Nash Equilibria in Concurrent Priced Games

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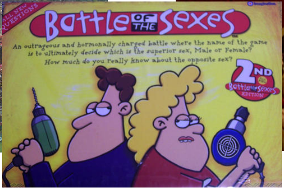
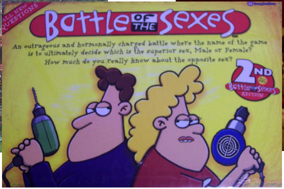
Battle of the Sexes



Battle of the Sexes



	F	**O**
F	$(1, 2)$	$(4, 4)$
O	$(6, 6)$	$(2, 1)$

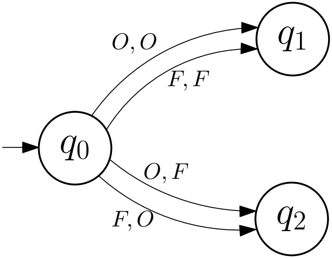
Game Characterization

- games on finite graphs with reachability objectives
- turn-based vs. **concurrent**
  - players take turns vs. *take actions simultaneously*
  - turn-based can be modelled by concurrent games
- zero-sum vs. **non-zero-sum**
  - opposite vs. *independent* objectives
- qualitative vs. **quantitative**
  - binary objectives vs. payoffs or *costs*
- object of study
  - who has a winning strategy vs. *(pure) Nash equilibria*

Overview

1. Priced Concurrent Game Structures
   Nash Equilibria
2. Algorithm for finding Nash equilibria
3. Complexity results

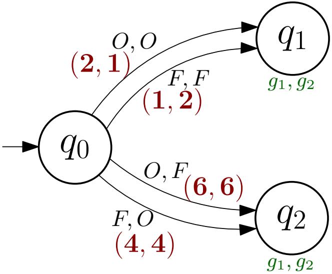
Concurrent Game Structure



Concurrent Game Structure

- K players, set of moves $\mathbb{M}$
- transition function $\delta : Q \times \mathbb{M}^K \to Q$
- δ total and deterministic for *enabled moves*
- a *computation* is a finite or infinite word over $\mathbb{M}^K$

Priced Concurrent Game Structure



Priced Concurrent Game Structure (PCGS)

- K-tuples of nonnegative *prices* on transitions
- goal states (independent for each player)

Preliminaries

Strategy

- $(\mathbb{M}^K)^* \to \mathbb{M}$
- history-dependent strategies, observing history of moves

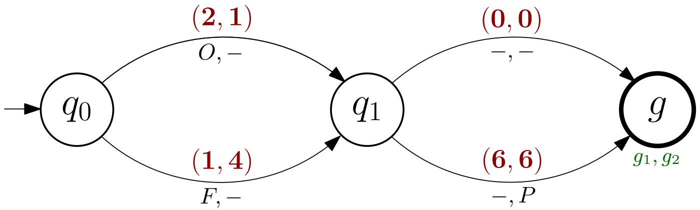
Strategy profile

- a *strategy profile* is a K-tuple of strategies (one for each player)

Cost

- $(\mathbb{M}^K)^* \times \{1 \dots K\} \to \mathbb{N} \cup \infty$
- cumulative price of transitions until the first goal state of the player
- if there is no goal state, the cost is ∞

Example



Strategy examples:

$$f_O(\Lambda) = \begin{cases} O & \text{if } \Lambda = \epsilon \\ - & \text{otherwise} \end{cases} \qquad g_P(\Lambda) = \begin{cases} P & \text{if } \Lambda = (F, -) \\ - & \text{otherwise} \end{cases}$$

$$f_F(\Lambda) = \begin{cases} F & \text{if } \Lambda = \epsilon \\ - & \text{otherwise} \end{cases} \qquad g_U(\Lambda) = \begin{cases} - & \text{always} \end{cases}$$

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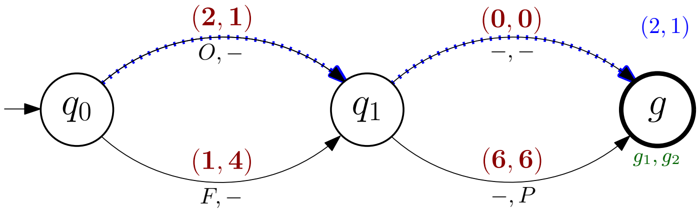

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the outcome of (f_O,g_P) is $(O, -)(-, -)$ with costs **(2,1)**

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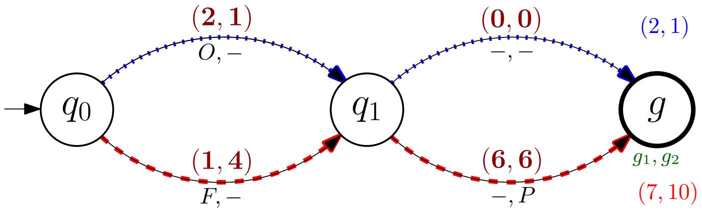

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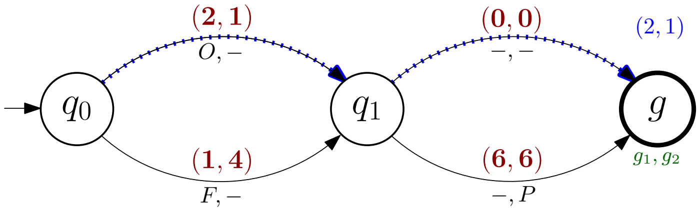
the outcome of (f_F,g_P) is $(F, -)(-, P)$ with costs **(7,10)**

Nash Equilibrium

- a stable strategy profile: no player can lower her cost by changing her strategy
- not necessarily optimal
- may not exist, or there can be more

- bounds vector $\mathbb{B} \in (\mathbb{N} \cup \infty)^K$
- the decision problem: is there a Nash equilibrium satisfying bounds $\mathbb{B}$?
- main problem: find all Nash equilibria satisfying bounds $\mathbb{B}$

Equilibrium Example



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Profile (f_O, g_P) is a Nash equilibrium as no player can reduce their cost. If player 1 uses f_F he gets a lower cost on the first step, but suffers a penalty in the second.

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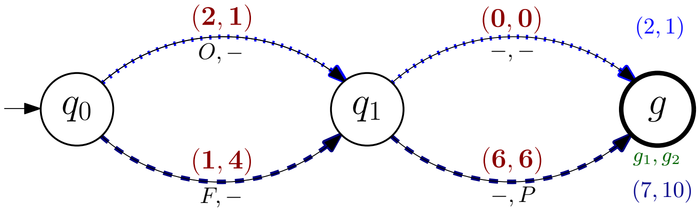

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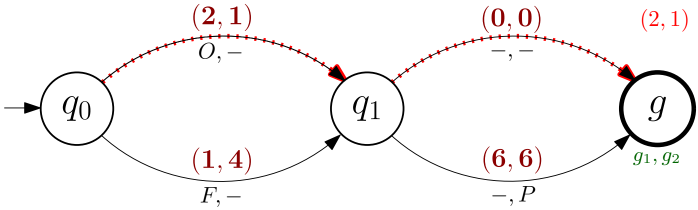

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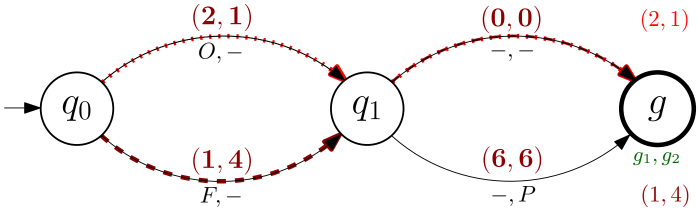

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Algorithm Overview

How to represent equilibria?

- strategies are infinite - how to represent and characterize them?
- *outcomes* of *all* equilibria form an ω-regular set
- we represent those outcomes by a Büchi automaton

Outcomes are as good as strategies!

- for each equilibrium outcome, we can find a strategy profile

Construction outline

1. we calculate *temptation* and *punishment* values for the game
2. we construct a Büchi automaton accepting outcomes of Nash equilibria

Temptation and Punishment

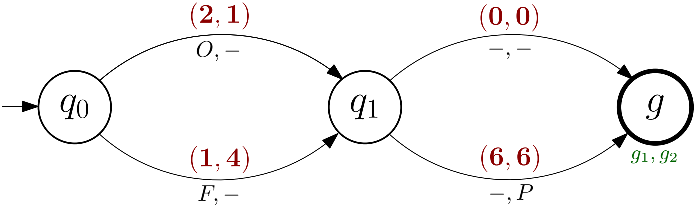
Why an outcome is not an equilibrium?

- a player betrays if she can do better - *temptation*

How to help players resist the temptation?

- strategies of other players try to make this payoff worse - *punishment*
- the punishment is announced in advance and its role is prevention
- strategies can start punishing one step after the defecting step

Punishment Values Example



Punishment - $\pi : Q \times \{1 \dots K\} \to \mathbb{N} \cup \infty$

Given a state and a defecting player, *punishment* π is the worst cost the remaining players can enforce for her.

Temptation - $\tau : Q \times \mathbb{M}^K \times \{1 \dots K\} \to \mathbb{N} \cup \infty$

Given a move vector from a state and a player, *temptation* τ for that player is the best cost she can achieve if she defects and chooses another move.

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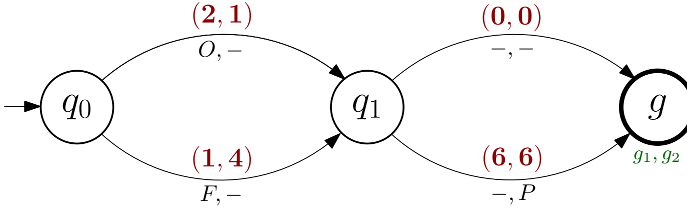

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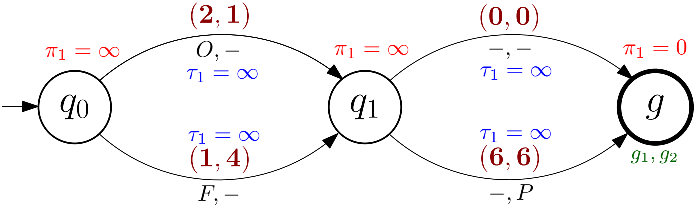

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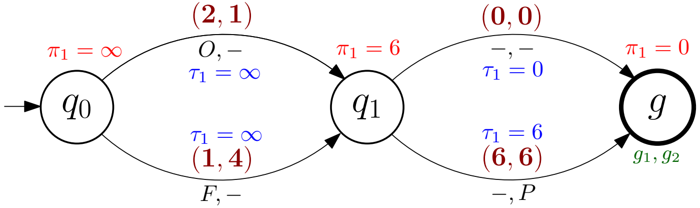

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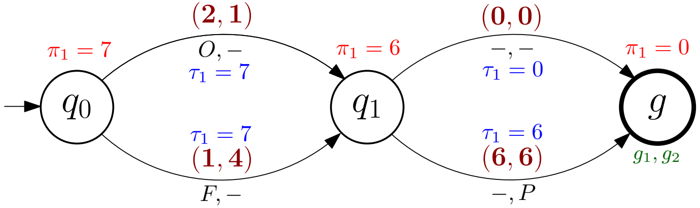

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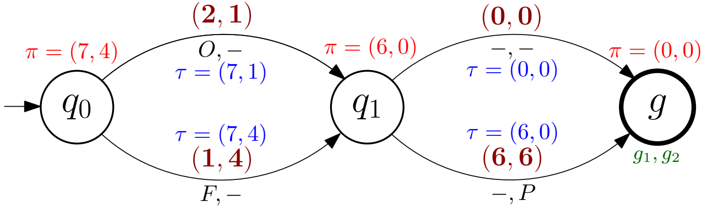

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Equilibrium Automaton

Local bounds construction

- with each state, we remember the remaining possible costs
- transitions reduce these costs with their costs and temptations

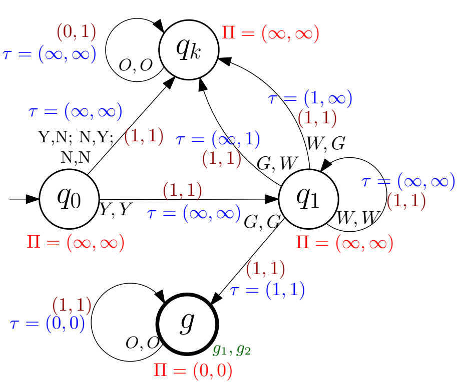
Büchi automaton

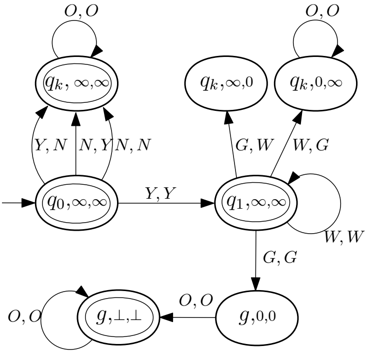
- accepting all equilibria outcomes
- number of states might be exponential

From equilibrium outcomes to strategy profiles

- strategies follow the outcome until someone betrays
- if that happens everybody starts only punishing

Equilibrium Automaton Example





Complexity of the Decision Variant

Recall the decision problem:

- Is there a Nash equilibrium satisfying bounds?

We prove that the decision problem is NP-complete

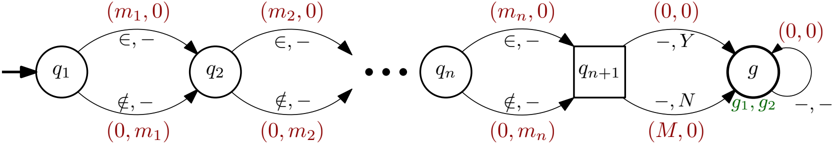
- in NP: Guessing an accepting lasso of polynomial length
- NP-hardness: Reduction from the subset sum problem

turn-based PCGS without bounds

- a Nash equilibrium always exists

NP-hardness

- Reduction from the subset sum problem.
- For input instance $(\{m_1 \ldots m_n\}, m)$ of the Subset sum problem, construct *two-player turn-based* game:



Lemma

There is a Nash Equilibrium satisfying the bounds $(m, M - m)$ if and only if there is a solution to the subset sum problem. $(M = \sum m_i)$

NP-hardness

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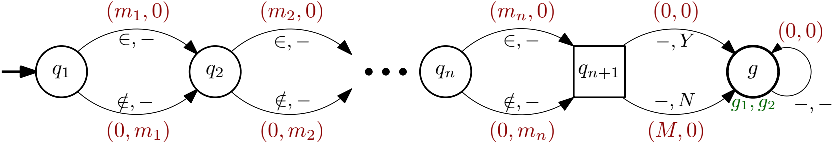

Lemma

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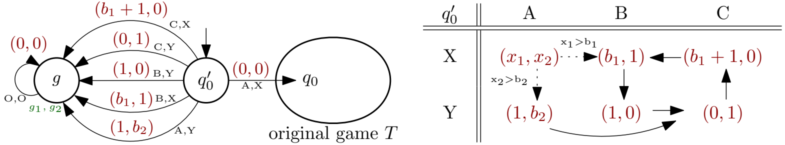
Player 1 In q_i, choose $\in$ if $m_i \in S'$, otherwise choose $\notin$.

Player 2 In q_{n+1}, choose Y if the accumulated costs so far are $(m, M - m)$, otherwise choose N.

Omitting Bounds

- we reduce the decision problem with bounds to the problem with no bounds
- for two-player PCGS and bounds (b_1, b_2), construct new PCGS



- equilibria satisfying the bounds are preserved
- equilibria not satisfying the bounds are suppressed by added edges
- no new added edge from q'_0 to g is a part of an equilibrium outcome

Complexity overview

Complexity results for the problem of deciding an existence of Nash equilibrium:

- PCGS and with/without bounds $\in$ NP
- Subset sum $\leq$ Two-player turn-based games with bounds $\leq$ PCGS with bounds
- Two-player PCGS with bounds $\leq$ PCGS without bounds

	full PCGS	**turn-based**
with bounds	NP-complete	NP-complete
without bounds	NP-complete	Trivial

Conclusion

1. Priced CGS with individual reachability objectives
   - Non-negative integer prices on transitions for each player
   - Cost for a player is accumulated sum of prices before reaching their goal state

2. Characterization of Nash Equilibria in PCGS
   - Set of Nash equilibrium outcomes is ω-regular language
   - We can extract strategies for these outcomes

3. Complexity of the decision variant of the problem
   - NP-complete problem

4. Ideas for future work
   - Mixed (probabilistic) strategies
   - Partial observability
   - Negative costs