

HoCHC: a Refutationally-complete and Semantically-invariant System of Higher-order Logic Modulo Theories

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We present a simple resolution proof system for *higher-order constrained Horn clauses* (HoCHC)—a system of higher-order logic modulo theories—and prove its soundness and refutational completeness w.r.t. the standard semantics. As corollaries, we obtain the compactness theorem and semi-decidability of HoCHC for semi-decidable background theories, and we prove that HoCHC satisfies a canonical model property. Moreover a variant of the well-known translation from higher-order to 1st-order logic is shown to be sound and complete for HoCHC in standard semantics. We illustrate how to transfer decidability results for (fragments of) 1st-order logic modulo theories to our higher-order setting, using as example the Bernays-Schönfinkel-Ramsey fragment of HoCHC modulo a restricted form of Linear Integer Arithmetic.

1 Introduction

Cathcart Burn et al. [1] recently advocated an automatic, programming-language independent approach¹ to verify safety properties of higher-order programs by framing them as solvability problems for systems of higher-order constraints. These systems consist of Horn clauses of higher-order logic, containing constraints expressed in some suitable background theory. Consider the functional program:

$$\begin{aligned} &\text{let } add\,xy = x + y \\ &\text{letrec } iter\,f\,s\,n = \text{if } n \leq 0 \text{ then } s \text{ else } f\,n\,(iter\,f\,s\,(n-1)) \\ &\text{in } \lambda n.\text{assert } (n \geq 1 \rightarrow (iter\,add\,nn > n + n)) \end{aligned}$$

Thus $(iter\,add\,nn)$ computes the value $n + \sum_{i=0}^n i$.

To verify that the program is *safe* (i.e. the assertion is never violated), it suffices to find an invariant that implies the required property. The idea then is to express the problem of finding such a program invariant, *logically*, as a satisfiability problem for the following higher-order constrained system:

Example 1 (Invariant as system of higher-order constraints).

$$\begin{aligned} &\forall x, y, z. (z = x + y \rightarrow \text{Add } xyz) \\ &\forall f, s, n, x. (n \leq 0 \wedge s = x \rightarrow \text{Iter } f\,s\,n\,x) \\ &\forall f, s, n, x. (n > 0 \wedge \exists y. (\text{Iter } f\,s\,(n-1)\,y \wedge f\,n\,y\,x) \rightarrow \text{Iter } f\,s\,n\,x) \\ &\forall n, x. (n \geq 1 \wedge \text{Iter } add\,nn\,x \rightarrow x > n + n) \end{aligned}$$

The above are Horn clauses of higher-order logic, obtained by transformation from the preceding program; $\text{Add} : \iota \rightarrow \iota \rightarrow \iota \rightarrow o$ and $\text{Iter} : (\iota \rightarrow \iota \rightarrow \iota \rightarrow o) \rightarrow \iota \rightarrow \iota \rightarrow \iota \rightarrow o$ are higher-order relations, and the binary predicates $(\leq, >, \dots)$ are formulas of the background theory, Linear Integer Arithmetic (LIA).

¹See the tool implementation *Horus* at <http://mjolnir.cs.ox.ac.uk/horus/>

Is higher-order logic modulo theories a sensible algorithmic approach to verification? Is it well-founded?

To set the scene, recall that 1st-order logic is semi-decidable: 1st-order validities² are r.e.; moreover if a formula is unsatisfiable then it is provable by resolution [2, 3]. By contrast, higher-order logic in standard semantics is wildly undecidable. E.g. the set² $\mathbf{V}^2(=)$ of valid sentences of the 2nd-order language of equality is not even analytical [4].

This does not necessarily spell doom for the higher-order logic approach. One could consider higher-order logic in *Henkin semantics* [5], which is, after all, “nothing but many-sorted 1st-order logic with comprehension axioms” [4] (see also [6, 7]). Cathcart Burn et al. [1], however, posit that standard semantics is the appropriate semantics for higher-order constrained Horn clauses; they argue that standard semantics is simple and natural, and hence the semantics of choice for higher-order logic in verification from the perspective of specification; moreover standard semantics is common in practice (e.g. HOL theorem prover [8, 9] in automated deduction, and monadic 2nd-order logic in algorithmic verification). Yet, desirable properties such as completeness [5], compactness, and sound-and-complete translation to first-order logic hold only for higher-order logic in Henkin semantics, and not standard semantics [10].

In this paper, we study the algorithmic, model-theoretic and semantical properties of higher-order Horn clauses with a 1st-order background theory.

A Complete Resolution Proof System for HoCHCs The main technical contribution of this paper is the design of a simple resolution proof system for *higher-order constrained Horn clauses* (HoCHC) where the background theory has a unique model [1], and its refutational completeness proof with respect to the standard semantics (Sec. 3). The proof system and its refutational completeness proof are generalised in Sec. 6 to arbitrary *compact* background theories, which may have more than one model.

The completeness proof hinges on an important model-theoretic insight: we extend a result of [1] to prove that every set of HoCHCs which is unsatisfiable in standard semantics is also unsatisfiable in *continuous* semantics. This allows us to derive a *finitary syntactic* explanation for unsatisfiability by considering the structure obtained by iterating the immediate consequence operator.

Semantic Invariance The soundness and completeness of our resolution proof system has the following pleasing corollary: in contrast to (full) higher-order logic, satisfiability of HoCHCs with respect to standard semantics, continuous semantics and Henkin semantics coincide.

Compactness Theorem and Semi-decidability of HoCHC A well-known feature of higher-order logic in standard semantics is failure of the compactness theorem. Perhaps surprisingly, it follows from the refutational completeness proof that the compactness theorem *does* hold for HoCHCs (in standard semantics): for every unsatisfiable set S of HoCHCs, there is a finite subset $S' \subseteq S$ which is unsatisfiable.

Moreover, if the consistency of conjunctions of atoms in the background theory is semi-decidable, so is HoCHC unsatisfiability. Crucially this underpins the *practicality* of the HoCHC-based approach to program verification.

Canonical Model Property As shown in [1], a disadvantage of the standard semantics is failure of the least model property (w.r.t. pointwise ordering). However, we prove in Sec. 4 that, given a satisfiable set of HoCHCs, the canonical structure obtained by iterating the immediate consequence operator always constitutes a model.

²Define $\mathbf{V}^n(P)$ to be the set of valid sentences of n th-order logic with 2-place predicate P . Then $\mathbf{V}^1(=)$ is r.e.

Complete 1st-order Translation Already adumbrated by the equivalence of standard and Henkin semantics, we show that there is a variant of the standard translation of higher-order logic into 1st-order logic which is sound and complete also for standard semantics, when restricted to HoCHC (Sec. 7). Precisely we show that a finite set S of HoCHCs is satisfiable iff its translation into 1st-order Horn constraints $\lfloor S \rfloor$ is satisfiable. Moreover if the unsatisfiability of goal clauses of the background theory is semi-decidable, then unsatisfiability of $\lfloor S \rfloor$ is also semi-decidable.

Decidable Fragments of HoCHC In Sec. 8, we put the canonical model property of the canonical structure to good use by leveraging the fact that the structure is a model of satisfiable sets of HoCHCs which can be finitely computed for background theories with finite domains. We can identify two fragments of HoCHC with a decidable satisfiability problem by showing equi-satisfiability to clauses w.r.t. a finite number of such background theories.

Outline We begin with some key definitions in Sec. 2. In Sec. 3, we present the resolution proof system for HoCHC and prove its completeness. We then prove, in Sec. 4, that the standard semantics satisfies a canonical model property. In Sec. 5 we show that HoCHC satisfiability is independent of the choice of semantics. In Sec. 6 we generalise the refutational completeness proof to arbitrary compact background theories, which may have more than one model. In Sec. 7 we present a 1st-order translation of higher-order logic and prove it complete when restricted to HoCHC. In Sec. 8 we exhibit decidable fragments of HoCHCs. Finally, we discuss related work in Sec. 9, and conclude in Sec. 10.

2 Technical Preliminaries

2.1 Relational Higher-order Logic

This subsection introduces the syntax and semantics of a restricted form of higher-order logic.

2.1.1 Syntax

For a fixed set \mathcal{I} (intuitively the types of individuals), the set of *argument types*, *relational types*, *1st-order types* and *types* generated by \mathcal{I} are mutual recursively defined by

$$\begin{array}{ll} \text{Argument type} & \tau ::= \iota \mid \rho \\ \text{Relational type} & \rho ::= o \mid \tau \rightarrow \rho \\ \text{1st-order type} & \sigma_{\text{FO}} ::= \iota \rightarrow o \mid \iota \mid \iota \rightarrow \sigma_{\text{FO}} \\ \text{Type} & \sigma ::= \rho \mid \sigma_{\text{FO}}, \end{array}$$

where $\iota \in \mathcal{I}$. We sometimes abbreviate the (1st-order) type $\underbrace{\iota \rightarrow \cdots \rightarrow \iota}_n \rightarrow \iota$ to $\iota^n \rightarrow \iota$ (similarly for $\iota^n \rightarrow o$). For types $\tau_1 \rightarrow \cdots \rightarrow \tau_n \rightarrow \sigma$ we also write $\bar{\tau} \rightarrow \sigma$. Intuitively, o is the type of the truth values (or Booleans). Besides, σ_{FO} contains all (1st-order) types of the form $\iota^n \rightarrow \iota$ or $\iota^n \rightarrow o$, i.e. all arguments are of type ι . Moreover, each relational type has the form $\bar{\tau} \rightarrow o$.

A *type environment* (typically Δ) is a function mapping variables to argument types; for $x \in \text{dom}(\Delta)$, we write $x : \tau \in \Delta$ to mean $\Delta(x) = \tau$. A *signature* is a set of distinct typed *symbols* $c : \sigma$, where $c \notin \text{dom}(\Delta)$ and c is not one of the *logical symbols* \neg, \wedge, \vee and \exists_τ (for argument types τ). It is *1st-order* if for each $c : \sigma \in \Sigma$, σ is 1st-order. We often write $c \in \Sigma$ if $c : \sigma \in \Sigma$ for some σ .

The set of Σ -pre-terms is given by

$$M ::= x \mid c \mid \neg \mid \wedge \mid \vee \mid \exists_\tau \mid MM \mid \lambda x.M$$

where $c \in \Sigma$. Following the usual conventions we assume that application associates to the left and the scope of abstractions extend as far to the right as possible. We also write $M\bar{N}$ and $\lambda\bar{x}.M'$ for $MN_1 \cdots N_n$ and $\lambda x_1. \cdots \lambda x_n.M'$, respectively, assuming implicitly that M is not an application. Besides, we abbreviate $\exists_\tau(\lambda x.M)$ as $\exists_\tau x.M$ and sometimes we even drop the subscript. Moreover, we identify terms up to α -equivalence and adopt Barendregt's *variable convention* [11].

The typing judgement $\Delta \vdash M : \sigma$ is defined by

$$\begin{array}{c} \frac{}{\Delta \vdash x : \Delta(x)} \text{ (Var)} \quad \frac{c : \sigma \in \Sigma}{\Delta \vdash c : \sigma} \text{ (Cst)} \\[10pt] \frac{\circ \in \{\wedge, \vee\}}{\Delta \vdash \circ : o \rightarrow o \rightarrow o} \text{ (And/Or)} \quad \frac{\Delta \vdash M : o}{\Delta \vdash \neg M : o} \text{ (Neg)} \quad \frac{}{\Delta \vdash \exists_\tau : (\tau \rightarrow o) \rightarrow o} \text{ (Ex)} \\[10pt] \frac{\Delta \vdash M_1 : \sigma_1 \rightarrow \sigma_2 \quad \Delta \vdash M_2 : \sigma_1}{\Delta \vdash M_1 M_2 : \sigma_2} \text{ (App)} \quad \frac{\Delta \vdash M : \rho}{\Delta \vdash \lambda x.M : \Delta(x) \rightarrow \rho} \text{ (Abs)} \end{array}$$

we say that M is *well-typed* if $\Delta \vdash M : \sigma$ for some σ . A well-typed Σ -pre-term is called a Σ -term, and a Σ -term of type o is a Σ -formula. A Σ -term (Σ -formula) M is a *1st-order Σ -term* (*1st-order Σ -formula*) if its construction is restricted to symbols $c : \sigma_{FO} \in \Sigma$ and variables $x : \iota \in \Delta$, and uses no λ -abstraction. Finally for a Σ -term M , $\text{fv}(M)$ is the set of free variables, and M is a *closed Σ -term* if $\text{fv}(M) = \emptyset$.

Remark 2. It follows from the definitions that (i) each term $\Delta \vdash M : \iota^n \rightarrow \iota$ can only contain variables of type ι and constants of non-relational 1st-order type, and contains neither λ -abstractions nor logical symbols; a similar approach is adopted in [12] (ii) \neg can only occur in a term if applied to another term (and not in pre-terms of the form $R \neg$).

The following kind of terms is particularly significant:

Definition 3. A Σ -term (resp. Σ -formula) is *positive existential* if the logical constant “ \neg ” is not a sub-term.

For Σ -terms M, N_1, \dots, N_n and variables x_1, \dots, x_n satisfying $\Delta \vdash N_i : \Delta(x_i)$, the (*simultaneous*) *substitution* $M[N_1/x_1, \dots, N_n/x_n]$ is defined in the standard way.

2.1.2 Semantics

A subset A of a poset P ordered by \sqsubseteq is *directed* if for every $x, y \in A$ there exists $z \in A$ satisfying $x \sqsubseteq z$ and $y \sqsubseteq z$. A poset in which every directed subset has a supremum is a *directed-complete partial order*. For directed-complete partial orders D and E , a function $f : D \rightarrow E$ is (*Scott-*) *continuous* if it is monotone and for every directed $A \subseteq D$, $f(\bigsqcup A) = \bigsqcup \{f(a) \mid a \in A\}$ and we denote the set of continuous functions $D \rightarrow E$ by $[D \xrightarrow{c} E]$.

For every $\iota \in \mathcal{I}$, we fix an arbitrary non-empty set D_ι . We define the *standard frame* \mathcal{S} recursively by

$$\mathcal{S}[\![o]\!] := \mathbb{B} \quad \mathcal{S}[\![\iota]\!] := D_\iota \quad \mathcal{S}[\![\tau \rightarrow \sigma]\!] := [\mathcal{S}[\![\tau]\!] \rightarrow \mathcal{S}[\![\sigma]\!]]$$

for $\iota \in \mathcal{I}$, where $[\mathcal{S}[\![\tau]\!] \rightarrow \mathcal{S}[\![\sigma]\!]]$ is the sets of functions from $\mathcal{S}[\![\tau]\!]$ to $\mathcal{S}[\![\sigma]\!]$.

Moreover, for types σ , let $\sqsubseteq_{\sigma}^{\mathcal{S}} \subseteq \mathcal{S}[\sigma] \times \mathcal{S}[\sigma]$ be the usual partial order defined pointwise for higher types, and which is the discrete order on $\mathcal{S}[\iota^n \rightarrow \iota]$. For $\mathfrak{R} \subseteq \mathcal{S}[\sigma]$, the supremum, $\bigsqcup_{\rho} \mathfrak{R}$, is defined pointwise, by recursion on (relational type) ρ (N.B. $0 < 1$).

Similarly, the *continuous frame* \mathcal{C} is defined recursively by

$$\mathcal{C}[\iota] := \mathbb{B} \quad \mathcal{C}[\iota] := D_{\iota} \quad \mathcal{C}[\tau \rightarrow \sigma] := [\mathcal{C}[\tau] \xrightarrow{\mathcal{C}} \mathcal{C}[\sigma]],$$

for $\iota \in \mathcal{I}$ and the analogous pointwise order $\sqsubseteq_{\sigma}^{\mathcal{C}} \subseteq \mathcal{C}[\sigma] \times \mathcal{C}[\sigma]$. It is easy to see that each $\mathcal{C}[\sigma]$ is directed-complete (indeed even complete for relational σ), i.e. that the definition is well-defined.

Throughout the paper, we omit type subscripts or frame superscripts to reduce clutter if they can be inferred or are clear from the context.

Let Σ be a signature and \mathcal{H} be one of \mathcal{S} or \mathcal{C} . We define $\perp_{\tau \rightarrow o}^{\mathcal{H}}(\bar{r}) := 0$ and $\top_{\tau \rightarrow o}^{\mathcal{H}}(\bar{r}) := 1$. Clearly, for each relational ρ , $\perp_{\rho}^{\mathcal{H}}, \top_{\rho}^{\mathcal{H}} \in \mathcal{H}[\rho]$. A (Σ, \mathcal{H}) -*structure* \mathcal{A} assigns to each $c : \sigma \in \Sigma$ an element $c^{\mathcal{A}} \in \mathcal{H}[\sigma]$ and we set $\mathcal{A}[\sigma] = \mathcal{H}[\sigma]$ for types σ . A (Δ, \mathcal{H}) -*valuation* α is a function such that for every $x : \tau \in \Delta$, $\alpha(x) \in \mathcal{H}[\tau]$. For a (Δ, \mathcal{H}) -valuation α ; variables x_1, \dots, x_n ; and $r_1 \in \mathcal{H}[\Delta(x_1)], \dots, r_n \in \mathcal{H}[\Delta(x_n)]$, $\alpha[x_1 \mapsto r_1, \dots, x_n \mapsto r_n]$ is defined in the usual way. Moreover, for a type environment Δ let $\top_{\Delta}^{\mathcal{H}}$ be the valuation satisfying $\top_{\Delta}^{\mathcal{H}}(x) = \top_{\rho}^{\mathcal{H}}$ for $x : \rho \in \Delta$ and $\top_{\Delta}^{\mathcal{H}}(x) = h_{\iota}$ for $x : \iota \in \Delta$ and some $h_{\iota} \in \mathcal{H}[\iota]$. Finally, we lift \sqsubseteq and the lub \bigsqcup to valuations and structures and the notion of directedness to sets thereof.

The *denotation* $\mathcal{A}[M](\alpha)$ of a term M with respect to \mathcal{A} and α is defined recursively by

$$\begin{aligned} \mathcal{A}[x](\alpha) &:= \alpha(x) & \mathcal{A}[c](\alpha) &:= c^{\mathcal{A}} \\ \mathcal{A}[\wedge](\alpha) &:= \text{and} & \mathcal{A}[\vee](\alpha) &:= \text{or} \\ \mathcal{A}[\exists_{\tau}](\alpha) &:= \text{exists}_{\tau}^{\mathcal{H}} & \mathcal{A}[\neg M](\alpha) &:= 1 - \mathcal{A}[M](\alpha) \\ \mathcal{A}[M_1 M_2](\alpha) &:= \mathcal{A}[M_1](\alpha)(\mathcal{A}[M_2](\alpha)) \\ \mathcal{A}[\lambda x. M](\alpha) &:= \lambda r \in \mathcal{A}[\Delta(x)]. \mathcal{A}[M](\alpha[x \mapsto r]) \end{aligned}$$

where $[\mathcal{H}[\sigma_1] \rightarrow \mathcal{H}[\sigma_2]]$ is the set of functions $\mathcal{H}[\sigma_1] \rightarrow \mathcal{H}[\sigma_2]$; $\perp_{\tau \rightarrow o}^{\mathcal{H}}(\bar{r}) := 0$; $\top_{\tau \rightarrow o}^{\mathcal{H}}(\bar{r}) := 1$;

and $(b_1)(b_2) := \min\{b_1, b_2\}$ $\text{or}(b_1)(b_2) := \max\{b_1, b_2\}$ $\text{exists}_{\tau}^{\mathcal{H}}(r) := \max\{r(s) \mid s \in \mathcal{H}[\tau]\}$;

By the following lemma, the denotation is well-defined:

Lemma 4. *Let \mathcal{B} be a (Σ', \mathcal{C}) -structure and \mathfrak{B}' be a directed set of (Σ', \mathcal{C}) -structures satisfying $\mathcal{B} \sqsubseteq \bigsqcup \mathfrak{B}'$, let α be a (Δ, \mathcal{C}) -valuation and \mathfrak{A}' be a directed set of (Δ, \mathcal{C}) -valuations satisfying $\alpha \sqsubseteq \bigsqcup \mathfrak{A}'$, and let $\Delta \vdash M : \sigma$ be a positive existential term.*

Then $\mathcal{B}[M](\alpha) \sqsubseteq \bigsqcup \{\mathcal{B}'[M](\alpha') \mid \mathcal{B}' \in \mathfrak{B}' \wedge \alpha' \in \mathfrak{A}'\}$, the expression on the right-hand side is well-defined and $\mathcal{B}[M](\alpha) \in \mathcal{C}[\sigma]$.

If M is closed, its denotation does not depend on the valuation, and we set $\mathcal{A}[M] := \mathcal{A}[M](\top_{\Delta}^{\mathcal{H}})$. For each $\mathcal{H} \in \{\mathcal{S}, \mathcal{C}\}$, (Σ, \mathcal{H}) -structure \mathcal{A} , Σ -formula F , and (\mathcal{H}, Δ) -valuation α , we write $\mathcal{A}, \alpha \models F$ if $\mathcal{A}[F](\alpha) = 1$, and we write $\mathcal{A} \models F$ if for all (Δ, \mathcal{H}) -valuations α' , $\mathcal{A}, \alpha' \models F$. We extend \models in the usual way to sets of formulas.

1st-order Structures Let Σ be a 1st-order signature. A *1st-order Σ -structure* is a (Σ, \mathcal{S}) -structure. Note that $\mathcal{S}[\iota^n \rightarrow \iota] = \mathcal{C}[\iota^n \rightarrow \iota]$ and $\mathcal{S}[\iota^n \rightarrow o] = \mathcal{C}[\iota^n \rightarrow o]$. Therefore, any 1st-order structure is also a (Σ, \mathcal{C}) -structure.

Example 5. In the examples we will primarily be concerned with the signature of *Linear Integer Arithmetic*³ $\Sigma_{\text{LIA}} := \{0, 1, +, -, <, \leq, =, \neq, \geq, >\}$ and its standard model \mathcal{A}_{LIA} .

2.2 Higher-order Constrained Horn Clauses

Assumption 1. Henceforth, we fix a 1st-order signature Σ over a single type of individuals ι (for which we can assume an equality symbol) and a 1st-order Σ -structure \mathcal{A} .

Moreover, we fix a signature Σ' extending Σ with (only) symbols of relational type, and a type environment Δ such that $\Delta^{-1}(\tau)$ is infinite for each argument type τ .

Intuitively, Σ and \mathcal{A} correspond to the language and interpretation of the background theory such as Σ_{LIA} together with its standard model \mathcal{A}_{LIA} . In particular, we first focus on background theories with a single model. In Sec. 6 we extend our results to a more general setting.

We are interested in whether 1st-order structures can be expanded to larger (higher-order) signatures. This is made precise by the following:

Definition 6. A (Σ', \mathcal{H}) -structure \mathcal{B} is a (Σ', \mathcal{H}) -expansion of \mathcal{A} if $c^{\mathcal{A}} = c^{\mathcal{B}}$ for all $c \in \Sigma$.

Next, we introduce higher-order constrained Horn clauses and their satisfiability problem.

Definition 7. 1) An *atom* is a Σ' -formula that does not contain a logical symbol.

2) An atom is a *background atom* if it is also a 1st-order Σ -formula. Otherwise it is a *foreground atom*.

3) An *easy foreground atom* is a foreground atom of the form $x\overline{M}$.

Note that a foreground atom has one of the following forms: 1. $R\overline{M}$ where $R \in (\Sigma' \setminus \Sigma)$, 2. $x\overline{M}$, or 3. $(\lambda y.N)\overline{M}$.

We use ϕ and A (and variants thereof) to refer to background atoms and general atoms, respectively.

Definition 8 (HoCHC). 1) A *goal clause* is a disjunction $\neg A_1 \vee \dots \vee \neg A_n$, where each A_i is an atom.

We write \perp to mean the empty (goal) clause.

2) G is an *easy goal clause* if each A_i is an easy foreground atom.

3) If G is a goal clause, $R \in (\Sigma' \setminus \Sigma)$ and the variables in \overline{x} are distinct, then $G \vee R\overline{x}$ is a *definite clause*.

4) A *(higher-order) constrained Horn clause (HoCHC)* is a goal or definite clause.

In the following we transform the higher-order sentences in Ex. 1 into HoCHCs (by first converting to prenex normal form and then omitting the universal quantifiers).

Example 9 (A system of HoCHCs). Let $\Sigma' = \Sigma_{\text{LIA}} \cup \{\text{Add} : \iota \rightarrow \iota \rightarrow \iota \rightarrow o, \text{Iter} : (\iota \rightarrow \iota \rightarrow \iota \rightarrow o) \rightarrow \iota \rightarrow \iota \rightarrow \iota \rightarrow o\}$ and let Δ be a type environment satisfying $\Delta(x) = \Delta(y) = \Delta(z) = \Delta(n) = \Delta(s) = \iota$ and $\Delta(f) = \iota \rightarrow \iota \rightarrow \iota \rightarrow o$.

$$\begin{aligned} & \neg(z = x + y) \vee \text{Add } xyz \\ & \neg(n \leq 0) \vee \neg(s = x) \vee \text{Iter } f s n x \\ & \neg(n > 0) \vee \neg \text{Iter } f s (n - 1) y \vee \neg(f n y x) \vee \text{Iter } f s n x \\ & \neg(n \geq 1) \vee \neg \text{Iter Add } n n x \vee \neg(x \leq n + n). \end{aligned}$$

³with the usual types $0, 1 : \iota$; $+, - : \iota \rightarrow \iota \rightarrow \iota$ and $\triangleleft : \iota \rightarrow \iota \rightarrow o$ for $\triangleleft \in \{<, \leq, =, \neq, \geq, >\}$; and we use the common abbreviation n for $\underbrace{1 + \dots + 1}_n$, where $1 \leq n \in \mathbb{N}$

Definition 10. Let S be a set of HoCHCs.

- 1) S is \mathcal{A} -standard-satisfiable if there exists a (Σ', \mathcal{S}) -expansion \mathcal{B} of \mathcal{A} satisfying $\mathcal{B} \models S$.
- 2) S is \mathcal{A} -continuous-satisfiable if there exists a (Σ', \mathcal{C}) -expansion \mathcal{B} of \mathcal{A} satisfying $\mathcal{B} \models S$.

2.3 Programs

Whilst HoCHCs have a simple syntax (thus yielding a simple proof system), our completeness proof relies on programs, which are syntactically slightly more complex.

Definition 11. A *program* (usually denoted by P) is a set of Σ' -formulas $\{\neg F_R \vee R\bar{x}_R \mid R \in (\Sigma' \setminus \Sigma)\}$ such that for each $R \in \Sigma' \setminus \Sigma$, F_R is positive existential, the variables in \bar{x}_R are distinct, and $\text{fv}(F_R) \subseteq \text{fv}(R\bar{x}_R)$.

For each goal clause G there is a closed positive existential formula⁴ $\text{posex}(G)$ such that for each $\mathcal{H} \in \{\mathcal{S}, \mathcal{C}\}$ and (Σ', \mathcal{H}) -structure \mathcal{B} , $\mathcal{B} \not\models G$ iff $\mathcal{B} \models \text{posex}(G)$. Similarly, for each finite set of HoCHCs S , there exists a program⁴ P_S such that for each frame $\mathcal{H} \in \{\mathcal{S}, \mathcal{C}\}$ and (Σ', \mathcal{H}) -structure \mathcal{B} , $\mathcal{B} \models \{D \in S \mid D \text{ definite}\}$ iff $\mathcal{B} \models P_S$.

Example 12 (Program). The following program corresponds to the set of (definite) HoCHCs of Ex. 9 (modulo renaming of variables):

$$\begin{aligned} & \neg(z = x + y) \vee \text{Add } xyz \\ & \neg((n \leq 0 \wedge s = x) \vee (\exists y. n > 0 \wedge \text{Iter } fs(n-1)y \wedge fnyx)) \vee \text{Iter } fsnx. \end{aligned}$$

3 Resolution Proof System

Assumption 2. Let S be a finite set of HoCHCs and set $P = P_S$ (the program corresponding to S).

If no confusion arises, we refrain from mentioning Σ' and Δ explicitly. Our resolution proof system consists of the following rules:

$$\begin{array}{ll} \textbf{Resolution} & \frac{\neg R\bar{M} \vee G \quad G' \vee R\bar{x}}{G \vee (G'[\bar{M}/\bar{x}])} \\ \textbf{\beta-Reduction} & \frac{\neg(\lambda x. L)M\bar{N} \vee G}{\neg L[M/x]\bar{N} \vee G} \\ \textbf{Constraint Refutation} & \frac{G \vee \neg\varphi_1 \vee \dots \vee \neg\varphi_n}{\perp} \end{array}$$

provided that G is easy, each φ_i is a background atom, and there exists a valuation α such that $\mathcal{A}, \alpha \models \varphi_1 \wedge \dots \wedge \varphi_n$.

Notice that resolution is always between a (compatible) pair of goal and definite clause, and the rules β -Reduction and Constraint Refutation are only applicable to goal clauses. Clearly the conclusion of each rule is a goal clause. Moreover, though undecidable in general for higher-order logic [13, 14, 15], unification is trivial when restricted to HoCHC.

Example 13 (Refutation proof). A refutation of the set of HoCHCs from Ex. 9 (named D_1 to D_4 respectively) is given by

⁴see App. A.2 for details

$$\begin{array}{c}
\text{Resolution} \frac{\neg(n \geq 1) \vee \boxed{\neg \text{Iter Add } n n x} \vee \neg(x \leq n + n) \quad D_3}{\neg(n \geq 1) \vee \neg(n > 0) \vee \neg \text{Iter Add } n(n-1)y \vee \boxed{\neg \text{Add } n y x} \vee \neg(x \leq n + n) \quad D_1} \\
\text{Resolution} \frac{\neg(n \geq 1) \vee \neg(n > 0) \vee \boxed{\neg \text{Iter Add } n(n-1)y} \vee \neg(x = n + y) \vee \neg(x \leq n + n) \quad D_2}{\neg(n \geq 1) \vee \neg(n > 0) \vee \neg(n-1 \leq 0) \vee \neg(n = y) \vee \neg(x = n + y) \vee \neg(x \leq n + n)} \\
\text{Constraint Refutation} \frac{}{\perp}
\end{array}$$

(Atoms involved in resolution steps are boxed; atoms that are added are underlined.) The last inference is admissible because for any valuation satisfying $\alpha(n) = \alpha(y) = 1$ and $\alpha(x) = 2$,

$$\mathcal{A}_{\text{LIA}}, \alpha \models (n \geq 1) \wedge (n > 0) \wedge (n-1 \leq 0) \wedge (n = y) \wedge (x = n + y) \wedge (x \leq n + n).$$

The rules have to be applied modulo the renaming of (free) variables, in general:

Definition 14. Let $S' \cup \{C\}$ be a set of HoCHCs. We write $S' \Rightarrow_{\text{Res}, \mathcal{A}} S' \cup \{C\}$ if there exist $C_1, C_2 \in S'$ and HoCHCs C'_1, C'_2 satisfying 1) C'_1 and C'_2 are obtained from C_1 and C_2 , respectively, by renaming (free) variables and 2) C is derived from C'_1 (and C'_2) by one of the rules Resolution, β -Reduction or Constraint Refutation. Furthermore, let $\Rightarrow_{\text{Res}, \mathcal{A}}^*$ be the reflexive, transitive closure of $\Rightarrow_{\text{Res}, \mathcal{A}}$.

Next, recall that background atoms only contain variables of type ι ; and $\mathcal{B} \models S$ implies $\mathcal{B}, \alpha \models G$ for all goal clauses $G \in S$ and valuations α satisfying $\alpha(x) = \top_{\rho}^{\mathcal{H}}$ for $x : \rho \in \Delta$. Consequently, we get:

Lemma 15. *Let G be an easy goal clause and $\varphi_1, \dots, \varphi_n$ be background atoms.*

$$\text{Then } G \vee \neg\varphi_1 \vee \dots \vee \neg\varphi_n \models \neg\varphi_1 \vee \dots \vee \neg\varphi_n.$$

Now, soundness of the proof system is straightforward:

Proposition 16 (Soundness). *Let S be a set of HoCHCs. If $S \Rightarrow_{\text{Res}, \mathcal{A}}^* S' \cup \{\perp\}$ for some S' then S is both standard- and continuous-unsatisfiable.*

The following completeness theorem is significantly more difficult. In fact, we will not prove it until Sec. 3.6.

Theorem 17 (Completeness). *If S is \mathcal{A} -standard-unsatisfiable or \mathcal{A} -continuous-unsatisfiable then $S \Rightarrow_{\text{Res}, \mathcal{A}}^* \{\perp\} \cup S'$ for some S' .*

Consequently, the resolution proof system gives rise to a semi-decision procedure for the standard-unsatisfiability problem provided it is (semi-)decidable whether conjunctions of background atoms are satisfiable in the background theory⁵.

3.1 Outline of the Completeness Proof

Next, we give a brief outline of the completeness proof.

- (S1) First, we iteratively construct a canonical model⁶ of the definite clauses in S (Sec. 3.2).
- (S2) Then, we show that S is continuous-unsatisfiable if S is standard-unsatisfiable. Hence, if a goal clause is falsified by the canonical continuous structure then it is already falsified after a finite number of iterations (Sec. 3.3).

⁵i.e. whether there exists a valuation α such that $\mathcal{A}, \alpha \models \varphi_1 \wedge \dots \wedge \varphi_n$

⁶in fact the “least” model (in a non-standard sense) of the definite clauses (Sec. 4)

- (S3) We then use this insight to argue that the reason why the goal clause is falsified can be captured syntactically by essentially “unfolding definitions” (Sec. 3.4).
- (S4) Finally, we prove that the “unfolding” actually only needs to take place at the leftmost positions of atoms, which can be captured by the resolution proof system (Sec. 3.5 and 3.6).

Observe that Proof Steps (S1) and (S2) are model theoretic / semantic, whilst Proof Steps (S3) and (S4) are proof theoretic / syntactic.

3.2 Canonical Structure

We define a canonical structure for the program P associated to S (following e.g. [12]). Given a Σ' -expansion \mathcal{B} of \mathcal{A} , the *immediate consequence operator* $T_P^{\mathcal{H}}$ returns the Σ' -expansion $T_P^{\mathcal{H}}(\mathcal{B})$ of \mathcal{A} defined by $R^{T_P^{\mathcal{H}}(\mathcal{B})} := \mathcal{B} \llbracket \lambda \bar{x}_R. F_R \rrbracket$, for relational symbols $R \in \Sigma' \setminus \Sigma$. (Recall that F_R is the unique positive existential formula such that $\neg F_R \vee R \bar{x}_R \in P$.) Using $T_P^{\mathcal{H}}$ we can define $\mathcal{A}_{P,0}^{\mathcal{H}} := \perp_{\Sigma'}^{\mathcal{H}}$;

$$\begin{aligned} \mathcal{A}_{P,\beta+1}^{\mathcal{H}} &:= T_P^{\mathcal{H}}(\mathcal{A}_{P,\beta}^{\mathcal{H}}) && \text{for } \beta \in \mathbf{On} \\ \mathcal{A}_{P,\gamma}^{\mathcal{H}} &:= \bigsqcup_{\beta < \gamma} \mathcal{A}_{P,\beta}^{\mathcal{H}} && \text{for } \gamma \in \mathbf{Lim} \\ \mathcal{A}_P^{\mathcal{H}} &:= \bigsqcup_{\beta \in \mathbf{On}} \mathcal{A}_{P,\beta}^{\mathcal{H}} \end{aligned}$$

where $\perp_{\Sigma'}^{\mathcal{H}}$ and $\bigsqcup \mathfrak{B}$ (for a set \mathfrak{B} of expansions of \mathcal{A}) are the Σ' -expansion of \mathcal{A} defined by $R^{\perp_{\Sigma'}^{\mathcal{H}}} := \perp_{\rho}^{\mathcal{H}}$ and $R^{\bigsqcup \mathfrak{B}} := \bigsqcup \{R^{\mathcal{B}} \mid \mathcal{B} \in \mathfrak{B}\}$, respectively, for relational symbols $R : \rho \in \Sigma' \setminus \Sigma$. It is easy to see that all $\mathcal{A}_{P,\beta}^{\mathcal{H}}$ are indeed (Σ', \mathcal{H}) -structures and expansions of \mathcal{A} . If no confusion arises, we abbreviate $\mathcal{A}_{P,\beta}^{\mathcal{H}}$ to $\mathcal{A}_{\beta}^{\mathcal{H}}$ in the following.

Intuitively, we start from the (\sqsubseteq) -minimal structure $\perp_{\Sigma'}^{\mathcal{H}}$ and “grow” the current structure to satisfy more of the program. Eventually we end up with a structure satisfying all of P :

Proposition 18 (Properties of $\mathcal{A}_P^{\mathcal{H}}$). 1) *There is an ordinal γ satisfying $\mathcal{A}_P^{\mathcal{H}} = \mathcal{A}_{\gamma}^{\mathcal{H}}$.*
 2) *$\mathcal{A}_P^{\mathcal{H}} \models P$ and $\mathcal{A}_P^{\mathcal{H}} \models \{D \in S \mid D \text{ definite}\}$.*

Unlike the 1st-order case [16], stage ω is not a fixed point of $T_P^{\mathcal{H}}$ in general for $\mathcal{H} = \mathcal{S}$, as the following example illustrates:

Example 19. Consider the following program:

$$\neg(x_R = 0 \vee R(x_R - 1)) \vee R x_R \qquad \neg(x_U R) \vee U x_U$$

where $\Sigma' = \Sigma_{\text{LIA}} \cup \{R : \iota \rightarrow o, U : ((\iota \rightarrow o) \rightarrow o) \rightarrow o\}$, $\Delta(x_R) = \iota$ and $\Delta(x_U) = (\iota \rightarrow o) \rightarrow o$. Let \mathcal{A} be the standard model of Linear Integer Arithmetic \mathcal{A}_{LIA} and let $\mathcal{H} = \mathcal{S}$ be the standard frame. For ease of notation, we introduce functions $r_{\alpha} : \mathcal{S} \llbracket \iota \rrbracket \rightarrow \mathbb{B}$ such that $r(n) = 1$ iff $0 \leq n < \alpha$, and $\delta_{\alpha} : \mathcal{S} \llbracket \iota \rightarrow o \rrbracket \rightarrow \mathbb{B}$ such that $\delta_{\alpha}(r) = 1$ iff $r = r_{\alpha}$, where $\alpha \in \omega \cup \{\omega\}$. Then it holds $R^{\mathcal{A}_n^{\mathcal{S}}} = r_n$, $U^{\mathcal{A}_0^{\mathcal{S}}} = \perp_{(\iota \rightarrow o) \rightarrow o}^{\mathcal{S}}$ and $U^{\mathcal{A}_n^{\mathcal{S}}}(s) = s(r_{n-1})$ for $n > 0$. Therefore $R^{\mathcal{A}_{\omega}^{\mathcal{S}}} = r_{\omega}$ and $U^{\mathcal{A}_{\omega}^{\mathcal{S}}}(s) = 1$ iff there exists $n < \omega$ satisfying $s(r_n) = 1$. In particular, $U^{\mathcal{A}_{\omega}^{\mathcal{S}}}(\delta_{\omega}) = 0$. On the other hand, $\mathcal{A}_{\omega}^{\mathcal{S}}, \top_{\Delta}^{\mathcal{S}}[x_U \mapsto \delta_{\omega}] \models x_U R$ holds and therefore $U^{\mathcal{A}_{\omega+1}^{\mathcal{S}}}(\delta_{\omega}) = 1$. Consequently, $\mathcal{A}_{\omega} \neq \mathcal{A}_{\omega+1}$.

3.3 Continuous-satisfiability implies Standard-satisfiability

[1] introduce *monotone* interpretations, in which function types are interpreted by monotone instead of continuous functions, and it is shown that the satisfiability problem of HoCHC coincides for standard and monotone semantics. It has been discovered in as yet unpublished work [17] that the proof can be adapted to continuous interpretations. In this section we briefly sketch the argument.

For types σ , we define recursively functions $I_\sigma : \mathcal{C}[\![\sigma]\!] \rightarrow \mathcal{S}[\![\sigma]\!]$ such that it is right adjoint to (a unique) $L_\sigma : \mathcal{S}[\![\sigma]\!] \rightarrow \mathcal{C}[\![\sigma]\!]$:

$$I_{\iota^n \rightarrow \iota}(f) := f \qquad I_o(b) := b \qquad I_{\tau \rightarrow \rho}(r) := I_\rho \circ r \circ L_\tau$$

The following are well-known facts about adjunctions:

Lemma 20. *Let P and $Q \subseteq Q'$ be directed-complete partial orders and let $f : P \rightarrow Q$, $g : Q \rightarrow P$ be such that $f \vdash g$. Then*

- 1) f and g are monotone,
- 2) $f \circ g \sqsubseteq 1_Q$ and $1_P \sqsubseteq g \circ f$,
- 3) f is continuous and
- 4) there exists $g' : Q' \rightarrow P$ satisfying $f \vdash g'$.

Lemma 21. *Let P_1, P_2, Q_1 and Q_2 be directed-complete partial orders and $f_i : P_i \rightarrow Q_i$ and $g_i : Q_i \rightarrow P_i$ ($i \in \{1, 2\}$) be functions such that $f_i \vdash g_i$ and g_i is continuous ($i \in \{1, 2\}$). Then*

$$\begin{aligned} f : [P_1 \xrightarrow{c} P_2] &\rightarrow [Q_1 \xrightarrow{c} Q_2] & g : [Q_1 \xrightarrow{c} Q_2] &\rightarrow [P_1 \xrightarrow{c} P_2] \\ h &\mapsto f_2 \circ h \circ g_1 & k &\mapsto g_2 \circ k \circ f_1 \end{aligned}$$

are well-defined adjoints $f \vdash g$ and g is continuous.

Lemma 22. *For each type σ , $I_\sigma : \mathcal{C}[\![\sigma]\!] \rightarrow \mathcal{S}[\![\sigma]\!]$ is well-defined and has a left adjoint $L_\sigma \dashv I_\sigma$. Moreover, both I_σ and L_σ are continuous.*

Proof. We prove the lemma by induction on the type σ . For $\iota^n \rightarrow \iota$ and o this is trivial and for $\sigma = \tau \rightarrow \rho$ this is a consequence of Lemmas 20.4) and 21. \square

We write I for the (unique) function extending each I_σ , and $I(\mathcal{B})$ (given a (Σ', \mathcal{C}) -structure \mathcal{B}) for the (Σ', \mathcal{S}) -structure defined by $R^{I(\mathcal{B})} = I(R^\mathcal{B})$. Besides, we assume analogous definitions for L .

The function I provides us with a method to move from continuous models to standard models:

Lemma 23. *Let \mathcal{B} be a (Σ', \mathcal{C}) -expansion of \mathcal{A} , α be a (Δ, \mathcal{S}) -valuation and M be a positive existential term. Then $I(\mathcal{B})[\![M]\!](\alpha) \sqsubseteq I(\mathcal{B})[\![M]\!](L \circ \alpha)$.*

Theorem 24. *Let \mathcal{B} be a (Σ', \mathcal{C}) -expansion of \mathcal{A} satisfying $\mathcal{B} \models S$. Then $I(\mathcal{B}) \models S$.*

Proof. Suppose $\mathcal{B} \models S$. Let $G \in S$ be a goal clause and α be a valuation. By Lemma 23, $I(\mathcal{B})[\![\text{posex}(G)]\!](\alpha) \leq \mathcal{B}[\![\text{posex}(G)]\!](L \circ \alpha)$ and hence $I(\mathcal{B}), \alpha \models G$.

Besides, let $R \in \Sigma' \setminus \Sigma$. Due to $\mathcal{B} \models \neg F_R \vee R\bar{x}_R$, $\mathcal{B}[\![\lambda \bar{x}_R. F_R]\!] \sqsubseteq R^\mathcal{B}$ and therefore by monotonicity of I , $I(\mathcal{B})[\![\lambda \bar{x}_R. F_R]\!] \sqsubseteq R^{I(\mathcal{B})}$. Again by Lemma 23,

$$I(\mathcal{B})[\![\lambda \bar{x}_R. F_R]\!](\top_\Delta^\mathcal{S}) \sqsubseteq I(\mathcal{B})[\![\lambda \bar{x}_R. F_R]\!](\top_\Delta^\mathcal{C}) \sqsubseteq R^{I(\mathcal{B})}$$

and hence, $I(\mathcal{B}) \models \neg F_R \vee R\bar{x}_R$. Consequently, $I(\mathcal{B}) \models S$. \square

Corollary 25. *If S is \mathcal{A} -continuous-satisfiable then S is \mathcal{A} -standard-satisfiable.*

Now, by Lemma 4, it is easy to see that $\mathcal{A}_P^{\mathcal{C}} = \mathcal{A}_\omega^{\mathcal{C}}$ and therefore by the same lemma we conclude:

Theorem 26. *If $\mathcal{A}_P^{\mathcal{C}} \not\models G$ then there exists $n \in \omega$ satisfying $\mathcal{A}_n^{\mathcal{C}} \not\models G$.*

3.4 Syntactic Unfolding

Having established Proof Steps (S1) and (S2) we need to devise a method to capture (syntactically) that $\mathcal{A}_n^{\mathcal{C}} \not\models G$ does indeed hold (Proof Step (S3)).

To that end we consider a functional relation \rightarrow_{\parallel} on positive existential terms. The idea is that $M \rightarrow_{\parallel} N$ if N is obtained from M by replacing all occurrences of symbols $R \in \Sigma' \setminus \Sigma$ by $\lambda \bar{x}_R. F_R$, which is reminiscent of the definition of the immediate consequence operator (specifically of $R^{T_P^{\mathcal{H}}(\mathcal{B})}$). A similar idea is exploited in [12]. Formally, this relation is defined in App. B.3. Writing $M \rightarrow_{\parallel}^n N$ to mean $\underbrace{M \rightarrow_{\parallel} \cdots \rightarrow_{\parallel} N}_n$, the intuition that \rightarrow_{\parallel} allows us to capture the effect of the immediate consequence operator syntactically is made precise by the following:

Proposition 27. *Let \mathcal{B} be an expansion of \mathcal{A} and let M and N be positive existential terms satisfying $M \rightarrow_{\parallel} N$. Then for all valuations α , $T_P^{\mathcal{H}}(\mathcal{B})\llbracket M \rrbracket(\alpha) = \mathcal{B}\llbracket N \rrbracket(\alpha)$.*

Let $v := \{(R, \lambda \bar{x}_R. F_R) \mid R \in \Sigma' \setminus \Sigma\}$ and $\beta v := \beta \cup v$. Besides, let $\rightarrow_{\beta v}$ be the compatible closure [11, p. 51] of βv . It is easy to see that $\rightarrow_{\parallel} \subseteq \rightarrow_{\beta v}$, where $\rightarrow_{\beta v}$ is the reflexive, transitive closure of $\rightarrow_{\beta v}$.

3.5 Leftmost Reduction

Next, the notion of leftmost reduction ($\xrightarrow{\ell}$) is made precise by the relation defined inductively by

$$\begin{array}{c} \frac{}{R\bar{M} \xrightarrow{\ell} (\lambda \bar{x}_R. F_R)\bar{M}} \quad R \in \Sigma' \setminus \Sigma \qquad \frac{}{(\lambda x. L)M\bar{N} \xrightarrow{\ell} L[M/x]\bar{N}} \qquad \frac{M_1 \xrightarrow{\ell} N_1 \quad M_2 \xrightarrow{\ell} N_2}{M_1 \circ M_2 \xrightarrow{\ell} N_1 \circ N_2} \quad \circ \in \{\wedge, \vee\} \\ \\ \frac{M \xrightarrow{\ell} N}{\exists x. M \xrightarrow{\ell} \exists x. N} \qquad \frac{}{M \xrightarrow{\ell} M} \quad 0 \qquad \frac{L \xrightarrow{\ell} M \quad M \xrightarrow{\ell} N}{L \xrightarrow{\ell} N} \quad m_1 + m_2 \end{array}$$

and we write $M \xrightarrow{\ell}^m N$ if $M \xrightarrow{\ell} N$ for some m . Intuitively, $M \xrightarrow{\ell}^m N$ holds if m -many leftmost $\rightarrow_{\beta v}$ -reductions have been performed.

It is easy to see that $\rightarrow_{\parallel} \not\subseteq \xrightarrow{\ell}$, i.e. we also have to do $\rightarrow_{\beta v}$ -reductions at “non-leftmost positions” to mimic the effect of \rightarrow_{\parallel} . Therefore, we define the relation \xrightarrow{s} by

$$\frac{\bar{M} \xrightarrow{s} \bar{N}}{L \xrightarrow{s} c\bar{N}} \quad L \xrightarrow{\ell} c\bar{M}, c \in \Sigma' \cup \{\wedge, \vee, \exists \tau\} \qquad \frac{\bar{M} \xrightarrow{s} \bar{N}}{L \xrightarrow{s} x\bar{N}} \quad L \xrightarrow{\ell} x\bar{M} \qquad \frac{M' \xrightarrow{s} N' \quad \bar{M} \xrightarrow{s} \bar{N}}{L \xrightarrow{s} (\lambda x. N')\bar{N}} \quad L \xrightarrow{\ell} (\lambda x. M')\bar{M}.$$

By $\bar{L} \xrightarrow{\ell} \bar{M}$ we mean $L_j \xrightarrow{\ell} M_j$ for each $1 \leq j \leq n$ assuming $\bar{L} = (L_1, \dots, L_n)$ and $\bar{M} = (M_1, \dots, M_n)$; similarly for $\bar{M} \xrightarrow{s} \bar{N}$. The idea is that $L \xrightarrow{s} N$ if for some M , $L \xrightarrow{\ell} M$ and we can obtain N from M by performing $\rightarrow_{\beta v}$ -reductions only on “non-leftmost positions”.

It turns out that if $M \beta v$ -reduces to N (in particular if $M \rightarrow_{\parallel}^n M$) then there exists a βv -reduction sequence that first only applies to redexes in “leftmost positions” and then only in “non-leftmost positions”.

Proposition 28. *If $K \xrightarrow{s} M \rightarrow_{\beta v} N$ then $K \xrightarrow{s} N$.*

The proof of this proposition is very similar to the proof of the standardisation theorem in the λ -calculus as presented in [18] and it relies on the insight that $\bar{K} \xrightarrow{s} \bar{M}$ and $K' \xrightarrow{s} M'$ and $O \xrightarrow{s} Q$ imply $K'[O/x]\bar{K} \xrightarrow{s} M'[Q/x]\bar{M}$.

Corollary 29. *Let M and N be positive existential terms such that $M \rightarrow_{\beta v} N$. Then $M \xrightarrow{s} N$.*

Next, we consider the relation \triangleright on positive existential formulas and valuations inductively defined by:

$$\frac{}{\alpha \triangleright x\bar{M}} \quad \frac{\mathcal{A}, \alpha \models \varphi}{\alpha \triangleright \varphi} \quad \frac{r \in \mathcal{H}[\tau] \quad \alpha[x \mapsto r] \triangleright M}{\alpha \triangleright \exists_{\tau} x. M} \quad \frac{i \in \{1, 2\} \quad \alpha \triangleright M_i}{\alpha \triangleright M_1 \vee M_2} \quad \frac{\alpha \triangleright M_1 \quad \alpha \triangleright M_2}{\alpha \triangleright M_1 \wedge M_2}$$

Intuitively, $\alpha \triangleright F$ if for some α' (agreeing with α on $\Delta^{-1}(\iota)$), $\mathcal{A}_0^{\mathcal{H}}, \alpha' \models M$ and there are no λ -abstractions in relevant leftmost positions.

Lemma 30. *Let G be a goal clause, F be a β -normal positive existential formula and α be a valuation such that $\mathcal{A}_0^{\mathcal{H}}, \alpha \models F$ and $\text{posex}(G) \xrightarrow{s} F$. Then there exists a positive existential formula F' satisfying $\text{posex}(G) \xrightarrow{\ell} F'$ and $\alpha \triangleright F'$.*

3.6 Refutational Completeness

Finally, we establish a connection between the (abstract) $\xrightarrow{\ell}$ -relation on positive existential terms and the resolution proof system on clauses. For a set $S' \supseteq S$ of HoCHCs we define a measure $\mu(S')$ by

$$\mu(S') := \min(\{\omega\} \cup \{m \mid \text{exists } G \in S', F \text{ and } \alpha \text{ s.t. } \text{posex}(G) \xrightarrow{m}_{\ell} F \text{ and } \alpha \triangleright F\}).$$

It turns out that we can use the resolution proof system to derive a set of HoCHCs S'' with a strictly smaller measure by mimicking a $\xrightarrow{1}_{\ell}$ -reduction step:

Proposition 31. *Let $S' \supseteq S$ be a set of HoCHCs satisfying $0 < \mu(S') < \omega$. Then there exists $S'' \supseteq S$ satisfying $S' \Rightarrow_{\text{Res}, \mathcal{A}} S''$ and $\mu(S'') < \mu(S')$.*

Example 32. Consider the set of HoCHCs $S = \{\neg(x_R \geq 5) \vee R x_R, \neg R(x_R - 5) \vee R x_R, \neg R 5\}$. It holds that $R 5 \xrightarrow{1}_{\ell} (\lambda x_R. x_R \geq 5 \vee R(x_R - 5)) 5 \xrightarrow{1}_{\ell} 5 \geq 5 \vee R(5 - 5)$ and $\mu(S) = 2$. Furthermore, $S \Rightarrow_{\text{Res}, \mathcal{A}} S \cup \{\neg 5 \geq 5\}$ and $\mu(S \cup \{\neg 5 \geq 5\}) = 0$.

Moreover, a set of HoCHCs S can be refuted in one step by the Constraint Refutation rule if it contains a goal clause G such that $\alpha \triangleright \text{posex}(G)$. Therefore, we finally get:

Theorem 17 (Completeness). *If S is \mathcal{A} -standard-unsatisfiable or \mathcal{A} -continuous-unsatisfiable then $S \Rightarrow_{\text{Res}, \mathcal{A}}^* \{\perp\} \cup S'$ for some S' .*

Proof. By Cor. 25 we can assume that S is \mathcal{A} -continuous unsatisfiable and by Prop. 18.2), $\mathcal{A}_P^{\mathcal{C}} \models D$ for all definite clauses $D \in S$. Since S is \mathcal{A} -continuous-unsatisfiable there exists a goal clause $G \in S$ satisfying $\mathcal{A}_P^{\mathcal{C}} \not\models G$. By Thm. 26 there exists $n \in \omega$ such that $\mathcal{A}_n^{\mathcal{C}} \not\models G$. Let F_n be such that $\text{posex}(G) \rightarrow_{\parallel}^n F_n$.

F_n . By Prop. 27, $\mathcal{A}_0^{\mathcal{C}}, \top_{\Delta}^{\mathcal{C}} \models F_n$. Let F'_n be the β -normal form of F_n . By Cor. 29 and Lemma 30 there exists F' such that $\text{posex}(G) \xrightarrow{\ell} F'$ and $\top_{\Delta}^{\mathcal{C}} \triangleright F'$. Consequently, $\mu(S) < \omega$. By Prop. 31 there exists $S' \supseteq S$ satisfying $S \Rightarrow_{\text{Res}, \mathcal{A}}^* S'$ and $\mu(S') = 0$.

Hence, there exists $G \in S'$ and α such that $\text{posex}(G) \xrightarrow{\ell} F'$ and $\alpha \triangleright F'$. Clearly, this implies $F' = \text{posex}(G)$, and G has the form $G' \vee \neg\varphi_1 \vee \dots \vee \neg\varphi_n$, where G' is easy and the φ_i are background atoms such that there exists a valuation α' satisfying $\mathcal{A}, \alpha' \models \varphi_1 \wedge \dots \wedge \varphi_n$. Therefore, the rule Constraint Refutation is applicable to G and hence, $S \Rightarrow_{\text{Res}, \mathcal{A}}^* S' \Rightarrow_{\text{Res}, \mathcal{A}} \{\perp\} \cup S'$. \square

3.7 Compactness of HoCHC

The reason why we restrict S to be finite in Assumption 2 is to achieve correspondence with programs (Def. 11), which are finite expressions. If we simply extend programs with infinitary disjunctions (but keep HoCHCs finitary) we can carry out exactly the same reasoning to derive that also every *infinite*, \mathcal{A} -standard-unsatisfiable set of HoCHCs can be refuted in the proof system. Consequently:

Theorem 33 (Compactness). *For every \mathcal{A} -standard-unsatisfiable set S of HoCHCs there exists a finite subset $S' \subseteq S$ which is \mathcal{A} -standard-unsatisfiable.*

4 Canonical Model Property of $\mathcal{A}_P^{\mathcal{H}}$

One of the main reasons to consider monotone semantics in [1] was the failure of the least model property of HoCHC w.r.t. the pointwise ordering \sqsubseteq in standard semantics:

Example 34. Consider the program P

$$\neg x_R U \vee R x_R \qquad \neg x_U \neq x_U \vee U x_U$$

with signature $\Sigma' = \Sigma_{\text{LIA}} \cup \{R : ((\iota \rightarrow o) \rightarrow o) \rightarrow o, U : \iota \rightarrow o\}$, a type environment Δ satisfying $\Delta(x_R) = (\iota \rightarrow o) \rightarrow o$ and $\Delta(x_U) = \iota$ taken from [1]. Let $\mathcal{H} = \mathcal{S}$ be the standard frame and let $\text{neg} \in \mathcal{S}[\iota \rightarrow o \rightarrow o]$ be such that $\text{neg}(s) = 1$ iff $s = \perp_{\iota \rightarrow o}$.

There are (at least) two expansions \mathcal{B}_1 and \mathcal{B}_2 defined by $U^{\mathcal{B}_1} = \perp_{\iota \rightarrow o}^{\mathcal{S}}$ and $R^{\mathcal{B}_1}(s) = 1$ iff $s(\perp_{\iota \rightarrow o}^{\mathcal{S}}) = 1$, and $U^{\mathcal{B}_2} = \top_{\iota \rightarrow o}^{\mathcal{S}}$ and $R^{\mathcal{B}_2}(s) = 1$ iff $s(\top_{\iota \rightarrow o}^{\mathcal{S}}) = 1$, respectively.

Note that both $\mathcal{B}_1 \models P$ and $\mathcal{B}_2 \models P$ and there are no models smaller than any of these with respect to the pointwise ordering \sqsubseteq . Furthermore, neither $\mathcal{B}_1 \sqsubseteq \mathcal{B}_2$ nor $\mathcal{B}_2 \sqsubseteq \mathcal{B}_1$ holds because $R^{\mathcal{B}_1}(\text{neg}) = 1 > 0 = R^{\mathcal{B}_2}(\text{neg})$ and for any $n \in \mathcal{S}[\iota]$, $U^{\mathcal{B}_2}(n) = 1 > 0 = U^{\mathcal{B}_1}(n)$.

In this section, we show that the canonical structure $\mathcal{A}_P^{\mathcal{H}}$ is however always a model of satisfiable sets of HoCHCs (for $\mathcal{H} \in \{\mathcal{S}, \mathcal{C}\}$). Thus, also HoCHC w.r.t. standard semantics has a canonical model property.

Definition 35. We define a relation $\sqsubseteq_{m, \sigma}^{\mathcal{H}} \subseteq \mathcal{H}[\sigma] \times \mathcal{H}[\sigma]$ as follows by recursion on the type σ :

$$\begin{aligned} f \sqsubseteq_{m, \iota^n \rightarrow \iota}^{\mathcal{H}} f' &:= f = f' & (f, f' \in \mathcal{H}[\iota^n \rightarrow \iota]) \\ b \sqsubseteq_{m, o}^{\mathcal{H}} b' &:= b \leq b' & (b, b' \in \mathcal{H}[o]) \\ r \sqsubseteq_{m, \tau \rightarrow \rho}^{\mathcal{H}} r' &:= \forall s, s' \in \mathcal{H}[\tau]. s \sqsubseteq_{m, \tau}^{\mathcal{H}} s' \rightarrow r(s) \sqsubseteq_{m, \rho}^{\mathcal{H}} r'(s') & (r, r' \in \mathcal{H}[\tau \rightarrow \rho]) \end{aligned}$$

Again, we frequently abbreviate $\sqsubseteq_{m, \rho}^{\mathcal{H}}$ as \sqsubseteq_m and lift all notions to valuations and structures in the obvious pointwise manner. The relation \sqsubseteq_m is transitive but neither reflexive nor antisymmetric.

Example 36. For the structures \mathcal{B}_1 and \mathcal{B}_2 of Ex. 34 it holds that $\mathcal{B}_1 = \mathcal{A}_P^{\mathcal{I}}$ and $\mathcal{B}_1 \sqsubseteq_m \mathcal{B}_2$ because due to $\perp_{i \rightarrow o} \sqsubseteq_m \top_{i \rightarrow o}$, for any $s \sqsubseteq_m s'$, $s(\perp_{i \rightarrow o}) \leq s'(\top_{i \rightarrow o})$ and therefore $R^{\mathcal{B}_1}(s) \leq R^{\mathcal{B}_2}(s')$. In particular, $\text{neg} \sqsubseteq_m \text{neg}$ does not hold and therefore the fact that $R^{\mathcal{B}_1}(\text{neg}) > R^{\mathcal{B}_2}(\text{neg})$ is not a concern.

All definitions have been set up in such a way that the denotation of positive existential terms is monotone with respect to \sqsubseteq_m :

Proposition 37. *Let $\mathcal{B} \sqsubseteq_m \mathcal{B}'$ be (Σ', \mathcal{H}) -expansions of \mathcal{A} , $\alpha \sqsubseteq_m \alpha'$ be valuations and let M be a positive existential term. Then $\mathcal{B} \llbracket M \rrbracket(\alpha) \sqsubseteq_m \mathcal{B}' \llbracket M \rrbracket(\alpha')$.*

Consequently, for each ordinal β and $\mathcal{B} \models P$, $\mathcal{A}_\beta^{\mathcal{H}} \sqsubseteq_m \mathcal{B}$ and therefore:

Theorem 38. *If S is \mathcal{A} -standard-satisfiable then $\mathcal{A}_P^{\mathcal{I}} \models S$; if S is \mathcal{A} -continuous-satisfiable then $\mathcal{A}_P^{\mathcal{C}} \models S$.*

5 Semantic Invariance

Since proof systems for higher-order logic are bound to be incomplete w.r.t. standard semantics, the literature typically considers *Henkin* semantics. Informally speaking, in Henkin semantics function types $\sigma_1 \rightarrow \sigma_2$ can be interpreted by arbitrary subsets $\mathcal{H} \llbracket \sigma_1 \rightarrow \sigma_2 \rrbracket \subseteq [\mathcal{H} \llbracket \sigma_1 \rrbracket \rightarrow \mathcal{H} \llbracket \sigma_2 \rrbracket]$ such that certain closure properties are satisfied [5, 6, 7, 4]. Every model in standard semantics (which exists in case of continuous-satisfiability, Cor. 25) is also a model in Henkin semantics. Intuitively, the closure properties enforce that natural properties such as $\mathcal{A} \llbracket M \rrbracket = \mathcal{A} \llbracket M' \rrbracket$ if M and M' are β - or η -equivalent hold.

In particular, the resolution proof system, which is complete for both standard and continuous semantics, is also sound w.r.t. Henkin semantics. Together with the equivalence with monotone semantics in [1], we conclude:

Theorem 39 (Semantic Invariance). *Let S be a set of HoCHCs. Then the following are equivalent:*

- 1) S is \mathcal{A} -standard-satisfiable,
- 2) S is \mathcal{A} -continuous-satisfiable,
- 3) S is \mathcal{A} -monotone-satisfiable,
- 4) S is \mathcal{A} -Henkin-satisfiable,

We call a set S of HoCHCs \mathcal{A} -satisfiable if it satisfies any of the equivalent conditions of Thm. 39.

Remark 40. It is possible to strengthen the Completeness Thm. 17 to arbitrary frames as long as they are closed under taking lubs and interpret λ -abstractions in the usual way, which subsumes standard, monotone and continuous frames.⁷ Thus, one could obtain a slightly stronger result than Thm. 39.

6 Compact Theories

Next, we generalise our setting to *compact* theories, which may have more than one model:

Definition 41. Let \mathfrak{A} be a set of 1st-order Σ -structures.

- 1) A set S of HoCHCs is \mathfrak{A} -satisfiable if it is \mathcal{A} -satisfiable for some $\mathcal{A} \in \mathfrak{A}$. Otherwise it is \mathfrak{A} -unsatisfiable.

⁷The proof exploits that it is possible to “internatlise” the notion of continuity in these frames.

- 2) \mathfrak{A} is *compact* if for all sets S of goal clauses of background atoms, whenever S is \mathfrak{A} -unsatisfiable then there exists a finite $S' \subseteq S$ which is \mathfrak{A} -unsatisfiable. (Note that \mathfrak{A} -satisfiability here is not about the existence of $\mathcal{A} \in \mathfrak{A}$ and an *expansion* \mathcal{B} of \mathcal{A} such that $\mathcal{B} \models S$ but only about the existence of $\mathcal{A} \in \mathfrak{A}$ such that $\mathcal{A} \models S$.)

Thus every finite set of 1st-order Σ -structures is compact and for every set of 1st-order formulas the set of its models is compact.

Consider the proof system in which we replace the rule Constraint Refutation with the following:

$$\begin{array}{c} \textbf{Compact Constraint} \quad G_1 \vee \bigvee_{i=1}^{m_1} \neg \varphi_{1,i} \quad \dots \quad G_n \vee \bigvee_{i=1}^{m_n} \neg \varphi_{n,i} \\ \textbf{Refutation} \quad \hline \perp \end{array}$$

provided that each G_i is easy, each $\varphi_{i,j}$ is a background atom and $\{\neg \varphi_{j,1} \vee \dots \vee \neg \varphi_{j,m_j} \mid 1 \leq j \leq n\}$ is \mathfrak{A} -unsatisfiable.

and let $\Rightarrow_{\text{Res-C}, \mathfrak{A}}$ be defined accordingly. Note that the rule Constraint Refutation is a special case of Compact Constraint Refutation.

Theorem 42 (Soundness and Completeness). *Let \mathfrak{A} be a compact set of 1st-order Σ -structures and let S be a set of HoCHCs. S is \mathfrak{A} -unsatisfiable iff $S \Rightarrow_{\text{Res-C}, \mathfrak{A}}^* S' \cup \{\perp\}$ for some S' .*

As an interesting special case, this shows that the proof system is also sound and complete in the unconstrained setting: the set of 1st-order Σ -structures (interpreting the equality symbol as identity) is compact by the compactness theorem for 1st-order logic. Consequently, there does not exist a Σ' -structure \mathcal{B} (interpreting equality as identity) satisfying $\mathcal{B} \models S$ iff there exists S' such that $S \Rightarrow_{\text{Res-C}, \mathfrak{A}}^* S' \cup \{\perp\}$.

7 1st-order Translation

We present a method to reduce the HoCHC \mathfrak{A} -satisfiability problem to the satisfiability problem of 1st-order logic modulo a theory, using a translation in the spirit of e.g. [19, 20, 21]. We prove that this translation is sound and complete, even for standard semantics. Fortunately, the target fragment is still semi-decidable.

It turns out that we do not need to consider HoCHCs containing λ -abstractions because we can replace every λ -abstraction $\lambda y.M$, where $\Delta \vdash \lambda y.M : \tau' \rightarrow \bar{\tau} \rightarrow o$ and $\text{fv}(\lambda y.M) = \bar{x}$, by $R_M \bar{x}$, where $R_M : \Delta(\bar{x}) \rightarrow \tau' \rightarrow \bar{\tau} \rightarrow o$ is fresh, and add the clause $\neg M \bar{z} \vee R_M \bar{x} y \bar{z}$, where $\Delta(\bar{z}) = \bar{\tau}$ and all of \bar{x} , y and \bar{z} are distinct.

Lemma 43. *Let \mathfrak{A} be a set of 1st-order Σ -structures and S be a finite set of HoCHCs. Then there exists a set of HoCHCs (over an extended signature) which does not contain λ -abstractions and which is \mathfrak{A} -satisfiable iff S is \mathfrak{A} -satisfiable.*

This constitutes a considerable generalisation of the “polarity-dependent renaming” for 1st-order logic [22, 23].

Assumption 3. *Henceforth, we fix a finite set S of HoCHCs which does not contain λ -abstractions and a set \mathfrak{A} of 1st-order Σ -structures.*

Let $\mathfrak{I} = \{\iota\} \cup \{[\rho] \mid \rho \text{ relational}\}$ (and we set $[\iota^n \rightarrow \iota] = \iota^n \rightarrow \iota$). Clearly, we can regard Σ and each $\mathcal{A} \in \mathfrak{A}$ as a 1st-order signature and structure, respectively, over the extended set of types of individuals.

Let $[\Sigma']$ be the 1st-order signature, which extends Σ with symbols 1) $c_R : [\rho]$, for each $R : \rho \in \Sigma' \setminus \Sigma$; 2) $c_\rho : [\rho]$, for each relational ρ ; 3) $@_{\tau, \rho} : [\tau \rightarrow \rho] \rightarrow [\tau] \rightarrow [\rho]$, for each relational $\tau \rightarrow \rho$, and 4) $H : [o] \rightarrow o$. To reduce clutter, we often omit the subscripts from $@$.

Besides, let $[\Delta]$ be a type environment such that for $x : \tau \in \Delta$, $[\Delta](x) = [\tau]$ and for each relational $\tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow o$ and $1 \leq i \leq n$, let $x_{\tau_i}^{(i)}$ be some variable of type $[\tau_i]$. For a Σ' -term M containing neither logical symbols nor λ -abstractions, we define $[M]'$ by structural recursion:

$$\begin{aligned} [x]' &:= x \\ [R]' &:= c_R && \text{if } R \in \Sigma' \setminus \Sigma \\ [c\bar{N}]' &:= c\bar{N} && \text{if } c \in \Sigma \\ [M\bar{N}N']' &:= @ [M\bar{N}]' [N']' && \text{if } M \notin \Sigma \end{aligned}$$

Note that because of Remark 2, for each Σ' -term $\Delta \vdash M : \sigma$ which is not a background atom, $[\Delta] \vdash [M]'$: $[\sigma]$. For higher-order constrained Horn clauses we define

$$\begin{aligned} [\neg A_1 \vee \dots \vee \neg A_n] &:= \neg [A_1] \vee \dots \vee \neg [A_n] \\ [\neg A_1 \vee \dots \vee \neg A_n \vee R\bar{x}] &:= \neg [A_1] \vee \dots \vee \neg [A_n] \vee [R\bar{x}] \end{aligned}$$

where

$$[A] := \begin{cases} A & \text{if } A = c\bar{N} \text{ with } c \in \Sigma \\ H[A]' & \text{otherwise (} A \text{ is a foreground atom).} \end{cases}$$

Moreover, for relational $\rho = \tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow o$, we define

$$\text{Comp}_\rho := H (@ (\dots (@ (@ c_\rho x_{\tau_1}^{(1)}) x_{\tau_2}^{(2)}) \dots) x_{\tau_n}^{(n)})$$

and for S we set

$$[S] := \{[C] \mid C \in S\} \cup \{\text{Comp}_\rho \mid x : \rho \in \Delta \text{ occurs in } S\}.$$

Note that $[S]$ is a finite set of 1st-order Horn clauses⁸ of the (1st-order) language of $[\Sigma']$.

Example 44 (1st-order translation $[\cdot]$). Consider again the set S of HoCHCs from Ex. 9. Applying the translation $[\cdot]$ to S we get

$$\begin{aligned} [D_1] &= \neg(z = x + y) \vee H (@ (@ (@ \text{Add } x) y) z) \\ [D_2] &= \neg(n \leq 0) \vee \neg(s = x) \vee H (@ (@ (@ (@ \text{Iter } f) s) n) x) \\ [D_3] &= \neg(n > 0) \vee \neg H (@ (@ (@ (@ \text{Iter } f) s) (n - 1)) y) \vee \neg H (@ (@ (@ f n) y) x) \\ &\quad \vee H (@ (@ (@ (@ \text{Iter } f) s) n) x) \\ [G] &= \neg(n \geq 1) \vee \neg H (@ (@ (@ (@ \text{Iter } \text{Add}) n) n) x) \vee \neg(x \leq n + n) \\ \text{Comp}_{t^3 \rightarrow o} &= H (@ (@ (@ c_{t^3 \rightarrow o} x_1) x_2) x_3) \end{aligned}$$

For $\mathcal{A} \in \mathfrak{A}$ and a (Σ', \mathcal{S}) -expansion \mathcal{B} of \mathcal{A} , let $[\mathcal{B}]$ be the 1st-order $([\Sigma], \mathcal{S})$ -expansion of \mathcal{A} defined by 1) $[\mathcal{B}][[\rho]] := \mathcal{B}[\rho]$ for relational ρ , 2) $c_R^{[\mathcal{B}]} := R^\mathcal{B}$ for $R \in \Sigma' \setminus \Sigma$, 3) $c_\rho^{[\mathcal{B}]} := \top_\rho^\mathcal{S}$ for relational ρ , 4) $@_{\tau, \rho}(r)(s) := r(s)$ for relational $\tau \rightarrow \rho$, $r \in [\mathcal{B}][[\tau \rightarrow \rho]]$ and $s \in [\mathcal{B}][[\tau]]$, and 5) $H^{[\mathcal{B}]}(b) := b$ for $b \in [\mathcal{B}][[o]] = \mathbb{B}$. It is easy to see that $\mathcal{B} \models S$ implies $[\mathcal{B}] \models [S]$. Consequently:

⁸in the standard sense

Proposition 45. *If S is \mathfrak{A} -satisfiable then $\lfloor S \rfloor$ is \mathfrak{A} -satisfiable.*

It turns out that applications of the (higher-order) Resolution rule can be matched by 1st-order resolution inferences between the corresponding translated clauses. Besides, the 1st-order translation contains comprehension axioms Comp_ρ , which are implicit in the proof of the Soundness Prop. 16. Therefore, we get:

Lemma 46. *Let S' be a set of HoCHCs not containing λ -abstractions and suppose $S' \Rightarrow_{\text{Res-C}, \mathfrak{A}} S' \cup \{G\}$. Then*

- 1) G does not contain λ -abstractions
- 2) if $G \neq \perp$ then $\lfloor S' \rfloor \models \lfloor S' \cup \{G\} \rfloor$
- 3) if $G = \perp$ then $\lfloor S' \rfloor$ is \mathfrak{A} -unsatisfiable.

By the Completeness Thm. 42 we conclude:

Corollary 47. *If \mathfrak{A} is compact and S is \mathfrak{A} -unsatisfiable then $\lfloor S \rfloor$ is \mathfrak{A} -unsatisfiable.*

Theorem 48. *Assuming that \mathfrak{A} is compact, S is \mathfrak{A} -satisfiable if and only if $\lfloor S \rfloor$ is \mathfrak{A} -satisfiable.*

If \mathfrak{A} is compact, *definable*⁹ and \mathfrak{A} -unsatisfiability of goal clauses of background atoms is semi-decidable, then \mathfrak{A} -unsatisfiability of $\lfloor S \rfloor$ is also semi-decidable [24, Thm. 24].

8 Decidable Fragments

Satisfiability of HoCHC is undecidable in general because already its 1st-order fragments are highly undecidable for Linear Integer Arithmetic [25, 26] or the unconstrained setting¹⁰ [27].

Remark 49. Despite these negative results, \mathfrak{A} -satisfiability of finite S is decidable if \mathfrak{A} is a finite set of Σ -structures such that for each $\mathcal{A} \in \mathfrak{A}$ and type σ , $\mathcal{A} \models \llbracket \sigma \rrbracket$ is finite. This is a consequence of Thm. 38 and the fact that we can compute each \mathcal{A}_{P_S} explicitly and check whether $\mathcal{A}_{P_S} \models S$ holds.

Thanks to this insight, we have identified two decidable fragments of HoCHCs, one of which is presented as follows; we leave the other to App. F.1.

8.1 Combining the Bernays-Schönfinkel-Ramsey Fragment of HoCHC with Simple Linear Integer Arithmetic

Some authors [28, 29] have studied 1st-order clauses without function symbols (the so-called *Bernays-Schönfinkel-Ramsey class*¹¹) extended with a restricted form of Linear Integer Arithmetic. The fragment enjoys the attractive property that every clause set is equi-satisfiable with a finite set of its ground instances, which implies decidability [28, 29].

In this section, we transfer this result to our higher-order Horn setting.

Assumption 4. *Let Σ be a (1st-order) signature extending Σ_{LIA} with constant symbols $c : \iota$, and let $\Sigma' \supseteq \Sigma$ be a relational extension of Σ .*

⁹or *term-generated* [24], i.e. for every $\mathcal{A} \in \mathfrak{A}$ and $a \in \mathcal{A} \models \llbracket \iota \rrbracket$ there exists a closed Σ -term M such that $\mathcal{A} \models \llbracket M \rrbracket = a$

¹⁰i.e. the background theories imposes no restrictions at all

¹¹Precisely the set of sentences that, when written in prenex normal form, have a $\exists^* \forall^*$ -quantifier prefix and contain no function symbols.

Definition 50. 1) A Σ -atom is *simple* if it has the form $x \leq M$, $M \leq x$ or $x \leq y$, where M is closed¹².
 2) A HoCHC is a *higher-order simple linear arithmetic Bernays-Schönfinkel-Ramsey Horn clause* (*HoBHC(SLA)*) if it has the form $\neg\varphi_1 \vee \dots \vee \neg\varphi_n \vee C$, where each φ_i is a simple (linear arithmetic) background atom and C is \perp or it does not contain symbols from Σ .

Note that we could also have allowed background atoms of the form $M \triangleleft N$, $x \triangleleft M$ and $x \trianglelefteq y$, where M, N are closed, $\triangleleft \in \{<, \leq, =, \neq, \geq, >\}$ and $\trianglelefteq \in \{\leq, =, \geq\}$ [29].

Example 51. Let $\Sigma = \Sigma_{LIA} \cup \{c, d : \iota\}$, $\Sigma' = \Sigma \cup \{R : \iota \rightarrow o, U : (\iota \rightarrow o) \rightarrow \iota \rightarrow o\}$, $\Delta(x) = \Delta(y) = \Delta(z) = \iota$ and $\Delta(f) = \iota \rightarrow o$. The following is a set of HoBHC(SLA).

$$\begin{aligned} & \neg(x \leq c + d - 5) \vee Rx \\ & \neg fx \vee \neg(y \leq x) \vee \neg(x \leq d) \vee Ufy \\ & \neg(c \leq x) \vee \neg(x \leq -1) \\ & \neg(x \leq d - 5) \vee \neg(d - 5 \leq x) \vee \neg(y \leq c - 10) \vee \neg(c - 10 \leq y) \vee \neg U(\lambda z. Rx)y. \end{aligned}$$

Assumption 5. Let \mathfrak{A} be the set of Σ -expansions of \mathcal{A}_{LIA} , and S be a finite set of HoBHC(SLA). Let $\text{gt}_\iota(S)$ be the set of closed terms of type ι occurring in S .

Lemma 52. Let S be a set of HoBHC(SLA). If $S \Rightarrow_{\text{Res-C}, \mathfrak{A}} S \cup \{G\}$ then G is a HoBHC(SLA) and $\text{gt}_\iota(G) \subseteq \text{gt}_\iota(S)$.

We define $\widehat{\Sigma} := \{\leq : \iota \rightarrow \iota \rightarrow o\} \cup \{c_M : \iota \mid M \in \text{gt}_\iota(S)\}$ and $\widehat{\Sigma}' := \widehat{\Sigma} \cup (\Sigma' \setminus \Sigma)$. For $\mathcal{A} \in \mathfrak{A}$, let $\widehat{\mathcal{A}}$ be the $(\widehat{\Sigma}, \mathcal{S})$ -structure defined by 1) $\widehat{\mathcal{A}}[\iota] := \text{gt}_\iota(S)$, 2) $\leq^{\widehat{\mathcal{A}}}(M)(N) := \mathcal{A}[M \leq N]$ and 3) $c_M^{\widehat{\mathcal{A}}} := M$ for $M \in \text{gt}_\iota(S)$, and let $\widehat{\mathfrak{A}} := \{\widehat{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{A}\}$.

Furthermore, for simple atoms $x \leq M$ and $M \leq x$, we set $\widehat{x \leq M} := x \leq c_M$ and $\widehat{M \leq x} := c_M \leq x$. For all other atoms A (i.e. $x \leq y$ or foreground atoms) we set $\widehat{A} := A$; we lift $\widehat{\cdot}$ in the obvious way to clauses¹³ and define $\widehat{S} := \{\widehat{C} \mid C \in S\}$. Note that \widehat{S} is a set of HoCHCs for $\widehat{\Sigma}$ and $\widehat{\Sigma}'$, and that $\widehat{\mathfrak{A}}$ is finite.

Lemma 53. Let $\mathcal{A} \in \mathfrak{A}$ and $\varphi_1, \dots, \varphi_n$ be simple background atoms satisfying $\text{gt}_\iota(\varphi_1) \cup \dots \cup \text{gt}_\iota(\varphi_n) \subseteq \text{gt}_\iota(S)$. Then there exists a valuation α satisfying $\mathcal{A}, \alpha \models \varphi_1 \wedge \dots \wedge \varphi_n$ if and only if there exists a valuation α' satisfying $\widehat{\mathcal{A}}, \alpha' \models \widehat{\varphi}_1 \wedge \dots \wedge \widehat{\varphi}_n$.

Proof. First, suppose that α' satisfies $\widehat{\mathcal{A}}, \alpha' \models \widehat{\varphi}_1 \wedge \dots \wedge \widehat{\varphi}_n$. Let α be such that for $x : \iota \in \Delta$, $\alpha(x) = \mathcal{A}[\alpha'(x)]$. It is easy to see that $\mathcal{A}, \alpha \models \varphi_1 \wedge \dots \wedge \varphi_n$.

Conversely, suppose that α satisfies $\mathcal{A}, \alpha \models \varphi_1 \wedge \dots \wedge \varphi_n$. For $x : \iota \in \Delta$, we define

$$\alpha'(x) := \begin{cases} \arg \max_{M \in \text{gt}_\iota(S)} \mathcal{A}[M] & \text{if } \{M \in \text{gt}_\iota(S) \mid \mathcal{A}[M] \geq \alpha(x)\} = \emptyset \\ \arg \min_{M \in \text{gt}_\iota(S) \wedge \mathcal{A}[M] \geq \alpha(x)} \mathcal{A}[M] & \text{otherwise} \end{cases}$$

It is easy to see that $\mathcal{A}[\varphi_i](\alpha) \leq \widehat{\mathcal{A}}[\widehat{\varphi}_i](\alpha')$ and therefore, $\widehat{\mathcal{A}}, \alpha' \models \widehat{\varphi}_1 \wedge \dots \wedge \widehat{\varphi}_n$. □

Corollary 54. Let S' be a set of simple background goal clauses satisfying $\text{gt}_\iota(S') \subseteq \text{gt}_\iota(S)$.

Then S' is \mathfrak{A} -satisfiable if and only if \widehat{S}' is $\widehat{\mathfrak{A}}$ -satisfiable.

¹²or *ground* because atoms do not contain (existential) quantifiers by definition

¹³i.e. $\neg A_1 \vee \dots \vee \neg A_n \vee (\neg)A = \neg \widehat{A}_1 \vee \dots \vee \neg \widehat{A}_n \vee (\neg)\widehat{A}$

Lemma 55. *Let S' be a set of $\text{HoBHC}(\text{SLA})$ satisfying $\text{gt}_1(S') \subseteq \text{gt}_1(S)$. Then*

- 1) *if $S' \Rightarrow_{\text{Res-C}, \mathfrak{A}} S' \cup \{G\}$ then $\widehat{S'} \Rightarrow_{\text{Res-C}, \widehat{\mathfrak{A}}} \widehat{S'} \cup \{\widehat{G}\}$*
- 2) *if $\widehat{S'} \Rightarrow_{\text{Res-C}, \widehat{\mathfrak{A}}} \widehat{S'} \cup \{\widehat{G'}\}$ then $S' \Rightarrow_{\text{Res-C}, \mathfrak{A}} S' \cup \{G\}$ for some G satisfying $\widehat{G} = G'$.*

The proof of the Completeness Thm. 42 can be strengthened (Thm. 88 in App. F.2) to yield:

Proposition 56. *If S is \mathfrak{A} -unsatisfiable then $S \Rightarrow_{\text{Res-C}, \mathfrak{A}}^* S' \cup \{\perp\}$ for some S' .*

Proposition 57. *S is \mathfrak{A} -satisfiable iff \widehat{S} is $\widehat{\mathfrak{A}}$ -satisfiable.*

Proof. First, suppose S is \mathfrak{A} -unsatisfiable. By Prop. 56 $S \Rightarrow_{\text{Res-C}, \mathfrak{A}}^* S' \cup \{\perp\}$ for some S' , by Lemmas 55.1) and 52, $\widehat{S} \Rightarrow_{\text{Res-C}, \widehat{\mathfrak{A}}}^* \widehat{S'} \cup \{\perp\}$ and therefore by Prop. 16, \widehat{S} is $\widehat{\mathfrak{A}}$ -unsatisfiable.

The converse is similar and omitted. □

Finally, note that $\widehat{\mathfrak{A}}$ is finite and by the decidability of Linear Integer Arithmetic (or *Presburger arithmetic*) [30] it can be effectively obtained. Moreover, for every $\mathcal{A} \in \widehat{\mathfrak{A}}$ and type σ , $\mathcal{A} \llbracket \sigma \rrbracket$ is finite. Consequently, by Remark 49, we conclude:

Theorem 58. *Let S be a finite set of $\text{HoBHC}(\text{SLA})$. It is decidable if there is a Σ' -expansion \mathcal{B} of \mathcal{A}_{LIA} satisfying $\mathcal{B} \models S$.*

9 Related Work

Higher-order Automated Theorem Proving There is a long history of resolution-based procedures for higher-order logic *without* background theories which are refutationally complete for Henkin semantics e.g. [31, 32, 33, 34]. Furthermore, a tableau-style proof system has been proposed [35]. Their completeness proofs construct *countable* Henkin models out of terms in case the proof system is unable to refute a problem. Hence, these proofs do not seem to be extendable to provide standard models when restricted to HoCHCs.

Theorem Proving for 1st-order Logic Modulo Theories In the 1990s, superposition [36]—the basis of most state-of-the-art theorem provers [37, 38]—was extended to a setting with background theories [24, 39]. The proof system is sound and complete, assuming a compact background theory and some technical conditions. Abstractly, their proof system is very similar to ours: there is a clear separation between logical / foreground reasoning and reasoning in the background theory. Moreover, the search is directed purely by the former whilst the latter is only used in a final step to check satisfiability of a conjunction of theory atoms.

Defunctionalisation Our translation to 1st-order logic (Sec. 7) resembles Reynolds' *defunctionalisation* [40]. A whole-program transformation, defunctionalisation reduces higher-order functional programs to 1st-order ones. It eliminates higher-order features, such as partial applications and λ -abstractions, by storing arguments in data types and recovering them in an application function, which performs a matching on the data type.

Recently [41] has adapted the approach to solve the satisfiability problem for HoCHCs: given a set of HoCHCs, it generates an equi-satisfiable set of 1st-order Horn clauses over the original background theory and additionally the theory of data types. By contrast, our translation is purely logical, directly yielding 1st-order Horn clauses, without recourse to inductive data types.

Extensional Higher-order Logic Programming The aim of higher-order logic programming is not only to establish satisfiability of a set of Horn clauses without background theories but also to find (representatives of) “answers to queries”, i.e. witnesses that goal clauses are falsified in every model of the definite clauses. Thus [12] propose a rather complicated domain-theoretic semantics (equivalent to the continuous semantics [12, Prop. 5.14]). They design a resolution-based proof system that supports a strong notion of completeness ([12, Thm. 7.38]) with respect to this semantics.

Their proof system is more complicated because it operates on more general formulas (which are nonetheless translatable to clauses). Moreover it requires the instantiation of variables with certain terms, which we avoid by implicitly instantiating all remaining relational variables with $\top_{\rho}^{\mathcal{H}}$ in the Constraint Refutation rule.

Refinement Type Assignments [1] additionally introduce a refinement type system, the aim of which is to automate the search for models. In this respect, the approach is orthogonal to our resolution proof system, which can be used to refute all unsatisfiable problems (but might fail on satisfiable instances). However, for satisfiable clause sets the method by [1] may also be unable to generate models.

10 Conclusion and Future Directions

In sum HoCHC lies at a “sweet spot” in higher-order logic, semantically robust and useful for algorithmic verification.

Future work Is HoCHC the *largest* fragment of higher-order logic having these desirable properties? Whilst most obvious proper extensions add too much expressivity, the situation is unclear for non-Horn clauses in which the positive atoms satisfy a linearity constraint¹⁴. A general fixpoint extension of HoCHC is another interesting problem.

Finally, it would be desirable to combine the search for a refutation (via the proof system in this paper) with the search for a model using techniques such as presented in [1, 42, 43] because the resolution proof system fails to detect many satisfiable instances.¹⁵

References

- [1] T. Cathcart Burn, C.-H. L. Ong, and S. J. Ramsay, “Higher-order constrained Horn clauses for verification,” *PACMPL*, vol. 2, no. POPL, pp. 11:1–11:28, 2018. [Online]. Available: <http://doi.acm.org/10.1145/3158099>
- [2] M. Davis and H. Putnam, “A computing procedure for quantification theory,” *J. ACM*, vol. 7, no. 3, pp. 201–215, 1960. [Online]. Available: <http://doi.acm.org/10.1145/321033.321034>
- [3] J. A. Robinson, “A machine-oriented logic based on the resolution principle,” *J. ACM*, vol. 12, no. 1, pp. 23–41, 1965. [Online]. Available: <http://doi.acm.org/10.1145/321250.321253>
- [4] H. B. Enderton, *A Mathematical Introduction to Logic*, 2nd ed. Academic Press, 2001.
- [5] L. Henkin, “Completeness in the theory of types,” *J. Symb. Log.*, vol. 15, no. 2, pp. 81–91, 1950. [Online]. Available: <https://doi.org/10.2307/2266967>
- [6] V. B. J. and D. K., “Higher-order logic,” in *Handbook of Philosophical Logic*, ser. Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science), G. D. and G. F., Eds. Springer, Dordrecht, 1983, vol. 164.

¹⁴i.e. each variable occurs at most once in all positive atoms of a clause

¹⁵Of course, due to undecidability this cannot always succeed.

- [7] D. Leivant, “Handbook of logic in artificial intelligence and logic programming,” D. M. Gabbay, C. J. Hogger, and J. A. Robinson, Eds. New York, NY, USA: Oxford University Press, Inc., 1994, ch. Higher Order Logic, pp. 229–321. [Online]. Available: <http://dl.acm.org/citation.cfm?id=185705.185718>
- [8] M. J. C. Gordon and T. F. Melham, *Introduction to HOL: A theorem proving environment for higher order logic*. Cambridge University Press, 1993.
- [9] M. J. C. Gordon and A. M. Pitts, “The HOL logic and system,” in *Towards Verified Systems*, J. Bowen, Ed. Elsevier, 1994, pp. 49–70.
- [10] J. A. Väänänen, “Second-order logic and foundations of mathematics,” *Bulletin of Symbolic Logic*, vol. 7, no. 4, pp. 504–520, 2001. [Online]. Available: <http://www.math.ucla.edu/%7Easl/bsl/0704/0704-003.ps>
- [11] H. P. Barendregt, *The lambda calculus, its syntax and semantics*, ser. Studies in Logic (London). College Publications, London, 2012, vol. 40, [Reprint of the 1984 revised edition, MR0774952], With addenda for the 6th imprinting, Mathematical Logic and Foundations.
- [12] A. Charalambidis, K. Handjopoulos, P. Rondogiannis, and W. W. Wadge, “Extensional higher-order logic programming,” *ACM Trans. Comput. Log.*, vol. 14, no. 3, pp. 21:1–21:40, 2013. [Online]. Available: <http://doi.acm.org/10.1145/2499937.2499942>
- [13] C. Lucchesi, “The undecidability of the unification problem for third order languages,” *Report CSRR*, vol. 2059, pp. 129–198, 1972.
- [14] G. P. Huet, “The undecidability of unification in third order logic,” *Information and control*, vol. 22, no. 3, pp. 257–267, 1973.
- [15] W. D. Goldfarb, “The undecidability of the second-order unification problem,” *Theoretical Computer Science*, vol. 13, no. 2, pp. 225 – 230, 1981. [Online]. Available: <http://www.sciencedirect.com/science/article/pii/0304397581900402>
- [16] N. Bjørner, A. Gurfinkel, K. L. McMillan, and A. Rybalchenko, “Horn clause solvers for program verification,” in *Fields of Logic and Computation II - Essays Dedicated to Yuri Gurevich on the Occasion of His 75th Birthday*, 2015, pp. 24–51. [Online]. Available: https://doi.org/10.1007/978-3-319-23534-9_2
- [17] J. Jochems, “HORS safety verification by reduction to HoCHC,” July 2018, working draft.
- [18] R. Kashima, “A proof of the standardization theorem in lambda-calculus,” Tokyo Institute of Technology, Research Reports on Mathematical and Computing Sciences C-145, 2000.
- [19] J. Van Benthem and K. Doets, *Higher-Order Logic*. Dordrecht: Springer Netherlands, 1983, pp. 275–329. [Online]. Available: https://doi.org/10.1007/978-94-009-7066-3_4
- [20] M. Kerber, “How to prove higher order theorems in first order logic,” in *Proceedings of the 12th International Joint Conference on Artificial Intelligence. Sydney, Australia, August 24-30, 1991*, 1991, pp. 137–142. [Online]. Available: <http://ijcai.org/Proceedings/91-1/Papers/023.pdf>
- [21] J. C. Blanchette, C. Kaliszyk, L. C. Paulson, and J. Urban, “Hammering towards QED,” *J. Formalized Reasoning*, vol. 9, no. 1, pp. 101–148, 2016. [Online]. Available: <https://doi.org/10.6092/issn.1972-5787/4593>
- [22] D. A. Plaisted and S. Greenbaum, “A structure-preserving clause form translation,” *J. Symb. Comput.*, vol. 2, no. 3, pp. 293–304, 1986. [Online]. Available: [https://doi.org/10.1016/S0747-7171\(86\)80028-1](https://doi.org/10.1016/S0747-7171(86)80028-1)
- [23] A. Nonnengart and C. Weidenbach, “Computing small clause normal forms,” in *Handbook of Automated Reasoning (in 2 volumes)*, 2001, pp. 335–367.
- [24] L. Bachmair, H. Ganzinger, and U. Waldmann, “Refutational theorem proving for hierarchic first-order theories,” *Appl. Algebra Eng. Commun. Comput.*, vol. 5, pp. 193–212, 1994. [Online]. Available: <https://doi.org/10.1007/BF01190829>
- [25] P. J. Downey, “Undecidability of presburger arithmetic with a single monadic predicate letter,” Center for Research in Computer Technology, Harvard University, Technical Report TR-18-72, 1972.

- [26] M. Horbach, M. Voigt, and C. Weidenbach, “The universal fragment of Presburger arithmetic with unary uninterpreted predicates is undecidable,” *CoRR*, vol. abs/1703.01212, 2017. [Online]. Available: <http://arxiv.org/abs/1703.01212>
- [27] Z. Manna, *Mathematical Theory of Computation*. New York, NY, USA: Dover Publications, Inc., 2003.
- [28] Y. Ge and L. M. de Moura, “Complete instantiation for quantified formulas in satisfiability modulo theories,” in *Computer Aided Verification, 21st International Conference, CAV 2009, Grenoble, France, June 26 - July 2, 2009. Proceedings*, 2009, pp. 306–320. [Online]. Available: https://doi.org/10.1007/978-3-642-02658-4_25
- [29] M. Horbach, M. Voigt, and C. Weidenbach, “On the combination of the Bernays-Schönfinkel-Ramsey fragment with simple linear integer arithmetic,” *CoRR*, vol. abs/1705.08792, 2017. [Online]. Available: <http://arxiv.org/abs/1705.08792>
- [30] M. Presburger, “über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt,” in *Comptes Rendus du I Congrès de Mathématiciens des Pays Slaves*, 1929, p. 92–101.
- [31] P. B. Andrews, “Resolution in type theory,” *Journal of Symbolic Logic*, vol. 36, no. 3, pp. 414–432, 1971.
- [32] G. P. Huet, “Constrained resolution: A complete method for higher-order logic.” Ph.D. dissertation, Cleveland, OH, USA, 1972, aAI7306307.
- [33] C. Benzmüller and M. Kohlhase, “Extensional higher-order resolution,” in *Automated Deduction - CADE-15, 15th International Conference on Automated Deduction, Lindau, Germany, July 5-10, 1998, Proceedings*, 1998, pp. 56–71. [Online]. Available: <https://doi.org/10.1007/BFb0054248>
- [34] A. Bentkamp, J. C. Blanchette, S. Cruanes, and U. Waldmann, “Superposition for lambda-free higher-order logic,” in *Automated Reasoning - 9th International Joint Conference, IJCAR 2018, Held as Part of the Federated Logic Conference, FloC 2018, Oxford, UK, July 14-17, 2018, Proceedings*, 2018, pp. 28–46. [Online]. Available: https://doi.org/10.1007/978-3-319-94205-6_3
- [35] C. E. Brown, “Reducing higher-order theorem proving to a sequence of SAT problems,” in *Automated Deduction - CADE-23 - 23rd International Conference on Automated Deduction, Wroclaw, Poland, July 31 - August 5, 2011. Proceedings*, 2011, pp. 147–161. [Online]. Available: https://doi.org/10.1007/978-3-642-22438-6_13
- [36] L. Bachmair and H. Ganzinger, “On restrictions of ordered paramodulation with simplification,” in *10th International Conference on Automated Deduction, Kaiserslautern, FRG, July 24-27, 1990, Proceedings*, 1990, pp. 427–441. [Online]. Available: https://doi.org/10.1007/3-540-52885-7_105
- [37] C. Weidenbach, D. Dimova, A. Fietzke, R. Kumar, M. Suda, and P. Wischniewski, “SPASS version 3.5,” in *Automated Deduction - CADE-22, 22nd International Conference on Automated Deduction, Montreal, Canada, August 2-7, 2009. Proceedings*, 2009, pp. 140–145. [Online]. Available: https://doi.org/10.1007/978-3-642-02959-2_10
- [38] L. Kovács and A. Voronkov, “First-order theorem proving and vampire,” in *Computer Aided Verification - 25th International Conference, CAV 2013, Saint Petersburg, Russia, July 13-19, 2013. Proceedings*, 2013, pp. 1–35. [Online]. Available: https://doi.org/10.1007/978-3-642-39799-8_1
- [39] E. Althaus, E. Kruglov, and C. Weidenbach, “Superposition modulo linear arithmetic SUP(LA),” in *Frontiers of Combining Systems, 7th International Symposium, FroCoS 2009, Trento, Italy, September 16-18, 2009. Proceedings*, 2009, pp. 84–99. [Online]. Available: https://doi.org/10.1007/978-3-642-04222-5_5
- [40] J. C. Reynolds, “Definitional interpreters for higher-order programming languages,” in *Proceedings of the ACM annual conference - Volume 2*. ACM, 1972, pp. 717–740.
- [41] L. T. Pham, “Defunctionalization of higher-order constrained Horn clauses,” 2018, third Year Project.
- [42] H. Unno, T. Terauchi, and N. Kobayashi, “Automating relatively complete verification of higher-order functional programs,” in *The 40th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL ’13, Rome, Italy - January 23 - 25, 2013*, 2013, pp. 75–86. [Online]. Available: <http://doi.acm.org/10.1145/2429069.2429081>

- [43] N. Kobayashi, R. Sato, and H. Unno, “Predicate abstraction and CEGAR for higher-order model checking,” in *Proceedings of the 32nd ACM SIGPLAN Conference on Programming Language Design and Implementation, PLDI 2011, San Jose, CA, USA, June 4-8, 2011*, 2011, pp. 222–233. [Online]. Available: <https://doi.org/10.1145/1993498.1993525>
- [44] J. A. Robinson, “A machine-oriented logic based on the resolution principle,” *J. ACM*, vol. 12, no. 1, pp. 23–41, 1965. [Online]. Available: <http://doi.acm.org/10.1145/321250.321253>
- [45] M. Fitting, *First-Order Logic and Automated Theorem Proving, Second Edition*, ser. Graduate Texts in Computer Science. Springer, 1996. [Online]. Available: <https://doi.org/10.1007/978-1-4612-2360-3>
- [46] H. Barendregt, W. Dekkers, and R. Statman, *Lambda Calculus with Types*. New York, NY, USA: Cambridge University Press, 2013.

A Supplementary Materials for Sec. 2

A.1 Supplementary Materials for Sec. 2.1

Lemma 4. *Let \mathcal{B} be a (Σ', \mathcal{C}) -structure and \mathfrak{B}' be a directed set of (Σ', \mathcal{C}) -structures satisfying $\mathcal{B} \sqsubseteq \sqcup \mathfrak{B}'$, let α be a (Δ, \mathcal{C}) -valuation and \mathfrak{A}' be a directed set of (Δ, \mathcal{C}) -valuations satisfying $\alpha \sqsubseteq \sqcup \mathfrak{A}'$, and let $\Delta \vdash M : \sigma$ be a positive existential term.*

Then $\mathcal{B} \llbracket M \rrbracket(\alpha) \sqsubseteq \sqcup \{ \mathcal{B}' \llbracket M \rrbracket(\alpha') \mid \mathcal{B}' \in \mathfrak{B}' \wedge \alpha' \in \mathfrak{A}' \}$, the expression on the right-hand side is well-defined and $\mathcal{B} \llbracket M \rrbracket(\alpha) \in \mathcal{C} \llbracket \sigma \rrbracket$.

Proof. We prove the lemma by structural induction on M .

- If M is a logical symbol (other than \neg), a variable or a symbol from Σ' this is obvious.
- Next, suppose M is an application $N_1 N_2$ and $\Delta \vdash N_1 : \sigma_1 \rightarrow \sigma_2$. By the inductive hypothesis

$$\mathcal{B} \llbracket N_1 \rrbracket(\alpha) \sqsubseteq \sqcup \{ \mathcal{B}_1 \llbracket N_1 \rrbracket(\alpha_1) \mid \mathcal{B}_1 \in \mathfrak{B}' \wedge \alpha_1 \in \mathfrak{A}' \} \quad (1)$$

$$\mathcal{B} \llbracket N_2 \rrbracket(\alpha_2) \sqsubseteq \sqcup \{ \mathcal{B}_2 \llbracket N_2 \rrbracket(\alpha_2) \mid \mathcal{B}_2 \in \mathfrak{B}' \wedge \alpha_2 \in \mathfrak{A}' \} \quad (2)$$

and the sets on the right-hand side are directed. Note that by directedness of \mathfrak{B}' and \mathfrak{A}' ,

$$\sqcup \{ \mathcal{B}_1 \llbracket N_1 \rrbracket(\alpha_1) (\mathcal{B}_2 \llbracket N_2 \rrbracket(\alpha_2)) \mid \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{B}' \wedge \alpha_1, \alpha_2 \in \mathfrak{A}' \} \quad (3)$$

$$= \sqcup \{ \mathcal{B}' \llbracket N_1 \rrbracket(\alpha') (\mathcal{B}' \llbracket N_2 \rrbracket(\alpha')) \mid \mathcal{B}' \in \mathfrak{B}' \wedge \alpha' \in \mathfrak{A}' \}. \quad (4)$$

Therefore,

$$\begin{aligned} & \mathcal{B} \llbracket M \rrbracket(\alpha) \\ &= \mathcal{B} \llbracket N_1 \rrbracket(\alpha) (\mathcal{B} \llbracket N_2 \rrbracket(\alpha)) \\ &\sqsubseteq \mathcal{B} \llbracket N_1 \rrbracket(\alpha) \left(\sqcup \{ \mathcal{B}_2 \llbracket N_2 \rrbracket(\alpha_2) \mid \mathcal{B}_2 \in \mathfrak{B}' \wedge \alpha_2 \in \mathfrak{A}' \} \right) && \text{Eq. (2)} \\ &\sqsubseteq \left(\sqcup \{ \mathcal{B}_1 \llbracket N_1 \rrbracket(\alpha_1) \mid \mathcal{B}_1 \in \mathfrak{B}' \wedge \alpha_1 \in \mathfrak{A}' \} \right) \left(\sqcup \{ \mathcal{B}_2 \llbracket N_2 \rrbracket(\alpha_2) \mid \mathcal{B}_2 \in \mathfrak{B}' \wedge \alpha_2 \in \mathfrak{A}' \} \right) && \text{Eq. (1)} \\ &= \sqcup \{ \mathcal{B}' \llbracket N_1 \rrbracket(\alpha') (\mathcal{B}' \llbracket N_2 \rrbracket(\alpha')) \mid \mathcal{B}' \in \mathfrak{B}' \wedge \alpha' \in \mathfrak{A}' \} && \text{Eq. (3)} \\ &= \sqcup \{ \mathcal{B}' \llbracket M \rrbracket(\alpha') \mid \mathcal{B}' \in \mathfrak{B}' \wedge \alpha' \in \mathfrak{A}' \} \end{aligned}$$

Moreover, by the inductive hypothesis $\mathcal{B} \llbracket N_1 \rrbracket(\alpha) \in \mathcal{C} \llbracket \sigma_1 \rightarrow \sigma_2 \rrbracket \subseteq [\mathcal{C} \llbracket \sigma_1 \rrbracket \rightarrow \mathcal{C} \llbracket \sigma_2 \rrbracket]$ and $\mathcal{B} \llbracket N_2 \rrbracket(\alpha) \in \mathcal{C} \llbracket \sigma_2 \rrbracket$. Therefore, $\mathcal{B} \llbracket M \rrbracket(\alpha) \in \mathcal{C} \llbracket \sigma_2 \rrbracket$.

- Finally, suppose M is a λ -abstraction $\Delta \vdash \lambda x. N : \tau \rightarrow \rho$. Let $s \in \mathcal{C}[\tau]$ be arbitrary. Then

$$\begin{aligned} \mathcal{B}[\![M]\!](\alpha)(s) &= \mathcal{B}[\![N]\!](\alpha[x \mapsto s]) \\ &\sqsubseteq \bigsqcup \{ \mathcal{B}'[\![N]\!](\alpha'[x \mapsto s]) \mid \mathcal{B}' \in \mathfrak{B}' \wedge \alpha' \in \mathfrak{A}' \} \quad \text{IH (and this is well-defined)} \\ &= \left(\bigsqcup \{ \mathcal{B}'[\![M]\!](\alpha') \mid \mathcal{B}' \in \mathfrak{B}' \wedge \alpha' \in \mathfrak{A}' \} \right)(s). \end{aligned}$$

Hence, $\mathcal{B}[\![M]\!](\alpha) \sqsubseteq (\bigsqcup \{ \mathcal{B}'[\![M]\!](\alpha') \mid \mathcal{B}' \in \mathfrak{B}' \wedge \alpha' \in \mathfrak{A}' \})$. Moreover, for directed $\mathfrak{S}' \subseteq \mathcal{C}[\tau]$ satisfying $s \sqsubseteq \bigsqcup \mathfrak{S}'$, by the inductive hypothesis,

$$\mathcal{B}[\![M]\!](\alpha)(s) = \mathcal{B}[\![N]\!](\alpha[x \mapsto s]) \sqsubseteq \bigsqcup \{ \mathcal{B}[\![N]\!](\alpha[x \mapsto s']) \mid s' \in \mathfrak{S}' \} = \bigsqcup \{ \mathcal{B}[\![M]\!](\alpha)(s') \mid s' \in \mathfrak{S}' \}$$

and therefore, $\mathcal{B}[\![M]\!](\alpha) \in \mathcal{C}[\tau \rightarrow \rho] = [\mathcal{C}[\tau] \xrightarrow{c} \mathcal{C}[\rho]]$. \square

The following lemma is completely standard and can be proven by an easy structural induction (exploiting the variable convention).

Lemma 59 (Substitution). *Let $\mathcal{H} \in \{\mathcal{S}, \mathcal{C}\}$, \mathcal{A} be a (Σ, \mathcal{H}) -structure and α be a (Δ, \mathcal{H}) -valuation. Furthermore, let $x \in \text{dom}(\Delta)$ and let M and N be terms such that $\Delta \vdash N : \Delta(x)$. Then $\mathcal{A}[\![M[N/x]]\!](\alpha) = \mathcal{A}[\![M]\!](\alpha[x \mapsto \mathcal{A}[\![N]\!](\alpha)])$.*

The following lemma states that in the standard and continuous frames, the denotation is stable under β - and η -conversion.

Lemma 60. *Let \mathcal{H} be a standard or continuous frame, let M and M' be Σ -terms, \mathcal{A} be a (Σ, \mathcal{H}) -structure and let α be a (Δ, \mathcal{H}) -valuation. Then*

- 1) if $M \rightarrow_\beta M'$ then $\mathcal{A}[\![M]\!](\alpha) = \mathcal{A}[\![M']\!](\alpha)$;
- 2) if $M \rightarrow_\eta M'$ then $\mathcal{A}[\![M]\!](\alpha) = \mathcal{A}[\![M']\!](\alpha)$.

Proof. We prove the lemma by induction on the compatible closure. The only interesting cases are the base cases $((\lambda x. N)N', N[N'/x]) \in \beta$ and $(\lambda y. Ly, L) \in \eta$, respectively. Then

$$\mathcal{A}[\![\lambda x. N]N'](\alpha) = \mathcal{A}[\![N]\!](\alpha[x \mapsto \mathcal{A}[\![N']\!](\alpha)]) = \mathcal{A}[\![N[N'/x]]\!](\alpha)$$

by Lemma 59, and

$$\mathcal{A}[\![\lambda y. Ly]\!](\alpha) = \lambda r \in \mathcal{H}[\![\Delta(y)]\!]. \mathcal{A}[\![L]\!](\alpha)(r) = \mathcal{A}[\![L]\!](\alpha). \quad \square$$

A.2 Supplementary Materials for Sec. 2.3

Let S be a finite set of HoCHCs. Without loss of generality, we can assume that for each $R \in \Sigma' \setminus \Sigma$ there is at least one Horn clause $G \vee R\bar{x}_R$ and each definite clause has this form.

For a goal clause $G = \neg A_1 \vee \dots \vee \neg A_n$ let $\text{posex}(G, V) := \exists y_1, \dots, y_m. A_1 \wedge \dots \wedge A_n$, where $\{y_1, \dots, y_m\} = \text{fv}(G) \setminus V$ and $\text{posex}(G) := \text{posex}(G, \emptyset)$. Clearly, $\text{posex}(G)$ is a positive existential closed formula. Let P_S be the set of definite formulas

$$\neg(\text{posex}(G_{R,1}, \bar{x}_R) \vee \dots \vee \text{posex}(G_{R,n}, \bar{x}_R)) \vee R\bar{x}_R,$$

where $G_{R,1} \vee R\bar{x}_R, \dots, G_{R,n} \vee R\bar{x}_R$ are the unnegated occurrences of $R\bar{x}_R$ in S and $R \in \Sigma' \setminus \Sigma$. Clearly, P_S is a program. Furthermore, the following is obvious by definition:

Lemma 61. *Let S be a finite set of HoCHCs, $\mathcal{H} \in \{\mathcal{S}, \mathcal{C}\}$ and \mathcal{B} a (Σ', \mathcal{H}) -expansion of \mathcal{A} .*

Then $\mathcal{B} \models \{D \in S \mid D \text{ definite}\}$ iff $\mathcal{B} \models P_S$.

B Supplementary Materials for Sec. 3

Prop. 16 is a simple consequence of the following:

Lemma 62. *Let S' be a set of HoCHCs and suppose that $S' \Rightarrow_{\text{Res}, \mathcal{A}} S' \cup \{C\}$. Then*

- 1) *if $C \neq \perp$ then $S' \models C$;*
- 2) *if \mathcal{B} is an expansion of \mathcal{A} and $\mathcal{B} \models S'$ then $\mathcal{B} \models C$.*

Proof. 1) Note that by assumption the rule Constraint Refutation cannot have been applied. Besides, for β -Reduction this is a consequence of Lemma 60.1). Finally suppose that $\neg R\overline{M} \vee G$ and $G' \vee R\overline{x}$ are in S' (modulo renaming of variables). The proof for this case uses the same ideas as the classic one for 1st-order logic (see e.g. [44, 45]):

- 2) Let \mathcal{B} be an expansion of \mathcal{A} satisfying $\mathcal{B} \models S'$. By Part 1) it suffices to consider the case when the rule Constraint Refutation is applicable to some goal clause $G \vee \neg\varphi_1 \vee \dots \vee \neg\varphi_n$, where G is easy, each φ_i is a background atom and there exists a valuation α such that $\mathcal{A}, \alpha \models \varphi_1 \wedge \dots \wedge \varphi_n$. However, by Lemma 15, $\mathcal{B}, \alpha \models \neg\varphi_1 \vee \dots \vee \neg\varphi_n$, which is clearly a contradiction to the fact that \mathcal{B} is an expansion of \mathcal{A} . \square

B.1 Supplementary Materials for Sec. 3.2

Proposition 18 (Properties of $\mathcal{A}_P^{\mathcal{H}}$). 1) *There is an ordinal γ satisfying $\mathcal{A}_P^{\mathcal{H}} = \mathcal{A}_\gamma^{\mathcal{H}}$.*

- 2) *$\mathcal{A}_P^{\mathcal{H}} \models P$ and $\mathcal{A}_P^{\mathcal{H}} \models \{D \in S \mid D \text{ definite}\}$.*

Proof. 1) First note that $\mathfrak{B} = \{\mathcal{A}_\beta^{\mathcal{H}} \mid \beta \in \mathbf{On}\}$ is a set. For $\mathcal{B} \in \mathfrak{B}$ let $\beta_{\mathcal{B}}$ be the minimal ordinal such that $\mathcal{B} = \mathcal{A}_{\beta_{\mathcal{B}}}^{\mathcal{H}}$. By the replacement axiom, $\mathfrak{D} = \{\beta_{\mathcal{B}} \mid \mathcal{B} \in \mathfrak{B}\}$ is a set of ordinals. Hence, $\bigcup \mathfrak{D}$ is an ordinal. Let $\gamma \geq \bigcup \mathfrak{D}$ be a limit ordinal. Then it holds

$$\mathcal{A}_\gamma^{\mathcal{H}} = \bigsqcup \{\mathcal{A}_\beta^{\mathcal{H}} \mid \beta \in \mathfrak{D}\} = \bigsqcup \mathfrak{B} = \mathcal{A}_P^{\mathcal{H}}.$$

- 2) Assume towards contradiction that $\mathcal{A}_P^{\mathcal{H}} \not\models P$. Then there exists $\neg F_R \vee R\overline{x}_R \in P$ and \overline{s} satisfying $\mathcal{A}_P^{\mathcal{H}}, \top_\Delta^{\mathcal{H}}[\overline{x}_R \mapsto \overline{s}] \not\models \neg F_R \vee R\overline{x}_R$ (because $\text{fv}(F_R) = \overline{x}_R$). By Part 1), there is an ordinal β such that $\mathcal{A}_P^{\mathcal{H}} = \mathcal{A}_\beta^{\mathcal{H}}$. Hence, $\mathcal{A}_\beta^{\mathcal{H}}, \top_\Delta^{\mathcal{H}}[\overline{x}_R \mapsto \overline{s}] \models F_R$ and therefore clearly, $\mathcal{A}_\beta^{\mathcal{H}} \models \llbracket \lambda \overline{x}_R. F_R \rrbracket (\top_\Delta^{\mathcal{H}})(\overline{s}) = 1$. Consequently, $R^{T_P^{\mathcal{H}}(\mathcal{A}_\beta^{\mathcal{H}})}(\overline{s}) = 1$, which implies $R^{\mathcal{A}_P^{\mathcal{H}}}(\overline{s}) = 1$. Clearly, this is a contradiction to $\mathcal{A}_P^{\mathcal{H}}, \top_\Delta^{\mathcal{H}}[\overline{x}_R \mapsto \overline{s}] \not\models \neg F_R \vee R\overline{x}_R$. \square

B.2 Supplementary Materials for Sec. 3.3

Lemma 23. *Let \mathcal{B} be a (Σ', \mathcal{C}) -expansion of \mathcal{A} , α be a (Δ, \mathcal{S}) -valuation and M be a positive existential term. Then $I(\mathcal{B})\llbracket M \rrbracket(\alpha) \sqsubseteq I(\mathcal{B}\llbracket M \rrbracket(L \circ \alpha))$.*

Proof. We prove the lemma by structural induction on M .

- For logical symbols (other than \neg) this is easy to see.
- For symbols $c \in \Sigma'$ it trivially holds $I(\mathcal{B})\llbracket R \rrbracket(\alpha) = R^{I(\mathcal{B})} \sqsubseteq I(R^{\mathcal{B}}) = I(\mathcal{B}\llbracket R \rrbracket(L \circ \alpha))$.
- For variables x by Lemma 20.2) it holds $I(\mathcal{B})\llbracket M \rrbracket(\alpha) = \alpha(x) \sqsubseteq I(L(\alpha(x))) = I(\mathcal{B}\llbracket M \rrbracket(L \circ \alpha))$.

- Next, suppose M is an application $N_1 N_2$. Then

$$\begin{aligned}
& I(\mathcal{B} \llbracket N_1 N_2 \rrbracket (L \circ \alpha)) \\
&= I(\mathcal{B} \llbracket N_1 \rrbracket (L \circ \alpha) (\mathcal{B} \llbracket N_2 \rrbracket (L \circ \alpha))) \\
&\sqsubseteq I(\mathcal{B} \llbracket N_1 \rrbracket (L \circ \alpha) (L(I(\mathcal{B} \llbracket N_2 \rrbracket (L \circ \alpha)))) \quad \text{Lemma 20.2), } I \text{ and } \mathcal{B} \llbracket N_1 \rrbracket (L \circ \alpha) \text{ are monotone} \\
&= I(\mathcal{B} \llbracket N_1 \rrbracket (L \circ \alpha)) (I(\mathcal{B} \llbracket N_2 \rrbracket (L \circ \alpha))) \\
&\sqsubseteq I(\mathcal{B} \llbracket N_1 \rrbracket (L \circ \alpha)) (I(\mathcal{B}) \llbracket N_2 \rrbracket (\alpha)) \quad \text{range of } I \text{ is monotone and IH} \\
&\sqsubseteq I(\mathcal{B}) \llbracket N_1 \rrbracket (\alpha) (I(\mathcal{B}) \llbracket N_2 \rrbracket (\alpha)) \quad \text{inductive hypothesis} \\
&= I(\mathcal{B}) \llbracket N_1 N_2 \rrbracket (\alpha)
\end{aligned}$$

- Finally consider λ -abstractions $\Delta \vdash \lambda x. N : \tau \rightarrow \rho$. Let $s \in \mathcal{C} \llbracket \tau \rrbracket$. Then

$$\begin{aligned}
I(\mathcal{B} \llbracket \lambda x. N \rrbracket (L \circ \alpha))(s) &= I(\mathcal{B} \llbracket \lambda x. N \rrbracket (L \circ \alpha) (L(s))) \\
&= I(\mathcal{B} \llbracket N \rrbracket (L \circ (\alpha[x \mapsto s]))) \\
&\sqsubseteq I(\mathcal{B}) \llbracket N \rrbracket (\alpha[x \mapsto s]) \quad \text{inductive hypothesis} \\
&= (I(\mathcal{B}) \llbracket \lambda x. N \rrbracket (\alpha))(s)
\end{aligned}$$

Hence, $I(\mathcal{B}) \llbracket \lambda x. N \rrbracket (\alpha) \sqsubseteq I(\mathcal{B} \llbracket \lambda x. N \rrbracket (L \circ \alpha))$.

□

B.3 Supplementary Materials for Sec. 3.4

The relation \rightarrow_{\parallel} is defined by:

$$\begin{array}{c}
\frac{}{R \rightarrow_{\parallel} \lambda \bar{x}_R. F_R} \quad R \in \Sigma' \setminus \Sigma \qquad \frac{}{c \rightarrow_{\parallel} c} \quad c \in \Sigma \cup \{\wedge, \vee, \exists\} \qquad \frac{}{x \rightarrow_{\parallel} x} \\
\frac{M_1 \rightarrow_{\parallel} N_1 \quad M_2 \rightarrow_{\parallel} N_2}{M_1 M_2 \rightarrow_{\parallel} N_1 N_2} \qquad \frac{M \rightarrow_{\parallel} N}{\lambda x. M \rightarrow_{\parallel} \lambda x. N}
\end{array}$$

Proposition 27. *Let \mathcal{B} be an expansion of \mathcal{A} and let M and N be positive existential terms satisfying $M \rightarrow_{\parallel} N$. Then for all valuations α , $T_P^{\mathcal{H}}(\mathcal{B}) \llbracket M \rrbracket (\alpha) = \mathcal{B} \llbracket N \rrbracket (\alpha)$.*

Proof. We prove the lemma by induction on \rightarrow_{\parallel} :

- For variables, symbols from Σ and logical constants (other than \neg) this is trivial.
- If M is a symbol $R \in \Sigma' \setminus \Sigma$ then $T_P^{\mathcal{H}}(\mathcal{B}) \llbracket R \rrbracket (\alpha) = \mathcal{B} \llbracket \lambda \bar{x}_R. F_R \rrbracket (\alpha)$.
- Next, if M is an application $M_1 M_2$, $M_1 \rightarrow_{\parallel} N_1$ and $M_2 \rightarrow_{\parallel} N_2$ then

$$\begin{aligned}
T_P^{\mathcal{H}}(\mathcal{B}) \llbracket M_1 M_2 \rrbracket (\alpha) &= T_P^{\mathcal{H}}(\mathcal{B}) \llbracket M_1 \rrbracket (\alpha) (T_P^{\mathcal{H}}(\mathcal{B}) \llbracket M_2 \rrbracket (\alpha)) \\
&= \mathcal{B} \llbracket N_1 \rrbracket (\alpha) (\mathcal{B} \llbracket N_2 \rrbracket (\alpha)) \\
&= \mathcal{B} \llbracket N_1 N_2 \rrbracket (\alpha),
\end{aligned}$$

using the inductive hypothesis in the second step.

- Finally, if M is a λ -abstraction $\lambda x.M'$ and $M' \rightarrow_{\parallel} N'$ then

$$\begin{aligned} T_P^{\mathcal{H}}(\mathcal{B})[\![\lambda x.M']\!](\alpha) &= \lambda r \in \mathcal{H}[\![\Delta(x)]\!]. T_P^{\mathcal{H}}(\mathcal{B})[\![M']\!](\alpha[x \mapsto r]) \\ &= \lambda r \in \mathcal{H}[\![\Delta(x)]\!]. \mathcal{B}[\![N']\!](\alpha[x \mapsto r]) \\ &= \mathcal{B}[\![\lambda x.N']\!](\alpha), \end{aligned}$$

exploiting the inductive hypothesis. \square

Definition 63. A positive existential formula F is *ex-normal* if for all subterms $\exists M$ of F , M is a λ -abstraction.

Lemma 64 (Basic properties of $\rightarrow_{\beta v}$). *Suppose $M \rightarrow_{\beta v} N$. Then*

- 1) $\rightarrow_{\parallel} \subseteq \rightarrow_{\beta v}$.
- 2) $\text{fv}(N) \subseteq \text{fv}(M)$;
- 3) if M is ex-normal then N is ex-normal, too.

Proof. 1) Straightforward induction on the definition of \rightarrow_{\parallel} .

- 2) We prove the first part of the lemma by induction on the compatible closure of $\rightarrow_{\beta v}$. If $(M, N) \in \beta$ this is a standard fact of β -reduction. If $(M, N) \in v$ then $\text{fv}(M) = \text{fv}(N) = \emptyset$. In the inductive cases the claim immediately follows from the inductive hypothesis.

- 3) We prove the claim by induction on the compatible closure of $\rightarrow_{\beta v}$.

- First, suppose that $(R, \lambda \bar{x}_R.F_R) \in v$. Obviously, F_R is ex-normal and hence, $\lambda \bar{x}_R.F_R$ is ex-normal, too.
- Next, suppose that $((\lambda x.M)M', M[M'/x]) \in \beta$. Clearly, M and M' must be ex-normal. We prove by induction on M that $M[M'/x]$ is ex-normal. If M is a variable this is obvious (because M' is ex-normal). The cases for (logical) constants and λ -abstractions are straightforward. Finally, suppose that M is an application and let $\exists L$ be a subterm of

$$M_1[M'/x]M_2[M'/x]$$

By the inductive hypothesis, both $M_1[M'/x]$ and $M_2[M'/x]$ are ex-normal. Hence, if $\exists K$ is a subterm of either $M_1[M'/x]$ or $M_2[M'/x]$ then K must be a λ -abstraction. Otherwise $M_1 = \exists$ and $K = M_2[M'/x]$. Then by assumption M_2 is a λ -abstraction and clearly, $M_2[M'/x]$ is a λ -abstraction, too.

- Next, suppose that $M_1M_2 \rightarrow_{\beta v} N_1M_2$ because $M_1 \rightarrow_{\beta v} N_1$. Clearly, M_1 is ex-normal. Therefore, by the inductive hypothesis, N_1 is ex-normal. Note that $N_1 = \exists$ is impossible. Therefore, any subterm $\exists L$ of N_1M_2 is either a subterm of N_1 or M_2 , which are both ex-normal. Hence, L is a λ -abstraction.
- Suppose $M_1M_2 \rightarrow_{\beta v} M_1N_2$ because $M_2 \rightarrow_{\beta v} N_2$. Clearly, M_2 is ex-normal. Therefore, by the inductive hypothesis, N_2 is ex-normal. Let $\exists L$ be a subterm of M_1N_2 . If $\exists L$ is a subterm of M_1 or N_2 the argument is as in the previous case. Hence, suppose $M_1 = \exists$ and $L = N_2[M'/x]$. By assumption M_2 is a λ -abstraction. Due to $M_2 \rightarrow_{\beta v} L$, L is a λ -abstraction, too.
- Finally, suppose that $\lambda x.M \rightarrow_{\beta v} \lambda x.N$ because $M \rightarrow_{\beta v} N$. Clearly, M is ex-normal and hence by the inductive hypothesis N is ex-normal. Let $\exists L$ be a subterm of $\lambda x.N$. Obviously, $\exists L$ must be a subterm of M , which is ex-normal. Hence, L is a λ -abstraction. \square

Lemma 65 (Subject Reduction). *Let $\Delta \vdash M : \sigma$ be a term such that $M \rightarrow_{\beta v} N$. Then*

- 1) $\Delta \vdash N : \sigma$ and
- 2) σ is a relational type.

Proof. 1) We prove the lemma by induction on the compatible closure of βv . For $(M, N) \in \beta$ this is [46, Proposition 1.2.6]. If $(R, \lambda \bar{x}_R. F_R) \in v$ and $R : \bar{\tau} \rightarrow o \in \Sigma' \setminus \Sigma$ then by convention $\Delta(\bar{x}_R) = \bar{\tau}$ and hence, $\Delta \vdash \lambda \bar{x}_R. F_R : \bar{\tau} \rightarrow o$, too. The proofs for the recursive cases are exactly as in the proof of [46, Proposition 1.2.6].

- 2) Clearly, it suffices to prove by induction on the compatible closure of βv that $M \rightarrow_{\beta v} N$ implies $\Delta \nVdash M : \iota^n \rightarrow \iota$ for all $n \in \mathbb{N}$.
 - If $(M, N) \in \beta v$ then clearly $\Delta \nVdash M : \iota^n \rightarrow \iota$ for all $n \in \mathbb{N}$.
 - Next, suppose $M_1 M_2 \rightarrow_{\beta v} N_1 N_2$ due to $M_1 \rightarrow_{\beta v} N_1$. Then by the inductive hypothesis $\Delta \nVdash M_1 : \iota^n \rightarrow \iota$ for all $n \in \mathbb{N}$. Hence, clearly $\Delta \nVdash M_1 M_2 : \iota^m \rightarrow \iota$ for all $m \in \mathbb{N}$.
 - Suppose $M_1 M_2 \rightarrow_{\beta v} N_1 N_2$ due to $M_2 \rightarrow_{\beta v} N_2$ and assume towards contradiction that $\Delta \vdash M_1 M_2 : \iota^n \rightarrow \iota$. Then $\Delta \vdash M_1 : \sigma \rightarrow \iota^n \rightarrow \iota$ and $\Delta \vdash M_2 : \sigma$ for some σ . However, by the definition of types this implies $\sigma = \iota$, which contradicts the inductive hypothesis.
 - Finally, if $\lambda x. M' \rightarrow_{\beta v} \lambda x. N$ then clearly $\Delta \nVdash \lambda x. M' : \iota^n \rightarrow \iota$ for all $n \in \mathbb{N}$. □

B.4 Supplementary Materials for Sec. 3.5

Lemma 66 (Basic Properties of $\xrightarrow{\ell}$). *Let L, M, N and Q be terms. Then*

- 1) $\xrightarrow{\ell}$ is reflexive and transitive;
- 2) $\xrightarrow{\ell} \subseteq \rightarrow_{\beta v}$.
- 3) if $M \xrightarrow[0]{\ell} N$ then $M = N$;
- 4) if $L \xrightarrow[m+1]{\ell} N$ then there exists M satisfying $L \xrightarrow[1]{\ell} M \xrightarrow[m]{\ell} N$;
- 5) if $M Q$ is well-typed and $M \xrightarrow{\ell} N$ then $M Q \xrightarrow{\ell} N Q$;
- 6) if $M[Q/z]$ is well-typed and $M \xrightarrow{\ell} N$ then $M[Q/z] \xrightarrow{\ell} N[Q/z]$.

Proof. 1) Completely trivial.

- 2) Straightforward induction on the definition of $\xrightarrow[m]{\ell}$
- 3) Straightforward induction on the definition of $\xrightarrow[0]{\ell}$
- 4) Straightforward induction on the definition of $\xrightarrow[m+1]{\ell}$
- 5) Straightforward induction on the definition of $M \xrightarrow[m]{\ell} N$ noting that for $\circ \in \{\wedge, \vee\}$ the cases $M_1 \circ M_2 \xrightarrow[m_1+m_2]{\ell} N_1 \circ N_2$ and $\exists x. M' \xrightarrow[m]{\ell} \exists x. N'$ cannot occur because $(M_1 \circ M_2) Q$ and $(\exists x. M') Q$ are not well-typed.
- 6) We prove by induction on $M \xrightarrow[m]{\ell} N$ that $M[Q/z] \xrightarrow[m]{\ell} N[Q/z]$.
 - If $M = N$ and $m = 0$ then also $M[Q/z] \xrightarrow[0]{\ell} N[Q/z]$.
 - If there exist L, m_1 and m_2 such that $M \xrightarrow[m_1]{\ell} L \xrightarrow[m_2]{\ell} N$ and $m = m_1 + m_2$ then by the inductive hypothesis $M[Q/z] \xrightarrow[m_1]{\ell} L[Q/z] \xrightarrow[m_2]{\ell} N[Q/z]$. Consequently, $M[Q/z] \xrightarrow[m]{\ell} N[Q/z]$.
 - Next, suppose that M is $M_1 \circ M_2$ for $\circ \in \{\wedge, \vee\}$ and that there exist m_1 and m_2 such that $m = m_1 + m_2$ and $M_j \xrightarrow[m_j]{\ell} N_j$ for $j \in \{1, 2\}$. By the inductive hypothesis, $M_j[Q/z] \xrightarrow[m_j]{\ell} N_j[Q/z]$.

Consequently,

$$(M_1 \circ M_2)[Q/z] = (M_1[Q/z] \circ M_2[Q/z]) \xrightarrow[m_1+m_2]{\ell} (N_1[Q/z] \circ N_2[Q/z]) = (N_1 \circ N_2)[Q/z].$$

- Suppose that M is $\exists x.M'$ and that $\exists x.M' \xrightarrow[m]{\ell} \exists x.N'$. By the inductive hypothesis, $M'[Q/z] \xrightarrow[m]{\ell} N'[Q/z]$. By the variable convention, $x \neq z$. Hence,

$$(\exists x.M')[Q/z] = \exists x.M'[Q/z] \xrightarrow[m]{\ell} \exists x.N'[Q/z] = (\exists x.N')[Q/z].$$

- Suppose that M is $R\overline{M}'$ for $R \in \Sigma' \setminus \Sigma$ and that $R\overline{M}' \xrightarrow[1]{\ell} (\lambda \overline{x}_R.F_R)\overline{M}'$. Clearly,

$$(R\overline{M}')[Q/z] = R\overline{M}'[Q/z] \xrightarrow[1]{\ell} (\lambda \overline{x}_R.F_R)\overline{M}'[Q/z] = ((\lambda \overline{x}_R.F_R)\overline{M}')[Q/z]$$

using the variable convention and the fact that $\lambda \overline{x}_R.F_R$ is closed.

- Finally, suppose that M is $(\lambda x.M')M''\overline{M}'''$ and that

$$(\lambda x.M')M''\overline{M}''' \xrightarrow[1]{\ell} M'[M''/x]\overline{M}'''.$$

By the variable convention $x \neq z$. Hence

$$\begin{aligned} ((\lambda x.M')M''\overline{M}''')[Q/z] &= (\lambda x.M'[Q/z])M''[Q/z]\overline{M}'''[Q/z] \\ &\xrightarrow[1]{\ell} M'[Q/z][M''[Q/z]/x]\overline{M}'''[Q/z] \\ &= (M'[M''/x]\overline{M}''')[Q/z] \end{aligned}$$

using the Nested Substitution Lemma from [11, 2.1.16. Substitution Lemma]. \square

Besides, the following Inversion Lemma is immediate by definition.

- Lemma 67** (Inversion). 1) If $\exists \overline{x}.E \xrightarrow[m]{\ell} F$ then there exists F' such that $F' = \exists \overline{x}.F'$ and $E \xrightarrow[m]{\ell} F'$.
- 2) If $E_1 \circ \dots \circ E_n \xrightarrow[m]{\ell} F$, where $\circ \in \{\wedge, \vee\}$, then there exist F_1, \dots, F_n and m_1, \dots, m_n satisfying $F = F_1 \circ \dots \circ F_n$, $m = \sum_{i=1}^n m_i$ and $E_j \xrightarrow[m_j]{\ell} F_j$ for each $1 \leq j \leq n$.
- 3) If $\exists \overline{x}.A_1 \wedge \dots \wedge A_n \xrightarrow[m]{\ell} F$ then there exist F_1, \dots, F_n and m_1, \dots, m_n satisfying $F = \exists \overline{x}.F_1 \wedge \dots \wedge F_n$, $m = \sum_{i=1}^n m_i$ and $A_j \xrightarrow[m_j]{\ell} F_j$ for each $1 \leq j \leq n$.
- 4) If $(\lambda x.K)L\overline{M} \xrightarrow[1]{\ell} N$ then $N = K[L/x]\overline{M}$.

- Lemma 68** (Basic Properties of \xrightarrow{s}). 1) \xrightarrow{s} is reflexive (on positive existential terms).

- 2) $\xrightarrow{s} \subseteq \rightarrow_{\beta v}$.
- 3) If $L \xrightarrow{s} N$ and $\overline{O} \xrightarrow{s} \overline{Q}$ then $L\overline{O} \xrightarrow{s} N\overline{Q}$.
- 4) If $K \xrightarrow[\ell]{\ell} L \xrightarrow{s} N$ then $K \xrightarrow{s} N$.
- 5) If $L \xrightarrow{s} N$ and $O \xrightarrow{s} Q$ then $L[O/z] \xrightarrow{s} N[Q/z]$.

Proof. 1) We prove by structural induction on M that $M \xrightarrow{s} M$. M has the form $M_1 \dots M_n$, where M_1 is either a variable, a symbol from $\Sigma' \cup \{\wedge, \vee, \exists, \tau\}$ or a λ -abstraction. In any case the inductive hypothesis and the reflexivity of $\xrightarrow[\ell]{\ell}$ immediately yield that $M_1 \dots M_n \xrightarrow{s} M_1 \dots M_n$.

We prove the remaining four parts by induction on the definition of \xrightarrow{s} . We only show the detailed proof for the case $L \xrightarrow{s} x\overline{N}$ due to $L \xrightarrow[\ell]{\ell} x\overline{M}$ and $\overline{M} \xrightarrow{s} \overline{N}$ for some \overline{M} (the other cases are analogous).

- 2) By Lemma 66.2) and the inductive hypothesis, $L \rightarrow_{\beta v} x\overline{M}$ and $\overline{M} \rightarrow_{\beta v} \overline{N}$. Therefore clearly, $L \rightarrow_{\beta v} x\overline{M} \rightarrow_{\beta v} x\overline{N}$ and hence also, $L \rightarrow_{\beta v} x\overline{N}$.
- 3) By Lemma 66.5), $L\overline{O} \rightarrow_{\ell} x\overline{M}\overline{O}$ and hence by definition $L\overline{O} \rightarrow_s x\overline{N}\overline{O}$.
- 4) By transitivity of \rightarrow_{ℓ} (Lemma 66.1)), $K \rightarrow_{\ell} x\overline{M}$ and hence by definition $K \rightarrow_s x\overline{N}$.
- 5) By the inductive hypothesis, $\overline{M}[O/z] \rightarrow_s \overline{N}[Q/z]$ and by assumption or Part 1), $x[O/z] \rightarrow_s x[Q/z]$. Therefore by Part 3) and Lemma 66.6),

$$L[O/z] \rightarrow_{\ell} x[O/z]\overline{M}[O/z] \rightarrow_s x[Q/z]\overline{N}[Q/z],$$

which proves $L[O/z] \rightarrow_s (x\overline{N})[Q/z]$ by Part 4). \square

Lemma 69 (Inversion). *Let E be an ex-normal formula.*

- 1) If $E \rightarrow_s x\overline{N}$ then there exists \overline{M} such that $E \rightarrow_{\ell} x\overline{M}$.
- 2) If $E \rightarrow_s c\overline{N}$, where $c \in \Sigma' \cup \{\wedge, \vee, \exists\tau\}$, then there exists \overline{M} such that $E \rightarrow_{\ell} c\overline{M}$ and $\overline{M} \rightarrow_s \overline{N}$.
- 3) If $E \rightarrow_s \exists N$ then there exist x, N' and M such that $N = (\lambda x.N')$, $E \rightarrow_{\ell} \exists x.M$ and $M \rightarrow_s N'$.

Proof. The first two parts are obvious by definition of \rightarrow_s . Hence, suppose that $E \rightarrow_s \exists N$.

By Lemmas 68.2) and 64.3), N has the form $\lambda x.N'$ for some N' . Furthermore, by Part 2) there exists L such that $E \rightarrow_{\ell} \exists\tau L$ and $L \rightarrow_s \lambda x.N'$. Again, by Lemmas 66.2) and 64.3), L has the form $\lambda y.L'$. By definition of \rightarrow_s , $(\lambda y.L') \rightarrow_s (\lambda x.N')$ implies that there exists M such that $(\lambda y.L') \rightarrow_{\ell} (\lambda x.M)$ and $M \rightarrow_s N'$. However, $(\lambda y.L') \rightarrow_{\ell} (\lambda x.M)$ clearly implies $y = x$ and $L' = M$. Consequently, $E \rightarrow_{\ell} \exists\tau x.M$ and $M \rightarrow_s N'$. \square

Proposition 28. *If $K \rightarrow_s M \rightarrow_{\beta v} N$ then $K \rightarrow_s N$.*

Proof. We prove the lemma by induction on $K \rightarrow_s M$.

- First, suppose $K \rightarrow_s xM_1 \cdots M_n$ because for some L_1, \dots, L_n , $K \rightarrow_{\ell} xL_1 \cdots L_n$ and $L_i \rightarrow_s M_i$ for each i . Clearly, $xM_1 \cdots M_n \rightarrow_{\beta v} xN_1 \cdots N_n$, because of $M_j \rightarrow_{\beta v} N_j$ for some j and $M_i = N_i$ for $i \neq j$ are the only possible βv -reductions. By the inductive hypothesis, $L_j \rightarrow_s N_j$ and therefore by definition, $L \rightarrow_s xN_1 \cdots N_n$.
- Next, suppose $K \rightarrow_s c\overline{M}$ because for some \overline{L} , $K \rightarrow_{\ell} c\overline{L}$ and $\overline{L} \rightarrow_s \overline{M}$. If $c = R \in \Sigma' \setminus \Sigma$ and $R\overline{M} \rightarrow_{\beta v} (\lambda \overline{x}_R.F_R)\overline{M}$ then $K \rightarrow_{\ell} R\overline{L} \xrightarrow{1}_{\ell} (\lambda \overline{x}_R.F_R)\overline{L}$. Therefore, by reflexivity of \rightarrow_s (Lemma 68.1)), $K \rightarrow_s (\lambda \overline{x}_R.F_R)\overline{M}$.

Otherwise, \overline{M} is reduced and the argument is analogous to the case for $K \rightarrow_s x\overline{M}$.

- Finally, suppose $K \rightarrow_s (\lambda x.M')\overline{M}$ because for some L' and \overline{L} , $K \rightarrow_{\ell} (\lambda x.L')\overline{L}$, $L' \rightarrow_s M'$ and $\overline{L} \rightarrow_s \overline{M}$. Let $\overline{L} = (L_1, \dots, L_n)$ and $\overline{M} = (M_1, \dots, M_n)$.

First, suppose $(\lambda x.M')\overline{M} \rightarrow_{\beta v} (\lambda x.N')\overline{M}$, where $M' \rightarrow_{\beta v} N'$. By the inductive hypothesis, $L' \rightarrow_s N'$. Therefore, by definition, $(\lambda x.K')\overline{L} \rightarrow_s (\lambda x.N')\overline{M}$.

The argument for the case $(\lambda x.M')M_1 \cdots M_n \rightarrow_{\beta v} (\lambda x.M')N_1 \cdots N_n$, where for some j , $M_j \rightarrow_{\beta v} N_j$ and $M_k = N_k$ for all $k \neq j$, is very similar.

Finally, assume that $n \geq 1$ and $(\lambda x.M')M_1 \cdots M_n \rightarrow_{\beta v} M'[M_1/x]M_2 \cdots M_n$. Then

$$L \rightarrow_{\ell} (\lambda x.L')L_1 \cdots L_n \xrightarrow{1}_{\ell} L'[L_1/x]L_2 \cdots L_n \rightarrow_s M'[M_1/x]M_2 \cdots M_n,$$

by Lemmas 68.3) and 68.5), which proves $L \rightarrow_s M'[M_1/x]M_2 \cdots M_n$ by Lemmas 66.1) and 68.4). \square

Lemma 30. *Let G be a goal clause, F be a β -normal positive existential formula and α be a valuation such that $\mathcal{A}_0^{\mathcal{H}}, \alpha \models F$ and $\text{posex}(G) \xrightarrow{s} F$. Then there exists a positive existential formula F' satisfying $\text{posex}(G) \xrightarrow{\ell} F'$ and $\alpha \triangleright F'$.*

Proof. We prove the lemma by induction on the structure of F . Note that the case $(\lambda x.N')\bar{N}$ cannot occur for otherwise F is not in β -normal form or does not have type o . If F has the form $x\bar{N}$ then by the Inversion Lemma 69 there exists \bar{M} such that $\text{posex}(G) \xrightarrow{\ell} x\bar{M}$, and clearly, $\alpha \triangleright x\bar{M}$.

Hence, the only remaining case is that F has the form $c\bar{N}$. By the Inversion Lemma 69 there exist \bar{M} such that $\text{posex}(G) \xrightarrow{\ell} c\bar{M}$ and $\bar{M} \xrightarrow{s} \bar{N}$. Note that $c \in \Sigma'$ implies $c : \iota^n \rightarrow o \in \Sigma$ for otherwise $\mathcal{A}_0^{\mathcal{H}}, \alpha \not\models c\bar{N}$. By Lemmas 65.2) and 68.2), $M = N$ and thus $\mathcal{A}_0^{\mathcal{H}}, \alpha \models c\bar{M}$. Consequently, $\alpha \triangleright c\bar{M}$.

Next, suppose that c is \wedge . Then F is $N_1 \wedge N_2$ and $c\bar{M}$ has the form $M_1 \wedge M_2$. By Lemma 66.2) and the Subject Reduction Lemma 65, M_j is a positive existential formula and clearly, by assumption, N_j is in β -normal form and $\mathcal{A}_0^{\mathcal{H}}, \alpha \models N_j$ for all $j \in \{1, 2\}$. By the inductive hypothesis, there are N'_1 and N'_2 satisfying $M_j \xrightarrow{\ell} N'_j$ and $\alpha \triangleright N'_j$. Consequently, $\alpha \triangleright N'_1 \wedge N'_2$ and by definition, $\text{posex}(G) \xrightarrow{\ell} N'_1 \wedge N'_2$.

The case where c is \vee is very similar.

Finally, suppose that c is \exists_τ and that F is $\exists_\tau N_1$. By the Inversion Lemma 69 there exist x, N' and M such that $N_1 = (\lambda x.N')$, $\text{posex}(G) \xrightarrow{\ell} \exists_\tau x.M$ and $M \xrightarrow{s} N'$. By Lemma 66.2) and the Subject Reduction Lemma 65, M is a positive existential formula and clearly N' is in β -normal form and $\mathcal{A}_0^{\mathcal{H}}, \alpha[x \mapsto r] \models N'$ for some $r \in \mathcal{H}[\tau]$. By the inductive hypothesis, there exists N'' satisfying $M \xrightarrow{\ell} N''$ and $\alpha[x \mapsto r] \triangleright N''$. Consequently, $\alpha \triangleright \exists_\tau x.N''$ and by definition, $\text{posex}(G) \xrightarrow{\ell} \exists_\tau x.N''$. \square

B.5 Supplementary Materials for Sec. 3.6

The proof of the following lemma is a straightforward induction on the definition of \triangleright :

Lemma 70. *Let α, α' be valuations and F be positive existential formulas satisfying $\alpha \triangleright F$. If $\alpha(x) = \alpha'(x)$ for all $x \in \text{fv}(F)$ then $\alpha' \triangleright F$.*

Proposition 31. *Let $S' \supseteq S$ be a set of HoCHCs satisfying $0 < \mu(S') < \omega$. Then there exists $S'' \supseteq S$ satisfying $S' \Rightarrow_{\text{Res}, \mathcal{A}} S''$ and $\mu(S'') < \mu(S')$.*

Proof. Let $G \in S'$ be goal clause, F be a (closed) positive existential formula, α be a valuation and let $m = \mu(G) > 0$ be such that $\text{posex}(G) \xrightarrow[\ell]{m} F$ and $\alpha \triangleright F$. Without loss of generality we can assume that

$$\text{fv}(G) \cap \text{fv}(C) = \emptyset \quad \text{for all } C \in S'. \quad (5)$$

(Otherwise, rename all variables occurring in G to obtain \tilde{G} satisfying Eq. (5) and clearly, by definition of $\Rightarrow_{\text{Res}, \mathcal{A}}$, $S' \cup \{\tilde{G}\} \Rightarrow_{\text{Res}, \mathcal{A}} S' \cup \{\tilde{G}, G'\}$ implies $S' \Rightarrow_{\text{Res}, \mathcal{A}} S' \cup \{G'\}$.)

Furthermore, suppose that $G = \neg A_1 \vee \dots \vee \neg A_n$ and $\text{posex}(G) = \exists \bar{x}. \bigwedge_{i=1}^n A_i$. By the Inversion Lemma 67, there exist F_1, \dots, F_n and m_1, \dots, m_n such that $F = \exists \bar{x}. \bigwedge_{i=1}^n F_i$, $m = \sum_{i=1}^n m_i$ and $A_j \xrightarrow[\ell]{m_j} F_j$ for each $1 \leq j \leq m$. Note that due to $\alpha \triangleright F$ we can assume without loss of generality that also $\alpha \triangleright F_j$ for each $1 \leq j \leq n$, and furthermore we can assume that $m_1 > 0$. By Lemma 66.4), there exists E such that $A_1 \xrightarrow[\ell]{1} E \xrightarrow[\ell]{m_1-1} F_1$. Since A_1 is an atom there are exactly two cases:

- 1) $A_1 = (\lambda y.L)M\bar{N}$ and $E = L[M/y]\bar{N}$ or

2) $A_1 = R\overline{M}$ and $E = (\lambda\overline{x}_R.F_R)\overline{M}$.

The first case is easy because for $G' = \neg L[M/y]\overline{N} \vee \bigvee_{i=2}^n \neg A_i$, $S' \cup \{G\} \Rightarrow_{\text{Res}, \mathcal{A}} S' \cup \{G, G'\}$ and $\text{posex}(G') = \exists \overline{x}. L[M/y]\overline{N} \wedge \bigwedge_{i=2}^n A_i$.

In the second case, note that $\frac{1}{\ell}$ is functional on applied λ -abstractions (by the Inversion Lemma 67). Hence, we can assume that

$$(\lambda\overline{x}_R.F_R)\overline{M} \twoheadrightarrow_{\ell} F_R[\overline{M}/\overline{x}_R] \xrightarrow{\ell}^{m_1^*} F_1,$$

where $m_1^* \leq m_1 - 1$ for otherwise $\alpha \triangleright F_1$ would clearly not hold.

F_R has the form $\text{posex}(G_{R,1}, \overline{x}_R) \vee \dots \vee \text{posex}(G_{R,k}, \overline{x}_R)$, where each $G_{R,j}$ is a goal clause and $G_{R,j} \vee R\overline{x}_R \in S'$. Let $\overline{y}_1, \dots, \overline{y}_k$ and E'_1, \dots, E'_k be such that for each j , $\text{posex}(G_{R,j}, \overline{x}_R) = \exists \overline{y}_j. E'_j$. Note that by Eq. (5), $\text{posex}(G_{R,j}, \overline{x}_R)[\overline{M}/\overline{x}_R] = \exists \overline{y}_j. E'_j[\overline{M}/\overline{x}_R]$ for each j and by the Inversion Lemma 67, there exist F'_1, \dots, F'_k and m'_1, \dots, m'_k such that $F_1 = \bigvee_{j=1}^k (\exists \overline{y}_j. F'_j)$, $E'_j[\overline{M}/\overline{x}_R] \xrightarrow{\ell}^{m'_j} F'_j$ and $m'_j \leq m_1^*$ for each j .

Next, because of $\alpha \triangleright F_1$ there exists $1 \leq j \leq k$ and $\overline{r} \in \mathcal{H}[\Delta(\overline{y}_j)]$ satisfying $\alpha[\overline{y}_j \mapsto \overline{r}] \triangleright F'_j$. Furthermore, because of Eq. (5) and Lemmas 64.2) and 70, $\alpha[\overline{y}_j \mapsto \overline{r}] \triangleright F_i$ for all $2 \leq i \leq n$. Therefore,

$$\alpha[\overline{y}_j \mapsto \overline{r}] \triangleright F'_j \wedge \bigwedge_{i=2}^n F_i. \quad (6)$$

Clearly, it holds that

$$S' \cup \{G\} \Rightarrow_{\text{Res}, \mathcal{A}} S' \cup \left\{ G, G_{R,j}[\overline{M}/\overline{x}_R] \vee \bigvee_{i=2}^n \neg A_i \right\} \quad (7)$$

$$E'_j[\overline{M}/\overline{x}_R] \wedge \bigwedge_{i=2}^n A_i \xrightarrow{\ell}^{m'_j + \sum_{i=2}^n m_i} F'_j \wedge \bigwedge_{i=2}^n F_i \quad (8)$$

and $\text{fv}(G') \subseteq \overline{x} \cup \overline{y}$. Let $\overline{x}' \subseteq \overline{x}$ and $\overline{y}' \subseteq \overline{y}$ be such that $\text{fv}(G') = \overline{x}' \cup \overline{y}'$. Hence, $\text{posex}(G') = \exists \overline{x}', \overline{y}'. E'_j[\overline{M}/\overline{x}_R] \wedge \bigwedge_{i=2}^n A_i$. We define

$$G' := G_{R,j}[\overline{M}/\overline{x}_R] \vee \bigvee_{i=2}^n \neg A_i \quad F' := \exists \overline{x}', \overline{y}'. E'_j \wedge \bigwedge_{i=2}^n F_i \quad m' := m'_j + \sum_{i=2}^n m_i.$$

By Eqs. (6) to (8), all of the following holds: 1) $S' \Rightarrow_{\text{Res}, \mathcal{A}} S' \cup \{G'\}$, 2) $\text{posex}(G') \xrightarrow{\ell}^{m'} F'$, 3) $\triangleright F'$ and 4) $m' \leq m_1^* + \sum_{i=2}^n m_i < \sum_{i=1}^n m_i = m$. Consequently, also $\mu(S' \cup \{G'\}) < \mu(S')$ \square

C Supplementary Materials for Sec. 4

Example 71. 1) It holds or \sqsubseteq_m or, and \sqsubseteq_m and for every $s \in \mathcal{H}[\rho]$, $\perp_{\rho}^{\mathcal{H}} \sqsubseteq_c s \sqsubseteq_c \top_{\rho}^{\mathcal{H}}$.

2) Next, suppose $r, r' \in \mathcal{H}[\tau \rightarrow o]$ are such that $r \sqsubseteq_m r'$ and $\text{exists}_{\tau}(r) = 1$. Hence, there exists $s \in \mathcal{H}[\tau]$ satisfying $r(s) = 1$. If $\tau = \iota$ then $r'(s) = 1$ and otherwise $r'(\top_{\tau}^{\mathcal{H}}) = 1$ because $s \sqsubseteq_m \top_{\tau}^{\mathcal{H}}$. Consequently, $\text{exists}_{\tau}^{\mathcal{H}}(r') = 1$ and $\text{exists}^{\mathcal{H}} \sqsubseteq_m \text{exists}^{\mathcal{H}}$, too.

- 3) Let $\mathcal{H} = \mathcal{S}$ be the standard frame and let $\widetilde{\text{neg}} : \mathbb{B} \rightarrow \mathbb{B}$ be defined by $\widetilde{\text{neg}}(b) = 1 - b$ for $b \in \mathbb{B}$. Clearly, $\widetilde{\text{neg}} \in \mathcal{S}[[o \rightarrow o]]$ and $0 \sqsubseteq_m 1$. However, $\widetilde{\text{neg}}(0) = 1 > 0 = \widetilde{\text{neg}}(1)$. This shows that \sqsubseteq_m is not reflexive.

Lemma 72. *Let ρ a relational type and $\mathfrak{R}, \mathfrak{R}' \subseteq \mathcal{H}[[\rho]]$ be sets such that for each $r \in \mathfrak{R}$ there exists $r' \in \mathfrak{R}'$ satisfying $r \sqsubseteq_m r'$. Then $\sqcup \mathfrak{R} \sqsubseteq_m \sqcup \mathfrak{R}'$.*

Proof. We prove the lemma by induction on the relational type ρ . For $\rho = o$ this is obvious. Hence, suppose $\rho = \tau \rightarrow \rho'$. To show $\sqcup \mathfrak{R} \sqsubseteq_c \sqcup \mathfrak{R}'$, let $s \sqsubseteq_m s' \in \mathcal{H}[[\tau]]$. We define $\mathfrak{T} = \{r(s) \mid r \in \mathfrak{R}\}$ and $\mathfrak{T}' = \{r'(s') \mid r' \in \mathfrak{R}'\}$. Let $r \in \mathfrak{R}$. By assumption, there exists $r' \in \mathfrak{R}'$ such that $r \sqsubseteq_m r'$. Hence, due to $s \sqsubseteq_m s'$, $r(s) \sqsubseteq_m r'(s')$. Therefore, the inductive hypothesis is applicable to \mathfrak{T} and \mathfrak{T}' , which yields $(\sqcup \mathfrak{R})(s) = \sqcup \mathfrak{T} \sqsubseteq_m \sqcup \mathfrak{T}' = (\sqcup \mathfrak{R}')(s')$. As a consequence, $\sqcup \mathfrak{R} \sqsubseteq_m \sqcup \mathfrak{R}'$. \square

Corollary 73. *Let \mathfrak{B} be a set of expansions of \mathcal{A} . Then*

- 1) $\mathcal{B} \sqsubseteq_m \mathcal{B} \in \mathfrak{B}$ implies $\mathcal{B} \sqsubseteq_m \sqcup \mathfrak{B}$, and
- 2) if \mathcal{B}' is an expansion of \mathcal{A} such that for all $\mathcal{B} \in \mathfrak{B}$, $\mathcal{B} \sqsubseteq_m \mathcal{B}'$, then $\sqcup \mathfrak{B} \sqsubseteq_m \mathcal{B}'$.

Corollary 74. *For each ordinal β , $\mathcal{A}_\beta^{\mathcal{H}} \sqsubseteq_m \mathcal{A}_P^{\mathcal{H}}$.*

Proposition 37. *Let $\mathcal{B} \sqsubseteq_m \mathcal{B}'$ be (Σ', \mathcal{H}) -expansions of \mathcal{A} , $\alpha \sqsubseteq_m \alpha'$ be valuations and let M be a positive existential term. Then $\mathcal{B}[[M]](\alpha) \sqsubseteq_m \mathcal{B}'[[M]](\alpha')$.*

Proof. We prove the claim by induction on the structure of M .

- If M is a variable x then $\mathcal{B}[[M]](\alpha) = \alpha(x) \sqsubseteq_m \alpha'(x) = \mathcal{B}'[[M]](\alpha')$ because of $\alpha \sqsubseteq_m \alpha'$.
- If M is a logical symbol (other than \rightarrow) then this is a consequence of Examples 71.1) and 71.2).
- If M is a (1st-order) symbol $c \in \Sigma$ then $R^{\mathcal{B}} = R^{\mathcal{A}} = R^{\mathcal{B}'}$. Otherwise, it is a (relational) symbol $R \in \Sigma' \setminus \Sigma$ and $\mathcal{B}[[M]](\alpha) = R^{\mathcal{B}} \sqsubseteq_m R^{\mathcal{B}'} = \mathcal{B}'[[M]](\alpha')$ because of $\mathcal{B} \sqsubseteq_m \mathcal{B}'$.
- If M is an application NN' then by the inductive hypothesis $\mathcal{B}[[N]](\alpha) \sqsubseteq_m \mathcal{B}'[[N]](\alpha')$ and $\mathcal{B}[[N']](\alpha) \sqsubseteq_m \mathcal{B}'[[N']](\alpha')$. First, suppose that $\Delta \vdash N : \iota^{n+1} \rightarrow \iota$. Then $\mathcal{B}[[N]](\alpha) = \mathcal{B}'[[N]](\alpha')$ and $\mathcal{B}[[N']](\alpha) = \mathcal{B}'[[N']](\alpha')$. Therefore,

$$\mathcal{B}[[M]](\alpha) = \mathcal{B}[[N]](\alpha)(\mathcal{B}[[N']](\alpha)) = \mathcal{B}'[[N]](\alpha')(\mathcal{B}'[[N']](\alpha')) = \mathcal{B}'[[M]](\alpha')$$

and thus $\mathcal{B}[[M]](\alpha) \sqsubseteq_m \mathcal{B}'[[M]](\alpha')$. Otherwise, $\Delta \vdash \tau \rightarrow \rho$ and hence,

$$\mathcal{B}[[M]](\alpha) = \mathcal{B}[[N]](\alpha)(\mathcal{B}[[N']](\alpha)) \sqsubseteq_m \mathcal{B}'[[N]](\alpha')(\mathcal{B}'[[N']](\alpha')) = \mathcal{B}'[[M]](\alpha')$$

by definition of \sqsubseteq_m .

- Finally, suppose M is an abstraction $\lambda x. N$. Let $s \sqsubseteq_m s'$. By the inductive hypothesis $\mathcal{B}[[N]](\alpha[x \mapsto s]) \sqsubseteq_m \mathcal{B}'[[N]](\alpha'[x \mapsto s'])$ and hence,

$$\mathcal{B}[[M]](\alpha)(s) = \mathcal{B}[[N]](\alpha[x \mapsto s]) \sqsubseteq_m \mathcal{B}'[[N]](\alpha'[x \mapsto s']) = \mathcal{B}'[[M]](\alpha')(s').$$

Due to the fact that this holds for every $s \sqsubseteq_m s'$, $\mathcal{B}[[M]](\alpha) \sqsubseteq_m \mathcal{B}'[[M]](\alpha')$. \square

Consequently, also the immediate consequence operator is monotone with respect to \sqsubseteq_m :

Corollary 75. *If $\mathcal{B} \sqsubseteq_m \mathcal{B}'$ then $T_P^{\mathcal{H}}(\mathcal{B}) \sqsubseteq_m T_P^{\mathcal{H}}(\mathcal{B}')$.*

Proof. It holds $\top_{\Delta}^{\mathcal{H}} \sqsubseteq_m \top_{\Delta}^{\mathcal{H}}$ (Ex. 71.1)). Hence, by Prop. 37, for every $R \in \Sigma' \setminus \Sigma$,

$$R^{T_P^{\mathcal{H}}(\mathcal{B})} = \mathcal{B}[\llbracket \lambda \bar{x}_R. F_R \rrbracket(\top_{\Delta}^{\mathcal{H}})] \sqsubseteq_m \mathcal{B}'[\llbracket \lambda \bar{x}_R. F_R \rrbracket(\top_{\Delta}^{\mathcal{H}})] = R^{T_P^{\mathcal{H}}(\mathcal{B}')} \square$$

The following proof is a straightforward transfinite induction:

Lemma 76. *Let β be an ordinal. Then for all ordinals $\beta' \geq \beta$, $\mathcal{A}_{\beta'}^{\mathcal{H}} \sqsubseteq_m \mathcal{A}_{\beta}^{\mathcal{H}}$.*

Proof. We proceed by (transfinite) induction on β .

- If $\beta = 0$ this is obvious by Ex. 71.1).
- Next, suppose $\beta = \tilde{\beta} + 1$ is a successor ordinal. Note that $\beta' = 0$ is impossible. If $\beta' = \tilde{\beta}' + 1$ then

$$\mathcal{A}_{\beta'}^{\mathcal{H}} = T_P^{\mathcal{H}}(\mathcal{A}_{\beta'}^{\mathcal{H}}) \sqsubseteq_m T_P^{\mathcal{H}}(\mathcal{A}_{\tilde{\beta}}^{\mathcal{H}}) = \mathcal{A}_{\tilde{\beta}}^{\mathcal{H}}$$

using the inductive hypothesis and Cor. 75.

If β' is a limit ordinal then by Cor. 73.1), $\mathcal{A}_{\beta'}^{\mathcal{H}} \sqsubseteq_m \mathcal{A}_{\beta'}^{\mathcal{H}}$.

- Finally, suppose β is a limit ordinal. By the inductive hypothesis, for each $\tilde{\beta} < \beta$, $\mathcal{A}_{\tilde{\beta}}^{\mathcal{H}} \sqsubseteq_m \mathcal{A}_{\beta'}^{\mathcal{H}}$. Therefore, by Cor. 73.2), $\mathcal{A}_{\beta}^{\mathcal{H}} \sqsubseteq_m \mathcal{A}_{\beta'}^{\mathcal{H}}$. \square

Next, we prove the key property to establish the \sqsubseteq_m -“leastness” of the canonical structure $\mathcal{A}_P^{\mathcal{H}}$:

Proposition 77. *Let \mathcal{B} be an expansion of \mathcal{A} satisfying $\mathcal{B} \models P$ and let β be an ordinal.*

Then $\mathcal{A}_{\beta}^{\mathcal{H}} \sqsubseteq_m \mathcal{B}$.

Proof. We prove the lemma by induction on β .

- If $\beta = 0$ this is obvious because both $\mathcal{A}_0^{\mathcal{H}}$ and \mathcal{B} are expansions of \mathcal{A} .
- Next, suppose $\beta = \beta' + 1$ is a successor ordinal. Let $R : \bar{\tau} \rightarrow o \in \Sigma' \setminus \Sigma$ and $\bar{s}, \bar{s}' \in \mathcal{H}[\bar{\tau}]$ be such that $\bar{s} \sqsubseteq_m \bar{s}'$. Assume that $R^{\mathcal{A}_{\beta'}^{\mathcal{H}}}(\bar{s}) = 1$. Then $\mathcal{A}_{\beta'}^{\mathcal{H}}[\llbracket F_R \rrbracket(\top_{\Delta}^{\mathcal{H}}[\bar{x}_R \mapsto \bar{s}])] = 1$. By the inductive hypothesis and Prop. 37,

$$1 = \mathcal{A}_{\beta'}^{\mathcal{H}}[\llbracket F_R \rrbracket(\top_{\Delta}^{\mathcal{H}}[\bar{x}_R \mapsto \bar{s}])] \leq \mathcal{B}[\llbracket F_R \rrbracket(\top_{\Delta}^{\mathcal{H}}[\bar{x}_R \mapsto \bar{s}'])]$$

Consequently, due to $\mathcal{B} \models P$, $R^{\mathcal{B}}(\bar{s}') = 1$.

- Finally, if β is a limit ordinal, by the inductive hypothesis, $\mathcal{A}_{\beta'}^{\mathcal{H}} \sqsubseteq_m \mathcal{B}$ for each $\beta' \in \beta$. Thus, by Cor. 73.2), $\mathcal{A}_{\beta}^{\mathcal{H}} \sqsubseteq_m \mathcal{B}$. \square

Corollary 78. *If $\mathcal{B} \models P$ is a (Σ', \mathcal{H}) -expansion of \mathcal{A} then $\mathcal{A}_P^{\mathcal{H}} \sqsubseteq_m \mathcal{B}$.*

Theorem 38. *If S is \mathcal{A} -standard-satisfiable then $\mathcal{A}_P^{\mathcal{S}} \models S$; if S is \mathcal{A} -continuous-satisfiable then $\mathcal{A}_P^{\mathcal{S}} \models S$.*

Proof. Let \mathcal{B} be a (Σ', \mathcal{S}) -expansion of \mathcal{A} satisfying $\mathcal{B} \models S$. By Prop. 37 and Cor. 78, $\mathcal{A}_P^{\mathcal{S}}[\llbracket G \rrbracket(\alpha)] \geq \mathcal{B}[\llbracket G \rrbracket(\top_{\Delta}^{\mathcal{S}})]$ for every goal clause $G \in S$ and valuation α . Hence, by assumption, $\mathcal{A}_P^{\mathcal{S}} \models G$ and therefore by Prop. 18.2) $\mathcal{A}_P^{\mathcal{S}} \models S$.

The proof in the case S is \mathcal{A} -continuous-satisfiable is analogous. \square

D Supplementary Materials for Sec. 6

Theorem 42 (Soundness and Completeness). *Let \mathfrak{A} be a compact set of 1st-order Σ -structures and let S be a set of HoCHCs. S is \mathfrak{A} -unsatisfiable iff $S \Rightarrow_{\text{Res-C}, \mathfrak{A}}^* S' \cup \{\perp\}$ for some S' .*

Proof. The “if”-direction is straightforward. For the converse, suppose that S is \mathfrak{A} -unsatisfiable. By the Completeness Thm. 17, for each $\mathcal{A} \in \mathfrak{A}$ there exist easy $G_{\mathcal{A}}$ and background atoms $\varphi_{\mathcal{A},i}$ and $S_{\mathcal{A}}$ such that $\mathcal{A} \not\models \neg\varphi_{\mathcal{A},1} \vee \dots \vee \neg\varphi_{\mathcal{A},m_{\mathcal{A}}}$ and $S \Rightarrow_{\text{Res}, \mathcal{A}}^* S_{\mathcal{A}} \cup \{G_{\mathcal{A}} \vee \neg\varphi_{\mathcal{A},1} \vee \dots \vee \neg\varphi_{\mathcal{A},m_{\mathcal{A}}}\} = S'_{\mathcal{A}}$. Hence, $\{\neg\varphi_{\mathcal{A},1} \vee \dots \vee \neg\varphi_{\mathcal{A},m_{\mathcal{A}}} \mid \mathcal{A} \in \mathfrak{A}\}$ is \mathfrak{A} -unsatisfiable and by compactness of \mathfrak{A} there exists finite $\mathfrak{A}' \subseteq \mathfrak{A}$ such that $\{\neg\varphi_{\mathcal{A},1} \vee \dots \vee \neg\varphi_{\mathcal{A},m_{\mathcal{A}}} \mid \mathcal{A} \in \mathfrak{A}'\}$ is \mathfrak{A} -unsatisfiable. Consequently, $S \Rightarrow_{\text{Res-C}, \mathfrak{A}}^* \{S'_{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{A}'\} \Rightarrow_{\text{Res-C}, \mathfrak{A}} \{\perp\} \cup \{S'_{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{A}'\}$. \square

E Supplementary Materials for Sec. 7

E.1 Elimination of λ -abstractions

In this section, we examine how to eliminate λ -abstractions. We make use of the notion of terms with holes (cf. [11, p. 29], [46]).

First, note that the proof of Prop. 37 can be adapted in a straightforward manner to obtain:

Lemma 79. *Let \mathcal{B} be a Σ' - and \mathcal{B}' be a (Σ'', \mathcal{C}) -structure, where $\Sigma'' \supseteq \Sigma'$, let $M[-]$ be a Σ' -term with a hole of type τ , let N be a Σ' - and N' be a Σ'' -formula satisfying*

- 1) $M[N]$ is a positive existential formula,
- 2) $\Delta \vdash N : \tau$ and $\Delta \vdash N' : \tau$,
- 3) for all $c \in \Sigma'$, $c^{\mathcal{B}} \sqsubseteq_m c^{\mathcal{B}'}$
- 4) for all valuations $\alpha \sqsubseteq_m \alpha'$, $\mathcal{B} \llbracket N \rrbracket(\alpha) \sqsubseteq_m \mathcal{B}' \llbracket N' \rrbracket(\alpha')$.

Then for all valuations $\alpha \sqsubseteq_m \alpha'$, $\mathcal{B} \llbracket M[N] \rrbracket(\alpha) \sqsubseteq_m \mathcal{B}' \llbracket M[N'] \rrbracket(\alpha')$.

Let $\lambda y.M$ be a positive existential Σ' -term not containing logical symbols with free variables \bar{x} such that $\Delta \vdash M : \tau' \rightarrow \bar{\tau} \rightarrow o$, let $\tilde{S}[-]$ be a set of terms with a hole of type $\tau' \rightarrow \bar{\tau} \rightarrow o$ such that $S[\lambda y.M]$ is a set of HoCHCs.

Let \bar{z} be distinct variables (different from \bar{x}, y) satisfying $\Delta(\bar{z}) = \bar{\tau}$. We define a signature $\Sigma'' = \Sigma' \cup \{R_M : \Delta(\bar{x}) \rightarrow \tau' \rightarrow \bar{\tau} \rightarrow o\}$ and

$$\begin{aligned} S &= \tilde{S}[\lambda y.M] \\ S' &= \tilde{S}[R_M \bar{x}] \cup \{\neg M \bar{z} \vee R_M \bar{x} y \bar{z}\}. \end{aligned}$$

These are sets of HoCHCs. Besides, we set $P = P_S$ and $P' = P_{S'}$. In the following, we prove that S and S' are in fact \mathcal{A} -continuous-equi-satisfiable for every fixed $\mathcal{A} \in \mathfrak{A}$.

First, we prove in the following lemma that although $\mathcal{A}_{P',n}^{\mathcal{C}}$ “grows slower” (with increasing $n \in \omega$) than $\mathcal{A}_{P,n}^{\mathcal{C}}$, $\mathcal{A}_{P',2n+1}^{\mathcal{C}}$ still roughly “catches up” with $\mathcal{A}_{P,n}^{\mathcal{C}}$.

Lemma 80. *For every $n \in \omega$,*

- 1) $R^{\mathcal{A}_{P,n}^{\mathcal{C}}} \sqsubseteq_m R^{\mathcal{A}_{P',2n}^{\mathcal{C}}}$ for $R \in \Sigma' \setminus \Sigma$,

- 2) $R^{\mathcal{A}_{P,n}^{\mathcal{C}}} \sqsubseteq_m R^{\mathcal{A}_{P',2n+1}^{\mathcal{C}}}$ for $R \in \Sigma' \setminus \Sigma$ and
- 3) $\mathcal{A}_{P,n}^{\mathcal{C}}[\![\lambda \bar{x}.y.M]\!](\top_{\Delta}^{\mathcal{C}}) \sqsubseteq_m R_M^{\mathcal{A}_{P',2n+1}^{\mathcal{C}}}$.

Proof. We prove the lemma by induction on $n \in \omega$.

- If $n = 0$ then the first two parts are obvious. Furthermore,

$$\mathcal{A}_{P,0}^{\mathcal{C}}[\![\lambda \bar{x}.y.M]\!](\top_{\Delta}^{\mathcal{C}}) \sqsubseteq_m \mathcal{A}_{P',0}^{\mathcal{C}}[\![\lambda \bar{x}.y.M]\!](\top_{\Delta}^{\mathcal{C}}) = \mathcal{A}_{P',0}^{\mathcal{C}}[\![\lambda \bar{x}.y.\bar{z}.M\bar{z}]\!](\top_{\Delta}^{\mathcal{C}}) = R_M^{\mathcal{A}_{P',1}^{\mathcal{C}}}.$$

- Next, to show the induction step from n to $n+1$ suppose that $n \geq 0$. By the inductive hypothesis,

$$1) R^{\mathcal{A}_{P,n}^{\mathcal{C}}} \sqsubseteq_m R^{\mathcal{A}_{P',2n}^{\mathcal{C}}} \text{ for } R \in \Sigma' \setminus \Sigma, 2) R^{\mathcal{A}_{P,n}^{\mathcal{C}}} \sqsubseteq_m R^{\mathcal{A}_{P',2n+1}^{\mathcal{C}}} \text{ for } R \in \Sigma' \setminus \Sigma \text{ and } 3) \mathcal{A}_{P,n}^{\mathcal{C}}[\![\lambda \bar{x}.y.M]\!](\top_{\Delta}^{\mathcal{C}}) \sqsubseteq_m R_M^{\mathcal{A}_{P',2n+1}^{\mathcal{C}}}.$$

- 1) Therefore, by Parts 2) and 3) of the inductive hypothesis and Lemma 79,

$$R^{\mathcal{A}_{P,n+1}^{\mathcal{C}}} = \mathcal{A}_{P,n}^{\mathcal{C}}[\![\lambda \bar{x}_R.F_R[\lambda y.M]]\!](\top_{\Delta}^{\mathcal{C}}) \sqsubseteq_m \mathcal{A}_{P',2n+1}^{\mathcal{C}}[\![\lambda \bar{x}_R.F_R[R_M\bar{x}]]\!](\top_{\Delta}^{\mathcal{C}}) = R^{\mathcal{A}_{P',2n+2}^{\mathcal{C}}}$$

for $R \in \Sigma' \setminus \Sigma$.

- 2) By Part 1) and Lemma 76, $R^{\mathcal{A}_{P,n+1}^{\mathcal{C}}} \sqsubseteq_m R^{\mathcal{A}_{P',2n+2}^{\mathcal{C}}} \sqsubseteq_m R^{\mathcal{A}_{P',2(n+1)+1}^{\mathcal{C}}}$ for $R \in \Sigma' \setminus \Sigma$.
- 3) Using Part 1) for $n+1$, Prop. 37 and Lemma 60.2) we obtain

$$\begin{aligned} \mathcal{A}_{P,n+1}^{\mathcal{C}}[\![\lambda \bar{x}.y.M]\!](\top_{\Delta}^{\mathcal{C}}) &\sqsubseteq_m \mathcal{A}_{P',2(n+1)}^{\mathcal{C}}[\![\lambda \bar{x}.y.M]\!](\top_{\Delta}^{\mathcal{C}}) \\ &= \mathcal{A}_{P',2(n+1)}^{\mathcal{C}}[\![\lambda \bar{x}.y.\bar{z}.M\bar{z}]\!](\top_{\Delta}^{\mathcal{C}}) \\ &= R_M^{\mathcal{A}_{P',2(n+1)+1}^{\mathcal{C}}}. \end{aligned} \quad \square$$

As a consequence we get:

Lemma 81. *S is \mathcal{A} -continuous-satisfiable if and only if S' is \mathcal{A} -continuous-satisfiable.*

Proof. • First, suppose that there exists a Σ' -expansion \mathcal{B} of \mathcal{A} satisfying $\mathcal{B} \models S$. We define a Σ'' -expansion \mathcal{B}' of \mathcal{A} by setting $R^{\mathcal{B}'} = R^{\mathcal{B}}$ for $R \in \Sigma' \setminus \Sigma$ and $R_M^{\mathcal{B}'} = \mathcal{B}[\![\lambda \bar{x}.y.\bar{z}.M\bar{z}]\!](\top_{\Sigma'}^{\mathcal{C}})$. By definition, $\mathcal{B}' \models \neg M\bar{z} \vee R_M\bar{x}y\bar{z}$. Furthermore for every positive existential Σ' -formula E and valuation α , $\mathcal{B}[E[\lambda y.M]](\alpha) = \mathcal{B}'[E[R_M\bar{x}]](\alpha)$. Consequently, $\mathcal{B}' \models S'$.

• Conversely, suppose that there exists a Σ'' -expansion \mathcal{B}' of \mathcal{A} satisfying $\mathcal{B}' \models S'$. Assume towards contradiction that $\mathcal{A}_P^{\mathcal{C}} \not\models S$. Then, by Thm. 26, $\mathcal{A}_{P,n}^{\mathcal{C}} \not\models G$ for some $n \in \omega$ and goal clause $G \in S$. By Lemmas 79 and 80, $\mathcal{A}_{P',2(n+1)}^{\mathcal{C}} \not\models G'$ (for the goal clause G' corresponding to G in S') and therefore, by Prop. 37 and Cor. 74, $\mathcal{A}_{P'}^{\mathcal{C}} \not\models G'$. Note that by Cor. 78, $\mathcal{A}_{P'}^{\mathcal{C}} \sqsubseteq_m \mathcal{B}'$ and therefore, by Prop. 37, $\mathcal{B}' \not\models G'$, which is clearly a contradiction. Consequently, $\mathcal{A}_P^{\mathcal{C}} \models S$. \square

Hence, we get Lemma 43 as a corollary.

E.2 Proof of Lemma 46

Before turning to Lemma 46, we prove the following auxiliary lemma:

Lemma 82. *Let M, M_1, \dots, M_n and \bar{L} be terms neither containing logical symbols nor λ -abstractions. Then:*

- 1) If $R \in \Sigma' \setminus \Sigma$, $RM_1 \cdots M_n$ and $Rx_1 \cdots x_n$ are terms such that $\text{fv}(RM_1 \cdots M_n) \cap \text{fv}(Rx_1 \cdots x_n) = \emptyset$ then $\llbracket M_1 \rrbracket' / x_1, \dots, \llbracket M_n \rrbracket' / x_n$ is a unifier of $\llbracket RM_1 \cdots M_n \rrbracket'$ and $\llbracket Rx_1 \cdots x_n \rrbracket'$.
- 2) If $\Delta(y) = \rho = \tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow o$ and $yM_1 \cdots M_n$ is a formula such that $\text{fv}(\text{Comp}_\rho) \cap \text{fv}(yM_1 \cdots M_n)$ then $\llbracket c_\rho / y, \llbracket M_1 \rrbracket' / x_{\tau_1}^{(1)}, \dots, \llbracket M_n \rrbracket' / x_{\tau_n}^{(n)} \rrbracket$ is a unifier of Comp_ρ and $\llbracket yM_1 \cdots M_n \rrbracket$.
- 3) If M is a term neither containing logical symbols nor λ -abstractions then $\llbracket M[\bar{L}/\bar{x}] \rrbracket' = \llbracket M \rrbracket'[\llbracket \bar{L} \rrbracket' / \bar{x}]$.
- 4) If G is a goal clause then $\llbracket G[M_1/x_1, \dots, M_n/x_n] \rrbracket = \llbracket G \rrbracket[\llbracket M_1 \rrbracket' / x_1, \dots, \llbracket M_n \rrbracket' / x_n]$.

Proof. 1) We prove Part 1) by induction on n . For $n = 0$ this is trivial. Hence, suppose $n \geq 0$. By the inductive hypothesis,

$$\llbracket RM_1 \cdots M_n \rrbracket'[\llbracket M_1 \rrbracket' / x_1, \dots, \llbracket M_n \rrbracket' / x_n] = \llbracket Rx_1 \cdots x_n \rrbracket'[\llbracket M_1 \rrbracket' / x_1, \dots, \llbracket M_n \rrbracket' / x_n]. \quad (9)$$

Consequently,

$$\begin{aligned} & \llbracket RM_1 \cdots M_{n+1} \rrbracket'[\llbracket M_1 \rrbracket' / x_1, \dots, \llbracket M_{n+1} \rrbracket' / x_{n+1}] \\ &= (@ \llbracket RM_1 \cdots M_n \rrbracket' \llbracket M_{n+1} \rrbracket')[\llbracket M_1 \rrbracket' / x_1, \dots, \llbracket M_{n+1} \rrbracket' / x_{n+1}] \\ &= @ \llbracket RM_1 \cdots M_n \rrbracket'[\llbracket M_1 \rrbracket' / x_1, \dots, \llbracket M_n \rrbracket' / x_n] \llbracket M_{n+1} \rrbracket' \\ &= (@ \llbracket Rx_1 \cdots x_n \rrbracket' x_{n+1})[\llbracket M_1 \rrbracket' / x_1, \dots, \llbracket M_{n+1} \rrbracket' / x_{n+1}] \\ &= \llbracket Rx_n \cdots x_1 \rrbracket'[\llbracket M_1 \rrbracket' / x_1, \dots, \llbracket M_{n+1} \rrbracket' / x_{n+1}], \end{aligned}$$

using that $x_i \notin \text{fv}(M_j)$ in the second and Eq. (9) in the third step.

- 2) Similar to Part 1).
- 3) We prove the claim by structural induction. For variables, and symbols from $\Sigma' \setminus \Sigma$ this is obvious. Next, consider a term of the form $c\bar{N}$, where $c \in \Sigma$. By Remark 2, $c\bar{N}$ only contains variables $y : \iota$ and for each term $\Delta \vdash K : \iota^n \rightarrow \iota$, $\llbracket K \rrbracket' = K$. Hence,

$$\llbracket (c\bar{N})[\bar{L}/\bar{x}] \rrbracket' = (c\bar{N})[\bar{L}/\bar{x}] = \llbracket c\bar{N} \rrbracket'[\llbracket \bar{L} \rrbracket' / \bar{x}].$$

Finally, consider a term of the form $M\bar{N}N'$, where $M \notin \Sigma$. Then,

$$\begin{aligned} \llbracket (M\bar{N}N')[\bar{L}/\bar{x}] \rrbracket' &= \llbracket (M\bar{N})[\bar{L}/\bar{x}]N'[\bar{L}/\bar{x}] \rrbracket' \\ &= @ \llbracket (M\bar{N})[\bar{L}/\bar{x}] \rrbracket' \llbracket N'[\bar{L}/\bar{x}] \rrbracket' \\ &= @ \llbracket M\bar{N} \rrbracket'[\llbracket \bar{L} \rrbracket' / \bar{x}] \llbracket N' \rrbracket'[\llbracket \bar{L} \rrbracket' / \bar{x}] \\ &= (@ \llbracket M\bar{N} \rrbracket' \llbracket N' \rrbracket')[\llbracket \bar{L} \rrbracket' / \bar{x}] \\ &= \llbracket M\bar{N}N' \rrbracket'[\llbracket \bar{L} \rrbracket' / \bar{x}], \end{aligned}$$

using the inductive hypothesis in the third step.

- 4) Immediate from Part 3). □

Lemma 46. Let S' be a set of HoCHCs not containing λ -abstractions and suppose $S' \Rightarrow_{\text{Res-C}, \mathfrak{A}} S' \cup \{G\}$. Then

- 1) G does not contain λ -abstractions
- 2) if $G \neq \perp$ then $\llbracket S' \rrbracket \models \llbracket S' \cup \{G\} \rrbracket$
- 3) if $G = \perp$ then $\llbracket S' \rrbracket$ is \mathfrak{A} -unsatisfiable.

Proof. 1) Obvious.

- 2) Note that by assumption the rule β -Reduction is not applicable. Next, let $\neg R\overline{M} \vee G_1$ and $G_2 \vee R\overline{x}_R$ be clauses in S' modulo renaming of variables (such that they are variable-disjoint) and suppose $G = G_1 \vee (G_2[\overline{M}/\overline{x}_R])$. By Lemma 82.1), $[R\overline{M}][[\overline{M}]'/\overline{x}_R] = [R\overline{x}][[\overline{M}]'/\overline{x}_R]$, by soundness of 1st-order resolution [44, 45],

$$[S'] \models ([G_1] \vee [G_2])[[\overline{M}]'/\overline{x}_R]$$

and by Lemma 82.4),

$$([G_1] \vee [G_2])[[\overline{M}]'/\overline{x}_R] = [G_1 \vee (G_2[\overline{M}/\overline{x}_R])].$$

Furthermore, $\text{vars}(G) \subseteq \text{vars}(\neg R\overline{M} \vee G_1) \cup \text{vars}(G_2)$ and therefore $[S' \cup \{G\}] = [S'] \cup \{[G]\}$.

- 3) Finally, suppose that there exists $\{\bigvee_{i=1}^{m'_j} \neg x_{j,i} M_{j,i} \vee \bigvee_{i=1}^{m_j} \neg \phi_{j,i} \mid 1 \leq j \leq n\} \subseteq S'$ such that $\{\bigvee_{i=1}^{m_j} \neg \phi_{j,i} \mid 1 \leq j \leq n\}$ is \mathfrak{A} -inconsistent. Note that each $x_{j,i}$ does not occur in any of the $\phi_{j',i'}$. Therefore, by Lemma 82.2), the fact that $\text{Comp}_{\Delta(x_{j,i})} \in [S']$ and the soundness of 1st-order resolution, $[S'] \models \{\bigvee_{i=1}^{m_j} \neg \phi_{j,i} \mid 1 \leq j \leq n\}$. Hence, by assumption $[S']$ is \mathfrak{A} -unsatisfiable. \square

F Supplementary Materials for Sec. 8

F.1 Higher-order Datalog

First consider the higher-order extension of datalog.

Assumption 6. Let $\Sigma \supseteq \{\approx, \not\approx : \iota \rightarrow \iota \rightarrow o, c_0 : \iota\}$ be a finite 1st-order signature containing $\approx, \not\approx$ and symbol(s) of type ι (but nothing else); let Σ' be a relational extension of Σ and S be a finite set of HoHCs.

Besides, let \mathfrak{A} be the set of (first-order) Σ -structures \mathcal{A} satisfying $\approx^{\mathcal{A}}(a)(b) = 1$ iff $\not\approx^{\mathcal{A}}(a)(b) = 0$ iff $a = b$, for $a, b \in \mathcal{A}[\iota]$.

In this setting we refer to HoHCs as *higher-order datalog clauses (HoDC)*. For $\mathcal{A} \in \mathfrak{A}$ we define $\widehat{\mathcal{A}}$ by 1) $\widehat{\mathcal{A}}[\iota] := \{c : \iota \in \Sigma\}$, 2) $c^{\widehat{\mathcal{A}}} := c$ for $c : \iota \in \Sigma$, 3) $\approx^{\widehat{\mathcal{A}}}(c)(d) := \mathcal{A}[c \approx d]$ and 4) $\not\approx^{\widehat{\mathcal{A}}}(c)(d) := \mathcal{A}[c \not\approx d]$ for $c, d \in \widehat{\mathcal{A}}[\iota]$. and we set $\widehat{\mathfrak{A}} := \{\widehat{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{A}\}$. Clearly, $\widehat{\mathfrak{A}}$ is finite and for each $\mathcal{A} \in \mathfrak{A}$ and type σ , $\widehat{\mathcal{A}}[\sigma]$ is finite.

Lemma 83. Let ϕ be a background atom and α' be a valuation satisfying $\widehat{\mathcal{A}}, \alpha' \models \phi$ then $\mathcal{A}, \alpha \models \phi$, where α is a valuation such that for each $x : \iota \in \Delta$, $\alpha(x) = \mathcal{A}[\alpha'(x)]$.

Corollary 84. Let S' be a set of background goal clauses. S' is $\widehat{\mathfrak{A}}$ -satisfiable if S' is \mathfrak{A} -satisfiable.

Lemma 85. Let S' be a set of background goal clauses. S' is \mathfrak{A} -satisfiable if S' is $\widehat{\mathfrak{A}}$ -satisfiable.

Proof. Let $\widehat{\mathcal{A}} \in \widehat{\mathfrak{A}}$ be such that $\widehat{\mathcal{A}} \models S'$. Consider the element $\widehat{\mathcal{A}}/\approx$ of \mathfrak{A} with domain

$$(\widehat{\mathcal{A}}/\approx)[\iota] := \{\{d \in \widehat{\mathcal{A}}[\iota] \mid \mathcal{A} \models c \approx d\} \mid c \in \widehat{\mathcal{A}}[\iota]\},$$

i.e. the quotient of $\widehat{\mathcal{A}}[\iota]$ over $\approx^{\widehat{\mathcal{A}}}$. It is easy to see that $\widehat{\mathcal{A}}/\approx \models S'$. \square

Note that both \mathfrak{A} and $\widehat{\mathfrak{A}}$ are compact. Hence, by soundness and completeness of the proof system (Thm. 17 and Prop. 16) we obtain:

Proposition 86. S is \mathfrak{A} -satisfiable iff S is $\widehat{\mathfrak{A}}$ -satisfiable.

Consequently, by Remark 49, we conclude:

Theorem 87. It is decidable whether there exists a Σ' -structure \mathcal{B} satisfying $\mathcal{B} \models S$ and $\approx^{\mathcal{B}}(a)(b) = 1$ iff $\not\approx^{\mathcal{B}}(a)(b) = 0$ iff $a = b$, for $a, b \in \mathcal{B}[\iota]$.

F.2 Supplementary Materials for Sec. 8.1

Lemma 52. *Let S be a set of $\text{HoBHC}(\text{SLA})$. If $S \Rightarrow_{\text{Res-C}, \mathfrak{A}} S \cup \{G\}$ then G is a $\text{HoBHC}(\text{SLA})$ and $\text{gt}_1(G) \subseteq \text{gt}_1(S)$.*

Proof. For the Compact Constraint Refutation rule this is trivial.

Next, if $\neg(\lambda x.L)M\bar{N} \vee G$ is a $\text{HoBHC}(\text{SLA})$ then neither $(\lambda x.L)M\bar{N}$ nor $L[M/x]\bar{N}$ contain symbols from Σ and $\text{gt}_1((\lambda x.L)M\bar{N}) = \text{gt}_1(L[M/x]\bar{N}) = \emptyset$. Hence, $\neg L[M/x]\bar{N} \vee G$ is a $\text{HoBHC}(\text{SLA})$ and $\text{gt}_1(\neg L[M/x]\bar{N} \vee G) = \text{gt}_1(\neg(\lambda x.L)M\bar{N} \vee G)$.

Finally, suppose $\neg R\bar{M} \vee G$ and $G' \vee R\bar{x}$ are $\text{HoBHC}(\text{SLA})$ s. Note that all terms in \bar{M} of type ι must be variables. Therefore $G \vee G'[\bar{M}/\bar{x}]$ is a $\text{HoBHC}(\text{SLA})$ and $\text{gt}_1(G \vee G'[\bar{M}/\bar{x}]) = \text{gt}_1(\neg R\bar{M} \vee G) \cup \text{gt}_1(G' \vee R\bar{x})$. \square

Lemma 55. *Let S' be a set of $\text{HoBHC}(\text{SLA})$ satisfying $\text{gt}_1(S') \subseteq \text{gt}_1(S)$. Then*

- 1) *if $S' \Rightarrow_{\text{Res-C}, \mathfrak{A}} S' \cup \{G\}$ then $\widehat{S'} \Rightarrow_{\text{Res-C}, \widehat{\mathfrak{A}}} \widehat{S'} \cup \{\widehat{G}\}$*
- 2) *if $\widehat{S'} \Rightarrow_{\text{Res-C}, \widehat{\mathfrak{A}}} \widehat{S'} \cup \{\widehat{G'}\}$ then $S' \Rightarrow_{\text{Res-C}, \mathfrak{A}} S' \cup \{G\}$ for some G satisfying $\widehat{G} = G'$.*

Proof. For the rule Compact Constraint Refutation this is due to Cor. 54 and for the rule β -Reduction this is obvious because $\widehat{(\lambda x.L)M\bar{N}} = (\lambda x.L)M\bar{N}$.

Finally, suppose that $\{\neg R\bar{M} \vee G, G' \vee R\bar{x}\} \subseteq S'$. It holds that $\widehat{R\bar{M}} = R\bar{M}$, $\widehat{R\bar{x}} = R\bar{x}$ and for every atom A , $\widehat{A[\bar{M}/\bar{x}]} = A[\bar{M}/\bar{x}]$. Consequently, $\widehat{G'[\bar{M}/\bar{x}]} = G'[\bar{M}/\bar{x}]$ and the lemma also holds for applications of the resolution rule. \square

Theorem 88. *Let Φ be a predicate on atoms¹⁶ satisfying*

- 1) $\Phi(S) = 1$,
- 2) *if $S' \Rightarrow_{\text{Res}, \mathfrak{A}} S''$ then $\Phi(S'') \geq \Phi(S')$ and*
- 3) *if S' is an \mathfrak{A} -unsatisfiable set of HoCHCs satisfying $\Phi(S') = 1$ then there exists a finite subset $S'' \subseteq S'$ which is \mathfrak{A} -unsatisfiable.*

Then S is \mathfrak{A} -unsatisfiable iff $S \Rightarrow_{\text{Res-C}, \mathfrak{A}}^ S' \cup \{\perp\}$ for some S' .*

Proof. Similar to the proof of Thm. 42. \square

Proposition 56. *If S is \mathfrak{A} -unsatisfiable then $S \Rightarrow_{\text{Res-C}, \mathfrak{A}}^* S' \cup \{\perp\}$ for some S' .*

Proof. Define $\Phi(A) = 1$ just if $\text{gt}_1(A) \subseteq \text{gt}_1(S)$ and $\neg A$ is a $\text{HoBHC}(\text{SLA})$. By Lemma 52 and Cor. 54, Thm. 88 is applicable, which yields the theorem. \square

¹⁶which is lifted to clauses by setting $\Phi(\neg A_1 \vee \dots \vee \neg A_n \vee (\neg)A) := \min\{\Phi(A_1), \dots, \Phi(A_n), \Phi(A)\}$ and to sets of HoCHCs S' by setting $\Phi(S') = \min\{\Phi(C) \mid C \in S'\}$

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