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200

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Abstract

We first present the arithmetic of Ore polynomials, then describe how they can be viewed as different types of linear operators, and finally give factorisation algorithms for linear differential and difference operators, emphasizing the similarities between the two.

Introduction

Linear ordinary differential equations are equations of the form

$$A_n(t)\frac{d^n y(t)}{dt^n} + \ldots + A_1(t)\frac{dy(t)}{dt} + A_0(t)y(t) = B(t)$$

and linear ordinary difference equations are equations of the form

$$A_n(t)y(t+n) + \ldots + A_1(t)y(t+1) + A_0(t)y(t) = B(t)$$

where in both cases the unknown y and the coefficients are functions of the (continuous or discrete) variable t. Those two types of equations are mathematically linked, the various algorithms for solving or otherwise manipulating them have some interesting similarities [1], and on occasion methods devised for one type of equation can be used on the other type [16]. A comparison of the algebraic properties of those equations points to the existence of some common mathematical abstraction behind them. It turns out that Ore polynomial rings [14] are an appropriate mathematical setting in which to manipulate the above types of linear equations, as well as other similar ones like q-difference equations. While Ore polynomials are classical objects in non–commutative ring theory [5], they are not supported by the standard libraries that come with the usual commercial computer algebra systems (although there are implementations of particular types of Ore polynomials, namely linear differential and difference operators), and are not described in computer algebra textbooks.

In the first section, we introduce Ore polynomials in a constructive way, by giving detailed algorithms for their basic arithmetic operations up to left and right greatest common divisors and least common multiples. We then show how those polynomials can be viewed as linear operators, and give some results which are valid for arbitrary Ore polynomials. We finally outline factoring algorithms for linear differential and difference operators, emphasizing the similarities (and differences) between the two. The question of generalizing the factoring algorithm to more general Ore rings remains open.

1 Rings of Ore polynomials

We begin with the formal definition of a left Ore ring, which is essentially a nontrivial ring without zero divisors in which any two nonzero elements have a nonzero left common multiple.

Definition 1 ([5], §0.5) *A ring R is a* left Ore ring *if $0 \neq 1$ in R, and for any $a, b \in R \setminus \{0\}$, $ab \neq 0$ and $Ra \cap Rb \neq \{0\}$.*

Right Ore rings are defined in a similar way. We describe now the algebra of Ore polynomials, mostly following chapter I of [14]. As expected, rings of Ore polynomials are left Ore rings, as will be established later in this section.

Let k be a field, x be an indeterminate over k, σ be a nonzero field–endomorphism of k, and $\delta : k \to k$ be a map satisfying

$$\delta(a+b) = \delta a + \delta b \quad \text{and} \quad \delta(ab) = \sigma(a)\, \delta b + \delta a\, b \qquad \text{for any } a, b \in k. \tag{1}$$

The *left skew polynomial ring* given by σ and δ is the ring $(k[x], +, \cdot)$ of polynomials in x over k with the usual polynomial addition, and the multiplication given by

$$xa = \sigma(a)x + \delta a \qquad \text{for any } a \in k.$$

This multiplication is extended uniquely to monomials by

$$(ax^n)(bx^m) = (ax^{n-1})(xb)x^m = (ax^{n-1})(\sigma(b)x^{m+1} + \delta b\, x^m) \text{ for } n > 0 \tag{2}$$

and to arbitrary polynomials by distributivity:

$$\left(\sum_i a_i x^i\right)\left(\sum_j b_j x^j\right) = \sum_i \sum_j (a_i x^i)(b_j x^j).$$

To avoid confusing it with the usual commutative polynomial ring $k[x]$, the left skew polynomial ring is denoted $k[x;\sigma,\delta]$, and we call its elements the *Ore polynomials.*

Definition 2 *The* constant subfield of k *(with respect* σ *and* δ*) is*

$$\mathrm{Const}_{\sigma,\delta}(k) = \{a \in k \textit{ such that } \sigma(a) = a \textit{ and } \delta a = 0\}.$$

It is easily checked that $\mathrm{Const}_{\sigma,\delta}(k)$ is a subfield of k.

Let $A, B \in k[x;\sigma,\delta] \setminus \{0\}$, ax^n and bx^m be the leading monomials of A and B respectively, where $a \neq 0$, $b \neq 0$, and $n, m \geq 0$. Then, by (2), the leading monomial of AB is $a\sigma^n(b)x^{n+m}$. Since σ is injective, $a\sigma^n(b) \neq 0$, so

$$\deg(AB) = \deg(A) + \deg(B) \geq \max(\deg(A), \deg(B)).$$

With the convention that $\deg(0) = -\infty$, the above equality holds for any A and B. In particular, it implies that $k[x;\sigma,\delta]$ has no zero divisors. It also means that the degree function satisfies the inequality of a Euclidean norm, and there is indeed a right Euclidean division algorithm in $k[x;\sigma,\delta]$: with the notation as above, suppose that $n \geq m$ and let

$$Q_0 = \frac{a}{\sigma^{n-m}(b)} x^{n-m}.$$

The leading monomial of Q_0B is then ax^n, so $d = \deg(A - Q_0B) < n$. If $d < m$ then letting $R = A - Q_0B$ and $Q = Q_0$, we have $A = BQ + R$ where $\deg(R) < \deg(B)$. Otherwise, we divide $A - Q_0B$ by B by induction, obtaining $Q_1, R \in k[x;\sigma,\delta]$ such that $A - Q_0B = Q_1B + R$ and $\deg(R) < m$. We then have $A = QB + R$, where $Q = Q_0 + Q_1$ and $\deg(R) < \deg(B)$. R is called the *right-remainder* of A by B and is denoted $R = \mathrm{rrem}(A, B)$, while Q is called the *right-quotient* of A by B and is denoted by $Q = \mathrm{rquo}(A, B)$.

If σ is an automorphism of k, then there is also a left Euclidean algorithm: let

$$Q_0 = \sigma^{-m}\left(\frac{a}{b}\right) x^{n-m}$$

then the leading monomial of BQ_0 is ax^n, which implies that $d = \deg(A - BQ_0) < n$. If $d < m$ then letting $R = A - BQ_0$ and $Q = Q_0$, we have $A = QB + R$ where $\deg(R) < \deg(B)$. Otherwise, we divide $A - BQ_0$ by B by induction, obtaining $Q_1, R \in k[x; \sigma, \delta]$ such that $A - BQ_0 = BQ_1 + R$ and $\deg(R) < m$. We then have $A = BQ + R$, where $Q = Q_0 + Q_1$ and $\deg(R) < \deg(B)$. R is called the *left-remainder* of A by B and is denoted $R = \mathrm{lrem}(A, B)$, while Q is called the *left-quotient* of A by B and is denoted by $Q = \mathrm{lquo}(A, B)$.

One can then compute the *right (resp. left) Euclidean remainder sequence* given by $R_0 = A, R_1 = B$ and $R_i = \mathrm{rrem}(R_{i-2}, R_{i-1})$ (resp. $\mathrm{lrem}(R_{i-2}, R_{i-1})$) for $i \geq 2$, and the *greatest common right (resp. left) divisor* of A and B which is the last nonzero element of that sequence.

In order to show that $k[x; \sigma, \delta]$ is a left Ore ring, we need to establish that any two $A, B \in k[x; \sigma, \delta] \setminus \{0\}$ have a nonzero common left multiple. Consider the right extended Euclidean algorithm with A and B:

$R_0 = A,\ R_1 = B$
$A_0 = 1,\ A_1 = 0$
$B_0 = 0,\ B_1 = 1$
$i = 1$
while $R_i \neq 0$ do
    increment i
    $Q_{i-1} = \mathrm{rquo}(R_{i-2}, R_{i-1})$
    $R_i = \mathrm{rrem}(R_{i-2}, R_{i-1})$
    $A_i = A_{i-2} - Q_{i-1}A_{i-1}$
    $B_i = B_{i-2} - Q_{i-1}B_{i-1}$
$n = i.$

It is easy to see by induction on i that

$$R_i = A_i A + B_i B \tag{3}$$

and, running induction backwards, that R_{n-1} right-divides R_i for $n \geq i \geq 0$. It follows that

$$R_{n-1} = A_{n-1}A + B_{n-1}B = \mathrm{gcrd}(A, B)\,.$$

Since $R_n = 0$ (the terminating condition of the *while* loop), we have

$$A_n A + B_n B = 0$$

hence $A_n A = -B_n B$ is a common left multiple of A and B. From the above algorithm we have

$$\deg(R_i) < \deg(R_{i-1}) \quad \text{and} \quad \deg(Q_{i-1}) = \deg(R_{i-2}) - \deg(R_{i-1}) \quad \text{for} \quad 2 \leq i \leq n,$$

hence by induction on i we see that

$$\deg(A_i) = \deg(B) - \deg(R_{i-1}) \quad \text{and} \quad \deg(B_i) = \deg(A) - \deg(R_{i-1})$$

for $2 \leq i \leq n$. It follows that $\deg(A_n) = \deg(B) - \deg(R_{n-1})$ and $\deg(B_n) = \deg(A) - \deg(R_{n-1})$, so $A_n \neq 0$ and $B_n \neq 0$. Hence $A_n A = -B_n B$ is a nonzero common left multiple of A and B. In fact, it is a *least* such multiple. To see this, assume that $CA = -DB$ is some common left multiple of A and B. Now let (cf. [4] and [11, Ex. 4.6.1.18])

$$\begin{array}{l} C_0 = -D \\ C_1 = C \\ \text{for } i = 2, 3, \ldots, n \text{ do} \\ \qquad C_i = C_{i-2} - C_{i-1} Q_{i-1}. \end{array}$$

An easy induction on i shows that

$$\begin{aligned} C_{i-1} R_i - C_i R_{i-1} &= 0 \\ C_{i-1} A_i - C_i A_{i-1} &= (-1)^i C \\ C_{i-1} B_i - C_i B_{i-1} &= (-1)^i D \end{aligned}$$

for $1 \leq i \leq n$. It follows that $C_n R_{n-1} = C_{n-1} R_n = 0$, hence that $C_n = 0$. Therefore A_n right-divides C, and B_n right-divides D. Thus

$$A_n A = -B_n B = \operatorname{lclm}(A, B).$$

When σ is an automorphism of k, the left extended Euclidean algorithm gives us the greatest common left divisor and least common right multiple of A and B in a completely analogous way, which proves the following result.

Theorem 1 *$k[x; \sigma, \delta]$ is a left Ore ring. If σ is an automorphism of k, then it is also a right Ore ring.*

Note that there are examples of left skew polynomial rings which are not right Ore rings, and that $k[x; \sigma, \delta]$ is a right Ore ring if and only if σ is an automorphism [5, Ex. 0.8.2].

2 Ore polynomials as linear operators

We describe in this section how arbitrary Ore polynomials can act on vector spaces over k, and thus be viewed as k-pseudo linear operators, or as $\mathrm{Const}_{\sigma,\delta}(k)$-linear operators. Throughout this section, let k, σ, δ and $k[x; \sigma, \delta]$ be as in the previous section.

Definition 3 ([10]) *Let V be a vector space over k. A map $\theta : V \longrightarrow V$ is called* k-pseudo linear (with respect to σ and δ) *if*

$$\theta(u + v) = \theta u + \theta v \quad \textit{and} \quad \theta(au) = \sigma(a)\, \theta u + \delta a\, u \quad \textit{for any } u, v \in V \textit{ and } a \in k. \tag{4}$$

Lemma 1 *Any k-pseudo linear map is $\mathrm{Const}_{\sigma,\delta}(k)$-linear.*

Proof. Let $\theta : V \to V$ be k-pseudo linear, $c \in \mathrm{Const}_{\sigma,\delta}(k)$ and $u, v \in V$. Then,

$$\theta(cu+v) = \theta(cu) + \theta v = (\sigma(c)\,\theta u + \delta c\, u) + \theta v = c\,\theta u + \theta v\,.$$

□

Given a vector space V over k and a k-pseudo linear map $\theta : V \to V$, we can make $k[x;\sigma,\delta]$ act k-pseudo linearly, and linearly with respect to $\mathrm{Const}_{\sigma,\delta}(k)$, on V via the action $*_\theta : k[x;\sigma,\delta] \times V \to V$ given by

$$\left(\sum_{i=0}^{n} a_i x^i\right) *_\theta u = \sum_{i=0}^{n} a_i \theta^i u \qquad \text{for any } u \in V,$$

and thus can view the elements of $k[x;\sigma,\delta]$ as operators acting on V. When there is no ambiguity from the context, we write $*$ instead of $*_\theta$. It turns out that the multiplication in $k[x;\sigma,\delta]$ is in fact operator composition.

Theorem 2 *$(AB) * u = A * (B * u)$ for any $A, B \in k[x;\sigma,\delta]$ and $u \in V$.*

Proof. We first prove by induction on n that

$$(ax^n bx^m) * u = ax^n * (bx^m * u) \tag{5}$$

for any $n, m \geq 0$, $a, b \in k$, and $u \in V$. If $n = 0$, then

$$(ax^0 bx^m) * u = (abx^m) * u = ab\,\theta^m u = a * (b\,\theta^m u) = a * (bx^m * u).$$

Suppose now that $n > 0$ and that (5) holds for $n-1$. Then, using (2) and (4),

$$\begin{aligned}
(ax^n bx^m) * u &= \left((ax^{n-1})\,(\sigma(b)\,x^{m+1} + \delta b\,x^m)\right) * u \\
&= (ax^{n-1}\sigma(b)\,x^{m+1}) * u + (ax^{n-1}\delta b\,x^m) * u \\
&= ax^{n-1} * (\sigma(b)\,x^{m+1} * u) + ax^{n-1} * (\delta b\,x^m * u) \\
&= ax^{n-1} * \left(\sigma(b)\,\theta^{m+1} u + \delta b\,\theta^m u\right) \\
&= ax^{n-1} * \theta(b\,\theta^m u) = ax^{n-1} * (x * (bx^m * u)) \\
&= ax^n * (bx^m * u).
\end{aligned}$$

Writing now $A = \sum_i a_i x^i$ and $B = \sum_j b_j x^j$, we have

$$\begin{aligned}
(AB) * u &= \left(\sum_i \sum_j (a_i x^i)(b_j x^j)\right) * u = \sum_i \sum_j \left((a_i x^i b_j x^j) * u\right) \\
&= \sum_i \sum_j \left(a_i x^i * (b_j x^j * u)\right) = \sum_i \left(a_i x^i * \sum_j (b_j x^j * u)\right) = A * (B * u).
\end{aligned}$$

□

The previous discussion described how to make Ore polynomials act on a vector space given a pseudo–linear transformation. We show now that such a transformation can always be given on k or any field extension of k to which σ and δ can be extended.

Definition 4 *We say that a field extension K of k is* compatible with k *if σ can be extended to a field homomorphism from K to K, and δ can be extended to a map from K to K in such a way that (1) remains satisfied.*

Note that k itself is compatible with k, and that if K is a compatible field extension of k, then $\mathrm{Const}_{\sigma,\delta}(k) \subseteq \mathrm{Const}_{\sigma,\delta}(K)$. For the rest of this section, let K be a compatible field extension of k, and $C = \mathrm{Const}_{\sigma,\delta}(K)$.

Lemma 2 *For any $c \in K$, the map $\theta_c : K \to K$ given by*

$$\theta_c a = c\,\sigma(a) + \delta a \tag{6}$$

is C-linear and K-pseudo linear.

Proof. Let $a, b \in K$. Using (1) and that σ is a field homomorphism, we get

$$\theta_c(a+b) = c\,\sigma(a+b) + \delta(a+b) = c\,(\sigma(a) + \sigma(b)) + \delta a + \delta b = \theta_c a + \theta_c b$$

and

$$\theta_c(ab) = c\,\sigma(ab) + \delta(ab) = c\,\sigma(a)\,\sigma(b) + (\sigma(a)\,\delta b + \delta a\ b) = \sigma(a)\,\theta_c b + \delta a\ b$$

which proves that θ_c is K-pseudo linear, hence C-linear by Lemma 1. □

Thus we can view the elements of $k[x;\sigma,\delta]$ as operators acting on K. Furthermore, since the action is C-linear, the solutions in K of $A *_{\theta_c} \alpha = 0$ form a vector space over C, and in particular $A *_{\theta_c} 0 = 0$ for any $A \in k[x;\sigma,\delta]$.

Note that when k is viewed as a subring of $k[x;\sigma,\delta]$, then the product of A in $k[x;\sigma,\delta]$ by $a \in k$ is *not* the same as the action of A on a. For example, with $A = x$ and $a \in k^*$, we have

$$xa = \sigma(a)x + \delta a \in k[x;\sigma,\delta]$$

which has degree 1 since σ is injective, while $x * a \in k$, so the two are different. Thus, we keep throughout this paper the $*$ notation to denote operator application, and use juxtaposition for the product of Ore polynomials.

3 Factors of degree 1

We now connect the existence of right hand factors of degree 1 to some special type of solution of the operator. Those solutions correspond to solutions with rational logarithmic derivatives or hypergeometric solutions in the classical theories of linear differential and difference equations. For a given $c \in k$, we use the notation θ_c to indicate the map given by (6) and $*_c$ for the action $*_{\theta_c}$ on any compatible field extension of k.

Definition 5 *We say that $A \in k[x;\sigma,\delta]$* has an hyperexponential solution *(over k) if there exists $c \in k$, a compatible field extension K of k, and $\alpha \in K^*$ such that θ_c is not identically 0 on K, $\theta_c\alpha/\alpha \in k$ and $A *_c \alpha = 0$.*

We say that α is a c-hyperexponential solution of A to indicate a particular coefficient c for which $\theta_c \neq 0$, $\theta_c\alpha/\alpha \in k$ and $A *_c \alpha = 0$.

Theorem 3 *If α is a c-hyperexponential solution of $A \in k[x;\sigma,\delta]$, then $x - \theta_c\alpha/\alpha$ is a right hand factor of A in $k[x;\sigma,\delta]$.*

Proof. Let α be a c-hyperexponential solution of A, and $u = \theta_c\alpha/\alpha \in k$. By the right Euclidean division of A by $x - u$, we get $A = Q(x-u) + R$ where $Q, R \in k[x;\sigma,\delta]$ and $\deg(R) < 1$, hence $R \in k$, so $R *_c \alpha = R\alpha$. We then have

$$0 = A *_c \alpha = Q(x-u) *_c \alpha + R *_c \alpha = Q *_c (\theta_c\alpha - u\alpha) + R\alpha = Q *_c 0 + R\alpha = R\alpha$$

which implies that $R = 0$, hence that $x - u$ is a right hand factor of A. □

Thus, if we can find an hyperexponential solution of A, then it must have a right hand factor of degree 1, something which is independent of the choice of the coefficient c chosen for the action of $k[x;\sigma,\delta]$.

A converse to Theorem 3 would allow one to prove that an Ore polynomial does not have a right hand factor of degree 1. We need an extra hypothesis on our rings for that, namely that we can construct solutions of polynomials of degree 1.

Definition 6 *Let $c \in k$. We say that $k[x;\sigma,\delta]$ is* c-solvable *if for any $a, b \in k^*$, there exists a compatible field extension K of k and $\alpha \in K^*$ such that $(ax+b) *_c \alpha = 0$.*

Theorem 4 *Let $c \in k$. If $k[x;\sigma,\delta]$ is c-solvable and $A \in k[x;\sigma,\delta]$ has a degree 1 right hand factor, then either x is a right hand factor of A, or A has a c-hyperexponential solution.*

Proof. Suppose that $k[x;\sigma,\delta]$ is c-solvable for some $c \in k$, and that $A = Q(ax+b)$ for some $a, b \in k$ with $a \neq 0$. If $b \neq 0$, then there exists a compatible field extension K of k, and $\alpha \in K^*$ such that $(ax+b) *_c \alpha = 0$. We then have $a\theta_c\alpha + b\alpha = 0$, hence $\theta_c\alpha = (-b/a)\,\alpha \neq 0$, which implies that θ_c is not identically 0 on K and that α is c-hyperexponential over k. Otherwise $b = 0$, so x is a right hand factor of A. □

Thus, if x is not a right hand factor of A and we can produce $c \in k$ such that $k[x;\sigma,\delta]$ is c-solvable and A has no c-hyperexponential solution, then A has no right hand factor of degree 1 in $k[x;\sigma,\delta]$.

4 Linear ordinary differential operators

Let k be a differential field of characteristic 0 with derivation D, σ be the identity on k and δ be D. Then $k[x;\sigma,\delta]$ is isomorphic as a left skew polynomial ring to the usual linear differential operator ring $k[D]$ via the isomorphism which takes x to D. Any differential extension K of k is a compatible field extension of k with σ being the identity on K and δ being D, so $\mathrm{Const}_{\sigma,\delta}(K) = \mathrm{Const}_D(K)$. In addition, for $c \in k$, the corresponding action of $k[x;\sigma,\delta]$ on K is given by $\theta_c y = c\,y + D*y$, so $y \in K^*$ is hyperexponential over k if and only if $(cy + D*y)/y \in k$ for some $c \in k$ for which θ_c is not identically 0, which is equivalent to $(D*y)/y \in k$. If we fix the coefficient c to be 0 for the action of $k[x;\sigma,\delta]$ on K, then the action $*_{\theta_0}$ is the same as the action of $k[D]$, namely $\theta_0 y = D*y$, so we write $*$ for $*_{\theta_0}$ in the rest of this section.

Let t be transcendental over k. Given $a, b \in k^*$, there is a unique extension of D to $k(t)$ such that $(k(t), D)$ is a differential extension of (k, D) and $D*t = -(b/a)t$ [13, ch. X, §7]. We then have

$$(ax+b)*t = a\theta_0 t + bt = a\,(D*t) + bt = 0$$

which shows that $k[x;\sigma,\delta]$ is 0-solvable. If D is a right hand factor of $L \in k[D]$, then 1 is a solution of $L * y = 0$ whose logarithmic derivative is in k, hence Theorems 3 and 4 imply that L in $k[D]$ has a right hand factor of degree 1 if and only if the differential equation $L * y = 0$ has a solution whose logarithmic derivative is in k. In the case where k is a Liouvillian extension of $\mathrm{Const}_D(k)$, there is an algorithm for finding the solutions whose logarithmic derivative is in k of a linear ordinary differential equation with coefficients in k [19], so we can decide whether elements of $k[x;\sigma,\delta]$ (and $k[D]$) have right hand factors of degree 1 and compute such a factor if they exist. With the same algorithm we can also decide existence of *left hand* factors of degree 1 (and compute them) by applying it to the adjoint operator, which is defined by

$$\left(\sum_{i=0}^{n} a_i D^i\right)^* = \sum_{i=0}^{n} (-1)^i D^i a_i .$$

It can be shown that for any $L_1, L_2 \in k[D]$,

$$\begin{aligned} (L_1^*)^* &= L_1 \\ \deg(L_1^*) &= \deg(L_1) \\ (L_1 + L_2)^* &= L_1^* + L_2^* \\ (L_1 L_2)^* &= L_2^* L_1^* . \end{aligned}$$

Hence left hand factors of L correspond to right hand factors of L^*, and vice versa.

It is well known that factorisation in $k[D]$ is not unique. For example [12, 18],

$$D^2 - \frac{2}{t}D + \frac{2}{t^2} = \left(D - \frac{1}{t(1+at)}\right)\left(D - \frac{1+2at}{t(1+at)}\right)$$

for any $a \in \mathbb{Q}$ where $k = \mathbb{Q}(t)$ and $D = d/dt$.

4.1 Factors of higher degree Let $L = D^n + \sum_{i=0}^{n-1} a_i D^i \in k[D]$ where $n \geq 4$, and assume that $L = L_1 L_2$ where

$$L_2 = D^e + \sum_{i=0}^{e-1} b_i D^i \tag{7}$$

is an unknown monic right factor of L of degree e, $2 \leq e \leq n-2$. Let $\{y_1, y_2, \ldots, y_e\}$ be a basis of the space of solutions of $L_2 * y = 0$ in an appropriate Picard–Vessiot extension K of k. The *Wronskian* of $(y_1, y_2, \ldots, y_e)$ is defined by

$$\mathrm{W}(y_1, y_2, \ldots, y_e) = \begin{vmatrix} y_1 & y_2 & \cdots & y_e \\ D * y_1 & D * y_2 & \cdots & D * y_e \\ \vdots & \vdots & \ddots & \vdots \\ D^{e-1} * y_1 & D^{e-1} * y_2 & \cdots & D^{e-1} * y_e \end{vmatrix} \in K.$$

The following Lemma about the derivative of a determinant can be checked easily by expressing $\det(\mathcal{M})$ as a sum over a permutation group, and using the identity $D * (u_1 \cdots u_m) = \sum_{i=1}^{m} u_1 \cdots u_{i-1}\,(D * u_i)\,u_{i+1} \cdots u_m$.

Lemma 3 *Let $\mathcal{M}$ be any $m \times m$ matrix with coefficients in k. Then,*

$$D * \det(\mathcal{M}) = \sum_{i=1}^{m} \det(\mathcal{M}_i)$$

where $\mathcal{M}_i$ is the result of applying D to the i^{th} row of $\mathcal{M}$.

Since $L_2 * y_i = 0$ and L_2 is monic it follows that

$$D^e * y_i = -\sum_{j=0}^{e-1} b_j D^j * y_i \tag{8}$$

and by Lemma 3,

$$D * \mathrm{W}(y_1, y_2, \ldots, y_e) = \begin{vmatrix} D * y_1 & D * y_2 & \cdots & D * y_e \\ D * y_1 & D * y_2 & \cdots & D * y_e \\ \vdots & \vdots & \ddots & \vdots \\ D^{e-1} * y_1 & D^{e-1} * y_2 & \cdots & D^{e-1} * y_e \end{vmatrix} +$$

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_e \\ D^2 * y_1 & D^2 * y_2 & \cdots & D^2 * y_e \\ \vdots & \vdots & \ddots & \vdots \\ D^{e-1} * y_1 & D^{e-1} * y_2 & \cdots & D^{e-1} * y_e \end{vmatrix} + \cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_e \\ D * y_1 & D * y_2 & \cdots & D * y_e \\ \vdots & \vdots & \ddots & \vdots \\ D^{e} * y_1 & D^{e} * y_2 & \cdots & D^{e} * y_e \end{vmatrix}.$$

All the determinants in the above sum are 0 except the last one, which by (8) is

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_e \\ D * y_1 & D * y_2 & \cdots & D * y_e \\ \vdots & \vdots & \ddots & \vdots \\ -b_{e-1} D^{e-1} * y_1 & -b_{e-1} D^{e-1} * y_2 & \cdots & -b_{e-1} D^{e-1} * y_e \end{vmatrix} = -b_{e-1} \mathrm{W}(y_1, y_2, \ldots, y_e).$$

Thus the Wronskian is annihilated by the operator $D + b_{e-1}$ and is therefore hyperexponential over k. The equation $D * \mathrm{W} + b_{e-1}\mathrm{W} = 0$ is called *Liouville's relation.*

Now let $\mathcal{M}$ be the $n \times e$ matrix of elements of k given by

$$\mathcal{M} = \begin{pmatrix} y_1 & y_2 & \cdots & y_e \\ D * y_1 & D * y_2 & \cdots & D * y_e \\ \vdots & \vdots & & \vdots \\ D^e * y_1 & D^e * y_2 & \cdots & D^e * y_e \\ \vdots & \vdots & & \vdots \\ D^{n-1} * y_1 & D^{n-1} * y_2 & \cdots & D^{n-1} * y_e \end{pmatrix}$$

and let $w_0, w_1, \ldots, w_{N-1}$ be all the $e \times e$ minors of $\mathcal{M}$ where $N = \binom{n}{e}$. We enumerate these minors in such a way that

$$w_i = \begin{vmatrix} y_1 & y_2 & \cdots & y_e \\ \vdots & \vdots & & \vdots \\ D^{i-1} * y_1 & D^{i-1} * y_2 & \cdots & D^{i-1} * y_e \\ D^{i+1} * y_1 & D^{i+1} * y_2 & \cdots & D^{i+1} * y_e \\ \vdots & \vdots & & \vdots \\ D^e * y_1 & D^e * y_2 & \cdots & D^e * y_e \end{vmatrix}$$

for $0 \le i \le e$. Obviously, $w_e = \mathrm{W}(y_1, y_2, \ldots, y_e)$, and using (8), we find that for $0 \le i < e$,

$$w_i = \begin{vmatrix} y_1 & y_2 & \cdots & y_e \\ \vdots & \vdots & & \vdots \\ D^{i-1} * y_1 & D^{i-1} * y_2 & \cdots & D^{i-1} * y_e \\ D^{i+1} * y_1 & D^{i+1} * y_2 & \cdots & D^{i+1} * y_e \\ \vdots & \vdots & & \vdots \\ -b_i D^i * y_1 & -b_i D^i * y_2 & \cdots & -b_i D^i * y_e \end{vmatrix} = (-1)^{e-i} b_i w_e \tag{9}$$

which implies that each nonzero w_i is hyperexponential over k.

The above relations would allow us to compute the b_i's if the various w_l's were known. We now describe how from the a_i's we can compute operators which annihilate the w_l's. Since L_2 is a right factor of L, we also have $L * y_i = 0$ and hence

$$D^n * y_i = -\sum_{j=0}^{n-1} a_j D^j * y_i \,. \tag{10}$$

Let $w_l = \det(W_l)$ be any $e \times e$ minor of $\mathcal{M}$. If the last row of $\mathcal{M}$ does not appear in w_l, then by Lemma 3, $D * w_l$ is a sum of e determinants, the nonzero ones being $e \times e$ minors of $\mathcal{M}$. If the last row of $\mathcal{M}$ does appear in W_l, then by Lemma 3 again, $D * w_l$ is a sum of $e - 1$ terms, all of which are either 0 or $e \times e$ minors, plus the determinant of a matrix whose last row is $D^n y_1, \ldots, D^n y_e$. Using (10) this last determinant can also be expressed as a linear combination of $e \times e$ minors with coefficients in k. By induction, we can express $D^j * w_l$ for $j \ge 0$ as a linear combination of $w_0, w_1, \ldots, w_{N-1}$ with coefficients in k:

$$D^j * w_l = \sum_{i=0}^{N-1} r_{i,j,l} w_i \,. \tag{11}$$

For a fixed l, the equations (11) for $j = 1, \ldots, N$ yield a system of linear equations for the unknowns $w_0, w_1, \ldots, w_{N-1}$ with determinant $\mathcal{D} = \det[r_{i-1,j,l}]_{i,j=1}^{N}$. In the "generic" case, $\mathcal{D}$ will be nonzero and we can solve this system for $w_0, w_1, \ldots, w_{N-1}$ in terms of $D * w_l, D^2 * w_l, \ldots, D^N * w_l$. This gives us an operator $L^{(l)} \in k[D]$ such that $\deg(L^{(l)}) = N$ and $L^{(l)} * w_l = 0$, as well as linear expressions for $w_0, w_1, \ldots, w_{N-1}$ in terms of w_l and its derivatives. The operator $L^{(l)}$ is called the l^{th} *associated operator of* L.

Now choose $l = e$ and use one of the algorithms for hyperexponential solutions [3, 19] to compute a finite set $\{h_1, \ldots, h_n, z_1, \ldots, z_m\}$ where the z_j's are in k and the h_i's are hyperexponential over k, such that any hyperexponential solution of $L^{(e)} * y = 0$ must be of the form

$$h_i \sum_{j=0}^{m} c_j z_j$$

for some constants c_j's. Since $L^{(e)} * w_e = 0$, w_e must be of the above form, hence we get from Liouville's relation an expression for b_{e-1} in terms of the constants c_j's. Since the remaining w_l's are expressed in terms of w_e and its derivatives, we can express w_i for $0 \leq i < e$ in terms of the constants c_j's, so each h_i yields one candidate right hand factor of the form (7) where

$$b_{e-1} = -\frac{D * w_e}{w_e} \qquad \text{and} \qquad b_i = (-1)^{e-i}\frac{w_i}{w_e} \quad \text{for} \quad 0 \leq i < e-1\,.$$

We test this candidate by dividing L with it from the right, equating the remainder to zero, and solving the resulting system of algebraic equations for the unknown constants. If all the remainders are nonzero for each h_i, then L has no right hand factor of degree e.

In the "degenerate" case, when $\mathcal{D} = 0$, there still exists a nontrivial associated operator for w_e, but it is of degree less than N. This operator can be found by the linear elimination of all the other w_l's from (11). However, in this case we may be unable to express all of the w_l's in terms of w_e and its derivatives. Therefore we need to compute and solve not just $L^{(e)}$, but several associated equations (for the other w_i's also), and test all possible combinations of the obtained solutions. Since we know that $w_i = (-1)^{e-i}b_i w_e$, we only need to search for the hyperexponential solutions of $L^{(e)} * y = 0$; for the others, we can make the substitutions $y = (-1)^{e-i}z\ w_e$ and then use algorithms that find only the solutions z which are in k [1, 2]. We note here that the associated operator method appeared already in the 19th century [17].

5 Linear ordinary difference operators

Let k be a difference field of characteristic 0 with transform E (see [6] for the relevant definitions), let σ be E and δ be 0 on k. Then $k[x;\sigma,\delta]$ is isomorphic as a left skew polynomial ring to the usual linear difference operator ring $k[E]$ via the isomorphism which takes x to E. Any difference extension K of k is a compatible field extension of k with σ being E and δ being 0, so $\mathrm{Const}_{\sigma,\delta}(K) = \mathrm{Const}_E(K)$. In addition, for $c \in k$, the corresponding action of $k[x;\sigma,\delta]$ on K is given by $\theta_c y = c\ E{*}y$, so $y \in K^*$ is hyperexponential over k if and only if $c\ (E{*}y)/y \in k$ for some $c \in k$ for which θ_c is not identically 0, which is equivalent to $(E * y)/y \in k$. In this case y is called an *hypergeometric term* (over k) (cf. [8] and [9]). If we fix the coefficient c to be 1 for the action of $k[x;\sigma,\delta]$ on K, then the action $*_{\theta_1}$ is the same as the action of $k[E]$, namely $\theta_1 y = E * y$, so we write $*$ for $*_{\theta_1}$ in the rest of this section.

Let t be transcendental over k. Given $a, b \in k^*$, there is a unique extension of E to $k(t)$ such that $(k(t), E)$ is a difference extension of (k, E) and $E{*}t = -bt/a \neq 0$ (see [6], Ex. 2, p. 59). We then have

$$(ax+b) * t = a\theta_1 t + bt = a\ (E * t) + bt = 0$$

which shows that $k[x;\sigma,\delta]$ is 1-solvable. Theorems 3 and 4 state that $L \in k[x;\sigma,\delta]$ given by

$$L = \sum_{i=0}^{n} a_i x^i \tag{12}$$

has a right hand factor of degree 1 in $k[x;\sigma,\delta]$ if and only if either $a_0 = 0$, or the difference equation $L * y = 0$ has an hypergeometric solution. In the case where $k = F(t)$ and F is a field of characteristic zero, E is constant on F, t is transcendental over F, and $E * t = t + 1$, there is an algorithm for finding the hypergeometric solutions of a linear difference equation with coefficients in k [15], so we can decide whether elements of $k[x; \sigma, \delta]$ (and $k[E]$) have right hand factors of degree 1 and compute such a factor if one exists. With the same algorithm we can also decide existence of *left hand* factors of degree 1 (and compute them) by applying it to the adjoint operator, which is defined by

$$\left(\sum_{i=0}^{n} a_i E^i\right)^* = \sum_{i=0}^{n} E^i a_{n-i}$$

when $a_n, a_0 \neq 0$. A direct computation shows that for any $L_1, L_2 \in k[E]$,

$$\begin{aligned}
(L_1^*)^* &= E^n L_1 E^{-n} \\
\deg(L_1^*) &= \deg(L_1) \\
(L_1 + L_2)^* &= L_1^* + L_2^* \\
(L_1 L_2)^* &= (E^n L_2^* E^{-n}) L_1^*
\end{aligned}$$

where $n = \deg(L_1)$. Hence left hand factors of L correspond to right hand factors of L^*, and vice versa.

Of course, factorisation in $k[E]$ is not unique. For example, when $k = \mathbb{Q}(t)$ the equation

$$E^2 - 2E + 1 = (E - \frac{at+b}{at+a+b})(E - \frac{at+a+b}{at+b}),$$

where $a, b \in \mathbb{Q}$ are not both zero but otherwise arbitrary, describes all possible factorisations of $L = E^2 - 2E + 1$ into monic factors of degree 1.

5.1 Factors of higher degree In this section we outline an algorithm for finding right hand factors of linear difference operators. Given an operator $L \in k[E]$ of degree $n \geq 4$, the algorithm will return all its right hand factors of degree e where $2 \leq e \leq n-2$.

Assume that $L = L_1 L_2$ where

$$\begin{aligned}
L &= E^n + \sum_{i=0}^{n-1} a_i E^i, \\
L_2 &= E^e + \sum_{i=0}^{e-1} b_i E^i,
\end{aligned} \tag{13}$$

and $a_0 \neq 0$. Let K be a difference extension field of k in which $L_2 * y = 0$ has e solutions, linearly independent over $\mathrm{Const}_E(K)$. Such an extension always exists [6, Thm. XII, p. 272], although for some operators $\mathrm{Const}_E(K)$ will be strictly larger than $\mathrm{Const}_E(k)$ for any such K [7]. Let $\{y_1, y_2, \ldots, y_e\}$ be a basis of the $\mathrm{Const}_E(K)$–linear space of solutions of $L_2 * y = 0$ in K. The corresponding *Casoratian determinant* $\mathrm{Cas}(y_1, y_2, \ldots, y_e) \in K$ is defined by

$$\mathrm{Cas}(y_1, y_2, \ldots, y_e) = \begin{vmatrix} y_1 & y_2 & \cdots & y_e \\ E * y_1 & E * y_2 & \cdots & E * y_e \\ \vdots & \vdots & \ddots & \vdots \\ E^{e-1} * y_1 & E^{e-1} * y_2 & \cdots & E^{e-1} * y_e \end{vmatrix}.$$

Since $L_2 * y_i = 0$ and L_2 is monic it follows that

$$E^e * y_i = -\sum_{j=0}^{e-1} b_j E^j * y_i \tag{14}$$

and

$$E * \mathrm{Cas}(y_1, y_2, \ldots, y_e) = \begin{vmatrix} E * y_1 & E * y_2 & \cdots & E * y_e \\ E^2 * y_1 & E^2 * y_2 & \cdots & E^2 * y_e \\ \vdots & \vdots & \ddots & \vdots \\ E^e * y_1 & E^e * y_2 & \cdots & E^e * y_e \end{vmatrix} =$$

$$= \begin{vmatrix} E * y_1 & E * y_2 & \cdots & E * y_e \\ E^2 * y_1 & E^2 * y_2 & \cdots & E^2 * y_e \\ \vdots & \vdots & \ddots & \vdots \\ E^{e-1} * y_1 & E^{e-1} * y_2 & \cdots & E^{e-1} * y_e \\ -b_0\, y_1 & -b_0\, y_2 & \cdots & -b_0\, y_e \end{vmatrix} = (-1)^e b_0 \mathrm{Cas}(y_1, y_2, \ldots, y_e)\,.$$

Thus the Casoratian is annihilated by the operator $E - (-1)^e b_0$ and is therefore hypergeometric over k. Up to a constant factor, the Casoratian is independent of the choice of a particular basis, and can be associated directly with L.

Now let $\mathcal{M}$ be the $n \times e$ matrix of elements of K given by

$$\mathcal{M} = \begin{pmatrix} y_1 & y_2 & \cdots & y_e \\ E * y_1 & E * y_2 & \cdots & E * y_e \\ \vdots & \vdots & & \vdots \\ E^e * y_1 & E^e * y_2 & \cdots & E^e * y_e \\ \vdots & \vdots & & \vdots \\ E^{n-1} * y_1 & E^{n-1} * y_2 & \cdots & E^{n-1} * y_e \end{pmatrix}$$

and let $w_0, w_1, \ldots, w_{N-1}$ be all the $e \times e$ minors of $\mathcal{M}$ where $N = \binom{n}{e}$. We enumerate these minors in such a way that

$$w_i = \begin{vmatrix} y_1 & y_2 & \cdots & y_e \\ \vdots & \vdots & & \vdots \\ E^{i-1} * y_1 & E^{i-1} * y_2 & \cdots & E^{i-1} * y_e \\ E^{i+1} * y_1 & E^{i+1} * y_2 & \cdots & E^{i+1} * y_e \\ \vdots & \vdots & & \vdots \\ E^e * y_1 & E^e * y_2 & \cdots & E^e * y_e \end{vmatrix}$$

for $0 \le i \le e$. Obviously, $w_e = \mathrm{W}(y_1, y_2, \ldots, y_e)$, and using (14), we find that for $0 \le i < e$,

$$w_i = \begin{vmatrix} y_1 & y_2 & \cdots & y_e \\ \vdots & \vdots & & \vdots \\ E^{i-1} * y_1 & E^{i-1} * y_2 & \cdots & E^{i-1} * y_e \\ E^{i+1} * y_1 & E^{i+1} * y_2 & \cdots & E^{i+1} * y_e \\ \vdots & \vdots & & \vdots \\ -b_i E^i * y_1 & -b_i E^i * y_2 & \cdots & -b_i E^i * y_e \end{vmatrix} = (-1)^{e-i} b_i w_e \tag{15}$$

which implies that each nonzero w_i is hypergeometric over k. Since L_2 is a right factor of L, we also have $L * y_i = 0$ for $i = 1, 2, \ldots, e$ and hence

$$E^n * y_i = -\sum_{j=0}^{n-1} a_j E^j * y_i \,. \tag{16}$$

Using (16) we can express any $E^j * w_l$ as a linear combination of $w_0, w_1, \ldots, w_{N-1}$ with coefficients in k:

$$E^j * w_l = \sum_{i=0}^{N-1} r_{i,j,l} w_i \,. \tag{17}$$

For a fixed l, equations (17) for $j = 1, 2, \ldots, N$ constitute a system of linear equations for unknowns $w_0, w_1, \ldots, w_{N-1}$ with determinant $\mathcal{D} = \det[r_{i-1,j,l}]_{i,j=1}^N$. In the "generic" case, $\mathcal{D}$ will be nonzero and we can solve this system for $w_0, w_1, \ldots, w_{N-1}$ in terms of $E * w_l, E^2 * w_l, \ldots, E^N * w_l$. This gives us an operator $L^{(l)} \in k[E]$ such that $\deg(L^{(l)}) = N$ and $L^{(l)} * w_l = 0$, as well as linear expressions for $w_0, w_1, \ldots, w_{N-1}$ in terms of w_l and its transforms. The operator $L^{(l)}$ is called the l^{th} *associated operator of L*.

Now choose $l = e$. Assume that $k = F(t)$ is the difference field of rational functions over some field F of characteristic 0. In this case, we can use the algorithm of [15] to find all hypergeometric solutions of $L^{(e)} * y = 0$. In general, we will obtain a finite set of functions $\{h_1, \ldots, h_n, z_1, \ldots, z_m\}$ where the z_j's are in k and the h_i's are hypergeometric over k, such that any hypergeometric solution of $L^{(e)} * y = 0$ must be of the form

$$h_i \sum_{j=0}^{m} c_j z_j$$

for some constants c_j's. Since the remaining w_l's are expressed in terms of w_e and its transforms, we can express w_i for $0 \le i < e$ in terms of the constants c_j's, so each h_i yields one candidate right hand factor of the form (13) where

$$b_0 = (-1)^e \frac{E * w_e}{w_e} \qquad \text{and} \qquad b_i = (-1)^{e-i} \frac{w_i}{w_e} \quad \text{for} \quad 0 < i \le e-1 \,.$$

We test this candidate by dividing L with it from the right, equating the remainder to zero, and solving the resulting system of algebraic equations for the unknown constants. If all the remainders are nonzero for each h_i, then L has no right hand factor of degree e.

In the "degenerate" case, when $\mathcal{D} = 0$, there still exists a nontrivial associated operator for w_e, but it is of degree less than N. This operator can be found by the linear elimination of all the other w_l's from (17). However, in this case we may be unable to express all of the w_l's in terms of w_e and its transforms. Therefore we need to compute and solve not just $L^{(e)}$, but several associated equations (for the other w_i's also), and test all possible combinations of the obtained solutions. Since we know that $w_i = (-1)^{e-i} b_i w_e$, we only need to search for the hypergeometric solutions of $L^{(e)} * y = 0$; for the others, we can make the substitutions $y = (-1)^{e-i} z\, w_e$ and then use algorithms that find only the solutions z which are in k [1].

Example

The case $n = 4$ and $e = 2$ is the simplest one which cannot be handled by the algorithm of [15] alone. Let

$$L = E^4 + d(t)E^3 + c(t)E^2 + b(t)E + a(t)$$

be the operator we wish to factorise, and let

$$L_2 = E^2 + f(t)E + g(t)$$

be a right factor of L. Here

$$\mathcal{M} = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1(t+1) & y_2(t+1) \\ y_1(t+2) & y_2(t+2) \\ y_1(t+3) & y_2(t+3) \end{pmatrix}.$$

Let $w(t)$ denote the Casoratian of L_2. As in (17), we express $w(t+1), w(t+2), \ldots, w(t+6)$ as linear combinations of the 2×2 minors of $\mathcal{M}$. The determinant of the resulting system of equations turns out to be

$$a^3 a_1^2 a_2 (b_2 b_3 - a_3 d_1 d_2) \tag{18}$$

(we write a for $a(t)$, a_i for $a(t+i)$ etc.). If $b_2 b_3 \neq a_3 d_1 d_2$ then by solving this system we find that $w(t)$ satisfies the following difference equation of degree 6:

$$\begin{aligned}
&w(t) a a_1 a_2 (a_3 d_1 d_2 - b_2 b_3) + \\
&w(t+1) a_1 a_2 (b_2 b_3 c - a_2 b_3 d + a_3 b_1 d_2 - a_3 c d_1 d_2) + \\
&w(t+2) a_2 (a_3 b_1 b_2 - b_1 b_2 b_3 d + a_2 b_3 c_1 d - a_2 a_3 d d_1 - a_3 b_1 c_1 d_2 + a_3 b_1 d d_1 d_2) + \\
&w(t+3)(a_2 a_3 c_1 d_1 d_2 - a_2 b_2 b_3 c_1 - a_3 b_1 b_2 c_2 + a_2 a_3 c_2 d d_1 + b_1 b_2^2 b_3 - a_2 a_3 d d_1^2 d_2) + \\
&w(t+4)(a_2 b_2 b_3 - a_2 b_3 c_2 d - b_1 b_2 b_3 d_2 + a_3 b_1 c_2 d_2 - a_2 a_3 d_1 d_2 + a_2 b_3 d d_1 d_2) + \\
&w(t+5)(b_1 b_2 c_3 - a_2 c_3 d d_1 - a_3 b_1 d_2 + a_2 b_3 d) + \\
&w(t+6)(a_2 d d_1 - b_1 b_2) = 0\,.
\end{aligned}$$

From the same system we obtain the coefficients of L_2 expressed in terms of $w(t)$:

$$\begin{aligned}
g &= r \\
f &= (a a_1 b_2 + (b b_1 b_2 - a_1 b_2 c + a_1 a_2 d - a_2 b d d_1) r + a_2 d (d d_1 - c_1) r r_1 + \\
&\quad b_2 (c_1 - d d_1) r r_1 r_2 + (c_2 d - b_2) r r_1 r_2 r_3 - d r r_1 r_2 r_3 r_4) / (a (b_1 b_2 - a_2 d d_1))
\end{aligned}$$

where $r(t) = w(t+1)/w(t)$.

When $b_2 b_3 = a_3 d_1 d_2$ the difference equation for $w(t)$ is of lower degree, $g = r$ as before, but in order to obtain f another associated equation has to be constructed.

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