BETWEENNESS GROUPS

J. A. H. SHEPPERD*.

Betweenness is a ternary relation, "b lies between a and c", and describes the order of the points on an undirected straight line, in the same way as linear order describes the order of the points on a directed straight line. Ordered groups may be generalized to betweenness groups, possessing a betweenness relation invariant under the group operation. The axioms of order, betweenness and separation of point pairs have already been discussed by the author [9] and definitions given of betweenness groups and separation groups. In this paper, the structures of betweenness groups are completely described in terms of ordered groups: the corresponding results for separation groups will be given in a later paper.

The class of betweenness groups is only slightly wider than that of ordered groups. A betweenness group may be of finite order, in which case it is one of the groups of order 4, both of which may be made betweenness groups. An infinite betweenness group may contain one (but only one) element of finite order, which is then of order 2. In this case, the cycle of order 2 is a direct factor, the complementary factor being an ordered group, convex with respect to, and naturally ordered by, the betweenness relation. A locally infinite betweenness group is an O-group (*i.e.* a group which can be ordered) but its betweenness relation is not necessarily that naturally induced by an order under which it is an ordered group. If it is not, then there exists a convex ordered normal sub-group of index 2, whose order determines the betweenness of the whole group in a simple manner. An ordered group is naturally a betweenness group. A group, with a normal sub-group of index 2 that is an ordered group, may be made a betweenness group provided that every automorphism of the sub-group, induced by an element outside the sub-group, is an order automorphism.

§1. *Definitions.*

The groups are written multiplicatively throughout with identity 1, and the betweenness relation "b lies between a and c" is written $[a, b, c]$†. The discussion of alternative systems of axioms in [9] led us

* Received 17 July, 1956; read 15 November, 1956.

† The notation $B[a, b, c]$ was used in [9], in order to distinguish betweenness from separation relations, but this is unnecessary here.

to a definition of betweenness in terms of the following axioms:

B1. *For any* a, b, c *either* $[a, b, c]$ *or* $[b, c, a]$ *or* $[c, a, b]$;

B2. $[a, b, c]$ *and* $[a, c, b]$ *if, and only if,* $b = c$;

B3. $[a, b, c]$ *implies* $[c, b, a]$;

B4. $[a, b, c]$ *and* $[a, c, d]$ *implies* $[b, c, d]$;

B5. $[a, b, c]$ *implies* $[gah, gbh, gch]$ *for all* g, h *in the group.*

A set whose elements possess a ternary relation satisfying the four axioms B1–B4 is a *betweenness set,* and a group whose elements form a betweenness set and satisfy in addition the axiom B5 is a *betweenness group.*

An *ordered set* is a set with a full (or linear) order relation "$<$" such that: for any a, b in the set either $a < b$ or $b < a$; $a < b$ and $b < a$ if, and only if, $a = b$; $a < b$ and $b < c$ implies $a < c$. Such an order relation naturally defines a betweenness in an ordered set, by the definition: $[a, b, c]$ if, and only if, $a < b < c$ or $c < b < a$. If the relation of a betweenness set arises in this way from an order of the set, then the betweenness is *linear* and is *induced* by the order relation*.

An *ordered group* is an ordered set whose order is invariant under the group operation (*i.e.* $a < b$ implies $gah < gbh$ for all g, h in the group), and it becomes a betweenness group under the betweenness induced by its order relation. Such a group is a *trivial betweenness group.* Following Neumann [5], we define an O-group as a group which can be ordered and a B-group as a group which can be made a betweenness group by some betweenness relation. A proper O-group contains at least 2 elements (and is therefore infinite) and a proper B-group contains at least 3 elements†.

The notation $[a_1, a_2, a_3, \ldots, a_n]$ is used for the linear betweenness relation on a set of n elements, induced by the order of the ordered set $(a_1, a_2, a_3, \ldots, a_n)$, and the notation $Cy[a, b, c, d]$ is used for the *cyclic betweenness* on four elements defined by $[a, b, c]$, $[b, c, d]$, $[c, d, a]$, $[d, a, b]$.

From the above definitions, it is deduced [9] that:

1.1. *A betweenness set either has linear betweenness or has cyclic betweenness and contains just four elements.*

---

* Partial betweenness relations (when B1 fails) are considered by Pitcher and Smiley [7] and Sholander [10]. They arise in a similar manner as generalizations of partial orders and could be used to define partial-betweenness groups.

† Thus excluding the group of order 2 with relations $[1, 1, 1]$, $[1, 1, a]$, $[1, a, a]$, $[a, a, a]$, $[a, a, 1]$, $[a, 1, 1]$.

There follow some useful transitivity laws:

1.2. $[a, b, c]$ *and* $[a, c, d]$ *implies* $[a, b, c, d]$;

1.3. $[a, b, c], [b, c, d]$ *and* $b \neq c$ *implies either* $[a, b, c, d]$ *or* $Cy\,[a, b, c, d]$.

§2. *Finite betweenness groups.*

A proper O-group is necessarily locally infinite [**3**], but this condition is relaxed slightly for B-groups.

2.1. *An element of finite order in a B-group is of order* 2 *or* 4.

Proof. Let g be an element in a B-group G, then for some betweenness of G either $[1, g, g^2]$ or $[g, 1, g^2]$ or $[1, g^2, g]$. These relations determine the relations between all the powers of g when $g^2 \neq 1$, as may be seen by considering the three cases.

(i) If $[1, g, g^2]$ then $[g, g^2, g^3]$ and hence either $Cy[1, g, g^2, g^3]$ and G is the cycle $\{g\}$ of order 4, or $[1, g, g^2, g^3]$. In this case assume also that for some $r \geqslant 3$ $[1, g, g^2, \ldots, g^{r-1}, g^r]$ when $[g, g^2, g^3, \ldots, g^r, g^{r+1}]$ and hence $[1, g, g^2, \ldots, g^r, g^{r+1}]$. An induction now shows that

$$[\ldots, g^{-r}, g^{-r+1}, \ldots, g^{-2}, g^{-1}, 1, g, g^2, \ldots, g^{r-1}, g^r, \ldots],$$

the betweenness relations for the powers of g are those induced by their natural order, and hence g is of infinite order.

(ii) If $[g, 1, g^2]$ then $[1, g^{-1}, g], [g^{-1}, g^{-2}, 1]$ and hence $[g, g^{-1}, g^{-2}, 1, g^2]$. Again it follows by induction that

$$[\ldots, g^{2r-1}, \ldots, g, g^{-1}, \ldots, g^{-2r+1}, \ldots, g^{-2r}, \ldots, g^{-2}, 1, g^2, \ldots, g^{2r}, \ldots]$$

and g is of infinite order.

(iii) If $[1, g^2, g]$ then $[g^{-2}, 1, g^{-1}]$ so that g^{-1} satisfies case (ii) and g is of infinite order, with

$$[\ldots, g^{-2r}, \ldots, g^{-2}, 1, g^2, \ldots, g^{2r}, \ldots, g^{2r-1}, \ldots, g, g^{-1}, \ldots, g^{-2r+1}, \ldots].$$

Clearly from the above proof, the cycle $\{g\}$ of order 4 is a B-group and is a betweenness group in just one way with cyclic betweenness $Cy[1, g, g^2, g^3]$. Furthermore, if a B-group G contains an element g of order 4, then under any betweenness in G the sub-group $\{g\}$ is also a betweenness group and hence has cyclic betweenness, therefore, by 1.1, $G = \{g\}$.

Consider now an element a of order 2 in a betweenness group G. Let g be any element in G, then if $[ag, ga, a]$ also $[aga, g, 1]$ and $[ga, ag, a]$, whence $ag = ga$. If however $[ag, a, ga]$, then $[1, ag, a, ga]$ implies $[a, g, 1]$, $[a, 1, g]$ and $g = 1$, whilst $[ag, 1, a, ga]$ implies $[g, a, 1]$, $[a, 1, g]$

and $a = 1$. Similarly $[ag, a, 1, ga]$ implies $a = 1$ and $[ag, a, ga, 1]$ implies $g = 1$. This proves that:

2.2. *An element of order 2 in a B-group lies in the centre.*

If a B-group G contains two distinct elements of order 2, say a and b, then they lie in the centre of G and hence $ab = c$ is also of order 2. Without loss of generality, suppose in some betweenness of G that $[a, b, c]$, then also $[1, c, b]$, $[c, 1, a]$, $[b, a, 1]$ so that $Cy[1, a, b, c]$ and, by 1.1, G is the four group. Conversely, the four group is a B-group and is a betweenness group in three possible ways. This completes the characterization of finite B-groups; the results are summarized in the following

2.3. THEOREM. *A finite group G is a proper B-group if, and only if,*

either (i) *G is the cycle $\{g\}$ of order 4, the only possible betweenness being cyclic, i.e. $Cy[1, g, g^2, g^3]$;*

or (ii) *G is the four group $\{1, a, b;\ a^2 = 1 = b^2,\ ab = ba = c\}$, there being three possible sets of betweenness relations in G, each of cyclic type, e.g. $Cy[1, a, b, c]$.*

The two finite B-groups arise from the adoption of the transitivity B4, which gives a "pseudo-betweenness" in Huntington's terminology [**1**]. In fact, B4 does not apply to cyclic betweenness, and if a B-group is to satisfy B4 in the positive sense that four elements may be found which satisfy the transitivity, then the group must be infinite, and locally infinite except for possibly one element of order 2, and any betweenness of the group must be linear. In this case, the betweenness transitivity B4 is no more general than Huntington's postulate 9 [**2**].

§3. *Infinite betweenness groups.*

With the possible exception of one element of order 2, an infinite betweenness group is locally infinite. The betweenness is linear so that the transitivities 1.2 and 1.3 become

3.1. $[a, b, c]$, $[a, c, d]$ *and* $[a, b, c]$, $[b, c, d]$, $b \neq c$ *both imply* $[a, b, c, d]$.

This is frequently used in subsequent work as is the next lemma, that the identity cannot lie between a conjugate pair.

3.2. *$[g, 1, f^{-1}gf]$ is false for all $g \neq 1$ and all f in an infinite betweenness group G.*

Proof. If $f^2 = 1$, then by 2.2 $f^{-1}gf = g$ and $[g, 1, g]$ is false for $g \neq 1$. If $f^2 \neq 1$ and $[g, 1, f^{-1}gf]$, then $[fg, f, gf]$. Now $[f, f^2, fg]$ implies $[1, f, g]$,

$[f, f^2, gf]$ and hence $[fg, f^2, f, f^2, gf]$ which is false. Similarly $[f, fg, f^2]$ implies $[f, gf, f^2]$ whilst $[fg, f, f^2]$ implies $[gf, f, f^2]$, each of which is inconsistent with $[fg, f, gf]$. Hence $[g, 1, f^{-1}gf]$ is false unless $g = 1$.

In the proof of 2.1, it is seen that in an infinite betweenness group G, $[1, g, g^2]$ if, and only if, the natural order of the powers of g induces their betweenness relations. This property serves to divide G into two sub-sets. Let H be the sub-set of all elements $h \varepsilon G$ such that $[1, h, h^2]$ and let $F = G - H$. Any element which is a square (including the identity) will belong to H, as will be seen from the proof of 2.1 [*cf.* (ii) and (iii)].

Both sub-sets H and F are convex with respect to the betweenness of G*. For suppose that $h, k \varepsilon H$ and that $[h, g, k]$ for some $g \varepsilon G$, then either $[1, g, k]$ or $[h, g, 1]$. Therefore assume, without loss of generality, that $[1, g, h]$ so that $[h, gh, h^2]$, $[g, g^2, gh]$ and hence $[1, g, g^2, gh, h^2]$ and $g \varepsilon H$. Furthermore, if $e, f \varepsilon F$ and $[e, h, f]$, then, since H is convex and contains the identity, $[e, 1, f]$ and so $[e^2, e, ef]$ and $[ef, f, f^2]$. But $e^2, f^2 \varepsilon H$ and hence $[1, e, ef]$, $[1, f, ef]$ which is inconsistent with $[e, 1, f]$, therefore $[e, h, f]$ is false and F is also convex.

The inverse (and in fact all the powers) of an element of H also belong to H, for $[1, h, h^2]$ implies $[h^{-2}, h^{-1}, 1]$. If $g, h \varepsilon H$, then $[g, 1, h]$ implies $[g^2, g, gh]$, $[gh, h, h^2]$ so that $[g^2, g, gh, h, h^2]$ and $gh \varepsilon H$ since H is convex, whilst if $[g, 1, h]$ is false, assume, without loss of generality, that $[1, g, h]$ when $[h, gh, h^2]$ and again $gh \varepsilon H$. If, on the other hand, $e, f \varepsilon F$, assume, without loss of generality, that $[1, e, f]$, then $[e, e^2, ef]$ and, since $e^2 \varepsilon H$ and F is convex, $ef \varepsilon H$. Therefore H is a sub-group of G and F is a single coset modulo H. Hence

3.3. *If F is not empty, H is a normal sub-group of index 2 in G.*

The group G, and therefore its sub-group H, is an ordered set under the linear order which induces its betweenness. However H is in fact an ordered group under this order.

3.4. *H is an ordered group under the order relation defined by : for any $g, h \varepsilon H$, and a fixed element $b \varepsilon H$, $g < h$ if, and only if, either $[1, b, g, h]$ or $[1, g, b, h]$ or $[1, g, h, b]$ or $[g, 1, b, h]$ or $[g, 1, h, b]$ or $[g, h, 1, b]$; and this order induces in H the betweenness of G.*

Proof. The definition of $g < h$ given in 3.4 is that used in [**9**] (§2.83) to make H an ordered set under an order which induces its betweenness. To show the invariance of the order relation, assume $g < h$ and $k \varepsilon H$, then either $[1, g, h]$ in which case $[h^{-1}, gh^{-1}, 1, g, h]$, or $[g, 1, h]$ implying $[gh^{-1}, h^{-1}, 1, h]$, or $[g, h, 1]$ implying $[gh^{-1}, 1, h^{-1}]$, and in each case $[gh^{-1}, 1, b]$ so that $gh^{-1} < 1$. It is easily seen that in fact $g < h$ if, and

* Convexity with respect to betweenness is used in the sense of Levi [**4**], that H is convex if, and only if, all the elements between two elements of H also belong to H.

only if, $gh^{-1} < 1$ and $h^{-1}g < 1$; but $gh^{-1} = gk(hk)^{-1}$ and $h^{-1}g = (kh)^{-1}kg$ so that $g < h$ implies $gk < hk$ and $kg < kh$, and G is an ordered group.

If F is empty, then $G = H$ is a (naturally) fully ordered group and is therefore a trivial betweenness group. When F is not empty, the results of 3.3 and 3.4 prove the following

3.5. THEOREM. *The elements h, such that $[1, h, h^2]$, in a non-trivial betweenness group G, form a (naturally) fully ordered normal sub-group H of index 2 in G.*

An infinite B-group G containing an element a of order 2 is the direct product of an O-group with the cycle of order 2 generated by a; for, in any betweenness of G, $a \in F = aH$ and by 2.2 a lies in the centre of G. It will follow from 3.8 that, conversely, the direct product of an O-group with a cycle of order 2 is a B-group. This is an example of a linear betweenness group which is not an O-group.

In a non-trivial betweenness group G the order of the sub-group H may be defined, more easily than in 3.4, by: $g < h$ if, and only if, $[g, h, f]$ for any $g, h \in H$ and $f \in F$. This order, which is either the same or the reverse of the order defined in 3.4, is defined as the *induced* order of H. The induced order of H can be extended to an order which induces the betweenness of G by defining: $h < f$ for all $h \in H$ and $f \in F$; $f_1 < f_2$ if, and only if, $ff_2 < ff_1$ in H, for $f, f_1, f_2 \in F$; this order may be described verbally as placing the elements of H, in their induced order, before all the elements of F, which when written as Hf are in the corresponding order reversed. Clearly, such a betweenness group is not a cyclically ordered group under a naturally induced relation*.

Since an ordered group contains no elements of finite order, it is not possible to extend the order of the ordered sub-group H, of a betweenness group G which contains an element of order 2, to form an ordered group containing any elements of $F = G - H$. However, if a betweenness group G contains no elements of finite order, the induced order of H can be extended to make the whole group G an ordered group.

3.6. THEOREM. *A non-trivial locally infinite betweenness group G is an ordered group under the extension of the induced order of its normal sub-group H defined by: $g < k$ if, and only if, $[(gk^{-1})^2, 1, f]$ for all $g, k \in G$ and $f \in F$.*

Proof. If $h \in H$, then $[1, h, h^2]$ so that $[h, 1, f]$ if, and only if, $[h^2, 1, f]$; also if $g, h \in H$, then $[gh^{-1}, 1, f]$ if, and only if, $[g, h, f]$. There-

* Cyclically ordered groups (with order relation represented by the order of the points on a directed circle) are studied by Rieger [8].

fore the above definition gives the induced order in H, for which H is an ordered group. For any $g, k \varepsilon G$, $[(gk^{-1})^2, 1, (kg^{-1})^2]$ since all squares lie in H, therefore either $g < k$ or $k < g$; also if $g < k$ and $k < g$ then $(gk^{-1})^2 = 1$ and, since G is locally infinite, $g = k$. The transitivity of the relation requires separate proofs in the eight possible cases, of which we give two as examples.

(i) If $g < h$, $h < f$, $g, h \varepsilon H$, $f \varepsilon F$, then $[gh^{-1}, 1, f]$ giving $[g, h, fh]$, $[gf^{-1}gf^{-1}, gf^{-1}hf^{-1}, ghf^{-1}]$ and hence $[(gf^{-1})^2, gf^{-1}hf^{-1}, f]$. Also $[g, h, f]$ so that $[gf^{-1}hf^{-1}, hf^{-1}hf^{-1}, hf^{-1}]$ and then $[gf^{-1}hf^{-1}, (hf^{-1})^2, f]$. But $[(hf^{-1})^2, 1, f]$ and hence $[(gf^{-1})^2, gf^{-1}hf^{-1}, (hf^{-1})^2, 1, f]$ and $g < f$.

(ii) If $f < g$, $g < h$, $g, h \varepsilon H$, $f \varepsilon F$, then $f < h$, for otherwise $h < f$ implies $g < f$ by (i) and then $g = h = f$.

The other cases are proved similarly and the transitivity of the order relation follows. Finally, suppose that in G $k < h$ and g is any element, then $[(kh^{-1})^2, 1, f]$ for $f \varepsilon F$, therefore $\left[\left(kg(hg)^{-1}\right)^2, 1, f\right]$ and $kg < hg$, whilst by 3.2 $[g(kh^{-1})^2 g^{-1}, 1, f]$, but $g(kh^{-1})^2 g^{-1} = \left(gk(gh)^{-1}\right)^2$ and so $gk < gh$. Thus G is an ordered group under the order defined.

It is now clear that an infinite B-group is either an O-group or the direct product of an O-group with a cycle of order 2. The existence in a non-trivial betweenness group of a fully ordered normal sub-group, of index 2, was established in 3.5, but a group containing such a sub-group is not necessarily a B-group. The group $\{a, b;\ b^{-1}ab = a^{-1}, b^2 = 1\}$ contains the naturally fully ordered normal sub-group of index 2, generated by a, but is not a B-group since b is not in the centre although $b^2 = 1$.

The characterization of the infinite B-groups is completed by a sufficient (and necessary) set of conditions for an infinite group to be a B-group, derived from similar conditions for an O-group given by Neumann [5] which we quote first in 3.7.

3.7. LEMMA. *The group G is an O-group if (and only if) it contains two sub-sets s^+ and s^-, such that*

(i) $s^+ \cup s^- = G - \{1\}$, *i.e. every element of G, except the identity, lies in s^+ or in s^-;*

(ii) $s^+ . s^+ \subset s^+$ *and* $s^- . s^- \subset s^-$, *i.e. s^+ and s^- are semi-groups;*

(iii) $t^{-1} s^+ t \subset s^+$ *for all $t \varepsilon G$, i.e. s^+ (and therefore also s^-) is self-conjugate in G.*

3.8. THEOREM. *The group G is an infinite B-group if, and only if, G contains a normal sub-group H, of index 1 or 2, and H contains two sub-sets*

s^+ and s^- such that

(i) $s^+ \cup s^- = H - \{1\}$;

(ii) $s^+ . s^+ \subset s^+$ *and* $s^- . s^- \subset s^-$;

(iii) $t^{-1} s^+ t \subset s^+$ *for all* $t \in G$, *i.e.* s^+ (*and therefore also* s^-) *is self-conjugate in* G.

Proof. Suppose that $[G:H] \leqslant 2$ and that the conditions (i), (ii) and (iii) are satisfied; then H is an O-group by 3.7, so let "$<$" be the order relation under which H is an ordered group, defined by

$$a < b \text{ if, and only if, } a^{-1} b \in s^+ \cup \{1\}.$$

If $H = G$, then G is a trivial betweenness group, with relation induced by this order.

If $H \neq G$, then let f be a fixed element in the set $F = G - H = Hf$, and define an extension to the order of H by

$$a < b \text{ if, and only if, either } a \in H \text{ and } b \in F$$
$$\text{or } a, b \in F \text{ and } bf < af \text{ in } H.$$

For the order now defined in G, if $a < b$ and $b < a$ then either $a, b \in H$ and $a = b$, or $a, b \in F$ and $bf < af$ and $af < bf$ in H, so that $af = bf$ and $a = b$. Also if $a < b$ and $b < c$ then either $a, b, c \in H$, or $a \in H$ and $c \in F$, or $a, b, c \in F$ and $cf < bf < af$ in H, and in each case $a < c$. Thus G is an ordered set and therefore also a betweenness set with induced betweenness defined by

$$[a, b, c] \text{ if, and only if, either } a < b < c \text{ or } c < b < a.$$

The elements of H that are greater than the identity are the elements of s^+ and hence form a self-conjugate set in G. Therefore if $y < z$ in H and $g \in G$, then $1 < zy^{-1}$ so that $1 < g^{-1} z g g^{-1} y^{-1} g$ and $g^{-1} y g < g^{-1} z g$. This preservation of order in H by any automorphism of H induced by an element outside H is used to show that the order in G is preserved on multiplication by an element of H and is reversed on multiplication by an element of F. Let $a < b$ and $h \in H$, then either $a, b \in H$, or $a \in H$ and $b \in F$ when $ah, ha \in H$ and $bh, hb \in F$, or else $a, b \in F$, when $bf < af$ and $fb < fa$ in H, so that $hbf < haf$ and $fbh < fah$ giving also $bhf < ahf$; in each case $ah < bh$ and $ha < hb$. Now let $a < b$ and $g \in F$, then either $a, b \in H$, $agf < bgf$ and $fga < fgb$ so that $gaf < gbf$, or $a \in H$ and $b \in F$ when $ag, ga \in F$ and $bg, gb \in H$, or else $a, b \in F$, $bf < af$ and $fb < fa$ giving $bg = bff^{-1}g < ag$ and $gb = gf^{-1}fb < ga$; in each case $bg < ag$ and $gb < ga$.

If $[a, b, c]$ and $g \in G$, then either $a < b < c$ or $c < b < a$, so that either $ag < bg < cg$ and $ga < gb < gc$ or $cg < bg < ag$ and $gc < gb < ga$, and hence $[ag, bg, cg]$, $[ga, gb, gc]$ and G is a betweenness group.

Conversely, if G is an infinite B-group, consider a relation for which G is a betweenness group. Then either G is a trivial betweenness group and $H = G$ is an ordered group, or G is non-trivial and, by 3.5, contains a fully ordered normal sub-group H of index 2. In that case, if $1 < h$ in the induced order of H and $f \varepsilon F = G - H$, then $[1, h, f]$ and hence $[1, f^{-1}hf, f]$ and $1 < f^{-1}hf$. In both cases the conditions of Theorem 3.8 are satisfied, and this completes the proof.

It is of interest to deduce from 3.6, 3.7 and 3.8 the following set of necessary and sufficient conditions for a group with a normal sub-group of index 2 to be an O-group.

3.9. THEOREM. *A locally infinite group G, with a normal sub-group H of index 2, is an O-group if, and only if, H contains two sub-sets s^+ and s^- such that*

(i) $s^+ \cup s^- = H - \{1\}$;

(ii) $s^+ . s^+ \subset s^+$ *and* $s^- . s^- \subset s^-$;

(iii) $t^{-1} s^+ t \subset s^+$ *for all* $t \varepsilon G$.

This theorem on ordered groups, which is obtained as a corollary of the results on betweenness groups, is however just a special case of a more general theorem: The statement of this theorem remains true when the index of H in G is finite (*cf.* [6]).

References.

1. E. V. Huntington and J. R. Kline, " Sets of independent postulates for betweenness ", *Trans. American Math. Soc.*, 18 (1917), 301–325.
2. ———, " A new set of postulates for betweenness with proof of complete independence", *Trans. American Math. Soc.*, 26 (1924), 257–282.
3. F. W. Levi, " Ordered groups ", *Proc. Indian Acad. Sc.*, 16 (1942), 256–263.
4. ———, " On Helly's Theorem and the axioms of convexity ", *J. Indian Math. Soc.*, 15 (1951), 65–76.
5. B. H. Neumann, " On ordered groups ", *American J. of Math.*, 71 (1949), 1–18.
6. ——— and J. A. Shepperd, " Finite extensions of fully ordered groups ", *Proc. Royal Soc.* (A).
7. E. Pitcher and M. F. Smiley, " Transitivities of betweenness ", *Trans. American Math. Soc.*, 52 (1942), 95–114.
8. L. S. Rieger, " O uspořádaných a cyklicky uspořádaných grupach " (On the ordered and cyclically ordered groups), *Věstnik Kralovské České Společnosti Nauk. Třida Matemat.-Přirodověd*, 1946 (6), 1947 (1), 1948 (1).
9. J. A. H. Shepperd, " Transitivities of betweenness and separation and the definitions of betweenness and separation groups ", *J. London Math. Soc.*, 31 (1956), 240–248.
10. M. Sholander, " Trees, lattices, order and betweenness " *Proc. American Math. Soc.* 3 (1952), 369–381.

The University,
Hull.