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Markov chain sensitivity measured by mean first passage times

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Abstract

The purpose of this article is to present results concerning the sensitivity of the stationary probabilities for an n-state, time-homogeneous, irreducible Markov chain in terms of the mean first passage times in the chain. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

In studying the effects of perturbations in n-state, time-homogeneous, irreducible Markov chains there are two approaches, and the appropriateness of each depends on the underlying application and what the modeler hopes to derive from a perturbation analysis. One approach, initiated in [20] and expanded upon in [5–9,12,14,17,18,21–25], analyzes the effects on the stationary probabilities to *absolute* perturbations in transition probabilities in the sense that the size of a perturbation is always measured relative to 1. The other approach, presented in [19], considers relative entry-wise perturbations in the sense that a perturbation to a transition probability p_{ij} is measured relative to the size of p_{ij}. It is established in [19] that small relative entry-wise

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perturbations do not unduly affect the stationary probabilities of an irreducible chain. But even small absolute perturbations can drastically affect the stationary probabilities, and this situation is the concern of this article.

Nearly all prior results concerning absolute perturbations can be characterized as follows. Suppose that an irreducible row-stochastic matrix $\mathbf{P}$ is perturbed by a matrix $\mathbf{E}$ such that $\tilde{\mathbf{P}} = \mathbf{P} - \mathbf{E}$ is also an irreducible row-stochastic matrix such that $\|\mathbf{E}\|$ is small relative to 1 for some appropriate matrix norm, and let $\pi^{\mathrm{T}} = (\pi_1, \pi_2, \ldots, \pi_n)$ and $\tilde{\pi}^{\mathrm{T}} = (\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_n)$ denote the respective stationary probability vectors for $\mathbf{P}$ and $\tilde{\mathbf{P}}$. Past sensitivity results concerning absolute perturbations have been phrased in terms of bounds of the form

$$\|\pi^{\mathrm{T}} - \tilde{\pi}^{\mathrm{T}}\| \leqslant \kappa\|\mathbf{E}\| \;\text{ or }\; |\pi_j - \tilde{\pi}_j| \leqslant \kappa_j\|\mathbf{E}\| \;\text{ or }\; \left|\frac{\pi_j - \tilde{\pi}_j}{\pi_j}\right| \leqslant \kappa_j\|\mathbf{E}\| \qquad (1.1)$$

in which *condition numbers* κ or κ_j are used as measures of sensitivity. (Examples of condition numbers κ or κ_j can be found in the afore-mentioned references.)

While several of the known bounds can provide a good numerical measure of the maximal extent to which the magnitude of the perturbation can be amplified, they all suffer from at least two shortcomings. First, while it is theoretically possible to compute good condition numbers κ or κ_j in (1.1), it is usually expensive to do so relative to computing π^{T} itself. Second, there is little qualitative information conveyed by a computed condition number. In other words, known bounds of the form (1.1) provide little or no information about how the structure of a given Markov chain might be used a priori to suggest the degree to which specified stationary probabilities might be sensitive (or insensitive) to perturbations.

The purpose of this article is to provide a remedy for these shortcomings by showing how to measure the sensitivity of stationary probabilities by using only the mean first passage times in the chain. Viewing sensitivity in this manner can sometimes help practitioners decide whether or not to expect sensitivity in their Markov chain models merely by observing the structure of the chain, thus obviating the need for computing or estimating condition numbers.

2. Sensitivity in terms of mean first passage times

The primary result of this article is given in the following theorem that presents perturbation bounds for individual stationary probabilities in terms of mean first passage times.

Theorem 2.1. *Let* $\mathbf{P}$ *and* $\tilde{\mathbf{P}} = \mathbf{P} - \mathbf{E}$ *be transition probability matrices for two irreducible n-state Markov chain with respective stationary distributions* π^{T} *and* $\tilde{\pi}^{\mathrm{T}}$, *and let* M_{ij} *denote the mean first passage time from the* ith *state to the* jth *state in the unperturbed chain* (M_{jj} *is the mean return time for the* jth *state*). *The absolute change in the* jth *stationary probability is given by*

$$|\pi_j - \tilde{\pi}_j| \leqslant \frac{\|\mathbf{E}\|_\infty}{2} \frac{\max_{i \neq j} M_{ij}}{M_{jj}}, \tag{2.1}$$

which is equivalent to saying that the relative change in π_j is

$$\frac{|\pi_j - \tilde{\pi}_j|}{\pi_j} \leqslant \frac{\|\mathbf{E}\|_\infty}{2} \max_{i \neq j} M_{ij}. \tag{2.2}$$

Proof. By permuting the order in which the states are listed, it is clear that it suffices to prove the result for $j = n$. A standard way of determining M_{in} is to consider state n as an absorbing state and compute the mean time to absorption from state i. That is, if

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_n & \mathbf{p} \\ \mathbf{q}^{\mathrm{T}} & p_{nn} \end{pmatrix}$$

is the transition matrix for the original irreducible chain, then

$$\mathbf{Q} = \begin{pmatrix} \mathbf{P}_n & \mathbf{p} \\ \mathbf{0} & 1 \end{pmatrix}$$

is the transition matrix for the associated absorbing chain, and it is well known [10] that the mean time to absorption from state i is

$$M_{in} = \left[(\mathbf{I} - \mathbf{P}_n)^{-1} \mathbf{e} \right]_i = \mathbf{e}_i^{\mathrm{T}} \mathbf{A}_n^{-1} \mathbf{e}, \tag{2.3}$$

where $\mathbf{e} = (1, 1, \ldots, 1)^{\mathrm{T}}$ is a column of 1’s, $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}}$ is the ith unit column, and $\mathbf{A}_n$ is the leading $(n-1) \times (n-1)$ principal submatrix of $\mathbf{A} = \mathbf{I} - \mathbf{P}$. The proof of the theorem involves expressing (2.3) in terms of entries from the group inverse, $\mathbf{A}^{\#}$, of $\mathbf{A}$, which can be characterized as the unique matrix satisfying the three equations

$$\mathbf{A}\mathbf{A}^{\#}\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#} \quad \text{and} \quad \mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}\mathbf{A}. \tag{2.4}$$

General properties of the group inverse and applications to the theory of finite Markov chains are well documented – see [1–4,11–13,15,16,18]. Some special properties of $\mathbf{A}^{\#}$ that are needed to prove this theorem are summarized below. Observe that if $\mathbf{A}$ and π^{T} are respectively partitioned as

$$\mathbf{A} = \mathbf{I} - \mathbf{P} = \begin{pmatrix} \mathbf{A}_n & \mathbf{c} \\ \mathbf{d}^{\mathrm{T}} & a_{nn} \end{pmatrix} \quad \text{and} \quad \pi^{\mathrm{T}} = (\overline{\pi}^{\mathrm{T}}, \pi_n),$$

then the group inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{\#} = \begin{pmatrix} (\mathbf{I} - \mathbf{e}\overline{\pi}^{\mathrm{T}})\mathbf{A}_n^{-1}(\mathbf{I} - \mathbf{e}\overline{\pi}^{\mathrm{T}}) & -\pi_n(\mathbf{I} - \mathbf{e}\overline{\pi}^{\mathrm{T}})\mathbf{A}_n^{-1}\mathbf{e} \\ -\overline{\pi}^{\mathrm{T}}\mathbf{A}_n^{-1}(\mathbf{I} - \mathbf{e}\overline{\pi}^{\mathrm{T}}) & \pi_n\overline{\pi}^{\mathrm{T}}\mathbf{A}_n^{-1}\mathbf{e} \end{pmatrix}. \tag{2.5}$$

This formula for $\mathbf{A}^{\#}$ appears in [9], and it is easily verified by using $a_{nn} - \mathbf{d}^{\mathrm{T}}\mathbf{A}_n^{-1}\mathbf{c} = 0$ (because $\mathrm{rank}(\mathbf{A}_n) = n - 1$) to check that (2.5) satisfies the three defining equations (2.4). By examining the last column of $\mathbf{A}^{\#}$ in (2.5), it is apparent that if $i \neq n$, then

$$\begin{aligned}\mathbf{A}^{\#}_{in} &= -\pi_n \left[(\mathbf{I} - \mathbf{e}\overline{\pi}^{\mathrm{T}})\mathbf{A}_n^{-1}\mathbf{e}\right]_i \\ &= -\pi_n \mathbf{e}_i^{\mathrm{T}}(\mathbf{I} - \mathbf{e}\overline{\pi}^{\mathrm{T}})\mathbf{A}_n^{-1}\mathbf{e} \\ &= -\pi_n \mathbf{e}_i^{\mathrm{T}}\mathbf{A}_n^{-1}\mathbf{e} + \pi_n\overline{\pi}^{\mathrm{T}}\mathbf{A}_n^{-1}\mathbf{e}, \end{aligned} \tag{2.6}$$

and

$$\mathbf{A}^{\#}_{nn} = \pi_n\overline{\pi}^{\mathrm{T}}\mathbf{A}_n^{-1}\mathbf{e}. \tag{2.7}$$

Eqs. (2.6) and (2.7) together with (2.3) imply that

$$\mathbf{A}^{\#}_{in} = \mathbf{A}^{\#}_{nn} - \pi_n M_{in} \tag{2.8}$$

and

$$\max_{1\leqslant i\leqslant n} \mathbf{A}^{\#}_{in} = \mathbf{A}^{\#}_{nn}. \tag{2.9}$$

Next, the following facts need to be realized:

Proposition 2.1 [14]. *The difference between* π^{T} *and* $\tilde{\pi}^{\mathrm{T}}$ *is given by*

$$\pi^{\mathrm{T}} - \tilde{\pi}^{\mathrm{T}} = \tilde{\pi}^{\mathrm{T}}\mathbf{E}\mathbf{A}^{\#}, \tag{2.10}$$

and, *in particular*,

$$\pi_n - \tilde{\pi}_n = \tilde{\pi}^{\mathrm{T}}\mathbf{E}\mathbf{A}^{\#}_{*n} \tag{2.11}$$

where $\mathbf{A}^{\#}_{*n}$ *denotes the* nth *column of* $\mathbf{A}^{\#}$.

Proposition 2.2 [7]. *If* $\mathbf{x} \in \mathbb{R}^{n\times 1}$ *is a vector such that* $\mathbf{x}^{\mathrm{T}}\mathbf{e} = 0$, *then*

$$|\mathbf{x}^{\mathrm{T}}\mathbf{y}| \leqslant \|\mathbf{x}\|_1 \left(\frac{y_{\max} - y_{\min}}{2}\right) \quad \textit{for all } \mathbf{y} \in \mathbb{R}^{n\times 1}. \tag{2.12}$$

This is a consequence of Hölder's inequality because $|\mathbf{x}^{\mathrm{T}}\mathbf{y}| = |\mathbf{x}^{\mathrm{T}}(\mathbf{y} - \alpha\mathbf{e})| \leqslant \|\mathbf{x}\|_1\|\mathbf{y} - \alpha\mathbf{e}\|_\infty$ *for all* α, *and* $\min_\alpha \|\mathbf{y} - \alpha\mathbf{e}\|_\infty = (y_{\max} - y_{\min})/2$ (*the minimum is attained at* $\alpha = (y_{\max} + y_{\min})/2$).

Since $\tilde{\mathbf{P}}$ is row stochastic, it follows that $\mathbf{E}\mathbf{e} = \mathbf{0}$, so (2.12) can be applied to (2.11) with $\mathbf{x}^{\mathrm{T}} = \tilde{\pi}^{\mathrm{T}}\mathbf{E}$ and $\mathbf{y} = \mathbf{A}^{\#}_{*n}$. This together with (2.9) yields:

$$\begin{aligned}|\pi_n - \tilde{\pi}_n| = |\tilde{\pi}^{\mathrm{T}}\mathbf{E}\mathbf{A}^{\#}_{*n}| &\leqslant \|\tilde{\pi}^{\mathrm{T}}\mathbf{E}\|_1 \left(\frac{\max_i \mathbf{A}^{\#}_{in} - \min_i \mathbf{A}^{\#}_{in}}{2}\right) \\ &\leqslant \|\tilde{\pi}^{\mathrm{T}}\|_1\|\mathbf{E}\|_\infty \left(\frac{\max_i \mathbf{A}^{\#}_{in} - \min_i \mathbf{A}^{\#}_{in}}{2}\right) \\ &= \|\mathbf{E}\|_\infty \left(\frac{\mathbf{A}^{\#}_{nn} - \min_i \mathbf{A}^{\#}_{in}}{2}\right)\end{aligned}$$

$$= \|\mathbf{E}\|_\infty \left(\frac{\mathbf{A}^\#_{nn} - \min_{i\neq n} \mathbf{A}^\#_{in}}{2} \right). \tag{2.13}$$

It follows from (2.8) that $\min_{i\neq n} \mathbf{A}^\#_{in} = \mathbf{A}^\#_{nn} - \pi_n \max_{i\neq n} M_{in}$, so (2.13) can be reformulated to state that

$$|\pi_n - \tilde{\pi}_n| \leqslant \frac{\|\mathbf{E}\|_\infty}{2} \pi_n \max_{i\neq n} M_{in},$$

which produces (2.1) (using the fact that $\pi_n = 1/M_{nn}$) and (2.2) for $j = n$. A permutation argument shows that these results hold for all values of j. $\square$

3. How good is the bound?

For many chains, the bound given in Theorem 2.1 is the best bound possible in the sense that equality in (2.1) and (2.2) will be attained for some perturbation $\mathbf{E}$.

Theorem 3.1. *Under the hypothesis of Theorem* 2.1, *equality in* (2.2) *can be attained for each stationary probability corresponding to a positive column of* $\mathbf{P}$. *That is*, *if the* jth *column of* $\mathbf{P}$ *is positive*, *then there exists a perturbation* $\mathbf{E}$ *such that* $\tilde{\mathbf{P}} = \mathbf{P} - \mathbf{E}$ *is also the transition probability matrix of an irreducible chain*, *and*

$$\pi_j - \tilde{\pi}_j = \frac{\|\mathbf{E}\|_\infty}{2} \frac{\max_{i\neq j} M_{ij}}{M_{jj}}, \tag{3.1}$$

or equivalently,

$$\frac{\pi_j - \tilde{\pi}_j}{\pi_j} = \frac{\|\mathbf{E}\|_\infty}{2} \max_{i\neq j} M_{ij}. \tag{3.2}$$

Proof. Again, it suffices to prove the theorem for $j = n$ because a permutation of the states will then make the result valid for any other value of j. Suppose that $\max_{i\neq n} M_{in}$ is attained when $i = k$, and use (2.8) to write

$$\max_{i\neq n} M_{in} = M_{kn} = \frac{\mathbf{A}^\#_{nn} - \mathbf{A}^\#_{kn}}{\pi_n}. \tag{3.3}$$

It is clear that if $p_{in} > 0$ for each i, then there exists a positive number ϵ such that

$$0 < p_{in} - \epsilon < 1 \quad \text{and} \quad 0 < p_{ik} + \epsilon < 1 \quad \text{for all } i = 1, 2, \ldots, n.$$

For any such ϵ, define the perturbation term to be $\mathbf{E} = \epsilon\mathbf{e}(\mathbf{e}_n^{\mathrm{T}} - \mathbf{e}_k^{\mathrm{T}})$, and notice that $\|\mathbf{E}\|_\infty = \epsilon/2$. The matrix $\tilde{\mathbf{P}} = \mathbf{P} - \mathbf{E}$ obtained by subtracting ϵ from the nth column of $\mathbf{P}$ and adding ϵ to the kth column of $\mathbf{P}$ results in another irreducible stochastic matrix, and if $\tilde{\pi}^{\mathrm{T}}$ is the stationary distribution of $\tilde{\mathbf{P}}$, then

$$\tilde{\pi}^{\mathrm{T}}\mathbf{E} = \epsilon\tilde{\pi}^{\mathrm{T}}\mathbf{e}(\mathbf{e}_n^{\mathrm{T}} - \mathbf{e}_k^{\mathrm{T}}) = \epsilon(\mathbf{e}_n^{\mathrm{T}} - \mathbf{e}_k^{\mathrm{T}}).$$

It follows from (2.10) and (3.3) that

$$\pi_n - \tilde{\pi}_n = \tilde{\pi}^{\mathrm{T}}\mathbf{E}\mathbf{A}^{\#}_{*n} = \epsilon(\mathbf{A}^{\#}_{nn} - \mathbf{A}^{\#}_{kn}) = \epsilon\pi_n M_{kn}.$$

Since $\epsilon = \|\mathbf{E}\|_\infty/2$, this yields

$$\frac{\pi_n - \tilde{\pi}_n}{\pi_n} = \frac{\|\mathbf{E}\|_\infty}{2} \max_{i \neq n} M_{in},$$

and thus the theorem is proven. □

4. Interpretations and consequences

There are some immediate corollaries of the previous results which have useful implications. The first observation given below is an interpretation of the bound in (2.2).

> Consider a state to be *centrally located* in an irreducible chain if every other state is "close" to it in the sense of mean first passage times. Many models reveal centrally located states merely by inspection. Theorem 2.1 guarantees that if state j is centrally located, then π_j has to be well-behaved. And, for many chains, the results of Theorem 3.1 imply that if at least one state is "far removed" from state j, then π_j will exhibit sensitivities.

The statement above pertains to general perturbations as defined in Theorem 2.1, but more can be said if the nature of the perturbation is more specific. In particular, consider a *minimal perturbation* involving states i, j, and k to be the situation in which the probability of moving from i to j is slightly perturbed while the probability of moving from i to k is changed by the same amount but in an opposite direction. The effects on the stationary probabilities can be made explicit as described below.

Corollary 4.1. *Suppose that the transition probability p_{ij} in an irreducible chain is changed by an amount ϵ while p_{ik} is changed by an amount $-\epsilon$, and suppose that such a perturbation results in another irreducible chain. The relative change in each stationary probability is given by*

$$\frac{\pi_r - \tilde{\pi}_r}{\pi_r} = \epsilon\tilde{\pi}_i(M_{jr} - M_{kr}) < \epsilon(M_{jr} - M_{kr}) \quad \textit{for } r = 1, 2, \ldots, n.$$

In other words, if the "distance" from j to r is comparable to the "distance" from k to r, then π_r is relatively insensitive to such a minimal perturbation.

Proof. In the context of Theorem 2.1, the perturbation matrix $\mathbf{E}$ corresponding to the described perturbation has the form $\mathbf{E} = \epsilon(\mathbf{e}_i\mathbf{e}_k^{\mathrm{T}} - \mathbf{e}_i\mathbf{e}_j^{\mathrm{T}})$. So, (2.8) together with (2.10) says that the effect on the rth stationary probability is given by

$$
\begin{aligned}
\pi_r - \tilde{\pi}_r &= \epsilon \tilde{\pi}^{\mathrm{T}} \left(\mathbf{e}_i \mathbf{e}_k^{\mathrm{T}} - \mathbf{e}_i \mathbf{e}_j^{\mathrm{T}} \right) \mathbf{A}^{\#}_{*r} \\
&= \epsilon \tilde{\pi}_i \left(\mathbf{A}^{\#}_{kr} - \mathbf{A}^{\#}_{jr} \right) \\
&= \epsilon \tilde{\pi}_i \pi_r (M_{jr} - M_{kr}). \qquad \square
\end{aligned}
$$

As a slightly more special case, consider the effect on the jth stationary probability when a perturbation involves only states j and k.

Corollary 4.2. *Suppose that the transition probability p_{jk} in an irreducible chain is changed by an amount ϵ while p_{jj} is changed by an amount $-\epsilon$, and suppose that such a perturbation results in another irreducible chain. The relative change in π_j is given by*

$$
\frac{\pi_j - \tilde{\pi}_j}{\pi_j} = \tilde{\pi}_j \epsilon M_{kj} < \epsilon M_{kj}.
$$

In other words, the "closer" state k is to state j, the less sensitive π_j is to a perturbation involving states k and j.

Proof. Similar to the preceding result, the perturbation matrix $\mathbf{E}$ corresponding to the described perturbation is $\mathbf{E} = \epsilon(\mathbf{e}_j \mathbf{e}_j^{\mathrm{T}} - \mathbf{e}_j \mathbf{e}_k^{\mathrm{T}})$, so, (2.8) together with (2.10) says that

$$
\begin{aligned}
\pi_j - \tilde{\pi}_j &= \epsilon \tilde{\pi}^{\mathrm{T}} \left(\mathbf{e}_j \mathbf{e}_j^{\mathrm{T}} - \mathbf{e}_j \mathbf{e}_k^{\mathrm{T}} \right) \mathbf{A}^{\#}_{*j} \\
&= \epsilon \tilde{\pi}_j \left(\mathbf{A}^{\#}_{jj} - \mathbf{A}^{\#}_{jk} \right) \\
&= \epsilon \tilde{\pi}_j (\pi_j M_{kj}). \qquad \square
\end{aligned}
$$

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