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Better-quasi-orderings and coinduction

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Abstract

We can characterise the class of BQOs as the largest class of well-founded quasiorders closed under the Hoare powerdomain construction and colimits.

1. Introduction

Computer scientists do not need to be reminded of the importance of wellfounded relations in their subject: their utility in proofs of termination is enough by itself to compel attention. The typical way for a wellfounded relation to arise is from declarations of recursive datatypes, but some seem to have different roots, and an important class of relations that can be wellfounded (or have natural wellfounded parts) is the class of wellquasiorders, and a special subclass of that family is the class of better quasiorders.

The aim of this note is to present an alternative definition of BQOs which has an algebraic flavour rather than the combinatorial flavour of the standard definition. In order to accommodate all the needed motivation and to make the paper self-contained, the material will be given *in extenso*—but even so the result will not be the lo-stress introduction to WQO theory and BQO theory for which the world has been waiting because the only elementary topics it treats are those needed to motivate the new definition. Various other introductions are available in the literature, for example Laver [3] and Prömel and Voigt [8].

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2. Definitions and notation

If X is a set, $\mathscr{P}(X)$ is the power set of X, $\mathscr{P}_{\aleph_0}(X)$ is the set of finite subsets of X and $\mathscr{P}_{\aleph_1}(X)$ is the set of countable subsets of X.

A binary relation that is transitive and reflexive will here be called a *quasiorder*. (The expressions "quasi-order", "quasiordering" and "preorder" are also to be seen in the literature). A *partial order* is a quasiorder that is antisymmetric. $\langle Q, \leqslant_Q\rangle$ is a *complete partial order* if every subset has a lub. $\langle Q, \leqslant_Q\rangle$ is a *chain-complete poset* if every chain has a lub. An antichain (in a poset) is set of elements no two of which are $\leqslant$-comparable.

The intersection of a quasiorder with its converse is an equivalence relation and will be called the *corresponding* equivalence relation. The quotient is a partial order and will be called the *corresponding* partial order.

Why bother with quasiorders? Why not just deal with the corresponding quotient partial order? Antisymmetry makes life so much easier! The answer is that although for most naturally occurring quasiorders the corresponding strict partial order is an even more natural object and gives rise to a sensible degree of abstraction there are cases where this is not so. For example order types of total orders do not form a partial order under embedding: each of the open and closed intervals $(0,1)$ and $[0,1]$ embeds in the other but they are not isomorphic and the corresponding equivalence relation does not seem particularly illuminating. Two further examples of quasiorders that are not obviously antisymmetrical are the graph minor and subgraph relations on the class of infinite graphs. There is another reason which will look mysterious at this stage, but to which we will return: the poset of partial orders under inclusion is chain-complete but not complete (a pair of partial orders that disagree has no upper bound) but the poset of quasiorders is complete.

The relation $x \leqslant y \not\leqslant x$ is *the strict part* of $\leqslant$ and will be written '$x<y$'. We will continue with the widespread bad habit of referring to a reflexive relation as wellfounded when we really mean that its strict part is wellfounded (when this is obvious from context).

Although '$\leqslant$' is usually used for partial orders, we will here use it for quasiorders as well. Worse still, in what follows we will use '$<$' for both the (strict part of the) quasiorder relation *and* the strict order on $\mathbb{N}$. The reader is warned!

We will assume the axiom of dependent choices throughout.

3. Lifts

There are many ways of lifting quasiorders or partial orders from a set Q to $\mathscr{P}(Q)$ or the set of partitions of Q, or lists, multisets, streams over Q, etc. etc. Let us collect some of them here. (See also Marcone [6]).

First come three ways to lift a quasiorder from a set to its power set.

Definition 1. Let $\langle Q, \leqslant, \rangle$ be a quasiorder. For Y and Z subsets of Q say
$Y \leqslant^+ Z$ iff $(\forall y \in Y)(\exists z \in Z)(y \leqslant z)$,

$Y \leqslant^* Z$ iff $(\forall z \in Z)(\exists y \in Y)(y \leqslant z)$,
$Y \leqslant_1 Z$ iff there is an injection $f : Y \hookrightarrow Z$ such that $(\forall y \in Y)(y \leqslant f(y))$.

Notice that these three constructors preserve reflexivity and transitivity. The first two preserve connexity ($\leqslant$ is *connected* if $(\forall xy)(x \leqslant y \vee y \leqslant x)$) but the third does not. Among computer scientists the first is commonly known as the *Hoare* powerdomain construction. The second is the dual construction called the *Smyth* powerdomain construction. The third relation will play no further part here.

Notice that for any quasiorder $\langle Q, \leqslant \rangle$, $\langle \mathscr{P}(Q), \leqslant^+ \rangle$ is what one might call a *complete quasiorder*, which is to say that every subset of $\mathscr{P}(Q)$ has sups and infs that are unique up to the corresponding equivalence relation. However it turns out that this is not the notion of complete quasiorder that we need.

The operations in Definition 1 are all monotone operators on the CPO of all quasiorders of V. The least fixed point of $+$ is '$\rho(x) \leqslant \rho(y)$', where ρ is set-theoretic rank. We shall meet its greatest fixed point too, but later.

In computer science ω-sequences from Q are usually called *streams* and finite sequences *lists*. I shall use that terminology here, and I shall write the set of Q-streams as Q^ω and the set of Q-lists as $Q^{<\omega}$. There is an obvious way of lifting a quasiorder on a set Q to a quasiorder on Q-lists or Q-streams. Following Mathias (unpublished) we use the word *stretching* to denote the relation that holds between two Q-lists (or Q-streams) l_1 and l_2 if there is a 1-1 increasing map f from the addresses of l_1 to the addresses of l_2 such that for all addresses a, $a \leqslant f(a)$. We write this '$l_1 \leqslant_l l_2$' with a subscript 'l' for '*l*ist', and we say l_1 *stretches* into l_2. Stretching on lists has an inductive definition and stretching on streams has a coinductive definition, though we will not be making much use of these facts.

The list constructor has finite character. We shall see later that it preserves wellfoundedness but it does not preserve connexity: $\langle \mathbb{N}, \leqslant \rangle$ is a total ordering but neither of the two-membered lists [1,2] and [2,1] stretch into the other.

If Q is a quasiorder, a Q-tree can be thought of as a kind of lower-semilattice where no two incomparable points have a common upper bound, equipped with labels from Q. Say $T_i \leqslant_t T_j$ if there is an injective lower-semilattice homomorphism f from the 'skeleton' of T_i to the skeleton of T_j such that the label at some node n of T_i is $\leqslant$ the label at node $f(n)$ of T_j.

We classify lifts into those with *finite character* and those without. The idea is that on the whole the constructors of finite character preserve wellfoundedness, but that those of infinite character do not. Virtue is to be gained by meditating on how to strengthen wellfoundedness appropriately into a stronger condition that is preserved by the constructors of infinite character.

The following lifts (among others) have finite character:
(i) $Q \mapsto Q$-lists under stretching;
(ii) $Q \mapsto$ finite Q-trees under $\leqslant_t$;
(iii) $Q \mapsto \langle \mathscr{P}_{\aleph_0}(Q), \leqslant^* \rangle$;
(iv) $Q \mapsto \langle \mathscr{P}_{\aleph_0}(Q), \leqslant^+ \rangle$.

Let us try to prove that these lifts preserve wellfoundedness. The claims range from easy-and-obvious to downright false.

(i) If $\langle Q, \leqslant \rangle$ is a wellfounded quasiorder, then Q-lists are wellfounded under stretching. Suppose not, and we had an infinite descending sequence of Q-lists under stretching. They can get shorter only finitely often, so without loss of generality we may assume that they are all the same length. But the entries at each coefficient can get smaller only finitely often, so they must eventually be constant.

(ii) Suppose $\langle Q, \leqslant \rangle$ is a wellfounded quasiorder and let $\langle t_i\colon\ i < \omega \rangle$ be a descending $>_t$-sequence of Q-trees. We will derive a contradiction. The number of children of t_i is a nonincreasing function of i and must be eventually constant: indeed the trees will be of eventually constant shape, and we can delete the initial segment of the sequence where they are settling down. Because the shape is eventually constant there are unique maps at each stage, so for any one address the sequence of elements appearing at that address gets smaller as i gets bigger.

(iii) Suppose $Q = \langle q_i\colon\ i \in \mathbb{N} \rangle$ is the identity quasiorder on a countable set. Then if we set $Q_i =: \{q_0 \dots q_i\}$ we find that $Q_i >^* Q_{i+1}$ for all i.

(iv) Suppose we have an infinitely descending sequence $\langle Q_i\colon\ i \in \mathbb{N} \rangle$ of finite subsets of Q under $<^+$. Without loss of generality we can assume that all the Q_i are antichains, by throwing away from each Q_i all elements that are not maximal. This will ensure that any x that appears in both Q_i and in Q_j with $j > i$ must appear in all intermediate levels: if $x \in Q_j$ then it must be $\leqslant$ something in Q_{j-1} and so on up to Q_i. Since Q_i is an antichain this thing can only be x itself (or something equivalent to it, which will do!) So any x that appears in infinitely many Q_i must appear in cofinitely many of them. But then it can be deleted altogether. So we can assume that each q appears in at most finitely many Q_i.

For each $x \in Q_0$ we can build a tree whose paths are sequences s where the ith representative comes from Q_i and for all i, $s(i+1) \leqslant s(i)$. We need to show that all these paths are finite. If they were not, they would have to be eventually constant, and we have just seen that we can assume that each q can be assumed to appear only finitely often. So the tree whose paths are these sequences is a finite branching tree all of whose paths are finite, so it has only finitely many levels. But there are only finitely many things in Q_0, so eventually the Q_i are empty. $\square$

Our counterexample to (iii) is useful in connection with $\leqslant^+$ too. It enables us to show that $\langle \mathscr{P}(Q), \leqslant^+ \rangle$ need not be wellfounded even if $\langle Q, \leqslant \rangle$ is. Take the same quasiorder, namely $\langle \mathbb{N}, = \rangle$. Set $Q_n =: \{m \in \mathbb{N}\colon\ m >_{\mathbb{N}} n\}$. Then $\langle Q_n\colon\ n > 0 \rangle$ is an infinite descending sequence under $=^+$. Clearly the source of the trouble is the infinite antichain. This bears thinking about, and will lead us to a new definition.

4. WQOs

It seems that wellfounded quasiorders without infinite antichains are going to be objects of interest, since it seems that—and we will prove this in remark 5—it is the absence of infinite antichains in a wellfounded quasiorder $\langle Q, \leqslant \rangle$ that enables us to show that $\langle \mathscr{P}(Q), \leqslant^+ \rangle$ is wellfounded.

This motivates the following definition:

Definition 2. $\langle Q, \leqslant \rangle$ is a *wellquasiorder* (hereafter 'WQO') iff whenever $\langle x_i \colon i \in \mathbb{N} \rangle$ is an infinite sequence of elements from Q then there are $i < j \in \mathbb{N}$ s.t. $x_i \leqslant x_j$.

A natural example of a WQO is the set of (unordered) pairs of natural numbers with $\{x, y\}$ related to $\{n, m\}$ if $\max(x, y) \leqslant \min(n, m)$. It is the wellfoundedness of this quasiorder that ensures termination of Euclid's algorithm.

Definition 3. A *bad sequence* (over $\langle Q, \leqslant \rangle$) is a sequence $\langle x_i \colon i \in \mathbb{N} \rangle$ such that for no $i < j$ is it the case that $x_i \leqslant x_j$. A sequence that is not bad is *good*. A sequence $\langle x_n \colon n \in \mathbb{N} \rangle$ is *perfect* if $i \leqslant j \to x_i \leqslant x_j$.

Thus a wellquasiorder is a quasiorder with no bad sequences. With the help of Ramsey's theorem we can prove that in a WQO not only is every sequence good but that it must have a perfect subsequence. (Notice that this is not the same as saying that in any quasiorder every good sequence has a perfect subsequence!)

Now some facts with an algebraic flavour, but without proof.

Remark 4.

Substructures of WQOs are WQO;
Homomorphic images of WQOs are WQO;
The pointwise product of finitely many WQOs is WQO;
The intersection of finitely many WQOs is WQO;
Disjoint unions of finitely many WQO are WQO.

However the class of WQOs is not closed under direct limits, even under end-extension. Consider the following sequence of Hasse diagrams of WQOs:

```
                                                .   x
                                                 \ /
                                .   x         .   .
                                 \ /           \ /
                   .   x      .   .         .   .
                    \ /        \ /           \ /
       .   x     .   .      .   .         .   .
        \ /       \ /        \ /           \ /
x        .         .          .             .
```

Each WQO in this sequence is obtained from its neighbour to the left by replacing an 'x' by a

```
.   x
 \ /
  .
```

The direct limit is not a WQO.

This is in contrast to the situation with wellfounded structures, the class of which *is* closed under direct limits of end-extensions. This being the case, the forthcoming fact that a quasiorder is a wellquasiorder iff its power set is wellfounded may turn out to be useful.

It is easy to find an inverse limit of WQOs that is not WQO. Indeed we can find an inverse limit of wellfounded quasiorders that is not a wellfounded quasiorder. Let A_n be $\{0, 1, \ldots n\}$ in their natural order, and $\lambda m.(m \dot{-} 1) : A_{m+1} \twoheadrightarrow A_m$: then the inverse limit is of order type $1 + \omega^*$.

We have already checked that if $\leqslant$ is a quasiorder, so is $\leqslant^+$. We are now in a position to come clean on the authorial omniscience with which this section began.

Remark 5. Let $\langle Q, \leqslant \rangle$ be a quasiorder. Then $\langle Q, \leqslant \rangle$ is WQO iff $\langle \mathscr{P}(Q), \leqslant^+ \rangle$ is wellfounded.

Proof. (R)$\to$(L): If $\langle q_i\colon\ i \in \mathbb{N} \rangle$ is a bad sequence then set $Q_i =: \{q_j\colon\ j > i\}$ for each $i \in \mathbb{N}$. Then $Q_1 >^+ Q_2 >^+ Q_3 \ldots$.

(L)$\to$(R): Suppose $\langle Q, \leqslant \rangle$ is wellfounded, and let $Q_0 >^+ Q_1 >^+ \ldots$ be a $>^+$ descending chain of subsets of Q. We shall find an infinite antichain $\subseteq Q$. For each i pick $q_i \in Q_i \not\leqslant$ anything in Q_{i+1}. So in particular, we immediately have $q_i \not\leqslant q_{i+1}$. But since $Q_j <^+ Q_i$ for $j > i$ it follows that if $j > i$ we cannot have $q_i \leqslant q_j$ since q_j must be less than something in Q_{i+1}, and q_i has been chosen not to be $\leqslant$ anything in q_{i+1}. An application of Ramsey's theorem to the set $\{q_i\colon\ i \in \mathbb{N}\}$ gives either a set of representatives which form an infinite descending sequence under $<$, which is impossible by wellfoundedness, or an antichain, which was what we wanted. $\square$

What is appealing about this characterisation of WQOs in terms of the induced quasiorder on the power set being wellfounded is that it takes us away from talk about ω-sequences. There is a definition of wellfoundedness in terms of ω-sequences (the descending chain condition) and it is the *wrong* definition because in order to prove the recursion theorem from it one needs DC. One might hope that the characterisation of WQOs in terms of the induced order on the power set should have the same advantages over that in terms of ω-sequences as the correct definition of wellfoundedness has over the descending chain condition.

4.1. Finite character again

Now we can return to the constructors of finite character and ask whether or not they preserve WQO-ness.

First $\leqslant^+$. Let $\langle Q, \leqslant \rangle$ be a WQO and suppose $\langle Q_i\colon\ i \in \mathbb{N} \rangle$ is a bad sequence of finite subsets under $\leqslant^+$. For each $i > 0$ there is $x_{0,i} \in Q_0$ with $x_{0,i} \not\leqslant^+ y$ for all $y \in Q_i$. But because Q_0 is finite, infinitely many of these $x_{0,i}$ are the same. Pick x_0 from $\{x_{0,i}\colon\ i > 0\}$ such that for some infinite $J \subseteq \mathbb{N}^+$ we have $x_0 \not\leqslant^+ y$ for all $y \in Q_j$ and all $j \in J$. Discard all the Q_i whose subscripts are not in J and renumber, giving the first one the subscript 1. Now repeat what we have just done to obtain x_1, x_2 and so on.

Then we use DC to construct a bad sequence $\langle x_i\colon\ i \in \mathbb{N}\rangle$. So this constructor preserves WQO-ness.

Kruskal showed [2] that if $\langle Q, \leqslant\rangle$ is WQO then Q-lists under stretching and Q-trees under $\leqslant_t$ are both WQO too. Nash-Williams gave a beautiful proof of this fact (see Laver [4]) but it will not be treated here, since too many of the ideas needed for its development are not germane to the purpose of this note.

Next $\leqslant^*$. Suppose $\langle Q_i\colon\ i \in \mathbb{N}\rangle$ is a $>^*$-descending chain of finite subsets of Q, where $\langle Q, \leqslant\rangle$ is WQO. Let x_0 be anything in Q_0 and thereafter pick $x_i \in Q_i$ s.t. $(\forall y \in Q_{i-1})(x_i \ngeqslant y)$. The x_is then form a bad sequence. This shows that if $\leqslant$ is WQO, then $\leqslant^*$ is at least wellfounded. (We proved the converse to this by considering the identity quasiorder on a countable set). Jančar (personal communication but see Jančar [1]) claims that RADO (which we shall see later) is a counterexample that shows that it need not be WQO.

What about constructors of infinite character such as $\leqslant^+$ on the whole of the power set, not just the finite sets?

5. BQOs

We saw in Remark 5 that the '+' operation preserves reflexivity and transitivity. It would be nice if in addition it were to preserve the condition on ω-sequences so that $\langle \mathscr{P}(A), \leqslant^+\rangle$ is WQO as long as $\langle A, \leqslant\rangle$ is. We shall see a counterexample due to Rado which will show that the Hoare ordering of the power set of a WQO is not always a WQO. It is natural to ask what extra conditions one has to add to those comprising WQOness to get a property that is preserved under this construction.

First let us suppose that $\langle \mathscr{P}(Q), \leqslant_Q{}^+\rangle$ is not WQO. So there is a bad sequence $\langle Q_i\colon\ i \in \mathbb{N}\rangle$ where for $i<j$, $Q_i \nleqslant^+_Q Q_j$.

It would be nice if for each i we could pick a member q_i in Q_i to get a bad sequence on Q, but of course we cannot. What we can do is pick, for each $i<j$, an element $q_{ij} \in Q_j$ s.t. $(\forall q \in Q_j)(q_{ij} \nleqslant_Q q)$. So, using DC we can pick a family of elements of Q indexed by pairs of distinct natural numbers, such that $(\forall i<j<k)(q_{ij} \nleqslant_Q q_{jk})$. Such a family we will call a bad *array*.

Now given a bad array on Q we can construct a bad sequence on $\mathscr{P}(Q)$ all of whose elements are countable sets: simply set $Q_i =: \{q_{ij}\colon\ j>i\}$. The idea here "If there is a bad sequence of subsets there is a bad sequence of *countable* subsets" is important and will be exploited later in the proof of Theorem 9.

If we assume that $\langle \mathscr{P}(\mathscr{P}(Q)), (\leqslant^+_Q)^+\rangle$ is not WQO we get, by a two stage descent, an analogous condition on increasing triples, namely: $(\forall i<j<k<l)(q_{ijk} \nleqslant_Q q_{jkl})$, and given a bad array of triples we can get a bad sequence on $\mathscr{P}^2(Q)$ of countable sets of countable subsets of Q.

Indeed for any finite n if we assume that $\langle \mathscr{P}^n(Q), (\leqslant^{n+})\rangle$ is not WQO we get an analogous condition on increasing n-tuples, namely: $(\forall i_0 < i_1 < \cdots < i_n \in \mathbb{N})(q_{i_0 i_1 \cdots i_{n-1}} \nleqslant_Q q_{i_1 i_2 \ldots i_n})$, and given a bad array of n-tuples we can get a bad sequence on $\mathscr{P}^n(Q)$ of sets that are hereditarily countable over Q.

Let us now see Rado's counterexample:

Quasiorder $\{\langle i,j\rangle\colon\ i<j\in\mathbb{N}\}$ by $\langle i,j\rangle\leqslant\langle i',j'\rangle$ iff $((i=i')\wedge(j\leqslant j'))\vee(j<i')$. Call this structure $RADO$.

We shall not need the following here so no proof will be given, but it is noteworthy. See e.g., Laver [4].

Remark 6.

(1) RADO is WQO;
(2) RADO^{ω} is not WQO under $\leqslant_l$;
(3) $\mathscr{P}(\mathrm{RADO})$ is not WQO under $\leqslant^+$;
(4) RADO $\hookrightarrow$ any other WQO with properties 1–3.

It is possible to show that there are generalisations of RADO in the following sense. A wellfounded quasiorder is WQO iff the Hoare ordering of its power set is wellfounded. We could say that a wellfounded quasiordering on Q is n-WQO if the Hoare ordering on $\mathscr{P}^n(Q)$ is wellfounded. It will turn out that for each n there is a special quasiorder one might call n-RADO such that the Hoare ordering on $\mathscr{P}^n(Q)$ is wellfounded iff we cannot embed n-RADO into Q. This was shown in Pouzet [7].

The reader should think of these equivalences as being generalisations of the equivalence of wellfoundedness of a binary structure $\langle X,R\rangle$ with there being no embedding of ω^* (the order type of the negative integers) into it. In fact we can extend the sequence of n-RADO structures and concepts of n-WQO into the transfinite, though we have no space to do this here, and this will have to be a topic for a future paper. Another topic for that future paper is the question of which natural notions in mathematics can be characterised in terms of excluded substructures in the way we have characterised WQOs. It may just be coincidence that Laver's beautiful theorem (Laver [3] which should be read in conjunction with Simpson [9]) says (*inter alia*) that the embedding relation between scattered order types is BQO, where an order type is *scattered* iff one cannot embed the rationals in it.

At this point we could direct our attention to the class of quasiorders $\langle Q,\leqslant\rangle$ such that, for all $n\in\mathbb{N}$, the result of doing $+$ n times to it is a WQO, and notice that this class is closed under $+$. This would give us a coinductive definition of a distinguished class of WQOs as the largest class of WQOs closed under $+$, and one could hope that this would turn out to be the class of BQOs. However we have to wring this idea out a little further, since there remains much to be gained by considering transfinite iterations of $+$. This is because one will then be able to generalise the array condition to something that has no finite bound on the length of the sequences.

In the discussion of the way bad sequences on $\mathscr{P}^n(Q)$ can be pulled down into funny arrays on Q we did not consider the possibility of anything being a member of both $\mathscr{P}^n(Q)$ and $\mathscr{P}^m(Q)$ for $m\neq n$. The $\mathscr{P}^n(Q)$ were all be taken to be formally disjoint. If we wish to iterate $+$ transfinitely we need the $\mathscr{P}^n(Q)$ to be *cumulative* not disjoint. The simplest way to make the iterated power sets cumulative is to identify each $q\in Q$ with its singleton. Notice that $q\leqslant q'$ iff $\{q\}\leqslant^+\{q'\}$. Notice also the important triviality that if f is an injection from $\langle Q_1,\leqslant_{Q_1}\rangle$ into $\langle Q_2,\leqslant_{Q_2}\rangle$ then $\lambda x.f``x$ is an injection from $\langle\mathscr{P}(Q_1),(\leqslant_{Q_1})^+\rangle$ into $\langle\mathscr{P}(Q_2),(\leqslant_{Q_2})^+\rangle$. Putting these two together enables us to think of $\langle\mathscr{P}^m(Q),(\leqslant_Q)^{m+}\rangle$ as an end-extension of $\langle\mathscr{P}^n(Q),(\leqslant_Q)^{n+}\rangle$ whenever $n\leqslant m$.

This makes $+$ into a continuous inflationary function from the complete lattice of all quasiorders of V into itself that can be iterated transfinitely.

We now need some more definitions.

Definition 7. $V(Q)$ is the largest fixed point for $\lambda X.(\mathscr{P}(X) \cup Q)$;
$V_\Omega(Q)$ is the wellfounded part of $V(Q)$;
$H_{\aleph_1}(Q)$ is the least fixed point for $\lambda X.(\mathscr{P}_{\aleph_1}(X) \cup Q)$.

Things in $V(Q)$ can be thought of as downward-branching trees (possibly with infinite branches) all of whose leaves are labelled with members of Q. (They satisfy various extensionality conditions which it is not illuminating to dwell on here.) Things in $V_\Omega(Q)$ can be thought of as downward-branching trees—with finite branches only—all of whose leaves are labelled with elements of Q (and they must satisfy the same extensionality conditions). $V_\Omega(Q)$ is of course the union of the Von Neumann hierarchy of wellfounded sets over Q and is sometimes called the *Zermelo Cone* over Q.

Since $H_{\aleph_1}(Q)$ is the *least* fixed point for $\lambda X.(\mathscr{P}_{\aleph_1}(X) \cup Q)$, it contains the *wellfounded* hereditarily countable sets over Q. In the tree picture of things its elements are countable branching trees that have no infinite branches and all of whose endpoints are labelled with elements of Q, with an extensionality condition.

There is a reason for the appearance of $H_{\aleph_1}(Q)$ here. Seminar expositions of this material have taught me that people brought up to believe in the axiom of foundation have difficulty coping with greatest-fixed point construction: they fall prey to mathematically insubstantial but nevertheless persistent and nagging worries: 'Is this thing a set?' they keep asking. The point about $H_{\aleph_1}(Q)$ is that it is always a set as long as Q is (and this is provable in ZF) and it can be substituted for $V(Q)$ in all the results to follow. However, as we shall see, it has independent interest.

$\lambda R.(R^+ \cup \leqslant_Q)$ is a map from the complete lattice of quasiorders of $V(Q)$ partially ordered by inclusion into itself. It is monotone, and so has a complete lattice of fixed points by the Tarski-Knaster theorem. In particular there is a greatest fixed point, which we notate $\leqslant_\infty$. (It is at this point that we use the fact that '$+$' is being applied to quasiorders not to partial orders: the collection of partial orders of $V(Q)$ is not a complete lattice under inclusion but only a chain-complete poset, and we cannot appeal to Tarski-Knaster.) It is nowadays widely understood that there is a connection between greatest fixed points and open games, and we can indeed characterise $\leqslant_\infty$ by means of a game. Define $G_{X \leqslant_\infty Y}$ as follows.

`false` picks a member X' of X, `true` picks a member Y' of Y. If their two choices are both in Q, `true` wins if $X' \leqslant_Q Y'$ and `false` wins if not. If one of them is not in Q they continue, playing $G_{X' \leqslant_\infty Y'}$. If the game goes on forever player `true` wins. The game is open so one or the other player has a winning strategy.

Let us say $X \leqslant_\infty Y$ iff player `true` has a winning strategy.

We are now going to turn our attention to identifying those WQOs $\langle Q, \leqslant_Q \rangle$ such that $\langle V(Q), \leqslant_\infty \rangle$ is also a WQO. It will turn out that they have a nice combinatorial characterisation.

We start out by noticing that if x is an illfounded member of $V(Q)$ then $(\forall y \in V(Q))$ $(y \leqslant_\infty x)$. This means that $\langle V(Q), \leqslant_\infty \rangle$ has (up to equivalence) only one more element than $\langle V_\Omega(Q), \leqslant_\infty\rangle$. This is not going to make one a WQO when the other is not. Accordingly we can restrict our attention to $\langle V_\Omega(Q), \leqslant_\infty\rangle$.

Now suppose that $\leqslant$ wellquasiorders Q but $\leqslant_\infty$ does not wellquasiorder $V(Q)$. Let us see if we can simplify this to something sensible.

We start with a bad sequence $\langle X_i\colon\ i\in\mathbb{N}\rangle$ of members of $V(Q)$. Some of these elements might be members of Q. They cannot all be, because Q is WQO by $\leqslant$, by hypothesis. We are going to leave alone all X_i that are in Q, and elaborate the others until they, too, turn into members of Q. (The complication in this transfinite case is that we do not know in advance how often we are going to have to unwrap each set).

Start off with $\{X_i\colon\ i\in\mathbb{N}\}$, and a digraph which initially is simply the usual wellordering on $\mathbb{N}$, so there is an arrow from X_i to X_j iff $i<j$. We will make ω passes.

When we consider x_s we first check to see if it is a member of Q. If it is, it is then *ratified* which means it will never be replaced. If it is not a member of Q life is a bit more complicated. For each X_t such that there is an arrow from X_s to X_t we choose a member of X_s that is not $\leqslant_\infty$ anything in X_t, and we call it $X_{(s;t)}$ (for the moment at least). We discard X_s and redirect all arrows ending at X_s to $X_{(s;t)}$ (so we replace each old arrow by a host of new ones) and we replace the arrow from X_s to X_t with a new arrow from $X_{(s;t)}$ to X_t.

After ω passes everything has been ratified or discarded. The wellfoundedness of $\langle V_\Omega(Q),\in\rangle$ ensures that there can be no infinite sequence of Xs with later subscripts always end-extensions of earlier subscripts.

The subscripts are a bit of a mess at the moment: every subscript is an ordered pair of earlier subscripts. Notice that at stage one the only new subscripts we construct are pairs of natural numbers where the first component is smaller than the second, and the only new arrows we generate are things like $X_{(1;3)}\not\leqslant_\infty X_{(3;5)}$. So there must be a member of $X_{(1;3)}$ that $\not\leqslant_\infty X_{(3;5)}$ and we call it $X_{((1,3);(3,5))}$. Since this is the only way we can invent new things at this level, we might as well rewrite it as '$X_{1,3,5}$' to remove the duplication of the '3'. The second component of the first pair and the first component of the second pair are always the same!

Now for what subscripts s do we know that $X_{1,3,5}\not\leqslant_\infty X_s$? (All arrows going *into* $X_{1,3,5}$ arose from arrows going into $X_{1,3}$.) The only arrows going *from* $X_{1,3,5}$ go to $X_{3,5}$ in the first instance, and thereafter to things with subscripts that are end-extensions of $\{3,5\}$ should $X_{3,5}$ not be a member of Q and have to be replaced.

The upshot is that we can take subscripts to be increasing finite sequences of natural numbers, and we only ever arrange for an arrow from X_s to X_t when t is an end-extension of the tail of s.

Now consider a set S of finite sequences from $\mathbb{N}$ that arises from a bad Q-sequence in this way. We will show that every increasing ω-sequence from $\mathbb{N}$ has a unique initial segment in S. Let $f:\mathbb{N}\to\mathbb{N}$ be increasing. Is $\langle f(0)\rangle$ in S? It will be if $X_{f(0)}\in Q$, and if that happens we are done. If $\langle f(0)\rangle$ is not in S this must be because $X_{f(0)}\notin Q$ and in these circumstances we have discarded $X_{f(0)}$ and replaced it by $X_{f(0),j}$ for each $j>f(0)$. In particular we will have done this for $j=f(1)$. So is $\langle f(0), f(1)\rangle\in S$? It will be unless $X_{f(0),f(1)}\notin Q$. In those circumstances we discarded $X_{f(0),f(1)}$ and replaced it by each of $X_{f(0),f(1),j}$ for each $j>f(1)$. And so on. Eventually we hit a member of Q and at that point we have an element of s that is an initial segment of f. Notice that we only ever put into S a sequence s if we have already discarded all initial segments of s, so that the initial segment in S of our infinite sequence is unique.

This motivates the definitions which follow.

A *block* is a set B of strictly increasing finite sequences of naturals with the property that every strictly increasing ω-sequence of natural numbers has a unique initial segment in B. We write $s \lhd t$ if t is the tail of an end-extension of s. Notice also that $\lhd$ is not transitive.

Definition 8. Let $\langle Q, \leqslant_Q \rangle$ be a quasiorder and B a block. A map $f : B \to Q$ is an *array*. An array is *good* if there are $s \lhd t \in B$ such that $f(s) \leqslant_Q f(t)$.

Then $\langle Q, \leqslant_Q \rangle$ is a *better-quasiorder* (hereafter "*BQO*") iff for every block B every array $f : B \to Q$ is good.

This is (almost) the standard combinatorial definition in the literature. However we will abort the development at this point and not derive the fully refined standard definition because our purpose here is to derive an alternative definition, hinted at in the following theorem.

The vernacular will need an expression for this process of turning a bad array on $\mathscr{P}^{\alpha}(Q)$ into a bad array on Q. I propose to call it *sifting*.

Theorem 9. *The following are equivalent for a quasiorder* $\langle Q, \leqslant_Q \rangle$:
(i) $\langle Q, \leqslant \rangle$ *is BQO*;
(ii) $\langle V_\Omega(Q), \leqslant_\infty \rangle$ *is wellfounded*;
(iii) $\langle H_{\aleph_1}(Q), \leqslant_\infty \rangle$ *is wellfounded*;

Proof. We have already shown (i) implies (ii), and (ii) implies (iii) because any substructure of a wellfounded quasiorder is wellfounded. It remains only to show that (iii) implies (i). We will prove the contrapositive. Suppose $\langle Q, \leqslant \rangle$ is not BQO. Then there is a bad array, whose elements are to be notated $\{q_s\colon\ s \in S\}$ where S is a block. We define qs with shorter subscripts by setting

$$q_s =: \{q_t\colon\ (\exists n)(t = s \frown \{n\}\}.$$

For example: $q_1 =: \{q_{1,2}, q_{1,3} \ldots\})$. If this recursion fails it is because there is an infinite sequence $\langle s_i\colon\ i \in \mathbb{N} \rangle$ of initial segments of S totally ordered by end-extension. But this is impossible, because no infinite sequence of naturals has more than one initial segment in S (let alone infinitely many!)

We then find that the sequence $\langle x_i\colon\ i \in \mathbb{N} \rangle$ is a bad sequence in $H_{\aleph_1}(Q)$. But a bad sequence in $\mathscr{P}_{\aleph_1}(X)$ always gives rise to an infinite descending sequence in X, and $H_{\aleph_1}(Q) = \mathscr{P}_{\aleph_1}(H_{\aleph_1}(Q))$, so $H_{\aleph_1}(Q)$ must be actually illfounded as well as not being a WQO. $\square$

In fact we can strengthen this further.

Theorem 10. *If* $\langle Q, \leqslant_Q\rangle$ *is BQO, so is* $\langle V(Q), \leqslant_\infty\rangle$.

Proof. Suppose there is a bad array over $V(Q)$. We will show how to refine it into a bad array Q. This is merely a more developed version of the process we applied to $\mathscr{P}^n(Q)$ earlier on.

Let $\{X_s\colon\ s \in B\}$ be a bad array over $V(Q)$. For each pair s, t in B with $s \lhd t$ we have $X_s \not\leqslant_\infty X_t$. Player `false` has a winning strategy $\sigma_{X_s \not\leqslant_\infty X_t}$ in the game $G_{X_s \leqslant_\infty X_t}$.

All the games $G_{X_s \leqslant_\infty X_t}$ will be played simultaneously. Indeed many *plays* of these games will be going on simultaneously. To be precise, there is a play for each infinite ascending $\lhd$-sequence.

It is convenient to describe what happens in terms of an ω-sequence of what one might as well call *passes*.

At the first pass, in each game $G_{X_s \leqslant_\infty X_t}$, `false` uses his strategy to pick a member of X_s. This will become $X_{s;t}$. At the second pass (and all subsequent passes) each play of $G_{X_s \leqslant_\infty X_t}$ multifurcates. At the first pass there was only one play of each game. For `false` to decide what to do as his second move in $G_{X_s \leqslant_\infty X_t}$ he deems `true`'s move in this game to be `false`'s move in $G_{X_t \leqslant_\infty X_u}$, for $t \lhd u$. Thus he deems `true` to have played $X_{t;u}$. Since he does this for *each u* such that $t \lhd u$, the one play of $G_{X_s \leqslant_\infty X_t}$ which was proceeding at pass one has become infinitely many. In each play he continues to use $\sigma_{X_s \not\leqslant_\infty X_t}$ and—since this strategy is winning—each play will terminate with a win for player `false`. This tells us that after ω passes every play of every game will have terminated in a win for player `false`.

Of course, since there is an entire bad array out there, we must expect to have to deal with $X_t \not\leqslant_\infty X_u$ for various u as well. For each game $G_{X_t \leqslant_\infty X_u}$ where $t \lhd u$ player `false` in that game uses his winning strategy to pick $X_{t;u}$. Player `false` in the game $G_{X_s \leqslant_\infty X_t}$ now has infinitely many replies to contend with, but he uses $\sigma_{X_s \not\leqslant_\infty X_t}$ to reply to each, and the play multifurcates, but `false` can continue to use $\sigma_{X_s \not\leqslant_\infty X_t}$ in each.

Since all the strategies $\sigma_{X_s \not\leqslant X_t}$ are winning for `false`, this process must halt with player `true` picking elements of Q. This gives us a bad array on Q. $\square$

This implies that $\langle Q, \leqslant_Q\rangle$ is BQO iff $\langle Q, \leqslant_Q\rangle$ belongs to the largest class of WQOs closed under the operation taking $\langle X, \leqslant\rangle$ to $\langle H_{\aleph_1}(X), \leqslant_\infty\rangle$. (Or, equivalently, to $\langle V(X), \leqslant_\infty\rangle$ or $\langle V_\Omega(X), \leqslant_\infty\rangle$.) Indeed, since $\langle Q, \leqslant_Q\rangle$ is WQO iff $\langle \mathscr{P}(Q), \leqslant^+\rangle$ is wellfounded we can strengthen this to the remarkable

Corollary 11. $\langle Q, \leqslant_Q\rangle$ *is BQO iff* $\langle Q, \leqslant_Q\rangle$ *belongs to the largest class of wellfounded quasiorders closed under the operation taking* $\langle X, \leqslant\rangle$ *to* $\langle H_{\aleph_1}(X), \leqslant_\infty\rangle$. (*Or, equivalently, to* $\langle V(X), \leqslant_\infty\rangle$ *or* $\langle V_\Omega(X), \leqslant_\infty\rangle$.)

6. Applications

Other than the graph minor relation on finite graphs (for which the question is still open) all naturally occurring WQOs seem to be BQOs. This has often excited comment, as has the fact there is no apparent explanation for it. What might help would be a technique that proves that certain things are BQO rather than proving that BQOs have certain properties. But this is just what coinduction does. A recursive datatype comes equipped with a principle of induction that enables one to show that its members have certain properties. A corecursive datatype comes equipped with a principle of coinduction that enables one to show that things with certain properties belong to it. Accordingly we should expect this coinductive characterisation of BQOs to give rise to proofs that lots of things are BQO.

Here is how some of these proofs could work.

Lemma 12. *Suppose C is a constructor taking BQOs to wellfounded quasiorders and C commutes with the $\langle V(Q), \leqslant_\infty\rangle$ constructor ('V' for short) in the sense that whenever B is a BQO then*

$$C(V(B)) \simeq V(C(B)).$$

Then the class of BQOs is closed under C.

Proof. Let BQO be the class of all BQOs, and W the class of all wellfounded quasiorders. C"BQO $\subseteq W$ by assumption. We want C"BQO to be closed under V but V"C"BQO $=$ C"V"BQO $\subseteq$ C"BQO.

C"*BQO* is now a class of wellfounded quasiorders closed under V and is therefore a class of BQOs as desired. □

Of course any substructure of a BQO is BQO, so any constructor F such that $F(Q)$ can be coded inside $V(Q)$ will preserve BQOs. This looks hopeful, because $V(Q)$ is clearly enormous, so there should be lots of room to code everything we want. However, if the quasiorder on Q is connected, so too will be the quasiorder $\leqslant_\infty$ on $V(Q)$, since $+$ preserves connexity. Stretching does not, as we have noted. So the set of Q-lists is not a substructure of $\langle V(Q), \leqslant_\infty\rangle$. It may be that proofs along the above lines of results like Nash-Williams's tree theorem [10] will require a re-working of these ideas rather than a straightforward application of them.

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