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Non-primitive recursive decidability of products of modal logics with expanding domains

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Abstract

We show that—unlike products of 'transitive' modal logics which are usually undecidable—their 'expanding domain' relativisations can be decidable, though not in primitive recursive time. In particular, we prove the decidability and the finite expanding product model property of bimodal logics interpreted in two-dimensional structures where one component—call it the 'flow of time'—is

- a finite linear order or a finite transitive tree

and the other is composed of structures like

- transitive trees/partial orders/quasi-orders/linear orders or only finite such structures

expanding over time. (It is known that none of these logics is decidable when interpreted in structures where the second component does not change over time.) The decidability proof is based on Kruskal's tree theorem, and the proof of non-primitive recursiveness is by reduction of the reachability problem for lossy channel systems. The result is used to show that the dynamic topological logic interpreted in topological spaces with continuous functions is decidable (in non-primitive recursive time) if the number of function iterations is assumed to be finite.

Keywords: Modal logic; Temporal logic; Decidability; Dynamic topological logic

1. Introduction

Started in the 1970s [40,41], the research programme of investigating and using *products of modal logics*[1] as a multi-dimensional formalism for a variety of promising applications in mathematical logic, computer science and

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[1] For the definition of *products* of modal logics see Section 5 below.

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artificial intelligence (see, e.g., [2,36,9,4,37,13,7,45]) has recently culminated in a series of interesting decidability and complexity results.

Decidability: Roughly, a two-dimensional product of modal logics can be decidable only if, in order to check satisfiability of a formula φ in product frames for the logic, it suffices to consider those of them where the depth of one of the component frames is bounded by some finite number depending on φ. In other words, only products of standard modal logics with **K**-like or **S5**-like[2] logics are decidable [13,44,11]. Three-dimensional products and products of transitive logics with arbitrary finite or infinite frames are *not* decidable [31,17,38, 14].

Complexity: The computational complexity of decidable product logics turns out to be much higher than the complexity of their components. For example, it is shown in [32] that *all* product logics between $\mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5}$ are coNExpTime-hard (while **K** is known to be PSpace-complete and **S5** coNP-complete). According to [33], even the satisfiability problem for formulas of modal depth 2 in $\mathbf{K} \times \mathbf{K}$-frames is NExpTime-hard. $\mathsf{Log}(\mathbb{N}, <) \times \mathbf{S5}$ is ExpSpace-hard, while $\mathbf{PTL} \times \mathbf{K}$ is not elementary [16,18,11].

Such is the price we have to pay for the strong interaction between the modal operators of the component logics of a product, which is syntactically reflected by the (seemingly harmless) commutativity and Church–Rosser axioms

$$\Diamond_1 \Diamond_2 p \leftrightarrow \Diamond_2 \Diamond_1 p \qquad \text{and} \qquad \Diamond_2 \Box_1 p \to \Box_1 \Diamond_2 p.$$

The general research problem we are facing now can be formulated as follows: *is it possible to reduce the computational complexity of product logics by relaxing the interaction between their components and yet keeping some of the useful and attractive features of the product construction*?

One approach to this problem is motivated by structures often used in such areas as temporal and modal first-order logics, temporal data or knowledge bases (say, temporal description logics) or logical modelling of dynamical systems. What we mean is models/structures with *expanding domains*: if at a certain time point (or in a world) w we have a 'population' Δ_w of elements (objects), then at every later point (in every accessible world) u the population Δ_u cannot be smaller but can grow—i.e., $\Delta_w \subseteq \Delta_u$. Standard product logics respect the stronger *constant domain assumption* according to which $\Delta_w = \Delta_u$ for all u and w.

In the case of *dynamic topological logics* [27,21], expanding domains correspond to the condition that the function describing movements of points in topological spaces is *continuous* (while constant domains correspond to *homeomorphisms*).

Models with expanding domains naturally arise also in the context of tableau- and resolution-based decision procedures that have been developed and implemented for certain monodic fragments of first-order temporal logic and some modal description logics [15,24,20] which include, in particular, the (expanding) products of the corresponding temporal and modal logics with **S5**. One of the most difficult problems in the development and implementation was the conflict between modularity and the necessity to backtrack after introducing every new element; in fact, the systems developed so far are considerably more efficient for expanding domain than for constant domain interpretations.

Products of modal logics with expanding domains were introduced in [30], where it was shown that they cannot be more complex than (in fact, are reducible to) products. But can they be simpler? For example, is it possible that a product logic is undecidable while its expanding relativisation is decidable? A similar question was asked in [12] where it was shown that the two-variable fragment of most first-order modal logics with constant domains is undecidable.

The main achievement of this paper is the discovery of the first pairs of 'standard' modal logics whose product with expanding domains is indeed simpler than their usual product. For example, we show that the expanding product of **GL.3** and **GL** is decidable and has the expanding product finite model property—in contrast to the product $\mathbf{GL.3} \times \mathbf{GL}$ which is undecidable and does not even have the (abstract) finite model property [14]. As a consequence of our results on expanding products, we also prove that the dynamic topological logic with continuous functions and finitely many iterations is decidable—again in contrast to the undecidability in the case of dynamic topological structures with homeomorphisms [21].

Our main results can be summarised as follows. Bimodal logics interpreted in expanding product frames where the first component consists of

[2] The definitions of some standard modal logics like **K**, **S5**, etc., can be found in Section 2.

- finite linear orders or finite transitive trees

and the second is composed of frames like

- transitive trees/partial orders/quasi-orders/linear orders or only finite such structures

are *decidable* and *have the expanding product finite model property*. If the second ('vertical') component is Noetherian (say, frames for **GL.3** or **GL**), then we may also allow infinite Noetherian first ('horizontal') components. None of these logics is decidable when interpreted in models with constant domains [14].

The decidability proof is based on Kruskal's tree theorem [29] and does not establish any elementary upper bound for the time/space complexity of the decision algorithm. We show that indeed no such upper bound exists by proving that there is *no primitive recursive decision algorithm* for such logics. The proof uses a recent result of Schnoebelen [39] according to which reachability in lossy channel systems is not decidable in time bounded by a primitive recursive function. This actually explains why numerous attempts to prove decidability of expanding products failed: quite often the idea was to reduce the decision problem to $S\omega S$ which is not elementary yet primitive recursive [6]. As a consequence, we also obtain that the dynamic topological logic with continuous functions cannot be decided in primitive recursive time, no matter whether the number of function iterations is assumed to be finite or infinite.

The structure of the paper is as follows. In Section 2 we introduce our central notions of two-dimensional expanding domain frames and the interpretation of bimodal formulas in them. In Section 3 we formulate and prove the main decidability results. This is done in three steps. First, in Section 3.1, we use the maximal point technique of [10] to show that the logics under consideration enjoy the expanding product finite model property. Then, in Section 3.2, Kruskal's tree theorem and König's infinity lemma are employed for proving decidability of these logics. Finally, in Section 3.3, we encode the reachability problem for lossy channel systems to establish the non-primitive recursive lower bound. Section 4 shows how the obtained results can be used for investigating the computational behaviour of dynamic topological logics. In Section 5 we compare the expanding domain products introduced in Section 2 with expanding relativised products of [30]. We conclude in Section 6 with a discussion of the obtained results and open problems.

2. Two-dimensional frames with expanding domains

Let $\mathcal{ML}_2$ be the usual propositional bimodal language with two diamonds $\diamondminus$, $\diamondvert$ (and their dual boxes $\boxminus$, $\boxvert$) and the Boolean connectives. The intended 'expanding domain semantics' for this language is defined as follows.

Let $\mathfrak{F} = (W, R)$ be a ('horizontal') frame[3] and let f be a function associating with every $x \in W$ a ('vertical') frame

$$f(x) = (W_x, R_x)$$

in such a way that whenever xRy in $\mathfrak{F}$ then $f(x)$ is a subframe of $f(y)$ in the sense that

- $W_x \subseteq W_y$ and
- for all $u, v \in W_x$, we have uR_xv iff uR_yv.

Then the pair $\mathfrak{H} = (\mathfrak{F}, f)$ is called an *expanding domain frame*, or simply an *e-frame* (see Fig. 1 for an example).

The following definition shows how to interpret $\mathcal{ML}_2$-formulas in e-frames. A *valuation* $\mathfrak{V}$ in an e-frame $\mathfrak{H} = (\mathfrak{F}, f)$ is a set $(\mathfrak{V}_w)_{w\in W}$ of valuations $\mathfrak{V}_w$ in the frames $f(w)$. The pair $\mathfrak{M} = (\mathfrak{H}, \mathfrak{V})$ is called an *expanding domain model based on* $\mathfrak{H}$. The *truth relation* $(\mathfrak{M}, (x, u)) \models \varphi$, where $\varphi \in \mathcal{ML}_2$, $x \in W$ and $u \in W_x$, is defined inductively as follows:

- $(\mathfrak{M}, (x, u)) \models p$ iff $u \in \mathfrak{V}_x(p)$, where p is a propositional variable,
- $(\mathfrak{M}, (x, u)) \models \diamondminus\psi$ iff there is $y \in W$ such that xRy and $(\mathfrak{M}, (y, u)) \models \psi$,
- $(\mathfrak{M}, (x, u)) \models \diamondvert\psi$ iff there is $v \in W_x$ such that uR_xv and $(\mathfrak{M}, (x, v)) \models \psi$

[3] We remind the reader that a pair $\mathfrak{F} = (W, R)$ is called a (unimodal) *Kripke frame* if W is a nonempty set and R is a binary relation on W. A *valuation* in $\mathfrak{F}$ is a function $\mathfrak{V}$ mapping propositional variables to subsets of W.

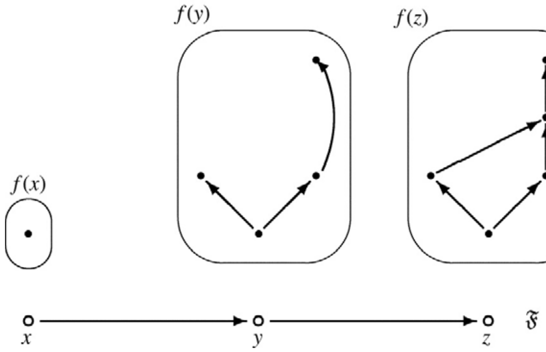

Fig. 1. An e-frame $(\mathfrak{F}, f)$.

(plus the standard clauses for the Boolean connectives). We say that φ is *valid* in $\mathfrak{H}$ ($\mathfrak{H} \models \varphi$, in symbols) if $(\mathfrak{M}, (x, u)) \models \varphi$ holds for all $x \in W$, $u \in W_x$ and all models $\mathfrak{M}$ based on $\mathfrak{H}$. Note that every e-frame validates the left commutativity and Church–Rosser axioms

$$\diamondsuit_v \diamondsuit_h p \to \diamondsuit_h \diamondsuit_v p \qquad \text{and} \qquad \diamondsuit_h \Box_v p \to \Box_v \diamondsuit_h p$$

but not the right commutativity $\diamondsuit_h \diamondsuit_v p \to \diamondsuit_v \diamondsuit_h p$ (see Fig. 1).

Given two classes $\mathcal{C}_1, \mathcal{C}_2$ of unimodal frames, denote by

$$(\mathcal{C}_1 \times \mathcal{C}_2)^{\mathsf{e}}$$

the class of all e-frames $\mathfrak{H} = (\mathfrak{F}, f)$ such that $\mathfrak{F} \in \mathcal{C}_1$ and $f(x) \in \mathcal{C}_2$ for every point x from $\mathfrak{F}$, and let

$$\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)^{\mathsf{e}} = \{\varphi \in \mathcal{ML}_2 \mid \forall \mathfrak{H} \in (\mathcal{C}_1 \times \mathcal{C}_2)^{\mathsf{e}} \;\; \mathfrak{H} \models \varphi\}.$$

Remark 1. Observe that $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)^{\mathsf{e}}$ is always a Kripke complete normal bimodal logic. Indeed, given an expanding domain model $\mathfrak{M} = (\mathfrak{H}, \mathfrak{V})$ as above, we can 'represent' it as a usual Kripke model $\overline{\mathfrak{M}} = (\overline{\mathfrak{H}}, \overline{\mathfrak{V}})$ based on the bimodal frame

$$\overline{\mathfrak{H}} = (\{(x, u) \mid x \in W,\; u \in W_x\}, R_h, R_v),$$

where

$$\begin{aligned} (x, u) R_h (y, v) \quad &\text{iff} \quad u = v \text{ and } x R y, \\ (x, u) R_v (y, v) \quad &\text{iff} \quad x = y \text{ and } u R_x v, \\ \overline{\mathfrak{V}}(p) &= \{(x, u) \mid u \in \mathfrak{V}_x(p)\}. \end{aligned}$$

Then, for every $\mathcal{ML}_2$-formula φ, we have $(\mathfrak{M}, (x, u)) \models \varphi$ iff $(\overline{\mathfrak{M}}, (x, u)) \models \varphi$.

Note that if the e-frame $\mathfrak{H} = (\mathfrak{F}, f)$ is such that $f(x) = \mathfrak{G}$ for all x in $\mathfrak{F}$, then $\overline{\mathfrak{H}}$ coincides with what is called the *product* of frames $\mathfrak{F}$ and $\mathfrak{G}$; for more details see Section 5.

Let L_1 be a normal unimodal logic in the language with the diamond $\diamondsuit_h$. Let L_2 be a normal unimodal logic in the language with the diamond $\diamondsuit_v$. Assume also that both L_1 and L_2 are Kripke complete. Then the *expanding domain product* (or *e-product*, for short) of the logics L_1 and L_2 is

$$(L_1 \times L_2)^{\mathsf{e}} = \mathsf{Log}\,(\mathsf{Fr}\,L_1 \times \mathsf{Fr}\,L_2)^{\mathsf{e}},$$

where $\mathsf{Fr}\,L_i$ is the class of all Kripke frames for L_i, $i = 1, 2$. Note that $(L_1 \times L_2)^{\mathsf{e}}$ is a conservative extension of both L_1 and L_2.

In order to make the paper self-contained, here we give a list of the standard modal logics we deal with. All logics L in this list are complete with respect to the classes $\mathsf{Fr}\,L$ of their Kripke frames:

- $\mathsf{Fr}\,\mathbf{K}$ is the class of all frames (W, R),
- $\mathbf{K4} = \mathbf{K} \oplus \Box p \to \Box\Box p$ and $\mathsf{Fr}\,\mathbf{K4}$ is the class of all frames (W, R) with *transitive* R,

- **S4** $=$ **K4** $\oplus \Box p \to p$ and Fr **S4** is the class of frames (W, R) with *transitive*, *reflexive* R,
- **S5** $=$ **S4** $\oplus \Diamond p \to \Box\Diamond p$ and Fr **S5** is the class of frames (W, R) where R is an *equivalence relation*,
- **GL** $=$ **K4** $\oplus \Box(\Box p \to p) \to \Box p$ and Fr **GL** is the class of all frames (W, R) such that R is *transitive, irreflexive* and *Noetherian* in the sense that there is no infinite sequence $x_0 R x_1 R x_2 \ldots$ where $x_i \neq x_{i+1}$ for $i < \omega$,
- **Grz** $=$ **S4** $\oplus \Box(\Box(p \to \Box p) \to p) \to p$ and Fr **Grz** is the class of all frames (W, R) such that R is *transitive, reflexive* and *Noetherian*,
- **K4.3** $=$ **K4**$\oplus\Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$ and Fr **K4.3** is the class of frames (W, R) such that R is *transitive* and *weakly connected* in the sense that whenever xRy, xRz and $y \neq z$ then either yRz or zRy. Rooted[4] transitive and weakly connected frames will be called *linear*. Note that linear frames can have clusters[5] of any kind, in particular, proper and degenerate ones. The logics **S4.3**, **GL.3**, and **Grz.3** are defined analogously.

Here $\oplus$ means 'add the axiom and take the closure under modus ponens, substitution and necessitation $\varphi/\Box\varphi$,' and $\Box^+\psi = \psi \wedge \Box\psi$.

3. Decidability and complexity

As e-products are known to be reducible to standard product logics (see [11, Theorem 9.12] or Proposition 5 below), e-product logics are usually decidable if one of their components is an **S5**- or **K**-like logic [13,44,11]. On the other hand, products of 'transitive' logics with frames of arbitrarily large finite or infinite depth are undecidable and do not have the finite model property [14].

In this section we show that logics of e-frames with arbitrarily large finite transitive components can be decidable, and can even have the following strong version of the finite model property. A bimodal logic L is said to have the *expanding product finite model property* (*e-product fmp*, for short) if, for every $\mathcal{ML}_2$-formula $\varphi \notin L$, there is a finite e-frame for L that refutes φ.

The main results of this paper are the following:

Theorem 1. *Let $\mathcal{C}_h$ be any of the following classes of frames:*

(C1) *all finite transitive antisymmetric frames,*
(C2) *all reflexive or all irreflexive members of* (C1)*,*
(C3) *all linear members of any of the classes in* (C1) *and* (C2)*.*

Let $\mathcal{C}_v$ be any of the classes:

(C4) *all transitive frames,*
(C5) *all reflexive and transitive frames,*
(C6) *all linear members of* (C4) *or* (C5)*.*

Then the logic $\mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ has the e-product fmp and is decidable, but not in time bounded by a primitive recursive function.

Theorem 2. *Let $\mathcal{C}_h$ and $\mathcal{C}_v$ be any of the following classes:*

(C7) *all Noetherian irreflexive transitive frames,*
(C8) *all Noetherian reflexive transitive frames,*
(C9) *all linear members of* (C7) *or* (C8)*.*

*Then the logic $\mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ has the e-product fmp and is decidable, but not in time bounded by a primitive recursive function. In other words, if $L_1, L_2 \in \{$**GL**, **Grz**, **GL.3**, **Grz.3**$\}$ then $(L_1 \times L_2)^{\mathsf{e}}$ has the e-product fmp and is decidable, but not in time bounded by a primitive recursive function.*

We give a common proof of Theorems 1 and 2 via a sequence of lemmas, where we assume $\mathcal{C}_h$ and $\mathcal{C}_v$ to be as in the formulations of the theorems.

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[4] We remind the reader that a frame (W, R) is called *rooted* if there exists $r \in W$ such that $W = \{u \in W \mid rR^*u\}$, where R^* is the reflexive and transitive closure of R.

[5] Recall that a set $X \subseteq W$ is called a *cluster* in $\mathfrak{F}$ if there is some $x \in W$ such that $X = \{x\} \cup \{y \in W \mid xRy \text{ and } yRx\}$. A cluster X is *proper* if $|X| \geq 2$, it is *simple* if $X = \{x\}$ and xRx; otherwise the cluster is called *degenerate*.

3.1. The expanding domain product fmp

Fix some $\mathcal{ML}_2$-formula φ.

Lemma 2.1. *If $\varphi \notin \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ then φ is refuted in a model $\mathfrak{M} = (\mathfrak{H}, \mathfrak{V})$ based on an e-frame $\mathfrak{H} = (\mathfrak{F}, f)$ such that*

- $\mathfrak{F} = (W, R) \in \mathcal{C}_h$,
- $f(x) = (W_x, R_x) \in \mathcal{C}_v$ $(x \in W)$ *and,*
- *for all $x \in W$, $v \in W_x$ and all $\mathcal{ML}_2$-formulas ψ with $(\mathfrak{M}, (x, v)) \models \psi$, the set*

$$A_{x,v,\psi} = \{u \in W_x \mid vR_x u \text{ and } (\mathfrak{M}, (x, u)) \models \psi\} \cup \{v\}$$

contains an R_x-maximal point (i.e., a point w such that if $wR_x w'$ for some $w' \in A_{x,v,\psi}$ then $w' R_x w$).

Proof. Clearly, the lemma holds if $\mathcal{C}_v$ is as in Theorem 2 (that is, consists of Noetherian frames only). So suppose that $\mathcal{C}_h$ and $\mathcal{C}_v$ are as in the formulation of Theorem 1, that is, $\mathcal{C}_h$ is one of (C1)–(C3) (and so contains only *finite* frames) and $\mathcal{C}_v$ is one of (C4)–(C6).

Suppose that $(\mathfrak{N}, (x_0, v_0)) \not\models \varphi$ for some model $\mathfrak{N} = (\mathfrak{G}, \mathfrak{U})$ based on an e-frame $\mathfrak{G} = (\mathfrak{F}, f)$, where $\mathfrak{F} = (W, R) \in \mathcal{C}_h$, $f(x) = (W_x, R_x) \in \mathcal{C}_v$, $x_0 \in W$ and $v_0 \in W_{x_0}$. By Remark 1, we may assume that x_0 is a root of $\mathfrak{F}$ and v_0 is a root of $f(x_0)$. Define a new model $\mathfrak{M} = (\mathfrak{H}, \mathfrak{V})$ based on an e-frame $\mathfrak{H} = (\mathfrak{F}, f^{ue})$ as follows. Take the set U of ultrafilters over $V = \bigcup_{x \in W} W_x$, and set $f^{ue}(x) = (W_x^{ue}, R_x^{ue})$, where

$$W_x^{ue} = \{\boldsymbol{u} \in U \mid W_x \in \boldsymbol{u}\}$$

and

$$\boldsymbol{u}_1 R_x^{ue} \boldsymbol{u}_2 \qquad \text{iff} \qquad \text{for all } A \in \boldsymbol{u}_2,\ \{v \in W_x \mid \exists v' \in A\ vR_x v'\} \in \boldsymbol{u}_1.$$

It is not hard to show that $\mathfrak{H}$ is indeed an e-frame. Note that $f^{ue}(x)$ does not necessarily coincide with the usual 'ultrafilter extension' of $f(x)$, as it may contain several different extensions of each ultrafilter over W_x. However, it is straightforward to check that $f^{ue}(x)$ is a transitive rooted frame for every $x \in W$ (the principal ultrafilter $\boldsymbol{u}_0$ containing $\{v_0\}$ is a root of $f^{ue}(x)$), and R_x^{ue} is reflexive (irreflexive, weakly connected) if R_x is reflexive (irreflexive, weakly connected). Therefore, $\mathfrak{H}$ belongs to $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$.

Define a valuation $\mathfrak{V}$ as the set $(\mathfrak{U}_x^{ue})_{x \in W}$, where

$$\mathfrak{U}_x^{ue}(p) = \{\boldsymbol{u} \in W_x^{ue} \mid \mathfrak{U}_x(p) \in \boldsymbol{u}\}.$$

We claim that, for all $x \in W$, $\boldsymbol{u} \in W_x^{ue}$, and all formulas ψ

$$(\mathfrak{M}, (x, \boldsymbol{u})) \models \psi \qquad \text{iff} \qquad \{v \in W_x \mid (\mathfrak{N}, (x, v)) \models \psi\} \in \boldsymbol{u}. \tag{1}$$

The proof is by induction on ψ. Here we show the only 'non-standard' step of $\psi = \diamondsuit\chi$. Suppose first that $(\mathfrak{M}, (x, \boldsymbol{u})) \models \diamondsuit\chi$. Then, by IH, there is some $y \in W$ such that xRy and

$$\{v \in W_y \mid (\mathfrak{N}, (y, v)) \models \chi\} \in \boldsymbol{u}.$$

Since $\boldsymbol{u} \in W_x^{ue}$, we have

$$\{v \in W_x \mid (\mathfrak{N}, (x, v)) \models \diamondsuit\chi\} \supseteq \{v \in W_x \mid (\mathfrak{N}, (y, v)) \models \chi\} \in \boldsymbol{u},$$

as required. Conversely, suppose $B_{x,\diamondsuit\chi} = \{v \in W_x \mid (\mathfrak{N}, (x, v)) \models \diamondsuit\chi\} \in \boldsymbol{u}$. Since $\mathfrak{F}$ is finite,[6] there are $y_1, \ldots, y_n$ in W such that, for each $i = 1, \ldots, n$, we have xRy_i, $B_{y_i,\chi} = \{v \in W_x \mid (\mathfrak{N}, (y_i, v)) \models \chi\} \neq \emptyset$ and $B_{x,\diamondsuit\chi} = \bigcup_{i=1}^n B_{y_i,\chi}$. It follows that there is some i such that $1 \leq i \leq n$ and

$$\{v \in W_{y_i} \mid (\mathfrak{N}, (y_i, v)) \models \chi\} \supseteq B_{y_i,\chi} \in \boldsymbol{u},$$

and so, by IH, $(\mathfrak{M}, (x, \boldsymbol{u})) \models \diamondsuit\chi$ holds.

As a consequence of (1) we obtain that $(\mathfrak{M}, (x_0, \boldsymbol{u}_0)) \not\models \varphi$.

[6] This step of the proof would not work for infinite $\mathfrak{F}$. In fact, as is shown in item 1 of Section 6, Theorem 1 does not even hold in this case.

The existence of R_x^{ue}-maximal points in sets of form $A_{x,\boldsymbol{u},\psi}$ in $\mathfrak{M}$ follows from a well-known result of Fine [10]. Here is a sketch of the proof. Consider the family

$$\mathcal{X} = \{X \subseteq A_{x,\boldsymbol{u},\psi} \mid R_x^{ue} \cap (X \times X) \text{ is linear, with smallest element } \boldsymbol{u}\}.$$

Let C be a $\subseteq$-maximal set in $\mathcal{X}$ (i.e., for every $C' \in \mathcal{X}$, $C \subseteq C'$ implies $C' = C$); its existence can be readily proved with the help of Zorn's lemma. Now take the set

$$\boldsymbol{y}_0 = \{A \subseteq W_x \mid \exists \boldsymbol{z} \in C\ \forall \boldsymbol{z}' \in C\ (\boldsymbol{z} R_x^{ue} \boldsymbol{z}' \to A \in \boldsymbol{z}')\}.$$

This set is not empty, since $\{v \in W_x \mid (\mathfrak{N}, (x, v)) \models \psi\} \in \boldsymbol{y}_0$, and clearly $\boldsymbol{y}_0$ has the finite intersection property. Hence we can find an ultrafilter $\boldsymbol{y} \in W_x^{ue}$ containing $\boldsymbol{y}_0$. Then it is easy to see, using the definition of R_x^{ue}, that

$$\forall \boldsymbol{z} \in C\ \boldsymbol{z} R_x^{ue} \boldsymbol{y}. \tag{2}$$

We claim that $\boldsymbol{y}$ is R_x^{ue}-maximal in $A_{x,\boldsymbol{u},\psi}$. Indeed, take some $\boldsymbol{y}' \in A_{x,\boldsymbol{u},\psi}$ such that $\boldsymbol{y} R_x^{ue} \boldsymbol{y}'$. If $\boldsymbol{y}' \in C$ then $\boldsymbol{y}' R_x^{ue} \boldsymbol{y}$ holds by (2). If $\boldsymbol{y}' \notin C$ then, by the $\subseteq$-maximality of C in $\mathcal{X}$, R_x^{ue} is not linear on $C \cup \{\boldsymbol{y}'\}$. Since by (2) and $\boldsymbol{y} R_x^{ue} \boldsymbol{y}'$, we have $\boldsymbol{z} R_x^{ue} \boldsymbol{y}'$ for all $\boldsymbol{z} \in C$, there exists a $\boldsymbol{z}' \in C$ such that $\boldsymbol{y}' R_x^{ue} \boldsymbol{z}'$, and so, again by (2), $\boldsymbol{y}' R_x^{ue} \boldsymbol{y}$ as required. $\square$

We will use Lemma 2.1 to show that $\mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ has the e-product fmp. To formulate the next lemma, we require the following notions.

We say that a transitive frame $\mathfrak{F} = (W, R)$ is a *quasi-tree of clusters* if $\mathfrak{F}$ is rooted and R is weakly connected on the set $\{y \in W \mid yRx\}$ for every $x \in W$. If in addition $\mathfrak{F}$ is antisymmetric (that is, does not contain proper clusters), then we call $\mathfrak{F}$ simply a *quasi-tree*. If a quasi-tree of clusters is well-founded (i.e., there are no infinite descending R-chains $\ldots Rx_2Rx_1Rx_0$ of points from distinct clusters) then we call $\mathfrak{F}$ a *tree of clusters*. Finally, a tree of clusters without proper clusters is called a *tree*.[7] Note that since Noetherian frames do not have proper clusters, a Noetherian tree (quasi-tree) of clusters is always just a tree (quasi-tree).

The *co-depth* $cd(x)$ of a point x in a quasi-tree $\mathfrak{F}$ is defined to be the R-distance of x from the root. More precisely, the co-depth of the root is 0, and the co-depth of immediate R-successors of a point of co-depth n is $n + 1$. If for no $n < \omega$ the point x is of co-depth n, then we say that x is of *infinite co-depth*. The *depth* of a finite tree $\mathfrak{F} = (W, R)$ is the maximum of $cd(x)$, for $x \in W$.

Remark 2. By a standard unravelling argument one can show that every rooted transitive frame $\mathfrak{F}$ that belongs to one of the classes (C1)–(C9) above is a p-morphic image of a quasi-tree $\mathfrak{G}$ of clusters belonging to the same class. It can also be shown that this unravelling 'commutes' with the formation of e-frames in both 'coordinates' in the following sense. On the one hand, if $(\mathfrak{F}, f)$ is an e-frame and $\mathfrak{F}$ is the π-image of a quasi-tree $\mathfrak{G}$ for some p-morphism π, then $(\mathfrak{F}, f)$ is a p-morphic image of the e-frame $(\mathfrak{G}, g)$ defined by taking $g(x) = f(\pi(x))$ (x in $\mathfrak{G}$). On the other hand, if $(\mathfrak{F}, f)$ is a rooted e-frame then for every x in $\mathfrak{F}$ there exists a quasi-tree $g(x)$ of clusters such that $(\mathfrak{F}, g)$ is an e-frame and $(\mathfrak{F}, f)$ is a p-morphic image of it. Moreover, if $(\mathfrak{F}, f)$ satisfies the 'maximal points' condition of Lemma 2.1 then the $g(x)$ can be chosen in such a way that $(\mathfrak{F}, g)$ satisfies this condition as well.

Denote by $\ell(\varphi)$ the *length* of φ, say, $\ell(\varphi) = |sub\,\varphi|$ where $sub\,\varphi$ is the set of all subformulas of φ.

Lemma 2.2. *If $\varphi \notin \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ then φ is refuted in a model $\mathfrak{M} = (\mathfrak{H}, \mathfrak{V})$ based on an e-frame $\mathfrak{H} = (\mathfrak{F}, f)$, where*

- *$\mathfrak{F} = (W, R) \in \mathcal{C}_h$ is a finite transitive tree*

and, for every $x \in W$,

- *$f(x) = (W_x, R_x) \in \mathcal{C}_v$ is a finite transitive tree of clusters,*
- *$|W_x| \leq \big(\ell(\varphi) + 1\big)!^{\,cd(x)+1}$, and*
- *x has at most $\ell(\varphi) \cdot \big(\ell(\varphi) + 1\big)!^{\,cd(x)+1}$ immediate R-successors in $\mathfrak{F}$.*

[7] Here we slightly deviate from the usual notion of a *transitive tree*, as our trees may contain both reflexive and irreflexive points.

Proof. Suppose that $(\mathfrak{M},(x,w)) \not\models \varphi$ for some model $\mathfrak{M} = (\mathfrak{H},\mathfrak{V})$ based on an e-frame $\mathfrak{H} = (\mathfrak{F}, f)$, where $\mathfrak{F} = (W, R) \in \mathcal{C}_h$, $f(x) = (W_x, R_x) \in \mathcal{C}_v$, $x \in W$ and $w \in W_x$. According to Remark 2, we may assume that $\mathfrak{M}$ satisfies the conditions of Lemma 2.1, $\mathfrak{F} = (W, R)$ is a (possibly infinite) Noetherian quasi-tree, and (W_x, R_x) is a quasi-tree of clusters, for every $x \in W$.

Now we take the closure Y of the set $X = \{(x,w)\}$ under the following three rules:

- $\Diamond$**-rule:** if $(y,v) \in X$, $(\mathfrak{M},(y,v)) \models \Diamond\psi$, for some $\Diamond\psi \in sub\,\varphi$, and there is no $(y',v) \in X$ such that yRy' and $(\mathfrak{M},(y',v)) \models \psi$, then choose an R-maximal point $y' \in W$ such that yRy', $(\mathfrak{M},(y',v)) \models \psi$ (such a point exists because $\mathfrak{F}$ is Noetherian), and set $X := X \cup \{(y',v)\}$.
- $\Diamond$**-rule:** if $(y,v) \in X$, $(\mathfrak{M},(y,v)) \models \Diamond\psi$, for some $\Diamond\psi \in sub\,\varphi$, and there is no $(y,v') \in X$ such that vR_yv' and $(\mathfrak{M},(y,v')) \models \psi$, then choose an R_y-maximal v' in $f(y)$ such that vR_yv', $(\mathfrak{M},(y,v')) \models \psi$ (such a point exists by Lemma 2.1), and set $X := X \cup \{(y,v')\}$.
- **Square-rule:** if $(y,v) \in X$, yRy' and $(y',v) \notin X$, then set $X := X \cup \{(y',v)\}$.

Consider the restriction $\mathfrak{H}' = (\mathfrak{F}', f')$ of $\mathfrak{H}$ to Y, where $\mathfrak{F}' = (W', R')$, $W' = W \cap \{x \mid (x,w) \in Y\}$, $R' = R \restriction W'$, and $f'(x) = (W'_x, R'_x)$ where $W'_x = \{v \mid (x,v) \in Y\}$ and $R'_x = R_x \restriction W'_x$ for $x \in W'$.

Since $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$, $f'(x)$ is a subframe of $f(x)$ for $x \in W'$, and the classes $\mathcal{C}_h$ and $\mathcal{C}_v$ are closed under taking subframes in all the cases (C1)–(C9), $\mathfrak{F}'$ is a Noetherian quasi-tree in $\mathcal{C}_h$ and the $f'(x)$ are quasi-trees of clusters in $\mathcal{C}_v$.

CLAIM 2.2.1. *If x is of finite co-depth in $\mathfrak{F}'$, then* $|W'_x| \le \big(\ell(\varphi)+1\big)!^{\,cd(x)+1}$.

PROOF. The proof is by induction on n. If $n = 0$, then by applying the $\Diamond$-rule to the root (x,w) of $\mathfrak{H}'$, we can obtain $\le \ell(\varphi)$ immediate R'_x-successors of the form (x,v). In view of maximality, at each of these points the number of formulas of the form $\Diamond\psi \in sub\,\varphi$ to which the $\Diamond$-rule still applies is $\le \ell(\varphi) - 1$. We proceed with the same kind of argument and finally get

$$|W'_x| \quad\le\quad 1 + \ell(\varphi) + \ell(\varphi)\cdot(\ell(\varphi)-1) + \cdots + \ell(\varphi)! \quad\le\quad (\ell(\varphi)+1)!.$$

The induction step for y of co-depth $n+1$ is considered analogously. The only difference is that instead of one 'starting' point in the root W'_x, we should start applying the $\Diamond$-rule to all points of the form (y,v) such that $v \in W'_z$ for the unique point z with $cd(z) = n$ and $zR'y$, that is to $|W'_z| \le \big(\ell(\varphi)+1\big)!^{\,n+1}$ many points. $\square$

CLAIM 2.2.2. *Every point x of finite co-depth in $\mathfrak{F}'$ has*

$$\le\; \ell(\varphi)\cdot\big(\ell(\varphi)+1\big)!^{\,cd(x)+1}$$

immediate R'-successors.

PROOF. Follows from the previous claim and the fact that the $\Diamond$-rule can be applied at most $\ell(\varphi)$ times to a point (x,v). $\square$

CLAIM 2.2.3. *Every point in $\mathfrak{F}'$ is of finite co-depth, that is, $\mathfrak{F}'$ is a tree.*

PROOF. Since $\mathfrak{F}'$ is Noetherian, we cannot have infinite ascending chains of distinct points in $\mathfrak{F}'$. Suppose $\mathfrak{F}'$ still contains a point x of infinite co-depth. This means that there is an infinite descending chain $\dots R'x_2R'x_1R'x$. Let y be an R'-maximal point of finite co-depth such that $yR'x$. It exists because $\mathfrak{F}'$ is Noetherian. By Claim 2.2.1, W'_y is finite. Therefore, we may apply the $\Diamond$-rule to points in W'_y finitely many times only, and so there exists an immediate R'-successor y' of y located properly between y and x. But then $cd(y') = cd(y) + 1$, and so the co-depth of y' is finite, which is a contradiction. $\square$

Thus, $\mathfrak{F}'$ is a Noetherian tree with finite branching. Therefore, by König's lemma, it must be finite. This completes the proof of Lemma 2.2. $\square$

3.2. Decidability

We are now in a position to prove that $\mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ is decidable. It is to be noted that the e-product fmp does not give decidability automatically because (i) we do not have an effective upper bound for the size of a model refuting a given formula $\varphi \notin \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$, nor (ii) do we know that $\mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ is finitely axiomatisable.

We will use a version of Kruskal's tree theorem [29]. Given a finite set Σ, a *labelled Σ-tree* is a triple $\mathfrak{T} = (T, <, l)$, where $(T, <)$ is a transitive tree and l is a function from T to Σ. Given two finite labelled Σ-trees $\mathfrak{T}_i = (T_i, <_i, l_i)$, $i = 1, 2$, we say that $\mathfrak{T}_1$ is *embeddable* into $\mathfrak{T}_2$ if there exists an injective map $\iota : T_1 \to T_2$ such that, for all $u, v \in T_1$,

- $u <_1 v$ iff $\iota(u) <_2 \iota(v)$,
- $l_2(\iota(u)) = l_1(u)$.

Theorem (*Kruskal*).[8] *For every infinite sequence $\mathfrak{T}_1, \mathfrak{T}_2, \ldots$ of finite labelled Σ-trees, there exist $i < j < \omega$ such that $\mathfrak{T}_i$ is embeddable into $\mathfrak{T}_j$.*

In order to use this theorem, we represent expanding domain models in a slightly different form. Roughly, the idea is as follows. By Lemma 2.2, we may assume that the 'vertical components' of e-frames are finite trees of clusters. We take the 'skeleton-tree' of such a tree of clusters, and label each node of this skeleton with the set of Boolean types of points from the cluster represented by the node.

To this end, denote by $\boldsymbol{T}_\varphi$ the set of *Boolean types* $\boldsymbol{t}$ over $sub\,\neg\varphi$, where

- $\neg\psi \in \boldsymbol{t}$ iff $\psi \notin \boldsymbol{t}$, for every $\neg\psi \in sub\,\neg\varphi$, and
- $\chi \wedge \psi \in \boldsymbol{t}$ iff $\chi \in \boldsymbol{t}$ and $\psi \in \boldsymbol{t}$, for every $\chi \wedge \psi \in sub\,\neg\varphi$.

Let $\mathcal{P}(\boldsymbol{T}_\varphi)^+$ be the set of all nonempty subsets of $\boldsymbol{T}_\varphi$. A pair $\mathfrak{Q} = (\mathfrak{F}, f)$ is called a *pre-quasimodel* (for φ) if

- $\mathfrak{F} = (W, R)$ is a transitive tree, and
- $f(x) = (T_x, <_x, l_x)$, for $x \in W$, is a finite labelled $\mathcal{P}(\boldsymbol{T}_\varphi)^+$-tree.

We call such a pre-quasimodel *small* if, for all $x, y \in W$,

(sm1) $|T_x| \leq \big(\ell(\varphi) + 1\big)!^{\,cd(x)+1}$,

(sm2) x has at most $\ell(\varphi) \cdot \big(\ell(\varphi) + 1\big)!^{\,cd(x)+1}$ immediate R-successors in $\mathfrak{F}$,

(sm3) if xRy and $x \neq y$ then $f(x)$ is not embeddable into $f(y)$.

For every $n < \omega$, let Q_n be the set of all small pre-quasimodels $(\mathfrak{F}, f)$ such that $\mathfrak{F}$ is a finite tree of depth n.

Lemma 2.3. *There is an $n < \omega$ such that $Q_n = \emptyset$, and so the set of small pre-quasimodels for φ is finite and can be constructed effectively from φ.*

Proof. Suppose otherwise. Define a relation E on the set Q of all small pre-quasimodels as follows. For $\mathfrak{Q} = (\mathfrak{F}, f)$, $\mathfrak{Q}' = (\mathfrak{F}', f')$ in Q, set $\mathfrak{Q} E \mathfrak{Q}'$ iff $\mathfrak{F}$ is an 'initial subtree' of $\mathfrak{F}'$ and f coincides with f' on the points of $\mathfrak{F}$. Clearly, for every $\mathfrak{Q}' \in Q_{n+1}$, there is some $\mathfrak{Q} \in Q_n$ such that $\mathfrak{Q} E \mathfrak{Q}'$. Therefore, by König's infinity lemma, there is an infinite E-chain $\mathfrak{Q}_0 E \mathfrak{Q}_1 E \ldots E \mathfrak{Q}_n E \ldots$ in Q such that $\mathfrak{Q}_n \in Q_n$ for $n < \omega$. Since $\mathfrak{Q}_{n+1}$ is always an extension of $\mathfrak{Q}_n$, their union $\mathfrak{Q} = \bigcup_{n<\omega} \mathfrak{Q}_n$ is also a pre-quasimodel. Let $\mathfrak{Q} = (\mathfrak{F}, f)$ and $\mathfrak{F} = (W, R)$. Then $\mathfrak{F}$ is an infinite tree with finite branching. By König's lemma, it must have an infinite branch $x_0 R x_1 R \ldots$. Then, by Kruskal's theorem, there exist $i < j < \omega$ such that $f(x_i)$ is embeddable into $f(x_j)$. But x_i and x_j already belonged to the underlying tree of $\mathfrak{Q}_j$, contrary to $\mathfrak{Q}_j$ being in Q_j. $\square$

What is left is to establish a connection between expanding domain models and pre-quasimodels. A *run r through* a pre-quasimodel $(\mathfrak{F}, f)$ (where $\mathfrak{F} = (W, R)$ and $f(x) = (T_x, <_x, l_x)$, for $x \in W$) is a partial function from W into $(\bigcup_{x \in W} T_x) \times \boldsymbol{T}_\varphi$ such that, for all $x \in W$,

[8] In the usual treatments of Kruskal's tree theorem, trees are meant to be either irreflexive [29] or reflexive [34]. However, it is easy to see that the theorem also holds without any such restriction, as we can add the information about reflexivity/irreflexivity of a tree-node to its label.

- if $x \in \operatorname{dom} r$ and $r(x) = (w_{r(x)}, \boldsymbol{t}_{r(x)})$, then $w_{r(x)} \in T_x$ and $\boldsymbol{t}_{r(x)} \in l_x(w_{r(x)})$,
- if $x \in \operatorname{dom} r$ and xRy then $y \in \operatorname{dom} r$,
- for all $\Diamond\psi \in sub\,\neg\varphi$, we have $\Diamond\psi \in \boldsymbol{t}_{r(x)}$ iff there exists $y \in W$ such that xRy and $\psi \in \boldsymbol{t}_{r(y)}$.

We call a triple $(\mathfrak{F}, f, \mathcal{R})$ a $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$-*quasimodel* (for φ) if the following conditions are satisfied:

(q0) $(\mathfrak{F}, f)$ is a pre-quasimodel, $\mathcal{R}$ is a set of runs through $(\mathfrak{F}, f)$, $\mathfrak{F} \in \mathcal{C}_h$ and $(T_x, <_x) \in \mathcal{C}_v$ for all $x \in W$;
(q1) $\neg\varphi \in l_r(w)$ for the root $r \in W$ of $\mathfrak{F}$ and the root w of $f(r)$;
(q2) for all $x \in W$, $w \in T_x$ and $\Diamond\psi \in sub\,\neg\varphi$, the following conditions are equivalent:

- there exists a $\boldsymbol{t} \in l_x(w)$ with $\Diamond\psi \in \boldsymbol{t}$;
- there exists a v with $w <_x v$ and $\boldsymbol{t}' \in l_x(v)$ such that $\psi \in \boldsymbol{t}'$;

(q3) for all $x \in W$, $w \in T_x$ and $\boldsymbol{t} \in l_x(w)$, there is $r \in \mathcal{R}$ such that $r(x) = (w, \boldsymbol{t})$;
(q4) for all $r, r' \in \mathcal{R}$ and for all $x, y \in \operatorname{dom} r \cap \operatorname{dom} r'$, $w_{r(x)} <_x w_{r'(x)}$ iff $w_{r(y)} <_y w_{r'(y)}$.

We call a quasimodel *small* if the underlying pre-quasimodel is small.

Lemma 2.4. $\varphi \notin \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$ *iff there is a small* $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$-*quasimodel for* φ.

Proof. Suppose that there is a $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$-quasimodel $(\mathfrak{F}, f, \mathcal{R})$ for φ (where $\mathfrak{F} = (W, R)$ and $f(x) = (T_x, <_x, l_x)$, for $x \in W$). Then we let, for all $x \in W$,

$$W_x = \{r \in \mathcal{R} \mid x \in \operatorname{dom} r\},$$
$$r R_x r' \quad \text{iff} \quad w_{r(x)} <_x w_{r'(x)},$$
$$g(x) = (W_x, R_x).$$

It is straightforward to check that $\mathfrak{H} = (\mathfrak{F}, g)$ is an e-frame in $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$. Moreover, by taking, for all $x \in W$ and propositional variables p,

$$\mathfrak{V}_x(p) = \{r \in W_x \mid p \in \boldsymbol{t}_{r(x)}\},$$

we obtain an expanding domain model $(\mathfrak{H}, \mathfrak{V})$ refuting φ.

Conversely, suppose that $\varphi \notin \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$. We may assume that φ is refuted in a model $\mathfrak{M} = (\mathfrak{H}, \mathfrak{V})$ based on an e-frame $\mathfrak{H} = (\mathfrak{F}, f)$ satisfying the conditions of Lemma 2.2. We can turn $\mathfrak{M}$ into a $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$-quasimodel $(\mathfrak{F}, g, \mathcal{R})$ as follows. Suppose that $\mathfrak{F} = (W, R)$ and $f(x) = (W_x, R_x)$ for $x \in W$. For every $x \in W$, define an equivalence relation $\sim_x$ on W_x by taking, for all $u, v \in W_x$,

$$u \sim_x v \quad \text{iff} \quad \text{either } u = v, \text{ or } uR_x v \text{ and } vR_x u,$$

that is, iff u and v are in the same R_x-cluster. Let $[u]_x$ denote the $\sim_x$-class of u. For all $x \in W$, $w \in W_x$, we let

$$\boldsymbol{t}_x^{\mathfrak{M}}(w) = \{\psi \in sub\,\neg\varphi \mid (\mathfrak{M}, (x, w)) \models \psi\}.$$

For every $x \in W$, let $g(x) = (T_x, <_x, l_x)$, where

$$T_x = \{[u]_x \mid u \in W_x\}$$
$$[u]_x <_x [v]_x \quad \text{iff} \quad \exists u' \in [u]_x\, \exists v' \in [v]_x\; u' R_x v'$$
$$l_x([u]_x) = \{\boldsymbol{t}_x^{\mathfrak{M}}(u') \mid u' \in [u]_x\}.$$

Finally, for every $w \in \bigcup_{x \in W} W_x$ define a run r_w through $(\mathfrak{F}, g)$ by taking

$$\operatorname{dom} r_w = \{x \in W \mid w \in W_x\}$$

and for every $x \in \operatorname{dom} r_w$,

$$r_w(x) = \big([w]_x, \boldsymbol{t}_x^{\mathfrak{M}}(w)\big).$$

Let $\mathcal{R} = \{r_w \mid w \in \bigcup_{x \in W} W_x\}$. It is straightforward to check that $(\mathfrak{F}, g, \mathcal{R})$ is indeed a $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$-quasimodel for φ. Moreover, by the assumption on $\mathfrak{M}$, the pre-quasimodel $(\mathfrak{F}, g)$ is finite. To show that we can turn it to a pre-quasimodel satisfying **(sm3)**, suppose that there are $x, y \in W$ such that xRy and $g(x)$ is embeddable into $g(y)$ by an embedding ι. Then we replace in $\mathfrak{F}$ the subtree generated by x with the subtree generated by y, thus obtaining some tree $\mathfrak{F}' = (W', R')$. Let g' be the restriction of g to W'. We define new runs through $(\mathfrak{F}', g')$ by taking, for all $r, r' \in \mathcal{R}$ such that $x \in \operatorname{dom} r$, $y \in \operatorname{dom} r'$, $\iota(w_{r(x)}) = w_{r'(y)}$, $\boldsymbol{t}_{r(x)} = \boldsymbol{t}_{r'(y)}$, and for all $z \in W'$, $z \in \operatorname{dom} r$,

$$(r + r')(z) = \begin{cases} r(z), & \text{if } zRx, \\ r'(z), & \text{if } z = y \text{ or } yRz. \end{cases}$$

Let $\mathcal{R}'$ be the collection of these new runs together with those runs from $\mathcal{R}$ that 'start at' a point z with yRz. It is straightforward to check that $(\mathfrak{F}', g', \mathcal{R}')$ is a $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$-quasimodel for φ. Since $\mathfrak{F}$ is finite, after finitely many repetitions of this procedure the underlying pre-quasimodel will satisfy **(sm3)**. To comply with the cardinality conditions **(sm1)** and **(sm2)**, we can use the construction from the proof of Lemma 2.2. Then, again we can get rid of the embeddable pairs as above, and so on. As at each step the underlying tree can get only smaller, we will end up with a small $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$-quasimodel for φ. □

Now we can describe the decision algorithm for $\mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$ as follows. Given a formula φ, by Lemma 2.3, we can effectively construct the set of all small pre-quasimodels for φ. Then for each such small pre-quasimodel, we check whether it is a $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$-quasimodel for φ (that is, whether conditions **(q0)**–**(q4)** hold). By Lemma 2.4, this way we find a quasimodel for φ iff $\varphi \notin \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathrm{e}}$.

3.3. Complexity

Now we complete the proof of Theorems 1 and 2 by showing that no algorithm can decide whether a given $\mathcal{ML}_2$-formula φ is satisfiable in an e-frame from $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ in *primitive recursive* time or space. To understand the meaning of this result, let us recall that every primitive recursive function $f : \omega \to \omega$ is (eventually) dominated by one of the (primitive recursive) functions h_n which are defined inductively as follows

$$\mathsf{h}_0(k) = 2k, \qquad \mathsf{h}_{n+1}(k) = \mathsf{h}_n^{(k)}(1),$$

where $\mathsf{h}_n^{(k)}$ denotes the result of k successive applications of h_n; see, e.g., [35] and references therein. For example,

$$\mathsf{h}_1(k) = 2^k, \qquad \mathsf{h}_2(k) = 2^{2^{\cdot^{\cdot^{2}}}} \Big\} \, k \text{ times}.$$

(In particular, all elementary functions are dominated by h_2.) The diagonal $\mathsf{h}_n(n)$—a variant of the Ackermann function—is not primitive recursive. We are about to prove that the decision problem for our logics is at least as hard as termination of Turing machines running in Ackermann time or space. It seems that these expanding products as well as some relevance logics [43] are the most complex natural and mathematically interesting decidable theories known so far (cf. [6]).

We will use a reduction of the *reachability problem for lossy channel systems* which was shown to have non-primitive recursive complexity by Schnoebelen [39], even for systems with a single channel. A *single channel system* is a triple $S = (Q, \Sigma, \Delta)$, where $Q = \{q_1, \ldots, q_n\}$ is a finite set of *control states*, $\Sigma = \{a, b, \ldots\}$ is a finite alphabet of *messages*, and $\Delta \subseteq Q \times \{?, !\} \times \Sigma \times Q$ is a finite set of *transitions*. A *configuration* of S is a pair $\gamma = (q, \boldsymbol{w})$, where $q \in Q$ and $\boldsymbol{w}$ is a finite nonempty[9] Σ-word. Say that a configuration $\gamma' = (q', \boldsymbol{w}')$ is the result of a *perfect transition* of S from $\gamma = (q, \boldsymbol{w})$ and write $\gamma \xrightarrow{S}_p \gamma'$ if

- there is $(q, !, a, q') \in \Delta$ such that $\boldsymbol{w}' = a\boldsymbol{w}$, or
- there is $(q, ?, a, q') \in \Delta$ such that $\boldsymbol{w} = \boldsymbol{w}'a$.

[9] In the standard definition, empty words are permitted. However, it is not hard to see that the computational behaviour of channel systems does not depend on whether empty words are permitted or not.

We say that γ' is a result of a *lossy transition* from γ and write $\gamma \xrightarrow{S}_{\ell} \gamma'$ if

$$\gamma \;\sqsupseteq\; \gamma_1 \xrightarrow{S}_{p} \gamma_2 \;\sqsupseteq\; \gamma'$$

for some γ_1 and γ_2, where $(q, \boldsymbol{w}) \sqsupseteq (q', \boldsymbol{w}')$ iff $\boldsymbol{w}'$ is a subword of $\boldsymbol{w}$ and $q = q'$. Denote by $\xrightarrow{S}{}^{*}_{\ell}$ and $\xrightarrow{S}{}^{*}_{p}$ the transitive and reflexive closures of $\xrightarrow{S}_{\ell}$ and $\xrightarrow{S}_{p}$, respectively.

As was proved by Schnoebelen [39], the following problem is not decidable in primitive recursive time: 'given a channel system S, two configurations γ_0 and γ_f, and any relation $\to$ in the interval

$$\xrightarrow{S}{}^{*}_{p} \;\subseteq\; \to \;\subseteq\; \xrightarrow{S}{}^{*}_{\ell},$$

decide whether $\gamma_0 \to \gamma_f$.' So in order to establish the non-primitive recursive lower bound for our logics, it is enough to prove the following:

Lemma 2.5. *For every channel system S and all configurations γ_0, γ_f, one can construct an $\mathcal{ML}_2$-formula $\varphi_{S,\gamma_0,\gamma_f}$ which is polynomial in the size of S, γ_0, γ_f and satisfies the following two properties:*

(a) *if $\varphi_{S,\gamma_0,\gamma_f}$ is satisfiable in an e-frame from $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ then $\gamma_0 \xrightarrow{S}{}^{*}_{\ell} \gamma_f$,*

(b) *if $\gamma_0 \xrightarrow{S}{}^{*}_{p} \gamma_f$ then $\varphi_{S,\gamma_0,\gamma_f}$ is satisfiable in an e-frame from $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$.*

Proof. To construct the required formula $\varphi_{S,\gamma_0,\gamma_f}$, we will need modal operators interpreted via accessibility relations that are irreflexive on certain points of e-frames. So, similarly to the undecidability proofs of [42,11,14,38], we fix two propositional variables h and v, and define new modal operators by setting, for every $\mathcal{ML}_2$-formula ψ,

$$\begin{aligned}
⬘\psi &= \big[\mathsf{h} \to \diamondminus\big(\neg\mathsf{h} \wedge (\psi \vee \diamondminus\psi)\big)\big] \wedge \big[\neg\mathsf{h} \to \diamondminus\big(\mathsf{h} \wedge (\psi \vee \diamondminus\psi)\big)\big],\\
⬗\psi &= \big[\mathsf{v} \to \diamondbar\big(\neg\mathsf{v} \wedge (\psi \vee \diamondbar\psi)\big)\big] \wedge \big[\neg\mathsf{v} \to \diamondbar\big(\mathsf{v} \wedge (\psi \vee \diamondbar\psi)\big)\big],\\
⬒\psi &= \neg⬘\neg\psi, \qquad \text{and} \qquad ◨\psi = \neg⬗\neg\psi.
\end{aligned}$$

We will use the following abbreviations. For every formula ψ, $\Box \in \{\boxminus, \boxbar\}$, and every $n < \omega$,

$$\begin{aligned}
&\Box^{+}\psi = \psi \wedge \Box\psi,\\
&⬗^{0}\psi = ◨^{0}\psi = \psi, && ⬗^{n+1}\psi = ⬗⬗^{n}\psi,\\
&◨^{n+1}\psi = ◨◨^{n}\psi, && ⬗^{=\mathbf{n}}\psi = ⬗^{n}\psi \wedge ◨^{n+1}\neg\psi.
\end{aligned}$$

The last formula says: 'see ψ vertically in n steps, but not in $n+1$ steps'.

With a slight abuse of notation, we also introduce propositional variables

- δ, for every transition $\delta \in \Delta$,
- a, for every $a \in \Sigma$,
- q, for every $q \in Q$,

and use the abbreviation $\mathsf{w} \leftrightarrow \bigvee_{a\in\Sigma} a$.

Now suppose that a channel system S and two configurations

$$\gamma_0 = (q_0, b_1 \ldots b_k), \qquad \gamma_f = (q_f, a_1 \ldots a_m)$$

are given. Define $\varphi_{S,\gamma_0,\gamma_f}$ to be the conjunction of formulas (3)–(12):

$$\boxminus^{+}\big((\mathsf{h} \to \boxbar\mathsf{h}) \wedge (\neg\mathsf{h} \to \boxbar\neg\mathsf{h})\big) \tag{3}$$

$$\boxminus^{+}\boxbar^{+}\big((\mathsf{v} \to \boxminus\mathsf{v}) \wedge (\neg\mathsf{v} \to \boxminus\neg\mathsf{v})\big) \tag{4}$$

$$\boxminus^{+}\boxbar^{+}\big((\mathsf{w} \to \boxbar\mathsf{w}) \wedge (\neg\mathsf{w} \to \boxminus\neg\mathsf{w})\big) \tag{5}$$

$$\boxminus^+\boxbar^+\Big(\bigwedge_{a\in\Sigma}\big(a\to\boxminus(\mathsf{w}\to a)\big)\wedge\bigwedge_{a\neq a'}(a\to\neg a')\Big) \tag{6}$$

$$\boxminus^+\boxbar^+\Big(\bigvee_{q\in Q}q\wedge\bigwedge_{q\neq q'}(q\to\neg q')\wedge\bigwedge_{q\in Q}(q\to\boxbar q)\Big) \tag{7}$$

$$\boxminus^+\boxbar^+\Big[\text{⬘}\top\to\Big(\bigvee_{\delta\in\Delta}\delta\wedge\bigwedge_{\delta\neq\delta'}(\delta\to\neg\delta')\wedge\bigwedge_{\delta\in\Delta}(\delta\to\boxbar\delta)\Big)\Big] \tag{8}$$

$$q_f\wedge\neg\mathsf{w}\wedge\text{⬖}^{=\mathbf{m}}\top\wedge\boxbar\bigwedge_{0\le i<m}(\text{⬖}^{=\mathbf{i}}\top\to a_{m-i}) \tag{9}$$

$$\boxminus\Big[\text{⬒}\bot\to\Big(q_0\wedge\boxbar^+\big((\text{⬖}^k\top\to\neg\mathsf{w})\wedge\bigwedge_{0\le i<k}(\text{⬖}^{=\mathbf{i}}\top\to b_{k-i})\big)\Big)\Big] \tag{10}$$

$$\bigwedge_{\delta=(q,!,a,q')}\boxminus^+\boxbar^+\Big[\delta\to\Big(q'\wedge\big(\mathsf{w}\to\text{◨}\text{⬘}(\mathsf{w}\wedge q)\big)\wedge \big(\mathsf{w}\wedge\text{◨}\bot\to\text{⬘}(\mathsf{w}\wedge q)\big)\wedge\big(\mathsf{w}\wedge\neg\text{⬘}(\mathsf{w}\wedge q)\to a\big)\Big)\Big] \tag{11}$$

$$\bigwedge_{\delta=(q,?,a,q')}\boxminus^+\boxbar^+\Big[\delta\to \Big(q'\wedge\big(\mathsf{w}\to\text{⬘}\big(\mathsf{w}\wedge q\wedge\boxbar^+(\text{◨}\bot\to a)\big)\big)\wedge\boxbar^+(\text{◨}\bot\to\text{⬒}\text{⬖}\top)\Big)\Big]. \tag{12}$$

The intended meaning of these conjuncts will be clear from the proof below.

Proof of (a). Suppose that $\varphi_{S,\gamma_0,\gamma_f}$ is satisfied at some point (x_0,u_0) of an expanding domain model $\mathfrak{M}=(\mathfrak{H},\mathfrak{V})$ that is based on an e-frame $\mathfrak{H}=(\mathfrak{F},f)$ from $(\mathcal{C}_h\times\mathcal{C}_v)^{\mathsf{e}}$, where $\mathfrak{F}=(W,R)$ and $f(x)=(W_x,R_x)$, for $x\in W$. By Lemma 2.2, we may assume that $\mathfrak{H}$ is finite, and (x_0,u_0) is a root of $\mathfrak{H}$.

Define new relations $\bar{R}$ and $\bar{R}_x$ $(x\in W)$ by taking, for all $y,y'\in W$, $u,u'\in W_x$,

$$\begin{aligned} y\bar{R}y' \quad\text{iff}\quad & \exists y''\in W\ \big[yRy'' \text{ and} \\ & \big((\mathfrak{M},(y,u_0))\models\mathsf{h}\iff(\mathfrak{M},(y'',u_0))\models\neg\mathsf{h}\big)\text{ and} \\ & (\text{either } y''=y' \text{ or } y''Ry')\big], \end{aligned} \tag{13}$$

$$\begin{aligned} u\bar{R}_xu' \quad\text{iff}\quad & \exists u''\in W_x\ \big[uR_xu'' \text{ and} \\ & \big((\mathfrak{M},(x,u))\models\mathsf{v}\iff(\mathfrak{M},(x,u''))\models\neg\mathsf{v}\big)\text{ and} \\ & (\text{either } u''=u' \text{ or } u''R_xu')\big]. \end{aligned} \tag{14}$$

It is readily checked that all of the $\bar{R}$ and $\bar{R}_x$, $x\in W$, are transitive, $\bar{R}\subseteq R$, $\bar{R}_x\subseteq R_x$, and for all $x\in W$, $u\in W_x$,

$$\begin{aligned} (\mathfrak{M},(x,u))\models\text{⬘}\psi \quad&\text{iff}\quad \exists y\in W\ (x\bar{R}y \text{ and } (\mathfrak{M},(y,u))\models\psi),\\ (\mathfrak{M},(x,u))\models\text{⬖}\psi \quad&\text{iff}\quad \exists v\in W_x\ (u\bar{R}_xv \text{ and } (\mathfrak{M},(x,v))\models\psi). \end{aligned}$$

Note that $((W,\bar{R}),\bar{f})$ where $\bar{f}=(W_x,\bar{R}_x)$ $(x\in W)$ is not necessarily an e-frame, because we can have $x,y\in W$, $u,v\in W_x$ such that $x\bar{R}y$, $u\bar{R}_yv$, but u is not $\bar{R}_x$-related to v. Nevertheless, for all $x,y\in W$, $u,v\in W_x$, we always have that

$$\text{if } x\bar{R}y \text{ and } u\bar{R}_xv \text{ then } u\bar{R}_yv. \tag{15}$$

Since there are no proper clusters in $\mathfrak{F}$, $\bar{R}$ is irreflexive. The $\bar{R}_x$ are not necessarily irreflexive, but all non-degenerate $\bar{R}_x$-clusters are necessarily 'blank' (i.e., make $\neg\mathsf{w}$ true):

CLAIM 2.5.1. *Let $y\in W$ and $v\in W_y$ be such that $(\mathfrak{M},(y,v))\models\mathsf{w}$. Then $v\bar{R}_yv$ does not hold.*

PROOF. Suppose otherwise, that is $v\bar{R}_yv$ and $(\mathfrak{M},(y,v))\models\mathsf{w}$. Then we have $(\mathfrak{M},(y,v))\models\text{⬘}\top$, since otherwise $(\mathfrak{M},(y,u_0))\models\text{⬒}\bot$ would hold, and so $(\mathfrak{M},(y,v))\models\neg\mathsf{w}$ by (10). Hence it follows from (8) that $(\mathfrak{M},(y,v))\models\delta$ for some $\delta\in\Delta$. Now we obtain $(\mathfrak{M},(y,v))\models\text{⬘}(\mathsf{w}\wedge q)$, by (11) and (12). Thus there exists $y_1\in W$ such that $y\bar{R}y_1$

and $(\mathfrak{M}, (y_1, v)) \models \mathsf{w}$. Since $\bar{R}$ is irreflexive, $y_1 \neq y$. By (15), we have $v\bar{R}_{y_1}v$. By repeating the above argument, we must have $(\mathfrak{M}, (y_1, v)) \models \text{⬘}\top$ again. Therefore, we can continue in this manner to obtain an infinite ascending chain $y\bar{R}y_1\bar{R}y_2\ldots$, contrary to $\mathfrak{F}$ being Noetherian. $\square$

For a finite sequence $\vec{v} = (v_1, v_2, \ldots, v_n)$ of elements of W_y with $v_i\bar{R}_y v_{i+1}$ and $y \in W$, we write

$$val_y(\vec{v}) \;=\; d_1 \ldots d_n$$

if, for all i, $1 \leq i \leq n$, we have $(\mathfrak{M}, (y, v_i)) \models d_i$ for some $d_i \in \Sigma \cup \{\neg\mathsf{w}\}$. Say that $\vec{u} = (u_1, u_2, \ldots, u_r)$ is an *extension* of $\vec{v}$, if $u_i \in W_y$, $u_i\bar{R}_y u_{i+1}$, and there are $i_1 < i_2 < \cdots < i_n \leq r$ such that $u_{i_j} = v_j$ for $1 \leq j \leq n$. Say that $\vec{v}$ *carries a Σ-word in y* if there are $d_1, \ldots, d_n \in \Sigma$ such that $val_y(\vec{v}) = d_1 \ldots d_n$. A sequence $\vec{v}$ is said to be *maximal carrying a Σ-word in y* if no extension of $\vec{v}$ carries a Σ-word in y.

CLAIM 2.5.2. *For all $x \in W$ and $q' \in Q$ such that $(\mathfrak{M}, (x, u_0)) \models q' \wedge \text{⬘}\top$, if a nonempty sequence $\vec{v}$ is maximal carrying a Σ-word in x then there exist $y \in W$, $q \in Q$, and a nonempty sequence $\vec{u}$ that is maximal carrying a Σ-word in y such that $x\bar{R}y$, $(\mathfrak{M}, (y, u_0)) \models q$, and*

$$(q, val_y(\vec{u})) \xrightarrow{S}_{\ell} (q', val_x(\vec{v})).$$

PROOF. Suppose that $\vec{v} = (v_1, \ldots, v_n)$ and $val_x(\vec{v}) = c_1 \ldots c_n$ for some $c_i \in \Sigma$. By (8), there exists a unique $\delta \in \Delta$ such that $(\mathfrak{M}, (x, u_0)) \models \delta$. By (11) and (12), δ is of the form $(q, !, a, q')$ or $(q, ?, a, q')$ for some $q \in Q$, $a \in \Sigma$.
<u>Case 1:</u> $\delta = (q, !, a, q')$. Then, by (11),

$$(\mathfrak{M}, (x, v_1)) \models \text{◨}\text{⬘}(\mathsf{w} \wedge q)$$

and there exists a minimal $i \leq n$ such that

$$(\mathfrak{M}, (x, v_i)) \models \text{⬘}(\mathsf{w} \wedge q).$$

Clearly, $1 \leq i \leq 2$. Take y such that $x\bar{R}y$ and $(\mathfrak{M}, (y, v_i)) \models \mathsf{w} \wedge q$. By (5), we have $(\mathfrak{M}, (y, v_j)) \models \mathsf{w}$, for all $j \geq i$. As we have $v_i\bar{R}_y \ldots \bar{R}_y v_n$ by (15),

$$val_x(v_i, \ldots, v_n) \;=\; val_y(v_i, \ldots, v_n).$$

follows from (6). Take any maximal extension $\vec{u}$ of $(v_i, \ldots, v_n)$ carrying a Σ-word in y. That such an extension exists in the finite e-frame $(\mathfrak{F}, f)$ follows from Claim 2.5.1. Assume first that $i = 2$. Then, by (11), we have $(\mathfrak{M}, (x, v_1)) \models a$. It follows that

$$(q, val_y(\vec{u})) \;\sqsupseteq\; (q, val_y(v_2, \ldots, v_n)) \xrightarrow{S}_{p} (q', a\, val_y(v_2, \ldots, v_n)) \;=\; (q', val_x(\vec{v})).$$

If $i = 1$ then

$$(q, val_y(\vec{u})) \xrightarrow{S}_{p} (q', a\, val_y(\vec{u})) \;\sqsupseteq\; (q', val_y(\vec{v})) \;=\; (q', val_x(\vec{v})).$$

<u>Case 2:</u> $\delta = (q, ?, a, q')$. By (12), there exists $y \in W$ such that $x\bar{R}y$ and

$$(\mathfrak{M}, (y, v_1)) \models \mathsf{w} \wedge q \wedge \text{⊡}^{+}(\text{◨}\bot \to a).$$

By (5) and Claim 2.5.1, $(\mathfrak{M}, (x, v_n)) \models \text{◨}\bot$. Therefore, by (12), we have $(\mathfrak{M}, (y, v_n)) \models \text{⬗}\top$. Since W_y is finite, by (5) and Claim 2.5.1 again, we find $v_{n+1} \in W_y$ with $v_n\bar{R}_y v_{n+1}$ and $(\mathfrak{M}, (y, v_{n+1})) \models \text{◨}\bot$. By (12), we have $(\mathfrak{M}, (y, v_{n+1})) \models a$. By (15), we have $v_1\bar{R}_y \ldots \bar{R}_y v_n$. Therefore, by (5), we have $val_x(\vec{v}) = val_y(\vec{v})$. Take any maximal extension $\vec{u}$ of $(v_1, \ldots, v_n, v_{n+1})$ carrying a Σ-word in y. By Claim 2.5.1, such an extension exists and

$$val_y(\vec{u}) \;=\; wa$$

for some Σ-word w having $val_y(\vec{v})$ as a subword. But then

$$(q, val_y(\vec{u})) \xrightarrow{S}_{p} (q', w) \;\sqsupseteq\; (q', val_y(\vec{v})) \;=\; (q', val_x(\vec{v})),$$

which completes the proof of Claim 2.5.2. $\square$

Now we can find a 'lossy run' from γ_0 to γ_f as follows. By (9), we have $(\mathfrak{M}, (x_0, u_0)) \models q_f$, and there exists a sequence $\vec{w}$ that is maximal carrying a Σ-word in x_0 and such that

$$val_{x_0}(\vec{w}) \;=\; a_1 \ldots a_k.$$

Since $\mathfrak{F}$ is finite and $\bar{R}$ is irreflexive, it follows from Claim 2.5.2 that there exist $x_1, \ldots, x_n \in W$, $q_1, \ldots, q_n \in Q$, nonempty sequences $\vec{w}_1, \ldots, \vec{w}_n$ such that $x_0 \bar{R} x_1 \bar{R} \ldots \bar{R} x_n$, $(\mathfrak{M}, (x_i, u_0)) \models q_i$, $\vec{w}_i$ is maximal carrying a Σ-word in x_i, $1 \le i \le n$,

$$(q_n, val_{x_n}(\vec{w}_n)) \xrightarrow{S}_{\ell} \;\ldots\; \xrightarrow{S}_{\ell} \;(q_1, val_{x_1}(\vec{w}_1)) \xrightarrow{S}_{\ell} \;(q_f, val_{x_0}(\vec{w})) \;=\; \gamma_f$$

and $(\mathfrak{M}, (x_n, u_0)) \models \boxminus\bot$. By (10), $q_n = q_0$ and $val_{x_n}(\vec{w}_n)$ is a subword of $b_1 \ldots b_k$. Therefore,

$$(q_0, b_1 \ldots b_k) \xrightarrow{S}_{\ell} (q_{n-1}, val_{x_{n-1}}(\vec{w}_{n-1})), \quad \text{and so} \quad \gamma_0 \xrightarrow{S}{}^{*}_{\ell}\, \gamma_f.$$

Proof of (b). Suppose that $\gamma_0 \xrightarrow{S}{}^{*}_{p}\, \gamma_f$, i.e., there exists a finite sequence

$$\gamma_0 \xrightarrow{S}_{p} \gamma_1 \xrightarrow{S}_{p} \ldots \xrightarrow{S}_{p} \gamma_n \;=\; \gamma_f$$

of perfect transitions, where $\gamma_i = (q_i, d_1^i \ldots d_{\ell_i}^i)$, for $i \le n$. Let δ_i denote the transition from γ_{i-1} to γ_i, $1 \le i \le n$, that is,

$$\delta_i \;=\; \begin{cases} (q_{i-1}, !, a, q_i), & \text{if } d_1^i \ldots d_{\ell_i}^i = a d_1^{i-1} \ldots d_{\ell_{i-1}}^{i-1}, \\ (q_{i-1}, ?, a, q_i), & \text{if } d_1^{i-1} \ldots d_{\ell_{i-1}}^{i-1} = d_1^i \ldots d_{\ell_i}^i a. \end{cases}$$

We show that the formula $\varphi_{S,\gamma_0,\gamma_f}$ is satisfiable in an e-frame from $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$. First, for each $i \le n$, we define inductively a number $N_i < \omega$ by taking $N_0 = \ell_n$ and, for $0 < i \le n$,

$$N_i \;=\; \begin{cases} N_{i-1}, & \text{if } \delta_{n-i+1} = (q_{n-i}, !, a, q_{n-i+1}) \in \Delta \text{ for some } a \in \Sigma, \\ N_{i-1} + 1, & \text{if } \delta_{n-i+1} = (q_{n-i}, ?, a, q_{n-i+1}) \in \Delta \text{ for some } a \in \Sigma. \end{cases}$$

Now we define an e-frame $\mathfrak{H} = (\mathfrak{F}, f)$ as follows. Let $W = \{0, \ldots, n\}$ and let $\mathfrak{F} = (W, \le)$ if $\mathcal{C}_h$ contains only reflexive frames, and $\mathfrak{F} = (W, <)$ otherwise. For each $i \in W$, let $W_i = \{0, \ldots, N_i\}$ and $f(i) = (W_i, \le)$ if $\mathcal{C}_v$ contains only reflexive frames, and $f(i) = (W_i, <)$ otherwise. Define valuations for the propositional variables by taking, for $i \le n$, $a \in \Sigma$, $q \in Q$, $\delta \in \Delta$,

$$\mathfrak{V}_i(\mathsf{h}) \;=\; \begin{cases} W_i, & \text{if } i \text{ is even}, \\ \emptyset, & \text{if } i \text{ is odd}; \end{cases}$$

$$\mathfrak{V}_i(\mathsf{v}) \;=\; \{j \le N_i \mid j \text{ is even}\};$$

$$\mathfrak{V}_i(a) \;=\; \{N_i - \ell_{n-i} + j \mid 1 \le j \le \ell_{n-i},\ d_j^{n-i} = a\};$$

$$\mathfrak{V}_i(q) \;=\; \begin{cases} W_i, & \text{if } q = q_{n-i}, \\ \emptyset, & \text{otherwise}; \end{cases}$$

$$\mathfrak{V}_i(\delta) \;=\; \begin{cases} W_i, & \text{if } i < n \text{ and } \delta = \delta_{n-i}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Finally, let $\mathfrak{M} = (\mathfrak{H}, (\mathfrak{V}_i)_{i \le n})$. It is easy to check that $(\mathfrak{M}, (0, 0)) \models \varphi_{S,\gamma_0,\gamma_f}$ holds. $\square$

4. An application to dynamic topological logic

Dynamic topological logic was introduced in 1997 (see, e.g., [25,26,28,3,27]) as a logical formalism for describing the behaviour of *dynamical systems*, e.g., in order to specify liveness and safety properties of hybrid systems [8]. Roughly, the idea is to model (some aspects of) these systems by means of *dynamic topological structures* (DTS)

$\mathfrak{D} = (\mathfrak{T}, g)$, where $\mathfrak{T} = (\Delta, \mathbb{I})$ is a topological space with an interior operator $\mathbb{I}$ and g is a continuous[10] function on $\mathfrak{T}$ which ‘moves’ the points of $\mathfrak{T}$ in each discrete unit of time. What we are interested in is the asymptotic behaviour of iterations of g, in particular, the orbits $\{w, g(w), g^2(w), \dots\}$ of states $w \in \Delta$. A natural formalism for speaking about such iterations is obtained by interpreting the previously introduced modal operator $\boxminus$ as ‘always in the future’, its dual $\diamondminus$ as ‘eventually’, the operator $\boxbar$ as topological interior and $\diamondbar$ as topological closure, by taking, for every $X \subseteq \Delta$,

$$\boxminus X = \bigcap_{0<n<\omega} g^{-n}(X), \qquad \diamondminus X = \bigcup_{0<n<\omega} g^{-n}(X),$$
$$\boxbar X = \mathbb{I}X, \qquad \diamondbar X = \Delta - \mathbb{I}(\Delta - X)$$

and adding the ‘next time’ operator $\bigcirc$:

$$\bigcirc X = g^{-1}(X).$$

The resulting language will be denoted by $\mathcal{ML}_2^{\circ}$.

By a *dynamic topological model with $N \le \omega$ iterations* (DTM$_N$, for short) we understand a triple $\mathfrak{M} = (\mathfrak{D}, \mathfrak{V}, N)$, where $\mathfrak{D} = (\mathfrak{T}, g)$ is a DTS with $\mathfrak{T} = (\Delta, \mathbb{I})$, and $\mathfrak{V}$, a *valuation*, associates with each propositional variable p a subset $\mathfrak{V}(p)$ of Δ. The truth of a formula φ at a state w depends on how many iterations of g we consider and at which iteration step we evaluate φ. Let $N' = N + 1$ if $N < \omega$ and $N' = \omega$ otherwise. For every $m < N'$, define inductively the *truth relation* $(\mathfrak{M}, w) \models_m \varphi$ (‘in model $\mathfrak{M}$, φ is true at w after m iterations of g’) as follows:

$$\begin{array}{lll}
(\mathfrak{M}, w) \models_m p & \text{iff} & w \in \mathfrak{V}(p),\ p \text{ a propositional variable,}\\
(\mathfrak{M}, w) \models_m \boxbar\varphi & \text{iff} & w \in \mathbb{I}\{v \in \Delta \mid (\mathfrak{M}, v) \models_m \varphi\},\\
(\mathfrak{M}, w) \models_m \diamondbar\varphi & \text{iff} & w \in \mathbb{C}\{v \in \Delta \mid (\mathfrak{M}, v) \models_m \varphi\},\\
(\mathfrak{M}, w) \models_m \bigcirc\varphi & \text{iff} & m + 1 < N' \text{ and } (\mathfrak{M}, g(w)) \models_{m+1} \varphi,\\
(\mathfrak{M}, w) \models_m \boxminus\varphi & \text{iff} & (\mathfrak{M}, g^n(w)) \models_{m+n} \varphi \text{ for all } n > 0 \text{ with } m + n < N',\\
(\mathfrak{M}, w) \models_m \diamondminus\varphi & \text{iff} & (\mathfrak{M}, g^n(w)) \models_{m+n} \varphi \text{ for some } n > 0 \text{ with } m + n < N'.
\end{array}$$

Here $g^n(w) = \overbrace{g \dots g}^{n}(w)$ and $\mathbb{C}$ is the closure operator on $\mathfrak{T}$. Note that if a formula ψ contains no ‘temporal’ operators or if $N = \omega$ then the truth relation $(\mathfrak{M}, w) \models_m \psi$ does not depend on m. Say that φ is *satisfiable* if there exist a DTM$_N$ $\mathfrak{M}$ and a state w in it such that $(\mathfrak{M}, w) \models_0 \varphi$. We also say that φ is *satisfiable in models with finite iterations* if φ is satisfied in a DTM$_N$ for some $N < \omega$. It is worth noting that for various natural properties it is sufficient to consider finitely many iterations only. For example, a *safety property* like ‘w will never visit some danger zone P’ is satisfiable iff it is satisfiable in models with finite iterations.

The language $\mathcal{ML}_2^{\circ}$ can also be interpreted in expanding domain models $\mathfrak{N}$ based on e-frames $\mathfrak{H} = (\mathfrak{F}, f)$, where $\mathfrak{F} = (W, <)$ is a *finite strict linear order* (that is, a finite irreflexive linear frame) and, for every $x \in W$, $f(x) = (\Delta_x, R_x)$ is a reflexive and transitive frame. Indeed, given such an $\mathfrak{N}$, we set

- $(\mathfrak{N}, (x, u)) \models \bigcirc\varphi$ iff there exists an immediate $<$-successor x' of x and $(\mathfrak{N}, (x', u)) \models \varphi$,

and leave all the other truth conditions from Section 2 unchanged. Then it is not hard to see that the proof of Theorem 1 can be generalised to show the following:

Theorem 3. *Let $\mathcal{C}_h$ be the class of all finite strict linear orders and let $\mathcal{C}_v$ be the class of all transitive and reflexive frames. Then the logic*

$$\{\varphi \in \mathcal{ML}_2^{\circ} \mid \forall \mathfrak{H} \in (\mathcal{C}_1 \times \mathcal{C}_2)^{\mathsf{e}}\ \ \mathfrak{H} \models \varphi\}$$

has the e-product fmp and is decidable, but not in time bounded by a primitive recursive function.

It is a challenging open question whether the satisfiability problem for $\mathcal{ML}_2^{\circ}$-formulas in dynamic topological structures is decidable. The known partial results are as follows. In [21] it is proved that the problem is *undecidable*, even for models with finite iterations, if we consider DTSs with *homeomorphisms*. In [22] it is shown that the problem

[10] Recall that a set $X \subseteq \Delta$ is called *open in* $\mathfrak{T}$ if $\mathbb{I}X = X$. A function g between topological spaces is called *continuous* if the inverse image $g^{-1}(X)$ of every open set X is open.

is again *undecidable* if we consider DTSs with continuous mappings but based on *Aleksandrov* topological spaces only (see below for definition). Here we prove—using Theorem 3 above—that the satisfiability problem for $\mathcal{ML}_2^{\circ}$-formulas in models with finite iterations is decidable, but not in primitive recursive time. It is not hard to see (using the relativisation technique of, say, [11]) that satisfiability in models with finite iterations is polynomially reducible to general satisfiability. Thus we obtain that the general satisfiability problem cannot be decided in primitive recursive time either.

Theorem 4. *The satisfiability problem for $\mathcal{ML}_2^{\circ}$-formulas in dynamic topological models with finite iterations is decidable, but not in primitive recursive time.*

Proof. We remind the reader that every reflexive and transitive frame (i.e., frame for modal logic **S4**) $\mathfrak{G} = (\Delta, R)$ gives rise to a topological space $\mathfrak{T}_{\mathfrak{G}} = (\Delta, \mathbb{I}_{\mathfrak{G}})$, where, for every $X \subseteq \Delta$,

$$\mathbb{I}_{\mathfrak{G}}(X) = \{x \in X \mid \forall y \in \Delta\ (xRy \to y \in X)\}.$$

Such spaces are known as *Aleksandrov spaces*. Alternatively they can be defined as topological spaces where arbitrary (not only finite) intersections of open sets are open; for details see [1,5]. The next lemma follows immediately from [3, 28,27]:

Lemma 4.1. *For every $N < \omega$, an $\mathcal{ML}_2^{\circ}$-formula is satisfiable in a DTM$_N$ iff it is satisfiable in a DTM$_N$ that is based on a (finite) Aleksandrov space.*

Thus, it is enough to consider DTMs of the form $\mathfrak{M} = ((\mathfrak{T}_{\mathfrak{G}}, g), \mathfrak{V}, N)$, where $\mathfrak{G} = (\Delta, R)$ is a reflexive and transitive frame. In this case we can rewrite the truth conditions for the operators $\boxdot$ and $\diamondtimes$ in a more familiar way:

$$(\mathfrak{M}, w) \models_m \boxdot\varphi \quad \text{iff} \quad (\mathfrak{M}, v) \models_m \varphi \ \text{ for every } v \in \Delta \text{ with } wRv,$$
$$(\mathfrak{M}, w) \models_m \diamondtimes\varphi \quad \text{iff} \quad (\mathfrak{M}, v) \models_m \varphi \ \text{ for some } v \in \Delta \text{ such that } wRv.$$

It is not hard to see that for any function $g\colon \Delta \to \Delta$,

$$g \text{ is continuous on } \mathfrak{T}_{\mathfrak{G}} \quad \text{iff} \quad \forall w, v \in \Delta\ \big(wRv \to g(w)Rg(v)\big). \tag{16}$$

Indeed, suppose first that g is continuous and wRv. Then

$$w \in \{u \in \Delta \mid g(w)Rg(u)\} = g^{-1}\big(\{u \in \Delta \mid g(w)Ru\}\big)$$

is open, and so $g(w)Rg(v)$ follows. Conversely, take any open set X in $T_{\mathfrak{G}}$ and let $w \in g^{-1}(X)$, wRv. Then $g(w) \in X$ and $g(w)Rg(v)$, from which $g(v) \in X$ follows.

Moreover, we have the following:

Lemma 4.2. *An $\mathcal{ML}_2^{\circ}$-formula φ is satisfiable in an e-frame $\mathfrak{H} = (\mathfrak{F}, f)$ where $\mathfrak{F}$ is a finite strict linear order and the $f(x)$ are reflexive and transitive frames iff φ is satisfiable in some DTM$_N$ with $N < \omega$.*

Proof. ($\Rightarrow$) Suppose that φ is satisfied in a model $\mathfrak{N} = (\mathfrak{H}, \mathfrak{V})$ based on an e-frame $\mathfrak{H} = (\mathfrak{F}, f)$, where $\mathfrak{F} = (W, <)$ is a finite strict linear order and each $f(x) = (\Delta_x, R_x)$ is a reflexive and transitive frame, for $x \in W$. We may assume that

$$\mathfrak{F} = (\{0, \ldots, N\}, <)$$

for some $N < \omega$, and $(\mathfrak{N}, (0, r)) \models \varphi$ for a root r of $f(0)$. Define a DTM$_N$ $\mathfrak{M} = (\mathfrak{D}, \mathfrak{U}, N)$ based on the DTS $\mathfrak{D} = ((\Delta, \mathbb{I}_{\mathfrak{G}}), g)$ with $\mathfrak{G} = (\Delta, R)$ and the valuation $\mathfrak{V}$ by taking

$$\Delta = \bigcup_{n \le N} (\{n\} \times \Delta_n),$$

for each $(n, w) \in \Delta$

$$g(n, w) = \begin{cases} (n+1, w), & \text{if } n < N, \\ (n, w), & \text{if } n = N, \end{cases}$$

for all $(n_1, w_1), (n_2, w_2) \in \Delta$

$$(n_1, w_1) R (n_2, w_2) \quad \text{iff} \quad n_1 = n_2 \text{ and } w_1 R_{n_1} w_2,$$

and, for every propositional variable p,

$$\mathfrak{U}(p) = \{(n, w) \in \Delta \mid w \in \mathfrak{V}_n(p)\}.$$

Clearly, $\mathfrak{M}$ is a DTM$_N$ (in particular, g is continuous by (16)). Moreover, it is easy to show by induction that for every $\mathcal{ML}_2^{\circ}$-formula ψ, every $n \leq N$ and every $w \in \Delta_n$,

$$(\mathfrak{N}, (n, w)) \models \psi \quad \text{iff} \quad (\mathfrak{M}, (n, w)) \models_n \psi.$$

($\Leftarrow$) Conversely, by Lemma 4.1 we may suppose that φ is satisfied in a DTM$_N$

$$\mathfrak{M} = ((\mathfrak{T}_{\mathfrak{G}}, g), \mathfrak{V}, N),$$

where $N < \omega$ and $\mathfrak{G} = (\Delta, R)$ is a reflexive and transitive frame. So, we can find a $v_0 \in \Delta$ such that $(\mathfrak{M}, v_0) \models_0 \varphi$.

Note first that without loss of generality we may assume that g is 'onto'. Indeed, if this is not the case, then we take the model $\mathfrak{M}' = ((\mathfrak{T}_{\mathfrak{G}'}, g'), \mathfrak{V}', N)$ with $\mathfrak{G}' = (\Delta', R')$, where

- $\Delta' = \mathbb{N} \times \Delta$;
- $(n_1, w_1) R' (n_2, w_2)$ iff $n_1 = n_2$ and $w_1 R w_2$;
- $g'(0, w) = (0, g(w))$ and, for any $n \in \mathbb{N}$, $g'(n+1, w) = (n, w)$;
- $(\mathfrak{M}', (n, w)) \models p$ iff $(\mathfrak{M}, w) \models p$.

Then, for every ψ and every $m \leq N$, we have

$$(\mathfrak{M}', (0, w)) \models_m \psi \quad \text{iff} \quad (\mathfrak{M}, w) \models_m \psi.$$

Now, for every $n \leq N$ and every propositional variable p, let

- $\Delta_n = \Delta$,
- $u R_n v$ iff $g^n(u) R g^n(v)$,
- $\mathfrak{U}_n(p) = \{(n, w) \mid g^n(w) \in \mathfrak{V}(p)\}$,

and let $\mathfrak{H} = ((\{0, \ldots, N\}, <), f)$ with $f(n) = (\Delta_n, R_n)$, and $\mathfrak{N} = (\mathfrak{H}, (\mathfrak{U}_n)_{n \leq N})$. It is not difficult to prove by induction that, for all $w \in \Delta$ and $m \leq N$,

$$(\mathfrak{M}, g^m(w)) \models_m \psi \quad \text{iff} \quad (\mathfrak{N}, (m, w)) \models \psi.$$

Note that we use that g is 'onto' in the induction step for $\boxdot$.

In general, $\mathfrak{H}$ is not an e-frame because, in view of (16), we only have $u R_n v \to u R_{n+1} v$ but not the other way round. However, we can take the transitive unravelling $f^*(n) = (\Delta_n^*, R_n^*)$ of $f(n) = (\Delta_n, R_n)$, where

$$\Delta_n^* = \{(v_0, v_1, \ldots, v_k) \mid v_i R_n v_{i+1} \text{ and } v_i \neq v_{i+1}\}$$

and R_n^* is the transitive and reflexive closure of the relation R_n' defined by taking

$$(v_0, \ldots, v_k) R_n' (v_0, \ldots, v_k, v_{k+1}) \quad \text{iff} \quad v_k R_n v_{k+1}.$$

The frame $\mathfrak{H}^* = ((\{0, \ldots, N\}, <), f^*)$ is an e-frame. Indeed, suppose that both $(v_0, \ldots, v_k)$ and $(v_0, \ldots, v_k, v_{k+1}, \ldots, v_m)$ are in W_n^*. Then, by the definition of R_n^*, we have $v_k R_n v_{k+1} R_n \ldots R_n v_m$ and so $(v_0, \ldots, v_k) R_n^* (v_0, \ldots, v_k, v_{k+1}, \ldots, v_m)$.

Now consider the model $\mathfrak{N}^* = (\mathfrak{H}^*, \mathfrak{U}^*)$, where $\mathfrak{U}^* = (\mathfrak{U}_n^*)_{n \leq N}$ and

$$\mathfrak{U}_n^*(p) = \{(v_0, v_1, \ldots, v_m) \in W_n^* \mid v_m \in \mathfrak{U}_n(p)\}.$$

By the unravelling theorem of classical modal logic, we have

$$(\mathfrak{N}, (n, v_0)) \models \psi \quad \text{iff} \quad (\mathfrak{N}^*, (n, (v_0))) \models \psi$$

for every formula ψ. $\square$

Now Theorem 4 follows immediately from Lemma 4.2 and Theorem 3. $\square$

5. Expanding domain products vs expanding relativisations

The original definition of ‘expanding product’ frames and logics from [30] was motivated by the idea of *relativising* the standard product construction.

Given unimodal Kripke frames $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$, their *product* is defined to be the bimodal frame

$$\mathfrak{F}_1 \times \mathfrak{F}_2 = (W_1 \times W_2, \bar{R}_1, \bar{R}_2),$$

where $W_1 \times W_2$ is the Cartesian product of W_1 and W_2 and, for all $u, u' \in W_1$, $v, v' \in W_2$,

$$(u, v)\bar{R}_1(u', v') \quad \text{iff} \quad uR_1u' \text{ and } v = v',$$
$$(u, v)\bar{R}_2(u', v') \quad \text{iff} \quad vR_2v' \text{ and } u = u'.$$

Let L_1 be a normal modal logic in the language with $\boxminus$, $\diamondminus$ and let L_2 be a normal modal logic in the language with $\boxbar$, $\diamondbar$. Assume also that both L_1 and L_2 are Kripke complete. Then the *product* of L_1 and L_2 is the normal bimodal logic $L_1 \times L_2$ in the language $\mathcal{ML}_2$ with the boxes $\boxminus$, $\boxbar$ and the diamonds $\diamondminus$, $\diamondbar$ which is characterised by the class of product frames $\mathfrak{F}_1 \times \mathfrak{F}_2$, where $\mathfrak{F}_i$ is a frame for L_i, $i = 1, 2$. (Here we assume that $\boxminus$ and $\diamondminus$ are interpreted by $\bar{R}_1$, while $\boxbar$ and $\diamondbar$ are interpreted by $\bar{R}_2$.)

According to the definition in [30], a frame $\mathfrak{G} = (W, R'_1, R'_2)$ is an *expanding relativised product frame* if there exist frames $\mathfrak{F}_1 = (U_1, R_1)$ and $\mathfrak{F}_2 = (U_2, R_2)$ such that

- $\mathfrak{G}$ is a subframe of $\mathfrak{F}_1 \times \mathfrak{F}_2$ (that is, $W \subseteq U_1 \times U_2$ and $R'_i = \bar{R}_i \restriction W$ for $i = 1, 2$), and
- for all $(w_1, w_2) \in W$ and $u \in U_1$, if w_1R_1u then $(u, w_2) \in W$.

Given two classes $\mathcal{C}_1, \mathcal{C}_2$ of unimodal frames, denote by

$$(\mathcal{C}_1 \times \mathcal{C}_2)^{\mathsf{ex}}$$

the class of all expanding relativised product frames that are subframes of some $\mathfrak{F}_1 \times \mathfrak{F}_2$, for some $\mathfrak{F}_i \in \mathcal{C}_i$, $i = 1, 2$, and let

$$\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)^{\mathsf{ex}} = \{\varphi \in \mathcal{ML}_2 \mid \forall \mathfrak{G} \in (\mathcal{C}_1 \times \mathcal{C}_2)^{\mathsf{ex}}\ \mathfrak{G} \models \varphi\}.$$

Given Kripke complete unimodal logics L_1 and L_2, let

$$(L_1 \times L_2)^{\mathsf{ex}} = \mathsf{Log}\,(\mathsf{Fr}\,L_1 \times \mathsf{Fr}\,L_2)^{\mathsf{ex}}$$

be the *expanding relativised product* of L_1 and L_2. We obviously have

$$(L_1 \times L_2)^{\mathsf{ex}} \subseteq L_1 \times L_2.$$

As is shown in [30], if both L_1 and L_2 are *subframe logics* (that is, each $\mathsf{Fr}\,L_i$ is closed under—not necessarily generated—subframes), then $(L_1 \times L_2)^{\mathsf{ex}}$ is a conservative extension of both L_1 and L_2. Note that all of the logics listed at the end of Section 2 are subframe logics.

Further, it is not hard to see that expanding relativised products are reducible to products. Indeed, let φ be an $\mathcal{ML}_2$-formula and e a propositional variable which does not occur in φ. Define by induction on the construction of φ an $\mathcal{ML}_2$-formula φ^e as follows:

$$\begin{aligned}
p^e &= p \quad (p \text{ a propositional variable}),\\
(\psi \wedge \chi)^e &= \psi^e \wedge \chi^e,\\
(\neg\psi)^e &= \neg\psi^e,\\
(\boxminus\psi)^e &= \boxminus\psi^e,\\
(\boxbar\psi)^e &= \boxbar(e \to \psi^e).
\end{aligned}$$

Let $md(\varphi)$ denote the *modal depth* of φ, that is, the maximal number of nested modal operators in φ. By a structural induction on φ, one can easily prove the following:

Proposition 5. *For all Kripke complete unimodal logics L_1 and L_2 and all $\mathcal{ML}_2$-formulas φ,*

$$\varphi \in (L_1 \times L_2)^{\mathsf{ex}} \quad \textit{iff} \quad \Big(e \wedge \boxminus^{\leq md(\varphi)} \boxdot^{\leq md(\varphi)} (e \to \boxminus e)\Big) \to \varphi^e \in L_1 \times L_2,$$

where $\Box^{\leq n}\psi = \bigwedge_{k \leq n} \Box^k \psi$, *for* $\Box \in \{\boxminus, \boxdot\}$.

The following proposition connects expanding domain products with expanding domain relativisations:

Proposition 6.

(i) *If both $\mathcal{C}_h$ and $\mathcal{C}_v$ are closed under subframes then*

$$\mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}} \subseteq \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{ex}}.$$

(ii) *Let $\mathcal{C}_h$ and $\mathcal{C}_v$ be as in the formulations of Theorem 1 or 2. Then*

$$\mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}} = \mathsf{Log}(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{ex}}.$$

Proof. To prove (i), let us assume that a formula φ is refuted in an expanding relativised product frame $\mathfrak{G} \subseteq \mathfrak{F}_1 \times \mathfrak{F}_2$ such that $\mathfrak{F}_1 \in \mathcal{C}_h$ and $\mathfrak{F}_2 \in \mathcal{C}_v$. Assume also that $\mathfrak{G} = (W, R'_1, R'_2)$ and $\mathfrak{F}_i = (U_i, R_i)$, $i = 1, 2$. Now let

$$X = \{u \in U_1 \mid \exists v \in U_2\ (u, v) \in W\},$$
$$\mathfrak{F} = \big(X, R_1 \cap (X \times X)\big).$$

For every $x \in X$, let

$$W_x = \{v \in U_2 \mid (x, v) \in W\},$$
$$f(x) = \big(W_x, R_2 \cap (W_x \times W_x)\big).$$

Since both $\mathcal{C}_h$ and $\mathcal{C}_v$ are closed under subframes, it is straightforward to see that $(\mathfrak{F}, f)$ is an e-frame in $(\mathcal{C}_h \times \mathcal{C}_v)^{\mathsf{e}}$ and φ can be refuted in it.

The inclusion $\subseteq$ of (ii) follows from (i) and from the fact that all the classes in the formulations of Theorems 1 and 2 are closed under subframes. To prove $\supseteq$, let us assume that some formula φ is refuted in an e-frame $(\mathfrak{F}, f)$, where $\mathfrak{F} = (W, R) \in \mathcal{C}_h$, and $f(x) = (W_x, R_x) \in \mathcal{C}_v$ for all $x \in W$. By Lemma 2.2, we may assume that $\mathfrak{F}$ is a (finite) transitive tree. It is not hard to see (using the fact that $\mathfrak{F}$ is a tree) that by renaming the points of the frames $f(x)$, $x \in W$, we can always end up with an e-frame having the following property: for all $x \neq y \in W$, $u \in W_x \cap W_y$,

$$\text{either } xRy \text{ or } yRx \text{ or there is } z \in W \text{ such that } zRx,\ zRy \text{ and } u \in W_z. \tag{17}$$

Now if $\mathcal{C}_v$ is not a class of linear frames (that is, it is not like in the cases (C6) of Theorem 1 or (C9) of Theorem 2), then define a frame $\mathfrak{G} = (U, S)$ by taking $U = \bigcup_{x \in W} W_x$ and S to be the *transitive closure* of $\bigcup_{x \in W} R_x$. If $\mathcal{C}_v$ is as in (C6) or (C9), then define S to be the *minimal transitive and linear extension* of $\bigcup_{x \in W} R_x$ instead.

CLAIM 6.1. *For all $x \in W$, $u, v \in W_x$,*

$$uSv \qquad \textit{iff} \qquad uR_xv.$$

PROOF. The ($\Leftarrow$) direction is obvious. The proof of the ($\Rightarrow$) direction is by induction on the length n of a minimal chain

$$uR_{x_1}u_1R_{x_2}\ldots R_{x_n}u_n = v. \tag{18}$$

We prove the general case only, and leave its modification to the linear case to the reader. The case $n = 1$ follows by (17), given that $(\mathfrak{F}, f)$ is an e-frame and $\mathfrak{F}$ is a tree. Now suppose that $n > 1$ and the claim holds for all $k < n$. If $x = x_1$ then $u_1 \in W_x$, so uR_xv follows by IH and transitivity of R_x. So suppose $x \neq x_1$. As $u \in W_x \cap W_{x_1}$, we can apply (17). There are several cases; we discuss only the most complex one, that is, when there is $z \in W$ such that zRx, zRx_1 and $u \in W_z$. By the minimality of the chain (18), we have $x_1 \neq x_2$. As $u_1 \in W_{x_1} \cap W_{x_2}$, we can apply (17) again. Again, we consider only the case when there is $z' \in W$ such that $z'Rx_1$, $z'Rx_2$ and $u_1 \in W_{z'}$. As $\mathfrak{F}$ is a tree, either $z = z'$, or zRz' or $z'Rz$. The first two cases cannot happen, otherwise $uR_{x_2}u_2$ which contradicts the minimality of the chain (18). Thus $z' R z$, and so we have $u R_x u_1$ because $(\mathfrak{F}, f)$ is an e-frame. Finally, $u R_x v$ follows by IH and transitivity of R_x. $\square$

By Claim 6.1, the representation $\overline{\mathfrak{H}}$ of the e-frame $\mathfrak{H}$ defined in Remark 1 is a subframe of $\mathfrak{F} \times \mathfrak{G}$. It remains to show that $\mathfrak{G}$ belongs to $\mathcal{C}_v$. By definition, $\mathfrak{G}$ is transitive. By Claim 6.1, $\mathfrak{G}$ is reflexive (irreflexive, linear) iff all the $f(x)$ $(x \in W)$ are reflexive (irreflexive, linear). So we only need to show that $\mathfrak{G}$ is Noetherian whenever all the $f(x)$ $(x \in W)$ are Noetherian. Since U is finite, it is enough to show that there are no proper S-clusters in $\mathfrak{G}$.

Suppose otherwise, that is there are $u \neq v \in U$, $x \in W$ such that $u S v R_x u$. By Claim 6.1, we have $u R_x v$, which is a contradiction as there are no proper R_x-clusters in $f(x)$. $\square$

As a consequence of Proposition 6(i) we obtain that if both L_1 and L_2 are subframe logics then

$$(L_1 \times L_2)^{\mathsf{e}} \subseteq (L_1 \times L_2)^{\mathsf{ex}}.$$

Moreover, a proof similar to that of Proposition 6(ii) shows that in fact

$$(L_1 \times L_2)^{\mathsf{e}} = (L_1 \times L_2)^{\mathsf{ex}},$$

whenever $L_1,\ L_2 \in \{\mathbf{K}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \mathbf{K4.3}, \mathbf{S4.3}\}$.

It is to be noted, however, that Proposition 6 does not hold for arbitrary subframe logics L_1 and L_2. Consider, for example, the formula

$$\chi \quad = \quad \Box_v \bot \wedge \Box_h^{+} \Box_v^{+} (\Box_v \bot \to \Diamond_h \Diamond_v \Box_v \bot). \tag{19}$$

It is clearly satisfied (under any valuation) in the e-frame $(\mathfrak{F}, f)$ in which $\mathfrak{F} = (\mathbb{N}, <)$ and $f(n) = (\{0, 1, \dots, n\}, <)$. Obviously, $\mathfrak{F} \models \mathbf{K4}$ and $f(n) \models \mathbf{GL}$ for each $n \in \mathbb{N}$. However it is impossible to 'embed' $(\mathfrak{F}, f)$ into a real product without an infinite ascending chain in the vertical component (although all the vertical components $f(n)$ of $(\mathfrak{F}, f)$ itself are finite). In fact, one can readily show that if χ is satisfied in an expanding relativised product frame $\mathfrak{G} = (W, R_1, R_2)$ where R_1 is transitive and R_2 is irreflexive, then W contains an infinite ascending R_2-chain. This means that χ is not satisfiable in any expanding relativised product frame for $(\mathbf{K4} \times \mathbf{GL})^{\mathsf{ex}}$, and so

$$(\mathbf{K4} \times \mathbf{GL})^{\mathsf{e}} \neq (\mathbf{K4} \times \mathbf{GL})^{\mathsf{ex}}.$$

6. Discussion

In this paper, we have presented first examples of products of modal logics with expanding domains which are

- decidable, but
- not in primitive recursive time,

while the corresponding product logics (with constant domains) are

- undecidable.

Numerous interesting problems concerning logics of expanding domain frames remain open:
1. Our decidability proofs make use of the e-product fmp. Unfortunately, if we relax the conditions of Theorems 1 and 2, then the resulting logics do not have the e-product fmp any more. It is easy to see using, for instance, the formula

$$\Box_v^{+} \Diamond_v \top \wedge \Box_v^{+} \Diamond_h (p \wedge \Box_v \neg p) \tag{20}$$

that $(\mathbf{GL} \times \mathbf{K4})^{\mathsf{e}}$ does not have the e-product fmp. In fact, a similar formula that has $\blacklozenge$ and $\blacksquare$ (see the proof of Lemma 2.5) in place of $\Diamond_v$ and $\Box_v$ shows the lack of the e-product fmp for $(L_1 \times L_2)^{\mathsf{e}}$, whenever L_1 is any logic that has a frame containing a point with infinitely many successors, and $\mathsf{Fr}\, L_2$ is any class of transitive frames containing an infinite ascending chain of distinct points. Note that $\mathbf{GL}$ is determined by the class $\mathcal{C}$ of all finite irreflexive and transitive frames, and so $\mathsf{Log}\,(\mathcal{C} \times \mathsf{Fr}\,\mathbf{K4})^{\mathsf{e}}$ has the e-product fmp (and is decidable) by Theorem 1. Thus (20) also shows that even if each component logic L_i is determined by a class $\mathcal{C}_i$ of frames $(i = 1, 2)$, the logics $(L_1 \times L_2)^{\mathsf{e}} = \mathsf{Log}\,(\mathsf{Fr}\, L_1 \times \mathsf{Fr}\, L_2)^{\mathsf{e}}$ and $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)^{e}$ are not necessarily the same.

It is also possible to 'force' an infinite ascending chain 'horizontally': the formula

$$\Box_h^+ \Diamond_v (p \wedge \Diamond_h \Box_h^+ \neg p)$$

shows the lack of e-product fmp for $(L_1 \times L_2)^{\mathsf{e}}$, whenever $\mathsf{Fr}\, L_1$ is any class of transitive frames containing an infinite ascending chain of distinct points, and L_2 is any logic that has a frame containing a point with infinitely many successors.

Moreover, as is shown in [22], the logic

$$\mathsf{Log}\left(\{(\mathbb{N}, <)\} \times \mathcal{C}\right)^{\mathsf{e}}$$

becomes undecidable, whenever $\mathcal{C}$ is any of the classes (C1)–(C6) listed in Theorem 1 above. It follows that the satisfiability problem for $\mathcal{ML}_2^{\circ}$-formulas in DTM$_\omega$s based on Aleksandrov spaces with continuous mappings is undecidable as well. Decidability of other e-products without the e-product fmp (such as, say, $(\mathbf{K4} \times \mathbf{K4})^{\mathsf{e}}$ and $(\mathbf{K4.3} \times \mathbf{K4.3})^{\mathsf{e}}$) remains open.

2. As is shown in [11, Section 9.1], logics of the form $(L \times (\mathbf{S5} \times \mathbf{S5}))^{\mathsf{ex}}$ are reducible to the two-variable fragment of quantified L with expanding domains. According to [23], these first-order modal logic fragments are actually undecidable, whenever L has a frame containing a point with infinitely many successors. (For the constant domain case this was proved in [12].) We conjecture that the proof techniques of [23] and [19] can be combined to show undecidability of all logics of the form $(L_1 \times (L_2 \times L_3))^{\mathsf{ex}}$, where L_1, L_2 and L_3 are any Kripke complete modal logics between $\mathbf{K}$ and $\mathbf{S5}$.

3. We did not consider the problem of finding axiomatisations for e-product logics. Here we just list a selection of open questions. Denote by $[L_1, L_2]^{\mathsf{e}}$ the bimodal logic obtained by adding to the independent fusion of L_1 and L_2 the axioms

$$\Diamond_v \Diamond_h p \to \Diamond_h \Diamond_v p \qquad \text{and} \qquad \Diamond_h \Box_v p \to \Box_v \Diamond_h p,$$

and call it the *expanding commutator* of L_1 and L_2. It is easy to see that

$$[L_1, L_2]^{\mathsf{e}} \subseteq (L_1 \times L_2)^{\mathsf{e}},$$

and if L_1 and L_2 are subframe logics then

$$[L_1, L_2]^{\mathsf{e}} \subseteq (L_1 \times L_2)^{\mathsf{ex}}.$$

As is shown in [11, Theorem 9.10], $(L_1 \times L_2)^{\mathsf{ex}} = [L_1, L_2]^{\mathsf{e}}$ whenever $L_1 \in \{\mathbf{K}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$ and L_2 is axiomatisable by modal formulas with a universal Horn first-order translation. It would be interesting to find pairs of logics such that $(L_1 \times L_2)^{\mathsf{ex}} \neq [L_1, L_2]^{\mathsf{e}}$, but $(L_1 \times L_2)^{\mathsf{ex}}$ (or $(L_1 \times L_2)^{\mathsf{e}}$) is still finitely axiomatisable. Are there any pairs of logics such that

$$(L_1 \times L_2)^{\mathsf{ex}} = [L_1, L_2]^{\mathsf{e}}, \qquad \text{but} \qquad (L_1 \times L_2) \neq [L_1, L_2],$$

where $[L_1, L_2] = ([L_1, L_2]^{\mathsf{e}} + \Diamond_h \Diamond_v p \to \Diamond_v \Diamond_h p)$?

Further, as is shown in [14], the product logics (such as, say, $\mathbf{GL} \times \mathbf{GL}$) whose 'expanding domain' versions are decidable by Theorem 2 are not even recursively enumerable. It is also shown in [14] that commutators like $[\mathbf{GL}, \mathbf{GL}]$ are (though also undecidable) Kripke incomplete, so cannot coincide with the corresponding product logics (which are Kripke complete by definition). Does any of these decidable e-products coincide with the corresponding expanding commutator? If not, are they finitely axiomatisable? Are these expanding commutators decidable or Kripke complete? Note that the formula (19) actually shows that

$$[\mathbf{K4}, \mathbf{GL}]^{\mathsf{e}} \neq (\mathbf{K4} \times \mathbf{GL})^{\mathsf{ex}},$$

but it is not known whether $[\mathbf{K4}, \mathbf{GL}]^{\mathsf{e}}$ and $(\mathbf{K4} \times \mathbf{GL})^{\mathsf{e}}$ are different.

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