

The Complexity of Subgame Perfect Equilibria in Quantitative Reachability Games

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Abstract

We study multiplayer quantitative reachability games played on a finite directed graph, where the objective of each player is to reach his target set of vertices as quickly as possible. Instead of the well-known notion of Nash equilibrium (NE), we focus on the notion of subgame perfect equilibrium (SPE), a refinement of NE well-suited in the framework of games played on graphs. It is known that there always exists an SPE in quantitative reachability games and that the constrained existence problem is decidable. We here prove that this problem is PSPACE-complete. To obtain this result, we propose a new algorithm that iteratively builds a set of constraints characterizing the set of SPE outcomes in quantitative reachability games. This set of constraints is obtained by iterating an operator that reinforces the constraints up to obtaining a fixpoint. With this fixpoint, the set of SPE outcomes can be represented by a finite graph of size at most exponential. A careful inspection of the computation allows us to establish PSPACE membership.

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1 Introduction

While two-player zero-sum games played on graphs are the most studied model to formalize and solve the reactive synthesis problem [25], recent work has considered non-zero-sum extensions of this mathematical framework, see e.g. [15, 19, 7, 21, 6, 5, 17, 4, 1], see also the surveys [20, 3, 13]. In the zero-sum game approach, the system and the environment are considered as *monolithic* and fully *adversarial* entities. Unfortunately, both assumptions may turn to be too strong. First, the reactive system may be composed of several components that execute concurrently and have their own purpose. So, it is natural to model such systems with *multiplayer* games with each player having his own objective. Second, the environment usually has its own objective too, and this objective is usually not the negation of the objective of the reactive system as postulated in the zero-sum case. Therefore, there are instances of the reactive synthesis problem for which no solution exists in the zero-sum setting, i.e. no winning strategy for the system against a completely antagonistic environment, while there exists a strategy for the system which enforces the desired properties against all *rational* behaviors of the environment pursuing its own objective.

While the central solution concept in zero-sum games is the notion of *winning strategy*, it is well known that this solution concept is not sufficient to reason about non-zero-sum games. In non-zero-sum games, notions of equilibria are used to reason about the rational behavior of players. The celebrated notion of *Nash equilibrium* (NE) [23] is one of the most studied. A profile of strategies is an NE if no player has an incentive to deviate, i.e. change his strategy and obtain a better reward, when this player knows that the other players will be playing their respective strategies in the profile. A well-known weakness of NE in sequential games, which include infinite duration games played on graphs, is that they are subject to *non-credible threats*: decisions in subgames that are irrational and used to threaten the other players and oblige them to follow a given behavior. To avoid this problem, the concept of *subgame perfect equilibria* (SPE) has been proposed, see e.g. [24]. SPEs are NEs with the additional property that they are also NEs in all subgames of the original game. While it is now quite well understood how to handle NEs algorithmically in games played on graphs [27, 28, 12, 17], this is not the case for SPEs.

Contributions In this paper, we provide an algorithm to decide in *polynomial space* the constrained existence problem for SPEs in *quantitative reachability games*. A quantitative reachability game is played by n players on a finite graph in which each player has his own reachability objective. The objective of each player is to reach his target set of vertices as quickly as possible. In a series of papers, it has been shown that SPEs always exist in quantitative reachability games [8], and that the set of outcomes of SPEs in a quantitative reachability game is a regular language which is effectively constructible [11]. As a consequence of the latter result, the constrained existence problem for SPEs is decidable.

Unfortunately, the proof that establishes the regularity of the set of possible outcomes of SPEs in [11] exploits a *well-quasi order* for proving termination and it cannot be used to obtain a good upper bound on the complexity for the algorithm. Here, we propose a new algorithm and we show that this set of outcomes can be represented using an automaton of size at most exponential. It follows that the constrained existence problem for SPEs can be decided in PSPACE. We also provide a matching lower-bound showing that this problem is PSPACE-complete.

Our new algorithm iteratively builds a set of constraints that exactly characterize the set of SPEs in quantitative reachability games. This set of constraints is obtained by iterating an operator that reinforces the constraints up to obtaining a fixpoint. A careful inspection of the computation allows us to establish PSPACE membership.

Related work Algorithms to reason on NEs in graph games are studied in [27] for ω -regular objectives and in [28, 12] for quantitative objectives. Algorithms to reason on SPEs are given in [26] for ω -regular objectives. Quantitative reachability objectives are not ω -regular objectives. Reasoning about NEs and SPEs for ω -regular specifications can also be done using strategy logics [16, 22].

Other notions of rationality and their use for reactive synthesis have been studied in the literature: rational synthesis in cooperative [19] and adversarial [21] setting, and their algorithmic complexity has been studied in [17]. Extensions with imperfect information have been investigated in [18]. Synthesis rules based on the notion of admissible strategies have been studied in [2, 7, 6, 5, 4, 1]. Weak SPEs have been studied in [11, 14, 9] and shown to be equivalent to SPEs for quantitative reachability objectives.

Structure of the paper In Section 2, we recall the notions of n -player graph games and (very weak/weak) SPEs, we introduce the notion of extended games and we state the studied constrained existence problem. In Section 3, we provide a way to characterize the set of plays that are outcomes of SPEs and give an algorithm to construct this set. This algorithm relies on the computation of a sequence of labeling functions $(\lambda_k)_{k \in \mathbb{N}}$ until reaching a fixpoint λ^* such that the plays which are λ^* -consistent are exactly the plays which are outcomes of SPEs. In Section 4, given a labeling function λ , we introduce the notion of counter graph in which infinite paths correspond to λ -consistent plays. We also show that each such graph has an exponential size. In Section 5, using counter graphs, we prove the PSPACE-easiness of the constrained existence problem. We conclude by proving the PSPACE-hardness of this problem.

2 Preliminaries

In this section, we recall the notions of quantitative reachability game and subgame perfect equilibrium. We also state the problem studied in this paper and our main result.

2.1 Quantitative reachability games

An *arena* is a tuple $G = (\Pi, V, (V_i)_{i \in \Pi}, E)$ where $\Pi = \{1, 2, \dots, n\}$ is a finite set of n players, V is a finite set of vertices, $(V_i)_{i \in \Pi}$ is a partition of V between the players, and $E \subseteq V \times V$ is a set of edges such that for all $v \in V$ there exists $v' \in V$ such that $(v, v') \in E$. Without loss of generality, we suppose that $|\Pi| \leq |V|$.

A *play* in G is an infinite sequence of vertices $\rho = \rho_0 \rho_1 \dots$ such that for all $k \in \mathbb{N}$, $(\rho_k, \rho_{k+1}) \in E$. A *history* is a finite sequence $h = h_0 h_1 \dots h_k$ with $k \in \mathbb{N}$ defined similarly. The *length* $|h|$ of h is the number k of its edges. We denote the set of plays by Plays and the set of histories by Hist (when it is necessary, we use notation Plays_G and Hist_G to recall the underlying arena G). Moreover, the set Hist_i is the set of histories such that their last vertex v is a vertex of player i , i.e. $v \in V_i$.

Given a play $\rho = \rho_0 \rho_1 \dots \in \text{Plays}$ and $k \in \mathbb{N}$, the prefix $\rho_0 \rho_1 \dots \rho_k$ of ρ is denoted by $\rho_{\leq k}$ and its suffix $\rho_k \rho_{k+1} \dots$ is denoted by $\rho_{\geq k}$. A play ρ is called a *lasso* if it is of the form $\rho = h\ell^\omega$ with $h\ell \in \text{Hist}$. Notice that ℓ is not necessarily a simple cycle. The *length* of a lasso $h\ell^\omega$ is the length of $h\ell$.

Given an arena G , we denote by $\text{Succ}(v) = \{v' \mid (v, v') \in E\}$ the set of *successors* of v , for $v \in V$, and by Succ^* the transitive closure of Succ .

A *quantitative game* $\mathcal{G} = (G, (\text{Cost}_i)_{i \in \Pi})$ is an arena equipped with a cost function profile $\text{Cost} = (\text{Cost}_i)_{i \in \Pi}$ such that each function $\text{Cost}_i : \text{Plays} \rightarrow \mathbb{R} \cup \{+\infty\}$ assigns a cost to each play. In a quantitative game \mathcal{G} , an initial vertex $v_0 \in V$ is often fixed, and we call (\mathcal{G}, v_0) an *initialized game*. A play (resp. a history) of (\mathcal{G}, v_0) is then a play (resp. a history) of \mathcal{G} starting in v_0 . The set of such plays (resp. histories) is denoted by $\text{Plays}(v_0)$ (resp. $\text{Hist}(v_0)$). We also use notation $\text{Hist}_i(v_0)$ when these histories end in a vertex $v \in V_i$.

In this article we are interested in *quantitative reachability games* such that each player has a target set of vertices that he wants to reach. The cost to pay is equal to the number of edges to reach the target set, and each player aims at minimizing his cost.

► **Definition 1** (Quantitative reachability game). *A quantitative reachability game is a tuple $\mathcal{G} = (G, (F_i)_{i \in \Pi}, (\text{Cost}_i)_{i \in \Pi})$ such that*

- *G is an arena;*
- *for each $i \in \Pi$, $F_i \subseteq V$ is the target set of player i ;*

- for each $i \in \Pi$ and each $\rho = \rho_0\rho_1 \dots \in \text{Plays}$, $\text{Cost}_i(\rho)$ is equal to the least index k such that $\rho_k \in F_i$, and to $+\infty$ if no such index exists.

In the sequel of this document, we simply call such a game a *reachability game*. Notice that the cost function used for reachability games can be supposed to be continuous in the following sense [8]. With V endowed with the discrete topology and V^ω with the product topology, a sequence of plays $(\rho_n)_{n \in \mathbb{N}}$ converges to ρ if every prefix of ρ is a prefix of all ρ_n except, possibly, finitely many of them. A cost function Cost_i is *continuous* if whenever $\lim_{n \rightarrow +\infty} \rho_n = \rho$, we have that $\lim_{n \rightarrow +\infty} \text{Cost}_i(\rho_n) = \text{Cost}_i(\rho)$. In reachability games, the function Cost_i can be transformed into a continuous one as follows: $\text{Cost}'_i(\rho) = 1 - \frac{1}{\text{Cost}_i(\rho)+1}$ if $\text{Cost}_i(\rho) < +\infty$, and $\text{Cost}'_i(\rho) = 1$ otherwise.

Given a quantitative game \mathcal{G} , a *strategy* of a player $i \in \Pi$ is a function $\sigma_i : \text{Hist}_i \rightarrow V$. This function assigns to each history hv , with $v \in V_i$, a vertex v' such that $(v, v') \in E$. In an initialized game (\mathcal{G}, v_0) , σ_i needs only to be defined for histories starting in v_0 . A play $\rho = \rho_0\rho_1 \dots$ is *consistent* with σ_i if for all $\rho_k \in V_i$ we have that $\sigma_i(\rho_0 \dots \rho_k) = \rho_{k+1}$. A strategy σ_i is *positional* if it only depends on the last vertex of the history, i.e., $\sigma_i(hv) = \sigma_i(v)$ for all $hv \in \text{Hist}_i$. It is *finite-memory* if it can be encoded by a finite-state machine $\mathcal{M} = (M, m_0, \alpha_u, \alpha_n)$ where M is a finite set of states (the memory of the strategy), $m_0 \in M$ is the initial memory state, $\alpha_u : M \times V \rightarrow M$ is the update function, and $\alpha_n : M \times V_i \rightarrow V$ is the next-action function. The machine \mathcal{M} defines a strategy σ_i such that $\sigma_i(hv) = \alpha_n(\hat{\alpha}_u(m_0, h), v)$ for all histories $hv \in \text{Hist}_i$, where $\hat{\alpha}_u$ extends α_u to histories as expected. The *size* of the strategy σ_i is the size $|M|$ of its machine \mathcal{M} . Note that σ_i is positional when $|M| = 1$.

A *strategy profile* is a tuple $\sigma = (\sigma_i)_{i \in \Pi}$ of strategies, one for each player. It is called positional (resp. finite-memory) if for all $i \in \Pi$, σ_i is positional (resp. finite-memory). Given an initialized game (\mathcal{G}, v_0) and a strategy profile σ , there exists a unique play from v_0 consistent with each strategy σ_i . We call this play the *outcome* of σ and it is denoted by $\langle \sigma \rangle_{v_0}$. Let $c = (c_i)_{i \in \Pi} \in (\mathbb{N} \cup \{+\infty\})^{|\Pi|}$, we say that σ is a strategy profile *with cost* c or that $\langle \sigma \rangle_{v_0}$ *has cost* c if $c_i = \text{Cost}_i(\langle \sigma \rangle_{v_0})$ for all $i \in \Pi$.

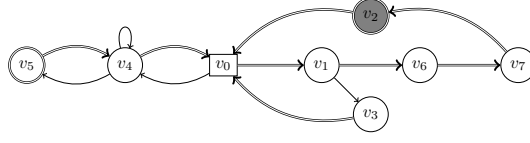
2.2 Solution concepts and constraint problem

In the multiplayer game setting, the solution concepts usually studied are *equilibria* (see [20]). We here recall the concepts of Nash equilibrium and subgame perfect equilibrium.

Let $\sigma = (\sigma_i)_{i \in \Pi}$ be a strategy profile in an initialized game (\mathcal{G}, v_0) . When we highlight the role of player i , we denote σ by (σ_i, σ_{-i}) where σ_{-i} is the profile $(\sigma_j)_{j \in \Pi \setminus \{i\}}$. A strategy $\sigma'_i \neq \sigma_i$ is a *deviating* strategy of player i , and it is a *profitable deviation* for him if $\text{Cost}_i(\langle \sigma \rangle_{v_0}) > \text{Cost}_i(\langle \sigma'_i, \sigma_{-i} \rangle_{v_0})$.

The notion of Nash equilibrium is classical: a strategy profile σ in an initialized game (\mathcal{G}, v_0) is a *Nash equilibrium* (NE) if no player has an incentive to deviate unilaterally from his strategy, i.e. no player has a profitable deviation. Formally, σ is an NE if for each $i \in \Pi$ and each deviating strategy σ'_i of player i , we have $\text{Cost}_i(\langle \sigma \rangle_{v_0}) \leq \text{Cost}_i(\langle \sigma'_i, \sigma_{-i} \rangle_{v_0})$.

When considering games played on graphs, a useful refinement of NE is the concept of *subgame perfect equilibrium* (SPE) which is a strategy profile being an NE in each subgame. It is well-known that contrarily to NEs, SPEs avoid non-credible threats [20]. Formally, given a quantitative game $\mathcal{G} = (G, \text{Cost})$, an initial vertex v_0 , and a history $hv \in \text{Hist}(v_0)$, the initialized game $(\mathcal{G}_{|h}, v)$ is called a *subgame* of (\mathcal{G}, v_0) such that $\mathcal{G}_{|h} = (G, \text{Cost}_{|h})$ and $\text{Cost}_{i|h}(\rho) = \text{Cost}_i(h\rho)$ for all $i \in \Pi$ and $\rho \in V^\omega$. Notice that (\mathcal{G}, v_0) is subgame of itself. Moreover if σ_i is a strategy for player i in (\mathcal{G}, v_0) , then $\sigma_{i|h}$ denotes the strategy in $(\mathcal{G}_{|h}, v)$



■ **Figure 1** A quantitative reachability game: player 1 (resp. player 2) owns circle (resp. square) vertices and $F_1 = \{v_2\}$ and $F_2 = \{v_2, v_5\}$.

such that for all histories $h' \in \text{Hist}_i(v)$, $\sigma_i(h)(h') = \sigma_i(hh')$. Similarly, from a strategy profile σ in (\mathcal{G}, v_0) , we derive the strategy profile $\sigma_{\upharpoonright h}$ in $(\mathcal{G}_{\upharpoonright h}, v)$.

► **Definition 2** (Subgame perfect equilibrium). *Let (\mathcal{G}, v_0) be an initialized game. A strategy profile σ is a subgame perfect equilibrium in (\mathcal{G}, v_0) if for all $hv \in \text{Hist}(v_0)$, $\sigma_{\upharpoonright h}$ is an NE in $(\mathcal{G}_{\upharpoonright h}, v)$.*

It is proved in [8] that there always exists an SPE in reachability games. In this paper, we are interested in solving the following *constraint problem*.

► **Definition 3** (Constraint problem). *Given (\mathcal{G}, v_0) an initialized reachability game and two threshold vectors $x, y \in (\mathbb{N} \cup \{+\infty\})^{|\Pi|}$, the constraint problem is to decide whether there exists an SPE in (\mathcal{G}, v_0) with cost c such that $x \leq c \leq y$, that is, $x_i \leq c_i \leq y_i$ for all $i \in \Pi$.*

Our main result is the following one:

► **Theorem 4.** *The constraint problem for initialized reachability games is PSPACE-complete.*

The sequel of the paper is devoted to the proof of this result. Let us first illustrate the introduced concepts with an example.

► **Example 5.** A reachability game $\mathcal{G} = (G, (F_i)_{i \in \Pi}, (\text{Cost}_i)_{i \in \Pi})$ with two players is depicted in Figure 1. The circle vertices are owned by player 1 whereas the square vertices are owned by player 2. The target sets of both players are respectively equal to $F_1 = \{v_2\}$ (grey vertices), $F_2 = \{v_2, v_5\}$ (double circled vertices).

The positional strategy profile $\sigma = (\sigma_1, \sigma_2)$ is depicted by double arrows, its outcome in (\mathcal{G}, v_0) is equal to $\langle \sigma \rangle_{v_0} = (v_0 v_1 v_6 v_7 v_2)^\omega$ with cost $(4, 4)$. Let us explain that σ is an NE. Player 1 reaches his target set as soon as possible and has thus no incentive to deviate. Player 2 has no profitable deviation that allows him to reach v_5 . For instance if he uses a deviating positional strategy σ'_2 such that $\sigma'_2(v_0) = v_4$, then the outcome of (σ_1, σ'_2) is equal to $(v_0 v_4)^\omega$ with cost $(+\infty, +\infty)$ which is not profitable for player 2.

One can verify that the strategy profile σ is also an SPE. For instance in the subgame $(\mathcal{G}_{\upharpoonright h}, v_5)$ with $h = v_0 v_4$, we have $\rho = \langle \sigma_{\upharpoonright h} \rangle_{v_5} = v_5 v_4 (v_0 v_1 v_6 v_7 v_2)^\omega$ such that $\text{Cost}_{\upharpoonright h}(\rho) = \text{Cost}(h\rho) = (8, 2)$. In this subgame, with ρ , both players reach their target set as soon as possible and have thus no incentive to deviate.

Consider now the positional strategy profile $\sigma' = (\sigma'_1, \sigma'_2)$ such that $\sigma'_1(v_4) = v_0$, $\sigma'_1(v_1) = v_3$, and $\sigma'_2(v_4) = v_0$. Its outcome in (\mathcal{G}, v_0) is equal to $(v_0 v_4)^\omega$ with cost $(+\infty, +\infty)$. Let us explain that σ' is an NE. On one hand, since player 2 never goes to v_1 , player 1 has no incentive to deviate, as his target is only accessible from v_1 . On the other hand, if player 2 deviates and chooses to go to v_1 , his cost is still $+\infty$ since player 1 goes to v_3 . However, the strategy profile σ' is *not* an SPE. Indeed, it features an *non-credible threat* by player 1: consider the history $v_0 v_1$ and the corresponding subgame $(\mathcal{G}_{\upharpoonright v_0, v_1})$. In that case, player 1 has an incentive to deviate to reach v_2 to yield a cost of 4. Thus, the strategy profile $\sigma'_{\upharpoonright v_0}$ is not a NE in the subgame $(\mathcal{G}_{\upharpoonright v_0, v_1})$, and thus is not an SPE in (\mathcal{G}, v_0) . ◀

2.3 Weak SPE, very weak SPE and extended game

In this section, we present two important tools that will be repeatedly used in the sequel. First, we explain that in reachability games, the notion of SPE can be replaced by the simpler notion of very weak SPE. Second we present an extended version of a reachability game where the vertices are enriched with the set of players that have already visited their target sets along a history. Working with this extended game is essential to prove that the constraint problem for reachability games is in PSPACE.

We begin by recalling the concepts of weak and very weak SPE introduced in [11, 14]. Let (\mathcal{G}, v_0) be an initialized game and $\sigma = (\sigma_i)_{i \in \Pi}$ be a strategy profile. Given $i \in \Pi$, we say that a strategy σ'_i is *finitely deviating* from σ_i if σ'_i and σ_i only differ on a finite number of histories, and that σ'_i is *one-shot deviating* from σ_i if σ'_i and σ_i only differ on v_0 . A strategy profile σ is a *weak NE* (resp. *very weak NE*) in (\mathcal{G}, v_0) if, for each player $i \in \Pi$, for each finitely deviating (resp. one-shot) strategy σ'_i of player i from σ_i , we have $\text{Cost}_i(\langle \sigma \rangle_{v_0}) \leq \text{Cost}_i(\langle \sigma'_i, \sigma_{-i} \rangle_{v_0})$. A strategy profile σ is a *weak SPE* (resp. *very weak SPE*) in (\mathcal{G}, v_0) if, for all $h v \in \text{Hist}(v_0)$, $\sigma|_h$ is a weak (resp. very weak) NE in $(\mathcal{G}|_h, v)$.

From the given definitions, every SPE is a weak SPE, and every weak SPE is a very weak SPE. It is known that weak SPE and very weak SPE are equivalent notions and that there exist initialized games that have a weak SPE but no SPE; nevertheless, all three concepts are equivalent for initialized reachability games [11, 14].

► **Proposition 6** ([11, 14]). *Let (\mathcal{G}, v_0) be a initialized reachability game and σ be a strategy profile in (\mathcal{G}, v_0) . Then σ is an SPE if and only if σ is a weak SPE if and only if σ is a very weak SPE.*

Let us now recall the notion of extended game for a given reachability game \mathcal{G} (see e.g. [9]). The vertices (v, I) of the extended game store a vertex $v \in V$ as well as a subset $I \subseteq \Pi$ of players that have already visited their target sets.

► **Definition 7** (Extended game). *Let $\mathcal{G} = (G, (F_i)_{i \in \Pi}, (\text{Cost}_i)_{i \in \Pi})$ be a reachability game with an arena $G = (\Pi, V, (V_i)_{i \in \Pi}, E)$, and let v_0 be an initial vertex. The extended game of \mathcal{G} is equal to $\mathcal{X} = (X, (F_i^X)_{i \in \Pi}, (\text{Cost}_i^X)_{i \in \Pi})$ with the arena $X = (\Pi, V^X, (V_i^X)_{i \in \Pi}, E^X)$, such that:*

- $V^X = V \times 2^\Pi$
- $((v, I), (v', I')) \in E^X$ if and only if $(v, v') \in E$ and $I' = I \cup \{i \in \Pi \mid v' \in F_i\}$
- $(v, I) \in V_i^X$ if and only if $v \in V_i$
- $(v, I) \in F_i^X$ if and only if $i \in I$
- for each $\rho \in \text{Plays}_X$, $\text{Cost}_i^X(\rho)$ is equal to the least index k such that $\rho_k \in F_i^X$, and to $+\infty$ if no such index exists.

The initialized extended game (\mathcal{X}, x_0) associated with the initialized game (\mathcal{G}, v_0) is such that $x_0 = (v_0, I_0)$ with $I_0 = \{i \in \Pi \mid v_0 \in F_i\}$.

Notice the way each target set F_i^X is defined: if $v \in F_i$, then $(v, I) \in F_i^X$ but also $(v', I') \in F_i^X$ for all $(v', I') \in \text{Succ}^*(v, I)$. In the sequel, to avoid heavy notations, each cost function Cost_i^X will be simply written as Cost_i .

The extended game of the reachability game of Figure 1 is depicted in Figure 2. We will come back to this example at the end of this section.

Let us state some properties of the extended game. First, notice that for each $\rho = (v_0, I_0)(v_1, I_1) \dots \in \text{Plays}_X(x_0)$, we have the next property called *I-monotonicity*:

$$I_k \subseteq I_{k+1} \quad \text{for all } k \in \mathbb{N}. \quad (1)$$

Second, given an initialized game (\mathcal{G}, v_0) and its extended game (\mathcal{X}, x_0) , there is a one-to-one correspondence between plays in $\text{Plays}_G(v_0)$ and plays in $\text{Plays}_X(x_0)$:

- from $\rho = \rho_0 \rho_1 \dots \in \text{Plays}_G(v_0)$, we derive $\rho^X = (\rho_0, I_0)(\rho_1, I_1) \dots \in \text{Plays}_X(x_0)$ such that I_k is the set of players i that have seen their target set F_i along $\rho_{\leq k}$;
- from $\rho = (v_0, I_0)(v_1, I_1) \dots \in \text{Plays}_X(x_0)$, we derive $\rho^G = v_0 v_1 \dots \in \text{Plays}_G(v_0)$ such that the second components I_k , $k \in \mathbb{N}$, are omitted.

Third, given $\rho \in \text{Plays}_G(v_0)$, we have that $\text{Cost}(\rho^X) = \text{Cost}(\rho)$, and conversely given $\rho \in \text{Plays}_X(x_0)$, we have that $\text{Cost}(\rho^G) = \text{Cost}(\rho)$. It follows that outcomes of SPE can be equivalently studied in (\mathcal{G}, v_0) and in (\mathcal{X}, x_0) , as stated in the next lemma.

► **Lemma 8.** *If ρ is the outcome of an SPE in (\mathcal{G}, v_0) , then ρ^X is the outcome of an SPE in (\mathcal{X}, x_0) with the same cost. Conversely, if ρ is the outcome of an SPE in (\mathcal{X}, x_0) , then ρ^G is the outcome of an SPE in (\mathcal{G}, v_0) with the same cost.*

By construction, the arena X of the initialized extended game is divided into different regions according to the players who have already visited their target set. Let us provide some useful notions with respect to this decomposition. We will often use them in the following sections. Let $\mathcal{I} = \{I \subseteq \Pi \mid \text{there exists } v \in V \text{ such that } (v, I) \in \text{Succ}^*(x_0)\}$ be the set of sets I accessible from the initial state x_0 , and let $N = |\mathcal{I}|$ be its size. For $I, I' \in \mathcal{I}$, if there exists $((v, I), (v', I')) \in E^X$, we say that I' is a *successor* of I and we write $I' \in \text{Succ}(I)$. Given $I \in \mathcal{I}$, $X^I = (V^I, E^I)$ refers to the sub-arena of X restricted to the vertices $\{(v, I) \in V^X \mid v \in V\}$. We say that X^I is the *region*¹ associated with I . Such a region X^I is called a *bottom region* whenever $\text{Succ}(I) = \emptyset$.

There exists a partial order on \mathcal{I} such that $I < I'$ if and only if $I' \in \text{Succ}^*(I) \setminus \{I\}$. We fix an arbitrary *total order* on \mathcal{I} that extends this partial order $<$ as follows:

$$J_1 < J_2 < \dots < J_N. \quad (2)$$

(with X^{J_N} a bottom region).² With respect to this total order, given $n \in \{1, \dots, N\}$, we denote by $X^{\geq J_n} = (V^{\geq J_n}, E^{\geq J_n})$ the sub-arena of X restricted to the vertices $\{(v, I) \in V^X \mid I \geq J_n\}$.

The total order given in (2) together with the I-monotonicity (see (1)) leads to the following lemma.

► **Lemma 9** (Region decomposition and section). *Let π be a (finite or infinite) path in \mathcal{X} . Then there exists a region decomposition of π as*

$$\pi[\ell]\pi[\ell+1] \dots \pi[m]$$

with $1 \leq \ell \leq m \leq N$, such that for each n , $\ell \leq n \leq m$:

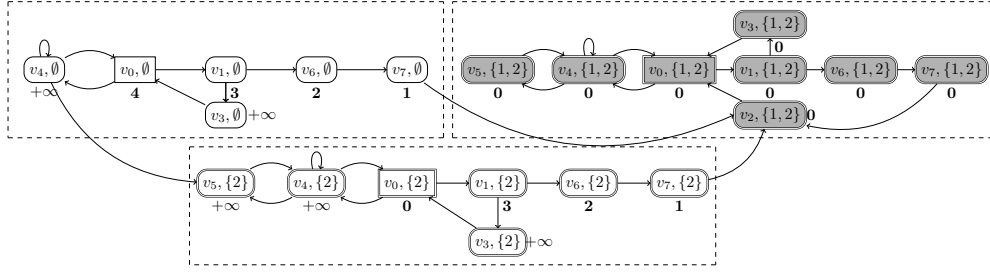
- $\pi[n]$ is a (possibly empty) path in \mathcal{X} ,
- every vertex of $\pi[n]$ are of the form (v, J_n) for some $v \in V$.

Each path $\pi[n]$ is called a *section*. The last section $\pi[m]$ is infinite if and only if π is infinite.

► **Example 10.** Let us come back to the initialized game (\mathcal{G}, v_0) of Figure 1. Its extended game (\mathcal{X}, x_0) is depicted in Figure 2 (only the part reachable from the initial vertex $x_0 = (v_0, \emptyset)$ is depicted; for the moment the reader should not consider the labeling indicated

¹ In the sequel, we indifferently call region either X^I , or V^I , or I .

² We use notation J_n , $n \in \{1, \dots, N\}$, to avoid any confusion with the sets I_k appearing in a play $\rho = (v_0, I_0)(v_1, I_1) \dots$.



■ **Figure 2** The extended game (\mathcal{X}, x_0) for the initialized game (G, v_0) of Figure 1. The values of a labeling function λ are indicated close to each vertex.

under or next to the vertices). As we can see, the extended game is divided into three different regions: one region associated to $I = \emptyset$ that contains the initial vertex x_0 , a second region associated to $I = \{2\}$, and a third bottom region associated to $I = \{1, 2\}$. Hence the set $\mathcal{I} = \{\emptyset, \{2\}, \Pi\}$ is totally ordered as $J_1 = \emptyset < J_2 = \{2\} < J_3 = \Pi$.

For all the vertices (v, I) of the region associated with $I = \{2\}$, we have $(v, I) \notin F_1^X$ and $(v, I) \in F_2^X$, and for those of the region associated with $I = \Pi$, we have $(v, I) \in F_1^X \cap F_2^X$.

From the SPE σ given in Example 5 with outcome $\rho = (v_0 v_1 v_6 v_7 v_2)^\omega \in \text{Plays}_G(v_0)$ and cost $(4, 4)$, we derive the SPE outcome $\rho^X \in \text{Plays}_X(x_0)$ equal to

$$(v_0, \emptyset)(v_1, \emptyset)(v_6, \emptyset)(v_7, \emptyset)((v_2, \Pi)(v_0, \Pi)(v_1, \Pi)(v_6, \Pi)(v_7, \Pi))^\omega$$

with the same cost $(4, 4)$. The region decomposition of ρ^X is equal to $\rho^X[1]\rho^X[2]\rho^X[3]$ such that $\rho^X[1] = (v_0, \emptyset)(v_1, \emptyset)(v_6, \emptyset)(v_7, \emptyset)$, $\rho^X[2]$ is empty, and $\rho^X[3] = ((v_2, \Pi)(v_0, \Pi)(v_1, \Pi)(v_6, \Pi)(v_7, \Pi))^\omega$. ◀

3 Characterization

In this section, given an initialized reachability game (\mathcal{G}, v_0) , we characterize the set of plays that are outcomes of SPEs, and we provide an algorithm to construct this set. For this characterization, by Lemma 8, we can work on the extended game (\mathcal{X}, x_0) instead of (\mathcal{G}, v_0) . Moreover by Proposition 6, we can focus on very weak SPEs only since they are equivalent to SPEs. Such a characterization already appears in [11] however with a different algorithm that cannot be used to obtain good complexity upper bounds for the constraint problem.

All along this section, when we refer to a vertex of V^X , we use notation v (instead of (u, I)) and notation $I(v)$ means the second component I of this vertex.

Our algorithm iteratively builds a set of *constraints* imposed by a *labeling function* $\lambda : V^X \rightarrow \mathbb{N} \cup \{+\infty\}$ such that the plays of the extended game satisfying those constraints are exactly the SPE outcomes. Let us provide a formal definition of such a function λ with the constraints that it imposes on plays.

► **Definition 11** (λ -consistent play). *Let \mathcal{X} be the extended game of a reachability game \mathcal{G} , and $\lambda : V^X \rightarrow \mathbb{N} \cup \{+\infty\}$ be a labeling function. Given $v \in V^X$, for all plays $\rho \in \text{Plays}_X(v)$, we say that $\rho = \rho_0 \rho_1 \dots$ is λ -consistent if for all $n \in \mathbb{N}$ and $i \in \Pi$ such that $\rho_n \in V_i$:*

$$\text{Cost}_i(\rho_{\geq n}) \leq \lambda(\rho_n). \quad (3)$$

We denote by $\Lambda(v)$ the set of plays $\rho \in \text{Plays}_X(v)$ that are λ -consistent.

Thus, a play ρ is λ -consistent if for all its suffixes $\rho_{\geq n}$, if player i owns ρ_n then the number of edges to reach his target set along $\rho_{\geq n}$ is bounded by $\lambda(\rho_n)$. Before going into the details of our algorithm, let us intuitively explain on an example how a well-chosen labeling function characterizes the set of SPE outcomes.

► **Example 12.** We consider the extended game (\mathcal{X}, x_0) of Figure 2, and a labeling function λ whose values are indicated under or next to each vertex.

If $v \in V_i^X$ is labeled by $\lambda(v) = c$, then if $c \in \mathbb{N}_0$, this means that player i will only accept outcomes in (\mathcal{X}, v) that reach his target set within c steps, otherwise he would have a profitable deviation. If $\lambda(v) = 0$, this means that player i has already reached his target set, and if $\lambda(v) = +\infty$, player i has no profitable deviation whatever outcome is proposed to him.

In Example 10 was given the SPE outcome equal to

$$\rho = (v_0, \emptyset)(v_1, \emptyset)(v_6, \emptyset)(v_7, \emptyset)((v_2, \Pi)(v_0, \Pi)(v_1, \Pi)(v_6, \Pi)(v_7, \Pi))^\omega$$

and with cost $(4, 4)$. We have $\lambda(v_0, \emptyset) = 4$ and player 2 reaches his target set from (v_0, \emptyset) within exactly 4 steps. The constraints imposed by λ on the other vertices of ρ are respected too.

Recall now the strategy profile σ' with outcome $\rho' = ((v_0, \emptyset)(v_4, \emptyset))^\omega$ described in Example 5. The outcome ρ' is not λ -consistent since player 2 does not reach his target set, and so in particular he does not reach his target set within 4 steps. We already know that σ' is not an SPE from Example 5. In fact all the profiles that yield ρ' as an outcome are not SPEs. ◀

Our algorithm roughly works as follows: the labeling function λ that characterizes the set of SPE outcomes is obtained from an initial labeling function that imposes no constraints, by iterating an operator that reinforces the constraints step after step, up to obtaining a fixpoint which is the required function λ . Thus, if λ^k is the labeling function computed at step k and $\Lambda^k(v)$, $v \in V^X$, the related sets of λ^k -consistent plays, initially we have $\Lambda^0(v) = \text{Plays}_X(v)$, and step by step, the constraints imposed by λ^k become stronger and the sets $\Lambda^k(v)$ become smaller, until a fixpoint is reached.

Initially, we want a labeling function λ^0 that imposes no constraint on the plays. We could define λ^0 as the constant function $+\infty$. We proceed a little bit differently. Indeed recall the definition of the target sets $(F_i^X)_{i \in \Pi}$ in an extended game (see Definition 7): $v \in F_i^X$ if and only if $i \in I(v)$. Hence, given $\rho = \rho_0 \rho_1 \dots$, once $\rho_k \in F_i^X$ for some $k \in \mathbb{N}$ then $\rho_n \in F_i^X$ for all $n \geq k$. It follows that for all $n \geq k$, $\text{Cost}_i(\rho_{\geq n}) = 0$ and the inequality (3) is trivially true. (See also Example 12.) Therefore we define the labeling function λ^0 as follows.

► **Definition 13** (Initial labeling). *For all $v \in V^X$, let $i \in \Pi$ be such that $v \in V_i^X$,*

$$\lambda^0(v) = \begin{cases} 0 & \text{if } i \in I(v) \\ +\infty & \text{otherwise.} \end{cases}$$

► **Lemma 14.** $\rho \in \Lambda_0(v)$ if and only if $\rho \in \text{Plays}_X(v)$.

Let us now explain how our algorithm computes the labeling functions λ^k , $k \geq 1$, and the related sets $\Lambda^k(v)$, $v \in V^X$. It works in a *bottom-up* manner, according to the total order $J_1 < J_2 < \dots < J_N$ of \mathcal{I} given in (2): it first iteratively updates the labeling function for all vertices v of the arena X^{J_N} until reaching a fixpoint in this arena, it then repeats this procedure in $X^{\geq J_{N-1}}$, $X^{\geq J_{N-2}}$, \dots , $X^{\geq J_1} = X$. Hence, suppose that we currently treat the arena $X^{\geq J_n}$ and that we want to compute λ^{k+1} from λ^k . We define the updated function λ^{k+1} as follows (we use the convention that $1 + (+\infty) = +\infty$).

► **Definition 15** (Labeling update). Let $k \geq 0$ and suppose that we treat the arena $X^{\geq J_n}$, with $n \in \{1, \dots, N\}$. For all $v \in V^X$,

■ if $v \in V^{\geq J_n}$, let $i \in \Pi$ be such that $v \in V_i^X$, then

$$\lambda^{k+1}(v) = \begin{cases} 0 & \text{if } i \in I(v) \\ 1 + \min_{(v, v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v')\} & \text{otherwise} \end{cases}$$

■ if $v \notin V^{\geq J_n}$, then

$$\lambda^{k+1}(v) = \lambda^k(v).$$

Let us provide some explanations. As this update concerns the arena $X^{\geq J_n}$, we keep $\lambda^{k+1} = \lambda^k$ outside of this arena. Suppose now that v belongs to the arena $X^{\geq J_n}$ and $v \in V_i^X$. We define $\lambda^{k+1}(v) = 0$ whenever $i \in I(v)$ (as already explained for the definition of λ^0). When it is updated, the value $\lambda^{k+1}(v)$ represents what is the best cost that player i can ensure for himself with a “one-shot” choice by only taking into account plays of $\Lambda^k(v')$ with $v' \in \text{Succ}(v)$.

Notice that it makes sense to run the algorithm in a bottom-up fashion according to the total ordering $J_1 < \dots < J_N$ since given a play $\rho = \rho_0 \rho_1 \dots$, if ρ_0 is a vertex of $V^{\geq J_n}$, then for all $k \in \mathbb{N}$, ρ_k is a vertex of $V^{\geq J_n}$ (by I -monotonicity). Moreover running the algorithm in this way is essential to prove that the constraint problem for reachability games is in PSPACE.

► **Example 16.** We consider again the extended game (\mathcal{X}, x_0) of Figure 2 with the total order $J_1 = \emptyset < J_2 = \{2\} < J_3 = J_N = \Pi$ of its set \mathcal{I} .

Let us illustrate Definition 15 on the arena $X^{\geq J_2}$. Let $k > 0$ and suppose that the labeling function λ^k has been computed such that $\lambda^k(v_0, J_2) = +\infty$, $\lambda^k(v_1, J_2) = 3$ and $\lambda^k(v_4, J_2) = +\infty$ (notice λ^k is not the labeling indicated in Figure 2). Let us show how to compute $\lambda^{k+1}(v_0, J_2)$. We need to compute $\sup\{\text{Cost}_2(\rho) \mid \rho \in \Lambda^k(v')\}$ for the two successors v' of (v_0, J_2) , that is, respectively $v' = (v_1, J_2)$ and $v' = (v_4, J_2)$. Recall that $\Lambda^k(v')$ is the set of all plays λ^k -consistent from v' . Thus, as $\lambda^k(v_1, J_2) = 3$, the set $\Lambda^k(v_1, J_2)$ consists only of the plays that go to (v_6, J_2) , as any play that goes to (v_3, J_2) has at least cost 6 for player 1. All the plays in $\Lambda^k(v_1, J_2)$ have cost 3 for player 1 and player 2. Thus, $\sup\{\text{Cost}_2(\rho) \mid \rho \in \Lambda^k(v_1, J_2)\} = 3$. On the other hand, as $\lambda^k(v_4, J_2) = +\infty$, the play $(v_4, J_2)^\omega$ is λ^k -consistent and belongs to $\Lambda^k(v_4, J_2)$. Thus, $\sup\{\text{Cost}_2(\rho) \mid \rho \in \Lambda^k(v_4, J_2)\} = +\infty$. Hence, the minimum is attained with successor (v_1, J_2) , and $\lambda^{k+1}(v_0, J_2) = 4$. ◀

We can now provide our algorithm that computes the sequence $(\lambda^k(v))_{k \in \mathbb{N}}$ until a fixpoint is reached (see Proposition 17 below). Initially, the labeling function is λ^0 (see Definition 13). For the next steps $k > 0$, we begin with the bottom region X^{J_N} of \mathcal{X} and update λ^{k-1} to λ^k as described in Definition 15. At some point, the values of λ^k do not change anymore in X^{J_N} and $(\lambda^k)_{k \in \mathbb{N}}$ reaches locally (on X^{J_N}) a fixpoint (see again Proposition 17). Now, we consider the arena $X^{\geq J_{N-1}}$ and in the same way, we continue to update locally the values of λ^k in $X^{\geq J_{N-1}}$. We repeat this procedure with arenas $X^{\geq J_{N-2}}, X^{\geq J_{N-3}}, \dots$ until the arena $X^{\geq J_1} = X$ is completely processed. From the last computed λ^k , we derive the sets $\Lambda^k(v)$, $v \in V^X$, that we need for the characterization of outcomes of SPEs (see Theorem 20 below). An example illustrating the execution of this algorithm is given at the end of this section.

We now state that the sequence $(\lambda^k)_{k \in \mathbb{N}}$ computed by this algorithm reaches a fixpoint - locally on each arena $X^{\geq J_n}$ and globally on X - in the following meaning:

► **Proposition 17.** *There exists a sequence of integers $0 = k_N^* < k_{N-1}^* < \dots < k_1^* = k^*$ such that*

Algorithm 1: Fixpoint

```

compute  $\lambda^0$ 
 $k \leftarrow 0$ 
 $n \leftarrow N$ 
while  $n \neq 0$  do
  repeat
     $k \leftarrow k + 1$ 
    compute  $\lambda^k$  from  $\lambda^{k-1}$  with respect to  $X^{\geq J_n}$ 
  until  $\lambda^k = \lambda^{k-1}$ 
   $n \leftarrow n - 1$ 
end
return  $\lambda^k$ .

```

- **Local fixpoint:** for all $J_n \in \mathcal{I}$, all $m \in \mathbb{N}$ and all $v \in V^{\geq J_n}$,
 $\lambda^{k_n^*+m}(v) = \lambda^{k_n^*}(v)$. (4)
- **Global fixpoint:** in particular, with $k^* = k_1^*$, for all $m \in \mathbb{N}$ and all $v \in V^X$,
 $\lambda^{k^*+m}(v) = \lambda^{k^*}(v)$.

The global fixpoint λ^{k^*} is also simply denoted by λ^* , and the set Λ^{k^*} is denoted by Λ^* .

This proposition indicates that Algorithm 1 terminates. Indeed for each $J_n \in \mathcal{I}$, taking the *least index* k_n^* which makes Equality (4) true shows that the repeat loop is broken and the variable n decremented by 1. The value $n = 0$ is eventually reached and the algorithm stops with the global fixpoint λ^* . Notice that the first local fixpoint is reached with $k_N^* = 0$ because X^{J_N} is a bottom region.

Proposition 17 also shows that when a local fixpoint is reached in the arena $X^{\geq J_{n+1}}$ and the algorithm updates the labeling function λ^k in the arena $X^{\geq J_n}$, the values of $\lambda^k(v)$ do not change anymore for any $v \in V^{\geq J_{n+1}}$ but can still be modified for some $v \in V^{J_n}$. Recall also that outside of $X^{\geq J_n}$, the values of $\lambda^k(v)$ are still equal to the initial values $\lambda^0(v)$. These properties will be useful when we will prove that the constraint problem for reachability games is in PSPACE. They are summarized in the next lemma.

► **Lemma 18.** Let $k \in \mathbb{N}$ be a step of the algorithm and let J_n with $n \in \{1, \dots, N\}$. For all $v \in V^{J_n}$:

- if $k \leq k_{n+1}^*$, then $\lambda^{k+1}(v) = \lambda^k(v) = \lambda^0(v)$,
- if $k_n^* \leq k$, then $\lambda^{k+1}(v) = \lambda^k(v) = \lambda^{k_n^*}(v)$,

Hence the values of $\lambda^k(v)$ and $\lambda^{k+1}(v)$ may be different only when $k_{n+1}^* < k < k_n^*$.

To prove Proposition 17, we have to prove that the sequences $(\lambda^k(v))_{k \in \mathbb{N}}$, with $v \in V^X$, are non increasing.

► **Lemma 19.** For all $v \in V^X$, the sequences $(\lambda^k(v))_{k \in \mathbb{N}}$ and $(\Lambda^k(v))_{k \in \mathbb{N}}$ are non increasing.

Proof. Let us prove by induction on k that for all $v \in V^X$

$$\lambda^{k+1}(v) \leq \lambda^k(v). \quad (5)$$

We will get that $\Lambda^{k+1}(v) \subseteq \Lambda^k(v)$ by Definition 11.

First, notice that $\lambda^k(v) = 0$ if and only if $i \in I(v)$, where i is the player owning v . So, in this case, $\lambda^{k+1}(v) = \lambda^k(v) = 0$ and assertion (5) is proved. It remains to prove this assertion when $i \notin I(v)$.

For $k = 0$, let $v \in V_i^X$ such that $i \notin I(v)$, then $\lambda^0(v) = +\infty$ and obviously $\lambda^1(v) \leq \lambda^0(v)$.

Suppose that assertion (5) is true for k and let us prove it for $k+1$. We know by induction hypothesis that for all $v' \in V^X$, $\lambda^{k+1}(v') \leq \lambda^k(v')$, and thus also

$$\Lambda^{k+1}(v') \subseteq \Lambda^k(v'). \quad (6)$$

Let us prove that for all $v \in V_i^X$ such that $i \notin I(v)$, $\lambda^{k+2}(v) \leq \lambda^{k+1}(v)$. If $\lambda^{k+2}(v) = \lambda^{k+1}(v)$ or $\lambda^{k+1}(v) = +\infty$, then the assertion is proved. Otherwise,

$$\lambda^{k+1}(v) = 1 + \min_{(v,v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v')\}.$$

By (6), it follows that

$$\lambda^{k+1}(v) \geq 1 + \min_{(v,v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^{k+1}(v')\} = \lambda^{k+2}(v).$$

And so, the assertion again holds. \blacktriangleleft

We can now prove Proposition 17.

Proof of Proposition 17. The base case is easily proved. As V^{J_N} is a bottom region, we have $\lambda_1(v) = \lambda_0(v)$ for all $v \in V^{J_N}$, and thus the local fixpoint is immediately reached on V^{J_N} . Hence with $k_N^* = 0$, for all $m \in \mathbb{N}$ and all $v \in V^{J_N}$, $\lambda^{k_N^*+m}(v) = \lambda^{k_N^*}(v)$.

Let J_n be an element of \mathcal{I} , with $n \in \{1, \dots, N-1\}$. Suppose that a fixpoint has been reached in the arena $X^{\geq J_{n+1}}$ and that the labeling function λ^k is updated on the arena $X^{\geq J_n}$ as described in Definition 15. Recall that the previously computed values of λ^k do no longer change on $X^{\geq J_{n+1}}$ (a fixpoint is reached) and they do not change outside of $V^{\geq J_n}$ (by construction). However they can be modified on X^{J_n} . In this region X^{J_n} , there are $|V|$ sequences $(\lambda^k(v))_{k \in \mathbb{N}}$, $v \in V^{J_n}$, since the vertices v are of the form $v = (u, J_n)$ where J_n is fixed. These sequences are non increasing by Lemma 19. As the component-wise ordering over $(\mathbb{N} \cup \{+\infty\})^{|V|}$ is a well-quasi-ordering, there exists a natural number k_n^* that we choose as small as possible such that for all $v \in V^{J_n}$, $\lambda^{k_n^*+1}(v) = \lambda^{k_n^*}(v)$. This equality also holds for all $v \in V^{\geq J_n}$ (and not only for V^{J_n}), and it follows that for all $m \in \mathbb{N}$ and all $v \in V^{\geq J_n}$, $\lambda^{k_n^*+m}(v) = \lambda^{k_n^*}(v)$.

Notice that when n is decremented in Algorithm 1, k is incremented at least once showing that the sequence $0 < k_N^* < k_{N-1}^* < \dots < k_1^* = k^*$ is strictly increasing.

Finally, the last arena processed by the algorithm is $X^{\geq J_1} = X$. So with $k^* = k_1^*$, we have that for all $v \in V$ and all $m \in \mathbb{N}$, $\lambda^{k^*+m}(v) = \lambda^{k^*}(v)$. \blacktriangleleft

We are ready to state how we characterize plays that are outcomes of SPEs. This is possible with the fixpoint computed by Algorithm 1.

► Theorem 20 (Characterization). *Let (\mathcal{G}, v_0) be an initialized quantitative game and (\mathcal{X}, x_0) be its extended game. Let ρ be a play in $\text{Plays}_{\mathcal{X}}(x_0)$. Then ρ is the outcome of an SPE in (\mathcal{X}, x_0) if and only if for all $v \in \text{Succ}^*(x_0)$, $\Lambda^*(v) \neq \emptyset$ and $\rho \in \Lambda^*(x_0)$.*

Notice that this theorem also provides a characterization of the outcomes of SPEs in (\mathcal{G}, v_0) by Lemma 8.

The two implications of Theorem 20 respectively follow from Proposition 23 and Proposition 24 given below. We first need to establish an important property satisfied by the sets $\Lambda^k(v)$: when for some player i , the costs $\text{Cost}_i(\rho)$ associated with the plays ρ in $\Lambda^k(v)$ are unbounded, there actually exists a play in this set that has an infinite cost. In other terms, either $\Lambda^k(v)$ contains at least one play ρ with an infinite cost $\text{Cost}_i(\rho)$ or there exists a finite maximal cost for all ρ in $\Lambda^k(v)$.

► **Proposition 21.** *For every $k \in \mathbb{N}$, for every $v \in V^X$ and for every $i \in \Pi$, the following implication holds: if $\sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v)\} = +\infty$, then there exists a play $\rho \in \Lambda^k(v)$ such that $\text{Cost}_i(\rho) = +\infty$.*

Proof. Let $k \in \mathbb{N}$ and let $v \in V^X$, we assume that $\sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v)\} = +\infty$, that is, for all $n \in \mathbb{N}$, there exists $\rho^n \in \Lambda^k(v)$ such that $\text{Cost}_i(\rho^n) > n$. By König's lemma, there exist $(\rho^{n_\ell})_{\ell \in \mathbb{N}}$ a subsequence of $(\rho^n)_{n \in \mathbb{N}}$ and $\rho \in \text{Plays}_X(v)$ such that $\rho = \lim_{\ell \rightarrow +\infty} \rho^{n_\ell}$. Moreover, $\text{Cost}_i(\rho) = +\infty$ as Cost_i can be transformed into a continuous function. Let us prove that

$$\rho \in \Lambda^k(v).$$

This will establish Proposition 21.

We prove by induction on t with $0 \leq t \leq k$ that $\rho \in \Lambda^t(v)$. If $t = 0$, as $\Lambda^0(v) = \text{Plays}_X(v)$ by Lemma 14, then $\rho = \lim_{\ell \rightarrow +\infty} \rho^{n_\ell} \in \text{Plays}_X(v)$. It follows that $\rho \in \Lambda^0(v)$.

Let $t > 0$ and assume that the assertion is true for $t - 1 < k$. Suppose by contradiction that $\rho \notin \Lambda^t(v)$. For all $\ell \in \mathbb{N}$, as $\rho^{n_\ell} \in \Lambda^k(v)$, we have $\rho^{n_\ell} \in \Lambda^t(v)$ by Lemma 19. Moreover by induction hypothesis we have $\rho \in \Lambda^{t-1}(v)$. It follows that there exists $m \in \mathbb{N}$ and $j \in \Pi$ with $\rho_m \in V_j$ such that

$$\text{Cost}_j(\rho_{\geq m}) > \lambda^t(\rho_m) \quad \text{and} \quad \text{Cost}_j(\rho_{\geq m}) \leq \lambda^{t-1}(\rho_m).$$

In particular, $\lambda^t(\rho_m) < \lambda^{t-1}(\rho_m)$ and thus $\lambda^t(\rho_m) < +\infty$, and player j does not reach his target set along $\pi = \rho_0 \dots \rho_m \dots \rho_{m+\lambda^t(\rho_m)}$. We choose n_ℓ large enough such that ρ and ρ^{n_ℓ} share a common prefix of length at least $|\pi|$. As $\text{Cost}_j(\rho_{\geq m}) > \lambda^t(\rho_m)$, it follows that $\text{Cost}_j(\rho_{\geq m}^{\ell_{n_\ell}}) > \lambda^t(\rho_m)$. We can conclude that $\rho^{n_\ell} \notin \Lambda^t(v)$ which leads to a contradiction. ◀

The next corollary is a direct consequence of Proposition 21 with the convention that the max belongs to $\mathbb{N} \cup \{+\infty\}$.

► **Corollary 22.** $\sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v)\} = \max\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v)\}.$

The two implications of Theorem 20 are proved in the following Proposition 23 and Proposition 24.

► **Proposition 23.** *If σ is an SPE in (\mathcal{X}, x_0) then for all $v \in \text{Succ}^*(x_0)$, $\Lambda^*(v) \neq \emptyset$ and $\langle \sigma \rangle_{x_0} \in \Lambda^*(x_0)$.*

Proof. Suppose that σ is an SPE in (\mathcal{X}, x_0) and let us prove by induction on k that for all $k \in \mathbb{N}$ and all $hv \in \text{Hist}_X(x_0)$,

$$\langle \sigma \rangle_{hv} \in \Lambda^k(v).$$

For case $k = 0$, this is true by definition of $\Lambda^0(v)$ and Lemma 14.

Suppose that this assertion is satisfied for $k \geq 0$ and by contradiction, assume that there exists $hv \in \text{Hist}_X(x_0)$ such that $\langle \sigma \rangle_{hv} \notin \Lambda^{k+1}(v)$. Let $\rho = \langle \sigma \rangle_{hv}$, by Definition 11, it means that there exist $n \in \mathbb{N}$ and $i \in \Pi$ such that $\rho_n \in V_i$ and

$$\text{Cost}_i(\rho_{\geq n}) > \lambda^{k+1}(\rho_n). \tag{7}$$

But by induction hypothesis, we know that

$$\text{Cost}_i(\rho_{\geq n}) \leq \lambda^k(\rho_n). \tag{8}$$

It follows by (7) and (8) that $\lambda^{k+1}(\rho_n) < \lambda^k(\rho_n)$ and that

$$\lambda^{k+1}(\rho_n) < +\infty, \lambda^k(\rho_n) \neq 0 \text{ and } \lambda^{k+1}(\rho_n) \neq 0.$$

In regards of Definition 15, these three relations allow us to conclude that

$$\lambda^{k+1}(\rho_n) = 1 + \min_{(\rho_n, v') \in E^X} \sup\{\text{Cost}_i(\rho') \mid \rho' \in \Lambda^k(v')\}.$$

Let $v' \in V$ be such that

$$\lambda^{k+1}(\rho_n) - 1 = \sup\{\text{Cost}_i(\rho') \mid \rho' \in \Lambda^k(v')\}. \quad (9)$$

Let $h' = h\rho_0 \dots \rho_{n-1} \in \text{Hist}_X(x_0)$, and let us define the one-shot deviating strategy τ_i from $\sigma_{i|h'}$ such that $\tau_i(\rho_n) = v'$. Let us prove that τ_i is a one-shot profitable deviation for player i in $(\mathcal{X}_{|h'}, \rho_n)$.

$$\begin{aligned} \text{Cost}_i(h' \langle \tau_i, \sigma_{-i|h'} \rangle_{\rho_n}) &= \\ &= \text{Cost}_i(h' \rho_n \langle \sigma_{|h' \rho_n} \rangle_{v'}) && (\text{as } \tau_i \text{ is a one-shot deviating strategy}) \\ &= |h' \rho_n v'| + \text{Cost}_i(\langle \sigma_{|h' \rho_n} \rangle_{v'}) && (\text{as } \lambda^{k+1}(\rho_n) \neq 0) \\ &\leq |h' \rho_n v'| + \lambda^{k+1}(\rho_n) - 1 && (\text{by (9) and as } \langle \sigma_{|h' \rho_n} \rangle_{v'} \in \Lambda^k(v') \text{ by induction hypothesis}) \\ &< |h' \rho_n| + \text{Cost}_i(\rho_{\geq n}) && (\text{by (7)}) \\ &= \text{Cost}_i(h' \langle \sigma_{|h'} \rangle_{\rho_n}) \end{aligned}$$

This proves that σ is not a very weak SPE and so not an SPE by Proposition 6, which is a contradiction. This concludes the proof. \blacktriangleleft

► **Proposition 24.** *Suppose that for all $v \in \text{Succ}^*(x_0)$, $\Lambda^*(v) \neq \emptyset$ and let $\rho \in \Lambda^*(x_0)$, then ρ is the outcome of an SPE in (\mathcal{X}, x_0) .*

Proof. We show how to construct a very weak SPE σ (and so an SPE by Proposition 6) with outcome ρ in (\mathcal{X}, x_0) . We define σ step by step by induction on the subgames of (\mathcal{X}, x_0) . We first partially build σ such as it produces ρ , i.e., $\langle \sigma \rangle_{x_0} = \rho$. Now, we define a set of plays which will be useful to define σ in the subgames. For all (i, v') such that $(v, v') \in E^X$ with $v, v' \in \text{Succ}^*(x_0)$ and $v \in V_i^X$, we define the play $\rho_{i,v}$

$$\rho_{i,v'} = \arg\max_{\rho \in \Lambda^{k^*}(v)} (\text{Cost}_i(\rho)). \quad (10)$$

Notice that such a play exists by Corollary 22. The construction of σ is done by induction as follows. Consider $hvv' \in \text{Hist}_X(x_0)$ with $v \in V_i^X$ such that $\langle \sigma_{|h} \rangle_v$ is already defined but not yet $\langle \sigma_{|hv} \rangle_{v'}$. We extend the definition of σ as follows:

$$\langle \sigma_{|hv} \rangle_{v'} = \rho_{i,v'}$$

Let us prove that σ is a very weak SPE. Consider the subgame $(\mathcal{X}_{|h}, v)$ for a given $hv \in \text{Hist}_X(x_0)$ with $v \in V_i^X$ and the one-shot deviating strategy σ'_i from $\sigma_{i|h}$ such that $\sigma'_i(v) = v'$. By construction, there exists $\rho_{j,u}$ and $\rho_{i,v'}$ as defined previously and $g \in \text{Hist}_X(x_0)$ such that $h \langle \sigma_{|h} \rangle_v = g\rho_{j,u}$ and $\langle \sigma_{|hv} \rangle_{v'} = \rho_{i,v'}$.

We have that $\rho = \langle \sigma_{|h} \rangle_v$ is suffix of $\rho_{j,u}$, that is, $\rho = \rho_{j,u,\geq n}$ for some $n \in \mathbb{N}$ and that $\rho_{j,u} \in \Lambda^{k^*}(u) = \Lambda^{k^*+1}(u)$ (by (10) and the fixpoint). It follows that:

$$\begin{aligned} \text{Cost}_i(\rho) &= \text{Cost}_i(\rho_{j,u,\geq n}) \\ &\leq \lambda^{k^*+1}(v) \\ &= 1 + \min_{(v,w) \in E^X} \sup\{\text{Cost}_i(\rho') \mid \rho' \in \Lambda^{k^*}(w)\} \\ &\leq 1 + \sup\{\text{Cost}_i(\rho') \mid \rho' \in \Lambda^{k^*}(v')\} \\ &= 1 + \text{Cost}_i(\rho_{i,v'}). \end{aligned} \quad (11)$$

Region	{1, 2}	{2}							\emptyset					
	v	v_0	v_1	v_6	v_7	v_3	v_4	v_5	v_0	v_1	v_6	v_7	v_3	v_4
$\lambda^0 = \lambda^1$	0	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\lambda^2 = \lambda^3$	0	0	3	2	1	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
λ^4	0	0	3	2	1	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3	2	1	$+\infty$	$+\infty$
$\lambda^5 = \lambda^*$	0	0	3	2	1	$+\infty$	$+\infty$	$+\infty$	4	3	2	1	$+\infty$	$+\infty$

■ **Table 1** The different steps of the algorithm computing λ^* for the extended game of Figure 2

We now prove that σ'_i is not a one-shot profitable deviation by proving that $\text{Cost}_i(h\langle\sigma_{\uparrow h}\rangle_v) \leq \text{Cost}_i(hv\langle\sigma_{\uparrow hv}\rangle_{v'})$. If $i \in I(v)$, then obviously $\text{Cost}_i(h\langle\sigma_{\uparrow h}\rangle_v) = \text{Cost}_i(hv\langle\sigma_{\uparrow hv}\rangle_{v'}) = 0$. Otherwise, $i \notin I(v)$ and

$$\begin{aligned}
\text{Cost}_i(h\langle\sigma_{\uparrow h}\rangle_v) &= |hv| + \text{Cost}_i(\rho) \\
&\leq |hv| + 1 + \text{Cost}_i(\rho_{i,v'}) && \text{(by (11))} \\
&= \text{Cost}_i(hv\rho_{i,v'}) \\
&= \text{Cost}_i(hv\langle\sigma_{\uparrow hv}\rangle_{v'}).
\end{aligned}$$

This concludes the proof. ◀

► **Example 25.** Let us come back to the running example of Figure 2. The different steps of Algorithm 1 are given in Table 1. The columns indicate the vertices according to their region, respectively Π , $\{2\}$, and \emptyset . Notice that for the region Π , we only write one column v as for all vertices (v, Π) the value of λ is equal to 0 all along the algorithm.

Recall that $J_1 = \emptyset < J_2 = \{2\} < J_3 = \Pi = \{1, 2\}$. The algorithm begins with the arena X^{J_3} . A fixpoint ($\lambda^1 = \lambda^0$) is immediately reached because all vertices belong to the target set of both players in X^{J_3} . Thus the first local fixpoint is reached with $k_3^* = 0$.

The algorithm then treats the arena $X^{\geq J_2}$. By Lemma 18, it is enough to consider the region X^{J_2} . Let us explain how to compute $\lambda^2(v)$ from $\lambda^1(v)$ on this region. For $v = (v_7, \{2\})$, we have that $\lambda^2(v) = 1 + \min_{(v,v') \in E^X} \sup\{\text{Cost}_1(\rho) \mid \rho \in \Lambda^1(v')\}$. As the unique successor of v is $(v_2, \{1, 2\})$, all λ^1 -consistent plays beginning in this successor have cost 0. So, we have that $\lambda^2(v) = 1$. For the computation of $\lambda^2(v_6, \{2\})$, the same argument holds since $(v_6, \{2\})$ has the unique successor $(v_7, \{2\})$. The vertex $(v_1, \{2\})$ has two successors: $(v_6, \{2\})$ and $(v_3, \{2\})$. Again, we know that all λ^1 -consistent plays beginning in $(v_6, \{2\})$ have cost 2. From $(v_3, \{2\})$ however, one can easily check that the play $(v_3, \{2\})(v_0, \{2\})((v_4, \{2\}))^\omega$ is λ^1 -consistent and has cost $+\infty$ for player 1. Thus, we obtain that $\lambda^2(v_1, \{2\}) = 3$. For the other vertices of X^{J_2} , one can see that $\lambda^2(v) = \lambda^1(v)$.

Finally, we can check that the local fixed point is reached in the arena $X^{\geq J_2}$ (resp. $X^{\geq J_1}$) with $\lambda^3 = \lambda^2$ (resp. $\lambda^6 = \lambda^5 = \lambda^*$). Therefore the respective fixpoints are reached with $k_2^* = 2$ and $k_1^* = 5$. The labeling function indicated in Figure 2 is the one of λ^* . ◀

4 Counter graph

In the previous section, we have introduced the concept of labeling function λ and the constraints that it imposes on plays. We have also proposed an algorithm that computes a sequence of labeling functions $(\lambda^k)_{k \in \mathbb{N}}$ until reaching a fixpoint λ^* such that the plays that are λ^* -consistent are exactly the SPE outcomes. In this section, given a labeling function λ , we introduce the concept of *counter graph* such that its infinite paths coincide with the plays that are λ -consistent. We then show that the counter graph associated with the fixpoint

function λ^* has an exponential size, an essential step to prove PSPACE membership of the constraint problem.

For the entire section, we fix a reachability game $\mathcal{G} = (G, (F_i)_{i \in \Pi}, (\text{Cost}_i)_{i \in \Pi})$ with an arena $G = (\Pi, V, (V_i)_{i \in \Pi}, E)$, and v_0 be an initial vertex. Let $\mathcal{X} = (X, (F_i^X)_{i \in \Pi}, (\text{Cost}_i)_{i \in \Pi})$ with the arena $X = (\Pi, V^X, (V_i^X)_{i \in \Pi}, E^X)$ be its associated extended game. Furthermore, when we speak about a play ρ we always mean a play in the extended game \mathcal{X} .

A labeling function λ give constraints on costs of plays from each vertex in X , albeit only for the player that owns this vertex. However, by the property of λ -consistence, constraints for a player carry over all the successive vertices, whether they belong to him or not. In order to check efficiently this property, we introduce the counter graph to keep track *explicitly* of the accumulation of constraints for all players at each step of a play. Let us first fix some notation.

► **Definition 26** (Restriction and maximal finite range). *Let $\lambda : V^X \rightarrow \mathbb{N} \cup \{+\infty\}$ be a labeling function.*

- We consider restrictions of λ to sub-arenas of V^X as follows. Let $n \in \{1, \dots, N\}$, we denote by $\lambda_n : V^{J_n} \rightarrow \mathbb{N} \cup \{+\infty\}$ the restriction of λ to V^{J_n} . Similarly we denote by $\lambda_{\geq n}$ (resp. $\lambda_{> n}$) the restriction of λ to $V^{\geq J_n}$ (resp. $V^{> J_n}$).
- The maximal finite range of λ , denoted by $\text{mR}(\lambda)$, is equal to

$$\text{mR}(\lambda) = \max\{c \in \mathbb{N} \mid \lambda(v) = c \text{ for some } v \in V^X\}$$

with the convention that $\text{mR}(\lambda) = 0$ if λ is the constant function $+\infty$. We also extend this notion to restrictions of λ with the convention that $\text{mR}(\lambda_{> n}) = 0$ if J_n is a bottom region.

Notice that in the definition of maximal finite range, we only consider the *finite* values of λ (and not the value $+\infty$).

► **Definition 27** (Counter Graph). *Let $\lambda : V^X \rightarrow \mathbb{N} \cup \{+\infty\}$ be a labeling function.*

Let $\mathcal{K} := \{0, \dots, K\} \cup \{+\infty\}$ with $K = \text{mR}(\lambda)$. The counter graph $\mathbb{C}(\lambda)$ for \mathcal{G} and λ is equal to $\mathbb{C}(\lambda) = (\Pi, V^C, (V_i^C)_{i \in \Pi}, E^C)$, such that:

- $V^C = V^X \times \mathcal{K}^{|\Pi|}$
- $(v, (c_i)_{i \in \Pi}) \in V_j^C$ if and only if $v \in V_j^X$
- $((v, (c_i)_{i \in \Pi}), (v', (c'_i)_{i \in \Pi})) \in E^C$ if and only if:
 - $(v, v') \in E^X$, and
 - for every $i \in \Pi$

$$c'_i = \begin{cases} 0 & \text{if } i \in I(v') \\ c_i - 1 & \text{if } i \notin I(v'), v' \notin V_i^X \text{ and } c_i > 1 \\ \min(c_i - 1, \lambda(v')) & \text{if } i \notin I(v'), v' \in V_i^X \text{ and } c_i > 1. \end{cases}$$

Intuitively, the counter graph is constructed such that once a value $\lambda(v)$ is finite for a vertex $v \in V_i^X$ along a play in \mathcal{X} , the corresponding path in $\mathbb{C}(\lambda)$ keeps track of the induced constraint by (i) decrementing the counter value c_i for the concerned player i by 1 at every step, (ii) updating this counter if a stronger constraint for player i is encountered by visiting a vertex v' with a smaller value $\lambda(v')$, and (iii) setting the counter c_i to 0 if player i has reached his target set.

Note that in the counter graph, there may be some vertices with *no outgoing edges*. Indeed, consider a vertex $(v, (c_i)_{i \in \Pi}) \in V^C$ such that $c_j = 1$ for some player j . By construction

of $\mathbb{C}(\lambda)$, the only outgoing edges from $(v, (c_i)_{i \in \Pi})$ must link to vertices $(v', (c'_i)_{i \in \Pi})$ such that $(v, v') \in E^X$, $c'_j = 0$ and $j \in I(v')$. However, it may be the case that no successor v' of v in X is such that $j \in I(v')$.

Note as well that for each vertex $v \in V^X$, there exist many different vertices $(v, (c_i)_{i \in \Pi})$ in $\mathbb{C}(\lambda)$, one for each counter values profile. However, the intended goal of the counter graph is to monitor explicitly the constraints accumulated by each player along a play in \mathcal{X} regarding the function λ . Thus, we will only consider paths in $\mathbb{C}(\lambda)$ that start in vertices $(v, (c_i)_{i \in \Pi})$ such that the counter values correspond indeed to the constraint at the *beginning* of a play in \mathcal{X} regarding λ :

► **Definition 28** (Starting vertex in $\mathbb{C}(\lambda)$). *Let $v \in V^X$. We distinguish one vertex $v^C = (v, (c_i)_{i \in \Pi})$ in V^C , such that for every $i \in \Pi$:*

$$c_i = \begin{cases} 0 & \text{if } i \in I(v) \\ \lambda(v) & \text{if } i \notin I(v) \text{ and } v \in V_i^X \\ +\infty & \text{otherwise.} \end{cases}$$

We call v^C the starting vertex associated with v , and denote by $\text{SV}(\lambda)$ the set of all starting vertices in $\mathbb{C}(\lambda)$.

► **Example 29.** Recall the extended game (\mathcal{X}, x_0) of Figure 2, and the labeling function λ whose values are indicated under or next to each vertex. In Example 12, we have shown that the play

$$\rho = (v_0, \emptyset)(v_1, \emptyset)(v_6, \emptyset)(v_7, \emptyset)((v_2, \Pi)(v_0, \Pi)(v_1, \Pi)(v_6, \Pi)(v_7, \Pi))^\omega$$

was λ -consistent. Let us show that there is a corresponding infinite path π that starts in $(v_0, \emptyset)^C = (v_0, \emptyset, (+\infty, 4))$ in $\mathbb{C}(\lambda)$. Following Definition 27, we see that in $\mathbb{C}(\lambda)$, there exists an edge between $(v_0, \emptyset, (+\infty, 4))$ and $(v_1, \emptyset, (3, 3))$ and that

$$\pi = (v_0, \emptyset)^C (v_1, \emptyset, (3, 3))(v_6, \emptyset, (2, 2))(v_7, \emptyset, (1, 1))\pi'^\omega$$

with

$$\pi' = (v_2, \Pi, (0, 0))(v_0, \Pi, (0, 0))(v_1, \Pi, (0, 0))(v_6, \Pi, (0, 0))(v_7, \Pi, (0, 0)).$$

Recall now the play $\rho' = ((v_0, \emptyset)(v_4, \emptyset))^\omega$ described in Example 12, which is not λ -consistent. From $(v_0, \emptyset)^C = (v_0, \emptyset, (+\infty, 4))$, there is an edge to $(v_4, \emptyset, (+\infty, 3))$, then to $(v_0, \emptyset, (+\infty, 2))$ and to $(v_4, \emptyset, (+\infty, 1))$. For the latter vertex, there is no outgoing edge back to $(v_0, \emptyset, (+\infty, 0))$ because $2 \notin I(v_0)$. Therefore there is no infinite path starting in $(v_0, \emptyset)^C$ in $\mathbb{C}(\lambda)$ that corresponds to ρ' . ◀

There exists a correspondence between λ -consistent plays in \mathcal{X} and *infinite* paths from starting vertices in $\mathbb{C}(\lambda)$ in the following way. On one hand, every play ρ in \mathcal{X} that is not λ -consistent does not appear in the counter graph: the first constraint regarding λ that is violated along ρ is reflected by a vertex in $\mathbb{C}(\lambda)$ with a counter value getting to 1 and no outgoing edges. On the other hand, λ -consistent plays in \mathcal{X} have a corresponding infinite path in the counter graph $\mathbb{C}(\lambda)$. This correspondence is formalized in the two following lemmas:

► **Lemma 30.** *Let $\rho = \rho_0\rho_1 \dots$ be a play in $\text{Plays}_X(v)$. Then there exists an associated infinite path $\pi = \pi_0\pi_1 \dots$ in $\mathbb{C}(\lambda)$ such that:*

■ $\pi_0 = v^C$,

- ρ is the projection of π on V^X , that is, π_n is of the form $(v', (c'_i)_{i \in \Pi})$ with $v' = \rho_n$, for every $n \in \mathbb{N}$.

Proof. Let $v \in V^X$ and $k \in \mathbb{N}$. Let ρ be a play in \mathcal{X} such that $\rho \in \lambda(v)$ and $\rho_0 = v$. We build the corresponding infinite path π in $\mathbb{C}(\lambda)$ iteratively as follows: Let $\pi_0 := v^C$. Let $n \in \mathbb{N}$, $n \geq 1$. Suppose that $\pi_{<n}$ has been already constructed, we show how to choose π_n . Suppose $\pi_{n-1} = (v', (c'_i)_{i \in \Pi})$. Then $\pi_n := (v'', (c''_i)_{i \in \Pi})$, where:

- $v'' = \rho_n$,
- for every $i \in \Pi$:

$$c''_i = \begin{cases} 0 & \text{if } i \in I(v''), \\ c'_i - 1 & \text{if } i \notin I(v'') \text{ and } v'' \notin V_i^X, \\ \min(c'_i - 1, \lambda(v'')) & \text{if } i \notin I(v'') \text{ and } v'' \in V_i^X \end{cases}$$

Let us show that $((v', (c'_i)_{i \in \Pi}), (v'', (c''_i)_{i \in \Pi})) \in E^C$:

- As ρ is a play in \mathcal{X} , we clearly have $(v', v'') \in E^X$,
- Assume now, towards contradiction, that $((v', (c'_i)_{i \in \Pi}), (v'', (c''_i)_{i \in \Pi})) \notin E^C$. Since $(v', v'') \in E^X$, it means that at least one counter value c''_i does not respect the constraints of the edge relation of $\mathbb{C}(\lambda)$. For every player $i \in I(v'')$, we have indeed counter value c''_i at π_n equal to 0, so the violation of the constraints must come from a player i such that $i \notin I(v'')$. Furthermore, if for every such player the counter value $c'_i > 1$, then by construction of $\mathbb{C}(\lambda)$ and definition of π_n , we have $((v', (c'_i)_{i \in \Pi}), (v'', (c''_i)_{i \in \Pi})) \in E^C$. Thus, there exists a player i such that $c'_i = 1$ (and $i \notin I(v'')$).
 - * if $v' \in V_i^X$ and $c'_i = \lambda(v') = 1$, then, since ρ is λ -consistent, player i sees his objective along ρ in one step, that is at vertex v'' , and thus $i \in I(v'')$, which is a contradiction.
 - * if $v' \in V_i^X$ and $\lambda(v') \neq 1$, then it means that there exists an index $m < n - 1$ such that the vertex $\rho_m \in V_i^X$ and such that $\lambda(\rho_m) = c^m$ with $c^m \in \mathbb{N} \setminus \{0\}$ and such that the counter value for player i decreases by exactly 1 at each step until v' . Indeed, along ρ , at each vertex w owned by player i , the counter value is the minimum of $\lambda(w)$ and the counter value at the predecessor minus 1. Thus, there must be before step $n - 1$ a last occurrence of the counter value being set to $\lambda(w)$. Let $w = \rho_m$ and $d := \lambda(w)$. Since ρ is λ -consistent, player i sees his objective along ρ in at most d steps, thus at most at vertex v'' , and thus $i \in I(v'')$, which is a contradiction.
 - * finally, if $c'_i = 1$ and $v' \notin V_i^X$, we are in a similar case: there exists an index $m < n - 1$ such that the vertex $\rho_m \in V_i^X$ and such that $\lambda(\rho_m) = c^m$ with $c^m \in \mathbb{N} \setminus \{0\}$ and such that the counter value for player i decreases by exactly 1 at each step until v' . Let $w = \rho_m$ and $d := \lambda(w)$. Since ρ is λ -consistent, player i sees his objective along ρ in at most d steps, thus at most at vertex v'' , and thus $i \in I(v'')$, which is a contradiction.

◀

► **Lemma 31.** Let $v^C = (v, (c_i)_{i \in \Pi})$ be a starting vertex in $\text{SV}(\lambda)$. Let $\pi = \pi_0 \pi_1 \dots$ be an infinite path in $\mathbb{C}(\lambda)$ such that $\pi_0 = v^C$. Then there exists a corresponding play $\rho = \rho_0 \rho_1 \dots$ in \mathcal{X} such that:

- ρ is λ -consistent,

■ ρ is the projection of π on V^X , that is, $\rho_n = v'$ with $\pi_n = (v', (c'_i)_{i \in \Pi})$, for every $n \in \mathbb{N}$.

Proof. Let $v^C = (v, (c_i)_{i \in \Pi})$ be a starting vertex in $\text{SV}(\lambda)$. Let π be an infinite path in $\mathbb{C}(\lambda)$ such that $\pi_0 = v^C$. Let ρ be the projection of π on V^X , that is, $\rho_n = v'$ with $\pi_n = (v', (c'_i)_{i \in \Pi})$, for every $n \in \mathbb{N}$.

- Clearly, ρ is a play in \mathcal{X} : by construction of $\mathbb{C}(\lambda)$, there exists an edge between two vertices $(v, (c_i)_{i \in \Pi})$ and $(v', (c'_i)_{i \in \Pi})$ in $\mathbb{C}(\lambda)$ only if (v, v') is an edge in E^X .
- Furthermore, the play ρ is λ -consistent: Assume, towards contradiction, that it is not. Thus, there exists $n \in \mathbb{N}$ and $i \in \Pi$ such that $\rho_n \in V_i^X$ and $\text{Cost}_i(\rho_{\geq n}) > \lambda(\rho_n)$. Consider now π_n and the value c_i of the counter for player i at this vertex. Since $\rho_n \in V_i^X$, we know that $c_i \leq \lambda(\rho_n)$. From this vertex, the counter value for player i decreases at least by 1 at each step along $\pi_{\geq n}$. and hits the value 0 before $\lambda(\rho_n)$ steps. However, this means that in $\rho_{\geq n}$, player i sees his objective sooner than expected and the cost is less than $\lambda(\rho_n)$, which is a contradiction. ◀

Since the edge relation E^C in the counter graph respects the edge relation E^X in the extended game, the region decomposition of a path in \mathcal{X} given in Lemma 9 can also be applied to a path in $\mathbb{C}(\lambda)$. We will often use such path region decompositions in the proofs of this section.

In order to prove the PSPACE membership for the constraint problem, we need to show that the counter graph $\mathbb{C}(\lambda^*)$, with λ^* the fixpoint function computed by Algorithm 1, has an exponential size. To this end it is enough to show an exponential upper bound on the maximal finite range $\text{mR}(\lambda^*)$ of λ^* . We proceed in two steps: First, with the next proposition, given a labeling function λ and its restriction $\lambda_{\geq \ell}$ to $V^{\geq \ell}$, we exhibit a bound on the supremum of the cost of λ -consistent plays for each player, in terms of the maximal finite range $\text{mR}(\lambda_{\geq \ell})$. Second, we consider the actual sequence of functions $(\lambda^k)_{k \in \mathbb{N}}$ defined in Definitions 13 and 15, as implemented by Algorithm 1. With Theorem 34, we show that $\text{mR}(\lambda^*)$ is bounded by an exponential in the size of the original game \mathcal{G} . The proof is by induction on k and uses Proposition 32.

► **Proposition 32** (Bound on finite supremum). *Let λ be a labeling function. Let $v \in V^X$ such that $I(v) = J_\ell$ with $\ell \in \{1, \dots, N\}$. Suppose there exists $c \in \mathbb{N}$ such that $\sup \{\text{Cost}_i(\rho) \mid \rho \in \Lambda(v)\} = c$. Then,*

$$c \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell})).$$

Before proving Proposition 32, we need the following technical lemma:

► **Lemma 33.** *Let v^C be a starting vertex in $\text{SV}(\lambda)$ associated with $v \in V^X$ such that $I(v) = J_\ell$. Let π be a finite prefix of a valid path in $\mathbb{C}(\lambda)$ such that:*

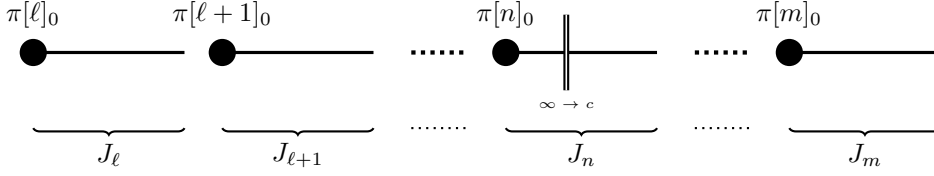
- $\pi_0 = v^C$,
- π does not contain any cycle.

Then,

$$|\pi| \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell})).$$

Proof sketch. The full proof of this lemma is quite technical, so we give here the main ideas and refer to the full proof for more details.

Let π be a finite prefix of a valid path in $\mathbb{C}(\lambda)$ as in the statement. Let $\pi[\ell] \dots \pi[m]$ be its region decomposition according to Lemma 9, graphically represented in Figure 3. Let ρ be the corresponding path in \mathcal{X} and $\rho[\ell] \dots \rho[m]$ be its region decomposition. Let us consider a fixed section $\pi[n]$.



■ **Figure 3** Region decomposition of π

Suppose first that the counter values at $\pi[n]_0$ are either 0 or $+\infty$. Let us prove that along $\pi[n]$, there can be at most $|V|$ steps before reaching a vertex with a finite positive value of λ :

- assume there is a cycle in the corresponding section $\rho[n]$ in \mathcal{X} such that from $\rho[n]_0$ and along the cycle, all the values of λ are either 0 or $+\infty$,
- by construction of $\mathbb{C}(\lambda)$, the counter values in the corresponding prefix of $\pi[n]$ remain fixed for each vertex of this prefix: as no value of λ is positive and finite, no counter value can be decremented,
- thus, the cycle in $\rho[n]$ is also a cycle in $\pi[n]$ which is impossible by hypothesis,
- thus there is no such cycle in $\rho[n]$, and as there are at most $|V|$ vertices in region X^{J_n} , $\rho[n]$ can have a prefix of length at most $|V|$ with only values 0 or $+\infty$ for λ , implying that this is also the case for $\pi[n]$.

Therefore, we can decompose $\pi[n]$ into a (possibly empty) prefix of length at most $|V|$, and a (possibly empty) suffix where at least one counter value c'_i , for some i , is a positive finite value in its first vertex v' . This frontier between prefix and suffix of $\pi[n]$ is represented by a vertical double bar $||$ with caption $\infty \rightarrow c$ in Figure 3. This value c'_i is bounded by $\text{mR}(\lambda_n)$, the maximal finite range of λ_n . From there, as the corresponding ρ is λ -consistent, player i reaches his target set in at most c'_i steps, and ρ enters a new region, which means that the section $\pi[n]$ is over. So, in that case, the length of $\pi[n]$ can be bounded by $|V| + \text{mR}(\lambda_n)$.

Suppose now that at vertex $\pi[n]_0$, there exists a counter value c_i for some player i that is neither 0 nor $+\infty$. This means that there was a constraint for player i initialized in a previous section $\pi[n']$, with $n' < n$, that has carried over to $\pi[n]_0$, via decrements of at least 1 per step. We know that the initial finite counter value is bounded by $\text{mR}(\lambda_{n'})$, and appeared before the end of section $\pi[n']$. Thus the length from the end of section $\pi[n']$ to the end of section $\pi[n]$ is bounded by $\text{mR}(\lambda_{n'})$, as again, once the counter value attains 0 for player i , the path π has entered the next section.

Therefore, considering the possible cases for each section, we can bound the total length of π as follows:

$$|\pi| \leq \sum_{j=\ell}^m |V| + 2 \cdot \text{mR}(\lambda_j).$$

Finally, remark that by I -monotonicity, it is actually the case that only (and at most) $|\Pi|$ different sections can appear in the decomposition of π . Furthermore, for each $n \in \{\ell+1, \dots, N\}$, we have $\text{mR}(\lambda_n) \leq \text{mR}(\lambda_{>\ell})$ by Definition 26. Thus, we have the following bound:

$$|\pi| \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \sum_{r=2}^{|\Pi|} |V| + 2 \cdot \text{mR}(\lambda_{>\ell})$$

which is the bound stated in Lemma 33. ◀

Proof of Lemma 33. Let v^C be a starting vertex in $\text{SV}(\lambda)$. Let π be a finite prefix of a valid path in $\mathbb{C}(\lambda)$ such that:

- $\pi_0 = v^C$,
- π does not contain any cycle.

The following proof is quite technical, thus we alleviate some of the difficulties by proving a first upper bound on the length of finite paths without cycles in $\mathbb{C}(\lambda)$, then by showing how to obtain the desired bound. The main idea is to bound iteratively the length of prefixes of π , adding at each step of the reasoning the section for the next region traversed by π .

Let π' be a valid path in $\mathbb{C}(\lambda)$ such that π is a prefix of π' . By Lemma 9, we know that there exist two natural numbers $\ell, m' \leq N$ and $m' - \ell$ (possibly empty) paths $\pi'^\ell, \dots, \pi'^{m'}$ in $\mathbb{C}(\lambda)$ such that:

- $\pi' = \pi'^\ell \dots \pi'^{m'}$,
- for each $\ell \leq j \leq m'$, each vertex in $\pi'^j[n]$ is of the form $(w, (c_i)_{i \in \Pi})$ with $I(w) = J_j$.

The finite path π is a finite prefix of π' . Thus there exists a natural number $m \leq m'$ and $m - \ell$ (possibly empty) paths π^ℓ, \dots, π^m in $\mathbb{C}(\lambda)$ such that:

- $\pi = \pi^\ell \dots \pi^m$,
- for each $\ell \leq j < m$, $\pi[j] = \pi'^j[j]$,
- $\pi[m]$ if a finite prefix of π'^m .

By Lemma 31, there exist a corresponding λ -consistent play ρ' in \mathcal{X} and a corresponding history ρ . Furthermore, since π contains no cycle, the sections $\pi[n]$ do not either.

We first treat the case where $m < m'$.

We first bound the length of the first section π^ℓ . Recall that since $\pi^\ell_0 = v^C \in \text{SV}(\lambda)$, we have that $v^C = (v, (c_i)_{i \in \Pi})$ with $c_i = 0$ if $i \in J_\ell$, $c_i = \lambda(v)$ if $v \in V_i$ and $c_i = +\infty$ otherwise.

Suppose $0 < \lambda(v) < +\infty$. In that case, we know that along ρ' , which is λ -consistent, player i reaches his target after at most $\lambda(v)$ steps. Thus, there exists $n \leq m'$, $n \neq \ell$ such that $I(\rho_{\lambda(v)}) = J_n$, $J_\ell \subsetneq J_n$, and $\pi_{\lambda(v)}$ belongs to section $\pi[n]$. This means that section π^ℓ is shorter than $\lambda(v)$. Since $I(v) = J_\ell$, we have that $\lambda(v) \leq \text{mR}(\lambda_\ell)$, and thus $|\pi^\ell| \leq \text{mR}(\lambda_\ell)$.

Suppose now that $\lambda(v) = 0$ or $\lambda(v) = +\infty$. The counter values at $(v)^C$ are thus either 0 or $+\infty$. Along π' , they can stay stable for at most $|V|$ steps (see Proof sketch of Lemma 33). Otherwise, the cycle induced in ρ' is also a cycle in π' as the counter values are fixed. If $\pi'_{|V|+1}$ is in section π'^n with $n > \ell$, then we immediately get that $|\pi^\ell| \leq |V|$. If $\pi'_{|V|+1}$ is in section π'^ℓ , then it means that a counter value for some player i has become finite along the first $|V| + 1$ vertices of π'^ℓ . Let t be the first index it does so along π'^ℓ . By the same argument as in the previous case, we know that from π'^ℓ_t , there is at most $\text{mR}(\lambda_\ell)$ vertices before entering the next section of π' . Thus, we obtain that:

$$\begin{aligned} |\pi^\ell| &\leq |\pi'^\ell| \\ &\leq |V| + \text{mR}(\lambda_\ell) \\ &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell). \end{aligned}$$

If $\ell = m$, we can already conclude that:

$$\begin{aligned} |\pi| &\leq |\pi^\ell| \\ &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) \\ &\leq \sum_{j=\ell}^m |V| + 2 \cdot \text{mR}(\lambda_j). \end{aligned}$$

Suppose now that $\ell < m$. We show that for each $n > \ell$, we have:

$$|\pi[\ell] \dots \pi[n]| \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + \text{mR}(\lambda_n).$$

Let $n = \ell + 1$. We assume $\pi[n]$ is not empty, otherwise its length is 0. Consider now $\pi[n]_0$.

Suppose there exists a player i such that his counter value c_i at $\pi[n]_0 = (w, J_n, (c_i)_{i \in \Pi})$ is a finite non-zero value. If $w \notin V_i$, it means that the counter value has decreased since a vertex $(w', (c'_i)_{i \in \Pi})$ in the previous section $\pi[\ell]$ such that $w' \in V_i$ and $c'_i = \lambda(w)$. Thus, the length of the path from this vertex $(w', (c'_i)_{i \in \Pi})$ to the next section $\pi[n+1]$ is smaller or equal than $\lambda(w')$. In particular, the whole section $\pi[n]$ is "covered" by this path. Therefore, we can conclude that $|\pi[n]| \leq \lambda(w') \leq \text{mR}(\lambda_\ell)$. Since we already know that $|\pi[\ell]| \leq |V| + \text{mR}(\lambda_\ell)$, we obtain that:

$$\begin{aligned} |\pi[\ell]\pi[n]| &\leq |V| + \text{mR}(\lambda_\ell) + \text{mR}(\lambda_\ell) \\ &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell). \end{aligned}$$

If $w \in V_i$, it means that either the counter value c_i is equal to $\lambda(w)$ or has decreased since a vertex $(w', (c'_i)_{i \in \Pi})$ in the previous section $\pi[\ell]$ such that $w' \in V_i$ and $c'_i = \lambda(w')$. Thus, we have that $c_i \leq \text{mR}(\lambda_n)$ or $c_i \leq \text{mR}(\lambda_\ell)$, and, in turn, $|\pi[n]| \leq \text{mR}(\lambda_n)$ or $|\pi[n]| \leq \text{mR}(\lambda_n)$. Therefore, we can conclude that:

$$\begin{aligned} |\pi[\ell]\pi[n]| &\leq |V| + \text{mR}(\lambda_\ell) + \text{mR}(\lambda_\ell) + \text{mR}(\lambda_n) \\ &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \text{mR}(\lambda_n). \end{aligned}$$

Suppose now that for every player i , his counter value c_i at $\pi[n]_0 = (w, (c_i)_{i \in \Pi})$ is either 0 or $+\infty$. In that case, we are in a similar case than for section $\pi[\ell]$, thus we can conclude that $|\pi[n]| \leq |v| + \text{mR}(\lambda_n)$. Thus, we have indeed:

$$|\pi[\ell]\pi[n]| \leq |V| + \text{mR}(\lambda_\ell) + |V| + \text{mR}(\lambda_n),$$

and also

$$|\pi[\ell]\pi[n]| \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + |V| + \text{mR}(\lambda_n).$$

If $m = \ell + 1$, we are done.

Suppose now that $m > \ell + 1$. Let n be such that $\ell + 1 < n \leq m$. Assume that for each $n' < n$, it holds that

$$|\pi[\ell] \dots \pi[n']| \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n'-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + \text{mR}(\lambda_{n'}).$$

We assume $\pi[n]$ is not empty, otherwise its length is 0. Consider now $\pi[n]_0$.

Suppose there exists a player i such that his counter value c_i at $\pi[n]_0 = (w, (c_i)_{i \in \Pi})$ is a finite non-zero value. If $w \notin V_i$, it means that the counter value has decreased since a vertex $(w', (c'_i)_{i \in \Pi})$ in a previous section $\pi[n']$ such that $w' \in V_i$ and $c'_i = \lambda(w')$. Thus, the

length of the path from this vertex $(w', (c'_i)_{i \in \Pi})$ to the next section $\pi[n+1]$ is smaller or equal than $\lambda(w')$. In particular, the whole sections from $\pi[n'+1]$ to $\pi[n]$ are "covered" by this path. Therefore, we can conclude that $|\pi[n'+1] \dots \pi[n]| \leq \lambda(w') \leq \text{mR}(\lambda_{n'})$. Since we already know that:

$$|\pi[\ell] \dots \pi[n']| \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n'-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + \text{mR}(\lambda_{n'}),$$

we obtain that:

$$\begin{aligned} |\pi[\ell] \dots \pi[n]| &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n'-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + \text{mR}(\lambda_{n'}) + \text{mR}(\lambda_{n'}) \\ &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n'-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + 2 \cdot \text{mR}(\lambda_{n'}) \\ &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + \text{mR}(\lambda_n). \end{aligned}$$

If $w \in V_i$, it means that either the counter value c_i is equal to $\lambda(w)$ or has decreased since a vertex $(w', (c'_i)_{i \in \Pi})$ in a previous section $\pi[n']$ such that $w' \in V_i$ and $c'_i = \lambda(w')$. Thus, we have that $c_i \leq \text{mR}(\lambda_n)$ or $c_i \leq \text{mR}(\lambda_{n'})$, and, in turn, $|\pi[n]| \leq \text{mR}(\lambda_n)$ or $|\pi[n]| \leq \text{mR}(\lambda_{n'})$.

If $|\pi[n]| \leq \text{mR}(\lambda_n)$, we can conclude that:

$$\begin{aligned} |\pi[\ell] \dots \pi[n]| &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + \text{mR}(\lambda_n) \\ &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + \text{mR}(\lambda_n). \end{aligned}$$

If $|\pi[n]| \leq \text{mR}(\lambda_{n'})$, we can conclude that:

$$\begin{aligned} |\pi[\ell] \dots \pi[n]| &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n'-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + \text{mR}(\lambda_{n'}) + \text{mR}(\lambda_{n'}) \\ &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + \text{mR}(\lambda_n). \end{aligned}$$

Suppose now that for every player i , his counter value c_i at $\pi[n]_0 = (w, (c_i)_{i \in \Pi})$ is either 0 or $+\infty$. In that case, we are in a similar case than for section $\pi[\ell]$, thus we can conclude that $|\pi[n]| \leq |v| + \text{mR}(\lambda_n)$. Thus, we have indeed:

$$|\pi[\ell] \dots \pi[n]| \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{n-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| + \text{mR}(\lambda_n)$$

and thus finally:

$$|\pi| \leq |\pi[\ell] \dots \pi[m]| \leq \sum_{j=\ell}^m |V| + 2 \cdot \text{mR}(\lambda_j).$$

Assume now that $m = m'$. In that case, we cannot rely on the section π'^m to be finite, as the play ρ' never reaches another region. However, in this situation, it is guaranteed that the counter values in π'^m are fixed and are equal to either 0 or $+\infty$: indeed, a finite counter value would imply that some player i such that $i \notin J_m$ reaches his target in ρ' exactly when his counter value becomes 0 in π' . But π'^m is the last section of ρ' , thus no new region is reached after J_m and no new player can see his objective than the players $i \in J_m$. Therefore, finite prefix of π'^m that contains no cycle has its length bounded by $|V|$. Thus, we can conclude:

$$\begin{aligned} |\pi| &= |\pi[\ell] \dots \pi[m]| \\ &\leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \left(\sum_{j=\ell+1}^{m-1} |V| + 2 \cdot \text{mR}(\lambda_j) \right) + |V| \\ &\leq \sum_{j=\ell}^m |V| + 2 \cdot \text{mR}(\lambda_j). \end{aligned}$$

In fact, the bound given above can be slightly changed to give the desired bound. Indeed, the bound above relies on the fact that $m \leq N$ and covers the case where a path traverses every region from J_ℓ to J_N . In all generality, the number N of different regions can be exponential in the number $|\Pi|$ of players. However, by the I -monotonicity property, we know that a path can actually traverse at most $|\Pi|$ regions. Thus, in the region decomposition of a path, only (and at most) $|\pi|$ sections are relevant and of length greater than 0. Therefore, we can define a subsequence of indices $(n_r)_{r \leq |\pi|}$, with $n_1 = \ell$, such that in fact $\pi = \pi[n_1] \dots \pi[n_{|\pi|}]$. Hence, we obtain the following bound on the length t of π :

$$t \leq \sum_{r=1}^{|\Pi|} |V| + 2 \cdot \text{mR}(\lambda_{n_r})$$

Finally, as for every $r \leq |\pi|$, $r > 1$, we have $\text{mR}(\lambda_{n_r}) \leq \text{mR}(\lambda_{>\ell})$, we obtain the desired bound:

$$t \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + \sum_{r=2}^{|\Pi|} |V| + 2 \cdot \text{mR}(\lambda_{>\ell})$$

That is,

$$t \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell})).$$

◀

We are now ready to prove Proposition 32.

Proof of Proposition 32. Let $v \in V^X$ with $I(v) = J_\ell$ and $i \in \Pi$ be such that there exists $c \in \mathbb{N}$ with $\sup \{\text{Cost}_i(\rho) \mid \rho \in \Lambda(v)\} = c$.

If $i \in J_\ell$, then $c = 0$, as every play starting in v has a cost 0 for player i . Hence the statement of Proposition 32 trivially holds.

Suppose now that $i \notin J_\ell$. Consider $\rho \in \Lambda^k(v)$ such that $\text{Cost}_i(\rho) = c < +\infty$. As player i eventually reaches his target set along ρ , this means that ρ eventually leaves region J_ℓ and eventually reaches another region J_n such that $i \in J_n$. Consider the prefix $\rho_{\leq c}$ of ρ of length c . Let $\pi := \pi[\ell] \dots \pi[m]$ be the associated path and region decomposition of $\rho_{\leq c}$ in $\mathbb{C}(\lambda)$.

Notice that $\pi[m]$ consists only of one vertex corresponding to ρ_c , and that for every $n < m$, we have $i \notin J_n$.

Suppose that π contains a cycle. By construction of $\mathbb{C}(\lambda)$, this cycle is included in one single section $\pi[n]$, where $n < m$ (as $\pi[m]$ contains only one vertex), and thus $i \notin J_n$. Consider the infinite path π' in $\mathbb{C}(\lambda)$ that follows π until the cycle and then repeats the cycle forever. By Lemma 31, there exists a corresponding λ -consistent play ρ' in \mathcal{X} . We have $\text{Cost}_i(\rho') = +\infty$ for player i , as ρ' never reaches a region where player i reaches his target set. This is a contradiction with the fact that $\sup \{\text{Cost}_i(\rho) \mid \rho \in \Lambda(v)\} = c$ is finite.

Therefore π contains no cycle, and by Lemma 33,

$$|\pi| \leq |V| + 2 \cdot \text{mR}(\lambda_\ell) + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell})).$$

Since $|\pi| = c = \text{Cost}_i(\rho)$, we obtain the statement of Proposition 32. \blacktriangleleft

We now come back to the labeling functions $(\lambda^k)_{k \in \mathbb{N}}$ computed by Algorithm 1. Recall that this algorithm works in a bottom-up manner (see Proposition 17): it first computes the local fixpoint $\lambda^{k_N^*}$ on region V^{J_N} , then the local fixpoint $\lambda^{k_{N-1}^*}$ on $V^{\geq J_{N-1}}$, ..., until finally computing the global fixpoint λ^* on $V^{\geq J_1} = V^X$. Recall also that when the algorithm computes the local fixpoint in the arena $X^{\geq J_n}$, the values of $\lambda^k(v)$ may only change in the region V^{J_n} (Lemma 18). We are now ready to show an exponential bound (in the size of \mathcal{G}) on the maximal finite ranges $\text{mR}(\lambda_\ell^{k_\ell^*})$ for each region X^{J_ℓ} and $\text{mR}(\lambda_{\geq \ell}^{k_\ell^*})$ for each arena $X^{\geq J_\ell}$. With $\ell = 1$, we get that $\text{mR}(\lambda^*)$ is of exponential size, and thus also the size of the counter graph.

► **Theorem 34** (Bound on maximal finite range). *For each $\ell \in \{1, \dots, N\}$, we have*

$$\text{mR}(\lambda_\ell^{k_\ell^*}) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_\ell| + 1) \cdot (|\Pi \setminus J_\ell| + |V|)})$$

and also:

$$\text{mR}(\lambda_{\geq \ell}^{k_\ell^*}) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_\ell| + 1) \cdot (|\Pi \setminus J_\ell| + |V|)}).$$

In particular for the global fixpoint λ^* we have

$$\text{mR}(\lambda^*) \leq \mathcal{O}(|V|^{(|\Pi| + 1) \cdot (|\Pi| + |V|)}).$$

Proof sketch. We follow the bottom-up order of Algorithm 1 to prove the local bounds on $\text{mR}(\lambda_\ell^{k_\ell^*})$, starting by $\text{mR}(\lambda_N^{k_N^*})$ and making our way up to $\text{mR}(\lambda_1^{k_1^*})$, assuming for each region V^{J_ℓ} such that $\ell < N$ that the bound is true for every region already treated by Algorithm 1. The base case (which is actually similar to the case of each bottom region) is simple: for region V^{J_N} , we know by Proposition 17 that $k_N^* = 0$. The values are thus either 0 or $+\infty$, and thus clearly $\text{mR}(\lambda_N^{k_N^*}) = 0$.

Given a non-bottom region V^{J_ℓ} and assuming the bound is true for every already treated region, we proceed to show the local bound for V^{J_ℓ} by induction on the number k of steps in the computation, which corresponds to the values of function λ^k in the sequence of functions leading to the fixpoint, up to step k_ℓ^* , where the values stabilize on region V^{J_ℓ} . The base case is again simple: the values of λ_ℓ^0 are either 0 or $+\infty$, thus clearly $\text{mR}(\lambda_\ell^0) = 0$. To treat the general case, we rely on a key property of the sequence of functions λ^k . Recall the update rule from Definition 15:

$$\lambda^{k+1}(v) = \begin{cases} 0 & \text{if } i \in I(v) \\ 1 + \min_{(v, v') \in E^X} \sup \{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v')\} & \text{otherwise} \end{cases}$$

Notice that the maximal finite range on V^{J_ℓ} can only increase from step k to step $k+1$ if the value of a vertex $v \in V^{J_\ell}$ went from $+\infty$ to a finite value $c > 0$. Furthermore, note that this increase can only happen at steps $k > k_{\ell+1}^*$. By Proposition 32, we have that, for every $v \in V^{J_\ell}$ such that $\lambda_\ell^k(v) = +\infty$ and $\lambda_\ell^{k+1}(v) \neq +\infty$:

$$\lambda_\ell^{k+1}(v) \leq 1 + |V| + 2 \cdot \text{mR}(\lambda_\ell^k) + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell}^*)) \quad (12)$$

Since the number of vertices in each region of the extended game \mathcal{X} is bounded by the number of vertices $|V|$ in the original game \mathcal{G} , this increase phenomenon can happen at most at $|V|$ many steps in the computation of the fixpoint values on region V^{J_ℓ} . Thus, we can focus on the (at most) $|V|$ induction steps where an increase actually occurs. The induction hypothesis on k is that $\text{mR}(\lambda_\ell^k) \leq n \cdot (1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| + n - 1 + |\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| + |V|)}))$, where n is the number of increasing steps up to step k . Together with the bound $\text{mR}(\lambda_{>\ell}^*)$ on already treated regions, it allows us to bound the sum given by Inequality (12) and obtain the desired bound for step $k+1$. As the number of increases is at most $|V|$, at the stabilizing step k_ℓ^* for region V^{J_ℓ} we obtain:

$$\text{mR}(\lambda_\ell^{k_\ell^*}) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_\ell| + 1) \cdot (|\Pi \setminus J_\ell| + |V|)}).$$

Furthermore, as for every $J_{\ell'}$ such that $J_\ell \subsetneq J_{\ell'}$, we have $|J_{\ell'}| > |J_\ell|$ and $\text{mR}(\lambda_{\ell'}^{k_{\ell'}^*}) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_{\ell'}| + 1) \cdot (|\Pi \setminus J_{\ell'}| + |V|)})$, we also obtain that $\text{mR}(\lambda_{\geq \ell}^{k_\ell^*}) \leq \text{mR}(\lambda_\ell^{k_\ell^*})$.

Finally, with $\ell = 1$, as $|J_1| \leq |\Pi|$, we get the following bound on the maximal finite range $\text{mR}(\lambda^*) (= \text{mR}(\lambda_{\geq 1}^{k_1^*}))$:

$$\text{mR}(\lambda^*) \leq \mathcal{O}(|V|^{(|\Pi| + 1) \cdot (|\Pi| + |V|)}).$$

◀

Proof of Theorem 34. The proof is done by a *double* induction: First, we exploit the fact that Algorithm 1 treats every region one after the other, following the total order on regions in reverse. That is, to compute the values of the fixpoint function λ^{k^*} over V^X , Algorithm 1 computes first the values of λ^{k^*} on region V^{J_N} , then on region $V^{J_{N-1}}$ etc... until finally on region V^{J_1} . Thus, we follow this order to prove the local bounds on $\text{mR}(\lambda_\ell^{k_\ell^*})$, starting by $\text{mR}(\lambda_N^{k_N^*})$ and making our way up to $\text{mR}(\lambda_1^{k_1^*})$, assuming for each region V^{J_ℓ} such that $\ell < N$ that the bound is true for every region already treated by Algorithm 1. Given a non-bottom region V^{J_ℓ} and assuming the bound is true for every already treated region, we proceed to show the local bound for V^{J_ℓ} by induction on the number k of steps in the computation, which corresponds to the values of function λ^k in the sequence of functions leading to the fixpoint, up to step k_ℓ^* , where the values stabilize on region V^ℓ .

Bottom-up induction on the regions J_ℓ :

Let $\ell \in \{1, \dots, N\}$. We show that $\text{mR}(\lambda_\ell^{k_\ell^*}) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_\ell| + 1) \cdot (|\Pi \setminus J_\ell| + |V|)})$.

$\ell = N$: If $J_\ell = J_N$, then J_ℓ is a bottom region: every path that starts in J_ℓ cannot go to another region. In other terms, all the players that are not in J_ℓ will never see their objective. Thus, $\lambda_\ell^{k_\ell^*} = \lambda_\ell^0$, and thus $\text{mR}(\lambda_\ell^{k_\ell^*}) = 0$.

$\ell < N$: If J_ℓ is a bottom region, we are in fact in a case similar to the case $\ell = N$, that is, $\lambda_\ell^{k_\ell^*} = \lambda_\ell^0$, and thus $\text{mR}(\lambda_\ell^{k_\ell^*}) = 0$.

Suppose now that J_ℓ is not a bottom region.

Induction hypothesis: We assume that for every ℓ' such that $\ell < \ell' \leq N$, the following bound is true:

$$\text{mR}(\lambda_{\ell'}^{k_*}) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_{\ell'}|+1) \cdot (|\Pi \setminus J_{\ell'}|+|V|)}) \quad (13)$$

Induction on the number of steps in Algorithm 1:

We now proceed to prove the desired bound for J_ℓ by induction on the number k of iterations of the algorithm.

First, recall Definition 15, which specifies how to update λ^{k+1} according to the values of λ :

$$\lambda^{k+1}(v) = \begin{cases} 0 & \text{if } i \in I(v) \\ 1 + \min_{(v,v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v')\} & \text{otherwise} \end{cases}$$

Notice that $\text{mR}(\lambda_\ell^{k+1}) > \text{mR}(\lambda_\ell^k)$ only if there exists $v \in V^{J_\ell}$ such that $\lambda^k(v) = +\infty$ and $\lambda^{k+1}(v) < +\infty$. Indeed, suppose that for every $v \in V^{J_\ell}$, we have either $\lambda^k(v) = +\infty$ and $\lambda^{k+1}(v) = +\infty$, or $\lambda^k(v) < +\infty$. Let $v \in V^{J_\ell}$, we show that $\lambda^{k+1}(v) \leq \lambda^k(v)$:

- $\lambda^k(v) = +\infty$ and $\lambda^{k+1}(v) = +\infty$: we immediately have $\lambda^{k+1}(v) \leq \lambda^k(v)$,
- $\lambda^k(v) < +\infty$: If $k = 0$, then it means that $\lambda^k(v) = 0$. Thus, we have $i \in J_\ell$ for the player i that owns vertex v , and thus we also have $\lambda^{k+1}(v) = 0$. Otherwise, recall that for every $v' \in V^X$, we have that $\Lambda^k(v') \subseteq \Lambda^{k-1}(v')$. In particular this holds for every successor v' of v in X . Thus, we have, for every $k > 0$:

$$\min_{(v,v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v')\} \leq \min_{(v,v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^{k-1}(v')\}$$

and thus immediately:

$$1 + \min_{(v,v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v')\} \leq 1 + \min_{(v,v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^{k-1}(v')\}$$

that is, $\lambda^{k+1}(v) \leq \lambda^k(v)$.

As this holds for every $v \in V^{J_\ell}$, we have $\text{mR}(\lambda_\ell^{k+1}) \leq \text{mR}(\lambda_\ell^k)$.

Therefore, we have that $\text{mR}(\lambda_\ell^{k+1}) > \text{mR}(\lambda_\ell^k)$ only if there exists $v \in V^{J_\ell}$ such that $\lambda^k(v) = +\infty$ and $\lambda^{k+1}(v) < +\infty$.

These changes from $+\infty$ to a finite value can happen at most $|V|$ times during the computation of the successive λ over a region J_ℓ . Recall now Proposition 17: there exists a local fixpoint index k_ℓ^* such that for all $n \in \mathbb{N}$ and $v \in V^{J_\ell}$, we have $\lambda_\ell^{k_\ell^*+n} = \lambda_\ell^{k_\ell^*}$, and thus $\text{mR}(\lambda_\ell^{k_\ell^*+n}) = \text{mR}(\lambda_\ell^{k_\ell^*})$.

Consider the computation iterations $0, 1, \dots, k_\ell^*$. There exist $m \in \mathbb{N}$, $m \leq |V|$, and a subsequence j_0, j_1, \dots, j_m of indices such that:

- $j_0 = 0$,
- $j_1 > k_{\ell+1}^*$,
- $j_n < j_{n+1}$ for every $n < m$,
- for every $k \leq k_\ell^*$ such that $k+1 \neq j_n$ for every $n \leq m$, we have $\text{mR}(\lambda_\ell^{k+1}) \leq \text{mR}(\lambda_\ell^k)$,
- for every $0 < n \leq m$, there exists $v \in V^{J_\ell}$ such that $\lambda^{(j_n)-1}(v) = +\infty$ and $\lambda^{j_n}(v) < +\infty$.

We can now start treating the steps of the induction:

k = 0: In that case, we already know that $\text{mR}(\lambda_\ell^k) = \text{mR}(\lambda_\ell^0) = 0$.

k > 0: We assume that $\mathbf{k} \geq \mathbf{j}_1 > \mathbf{k}_{\ell+1}^*$, otherwise by Definition 15, we know that $\text{mR}(\lambda_\ell^k) = 0$. We first bound $\text{mR}(\lambda_\ell^k)$ by bounding the values of $\lambda^k(v)$ for every $v \in V^{J_\ell}$. Let $v \in V^{J_\ell}$ and $i \in \Pi$ such that $v \in V_i^X$.

If $\mathbf{k} = \mathbf{j}_1$, then we can bound the values of $\lambda^k(v) = \lambda^{j_1}(v)$ for every $v \in X^{J_\ell}$:

- If $\lambda^0(v) = 0$, then $\lambda^{j_1}(v) = 0$.
- If $\lambda^0(v) = +\infty$ and $\lambda^{j_1}(v) = +\infty$, then it does not impact $\text{mR}(\lambda_\ell^{j_1})$.
- If $\lambda^0(v) = +\infty$ and $\lambda^{j_1}(v) \neq +\infty$, then we know, by definition of the sequence $(j_n)_{n \leq m}$, that $\lambda^{(j_1)-1}(v) = +\infty$ as well, and we have the following:

$$\lambda^{j_1}(v) = 1 + \min_{(v, v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^{(j_1)-1}(v')\}$$

Since $\lambda^{j_1}(v) \neq +\infty$, we know that $\sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^{(j_1)-1}(v')\}$ is finite, for at least one successor v' of v . Thus, by Proposition 32, we obtain:

$$\lambda^{j_1}(v) \leq 1 + |V| + 2 \cdot \text{mR}(\lambda_\ell^{(j_1)-1}) + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell}^{(j_1)-1}))$$

As $\text{mR}(\lambda_\ell^{(j_1)-1}) \leq \text{mR}(\lambda_\ell^{j_0}) = 0$, we have:

$$\lambda^{j_1}(v) \leq 1 + |V| + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell}^{(j_1)-1}))$$

By (13) and as $(j_1) - 1 \geq k_{\ell+1}^*$, we know that for each ℓ' such that $J_\ell \subsetneq J_{\ell'}$, we have $\text{mR}(\lambda_{\ell'}^{(j_1)-1}) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_{\ell'}|+1) \cdot (|\Pi \setminus J_{\ell'}|+|V|)})$. As $|J_{\ell'}| \geq |J_\ell| + 1$, we obtain:

$$\text{mR}(\lambda_{>\ell}^{(j_1)-1}) \leq \mathcal{O}(|V|^{|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell|+|V|)}) \quad (14)$$

And thus,

$$\lambda^{j_1}(v) \leq 1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell|+|V|)+1})$$

Hence we obtain, as $|\Pi \setminus J_\ell| \geq 1$:

$$\begin{aligned} \text{mR}(\lambda_\ell^{j_1}) &\leq 1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell|+|V|)+1}) \\ &\leq 1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell|+|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell|+|V|)}) \end{aligned}$$

Induction Hypothesis: Let $n \leq m$, $n > 0$. Assume that for every $n' < n$, $n' > 0$, we have:

$$\text{mR}(\lambda_\ell^{j_{n'}}) \leq n' \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell|+n'-1+|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell|+|V|)})\right) \quad (15)$$

If $\mathbf{k} \neq \mathbf{j}_n$ for every $n \leq m$ and $k < j_m$, then we know that there exists $n' < m$, $n' > 1$, such that $j_{n'} < k < j_{n'+1}$. Furthermore, we have that $\text{mR}(\lambda_\ell^k) \leq \text{mR}(\lambda_\ell^{j_{n'}})$. Thus, by the induction hypothesis (15), we have:

$$\text{mR}(\lambda_\ell^k) \leq n' \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell|+n'-1+|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell|+|V|)})\right) \quad (16)$$

If $\mathbf{k} = \mathbf{j}_n$ for some $1 < n \leq m$, then we can bound the values of $\lambda^k(v) = \lambda^{j_n}(v)$ for every $v \in V^{J_\ell}$. Let $v \in V^{J_\ell}$ and $i \in \Pi$ such that $v \in V_i^X$.

■ If $\lambda^{j_{n-1}}(v) \neq +\infty$, then we immediately have:

$$\begin{aligned} \lambda^{j_n}(v) &\leq \lambda^{j_{n-1}}(v) \\ &\leq \text{mR}(\lambda_\ell^{j_{n-1}}) \\ &\leq (n-1) \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| + n - 2 + |\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| \cdot |V|)})\right) \\ &\leq n \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| + n - 1 + |\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| \cdot |V|)})\right) \end{aligned}$$

- If $\lambda^{j_{n-1}}(v) = +\infty$ and $\lambda^{j_n}(v) = +\infty$, then this value does not impact $\text{mR}(\lambda_\ell^{j_n})$.
- If $\lambda^{j_{n-1}}(v) = +\infty$ and $\lambda^{j_n}(v) \neq +\infty$, we know, by definition of the sequence $(j_n)_{n \leq m}$, that $\lambda^{(j_n)-1}(v) = +\infty$ as well, and we have the following:

$$\lambda^{j_n}(v) = 1 + \min_{(v, v') \in E^X} \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^{(j_n)-1}(v')\}$$

Since $\lambda^{j_n}(v) \neq +\infty$, we know that $\sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^{(j_n)-1}(v')\}$ is finite, for at least one successor v' of v . Thus, by Proposition 32, we obtain:

$$\lambda^{j_n}(v) \leq 1 + |V| + 2 \cdot \text{mR}(\lambda_\ell^{(j_n)-1}) + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell}^{(j_n)-1}))$$

The sum above can be decomposed as follows:

- As we have $j_{n-1} \leq (j_n) - 1 < j_n$, we know that $\text{mR}(\lambda_\ell^{(j_n)-1}) \leq \text{mR}(\lambda_\ell^{j_{n-1}})$, thus, we obtain by the induction hypothesis (15):

$$\text{mR}(\lambda_\ell^{(j_n)-1}) \leq (n-1) \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| + n - 2 + |\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| \cdot |V|)})\right) \quad (17)$$

- By (13) and as $(j_n) - 1 \geq k_{\ell+1}^*$, we know that for each ℓ' such that $J_\ell \subsetneq J_{\ell'}$, we have $\text{mR}(\lambda_{\ell'}^{(j_n)-1}) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_{\ell'}| + 1) \cdot (|\Pi \setminus J_{\ell'}| + |V|)})$. As $|J_{\ell'}| \geq |J_\ell| + 1$, we obtain:

$$\text{mR}(\lambda_{>\ell}^{(j_n)-1}) \leq \mathcal{O}(|V|^{|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| + |V|)}) \quad (18)$$

And thus we have:

- By (17):

$$2 \cdot \text{mR}(\lambda_\ell^{(j_n)-1}) \leq (n-1) \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| + n - 2 + |\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| \cdot |V|)})\right) \quad (19)$$

- By (18):

$$\begin{aligned} (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell}^{(j_n)-1})) &\leq (|\Pi| - 1) \cdot \mathcal{O}(|V|^{|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| + |V|)}) \\ &\leq \mathcal{O}(|V|^{|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| + |V|) + 1}) \end{aligned} \quad (20)$$

By (19) and (20), we obtain:

$$\begin{aligned}
\lambda^{j_n}(v) &= 1 + |V| + 2 \cdot \text{mR}(\lambda_{\ell}^{(j_n)-1}) + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{\ell}^{(j_n)-1})) \\
&\leq 1 + |V| + (n - 1) \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_{\ell}| + n - 2 + |\Pi \setminus J_{\ell}| \cdot (|\Pi \setminus J_{\ell}| \cdot |V|)}) \right) + \\
&\quad \mathcal{O}(|V|^{|\Pi \setminus J_{\ell}| \cdot (|\Pi \setminus J_{\ell}| + |V|) + 1}) \\
&\leq 1 + |V| + (n - 1) \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_{\ell}| + n - 2 + |\Pi \setminus J_{\ell}| \cdot (|\Pi \setminus J_{\ell}| \cdot |V|)}) \right) + \\
&\quad \mathcal{O}(|V|^{|\Pi \setminus J_{\ell}| + n - 2 + |\Pi \setminus J_{\ell}| \cdot (|\Pi \setminus J_{\ell}| + |V|)})
\end{aligned}$$

That is:

$$\lambda^{j_n}(v) \leq n \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_{\ell}| + n - 1 + |\Pi \setminus J_{\ell}| \cdot (|\Pi \setminus J_{\ell}| \cdot |V|)}) \right) \quad (21)$$

Hence we get:

$$\text{mR}(\lambda_{\ell}^{j_n}) \leq n \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_{\ell}| + n - 1 + |\Pi \setminus J_{\ell}| \cdot (|\Pi \setminus J_{\ell}| \cdot |V|)}) \right) \quad (22)$$

Finally, since $n \leq |V|$:

$$\begin{aligned}
\text{mR}(\lambda_{\ell}^{k_{\ell}^*}) &\leq |V| \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_{\ell}| + |V| - 1 + |\Pi \setminus J_{\ell}| \cdot (|\Pi \setminus J_{\ell}| \cdot |V|)}) \right) \\
&\leq \mathcal{O}(|V|^{(|\Pi \setminus J_{\ell}| + 1) \cdot (|\Pi \setminus J_{\ell}| + |V|)})
\end{aligned} \quad (23)$$

Furthermore, as for every $J_{\ell'}$ such that $J_{\ell} \subsetneq J_{\ell'}$, we have $|J_{\ell'}| > |J_{\ell}|$ and $\text{mR}(\lambda_{\ell'}^{k_{\ell'}^*}) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_{\ell'}| + 1) \cdot (|\Pi \setminus J_{\ell'}| + |V|)})$, we also obtain that $\text{mR}(\lambda_{\geq \ell}^{k_{\ell}^*}) \leq \text{mR}(\lambda_{\ell}^{k_{\ell}^*})$.

Finally, as $|J_1| \leq |\Pi|$, for the global fixpoint λ^* we have:

$$\begin{aligned}
\text{mR}(\lambda^*) &= \text{mR}(\lambda_{\geq 1}^{k_{\geq 1}^*}) \\
&\leq \mathcal{O}(|V|^{(|\Pi| + 1) \cdot (|\Pi| + |V|)})
\end{aligned}$$

◀

The next corollary is a direct consequence of Theorem 34.

► **Corollary 35.** *The counter graph $\mathbb{C}(\lambda^*)$ has a size $|\mathbb{C}(\lambda^*)|$ exponential in the size of the game \mathcal{G} .*

We conclude this section with two other corollaries that we be useful in the next section.

► **Corollary 36.** *For every $k \in \mathbb{N}$ and region $V^{J_{\ell}}$, we have*

$$\text{mR}(\lambda_{\ell}^k) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_{\ell}| + 1) \cdot (|\Pi \setminus J_{\ell}| + |V|)})$$

and also

$$\text{mR}(\lambda_{\geq \ell}^k) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_{\ell}| + 1) \cdot (|\Pi \setminus J_{\ell}| + |V|)})$$

Proof. Let $k \in \mathbb{N}$. Using the terminology of the proof of Theorem 34, we know that there exists $n \leq |V|$, such that $\text{mR}(\lambda_{\ell}^k) \leq \text{mR}(\lambda_{\ell}^{j_n})$. Assume, without loss of generality, that J_{ℓ} is not a bottom region. By inequality (22) in the same proof, we know that:

$$\text{mR}(\lambda_{\ell}^{j_n}) \leq n \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_{\ell}| + n - 1 + |\Pi \setminus J_{\ell}| \cdot (|\Pi \setminus J_{\ell}| \cdot |V|)}) \right)$$

Thus, we have:

$$\text{mR}(\lambda_\ell^k) \leq n \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| + n - 1 + |\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| + |V|)})\right)$$

As $n \leq |V|$, we obtain:

$$\text{mR}(\lambda_\ell^k) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_\ell| + 1) \cdot (|\Pi \setminus J_\ell| + |V|)})$$

Furthermore, as for every $J_{\ell'}$ such that $J_\ell \subsetneq J_{\ell'}$, we have $|J_{\ell'}| > |J_\ell|$ and $\text{mR}(\lambda_{\ell'}^k) \leq \mathcal{O}(|V|^{(|\Pi \setminus J_{\ell'}| + 1) \cdot (|\Pi \setminus J_{\ell'}| + |V|)})$, we also obtain that $\text{mR}(\lambda_{\geq \ell}^k) \leq \text{mR}(\lambda_\ell^k)$. \blacktriangleleft

► **Corollary 37.** *Let $v \in V^X$ with $I(v) = J_\ell$ with $\ell < N$. Let $k \in \mathbb{N}$ such that $k > k_{\ell+1}^*$. Suppose there exists $c \in \mathbb{N}$ such that $\sup \{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v)\} = c$. Then, the following holds:*

$$c \leq \mathcal{O}(|V|^{(|\Pi \setminus J_\ell| + 1) \cdot (|\Pi \setminus J_\ell| + |V|)}).$$

Proof. Let $v \in V^X$ with $I(v) = J_\ell$ and $k \in \mathbb{N}$. Suppose there exists $c \in \mathbb{N}$ such that $\sup \{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v)\} = c$. Then, by Proposition 32, we know that:

$$c \leq |V| + 2 \cdot \text{mR}(\lambda_\ell^k) + (|\Pi| - 1) \cdot (|V| + 2 \cdot \text{mR}(\lambda_{>\ell}^k)).$$

Using the terminology of the proof of Theorem 34, we know that there exists $n \leq |V|$, such that $\text{mR}(\lambda_\ell^k) \leq \text{mR}(\lambda_\ell^{j_n})$. By inequalities (19) and 20 in the same proof, we obtain that:

$$c \leq |V| + n \cdot (1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| + n - 1 + |\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| + |V|)})) + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| + |V|) + 1})$$

and thus:

$$\begin{aligned} c &\leq (n + 1) \cdot \left(1 + |V| + \mathcal{O}(|V|^{|\Pi \setminus J_\ell| + n - 1 + |\Pi \setminus J_\ell| \cdot (|\Pi \setminus J_\ell| + |V|)})\right) \\ &\leq \mathcal{O}(|V|^{(|\Pi \setminus J_\ell| + 1) \cdot (|\Pi \setminus J_\ell| + |V|)}) \end{aligned}$$

\blacktriangleleft

5 PSPACE completeness

In this section, we prove Theorem 4. We first prove that the constraint problem is in PSPACE and then that it is PSPACE-hard.

5.1 PSPACE easyness

The purpose of this section is to prove that determining if, given a reachability game (\mathcal{G}, v_0) and two thresholds $x, y \in (\mathbb{N} \cup \{+\infty\})^{|\Pi|}$, there exists an SPE σ in this game such that for all $i \in \Pi$, $x_i \leq \text{Cost}_i(\langle \sigma \rangle_{v_0}) \leq y_i$ can be done in PSPACE (see Theorem 4).

► **Proposition 38.** *The constraint problem for quantitative reachability games is in PSPACE.*

Let us provide a high level sketch of the proof of our PSPACE procedure for this constraint problem. Thanks to Theorem 20, solving the constraint problem reduced in finding a λ^* -consistent play ρ in (\mathcal{X}, x_0) satisfying the constraints. By Lemmas 30 and 31, the latter problem reduces in finding a valid path ρ^C in the counter graph $\mathbb{C}(\lambda^*)$, satisfying the constraints. Roughly speaking, in order to solve our initial problem, it suffices to decide the existence of a valid path that is a lasso and satisfies the constraints in the counter graph, which is exponential in the size of our original input (by Corollary 35). Classical arguments, using Savitch's Theorem can thus be used to prove the PSPACE membership. Nevertheless, the detailed proof is more intricate for two reasons. The first reason is that the counter graph is based on the labeling function λ^* . We thus also have to prove that the labeling function λ^* can be computed in PSPACE. The second reason is that, a priori, although we know that the counter graph is of exponential size, we do not know explicitly its size. This is problematic when using classical NPSpace algorithms that guess some path in a graph of exponential size, where a counter bounded by the size of the graph is needed to guarantee the termination of the procedure. In order to overcome this, we also need a PSPACE procedure to obtain the actual size of the counter graph.

The whole procedure works by induction on k , the steps in the computation of the labeling function λ^* . Moreover, it exploits the structural evolution of the local fixpoints formalized in Proposition 17 and especially in Lemma 18. Let X^{J_ℓ} be a region, we aim at proving that $\lambda^{k+1}(v)$ is computable in PSPACE for each $v \in V^{J_\ell}$. Let k_ℓ^* (resp. $k_{\ell+1}^*$) be the step where the fixpoint is reached for region X^{J_ℓ} (resp. $X^{J_{\ell+1}}$). Recall that $k_{\ell+1}^* < k_\ell^*$. When $k \leq k_{\ell+1}^*$ (resp. $k > k_\ell^*$), we have that $\lambda^{k+1}(v) = \lambda^k(v)$, for each $v \in V^{J_\ell}$, by Lemma 18. The tricky case is when $k_{\ell+1}^* < k \leq k_\ell^*$. (Notice the little difference with the inequalities given in Lemma 18. When $k = k_\ell^*$, we still need to compute λ^{k+1} to realize that the fixpoint is effectively reached.)

In the latter case, the computation of $\lambda^{k+1}(v)$ from $\lambda^k(v)$, for all $v \in V^{J_\ell}$, relies on the maximum of the cost of the λ^k -consistent plays (see Definition 15). The latter quantity can be bounded, once the size of the counter graph $\mathbb{C}(\lambda_{\geq \ell}^k)$ is known explicitly. Note that the counter graph $\mathbb{C}(\lambda_{\geq \ell}^k)$ has a size $|\mathbb{C}(\lambda_{\geq \ell}^k)|$ equal to $|V| \cdot 2^{|\mathbb{I}|} \cdot (K+1)^{|\mathbb{I}|}$ where $K = \text{mR}(\lambda_{\geq \ell}^k)$. Hence to know its size, we have to compute a bound on K . Under the induction hypothesis that (i) $\{\lambda^k(v) \mid v \in V^{J'} \text{ with } J' \geq J_\ell\}$, (ii) the maximum of the cost of the λ^k -consistent plays starting in a successor of $v \in V^{J'}$, with $J' \geq J_\ell$, and (iii) the size of $\mathbb{C}(\lambda_{\geq \ell}^k)$ can be computed in PSPACE, we manage to prove that $\lambda^{k+1}(v)$ is indeed computable in PSPACE for each $v \in V^{J_\ell}$. The correctness of the approach relies on Proposition 17 which ensures that the values of λ^* have already reached a fixpoint for all the $v \in V^{J'}$ such that $J' > J_\ell$.

Let us state and prove Proposition 39

► **Proposition 39.** *Given a quantitative game (\mathcal{G}, v_0) , for all $k \in \mathbb{N}$, for all $J_\ell \in \mathcal{I}$ (for some $\ell \in \{1, \dots, N\}$), the set $\{\lambda^k(v) \mid v \in V^{J_\ell}\}$ and the maximal finite range $\text{mR}(\lambda_{\geq \ell}^k)$ can be computed in PSPACE.*

We proceed by induction on the step $k \in \mathbb{N}$ of the algorithm and, once a step is fixed, we proceed region by region, beginning with the bottom region J_N and then proceeding bottom-up by following the total order on \mathcal{I} .

In the following proof, we have to deal with regions of the form X^J but we only have the game \mathcal{G} in input. Clearly, as there are at most $|V|$ vertices of the form (v, J) we can recover and encode the arena X^J in polynomial size memory.

Proof of Proposition 39. We proceed by induction on $k \in \mathbb{N}$.

For $k=0$, we have to prove that for all $J_\ell \in \mathcal{I}$ (for some $\ell \in \{1, \dots, N\}$): $\{\lambda^k(v) \mid v \in V^{J_\ell}\}$ and $\text{mR}(\lambda_{\geq \ell}^k)$ can be both computed in PSPACE. Let $J_\ell \in \mathcal{I}$, thanks to Definition 13, we have that either $\lambda^0(v) = 0$ if $v \in V_i$ and $i \in J_\ell$ or $\lambda^0(v) = +\infty$ otherwise. So, we only have to check if $i \in J_\ell$ and this can be done in PSPACE. Finally, $\text{mR}(\lambda_{\geq \ell}^0) = 0$ because we are at the initial step.

Now, assume that for all $0 \leq n \leq k$, for all $J_\ell \in \mathcal{I}$ (for some $\ell \in \{1, \dots, N\}$) the set $\{\lambda^n(v) \mid v \in V^{J_\ell}\}$ and the value $\text{mR}(\lambda_{\geq \ell}^n)$ can be computed in PSPACE. And we prove that it remains true for $n = k+1$.

We proceed by induction on the region X^{J_ℓ} . If $J_\ell = J_N$, then it is a bottom region. Then for all $v \in V^{J_N}$, $\lambda^{k+1}(v) = \lambda^0(v)$ and so $\{\lambda^{k+1}(v) \mid v \in V^{J_N}\} = \{\lambda^0(v) \mid v \in V^{J_N}\}$ (local fixpoint $k_N^* = 0$ in Proposition 17), and we know that this latter set can be computed in PSPACE. Moreover, as J_N is a bottom region, we have that $\text{mR}(\lambda_{\geq N}^{k+1}) = \text{mR}(\lambda_N^{k+1}) = 0$.

Now, let $J_\ell \in \mathcal{I}$ be a region different of J_N . If it is a bottom region, the previous arguments hold. Otherwise, we assume that we know (i) $\{\lambda^k(v) \mid v \in V^{J_\ell}\}$, and we can store it in polynomial size memory (ii) by induction hypothesis, we can compute $\{\lambda^k(v) \mid v \in V^{J'}\}$, for all $J' \geq J_\ell$, and $\text{mR}(\lambda_{\geq \ell}^k)$ in PSPACE (see Figure 4) and we prove that we can compute

$$\{\lambda^{k+1}(v) \mid v \in V^{J_\ell}\} \text{ and } \text{mR}(\lambda_{\geq \ell}^{k+1}) \text{ in PSPACE} \quad (24)$$

First, let us recall that thanks to Proposition 17, we know that Algorithm 1 reaches for each region a local fixpoint. We denote by k_ℓ^* (resp. $k_{\ell+1}^*$), the step after which the region X^{J_ℓ} (resp. $X^{J_{\ell+1}}$) has reached its fixpoint. Moreover, we know that $k_{\ell+1}^* < k_\ell^*$.

If $k \leq k_{\ell+1}^*$, then by Proposition 17 the region $J_{\ell+1}$ has not reached his local fixpoint yet. So, it implies that the values of the vertices in J_ℓ have not change since initialization (see Lemma 18). More formally, for all $v \in V^{J_\ell}$, either $\lambda^{k+1}(v) = 0$ if $i \in J_\ell$ or $\lambda^{k+1}(v) = +\infty$ otherwise. So $\{\lambda^{k+1}(v) \mid v \in V^{J_\ell}\}$ can be computed and stored in polynomial size memory. It remains to prove that $\text{mR}(\lambda_{\geq \ell}^{k+1})$ can be computed in PSPACE. Clearly we have that $\text{mR}(\lambda_\ell^{k+1}) = 0$ and we can compute $\text{mR}(\lambda_{\geq \ell}^{k+1}) = \text{mR}(\lambda_{\geq \ell+1}^{k+1})$ in PSPACE by induction hypothesis and this value is at most exponential in the size of the input (by Corollary 36). So, as $\text{mR}(\lambda_{\geq \ell}^{k+1}) = \max(\text{mR}(\lambda_\ell^{k+1}), \text{mR}(\lambda_{\geq \ell+1}^{k+1}))$, assertion (24) is proved.

If $k_{\ell+1}^* < k \leq k_\ell^*$, the step $k+1$ of the iterative computation of the operator λ corresponds to a step during which the value of λ for the vertices in V^{J_ℓ} can change or if $k = k_\ell^*$ we realize that the fixpoint is effectively reached. The fixed point is reached for all the regions $J' > J_\ell$ and is not reached yet for J^ℓ . This is the most difficult case of this induction.

For all $v \in V^{J_\ell}$, if $v \in V_i$, then, by definition, $\lambda^{k+1}(v)$ is either equal to 0 (if $i \in J_\ell$) or depends on the computation of $\sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v')\}$ with $v' \in \text{Succ}(v)$. We know by Corollary 36 that for all $v \in V^{J_\ell}$ the value λ^{k+1} is at most exponential in the size of the input. It remains to prove that $\sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v')\}$ can be computed in PSPACE.

In order to prove this we need to guess lassoes of length at most $2 \cdot |\mathbb{C}(\lambda_{\geq \ell}^k)|$ such that $|\mathbb{C}(\lambda_{\geq \ell}^k)| = (|V| \cdot 2^{|V|} \cdot (K+1)^{|\Pi|})$ where $K = \text{mR}(\lambda_{\geq \ell}^k)$. (Because a path beginning in X^{J_ℓ} will only reach vertices in $X^{\geq J_\ell}$ so we only have to consider the part of the counter graph which correspond to $X^{\geq J_\ell}$.) We know by induction hypothesis that we can compute K in PSPACE and that its value is at most exponential in the size of the input (Corollary 36), so we can encode K with a polynomial space memory.

We proceed as follows: first recall by Corollary 22 that $M = \sup\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v)\} = \max\{\text{Cost}_i(\rho) \mid \rho \in \Lambda^k(v)\}$. We guess this value M , for which we know that either $M = +\infty$ or $M = c < +\infty$. We first test if $M = +\infty$, and then if $M = c$.

- If $M = +\infty$: by Proposition 21, we can find a lasso $\pi = hg^\omega$ of length at most $2 \cdot |\mathbb{C}(\lambda_{\geq \ell}^k)|$ in $\mathbb{C}(\lambda_{\geq \ell}^k)$ such that if ρ^X is the corresponding λ^k -consistent play in the extended game, then $\text{Cost}_i(\rho) = +\infty$. To guess π we need to know the values of λ^k for all vertices $v \in X^{\geq J_\ell}$. It is not possible to maintain all these values in memory because there is maybe an exponential number of them. So, we have to proceed in a smarter way. We know by Lemma 9 that π can be decomposed according to the regions, *i.e.*, $\pi = \pi[\ell]\pi[\ell+1] \dots \pi[t]$ for some $t \in \{\ell, \dots, N\}$, where possibly some sections are empty. So, we first guess the first vertex of g , the vertex on which there will be a cycle. And then we guess successively $\pi[\ell], \pi[\ell+1]$ and so on. Additionally, we keep a counter C_L (which is initialized to 0 and incremented by 1 each time we guess a new vertex of π) and a counter C_T (which is equal to 0 as long as player i does not reach his target set and is equal to 1 after that). To guess a vertex $(v', J_m, (c'_i)_{i \in \Pi})$ of $\pi[m]$ with $\ell \leq m \leq t$, assuming $\pi[m]$ is not empty, we only have to keep in memory its predecessor $(v, J_n, (c_i)_{i \in \Pi})$ (with $n \leq m$) and to know the value of $\lambda^k(v', J_m)$. If $J_m = J_\ell$ it is done because we have $\{\lambda^k(v) \mid v \in V^{J_\ell}\}$ in memory. Otherwise, by induction hypothesis, we can compute $\{\lambda^k(u) \mid u \in V^{J_m}\}$ in PSPACE. Once $\pi[m]$ is guessed we can forget $\{\lambda^k(u) \mid u \in V^{J_m}\}$. We stop if we have found the lasso or if $C_L = 2 \cdot |\mathbb{C}(\lambda_{\geq \ell}^k)| + 1$. If $C_L \leq 2 \cdot |\mathbb{C}(\lambda_{\geq \ell}^k)|$ and $C_T = 0$, we have found a good lasso *i.e.*, one such that player i does not reach his target set along its corresponding play in the extended game.

To summarize, during this procedure we only keep in memory: (i) the vertex on which the lasso will loop; (ii) the vertex we are guessing and its predecessor; (iii) $\{\lambda^k(v) \mid v \in V^{J_\ell}\}$. It is at most $|V|$ values less or equal to K where K is at most exponential in the size of the input (Corollary 36); (iv) $\{\lambda^k(u) \mid u \in V^{J_m}\}$ if we are guessing $\pi[m]$. Once again at most $|V|$ values less or equal to K ; (v) a counter $C_L \leq 2 \cdot |\mathbb{C}(\lambda_{\geq \ell}^k)|$; (vi) a counter C_T which is equal either to 0 or 1. All these different values can be encoded in polynomial size memory. Notice that for (i) and (ii) a vertex is only a vertex in the game \mathcal{G} , a subset of Π and $|\Pi|$ values in $\{0, \dots, K\} \cup \{+\infty\}$.

- If $M = c < +\infty$, we know, by Corollary 37, that the size of c cannot exceed an exponential in the size of the input. We have to verify that: (i) there exists a lasso π such that the corresponding play ρ^X in the extended game is such that $\text{Cost}_i(\rho^X) = c$ and the length of π is at most $2 \cdot |\mathbb{C}(\lambda_{\geq \ell}^k)|$; (ii) there does not exist a lasso π' such that the corresponding play ρ'^X in the extended game is such that $\text{Cost}_i(\rho'^X) > c$ and the length of π' is at most $2 \cdot |\mathbb{C}(\lambda_{\geq \ell}^k)|$.

We proceed iteratively as follows: we first guess a lasso π such that $\text{Cost}_i(\rho^X) \geq d$ with $d = 0$. If it exists, we repeat the same procedure with $d = 1$. At some point, there will exist a lasso π such that $\text{Cost}_i(\rho^X) \geq d$ for some $d \in \mathbb{N}$ but there does not exist a lasso π' such that $\text{Cost}_i(\rho'^X) \geq d+1$. If d is equal to the value M that we have guessed, then we have guessed the good maximum.

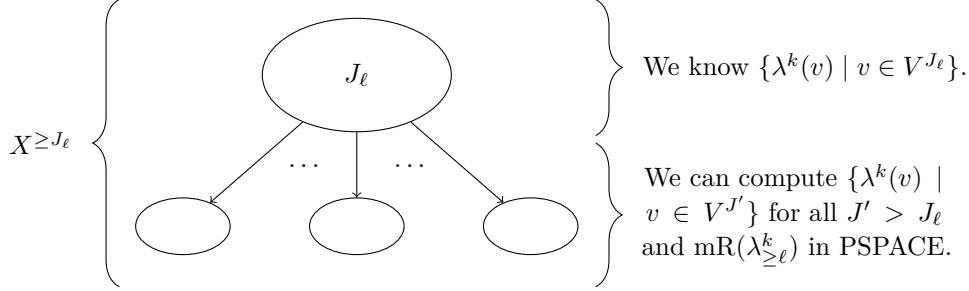
To guess the different lassoes π , we proceed as for the case $M = +\infty$ but we keep an additionnal counter C which keeps track of the cost of player i for the lasso (and so indirectly for the corresponding play in the extended game) that we are guessing. But, once again, this value is at most exponential in the size of the input.

This conclude the proof that $\{\lambda^{k+1}(v) \mid v \in V^{J_\ell}\}$ can be computed in PSPACE.

To conclude the global proof, it remains to proof that $\text{mR}(\lambda_{\geq \ell}^{k+1})$ can be computed in PSPACE. Clearly we can compute $\text{mR}(\lambda_{\geq \ell}^{k+1})$ in PSPACE as we now have $\{\lambda^{k+1}(v) \mid v \in V^{J_\ell}\}$ in memory and we can compute $\text{mR}(\lambda_{\geq \ell}^{k+1}) = \text{mR}(\lambda_{\geq \ell+1}^{k+1}) = \text{mR}(\lambda_{\geq \ell+1}^k)$ (as $k > k_{\ell+1}^*$ the region $X^{J_{\ell+1}}$ has already reached its local fixpoint) in PSPACE by induction hypothesis and this value is at most exponential in the size of the input (by Corollary 36). So, as

$\text{mR}(\lambda_{\geq \ell}^{k+1}) = \max(\text{mR}(\lambda_{\ell}^{k+1}), \text{mR}(\lambda_{\geq \ell}^{k+1}))$, assertion (24) is proved.

If $k > k_{\ell}^*$, the local fixpoint of region $X^{J_{\ell}}$ is reached and then by Proposition 17 $\{\lambda^{k+1}(v) \mid v \in V^{J_{\ell}}\} = \{\lambda^k(v) \mid v \in V^{J_{\ell}}\}$ and $\text{mR}(\lambda_{\geq \ell}^{k+1}) = \text{mR}(\lambda_{\geq \ell}^k)$ and (24) is proved. ◀



■ **Figure 4** Induction hypothesis for the computation of λ^{k+1} .

We are now able to prove Proposition 38.

Proof of Proposition 38. Let (\mathcal{G}, v_0) be an initialized reachability game and let $x, y \in (\mathbb{N} \cup \{+\infty\})^{|\Pi|}$ be two thresholds. Let (\mathcal{X}, x_0) be its extended game with $x_0 = (v_0, I_0)$. Let $\mathbb{C}(\lambda_{\geq I_0}^*)$ be the counter graph restricted to the arena $X \geq I_0$ and so $|\mathbb{C}(\lambda_{\geq I_0}^*)| = |V| \cdot 2^{|\Pi|} \cdot (K^0 + 1)^{|\Pi|}$ with $K^0 = \text{mR}(\lambda_{\geq I_0}^*)$. Let us recall that K^0 can be computed (resp. encoded) in PSPACE thanks to Proposition 39 (resp. Corollary 36).

We have to guess a lasso $\pi = hg^\omega$ in the counter graph $\mathbb{C}(\lambda_{\geq I_0}^*)$ such that its corresponding play ρ^X in the extended game satisfies the constraints, *i.e.*, $x_i \leq \text{Cost}_i(\rho^X) \leq y_i$ for all $i \in \Pi$. The length L of π is at most $\max(\max\{y_i \mid y_i < +\infty\}, |\mathbb{C}(\lambda_{\geq I_0}^*)|) + 2 \cdot |\mathbb{C}(\lambda_{\geq I_0}^*)|$ as the cycle of π can appear after the constraints $y_i < +\infty$ are satisfied.

In order to have a PSPACE procedure, we cannot guess π entirely and we have to proceed region by region. If $I_0 = J_{\ell}$ for some $\ell \in \{1, \dots, N\}$, we consider the region decomposition $\pi[\ell]\pi[\ell+1] \dots \pi[t]$ of π , with $t \in \{\ell, \dots, N\}$, where some sections $\pi[m]$ may be empty.

We first guess the first vertex of the cycle g and then we guess successively $\pi[\ell]\pi[\ell+1]$ and so on. To guess $\pi[m]$ with $\ell \in \{m, \dots, t\}$, assuming it is not empty, we guess one by one its vertices. To guess a vertex $(v', J_m, (c'_i)_{i \in \Pi})$ we only have to keep its predecessor in memory and to know the value $\lambda^*(v', J_m)$. So, we need to know $\{\lambda^*(v) \mid v \in V^{J_m}\}$. Thanks to Proposition 39, we can obtain this set in polynomial size memory and once we move to another region, we can forget this set and compute the new one.

Additionally, we have a counter C_L to count the length of π and for each player $i \in \Pi$ we have a counter C_i keeping track the current cost of player i along π . For all $i \in \Pi$, we have that $C_i, C_L \leq L$. Henceforth, all these counters can be encoded with polynomial size memory. Moreover, in addition to these counters, we currently maintain: (i) the first vertex of g , (ii) the current vertex we are guessing, (iii) its predecessor and, (iv) the set $\{\lambda^*(v) \mid v \in V^{J_m}\}$ (only $|V|$ values which can be encoded in polynomial size memory by Corollary 36) if we are guessing $\pi[m]$. As for (ii) and (iii) a vertex in the counter graph is composed of a vertex of V , a subset I of Π and $|\Pi|$ counter values which are at most exponential in the input (Corollary 36), all this procedure can be done in PSPACE. ◀

5.2 PSPACE hardness

We now prove that the constraint problem is PSPACE-hard for quantitative reachability games.

► **Proposition 40.** *The constraint problem for quantitative reachability games is PSPACE-hard.*

The proof of this proposition is based on a polynomial reduction from the QBF problem which is PSPACE-complete. It is close to the proof given in [10] for the PSPACE-hardness of the constraint problem for Boolean games with reachability objectives – either a player reaches his objective or not. The main difference is here to manipulate costs instead of considering qualitative reachability.

The QBF problem is to decide whether a fully quantified Boolean formula ψ is true. The formula ψ can be assumed to be in prenex normal form $Q_1x_1Q_2x_2\ldots Q_mx_m\phi(X)$ such that the quantifiers are alternating existential and universal quantifiers ($Q_1 = \exists$, $Q_2 = \forall$, $Q_3 = \exists, \dots$), $X = \{x_1, x_2, \dots, x_m\}$ is the set of quantified variables, and $\phi(X) = C_1 \wedge \dots \wedge C_n$ is an unquantified Boolean formula over X equal to the conjunction of the clauses C_1, \dots, C_n .

Such a formula ψ is true if there exists a value of x_1 such that for all values of x_2 , there exists a value of $x_3 \dots$, such that the resulting valuation ν of all variables of X evaluates $\phi(X)$ to true. Formally, for each odd (resp. even) k , $1 \leq k \leq m$, let us denote by $f_k : \{0, 1\}^{k-1} \rightarrow \{0, 1\}$ (resp. $g_k : \{0, 1\}^{k-1} \rightarrow \{0, 1\}$) a valuation of variable x_k given a valuation of previous variables x_1, \dots, x_{k-1} ³. Given these sequences $f = f_1, f_3, \dots$ and $g = g_2, g_4, \dots$, let us denote by $\nu = \nu_{(f,g)}$ the valuation of all variables of X such that $\nu(x_1) = f_1$, $\nu(x_2) = g_2(\nu(x_1))$, $\nu(x_3) = f_3(\nu(x_1)\nu(x_2))$, \dots . Then

$$\psi = Q_1x_1Q_2x_2\ldots Q_mx_m\phi(X) \text{ is true} \\ \text{if and only if}$$

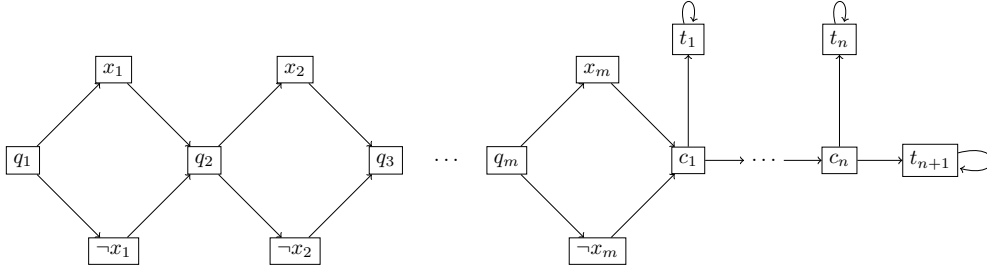
there exist $f = f_1, f_3, \dots$ such that for all $g = g_2, g_4, \dots$, the valuation $\nu_{f,g}$ evaluates $\phi(X)$ to true.

Proof of Proposition 40. Let us detail a polynomial reduction from the QBF problem to the constraint problem for quantitative reachability games. Let $\psi = Q_1x_1Q_2x_2\ldots Q_mx_m\phi(X)$ with $\phi(X) = C_1 \wedge \dots \wedge C_n$ be a fully quantified Boolean formula in prenex normal form. We build the following quantitative reachability game $\mathcal{G}_\psi = (\Pi, V, (V_i)_{i \in \Pi}, E, (\text{Cost}_i)_{i \in \Pi}, (F_i)_{i \in \Pi})$ (see Figure 5):

- the set V of vertices:
 - for each variable $x_k \in X$ under quantifier Q_k , there exist vertices x_k , $\neg x_k$ and q_k ;
 - for each clause C_k , there exist vertices c_k and t_k ;
 - there exists an additional vertex t_{n+1} ;
- the set E of edges:
 - from each vertex q_k there exist an edge to x_k and an edge to $\neg x_k$;
 - from each vertex x_k and $\neg x_k$, there exists an edge to q_{k+1} , except for $k = m$ where this edge is to c_1 ;
 - from each vertex c_k , there exist an edge to t_k and an edge to c_{k+1} , except for $k = n$ where there exist an edge to t_n and an edge to t_{n+1} ;
 - there exists a loop on each t_k ;

³ Notice that $f_1 : \emptyset \rightarrow \{0, 1\}$.

- the set Π of $n + 2$ players:
 - each player i , $1 \leq i \leq n$, owns vertex c_i ;
 - player $n + 1$ (resp. $n + 2$) is the player who owns the vertices q_i for each existential (resp. universal) quantifier Q_i ;
 - as all other vertices have only one outgoing edge, it does not matter which player owns them;
- each function Cost_i is associated with the target set F_i defined as follows:
 - for all i , $1 \leq i \leq n$, $F_i = \{\ell \in V \mid \ell \text{ is a literal of clause } C_i\} \cup \{t_i\}$;
 - $F_{n+1} = \{t_{n+1}\}$;
 - $F_{n+2} = \{t_1, \dots, t_n\}$.



■ **Figure 5** Reduction from the formula ψ to the quantitative reachability game \mathcal{G}_ψ

► **Remark 41.** (1) Notice that a sequence f of functions $f_k : \{0, 1\}^{k-1} \rightarrow \{0, 1\}$, with k odd, $1 \leq k \leq m$, as presented above, can be translated into a strategy σ_{n+1} of player $n + 1$ in the initialized game (\mathcal{G}_ψ, q_1) , and conversely. Similarly, a sequence g of functions $g_k : \{0, 1\}^{k-1} \rightarrow \{0, 1\}$, with k even, $1 \leq k \leq m$ is nothing else than a strategy σ_{n+2} of player $n + 2$. (2) Notice also that if ρ is a play in (\mathcal{G}_ψ, q_1) , then $\text{Cost}_{n+1}(\rho) < +\infty$ if and only if $\text{Cost}_{n+2}(\rho) = +\infty$. Moreover, suppose that ρ visits t_{n+1} , then for all i , $1 \leq i \leq n$, $\text{Cost}_i(\rho) \leq 2 \cdot m$ if and only if for all i , $1 \leq i \leq n$, ρ visits a vertex that is a literal of C_i , and that is the case if and only if there is a valuation of all variables of X that evaluates $\phi(X)$ to true.

Consider the game \mathcal{G}_ψ and the bound $x = (2 \cdot m, \dots, 2 \cdot m, 2 \cdot m + n, +\infty)$. Both can be constructed from ψ in polynomial time. Let us now show that ψ is true if and only if there exists an SPE in (\mathcal{G}_ψ, q_1) with cost $\leq x$.

(\Rightarrow) Suppose that ψ is true. Then there exists a sequence f of functions $f_k : \{0, 1\}^{k-1} \rightarrow \{0, 1\}$, with k odd, $1 \leq k \leq m$, such that for all sequences g of functions $g_k : \{0, 1\}^{k-1} \rightarrow \{0, 1\}$, with k even, $1 \leq k \leq m$, the valuation $\nu_{f,g}$ evaluates $\phi(X)$ to true. We define a strategy profile σ as follows:

- for player $n + 1$, his strategy σ_{n+1} is the strategy corresponding to the sequence f (by Remark 41);
- for player $n + 2$, his strategy is an arbitrary strategy σ_{n+2} ; we denote by g the corresponding sequence $g_k : \{0, 1\}^{k-1} \rightarrow \{0, 1\}$, with k even, $1 \leq k \leq m$ (by Remark 41);
- for each player i , $1 \leq i \leq n$,
 - if $hv \in \text{Hist}_i(q_1)$ with $v = c_i$, is consistent with σ_{n+1} , then $\sigma_i(hv) = c_{i+1}$ if $i \neq n$ and t_{n+1} otherwise
 - else $\sigma_i(hv) = t_i$.

Let us first prove that the play $\rho = \langle \sigma \rangle_{q_1}$ has a cost $\leq x = (2 \cdot m, \dots, 2 \cdot m, 2 \cdot m + n, +\infty)$. By hypothesis, the valuation $\nu_{f,g}$ evaluates $\phi(X)$ to true, that is, it evaluates all clauses C_i to true. Hence by Remark 41, ρ visits a vertex of F_i for all i , $1 \leq i \leq n$, and by definition of σ , ρ eventually loops on t_{n+1} . It follows that $\text{Cost}_i(\rho) \leq 2 \cdot m$ for all i , $1 \leq i \leq n$, $\text{Cost}_{n+1}(\rho) \leq 2 \cdot m + n$, and $\text{Cost}_{n+2}(\rho) = +\infty$. Hence $\text{Cost}(\rho) \leq x$.

Let us now prove that σ is an SPE, that is, for each history $hv \in \text{Hist}(q_1)$, there is no one-shot deviating strategy in the subgame $(\mathcal{G}_{\psi|h}, v)$ that is profitable to the player who owns vertex v (by Proposition 6). This is clearly true for all $v = t_i$, $1 \leq i \leq n+1$, since t_i has only one outgoing edge. For the other vertices v , we study two cases:

- hv is consistent with σ_{n+1} : Notice that hv is maybe not consistent with σ_{n+2} , but with another arbitrary strategy σ'_{n+2} . Let g' be the sequence corresponding to σ'_{n+2} by Remark 41. By hypothesis, the valuation $\nu_{f,g'}$ evaluates $\phi(X)$ to true. Hence as explained previously for $\langle \sigma \rangle_{q_1}$, the cost of play $\rho = h\langle \sigma|_h \rangle_v$ is such that $\text{Cost}_i(\rho) \leq 2 \cdot m$ for all i , $1 \leq i \leq n$, $\text{Cost}_{n+1}(\rho) \leq 2 \cdot m + n$, and $\text{Cost}_{n+2}(\rho) = +\infty$. If v belongs to player i , $1 \leq i \leq n$, this player has no incentive to deviate since he has already visited his target set along h and thus cannot decrease his cost. If v belongs to player $n+1$, a one-shot deviation will lead to a play eventually looping on t_1 by definition of σ , thus leading to a cost $+\infty$ which is not profitable for player $n+1$. Finally if v belongs to player $n+2$, a one-shot deviation will not decrease his cost by definition of σ (σ_{n+2} is arbitrary).
- hv is not consistent with σ_{n+1} : Suppose that $v = c_k$. Then by definition of σ , the play $h\langle \sigma|_h \rangle_v$ eventually loops on t_k leading to a cost $\leq 2 \cdot m + k$ for player k . In fact, if player k has already seen his target set along hv , using a one-shot deviation in the subgame $(\mathcal{G}_{\psi|h}, v)$ leads to the same cost for him. Otherwise, it leads to a cost equal to $+\infty$: indeed, deviating here means going to the state c_{k+1} (or t_{n+1} if $k = n$, which leads to a cost of $+\infty$ for player n), and since hv is not consistent with σ_{n+1} , by definition of σ_{k+1} , player $k+1$ will choose to go to t_{k+1} . This player has thus no incentive to deviate. Suppose that $v = q_k$. Then by definition of σ , the play $\rho = h\langle \sigma|_h \rangle_v$ eventually loops on t_1 . It follows that $\text{Cost}_{n+1}(\rho) = +\infty$ and $\text{Cost}_{n+2}(\rho) = 2 \cdot m + 1$. Due to the structure of the game graph, $2 \cdot m + 1$ is the smallest cost that player $n+2$ is able to obtain. So if $q_k \in V_{n+2}$, player $n+2$ has no incentive to deviate. And if $q_k \in V_{n+1}$, player $n+1$ could try to use a one-shot deviating strategy, however the resulting play still eventually loops on t_1 .

This proves that σ is an SPE and we already showed that its cost was bounded by x .

(\Leftarrow) Suppose that there exists an SPE σ in (\mathcal{G}_ψ, q_1) with outcome ρ such that $\text{Cost}(\rho) \leq x$. In particular $\text{Cost}_{n+1}(\rho) < +\infty$. By Remark 41, it follows that $\text{Cost}_{n+2}(\rho) = +\infty$. We have to prove that ψ is true. To this end, consider the sequence f of functions $f_k : \{0, 1\}^{k-1} \rightarrow \{0, 1\}$, with k odd, $1 \leq k \leq m$, that corresponds to strategy σ_{n+1} of player $n+1$ by Remark 41. Let us show that for all sequences g of functions $g_k : \{0, 1\}^{k-1} \rightarrow \{0, 1\}$, with k even, $1 \leq k \leq m$, the valuation $\nu_{f,g}$ evaluates $\phi(X)$ to true.

By contradiction assume that it is not the case for some sequence g' and consider the related strategy σ'_{n+2} of player $n+2$ by Remark 41. Notice that σ'_{n+2} is a finitely deviating strategy. Let us consider the outcome ρ' of the strategy profile $(\sigma'_{n+2}, \sigma_{-(n+2)})$ from q_1 . As $\text{Cost}_{n+2}(\rho) = +\infty$, we must have $\text{Cost}_{n+2}(\rho') = +\infty$, otherwise σ'_{n+2} is a profitable deviation for player $n+2$ whereas σ is an SPE. It follows that $\text{Cost}_{n+1}(\rho') < +\infty$ by Remark 41, that is, ρ' eventually loops on t_{n+1} .

Now recall that the valuation $\nu_{f,g'}$ evaluates $\phi(X)$ to false, which means that it evaluates some clause C_k of $\phi(X)$ to false. Consider the history $hc_k < \rho'$. As strategy σ'_{n+2} only acts on the left part of the underlying graph of \mathcal{G}_ψ , we have $\rho' = \langle \sigma'_{n+2}, \sigma_{-(n+2)} \rangle_{q_1} = h\langle \sigma_{\uparrow h} \rangle_{c_k}$. In the subgame $(\mathcal{G}_{\psi \uparrow h}, c_k)$, the outcome of $\sigma_{\uparrow h}$ gives a cost of $+\infty$ to player k because $\rho' = h\langle \sigma_{\uparrow h} \rangle_{c_k}$ does not visit t_k and $\nu_{f,g'}$ evaluates C_k to false. In this subgame, player k has thus a profitable one-shot deviation that consists to move to t_k . It follows that σ is not an SPE which is impossible. Then ψ is true. ◀

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