T(A) = T(B)?

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Abstract. The equivalence problem for deterministic pushdown transducers with inputs in a free monoid X^* and outputs in a linear group $H = \mathrm{GL}_n(\mathbb{Q})$, is shown to be *decidable*.

Keywords: deterministic pushdown transducers; rational series; finite dimensional vector spaces; matrix semi-groups; test-sets; complete formal systems.

1 Introduction

We show here that, given two deterministic pushdown *transducers* (dpdt's for short) A, B from a free monoid X^* into a *linear group* $H = \mathrm{GL}_n(\mathbb{Q})$, one can decide whether $\mathrm{S}(A) = \mathrm{S}(B)$ or not (i.e. whether A, B compute the same function $f : X^* \to H$).

This main result generalizes the decidability of the equivalence problem for deterministic pushdown *automata* ([Sén97],[Sén98b]). It immediately implies that the same problem is decidable for any group (or monoid) H, as soon as H is embeddable in a linear group $\mathrm{GL}_n(\mathbb{Q})$. Hence we obtain as a corollary the decidability of the equivalence problem for dpdt's A, B from a free monoid X^* into a *free group* $H = \mathrm{F}(Y)$, or a *free monoid* $H = Y^*$.

Our main result generalizes several other known results about transducers:

- the case where $H = Y^*$ had been adressed in previous works [IR81,CK86b, TS89] and was known to be decidable in the case where A is a *strict-real-time* dpdt while B is a general dpdt [TS89].

- the case where H is an *abelian* group was known to be decidable by ([Sén98b, section 11], [Sén98a]).

Our solution leans on a combination of the methods developped in ([Sén97], [Sén98b]) for the equivalence problem for dpda's, with the methods developped in ([Gub85], [CK86a]) for Ehrenfeucht's conjecture (about the existence and computability of test-sets).

The full proofs corresponding to this extended abstract can be found in [Sén99]. More general information about equivalence problems for transducers can be found in [Cul90],[Lis96].

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2 Preliminaries

2.1 Formal Power Series

The reader is refered to [BR88] for formal power series. Let us just review some vocabulary. Let $(\mathsf{B}, +, \cdot, 0, 1)$ where $\mathsf{B} = \{0, 1\}$ denote the semi-ring of "booleans". Let M be some monoid. By $(\mathsf{B}\langle\langle\ M\ \rangle\rangle, +, \cdot, \emptyset, 1_M)$ we denote the semi-ring of *boolean series* over M: the set $\mathsf{B}\langle\langle\ M\ \rangle\rangle$ is defined as B^M; the sum and product are defined as usual; $\mathsf{B}\langle\langle\ M\ \rangle\rangle$ is isomorphic with $(\mathcal{P}(M), \cup, \cdot, \emptyset, \{1_M\})$. The usual ordering $\leq$ on B extends to $\mathsf{B}\langle\langle\ M\ \rangle\rangle$ by: $S \leq S'$ iff $\forall w \in M, S_w \leq S'_w$.
We focus here on monoids of the form $M = K \times W^*$ (the direct product of the group K by the free monoid W^*) and $M = K * W^*$ (the free product of K by W^*, see [LS77, p.174-178] for more information on free products). Let $M = K * W^*$ and M' be another monoid containing K. A map $\psi : \mathsf{B}\langle\langle\ M\ \rangle\rangle \to \mathsf{B}\langle\langle\ M'\ \rangle\rangle$ is called a *substitution* iff it is a semi-ring homomorphism which is σ-additive and which induces the identity map on K (i.e. $\forall k \in K, \psi(k) = k$).

2.2 Finite *K*-Automata

***K*-automata** Let $(K, \cdot)$ be some group. We call a finite K-automaton over the finite alphabet W any 7-tuple $\mathcal{M} =< K, W, Q, \delta, k_0, q_0, Q' >$ such that Q is the finite set of states, δ, the set of transitions, is a finite subset of $Q \times W \times K \times Q$, $k_0 \in K$, $q_0 \in Q$ and $Q' \subseteq Q$. The series recognized by $\mathcal{M}$, $\mathrm{S}(\mathcal{M})$, is the element of $\mathsf{B}\langle\langle\ K * W^*\ \rangle\rangle$defined by: $\mathrm{S}(\mathcal{M}) = k_0 \cdot A \cdot B^* \cdot C$, where $A \in \mathsf{B}_{1,Q}\langle\langle\ K * W^*\ \rangle\rangle, B \in \mathsf{B}_{Q,Q}\langle\langle\ K * W^*\ \rangle\rangle$, and $C \in \mathsf{B}_{Q,1}\langle\langle\ K * W^*\ \rangle\rangle$ are given by: $A_{1,q} = \emptyset(\text{ if } q \neq q_0)$, $A_{1,q_0} = \epsilon$, $B_{q,q'} = \sum_{(q,v,k,q') \in \delta} v \cdot k$, $C_{q,1} = \emptyset\ (\text{ if } q \notin Q'), C_{q,1} = \epsilon\ (\text{ if } q \in Q')$. $\mathcal{M}$ is said W-*deterministic* iff,

$$\forall r \in Q, \forall v \in W, \mathrm{Card}(\{(k, r') \in K \times Q \mid (r, v, k, r') \in \delta\}) \leq 1. \tag{1}$$

2.3 Pushdown *H*-Automata

Let $(H, \cdot)$ be some group. We call a *pushdown H-automaton* over the finite alphabet X any 7-tuple

$$\mathcal{M} =< H, X, Z, Q, \delta, q_0, z_0 >$$

where Z is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, z_0 is the initial stack-symbol and $\delta : QZ \times (X \cup \{\epsilon\}) \to \mathcal{P}_f(H \times QZ^*)$, is the transition mapping. Let $q, q' \in Q, \omega, \omega' \in Z^*, z \in Z, h \in H, w \in X^*$ and $a \in X \cup \{\epsilon\}$; we note $(qz\omega, h, aw) \longmapsto_{\mathcal{M}} (q'\omega'\omega, h \cdot h', w)$ if $(h', q'\omega') \in \delta(qz, a)$. $\overset{*}{\longmapsto}_{\mathcal{M}}$ is the reflexive and transitive closure of $\longmapsto_{\mathcal{M}}$. For every $q\omega, q'\omega' \in QZ^*$ and $h \in H, w \in X^*$, we note $q\omega \overset{(h,w)}{\longrightarrow}_{\mathcal{M}} q'\omega'$ iff $(q\omega, 1_H, w) \overset{*}{\longmapsto}_{\mathcal{M}} (q'\omega', h, \epsilon)$. $\mathcal{M}$ is said *deterministic* iff it fulfills the following disjunction:

$$\text{either } \mathrm{Card}(\delta(qz, \epsilon)) = 1 \text{ and for every } x \in X, \mathrm{Card}(\delta(qz, x)) = 0, \tag{2}$$

$$\text{or } \mathrm{Card}(\delta(qz,\epsilon)) = 0 \text{ and for every } x \in X, \mathrm{Card}(\delta(qz,x)) \le 1. \tag{3}$$

We call *mode* every element of $QZ \cup \{\epsilon\}$. For every $q \in Q, z \in Z$, qz is said *ϵ-bound* (respectively *ϵ-free*) iff condition (2) (resp. condition (3)) in the above definition of deterministic H-automata is realized. The mode ϵ is said ϵ-free.
A H-dpda $\mathcal{M}$ is said *normalized* iff, for every $qz \in QZ, x \in X$:

$$q'\omega' \in \delta_2(qz,x) \Rightarrow \mid \omega' \mid \le 2, \text{ and } q'\omega' \in \delta_2(qz,\epsilon) \Rightarrow \mid \omega' \mid = 0, \tag{4}$$

where $\delta_2 : QZ \times (X \cup \{\epsilon\}) \to \mathcal{P}_f(QZ^*)$, is the second component of the map δ.
Given some deterministic pushdown H-automaton $\mathcal{M}$ and a finite set $F \subseteq QZ^*$ of configurations, the *series* (in $\mathsf{B}\langle\langle\ H \times X^*\ \rangle\rangle$) *recognized by $\mathcal{M}$ with final configurations F* is defined by

$$\mathrm{S}(\mathcal{M},F) = \sum_{c \in F} \sum_{q_0 z_0 \xrightarrow{(h,w)}_{\mathcal{M}} c} (h,w).$$

For every pair (h,w) having a coefficient 1 in the series $\mathrm{S}(\mathcal{M},F)$, h can be seen as the "output" of the automaton $\mathcal{M}$ on the "input" w. $\mathcal{M}$ can then be named a deterministic pushdown *transducer* from X^* to H.
We suppose that Z contains a special symbol e subject to the property:

$$\forall q \in Q, \delta(qe,\epsilon) = \{(1_H,q)\} \text{ and } \mathrm{im}(\delta_2) \subseteq \mathcal{P}_f(Q(Z - \{e\})^*). \tag{5}$$

2.4 Monoids Acting on Semi-rings

Actions of monoids The general notions of right-action and σ-right-action of a monoid over a semi-ring is the same as in [Sén97, §2.3.2].

***M* acting on $\mathsf{B}\langle\langle\ M\ \rangle\rangle$ (residuals action)** We recall the following classical σ-right-action $\bullet$ of the monoid M over the semi-ring $\mathsf{B}\langle\langle\ M\ \rangle\rangle$: for all $S, S' \in \mathsf{B}\langle\langle\ M\ \rangle\rangle, u \in M$

$$S \bullet u = S' \Leftrightarrow \forall w \in M, \ \ S'_w = S_{u \cdot w}. \tag{6}$$

(i.e. $S \bullet u$ is the *left-quotient* of S by u , or the *residual* of S by u). For every $S \in \mathsf{B}\langle\langle\ M\ \rangle\rangle$ we denote by $\mathsf{Q}(S)$ the set of residuals of S: $\mathsf{Q}(S) = \{S \bullet u \mid u \in M\}$.
Let us denote by $\mathsf{B}_{n,m}\langle\langle\ M\ \rangle\rangle$ the set of matrices of dimension n, m with entries in $\mathsf{B}\langle\langle\ M\ \rangle\rangle$. The right-action $\bullet$ on $\mathsf{B}\langle\langle\ M\ \rangle\rangle$ is extended componentwise to $\mathsf{B}_{n,m}\ \langle\langle\ M\ \rangle\rangle$: for every $S = (s_{i,j})$, $u \in M$, the matrix $T = S \bullet u$ is defined by $t_{i,j} = s_{i,j} \bullet u$. The notation $\mathsf{Q}(S) = \{S \bullet u \mid u \in M\}$, is extended to matrices as well. Given $n \ge 1, m \ge 1$, and $S \in \mathsf{B}_{n,m}\langle\langle\ M\ \rangle\rangle$ we denote by $\mathsf{Q}_r(S)$ the set of *row-residuals* of S:

$$\mathsf{Q}_r(S) = \bigcup_{1 \le i \le n} \mathsf{Q}(S_{i,*}).$$

The ordering $\le$ on B is extended componentwise to $\mathsf{B}_{n,m}\langle\langle\ M\ \rangle\rangle$.

$K \times X^*$ acting on $\mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle$ (automaton action) Let us fix now a deterministic (normalized) H-dpda $\mathcal{M}$ and a group K containing H.

H-grammar The *variable* alphabet $V_{\mathcal{M}}$ associated with $\mathcal{M}$ is defined as: $V_{\mathcal{M}} = \{[p,z,q] \mid p,q \in Q, z \in Z\}$. The context-free H-grammar associated with $\mathcal{M}$ is then $G_{\mathcal{M}} =< H, X, V_{\mathcal{M}}, P_{\mathcal{M}} >$ where $P_{\mathcal{M}} \subseteq V_{\mathcal{M}} \times (H * (X \cup V_{\mathcal{M}})^*)$ is the set of all the pairs of one of the following forms:

$$([p,z,q], x{\cdot}h{\cdot}[p',z_1,p''][p'',z_2,q]) \text{ or } ([p,z,q], x{\cdot}h{\cdot}[p',z',q]) \text{ or } ([p,z,q], a{\cdot}h) \quad (7)$$

where $p,q,p',p'' \in Q, x \in X, a \in X \cup \{\epsilon\}, (h, p'z_1z_2) \in \delta(pz,x), (h,p'z') \in \delta(pz,x), (h,q) \in \delta(pz,a)$.

Action $\otimes$ As long as the automaton $\mathcal{M}$ is fixed, we can safely skip the indexes in $V_{\mathcal{M}}, P_{\mathcal{M}}$. We define a σ-right-action $\otimes$ of the monoid $K \times (X \cup \{e\})^*$ over the semi-ring $\mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle$by: for every $p,q \in Q, z \in Z, x \in X, h \in H, k \in K$:

$$[p,z,q] \otimes x = \sum_{([p,z,q],m)\in P} m \bullet (1_H, x), \quad (8)$$

$$[p,z,q] \otimes e = h \text{ iff } ([p,z,q],h) \in P, \quad (9)$$

$$[p,z,q] \otimes e = \emptyset \text{ iff } (\{[p,z,q]\} \times H) \cap P = \emptyset, \quad (10)$$

$$k \otimes x = \emptyset, \ \ k \otimes e = \emptyset. \quad (11)$$

The action is extended by: for every $k \in K, \beta \in K * V^*, y \in X \cup \{e\}, S \in \mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle, k \in K$,

$$(k \cdot [p,z,q] \cdot \beta) \otimes y = k \cdot ([p,z,q] \otimes y) \cdot \beta, \ \ S \otimes k = k^{-1} \cdot S. \quad (12)$$

Action $\odot$ We define the map $\rho_\epsilon : \mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle \to \mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle$ as the unique σ-additive map such that,

$$\rho_\epsilon(\emptyset) = \emptyset, \ \ \rho_\epsilon(\epsilon) = \epsilon,$$

for every $k \in K, S \in \mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle$,

$$\rho_\epsilon(k \cdot S) = k \cdot \rho_\epsilon(S),$$

and for every $p \in Q, z \in Z, q \in Q, \beta \in K * V^*$,

$$\rho_\epsilon([p,z,q] \cdot \beta) = \rho_\epsilon(([p,z,q] \otimes e) \cdot \beta) \text{ if } pz \text{ is } \epsilon-\text{bound and,}$$

$$\rho_\epsilon([p,z,q] \cdot \beta) = [p,z,q] \cdot \beta \text{ if } pz \text{ is } \epsilon-\text{free.}$$

We call ρ_ϵ the ϵ-reduction map. We then define $\odot$ as the unique right-action of the monoid $K \times X^*$ over the semi-ring $\mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle$ such that: for every $S \in \mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle, k \in K, x \in X$,

$$S \odot (k,x) = \rho_\epsilon(\rho_\epsilon(S) \otimes (k,x)).$$

Let us consider the unique substitution $\varphi : \mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle \ \to \mathsf{B}\langle\langle\ K \times X^*\ \rangle\rangle$ fulfilling: for every $v \in V$,

$$\varphi(v) = \sum_{\substack{k \in K, u \in X^* \\ v \odot (k,u) = \epsilon}} (k, u)$$

(in other words, φ maps every subset $L \subseteq K * V^*$ on the set generated by the grammar G from the set of axioms L).

Lemma 21 *For every $S \in \mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle, k \in K, u \in X^*$,*

1. $\varphi(\rho_\epsilon(S)) = \varphi(S)$
2. $\varphi(S \odot (k, u)) = \varphi(S) \bullet (k, u)$, *i.e. φ is a morphism of right-actions.*

We denote by $\equiv$ the kernel of φ i.e.: for every $S, T \in \mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle$,

$$S \equiv T \Leftrightarrow \varphi(S) = \varphi(T).$$

3 Effective Test-Sets for Morphic Sets

3.1 Morphic Sets

Let $(M, \cdot, 1_M)$ be a monoid.

Definition 31 *A subset $L \subseteq M$ is said morphic iff there exists an element $u \in M$, a finite sequence $\psi_1, \psi_2, \ldots, \psi_m (m \geq 1)$ of homomorphisms $\psi_i : M \to M$ and a rational subset $\mathcal{R}$ of $\{\psi_1, \psi_2, \ldots, \psi_m\}^*$ such that:*

$$L = \{\psi(u) \mid \psi \in \mathcal{R}\}.$$

Remark 32 *In the particular case where M is a finitely generated free monoid X^*, and $\mathcal{R} = \{\psi_1, \psi_2, \ldots, \psi_m\}^*$ the notion of "morphic subset" coincides with the classical notion of "DT0L language" ([RS80]).*

3.2 Test-Sets

Definition 33 *Let $(M, \cdot, 1_M), (N, \cdot, 1_N)$ be two monoids and $L \subseteq M$. A subset $F \subseteq L$ is called a* test-set *for L with respect to N iff, F is finite and, for every pair of homorphisms $\eta, \eta' : M \to N$, if η agrees with η' on F, then it also agrees on L:*

$$[\forall x \in F, \eta(x) = \eta'(x)] \Rightarrow [\forall x \in L, \eta(x) = \eta'(x)].$$

Theorem 34 *Let $(M, \cdot, 1_M)$ be a finitely generated monoid, L be a morphic subset of M, n a non-negative integer and $H = \mathrm{M}_{n,n}(\mathbb{Q})$ (the monoid of square n by n matrices with entries in $\mathbb{Q}$). Then L admits a test-set F with respect to H and such a test-set can be computed from any $(m+2)$-tuple $u, \psi_1, \ldots, \psi_m, \mathcal{R}$ defining L.*

We use here the arguments of [CK86a, p.79], combined with the main idea of [Gub85] for establishing the *existence* of a test-set. We then use the algorithm of [Buc85, p.11-13] to *construct* such a test-set.

4 Series and Matrices

4.1 Deterministic Series and Matrices

Let us fix a group $(K, \cdot)$ and a structured alphabet $(W, \smile)$.(We recall it just means that $\smile$ is an equivalence relation over the set W).

Definition 41 *Let $n, m \in \mathbb{N}, S, T \in \mathsf{B}_{n,m}\langle\langle\ K * W^*\ \rangle\rangle$. S, T are said* proportional *and we note $S \approx T$, if and only if, there exists $k \in K$ such that $S = k \cdot T$.*

Definition 42 *Let $m \in \mathbb{N}, S \in \mathsf{B}_{1,m}\langle\langle\ K * W^*\ \rangle\rangle : S = (S_1, \cdots, S_m)$. S is said* left-deterministic *iff either*
(1) $\forall i \in [1, m], S_i = \emptyset$ *or*
(2) $\exists i_0 \in [1, m], S_{i_0} \approx \epsilon$ *and* $\forall i \neq i_0, S_i = \emptyset$ *or*
(3) $\exists i_0 \in [1, m], S_{i_0} \not\approx \emptyset$ *and* $S_{i_0} \not\approx \epsilon$ *and* $\forall i, j \in [1, m], \forall k \in K, A \in W, \beta, \gamma \in K*W^*, [k \cdot A \cdot \beta \leq S_i, \gamma \leq S_j] \Rightarrow \exists A' \in W, \beta' \in K*W^*, A \smile A'$ *and* $\gamma = k \cdot A' \cdot \beta'$.

Definition 43 *Let $m \geq 1, S \in \mathsf{B}_{1,m}\langle\langle\ K * W^*\ \rangle\rangle$. S is said* deterministic *iff, for every $u \in K * W^*$, $S \bullet u$ is left-deterministic.*

Definition 44 *Let $S \in \mathsf{B}_{n,m}\langle\langle\ K * W^*\ \rangle\rangle$. S is said* deterministic *iff, for every $i \in [1, n]$, $S_{i,*}$ is a deterministic row-vector.*

We denote by $\mathsf{DB}_{n,m}\langle\langle\ K * W^*\ \rangle\rangle$ the subset of deterministic matrices of dimension (n, m) over $\mathsf{B}\langle\langle\ K * W^*\ \rangle\rangle$.

Lemma 45 *For every $S \in \mathsf{DB}_{n,m}\langle\langle\ K * W^*\ \rangle\rangle, T \in \mathsf{DB}_{m,s}\langle\langle\ K * W^*\ \rangle\rangle$, $S \cdot T \in \mathsf{DB}_{n,s}\langle\langle\ K * W^*\ \rangle\rangle$.*

Let us call a matrix $S \in \mathsf{B}_{n,m}\langle\langle\ K * W^*\ \rangle\rangle$ *rational* iff every component $S_{i,j}$ for $i \in [1, n], j \in [1, m]$ is rational.

Proposition 46 *Let $m \geq 1$, $S \in \mathsf{DB}_{1,m}\langle\langle\ K * W^*\ \rangle\rangle$. Then S is rational if and only if $\mathsf{Q}(S)/\approx$ is finite*

Norm Proposition 46 suggests the following notion of *norm*. For every $S \in \mathsf{DB}_{n,m}\langle\langle\ K * W^*\ \rangle\rangle$, the norm of S is defined by:

$$\|S\| = \mathrm{Card}(\mathsf{Q}_r(S)/\approx) \in \mathsf{N} \cup \{\infty\}.$$

It follows from proposition 46 that a deterministic matrix $S \in \mathsf{DB}_{n,m}\langle\langle\ K*W^*\ \rangle\rangle$ is rational iff it has a finite norm. As well, a special notion of deterministic "finite m-K-automata" can be devised, such that these automata recognize exactly the deterministic rational $(1, m)$-row-vectors (see [Sén99, definition 4.11]).

Lemma 47 *Let $S \in \mathsf{DB}_{n,m}\langle\langle\ K * W^*\ \rangle\rangle, T \in \mathsf{DB}_{m,s}\langle\langle\ K * W^*\ \rangle\rangle$. Then $\|S \cdot T\| \leq \|S\| + \|T\|$.*

4.2 Algebraic Properties

Let $(W, \smile)$ be the structured alphabet $(V, \smile)$ associated with the H-dpda $\mathcal{M}$ and let K be a group containing H. The notion of *linear combination* of series is defined as in [Sén97, §3.2.1]. The subsequent notions of *space* of series and *linear independence* of series can be easily adapted to $\mathsf{DRB}\langle\langle\ K * W^*\ \rangle\rangle$.(We recall this last notion originated in [Mei89, lemma 11 p.589 [1]]).

5 Deduction Systems

5.1 General Systems

We use here a notion of *deduction system* which was inspired by [Cou83]. The reader is referred to [Sén97, section 4] for a precise definition of this notion and of the related notion of *strategy*.

5.2 Systems $\mathcal{K}_0, \mathcal{H}_0$

Let $H = \mathrm{GL}_n(\mathbb{Q})$. We define here a particular deduction system $\mathcal{H}_0$ "Taylored for the equivalence problem for H-dpda's" and also an auxiliary more general deduction system $\mathcal{K}_0$.

Given a fixed H-dpda $\mathcal{M}$ over the terminal alphabet X, we consider the variable alphabet V associated to $\mathcal{M}$ (see §2.4), a denumerable alphabet U (we call it the alphabet of *parameters*), the group $K = \mathrm{F}(U) * H$ [2] and the set $\mathsf{DRB}\langle\langle\ K * V^*\ \rangle\rangle$ (the set of Deterministic Rational Boolean series over $K * V^*$).

The set of assertions is defined by :

$$\mathcal{A} = \mathbb{N} \times \mathsf{DRB}\langle\langle\ K * V^*\ \rangle\rangle \times \mathsf{DRB}\langle\langle\ K * V^*\ \rangle\rangle$$

i.e. an assertion is here a *weighted equation* over $\mathsf{DRB}\langle\langle\ K * V^*\ \rangle\rangle$.
The "cost-function" $J : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is defined by :

$$J(n, S, S') = n + 2 \cdot \mathrm{Div}(S, S'),$$

where $\mathrm{Div}(S, S')$, the divergence between S and S', is defined by:
$\mathrm{Div}(S, S') = \inf\{|u|, u \in X^*, \exists k \in K, (k, u) \leq \varphi(S) \Leftrightarrow (k, u)) \not\leq \varphi(S')\}$.
(Notice that: $J(n, S, S') = \infty \iff S \equiv S'$).

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[1] numbering of the english version
[2] these values of H, K are fixed, up to corollary 63

We define a binary relation $|\!\!\vdash\!\!-\ \subset \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

$$
\begin{array}{ll}
(K0)\ \{(p,S,T)\} & |\!\!\vdash\!\!-\ (p+1,S,T) \\
(K1)\ \{(p,S,T)\} & |\!\!\vdash\!\!-\ (p,T,S) \\
(K2)\ \{(p,S,S'),(p,S',S'')\} & |\!\!\vdash\!\!-\ (p,S,S'') \\
(K3)\ \emptyset & |\!\!\vdash\!\!-\ (0,S,S) \\
(K'3)\ \emptyset & |\!\!\vdash\!\!-\ (0,S,T) \text{ for } T \in \{\emptyset,\epsilon\}, S \equiv T \\
(K4)\ \{(p+1,S\odot x,T\odot x) \mid x \in X\} & |\!\!\vdash\!\!-\ (p,S,T) \\
\ \text{where } (\forall k \in K, S \not\equiv k \wedge T \not\equiv k) & \\
(K5)\ \{(p,S,S')\} & |\!\!\vdash\!\!-\ (p+2,S\odot x,S'\odot x) \\
\ \text{for } x \in X & \\
(K6)\ \{(p,S_1\cdot T+S_2,T)\} & |\!\!\vdash\!\!-\ (p,S_1^*\cdot S_2,T) \\
\ \text{where } (\forall k \in K, S_1 \not\equiv k) & \\
(K7)\ \{(p,S_1,T_1),(p,S_2,T_2)\} & |\!\!\vdash\!\!-\ (p,S_1+S_2,T_1+T_2) \\
(K8)\ \{(p,S,S')\} & |\!\!\vdash\!\!-\ (p,S\cdot T,S'\cdot T) \\
(K9)\ \{(p,T,T')\} & |\!\!\vdash\!\!-\ (p,S\cdot T,S\cdot T') \\
(K10)\ \emptyset & |\!\!\vdash\!\!-\ (0,S,\rho_\epsilon(S)) \\
(K11)\ \emptyset & |\!\!\vdash\!\!-\ (0,S,\rho_e(S)),
\end{array}
$$

where $p \in \mathbb{N}, S,S',T,T' \in \mathsf{DRB}\langle\langle\ K * V^*\ \rangle\rangle, (S_1,S_2),(T_1,T_2) \in \mathsf{DRB}_{1,2}\langle\langle\ K * V^*\ \rangle\rangle$. The map ρ_ϵ involved in rule $(K10)$ was defined in §2.4 and we define the new map ρ_e involved in rule $(K11)$ as the unique substitution $\mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle \to \mathsf{B}\langle\langle\ K * V^*\ \rangle\rangle$ such that, for every $p,q \in Q, z \in Z$,

$$\rho_e([p,e,q] = \emptyset(ifp \neq q), \quad \rho_e([p,e,q] = \epsilon(ifp = q), \quad \rho_e([p,z,q] = [p,z,q](ifz \neq e),$$

where e is the "dummy" symbol introduced in (5). ρ_e maps every $S \in \mathsf{DRB}\langle\langle\ K * V^*\ \rangle\rangle$ into an image $\rho_e(S) \in \mathsf{DRB}\langle\langle\ K * V^*\ \rangle\rangle$.

Let us define $\vdash\!\!-$ by : for every $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$,

$$P \vdash\!\!- A \Longleftrightarrow P\ \overset{<*>}{|\!\!\vdash\!\!-} \circ \overset{[1]}{|\!\!\vdash\!\!-}_{0,3,4,10,11} \circ\ \overset{<*>}{|\!\!\vdash\!\!-}\ \{A\},$$

where $|\!\!\vdash\!\!-_{0,3,4,10,11}$ is the relation defined by $K0, K3, K'3, K4, K10, K11$ only. We let $\mathcal{K}_0 =< \mathcal{A}, J, \vdash\!\!- >$. We define $\mathcal{H}_0$ as the system obtained by replacing K by H in the above definitions.

Lemma 51 *: $\mathcal{K}_0, \mathcal{H}_0$ are deduction systems.*

By $\mathrm{Hom}_H(K,K)$ we denote the set of homomorphisms $\psi : K \to K$ which leave H pointwise invariant. $\mathcal{K}_0$ is "compatible with homomorphisms" in the following sense

Lemma 52 *For every $P \in \mathcal{P}(\mathcal{A})$ and every homomorphism $\psi \in \mathrm{Hom}_H(K,K)$, if P is a $\mathcal{K}_0$-proof then $\psi(P)$ is a $\mathcal{K}_0$-proof too.*

For every integer $t \in \mathbb{N}$, we denote by $\tau_t : \mathcal{A} \to \mathcal{A}$ the translation on the weights: $\forall p \in \mathbb{N}, S,T \in \mathsf{DRB}\langle\langle\ K * V^*\ \rangle\rangle, \tau_t(p,S,T) = (p+t,S,T)$.

5.3 Regular Proofs

Let us use the notation $\mathrm{Par}(S) \subseteq U$ for the set of parameters occuring in a given series S. (The notation is extended to assertions and sets of assertions in a natural way).

Definition 53 (germs) *We call a H-germ any 9-tuple $G = (n, \alpha, \beta, \gamma, (P_i)_{0 \le i \le n}, (A_i)_{0 \le i \le n}, (B_{i,j})_{0 \le i \le n, 0 \le j \le \alpha(i)}, (C_{i,k})_{0 \le i \le n, 0 \le k \le \beta(i)}, (\psi_{i,j})_{0 \le i \le n, 0 \le j \le \alpha(i)})$ such that*

1. *n is a non-negative integer,*
2. *α, β are two integer mappings: $[0, n] \to \mathbb{N}$,*
3. *γ is a mapping: $\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 0 \le i \le n, 0 \le j \le \alpha(n)\} \to [0, n]$,*
4. *every P_i is a finite subset of $\mathcal{A}$,*
5. *$A_i, B_{i,j}, C_{i,k}$ are assertions belonging to P_i ,*
6. *let $U_i = \mathrm{Par}(A_i)$, $U_G = \cup_{0 \le i \le n} U_i$,$K_G = \mathrm{F}(U_G) * H$; $U_0 = \emptyset$ and every assertion of P_i belongs to $\mathbb{N} \times \mathsf{DRB}\langle\langle\ K_i * V^*\ \rangle\rangle \times \mathsf{DRB}\langle\langle\ K_i * V^*\ \rangle\rangle$*
7. *every assertion $C_{i,k}$ has the form: $C_{i,k} = (\pi_{i,k}, S_{i,k}, T_{i,k})$ where $S_{i,k} \in K, T_{i,k} \in K$,*
8. *$A_i \notin \{B_{i,j} \mid 0 \le j \le \alpha(i)\} \cup \{C_{i,k} \mid 0 \le k \le \beta(i)\}$,*
9. *P_i is a proof relative to the set of hypotheses $\{B_{i,j} \mid 0 \le j \le \alpha(i)\} \cup \{C_{i,k} \mid 0 \le k \le \beta(i)\}$,*
10. *$\psi_{i,j} \in \mathrm{Hom}_H(K_{\gamma(i,j)}, K_i)$ and there exists some non-negative integer $t \in \mathbb{N}$ such that $\tau_t(\psi_{i,j}(A_{\gamma(i,j)})) = B_{i,j}$.*

Definition 54 (rational sets of homomorphisms) *Let G be a H-germ defined as in definition 53. We define rational subsets $(\mathcal{R}_i)_{0 \le i \le n}$ of $\mathrm{Hom}_H(K, K)$ by $\mathcal{R}_i = \{\psi_{i_0,j_0} \circ \psi_{i_1,j_1} \cdots \circ \psi_{i_\ell,j_\ell} \mid i_0 = 0, \ell \ge 0, \forall k \in [0, \ell], 0 \le j_k \le \alpha(i_k)$, $\forall k \in [0, \ell - 1] i_{k+1} = \gamma(i_k, j_k)$ and $i = \gamma(i_\ell, j_\ell)\}$.*

Definition 55 *Let G be a H-germ. We define the* set of assertions *associated with G, as the set: $\mathsf{P}(G) = \bigcup_{\substack{\psi_i \in \mathcal{R}_i \\ 0 \le i \le n}} \psi_i(P_i)$.*

Definition 56 (germs of proofs) *Let G be a H-germ. G is called a* germ of proof *iff, for every $i \in [0, n], k \in [0, \beta(i)], \psi_i \in \mathcal{R}_i$, $\psi_i(S_{i,k}) = \psi_i(T_{i,k})$.*

Definition 57 (regular proofs) *Let $P \subseteq \mathbb{N} \times \mathsf{DRB}\langle\langle\ H * V^*\ \rangle\rangle \times \mathsf{DRB}\langle\langle\ H * V^*\ \rangle\rangle$. P is called a* regular *proof iff there exists some germ of proof G such that $P = \mathsf{P}(G)$.*

(One can check that, due to lemma 52, "regular proofs" are indeed $\mathcal{H}_0$-proofs).

Theorem 58 *The set of all germs of proof is recursively enumerable.*

The proof of theorem 58 leans essentially on theorem 34 applied on the monoid $M = K_G$ (see point (6) of definition 53).

6 Completeness of $\mathcal{H}_0$

Theorem 61 *Let* $A_0 \in \mathbb{N} \times \mathsf{DRB}\langle\langle\ H * V^*\ \rangle\rangle \times \mathsf{DRB}\langle\langle\ H * V^*\ \rangle\rangle$*. If* A_0 *is true (i.e.* $J(A_0) = \infty$*) then* A_0 *has some regular* $\mathcal{H}_0$*-proof.*

Which might be rephrased as: the system $\mathcal{H}_0$ is "regularly"-complete. Let us sketch the main ideas of the proof ([Sén99, section 10]). There exists a constant $D_2 \in \mathbb{N}$ such that $\parallel A_0 \parallel \leq D_2$ and

- with the help of §4.2, we can devise a *strategy* $\mathcal{S}$, producing from every assertion A, with $\parallel A \parallel \leq D_2$, a finite $\mathcal{K}_0$-proof P, whose hypotheses still have a norm $\leq D_2$;
- we consider the set $\mathcal{F}(D_2)$ of all assertions $A' = (p', S', T')$ with a norm $\leq D_2$ and where S', T' are *generic* (this notion is close to that of *transducer schema* defined in [CK86a]); this set is finite: $\mathcal{F}(D_2) = \{A_i \mid 1 \leq i \leq n\}$;
- the $(n+1)$-tuple $(P_i)_{0 \leq i \leq n}$ of proofs produced by $\mathcal{S}$ from all the A_i, $0 \leq i \leq n$, can be extended into a H-germ $G = (n, *, *, *, (P_i)_{0 \leq i \leq n}, (A_i)_{0 \leq i \leq n}, *, *, *)$ and $\mathsf{P}(G)$ is a regular proof of A_0.

Theorem 62 *Let* $H = \mathrm{GL}_n(\mathbb{Q})$ *for some* $n \in \mathbb{N}$*. The equivalence problem for deterministic* H*-pushdown automata is decidable.*

Corollary 63 *Let* H *be a finitely generated free group or free monoid. The equivalence problem for deterministic* H*-pushdown automata is decidable.*

Proof: It suffices to notice that $Y^* \hookrightarrow \mathrm{F}(Y) \hookrightarrow \mathrm{GL}_2(\mathbb{Q})$ (see [LS77, prop.12.3 p.167]) and to apply theorem 62.

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