Ultimate Positivity is Decidable for Simple Linear Recurrence Sequences

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Abstract

We consider the decidability and complexity of the Ultimate Positivity Problem, which asks whether all but finitely many terms of a given rational linear recurrence sequence (LRS) are positive. Using lower bounds in Diophantine approximation concerning sums of S-units, we show that for simple LRS (those whose characteristic polynomial has no repeated roots) the Ultimate Positivity Problem is decidable in polynomial space. If we restrict to simple LRS of a fixed order then we obtain a polynomial-time decision procedure. As a complexity lower bound we show that Ultimate Positivity for simple LRS is hard for co$\exists\mathbb{R}$: the class of problems solvable in the universal theory of the reals, which lies between **coNP** and **PSPACE**.

1 Introduction

A ***linear recurrence sequence (LRS)*** is an infinite sequence $\mathbf{u} = \langle u_0, u_1, u_2, \ldots \rangle$ of rational numbers satisfying a recurrence relation

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n \tag{1}$$

for all $n \geq 0$, where $a_1, a_2, \ldots, a_k$ are fixed rational numbers with $a_k \neq 0$. Such a sequence is determined by its initial values $u_0, \ldots, u_{k-1}$ and the recurrence relation. We say that the recurrence has ***characteristic polynomial***

$$f(x) = x^n - a_1 x^{n-1} - \ldots - a_{k-1} x - a_k \, .$$

The least k such that $\mathbf{u}$ satisfies a recurrence of the form (1) is called the ***order*** of $\mathbf{u}$. If the characteristic polynomial of this (unique) recurrence has no repeated roots then we say that $\mathbf{u}$ is ***simple***.

A basic fact about LRS is that there are polynomials $p_1, \ldots, p_k \in \mathbb{C}[x]$ such that

$$u_n = p_1(n)\gamma_1^n + \ldots + p_k(n)\gamma_k^n \, ,$$

where $\gamma_1, \ldots, \gamma_k$ are the roots of characteristic polynomial. Then $\mathbf{u}$ is simple if and only if it admits such a representation in which each polynomial p_i is a constant [10]. Simple LRS are a natural subclass whose analysis nevertheless remains extremely challenging [1, 10, 11].

Motivated by questions in formal languages and power series, Rozenberg, Salomaa, and Soittola [33, 34, 38] highlight the following four decision problems concerning LRS. Given an LRS $\langle u_n \rangle_{n=0}^{\infty}$ (represented by a linear recurrence and sequence of initial values):

1. Does $u_n = 0$ for some n?
2. Does $u_n = 0$ for infinitely many n?
3. Is $u_n \geq 0$ for all n?
4. Is $u_n \geq 0$ for all but finitely many n?

Linear recurrence sequences are ubiquitous in mathematics and computer science, and the above four problems (and assorted variants) arise in a variety of settings; see [30] for references. For example, an LRS modelling population size is biologically meaningful only if it never becomes negative.

Problem 1 is known as ***Skolem's Problem***, after the Skolem-Mahler-Lech Theorem [21, 23, 37], which characterises the set $\{n \in \mathbb{N} : u_n = 0\}$ of zeros of an LRS $\mathbf{u}$ as an ultimately periodic set. The proof of the Skolem-Mahler-Lech Theorem is non-effective, and the decidability of Skolem's Problem is open. Blondel and Tsitsiklis [5] remark that "the present consensus among number theorists is that an algorithm [for Skolem's problem] should exist". However, so far decidability is known only for LRS of order at most 4: a result due independently to Vereschagin [42] and Mignotte, Shorey and Tijdeman [27]. At order 5 decidability is not known, even for simple LRS [28]. Decidability of Skolem's Problem is also listed as an open problem and discussed at length by Tao [39, Section 3.9]. The problem can also be seen as a far-reaching generalisation of the Orbit Problem, studied by Kannan and Lipton [19, Section 5].

In contrast to the situation with Skolem's Problem, Problem 2—reaching zero infinitely often—was shown to be decidable for arbitrary LRS by Berstel and Mignotte [4].

Problems 3 and 4 are respectively known as the ***Positivity*** and ***Ultimate Positivity*** Problems. In [38] they are described, along with Skolem's Problem, as being "very difficult", while in [33] the authors assert that the problems are "generally conjectured [to be] decidable". The problems are moreover stated as open in [2, 16, 20, 22, 40], among others.[1] In fact decidability of

[1] It is straightforward that the Positivity and Ultimate Positivity Problems are polynomial-time interreducible respectively with the corresponding problems with the strict inequality $u_n > 0$ in place of $u_n \geq 0$.

Positivity entails decidability of Skolem's Problem via a straightforward algebraic transformation of LRS (which however does not preserve the order) [16].

Hitherto, all decidability results for Positivity and Ultimate Positivity have been for low-order sequences. The paper [30] gives a detailed account of these results, obtained over a period of time stretching back some 30 years, and proves decidability of both problems for sequences of order at most 5. It is moreover shown in [30] that obtaining decidability for either Positivity or Ultimate Positivity at order 6 would necessarily entail major breakthroughs in Diophantine approximation.

The main result of this paper is that the Ultimate Positivity Problem for simple LRS of arbitrary order is decidable. The restriction to simple LRS allows us to circumvent the strong "mathematical hardness" result for sequences of order 6 alluded to above. However, our decision procedure is non-constructive: given an ultimately positive LRS $\langle u_n \rangle_{n=0}^{\infty}$, the procedure does not compute a threshold N such that $u_n \geq 0$ for all $n \geq N$. Indeed the ability to compute such a threshold would immediately yield an algorithm for the Positivity Problem for simple LRS since the positivity of $u_0, \ldots, u_{N-1}$ can be checked directly. In turn this would yield decidability of Skolem's problem for simple LRS. But both problems are open: recall that Skolem's Problem is open for simple LRS of order 5, while (as discussed below) Positivity for simple LRS is only known to be decidable up to order 9.

The non-constructive aspect of our results arises from our use of lower bounds in Diophantine approximation concerning sums of S-*units*. These bounds were proven in [12, 41] using Schlickewei's p-adic generalisation [36] of Schmidt's Subspace theorem (itself a far-reaching generalisation of the Thue-Siegel-Roth Theorem), and therein applied to study the asymptotic growth of LRS in absolute value. By contrast, in [29] we use Baker's theorem on linear forms in logarithms to show decidability of Positivity for simple LRS of order at most 9. Unfortunately, while Baker's theorem yields effective Diophantine-approximation lower bounds, it appears only to be applicable to low-order LRS. In particular, the analytic and geometric arguments that are used in [29] to bring Baker's theorem to bear (and which give that work a substantially different flavour to the present paper) do not apply beyond order 9.

Is is worth remarking that non-effective results are quite common for LRS. The Skolem-Mahler-Lech Theorem is one instance; other examples arise in connection with analysis of growth rates and multiplicity of zeros [10].

Relying on complexity bounds for the decision problem for first-order formulas over the field of real numbers, we show that our procedure for deciding Ultimate Positivity requires polynomial space in general and polynomial time for LRS of each fixed order. As a complexity lower bound, we show that Ultimate Positivity for simple LRS is hard for co$\exists\mathbb{R}$. Here co$\exists\mathbb{R}$ is the class of problems solvable in the universal theory of the reals. It is straightforward that this class contains **coNP** and, from the work of Canny [7], it is contained in **PSPACE**. Hitherto the best lower bound known for either Positivity or Ultimate Positivity was **coNP**-hardness [3]. The new lower bound reflects the close relationship of the Positivity and Ultimate Positivity problems to questions in real algebraic geometry. In the Conclusion we discuss the prospects for obtaining matching upper and lower complexity bounds for the Ultimate Positivity Problem for simple LRS.

2 Background

In this section we state without proof some basic results on linear recurrence sequences. Details can be found in the recent monograph of Everest *et al.* [10] as well as the survey [17] on Skolem's Problem. We also review relevant notions in number theory and results on decision procedures for first-order logic over the field of real numbers.

2.1 Number Theory

A complex number α is ***algebraic*** if it is a root of a univariate polynomial with integer coefficients. The ***defining polynomial*** of α, denoted p_α, is the unique integer polynomial of least degree, whose coefficients have no common factor, that has α as a root. The ***degree*** of α is the degree of p_α, and the ***height*** of α is the maximum absolute value of the coefficients of p_α. If p_α is monic then we say that α is an ***algebraic integer***.

We represent a real algebraic number α by a polynomial f that has α as a root, along with an interval (q_1, q_2), with rational endpoints, in which α is the only root of f. A separation bound due to Mignotte [26] states that for roots $\alpha_i \neq \alpha_j$ of a polynomial $f(x)$,

$$|\alpha_i - \alpha_j| > \frac{\sqrt{6}}{d^{(d+1)/2} H^{d-1}} , \tag{2}$$

where d and H are the degree and height of f, respectively. Thus the representation is well-defined if $q_1 < \alpha < q_2$ and $q_2 - q_1$ is at most the root separation bound. A complex algebraic number α is represented in terms of its real and imaginary parts. We denote by $||\alpha||$ the length of this representation.[2] Given a univariate polynomial f, it is known how to obtain representations of each of its roots in time polynomial in $||f||$, see [31].

A ***number field*** K is a finite-dimensional extension of $\mathbb{Q}$. The set of algebraic integers in K forms a ring, denoted $\mathcal{O}$. Given two ideals I, J in $\mathcal{O}$, the product IJ is the ideal generated by the elements ab, where $a \in I$ and $b \in J$. An ideal P of $\mathcal{O}$ is ***prime*** if $ab \in P$ implies $a \in P$ or $b \in P$. The fundamental theorem of ideal theory states that any non-zero ideal in $\mathcal{O}$ can be written as the product of prime ideals, and the representation is unique if the order of the prime ideals is ignored.

We will need the following classical result of Dirichlet on primes in arithmetic progressions [13].

Theorem 1 (Dirichlet) *Let P be the set of primes and $P_{a,b}$ the set of primes congruent to a mod b, where* $\gcd(a, b) = 1$. *Then*

$$\lim_{n\to\infty} \frac{|P_{a,b} \cap \{1, \ldots, n\}|}{|P \cap \{1, \ldots, n\}|} = \frac{1}{\varphi(b)} ,$$

where φ denotes Euler's totient function.

2.2 Linear Recurrence Sequences

Let $\mathbf{u} = \langle u_n \rangle_{n=0}^\infty$ be a sequence of rational numbers satisfying the recurrence relation $u_{n+k} = a_1 u_{n+k-1} + \ldots + a_k u_n$. We represent such an LRS as a $2k$-tuple $(a_1, \ldots, a_k, u_0, \ldots, u_{k-1})$ of rational numbers (encoded in binary). Given an arbitrary representation of $\mathbf{u}$, we can compute the coefficients of the unique minimal-order recurrence satisfied by $\mathbf{u}$ in polynomial time by straightforward linear algebra. Henceforth we will always assume that an LRS is presented in terms of its minimal-order recurrence. By the characteristic polynomial of an LRS we mean the characteristic polynomial of the minimal-order recurrence. The roots of this polynomial are called the ***characteristic roots***. The characteristic roots of maximum modulus are said to be ***dominant***.

It is well-known (see, e.g., [2, Thm. 2]) that if an LRS $\mathbf{u}$ has no real positive dominant characteristic root then there are infinitely many n such that $u_n < 0$ and infinitely many n such that $u_n > 0$. Clearly such an LRS cannot be ultimately positive.

Since the characteristic polynomial of $\mathbf{u}$ has real coefficients, its set of roots can be written in the form $\{\rho_1, \ldots, \rho_\ell, \gamma_1, \overline{\gamma_1}, \ldots, \gamma_m, \overline{\gamma_m}\}$, where each $\rho_i \in \mathbb{R}$. If $\mathbf{u}$ is simple then there are non-zero real algebraic constants $b_1, \ldots, b_\ell$ and complex algebraic constants $c_1, \ldots, c_m$ such that, for all $n \geq 0$,

$$u_n = \sum_{i=1}^{\ell} b_i \rho_i^n + \sum_{j=1}^{m} \left(c_j \gamma_j^n + \overline{c_j \gamma_j}^n \right) . \tag{3}$$

Conversely, a sequence $\mathbf{u}$ that admits the representation (3) is a simple LRS over $\mathbb{R}$, with characteristic roots among $\rho_1, \ldots, \rho_\ell, \gamma_1, \overline{\gamma_1}, \ldots, \gamma_m, \overline{\gamma_m}$. Arbitrary LRS admit a more general "exponential-polynomial" representation in which the coefficients b_i and c_j are replaced by polynomials in n.

[2] In general we denote by $||X||$ the length of the binary representation of a given object X.

An LRS is said to be ***non-degenerate*** if it does not have two distinct characteristic roots whose quotient is a root of unity. A non-degenerate LRS over any field of characteristic zero is either identically zero or only has finitely many zeros. The study of arbitrary LRS can effectively be reduced to that of non-degenerate LRS using the following result from [10].

Proposition 1 *Let $\langle u_n\rangle_{n=0}^{\infty}$ be an LRS of order k over $\mathbb{Q}$. There is a constant $M = 2^{O(k\sqrt{\log k})}$ such that each subsequence $\langle u_{Mn+l}\rangle_{n=0}^{\infty}$ is non-degenerate for $0 \leq l < M$.*

The constant M in Proposition 1 is the least common multiple of the orders of all roots of unity appearing as quotients of characteristic roots of $\mathbf{u}$. This number can be computed in time polynomial in $||\mathbf{u}||$ since determining whether an algebraic number α is a root of unity (and computing the order of the root) can be done in polynomial time in $||\alpha||$ [17]. From the representation (3) we see that if the original LRS is simple with characteristic roots $\lambda_1, \ldots, \lambda_k$, then each subsequence $\langle u_{Mn+l}\rangle_{n=0}^{\infty}$ is also simple, with set of characteristic roots $\{\lambda_1^M, \ldots, \lambda_k^M\}$.

The following is a celebrated result on LRS [21, 23, 37].

Theorem 2 (Skolem-Mahler-Lech) *The set $\{n : u_n = 0\}$ of zeros of an LRS $\mathbf{u}$ comprises a finite set together with a finite number of arithmetic progressions. If $\mathbf{u}$ is non-degenerate and not identically zero, then its set of zeros is finite.*

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are LRS of orders k and l respectively, then the pointwise sum $\langle u_n+v_n\rangle_{n=0}^{\infty}$ is an LRS of order at most $k + l$, and the pointwise product $\langle u_n v_n\rangle_{n=0}^{\infty}$ is an LRS of order at most kl. Given representations of $\mathbf{u}$ and $\mathbf{v}$ we can compute representations of the sum and product in polynomial time by straightforward linear algebra.

2.3 First-Order Theory of the Reals

We now turn to the ***first-order theory of the reals***. Let $\boldsymbol{x} = x_1, \ldots, x_m$ be a list of m real-valued variables, and let $\sigma(\boldsymbol{x})$ be a Boolean combination of atomic predicates of the form $g(\boldsymbol{x}) \sim 0$, where each $g(\boldsymbol{x})$ is a polynomial with integer coefficients in the variables $\boldsymbol{x}$, and $\sim$ is either $>$ or $=$. We consider the problem of deciding the truth over the field $\mathbb{R}$ of sentences φ in the form

$$Q_1 x_1 \ldots Q_m x_m \, \sigma(\boldsymbol{x}) \,, \tag{4}$$

where each Q_i is one of the quantifiers $\exists$ or $\forall$. We write $||\varphi||$ for the length of the syntactic representation of φ.

The collection of true sentences of the form (4) is called the first-order theory of the reals. Tarski famously showed that this theory admits quantifier elimination and is therefore decidable. In this paper we rely on decision procedures for two fragments of this theory. We use the result of Canny [7] that if each Q_i is a universal quantifier, then the truth of φ can be decided in space polynomial in $||\varphi||$. We also use the result of Renegar [32] that for each fixed $M \in \mathbb{N}$, if the number of variables in φ is at most M, then the truth of φ can be determined in time polynomial in $||\varphi||$.

Observe that the representation of a real algebraic number α, as described in Section 2.1, can be directly translated to a quantifier-free formula $\varphi(x)$ of size polynomial in $||\alpha||$ that is uniquely satisfied by α.

3 Multiplicative Relations

Throughout this section let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s)$ be a tuple of algebraic numbers, each of height at most H and degree at most d. Assume that each λ_i is represented in the manner described in Section 2.1. Note that s, d, and $\log H$ are all at most $||\boldsymbol{\lambda}||$.

We define the group of multiplicative relations holding among the λ_i to be the subgroup $L(\boldsymbol{\lambda})$ of $\mathbb{Z}^s$ defined by

$$L(\boldsymbol{\lambda}) = \{\boldsymbol{v} = (v_1, \ldots, v_s) \in \mathbb{Z}^s : \lambda_1^{v_1} \ldots \lambda_s^{v_s} = 1\} \,.$$

Bounds on the complexity of computing a basis of $L(\boldsymbol{\lambda})$, considered as a free abelian group, can be obtained from the following deep result of Masser [24] which gives an upper bound on the magnitude of the entries of the vectors in such a basis.

Theorem 3 (Masser) *The free abelian group $L(\boldsymbol{\lambda})$ has a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l \in \mathbb{Z}^s$ for which*

$$\max_{1 \le i \le l,\, 1 \le j \le s} |\boldsymbol{v}_{i,j}| \le (d \log H)^{O(s^2)} .$$

Corollary 1 *A basis of $L(\boldsymbol{\lambda})$ can be computed in space polynomial in $||\boldsymbol{\lambda}||$. If s and d are fixed, such a basis can be computed in time polynomial in $||\boldsymbol{\lambda}||$.*

Proof. Masser's bound entails that there is a basis B such that $||B||$ is polynomial in s, $\log d$ and $\log\log H$. As noted, these are all polynomial in $||\boldsymbol{\lambda}||$; moreover the membership problem "$\lambda_1^{v_1} \ldots \lambda_s^{v_s} = 1$?" for a potential basis vector $\boldsymbol{v} \in \mathbb{Z}^s$ is decidable in space polynomial in $||\boldsymbol{\lambda}||$ by reduction to the decision problem for existential sentences over the reals. Thus we can compute a basis of $L(\boldsymbol{\lambda})$ in space polynomial in $||\boldsymbol{\lambda}||$ by brute-force search.

If s and d are fixed then the same brute-force search can be done in time polynomial in $||\boldsymbol{\lambda}||$, noting that the number of possible bases is polynomial in $||\boldsymbol{\lambda}||$ and the membership problem "$\lambda_1^{v_1} \ldots \lambda_s^{v_s} = 1$?" is decidable in time polynomial in $||\boldsymbol{\lambda}||$ by reduction to the decision problem for existential sentences over the reals in a fixed number of variables. □

For later use we note the following easy consequence of Corollary 1.

Corollary 2 *Given $M \in \mathbb{N}$, a basis of $L(\lambda_1^M, \ldots, \lambda_s^M)$ can be computed in space polynomial in $||M||$ and $||\boldsymbol{\lambda}||$.*

Proof. Observe that

$$L(\lambda_1^M, \ldots, \lambda_s^M) = \tfrac{1}{M}(L(\lambda_1, \ldots, \lambda_s) \cap M\mathbb{Z}^s) .$$

But the the intersection of two subgroups of $\mathbb{Z}^s$, represented by their respective bases, is well-known to be computable in polynomial time (e.g., by combining algorithms for computing unions and duals of lattices, see [25]). □

Remark 1 *Let K be a number field containing $\lambda_1, \ldots, \lambda_s$. Ge [14] showed that one can compute a basis of $L(\boldsymbol{\lambda})$ in time polynomial in the representation of K as a finite-dimensional algebra over $\mathbb{Q}$ and of the λ_i as elements of K. Unfortunately we cannot directly use this result to improve the polynomial-space bound in Corollary 1 since a field containing $\lambda_1, \ldots, \lambda_s$ may have degree over $\mathbb{Q}$ that is exponential in s.*

Another approach to compute a basis of $L(\boldsymbol{\lambda})$ is to use the algorithm of Håstad et al. *[18] to compute small integer linear relations among $\log \lambda_1, \ldots, \log \lambda_s$, and 2π. However, the seeming need for exponentially many bits of precision, combined with the use of the nearest-integer function in the algorithm, make it unclear whether one can improve the polynomial-space bound in Corollary 1 by this method.*

Next we relate $L(\boldsymbol{\lambda})$ to the ***orbit*** $\{(\lambda_1^n, \ldots, \lambda_s^n) \mid n \in \mathbb{N}\}$ of $\boldsymbol{\lambda}$. Recall from [8] the following classical theorem of Kronecker on inhomogeneous Diophantine approximation.

Theorem 4 (Kronecker) *Let $\theta_1, \ldots, \theta_s$ and $\psi_1, \ldots, \psi_s$ be real numbers. Suppose moreover that for all integers $u_1, \ldots, u_s$, if $u_1\theta_1 + \ldots + u_s\theta_s \in \mathbb{Z}$ then also $u_1\psi_1 + \ldots + u_s\psi_s \in \mathbb{Z}$, i.e., all integer relations among the θ_i also hold among the ψ_i (modulo $\mathbb{Z}$). Then for each $\varepsilon > 0$, there exist integers $p_1, \ldots, p_s$ and a non-negative integer n such that*

$$|n\theta_i - p_i - \psi_i| \le \varepsilon .$$

Write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and consider the s-dimensional torus $\mathbb{T}^s$ as a group under coordinatewise multiplication. The following can be seen as a multiplicative formulation of Kronecker's theorem.

Proposition 2 *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \mathbb{T}^s$ and consider the group $L(\boldsymbol{\lambda})$ of multiplicative relations among the λ_i. Define a subgroup $T(\boldsymbol{\lambda})$ of the torus $\mathbb{T}^s$ by*

$$T(\boldsymbol{\lambda}) = \{(\mu_1, \ldots, \mu_s) \in \mathbb{T}^s \mid \mu_1^{v_1} \ldots \mu_s^{v_s} = 1 \text{ for all } \boldsymbol{v} \in L(\boldsymbol{\lambda})\} .$$

Then the orbit $S = \{(\lambda_1^n, \ldots, \lambda_s^n) \mid n \in \mathbb{N}\}$ is a dense subset of $T(\boldsymbol{\lambda})$.

Proof. For $j = 1, \dots, s$, let $\theta_j \in \mathbb{R}$ be such that $\lambda_j = e^{2\pi i \theta_j}$. Notice that multiplicative relations $\lambda_1^{v_1} \dots \lambda_s^{v_s} = 1$ are in one-to-one correspondence with additive relations $\theta_1 v_1 + \dots + \theta_s v_s \in \mathbb{Z}$. Let $(\mu_1, \dots, \mu_s)$ be an arbitrary element of $T(\boldsymbol{\lambda})$, with $\mu_j = e^{2\pi i \psi_j}$ for some $\psi_j \in \mathbb{R}$. Then the hypotheses of Theorem 4 apply to $\theta_1, \dots, \theta_s$ and $\psi_1, \dots, \psi_s$. Thus given $\varepsilon > 0$, there exists $n \geq 0$ and $p_1, \dots, p_s \in \mathbb{Z}$ such that $|n\theta_j - p_j - \psi_j| \leq \varepsilon$ for $j = 1, \dots, s$. Whence for $j = 1, \dots, s$,

$$|\lambda_j^n - \mu_j| = |e^{2\pi i (n\theta_j - p_j)} - e^{2\pi i \psi_j}| \leq |2\pi(n\theta_j - p_j - \psi_j)| \leq 2\pi\varepsilon \,.$$

It follows that $(\mu_1, \dots, \mu_s)$ lies in the closure of S. □

4 Algorithm for Ultimate Positivity

In this section we give a procedure for deciding Ultimate Positivity for simple LRS.

Let K be a number field of degree d over $\mathbb{Q}$. Recall that there are d distinct embeddings $\sigma_1, \dots, \sigma_d : K \to \mathbb{C}$ [13]. Given a finite set S of prime ideals in the ring of integers $\mathcal{O}$ of K, we say that $\alpha \in \mathcal{O}$ is an S-***unit*** if the principal ideal (α) is a product of prime ideals in S. The following lower bound on the magnitude of sums of S-units, whose key ingredient is Schlickewei's p-adic generalisation [36] of Schmidt's Subspace theorem, was established in [12, 41] to analyse the growth of LRS.

Theorem 5 (Evertse, van der Poorten, Schlickewei) *Let m be a positive integer and S a finite set of prime ideals in $\mathcal{O}$. Then for every $\varepsilon > 0$ there exists a constant C, depending only on m, K, S, and ε with the following property. For any set of S-units $x_1, \dots, x_m \in \mathcal{O}$ such that $\sum_{i \in I} x_i \neq 0$ for all non-empty $I \subseteq \{1, \dots, m\}$, it holds that*

$$|x_1 + \dots + x_m| \geq CXY^{-\varepsilon} \tag{5}$$

where $X = \max\{|x_i| : 1 \leq i \leq m\}$ and $Y = \max\{|\sigma_j(x_i)| : 1 \leq i \leq m, 1 \leq j \leq d\}$.

We first consider how to decide Ultimate Positivity in the case of a non-degenerate simple LRS $\mathbf{u}$. As explained in Section 2.2, we can assume without loss of generality that $\mathbf{u}$ has a positive real dominant root. Furthermore, by considering the LRS $\langle p^n u_n \rangle_{n=0}^{\infty}$ for a suitable integer $p \geq 1$, we may assume that the characteristic roots and coefficients in the closed-form solution (3) are all algebraic integers.

Suppose that $\mathbf{u}$ has dominant characteristic roots $\rho, \gamma_1, \overline{\gamma_1}, \dots, \gamma_s, \overline{\gamma_s}$, where ρ is real and positive. Then we can write $\mathbf{u}$ in the form

$$u_n = b\rho^n + c_1\gamma_1^n + \overline{c_1\gamma_1}^n + \dots + c_s\gamma_s^n + \overline{c_s\gamma_s}^n + r(n) \,, \tag{6}$$

where $r(n) = o(\rho^{n(1-\varepsilon)})$ for some $\varepsilon > 0$. Now let $\lambda_i = \gamma_i/\rho$ for $i = 1, \dots, s$. Then we can write

$$u_n = \rho^n f(\lambda_1^n, \dots, \lambda_s^n) + r(n) \,, \tag{7}$$

where $f : \mathbb{T}^s \to \mathbb{R}$ is defined by $f(z_1, \dots, z_s) = b + c_1 z_1 + \overline{c_1 z_1} + \dots + c_s z_s + \overline{c_s z_s}$.

Proposition 3 *The LRS $\langle u_n \rangle_{n=0}^{\infty}$ is ultimately positive if and only if $f(\boldsymbol{z}) \geq 0$ for all $\boldsymbol{z} \in T(\boldsymbol{\lambda})$.*

Proof. Consider the expression (6). Let K be the number field generated over $\mathbb{Q}$ by the characteristic roots of $\mathbf{u}$ and let S be the set of prime ideal divisors of the dominant characteristic roots $\rho, \gamma_1, \overline{\gamma_1}, \dots, \gamma_s, \overline{\gamma_s}$ and the associated coefficients $b, c_1, \overline{c_1}, \dots, c_s, \overline{c_s}$. (These coefficients lie in K by straightforward linear algebra.) Then the term

$$b\rho^n + c_1\gamma_1^n + \overline{c_1\gamma_1}^n + \dots + c_s\gamma_s^n + \overline{c_s\gamma_s}^n$$

is a sum of S-units. Applying Theorem 5 to this term, we have $X = C_1\rho^n$ for some constant $C_1 > 0$ and $Y = C_2\rho^n$ for some constant $C_2 > 0$ (since an embedding of K into $\mathbb{C}$ maps characteristic roots

Figure 1: Decision Procedure for Ultimate Positivity of a Simple Integer LRS $\mathbf{u}$

1. Compute the characteristic roots $\{\rho_1, \ldots, \rho_\ell, \gamma_1, \overline{\gamma_1}, \ldots, \gamma_m, \overline{\gamma_m}\}$ of $\mathbf{u}$. Writing $\alpha \sim \beta$ if α/β is a root of unity, let $M = \mathrm{lcm}\{\mathrm{ord}(\alpha/\beta) : \alpha \sim \beta \text{ are characteristic roots}\}$. Moreover let $\{\rho_i : i \in I\} \cup \{\gamma_j, \overline{\gamma_j} : j \in J\}$ contain a unique representative from each equivalence class.

2. For $l = 0, \ldots, M-1$, check ultimate positivity of the non-degenerate subsequence $v_n = u_{Mn+l}$ as follows:

   2.1. Compute the coefficients b_i and c_j in the closed-form solution
   $$v_n = \sum_{i \in I} b_i \rho_i^{Mn} + \sum_{j \in J} \left(c_j \gamma_j^{Mn} + \overline{c_j \gamma_j}^{Mn} \right) . \tag{8}$$

   2.2. If $\mathbf{v} \not\equiv 0$ and there is no dominant real characteristic root in (8) then $\mathbf{v}$ is not ultimately positive.

   2.3. Let $\rho_1, \gamma_1, \overline{\gamma_1}, \ldots, \gamma_s, \overline{\gamma_s}$ be dominant among the characteristic roots appearing in (8). Define $\lambda_1 = \gamma_1/\rho_1, \ldots, \lambda_s = \gamma_s/\rho_1$ and compute a basis of $L(\lambda_1^M, \ldots, \lambda_s^M)$.

   2.4. Let $f : \mathbb{T}^s \to \mathbb{R}$ be given by
   $$f(z_1, \ldots, z_s) = b_1 + c_1 z_1 + \overline{c_1 z_1} + \ldots + c_s z_s + \overline{c_s z_s} .$$
   Then $\mathbf{v}$ is ultimately positive if and only if $f(\boldsymbol{z}) \geq 0$ for all $\boldsymbol{z} \in T(\boldsymbol{\lambda}^M)$.

to characteristic roots). The theorem tells us that for each $\varepsilon > 0$ there exists a constant $C > 0$ such that
$$|b\rho^n + c_1\gamma_1^n + \overline{c_1\gamma_1}^n + \ldots + c_s\gamma_s^n + \overline{c_s\gamma_s}^n| \geq C\rho^{n(1-\varepsilon)}$$
for all but finitely many values of n. (Since $\mathbf{u}$ is non-degenerate, it follows from the Skolem-Mahler-Lech Theorem that each non-empty sub-sum of the left-hand side vanishes for finitely many n.)

Now choose $\varepsilon > 0$ such that $r(n) = o(\rho^{n(1-\varepsilon)})$ in (6). Then for all sufficiently large n, $u_n \geq 0$ if and only if $b\rho^n + c_1\gamma_1^n + \overline{c_1\gamma_1}^n + \ldots + c_s\gamma_s^n + \overline{c_s\gamma_s}^n > 0$. Equivalently, looking at (7), it holds that for all sufficiently large n, $u_n \geq 0$ if and only if $f(\lambda_1^n, \ldots, \lambda_s^n) \geq 0$. But the orbit $\{(\lambda_1^n, \ldots, \lambda_s^n) : n \in \mathbb{N}\}$ is a dense subset of $T(\boldsymbol{\lambda})$ by Proposition 2. Thus u_n is ultimately positive if and only if $f(\boldsymbol{z}) \geq 0$ for all $\boldsymbol{z} \in T(\boldsymbol{\lambda})$. □

Proposition 3 reduces the Ultimate Positivity Problem for simple non-degenerate LRS to the decision problem for universal sentences over the reals, by separately handling real and complex parts. We treat arbitrary simple LRS by reduction to the non-degenerate case using Proposition 1. The resulting procedure is summarised in Figure 1. Below we account for the complexity of each step.

As noted in Section 2.2, Step 1 can be accomplished in time polynomial in $||\mathbf{u}||$.

For LRS of fixed order, there is an absolute bound on M in Step 2, while for LRS of arbitrary order, M is exponentially bounded in $||\mathbf{u}||$ by Proposition 1. We show that for each subsequence $\mathbf{v}$, Steps 2.1–2.4 require polynomial time for fixed-order LRS, and polynomial space in general.

Using iterated squaring, the coefficients b_i and c_j in the closed-form expression for $\mathbf{v}$ are definable in terms of the characteristic roots of $\mathbf{u}$ and the corresponding coefficients in the closed-form expression for $\mathbf{u}$ by a polynomial-size first-order formula that uses only universal quantifiers. This accomplishes Step 2.1.

Combining Corollaries 1 and 2, Step 2.3 can be done in polynomial space for arbitrary LRS and polynomial time for LRS of fixed order.

Step 2.4 is performed by reduction to the decision problem for universal sentences over the reals, having already noted that the coefficients b_i and c_j are first-order definable. By the results described in Section 2.3 this can be done in polynomial space for arbitrary LRS and polynomial time for LRS of fixed order.

We thus have our main result.

Theorem 6 *The Ultimate Positivity Problem for simple LRS is decidable in polynomial space in general, and in polynomial time for LRS of fixed order.*

5 Complexity Lower Bound

Motivated by problems in fixed-point computation, game theory, graph drawing, and geometry, Schaefer [35] has recently introduced the complexity class $\exists\mathbb{R}$, comprising all problems polynomial-time reducible to the decision problem for existential sentences over the reals. It is known that $\mathbf{NP} \subseteq \exists\mathbb{R} \subseteq \mathbf{PSPACE}$ [35]. In this section we show that the Positivity and Ultimate Positivity Problems for simple integer LRS are co$\exists\mathbb{R}$-hard, that is, at least as hard as the decision problem for universal sentences over the reals.

It is known that the problem of whether a degree-four polynomial $f(x_1,\ldots,x_n)$ with rational coefficients has a root in the region $[0,1]^n$ is complete for $\exists\mathbb{R}$ [6]. Here we consider an analogous problem about the positivity of f.

> POS-d. Determine whether a degree-d polynomial $f(x_1,\ldots,x_n)$ with rational coefficients assumes a strictly positive value on the region $[0,1]^n$.

We show that POS-8 is complete for $\exists\mathbb{R}$. To the best of our knowledge this fact has not been noted before. Indeed it was conjectured in [9] that, for sufficiently large d, POS-d might be intermediate between $\mathbf{P}_\mathbb{R}$ and $\mathbf{NP}_\mathbb{R}$ in the Blum-Shub-Smale model of real computation, which would imply that it is not $\exists\mathbb{R}$-complete unless $\mathbf{P}_\mathbb{R} = \mathbf{NP}_\mathbb{R}$.[3] The closest result in the literature appears to be [35, Theorem 2.6], which shows $\exists\mathbb{R}$-hardness of the problem whether, given a system of polynomials $f_1,\ldots,f_s : \mathbb{R}^n \to \mathbb{R}$, there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $f_i(\boldsymbol{x}) > 0$ for $i = 1,\ldots,s$. Here we strengthen the result to the case of a single polynomial of fixed degree.

Like [35] we rely on the following separation bound from [15].

Theorem 7 *Let $S_1, S_2 \subseteq \mathbb{R}^n$ be respectively defined by quantifier-free formulas $\sigma_1(\boldsymbol{x})$ and $\sigma_2(\boldsymbol{x})$. If S_1 and S_2 have strictly positive distance, then they have distance at least $2^{-(||\sigma_1||+||\sigma_2||)^{cn^3}}$ for some absolute constant $c > 0$.*

Theorem 8 *POS-8 is $\exists\mathbb{R}$-complete.*

Proof. Membership in $\exists\mathbb{R}$ is immediate. To prove hardness we reduce from the problem of whether a polynomial $f(x_1,\ldots,x_n)$ of degree 4 has a zero in $[0,1]^n$. If f does not have such a zero then f^2 is strictly positive on $[0,1]^n$. Applying Theorem 7 to $S_1, S_2 \subseteq \mathbb{R}^{n+1}$ given by

$$S_1 = \{(\boldsymbol{x}, f^2(\boldsymbol{x})) : \boldsymbol{x} \in [0,1]^n\} \quad \text{and} \quad S_2 = \{(\boldsymbol{x}, 0) : \boldsymbol{x} \in [0,1]^n\},$$

there is an absolute constant c such that f^2 is bounded below by $2^{-||f||^{cn^3}}$ on $[0,1]^n$. Our strategy is now to manufacture a degree-8 polynomial $g(y_1,\ldots,y_m)$, whose variables $y_1,\ldots,y_m$ are distinct from $x_1,\ldots,x_n$, such that the supremum of g on $[0,1]^m$ is strictly positive but less than $2^{-||f||^{cn^3}}$. Then it will be the case that f has a zero on $[0,1]^{n+m}$ if and only if $g - f^2$ assumes a strictly positive value over this region.

Before defining g, we first consider the values attained by the polynomial

$$h(u_1,\ldots,u_m) = (u_1/2)^4 - \sum_{i=1}^{m-1} \left(u_i - (u_{i+1}/2)^2\right)^2$$

[3] Strictly speaking [9] considered a version of POS-d for polynomials over $\mathbb{R}^n$ rather than $[0,1]^n$. However it is straightforward that the reduction in Theorem 8 can be adapted *mutatis mutandis* show $\exists\mathbb{R}$-hardness in this case also.

on the region

$$R = \{(u_1, \dots, u_m) \in [0,1]^m : u_1 \le u_2, \dots, u_1 \le u_m\}.$$

If $u_i = (u_{i+1}/2)^2$ for $i = m-1, \dots, 1$, then $h(u_1, \dots, u_m) = (u_1/2)^4$. Thus h attains a strictly positive value on R. On the other hand, for all $(u_1, \dots, u_m) \in R$ such that $h(u_1, \dots, u_m) \ge 0$ we have

$$\left(u_i - (u_{i+1}/2)^2\right)^2 \le (u_1/2)^4$$

for $i = 1, \dots, m-1$. Taking square roots, it follows that

$$u_i - (u_{i+1}/2)^2 \le (u_1/2)^2$$

and, since $u_1 \le u_{i+1}$,

$$u_i - (u_{i+1}/2)^2 \le (u_{i+1}/2)^2.$$

Simplifying, this yields $u_i \le u_{i+1}^2/2$.

Now, starting from $u_m \le 1$, and reasoning inductively for $i = m-1, \dots, 1$ using the inequality $u_i \le u_{i+1}^2/2$, we deduce that $0 \le u_1 \le 2^{1-2^m}$. Hence $\max_{\mathbf{u} \in R} h(\mathbf{u}) \le 2^{-2^{m+2}}$. It follows that for some choice of $m = ||f||^{O(1)}$ we can arrange that $0 < \max_{\mathbf{u} \in R} h(\mathbf{u}) < 2^{-||f||^{cn}}$.

Now define the required polynomial g by

$$g(y_1, \dots, y_m) = h(y_1, 1 - y_2 + y_2 y_1, \dots, 1 - y_m + y_m y_1).$$

Then the range of g over $[0,1]^m$ equals the range of h over R. Thus the maximum of g on $[0,1]^m$ is strictly positive but less than $2^{-||f||^{cn}}$. □

We now reduce POS-8 to the complements of the Positivity and Ultimate Positivity Problems. These are respectively (equivalent to) the problem of whether $u_n > 0$ for some term u_n of a given LRS $\mathbf{u}$ and whether $u_n > 0$ for infinitely many terms u_n.

To set up the reduction we first show how to compute a collection of s multiplicatively independent algebraic numbers of modulus 1 in time polynomial in s.

By a classical result of Lagrange, a prime number is congruent to 1 modulo 4 if and only if it can be written as the sum of two squares [13]. By Theorem 1, the class of such primes has asymptotic density 1/2 in the set of all primes, and therefore, by the Prime Number Theorem, asymptotic density $1/(2\log n)$ in the set of natural numbers. It follows that we can compute the first s such primes $p_1, \dots, p_s$ and their decomposition as sums of squares in time polynomial in s. Writing $p_j = a_j^2 + b_j^2$, where $a_j, b_j \in \mathbb{Z}$, define $\lambda_j = \frac{(a_j + ib_j)^2}{p_j}$. Clearly λ_j is an algebraic number of degree 2 and modulus 1.

Proposition 4 *$\lambda_1, \dots, \lambda_s$ are multiplicatively independent.*

Proof. The ring of algebraic integers in $\mathbb{Q}(i)$ is the ring $\mathbb{Z}(i)$ of *Gaussian integers* $\mathbb{Z}(i)$. Recall that the latter is a unique factorisation domain and that $a + ib \in \mathbb{Z}(i)$ is prime if $a^2 + b^2$ is a rational prime [13]. Now $\lambda_1^{n_1} \dots \lambda_s^{n_s} = 1$ if and only if

$$(a_1 + ib_1)^{2n_1} \dots (a_s + ib_s)^{2n_s} = p_1^{n_1} \dots p_s^{n_s}.$$

But $a_j + ib_j$ is prime by construction, and p_j factors as a product of primes $p_j = (a_j + ib_j)(a_j - ib_j)$. Thus by unique factorisation we must have $n_1 = \dots = n_s = 0$. □

Proposition 5 *There are polynomial-time reductions from POS-8 to the problem of whether $u_n > 0$ for some index n of a given simple rational LRS $\langle u_n \rangle_{n=0}^{\infty}$ and to the problem of whether $u_n > 0$ for infinitely many n.*

Proof. Suppose we are given an instance of POS-8, consisting of a polynomial $f(x_1,\ldots,x_s)$. Let $\lambda_1,\ldots,\lambda_s$ be multiplicatively independent algebraic numbers, constructed as in Proposition 4. For $j=1,\ldots,s$, the sequence $\langle y_{j,n} : n \in \mathbb{N}\rangle$ defined by $y_{j,n} = \frac{1}{2}(\lambda_j^n + \overline{\lambda_j}^n)$ satisfies a second-order linear recurrence $y_{j,n+2} = (2a_j/p_j)y_{j,n+1} - y_{j,n}$ with rational coefficients.

Recall, moreover, that given two simple LRS of respective orders l and m, their sum is a simple LRS of order at most $l+m$, their product is a simple LRS of order at most lm, and representations of both can be computed in polynomial time in the size of the input LRS. Thus the sequence $\boldsymbol{u} = \langle u_n : n \in \mathbb{N}\rangle$ given by $u_n = f(y_{1,n}^2,\ldots,y_{s,n}^2)$ is a simple LRS over the rationals. Since f has degree at most 8, the order of $\boldsymbol{u}$ is at most 4^8 times the number of monomials in f, and the recurrence satisfied by $\boldsymbol{u}$ can be computed in time polynomial in $||f||$. (Observe that if the degree of f were not fixed, then the above reasoning would yield an upper bound on the order of $\boldsymbol{u}$ that is exponential in the degree of f.)

From Propositions 2 and 4 it follows that the orbit $\{(\lambda_1^n,\ldots,\lambda_s^n) : n \in \mathbb{N}\}$ is dense in the torus $\mathbb{T}^s$. Thus the set $\{(y_{1,n}^2,\ldots,y_{s,n}^2) : n \in \mathbb{N}\}$ is dense in $[0,1]^s$ and f assumes a strictly positive value on $[0,1]^s$ if and only if $u_n > 0$ for some (equivalently infinitely many) n. This completes the reduction. ☐

Combining Theorem 8 and Proposition 5 we get that the Positivity and Ultimate Positivity Problems are co∃ℝ-hard.

Corollary 3 *There are polynomial-time reductions of the decision problem for universal sentences over the reals to the Positivity and Ultimate Positivity Problems for simple integer LRS.*

6 Conclusion

We have shown that the Ultimate Positivity Problem for simple LRS is decidable in polynomial space and as hard as the decision problem for universal sentences over the field of real numbers. A more careful accounting of the complexity of the decision procedure yields a $\mathbf{coNP}^{\exists\mathbb{R}}$ upper bound. Thus a **PSPACE**-hardness result would have non-trivial consequences for the complexity of decision problems for first-order logic over the reals. On the other hand, the obstacle to improving the polynomial-space upper bound is the complexity of computing a basis of the group of multiplicative relations among characteristic roots, see Remark 1.

Without the hypothesis of simplicity, obtaining decidability of the Ultimate Positivity Problem would require much tighter bounds in Diophantine approximation than are currently available—even for LRS of order at most 6 (see [30, Theorem 8] for a precise statement).

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