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COUNTEREXAMPLES TO TERMINATION FOR THE DIRECT SUM OF TERM REWRITING SYSTEMS

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The direct sum of two term rewriting systems is the union of systems having disjoint sets of function symbols. It is shown that the direct sum of two term rewriting systems is not terminating, even if these systems are both terminating.

Keywords: Term rewriting system, termination

1. Introduction

A term rewriting system R is a set of rewriting rules $M \to N$, where M and N are terms [1,3,5]. The direct sym system $R_1 \oplus R_2$ is defined as the union of two term rewriting systems with disjoint function symbols [8]. The following was proved in [8], for any two term rewriting systems R_1 and R_2.

Proposition. $R_1 \oplus R_2$ *is confluent iff* R_1 *and* R_2 *are confluent.*

By replacing 'confluent' with 'terminating' in the above proposition, the analogous conjecture for the terminating property has the following form.

Conjecture. $R_1 \oplus R_2$ *is terminating iff* R_1 *and* R_2 *are terminating.*

However, the answer to this Conjecture is negative against our expectation. We show counterexamples to this Conjecture and its modifications.

2. Counterexamples

A counterexample to the above Conjecture is obtained by R_1 and R_2 having the following rewriting rules [8]:

$$R_1 \quad \{F(0, 1, x) \to F(x, x, x),$$

$$R_2 \quad \begin{cases} G(x, y) \to x, \\ G(x, y) \to y. \end{cases}$$

It is trivial that R_1 and R_2 are terminating. However, $R_1 \oplus R_2$ is not terminating, because $R_1 \oplus R_2$ has the following infinite reduction sequence:

$$\begin{aligned} &F(G(0, 1), G(0, 1), G(0, 1)) \\ &\quad \to F(0, G(0, 1), G(0, 1)) \\ &\quad \to F(0, 1, G(0, 1)) \\ &\quad \to F(G(0, 1), G(0, 1), G(0, 1)) \to \cdots. \end{aligned}$$

This counterexample also provides a negative answer to the same question for the direct sum of recursive program schemes suggested by Klop [6].

Dershowitz [1,2,3] showed the following theorem for termination of the union system.

Theorem (Dershowitz, 1981). *Let R_1 and R_2 be two term rewriting systems. Suppose that R_1 is left linear and R_2 is right linear, and that there is no overlap between the left-hand sides of R_1 and the right-hand sides of R_2. Then, the union of the two systems is terminating iff both R_1 and R_2 are terminating.*

However, Dershowitz's Theorem [2,3] is not correct, because the above counterexample refutes his theorem. [1]

In this counterexample, note that R_2 is not confluent. Hence, the present author [8] conjectured that, under the assumption of confluence for R_1 and R_2, $R_1 \oplus R_2$ is terminating iff R_1 and R_2 are terminating. Since the direct sum of two term rewriting systems always preserves their confluence, this conjecture can be stated in the following form.

Conjecture. $R_1 \oplus R_2$ *is canonical iff* R_1 *and* R_2 *are canonical.*

Here, canonical means confluent and terminating.

However, this Conjecture is also not true. Klop and Barendregt showed a counterexample [7] by extending Toyama's counterexample. Consider R_1 and R_2 having the following rewriting rules:

$$R_1 \begin{cases} F(4, 5, 6, x) \to F(x, x, x, x), \\ F(x, y, z, w) \to 7, \end{cases}$$

$$\begin{tikzcd} 1 \arrow[d] \arrow[dr] & 2 \arrow[dl] \arrow[dr] & 3 \arrow[dl] \arrow[d] \\ 4 \arrow[dr] & 5 \arrow[d] & 6 \arrow[dl] \\ & 7 & \end{tikzcd}$$

$$R_2 \begin{cases} G(x, x, y) \to x, \\ G(x, y, x) \to x, \\ G(y, x, x) \to x. \end{cases}$$

Then, R_1 is confluent, because any term can be reduced to 7. R_1 is also terminating; no term can be reduced to 4, 5, and 6; hence, the first rule cannot be applied infinitely. Thus, R_1 is canonical. Clearly, R_2 is canonical.

However, $R_1 \oplus R_2$ is not canonical, since $F(t, t, t, t)$ with $t \equiv G(1, 2, 3)$ reduces to itself:

$$\begin{aligned} &F(t, t, t, t) \to \cdots \\ &\quad \to F(G(4, 4, 3), G(5, 2, 5), G(1, 6, 6), t) \\ &\quad \to \cdots \\ &\quad \to F(4, 5, 6, t) \\ &\quad \to F(t, t, t, t) \\ &\quad \to \cdots. \end{aligned}$$

We say that R is irreducible if, for any rule $M \to N$ in R, M and N are normal forms in $R - \{M \to N\}$. In Klop and Barendregt's counterexample, R_1 is not irreducible, since the left-hand side $F(4,5,6,x)$ and the right-hand side $F(x, x, x, x)$ of the first rule can be reduced by using other rules. Hence, Hsiang [4] conjectured that, for irreducible term rewriting systems R_1 and R_2, $R_1 \oplus R_2$ is canonical iff R_1 and R_2 are canonical. Clearly, the direct sum of two systems always preserves their irreduciblility. Hence, Hsiang's conjecture can be shown in the following form.

Conjecture (Hsiang). $R_1 \oplus R_2$ *is canonical and irreducible iff* R_1 *and* R_2 *are canonical and irreducible.*

However, Hsiang's conjecture is also not true. We can find the following counterexample to his conjecture by extending Klop and Barendregt's counterexample. Let R_1 and R_2 have the following rewriting rules:

[1] The version of Dershowitz's Theorem in [1] is correct since the definition of overlap in [1] is different from it in [2,3]. However, the examples in [1] are wrong, since the definition of overlap is in [2,3]. This remark is based on letters from Leo Bachmair (on October 24, 1986) and from Nachum Dershowitz (on November 11, 1986).

$$
R_1 \begin{cases} F(f_4(x,x), f_5(x,x), f_6(x,x), y, x) \\ \quad \to F(y, y, y, x), \\ F(x, y, z, u, 0) \to 1, \\ f_1(0, x) \to f_4(0, x), \\ f_1(x, 0) \to f_5(x, 0), \\ f_2(0, x) \to f_4(0, x), \\ f_2(x, 0) \to f_6(x, 0), \\ f_3(0, x) \to f_5(0, x), \\ f_3(x, 0) \to f_6(x, 0), \\ f_4(0, 0) \to 1, \\ f_5(0, 0) \to 1, \\ f_6(0, 0) \to 1, \end{cases}
$$

$$
R_2 \begin{cases} G(x, x, y) \to x, \\ G(x, y, x) \to x, \\ G(y, x, x) \to x. \end{cases}
$$

Then, we can show that R_1 and R_2 are canonical and irreducible. However, $R_1 \oplus R_2$ is not canonical, since $F(t, t, t, t, 0)$ with $t \equiv G(f_1(0, 0), f_2(0, 0), f_3(0, 0))$ reduces to itself:

$$
\begin{aligned}
&F(t, t, t, t, 0) \to \cdots \\
&\quad \to F(G(f_4(0,0), f_4(0,0), f_3(0,0)), \\
&\qquad\quad G(f_5(0,0), f_2(0,0), f_5(0,0)), \\
&\qquad\quad G(f_1(0,0), f_6(0,0), f_6(0,0)), t, 0) \\
&\quad \to \cdots \\
&\quad \to F(f_4(0,0), f_5(0,0), f_6(0,0), t, 0) \\
&\quad \to F(t, t, t, t, 0) \to \cdots .
\end{aligned}
$$

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