Transfinite Rewriting Semantics for Term Rewriting Systems*

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Abstract. We provide some new results concerning the use of transfinite rewriting for giving semantics to rewrite systems. We especially (but not only) consider the computation of possibly infinite constructor terms by transfinite rewriting due to their interest in many programming languages. We reconsider the problem of compressing transfinite rewrite sequences into shorter (possibly finite) ones. We also investigate the role that (finitary) confluence plays in transfinite rewriting. We consider different (quite standard) rewriting semantics (mappings from input terms to sets of reducts obtained by –transfinite– rewriting) in a unified framework and investigate their algebraic structure. Such a framework is used to formulate, connect, and approximate different properties of TRSs.

Keywords: Declarative programming, semantics, infinitary rewriting.

1 Introduction

Rewriting that considers infinite terms and reduction sequences of any ordinal length is called *transfinite rewriting*; rewriting sequences of length ω are often called *infinitary*. The motivation to distinguish between them is clear: reduction sequences of length of at most ω seem more adequate for real applications (but transfinite rewriting is suitable for modeling rewriting with finite cyclic graphs). There are two main frameworks for transfinite rewriting: [DKP91] considers standard *Cauchy convergent* rewriting sequences; [KKSV95] only admits *strongly convergent* sequences which are Cauchy convergent sequences in which redexes are contracted at deeper and deeper positions. Cauchy convergent sequences are more powerful than strongly convergent ones w.r.t. their *computational strength*, i.e., the ability to compute canonical forms of terms (normal forms, values, etc.).

Example 1. Consider the TRS (see [KKSV95]):

$$\begin{array}{ll} \mathtt{f(x,g)} \to \mathtt{f(c(x),g)} & \mathtt{g} \to \mathtt{a} \\ \mathtt{f(x,a)} \to \mathtt{c(x)} & \mathtt{h} \to \mathtt{c(h)} \end{array}$$

and the derivation of length $\omega + 2$:

$$\underline{\mathtt{f(a,g)}} \to \underline{\mathtt{f(c(a),g)}} \to \cdots\ \mathtt{f(c^{\omega},\underline{g})} \to \underline{\mathtt{f(c^{\omega},a)}} \to \mathtt{c}^{\omega}$$

No strongly convergent reduction rewrites $\mathtt{f(a,g)}$ into the infinite term $\mathtt{c}^{\omega}$.

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Transfinite, strongly convergent sequences can be *compressed* into infinitary ones when dealing with left-linear TRSs [KKSV95]. This is not true for Cauchy convergent sequences: e.g., there is no Cauchy convergent *infinitary* sequence reducing `f(a,g)` into $\mathtt{c}^\omega$ in Example 1. Because of this, Kennaway et al. argue that strongly convergent transfinite rewriting is the best basis for a theory of transfinite rewriting [KKSV95]. However, Example 1 shows that the restriction to strongly convergent sequences may lose computational power, since many possibly useful sequences are just disallowed. Thus, from the semantic point of view, it is interesting to compare them further.

In this paper, we especially consider the computation of possibly infinite constructor terms by (both forms of) transfinite rewriting. In algebraic and functional languages, constructor terms play the role of completely meaningful pieces of information that (defined) functions take as their input and produce as the outcome. We prove that every infinitary rewrite sequence leading to a constructor term is strongly convergent. We prove that, for left-linear TRSs, transfinite rewrite sequences leading to finite terms can always be compressed into *finite* rewrite sequences and that infinite terms obtained by transfinite rewrite sequences can always be finitely (but arbitrarily) approximated by finitary rewrite sequences. We have investigated the role of *finitary* confluence in transfinite rewriting. We prove that for left-linear, (finitary) confluent TRSs, Cauchy convergent transfinite rewritings leading to infinite constructor terms can always be compressed into infinitary ones. We also prove that finitary confluence ensures the uniqueness of infinite constructor terms obtained by infinitary rewriting. We use our results to define and compare *rewriting semantics*. By a rewriting semantics we mean a mapping from input terms to sets of reducts obtained by finite, infinitary, or transfinite rewriting. We study different rewriting semantics and their appropriateness for computing different kinds of interesting semantic values in different classes of TRSs. We also investigate two orderings between semantics that provide an algebraic framework for approximating semantic properties of TRSs. We motivate this framework using some well-known problems in term rewriting.

Section 2 introduces transfinite rewriting. Section 3 deals with compression. Section 4 investigates the role of finitary confluence in transfinite rewriting computations. Section 5 introduces the semantic framework and Section 6 discusses its use in approximating properties of TRSs. Section 7 compares different semantics studied in the paper. Section 8 discusses related work.

2 Transfinite rewriting

Terms are viewed as labelled trees in the usual way. An (infinite) term on a signature Σ is a finite (or infinite) ordered tree such that each node is labeled by a symbol $f \in \Sigma$ and has a tuple of descendants, and the size of such a tuple is equal to $ar(f)$ (see [Cou83] for a formal definition). When considering a denumerable set of variables $\mathcal{X}$, we obtain terms with variables in the obvious way. The set of (ground) infinite terms is denoted by $\mathcal{T}^\omega(\Sigma, \mathcal{X})$ (resp. $\mathcal{T}^\omega(\Sigma)$) and $\mathcal{T}(\Sigma, \mathcal{X})$

(resp. $\mathcal{T}(\Sigma)$) is the set of (resp. ground) finite terms. Notations $\mathcal{T}^\infty(\Sigma,\mathcal{X})$ or $\mathcal{T}^\infty(\Sigma)$ are more frequent; by using ω, we emphasize that transfinite terms are not considered. This is more consistent with the use of ω and ∞ in the paper.

By $\mathcal{P}os(t)$, we denote the set of positions of a term t, and $|p|$ is the length of a position p. By Λ, we denote the empty chain. The height d_t of $t \in \mathcal{T}(\Sigma,\mathcal{X})$ is given by $\mathsf{d}_t = 1+max(\{|p| \mid p \in \mathcal{P}os(t)\})$. We use the following distance d among terms: $d(t,s) = 0$ if $t = s$; otherwise, $d(t,s) = 2^{-p(t,s)}$ (where $p(t,s)$ is the length $|p|$ of the shortest position $p \in \mathcal{P}os(t) \cap \mathcal{P}os(s)$ such that $root(t|_p) \neq root(s|_p)$ [Cou83]). Therefore, $(\mathcal{T}^\omega(\Sigma,\mathcal{X}),d)$ is a metric space. Note that, if $t \in \mathcal{T}(\Sigma)$ and $\epsilon \leq 2^{-\mathsf{d}_t}$, then $\forall s \in \mathcal{T}^\omega(\Sigma)$, $d(t,s) < \epsilon \Leftrightarrow t = s$. A *substitution* is a mapping $\sigma : \mathcal{X} \to \mathcal{T}^\omega(\Sigma,\mathcal{X})$ which we homomorphically extend to a mapping $\sigma : \mathcal{T}^\omega(\Sigma,\mathcal{X}) \to \mathcal{T}^\omega(\Sigma,\mathcal{X})$ by requiring that it be continuous w.r.t. the metric d. A rewrite rule is an ordered pair (l,r), written $l \to r$, with $l,r \in \mathcal{T}(\Sigma,\mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side[1] (*rhs*). A TRS is a pair $\mathcal{R} = (\Sigma,R)$ where R is a set of rewrite rules. Given $\mathcal{R} = (\Sigma,R)$, we consider Σ as the disjoint union $\Sigma = \mathcal{C} \uplus \mathcal{F}$ of symbols $c \in \mathcal{C}$, called *constructors* and symbols $f \in \mathcal{F}$, called *defined functions*, where $\mathcal{F} = \{f \mid f(\bar{l}) \to r \in R\}$ and $\mathcal{C} = \Sigma - \mathcal{F}$. Then, $\mathcal{T}(\mathcal{C},\mathcal{X})$ (resp. $\mathcal{T}(\mathcal{C}),\mathcal{T}^\omega(\mathcal{C})$) is the set of (resp. ground, possibly infinite, ground) constructor terms. A term $t \in \mathcal{T}^\omega(\Sigma,\mathcal{X})$ is a redex if there exist a substitution σ and a rule $l \to r$ such that $t = \sigma(l)$. A term $t \in \mathcal{T}^\omega(\Sigma,\mathcal{X})$ rewrites to s (at position p), written $t \to_\mathcal{R} s$ (or just $t \to s$), if $t|_p = \sigma(l)$ and $s = t[\sigma(r)]_p$, for some rule $\rho : l \to r \in R$, $p \in \mathcal{P}os(t)$ and substitution σ. This can eventually be detailed by writing $t \stackrel{[p,\rho]}{\to} s$ (substitution σ is uniquely determined by $t|_p$ and l). A normal form is a term without redexes. By $\mathsf{NF}_\mathcal{R}$ ($\mathsf{NF}^\omega_\mathcal{R}$) we denote the set of finite (resp. possibly infinite) ground normal forms of $\mathcal{R}$. A term t is root-stable (or a head normal form) if $\forall s$, if $t \to^* s$, then s is not a redex. By $\mathsf{HNF}_\mathcal{R}$ ($\mathsf{HNF}^\omega_\mathcal{R}$) we denote the set of ground root-stable finite (resp. possibly infinite) terms of $\mathcal{R}$. In the following, we are mainly concerned with ground terms.

A *transfinite* rewrite sequence of a TRS $\mathcal{R}$ is a mapping A whose domain is an *ordinal* α such that A maps each $\beta < \alpha$ to a reduction step $A_\beta \to A_{\beta+1}$ [KKSV95]. If α is a limit ordinal, A is called *open*; otherwise, it is *closed*. If $\alpha \in \{\omega,\omega+1\}$, we will say that A is *infinitary*, rather than transfinite. The *length* of the sequence is α if α is a limit ordinal; otherwise, it is $\alpha - 1$. For limit ordinals $\beta < \alpha$, the previous definition of transfinite rewrite sequence does not stipulate any relationship between A_β and the earlier terms in the sequence. Thus, the following notion is helpful [KKSV95]: Given a distance d on terms, a rewriting sequence A is said to be Cauchy continuous if for every ordinal limit $\lambda < \alpha$, $\forall \epsilon > 0, \exists \beta < \lambda, \forall \gamma\ (\beta < \gamma < \lambda \Rightarrow d(A_\gamma, A_\lambda) < \epsilon)$. Given a reduction sequence A, let $\mathfrak{d}_\beta$ be the length of the position of the redex reduced in the step from A_β to $A_{\beta+1}$. A Cauchy continuous closed sequence A is called *Cauchy convergent*. A Cauchy convergent sequence is called *strongly continuous* if for every limit ordinal $\lambda < \alpha$, the sequence $(\mathfrak{d}_\beta)_{\beta<\lambda}$ tends to infinity. If A is strongly

[1] In [KKSV95], infinite right-hand sides are also allowed.

continuous and closed, then S is strongly convergent. We write: $A : t \hookrightarrow^{\alpha} s$ (resp. $A : t \to^{\alpha} s$) for a Cauchy (resp. strongly) convergent sequence of length α starting from t and ending at s. We write: $A : t \hookrightarrow^{\leq\alpha} s$ (resp. $A : t \to^{\leq\alpha} s$) for a Cauchy (resp. strongly) convergent sequence starting from t and ending at s whose length is less than or equal to α; moreover, we often take the initial term t of A as its name and write t_β for an ordinal $\beta < \alpha$ rather than A_β. We write $t \hookrightarrow^{\infty} s$ (resp. $t \to^{\infty} s$) if we do not wish to explicitly indicate the length of the sequence.

3 Compression of transfinite rewrite sequences

The following result shows that, for computing *constructor terms*, Cauchy convergent and strongly convergent *infinitary* rewriting are equivalent.

Theorem 1. *Let $\mathcal{R}$ be a TRS, $t \in \mathcal{T}^{\omega}(\Sigma)$ and $\delta \in \mathcal{T}^{\omega}(\mathcal{C})$. Every Cauchy convergent infinitary rewrite sequence from t to δ is strongly convergent.*

Proof. Let $t \hookrightarrow^{\omega} \delta$. Since no rule may overlap any constructor prefix of δ, $\mathfrak{d}_\beta$ for redexes contracted in each step $t_\beta \to t_{\beta+1}$ must tend to infinity as the normal form does.

Example 1 shows that, in general, Theorem 1 does not hold for transfinite sequences. Moreover, Theorem 1 does not hold for arbitrary normal forms.

Example 2. Consider the orthogonal (infinite) TRS [KKSV95]:

$$\begin{array}{lll} \mathtt{f(g}^{n}\mathtt{(c))} & \to \mathtt{f(g}^{n+1}\mathtt{(c))} & \text{for } n \geq 0 \\ \mathtt{a} & \to \mathtt{g(a)} & \end{array}$$

Thus, $\mathtt{f(c)} \hookrightarrow^{\omega} \mathtt{f(g}^{\omega}\mathtt{)} \in \mathsf{NF}^{\omega}_{\mathcal{R}}$, but $\mathtt{f(c)} \not\to^{\omega} \mathtt{f(g}^{\omega}\mathtt{)}$.

Kennaway et al. proved the following:

Theorem 2. [KKSV95] *Let $\mathcal{R}$ be a left-linear TRS and $t, s \in \mathcal{T}^{\omega}(\Sigma)$. If $t \to^{\infty} s$, then $t \to^{\leq\omega} s$.*

Theorem 2 does not hold for Cauchy convergent reductions.

Remark 1. The compression of Cauchy convergent sequences into infinitary ones has been studied[2] in [DKP91], where *top-termination* (i.e., no infinitary reduction sequence performs infinitely many rewrites at Λ), is required to achieve it. Kennaway et al. noticed that it implies that "*every reduction starting from a finite term is strongly convergent*" thus arising as a consequence of Theorem 2.

For Cauchy convergent sequences, we prove several restricted compression properties.

Proposition 1. *Let $\mathcal{R}$ be a TRS, $t \in \mathcal{T}^{\omega}(\Sigma)$ and s be a finite term. If $t \hookrightarrow^{\lambda} s$ for a limit ordinal λ, then $t \hookrightarrow^{\beta} s$ for some $\beta < \lambda$.*

[2] In the terminology of [DKP91], ω-closure.

Proof. If $t \hookrightarrow^{\lambda} s$, then $\forall \epsilon > 0, \exists \beta_\epsilon, \forall \gamma, \beta_\epsilon < \gamma < \lambda, d(t_\gamma, s) < \epsilon$. Since s is finite, we let $\epsilon = 2^{-\mathsf{d}_s}$. Then, $\forall \gamma, \beta_\epsilon < \gamma < \lambda$, it must be $t_\gamma = s$. In particular, $t \hookrightarrow^{\beta} s$ for $\beta = \beta_\epsilon + 1 < \lambda$ (since λ is a limit ordinal).

In Theorem 3 below, we prove that, for left-linear TRSs, finite pieces of information obtained from transfinite rewriting can always be obtained from finitary sequences. In the following auxiliary result, we use $\mathsf{d}_\rho = max(\mathsf{d}_l, \mathsf{d}_r)$ for a rule $\rho : l \to r$ (note that d_ρ only makes sense for rules whose *rhs*'s are finite).

Proposition 2. *Let $\mathcal{R}$ be a left-linear TRS. Let $t, s \in \mathcal{T}^\omega(\Sigma)$ be such that $t = t_1 \stackrel{[p_1,\rho_1]}{\to} t_2 \to \cdots \to t_n \stackrel{[p_n,\rho_n]}{\to} t_{n+1} = s$. For all $\kappa \in]0,1]$, and $t' \in \mathcal{T}^\omega(\Sigma)$ such that $d(t', t) < 2^{-(\sum_{i=1}^n |p_i| + \mathsf{d}_{\rho_i}) + log_2(\kappa)}$, there exists $s' \in \mathcal{T}^\omega(\Sigma)$ such that $t' \to^* s'$ and $d(s', s) < \kappa$.*

Proof. By induction on n. If $n = 0$, then $t = s$ and $d(t', s) < 2^{log_2(\kappa)} = \kappa$. We let $s' = t'$ and the conclusion follows. If $n > 0$, then, since $0 < \kappa \leq 1$, we have $d(t', t) < 2^{-(\sum_{i=1}^n |p_i| + \mathsf{d}_{\rho_i}) + log_2(\kappa)} \leq 2^{-|p_1| - \mathsf{d}_{l_1}}$. Since $\mathcal{R}$ is left-linear, it follows that $t'|_{p_1}$ is a redex of ρ_1. Thus, $t' \to t'[\sigma_1'(r_1)]_{p_1}$ and for all $x \in \mathcal{V}ar(l_1)$, $d(\sigma_1'(x), \sigma_1(x)) < 2^{-(\sum_{i=2}^n |p_i| + \mathsf{d}_{\rho_i}) + log_2(\kappa)}$. Hence, since $\mathcal{V}ar(r_1) \subseteq \mathcal{V}ar(l_1)$ and $\mathsf{d}_{\rho_1} = max(\mathsf{d}_{l_1}, \mathsf{d}_{r_1})$, we have $d(t'[\sigma_1'(r_1)]_{p_1}, t_2) \leq 2^{-(\sum_{i=2}^n |p_i| + \mathsf{d}_{\rho_i}) - |p_1| - \mathsf{d}_{r_1} + log_2(\kappa)}$ $< 2^{-(\sum_{i=2}^n |p_i| + \mathsf{d}_{\rho_i}) + log_2(\kappa)}$. By the induction hypothesis, $t'[\sigma_1'(r_1)]_{p_1} \to^* s'$ and $d(s', s) < \kappa$; hence $t' \to^* s'$ and the conclusion follows.

Proposition 3. *Let $\mathcal{R}$ be a left-linear TRS, $t, s \in \mathcal{T}^\omega(\Sigma)$, and $t' \in \mathcal{T}^\omega(\Sigma) - \mathcal{T}(\Sigma)$. If $t \hookrightarrow^{\lambda} t' \to^* s$ for a limit ordinal λ, then for all $\kappa \in]0,1]$, there exist $s' \in \mathcal{T}^\omega(\Sigma)$ and $\beta < \lambda$ such that $t \hookrightarrow^{\beta} s'$ and $d(s', s) < \kappa$.*

Proof. Since λ is a limit ordinal and t' is infinite, for all $\epsilon > 0$, there exists $\beta_\epsilon < \lambda$ such that, for all γ, $\beta_\epsilon < \gamma < \lambda$, $d(t_\gamma, t') < \epsilon$. Let $t' = t_1 \stackrel{[p_1,\rho_1]}{\to} t_2 \to \cdots \to t_n \stackrel{[p_n,\rho_n]}{\to} t_{n+1} = s$, $\epsilon = 2^{-(\sum_{i=1}^n |p_i| + \mathsf{d}_{\rho_i}) + log_2(\kappa)}$, and $\beta_0 = \beta_\epsilon + 1$. Thus, $d(t_{\beta_0}, t') < \epsilon$ and, by Proposition 2, $t_{\beta_0} \to^* s'$ and $d(s', s) < \kappa$. Thus, $t \hookrightarrow^{\beta} s'$ for $\beta = \beta_0 + n + 1 < \lambda$ and the conclusion follows.

Theorem 3. *Let $\mathcal{R}$ be a left-linear TRS and $t, s \in \mathcal{T}^\omega(\Sigma)$. If $t \hookrightarrow^{\infty} s$, then for all $\kappa \in]0,1]$, there exists $s' \in \mathcal{T}^\omega(\Sigma)$ such that $t \to^* s'$ and $d(s', s) < \kappa$.*

Proof. Let $t \hookrightarrow^{\alpha} s$ for an arbitrary ordinal α. We proceed by transfinite induction. The finite case $\alpha < \omega$ (including the base case $\alpha = 0$) is immediate since $t \to^* s$ and $d(s, s) = 0 < \kappa$, for all $\kappa \in]0,1]$. Assume $\alpha \geq \omega$. We can write $t \hookrightarrow^{\lambda} t' \to^* s$ for some limit ordinal $\lambda \leq \alpha$. If t' is finite, by Proposition 1, $\exists \beta < \lambda$ such that $t \hookrightarrow^{\beta} t'$. Thus, we have $t \hookrightarrow^{\beta+n} s$ where n is the length of the finite derivation $t' \to^* s$. Since λ is a limit ordinal, $\beta < \lambda$, and $n \in \mathbb{N}$, we have $\beta + n < \lambda$ and by the induction hypothesis, the conclusion follows. If t' is not finite, by Proposition 3, there exist $s' \in \mathcal{T}^\omega(\Sigma)$ and $\beta < \lambda$ such that $t \hookrightarrow^{\beta} s'$ and $d(s', s) < \kappa$. Since $\beta < \alpha$, by the induction hypothesis, $t \to^* s'$ and the conclusion follows.

Corollary 1. *Let $\mathcal{R}$ be a left-linear TRS, $t \in \mathcal{T}^\omega(\Sigma)$, and s be a finite term. If $t \hookrightarrow^\infty s$, then $t \to^* s$.*

The following example shows the need for left-linearity to ensure Corollary 1.

Example 3. Consider the TRS [DKP91]:

$$\mathtt{a} \to \mathtt{g(a)} \qquad \mathtt{b} \to \mathtt{g(b)} \qquad \mathtt{f(x,x)} \to \mathtt{c}$$

Note that $\mathtt{f(a,b)} \to^\omega \mathtt{f(g^\omega,g^\omega)} \to \mathtt{c}$, but $\mathtt{f(a,b)} \not\to^* \mathtt{c}$.

4 Finitary confluence and transfinite rewriting

A TRS $\mathcal{R}$ is (finitary) confluent if $\to$ is confluent, i.e., for all $t, s, s' \in \mathcal{T}(\Sigma, \mathcal{X})$, if $t \to^* s$ and $t \to^* s'$, there exists $u \in \mathcal{T}(\Sigma, \mathcal{X})$ such that $s \to^* u$ and $s' \to^* u$. Finitary confluence of $\to$ for finite terms extends to infinite terms.

Proposition 4. *Let $\mathcal{R}$ be a confluent TRS and $t, s, s' \in \mathcal{T}^\omega(\Sigma, \mathcal{X})$. If $t \to^* s$ and $t \to^* s'$, then there exists $u \in \mathcal{T}^\omega(\Sigma, \mathcal{X})$ such that $s \to^* u \;{}^*\!\!\leftarrow s'$.*

Proof. (sketch) If t is finite, then it is obvious. If t is infinite, we consider the derivations $t = t_1 \overset{[p_1,\rho_1]}{\to} t_2 \to \cdots \to t_n \overset{[p_n,\rho_n]}{\to} t_{n+1} = s$ and $t = t'_1 \overset{[p'_1,\rho'_1]}{\to} t'_2 \to \cdots \to t'_m \overset{[p'_m,\rho'_m]}{\to} t'_{m+1} = s'$. Let $P = \frac{1}{2} + \sum_{i=1}^n |p_i| + \mathsf{d}_{\rho_i} + \sum_{i=1}^m |p'_i| + \mathsf{d}_{\rho'_i}$ and $\widehat{t}$ be the finite term obtained by replacing in t all subterms at positions $q_1, \ldots, q_k$ such that $|q_i| = P$, for $1 \le i \le k$, by new variables $x_1, \ldots, x_\ell$, $\ell \le k$. Let $\nu : \{1, \ldots, k\} \to \{1, \ldots, \ell\}$ be a surjective mapping that satisfies $\nu(i) = \nu(j) \Leftrightarrow t|_{p_i} = t|_{p_j}$ for $1 \le i \le j \le k$. Thus, each variable $x \in \{x_1, \ldots, x_\ell\}$ will name equal subterms at possibly different positions. It is not difficult to show that $\exists \widehat{s}, \widehat{s'} \in \mathcal{T}(\Sigma, \mathcal{X})$ such that $\widehat{t} \to^* \widehat{s}$, $\widehat{t} \to^* \widehat{s'}$, and $s = \sigma(\widehat{s})$, $s' = \sigma(\widehat{s'})$ for σ defined by $\sigma(x_{\nu(i)}) = t|_{q_i}$ for $1 \le i \le k$. By confluence, $\exists \widehat{u} \in \mathcal{T}(\Sigma, \mathcal{X})$ such that $\widehat{s} \to^* \widehat{u} \;{}^*\!\!\leftarrow \widehat{s'}$ and by stability of rewriting, $s = \sigma(\widehat{s}) \to^* \sigma(\widehat{u}) \;{}^*\!\!\leftarrow \sigma(\widehat{s'}) = s'$. Thus, $u = \sigma(\widehat{u})$ is the desired (possibly infinite) term.

Proposition 4 could fail for TRSs whose *rhs*'s are infinite (see [KKSV95], Counterexample 6.2). Concerning the computation of infinite constructor terms, we prove that, for left-linear, confluent TRSs, infinitary sequences suffice. First we need an auxiliary result.

Proposition 5. *Let $\mathcal{R}$ be a left-linear, confluent TRS, $t \in \mathcal{T}^\omega(\Sigma)$, and $\delta \in \mathcal{T}^\omega(\mathcal{C}) - \mathcal{T}(\mathcal{C})$. If $t \hookrightarrow^\infty \delta$, then (1) $\nexists s \in \mathsf{NF}_\mathcal{R}$ such that $t \to^* s$, (2) $\nexists \delta' \in \mathcal{T}^\omega(\mathcal{C})$, $\delta \neq \delta'$ such that $t \hookrightarrow^\infty \delta'$.*

Proof. (1) Assume that $t \to^* s \in \mathsf{NF}_\mathcal{R}$ and let $\kappa = 2^{-\mathsf{d}_s - 1}$. Since $t \hookrightarrow^\infty \delta$, by Theorem 3, there exists s' such that $t \to^* s'$ and $d(s', \delta) < \kappa$. Since $s \in \mathsf{NF}_\mathcal{R}$, by confluence, $s' \to^* s$ which is not possible as s' has a constructor prefix whose height is greater than the height of s. (2) Assume that $t \hookrightarrow^\infty \delta'$ for some $\delta' \in \mathcal{T}^\omega(\mathcal{C})$, $\delta \neq \delta'$. Let $\kappa = d(\delta, \delta')/2$. By Theorem 3, $\exists s, s'$ such that $t \to^* s$, $t \to^* s'$, $d(s, \delta) < \kappa$, and $d(s', \delta') < \kappa$. By Proposition 4, there exists u such that $s \to^* u \;{}^*\!\!\leftarrow s'$. However, this is not possible since there exists $p \in \mathcal{P}os(s) \cap \mathcal{P}os(s')$ such that $root(s|_p) = c \neq c' = root(s'|_p)$ and $c, c' \in \mathcal{C}$.

Theorem 4. *Let $\mathcal{R}$ be a left-linear, confluent TRS, $t \in \mathcal{T}^\omega(\Sigma)$, and $\delta \in \mathcal{T}^\omega(\mathcal{C})$. If $t \hookrightarrow^\infty \delta$, then $t \hookrightarrow^{\leq\omega} \delta$.*

Proof. If $\delta \in \mathcal{T}(\mathcal{C})$, it follows by Corollary 1. Let $\delta \in \mathcal{T}^\omega(\mathcal{C}) - \mathcal{T}(\mathcal{C})$. Assume that $t \not\hookrightarrow^{\leq\omega} \delta$. Then, by Proposition 5(2), there is no other infinite constructor term reachable from t. Thus, we can associate a number $M_\mathcal{C}(A) \in \mathbb{N}$ to every (open or closed) infinitary sequence A starting from t, as follows: $M_\mathcal{C}(A) = max(\{m_\mathcal{C}(s) \mid s \in A\})$ where $m_\mathcal{C}(s)$ is the minimum length of maximal sub-constructor positions of s (i.e., positions $p \in \mathcal{P}os(s)$ such that $\forall q < p, root(s|_q) \in \mathcal{C}$). Note that, since we assume that no infinitary derivation A converges to δ, $M_\mathcal{C}(A)$ is well defined. We also associate a number $n(A) \in \mathbb{N}$ to each infinite derivation A to be $n(A) = min(\{i \in \mathbb{N} \mid m_\mathcal{C}(A_i) = M_\mathcal{C}(A)\})$, i.e., from term $A_{n(A)}$ on, terms of derivation A does not increase its $m_\mathcal{C}$ number. We claim that $\mathsf{M}_\mathcal{C}(t) = \{M_\mathcal{C}(A) \mid A \text{ is an infinitary derivation starting from } t\}$ is bounded by some $\mathsf{N} \in \mathbb{N}$. In order to prove this, we use a kind of 'diagonal' argument. If $\mathsf{M}_\mathcal{C}(t)$ is not bounded, there would exist infinitely many classes $\mathcal{A}$ of infinitary derivations starting from t such that $\forall A, A' \in \mathcal{A}$, $M_\mathcal{C}(A) = M_\mathcal{C}(A')$ (that we take as $M_\mathcal{C}(\mathcal{A})$ for the class $\mathcal{A}$) that can be ordered by $\mathcal{A} < \mathcal{A}'$ iff $M_\mathcal{C}(\mathcal{A}) < M_\mathcal{C}(\mathcal{A}')$. Thus, classes of infinite derivations starting from t can be enumerated by $0, 1, \ldots$ according to their $M_\mathcal{C}(\mathcal{A})$. Without losing generality, we can assume $M_\mathcal{C}(\mathcal{A}_0) > m_\mathcal{C}(t)$. Consider an arbitrary $A \in \mathcal{A}_i$ for some $i \geq 0$. By confluence and Proposition 5(1), there exist a least $j > i$ and a derivation $A' \in \mathcal{A}_j$ such that the first $n(A)$ steps of A and A' coincide. We say that A *continues into* A'. Obviously, since $M_\mathcal{C}(\mathcal{A}_i) < M_\mathcal{C}(\mathcal{A}_j)$, and the first $n(A)$ steps of A and A' coincide, we have $n(A) < n(A')$. Once $A^0 \in \mathcal{A}_0$ has been fixed, by induction, we can define an infinite sequence $\mathsf{A} : A^0, A^1, \ldots, A^n, \ldots$ of infinitary derivations such that, for all $n \in \mathbb{N}$, A^n continues into A^{n+1}. For such a sequence A, we let $\iota : \mathbb{N} \to \mathbb{N}$ be as follows: $\iota(n) = min(\{m \mid n < n(A^m)\})$. We define an infinite derivation A starting from t as follows: $\forall \beta < \omega, A^{\iota(\beta)}_\beta \to A^{\iota(\beta)}_{\beta+1}$. It is not difficult to see that A is well defined and, by construction, $M_\mathcal{C}(A)$ is not finite thus contradicting that $t \not\hookrightarrow^{\leq\omega} \delta$. Thus, $\mathsf{M}_\mathcal{C}(t) = \{M_\mathcal{C}(A) \mid A \text{ is an infinitary derivation starting from } t\}$ is bounded by some $\mathsf{N} \in \mathbb{N}$. Let $\kappa = 2^{-\mathsf{N}-1}$. By Theorem 3, there exists s such that $t \to^* s$ and $d(s, \delta) < \kappa$. Since Proposition 5(1) ensures that every finite sequence $t \to^* s$ can be extended into an infinitary one, it follows that $A : t \to^* s \hookrightarrow^\omega \cdots$ satisfies $M_\mathcal{C}(A) > \mathsf{N}$, thus contradicting that N is an upper bound of $\mathsf{M}_\mathcal{C}(t)$.

Since $\mathcal{R}$ in Example 1 is not confluent, it shows the need for confluence in order to ensure Theorem 4. Theorem 4 does not generalize to arbitrary normal forms.

Example 4. Consider the following left-linear, confluent TRS:

```
h(f(a(x),y)) → h(f(x,b(y)))       f(a(x),b(y)) → g(f(x,y))
h(f(x,b(y))) → h(f(a(x),y))       g(f(x,y))    → f(a(x),b(y))
```

The transfinite rewrite sequence

$$\mathtt{h(f(a^\omega,y))} \hookrightarrow^\omega \mathtt{h(f(a^\omega,b^\omega))} \hookrightarrow^\omega \mathtt{h(g^\omega)}$$

cannot be compressed into an infinitary one.

With regard to Remark 1, we note that the conditions of Theorem 4 do not forbid 'de facto' non-strongly convergent transfinite sequences (as in [DKP91]).

Example 5. Consider TRS $\mathcal{R}$ of Example 2 plus the rule $\mathtt{h(x)} \to \mathtt{b(h(x))}$, and the Cauchy convergent reduction sequence of length $\omega \cdot 2$:

$$\mathtt{h(\underline{f(c)})} \to \mathtt{h(\underline{f(g(c))})} \to \mathtt{h(f(g(g(c))))} \hookrightarrow^{\omega} \mathtt{h(f(g^{\omega}))} \to^{\omega} \mathtt{b}^{\omega}$$

which is not strongly convergent. Theorem 4 ensures the existence of an (obvious) Cauchy convergent infinitary sequence leading to $\mathtt{b}^{\omega}$ (which, by Theorem 1, is strongly convergent). However, results in [DKP91] do not apply here since the TRS is not top-terminating.

A TRS $\mathcal{R} = (\Sigma, R)$ is left-bounded if the set $\{\mathsf{d}_l \mid l \to r \in R\}$ is bounded.

Theorem 5. [KKSV95] *Let $\mathcal{R}$ be a left-bounded, orthogonal TRS, $t \in \mathcal{T}^{\omega}(\Sigma)$ and $s \in \mathsf{NF}^{\omega}_{\mathcal{R}}$. If $t \hookrightarrow^{\infty} s$, then $t \to^{\infty} s$.*

By Theorem 2, we conclude that if $t \hookrightarrow^{\infty} s \in \mathsf{NF}^{\omega}_{\mathcal{R}}$, then $t \to^{\leq\omega} s$. Thus, for TRSs with finite rhs's, Theorem 4 along with Theorem 1 improves Theorem 5 with regard to normal forms of a special kind: constructor terms. In particular, note that Theorem 5 along with Theorem 2 does not apply to Example 5.

The uniqueness of infinite normal forms obtained by transfinite rewriting is a consequence of the confluence of transfinite rewriting. Even though orthogonality does *not* imply confluence of transfinite rewriting, Kennaway et al. proved that it implies uniqueness of (possibly infinite) normal forms obtained by strongly convergent reductions. We prove that finitary confluence implies the uniqueness of *constructor* normal forms obtained by infinitary rewriting.

Theorem 6. *Let $\mathcal{R}$ be a confluent TRS, $t \in \mathcal{T}^{\omega}(\Sigma)$, and $\delta, \delta' \in \mathcal{T}^{\omega}(\mathcal{C})$. If $t \hookrightarrow^{\leq\omega} \delta$ and $t \hookrightarrow^{\leq\omega} \delta'$, then $\delta = \delta'$.*

Proof. If $t \to^{*} \delta$ and $t \to^{*} \delta'$, it follows by Proposition 4. Assume that $t \hookrightarrow^{\omega} \delta$ and $\delta \neq \delta'$. In this case, δ is necessarily infinite. Let $\epsilon = d(\delta, \delta')$ and $p \in \mathcal{P}os(\delta)$ be such that $|p| = -log_2(\epsilon)$. Since $t \hookrightarrow^{\omega} \delta$, there exists $n_{\epsilon} \in \mathbb{N}$ such that, $\forall m$, $n_{\epsilon} < m < \omega$, $d(t_m, \delta) < \epsilon$. Let $m = n_{\epsilon} + 1$; thus, $t \to^{*} t_m$. If $t \to^{*} \delta'$, then, by Proposition 4, $t_m \to^{*} \delta'$. However, since $root(t_m|_p) = root(\delta|_p) \in \mathcal{C}$, and $root(\delta|_p) \neq root(\delta'|_p)$, rewriting steps issued from t_m cannot rewrite t_m into δ', thus leading to a contradiction. On the other hand, if $t = t' \hookrightarrow^{\omega} \delta'$, we consider a term $t'_{m'}$ such that $d(t'_{m'}, \delta') < \epsilon$ whose existence is ensured by the convergence of $t = t'$ into δ'. The confluence of $t'_{m'}$ and t_m into a common reduct s is also impossible for similar reasons.

Theorem 6 does not hold for arbitrary normal forms.

Example 6. Consider the (ground) TRS $\mathcal{R}$:

$$\mathtt{f(a)} \to \mathtt{a} \qquad\qquad \mathtt{f(a)} \to \mathtt{f(f(a))}$$

Since $t \to^{*} \mathtt{a}$ for every ground term t, $\mathcal{R}$ is ground confluent and hence confluent (see [BN98]). We have $\underline{\mathtt{f(a)}} \to \mathtt{f(\underline{f(a)})} \to \mathtt{f(f(f(a)))} \to^{\omega} \mathtt{f}^{\omega} \in \mathsf{NF}^{\omega}_{\mathcal{R}}$, but also $\underline{\mathtt{f(a)}} \to \mathtt{a} \in \mathsf{NF}^{\omega}_{\mathcal{R}}$.

For left-linear TRSs, Theorem 6 generalizes to transfinite rewriting.

Theorem 7. *Let $\mathcal{R}$ be a left-linear, confluent TRS, $t \in \mathcal{T}^\omega(\Sigma)$, and $\delta, \delta' \in \mathcal{T}^\omega(\mathcal{C})$. If $t \hookrightarrow^\infty \delta$ and $t \hookrightarrow^\infty \delta'$, then $\delta = \delta'$.*

5 Term and rewriting semantics

Giving semantics to a programming language (e.g., TRSs) is the first step for discussing the properties of programs. Relating semantics of (the same or different TRSs) is also essential in order to effectively analyze properties (via approximation). We provide a notion of semantics which can be used with term rewriting, and investigate two different orderings between semantics aimed at approximating semantics and (hence) at properties of programs. Our notion of semantics is aimed at couching both operational and denotational aspects in the style of *computational*, *algebraic*, or *evaluation* semantics [Bou85,Cou90,Pit97]. Since many definitions and relationships among our semantics do not depend on any computational mechanism, we consider rewriting issues later (Section 5.1).

Definition 1. *A (ground)* term semantics *for a signature Σ is a mapping* $\mathsf{S} : \mathcal{T}(\Sigma) \to \mathcal{P}(\mathcal{T}^\omega(\Sigma))$.

A trivial example of term semantics is empty given by $\forall t \in \mathcal{T}(\Sigma), \mathsf{empty}(t) = \varnothing$. We say that a semantics S is *deterministic* (resp. *defined*) if $\forall t \in \mathcal{T}(\Sigma), |\mathsf{S}(t)| \leq 1$ (resp. $|\mathsf{S}(t)| \geq 1$). Partial order $\sqsubseteq$ among semantics is the pointwise extension of partial order $\subseteq$ on sets of terms: $\mathsf{S} \sqsubseteq \mathsf{S}'$ if and only if $\forall t \in \mathcal{T}(\Sigma), \mathsf{S}(t) \subseteq \mathsf{S}'(t)$. Another partial order $\preceq$ is defined: $\mathsf{S} \preceq \mathsf{S}'$ if there exists $T \subseteq \mathcal{T}^\omega(\Sigma)$ such that, for all $t \in \mathcal{T}(\Sigma)$, $\mathsf{S}(t) = \mathsf{S}'(t) \cap T$. Given semantics S and S' such that $\mathsf{S} \preceq \mathsf{S}'$, any set $T \subseteq \mathcal{T}^\omega(\Sigma)$ satisfying $\mathsf{S}(t) = \mathsf{S}'(t) \cap T$ is called a *window set* of S' w.r.t. S. Clearly, $\mathsf{S} \preceq \mathsf{S}'$ implies $\mathsf{S} \sqsubseteq \mathsf{S}'$ (but not vice versa). Note that $\mathsf{empty} \preceq \mathsf{S}$ for all semantics S. The *range* $W_\mathsf{S} = \bigcup_{t \in \mathcal{T}(\Sigma)} \mathsf{S}(t)$ of a semantics S suffices to compare it with all the others.

Theorem 8. *Let S, S' be semantics for a signature Σ. Then, $\mathsf{S} \preceq \mathsf{S}'$ if and only if $\forall t \in \mathcal{T}(\Sigma)$, $\mathsf{S}(t) = \mathsf{S}'(t) \cap W_\mathsf{S}$.*

Proposition 6. *Let S, S' be semantics for a signature Σ. (1) If $\mathsf{S} \sqsubseteq \mathsf{S}'$, then $W_\mathsf{S} \subseteq W_{\mathsf{S}'}$. (2) If $\mathsf{S} \prec \mathsf{S}'$, then $W_\mathsf{S} \subset W_{\mathsf{S}'}$.*

Remark 2. As W_S is the union of possible outputs of semantics S, it collects the 'canonical values' we are interested in. Hence, it provides a kind of computational reference. S' is 'more powerful' than S whenever $\mathsf{S} \sqsubseteq \mathsf{S}'$. However, if $\mathsf{S} \sqsubseteq \mathsf{S}'$ but $\mathsf{S} \not\preceq \mathsf{S}'$, there will be input terms t for which S is not able to compute interesting values (according to W_S!) which, in turn, will be available by using S'. Thus, $\mathsf{S} \preceq \mathsf{S}'$ ensures that (w.r.t. semantic values in W_S) we do *not* need to use S' to compute them.

The following results that further connect $\preceq$ and $\sqsubseteq$ will be used later.

Proposition 7. *Let* $\mathsf{S}_1, \mathsf{S}'_1, \mathsf{S}_2, \mathsf{S}'_2$ *be semantics for a signature* Σ*. If* $\mathsf{S}_1 \not\preceq \mathsf{S}_2$*,* $W_{\mathsf{S}_1} \subseteq W_{\mathsf{S}_2}$*,* $\mathsf{S}_1 \preceq \mathsf{S}'_1$*, and* $\mathsf{S}_2 \preceq \mathsf{S}'_2$*, then* $\mathsf{S}'_1 \not\preceq \mathsf{S}'_2$*.*

Proposition 8. *Let* $\mathsf{S}, \mathsf{S}', \mathsf{S}''$ *be semantics for a signature* Σ*. If* $\mathsf{S} \preceq \mathsf{S}'$ *and* $\mathsf{S} \sqsubseteq \mathsf{S}'' \sqsubseteq \mathsf{S}'$*, then* $\mathsf{S} \preceq \mathsf{S}''$*.*

5.1 Rewriting semantics

Definition 1 does not indicate how to associate a set of terms $\mathsf{S}(t)$ to each term t. Rewriting can be used for setting up such an association.

Definition 2. *A (ground)* rewriting semantics *for a TRS* $\mathcal{R} = (\Sigma, R)$ *is a term semantics* S *for* Σ *such that for all* $t \in \mathcal{T}(\Sigma)$ *and* $s \in \mathsf{S}(t)$*,* $t \hookrightarrow^{\infty} s$

A *finitary* rewriting semantics S only considers reducts reached by means of a finite number of rewriting steps, i.e., $\forall t \in \mathcal{T}(\Sigma), s \in \mathsf{S}(t) \Rightarrow t \to^* s$. Given a TRS $\mathcal{R}$, semantics red, hnf, nf, and eval (where $\forall t \in \mathcal{T}(\Sigma)$, $\mathsf{red}(t) = \{s \mid t \to^* s\}$, $\mathsf{hnf}(t) = \mathsf{red}(t) \cap \mathsf{HNF}_{\mathcal{R}}$, $\mathsf{nf}(t) = \mathsf{hnf}(t) \cap \mathsf{NF}_{\mathcal{R}}$, and $\mathsf{eval}(t) = \mathsf{nf}(t) \cap \mathcal{T}(\mathcal{C})$) are the most interesting finitary rewriting semantics involving reductions to arbitrary (finitely reachable) reducts, head-normal forms, normal forms and values, respectively. In general, if no confusion arises, we will not make the underlying TRS $\mathcal{R}$ explicit in the notations for these semantics (by writing $\mathsf{red}_{\mathcal{R}}$, $\mathsf{hnf}_{\mathcal{R}}$, $\mathsf{nf}_{\mathcal{R}}$, ...). Concerning non-finitary semantics, the corresponding (Cauchy convergent) infinitary counterparts are: $\omega\mathsf{-red}$, $\omega\mathsf{-hnf}$, $\omega\mathsf{-nf}$, and $\omega\mathsf{-eval}$ using $\hookrightarrow^{\leq\omega}$, $\mathsf{HNF}^{\omega}_{\mathcal{R}}$, $\mathsf{NF}^{\omega}_{\mathcal{R}}$, and $\mathcal{T}^{\omega}(\mathcal{C})$. For a given TRS $\mathcal{R}$, we have $\mathsf{eval} \preceq \mathsf{nf} \preceq \mathsf{hnf} \preceq \mathsf{red}$ and $\omega\mathsf{-eval} \preceq \omega\mathsf{-nf} \preceq \omega\mathsf{-hnf} \preceq \omega\mathsf{-red}$ with $W_{\mathsf{red}} = \{s \in \mathcal{T}(\Sigma) \mid \exists t \in \mathcal{T}(\Sigma), t \to^* s\}$, $W_{\mathsf{hnf}} = \mathsf{HNF}_{\mathcal{R}}$, $W_{\mathsf{nf}} = \mathsf{NF}_{\mathcal{R}}$, and $W_{\mathsf{eval}} = \mathcal{T}(\mathcal{C})$. Also, $W_{\omega-\mathsf{red}} = \{s \in \mathcal{T}^{\omega}(\Sigma) \mid \exists t \in \mathcal{T}(\Sigma), t \hookrightarrow^{\leq\omega} s\}$, $W_{\omega-\mathsf{hnf}} = W_{\omega-\mathsf{red}} \cap \mathsf{HNF}^{\omega}_{\mathcal{R}}$, $W_{\omega-\mathsf{nf}} = W_{\omega-\mathsf{red}} \cap \mathsf{NF}^{\omega}_{\mathcal{R}}$, and $W_{\omega-\mathsf{eval}} = W_{\omega-\mathsf{red}} \cap \mathcal{T}^{\omega}(\mathcal{C})$. We also consider the strongly convergent versions $\omega\mathsf{-Sred}, \omega\mathsf{-Shnf}, \omega\mathsf{-Snf}$, and $\omega\mathsf{-Seval}$ using $\to^{\leq\omega}$. As a simple consequence of Theorem 1, we have:

Theorem 9. *For all TRS,* $\omega\mathsf{-eval} = \omega\mathsf{-Seval}$*.*

We consider the following (Cauchy convergent) *transfinite semantics*: $\infty\mathsf{-red}(t) = \{s \in \mathcal{T}^{\omega}(\Sigma) \mid t \hookrightarrow^{\infty} s\}$, $\infty\mathsf{-hnf}(t) = \infty\mathsf{-red}(t) \cap \mathsf{HNF}^{\omega}_{\mathcal{R}}$, $\infty\mathsf{-nf}(t) = \infty\mathsf{-hnf}(t) \cap \mathsf{NF}^{\omega}_{\mathcal{R}}$, and $\infty\mathsf{-eval}(t) = \infty\mathsf{-nf}(t) \cap \mathcal{T}^{\omega}(\mathcal{C})$ (note that these definitions implicitly say that $\infty\mathsf{-eval} \preceq \infty\mathsf{-nf} \preceq \infty\mathsf{-hnf} \preceq \infty\mathsf{-red}$). The '*strongly convergent*' versions are $\infty\mathsf{-Sred}, \infty\mathsf{-Shnf}, \infty\mathsf{-Snf}$, and $\infty\mathsf{-Seval}$ using $\to^{\infty}$ instead of $\hookrightarrow^{\infty}$.

6 Semantics, program properties, and approximation

Semantics of programming languages and partial orders between semantics can be used to express and analyze properties of programs. For instance, definedness of (term) semantics is obviously monotone w.r.t. $\sqsubseteq$ (i.e., if S is defined and $\mathsf{S} \sqsubseteq \mathsf{S}'$, then S' is defined); on the other hand, determinism is antimonotone (i.e., if S' is deterministic and $\mathsf{S} \sqsubseteq \mathsf{S}'$, then S is deterministic). Given a TRS $\mathcal{R}$, definedness of $\mathsf{nf}_\mathcal{R}$ is usually known as ($\mathcal{R}$) being *normalizing* [BN98]; definedness of $\mathsf{eval}_\mathcal{R}$ corresponds to the standard notion of 'completely defined'; definedness of $\omega{-}\mathsf{eval}_\mathcal{R}$ is 'sufficient completeness' of [DKP91]. Determinism of $\mathsf{nf}_\mathcal{R}$ is the standard *unique normal form (w.r.t. reductions)* property ($UN^\rightarrow$); determinism of $\infty{-}\mathsf{Snf}_\mathcal{R}$ is the *unique normal form (w.r.t. strongly convergent transfinite reductions)* property of [KKSV95].

A well-known property of TRSs that we can express in our framework is neededness [HL91]. In [DM97], Durand and Middeldorp define neededness (for normalization) without using the notion of residual (as in [HL91]). In order to formalize it, the authors use an augmented signature $\Sigma \cup \{\bullet\}$ (where $\bullet$ is a new constant symbol). TRSs over a signature Σ are also used to reduce terms in $\mathcal{T}(\Sigma \cup \{\bullet\})$. According to this, neededness for normalization can be expressed as follows (we omit the proof which easily follows using the definitions in [DM97]):

Theorem 10. *Let $\mathcal{R} = (\Sigma, R)$ be an orthogonal TRS. A redex $t|_p$ in $t \in \mathcal{T}(\Sigma)$ is needed for normalization if and only if $\mathsf{nf}_\mathcal{R}(t[\bullet]_p) \subseteq \mathcal{T}(\Sigma \cup \{\bullet\}) - \mathcal{T}(\Sigma)$.*

Theorem 10 suggests the following *semantic* definition of neededness.

Definition 3. *Let S be a term semantics for the signature $\Sigma \cup \{\bullet\}$. Let $t \in \mathcal{T}(\Sigma) - W_\mathsf{S}$ and $p \in \mathcal{P}os(t)$. Subterm $t|_p$ is S-needed in t if $\mathsf{S}(t[\bullet]_p) \subseteq \mathcal{T}^\omega(\Sigma \cup \{\bullet\}) - \mathcal{T}^\omega(\Sigma)$.*

We do not consider any TRS in our definition but only a term semantics; thus, we do not require that $t|_p$ be a redex but just a subterm of t. The restriction $t \in \mathcal{T}(\Sigma) - W_\mathsf{S}$ in Definition 3 is natural since terms in W_S already have complete semantic meaning according to S. Our notion of $\mathsf{nf}_\mathcal{R}$-neededness coincides with the standard notion of neededness for normalization when considering an orthogonal TRS $\mathcal{R}$ and we restrict the attention to redexes within terms. S-neededness applies in other cases.

Example 7. Consider the TRS $\mathcal{R}$:

$$\mathtt{g(x)} \to \mathtt{c(g(x))} \qquad\qquad \mathtt{a} \to \mathtt{b}$$

Redex $\mathtt{a}$ in $\mathtt{g(a)}$ is $\mathsf{nf}_\mathcal{R}$-needed, since $\mathsf{nf}_\mathcal{R}(\mathtt{g}(\bullet)) = \varnothing$. However, it is *not* $\omega{-}\mathsf{Seval}_\mathcal{R}$-needed, since $\omega{-}\mathsf{Seval}_\mathcal{R}(\mathtt{g}(\bullet)) = \{\mathtt{c}^\omega\}$.

Neededness of redexes for infinitary or transfinite normalization has been studied in [KKSV95] for strongly convergent sequences. As remarked by the authors, their definition is closely related to standard Huet and Lévy's finitary one. In fact, it is not difficult to see (considering Theorems 4 and 1), that, for every orthogonal TRS $\mathcal{R}$, $\omega{-}\mathsf{Seval}_\mathcal{R}$-needed redexes are needed in the sense of [KKSV95].

Unfortunately, S-neededness does not always 'naturally' coincide with other well-known notions of neededness such as, e.g., root-neededness [Mid97].

Example 8. Consider the TRS $\mathcal{R}$:

$$\mathtt{a} \to \mathtt{b} \qquad\qquad \mathtt{c} \to \mathtt{b} \qquad\qquad \mathtt{f(x,b)} \to \mathtt{g(x)}$$

and $t = \mathtt{f(a,c)}$ which is not root-stable (hence, $\mathtt{f(a,c)} \notin W_{\mathsf{hnf}}$). Since derivation $\mathtt{f(a,\underline{c})} \to \underline{\mathtt{f(a,b)}} \to \mathtt{g(a)}$ does not reduce redex $\mathtt{a}$ in t, $\mathtt{a}$ is not root-needed. However, $\mathtt{f(\bullet,\underline{c})} \to \underline{\mathtt{f(\bullet,b)}} \to \mathtt{g(\bullet)}$, which means that $\mathtt{a}$ is hnf-needed.

This kind of problem has already been noticed in [DM97]. A deep comparison between semantic neededness of Definition 3 and other kinds of neededness is out of the scope of this paper. However, it allows us to show the use of $\sqsubseteq$ in *approximation*. For instance, S-neededness is antimonotone with regard to $\sqsubseteq$.

Theorem 11. *Let* S, S' *be term semantics for a signature* Σ *such that* $\mathsf{S} \sqsubseteq \mathsf{S}'$. *If* $t|_p$ *is* S'*-needed in* $t \in \mathcal{T}(\Sigma) - W_{\mathsf{S}'}$, *then* $t|_p$ *is* S*-needed in* t.

Theorem 11 suggests using $\sqsubseteq$ for approximating the neededness of a TRS, either by using semantics for *other* TRSs or other semantics for the *same* TRS. Given TRSs $\mathcal{R}$ and $\mathcal{S}$ over the same signature, we say that $\mathcal{S}$ *approximates* $\mathcal{R}$ if $\to^*_{\mathcal{R}} \subseteq \to^*_{\mathcal{S}}$ and $\mathsf{NF}_{\mathcal{R}} = \mathsf{NF}_{\mathcal{S}}$ [DM97,Jac96]. An approximation of TRSs is a mapping α from TRSs to TRSs with the property that $\alpha(\mathcal{R})$ approximates $\mathcal{R}$ [DM97]. By using approximations of TRSs we can *decide* a property of $\alpha(\mathcal{R})$ (e.g., neededness) which is valid (but often undecidable) for the 'concrete' TRS $\mathcal{R}$. In [DM97], four approximations, namely $\mathsf{s}, \mathsf{nv}, \mathsf{sh}$, and g, have been studied[3] and neededness proved decidable w.r.t. these approximations. Since $\mathsf{nf}_{\mathcal{R}} \sqsubseteq \mathsf{nf}_{\alpha(\mathcal{R})}$, by Theorem 11 $\mathsf{nf}_{\alpha(\mathcal{R})}$ correctly approximates $\mathsf{nf}_{\mathcal{R}}$-neededness (as proved in [DM97]).

A final example is *redundancy* of arguments. Given a term semantics S for the signature Σ, the i-th argument of $f \in \Sigma$ is redundant w.r.t. S if, for all context $C[\,]$, $t \in \mathcal{T}(\Sigma)$ such that $root(t) = f$, and $s \in \mathcal{T}(\Sigma)$, $\mathsf{S}(C[t]) = \mathsf{S}(C[t[s]_i])$. Redundancy is antimonotone with regard to $\preceq$ but *not* w.r.t. $\sqsubseteq$ [AEL00].

7 Relating transfinite, infinitary, and finitary semantics

Following our previous discussion about the usefulness of relating different semantics, in this section, we investigate orders $\sqsubseteq$ and $\preceq$ among semantics of Section 5.1. Strongly convergent sequences are Cauchy convergent; thus, $\infty{-}\mathsf{S}\varphi \sqsubseteq \infty{-}\varphi$ for $\varphi \in \{\mathsf{red}, \mathsf{hnf}, \mathsf{nf}, \mathsf{eval}\}$. In general, these inequalities are strict: consider $\mathcal{R}$ in Example 1. We have $\infty{-}\mathsf{Seval} \not\preceq \infty{-}\mathsf{eval}$; otherwise, $\infty{-}\mathsf{Seval}(\mathtt{f(a,g)}) = \infty{-}\mathsf{eval}(\mathtt{f(a,g)}) \cap W_{\infty-\mathsf{Seval}}$. Since $\mathsf{c}^\omega \in \infty{-}\mathsf{eval}(\mathtt{f(a,g)})$ but we have that $\mathsf{c}^\omega \notin \infty{-}\mathsf{Seval}(\mathtt{f(a,g)})$, it follows that $\mathsf{c}^\omega \notin W_{\infty-\mathsf{Seval}}$. However, $\mathsf{c}^\omega \in \infty{-}\mathsf{Seval}(\mathtt{h}) = \infty{-}\mathsf{eval}(\mathtt{h}) \cap W_{\infty-\mathsf{Seval}}$ contradicts the latter. Thus, in general, $\infty{-}\mathsf{Seval} \not\preceq \infty{-}\mathsf{eval}$, i.e., $\infty{-}\mathsf{Seval} \sqsubset \infty{-}\mathsf{eval}$. By Proposition 6, $W_{\infty-\mathsf{Seval}} \subseteq W_{\infty-\mathsf{eval}}$; thus, by Proposition 7 we also have $\infty{-}\mathsf{S}\varphi \not\preceq \infty{-}\varphi$, and hence $\infty{-}\mathsf{S}\varphi \sqsubset \infty{-}\varphi$ for $\varphi \in \{\mathsf{red}, \mathsf{hnf}, \mathsf{nf}, \mathsf{eval}\}$. By Theorem 5, for left-bounded, orthogonal TRSs, $\infty{-}\mathsf{nf} = \infty{-}\mathsf{Snf}$ and $\infty{-}\mathsf{eval} = \infty{-}\mathsf{Seval}$.

With regard to *infinitary semantics*, the situation is similar to the transfinite case (but consider Theorem 9). We also have the following:

[3] Names $\mathsf{s}, \mathsf{nv}, \mathsf{sh}$, and g correspond to *strong*, *NV*, *shallow*, and *growing* approximations, respectively.

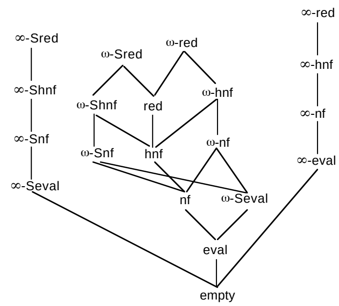

Fig. 1. Semantics for a TRS ordered by $\preceq$

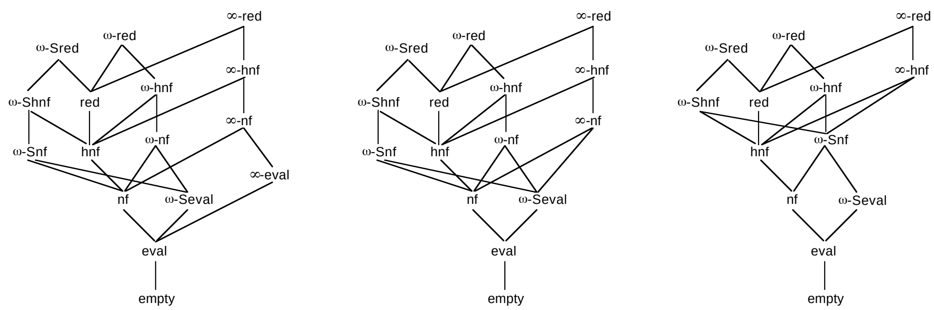

Fig. 2. Semantics for a left-linear / left-linear confluent / left-bounded orthogonal TRS (ordered by $\preceq$)

Proposition 9. *For all TRS, $\varphi \preceq \omega{-}\varphi$ for $\varphi \in \{\mathsf{red}, \mathsf{hnf}, \mathsf{nf}, \mathsf{eval}\}$.*

By Propositions 9 and 8, $\varphi \preceq \omega{-}\mathsf{S}\varphi$ for $\varphi \in \{\mathsf{red}, \mathsf{hnf}, \mathsf{nf}, \mathsf{eval}\}$. However (consider the TRS in Example 3), in general $\mathsf{eval} \not\preceq \infty{-}\mathsf{Seval}$ (and $\mathsf{eval} \not\preceq \infty{-}\mathsf{eval}$)! By Proposition 7, no comparison (with $\preceq$) is possible between (considered here) transfinite semantics and infinitary or finitary ones (except empty). Figure 1 shows the hierarchy of semantics for a TRS ordered by $\preceq$.

By Theorem 2, for *left-linear* TRSs, we have $\infty{-}\mathsf{S}\varphi = \omega{-}\mathsf{S}\varphi$ for $\varphi \in \{\mathsf{red}, \mathsf{hnf}, \mathsf{nf}, \mathsf{eval}\}$. Example 1 shows that $\omega{-}\mathsf{eval} \not\preceq \infty{-}\mathsf{eval}$ and, since $\omega{-}\mathsf{eval} \sqsubseteq \infty{-}\mathsf{eval}$, by Propositions 6 and 7, $\omega{-}\varphi \not\preceq \infty{-}\varphi$ for $\varphi \in \{\mathsf{red}, \mathsf{hnf}, \mathsf{nf}, \mathsf{eval}\}$. As a simple consequence of Corollary 1, for every left-linear TRS, $\varphi \preceq \infty{-}\varphi$ for $\varphi \in \{\mathsf{red}, \mathsf{hnf}, \mathsf{nf}, \mathsf{eval}\}$. By additionally requiring (finitary) *confluence*, Theorem 4 entails that, for every left-linear, confluent TRS, $\infty{-}\mathsf{eval} = \omega{-}\mathsf{eval}$. By using Theorem 2, we have that, for every *left-bounded, orthogonal* TRS, $\infty{-}\mathsf{nf} = \omega{-}\mathsf{Snf}$. Diagrams of Figure 2 summarize these facts. Table 1 shows the appropriateness of different semantics for computing different kinds of interesting semantic values (see Remark 2). Left-linear, confluent TRSs provide the best framework for computing constructor terms, since infinitary, strongly convergent sequences suffice and determinism of computations is guaranteed.

	$\mathcal{T}(\mathcal{C})$	$\mathsf{NF}_{\mathcal{R}}$	$\mathcal{T}^{\omega}(\mathcal{C})$	$\mathsf{NF}^{\omega}_{\mathcal{R}}$
arbitrary TRSs	$\infty-$eval	$\infty-$nf	$\infty-$eval	$\infty-$nf
left-linear TRSs	eval	nf	$\infty-$eval	$\infty-$nf
left-linear, confluent TRSs	eval (!)	nf (!)	$\omega-$Seval (!)	$\infty-$nf
left-bounded, orthogonal TRSs	eval (!)	nf (!)	$\omega-$Seval (!)	$\omega-$Snf (!)

Table 1. Semantics for computing different canonical forms; (!) means determinism

8 Related work

Our rewriting semantics are related to other (algebraic) approaches to semantics of recursive program schemes [Cou90] and TRSs [Bou85]. For instance, a *computational semantics*, $Comp_{\langle\mathcal{R},\mathcal{A}\rangle}(t)$, of a ground term t in a TRS $\mathcal{R} = (\Sigma, R)$ is given in [Bou85] as the collection of lub's of increasing partial information obtained along maximal computations starting from t (see [Bou85], page 212). The partial information associated to each component of a rewrite sequence is obtained by using an interpretation $\mathcal{A}$ of $\mathcal{R}$ which is a complete Σ-algebra[4] satisfying $l_{\mathcal{A}} = \bot$ for all rules $l \to r \in R$. Consider $\mathcal{R}$ in Example 6 and let $\mathcal{A}$ be a Σ-algebra interpreting $\mathcal{R}$. Since $\mathtt{f(a)} \to \mathtt{a}$ is a rule of $\mathcal{R}$, then $f(a) = \bot$, where f and a interpret $\mathtt{f}$ and $\mathtt{a}$, respectively. Since $\bot \sqsubseteq a$ and f is monotone, we have $f(f(a)) = f(\bot) \sqsubseteq f(a) = \bot$, i.e., $f(f(a)) = \bot$. In general, $f^n(a) = \bot$, for $n \geq 1$, and thus $Comp_{\langle\mathcal{R},\mathcal{A}\rangle}(\mathtt{f(a)}) = \{\bot, a\}$. However, $\omega-\mathsf{Snf}(\mathtt{f(a)}) = \{\mathtt{a}, \mathtt{f}^{\omega}\}$. Moreover, since $\mathcal{R}$ in Example 6 is confluent, according to [Bou85], we can provide another semantics $Val_{\langle\mathcal{R},\mathcal{A}\rangle}(t)$ which is the lub of the interpretations of all finite reducts of t. In particular, $Val_{\langle\mathcal{R},\mathcal{A}\rangle}(\mathtt{f(a)}) = \bigsqcup\{\bot, a\} = a$, which actually corresponds to $\mathsf{nf}(\mathtt{f(a)})$, since there is no reference to the infinitary term $\mathtt{f}^{\omega}$ obtained from $\mathtt{f(a)}$ by strongly convergent infinitary rewriting.

Closer to ours, in [HL01,Luc00], an *observable semantics* is given to (computations of) terms without referring its meaning to any external semantic domain. An observation mapping is a lower closure operator on Ω-terms ordered by a partial order $\sqsubseteq$ on them. An *adequate* observation mapping $(\!|\ |\!)$ permits us to describe computations which are issued from a term t as the set of all observations $(\!|s|\!)$ of finitary reducts s of t. An observation mapping $(\!|\ |\!)$ is *adequate* for observing rewriting computations if the observations of terms according to $(\!|\ |\!)$ are refined as long as the computation proceeds. For instance, the observation mapping $(\!|\ |\!)_{\omega}$ that yields the normal form of a term w.r.t. Huet and Lévy's Ω-reductions [HL91] is adequate for observing rewriting computations [Luc00]. However, using $(\!|\ |\!)_{\omega}$ to observe computations issued from $\mathtt{f(a)}$ in the TRS of Example 6 does not provide any information about $\mathtt{f}^{\omega}$, since only the set $\{\Omega, \mathtt{a}\}$ is obtained.

In [DKP91], continuous Σ-algebras (based on quasi-orders) are used to provide models for ω-canonical TRSs, i.e., ω-confluent and ω-normalizing (that is,

[4] By a complete Σ-algebra we mean $\mathcal{A} = (D, \sqsubseteq, \bot, \{f_{\mathcal{A}} \mid f \in \Sigma\})$ where $(D, \sqsubseteq, \bot)$ is a *cpo* with least element $\bot$ and each $f_{\mathcal{A}}$ is continuous (hence monotone) [Bou85].

every term has a normal form[5] which is reachable in at most ω-steps) TRSs. Models are intended to provide *inequality* of left- and right-hand sides rather than equality, as usual. Also, in contrast to [Bou85,HL01,Luc00], semantics is given to symbols (via interpretations) rather than terms (via computations). This semantic framework does not apply to $\mathcal{R}$ in Example 6 since it is not ω-canonical (it lacks ω-confluence).

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[5] Notice that the notion of normal form of [DKP91] (a term t such that $t = t'$ whenever $t \to t'$) differs from the standard one.