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Approximation Schemes for Viscosity Solutions of Hamilton–Jacobi Equations

PANAGIOTIS E. SOUGANIDIS*

Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

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Equations of Hamilton–Jacobi type arise in many areas of applications, including the calculus of variations, control theory and differential games. Recently M. G. Crandall and P.-L. Lions established the correct notion of generalized solutions for these equations. This article discusses the convergence of general approximation schemes to this solution and gives, under certain hypotheses, explicit error estimates. These results are then applied to obtain various representations as limits of solutions of general explicit and implicit finite difference schemes, with error estimates.

INTRODUCTION

Recently Crandall and Lions ([3], also see Crandall, Evans and Lions [2]) introduced the notion of the viscosity solution of nonlinear first order partial differential equations. They used this notion to prove uniqueness and stability results for Hamilton–Jacobi type equations, in particular the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + H(t, x, u, Du) = 0 && \text{in} \quad \mathbb{R}^N \times (0, T] \\ &u(x, 0) = u_0(x) && \text{in} \quad \mathbb{R}^N \end{aligned} \tag{0.1}$$

where $H: [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous and $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$ denotes the gradient of u. The existence of this solution for the problem (0.1) was established by Crandall and Lions [3], Lions [6, 7], Souganidis [8] and Barles [1]. Moreover, recently Crandall and Lions

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([4]) proved the convergence of a general class of finite difference schemes to the viscosity solution of the model problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + H(Du) = 0 \qquad && \text{in} \quad \mathbb{R}^N \times (0, T] \\ &u(x, 0) = u_0(x) && \text{in} \quad \mathbb{R}^N \end{aligned} \tag{0.2}$$

and gave an explicit error estimate.

This paper discusses the convergence of general approximation schemes to the viscosity solution of (0.1). In particular, it contains a general theorem which roughly says that any "reasonable" scheme converges to the viscosity solution of (0.1). Under certain hypotheses explicit error estimates are also given. (Some of the arguments in the proof of these estimates parallel the ones in [4].) We then use this abstract theorem to establish several results concerning the convergence of explicit and implicit finite difference schemes (with error estimate). For other applications of this theorem, in particular, for convergence of Trotter products as well as special representations (min-max), related to the theory of differential games, we refer to [9, 10].

The statement of the abstract results as they apply to (0.1) is rather lengthy and complicated. We therefore defer it to a later section. Here we describe a simple version of these results related to the model problem (0.2) and show how one can use them to obtain the convergence of implicit finite difference schemes.

To this end, for $\rho \geqslant 0$ we introduce a mapping $F(\rho)$: $BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$[1] such that for every $u, \hat{u} \in BUC(\mathbb{R}^N)$

(F1) $F(0)\, u = u$.

(F2) The mapping $\rho \to F(\rho)\, u$ is continuous in $BUC(\mathbb{R}^N)$.

(F3) There is a constant $C_1 \geqslant 0$ such that

$$\|F(\rho)\, u\| \leqslant C_1 \rho + \|u\|.^2$$

(F4) $F(\rho)(u + k) = F(\rho)\, u + k$ for every $k \in \mathbb{R}$.

(F5) $\|F(\rho)\, u - F(\rho)\, \hat{u}\| \leqslant \|u - \hat{u}\|$.

(F6) If $u \in C_b^{0,1}(\mathbb{R}^N)$,[3] then $F(\rho) u \in C_b^{0,1}(\mathbb{R}^N)$ and

$$\|DF(\rho)\, u\| \leqslant \|Dv\|^3$$

[1] $BUC(\mathcal{O})$ is the Banach space of bounded real valued uniformly continuous functions defined on $\mathcal{O}$.

[2] For u: $\mathcal{O} \to \mathbb{R}$, $\|u\| = \sup_{x \in \mathcal{O}} |u(x)|$.

[3] $C_{(b)}^{0,1}(\mathcal{O})$ is the space of (bounded) real valued Lipschitz continuous functions defined on $\mathcal{O}$. For $u \in C_{(b)}^{0,1}(\mathcal{O})$, $\|Du\|$ denotes the Lipschitz constant of u.

Moreover

$$\|F(\rho)\,u-u\| \leqslant C_2\rho$$

where C_2 is a constant which depends only on $\|Du\|$.

Finally, we want to assume that, when applied to smooth functions, $F(\rho)$ behaves as a "generator." We have

(F7) For every $\phi \in C_b^2(\mathbb{R}^N)$[4]

$$\left\|\frac{F(\rho)\,\phi-\phi}{\rho}+H(D\phi)\right\| \to 0$$

as $\rho \to 0$. Moreover, for each $R>0$ the limit is uniform in ϕ provided that $\|D\phi\|, \|D^2\phi\| \leqslant R$.[5]

Now for every partition $P=\{0=t_0<t_1<\cdots<t_{n(P)}=T\}$ of $[0, T]$ and for $u_0 \in BUC(\mathbb{R}^N)$ define $u_P\colon \bar{Q}_T \to \mathbb{R}$ by[6]

$$\begin{aligned} &u_P(x, 0)=u_0(x), \\ &u_P(x, \tau)=F(\tau-t_{i-1})\,u_P(\cdot, t_{i-1})(x) \\ &\qquad\qquad \text{if } \tau \in (t_{i-1}, t_i] \text{ for some } i=1,\dots, n(P). \end{aligned} \tag{0.3}$$

We have

THEOREM. *Let $H\colon \mathbb{R}^N \to \mathbb{R}$ be continuous and assume that for every $\rho \geqslant 0$ $F(\rho)\colon BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$ satisfies* (F1), (F2), (F3), (F4), (F5), (F6) *and* (F7). *If, for $u_0 \in BUC(\mathbb{R}^N)$ and a partition P of* $[0, T]$, *$u \in BUC(\bar{Q}_T)$ is the viscosity solution of* (0.2) *and $u_P\colon \bar{Q}_T \to \mathbb{R}$ is given by* (0.3), *then*

$$\sup_{(x,t)\in \bar{Q}_T} |u(x, t)-u_P(x, t)| \to 0 \qquad \text{as} \quad |P| \to 0,^{7} \tag{0.4}$$

If, moreover, $H \in C^{0,1}(\mathbb{R}^N)$ and $F(\rho)$ satisfies:

(F8) *There is a constant $C_3>0$ such that for every $\phi \in C_b^2(\mathbb{R}^N)$*

$$\left\|\frac{F(\rho)\,\phi-\phi}{\rho}+H(D\phi)\right\| \leqslant C_3\rho(1+\|D\phi\|+\|D^2\phi\|)$$

[4] $C^k_{(b)}(\mathcal{O})$ is the space of k times continuously differentiable functions defined on $\mathcal{O}$ (which together with their k derivatives are bounded).

[5] For $\phi\colon \mathcal{O} \to \mathbb{R}$ such that $\partial^2\phi/\partial x_i \partial x_j$ exist, $\|D^2\phi\|=\sum_{i,j} \|\partial^2\phi/\partial x_i \partial x_j\|$.

[6] $Q_T=\mathbb{R}^N \times (0, T]$, $\bar{Q}_T=\mathbb{R}^N \times [0, T]$.

[7] For a partition $P=\{0=t_0<t_1<\cdots<t_{n(P)}=T\}$ of $[0, T]$, $|P|=\max_i (t_i-t_{i-1})$.

then for every $u_0 \in C_b^{0,1}(\mathbb{R}^N)$

$$\sup_{(x,t)\in \bar{Q}_T} |u(x,t) - u_P(x,t)| \leqslant K|P|^{1/2} \tag{0.5}$$

where K *is a constant which depends only on* $\|u_0\|$ *and* $\|Du_0\|$.

Next we describe how one can obtain, using the above theorem, the convergence of some implicit finite difference schemes related to (0.2) for $N=1$. In particular, let $H\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous nonincreasing function. For $\rho \geqslant 0$ and $v \in BUC(\mathbb{R})$ let $F(\rho)\, v\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$F(\rho)\, v = \begin{array}{l} v, \qquad \text{if} \quad \rho = 0, \\ \text{the unique in } BUC(\mathbb{R}) \text{ solution of} \\ \displaystyle u(x) + \rho H\left(\frac{u(x+\rho\alpha) - u(x)}{\rho\alpha}\right) = v(x) \\ \text{where } \alpha > 0 \text{ is some constant, if } \rho > 0. \end{array}$$

One can check (and we do so later) that $F(\rho)$ satisfies (F1), (F2), (F3), (F4), (F5), (F6) and (F8). In view of the theorem, if, for $u_0 \in BUC(\mathbb{R})$, $u \in BUC(\bar{Q}_T)$ is the viscosity solution of

$$\frac{\partial u}{\partial t} + H(Du) = 0 \qquad \text{in} \quad Q_T$$

$$u(x,0) = u_0(x) \qquad \text{in} \quad \mathbb{R}$$

and $u_P\colon \bar{Q}_T \to \mathbb{R}$ is defined by (0.3) then

$$u(x,t) = \lim_{|P| \to 0} u_P(x,t). \tag{0.6}$$

Moreover, if $u_0 \in C_b^{0,1}(\mathbb{R})$, for $|P|$ sufficiently small,

$$|u(x,t) - u_P(x,t)| < K|P|^{1/2} \tag{0.7}$$

where K depends only on $\|u_0\|$ and $\|Du_0\|$.

The paper is organized as follows. Section 1 recalls the definition and some properties of the viscosity solution as they are stated in [2] and [3]. Moreover, it includes the existence results and some further properties of the solution as they are stated in [1] and [8] as well as the general assumptions made on H. Section 2 is devoted to the abstract convergence theorems. In particular, two theorems are given. The first deals with schemes which satisfy an (F7) type assumption (such an assumption is identified as a "generator" property). The second theorem corresponds to schemes which do not satisfy such an assumption directly. In Section 3 we obtain the convergence of several explicit finite difference schemes and give error estimates. Section 4 is devoted to fully implicit finite difference schemes. Detailed references for all the above are given in each section.

1

We begin this section by describing the assumptions on H. Throughout this discussion we will assume:

(H1) $H \in C([0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ is uniformly continuous on $[0, T] \times \mathbb{R}^N \times [-R, R] \times B_N(0, R)$ for each $R > 0$.[8]

(H2) There is a constant $C > 0$ such that

$$C = \sup_{(x,t) \in \bar{Q}_T} |H(t, x, 0, 0)| < \infty.$$

Moreover, we require some monotonicity of H with respect to u. More precisely, we assume:

(H3) For $R > 0$ there is a $\gamma_R \in \mathbb{R}$ such that

$$H(t, x, r, p) - H(t, x, s, p) \geqslant \gamma_R(r - s)$$

for $x \in \mathbb{R}^N$, $-R \leqslant s \leqslant r \leqslant R$, $0 \leqslant t \leqslant T$ and $p \in \mathbb{R}^N$.

Finally, we will have to restrict the nature of the joint continuity of H. The following Lipschitz-type assumption will be used:

(H4) For $R > 0$ there is a constant $C_R > 0$ such that

$$|H(t, x, r, p) - H(t, y, r, p)| \leqslant C_R(1 + |p|)\,|x - y|$$

for $t \in [0, T]$, $|r| \leqslant R$ and $x, y, p \in \mathbb{R}^N$.

Next we state some assumptions on H which we are going to use later in addition to the above. In particular, occasionally we will assume:

(H5) For $R > 0$ there is a $\bar{L}_R > 0$ such that

$$|H(t, x, r, p) - H(t, x, s, p)| \leqslant \bar{L}_R |r - s|$$

for $x \in \mathbb{R}^N$, $-R \leqslant s \leqslant r \leqslant R$, $0 \leqslant t \leqslant T$ and $p \in \mathbb{R}^N$.

[8] $B_N(x_0, R) = \{x \in \mathbb{R}^N : |x - x_0| \leqslant R\}$.

(H6) For $R > 0$ there is a $N_R > 0$

$$|H(t, x, r, p) - H(\bar{t}, x, r, p)| \leqslant N_R(1 + |p|)\, |t - \bar{t}|$$

for $t, \bar{t} \in [0, T]$, $|r| \leqslant R$ and $x, p \in \mathbb{R}^N$.

Finally:

(H7) For $R > 0$ there is a $M_R > 0$ such that

$$|H(t, x, r, p) - H(t, x, r, q)| \leqslant M_R |p - q|$$

for $t \in [0, T]$, $x \in \mathbb{R}^N$, $|r| \leqslant R$ and $p, q \in \mathbb{R}^N$ with $|p|, |q| \leqslant R$.

We continue now with the definition of the viscosity solution of (0.1). We have

DEFINITION 1.1 ([2, 3]). Let $H \in C([0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$. A function $u \in C(Q_T)$ is a viscosity solution of

$$\frac{\partial u}{\partial t} + H(t, x, u, Du) = 0$$

if for every $\phi \in C^\infty(Q_T)$[9]

if $u - \phi$ attains a local maximum at $(x_0, t_0) \in Q_T$, then

$$\frac{\partial \phi}{\partial t}(x_0, t_0) + H(t_0, x_0, u(x_0, t_0), D\phi(x_0, t_0)) \leqslant 0 \tag{1.1}$$

and

if $u - \phi$ attains a local minimum at $(x_0, t_0) \in Q_T$, then

$$\frac{\partial \phi}{\partial t}(x_0, t_0) + H(t_0, x_0, u(x_0, t_0), D\phi(x_0, t_0)) \geqslant 0. \tag{1.2}$$

If, moreover, $u \in C(\bar{Q}_T)$ and $u(x, 0) = u_0(x)$ in $\mathbb{R}^N$, we say that u is a viscosity solution of (0.1) on $\bar{Q}_T$.

Remark 1.1. Definition 1.1 is a combination of Definition 2 and Lemma 4.1 of [2].

Next we state the theorems about the uniqueness and existence of the viscosity solution of (0.1) as well as some other results of [1, 2, 3, 8] concerning this solution.

[9] $C^\infty_{(0)}(\mathcal{O})$ is the space of infinitely many times continuously differentiable functions (of compact support).

THEOREM 1.1 ([1, 3, 8]). *Let* $H\colon [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfy* (H1), (H2), (H3) and (H4). *For every* $u_0 \in BUC(\mathbb{R}^N)$ *there is a* $T = T(\|u_0\|) > 0$ *and* $u \in BUC(\bar{Q}_T)$ *such that u is the unique viscosity solution of* (0.1) *on* $\bar{Q}_T$. *If, moreover,* γ_R *in* (H2) *is independent of R, then* (0.2) *has a unique viscosity solution on* $\bar{Q}_T$ *for every* $T > 0$.

PROPOSITION 1.1 (I.3 [3]). *Let* $u \in C(Q_T)$ *be a viscosity solution of*

$$\frac{\partial u}{\partial t} + H(t, x, u, Du) = 0 \qquad in \quad Q_T.$$

If for $\phi \in C^1(Q_T)$ *with* $\phi \geqslant 0$ *and* $\psi \in C^1(Q_T)$:

$\phi(u - \psi)$ *attains a positive maximum at* $(x_0, t_0) \in Q_T$, *then*

$$\begin{aligned} &-\frac{(u(x_0, t_0) - \psi(x_0, t_0))}{\phi(x_0, t_0)} \frac{\partial \phi}{\partial t}(x_0, t_0) + \frac{\partial \psi}{\partial t}(x_0, t_0) \\ &+ H(t_0, x_0, u(x_0, t_0), \\ &-\frac{u(x_0, t_0) - \psi(x_0, t_0)}{\phi(x_0, t_0)} D\phi(x_0, t_0) + D\psi(x_0, t_0)) \leqslant 0. \end{aligned} \tag{1.3}$$

If for $\phi \in C^1(Q_T)$ *with* $\phi \geqslant 0$ *and* $\psi \in C^1(Q_T)$

$\phi(u - \psi)$ *attains a negative minimum at* $(x_0, t_0) \in Q_T$, *then*

$$\begin{aligned} &-\frac{(u(x_0, t_0) - \psi(x_0, t_0))}{\phi(x_0, t_0)} \frac{\partial \phi}{\partial t}(x_0, t_0) + \frac{\partial \psi}{\partial t}(x_0, t_0) \\ &+ H\Bigg(t_0, x_0, u(x_0, t_0), \\ &-\frac{u(x_0, t_0) - \psi(x_0, t_0)}{\phi(x_0, t_0)} D\phi(x_0, t_0) + D\psi(x_0, t_0)\Bigg) \geqslant 0. \end{aligned} \tag{1.4}$$

The following results of [8] give a priori bounds on the norm, the Lipschitz constant (in the x variable) of the solution u and the difference $\|u - u_0\|$.

We have:

PROPOSITION 1.2 (1.5 [8]). *Let* $H\colon [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfy* (H1) *and* (H3) *with* $\gamma_R \leqslant 0$ *for every* $R > 0$. *If, for* $u_0 \in BUC(\mathbb{R}^N)$, $u \in BUC(\bar{Q}_T)$ *is the viscosity solution of* (0.1) *in* $\bar{Q}_T$, *let* $R > \|u\|$ *and* $\gamma = \gamma_R$. *The following are true for every* τ, $s \in [0, T]$:

(a) *If H satisfies* (H2), *then*

$$\|u(\cdot,\tau)\| \leqslant e^{-\gamma\tau}(\tau C + \|u_0\|) \tag{1.5}$$

where C is given by (H2).

(b) *If H satisfies* (H4) *and $u(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N)$ for every $\tau \in [0,T]$ with $L = \sup_{0 \leqslant \tau \leqslant T} \|Du(\cdot,\tau)\|$, then*

$$\|Du(\cdot,\tau)\| \leqslant e^{-\gamma\tau}(L_0 + \tau[C_R(1+L)]) \tag{1.6}$$

for every $\tau \in [0,T]$ where $L_0 = \|Du_0\|$ and C_R is given by (H4). *Moreover,*

$$L \leqslant e^{T(2C_R e^{-\gamma T} - \gamma)}(L_0 + TC_R). \tag{1.7}$$

(c) *If $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, then*

$$\|u(\cdot,\tau) - u_0\| \leqslant \tau e^{-\gamma\tau} \sup_{\substack{(x,t) \in \bar{Q}_T \\ |r| \leqslant \|u_0\| \\ |p| \leqslant \|Du_0\|}} |H(t,x,r,p)|.$$

(d) *If $u(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N)$ for every $\tau \in [0,T]$ and $\sup_{0 \leqslant \tau \leqslant T} \|Du(\cdot,\tau)\| \leqslant L$, then $u \in C_b^{0,1}(\bar{Q}_T)$ and*

$$\|u(\cdot,\tau) - u(\cdot,s)\| \leqslant |\tau - s|\, e^{-\gamma T} \sup_{\substack{(x,t) \in \bar{Q}_T \\ |r| \leqslant \|u\| \\ |p| \leqslant L}} |H(t,x,r,p)|. \tag{1.9}$$

We conclude this section with some results concerning the behavior of the viscosity solution of (0.1) in the case that H satisfies (H4) and $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. We have

PROPOSITION 1.3 (2.2 [8]). *If, for H: $[0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying* (H1), (H2), (H3) *and* (H4) *and $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, $u \in BUC(\bar{Q}_T)$ is the viscosity solution of* (0.1) *in $\bar{Q}_T$, then $u \in C_b^{0,1}(\bar{Q}_T)$. Moreover, the Lipschitz constant is estimated by Proposition* 1.2.

2

In this section we deal with the convergence of general approximation schemes to the viscosity solution of (0.1). Moreover, under certain assumptions explicit error estimates are given. In particular, we prove two theorems, which, in the applications we examine later, lead to the same conclusion, in the case that H is independent of u. The first theorem is concerned with schemes, which satisfy a "generator" type assumption (like (F7), (F8) in the Introduction). In particular, we have

THEOREM 2.1. (a) *For* $H: [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfying* (H1), (H2) (*with constant* C), (H5) (*with constant* $\bar{L}$ *independent of* R) *and* (H4), (H6), (H7) (*with constants* C_R, N_R, M_R, *respectively, for* $R \geqslant 0$) *and for* $u_0 \in C_b^{0,1}(\mathbb{R}^N)$ *let* $u \in C_b^{0,1}(\bar{Q}_T)$ *be the viscosity solution of* (0.1) *in* $\bar{Q}_T$. *For* $(t, \rho) \in K = \{(t, \rho) \in [0, T] \times [0, \rho_0]: 0 \leqslant \rho \leqslant t\}$, *where* $\rho_0 = \rho_0(\|u_0\|) > 0$, *let* $F(t, \rho, \cdot, \cdot): C_b^{0,1}(\mathbb{R}^N) \times C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ *be such that for every* $u, \bar{u}, v, \bar{v} \in C_b^{0,1}(\mathbb{R}^N)$

(F1) $F(t, 0, u, v) = v$.

(F2) *The mapping* $(t, \rho) \to F(t, \rho, u, u)$ *is continuous with respect the* $\| \ \|$ *norm.*

(F3) $F(t, \rho, u, v + k) = F(t, \rho, u, v) + k$ *for every* $k \in \mathbb{R}$.

(F4) $\|F(t, \rho, u, u) - u\| \leqslant C_1$, *where* $C_1 = C_1(\|u\|, \|Du\|) \geqslant 0$.

(F5) *There exists an* $r > 0$ *and* $L_1 > 0$ *such that if* $v(x) \leqslant \bar{v}(x)$ *for every* $x \in \mathbb{R}^N$, *then for any* $y \in \mathbb{R}^N$, *such that*

$$|v(y + w) - v(y + \bar{w})|,\ |\bar{v}(y + w) - \bar{v}(y + \bar{w})| \leqslant \bar{\bar{L}}\,|w - \bar{w}|$$

for every $w, \bar{w} \in B_N(0, \rho r)$,

$$F(t, \rho, u, v)(y) \leqslant F(t, \rho, u, \bar{v})(y)$$

where $L = \sup_{0 \leqslant \tau \leqslant T} \|Du(\cdot, \tau)\|$ *and* $\bar{\bar{L}} = \max(L_1, L) + 1$.

(F6) *There exists a constant* $C_2 > 0$ *such that*

$$\|F(t, \rho, u, u)\| \leqslant e^{\rho C_2}(\|u\| + \rho C_2)$$

provided that $\|Du\| \leqslant \bar{\bar{L}}$.

(F7) *There exist constants* $C_3, C_4 > 0$ *such that*

$$e^{T(C_3 + C_4)}(\|Du_0\| + TC_4) \leqslant \bar{\bar{L}}$$

and

$$\|DF(t, \rho, u, u)\| \leqslant e^{\rho(C_3 + C_4)}(\|Du\| + \rho C_4)$$

provided that $\|u\| \leqslant e^{TC_2}(\|u_0\| + TC_2)$ *and* $\|Du\| \leqslant \bar{\bar{L}}$.

(F8) *For every* $\phi \in C_b^2(\mathbb{R}^N)$ *and* $x \in \mathbb{R}^N$ *such that* $|D\phi(x)| < L + 1$,

$$\left|\frac{F(t, \rho, u, \phi)(x) - \phi(x)}{\rho} + H(t, x, u(x), D\phi(x))\right| \leqslant C_5(1 + \|D\phi\| + \|D^2\phi\|)\,\rho$$

where $L = \sup_{0 \leqslant \tau \leqslant T} \|Du(\cdot, \tau)\|$ *and* $C_5 = C_5(\|u\|, \|Du\|, L)$.

For a partition $P=\{0=t_0<t_1<\cdots<t_{n(P)}=T\}$ *of* $[0, T]$, *let* $u_P\colon \bar{Q}_T\to\mathbb{R}$ *be defined by*

$$u_P(x,0)=u_0(x),$$
$$u_P(x,t)=F(t, t-t_{i-1}\, u_P(\cdot, t_{i-1}), u_P(\cdot, t_{i-1}))(x)$$
$$\text{if } t\in(t_{i-1}, t_i] \text{ for some } i=1,..., n(P). \tag{2.1}$$

Then there exists a constant K *depending only on* $\|u_0\|$ *and* $\|Du_0\|$ *such that*

$$\|u_P-u\|\leqslant K|P|^{1/2} \tag{2.2}$$

for $|P|$ *sufficiently small.*

(b) *For* $H\colon [0, T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ *satisfying* (H1), (H2), (H4) (*with constant* C_R *for* $R\geqslant 0$) *and* (H5) (*with a constant* $\bar{L}$ *independent of* R) *and for* $u_0\in BUC(\mathbb{R}^N)$ *let* $u\in BUC(\bar{Q}_T)$ *be the viscosity solution of* (0.1) *in* $\bar{Q}_T$. *For* $(t,\rho)\in K=\{(t,\rho)\in[0,T]\times[0,\rho_0]: 0\leqslant t\leqslant\rho\}$, *where* $\rho_0=\rho_0(\|u_0\|)>0$, *let* $F(t,\rho,\cdot,\cdot)\colon BUC(\mathbb{R}^N)\times BUC(\mathbb{R}^N)\to BUC(\mathbb{R}^N)$ *be such that, for every* $u, \bar{u}, v, \bar{v}\in BUC(\mathbb{R}^N)$, *it satisfies* (F1), (F2) (*for* $u\in C_b^{0,1}(\mathbb{R}^N)$), (F3), (F4) (*for* $u\in C_b^{0,1}(\mathbb{R}^N)$) *and moreover*:

(F9) *There exists a constant* $C_6\geqslant 0$ *such that*

$$\|F(t,\rho,u,v)-F(t,\rho,\bar{u},\bar{v})\|\leqslant\|v-\bar{v}\|+\rho C_6\|u-\bar{u}\|$$

provided that $\bar{u}, \bar{v}\in C_b^{0,1}(\mathbb{R}^N)$.

(F10) *There exists a constant* $C_7\geqslant 0$ *such that*

$$\|F(t,\rho,u,u)\|\leqslant e^{\rho C_7}(\|u\|+\rho C_7).$$

(F11) *If* $u\in C_b^{0,1}(\mathbb{R}^N)$, *then* $F(t,\rho,u,u)\in C_b^{0,1}(\mathbb{R}^N)$ *and*

$$\|DF(t,\rho,u,u)\|\leqslant e^{\rho(C_8+C_9)}(\|Du\|+\rho C_9)$$

where $C_8\geqslant 0$ *and* $C_9=C_9(\|u\|)\geqslant 0$.

(F12) *For* $u\in C_b^{0,1}(\mathbb{R}^N)$ *and* $\phi\in C_b^2(\mathbb{R}^N)$

$$\left\|\frac{F(t,\rho,u,\phi)-\phi}{\rho}+H(t,\cdot,u,D\phi)\right\|\to 0 \qquad \text{as} \quad \rho\to 0.$$

Moreover, for each $R>0$ *the above limit is uniform on* u, ϕ, *provided that* $\|u\|, \|Du\|, \|D\phi\|, \|D^2\phi\|\leqslant R$.

If, for a partition P *of* $[0, T]$, $u_P\colon\bar{Q}_T\to\mathbb{R}$ *is defined by* (2.1), *then*

$$\|u_P-u\|\to 0 \qquad \text{as} \quad |P|\to 0. \tag{2.3}$$

Before we give the proof of the theorem we discuss some of its assumptions. In particular, since non-expansive mappings commuting with the addition of constants are order-preserving and vice versa (Crandall and Tartar [5]), (F5) is implied by (F3) and

(F13) $\|F(t, \rho, u, v) - F(t, \rho, u, \bar{v})\| \leqslant \|v - \bar{v}\|$ for $u, v, \bar{v} \in C_b^{0,1}(\mathbb{R}^N)$.

Similarly, (F5) together with (F3) implies that, for fixed (t, ρ) and u, $F(t, \rho, u, \cdot)$ is non-expansive on $\{u \in C_b^{0,1}(\mathbb{R}^N) \colon \|Du\| \leqslant \bar{\bar{L}} + 1\}$. In several applications we are going to have (F3) and (F13), in which case the conditions on u in (F6), (F7) are irrelevant. Moreover, instead of (F8), occasionally we will assume

(F14) For every $u \in C_b^{0,1}(\mathbb{R}^N)$ and $\phi \in C_b^2(\mathbb{R}^N)$

$$\left\| \frac{F(t, \rho, u, \phi) - \phi}{\rho} + H(t, \cdot, u, D\phi) \right\| < C_{10}(1 + \|D\phi\| + \|D^2\phi\|)\, \rho$$

where $C_{10} = C_{10}(\|u\|, \|Du\|)$.

This of course implies (F8). Finally, we want to remark that the important hypotheses are the ones on F. In particular, in part (b) one can assume a more general condition than (H4) and the result is still true. However, in applications, most of the time, one needs (H4) to check (F11) and (F12). Moreover, the assumption that the constant in (H5) is independent of R has been made only for simplicity. In fact in the applications one can always reduce to this case.

Proof of Theorem 2.1. (a) We begin with a lemma, which records some of the properties of u_P. In particular, we have:

LEMMA 2.1. *For a partition* $P = \{0 = t_0 < \cdots < t_{n(P)} = T\}$ *of* $[0, T]$ *and* $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, *let* $u_P \colon \bar{Q}_T \to \mathbb{R}$ *be defined by* (2.1). *The following are true:*

(a) *For every* $\tau \in [0, T]$

$$\|u_P(\cdot, \tau)\| \leqslant e^{\tau C_2}(\tau C_2 + \|u_0\|) \tag{2.4}$$

and

$$u_P(\cdot, \tau) \in C_b^{0,1}(\mathbb{R}^N) \quad \textit{with} \quad \|Du_P(\cdot, \tau)\| \leqslant e^{\tau(C_3 + C_4)}(\|Du_0\| + \tau C_4). \tag{2.5}$$

Moreover, if $\tau \in (t_{i-1}, t_i]$ *for some* $i = 1, ..., n(P)$, *then*

$$\|u_P(\cdot, \tau) - u_P(\cdot, t_{i-1})\| \leqslant \bar{C}_1(\tau - t_{i-1}) \tag{2.6}$$

where $\bar{C}_1 = C_1(e^{TC_2}(\|u_0\| + TC_2), \bar{\bar{L}})$.

(b) $u_P \in BUC(\bar{Q}_T)$.

The proof is rather simple and we leave it to the reader to supply the details.

We continue now with the proof of (2.2). It is obvious that it suffices to show that there exists a constant K_1, which depends only on $\|u_0\|$ and $\|Du_0\|$, such that

$$\sup_{(x,\tau)\in\bar{Q}_T} (e^{-\bar{L}\tau}(u_P(x,\tau)-u(x,\tau))^{\pm}) \leqslant K_1 |P|^{1/2} \text{[10]} \qquad (2.7)^{\pm}$$

for $|P|$ sufficiently small. Here we prove only $(2.7)^+$, since the proof of $(2.7)^-$ is identical. To this end, let M_P be defined by

$$M_P = \sup_{(x,\tau)\in\bar{Q}_T} [e^{-\bar{L}\tau}(u_P(x,\tau)-u(x,\tau))^+].$$

Without any loss of generality we may assume

$$M_P > 0. \qquad (2.8)$$

In view of Lemma 2.1(a), we know that there is an $R_1 > 0$, independent of the partition P, such that

$$\|u_P\| \leqslant R_1.$$

For $R = \max(R_1, \|u\|)$ and $\varepsilon = |P|^{1/4}$, let $\Phi\colon \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T] \to \mathbb{R}$ be defined by

$$\Phi(x,y,\tau,s) = e^{-\bar{L}[(\tau+s)/2]}(u_P(x,\tau)-u(y,s))^+ + 3(R+1)\,\beta_\varepsilon(x-y) + 3(R+1)\,\gamma_\varepsilon(\tau-s) - \frac{\tau+s}{4T} M_P$$

where $\beta_\varepsilon(\cdot) = \beta(\cdot/\varepsilon)$ and $\gamma_\varepsilon(\cdot) = \gamma(\cdot/\varepsilon)$ with

$$\begin{gathered} \beta \in C_0^\infty(\mathbb{R}^N),\ \beta(0)=1,\ 0 \leqslant \beta \leqslant 1, \\ |D\beta| \leqslant 2,\ |D^2\beta| \leqslant 4,\ \beta(w)=0 \quad \text{if} \quad |w|>1, \\ \beta(w) = 1-|w|^2 \qquad \text{for} \quad |w| \leqslant \sqrt{3/2}, \\ \beta(w) < 1/2 \qquad \text{for} \quad |w| > \sqrt{3/2} \end{gathered} \qquad (2.9)$$

and

$$\begin{gathered} \gamma \in C_0^\infty(R),\ \gamma(0)=1,\ 0 \leqslant \gamma \leqslant 1, \\ |D\gamma| \leqslant 2,\ |D^2\gamma| \leqslant 4,\ \gamma(t)=0 \quad \text{if} \quad |t|>1, \\ \gamma(t) = 1-t^2 \qquad \text{for} \quad |t| \leqslant \sqrt{3/2}, \\ \gamma(t) < 1/2 \qquad \text{for} \quad |t| > \sqrt{3/2}. \end{gathered} \qquad (2.10)$$

[10] $r^+ = \max(r, 0)$, $r^- = \max(-r, 0)$.

Since Φ is bounded on $\mathbb{R}^N \times \mathbb{R}^N \times [0, T] \times [0, T]$, for every $\delta > 0$ there is a point $(x_1, y_1, \tau_1, s_1) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T] \times [0, T]$ such that

$$\Phi(x_1, y_1, \tau_1, s_1) > \sup_{(x,y,\tau,s) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T]} \Phi(x, y, \tau, s) - \delta.$$

Next choose $\zeta \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ so that $0 \leqslant \zeta \leqslant 1$, $\zeta(x_1, y_1) = 1$, $|D\zeta| \leqslant 1$, $|D^2\zeta| \leqslant 1$ and define Ψ: $\mathbb{R}^N \times \mathbb{R}^N \times [0, T] \times [0, T] \to \mathbb{R}$ by

$$\Psi(x, y, \tau, s) = \Phi(x, y, \tau, s) + 2\delta\zeta(x, y).$$

$\Psi = \Phi$ off the support of ζ and

$$\begin{aligned}\Psi(x_1\ y_1, \tau_1, s_1) &= \Phi(x_1, y_1, \tau_1, s_1) + 2\delta \\ &> \sup_{(x,y,\tau,s) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,T] \times [0,T]} \Phi(x, y, \tau, s) + \delta\end{aligned}$$

therefore there is a point $(x_0, y_0, \tau_0, s_0) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T] \times [0, T]$ such that

$$\begin{gathered}\Psi(x_0, y_0, \tau_0, s_0) \geqslant \Psi(x, y, \tau, s) \\ \text{for every } (x, y, \tau, s) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T] \times [0, T].\end{gathered} \tag{2.11}$$

Moreover, for $\delta < \min(\frac{1}{24}, \frac{1}{8}M_P)$,

$$\begin{gathered}|x_0 - y_0| \leqslant \varepsilon,\ |\tau_0 - s_0| \leqslant \varepsilon, \\ u_P(x_0, \tau_0) - u(y_0, s_0) > 0,\ \Psi(x_0, y_0, \tau_0, s_0) > 0, \\ |x_0 - y_0| \leqslant (L + 2\delta)\,\varepsilon^2, \\ \gamma_\varepsilon(\tau_0 - s_0) = 1 - \frac{(\tau_0 - s_0)^2}{\varepsilon^2}, \qquad \beta_\varepsilon(x_0 - y_0) = 1 - \frac{|x_0 - y_0|^2}{\varepsilon^2}.\end{gathered} \tag{2.12}$$

Indeed observe that, if for $\delta < \frac{1}{2}$ either $|x_0 - y_0| > \varepsilon$ or $|\tau_0 - s_0| > \varepsilon$, then (2.11) implies

$$\begin{aligned}2(R+1) + 3(R+1) + 2\delta &\geqslant \Psi(x_0, y_0, \tau_0, s_0) \\ &\geqslant \Psi(x, x, \tau, \tau) \geqslant e^{-\bar{L}\tau}(u_P(x, \tau) - u(x, \tau)) + 3(R+1) - \tfrac{1}{2}M_P\end{aligned}$$

and therefore

$$2\delta \geqslant \tfrac{1}{2}M_P + R + 1$$

which is a contradiction. Moreover, the above argument also shows that

$$\Psi(x_0, y_0, \tau_0, s_0) \geqslant \tfrac{1}{2}M_P + 6(R+1) > 0.$$

Next observe that

$$e^{-\bar{L}[(\tau_0+s_0)/2]}(u_P(x_0,\tau_0)-u(y_0,s_0))^+ + 2\delta \geqslant \tfrac{1}{2}M_P > 0$$

therefore for $\delta < \frac{1}{8}M_P$

$$e^{-\bar{L}[(\tau_0+s_0)/2]}(u_P(x_0,\tau_0)-u(y_0,s_0))^+ \geqslant \tfrac{1}{4}M_P > 0$$

i.e.,

$$u_P(x_0,\tau_0)-u(y_0,s_0)>0.$$

Moreover, (2.1) yields

$$\begin{aligned}&2(R+1)+3(R+1)\,\beta_\varepsilon(x_0-y_0)+3(R+1)+2\delta\\ &\quad\geqslant \Psi(x_0,y_0,\tau_0,s_0)\geqslant \Psi(x,x,\tau,\tau)\geqslant \tfrac{1}{2}M_P+3(R+1)+3(R+1)\end{aligned}$$

therefore, since $\delta < 1/24$,

$$\beta_\varepsilon(x_0-y_0)\geqslant \frac{1}{3}-\frac{2\delta}{3(R+1)}\geqslant\frac{1}{4}$$

i.e.,

$$\beta_\varepsilon(x_0-y_0)=1-\frac{|x_0-y_0|^2}{\varepsilon^2}. \tag{2.13}$$

A similar argument implies

$$\gamma_\varepsilon(\tau_0-s_0)=1-\frac{|\tau_0-s_0|^2}{\varepsilon^2}. \tag{2.14}$$

Finally y_0 is a minimum point of the mapping

$$y\to e^{-(\bar{L}/2)(\tau_0+s_0)}u(y,s_0)-3(R+1)\,\beta_\varepsilon(x_0-y)-2\delta\zeta(x_0,y)$$

therefore for every $y\in\mathbb{R}^N$

$$\begin{aligned}&3(R+1)\,\beta_\varepsilon(x_0-y)+2\delta\zeta(x_0,y)-3(R+1)\,\beta_\varepsilon(x_0-y_0)-2\delta\zeta(x_0,y_0)\\ &\quad\leqslant e^{-(\bar{L}/2)(\tau_0+s_0)}(u(y,s_0)-u(y_0,s_0))\\ &\quad\leqslant e^{-(\bar{L}/2)(\tau_0+s_0)}L\,|x_0-y_0|\leqslant 6(R+1)\,L\,|y-y_0|.\end{aligned}$$

This implies

$$|-3(R+1)\, D\beta_\varepsilon(x_0-y_0)+2\delta D_y\zeta(x_0,\, y_0)|\leqslant 6(R+1)\, L$$

and, in view of the choice of ζ and (2.13),

$$|x_0-y_0|\leqslant (L+2\delta)\,\varepsilon^2.$$

There are several cases to be considered. These are: $\tau_0\geqslant 0$ and $s_0=0$, $\tau_0=0$ and $s_0\geqslant 0$ and $\tau_0>0$ and $s_0>0$. We begin with the case $\tau_0\geqslant 0$ and $s_0=0$.

1*st case*. $\tau_0\geqslant 0$ *and* $s_0=0$. From (2.11), (2.12) and

$$\Psi(x_0,\, y_0,\, \tau_0,\, 0)\geqslant \Psi(x_0,\, y_0,\, 0,\, 0)$$

it follows that

$$\begin{aligned} &u_P(x_0,\, \tau_0)-u_0(y_0)+3(R+1)\,\beta_\varepsilon(x_0-y_0)+3(R+1)(1-\tau_0^2/\varepsilon^2)\\ &\qquad +2\delta\zeta(x_0,\, y_0)\\ &\quad \geqslant \Psi(x_0,\, y_0,\, \tau_0,\, 0)\geqslant u_0(x_0)-u_0(y_0)+3(R+1)\,\beta_\varepsilon(x_0-y_0)\\ &\qquad +3(R+1)+2\delta\zeta(x_0,\, y_0). \end{aligned}$$

This yields

$$\begin{aligned} &3(R+1)\,|u_P(x_0,\, \tau_0)-u_0(x_0)|+3(R+1)(1-\tau_0^2/\varepsilon^2)\\ &\quad \geqslant 3(R+1). \end{aligned}$$

Using Lemma 2.1(a) we obtain

$$\tau_0^2/\varepsilon^2\leqslant \bar{C}_1\tau_0$$

and thus

$$\tau_0\leqslant \bar{C}_1\varepsilon^2 \tag{2.15}$$

where $\bar{C}_1$ is a constant which depends only on $\|u_0\|$ and $\|Du_0\|$ and is given by Lemma 2.1(a). Then (2.11) implies

$$\begin{aligned} &|u_P(x_0,\, \tau_0)-u_0(x_0)|+|u_0(x_0)-u_0(y_0)|+3(R+1)+3(R+1)+2\delta\\ &\quad \geqslant \Psi(x_0,\, y_0,\, \tau_0,\, 0)\geqslant \tfrac{1}{2}M_P+3(R+1)+3(R+1), \end{aligned}$$

i.e.,

$$M_P\leqslant 2[(\bar{C}_1)^2+L\,\|Du_0\|]\,\varepsilon^2+4\delta(\varepsilon^2+1). \tag{2.16}$$

2nd case. $\tau_0 = 0$ *and* $s_0 \geqslant 0$. From (2.11) and (2.12) it follows that

$$\begin{aligned}u_0(x_0) - u(y_0, s_0) + 3(R+1)\,\beta_\varepsilon(x_0 - y_0) + 3(R+1)(1 - s_0^2/\varepsilon^2) + 2\delta\zeta(x_0, y_0)\\ \geqslant \Psi(x_0, y_0, 0, s_0) \geqslant \Psi(x_0, y_0, 0, 0)\\ \geqslant u_0(x_0) - u_0(y_0) + 3(R+1)\,\beta_\varepsilon(x_0 - y_0) + 3(R+1) + 2\delta\zeta(x_0, y_0)\end{aligned}$$

therefore

$$3(R+1)\,|u(y_0, s_0) - u_0(y_0)| + 3(R+1)(1 - s_0^2/\varepsilon^2) \geqslant 3(R+1).$$

But, since $u \in C^{0,1}(\bar{Q}_T)$ with Lipschitz constant $\|Du\|$,

$$s_0^2/\varepsilon^2 \leqslant \|Du\|\, s_0$$

i.e.

$$s_0 \leqslant \|Du\|\, \varepsilon^2. \tag{2.17}$$

Then (2.11) implies

$$\begin{aligned}|u_0(x_0) - u_0(y_0)| + |u_0(y_0) - u(y_0, s_0)| + 3(R+1) + 3(R+1) + 2\delta\\ \geqslant \Psi(x_0, y_0, 0, s_0) \geqslant \tfrac{1}{2} M_P + 3(R+1) + 3(R+1)\end{aligned}$$

and therefore

$$M_P \leqslant 2(\|Du\|^2 + L\,\|Du_0\|)\,\varepsilon^2 + 4\delta(\varepsilon^2 + 1). \tag{2.18}$$

3rd case. $\tau_0 > 0$ *and* $s_0 > 0$. It follows for (2.11) and (2.12) that s_0 is a minimum point of the mapping $s \to e^{-(\bar{L}/2)(\tau_0 + s)} u(y_0, s) - 3(R+1)\,\gamma_\varepsilon(\tau_0 - s) + (s/4T)\, M_P$, therefore for $s \in [0, T]$

$$\begin{aligned}3(R+1)\,\gamma_\varepsilon(\tau_0 - s) - \frac{s}{4T} M_P - 3(R+1)\,\gamma_\varepsilon(\tau_0 - s_0) + \frac{s_0}{4T} M_P\\ \leqslant e^{-(\bar{L}/2)(\tau_0 + s)} u(y_0, s) - e^{-(\bar{L}/2)(\tau_0 + s_0)} u(y_0, s_0)\\ \leqslant \left(\frac{R\bar{L}}{2} + \|Du\|\right) |s - s_0| \leqslant 6(R+1)\left(\frac{R\bar{L}}{2} + \|Du\|\right) |s - s_0|.\end{aligned}$$

But in view of (2.10),

$$\begin{aligned}6(R+1)\,\frac{|\tau_0 - s_0}{\varepsilon^2} \leqslant 6(R+1)\left(\frac{R\bar{L}}{2} + \|Du\|\right) + \frac{1}{4T} M_P\\ \leqslant 6(R+1)\left(\frac{R\bar{L}}{2} + \|Du\| + \frac{1}{2T}\right)\end{aligned}$$

i.e.

$$|\tau_0 - s_0| \leqslant \left(\frac{R\bar{L}}{2} + \|Du\| + \frac{1}{2T}\right)\varepsilon^2. \tag{2.19}$$

Next observe that $(x_0, \tau_0) \in Q_T$ is a point where $\phi(u_P - \psi)$ attains a positive maximum and $(y_0, s_0) \in Q_T$ is a point where $\bar{\phi}(u - \bar{\psi})$ attains negative minimum where

$$\begin{aligned}
&\phi(x, \tau) = e^{-(\bar{L}/2)(\tau + s_0)},\\
&\psi(x, \tau) = u(y_0, s_0) - 3(R+1)\, e^{(\bar{L}/2)(\tau + s_0)}\beta_\varepsilon(x - y_0)\\
&\qquad - 3(R+1)\, e^{(\bar{L}/2)(\tau + s_0)}\gamma_\varepsilon(\tau - s_0)\\
&\qquad - 2\delta e^{(\bar{L}/2)(\tau + s_0)}\zeta(x, y_0) + \frac{(\tau + s_0)}{4T} M_P e^{(\bar{L}/2)(\tau + s_0)},\\
&\bar{\phi}(y, s) = e^{-(\bar{L}/2)(\tau_0 + s)},\\
&\bar{\psi}(y, s) = u_P(x_0, \tau_0) + 3(R+1)\, e^{(\bar{L}/2)(\tau_0 + s)}\beta_\varepsilon(x_0 - y)\\
&\qquad + 3(R+1)\, e^{(\bar{L}/2)(\tau_0 + s)}\gamma_\varepsilon(\tau_0 - s)\\
&\qquad + 2\delta e^{(\bar{L}/2)(\tau_0 + s)}\zeta(x_0, y) - \frac{(\tau_0 + s)}{4T} M_P e^{(\bar{L}/2)(\tau_0 + s)}.
\end{aligned} \tag{2.20}$$

Using (2.20) and Proposition 1.1, we obtain

$$\begin{aligned}
0 \leqslant &\frac{\bar{L}}{2}(u(y_0, s_0) - \bar{\psi}(y_0, s_0)) + \frac{\bar{L}}{2} 3(R+1)\, e^{(\bar{L}/2)(\tau_0 + s_0)}\beta_\varepsilon(x_0 - y_0)\\
&+ \frac{\bar{L}}{2} 3(R+1)\, e^{(\bar{L}/2)(\tau_0 + s_0)}\gamma_\varepsilon(\tau_0 - s_0) - 3(R+1)\, e^{(\bar{L}/2)(\tau_0 + s_0)}\gamma'_\varepsilon(\tau_0 - s_0)\\
&\frac{\bar{L}}{2} 2\delta e^{(\bar{L}/2)(\tau_0 + s_0)}\zeta(x_0, y_0) - \frac{1}{4T} M_P e^{(\bar{L}/2)(\tau_0 + s_0)}\\
&- \frac{\bar{L}}{2}\frac{(\tau_0 + s_0)}{4T} M_P e^{(\bar{L}/2)(\tau_0 + s_0)} + H(s_0, y_0, u(y_0, s_0), D\bar{\psi}(y_0, s_0))
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{1}{4T} M_P e^{(\bar{L}/2)(\tau_0 + s_0)} + 3(R+1)\, e^{(\bar{L}/2)(\tau_0 + s_0)}\gamma'_\varepsilon(\tau_0 - s_0)\\
&\quad \leqslant \frac{\bar{L}}{2}(u(y_0, s_0) - u_P(x_0, \tau_0)) + H(s_0, y_0, u(y_0, s_0), D\bar{\psi}(y_0, s_0)).
\end{aligned} \tag{2.21}$$

Moreover, since $\tau_0 \in (t_{i-1}, t_i]$ for some $i = 1,..., n(P)$, we also have

$$u_P(x, t_{i-1}) \leqslant \psi(x, t_{i-1}) + e^{-(\bar{L}/2)(\tau_0 - t_{i-1})}(u_P(x_0, \tau_0) - \psi(x_0, \tau_0))$$
$$\text{for every } x \in \mathbb{R}^N. \qquad (2.22)$$

In view of (2.9), (2.10), (2.11), (2.20) and the choice of ζ, the following are true:

$$|D\psi(x_0, t_{i-1}) - D\psi(x_0, \tau_0)| \leqslant \frac{\bar{L}}{2} |D\psi(x_0, \tau_0)|\ |P|,$$
$$\|D\psi(\cdot, t_{i-1})\| \leqslant 6(R+1)\, e^{\bar{L}T}\left(\frac{1}{\varepsilon} + \delta\right), \qquad (2.23)$$
$$\|D^2\psi(\cdot, t_{i-1})\| \leqslant 12(R+1)\, e^{\bar{L}T}\left(\frac{1}{\varepsilon^2} + \delta\right).$$

Finally, in the course of the proof of (2.12) we established that

$$|D\bar{\psi}(y_0, s_0)| = |-3(R+1)\, e^{(\bar{L}/2)(\tau_0 + s_0)}\, D\beta_\varepsilon(x_0 - y_0)$$
$$+ 2\delta e^{(\bar{L}/2)(\tau_0 + s_0)}\, D_y\zeta(x_0, y_0)| \leqslant L$$

and thus

$$|D\psi(x_0, \tau_0)| < L + \tfrac{1}{2} \qquad (2.24)$$

for $\delta < e^{-\bar{L}T}/4$, since

$$D\psi(x_0, \tau_0) = D\psi(y_0, s_0) - 2\delta e^{(\bar{L}/2)(\tau_0 + s_0)}(D_y\zeta(x_0, y_0) + D_x\zeta(x_0, y_0)).$$

Next observe that for $w, \bar{w} \in B_N(0, (\tau_0 - t_{i-1})\, r)$ (where $r > 0$ is given by (F5))

$$|\psi(x_0 + w, t_{i-1}) - \psi(x_0 + \bar{w}, t_{i-1})|$$
$$\leqslant |\psi(x_0 + w, t_{i-1}) - \psi(x_0 + \bar{w}, t_{i-1}) - D\psi(x_0, t_{i-1}) \cdot (w - \bar{w})|$$
$$+ |D\psi(x_0, t_{i-1}) - D\psi(x_0, \tau_0)) \cdot (w - \bar{w})| + |D\psi(x_0, \tau_0) \cdot (w - \bar{w})|.$$

(2.9), (2.23), (2.24) and the choice of ζ imply

$$|\psi(x_0 + w, t_{i-1}) - \psi(x_0 + \bar{w}, t_{i-1})|$$
$$\leqslant \left(12(\tau_0 - t_{i-1})\, r\left(\frac{1}{\varepsilon^2} + \delta\right) e^{\bar{L}T} + \frac{\bar{L}}{2}(L + 1/2)\, |P| + L + 1/2\right) |w - \bar{w}|.$$

Since $\varepsilon=|P|^{1/4}$, for $\delta<1$ and $|P|$ so small that

$$|P|^{1/2}(12r(1+|P|^{1/2})\,e^{\bar{L}T}+\frac{\bar{L}}{2}(L+1/2)\,|P|^{1/2})<1/2 \tag{2.25}$$

we have

$$|\psi(x_0+w,t_{i-1})-\psi(x_0+\bar{w},t_{i-1})|<\bar{\bar{L}}\,|w-\bar{w}|.$$

Moreover,

$$|u_P(x_0+w,t_{i-1})-u_P(x_0+\bar{w},t_{i-1})|<\bar{\bar{L}}\,|w-\bar{w}|.$$

Thus (F3), (F5) and (2.22) imply that

$$\begin{aligned}u_P(x_0,\tau_0)&=F(\tau_0,\tau_0-t_{i-1},u_P(\cdot,t_{i-1}),u_P(\cdot,t_{i-1}))(x_0)\\&\leqslant F(\tau_0,\tau_0-t_{i-1},u_P(\cdot,t_{i-1}),\psi(\cdot,t_{i-1}))(x_0)\\&\quad+e^{-(\bar{L}/2)(\tau_0-t_{i-1})}(u_P(x_0,\tau_0)-\psi(x_0,\tau_0))\end{aligned}$$

and

$$\begin{aligned}0\leqslant&\frac{F(\tau_0,\tau_0-t_{i-1},u_P(\cdot,t_{i-1}),\psi(\cdot,t_{i-1}))(x_0)-\psi(x_0,t_{i-1})}{\tau_0-t_{i-1}}\\&+\frac{\psi(x_0,t_{i-1})-\psi(x_0,\tau_0)}{\tau_0-t_{i-1}}\\&+\frac{(e^{-(\bar{L}/2)(\tau_0-t_{i-1})}-1)(u_P(x_0,\tau_0)-\psi(x_0,\tau_0))}{\tau_0-t_{i-1}}.\end{aligned}$$

But

$$\frac{e^{-(\bar{L}/2)(\tau_0-t_{i-1})}-1}{\tau_0-t_{i-1}}\leqslant-\frac{\bar{L}}{2}+\frac{(\bar{L})^2}{8}|P|,$$

$$0<u_P(x_0,\tau_0)-\psi(x_0,\tau_0)=e^{(\bar{L}/2)(\tau_0+s_0)}\psi(x_0,y_0,\tau_0,s_0)\leqslant 9(R+1)\,e^{\bar{L}T}$$

and

$$\begin{aligned}&\frac{\psi(x_0,t_{i-1})-\psi(x_0,\tau_0)}{\tau_0-t_{i-1}}\\&\quad\leqslant-\frac{1}{4T}M_Pe^{(\bar{L}/2)(\tau_0+s_0)}+\frac{\bar{L}}{2}(u(y_0,s_0)-\psi(x_0,\tau_0))\\&\qquad+\bar{L}\left(\frac{R}{4T}+\frac{\bar{L}}{8}\left(6(R+1)+1+\frac{R}{2}\right)+3(R+1)\right)e^{\bar{L}T}|P|\\&\qquad+3(R+1)\,e^{(\bar{L}/2)(\tau_0+s_0)}\frac{\gamma_\varepsilon(\tau_0-s_0)-\gamma_\varepsilon(t_{i-1}-s_0)}{\tau_0-t_{i-1}}\end{aligned}$$

(F8), (H5), (2.23), (2.24) and the last inequality imply

$$
\begin{aligned}
&\frac{1}{4T} M_P e^{(\bar{L}/2)(\tau_0+s_0)} - 3(R+1)\, e^{(\bar{L}/2)(\tau_0+s_0)} \frac{\gamma_\varepsilon(\tau_0-s_0)-\gamma_\varepsilon(t_{i-1}-s_0)}{\tau_0-t_{i-1}} \\
&\qquad + \frac{\bar{L}}{2}\,(u_P(x_0,\tau_0)-u(y_0,s_0)) \\
&\quad \leqslant -H(\tau_0\, x_0, u_P(x_0,\tau_0), D\psi(x_0,t_{i-1})) + \bar{L}\bar{C}_1\,|P| \\
&\qquad + \bar{C}_5(1+\|D\psi(\cdot,t_{i-1})\| + \|D^2\psi(\cdot,t_{i-1})\|)\,|P| \\
&\qquad + \bar{L}\left(\frac{R}{4T} + (\bar{L}+3)(R+1)\right) e^{LT}\,|P| \qquad (2.28)
\end{aligned}
$$

where $\bar{C}_5 = C_5(R, \bar{\bar{L}})$ is given by (F8), provided that

$$\frac{\bar{L}}{2}\left(L+\frac{1}{2}\right)|P| < \frac{1}{2}. \qquad (2.29)$$

Now we add (2.21) and (2.28). Using (2.20), (2.23), (2.24), (H7) with $\bar{M} = M_{\max(R, L+1)}$ and

$$|D\psi(x_0,\tau_0) - D\psi(y_0,s_0)| \leqslant 4\delta e^{LT}$$

we obtain

$$
\begin{aligned}
&\frac{1}{2T} M_P e^{(\bar{L}/2)(\tau_0+s_0)} + \bar{L}(u_P(x_0,\tau_0)-u(y_0,s_0)) \\
&\quad \leqslant H(s_0, y_0, u(y_0,s_0)\, D\psi(y_0,s_0)) \\
&\qquad - H(\tau_0, x_0, u_P(x_0,\tau_0), D\bar{\psi}(y_0,s_0)) \\
&\qquad + \bar{L}\left(\frac{R}{4T} + (\bar{L}+3)(R+1)\right) e^{LT}\,|P| + 3(R+1)\, e^{(\bar{L}/2)(\tau_0+s_0)} \\
&\qquad \times \left(-\gamma'_\varepsilon(\tau_0-s_0) + \frac{\gamma_\varepsilon(\tau_0-s_0)-\gamma_\varepsilon(t_{i-1}-s_0)}{\tau_0-t_{i-1}}\right) \\
&\qquad + \bar{C}_5(1+\|D\psi(\cdot,t_{i-1})\| + \|D^2\psi(\cdot,t_{i-1})\|\ |P| \\
&\qquad + \bar{L}\bar{C}_1\,|P| + \bar{M}\left(4\delta e^{LT} + \frac{\bar{L}}{2}\left(L+\frac{1}{2}\right)|P|\right)
\end{aligned}
$$

and in view of (H3), (H5), (H4), (H6), (2.12) and (2.19),

$$\frac{1}{2T} M_P e^{(\bar{L}/2)(\tau_0+s_0)}$$

$$\leqslant \left(4\delta e^{\bar{L}T}+\frac{\bar{L}}{2}\left(L+\frac{1}{2}\right)|P|\right)\bar{M}+\bar{L}\bar{C}_1|P|$$

$$+\bar{C}_5\left(1+6(R+1)\,e^{\bar{L}T}\left(\frac{1}{\varepsilon}+\delta\right)+12(R+1)\,e^{\bar{L}T}\left(\frac{1}{\varepsilon^2}+\delta\right)\right)|P|$$

$$+(C_R+N_R)(1+Le^{\bar{L}T})\left(\frac{R\bar{L}}{2}+\frac{1}{2T}+2\delta+L+\|Du\|\right)\varepsilon^2$$

$$+\frac{3(R+1)\,e^{\bar{L}T}}{\varepsilon^2}|P|+\bar{L}\left(\frac{R}{4T}+(\bar{L}+3)(R+1)\right)e^{\bar{L}T}|P|. \tag{2.30}$$

Combining (2.16), (2.18) and (2.30), using the fact that $\varepsilon=|P|^{1/4}$, assuming $|P|<1$ and letting $\delta\downarrow 0$ we obtain

$$M_P \leqslant K_1|P|^{1/2}$$

and therefore

$$\sup_{(x,\tau)\in \bar{Q}_T}(u_P(x,\tau)-u(x,\tau))^+\leqslant K|P|^{1/2}$$

where

$$K=e^{\bar{L}T}K_1=2((\bar{C}_1)^2+L\,\|Du_0\|+\|Du\|^2+L\,\|Du_0\|)\,e^{\bar{L}T}$$

$$+2Te^{\bar{L}T}\left(\frac{\bar{L}}{2}\left(L+\frac{1}{2}\right)\bar{M}+\bar{L}\bar{C}_1+\bar{C}_5(1+18(R+1)\,e^{\bar{L}T})\right.$$

$$\left.+(C_R+N_R)(1+L)\left(\frac{R\bar{L}}{2}+\frac{1}{2T}+L+\|Du\|\right)+3(R+1)\,e^{\bar{L}T}\right) \tag{2.31}$$

provided that

$$|P|<\min\left\{1,\frac{1}{\bar{L}(L+\frac{1}{2})},\frac{1}{4}\frac{1}{(24re^{\bar{L}T}+(\bar{L}/2)(L+\frac{1}{2}))^2}\right\}. \tag{2.32}$$

(b) We begin with the observation that it suffices to assume $u_0\in C_b^{0,1}(\mathbb{R}^N)$. Indeed, if $u_0\in BUC(\mathbb{R}^N)$, we can find a sequence $\{u_{0m}\}$ in $C_b^{0,1}(\mathbb{R}^N)$ such that $\|u_{0m}-u_0\|\to 0$ as $m\to\infty$. Then, in view of Proposition 1.4,

$$\|u-u_m\|\leqslant e^{\bar{L}T}\|u_0-u_{0m}\|$$

where u_m is the viscosity solution of (0.1) for u_{0m}. Moreover, if $u_{P,m}$: $\bar{Q}_T \to \mathbb{R}$ is defined by (2.1) for u_{0m}, then,

$$\begin{aligned}\|u_{P,m}(\cdot,\tau) - u_P(\cdot,\tau)\| &\leqslant (1 + (\tau - t_{i-1})\, C_6)\, \|u_{P,m}(\cdot, t_{i-1}) - u_P(\cdot, t_{i-1})\| \\ &\leqslant e^{(\tau - t_{i-1})C_6}\, \|u_{P,m}(\cdot, t_{i-1}) - u_P(\cdot, t_{i-1})\|\end{aligned}$$

where t_{i-1} is such that $\tau \in (t_{i-1}, t_i]$.

Combining all the above and using a simple inductive argument we obtain

$$\|u_P - u\| \leqslant (e^{T\bar{L}} + e^{TC_6})\, \|u_{0m} - u_0\| + \|u_{P,m} - u_m\|$$

which proves the claim. For the rest we are going to assume that $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. In this case, we need the following lemma

LEMMA 2.2. *For a partition $P = \{0 = t_0 < \cdots < t_{n(P)} = T\}$ of $[0, T]$ and $u_0 \in C_b^{0,1}(\mathbb{R}^N)$ let $u_P : \bar{Q}_T \to \mathbb{R}$ be defined by* (2.1). *The following are true*:

(a) *for every $\tau \in [0, T]$*

(i) $\|u_P(\cdot, \tau)\| \leqslant e^{\tau C_7}(\|u_0\| + TC_7)$.

(ii) $u_P(\cdot, \tau) \in C_b^{0,1}(\mathbb{R}^N)$ *and* $\|Du_P(\cdot, \tau)\| \leqslant e^{\tau(C_8 + C_9)}(\|Du_0\| + \tau\bar{C}_9)$ *where* $\bar{C}_9 = C_9(e^{TC_7}(\|u_0\| + TC_7))$.

(iii) *If* $\tau \in (t_{i-1}, t_i]$ *for some* $i = 1, \ldots, n(P)$, *then* $\|u_P(\cdot, \tau) - u_P(\cdot, t_{i-1})\| \leqslant \bar{\bar{C}}_1(\tau - t_{i-1})$ *where* $\bar{\bar{C}}_1 = C_1(e^{TC_7}(\|u_0\| + TC_7), e^{T(C_8 + \bar{C}_9)}(\|Du_0\| + T\bar{C}_9))$.

(b) $u_P \in BUC(\bar{Q}_T)$.

Since the lemma is proved in exactly the same way as Lemma 2.1, we omit its proof and we continue with the proof of Theorem 2.2(b). It suffices to show that

$$\sup_{(x,\tau) \in \bar{Q}_T} \{e^{-\bar{L}\tau}(u_P(x,\tau) - u(x,\tau))^{\pm}\} \to 0 \qquad \text{as} \quad |P| \to 0.$$

Without any loss of generality here we prove only that if

$$M_P = \sup_{(x,\tau) \in \bar{Q}_T} (e^{-\bar{L}\tau}(u_P(x,\tau) - u(x,\tau))^+)$$

then

$$M_P \to 0 \qquad \text{as} \quad |P| \to 0.$$

The proof has many similarities with that of part (a); therefore we omit some of the details.

To this end, we claim that for every $\alpha>0$ there is a $\rho_0=\rho_0(\|u_0\|, \alpha)>0$ so that if $|P|<\rho_0$, then

$$M_P<\alpha.$$

For $\alpha>0$ fixed but arbitrary, let $\varepsilon>0$ be so that

$$2(\|Du_0\|+\|Du\|+\bar{\bar{C}}_1)\,\varepsilon+2T\omega_{H,\max(\|Du\|+1,\|u\|)}(\varepsilon)<\frac{\alpha}{2} \tag{2.33}$$

where $\omega_{H,R}$ is the modulus of continuity of H on $[0, T]\times\mathbb{R}^N\times[-R, R]\times B_N(0, R)$, $\bar{\bar{C}}_1$ is given by Lemma 2.2(a)(iii) and $\|Du\|=\sup_{0\leqslant\tau\leqslant T}\|Du(\cdot, \tau)\|$. (Note that Proposition 1.2 implies that $u(\cdot, \tau)\in C_b^{0,1}(\mathbb{R}^N)$ for every $\tau\in[0, T]$). For such ε choose $\rho_0>0$ so that if $\rho<\rho_0$, then

$$\begin{aligned}
2T\Bigg[&\omega_{H,\mathrm{mac}(\|Du\|+1,\|u\|)}\left(\frac{\bar{L}}{2}\left(L_1+\frac{1}{2}\right)\rho\right)+\rho\bar{L}\bar{\bar{C}}_1\\
&+\bar{L}\left(\frac{R}{4T}+(\bar{L}+3)(R+1)\right)e^{\bar{L}T}\rho\\
&+\left\|\frac{F(t, \rho, u, \phi)-\phi}{\rho}+H(t, \cdot, u, D\phi)\right\|\Bigg]<\frac{\alpha}{2}
\end{aligned} \tag{2.34}$$

with $R\geqslant\max(\|u\|, e^{TC_7}(TC_7+\|u_0\|))$ and

$$\|u\|, \|Du\|, \|D\phi\|, \|D^2\phi\|$$
$$\leqslant\max\left(R, L_1, \left(\frac{6(R+1)}{\varepsilon}+1\right)e^{\bar{L}T}, \left(\frac{12(R+1)}{\varepsilon^2}+1\right)e^{\bar{L}T}\right)$$

where L_1 is given by Lemma 2.1(a)(ii) so that

$$\|Du\|, \sup_{0\leqslant\tau\leqslant T}\|Du_P(\cdot, \tau)\|\leqslant L_1$$

and $\rho_0>0$ is such that

$$\frac{\bar{L}}{2}\left(L_1+\frac{1}{2}\right)\rho_0<\frac{1}{2}.$$

We are going to show that if $|P|<\rho_0$, then

$$M_P<\alpha. \tag{2.35}$$

Indeed, let $P=\{0=t_0<t_1<\cdots<t_{n(P)}=T\}$ be a partition of $[0,T]$ with $|P|<\rho_0$. Without any loss of generality we assume that

$$M_P>0.$$

In this case, as in the proof of part (a), for every $\delta>0$ we can define a continuous function $\Psi\colon \mathbb{R}^N\times\mathbb{R}^N\times[0,T]\times[0,T]\to\mathbb{R}$ by

$$\begin{aligned}\Psi(x,y,\tau,s)&=e^{-(\bar{L}/2)(\tau+s)}(u_P(x,\tau)-u(y,s))^+ +3(R+1)\,\beta_\varepsilon(x-y)\\&\quad+3(R+1)\,\gamma_\varepsilon(\tau-s)+2\delta\zeta(x,y)-\frac{(\tau+s)}{4T}M_P\end{aligned}$$

where $\zeta\in C_0^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, $0\leqslant\zeta\leqslant 1$, $|D\zeta|\leqslant 1$, $|D^2\zeta|\leqslant 1$, $\beta_\varepsilon(\cdot)=\beta(\cdot/\varepsilon)$, $\gamma_\varepsilon(\cdot)=\gamma(\cdot/\varepsilon)$, β and γ as in (2.9), (2.10) respectively,

$$\gamma\in C_0^\infty(\mathbb{R}),\, 0\leqslant\gamma\leqslant 1,\, \gamma(0)=1,\, |\gamma'|\leqslant 2,\, |\gamma''|\leqslant 4$$

such that there exists a point $(x_0,y_0,\tau_0,s_0)\in\mathbb{R}^N\times\mathbb{R}^N\times[0,T]\times[0,T]$ with the property that

$$\begin{aligned}&\Psi(x_0,y_0,\tau_0,s_0)\geqslant\Psi(x,y,\tau,s)\\&\text{for every } (x,y,\tau,s)\in\mathbb{R}^N\times\mathbb{R}^N\times[0,T]\times[0,T].\end{aligned}\tag{2.36}$$

Moreover, as in part (a), for $\delta<\delta_0=\min(\frac{1}{24},\frac{1}{8}M_P)$ we have

$$|x_0-y_0|\leqslant\varepsilon,\ |\tau_0-s_0|\leqslant\varepsilon,\ u_P(x_0,\tau_0)-u(y_0,s_0)>0$$

and (2.37)

$$\Psi(x_0,y_0,\tau_0,s_0)>0.$$

We have to consider the following three cases: $\tau_0\geqslant 0$ and $s_0=0$, $\tau_0=0$ and $s_0\geqslant 0$ and $\tau_0>0$ and $s_0>0$. We begin with the case $\tau_0\geqslant 0$ and $s_0=0$.

1*st case.* $\tau_0\geqslant 0$ *and* $s=0$. It follows from (2.36) that

$$\begin{aligned}&e^{-(\bar{L}/2)\tau_0}(u_P(x_0,\tau_0)-u_0(y_0))^+ +6(R+1)+2\delta\\&\quad\geqslant\Psi(x_0,y_0,\tau_0,0)\geqslant\tfrac{1}{2}M_P+6(R+1).\end{aligned}$$

So, in view of (2.37), for $\delta<\delta_0$

$$M_P\leqslant 2(\|Du_0\|+\bar{\bar{C}}_1)\,\varepsilon+4\delta.\tag{2.38}$$

2nd case. $\tau_0 = 0$ *and* $s_0 \geqslant 0$. Again (2.36) implies that

$$e^{-(\bar{L}/2)s_0}(u_0(x_0) - u(y_0, s_0))^+ + 6(R+1) + 2\delta$$
$$\geqslant \Psi(x_0, y_0, 0, s_0) \geqslant \tfrac{1}{2}M_P + 6(R+1)$$

and so,

$$M_P \leqslant 2\,\|Du\|\,\varepsilon + 4\delta. \tag{2.39}$$

3rd case. $\tau_0 > 0$ *and* $s_0 > 0$. As in the 3rd case of the proof of part (a) for $\delta < \delta_0$ we have

$$\frac{1}{2T} M_P e^{(\bar{L}/2)(\tau_0 + s_0)} \leqslant H(s_0, y_0, u(y_0, s_0), D\bar{\psi}(y_0, s_0))$$
$$- H(\tau_0, x_0, u(y_0, s_0), D\bar{\psi}(y_0, s_0))$$
$$+ \omega_{H, \max(R, L_1 + 1)}\left(4\delta e^{\bar{L}T} + \frac{\bar{L}}{2}\left(L_1 + \frac{1}{2}\right)|P|\right) + \frac{\alpha}{4T}$$

where $\bar{\psi}$ is given by (2.20). This yields

$$M_P \leqslant 2T\left(\omega_{H, \max(\|u\|, \|Du\| + 1)}(\varepsilon)\right.$$
$$\left. + \omega_{H, \max(R, L_1 + 1)}\left(4\delta e^{\bar{L}T} + \frac{\bar{L}}{2}\left(L_1 + \frac{1}{2}\right)|P|\right) + \frac{\alpha}{4T}\right). \tag{2.40}$$

Adding (2.38), (2.39) and (2.40), letting $\delta \downarrow 0$, and using (2.33), we obtain (2.35).

Remark 2.1. It follows from the proof of Theorem 2.1(b), that part (b) is really a result concerning Lipschitz continuous functions. In particular, if for $(t, \rho) \in K$, $F(t, \rho, \cdot, \cdot)$: $C_b^{0,1}(\mathbb{R}^N) \times C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ satisfies all the assumptions of Theorem 2.1(b) with (F9) replaced by (F13), then the conclusion of Theorem 2.1(b) holds for every $u_0 \in C_b^{0,1}(\mathbb{R}^N)$; (F9) was used only to show that if (2.2) is true for every $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, then it is true for $u_0 \in BUC(\mathbb{R}^N)$ too.

The next theorem is concerned with schemes which, although they do not satisfy a generator type assumption, can be approximated in a suitable way by schemes of the type considered in Theorem 2.1. More precisely:

THEOREM 2.2. (a) *For* H: $[0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfying* (H1), (H2) (*with a constant* C), (H5) (*with constant* $\bar{L}$ *independent of* R) *and*

(H4), (H6), (H7) (*with constants* C_R, N_R, M_R *respectively for* $R \geqslant 0$) *and for* $u_0 \in C_b^{0,1}(\mathbb{R}^N)$ *let* $u \in C_b^{0,1}(\bar{Q}_T)$ *be the viscosity solution of* (0.1) *in* $\bar{Q}_T$. *For* $(t, \rho) \in K = \{(t, \rho) \in [0, T] \times [0, \rho_0]: 0 \leqslant \rho \leqslant t\}$, *where* $\rho_0 = \rho_0(\|u_0\|) > 0$, *let* $F(t, \rho, \cdot): C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ *be such that for every* $u, \bar{u} \in C_b^{0,1}(\mathbb{R}^N)$:

(F15) *There exists a constant* $C_{11} \geqslant 0$ *such that*

$$\|F(t, \rho, u) - F(t, \rho, \bar{u})\| \leqslant e^{\rho c_{11}} \|u - \bar{u}\|.$$

Moreover, suppose that for every $(t, \rho) \in K$ *there exists a mapping* $\bar{F}(t, \rho, \cdot, \cdot): C_b^{0,1}(\mathbb{R}^N) \times C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$, *which satisfies the assumptions of Theorem* 2.1(a) *with* (F13) *instead of* (F5) *and also*

(F16) *For every* $u \in C_b^{0,1}(\mathbb{R}^N)$

$$\|F(t, \rho, u) - \bar{F}(t, \rho, u, u)\| \leqslant C_{12} \rho^2$$

where $C_{12} = C_{12}(\|u\|, \|Du\|)$.

For a partition $P = \{0 = t_0 < t_1 < \cdots < t_{n(P)} = T\}$ *of* $[0, T]$, *let* $u_P: \bar{Q}_T \to R$ *be defined by*

$$\begin{aligned} &u_P(x, 0) = u_0(x), \\ &u_P(x, t) = F(t, t - t_{i-1}, u_P(\cdot, t_{i-1}))(x) \\ &\qquad\qquad \text{if } t \in (t_{i-1}, t_i] \text{ for some } i = 1, \ldots, n(P). \end{aligned} \tag{2.41}$$

There exists a constant K, which depends only on $\|u_0\|$ *and* $\|Du_0\|$, *such that*

$$\|u_P - u\| \leqslant K |P|^{1/2} \tag{2.42}$$

for $|P|$ *sufficiently small.*

(b) *For* H: $[0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfying* (H1), (H2), (H4) *and* (H5) (*with a constant* $\bar{L}$ *independent of* R) *and for* $u_0 \in BUC(\mathbb{R}^N)$ *let* $u \in BUC(\bar{Q}_T)$ *be the viscosity solution of* (0.1) *in* $\bar{Q}_T$. *For* $(t, \rho) \in K = \{(t, \rho) \in [0, T] \times [0, \rho_0]: 0 \leqslant \rho \leqslant t\}$, *where* $\rho_0 = \rho_0(\|u_0\|) > 0$, *let* $F(t, \rho, \cdot)$: $BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$ *be such that it satisfies* (F15) *for every* $u, \bar{u} \in BUC(\mathbb{R}^N)$. *Moreover, suppose that for every* $(t, \rho) \in K$ *there exists a mapping* $\bar{F}(t, \rho, \cdot, \cdot)$: $BUC(\mathbb{R}^N) \times BUC(\mathbb{R}^N) \to BUC(\mathbb{R}^N)$, *which satisfies the assumption of Theorem* 2.1(b) *and also*

(F17) *For every* $u \in C_b^{0,1}(\mathbb{R}^N)$

$$\|F(t, \rho, u) - \bar{F}(t, \rho, u, u)\| = o(\rho)$$

where $o(\rho)$ *depends only on* $\|u\|$ *and* $\|Du\|$ *and* $o(\rho)/\rho \to 0$ *as* $\rho \to 0$.

If, for a partition P of $[0, T]$, $u_P\colon \bar{Q}_T \to R$ *is defined by* (2.41), *then*

$$\|u_P - u\| \to 0 \qquad \textit{as} \quad |P| \to 0. \tag{2.43}$$

Remark 2.2. A remark analogous to the ones following Theorem 2.1 applies here too.

Proof of Theorem 2.2. (b) For $u_0, v_0 \in BUC(\mathbb{R}^N)$ let $u_P, v_P\colon \bar{Q}_T \to \mathbb{R}$ be defined by (2.41). A simple inductive argument, in view of (F15), implies that

$$\|u_P - v_P\| \leqslant e^{TC_{11}} \|u_0 - v_0\|.$$

But then it is easy to see, using the arguments at the beginning of the proof of Theorem 2.1(b), that it suffices to prove (2.43) for $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. In this case let $\bar{u}_P\colon \bar{Q}_T \to \mathbb{R}$ be defined by (2.1) for the above $\bar{F}$ and u_0. There exist constants R_1 and L_1 which depend only on $\|u_0\|$ and $\|Du_0\|$ such that for every partition P of $[0, T]$

$$\|\bar{u}_P\| \leqslant R_1 \qquad \text{and} \qquad \sup_{0 \leqslant \tau \leqslant T} \|D\bar{u}_P(\cdot, \tau)\| < L_1.$$

Next for $\alpha > 0$ fixed but arbitrary, let $\rho_1 = \rho_1(\alpha) > 0$ be so that if $\rho \leqslant \rho_1$ then

$$o(\rho) < \frac{\alpha}{2Te^{C_{11}T}} \rho$$

where $o(\rho)$ is given by (F17) and corresponds to R_1 and L_1. If $\tau \in (t_{i-1}, t_i]$ for some $i = 1, \dots, n(P)$,

$$\begin{aligned} \|u_P(\cdot, \tau) - \bar{u}_P(\cdot, \tau)\| \leqslant{} & \|F(\tau, \tau - t_{i-1}, u_P(\cdot, t_{i-1})) \\ & - F(\tau, \tau - t_{i-1}, \bar{u}_P(\cdot, t_{i-1})\| \\ & + \|F(\tau, \tau - t_{i-1}, \bar{u}_P(\cdot, t_{i-1})) \\ & - \bar{F}(\tau, \tau - t_{i-1}, \bar{u}_P(\cdot, t_{i-1}), \bar{u}_P(\cdot, t_{i-1}))\| \end{aligned}$$

i.e.

$$\begin{aligned} \|u_P(\cdot, \tau) - \bar{u}_P(\cdot, \tau)\| \leqslant{} & e^{(\tau - \tau_{i-1})C_{11}}(\|u_P(\cdot, t_{i-1}) - \bar{u}_P(\cdot, t_{i-1})\| \\ & + \frac{\alpha}{2Te^{C_{11}T}} (\tau - t_{i-1}). \end{aligned}$$

An inductive argument then implies

$$\|u_P(\cdot,\tau)-\bar{u}_P(\cdot,\tau)\| \leqslant Te^{TC_{11}}\frac{\alpha}{2Te^{C_{11}T}}=\frac{\alpha}{2}.$$

Finally, since by Theorem 2.1(b) $\bar{u}_P \to u$ as $|P| \to 0$, let $\rho_2=\rho_2(\alpha)>0$ be such that if $\rho \leqslant \rho_0$, then

$$\|\bar{u}_P-u\| < \frac{\alpha}{2}.$$

For $\rho \leqslant \min\{\rho_1, \rho_2\}$ we have

$$\|u_P-u\| < \alpha$$

which proves the result.

(a) Using Theorem 2.1(a), the above relations, appropriately modified so that they apply to this case, and (F16), for $|P|$ sufficiently small, we have

$$\|u_P-u\| < \bar{K}|P|^{1/2}+TC_{12}e^{TC_{11}}|P$$

where $\bar{K}$ and C_{12} depend only on $\|u_0\|$ and $\|Du_0\|$. For $|P|<1$ this implies (2.42) with

$$K=\bar{K}+TC_{12}e^{TC_{11}}.$$

Remark 2.3. A remark analogous to Remark 2.1 applies to Theorem 2.2. In particular, if $F(t,\rho,\cdot)$: $C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ satisfies all the assumptions of Theorem 2.2(b), then the results hold for every $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. Moreover, it follows from the proof of part (b) that we may assume $\bar{F}(t,\rho,\cdot,\cdot)$: $C_b^{0,1}(\mathbb{R}^N)\times C_b^{0,1}(\mathbb{R}^N) \to C_b^{0,1}(\mathbb{R}^N)$ satisfying all the properties. Then the result still holds.

3

In this section we prove the convergence of explicit finite difference schemes to the viscosity solution of (0.1) and give explicit error estimates. As mentioned in the introduction such a result was first proved by Crandall and Lions [4] for the problem (0.2). The theorem stated below is a generalization of their result.

We now describe the class of difference schemes to be considered here. For notational simplicity only, we will assume $N=2$. The definitions and results for general N will be clear from this special case and we will not state them. A generic point in $\mathbb{R}^2$ will be denoted by (x, y) and we will write $Du=(u_x, u_y)$. Let α, β be some given positive numbers. For $\rho>0$, $u: \mathbb{R}^2 \to \mathbb{R}$ and $(j, k) \in \mathbb{Z} \times \mathbb{Z}$,[11] we define $u^\rho_{j,k}: \mathbb{R}^2 \to \mathbb{R}$, $\Delta^x_+ u^\rho_{j,k}: \mathbb{R}^2 \to \mathbb{R}$ and $\Delta^y_+ u^\rho_{j,k}: \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} u^\rho_{j,k}(x, y) &= u(x+j\alpha\rho, y+k\beta\rho), \\ \Delta^x_+ u^\rho_{j,k}(x, y) &= u^\rho_{j+1,k}(x, y) - u^\rho_{j,k}(x, y), \\ \Delta^y_+ u^\rho_{j,k}(x, y) &= u^\rho_{j,k+1}(x, y) - u^\rho_{j,k}(x, y). \end{aligned} \tag{3.1}$$

Moreover, for p, q, r, s fixed nonnegative integers and $u: \mathbb{R}^2 \to \mathbb{R}$ let $(\Delta^x_+ u)^\rho: \mathbb{R}^2 \to \mathbb{R}^{(p+q+1)(r+s+2)}$ and $(\Delta^y_+ u)^\rho: \mathbb{R}^2 \to \mathbb{R}^{(p+q+2)(r+s+1)}$ be defined by

$$(\Delta^x_+ u)^\rho (x, y) = (\Delta^x_+ u^\rho_{-p,-r}(x, y), ..., \Delta^x_+ u^\rho_{q,s+1}(x, y))$$

and (3.2)

$$(\Delta^y_+ u)^\rho (x, y) = (\Delta^y_+ u^\rho_{-p,-r}(x, y), ..., \Delta^y_+ u^\rho_{q+1,s}(x, y)).$$

If $u \in C^{0,1}(\mathbb{R}^2)$, it is easy to see that for every $(x, y) \in \mathbb{R}^2$,

$$\frac{|\Delta^x_+ u^\rho_{j,k}(x, y)|}{\rho\alpha}, \frac{|\Delta^y_+ u^\rho_{j,k}(x, y)|}{\rho\beta} < \|Du\| \tag{3.3}$$

and

$$\frac{|(\Delta^x_+ u)^\rho (x, y)|}{\rho\alpha}, \frac{|(\Delta^y_+ u)^\rho (x, y)|}{\rho\beta} \leqslant A \|Du\| \tag{3.4}$$

where $A=\sqrt{2(p+q+2)(r+s+2)}$ and $|\ \ |$ denotes the usual metric in any $\mathbb{R}^m$. Finally, for $u, v: \mathbb{R}^2 \to \mathbb{R}$ and $0 \leqslant \rho \leqslant t \leqslant T$, let $F(t, \rho, u, v): \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\begin{aligned} &F(t, \rho, u, v)(x, y) \\ &\quad = v(x, y) - \rho g\left(t, x, y, u(x, y), \frac{(\Delta^x_+ v)^\rho}{\rho\alpha}(x, y), \frac{(\Delta^y_+ v)^\rho}{\rho\beta}(x, y)\right) \quad \text{if } \rho>0, \\ &F(t, 0, u, v)(x, y) = v(x, y) \end{aligned} \tag{3.5}$$

where $g: [0, T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)} \to \mathbb{R}$ satisfies

(G1) g is uniformly continuous on $[0, T] \times \mathbb{R}^2 \times [-R, R] \times B_{(p+q+1)(r+s+2)}(0, R) \times B_{(p+q+2)(r+s+1)}(0, R)$ for every $R>0$.

[11] $\mathbb{Z}$ is the set of integers.

(G2) There exists a constant $C \geqslant 0$ such that

$$\sup_{(x,t) \in \bar{Q}_T} |g(t, x, 0, 0, \ldots, 0)| \leqslant C.$$

(G3) For every $R > 0$ there exists a constant $\bar{L}_R > 0$ such that

$$|g(t, x, y, r, w, z) - g(t, x, y, \bar{r}, w, z)| \leqslant \bar{L}_R |r - \bar{r}|$$

for every $t \in [0, T]$, $(x, y) \in \mathbb{R}^2$, $r, \bar{r} \in [-R, R]$ and $(w, z) \in \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)}$.

(G4) For every $R > 0$ there is a constant C_R such that

$$\begin{aligned} &|g(t, x, y, r, w, z) - g(t, \bar{x}, \bar{y}, r, w, z)| \\ &\qquad \leqslant C_R(1 + |(w, z)|)(|t - \bar{t}| + |(x, y) - (\bar{x}, \bar{y})|) \end{aligned}$$

for $t, \bar{t} \in [0, T]$, (x, y), $(\bar{x}, \bar{y}) \in \mathbb{R}^2$, $|r| \leqslant R$ and $(w, z) \in \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)}$.

(G5) For every $R > 0$ there is a constant $M_R > 0$ such that $|g(t, x, y, r, w, z) - g(t, x, y, r, \bar{w}, \bar{z})| \leqslant M_R |(w, z) - (\bar{w}, \bar{z})|$ for $t \in [0, T]$, $(x, y) \in \mathbb{R}^2$, $|r| \leqslant R$ and (w, z), $(\bar{w}, \bar{z}) \in \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)}$ with $|(w, z)|$, $|(\bar{w}, \bar{z})| \leqslant R$.

The explicit finite difference schemes of interest here are generated by (3.5). We say that (3.5) is *consistent* with the equation $u_t + H(t, x, y, u, u_x, u_y) = 0$ occurring in (0.1), if

$$g(t, x, y, r, a, \ldots, a, b, \ldots, b) = H(t, x, y, r, a, b) \qquad \text{for } t \in [0, T], (x, y) \in \mathbb{R}^2, r \in \mathbb{R}, a, b \in \mathbb{R}. \tag{3.6}$$

Moreover, we call (3.5) *monotone* on $[-R, R]$, if

For every $u: \mathbb{R}^2 \to \mathbb{R}$, if $v(x, y) \leqslant w(x, y)$ for every $(x, y) \in \mathbb{R}^2$, then, for any $(a, b) \in \mathbb{R}^2$, such that

$$\begin{aligned} &\frac{|\Delta_+^x v_{j,k}^\rho(a, b)|}{\rho\alpha}, \frac{|\Delta_+^y v_{j',k'}^\rho(a, b)|}{\rho\beta}, \\ &\frac{|\Delta_+^x w_{j,k}^\rho(a, b)|}{\rho\alpha}, \frac{|\Delta_+^y w_{j',k'}^\rho(a, b)|}{\rho\beta} \leqslant R \end{aligned} \tag{3.7}$$

for $-p \leqslant j' \leqslant q+1$, $-r \leqslant k' \leqslant s$, $-p \leqslant j \leqslant q$, $-r \leqslant k \leqslant s+1$,

$$F(t, \rho, u, v)(a, b) \leqslant F(t, \rho, u, w)(a, b).$$

The main result is

THEOREM 3.1. *Let H: $[0, T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $u_0 \in C_b^{0,1}(\mathbb{R}^2)$. Let g: $[0, T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{(p+q+1)(r+s+2)} \times \mathbb{R}^{(p+q+2)(r+s+1)} \to \mathbb{R}$ satisfy* (G1), (G2), (G3), (G4) *and* (G5) *and suppose that* (3.5) *is consistent with* (0.1) *and monotone on* $[-[e^{T(2\bar{C}e^{\bar{L}T}+\bar{L})}(\|Du_0\| + \bar{C}T) + 1), e^{T(2\bar{C}e^{\bar{L}T}+\bar{L})}(\|Du_0\| + \bar{C}T) + 1]$, *where, if* $R = e^{\bar{L}T}(\|u_0\| + TC)$, *then* $\bar{C} = C_R$. *For a partition P of* $[0, T]$, *define* $u_P: \bar{Q}_T \to \mathbb{R}$ *by* (2.1) *and* (3.5). *Let u be the viscosity solution of* (0.1). *Then there is a constant K, which depends only on* $\|u_0\|$, $\|Du_0\|$, *g and T, such that, for sufficiently small* $|P|$,

$$\|u_P - u\| \leqslant K|P|^{1/2}. \tag{3.8}$$

Proof. It suffices to check the assumptions of Theorem 2.1(a). It is obvious that, for every (t, ρ) and $u, v \in C_b^{0,1}(\mathbb{R}^2)$, $F(t, \rho, u, v) \in C_b^{0,1}(\mathbb{R}^2)$. Moreover, (F1) and (F3) follow from (3.5). Next, for $u \in C_b^{0,1}(\mathbb{R}^2)$ observe that

$$\begin{aligned} &|F(t, \rho, u, u)(x, y) - u(x, y)| \\ &\quad = \rho \left| g\left(t, x, y, u(x, y), \frac{(\Delta_+^x u)^\rho}{\rho\alpha}(x, y), \frac{(\Delta_+^y u)^\rho}{\rho\beta}(x, y)\right)\right| \end{aligned}$$

and therefore

$$\|F(t, \rho, u, u) - u\| \leqslant \rho C_4(\|u\|, \|Du\|)$$

where

$$C_4(\|u\|, \|Du\|) = \sup_{\substack{t \in [0,T] \\ \xi \in R^2 \\ |r| \leqslant \|u\| \\ |w| \leqslant A\|Du\| \\ |z| \leqslant A\|Du\|}} |g(t, \xi, r, w, z)|$$

with A given by (3.4). The fact that $(t, \rho) \to F(t, \rho, u, u)$ is continuous in the $\|\ \|$-norm for $u \in C_b^{0,1}(\mathbb{R}^2)$ is a consequence of the above inequality and (G2), (G3), (G4), (G5).

Now we want to verify (F5). To this end, let

$$r = \sqrt{2}\,(\max\{p, r, q+1, s+1\} + 1)\max\{\alpha, \beta\}$$

and assume that $v(x, y) \leqslant w(x, y)$ for every $(x, y) \in \mathbb{R}^2$. If for some $(\bar{x}, \bar{y}) \in \mathbb{R}^2$

$$\begin{aligned} &|v(\bar{x} + a, \bar{y} + b) - v(\bar{x} + \bar{a}, \bar{y} + \bar{b})|, |w(\bar{x} + a, \bar{y} + b) - w(\bar{x} + \bar{a}, \bar{y} + \bar{b})| \\ &\qquad \leqslant \bar{\bar{L}}\,|(a, b) - (\bar{a}, \bar{b})| \end{aligned} \tag{3.9}$$

for $(a, b), (\bar{a}, \bar{b}) \in B_2(0, \rho r)$ and $\bar{\bar{L}} = e^{T(\bar{L} + 2\bar{C}e^{\bar{L}T})}(\|Du_0\| + \bar{C}T) + 1$, we claim that, for any $u \in C_b^{0,1}(\mathbb{R}^N)$,

$$F(t, \rho, u, v)(\bar{x}, \bar{y}) \leqslant F(t, \rho, u, \bar{v})(\bar{x}, \bar{y}). \tag{3.10}$$

Condition (F5) follows from (3.10) and the fact that (G2), (G3), (G4), (5.6) and Proposition 1.3(c) imply that $u(\cdot, \tau) \in C_b^{0,1}(R^2)$ for every τ with

$$\sup_{0 \leqslant \tau \leqslant T} \|Du(\cdot, \tau)\| \leqslant e^{T(2\bar{C}e^{\bar{L}T} + \bar{L})}(\|Du_0\| + \bar{C}T).$$

To prove (3.10) we use the monotonicity of the scheme. In particular, for some j, k with $-p \leqslant j \leqslant q$, $-r \leqslant k \leqslant s+1$ we have

$$\left|\frac{\Delta_+^x v_{j,k}^\rho}{\rho\alpha}(\bar{x}, \bar{y})\right| = \left|\frac{v(\bar{x} + \rho(j+1)\,\alpha, \bar{y} + \rho k\beta) - v(\bar{x} + \rho j\alpha, \bar{y} + \rho k\beta)}{\rho\alpha}\right|.$$

But $(\rho(j+1)\,\alpha, \rho k\beta), (\rho j\alpha, \rho k\beta) \in B_2(0, \rho r)$; therefore by (3.9)

$$|\Delta_+^x v_{j,k}^\rho(\bar{x}, \bar{y})| \leqslant \bar{\bar{L}}\rho\alpha$$

and similarly

$$|\Delta_+^x w_{j,k}^\rho(\bar{x}, \bar{y})| \leqslant \bar{\bar{L}}\rho\alpha, \ |\Delta_+^y v_{j',k'}^\rho(\bar{x}, \bar{y})|, \ |\Delta_+^y w_{j',k'}^\rho(\bar{x}, \bar{y})| \leqslant \bar{\bar{L}}\rho\beta.$$

Inequality (3.10) then follows from (3.7).

For (F6) observe that the discussion after the statement of Theorem 2.1 implies that, if $u \in C_b^{0,1}(\mathbb{R}^2)$ with $\|Du\| \leqslant \bar{\bar{L}} + 1$, then

$$\|F(t, \rho, u, u) - F(t, \rho, u, 0)\| \leqslant \|u\|$$

and therefore

$$\|F(t, \rho, u, u)\| \leqslant \|u\| + \|F(t, \rho, u, 0)\|.$$

But

$$\begin{aligned} &|F(t, \rho, u, 0)(x, y)| \\ &\quad = |-\rho g(t, x, y, u(x, y), 0 \cdots 0, 0 \cdots 0)| \\ &\quad \leqslant \rho(\bar{L}\,\|u\| + C) \end{aligned}$$

thus

$$\|F(t, \rho, u, u)\| \leqslant e^{\rho\bar{L}}(\|u\| + \rho C).$$

Next, for $u \in C_b^{0,1}(\mathbb{R}^2)$ such that $\|u\| \leqslant e^{T\bar{L}}(\|u\| + TC)$ and $\|Du\| \leqslant \bar{\bar{L}}$, let $\bar{u}\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\bar{u}(x, y) = u(x+\eta, y+\xi)$$

for some $(\eta, \xi) \in \mathbb{R}^2$. We have

$$\begin{aligned}
&|F(t, \rho, u, u)(x, y) - F(t, \rho, u, u)(x+\eta, y+\xi)| \\
&\quad \leqslant \|F(t, \rho, u, u) - F(t, \rho, u, \bar{u})\| + \rho \left| g\left(t, x, y, u(x, y),\right.\right. \\
&\quad \left.\frac{(\Delta_+^x u)^\rho}{\rho\alpha}(x+\eta, y+\xi), \frac{(\Delta_+^y u)^\rho}{\rho\beta}(x+\eta, y+\xi)\right) \\
&\quad - g\left(t, x+\eta, y+\xi, u(x+\eta, y+\xi),\right. \\
&\quad \left.\left.\frac{(\Delta_+^x u)^\rho}{\rho\alpha}(x+\eta, y+\xi), \frac{(\Delta_+^y u)^\rho}{\rho\beta}(x+\eta, y+\xi)\right)\right|
\end{aligned}$$

therefore

$$\begin{aligned}
&|F(t, \rho, u, u)(x, y) - F(t, \rho, u, u)(x+\eta, y+\xi)| \\
&\quad \leqslant \|Du\|\,|(\eta, \xi)| + \rho\bar{L}\,\|Du\|\,|(\eta, \xi)| + \rho\bar{C}(1 + \|Du\|)\,|(\eta, \xi)|
\end{aligned}$$

which implies

$$\|DF(t, \rho, u, u)\| \leqslant e^{\rho(\bar{L} + \bar{C})}(\|Du\| + \bar{C}\rho).$$

Since

$$e^{T(\bar{L} + \bar{C})}(\|Du_0\| + \bar{C}T) < \bar{\bar{L}}$$

(F7) holds.

Finally, for $u \in C_b^{0,1}(\mathbb{R}^N)$ and $\phi \in C_b^2(\mathbb{R}^N)$, if $(x, y) \in \mathbb{R}^2$ is such that

$$|D\phi(x, y)| < \bar{\bar{L}} + 1$$

then, for $\rho > 0$,

$$\begin{aligned}
&\left|\frac{F(t, \rho, u, \phi)(x, y) - \phi(x, y)}{\rho} + H(t, x, y, u(x, y), \phi_x(x, y), \phi_y(x, y))\right| \\
&\quad = \left| g\left(t, x, y, u(x, y), \frac{(\Delta_+^x \phi)^\rho}{\rho\alpha}(x, y), \frac{(\Delta_+^y \phi)^\rho}{\rho\alpha}(x, y)\right)\right. \\
&\qquad \left. - g(t, x, y, u(x, y), \phi_x(x, y), \ldots, \phi_x(x, y), \phi_y(x, y), \ldots, \phi_y(x, y))\right|
\end{aligned}$$

and therefore

$$\left|\frac{F(t, \rho, u, \phi)(x, y) - \phi(x, y)}{\rho} + H(t, x, y, u(x, y), \phi_x(x, y), \phi_y(x, y))\right|$$
$$\leqslant \bar{M} \|D^2\phi\| \rho$$

where $\bar{M} = 2AM_{\max(\|u\|, \bar{L}+1)}r$, thus (F8).

4

This section is devoted to the convergence of certain fully implicit finite difference schemes to the viscosity solution of (0.1). We now describe the class of difference schemes to be considered here. For notational simplicity only, we will assume $N = 2$. The definitions and results for general N will be clear from this special case and we will not state them. A generic point in $\mathbb{R}^2$ will be denoted by (x, y) and we will write $Du = (u_x, u_y)$. Let $\alpha, \beta > 0$ be some given positive numbers. For $\rho > 0$ and $u\colon \mathbb{R}^2 \to \mathbb{R}$ we define $\Delta^{x,\pm,\rho}u\colon \mathbb{R}^2 \to \mathbb{R}$ and $\Delta^{y,\pm,\rho}u\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned}
\Delta^{x,+,\rho}u(x, y) &= u(x + \alpha\rho, y) - u(x, y),\\
\Delta^{x,-,\rho}u(x, y) &= u(x, y) - u(x - \alpha\rho, y),\\
\Delta^{y,+,\rho}u(x, y) &= u(x, y + \rho\beta) - u(x, y),\\
\Delta^{y,-,\rho}u(x, y) &= u(x, y) - u(x, y - \rho\beta).
\end{aligned} \tag{4.1}$$

If $u \in C^{0,1}(\mathbb{R}^2)$, it is easy to see that for every $(x, y) \in \mathbb{R}^2$

$$\frac{|\Delta^{x,\pm,\rho}u(x, y)|}{\rho\alpha}, \frac{|\Delta^{y,\pm,\rho}u(x, y)|}{\rho\beta} \leqslant \|Du\|. \tag{4.2}$$

As far as $H\colon [0, T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is concerned, here we assume that it satisfies (H1), (H2),

(H8) There is a constant $L > 0$ such that

$$\begin{aligned}
&|H(t, (x, y), r, (p, q)) - H(\bar{t}, (\bar{x}, \bar{y}), \bar{r}, (\bar{p}, \bar{q}))|\\
&\qquad \leqslant L(|t - \bar{t}| + |(x, y) - (\bar{x}, \bar{y})| + |r - \bar{r}| + |(p, q) - (\bar{p}, \bar{q})|)
\end{aligned}$$

for every $t, \bar{t} \in [0, T]$, $r, \bar{r} \in \mathbb{R}$ and $(x, y), (\bar{x}, \bar{y}), (p, q), (\bar{p}, \bar{q}) \in \mathbb{R}^2$, and

(H9) H is monotone with respect to p and q for every $t \in [0, T]$, $r \in \mathbb{R}$ and $(x, y), (p, q) \in \mathbb{R}^2$.

For $u, v, w \in BUC(\mathbb{R}^2)$ and $0<\rho \leqslant t \leqslant T$ let $T(t, \rho, u, v)\, w\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$T(t, \rho, u, v)\, w(x, y)$$
$$= v(x, y) - \rho H\left(t, x, y, u(x, y), \frac{\Delta^{x, \pm, \rho} w}{\rho\alpha}(x, y), \frac{\Delta^{y, \pm, \rho} w}{\rho\beta}(x, y)\right) \tag{4.3}$$

where we use $\Delta^{x,+,\rho}(\Delta^{y,+,\rho})$, if H is nonincreasing with respect to $p(q)$, and $\Delta^{x,-,\rho}(\Delta^{y,-,\rho})$, if H is nondecreasing with respect to $p(q)$. In view of (4.1), (H1), (H2) and (H8), it is obvious that $T(t, \rho, u, v)\, w \in BUC(\mathbb{R}^2)$. Moreover, we have

LEMMA 4.1. *For α, β sufficiently large, $T(t, \rho, u, v)$ has a fixed point in $BUC(\mathbb{R}^2)$. If, moreover, $u, v \in C_b^{0,1}(\mathbb{R}^2)$, then the fixed point is in $C_b^{0,1}(\mathbb{R}^2)$.*

Proof. We first show that, for α, β sufficiently large, $T(t, \rho, u, v)$ is a strict contraction in the $\|\ \|$-norm. Indeed, if $w, z \in BUC(\mathbb{R}^2)$, then

$$\begin{aligned} &|T(t, \rho, u, v)\, w(x, y) - T(t, \rho, u, v)\, z(x, y)| \\ &\quad\leqslant \rho \left| H\left(t, x, y, u(x, y), \frac{\Delta^{x, \pm, \rho} w}{\rho\alpha}(x, y), \frac{\Delta^{y, \pm, \rho} w}{\rho\beta}(x, y)\right) \right. \\ &\qquad \left. - H\left(t, x, y, u(x, y), \frac{\Delta^{x, \pm, \rho} z}{\rho\alpha}(x, y), \frac{\Delta^{y, \pm, \rho} z}{\rho\beta}(x, y)\right)\right| \\ &\quad\leqslant 2\sqrt{2}\, L\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \|w - z\|. \end{aligned}$$

So if $C_0 = 2\sqrt{2}\, L(1/\alpha + 1/\beta) < \frac{1}{2}$,

$$\|T(t, \rho, u, v)\, w - T(t, \rho, u, v)\, z\| \leqslant C_0 \|w - z\|. \tag{4.4}$$

By the contraction mapping principle $T(t, \rho, u, v)$ has a unique fixed point in $BUC(\mathbb{R}^2)$.

If $u, v, w \in C_b^{0,1}(\mathbb{R}^2)$, it follows directly from (H8) and (4.3) that

$$\|DT(t, \rho, u, v)\, w\| \leqslant \|Dv\| + \rho L(1 + \|Du\|) + C_0 \|Dw\|. \tag{4.5}$$

So, if w is such that

$$\|Dw\| \leqslant \frac{\|Dv\| + \rho L(1 + \|Du\|) + C_0}{1 - C_0}$$

we have

$$\|DT(t, \rho, u, v)\, w\| \leqslant \frac{\|Dv\| + \rho L(1 + \|Du\|) + C_0}{1 - C_0}. \tag{4.6}$$

It follows that $T(t, \rho, u, v)\colon C_b^{0,1}(\mathbb{R}^2) \to C_b^{0,1}(\mathbb{R}^2)$ has unique fixed point $\tilde{w} \in C_b^{0,1}(\mathbb{R}^2)$, which satisfies

$$\|D\tilde{w}\| \leqslant \frac{\|Dv\| + \rho L(1 + \|Du\|) + C_0}{1 - C_0}$$

and (4.7)

$$\|\tilde{w}\| \leqslant \frac{\|v\| + \rho L(C + \|u\|) + C_0}{1 - C_0}$$

where the second inequality is a result of (H2) (C is the constant in (H2)), (H8) and

$$\|T(t, \rho, u, v)\, w\| \leqslant \|v\| + \rho(C + \|u\|) + C_0\|w\| \tag{4.8}$$

and it is valid even when $u, v, w \in BUC(\mathbb{R}^2)$.

Next for $0 \leqslant \rho \leqslant t \leqslant T$ let $F(t, \rho, \cdot, \cdot)\colon BUC(\mathbb{R}^2) \times BUC(\mathbb{R}^2) \to BUC(\mathbb{R}^2)$ be defined by:

If $\rho = 0$, then $F(t, \rho, u, v) = v$,

If $\rho > 0$, then $F(t, \rho, u, v)$ is the unique fixed point (4.9)

of $T(t, \rho, u, v)\colon BUC(\mathbb{R}^2) \to BUC(\mathbb{R}^2)$.

We have:

THEOREM 4.1. (a) *Let $H\colon [0, T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy* (H1), (H2), (H8), (H9) *and $u_0 \in BUC(\mathbb{R}^N)$. For a partition P of $[0, T]$, define $u_P\colon \bar{Q}_T \to \mathbb{R}$ by* (2.1) *and* (4.9). *Let u be the viscosity solution of* (0.1). *Then*

$$\|u_P - u\| \to 0 \qquad \text{as} \quad |P| \to 0. \tag{4.10}$$

(b) *If $u_0 \in C_b^{0,1}(\mathbb{R}^2)$, then, for sufficiently small $|P|$,*

$$\|u_P - u\| \leqslant K|P|^{1/2}$$

where K is a constant which depends only on $\|u_0\|$, $\|Du_0\|$.

Proof. (a) It suffices to check the assumptions of Theorem 2.1(b).

Conditions (F1) and (F3) are immediate consequences of the definition of $F(t, \rho, u, v)$ and $T(t, \rho, u, v)$. To check (F4) observe that (4.7) yields

$$\|F(t, \rho, u, u) - u\| \leqslant \rho \sup_{\substack{((x,y),\tau) \in \bar{Q}_T \\ |(p,q)| \leqslant (\|Du\| + TL(1+\|Du\|) + C_0)/(1-C_0)}} |H(\tau, x, y, u(x, y), p, q)|.$$

The continuity of $(t, \rho) \to F(t, \rho, u, u)$ for $u \in C_b^{0,1}(\mathbb{R}^2)$ follows from the above inequality for $\rho = 0$ and from the properties of $T(t, \rho, u, u)$ and (4.6), (4.7), (4.8), in the case that $\rho > 0$.

For (F9) we need to specify the monotonicity of H. In particular, here we assume that H is nonincreasing with respect to p and nondecreasing with respect to q. If another combination is true, then one has to modify what follows in an appropriate way. If $\rho = 0$, then

$$\|F(t, 0, u, v) - F(t, 0, u, \bar{v})\| = \|v - \bar{v}\|.$$

If $\rho > 0$, then $w = F(t, \rho, u, v)$ and $\bar{w} = F(t, \rho, u, \bar{v})$ satisfy

$$\begin{aligned} w(x, y) + \rho H\left(t, x, y, u(x, y), \frac{\Delta^{x,+,\rho} w}{\rho\alpha}(x, y), \frac{\Delta^{y,-,\rho} w}{\rho\beta}(x, y)\right) &= v(x, y), \\ \bar{w}(x, y) + \rho H\left(t, x, y, \bar{u}(x, y), \frac{\Delta^{x,+,\rho} \bar{w}}{\rho\alpha}(x, y), \frac{\Delta^{y,-,\rho} \bar{w}}{\rho\beta}(x, y)\right) &= \bar{v}(x, y). \end{aligned} \tag{4.11}$$

We show that

$$\sup_{(x,y) \in \mathbb{R}^2} (w(x, y) - \bar{w}(x, y))^+ \leqslant \|v - \bar{v}\| + \rho L \|u - \bar{u}\|. \tag{4.12}$$

The above, together with a similar inequality for $\sup_{(x,y) \in \mathbb{R}^2} (\bar{w}(x, y) - w(x, y))^-$, which is proved exactly as (4.12), implies (F9). Without any loss of generality we assume that

$$\sup_{(x,y) \in \mathbb{R}^2} (w(x, y) - \bar{w}(x, y))^+ > 0. \tag{4.13}$$

In this case let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\Phi(x, y) = (w(x, y) - \bar{w}(x, y))^+.$$

Since Φ is bounded, for every $\delta > 0$ there is a $(x_1, y_1) \in \mathbb{R}^2$ such that

$$\Phi(x_1, y_1) > \sup_{(x,y) \in \mathbb{R}^2} (w(x, y) - \bar{w}(x, y))^+ - \delta.$$

Next choose $\zeta \in C_0^2(\mathbb{R}^2)$ such that $0 \leqslant \zeta \leqslant 1$, $|D\zeta| \leqslant 1$, $\zeta(x_1, y_1) = 1$ and define $\Psi: \mathbb{R}^2 \to \mathbb{R}$ by

$$\Psi(x, y) = \Phi(x, y) + 2\delta\zeta(x, y).$$

$\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1, y_1) = \Phi(x_1, y_1) + 2\delta > \sup_{(x,y) \in \mathbb{R}^2} \Phi(x, y) + \delta$$

therefore there is a point $(x_0, y_0) \in \mathbb{R}^2$ such that

$$\Psi(x_0, y_0) \geqslant \Psi(x, y) \qquad \text{for every} \quad (x, y) \in \mathbb{R}^2. \tag{4.14}$$

Moreover, it is easy to see that, for $\delta < \frac{1}{2} \sup_{(x,y) \in \mathbb{R}^2} (w(x, y) - \bar{w}(x, y))^+$,

$$\begin{aligned} & w(x_0, y_0) - \bar{w}(x_0, y_0) > 0 \\ \text{and} \quad & \\ & \sup_{(x,y) \in \mathbb{R}^2} (w(x, y) - \bar{w}(x, y))^+ \leqslant (w(x_0, y_0) - \bar{w}(x_0, y_0))^+ + 2\delta. \end{aligned} \tag{4.15}$$

Using (4.11) and (4.15) we obtain

$$\begin{aligned} & \sup_{(x,y) \in \mathbb{R}^2} (w(x, y) - \bar{w}(x, y))^+ \\ & \leqslant \|v - \bar{v}\| + 2\delta + \rho H\Bigg(t, x_0, y_0, \bar{u}(x_0, y_0), \\ & \qquad \frac{\Delta^{x,+,\rho}\bar{w}}{\rho\alpha}(x_0, y_0), \frac{\Delta^{y,-,\rho}\bar{w}}{\rho\beta}(x_0, y_0)\Bigg) \\ & \qquad - \rho H\Bigg(t, x_0, y_0, u(x_0, y_0), \frac{\Delta^{x,+,\rho}w}{\rho\alpha}(x_0, y_0), \frac{\Delta^{y,-,\rho}w}{\rho\beta}(x_0, y_0)\Bigg) \end{aligned}$$

and therefore

$$\begin{aligned} & \sup_{(x,y) \in \mathbb{R}^2} (w(x, y) - \bar{w}(x, y))^+ \\ & \qquad \leqslant \|v - \bar{v}\| + 2\delta + \rho L \|u - \bar{u}\| + \rho H\Bigg(t, x_0, y_0, u(x_0, y_0), \\ & \qquad \frac{\Delta^{x,+,\rho}\bar{w}}{\rho\alpha}(x_0, y_0), \frac{\Delta^{y,-,\rho}\bar{w}}{\rho\beta}(x_0, y_0)\Bigg) \\ & \qquad - \rho H\Bigg(t, x_0, y_0, u(x_0, y_0), \frac{\Delta^{x,+,\rho}w}{\rho\alpha}(x_0, y_0), \frac{\Delta^{y,-,\rho}w}{\rho\beta}(x_0, y_0)\Bigg). \end{aligned}$$

But (4.14) and (4.15) yield

$$\frac{\Delta^{x,+,\rho}w}{\rho\alpha}(x_0, y_0) \leqslant \frac{\Delta^{x,+,\rho}\bar{w}}{\rho\alpha}(x_0, y_0) - 2\delta\,\frac{\Delta^{x,+,\rho}\zeta}{\rho\alpha}(x_0, y_0)$$

and

$$\frac{\Delta^{y,-,\rho}w}{\rho\beta}(x_0, y_0) \geqslant \frac{\Delta^{y,-,\rho}\bar{w}}{\rho\beta}(x_0, y_0) - 2\delta\,\frac{\Delta^{y,-,\rho}\zeta}{\rho\beta}(x_0, y_0).$$

The monotonicity of H implies

$$\begin{aligned} &H\left(t, x_0, y_0, u(x_0, y_0), \frac{\Delta^{x,+,\rho}w}{\rho\alpha}(x_0, y_0), \frac{\Delta^{y,-,\rho}w}{\rho\beta}(x_0, y_0)\right) \\ &\quad - H\left(t, x_0, y_0, u(x_0, y_0), \frac{\Delta^{x,+,\rho}\bar{w}}{\rho\alpha}(x_0, y_0)\right. \\ &\quad \left. - 2\delta\,\frac{\Delta^{x,+,\rho}\zeta}{\rho\alpha}(x_0, y_0), \frac{\Delta^{x,-,\rho}\bar{w}}{\rho\beta}(x_0, y_0) - 2\delta\,\frac{\Delta^{y,-,\rho}\zeta}{\rho\beta}(x_0, y_0)\right) \geqslant 0. \end{aligned}$$

Combining all the above and letting $\delta \downarrow 0$ we obtain (4.12).

Next we check (F10). In view of (F9), we have

$$\begin{aligned} \|F(t, \rho, u, u)\| &\leqslant \|F(t, \rho, u, u) - F(t, \rho, 0, 0)\| + \|F(t, \rho, 0, 0)\| \\ &\leqslant (1 + \rho L)\, \|u\| + \|F(t, \rho, 0, 0)\|. \end{aligned}$$

Moreover, if $w = F(t, \rho, 0, 0)$, then

$$w(x, y) = -\rho H\left(t, (x, y), 0, \frac{\Delta^{x,+,\rho}w}{\rho\alpha}(x, y), \frac{\Delta^{y,-,\rho}w}{\rho\beta}(x, y)\right).$$

But (4.7) and (4.8) yield

$$\|Dw\| \leqslant \frac{TL + C_0}{1 - C_0}$$

thus

$$\|F(t, \rho, 0, 0)\| \leqslant \rho \sup_{\substack{(x,y) \in \mathbb{R}^2 \\ \tau \in [0,T] \\ |(p,q)| \leqslant (TL + C_0)/(1 - C_0)}} |H(\tau, (x, y), 0, p, q)|$$

and

$$\|F(t, \rho, u, u)\| \leqslant e^{\rho L}(\|u\| + \rho C_7)$$

where

$$C_7 = \sup_{\substack{((x,y),\tau)\in \bar{Q}_T \\ |(p,q)| \leqslant (TL + C_0)/(1-C_0)}} |H(\tau, x, y, 0, p, q)|.$$

For (F11) observe that, if $u \in C_b^{0,1}(\mathbb{R}^2)$, then $F(t, \rho, u, u) \in C_b^{0,1}(\mathbb{R}^2)$. Let $(\xi, \eta) \in \mathbb{R}^2$. If $w = F(t, \rho, u, u)$, let $\bar{w}$: $\mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\bar{w}(x, y) = w(x + \xi, y + \eta).$$

It is easy to check that

$$\bar{w} = F(t, \rho, \bar{u}, \bar{u} + \rho f)$$

where

$$\bar{u}(x, y) = u(x + \xi, y + \eta)$$

and

$$f(x, y) = H\left(t, x, y, \bar{u}(x, y), \frac{\Delta^{x,+,\rho}\bar{w}}{\rho\alpha}(x, y), \frac{\Delta^{y,-,\rho}\bar{w}}{\rho\beta}(x, y)\right)$$
$$- H\left(t, x + \xi, y + \eta, \bar{u}(x, y), \frac{\Delta^{x,+,\rho}\bar{w}}{\rho\alpha}(x, y), \frac{\Delta^{y,-,\rho}\bar{w}}{\rho\beta}(x, y)\right).$$

But then (F9) and (H8) imply

$$\|w - \bar{w}\| \leqslant \|u - (\bar{u} + \rho f)\| + \rho L \|u - \bar{u}\|$$

and

$$\|Dw\| \leqslant \|Du\| + \rho L + \rho L \|Du\| \leqslant e^{\rho L}(\|Du\| + \rho L)$$

thus (F11).

Finally, we want to verify (F12). We show that

$$\sup_{(x,y)\in\mathbb{R}^2} (F(t, \rho, u, \phi)(x, y) - \phi(x, y) + \rho H(t, x, y, u(x, y), D\phi(x, y))))^{\pm}$$
$$\leqslant C_5(1 + \|D^2\phi\| + \|D\phi\|)\,\rho^2 \tag{4.16$^{\pm}$}$$

where $C_5 = C_5(\|Du\|)$. Here we prove only $(4.16)^+$; $(4.16)^-$ can be shown in exactly the same way. Let $w = F(t, \rho, u, \phi)$. Without any loss of generality, we assume that

$$\sup_{(x,y)\in\mathbb{R}^2} (w(x, y) - \phi(x, y) + \rho H(t, x, y, u(x, y), D\phi(x, y))^+ > 0.$$

Let Φ: $\mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\Phi(x, y) = (w(x, y) - \phi(x, y) + \rho H(t, x, y, u(x, y), D\phi(x, y)))^+.$$

Since Φ is bounded, for every $\delta > 0$ there is a point $(x_1, y_1) \in \mathbb{R}^2$ such that

$$\Phi(x_1, y_1) > \sup_{(x,y) \in \mathbb{R}^2} \Phi(x, y) - \delta.$$

Next choose $\zeta \in C_0^\infty(\mathbb{R}^2)$ such that $0 \leqslant \zeta \leqslant 1$, $|D\zeta| \leqslant 1$, $|D^2\zeta| \leqslant 1$, $\zeta(x_1, y_1) = 1$ and define $\Psi: \mathbb{R}^2 \to \mathbb{R}$ by

$$\Psi(x, y) = \Phi(x, y) + 2\delta\zeta(x, y).$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1, y_1) = \Phi(x_1, y_1) + 2\delta > \sup_{(x,y) \in \mathbb{R}^2} \Phi(x, y) + \delta$$

there is a point $(x_0, y_0) \in \mathbb{R}^2$ such that

$$\Psi(x_0, y_0) \geqslant \Psi(x, y) \qquad \text{for every} \quad (x, y) \in \mathbb{R}^2. \tag{4.17}$$

It is easy to check that if

$$\delta < \tfrac{1}{2} \sup_{(x,y) \in \mathbb{R}^2} (w(x, y) - \phi(x, y) + \rho H(t, x, y, u(x, y), D\phi(x, y))^+$$

then

$$w(x_0, y_0) - \phi(x_0, y_0) + \rho H(t, x_0, y_0, u(x_0, y_0), D\phi(x_0, y_0)) > 0$$

and (4.18)

$$\sup_{(x,y) \in \mathbb{R}^2} (w(x, y) - \phi(x, y) + \rho H(t, x, y, u(x, y), D\phi(x, y))^+$$
$$\leqslant (w(x_0, y_0) - \phi(x_0, y_0) + \rho H(t, x_0, y_0, u(x_0, y_0), D\phi(x_0, y_0)) + 2\delta.$$

In this case we have

$$w(x_0, y_0) - \phi(x_0, y_0) + \rho H(t, x_0, y_0, u(x_0, y_0), D\phi(x_0, y_0))$$
$$= \rho \Bigg[H(t, x_0, y_0, u(x_0, y_0), \phi_x(x_0, y_0), \phi_y(x_0, y_0))$$
$$- H\left(t, x_0, y_0, u(x_0, y_0), \frac{\Delta^{x,+,\rho} w}{\rho\alpha}(x_0, y_0), \frac{\Delta^{y,-,\rho} w}{\rho\beta}(x_0, y_0)\right)\Bigg]. \tag{4.19}$$

But (4.17) implies

$$\frac{\Delta^{x,+,\rho} w}{\rho\alpha}(x_0, y_0)$$
$$\leqslant \frac{1}{\rho\alpha} \Delta^{x,+,\rho}(\phi - \rho H(t, \cdot, \cdot, u(\cdot, \cdot), D\phi(\cdot, \cdot)) - 2\delta\zeta(\cdot, \cdot))(x_0, y_0)$$

and

$$\frac{\Delta^{y,-,\rho}w}{\rho\beta}(x_0, y_0)$$
$$\geqslant \frac{1}{\rho\beta}\Delta^{y,-,\rho}(\phi - \rho H(t,\cdot,\cdot,u(\cdot,\cdot), D\phi(\cdot,\cdot)) - 2\delta\zeta(\cdot,\cdot))(x_0, y_0).$$

The above inequalities together with (H8), the monotonicity of H, (4.3), (4.18) and (4.19) yield

$$\sup_{(x,y)\in\mathbb{R}^2}(w(x,y) - \phi(x,y) + \rho H(t,x,y,u(x,y), D\phi(x,y))^+)$$
$$\leqslant 2\delta + \rho\Big[H(t, x_0, y_0, u(x_0, y_0), \phi_x(x_0, y_0), \phi_y(x_0, y_0))$$
$$- H\Big(t, x_0, y_0, u(x_0, y_0), \frac{1}{\rho\alpha}\Delta^{x,+,\rho}(\phi - \rho H(t,\cdot,\cdot,u(\cdot,\cdot), D\phi(\cdot,\cdot))$$
$$- 2\delta\zeta)(x_0, y_0), \frac{1}{\rho\beta}\Delta^{y,-,\rho}(\phi - \rho H(t,\cdot,\cdot,u, D\phi) - 2\delta\zeta)(x_0, y_0)\Big)\Big]$$
$$\leqslant 2\delta + \sqrt{2}\,(\max(\alpha,\beta)\,\|D^2\phi\| + L(1 + \|Du\| + \|D^2\phi\|) + 2\delta)\,L\rho^2.$$

Letting $\delta \downarrow 0$ we obtain $(4.16)^+$.

(b) It follows from Theorem 2.1(a), since in part (a) above we checked all of its hypotheses.

Remark 4.1. One can prove the same result in the case that H satisfy (H4) type assumptions.

Remark 4.2. Assumption (H9) is not really restrictive. In particular, it is easy to check that, if $u \in BUC(\bar{Q}_T)$ is the viscosity solution of

$$\begin{aligned} u_t + H(t, x, u, Du) &= 0 \qquad \text{in } Q_T \\ u(x, 0) &= u_0(x) \qquad \text{in } \mathbb{R}^N \end{aligned}$$

then $v(x,t) = u(x - tc, t)$, where $c \in \mathbb{R}^N$, is the viscosity solution of

$$\begin{aligned} v_t + H(t, x - tc, Dv) + c\cdot Dv &= 0 \qquad \text{in } Q_T \\ v(x, 0) &= u_0(x) \qquad \text{in } \mathbb{R}^N \end{aligned}$$

where $c \cdot Dv$ denotes the usual inner product $\mathbb{R}^N$. In view of (H9), we see that, with an appropriate choice of c, we can always achieve (H10).

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