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Inverse problems for multiple invariant curves

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Planar polynomial vector fields which admit invariant algebraic curves, Darboux integrating factors or Darboux first integrals are of special interest. We solve the inverse problem for invariant algebraic curves with a given multiplicity and for integrating factors, under generic assumptions regarding the (multiple) invariant algebraic curves involved. In particular, we prove, in this generic scenario, that the existence of a Darboux integrating factor implies Darboux integrability. Furthermore, we construct examples in which the genericity assumption does not hold and indicate that the situation is different for these.

1. Introduction

In 1878, Darboux [8] published his original work on the integrability of polynomial differential equations in the plane. He showed how the existence of sufficiently many invariant algebraic curves forces the integrability of a polynomial system. Darboux's idea was to use the invariant algebraic curves $\{f_i = 0\}$ of the system for constructing an integrating factor of the form

$$\prod_{i=1}^{r} f_i^{l_i}.$$

It was shown in [11] that such integrating factors account for all polynomial differential systems with elementary first integrals. (For precise definitions and basic results on the Darboux theory see § 2.)

Darboux's integrability theory has been extended to take into account the multiplicity or coalescence of invariant algebraic curves through the existence of exponential factors [2, 4, 5, 7]. That is, the Darboux integrating factors are now considered to be of the form

$$\prod_{i=1}^{r} f_i^{l_i} \prod_{j=1}^{s} \exp\left(\frac{g_j}{h_j}\right),$$

where the functions $\exp(g_i/h_i)$ (i.e. the exponential factors) satisfy an equation similar to that for the f_i defining the invariant algebraic curves. Such functions are called *Darboux functions*. It has been shown by Singer [13] (see additional comments in [3], for example) that polynomial differential systems with Darboux integrating factors are exactly those with Liouvillian first integrals.

Among other results, [7] contains a study of a kind of inverse problem in the Darboux theory of integrability. More precisely, assuming certain algebraic conditions, [7] contains an explicit basis of the polynomial vector fields which have a given set of algebraic curves with a given multiplicity as invariant curves for their flow. Clearly, the study of such inverse problems can also greatly increase our knowledge of the direct applications of Darboux integrability.

In this paper, we wish to study this inverse problem in greater depth and detail, and show that this does indeed lead us to a new understanding of the consequences of Darboux integrability.

In § 3, assuming certain (generic) geometric conditions on the curves, we give precise bounds for the degrees of the polynomials in the defining equations of polynomial vector fields with given collections of multiple invariant algebraic curves (theorems 3.2 and 3.5). We refer the reader to § 3 for a precise definition of the multiplicity of an invariant algebraic curve.

In § 4, we study another type of 'inverse problem', in that we seek polynomial vector fields which have a given Darboux function as a first integral or integrating factor. Such studies have been carried out before [9, 14, 15]. Here, we extend the study to multiple curves. In theorem 4.2 we show, under generic conditions, that a polynomial vector field with a Darboux inverse integrating factor must admit a Darboux first integral.

Finally, in § 5 we construct several examples to show that the results of the previous sections do not hold without the imposition of some genericity assumptions.

Our approach is mainly algebraic and formal. A different (analytic-topological) approach to some of these questions is given by Żołądek [16]. We briefly sketch this for curves of multiplicity 1. Suppose that D is a Darboux integrating factor, without exponential terms, for the differential equation (2.1), below. Then

$$\phi = \int D(P\,\mathrm{d}y - Q\,\mathrm{d}x)$$

defines a multi-valued integral outside the set Z of zeros of the curves $\{f_i = 0\}$. The effect of passing around a non-trivial loop γ in the complement of Z takes ϕ to $h_\gamma(\phi) = a_\gamma \phi + b_\gamma$ for some constants a_γ and b_γ. Under the conditions of theorem 3.6 (all the curves being non-multiple), the set Z, together with the line at infinity, forms a normal crossing divisor of $\mathbb{P}^2$, and it is well known that in this case its complement has an abelian fundamental group. This means that the maps h_γ commute. It is then straightforward to show that either $a_\gamma = 1$ for all γ or the addition of a constant to ϕ makes all the b_γ vanish. In the first case, $\phi - \sum c_i \log(f_i)$

is single valued for some constants c_i determined by the b_γ, and in the second, $\phi/\prod f_i^{m_i}$ is single valued for some constants m_i determined by the a_γ. Growth estimates show that these functions are rational and hence one obtains either $\exp(\phi)$ or ϕ as a Darboux first integral. When D contains exponential terms (or equivalently, algebraic curves with multiplicity greater than 1), it is necessary to discuss the extended monodromy group in Żołądek's approach, which seems to be more difficult to understand and to work with.

An algebraic approach to questions of Darboux integrability was given in [6] in the case of curves of multiplicity 1; it was extended to curves of multiplicity greater than 1 in [7], and is further extended here. It works on a more elementary and explicit level, and seems to be more appropriate for concrete computations. It also allows the clarification of the geometric non-degeneracy conditions which underlie the theorems; see theorem 3.6 and § 4. Moreover, algebraic tools are naturally suited for the (counter)examples in the final section. Thus, it seems reasonable and useful to pursue the algebraic approach in its own right.

2. Basic notions

We consider planar polynomial differential systems of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = P(x,y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y} = Q(x,y), \tag{2.1}$$

where $P, Q \in \mathbb{C}[x,y]$ and $t \in \mathbb{C}$ (although in applications we may wish to restrict t to only real values). We associate with the polynomial differential system (2.1) in $\mathbb{C}^2$ the polynomial vector field

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{2.2}$$

Sometimes, the polynomial vector field X will be denoted simply by (P,Q).

The *degree* m of the polynomial differential system (2.1) or of the polynomial vector field X will be denoted by δX and is the maximum of the degrees of the polynomials P and Q. The degree of a polynomial P is denoted by δP. The degree of a rational function P/Q is defined as $\delta(P/Q) = \max\{\delta P, \delta Q\}$.

A *first integral* of system (2.1) on an open subset U of $\mathbb{C}^2$ is an analytic function $H : U \to \mathbb{C}$ which is not constant on any connected component of U but is constant on every solution curve $(x(t), y(t))$ of (2.1) in U. We say that the polynomial system (2.1) is *integrable* on U if there is a first integral on U. Equivalently, we seek a function H on U such that $X(H)$ vanishes identically. This latter version is more useful, as it is easier to work with algebraically.

As usual, for an analytic function f, we denote by f_x and f_y the partial derivatives with respect to x and y, respectively. The Hamiltonian vector field X_f is defined by

$$X_f = -f_y\frac{\partial}{\partial x} + f_x\frac{\partial}{\partial y}, \tag{2.3}$$

and clearly admits f as a first integral.

An analytic function $R : U \to \mathbb{C}$ which is not identically zero on U is called an *integrating factor* of system (2.1) if it satisfies

$$X(R) = -R\,\mathrm{div}(X),$$

in U. As usual, the *divergence* of the vector field X is defined by

$$\mathrm{div}(X) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

If $R : U \to \mathbb{C}$ is an integrating factor of system (2.1) and $W = U \setminus \{R = 0\}$, then we call $V = 1/R : W \to \mathbb{C}$ an *inverse integrating factor* of (2.1).

We are interested in invariant algebraic sets for polynomial vector fields. The following result is well known, but we include it for completeness.

LEMMA 2.1. *Let X be a polynomial vector field and let $f_1, \dots, f_r$ be polynomials.*

(i) *If $X(f_i) \in \langle f_1, \dots, f_r\rangle$ for all i, then the common zero set Z of $f_1, \dots, f_r$ is invariant for the local flow of X.*

(ii) *Conversely, if the common zero set of $f_1, \dots, f_r$ is invariant for the local flow of X, then $X(f_i) \in \mathrm{rad}\langle f_1, \dots, f_r\rangle$ for all i, and in particular $X(f_i) \in \langle f_1, \dots, f_r\rangle$ for all i if the ideal is radical.*

Proof. (i) Since $X(f_i) \in \langle f_1, \dots, f_r\rangle$ for all i, there exist $W_{ij} \in \mathbb{C}[x, y]$ such that

$$X(f_i) = \sum_{j=1}^{r} W_{ij} f_j.$$

Let $v(t)$ be a solution of the differential equation such that $v(0) \in Z$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t} f_i(v(t)) = X(f_i)(v(t)) = \sum_{j=1}^{r} W_{ij}(v(t)) f_j(v(t)),$$

the $f_i(v(t))$ satisfy a homogeneous linear system of differential equations with coefficient matrix $(W_{ij}(v(t)))_{i,j}$, and initial value zero. By uniqueness, the $f_i(v(t))$ are identically zero, and thus $v(t) \in Z$.

(ii) Conversely, let $v(t)$ be a solution contained in Z, and $1 \leqslant i \leqslant r$. Then,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} f_i(v(t)) = X(f_i)(v(t)),$$

shows that $X(f_i)(v(0)) = 0$ and therefore $X(f_i)$ vanishes on Z. By the Hilbert Nullstellensatz (see, for example, [12]) we have $X(f_i) \in \mathrm{rad}\langle f_1, \dots, f_r\rangle$, and this ideal equals $\langle f_1, \dots, f_r\rangle$ if the latter is radical. □

In particular, let $f \in \mathbb{C}[x, y]$ be an irreducible polynomial. The algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of the polynomial vector field X if and only if, for some polynomial $K \in \mathbb{C}[x, y]$, we have

$$X(f) = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. If the polynomial vector field has degree m, any cofactor must have degree at most $m - 1$.

The following consequence of lemma 2.1 will be useful for constructions later on.

LEMMA 2.2. *Let X be a planar polynomial vector field, and let f_0 and f_1 be relatively prime polynomials. Moreover, assume that $X(f_0) = L_0 f_0$ for some polynomial L_0. Then, $X(f_1) = L_2 f_1 + L_1 f_0$ for suitable polynomials L_1, L_2 only if every point of the intersection of $\{f_0 = 0\}$ and $\{f_1 = 0\}$ is a critical point of X.*

In addition, if all intersections of $\{f_0 = 0\}$ and $\{f_1 = 0\}$ are transversal (i.e. the derivatives are linearly independent at all intersection points) then the condition is also sufficient.

Proof. The first assumption is a direct consequence of lemma 2.1, since the connected components of $\{f_0 = 0\} \cap \{f_1 = 0\}$ are points, due to relative primeness. For the second assertion, recall that the transversality condition holds if and only if the ideal $\langle f_0, f_1 \rangle$ is radical. (This may be proven using the Chinese remainder theorem: the ideal is radical if and only if $\mathbb{C}[x, y]/\langle f_0, f_1 \rangle$ is a direct sum of fields; see the arguments in [14], for instance.) $\square$

If two invariant algebraic curves coalesce, then we obtain in a natural way the notion of exponential factors (see [2, 7]).

Let $g, h \in \mathbb{C}[x, y]$ and assume that g and h are relatively prime. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial vector field X if, for some polynomial $L \in \mathbb{C}[x, y]$,

$$X\left(\exp\left(\frac{g}{h}\right)\right) = L \exp\left(\frac{g}{h}\right).$$

As above we call L the *cofactor* of the exponential factor $\exp(g/h)$. From [2] the following result is known.

PROPOSITION 2.3. *If $\exp(g/h)$ is an exponential factor for the polynomial vector field X, then $h = 0$ defines an invariant algebraic curve of X.*

The next result is due essentially to Darboux [8] (at least in the case without the exponential factors).

THEOREM 2.4. *Suppose that a polynomial vector field X admits p irreducible invariant algebraic curves $\{f_i = 0\}$ with cofactors K_i for $i = 1, \dots, p$ and q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$.*

(i) *There then exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = \operatorname{div}(X),$$

if and only if the (multi-valued) function

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q}, \tag{2.4}$$

is an inverse integrating factor of X.

(ii) *There exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0,$$

if and only if the function given in (2.4) is a first integral of X.

The precise relationship between exponential factors and multiple invariant algebraic curves will be discussed in the next section.

3. Generalized invariant algebraic curves

We refer the reader to [7] for more details about the contents of this section. However, we have altered the notation slightly in order to facilitate the exposition here.

Consider a polynomial vector field X, with an invariant algebraic curve $\{f_0 = 0\}$. We shall assume throughout this section that f_0 is irreducible as a polynomial. We say that a polynomial $f_0 \in \mathbb{C}[x, y]$ admits a *generalized invariant algebraic curve of order* n *based on* $\{f_0 = 0\}$ if there are polynomials $f_1, \dots, f_{n-1}$ with $\delta f_i \leqslant \delta f_0$, and $L_0, L_1, \dots, L_{n-1} \in \mathbb{C}[x, y]$ such that

$$\left.\begin{aligned} X(f_0) &= L_0 f_0, \\ X(f_1) &= L_0 f_1 + L_1 f_0, \\ &\vdots \\ X(f_{n-1}) &= L_0 f_{n-1} + L_1 f_{n-2} + \cdots + L_{n-1} f_0. \end{aligned}\right\} \tag{3.1}$$

Condition (3.1) can be conveniently rewritten using matrices. We denote by I_n the $n \times n$ unit matrix, and let

$$\varepsilon = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ & & & 0 & 1 \\ 0 & & \cdots & & 0 \end{pmatrix}.$$

Then we represent the generalized invariant algebraic curve by the matrix

$$F = f_0 I_n + \varepsilon f_1 + \cdots + f_{n-1}\varepsilon^{n-1} = \begin{pmatrix} f_0 & f_1 & f_2 & \cdots & f_{n-1} \\ 0 & f_0 & f_1 & \cdots & f_{n-2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & f_0 & f_1 \\ 0 & \cdots & \cdots & 0 & f_0 \end{pmatrix}, \tag{3.2}$$

whence condition (3.1) becomes

$$X(B) = FL, \tag{3.3}$$

with

$$B = \begin{pmatrix} f_{n-1} \\ \vdots \\ f_0 \end{pmatrix}, \qquad L = \begin{pmatrix} L_{n-1} \\ \vdots \\ L_0 \end{pmatrix}.$$

The generalized curve is said to be *non-degenerate* if f_1 is not a multiple of f_0. Degenerate generalized invariant algebraic curves are of little interest as they can always be recast as non-degenerate ones (possibly of lower order) unless they are multiples of simple curves (i.e. just $k(\varepsilon)f_0$, for some polynomial k). From now on we will always assume non-degeneracy.

The *multiplicity* of $f_0 = 0$ with respect to X is the maximal order of all non-degenerate generalized algebraic curves based on $f_0 = 0$. Other notions of multiplicity and their interrelations are discussed in [7].

Generalized invariant algebraic curves occur naturally when one considers exponential factors. Using relation (3.2) and the logarithmic series for

$$\log\left(I_n + \left(\varepsilon\frac{f_1}{f_0} + \varepsilon^2\frac{f_2}{f_0} + \cdots\right)\right)$$

we get

$$\begin{aligned} \log(F) &= \log(f_0)I_n + \varepsilon\frac{f_1}{f_0} + \varepsilon^2\left(\frac{f_2}{f_0} - \frac{f_1^2}{2{f_0}^2}\right) + \cdots \\ &= F_0 I_n + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots + \varepsilon^{n-1}F_{n-1}. \end{aligned} \tag{3.4}$$

Here $F_0 = \log(f_0)$ and the F_i are rational functions in x and y of degree at most $i \cdot \delta f_0$ for $i \geqslant 1$. (To justify the notation: if one views $\log f_0$ as some analytic branch, then $\exp(\log(F)) = F$ is readily verified.) It is easy to show that the expressions $\exp(F_i)$ are exponential factors of the vector field X with cofactors L_i for $i = 1, \ldots, n-1$. Thus, a generalized invariant algebraic curve of order n gives rise to $n-1$ exponential factors $\exp(g_j/f_0^j)$ for $1 \leqslant j \leqslant n-1$, with g_i coprime to f_0. Conversely, exponential factors give rise to generalized invariant algebraic curves. Once again, more details can be found in [7].

The main topic of the present paper is the inverse problem for generalized and multiple invariant algebraic curves, and applications to Darboux integrals and integrating factors. Thus, given generalized invariant algebraic curves $F^{(i)}$, we wish to determine all polynomial vector fields X such that relations (3.3) hold for suitable vectors $L^{(i)}$. A solution of the inverse problem for generalized invariant algebraic curves was given in [7, theorem 5.11]. We will improve on this theorem to obtain bounds on the degrees of the arbitrary polynomials which appear in the solution. This will form an essential part of our proof of the main result about Darboux integrals and integrating factors. Moreover, we will explore the meaning of the ideal I introduced in [7], and show that it is trivial under some natural genericity conditions, answering a question posed in that paper.

We first consider the case of a single generalized invariant algebraic curve

$$F = f_0 I_n + \varepsilon f_1 + \cdots + \varepsilon^{n-1} f_{n-1}$$

based on a curve $\{f_0 = 0\}$ of degree k and order n in a polynomial vector field X, so that (3.3) holds for some appropriate matrix L.

We introduce the notation

$$X_F = \begin{pmatrix} X_{f_{n-1}} \\ \vdots \\ X_{f_0} \end{pmatrix}, \qquad X_{F'} = \begin{pmatrix} X_{F_{n-1}} \\ \vdots \\ X_{F_0} \end{pmatrix}, \qquad B' = \begin{pmatrix} F_{n-1} \\ \vdots \\ F_0 \end{pmatrix},$$

where X_f is the Hamiltonian vector field described in (2.3). Thus, X_F and $X_{F'}$ are vectors of operators.

Let $|F|$ be the determinant of the matrix F; thus, $|F| = f_0^n$ by (3.2). Moreover, let $F^{\mathrm{a}} = |F|F^{-1}$ be the adjoint matrix of F.

The Poisson bracket of two functions $f(x, y)$ and $g(x, y)$ is given by

$$\{f, g\} = f_x g_y - f_y g_x.$$

We denote by I the ideal generated in $\mathbb{C}[x, y]$ by $|F|$, $|F|F_{ix}$, $|F|F_{iy}$ and $|F|\{F_i, F_j\}$ for $i, j = 0, \dots, n-1$. Due to a non-trivial result of [7], each of these terms is a polynomial. In [7, theorem 5.12], the following characterization is given for polynomial vector fields which admit given generalized invariant curves.

Theorem 3.1. *Let polynomials $f_0, \dots, f_{n-1}$ be given, with associated matrix F, and let X_F, $X_{F'}$ and I be as above. If X is a polynomial vector field such that (3.1) holds for suitable polynomials L_i and $h \in I$, the vector field X admits a representation*

$$hX = |F|X_0 + C^{\mathrm{T}} F^{\mathrm{a}} X_F = |F|X_0 + C^{\mathrm{T}}|F|X_{F'}, \tag{3.5}$$

with some polynomial vector field X_0 and a vector of polynomials

$$C = (C_{n-1}, \dots, C_0)^{\mathrm{T}}; \quad C_i \in \mathbb{C}[x, y].$$

In particular, if $I = \langle 1 \rangle$, then we can write X in the form

$$X = |F|X_0 + C^{\mathrm{T}} F^{\mathrm{a}} X_F = |F|X_0 + C^{\mathrm{T}}|F|X_{F'}. \tag{3.6}$$

Theorem 3.1 is a generalization of [6], Theorem 1; see also [14] for vector fields with one invariant algebraic curve. This theorem does not provide information on the degrees of the arbitrary polynomials which appear in (3.6). In the following theorem, given additional conditions, we determine degree bounds for these. These bounds will be an essential point in the analysis in the following section.

Theorem 3.2. *Let $f_0 = 0$ be an invariant algebraic curve of multiplicity n with $\delta f_0 = k$ and let F be its associated matrix (3.2). Assume that the following hold:*

(i) *$I = \langle 1 \rangle$;*

(ii) *f_0 has distinct factors in the highest-order terms;*

(iii) *f_0 and f_1 have no common factors in the highest-order terms.*

Then all the polynomial vector fields X having F as a multiple invariant algebraic curve can be written in the form (3.6) with $\delta X_0 \leqslant \delta X - nk$ and $\delta C_i \leqslant \delta X - nk + 1$ for $i = 0, \dots, n-1$.

As a preliminary step before proving this result, we state and prove an auxiliary result. We denote by f^* the terms of highest degree of the polynomial $f \in \mathbb{C}[x, y]$. Moreover, F^* is determined for F in (3.2) by replacing f_i by f_i^* for $i = 0, \ldots, n-1$; and in a similar way we define C^*, X_0^* and X_{F^*}.

LEMMA 3.3. *Let $f_0 = 0$ define an invariant algebraic curve of multiplicity n for X, and let F be given by (3.2), and C, X_0 and X_F as in (3.6). If f_0 satisfies the assumptions of theorem 3.2 and the equality*

$$|F^*|X_0^* + C^{*\mathrm{T}} F^{*a} X_{F^*} = 0 \tag{3.7}$$

holds, then there exists an $n \times 1$ matrix $\bar{C}$ such that $C^{*\mathrm{T}} = \bar{C}^{*\mathrm{T}} F^*$ and $X_0^* = -\bar{C} X_{F^*}$.

Proof.

STEP 1. For $n > 1$ we have

$$F^* = f_0^* \left(I_n + \frac{f_1^*}{f_0^*} \varepsilon + \cdots + \frac{f_{n-1}^*}{f_0^*} \varepsilon^{n-1} \right) = f_0^* (I_n + N),$$

and hence we get

$$F^{*-1} = f_0^{*-1} (I_n - N + N^2 - \cdots + (-1)^{n-1} N^{n-1}).$$

Therefore, the only entry of $F^{*a} = f_0^{*n} F^{*-1}$ that is not a multiple of f_0^* is the upper-right one, since the contributions of $N, \ldots, N^{n-2}$ provide only multiples of f_0^*, and the only non-zero entry of N^{n-1} is $(f_1^*/f_0^*)^{n-1}$ in the upper-right position. Hence, the unique element in $C^{*\mathrm{T}} F^{*a} X_{F^*}$ that is not automatically divisible by f_0^* is $C_{n-1}^* f_1^{*n-1} X_{f_0^*}$.

STEP 2. After this technical preparation, we claim that $C^{*\mathrm{T}} = \bar{C}^{*\mathrm{T}} F^*$ for some $n \times 1$ matrix $\bar{C}$. For $n = 1$ we have

$$f_0^* X_0^* + C^* X_{f_0^*} = 0.$$

Since f_0^* has no multiple prime factors, no prime factor of f_0^* divides both entries of $X_{f_0^*}$, and $C^* \in \langle f_0^* \rangle$ follows. Proceed by induction. Condition (3.7), together with step 1, implies that f_0^* divides $C_{n-1}^* f_1^{*n-1} X_{f_0^*}$. Since f_0^* has no multiple factors, and f_0^* and f_1^* are relatively prime, we find that f_0^* divides C_{n-1}^*. We may therefore write

$$(C_{n-1}^*, C_{n-2}^*, \ldots, C_0^*) = (\tilde{C}_{n-1}, \tilde{C}_{n-2}, \ldots, \tilde{C}_0) \begin{pmatrix} f_0^* & f_1^* & \cdots & f_{n-1}^* \\ 0 & & I_{n-1} & \end{pmatrix},$$

for some polynomials $\tilde{C}_{n-1}, \ldots, \tilde{C}_0$. Substituting this in (3.7), we obtain

$$\begin{aligned} 0 &= f_0^{*n} X_0^* + \tilde{C}^{\mathrm{T}} \begin{pmatrix} f_0^* & f_1^* & \cdots & f_{n-1}^* \\ 0 & & I_{n-1} & \end{pmatrix} F^{*a} X_{F^*} \\ &= f_0^{*n} X_0^* + \tilde{C}^{\mathrm{T}} \begin{pmatrix} f_0^* & * & \cdots & * \\ 0 & & f_0^* F^*|_{n-1}^{\mathrm{a}} & \end{pmatrix} X_{F^*}, \end{aligned}$$

using the fact that the first row of F^*F^{*a} equals $(f_0^n, 0, \dots, 0)$, and abbreviating $F^*|_{n-1}$ for the matrix obtained from F^* by discarding the first row and column. Moreover, one can verify from step 1 that $F^*|_{n-1}^{\mathrm{a}} = f_0^{*-1} F^{*a}|_{n-1}$. Discarding the first element of X_0^*, X_{F^*} and $\tilde{C}$, we have

$$f_0^{*n-1} X_0^*|_{n-1} + \tilde{C}^{\mathrm{T}}|_{n-1} F^*|_{n-1}^{\mathrm{a}} X_{F^*}|_{n-1} = 0,$$

and the induction hypothesis shows that

$$\tilde{C}^{\mathrm{T}} = \hat{C}^{\mathrm{T}} F^*|_{n-1},$$

for a suitable $\hat{C}$. In summary,

$$\tilde{C}^{\mathrm{T}} = (\hat{C}_{n-1}, \hat{C}_{n-2}, \dots, \hat{C}_0) \begin{pmatrix} 1 & 0 \\ 0 & F^*|_{n-1} \end{pmatrix},$$

and therefore

$$C^{*\mathrm{T}} = \tilde{C}^{\mathrm{T}} \begin{pmatrix} f_0^* & f_1^* & \cdots & f_{n-1}^* \\ 0 & & I_{n-1} & \end{pmatrix} = \hat{C}^{\mathrm{T}} F^*.$$

Step 3. As for the last claim, note that

$$\begin{aligned} 0 &= |F^*| X_0^* + C^{*\mathrm{T}} F^{*a} X_{F^*} \\ &= |F^*| X_0^* + \bar{C}^{*\mathrm{T}} F^* F^{*a} X_{F^*} \\ &= |F^*| (X_0^* + \bar{C}^{\mathrm{T}} X_{F^*}), \end{aligned}$$

because $F^*F^{*a} = |F^*| I_n$. □

Proof of theorem 3.2. We assume that the vector field X is of the form (3.6) and let $r = \max\{\delta X_0, \delta C_i - 1\}$. If the degree bounds for the polynomial vector field X_0 and for the polynomials C_i are not satisfied, then (3.7) holds. By lemma 3.3 there exists a $\bar{C}$ such that $X_0^* = -\bar{C}^{*\mathrm{T}} X_{F^*}$ and $C^{*\mathrm{T}} = \bar{C^*}^{\mathrm{T}} F^*$. We take $C'^{\mathrm{T}} = C^{\mathrm{T}} - \bar{C^*}^{\mathrm{T}} F$ and $X_0' = X_0 + \bar{C^*}^{\mathrm{T}} X_F$. Then

$$\begin{aligned} |F| X_0' + C'^{\mathrm{T}} F^{\mathrm{a}} X_F &= |F| X_0 + |F| \bar{C}^{*\mathrm{T}} X_F + C^{\mathrm{T}} F^{\mathrm{a}} X_F - \bar{C^*}^{\mathrm{T}} F F^{\mathrm{a}} X_F \\ &= |F| X_0 + C^{\mathrm{T}} F^{\mathrm{a}} X_F \\ &= X. \end{aligned}$$

Moreover, $C'^*T = C^{*\mathrm{T}} - \bar{C}^{*\mathrm{T}} F^* = 0$ and $X_0^* + \bar{C}^{*\mathrm{T}} X_{F^*} = 0$, and hence $\delta X_0' \leqslant r$ and $\delta C_i' \leqslant r$. Therefore, the vector field $X = |F| X_0' + C'^{\mathrm{T}} F^{\mathrm{a}} X_F$ is of the form (3.6) and we have reduced the maximal degrees of X_0' and C' which appear in the expression. We continue this process until the bounds given in theorem 3.2 are reached. □

Now let us consider the case of several multiple invariant algebraic curves. We assume that the polynomial vector field X admits r generalized invariant algebraic curves $F^{(i)}$ of order n_i based on curves $\{f_0^{(i)} = 0\}$ for $i = 1, \dots, r$ and we define $n = n_1 + \cdots + n_r$. We then define the matrix F to be the block diagonal matrix with the blocks $F^{(i)}$, i.e.

$$F = \begin{pmatrix} F^{(1)} & & 0 \\ & \ddots & \\ 0 & & F^{(r)} \end{pmatrix}, \quad \text{where } F^{(i)} = \begin{pmatrix} f_0^{(i)} & f_1^{(i)} & \cdots & f_{n_i-1}^{(i)} \\ 0 & f_0^{(i)} & \cdots & f_{n_i-2}^{(i)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & f_0^{(i)} \end{pmatrix}. \tag{3.8}$$

The notions defined for one multiple invariant algebraic curve can be extended as follows. Let

$$B = \begin{pmatrix} B^{(1)} \\ \vdots \\ B^{(r)} \end{pmatrix}, \qquad B^{(i)} = \begin{pmatrix} f_{n_i-1}^{(i)} \\ \vdots \\ f_0^{(i)} \end{pmatrix},$$

and

$$L = \begin{pmatrix} L^{(1)} \\ \vdots \\ L^{(r)} \end{pmatrix}, \qquad L^{(i)} = \begin{pmatrix} L_{n_i-1}^{(i)} \\ \vdots \\ L_0^{(i)} \end{pmatrix}.$$

In a similar way, we define the vector B' by

$$B' = \begin{pmatrix} B'^{(1)} \\ \vdots \\ B'^{(r)} \end{pmatrix}, \quad \text{where } B'^{(i)} = \begin{pmatrix} F_{n_i-1}^{(i)} \\ \vdots \\ F_0^{(i)} \end{pmatrix},$$

and the $F_j^{(i)}$ are defined by $\log F^{(i)} = F_0^{(i)} + \varepsilon F_1^{(i)} + \cdots$. We note that

$$|F| = |F^{(1)}| \cdots |F^{(r)}| = (f_0^{(1)})^{n_1} \cdots (f_0^{(r)})^{n_r}$$

and again call F^{a} the adjoint matrix of F. In addition, we define the vectors

$$X_F = \begin{pmatrix} X_{F^{(1)}} \\ \vdots \\ X_{F^{(r)}} \end{pmatrix}, \qquad X_{F^{(i)}} = \begin{pmatrix} X_{f_{n_i-1}^{(i)}} \\ \vdots \\ X_{f_0^{(i)}} \end{pmatrix},$$

and

$$X_{F'} = \begin{pmatrix} X_{F'^{(1)}} \\ \vdots \\ X_{F'^{(r)}} \end{pmatrix}, \qquad X_{F'^{(i)}} = \begin{pmatrix} X_{F_{n_i-1}^{(i)}} \\ \vdots \\ X_{F_0{}^{(i)}} \end{pmatrix}.$$

In this case, let $I \subseteq \mathbb{C}[x, y]$ be the ideal generated by the polynomials

$$|F|, \quad |F|F_{kx}^{(i)}, \quad |F|F_{ky}^{(i)}, \quad |F|\{F_k^{(i)}, F_j^{(l)}\},$$

for $k, j = 1, \ldots, n_i - 1$ and $i, l = 1, \ldots, r$, where $F^{(i)}$ is defined in (3.8).

In what follows we use the notation

$$C^{\mathrm{T}} = ({C^{(1)}}^{\mathrm{T}}, \dots, {C^{(r)}}^{\mathrm{T}}), \quad \text{where } {C^{(i)}}^{\mathrm{T}} = (C^{(i)}_{n_i-1}, \dots, C^{(i)}_0) \in (\mathbb{C}[x,y])^{n_i}. \tag{3.9}$$

The following generalization of theorem 3.1 also appears in [7].

Theorem 3.4. *Let F be the matrix (3.8). Then, for each $h \in I$, any polynomial vector field X having the generalized invariant algebraic curves $F^{(i)}$, of order n_i for $i = 1, \dots, r$, can be written in the form*

$$hX = |F|X_0 + C^{\mathrm{T}} F^{\mathrm{a}} X_F = |F|X_0 + |F|C^{\mathrm{T}} X_{F'}, \tag{3.10}$$

where X_0 is some polynomial vector field and C is a vector of polynomials of the form (3.9). In particular, if $1 \in I$, then

$$X = |F|X_0 + C^{\mathrm{T}} F^{\mathrm{a}} X_F = |F|X_0 + |F|C^{\mathrm{T}} X_{F'}. \tag{3.11}$$

Again, under some additional conditions we obtain degree bounds for the vector fields in (3.11), as follows.

Theorem 3.5. *Let F be as in (3.8), and assume the following conditions:*

(i) *$I = \langle 1 \rangle$;*

(ii) *all the highest degree factors of $f_0^{(1)}, \dots, f_0^{(r)}$ are pairwise relatively prime;*

(iii) *for all i, $f_0^{(i)}$ and $f_1^{(i)}$ have no common factors in the highest-order terms for $i = 1, \dots, r$.*

Then all polynomial vector fields X having the generalized invariant algebraic curves $F^{(i)}$ of order n_i for $i = 1, \dots, r$ can be written in the form (3.11) such that X_0 is a polynomial vector field with $\delta X_0 \leqslant \delta X - k$ and C is as in (3.9) with $\delta C^{(i)}_{j_i} \leqslant \delta X - k + 1$ for all $i = 1, \dots, r$ and $j_i = 0, 1, \dots, n_i - 1$, where $k = n_1 \cdot \delta f_0^{(1)} + \dots + n_r \cdot \delta f_0^{(r)}$.

Proof. Let s be the maximum of δX_0 and the $\delta C^{(i)}_j - 1$, so that the highest degree occurring on the right-hand side of (3.11) will be $s^* = s + k$. As above, we indicate homogeneous highest degree terms of polynomials and polynomial vector fields by an asterisk.

Suppose the degree condition is not satisfied, and thus $s^* > \delta X$. Considering highest degree terms we find that

$$\vartheta_0 |F^*| X_0^* + \sum_{i=1}^{r} \vartheta_i {C^{*(i)}}^{\mathrm{T}} \frac{|F^*|}{|F^{*(i)}|} (F_i^*)^{\mathrm{a}} X_{F_i^*} = 0,$$

where we have used the block diagonal form

$$(F^*)^{\mathrm{a}} = \operatorname{diag}\left(\frac{|F^*|}{|F^{*(1)}|} {F^{*(1)}}^{\mathrm{a}}, \dots, \frac{|F^*|}{|F^{*(r)}|} {F^{*(r)}}^{\mathrm{a}} \right),$$

and set $\vartheta_i = 1$ if the corresponding term has degree s^*, $\vartheta_i = 0$ otherwise.

Assume that $\vartheta_k = 1$, $k \geqslant 1$. Then the only entry of $(|F^*|/|F^{*(k)}|){F^{*(k)}}^{\mathrm{a}}$ which is not divisible by $f_0^{*(k)}$ is in the upper-right position; cf. the proof of lemma 3.3 and note that the highest degree terms of the $f_0^{(i)}$ have mutually relatively prime linear factors. As in the proof of lemma 3.3, we may conclude that $f_0^{(k)}$ divides $C_{n_k-1}^{(k)}$, and this allows us to prove $C^{*(k)} = \bar{C}^{*(k)\mathrm{T}} F^{*(k)}$ for suitable $\bar{C}^{*(k)}$, by induction. The induction starts with all $n_i = 1$, and the relation

$$\vartheta_0 \prod_{j=1}^{r} f_0^{*(j)} X_0^* + \sum_{i=1}^{r} \vartheta_i C^{*(i)} \prod_{j \neq i} f_0^{*(j)} X_{f_i^*} = 0,$$

which forces $f_0^{*(k)}$ to divide C_k^* in the case when $\vartheta_k = 1$. Then the argument runs analogously to the proof of theorem 3.2. □

In order to appreciate the conditions underlying theorems 3.2 and 3.5, we provide a geometric interpretation of the condition '$1 \in I$', answering a question posed in [7]. We will give a set of sufficient conditions, which turns out to be generic for the polynomials and curves involved. We will use the matrix F in (3.8), and the subsequent notation.

THEOREM 3.6. *Consider generalized invariant curves $F^{(i)}$ of orders n_i based on the curves $\{f_0^{(i)} = 0\}$, for $1 \leqslant i \leqslant r$. Assume that the following conditions are satisfied.*

(i) *Each $f_0^{(i)}$ is irreducible and each curve $\{f_0^{(i)} = 0\}$ is non-singular, $1 \leqslant i \leqslant r$.*

(ii) *If $n_i > 1$, $1 \leqslant i \leqslant r$, then $f_0^{(i)}$ and $f_1^{(i)}$ are relatively prime and the curves $\{f_0^{(i)} = 0\}$ and $\{f_1^{(i)} = 0\}$ intersect transversally.*

(iii) *If $n_i > 2$, then $f_0^{(i)}$, $f_1^{(i)}$ and $f_2^{(i)}$ have no common zeros, $1 \leqslant i \leqslant r$.*

(iv) *If $r > 1$, then, for all distinct $i, j \in \{1, \ldots, r\}$, $f_0^{(i)}$ and $f_0^{(j)}$ are relatively prime, and the curves*

$$\{f_0^{(i)} = 0\} \quad \textit{and} \quad \{f_0^{(j)} = 0\}$$

intersect transversally. Moreover, if $r > 2$, then $f_0^{(i)}$, $f_0^{(j)}$ and $f_0^{(k)}$ have no common zeros for any three distinct i, j and k.

(v) *If $r > 1$, $i, j \in \{1, \ldots, r\}$ are distinct and $n_i > 1$, then $f_0^{(i)}$, $f_1^{(i)}$ and $f_0^{(j)}$ have no common zeros.*

Then $1 \in I$ holds.

Proof. By the Hilbert Nullstellensatz it is sufficient to exhibit certain generators of I which do not have a common zero. We will first consider the case of one generalized invariant curve F based on $\{f_0 = 0\}$ of order n, using the notation introduced prior to theorem 3.1. Here we have to verify the assertion if conditions (i)–(iii) hold. Then we will turn to the setting of several curves.

STEP 1. From (3.2) we have

$$F = f_0 \cdot (I + g_1 \varepsilon + \cdots + g_{n-1} \varepsilon^{n-1}),$$

with $g_i = f_i / f_0$, and therefore

$$\begin{aligned} \log F &= F_0 \cdot I + F_1 \cdot \varepsilon + \cdots + F_{n-1} \varepsilon^{n-1} \\ &= \log f_0 \cdot I + N - \tfrac{1}{2} N^2 + \tfrac{1}{3} N^3 \cdots + \frac{(-1)^n}{n-1} N^{n-1} + \cdots \end{aligned}$$

with $N := \sum_{i=1}^{n-1} g_i\varepsilon^i$. Since we can write N^k as

$$N^k = \sum_{\ell}\biggl(\sum_{i_1+\cdots+i_k=\ell} g_{i_1}\cdots g_{i_k}\biggr)\varepsilon^{\ell},$$

with summation over tuples $(i_1,\dots,i_k)\in\mathbb{Z}_+^k$, rearranging yields

$$F_\ell = \sum_{k}^{n-1}\frac{(-1)^{k+1}}{k}\biggl(\sum_{i_1+\cdots+i_k=\ell} g_{i_1}\cdots g_{i_k}\biggr). \tag{3.12}$$

We note that

$$(g_{i_1}\cdots g_{i_k})_x = \frac{1}{f_0^{k+1}}\sum_{m=1}^{k}(f_{i_1}\cdots\hat{f}_{i_m}\cdots f_{i_k})(f_{i_m x}f_0 - f_{0x}f_{i_m}), \tag{3.13}$$

where a '$\hat{\cdot}$' symbolizes deletion, and a similar expression holds for the partial derivatives with respect to y. Moreover, one verifies

$$\{g_i,g_j\} = \frac{1}{f_0^3}(f_0\{f_i,f_j\} - f_j\{f_i,f_0\} - f_i\{f_0,f_j\}), \tag{3.14}$$

$$\{\log f_0, g_i\} = \frac{1}{f_0^2}\{f_0,f_i\}, \tag{3.15}$$

for all $i,j\in\{1,\dots,n\}$.

Step 2. Now assume that z is a zero of I. Then $f_0(z)^n = 0$, hence $f_0(z) = 0$. From (3.12) we see that

$$F_{n-1} = \frac{(-1)^n}{n-1}g_1^{n-1} + \frac{1}{f_0^{n-2}}\cdot q,$$

for some polynomial q, and hence

$$\begin{aligned} 0 &= (|F|\cdot F_{n-1,x})(z) = (-1)^{n+1}f_1(z)^{n-1}f_{0x}(z),\\ 0 &= (|F|\cdot F_{n-1,y})(z) = (-1)^{n+1}f_1(z)^{n-1}f_{0y}(z), \end{aligned}$$

by (3.13). Since the curve $\{f_0 = 0\}$ is non-singular, we find that $f_1(z) = 0$.

Step 3. Now consider $|F|\cdot\{F_{n-1},F_{n-2}\}$, which is a polynomial by [7, lemma 5.9]. Since F_ℓ has the representation (3.12), we have to deal with Poisson brackets of type

$$p = \{g_{i_1}\cdots g_{i_k}, g_{j_1}\cdots g_{j_m}\}.$$

By the Leibniz rule and (3.14), expanding this term will produce a rational function with denominator f_0^{k+m+1}. We are only interested in the case $k+m\geqslant n-1$, since $|F|\cdot p$ is a polynomial multiple of f_0 in case $k+m<n-1$, and such terms will vanish upon substituting a zero z of I.

STEP 4. In the case when $n = 2$ we have

$$|F| \cdot \{F_1, F_0\} = \{f_1, f_0\},$$

by (3.12) and (3.15). Since any zero of I must be a common zero of f_0, f_1 and $\{f_1, f_0\}$ and this contradicts the transversality condition, we conclude that I admits no zero.

Now assume that $n > 2$ is even. Then the representation of F_{n-1} contains the term $g_1 g_2^{(n-2)/2}$ (with $n/2$ factors) and that of F_{n-2} contains the term $g_2^{(n-2)/2}$ (with $(n-2)/2$ factors), both with non-zero coefficients. By (3.14) we have

$$\begin{aligned}|F| \cdot \{g_1 g_2^{(n-2)/2}, g_2^{(n-2)/2}\} &= |F| \cdot g_2^{(n-2)/2} \cdot \frac{n-2}{2} g_2^{(n-4)/2} \cdot \{g_1, g_2\} \\ &= \frac{n-2}{2} f_2^{n-3}(f_0\{f_1, f_2\} - f_2\{f_1, f_0\} - f_1\{f_0, f_2\}).\end{aligned}$$

Substituting a zero z of I yields $f_2(z)\{f_1, f_0\}(z) \neq 0$, since z is a transversal intersection point of $\{f_0 = 0\}$ and $\{f_1 = 0\}$, and f_2 does not vanish at z.

We will show next that all other relevant Poisson brackets p (as in step 3) which involve a total of $n-1$ or more factors will contain a term g_1^d with some $d > 1$ in one of the entries. By the Leibniz rule we then have

$$|F| \cdot p = g_1 \cdot p^* = f_1 \cdot q,$$

with a suitable polynomial p^* and a suitable rational function q. Substitution of a zero z of I will thus yield 0 for each such bracket, and in total we find

$$(|F| \cdot \{F_{n-1}, F_{n-2}\})(z) \neq 0,$$

which is in contradiction with the assumption that I admits the zero z. We will be finished when the two claims below are proven.

CLAIM 3.7. *In (3.12) for F_{n-1}, every term with more than $n/2$ factors, as well as every term with $n/2$ factors that is different from $g_1 g_2^{(n-2)/2}$, contains a factor g_1^d with $d > 1$.*

To prove this, assume that $i_1 + \cdots + i_k = n-1$. If all $i_s \geqslant 2$, then $n-1 \geqslant 2k$ and $n-2 \geqslant 2k$ because n is even. So, $(n-2)/2 \geqslant k$. Thus, for $k \geqslant n/2$ there is at least one factor g_1. Furthermore, assume that, for instance, $i_1 = 1$ and all other $i_s \geqslant 2$. Then $n-2 \geqslant 2(k-1)$, so $n/2 \geqslant k$. Thus, for $k > n/2$ we must have a factor g_1^d with some $d > 1$. If $k = n/2$ and there is a simple factor g_1 (thus all $i_s \geqslant 2$ for $s > 1$), we find that

$$n-2 = i_2 + \cdots + i_{n/2} \geqslant ((n-2)/2) \cdot 2 = n-2,$$

which is possible only for $i_2 = \cdots = i_{n/2} = 2$. Hence, claim 3.7 is proved.

CLAIM 3.8. *In (3.12) for F_{n-2}, every term with more than $(n-2)/2$ factors, as well as every term with $(n-2)/2$ factors that is different from $g_2^{(n-2)/2}$, contains a factor g_1^d with $d > 1$.*

The proof is an obvious variant of the proof of claim 3.7.

Step 5. If $n > 1$ is odd, then repeat step 4 with the terms $g_2^{(n-1)/2}$ in F_{n-1} and $g_1 \cdot g_2^{(n-3)/2}$ in F_{n-2}, with straightforward modifications.

Thus, the case of one multiple curve is settled: if (i)–(iii) hold, then $1 \in I$.

Step 6. Now we turn to several multiple curves. Clearly, a zero z of I is also a zero of $f_0^{(1)} \cdots f_0^{(r)}$, and substituting z into

$$|F| \cdot F^{(i)}_{n_i-1,x} \quad \text{and} \quad |F| \cdot F^{(i)}_{n_i-1,y}$$

shows that a zero z of I with $f_0^{(i)}(z) = 0$ satisfies $f_0^{(j)}(z) = 0$ for some $j \neq i$ or $f_1^{(i)}(z) = 0$; cf. the argument in step 2. In the latter case we have

$$\prod_{j \neq i} f_0^{(j)}(z) \neq 0,$$

and, just as in steps 3–5, we obtain a contradiction.

In the first case, assume that $f_0^{(i)}(z) = f_0^{(j)}(z) = 0$ and consider

$$|F| \cdot \left\{ \left(\frac{f_1^{(i)}}{f_0^{(i)}} \right)^{n_i-1}, \left(\frac{f_1^{(j)}}{f_0^{(j)}} \right)^{n_j-1} \right\}$$

if $n_i > 1$ and $n_j > 1$. This term is equal to

$$\prod_{k \neq i,j} |F^{(k)}| \cdot ((n_i-1)(n_j-1) \cdot {f_1^{(i)}}^{n_i-2} {f_1^{(j)}}^{n_j-2} \cdot \{f_0^{(i)}, f_0^{(j)}\} + f_0^{(i)} \cdot (\cdots) + f_0^{(j)} \cdot (\cdots)).$$

Upon substituting z, we obtain a non-zero first factor, and a non-zero first term inside the bracket, while the second and third terms vanish. Again the assumption that I admits a zero leads to a contradiction.

In the case when $n_i = 1$ or $n_j = 1$, replace the corresponding term in the Poisson bracket by $\log f_0^{(i)}$ or $\log f_1^{(i)}$, respectively, and an obvious modification of the above argument applies. $\square$

4. Darboux integrals and integrating factors

We now turn to a more special inverse problem: describe the vector space of polynomial vector fields which admit a given Darboux integrating factor. Using the notation introduced in the previous section, a Darboux inverse integrating factor formed by r multiple curves based on $f^{(1)} \cdots f^{(r)}$ can be written in the form

$$\begin{aligned} V &= \exp(\lambda \cdot B') \\ &= \exp(\lambda^{(1)} \cdot B'^{(1)}) \cdots \exp(\lambda^{(r)} \cdot B'^{(r)}) \\ &= (f_0^{(1)})^{\lambda_0^{(1)}} \prod_{j=1}^{n_1-1} \exp(\lambda_j^{(1)} F_j{}^{(1)}) \cdots (f_0^{(r)})^{\lambda_0^{(r)}} \prod_{j=1}^{n_r-1} \exp(\lambda_j^{(r)} F_j{}^{(r)}), \end{aligned} \tag{4.1}$$

with $n_1 \geqslant 1, \ldots, n_r \geqslant 1$, where a product over an empty index set equals 1. We have introduced the abbreviations

$$\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}),$$

where $\lambda^{(i)} = (\lambda^{(i)}_{n_i-1}, \ldots, \lambda^{(i)}_0) \in \mathbb{C}^{n_i}$,

$$\alpha = (\alpha^{(1)}, \ldots, \alpha^{(r)}),$$

where $\alpha^{(i)} = (0, \ldots, 0, n_i) \in \mathbb{C}^{n_i}$, and

$$e_k = (e^{(1)}_k, \ldots, e^{(r)}_k),$$

where $e^{(i)}_k = (0, \ldots, 0, \delta_{kj}) \in \mathbb{C}^{n_j}$, for $1 \leqslant k \leqslant r$. We always require $\lambda^{(k)}_{n_k-1} \neq 0$ if $n_k > 1$. Clearly, this involves no loss of generality.

As a first step we exhibit some polynomial vector fields that admit the inverse integrating factor V from (4.1). For a given polynomial g, denote by

$$Z^{\lambda}_g = X_{g \exp[(\alpha-\lambda)\cdot B']}$$

the Hamiltonian vector field of the function $g \exp[(\alpha - \lambda) \cdot B']$. We collect some properties of such vector fields, as follows.

Lemma 4.1.

(i) *One has*

$$Z^{\lambda}_g = \exp(-\lambda \cdot B')(|F| X_g + g|F|(\alpha - \lambda) \cdot X_{F'}), \tag{4.2}$$

and therefore $\exp(\lambda \cdot B') Z^{\lambda}_g$ *is a polynomial vector field which admits the Darboux inverse integrating factor* V *and the Darboux first integral* $g \exp((\alpha - \lambda) \cdot B')$.

(ii) *Given integers* $d_1, \ldots, d_r \geqslant 0$*, the identity*

$$Z^{\lambda - \sum d_k e_k}_g = Z^{\lambda}_{g^*}, \tag{4.3}$$

holds for $g^* = g \cdot (f^{(1)}_0)^{d_1} \cdots (f^{(r)}_0)^{d_r}$.

Proof. (i) Using

$$\exp[(\alpha - \lambda) \cdot B'] = \exp(\alpha \cdot B') \exp(-\lambda \cdot B') = |F| \exp(-\lambda \cdot B'),$$

we find

$$\begin{aligned} Z^{\lambda}_g &= \exp[(\alpha - \lambda) \cdot B'] X_g + g \exp[(\alpha - \lambda) \cdot B'] X_{(\alpha-\lambda)\cdot B'} \\ &= \exp(-\lambda \cdot B')(|F| X_g + g|F| X_{(\alpha-\lambda)\cdot B'}), \\ &= \exp(-\lambda \cdot B')(|F| X_g + g|F|(\alpha - \lambda) \cdot X_{F'}), \end{aligned}$$

by various definitions. As noted earlier, all $|F| X_{F^{(i)}_j}$ are polynomial vector fields, and the first assertion is proved. The last two assertions are obvious.

(ii) Since $Z^{\lambda - e_k}_g$ is the Hamiltonian vector field of

$$g \exp[(\alpha + e_k - \lambda) \cdot B'] = g f^{(k)}_0 |F| \exp(-\lambda \cdot B') = (g f^{(k)}_0) \exp((\alpha - \lambda) \cdot B'),$$

we have

$$Z^{\lambda - e_k}_g = Z^{\lambda}_{g f^{(k)}_0}.$$

The assertion now follows by simple induction. □

THEOREM 4.2. *Assume that conditions (i)–(v) of theorem 3.6 hold, and X is a polynomial vector field which admits the inverse integrating factor V given by (4.1).*

(i) *If some $n_k > 1$, or $n_1 = \cdots = n_r = 1$ but some $\lambda_0^{(k)}$ is not a positive integer, and the additional conditions of theorem 3.5 are satisfied, then $X = \exp(\lambda \cdot B')Z_g^\lambda$ for some polynomial g, and X admits the Darboux first integral $g\exp((\alpha - \lambda) \cdot B')$.*

(ii) *If $n_1 = \cdots = n_r = 1$ and all $\lambda_0^{(j)}$ are positive integers, then*

$$X = |F|Z_h^\lambda + |F|X_{\tilde{G}} - |F|\sum_{i=1}^{r}\beta_i\frac{X_{f_0^{(i)}}}{f_0^{(i)}},$$

for certain polynomials h and $\tilde{G}$. In this case, X admits a Darboux first integral

$$\exp(G)\prod_{i=1}^{r}(f_0^{(i)})^{\beta_i},$$

with some rational function G. The denominator of G is a product of powers of the $f_0^{(i)}$.

Proof. According to theorem 3.4 we can write X in the form

$$\begin{aligned} X &= |F|X_0 + |F|C^{\mathrm{T}}X_{F'} \\ &= |F|X_0 + |F|\sum_{i=1}^{r}\sum_{j=0}^{n_i-1}C_j^{(i)}X_{F_j^{(i)}} \end{aligned}$$

with degree bounds on the $C_j^{(i)}$ if the additional hypotheses of theorem 3.5 hold.

STEP 1. We first establish a reduction principle.

If some $n_l > 1$ (thus $\lambda_{n_l-1}^{(l)} \neq 0$), define

$$g = \frac{C_{n_{k-1}}^{(k)}}{\lambda_{n_k-1}^{(k)}}$$

to be such that $n_k > 1$ and $C_{n_k-1}^{(k)}$ has the smallest degree among the $C_{n_l-1}^{(l)}$ with $n_l > 1$.

If $n_1 = \cdots = n_r = 1$, but some $\lambda_0^{(k)} \neq 1$, define

$$g = \frac{C_0^{(k)}}{\lambda_0^{(k)} - 1}.$$

Then

$$\begin{aligned} \exp(\lambda \cdot B')Z_g^\lambda &= |F|X_g + g|F|\sum_{i,j}(\alpha_j^{(i)} - \lambda_j^{(i)})X_{F_j^{(i)}} \\ &= |F|X_g - (\alpha_{n_k-1}^{(k)} + \lambda_{n_k-1}^{(k)})g|F|X_{F_{n_k-1}^{(k)}} \\ &\quad + g|F|\sum_{(i,j)\neq(k,n_k-1)}(\alpha_j^{(i)} - \lambda_j^{(i)})X_{F_j^{(i)}}. \end{aligned}$$

Now

$$\alpha^{(k)}_{n_k-1} - \lambda^{(k)}_{n_k-1} = -\lambda^{(k)}_{n_k-1} \quad \text{if } n_k > 1$$

and thus $(\alpha^{(k)}_{n_k-1} - \lambda^{(k)}_{n_k-1})g = -C^{(k)}_{n_k-1}$.

In the case when $n_k = 1$, $\lambda^{(k)}_0 \neq 1$, we have

$$\alpha^{(k)}_{n_k-1} - \lambda^{(k)}_{n_k-1} = 1 - \lambda^{(k)}_0,$$

and again $(\alpha^{(k)}_{n_k-1} - \lambda^{(k)}_{n_k-1})g = -C^{(k)}_{n_k-1}$. Thus, in both cases we arrive at a polynomial vector field

$$X + \exp(\lambda \cdot B')Z^{\lambda}_g = |F|(X_0 + X_g) + |F| \sum_{(i,j)\neq(k,n_k-1)} \tilde{C}^{(i)}_j X_{F^{(i)}_j}, \tag{4.4}$$

with $\tilde{C}^{(i)}_j = C^{(i)}_j + g(\alpha^{(i)}_j - \lambda^{(i)}_j)$.

If the hypotheses of theorem 3.5 hold, then, due to the degree bounds on the $C^{(i)}_j$ and the choice of $C^{(k)}_{n_k-1}$, the degree of $X + \exp(\lambda \cdot B')Z^{\lambda}_g$ is not greater than the degree of X.

Furthermore, $f^{(k)}_0$ divides each term on the right-hand side, since the denominator of $X_{F^{(i)}_j}$ does not contain $f^{(k)}_0$ for $i \neq k$, and the denominator of $X_{F^{(k)}_j}$ contains $f^{(k)}_0$ to a power less than n_k unless $j = n_k$ (see [7, remark 5.10]). Therefore, we obtain

$$X + \exp(\lambda \cdot B')Z^{\lambda}_g = f^{(k)}_0 \hat{X},$$

and $\hat{X}$ is a polynomial vector field which obviously admits the inverse integrating factor $\hat{V} \cdot (f^{(k)}_0)^{-1} = \exp((\lambda - e_k) \cdot B')$. Moreover, $\delta\hat{X} < \delta X$ if the hypotheses of theorem 3.5 hold.

Step 2. Assume that some $n_j > 1$ or $n_1 = \cdots = n_r = 1$ but some $\lambda^{(k)}_0$ is not a positive integer. Then we can repeat the reduction procedure an arbitrary number of times. Since the hypotheses of theorem 3.5 continue to hold for the vector fields remaining after the reduction step, the strict descent of degrees ensures that one will eventually arrive at the zero vector field. All vector fields added in the process are of type $\exp(\lambda \cdot B')Z^{\lambda}_h$ for some polynomial h, by lemma 4.1. Thus, part (i) is proven.

Step 3. Now assume that $n_1 = \cdots = n_r = 1$ and all $\lambda^{(k)}_0 = d_k$ are positive integers. Here we can reduce the $\lambda^{(k)}_0$ until all $\lambda^{(k)}_0 = 1$, and then, abbreviating $f_i = f^{(i)}_0$, we have

$$X = |F|Z^{\lambda}_h + \prod_{i=1}^{r} f_i^{d_i-1}\tilde{X},$$

where h is some polynomial and $\tilde{X}$ is a polynomial vector field with integrating factor $(f_1 \cdots f_r)^{-1}$.

Since $1 \in I$, the vector field $\tilde{X}$ can be written in the form (3.10), i.e.

$$\tilde{X} = f_1 \cdots f_r X_0 + f_1 \cdots f_r \sum_{i=1}^{r} C_i \frac{X_{f_i}}{f_i}, \tag{4.5}$$

and $\tilde{X}$ has the inverse integrating factor $V = f_1 \cdots f_r$, whence $\operatorname{div}(X/(f_1 \cdots f_r)) = 0$, or

$$0 = \operatorname{div}(X_0) + \sum_{i=1}^{r} \frac{1}{f_i}\{f_i, C_i\}. \tag{4.6}$$

Since the f_i are relatively prime polynomials, each f_i must divide $\{f_i, C_i\}$. Hence, the Hamiltonian vector field $X_C = (-C_{iy}, C_{ix})$ has $f_i = 0$ as invariant algebraic curve. Therefore, $\{f_i = 0\} \subseteq \{C_i - \beta_i = 0\}$ for some constant $\beta_i \in \mathbb{C}$, for all $i = 1, \dots, r$. From the Hilbert Nullstellensatz, and because f_i is irreducible we get $C_i - \beta_i = M_i f_i$ for some $M_i \in \mathbb{C}[x, y]$, $i = 1, \dots, r$. Using

$$\operatorname{div}\left(\sum_{i=1}^{r} M_i X_{f_i}\right) = \sum_{i=1}^{r} \{f_i, M_i\} = \sum_{i=1}^{r} \frac{1}{f_i}\{f_i, C_i\},$$

(4.6) becomes

$$\operatorname{div}\left(X_0 + \sum_{i=1}^{r} M_i X_{f_i}\right) = 0,$$

and we conclude that

$$X_0 + \sum_{i=1}^{r} M_i X_{f_i} = X_{\tilde{G}},$$

for some polynomial $\tilde{G}$. Substituting X_0 into (4.5), we obtain

$$\tilde{X} = f_1 \cdots f_r X_{\tilde{G}} - f_1 \cdots f_r \sum_{i=1}^{r} \beta_i \frac{X_{f_i}}{f_i},$$

and thus

$$X = |F| Z_h^{\lambda} + |F| X_{\tilde{G}} - |F| \sum_{i=1}^{r} \beta_i \frac{X_{f_i}}{f_i}.$$

Therefore, X admits the first integral

$$\begin{aligned} H &= h \exp((\alpha - \lambda) \cdot B') + \tilde{G} - \sum \beta_i \log f_i \\ &= h f_1^{1-d_1} \cdots f_r^{1-d_r} + \tilde{G} - \sum \beta_i \log f_i \\ &=: G - \sum \beta_i \log f_i \end{aligned}$$

with a rational function G whose denominator is a product of powers of the f_i, and hence the Darboux first integral $\exp(H) = \exp(G) f_1^{-\beta_1} \cdots f_r^{-\beta_r}$. □

In case (ii) of the theorem, a more general result can be stated and proven.

Proposition 4.3. *If the vector field given by (3.10) admits an inverse integrating factor of the form $V = f_1 \cdots f_r$ with irreducible and pairwise relatively prime f_i, then it has a Darboux first integral of the form $\exp(G) \prod_{i=1}^{r} f_i^{\alpha_i}$, where G is a rational function and $\alpha_i \in \mathbb{C}$.*

Proof. The vector field admits the invariant algebraic curves $\{f_i = 0\}$ for all i. Since the ideal I contains the elements

$$\left(\prod_{i\neq k} f_i\right)\cdot f_{kx}, \qquad \left(\prod_{i\neq k} f_i\right)\cdot f_{ky},$$

and each f_k is irreducible, there exists a polynomial $h \in I$ which is not a multiple of any f_k. For this h, theorem 3.4 shows the existence of a polynomial vector field X_0 and polynomials C_i such that

$$hX = VX_0 + V\sum_{i=1}^{r} C_i \frac{X_{f_i}}{f_i}$$

or

$$X = VX_0' + V\sum_{i=1}^{r} C_i' \frac{X_{f_i}}{f_i}, \tag{4.7}$$

with $X_0' = X_0/h$ and $C_i' = C_i/h$. In the following we use familiar properties of the ring $R := \mathbb{C}[x, y][1/h]$, which is the coordinate ring of an affine variety, namely the principal open subset defined by $h \neq 0$ (see [12, ch. 1, § 4]). In particular, the Hilbert Nullstellensatz continues to hold over this ring, and the ideal generated by f_i remains prime in R by the choice of h. Since the vector field X has the inverse integrating factor V, we get $\operatorname{div}(X/V) = 0$, or equivalently,

$$0 = \operatorname{div}(X_0') + \sum_{i=1}^{r} \frac{1}{f_i}\{f_i, C_i'\}. \tag{4.8}$$

Repeating the argument in the proof of theorem 4.2(iii), we obtain $C_i' = \theta_i + M_i' f_i$ with suitable $\theta_i \in \mathbb{C}$ and $M_i' \in R$. Because

$$\operatorname{div}\left(\sum_{i=1}^{r} M_i' X_{f_i}\right) = \sum_{i=1}^{r}\{f_i, M_i'\} = \sum_{i=1}^{r}\frac{1}{f_i}\{f_i, C_i'\},$$

(4.8) becomes

$$\operatorname{div}\left(X_0' + \sum_{i=1}^{r} M_i' X_{f_i}\right) = 0,$$

and $\hat{X} = X_0' + \sum_{i=1}^{r} M_i' X_{f_i}$ is a rational vector field. By a well-known result (see, for example, [3]), we have $\hat{X} = X_{\log F}$, where F is a Darboux function of the form

$$F = \left(\prod_{i=1}^{r} f_i^{\beta_i}\right)\exp(G) \quad \text{with } \beta_i \in \mathbb{C},$$

and G is a rational function. Thus,

$$\hat{X} = \sum_{i=1}^{r} \beta_i \frac{X_{f_i}}{f_i} + X_G,$$

and moreover

$$X_0' + \sum_{i=1}^{r} M_i' X_{f_i} = X_G + \sum_{i=1}^{r} \beta_i \frac{X_{f_i}}{f_i}.$$

Substituting X_0' into (4.7), we obtain

$$X = VX_G - V\sum_{i=1}^{r} \theta_i \frac{X_{f_i}}{f_i} + V\sum_{i=1}^{r} \beta_i \frac{X_{f_i}}{f_i}. \tag{4.9}$$

This vector field X has the Darboux first integral $\exp(G)\prod_{i=1}^{r} f_i^{\theta_i-\beta_i}$. $\square$

5. Examples and counterexamples

In this section we use lemma 2.2 for an approach to the construction of vector fields with multiple invariant curves. We are particularly interested in cases when the genericity condition '$1 \in I$' from § 3 is not satisfied.

We observe that the form of the relations (3.1) suggests a step-by-step approach. First, find all polynomial vector fields that satisfy $X(f_0) = L_0 f_0$ for some polynomial L_0. Then rewrite the second condition as

$$X(f_1) - L_0 f_1 \in \langle f_0 \rangle,$$

which will impose further restrictions on X and L_0, and proceed. The intermediate condition

$$X(f_1) = L_2 f_1 + L_1 f_0 \in \langle f_0, f_1 \rangle$$

yields some properties of X that facilitate the subsequent work. Here lemma 2.2 is useful.

In our first example, f_0 is reducible and the components of $\{f_0 = 0\}$ intersect.

Proposition 5.1. *For $f_0 = xy$, $f_1 = 1 + x + y$ the following hold.*

(i) *A polynomial vector field X satisfies*

$$X(f_0) = L_0 f_0,$$
$$X(f_1) = L_0 f_1 + L_1 f_0,$$

for suitable polynomials L_0 and L_1 if and only if

$$X = \alpha \begin{pmatrix} x^2 y \\ 0 \end{pmatrix} + B(f_1 X_{f_0} - f_0 X_{f_1}) + C f_0 X_{f_0} + f_0^2 X_0,$$

with $\alpha \in \mathbb{C}$, and where B and C are arbitrary polynomials and X_0 is an arbitrary polynomial vector field. For $\alpha = 0$ this includes only the terms given in theorem 3.1, but for $\alpha \neq 0$ the vector field has no such representation.

(ii) *If X admits an integrating factor of type*

$$f_0^{-\lambda_0}\left(\exp\left(\frac{f_1}{f_0}\right)\right)^{-\lambda_1},$$

then $\lambda_1 = 0$ or $\alpha = 0$. In both cases there is a Darboux first integral.

Proof.

STEP 1. Since $f_0 = xy$, and $\{x = 0\}$ and $\{y = 0\}$ intersect transversally, we have $X(f_0) \in \langle f_0 \rangle$ if and only if

$$X = A_1 y \frac{\partial}{\partial y} - A_2 x \frac{\partial}{\partial x} + xy\tilde{X}_0$$

(see [6, lemma 7(a)]). Moreover, $\{f_0 = 0\} \cap \{f_1 = 0\} = \{(-1, 0), (0, -1)\}$. Stationarity of these points, as required by lemma 2.2, is equivalent to $A_1(0, -1) = A_2(-1, 0) = 0$, so

$$A_1 = (y + 1)\tilde{A}_1, \qquad A_2 = (x + 1)\tilde{A}_2,$$

which implies that

$$X = y(y + 1)\tilde{A}_1 \frac{\partial}{\partial y} - x(x + 1)\tilde{A}_2 \frac{\partial}{\partial x} + xy\tilde{X}_0.$$

The cofactor of f_0 is equal to

$$L_0 = (y + 1)\tilde{A}_1 - (x + 1)\tilde{A}_2 + y\tilde{P}_0 + x\tilde{Q}_0,$$

with $\tilde{X}_0 = (\tilde{P}_0, \tilde{Q}_0)$.

STEP 2. For (i) the necessary and sufficient condition is

$$X(f_1) - L_0 f_1 \in \langle f_0 \rangle,$$

or, equivalently,

$$-y(1 + x + y)\tilde{P}_0 - x(1 + x + y)\tilde{Q}_0 - (1 + x)(1 + y)\tilde{A}_1 + (1 + x)(1 + y)\tilde{A}_2 \in \langle xy \rangle,$$

as a brief computation shows. This is equivalent to

$$-y(1 + x + y)\tilde{P}_0 - x(1 + x + y)\tilde{Q}_0 - (1 + x + y)\tilde{A}_1 + (1 + x + y)\tilde{A}_2 \in \langle xy \rangle,$$

and we have the necessary and sufficient condition

$$\tilde{A}_2 = \tilde{A}_1 + y\tilde{P}_0 + x\tilde{Q}_0 + xyR,$$

for some polynomial R. Substituting this in $X = (P, Q)$, we obtain

$$\begin{aligned}\begin{pmatrix} P \\ Q \end{pmatrix} &= \tilde{A}_1 \begin{pmatrix} -x(x+1) \\ y(y+1) \end{pmatrix} - \tilde{P}_0 \begin{pmatrix} x^2 y \\ 0 \end{pmatrix} + \tilde{Q}_0 \begin{pmatrix} -x^3 + x^2 \\ xy \end{pmatrix} + (-R + \tilde{P}_0) \begin{pmatrix} x^3 y \\ 0 \end{pmatrix} \\ &= A_1 \begin{pmatrix} -x(x+1) \\ y(y+1) \end{pmatrix} + \hat{P}_0 \begin{pmatrix} x^2 y \\ 0 \end{pmatrix} + \hat{Q}_0 \begin{pmatrix} 0 \\ xy^2 \end{pmatrix} + \hat{R} \begin{pmatrix} x^3 y \\ 0 \end{pmatrix},\end{aligned}$$

with $\tilde{A}_1 = A_1 - x\tilde{Q}_0$ and obvious renaming.

STEP 3. Vector fields of the form $X^* = (P^*, Q^*)$ with

$$\begin{pmatrix} P^* \\ Q^* \end{pmatrix} = B \begin{pmatrix} -x(x+1) \\ y(y+1) \end{pmatrix} + C \begin{pmatrix} -x^2 y \\ xy^2 \end{pmatrix} + x^2 y^2 \begin{pmatrix} P_0^* \\ Q_0^* \end{pmatrix} \tag{5.1}$$

clearly admit $f_0 + \varepsilon f_1$ as a generalized invariant algebraic curve, being in the form of theorem 3.1.

We claim that the vector field $(x^2y, 0)$ cannot be written in the form (5.1). To prove this claim, assume that $(x^2y, 0)$ is written in such a form. Thus, we have

$$\begin{aligned} x^2y &= -Bx - Bx^2 - Cx^2y + x^2y^2P_0^*, \\ 0 &= By + By^2 + Cxy^2 + x^2y^2Q_0^*, \end{aligned}$$

or equivalently

$$\left.\begin{aligned} xy &= -B - Bx - Cxy + xy^2P_0^*, \\ 0 &= B + By + Cxy + x^2yQ_0^*. \end{aligned}\right\} \tag{5.2}$$

From $(5.2)_1$ we find $B = \hat{B}x$, and so we can rewrite $(5.2)_2$ as

$$0 = \hat{B} + \hat{B}y + Cy + xyQ_0^*,$$

which implies that $\hat{B} = \hat{\hat{B}}y$ as well as $\hat{\hat{B}} = -C(\mathrm{mod}\langle x, y\rangle)$. Thus, we have $B = -Cxy +$ higher-order terms. Substituting this in $(5.2)_1$ yields a contradiction, and the claim is proved.

Since

$$\begin{aligned} \begin{pmatrix} x^3y \\ 0 \end{pmatrix} &= x \begin{pmatrix} x^2y \\ -xy^2 \end{pmatrix} + x^2y^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \begin{pmatrix} 0 \\ xy^3 \end{pmatrix} &= -y \begin{pmatrix} x^2y \\ -xy^2 \end{pmatrix} + (xy)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{pmatrix} 0 \\ xy^2 \end{pmatrix} = -\begin{pmatrix} x^2y \\ -xy^2 \end{pmatrix} + \begin{pmatrix} x^2y \\ 0 \end{pmatrix},$$

every vector field determined in step 2 can be expressed in the form

$$\alpha x^2y \frac{\partial}{\partial x} + X^*,$$

with X^* as in (3.6). Thus, part (i) is proven.

STEP 4. Now let X be as in part (i). Routine computations show that

$$\begin{aligned} L_0 &= B(y - x) + \alpha xy + xy(yP_0 + xQ_0), \\ L_1 &= -\alpha(1 + y) + C(y - x) - y(1 + y)P_0 - x(1 + x)Q_0, \\ \operatorname{div} X &= 2\alpha xy + (-x(x + 1)B_x + y(y + 1)B_y) + 2B(y - x) - x^2yC_x \\ &\quad + xy^2C_y + 2xy(yP_0 + xQ_0) + (xy)^2(P_{0x} + Q_{0y}). \end{aligned}$$

Note that L_1 is the cofactor of $F = \exp(f_1/f_0)$. The integrating factor condition, according to theorem 2.4, is

$$\lambda_0 L_0 + \lambda_1 L_1 = \operatorname{div} X.$$

Evaluation at $x = y = 0$ yields $\lambda_1\alpha = 0$. According to lemma 4.1, or theorem 4.2 applied to $f_0^{(1)} = x$, $f_0^{(2)} = y$, there exists a Darboux first integral. This proves part (ii). □

Note that the Darboux integral is not necessarily of the form determined in theorem 4.2; rather than powers of f_0, one has products of powers of its prime factors, x and y.

For our second example, we consider algebraic curves that intersect non-transversally.

PROPOSITION 5.2. *For $f_0 = x^2 - y$ and $f_1 = y$ the following hold.*

(i) *A polynomial vector field $X = (P, Q)$ satisfies*

$$X(f_0) = L_0 f_0,$$
$$X(f_1) = L_0 f_1 + L_1 f_0,$$

for suitable polynomials L_0 and L_1 if and only if

$$\begin{pmatrix} P \\ Q \end{pmatrix} = A \begin{pmatrix} x \\ 2y \end{pmatrix} + (x^2 - y)B \begin{pmatrix} 1 \\ 2x \end{pmatrix} + (x^2 - y)^2 C \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with arbitrary polynomials A, B and C.

(ii) *The particular vector fields of type*

$$\begin{pmatrix} P \\ Q \end{pmatrix} = (\alpha x + \beta y) \begin{pmatrix} x \\ 2y \end{pmatrix} + (x^2 - y) \begin{pmatrix} 1 \\ 2x \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$ and $\beta \neq 0$, admit the Darboux integrating factor

$$f_0^{-5/2} \exp \left(\frac{f_1}{f_0} \right)^{\alpha/2}$$

but have no Darboux first integral.

Proof.

STEP 1. By [6, lemma 6(a)], all polynomial vector fields having $f_0 = 0$ as an invariant algebraic curve can be written as

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \tilde{R} \begin{pmatrix} 1 \\ 2x \end{pmatrix} + (x^2 - y) \begin{pmatrix} \tilde{P}_0 \\ \tilde{Q}_0 \end{pmatrix}.$$

The stationarity of $(0, 0)$ (note that $\{f_0 = 0\} \cap \{f_1 = 0\} = \{(0, 0)\}$) forces that $\tilde{R} = R_1 x + R_2 y$, and

$$\begin{aligned} \begin{pmatrix} P \\ Q \end{pmatrix} &= R_1 \begin{pmatrix} x \\ 2x^2 \end{pmatrix} + R_2 \begin{pmatrix} y \\ 2xy \end{pmatrix} + (x^2 - y) \begin{pmatrix} \tilde{P}_0 \\ \tilde{Q}_0 \end{pmatrix} \\ &= R_1 \begin{pmatrix} x \\ 2y \end{pmatrix} + R_2 x \begin{pmatrix} x \\ 2y \end{pmatrix} + (x^2 - y) \left[\begin{pmatrix} \tilde{P}_0 \\ \tilde{Q}_0 \end{pmatrix} + R_1 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + R_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &= A \begin{pmatrix} x \\ 2y \end{pmatrix} + (x^2 - y) \begin{pmatrix} B \\ Q_0 \end{pmatrix}. \end{aligned}$$

STEP 2. The cofactor of f_0 equals $L_0 = 2A + 2xB - Q_0$, and the condition $X(f_1) - L_0 f_1 \in \langle f_0 \rangle$ is equivalent to

$$(x^2 - y)Q_0 - (2xB - Q_0)y \in \langle f_0 \rangle,$$

which gives $2xB - Q_0 \in \langle f_0 \rangle$, and so $Q_0 = 2xB + Cf_0$. Substituting this expression in the vector field X yields part (i).

STEP 3. For a vector field as in part (i), the algebraic curve $f_0 = 0$ is invariant and has cofactor $L_0 = 2A - f_0 C$. The exponential factor $\exp(f_1/f_0)$ is also invariant and has cofactor $L_1 = x(2B + Cx)$. In particular, for $A = \alpha x + \beta y$, $B = 1$ and $C = 0$, we have $L_0 = 2\alpha x + 2\beta y$ and $L_1 = 2x$. The divergence of the vector field is equal to $4\alpha x + 5\beta y$. Therefore, the relation

$$-\tfrac{5}{2} L_0 + \tfrac{1}{2} \alpha L_1 = -\operatorname{div}(X)$$

holds, and by theorem 2.4 the vector field X has the Darboux integrating factor $f_0^{-5/2} f_1^{\alpha/2}$. The relation $\lambda_0 L_0 + \lambda_1 L_1 = 0$ for $\beta \neq 0$ gives $\lambda_0 = \lambda_1 = 0$. Hence, X cannot have a Darboux first integral formed only by the polynomial f_0 and the exponential factor $\exp(f_1/f_0)$. To prove that X cannot admit any Darboux first integral, we need two more steps.

STEP 4. First we claim that $\{f_0 = 0\}$ is the only invariant algebraic curve admitted by X. As a consequence, the only candidates for Darboux first integrals are of type $f_0^\theta \cdot \exp(g/f_0^n)$ with a polynomial g, a non-negative integer n and some complex θ. As a second consequence, the vector field admits no polynomial first integral. To prove this claim we consider stationary points at infinity (see [10] and the computations in [15]).

The leading term of X is $2x^3 \cdot \partial/\partial y$. Hence, the stationary points at infinity are determined by the prime factors of the homogeneous polynomial

$$\det \begin{pmatrix} x & 0 \\ y & 2x^3 \end{pmatrix} = 2x^4.$$

Thus, there is only one stationary point at infinity, namely $(0 : 1 : 0)$. We determine the corresponding Poincaré transform by the procedure outlined in [15, § 3]. First homogenize the vector field to obtain

$$g = \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -yz^2 + (1+\alpha)x^2 z + \beta xyz \\ (2\alpha - 2)xyz + 2\beta y^2 z + 2x^3 \\ 0 \end{pmatrix},$$

then take the projection with respect to y, i.e.

$$\begin{aligned} \tilde{g} = \begin{pmatrix} \tilde{g}_1 \\ 0 \\ \tilde{g}_3 \end{pmatrix} &= -g_2 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + y \cdot g \\ &= \begin{pmatrix} (3-\alpha)x^2 yz - \beta xy^2 z - 2x^4 - y^2 z^2 \\ 0 \\ -2\beta y^2 z^2 - (2\alpha - 2)xyz^2 - 2x^3 z \end{pmatrix}. \end{aligned}$$

We obtain the Poincaré transform

$$\begin{pmatrix} P^*(x,z) \\ Q^*(x,z) \end{pmatrix} := \begin{pmatrix} \tilde{g}_1(x,1,z) \\ \tilde{g}_3(x,1,z) \end{pmatrix} = \begin{pmatrix} -\beta xz - z^2 + (3-\alpha)x^2 z - 2x^4 \\ -2\beta z^2 - (2\alpha-2)xz^2 - 2x^3 z \end{pmatrix},$$

and the stationary point of X at infinity corresponds to the stationary point 0 for this system.

This stationary point is not elementary; therefore, we use a blow-up (see, for example, [1, ch. 1, § 2], where this is called a σ-process) to obtain the vector field

$$\begin{aligned} z \cdot Y &:= \frac{1}{z} \cdot \begin{pmatrix} P^*(xz,z) - xQ^*(xz,z) \\ zQ^*(xz,z) \end{pmatrix} \\ &= z \cdot \begin{pmatrix} -1 + \beta x + (1+\alpha)x^2 z - 4x^4 z^2 \\ -2\beta z - (2\alpha-2)xz^2 - 2x^3 z^3 \end{pmatrix}. \end{aligned}$$

The invariant line $\{z = 0\}$ of Y is the blow-up of $(0,0)$; hence, we investigate the stationary points on this line. Obviously, $(\beta^{-1}, 0)$ is the only such stationary point, and its Jacobian is easily computed as

$$\begin{pmatrix} \beta & * \\ 0 & -2\beta \end{pmatrix}.$$

The eigenvalue ratio equals -2, and thus (for example, by [15, theorem 2.3]), there exist exactly two irreducible local analytic invariant curves containing this point. Going back again, we see that the stationary point 0 of the Poincaré transform is contained in exactly two local analytic invariant curves, and one of these is the line at infinity. But every invariant algebraic curve induces at least one irreducible local invariant curve for some stationary point at infinity. Therefore, apart from $\{f_0 = 0\}$ there can be no others.

STEP 5. To preclude the existence of a Darboux first integral $f_0^{\mu} \cdot \exp(g)$, note that the cofactor of such an expression is equal to

$$K := \mu \cdot L_0 + X(g) = \mu \cdot L_0 + P \cdot g_x + Q \cdot g_y.$$

The assumption $K = 0$ implies $K(x,0) = 0$ in particular, and evaluation yields

$$0 = 2\alpha\mu \cdot x + x^2 \cdot (\cdots)$$

and thus $\mu = 0$. Since X has no polynomial first integral, this is impossible.

STEP 6. To preclude the existence of a Darboux first integral $f_0^{\mu} \cdot \exp(g/f_0^n)$ with some integer $n > 1$, we will show that $\{f_0 = 0\}$ has exact multiplicity 2. Assume that there are polynomials f_2 and L_2 such that

$$X(f_2) = L_0 f_2 + L_1 f_1 + L_2 f_0. \tag{5.3}$$

Consider the degree function deg defined by $\deg(x) = 1$ and $\deg(y) = 2$. Then f_0 and f_1 are homogeneous of degree 2 with respect to deg, and L_1 is homogeneous of degree 1. The vector fields

$$X_1 := \beta y \cdot \begin{pmatrix} x \\ 2y \end{pmatrix}, \qquad X_2 := \alpha x \cdot \begin{pmatrix} x \\ 2y \end{pmatrix} + (x^2 - y) \cdot \begin{pmatrix} 1 \\ 2x \end{pmatrix}$$

induce deg-homogeneous derivations of degrees 3 and 2, respectively; thus,

$$\deg(X_1(g)) = \deg(g) + 2 \quad \text{and} \quad \deg(X_2(g)) = \deg(g) + 1$$

for all homogeneous g. Moreover, $X = X_1 + X_2$. We will write a polynomial

$$g = g^{(0)} + g^{(1)} + \cdots + g^{(r)},$$

as the sum of its deg-homogeneous terms; in particular $L_0 = L_0^{(1)} + L_0^{(2)}$ with $L_0^{(1)} = 2\alpha x$ and $L_0^{(2)} = 2\beta y$. With $f_2 = f_2^{(0)} + \cdots$ and $L_2 = L_2^{(0)} + \cdots$, compare terms of smallest degree in (5.3). At degree 1 one has

$$0 = L_0^{(1)} f_2^{(0)},$$

and therefore $f_2^{(0)} = 0$. Using this and $f_2^{(1)} = \mu x$ for some μ, the degree-2 terms yield

$$X_2(f_2^{(1)}) = L_0^{(1)} f_2^{(1)} + (x^2 - y) \cdot L_2^{(0)}$$

or

$$\alpha\mu x^2 + \mu(x^2 - y) = 2\alpha\mu x^2 + L_2^{(0)}(x^2 - y),$$

which forces $\mu = L_2^{(0)} = 0$ in view of $\alpha \neq 0$. Using $f_2^{(1)} = 0$ at degree 3, we obtain

$$X_2(f_2^{(2)}) = 2\alpha x f_2^{(2)} + 2xy + (x^2 - y) \cdot L_2^{(1)}.$$

Since the terms $2\alpha x f_2^{(2)}$ on the left- and right-hand sides cancel, we obtain the contradiction that $x^2 - y$ divides $2xy$. Thus, relation (5.3) is impossible. □

As a final example we discuss a system admitting an invariant curve defined by $f = y^2 + x^3$. Since the curve $\{f = 0\}$ has a cusp at the origin, it does not satisfy the criteria in theorem 3.6. The form of the first integral follows the pattern for these exceptional cases conjectured by Żołądek in [16], where more examples are given.

Proposition 5.3. *The system*

$$\dot{x} = \tfrac{1}{3}\alpha y + 2\beta x, \qquad \dot{y} = 3\beta y - \tfrac{1}{2}\alpha x^2 \tag{5.4}$$

admits an inverse integrating factor $(y^2 + x^3)^{5/6}$, and a first integral

$$\phi = \alpha(y^2 + x^3)^{1/6} + \beta \int^{y^2/x^3} \frac{\mathrm{d}z}{(1+z)^{5/6} z^{1/2}}.$$

But the system does not admit a Darboux first integral if $\alpha \neq 0$ and $\beta \neq 0$.

Proof. Let X be the vector field corresponding to the system. Note that $X(f) = 6\beta f$.

Step 1. Assume that X admits the polynomial g, so that $X(g) = L \cdot g$ for some polynomial L, and g is not a multiple of f. Comparing homogeneous terms of highest degree shows that the highest-order terms of g are a multiple of x^n for some n. The next highest terms show that L is a constant. There is a unique representation

$$g = A(x) + y \cdot B(x) + f \cdot \tilde{g}$$

and one obtains the corresponding representation

$$\begin{aligned}X(g) &= (2\beta x + \tfrac{1}{3}\alpha y) \cdot (A'(x) + yB'(x)) + (3\beta y - \tfrac{1}{2}\alpha x^2) \cdot B(x) + f \cdot h \\ &= 2\beta x A'(x) - \tfrac{1}{2}\alpha x^2 B(x) - \tfrac{1}{3}\alpha x^3 B'(x) \\ &\quad + y \cdot (2\beta x B'(x) + \tfrac{1}{3}\alpha A'(x) + 3\beta B(x)) + f \cdot \tilde{h}\end{aligned}$$

for some polynomials h and $\tilde{h}$, since $y^2 \equiv x^3 \bmod f$. Evaluating $X(g) = L \cdot g$ for some constant L yields

$$2\beta A' - \tfrac{1}{2}\alpha x^2 B - \tfrac{1}{3}\alpha x^3 B' = L \cdot A, \tag{5.5}$$

$$2\beta x B' + \tfrac{1}{3}\alpha A' + 3\beta B = L \cdot B. \tag{5.6}$$

Assuming that $B \neq 0$, with $\delta B = m$, (5.6) shows that $\delta A \leqslant 1+m$. Comparing terms of highest degree in (5.5) then yields $\frac{1}{2}x^{m+2} + \frac{1}{3}mx^{m+2} = 0$; this contradiction forces $B = A = 0$. But this means that g is a multiple of f, contrary to our assumption. In particular there is no polynomial first integral.

Step 2. The only possible exponential factors are therefore of the form $\exp(g/f^n)$ for some n. There is one obvious such factor: $\exp(x)$ with cofactor $\frac{1}{3}\alpha y + 2\beta x$. We shall show that there are no others. We first exclude the case $n = 0$. For, suppose that there is an exponential factor of X of the form $\exp(g)$ for some polynomial g. This would satisfy $X(g) = L$ for some polynomial L of degree 1. Replacing g by $g - kx$ for some suitably chosen constant k we can assume that $\exp(g)$ now has a cofactor of the form $ax + b$, for some constants a and b. A calculation along the lines of step 1 is performed, giving (5.5) and (5.6) with $L \cdot A$ and $L \cdot B$ replaced by $ax + b$ and 0, respectively. As above, there are no non-trivial solutions and so $g = f\tilde{g}$ for some polynomial $\tilde{g}$. But then $X(g) = f(6\beta g + X(g)) = ax + b$, which is not possible unless $a = b = 0$, but this implies that there is a polynomial first integral which we excluded above.

Step 3. Finally, we exclude exponential factors of the form $\exp(g/f^n)$ for some $n > 1$, where we assume that f does not divide g. We therefore have $X(g) - 6\beta ng = f^n\tilde{L}$ for some cofactor L. Exactly as in step 1 we obtain (5.5) and (5.6) with $L \cdot A$ and $L \cdot B$ replaced by $6\beta nA$ and $6\beta nB$, respectively, and again find that g must be a multiple of f, which we have excluded.

Thus, the only exponential factors or invariant algebraic curves supported by X from which we can construct our Darboux first integral are f and $\exp(x)$. However, such a construction is not possible, as their cofactors are independent over $\mathbb{C}$. □

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