Linear Algebra over Polynomial Rings

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Introduction

The main question I will address in this talk is

> *How does the rank of a matrix A with entries in a ring of polynomials $\mathbb{F}[x_1, \ldots, x_k]$ depend on the parameters?*

In the simplest case of *no parameters* (matrices over the field $\mathbb{F}$), the question is answered by Gaussian elimination, which allows us to compute the *row canonical form* (RCF) of the matrix.

In the case of *one parameter*, the polynomial ring $\mathbb{F}[x_1]$ is a PID, and Gaussian elimination combined with the Euclidean algorithm for GCDs allows us to compute the *Hermite normal form* (HNF).

For *two or more parameters*, we need the useful fact that

> $\operatorname{rank}(A) = r$ *if and only if at least one* $r \times r$ *minor is nonzero but every* $(r+1) \times (r+1)$ *minor is zero.*

Minors of a fixed size r in a given polynomial matrix A generate determinantal ideals DI_r of A in the polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$.

To find the zero sets $V(DI_r)$ of these ideals we compute their Gröbner bases and possibly Gröbner bases for the radicals $\sqrt{DI_r}$.

Large determinants are difficult to compute, especially with more than two parameters, since we cannot use Gaussian elimination.

This leads us to search for canonical or at least reduced forms of matrices to make the determinantal ideals easier to compute.

Canonical forms of matrices over $\mathbb{F}[x_1, \ldots, x_k]$ are very close to Gröbner bases for submodules of free modules over $\mathbb{F}[x_1, \ldots, x_k]$.

Matrices over a field $\mathbb{F}$

- A is an $m \times n$ matrix over the field $\mathbb{F}$
- rowspace(A) = subspace of $\mathbb{F}^n$ generated by rows of A
- $\mathrm{rank}(A) = \dim(\mathrm{rowspace}(A))$
- modules over a field are free (vector spaces with dimension)
- perform Gaussian elimination, use elementary row operations
- compute RCF (row canonical form) of matrix
- rank is the number of nonzero rows in the RCF
- colspace(A) = subspace of $\mathbb{F}^m$ generated by columns of A
- $\mathrm{rank}(A) = \dim(\mathrm{colspace}(A))$
- simultaneous row-column reduction: Smith normal form

Matrices over a PID

- A is an $m \times n$ matrix over the PID R, say $R = \mathbb{F}[x]$
- rowmodule(A) = submodule of R^n generated by rows of A
- structure theory for finitely generated R-modules $\Longrightarrow$ rowmodule(A) = free $\oplus$ torsion, but rowmodule$(A) \subseteq R^n$, and R^n is free R-module, so torsion $= 0$, rowmodule(A) = free.
- $\mathrm{rank}(A) = \mathrm{freerank}(\mathrm{rowmodule}(A))$
- assuming R is a Euclidean domain, we have an algorithm:
  - perform Gaussian elimination using elementary row operations
  - to compute pivots, perform Euclidean algorithm for GCDs using row operations to put GCD in pivot position
  - result is the HNF (Hermite normal form) of matrix
  - rank is the number of nonzero rows in the HNF
- same works for rows and columns (Smith normal form)

HNF algorithm in detail

Input: An $m \times n$ matrix A with entries in a Euclidean domain R.

1. Set $H \leftarrow A$ and $i \leftarrow 1$ and $j \leftarrow 1$.
2. While $i \leq m$ and $j \leq n$ do:
   If $h_{kj} = 0$ for $k = i, \ldots, m$ then (all entries 0 at/below pivot)
   - Set $j \leftarrow j + 1$

   else
   - While there is a nonzero entry (strictly) below the pivot:
     1. Find k with $i \leq k \leq m$ such that h_{kj} has minimal degree (depending on R) among nonzero entries at/below pivot.
     2. If $i \neq k$ then interchange rows i and k.
     3. Normalize h_{ij} depending on R (e.g. monic for polynomials).
     4. Use add-multiple row operations to reduce entries below pivot to their remainders modulo h_{ij}.
   - Use add-multiple row operations to reduce entries above pivot to their remainders modulo h_{ij}.
   - Set $i \leftarrow i + 1$ and $j \leftarrow j + 1$.
3. Return H.

Matrices over $\mathbb{F}[x_1, \ldots, x_k]$, $\mathbb{F}$ field, $k \geq 2$

> *"Linear algebra over rings is lots more fun than over fields." (R. Wiegand, "What is . . . a syzygy?", Notices of the American Mathematical Society, April 2006)*

One reason it's more fun is that we have to ask ourselves what we mean by the rank in this case, and there are different answers.

The simplest answer: The integral domain $R = \mathbb{F}[x_1, \ldots, x_k]$ is contained in its field of quotients $Q(R)$, the rational functions:

$$R = \mathbb{F}[x_1, \ldots, x_k] \subset \mathbb{F}(x_1, \ldots, x_k) = Q(R).$$

We can regard a matrix over R as a matrix over $Q(R)$ and apply Gaussian elimination to find its rank over $Q(R)$.

But when we divide by $f \in R$, we erase the information contained in the zeros of f, so the results will be not be valid in general.

Rank and determinants

A more useful notion of the rank of a matrix over $\mathbb{F}[x_1, \ldots, x_k]$ is given by taking the following characterization of the rank in the field case as the definition of the rank in the polynomial case.

Definition

Let A be an $m \times n$ matrix over the commutative (associative) unital ring R. For $1 \leq r \leq \min(m, n)$, by a **minor of rank** r we mean the determinant of any $r \times r$ submatrix of A.

Theorem

Let A be an $m \times n$ matrix over the field $\mathbb{F}$. Then the rank of A is r if and only if at least one minor of rank r is not 0, and every minor of A of rank $r + 1$ is 0.

If we replace the field $\mathbb{F}$ by the polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$, then

- "at least one minor of rank r is not 0" becomes "the ideal generated by the minors of rank r is not $\{0\}$", and
- "every minor of A of rank $r+1$ is 0" becomes "the ideal generated by the minors of rank $r+1$ is $\{0\}$".

Definition

Let A be an $m \times n$ matrix over the polynomial ring $\mathbb{F}[x_1, \ldots, x_k]$. For $0 \leq r \leq \min(m, n)$, the ideal $DI_r(A)$ generated by the minors of A of rank r is called the r-th **determinantal ideal** of A.

For $r = 0$, there is $\binom{m}{0}\binom{n}{0} = 1$ minor of rank 0, and it is nonzero when every 1×1 minor (every entry) of A is zero ($A = O$).
The 0×0 minor of any matrix A is 1, so $DI_0(A) = \mathbb{F}[x_1, \ldots, x_k]$.
The 1^{st} determinantal ideal $DI_1(A)$ is generated by the entries of A.

Definition

Let I be an ideal in $\mathbb{F}[x_1, \ldots, x_k]$. The **zero set** of I, denoted $V(I)$, is the set of all points in $\mathbb{F}^k$ which satisfy every polynomial $f \in I$:

$$V(I) = \{ (a_1, \ldots, a_k) \in \mathbb{F}^k \mid f(a_1, \ldots, a_k) = 0, \ \forall f \in I \}.$$

The special cases of A with rank $< r$ (strictly less than) are obtained by substituting the values $(a_1, \ldots, a_k)$ in the zero set $Z(DI_r(A))$ for the parameters $x_1, \ldots, x_k$ in A.

(Recall that rank $< r$ means every $r \times r$ minor is 0.)

The special cases of A with rank $= r$ correspond to the values

$$(a_1, \ldots, a_k) \in Z(DI_{r+1}(A)) \setminus Z(DI_r(A)).$$

(Rank $< r + 1$, but not $< r$.)

The relation between ideals and their zero sets is order-reversing:

$$I \subseteq J \iff V(I) \supseteq V(J).$$

(A smaller set of equations produces a larger set of solutions.)

Applying this to determinantal ideals, we first note that for any A,

$$V(DI_0(A)) = V(\mathbb{F}[x_1, \ldots, x_k]) = \emptyset.$$

We have a weakly increasing sequence of algebraic varieties in $\mathbb{F}^k$:

$$\emptyset = V(DI_0(A)) \subseteq V(DI_1(A)) \subseteq \cdots \subseteq V(DI_{\min(m,n)}(A)).$$

If at any step we have equality, $V(DI_r(A)) = V(DI_{r+1}(A))$, then there are no solutions of rank r. *Notation:* $V_r = V(DI_r(A))$.

Example 1

Consider this 4×4 matrix A with entries in $\mathbb{F}[x, y]$:

$$A = \begin{bmatrix} 0 & x & x & 1 \\ 0 & y & 1 & 1 \\ x & y & y & 1 \\ 0 & x & y & 0 \end{bmatrix}$$

- We have $DI_0 = \mathbb{F}[x, y]$ and $V_0 = \emptyset$.
- Since 1 is an entry of A we have $DI_1(A) = \mathbb{F}[x, y]$ and $V_1 = \emptyset$.
- The monic 2×2 minors of A are

$$\begin{array}{llllllll} x, & x-1, & y, & y-1, & y-x, & x^2, & yx, \\ yx-x, & yx-x^2, & y^2-x, & y^2-y, & y^2-yx. \end{array}$$

Hence $1 \in DI_2(A)$ giving $DI_2(A) = \mathbb{F}[x, y]$ and $V_2 = \emptyset$.

- The monic 3×3 minors of A are

$$\begin{array}{l} x^2, \quad x^2 - x, \quad yx, \quad yx - x, \quad yx - x^2, \\ y^2 - yx - y + x, \quad y^2 - 2yx + x^2, \quad y^2 - yx + x^2 - x, \\ yx^2 - x^2, \quad yx^2 - x^3, \quad y^2x - x^2, \end{array}$$

  Hence a Gröbner basis for $DI_3(A)$ is $\{x, y\}$ and $V_3 = \{(0, 0)\}$.

- The only 4×4 minor is the determinant

$$y^2x - yx^2 + x^3 - x^2 = x(y^2 - yx + x^2 - x).$$

  This has the solutions

$$V_4 = \begin{array}{l} \left\{(0, y) \mid y \in \mathbb{F}\right\} \cup \\ \left\{(x, y) \mid x \in \mathbb{F},\, y = \frac{1}{2}\left(x \pm \sqrt{x(4 - 3x)}\right)\right\}. \end{array}$$

Conclusions for rank(A):

- Since $V_1 = V_2 = \emptyset$, the matrix A never has rank 0 or 1.
- Since $V_3 = \{(0,0)\}$, the rank is 2 if and only if $x = y = 0$.
- The rank is 3 if and only if $(x, y) \neq (0,0)$ and $(x, y) \in V_4$.
- The rank is 4 if and only if $(x, y) \notin V_4$.
- Full rank occurs on a Zariski dense subset of $\mathbb{F}^2$.

This example illustrates what we mean by finding the rank of a matrix with entries in a polynomial ring: *finding explicitly how the rank depends on the values of the parameters*.

Over the field of rational functions $\mathbb{F}(x, y)$, the rank of A is 4, which is the maximal rank obtained from values of the parameters. This is usually called the *generic rank* of the matrix.

Generators for determinantal ideals can be hard to compute.

We need to calculate arbitrarily large determinants, but since the matrix entries are polynomials in more than one parameter, . . . *we can't use Gaussian elimination.*

A small example: an 8×24 matrix over $\mathbb{F}[x_1, \ldots, x_k]$ with $k \geq 2$. The last column contains the number of $r \times r$ minors:

r	$\binom{8}{r}$	$\binom{24}{r}$	$\binom{8}{r}\binom{24}{r}$
1	8	24	192
2	28	276	7728
3	56	2024	113344
4	70	10626	743820
5	56	42504	2380224
6	28	134596	3768688
7	8	346104	2768832
8	1	735471	735471

For each $r = 1, \ldots, \min(m, n)$, after we have computed all the minors which generate the determinantal ideal $DI_r(A)$, we then:

- compute the Gröbner basis for $DI_r(A)$ with respect to some monomial order, which may be difficult if the generating set consists of millions of polynomials of high degrees, and
- solve the system of polynomial equations (obtained by setting every Gröbner basis element to zero) to find the zero set of $DI_r(A)$; at this point it may (or may not) be helpful to first compute a Gröbner basis for the radical $\sqrt{DI_r(A)}$.

Since there are so many minors, we want to reduce the size of the matrix as much as possible before computing the minors.

We first recall the *Smith normal form* over a field or a PID.

Remark

Henry J. S. Smith was born in Dublin in 1826. His paper on normal forms is "On systems of linear indeterminate equations and congruences", Phil. Trans. R. Soc. Lond. 151 (1) (1861) 293–326.

Theorem

Let A be an $m \times n$ matrix over a field $\mathbb{F}$, or a PID R. There exist:

- *invertible matrices U $(m \times m)$ and V $(n \times n)$, and*
- *an $r \times r$ diagonal matrix D where $r = \operatorname{rank}(A)$,*

such that

$$UAV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, writing $D = \operatorname{diag}(d_1, \ldots, d_r)$ we may assume that $d_i \mid d_{i+1}$ for $i = 1, \ldots, r-1$ and $d_1, \ldots, d_r$ are invariant up to multiplication by units (so in the case of a field they are all 1).

Definition

The matrix UAV is the **Smith normal form** of the matrix A.

The Smith normal form can be computed using elementary row and column operations: a "two-sided" version of Gaussian elimination.

Now consider a matrix A with entries in $\mathbb{F}[x_1, \ldots, x_k]$ for $k \geq 2$:

- Typically, many of the entries of A will be nonzero scalars.
- Using elementary row and column operations, we move these nonzero scalars into the upper left corner of the matrix:
  - We put the nonzero scalars on the main diagonal.
  - We scale them to be leading 1s.
  - We use these leading 1s with row and column operations to eliminate the nonzero elements below and to the right.
- This creates the largest possible identity matrix I in the upper left corner, and forces all the information of the original matrix A into a (much smaller) block B in the lower right corner.
- $\text{rank}(A) = \text{rank}(I) + \text{rank}(B)$
- The size of I gives a lower bound on $\text{rank}(A)$.
- We need to compute the determinantal ideals only for B.

Definition

This process — using elementary row and column operations to reduce the original matrix A to an upper left identity block I and a lower right block B with *no nonzero scalar entries* — is called computing a **partial Smith form** of A.

If A has size $m \times n$ and entries in $R = \mathbb{F}[x_1, \ldots, x_k]$, then any partial Smith form of A belongs to the orbit of A under the action of the group $E_m(R) \times E_n(R)$ where E stands for the group generated by the elementary matrices:

$$A \longrightarrow UAV = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}, \quad U \in E_m(R), \quad V \in E_n(R).$$

Note: $GL_n(R)$ is not necessarily equal to $E_n(R)$ for general rings.

Example 2

Consider this 8×12 matrix over $\mathbb{F}[a, b]$, writing dot for zero to highlight the nonzero entries:

$$R = \begin{bmatrix} 1 & a & b & . & . & . & . & . & . & . & . & . \\ 1 & . & . & a & b & . & . & . & . & . & . & . \\ . & 1 & . & . & . & a & . & . & b & . & . & . \\ . & . & 1 & . & . & . & a & . & . & b & . & . \\ . & . & . & 1 & . & . & . & a & . & . & b & . \\ . & . & . & . & 1 & . & . & . & a & . & . & b \\ . & . & . & . & . & 1 & a & b & . & . & . & . \\ . & . & . & . & . & . & . & . & 1 & a & b & \end{bmatrix}$$

Every row has a leading 1; there are two leading 1s in column 1; there is a sequence of leading 1s just below the diagonal.

A partial Smith form for this matrix is as follows:

$$\left[\begin{array}{ccccccc|ccccc} 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . & . & . \\ \hline . & . & . & . & . & . & . & -a^2b-a^2 & . & -a^3+ab & -ab^2-ab & -b^3-b^2 \end{array}\right]$$

The upper left identity block has size 7; the lower right block B has size 1×5, with only four nonzero entries; in factored form:

$$-a^2(b+1), \qquad -a(a^2-b), \qquad -ab(b+1), \qquad -b^2(b+1).$$

The ideal $DI_1(B) \subset \mathbb{F}[a, b]$ generated by these four nonzero entries has this pure lex Gröbner basis ($a \prec b$):

$$a^2(a^2+1), \qquad a(b-a^2), \qquad b^2(b+1).$$

This is a zero-dimensional ideal, and its finite solution set is

$$(a, b,) \quad = \quad (0,0), \quad (0,-1), \quad (\pm i, -1).$$

For these values of (a, b) the original matrix R has rank 7.

For all other pairs $(a, b) \in \mathbb{F}^2$ the rank of R is 8.

Where did the last example come from?

Start with a ternary operation $(-,-,-)$ satisfying this relation R:

$$((-,-,-),-,-) + a(-,(-,-,-),-) + b(-,-,(-,-,-)) \equiv 0.$$

- In every monomial of arity (degree) n, the n dashes represent the n arguments $x_1, \ldots, x_n$ which always occur in the same order from left to right (which is why we can omit them).
- In other words, only the identity permutation of the subscripts can occur (this is what is known as a *nonsymmetric* operad).
- That is, $(-,-,-) = (x_1, x_2, x_3)$ and

$$\begin{aligned}((-,-,-),-,-) &= ((x_1,x_2,x_3),x_4,x_5),\\ (-,(-,-,-),-) &= (x_1,(x_2,x_3,x_4),x_5),\\ (-,-,(-,-,-)) &= (x_1,x_2,(x_3,x_4,x_5)).\end{aligned}$$

Consider the consequences of $R(-,-,-,-,-)$ obtained by substituting $(-,-,-)$ for one argument of R, for example

$$R(-,-,(-,-,-),-,-),$$

or R for one argument of $(-,-,-)$, for example

$$(-,R(-,-,-,-,-),-).$$

We obtain altogether $5+3=8$ different relations in arity 7:

$$\begin{array}{l}
(((---)--)--)+a((---)(---)-)+b((---)-(---)),\\
((-(---)-)--)+a(-((---)--)-)+b(-(---)(---)),\\
((--(---))--)+a(-(-(---)-)-)+b(--((---)--)),\\
((---)(---)-)+a(-(--(---))-)+b(--(-(---)-)),\\
((---)-(---))+a(-(---)(---))+b(--(--(---))),\\
(((---)--)--)+a((-(---)-)--)+b((--(---))--),\\
(-((---)--)-)+a(-(-(---)-)-)+b(-(--(---))-),\\
(--((---)--))+a(--(-(---)-))+b(--(--(---))).
\end{array}$$

There are 12 distinct ternary monomials in arity 7, which we can order as follows, to verify that we have them all:

$$
\begin{array}{lll}
(((---)--)--), & ((-(---)-)--), & ((--(---))--), \\
(-((---)--)-), & (-(-(---)-)-), & (-(--(---))-), \\
(--((---)--)), & (--(-(---)-)), & (--(--(---))), \\
((---)(---)-), & ((---)-(---)), & (-(---)(---)).
\end{array}
$$

Hence the space of relations in arity 7 which are consequences of the relation R in arity 5 is the row space of an 8×12 matrix, which is the matrix considered in the last Example.

Example 3

- For this, I'll start with the motivation from algebraic operads.
- The **associativity relation** $(ab)c \equiv a(bc)$ implies that we can omit parentheses in every degree without causing ambiguity.
- There are obvious and non-obvious analogues of associativity for two operations; the best known of the latter are the **diassociative relations** for **left** and **right** operations $\dashv$ and $\vdash$:

$$(a \dashv b) \dashv c \equiv a \dashv (b \dashv c), \qquad a \dashv (b \dashv c) \equiv a \dashv (b \vdash c),$$
$$(a \vdash b) \vdash c \equiv a \vdash (b \vdash c), \qquad (a \vdash b) \vdash c \equiv (a \dashv b) \vdash c,$$
$$(a \vdash b) \dashv c \equiv a \vdash (b \dashv c).$$

- On the left we have **left, right, and inner associativity**.
- On the right we have the **bar relations**: on the bar side of the operation symbols, the operation doesn't matter.

Theorem

These relations imply that any monomial m of degree n in the variables $a_1, \ldots, a_n$ (from left to right) with any placement of parentheses and any choice of operation symbols, has a uniquely defined **center** *a_i such that m is equal to its* **normal form**:

$$m = (a_1 \vdash \cdots \vdash a_{i-1}) \vdash a_i \dashv (a_{i+1} \dashv \cdots \dashv a_n).$$

Corollary

In the free diassociative algebra, there are n distinct normal forms in degree n for the monomial with the identity permutation of the variables (just as in the free associative algebra there is only one distinct normal form in every degree for the monomial with the identity permutation of the variables).

Question

- Are there any other sets of nonsymmetric relations in degree 3 for two operations (associative or nonassociative) which produce exactly n normal forms in degree n for all $n \geq 1$?
- In other words, how special are the diassociative relations?

For the general case, we'll use the operation symbols $\{\bullet, \circ\}$.

- There is always one normal form in degree 1, namely a_1.
- There are always two normal forms in degree 2: $a_1 \bullet a_2$, $a_1 \circ a_2$.

In degree 3, there are 8 distinct monomials: 2 placements of parentheses, and 2 choices of operations in each of 2 positions:

$$(a_1 \bullet a_2) \bullet a_3, \quad (a_1 \bullet a_2) \circ a_3, \quad (a_1 \circ a_2) \bullet a_3, \quad (a_1 \circ a_2) \circ a_3,$$
$$a_1 \bullet (a_2 \bullet a_3), \quad a_1 \bullet (a_2 \circ a_3), \quad a_1 \circ (a_2 \bullet a_3), \quad a_1 \circ (a_2 \circ a_3).$$

- $\exists$ 3 normal forms in degree 3 $\iff$ $\exists$ 5 independent relations.
- Represent relations by coefficient vectors in monomial basis.
- Span of relations is row space of unique 5×8 matrix in RCF.
- This is the *relation matrix* denoted R, and $\{R\} \leftrightarrow Gr(8,5)$.
- Grassmannian $Gr(8,5)$ of 5-dim'l subspaces of 8-dim'l space.
- Partition $Gr(8,5)$ by column indices of leading 1s in R.
- $\binom{8}{5} = 56$ cases for columns $J = \{j_1, \ldots, j_5\}$ of leading 1s.
- $R_{i,j_i} = 1$ $(1 \le i \le 5)$, $R_{i,k} = 0$ $(k < j_i)$, $R_{k,j_i} = 0$ $(k \ne i)$.
- If $j > j_i$ and $j \notin \{j_1, \ldots, j_5\}$ then $R_{i,j}$ is an indeterminate.
- $J = \{1, \ldots, 5\}$ (15 indets); $J = \{4, \ldots, 8\}$ (no indets).

I'll discuss in detail $J = \{1, 4, 5, 6, 8\}$, since it is a case for which

- the results are non-trivial, and
- the computations fit on the screen.

For $J = \{1, 4, 5, 6, 8\}$ the relation matrix is

$$R = \begin{bmatrix} 1 & x_1 & x_2 & . & . & . & x_3 & . \\ . & . & . & 1 & . & . & x_4 & . \\ . & . & . & . & 1 & . & x_5 & . \\ . & . & . & . & . & 1 & x_6 & . \\ . & . & . & . & . & . & . & 1 \end{bmatrix}$$

The rows of R represent these relations in degree 3:

$$\begin{aligned} (a_1 \bullet a_2) \bullet a_3 + x_1 (a_1 \bullet a_2) \circ a_3 + x_2 (a_1 \circ a_2) \bullet a_3 + x_3 a_1 \circ (a_2 \bullet a_3) &\equiv 0, \\ (a_1 \circ a_2) \circ a_3 + x_4 a_1 \circ (a_2 \bullet a_3) &\equiv 0, \\ a_1 \bullet (a_2 \bullet a_3) + x_5 a_1 \circ (a_2 \bullet a_3) &\equiv 0, \\ a_1 \bullet (a_2 \circ a_3) + x_6 a_1 \circ (a_2 \bullet a_3) &\equiv 0, \\ a_1 \circ (a_2 \circ a_3) &\equiv 0. \end{aligned}$$

We want to generate all their consequences in degree 5.

There are 40 monomials in degree 4:

- 5 ways to place parentheses (Catalan number),
- 2 choices for each of 3 operations,
- $5 \cdot 2^3 = 40$.

Any relation $f(a, b, c)$ in degree 3 has 10 consequences in degree 4:

$$f(a*d, b, c), \quad f(a, b*d, c), \quad f(a, b, c*d), \quad f(a, b, c)*d, \quad d*f(a, b, c),$$

where $* \in \{\bullet, \circ\}$.

The five relations for $J = \{1, 4, 5, 6, 8\}$ produce 49 distinct consequences (one repetition).

The consequences in degree 4 of the relations R in degree 3 are represented by a matrix of size 49×40.

Its partial Smith form has an identity block of size 35 and a lower right block of size 14×5.

The lower right block contains 6 zero rows and 1 zero column.

After deleting the zero rows and column we obtain (the transpose of) this 4×8 matrix B:

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & x_1 - x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_5 - x_1 & 0 & -x_6 & -x_3 & x_4 \\ -x_6 & -x_1x_2x_6 & x_4 & 0 & x_1^2x_2x_4 & 0 & 0 & 0 \\ 0 & 0 & -x_4 & 0 & -x_1x_2^2x_4 & 0 & -x_1x_2x_6 & -x_6 \end{bmatrix}$$

We want nullity 4 for the original 49×40 matrix, hence rank 36, and since the identity block has size 35, we want rank 1 for B:

$$DI_1(B) \neq \{0\}, \qquad DI_2(B) = \{0\}.$$

$DI_1(B)$ is the ideal generated by the entries of the matrix; with $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6$, its degrevlex Gröbner basis is

$$DI_1(B) = (\quad x_2 - x_1, \quad x_3, \quad x_4, \quad x_5 - x_1, \quad x_6 \quad) = \sqrt{DI_1(B)}.$$

Thus $B = O$ if and only if $x_1 = x_2 = x_5$ and $x_3 = x_4 = x_6 = 0$.

For $D_2(B)$ we need to compute all 2×2 minors; ignoring 0 and making the rest monic, we obtain these 29 polynomials, sorted according to the monomial order:

$$x_3x_2 - x_3x_1, \quad x_4x_2 - x_4x_1, \quad x_5x_2 - x_5x_1 - x_2x_1 + x_1^2,$$
$$x_6x_2 - x_6x_1, \quad x_4x_3, \quad x_6x_3, \quad x_4^2, \quad x_5x_4 - x_4x_1, \quad x_6x_4,$$
$$x_6x_5 - x_6x_1, \quad x_6^2, \quad x_6x_2^2x_1 - x_6x_2x_1^2, \quad x_6x_3x_2x_1, \quad x_6x_4x_2x_1,$$
$$x_6x_4x_2x_1 + x_6x_3, \quad x_6x_5x_2x_1 - x_6x_2x_1^2, \quad x_6^2x_2x_1, \quad x_4x_3x_2x_1^2,$$
$$x_4^2x_2x_1^2, \quad x_5x_4x_2x_1^2 - x_4x_2x_1^3, \quad x_6x_4x_2x_1^2, \quad x_4x_3x_2^2x_1,$$
$$x_4^2x_2^2x_1, \quad x_4^2x_2^2x_1 - x_4^2x_2x_1^2, \quad x_5x_4x_2^2x_1 - x_4x_2^2x_1^2, \quad x_6x_4x_2^2x_1,$$
$$x_6^2x_2^2x_1^2, \quad x_6x_4x_2^2x_1^3, \quad x_6x_4x_2^3x_1^2.$$

The degrevlex Gröbner basis (unfactored) for $DI_2(B)$ is

$$x_3x_2 - x_3x_1, \quad x_4x_2 - x_4x_1, \quad x_5x_2 - x_5x_1 - x_2x_1 + x_1^2,$$
$$x_6x_2 - x_6x_1, \quad x_4x_3, \quad x_6x_3, \quad x_4^2, \quad x_5x_4 - x_4x_1, \quad x_6x_4,$$
$$x_6x_5 - x_6x_1, \quad x_6^2.$$

$DI_2(B)$ is not a radical ideal; the Gröbner basis for $\sqrt{DI_2(B)}$ is

$$x_4, \quad x_6, \quad (x_2 - x_1)x_3, \quad (x_5 - x_1)(x_2 - x_1).$$

This gives two families of solutions:

$$x_1 = x_5, \quad x_2 = \text{free}, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = \text{free}, \quad x_6 = 0$$
$$x_1 = x_2, \quad x_2 = \text{free}, \quad x_3 = \text{free}, \quad x_4 = 0, \quad x_5 = \text{free}, \quad x_6 = 0$$

For these values of the parameters we have $\text{rank}(B) \leq 1$.

To get $\mathrm{rank}(B) = 1$ we must exclude the solutions in the zero set of $DI_0(B)$, namely $x_1 = x_2 = x_5$ and $x_3 = x_4 = x_6 = 0$.

We have only checked degree 4; it remains to check degrees $n \geq 5$.

This is only one of the easier cases out of a total of 56 cases for five relations in degree 3 on two binary operations.

So the original question of the uniqueness of the diassociative relations is still open!

Matrices and Submodules of Free Modules

Let A be an $m \times n$ matrix over $P = \mathbb{F}[x_1, \ldots, x_k]$, $k \geq 2$.

The row vectors of A belong to P^n, the free P-module of rank n.

Row module of A = submodule of P^n generated by rows of A.

Hence the row module of A is a submodule of a free P-module.

Submodules of a free P-module of rank 1 are simply ideals in P.

The very useful theory of Gröbner bases can be regarded as a theory of submodules of free P-modules of rank 1.

This was extended by Möller & Mora in 1986 to Gröbner bases for submodules of free P-modules of rank n (J. Algebra 100, 138–178).

Computing the Gröbner basis for the row module of the matrix A is essentially the same as computing a row canonical form for A (with respect to a given monomial order and order of the columns).

4. For every pair of indices k, k' such that $i \leq k \neq k' \leq m$ and the entries in positions (i, k) and (i, k') produce an S-polynomial with a nonzero reduced form modulo the entries in rows i through m, do the following:
   - Set $m \leftarrow m + 1$; add a new zero row at the bottom.
   - Use row operations to construct the S-polynomial in position $(m+1, j)$.
   - Compute its nonzero reduced form modulo the entries in rows i through m.
5. Repeat steps [1]–[4] until the entries at and below the pivot form a reduced Gröbner basis for the ideal they generate.
6. Delete any zero rows and modify m accordingly.
7. Use the Gröbner basis at and below the pivot to reduce the entries above the pivot to their normal forms.
8. Set $i \leftarrow i + 1$, $j \leftarrow j + 1$.

Example 4

Consider the 10×14 matrix A displayed in two parts below:

$$\begin{bmatrix}
0 & 0 & 0 & -ba^2-a^2 & 0 & ba-a^3 & 0 \\
b^2a^2+ba^2 & b^3a+b^2a & b^2a^2-2ba^4+a^6 & b^2a^4+b^2a^2 & b^3a^3-ba^3 & b^3a-b^2a^3+ba^5+ba^3 & 0 \\
0 & 0 & 0 & 0 & -ba^2-a^2 & 0 & 0 \\
0 & 0 & -ba^2+a^4 & 0 & 0 & -b^2a+ba^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ba-a^3 \\
0 & 0 & b^2a^2-2ba^4+ba^2+a^6-a^4 & b^2a^4+ba^4 & b^3a^3+b^2a^3 & b^3a-b^2a^3+b^2a+ba^5 & -b^2a+ba^3 \\
0 & 0 & ba-a^3 & ba^3+a^3 & b^2a^2+ba^2 & 0 & 0 \\
0 & 0 & ba-a^3 & -b^2a-ba & -b^3-b^2 & -ba^2-a^2 & 0 \\
-b^2a-ba & -b^3-b^2 & ba^3-a^5 & -b^2a^3-b^2a & -b^3a^2+ba^2 & -ba^4-ba^2 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
0 & -b^2a-ba & 0 & 0 & 0 & b^3a+b^2a & b^4+b^3 \\
0 & -b^4a+b^3a^3 & b^4a^2-b^2a^2 & -b^4a-b^3a & b^2a^2-ba^4+ba^2-a^4 & b^5a-b^3a & b^6-b^4 \\
0 & 0 & -b^2a-ba & 0 & ba-a^3 & -b^3-b^2 & 0 \\
0 & b^3a-b^2a^3 & -b^3a^2+ba^2 & b^3a+b^2a & -ba^2-a^2 & -b^4a+b^2a & -b^5+b^3 \\
0 & ba-a^3 & -ba^2-a^2 & 0 & 0 & -b^2a-ba & -b^3-b^2 \\
-ba^2-a^2 & 0 & 0 & -b^2a-ba & 0 & 0 & -b^3-b^2 \\
b^2a^2+ba^2 & -b^4a+b^3a^3-b^3a+b^2a^3 & b^4a^2+b^3a^2 & -b^4a-b^3a & b^2a^2-ba^4 & b^5a+b^4a & b^6+b^5 \\
0 & b^2a^2+ba^2 & b^3a+b^2a & -b^3-b^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ba^3+a^3 & 0 & 0
\end{bmatrix}$$

Column 1. The first column has two nonzero entries which generate the principal ideal $(ab^2 + ab)$.
We interchange rows 1 and 10, multiply row 1 by -1, and then subtract a times row 1 from row 2.
Column 1 is now zero except for the ideal generator in row 1.

Column 2. At this point column 2 has $b^3 + b^2$ in row 1 and zeros in the other rows, so it is already reduced.

Column 3. The entries in column 3 in row 2 and below generate the principal ideal $(ba - a^3)$:

$$ba - a^3, \quad -ba^2 + a^4, \quad b^2a^2 - ba^4, \quad b^2a^2 - 2ba^4 + ba^2 + a^6 - a^4.$$

Conveniently, the generator appears in row 8, so we swap it up to row 2 and use row operations to eliminate the entries below it.
The entry in row 1 is $-ba^3 + a^5$, which is $-a^2$ times the generator; we use one more row operation to make this entry zero too.

Column 4. The entries in column 4 in row 3 and below generate our first non-principal ideal:

$$-ba^2 - a^2, \qquad ba^4 + a^4, \qquad ba^6 - ba^4 + a^6 - a^4,$$
$$-b^2a - ba^3 - ba - a^3, \qquad -b^2a^4 - ba^4.$$

The pure lex ($a \prec b$) Gröbner basis for this ideal is

$$ba^2 + a^2, \qquad b^2a + ba.$$

The first of these polynomials already appears in the column so we swap it up to row 3 and use row operations to replace each of the lower entries by their remainders modulo this entry.
This makes all the lower entries zero except for $-b^2a - ba$ (row 9); we swap it up to row 4 and change its sign.
We now have the Gröbner basis in rows 3 and 4; we use it to reduce the entries in rows 1 and 2.

Column 5. The entries in column 5, in row 5 and below, generate the principal ideal $(ba^2 + a^2)$. The negative of the generator is the entry in row 10, so we swap it up to row 5, change its sign, and use row operations to eliminate the entries below it and reduce the entries above it.

The calculations get significantly more complicated at this point, so we record the state of the reduced part of the matrix after the reduction of column 5. The upper left 5×5 block is as follows, and the 5×5 block below it is the zero matrix:

$$\begin{bmatrix} b^2a + ba & b^3 + b^2 & 0 & -ba + a^3 & -b^3 - b^2 \\ 0 & 0 & ba - a^3 & 0 & 0 \\ 0 & 0 & 0 & ba^2 + a^2 & 0 \\ 0 & 0 & 0 & b^2a + ba & b^3 + b^2 \\ 0 & 0 & 0 & 0 & ba^2 + a^2 \end{bmatrix}$$

Column 6. The nonzero entries at or below the current pivot (6,6) are as follows, appearing once each in rows 9, 8, 7 respectively :

$$f = -b^2a + 2ba^3 - a^5,$$
$$g = b^3a - 2b^2a^3 + ba^5,$$
$$h = b^3a - b^2a^3 + b^2a + 2ba^5 - ba^3 - a^7 + a^5.$$

Clearly $g = -bf$ so we eliminate g by a row operation with f. Replacing f by $-f$, we get these generators of the column ideal:

$$f = b^2a - 2ba^3 + a^5, \qquad g = b^3a - b^2a^3 + b^2a + 2ba^5 - ba^3 - a^7 + a^5.$$

The normal form of g modulo f is $3ba^5 + ba^3 - 2a^7$, so the new generators are (renaming again):

$$f = 3ba^5 + ba^3 - 2a^7, \qquad g = b^2a - 2ba^3 + a^5.$$

We now have a self-reduced set, so we consider S-polynomials.

There is one, denoted s, corresponding to the overlap ab.

We also give its reduced form s' modulo f and g:

$$s = b^2a^3 + 4ba^7 - 3a^9, \qquad s' = 2ba^3 + 3a^9 + 5a^7.$$

Computing the reduced forms of f and g modulo s' gives the polynomials f' and g':

$$f' = a^{11} + 2a^9 + a^7, \qquad g' = b^2a + 3a^9 + 5a^7 + a^5.$$

We now verify that the ordered set $\{f', s', g'\}$ is a reduced pure lex Gröbner basis for the column ideal in this case.

Let us see how this can be translated into matrix terms.

Before computing the S-polynomial, we do these row operations:

- Interchange rows 6 and 9.
- Multiply row 6 by -1.
- Add $-b$ times row 6 to row 8.
- Add $-(a^2 + b + 1)$ times row 6 to row 7.
- Interchange rows 6 and 7.

At this point rows 6 and 7 contain (the last values of) f and g.

In order to construct the S-polynomial we need either

- to have a zero row in the matrix, or
- to add a new zero row at the bottom of the matrix.

Conveniently, row 8 is zero, and although this is not necessary, we start by swapping this zero row to the bottom of the matrix so that we can do our calculations there.

Recall that the S-polynomial is

$$b(3ba^5 + ba^3 - 2a^7) - 3a^4(b^2a - 2ba^3 + a^5) = b^2a^3 + 4ba^7 - 3a^9.$$

To compute the S-polynomial and the Gröbner basis, we perform these row operations:

- Interchange rows 8 and 10.
- Add $b \times$ (row 6) to row 10; add $-3a^4 \times$ (row 7) to row 10.
- Add $(-\frac{4}{3}a^2 - \frac{2}{9})$(row 6) to row 10; add $-a^2$(row 7) to row 10.
- Multiply row 10 by -9.
- Interchange rows 7 and 8, 6 and 7, 10 and 6.
- Add $(-\frac{3}{2}a^2 - \frac{1}{2})$(row 6) to row 7; add $-$(row 6) to row 8.
- Multiply row 7 by $-\frac{2}{9}$; interchange rows 6 and 7.

It remains to reduce the entries above the pivot with respect to the Gröbner basis.

These are the entries in the upper right corner of the following array, which is the upper left 8×6 block of the current matrix:

b^2a+ba	b^3+b^2	0	$-ba+a^3$	$-b^3-b^2$	$b^2a^2+ba^4-2ba^2-a^6+2a^4-a^2$
0	0	$ba-a^3$	0	0	ba^2-a^4
0	0	0	ba^2+a^2	0	$-ba+a^3$
0	0	0	b^2a+ba	b^3+b^2	$2ba^2-a^4+a^2$
0	0	0	0	ba^2+a^2	0
0	0	0	0	0	$a^{11}+2a^9+a^7$
0	0	0	0	0	$2ba^3+3a^9+5a^7$
0	0	0	0	0	$b^2a+3a^9+5a^7+a^5$

Column 7. Rows 1–5 and 10 contain 0, and row 9 contains $ab - a^3$. Rows 6–8 contain two (one is repeated) polynomials which are a direct result of the S-polynomial calculation from column 6. However, these polynomials are multiples of $ab - a^3$ and so the column ideal is principal:

$$-b(4a^4 - 3a^2b + 2a^2 - b)(ab - a^3), \qquad -b(12a^2 - 9b + 2)(ab - a^3).$$

These multipliers show us how to use row operations to use the leading entry of row 9 to make every other entry in column 7 zero.

Columns 8–14. Row 10 is not zero, so there is only one remaining leading entry. Finishing the reduction of the matrix is an easy exercise with the help of a computer algebra system.

Using column operations as well, we can show that the submodule generated by the rows of the matrix is free of rank 9.