



Classifying the phase transition threshold for Ackermannian functions

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ABSTRACT

It is well known that the Ackermann function can be defined via diagonalization from an iteration hierarchy (of Grzegorzczuk type) which is built on a start function like the successor function. In this paper we study for a given start function g iteration hierarchies with a sub-linear modulus h of iteration. In terms of g and h we classify the phase transition for the resulting diagonal function from being primitive recursive to being Ackermannian.

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1. Introduction

This paper is part of a general program on phase transitions in logic and combinatorics. In general terms phase transition is a type of behavior wherein small changes of a parameter of a system cause dramatic shifts in some globally observed behavior of the system, such shifts being usually marked by a sharp ‘threshold point’. (Everyday life examples of such thresholds are ice melting and water boiling temperatures.) Phenomena of this kind nowadays occur throughout many mathematical and computational disciplines: statistical physics, evolutionary graph theory, percolation theory, computational complexity, artificial intelligence etc.

The last few years have seen an unexpected series of results that bring together independence results in logic, analytic combinatorics and Ramsey Theory. These results can be described intuitively as phase transitions from provability to unprovability of an assertion by varying a threshold parameter [1,3,6,10,13,14,16–18]. Another face of this phenomenon is the transition from slow-growing to fast-growing computable functions [11,12,15].

In this paper we investigate phase transition phenomena which are related to natural subclasses of the recursive functions. In particular we take a closer look at the Grzegorzczuk hierarchy from the phase transition perspective. For this purpose let us assume that we are given two functions $g, h : \mathbb{R} \cap [0, \infty) \rightarrow \mathbb{R} \cap [0, \infty)$. Further, for $r \in \mathbb{R}$, let $\lfloor r \rfloor$ denote the largest integer not exceeding r .

Define for $x \in \mathbb{N}$

$$B(g, h)_0(x) := g(x),$$

$$B(g, h)_{k+1}(x) := B(g, h)_k^{\lfloor h(x) \rfloor}(x) \quad \text{i.e. } \lfloor h(x) \rfloor \text{ many iterations,}$$

$$B(g, h)_\omega(x) := B(g, h)_{\lfloor x \rfloor}(x).$$

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We allow here for real number values in the range of $B(g, h)_k$ to avoid messy rounding to integers at every step of the calculation. This would be necessary if we would deal with number-theoretic functions only.

We recall that the Ackermann function is defined as $\text{Ack}(n) = B(g, h)_\omega(n)$ where $g(x) = x + 1$ and $h = \text{Id}$, and that $A_i(n) = B(g, h)_i(n)$ is called the i th approximation of the Ackermann function. It is well known (see e.g. [4]) that each approximation A_i is *primitive recursive* and that every primitive recursive function is eventually dominated by some A_i . Thus the Ackermann function eventually dominates every primitive recursive function. We use the term “Ackermannian” to mean “eventually faster than every primitive recursive function”. There is no “smallest” Ackermannian function; if $B : \mathbb{N} \rightarrow \mathbb{N}$ is Ackermannian, then so is $B/2$ or $B^{1/2}$, etc. If the composition $f \circ g$ of two non-decreasing functions is Ackermannian and one of $\{f, g\}$ is primitive recursive, then the other is Ackermannian. It is also important to note that there are functions $B : \mathbb{N} \rightarrow \mathbb{N}$ which are neither Ackermannian nor bounded by any primitive recursive function. [In our paper we only consider Ackermannian functions which are elementary recursive in the standard Ackermann function, so this extra requirement might have been added safely to the definition of being Ackermannian.]

For an unbounded function $g : \mathbb{N} \rightarrow \mathbb{N}$ define the inverse function $g^{-1} : \mathbb{N} \rightarrow \mathbb{N}$ by $g^{-1}(m) := \min\{n : g(n) \geq m\}$. Let us remark that although Ack is not primitive recursive, its inverse Ack^{-1} is primitive recursive.

To avoid trivialities we assume that for some $\varepsilon > 0$ we have $g(x) \geq x + \varepsilon$ for all but finitely many x [an iteration of the identity map would in our context of course be senseless] and we assume that h is weakly increasing and unbounded. Now, fixing g , one may ask for which h the function $B(g, h)_\omega$ becomes Ackermannian. Similarly, fixing h , one may ask for which g the function $B(g, h)_\omega$ becomes Ackermannian. So in contrast to the situations previously considered the phase transition depends on two order parameters and we will indicate that the phase transition has a surprisingly rich structure.

2. Iteration hierarchies for $g(x) := x + 1$

In this section we fix $g(x) := x + 1$. This particular case was considered and partially solved in [7]. The result of [7] was later on improved in [5]. The results given in these two papers were rather indirect and involved the phase transition for the Kanamori McAloon result for pairs. Nevertheless, they have independent interest since they show how regressive Ramsey functions are intrinsically related to parameterized iteration hierarchies. The following yields a rather sharp threshold on the behavior of such function hierarchies. Using the notation of [7,5] we denote $B(g, x^{1/t})$, where $t \in \mathbb{N}$ is a constant, by (f_t) . Namely, $(f_t)_i^j(x) = B(g, h)_i^j(x)$ for all i, j and x , where $g(x) = x + 1$ and $h(x) = x^{1/t}$. Moreover let $|x|$ be the non-negative part of the logarithm function with respect to base two. Thus $|x| = \max\{\log_2(x), 0\}$. Alternatively we could have used the binary length instead but this would have caused certain extra complications. Moreover let $\|x\| := \lfloor |x| \rfloor$.

Claim 1. For every $t > 0$ and $n > \max(\{4, 3^t, t^t\})$ it holds that

$$(f_t)_{i+t^2+2t+2}(n) > A_i(n).$$

Proof. See Claim 2.12 in [5]. \square

Claim 2. For every $i \in \mathbb{N}$ and for every $n \in \mathbb{N}$ such that:

- (1) $n > i + (\|n\|)^2 + 2\|n\| + 2$ and
- (2) $\text{Ack}(\|n\|) > A_i(n)$

it holds for $h_{\text{Ack}}(n) := n^{\frac{1}{\text{Ack}^{-1}(n)}}$ that

$$B(g, h_{\text{Ack}})_{i+(\|n\|)^2+2\|n\|+2}(n) > A_i(n).$$

Proof. To show that, we examine two cases. First, if it holds that

$B(g, h_{\text{Ack}})_{i+(\|n\|)^2+2\|n\|+2}(n) \geq \text{Ack}(\|n\|)$, then we are done by demand 2. Otherwise, we may fix $t := \|n\|$ and we have that

for all $y \in \{0, \dots, \text{Ack}(t) - 1\}$ it holds that $y^{\frac{1}{t}} < y^{\frac{1}{\text{Ack}^{-1}(y)}}$. Since h_{Ack} is non-decreasing, we have that $B(g, h_{\text{Ack}})_{i+t^2+2t+2}$ is also non-decreasing. Thus, it holds that

$$B(g, h_{\text{Ack}})_{i+t^2+2t+2}(n) \geq (f_t)_{i+t^2+2t+2}(n)$$

which by Claim 1 is larger than $A_i(n)$. \square

We remark that the choice of $t = \|n\|$ is arbitrary and any α^{-1} , such that α is a monotone increasing primitive recursive function and $\alpha(x) > x^x$ for large enough x , would do the job.

Theorem 1. Let $g(x) := x + 1$ and $h_\alpha(x) := x^{\frac{1}{B(g, \text{id})_\alpha^{-1}(x)}}$. Then $B(g, h_\alpha)_\omega$ is Ackermannian iff $\alpha = \omega$.

Proof. The ‘if’ direction is in fact the claim that if $h_\alpha(x) = x^{\frac{1}{\text{Ack}^{-1}(x)}}$, then $B(g, h_\alpha)_\omega$ eventually grows faster than any primitive recursive function. It would suffice to show that for every $i \in \mathbb{N}$, there exists x_0 such that for all $x > x_0$, it holds that $B(g, h_\alpha)_\omega(x) > A_i(x)$. Now, this is a direct corollary of Claim 2, since it is clear that for every such i there exists some $x_0 \in \mathbb{N}$ such that for all $x > x_0$ it holds that $\text{Ack}(\|x\|) > A_i(x)$ and such that $B(g, h_\alpha)_x(x) \geq B(g, h_\alpha)_{i+(\|x\|)^2 + 2\|x\| + 2}(x)$ which by Claim 2 is larger than $A_i(x)$. In other words, for every primitive recursive function f , $B(g, h_\alpha)_x(x)$ eventually dominates f .

The ‘only if’ direction is the claim that if $\alpha = i$ for some $i \in \mathbb{N}$, and therefore $h_\alpha(x) = x^{\frac{1}{A_i^{-1}(x)}}$, then $B(g, h_\alpha)_\omega(x)$ is not Ackermannian in terms of x . Note this implies the same for any h_α of the form $h_\alpha(x) = x^{\frac{1}{\beta^{-1}(x)}}$ where β is a non-decreasing unbounded primitive recursive function. To show this direction, for $\alpha = i > 3$ and $h_\alpha(x) = x^{\frac{1}{A_i^{-1}(x)}}$, fix $h_\beta(x) := 4(h_\alpha(x))^2 = x^{\frac{1}{\beta^{-1}(x)}}$ where $\beta^{-1}(x) = \frac{|x|A_i^{-1}(x)}{2|x|+2A_i^{-1}(x)}$. We again refer to [5]. Corollary 2.3 in [5] states that the h_β -regressive Ramsey number $R_{h_\beta}^{\text{reg}}(k)$ is primitive recursive in k since A_i is primitive recursive. On the other hand, Corollary 2.25 in [5] asserts that if $B(g, h_\alpha)_\omega(k)$ is Ackermannian in k , using the function $\mu_{h_\beta}(k) := k^k$, we may obtain an Ackermannian lower bound also for $R_{h_\beta}^{\text{reg}}(k)$, but this would be a contradiction. For the case of $\alpha \leq 3$, observe that $h_\alpha \leq h_{\alpha+1}$ and thus $B(g, h_\alpha)_\omega(k) \leq B(g, h_{\alpha+1})_\omega(k)$. \square

3. Slow-growing iteration hierarchies

For the rest of this section let $F_0(x) := 2^x$ and $F_{k+1}(x) := F_k^x(x)$. Then F_k is primitive recursive (in each k). Further let $F(x) := F_x(x)$. Then F is a slight variant of the Ackermann function, hence Ackermannian and of course not primitive recursive.

In addition let $2_l(x) := F_0^l(x)$. Let $|x|_{l+1} := \|x\|_l$ where $|x|_0 := x$. Then $|\cdot|_l$ is the l th iterate of $|\cdot|$ so that $|2_l(x)|_l = x$.

For the rest of the paper fix $\varepsilon > 0$, let $g_0(x) := x + \varepsilon$ and define recursively $g_{k+1}(x) := 2^{g_k(|x|)}$. Then

$$g_l(x) = 2_l(|x|_l + \varepsilon).$$

These scaling functions grow faster and faster when l becomes larger but no g_l is of exponential growth.

The following result classifies slow-growing iteration hierarchies for a rather large class of order parameters.

Theorem 2. Let $1 \geq \varepsilon > 0$ and let d be a natural number.

Define $h[d, l](x) := |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$ and

$$B[d, l]_k(x) := B(g_l, h[d, l])_k(x).$$

Let $C := \max\{2_l(F_d(2^{k+2}))\}$. Then for all $x \geq C$ and all $i \leq |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$ we have

$$B[d, l]_k^i(x) \leq 2_l \left(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i \right).$$

Hence the diagonal function $B[d, l]$ is elementary recursive.

Proof. Since g_l and hence $B[d, l]_k$ are monotone in ε we may assume that $\varepsilon = 1$. We prove the claim by main induction on k . If $k = 0$ then $B[d, l]_0^i(x) = g_l^i(x)$. We prove the claim by subsidiary induction on i . Assume first that $i = 1$. We prove the claim by another subsidiary induction on l . Assume $l = 0$. Then for $x \geq C$:

$$\begin{aligned} B[d, 0]_0^1(x) &= g_0(x) = x + 1 \\ &\leq 2_0 \left(|x|_0 + |x|_0^{\frac{2^1}{F_d^{-1}(|x|_0)}} \right). \end{aligned}$$

Assume now $l > 0$. Then the induction hypothesis for $l - 1$ yields for $x \geq C$:

$$\begin{aligned} B[d, l]_0^1(x) &= g_l(x) \\ &= 2^{g_{l-1}(|x|)} \\ &\leq 2^{2_{l-1} \left(\|x\|_{l-1} + \|x\|_{l-1}^{\frac{2}{F_d^{-1}(\|x\|_{l-1})}} \right)} \\ &\leq 2 \left(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}} \right). \end{aligned}$$

Now consider the case $1 \leq i < |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$. Then we obtain by the subsidiary induction hypothesis

$$\begin{aligned} B[d, \Gamma_0^{i+1}(x)] &= B[d, \Gamma_0] (B[d, \Gamma_0^i(x)]) \\ &\leq B[d, \Gamma_0] \left(2_l \left(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}} \cdot i \right) \right) \\ &= 2_l \left(\left| 2_l \left(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}} \cdot i \right) \right|_l + 1 \right) \\ &= 2_l \left(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}} \cdot i + 1 \right) \\ &\leq 2_l \left(|x|_l + |x|_l^{\frac{2}{F_d^{-1}(|x|_l)}} \cdot (i + 1) \right) \end{aligned}$$

since by assumption $x \geq C = 2_l(F_d(2^{k+2}))$.

Now assume that $k > 0$. We prove the claim by subsidiary induction on i .

If $i = 1$ then the main induction hypothesis yields

$$\begin{aligned} B[d, \Gamma_k(x)] &= B[d, \Gamma_{k-1}^{\lfloor |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}} \rfloor}(x) \\ &\leq 2_l \left(|x|_l + |x|_l^{\frac{2^k}{F_d^{-1}(|x|_l)}} \cdot \left\lfloor |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}} \right\rfloor \right) \\ &\leq 2_l \left(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \right). \end{aligned}$$

If $1 \leq i < |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$ then we obtain by the subsidiary induction hypothesis

$$\begin{aligned} B[d, \Gamma_k^{i+1}(x)] &= B[d, \Gamma_k] (B[d, \Gamma_k^i(x)]) \\ &\leq B[d, \Gamma_k] \left(2_l \left(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i \right) \right). \end{aligned}$$

Now set $y := 2_l(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i)$. Then we obtain from the main induction hypothesis and $i < |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$ that

$$\begin{aligned} B[d, \Gamma_k^{i+1}(x)] &\leq B[d, \Gamma_{k-1}^{\lfloor |y|_l^{\frac{1}{F_d^{-1}(|y|_l)}} \rfloor}(y) \\ &\leq 2_l \left(|y|_l + |y|_l^{\frac{2^k}{F_d^{-1}(|y|_l)}} \cdot \left\lfloor |y|_l^{\frac{1}{F_d^{-1}(|y|_l)}} \right\rfloor \right) \\ &\leq 2_l \left(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i + |y|_l^{\frac{2^{k+1}}{F_d^{-1}(|y|_l)}} \right). \end{aligned}$$

The claim would now follow from

$$|y|_l^{\frac{2^{k+1}}{F_d^{-1}(|y|_l)}} \leq |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}}.$$

Since $F_d^{-1}(|x|_l + |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \cdot i) \geq F_d^{-1}(|x|_l)$ and $i < |x|_l^{\frac{1}{F_d^{-1}(|x|_l)}}$ this would follow from

$$\left(|x|_l + |x|_l^{\frac{2^{k+1}+1}{F_d^{-1}(|x|_l)}} \right)^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}} \leq |x|_l^{\frac{2^{k+1}}{F_d^{-1}(|x|_l)}}$$

hence from

$$|x|_l + |x|_l^{\frac{2^{k+1}+1}{F_d^{-1}(|x|_l)}} \leq |x|_l^{\frac{2^{k+1}}{2^{k+1}}}.$$

This finally follows from the assumption that $x \geq C = 2_l(F_d(2^{k+2}))$. \square

4. Fast-growing iteration hierarchies

In this section we show that replacing the functions $h[d, l]$ from [Theorem 2](#) by slightly faster-growing functions yields Ackermannian growth of the induced iteration hierarchies. Let us recall the definition of the Ackermann hierarchy from [Section 1](#). We put $A_0(x) := x + 1$ and $A_{k+1}(x) := A_k^x(x)$. Thus, if we put $\text{Ack}(x) := A_x(x)$, then Ack is the Ackermann function which eventually dominates every primitive recursive function. Further recall that our scale functions are defined as follows: $g_0(x) := x + \varepsilon$ and $g_{k+1}(x) := 2^{g_k(|x|)}$. Let us further assume from now on that $d > 0$.

Let us fix constants $C_{k,l}$ for $k > 0$ and $l \geq 0$ such that

$$\lfloor |x|_l^{\frac{1}{d}} \rfloor \cdot |x|_l^{\frac{k-1}{d}} \geq |x|_l^{\frac{k}{d}} \cdot \frac{1}{2}$$

for $x \geq C_{k,l}$. We may assume that the function $k \mapsto C_{k,l}$ is primitive recursive in k for any fixed l .

Theorem 3. Assume $1 \geq \varepsilon > 0$ and let d be a natural number.

Let

$$C[d] := \max \left\{ C_{3,d,l}, 2_l \left(\left\lfloor \frac{2^{3 \cdot d}}{\varepsilon} \right\rfloor + 1 \right) \right\}.$$

Define

$$h[[d, l]](x) := \sqrt[d]{|x|_l}$$

and

$$B[[d, l]]_k(x) := B(g_l, h[[d, l]])_k(x).$$

Then we have

$$B[[d, l]]_{3 \cdot d + i + 1}(2_l(x^d)) \geq 2_l((A_i(x))^d)$$

for $x \geq C[d]$.

Proof. Recall that $g_l(x) = 2_l(\varepsilon + |x|_l)$. By induction on i one verifies $B[[d, l]]_0^i(x) = g_l^i(x) = 2_l(\varepsilon \cdot i + |x|_l)$. Let $\varepsilon_k := \frac{\varepsilon}{2^k}$. Now we claim

$$B[[d, l]]_k^i(x) \geq 2_l(\varepsilon_k \cdot i \cdot |x|_l^{\frac{k}{d}} + |x|_l) \quad (1)$$

for $i, k \geq 1$ and $x \geq C_{k,l}$. We prove claim (1) by main induction on k and subsidiary induction on i . Assume that $k = 1$. Then we obtain for $i = 1$ that

$$\begin{aligned} B[[d, l]]_1^1(l)(x) &= B[[d, l]]_0^{\lfloor |x|_l^{\frac{1}{d}} \rfloor}(x) \\ &\geq 2_l(\varepsilon \cdot \lfloor |x|_l^{\frac{1}{d}} \rfloor + |x|_l) \\ &\geq 2_l(\varepsilon_1 \cdot |x|_l^{\frac{1}{d}} + |x|_l) \end{aligned}$$

since $x \geq C_{1,l}$. The subsidiary induction hypothesis yields

$$\begin{aligned} B[[d, l]]_1^{i+1}(x) &= B[[d, l]]_1^1(B[[d, l]]_1^i(x)) \\ &\geq B[[d, l]]_1^1(2_l(\varepsilon_1 \cdot i \cdot |x|_l^{\frac{1}{d}} + |x|_l)) \\ &\geq 2_l(\varepsilon_1 \cdot (2_l(\varepsilon_1 \cdot i \cdot |x|_l^{\frac{1}{d}} + |x|_l))_l^{\frac{1}{d}} + 2_l(\varepsilon_1 \cdot i \cdot |x|_l^{\frac{1}{d}} + |x|_l)_l) \\ &\geq 2_l(\varepsilon_1 \cdot |x|_l^{\frac{1}{d}} + \varepsilon_1 \cdot i \cdot |x|_l^{\frac{1}{d}} + |x|_l). \end{aligned}$$

Assuming claim (1) for k we show it for $k + 1$ by subsidiary induction on i as follows: First let $i = 1$. Then

$$\begin{aligned} B[[d, l]]_{k+1}^1(x) &= B[[d, l]]_k^{\lfloor |x|_l^{\frac{1}{d}} \rfloor}(x) \\ &\geq 2_l(\varepsilon_k \cdot \lfloor |x|_l^{\frac{1}{d}} \rfloor \cdot |x|_l^{\frac{k}{d}} + |x|_l) \\ &\geq 2_l(\varepsilon_{k+1} \cdot |x|_l^{\frac{k+1}{d}} + |x|_l) \end{aligned}$$

since $x \geq C_{k+1,l}$. For the induction step of the subsidiary induction we obtain

$$\begin{aligned} B[[d, l]]_{k+1}^{i+1}(x) &= B[[d, l]]_{k+1}(B[[d, l]]_{k+1}^i(x)) \\ &\geq B[[d, l]]_{k+1}(2_l(\varepsilon_{k+1} \cdot i \cdot |x|_l^{\frac{k+1}{d}} + |x|_l)) \\ &\geq 2_l(\varepsilon_{k+1} \cdot (2_l(\varepsilon_{k+1} \cdot i \cdot |x|_l^{\frac{k+1}{d}} + |x|_l)|_l)^{\frac{k+1}{d}} + |2_l(\varepsilon_{k+1} \cdot i \cdot |x|_l^{\frac{k+1}{d}} + |x|_l)|_l) \\ &\geq 2_l(\varepsilon_{k+1} \cdot |x|_l^{\frac{k+1}{d}} + \varepsilon_{k+1} \cdot i \cdot |x|_l^{\frac{k+1}{d}} + |x|_l). \end{aligned}$$

Claim (1) yields $B[[d, l]]_{3,d}(x) \geq 2_l(|x|_l^2)$ for $x \geq C[d]$.

By induction on i this yields

$$B[[d, l]]_{3,d}^i(x) \geq 2_l(|x|_l^{2^i}) \quad (2)$$

for $x \geq C[d]$.

We claim now that

$$B[[d, l]]_{d-3+i+1}(2_l(x^d)) \geq 2_l((A_i(x))^d)$$

for $x \geq C[d]$. The proof is by induction on i . For $i = 0$ we find by (2)

$$\begin{aligned} B[[d, l]]_{3,d+1}(2_l(x^d)) &\geq B[[d, l]]_{3,d}^x(2_l(x^d)) \\ &\geq 2_l((2_l(x^d)|_l)^{2^x}) \\ &\geq 2_l((A_0(x))^d). \end{aligned}$$

Assuming the claim for i we obtain it for $i + 1$ as follows:

$$\begin{aligned} B[[d, l]]_{3,d+1+i}(2_l(x^d)) &\geq B[[d, l]]_{3,d+i}^x(2_l(x^d)) \\ &\geq 2_l((A_i^*(x))^d) \\ &= 2_l((A_{i+1}(x))^d). \quad \square \end{aligned}$$

Theorem 4. Assume $1 \geq \varepsilon > 0$. Let $C[d] := \max\{C_{3,d,l}, 2_l(\lfloor \frac{2^{3 \cdot d}}{\varepsilon_k} \rfloor + 1)\}$.

Define $h[[l]]^*(x) := |x|_l^{\frac{1}{\text{Ack}^{-1}(x)}}$. Let

$$B[[l]]_k^*(x) := B(g_l, h[[l]]^*)_k(x)$$

and

$$B[[l]]^*(x) := B[[d, l]]_{[x]}^*(x).$$

Then we have

$$B[[l]]^*(2_l((4 \cdot d + C[d])^d)) > \text{Ack}(d).$$

Hence $B[[l]]^*$ is not primitive recursive.

Proof. Assume for a contradiction that $\text{Ack}(d) \geq B[[l]]^*(2_l((4 \cdot d + C[d])^d))$. Then for any $i \leq B[[l]]_{4 \cdot d + C[d]}^*(2_l((4 \cdot d + C[d])^d))$

we have $\text{Ack}^{-1}(i) \leq d$ hence $|i|_l^{\frac{1}{d}} \leq |i|_l^{\frac{1}{\text{Ack}^{-1}(i)}}$ and therefore by Theorem 3

$$\begin{aligned} B[[l]]^*(2_l((4 \cdot d + C[d])^d)) &\geq B[[d, l]]_{4 \cdot d + C[d]}^*(2_l(4 \cdot d + C[d])^d) \\ &\geq B[[d, l]]_{4 \cdot d + C[d]}(2_l(4 \cdot d + C[d])^d) \\ &> 2_l(A_d(4 \cdot d + C[d]))^d \\ &> \text{Ack}(d). \end{aligned}$$

Contradiction! Hence $B[[l]]^*$ is not primitive recursive since $d \mapsto C[d]$ is primitive recursive. \square

It seems plausible that Theorems 2–4 hold for all start functions g_l where $x + \varepsilon \leq g_0(x) \leq x + x^c$ for some fixed $c < 1$ and the same functions $h(d)_l$ and $h(l)^*$. So we expect that our phase transition results will be structurally stable under small perturbations of the starting function g .

For the record let us consider the situation when one starts with an exponential or double exponential function. This leads rather quickly to Ackermannian growth

Theorem 5. (1) Let $g(x) := 2^x$ and $h(x) = |x|_k$. Then $B(g, h)_\omega$ is Ackermannian.

(2) Let $g(x) := 2^{2^x}$ and $h(x) := \min\{l : |x|_l \leq 1\}$. Then $B(g, h)_\omega$ is Ackermannian.

Proof. 1. By induction on k one easily shows $B(g, h)_k(2_k(x)) \geq 2_k(A_k(x))$.

2. By induction on k one easily shows $B(g, h)_k(2_k(x)) \geq 2_{A_k(x)}(A_k(x))$. \square

In general we expect that sharp phase transition thresholds can be obtained for any start function $g(x) = A_d(x)$ and we expect that the resulting thresholds are all different. We intend to cover this material and structural stability of resulting phase transitions in a sequel paper. We intend also to cover phase transition thresholds for the transfinite extensions of the Ackermann hierarchy which is also known as the Schwichtenberg–Wainer hierarchy [2,8,9]. We expect essentially that stepping up in the ordinals by one power of ω will allow for one additional iteration of the binary logarithm function in the threshold function.

Problem: The functions g_l considered in this paper figure prominently in weak arithmetic. It seems to be of general interest to explore possible connections.

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