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Quickly Excluding a Planar Graph

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In an earlier paper, the first two authors proved that for any planar graph H, every graph with no minor isomorphic to H has bounded tree width; but the bound given there was enormous. Here we prove a much better bound. We also improve the best known bound on the tree-width of planar graphs with no minor isomorphic to a $g \times g$ grid. © 1994 Academic Press, Inc.

1. INTRODUCTION

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. (In this paper, all graphs are finite and may have loops or multiple edges.) A *tree-decomposition* of a graph G is a pair (T, W), where T is a tree and $W = (W_t : t \in V(T))$ is a family of subsets of $V(G)$, such that

(i) $\bigcup_{t \in V(T)} W_t = V(G)$, and for every edge $e \in E(G)$ there exists $t \in V(T)$ such that W_t contains both ends of e

(ii) if $t, t', t'' \in V(T)$ and t' lies on the path of T between t and t'' then $W_t \cap W_{t''} \subseteq W_{t'}$.

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($V(G)$ and $E(G)$ denote the vertex- and edge-sets of G.) The *width* of (T, W) is $\max_{t \in V(T)}(|W_t| - 1)$, and the *tree-width* of G is the minimum width of a tree-decomposition of G. The following was proved in [6]:

(1.1). *For any planar graph H there is a number N such that every graph with no minor isomorphic to H has tree-width $\leqslant N$.*

An explicit formula for a number N with this property was given (in terms of $|V(H)|$, $|E(H)|$) but it was enormous; it involved iterated exponentiation where the number of iterations also involved iterated exponentiation (and so on, twice more). Our main objective here is to give a much better bound on N in the following:

(1.2). *Let H be a planar graph and let $n = 2|V(H)| + 4|E(H)|$. Then every graph with no minor isomorphic to H has tree-width $\leqslant 20^{2n^5}$.*

To prove this, we first reduce the problem to the case when H is a $g \times g$ grid, that is, the simple graph with g^2 vertices $\{(i, j) : 1 \leqslant i, j \leqslant g\}$, where (i, j) and (i', j') are adjacent if $|i' - i| + |j' - j| = 1$. This reduction will occupy the remainder of this section, and Sections 2–5 are concerned with the grid case. Section 6 gives a similar improvement of a theorem of [5], concerning the tree-width of planar graphs with no minor isomorphic to a $g \times g$ grid.

The main part of the proof of (1.2), in Sections 2–5, relies heavily on the concept of a "tangle" in a graph. This is defined in Section 2, but, roughly, a tangle in G is a part of G which is highly coherent, so that for every small-order separation of G the tangle clearly resides on one side of it or the other. It is shown in [8] that a graph has large tree-width if and only if it has a large tangle, and so our problem is to show that a graph with a large tangle has a large grid minor. The way we construct the large grid minor is rather like the way one knits a scarf, stitch by stitch and row by row. The main lemmas assert that if part of the minor has been knitted and we still have a large tangle in its complement, then we can add another stitch to our minor without destroying too much of the remaining tangle.

Now let us reduce (1.2) to the case when H is a grid. We begin with the following:

(1.3). *If H is a simple planar Hamiltonian graph with g vertices then H is isomorphic to a minor of a $g \times g$ grid.*

Proof. We may assume that H is a triangulation, that is, every region of H is bordered by a three-edge circuit. Let $V(H) = \{v_1, ..., v_g\}$, where v_i and v_{i+1} are adjacent $(1 \leqslant i \leqslant g)$, where v_{g+1} means v_1. Let C be the corresponding Hamiltonian circuit and take a drawing of H in a sphere. Let A, B be the subgraphs of H drawn in the two closed discs bounded by C (thus, A and B are both "outerplanar" graphs and C is a Hamiltonian circuit of both of them). Let G be the $g \times g$ grid with vertex set $\{(i, j) : 1 \leqslant i, j \leqslant g\}$ as before. For $2 \leqslant k \leqslant g$, choose $i(k)$, $j(k)$ with $1 \leqslant i(k)$, $j(k) \leqslant k-1$, minimum such that $v_{i(k)}$, v_k are adjacent in A, and $v_{j(k)}$, v_k are adjacent in B. For $1 \leqslant k \leqslant g-1$, choose $i'(k)$, $j'(k)$ with $k+1 \leqslant i'(k)$, $j'(k) \leqslant g$, maximum such that v_k, $v_{j'(k)}$ are adjacent in A and v_k, $v_{i'(k)}$ are adjacent in B. For $2 \leqslant k \leqslant g-1$, let X_k be the subgraph of G induced by

$$\{(i, k) : i(k) < i < i'(k)\} \cup \{(k, j) : j(k) < j < j'(k)\},$$

and let X_1, X_g be the subgraphs induced respectively by

$$\{(i, 1) : 1 \leqslant i < i'(1)\} \cup \{(1, j) : 1 \leqslant j < j'(1)\}$$
$$\{(i, g) : i(g) \leqslant i \leqslant g\} \cup \{(g, j) : j(g) \leqslant j \leqslant g\}.$$

Then each X_k is connected, and $X_1, ..., X_g$ are mutually disjoint, as is easily seen. Moreover, if $k < k'$ and $j'(k) = k'$ and $i(k') = k$ then $k = 1$ and $k' = g$, since H is a triangulation; and the same holds if $i'(k) = k'$ and $j(k') = k$. It follows that if v_k, $v_{k'}$ are adjacent in H, where $k' > k$, then there is an edge of G between $V(X_k)$ and $V(X_{k'})$. (For instance, if v_k, $v_{k'}$ are adjacent in A then $(k+1, k') \in V(X_{k'})$, $(k, k'-1) \in V(X_k)$, and (k, k') belongs to one of $V(X_k)$, $V(X_{k'})$.) Thus G has a minor isomorphic to H, as required. ∎

(1.4). *If $n \geqslant 2$ and H is a simple planar graph with $\leqslant n$ vertices, then there is a simple planar Hamiltonian graph G with $\leqslant 2n$ vertices such that G has a minor isomorphic to H.*

Proof. We may assume that $|V(H)| \geqslant 4$ and H is a triangulation. It follows easily by induction that a triangulation T with $n \geqslant 4$ vertices has at most $n-4$ "separating triangles," that is, circuits of length 3 whose deletion disconnects T. Choose an edge e from a separating triangle of H if there is one, and let r_1, r_2 be the two regions bordered by e. Let the four vertices incident with r_1 or r_2 be v_1, v_2, v_3, v_4. Delete e and add a new vertex adjacent to v_1, v_2, v_3, v_4. This procedure reduces the number of separating triangles by at least one, and so by applying this procedure at most $n-4$ times, we obtain a triangulation G with no separating triangle which has a minor isomorphic to H. By Whitney's theorem [9] G is Hamiltonian, as required. ∎

(1.5). *If H is a planar graph with $|V(H)| + 2\,|E(H)| \leqslant n$, then H is isomorphic to a minor of the $2n \times 2n$ grid.*

Proof. We may assume that $n \geqslant 2$. By subdividing each edge of H twice, we see that H is isomorphic to a minor of a simple planar graph H_1 with $|V(H_1)| \leqslant n$. By (1.4) H_1 is isomorphic to a minor of a simple planar Hamiltonian graph with $\leqslant 2n$ vertices. By (1.3) the result follows. ∎

Thus, to prove (1.2) it suffices to prove the following.

(1.6) *For $g \geqslant 1$, every graph with no minor isomorphic to the $g \times g$ grid has tree-width $\leqslant 20^{2g^5}$.*

Although (1.6) gives a much better bound than was previously known, there is still no reason to think that it is close to the right answer. By a probabilistic argument based on girth considerations (a slight variant on a construction of Erdös [2]), we have proved the existence of graphs which have no $g \times g$ grid minor and have tree-width at least proportional to $g^2 \log g$. We suspect that $O(g^2 \log g)$ is closer to the right answer than the bound of (1.6); indeed, it may even be the right answer.

2. Tangles

The proof of (1.6) is based on the concept of a "tangle," developed in [8]. A *separation* of a graph G is a pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$; its *order* is $|V(A \cap B)|$. If $\theta \geqslant 1$ is an integer, a *tangle of order* θ in G is a set $\mathscr{T}$ of separations of order $< \theta$, such that

(i) if (A, B) is a separation of G of order $< \theta$, then $\mathscr{T}$ contains one of (A, B), (B, A)

(ii) if (A_1, B_1), (A_2, B_2), $(A_3, B_3) \in \mathscr{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$

(iii) if $(A, B) \in \mathscr{T}$ then $V(A) \neq V(G)$.

(We refer to these as the *first*, *second*, and *third axioms*.) We define $\mathrm{ord}(\mathscr{T}) = \theta$.

In this section we shall present some preliminary lemmas about tangles. It is proved in [8, Theorem (5.2)] that

(2.1) *If G has a tangle of order $\geqslant \theta$, then the tree-width of G is at least $\theta - 1$. Conversely, if G has no tangle of order $\geqslant \theta$, then its tree-width is at most $3\theta/2$.*

This will enable us to use the presence of tangles rather than the absence of tree-decompositions to prove the existence of a large grid minor. We shall prove the following, from which (1.6) follows via (2.1).

(2.2) *For $g \geqslant 1$, every graph with a tangle of order $\geqslant 20^{g^4(2g-1)}$ has a minor isomorphic to the $g \times g$ grid.*

Let H be a $g \times g$ grid minor of G, where $g \geqslant 2$, with vertex set $\{(i, j) : 1 \leqslant i, j \leqslant g\}$, labelled in the natural way. For $1 \leqslant i \leqslant g$ the set of edges of H joining two vertices of H with first coordinate i is called a *row* of H, and the *columns* of H are defined similarly. Since $E(H) \subseteq E(G)$, the rows and columns of H are subsets of $E(G)$. Let $\mathcal{T}_H$ be the set of all separations (A, B) of G of order $<g$ such that $E(B)$ includes a row of H (or equivalently, by [8, Theorem (7.1)], includes a column of H). It follows easily from [8, Theorems (7.3) and (6.1)] that $\mathcal{T}_H$ is a tangle in G of order g. A tangle $\mathcal{T}$ in G *dominates* H if $\mathcal{T}_H \subseteq \mathcal{T}$.

For applications in future papers of the Graph Minors series, it is convenient to prove a strengthening of (2.2), the following:

(2.3). *Let $g \geqslant 2$, and let $\mathcal{T}$ be a tangle in G of order $\geqslant 20^{g^4(2g-1)}$. Then $\mathcal{T}$ dominates a $g \times g$ grid minor.*

A (numerically weaker) form of (2.3) was included in an early version of [8], but it turns out to be easier to prove it here instead. We see that (2.3) implies (2.2), because (2.2) is obvious when $g = 1$. We shall prove (2.3) in Section 5.

Let $\mathcal{T}$ be a tangle in G. For $X \subseteq V(G)$, we define $\mathrm{rk}_{\mathcal{T}}(X)$ to be the minimum order of a separation $(A, B) \in \mathcal{T}$ with $X \subseteq V(A)$, if there is such an (A, B), and otherwise $\mathrm{rk}_{\mathcal{T}}(X) = \mathrm{ord}(\mathcal{T})$. It follows from the third axiom that $\mathrm{rk}_{\mathcal{T}}(V(G)) = \mathrm{ord}(\mathcal{T})$. Let us say that $X \subseteq V(G)$ is *free* (relative to $\mathcal{T}$) if $|X| \leqslant \mathrm{ord}(\mathcal{T})$ and there is no $(A, B) \in \mathcal{T}$ of order $<|X|$ with $X \subseteq V(A)$. From [8, Theorems (12.1) and (12.2)] and the fact that $\mathrm{rk}_{\mathcal{T}}(V(G)) = \mathrm{ord}(\mathcal{T})$, it follows that

(2.4). $\mathrm{rk}_{\mathcal{T}}$ *is the rank function of a matroid on $V(G)$ with rank* $\mathrm{ord}(\mathcal{T})$, *and its independent sets are the free sets.*

Let H be a subgraph of G and let $\mathcal{T}'$ be a tangle in H. Let $\mathcal{T}$ be the set of all separations (A, B) of G of order $<\mathrm{ord}(\mathcal{T}')$ such that $(A \cap H, B \cap H) \in \mathcal{T}'$. It is easy to see that

(2.5). *$\mathcal{T}$ thus defined is a tangle in G of order* $\mathrm{ord}(\mathcal{T}')$.

We call $\mathcal{T}$ the tangle in G *induced* by $\mathcal{T}'$. If $\mathcal{T}''$ is any tangle in G with $\mathcal{T} \subseteq \mathcal{T}''$, we say that $\mathcal{T}'$ is *conformal* with $\mathcal{T}''$. Certainly $\mathcal{T}'$ is conformal with $\mathcal{T}$ itself.

Since we shall have many occasions to verify that some tangle is conformal with another, let us point out the following short-cut.

(2.6). *Let H be a subgraph of G, let $\mathcal{T}$ be a tangle in G, and let $\mathcal{T}'$ be a tangle in H. Then $\mathcal{T}'$ is conformal with $\mathcal{T}$ if and only if* $\mathrm{ord}(\mathcal{T}') \leqslant \mathrm{ord}(\mathcal{T})$ *and $(A \cap H, B \cap H) \in \mathcal{T}'$ for every $(A, B) \in \mathcal{T}$ of order* $<\mathrm{ord}(\mathcal{T}')$.

Proof. We assume $\mathrm{ord}(\mathcal{T}') \leqslant \mathrm{ord}(\mathcal{T})$, for otherwise $\mathcal{T}'$ is not conformal with $\mathcal{T}$. Now $\mathcal{T}'$ is conformal with $\mathcal{T}$ if and only if for every separation (A, B) of G of order $< \mathrm{ord}(\mathcal{T}')$, if $(A \cap H, B \cap H) \in \mathcal{T}'$ then $(A, B) \in \mathcal{T}$; or equivalently (since exactly one of $(A \cap H, B \cap H)$, $(B \cap H, A \cap H) \in \mathcal{T}'$, and exactly one of (A, B), $(B, A) \in \mathcal{T}$ because $\mathrm{ord}(\mathcal{T}') \leqslant \mathrm{ord}(\mathcal{T})$) if $(B, A) \in \mathcal{T}$ then $(B \cap H, A \cap H) \in \mathcal{T}'$. The result follows. ∎

We shall also need the following.

(2.7). *Let $\mathcal{T}$ be a tangle in G, let H be a subgraph of G, and let $\mathcal{T}'$ be a tangle in H conformal with $\mathcal{T}$. For $X \subseteq V(H)$, $\mathrm{rk}_{\mathcal{T}'}(X) \leqslant \mathrm{rk}_{\mathcal{T}}(X)$.*

Proof. Since $\mathrm{rk}_{\mathcal{T}'}(X) \leqslant \mathrm{ord}(\mathcal{T}')$, we may assume that $\mathrm{rk}_{\mathcal{T}}(X) < \mathrm{ord}(\mathcal{T}')$, and since $\mathrm{ord}(\mathcal{T}') \leqslant \mathrm{ord}(\mathcal{T})$, there exists $(A, B) \in \mathcal{T}$ of order $\mathrm{rk}_{\mathcal{T}}(X)$ with $X \subseteq V(A)$. Now (A, B) has order $\mathrm{rk}_{\mathcal{T}}(X) < \mathrm{ord}(\mathcal{T}')$, and by (2.6), $(A \cap H, B \cap H) \in \mathcal{T}'$. Since $X \subseteq V(A \cap H)$ it follows that

$$\mathrm{rk}_{\mathcal{T}'}(X) \leqslant |V((A \cap H) \cap (B \cap H))| \leqslant |V(A \cap B)| = \mathrm{rk}_{\mathcal{T}}(X)$$

as required. ∎

The next result is proved in [8, Theorem (2.9)].

(2.8). *Let $\mathcal{T}$ be a tangle in a graph G, let $(A_1, B_1) \in \mathcal{T}$, and let (A_2, B_2) be a separation of G of order $< \mathrm{ord}(\mathcal{T})$. If either $V(B_1) \subseteq V(B_2)$ or $V(A_2) \subseteq V(A_1)$ then $(A_2, B_2) \in \mathcal{T}$.*

We denote by $G \backslash Z$ the graph obtained from G by deleting Z (here Z may be a vertex or edge, or a set of vertices or edges). Let $\mathcal{T}$ be a tangle in G, and let $Z \subseteq V(G)$ with $|Z| < \mathrm{ord}(\mathcal{T})$. We denote by $\mathcal{T} \backslash Z$ the set of all separations (A', B') of $G \backslash Z$ such that there exists $(A, B) \in \mathcal{T}$ with $Z \subseteq V(A \cap B)$ and $A \backslash Z = A'$, $B \backslash Z = B'$.

(2.9). *$\mathcal{T} \backslash Z$ is a tangle in $G \backslash Z$ of order $\mathrm{ord}(\mathcal{T}) - |Z|$, conformal with $\mathcal{T}$.*

Proof. The first assertion is proved in [8, Theorem (8.5)]. For the second, we use (2.6). Let $(A, B) \in \mathcal{T}$ have order $< \mathrm{ord}(\mathcal{T}) - |Z|$. Choose a separation (A^+, B^+) of G with $Z \subseteq V(A^+ \cap B^+)$, $A \subseteq A^+$, $B \subseteq B^+$ and $A^+ \backslash Z = A \cap (G \backslash Z)$, $B^+ \backslash Z = B \cap (G \backslash Z)$. Then (A^+, B^+) has order $< \mathrm{ord}(\mathcal{T})$. Now $V(B) \subseteq V(B^+)$, and so by (2.8), $(A^+, B^+) \in \mathcal{T}$. Hence $(A^+ \backslash Z, B^+ \backslash Z) \in \mathcal{T} \backslash Z$ by definition of $\mathcal{T} \backslash Z$, and so $(A \cap (G \backslash Z), B \cap (G \backslash Z)) \in \mathcal{T} \backslash Z$. By (2.6), $\mathcal{T} \backslash Z$ is conformal with $\mathcal{T}$. ∎

(2.10). *Let* $\mathscr{T}$ *be a tangle in a graph* G, *and let* $z_1, ..., z_k \in V(G)$ *be distinct, such that* $\{z_1, ..., z_k\}$ *is free relative to* $\mathscr{T}$ *and* $\mathrm{ord}(\mathscr{T}) \geqslant 2k$. *Then there exist* $y_1, ..., y_k \in V(G \backslash \{z_1, ..., z_k\})$, *distinct, such that*

(i) *for* $1 \leqslant i \leqslant k$, y_i *is adjacent in* G *to* z_i, *and*

(ii) $\{y_1, ..., y_k\}$ *is free relative to* $\mathscr{T} \backslash \{z_1, ..., z_k\}$.

Proof. Let $Z = \{z_1, ..., z_k\}$, let $\mathscr{T}' = \mathscr{T} \backslash Z$, let $G' = G \backslash Z$ and for $1 \leqslant i \leqslant k$, let Y_i be the set of vertices of G' adjacent to z_i in G. First we show the following:

(1) *For all* $I \subseteq \{1, ..., k\}$, $\mathrm{rk}_{\mathscr{T}'}(\bigcup (Y_i : i \in I)) \geqslant |I|$.

Subproof. Suppose not. Since $\mathrm{ord}(\mathscr{T}') \geqslant k \geqslant |I|$, there exists $(A', B') \in \mathscr{T}'$ of order $< |I|$ such that $\bigcup (Y_i : i \in I) \subseteq V(A')$. From the definition of $\mathscr{T} \backslash Z$ there exists $(A, B) \in \mathscr{T}$ with $Z \subseteq V(A \cap B)$ and $A \backslash Z = A'$, $B \backslash Z = B'$. By choosing A maximal, it follows from (2.8) that for each $i \in I$, every edge incident with z_i belongs to $E(A)$. Hence (A, B'') is a separation of G, where $B'' = B \backslash \{z_i : i \in I\}$. Its order is

$$|V(A \cap B)| - |I| = |V(A' \cap B')| + |Z| - |I| < |Z|;$$

and $(B'', A) \notin \mathscr{T}$, since $(A, B) \in \mathscr{T}$ and $A \cup B'' = G$. Thus $(A, B'') \in \mathscr{T}$. But that contradicts the hypothesis that Z is free relative to $\mathscr{T}$. Thus there is no such (A', B'). This proves (1). ∎

From a theorem of Rado [4], (2.4), and (1), it follows that there exist $y_i \in Y_i$ $(1 \leqslant i \leqslant k)$ such that $\mathrm{rk}_{\mathscr{T}'}\{y_1, ..., y_k\} = k$, and so, by (2.4), $\{y_1, ..., y_k\}$ is free relative to $\mathscr{T}'$. ∎

(2.11). *Let* $\mathscr{T}$ *be a tangle in* G *and let* $(C, D) \in \mathscr{T}$ *with order* 0. *Let*

$$\mathscr{T}' = \{(A \cap D, B \cap D) : (A, B) \in \mathscr{T}\}.$$

Then $\mathscr{T}'$ *is a tangle in* D *with the same order as* $\mathscr{T}$, *and it is conformal with* $\mathscr{T}$.

Proof. We must verify the three axioms. Let (A', B') be a separation of D of order $< \mathrm{ord}(\mathscr{T})$. Then $(A' \cup C, B')$ is a separation of G of the same order, and so one of $(A' \cup C, B')$, $(B', A' \cup C) \in \mathscr{T}$. Hence one of (A', B'), $(B', A') \in \mathscr{T}'$. This verifies the first axiom.

Let $(A_i', B_i') \in \mathscr{T}'$ $(i = 1, 2, 3)$ and suppose that $A_1' \cup A_2' \cup A_3' = D$. There exists $(A_i, B_i) \in \mathscr{T}$ with $A_i' = A_i \cap D$, $B_i' = B_i \cap D$ $(i = 1, 2, 3)$. Now $(A_i \cup C, B_i \cap D)$ has order at most that of (A_i, B_i), and $(B_i \cap D, A_i \cup C) \notin \mathscr{T}$ since

$(C, D) \in \mathscr{T}$ and $A_i \cup C \cup (B_i \cap D) = G$. Thus $(A_i \cup C, B_i \cap D) \in \mathscr{T}$ ($i = 1, 2, 3$). But $(A_1 \cup C) \cup (A_2 \cup C) \cup (A_3 \cup C) = G$, a contradiction. This verifies the second axiom.

Let $(A', B') \in \mathscr{T}'$ and choose $(A, B) \in \mathscr{T}$ with $A \cap D = A'$, $B \cap D = B'$. Then $(A \cup C, B \cap D) \in \mathscr{T}$ as before, and so $V(A \cup C) \neq V(G)$, that is, $V(A) \neq V(D)$. This verifies the third axiom.

Thus $\mathscr{T}'$ is a tangle in D; and by (2.6) it is conformal with $\mathscr{T}$, as required. ▌

(2.12). *Let $\mathscr{T}$ be a tangle in G, and let $Z \subseteq V(G)$ be free relative to $\mathscr{T}$, with $|Z| < \mathrm{ord}(\mathscr{T})$. Then there is a component C of $G \backslash Z$ such that each vertex in Z has a neighbour in $V(C)$.*

Proof. Choose $(A, B) \in \mathscr{T}$ with $V(A \cap B) = Z$ and with B minimal. (This is possible, for setting $B = G$ and $V(A) = Z$, $E(A) = \emptyset$ satisfies $(A, B) \in \mathscr{T}$, by the first and third axioms, since $|Z| < \mathrm{ord}(\mathscr{T})$.)

(1) *There do not exist subgraphs B_1, B_2 of B with $B_1 \cup B_2 = B$, $E(B_1 \cap B_2) = \emptyset$, $V(B_1 \cap B_2) = Z$, and $V(B_1)$, $V(B_2) \neq Z$.*

Subproof. If B_1, B_2 have these properties, then $(A \cup B_1, B_2)$ is a separation of G, and $V((A \cup B_1) \cap B_2) = Z$. From the minimality of B, $(A \cup B_1, B_2) \notin \mathscr{T}$, and so $(B_2, A \cup B_1) \in \mathscr{T}$. Similarly $(B_1, A \cup B_2) \in \mathscr{T}$; but $(A, B) \in \mathscr{T}$ and $A \cup B_1 \cup B_2 = G$, contrary to the second axiom.

Let $C = B \backslash Z$. Then C is non-null, from the third axiom. From (1), (2.8) and the minimality of B, we deduce that C is connected and $V(C)$ meets every edge of B. Now let $v \in Z$; to complete the proof, we must show that v has a neighbour in $V(C)$. Suppose not. Since $V(C)$ meets every edge of B, v is an isolated vertex of B, and so $(A, B \backslash v)$ is a separation of G. By (2.8), $(A, B \backslash v) \in \mathscr{T}$. But $Z \subseteq V(A)$, and Z is free relative to $\mathscr{T}$, a contradiction, as required. ▌

3. Linking Two Vertices

The object of this section is to prove that if G has a large order tangle and $s, t \in V(G)$, and $\{s, t\}$ is free, then there is a path P between s and t such that $G \backslash V(P)$ still has a large order tangle (not as large as the original, but of order at least $\frac{1}{18}$ the original). If H is a subgraph of G we write $H \subseteq G$. We begin with the following lemma.

(3.1). *Let $k \geqslant 1$ be an integer, let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 18k$. Suppose that for all X, $Y \subseteq Z$ with $|X| = |Y|$ there are $|X|$ mutually vertex-disjoint paths between X and Y. Let (S, T) be a partition of Z with $|S| = |T| = 9k$. Then there is a path $P \subseteq G$ with one end in S and the* *other in T, and with no internal vertex in Z, such that there is no separation (A, B) of $G \backslash V(P)$ of order $<k$ with $|V(A) \cap Z|$, $|V(B) \cap Z| \geqslant 6k$.*

Proof. Let $\mathscr{T}$ be the tangle of order $6k$ consisting of all separations (A, B) of order $<6k$ with $|V(A) \cap Z| \leqslant |V(A \cap B)|$. (We observe that this is a tangle, for if (A, B) is a separation then one of $|V(A) \cap Z|$, $|V(B) \cap Z| \leqslant |V(A \cap B)|$ from our hypothesis.)

(1) *For any path P from S to T with no internal vertex in Z, if* $\mathrm{rk}_{\mathscr{T}}(V(P)) \leqslant 5k$ *then P satisfies the theorem.*

Subproof. Let $(C, D) \in \mathscr{T}$ with $V(P) \subseteq V(C)$ and $|V(C \cap D)| \leqslant 5k$. Suppose that P does not satisfy the theorem; then there is a separation (A, B) of G with $V(P) \subseteq V(A \cap B)$ and $|V(A \cap B) - V(P)| < k$ such that $|V(A) \cap Z|$, $|V(B) \cap Z| \geqslant 6k + 2$. Now since $(C, D) \in \mathscr{T}$ it follows that $|V(C) \cap Z| \leqslant |V(C \cap D)| \leqslant 5k$, and so $|V(D) \cap Z| \geqslant 13k$. From the symmetry between A and B, we may assume that

$$|V(A) \cap V(D) \cap Z| \geqslant 13k/2 > 6k.$$

Now $(C', D') = (A \cap D, B \cup C)$ is a separation of G of order $<6k$, because

$$\begin{aligned} V((A \cap D) \cap (B \cup C)) &\subseteq V(C \cap D) \cup (V(A \cap B) - V(C)) \\ &\subseteq V(C \cap D) \cup (V(A \cap B) - V(P)). \end{aligned}$$

But $|V(C') \cap Z| > 6k$, and $|V(D') \cap Z| \geqslant |V(B) \cap Z| > 6k$. Thus (C', D'), $(D', C') \notin \mathscr{T}$, a contradiction. This proves (1).

Let $\mathscr{P}$ be a set of $9k$ mutually disjoint paths between S and T. Then no member of $\mathscr{P}$ has an internal vertex in Z, and we shall prove that one of them satisfies the theorem. If P_1, P_2, $P_3 \in \mathscr{P}$ are distinct, let us say that P_2 *splits* P_1 and P_3 if there do not exist k mutually disjoint paths of G from $V(P_1)$ to $V(P_3)$ all disjoint from $V(P_2)$. We claim that we may assume that

(2) *If P_1, P_2, $P_3 \in \mathscr{P}$ are distinct and P_2 splits P_1 and P_3, then P_3 does not split P_1 and P_2.*

Subproof. By (1) we may assume that $\mathrm{rk}_{\mathscr{T}}(V(P_1))$, $\mathrm{rk}_{\mathscr{T}}(V(P_2)) > 5k$, and so there are $>5k$ mutually disjoint paths of G from $V(P_1)$ to $V(P_2)$. Thus, either there are $>5k/2$ disjoint paths between $V(P_1)$ and $V(P_2)$ avoiding $V(P_3)$, or $>5k/2$ disjoint paths between $V(P_1)$ and $V(P_3)$ avoiding $V(P_2)$. The result follows.

By (2), there are at most $\binom{9k}{3}$ pairs $(P_2, \{P_1, P_3\})$, where P_1, P_2, $P_3 \in \mathscr{P}$ are distinct and P_2 splits P_1 and P_3. Hence the average number of unordered pairs $\{P_1, P_3\}$ split by $P_2 \in \mathscr{P}$ (averaged over all $P_2 \in \mathscr{P}$) is at most

$\binom{9k}{3}/9k < 55k^2/4$. Consequently, there exists $P \in \mathscr{P}$ which splits fewer than $55k^2/4$ unordered pairs P_1, P_2. We claim that P satisfies the theorem. For suppose not, and let (A, B) be a separation of G with $V(P) \subseteq V(A \cap B)$ and $|V(A \cap B)| - |V(P)| < k$, such that $|V(A) \cap Z|$, $|V(B) \cap Z| \geq 6k+2$. Let a be the number of members $P' \in \mathscr{P}$ with $V(P') \subseteq V(A)$ and $P' \neq P$, and define b similarly. Then $a+b+k \geq 9k$ since at most k members of $\mathscr{P}$ intersect $V(A \cap B)$. Moreover, since some path of $\mathscr{P}$ begins at each vertex of $V(A) \cap Z$, and at most $k-1$ of them leave $V(A)$, and all the rest (except for P) contribute to a, it follows that

$$|V(A) \cap Z| \leq 2(a+1)+k-1.$$

Since $|V(A) \cap Z| \geq 6k+2$, we deduce that $2a \geq 5k$, and similarly $2b \geq 5k$. We have shown then that $a+b \geq 8k$, $a \geq 5k/2$, $b \geq 5k/2$, and $ab < 55k^2/4$ from the choice of P. But these four inequalities are contradictory. Thus P satisfies the theorem as required. ▮

The main result of this section is the following.

(3.2). *Let $k \geq 1$ be an integer, let s, t be distinct vertices of a graph G, and let $\mathscr{T}$ be a tangle in G of order $\geq 18k+3$, such that $\{s, t\}$ is free relative to $\mathscr{T}$. Then there is a path P between s and t, and a tangle in $G \backslash V(P)$ of order k, conformal with $\mathscr{T}$.*

Proof. Choose $(A, B) \in \mathscr{T}$ with A maximal such that there is a partition (S, T) of $V(A \cap B)$, and two disjoint connected subgraphs J, K of A, satisfying the following conditions (1)–(4):

(1) *There is no $(A', B') \in \mathscr{T}$ of order $< |V(A \cap B)|$ such that $A \subseteq A'$ and $B' \subseteq B$.*

(2) $|S|$, $|T| \leq 9k+1$.

(3) *$s \in V(J)$, $t \in V(K)$, $V(J) \cap S \neq \varnothing \neq V(K) \cap T$, and J, K both contain exactly one vertex of $V(A \cap B)$.*

(4) *For each $v \in V(A \cap B) - V(J \cup K)$, if $v \in S$ then v is adjacent in A to a vertex of J, and if $v \in T$ then v is adjacent in A to a vertex of K.*

Such a choice of (A, B) is possible, because setting $B = G$ and $V(A) = \{s, t\}$, $E(A) = \varnothing$, $V(J) = \{s\}$, $V(K) = \{t\}$ satisfies (1)–(4). Let $Z = V(A \cap B)$.

(5) *There is no $(A', B') \in \mathscr{T}$ of order $|Z|$ such that $A' \neq A$, $A \subseteq A'$ and $B' \subseteq B$.*

Subproof. Suppose that such a choice of (A', B') exists. Let $|Z| = n$. Suppose first that there is a separation (C, D) of $B \cap A'$ of order $< n$ with

$Z \subseteq V(C)$ and $V(A' \cap B') \subseteq V(D)$. Let $A'' = A \cup C$, $B'' = B' \cup D$; then (A'', B'') is a separation of G of order $<n$ with $A \subseteq A'' \subseteq A'$ and $B' \subseteq B'' \subseteq B$. But $(A', B') \in \mathscr{T}$ and so $(A'', B'') \in \mathscr{T}$ by (2.8), contrary to (1). Thus there is no such (C, D); and so by Menger's theorem, there are mutually disjoint paths $P_1, ..., P_n$ of $B \cap A'$ from Z to $V(A' \cap B')$. Let P_i have first vertex v_i and last vertex v_i' $(1 \leqslant i \leqslant n)$. Let J' be the union of J, the unique path P_i such that $v_i \in V(J)$, and for each j such that $v_j \in S - V(J)$ and $v_j \neq v_j'$, the path $P_j \backslash v_j'$ and an edge of A joining v_j to a vertex of J (which exists by (4)). Define K' similarly. Let $S' = \{v_i' : 1 \leqslant i \leqslant n,\ v_i \in S\}$, and define T' similarly. Then (A', B'), S', T', J', K' satisfy (1)–(4), contrary to the maximality of A. This proves (5).

(6) $|S| = |T| = 9k+1$, *and hence* $|Z| = 18k+2$.

Subproof. Suppose that $|S| < 9k+1$, say. Let $V(J) \cap Z = \{v\}$. If $A \cup (B\backslash v) = G$, then $(A, B\backslash v)$ is a separation of order smaller than $|Z|$, and by (2.8), $(A, B\backslash v) \in \mathscr{T}$, contrary to (1). Hence $A \cup (B\backslash v) \neq G$, and so there is an edge e of B incident with v. Some end, u, say, of e is not in $V(A)$, by (2.8) and the maximality of A. Let $V(A') = V(A) \cup \{u\}$, $E(A') = E(A) \cup \{e\}$, $B' = B\backslash e$; then (A', B') has order $1 + |Z| \leqslant 18k+2$ and $(A', B') \in \mathscr{T}$ by (2.8). Let $S' = S \cup \{u\}$; then A', B', S', T, J, K satisfy (1)–(4), because of (5), contrary to the maximality of A. This proves (6).

(7) *If* $X, Y \subseteq Z$ *with* $|X| = |Y|$, *there are* $|X|$ *mutually disjoint paths of* B *between* X *and* Y, *each with no internal vertex in* Z.

Subproof. Suppose not. Then by Menger's theorem, there is a separation (C, D) of B with $Z - (X \cup Y) \subseteq V(C \cap D)$ and

$$|V(C \cap D) - (Z - (X \cup Y))| < |X|$$

such that $X \subseteq V(C)$ and $Y \subseteq V(D)$. Now $(A \cup C, D)$ has order $< |Z|$, for $V(A \cap D) - V(C) \subseteq Y - X$, and so

$$\begin{aligned} |V((A \cup C) \cap D)| &= |V(C \cap D) - (Z - (X \cup Y))| + |Z - (X \cup Y)| \\ &\quad + |V(A \cap D) - V(C)| \\ &< |X| + |Z - (X \cup Y)| + |Y - X| = |Z|. \end{aligned}$$

From (1), it follows that $(A \cup C, D) \notin \mathscr{T}$, and so $(D, A \cup C) \in \mathscr{T}$. Similarly, $(C, A \cup D) \in \mathscr{T}$; but $(A, B) \in \mathscr{T}$, and $A \cup C \cup D = G$, contrary to the second axiom. This proves (7).

Let $V(J) \cap Z = \{v\}$ and $V(K) \cap Z = \{w\}$, and let $G' = B\backslash\{v, w\}$, $Z' = Z - \{v, w\}$, $S' = S - \{v\}$, $T' = T - \{w\}$. By (6), (7), and (3.1) applied to G', Z', S', T' we deduce

(8) *There is a path P' of G' from S' to T' with no internal vertex in Z', such that there is no separation (A', B') of $G'\backslash V(P')$ of order $<k$ with $|V(A')\cap Z'|, |V(B')\cap Z'| \geqslant 6k$.*

Now let P be a path of G from s to t with $P\cap G' = P'$ (this exists and can be chosen with $V(P)\subseteq V(J\cup K\cup P')$ because of (4)). Let $\mathscr{T}'$ be the set of all separations (A', B') of $G\backslash V(P)$ of order $<k$ such that $|V(A')\cap Z'| < 6k$. We claim that $\mathscr{T}'$ is a tangle in $G\backslash V(P)$ of order k. For the second and third axioms are clearly satisfied, since $|Z'\cap V(G\backslash V(P))| = 18k-2 > 3(6k-1)$. To see the first axiom, let (A', B') be a separation of $G\backslash V(P)$ of order $<k$; then $(A'\cap G', B'\cap G')$ is a separation of $G'\backslash V(P')$ of order $<k$, and so by (8) one of $|V(A'\cap G')\cap Z'|$, $|V(B'\cap G')\cap Z'| < k$; that is, one of $|V(A')\cap Z'|$, $|V(B')\cap Z'| < k$. It follows that one of (A', B'), (B', A') belongs to $\mathscr{T}'$, and this verifies the first axiom. Hence $\mathscr{T}'$ is a tangle in $G\backslash V(P)$ of order k.

(9) *$\mathscr{T}'$ is conformal with $\mathscr{T}$.*

Subproof. Let (A_0, B_0) be a separation of G of order $<k$, such that $(A', B')\in\mathscr{T}'$, where $A' = A_0\cap(G\backslash V(P))$, $B' = B_0\cap(G\backslash V(P))$. We must show that $(A_0, B_0)\in\mathscr{T}$. Now

$$V(P)\cap Z' = V(P\cap G')\cap Z' = V(P')\cap Z',$$

and so $|V(P)\cap Z'| \leqslant 2$ by (8). But $V(P)\cap Z\subseteq (V(P)\cap Z')\cup\{v, w\}$, and so $|V(P)\cap Z| \leqslant 4$. Since $|V(A')\cap Z'| \leqslant 6k-1$ from the definition of $\mathscr{T}'$ and

$$Z - V(B_0)\subseteq V(A_0)\cap Z\subseteq (V(A')\cap Z)\cup(V(P)\cap Z),$$

it follows that $|Z - V(B_0)| \leqslant 6k+3$. But

$$V((A\cup B_0)\cap B\cap A_0)\subseteq V(A_0\cap B_0)\cup(Z - V(B_0))$$

and $|V(A_0\cap B_0)| \leqslant k-1$, and so $(A\cup B_0, B\cap A_0)$ has order $\leqslant (6k+3)+(k-1) = 7k+2$. By (1) and (6), $(A\cup B_0, B\cap A_0)\notin T$, and so $(B\cap A_0, A\cup B_0)\in\mathscr{T}$ by the first tangle axiom. It follows that $(B_0, A_0)\notin\mathscr{T}$, by the second axiom, because (A, B), $(B\cap A_0, A\cup B_0)\in\mathscr{T}$ and $A\cup(B\cap A_0)\cup B_0 = G$. Hence $(A_0, B_0)\in\mathscr{T}$, as required. ▌

From (9), the result follows. ▌

4. Preserving Freedom

As we said earlier, the purpose of (3.2) is to enable us to "knit" a grid minor, pulling out one stitch at a time from a tangle in the remainder of the graph, in such a way that after each stitch there still remains a large order tangle in the remainder, so that we can continue. However, it is important that the part of the grid that we have already knitted (or at least the most recently knitted row of it) remain well connected to the tangle in the complement, for otherwise we shall not be able to add subsequent stitches where we want them. This is a serious problem (it must be, for otherwise one could construct clique minors just as well as grid minors), and to resolve it we shall modify our method of specifying exactly where the next stitch should attach. To this end, we notice that if the next stitch cannot be placed where we want it, we can infer a "non-planarity" in the graph, and enough non-planarities will give us a large clique minor. So either the stitches mostly come along where we want them, and we knit the grid we want, or we produce a large clique minor (which, of course, contains a large grid anyway).

Let G be a graph, $\mathscr{T}$ a tangle in G, and $Z \subseteq V(G)$, free relative to $\mathscr{T}$. We call the triple G, $\mathscr{T}$, Z a *nodule* of *order* k, where $k = \operatorname{ord}(\mathscr{T})$. Let G, $\mathscr{T}$, Z and G', $\mathscr{T}'$, Z' be nodules. We say that the second is a *residual* nodule of the first if G' is a subgraph of $G \backslash Z$, $\mathscr{T}'$ is conformal with $\mathscr{T}$, $|Z'| = |Z|$, and there are $|Z|$ mutually vertex-disjoint paths of G between Z and Z', each with no vertex in G' except its last vertex. We call such a set of paths a *linkage* (*for* G' $\mathscr{T}'$, Z' *from* G, $\mathscr{T}$, Z). We observe that, by (2.6), if G', $\mathscr{T}'$, Z' is a residual nodule of G, $\mathscr{T}$, Z and G'', $\mathscr{T}''$, Z'' is a residual nodule of G', $\mathscr{T}'$, Z' then G'', $\mathscr{T}''$, Z'' is a residual nodule of G, $\mathscr{T}$, Z.

If $e \in E(G)$, G/e denotes the graph obtained by contracting e.

(4.1). *Let G be a graph, let $Z \subseteq V(G)$, and let C be a connected subgraph of G with $V(C) \cap Z \neq \varnothing$. Let $\mathscr{T}$ be a tangle in $G \backslash V(C)$ of order $\geqslant 2\,|Z|$, inducing a tangle $\mathscr{T}_0$ in G such that G, $\mathscr{T}_0$, Z is a nodule. Then there is a residual nodule G', $\mathscr{T}'$, Z' of order* $\operatorname{ord}(\mathscr{T}) - |Z|$ *such that $C \cap G'$ is null.*

Proof. We proceed by induction on $|V(C) - Z|$. Let $|Z| = h$, and choose $(A, B) \in \mathscr{T}_0$ of order h with $Z \subseteq V(A)$ and A maximal. (This is possible, for $V(A) = Z$, $E(A) = \varnothing$, $B = G$ is one such choice.) There are two cases.

Case 1. $E(C) \nsubseteq E(A)$. Choose $e \in E(C) - E(A)$. Let $\mathscr{T}'_0$ be the tangle in G/e induced by $\mathscr{T}$ (we see that $G \backslash V(C)$ is a subgraph of G/e since $e \in E(C)$). Now not both ends of e belong to Z, by the maximality of A and (2.8), and so no two members of Z are identified in forming G/e. Thus we may regard Z as a subset of $V(G/e)$. We claim that Z is free relative to $\mathscr{T}'_0$.

For if not, there is a separation $(A'_0, B'_0) \in \mathscr{T}'_0$ of order $< h$ with $Z \subseteq V(A'_0)$. Let w be the vertex of G/e formed by identifying the ends of e. If $w \in V(A'_0) - V(B'_0)$ let $B' = B'_0$, and let A' be the subgraph of G with $e \in E(A')$ and $A'/e = A'_0$. If $w \in V(B'_0) - V(A'_0)$ let $A' = A'_0$ and let B' be the subgraph of G with $e \in E(B')$ and $B'/e = B'_0$. If $w \in V(A'_0 \cap B'_0)$ let A', B' be the subgraphs of G with $e \in E(A')$, $A'/e = A'_0$, $e \notin E(B')$, $(B'+e)/e = B'_0$, with the natural definition of $B'+e$. In each case (A', B') is a separation of G and

$$(A' \cap (G \backslash V(C)),\ B' \cap (G \backslash V(C))) = (A'_0 \cap (G \backslash V(C)),\ B'_0 \cap (G \backslash V(C))) \in \mathscr{T}.$$

Thus $(A', B') \in \mathscr{T}_0$, because the order of (A', B') is at most one more than the order (A'_0, B'_0) and hence at most $h < \operatorname{ord}(\mathscr{T}_0)$. Since Z is free with respect to $\mathscr{T}_0$ and $Z \subseteq V(A')$ it follows that $|V(A' \cap B')| \geqslant h$; hence, $|V(A' \cap B')| = h$ and $e \in E(A')$. Yet $(A, B) \in \mathscr{T}_0$, and $(A \cup A',\ B \cap B')$ has order $> h$; for otherwise it belongs to $\mathscr{T}_0$, contrary to the maximality of A, since $e \in E(A') - E(A)$. Since the sum of the orders of $(A \cup A',\ B \cap B')$ and $(A \cap A',\ B \cup B')$ equals the sum of the orders of (A, B) and (A', B') and, hence, is $2h$, it follows that $(A \cap A',\ B \cup B')$ has order $< h$. Now $(A, B) \in \mathscr{T}_0$ and so by (2.8), $(A \cap A',\ B \cup B') \in \mathscr{T}_0$, contrary to the freedom of Z relative to $\mathscr{T}_0$. This proves the freedom of Z relative to $\mathscr{T}'_0$.

From the inductive assumption applied to G/e, C/e and $\mathscr{T}$, there exists a residual nodule G', $\mathscr{T}'$, Z' of G/e, $\mathscr{T}'_0$, Z of order $\operatorname{ord}(\mathscr{T}) - |Z|$, with $G' \cap (C/e)$ null. We claim that $\mathscr{T}'$ is conformal with $\mathscr{T}_0$. For let (A', B') be a separation of G of order $< \operatorname{ord}(\mathscr{T}') = \operatorname{ord}(\mathscr{T}) - |Z|$, such that $(A' \cap G',\ B' \cap G') \in \mathscr{T}'$. We must show that $(A', B') \in \mathscr{T}_0$. Let (A^*, B^*) be the separation of G/e, where $A^* = A'/e$ if $e \in E(A')$, $A^* = A'$ if some end of e is not in $V(A')$, and A^* is obtained from A' by identifying the ends of e if $e \in E(B')$ and both ends of e are in $V(A')$ (and B^* is defined similarly). Now $A' \cap G' = A^* \cap G'$ and $B' \cap G' = B^* \cap G'$, and (A^*, B^*) has order $< \operatorname{ord}(\mathscr{T}')$; and so $(A^*, B^*) \in \mathscr{T}'_0$ since $(A' \cap G',\ B' \cap G') \in \mathscr{T}'$ and $\mathscr{T}'$ is conformal with $\mathscr{T}'_0$. By the definition of $\mathscr{T}'_0$,

$$(A' \cap (G \backslash V(C)),\ B' \cap (G \backslash V(C))) = (A^* \cap (G \backslash V(C)),\ B^* \cap (G \backslash V(C))) \in \mathscr{T},$$

and so $(A', B') \in \mathscr{T}_0$. Thus $\mathscr{T}'$ is conformal with $\mathscr{T}_0$. Hence G', $\mathscr{T}'$, Z' is a residual nodule of G, $\mathscr{T}_0$, Z satisfying the theorem.

Case 2. $E(C) \subseteq E(A)$. Let $V(A \cap B) = X = \{x_1, ..., x_h\}$. We claim that X is free relative to $\mathscr{T}_0$. For suppose not and let $(A_0, B_0) \in \mathscr{T}_0$ with order $< h$, such that $X \subseteq V(A_0)$. Now

$$V((A \cup A_0) \cap (B \cap B_0)) \subseteq V(A_0 \cap B_0) \cup (X \cap V(B_0)) = V(A_0 \cap B_0)$$

and so $(A \cup A_0,\ B \cap B_0)$ has order $< h$. Since $Z \subseteq V(A \cup A_0)$ we deduce from the freedom of Z relative to $\mathscr{T}_0$ that $(A \cup A_0,\ B \cap B_0) \notin \mathscr{T}_0$. Thus $(B \cap B_0,\ A \cup A_0) \in \mathscr{T}_0$; but (A, B), $(A_0, B_0) \in \mathscr{T}_0$ and $A \cup A_0 \cup (B \cap B_0) = G$, contrary to the second axiom. Thus X is free relative to $\mathscr{T}_0$.

By (2.10) there exist distinct $y_1, ..., y_h \in V(G) - X$ such that $\{y_1, ..., y_h\}$ is free relative to $\mathscr{T}_0 \backslash X$ and y_i is adjacent to x_i $(1 \leqslant i \leqslant h)$. Since $(A \backslash X, B \backslash X) \in \mathscr{T}_0 \backslash X$ and, hence,

$$\mathrm{rk}_{\mathscr{T}_0 \backslash X}(V(A \backslash X)) = 0,$$

it follows by (2.4) that $y_1, ..., y_h \in V(B)$. Let $G' = B \backslash X$. Since C is connected, $V(C) \cap Z \neq \varnothing$ and $E(C) \subseteq E(A)$, it follows that $V(C \cap G') = \varnothing$. By (2.9) and (2.11) there is a tangle $\mathscr{T}'$ in G' of order $\mathrm{ord}(\mathscr{T}) - h$, conformal with $\mathscr{T}_0 \backslash X$ and hence with $\mathscr{T}_0$ (since "conformality" is transitive and $\mathscr{T}_0 \backslash X$ is conformal with $\mathscr{T}_0$).

We claim that $\{y_1, ..., y_h\}$ is free relative to $\mathscr{T}'$. For suppose not; then since $\mathrm{ord}(\mathscr{T}') = \mathrm{ord}(\mathscr{T}) - h \geqslant h$, there exists $(A', B') \in \mathscr{T}'$ of order $< h$ with $\{y_1, ..., y_h\} \subseteq V(A')$. Since $\mathscr{T}'$ is conformal with $\mathscr{T}_0 \backslash X$ it follows that $(A' \cup (A \backslash X), B') \in \mathscr{T}_0 \backslash X$, and yet this separation has the same order as (A', B'), contrary to the freedom of $\{y_1, ..., y_h\}$ relative to $\mathscr{T}_0 \backslash X$. Thus $\{y_1, ..., y_h\}$ is free relative to $\mathscr{T}'$, and so G', $\mathscr{T}'$, $\{y_1, ..., y_h\}$ is a nodule.

We claim that there are h disjoint paths of A between Z and X. For suppose not; then by Menger's theorem there is a separation (A_1, B_1) of A of order $< h$ with $Z \subseteq V(A_1)$ and $X \subseteq V(B_1)$. Then $(A_1, B_1 \cup B)$ is a separation of G of order $< h$, and since $(A, B) \in \mathscr{T}_0$ it follows from (2.8) that $(A_1, B_1 \cup B) \in \mathscr{T}_0$, contrary to the freedom of Z relative to $\mathscr{T}_0$. This proves our claim that there are h disjoint paths $P_1, ..., P_h$ of A between Z and X. Let the vertex of P_i in X be x_i $(1 \leqslant i \leqslant h)$; then by extending P_i by an edge joining x_i and y_i, for each i, we deduce that there is a linkage for G', $\mathscr{T}'$, $\{y_1, ..., y_n\}$. The result follows. ∎

From (4.1) and (3.2) we deduce the following.

(4.2. *Let G, $\mathscr{T}$, Z be a nodule with $|Z| = h$, of order $\geqslant 18(h+k)+3$, where $k \geqslant h$, and let $s, t \in Z$ be distinct. Then there is a residual nodule G', $\mathscr{T}'$, Z' of order k, and a linkage $\mathscr{P}$ for it from G, $\mathscr{T}$, Z, and a path Q of G from s to t disjoint from G', such that for each $P \in \mathscr{P}$, $P \cap Q$ is either null or a path.*

Proof. By (3.2) there is a path C between s and t and a tangle $\mathscr{T}_1$ in $G \backslash V(C)$ of order $h+k$, conformal with $\mathscr{T}$. Let $\mathscr{T}_0$ be the tangle in G induced by $\mathscr{T}_1$; then $\mathscr{T}_0 \subseteq \mathscr{T}$. Since $|Z| \leqslant \mathrm{ord}(\mathscr{T}_0)$ and $\mathscr{T}_0 \subseteq \mathscr{T}$ and G, $\mathscr{T}$, Z is a nodule, it follows that G, $\mathscr{T}_0$, Z is a nodule. Since $\mathrm{ord}(\mathscr{T}_1) \geqslant 2|Z|$, we may apply (4.1). We deduce that there is a residual nodule G', $\mathscr{T}'$, Z' of G, $\mathscr{T}_0$, Z and hence of G, $\mathscr{T}$, Z, of order $\mathrm{ord}(\mathscr{T}_1) - h = k$, such that $C \cap G'$ is null. Let $\mathscr{P}$ be a linkage for it from G, $\mathscr{T}$, Z and choose a path Q of G from s to t, disjoint from G', with $Q \cup \bigcup (P : P \in \mathscr{P})$ minimal. (Such a choice is possible because $Q=C$ is one such choice.) It follows that $P \cap Q$ is either null or a path for each $P \in \mathscr{P}$, as required. ∎

(4.3). *Let G, $\mathscr{T}$, Z be a nodule of order $\geqslant k20^n$, where $|Z|=n\geqslant 2$ and $k \geqslant n/2$. Then there is a residual nodule G', $\mathscr{T}'$, Z' of order $\geqslant k$ such that Z is a subset of the vertex set of one component of $G\backslash V(G')$.*

Proof. Choose r with $1 \leqslant r \leqslant n-1$, minimum such that there is a residual nodule G', $\mathscr{T}'$, Z' of order $\geqslant k20^r$ and Z is a subset of the union of the vertex sets of at most r components of $G\backslash V(G')$. Such a choice of r is possible, because by (4.2) such a residual nodule exists when $r=n-1$. Suppose that $r \geqslant 2$, and let s, $t \in Z$ belong to different components of $G\backslash V(G')$. Choose a linkage for G', $\mathscr{T}'$, Z', and let s', $t' \in Z'$ be the last vertices of the paths in this linkage with first vertices s, t. Let $h'=n$, $k'=k20^{r-1}$; then by applying (4.2) to G', $\mathscr{T}'$, Z', s', t', h', k', we obtain a contradiction to the minimality of r. Hence $r=1$, and the result follows. ∎

(4.4). *Let $2 \leqslant h \leqslant n \leqslant 2k$, and let G, $\mathscr{T}$, Z be a nodule of order $\geqslant k20^{nh}$, with $|Z|=h$. Then there is a nodule G', $\mathscr{T}'$, Y of order $\geqslant k$, with G' a subgraph of $G\backslash Z$ and with $|Y|=n$, such that G', $\mathscr{T}'$, Z' is a residual nodule of G, $\mathscr{T}$, Z for every $Z' \subseteq Y$ with $|Z'|=h$.*

Proof. By (2.4), since $\mathrm{ord}(\mathscr{T}) \geqslant n$ there is a set $Z_0 \subseteq V(G)$ with $Z \subseteq Z_0$ and $|Z_0|=n$, free relative to $\mathscr{T}$. By h applications of (4.3), we deduce that there are nodules G_r, $\mathscr{T}_r$, Z_r $(0 \leqslant r \leqslant h)$, where $G_0=G$ and $\mathscr{T}_0=\mathscr{T}$, such that for $1 \leqslant r \leqslant h$

(i) G_r, $\mathscr{T}_r$, Z_r is a residual nodule of G_{r-1}, $\mathscr{T}_{r-1}$, Z_{r-1}

(ii) $\mathscr{T}_r$ has order $\geqslant k20^{n(h-r)}$, and

(iii) Z_{r-1} is a subset of the vertex set of one component C_r of $G_{r-1}\backslash V(G_r)$.

We show that setting $G'=G_h$, $\mathscr{T}'=\mathscr{T}_h$, $Y=Z_h$ satisfies the theorem. For let $Z' \subseteq Z_h$ with $|Z'|=h$, and let J be the graph $G\backslash(V(G')-Z')$. Certainly $\mathscr{T}'$ is conformal with $\mathscr{T}$, so it suffices to show that there is a linkage for G', $\mathscr{T}'$, Z' from G, $\mathscr{T}$, Z; that is, there are h disjoint ZZ'-paths in J. To see this, let $X \subseteq V(J)$ with $|X|<h$. For $r=1, ..., h$, by (i) G_r, $\mathscr{T}_r$, Z_r is a residual nodule of G_{r-1}, $\mathscr{T}_{r-1} Z_{r-1}$; let $\mathscr{P}_r$ be a corresponding linkage. Then $\mathscr{P}_r$ is a set of n disjoint Z_{r-1} Z_r-paths in G_{r-1}, each with just one vertex in $V(G_r)$, namely its last vertex. Since $|Z_0|=|Z_1|=\cdots=|Z_h|=n$, there are n disjoint Z_0Z_h-paths $P_1, ..., P_n$ in G such that each P_i is of the form $P_i^1 \cup P_i^2 \cup \cdots \cup P_i^h$, where $P_i^r \in \mathscr{P}_r$ $(r=1, ..., h)$. Since $Z \subseteq Z_0$, $Z' \subseteq Z_h$

and $|Z| = |Z'| = h > |X|$, there exist $i, j \in \{1, ..., h\}$ such that P_i is a ZZ_h-path, P_j is a Z_0Z'-path, and neither P_i nor P_j meets X. Since $C_1, ..., C_h$ are disjoint and $|X| < h$ there exists r with $1 \leqslant r \leqslant h$ such that $X \cap V(C_r) = \emptyset$. Let the last vertex of P_i be v. Then $P_i \backslash v$, P_j and C_r are all subgraphs of J, they are all connected, and both $P_i \backslash v$ and P_j meet $V(C_r)$; and so $(P_i \backslash v) \cup P_j \cup C_r$ is a connected subgraph of J meeting Z and Z' and not meeting X. Since this holds for all choices of X, it follows by Menger's theorem that there are h disjoint ZZ'-paths in J; that is, G', $\mathscr{T}'$, Z' is a residual nodule of G, $\mathscr{T}$, Z, as required. ∎

5. Producing a Grid

Now we use (4.2) to deduce properties of tangles not dominating a large grid minor. We begin with the following lemma.

(5.1). *Let $\mathscr{T}$ be a tangle in G of order $\geqslant g \geqslant 2$, and for $1 \leqslant i, j \leqslant g$ let $W_{ij} \subseteq V(G)$ be such that*

(i) *the sets W_{ij} are mutually disjoint, and each is the vertex set of a connected subgraph of G*

(ii) *for all i, j, i', j' with $|i'-i| + |j'-j| = 1$ there is an edge of G joining W_{ij} and $W_{i'j'}$*

(iii) *for $1 \leqslant j \leqslant g$, $\mathrm{rk}_{\mathscr{T}}(W_{1j} \cup \cdots \cup W_{gj}) \geqslant g$.*

Let H be the corresponding $g \times g$ grid minor; then $\mathscr{T}$ dominates H.

Proof. Let $(A, B) \in \mathscr{T}_H$ (we recall that $\mathscr{T}_H$ was defined before (2.3)); we must show that $(A, B) \in \mathscr{T}$. For $1 \leqslant j \leqslant g$ let $W_j = W_{1j} \cup \cdots \cup W_{gj}$. Now $E(B)$ includes a row of H, say the ith row. Thus for $1 \leqslant j \leqslant g$, $W_{ij} \cap V(B) \neq \emptyset$, and so $W_j \cap V(B) \neq \emptyset$. Since $|V(A \cap B)| < g$ and the sets $W_1, ..., W_g$ are mutually disjoint, there exists j such that $W_j \cap V(A \cap B) = \emptyset$. Since the restriction of G to W_j is connected and $W_j \cap V(B) \neq \emptyset$, it follows that $W_j \subseteq V(B) - V(A)$. Since $\mathrm{rk}_{\mathscr{T}}(W_j) \geqslant g$ and $|V(A \cap B)| < g$, we deduce that $(B, A) \notin \mathscr{T}$, and so $(A, B) \in \mathscr{T}$, as required. ∎

(5.2). *If $\mathscr{T}$ is a tangle in G dominating no $g \times g$ grid minor, and $\mathscr{T}'$ is a tangle in a subgraph $G' \subseteq G$ which is conformal with $\mathscr{T}$, then $\mathscr{T}'$ dominates no $g \times g$ grid minor of G'.*

Proof. Let H be a $g \times g$ grid minor of G' and suppose that $\mathscr{T}'$ dominates H. Then H is a $g \times g$ grid minor of G, and we claim that $\mathscr{T}$

dominates H. For let (A, B) be a separation of G of order $<g$ such that $E(B)$ includes a row of H. Then $(A \cap G', B \cap G')$ is a separation of G' of order $<g$, and $E(B \cap G')$ includes a row of H. Since $\mathscr{T}'$ dominates H, it follows that $(A \cap G', B \cap G') \in \mathscr{T}'$, and hence $(A, B) \in \mathscr{T}$ since $\mathscr{T}'$ is conformal with $\mathscr{T}$. Thus $\mathscr{T}$ dominates H, a contradiction. ∎

(5.3). *Let g, k, n be integers with $2k \geqslant n \geqslant 2g-1 \geqslant 3$. Let $G, \mathscr{T}, Z$ be a nodule of order $\geqslant k20^{2g}$, such that $\mathscr{T}$ dominates no $g \times g$ grid minor and $|Z| = n$. Let $Z = \{z_1, ..., z_n\}$. Then there is a residual nodule $G', \mathscr{T}', Z'$ of order $\geqslant k$ and a linkage $\{P_1, ..., P_n\}$, where P_i has first vertex z_i $(1 \leqslant i \leqslant n)$, and a path Q of $G \backslash V(G')$, and there exist i, j with $2 \leqslant i \leqslant 2g-2$, $1 \leqslant j \leqslant n$, and $j \neq i-1, i, i+1$, such that Q has first vertex in P_i, last vertex in P_j, and no other vertex in $P_1 \cup \cdots \cup P_n$.*

Proof. Let $G_0 = G$, $\mathscr{T}_0 = \mathscr{T}$, $Z_0 = Z$, and $z_i^0 = z_i$ $(1 \leqslant i \leqslant n)$. By $2g-1$ applications of (4.2), we may define inductively for all r with $1 \leqslant r \leqslant 2g-1$:

(i) a nodule $G_r, \mathscr{T}_r, Z_r$ of order $\geqslant k20^{2g-r}$ which is a residual nodule of $G_{r-1}, \mathscr{T}_{r-1}, Z_{r-1}$ (and hence of $G, \mathscr{T}, Z$, by transitivity), and a linkage $\{P_1^r, ..., P_n^r\}$ for it from $G_{r-1}, \mathscr{T}_{r-1}, Z_{r-1}$

(ii) an ordering $\{z_1^r, ..., z_n^r\}$ of Z_r such that P_i^r has ends z_i^{r-1}, z_i^r for $1 \leqslant i \leqslant n$, and

(iii) a path Q_r of $G_{r-1} \backslash V(G_r)$ from z_g^{r-1} to z_1^{r-1}, such that for $1 \leqslant i \leqslant n$, $P_i^r \cap Q_r$ is null or a path.

For $1 \leqslant i \leqslant n$, let $P_i = P_i^1 \cup \cdots \cup P_i^{2g-1}$.

Case 1. *For at least g values of r with $1 \leqslant r \leqslant 2g-1$, Q_r meets $P_g^r, P_{g-1}^r, ..., P_1^r$ in order before meeting any other P_i^r.*

Let $1 \leqslant r_1 < r_2 < \cdots < r_g \leqslant 2g-1$ be such that for $r = r_1, ..., r_g$, Q_r meets $P_g^r, P_{g-1}^r, ..., P_1^r$ in order before meeting any other P_i^r. Let $r_0 = 0$, and for $1 \leqslant i, j \leqslant g$ let W_{ij} be the union of the following two sets:

(a) $V(P_i) \cap (V(G_{r_{j-1}}) - V(G_{r_j}))$

(b) all vertices of Q_{r_j} which lie in Q_{r_j} strictly between the intersections with P_{i-1} and P_i (or $\varnothing$, if $i = 1$).

We see that W_{ij} satisfy the hypothesis of (5.1), for if $1 \leqslant j \leqslant g$ then since $\mathscr{T}_r$ is conformal with $\mathscr{T}$ by (i), it follows from (2.7) (writing r for r_{j-1}) that

$$\mathrm{rk}_{\mathscr{T}}(W_{1j} \cup \cdots \cup W_{gj}) \geqslant \mathrm{rk}_{\mathscr{T}}\{z_1^r, ..., z_g^r\} \geqslant \mathrm{rk}_{\mathscr{T}_r}\{z_1^r, ..., z_g^r\} = g.$$

Thus, by (5.1), $\mathscr{T}$ dominates a $g \times g$ grid minor, a contradiction.

Case 2. *For at least g values of r with $1 \leqslant r \leqslant 2g-1$, Q_r meets P^r_g, $P^r_{g+1}, ..., P^r_{2g-1}$ in order, before meeting any other P^r_i.*

Again, $\mathscr{T}$ dominates a $g \times g$ grid minor; a contradiction.

Case 3. *Neither Case* 1 *nor Case* 2 *applies.*

Then there exists r with $1 \leqslant r \leqslant 2g-1$ such that for some i with $2 \leqslant i \leqslant 2g-2$, Q_r meets P^r_i and next meets P^r_j where $j \neq i-1$, i, $i+1$. Let Q be a minimal subpath of Q_r between P^r_i and P^r_j; then the theorem is satisfied. ▮

A *leaf* of a tree is a vertex of valency 1. Two disjoint subgraphs A, B of G are said to *touch* if there is a vertex of A adjacent in G to some vertex of B. Let G', $\mathscr{T}'$, Z' be a residual nodule of G, $\mathscr{T}$, Z, where $|Z| = m$. A graph S with $V(S) = Z$ is a *shadow* of G', $\mathscr{T}'$, Z' if there are m disjoint connected subgraphs $C_1, ..., C_m$ of $G \backslash (V(G') - Z')$ such that

(i) for $1 \leqslant i \leqslant m$, $|V(C_i) \cap Z| = 1 = |V(C_i) \cap Z'|$; let $V(C_i) \cap Z = \{z_i\}$

(ii) for $1 \leqslant i < j \leqslant m$, if z_i and z_j are adjacent in S then $C_i \backslash Z'$ and $C_j \backslash Z'$ touch.

(5.4). *Let g, m, k be integers with $g \geqslant 2$, $m \geqslant 3$, and $k \geqslant n/2$, where $n = (2g+1)(2m-5)+2$. Let G, $\mathscr{T}$, Z be a nodule of order $\geqslant k20^{2g(m-2)+n}$, such that $\mathscr{T}$ dominates no $g \times g$ grid minor and $|Z| = n$. Then there is a residual nodule G', $\mathscr{T}'$, Z' of order $\geqslant k$ with a shadow which is a tree with $\geqslant m$ leaves.*

Proof. First we show the following.

(1) *There is a residual nodule of G, $\mathscr{T}$, Z of order $\geqslant k20^{2g(m-2)}$ with a shadow which is a tree with $\geqslant 2$ leaves.*

Subproof. By (4.3) there is a residual nodule G', $\mathscr{T}'$, Z' of order $\geqslant k20^{2g(m-2)}$ such that Z is a subset of one component C of $G \backslash V(G')$. Let $\{P_1, ..., P_n\}$ be a linkage for G', $\mathscr{T}'$, Z' from G, $\mathscr{T}$, Z, where P_i has first vertex z_i $(1 \leqslant i \leqslant n)$. Let $J = C \cup P_1 \cup \cdots \cup P_n$, and for $1 \leqslant i \leqslant n$ let C_i be a connected subgraph of J with $P_i \subseteq C_i$, such that $C_1, ..., C_n$ are mutually vertex-disjoint and their union is maximal. Since J is connected, it follows that $V(C_1 \cup \cdots \cup C_n) = V(J)$. Let S_0 be the graph with $V(S_0) = Z$ in which z_i and z_j are adjacent if $i \neq j$ and $C_i \backslash Z'$, $C_j \backslash Z'$ touch. Since C is connected it follows that S_0 is connected and, hence, has a spanning tree S, which has $\geqslant 2$ leaves since $n \geqslant 2$. But S is a shadow, and so (1) holds.

We shall prove the following statement (2) for all r with $2 \leqslant r \leqslant m$, by induction on r.

(2) *There is a residual nodule of G, $\mathscr{T}$, of Z of order $\geqslant k20^{2g(m-r)}$ with a shadow which is a tree with $\geqslant r$ leaves.*

Subproof. (2) holds when $r=2$; suppose then that (2) holds for some r with $2 \leqslant r < m$; we shall prove that it holds for $r+1$. Let G', $\mathscr{T}'$, Z' be the residual nodule in (2), and let S be the shadow and $C_1, ..., C_n$ the corresponding disjoint connected subgraphs of $G \backslash (V(G') - Z')$. Let $C_i \cap Z = \{z_i\}$, $C_i \cap Z' = \{z'_i\}$ $(1 \leqslant i \leqslant n)$. Now we may assume that S has exactly r leaves, for otherwise the nodule G', $\mathscr{T}'$, Z' satisfies (2) with r replaced by $r+1$; and so S is a subdivision of a tree with $\leqslant 2r-3$ edges. Since

$$|E(S)| = n - 1 > (2g+1)(2m-5),$$

and $2r - 3 \leqslant 2m - 5$, it follows that one of these edges is subdivided at least $2g+1$ times. Thus we may choose the numbering of $C_1, ..., C_n$ (and hence of $z_1, ..., z_n$), such that $z_n, z_1, z_2, ..., z_{2g}$ all have valency 2 in S and such that for $0 \leqslant i \leqslant 2g-1$, z_i is adjacent in S to z_{i+1} (where z_0 means z_n). By (5.2), $\mathscr{T}'$ dominates no $g \times g$ grid minor of G', since $\mathscr{T}'$ is conformal with $\mathscr{T}$. By (5.3) applied to G', $\mathscr{T}'$, Z', there is a residual nodule G'', $\mathscr{T}''$, Z'' of G', $\mathscr{T}'$, Z' of order $\geqslant k 20^{2g(m-(r+1))}$ and a linkage $\{P'_1, ..., P'_n\}$ for it from G', $\mathscr{T}'$, Z', where P'_i has first vertex z'_i $(1 \leqslant i \leqslant n)$, and a path Q of $G' \backslash V(G'')$, such that Q has first vertex in some P'_i with $2 \leqslant i \leqslant 2g-2$, last vertex in some P'_j where $j \neq i-1$, i, $i+1$, and it has no other vertex in $P'_1 \cup \cdots \cup P'_n$.

Let S' be obtained from S by adding the edge between z_i and z_j. Then S' is a shadow of G'', $\mathscr{T}''$, Z'' (regarded as a residual nodule of G, $\mathscr{T}$, Z). But S' has a spanning tree with $\geqslant r+1$ leaves (delete either the edge between z_{i-2} and z_{i-1}, or that between z_{i+1} and z_{i+2}, depending which gives a tree). Thus (2) holds for $r+1$. In particular, (2) holds with $r=m$, as required. ▮

(5.5). *Let g, m, k be integers with $g \geqslant 2$, $m \geqslant 3$, and $k \geqslant n/2$, where $n = (2g+1)(2m-5)+2$. Let $t = 2g(m-2) + n(m+1)$, and let G, $\mathscr{T}$, Z be a nodule of order $\geqslant k 20^t$, such that $\mathscr{T}$ dominates no $g \times g$ grid minor and $|Z| = m$. Then there is a residual nodule G', $\mathscr{T}'$, Z' of order $\geqslant k$ with a linkage $\mathscr{P}$, and a connected subgraph C of $G \backslash V(G')$ which touches but does not intersect $P \backslash Z'$ for each $P \in \mathscr{P}$.*

Proof. By (4.4), there is a nodule G'', $\mathscr{T}''$, Y of order $\geqslant k 20^{2g(m-2)+n}$, such that G'' is a subgraph of $G \backslash Z$, and $|Y| = n$, and G'', $\mathscr{T}''$, Z'' is a residual nodule of G, $\mathscr{T}$, Z for every $Z'' \subseteq Y$ with $|Z''| = m$. By (5.2), $\mathscr{T}''$ dominates no $g \times g$ grid minor of G''. By (5.4) applied to G'', $\mathscr{T}''$, Y we deduce that it has a residual nodule G', $\mathscr{T}'$, Y' of order $\geqslant k$ with a linkage $\mathscr{P}'$, and there is a connected subgraph C of $G'' \backslash V(G')$ which touches but does not meet $P \backslash Y'$ for at least m members P of $\mathscr{P}'$. Choose $Z' \subseteq Y'$ with $|Z'| = m$ such that C touches but does not meet $P \backslash Y'$ for each $P \in \mathscr{P}'$ with last vertex in Z'. Let Z'' be the set of first vertices of the members of $\mathscr{P}'$ with last vertex in Z'. Since $Z'' \subseteq Y$ and $|Z''| = m$, it follows that G'', $\mathscr{T}''$, Z'' is a residual nodule of G, $\mathscr{T}$, Z; let $\mathscr{P}''$ be a corresponding linkage. Each member of $\mathscr{P}''$ has its last vertex in Z'' and no other vertex in $V(G'')$, while each member of $\mathscr{P}'$ is a path of G''. Let $\mathscr{P}$ be the set of paths formed by taking, for each $P \in \mathscr{P}''$, the union of P and the path of $\mathscr{P}'$ which meets P; then $\mathscr{P}$ is a linkage for G', $\mathscr{T}'$, Z' satisfying the theorem. ∎

Let m, $n \geqslant 1$ be integers. An *$m \times n$ array* in a graph G is a family of subsets W_{ij} $(0 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ of $V(G)$ such that

(i) the sets W_{ij} are mutually disjoint, and each is the vertex set of a connected subgraph of G

(ii) for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ there is an edge of G between W_{0j} and W_{ij}; and for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n-1$ there is an edge of G between W_{ij} and $W_{i,j+1}$.

If $\mathscr{T}$ is a tangle in G, we say $\mathscr{T}$ *dominates* this array if for $0 \leqslant i \leqslant m$,

$$\mathrm{rk}_{\mathscr{T}}(W_{i1} \cup \cdots \cup W_{in}) \geqslant \min(m, n).$$

(5.6). *Let $g \geqslant 2$, $m \geqslant 3$, $n \geqslant 1$ be integers, and let*

$$t = 2g(m-2) + ((2g+1)(2m-5)+2)(m+1).$$

If $\mathscr{T}$ is a tangle of order $\geqslant 20^{nt}$ in a graph G and $\mathscr{T}$ dominates no $g \times g$ grid minor, then $\mathscr{T}$ dominates an $m \times n$ array in G.

Proof. Let $\mathscr{T}$ be a tangle of order $\geqslant 20^{nt}$. By (2.4) we can choose $Z \subseteq V(G)$ with $|Z| = m$, free relative to $\mathscr{T}$. Thus G, $\mathscr{T}$, Z is a nodule, and $\mathscr{T}$ dominates no $g \times g$ grid minor. By $n-1$ applications of (5.2) and (5.5) we deduce that there are nodules G_r, $\mathscr{T}_r$, Z_r $(0 \leqslant r \leqslant n-1)$, where $G_0 = G$, $\mathscr{T}_0 = \mathscr{T}$, $Z_0 = Z$, such that for $1 \leqslant r \leqslant n-1$,

(i) G_r, $\mathscr{T}_r$, Z_r is a residual nodule of G_{r-1}, T_{r-1}, Z_{r-1}

(ii) $\mathscr{T}_r$ has order $\geqslant 20^{(n-r)t}$ and dominates no $g \times g$ grid minor of G_r

(iii) there is a linkage $\mathscr{P}_r$ for G_r, $\mathscr{T}_r$, Z_r from G_{r-1}, $\mathscr{T}_{r-1}$, Z_{r-1} and a connected subgraph C_r of $G_{r-1} \backslash V(G_r)$, such that for each $P \in \mathscr{P}_r$, $C_r \cap P$ is null and some vertex of C_r is adjacent to some vertex of $P \backslash Z_r$.

Now Z_{n-1} is free relative to $\mathscr{T}_{n-1}$, and since $|Z_{n-1}| = m < \mathrm{ord}(\mathscr{T}_{n-1})$ there is a component C_n of $G_{n-1} \backslash Z_{n-1}$ such that each vertex in Z_{n-1} has a neighbour in $V(C_n)$, by (2.12). If $n = 1$, let $P_1, \ldots, P_m$ be the m one-vertex paths with a vertex in Z_0. If $n \geqslant 2$, let $P_1, \ldots, P_m$ be the components of the union of all the members of $\mathscr{P}_1 \cup \cdots \cup \mathscr{P}_{n-1}$. For $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n-1$ let

$$W_{ij} = V(P_i) \cap (V(G_{j-1}) - V(G_j));$$

for $1 \leqslant i \leqslant m$ let $W_{in} = V(P_i) \cap V(G_{n-1})$; and for $1 \leqslant j \leqslant n$ let $W_{0j} = V(C_j)$. Then the sets W_{ij} form an $m \times n$ array, and we claim that $\mathscr{T}$ dominates it. For let $0 \leqslant i \leqslant m$ and suppose that there exists $(A, B) \in \mathscr{T}$ of order $< \min(m, n)$ with

$$W_{i1} \cup \cdots \cup W_{in} \subseteq V(A).$$

Since $|V(A \cap B)| < n$, there exists j with $1 \leqslant j \leqslant n$ such that $X \cap V(A \cap B) = \varnothing$, where

$$X = W_{0j} \cup W_{1j} \cup \cdots \cup W_{mj}.$$

Since the restriction of G to X is connected and since $X \cap V(A) \neq \varnothing$ because $\varnothing \neq W_{ij} \subseteq X$, it follows that $X \subseteq V(A)$. But $Z_{j-1} \subseteq X$, and so $Z_{j-1} \subseteq V(A)$, and hence $\mathrm{rk}_{\mathscr{T}}(Z_{j-1}) \leqslant |V(A \cap B)| < m$, contrary to (2.7). Hence there is no such (A, B), as required. ∎

Now we can prove our main result (2.3), which we restate.

(5.7). *Let $g \geqslant 2$, and let $\mathscr{T}$ be a tangle in G of order $\geqslant 20^{g^4(2g-1)}$. Then $\mathscr{T}$ dominates a $g \times g$ grid minor.*

Proof. Any tangle of order $\geqslant 3$ dominates a 2×2 grid, as can be shown for instance as follows. By [8, Theorem (8.3)] we may assume that G is two-connected. But $|V(G)| \geqslant 4$ since $\mathscr{T}$ has order $\geqslant 3$, and so there is a circuit of length $\geqslant 4$. Hence G has a 2×2 grid minor, and $\mathscr{T}$ dominates it since G is two-connected.

We may therefore assume that $g \geqslant 3$. Let $t = \lfloor \frac{1}{2} g \rfloor$; then, since

$$g^4(2g-1) \geqslant 2g(g-2)\,gt + ((2g+1)(2g-5)+2)(g+1)\,gt,$$

we may assume from (5.6) (taking $m = g \geqslant 3$ and $n = gt$) that $\mathscr{T}$ dominates a $g \times gt$ array W_{ij} $(0 \leqslant i \leqslant g,\ 1 \leqslant j \leqslant gt)$. For $1 \leqslant i \leqslant g$ and $1 \leqslant j \leqslant g$, let

$$U_{ij} = \bigcup (W_{ik} : (j-1)t < k \leqslant jt)$$

if i is odd, and let

$$U_{ij} = \bigcup (W_{ik} : (j-1)t < k \leqslant jt) \cup W_{0,\,(j-1)t+i/2}$$

if i is even. Then for all i, j, i', j' with $|i-i'|+|j-j'|=1$ there is an edge between U_{ij} and $U_{i'j'}$, and each U_{ij} is the vertex set of a connected sugraph of G. Moreover, for $1 \leqslant i \leqslant g$,

$$U_{i1} \cup \cdots \cup U_{ig} \supseteq W_{i1} \cup \cdots \cup W_{i,\, gt}$$

and so $\mathrm{rk}_{\mathscr{T}}(U_{i1} \cup \cdots \cup U_{ig}) \geqslant \min(g, gt) = g$, since $\mathscr{T}$ dominates the array. From (5.1) (with rows and columns exchanged) $\mathscr{T}$ dominates a $g \times g$ grid minor, as required. ▮

In some circumstances, the maximum size of a complete minor of G is much smaller than the maximum size of a grid minor, and if so we can obtain an improved bound on tree-width, as follows.

(5.8). *If G has no $g \times g$ grid minor and no K_h minor, then the tree-width of G is at most 20^{5gh^3}.*

Proof. We assume that $g \geqslant 2$ and $h \geqslant 3$, for otherwise the result is trivial. Suppose that W_{ij} $(0 \leqslant i \leqslant h,\ 1 \leqslant j \leqslant h)$ is an $h \times h$ array in G. For $1 \leqslant i \leqslant h$, let

$$V_i = W_{0i} \cup (W_{i1} \cup \cdots \cup W_{ih}).$$

Then $V_1, ..., V_h$ are disjoint, each is the vertex set of a connected subgraph of G, and for $1 \leqslant i < j \leqslant h$ there is an edge between V_i and V_j. Hence G has a K_h minor, a contradiction. It follows that G has no $h \times h$ array, and so from (5.6), no tangle of order $\geqslant 20^k$ where

$$k = 2gh(h-2) + ((2g+1)(2h-5)+2)\, h(h+1) \leqslant 5gh^3 - 1.$$

The result follows from (2.1). ▮

6. Grids in Planar Graphs

So far, we have been using tangles to obtain a numerically improved version of the theorem of [6], concerning the tree-width of general graphs with no $g \times g$ grid minor. One can also use tangles to improve the theorem of [5], concerning the tree-width of planar graphs with no $g \times g$ grid minor. Since the "$r \times r$ cylinder" of [5] has the $r \times r$ grid as a subgraph, it follows from [5, Theorem (5.2)] that

(6.1). *Let $g \geqslant 1$ be an integer. Every planar graph with no $g \times g$ grid minor has tree-width $< \frac{3}{2}(g^2+2g)-2$.*

We shall prove the following improvement.

(6.2). *Let $g \geqslant 1$ be an integer. Every planar graph with no $g \times g$ grid minor has tree-width $\leqslant 6g-5$.*

(6.2) follows from (2.1) and the following.

(6.3). *Let $g \geqslant 1$ be an integer. Every planar graph with a tangle of order $4g-3$ has a $g \times g$ grid minor.*

Proof. We may assume that $g \geqslant 2$ and by (2.11) it suffices to prove the result for connected graphs. Thus, let G be a connected planar graph, drawn in a sphere Σ, and let $\mathscr{T}$ be a tangle in G of order $4g-3$. Choose closed discs $\Delta_1, \Delta_2 \subseteq \Sigma$ with $\Delta_1 \cup \Delta_2 = \Sigma$ and $\Delta_1 \cap \Delta_2 = bd(\Delta_1) = bd(\Delta_2)$, such that

(i) $\Delta_1 \cap \Delta_2$ meets the drawing only in vertices,

(ii) $(G \cap \Delta_1, G \cap \Delta_2) \in \mathscr{T}$, where $G \cap \Delta_i$ denotes the subgraph of G drawn in Δ_i

(iii) subject to (i) and (ii), $|V(G \cap \Delta_1)| + |E(G \cap \Delta_1)| - |V(G \cap \Delta_2)|$ is maximum.

(Certainly this is possible, for if we choose Δ_1, Δ_2 with $G \cap \Delta_1$ null then (i) and (ii) are satisfied.) We refer to (iii) as the *optimality* of Δ_1. A *line* in Σ is a subset homeomorphic to $[0, 1]$ and its *ends* are defined in the obvious way.

(1) *Let $a, b \in \Delta_1 \cap \Delta_2$ be distinct, and let F_1, F_2 be the two lines in $\Delta_1 \cap \Delta_2$ with ends a, b. Let $F \subseteq \Delta_2$ be a line with ends a, b, meeting the drawing only in vertices, and with $F \cap \Delta_1 = \{a, b\}$. Then*

$$|F \cap V(G)| \geqslant \min(|F_1 \cap V(G)|, |F_2 \cap V(G)|).$$

Subproof. Let D_i be the closed disc in Δ_2 bounded by $F \cup F_i$ ($i = 1, 2$). Then $D_1 \cup \Delta_1$, $D_2 \cup \Delta_1$ are both closed discs. From the second axiom, $\mathscr{T}$ does not contain both

$$(G \cap D_1, G \cap (D_2 \cup \Delta_1)), \qquad (G \cap D_2, G \cap (D_1 \cup \Delta_1)),$$

because $(G \cap \Delta_1) \cup (G \cap D_1) \cup (G \cap D_2) = G$. Consequently we may assume that $(G \cap D_1, G \cap (D_2 \cup \Delta_1)) \notin \mathscr{T}$. If $(G \cap (D_2 \cup \Delta_1), G \cap D_1) \in \mathscr{T}$, then from the optimality of Δ_1 we deduce that $V(G \cap \Delta_2) = V(G \cap D_1)$, and in particular $F_2 \cap V(G) \subseteq F \cap V(G)$ and the result holds. If $(G \cap (D_2 \cup \Delta_1), G \cap D_1) \notin \mathscr{T}$, then from the first axiom it has order $\geqslant 4g-3$ and, hence, greater than the order of $(G \cap \Delta_1, G \cap \Delta_2)$; and so

$$\begin{aligned} 0 &< |V(G) \cap ((D_2 \cup \Delta_1) \cap D_1)| - |V(G) \cap (\Delta_1 \cap \Delta_2)| \\ &= |V(G) \cap (F_1 \cup F)| - |V(G) \cap (F_1 \cup F_2)| \\ &= |V(G) \cap F| - |V(G) \cap F_2|, \end{aligned}$$

as required.

(2) $|\Delta_1 \cap \Delta_2 \cap V(G)| = 4g-4$.

Subproof. Certainly $|\Delta_1 \cap \Delta_2 \cap V(G)| \leqslant 4g-4$, since $(G \cap \Delta_1, G \cap \Delta_2) \in \mathscr{T}$. Suppose that $|\Delta_1 \cap \Delta_2 \cap V(G)| \leqslant 4g-5$. Let r be a region of G with $\Delta_1 \cap \Delta_2 \cap r \neq \varnothing$ (regions are open sets). If r is incident with some vertex v of G not in Δ_1, then by (2.8) we can enlarge Δ_1 within $r \cup \{v\}$ so that its boundary passes through v (and reduce Δ_2 correspondingly), contrary to the optimality of Δ_1. Thus there is no such v. By a similar argument and (2.8), there is no edge of $G \cap \Delta_2$ incident with r with both ends in $\Delta_1 \cap \Delta_2$. It follows that $\Delta_2 \subseteq r \cup (\Delta_1 \cap \Delta_2)$, and hence $V(G \cap \Delta_1) = V(G)$, contrary to the third axiom. This proves (2).

Let the vertices of G in $\Delta_1 \cap \Delta_2$ be

$$a_1, a_2, ..., a_g = b_1, b_2, ..., b_g = c_1, c_2, ..., c_g = d_1, d_2, ..., d_g = a_1$$

in order. By (1) and (for example) [7, Theorem (3.6)], there are vertex-disjoint paths $P_1, ..., P_g$ of $G \cap \Delta_2$ such that P_i has ends a_i, c_{g+1-i} for $1 \leqslant i \leqslant g$. Similarly, there are vertex-disjoint paths $Q_1, ..., Q_g$ of $G \cap \Delta_2$ such that Q_i has ends b_i, d_{g+1-i} for $1 \leqslant i \leqslant g$. By a straightforward modification of the proof of [5, Theorem (4.1)], we may choose $P_1, ..., P_g$ and $Q_1, ..., Q_g$ so that $P_i \cap Q_j$ is a path for all i, j. But then G has a $g \times g$ grid minor, as required. ∎

As a corollary, we have

(6.4). *If G is planar, with n vertices, then G has no tangle of order $\geqslant 4\lfloor n^{1/2} \rfloor + 1$.*

Proof. Let $g = 1 + \lfloor n^{1/2} \rfloor$. Since $g^2 > n$, it follows that G has no $g \times g$ grid minor, and so by (6.3), G has no tangle of order $4g-3$. The result follows. ∎

Consequently:

(6.5). *If G is planar, with n vertices, there is a separation (A, B) of G of order $\leqslant 4n^{1/2}$ with $|V(A) - V(B)|, |V(B) - V(A)| \leqslant \frac{2}{3} |V(G)|$.*

Proof. Let $\mathscr{T}$ be the set of all separations (A, B) of G of order $\leqslant 4n^{1/2}$ with $|V(B) - V(A)| > \frac{2}{3}|V(G)|$. By (6.4), $\mathscr{T}$ is not a tangle, and yet the second and third axioms are satisfied (for if $(A, B) \in \mathscr{T}$ then $|V(A)| < \frac{1}{3}|V(G)|$). Hence the first is not satisfied, and the result follows. ∎

(6.5) is a form of a theorem of Lipton and Tarjan [3], who in fact proved the same statement with $4n^{1/2}$ replaced by $2(2n)^{1/2}$, using a completely different argument. The method given above was derived from [1] where the same method is used to show that (6.5) holds with $4n^{1/2}$ replaced by $3(2n)^{1/2}/2$.

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