# Refutations of pebble minimization via output languages

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**Abstract.** Polyregular functions are the class of string-to-string functions definable by pebble transducers, an extension of finite-state automata with outputs and multiple two-way reading heads (pebbles) with a stack discipline. If a polyregular function can be computed with k pebbles, then its output length is bounded by a polynomial of degree k in the input length. But Bojańczyk has shown that the converse fails.

In this paper, we provide two alternative easier proofs. The first establishes by elementary means that some quadratic polyregular function requires 3 pebbles. The second proof – just as short, albeit less elementary – shows a stronger statement: for every k, there exists some polyregular function with quadratic growth whose output language differs from that of any k-fold composition of macro tree transducers (and which therefore cannot be computed by a k-pebble transducer). Along the way, we also refute a conjectured logical characterization of polyblind functions.

**Keywords:** polyregular functions, pebble transducers, macro tree transducers

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### 1. Introduction

Many works about *transducers* – automata that can produce output string/trees, not just recognize languages – are concerned with:

- either well-known classes of *linearly growing* functions, i.e. |f(x)| = O(|x|) in the string-to-string case, those are the sequential, rational and regular functions, see e.g. [MP19];
- or devices with possibly (hyper)exponential growth rates: HDT0L systems [FMS14, FR21, DTFG20], macro tree transducers (MTTs) [EV85], compositions of MTTs (see below)...

A middle ground is occupied by the *polyregular functions*, surveyed in [Boj22], which derive their name from their *polynomial* growth rate. They are the functions computed by string-to-string *pebble transducers*. Although pebble transducers have been around for two decades (starting with the tree-to-tree version in [MSV03], see also [EM03, EM02b, Eng15]), it is only in the late 2010s that several alternative characterizations of polyregular functions were introduced [Boj18, BKL19]. This had led to renewed interest in this robust function class (e.g. [CDTL23, Jor22, RDHR23]).

**Example 1.1.** The *inner squaring* function (from [Boj22, §6.2]) defined as

innsq: 
$$\{a, b, \#\}^*$$
  $\rightarrow$   $\{a, b, \#\}^*$   $w_0 \# \dots \# w_n \mapsto (w_0)^n \# \dots \# (w_n)^n \quad (w_0, \dots, w_n \in \{a, b\}^*)$ 

– for instance, innsq(aba#baa#bb) = abaaba#baaa#bbaaa#bbb – can be computed by the 3-pebble<sup>1</sup> transducer that we describe in Section 2. Essentially, it uses a stack of at most 3 pointers to the inputs, called pebbles, in order to simulate 3 nested for-loops. This means that in this example, the pebbles only move in a left-to-right direction; but in general, they can choose to move to the left or to the right depending on the current state, unlike the indices in a for-loop.

A straightforward property of k-pebble transducers is that their growth rate is  $O(n^k)$ , and this is the best possible bound. Hence the question of *pebble minimization*: if a polyregular function (that is, defined by some  $\ell$ -pebble transducer) has growth  $O(n^k)$ , is it always computable by some k-pebble transducer? Definitely not, as Bojańczyk recently showed [Boj23, Section 3]: no number of pebbles suffices to compute all polyregular functions with *quadratic* growth (but those of *linear* growth require a single pebble). Moreover, he uses the tools from [Boj23] to show in [Boj22, Theorem 6.3]<sup>3</sup> that the inner squaring function requires 3 pebbles even though  $|\text{innsq}(w)| = O(|w|^2)$ .

**Contributions** Bojańczyk's proof of the aforementioned results is long and technical. We propose shorter and easier arguments:

 $<sup>^1</sup>$ We follow the convention of newer papers for two-way string transducers are 1-pebble transducers, while some older papers would have called them 0-pebble transducers; thus,  $k \ge 1$  here.

<sup>&</sup>lt;sup>2</sup>This applies to strings; for trees, (output height) =  $O((input \, size)^k)$  [EM03, Lemma 7].

<sup>&</sup>lt;sup>3</sup>The paper [Boj23] proposes a slightly different quadratic example that requires 3 pebbles, called "block squaring".

- We reprove in Section 3 that inner squaring requires 3 pebbles. Our proof depends only on a few old and familiar properties of *regular functions* those computed by two-way transducers, i.e. 1-pebble transducers such as their closure under composition [CJ77] and an elementary pumping lemma [Roz86] for their *output languages* (their sets of possible output strings).
- In Section 5, we use first-order interpretations (recalled in Section 4) to construct a certain sequence  $(f_k)_{k\geq 1}$  of quadratic polyregular functions. Then, we show that  $f_k$  requires k+1 pebbles, as a consequence of a stronger result: the output language of  $f_k$  differs from that of any k-fold composition of macro tree transducers. (This composition hierarchy is quite canonical and well-studied, see §5.1.) Again, our proof is quite short,<sup>4</sup> and even arguably easier to check than our ad-hoc argument for innsq, though it is less elementary since its "trusted base" is larger: we use a powerful "bridge theorem" on MTTs from [EM02a].

### On subclasses of polyregular functions Some restrictions on pebble transducers ensure that:

- the computed function sits at a low level of the aforementioned composition hierarchy;
- a pebble minimization property holds: functions of growth  $O(n^k)$  require only k pebbles.

This is for instance the case for k-marble transducers – they can compute precisely the same string-to-string functions as MTTs<sup>5</sup> with growth  $O(n^k)$  [DTFG20, §5] – or for blind pebble transducers, which define polyblind<sup>6</sup> functions (cf. Theorem 3.8, taken from [NNP21]). Furthermore, Douéneau-Tabot has recently proved pebble minimization for "last pebble" transducers [Dou23], a model that subsumes both marble and blind pebble transducers; and a function computed by any such device can be obtained by a composition of two MTTs<sup>7</sup> (a consequence of [EHS21, Theorem 53 (in §15)]).

Therefore, counterexamples to pebble minimization cannot be computed by "blind pebble" or "last pebble" transducers. Based on this observation, and using Example 1.1, we refute the conjecture in [NNP21] about a logical characterization of polyblind functions (Theorem 4.6).

**Notations** The set of natural numbers is  $\mathbb{N} = \{0, 1, \dots\}$ . Alphabets are always finite sets.

Let  $\Sigma$  be an alphabet. We write  $\varepsilon$  for the empty string and  $\Sigma^*$  for the set of strings (or words) with letters in  $\Sigma$ , i.e. the free monoid over  $\Sigma$ ; the Kleene star  $(-)^*$  will also be applied to languages  $L \subseteq \Sigma^*$  as part of usual regular expression syntax.

Let  $w = w_1 \dots w_n \in \Sigma^*$  ( $w_i \in \Sigma$  for  $i \in \{1, \dots, n\}$ ); we also write  $w[i] = w_i$ . The length |w| of w is n, and  $|w|_c$  refers to the number of occurrences of  $c \in \Sigma$  in w.

The *output language* of a function  $f: X \to \Sigma^*$  is  $f(X) \subseteq \Sigma^*$ , also denoted by  $\operatorname{Im}(f)$ .

<sup>&</sup>lt;sup>4</sup>And mostly unoriginal: it consists of little adjustments to an argument by Engelfriet & Maneth [EM02b, §4]. In a followup paper [Eng15], Engelfriet mentions and corrects a mistake in [EM02b, §3], but it does not affect the section which is relevant for our purposes.

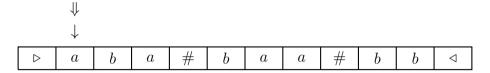
<sup>&</sup>lt;sup>5</sup>Using the fact that string-to-string MTTs are syntactically isomorphic, up to insignificant details, to the copyful streaming string transducers of [FR21].

<sup>&</sup>lt;sup>6</sup>A name given by Douéneau-Tabot [DT21, DT22, Dou23] to what Nguyễn, Noûs and Pradic [NNP21] originally called the "comparison-free" subclass of polyregular functions.

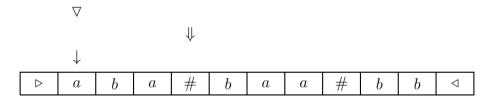
<sup>&</sup>lt;sup>7</sup>See also [Sé23] for a characterization of the string-to-string functions computed by compositions of two MTTs.

# 2. An example of string-to-string pebble transducer

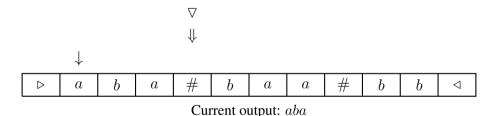
We describe informally a pebble transducer that computes the inner squaring function of Example 1.1. The machine has a finite-state control, and starts with a single pebble (pointer to the input)  $\downarrow$  initialized to the first position in the input. The first thing it does is to *push* a second pebble  $\downarrow$  on its *stack of pebbles*, resulting in the following configuration:



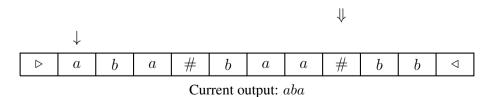
The goal of the second pebble is to count the #s in the input. The transducer moves it forward until it reaches a #, at which point it pushes the third pebble  $\nabla$ :



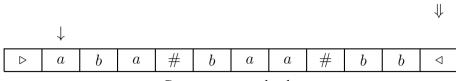
The goal of this third pebble  $\nabla$  is to copy the input block that the first pebble  $\downarrow$  points to. Therefore, it is going to move forward while copying each input letter to the output, until it reaches #:



Now, in order to count the number of #s – which is the number of copies of aba that need to be outputted – we would like to move the second pebble  $\Downarrow$  forward. The *stack restriction* on pebble transducers says that *only the topmost pebble can be moved*. Therefore, the transducer has to *pop* the third pebble  $\triangledown$ , *forgetting its position*, so that  $\Downarrow$  becomes free to move to the second #:

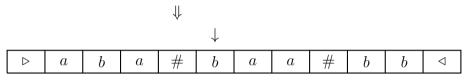


Next,  $\nabla$  is pushed again – reinitialized to the first input position – in order to output a copy of aba as before. After this,  $\nabla$  is popped and  $\downarrow$  moves forward, until it reaches the end of the word:



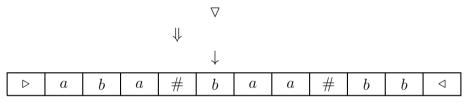
Current output: abaaba

When this happens, the transducer pops  $\downarrow$  and moves  $\downarrow$  forward to the beginning of the second block baa. Then  $\downarrow$  is pushed again to count the #s:



Current output: abaaba#

At this point, the third pebble  $\nabla$  is pushed again, its purpose being to copy the block baa to the output. To do so, it moves forward until it reaches the beginning of that block, pointed by  $\downarrow$ :



Current output: abaaba#

The transducer can know that it has reached the beginning of the block it wants to copy because it is allowed to *compare the positions* of  $\nabla$  and  $\downarrow$ . The third pebble  $\nabla$  can now move forward to output baa, and the execution continues as the reader might expect.

**Moral of the story** Note that every time an output letter is produced, the first pebble  $\downarrow$  must be on the leftmost position of the block in which the third pebble  $\triangledown$  is. Thus, the output positions can be morally parameterized by pairs of input positions, corresponding to  $\Downarrow$  and  $\triangledown$ ; this is why the transducer has only quadratic growth, despite its 3 pebbles. (This idea will make an appearance again in Example 4.3.) The first pebble  $\downarrow$  is redundant, but this redundancy is made necessary by the stack condition: the role of  $\downarrow$  could be seen as keeping some partial information about  $\triangledown$  while  $\Downarrow$  moves, since to be allowed to move  $\Downarrow$ , first  $\triangledown$  must be popped.

# 3. A simple proof that inner squaring requires 3 pebbles

The 3-pebble transducer described in the previous section computes the inner squaring function innsq (Example 1.1); we will see in Example 3.6 how to turn this into a more formal argument for the fact that innsq  $\in$  Pebble<sub>3</sub>. The goal of this section is to reprove [Boj22, Theorem 6.3]:

**Theorem 3.1.** Inner squaring cannot be computed with 2 pebbles: innsq  $\notin$  Pebble<sub>2</sub>.

**Remark 3.2.** However, one can show that the restriction of innsq to inputs in  $\{a, \#\}^*$  is in Pebble<sub>2</sub>.

**Remark 3.3.** Bojańczyk's proof sketch for [Boj22, Theorem 6.3] actually shows that the function  $a_1 \ldots a_n \in \mathbb{A}^* \mapsto (a_1)^n \ldots (a_n)^n$  cannot be computed by a 2-pebble *atom-oblivious* transducer (here the alphabet  $\mathbb{A}$  is an *infinite* set of *atoms*). Combining this with the Deatomization Theorem from [Boj23] – whose proof is rather complicated – shows that no  $f: \{\langle, \rangle, \bullet\}^* \to \{\langle, \rangle, \bullet\}^*$  in Pebble2 can satisfy  $f(\langle \bullet^{p_1} \rangle \ldots \langle \bullet^{p_n} \rangle) = (\langle \bullet^{p_1} \rangle)^n \ldots (\langle \bullet^{p_n} \rangle)^n$ , from which Theorem 3.1 can be deduced using the composition properties of pebble transducers [Eng15].

Instead of introducing the hierarchy  $(Pebble_k)_{k \in \mathbb{N}}$  by a concrete machine model, we define it in Section 3.2 using combinators (operators on functions). This abstract presentation depends on the *origin semantics* [Boj14] (see also [MP19, §5]) of regular functions, about which we say a few words in Section 3.1. We also recall in §3.2 a similar definition of polyblind functions; their only use in this section will be to state and prove Corollary 3.9, but they will appear again briefly in Section 4. After all this, Section 3.3 proves Theorem 3.1.

## 3.1. Regular functions with origin information

Regular functions are those computed by 1-pebble transducers, also known as *two-way transducers* (2DFTs). We will mostly avoid explicit manipulations of 2DFTs; this is why we do not recall a formal definition here, and refer to the survey [MP19] for more on regular functions and their origin semantics. A typical example of a regular string-to-string function is:

$$a^{m_0} \# \dots \# a^{m_n} \in \{a, \#\}^* \mapsto a^{m_n} b^{m_n} \# \dots \# a^{m_0} b^{m_0}$$

There are several natural ways to lift it to a regular function with origins, which reflect different ways to compute it with a 2DFT: for example aaa#aa could be mapped to

The second component indicates, for each output letter, which input position it "comes from" – that is, where the reading head was placed when the letter was outputted.

We shall write  $f^{\circ}, g^{\circ}, \ldots : \Gamma^* \to (\Sigma \times \mathbb{N})^*$  for regular functions with origin information and  $f, g, \ldots : \Gamma^* \to \Sigma^*$  for the corresponding regular string-to-string functions. Thus,  $f = (\pi_1)^* \circ f^{\circ}$  and  $(\pi_2)^* \circ f(w) \in \{1, \ldots, |w|\}^*$  for any  $w \in \Gamma^*$ , where  $\pi_i$  is the projection on the *i*-th component of a pair  $(i \in \{1, 2\})$ .

**Remark 3.4.** The notion of origin semantics can be a bit problematic when the empty string  $\varepsilon$  has a non-empty image. This is why, in this section, we consider that **all our (poly)regular functions map**  $\varepsilon$  **to**  $\varepsilon$  in order to avoid inessential inconveniences. This makes no difference concerning the strength of our result, since innsq( $\varepsilon$ ) =  $\varepsilon$ .

## 3.2. Pebble transducers in an abstract style and polyblind functions

We denote by  $\underline{\Sigma}$  a disjoint copy of  $\Sigma$ , made of "underlined letters", so that  $a \in \Sigma \mapsto \underline{a} \in \underline{\Sigma}$  is a bijection. We also write  $w \not \in i$  for  $w[1] \dots w[i-1]w[i]w[i+1] \dots w[n] \in (\Sigma \cup \underline{\Sigma})^*$  where n = |w|.

**Definition 3.5.** Let  $f^{\circ} : \Gamma^* \to (I \times \mathbb{N})^*$  be a regular function with origins (so  $f : \Gamma^* \to I^*$  is regular) and, for  $i \in I$ , let  $g_i : (\Gamma \cup \underline{\Gamma})^* \to \Sigma^*$  and  $h_i : \Gamma^* \to \Sigma^*$ . For  $w \in \Gamma^*$ , define

pebble
$$(f^{\circ}, (g_i)_{i \in I})(w) = g_{i_1}(w \not\in j_1) \cdot \ldots \cdot g_{i_n}(w \not\in j_n)$$
 where  $f^{\circ}(w) = (i_1, j_1) \cdot \ldots \cdot (i_n, j_n)$   
blind $(f, (h_i)_{i \in I})(w) = h_{i_1}(w) \cdot \ldots \cdot h_{i_n}(w)$ 

(blind is called "composition by substitution" in [NNP21, §4]). Using these combinators, we define the hierarchies of function classes  $Pebble_n$  and  $Blind_n$  inductively.  $Pebble_0 = Blind_0$  is the class of string-to-string functions with finite range (or equivalently, whose output has bounded length), and

$$\forall n \in \mathbb{N}$$
,  $\mathsf{Pebble}_{k+1} = \{\mathsf{pebble}(f^{\circ}, (g_i)) \mid f^{\circ} \text{ regular}, \ g_i \in \mathsf{Pebble}_k\}$   
 $\mathsf{Blind}_{k+1} = \{\mathsf{blind}(f, (g_i)) \mid f \text{ regular}, \ g_i \in \mathsf{Blind}_k\}$ 

Morally, the correspondence with the informal presentation of k-pebble transducers in Section 2 is that we can compute  $pebble(f^{\circ}, (g_i)_{i \in I})$  by tweaking some two-way transducer  $\mathcal{T}$  computing  $f^{\circ}$ : each time  $\mathcal{T}$  would output a letter i, we instead "call a subroutine" computing  $g_i(w \not\downarrow j)$  where j is the current position of the reading head. If every  $g_i$  is implemented using a stack of k pebbles, then we can implement the subroutine-calling machine with a stack of height k+1: the pebble at the bottom of the stack corresponds to the reading head of  $\mathcal{T}$ . Conversely, in any pebble transducer, pushing a pebble can be seen as initiating a subroutine call.

**Example 3.6.** Let us turn the 3-pebble transducer of Section 2 into an "abstract-style" proof that innsq  $\in$  Pebble<sub>3</sub>. Let  $f^{\circ}: \{a, b, \#\}^* \to (\{\bullet, \#\} \times \mathbb{N})^*$  be defined by

$$f^{\circ}(\underbrace{w[1]\dots w[i_1-1]}_{\text{each block is in }\{a,b\}^*}\#\dots\#w[i_m+1]\dots w[n]) = \underbrace{i_1}_{i_1} \underbrace{i_1+1}_{i_1+1} \dots \underbrace{i_m}_{i_m+1} i_{m+1}$$

and  $h(\dots \# cw \# \dots) = cw$  for  $c \in \{a,b\}$  and  $w \in \{a,b\}^*$ . Then

$$\underbrace{\mathsf{innsq}}_{\mathsf{Example } 1.1} = \underbrace{\mathsf{pebble}(\overbrace{f^{\circ}}, (g_i)_{i \in \{\bullet, \#\}})}_{\in \mathsf{Pebble}_3} \qquad g_{\bullet} = \underbrace{\mathsf{blind}(\overbrace{w \mapsto \bullet^{|w|_\#}, (h)_{j \in \{\bullet\}}})}_{\mathsf{EBlind}_2 \subset \mathsf{Pebble}_2}, \ g_{\#} \colon \underbrace{w \mapsto \#}_{\in \mathsf{Pebble}_0}$$

The fact that  $Blind_k$  consists of the functions computed by  $blind_k$ -pebble transducers – that cannot compare the positions of their pebbles – is stated in [NNP21, §5], with a detailed proof available in [NNP21, Appendix D] refining the intuitions on "subroutines" given above. By a straightforward adaptation of that proof:

**Proposition 3.7.** Pebble<sub>k</sub> is exactly the class of string-to-string functions computed by k-pebble transducers [Boj22, §2] for any  $k \ge 1$ .

Note that a similar definition of the "last pebble" transducers mentioned in the introduction is given in [Dou23, Definition 3.3], without explicit proof that they coincide with the expected machine model.

Finally, let us recall that unlike general pebble transducers, blind pebble transducers enjoy the pebble minimization property [NNP21, Theorem 7.1]. See also [Dou23] for an *effective* minimization proof<sup>8</sup> (i.e. that provides an algorithm to find, given a blind  $\ell$ -pebble transducer, its degree k of growth and an equivalent blind k-pebble transducer) and a generalization to "last pebble" transducers.

**Theorem 3.8.** For all 
$$k \in \mathbb{N}$$
,  $\mathsf{Blind}_k = \left\{ f \in \bigcup_{\ell \in \mathbb{N}} \mathsf{Blind}_\ell \ \middle| \ |f(w)| = O(|w|^k) \right\}$ .

**Corollary 3.9.** The inner squaring function (Example 1.1) is not polyblind.

#### **Proof:**

Since it has quadratic growth, if it were polyblind, it would be in  $Blind_2 \subset Pebble_2$ . This would contradict the main result of this section (Theorem 3.1).

### 3.3. Proof of Theorem 3.1

Our approach to prove innsq ∉ Pebble<sub>2</sub> goes through the output languages of regular functions.

**Definition 3.10.** Call a language  $L \subseteq \Sigma^*$  a *regular image* if there exists a regular function f with codomain  $\Sigma^*$  such that  $L = \operatorname{Im}(f)$ .

Not all regular images are regular or even context-free languages: consider e.g.  $\{a^nb^nc^nd^n \mid n \in \mathbb{N}\}$ . We will show that no function in Pebble<sub>2</sub> can coincide with innsq on the subset of inputs

$$(a^*b\#)^*\#^* = \{\text{strings with the shape } a \dots ab\# \dots \# a \dots ab\# \# \dots \# \}$$

For the sake of contradiction, assume the opposite.

**Claim 3.11.** Under this assumption, there exists a language  $L \subseteq b\{a,b\}^*b$ 

- · which is a regular image
- that contains only factors of words from innsq $((a^*b\#)^*\#^*)$

• and such that, for any 
$$N \in \mathbb{N}$$
, some word in  $L$  has the shape  $\overbrace{b \ a \ldots a \ b \ldots b \ a \ldots a \ b}$  every block of  $as$  has the same length  $n \ge N$ 

<sup>&</sup>lt;sup>8</sup>It is likely that the proof of [NNP21] could be made effective, but this is not explicit.

### **Proof:**

We start by an analysis of some output factors arising in the computation of innsq by a hypothetical 2-pebble transducer, which will later inform the construction of a suitable language L. Unfolding our formal definition of Pebble<sub>2</sub>, our assumption means that innsq coincides on  $(a^*b\#)^*\#^*$  with some function pebble $(g^{\circ}, (h_i)_{i \in I})$  where

- $g^{\circ}: \{a, b, \#\}^* \to (\{a, b, \#\} \times \mathbb{N})^*$  is a regular function with origins;
- for  $i \in I$ , the function  $h_i : \{a, b, \#, \underline{a}, \underline{b}, \#\}^* \to \{a, b, \#\}^*$  is in Pebble<sub>1</sub>, so it is regular.

The functions  $g^{\circ}$  and  $h_i$ , being regular, have linear growth: there is some constant  $C \in \mathbb{N}$  such that, for all input words u, v and  $i \in I$ , we have  $|g^{\circ}(u)| \leq C|u|$  and  $|h_i(v)| \leq C|v|$ .

Consider the input word  $u = (a^n b \#)^n \#^{nm}$  for some  $n, m \in \mathbb{N}$  (which shall be taken large enough to satisfy constraints that will arise during the proof). By definition,

$$\mathtt{innsq}(u) = v_1 \dots v_k$$
 where  $g^{\circ}(w) = (i_1, j_1) \dots (i_k, j_k)$  and  $v_{\ell} = h_{i_{\ell}}(u \not\in j_{\ell})$ 

For large enough n and m, we have that for all  $\ell \in \{1, \ldots, k\}$ ,

$$|v_\ell| \le C|u \not \downarrow j_\ell| = Cn(n+2+m) < (n+1)n(1+m) = \underbrace{|a^n b| \times |u|_\#}_{\text{distance between two consecutive non-adjacent occurrences of $\#$ in innsq($u$)}$$

Thanks to the above, if we call p the largest integer such that  $|v_1 \dots v_p|_{\#} < n$ , then:

- $|v_{\ell}|_{\#} \leq 1$  for every  $\ell \in \{1, \ldots, p\}$ ;
- $v_1 \dots v_p$  is a prefix of innsq(u) that contains all the n-1 first occurrences of #.

As a consequence of the latter item,  $((a^nb)^{n(1+m)})\#)^{n-1}$  is a prefix of  $v_1 \dots v_p$  and  $|v_1 \dots v_p|_b \ge n(1+m)(n-1)$ . Meanwhile, we also have  $p < k = |g^{\circ}(u)| \le C|u| = Cn(n+2+m)$ . Hence

$$\frac{|v_1 \dots v_p|_b}{n} \ge \frac{n(1+m)(n-1)}{Cn(n+2+m)} \xrightarrow[n,m\to+\infty]{} +\infty$$

Thus, by the pigeonhole principle, for an input u parameterized by large enough values of n and m, there is at least one output factor  $v_\ell$  containing 2N or more occurrences of b. Since  $|v_\ell|_\# \le 1$ , either its largest #-free prefix or its largest #-free suffix (or both) contains N or more occurrences of b. After applying the regular function  $a \dots abwba \dots a \mapsto bwb$  (for  $w \in \{a,b\}^*$ ) to trim this prefix or suffix, we get a factor of  $v_\ell$  – and therefore of  $\mathrm{innsq}(u)$  – in  $b(a^nb)^*$ , and we may take  $n \ge N$  to ensure that the blocks of as have length at least N.

To sum up, for every N, there are some  $n, r \geq N$  such that  $b(a^n b)^r$  belongs to the following language L, where  $p_i$  is the identity on  $\{u \not\downarrow j \mid u \in (a^*b\#)^*\#^*, (i,j) \text{ appears in } g^{\circ}(u)\}$  and sends every other word of  $\{a, b, \#, \underline{a}, \underline{b}, \#\}^*$  to  $\varepsilon$ :

$$L = \bigcup_{i \in I} \operatorname{Im}(\operatorname{trim} \circ \operatorname{largest} \operatorname{\#-free} \operatorname{prefix} \circ h_i \circ p_i) \cup \operatorname{Im}(\operatorname{trim} \circ \operatorname{largest} \operatorname{\#-free} \operatorname{suffix} \circ h_i \circ p_i)$$

The "projection"  $p_i$  is here to make sure that any  $v \in \text{Im}(h_i \circ p_i)$  is a factor of some word from  $\text{innsq}((a^*b\#)^*\#^*)$  – which, in turn, guarantees that L itself contains only such factors.

To complete the proof, we need to show that L is a regular image. One can check that regular images are closed under finite unions, so it is enough that  $\operatorname{Im}(-)$  is applied to regular functions in the above expression. That is indeed the case: regular functions are closed under composition [CJ77], and  $p_i$  is regular because it is computed by a two-way transducer that first checks in a single pass that its input is in  $(a^*b\#)^*\#^*$  when the underlining is ignored, then simulates the transducer for  $g^{\circ}$  – without producing output – until it observes that some i is indeed outputted with origin at the underlined position, and finally, if the previous checks have not failed, copies its input.

We shall now use the following pumping lemma due to Rozoy [Roz86, §4.1]:

**Lemma 3.12.** If L is a regular image, then for some  $k, K \in \mathbb{N}$ , every  $w \in L$  with  $|w| \geq K$  has a decomposition  $w = u_0 v_1 \dots u_{k-1} v_k u_k$  with

- $\exists i \in \{1,\ldots,k\} : v_i \neq \varepsilon$
- $\forall i \in \{1, ..., k\}, |v_i| \le K$
- $\{u_0(v_1)^n \dots u_{k-1}(v_k)^n u_k \mid n \in \mathbb{N}\} \subseteq L$

Let us apply this lemma to the language L given by Claim 3.11, yielding two constants  $k, K \in \mathbb{N}$ . One of the properties stated in Claim 3.11 is that there exist  $n \geq K$  and  $r \geq 2k+1$  such that  $b(a^nb)^r \in L$ . This string has length greater than K, so it is pumpable:

$$b(a^n b)^r = u_0 v_1 \dots u_{k-1} v_k u_k$$
 and  $w = u_0 (v_1)^2 \dots u_{k-1} (v_k)^2 u_k \in L$ 

Let us now show that  $ba^nb$  is a factor of w. Since  $|v_i| \le K \le n$  for every  $i \in \{1, ..., k\}$ , and any two occurrences of b in  $b(a^nb)^r$  are separated by  $a^n$ , each factor  $v_i$  can contain at most one b. So  $|v_1|_b + \cdots + |v_k|_b \le k$  which leads to  $|u_0|_b + \cdots + |u_k|_b \ge r + 1 - k \ge k + 2$ . One of the  $u_j$  must then contain two occurrences of b, and this  $u_j$  is also a factor of w.

Recall that L is comprised exclusively of #-free factors of words in  $\mathtt{innsq}((a^*b\#)^*\#^*)$ , so all the "blocks of as" in w must have the same size. Having seen above that this size must be n, we can now wrap up our proof by contradiction of Theorem 3.1 with a case analysis:

- First assume that  $|v_i|_b \ge 1$  for some  $i \in \{1, \ldots, k\}$ . We may write  $v_i = a^\ell b \ldots b a^m$  (where the first and last b can coincide) in this case. Then w contains  $(v_i)^2$  which in turn contains a factor  $ba^{\ell+m}b$ . So  $\ell+m=n$  since it is the size of a block of as in w; but at the same time, using the second item of Lemma 3.12,  $\ell+m < |v_i| \le K \le n$ .
- Otherwise,  $v_i \in \{a\}^*$  for all i, so pumping does not increase the number of bs. Therefore w has as many blocks of as as  $b(a^nb)^r$ , and we have also seen that its blocks have size n, so  $w = b(a^nb)^r$ . But since at least one  $v_i$  is nonempty,  $|w| > |b(a^nb)^r|$ .

<sup>&</sup>lt;sup>9</sup>This is a quite non-trivial result, whose use could be avoided in our proof at the price of more explicit manipulations of two-way transducers.

Remark 3.13. Rozoy proves Lemma 3.12 for deterministic two-way transducers (cf. Section 3.1). She also shows [Roz86, §4.2] that the output languages of *nondeterministic* two-way transducers enjoy a weaker version of the lemma without the bound on the length of the  $v_i$  (this bound is refuted by the example  $\{(w\#)^n \mid w \in \Sigma^*, n \in \mathbb{N}\}$ ) (a result later rediscovered by Smith [Smi14]); in general, the languages that satisfy this weaker pumping lemma are called k-iterative in the literature (see e.g. [Smi14, KKM<sup>+</sup>14]). For more on regular images, see [EY71, EH91]; several references are also given in [Gau20, p. 18].

# 4. First-order interpretations (in 2 dimensions)

In this section, which is entirely independent from the previous one, we recall another way to specify string-to-string functions, namely first-order (FO) interpretations. There are two main motivations:

- they are a convenient tool (but not strictly necessary, cf. Remark 5.3) for the next section;
- we wish to state and prove Theorem 4.6, refuting [NNP21, Conjecture 10.1].

We assume basic familiarity with first-order logic. A word  $w \in \Sigma^*$  can be seen as a structure whose domain is the set of *positions*  $\{1, \ldots, |w|\}$ , over the relational vocabulary consisting of:

- for each  $c \in \Sigma$ , a unary symbol c where c(i) is interpreted as true whenever w[i] = c,
- and a binary relation symbol  $\leq$ , interpreted as the total order on positions.

As an example, for  $\Sigma = \{a,b\}$ , let  $F(x) = b(x) \vee \forall y$ .  $(a(y) \vee x \leq y)$ . A string  $w \in \Sigma^*$  satisfies F(i) for a given  $i \in \{1,\ldots,w\}$  – notation:  $w \models F(i)$  – when either w[i] = b, or the position i contains an a which occurs before (at the left of) all the bs in the word w. Thus, the formula  $\forall x$ . F(x) evaluates to true exactly over the words in  $a^*b^*$ .

This model-theoretic perspective also leads to a way to specify string-to-string functions.

**Definition 4.1.** Let  $k \geq 1$  and  $\Gamma, \Sigma$  be alphabets. A two-dimensional first-order interpretation  $\mathcal{I}$  from  $\Gamma^*$  to  $\Sigma^*$  consists of several FO formulas over  $\Gamma$ :  $\mathcal{I}_c(x_1, x_2)$  for each  $c \in \Sigma$ , and  $\mathcal{I}_{\leq}(x_1, x_2, y_1, y_2)$ . For  $u \in \Gamma^*$ , let  $O_u^{\mathcal{I}}$  be the structure

- with domain  $\{\langle i_1, i_2 \rangle \in \{1, \dots, |u|\}^2 \mid \exists c \in \Sigma \colon u \models \mathcal{I}_c(i_1, i_2)\};$
- where  $c(\langle i_1, i_2 \rangle)$  iff  $u \models \mathcal{I}_c(i_1, i_2)$ , and  $\langle i_1, i_2 \rangle \leq \langle j_1, j_2 \rangle$  iff  $u \models \mathcal{I}_{\leq}(i_1, i_2, j_1, j_2)$ .

Then  $\mathcal I$  defines the function  $u\in\Gamma^*\mapsto \begin{cases} v & \text{if } O_u^\mathcal I\cong \text{the structure corresponding to } v\\ \varepsilon & \text{when}^{11} \text{ there is no such } v\in\Sigma^* \end{cases}$ 

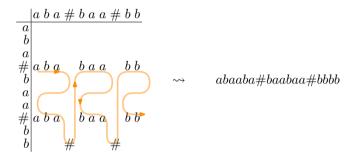
<sup>&</sup>lt;sup>10</sup>Instead of Latteux's unobtainable technical report cited by [Gau20], see [Lat79, Prop. I.2] & compare with [Raj72].

<sup>&</sup>lt;sup>11</sup>This is purely for convenience, to avoid having to consider partial functions. The language of input words u for which such a v exists is first-order definable, so a partial FO interpretation can always be completed to a total one.

**Example 4.2.**  $a^n \in \{a\}^* \mapsto (a^{n-1}b)^{n-1}$  is defined by a two-dimensional FO interpretation  $\mathcal{I}$  such that  $\mathcal{I}_{<}$  is the lexicographic order over pairs and

$$\mathcal{I}_a(x_1, x_2) = \neg \underbrace{\max(x_1)}_{\text{i.e. } \forall y. \ y \le x_1} \land \neg \max(x_2) \qquad \mathcal{I}_b(x_1, x_2) = \neg \max(x_1) \land \max(x_2)$$

**Example 4.3.** As a more subtle example, the inner squaring function (Example 1.1) admits a two-dimensional FO interpretation that we intuitively illustrate over the input aba#baa#bb as follows:



The array on the left handside indicates the output letter associated with each pair of positions in the input word (if they are in the domain) and the path represents the order in which these coordinates should be read to form the inputs. The interpretation is defined by the following formulas:

- $\mathcal{I}_a(x_1, x_2) = a(x_1) \land \#(x_2)$  and  $\mathcal{I}_b(x_1, x_2) = b(x_1) \land \#(x_2)$
- $\mathcal{I}_{\#}(x_1, x_2) = \#(x_1) \wedge \max(x_2)$
- $\mathcal{I}_{\leq}(x_1, x_2, y_1, y_2) = \text{either } \#(y_1) \land x_1 \leq y_1, \text{ or there exist } x_3, y_3 \text{ such that }$ 
  - $x_3 \le x_1$ , and there are no #s strictly in-between  $x_3$  and  $x_1$ ;
  - $y_3 \le y_1$ , and there are no #s strictly in-between  $y_3$  and  $y_1$ ;
  - neither  $x_3$  nor  $y_3$  has an immediate predecessor which is an a or a b;
  - $(x_3, x_2, x_1) \le (y_3, y_2, y_1)$  for the lexicographic order on 3-tuples.

Note the connection with the 3-pebble transducer of Section 2:  $x_3, x_2, x_1$  correspond to the positions of the respective pebbles  $\downarrow$ ,  $\Downarrow$ ,  $\triangledown$ .

### Theorem 4.4. (Bojańczyk, Kiefer & Lhote [BKL19])

Every function specified by a first-order interpretation is polyregular, i.e. in  $\bigcup_{k \in \mathbb{N}} \mathsf{Pebble}_k$ .

This holds for interpretations of arbitrary dimension  $k \in \mathbb{N}$ , even though we only defined the k=2 case; and also for MSO interpretations, where first-order logic is replaced by monadic second-order logic. A converse is also shown in [BKL19]: every polyregular function coincides, on inputs of length at least 2, with an MSO interpretation.

**Remark 4.5.** One can drop the condition "of length at least 2" using the "multi-component" variant of MSO interpretations [Boj22, §5.1]. These also have the more significant advantage that a polyregular function of growth  $O(n^k)$  can be expressed as a k-dimensional interpretation [Boj23, §2]: the natural counterpart to pebble minimization works for multi-component MSO interpretations.

Similarly to the characterization of polyregular functions by MSO interpretations, it would be desirable to have a logical formalism for polyblind functions (those that are in  $Blind_k$  for some k). Unfortunately – and this is a new result:

Theorem 4.6. The characterization of polyblind functions conjectured in [NNP21, §10] is wrong.

#### **Proof:**

For purely syntactic reasons, the two-dimensional FO interpretations such that:

- in every formula defining the interpretation, each variable can be given a sort in  $\{1,2\}$  in such a way that variables of different sorts are never compared (for equality or ordering)
- and in  $\mathcal{I}(x_1, x_2, y_1, y_2)$  and  $\mathcal{I}_c(x_1, x_2)$   $(c \in \Sigma)$ ,  $x_i$  and  $y_i$  have sort  $i \in \{1, 2\}$

can be seen as a special case of the logical interpretations considered in [NNP21, Conjecture 10.1]. Example 4.3 fits these criteria (with  $x_3$  and  $y_3$  having sort 1). If the conjecture were true, innsq would therefore be polyblind, contradicting Corollary 3.9.

# 5. Quadratic polyregular functions vs macro tree transducers

Our goal now is to show the following:

**Theorem 5.1.** For any  $k \ge 1$ , there exists a string-to-string function  $f_k$  such that:

- $f_k$  is computed by a two-dimensional first-order interpretation (see the previous section);
- for any k-tuple of functions  $(g_1, \ldots, g_k)$  such that each  $g_i$  is computed by some macro tree transducer,  $\operatorname{Im}(g_1 \circ \cdots \circ g_k) \neq \operatorname{Im}(f_k)$ .

In the second item, the domain of  $g_i$  must be equal to the codomain of  $g_{i+1}$  for the composition to be well-defined; and to make the comparison of output languages meaningful, the codomains of  $g_1$  and  $f_k$  should be equal. But  $g_1$  outputs ranked trees, whereas  $f_1$  outputs strings. To make sense of this, as usual, we identify  $\Sigma^*$  with the set of trees over the ranked alphabet that consists of a letter  $\hat{c}$  of rank 1 for each  $c \in \Sigma$ , plus a single letter  $\hat{c}$  of rank 0. Through this identification, the domain of  $g_k$  and  $f_k$  may also be equal; in that case we may conclude that  $g_1 \circ \cdots \circ g_k \neq f_k$ .

The relevance of Theorem 5.1 to the question of pebble minimization comes from:

# Theorem 5.2. (Engelfriet & Maneth [EM03, item (2) of the abstract])

Any tree-to-tree function computed by some k-pebble<sup>12</sup> tree transducer can also be expressed as a k-fold composition of macro tree transducers. (But the converse is false.)

<sup>&</sup>lt;sup>12</sup>As explained in Footnote 1 of the introduction, the indexing convention for the pebble transducer hierarchy in [EM03] is off by one compared to ours.

A k-pebble string transducer is none other than a k-pebble tree transducer working on the encodings of strings as unary trees described above. Thus it follows directly (without having to recall the definition of Pebble $_k$  from §3.2) that  $f_k \notin \text{Pebble}_k$ , and even that  $\text{Im}(f_k) \neq \text{Im}(h)$  for any  $h \in \text{Pebble}_k$ . On the other hand, any two-dimensional FO interpretation is polyregular (Theorem 4.4) with quadratic growth (an immediate consequence of the definition is that the output length is at most the square of the input length). So the  $f_k$  are indeed counterexamples to pebble minimization.

**Remark 5.3.** We have defined the  $f_k$  by first-order interpretations because they provide a convenient notion of "two-dimensional origin semantics", used in our inductive construction. But it would have been possible to show by a simple ad-hoc argument that each  $f_k$  is in some Pebble $\ell$  (as defined in Section 3.2). Therefore, we do not depend in an essential way on the difficult translation from FO interpretations to pebble transducers (Theorem 4.4) to refute pebble minimization.

We recall the required properties of macro tree transducers in Section 5.1, then we prove the main Theorem 5.1 in Section 5.2.

### **5.1.** Compositions of macro tree transducers (MTTs)

We do not formally define MTTs here, but only recall useful facts for our purposes. In this paper, we only consider *total deterministic* MTTs. Let MTT<sup>k</sup> be the class of tree-to-tree functions computed by some composition of  $k \ge 1$  macro tree transducers.

**Remark 5.4.** MTT<sup>k</sup> can be characterized equivalently as the class of functions computed by k-iterated pushdown transducers [EV86], or by "level-k" tree transducers [EV88]. For any k, the functions in MTT<sup>k</sup> with linear growth are precisely the regular tree functions [EIM21]<sup>13</sup> – this, combined with Theorem 5.2, proves pebble minimization for polyregular functions of linear growth, <sup>14</sup> showing that our counterexamples with quadratic growth are in some sense minimal.

For a class of tree-to-tree functions  $\mathcal{C}$ , we also write  $\mathrm{SO}(\mathcal{C})$  for the subclass of functions that output strings (via the identification described at the beginning of §5). The literature on tree transducers often refers to the classes  $\mathrm{SO}(\mathsf{MTT}^k)$  by equivalent descriptions involving a *yield* operation that maps trees to strings:  $\mathtt{yield}(t)$  is the word formed by listing the labels of the leaves of t in infix order.

**Claim 5.5.**  $SO(\mathsf{MTT}^1) = \{ \mathsf{yield} \circ f \mid f \text{ is computed by some } top\text{-}down \text{ tree transducer} \}, \text{ and for any } k \in \mathbb{N}, \text{ we also have } SO(\mathsf{MTT}^{k+2}) = \{ \mathsf{yield} \circ f \mid f \in \mathsf{MTT}^{k+1} \}.$ 

### Proof:

The claim about  $SO(\mathsf{MTT}^1)$  is well-known. For instance it is stated by Maneth [Man15, end of §5] as follows: "Note that macro tree transducers with monadic output alphabet are essentially the same as

<sup>&</sup>lt;sup>13</sup>The paper [EIM21] talks about compositions of tree-walking transducers (TWT), but MTT<sup>1</sup> is included in the class of functions obtained by composing 3 TWTs [EM03, Lemma 37].

<sup>&</sup>lt;sup>14</sup>This special case is also a consequence of dimension minimization for MSO interpretations (Remark 4.5), thanks to the equivalence between two-way (i.e. 1-pebble) transducers and MSO transductions [EH01].

top-down tree-to-string transducers" – where, as is the tradition in tree transducer papers, "top-down tree-to-string" means yield  $\circ$  (top-down tree-to-tree), *not* SO(top-down tree-to-tree)!

We can then deduce the second claim:

$$\begin{split} \mathrm{SO}(\mathsf{MTT}^{k+2}) &= \mathrm{SO}(\mathsf{MTT}^1) \circ \mathsf{MTT}^1 \circ \mathsf{MTT}^k \\ &= \mathsf{yield} \circ \underbrace{(\mathsf{top\text{-}down\ tree\ transducers}) \circ \mathsf{MTT}}_{= \,\mathsf{MTT},\ \mathsf{cf.}\ [\mathsf{EM02a}, \mathsf{Lemma\ 5}]} \circ \mathsf{MTT}^k \end{split}$$

(beware: in [EM02a], the notation  $\circ$  is flipped compared to ours).

This allows us to rephrase a key "bridge theorem" by Engelfriet and Maneth [EM02a] in a form that suits us better. For a class of string-valued functions C', let  $\operatorname{Im}(C') = \{\operatorname{Im}(f) \mid f \in C'\}$ .

### Theorem 5.6. ([EM02a, Theorem 18] + Claim 5.5)

Let  $k \ge 1$  and L, L' be string languages. If  $L' \in \operatorname{Im}(\operatorname{SO}(\mathsf{MTT}^{k+1}))$  and L' is *d-complete* (see below) for L, then  $L \in \operatorname{Im}(\operatorname{SO}(\mathsf{MTT}^k))$ .

**Definition 5.7.** Let  $L \subseteq \Sigma^*$  and  $L' \subseteq (\Sigma \cup \Delta)^*$  with  $\Sigma \cap \Delta = \emptyset$ . We say that L' is:

- $\delta$ -complete for L [EM02a, §5] when for every  $u \in L$  there exist  $w_0, \ldots, w_n \in \Delta^*$  such that
  - all the  $w_i$  for  $i \in \{1, ..., n-1\}$  are pairwise distinct words;
  - n = |u| and  $w_0u[1]w_1 \dots u[n]w_n \in L'$ ;
- d-complete for L [EM02b, §4] when it is  $\delta$ -complete for L and "conversely", by erasing the letters from  $\Delta$  in the words in L' one gets exactly L, i.e.  $\varphi(L') = L$  where  $\varphi$  is the monoid morphism mapping each letter in  $\Sigma$  to itself and  $\Delta$  to  $\varepsilon$ .

### 5.2. Proof of Theorem 5.1

Our strategy is a minor adaptation of Engelfriet and Maneth's proof [EM02b, §4] that the hierarchy of output languages of k-pebble string transducers is strict. (See also [EM03, Theorem 41] for a similar result on tree languages.) Writing  $\operatorname{Im}(\mathcal{I}) = \operatorname{Im}(f)$  when the first-order interpretation  $\mathcal{I}$  defines the string function f, we show that:

**Lemma 5.8.** From any two-dimensional first-order interpretation  $\mathcal{I}$ , one can build another 2D FO interpretation  $\Psi(\mathcal{I})$  such that  $\operatorname{Im}(\Psi(\mathcal{I}))$  is d-complete for  $\operatorname{Im}(\mathcal{I})$ .

Let us explain how this lemma leads to Theorem 5.1. For  $k \geq 1$ , let

$$\mathcal{I}_k = \Psi^{k-1}$$
 (the FO interpretation of Example 4.2 that defines  $a^n \mapsto (a^{n-1}b)^{n-1}$ )

We have  $^{15}$  Im( $\mathcal{I}_1$ )  $\notin$  Im(SO(MTT<sup>1</sup>)) by immediate application of [Eng82, Theorem 3.16]. <sup>16</sup> By induction on k, we then get Im( $\mathcal{I}_k$ )  $\notin$  Im(SO(MTT<sup>k</sup>)) for all k – which is precisely what we want – using Lemma 5.8 together with the contrapositive of Theorem 5.6 for the induction step.

<sup>15</sup> This implies  $(a^n \mapsto (a^{n-1}b)^{n-1}) \notin \mathsf{MTT}^1$ , a fact that was later reproved in [NNP21, Theorem 8.1(a)].

<sup>&</sup>lt;sup>16</sup>This old result of Engelfriet says that some superclass of  $\operatorname{Im}(\operatorname{SO}(\mathsf{MTT}^1))$  does not contain any language of the form  $\{(a^nb)^{f(n)} \mid n \in X\}$  where  $X \subseteq \mathbb{N} \setminus \{0\}$  is infinite and  $f: X \to \mathbb{N} \setminus \{0\}$  is injective.

Our only remaining task is to prove Lemma 5.8.

#### **Proof:**

Let  $\mathcal{I}$  be a 2D FO interpretation from  $\Gamma^*$  to  $\Sigma^*$ . Choose  $\clubsuit \notin \Gamma$  and  $\Box, \Diamond \notin \Sigma$  with  $\Box \neq \Diamond$ . We define a new function  $f': (\Gamma \cup \{\clubsuit\})^* \to (\Sigma \cup \{\Box, \lozenge\})^*$ , which coincides with f on  $\Gamma^*$ , in the following way. First, let us give an example: for the interpretation of Example 4.2 defining  $a^n \mapsto (a^{n-1}b)^{n-1}$ , the new function would map (the colors below serve to highlight blocks of matching sizes)

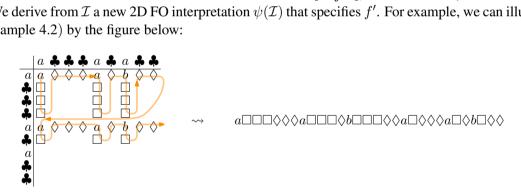
$$\clubsuit \dots \clubsuit a \clubsuit \clubsuit \clubsuit a \clubsuit a \clubsuit \Leftrightarrow$$
 to  $a \square \square \lozenge \lozenge \lozenge \lozenge a \square \square \lozenge b \square \square \lozenge \lozenge a \square \lozenge \lozenge \lozenge a \square \lozenge b \square \lozenge \lozenge$ .

In general, for any input  $u = u_1 \dots u_n \in \Gamma^*$ , let  $f(u) = v_1 \dots v_m$  (where the  $u_i$  and  $v_i$  are letters). Reusing the notation  $O_u^{\mathcal{I}}$  from Definition 4.1, let  $O_u^{\mathcal{I}} = \{\langle i_1, j_1 \rangle \leq \cdots \leq \langle i_m, j_m \rangle\}$  – morally,  $\langle i_p, j_p \rangle$  is the "origin" (as in Section 3.1) of the output letter  $v_p$ . Then, for any  $p \in \mathbb{N}^{\{0,\dots,n\}}$ , we take

$$f'\left( \clubsuit^{p[0]} u_1 \clubsuit^{p[1]} \dots u_n \clubsuit^{p[n]} \right) = v_1 \Box^{p[i_1]} \Diamond^{p[j_1]} \dots v_m \Box^{p[i_m]} \Diamond^{p[j_m]}$$

- note that the prefix in ♣\* is ignored. (Our use of the blocks of ♣s is inspired by the coding of atom-oblivious functions in the Deatomization Theorem of [Boj23], cf. Remark 3.3.)

We derive from  $\mathcal{I}$  a new 2D FO interpretation  $\psi(\mathcal{I})$  that specifies f'. For example, we can illustrate  $\Psi$ (Example 4.2) by the figure below:



Now let us give the general recipe for  $\Psi(\mathcal{I})$ . Given a formula F over the relational signature for  $\Gamma^*$ , let  $F^R$  be the relativized formula over  $(\Gamma \cup \{\clubsuit\})$  where all quantifiers  $\forall z.(\dots)$  and  $\exists z.(\dots)$  are replaced by  $\forall z. \neg \clubsuit(z) \Rightarrow (\dots)$  and  $\exists z. \neg \clubsuit(z) \land (\dots)$  respectively. Let P(x,y) be a formula stating that the restriction of the input to [x, y] is in  $\Gamma \clubsuit^*$ :

$$P(x,y) = x \leq y \land \neg \clubsuit(x) \land \forall z. \; (\neg(z \leq x) \land z \leq y) \Rightarrow \clubsuit(z)$$

We take the following definition for the interpretation  $\Psi(\mathcal{I})$ :

• 
$$\Psi(\mathcal{I})_c(x_1, x_2) = \neg \clubsuit(x_1) \wedge \neg \clubsuit(x_2) \wedge \mathcal{I}_c^{\mathbf{R}}(x_1, x_2)$$
 for  $c \in \Sigma$ 

• 
$$\Psi(\mathcal{I})_{\square}(x_1, x_2) = \clubsuit(x_1) \land \neg \clubsuit(x_2) \land \exists \widehat{x_1}. \ P(\widehat{x_1}, x_1) \land \bigvee_{c \in \Sigma} \mathcal{I}_c^{\mathbb{R}}(\widehat{x_1}, x_2)$$

• 
$$\Psi(\mathcal{I})_{\Diamond}(x_1, x_2) = \neg \clubsuit(x_1) \land \clubsuit(x_2) \land \exists \widehat{x_2}. \ P(\widehat{x_2}, x_2) \land \bigvee_{c \in \Sigma} \mathcal{I}_c^{\mathbf{R}}(x_1, \widehat{x_2})$$

• 
$$\Psi(\mathcal{I})_{\leq}(x_1,x_2,y_1,y_2) = \exists \widehat{x}_1,\widehat{x}_2,\widehat{y}_1,\widehat{y}_2. \ \bigwedge_{i=1,2} P(\widehat{x}_i,x_i) \land P(\widehat{y}_i,y_i)$$
 and

- either  $(\widehat{x}_1,\widehat{x}_2) \neq (\widehat{y}_1,\widehat{y}_2)$  and  $\mathcal{I}^{\mathrm{R}}_{\leq}(\widehat{x}_1,\widehat{x}_2,\widehat{y}_1,\widehat{y}_2)$
- or  $(\widehat{x}_1,\widehat{x}_2)=(\widehat{y}_1,\widehat{y}_2)$  and  $(x_2,x_1)\leq (y_2,y_1)$  lexicographically.

Finally, we claim that  $\operatorname{Im}(f')$  is d-complete for  $\operatorname{Im}(f)$ . The part concerning the erasing morphism is immediate. For  $\delta$ -completeness (Definition 5.7), let  $v \in \operatorname{Im}(f)$ , i.e.  $v = f(u) = f(u_1 \dots u_n)$  for some  $u \in \Gamma^*$ ; then  $f'(u_1 \cdot u_2 \cdot u_2 \cdot u_n \cdot$ 

**Remark 5.9.** The construction in our proof of Lemma 5.8 preserves the class of interpretations that verify the property described in the proof of Theorem 4.6.

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