Ehrenfeucht Games, the Composition Method, and the Monadic Theory of Ordinal Words

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Abstract. When Ehrenfeucht introduced his game theoretic characterization of elementary equivalence in 1961, the first application of these "Ehrenfeucht games" was to show that certain ordinals (considered as orderings) are indistinguishable in first-order logic and weak monadic second-order logic. Here we review Shelah's extension of the method, the "composition of monadic theories", explain it in the example of the monadic theory of the ordinal ordering $(\omega, <)$, and compare it with the automata theoretic approach due to Büchi. We also consider the expansion of ordinals by recursive unary predicates (which gives "recursive ordinal words"). It is shown that the monadic theory of a recursive ω^n-word belongs to the $2n$-th level of the arithmetical hierarchy, and that in general this bound cannot be improved.

1 Introduction

One of the most successful tools of mathematical logic, in particular of those parts of logic which are relevant to computer science, is the method of "Ehrenfeucht games", introduced by A. Ehrenfeucht in [Ehr61]. These games are a very clear and intuitive formulation of Fraïssé's characterization [Fr54] of elementary equivalence (indistinguishability of relational structures by first-order formulas). In [Ehr61] the method was extended to weak monadic second-order logic; later other logical systems were covered, such as infinitary logic, modal and temporal logics, transitive closure logic, and fixed point logics, to mention a few. Today one speaks of the "Ehrenfeucht-Fraïssé method". Already in [Ehr61], Ehrenfeucht provided first applications by showing that the first-order theory and the weak monadic second-order theory of an ordinal ordering $(\alpha, <)$ depends only on its "ω-tail". (If $\alpha = z + \omega^y + \omega^n c_n + \ldots + \omega^0 c_0$, where $+$ is ordinal addition, $y \geq \omega$, and $c_i < \omega$, then $\omega^n c_n + \ldots + \omega^0 c_0$ is the ω-tail of α.)

This work opened a series of hundreds of papers, especially in the theory of finite models, where the technique was applied to clarify the expressive power of logical systems, often connected with decidability proofs. The decidability of the first-order and weak monadic second-order theory of linear orderings were among the first results, shown by Ehrenfeucht [Ehr59] and Läuchli [Lä68], respectively.

In this paper, we take up the Ehrenfeucht-Fraïssé method in the context of full monadic second-order logic over orderings, explain its extension by Shelah (for the "composition of monadic theories"), and show some results on the monadic theory of "ordinal words" (expansions of ordinals by unary predicates).

At the time of the appearance of [Ehr61], Büchi showed in [Bü62] that the monadic second-order theory of the ordinal $(\omega, <)$ can be analyzed with concepts from automata theory. He proved that a monadic second-order formula can be converted effectively into an equivalent finite automaton (over ω-words) and concluded that the monadic theory of $(\omega, <)$ is decidable. Later papers (e.g., [Bü65]) extended the method also to greater countable ordinals, and it was shown, for example, that the (unrestricted) monadic second-order theory of each countable ordinal is decidable. The automata theoretic method was taken up by many researchers and turned out successful especially in the context of theoretical computer science, regarding areas like generalized formal language theory, program verification, and concurrency theory (see e.g. [Th90] and the articles in [MB96]).

In this track of research it was barely realized, however, that a model theoretic approach based on the Ehrenfeucht-Fraïssé technique, called "composition method", could as well be applied, avoiding the use of automata. This extension of [Fr54] and [Ehr61] was started by Läuchli in [Lä68] and was further developed by Shelah in his celebrated and difficult paper [Sh75]. Shelah gave alternative proofs for Büchi's results and proved many more, also on dense orderings. For example, he showed that the monadic theory of the real number ordering is decidable when the set quantifiers are restricted to countable sets only, but that without this restriction the theory is undecidable. More results were obtained by Gurevich and Shelah in [Gu79], [GS79], [GS83], [GS85], and with Magidor in [GMS83]. Other applications appeared in [Th80], [Gu82], [CFGS82], and [Ze94]. (This list is not claimed to be complete; in particular, there may be further work of Shelah on the topic which is not cited here.)

Although the subject was exposed in Gurevich's concise survey [Gu85], it did not attract much attention among theoretical computer scientists. Preference was (and still is) given to the automata theoretic method: by its connection with a computational model it looks more intuitive, it incorporates "programs" in the form of state-transition systems, and it does not involve frightening logical technicalities as one finds them in [Sh75]. Thus there is a tendency that the merits of the model theoretic approach are overlooked. This is unfortunate, because it excludes some interesting applications. For instance, the model theoretic treatment of dense orderings (which transcends the domain of discrete automata theory) may be of interest in the formal analysis of systems in which conditions on dense time are involved.

We take this as a motivation to review the composition method in the study of monadic theories of orderings (here in particular: ordinals) again, trying to make it more accessible to readers with a background in theoretical computer science. We give an intuitive introduction, including simple examples, in order to enable the reader to follow the results and the proofs of Shelah and Gurevich. We assume that the reader is familiar with the fundamentals of first-order logic, has some idea of first-order Ehrenfeucht games, as presented e.g. in [Ehr61] or in textbooks like [EFT84, EF95], and is acquainted with basic recursion theory, concerning relative recursiveness and the arithmetical hierarchy (see e.g. [Ro67], [Od89]).

After some preparations, we present three results: First, we (re)prove in a new form the "composition theorem" of [Sh75, Gu79] on monadic theories of concatenated orderings. Then we recapitulate Shelah's elegant (and "automata-free") proof of Büchi's theorem on the decidability of the monadic theory of the structure $(\omega, <)$. Finally, we analyze the situation where an ordinal ordering is expanded by unary predicates $P_1, \ldots, P_m$; such a structure $(\alpha, <, P_1, \ldots, P_m)$ can be considered as an *ordinal word*, or to be more precise, as an α-word over the alphabet $\{0,1\}^m$. We show that a recursive ω^n-word (where the underlying predicates P_i are recursive) can have an undecidable monadic theory (even first-order theory), but that its complexity is bounded: Such a monadic theory belongs to the $2n$-th level of the arithmetical hierarchy (Kleene-Mostowski-hierarchy); and we prove that in general this bound cannot be improved.

2 The monadic theory of labelled orderings

The structures considered in this paper are expansions of nonempty linear orderings $\mathcal{A} = (A, <^A)$ by subsets $P_1^A, \ldots P_m^A$. When no confusion arises we cancel the superscript ^{A}, use the abbreviating notation $\overline{P}$ for the set tuple $(P_1^A, \ldots, P_m^A)$, and write $(\mathcal{A}, \overline{P})$. Such a structure can be regarded as a labelled ordering with labels in $\{0,1\}^m$: the element $a \in A$ has the label $(b_1, \ldots, b_m)$ defined by $b_i = 1$ iff $a \in P_i^A$. Special structures of this type are labelled ordinal orderings (short: ordinal words); in this case A is an ordinal number, considered as the set containing all smaller ordinals as elements, and $<^A$ is the usual ordering of these elements.

The standard first-order and monadic second-order language for structures of this signature is built up as follows, using the relation symbols $<$ and $P_1, \ldots, P_m$: We have first-order variables $x, y, \ldots$ for elements of structures, monadic second-order variables $X, Y, \ldots$ for sets of elements of structures, and the atomic formulas are of the form $x = y$, $x < y$, $x \in P_i$, and $x \in Y$, with the canonical interpretation. First-order formulas are formed by using the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and by applying the quantifiers $\exists$, $\forall$ to first-order variables; if application of these quantifiers to monadic second-order variables is also allowed, one obtains the monadic second-order formulas. If $\varphi(x_1, \ldots, x_k, X_1, \ldots X_l)$ is a formula with (at most) the indicated free variables and $q_1, \ldots, q_k$ are elements and $Q_1, \ldots Q_l$ are subsets of A, we write

$$(A, <^A, P_1^A, \ldots, P_m^A, q_1, \ldots, q_k, Q_1, \ldots, Q_l) \models \varphi(x_1, \ldots, x_k, X_1, \ldots X_l)$$

if φ is satisfied when interpreting x_i by q_i and X_j by Q_j. A sentence is a formula without free variables.

For the analysis of monadic second-order theories it will be convenient to work with a slightly modified (but expressively equivalent) set-up, in which the first-order variables are cancelled. We allow only monadic second-order variables and take as atomic formulas the following: Nonempty$(X \cap Y)$, $X \subseteq Y$, $X < Y$, and $Y_1 \cup \ldots \cup Y_k =$ All (for distinct Y_i). These are interpreted, respectively, as "$X \cap Y$ is nonempty", "X is a subset of Y", "some element of X is smaller than some element of Y", and "the union of $Y_1, \ldots, Y_k$ is the universe". For better readability, we shall often write Nonempty(X) instead of Nonempty$(X \cap X)$. In the context of a fixed indexing $X_1, X_2, \ldots$ of variables, we allow formulas Nonempty$(X_{i_1} \cap X_{i_2})$ or $X_{i_1} \cup \ldots \cup X_{i_k} =$ All with distinct i_j only for $i_1 < i_2 < \ldots < i_k$, to spare some redundancies in the set of atomic formulas. By the particular interpretation of $<$ between sets we have, for example, $\{10, 11\} < \{10, 11\}$ but not $\emptyset < \emptyset$ and not $\{10\} < \{10\}$. In fact, we may use the defined predicate "X is a singleton" (written Sing(X)), introduced by

$$\mathrm{Sing}(X) :\leftrightarrow \ \mathrm{Nonempty}(X) \wedge \neg\, X < X.$$

The use of the unary relation symbols P_i will be avoided, by taking free set variables X_i instead. Thus, we shall use labelled orderings $(\mathcal{A}, \overline{P})$ as interpretations of monadic formulas $\varphi(\overline{X})$. For instance, the formula (from the standard language)

$$\varphi(Z) := \forall x(x \in Z \rightarrow \exists y(x < y \wedge y \in P_1))$$

will now be written as

$$\varphi'(X_1, Z) := \forall X(\mathrm{Sing}(X) \wedge X \subseteq Z \ \rightarrow \ \exists Y(\mathrm{Sing}(Y) \wedge X < Y \wedge Y \subseteq X_1)).$$

The monadic theory of an ordered structure $(\mathcal{A}, \overline{P})$, denoted $\mathrm{MTh}(\mathcal{A}, \overline{P})$, is now the set of formulas $\varphi(\overline{X})$ which are satisfied in $(\mathcal{A}, \overline{P})$ (when interpreting X_i by P_i).

In order to classify formulas by an appropriate measure of "complexity", we refer to the prenex normal form: Each monadic formula $\varphi(X_1, \ldots, X_m)$ can be written as

$$\mathrm{Qu}_1 \overline{Y_1} \mathrm{Qu}_2 \overline{Y_2} \ \ldots \ \mathrm{Qu}_n \overline{Y_n} \ \psi(\overline{X}, \overline{Y_1}, \overline{Y_2}, \ldots, \overline{Y_n}),$$

where ψ is quantifier-free, each Qu_i is either $\exists$ or $\forall$, $\overline{X}$ stands for $X_1, \ldots, X_m$ and $\overline{Y_i}$ for $Y_{i1}, \ldots, Y_{ik_i}$. Referring to the sequence $\overline{k} = (k_1, \ldots, k_n)$ of the lengths of the quantifier blocks, we call such a formula a *$\overline{k}$-formula*. The set of all $\overline{k}$-formulas which are satisfied in a structure is called its $\overline{k}$-theory. We write

$$(\mathcal{A}, \overline{P}) \equiv_{\overline{k}} (\mathcal{B}, \overline{R})$$

if the two labelled orderings $(\mathcal{A}, \overline{P})$, $(\mathcal{B}, \overline{R})$ satisfy the same $\overline{k}$-formulas $\varphi(\overline{X})$, i.e., have the same $\overline{k}$-theory.

3 The monadic Ehrenfeucht game

In the present context, an Ehrenfeucht game serves to verify that two structures are $\equiv_{\overline{k}}$-equivalent. Here we consider labelled linear orderings (until Section 7 not necessarily well-orderings), say $(\mathcal{A}, \overline{P})$ and and $(\mathcal{B}, \overline{R})$ with $\overline{P} = (P_1, \ldots, P_m)$ and $\overline{R} = (R_1, \ldots, R_m)$. The corresponding game $G_{\overline{k}}((\mathcal{A}, \overline{P}), (\mathcal{B}, \overline{R}))$ is played on the two structures by two players I and II, also called Spoiler and Duplicator. If $\overline{k} = (k_1, \dots, k_n)$ there are n rounds in a play of the game. In round i, Spoiler begins by picking subsets $P_{i1}, \dots, P_{ik_i}$ in A or subsets $R_{i1}, \dots, R_{ik_i}$ in B, to which Duplicator responds by picking k_i sets in the other structure (i.e, sets $R_{i1}, \dots, R_{ik_i}$ in B, respectively $P_{i1}, \dots, P_{ik_i}$ in A). After the n rounds, Duplicator has won the play if the truth of atomic formulas is preserved when passing from the P_i, P_{ij} to the corresponding sets R_i, R_{ij}. More formally (and writing P_{i0}, R_{i0} instead of P_i, R_i, respectively) we should have:

- $P_{ij} \cap P_{i'j'} \neq \emptyset$ iff $R_{ij} \cap R_{i'j'} \neq \emptyset$,
- $P_{ij} < P_{i'j'}$ iff $R_{ij} < R_{i'j'}$,
- $P_{ij} \subseteq P_{i'j'}$ iff $R_{ij} \subseteq R_{i'j'}$, and
- $P_{i_1j_1} \cup \dots \cup P_{i_kj_k} = A$ iff $R_{i_1j_1} \cup \dots \cup R_{i_kj_k} = B$.

We indicate that Duplicator has a winning strategy in this game by writing $(\mathcal{A}, \overline{P}) \sim_{\overline{k}} (\mathcal{B}, \overline{R})$.

Let us note two differences to the classical (first-order) Ehrenfeucht game: First, sets of elements are chosen in the moves of a play, rather than individual elements. This reflects the fact that we deal with set quantifiers. Similarly, in the weak second-order game of [Ehr61], Ehrenfeucht allowed the choice of finite sets by the two players in each move. Secondly, the quantifiers are not treated one at a time (i.e., sets are not chosen one at a time) but block-wise, in correspondence with the entries of a sequence $\overline{k} = (k_1, \dots, k_n)$. This approach will prove useful later on, when the usual induction parameter, the number of (nested) quantifiers, is replaced by the alternation depth of a quantifier sequence.

The basic result on the game, the "Ehrenfeucht-Fraïssé Theorem", states that existence of a winning strategy for Duplicator in the game $G_{\overline{k}}((\mathcal{A}, \overline{P}), (\mathcal{B}, \overline{R}))$ characterizes $\equiv_{\overline{k}}$-equivalence between $(\mathcal{A}, \overline{P})$ and $(\mathcal{B}, \overline{R})$. It is formulated here just for orderings expanded by unary predicates, but is valid for relational structures of any finite signature. The standard proof proceeds in two steps: First one applies Fraïssé's characterization ([Fr54]) of $\equiv_{\overline{k}}$-equivalence by the existence of families of partial isomorphisms with the "back-and-forth property", secondly one shows that this latter condition just means that Duplicator wins the corresponding Ehrenfeucht game. The details are supplied by a straightforward adaptation of the classical case (first-order logic), as treated e.g. in the textbooks [EFT84], [EF95], to the present context of monadic logic.

Theorem 1. (Ehrenfeucht-Fraïssé Theorem)
For any $\overline{k} = (k_1, \dots, k_n)$ and any two linear orderings $\mathcal{A} = (A, <^A)$ and $\mathcal{B} = (B, <^B)$ expanded by sequences $\overline{P} = (P_1, \dots, P_m)$, respectively $\overline{R} = (R_1, \dots, R_m)$ of subsets we have: $(\mathcal{A}, \overline{P}) \equiv_{\overline{k}} (\mathcal{B}, \overline{R})$ iff $(\mathcal{A}, \overline{P}) \sim_{\overline{k}} (\mathcal{B}, \overline{R})$.

4 Hintikka formulas

In this section we show how $\equiv_{\overline{k}}$-equivalence classes can be defined in monadic logic. The defining formulas are sometimes called *Hintikka formulas* (reminding of their first appearance, in [Hi53], in the framework of first-order logic).

As before, we use letter $\mathcal{A}$ for linear orderings $(A, <^A)$ and write expansions by tuples $\overline{P} = (P_1, \ldots, P_m)$ of subsets in the form $(\mathcal{A}, \overline{P})$. For such a structure and any sequence $\overline{k} = (k_1, \ldots, k_n)$, we define a formula $\varphi^{\overline{k}}_{(\mathcal{A},\overline{P})}(X_1, \ldots, X_m)$, which is satisfied precisely in those structures $(\mathcal{B}, \overline{R})$ that are $\equiv_{\overline{k}}$-equivalent to $(\mathcal{A}, \overline{P})$. In other words, the formula $\varphi^{\overline{k}}_{(\mathcal{A},\overline{P})}(\overline{X})$ has to describe which $\overline{k}$-formulas $\psi(\overline{X})$ are true, respectively false, in the structure $(\mathcal{A}, \overline{P})$.

The task is easy if $\overline{k}$ is the empty sequence λ: Here we have to state which atomic formulas (whose free variables are among $X_1, \ldots, X_m$) are true in $(\mathcal{A}, \overline{P})$ and which are false in this structure. Thus we take the formula

$$\varphi^{\lambda}_{(\mathcal{A},\overline{P})}(\overline{X}) := \bigwedge_{\substack{\varphi(\overline{X}) \text{ atomic,} \\ (\mathcal{A},\overline{P}) \models \varphi(\overline{X})}} \varphi(\overline{X}) \ \wedge \bigwedge_{\substack{\varphi(\overline{X}) \text{ atomic,} \\ (\mathcal{A},\overline{P}) \models \neg\varphi(\overline{X})}} \neg\varphi(\overline{X}).$$

Example 1. Consider the structure $\mathcal{A} = (\omega, <)$ expanded by $\overline{P} = (P_1, P_2)$ where $P_1 = \{10\}$ and $P_2 = \{11\}$. The relevant atomic formulas for $\varphi^{\lambda}_{(\mathcal{A},P_1,P_2)}(X_1, X_2)$ are the following: Nonempty(X_1), Nonempty(X_2), Nonempty$(X_1 \cap X_2)$, $X_i \subseteq X_j$ for $i, j \in \{1, 2\}$, $X_i < X_j$ for $i, j \in \{1, 2\}$, $X_1 =$ All, $X_2 =$ All, and $X_1 \cup X_2 =$ All. The conjunction of satisfied formulas is

$$\text{Nonempty}(X_1) \ \wedge \ \text{Nonempty}(X_2) \ \wedge \ X_1 \subseteq X_1 \ \wedge \ X_2 \subseteq X_2 \ \wedge \ X_1 < X_2,$$

whereas the other atomic formulas (Nonempty$(X_1 \cap X_2)$, $X_1 \subseteq X_2$, $X_2 \subseteq X_1$, $X_1 < X_1$, $X_2 < X_2$, $X_2 < X_1$, $X_1 =$ All, $X_2 =$ All, and $X_1 \cup X_2 =$ All) are false and assembled in negated form in the second conjunction. In the sequel we denote this formula $\varphi^{\lambda}_{(\mathcal{A},P_1,P_2)}(X_1, X_2)$ just by $\varphi_0(X_1, X_2)$. □

Let us define $\varphi^{\overline{k}}_{(\mathcal{A},\overline{P})}(\overline{X})$ for nonempty sequences $\overline{k}$. If $\overline{k}$ has just a single entry, say $\overline{k} = (k_1)$, then the desired Hintikka formula has to describe which different situations for the truth of atomic formulas can be generated by expanding the structure $(\mathcal{A}, \overline{P})$ by further predicates $\overline{Q} = (Q_1, \ldots Q_{k_1})$. The set of all such k_1-tuples of sets from the powerset of A will be denoted by $\mathcal{P}(A)^{k_1}$. For each choice of $\overline{Q} \in \mathcal{P}(A)^{k_1}$, we shall state the existence of $\overline{Y}$ with $\varphi^{\lambda}_{(\mathcal{A},\overline{P},\overline{Q})}(\overline{X}, \overline{Y})$; moreover, the existence of other tuples has to be excluded, in the sense that for each tuple $\overline{Y}$ one of the above formulas $\varphi^{\lambda}_{(\mathcal{A},\overline{P},\overline{Q})}(\overline{X}, \overline{Y})$ holds.

Example 2. (continued) Let us consider the simple case $\overline{k} = (1)$ and sketch the formula $\varphi^{(1)}_{(\mathcal{A},P_1)}$ where $\mathcal{A} = (\omega, <)$ and $P_1 = \{10\}$ are as above. We shall write down a conjunction of formulas in which, for example, $\exists X_2 \varphi_0(X_1, X_2)$ with $\varphi_0(X_1, X_2)$ from above occurs, representing the choice of $\{11\}$ for X_2. One sees that the *same* formula arises if we take any singleton $\{n\}$ with $n > 11$ instead. But we get different formulas if we consider (for X_2) one of the following eleven sets: $\emptyset$, $\{9\}$, $\{8, 9\}$, $\{9, 10\}$, $\{10\}$, $\{10, 11\}$, $\{11, 12\}$, $\{9, 11\}$, $\{9, 10, 11\}$, $\omega \setminus \{10\}$, and ω. One verifies that the twelve options altogether exhaust all different possibilities to satisfy the atomic formulas in X_1, X_2, given the set $P_1 = \{10\}$ as interpretation of X_1. Similarly to $\exists X_2\varphi_0(X_1, X_2)$ we can write down corresponding existence claims $\exists X_2\varphi_i(X_1, X_2)$ for the remaining eleven cases $i = 1, \ldots, 11$. (For instance, the formula $\varphi_3(X_1, X_2)$, which corresponds to the choice of $\{8, 9\}$ for X_2, is the conjunction

$$\text{Nonempty}(X_1) \wedge \text{Nonempty}(X_2) \wedge X_1 \subseteq X_1 \wedge X_2 \subseteq X_2 \wedge X_2 < X_2 \wedge X_2 < X_1,$$

while the other atomic formulas are adjoined in negated form.) Now we may set

$$\varphi^{(1)}_{(\mathcal{A},P_1)} := \bigwedge_{i=0}^{11} \exists X_2\varphi_i(X_1, X_2) \;\wedge\; \forall X_2 \bigvee_{i=0}^{11} \varphi_i(X_1, X_2).$$

Here the second part expresses that the list of twelve options is complete, i.e. that any choice of a subset occurs in the list. $\square$

The general step, from a sequence $\overline{k} = (k_1, \ldots, k_n)$ to its extension $\overline{k}^\wedge k_{n+1}$, is handled in precisely the same way, replacing λ by $\overline{k}$, and taking into account that k_{n+1} sets are chosen simultaneously. So we obtain the following definition:

$$\varphi^{\overline{k}^\wedge k_{n+1}}_{(\mathcal{A},\overline{P})}(\overline{X}) := \bigwedge_{\overline{Q}\in\mathcal{P}(A)^{k_{n+1}}} \exists\overline{Y}\varphi^{\overline{k}}_{(\mathcal{A},\overline{P},\overline{Q})}(\overline{X},\overline{Y}) \wedge \forall\overline{Y} \bigvee_{\overline{Q}\in\mathcal{P}(A)^{k_{n+1}}} \varphi^{\overline{k}}_{(\mathcal{A},\overline{P},\overline{Q})}(\overline{X},\overline{Y}).$$

The disjunction and conjunction in this formula are finite. To verify this, note that there are only finitely many atomic formulas involving variables from a finite set $\{X_1, \ldots, X_r\}$, and that (as verified by induction on the length of $\overline{k}$) the number of logically non-equivalent $\overline{k}$-formulas $\psi(X_1, \ldots, X_r)$ (for a given number of free variables) is finite. Thus the disjunction and the conjunction (over $\overline{Q} \in \mathcal{P}(A)^{k_{n+1}}$) in the definition of $\varphi^{\overline{k}^\wedge k_{n+1}}_{(\mathcal{A},\overline{P})}(\overline{X})$ both range only over finitely many formulas $\varphi^{\overline{k}}_{(\mathcal{A},\overline{P},\overline{Q})}(\overline{X},\overline{Y})$ and hence specify monadic formulas. By the same reason, for any $\overline{k}$ and any length of the tuple $\overline{X}$, only finitely many different Hintikka formulas $\varphi^{\overline{k}}_{(\mathcal{A},\overline{P})}(\overline{X})$ exist which arise from the infinitely many possible structures $(\mathcal{A}, \overline{P})$. Any such structure will satisfy precisely one Hintikka formula.

Let us formulate some further key properties of Hintikka formulas. First, a Hintikka formula $\varphi^{\overline{k}}_{(\mathcal{A},\overline{P})}(\overline{X})$ fixes truth, respectively falsehood, of any given $\overline{k}$-formula in $(\mathcal{A}, \overline{P})$.

Example 3. (continued) To illustrate this, consider the formula $\varphi^{(1)}_{(\mathcal{A},P_1)}(X_1)$ from the example above. We check that the (1)-formula

$$\psi(X_1) := \exists Y(X_1 \subseteq Y \wedge X_1 < Y)$$

is true in $(\mathcal{A}, P_1)$ $(= (\omega, <, \{10\}))$: From the conjunction member $\exists X_2\varphi_6(X_1, X_2)$ of $\varphi^{(1)}_{(\mathcal{A},P_1)}(X_1)$ (corresponding to the choice $\{10, 11\}$ for X_2) the formula ψ follows and hence is true in $(\mathcal{A}, P_1)$. On the other hand, $\forall Y(X_1 \subseteq Y \to X_1 < Y)$ is not true in $(\mathcal{A}, P_1)$: This universal statement holds if it is true in all twelve options listed above. By inspection of $\varphi_0, \ldots, \varphi_{11}$ we see that occurrence of the conjunction member $X_1 \subseteq X_2$ does not imply that also $X_1 < X_2$ occurs as a conjunction member; a counterexample is $\varphi_0(X_1, X_2)$ (arising from the choice of $\{10\}$ for X_2). □

So we obtain the following lemma (which is proved by induction on $\overline{k}$):

Lemma 2. (Hintikka formulas)
Let $(\mathcal{A}, \overline{P})$ be the expansion of the ordering $\mathcal{A}$ by unary predicates $P_1, \ldots, P_m$ and let $\overline{k} = (k_1, \ldots, k_n)$.
(a) *If $\psi(\overline{X})$ is a $\overline{k}$-formula, then*

$$(\mathcal{A}, \overline{P}) \models \psi(\overline{X}) \quad \textit{iff} \quad \varphi^{\overline{k}}_{(\mathcal{A}, \overline{P})}(\overline{X}) \models \psi(\overline{X})$$

(i.e., iff the second formula follows from the first); moreover, this can be checked effectively.
Conversely, $(\mathcal{A}, \overline{P}) \models \neg\psi(\overline{X})$ iff $\varphi^{\overline{k}}_{(\mathcal{A}, \overline{P})}(\overline{X}) \models \neg\psi(\overline{X})$.
(b) *The Hintikka formula $\varphi^{\overline{k}}_{(\mathcal{A}, \overline{P})}(\overline{X})$ characterizes the $\equiv_{\overline{k}}$-class of $(\mathcal{A}, \overline{P})$ in the sense that for any labelled ordering $(\mathcal{B}, \overline{R})$ with $\overline{R} = (R_1, \ldots, R_m)$ we have:*

$$(\mathcal{A}, \overline{P}) \equiv_{\overline{k}} (\mathcal{B}, \overline{R}) \quad \textit{iff} \quad (\mathcal{B}, \overline{R}) \models \varphi^{\overline{k}}_{(\mathcal{A}, \overline{P})}(\overline{X}).$$

Considering a $\overline{k}$-formula $\psi(\overline{X})$, we may collect those $\overline{k}$-Hintikka formulas from which the formula $\psi(\overline{X})$ follows (as in part (a) of the Lemma). Since each structure $(\mathcal{B}, \overline{R})$ satisfies precisely one Hintikka formula $\varphi^{\overline{k}}_{(\mathcal{A}, \overline{P})}(\overline{X})$ of appropriate signature, we obtain:

Lemma 3. (Distributive normal form of $\overline{k}$-formulas)
A $\overline{k}$-formula $\psi(\overline{X})$ is equivalent to the disjunction of those $\overline{k}$-Hintikka-formulas of appropriate signature from which $\psi(\overline{X})$ follows; this disjunction can be computed effectively from $\psi(\overline{X})$.

5 $\overline{k}$-types

Besides the Hintikka-formulas, there is an alternative description of $\equiv_{\overline{k}}$-classes, first used by Läuchli [Lä68] for weak monadic logic and then by Shelah [Sh75] for unrestricted monadic logic. The formalism amounts to a compact representation of Hintikka formulas. Instead of inductively piling up nested disjunctions and conjunctions, one collects the relevant information by iterative formations of subsets of the power set, starting from a set of atomic formulas in certain variables $X_1, \ldots, X_m$. The (hereditarily finite) sets built up in this way to define $\equiv_{\overline{k}}$-classes are called $\overline{k}$-types.

For a sequence $\overline{k} = (k_1, \ldots, k_n)$ and a number m, we define the finite set $\mathcal{T}^{\overline{k}}(m)$ of "all formally possible $\overline{k}$-types with m set parameters". Let Atom_m be the set of atomic formulas in the variables $X_1, \ldots, X_m$ and set

- $\mathcal{T}^{\lambda}(m) := \mathcal{P}(\mathrm{Atom}_m)$
- $\mathcal{T}^{\overline{k}^\wedge k_{n+1}}(m) := \mathcal{P}(\mathcal{T}^{\overline{k}}(m + k_{n+1}))$

Now we associate with a sequence $\overline{k} = (k_1, \ldots, k_n)$ and a labelled ordering $(\mathcal{A}, P_1, \ldots, P_m)$ an element of $\mathcal{T}^{\overline{k}}(m)$, called the *$\overline{k}$-type* $T^{\overline{k}}(\mathcal{A}, \overline{P})$, again defined inductively over the length of $\overline{k}$:

- $T^{\lambda}(\mathcal{A}, \overline{P}) := \{\varphi(\overline{X}) \mid \varphi(\overline{X}) \text{ atomic}, (\mathcal{A}, \overline{P}) \models \varphi(\overline{X})\}$
- $T^{\overline{k}^\wedge k_{n+1}}(\mathcal{A}, \overline{P}) :=$

$$\{T^{\overline{k}}(\mathcal{A}, \overline{P}, Q_1, \ldots, Q_{k_n+1}) \mid (Q_1, \ldots, Q_{k_n+1}) \in \mathcal{P}(A)^{k_{n+1}}\}.$$

Example 4. (continued) Consider again the structure $(\omega, <, P_1, P_2)$ with $P_1 = \{10\}$ and $P_2 = \{11\}$. The type $T^{\lambda}(\omega, P_1, P_2)$ contains the following formulas:

$$\mathrm{Nonempty}(X_1), \mathrm{Nonempty}(X_2), X_1 \subseteq X_1, X_2 \subseteq X_2, X_1 < X_2.$$

This information suffices for specifying the Hintikka formula $\varphi^{\lambda}_{(\omega,P_1,P_2)}(X_1, X_2)$. We call this λ-type τ_0 and now form the types $\tau_1, \ldots, \tau_{11}$ for other interpretations of X_2 than $P_2 = \{11\}$, in accordance with the eleven Hintikka formulas $\varphi_1(X_1, X_2), \ldots, \varphi_{11}(X_1, X_2)$ in the example above. For instance, we have

$$\tau_1 = \{X_1 \subseteq X_1, X_2 \subseteq X_2, X_2 \subseteq X_1\},$$

corresponding to the choice of $\emptyset$ for X_2. The (1)-type of the structure (ω, P_1) is the set $T^{(1)}(\omega, P_1) = \{\tau_0, \ldots, \tau_{11}\}$. □

Whereas the $\overline{k}$-type of $(\mathcal{A}, P_1, \ldots, P_m)$ belongs to $\mathcal{T}^{\overline{k}}(m)$, this set $\mathcal{T}^{\overline{k}}(m)$ contains other elements which are not $\overline{k}$-types of structures: There may be inconsistencies, for example when the formula $X_i < X_i$ occurs whereas $\mathrm{Nonempty}(X_i)$ is missing. So $\mathcal{T}^{\overline{k}}(m)$ is defined disregarding the question of satisfiability.

It is useful to note the precise correspondence between $\overline{k}$-types and $\overline{k}$-Hintikka formulas. The nesting of set formation in $\overline{k}$-types captures the nesting of conjunctions and disjunctions in Hintikka formulas; here the $\overline{k}$-types are slightly more "economic" than Hintikka formulas, because the possibilities of extension are collected as a set, whereas the description in a Hintikka formula has to refer to these possibilities twice: in the existence claim (conjunction of existential formulas) and in the completeness claim (universal formula over a disjunction).

One may regard the $(k_1, \ldots, k_n)$-type of a structure $(\mathcal{A}, <, P_1, \ldots, P_m)$ as a finite tree of height n, where the levels are numbered from 0 (for the leaves) to n (for the root). On the level 0 (of leaves), there are λ-types of structures $(A, <, \overline{P}, \overline{Q})$ where the length of $\overline{Q}$ is $k_1 + \ldots + k_n$. (These types are sets of atomic formulas in the variables $X_1, \ldots, X_l$ where $l = m + k_1 + \ldots + k_n$.) In general, level i contains $(k_1, \ldots, k_i)$-types of structures $(A, <, \overline{P}, \overline{Q})$ where the length of $\overline{Q}$ is $k_{i+1} + \ldots + k_n$, and at the root (level n) we find $T^{\overline{k}}(A, <, \overline{P})$. Such a tree of extension possibilities of a given structure is sometimes called the *Fraïssé tree* of the structure.

In analogy to Lemma 2 (b) above, the $\overline{k}$-type of a structure $(\mathcal{A}, \overline{P})$ determines effectively for any $\overline{k}$-formula $\varphi(\overline{X})$ whether $(\mathcal{A}, \overline{P}) \models \varphi(\overline{X})$. The algorithm to determine the truth value is best explained by an example.

Example 5. Consider the structure $(\omega, <, P_1, P_2)$ where $P_1 = \{10\}$ and P_2 is the set of even numbers. In this structure, the formula

$$\forall X_3(X_1 < X_3 \to \exists X_4(X_3 < X_4 \land X_4 \subseteq X_2))$$

is true; it expresses that for each set S containing a number > 10 there is a set of even numbers containing a number which is greater than some element of S. The prenex normal form of the formula is

$$\forall X_3 \exists X_4(X_1 < X_3 \ \to \ X_3 < X_4 \land X_4 \subseteq X_2),$$

and we verify its truth in $(\omega, <, P_1, P_2)$ by checking the following property of the type $T^{(1,1)}(\omega, P_1, P_2)$: For each element of this $(1,1)$-type (i.e. for each (1)-type occurring in it), there is an element (which is then a λ-type) which satisfies the following: if the formula $X_1 < X_3$ occurs in it, so do $X_3 < X_4$ and $X_4 \subseteq X_2$.

So the quantifiers over sets in a structure correspond to quantifiers over finite sets of types within a given type. It may also happen that a $(k_1, \ldots, k_n)$-type determines an atomic formula ψ to hold even for $n > 1$. This means that in all sets which occur as leaves in the associated Fraïssé tree the atomic formula ψ appears. We say in this case that the $\overline{k}$-type *induces* ψ. Altogether we note that the $\overline{k}$-theory of a structure $(\mathcal{A}, \overline{P})$ is obtained effectively from its $\overline{k}$-type. Thus we obtain the following connection with decidability:

Lemma 4. *The monadic second-order theory* $\mathrm{MTh}(\mathcal{A}, \overline{P})$ *of a labelled ordering* $(\mathcal{A}, \overline{P})$ *is decidable iff the function*

$$f_{(\mathcal{A},\overline{P})} : \overline{k} \ \mapsto \ T^{\overline{k}}(\mathcal{A}, \overline{P})$$

is computable; more generally, $\mathrm{MTh}(\mathcal{A}, \overline{P})$ *and* $f_{(\mathcal{A},\overline{P})}$ *are recursive in each other.*

For a given finite structure $(\mathcal{A}, \overline{P})$, it is clear that the types $T^{\overline{k}}(\mathcal{A}, \overline{P})$ can be generated effectively, so the function $f_{(\mathcal{A},\overline{P})}$ is computable and the theory $\mathrm{MTh}(\mathcal{A}, \overline{P})$ decidable.

6 Composition of $\overline{k}$-types

In this section we consider the concatenation of labelled orderings, and show how to obtain the $\overline{k}$-theory of such a composed labelled ordering from the $\overline{k}$-theories of the component orderings. Here a $\overline{k}$-theory will be represented by the corresponding $\overline{k}$-type. The possibility of such an algorithmic "composition of $\overline{k}$-types" provides the background for decidability results on monadic theories of orderings.

The composition of $\overline{k}$-types involves two aspects: First, it has to be shown that the $\overline{k}$-types of labelled orderings, say $(\mathcal{A}_\iota, \overline{P}_\iota)$ for $\iota = 1, \ldots, r$, determine uniquely the $\overline{k}$-type of the concatenation of these structures in the order given by the indices, written $(\mathcal{A}_1, \overline{P}_1) + \ldots + (\mathcal{A}_r, \overline{P}_r)$. Secondly, an effective procedure is required which produces the $\overline{k}$-type of the composed structure from the $\overline{k}$-types of the components.

The first claim means that $\equiv_{\overline{k}}$ is a congruence with respect to concatenation of labelled orderings. This can be shown elegantly by means of the Ehrenfeucht game: Given labelled orderings $(\mathcal{A}_\iota, \overline{P}_\iota)$ and $(\mathcal{B}_\iota, \overline{R}_\iota)$ with $(\mathcal{A}_\iota, \overline{P}_\iota) \sim_{\overline{k}} (\mathcal{B}_\iota, \overline{R}_\iota)$ for $\iota = 1, \ldots, r$ one has to show that

$$(\mathcal{A}_1, \overline{P}_1) + \ldots + (\mathcal{A}_r, \overline{P}_r) \ \sim_{\overline{k}} \ (\mathcal{B}_1, \overline{R}_1) + \ldots + (\mathcal{B}_r, \overline{R}_r).$$

This means that from winning strategies $S_1, \ldots, S_r$ for Duplicator in the games $G_{\overline{k}}((\mathcal{A}_\iota, \overline{P}_\iota), (\mathcal{B}_\iota, \overline{R}_\iota))$ one has to compose a winning strategy for Duplicator in the game $G_{\overline{k}}((\mathcal{A}_1, \overline{P}_1) + \ldots + (\mathcal{A}_r, \overline{P}_r), (\mathcal{B}_1, \overline{R}_1) + \ldots + (\mathcal{B}_r, \overline{R}_r))$. The new strategy is the obvious one: "play S_ι on the pair $((\mathcal{A}_\iota, \overline{P}_\iota), (\mathcal{B}_\iota, \overline{R}_\iota))$, simultaneously for all $\iota \in \{1, \ldots, r\}$". One can apply the same idea if the ordered index set is infinite. The idea has to be refined when the index orderings for the $(\mathcal{A}_\iota, \overline{P}_\iota)$ and $(\mathcal{B}_\iota, \overline{R}_\iota)$ are different.

Shelah's calculus of $\overline{k}$-types also covers this possibility of different index orderings and adds the aspect of computability to the composition process.

Such a composition of theories is familiar from first-order model theory in connection with direct (and reduced) products. The direct product $\mathcal{A} := \prod_{\iota \in I} \mathcal{A}_\iota$ of relational structures $\mathcal{A}_\iota$ (all of a fixed finite signature σ) has the cartesian product $\prod_{\iota \in I} A_\iota$ as universe, and for a relation symbol R from σ one defines $((a_\iota^1)_{\iota \in I}, \ldots, (a_\iota^n)_{\iota \in I}) \in R^{\mathcal{A}}$ iff $(a_\iota^1, \ldots, a_\iota^n) \in R^{\mathcal{A}_\iota}$ for all $\iota \in I$. The Feferman-Vaught Theorem ([FV59], see also [CK73]) shows, in the simplest case, that the first-order theory of a product $\prod_{\iota \in I} \mathcal{A}_\iota$ can be obtained effectively from the first-order theories of the factors $\mathcal{A}_\iota$ and from the theory of the Boolean algebra $(\mathcal{P}(I), \sim, \cup, \cap)$, as follows: Given a sentence ψ (whose truth value in the product is to be determined), one can can compute sentences $\psi_1, \ldots, \psi_r$ (describing "factor properties") and a formula $\beta(x_1, \ldots, x_r)$ of the first-order language of Boolean algebras such that

$$\prod_{\iota \in I} \mathcal{A}_\iota \models \psi \ \text{ iff } \ (\mathcal{P}(I), \sim, \cup, \cap, P_1, \ldots, P_r) \models \beta(x_1, \ldots, x_r)$$

where P_j is the set of indices $\iota \in I$ with $\mathcal{A}_\iota \models \psi_j$. In this sense, the truth of ψ in the product can be recovered from the distribution of the truth values of certain factor properties over the index set I.

Shelah [Sh75] developed an analogous composition formalism for monadic formulas; here the structures are labelled orderings $(\mathcal{A}_\iota, \overline{P}_\iota)$ with $\mathcal{A}_\iota = (A_\iota, <^{\mathcal{A}_\iota})$, combined by concatenation via an ordered index set I. One speaks of the *ordered sum* of the structures $(\mathcal{A}_\iota, \overline{P}_\iota)$ with respect to the ordering $(I, <^I)$. Formally, this ordered sum, denoted $\sum_{\iota \in I} (\mathcal{A}_\iota, \overline{P}_\iota)$, is defined as follows (assuming that the structures $\mathcal{A}_\iota$ are disjoint and that $\overline{P}_\iota = (P_{\iota 1}, \ldots, P_{\iota m})$):

$$\textstyle\sum_{\iota \in I} (\mathcal{A}_\iota, \overline{P}_\iota) \ = \ (\bigcup_{\iota \in I} A_\iota, \ <, \ \bigcup_{\iota \in I} \overline{P}_\iota)$$

where

- $a < b$ holds iff $a \in A_\iota$ and $b \in A_\kappa$ with $\iota <^I \kappa$, or $a, b \in A_\iota$ and $a <^{A_\iota} b$ for some $\iota \in I$,
- $\bigcup_{\iota \in I} \overline{P} := (\bigcup_{\iota \in I} P_{\iota 1}, \ldots, \bigcup_{\iota \in I} P_{\iota m})$.

The task of the desired composition algorithm is to produce the $\overline{k}$-type of an ordered sum

$$\sum_{\iota \in I}(\mathcal{A}_\iota, P_{\iota 1}, \ldots, P_{\iota m})$$

from a certain $\overline{r}$-type of the index ordering to which information about the $\overline{k}$-types of the components (summand structures) is added. These component $\overline{k}$-types are all from the set $\mathcal{T}^{\overline{k}}(m)$, and we assume a fixed listing of them in the form $\tau_1, \ldots, \tau_s$. Consider the expansion of the index ordering $(I, <^I)$ by the sets $Q_1, \ldots, Q_s$ where

$$Q_j = \{\iota \in I \mid T^{\overline{k}}(\mathcal{A}_\iota, \overline{P}_\iota) = \tau_j\}.$$

We call this structure $(I, <^I, Q_1, \ldots, Q_s)$ the *$\mathcal{T}^{\overline{k}}(m)$-expansion of $(I, <^I)$ with respect to $(\mathcal{A}_\iota, \overline{P}_\iota)_{\iota \in I}$*. Clearly, these Q_j define a partition of I.

It will turn out that, for a suitable $\overline{r}$, the $\overline{r}$-type of such an expansion suffices to determine $T^{\overline{k}}(\sum_{\iota \in I}(\mathcal{A}_\iota, \overline{P}_\iota))$. We present the result here at a more relaxed pace than in [Sh75] (where it is just stated) and [Gu79], [Gu85] (where the proofs are rather condensed and given in the more abstract framework of arbitrary signatures and general Feferman-Vaught-type theorems). Our specific choice of signature leads to a certain simplification.

Theorem 5. (Composition Theorem, [Sh75], [Gu79])
From a sequence $\overline{k} = (k_1, \ldots, k_n)$ and a number m one can compute a sequence $\overline{r} = (r_1, \ldots, r_n)$ such that for any ordered sum $\sum_{\iota \in I}(\mathcal{A}_\iota, P_{\iota 1}, \ldots, P_{\iota m})$ its $\overline{k}$-type is determined by (and can be computed from) the $\overline{r}$-type of the $\mathcal{T}^{\overline{k}}(m)$-expansion of $(I, <^I)$ with respect to $(\mathcal{A}_\iota, \overline{P}_\iota)_{\iota \in I}$.

Before turning to the proof, let us note an essential technical point in this theorem, namely the preservation of the length n in the step from the sequence $\overline{k}$ to the sequence $\overline{r}$. The *sum* of the $\overline{r}$-entries will in general be larger than that of the $\overline{k}$-entries. Speaking in terms of quantifiers this means: In order to determine which formulas are true in an ordered sum, one may have to know "more complicated" facts about the index ordering with respect to the number of individual quantifiers, however with respect to the number of quantifier alternations, no increase of formula complexity is involved. This is the reason why the present classification of formulas, Ehrenfeucht games, and types does not refer to quantifier depth but rather to quantifier alternation depth.

Proof. First we say how to find $\overline{r}$ from $\overline{k}$ and m. We define a corresponding function $\rho : (\overline{k}, m) \mapsto \overline{r}$ inductively over the length of $\overline{k}$: Let $\rho(\lambda, m) = \lambda$ for all m, and let $\rho(\overline{k}^\wedge k_{n+1}, m) = \rho(\overline{k}, m + k_{n+1})^\wedge |\mathcal{T}^{\overline{k}}(m + k_{n+1})|$. For $\overline{r} := \rho(\overline{k}, m)$ we verify the claim of the theorem inductively on the length n of $\overline{k}$.

For $n = 0$ we have to verify that the type $T^\lambda(\sum_{\iota \in I}(\mathcal{A}_\iota, \overline{P}_\iota))$ can be determined effectively from the λ-type of the $\mathcal{T}^\lambda(m)$-expansion of $(I, <^I)$ with respect to $(\mathcal{A}_\iota, \overline{P}_\iota)_{\iota \in I}$. This expansion is the structure $(I, <^I, Q_1, \ldots, Q_s)$ where Q_k assembles those indices $\iota \in I$ for which the ι-th component of the ordered sum has λ-type τ_k (in the listing of types).

Let us check first when a formula Nonempty$(X_i \cap X_j)$ is true in the ordered sum (i.e., belongs to $T^\lambda(\sum_{\iota \in I}(\mathcal{A}_\iota, \overline{P}_\iota))$). This holds iff some index ι exists such that in $(\mathcal{A}_\iota, \overline{P}_\iota)$ the formula Nonempty$(X_i \cap X_j)$ is true. So the type τ_k of this summand should contain Nonempty$(X_i \cap X_j)$, i.e. the corresponding subset Q_k of I, to which ι belongs, is nonempty. Altogether, Nonempty$(X_i \cap X_j)$ is true in the ordered sum iff for some $k \in \{1, \ldots, s\}$ with Nonempty$(X_i \cap X_j) \in \tau_k$, the formula Nonempty$(X_k)$ is in the type $T^\lambda(I, <^I, Q_1, \ldots, Q_s)$.

The other three kinds of atomic formulas are handled similarly: A formula $X_i \subseteq X_j$ is true in the ordered sum iff it is true for each summand iff only those Q_k are nonempty (i.e., we have Nonempty$(X_k) \in \mathcal{T}^\lambda(I, <^I, Q_1, \ldots, Q_s)$) where type τ_k contains $X_i \subseteq X_j$. A formula $X_i < X_j$ is true in the ordered sum iff

- either we have Nonempty$(X_k) \in \mathcal{T}^\lambda(I, <^I, Q_1, \ldots, Q_s)$ for some k such that type τ_k contains $X_i < X_j$ (i.e., $X_i < X_j$ holds already in some component, say of type τ_k),
- or we have $X_k < X_{k'} \in \mathcal{T}^\lambda(I, <^I, Q_1, \ldots, Q_s)$ for some pair (k, k') such that τ_k contains Nonempty(X_i) and $\tau_{k'}$ contains Nonempty(X_j) (this is the case where some component $\mathcal{A}_\iota$ (of type τ_k) exists before a component $\mathcal{A}_{\iota'}$ (of type $\tau_{k'}$), such that there is a X_i-element in $\mathcal{A}_\iota$ and a X_j-element in $\mathcal{A}_{\iota'}$).

Finally, a formula $X_{i_1} \cup \ldots \cup X_{i_l} =$ All is true in the ordered sum iff it holds in all summands iff Nonempty(X_k) occurs in $\mathcal{T}^\lambda(I, <^I, Q_1, \ldots, Q_s)$ only for those k where τ_k contains $X_{i_1} \cup \ldots \cup X_{i_l} =$ All.

Altogether we obtain an effective method to extract $T^\lambda(\sum_{\iota \in I}(\mathcal{A}_\iota, \overline{P}_\iota))$ from $T^\lambda(I, <^I, Q_1, \ldots, Q_s)$.

In the induction step we want to compute the $\overline{k}^\wedge k_{n+1}$-type of a structure $\sum_{\iota \in I}(\mathcal{A}_\iota, P_{\iota 1}, \ldots, P_{\iota m})$ from a certain $\overline{r}^\wedge r_{n+1}$-type of an expansion of the index ordering $(I, <^I)$, namely of the $\mathcal{T}^{\overline{k}^\wedge k_{n+1}}(m)$-expansion of $(I, <^I)$ with respect to $(\mathcal{A}_\iota, \overline{P}_\iota)_{\iota \in I}$. Following the inductive definition of ρ, we fix $\overline{r}$ and r_{k+1}:

$$\overline{r} := \rho(\overline{k}, m + k_{n+1}), \quad r_{n+1} := |\mathcal{T}^{\overline{k}}(m + k_{n+1})|.$$

The idea for the computation of $T^{\overline{k}^\wedge k_{n+1}}(\sum_{\iota \in I}(\mathcal{A}_\iota, P_{\iota 1}, \ldots, P_{\iota m}))$ is as follows: We recall that this type is the set of all types

$$T^{\overline{k}}(\textstyle\sum_{\iota \in I}(\mathcal{A}_\iota, P_{\iota 1}, \ldots, P_{\iota m}, R_{\iota 1}, \ldots, R_{\iota k_{n+1}}))$$

for all possible choices of $(R_{\iota 1}, \ldots, R_{\iota k_{n+1}})_{\iota \in I}$. By induction, we can compute each such type from the $\overline{r}$-type of the $\mathcal{T}^{\overline{k}}(m + k_{n+1})$-expansion of $(I, <^I)$ with respect to $(\mathcal{A}_\iota, P_{\iota 1}, \ldots, P_{\iota m}, R_{\iota 1}, \ldots, R_{\iota k_{n+1}})_{\iota \in I}$. It suffices to know the (finite) collection $\mathcal{C}$ of all these $\overline{r}$-types of such expansions of $(I, <^I)$, as induced by all possible choices of $(R_{\iota 1}, \ldots, R_{\iota k_{n+1}})_{\iota \in I}$.

We obtain this collection $\mathcal{C}$ from the object we are given by assumption: This object is

$$T^{\overline{r}^\wedge r_{n+1}}(I, <^I, Q_1, \ldots, Q_s)$$

where for $h = 1, \ldots, s$

$$Q_h = \{\iota \in I \mid T^{\overline{k}^\wedge k_{n+1}}(\mathcal{A}_\iota, P_{\iota 1}, \ldots, P_{\iota m}) = \tau_h\}$$

(referring to the list $\tau_1, \ldots, \tau_s$ of the types in $\mathcal{T}^{\overline{k}^\wedge k_{n+1}}(m)$).

The elements of this type $T^{\overline{r}^\wedge r_{n+1}}(I, <^I, Q_1, \ldots, Q_s)$ are, by definition, the types

$$T^{\overline{r}}(I, <^I, Q_1, \ldots, Q_s, Q'_1, \ldots, Q'_t)$$

where $t = r_{n+1} = |\mathcal{T}^{\overline{k}}(m + k_{n+1})|$ and $(Q'_1, \ldots, Q'_t) \in \mathcal{P}(I)^t$. The corresponding variables occurring in this type are written as $X_1, \ldots, X_s, X'_1, \ldots, X'_t$. By our choice of t, we refer to the correspondence between the sets $Q'_1, \ldots, Q'_t$ and the types in $\mathcal{T}^{\overline{k}}(m + k_{n+1})$ (which we assume listed, say as $\sigma_1, \ldots, \sigma_t$). Now, if $\iota \in Q'_j$ is to indicate that $T^{\overline{k}}(\mathcal{A}_\iota, \overline{P}_\iota, \overline{R}_\iota) = \sigma_j$, then the sets Q'_j have to define a partition of I and to meet a certain compatibility with the Q_h (and hence are no more arbitrary): Suppose $\iota \in Q_h$; then $T^{\overline{k}^\wedge k_{n+1}}(\mathcal{A}_\iota, \overline{P}_\iota) = \tau_h$, and $\iota \in Q'_j$ must mean that the type σ_j of the ι-th component originates from τ_h via some expansion $\overline{R}_\iota$ (so that $\tau_h = T^{\overline{k}^\wedge k_{n+1}}(\mathcal{A}_\iota, \overline{P}_\iota)$ and $\sigma_j = T^{\overline{k}}(\mathcal{A}_\iota, \overline{P}_\iota, \overline{R}_\iota)$). In other words: the $\overline{k}$-type σ_j is an element of the $\overline{k}^\wedge k_{n+1}$-type τ_h. So the compatibility condition says: The sets Q'_j form a partition of I, and if $Q_h \cap Q'_j$ is nonempty, then $\sigma_j \in \tau_h$. It is easy to see that this condition not only is necessary , but also sufficient for satisfiability by an appropriate ordered sum $\sum_{\iota \in I}(\mathcal{A}_\iota, \overline{P}_\iota, \overline{R}_\iota)$.

So from the elements of the given type $T^{\overline{r}^\wedge r_{n+1}}(I, <^I, Q_1, \ldots, Q_s)$ we find the desired collection $\mathcal{C}$ by assembling just those $\overline{r}$-types ($\in \mathcal{T}^{\overline{r}}(s + t)$) where

- the formula $X'_1 \cup \ldots \cup X'_t = \text{All}$ occurs but no formula $\text{Nonempty}(X'_{j_1} \cap X'_{j_2})$ occurs for distinct j_1, j_2 (the partition property),
- the occurrence of the formula $\text{Nonempty}(X_h \cap X'_j)$ implies that σ_j is an element of τ_h.

(Definability of the partition property on the quantifier-free level is the point where the atomic formulas $X_{i_1} \cup \ldots \cup X_{i_l} = \text{All}$ are useful. Clearly they are definable in terms of $\text{Nonempty}(X, Y)$ and $X \subseteq Y$ (even the latter suffices), however at the cost of quantifiers.)

For the case that $\overline{r}$ is not λ, the items above have to be reformulated: Rather than occurrence of atomic formulas in the $\overline{r}$-type under consideration we have to check whether these formulas are *induced* by the $\overline{r}$-type (i.e. they occur in all leaves of the Fraïssé tree associated with the type).

Strictly speaking, our induction step exhibited an algorithm for the computation of a type $T^{\overline{k}^\wedge k_{n+1}}(\sum_{\iota \in I}(\mathcal{A}_\iota, \overline{P}_\iota))$ from $T^{\overline{r}^\wedge r_{n+1}}(I, <^I, Q_1, \ldots, Q_s)$, using (by induction) a corresponding algorithm for the computation of types $T^{\overline{k}}(\sum_{\iota \in I}(\mathcal{A}_\iota, \overline{P}_\iota, \overline{R}_\iota))$ from types $T^{\overline{r}}(I, <^I, Q_1, \ldots, Q_s, Q'_1, \ldots, Q'_t)$. It should be clear from our description, however, that the different algorithms which enter here can be described uniformly and hence within a single global procedure, such that the induction step corresponds to a specific call of this procedure. □

We shall apply the Composition Theorem only in two rather special cases, namely for the two element index ordering $(\{1,2\},<)$ where the associated component types are arbitrary, and for the ordering $(\omega,<)$ where the component types all coincide.

The uniqueness claim of the Composition Theorem allows to write the $\overline{k}$-type of the ordered sum of two labelled orderings with $\overline{k}$-types σ_1, σ_2 (in this order) simply as *sum type* $\sigma_1 + \sigma_2$, independently of the underlying two labelled orderings which are used for concatenation. Now one uses the obvious fact (noted after Lemma 4) that for the finite ordering $(\{1,2\},<)$ and any expansion of it by given predicates $Q_1, \ldots, Q_s$, one can compute the $\overline{r}$-type of $(\{1,2\},<,Q_1,\ldots,Q_s)$. Hence, by the Composition Theorem, we have

Corollary 6. *From two types $\sigma_1, \sigma_2 \in \mathcal{T}^{\overline{k}}(m)$ one can compute the sum type $\sigma_1 + \sigma_2$.*

By an analogous argument, the $\overline{k}$-type of an ordered sum $\sum_{i\in\omega}(\mathcal{A}_i, \overline{P}_i)$ with $T^{\overline{k}}(\mathcal{A}_i, \overline{P}_i) = \sigma$ for all i only depends on σ and hence can be written as *ω-sum type* $\sum_{i\in\omega}\sigma$. Let us verify that this sum type is computable from σ. (The proof shows that the claim holds as well for any index ordering $(I,<)$ instead of $(\omega,<)$.)

Corollary 7. *From a type $\sigma \in \mathcal{T}^{\overline{k}}(m)$ and the $\overline{r}$-type of $(\omega,<)$, where $\overline{r}$ is chosen as in the Composition Theorem, one can effectively compute the ω-sum type $\sum_{i\in\omega}\sigma$.*

Proof. By the Composition Theorem, the computation of $\sum_{i\in\omega}\sigma$ is possible from $T^{\overline{r}}(\omega,<,Q_1,\ldots,Q_s)$, where Q_j is the set of indices carrying a structure of $\overline{k}$-type τ_j in the listing of $\mathcal{T}^{\overline{k}}(m)$. In the present case we have $Q_j = \omega$ for the unique j with $\tau_j = \sigma$, while otherwise $Q_j = \emptyset$. Our task is to compute $\sum_{i\in\omega}\sigma$ from $T^{\overline{r}}(\omega,<)$ alone (without the expansion). For this, we shall verify:

> Let $\overline{r} = (r_1, \ldots r_n)$. For any m and any structure $(\omega,<,\overline{P})$ with $\overline{P} = (P_1,\ldots,P_m)$ we have: If $Q_j \in \{\omega,\emptyset\}$ for $j \in \{1,\ldots,s\}$, then the type $T^{\overline{r}}(\omega,<,Q_1,\ldots,Q_s,\overline{P})$ is computable from $T^{\overline{r}}(\omega,<,\overline{P})$.

Then we obtain the claim of the Corollary when taking the empty tuple for $\overline{P}$.

The proof proceeds by induction on the length of $\overline{r}$, for all m simultaneously. The type $T^{\lambda}(\omega,<,Q_1,\ldots,Q_s,\overline{P})$ is easily derived from $T^{\lambda}(\omega,<,\overline{P})$, using the information which Q_j are empty and which are ω. In the induction step, we have to compute the $\overline{r}^\wedge r_{n+1}$-type of $(\omega,<,Q_1,\ldots,Q_s,\overline{P})$ from the $\overline{r}^\wedge r_{n+1}$-type of $(\omega,<,\overline{P})$, in other words: the set $\{T^{\overline{r}}(\omega,<,\overline{Q},\overline{P},\overline{R}) \mid \overline{R} \in \mathcal{P}(\omega)^{r_{n+1}}\}$ from the set $\{T^{\overline{r}}(\omega,<,\overline{P},\overline{R}) \mid \overline{R} \in \mathcal{P}(\omega)^{r_{n+1}}\}$. We can generate the first set from the second set, element by element, using the inductive assumption for the tuple length $m + r_{n+1}$. □

7 Decidability of the monadic theory of $(\omega, <)$

The previous results have shown how to compute certain types of structures from other given types. For a decidability result, however, an algorithm is desired to compute types without using auxiliary information. In the present section we show that such an algorithm exists for the computation of $T^{\overline{k}}(\omega, <)$ for given $\overline{k}$.

As a preparation, we need a result on types of finite orderings. Let $\mathrm{Fin}(m)$ be the class of labelled orderings $(A, <, P_1, \ldots, P_m)$ with finite universe A, and set

$$\mathcal{T}^{\overline{k}}(\mathrm{Fin}(m)) := \{T^{\overline{k}}(\mathcal{A}, \overline{P}) \mid (\mathcal{A}, \overline{P}) \in \mathrm{Fin}(m)\}.$$

Lemma 8. *There is an algorithm to compute $\mathcal{T}^{\overline{k}}(\mathrm{Fin}(m))$ for given $\overline{k}$ and m.*

Proof. We proceed by the increasing size of the structures $(\mathcal{A}, \overline{P})$. Since types are invariant under isomorphism we need just to compute $\{T^{\overline{k}}(\{1, \ldots, r\}, <, \overline{P}) \mid \overline{P} \in \mathcal{P}(\{1, \ldots, r\})^m\}$ for $r = 1$, $r = 2$, etc. For each r and m this can be done effectively, because only finitely many different structures occur. Since $\mathcal{T}^{\overline{k}}(m)$ is finite, one finds effectively the smallest number r_0 such that all types in

$$\{T^{\overline{k}}(\{1, \ldots, r_0 + 1\}, <, \overline{P}) \mid \overline{P} \in \mathcal{P}(\{1, \ldots, r_0 + 1\})^m\}$$

already occur as types of labelled orderings with $\leq r_0$ elements. It follows that for each labelled ordering $(\{1, \ldots, r\}, <, \overline{P})$ with $r > r_0$ there is a shorter one of same $\overline{k}$-type. (Namely, if $r > r_0 + 1$, write the structure as an ordered sum $(\{1, \ldots, r_0 + 1\}, <, \overline{P}) + (\mathcal{A}, \overline{R})$; now the first part may be shortened to a $\overline{k}$-equivalent one of length $\leq r_0$, and the claim follows by Corollary 6.) So the desired set of types is obtained by exhausting the labelled orderings up to cardinality r_0. □

For the proof that the monadic theory of $(\omega, <)$ is decidable, a reduction "from infinity to finiteness" is required. Büchi discovered in [Bü62] that for $\mathrm{MTh}(\omega, <)$ this reduction can be built on Ramsey's Theorem A ([Ra29]). Shelah [Sh75] developed a more abstract framework of "additive colorings" and showed also other combinatorial results, for instance over dense orderings. Our presentation below concentrates on $(\omega, <)$ and follows [Th81].

A (finite) *coloring* of ω is a map C from the set of unordered pairs of natural numbers to a finite set $\{c_1, \ldots, c_s\}$ of colors. When writing $C(i, j)$ we assume $i < j$. A coloring is *additive* if from $C(i, j) = C(i', j')$ and $C(j, k) = C(j', k')$ we can infer $C(i, k) = C(i', k')$. In this case we may introduce an addition operation $+$ on the set of colors and write $c + d = e$ if there are i, j, k with $C(i, j) = c$, $C(j, k) = d$, $C(i, k) = e$.

An example of an additive coloring, denoted $C_{\overline{k},\overline{P}}$, is obtained for any sequence $\overline{k}$ and any structure $(\omega, <, P_1, \ldots, P_m)$; here the color $C_{\overline{k},\overline{P}}(i, j)$ refers to the restriction of $\overline{P}$ to the segment $[i, j]$, written $\overline{P}|[i, j]$. Formally, define the map $C_{\overline{k},\overline{P}}$ by

$$C_{\overline{k},\overline{P}}(i, j) = T^{\overline{k}}([i, j], <, \overline{P}|[i, j]).$$

The additivity of this coloring is obvious from the summation result for $\overline{k}$-types (Corollary 6).

Referring to an additive coloring C, we call an infinite set $\{i_1 < i_2 < \ldots\}$ of natural numbers *(c,d)-homogeneous* if $C(0,i_1) = c$ and $C(i_k, i_l) = d$ for any pair $k < l$. Clearly, we have $d + d = d$ in this case.

Theorem 9. (Ramsey Theorem for additive colorings on ω)
For any finite additive coloring on ω there is, for a suitable pair (c,d) of colors, a (c,d)-homogeneous set.

Proof. We use the following definition, referring to the coloring C: Two numbers i,j *merge at* k $(> i,j)$ if $C(i,k) = C(j,k)$; in this case we write $i \sim_C j(k)$. This means, by additivity, that also $C(i,k') = C(j,k')$ for each $k' > k$ (just note that $C(i,k) + C(k,k') = C(j,k) + C(k,k')$). It follows that the merging relation $i \sim_C j$, which holds when $i \sim_C j(k)$ for some k, is an equivalence relation (of finite index, by the finiteness of the set of colors).

The first step of the proof is to verify the following claim: *There is a (c,d)-homogeneous set iff the following condition $H(c,d)$ holds:*

$$H(c,d): \quad \exists i(C(0,i) = c \wedge \forall l \exists j,k > l(C(i,j) = d \wedge i \sim_C j(k)))$$

The direction from left to right is easy: Given a (c,d)-homogeneous set $\{i_1, i_2, \ldots\}$, let $i = i_1$ and observe that for all $m > 1$, $C(i_1, i_m) = d$ and $i_1 \sim_C i_m(i_{m+1})$, which establishes $H(c,d)$. For the other direction, let us define a suitable sequence $i_1, i_2, \ldots$ inductively, preserving the following property for increasing m: $i_1 \sim_C i_r(k_r)$ for $r = 1, \ldots, m$ with suitable k_r. Let i_1 be the minimal i as guaranteed by $H(c,d)$. If $i_1, \ldots, i_m$ are defined with the above property, choose $l > k_1, \ldots k_m$ so that the i_r all merge pairwise at l. Applying $H(c,d)$, let i_{m+1} be the smallest $j > l$ with $C(i_1, j) = d$ and such that $i_1 \sim_C j(k)$ for suitable $k (= k_{m+1})$. Then $i_1 \sim_C i_r(k_r)$ for $r = 1, \ldots, m+1$. Clearly $C(i_1, i_2) = C(i_2, i_3) = d$ and by $i_1 \sim_C i_2(i_3)$ also $d + d = d$. Hence $C(i_r, i_{r+1}) = d$ for $r \geq 1$, which means that $\{i_1, i_2, \ldots\}$ is (c,d)-homogeneous.

Now it suffices to guarantee a pair (c,d) of colors such that $H(c,d)$ holds. Let M be an infinite $\sim_C$-equivalence class and i_1 its minimal element. Define $c = C(0, i_1)$ and choose d such that for infinitely many $i \in M$ we have $C(i_1, i) = d$. Then $H(c,d)$ holds, as was to be shown. □

Now all preparations for an "automata-free" proof of Büchi's Theorem are done:

Theorem 10. (Büchi's Theorem [Bü62])
The monadic theory MTh$(\omega, <)$ *is decidable.*

Proof. ([Sh75]) By Lemma 4, it suffices to compute the $\overline{k}$-type of $(\omega, <)$ for each $\overline{k}$. Since $T^\lambda(\omega, <) = \emptyset$, we consider only nonempty sequences, i.e. of the form $\overline{k}^\wedge m$ (where now $\overline{k}$ may be empty). We shall present an algorithm to compute, inductively over the length of $\overline{k}$ and simultaneously for all numbers m, the type $T^{\overline{k}^\wedge m}(\omega, <)$, which is the set

$$\{T^{\overline{k}}(\omega, <, \overline{P}) \mid \overline{P} \in \mathcal{P}(\omega)^m\}.$$

For $\overline{k} = \lambda$ it is tedious but easy to compile the corresponding sets of atomic formulas $\varphi(X_1, \ldots, X_m)$; the finitely many possibilities to satisfy these formulas by different set tuples $\overline{P}$ can be generated effectively (see the examples in Sections 4 and 5).

Next we have to compute the set $\{T^{\overline{k}}(\omega, <, \overline{P}) \mid \overline{P} \in \mathcal{P}(\omega)^m\}$ for any *nonempty* sequence $\overline{k} = (k_1, \ldots k_n)$ and any m. We can assume that we already know how to compute the corresponding set for any shorter sequence $(k_1', \ldots k_{n-1}')$ and arbitrary m'. In order to generate all possible $T^{\overline{k}}(\omega, <, \overline{P})$, we consider for each $\overline{P} \in \mathcal{P}(\omega)^m$ the finite additive coloring $C_{\overline{k},\overline{P}}$ with colors in $\mathcal{T}^{\overline{k}}(m)$. By Theorem 9, some pair (τ, σ) of types in this set exists such there is a (τ, σ)-homogeneous set $\{i_1, i_2, \ldots\}$, i.e. $(\omega, < \overline{P})$ is the ordered sum

$$([0, i_1], <, \overline{P}|[0, i_1]) \; + \; ([i_1 + 1, i_2], <, \overline{P}|[i_1 + 1, i_2]) \; + \; \ldots$$

with $T^{\overline{k}}([0, i_1], <, \overline{P}|[0, i_1]) = \tau$ and $T^{\overline{k}}([i_l + 1, i_{l+1}], <, \overline{P}|[i_l + 1, i_{l+1}]) = \sigma$ for $l \geq 1$. *Hence* $T^{\overline{k}}(\omega, <, \overline{P})$ *can be obtained as a sum type* $\tau + \sum_{i \in \omega} \sigma$ *where* $\tau, \sigma \in \mathcal{T}^{\overline{k}}(\mathrm{Fin}(m))$. Conversely, every such sum type clearly leads to some type $T^{\overline{k}}(\omega, <, \overline{P})$.

By Lemma 8, we can effectively generate all the types in $\mathcal{T}^{\overline{k}}(\mathrm{Fin}(m))$, i.e. all candidates for τ, σ. It remains to compute the sums $\tau + \sum_{i \in \omega} \sigma$. Considering a fixed pair (τ, σ), we apply the Composition Theorem. In order to obtain $\sum_{i \in \omega} \sigma$ it suffices, by Corollary 7, to compute $\mathcal{T}^{\overline{r}}(\omega, <)$, where $\overline{r} = (r_1, \ldots, r_n)$ is chosen as in the Composition Theorem. By definition, this type is

$$\{T^{(r_1, \ldots, r_{n-1})}(\omega, <, \overline{R}) \mid \overline{R} \in \mathcal{P}(\omega)^{r_n}\}.$$

Since $(r_1, \ldots, r_{n-1})$ is shorter than $\overline{k}$, we can compute this set by our assumption (setting $m' = r_n$). Thus $\sum_{i \in \omega} \sigma$ is computed, after which $\tau + \sum_{i \in \omega} \sigma$ is obtained by Corollary 6. □

Note that in the computation of types it was essential to proceed inductively by the length of quantifier alternation types (sequences $\overline{k}$): A type set

$$\{T^{(k_1, \ldots, k_n)}(\omega, <, \overline{P}) | \overline{P} \in \mathcal{P}(\omega)^m\}$$

turned out to be computable from

$$T^{(r_1, \ldots, r_n)}(\omega, <) = \{T^{(r_1, \ldots, r_{n-1})}(\omega, <, \overline{R}) \mid \overline{R} \in \mathcal{P}(\omega)^{r_n}\}$$

for suitable $(r_1, \ldots, r_n)$; this reduction from the sequence length n to $n-1$ supplied the inductive computation process for determining the $\overline{k}$-theories of $(\omega, <)$.

It is instructive to compare this proof of Shelah [Sh75] with the original one of Büchi [Bü83]. Both approaches rely on Ramsey's Theorem. They differ in the choice and application of additive colorings. We explain this in more detail.

Büchi's idea is to write monadic formulas $\varphi(X_1, \ldots, X_m)$ in "automata normal form"

$$\exists Y_1 \ldots \exists Y_k \psi(X_1, \ldots, X_m, Y_1, \ldots, Y_k);$$

here ψ is a *first-order* formula which says that $\overline{Y}$ is a run of a nondeterministic automaton on the input $\overline{X}$ (viewed as an ω-word over $\{0,1\}^m$) which visits infinitely often a final state. (Today we speak of *Büchi automata.*) Since the (first-order) quantifier depth of ψ is just 2, Büchi works with formulas of fixed quantifier alternation depth 3 and captures all the expressive power of monadic formulas by a growing length of the tuples $\overline{Y}$, corresponding to a growing number of states in Büchi automata. Where Shelah works upwards in quantifier alternation depth, Büchi shows that the automata normal form is closed under complement at the cost of more states. Indeed, this amounts to a reduction lemma for quantifier alternation depth: A formula $\forall\overline{Z}\exists\overline{Y}\psi(\overline{X},\overline{Y},\overline{Z})$, which is equivalent to $\neg\exists\overline{Z}\neg\exists Y\psi(\overline{X},\overline{Y},\overline{Z})$, can be written as $\exists\overline{Y'}\psi(\overline{X},\overline{Y'})$ using two such complementation steps.

Ramsey's Theorem is applied to establish this complementation (and hence, just as in Shelah's approach, for reducing quantifier alternation depth). In treating the negated automata normal form $\neg\exists Y_1 \ldots Y_k \psi(X_1, \ldots, X_m, \overline{Y})$ Büchi defines an additive coloring $C_{k,\overline{P}}$ on any input structure $(\omega, <, P_1, \ldots, P_m)$ which can serve as interpretation of the formula. Such a structure can be considered as an ω-word over $\{0,1\}^m$. The coloring $C_{k,\overline{P}}$ refers to the state set $\{0,1\}^k$ of the automaton described in ψ. While in Shelah's set-up two segments of $(\omega, <, P_1, \ldots, P_m)$ get the same color when their $\overline{k}$-types coincide, Büchi associates the same color to two segments u, v if the automaton described in ψ cannot distinguish them: For any pair p, q of states, the automaton should be able to pass from p to q via u iff this is possible via v, and a visit of a final state should be possible in such a run via u iff this is possible via v. The (finitely many) classes of this equivalence relation over finite words provide the colors for $C_{k,\overline{P}}$, and the additivity is clear from the way finite automata work. By Ramsey's Theorem, it turns out that the set of ω-words which satisfy $\neg\exists Y_1 \ldots Y_k \psi(X_1, \ldots, X_m, Y_1, \ldots, Y_k)$ can be generated as a union of "periodic sets" $U \cdot V^\omega$ where U, V are equivalence classes obtained from the automaton for ψ.

In the framework of $\overline{k}$-types, Shelah shows a similar fact: The structures $(\omega, <, \overline{P})$ of a given $\overline{k}$-type τ_0 can be obtained by collecting all sets $U_\tau \cdot V_\sigma^\omega$ with $\tau + \sum_{i \in \omega} \sigma = \tau_0$, where U_τ contains the finite segments of type τ and V_σ contains the finite segments of type σ.

At this point, where the periodic representation $U \cdot V^\omega$ of definable sets is reached, the decidability proof proceeds in two different ways: Büchi decides non-emptiness of these periodic sets, using the regularity of U and V (i.e., he decides non-emptiness of Büchi automata). Shelah generates the $\overline{k}$-theories of $(\omega, <)$ for increasing $\overline{k}$ by the composition process, and thus shows decidability in a synthetic rather than an analytic way.

There is a third approach to the decidability of $\mathrm{MTh}(\omega,<)$, due to Ladner [La77], which can be regarded as a combination of the methods of Büchi and Shelah. Ladner classifies finite segments of orderings by equivalence with respect to formulas of given quantifier depth (not alternation depth), using the corresponding Ehrenfeucht game, and applies Ramsey's Theorem to the associated finite additive coloring. So the equivalence is a logical one as in Shelah's proof. On the other hand, the resulting periodic representation of monadic second-order definable sets of ω-words is applied more in the spirit of Büchi's approach. As in the nonemptiness test for automata, it is analyzed which length of words u and v (depending on the quantifier depth under consideration) suffices to obtain an ω-word $u \cdot v^{\omega}$ as an element in such such a periodic set. The estimation is done directly, however, without passing to automata and using their number of states.

So there is a close relationship between the different proofs. Büchi went so far to detect "automata" also in Shelah's approach to the monadic theory of $(\omega,<)$ and of greater ordinals (cf. concluding section of [Bü83]). This seems to be a too liberal interpretation of "automata", which overrides the subtle use of the quantifier alternation depth measure in Shelah's proof. Rather it seems an open question to the present author whether reasonable "automata" exist on orderings different from ω and the ordinals. A logical view of "automata" (which also Büchi adopted) is to consider them as rather special formulas (of low quantifier alternation depth). Here it seems open whether for theories of dense orderings, where Shelah and Gurevich ([Sh75], [Gu79], [GS79]) showed decidability results with the calculus of $\overline{k}$-types, such "simple formulas" exist, which are as expressive as the full language and can play a similar role as the automata normal form over $(\omega,<)$ or larger ordinals.

8 Recursive ordinal words and their monadic theory

Ramsey's Theorem says that, given a finite additive coloring of ω, a (c,d)-homogeneous set exists for suitable colors c,d. In this section we consider again the colorings $C_{\overline{k},\overline{P}}$ and extend Ramsey's Theorem by a statement on the "difficulty" to determine a pair (c,d) from $\overline{k}$, assuming that $\overline{P}$ is fixed.

Here we use standard terminology from recursion theory, concerning relative computability and the arithmetical hierarchy (cf. [Ro67]). Given sets $P_1,\ldots,P_m$ of natural numbers, we denote by $\overline{P}'$ the *jump* of the recursion theoretic join of $P_1,\ldots,P_m$. This is a set of natural numbers with two properties (and we skip here the existence proof): First, $\overline{P}'$ is Σ_1 relative to $\overline{P}$, i.e. definable in the form

$$k \in \overline{P}' \Leftrightarrow \exists i R(i,k), \quad \text{with } R \text{ recursive in } \overline{P}.$$

Secondly, $\overline{P}'$ has a completeness property: Any set M definable in the form $M = \{k \mid \exists i R(i,n)\}$ with R recursive in $\overline{P}$ is itself recursive in $\overline{P}'$. Similarly, the *n-th jump* of $\overline{P}$, denoted $\overline{P}^{(n)}$, is a Σ_n-set relative to $\overline{P}$, and any such set M, i.e.

with a definition

$$k \in M \Leftrightarrow \exists i_1 \forall i_2, \ldots \exists/\forall i_n R(i_1, i_2, \ldots, i_n, k)\}, \text{ where } R \text{ is recursive in } \overline{P},$$

is recursive in $\overline{P}^{(n)}$.

The sets $\overline{P}', \overline{P}'', \ldots, \overline{P}^{(n)}, \ldots$ mark the levels of the *arithmetical hierarchy over* $\overline{P}$. The classical arithmetical hierarchy is obtained when $\overline{P}$ consists of recursive sets (or just the empty set).

Our aim is to determine on which level of this hierarchy over $\overline{P}$ the monadic theory $\mathrm{MTh}(\omega, <, \overline{P})$ is located. (Here one identifies a set of formulas with a set of natural numbers via an appropriate coding.)

Theorem 11. *The monadic theory* $\mathrm{MTh}(\omega, <, \overline{P})$ *is recursive in* $\overline{P}''$.

Proof. By Lemma 4, we have to show that the function

$$f_{(\omega,\overline{P})} : \overline{k} \mapsto T^{\overline{k}}(\omega, <, \overline{P})$$

is recursive in $\overline{P}''$. This means that we have to describe an algorithm which computes $f_{(\omega,\overline{P})}$ while during its computation has access to a $\overline{P}''$-oracle, i.e. obtains correct answers to questions about membership of concrete numbers in $\overline{P}''$. Of course, the algorithm can also determine, for any given numbers i, j, the type $T^{\overline{k}}([i,j], <, \overline{P}|[i,j])$ (for this, even the weaker $\overline{P}$-oracle would suffice).

In order to compute $T^{\overline{k}}(\omega, <, \overline{P})$, we can apply the fact, proved by means of Ramsey's Theorem, that this type originates as an ordered sum $\tau + \sum_{i \in \omega} \sigma$. Indeed, by the Composition Theorem we can compute $T^{\overline{k}}(\omega, <, \overline{P})$ once τ and σ are found. Here τ and σ are correctly chosen if there is a (τ, σ)-homogeneous set for the coloring $C_{\overline{k},\overline{P}}$, as defined in the proof of Theorem 10. By our proof of Ramsey's Theorem, τ and σ have this property iff the following condition holds (where we write C for $C_{\overline{k},\overline{P}}$):

$$H(\tau, \sigma): \quad \exists i(C(0,i) = \tau \ \wedge \ \forall l \exists j, k > l(C(i,j) = \sigma \wedge i \sim_C j(k)))$$

Our algorithm proceeds as follows, when supplied with input $\overline{k}$: It checks the condition

$$C(0,i) = \tau \ \wedge \ \forall l \exists j, k > l(C(i,j) = \sigma \wedge i \sim_C j(k))),$$

successively for $i = 1, 2, \ldots$, and for each i works through all (finitely many!) type pairs τ, σ from $\mathcal{T}^{\overline{k}}(\mathrm{Fin}(m))$. Each such test (for fixed i, τ, σ) involves two questions to the $\overline{P}''$-oracle. The first asks whether i satisfies $C(0,i) = \tau$ (for which even the weaker $\overline{P}$-oracle would suffice). The second question asks about the second conjunct above, which is again a condition on i. The set M of numbers satisfying this condition is a Π_2-set relative to $\overline{P}$, as seen from the formulation

$$i \in M \Leftrightarrow \forall l \exists k[k > l \wedge \exists j(i < j < k \wedge C(i,j) = \sigma \wedge C(i,k) = C(j,k))];$$

note that the relation in square brackets (in i, k, l) is recursive in $\overline{P}$. So M is the complement of a Σ_2-set relative to $\overline{P}$, and thus membership of a given number i in it is answered by the $\overline{P}''$-oracle.

By (the proof of) Ramsey's Theorem, $H(\tau, \sigma)$ holds for some pair (τ, σ), and hence a corresponding i exists as required in $H(\tau, \sigma)$. The algorithm will detect such i, τ, σ after finitely many steps. From the pair τ, σ it produces $\tau + \sum_{\iota \in I} \sigma$ and hence the desired type $T^{\overline{k}}(\omega, <, \overline{P})$. □

In a next step we lift this result to higher ordinals ω^n.

Theorem 12. *The monadic theory* $\mathrm{MTh}(\omega^n, <, \overline{P})$ *is recursive in* $\overline{P}^{(2n)}$.

Proof. We proceed by induction on $n \geq 1$. The case $n = 1$ is given by the previous theorem. Consider a labelled ordinal ordering $(\omega^{n+1}, <, \overline{P})$. It suffices to compute the function $\overline{k} \mapsto T^{\overline{k}}(\omega^{n+1}, <, \overline{P})$ by means of an algorithm with a $\overline{P}^{(2n+2)}$-oracle. For this purpose we decompose $(\omega^{n+1}, <, \overline{P})$ as the ω-sum of labelled orderings $(\mathcal{A}_i, \overline{P}_i) = (\omega^n, <, \overline{P}_i)$ where $i \in \omega$. Define a corresponding coloring C on ω by

$$C(i, j) = T^{\overline{k}}((\mathcal{A}_i, \overline{P}_i) + \ldots + (\mathcal{A}_j, \overline{P}_j)).$$

By inductive assumption, for any given i the type $T^{\overline{k}}(\mathcal{A}_i, \overline{P}_i)$ can be computed by means of a $\overline{P}^{(2n)}$-oracle. Applying the effective summation of a finite number types (via Corollary 6), we conclude that the function C is recursive in $\overline{P}^{(2n)}$. Now apply the algorithm of the previous theorem for the new coloring C of ω. This algorithm produces $T^{\overline{k}}(\omega^{n+1}, <, \overline{P})$ upon input $\overline{k}$. During computation it checks for $i = 0, 1, \ldots$ the same conditions as before, now for the new C. Since C is recursive in $\overline{P}^{(2n)}$, a $\overline{P}^{(2n+2)}$-oracle suffices where previously the $\overline{P}''$-oracle was asked. (We use here the fact that a set which is recursive in M'' where M is recursive in $N^{(n)}$, is recursive in $N^{(n+2)}$.) Hence $\mathrm{MTh}(\omega^{n+1}, <, \overline{P})$ is recursive in $\overline{P}^{(2n+2)}$. □

Let us consider the special case of recursive predicates on an ordinal ω^n. (The notion of a recursive predicate on an ordinal ω^n is canonical: Since its elements are representable in the form $\omega^{n-1} \cdot c_{n-1} + \ldots + \omega \cdot c_1 + c_0$ with $c_i \in \omega$, they are in effective 1-1-correspondence to the natural numbers; so the notion of recursiveness is transferred from ω to ω^n.) As mentioned in the first two sections, a labelled ordering $(\omega^n, <, \overline{P})$ is viewed as an ω^n-word; for recursive $\overline{P}$ we speak of a recursive ω^n-word. For this case the previous theorem says the following:

Corollary 13. *The monadic theory of a recursive* ω^n*-word is recursive in* $\emptyset^{(2n)}$*, the* $2n$*-th jump of the empty set.*

The previous results can be sharpened slightly. For this purpose, one transforms the condition $H(c, d)$ used in Ramsey's Theorem from the present $\exists\forall\exists$-form (where just the unbounded quantifiers are counted) to a boolean combination of $\exists\forall$-clauses. This transformation is a combinatorial analogue of McNaughton's Theorem in the theory of ω-automata (see Section 1 of [Th81] for details). Given the new form of $H(c,d)$, the reducibility relation "recursive in", as it appears in the results above, can be strengthened to "truth-table reducible in" (for definitions see [Ro67]). A further restriction in the reducibility notion (namely, to bounded truth-table reducibility) cannot be reached, however, as shown in [Th78].

As a final result, we show that the bound $\emptyset^{(2n)}$ in Corollary 13 cannot be improved.

Theorem 14. *For $n \geq 1$ there is a recursive ω^n-word $(\omega^n, <, P_n)$ such that $\emptyset^{(2n)}$ is recursive in* $\mathrm{MTh}(\omega^n, <, P_n)$ *(even in the first-order theory of $(\omega^n, <, P_n)$).*

Proof. It is convenient to work with the complement of $\emptyset^{(2n)}$ rather than $\emptyset^{(2n)}$ itself. These two sets are recursive in each other; so either of them can be applied for the claim. While $\emptyset^{(2n)}$ is Σ_{2n}-complete, the complement of $\emptyset^{(2n)}$ is Π_{2n}-complete. For Π_{2n}-sets we use a specific representation which involves n quantifiers $\exists^\omega$, meaning "there exist infinitely many". (For our purpose, it is convenient to index the n quantified variables from $n-1$ down to 0.)

Lemma 15. ([KSW60], see also section 14.8 of [Ro67])
A set M of natural numbers is a Π_{2n}-set iff it can be defined in the form

$$k \in M \;\Leftrightarrow\; \exists^\omega i_{n-1} \ldots \exists^\omega i_0 R(i_{n-1}, \ldots i_0, k)$$

where R is a recursive relation.

Let M_n be the complement of $\emptyset^{(2n)}$ and R_n the corresponding recursive relation according to the Lemma.

For the definition of the desired (recursive) set P_n in ω^n we first treat the case $n = 1$, using the representation of M_1 in the form

$$k \in M_1 \;\Leftrightarrow\; \exists^\omega i_0 R_1(i_0, k), \quad \text{where } R_1 \text{ is recursive.}$$

We build up the predicate P_1 in the form of an ω-sequence with letters 0 and 1, using an enumeration of the pairs $(i_0, k) \in \omega \times \omega$. Suppose that during the enumeration process the pair (i_0, k) is reached. Now check whether (i_0, k) is in R_1. If yes, then add $01^{k+1}0$ to the P_1-prefix built up so far, otherwise just add 0. Clearly the resulting predicate P_1 is recursive. The occurrence of such blocks of $k+1$ letters 1 (enclosed by two zeroes) is expressible even in first-order logic; we use the auxiliary formula "at y starts a $(k+1)$-block" which says "y and the next k successors belong to P_1, but the subsequent successor as well as the predecessor of y do not". (We work here with the standard first-order language and do not take the trouble of translating the formulas back into the monadic second-order framework as used in the previous sections.) We have

$$k \in M_1 \Leftrightarrow (\omega, <, P_1) \models \forall x \exists y (x < y \wedge \text{at } y \text{ starts a } k+1\text{-block}),$$

which shows that M_1 is recursive in the first-order theory of $(\omega, <, P_1)$.

The construction is now generalized to M_n. Recall the representation of M_n by

$$k \in M_n \Leftrightarrow \exists^\omega i_{n-1} \ldots \exists^\omega i_0 R_n(i_{n-1}, \ldots, i_0, k),$$

where R_n is recursive. On ω^n we define a suitable recursive predicate P_n. The elements of ω^n can be represented as $\omega^{n-1} \cdot i_{n-1} + \ldots + \omega \cdot i_1 + i_0$ where $i_k \in \omega$ (Cantor's normal form); we shall just write $(i_{n-1}, \ldots, i_1, i_0)$ instead. Note that the prefix $(i_{n-1}, \ldots, i_1)$ fixes a unique ω-copy within ω^n, consisting of the elements $(i_{n-1}, \ldots, i_1, j)$ where $j \in \omega$; we call $(i_{n-1}, \ldots, i_1)$ the *address* of this ω-copy. For the construction of P_n, in the form of an ω^n-word over $\{0, 1\}$, we now use an enumeration of the $(n+1)$-tuples over ω; while the enumeration proceeds, each ω-copy in the desired word $(\omega^n, <, P_n)$ is built up simultaneously. Suppose we just deal with the tuple $(i_{n-1}, \ldots, i_1, i_0, k)$ during the enumeration. If it does not belong to R_n, then attach 0 to each ω-copy (more precisely, to the P_n-prefix constructed so far on this ω-copy). If the tuple belongs to R_n, then one exception is made, namely on the unique copy with address $(i_{n-1}, \ldots, i_1)$ the word $01^{k+1}0$ is attached. Again, the construction directly shows that P_n is recursive.

To verify that M_n is recursive in the first-order theory of $(\omega^n, < P_n)$ we have to exhibit, for each k, a sentence φ_k such that

$$k \in M_n \Leftrightarrow (\omega^n, <, P_n) \models \varphi_k.$$

By the construction of P_n it will suffice to express the statement: "$\exists^\omega i_{n-1} \ldots \exists^\omega i_1$ such that on the ω-copy with address $(i_{n-1}, \ldots, i_1)$ there are infinitely many $(k+1)$-blocks".

This is reformulated as follows:

"there are infinitely many ω^{n-1}-copies C_{n-1} such that
on C_{n-1} there are infinitely many ω^{n-2}-copies C_{n-2} such that
...
on C_2 there are infinitely many ω-copies C_1 such that
on C_1 there are infinitely many $(k+1)$-blocks"

An ω^i-copy C_i is easily fixed by the ω^i-limit ordinal (or 0) which is its first element and by the ω^i-limit ordinal which succeeds the copy. It is well-known how to define the ω^i-limit ordinals in first-order logic (inductively on i). For an ω^i-copy C_i enclosed by y_i, z_i, the condition "on C_i there exist infinitely many ω^{i-1}-copies C_{i-1} such that ..." is formalized following the pattern

$$\begin{aligned}&\forall x_{i-1} \in [y_i, z_i)\, \exists y_{i-1} \in [x_{i-1}, z_i) \exists z_{i-1} \in [x_{i-1}, z_i) \\ &\qquad (y_{i-1} < z_{i-1} \;\wedge\; y_{i-1}, z_{i-1} \text{ are successive } \omega^{i-1}\text{-limit ordinals} \;\wedge \ldots)\end{aligned}$$

whence our statement is indeed expressible by a sentence φ_k in the monadic (even first-order) language of the structure $(\omega^n, <, P_n)$. □

The concatenation of all the ordinal words $(\omega^n, <, P_n)$ for $n = 1, 2, \ldots$ will result in a recursive ω^ω-word. Every set $\emptyset^{(2n)}$ (for $n = 1, 2, \ldots$) is recursive in the first-order theory of this ω^ω-word. Hence we obtain:

Corollary 16. *There is recursive ω^ω-word whose monadic (and even first-order) theory is not arithmetical.*

9 Conclusion

We have exposed a model theoretic approach to analyze the monadic theory of labelled orderings, based on the Ehrenfeucht-Fraïssé method and further developed into a powerful calculus by Shelah and Gurevich, and we presented some simple applications concerning the monadic theory of the ordinal ω and of ordinal words. The reader is invited to take this paper as a start to study [Sh75], [Gu79], [GS79] and subsequent work, where more powerful results are shown. Among the open fields for further research we mention just two: the extension of the method to more general structures and the investigation of the computational complexity of procedures which manipulate $\overline{k}$-types.

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