Finding closed form solutions of differential equations

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GIVEN: A linear ordinary differential equation with polynomial coefficients.

- For example

$$\begin{aligned}(16x^4 + 48x^3 + 48x^2 + 18x + 2)f''(x) \\ - (16x^4 + 48x^3 + 52x^2 + 32x + 9)f'(x) \\ + (4x^2 + 14x + 7)f(x) = 0.\end{aligned}$$

One independent variable x

One unknown function $f(x)$

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FIND: closed form solutions $f(x)$ of this equation.

- In the example: $f(x) = \exp(x)$ and $f(x) = \frac{\sqrt{1+3x+2x^2}}{x+1}$.

Some possible meanings of "closed form":

- polynomials e.g. $5x^2 + 3x - 2$
- rational functions e.g. $\big(5x-3\big)/\big(3x^2 - x + 5\big)$
- hyperexponential functions e.g. $\exp\big(\frac{2x+3}{x^2(x+1)}\big)\frac{(2x+5)^{1/3}}{(7x^2+x-3)^{1/2}}$
- algebraic functions e.g. $x - \sqrt{x^2+1}$
- elementary functions e.g. $\sin(x)/\sqrt{1+\log(1-\mathrm{e}^x)}$
- special functions e.g. $J_3(x^2{+}1) - {}_2\mathrm{F}_1(2,3;1)(\frac{1}{x})$
- holonomic functions

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- For example

$$(2x^3 - 9x^2 - 5)f^{(3)}(x) - (2x^3 - 9x^2 - 5)f''(x) \\ + (6x^2 - 24x + 18)f'(x) + (6 - 6x)f(x) = 0$$

FIND: its polynomial solutions.

- In the example, a basis of the vector space of all polynomial solutions is given by $x - 3$ and $x^3 + 5$. (A third solution, linearly independent of those two, is not polynomial.)

The problem is **easy** if we restrict to polynomials of fixed degree.

For example, suppose we are only interested in **cubic polynomials.**

The polynomial $f(x)$ solves the differential equation iff

$$\begin{aligned}&(2x^3 - 9x^2 - 5)6c_3 \\ &- (2x^3 - 9x^2 - 5)(2c_2 + 6c_3x) \\ &+ (6x^2 - 24x + 18)(c_1 + 2c_2x + 3c_3x^2) \\ &\quad + (6 - 6x)(c_0 + c_1x + c_2x^2 + c_3x^3) = 0\end{aligned}$$

The problem is **easy** if we restrict to polynomials of fixed degree.

For example, suppose we are only interested in **cubic polynomials.**

The polynomial $f(x)$ solves the differential equation iff

$$
\begin{aligned}
(6c_0 + 18c_1 + 10c_2 - 30c_3) &= 0 \\
(-6c_0 - 18c_1 + 36c_2 + 30c_3) &= 0 \\
-24c_2 &= 0 \\
+2c_2 &= 0
\end{aligned}
$$

The problem is **easy** if we restrict to polynomials of fixed degree.

For example, suppose we are only interested in **cubic polynomials.**

The polynomial $f(x)$ solves the differential equation iff

$$\begin{pmatrix} 6 & 18 & 10 & -30 \\ -6 & -18 & 36 & 30 \\ 0 & 24 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

The problem is **easy** if we restrict to polynomials of fixed degree.

For example, suppose we are only interested in **cubic polynomials.**

The polynomial $f(x)$ solves the differential equation iff

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \alpha \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

for some constants α, β.

The problem is **easy** if we restrict to polynomials of fixed degree.

For example, suppose we are only interested in **cubic polynomials.**

The polynomial $f(x)$ solves the differential equation iff

$$f(x) = \alpha(5 + 1x^3) + \beta(-3 + 1x) = (5\alpha - 3\beta) + \beta x + \alpha x^3$$

for some constants α, β.

At this point, we know all cubic polynomial solutions.

There could still be polynomial solutions of higher degree.

In order to find **all** polynomial solutions, we need to know in advance how large their degree can get.

Every polynomial has the form $f(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots$, where $c_d \neq 0$ and the $\cdots$ represent lower order terms.

No matter what the d and $c_d, c_{d-1}, \ldots$ are, we have

$$\begin{aligned} f(x) &= c_d\, x^d + \cdots \\ f'(x) &= c_d\, d x^{d-1} + \cdots \\ f''(x) &= c_d\, d(d-1) x^{d-2} + \cdots \\ f'''(x) &= c_d\, d(d-1)(d-2) x^{d-3} + \cdots \end{aligned}$$

In order to find **all** polynomial solutions, we need to know in advance how large their degree can get.

If $f(x) = c_d\, x^d + \cdots$ solves the differential equation, then

$$\begin{aligned} &(2x^3 - 9x^2 - 5)(c_d\, d(d-1)(d-2)x^{d-3} + \cdots) \\ &- (2x^3 - 9x^2 - 5)(c_d\, d(d-1)x^{d-2} + \cdots) \\ &+ (6x^2 - 24x + 18)(c_d\, dx^{d-1} + \cdots) \\ &+ (6 - 6x)(c_d\, x^d + \cdots) = 0 \end{aligned}$$

In order to find **all** polynomial solutions, we need to know in advance how large their degree can get.

If $f(x) = c_d\, x^d + \cdots$ solves the differential equation, then

$$\left(-2c_d d^2 + 8c_d d - 6c_d\right)x^d + \cdots = 0$$

In order to find **all** polynomial solutions, we need to know in advance how large their degree can get.

If $f(x) = c_d\, x^d + \cdots$ solves the differential equation, then

$$2c_d(d-3)(d-1)x^d = 0$$

In order to find **all** polynomial solutions, we need to know in advance how large their degree can get.

If $f(x) = c_d\, x^d + \cdots$ solves the differential equation, then

$$d = 3 \quad \text{or} \quad d = 1.$$

In general, plugging x^d with symbolic exponent d into an ODE gives $p(d)x^{d+i} + \cdots$ for some polynomial p (and some integer i).

$\boxed{\text{The possible degrees are integer roots of this polynomial.}}$

The polynomial p is called the **indicial polynomial** of the differential equation.

Algorithm summary

INPUT: A linear ordinary differential equation with polynomial coefficients.

OUTPUT: A basis of the vector space of all its polynomial solutions.

1. Determine the indicial polynomial p of the equation.
2. Let d be the greatest integer root of p.
3. Make an ansatz $f(x) = c_0 + c_1 x + \cdots + c_d x^d$.
4. Plug the ansatz into the equation and compare coefficients.
5. Solve the resulting linear system for $c_0, \ldots, c_d$.
6. The solutions of the system correspond to polynomial solutions of the equation.

GIVEN: A linear ordinary differential equation with polynomial coefficients.

- For example

$$(2x^4 - x^3 + 3x)f^{(3)}(x) - (2x^4 - 15x^3 + 15x^2 - 9x - 9)f''(x) \\ - (6x^3 - 30x^2 + 42x - 18)f'(x) + (6x - 18)f(x) = 0$$

FIND: its rational solutions.

- In the example, a basis of the vector space of all rational solutions is given by $(3-x)/x$ and $1/(1+x)^2$. (A third solution, linearly independent of those two, is not a rational function.)

The problem is **easy** if we prescribe the denominator.

For example, suppose we are only interested in solutions of the form $f(x) = u(x)/x$, where $u(x)$ is a polynomial.

No matter what $u(x)$ is, we have

$$\begin{aligned} f(x) &= \frac{u(x)}{x} \\ f'(x) &= \frac{u'(x)}{x} - \frac{u(x)}{x^2} \\ f''(x) &= \frac{u''(x)}{x} - 2\frac{u'(x)}{x^2} + 2\frac{u(x)}{x^3} \\ f'''(x) &= \frac{u'''(x)}{x} - 3\frac{u''(x)}{x^2} + 6\frac{u'(x)}{x^3} - 6\frac{u(x)}{x^4} \end{aligned}$$

The problem is **easy** if we prescribe the denominator.

For example, suppose we are only interested in solutions of the form $f(x) = u(x)/x$, where $u(x)$ is a polynomial.

Plug $f(x) = u(x)/x$ into the differential equation. This gives

$$\begin{gathered}(2x^7 - x^6 + 3x^4)u^{(3)}(x) - (2x^7 - 9x^6 + 12x^5 - 9x^4)u''(x) \\ - (2x^6 - 12x^5 + 18x^4)u'(x) + (2x^5 - 6x^4)u(x) = 0.\end{gathered}$$

Determine the polynomial solutions of this equation. This gives $u(x) = 3 - x$ (up to constant multiples).

It follows that $f(x) = (3 - x)/x$ is (up to constant multiples) the only rational solution of the original equation with denominator x.

In order to find **all** rational solutions, we need to know which factors can occur in a denominator, and with which multiplicity.

If $f(x) = \frac{u}{v\,p^e}$ is a rational function with $\gcd(p,u) = \gcd(p,v) = 1$ then

$$f'(x) = \frac{\blacksquare}{\blacksquare\, p^{e+1}},\quad f''(x) = \frac{\blacksquare}{\blacksquare\, p^{e+2}},\quad f'''(x) = \frac{\blacksquare}{\blacksquare\, p^{e+3}}, \text{etc.}$$

It can be shown that there can be no cancellation between the numerators and p.

Therefore, if $f(x)$ is a solution of a differential equation

$$a_0 \frac{\blacksquare}{\blacksquare\, p^{e}} + a_1 \frac{\blacksquare}{\blacksquare\, p^{e+1}} + a_2 \frac{\blacksquare}{\blacksquare\, p^{e+2}} + a_3 \frac{\blacksquare}{\blacksquare\, p^{e+3}} = 0$$

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$$a_0 \frac{\blacksquare}{\blacksquare\, p^e} + a_1 \frac{\blacksquare}{\blacksquare\, p^{e+1}} + a_2 \frac{\blacksquare}{\blacksquare\, p^{e+2}} + a_3 \frac{\blacksquare}{\blacksquare\, p^{e+3}} = 0 \quad \Big| \cdot \blacksquare\, p^{e+3}$$

In order to find **all** rational solutions, we need to know which factors can occur in a denominator, and with which multiplicity.

If $f(x) = \frac{u}{v\,p^e}$ is a rational function with $\gcd(p,u) = \gcd(p,v) = 1$ then

$$f'(x) = \frac{\blacksquare}{\blacksquare\, p^{e+1}},\ \ f''(x) = \frac{\blacksquare}{\blacksquare\, p^{e+2}},\ \ f'''(x) = \frac{\color{red}{\blacksquare}}{\blacksquare\, p^{e+3}}, \text{etc.}$$

It can be shown that there can be no cancellation between the numerators and p.

Therefore, if $f(x)$ is a solution of a differential equation

$$\big(a_0 p^2 \blacksquare + a_1 p \blacksquare + a_2 \blacksquare\big) p = a_3 \color{red}{\blacksquare}$$

$\boxed{\text{The factor } p \text{ must divide the leading coefficient } a_3 \text{ of the ODE.}}$

In order to find **all** rational solutions, we need to know which factors can occur in a denominator, and with which multiplicity.

- Suppose p is a factor of the leading coefficient of the ODE.
- Suppose $f = \frac{u}{v\,p^{e}}$ is a solution of the ODE.
- Without loss of generality, $\gcd(p,u) = \gcd(p,v) = 1$.
- Without loss of generality, $p = x - \alpha$.
- Without loss of generality, $\alpha = 0$, so that $p = x$.

In order to find **all** rational solutions, we need to know which factors can occur in a denominator, and with which multiplicity.

- Expand $\frac{u}{v}$ as a power series $c_0 + c_1 x + c_2 x^2 + \cdots$. Then

$$\begin{aligned} f(x) &= c_0 x^{-e} + \cdots \\ f'(x) &= c_0\,(-e) x^{-e-1} + \cdots \\ f''(x) &= c_0\,(-e)(-e-1) x^{-e-2} + \cdots \\ f'''(x) &= c_0\,(-e)(-e-1)(-e-2) x^{-e-3} + \cdots, \text{etc.} \end{aligned}$$

- Looks familiar…

In order to find **all** rational solutions, we need to know which factors can occur in a denominator, and with which multiplicity.

- Plug x^{-e} with symbolic exponent e into the equation.
- The **trailing** coefficient is a certain polynomial in e.
- If $f = \frac{u}{v\,x^e}$ is a rational solution, then $-e$ is an integer root of this polynomial.
- Also this polynomial is called **indicial polynomial.**

Algorithm summary.

INPUT: A linear ordinary differential equation with polynomial coefficients.

OUTPUT: A basis of the vector space of all its rational function solutions.

1. Determine the factors $p_1, \dots, p_n$ of the leading coefficient.
2. For each p_i, determine the maximum possible multiplicity e_i.
3. Make an ansatz $f(x) = u(x)/\big(p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}\big)$.
4. Determine an ODE for the numerator $u(x)$.
5. Find the polynomial solutions $u(x)$ of this equation.
6. Return the corresponding rational functions $f(x)$.

Definition.
$f(x)$ is called *hyperexponential* if there are polynomials $p(x), q(x)$ with

$$p(x)f'(x) - q(x)f(x) = 0.$$

Hyperexponential terms can be written in the form

$$f(x) = \exp\Big(\mathrm{rat}(x)\Big) \prod_{i=1}^{k} \mathrm{pol}_i(x)^{\gamma_i}$$

$\approx$ exponential part **$\approx$ rational part**

More precisely:

Definition.

- Two hyperexponential terms $f(x)$ and $g(x)$ are called *similar* if $f(x)/g(x)$ is a rational function.
- The equivalence classes of hyperexponential terms under this relation are called *exponential parts.*

Examples.

$$x^{\sqrt{2}}(x+1) \sim x^{\sqrt{2}+4}(x+1)^{-3} \qquad x^{\sqrt{2}} \not\sim x^2 \qquad x^2 \not\sim \exp(x).$$

GIVEN: A linear ordinary differential equation with polynomial coefficients.

- For example

$$\begin{aligned}&(6x^5 - 60x^4 + 225x^3 - 386x^2 + 301x - 84)f(x)\\&\quad + (x-1)^2(10x^5 - 86x^4 + 277x^3 - 411x^2 + 272x - 59)f'(x)\\&\qquad + (x-2)^2(x-1)^4(2x^2 - 8x + 7)f''(x) = 0.\end{aligned}$$

FIND: its hyperexponential solutions.

- In the example, there are two hyperexponential solutions $\exp\left(\frac{x-3}{(x-1)(x-2)}\right)$ and $\exp\left(\frac{1}{x-1}\right)\frac{x^3-3x^2+2x-1}{(x-1)^3}$. (Here, all solutions can be written as linear combinations of hyperexponential terms. In general, this is not possible.)

The problem is **easy** if we prescribe a specific exponential part.

For example, suppose we want to find solutions of the form $f(x) = \exp(\frac{1}{x-1})u(x)$, where $u(x)$ is a rational function.

Plug $f(x) = u(x)\exp\big(\frac{1}{x-1}\big)$ into the differential equation, divide by $\exp\big(\frac{1}{x-1}\big)$, and clear denominators. This gives the equation

$$\begin{aligned}&(x-2)^2(x-1)^4(2x^2-8x+7)u''(x)\\&+(x-1)^2(10x^5-90x^4+309x^3-505x^2+392x-115)u'(x)\\&-(8x^3-50x^2+92x-53)(x-1)u(x)=0.\end{aligned}$$

Find its rational solutions. This gives $u(x) = \frac{x^3-3x^2+2x-1}{(x-1)^3}$.

In order to find **all** hyperexponential solutions, we need to know which exponential parts can occur.

Fact. There is a way to compute the *"local solutions"* of a given ODE at a given point ξ.

These are series expansions of the form

$$\exp\Big(\frac{p(x)}{(x-\xi)^d}\Big)(x-\xi)^\alpha\Big(1+c_1(x-\xi)+c_2(x-\xi)^2+\cdots\Big),$$

where $d \in \mathbb{N}$, $p(x)$ is a polynomial of degree $< d$, and $\alpha, c_1, c_2, \ldots$ are constants.

In order to find **all** hyperexponential solutions, we need to know which exponential parts can occur.

Fact. There is a way to compute the *"local solutions"* of a given ODE at a given point ξ.

Example. For the ODE above and $\xi = 1$, we get

$$\exp\left(\tfrac{2}{x-1}\right)\Big(1 + (x-1) + \tfrac{3}{2}(x-1)^2 + \tfrac{13}{6}(x-1)^3 + \cdots\Big)$$
$$\exp\left(\tfrac{1}{x-1}\right)\Big((x-1)^{-3} + (x-1)^{-2} - 1 + \Big)$$

In order to find **all** hyperexponential solutions, we need to know which exponential parts can occur.

Fact. There is a way to compute the *"local solutions"* of a given ODE at a given point ξ.

Example. For the ODE above and $\xi = 2$, we get

$$\exp\left(\tfrac{-1}{x-2}\right)\Big(1 - 2(x-2) + 4(x-2)^2 - \tfrac{22}{3}(x-2)^2 + \cdots\Big)$$
$$\exp(0)\Big(1 - 6(x-2) + \tfrac{31}{2}(x-2)^2 - \tfrac{98}{3}(x-2)^3 + \cdots\Big)$$

In order to find **all** hyperexponential solutions, we need to know which exponential parts can occur.

Fact. There is a way to compute the *"local solutions"* of a given ODE at a given point ξ.

Fact. The exponential parts of a hyperexponential solution are *combinations* of exponential parts of local solutions at roots of the leading coefficient of the equation.

Example. For the ODE above, there are four candidates:

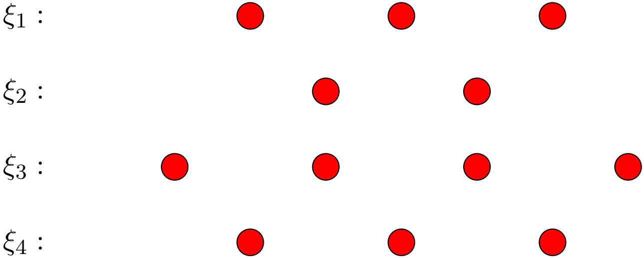
$$\begin{array}{ll} \exp\left(\frac{2}{x-1} - \frac{1}{x-2}\right) & \exp\left(\frac{2}{x-1} + 0\right) \\ \exp\left(\frac{1}{x-1} - \frac{1}{x-2}\right) & \exp\left(\frac{1}{x-1} + 0\right). \end{array}$$

Algorithm summary

INPUT: A linear ordinary differential equation with polynomial coefficients.

OUTPUT: A list of its hyperexponential solutions.

1. Let $\xi_1, \xi_2, \dots$ be the roots of the leading coefficient.
2. For each ξ_i, compute the exponential parts $\exp\left(\frac{p_j}{(x-\xi_i)^{d_j}}\right)$ $(j = 1, 2, \dots)$ of the local solutions at ξ_i.
3. For each combination $E := \exp\left(\frac{p_{j_1}}{(x-\xi_1)^{d_{j_1}}} + \frac{p_{j_2}}{(x-\xi_2)^{d_{j_2}}} + \cdots\right)$ do:
4. Make an ansatz $f(x) = u(x)\,E$
5. Construct an auxiliary equation for $u(x)$
6. Find its rational solutions
7. For each solution $u(x)$, output $f(x) = u(x)\,E$.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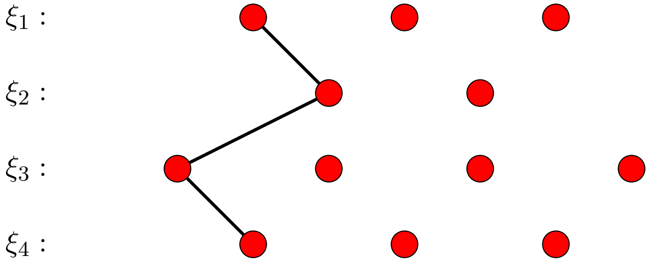
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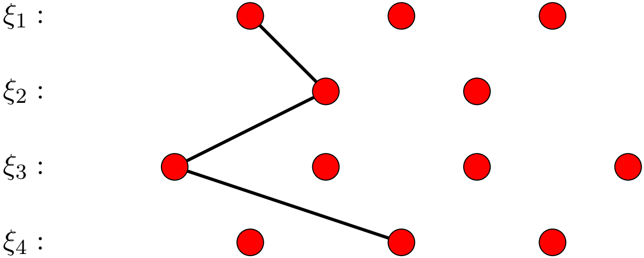
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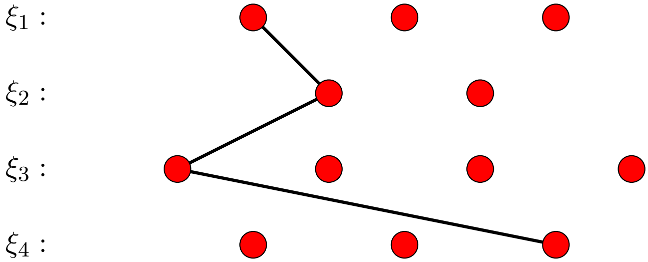
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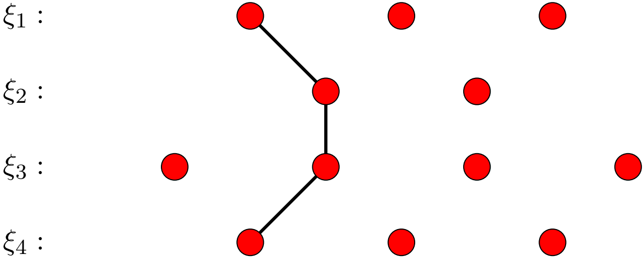
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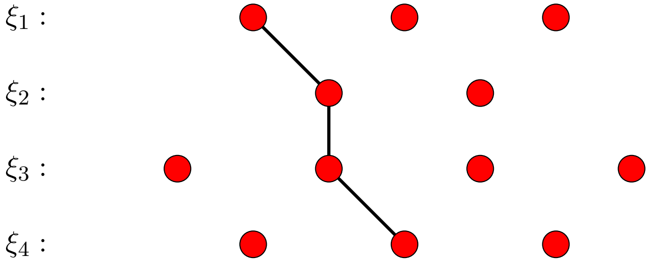
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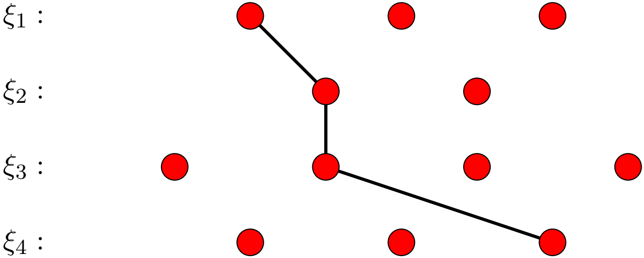
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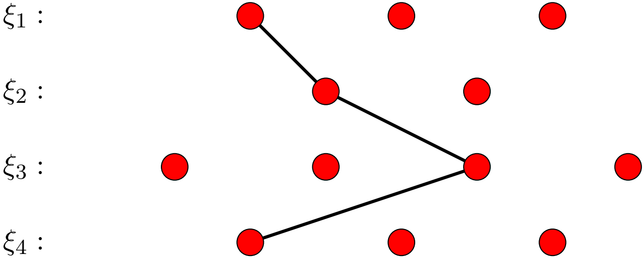
▸ Hyperexponential Solutions



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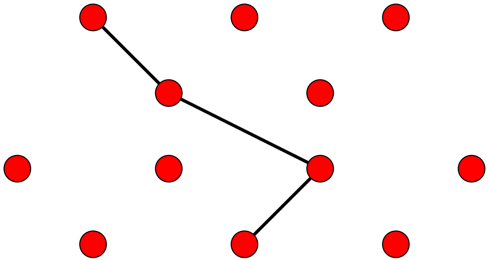

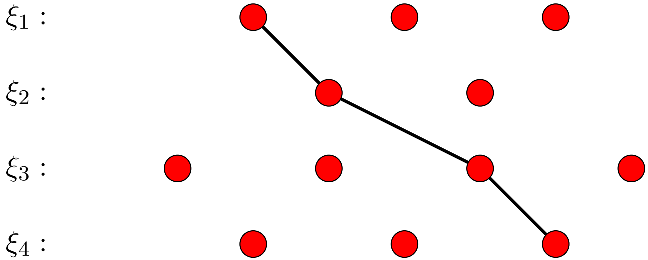

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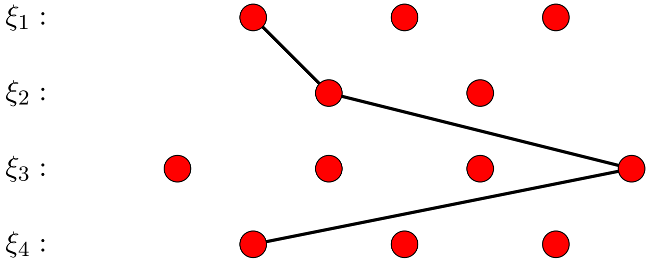
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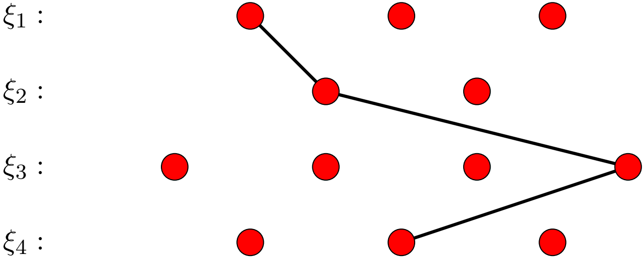
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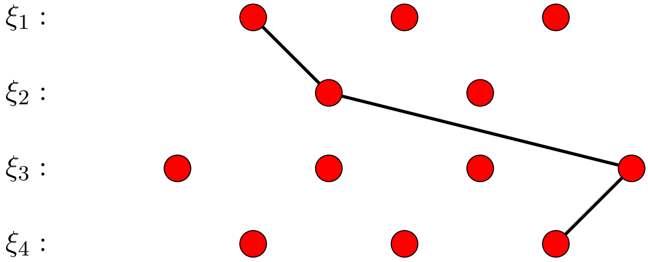
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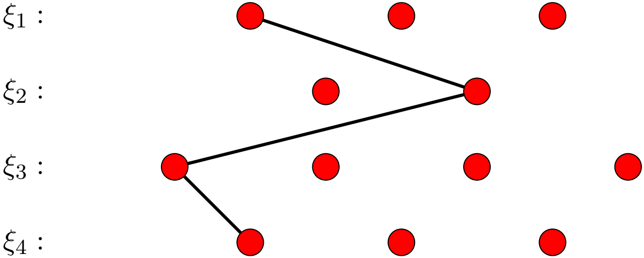
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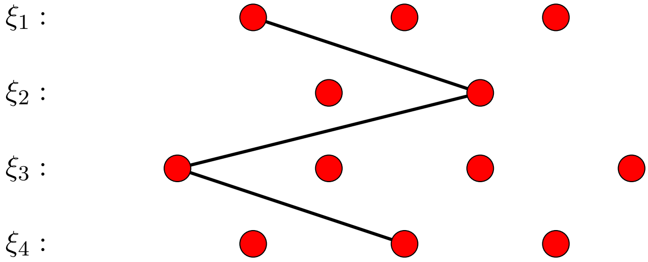
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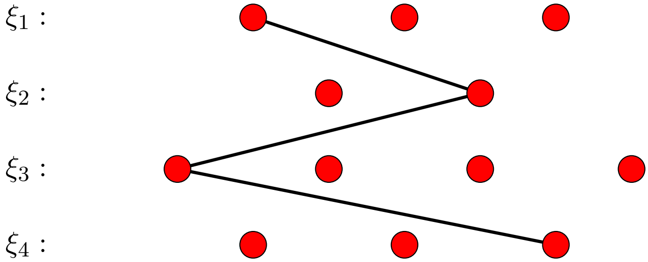
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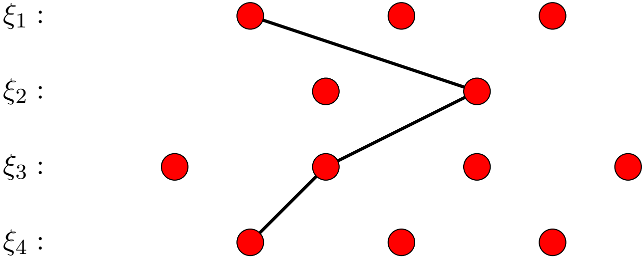
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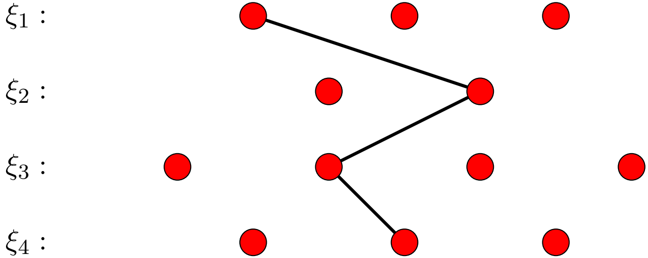
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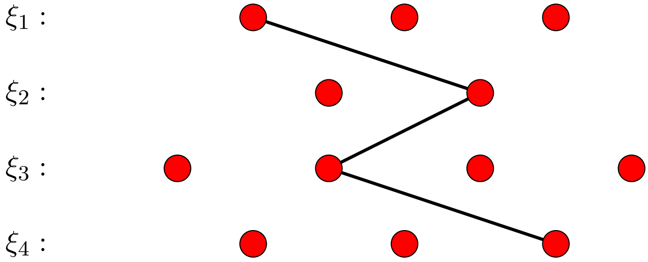
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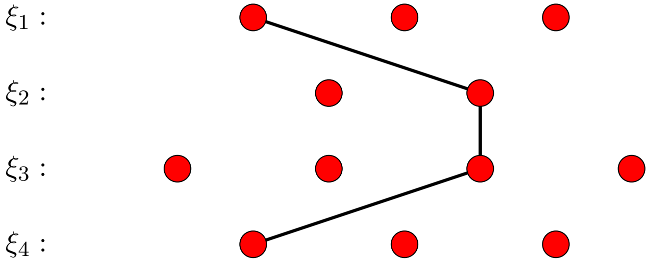
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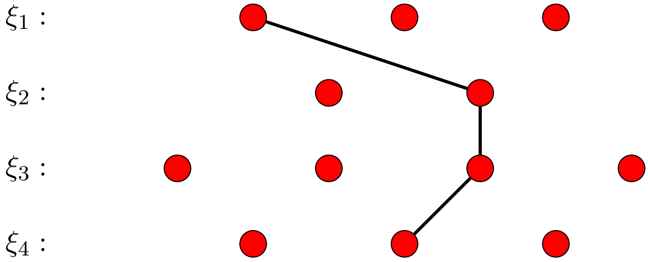
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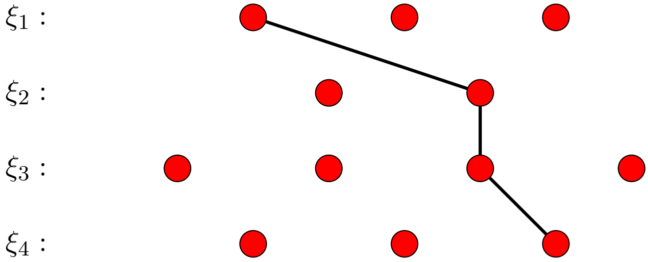

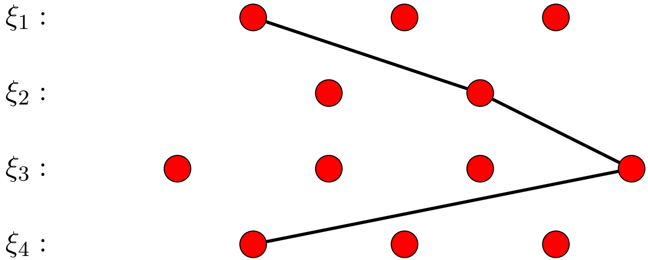
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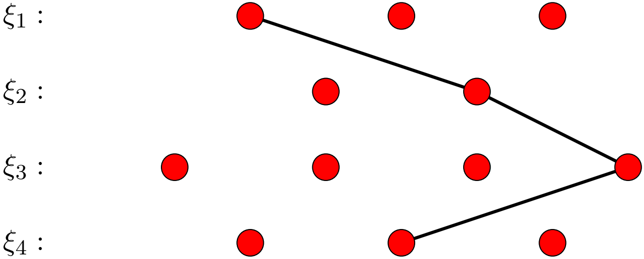
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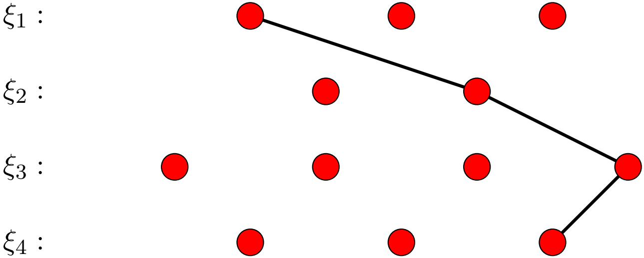
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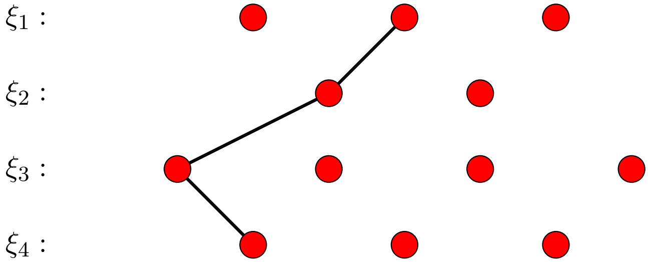
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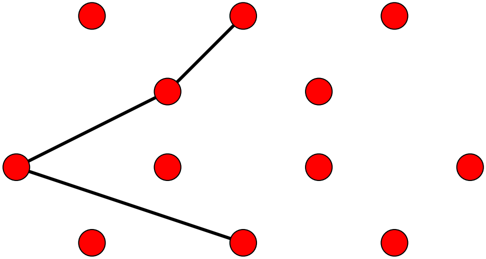

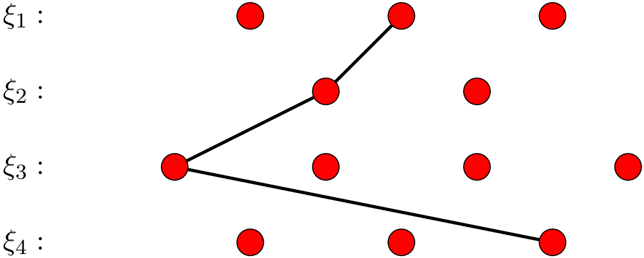
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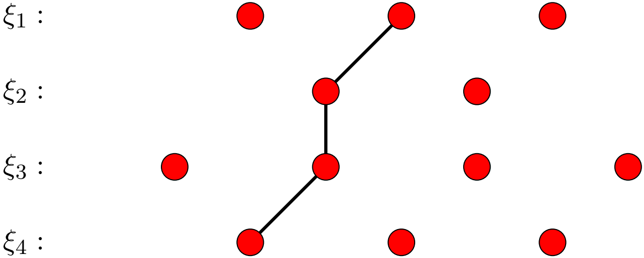
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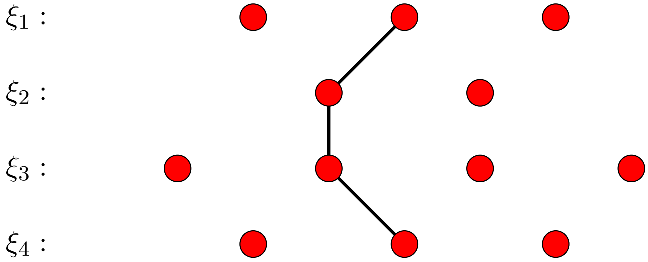
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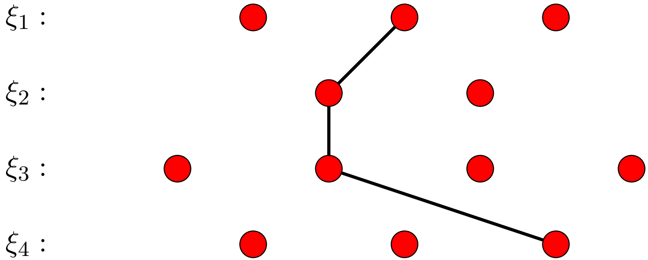
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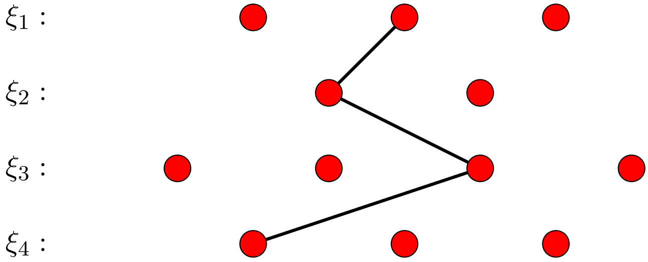
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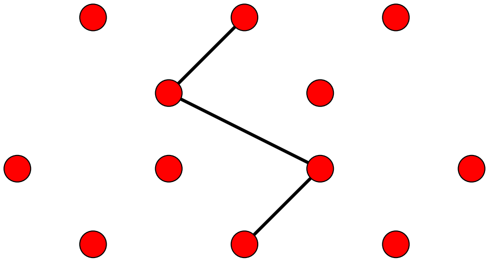

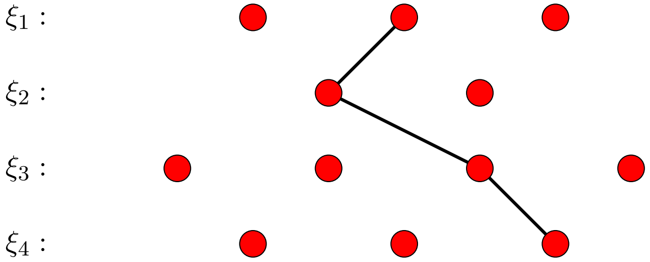
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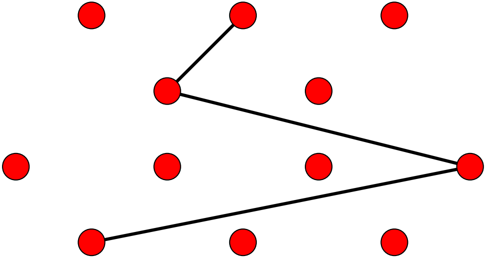

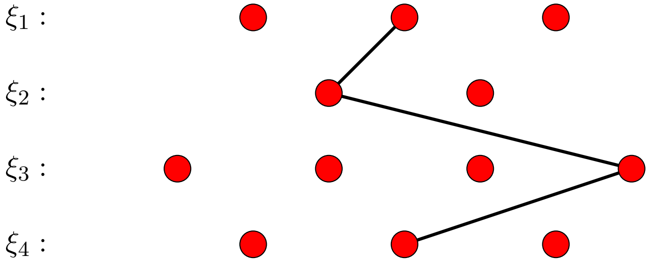
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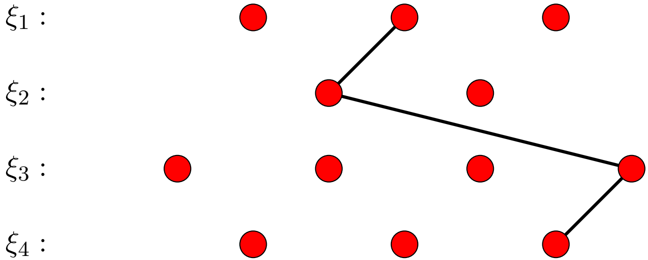
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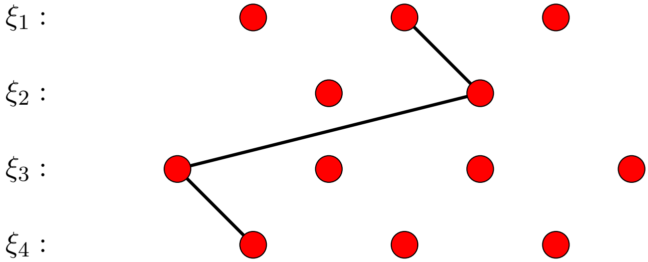
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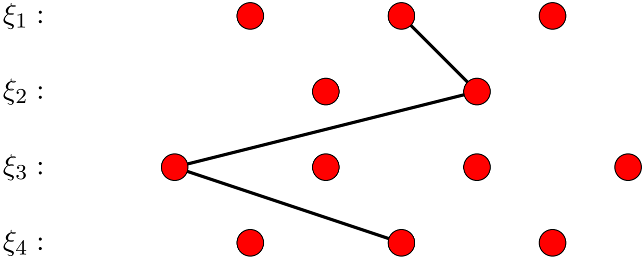
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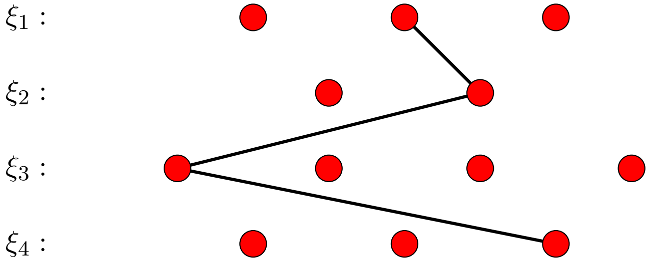

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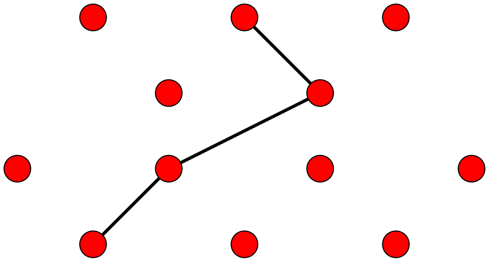

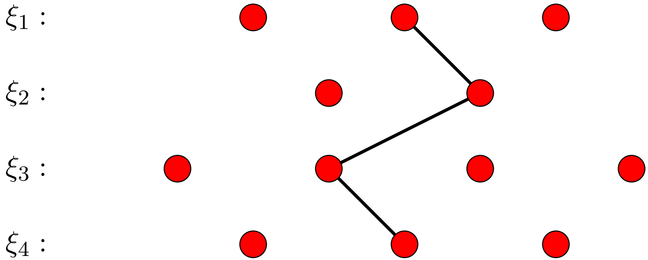
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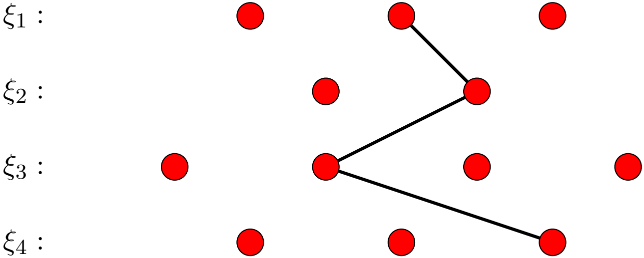
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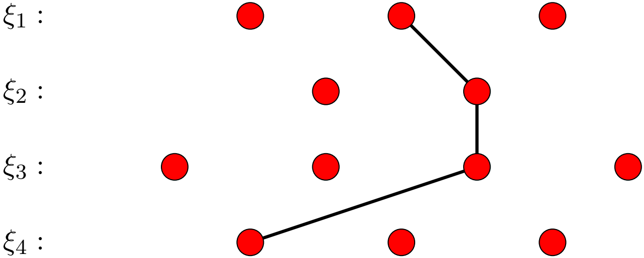
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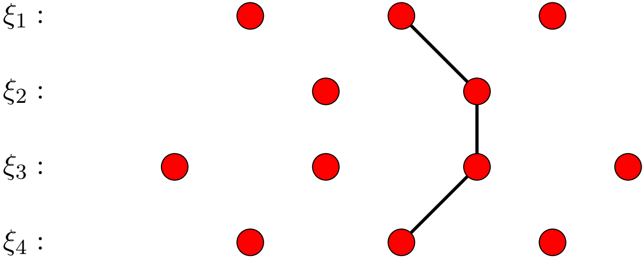

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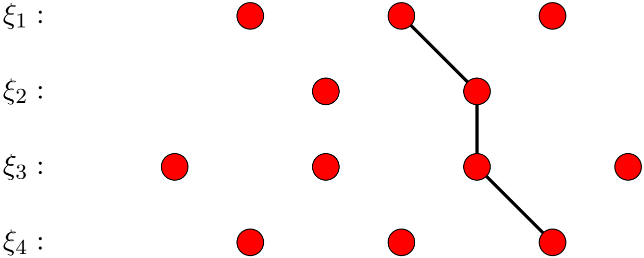
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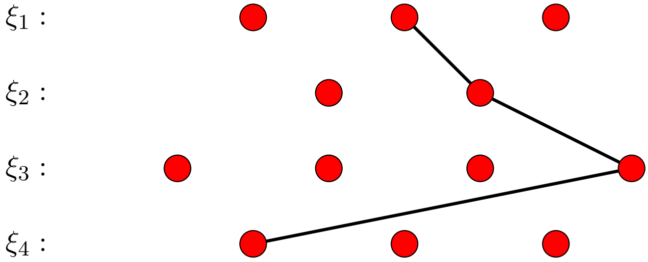
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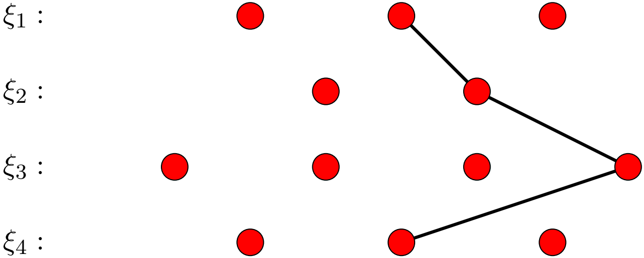

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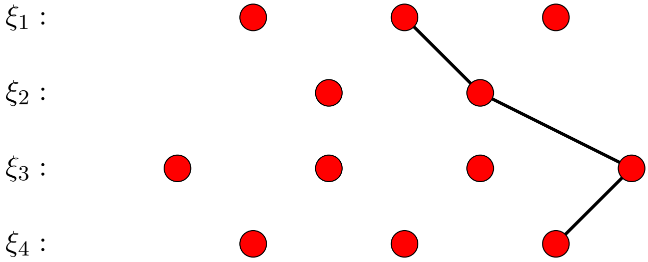
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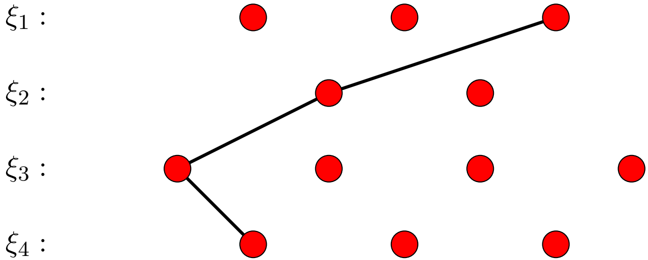

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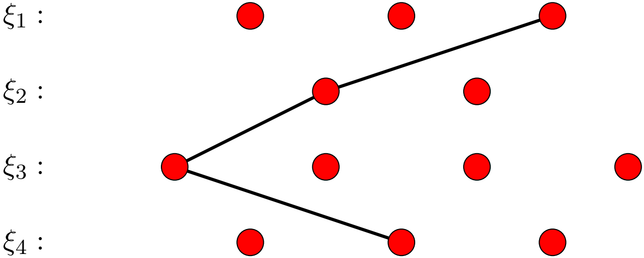
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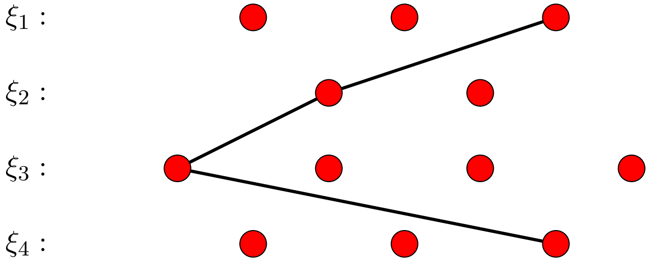
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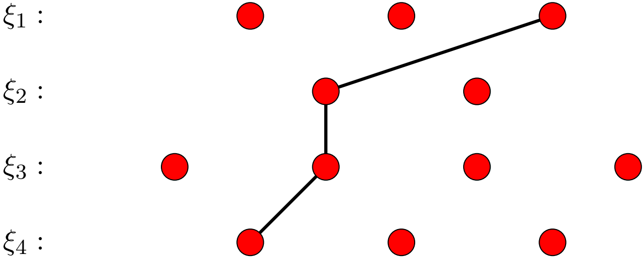
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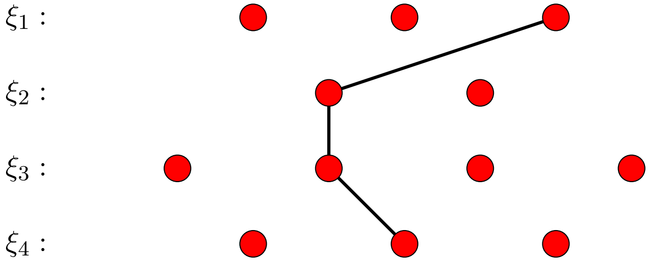

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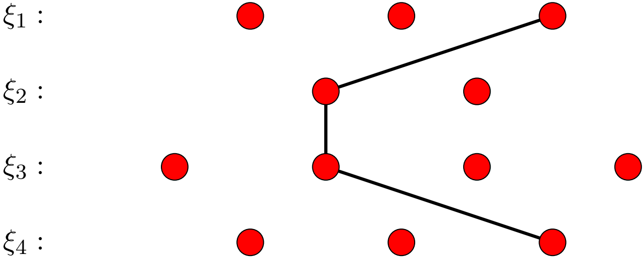
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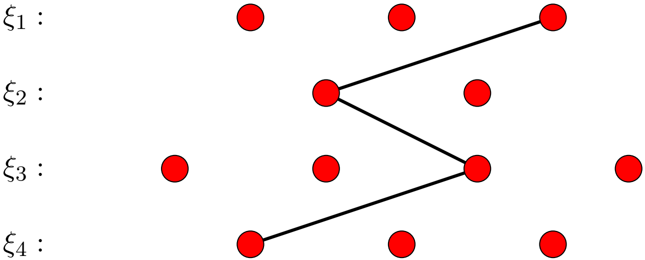
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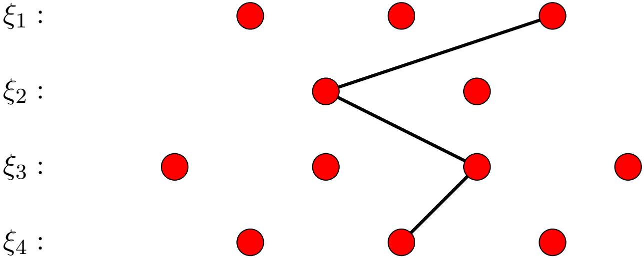
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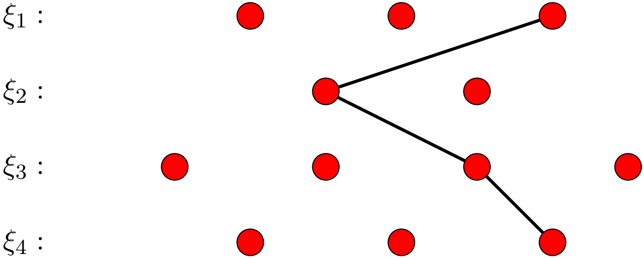
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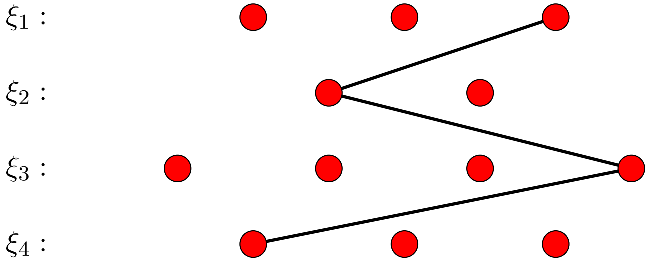
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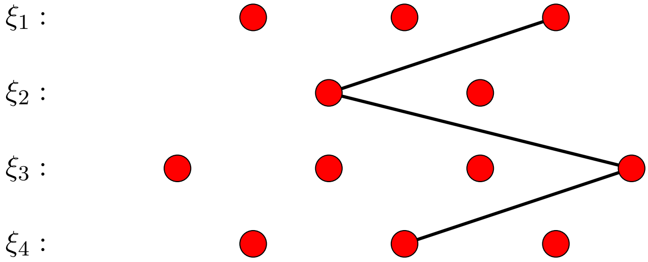
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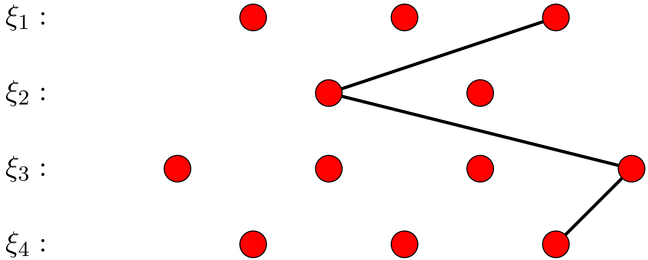
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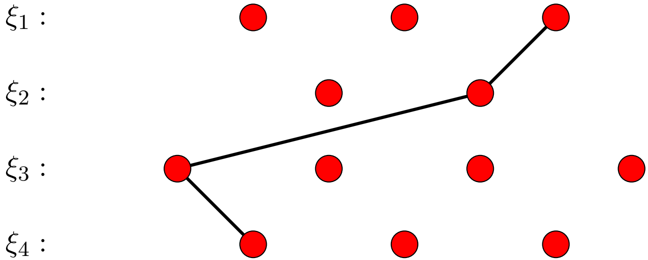
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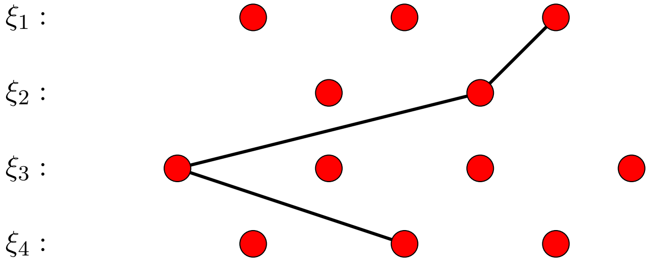
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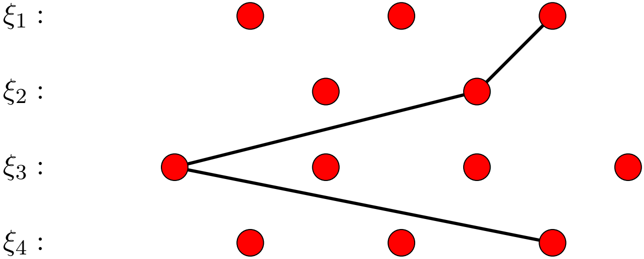
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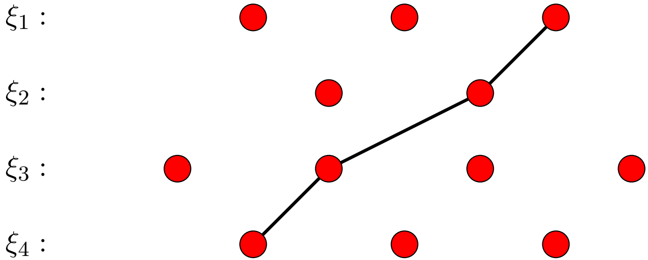
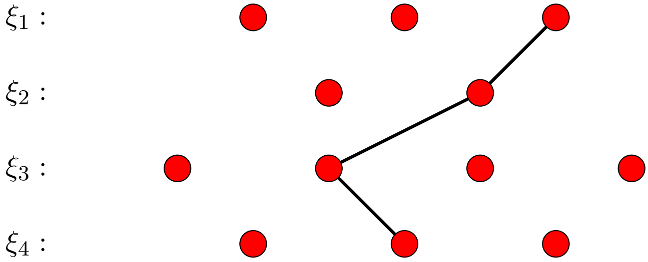
▸ Hyperexponential Solutions

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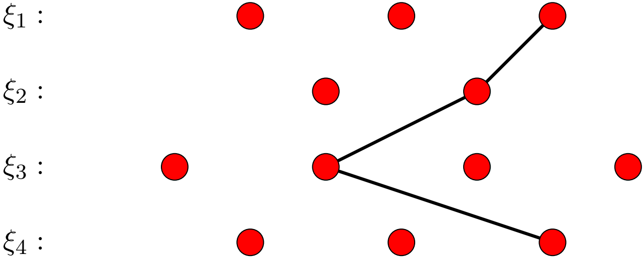
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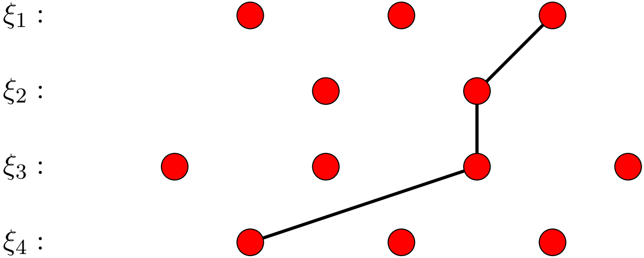
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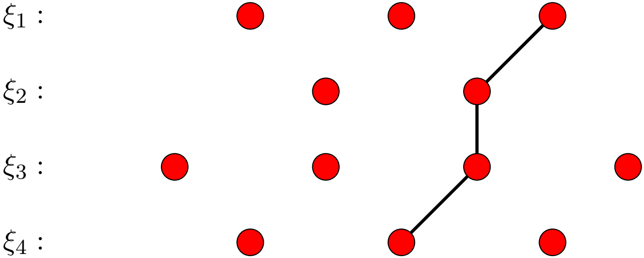
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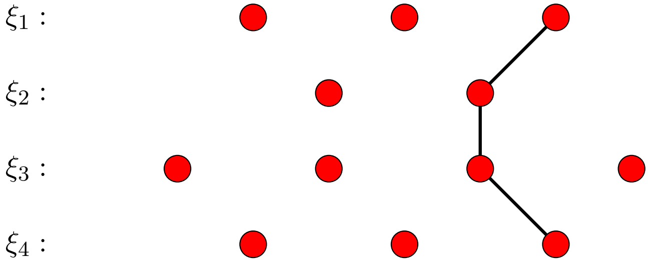
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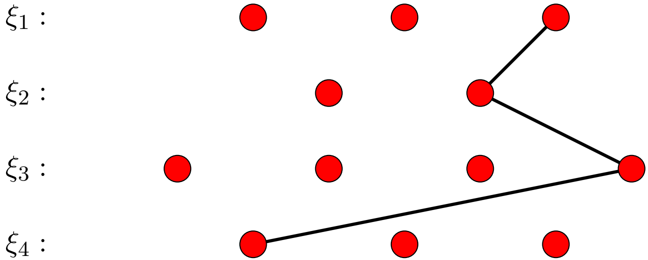
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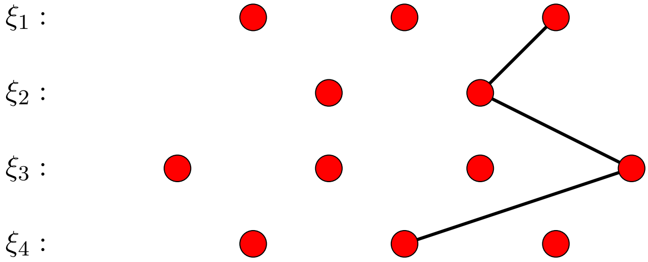
ξ_1 :
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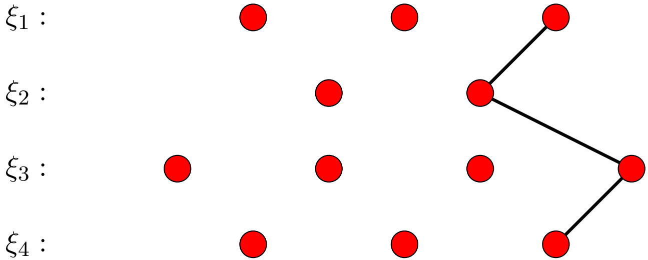
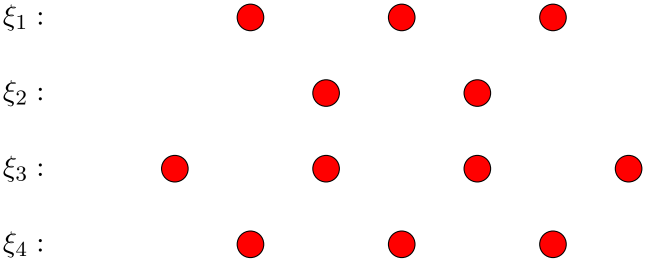
▸ Hyperexponential Solutions

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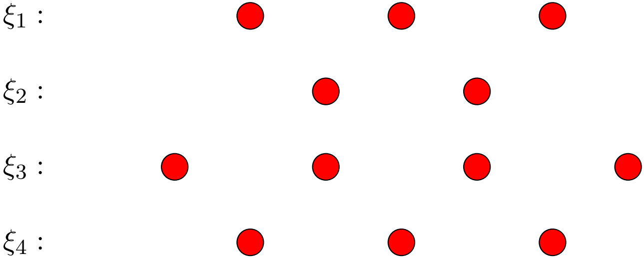
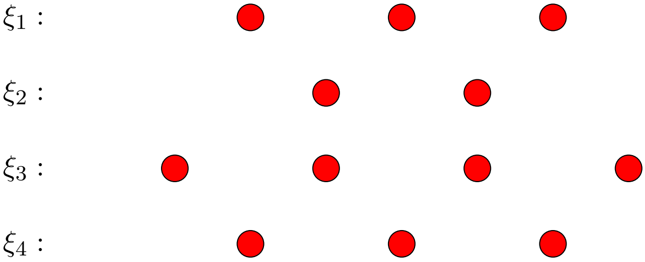


For an order r equation with n singular points, there are r^n combinations.



For an order r equation with n singular points, there are r^n combinations. **That's a lot.**

NEW

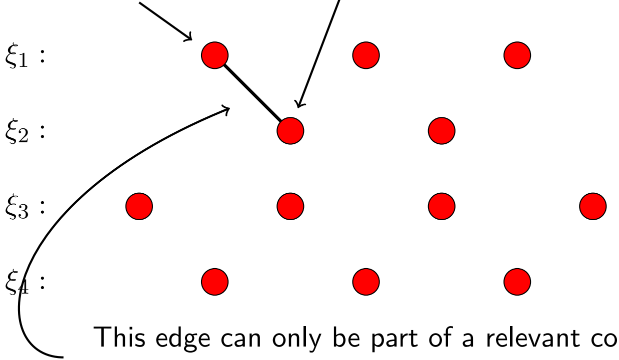
Our contribution (Johansson, MK, Mezzarobba; ISSAC'13):

- An algorithm for quickly finding the relevant combinations.
- Returns at most r candidates (instead of r^n).
- Needs at most $n^4 r$ arithmetic operations to find them.
- Is based on the principle of *dynamic programming.*
- Also requires *effective analytic continuation.*

NEW

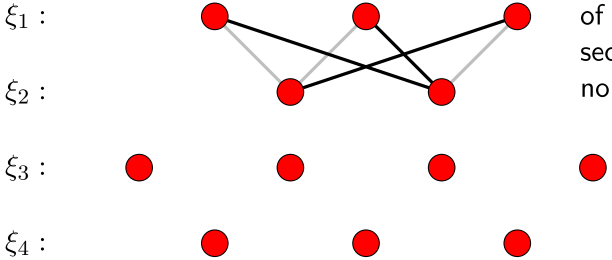
vector space of all
series solution at ξ_1 with
a certain exponential part

vector space of all
series solution at ξ_2 with
a certain exponential part



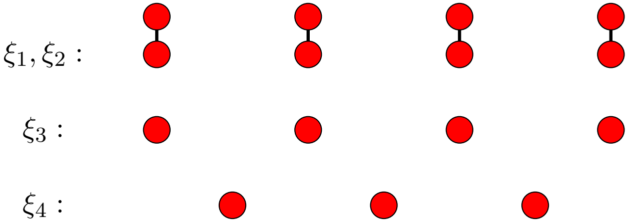
This edge can only be part of a relevant combination
if the intersection of the two vector spaces is nonempty

NEW



Fact. At most r of these $O(r^2)$ intersections can be nonempty.

NEW



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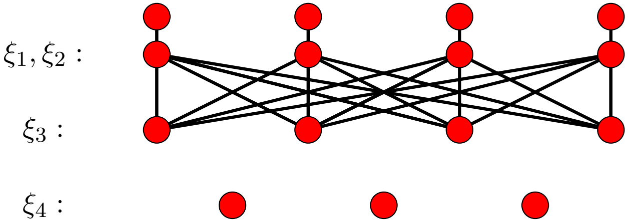

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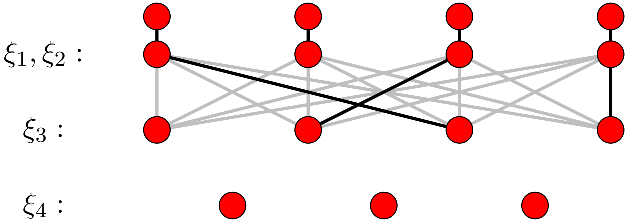
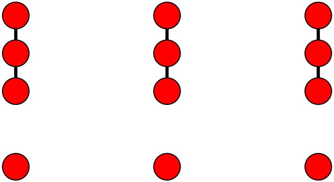


NEW

ξ_1, ξ_2, ξ_3 :

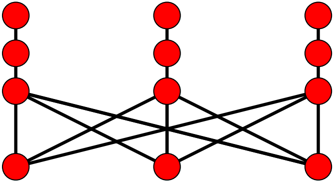
ξ_4 :



NEW

$\xi_1, \xi_2, \xi_3 :$

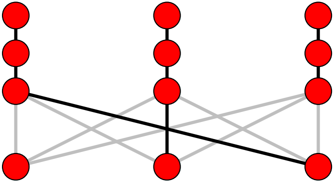
$\xi_4 :$



NEW

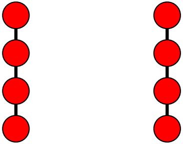
ξ_1, ξ_2, ξ_3 :

ξ_4 :



NEW

$\xi_1, \xi_2, \xi_3, \xi_4$:



NEW

How to carry out the required vector space intersections?

A priori, spaces for different ξ_i are not comparable.

Example: What is

$$\begin{aligned} &\Big[\exp\big(\tfrac{1}{x-1}\big)P_1(x-1), \exp\big(\tfrac{1}{x-1}\big)P_2(x-1)\Big] \\ \cap\ &\Big[\exp\big(\tfrac{1}{x-2}\big)Q_1(x-2), \exp\big(\tfrac{1}{x-2}\big)Q_2(x-2)\Big] \end{aligned}$$

supposed to mean?

Idea: Interpret the series as asymptotic expansions of actual complex functions, and determine their expansions at some fixed common reference point using certified numerical approximation.

NEW

How to carry out the required vector space intersections?

A priori, spaces for different ξ_i are not comparable.

Example: What is

$$\begin{aligned} &\left[\tilde{P}_1(x-0), \tilde{P}_2(x-0)\right] \\ \cap \ &\left[\tilde{Q}_1(x-0), \tilde{Q}_2(x-0)\right] \end{aligned}$$

supposed to mean?

Idea: Interpret the series as asymptotic expansions of actual complex functions, and determine their expansions at some fixed common reference point using certified numerical approximation.

NEW

How to carry out the required vector space intersections?

A priori, spaces for different ξ_i are not comparable.

Example: What is

$$\Big[\tilde{R}_1(x-0), \tilde{R}_2(x-0)\Big]$$

supposed to mean?

Idea: Interpret the series as asymptotic expansions of actual complex functions, and determine their expansions at some fixed common reference point using certified numerical approximation.

This is not an easy thing to do, but efficient algorithms for this task are known.