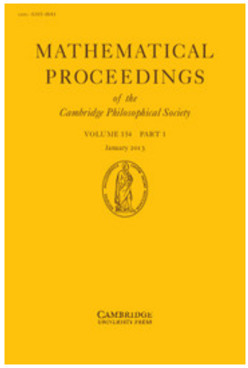

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Polynomial functors and polynomial monads

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Abstract

We study polynomial functors over locally cartesian closed categories. After setting up the basic theory, we show how polynomial functors assemble into a double category, in fact a framed bicategory. We show that the free monad on a polynomial endofunctor is polynomial. The relationship with operads and other related notions is explored.

Introduction

Background. Notions of polynomial functor have proved useful in many areas of mathematics, ranging from algebra [**37**, **44**] and topology [**10**, **53**] to mathematical logic [**18**, **48**] and theoretical computer science [**2**, **21**, **26**]. This paper deals with the notion of polynomial functor over locally cartesian closed categories. Before outlining our results, let us briefly motivate this level of abstraction.

Among the devices used to organise and manipulate numbers, polynomials are ubiquitous. While formally a polynomial is a sequence of coefficients, it can be viewed also as a function, and the fact that many operations on polynomial functions, including composition, can be performed in terms of the coefficients alone is a crucial feature. The idea underpinning the notion of a polynomial functor is to lift the machinery of polynomials and polynomial functions to the categorical level. An obvious notion results from letting the category of finite sets take the place of the semiring of natural numbers, and defining polynomial functors to be functors obtained by finite combinations of disjoint union and cartesian product. It is interesting and fruitful to allow infinite sets. One reason is the interplay between inductively defined sets and polynomial functors. For example, the set of natural numbers can be characterised as the least solution to the polynomial equation of sets

$$X \cong 1 + X ,$$

while the set of finite planar trees appears as least solution to the equation

$$X \cong 1 + \sum_{n \in \mathbb{N}} X^n .$$

Hence, one arrives at considering as polynomial functors on the category of sets all the functors of the form

$$X \longmapsto \sum_{a \in A} X^{B_a} , \tag{1}$$

where A is a set and $(B_a \mid a \in A)$ is an A-indexed family of sets, which we represent as a map $f : B \to A$ with $B_a = f^{-1}(a)$. It is natural to study also polynomial functors in many variables. A J-indexed family of polynomial functors in I-many variables has the form

$$(X_i \mid i \in I) \longmapsto \left(\sum_{a \in A_j} \prod_{b \in B_a} X_{s(b)} \mid j \in J \right) , \tag{2}$$

where the indexing refers to the diagram of sets

$$I \xleftarrow{\ s\ } B \xrightarrow{\ f\ } A \xrightarrow{\ t\ } J . \tag{3}$$

This expression reduces to (1) when I and J are singleton sets. The functor specified in (2) is the composite of three functors: pullback along s, the right adjoint to pullback along f, and the left adjoint to pullback along t. The categorical properties of these basic types of functors allow us to manipulate polynomial functors like (2) in terms of their representing diagrams (3); this is a key feature of the present approach to polynomial functors.

Although the theory of polynomial functors over Set is already rich and interesting, one final abstraction is called for: we may as well work in any category with finite limits where pullback functors have both adjoints. These are the locally cartesian closed categories, and we develop the theory in this setting, applicable not only to some current developments in operad theory and higher-dimensional algebra [**35**, **36**], but also in mathematical logic [**48**], and in theoretical computer science [**2**, **21**]. We hasten to point out that since the category of vector spaces is not locally cartesian closed, our theory does not immediately apply to various notions of polynomial functor that have been studied in that context [**44**, **53**].

Main results. Our general goal is to present a mathematically efficient account of the fundamental properties of polynomial functors over locally cartesian closed categories, which can serve as a reference for further developments. With this general aim, we begin our exposition by including some known results that either belong to folklore or were only available in the computer science literature (but not in their natural generality), giving them a unified treatment and streamlined proofs. These results mainly concern the diagram representation of strong natural transformations between polynomial functors, and versions of some of these results can be found in Abbott's thesis [**1**].

Having laid the groundwork, our first main result is to assemble polynomial functors into a double category, in fact a framed bicategory in the sense of Shulman [**55**], hence providing a convenient and precise way of handling the base change operation for polynomial functors. There are two biequivalent versions of this framed bicategory: one is the strict framed 2-category of polynomial functors, the other is the (nonstrict) bicategory of their representing diagrams.

Our second main result states that the free monad on a polynomial functor is a polynomial monad. This result extends to general polynomial functors the corresponding result for polynomial functors in a single variable [**17**] and for finitary polynomial functors on the category of sets [**35**, **36**]. We also observe that free monads enjoy a double-categorical universal property which is stronger than the bicategorical universal property that a priori characterises them.

The final section gives some illustration of the usefulness of the double-category viewpoint in applications. We give a purely diagrammatic comparison between Burroni P-spans [**12**], and polynomials over P (for P a polynomial monad). This yields in turn a concise equivalence between polynomial monads over P and P-multicategories [**12**, **43**], with base change (multifunctors) conveniently built into the theory. Operads are a special case of this.

Related work. Polynomial functors and closely related notions have been reinvented several times by workers in different contexts, unaware of the fact that such notions had already been considered elsewhere. To help unifying the disparate developments, we provide many pointers to the literature, although surveying the different developments in any detail is outside the scope of this paper.

We should say first of all that our notions of polynomial and polynomial functor are almost exactly the same as the notions of container and container functor introduced in theoretical computer science by Abbott, Altenkirch and Ghani [**1**, **2**, **3**, **4**] to provide semantics for recursive data types, and studied further in [**5**]. The differences, mostly stylistic, are explained in Section 2·18. A predecessor to containers were the shapely types of Jay and Cockett [**26**] which we revisit in Sections 3·16–3·17. The importance of polynomial functors for dependent type theory was first observed by Moerdijk and Palmgren [**48**], cf. Section 4·3. Their polynomial functors are what we call polynomial functors in one variable.

The use of polynomial functors in program semantics goes back at least to Manes and Arbib [**46**], and was recently explored from a different viewpoint under the name 'interaction systems' in the setting of dependent type theory by Hancock and Setzer [**21**] and by Hyvernat [**24**], where polynomials are also given a game-theoretic interpretation. The morphisms there are certain bisimulations, more general than the strong natural transformations used in the present work.

Within category theory, many related notions have been studied. In Section 1·18 we list six equivalent characterisations of polynomial functors over Set, and briefly comment on the contexts of the related notions: familially representable functors of Diers [**14**] and Carboni–Johnstone [**13**] (see also [**43**, appendix C]), and local right adjoints of Lamarche [**39**], Taylor [**60**], and Weber [**61**, **62**], a notion that in the present setting is equivalent to the notion of parametric right adjoint of Street [**57**]. We also comment on the relationship with species and analytic functors [**9**, **29**], and with Girard's normal functors [**18**].

Tambara [**59**] studied a notion of polynomial motivated by representation theory and group cohomology, where the three operations are, respectively, 'restriction', 'trace' (additive transfer), and 'norm' (multiplicative transfer). In Section 1·23, we give an algebraic-theory interpretation of one of his discoveries. Further study of *Tambara functors* has been carried out by Brun [**11**], with applications to Witt vectors.[1]

Most of the results of this paper generalise readily from locally cartesian closed categories to cartesian closed categories, as we briefly explain in Section 1·17, if just the 'middle

[1] Added in proof. Tambara functors have recently been investigated also by N. Strickland [**58**].

maps' $f : B \to A$ are individually required to be exponentiable.[2] This generalisation is useful: for example, covering maps are exponentiable in the category of compactly generated Hausdorff spaces, and in this way the theory would also include the notion of polynomial functor used by Bisson and Joyal [**10**] to give a geometric construction of Dyer–Lashof operations in bordism.

The name polynomial functor is often given to endofunctors of the category of vector spaces involving actions of the symmetric groups, cf. appendix A of Macdonald's book [**44**], a basic ingredient in the algebraic theory of operads [**37**]. The truncated version of such functors is a basic notion in functor cohomology, cf. the survey of Pirashvili [**53**]. As mentioned, these developments are not covered by our theory in its present form.

This paper was conceived in parallel to [**35**, **36**], to take care of foundational issues. Both papers rely on the double-categorical structures described in the present paper, and freely blur the distinction between polynomials and polynomial functors, as justified in Section 2 below. The paper [**36**] uses polynomial functors to establish the first purely combinatorial characterisation of the opetopes, the shapes underlying several approaches to higher-dimensional category theory [**42**], starting with the work of Baez and Dolan [**6**]. In [**35**], a new tree formalism based on polynomial functors is introduced, leading to a nerve theorem charactersising polynomial monads among presheaves on a category of trees. Further applications of polynomial functors in higher-dimensional category theory are presented in [**64**].

Outline of the paper. In Section 1 we recall the basic facts needed about locally cartesian closed categories, introduce polynomials and polynomial functors, give basic examples, and show that polynomial functors are closed under composition. We also summarise the known intrinsic characterisations of polynomial functors in the case $\mathscr{E} = \text{Set}$. In Section 2 we show how strong natural transformations between polynomial functors admit representation as diagrams connecting the polynomials. In Section 3 we assemble polynomial functors into a double category, in fact a framed bicategory. In Section 4 we recall a few general facts about free monads, and give an explicit construction of the free monad on a polynomial endofunctor, exhibiting it as a polynomial monad. Section 5 explores, in diagrammatic terms, the relationship between polynomial monads, multicategories, and operads.

1. *Polynomial functors*

1·1. Throughout we work in a locally cartesian closed category $\mathscr{E}$, assumed to have a terminal object [**16**]. In Section 4 we shall furthermore assume that $\mathscr{E}$ has sums and that these are disjoint (cf. also [**48**]), but we wish to stress that the basic theory of polynomial functors (Section 1, 2, 3 and 5) does not depend on this assumption. For $f : B \to A$ in $\mathscr{E}$, we write $\Delta_f : \mathscr{E}/A \to \mathscr{E}/B$ for pullback along f. The left adjoint to Δ_f is called the *dependent sum* functor along f and is denoted $\Sigma_f : \mathscr{E}/B \to \mathscr{E}/A$. The right adjoint to Δ_f is called the *dependent product* functor along f, and is denoted $\Pi_f : \mathscr{E}/B \to \mathscr{E}/A$. We note that both unit and counit for the adjunction $\Sigma_f \dashv \Delta_f$ are cartesian natural transformations (i.e. all their naturality squares are cartesian), whereas the unit and counit for $\Delta_f \dashv \Pi_f$ are generally not cartesian.

[2] Added in proof. This idea has been pursued recently by M. Weber [**63**], who also gives a deeper analysis of distributivity.

Following a well-established tradition in category theory [**45**], we will sometimes use the internal logic of $\mathscr{E}$ to manipulate objects and maps of $\mathscr{E}$ syntactically rather than diagrammatically, when this is convenient. This internal language is essentially the extensional dependent type theory presented in [**54**] and its use is justified by the results in [**15**, **23**]. The internal language allows us to manipulate objects and maps as if the category $\mathscr{E}$ were the category of sets. Thus, the reader not familiar with the internal language could interpret its use (notably in Section 4) as arguments valid in the category of sets. In the internal language, an object $f : X \to A$ of $\mathscr{E}/A$ is represented as a family $(X_a \mid a \in A)$, where we think of X_a as the fiber of f over $a \in A$. Note that the name of the map is only implicit in the family notation. The three functors associated to $f : B \to A$ take the following form:

$$\Delta_f(X_a \mid a \in A) = (X_{f(b)} \mid b \in B)\,,$$

$$\Sigma_f(Y_b \mid b \in B) = \left(\sum_{b \in B_a} X_b \mid a \in A\right),$$

$$\Pi_f(Y_b \mid b \in B) = \left(\prod_{b \in B_a} X_b \mid a \in A\right).$$

1·2. We shall make frequent use of the Beck–Chevalley isomorphisms and of the distributivity law of dependent sums over dependent products [**48**]. Given a cartesian square

$$\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

the Beck–Chevalley isomorphisms are

$$\Sigma_g\, \Delta_u \cong \Delta_v\, \Sigma_f \qquad \text{and} \qquad \Pi_g\, \Delta_u \cong \Delta_v\, \Pi_f\,.$$

To discuss the distributive law of dependent sums over dependent products, observe that, for maps $u\colon C \to B$ and $f\colon\; : B \to A$, we can construct the diagram

$$\begin{tikzcd} & N \arrow[r, "g"] \arrow[dl, "e"'] \arrow[dd, "w=\Delta_f(v)"] \arrow[ddr, phantom, "\lrcorner", very near start] & M \arrow[dd, "v=\Pi_f(u)"] \\ C \arrow[dr, "u"'] & & \\ & B \arrow[r, "f"'] & A\,, \end{tikzcd} \tag{4}$$

where $w = \Delta_f\, \Pi_f(u)$ and e is the counit of $\Delta_f \dashv \Pi_f$. Taking the mate of the Beck–Chevalley isomorphism

$$\Pi_g\, \Delta_e\, \Delta_u \cong \Delta_v\, \Pi_f$$

by the cartesian adjunctions $\Sigma_u \dashv \Delta_u$ and $\Sigma_v \dashv \Delta_v$, we obtain a natural transformation

$$\Sigma_v\, \Pi_g\, \Delta_e \Longrightarrow \Pi_f\, \Sigma_u\,, \tag{5}$$

which is again a cartesian natural transformation. The *distributive law* states that this natural transformation is an isomorphism. Since the transformation is cartesian, to see that this is the case, it is enough to consider its component at the terminal object of $\mathscr{E}/C$, in which case it is obvious.

To illustrate the use of the internal language, here is a proof in such terms, where the involved families refer to diagram (4):

$$\left(\prod_{b\in B_a}\sum_{c\in C_b} X_c \mid a \in A\right) \cong \left(\sum_{m\in M_a}\prod_{n\in N_m} X_{e(n)} \mid a \in A\right)$$
$$\cong \left(\sum_{m\in\prod\limits_{b\in B_a} C_b}\ \prod_{b\in B_a} X_{m(b)} \mid a \in A\right). \tag{6}$$

1·3. We recall some basic facts about enrichment, tensoring, and strength [**31**, **33**]. For any object $a : A \to I$ in $\mathscr{E}/I$, the diagram $A \xrightarrow{a} I \xrightarrow{u} 1$ defines a pair of adjoint functors

$$\Sigma_a \Delta_a \Delta_u \dashv \Pi_u \Pi_a \Delta_a .$$

The right adjoint provides enrichment of $\mathscr{E}/I$ over $\mathscr{E}$ by setting

$$\underline{\mathrm{Hom}}(a, x) = \Pi_u \Pi_a \Delta_a(x) \in \mathscr{E}, \qquad x \in \mathscr{E}/I .$$

The left adjoint makes $\mathscr{E}/I$ tensored over $\mathscr{E}$ by setting

$$K \otimes a = \Sigma_a \Delta_a \Delta_u(K) \in \mathscr{E}/I, \qquad K \in \mathscr{E} . \tag{7}$$

Explicitly, $K \otimes a$ is the object $K \times A \to A \to I$. In the internal language, the formulae are (for $a : A \to I$ and $x : X \to I$ in $\mathscr{E}/I$):

$$\underline{\mathrm{Hom}}(a, x) = \prod_{i\in I} X_i^{A_i}, \qquad K \otimes a = (K \times A_i \mid i \in I) .$$

Recall that a tensorial strength [**33**] on a functor $F : \mathscr{D} \to \mathscr{C}$ between categories tensored over $\mathscr{E}$ is a family of maps

$$\tau_{K,a} : K \otimes F(a) \longrightarrow F(K \otimes a)$$

natural in $K \in \mathscr{E}$ and in $a \in \mathscr{D}$, and satisfying two axioms expressing an associativity and a unit condition. A natural transformation between strong functors is called strong if it is compatible with the given strengths. When $\mathscr{E}$ is cartesian closed, giving a tensorial strength is equivalent to giving an enrichment, and a natural transformation is strong if and only if it is enriched.

For any $f : B \to A$, there is a canonical strength on each of the three functors Δ_f, Σ_f, and Π_f: writing out using (7) it is easily seen that the strength on Δ_f is essentially a Beck–Chevalley isomorphism, the strength of Σ_f is essentially trivial, whereas the strength of Π_f depends on distributivity and is essentially an instance of the unit for the $\Delta \dashv \Pi$ adjointness. It is also direct to verify that the natural transformations given by the units and counits for the adjunctions, as well as those expressing pseudo-functoriality of pullback and its adjoints, are all strong natural transformations. We shall work with strong functors and strong natural transformations, as a convenient alternative to the purely enriched viewpoint.

1·4. We define a *polynomial* over $\mathscr{E}$ to be a diagram F in $\mathscr{E}$ of shape

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J\,. \tag{8}$$

We define $P_F : \mathscr{E}/I \to \mathscr{E}/J$ as the composite

$$\mathscr{E}/I \xrightarrow{\Delta_s} \mathscr{E}/B \xrightarrow{\Pi_f} \mathscr{E}/A \xrightarrow{\Sigma_t} \mathscr{E}/J\,.$$

We refer to P_F as the *polynomial functor* associated to F, or the *extension* of F, and say that F *represents* P_F. In the internal language of $\mathscr{E}$, the functor P_F has the expression

$$P_F(X_i \mid i \in I) = \Big(\sum_{a \in A_j} \prod_{b \in B_a} X_{s(b)} \mid j \in J \Big)\,.$$

By a *polynomial functor* we understand any functor isomorphic to the extension of a polynomial. The distinction between polynomial and polynomial functor is similar to the usage in elementary algebra, where a polynomial defines a polynomial function. The bare polynomial is an abstract configuration of exponents and coefficients which can be interpreted by extension as a function. This extension is of course a crucial aspect of polynomials, and conversely it is a key feature of polynomial functions that they can be manipulated in terms of the combinatorial data. A similar interplay characterises the theory of polynomial functors. We shall shortly establish a result justifying the blur between polynomials and polynomial functors; only in the present paper do we insist on the distinction.

1·5. When $I = J = 1$, a polynomial is essentially given by a single map $B \to A$, and the extension reduces to

$$P(X) = \sum_{a \in A} X^{B_a}\,.$$

Endofunctors of this form, simply called polynomial functors in [**48**], will be referred to here as *polynomial functors in a single variable*.

Example 1·6.

(i) The identity functor $\mathrm{Id} : \mathscr{E}/I \to \mathscr{E}/I$ is polynomial, it is represented by

$$I \xleftarrow{=} I \xrightarrow{=} I \xrightarrow{=} I.$$

(ii) If $\mathscr{E}$ has an initial object $\varnothing$, then for any $A \in \mathscr{E}/J$, the constant functor $\mathscr{E}/I \to \mathscr{E}/J$ with value A is polynomial, represented by

$$I \xleftarrow{s} \varnothing \longrightarrow A \longrightarrow J.$$

(Indeed already Δ_s is constant $\varnothing$.)

Example 1·7. A span $I \overset{s}{\leftarrow} M \overset{t}{\rightarrow} J$ can be regarded as a polynomial

$$I \xleftarrow{s} M \xrightarrow{=} M \xrightarrow{t} J.$$

The associated polynomial functor

$$P_M(X_i \mid i \in I) = \Bigg(\sum_{m \in M_j} X_{s(m)} \mid j \in J \Bigg)$$

is called a *linear* functor, since it is given by the formula for matrix multiplication, and since P_M preserves sums. Hence polynomials can be seen as a natural 'non-linear' generalisation of spans.

Example 1·8. Let $C = (C_0 \underset{t}{\overset{s}{\leftleftarrows}} C_1)$ be a category object in $\mathscr{E}$. The polynomial

$$C_0 \xleftarrow{s} C_1 \xrightarrow{=} C_1 \xrightarrow{t} C_0$$

represents the polynomial (in fact linear) endofunctor $\mathscr{E}/C_0 \to \mathscr{E}/C_0$ which gives the free internal presheaf on a C_0-indexed family [**41**, Section V·7].

Example 1·9. The free-monoid monad, also known as the word monad or the list monad,

$$\begin{aligned} M : \mathrm{Set} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \sum_{n\in\mathbb{N}} X^n \end{aligned}$$

is polynomial, being represented by the diagram

$$1 \longleftarrow \mathbb{N}' \longrightarrow \mathbb{N} \longrightarrow 1,$$

where $\mathbb{N}' \to \mathbb{N}$ is such that the fibre over n has cardinality n, as given for example by the second projection from $\mathbb{N}' = \{(i, n) \in \mathbb{N} \times \mathbb{N} \mid i < n\}$.

Example 1·10. (Cf. [**35**]). A rooted tree defines a polynomial in Set:

$$A \xleftarrow{s} M \xrightarrow{f} N \xrightarrow{t} A$$

where A is the set of edges, N is the set of nodes, and M is the set of nodes with a marked incoming edge. The map t returns the outgoing edge of the node, the map f forgets the marked edge, and the map s returns the marked edge. It is shown in [**35**] that every polynomial is a colimit of trees in a precise sense.

1·11. We now define the operation of substitution of polynomials, and show that the extension of substitution is composition of polynomial functors, as expected. In particular, the composite of two polynomial functors is again polynomial. Given polynomials

$$\begin{tikzcd} & B \arrow[r, "f"] \arrow[dl, "s"'] & A \arrow[dr, "t"] & \\ I & & F & J \end{tikzcd} \qquad \begin{tikzcd} & D \arrow[r, "g"] \arrow[dl, "u"'] & C \arrow[dr, "v"] & \\ J & & G & K \end{tikzcd}$$

we say that F is a polynomial *from I to J* (and G from J to K), and we define $G \circ F$, the *substitution* of F into G, to be the polynomial $I \leftarrow N \to M \to K$ constructed via this diagram:

$$
\begin{tikzcd}
 & & & N \arrow[rr, "p"] \arrow[dl, "n"'] \arrow[dr, phantom, "(iv)"] & & D' \arrow[r, "q"] \arrow[dl, "\varepsilon"] \arrow[dr, phantom, "(ii)"] & M \arrow[dd, "w"] & \\
 & & B' \arrow[rr, "r"] \arrow[dl, "m"'] \arrow[dr, phantom, "(iii)"] & & A' \arrow[dl, "h"'] \arrow[dr, "k"] & & & \\
 & B \arrow[rr, "f"] \arrow[dl, "s"'] & & A \arrow[dr, "t"'] \arrow[rr, phantom, "(i)"] & & D \arrow[r, "g"] \arrow[dl, "u"] & C \arrow[dr, "v"] & \\
I & & & & J & & & K.
\end{tikzcd}
\tag{9}
$$

Square (i) is cartesian, and (ii) is a distributivity diagram like (4): w is obtained by applying Π_g to k, and D' is the pullback of M along g. The arrow $\varepsilon : D' \to A'$ is the k-component of the counit of the adjunction $\Sigma_g \dashv \Delta_g$. Finally, the squares (iii) and (iv) are cartesian.

PROPOSITION 1·12. *There is a natural isomorphism*

$$P_{G\circ F} \cong P_G \circ P_F.$$

Proof. Referring to diagram (9) we have the following chain of natural isomorphisms:

$$
\begin{aligned}
P_G \circ P_F &= \Sigma_v\, \Pi_g\, \Delta_u\, \Sigma_t\, \Pi_f\, \Delta_s \\
&\cong \Sigma_v\, \Pi_g\, \Sigma_k\, \Delta_h\, \Pi_f\, \Delta_s \\
&\cong \Sigma_v\, \Sigma_w\, \Pi_q\, \Delta_\varepsilon\, \Delta_h\, \Pi_f\, \Delta_s \\
&\cong \Sigma_v\, \Sigma_w\, \Pi_q\, \Pi_p\, \Delta_n\, \Delta_m\, \Delta_s \\
&\cong \Sigma_{(v\,w)}\, \Pi_{(q\,p)}\, \Delta_{(s\,m\,n)} \\
&= P_{G\circ F}.
\end{aligned}
$$

Here we used the Beck–Chevalley isomorphism for the cartesian square (i), the distributivity law for (ii), Beck–Chevalley isomorphism for the cartesian squares (iii) and (iv), and finally pseudo-functoriality of the pullback functors and their adjoints.

1·13. Let us also spell out the composition in terms of the internal language, to highlight the substitutional aspect. By definition, the composite functor is given by

$$P_G \circ P_F(X_i \mid i \in I) = \left(\sum_{c\in C_k} \prod_{d\in D_c} \sum_{a\in A_{u(d)}} \prod_{b\in B_a} X_{s(b)} \mid k \in K \right).$$

For fixed $c \in C$, by distributivity (6), we have

$$\prod_{d\in D_c} \sum_{a\in A_{u(d)}} \prod_{b\in B_a} X_{s(b)} \cong \sum_{m\in M_c} \prod_{d\in D_c} \prod_{b\in B_{m(d)}} X_{s(b)},$$

where we have put

$$M_c = \prod_{d\in D_c} A_{u(d)},$$

the w-fibre over c in diagram (9). If we also put, for $m \in M_c$,

$$N_{(c,m)} = \sum_{d \in D_c} B_{m(d)},$$

the $(q \circ p)$-fibre over $m \in M_c$, we can write

$$\sum_{m \in M_c} \prod_{d \in D_c} \prod_{b \in B_{m(d)}} X_{s(b)} \cong \sum_{m \in M_c} \prod_{(d,b) \in N_{(c,m)}} X_{s(b)}.$$

Summing now over $c \in C_k$, for $k \in K$, we conclude

$$P_G \circ P_F(X_i \mid i \in I) \cong \left(\sum_{(c,m) \in M_k} \prod_{(d,b) \in N_{(c,m)}} X_{s(b)} \mid k \in K \right),$$

(where $M_k = \sum_{c \in C_k} M_c$ is the $(v \circ w)$-fibre over $k \in K$).

COROLLARY 1·14. *The class of polynomial functors is the smallest class of functors between slices of $\mathscr{E}$ containing the pullback functors and their adjoints, and closed under composition and natural isomorphism.*

PROPOSITION 1·15. *Polynomial functors have a natural strength.*

Proof. Pullback functors and their adjoints have a canonical strength.

PROPOSITION 1·16. *Polynomial functors preserve connected limits. In particular, they are cartesian.*

Proof. Given a diagram as in (8), the functors $\Delta_s : \mathscr{E}/I \to \mathscr{E}/B$ and $\Pi_f : \mathscr{E}/B \to \mathscr{E}/A$ preserve all limits since they are right adjoints. A direct calculation shows that also the functor $\Sigma_t : \mathscr{E}/A \to \mathscr{E}/J$ preserves connected limits [**13**].

1·17. In this paper we have chosen to work with locally cartesian closed categories, since it is the most natural generality for the theory. However, large parts of the theory make sense also over cartesian closed categories, by considering only polynomials for which the 'middle map' $f : B \to A$ is exponentiable, or belongs to a subclass of the exponentiable maps having the same stability properties to ensure that Beck–Chevalley, distributivity, and composition of polynomial functors work just as in the locally cartesian closed case. Further results about polynomial functors in this generality can be deduced from the locally cartesian closed theory by way of the Yoneda embedding $y : \mathscr{E} \to \widehat{\mathscr{E}}$, where $\widehat{\mathscr{E}}$ denotes the category of presheaves on $\mathscr{E}$ with values in a category of sets so big that $\mathscr{E}$ is small relatively to it. The Yoneda embedding is compatible with slicing and preserves the three basic operations, so that basic results about polynomial functors in $\mathscr{E}$ can be proved by reasoning in $\widehat{\mathscr{E}}$. A significant example of this situation is the cartesian closed category of compactly generated Hausdorff spaces, where for example the covering maps constitute a stable class of exponentiable maps. Polynomial functors in this setting were used by Bisson and Joyal [**10**] to give a geometric construction of Dyer–Lashof operations in bordism. Another example is the category of small categories, where the Conduché fibrations are the exponentiable maps. In this setting, an example of a polynomial functor is the family functor, associating to a category X the category of families of objects in X.

1·18. For the remainder of this section, with the aim of putting the theory of polynomial functors in historical perspective, we digress into the special case $\mathscr{E} = \mathrm{Set}$, then make some remarks on finitary polynomial functors, and end with finite polynomials. This material is not needed in the subsequent sections.

The case $\mathscr{E} = \mathrm{Set}$ is somewhat special due to the equivalence $\mathrm{Set}/I \simeq \mathrm{Set}^I$, which allows for various equivalent characterisations of polynomial functors over Set.

For a functor $P : \mathrm{Set}/I \to \mathrm{Set}/J$, the following conditions are equivalent.

(i) P is polynomial.
(ii) P preserves connected limits (or, equivalently, pullbacks and cofiltered limits, or equivalently, wide pullbacks).
(iii) P is familially representable (i.e. a sum of representables).
(iv) The comma category $(\mathrm{Set}/J)\downarrow P$ is a presheaf topos.
(v) P is a local right adjoint (i.e. the slices of P are right adjoints).
(vi) P admits strict generic factorisations [**61**].
(vii) Every slice of $\mathrm{el}(P)$ has an initial object (Girard's normal-form property).

The equivalences (ii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) go back to Lamarche [**39**] and Taylor [**60**], who were motivated by the work of Girard [**18**], cf. below. They arrived at condition (vi) as the proper generalisation of (vii), itself a categorical reformulation of Girard's normal-form condition [**18**]. Below we give a direct proof of (i) $\Leftrightarrow$ (vii), to illuminate the relation with Girard's normal functors. The equivalence (ii) $\Leftrightarrow$ (iii) is due to Diers [**14**], and was clarified further by Carboni and Johnstone [**13**], who established in particular the equivalence (ii) $\Leftrightarrow$ (iv) as part of their treatment of Artin gluing. The equivalence (i) $\Leftrightarrow$ (iii) is also implicit in their work, the one-variable case explicit. The equivalence (i) $\Leftrightarrow$ (v) was observed by Weber [**62**], who also notes that on general presheaf toposes, local right adjoints need not be polynomial: for example the free-category monad on the category of directed graphs is a local right adjoint but not a polynomial functor.[3]

1·19. A polynomial functor $P : \mathrm{Set}/I \to \mathrm{Set}/J$ is *finitary* if it preserves filtered colimits. If P is represented by $I \leftarrow B \rightarrow A \rightarrow J$, this condition is equivalent to the map $B \to A$ having finite fibres.

1·20. Recall [**9**, **28**] that a *species* is a functor $F : \mathrm{FinSet}_{\mathrm{bij}} \to \mathrm{Set}$, or equivalently, a sequence $(F[n] \mid n \in \mathbb{N})$ of Set-representations of the symmetric groups. To a species is associated an *analytic functor*

$$\begin{aligned} \mathrm{Set} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \sum_{n\in\mathbb{N}} F[n] \times_{\mathfrak{S}_n} X^n \,. \end{aligned}$$

Species and analytic functors were introduced by Joyal [**29**], who also characterised analytic functors as those preserving weak pullbacks, cofiltered limits, and filtered colimits. It is the

[3] Added in proof. Recently A. Kock and J. Kock [**34**] have established a version of (i) $\Leftrightarrow$ (v) for general locally cartesian closed categories: the appropriate notion in this generality is that of a local *fibred* right adjoint.

presence of group actions that makes the preservation of pullbacks weak, in contrast to the polynomial functors, cf. (ii) above. Species for which the group actions are free are called *flat* species [**9**]; they encode rigid combinatorial structures, and correspond to ordinary generating functions rather than exponential ones. The analytic functor associated to a flat species preserves pullbacks strictly and is therefore the same thing as a finitary polynomial functor on Set. Explicitly, given a one-variable finitary polynomial functor $P(X) = \sum_{a \in A} X^{B_a}$ represented by $B \to A$, we can 'collect terms': let A_n denote the set of fibres of cardinality n, then there is a bijection

$$\sum_{a \in A} X^{B_a} \cong \sum_{n \in \mathbb{N}} A_n \times X^n.$$

The involved bijections $B_a \cong n$ are not canonical: the degree-n part of P is rather a $\mathfrak{S}_n$-torsor, denoted $P[n]$, and we can write instead

$$P(X) \cong \sum_{n \in \mathbb{N}} P[n] \times_{\mathfrak{S}_n} X^n, \tag{10}$$

which is the analytic expression of P.

As an example of the polynomial encoding of a flat species, consider the species C of binary planar rooted trees. The associated analytic functor is

$$X \mapsto \sum_{n \in \mathbb{N}} C[n] \times_{\mathfrak{S}_n} X^n\,,$$

where $C[n]$ is the set of ways to organise an n-element set as the set of nodes of a binary planar rooted tree; $C[n]$ has cardinality $n!\,c_n$, where c_n are the Catalan numbers $1, 1, 2, 5, 14, \ldots$ The polynomial representation is

$$1 \longleftarrow B \longrightarrow A \longrightarrow 1$$

where A is the set of isomorphism classes of binary planar rooted trees, and B is the set of isomorphism classes of binary planar rooted trees with a marked node.

1·21. Girard [**18**], aiming at constructing models for lambda calculus, introduced the notion of *normal functor*: it is a functor $\mathrm{Set}^I \to \mathrm{Set}^J$ which preserves pullbacks, cofiltered limits and filtered colimits, i.e. a finitary polynomial functor. Girard's interest was a certain normal-form property (reminiscent of Cantor's normal form for ordinals), which in modern language is (vii) above: the normal forms of the functor are the initial objects of the slices of its category of elements. Girard, independently of [**29**], also proved that these functors admit a power series expansion, which is just the associated (flat) analytic functor. From Girard's proof we can extract in fact a direct equivalence between (i) and (vii) (independent of the finiteness condition). The proof shows that, in a sense, the polynomial representation *is* the normal form. For simplicity we treat only the one-variable case.

PROPOSITION 1·22. *A functor* $P : \mathrm{Set} \to \mathrm{Set}$ *is polynomial if and only if every slice of* $\mathrm{el}(P)$ *has an initial object.*

Proof. Suppose P is polynomial, represented by $B \to A$. An element of P is a triple (X, a, s), where X is a set, $a \in A$, and $s : B_a \to X$. The set of connected components of $\mathrm{el}(P)$ is in bijection with the set $P(1) = A$. For each element $a \in A = P(1)$, it is clear that the triple $(B_a, a, \mathrm{Id}_{B_a})$ is an initial object of the slice $\mathrm{el}(P)/(1, a, u)$, where u is the map to the terminal object. These initial objects induce initial objects in all the slices, since every element (X, a, s) has a unique map to $(1, a, u)$.

Conversely, suppose every slice of $\mathrm{el}(P)$ has an initial object; again we only need the initial objects of the special slices $\mathrm{el}(P)/(1, a, u)$, for $a \in P(1)$. Put $A = P(1)$. It remains to construct B over A and show that the resulting polynomial functor is isomorphic to P. Denote by (B_a, b) the initial object of $\mathrm{el}(P)/(1, a, u)$. Let now X be any set. The unique map $X \to 1$ induces $P(X) \to P(1) = A$, and we denote by $P(X)_a$ the preimage of a. For each element $x \in P(X)_a$, the pair (X, x) is therefore an object of the slice $\mathrm{el}(P)/(1, a, u)$, so by initiality we get a map $B_a \to X$. Conversely, given any map $\alpha : B_a \to X$, define x to be the image under $P(\alpha)$ of the element b; clearly $x \in P(X)_a$. These two constructions are easily checked to be inverse to each other, establishing a bijection $P(X)_a \cong X^{B_a}$. These bijections are clearly natural in X, and since $P(X) = \sum_{a \in A} P(X)_a$ we conclude that P is isomorphic to the polynomial functor represented by the projection map $\sum_{a \in A} B^a \to A$.

1·23. Call a polynomial over Set

$$I \longleftarrow B \longrightarrow A \longrightarrow J \tag{11}$$

finite if the four involved sets are finite. Clearly the composite of two finite polynomials is again finite. The category $\mathbb{T}$ whose objects are finite sets and whose morphisms are the finite polynomials (up to isomorphism) was studied by Tambara [**59**], in fact in the more general context of finite G-sets, for G a finite group. His paper is probably the first to display and give emphasis to diagrams like (11). Tambara was motivated by representation theory and group cohomology, where the three operations Δ, Σ, Π are, respectively, 'restriction', 'trace' (additive transfer), and 'norm' (multiplicative transfer). We shall not go into the G-equivariant achievements of [**59**], but wish to point out that the following fundamental result about polynomial functors is implicit in Tambara's paper and should be attributed to him.

THEOREM 1·24. *The skeleton of $\mathbb{T}$ is the Lawvere theory for commutative semirings.*

The point is firstly that $m + n$ is the product of m and n in $\mathbb{T}$ (this is most easily seen by extension, where it amounts to $\mathrm{Set}/(m + n) \simeq \mathrm{Set}/m \times \mathrm{Set}/n$). And secondly that for the two Set-maps

$$0 \xrightarrow{e} 1 \xleftarrow{m} 2$$

the polynomial functor Σ_m, considered as a map in $\mathbb{T}$, represents addition, Π_m represents multiplication, and Σ_e and Π_e represent the additive and multiplicative neutral elements, respectively. Pullback provides the projection for the product in $\mathbb{T}$, and is also needed to account for distributivity, which in syntactic terms involves duplicating elements. It is a beautiful exercise to use the abstract distributive law (5) to compute

$$\Pi_m \circ \Sigma_k$$

where $k : 3 \to 2$ is the map pictured as  , recovering the distributive law $a(x + y) = ax + ay$ of elementary algebra.

2. *Morphisms of polynomial functors*

Since polynomial functors have a canonical strength, the natural notion of morphism between polynomial functors is that of strong natural transformation. We shall see that strong natural transformations between polynomial functors are uniquely represented by certain diagrams connecting the polynomials.

2·1. Given a diagram

$$
\begin{tikzcd}
F' : & I \arrow[d, equal] & B' \arrow[l, "s'"'] \arrow[r, "f'"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[r, "t'"] \arrow[d, "\alpha"'] & J \arrow[d, equal] \\
F : & I & B \arrow[l, "s"] \arrow[r, "f"'] & A \arrow[r, "t"'] & J
\end{tikzcd}
\tag{12}
$$

we define a cartesian strong natural transformation $\phi : P_{F'} \Rightarrow P_F$ by the pasting diagram

$$
\begin{tikzcd}
\mathscr{E}/I' \arrow[rr, "\Delta_{s'}"] \arrow[dr, "\Delta_s"'] & \arrow[d, phantom, "\cong"] & \mathscr{E}/B' \arrow[rr, "\Pi_{f'}"] & \arrow[d, phantom, "\cong"] & \mathscr{E}/A' \arrow[rr, "\Sigma_{t'}"] \arrow[dr, "\Sigma_\alpha"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & \mathscr{E}/J' \\
& \mathscr{E}/B \arrow[ur, "\Delta_\beta"'] \arrow[rr, "\Pi_f"'] & & \mathscr{E}/A \arrow[ur, "\Delta_\alpha"] \arrow[rr, equal] & & \mathscr{E}/A. \arrow[ur, "\Sigma_t"'] &
\end{tikzcd}
$$

It is cartesian and strong since its constituents are so.

In the internal language of $\mathscr{E}$, the component of $\phi : P_{F'} \Rightarrow P_F$ at $X = (X_i \mid i \in I)$ is the function

$$
\phi_X : \left(\sum_{a' \in A'_j} \prod_{b' \in B'_{a'}} X_{s'(b')} \mid j \in J \right) \to \left(\sum_{a \in A_j} \prod_{b \in B_a} X_{s(b)} \mid j \in J \right)
$$

defined by

$$
\phi_X(a', x') = \big(\alpha(a'), x' \cdot \beta_{a'}^{-1}\big),
$$

where $\beta_{a'} : B'_{a'} \to B_{\alpha(a')}$ is the isomorphism determined by the cartesian square in (12).

LEMMA 2·2. *Let $P : \mathscr{E}/I \to \mathscr{E}/J$ be a polynomial functor. If $Q \Rightarrow P$ is a cartesian natural transformation, then Q is also a polynomial functor.*

Proof. Assume P is represented by $I \leftarrow B \to A \to J$. Construct the diagram

$$
\begin{tikzcd}
I \arrow[d, equal] & B' \arrow[l, "s'"'] \arrow[r, "f'"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[r, "t'"] \arrow[d, "\alpha"'] & J \arrow[d, equal] \\
I & B \arrow[l, "s"] \arrow[r, "f"'] & A \arrow[r, "t"'] & J
\end{tikzcd}
$$

by setting $A' = Q(1)$, and taking $\alpha : A' \to A$ to be the map $\phi_1 : Q(1) \to P(1)$, and letting B' be the pullback. The top row represents a polynomial functor P', and the diagram defines a cartesian natural transformation to P. Since P' and Q both have a cartesian natural transformation to P which agree on the terminal object, they are naturally isomorphic. Hence Q is polynomial.

2·3. Recall that, for a category $\mathscr{C}$ with a terminal object 1 and a category $\mathscr{D}$ with pullbacks, the functor

$$
\begin{aligned}
[\mathscr{C}, \mathscr{D}] &\longrightarrow \mathscr{D} \\
P &\longmapsto P(1)
\end{aligned}
$$

is a Grothendieck fibration. The cartesian arrows for this fibration are precisely the cartesian natural transformations, while the vertical arrows are the natural transformations whose component on 1 is an identity map. We refer to such natural transformations as *vertical natural transformations*.

If $\mathscr{C}$ and $\mathscr{D}$ are enriched and tensored, then the above remark carries over to the case where $[\mathscr{C}, \mathscr{D}]$ denotes the category of strong functors and strong natural transformations. The verification of this involves observing that the cartesian lift of a strong functor has a canonical strength.

PROPOSITION 2·4. *Let I, $J \in \mathscr{E}$. The restriction of the Grothendieck fibration*

$$\begin{aligned} [\mathscr{E}/I, \mathscr{E}/J] &\longrightarrow \mathscr{E}/J \\ P &\longmapsto P(1) \end{aligned}$$

to the category of polynomial functors and strong natural transformations is again a Grothendieck fibration.

Proof. Lemma 2·2 implies that the cartesian lift of a polynomial functor is again polynomial.

2·5. Proposition 2·4 implies that every strong natural transformation between polynomial functors factors in an essentially unique way as a vertical strong natural transformation followed by a cartesian one. We proceed to establish representations of the two classes of strong natural transformations between polynomial functors. The key ingredient is the following version of the enriched Yoneda lemma.

LEMMA 2·6. *Let $u : I \to 1$ denote the unique arrow in $\mathscr{E}$ to the terminal object. For any $s : B \to I$ and $s' : B' \to I$ in $\mathscr{E}/I$, the natural map*

$$\mathrm{Hom}_{\mathscr{E}/I}(s, s') \longrightarrow \mathrm{StrNat}(\Pi_u \Pi_{s'} \Delta_{s'}, \Pi_u \Pi_s \Delta_s)$$

sending an I-map $w : B \to B'$ to the composite $\Pi_u \Pi_{s'} \Delta_{s'} \overset{\eta}{\Rightarrow} \Pi_u \Pi_{s'} \Pi_w \Delta_w \Delta_{s'} \cong \Pi_u \Pi_s \Delta_s$ is a bijection.

Proof. Just note that $\Pi_u \Pi_s \Delta_s = \underline{\mathrm{Hom}}_{\mathscr{E}/I}(s, -) : \mathscr{E}/I \to \mathscr{E}$, and the result is the usual enriched Yoneda lemma [**31**], remembering that since $\mathscr{E}/I$ is tensored over $\mathscr{E}$, a natural transformation (between strong functors) is enriched if and only if it is strong.

2·7. Given a diagram

$$\begin{tikzcd} F' : & I & B' \arrow[l, "s'"'] \arrow[r, "f'"] & A \arrow[r, "t"] & J \\ F : & I \arrow[u, equal] & B \arrow[l, "s"] \arrow[r, "f"'] \arrow[u, "w"'] & A \arrow[r, "t"'] \arrow[u, equal] & J \arrow[u, equal] \end{tikzcd} \tag{13}$$

we define a strong natural transformation $\phi : P_{F'} \Rightarrow P_F$ by the pasting diagram

$$\begin{tikzcd} & \mathscr{E}/B' \arrow[rr, equal] \arrow[dr, "\Delta_w"] \arrow[d, phantom, "\cong"] & & \mathscr{E}/B' \arrow[dr, "\Pi_{f'}"] \arrow[d, phantom, "\cong"] & & \\ \mathscr{E}/I \arrow[ur, "\Delta_{s'}"] \arrow[rr, "\Delta_s"'] & {} & \mathscr{E}/B \arrow[ur, "\Pi_w"] \arrow[u, phantom, "\Downarrow\eta"] \arrow[rr, "\Pi_f"'] & {} & \mathscr{E}/A \arrow[r, "\Sigma_t"'] & \mathscr{E}/J. \end{tikzcd}$$

In the internal language, the component of ϕ at $X = (X_i \mid i \in I)$ is given by the function

$$\phi_X : \left(\sum_{a \in A_j} \prod_{b' \in B'_a} X_{u(b)} \mid j \in J \right) \longrightarrow \left(\sum_{a \in A_j} \prod_{b \in B_a} X_{s(b)} \mid j \in J \right)$$

defined by

$$\phi_X(a, x) = \big(a, x \cdot w_a\big).$$

Clearly $\phi_1 = \mathrm{Id}_A$, so ϕ is vertical for the Grothendieck fibration.

PROPOSITION 2·8. *For F and F' as above, every vertical strong natural transformation $\phi : P_{F'} \Rightarrow P_F$ is uniquely represented by a diagram like* (13).

Proof. We already have the outline of the diagram (13), it remains to construct the map $w : B \to B'$ commuting with the rest. Since w must be an A-map, we can construct it fibrewise, so we need for each $a \in A$ a map $B'_a \to B_a$. This allows reduction to the case $A = 1$, and the result is a direct consequence of the above Yoneda lemma.

PROPOSITION 2·9. *Let $I, J \in \mathscr{E}$. Let $F : I \to J$ and $F' : I \to J$ be polynomials. Every cartesian strong natural transformation $\phi : P_{F'} \Rightarrow P_F$ is uniquely represented by a diagram of the form* (12).

Proof. We have $A' \cong P_{F'}(1)$ and $A \cong P_F(1)$. Define $\alpha : A' \to A$ to be the composite

$$A' \cong P_{F'}(1) \xrightarrow{\phi_1} P_F(1) \cong A.$$

We need to construct $\beta : B' \to B$, and since it has to be compatible with α, f' and f, it is enough to construct $B'_{a'} \to B_{\alpha(a')}$ for each $a' \in A'$. Thereby we can reduce to the case where $A' = A = 1$; in this case ϕ is invertible since it is simultaneously vertical and cartesian. But in this case the enriched Yoneda lemma above already ensures that the natural transformation is induced by a unique map $B \to B'$, which we furthermore know is invertible. Its inverse is what we need for $B'_{a'} \to B_{\alpha(a')}$. We have now constructed a diagram like (12), and it is routine to check that this diagram represents ϕ.

2·10. We give an example of a natural transformation that cannot be represented by diagrams. On the category $\mathrm{Set}^{\mathbb{Z}_2}$ of involutive sets, the identity functor is represented by $1 \leftarrow 1 \to 1 \to 1$. The twist natural transformation $\tau : \mathrm{Id} \Rightarrow \mathrm{Id}$, whose component on an object X is the involution of X, is both cartesian and vertical. It is clear that it cannot be represented by any diagram connecting $1 \leftarrow 1 \to 1 \to 1$, since any connecting arrows would have to be identities and thereby induce the trivial natural transformation. Observe that τ is not strong.

2·11. We can now combine the diagrams representing vertical and cartesian strong natural transformations. Given a diagram

$$\begin{tikzcd}
G: & I \arrow[dd, equal] & D \arrow[l, "u"'] \arrow[r, "g"] & C \arrow[r, "v"] \arrow[d, equal] & J \arrow[dd, equal] \\
 & & B' \arrow[u] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] & \\
F: & I & B \arrow[l, "s"] \arrow[r, "f"'] & A \arrow[r, "t"'] & J
\end{tikzcd} \tag{14}$$

there is induced, by (2·1) and (2·7), a strong natural transformation $P_\phi : P_G \Rightarrow P_F$. We refer to a diagram like (14) as a *morphism* from G to F. We arrive at the following result, a version of which appears as [**2**, theorem 3·4], where it is stated for polynomial functors between slice categories over discrete objects.

THEOREM 2·12. *Every strong natural transformation $P_G \Rightarrow P_F$ between polynomial functors is represented in an essentially unique way by a diagram like* (14).

Proof. By Proposition 2·4, every strong natural transformation factors as a vertical strong transformation followed by a cartesian strong natural transformation in an essentially unique way. The claim then follows from Proposition 2·8 and Proposition 2·9.

COROLLARY 2·13. *Every strong natural transformation between polynomial functors is a composite of units and counits of the basic adjunctions, their inverses when they exist, and coherence* 2*-cells for pullback and its adjoints.*

Proof. The ingredients of the constructions in (2·1) and (2·7) are units, counits, pseudo-functoriality 2-cells, as well as Beck-Chevalley isomorphisms, which in turn are constructed using units and counits (and inverses of their composites).

2·14. Polynomials from I to J and their morphisms form a category denoted $\mathrm{Poly}_{\mathscr{E}}(I, J)$. Vertical composition of diagrams like (14) involves a simple pullback construction that via extension amounts precisely to refactoring cartesian-followed-by-vertical into vertical-followed-by-cartesian, cf. the fibration property. This can also be described as the unique way of defining vertical composition of diagrams to make the assignment given by extension functorial. If we let $\mathrm{PolyFun}_{\mathscr{E}}(\mathscr{E}/I, \mathscr{E}/J)$ denote the category of polynomial functors from $\mathscr{E}/I$ to $\mathscr{E}/J$ and strong natural transformations, we can reformulate Theorem 2·12 as follows.

LEMMA 2·15. *For any I, J, the functor given by extension,*

$$\mathrm{Ext} : \mathrm{Poly}_{\mathscr{E}}(I, J) \longrightarrow \mathrm{PolyFun}_{\mathscr{E}}(\mathscr{E}/I, \mathscr{E}/J)\,,$$

is an equivalence of categories.

2·16. The involved categories are hom categories of appropriate bicategories of polynomials and polynomial functors, respectively, that we now describe, assembling the equivalences of the lemma into a biequivalence of bicategories (2·17). We define the 2-category of polynomial functors $\mathrm{PolyFun}_{\mathscr{E}}$ as the sub-2-category of Cat having slices of $\mathscr{E}$ as 0-cells, polynomial functors as 1-cells, and strong natural transformations as 2-cells.

We shall describe a bicategory $\mathrm{Poly}_{\mathscr{E}}$ which has objects of $\mathscr{E}$ as 0-cells, polynomials as 1-cells, and whose 2-cells are the morphisms of polynomials, i.e. diagrams like (14). The vertical composition of 2-cells has already been described, as has the horizontal composition of 1-cells. To define the horizontal composition of 2-cells we simply transport back the 2-cell structure from $\mathrm{PolyFun}_{\mathscr{E}}$ along the local equivalences of Lemma 2·15.

We begin by extending the family of functions mapping a pair of composable polynomials F and G to their composite $G \circ F$, which we defined in Paragraph (1·11), to a family of functors

$$\mathrm{Poly}_{\mathscr{E}}(J, K) \times \mathrm{Poly}_{\mathscr{E}}(I, J) \longrightarrow \mathrm{Poly}_{\mathscr{E}}(I, K) .$$

For this, let $\phi : F \Rightarrow F'$ be a morphism between polynomials from I to J, and let $\psi : G \Rightarrow G'$ be a morphism between polynomials from J to K. We define the morphism $\psi \circ \phi : G \circ F \Rightarrow G' \circ F'$ as the unique morphism of polynomials making the following diagram commute

$$\begin{tikzcd} P(G \circ F) \arrow[r, "P(\psi\circ\phi)"] \arrow[d, "\alpha_{G,F}"'] & P(G' \circ F') \arrow[d, "\alpha_{G',F'}"] \\ P(G)\, P(F) \arrow[r, "P(\psi)\, P(\phi)"'] & P(G')\, P(F') . \end{tikzcd}$$

Here $\alpha_{G,F}$ and $\alpha_{G',F'}$ are instances of the isomorphism of Theorem 1·12, and the diagram now expresses the naturality of α. We therefore get the following natural isomorphism of functors

$$\begin{tikzcd} \mathrm{Poly}_{\mathscr{E}}(J, K) \times \mathrm{Poly}_{\mathscr{E}}(I, J) \arrow[r] \arrow[d, "P_{J,K} \times P_{I,J}"'] \arrow[dr, phantom, "\cong"] & \mathrm{Poly}_{\mathscr{E}}(I, K) \arrow[d, "P_{I,K}"] \\ \mathrm{PolyFun}_{\mathscr{E}}(\mathscr{E}/J, \mathscr{E}/K) \times \mathrm{PolyFun}_{\mathscr{E}}(\mathscr{E}/I, \mathscr{E}/J) \arrow[r] & \mathrm{PolyFun}_{\mathscr{E}}(\mathscr{E}/I, \mathscr{E}/K) \end{tikzcd}$$

where the top horizontal functor is substitution of polynomials and the bottom horizontal map is composition of functors in $\mathrm{PolyFun}_{\mathscr{E}}$. The identity maps in $\mathrm{Poly}_{\mathscr{E}}$ are represented by the polynomials $\mathrm{Id}_I : I \to I$, and we have natural isomorphisms

$$\begin{tikzcd} & \mathrm{Poly}_{\mathscr{E}}(I, I) \arrow[d, "P_{I,I}"] \\ 1 \arrow[ur, "\mathrm{Id}_I"] \arrow[dr, "1_{\mathscr{E}/I}"'] \arrow[r, phantom, "\cong"] & {} \\ & \mathrm{PolyFun}_{\mathscr{E}}(\mathscr{E}/I, \mathscr{E}/I) . \end{tikzcd}$$

We define the associativity and unit isomorphisms. For associativity, given polynomials $F : I \to J$, $G : J \to K$, and $H : K \to L$, define

$$\theta_{H,G,F} : (H \circ G) \circ F \Longrightarrow H \circ (G \circ F)$$

to be the unique morphism of polynomials making the following diagram commute

$$
\begin{tikzcd}
P((H \circ G) \circ F) \arrow[r, "P(\theta_{H,G,F})"] \arrow[d, "\alpha_{H \circ G, F}"'] & P(H \circ (G \circ F)) \arrow[d, "\alpha_{H, G \circ F}"] \\
P(H \circ G)\, P(F) \arrow[d, "\alpha_{H,G}\, P(F)"'] & P(H)\, P(G \circ F) \arrow[d, "P(H)\, \alpha_{G,F}"] \\
\big(P(H)\, P(G)\big)\, P(F) \arrow[r, equal] & P(H)\, \big(P(G)\, P(F)\big) .
\end{tikzcd}
\tag{15}
$$

For the unit isomorphisms, given a polynomial $F : I \to J$, define

$$\lambda_F : \mathrm{Id}_J \circ F \Longrightarrow F\,, \qquad \rho_F : F \circ \mathrm{Id}_I \Longrightarrow F$$

to be the unique morphism of polynomials such that

$$
\begin{tikzcd}
P(\mathrm{Id}_J \circ F) \arrow[r, "P(\lambda_F)"] \arrow[d, "\alpha_{\mathrm{Id}_J, F}"'] & P(F) \arrow[d, equal] \\
P(\mathrm{Id}_J)\, P(F) \arrow[r, "\alpha_J\, P(F)"'] & 1_{\mathscr{E}/J}\, P(F)
\end{tikzcd}
\tag{16}
$$

and

$$
\begin{tikzcd}
P(F \circ \mathrm{Id}_I) \arrow[r, "P(\rho_F)"] \arrow[d, "\phi_{F, \mathrm{Id}_I}"'] & P(F) \arrow[d, equal] \\
P(F)\, P(\mathrm{Id}_I) \arrow[r, "P(F)\, \alpha_I"'] & P(F)\, 1_{\mathscr{E}/I}
\end{tikzcd}
\tag{17}
$$

commute. All the data of the bicategory $\mathrm{Poly}_{\mathscr{E}}$ have now been given. The naturality and coherence axioms for a bicategory can be verified by standard diagram-chasing arguments, which exploit the uniqueness properties used to define the components of θ, λ, and ρ. The interchange law of $\mathrm{PolyFun}_{\mathscr{E}}$ is used at several points. Let us remark that the definition of the bicategory $\mathrm{Poly}_{\mathscr{E}}$ is essentially determined by the requirement that we obtain a pseudo-functor

$$\mathrm{Ext} : \mathrm{Poly}_{\mathscr{E}} \longrightarrow \mathrm{PolyFun}_{\mathscr{E}}\,.$$

Indeed, the diagrams in (15), (16), (17) express exactly the coherence conditions for a pseudo-functor [**8**]. It is clear by construction that this pseudo-functor is bijective on objects, and it is locally an equivalence of categories by Lemma 2·15. Hence we have established the following.

THEOREM 2·17. *The extension pseudo-functor*

$$\mathrm{Ext} : \mathrm{Poly}_{\mathscr{E}} \longrightarrow \mathrm{PolyFun}_{\mathscr{E}}$$

is a biequivalence.

2·18. The notions of polynomial and polynomial functors are almost exactly the same as what is called *container* and *container functor* by Abbott, Altenkirch and Ghani [**1**, **2**, **3**, **4**]. One minor difference is that they only consider slices over discrete objects, i.e. of the form $\mathscr{E}/n \simeq \mathscr{E}^n$, where n denotes the sum of n copies of the terminal object, and for this they also need to assume finite sums. In our setting there is no reason for that restriction, and in fact Altenkirch and Morris [**5**] have been able to lift the restriction also from the container theory, introducing the notion of *indexed container*. Another difference, also quite minor, is that while we prefer to work with strength, the literature on containers considers fibred categories, fibred functors and fibred natural transformations. This involves replacing all slice categories $\mathscr{E}/I$ by the fibration over $\mathscr{E}$ whose K-fibre is $\mathscr{E}/(K \times I)$, and work with those instead. The two viewpoints are in fact equivalent, thanks to a result of Paré, who showed (cf. [**27**]) that if a strong functor preserves pullbacks then it is canonically indexed, i.e. fibred. (It is easy to see that a fibred functor has a strength.) We have chosen the viewpoint of tensorial strength for its simplicity. Modulo the above minor differences (and modulo Paré's theorem), Lemma 2·2, Theorem 2·12, and Theorem 2·17 were also proved in Abbott's thesis [**1**].

3. *The double category of polynomial functors*

3·1. It is important to be able to compare polynomial functors with different endpoints, and to base change polynomial functors along maps in $\mathscr{E}$. This need can been seen already for linear functors 1·7: a small category is a monad in the bicategory of spans [**8**], but in order to get functors between categories with different object sets, one needs maps between spans with different endpoints [**38**]. The most convenient framework for this is that of double categories, as it allows for diagrammatic representation. The base change structure is concisely captured in Shulman's notion of framed bicategory [**55**]: our double categories of polynomial functors will in fact be framed bicategories.

3·2. Recall that a double category $\mathbb{D}$ consists of a category of objects $\mathbb{D}_0$, a category of morphisms $\mathbb{D}_1$, together with structure functors

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, "" description] & \mathbb{D}_1 \arrow[l, shift right=2, "\partial_1"'] \arrow[l, shift left=2, "\partial_0"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\text{comp.}"'] \end{tikzcd}$$

subject to the usual category axioms. The objects of $\mathbb{D}_0$ are called *objects* of $\mathbb{D}$, the morphisms of $\mathbb{D}_0$ are called *vertical arrows*, the objects of $\mathbb{D}_1$ are called *horizontal arrows*, and the morphisms of $\mathbb{D}_1$ are called *squares*. As is custom [**19**], we allow the possibility for the horizontal composition to be associative and unital only up to specified coherent isomorphisms. Precisely, a double category is a pseudo-category [**47**] in the 2-category Cat; see also [**43**, Section 5·2].

3·3. A *framed bicategory* [**55**] is a double category for which the functor

$$(\partial_0, \partial_1) : \mathbb{D}_1 \longrightarrow \mathbb{D}_0 \times \mathbb{D}_0$$

is a bifibration. (In fact, if it is a fibration then it is automatically an opfibration, and vice versa.) The upshot of this condition is that horizontal arrows can be base changed and cobase changed along arrows in $\mathbb{D}_0 \times \mathbb{D}_0$ (i.e. pairs of vertical arrows).

3·4. We need to fix some terminology. The characteristic property of a fibration is that every arrow in the base category admits a cartesian lift, and that every arrow in the total space factors (essentially uniquely) as a vertical arrow followed by a cartesian one. In the present situation, the term 'cartesian' is already in use to designate cartesian natural transformations (which fibrationally speaking are vertical rather than cartesian), and the word 'vertical' already has a double-categorical meaning. For these reasons, instead of talking about 'cartesian arrow' for a fibration we shall say *transporter* arrow; this terminology goes back to Grothendieck [**20**]. Correspondingly, we shall say *cotransporter* instead of opcartesian. We shall simply refrain from using 'vertical' in the fibration sense. The arrows mapping to identity arrows by the fibration will be precisely the natural transformations of polynomial functors.

3·5. We want to extend the bicategories $\mathrm{Poly}_{\mathscr{E}}$ and $\mathrm{PolyFun}_{\mathscr{E}}$ to double categories. The objects of the double category $\mathrm{PolyFun}_{\mathscr{E}}$ are the slices of $\mathscr{E}$, and the horizontal arrows are the polynomial functors. The vertical arrows are the dependent sum functors (i.e. functors of the form Σ_u for some u), and the squares in $\mathrm{Poly}_{\mathscr{E}}$ are of the form

$$\begin{tikzcd} \mathscr{E}/I' \arrow[r, "P'"] \arrow[d, "\Sigma_u"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathscr{E}/J' \arrow[d, "\Sigma_v"] \\ \mathscr{E}/I \arrow[r, "P"'] & \mathscr{E}/J \end{tikzcd} \tag{18}$$

where P' and P are polynomial functors and ϕ is a strong natural transformation.

PROPOSITION 3·6. *The double category* $\mathrm{PolyFun}_{\mathscr{E}}$ *is a framed bicategory.*

Proof. The claim is that the functor sending a polynomial functor $P : \mathscr{E}/I \to \mathscr{E}/J$ to (I, J) is a bifibration. For each pair of arrows $u : I' \to I$, $v : J' \to J$ in $\mathscr{E}$ we have the following basic squares (companion pairs and conjoint pairs [**19**])

$$\begin{tikzcd} \mathscr{E}/I' \arrow[r, "\Sigma_u"] \arrow[d, "\Sigma_u"'] & \mathscr{E}/I \arrow[d, equal] \\ \mathscr{E}/I \arrow[r, equal] & \mathscr{E}/I \end{tikzcd} \quad \begin{tikzcd} \mathscr{E}/I' \arrow[r, equal] \arrow[d, "\Sigma_u"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathscr{E}/I' \arrow[d, equal] \\ \mathscr{E}/I \arrow[r, "\Delta_u"'] & \mathscr{E}/I' \end{tikzcd} \quad \begin{tikzcd} \mathscr{E}/J' \arrow[r, equal] \arrow[d, equal] & \mathscr{E}/J' \arrow[d, "\Sigma_v"] \\ \mathscr{E}/J' \arrow[r, "\Sigma_v"'] & \mathscr{E}/J \end{tikzcd} \quad \begin{tikzcd} \mathscr{E}/J \arrow[r, "\Delta_v"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon"] & \mathscr{E}/J' \arrow[d, "\Sigma_v"] \\ \mathscr{E}/J \arrow[r, equal] & \mathscr{E}/J \end{tikzcd}.$$

It is now direct to check that the pasted square

$$\begin{tikzcd} \mathscr{E}/I' \arrow[r, "\Sigma_u"] \arrow[d, "\Sigma_u"'] & \mathscr{E}/I \arrow[r, "P"] \arrow[d, equal] & \mathscr{E}/J \arrow[r, "\Delta_v"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon"] & \mathscr{E}/J' \arrow[d, "\Sigma_v"] \\ \mathscr{E}/I \arrow[r, equal] & \mathscr{E}/I \arrow[r, "P"'] & \mathscr{E}/J \arrow[r, equal] & \mathscr{E}/J \end{tikzcd}$$

is a transporter lift of (u, v) to P. We call $\Delta_v \circ P \circ \Sigma_u$ the *base change* of P along (u, v), and denote it $(u, v)^*(P)$.

Dually, it is direct to check that the pasted square

$$\begin{tikzcd} \mathscr{E}/I' \arrow[r, equal] \arrow[d, "\Sigma_u"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathscr{E}/I' \arrow[r, "P'"] \arrow[d, equal] & \mathscr{E}/J' \arrow[r, equal] \arrow[d, equal] & \mathscr{E}/J' \arrow[d, "\Sigma_v"] \\ \mathscr{E}/I \arrow[r, "\Delta_u"'] & \mathscr{E}/I' \arrow[r, "P'"'] & \mathscr{E}/J' \arrow[r, "\Sigma_v"'] & \mathscr{E}/J \end{tikzcd}$$

is a cotransporter lift of (u, v) to P'. We call $\Sigma_v \circ P' \circ \Delta_u$ the *cobase change* of P' along (u, v), and denote it $(u, v)_!(P')$.

The above procedure of getting a framed bicategory out of a bicategory is a general construction: one starts with a bicategory $\mathscr{C}$ with a subcategory $\mathscr{L}$ consisting of left adjoints and comprising all the objects of $\mathscr{C}$, and obtains a framed bicategory by taking as vertical arrows the arrows in $\mathscr{L}$. The details can be found in [**55**, appendix].

3·7. Via the biequivalence $\mathrm{Poly}_{\mathscr{E}} \simeq \mathrm{PolyFun}_{\mathscr{E}}$ between the bicategory of polynomials and the 2-category of polynomial functors, Proposition 3·6 gives us also a framed bicategory of polynomials $\mathrm{Poly}_{\mathscr{E}}$, featuring nice diagrammatic representations which we now spell out, extending the results of Section 2. The following result is the double-category version of Theorem 2·12.

THEOREM 3·8. *The squares* (18) *of* $\mathrm{PolyFun}_{\mathscr{E}}$ *are represented by diagrams of the form*

$$\begin{tikzcd}
P' : & I' \arrow[dd, "u"'] & B' \arrow[l] \arrow[r] & A' \arrow[r] & J' \arrow[dd, "v"] \\
& & \cdot \arrow[u] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[u, equal] \arrow[d] & \\
P : & I & B \arrow[l] \arrow[r] & A \arrow[r] & J .
\end{tikzcd} \tag{19}$$

This representation is unique up to choice of pullback in the middle. It follows that extension constitutes a framed biequivalence

$$\mathrm{Poly}_{\mathscr{E}} \xrightarrow{\sim} \mathrm{PolyFun}_{\mathscr{E}} .$$

Proof. By Theorem 2·12, diagrams like (19) (up to choice of pullback) are in bijective correspondence with strong natural transformations $\Sigma_v \circ P' \circ \Delta_u \Rightarrow P$, which by adjointness correspond to strong natural transformations $\Sigma_v \circ P' \Rightarrow P \circ \Sigma_u$, i.e. squares (18) in $\mathrm{PolyFun}_{\mathscr{E}}$.

3·9. The vertical composition of two diagrams

$$\begin{tikzcd}
\cdot \arrow[dd] & \cdot \arrow[l] \arrow[r] & \cdot \arrow[r] & \cdot \arrow[dd] \\
& \cdot \arrow[u] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[u, equal] \arrow[d] & \\
\cdot \arrow[dd] & \cdot \arrow[l] \arrow[r] & \cdot \arrow[r] & \cdot \arrow[dd] \\
& \cdot \arrow[u] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[u, equal] \arrow[d] & \\
\cdot & \cdot \arrow[l] \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

is performed by replacing the two middle squares

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \\
\cdot \arrow[u] \arrow[r] & \cdot \arrow[u, equal]
\end{tikzcd}$$

by a configuration

$$
\begin{tikzcd}
\cdot \arrow[r] & \cdot \\
\cdot \arrow[r] \arrow[u] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \cdot \arrow[u, equal] \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
$$

and then composing vertically. The replacement is a simple pullback construction, and checking that the composed diagram has the same extension as the vertical pasting of the extensions is a straightforward calculation.

3·10. At the level of polynomials, the bifibration $\mathrm{Poly}_{\mathscr{E}} \to \mathscr{E} \times \mathscr{E}$ is now the 'endpoints' functor, associating to a polynomial $I \leftarrow B \to A \to J$ the pair (I, J). With notation as in the proof of Proposition 3·6, we know the cobase change of P' along (u, v) is just $\Sigma_v \circ P' \circ \Delta_u$, and it is easy to see that

$$
\begin{tikzcd}
P' : & I' \arrow[d, "u"'] & B' \arrow[l] \arrow[r] \arrow[d, equal] & A' \arrow[r] \arrow[d, equal] & J' \arrow[d, "v"] \\
(u, v)_!(P') : & I & B' \arrow[l] \arrow[r] & A' \arrow[r] & J
\end{tikzcd}
$$

is a cotransporter lift of (u, v) to P'.

The transporter lift of (u, v) to P, which is the same thing as the base change of P along (u, v), is slightly more complicated to construct. It can be given by first base changing along (u, Id) and then along (Id, v):

$$
\begin{tikzcd}
(u, v)^*(P) : & I' \arrow[d, equal] & \cdot \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & J' \arrow[d, "v"] \\
(u, \mathrm{Id})^*(P) : & \cdot \arrow[d, equal] & \cdot \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \cdot \arrow[r] \arrow[dd] & \cdot \arrow[dd, equal] \\
& \cdot \arrow[d, "u"'] & \cdot \arrow[l] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] & & \\
P : & I & B \arrow[l] \arrow[r] & A \arrow[r] & J.
\end{tikzcd}
\tag{20}
$$

3·11. The intermediate polynomial $(u, \mathrm{Id})^*(P)$ is called the *source lift* of P along u, and we shall need it later on. Since ∂_0 (as well as ∂_1) is itself a bifibration, for which the source lift is the transporter lift, it enjoys the following universal property: every square

$$
\begin{tikzcd}
P' : & I' \arrow[dd, "u"'] & \cdot \arrow[l] \arrow[r] & \cdot \arrow[r] \arrow[d, equal] & J' \arrow[dd, "v"] \\
& & \cdot \arrow[r] \arrow[u] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \cdot \arrow[d] & \\
P : & I & \cdot \arrow[l] \arrow[r] & \cdot \arrow[r] & J
\end{tikzcd}
$$

factors uniquely through the source lift, like

$$
\begin{tikzcd}
P' : & I' \arrow[d, equal] & \cdot \arrow[l] \arrow[r] & \cdot \arrow[r] \arrow[d, equal] & J' \arrow[dd, "v"] \\
 & & \cdot \arrow[u] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d] & \\
(u,\mathrm{Id})^*(P) : & I' \arrow[uu, equal] \arrow[d, "u"'] & \cdot \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, equal] \\
P : & I & \cdot \arrow[l] \arrow[r] & \cdot \arrow[r] & J
\end{tikzcd}
$$

where the bottom part is as in (20).

3·12. All the constructions and arguments of this section apply equally well inside the cartesian fragment: starting with the 2-category $\mathrm{PolyFun}^{\mathrm{c}}_{\mathscr{E}}$ of polynomial functors and their cartesian strong natural transformations, a double category $\mathbb{P}\mathrm{olyFun}^{\mathrm{c}}_{\mathscr{E}}$ results, which is a framed bicategory. The only point to note is that all the constructions are compatible with the cartesian condition, since they all depend on the $\Sigma \dashv \Delta$ adjunction, which is cartesian. Note also that the transporter and cotransporter lifts belong to the cartesian fragment. The following two results follow readily.

PROPOSITION 3·13. *The double category* $\mathbb{P}\mathrm{olyFun}^{\mathrm{c}}_{\mathscr{E}}$ *whose objects are the slices of* $\mathscr{E}$*, whose horizontal arrows are the polynomial functors, whose vertical arrows are the dependent sum functors, and whose squares are cartesian strong natural transformations*

$$
\begin{tikzcd}
\mathscr{E}/I' \arrow[r, "P'"] \arrow[d, "\Sigma_u"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathscr{E}/J' \arrow[d, "\Sigma_v"] \\
\mathscr{E}/I \arrow[r, "P"'] & \mathscr{E}/J
\end{tikzcd}
$$

is a framed bicategory.

PROPOSITION 3·14. *The squares of* $\mathbb{P}\mathrm{olyFun}^{\mathrm{c}}_{\mathscr{E}}$ *are represented uniquely by diagrams*

$$
\begin{tikzcd}
I' \arrow[d] & B' \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[r] \arrow[d] & J' \arrow[d] \\
I & B \arrow[l] \arrow[r] & A \arrow[r] & J ,
\end{tikzcd}
\tag{21}
$$

hence extension constitutes a framed biequivalence

$$\mathbb{P}\mathrm{oly}^{\mathrm{c}}_{\mathscr{E}} \xrightarrow{\sim} \mathbb{P}\mathrm{olyFun}^{\mathrm{c}}_{\mathscr{E}} .$$

3·15. For the remainder of this paper, we shall only deal with the cartesian fragment, which is also what is needed in [**35**] and [**36**]. In those two papers, a central construction is to label trees by a polynomial endofunctor P. Trees are themselves seen as polynomial endofunctors (cf. Example 1·10), and the labelling amounts precisely to a cartesian 2-cell in the double category of polynomial functors. The importance of the cartesian condition (a bijection of certain fibres) is to ensure that a node in a tree is labelled by an operation of the same arity.

3·16. We finish this section with a digression on the relationship between polynomial functors and the shapely functors and shapely types of Jay and Cockett [**25**, **26**], since the double-category setting provides some conceptual simplification of the latter notion. We now assume $\mathscr{E}$ has sums.

A *shapely functor* [**26**] is a pullback-preserving functor $F : \mathscr{E}^m \to \mathscr{E}^n$ equipped with a strength. Since, for a natural number n, the discrete power $\mathscr{E}^n$ is equivalent to the slice $\mathscr{E}/n$, where n now denotes the n-fold sum of 1 in $\mathscr{E}$, it makes sense to compare shapely functors and polynomial functors. Since a polynomial functor preserves pullbacks and has a canonical strength, it is canonically a shapely functor. It is not true that every shapely functor is polynomial. For a counter example, let K be a set with a non-principal filter $\mathscr{D}$, and consider the filter-power functor

$$\begin{aligned} F : \mathrm{Set} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \operatorname{colim}_{D\in\mathscr{D}} X^D , \end{aligned}$$

which preserves finite limits since it is a filtered colimit of representables. Since every endofunctor on Set has a canonical strength, F is a shapely functor. However, F does not preserve all cofiltered limits, and hence, by (1·18) ((ii)) cannot be polynomial. For example, $\emptyset = \lim_{D\in\mathscr{D}} D$ itself is not preserved. This example is apparently at odds with [**2**, theorem 8·3].

3·17. Let $L : \mathscr{E} \to \mathscr{E}$ denote the *list endofunctor*, $L(X) = \sum_{n\in\mathbb{N}} X^n$, which is the same as what we called the free-monoid monad in Example 1·9. A *shapely type* [**26**] in one variable is a shapely functor equipped with a cartesian strong natural transformation to L. A morphism of shapely types is a natural transformation commuting with the structure map to L. The idea is that the shapely functor represents the template or the shape into which some data can be inserted, while the list holds the actual data; the cartesian natural transformation encodes how the data is to be inserted into the template. As emphasized in [**49**], the cartesian strong natural transformation is part of the structure of a shapely type. Since any functor with a cartesian natural transformation to L is polynomial by Lemma 2·2, it is clear that one-variable shapely types are essentially the same thing as one-variable polynomial endofunctors with a cartesian natural transformation to L, and that there is an equivalence of categories between the category of shapely types and the category $\mathrm{Poly}^{\mathrm{c}}_{\mathscr{E}}(1, 1)/L$.

According to Jay and Cockett [**26**], a shapely type in m input variables and n output variables is a shapely functor $\mathscr{E}^m \to \mathscr{E}^n$ equipped with a cartesian strong natural transformation to the functor $L_{m,n} : \mathscr{E}^m \to \mathscr{E}^n$ defined by

$$L_{m,n}(X_i \mid i \in m) = \big(L(\textstyle\sum_{i\in m} X_i) \mid j \in n\big) ,$$

and they motivate this definition by considerations on how to insert data into templates. With the double-category formalism, we can give a conceptual explanation of the formula: writing $u_m : m \to 1$ and $u_n : n \to 1$ for the maps to the terminal object, the functor $L_{m,n} : \mathscr{E}^m \to \mathscr{E}^n$ is nothing but the composite

$$\Delta_{u_n} \circ L \circ \Sigma_{u_m} = (u_m, u_n)^* L ,$$

the base change of L along (u_m, u_n). Hence we can say uniformly that a shapely type is an object in $\mathrm{Poly}^{\mathrm{c}}_{\mathscr{E}}/L$ with endpoints finite discrete objects.

4. *Polynomial monads*

4·1. Let $I \in \mathscr{E}$. A *polynomial monad* on $\mathscr{E}/I$ is a monad (T, η, μ) for which T is a polynomial functor and η and μ are cartesian strong natural transformations. From the point of view of the formal theory of monads [**56**], a polynomial monad is a monad in the 2-category $\mathrm{PolyFun}^{\mathrm{c}}_{\mathscr{E}}$. A basic example of a polynomial monad is the free-monoid monad of Example 1·9.

4·2. We are interested in the construction of the free monad on a polynomial endofunctor, and start by recalling from [**7**, **30**] some general facts about free monads. Let $\mathscr{C}$ be a category and $P : \mathscr{C} \to \mathscr{C}$ an endofunctor. The *free monad* on P is a monad (T, η, μ) on $\mathscr{C}$ together with a natural transformation $\alpha : P \Rightarrow T$ enjoying the following universal property: for any monad (T', η', μ') on $\mathscr{C}$ and any natural transformation $\phi : P \Rightarrow T'$ there exists a unique monad morphism $\phi^\sharp : T \Rightarrow T'$ making the following diagram commute:

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[dr, bend right, "\phi"'] & T \arrow[d, "\phi^\sharp"] \\ & T'. \end{tikzcd}$$

The following construction of the free monad on P is standard. Let P-alg denote the category of P-algebras and P-algebra morphisms. We denote P-algebras as pairs (X, sup_X) where X is the underlying object, and $\mathrm{sup}_X : PX \to X$ is the structure map, sometimes suppressed from the notation for brevity. If the forgetful functor $U : P\text{-alg} \to \mathscr{C}$ has a left adjoint, then the monad (T, η, μ) resulting from the adjunction is the free monad on P. If $\mathscr{C}$ has binary sums, a necessary and sufficient condition for the existence of the left adjoint to U is that, for every $X \in \mathscr{C}$, the endofunctor $X + P(-) : \mathscr{C} \to \mathscr{C}$ has an initial algebra. Indeed, in that case we can construct the free monad as follows. For $X \in \mathscr{C}$, we define TX as the initial algebra for $X + P(-) : \mathscr{C} \to \mathscr{C}$, and $\eta_X : X \to TX$ as the composite

$$X \xrightarrow{\iota_1} X + P(TX) \xrightarrow{t_X} TX$$

where ι_1 is the first sum inclusion and t_X is the structure map of TX as an $(X + P)$-algebra. Finally, since T^2X is the initial algebra for the functor $TX + P(-)$, we can define $\mu_X : T^2X \to TX$ as the unique map making the following diagram commute:

$$\begin{tikzcd} TX + P(T^2X) \arrow[r, "TX+P(\mu_X)"] \arrow[dd, "t_{TX}"'] & TX + P(TX) \arrow[d] \\ & TX + X + P(TX) \arrow[d, "(1_{TX}, t_X)"] \\ T^2X \arrow[r, "\mu_X"'] & TX. \end{tikzcd}$$

Functoriality, naturality, and the monad axioms follow readily from these definitions. Note that the X-component of the natural transformation $\alpha : P \Rightarrow T$ is given as the composite

$$PX \xrightarrow{P(\eta_X)} P(TX) \xrightarrow{\iota_2} X + P(TX) \xrightarrow{t_X} TX\,. \qquad (22)$$

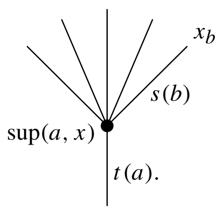
4·3. Let us now return to the locally cartesian closed category $\mathscr{E}$, now assumed to be extensive and in particular have finite sums. Recall from [**48**] that $\mathscr{E}$ is said to have W-types if every polynomial functor in a single variable on $\mathscr{E}$ has an initial algebra. This terminology is motivated by the fact that initial algebras for polynomial functors in a single variable are category-theoretic counterparts of Martin–Löf's types of wellfounded trees [**51**]. Every elementary topos with a natural numbers object has W-types [**48**]. If $\mathscr{E}$ has W-types, then every polynomial endofunctor, not just those in a single variable, has an initial algebra [**17**, theorem 14]. Initial algebras for general polynomial functors are category-theoretic counterparts of Petersson and Synek's general tree types [**52**]; see also [**50**, chapter 16].

Henceforth, we assume that $\mathscr{E}$ has W-types. For any polynomial endofunctor $P : \mathscr{E}/I \to \mathscr{E}/I$ and any $X \in \mathscr{E}/I$, the functors $X + P(-) : \mathscr{E}/I \to \mathscr{E}/I$ are again polynomial, hence have initial algebras. Therefore every polynomial endofunctor admits a free monad.

4·4. Theorem 4·5 below asserts that the free monad on a polynomial functor is polynomial. The proof exploits the possibility of recursively defining maps out of initial algebras for polynomial functors, and we need first to set up some notation to handle this. Let $P : \mathscr{E}/I \to \mathscr{E}/I$ be the polynomial functor represented by the diagram

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} I.$$

We regard such a diagram as a generalised many-sorted signature. This point of view is most easily illustrated by considering the case of $\mathscr{E} = \mathrm{Set}$. The object I provides the set of sorts of the signature. The set of terms of the signature is defined inductively by saying that we have a term $\sup_a(x)$ of sort $t(a)$ whenever $a \in A$ and $x = (x_b \mid b \in B_a)$ is a family of terms such that x_b has sort $s(b)$ for all $b \in B_a$. Such a term may be represented graphically as a one-node tree



The incoming edges are indexed by the elements of B_a and further labelled by elements of I, with the edge indexed by $b \in B_a$ labelled by $s(b) \in I$. The outgoing edge is labelled by $t(a) \in I$. We label the node $\sup(a, x)$ if the family $x = (x_b \mid b \in B_a)$ labels its incoming edges.

Let W be the initial algebra for P, with structure map $\sup_W : PW \to W$. Initiality of the algebra means that for any other algebra $(X, \sup_X)$, there exists a unique algebra map $\theta : W \to X$, thus making the following diagram commute

$$\begin{tikzcd}
PW \arrow[r, "P(\theta)"] \arrow[d, "\sup_W"'] & PX \arrow[d, "\sup_X"] \\
W \arrow[r, "\theta"'] & X\,.
\end{tikzcd}$$

In the internal language of $\mathscr{E}$, we can represent the structure map of W as the I-indexed family

$$\sup_{W_i} : \sum_{a \in A_i} \prod_{b \in B_a} W_{sb} \to W_i\,.$$

The initiality of W can be expressed by saying that there exists a unique family of maps $\theta_i : W_i \to X_i$ satisfying the recursive equation

$$\theta_i(\sup_{W_i}(a, h)) = \sup_{X_i}(a, (\lambda b \in B_a)\ \theta_{sb}(hb))\,,$$

where we employ the lambda calculus notation $(\lambda b \in B_a)\ \theta_{sb}(hb)$ to indicate the function $B_a \to X$ sending b to $\theta_{sb}(hb)$.

THEOREM 4·5. *The free monad on a polynomial endofunctor is a polynomial monad.*

Proof. Let $P : \mathscr{E}/I \to \mathscr{E}/I$ be the polynomial endofunctor represented by

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} I,$$

and let (T, η, μ) be the free monad on P. We need to show that $T : \mathscr{E}/I \to \mathscr{E}/I$ is a polynomial functor, and that $\eta : \mathrm{Id} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are cartesian strong natural transformations. We shall show that T is naturally isomorphic to the polynomial functor represented by the diagram

$$I \xleftarrow{u} D \xrightarrow{g} C \xrightarrow{v} I \tag{23}$$

whose constituents we now proceed to construct. Intuitively, C is the set of wellfounded trees with branching profile given by the polynomial endofunctor $1 + P : \mathscr{E}/I \to \mathscr{E}/I$, while D is the set of such trees but with a marked leaf. We construct these two objects as least fixpoints. Put $Q = 1 + P$; in the internal language we have

$$Q(X_i \mid i \in I) = \left(\{i\} + \sum_{a \in A_i} \prod_{b \in B_a} X_{sb} \mid i \in I \right).$$

Let $(C_i \mid i \in I)$ be the initial algebra for Q. Its structure map is given by the family of isomorphisms

$$\sup_{C_i} : \{i\} + \sum_{a \in A_i} \prod_{b \in B_a} C_{sb} \xrightarrow{\sim} C_i\,, \tag{24}$$

meaning that a Q-tree is either a trivial tree (of some type $i \in I$) or a one-node tree which is a term from P (that is the choice of $a \in A_i$) and whose incoming edges are labelled by Q-trees (that is the map $k : B_a \to C_{sb}$). We now define the polynomial endofunctor $R : \mathscr{E}/C \to \mathscr{E}/C$ by letting

$$R(X_c \mid c \in C) = \big(\widetilde{X}_c \mid c \in C\big)\,,$$

where

$$\widetilde{X}_c = \begin{cases} \{i\} & \text{if } c = \sup(i)\,, \\ \sum_{b \in B_a} X_{kb} & \text{if } c = \sup(a, k)\,. \end{cases}$$

This definition can be seen to be that of a polynomial functor using the isomorphisms in (24) and the extensivity of $\mathscr{E}$. Let $(D_c \mid c \in C)$ be the initial algebra for R. Its structure maps consist of the following isomorphisms:

$$\sup_{D_{\sup_C(i)}} : \{i\} \xrightarrow{\sim} D_{\sup_C(i)}\,, \quad \sup_{D_{\sup_C(a,h)}} : \sum_{b \in B_a} D_{hb} \xrightarrow{\sim} D_{\sup_C(a,h)}\,.$$

The idea here is that a tree with a marked leaf is either a trivial tree, with the unique leaf marked, or it is a pointed collection of trees, for which the distinguished tree has a marked leaf. We now define $u : D \to I$ recursively so that we have

$$u(d) = \begin{cases} i & \text{if } d = \sup_D(i)\,, \\ u(d') & \text{if } d = \sup_D(b, d')\,. \end{cases}$$

We have now constructed the polynomial in (23), and we proceed to verify that the associated polynomial functor is naturally isomorphic to T. To prove this, it is sufficient to show that for every $X = (X_i \mid i \in I)$, the object

$$\Big(\sum_{c \in C_i} \prod_{d \in D_c} X_{ud} \mid i \in I \Big)$$

enjoys the same universal property that characterises TX, namely that of being an initial algebra for the functor $X + P(-) : \mathscr{E}/I \to \mathscr{E}/I$. The required structure map is given by the following chain of isomorphisms:

$$\begin{aligned} X_i + \sum_{a \in A_i} \prod_{b \in B_a} \sum_{c \in C_{sb}} \prod_{d \in D_c} X_{ud} &\cong X_i + \sum_{a \in A_i} \sum_{k \in \prod_{b \in B_a} C_{sb}} \prod_{b \in B_a} \prod_{d \in D_{kb}} X_{ud} \\ &\cong X_i + \sum_{(a,k) \in \sum_{a \in A_i} \prod_{b \in B_a} C_{sb}} \; \prod_{(b,d) \in \sum_{b \in B_a} D_{kb}} X_{ud} \\ &\cong \sum_{c \in C_i} \prod_{d \in D_c} X_{ud}\,. \end{aligned}$$

The initiality of the algebra follows by the initiality of C and D via lengthy, but not difficult, calculations.

It remains to show that the unit and multiplication are cartesian. For the unit $\eta : \mathrm{Id} \Rightarrow T$, we construct a diagram

$$\begin{tikzcd} I \arrow[r, equal] \arrow[d, equal] & I \arrow[r, equal] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & I \arrow[r, equal] \arrow[d, "e"] & I \arrow[d, equal] \\ I & D \arrow[l, "u"] \arrow[r, "g"'] & C \arrow[r, "v"'] & I \end{tikzcd}$$

representing a cartesian strong natural transformation that coincides with η, modulo the isomorphism established above. For $i \in I$, we define $e_i : \{i\} \to C_i$ by letting $e_i(i) = \sup_C(i)$. With this definition, we have an isomorphism $\{i\} \cong D_{e_i(i)}$ for every $i \in I$, hence the middle square is cartesian. We proceed analogously for the multiplication. We will construct a diagram of the form

$$\begin{tikzcd} I \arrow[d, equal] & F \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[r] \arrow[d, "m"] & I \arrow[d, equal] \\ I & D \arrow[l, "u"] \arrow[r, "g"'] & C \arrow[r, "v"'] & I \end{tikzcd}$$

where the top polynomial represents $T^2 : \mathscr{E}/I \to \mathscr{E}/I$ and the diagram represents the multiplication. Direct calculations with the definition of substitution show that, for $i \in I$, we have

$$E_i = \sum_{c \in C_i} \prod_{d \in D_c} C_{ud}\,,$$

and that, for $(c, k) \in E_i$, we have

$$F_{(c,k)} = \sum_{d \in D_c} D_{kd} \,.$$

The family of maps $m_i : E_i \to C_i$ is defined recursively so that, for $(c, k) \in E_i$, we have

$$m_i(c, k) = \begin{cases} k(i) & \text{if } c = \sup_C(i) \\ \sup(a, (\lambda b \in B_a)\ m_{sb}(hb, kb)) & \text{if } c = \sup_C(a, h) \,. \end{cases}$$

To check that the second clause is well-defined, observe that if $\sup_C(a, h) \in C_i$ then, for $b \in B_a$, we have $hb \in C_{sb}$. Furthermore we have

$$\prod_{d \in D_{\sup(a,h)}} C_{ud} \;\cong \prod_{(b,d') \in \sum_{b \in B_a} D_{hb}} C_{u(b,d')} \;\cong \prod_{b \in B_a} \prod_{d' \in D_{hb}} C_{u(d')} \,.$$

Hence, for $b \in B_a$, we can regard kb as an element of

$$\prod_{d' \in D_{hb}} C_{u(d')}$$

so that $(hb, kb) \in E_{sb}$, and therefore $m_{sb}(hb, kb) \in C_{sb}$, as required. It is now easy to check that, for $(c, k) \in E_i$, we have an isomorphism

$$D_{m_i(c,k)} \cong F_{(c,k)} \,.$$

It remains to check that the natural transformation induced by the diagram above is indeed the multiplication of the free monad on P. This involves checking that its components satisfy the condition that determines $\mu_X : T^2 X \to TX$ uniquely. This is a lengthy calculation which we omit.

4·6. To conclude this section, we derive from Theorem 4·5 a stronger universal property of the free monad. Let $\mathrm{PolyEnd}_{\mathscr{E}}$ denote the category whose objects are pairs (I, P) consisting of an object $I \in \mathscr{E}$ and a polynomial endofunctor P on $\mathscr{E}/I$, and whose morphisms from (I, P) to (I', P') consist of a map $u : I' \to I$ in $\mathscr{E}$ and a cartesian strong natural transformation

$$\begin{tikzcd} \mathscr{E}/I' \arrow[r, "P'"] \arrow[d, "\Sigma_u"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathscr{E}/I' \arrow[d, "\Sigma_u"] \\ \mathscr{E}/I \arrow[r, "P"'] & \mathscr{E}/I \,. \end{tikzcd} \tag{25}$$

The category $\mathrm{PolyMnd}_{\mathscr{E}}$ of polynomial monads in $\mathscr{E}$ is defined in a similar way: the objects are pairs (I, T) consisting of an object $I \in \mathscr{E}$ and a polynomial monad T on $\mathscr{E}/I$. Maps from (I, T) to (I', T') are as in (25), but required now to satisfy the following monad map axioms:

$$\begin{tikzcd} \Sigma_u \arrow[r, "\Sigma_u \eta'"] \arrow[dr, bend right, "\eta \Sigma_u"'] & \Sigma_u T' \arrow[d, "\phi"] \\ & T \Sigma_u \end{tikzcd} \qquad \begin{tikzcd} \Sigma_u T'^2 \arrow[r, "\phi T'"] \arrow[d, "\Sigma_u \mu'"'] & T \Sigma_u T' \arrow[r, "T\phi"] & T^2 \Sigma_u \arrow[d, "\mu \Sigma_u"] \\ \Sigma_u T' \arrow[rr, "\phi"'] & & T \Sigma_u \,. \end{tikzcd} \tag{26}$$

Let us point out that the monad morphisms defined above are more special than those that would arise by instantiating the notion of a monad morphism between monads in a

2-category, as defined in [**56**], to the 2-category $\mathrm{PolyFun}^{\mathrm{c}}_{\mathscr{E}}$: we allow only functors of the form $\Sigma_u : \mathscr{E}/I' \to \mathscr{E}/I$, rather than arbitrary polynomial functors, as vertical maps in the diagram (25). Note also that our direction of 2-cells are the oplax monad maps rather than the lax ones.

COROLLARY 4·7. *The forgetful functor* $U : \mathrm{PolyMnd}_{\mathscr{E}} \to \mathrm{PolyEnd}_{\mathscr{E}}$ *has a left adjoint.*

Proof. Both $\mathrm{PolyEnd}_{\mathscr{E}}$ and $\mathrm{PolyMnd}_{\mathscr{E}}$ are fibred over $\mathscr{E}$ via the functors mapping an object $(I, -)$ to I, and U is a fibred functor. Therefore, to define a left adjoint to U, it is sufficient to define left adjoints to the forgetful functors

$$U_I : \mathrm{PolyMnd}_{\mathscr{E}}(\mathscr{E}/I) \longrightarrow \mathrm{PolyEnd}_{\mathscr{E}}(\mathscr{E}/I)\,,$$

where $\mathrm{PolyMnd}_{\mathscr{E}}(\mathscr{E}/I)$ and $\mathrm{PolyEnd}_{\mathscr{E}}(\mathscr{E}/I)$ denote the fibre categories over $I \in \mathscr{E}$. But each U_I has a left adjoint, sending P to the free monad on P, cf. Theorem 4·5. It remains to observe that the canonical natural transformation $\alpha : P \Rightarrow T$ ('insertion of generators') is strong and cartesian. But we even have a polynomial representation of it: with notation as in the proof of Theorem 4·5, α is given by the diagram

$$\begin{tikzcd} I \arrow[d, equal] & B \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r] \arrow[d, "\alpha_1"] & I \arrow[d, equal] \\ I & D \arrow[l] \arrow[r] & C \arrow[r] & I, \end{tikzcd}$$

cf. (22) for the description of α; the map α_1 takes a term in A and interprets it as a tree with one node. The map $B \to D$ is described similarly but with a marked leaf.

4·8. Observe that even if the forgetful functor $U : \mathrm{PolyMnd}_{\mathscr{E}} \to \mathrm{PolyEnd}_{\mathscr{E}}$ is fibred, its left adjoint is not. The situation is analogous to the one represented in the diagram

$$\begin{tikzcd} \mathrm{Cat} \arrow[rr] \arrow[dr] & & \mathrm{Grph} \arrow[dl] \\ & \mathrm{Set} & \end{tikzcd}$$

where Cat is the category of small categories, and Grph is the category of directed, non-reflexive graphs. The forgetful functor, mapping a category to its underlying graph, is a fibred functor, but its left adjoint, the free category functor, is not.

5. *P-spans, P-multicategories and P-operads*

5·1. Let $\mathrm{Span}_{\mathscr{E}}$ denote the bicategory of spans in $\mathscr{E}$, as introduced in [**8**]. Under the interpretation of spans as linear polynomials (cf. Example 1·7), composition of spans (resp. morphisms of spans) agrees with composition of polynomials (resp. morphisms of polynomials), so we can regard $\mathrm{Span}_{\mathscr{E}}$ as a locally full sub-bicategory of $\mathrm{Poly}^{\mathrm{c}}_{\mathscr{E}}$, and view polynomials as a natural 'non-linear' generalisation of spans.

5·2. There is another notion of 'non-linear' span, namely the P-spans of Burroni [**12**], which is a notion relative to is a cartesian monad P. This section is dedicated to a systematic comparison between the two notions, yielding (for a fixed polynomial monad P) an equivalence of framed bicategories between Burroni P-spans and polynomials over P in the double-category sense. We show how the comparison can be performed directly at the level of diagrams by means of some pullback constructions. Considering monads in these categories, we find an equivalence between P-multicategories (also called coloured P-operads) and polynomial monads over P, in the double-category sense.

In this section, strength plays no essential role: everything is cartesian relative to a fixed P, eventually assumed to be polynomial and hence strong, and for all the cartesian natural transformations into P there is a unique way to equip the domain with a strength in such a way that the natural transformation becomes strong.

5·3. We first need to recall some material on P-spans and their extension. To avoid clutter, and to place ourselves in the natural level of generality, we work in a cartesian closed category $\mathscr{C}$, and consider a fixed cartesian endofunctor $P : \mathscr{C} \to \mathscr{C}$. We shall later substitute $\mathscr{E}/I$ for $\mathscr{C}$, and assume that P is a polynomial monad on $\mathscr{E}/I$.

5·4. By definition, a *P-span* is a diagram in $\mathscr{C}$ of the form

$$P(D) \xleftarrow{d} N \xrightarrow{c} C\,, \tag{27}$$

A *morphism of P-spans* is a diagram like

$$\begin{tikzcd} P(D') \arrow[d, "P(f)"'] & N' \arrow[l, "d'"'] \arrow[r, "c"] \arrow[d, "g"] & C' \arrow[d, "h"] \\ P(D) & N \arrow[l, "d"] \arrow[r, "c"'] & C\,, \end{tikzcd} \tag{28}$$

We write P-Span for the category of P-spans and P-span morphisms in $\mathscr{C}$.

5·5. Let $\mathrm{Cart}_{\mathscr{C}}$ denote the category whose objects are cartesian functors between slices of $\mathscr{C}$ and whose arrows are diagrams of the form

$$\begin{tikzcd} \mathscr{C}/D' \arrow[r, "Q'"] \arrow[d, "\Sigma_u"'] \arrow[dr, phantom, "\Downarrow\psi"] & \mathscr{C}/C' \arrow[d, "\Sigma_v"] \\ \mathscr{C}/D \arrow[r, "Q"'] & \mathscr{C}/C\,, \end{tikzcd} \tag{29}$$

for $u : D' \to D$ and $v : C' \to C$ in $\mathscr{C}$, and ψ a cartesian natural transformation. Under the identification $\mathscr{C} = \mathscr{C}/1$, we can consider P as an object of $\mathrm{Cart}_{\mathscr{C}}$, so it makes sense to consider the slice category $\mathrm{Cart}_{\mathscr{C}}/P$: its objects are the cartesian functors $Q : \mathscr{C}/D \to \mathscr{C}/C$ equipped with a cartesian natural transformation

$$\begin{tikzcd} \mathscr{C}/D \arrow[r, "Q"] \arrow[d] \arrow[dr, phantom, "\Downarrow\phi"] & \mathscr{C}/C \arrow[d] \\ \mathscr{C} \arrow[r, "P"'] & \mathscr{C}\,. \end{tikzcd} \tag{30}$$

We now construct a functor $\mathrm{Ext} : P\text{-}\mathrm{Span}_{\mathscr{C}} \to \mathrm{Cart}_{\mathscr{C}}/P$. Its action on objects is defined by mapping a P-span $PD \xleftarrow{d} N \xrightarrow{c} C$ to the diagram

$$\begin{tikzcd}
\mathscr{C}/D \arrow[r, "P_{/D}"] \arrow[d] & \mathscr{C}/PD \arrow[r, "\Delta_d"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \mathscr{C}/N \arrow[r, "\Sigma_c"] \arrow[d] & \mathscr{C}/C \arrow[d] \\
\mathscr{C} \arrow[r, "P"'] & \mathscr{C} \arrow[r, equal] & \mathscr{C} \arrow[r, equal] & \mathscr{C} .
\end{tikzcd}$$

Here $P_{/D} : \mathscr{C}/D \to \mathscr{C}/PD$ sends $f : X \to D$ to $Pf : PX \to PD$, and the outer squares are commutative. The middle square is essentially given by the counit of the adjunction $\Sigma_d \dashv \Delta_d$, and is therefore a cartesian natural transformation. More precisely, it is the mate [**32**] of the commutative square

$$\begin{tikzcd}
\mathscr{C}/PD \arrow[d] & \mathscr{C}/N \arrow[l, "\Sigma_d"'] \arrow[d] \\
\mathscr{C} \arrow[r, equal] & \mathscr{C} .
\end{tikzcd}$$

The action of the functor $\mathrm{Ext} : P\text{-}\mathrm{Span}_{\mathscr{C}} \to \mathrm{Cart}_{\mathscr{C}}/P$ on morphisms is defined by mapping a diagram like (28) to the natural transformation

$$\begin{tikzcd}
\mathscr{C}/D' \arrow[r, "P_{/D'}"] \arrow[d, "\Sigma_f"'] & \mathscr{C}/PD' \arrow[r, "\Delta_{d'}"] \arrow[d, "\Sigma_{Pf}"'] \arrow[dr, phantom, "\Downarrow"] & \mathscr{C}/N' \arrow[r, "\Sigma_{c'}"] \arrow[d, "\Sigma_g"] & \mathscr{C}/C' \arrow[d, "\Sigma_h"] \\
\mathscr{C}/D \arrow[r, "P_{/D}"'] & \mathscr{C}/PD \arrow[r, "\Delta_d"'] & \mathscr{C}/N \arrow[r, "\Sigma_c"'] & \mathscr{C}/C ,
\end{tikzcd}$$

together with the structure maps to P. The outer squares are just commutative and the middle square (again cartesian) is the mate of the identity 2-cell

$$\begin{tikzcd}
\mathscr{C}/PD' \arrow[d, "\Sigma_{Pf}"'] & \mathscr{C}/N' \arrow[l, "\Sigma_{d'}"'] \arrow[d, "\Sigma_g"] \\
\mathscr{C}/PD & \mathscr{C}/N . \arrow[l, "\Sigma_d"]
\end{tikzcd}$$

PROPOSITION 5·6. *The functor* $\mathrm{Ext} : P\text{-}\mathrm{Span}_{\mathscr{C}} \to \mathrm{Cart}_{\mathscr{C}}/P$ *is an equivalence of categories.*

Proof. The quasi-inverse is defined by mapping

$$\begin{tikzcd}
\mathscr{C}/D \arrow[r, "Q"] \arrow[d] \arrow[dr, phantom, "\Downarrow\phi"] & \mathscr{C}/C \arrow[d] \\
\mathscr{C} \arrow[r, "P"'] & \mathscr{C}
\end{tikzcd}$$

to the P-span

$$PD \xleftarrow{\phi_D} QD \xrightarrow{Q(1_D)} C .$$

The verification of the details is straightforward.

5·7. Given a cartesian natural transformation $\theta : P \Rightarrow P'$, there is a shape-change functor

$$\begin{aligned} P\text{-Span}_{\mathscr{C}} &\longrightarrow P'\text{-Span}_{\mathscr{C}} \\ [PD \leftarrow N \rightarrow C] &\longmapsto [P'D \xleftarrow{\theta_D} PD \leftarrow N \rightarrow C]. \end{aligned}$$

We also have the functor

$$\begin{aligned} \mathrm{Cart}_{\mathscr{C}}/P &\longrightarrow \mathrm{Cart}_{\mathscr{C}}/P' \\ [Q \Rightarrow P] &\longmapsto [Q \Rightarrow P \overset{\theta}{\Rightarrow} P']. \end{aligned}$$

LEMMA 5·8. *The equivalence* Ext *of Proposition* 5·6 *is compatible with change of shape, in the sense that the following diagram commutes*:

$$\begin{tikzcd} P\text{-Span}_{\mathscr{C}} \arrow[r, "\mathrm{Ext}"] \arrow[d] & \mathrm{Cart}_{\mathscr{C}}/P \arrow[d] \\ P'\text{-Span}_{\mathscr{C}} \arrow[r, "\mathrm{Ext}"'] & \mathrm{Cart}_{\mathscr{C}}/P'. \end{tikzcd}$$

Proof. The claim amounts to checking

$$\Delta_{\theta_D} \circ P'_{/D} = P_{/D}$$

which follows from the assumption that θ is cartesian.

5·9. We now assume that P is a cartesian monad, so we have two natural transformations $\eta : 1 \Rightarrow P$ and $\mu : P \circ P \Rightarrow P$ at our disposal for shape-change. As is well known [**43**], this allows us to define horizontal composition of P-spans: given composable P-spans

$$\begin{tikzcd} & N \arrow[dl, "d"'] \arrow[dr, "c"] & & & U \arrow[dl, "s"'] \arrow[dr, "t"] & \\ PD & & C & PC & & B, \end{tikzcd}$$

we define their composite P-span by applying P to the first P-span, performing a pullback, and using the multiplication map:

$$\begin{tikzcd} & & & PN \times_{PC} U \arrow[dl] \arrow[dr] & & \\ & & PN \arrow[dl, "Pd"'] \arrow[dr, "Pc"] & & U \arrow[dl, "s"'] \arrow[dr, "t"] & \\ & PPD \arrow[dl, "\mu_D"'] & & PC & & B \\ PD & & & & & \end{tikzcd} \qquad (31)$$

Associativity of the composition law (up to coherent isomorphism) depends on that fact that P preserves pullbacks and that μ is cartesian. It further follows from the fact that η is cartesian that for each D the P-span

$$PD \xleftarrow{\eta_D} D \xrightarrow{1_D} D$$

is the identity P-spans for the composition law (up to coherent isomorphisms). It is clear that these constructions are functorial in vertical maps between P-spans, yielding altogether a double category of P-spans, denoted $P\text{-Span}_{\mathscr{C}}$: the objects and vertical morphisms are those of $\mathscr{C}$, the horizontal arrows are the P-spans, and the squares are diagrams like (28).

5·10. We also have a double-category structure on $\mathrm{Cart}_{\mathscr{C}}/P$: the horizontal composite of $Q \Rightarrow P$ with $R \Rightarrow P$ is $R \circ Q \Rightarrow P \circ P \Rightarrow P$, and the horizontal identity arrow is $\mathrm{Id} \Rightarrow P$. Let us verify that the extension of a horizontal composite is isomorphic to the composite of the extensions: in the diagram

$$
\begin{tikzcd}
 & & & \mathscr{C}/C \arrow[dr, "P_{/C}"] & & & \\
 & & \mathscr{C}/N \arrow[ur, "\Sigma_c"] \arrow[dr, "P_{/N}"'] & & \mathscr{C}/PC \arrow[dr, "\Delta_s"] & & \\
 & \mathscr{C}/PD \arrow[ur, "\Delta_d"] \arrow[dr, "P_{/PD}"] & & \mathscr{C}/PN \arrow[ur, "\Sigma_{(Pc)}"] \arrow[dr] \arrow[rr, phantom, "\text{B.C.}"] & & \mathscr{C}/U \arrow[dr, "\Sigma_t"] & \\
\mathscr{C}/D \arrow[ur, "P_{/D}"] \arrow[rr, "(PP)_{/D}"'] & & \mathscr{C}/PPD \arrow[ur, "\Delta_{(Pd)}"'] & & \mathscr{C}/PN \times_{PC} U \arrow[ur] & & \mathscr{C}/B
\end{tikzcd}
$$

the top path is the composite of the extension functors, and the bottom path is the extension of the composite span. The square marked B.C. is the Beck-Chevalley isomorphism for the cartesian square (31), and the other squares, as well as the triangle, are clearly commutative. The following proposition now follows from Proposition 5·6.

PROPOSITION 5·11. *The functor* $\mathrm{Ext} : P\text{-Span}_{\mathscr{C}} \to \mathrm{Cart}_{\mathscr{C}}/P$ *is an equivalence of double categories, in fact an equivalence of framed bicategories.*

We just owe to make explicit how the double category of P-spans is a framed bicategory: to each vertical map $u : D' \to D$ we associate the P-span

$$PD' \xleftarrow{\eta_{D'}} D' \xrightarrow{u} D\,.$$

This is a left adjoint; its right adjoint is the P-span

$$PD \xleftarrow{\eta_D \circ u} D' \xrightarrow{=} D'$$

as follows by noting that their extensions are respectively Σ_u and Δ_u. For this the important fact is that η is cartesian. With this observation it is clear that the equivalence is framed.

5·12. We now specialise to the case of interest, where $\mathscr{C} = \mathscr{E}/I$ and P is a polynomial monad on $\mathscr{E}/I$, represented by

$$I \longleftarrow B \longrightarrow A \longrightarrow I.$$

Since now all the maps involved in the P-spans are over I, a P-span can be interpreted as a commutative diagram

$$\begin{tikzcd}
& N \arrow[dl, "d"'] \arrow[dr, "c"] & \\
PD \arrow[dr] & & C \arrow[dl] \\
& I\,. &
\end{tikzcd}$$

If C is an object of $\mathscr{C}$, i.e. a map in $\mathscr{E}$ with codomain I, we shall write C also for its domain, and we have a natural identification of slices $\mathscr{C}/C \simeq \mathscr{E}/C$. That $P : \mathscr{E}/I \to \mathscr{E}/I$ is a polynomial monad, means thanks to Lemma 2·2, that all objects in $\mathrm{Cart}_{\mathscr{E}/I}/P$ are polynomial again, so $\mathrm{Cart}_{\mathscr{E}/I}/P \cong \mathrm{Poly}^{\mathrm{c}}_{\mathscr{E}}/P$, the category of polynomials cartesian over P in the double-category sense. In conclusion:

PROPOSITION 5·13. *The functor* Ext $: P\text{-Span}_{\mathscr{E}/I} \to \mathrm{Poly}^{\mathrm{c}}_{\mathscr{E}}/P$ *is an equivalence of framed bicategories.*

5·14. It is a natural question whether there is a direct comparison between P-spans and polynomials over P, without reference to their extensions. This is indeed the case, as we now proceed to establish, exploiting the framed structure. Given a polynomial over P, like

$$\begin{tikzcd}
Q: & D \arrow[d, "u"'] & M \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & N \arrow[r] \arrow[d] & C \arrow[d] \\
P: & I & B \arrow[l] \arrow[r] & A \arrow[r] & I\,.
\end{tikzcd}$$

Consider the canonical factorisation of this morphism through the source lift of P along u (cf. 3·11):

$$\begin{tikzcd}
Q: & D \arrow[d, equal] & M \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & N \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d] \\
(u,\mathrm{Id})^*P: & D \arrow[d, equal] & \cdot \arrow[l] \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & PD \arrow[r] \arrow[dd] & I \arrow[dd, equal] \\
& D \arrow[d, "u"'] & \cdot \arrow[l] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] & & \\
P: & I & B \arrow[l] \arrow[r] & A \arrow[r] & I\,.
\end{tikzcd} \tag{32}$$

Now we just read off the associated P-span:

$$\begin{tikzcd}
N \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d] \\
PD \arrow[r] & I\,.
\end{tikzcd}$$

Conversely, given such a P-span, place it on top of the rightmost leg of $P \circ \Sigma_\alpha = (u, \mathrm{Id})^* P$ (the middle row of the diagram, which depends only on α and P), and let M be the pullback of $N \to P(D)$ along the arrow labelled f. It is easy to see that these constructions are functorial, yielding an equivalence of hom categories $\mathrm{Poly}^{\mathrm{c}}_{\mathscr{E}}(D, C)/P \simeq P\text{-Span}_{\mathscr{E}/I}(D, C)$.

Example 5·15. Endo-P-spans $PC \leftarrow N \rightarrow C$, that is, polynomial endofunctors over P, are called *C-coloured P-collections*. If furthermore $C = I$ we simply call them *P-collections*. These are just polynomial endofunctors $Q : \mathscr{E}/I \rightarrow \mathscr{E}/I$ equipped with a cartesian natural transformation to P. This category is itself a slice of $\mathscr{E}$: it is easy to see that the functor

$$\begin{aligned} \mathrm{Poly}^{\mathrm{c}}_{\mathscr{E}}(I, I)/P &\longrightarrow \mathscr{E}/P1 \\ Q &\longmapsto [Q1 \rightarrow P1] \end{aligned}$$

is an equivalence of categories.

5·16. Burroni [**12**], Leinster [**43**] and Hermida [**22**] define P-multicategories (also called coloured P-operads) as monads in the bicategory of P-spans. P-multicategories are also monads in the double category of P-spans — this description also provides the P-multifunctors as (oplax) cartesian monad maps. P-multicategories based at the terminal object in $\mathscr{E}/I$ are called P-operads. If the base monad P is a polynomial monad, the equivalence of Proposition 5·13 induces an equivalence of the categories of monads, as summarised in the corollary below.

In the classical example, $\mathscr{E}$ is Set and P is the free-monoid monad M of Example 1·9. In this case, M-multicategories are the classical multicategories of Lambek [**40**], which are also called coloured nonsymmetric operads. In the one-object case, M-operads are the plain (nonsymmetric) operads. The other standard example is taking P to be the identity monad on Set. Then P-multicategories are just small categories and P-operads are just monoids. Hence small categories are essentially polynomial monads on some slice Set$/C$ with an oplax cartesian double-categorical monad map to Id, and monoids are essentially polynomial monads on Set with a cartesian monad map to Id. In summary, we have the following result.

COROLLARY 5·17. *There are natural equivalences of categories*

$$\begin{aligned} P\text{-Multicat} &\simeq \mathrm{PolyMnd}/P & P\text{-Operad} &\simeq \mathrm{PolyMnd}(1)/P \\ \mathrm{Multicat} &\simeq \mathrm{PolyMnd}/M & \mathrm{PlainOperad} &\simeq \mathrm{PolyMnd}(1)/M \\ \mathrm{Cat} &\simeq \mathrm{PolyMnd}/\mathrm{Id} & \mathrm{Monoid} &\simeq \mathrm{PolyMnd}(1)/\mathrm{Id}\,. \end{aligned}$$

5·18. The double category of polynomials is very convenient for reasoning with P-multicategories. The role of the base monad P for P-multicategories is to specify a profile for the operations. This involves specifying the shape of the input data, and it may also involve type constraints on input and output. In the classical case of $P = M$, the fibres of $\mathbb{N}' \rightarrow \mathbb{N}$ (Example 1·9) are finite ordinals, expressing the fact that inputs to an operation in a classical multicategory must be given as a finite list of objects. In this case there are no type constraints imposed by P on the operations.

For a more complicated example, let P : Set$/\mathbb{N} \rightarrow$ Set$/\mathbb{N}$ be the free-plain-operad monad, which takes a collection (i.e. an object in Set$/\mathbb{N}$) and returns the free plain operad on it [**43**, p.135, p.145, p.155]. This monad is polynomial (cf. [**36**]): it is represented by

$$\mathbb{N} \xleftarrow{s} \mathrm{Tr}^{\bullet} \xrightarrow{p} \mathrm{Tr} \xrightarrow{t} \mathbb{N}\,,$$

where Tr denotes the set of (isomorphism classes of) finite planar rooted trees, and $\mathrm{Tr}^{\bullet}$ denotes the set of (isomorphism classes of) finite planar rooted trees with a marked node. The map s returns the number of input edges of the marked node; the map p forgets the mark, and t returns the number of leaves. A P-multicategory Q has a set of objects and a set of operations. Each operation has its input slots organised as the set of nodes of some planar rooted tree, since this is how the p-fibres look like. Furthermore, there are type constraints: each object of Q must be typed in $\mathbb{N}$, via a number that we shall call the *degree* of the object, and a compatibility is required between the typing of operations and the typing of objects. Namely, the degree of the output object of an operation must equal the total number of leaves of the tree whose nodes index the input, and the degree of the object associated to a particular input slot must equal the number of incoming edges of the corresponding node in the tree. All this is displayed with clarity by the fact that Q is given by a diagram

$$\begin{tikzcd} Q: & D \arrow[d, "\alpha"'] & M \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & N \arrow[r] \arrow[d, "\beta"] & D \arrow[d, "\alpha"] \\ P: & \mathbb{N} & \mathrm{Tr}^{\bullet} \arrow[l] \arrow[r] & \mathrm{Tr} \arrow[r] & \mathbb{N}. \end{tikzcd}$$

The typing of the operations is concisely given by the map β, and the organisation of the inputs in terms of the fibres of the middle map of P is just the cartesian condition on the middle square. The typing of objects is encoded by α and the compatibility conditions, somewhat tedious to formulate in prose, are nothing but commutativity of the outer squares.

Finite planar rooted trees can be seen as M-trees, where M : Set $\to$ Set is the free-monoid monad (1·9). Abstract trees, in turn, can be seen as polynomial functors (1·10): to a tree is associated the polynomial functor

$$A \longleftarrow N' \longrightarrow N \longrightarrow A,$$

where A is the set of edges, N is the set of nodes, and N' is the set of nodes with a marked incoming edge. Formally, an M-tree is a tree over M in $\mathrm{Poly}^{\mathrm{c}}_{\mathscr{E}}$, that is to say a diagram

$$\begin{tikzcd} A \arrow[d] & N' \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & N \arrow[r] \arrow[d] & A \arrow[d] \\ 1 & \mathbb{N}' \arrow[l] \arrow[r] & \mathbb{N} \arrow[r] & 1\,. \end{tikzcd}$$

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