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Multiplicities: A deterministic view of nondeterminism

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Abstract

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We discuss four important problems in automata theory each of which has three natural variants: a deterministic one, a nondeterministic one, and as an in-between case a nondeterministic one with multiplicities. Our goal is to support a general view that the third variant is much closer to the first one than to the second one.

1. Introduction

As it is well known, there exist many cases when deterministic devices and nondeterministic devices have very different properties. A typical example is the equivalence problem for finite transducers: It is decidable for deterministic ones, e.g. for deterministic gsm's, while it is undecidable for nondeterministic finite transducers. Results like this emphasize the *difference* of determinism and nondeterminism, and has led to search for devices, such as unambiguous, single-valued or finite-valued automata, which would lie in between these two utmost classes.

Our purpose, however, is to look for the *similarity*, and not for the difference, between determinism and nondeterminism. By this we do not mean that we would be looking for properties with respect to which deterministic and nondeterministic devices would behave in the same way, as is exemplified by the language-generating power for several well-known classes of automata.

The similarity we want to point out is the following. Let $\mathscr{D}$ be a class of deterministic devices, $\mathscr{N}\mathscr{D}$ its nondeterministic counterpart and P a problem for (or a property of) $\mathscr{D}$. We would like to define a problem (or a property) P' such that

(i) in $\mathscr{D}$ P and P' coincide, and

(ii) $\mathscr{N}\mathscr{D}$ behaves with respect to P' as $\mathscr{D}$ behaves with respect to P.

In other words, we would like to extend P from $\mathscr{D}$ to $\mathscr{N}\mathscr{D}$, without "losing the nature of P".

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Let us illustrate our goals with an example. Let $\mathscr{D}$ be the class of finite deterministic two-tape automata and, hence, $\mathscr{N}\mathscr{D}$ is the class of finite nondeterministic two-tape automata. Further, let P be the problem of deciding the equivalence of two automata. If we would take $P'=P$, then (i), of course, would be satisfied. But (ii) would not hold true, since P for $\mathscr{N}\mathscr{D}$, as we already pointed out, is easily seen to be undecidable, while P for $\mathscr{D}$ was for a long time one of the important open problems in automata theory (cf. Problem P_2). Consequently, "P' for $\mathscr{N}\mathscr{D}$" and "P for $\mathscr{D}$" would not resemble each other.

Let us try another choice for P'. In P we are asking whether two automata $\mathscr{A}_1$ and $\mathscr{A}_2$ accept exactly the same pairs (u, v) of words. If $\mathscr{A}_i$'s are in $\mathscr{D}$ then each accepted pair (u, v) is accepted in a *unique way*. This is not true for automata in $\mathscr{N}\mathscr{D}$. However, even for a nondeterministic automaton $\mathscr{A}$ triples (n, u, v), when n tells how many times $\mathscr{A}$ accepts pair (u, v), are "accepted" in a unique way. This proposes to define P' as a problem asking to decide whether two given automata $\mathscr{A}_1$ and $\mathscr{A}_2$ in $\mathscr{N}\mathscr{D}$ accept each pair (u, v) of words equally many times, that is to say, whether $\mathscr{A}_1$ and $\mathscr{A}_2$ are *multiplicitly equivalent* (or *equivalent with multiplicities*). Then clearly P' coincides with P in $\mathscr{D}$, so that (i) holds.

The validity of condition (ii) is not easy to judge, since this condition is not precisely defined. However, it is immediately clear that the undecidability of P for $\mathscr{N}\mathscr{D}$ cannot be extended straightforwardly to the undecidability of P' for $\mathscr{N}\mathscr{D}$. Indeed, the decidability status of the latter problem was stated as an open and interesting question in [18]. As was shown in [15] (see also Section 3) both "P for $\mathscr{D}$" and "P' for $\mathscr{N}\mathscr{D}$" are decidable, and even in a similar way. Consequently, we are tempted to say that (ii) holds in this example.

What we did above can be interpreted as an attempt to define the deterministic behaviour of a nondeterministic process. Indeed, in a deterministic process (or, in fact, in an unambiguous process) for each input there exists a unique consequence, while in a nondeterministic process what is uniquely associated to an input is not its consequence but rather the set of all its consequences.

By the very definition, it is clear that in this sense determinism and nondeterminism with multiplicities correspond to each other. However, and this is the real goal of this note, we claim that this correspondence goes much deeper. We give a number of examples where we have "P for $\mathscr{D}$" and "P' for $\mathscr{N}\mathscr{D}$" such that the above conditions (i) and (ii) hold true, and P and P' are linked by the correspondence "determinism" vs. "nondeterminism with multiplicities". However, it has to be emphasized that in many cases it is not at all immediate to conclude (ii), although P' would be naturally defined. This is supported by several open problems we pose, as well as different techniques needed to establish that P' holds for $\mathscr{N}\mathscr{D}$.

In more detail, our goals are as follows:

(a) To support the above-mentioned correspondence of determinism and nondeterminism with multiplicities;

(b) To extend (as part of (a)) some results of the deterministic case to corresponding results of the nondeterministic case with multiplicities;

(c) To show the usefulness of the multiplicity considerations by using them to solve some classical problems in the deterministic case.

To achieve our goals we consider several problems with three variants: deterministic P(d), nondeterministic P(n) and nondeterministic with multiplicities P(m). We point out that the variants P(d) and P(m) resemble each other – unless the status of P(m) is not known – while P(n) behaves very differently. In addition, as a support for (c), we recall that the extension of a particular P(d) to P(m) and a final solution of the latter one lead, as is shown in [15], to a solution of a classical problem in (deterministic) automata theory.

To conclude this introduction we want to make the following two remarks.

Firstly, this note does not contain any new original results, and it has to be seen as a collection of results supporting our above view, and may be, in a minor scale, as a source of interesting open problems.

Secondly, the notion of multiplicity of a nondeterministic process is by no means new: It is exactly what is studied in the theory of formal power series, cf. [14, 27, 22, 5]. Consequently, our goals above are very much the same as normally stated for a support of the theory of formal power series. However, we try to provide a support for our view, i.e. for (a) above, without introducing any involved formalism and, thus, make this correspondence more intuitive and easier to accept.

2. Problems

In this section we introduce four central problems in automata theory, each of which has in a natural way the above-mentioned three variants: deterministic, nondeterministic and nondeterministic with multiplicities. Three of the questions are decidability questions, the remaining being an existential question. Also three of these, but not the same three as above, are connected directly to morphisms of monoids.

One of the simplest deterministic operations on a free monoid Σ^* is a *morphism* $h:\Sigma^*\to\Delta^*$ which translates deterministically words of Σ^* into words of Δ^*. Its nondeterministic variant is a *finite substitution* $\sigma:\Sigma^*\to\Delta^*$, which associates nondeterministically with a word of Σ^* a word of Δ^*. Of course, σ can be considered as a morphism from a free monoid Σ^* into the monoid of finite subsets of Δ^*, i.e. $\sigma:\Sigma^*\to 2^{\Delta^*}$. In between are finite substitutions with multiplicities, where one is not only listing all words in the images but also their multiplicities. Mathematically, such a finite substitution σ_M is a morphism from a free monoid Σ^* into the monoid of formal noncommutative polynomials $\mathbf{N}\langle\Delta\rangle$ with product of polynomials as the operation, i.e. $\sigma_M:\Sigma^*\to\mathbf{N}\langle\Delta\rangle$.

Let us go now to the problems:

$\mathbf{P_1}$: "Equivalence of morphisms on regular languages".

This problem asks to decide whether, for a given regular language $L\subseteq\Sigma^*$ and for two morphisms $h,g:\Sigma^*\to\mathcal{M}$, h and g *are equivalent* (or *agree*) on L, i.e. whether or not

$$h(w)=g(w)\quad\text{for all } w \text{ in } L \tag{1}$$

holds true. We abbreviate (1) as $h \overset{L}{\equiv} g$. The three variants of P_1 we are interested in are now obtained by choosing $\mathscr{M} = \Delta^*$, 2^{Δ^*} and $\mathbf{N}\langle \Delta \rangle$, respectively.

P_1 was first introduced in [12] for morphisms of free monoids, but with respect to any family $\mathscr{L}$ of language. Of course, also the family of mappings can be an arbitrary family Θ, so that we can define the problem $EP(\Theta, \mathscr{L})$ as a decision problem to decide whether two mappings from Θ are equivalent on a given L from $\mathscr{L}$. In these terms our three variants of the problem can be stated as

$$EP(\mathscr{H}, Reg), \ EP(\mathscr{F}\mathscr{S}, Reg), \ EP(\mathscr{F}\mathscr{S}_{\mathscr{M}}, Reg),$$

where $\mathscr{H}$, $\mathscr{F}\mathscr{S}$ and $\mathscr{F}\mathscr{S}_{\mathscr{M}}$ are the above-mentioned three classes of morphisms and *Reg* denotes the family of regular languages (over a considered alphabet Σ).

$\mathbf{P_2}$: "The equivalence problem for finite multitape automata".

This problem is selfexplanatory: Given two n-tape finite automata for some $n \geqslant 1$, decide whether they are equivalent. For deterministic and nondeterministic variants of the problem no further explanations are needed, for the multiplicity variant the equivalence has to be understood, of course, in the sense of the multiplicity equivalence. This problem is fundamental in automata theory, and was already presented (implicitly) in the classical paper of [26].

$\mathbf{P_3}$: "Injectivity of a morphism".

This problem is one of the most natural and fundamental problems of morphisms. It asks to decide whether a given morphism $h: \Sigma^* \to \mathscr{M}$ is injective, or in other words, whether a given finite subset X of $\mathscr{M}$ is uniquely decipherable. Of course, the three variants of the problem are clear: take $\mathscr{M}$ equal to Δ^*, 2^{Δ^*} or $\mathbf{N}\langle \Delta \rangle$, respectively. The deterministic variant was first studied in [28].

$\mathbf{P_4}$: "The Ehrenfeucht Problem for morphisms".

This is our only existential problem. It asks whether, for each language $L \subseteq \Sigma^*$, there exists a finite subset F of L such that, to test the equivalence of arbitrary two morphisms $h, g: \Sigma^* \to \mathscr{M}$ on L, it is enough to do so on F, i.e. whether

$$\forall h, g: \Sigma^* \to \mathscr{M}: h(w) = g(w), \ \forall w \in F \ \Rightarrow \ h(w) = g(w), \ \forall w \in L.$$

The problem was originally introduced by Ehrenfeucht around 1973 for morphisms of free monoids, i.e. for the case $\mathscr{M} = \Delta^*$, as a conjecture (so-called Ehrenfeucht's Conjecture). Again the three variants we are interested in are obvious, choose $\mathscr{M}$ equal to Δ^*, 2^{Δ^*} and $\mathbf{N}\langle \Delta \rangle$, respectively. Further variants are obtained by restricting the family of languages in some suitable way, for instance by considering only regular languages instead of all languages.

Table 1

	De/Tr	Open	Un/Fa
P_1	d, $\underline{\mathrm{m}}$	n	
P_2	$\underline{\mathrm{d}}$, $\underline{\mathrm{m}}$		n
P_3	d	m, n	
P_4	d	m	n

The subset F above is often referred to as a *test set* for L and, consequently, the whole problem can be called "the test-set problem for morphisms". In general setting it is clear that F cannot be found effectively; that is the reason we have here an existential problem. The Ehrenfeucht problem certainly proposes a very fundamental compactness question of languages and monoids. Its importance was even emphasized when it was noted in [9] (cf. also [17]) that the problem is equivalent to the following: Each system of equations (with a finite number of unknowns) over Δ^* has an equivalent finite subsystem, i.e. a subsystem having exactly the same solutions as the original one. This reformulation clearly holds for the monoids 2^{Δ^*} and $\mathbf{N}\langle \Delta \rangle$ as well.

We conclude this section with Table 1, which summarizes the results we are going to talk about in the next section.

In Table 1 we consider all three variants of all of our four problems. Letters d, m and n refer to the corresponding variants of the problems, respectively. For the decision problems P_1, P_2 and P_3 the possibilities are decidable (De), open or undecidable (Un), while for the existential problem P_4 they are true (Tr), open or false (Fa). Underlined cases were obtained rather recently, and form the heart of this note.

3. Results

In this section we present the results of Table 1 in more detail. Concerning the proofs, only some basic ideas, and detailed references, are repeated here. We consider each of our four problems separately.

Problem $\mathbf{P_1}$. A basic result here is the following, cf. e.g. [19].

Theorem 3.1. *$EP(\mathscr{H}, Reg)$ is decidable. More strongly, for any regular language $L \subseteq \Sigma^*$ and two morphisms $h, g : \Sigma^* \to \Delta^*$ we have*

$$h \overset{L}{\equiv} g \quad \textit{iff} \quad h \overset{F}{\equiv} g \quad \textit{with} \quad F = L \cap \Sigma^{\leqslant 2\|\mathscr{A}\|},$$

where $\|\mathscr{A}\|$ denotes the size of the state of an automaton accepting L.

Proof (*Outline*). In order to establish Theorem 3.1, the following two facts are needed. First, ordinary pumping property of regular languages, and second the following implication: For all words $x, y, u, v, \bar{x}, \bar{y}, \bar{u}$ and $\bar{v}$ we have

$$\left.\begin{aligned} xy &= \bar{x}\bar{y} \\ xuy &= \bar{x}\bar{u}\bar{y} \\ xvy &= \bar{x}\bar{v}\bar{y} \end{aligned}\right\} \Rightarrow xuvy = \bar{x}\bar{u}\bar{v}\bar{y}. \tag{2}$$

For words (2) is very easy to prove. □

In order to prove (2) and, hence, Theorem 3.1, actually the freeness of the monoid is not needed but, instead, it is enough if the monoid $\mathscr{M}$ satisfies the following two conditions:

(i) $\mathscr{M}$ is cancellative;

(ii) Whenever an equation $rs = tu$ holds in $\mathscr{M}$, there exists an element z in $\mathscr{M}$ such that $r = tz$ or $t = rz$.

Neither of these properties does hold in the monoid 2^{Δ^*} of a finite language. Consequently, we are not able to conclude a nondeterministic variant of Theorem 3.1. In fact, we have the following question.

Open Problem 1. Is $EP(\mathscr{F}\mathscr{S}, Reg)$ decidable?

This is a nice example of a very simply formulated problem which seems to be hard to solve. Note that each finite substitution is a composition of the form $h \circ c^{-1}$, where h is a nonerasing morphism and c is a strictly length-preserving morphism. As related problems, we recall the following results from [20]: The problem $EP(\mathscr{H} \circ \mathscr{H}^{-1}, Reg)$ is undecidable, while the problem $EP(\mathscr{H}^{-1} \circ \mathscr{H}, Reg)$ is decidable . (Here $\mathscr{H} \circ \mathscr{H}^{-1}$, for example, denotes the family of mappings of the form $h \circ g^{-1}$, where h and g are morphisms of free monoids.) Despite the first part of the above result, and our discussion after Theorem 3.8, our guess is that Open Problem 1 is decidable (cf. also [10]).

Finally, let us go to the third variant of Problem P_1. Now, let us try to consider the implication (2) in the monoid $\mathbf{N}\langle\Delta\rangle$ of polynomials. Clearly, this monoid is cancellative, so that (i) holds. Condition (ii) is more troublesome. It does not hold as such in this monoid, but it holds in the monoid $\mathbf{Q}\langle\langle\Delta\rangle\rangle$ of formal power series over Δ with rational coefficients, and since $\mathbf{N}\langle\Delta\rangle$ is naturally embeddable into that we can reformulate Theorem 3.1 for finite substitutions with multiplicities; for more details, cf. [19].

Theorem 3.2. *$EP(\mathscr{F}\mathscr{S}_{\mathscr{M}}, Reg)$ is decidable. More strongly, for any regular language $L \subseteq \Sigma^*$ and two morphisms $h, g: \Sigma^* \to \mathbf{N}\langle\Delta\rangle$ we have*

$$h \overset{L}{\equiv} g \text{ iff } h \overset{F}{\equiv} g \text{ with } F = L \cap \Sigma^{\leqslant 2\|\mathscr{A}\|},$$

where $\|\mathscr{A}\|$ denotes the size of the state set of an automaton accepting L.

Problem P_2. Now a natural starting point is the undecidability of the nondeterministic version of the problem, since this is indeed a standard example of undecidable problems in automata theory, cf. e.g. [3], where the problem is formulated for finite transducers.

Theorem 3.3. *The equivalence problem for two-tape finite automata is undecidable.*

The deterministic variant of the problem was considered one of the classical difficult open problems in automata theory. Indeed, it is almost as old as the whole theory, and although the two-tape case was settled affirmatively in [6] quite early, the general n-tape case seemed to resist all attacks to solve it, cf. e.g. [24, 21, 11].

Finally, an affirmative solution for the general n-tape case came from a surprising direction. Indeed, it turned out that instead of deterministic multitape automata it is more profitable to consider nondeterministic multitape automata with multiplicities. This, of course, leads to a much more difficult problem, but at the same time opens new mathematical tools to attack the problem and, thus, provides a beautiful example of the usefulness of the multiplicity considerations. All in all, this approach yields the following theorem (cf. [15]).

Theorem 3.4. *The multiplicity equivalence of two finite nondeterministic n-tape automata is decidable. More strongly, two such automata $\mathscr{A}_1$ and $\mathscr{A}_2$ are multiplicitly equivalent iff they are so on computations of the length at most $\|\mathscr{A}_1\| + \|\mathscr{A}_2\|$, where $\|\mathscr{A}_i\|$ denotes the size of the state set of $\mathscr{A}_i$.*

As a corollary of Theorem 3.4 we obtain a solution to a classical problem in automata theory.

Theorem 3.5. *The equivalence problem for finite deterministic multitape automata is decidable. More strongly two such automata $\mathscr{A}_1$ and $\mathscr{A}_2$ are equivalent iff they are so on computations of the length at most $\|\mathscr{A}_1\| + \|\mathscr{A}_2\|$, where $\|\mathscr{A}_i\|$ denotes the size of the state set of $\mathscr{A}_i$.*

Proof of Theorem 3.4 (*Outline*). First we have to emphasize that our n-tape automata are (as is most natural) *normalized* in the sense that in each transition they read exactly in one tape something and this something is a symbol. Consequently, we do not allow (as is normally allowed for transducers) transitions which would not read anything but empty words. This is to guarantee that multiplicities of accepted n-tuples are finite. Otherwise, the whole problem becomes undecidable, as is easy to see.

Now, the proof is based on the following three steps; for details, cf. [15].

First, the multiplicity equivalence of two n-tape automata with multiplicities in $\mathbf{N}$ is reduced to the multiplicity equivalence of two one-tape automata with multiplicities in the semiring $\mathbf{N}\langle \Sigma_1^* \times \cdots \times \Sigma_n^* \rangle$.

Second, using known results from algebra, mainly those of B.H. Neumann, we embed the above semiring into a division ring, cf. [15].

Third, we use the theory of vector spaces over division rings, in particular, dimension properties of such spaces, in the spirit of Eilenberg, when he proved the so-called equality theorem for finite nondeterministic automata, cf. [14]. □

Our method of proving Theorem 3.4 was recently extended in [29], or essentially already in [13], to cover rational formal power series over partially commutative free monoids. Moreover, as shown in [15], Theorem 3.4 can be used to introduce a class of linear CF grammars such that the equivalence problem remains undecidable but the multiplicity equivalence becomes decidable.

Finally, we note that actually our problem P_1 is a special case of P_2, and that the same holds for all the corresponding variants. Thus, the decidability results of P_2 would immediately imply those of P_1. However, we preferred to deal with problem P_1 separately, since it nicely illustrates and supports our view of the correspondence between the determinism and the nondeterminism with multiplicities.

Problem $\mathbf{P_3}$. Here the starting point is a result usually referred to as Sardinas–Patterson algorithm, cf. [28, 4].

Theorem 3.6. *It is decidable whether a given morphism* $h:\Sigma^* \to \Delta^*$ *is injective.*

The other two variants of the problem are open.

Open Problem 2. Is it decidable whether a given morphism $h:\Sigma^* \to \mathbf{N}\langle \Delta \rangle$ is injective?

Open Problem 3. Is it decidable whether a given morphism $h:\Sigma^* \to 2^{\Delta^*}$ is injective?

Although we had almost nothing to say about our problem P_3 we wanted to include it, since certainly the open problems are interesting and fits well to our theme. Moreover, in the light of two previous problems, Open Problem 2 might not be that hopeless after all. On the other hand, if Open Problem 3 is decidable, which is easier to believe, it is likely to be difficult.

Problem $\mathbf{P_4}$. Here a solution of the deterministic variant resembles that of problem P_2. Indeed, the Ehrenfeucht Problem was for a while very interesting and hardlooking open problem, until it was noticed by Albert and Lawrence [1], and Guba, cf. [25], that it actually can be solved with the help of known results in algebra; in particular, by using Hilbert's Basis Theorem.

Theorem 3.7. *The Ehrenfeucht Problem for morphisms has an affirmative answer, i.e. each language* $L \subseteq \Sigma^*$ *possesses a finite test set with respect to morphisms* $h:\Sigma^* \to \Delta^*$.

For finite substitutions the situation is completely different.

Theorem 3.8. *The Ehrenfeucht Problem for finite substitutions does not have an affirmative answer, i.e. there exists a language* $L \subseteq \Sigma^*$ *such that it does not possess a finite test set with respect to morphisms* $h: \Sigma^* \to 2^{\Delta^*}$.

Theorem 3.8 was shown in [23] with the following nice counterexample. The regular language ab^*c does not have a finite test set with respect to finite substitutions. In other words, the system of equations

$$xy^iz = uv^iw, \quad i \geqslant 0 \tag{3}$$

does not have any finite equivalent subsystem in the monoid 2^{Δ^*} of finite languages. In the monoid Σ^* of words (3) is, by implication (2), equivalent to the pair

$$\begin{cases} xz \;\; = uw, \\ xyz = uvw. \end{cases}$$

It follows that no natural modification of the proof of Theorem 3.1 works for finite substitutions, i.e. solve our Problem $P_1(n)$. This is one indication why the problem looks difficult.

Now, let us go to the third variant of the Ehrenfeucht Problem. Here again the situation seems to be much closer to the deterministic variant than the nondeterministic one. Indeed, the proof of Theorem 3.8 does not work at all for finite substitutions with multiplicities. On the contrary, by Theorem 3.2 we conclude that each regular language possesses a finite test with respect to morphisms $h: \Sigma^* \to \mathbf{N}\langle \Delta \rangle$ and, moreover, this test set is exactly the same as we have for morphisms $h: \Sigma^* \to \Delta^*$ given by Theorem 3.1. In general, this test set is of exponential size (in terms of underlying automata) but it was shown recently in [16] that it can be made of linear size.

Consequently, for *regular languages* all three variants of the Ehrenfeucht Problem are resolved. However, in general, we have the following problem.

Open Problem 4. Does the Ehrenfeucht Problem for morphisms $h: \Sigma^* \to \mathbf{N}\langle \Delta \rangle$ have an affirmative answer in general?

On the one hand, it is feasible to believe that this problem has a positive answer, since in the monoid $\mathbf{N}\langle \Delta \rangle$ there exist only relatively few identities, cf. [7, 8] and, hence, it is not "very far from being free". On the other hand, the identities in $\mathbf{N}\langle \Delta \rangle$ are difficult to handle, as shown by the simple equation $xy = yx$, which can be completely (and nicely) solved in $\mathbf{N}\langle \Delta \rangle$, but the proof of its correctness is not easy, cf. [2, 8].

4. Concluding remarks

We have discussed on four natural problems in automata theory and modified each of them into three different variants. The simplest one is always a deterministic variant and the most complex one is a nondeterministic one. The third one is an in-between case and tries to capture a deterministic behaviour of the nondeterministic one. This is achieved by putting together all the choices (including repetitions) allowed by a nondeterministic procedure. We refer to this variant of a problem as its multiplicity variant.

We have shown that in several cases the multiplicity variant has similar properties as the deterministic one. Here we want to mention still one more example of this kind of similarity. The equality theorem of Eilenberg, cf. [14], shows that to decide the equivalence of deterministic finite automata on one hand, and the multiplicity equivalence of nondeterministic finite automata on the other, one has to consider in both cases words of exactly the same lengths.

Even more importantly, we have given an example, when the study of a multiplicity version as a generalization of a deterministic one has led, via new tools it provides, to a solution of a classical deterministic problem in automata theory.

On the other hand, our several open problems suggest that even if one accepts our general view that determinism and nondeterminism with multiplicities correspond to each other, it is not always easy to show the correspondence in concrete results.

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