Mathematics 188 (1971) pp. 311–366.
[34] D.S. Scott and C. Strachey, Toward a mathematical semantics for computer languages, Proc. Symp. Computers and Automata (Polytechnic Institute of Brooklyn Press, 1971) pp. 19–46.
[35] E.A. Sonenberg, Fixed points of non-continuous functionals: Applications in programming language semantics, Res. Rept. 132, RAAF Academy, University of Melbourne, 1981.
[36] J.E. Stoy, Denotational Semantics: The Scott–Strachey Approach to Programming Language Theory (M.I.T. Press, Cambridge, MA, 1977).
[37] A. Tarski, A fixed point theorem for lattices and its applications (preliminary version), Bull. A.M.S. 55 (1949) 1051–1052; 1192.
[38] A. Tarski, A lattice theoretical fixed point theorem and its applications, Pacific J. Math. 5 (1955) 285–309.