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Fan-planarity: Properties and complexity ☆

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ABSTRACT

In a *fan-planar drawing* of a graph an edge can cross only edges with a common end-vertex. Fan-planar drawings have been recently introduced by Kaufmann and Ueckerdt [35], who proved that every n-vertex fan-planar drawing has at most $5n - 10$ edges, and that this bound is tight for $n \geq 20$. We extend their result from both the combinatorial and the algorithmic point of view. We prove tight bounds on the density of constrained versions of fan-planar drawings and study the relationship between fan-planarity and k-planarity. Also, we prove that testing fan-planarity in the variable embedding setting is NP-complete.
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1. Introduction

There is a growing interest in the study of non-planar drawings of graphs with forbidden crossing configurations. The idea is to relax the planarity constraint by allowing edge crossings that do not affect too much the drawing readability. Among the most popular types of non-planar drawings studied so far we recall:

- *k-planar drawings*, where an edge can have at most k crossings (see, e.g., [5,8,9,15,22,24,28,33,34,36,37,40]);
- *k-quasi-planar drawings*, which do not contain k mutually crossing edges (see, e.g., [1,3,4,21,30,41]);
- *RAC (Right Angle Crossing) drawings*, where edges can cross only at right angles (see, e.g., [25] and [26] for a survey);
- *ACE_α drawings* [2] and *ACL_α drawings* [6,20,27], which are generalizations of RAC drawings; namely, in an ACE_α drawing edges can cross only at an angle that is *exactly* α ($\alpha \in (0, \pi/2]$); in an ACL_α drawing edges can cross only at angles that are *at least* α (see also [26]);
- *fan-crossing free drawings*, where an edge cannot cross two other edges having a common end-vertex [16].

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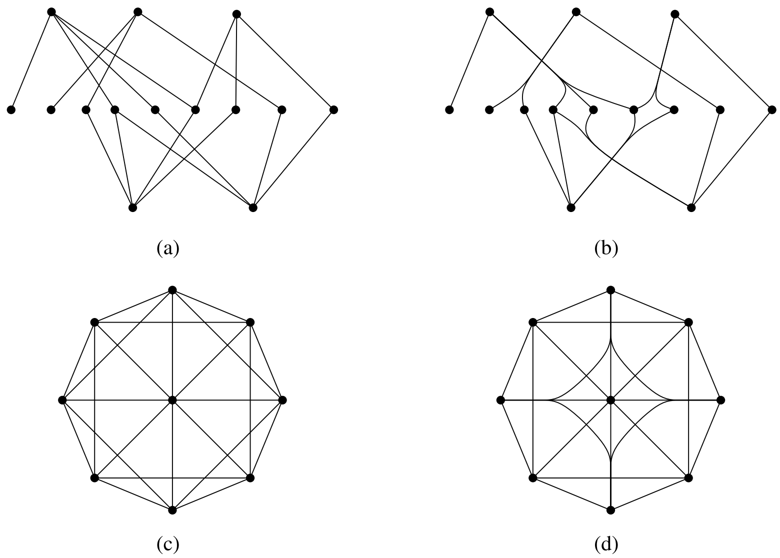

Fig. 1. (a) A fan-planar drawing of a graph G with 12 crossings. (b) A confluent drawing of G with 3 crossings. (c) A fan-planar drawing with 16 crossings of another graph G. (d) A confluent drawing of G with 8 crossings.

Given a desired type T of non-planar drawing with forbidden crossing configurations, a classical combinatorial problem is to establish bounds on the maximum number of edges that a drawing of type T can have; this problem is usually dubbed a Turán-type problem, and several tight bounds have been proved for the types of drawings mentioned above, both for edges drawn as straight-line segments and for edges drawn as polylines (see, e.g., [1,2,4,15,16,24,25,27,30,37,41]). From the algorithmic point of view, the complexity of testing whether a graph G admits a drawing of type T is one of the most interesting problem. Also for this problem several results have been shown, both in the *fixed* and in the *variable embedding setting* (see, e.g., [8,18,19,32,33,36]). In the fixed embedding setting, for each vertex v, the circular ordering of the edges incident to v is given as part of the input and cannot be changed; conversely, in the variable embedding setting such an ordering can be freely chosen.

In this paper we investigate *fan-planar drawings* of graphs, in which an edge cannot cross two independent edges, i.e., an edge can cross several edges provided that they have a common end-vertex. Fan-planar drawings have been recently introduced by Kaufmann and Ueckerdt [35]; they proved that every n-vertex graph without loops and multiple edges that admits a fan-planar drawing has at most $5n - 10$ edges, and that this bound is tight for $n \geq 20$. Fan-planar drawings are on the opposite side of fan-crossing free drawings mentioned above. Besides its intrinsic theoretical interest, we observe that fan-planarity can be also used in many cases for creating drawings with few edge crossings per edge in a confluent drawing style (see, e.g., [23,29]). For example, Fig. 1(a) shows a fan-planar drawing Γ with 12 crossings; Fig. 1(b) shows a new drawing with just 3 crossings obtained from Γ by bundling crossing "fans". Another example is shown in Figs. 1(c) and 1(d).

We prove both combinatorial properties and complexity results related to fan-planar drawings of graphs. The main contributions of our work are as follows:

- We study the density of constrained versions of fan-planar drawings, namely *outer fan-planar drawings*, where all vertices must lie on the external boundary of the drawing, and 2-*layer fan-planar drawings*, where vertices are placed on two distinct horizontal lines and edges are vertically monotone lines. We prove tight bounds for the edge density of these drawings. Namely, we show that n-vertex outer fan-planar drawings have at most $3n - 5$ edges (a tight bound for $n \geq 5$), and that n-vertex 2-layer fan-planar drawings have at most $2n - 4$ edges (a tight bound for $n \geq 3$). We remark that outer and 2-layer non-planar drawings have been previously studied in the 1-planarity setting [8,24,33] and in the RAC planarity setting [18,19].
- Since general fan-planar drawable graphs have at most $5n - 10$ edges and the same bound holds for 2-planar drawable graphs [37], we investigate the relationship between these two graph classes (observe that 1-planar graphs are always fan-planar by definition). More in general, we study the relationship between k-planarity and fan-planarity, proving that in fact for any $k \geq 2$ there exist fan-planar drawable graphs that are not k-planar, and vice versa.
- Finally, we show that testing whether a graph admits a fan-planar drawing in the variable embedding setting is NP-complete.

The rest of the paper is structured as follows: In Section 2 we give some preliminary definitions. Section 3 describes the tight bounds on the edge density of outer and 2-layer fan-planar drawable graphs. The relationship between k-planarity and fan-planarity is shown in Section 4, while Section 5 proves the NP-completeness of the fan-planarity testing problem.

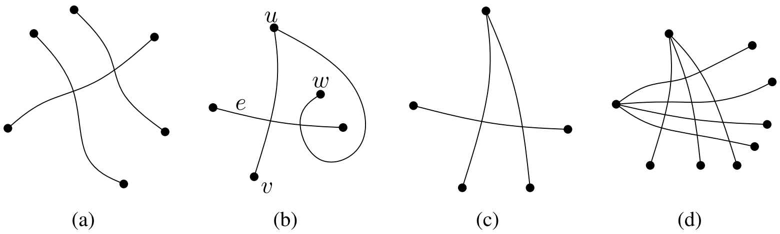

Fig. 2. (a)–(b) Forbidden configurations in a fan-planar drawing. (c)–(d) Allowed configurations in a fan-planar drawing.

A final discussion that analyzes further work on fan-planarity and that presents some open research directions is given in Section 6.

2. Preliminary definitions and results

A *drawing* Γ of a graph G maps each vertex to a distinct point of the plane and each edge is a non-self intersecting curve between the points corresponding to the end-vertices of the edge. For a subgraph G' of G, we denote by $\Gamma[G']$ the restriction of Γ to G'. Throughout the paper we consider only *simple graphs*, i.e., graphs with neither multiple edges nor self-loops; also, from now on we only consider *simple drawings*, i.e., drawings such that the arcs representing two edges have at most one point in common, which is either a common end-vertex or a common interior point where the two arcs properly cross. A drawing Γ is *straight-line* if every edge is represented as a straight-line segment. In a *polyline drawing* each edge is represented as a polyline, i.e., a sequence of consecutive straight-line segments. Clearly, a straight-line drawing is a special case of a polyline drawing.

For each vertex v of G, the set of edges incident to v is called the *fan of* v. Clearly, each edge (u, v) of G belongs to the fan of u and to the fan of v at the same time. Two edges that do not share a vertex are called *independent edges*, and always belong to distinct fans. A *fan-planar drawing* Γ of G is a drawing of G such that:

- (a) no edge is crossed by two independent edges (this forbidden configuration is depicted in Fig. 2(a));
- (b) there are not two adjacent edges (u, v), (u, w) that cross an edge e from different "sides" while moving from u to v and from u to w (see Fig. 2(b) for an illustration of this forbidden configuration). More formally, let p (respectively q) be the crossing point between e and (u, v) (respectively (u, w)). Let $\ell_{p,q}$ be the portion of edge e from p to q, $\ell_{u,p}$ be the portion of edge (u, v) from u to p, and $\ell_{u,q}$ be the portion of edge (u, w) from u to q. The union of $\ell_{p,q}$, $\ell_{u,p}$, and $\ell_{u,q}$ is a closed simple curve $\mathcal{C}$ that subdivides the plane into two topological connected regions having $\mathcal{C}$ as their boundary. In the forbidden configuration (b), v lies in one of these two regions, while w lies in the other one.

Figs. 2(c) and 2(d) show two allowed configurations of a fan-planar drawing. A *fan-planar graph* is a graph that admits a fan-planar drawing.

The next property immediately follows from the definition of fan-planar drawings.

Property 1. A fan-planar drawing does not contain 3-mutually crossing edges.

Let Γ be a non-planar drawing of G; the *planar enhancement* Γ' of Γ is the drawing obtained from Γ by replacing each crossing point with a dummy vertex. Drawing Γ' subdivides the plane into topologically connected regions, called *faces*. Exactly one of these faces is an infinite region, and is called the *outer face* of Γ'. The other faces are called *internal faces* of Γ'. The boundary of each face f' of Γ' consists of a sequence of real and dummy vertices (plus the edge segments that connect them); the connected region f of the plane that corresponds to f' in Γ consists of a sequence of vertices and crossing points. For simplicity we call f a *face* of Γ. The *outer face* of Γ is the face corresponding to the outer face of Γ'. A fan-planar drawing of G with all vertices on the outer face is called an *outer fan-planar drawing* of G. Observe that the configuration in Fig. 2(b) cannot occur in a drawing with all vertices on the outer face; hence, a drawing is outer fan-planar if and only if all vertices are on the outer face and it does not contain an edge crossed by two independent edges. An *outer fan-planar graph* is a graph that admits an outer fan-planar drawing. Fig. 3(a) shows an example of an outer fan-planar drawing of a graph. Fig. 3(b) shows a fan-planar graph that is not outer fan-planar.

An outer fan-planar graph G is *maximal* if no edge can be added to G without loosing the property that G remains outer fan-planar. An outer fan-planar graph G with n vertices is *maximally dense* if it has the maximum number of edges among all outer fan-planar graphs with n vertices. If G is maximally dense then it is also maximal, but not vice versa. We remark that maximally dense graphs are sometimes called "optimal" in the literature (see, e.g., [14,17,39]). The following property holds.

Lemma 1. *Let $G = (V, E)$ be a maximal outer fan-planar graph and let Γ be an outer fan-planar drawing of G. The outer face of Γ does not contain crossing points, i.e., it consists of $|V|$ uncrossed edges.*

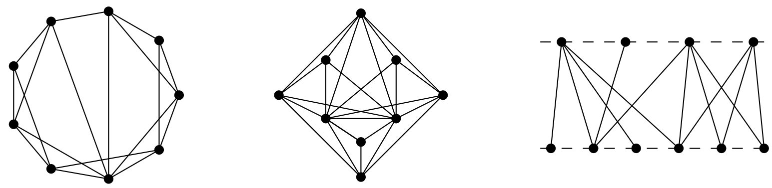

Fig. 3. (a) An example of an outer fan-planar graph. (b) A fan-planar drawing of a graph that is not outer fan-planar. (c) An example of a 2-layer fan-planar drawing.

Proof. Suppose by contradiction that the outer face of Γ contains at least one crossing point. Then there exist two vertices u and v such that walking clockwise along the boundary of the outer face from u to v at least one crossing is encountered. In other words, walking clockwise along the boundary of the outer face from u one encounters a non-empty set of crossing points, and finally v. We now prove that u and v are not connected by an edge. Suppose by contradiction that the edge (u, v) exists. Let $\gamma_{u,v}$ be the portion of the boundary of the outer face traversed when walking clockwise between u and v. We first show that $\gamma_{u,v}$ and the edge (u, v) do not share any point (except u and v). Namely, suppose that they share at least one point; since the drawing is simple, one of the shared points must be a crossing point c. Let (w, z) be the edge sharing c with (u, v). In order to have a single crossing between (u, v) and (w, z), one between w and z must be encountered between u and v when walking clockwise, and the other one must be encountered between v and u. Since neither w nor z can be encountered between u and v when walking clockwise (because otherwise it would belong to $\gamma_{u,v}$, but $\gamma_{u,v}$ does not contain any vertex), then (w, z) must cross (u, v) more than once. But this is not possible in a simple drawing.

Thus the union of the edge (u, v) and of $\gamma_{u,v}$ forms a simple closed curve $\mathcal{C}$. By definition $\gamma_{u,v}$ contains at least one crossing c'. Let e_1 and e_2 be the edges crossing at c'. At least two of the four distinct end-vertices of e_1 and e_2 are distinct from u and v. Let z_1 and z_2 be such vertices. Since z_1 and z_2 do not belong to $\gamma_{u,v}$, they must be encountered between v and u when walking clockwise along the boundary of the outer face. This implies that both e_1 and e_2 cross (u, v). Since e_1 and e_2 are independent edges, this contradicts the fan-planarity of the drawing. This concludes the proof that (u, v) does not exist.

Hence, one can draw in the outer face of Γ a simple curve connecting u and v without crossing any other edge of Γ, so that u and v remain on the outer face while $\gamma_{u,v}$ is no longer on the outer face. The resulting drawing is still outer fan-planar and contains one more edge than Γ, thus contradicting the hypothesis that G is maximal outer fan-planar. $\square$

Given an outer fan-planar drawing Γ of a maximal outer fan-planar graph G, the edges of G on the external boundary of Γ will be also called the *outer edges* of Γ.

A 2-*layer fan-planar drawing* is a fan-planar drawing such that: (i) each vertex is drawn on one of two distinct horizontal lines, called *layers*; (ii) each edge connects vertices of different layers and it is drawn as a curve that is monotone in the vertical direction (i.e., any horizontal line intersects the curve in at most one point). By definition, a 2-layer fan-planar drawing is also an outer fan-planar drawing. A 2-*layer fan-planar graph* is a graph that admits a 2-layer fan planar drawing. An example of a 2-layer fan-planar drawing is shown in Fig. 3(c).

3. Density of outer and 2-layer fan-planar graphs

We first prove that an n-vertex outer fan-planar graph G has at most $3n - 5$ edges. Then we describe how to construct outer fan-planar graphs with n vertices and $3n - 5$ edges. Let G be a graph and let Γ be a drawing of G. The *crossing graph* of Γ, denoted as $\mathrm{CR}(\Gamma)$, is a graph having a vertex for each edge of G and an edge between any two vertices whose corresponding edges cross in Γ. A cycle of $\mathrm{CR}(\Gamma)$ of odd length will be called an *odd cycle* of $\mathrm{CR}(\Gamma)$; similarly, an *even cycle* of $\mathrm{CR}(\Gamma)$ is a cycle of even length. We start by proving some interesting combinatorial properties of G related to the cycles of the crossing graph of outer-fan planar drawings of G.

Lemma 2. *Let $G = (V, E)$ be a maximal outer fan-planar graph with $n = |V|$ vertices and $m = |E|$ edges. Let Γ be an outer fan-planar drawing of G. If $\mathrm{CR}(\Gamma)$ does not have odd cycles then $m \le 3n - 6$.*

Proof. If $\mathrm{CR}(\Gamma)$ does not contain odd cycles, then it is bipartite and its vertices can be partitioned into two independent sets W_1 and W_2. Since by Lemma 1 the outer edges of Γ are not crossed, they correspond to n isolated vertices in $\mathrm{CR}(\Gamma)$. We can arbitrarily assign all these vertices to the same set, say W_1. Denote by E_i the set of edges of G corresponding to the vertices of W_i ($i \in \{1, 2\}$). Clearly, E_1 and E_2 partition the set E. Since no two edges of E_i cross in Γ, then the two subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are outerplanar graphs. Since an outerplanar graph has at most $2n - 3$ edges and E_2 does not contain the n outer edges of Γ, we have $|E_1| \le 2n - 3$ and $|E_2| \le 2n - 3 - n$. Thus, $m = |E| = |E_1| + |E_2| \le 3n - 6$. $\square$

The next lemma shows that the length of any odd cycle of $\mathrm{CR}(\Gamma)$ is at most 5.

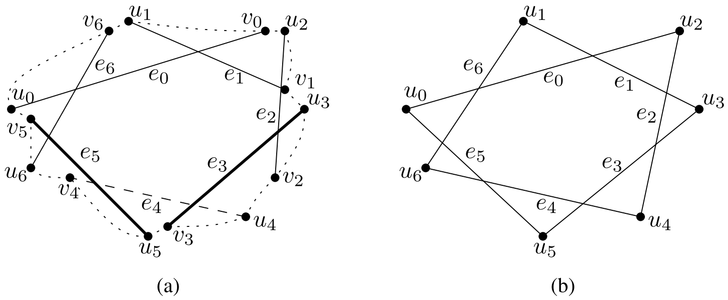

Fig. 4. Illustration for the proof of Lemma 3. (a) An edge set $E(C)$ with $\ell = 7$. If v_3 and u_5 do not coincide, e_4 (dashed) is crossed by the two independent edges e_3 and e_5 (in bold). (b) $E(C)$ with $\ell = 7$, where v_i coincides with u_{i+2}, for $i = 0, \ldots, 7$.

Lemma 3. *Let G be a maximally dense outer fan-planar graph with n vertices and let Γ be an outer fan-planar drawing of G.* CR(Γ) *does not contain odd cycles of length greater than* 5.

Proof. Let C be an odd cycle of length ℓ in CR(Γ). Let $E(C) = \{e_0 = (u_0, v_0), \ldots, e_{\ell-1} = (u_{\ell-1}, v_{\ell-1})\}$ be the set of ℓ edges of G corresponding to the vertices of C, such that e_i crosses e_{i+1} for $i = 0, \ldots, \ell - 1$, where indices are taken modulo ℓ. Recall that all vertices of G are on the outer face of Γ, which implies that the end-vertices of the edges in $E(C)$ are encountered in the following order when walking clockwise on the boundary of the outer face of Γ: u_i precedes v_{i-1} and v_i precedes u_{i+2} (see, e.g., Fig. 4(a)). Furthermore, vertices v_i and u_{i+2} must coincide, for $i = 0, \ldots, \ell - 1$. Indeed, if v_i and u_{i+2} are distinct, for some $i = 0, \ldots, \ell - 1$, then edge e_{i+1} is crossed by two independent edges (i.e., e_i and e_{i+2}), which contradicts the hypothesis that Γ is fan-planar. See also Fig. 4(a). Thus, we have that u_i precedes u_{i+1} while walking clockwise on the boundary of the outer face of Γ, for $i = 0, \ldots, \ell - 1$, as shown in Fig. 4(b). Moreover, it can be seen that the edges in $E(C)$ are not crossed by any edge not in $E(C)$, as otherwise the drawing would not be fan-planar.

Now, suppose by contradiction that ℓ is odd and greater than 5 (refer to Fig. 4(b) for an illustration). Consider a vertex u_i, for some $i = 0, \ldots, \ell - 1$, and denote by $\overline{V}$ the set of vertices encountered between u_{i+3} and u_{i-3} while walking clockwise on the boundary of the outer face of Γ (including u_{i+3} and u_{i-3}). Vertex u_i cannot be adjacent to any vertex in $\overline{V}$. Namely, if an edge $e = (u_i, u_j)$ existed, for some $u_j \in \overline{V}$, then it would be crossed by the two independent edges e_{i-1} and e_{j-1}. Thus, removing e_{i-1} from Γ, one can suitably connect u_i to all the vertices in $\overline{V}$, still obtaining a fan-planar drawing Γ^* with n vertices. Since the size of $\overline{V}$ is $\ell - 5$, and since by hypothesis $\ell \geq 7$, we have that Γ^* has at least two edges more than Γ, which contradicts the hypothesis that G is maximally dense. □

The following corollary is a consequence of Lemma 3 and Property 1.

Corollary 1. *Let G be a maximally dense outer fan-planar graph. Any odd cycle in the crossing graph of a fan-planar drawing of G has exactly length* 5.

The next lemma claims that odd cycles in the crossing graph correspond to K_5.

Lemma 4. *Let G be a maximally dense outer fan-planar graph and let Γ be an outer fan-planar drawing of G. If* CR(Γ) *contains a cycle C of length* 5*, then the subgraph of G induced by the end-vertices of the edges corresponding to the vertices of C is K_5 and the edges of the K_5 that do not correspond to the vertices of* CR(Γ) *are not crossed in Γ.*

Proof. Let $E(C) = \{e_0 = (u_0, v_0), \ldots, e_4 = (u_4, v_4)\}$ be the set of 5 edges of G corresponding to the vertices of C, such that e_i crosses e_{i+1} for $i = 0, \ldots, 4$, where indices are taken modulo 5.

With the same argument used in the proof of Lemma 3, vertices v_i and u_{i+2} must coincide, for $i = 0, \ldots, 4$. It follows that u_i precedes u_{i+1} walking clockwise on the boundary of the outer face of Γ, and that u_i is connected to u_{i+2}, for $i = 0, \ldots, 4$. Moreover, u_i and u_{i+1} are connected by an edge, for $i = 0, \ldots, 4$. Indeed, if there is no vertex of G between u_i and u_{i+1} walking clockwise on the boundary of the outer face of Γ, for some $i = 0, \ldots, 4$, then the edge (u_i, u_{i+1}) can be added to Γ without creating any crossing and so that all vertices remain on the outer face. If there is a vertex of G between u_i and u_{i+1} walking clockwise on the boundary of the outer face of Γ then it is easy to see that this vertex cannot be adjacent to any vertex u_j distinct from u_i and u_{i+1}, because this would cause a forbidden crossing (two independent edges crossed by an edge); it follows that edge (u_i, u_{i+1}) can be still added without creating crossing and so that all vertices of G remain on the outer face. Hence, the subgraph induced by $u_0, u_1, \ldots, u_4$ is K_5 and every edge (u_i, u_{i+1}) is not crossed in Γ. □

We now prove the upper bound on the density of outer fan-planar graphs. Clearly, it is sufficient to restrict to maximally dense outer fan-planar graphs.

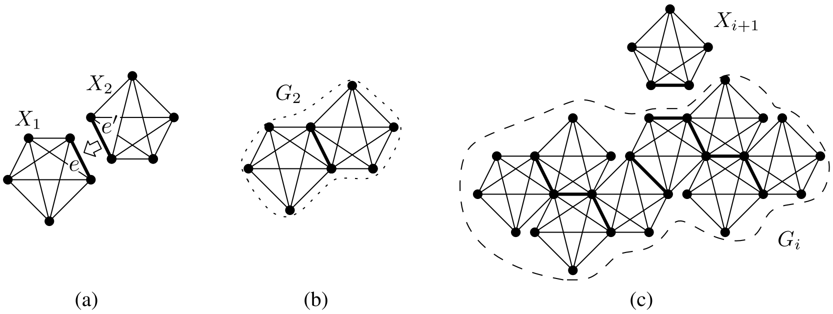

Fig. 5. Illustration for the proof of Lemma 6. (a) X_1 and X_2 before being merged. (b) Merging X_1 and X_2 into G_2. (c) G_i and X_{i+1}, the bold edges are used for merging.

Lemma 5. *Let G be a maximally dense outer fan-planar graph with n vertices and m edges. Then $m \leq 3n - 5$ edges.*

Proof. Since K_n with $n \leq 5$ is outer fan-planar and has at most $3n - 5$ edges, we prove the statement for $n > 5$. Let Γ be an outer fan-planar drawing of G. We first claim that G is biconnected. Suppose by contradiction that G is not biconnected, and let C_1 and C_2 be two distinct biconnected components of G that share a cut-vertex v. Let u be the first vertex of G encountered while moving from v clockwise on the external boundary of $\Gamma[C_1]$, and let w be the first vertex encountered while moving from v counterclockwise on the external boundary of $\Gamma[C_2]$. One can suitably add edge (u, w) in Γ, still getting an outer fan-planar drawing, which contradicts the hypothesis that G is maximally dense.

Now, by Corollary 1, $\mathrm{CR}(\Gamma)$ can only have either even cycles or cycles of length 5. Also, by Lemma 4, every cycle of length 5 in $\mathrm{CR}(\Gamma)$ corresponds to a subset of edges whose end-vertices induce K_5. We prove the statement by induction on the number h of K_5 subgraphs in G.

Base Case. If $h = 0$ then, by Lemma 2, G has at most $3n - 6$ edges.

Inductive Case. Suppose by induction that the claim is true for $h \geq 0$, and suppose G contains $h + 1$ subgraphs that are K_5. Let G^* be one of these $h + 1$ subgraphs. Let $e = (u, v)$ be an edge on the outer face of $\Gamma[G^*]$ that is not on the outer face of Γ. Notice that edge e exists because otherwise G would coincide with K_5 (but we are considering only graphs with more than 5 vertices). Vertices u and v are a separation pair of G, as otherwise either (i) edge (u, v) is crossed by some edge of Γ, or (ii) one between u or v is not on the outer face of Γ. We recall that two vertices are a separation pair if their removal disconnect the graph. However, case (i) is ruled out by Lemma 4 and case (ii) by the outer fan-planarity of Γ. Hence, we can split G into two biconnected subgraphs that share only edge e, one of them containing G^*. Let $G_1, G_2, \ldots, G_k$ ($k \leq 5$) be the biconnected subgraphs of G distinct from G^* such that each G_i shares exactly one edge with G^*. Each G_i ($i = 1, 2, \ldots, k$) contains at most h subgraphs that are K_5, and therefore it has at most $3n_i - 5$ edges by induction, where n_i denotes the number of vertices of G_i. On the other hand, G^* has $3n^* - 5 = 10$ edges, where $n^* = 5$ is the number of vertices of G^*. It follows that $m \leq 3(n^* + n_1 + \cdots + n_k) - 5(k + 1) - k$ ($k \leq 5$). Since $n^* + n_1 + \cdots + n_k \leq n + 2k$ we have $m \leq 3(n + 2k) - 5(k + 1) - k = 3n - 5$. □

The existence of an infinite family of outer fan-planar graphs that match the $3n - 5$ bound is proved in the next lemma. Refer to Fig. 5 for an illustration.

Lemma 6. *For any integer $h \geq 1$ there exists an outer fan-planar graph G with $n = 3h + 2$ vertices and $m = 3n - 5$ edges.*

Proof. Consider h graphs $X_1, \ldots, X_h$, such that each X_i is a K_5 graph, for $i = 1, \ldots, h$. We now describe how to construct G. The idea is to "glue" $X_1, \ldots, X_h$ together in such a way that they share single edges one to another. The proof is by induction on the number of merged graphs. Denote by G_i the graph obtained after merging $X_1, \ldots, X_i$, for $1 < i \leq h$. We prove by induction that G_i respects the following invariants: (I1) it is an outer fan-planar graph; (I2) it has $n_i = 3i + 2$ vertices and $m_i = 3n_i - 5$ edges. In the base case $i = 2$, we merge $G_1 = X_1$ and X_2 as follows. Pick an edge e on the outer face of X_1 and an edge e' on the outer face of X_2. Merge X_1 and X_2 by identifying e with e', see also Figs. 5(a) and 5(b). The new graph G_2 is clearly an outer fan-planar graph with $n_2 = 5 + 5 - 2 = 8$ vertices and $m_2 = 10 + 10 - 1 = 19$ edges. Thus, the two invariants hold. In the inductive case, suppose we constructed G_i for $2 < i < h$ and we want to attach X_{i+1} (see also Fig. 5(c)). Pick any edge e on the outer face of G_i and any edge e' on the outer face of X_{i+1}. Merge the two graphs in the same way as done in the base case. It is immediate to see that (I1) holds. Also, $n_{i+1} = n_i + 3 = (3i + 2) + 3 = 3(i + 1) + 2$ and $m_{i+1} = m_i + 9$. Since by induction $m_i = 3n_i - 5$, then $m_{i+1} = 3n_i - 5 + 9 = 3n_{i+1} - 5$. □

Lemmas 5 and 6 imply the following theorem.

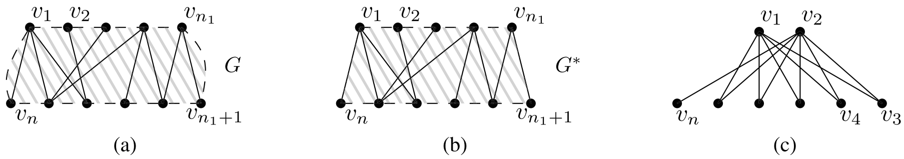

Fig. 6. Illustration for the proof of Theorem 2.

Theorem 1. *An outer fan-planar graph with n vertices has at most* $3n-5$ *edges, and this bound is tight for* $n \geq 5$.

An obvious consequence of Theorem 1 and of the definition of outer fan-planar graphs that are maximally dense is the following fact.

Corollary 2. *Every maximally dense outer fan-planar graph with* $n = 3h+2$ *vertices* ($h \geq 1$) *has* $3n-5$ *edges.*

Concerning 2-layer fan planar graphs, we already observed that a 2-layer fan planar graph G is an outer fan-planar graph. Also, since all vertices on the same layer form an independent set, G is bipartite.

Theorem 2. *A* 2*-layer fan-planar graph with n vertices has at most* $2n-4$ *edges, and this bound is tight for* $n \geq 3$.

Proof. Let G be a maximally dense 2-layer fan-planar graph with n vertices and m edges, and let Γ be a 2-layer fan-planar drawing of G. Denote by $V_1 = \{v_1, \dots, v_{n_1}\}$ and $V_2 = \{v_{n_1+1}, \dots, v_n\}$ the two independent sets of vertices of G. Without loss of generality, suppose that in $\Gamma\, v_i$ precedes v_{i+1} along the layer of V_1 (for $i = 1, \dots, n_1 - 1$), and v_j follows v_{j+1} along the layer of V_2 (for $j = n_1+1, \dots, n-1$). See Fig. 6(a). Construct from G a super-graph G^*, by adding an edge (v_i, v_{i+1}), for $i = 1, \dots, n_1 - 1$, and an edge (v_j, v_{j+1}), for $j = n_1+1, \dots, n$ (see Fig. 6(b)). Graph G^* is still outer fan-planar. Moreover, since G does not contain a K_5 subgraph (because it is bipartite), also G^* does not contain a K_5 subgraph, as otherwise at least three vertices of the same layer in G should form a 3-cycle in G^* (which does not happen by construction). Thus, by Lemma 3 and Property 1, the crossing graph of any outer fan-planar drawing of G^* contains only even cycles. Hence, denoted as m^* the number of edges of G^*, by Lemma 2 we have $m^* \leq 3n-6$, and therefore $m = m^* - (n-2) \leq 2n-4$. A family of 2-layer fan-planar graphs with $2n-4$ edges is the family of the bipartite complete graphs $K_{2,n-2}$ (see Fig. 6(c)). □

4. Fan-planar and *k*-planar graphs

A *k-planar drawing* is a drawing where each edge is crossed at most k times, and a *k-planar graph* is a graph that admits a k-planar drawing. Clearly, every 1-planar graph is also a fan-planar graph. Also, both the maximum number of edges of fan-planar graphs [35] and the maximum number of edges of 2-planar graphs [37] have been shown to be $5n-10$. Thus it is natural to ask what is the relationship between fan-planar and 2-planar graphs. More in general, we prove that there are fan-planar graphs that are not k-planar, for any $k \geq 1$, and that there are k-planar graphs (for $k > 1$) that are not fan-planar. The existence of fan-planar graphs that are not k-planar is proved with a counting argument on the minimum number of crossings of graph drawings. The *crossing number* $cr(G)$ of G is the smallest number of crossings required in any drawing of G. We give the following.

Theorem 3. *For any integer* $k \geq 1$ *there is a graph that is fan-planar but not k-planar.*

Proof. Consider the complete 3-partite graph $K_{1,3,h}$. This graph is fan-planar for every $h \geq 1$ (see Fig. 7(a)). It is known [7, 38] that $cr(K_{1,3,h}) = 2\left\lfloor \frac{h}{2} \right\rfloor \left\lfloor \frac{h-1}{2} \right\rfloor + \left\lceil \frac{h}{2} \right\rceil$. For $h = 4k+2$, we have $cr(K_{1,3,4k+2}) = 2\left\lfloor \frac{4k+2}{2} \right\rfloor \left\lfloor \frac{4k+1}{2} \right\rfloor + \left\lceil \frac{4k+2}{2} \right\rceil = 4k(2k+1) + 2k+1 = 8k^2+6k+1$. Thus, in every drawing of $K_{1,3,4k+2}$ there are at least $8k^2+6k+1$ crossings. On the other hand, in a k-planar drawing with m edges there can be at most $\frac{km}{2}$ crossings (because each edge is crossed at most k times and each crossing is shared by two edges). Since $K_{1,3,4k+2}$ has $16k+11$ edges, to be k-planar it should admit a drawing with at most $\frac{km}{2} = \frac{k(16k+11)}{2} = 8k^2 + \frac{11}{2}k$ crossings. Since $6k+1 > \frac{11}{2}k$ for every $k \geq 1$, $K_{1,3,4k+2}$ is not k-planar. □

To prove that for any $k > 1$ there are k-planar graphs that are not fan-planar (Theorem 4), we first give a technical result (Lemma 7), which will be also reused in Section 5. Let Γ be a fan-planar drawing of a graph. We may regard crossed edges of Γ as composed by *fragments*, where a fragment is the portion of the edge that is between two consecutive crossings or between one of the two end-vertices of the edge and the first crossing encountered while moving along the edge towards the other end-vertex. An edge that is not crossed does not have any fragment. Fig. 7(b) shows a fan-planar drawing of the K_7 graph and Fig. 7(c) shows the fragments of the drawing in Fig. 7(b). We consider two fragments *adjacent* if they share a common crossing or a common end-vertex. The next lemma provides an interesting and useful property.

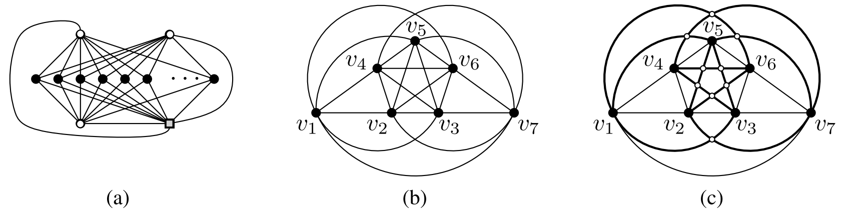

Fig. 7. (a) A fan-planar drawing of $K_{1,3,h}$. (b) A fan-planar drawing of the K_7 graph. (c) The fragments of the fan-planar drawing in (b) are the thicker lines.

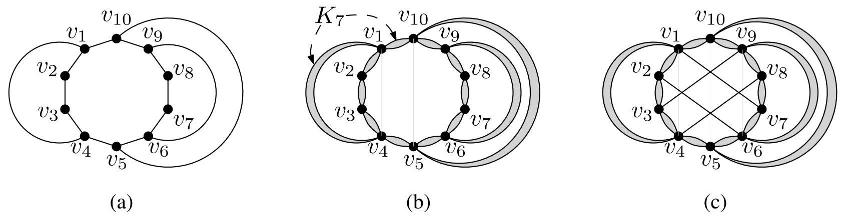

Fig. 8. (a)–(c) Illustration for the proof of Theorem 4: (a) graph G'; (b) graph G''; (c) graph G.

Lemma 7. *In any fan-planar drawing of the K_7 graph, any pair of vertices is joined by a sequence of adjacent fragments.*

Proof. Consider a fan-planar drawing Γ of the K_7 graph and consider any vertex v_i of it ($i = 1, 2, \dots, 7$). Vertex v_i must be incident to some fragment in Γ. Indeed, if vertex v_i had no incident fragment, all the edges incident to v_i were uncrossed in Γ, and removing v_i and all its incident edges from Γ would yield a fan-planar drawing of the K_6 graph where all vertices are on the same face (this would clearly imply the existence of an outer fan-planar drawing of K_6). This is however impossible by Lemma 5 (K_6 has 6 vertices and 15 edges, i.e., more than $3 \cdot 6 - 5$ edges). Since a fragment is originated by a crossed edge and since two crossing edges are not adjacent, we have that vertex v_i is linked by a sequence of fragments to at least other three distinct vertices. Therefore, the vertices of K_7 are linked by sequences of fragments in groups of at least four. Being seven vertices in total, this implies that all vertices of K_7 are linked together by sequences of fragments. □

Theorem 4. *For any integer $k > 1$ there is a graph that is k-planar but not fan-planar.*

Proof. Since 2-planar graphs are also k-planar graphs, for $k > 1$, it is sufficient to prove that there is a 2-planar graph that is not fan-planar. Let G' be a graph consisting of a cycle $C = (v_1, v_2, \dots, v_{10})$ and of the edges (v_1, v_4), (v_5, v_{10}), (v_6, v_9) (see Fig. 8(a)). Let G'' be the graph obtained from G' by replacing each edge (v_i, v_j) ($1 \le i, j \le 10$) with a copy of K_7, whose vertices are denoted as $u_1, u_2, \dots, u_7$, so that $v_i = u_1$ and $v_j = u_7$ (see Fig. 8(b)). The copy of K_7 that replaces (v_i, v_j) is denoted as $K_7^{i,j}$. Let G be the graph obtained from G'' by adding the four edges (v_1, v_7), (v_2, v_6), (v_3, v_9), and (v_4, v_8) (see Fig. 8(c)). Graph G is 2-planar. Namely, planarly embed G' as shown in Fig. 8(a). Construct a drawing Γ of G by replacing each edge of G' with a drawing of $K_7^{i,j}$ like the one in Fig. 7(a) (see Fig. 8(b)), and draw the edges (v_1, v_7), (v_2, v_6), (v_3, v_9), (v_4, v_8) inside C as in Fig. 8(c). Drawing Γ is 2-planar.

We now prove that G is not fan-planar. Suppose by contradiction that G has a fan-planar drawing Γ. By Lemma 7, for each $K_7^{i,j}$ ($1 \le i, j \le 10$) there is a sequence of fragments leading from $v_i = u_1$ to $v_j = u_7$; we call it the *spine* of $K_7^{i,j}$. Delete from Γ all fragments except those in the spine of each $K_7^{i,j}$; replace the endpoints of the fragments that were crossings of Γ with dummy vertices; finally, delete all non-crossed edges and isolated vertices. The remaining drawing Γ' is a planar drawing, because each spine cannot be crossed by any other fragment or edge, otherwise the drawing is no longer fan-planar. We denote by C' the cycle of spines corresponding to C, by S the set of spines of $K_7^{1,4}$, $K_7^{5,10}$, and $K_7^{6,9}$, and by F the set of edges (v_1, v_7), (v_2, v_6), (v_3, v_9), and (v_4, v_8). Since Γ' is planar each spine in S is either inside C' or outside C' in Γ', and therefore in Γ. Furthermore, since the edges of F cannot cross the spine of any $K_7^{i,j}$ ($1 \le i, j \le 10$) in Γ, each of them must be either inside or outside C' in Γ. Given two elements of $S \cup F$ we say that they are on the *same side* of C' if they are both inside or both outside C' in Γ, otherwise we say that they are on *opposite sides* of C'. Since there cannot be a crossing between an element of F and one of S, each of the two edges (v_2, v_6) and (v_3, v_9) must be on the opposite side of C' with respect to $K_7^{1,4}$. Analogously, each of the two edges (v_1, v_7) and (v_4, v_8) must be on the opposite side of C' with respect to $K_7^{6,9}$. Finally, $K_7^{5,10}$ must be on the opposite side of C' with respect to (v_1, v_7), (v_2, v_6), (v_3, v_9), and (v_4, v_8). It follows that the spines of S and the edges of F must be on opposite sides of C', which implies that each edge in F is crossed by two independent edges (see Fig. 8(c)), a contradiction. □

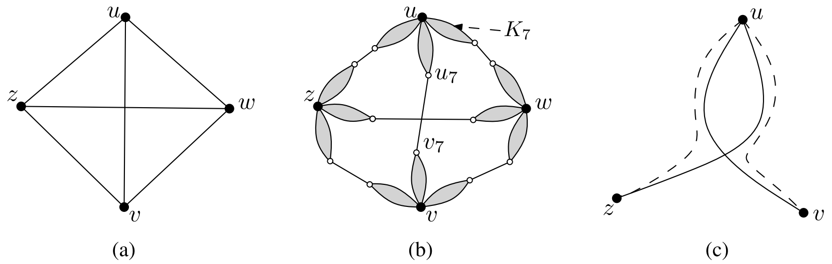

Fig. 9. Illustration of the reduction used in Theorem 5. (a) An instance G of 1-planarity testing. (b) The reduced instance G_f of fan-planarity testing. (c) Two adjacent edges of G that cross one to another in Γ; the crossing can be removed by rerouting the two edges as shown by the dashed lines.

5. Complexity of the fan-planarity testing problem

We exploit the results of Sections 3 and 4 to prove that testing whether a graph is fan-planar in the variable embedding setting is NP-complete. We call this problem the *fan-planarity testing*. We use a reduction from the *1-planarity testing*, which is NP-complete in the variable embedding setting [32,36]. The 1-planarity testing asks whether a given graph admits a 1-planar drawing. We prove the following.

Theorem 5. *Fan-planarity testing is NP-complete.*

Proof. Given a drawing of a graph it is immediate to check if it satisfies the two fan-planarity constraints. A non-deterministic algorithm to generate all drawings with k crossings first considers all possible k pairs of edges that cross (and the order of the crossings along the edges) with a technique similar to the one in [31]. Then it replaces crossings with dummy vertices and non-deterministically generates all possible planar embeddings of the obtained graphs. Hence, the problem belongs to NP.

We now prove the hardness. Given an instance $G = (V, E)$ of the 1-planarity testing we build an instance $G_f = (V_f, E_f)$ of the fan-planarity testing by replacing each edge $(u, v) \in E$ with two K_7 graphs with vertices $u = u_1, u_2, \ldots, u_7$ and $v = v_1, v_2, \ldots, v_7$, called *attachment gadgets* and joined by a *spanning* edge (u_7, v_7) (see Fig. 9 for an illustration). $G_f = (V_f, E_f)$ can be constructed in polynomial time, having $|V_f| = |V| + |E| \times 12$ vertices and $|E_f| = |E| \times 43$ edges, where $|E| \times 42$ of them belong to the attachment gadgets and the remaining $|E|$ are spanning edges that join different attachment gadgets. We show that G is 1-planar if and only if G_f is fan-planar. If G admits a 1-planar drawing, replace each edge (u, v) of G with two fan-planar drawings of K_7 like those depicted in Fig. 7(b) and with edge (u_7, v_7), in such a way that the possible crossing of (u, v) occurs on (u_7, v_7). The obtained drawing of G_f is fan-planar since each attachment gadget has a fan-planar drawing and each spanning edge has at most one crossing. Conversely, suppose G_f admits a fan-planar drawing Γ_f. By Lemma 7, for any attachment gadget of G_f attached to vertex u, there is at least a sequence of fragments leading from $u = u_1$ to u_7. As in the proof of Theorem 4, call such a sequence of fragments the *spine* of the attachment gadget. Delete from Γ_f all fragments except those in the spines. Delete from Γ_f all uncrossed edges except the spanning edges. Remove also isolated vertices. A drawing Γ of G is obtained, where the drawing of edge (u, v) is given by the spine from $u = u_1$ to u_7, the spanning edge (u_7, v_7), and the spine from v_7 to $v_1 = v$. Observe that, $u \neq v$, as otherwise there would be a self-loop in G. We claim that Γ is a 1-planar drawing of G. Indeed, fragments in the spines cannot be crossed by any other fragment or spanning edge of Γ_f. It follows that spanning edges can cross only among themselves in Γ_f. However, they can cross only once, as they are a matching of G_f and Γ_f is fan-planar. Hence, Γ is a 1-planar drawing, but not necessarily simple; indeed, it may happen that two crossing edges (u, v) and (w, z) in Γ share an end-vertex, say $u = w$ (this happens when in Γ_f there are two crossing spanning edges of two K_7 attached to u). The crossing between (u, v) and (u, z) in Γ can be easily removed by rerouting the edges (see Fig. 9(c)). □

6. Conclusions and open problems

We extended the study of fan-planar drawings started by Kaufmann and Ueckerdt [35]. We showed tight bounds on the density of constrained versions of fan-planar drawings and clarified the relationship between fan-planarity and k-planarity. Also, we proved that the fan-planarity testing in the variable embedding setting is NP-complete. A related work by Bekos et al. [10] proves that the fan-planarity testing problem is NP-hard also if the circular ordering of the edges around each vertex is given and cannot be changed, i.e., if the graph is given with a so-called *rotation system*; on the positive side, they prove that it is polynomial-time solvable to recognize graphs that are maximal outer fan-planar, i.e., graphs that admit an outer fan-planar drawing and that cannot be augmented with any edge without loosing this property (see also [11] for a technical report). Instead, the complexity of recognizing outer fan-planar drawable graphs in the general case remains an open problem.

Several other interesting research directions can be explored, including the following:

Research Direction 1. From the combinatorial point of view, it would be interesting to establish lower bounds on the number of edges of maximal fan-planar graphs, also in the outer and in the 2-layer constrained versions. Results on this kind of question have been established for example for maximal 1-planar and 2-planar graphs [9,15].

Research Direction 2. On the algorithmic side, it is still unknown the complexity of recognizing outer fan-planar or 2-layer fan-planar graphs that are not necessarily maximal. In this research line, efficient recognition algorithms have been provided for example for outer 1-planar graphs [8,33] and for 2-layer RAC graphs [19]. It is also still unknown the complexity of recognizing maximal or maximally dense fan-planar graphs.

Research Direction 3. From an application-oriented perspective, it would be interesting to develop algorithms that are able to combine fan-planarity and bundling techniques to create confluent drawings with few crossings (similarly to the examples of Figs. 1(b) and 1(c)).

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