## SYNTHESIS OF DATA WORD TRANSDUCERS

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ABSTRACT. In reactive synthesis, the goal is to automatically generate an implementation from a specification of the reactive and non-terminating input/output behaviours of a system. Specifications are usually modelled as logical formulae or automata over infinite sequences of signals ( $\omega$ -words), while implementations are represented as transducers. In the classical setting, the set of signals is assumed to be finite. In this paper, we consider data  $\omega$ -words instead, i.e., words over an infinite alphabet. In this context, we study specifications and implementations respectively given as automata and transducers extended with a finite set of registers. We consider different instances, depending on whether the specification is nondeterministic, universal or deterministic, and depending on whether the number of registers of the implementation is given or not.

In the unbounded setting, we show undecidability for both universal and nondeterministic specifications, while decidability is recovered in the deterministic case. In the bounded setting, undecidability still holds for nondeterministic specifications, but can be recovered by disallowing tests over input data. The generic technique we use to show the latter result allows us to reprove some known result, namely decidability of bounded synthesis for universal specifications.

LOGICAL METHODS IN COMPUTER SCIENCE

DOI:10.2168/LMCS-???

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Key words and phrases: Register Automata, Synthesis, Data words, Transducers.

This paper is the journal version of [EFR19]. ACM Classification: Theory of computation  $\rightarrow$  Logic and verification; Theory of computation  $\rightarrow$  Automata over infinite objects; Theory of computation  $\rightarrow$  Transducers.

Funded by a FRIA fellowship from the F.R.S.-FNRS..

Research associate of F.R.S.-FNRS. He is supported by the ARC Project Transform Fédération Wallonie-Bruxelles and the FNRS CDR J013116F and MIS F451019F projects.

Partly funded by the DeLTA project (ANR-16-CE40-0007).

## Introduction

Reactive synthesis is an active research domain whose goal is to design algorithmic methods able to automatically construct a reactive system from a specification of its admissible behaviours. Such systems are notoriously difficult to design correctly, and the main appealing idea of synthesis is to automatically generate systems that are correct by construction. Reactive systems are non-terminating systems that continuously interact with the environment in which they are executed, through input and output signals. At each time step, the system receives an input signal from a set In and produces an output signal from a set Out. An execution is then modelled as an infinite sequence alternating between input and output signals, i.e., an  $\omega$ -word in  $(\operatorname{In} \cdot \operatorname{Out})^{\omega}$ . Classically, the sets In and Out are assumed to be finite and reactive systems are modelled as (sequential) transducers. Transducers are simple finite-state machines with transitions of type  $States \times In \rightarrow States \times Out$ , which, at any state, can process any input signal and deterministically produce some output signal, while possibly moving, again deterministically, to a new state. A specification is then a language  $S \subseteq (\operatorname{In} \cdot \operatorname{Out})^{\omega}$  telling which are the acceptable behaviours of the system. It is also classically represented as an automaton, or as a logical formula then converted into an automaton. Some regular specifications may not be realisable by any transducer, and the realisability problem asks, given a regular specification S, whether there exists a transducer T whose behaviours satisfy S (i.e., are included in S). The synthesis problem asks to construct T if it exists.

A typical example of reactive system is that of a server granting requests from a finite set of clients C. Requests are represented as the set of input signals  $\mathsf{In} = \{(r,i) \mid i \in C\} \cup \{\mathsf{idle}\}\$  (client i requests the ressource) and grants by the set of output signals  $\mathsf{Out} = \{(g,i) \mid i \in C\} \cup \{\mathsf{idle}\}\$  (server grants client i's request). A typical constraint to be imposed on such a system is that every request is eventually granted, which can be represented by the LTL formula  $\bigwedge_{i \in C} G((r,i) \to F(g,i))$ . The latter specification is realisable for instance by the transducer which outputs (g,i) whenever it reads (r,i) and  $\mathsf{idle}$  whenever it reads  $\mathsf{idle}$ .

It is well-known that the realisability problem is decidable for  $\omega$ -regular specifications. It is ExpTime-complete when represented by parity automata [JL69, PR89, FJLW16]; and 2ExpTime-complete for LTL specifications [PR89]. Such positive results have triggered a recent and very active research interest in efficient symbolic methods and tools for reactive synthesis (see e.g. [BCJ18]). Extensions of this classical setting have been proposed to capture more realistic scenarios [BCJ18]. However, only a few works have considered infinite sets of input and output signals. In the previous example, the number of clients is assumed to be finite, and small. To the best of our knowledge, existing synthesis tools do not handle large alphabets, although it is more realistic to consider an unbounded (infinite) set of client identifiers, e.g.  $C = \mathbb{N}$ . The goal of this paper is to investigate how reactive synthesis can be extended to handle infinite sets of signals.

Data words are infinite sequences  $x_1x_2...$  of labelled data, i.e., pairs  $(\sigma, d)$  with  $\sigma$  a label from a finite alphabet and d is a data from a countably infinite alphabet  $\mathcal{D}$ . They can naturally model executions of reactive systems over an infinite set of signals. Among other models, register automata are one of the main extensions of automata recognising languages of data words [KF94, Seg06]. They can use a finite set of registers in which to store data that are read, and to compare the current data with the content of some of the registers (in this paper, we allow comparison of equality). Likewise, transducers can be extended to register transducers as a model of reactive systems over data words: a register

transducer is equipped with a set of registers, and when reading an input labelled data  $(\sigma, d)$ , it can test d for equality with the content of some of its registers, and depending on the result of this test, deterministically assign some of its registers to d and output a finite label  $\beta$  along with the content of one of its registers. Its executions are then data words alternating between input and output labelled data, and register automata can thus be used to represent specifications, as languages of such data words.

Contributions. We consider two classical semantics for register automata, nondeterministic and universal, both with a parity acceptance condition, which give two classes of register automata respectively denoted NRA and URA. We study the parity acceptance condition because it can express the other classical acceptance conditions; e.g., Büchi and co-Büchi can be expressed with a 2-colours parity condition. Since NRA are not closed under complement (already over finite data words), NRA and URA define incomparable classes of specifications. The request-grant specification, as defined above, can be generalised to an infinite number of clients, and it is then expressible by an URA [KMB18]: whenever a request is made by client i (labelled data (r,i)), universally trigger a run which stores i in some register and verifies that the labelled data (g,i) eventually occurs in the data word. In contrast, no NRA can define it. On the other hand, consider the specification  $S_0$ : "all input data but one are copied on the output, the missing one being replaced by some data which occurred before it", modelled as the set of data sequences  $d_1d_1d_2d_2\dots d_id_id_{i+1}d_{i+1}\dots$  for all  $i\geq 0$ and j < i (finite labels are irrelevant and not represented).  $S_0$  is not definable by any URA, as it would require to guess j, which can be arbitrarily smaller than i, but it is expressible by some NRA making this guess.

However, we show (unsurprisingly) that the realisability problem by register transducers of specifications defined by NRA is undecidable. The same negative result also holds for URA, solving an open question raised in [KMB18]. On the positive side, we show that decidability is recovered for deterministic (parity) register automata (DRA) in which the output is driven by the input (meaning that it is contained in some register). We call this class the DRA with input-driven outputs, denoted by DRA $_{ido}$ . One of the difficulties of register transducer synthesis is that the number of registers needed to realise the specification is, a priori, unbounded with regards to the number of registers of the specification. We show it is in fact not the case for DRA $_{ido}$ : any specification expressed as a DRA $_{ido}$  with r registers is realisable by a register transducer iff it is realisable by a transducer with r registers.

A way to obtain decidability is to fix a bound k and to target register transducers with at most k registers. This setting is called bounded synthesis in [KMB18], which establishes that bounded synthesis is decidable in 2ExpTime for URA. We show that unfortunately, bounded synthesis is still undecidable for NRA specifications (even when targetting implementations with a single register). To recover decidability for NRA, we disallow equality tests on the input data and add a syntactic requirement which entails that on any accepted word, each output data is the content of some register which has been assigned an input data occurring before. This defines a subclass of NRA that we call (input) test-free NRA (NRAff). NRAff can express how output data can be obtained from input data (by copying, moving or duplicating them), although they do not have the whole power of register automata on the input nor the output side. Note that the specification  $S_0$  given before is NRAff-definable. To show that bounded synthesis is decidable for NRAff, we establish a generic transfer property characterising realisable data word specifications in terms of

realisability of corresponding specifications over a finite alphabet, thus reducing to the well-known synthesis problem over a finite alphabet. Such property also allows us to reprove the result of [KMB18], with a rather short proof based on standard results from the theory of register automata, indicating that it might allow to establish decidability for other classes of data specifications. Our results are summarised in Table 1.

	DRA <sub>ido</sub>	NRA	URA	$NRA_{tf}$
Bounded	2ExpTime	Undecidable $(k \ge 1)$	2Exp $T$ ime	2ExpTime
Synthesis	(Thm. 4.6)	(Thm. 3.2)	([KMB18] and Thm. 4.6)	(Thm. 4.11)
General	ExpTime-c	Undecidable	Undecidable	Open
Case	(Thm. 3.7)	(Thm. 3.1)	(Thm. 3.3)	Open

Table 1. Decidability status of the problems studied. As observed in Corollary 3.8, the bounded synthesis for DRA<sub>ido</sub> is in ExpTime if the target number of registers is larger than or equal to the number of registers of the specification.

Related Work. As already mentioned, bounded synthesis of register transducers is considered in [KMB18] where it is shown to be decidable for URA. We reprove this result in a shorter way. Our proof bears some similarities with that of [KMB18], but it seems that our formulation benefits more from the use of existing results. The technique is also more generic and we instantiate it to NRA<sub>tf</sub>. NRA<sub>tf</sub> correspond to the one-way, nondeterministic version of the expressive transducer model of [DH16], which however does not consider the synthesis problem.

The synthesis problem over infinite alphabets is also considered in [ESK14], in which data represent identifiers and specifications (given as particular automata close to register automata) can depend on equality between identifiers. However, the class of implementations is very expressive: it allows for unbounded memory through a queue data structure. The synthesis problem is shown to be undecidable and a sound but incomplete algorithm is given.

Finally, classical reactive synthesis has strong connections with game theory on finite graphs. Some extension of games to infinite graphs whose vertices are valuations of variables in an infinite data domain have been considered in [FP18]. Such games are shown to be undecidable and a decidable restriction is proposed, which however does not seem to match our context.

### 1. Data Words and Register Automata

For a (possibly infinite) set S, we denote by  $S^{\omega}$  the set of infinite words over this alphabet. For  $1 \leq i \leq j$ , we let  $u[i:j] = u_i u_{i+1} \dots u_j$  and u[i] = u[i:i] the *i*th letter of u. For  $u, v \in S^{\omega}$ , we define their *interleaving*  $\langle u, v \rangle = u[1]v[1]u[2]v[2]\dots$  Data Words. Let  $\Sigma$  be a finite alphabet and  $\mathcal{D}$  a countably infinite set, denoting, all over this paper, a set of elements called data. We also distinguish an (arbitrary) data value  $\mathbf{d}_0 \in \mathcal{D}$ . Given a set R, let  $\tau_0^R$  be the constant function defined by  $\tau_0^R(r) = \mathbf{d}_0$  for all  $r \in R$ . A labelled data (or l-data for short) is a pair  $x = (\sigma, d) \in \Sigma \times \mathcal{D}$ , where  $\sigma$  is the label and d the data. We define the projections  $\mathsf{lab}(x) = \sigma$  and  $\mathsf{dt}(x) = d$ . A data word over  $\Sigma$  and  $\mathcal{D}$  is an infinite sequence of labelled data, i.e. a word  $w \in (\Sigma \times \mathcal{D})^\omega$ . We extend the projections  $\mathsf{lab}$  and  $\mathsf{dt}$  to data words naturally, i.e.  $\mathsf{lab}(w) \in \Sigma^\omega$  and  $\mathsf{dt}(w) \in \mathcal{D}^\omega$ . We denote the set of data words over  $\Sigma$  and  $\mathcal{D}$  by  $\mathsf{DW}(\Sigma, \mathcal{D})$  ( $\mathsf{DW}$  when clear from the context). A data word language is a subset  $L \subseteq \mathsf{DW}(\Sigma, \mathcal{D})$ . Note that in this paper, data words are infinite, otherwise they are called finite data words, and we denote by  $\mathsf{DW}_f(\Sigma, \mathcal{D})$  the set of finite data words.

Register Automata. Register automata are automata recognising data word languages. They were first introduced in [KF94] as finite-memory automata. Here, we define them in a spirit close to [LTV15], but over infinite words, with a parity acceptance condition. The current data can be compared for equality with the register contents via tests. Our tests are symbolic and defined via Boolean formulas of the following form. Given R a set of registers, a test is a formula  $\phi$  satisfying the following syntax:

$$\phi ::= \top \mid \bot \mid r^{=} \mid r^{\neq} \mid \phi \land \phi \mid \phi \lor \phi \mid \neg \phi$$

where  $r \in R$ . Given a valuation  $\tau : R \to \mathcal{D}$ , a test  $\phi$  and a data d, we denote by  $\tau, d \models \phi$  the satisfiability of  $\phi$  by d in valuation  $\tau$ , defined as  $\tau, d \models r^=$  if  $\tau(r) = d$  and  $\tau, d \models r^{\neq}$  if  $\tau(r) \neq d$ . The Boolean combinators behave as usual. We denote by  $\mathsf{Tst}_R$  the set of (symbolic) tests over R.

**Definition 1.1.** A register automaton (RA) is a tuple  $\mathcal{A} = (\Sigma, \mathcal{D}, Q, q_0, \delta, R, c)$ , where:

- $\Sigma$  is a finite alphabet of labels,  $\mathcal{D}$  is an infinite alphabet of data
- Q is a finite set of states and  $q_0 \in Q$  is the initial state
- R is a finite set of registers. We denote  $\mathsf{Asgn}_R = 2^R$ .
- $c: Q \to \{1, \ldots, d\}$ , where  $d \in \mathbb{N}$  is the number of priorities, is the colouring function, used to define the acceptance condition
- $\bullet \ \delta \subseteq Q \times \Sigma \times \mathsf{Tst}_R \times \mathsf{Asgn}_R \times Q \text{ is a set of transitions.}$

A transition  $(q, \sigma, \phi, \mathsf{asgn}, q')$  is also written  $q \xrightarrow{\sigma, \phi, \mathsf{asgn}} q'$ . We may omit  $\mathcal{A}$  in the latter notation. Intuitively such transition means that on input  $(\sigma, d)$  in state q the automaton:

- (1) checks that  $\phi$  is satisfied by the current register contents and the current data
- (2) assigns d to all the registers in asqn (asqn might be empty)
- (3) transitions to state q'.

 $\mathcal{A}$  is said to be deterministic if the tests are mutually exclusive, i.e., for any two distinct transitions of the form  $q \xrightarrow{\sigma,\phi,\mathsf{asgn}} q'$  and  $q \xrightarrow{\sigma',\phi',\mathsf{asgn}'} q''$ , then either  $\sigma \neq \sigma'$  or  $\phi \land \phi'$  is not satisfiable. The automaton  $\mathcal{A}$  is said to be *complete* if for any given state q, any label  $\sigma$ , any data d and any register valuation  $\tau$ , there exists a transition  $q \xrightarrow{\sigma,\phi,\mathsf{asgn}} q' \in \delta$  such that  $\tau,d \models \phi$ .

Configurations and Runs. A configuration is a pair  $(q,\tau) \in Q \times (R \to \mathcal{D})$ . Given a transition  $t = p \xrightarrow{\sigma,\phi,\mathsf{asgn}} p'$ , we say that  $(q,\tau)$  enables t on reading  $(\sigma',d)$  if  $q = p, \sigma' = \sigma$  and  $\tau,d \models \phi$ . Let  $\mathsf{next}(\tau,\mathsf{asgn},d)$  be the configuration  $\tau'$  defined by  $\tau'(i) = d$  if  $i \in \mathsf{asgn}$ , and  $\tau'(i) = \tau(i)$  otherwise. We extend this notation to configurations as follows: if  $\gamma = (q,\tau)$  enables t on input  $(\sigma,d)$ , the  $\mathsf{successor}$  configuration of  $(q,\tau)$  by t on input  $(\sigma,d)$  is  $\mathsf{next}(\gamma,\mathsf{asgn},d,t) = (p',\mathsf{next}(\tau,\mathsf{asgn},d))$ . We also write  $\mathsf{next}(\gamma,t,\sigma,d)$  to denote the successor of  $(q,\tau)$  by transition t when  $(q,\tau)$  enables t on input  $(\sigma,d)$ . The  $\mathsf{initial}$  configuration is  $(q_0,\tau_0^R)$ . Then, a  $\mathsf{run}$  over a data word  $(\sigma_1,d_1)(\sigma_2,d_2)\ldots$  is an infinite sequence of transitions  $t_0t_1\ldots$  such that there exists a sequence of configurations  $\gamma_0\gamma_1\cdots=(q_0,\tau_0)(q_1,\tau_1)\ldots$  such that  $\gamma_0$  is initial and for all  $t \geq 0$ ,  $\gamma_{t+1} = \mathsf{next}(\gamma_t,t_t,\sigma_t,d_t)$ . With a run  $\rho$ , we associate its sequence of states  $\mathsf{states}(\rho) = q_0q_1\ldots$ 

Languages Defined by RA. Given a run  $\rho$ , we denote, by a slight abuse of notation,  $c(\rho) = \max\{j \mid c(q_l) = j \text{ for infinitely many } q_l \in \text{states}(\rho)\}$  the maximum color that occurs infinitely often in  $\rho$ . Then, in the parity acceptance condition,  $\rho$  is accepting whenever  $c(\rho)$  is even. We consider two dual semantics for RA: nondeterministic (N) and universal (U). Given a RA A, depending on whether it is considered nondeterministic or universal, it recognises  $L_N(A) = \{w \mid \text{there exists an accepting run } \rho \text{ on } w\}$  or  $L_U(A) = \{w \mid \text{all runs } \rho \text{ on } w \text{ are accepting}\}$ . Note that those semantics are dual: for a RA A, by letting  $\overline{A}$  be a copy of A with colouring function  $\overline{c}: q \mapsto c(q) + 1$ , we have that  $L_U(\overline{A}) = \overline{L_N(A)}$ .

We denote by NRA (resp. URA) the class of register automata interpreted with a nondeterministic (resp. universal) parity acceptance condition, and given  $A \in \mathsf{NRA}$  (resp.  $A \in \mathsf{URA}$ ), we write L(A) instead of  $L_N(A)$  (resp.  $L_U(A)$ ). We also denote by DRA the class of deterministic parity register automata.

## 2. Synthesis of Register Transducers

Specifications, Implementations and the Realisability Problem. Let  $\Sigma_i$  and  $\Sigma_o$  be two finite alphabets of labels, and  $\mathcal{D}$  a countable set of data. A relational data word is an element of  $w \in [(\Sigma_i \times \mathcal{D}) \cdot (\Sigma_o \times \mathcal{D})]^\omega$ . Such a word is called relational as it defines a pair of data words in  $\mathsf{DW}(\Sigma_i, \mathcal{D}) \times \mathsf{DW}(\Sigma_o, \mathcal{D})$  through the following projections. If  $w = x_1^1 x_0^1 x_1^2 x_0^2 \dots$ , we let  $\mathsf{inp}(w) = x_1^1 x_1^2 \dots$  and  $\mathsf{out}(w) = x_0^1 x_0^2 \dots$  We denote by  $\mathsf{RW}(\Sigma_i, \Sigma_o, \mathcal{D})$  (just  $\mathsf{RW}$  when clear from the context) the set of relational data words. A specification is simply a language  $S \subseteq \mathsf{RW}(\Sigma_i, \Sigma_o, \mathcal{D})$ . An implementation is a total function  $I : (\Sigma_i \times \mathcal{D})^* \to \Sigma_o \times \mathcal{D}$ . From I, we define another function  $f_I : \mathsf{DW}(\Sigma_i, \mathcal{D}) \to \mathsf{DW}(\Sigma_o, \mathcal{D})$  which, with an input data word  $w_i = x_i^1 x_i^2 \dots \in \Sigma_i \times \mathcal{D}$ , associates the output data word  $f_I(w_i) = x_o^1 x_o^2 \dots$  such that  $\forall i \geq 1$ ,  $x_o^i = I(x_i^1 \dots x_i^{i-1})$ . I also defines a language of relational data words  $L(I) = \{\langle w_i, f_I(w_i) \rangle \mid w_i \in \mathsf{DW}(\Sigma_i, \mathcal{D})\}$ .

We say that I realises S when  $L(I) \subseteq S$ , and that S is realisable if there exists an implementation realising it. Note that since  $f_I$  is a total function, we have that if S is realisable, then in particular its domain is total, i.e. for all  $w_i \in \mathsf{DW}(\Sigma_i, \mathcal{D})$ , there exists  $w_0 \in \mathsf{DW}(\Sigma_0, \mathcal{D})$  such that  $\langle w_i, w_0 \rangle \in S$ . Therefore, any specification whose domain is not total is not realisable according to this definition. For a discussion on this definition, see Section 5

The realisability problem consists, given a (finite representation of a) specification S, in checking whether S is realisable. In general, we parameterise this problem by classes

of specifications  $\mathcal{S}$  and of implementations  $\mathcal{I}$ , defining the  $(\mathcal{S}, \mathcal{I})$ -realisability problem, denoted Real $(\mathcal{S}, \mathcal{I})$ . Given a specification  $S \in \mathcal{S}$ , it asks whether S is realisable by some implementation  $I \in \mathcal{I}$ . We now introduce the classes  $\mathcal{S}$  and  $\mathcal{I}$  we consider.

Specification Register Automata. In this paper, we consider specifications defined by register automata (hence alternately reading input and output labelled data). We assume that the set of states is partitioned into  $Q_i$  (called input states, reading only labels in  $\Sigma_i$ ) and  $Q_{\Phi}$  (called output states, reading only labels in  $\Sigma_{\Phi}$ ), where  $q_0 \in Q_i$ , and such that the transition relation  $\delta$  alternates between these two sets, i.e.  $\delta \subseteq \bigcup_{\alpha=i,\Phi} (Q_{\alpha} \times \Sigma_{\alpha} \times \mathsf{Tst}_{R} \times \mathsf{Asgn}_{R} \times Q_{\overline{\alpha}})$ , where  $\overline{i} = \Phi$  (resp.  $\overline{\Phi} = i$ ). We denote by DRA (resp. NRA, URA) the class of specifications defined by deterministic (resp. nondeterministic, universal) parity register automata.

**Example 2.1.** Remember the setting described in the introduction of a server granting requests from an unbounded set of clients C. The input (resp. output) finite alphabets are  $\Sigma_i = \{\text{req}, \text{idle}\}\$ and  $\Sigma_o = \{\text{grt}, \text{idle}\}\$ , while the set of data is any countably infinite set  $\mathcal{D}$  containing C. Without loss of generality,  $C \subseteq \mathbb{N}$  is a set of client ids, so we can take  $\mathcal{D} = \mathbb{N}$ . Then, as stated in the introduction, the specification that for all  $i \in C$ , every request of client i is eventually granted can be expressed with the URA of Figure 1.

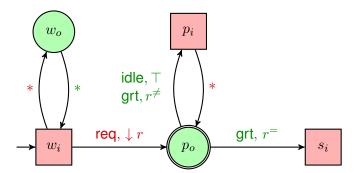


FIGURE 1. A universal register automaton checking that every request is eventually granted. Input is in red (states are squares), output is in green (states are circles). Finite labels are sans serif. All states have priority 0, except the doubly circled state  $p_o$ , which has priority 1. This corresponds to a co-Büchi acceptance condition with rejecting state  $p_o$ . The automaton always loops between  $w_i$  and  $w_o$  (the \* symbol means that the transition is taken, no matter the labelled-data received). Whenever it receives a request as input, it universally spawns a run which stores the corresponding id in its single register r (depicted as  $\downarrow r$ ), and transitions to  $p_0$ . Then, it loops between  $p_i$  and  $p_o$  while it does not receive the corresponding grant, with matching id, as output (i.e. either reads idle or receives a grant with wrong id:  $d \neq r$ ). When it receives a grant with right id (d = r), it transitions to  $s_i$ , then the run dies at the next step (which favors acceptance in the universal semantics).

Register Transducers As Implementations. We consider implementations represented as transducers processing data words. A register transducer is a tuple  $T = (\Sigma_i, \Sigma_o, Q, q_0, \delta, R)$  where Q is a finite set of states with initial state  $q_0$ , R is a finite set of registers, and  $\delta: Q \times \Sigma_i \times \mathsf{Tst}_R \to \mathsf{Asgn}_R \times \Sigma_o \times R \times Q$  is the transition function (as before,  $\mathsf{Asgn}_R = 2^R$ ), assumed to be complete in the sense that, as for RA, for all state q and label  $\sigma_i$ , all data d and register valuation  $\tau$ , there exists a transition  $\delta(q, \sigma_i, \phi) = (\mathsf{asgn}, \sigma_o, r, q')$  such that  $\tau, d \models \phi$ . When processing an l-data  $(\sigma_i, d)$ , T compares d with the content of some of its registers, and depending on the result, moves to another state, stores d in some registers, and outputs some label in  $\Sigma_{\Phi}$  along with the content of some register  $r \in R$ .

Let us formally define the semantics of a register transducer T, as an implementation  $I_T$ . First, for a finite input data word  $w=(\sigma_{\bf i}^1,d_{\bf i}^1)\dots(\sigma_{\bf i}^n,d_{\bf i}^n)$  in  $(\Sigma_{\bf i}\times \mathcal D)^*$ , we denote by  $(q_i,\tau_i)$  the ith configuration reached by T on w, where  $(q_0,\tau_0)$  is initial and for all  $0< i< n, (q_i,\tau_i)$  is the unique configuration such that there exists a transition  $\delta(q_{i-1},\sigma_{\bf i}^i,\phi)=(\mathbf{asgn},\sigma_{\bf o},r,q_i)$  such that  $\tau_{i-1},d_{\bf i}^i\models\phi$  and  $\tau_i=\mathrm{next}(\tau_{i-1},d_{\bf i}^i,\mathbf{asgn})$ . We let  $(\sigma_{\bf o}^i,d_{\bf o}^i)=(\sigma_{\bf o},\tau_i(r))$  and  $I_T(w)=(\sigma_{\bf o}^n,d_{\bf o}^n)$ . Then, we denote  $f_T=f_{I_T}$  and  $L(T)=L(I_T)$ . Note that if T is interpreted as a DRA with exactly one transition per output state and whose states are all accepting (i.e. have even maximal parity 0), then  $L(I_T)$  is indeed the language of such register automaton. We denote by  $\mathsf{RT}[k]$  the class of implementations defined by register transducers with at most k registers, and by  $\mathsf{RT}=\bigcup_{k\geq 0}\mathsf{RT}[k]$  the class of implementations defined by register transducers.

**Example 2.2.** Consider again the specification of Example 2.1. Such specification is realisable for instance by the transducer which outputs (grt, i) whenever it reads (req, i) and (idle, d) (d does not matter) whenever it reads idle, which is depicted in Figure 2.

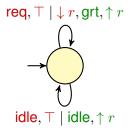


FIGURE 2. A register transducer immediately granting each request. The notations are the same as in Figure 1. Additionally, here,  $\uparrow r$  means that the transducer outputs the content of r.

Synthesis from Data-Free Specifications. If in the latter definitions of the problem, one considers specifications defined by RA with no registers (i.e. Mealy machines), and implementations defined by RT with no registers, then the data in data-words can be ignored and we are back to the classical reactive synthesis setting, for which important results are known:

**Theorem 2.3.** [JL69] Given a (data-free) specification S defined by some (register-free) nondeterministic parity automaton, the realisability problem of S by (register-free) transducers is ExpTime-complete.

*Proof.* Easiness was first established in [JL69] and [PR89]. Hardness is folklore, but a proof in the particular case of finite words (easily adapted to the  $\omega$ -word setting) can be found in [FJLW16, Proposition 6].

#### 3. Unbounded Synthesis

In this section, we consider the unbounded synthesis problem Real(RA, RT). Thus, we do not fix a priori the number of registers of the implementation.

3.1. Undecidability Results. Let us first consider the case of NRA and URA, which are, in our setting, the most natural devices to express data word specifications. Unfortunately, the two corresponding problems happen to be undecidable:

**Theorem 3.1.** Real(NRA, RT) is undecidable.

*Proof.* We reduce the problem from the universality of NRA over finite words, which is undecidable [NSV04]. Let A be a (finite data-word) NRA. Let S be a specification which first reads some finite data word w, then a separator # (its associated data is arbitrary and not represented), then allows for swapping the first and second l-data on any input read later on, while also allowing to behave like the identity whenever  $w \in L(A)$ . S is also equal to the identity over any word not containing # so that its domain is total. Formally, let  $S = S_1 \cup S_2 \cup T$ , where:

$$S_{1} = \{(w\#(\sigma_{1},d_{1})(\sigma_{2},d_{2})u,w\#(\sigma_{2},d_{1})(\sigma_{1},d_{2})u) \mid d_{1},d_{2} \in \mathcal{D}, \sigma_{1},\sigma_{2} \in \Sigma, w \in \mathsf{DW}_{f}, u \in \mathsf{DW}\}$$

$$S_{2} = \{(w\#u,w\#u) \mid w \in L(A), u \in \mathsf{DW}\}$$

$$T = \{(w,w) \mid w \notin \mathsf{DW}_{f}\#\mathsf{DW}\}$$

S is definable by a NRA running over relational data words, because each component is and NRA are closed under union. Recognising the interversion of the first two labels  $\sigma_1$  and  $\sigma_2$  after the # in  $S_1$  is easily done using nondeterminism, and the behaviour on data is the identity, so  $S_1$  is NRA-definable. Then, emulating the identity over some NRA-definable domain is easy, so  $S_2$  and T are also NRA-definable.

Now, if A is universal, ie  $L(A) = \mathsf{DW}_f$ , then the identity  $\mathsf{id}_{\mathsf{DW}}$  over  $\mathsf{DW}$  realises S, since then  $\mathsf{id}_{\mathsf{DW}} \subseteq S$  and has total domain. Conversely, if  $L(A) \subseteq \mathsf{DW}_f$ , assume by contradiction that S is realisable by a register transducer I. Let  $w \in \mathsf{DW}_f \backslash L(A)$ . Then, for any  $(\sigma_1, d_1)(\sigma_2, d_2)u \in \mathsf{DW}$ , we must have  $I(w\#(\sigma_1, d_1)(\sigma_2, d_2)u) = w\#(\sigma_2, d_1)(\sigma_1, d_2)u$ ; but this implies guessing the second label while having only read the first one, which is not doable by any transducer as soon as  $\sigma_1 \neq \sigma_2$ .

Actually, we can observe that such undecidability proof extends to REAL(NRA, RT[1]), and thus to all REAL(NRA, RT[k]) for  $k \ge 1$ . Indeed, A is universal iff S is realisable by the identity over data words, which is implementable using a 1-register transducer:

**Theorem 3.2.** For all  $k \geq 1$ , REAL(NRA, RT[k]) is undecidable.

Now, we can show that the unbounded synthesis problem is also undecidable for URA, answering a question left open in [KMB18].

**Theorem 3.3.** Real(URA, RT) is undecidable.

*Proof.* We present a reduction to our synthesis problem from the emptiness problem of URA over finite words. The latter is undecidable by a direct reduction from the universality problem of NRA, which is undecidable by [NSV04].

First, consider the relation  $S_1 = \{(u \# v, u \# w) \mid u \in \mathsf{DW}_f, v \in \mathsf{DW}, \text{ each data of } u$  appears infinitely often in  $w\}$ .  $S_1$  is recognised by a 1-register URA which, upon reading a data d in u, stores it in its register and checks that it appears infinitely often in w by visiting a state with maximal parity 2 every time it sees d (all other states have parity 1). Note that for all  $k \geq 1$ ,  $S_1 \cap \{(u \# v, u \# w) \mid u \in \mathsf{DW}_f, v, w \in \mathsf{DW} \text{ and } |\mathsf{dt}(u)| \leq k\}$  is realisable by a k-register transducer: on reading u, store each distinct data in one register, and after the # outputs them in turn in a round-robin fashion. However,  $S_1$  is not realisable: on reading the # separator, any implementation must have all the data of  $\mathsf{dt}(u)$  in its registers, but the size of  $\mathsf{dt}(u)$  is not bounded (u can have pairwise distinct data and be of arbitrary length).

Then, let A be a URA over finite data words. Consider the specification  $S = S_1 \cup S_2 \cup T$ , where  $S_2 = \{(u \# v, u \# w \# (a, \mathsf{d}_0)^\omega) \mid u \in \mathsf{DW}_f, v \in \mathsf{DW}, w \in L(A)\}$  and  $T = \{(u, w) \mid u \notin \mathsf{DW}_f \# \mathsf{DW}, w \in \mathsf{DW}\}$ . S has total domain, and is recognisable by a URA. Indeed, URA are closed under union, by the same product construction as for the intersection of NRA [KF94], and each part is URA-recognisable:  $S_1$  is, as described above,  $S_2$  is by simulating A on the output to check  $w \in L(A)$  then looping over  $(a, \mathsf{d}_0)$ , and T simply checks a regular property.

Now, if  $L(A) \neq \emptyset$ , let  $w \in L(A)$ , and k its number of distinct data. Then S is realisable by a k-register transducer realising  $S_1$  when the number of data in u is lower than or equal to k, and, when it is greater than k, by outputting  $u\#\widehat{w}\#(a,\mathbf{d}_0)^\omega$  where  $\widehat{w}$  is a membership-preserving renaming of w using k distinct data of u (this can always be done thanks to the so-called "indistinguishability property" stated in [KF94]). Conversely, if  $L(A) = \emptyset$ , then S is not realisable. If it were,  $S \cap \mathsf{DW}_f \# \mathsf{DW} = S_1$  would be too, as a regular domain restriction, but we have seen above that this is not the case. Thus, S is realisable iff  $L(A) = \emptyset$ .

3.2. A Decidable Subclass: DRA<sub>ido</sub>. However, we show that restricting to DRA allows to recover decidability, modulo one additional assumption, namely that the output data of a transition has to be the content of some register. We formally define this class as follows:

**Definition 3.4** (DRA<sub>ido</sub>). Let  $\mathcal{A} = (\Sigma, \mathcal{D}, Q, q_0, \delta, R, c)$  be a DRA. We say that  $\mathcal{A}$  is with input-driven outputs if for any output transition  $p \xrightarrow{\sigma, \phi, \mathsf{asgn}} q$ , the test  $\phi$  is of the form  $r^=$  for some  $r \in R$ . We denote by DRA<sub>ido</sub> the class of DRA with input-driven outputs.

Such assumption rules out pathological, and to our opinion uninteresting and technical cases stemming from the asymmetry between the class of specifications and implementations. E.g., consider the single-register DRA in Fig. 3a (finite labels are arbitrary and not depicted). It starts by reading one input data d and stores it in r, asks that the corresponding output data is different from the content d of r, then accepts any output over any input (transitions  $\top$  are always takeable). It is not realisable because transducers necessarily output the content of some register (hence producing a data which already appeared). On the other hand, having tests of the form  $\phi = r^{\neq}$  for instance does not imply unrealisability, as shown by the DRA of Fig. 3b: it starts by reading one data  $d_1$ , asks to copy it on the output, then reads another data  $d_2$ , and requires that the output is either distinct from  $d_1$  or equal to it, depending on whether  $d_2 \neq d_1$ . It happens that such specification is realisable by the identity.

We reduce the realisability of DRA<sub>ido</sub>-specifications to solving a finite parity game.

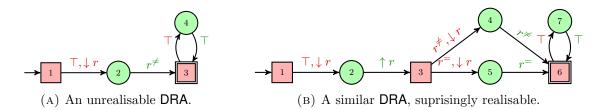


FIGURE 3. Pathological **DRA** specifications.

For any RA  $A = (\Sigma_i, \Sigma_o, \mathcal{D}, Q, q_0, \delta, R, c)$ , we define the finite parity automaton  $\mathsf{fin}(A) = (Q, q_0, \delta', c)$  over the alphabet  $\delta$  where  $t = (q, \sigma, \phi, \mathsf{asgn}, q') \in \delta$  iff  $(q, t, q') \in \delta'$ . Note that if A is deterministic, so is  $\mathsf{fin}(A)$ . A RA A is said to be trim if for all sequences of transitions  $\rho = t_1 t_2 \cdots \in L(\mathsf{fin}(A))$ , there exists a data word  $w \in L(A)$  such that  $\rho$  is an accepting run of A on w.

We say that a RA is in good form if

- (1) it is trim
- (2) it is complete on its input states
- (3) its tests  $\phi$  are maximally consistent conjunctions of atoms
- (4) any transition t whose test is different from  $\bigwedge_{r \in R} r^{\neq}$  does not conduct an assignment  $(\mathsf{asgn} = \varnothing)$

**Lemma 3.5.** For all RA A, there exists an equivalent RA A' in good form with exponentially many more states and transitions, and the same number of priorities and registers. Moreover if A is a DRA<sub>ido</sub>, so is A'.

*Proof.* Let  $A = (\Sigma, \mathcal{D}, Q, q_0, \delta, R, c)$  be a RA. In a first step, we enrich the states with information on the equalities between registers in the current register valuation.

Formally, we define constraints as equivalence relations on R. In the following, we denote by  $\mathsf{ER}(R)$  the set of equivalence relations on R. Given a valuation  $\tau$  of registers in R, we can associate to it an equivalence relation on R in the natural way (two registers  $r, r' \in R$  are equivalent iff  $\tau(r) = \tau(r')$ ). We denote it by  $[\tau]$ . We use the letter C to denote an element of  $\mathsf{ER}(R)$ , and we call it a constraint.

We let  $A' = (\Sigma, \mathcal{D}, Q', q'_0, \delta', R, c')$  be defined as follows:

- $Q' = Q \times \mathsf{ER}(R) \uplus \{s, s'\}$ , where s, s' are fresh states
- $q_0' = (q_0, [\tau_0^R])$
- c'(q,C) = c(q), for every  $(q,C) \in Q \times \mathsf{ER}(R)$ , and c'(s) = c'(s') = 1
- $\delta'$  will be defined in the sequel.

The new states s and s' are sink states, used to obtain a RA that is complete on its input states. They are not accepting. We add transitions  $(s, \sigma_i, \top, \varnothing, s')$  and  $(s, \sigma_{\odot}, \top, \varnothing, s')$  for all  $\sigma_i \in \Sigma_i, \sigma_{\odot} \in \Sigma_{\odot}$ .

Given a constraint C, and a set  $E \subseteq R$  corresponding to an equivalence class of C, we define a test corresponding to a maximally consistent conjunction of equalities and inequalities:  $\alpha_E = \bigwedge_{r \in E} r^= \wedge \bigwedge_{r \notin E} r^{\neq}$ . A data value satisfies this test iff it is equal to the (common) value stored in registers of R. We also consider the test  $\alpha_{\emptyset} = \bigwedge_{r \in R} r^{\neq}$  which corresponds to the case of a fresh data value, i.e. a data value distinct from all the values stored in registers.

Consider a transition  $(p, \sigma, \phi, \mathsf{asgn}, q) \in \delta$ . Given a formula  $\alpha_E$  as defined above, one can decide whether the formula  $\alpha_E \Rightarrow \phi$  is valid or not. If this is the case, then we add the

following transition to  $\delta'$ :

$$(p,C) \xrightarrow{\sigma,\alpha_E,\mathsf{asgn}} (q,C')$$

where C' is defined as follows: two registers r, r' are in relation w.r.t. C' if and only if one of the following cases holds:

- they are in relation in C, and not in asgn
- they are both in asgn
- r belongs to E and r' belongs to asqn, or vice versa.

Observe that by definition, A' satisfies item (3) of the definition of good form. In order to obtain property (2), one can add, for each symbol  $\sigma \in \Sigma_i$ , transitions to the sink state with a test expressing that no other  $\sigma$ -labelled transition can be satisfied.

It is easy to show by induction that runs of A and runs of A' avoiding the sink states, starting in the initial configurations, are in bijection. In addition, we can prove the two following properties:

- every run of A' reaching some configuration  $((p, C), \tau)$  satisfies  $C = [\tau]$ ,
- for every sequence of consecutive transitions  $t_1 ldots t_n$  of A' starting from the initial state of A' and avoiding the sink states, there exists an execution of A' along this sequence.

The second property follows from the first one, together with the fact that only valid transitions are defined in A' (we check validity of the choice of the data value w.r.t. the test labelling the transition).

As a consequence, in order to trim A', it is sufficient to eliminate states that are non-accessible.

The last step concerns property (4). Intuitively, the idea is that if the data read corresponds to a data stored in some register, then the assignment can be replaced by keeping in memory a relation between registers. This idea is merely an adaptation of the conversion from register automata ("M-automata", in their terminology) to finite-memory automata [KF94]. The states can be enriched with the right information to deal with these additional relations.

In order to solve the unbounded register synthesis problem, we resort to a synthesis problem for data-free specifications. In that framework, when specifications are described by means of parity automata, synthesis problems can be solved using reductions to parity games. We thus quickly recall the notion of parity game. For a complete presentation, we refer the reader to [AG11].

A two-play parity game is given as a finite graph, in which vertices are partitioned among the two players, together with an initial vertex. A colouring function associates with each vertex an integer. It is used to define the winning plays as follows: a play is winning iff the maximum colour appearing infinitely often is even.

In the sequel, we will use the parity game associated with a DRA A, which is denoted as  $G_A$ . It is defined as follows: its set of vertices is exactly that of A. Player Adam owns input vertices, and the associated input transitions, while player Eve owns output vertices/transitions. The colouring function is that of A, and the initial vertex is the initial state of A.

**Proposition 3.6.** Let A be a DRA<sub>ido</sub> in good form. Then L(A) is realisable by a register transducer iff Eve wins the parity game  $G_A$  associated with A. In addition, if L(A) is

realisable by a register transducer, then it is so by a register transducer with as many registers as A.

*Proof.* We start with a preliminary remark on  $\mathsf{DRA}_{\mathsf{ido}}$ . As A is a  $\mathsf{DRA}_{\mathsf{ido}}$ , every output transition has a test with at least one equality constraint ( $r^{=}$  for some r), and thus, as A is in good form (property (4)), the assignment of output transitions are all empty.

# From the parity game $G_A$ to the realisability of L(A).

Assume Eve wins the game  $G_A$ . Parity games admit memoryless strategies. We can thus consider a memoryless winning strategy for Eve, which we denote by a mapping  $\chi$  from output vertices to output symbols.

We detail how we define from  $\chi$  a register transducer  $T_{\chi}$  with  $R^A$  as set of registers as follows:

- States are those of A
- The initial state is that of A
- Transitions are defined as follows. Consider some input state p and some transition  $t_i$  from p to q. By definition of A, q is an output state, and we let  $t_{\Phi} = \chi(q)$  be the transition given by Eve's strategy.

We write  $t_{\rm i}=(p,\sigma,\phi,{\sf asgn},q)$  and  $t_{\rm o}=(q,\sigma',\phi',{\sf asgn}',q')$ . Thanks to our initial comment on the form of output transitions of  ${\sf DRA}_{\sf ido}$  in good form, there exists a register r appearing with an equality constraint in the test  $\phi'$  of the transition  $t_{\rm o}$ , and we have  ${\sf asgn}'=\varnothing$ . Then, we add to  $T_\chi$  the transition  $p\xrightarrow{\sigma,\phi|{\sf asgn},\sigma',r} q'$ .

Observe that T is indeed a register transducer as for each state p, it only uses transitions outgoing from p in A, hence it is deterministic as A was.

We claim that  $T_{\chi}$  realises L(A). Consider some input data word, and the behaviour of  $T_{\chi}$  on this data word. As A is in good form, it is complete on its input states. This entails that this run is infinite. It corresponds to a play in  $G_A$  compatible with Eve's strategy  $\chi$ . As  $\chi$  is a winning strategy, this entails that the run is accepting, hence corresponds to some accepting run of A, yielding the result.

# From the realisability of L(A) to the parity game $G_A$ .

We have to prove that if L(A) is realisable, then it is actually possible to play directly on the graph of A, and derive a winning strategy for Eve in  $G_A$ . To this end, we first show that if L(A) is realisable, then there exists an implementation which uses  $R^A$  as set of registers.

Simplifying the implementation of L(A). Let  $A = (\Sigma_i, \Sigma_0, \mathcal{D}, Q^A, q_0^A, \delta^A, R^A, c)$  be a DRA<sub>ido</sub> specification realisable by a register transducer  $I = (\Sigma_i, \Sigma_0, \mathcal{D}, Q^I, q_0^I, \delta^I, R^I)$ . As A is in good form, its tests are maximally consistent conjunctions of atoms. In this proof, we identify tests with subsets of  $R^A$  (given a maximally consistent conjunction of atom, we associate with it the set of registers appearing positively, i.e. with an equality constraint). To simplify notations, and without loss of generality (one can put any test in disjunctive normal form), we also assume that tests of I are represented as subsets of  $R^I$ . In addition, as A is in good form, it is both deterministic and complete on its input states. As a consequence, for each input state, there exists a partition of  $R^A$  which is in bijection with the set of outgoing transitions.

Our goal is to extract from A a transducer I' realising A with  $R^A$  registers, using I as a guide to make choices for the output. To this end, we simulate I synchronously with A.

Since we only have  $R^A$  registers, we cannot properly simulate I, as the registers are already used to simulate A. However, we will see that keeping constraints in memory is sufficient for our purpose.

Constraints. A constraint represents the equality relations between the registers in  $\mathbb{R}^A$  and those in  $R^{I}$  (note that such idea is pervasive in the study of register automata, e.g. to recognise the projection over finite labels). Thus, a constraint is a subset  $C \subseteq \mathbb{R}^A \times \mathbb{R}^I$ , which is intended to be an (exhaustive) set of equalities between the content of registers in  $R^A$  and in  $R^I$ , as formalised later in property 5. Then, knowing tests  $\phi^A$ ,  $\phi^I$  and assignments  $\operatorname{\mathsf{asgn}}^A$ ,  $\operatorname{\mathsf{asgn}}^I$  performed by A and I respectively allows to update the constraints: we define

$$\mathrm{next}(C,\phi^A,\mathsf{asgn}^A,\phi^I,\mathsf{asgn}^I) = C \backslash ((\mathsf{asgn}^A \times R^I) \cup (R^A \times \mathsf{asgn}^I)) \\ \cup ((\phi^A \cup \mathsf{asgn}^A) \times (\phi^I \cup \mathsf{asgn}^I))$$

For instance, assume  $R^I = \{r_1, r_2\}$  and  $R^A = \{s_1, s_2\}$ , and at some point in a run, we have  $C = \{(s_2, r_1), (s_2, r_2)\}, \text{ i.e. } s_2 = r_1 = r_2 \text{ and } s_1 \neq r_1 \text{ (inequalities are implicit, since } C$ is an exhaustive set of equalities). Now, A and I synchronously read some data d which satisfies the tests  $\phi^A = \{s_1\}$  in A and  $\phi^I = \emptyset$  in I (such tests are consistent because  $s_1 \neq r_1, r_2$ ), and conduct assignments  $\operatorname{asgn}^A = \emptyset$  and  $\operatorname{asgn}^I = \{r_2\}$ . Then, on the one hand,  $s_1 = r_2$  (both contain d), and on the other hand  $s_2 = r_1$  (since the content of those registers did not change). Moreover,  $r_1 \neq r_2$  since  $r_2$  has been reassigned and  $s_1 \neq r_1$  still holds. This is represented by the set of constraints  $C' = \{(s_1, r_2), (s_2, r_1)\}$ , and indeed,  $next(C, \{s_1\}, \emptyset, \emptyset, \{r_2\}) = C'.$ 

Abstracting the behaviour of I modifies its language (it somehow simplifies it), but we will see that what we build is still an implementation.

Definition of I'. We build  $I' = (\Sigma_{\mathbf{i}}, \Sigma_{\mathbf{o}}, \mathcal{D}, Q, q_0, \delta, R^A)$ , where  $Q = Q^A \times Q^I \times 2^{R^A \times R^I}$  and  $q_0 = (q_0^A, q_0^I, R^A \times R^I)$ ; we now define  $\delta$ . For each state  $(q_{\mathbf{i}}^A, q^I, C) \in Q$ , for each pair  $(\sigma_{\mathbf{i}}, \phi_{\mathbf{i}}^A) \in \Sigma_{\mathbf{i}} \times 2^{R^A}$ , we construct a transition  $t = (q_{\mathbf{i}}^A, q^I, C) \xrightarrow{\sigma_{\mathbf{i}}, \phi_{\mathbf{i}}^A, \operatorname{asgn}^A, \sigma_{\mathbf{o}}, s_{\mathbf{o}}} (q'_{\mathbf{i}}^A, q'^I, C')$ whenever there exist the following transitions of A and I:

$$t_{\mathrm{i}}^{A} = q_{\mathrm{i}}^{A} \xrightarrow[A]{\sigma_{\mathrm{i}}, \phi_{\mathrm{i}}^{A}, \mathrm{asgn}^{A}} q_{\mathrm{o}}^{A} \qquad t_{\mathrm{o}}^{A} = q_{\mathrm{o}}^{A} \xrightarrow[A]{\sigma_{\mathrm{o}}, \phi_{\mathrm{o}}^{A}} {q'_{\mathrm{i}}^{A}} \qquad t^{I} = q^{I} \xrightarrow[I]{\sigma_{\mathrm{i}}, \phi^{I}, \mathrm{asgn}^{I}, \sigma_{\mathrm{o}}, r_{\mathrm{o}}} {q'^{I}}$$

such that, for some fixed arbitrary order on  $\mathbb{R}^A$ , we have:

- $\begin{array}{l} (1) \ \ \phi^{I} = \{r \in R^{I} \mid \exists s \in \phi_{i}^{A}, (s,r) \in C\} \\ (2) \ \ C' = \operatorname{next}(C, \phi_{i}^{A}, \operatorname{asgn}^{A}, \phi^{I}, \operatorname{asgn}^{I}) \\ (3) \ \ \phi_{o}^{A} = \{s \in R^{A} \mid (s,r_{o}) \in C'\} \\ (4) \ \ s_{o} = \min \phi_{o}^{A} \ (\phi_{o}^{A} \text{ is non-empty as we consider } \mathsf{DRA}_{\mathsf{ido}}) \end{array}$

Item 1 ensures with the help of constraints that in any reachable configuration of I', there exists at least one input data which satisfies both  $\phi_i^A$  and  $\phi^I$ , which allows I' to synchronise A with I. Note that this does not mean that I' is the synchronous product of A and I on any input: since I' only has the registers of A, it cannot discriminate data as subtly as I, and might thus adopt a different behaviour. For instance, it can be that upon reading some input data word, at some point, I would store some input data d in some register r that A would not, and use it later on in a test  $\phi^I = \{r\}$  to take different actions, while neither I' nor A could discriminate between those choices: on reading d, I' simulates

<sup>&</sup>lt;sup>1</sup>For readability, we confuse a register with its content.

A with  $\phi_i^A = \emptyset$  and synchronously simulates in I the transition with input test  $\phi^I = \emptyset$ . Nevertheless, we show the *existence* of some relational data word common to I and I' for each run of I' (which is also a run of A). This is sufficient to conclude that I' realises A, because then each run of I', interpreted as a run of A, is accepting. Then, items 3 and 4 ensures the same property as item 1 does, but this time on output positions.

We shall see that for a state  $(q^A, q^I, C)$  accessible on some finite input data word, and for a test  $(\sigma_i, \phi_i^A)$  satisfied by the current register valuation, transitions  $t_i^A, t_{\scriptscriptstyle 0}^A$  and  $t^I$  exist and are unique, hence allowing to define the transition t in I'. So, for a run  $\rho = t_1 t_2 \ldots$  of I', we define  $\rho^A = t_{i1}^A t_{o1}^A t_{i2}^A t_{o2}^A \ldots$  and  $\rho^I = t_1^I t_2^I \ldots$ 

Proof of Correctness. Let us show that I' is indeed a transducer realising A: we show that for all  $u_i \in \mathsf{DW}(\Sigma_i, \mathcal{D})$ , there exists a unique sequence of transitions  $\rho$  in I', a unique output data word  $u_{\mathfrak{o}} \in \mathsf{DW}(\Sigma_{\mathfrak{o}}, \mathcal{D})$  (we denote  $u = \langle u_i, u_{\mathfrak{o}} \rangle$ ) and a  $w \in L(I)$  such that:

- (1)  $\rho^I$  is the run of I over w
- (2)  $\rho^A$  is the run of A over w
- (3)  $\rho$  is the run of I' over u
- (4)  $\rho^A$  is the run of A over u

Note that the above properties imply  $\mathsf{lab}(u) = \mathsf{lab}(w)$ , but it can be that  $u \neq w$ , which is consistent with the observations we made. Let us show that they entail the result we need: let  $u_i \in \mathsf{DW}(\Sigma_i, \mathcal{D})$  be some input data word. By property 3,  $f_{I'}(u_i)$  exists and is unique, so I' has total domain. Now, by denoting  $\rho^A$  the run of A over  $u = \langle u_i, f_{I'}(u_i) \rangle$ , we know by property 2 that there exists  $w \in L(I)$  such that  $\rho^A$  is the run of A over w. Then,  $\rho^A$  is accepting because I realises A so  $w \in L(A)$ , hence  $u \in L(A)$ . Thus, I' realises A.

We now establish properties 1-4. Let  $u_i \in \mathsf{DW}(\Sigma_i, \mathcal{D})$ . We build by induction on j the run  $\rho = \rho[1] \dots \rho[j] \dots$  in I' and  $w = w[1] \dots w[j] \dots \in L(I)$  satisfying properties 1, 2, 3 and 4, as well as two additional properties which will be needed in the induction: by letting  $(\tau_j^I)_{j \in \mathbb{N}}$  the sequence of register configurations associated with  $\rho^I$  upon reading w,  $(\tau_j^A)_{j \in \mathbb{N}}$  the sequence of configurations associated with the input states<sup>2</sup> of  $\rho^A$  upon reading w, and  $(\lambda_j)_{j \in \mathbb{N}}$  the sequence associated with (the input states of)  $\rho^A$  upon reading u, we have, for all  $j \in \mathbb{N}$ :

- (5) for all  $s \in \mathbb{R}^A$ , for all  $r \in \mathbb{R}^I$ ,  $(s,r) \in \mathbb{C}_j \Leftrightarrow \tau_i^A(s) = \tau_i^I(r)$
- (6) for all  $r, r' \in \mathbb{R}^A$ ,  $\tau_j^A(r) = \tau_j^A(r') \Leftrightarrow \lambda_j(r) = \lambda_j(r')$

More precisely, we show that for all j, assuming  $\rho[:j-1]$  and w[:j-1] have been constructed and satisfy<sup>3</sup> properties 1-6, we can extend  $\rho[:j-1]$  and w[:j-1] to  $\rho[:j]$  and w[:j].

Initialisation. The base case is when j=1. Then the prefix of u up to index 0 is  $\varepsilon$ , so properties 1, 2, 3 and 4 trivially hold. Then, note that initially,  $\tau_0^A = \lambda_0 = \tau_0^{R^A}$  and  $\tau_0^I = \tau_0^{R^I}$ , i.e. all registers are initially filled with  $\mathbf{d}_0$ , so  $C_0 = R^A \times R^I$  satisfies property 5, and property 6 indeed holds.

<sup>&</sup>lt;sup>2</sup>Since we assumed that A does not conduct assignments on the output, it is not necessary to take into account the register configurations associated with output states, because they are equal to the ones associated with the input states (i.e. for all  $j \in \mathbb{N}$ ,  $\tau_{ij}^A = \tau_{oj}^A$ ). Moreover, it allows to consider the same index j for  $\tau^A$  and  $\tau^I$ , which makes the proof more readable.

<sup>&</sup>lt;sup>3</sup>We defined those properties for infinite sequences, but they are defined analogously for finite ones.

Induction. Now, assume that we have built  $\rho$  and w up to position j-1, so that they satisfy properties 1-6. Let us extend them with  $\rho[j]$  and w[j]. Thus, let  $(\sigma_i^j, d_i^j) = u_i[j]$ . We first examine the behaviour of A on such input: in configuration  $(q_{ij-1}^A, \lambda_{j-1}), d_i^j$  passes the test  $\phi_{ij}^A = \{r \in R^A \mid \lambda_{j-1}(r) = d_i^j\}$ . Thus, we let  $t_{ij}^A$  the transition in A with input test  $(\sigma_i^j, \phi_{ij}^A)$  from  $q_j$  in A (it exists since A is complete and is unique since A is deterministic). Before determining the output transition  $t_{0j}^A$ , we first have to define the corresponding  $w_i[j]$  and to examine the behaviour of I on such input.

Definition of  $w_i[j]$ . By property 6 and the induction hypothesis, we know that for all  $r, r' \in \phi_{ij}^A, \tau_{j-1}^A(r) = \tau_{j-1}^A(r')$ , since it is the case for  $\lambda_{j-1}$ . So, if  $\phi_{ij}^A \neq \varnothing$ ,  $\tau_{j-1}^A(\phi_{ij}^A)$  is a singleton and we can define  $e_i^j = \tau_{j-1}^A(\phi_{ij}^A)$ . Otherwise, it means  $d_i^j$  is locally fresh w.r.t.  $\lambda_{j-1}$ , so take some locally fresh data  $e_i^j \notin \tau_{j-1}^A(R^A) \cup \tau_{j-1}^I(R^I)$ . Taking  $e_i^j$  locally fresh also w.r.t.  $\tau_{j-1}^I$  ensures that the transition taken in I also has input test  $\phi^I = \varnothing$ , otherwise I might adopt a behaviour different from I'. Then, define  $w_i[j] = (\sigma_i^j, e_i^j)$  (necessarily,  $\mathsf{lab}(w_i[j]) = \sigma_i^j$ , as noted above, otherwise u and w cannot have the same run in A). By construction, in configuration  $(q_{j-1}^A, \tau_{j-1}^A)$ , on reading  $w_i[j]$ , A takes transition  $t_{ij}^A$ .

Property 6. Then, property 6 holds at step j because A takes the same input transition  $t_{i\ j}^A$  when reading  $u_i[j]$  and  $w_i[j]$  (since they have the same finite label and pass the same test  $\phi_{i\ j}^A$ ), so A executes the same assignment  $\operatorname{asgn}_j^A$ , which affects the equalities in the same way for  $\tau_j^A$  and  $\lambda_j$  (recall that we do not need to examine the output transition of A, since it does not conduct an assignment, so it does not affect the register configuration).

Property 1. Now, in configuration  $(q_{j-1}^I, \tau_{j-1}^I)$ ,  $e_{\mathbf{i}}^j$  passes  $\phi^I = \{r \in R^I \mid \tau_{j-1}^I(r) = e_{\mathbf{i}}^j\} = \{r \in R^I \mid \exists s \in \phi_{\mathbf{i}j}^A, \tau_{j-1}^A(s) = \tau_{j-1}^I(r)\}$  by definition of  $e_{\mathbf{i}}^j$ . Property 5 yields  $\phi^I = \{r \in R^I \mid \exists s \in \phi_{\mathbf{i}j}^A, (s,r) \in C_{j-1}\}$ , so  $\phi^I = \phi_j^I$  as defined in item 1 of the construction of I'. Let  $t_j^I = q_{j-1}^I \xrightarrow{\sigma_j^i, \phi_j^I, \operatorname{asgn}_j^I, \sigma_o^j, r_o^j} q^I_j$  be the transition taken by I on reading  $w_{\mathbf{i}}[j]$ . Such transition exists and is unique since I has total domain and is deterministic over the input, so property 1 holds at step j (assuming the transition  $t_j$  in I' actually exists, which we show later). Let  $w_{\mathbf{0}}[j] = (\sigma_{\mathbf{0}}^j, e_{\mathbf{0}}^j)$ , where  $e_{\mathbf{0}}^j = \tau_j^I(r_{\mathbf{0}}^j)$ , be the output produced by I on such transition.

Property 5. Before examining what happens for the output, we need to show that property 5 holds at step j. Actually, this directly follows from the semantics of register automata and register transducers. Indeed, after executing assignments  $\operatorname{asgn}_j^A$  and  $\operatorname{asgn}_j^I$ , we have that for all  $s \in R^A$  and for all  $r \in R^I$ ,  $\tau_j^A(s) = \tau_j^I(r)$  iff either  $\tau_{j-1}^A(s) = \tau_{j-1}^I(r)$  and neither s nor r have been assigned, or both s and r have been assigned  $e_i^j$  or tested for equality w.r.t.  $e_i^j$ , so we indeed get that  $C_j$  satisfies the required property.

Property 2. Now, in configuration  $(q_{0j}^A, \tau_j^A)$ ,  $e_0^j$  passes  $\phi_0^A = \{s \in R^A \mid \tau_j^A(s) = e_0^j\} = \{s \in R^A \mid \tau_j^A(s) = \tau_j^I(r_0^j)\}$  by definition of  $e_0^j$ . By property 5, we get  $\phi_0^A = \{s \in R^A \mid (s, r_0^j) \in C_j\}$ , so  $\phi_0^A = \phi_{0j}^A$  as defined in item 3. Therefore,  $\rho^A$  up to step j is indeed the run of A over w up to index j, and property 2 holds at step j.

Property 3. To show that the transition  $t_i$  actually exists, it remains to show that there exists  $s_0^j \in \mathbb{R}^A$  such that  $(s_0^j, r_0^j) \in C_j$ . Assume by contradiction that it is not the case. Then, by property 5 it means that  $e_{\Phi}^{j} = \tau_{j}^{I}(r_{\Phi}^{j})$  is such that for all  $s \in \mathbb{R}^{A}, \tau_{j}^{A}(s) \neq e_{\Phi}^{j}$ , so upon reading  $w_{\mathbb{Q}}[j] = (\sigma_{\mathbb{Q}}^{j}, e_{\mathbb{Q}}^{j})$  in configuration  $q_{\mathbb{Q}j}^{A}, \tau_{j}^{A}$ , A transitions to a non-accepting, absorbing state by definition of the behaviour of DRA<sub>ido</sub> on taking transitions labelled with  $\emptyset$  tests. But this means that I does not realise A: since I is a transducer with total domain. the partial run  $\rho^I$  we built so far can be extended with any suffix  $v = \langle v_i, v_o \rangle \in \mathsf{RW}$  accepted by I from  $(q_i^I, \tau_i^I)$ , which yields some word  $wv \in L(I)$  such that  $wv \notin L(A)$ .

Overall, there indeed exists a transition  $(q_{ij-1}^A, q_{j-1}^I, C_{j-1}) \xrightarrow{\sigma_i^j, \phi_{ij}^A, \operatorname{asgn}_j^A, \sigma_o, s_o^j} (q_{ij}^A, q_j^I, C_j)$ in I on input  $(\sigma_i^j, d_i^j)$ . Let us summarise biefly why it is unique: once  $\phi_{ij}^A$  and  $C_{j-1}$  are fixed,

- (1)  $t_{ij}^A$  exists because A is complete, and it is unique since A is deterministic (2)  $t_j^I$  exists and is unique because  $\phi_j^I$  is uniquely defined, and I has total domain and is deterministic over the input
- (3)  $t_{ij}^A$  and  $t_j^I$  determine  $C_j$  which determines  $\phi_{0j}^A$  (by item 3 of the definition of I'), hence determining  $t_{\bullet j}^{A}$  (because again, A is complete and deterministic)
- (4) finally,  $s_{\bullet}^{j}$  is chosen canonically.

Thus, property 3 holds. Upon reading  $u_i[j]$ , I' takes such transition and produces output  $u_{\Phi}[j] = (\sigma_{\Phi}^{j}, d_{\Phi}^{j}) \text{ where } d_{\Phi}^{j} = \lambda_{i}(s_{\Phi}^{j}).$ 

Property 4. Finally, in configuration  $(q_{0j}^A, \lambda_j), d_0^j = \lambda_j(s_0^j)$  passes  $\phi'_0^A = \{s \in \mathbb{R}^A \mid \lambda_j(s) = 0\}$  $d_{\mathfrak{o}}^{j} = \{ s \in \mathbb{R}^{A} \mid \lambda_{j}(s) = \lambda_{j}(s_{\mathfrak{o}}^{j}) \} = \{ s \in \mathbb{R}^{A} \mid \tau_{j}^{A}(s) = \tau_{j}^{A}(s_{\mathfrak{o}}^{j}) \} \text{ by property 6, which implies}$  $\phi'_{\mathfrak{o}}^{A} = \phi_{\mathfrak{o}}^{A} = \phi_{\mathfrak{o}_{j}}^{A}$  (where  $\phi_{\mathfrak{o}}^{A}$  is the test validated by  $e_{\mathfrak{o}}^{j}$  in A, as defined above in the proof of property 2). Thus, on reading  $d_{\mathfrak{o}}^{j}$ , A takes transition  $t_{\mathfrak{o}j}^{A}$ , which means that  $\rho^{A}$  up to step j is indeed the run of A over u up to index j, i.e. property 4 holds.

A strategy in the parity game. We have thus exhibited an implementation I' using only  $R^A$ as set of registers. In addition, from the definition of I', one can easily come up with a winning strategy for Eve in the parity game  $G_A$ . This strategy has finite memory, given by pairs  $(q^I, C)$  composed of a state of I, and a constraint. Given a state  $q^A$  of A, and an input transition  $t_i^A = (q^A, \sigma_i, \phi, \mathsf{asgn}, p^A)$  outgoing from  $q^A$ , the strategy associates the output transition  $t_0^A$  used to define the corresponding transition t of I'. As we have shown, this transition exists and is unique, provided that we can reach a configuration in which transition  $t_i^A$  can be applied, and this is the case as we assume that A is trim (second property of being in good form).

# **Theorem 3.7.** Real(DRA<sub>ido</sub>, RT) is ExpTime-c.

*Proof.* First, we put A in good form thanks to Lemma 3.5, resulting in some DRA<sub>ido</sub> B exponentially bigger. Then, by Proposition 3.6, it suffices to solve the parity game  $G_B$ . It is well-known to be possible in time  $O(n^d)$  where n is the number of states and d the number of priorities. If  $n_A$  denotes the number of states of A and d its number of priorities, then B has  $n_A \times 2^{|R|^2}$  states and the same number of priorities d, hence checking the realisability of A can be done in time  $O(n_A^d \times 2^{d \cdot |R|^2})$ , which is exponential.

Hardness. The following proof is an adaptation of the one establishing PSPACE-hardness of the nonemptiness problem for DRA presented in [DL09, Theorem 5.1]. Here, we use the input part to simulate universal transitions, and the output part to simulate nondeterministic ones, hence simulating alternation, which yields an ExpTime lower bound.

Thus, we reduce from the halting problem for alternating Turing machines over a binary alphabet with linearly bounded tapes. An alternating Turing machine is a tuple  $\mathcal{M}=$  $\langle Q, q_i, \delta \rangle$ , where:

- Q is a finite set of states, partitioned into existential  $(Q_{\exists})$  and universal  $(Q_{\forall})$  states:  $Q = Q_{\exists} \uplus Q_{\forall}$ , where  $q_i \in Q_{\exists}$  is the *initial* state •  $\delta : Q \times \{0,1\} \to 2^{Q \times \{0,1\} \times \{-1,1\}}$  is the *transition function*.

Then, a configuration of  $\mathcal{M}$  is a triple (q, i, w), where  $q \in Q$  is the machine state,  $i \in \mathcal{M}$  $\{0,\ldots,|\mathcal{M}|-1\}$  is the head position, and  $w\in\{0,1\}^{|\mathcal{M}|}$  is the tape content. Then, a configuration (q', i', w') is a successor of (q, i, w) if there exists  $(p, a, m) \in \delta(q, w[i]), p = q'$  $i'=i+m\in\{0,\ldots,|M|-1\}$  and w' is such that  $\forall j\neq i,\ w'[j]=w[j]$  and w[i]=a. We then say that  $t = q \xrightarrow{w[i], a, m} p$  is the associated transition. Then, a configuration (q, i, w) is accepting if, either:

- (1)  $q \in Q_{\exists}$  is an existential state and at least one of the successor configurations is
- (2)  $q \in Q_{\forall}$  is a universal state and all the successor configurations are accepting Note that the base case of such inductive definition consists in universal configurations with no successor. The following problem is EXPTIME-hard [CKS81]: given an alternating Turing machine  $\mathcal{M}$ , decide whether its initial configuration  $(q_i, 0, 0^{|M|})$  is accepting.

Finally, a computation is a finite sequence of successive configurations. Let  $(q_0, i_0, w_0) \dots$  $(q_n, i_n, w_n)$  be a computation of  $\mathcal{M}$ , and  $t_0 \dots t_{n-1}$  the sequence of associated transitions. We encode such computation by the following data word over the alphabet  $Q \uplus \delta \uplus \{-\}$ :

$$(-,d_0)(-,d_1)a_0^0a_1^0\dots a_{|\mathcal{M}|-1}^0t_0a_0^1a_1^1\dots a_{|\mathcal{M}|-1}^1t_1\dots t_{n-1}a_0^na_1^n\dots a_{|\mathcal{M}|-1}^n$$

where  $d_0 \neq d_1 \in \mathcal{D}$  are two distinct data respectively encoding letters 0 and 1, and we have  $\mathsf{lab}(a_l^k) = q_k$  if  $l = i_k$  and  $\mathsf{lab}(a_l^k) = -$  otherwise. Then,  $\mathsf{dt}(a_l^k) = d_0$  if  $w_k[l] = 0$  and  $dt(a_l^k) = d_1$  if  $w_k[l] = 1$ .  $dt(t_k)$  does not matter.

Then, as in [DL09], we can construct a DRA  $A_{\mathcal{M}}$  which accepts a data word iff it has a prefix that encodes a computation of  $\mathcal{M}$  from the initial state to a state with no successor. Indeed, the transitions are part of the input, so they do not have to be guessed: neither nondeterministic nor universal branching is needed here (they will respectively be simulated by the output and input players). For completeness, we describe the construction:  $A_{\mathcal{M}}$  has memory Q, along with an  $|\mathcal{M}|$ -bounded counter l to keep track of the position of the reading head in  $w_k$ , a variable i taking its values in  $\{0, \ldots, |\mathcal{M}| - 1\}$  used to store the value of  $i_k$  and a variable t taking its values in  $\delta$  to memorise  $t_k$ ; which overall yields a  $O(|\mathcal{M}|^4)$  memory. Its finite alphabet is  $Q \uplus \delta \uplus \{-\}$ , and it has  $|\mathcal{M}| + 2$  registers:  $r_0$  and  $r_1$  respectively store  $d_0$  and  $d_1$ , and, for all  $0 \le l < |\mathcal{M}|$ ,  $r'_l$  successively stores the different values of  $w_k[l]$ for  $0 \leq k \leq n$ . Then, a run of  $A_{\mathcal{M}}$  is as follows: initially,  $A_{\mathcal{M}}$  stores  $d_0$  and  $d_1$ , while checking that they are distinct. Then, it checks that  $w_0 = 0^{|\mathcal{M}|}$ . To check successorship, while maintaining the invariant that at any step k,  $r'_l$  contains  $w_k[l]$ , the automaton, when reading  $t_k = q \xrightarrow{c,a,m} p$ , checks that  $q = q_k$  (it was stored as the target of  $t_{k-1}$ ),  $c = w_k[i_k]$ (i.e. that  $r'_{i_k}$  contains  $d_c$ ), and updates the value of  $i_k$  to  $i_{k+1} = i_k + m_k$ , while checking

that  $i_k \in \{0, \dots, |\mathcal{M}| - 1\}$ . Then, with the help of its registers and its counter l, it checks that  $w_{k+1}[l] = w_k[l]$  for all  $l \neq i_{k+1}$ , and that  $w_{k+1}[i_{k+1}] = d_a$ .

From such automaton, by adding #s to enforce the alternation between input and output, we can build a specification automaton such that the input player provides the encoding of the successive configurations, and resolves the universal branching, and the output player has to resolve nondeterminism (i.e. chooses which nondeterministic transition to take). Then, if the input player can force the computation to go on ad infinitum, he wins, otherwise (if either the provided encoding is not correct, or if the computation is finite), the output player wins. Formally:

$$S = \{(-, d_0)\#(-, d_1)\#\langle c_0, \#^{|M|}\rangle t_0\#\langle c_1, \#^{|M|}\rangle \#t_1\langle c_2, \#^{|M|}\rangle t_2\# \dots \langle c_n, \#^{|M|}\rangle \#^{\omega} \mid d_0 \neq d_1 \text{ and } c_0t_0c_1t_1c_2t_2\dots t_{n-1}c_n \text{ is the encoding of a computation of } \mathcal{M}\}$$

$$\cup \{\langle w, w' \rangle \mid \text{ there exists a prefix of } w \text{ which is not the encoding of a computation of } \mathcal{M}\}$$

$$\cup \{\langle (-, d_0)\#(-, d_1)w, w' \rangle \mid d_0 = d_1\}$$

The data corresponding to the # and  $t_i$  do not matter, and are not depicted. Note that the even (i.e. universal) transitions are picked by the input player, while the odd (i.e. nondeterministic) transitions are picked by the output player.

Now, if  $\mathcal{M}$  halts, A admits an implementation, which behaves as follows: it first checks that the  $d_0$  and  $d_1$  given as input are indeed distinct. Then, it checks on-the-fly that the given input is indeed an encoding of the initial configuration, while outputting #s. It then checks that  $c_1$  is indeed a successor of  $c_0$  following  $t_0$ , again while outputting #s. Then, if it receives as input a #, it picks some  $t_1$  which is a witness that  $c_0$  is indeed accepting, and so on. If, at some point, the given input is not a valid encoding, then it behaves arbitrarily (e.g. by outputting only #s).

Conversely, if  $\mathcal{M}$  does not halt, then, by choosing as input whose universal transitions are witnesses that  $c_0$  is not accepting, then either the implementation provides some non-admissible output at some point, or the computation goes ad infinitum, which breaks the specification.

As a consequence of the fact that if a  $\mathsf{DRA}_{\mathsf{ido}}$  is realisable, then it is so by a register transducer with the same number of registers, we obtain the following corollary:

Corollary 3.8. Let  $k \geq r$  be two integers. We denote by  $\mathsf{DRA}_{\mathsf{ido}}[r]$  the class of  $\mathsf{DRA}_{\mathsf{ido}}$  with r registers.  $\mathsf{REAL}(\mathsf{DRA}_{\mathsf{ido}}[r], \mathsf{RT}[k])$  is in  $\mathsf{EXPTIME}$ .

## 4. Bounded Synthesis: A Generic Approach

In this section, we study the setting where target implementations are register transducers in the class  $\mathsf{RT}[k]$ , for some  $k \geq 0$  that we now fix for the whole section. For the complexity analysis, we assume k is given as input, in unary. Indeed, describing a k-register automaton in general requires O(k) bits, and not  $O(\log k)$  bits. We prove the decidable cases of the first line of Table 1 (page 4), by reducing the problems to realisability problems for data-free specifications.

Abstract Actions. We let  $R_k = \{1, \ldots, k\}$  be a set of k registers. Our aim is to reduce the problem to a finite alphabet problem. First, since the set of test formulas over  $R_k$  is infinite and there are doubly exponentially many non-equivalent formulas over  $R_k$ , we rather synthesise transducers whose tests are maximally consistent conjunctions of atoms of the form  $r^=$  or  $r^{\neq}$ . Such conjunctions can be identified as subsets of  $R_k$  in a natural way, e.g. for k=3, the test  $r_1^= \wedge r_2^{\neq} \wedge r_3^=$  is identified with the set  $\{1,3\}$ . We call them explicit tests and denote them by the capital letter E. An explicit test  $E \subseteq R_k$  is converted into the (implicit) test  $\phi_E = \bigwedge_{r \in E} r^= \wedge \bigwedge_{r \notin E} r^{\neq}$ . Explicit tests are for instance used in [Seg06].

We let  $\mathsf{Tst}_k = \mathsf{Asgn}_k = 2^{R_k}$ . The finite input actions are  $A_1^k = \Sigma_i \times \mathsf{Tst}_k$  which corresponds to picking a label and a test over the k registers, and the output actions are  $A_0^k = \Sigma_0 \times \mathsf{Asgn}_k \times R_k$ , corresponding to picking some output symbol, some assignment and some register whose content is to be output.

An alternating sequence of actions  $\overline{a} = (\sigma_i^1, E_1)(\sigma_0^1, \mathsf{asgn}_1, r_1) \cdots \in (A_i^k A_0^k)^\omega$  abstracts a set of relational data words of the form  $w = (\sigma_i^1, d_i^1)(\sigma_0^1, d_0^1) \cdots \in \mathsf{RW}(\Sigma_i, \Sigma_0, \mathcal{D})$  via a compatibility relation that we now define. We say that w is compatible with  $\overline{a}$  if there exists a sequence of register configurations  $\tau_0\tau_1 \cdots \in (R_k \to \mathcal{D})^\omega$  such that  $\tau_0 = \tau_0^{R_k}$  and for all  $i \geq 1, \tau_i, d_i^i \models E_i, d_0^i = \tau_i(r_i)$  and  $\tau_{i+1} = \mathsf{next}(\tau_i, d_i^i, \mathsf{asgn}_i)$ . In other words, w is compatible with  $\overline{a}$  if there exists some k-register transducer and a run  $\rho = t_0t_1 \ldots$  such that for all i, is of the form  $t_i = q_i \xrightarrow{\sigma_i^i, E_i \mid \sigma_0^i, \mathsf{asgn}_i, r_i} q_{i+1}$  for some  $q_i, q_{i+1} \in Q_T$ . Note that this sequence is unique if it exists. We denote by  $\mathsf{Comp}(\overline{a})$  the set of relational data words compatible with  $\overline{a}$ . Given a specification S, we let  $W_{S,k} = \{\overline{a} \mid \mathsf{Comp}(\overline{a}) \subseteq S\}$ . The set  $W_{S,k}$  is then a specification over the finite input (resp. output) alphabets  $A_i^k$  (resp.  $A_0^k$ ).

**Theorem 4.1** (Transfer). Let S be a data word specification. The following are equivalent:

- (1) S is realisable by a transducer with k registers.
- (2) The (data-free) word specification  $W_{S,k}$  is realisable by a (register-free) finite transducer.

Proof. Let T be a transducer with k registers realising S. The tests of T are implicit tests, so in a first step we explicit them, possibly by adding new transitions to T. Formally, a transition  $q \xrightarrow{\sigma_{\mathbf{i}}, \phi | \sigma_{\mathbf{o}}, \mathsf{asgn}, r} q'$  is replaced by all the transitions  $q \xrightarrow{\sigma_{\mathbf{i}}, E | \sigma_{\mathbf{o}}, \mathsf{asgn}, r} q'$  for all  $E \subseteq R_k$  such that  $\phi_E \Rightarrow \phi$  is true. We call T the resulting transducer. It can be seen as a finite transducer T' over input alphabet  $A_{\mathbf{i}}^k$  and output alphabet  $A_{\mathbf{o}}^k$ . Moreover, since the transition function of T is complete, it is also the case of T' (this is required by the definition of transducer defining implementations).

Let us show that  $W_{S,k}$  is realisable by T', i.e.  $L(T') \subseteq W_{S,k}$ . Take a sequence  $\overline{a} = a_1e_1a_2e_2\cdots \in L(T')$ . We show that  $\mathsf{Comp}(\overline{a}) \subseteq S$ . Let  $w \in \mathsf{Comp}(\overline{a})$ . Then, there exists a run  $q_0q_1q_2\ldots$  of T' on  $\overline{a}$  since  $\overline{a} \in L(T')$ . By definition of compatibility for w, there exists a sequence of register configurations  $\tau_0\tau_1\cdots \in (R_k\to \mathcal{D})^\omega$  satisfying the conditions in the definition of compatibility. From this we can deduce that  $(q_0,\tau_0)(q_1,\tau_1)\ldots$  is an initial sequence of configurations of T over w, so  $w \in L(T)$ . Finally,  $L(T) \subseteq S$ , since T realises S.

Conversely, suppose that  $W_{S,k}$  is realisable by some finite transducer T' over the input (output) alphabets  $A_i^k$  ( $A_o^k$ ). Again, the transducer T can be seen as a transducer with k registers over data words with explicit tests. We show that T realises S, i.e.,  $L(T) \subseteq S$ . Let  $w \in L(T)$ . The run of T over w induces a sequence of actions  $\overline{a}$  in  $(A_i^k A_o^k)^\omega$  which, by definition of compatibility, satisfies  $w \in \mathsf{Comp}(\overline{a})$ . Moreover,  $\overline{a} \in L(T')$ . Hence, since T' realises  $W_{S,k}$ , we get  $\mathsf{Comp}(\overline{a}) \subseteq S$ , so  $w \in S$ , concluding the proof.

4.1. The case of URA specifications. In this section, we show that for any S a data word specification given as some URA, the language  $W_{S,k}$  is effectively  $\omega$ -regular, entailing the decidability of Real(URA, RT[k]), by Theorem 4.1 and the decidability of (data-free) synthesis. Let us first prove a series of intermediate lemmas.

We define an operation  $\otimes$  between relational data words  $w \in \mathsf{RW}(\Sigma_{\mathtt{i}}, \Sigma_{\mathtt{o}}, \mathcal{D})$  and sequences of actions  $\overline{a} \in (A^k_{\mathtt{i}} A^k_{\mathtt{o}})^{\omega}$  as follows:  $w \otimes \overline{a} \in \mathsf{RW}(A^k_{\mathtt{i}}, A^k_{\mathtt{o}}, \mathcal{D})$  is defined only if for all  $i \geq 1$ ,  $\mathsf{lab}(w[i]) = \mathsf{lab}(\overline{a}[i])$  where  $\mathsf{lab}(\overline{a}[i])$  is the first component of  $\overline{a}[i]$  (a label in  $\Sigma_{\mathtt{i}} \cup \Sigma_{\mathtt{o}}$ ), by  $(w \otimes \overline{a})[i] = (\overline{a}[i], \mathsf{dt}(w[i]))$ .

**Lemma 4.2.** The language  $L_k = \{w \otimes \overline{a} \mid w \in \textit{Comp}(\overline{a})\}$  is definable by some NRA.

*Proof.* We define an NRA with k registers which roughly follows the actions it reads on its input. Its set of states is  $\{q\} \cup \mathsf{Asgn}_R$ , with initial state q. In state q, it is only allowed to read labelled data in  $A_i^k \times \mathcal{D}$ . On reading  $(\sigma_i, \phi, d)$ , it guesses some assignment  $\mathsf{asgn}$ , performs the test  $\phi$  and the assignment  $\mathsf{asgn}$  and goes to state  $\mathsf{asgn}$ . In any state  $\mathsf{asgn} \in \mathsf{Asgn}_R$ , it is only allowed to read labelled data of the form  $(\sigma_0, \mathsf{asgn}, r, d)$ , for which it tests whether d is equal to the content of r. It does no assignment and moves back to state q. All states are accepting (i.e. have maximal even parity 0). Such NRA has size  $O(2^{k^2})$ .

Let S be a specification defined by some URA  $A_S$  with set of states Q. The following subset of  $L_k$  is definable by some NRA, where  $\overline{S}$  denotes the complement of S:

**Lemma 4.3.** The language  $L_{\overline{S},k} = \{w \otimes \overline{a} \mid w \in \textit{Comp}(\overline{a}) \cap \overline{S}\}$  is definable by some NRA.

*Proof.* Since S is definable by the URA  $A_S$ ,  $\overline{S}$  is NRA-definable with  $\overline{A_S}$ , a copy of  $A_S$  with colouring function  $\overline{c}: q \mapsto c(q)+1$ , interpreted as an NRA. Let B be some NRA defining  $L_k$  (it exists by Lemma 4.2). It now suffices to take a product of  $A_{\overline{S}}$  and B to get an NRA defining  $L_{\overline{S},k}$ .

Given a data word language L, we denote by  $lab(L) = \{lab(w) \mid w \in L\}$  its projection on labels. The language  $W_{S,k}$  is obtained as the complement of the label projection of  $L_{\overline{S},k}$ :

Lemma 4.4. 
$$W_{S,k} = \overline{lab(L_{\overline{S},k})}$$
.

structions [Pit07]

*Proof.* Let 
$$\overline{a} \in (A_{\overline{a}}^k A_{\overline{o}}^k)^\omega$$
. Then,  $\overline{a} \notin W_{S,k} \Leftrightarrow \mathsf{Comp}(\overline{a}) \not\subseteq S \Leftrightarrow \exists w \in \mathsf{RW}, w \in \mathsf{Comp}(\overline{a}) \cap \overline{S} \Leftrightarrow \exists w \in \mathsf{RW}, w \otimes \overline{a} \in L_{\overline{S},k} \Leftrightarrow \overline{a} \in \mathsf{lab}(L_{\overline{S},k})$ .

We are now able to show regularity of  $W_{S,k}$ .

**Lemma 4.5.** Let S be a data word specification,  $k \geq 0$ . If S is definable by some URA with n states and r registers, then  $W_{S,k}$  is effectively  $\omega$ -regular, definable some deterministic parity automaton with  $O(2^{n^216^{(r+k)^2}})$  states and  $O(n.4^{(r+k)^2})$  priorities.

Proof. First,  $L_{\overline{S},k}$  is definable by some NRA with  $O(2^{k^2}n)$  states and O(r+k) registers by Lemma 4.4, obtained as product between the NRA  $\overline{A_S}$  and the automaton obtained in Lemma 4.2, of size  $O(2^{k^2})$ . It is known that the projection on the alphabet of labels of a language of data words recognised by some NRA is effectively regular [KF94]. The same construction, which is based on extending the state space with register equality types, carries over to  $\omega$ -words, and one obtains a nondeterministic parity automaton with  $O(n.4^{(r+k)^2})$  states and d priorities recognising  $lab(L_{\overline{S},k})$ . It can be complemented into a deterministic parity automaton with  $O(2^{n^2.16^{(r+k)^2}})$  states and  $O(n.4^{(r+k)^2})$  priorities using standard constraints.

We are now able to reprove the following result, known from [KMB18]:

**Theorem 4.6.** For all  $k \geq 0$ , REAL(URA, RT[k]) is in 2EXPTIME.

*Proof.* By Lemma 4.5, we construct a deterministic parity automaton  $P_{S,k}$  for  $W_{S,k}$ . Then, according to Theorem 4.1, it suffices to check whether it is realisable by a (register-free) transducer. The way to decide it is to see  $P_{S,k}$  as a two-player parity game and check whether the protagonist has a winning strategy. Parity games can be solved in time  $O(m^{\log d})$  [CJK<sup>+</sup>17] where m is the number of states of the game and d the number of priorities. Overall, solving it requires doubly exponential time, more precisely in  $O(2^{n^316^{(r+k)^2}})$ .

4.2. The case of test-free NRA specifications. Unfortunately, by Theorem 3.2, the synthesis problem for specifications expressed as NRA is undecidable, even when the number of registers of the implementation is bounded. And indeed, if we mimic the reasoning of the previous section, we get that  $L_{\overline{S},k}$  is definable by a URA, but Lemma 4.4 does not allow to conclude because:

**Proposition 4.7.** There exists a data word language L which is URA-definable and whose string projection is not  $\omega$ -regular.

Proof. Consider  $L = \{(r, d_1) \dots (r, d_n)(g, d'_1) \dots (g, d'_m)(\#, d)^{\omega} \mid \forall i \neq j, d_i \neq d_j \land \forall 1 \leq i \leq n, \exists j, d'_j = d_i\}$ , which consists in a word  $w \in r^n$  with pairwise distinct data followed by a word  $w' \in g^m$  which contains at least all the data of w, and extended with  $(\#, d)^{\omega}$  to make it infinite (here, the choice of d does not matter). Such language can be interpreted as the request-grant specification, restricted to the case where all requests are made first, and are all made by pairwise distinct clients (plus a # infinite padding). L is recognised by an URA which, on reading  $(r, d_i)$ , universally triggers a run checking that

(1) Once a label g is read, only gs are read; and after the last g, only # are read (this is an  $\omega$ -regular property)

- (2)  $(r, d_i)$  does not appear again
- (3)  $(g, d_i)$  appears at least once.

Now, we have  $lab(L) = \{r^n g^m \#^\omega \mid m \geq n\}$ , which is not  $\omega$ -regular.

In this section, we consider the class of NRA which do not perform tests on input data, which we call test-free nondeterministic register automata (NRA<sub>tf</sub> for short). Such restriction is inspired from [DH16], which defines transformations of data words using MSO interpretations with an MSO origin relation. The MSO interpretation describes the transformation over the finite alphabet (called the *string transduction*), as in [Cou94], while the MSO origin relation describes the relation between input and output data. Such relation does not depend on (un)equalities between different input data: it uniquely maps each output position to an input position, expressing that the output data at this position is equal to the corresponding input data. They then show that such model is equivalent to two-way deterministic transducers with *data variables*<sup>4</sup>. Such data variables are used to implement the MSO origin relation: they are registers in which the transducer can store the input data values and output them, but it is not allowed to perform any test on the stored data, contrary to our model of register automata. To define NRA<sub>ff</sub>, we apply the same restriction to

<sup>&</sup>lt;sup>4</sup>Themselves equivalent to one-way streaming string transducers with data variables and parameters; such parameters are reminiscent of the guessing mechanism described in [KZ10].

NRA: they correspond to nondeterministic one-way transducers with data variables. Such machines can only rearrange input data (duplicate, erase, copy) regardless of the actual data values (as there are no tests). This way, as stated in Proposition 4.9, registers induce an origin relation between input and output data.

To avoid confusion between the nature of specifications and implementations, we prefer to define them as register automata, instead of transducers.

# **Definition 4.8** (Test-free register automaton). A NRA is test-free if:

- (1) Its input transitions do not depend on equality relations between input data: for all  $t \in \delta$ , if  $t = q \xrightarrow{\sigma, \phi, \mathsf{asgn}} q'$  is an input transition, then  $\phi = \top$ . (2) Its output transitions consist in outputting the content of some register: for all
- $t \in \delta$ , if  $t = q \xrightarrow{\sigma,\phi, \text{asgn}} q'$  is an output transition, then  $\phi = r^{=}$  for some  $r \in R$  and

We now make the relation with the notion of origin precise: as shown in [DFL18], there is a tight connection between origin graphs and data words. Here, the encoding is slightly different, as we do not necessarily ask that the data labelling input position n is equal to n. However, as long as the input data are all pairwise distinct, such encoding carries to our setting: the output data at position j is equal to  $d_i^i$ , where i is the (input) origin position. Thus, in the following, we let AllDiff denote the set of relational data words whose input data are pairwise distinct:

$$\text{AllDiff} = \{w = (\sigma^1_{\mathbf{i}}, d^1_{\mathbf{i}})(\sigma^1_{\mathbf{o}}, d^1_{\mathbf{o}}) \cdots \in \mathsf{RW} \mid \forall 0 \leq i < i', d^i_{\mathbf{i}} \neq d^{i'}_{\mathbf{i}} \}$$

where, by convention  $d_i^0 = \mathbf{d}_0$ . Then, as we will show, the behaviour of an NRA<sub>tf</sub> over

AllDiff determines its origin relation, and hence its behaviour over the entire data domain. To a run  $\rho = q_0 \xrightarrow{\sigma_i^1, \operatorname{asgn}^1, r^1, \sigma_o^1} q_1 \xrightarrow{\sigma_i^2, \operatorname{asgn}^2, r^2, \sigma_o^2} q_2 \dots$ , we associate the origin function  $o_\rho: j \mapsto \max\{i \leq j \mid r_j \in \operatorname{asgn}_i\}$ , with the convention  $\max \varnothing = 0$ . In other words,  $o_\rho(j)$ is the last input position at which the register output at position j was assigned, so the corresponding input data is the one which is output (if the register has never been assigned, it contains  $\mathbf{d}_0$ , which, by convention, is the data associated with input position 0).

Now, for an origin function  $o: \mathbb{N}\setminus\{0\} \to \mathbb{N}$  and for a relational data word  $w \in \mathbb{RW}$ , we say w is compatible with the origin function o, denoted  $w \models o$ , whenever for all  $j \geq 1$ ,  $\mathsf{dt}(\mathsf{out}(w)[j]) = \mathsf{dt}(\mathsf{inp}(w)[o(j)]), \text{ with the convention } \mathsf{dt}(\mathsf{inp}(w)[0]) = \mathsf{d}_0.$ 

The following proposition shows that actual data values in a word w do not matter with respect to membership in some NRA<sub>ff</sub>, only the compatibility with origin functions does:

**Proposition 4.9.** Let  $w \in RW$  and  $\rho$  a sequence of transitions of some NRA<sub>ff</sub>. Then,

- (1) If  $\rho$  is a run over w, then  $w \models o_{\rho}$ .
- (2) If  $\rho$  is a run over w and  $w \in AllDiff$ , then for all  $o : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ ,  $w \models o \Leftrightarrow o = o_{\rho}$ .
- (3) If w and  $\rho$  have the same finite labels and if  $w \models o_{\rho}$ , then  $\rho$  is a run over w.

*Proof.* 1 and 3 follow from the semantics of NRA<sub>ff</sub>, which do not conduct any test on the input data. The  $\Leftarrow$  direction of 2 is exactly 1. Now, assume  $w \in AllDiff$  admits  $\rho$  as a run, and let o such that  $w \models o$ . Then, let  $j \geq 1$  be such that  $\mathsf{dt}(\mathsf{out}(w)[j]) = \mathsf{dt}(\mathsf{inp}(w)[o(j)])$ . By 1 we know that  $\mathsf{dt}(\mathsf{out}(w)[j]) = \mathsf{dt}(\mathsf{inp}(w)[o_\varrho(j)])$ , so  $\mathsf{dt}(\mathsf{inp}(w)[o(j)]) = \mathsf{dt}(\mathsf{inp}(w)[o_\varrho(j)])$ . Since  $w \in AllDiff$ , this implies  $o(j) = o_{\rho}(j)$ , so, overall,  $o = o_{\rho}$ .

It is not clear whether  $W_{S,k}$  is regular for NRA<sub>ff</sub> specifications, but we show that it suffices to consider another set denoted  $W_{S,k}^{\mathsf{ff}}$  which is easier to analyse (and can be proven regular), which describes the behaviour of S over input with pairwise distinct data. Indeed, as expressed by the above proposition, since NRA<sub>ff</sub> cannot conduct test on input data, they behave the same on an input word whose data are all distinct, and such choice ensures that two equal input data will not ease the task of the implementation. An interesting side-product of this approach is that it implies that we can restrict to test-free implementations. A test-free transducer is a transducer whose transitions do not depend on tests over input data, i.e., for all transitions  $t = q \xrightarrow{\sigma_{\mathbf{i}}, \phi | \mathsf{asgn}, \sigma_{\mathbf{o}}, r} q' \in \delta$ , we have  $\phi = \top$ .

**Proposition 4.10.** Let S be a NRA<sub>tf</sub> specification, and  $A_{i}^{\varnothing} = \Sigma_{i} \times \{\varnothing\}$ . The following are equivalent:

- (1) S is realisable
- (2)  $W_{S,k}^{\mathsf{tf}} = \{ \overline{a} \in (A_{\mathsf{i}}^{\varnothing} A_{\mathsf{o}}^{k})^{\omega} \mid \mathsf{Comp}(\overline{a}) \cap S \cap \mathsf{AllDiff} \neq \varnothing \}$  is realisable by a (register-free) transducer with input alphabet  $A_{\mathsf{i}}^{\varnothing}$
- (3) S is realisable by a test-free transducer

*Proof.*  $3 \Rightarrow 1$  is trivial.

 $1\Rightarrow 2$ : If S is realisable, then, by Theorem 4.1,  $W_{S,k}$  is realisable by some transducer I. Now, since transducers are closed under regular domain restriction,  $W_{S,k}^{\varnothing}=W_{S,k}\cap (A_{\mathbf{i}}^{\varnothing}A_{\mathbf{0}}^{k})^{\omega}$  is realisable by I restricted to the input alphabet  $A_{\mathbf{i}}^{\varnothing}$ ; more precisely, by the transducer I' with the same set of states as I and transition function  $\delta'=\delta\cap\{Q_I\times\Sigma_{\mathbf{i}}\times\{\varnothing\}\to\mathsf{Asgn}_{R_k}\times\Sigma_{\mathbf{0}}\times R_k\times Q_I\}$ . Moreover,  $W_{S,k}^{\varnothing}\subseteq W_{S,k}^{\mathsf{tf}}$ . Indeed, let  $\overline{a}\in W_{S,k}^{\varnothing}$ . Then,  $\mathsf{Comp}(\overline{a})\subseteq S$ . It is easy to build by induction a data word  $w\in\mathsf{Comp}(\overline{a})\cap\mathsf{AllDiff}$ , so  $\mathsf{Comp}(\overline{a})\cap S\cap\mathsf{AllDiff}\neq\varnothing$ . Thus,  $W_{S,k}^{\mathsf{tf}}$  is realisable by any transducer realising  $W_{S,k}^{\varnothing}$ .

 $2\Rightarrow 3$ : Now, assume  $W^{\mathsf{ff}}_{S,k}$  is realisable by some transducer I. We show that I, when ignoring the  $\varnothing$  input tests, is actually an implementation of S. Thus, let I' be the same transducer as I except that all input tests  $\varnothing$  have been replaced with  $\top$ . Formally,  $q\xrightarrow[I']{\sigma_i,\top|\mathsf{asgn},\sigma_o,r} q'$  iff  $q\xrightarrow[I']{\sigma_i,\varnothing|\mathsf{asgn},\sigma_o,r} q'$  Note that I', interpreted as a register transducer, is test-free. Let  $w\in\mathsf{DW}$ , and  $\overline{a}_i=\mathsf{lab}(w)\times\varnothing^\omega$  be the input action in  $A_i^\varnothing$  with same finite labels as w. Let  $\overline{a}=I(\overline{a}_i)$ , and let  $w'\in\mathsf{Comp}(\overline{a})\cap S\cap\mathsf{AllDiff}$  (such w' exists because, as above,  $\mathsf{Comp}(\overline{a})\cap\mathsf{AllDiff}\neq\varnothing$ ). Then, since  $\mathsf{lab}(w)=\mathsf{lab}(w')$ , they admit the same run  $\rho^I$  in I, so  $w,w'\models o_{\rho^I}$ . Now,  $w'\in S$ , so it admits an accepting run  $\rho^S$  in S, which implies  $w'\models o_{\rho^S}$ . Moreover,  $w'\in\mathsf{AllDiff}$  so, by Proposition 4.9 2, we get  $o_{\rho^I}=o_{\rho^S}$ . Therefore,  $w\models o_{\rho^S}$ , so, by 3, w admits  $\rho^S$  as a run, i.e.  $w\in S$ . Overall,  $L(I)\subseteq S$ , meaning that I is a (test-free) implementation of S.

Finally,  $W^{\mathsf{tf}}_{S,k} = \{ \overline{a} \in (A^\varnothing_{\mathtt{i}} A^k_{\mathtt{o}})^\omega \mid \mathsf{Comp}(\overline{a}) \cap S \cap \mathsf{AllDiff} \neq \varnothing \}$  is regular. Indeed,  $W^{\mathsf{tf}}_{S,k} = \{ \overline{a} \in (A^\varnothing_{\mathtt{i}} A^k_{\mathtt{o}})^\omega \mid \mathsf{Comp}(\overline{a}) \cap S^\varnothing \neq \varnothing \}$ , where  $S^\varnothing$  is the same automaton as S except that all input transitions  $q \xrightarrow{\sigma_{\mathtt{i}}, \top, \mathsf{asgn}} q'$  have been replaced with  $q \xrightarrow{\sigma_{\mathtt{i}}, \bigwedge_{r \in R_k} r^{\neq}, \mathsf{asgn}} q'$ , because, for all  $\overline{a} \in (A^\varnothing_{\mathtt{i}} A^k_{\mathtt{o}})^\omega$ ,  $\mathsf{Comp}(\overline{a}) \cap S \cap \mathsf{AllDiff} \neq \varnothing \Leftrightarrow \mathsf{Comp}(\overline{a}) \cap S^\varnothing \neq \varnothing$  (the  $\Rightarrow$  direction is trivial, and the  $\Leftarrow$  stems from the fact that an  $\mathsf{AllDiff}$  input only takes  $\phi = \varnothing$  transitions).

Then,  $L_{S,k}^{\mathsf{tf}} = \{w \otimes \overline{a} \in \mathsf{RW} \otimes (A_{\mathtt{i}}^{\varnothing} A_{\mathtt{o}}^{k})^{\omega} \mid w \in \mathsf{Comp}(\overline{a}) \cap S^{\varnothing} \}$  is NRA-definable. Indeed, S is NRA-definable, so  $S^{\varnothing}$  is NRA-definable, and by Lemma 4.2,  $L_{k} = \{w \otimes \overline{a} \mid w \in \mathsf{Comp}(\overline{a})\}$  is NRA-definable, so their product recognises  $L_{S,k}^{\mathsf{tf}}$ . Finally,  $W_{S,k}^{\mathsf{tf}} = \mathsf{lab}(L_{S,k}^{\mathsf{tf}})$ , and the projection of a NRA over some finite alphabet is regular [KF94].

Overall, by Theorem 4.1, we finally get (the complexity analysis is the same as for URA):

**Theorem 4.11.** For all  $k \geq 0$ , REAL(NRA<sub>ff</sub>, RT[k]) is decidable and in 2EXPTIME.

#### 5. Synthesis and Uniformisation

In this section, we discuss the connection between synthesis and uniformisation of relations, which is a more general problem: as pointed out in Section 2, if S is realisable by a register transducer, then, in particular, it has a total domain, i.e.  $\mathsf{inp}(S) = \mathsf{DW}(\Sigma_i, \mathcal{D})$ , otherwise it cannot be that  $L(T) \subseteq S$  for T a register transducer, since by definition of transducers  $\mathsf{inp}(T) = \mathsf{DW}(\Sigma_i, \mathcal{D})$ . However, when defining a specification, the user might be interested only in a subset of behaviours (for instance, s/he knows that all input data will be pairwise distinct). In the finite alphabet setting, since the formalisms used to express specifications are closed under complement (whether it is LTL or  $\omega$ -automata), it is actually not a restriction to assume that the input domain of the specification is total: it suffices to complete the specification by allowing any behaviour on the input not considered. However, since register automata are not closed under complement, such approach is not possible here. Thus, it is relevant to generalise the realisability problem to the case where the domain of the specification is not total. This can be done by equipping register transducers with an acceptance condition. It is also necessary to adapt the notion of realisability; otherwise, any transducer accepting no words realises any specification. (since it is always the case that  $\varnothing \subseteq S$ ). A natural way is to consider synthesis as a uniformisation problem [FJLW16] An (implementation) function  $f: \mathsf{In} \to \mathsf{Out}$  is said to *uniformise* a (specification) relation  $R \subseteq \mathsf{In} \times \mathsf{Out}$  whenever:

- (1) dom(f) = dom(R) and
- (2) for all  $i \in \text{dom}(f), (i, f(i)) \in R$

Note that constraint 1 is the main difference with the notion of realisability.

In the context of reactive synthesis, where  $f = f_I$  is defined from an implementation I and R is given as a language of relational words, it can be rephrased as

- (1)  $\mathsf{inp}(L(I)) = \mathsf{inp}(R)$  and
- (2) for all  $w_i \in \mathsf{inp}(L(I)), \langle w_i, f_I(w_i) \rangle \in R$

Note that such definition coincides with the one of realisability of Section 2 when the class of implementations has total domain, because then it is equivalent to asking  $L(I) \subseteq R$ . In the following, we denote by  $\text{UNIF}(\mathcal{S}, \mathcal{I})$  the uniformisation problem from specifications in  $\mathcal{S}$  to implementations in  $\mathcal{I}$ . Unfortunately, this setting is actually much harder, as shown by the next two theorems:

**Theorem 5.1.** Given S a specification represented by a DRA, checking whether  $inp(S) = DW(\Sigma_i, \mathcal{D})$  is undecidable.

*Proof.* We reduce from the universality problem of NRA, which is undecidable [NSV04]. Let  $A = (\Sigma, \mathcal{D}, Q, q_0, \delta, R, c)$  be an NRA. We encode L(A) as the domain of some DRA

specification: the input transitions are the same as the transitions of the original automaton, but when there is some nondeterminism, its resolution is postponed to the corresponding output transition, whose finite label corresponds to the chosen transition. In the vocabulary of games, the input player chooses the finite input label and the equality relation of the input data to the registers of A, and the output player resolves the nondeterminism. Thus, we construct a DRA D accepting  $R(D) = \{((\sigma_1, d_1)(\sigma_2, d_2) \dots, (t_1, d_1)(t_2, d_2) \dots) \mid t_1 t_2 \dots$  is a run of A over  $(\sigma_1, d_1)(\sigma_2, d_2) \dots \}$ .

Thus, define  $D = (\Sigma \uplus \delta, \mathcal{D}, Q \uplus Q \times (\Sigma \times \mathsf{Tst}_R), q_0, \delta', R \uplus \{r_0\}, c')$ , where  $\delta'$  is defined as follows: for all  $q \in Q$ ,  $\sigma \in \Sigma$  and  $\phi \in \mathsf{Tst}_R$ , we define the input transition  $q \xrightarrow{\sigma,\phi,\{r_0\}} (q,(\sigma,\phi))$ . Then, for all  $t = q \xrightarrow{\sigma,\phi,\mathsf{asgn}} q' \in \delta$ , we define the output transition  $(q,(\sigma,\phi)) \xrightarrow{\sigma,\phi \wedge r_0^-,\mathsf{asgn}} q'$ . Then, let  $c': q \mapsto c(q)$  and  $(q,\bullet) \mapsto c(q)$ . Such automaton is indeed deterministic, and it recognises the relation  $R(D) = \{((\sigma_1,d_1)(\sigma_2,d_2)\dots,(t_1,d_1)(t_2,d_2)\dots) \mid t_1t_2\dots$  is a run of A over  $(\sigma_1,d_1)(\sigma_2,d_2)\dots\}$ . Then,  $\mathsf{inp}(R(D))$  is universal iff L(A) is universal.

Such result extends to NRA and URA, whose DRA are a special case. Note that the unbounded realisability problem for DRA is not reducible to deciding whether the domain is total: if the specification S is not realisable, it is not possible to determine whether it is because the domain of S is not total or because S is not realisable by a sequential machine (e.g. S asks to output right away a data that will only be input in the future).

Then, while the uniformisation setting obviously preserves the undecidability results from the synthesis setting, the above result allows to show that the somehow more general uniformisation problem is undecidable. For instance, we can prove:

**Theorem 5.2.** For all  $k \ge 1$ , UNIF(URA, RT[k]) is undecidable.

Proof. Consider some unrealisable URA specification  $S_u$  and the following specification S mapping  $w_1 \# w_2$  to  $w_1 \# w_2'$  such that  $(w_2, w_2') \in S_u$ , defined only when  $w_1$  is a finite data word accepted by some URA A. Clearly, S is URA-definable and realisable iff its domain is empty, i.e.  $L(A) = \emptyset$ . However, emptiness of URA is an undecidable problem.

If the domain of the specification is DRA-recognisable, it is possible to reduce the uniformisation problem to realisability, by allowing any behaviour on the complement on the domain (which is then DRA-recognisable). However, such property is undecidable as a direct corollary of Theorem 5.1.

## Conclusion

In this paper, we have given a picture of the decidability landscape of the synthesis of register transducers from register automata specifications. We studied the parity acceptance condition because of its generality, but our results allow to reduce the synthesis problem for register automata specifications to the one for finite automata while preserving the acceptance condition. We have also introduced and studied test-free NRA, which do not have the ability to test their input, but still have the power of duplicating, removing or copying the input data to form the output. We have shown that they allow to recover decidability in presence of non-determinism, in the bounded synthesis case. We leave open the unbounded case, which we conjecture to be decidable. As future work, we want to study synthesis problems for specifications given by logical formulae, for decidable data words logics such as two-variable fragments of FO [BMS<sup>+</sup>06, SZ12, DFL18].

### ACKNOWLEDGMENT

The authors would like to thank Ayrat Khalimov for his remarks and suggestions, which helped improve the quality of the paper.

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