Independence of Intuitionistic Propositional Connectives in Coq

Hai Dang

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Abstract

This report describes the contructive proof for the independence of intuitionistic propositional connectives, that is the undefinability of each connective by the other connectives. The proof is based on the counter-examples originally by McKinsey for Heyting algebra semantics of intuitionistic logic. The main result is the Coq formalization.

1 Introduction

While the classical logic, certain connectives can be defined by others, for example $a \wedge b = \neg(\neg a \vee \neg b)$, the situation is not the same for intuitionistic logic. McKinsey [1939] gave the counter-examples as Heyting algebras for the independence of intuitionistic propositional connectives. Heyting algebras can be used as one of the complete semantics for intuitionistic logic. In this report we formulate intuitionistic logic in the natural deduction style, prove soundness of the system with respect to the Heyting's semantics, and then prove the undefinability results of all four connectives and constants (conjuction, disjunction, implication and falsehood) using McKinsey's three algebras. The formalization was completed in Coq.

2 Definitions

Definition 1 (Propositional formulas)**.** The grammar of propositional formulas is

$$s, t ::= x | s \to t | s \wedge t | s \vee t | \bot$$

where x ranges over propositional variables. Define $\neg s = s \to \bot$ and $s \leftrightarrow t = s \to t \wedge t \to s$. Γ denotes a finite list of formulas.

Definition 2 (Natural deduction system)**.** The intuitionistic natural deduction system defines the entailment relation $\Gamma \vdash s$ by rules given in Figure 1.

$$\frac{s \in \Gamma}{\Gamma \vdash s}\ \mathrm{A} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash s}\ \mathrm{E}$$

$$\frac{s, \Gamma \vdash t}{\Gamma \vdash s \to t}\ {\to}\mathrm{I} \qquad \frac{\Gamma \vdash s \to t \qquad \Gamma \vdash s}{\Gamma \vdash t}\ {\to}\mathrm{E}$$

$$\frac{\Gamma \vdash s \qquad \Gamma \vdash t}{\Gamma \vdash s \wedge t}\ {\wedge}\mathrm{I} \qquad \frac{\Gamma \vdash s \wedge t \qquad s, t, \Gamma \vdash u}{\Gamma \vdash u}\ {\wedge}\mathrm{E}$$

$$\frac{\Gamma \vdash s}{\Gamma \vdash s \vee t}\ {\vee}\mathrm{IL} \qquad \frac{\Gamma \vdash t}{\Gamma \vdash s \vee t}\ {\vee}\mathrm{IR}$$

$$\frac{\Gamma \vdash s \vee t \qquad s, \Gamma \vdash u \qquad t, \Gamma \vdash u}{\Gamma \vdash u}\ {\vee}\mathrm{E}$$

Figure 1: Rules for the Intuitionistic Natural Deduction system

Definition 3 (Heyting algebra)**.** A *Heyting algebra* is a *preorder* $(H, \leq)$ with a smallest element $\bot$ and a largest element $\top$ and three operations $\wedge$, $\vee$, and $\to$ satisfying the following conditions for all $x, y, z \in H$:

(i) $x \leq \top$

(ii) $\bot \leq x$

(iii) $z \leq x \wedge y$ iff $z \leq x$ and $z \leq y$

(iv) $x \vee y \leq z$ iff $x \leq z$ and $y \leq z$

(v) $z \leq x \to y$ iff $z \wedge x \leq y$.

Definition 4 (Valuation)**.** A valuation of a Heyting algebra H is a function $V : P \mapsto H$ that assigns to each propositional variable a specific element of the algebra, where P is the infinite set of propositional variables. The valuation is extended to formulas recursively:

$$\begin{aligned} V(\bot) &= \bot \\ V(s \to t) &= V(s) \to V(t) \\ V(s \wedge t) &= V(s) \wedge V(t) \\ V(s \vee t) &= V(s) \vee V(t) \end{aligned}$$

Note that $\wedge$, $\vee$, $\to$, and $\bot$ on the left-hand side are the connectives and Falsehood constant of the logic, while on the right-hand side are the operations and smallest element of a Heyting algebra, respectively.

A valuation of a list of formulas $V(\Gamma)$ is defined as the valuation of the conjunction of the formulas in that list:

$$\begin{aligned} V(nil) &= \top \\ V(s, \Gamma) &= V(s) \wedge V(\Gamma) \end{aligned}$$

Definition 5 (Heyting entailment)**.** We say that s is H-entailed by Γ, denoted $\Gamma \vDash_H s$, if $V(\Gamma) \leq V(s)$ for *all* valuations V of H.

Lemma 6 (Soundness [1])**.** *Given H an arbitrary Heyting algebra, if $\Gamma \vdash s$, then $\Gamma \vDash_H s$.*

Proof. By induction on the derivation $\Gamma \vdash s$. All the cases make use of the properties of H (properties of the partial order and (i)-(v)). For the rule A, another nested induction on the list Γ is needed. □

Collorary 7 (Semantics Equivalence)**.** If $\vdash s \leftrightarrow t$, then for any Heyting algebra H and its valuation V, $V(s) \leq V(t)$ and $V(t) \leq V(s)$.

Proof. By the soundness lemma we have $\vDash_H s \leftrightarrow t$, therefore $V(nil) = \top \leq V(s \leftrightarrow t) = V(s \rightarrow t) \wedge V(t \rightarrow s)$. Then we have $\top \leq V(s \rightarrow t)$ and $\top \leq V(t \rightarrow s)$, which lead to $V(s) \leq V(t)$ and $V(t) \leq V(s)$. □

Lemma 8 (Completeness [2])**.** *If $\Gamma \vDash_H s$ for any Heyting algebra H, then $\Gamma \vdash s$.*

Proof. We prove completeness by proving an equivalent statement: *if $\top \leq V(s)$ for any Heyting algebra H and any valuation V of H, then $\vdash s$.*

We construct a Heyting algebra H_s of *formulas* where $s \leq t = \vdash s \rightarrow t$. The bottom element $\bot$ and operations $\wedge$, $\vee$, $\rightarrow$ of H_s are respectively the constant $\bot$ and connectives $\wedge$, $\vee$, $\rightarrow$ of formulas. It is easy to see that H_s is a Heyting algebra. We use the valuation $V(x) = x$. By induction $V(s) = s$. Now, if $\top \leq V(s)$, then $\top \leq s$, which means that $\vdash \top \rightarrow s$. Therefore $\vdash s$. □

Definition 9 ($\circ$-free)**.** A formula s is called $\circ$-free, where $\circ$ is a placeholder for some connective or constant, if $\circ$ does not appear in s. For example, that s is $\wedge$-free is defined recursively as:

(i) if s is a propositional variable, then s is $\wedge$-free

(ii) if $s = \bot$, then s is $\wedge$-free

(iii) if $s = s_1 \vee s_2$, then s is $\wedge$-free if s_1 and s_2 are $\wedge$-free

(iv) if $s = s_1 \rightarrow s_2$, then s is $\wedge$-free if s_1 and s_2 are $\wedge$-free.

McKinsey [1939] gave three counter-examples to prove independence of the 4 connectives of intuitionistic propositional logic, of which the first is for the negation, the second is for the disjunction, and the third is for implication and conjunction connectives. McKinsey considered negation as one of the primitive connective, while here we use the Falsehood constant and implication to define negation. The counter-example for negation, however, still works in the same way. The next sections follow the order of the three counter-examples.

[1] see *nd_soundHA* in the formalization.
[2] see *HA_iff_nd* in the formalization.

3 Independence of Falsehood

The first Heyting algebra H_1 by McKinsey is given as the lattice in Figure 2, and its corresponding operations in Table 1.

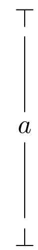

Figure 2: The partial order for H_1

Table 1: Operations for H_1

(a)

$\to$	$\top$	a	$\bot$
$\top$	$\top$	a	$\bot$
a	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\top$

(b)

$\wedge$	$\top$	a	$\bot$
$\top$	$\top$	a	$\bot$
a	a	a	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

(c)

$\vee$	$\top$	a	$\bot$
$\top$	$\top$	$\top$	$\top$
a	a	a	a
$\bot$	$\top$	a	$\bot$

Proposition 10. H_1 with its operations is a Heyting algebra.

Proof. We have to show that H_1 is a preorder, and its operations satisfy properties (i)-(v). This can be done by enumerating all the possible values of any $x, y, z \in H_1$. Coq can check this automatically. $\square$

Now, if we use a valuation V_1 that assigns every propositional variable to $\top$, then it is clear that for any $\bot$-free formula s, $V_1(s) = \top \neq \bot = V_1(\bot)$. Therefore, $\nvDash_{H_1} s \leftrightarrow \bot$, which means that there is no $\bot$-free formula s that can replace $\bot$.

Proposition 11. If $V_1 : P \mapsto H_1$ is a valuation of H_1 that assigns any variable to $\top$, then for any $\bot$-free formula s, $V_1(s) = \top$.

Proof. By induction on the structure of s. $\square$

Lemma 12 (Independence of Falsehood [3])**.** *There is no $\bot$-free formula s such that $\vdash s \leftrightarrow \bot$.*

Proof. Assume that there is a $\bot$-free s that $\vdash s \leftrightarrow \bot$. Using V_1 we must have by Collorary 7 $V_1(s) \leq V_1(\bot) = \bot$, which contradictions Proposition 11. $\square$

[3] see *FalIndependence_hey* in the formalization.

4 Independence of Disjunction

The Heyting algebra H_2 is given as the lattice in Figure 3, and its corresponding operations in Table 2.

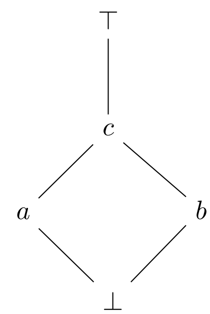

Figure 3: The partial order for H_2

Table 2: Operations for H_2

(a)

$\to$	$\top$	a	b	c	$\bot$
$\top$	$\top$	a	b	c	$\bot$
a	$\top$	$\top$	b	$\top$	b
b	$\top$	a	$\top$	$\top$	a
c	$\top$	a	b	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

(b)

$\wedge$	$\top$	a	b	c	$\bot$
$\top$	$\top$	a	b	c	$\bot$
a	a	a	$\bot$	a	$\bot$
b	b	$\bot$	b	b	$\bot$
c	c	a	b	c	$\bot$
$\bot$	$\bot$	a	$\bot$	$\bot$	$\bot$

(c)

$\vee$	$\top$	a	b	c	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\top$	a	c	c	a
b	$\top$	c	b	c	b
c	$\top$	c	c	c	c
$\bot$	$\top$	a	b	c	$\bot$

Proposition 13. H_2 with its operations is a Heyting algebra.

Proposition 14. If $V_2 : P \mapsto H_2$ is a valuation of H_2 that assigns any variable to either a or b, then for any $\vee$-free formula s, $V_2(s) \neq c$.

Proof. By induction on the structure of s. The possible values of $V_2(s)$ are marked as red in Table 2 (a) and (b). $\square$

McKinsey's original observation was that the set $\{\top, a, b, \bot\} \subseteq H_2$ is *closed* under the operations $\wedge$ and $\to$, while $a \vee b = c$ is not. Therefore V_2 is a counter-example for disjunction.

Lemma 15 (Independence of Disjunction[4])**.** *There is no $\vee$-free formula s such that $\vdash s \leftrightarrow x \vee y$ for two different variables x and y.*

Proof. Assume that there is a $\vee$-free s that $\vdash s \leftrightarrow x \vee y$. Using V_2 that assigns x to a and y to b, and any other variable to either a or b, we have $V_2(x \vee y) = V_2(a \vee b) = c$. By Collorary 7, we have $V_2(s) \leq c$ and $c \leq V_2(s)$, which contradicts Proposition 14. $\square$

[4]see *OrIndependence_hey* in the formalization.

5 Independence of Implication and Conjunction

The Heyting algebra H_3 is the direct product of H_1 to itself: $H_3 = H_1 \times H_1$. The lattice is given in Figure 4.

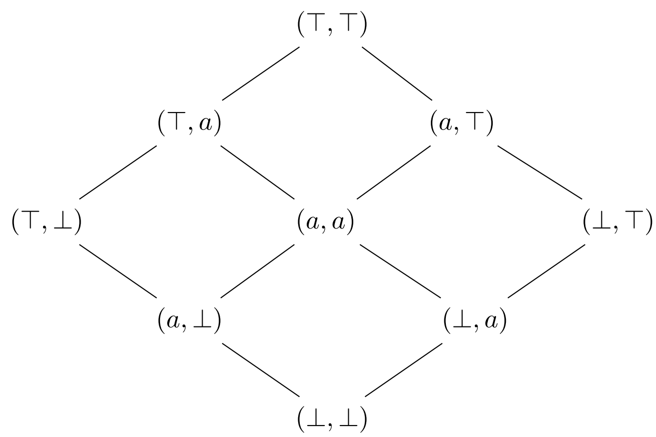

Figure 4: The partial order for H_3

The operations for $(x_1, y_1), (x_2, y_2) \in H_3$ are defined using the operations of H_1:

(i) $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$

(ii) $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$

(iii) $(x_1, y_1) \to (x_2, y_2) = (x_1 \to x_2, y_1 \to y_2)$

Proposition 16. H_3 with its operations is a Heyting algebra.

Proposition 17. The operations $\wedge$ and $\vee$ of H_3 are closed on the set $M_{\to} = \{(\top, \top), (\top, a), (a, a), (\bot, \bot)\}$.

Lemma 18 (Independence of Implication[5])**.** *There is no $\to$-free formula s such that $\vdash s \leftrightarrow x \to y$ for two different variables x and y.*

Proof. Assume that there is a $\to$-free s that $\vdash s \leftrightarrow x \to y$. Using $V_{3\to}$ that assigns x to $(\top, a)$ and y to (a, a), and any other variable to some value in the set $M_{\to}$, we have $V_{3\to}(x \to y) = (a, \top) \notin M_{\to}$. By Collorary 7, we have $V_{3\to}(s) \leq (a, \top)$ and $(a, \top) \leq V_{3\to}(s)$, which contradicts Proposition 17. $\square$

Proposition 19. The operations $\to$ and $\vee$ of H_3 are closed on the set $M_{\wedge} = \{(\top, \top), (\top, a), (\top, \bot), (\bot, \top), (\bot, \bot)\}$.

Lemma 20 (Independence of Conjunction[6])**.** *There is no $\wedge$-free formula s such that $\vdash s \leftrightarrow x \wedge y$ for two different variables x and y.*

[5] see *ImpIndependence_hey* in the formalization.
[6] see *AndIndependence_hey* in the formalization.

Proof. Assume that there is a $\wedge$-free s that $\vdash s \leftrightarrow x \wedge y$. Using $V_{3\wedge}$ that assigns x to $(\top, a)$ and y to $(\bot, \top)$, and any other variable to some value in the set $M_\wedge$, we have $V_{3\wedge}(x \wedge y) = (\bot, a) \notin M_\wedge$. By Collorary 7 we have $V_{3\wedge}(s) \leq (\bot, a)$ and $(\bot, a) \leq V_{3\wedge}(s)$ which contradicts Proposition 19. $\square$

6 Disjunction Property

In this section we give the proof of independence of disjunction without using Heyting's semantics, but the *disjunction property* [Negri et al., 2008, p. 41]. The proof, however, is done using Gentzen's sequent system for intuitionistic logic, instead of natural deduction.

Definition 21 (Intuitionistic sequent system)**.** The intuitionistic sequent system defines the entailment relation $\Gamma \Rightarrow s$ by rules given in Figure 5.

$$\frac{s \in \Gamma}{\Gamma \Rightarrow s}\,\mathrm{A} \qquad \frac{\bot \in \Gamma}{\Gamma \Rightarrow s}\,\mathrm{E}$$

$$\frac{s, \Gamma \Rightarrow t}{\Gamma \Rightarrow s \to t}\,{\to}\mathrm{R} \qquad \frac{s \to t \in \Gamma \qquad \Gamma \Rightarrow s}{\Gamma \Rightarrow t}\,{\to}\mathrm{L}$$

$$\frac{\Gamma \Rightarrow s \qquad \Gamma \Rightarrow t}{\Gamma \Rightarrow s \wedge t}\,{\wedge}\mathrm{R} \qquad \frac{s \wedge t \in \Gamma \qquad s, t, \Gamma \Rightarrow u}{\Gamma \Rightarrow u}\,{\wedge}\mathrm{L}$$

$$\frac{\Gamma \Rightarrow s}{\Gamma \Rightarrow s \vee t}\,{\vee}\mathrm{R1} \qquad \frac{\Gamma \Rightarrow t}{\Gamma \Rightarrow s \vee t}\,{\vee}\mathrm{R2}$$

$$\frac{s \vee t \in \Gamma \qquad s, \Gamma \Rightarrow u \qquad t, \Gamma \Rightarrow u}{\Gamma \Rightarrow u}\,{\vee}\mathrm{L}$$

Figure 5: Rules for the Intuitionistic Gentzen system

Lemma 22 (Admissible rules)**.** *The Weakening and Cut rules are admissible in the intuitionistic sequent system.*

$$\frac{\Gamma \Rightarrow s \qquad \Gamma \subseteq \Delta}{\Delta \Rightarrow s}\,\textit{Weakening}$$

$$\frac{\Gamma \Rightarrow s \qquad s, \Gamma \Rightarrow t}{\Gamma \Rightarrow t}\,\textit{Cut}$$

Definition 23 (Harrop formula)**.** A Harrop formula is defined recursively as:

(i) propositional variables and the Falsehood constant $\bot$ are Harrop formulas

(ii) $s \to t$ is a Harrop formula if t is also a Harrop formula

(iii) $s \wedge t$ is a Harrop formula if both s and t are Harrop formulas.

It is easy to see that if s is $\vee$-free, then s is a Harrop formula.

Lemma 24 (Disjunction property [7]). *If every formula in* Γ *is a Harrop formula, then* $\Gamma \Rightarrow s \vee t$ *iff* $\Gamma \Rightarrow s$ *or* $\Gamma \Rightarrow t$.

Proof. The direction from right to left is straightforward. The direction from left to right is by induction on the derivation $\Gamma \Rightarrow s \vee t$. ☐

Lemma 25 (Independence of Disjunction 2 [8]). *Given two inequivalent formulas* p *and* q*, i.e.* $\not\Rightarrow p \to q$ *and* $\not\Rightarrow q \to p$*, there does not exist a* $\vee$*-free formula* s *such that* $\Rightarrow p \vee q \leftrightarrow s$.

Proof. Assume that there is a $\vee$-free s such that $\Rightarrow p \vee q \leftrightarrow s$. It is easy to see that $\Rightarrow p \vee q \leftrightarrow s$ iff $\Rightarrow p \vee q \to s$ and $\Rightarrow s \to p \vee q$. By Cut and Weakening we have $p \vee q \Rightarrow s$ and $s \Rightarrow p \vee q$. From $p \Rightarrow p \vee q$ and $p \vee q \Rightarrow s$, by Cut we have $p \Rightarrow s$, and similarly, $q \Rightarrow s$. Since s is $\vee$-free, it is also a Harrop formula, therefore by the disjunction property $s \Rightarrow p$ *or* $s \Rightarrow q$. In either case, from $p \Rightarrow s$ and $q \Rightarrow s$, again by Cut we have either $p \Rightarrow q$ or $q \Rightarrow p$, which contradicts the assumption that p and q are inequivalent. ☐

7 Remarks

Remark 1 (Underivability results). H_1 and V_1 can be used to disprove $x \vee \neg x$ and $\neg\neg x \to x$. H_2 and V_2 can be used to disprove $\neg x \vee \neg\neg x$ and $\neg(x \wedge y) \to \neg x \vee \neg y$.

Remark 2. In the Heyting algebra definition, $\top$ and property (i) can be removed and replaced by the definition $\top = \bot \to \bot$. Property (i) then is implied from properties (iii) and (v).

Remark 3. In the original definition of Heyting algebra, the order is a partial order while here we use a preorder, since the soundness proof does not need antisymmetry. This is also observed by Brown [2014]. If antisymmetry is accepted, then in Collorary 7 we can replace $V(s) \leq V(t)$ and $V(t) \leq V(s)$ with $V(s) = V(t)$.

Remark 4. Here we have only formalized the completeness proof for preordered Heyting algebras basing on Troelstra and Dalen [1988], in which a stronger proof for partial-ordered Heyting algebras is provided. The authors also observe that completeness holds for finite Heyting algebras.

References

C. E. Brown. Semantics of intuitionistic propositional logic: Heyting algebras and Kripke models. 2014. URL https://www.ps.uni-saarland.de/HeytingKripke/description.html.

J. C. C. McKinsey. Proof of the Independence of the Primitive Symbols of Heyting's Calculus of Propositions. *Journal of Symbolic Logic*, 4(4):155–158, 1939.

Sara Negri, Jan von Plato, and Aarne Ranta. *Structural proof theory*. Cambridge University Press, 2008.

A.S. Troelstra and D. Dalen. *Constructivism in Mathematics: An Introduction*, volume 2. North-Holland, 1988. ISBN 9780444703583.

[7]see *Harrop* in the formalization.

[8]see *OrIndependence* in the formalization.