On Schanuel's conjectures

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In this paper proofs are given of conjectures of Schanuel on the algebraic relations satisfied by exponentiation in a differential-algebraic setting. The methods and results are then used to give new proofs and generalizations of the theorems of Chabauty, Kolchin, and Skolem.

1. Introduction

(i) *Statement of the conjectures and our main results.* S. Schanuel has made a conjecture [1, p. 30–31] concerning the exponential function which embodies all its known transcendentality properties such as the theorems of Lindemann [2, p. 225 or 1, Ch. VII, § 2, Th. 1], Baker [3, Cor. 1, 2, and 4, Th. 1, 2], and other results (e.g. [1, Ch. II, Th. 1; Ch. V, Th. 1]) and implies a whole collection of special conjectures (e.g. [1, p. 11, Remark], [5, p. 138, Problems 1, 7, 8] and the algebraic independence of π and e over $\mathbf{Q}$).

The conjecture runs as follows:

(S) Let $y_1, \cdots, y_n \in \mathbf{C}$ be $\mathbf{Q}$-linearly independent. Then

$$\dim_{\mathbf{Q}} \mathbf{Q}(y_1, \cdots, y_n, e^{y_1}, \cdots, e^{y_n}) \geqq n .$$

Here $\dim_E F$, for any extension of fields F/E, denotes the cardinality of a maximally E-algebraically independent subset of F.

Schanuel also made the analogous power series conjecture.

(SP) Let $y_1, \cdots, y_n \in t\mathbf{C}[[t]]$ be $\mathbf{Q}$-linearly independent. Then

$$\dim_{\mathbf{C}(t)} \mathbf{C}(t)(y_1, \cdots, y_n, \exp y_1, \cdots, \exp y_n) \geqq n .$$

In this paper we prove (SP) and obtain certain generalizations and related results.

Let us consider the hypothesis

(Σ) Let $y_1, \cdots, y_n \in \mathbf{C}[[t_1, \cdots, t_m]]$ be $\mathbf{Q}$-linearly independent. Then

$$\dim_{\mathbf{Q}} \mathbf{Q}(y_1, \cdots, y_n, \exp y_1, \cdots, \exp y_n) \geqq n + \operatorname{rank}\left(\frac{\partial y_\nu}{\partial t_\mu}\right)_{\substack{\nu=1,\cdots,n \\ \mu=1,\cdots,m}} .$$

Then (S) is the special case of (Σ) when $m = 0$ (or when each $y_\nu \in \mathbf{C}$). (SP)

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implies the special case when $m = 1$ and each y_ν is without constant term. The following is our main result on (Σ).

THEOREM 1. *(Σ) is true when the y_ν are without constant terms, or more generally when the $y_\nu - y_\nu(0)$ are $\mathbf{Q}$-linearly independent.*

Moreover by utilizing the results of the type of Theorem 1 we can prove

THEOREM 2. $(\mathrm{S}) \Leftrightarrow (\Sigma)$.

Our approach, through differential algebra, to these results had already been signaled by the following conjecture of Schanuel.

(SD) Let F be a field and D a derivation of F with constants $C \supseteq \mathbf{Q}$. Let $y_1, \cdots, y_n, z_1, \cdots, z_n \in F^*$ be such that

(a) $Dy_\nu = Dz_\nu/z_\nu$ for $\nu = 1, \cdots, n$, and

(b) the Dy_ν are $\mathbf{Q}$-linearly independent.

Then

$$\dim_C C(y_1, \cdots, y_n, z_1, \cdots, z_n) \geqq n + 1 .$$

Upon taking $C = \mathbf{C}$, $F = \mathbf{C}((t))$ and $D = d/dt$, we have that $(\mathrm{SD}) \Rightarrow (\mathrm{SP})$. We obtain the following result which implies (SD), (SP), and Theorem 1.

THEOREM 3. *Let $F \supseteq C \supseteq \mathbf{Q}$ be a tower of fields and Δ a set of derivations of F with $\bigcap_{D \in \Delta} \ker D = C$. Let $y_1, \cdots, y_n, z_1, \cdots, z_n \in F^*$ be such that*

(a) *for all $D \in \Delta$, $\nu = 1, \cdots, n$, $Dy_\nu = Dz_\nu/z_\nu$ and either*

(b) *no non-trivial power product of the z_ν is in C, or*

(b′) *the y_ν are $\mathbf{Q}$-linearly independent modulo C.*

Then

$$\dim_C C(y_1, \cdots, y_n, z_1, \cdots, z_n) \geqq n + \operatorname{rank}\,(Dy_\nu)_{\substack{\nu=1,\cdots,n \\ D \in \Delta}} .$$

COROLLARY 1. *Let $C \supseteq \mathbf{Q}$ and $y_1, \cdots, y_n \in C[[t_1, \cdots, t_r]]$ be power series without constant terms, $\mathbf{Q}$-linearly independent. Then*

$$\dim_C C(y_1, \cdots, y_n, \exp y_1, \cdots, \exp y_n) \geqq n + \operatorname{rank}\left(\frac{\partial y_\nu}{\partial t_\rho}\right)_{\substack{\nu=1,\cdots,n \\ \rho=1,\cdots,r}} .$$

In the following statement let C denote an algebraically closed field containing $\mathbf{Q}$ and complete with respect to a non-discrete absolute value.

COROLLARY 2. *Let $y_1, \cdots, y_n$ be analytic functions in some polydisk about the origin 0 in C^r for which the $y_\nu - y_\nu(0)$ are $\mathbf{Q}$-linearly independent. Then*

$$\dim_C C(y_1, \cdots, y_n, \exp y_1, \cdots, \exp y_n) \geqq n + \operatorname{rank}\left(\frac{\partial y_\nu}{\partial t_\rho}\right)_{\substack{\nu=1,\cdots,n \\ \rho=1,\cdots,r}} ,$$

assuming the $\exp y_\nu$ *are defined.*

We also establish the following relative version of (SD).

THEOREM 4. *Let $F \supseteq E \supseteq C \supseteq \mathbf{Q}$ be a tower of fields and Δ a set of derivations of F such that for all $D \in \Delta$ we have $DE \subseteq E$ and $\bigcap_{D \in \Delta} \ker D = C$. Let $y_1, \cdots, y_n, z_1, \cdots, z_n \in F^*$ and $x_1, \cdots, x_n \in E$ be such that*

(a) *for all $D \in \Delta$, $\nu = 1, \cdots, n$, $Dy_\nu = x_\nu + Dz_\nu/z_\nu$, and*

(b) *no non-trivial power product of the z_ν is algebraic over E.*

Then

$$\dim_E E(y_1, \cdots, y_n, z_1, \cdots, z_n) \geqq n .$$

(ii) *Statements of previous results and applications.* Schanuel and his student D. Brownawell have proven the following cases of (SD);

(1) when $n \leqq 2$;

(2) when $\dim_C C(y_1, \cdots, y_n) = 1$;

(3) when $\dim_C C(y_1, \cdots, y_n) = n$;

(4) when $\dim_C C(z_1, \cdots, z_n) = n$.

R. Risch in his work on elementary functions has proven a result [6, p. 5, Structure Theorem] which is equivalent to the special case of Theorem 4 where Δ contains a single element and where for each $i = 1, \cdots, n$,

$$\dim_{C(y_1,\cdots,y_{i-1},z_1,\cdots,z_{i-1})} C(y_1, \cdots, y_i, z_1, \cdots, z_i) \leqq 1 .$$

Risch had also obtained the corresponding special case of (SD).

In connection with Skolem's method [7] or [8, Ch. 4, § 6] for proving the finiteness of the set of solutions of certain diophantine problems, Borevich-Shafarevich [7, p. 300] raised the problem of proving the following statement.

(B-S) Let $C \supseteq \mathbf{Q}$ and $y_1, \cdots, y_n \in tC[[t]]$ be such that $n \geqq 2$ and

$$\operatorname{rank}_C (y_1, \cdots, y_n) + \operatorname{rank}_C (\exp y_1, \cdots, \exp y_n) \leqq n .$$

Then there exists distinct i and j for which $y_i = y_j$.

We show by means of examples the falsity of (B-S) in § 5 (ii). On the other hand, Corollary 1 to Theorem 3 contains as a special case the following result.

THEOREM 5. *Let $C \supseteq \mathbf{Q}$ and $y_1, \cdots, y_n \in tC[[t]]$ be such that*

$$\operatorname{rank}_C (y_1, \cdots, y_n) + \operatorname{rank}_C (\exp y_1, \cdots, \exp y_n) \leqq n .$$

Then $y_1, \cdots, y_n$ are $\mathbf{Q}$-linearly dependent.

Under the same hypothesis as in Theorem 5, (B-S) asserts the existence of a very special $\mathbf{Q}$-linear dependency $y_i = y_j$. Since this sort of conclusion is important for the applications we give in § 5 (iii) (Theorem 6), a result of this type containing the result of Skolem that (B-S) is true when rank $(\exp y_1, \cdots, \exp y_n) \leqq 2$.

In [9], Chabauty obtained results including Skolem's and penetrating considerably deeper. The basis of these results is a lower bound for the dimension of the intersection of certain analytic varieties, called μ-varieties, with algebraic varieties [9, Lemmas 2.1, 2.2, 2.3]; these are special cases of Theorem 3 as is shown in § 5 (i).

2. Dualization

(i) *The module of relative differentials.* Let A be a commutative ring and B a commutative A-algebra. Then there exists [10, Ch. 3, § 1, pp. 279–280] a B-module $\Omega_{B/A}$ and an A-derivation $d = d_{B/A}$: $B \to \Omega_{B/A}$ such that if M is any B-module and $B \xrightarrow{\lambda} M$ any A-derivation then there exists a unique B-homomorphism $\xi = \xi_\lambda$ making

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr] & \Omega_{B/A} \arrow[d, dashed, "\xi"] \\ & M \end{tikzcd}$$

commute. Thus $\mathrm{Hom}_B(\Omega_{B/A}, M)$ is canonically isomorphic to the B-module $\mathrm{Der}_A(B, M)$ of A-derivations of B into M. In particular, $\mathrm{Der}_A(B, B) \approx \hat{\Omega}_{B/A} = \mathrm{Hom}_B(\Omega_{B/A}, B)$. $\Omega_{B/A}$ can be realized by letting J be the free B-module on the set $\{\delta b \mid b \in B\}$ and letting M be the intersection of all B-submodules N of J for which the composed map

$$B \xrightarrow{\delta} J \longrightarrow J/N$$

is an A-derivation. Then we can take $\Omega_{B/A} = J/M$ and $d = [B \xrightarrow{\delta} J \longrightarrow J/M]$. Another realization of $\Omega_{B/A}$ is that of the kernel I of the A-algebra homomorphism $B \otimes_A B \to B$ (sending $b_1 \otimes b_2 \to b_1 b_2$) modulo I^2. Using either of these realizations it can be shown [15, Ch. II, § 1] or [16, p. 93, Prop.] that for all derivations D of B for which there exists a derivation D_A of A such that

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "D_A"'] & B \arrow[d, "D"] \\ A \arrow[r, "s"] & B \end{tikzcd}$$

commutes where s is the structural morphism there exists a unique derivation D^t of $\Omega_{B/A}$ satisfying

$$D^1(b_1 db_2) = (Db_1)db_2 + b_1 d(Db_2) .$$

This extension of the action of the derivations of B to $\Omega_{B/A}$ has also been considered in a special case by Manin [11, Ch. I, § 1.1].

(ii) *On the field of definition for linear relations among differentials.*

PROPOSITION 1. *Let $F \supseteq E \supseteq C$ be a tower of fields and Δ a set of derivations of F with $\bigcap_{D\in\Delta} \ker D = C$ and $DE \subseteq E$ for $D \in \Delta$. Then the canonical map*

$$F \otimes_C \bigcap\nolimits_{D\in\Delta} \ker D^1 \xrightarrow{\beta} \Omega_{F/E}$$

is injective; here $\beta(f \otimes \omega) = f\omega$.

Proof. If false there exist $\omega_1, \cdots, \omega_m \in \bigcap_{D\in\Delta} \ker D^1$ and $f_1, \cdots, f_m \in F$ not all zero such that

$$(\dagger) \qquad \sum\nolimits_{\mu=1}^{m} f_\mu \omega_\mu = 0 \; .$$

We assume that m is the minimal length of such relations, that $f_1 = 1$, and then show that for all μ, $f_\mu \in C$. Indeed, applying D^1 to ($\dagger$) for $D \in \Delta$ we get

$$0 = \sum\nolimits_{\mu=1}^{m} [(Df_\mu)\omega_\mu + f_\mu D^1\omega_\mu] = \sum\nolimits_{\mu=2}^{m} (Df_\mu)\omega_\mu \; .$$

By the minimality of m, we must have $Df_\mu = 0$ for all $D \in \Delta$, i.e. $f_\mu \in C$ for all μ. This proves the proposition.

The following lemma is well-known.

LEMMA 1. *If $F \supseteq E \supseteq C \supseteq \mathbf{Q}$ is a tower of fields and if $\dim_C E = m$, then the F-rank of the F-subspace $Fd_{F/C}E$ of $\Omega_{F/C}$ generated by $d_{F/C}E$ is m.*

Proof. If $f_1, \cdots, f_b \in F$ are algebraically dependent over C, then there exists a polynomial $P \in C[x_1, \cdots, x_b] - 0$ of minimal degree such that $P(f_1, \cdots, f_b) = 0$. But then applying $d_{F/C}$ we get

$$\sum\nolimits_{\beta=1}^{b} \frac{\partial P}{\partial x_\beta}(f_1, \cdots, f_b)df_\beta = 0 \qquad \text{in } \Omega_{F/C}$$

so that the df_β are F-linearly dependent. It follows that if $\{t_\mu : \mu \leqq m\}$ is a transcendence basis for E over C, then $\{dt_\mu : \mu \leqq m\}$ generates $Fd_{F/C}E$ over F. Now assume $\sum_{\mu=m} g_\mu dt_\mu = 0$ with $g_\mu \in F$. For each $\lambda \leqq m$ there exists a derivation D of F such that $D(t_\mu) = 0$ for $\mu \neq \lambda$ and $D(t_\lambda) = 1$. Let $\xi \in \hat{\Omega}_{F/C}$ correspond to D. Then applying ξ we get $g_\lambda = 0$, and the lemma follows.

LEMMA 2. *Let $F \supsetneqq C$ be an extension of fields with C relatively closed in F. Let W be the set of subfields $E \supseteq C$ of F with E relatively algebraically closed in F and $\dim_E F = 1$. Then*

$$\bigcap\nolimits_{E\in W} E = C \; .$$

Proof. Let $t \in F \sim C$. Then there exists a subset B of $F \sim C(t)$ such that $B \cup \{t\}$ is a transcendence basis for F/C. If E is the relative algebraic closure of $C(B)$ in F then $E \in W$ and $t \notin E$. The lemma follows.

We denote by dF the C-subspace of $\Omega_{F/C}$ consisting of the elements df for $f \in F$. We denote by dF/F the $\mathbf{Z}$-submodule of $\Omega_{F/C}$ consisting of the elements $df/f = (1/f)df$ for $f \in F^*$.

The canonical map referred to in the following statement is that induced by sending $c \otimes \omega$ to $c\omega$ modulo dF for $c \in C$ and $\omega \in \Omega_{F/C}$.

PROPOSITION 2. *Let $F \supseteq C \supseteq \mathbf{Q}$ be a tower of fields. Then the canonical map*

$$C \otimes_{\mathbf{Z}} dF/F \longrightarrow \Omega_{F/C}/dF$$

is injective.

Proof. Assume there exists $c_1, \cdots, c_m \in C$, $\mathbf{Q}$-linearly independent and $v_1, \cdots, v_m \in F$ and $v \in F$ such that

$$(*) \qquad \sum_{\mu=1}^{m} c_\mu dv_\mu/v_\mu = dv \ .$$

We will show that $(*)$ implies $dv_\mu = 0$ for all μ. We can assume $\dim_C F = t < \infty$. If $t = 0$, $dv = 0$ for all $v \in F$ and there is nothing to prove. Now let $t = 1$. We may assume C is relatively algebraically closed in F, so that F is a field of algebraic functions in one variable over C. Thus for each C-plane $\boldsymbol{p}$ of F there is a well-defined valuation

$$\mathrm{ord}_{\boldsymbol{p}}\colon F \longrightarrow \mathbf{Z} \cup \{\infty\}$$

and a C-linear map

$$\mathrm{res}_{\boldsymbol{p}}\, \Omega_{F/C} \longrightarrow C \ .$$

Moreover we have for all $v \in F^*$,

$$\mathrm{res}_{\boldsymbol{p}}\,(dv/v) = \mathrm{ord}_{\boldsymbol{p}}\, v \quad \text{and} \quad \mathrm{res}_{\boldsymbol{p}}\,(dv) = 0 \ .$$

Thus $(*)$ yields for all such $\boldsymbol{p}$

$$0 = \mathrm{res}_{\boldsymbol{p}} \sum_{\mu=1}^{m} c_\mu dv_\mu/v_\mu = \sum_{\mu=1}^{m} c_\mu \,\mathrm{ord}_{\boldsymbol{p}}\, v_\mu \ .$$

Since $\mathrm{ord}_{\boldsymbol{p}} v_\mu \in \mathbf{Z}$ and the c_μ are $\mathbf{Q}$-linearly independent we obtain that $\mathrm{ord}_{\boldsymbol{p}} v_\mu = 0$ for all $\boldsymbol{p}$ which implies $v_\mu \in C$ for all μ, as desired.

Next let $E \supseteq C$ be any relatively algebraically closed subfield of F for which $\dim_E F = 1$. Applying the canonical epimorphism $\Omega_{F/C} \rightarrow \Omega_{F/E}$ to $(*)$ we obtain

$$\sum_{\mu=1}^{m} c_\mu d_{F/E} v_\mu/v_\mu = d_{F/E} v$$

which, by what we have already established, implies $v_\mu \in E$ for all μ. Thus $v_\mu \in \bigcap E$ where the intersection is over the set of $E \supseteq C$ relatively algebraically closed in F and for which $\dim_E F = 1$. Thus $v_\mu \in C$ for all μ, by Lemma 2, and this completes the proof.

3. Proof of the main results

We will make use of the following fact.

LEMMA 3. *Let $F \supseteq C$ be fields, $y, z \in F$ and D a derivation of F such that $DC = 0$. Set $\omega = dy - dz/z \in \Omega_{F/C}$. Then $D^1\omega = d(Dy - Dz/z)$.*

Proof. $D^1\omega = D^1(dy - dz/z) = dDy - D(1/z)dz - 1/zdDz$. Also $D(1/z) = -Dz/z^2$ and $d(Dz/z) = 1/zdDz + -(Dz/z^2)dz$. The lemma follows.

(i) *Proof of Theorem* 3. Let $r = \operatorname{rank}(Dy_\nu)_{\substack{\nu=1,\cdots,n \\ D\in\Delta}}$. We may assume $D_1, \cdots, D_r \in \Delta$ and $y_1, \cdots, y_r$ are such that

$$0 \neq \det(D_\sigma y_\rho)_{\sigma,\rho=1,\cdots,r} .$$

Let $(a_{ij})_{i,j=1,\cdots,r} = (D_\sigma y_\rho)^{-1}_{\sigma,\rho=1,\cdots,r}$. Setting $E_i = \sum_{\sigma=1}^r a_{i\sigma}D_\sigma \in \operatorname{Der}_C(F, F)$ we have

$$(*) \qquad E_i(y_\rho) = \sum_{\sigma=1}^r a_{i\sigma}D_\sigma(y_\rho) = \delta_{i\rho} .$$

Now for each $D \in \Delta$ there exist unique $b_\rho(D) \in F$ such that $D(y_\sigma) = \sum_{\rho=1}^r b_\rho(D)D_\rho(y_\sigma)$ for $\sigma = 1, \cdots, r$ since $(D_\sigma y_\rho)$ is non-singular.
Set $D' = D - \sum_{\rho=1}^r b_\rho(D)D_\rho \in \operatorname{Der}_C(F, F)$ and

$$\Delta' = \{D' \mid D \in \Delta\} \cup \{E_1, \cdots, E_r\} .$$

Assume $f \in F$ is such that $E(f) = 0$ for all $E \in \Delta'$. Then $0 = E_i(f) = \sum_{\sigma=1}^r a_{i\sigma}D_\sigma(f)$ for $i = 1, \cdots, r$ and since $(a_{i\sigma})_{i,\sigma}$ is non-singular, we must have $D_\sigma(f) = 0$ for $\sigma = 1, \cdots, r$. Thus for all $D \in \Delta$,

$$0 = D'(f) = D(f) - \sum_{\rho=1}^r b_\rho(D)D_\rho(f) = D(f) ,$$

i.e., $f \in C$. Conversely, for all $c \in C$ and for all $E \in \Delta'$, $E(c) = 0$ since each such E is in the left F-subspace of $\operatorname{Der}_C(F, F)$ generated by Δ. For the same reason, (a) holds with Δ replaced Δ' and $\operatorname{rank}(Ey_\nu)_{\substack{\nu=1,\cdots,n \\ E\in\Delta}} \leqq r$. Equality holds since $E_iy_\nu = \delta_{i\nu}$ for $i, \nu = 1, \cdots, r$. This proves that we may without loss in generality augment the hypothesis of the theorem to include the existence of $D_1, \cdots, D_r \in \Delta$ such that $D_i(y_j) = \delta_{ij}$ for $i, j = 1, \cdots, r$ and that $D \in \Delta - \{D_1, \cdots, D_r\}$ implies $D(y_i) = 0$ for $i = 1, \cdots, r$.

If $\dim_C C(y_1, \cdots, y_n, z_1, \cdots, z_n) < n + r$ then by Lemma 1, there exist $f_1, \cdots, f_n, g_1, \cdots, g_r \in F$, not all zero such that with $\omega_\nu = d_{F/C}y_\nu - 1/z_\nu d_{F/C}z_\nu$ we have

$$(*) \qquad \sum_{\nu=1}^n f_\nu\omega_\nu + \sum_{\rho=1}^r g_\rho dy_\rho = 0 \qquad \text{in } \Omega_{F/C} .$$

For all $D \in \Delta$, we have by Lemma 3, $D^1(\omega_\nu) = D^1(dy_\rho) = 0$ for $\nu = 1, \cdots, n, \rho = 1, \cdots, r$ and so by Proposition 1, we may assume $f_\nu, g_\rho \in C$ for $\nu = 1, \cdots, n, \rho = 1, \cdots, r$. If some $f_\nu \neq 0$ then some C-linear combination of the $(1/z_\nu)dz_\nu$ is exact and hence by Proposition 2 the $(1/z_\nu)dz_\nu$ are $\mathbf{Z}$-linearly dependent, but this contradicts (b). It follows that $(*)$ is really of the form

$$(**) \qquad \sum_{\rho=1}^{r} g_\rho dy_\rho = 0$$

with some g_ρ, say $g_\sigma \neq 0$. But then applying the linear functional $\xi_{D_\sigma} \in \hat{\Omega}_{F/C}$ corresponding to D_σ to the relation $(**)$ we get

$$0 = \xi_{D_\sigma}\left(\sum_{\rho=1}^{r} g_\rho dy_\rho\right) = g_\sigma ,$$

a contradiction.

To use (b′) instead of (b) we observe that if (b) is false there exist $a_\nu \in \mathbf{Z}$ not all zero such that $z = \prod_{\nu=1}^{n} z_\nu^{a_\nu} \in C$. Hence for all $D \in \Delta$,

$$0 = Dz/z = \sum_{\nu=1}^{n} a_\nu Dz_\nu/z_\nu = \sum_{\nu=1}^{n} a_\nu Dy_\nu = D\left(\sum_{\nu=1}^{n} a_\nu y_\nu\right) ,$$

i.e., $\sum_{\nu=1}^{n} a_\nu y_\nu \in C$ in contradiction to (b′).

(ii) *Proof of Theorem* 4. Assume that (a) and (b) hold but that

$$\dim_E E(y_1, \cdots, y_n, z_1, \cdots, z_n) < n .$$

Then by Lemma 1, the

$$\omega_\nu = dy_\nu - (1/z_\nu)dz_\nu \in \Omega_{F/E}$$

are F-linearly dependent. For all $D \in \Delta$ we have, using (a) and Lemma 3, $D^1\omega_\nu = 0$.

By Proposition 1, the ω_ν are C-linearly dependent where $C = \bigcap_{D\in\Delta} \ker D \subseteq E$. But from a non-trivial C-linear relation

$$0 = \sum_\nu c_\nu\omega_\nu = \sum_\nu c_\nu dy_\nu - \sum c_\nu(1/z_\nu)dz_\nu$$

we conclude that a non-trivial C-linear combination $\sum c_\nu(1/z_\nu)dz_\nu$ is exact which by Proposition 2 implies that there exist $a_\nu \in \mathbf{Z}$, not all zero such that

$$0 = \sum (a_\nu/z_\nu)dz_\nu = d\prod_{\nu=1}^{n} z_\nu^{a_\nu} .$$

Hence $\prod_{\nu=1}^{n} z_\nu^{a_\nu}$ is algebraic over E in contradiction to (b).

(iii) *Proofs of remaining assertions of the introduction.*

Proof of Theorem 1. Let $C = \mathbf{C}$, $F =$ the quotient field of $C[[t_1, \cdots, t_m]]$, $\Delta = \{\partial/\partial t_1, \cdots, \partial/\partial t_m\}$, and $z_\nu = \exp y_\nu$ for $\nu = 1, \cdots, n$. Since in the hypothesis of Theorem 1, the $y_\nu - y_\nu(0)$ are $\mathbf{Q}$-linearly independent, the y_ν are $\mathbf{Q}$-linearly independent modulo C as required in the hypothesis of Theorem 3; so by that theorem

$$\dim_C C(y_1, \cdots, y_n, z_1, \cdots, z_n) \geqq n + \operatorname{rank}\left(\frac{\partial y_\nu}{\partial t_\mu}\right)$$

which implies Theorem 1.

The corollaries to Theorem 3 are similarly deduced.

Proof of Theorem 2. We need only assume (S) and deduce (Σ). We can assume that $y_1 - y_1(0), \cdots, y_p - y_p(0)$ are a $\mathbf{Q}$-basis for $\sum_{\nu=1}^{n} \mathbf{Q}(y_\nu - y_\nu(0))$. Hence there exist $r_{\nu\pi} \in \mathbf{Q}$ such that

$$y_\nu - y_\nu(0) = \sum_{\pi=1}^{p} r_{\nu\pi}(y_\pi - y_\pi(0)) \qquad \text{for } \nu = p+1, \cdots, n\,.$$

Replacing y_ν by $y_\nu - \sum_{\pi=1}^{p} r_{\nu\pi} y_\pi$ for $\nu = p+1, \cdots, n$ we have that the hypotheses of (Σ) still hold and in addition $y_1 - y_1(0), \cdots, y_p - y_p(0)$ are $\mathbf{Q}$-linearly independent while $y_{p+1}, \cdots, y_n \in \mathbf{C}$ are also $\mathbf{Q}$-linearly independent. Set $C = \mathbf{Q}(y_{p+1}, \cdots, y_n, \exp y_{p+1}, \cdots, \exp y_n)$; by (S), $\dim_{\mathbf{Q}} C \geqq n - p$. By the last line of the proof of Theorem 1,

$$\begin{aligned}\dim_{\mathbf{C}} \mathbf{C}(y_1, \cdots, y_p, \exp y_1, \cdots, y_p) &\geqq p + \operatorname{rank}\left(\frac{\partial y_\pi}{\partial t_\mu}\right)_{\substack{\pi=1,\cdots,p\\ \mu=1,\cdots,m}} \\ &= p + \operatorname{rank}\left(\frac{\partial y_\nu}{\partial t_\mu}\right)_{\substack{\nu=1,\cdots,n\\ \mu=1,\cdots,m}}\,.\end{aligned}$$

The two inequalities we have established together imply (Σ), thereby proving Theorem 2.

4. Some related results

(1) *Ostrowski's Theorem.* This theorem [12], has been generalized by Kolchin [13, § 2] as follows for $F \supseteq E \supseteq C \supseteq \mathbf{Q}$ and Δ as in Theorem 4.

THEOREM (Kolchin). *Let $y_1, \cdots, y_m, z_1, \cdots, z_n \in F^*$ be such that for all $D \in \Delta$, $Dy_\mu, Dz_\nu/z_\nu \in E$. Assume the y_μ are C-linearly independent modulo E and that no non-trivial power product of the z_ν is in E. Then $y_1, \cdots, y_m, z_1, \cdots, z_n$ are algebraically independent over E.*

Proof. We proceed as in the proof of Theorem 4. If false, the dy_μ and dz_ν/z_ν are F-linearly dependent in $\Omega_{F/E}$. Since by Lemma 3 for all $D \in \Delta$, $D^1(dy_\mu) = D^1(dz_\nu/z_\nu) = 0$ we have by Proposition 1 that the dy_μ and dz_ν/z_ν are C-linearly dependent. By Proposition 2, this implies the dy_μ are C-linearly dependent or the dz_ν/z_ν are $\mathbf{Z}$-linearly dependent. Letting E_1 equal the relative algebraic closure of E in F we have either

($*$) some non-trivial C-linear combination $\sum_{\mu=1}^{m} c_\mu y_\mu = y \in E_1$ or

($**$) some non-trivial power product $\prod_{\nu=1}^{n} z_\nu^{a_\nu} = z \in E_1$.

We can find a finite subextension E_0/E of E_1/E such that if ($*$) holds, then $y \in E_0$. Thus for all $D \in \Delta$, $Dy = \sum_{\mu=1}^{m} c_\mu D y_\mu \in E$, and so

$$[E_0\colon E_1]Dy = \operatorname{Trace}_{E_0/E}(Dy) = D(\operatorname{Trace}_{E_0/E}(y))\,,$$

i.e., $D(\sum_{\mu=1}^{m} c_\mu y_\mu) = Dy'$ with $y' = [E_0\colon E_1]^{-1} \operatorname{Trace}_{E_0/E}(y) \in E$. It follows, because $C \subseteq E$ that $\sum_{\mu=1}^{m} c_\mu y_\mu \in E$ as desired. A similar argument with Norm instead of Trace shows that in case (**) we have $\prod_{\nu=1}^{n} z_\nu^{a_\nu} \in E$. This completes the proof.

(ii) *Some examples related to Theorem* 4. The considerations of the end of our proof of Kolchin's Theorem in § 4 (i) might lead one to examine the necessity of the inclusion of hypothesis (b) in Theorem 4 rather than the weakened assumption that no non-trivial power product of the z_ν is in E (instead of the relative algebraic closure of E). For example, can both y and an exponential z of y be properly algebraic over E? The answer is in the affirmative as is seen by taking $E = C((t))$, $F = C((t^{1/2}))$, $D = d/dt$, $y = t^{1/2}$, and $z = \exp y$.

Another question that arises about Theorem 4 is whether the absolute results such as Theorem 3 or more simply (SD) can be "derived from it." The answer is again affirmative although the procedure is tedious; it cannot be done by taking $C = E$, i.e., in Theorem 4, n cannot be improved to $n + 1$. Indeed, let $n = 1$, $F = C((t))$, $E = C(t)$, $D = d/dt$, and let $y \in tC[[t]]$ be the solution of $1 + 2t = y + \exp y$. Setting $z = \exp y$ we have

$$\dim_{C(t)} C(t)(y, z) = 1 .$$

(iii) *The converse of a Schanuel-type statement.* Let $F \supseteq C \supseteq \mathbf{Q}$ be a tower of fields, and let $y_1, \cdots, y_n, z_1, \cdots, z_n \in F^*$. Assume the y_ν are $\mathbf{Q}$-linearly independent modulo C. It follows from Theorem 3 that if Δ is a set of derivations of F with $C = \bigcap_{D \in \Delta} \ker D$ and such that for all $D \in \Delta$, $Dy_\nu = Dz_\nu/z_\nu$, then

$$\dim_C C(y_1, \cdots, y_n, z_1, \cdots, z_n) \geqq n + 1 .$$

This implies that if Y is the $\mathbf{Z}$-submodule of F generated by the y_ν, if $m \geqq 1$ and if $y'_1, \cdots, y'_m \in Y$ and $z'_1, \cdots, z'_m$ are the corresponding power products of the z_μ (if $y'_\mu = \sum_{\nu=1}^{n} a_\nu y_\nu$, then $z'_\mu = \prod_{\nu=1}^{m} z_\nu^{a_\nu}$) then $\dim_C C(y'_1, \cdots, y'_m, z'_1, \cdots, z'_m) \geqq m + 1$.

We are going to show that these lower bounds *characterize* sets $y_1, \cdots y_n$, $z_1, \cdots, z_n$ for which the z_ν can be made exponentials of the y_ν.

Let $F \supseteq C \supseteq \mathbf{Q}$ be as above, with C relatively algebraically closed in F. Let Y be an additive subgroup of F such that $Y \cap C = \{0\}$. Let $e \colon Y \to F^*$ be a homomorphism.

THEOREM 6. *The following three conditions are equivalent.*

(I) *There exists a set Δ of derivations with $\bigcap_{D \in \Delta} \ker D = C$ and for all $y \in Y$, $De(y) = e(y)Dy$.*

(II) *For all $\mathbf{Z}$-linearly independent $y_1, \cdots, y_n \in Y$ we have*

$$\dim_C C(y_1, \cdots, y_n, e(y_1), \cdots, e(y_n)) \geqq n + 1 ;$$

(III) $\sum_{y \in Y} F(dy - de(y)/e(y)) \cap dF = \{0\}$ *in* $\Omega_{F/C}$.

Proof. By the remarks preceding this theorem, (I) $\Rightarrow$ (II).

(III) $\Rightarrow$ (I). Let L be the set of $\lambda \in \hat{\Omega}_{F/C} = \mathrm{Hom}_F(\Omega_{F/C}, F)$ such that $\lambda(\sum_{y \in Y} F(dy - de(y)/e(y))) = 0$ By (III), $\bigcap_{\lambda \in L} \ker \lambda \cap dF = \{0\}$. If D_λ is the C-derivation of F corresponding to λ, we have for $f \in F$ that $D_\lambda(f) = \lambda(df)$. Therefore $D_\lambda(f) = 0$ for all $\lambda \in L \Leftrightarrow df \in \bigcap_{\lambda \in L} \ker \lambda \Leftrightarrow df = 0 \Leftrightarrow f \in C$, i.e., $\bigcap_{\lambda \in L} \ker D_\lambda = C$. Also for all $y \in Y$,

$$0 = \lambda(dy - de(y)/e(y)) = D_\lambda y - (D_\lambda e(y))/e(y)$$

and this establishes (III) $\Rightarrow$ (I).

(II) $\Rightarrow$ (III). We assume (II) and that $f \in F$ is such that $df \in \sum_{y \in Y} F(dy - de(y)/e(y))$ and then show $df = 0$. Let

$$(*) \qquad adf = \sum_{\nu=1}^{n} f_\nu(dy_\nu - de(y_\nu)/e(y_\nu))a\,, \qquad f_1, \cdots, f_n \in F,\ a \neq 0\,, \quad f_1 = 1,\ y_\nu \in Y \text{ for } \nu = 1, \cdots, n\,.$$

We assume inductively that a relation of type $(*)$ with $df \neq 0$ implies, for $n < p$ and (all pairs F/C as above), that $\dim_C C(y_1, \cdots, y_n, e(y_1), \cdots, e(y_n)) \leqq n$ and show the same holds for $n = p$. We can assume the y_ν are $\mathbf{Z}$-linearly independent.

Applying the canonical epimorphism $\Omega_{F/C} \to \Omega_{F/E}$, where E is the relative algebraic closure of $C(f)$ in F, we get

$$(**) \qquad 0 = \sum_{\pi=1}^{p} f_\pi(d_{F/E}y_\pi - d_{F/E}e(y_\pi)/e(y_\pi))$$

which we can assume to be of minimal length.

If $p = 1$, this relation

$$0 = d_{F/E}y_1 - d_{F/E}e(y_1)/e(y_1)$$

implies by Proposition 2 that $y_1, e(y_1) \in E$ so that $\dim_C (y_1, e(y_1)) \leqq 1$ as desired. We now assume $p \geqq 2$. The canonical derivation $d_{F/E}\colon F \to \Omega_{F/E}$ extended to a C-derivation of the exterior F-algebra $\wedge\, \Omega_{F/E}$ built on $\Omega_{F/E}$ (similar to the classical case as in, e.g., [14, § 3.2]). Moreover a computation we omit shows that for all $y, z \in F$, $d_{F/E}$ and $d_{F/E}z/z$ are in the kernel of $d_{F/E}$ so that $(**)$ yields

$$(***) \qquad \begin{aligned} 0 &= \sum_{\pi=1}^{p} df_\pi \wedge (dy_\pi - de(y_\pi)/e(y_\pi)) \\ &= \sum_{\pi=2}^{p} df_\pi \wedge (dy_\pi - de(y_\pi)/e(y_\pi)) \qquad \text{in } \wedge\, \Omega_{F/E}\,. \end{aligned}$$

Wedging $(***)$ with $\bigwedge_{\pi=3}^{p} (dy_\pi - de(y_\pi)/e(y_\pi))$ (or leaving it alone if $p = 2$) yields $df_2 \wedge \bigwedge_{\pi=2}^{p} (dy_\pi - de(y_\pi)/e(y_\pi)) = 0$ and by the minimality of $(**)$ the $dy_\pi - de(y_\pi)/e(y_\pi)$ for $2 \leqq \pi \leqq p$ are F-linearly independent so we conclude

$$df_2 \in \sum_{\pi=2}^{p} F(dy_\pi - de(y_\pi)/e(y_\pi)) \qquad \text{in } \Omega_{F/E}\,.$$

If $d_{F/E}f_2 \neq 0$ by inductive hypothesis we have

$$\dim_E E\big((y_1, \cdots, y_p, e(y_2), \cdots, e(y_p)\big) \leqq p - 1$$

and so $\dim_C C\big(y_2, \cdots, y_p, e(y_2), \cdots, e(y_p)\big) \leqq p$ in contradiction to (II). Thus $df_2 = 0$, i.e., $f_2 \in E$. Likewise $f_\pi \in E$ for $\pi = 2, \cdots, n$, so that the $dy_\pi - de(y_\pi)/e(y_\pi)$ are E-linearly dependent in $\Omega_{F/E}$. By Proposition 2, the $de(y_\pi)/e(y_\pi)$ are $\mathbf{Z}$-linearly dependent, i.e., there exist $b_1, \cdots, b_p \in \mathbf{Z}$ and not all zero such that $\prod_{\pi=1}^{p} e(y_\pi)^{b_\pi} = e\big(\sum_{\pi=1}^{p} b_\pi y_\pi\big) \in E$. Setting $y = \sum_{\pi=1}^{p} b_\pi y_\pi \in Y$ we have $\dim_C C\big(y, e(y)\big) \leqq 1$ contradicting (II). Thus $df = 0$, proving (II) $\Rightarrow$ (III).

5. On the methods of Chabauty and Skolem

By a p-adic method [8, Ch. 4, § 6] due to Skolem [7] the problem of proving the finiteness of the number of solutions of certain diophantine equations is reduced to consideration of the algebraic relations satisfied by the exponential function. Skolem's results [7] on these relations are contained in those of Chabauty [9] and these in turn follow from Corollary 1 to Theorem 3 as we show next.

(i) *Chabauty's results.* Let C be an algebraically closed field containing $\mathbf{Q}$ and complete with respect to a non-discrete absolute value.

Let $b_{\mu\nu} \in C$ and $q_\nu \in C^*$ for $\mu = 1, \cdots, c$, $\nu = 1, \cdots, n$. Then, following Chabauty [9, p. 144], we say that the local analytic subvariety M of C^n at $q = (q_1, \cdots, q_n)$ defined by

$$(*) \qquad \sum_{\nu=1}^{n} b_{\gamma\nu} \log (x_\nu/q_\nu) = 0 \qquad \gamma = 1, \cdots, c$$

is a *μ-variety.* If we can choose the $b_{\mu\nu}$ to be in $\mathbf{Z}$ we shall call M an algebraic μ-variety for in this case M is the local analytic variety at q defined by the algebraic variety with defining equations

$$\prod_{\nu=1}^{n} (x_\nu/q_\nu)^{b_{\gamma\nu}} = 1\ , \qquad \gamma = 1, \cdots, c\ .$$

The following result is a restatement of [9, Lemmas 2.1, 2.2, 2.3].

THEOREM (Chabauty). *Let M be a μ-variety at q and W be an algebraic variety containing q. Then for each component I of $W \cap M$ there exists an algebraic μ-variety A such that $A \supseteq I$, and we have $a \leqq m + w - i$ where*

$$\begin{aligned} \dim A &= a \\ \dim I &= i \\ \dim M &= m \\ \dim W &= w\ . \end{aligned}$$

Proof. We can assume $q_\nu = 1$ for $\nu = 1, \cdots, n$ by applying to C^n the map

$$(x_1, \cdots, x_n) \longrightarrow (x_1/q_1, \cdots, x_n/q_n)\ .$$

Let I be an irreducible component of $M \cap W$ of dimension i. Then we can parameterize I at q; i.e., we can find $z_1, \cdots, z_n \in C[[t_1, \cdots, t_i]]$ convergent in a polydisk D about 0 in C^i such that $z_\nu(0) = 1$ for $\nu = 1, \cdots, n$ and such that for all $c \in I$ sufficiently close to q there exists $\tau \in D$ with $z(\tau) = (z_1(\tau), \cdots, z_n(\tau)) = c$. This implies that

$$\operatorname{rank}\left(\frac{\partial z_\nu}{\partial t_j}(\tau)\right)_{\substack{\nu=1,\cdots,n\\ j=1,\cdots,i}} = i \qquad \text{for some } \tau \in D$$

and hence that

$$\operatorname{rank}\left(\frac{\partial z_\nu}{\partial t_j}\right) = i\ .$$

Set $y_\nu = \log z_\nu$ for $\nu = 1, \cdots, n$; these y_ν are power series without constant terms. Let a be the $\mathbf{Z}$-rank of $y_1, \cdots, y_n$, say $y_1, \cdots, y_a$ are $\mathbf{Z}$-independent. We have

$$\operatorname{rank}_C \{y_1, \cdots, y_n\} \leqq \dim M = m$$

and

$$\dim_C C(z_1, \cdots, z_n) \leqq \dim W = w\ .$$

Thus $\dim_C C(y_1, \cdots, y_n, \exp y_1, \cdots, \exp y_n) \leqq m + w$. But by Corollary 1 to Theorem 3,

$$\begin{aligned} &\dim_C C(y_1, \cdots, y_n, \exp y_1, \cdots, \exp y_n)\\ &= \dim_C C(y_1, \cdots, y_a, \exp y_1, \cdots, \exp y_a) \geqq a + \operatorname{rank}\left(\frac{\partial y_\alpha}{\partial t_j}\right)_{\substack{\alpha=1,\cdots,a\\ j=1,\cdots,i}}\ . \end{aligned}$$

Since

$$\frac{\partial z_\alpha}{\partial t_j} = z_\alpha \frac{\partial y_\alpha}{\partial t_j}\ ,$$

we have

$$\begin{aligned} \operatorname{rank}\left(\frac{\partial y_\alpha}{\partial t_j}\right)_{\substack{\alpha=1,\cdots,a\\ j=1,\cdots,i}} &= \operatorname{rank}\left(\frac{\partial z_\alpha}{\partial t_j}\right)_{\substack{\alpha=1,\cdots,a\\ j=1,\cdots,i}}\\ &= \operatorname{rank}\left(\frac{\partial z_\nu}{\partial t_j}\right)_{\substack{\nu=1,\cdots,n\\ j=1,\cdots,i}} = i\ . \end{aligned}$$

Thus $m + w \geqq a + i$. I is contained in the algebraic μ-variety A of dimension a defined by the system $(*)$ where $(b_{\gamma\nu})_{\nu=1,\ldots,n}$ $\gamma = 1, \cdots, c = n - a$ is a basis for the set of $(b_1, \cdots, b_n) \in \mathbf{Z}^n$ such that

$$\textstyle\sum_{\nu=1}^n b_\nu y_\nu = 0\ .$$

This completes the proof.

(ii) *Counter-examples to* (B-S). Let N be a non-negative integer. Set $n = N(N+1)/2$ and $\{y_1, \cdots, y_n\} = \{a \log(1-t) + b \log(1+t) \mid a + b < N\}$. Then $\operatorname{rank}_C(y_1, \cdots, y_n) \leqq 2$. Also

$$\operatorname{rank}_C(\exp y_1, \cdots, \exp y_n) = \operatorname{rank}_C\big((1-t)^a(1+t)^b \colon a + b < N\big) \leqq N .$$

Hence for $N(N+1)/2 > N + 2$ we get counter-examples to (B-S), the smallest value $n = 6$ coinciding with its first unproven case.

(iii) *Skolem-type results.* We have already mentioned the affirmative results of Skolem on (B-S). Theorem 5 is a result in this direction. From the previous section it is clear how the **Q**-linear independence of the y_ν is the crucial point, and not their mere distinctness. Nevertheless, it seems important to find valid modifications of (B-S) including its known cases. The following is a result in this direction.

THEOREM 7. *Let* $C \supseteq \mathbf{Q}$ *and let* $0 = y_0, y_1, \cdots, y_{n-1} \in tC[[t]]$ *be such that* $n \geqq 2$ *and*

(α) $\exp y_1, \cdots, \exp y_s$ *are C-algebraically independent;*

(β) $y_{s+1}, \cdots, y_{n-1}$ *are C-linearly independent.*

Then

(γ) $\operatorname{rank}_C(y_0, \cdots, y_{n-1}) + \operatorname{rank}_C(\exp y_0, \cdots, \exp y_{n-1}) \leqq n$ *implies there exist distinct i and j for which* $y_i = y_j$.

Proof. (α), (β), and (γ) imply that $\exp y_0, \cdots, \exp y_s$ comprise a C-linear basis for $\sum_{\nu=0}^{n-1} C \exp y_\nu$ and $y_{s+1}, \cdots, y_{n-1}$ comprise a C-linear basis for $\sum_{\nu=0}^{n-1} C y_\nu$. Thus there exist unique $a_{\nu i} \in C$ for $\nu = 0, \cdots, s$ and $i = s+1, \cdots, n-1$ such that

$$y_\nu = \sum_{i=s+1}^{n-1} a_{\nu i} y_i , \qquad \nu = 0, \cdots, s . \tag{1}$$

Set $z_\nu = \exp y_\nu$ so that $z_\nu^{-1}(dz_\nu/dt) = dy_\nu/dt$ for $\nu = 0, \cdots, n-1$. Then differentiating (1) we get

$$0 = \sum_{i=0}^{n-1} f_{\nu i} z_i^{-1} \frac{dz_i}{dt} \qquad \text{for } \nu = 0, \cdots, s \tag{2}$$

with

$$f_{\nu i} = \delta_{\nu i}, \qquad \nu, i = 0, \cdots, s . \tag{3}$$

Again there exist unique $b_{i\sigma} \in C$ for $i = 0, \cdots, n-1, \sigma = 0, \cdots, s$ such that

$$\frac{dz_i}{dt} = \sum_{\sigma=0}^{s} b_{i\sigma} \frac{dz_\sigma}{dt} \qquad \text{for } i = 0, \cdots, n-1 \tag{4}$$

with

$$(5) \qquad b_{i\sigma} = \delta_{i\sigma}\,, \qquad i, \sigma = 0, \cdots, s\,.$$

Combining (2) and (4) we get

$$(6) \qquad 0 = \sum_{\sigma=0}^{s} \sum_{i=0}^{n-1} f_{\nu i} z_i^{-1} b_{i\sigma} \frac{dz_\sigma}{dt} \qquad \text{for } \nu = 0, \cdots, s\,.$$

Assuming, as we may that $s \geqq 1$ but that the y_i are distinct, some $dz_\sigma/dt \neq 0, \sigma = 0, \cdots, s$. Hence

$$(7) \qquad 0 = \det (c_{\nu\sigma})_{\nu,\sigma=0,\cdots,s}$$

where

$$(8) \qquad c_{\nu\sigma} = \sum_{i=0}^{n-1} f_{\nu i} z_i^{-1} b_{i\sigma}\,, \qquad \nu, \sigma = 0, \cdots, s\,.$$

Now $\det (c_{\nu\sigma}) = \sum_{\varphi \in P} \operatorname{sg} \varphi \prod_{\nu=0}^{s} c_{\nu\varphi}(\nu)$ where $P =$ the group of permutations of $\{0, \cdots, s\}$ and for $\varphi \in P$, $\operatorname{sg} \varphi$ is the sign of φ. Thus

$$(9) \qquad 0 = \sum_{\varphi \in P} \operatorname{sg} \varphi \prod_{\nu=0}^{s} \sum_{i=0}^{n-1} f_{\nu i} z_i^{-1} b_{i\varphi(\nu)}\,.$$

Let Q be the set of functions

$$\psi\colon \{0, \cdots, s\} \longrightarrow \{0, \cdots, n-1\}\,.$$

Then (9) yields

$$(10) \qquad \begin{aligned} 0 &= \sum_{\varphi \in P} \operatorname{sg} \varphi \sum_{\psi \in Q} \prod_{\nu=0}^{s} f_{\nu\psi(\nu)} z_{\psi(\nu)}^{-1} b_{\psi(\nu)\varphi(\nu)} \\ &= \sum_{\psi \in Q} \prod_{\nu=0}^{s} f_{\nu\psi(\nu)} z_{\psi(\nu)}^{-1} \det (b_{\psi(\nu)i})_{\nu,i=0,\cdots,s} \\ &= \sum_{\psi \in Q_1} \prod_{\nu=0}^{s} f_{\nu\psi(\nu)} z_{\psi(\nu)}^{-1} \det (b_{\psi(\nu)i})_{\nu,i=0,\cdots,s} \end{aligned}$$

where Q_1 is the set of injective maps $\psi \in Q$. Let M be the set of $\psi \in Q_1$ such that $0 \leqq i < j < s \Rightarrow \psi(i) < \psi(j)$ and for each $\psi \in M$, let P_ψ be the set of permutations of $\psi(\{0, \cdots, s\})$. Then for every $\psi \in Q_1$ there exist unique $\xi \in M$ and $\mu \in P_\xi$ such that $\psi = \mu \circ \xi$. Thus

$$(11) \qquad 0 = \sum_{\xi \in M} \Big(\prod_{\nu=0}^{s} z_{\xi(\nu)}^{-1}\Big) \sum_{\mu \in P_\xi} \det (b_{\mu(\xi(\nu))i}) \prod_{\nu=0}^{s} f_{\nu\mu(\xi(\nu))}\,.$$

Since $\det (b_{\mu(\xi(\nu))i})_{\nu,i=0,\cdots,s} = \operatorname{sg} \mu \det (b_{\xi(\nu)i})_{\nu,i=0,\cdots,s}$ and

$$\sum_{\mu \in P_\xi} \operatorname{sg} \mu \prod_{\nu=0}^{s} f_{\nu\mu(\xi(\nu))} = \det (f_{\nu\xi(i)})_{\nu,i=0,\cdots,s}\,,$$

equation (11) yields

$$(12) \qquad 0 = \sum_{\xi \in M} \prod_{\nu=0}^{s} z_{\xi(\nu)}^{-1} \det (b_{\xi(\nu)i}) \det (f_{\nu\xi(i)})\,.$$

In the summand corresponding to $\xi = \xi_1$ where $\xi_1(\nu) = \nu$ for $\nu = 0, \cdots, s$ both determinants are equal to 1 by (3) and (5). Thus we will complete the proof when we contradict the C-linear dependency (12). Then there exists unique $L_i \in C[X_1, \cdots, X_s] - 0$ such that $\deg L_i \leqq 1$ and $z_i = L_i(z_1, \cdots, z_s)$ for $i = 0, \cdots, n-1$. Moreover since the z_i are distinct and $z_i(0) = 1$, it follows that for each $i \neq j$, $z_i/z_j \notin C$ so that

$$L_i/L_j \notin C .$$

The non-trivial linear relation (12) shows that the $\prod_{\nu=0}^{s} L_{\xi(\nu)}^{-1}$ for $\xi \in M$ are C-linearly dependent. Let

$$H_i = Y_0 L_i(Y_1/Y_0, \cdots, Y_s/Y_0) \in C[Y_0, \cdots, Y_s] - 0$$

be the homogeneous linear form corresponding to L_i for $i = 0, \cdots, n-1$. Then we have that the $\prod_{\nu=0}^{s} H_{\xi(\nu)}^{-1}$, $\xi \in M$ are C-linearly dependent while for each pair of distinct i and j, $H_i/H_j \notin C$; say

$$\sum_{\xi \in M} e_\xi \prod_{\nu=0}^{s} H_{\xi(\nu)}^{-1} = 0 , \qquad e_\xi \in C,\ e_{\xi_1} = 1 . \tag{13}$$

If $M_0 = \{\xi \in M \mid \xi(0) = 0\}$, then (13) implies

$$\sum_{\xi \in M_0} e_\xi \Big(\prod_{\nu=1}^{s} H_{\xi(\nu)}^{-1}\Big) H_0^{-1} = 0$$

and so an inductive argument yields the contradiction $e_{\xi_1} = 0$.

Let us show how to deduce the following result of Skolem from Theorem 7.

THEOREM (Skolem). (B-S) *is true if* $\operatorname{rank}_C(\exp y_1, \cdots, \exp y_n) \leqq 2$.

Proof. Assume $\operatorname{rank}_C(\exp y_1, \cdots, \exp y_n) \leqq 2$. By subtracting y_n from each y_ν we can assume $y_0 = y_n = 0$ and

$$\operatorname{rank}_C(y_0, \cdots, y_{n-1}) + \operatorname{rank}_C(\exp y_0, \cdots, \exp y_{n-1}) \leqq n \tag{$*$}$$

and $\operatorname{rank}_C(\exp y_0, \cdots, \exp y_{n-1}) \leqq 2$. If for some ν, $\nu = 1, \cdots, n-1$ we have $\exp y_0$ and $\exp y_\nu$ being C-linearly dependent, then $\exp y_\nu \in C$, so $y_0 = y_\nu = 0$ and we are done. Hence we can assume $\operatorname{rank}_C(\exp y_0, \cdots, \exp y_{n-1}) = 2$, and that $1 = \exp y_0$ and $\exp y_\nu$ form a C-basis for $\sum_{\nu=0}^{n-1} C \exp y_\nu$ for every $\nu = 1, \cdots, n-1$. Inductively we can assume equality holds in $(*)$ and so there exists ν for which $\operatorname{rank}_C(\{y_1, \cdots, y_{n-1}\} - \{y_\nu\}) = n-2$, say $\nu = n-1$.

We can therefore apply Theorem 7 with $s = 1$ to complete the proof.

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