

# Fixed Points and Noetherian Topologies

Aliaume Lopez  

Université Paris Cité, CNRS, IRIF, F-75013, Paris, France

Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France.

---

## Abstract

This paper provides a canonical construction of a Noetherian least fixed point topology. While such least fixed point are not Noetherian in general, we prove that under a mild assumption one can use a *topological minimal bad sequence argument*. We then apply this fixed point theorem to rebuild known Noetherian topologies with a uniform proof.

In the case of spaces that are defined inductively (such as finite words, finite trees,...), we provide a uniform definition of a divisibility topology using our fixed point theorem. We then prove that the divisibility topology is a generalisation of the divisibility preorder introduced by Hasegawa in the case of well-quasi-orders.

**2012 ACM Subject Classification** Theory of computation → Finite Model Theory; Mathematics of computing → Discrete mathematics; Mathematics of computing → Point-set topology

**Keywords and phrases** Noetherian spaces, topology, well-quasi-orderings, initial algebras, Kruskal, Higman.

**Related Version** A full version of the paper is available at [TODO](#).

**Acknowledgements** I thank Jean Goubault-Larrecq and Sylvain Schmitz for their help and support in writing this paper. I thank Simon Halfon for his help on transfinite words.

## 1 Introduction

Introduced through Higman’s Lemma [11], the theory of **well-quasi-orderings** is a combinatorial tool that is frequently used or re-discovered [14]. It continues to be a thriving research topic as its application ranges from graph theory to the verification of infinite state transitions systems [13, 8, 2]. While being **better behaved than well-founded pre-orderings** with respect to most of the usual set constructions (products, words, trees, etc.), some natural ones fail to preserve well-quasi-orderings, such as the powerset construction as noticed by Rado [17].

There are two main possibilities to tackle this issue. The first one is to strengthen the definition of well-quasi-ordering to ensure that Rado’s construction cannot be built. This is the path leading to the theory of **better-quasi-orderings** [16, 15]. A different approach is taken by **Goubault-Larrecq** [4], who proceeds to **weaken the definition of a well-quasi-ordered to a notion of Noetherian topological space**. This shift from quasi-orders to topologies resolves these stability properties through careful adaptation of the topologies [5]. However, this approach suffers from the fact that many topologies correspond to a single quasi-ordering, and there usually does not exist a finest one that is Noetherian.

Notice that this lack of canonicity already arise from the definitions of the well-quasi-orderings on words or trees. Finding natural constructions for those quasi-orderings has been a recurrent research topic [10, 3]. We follow this road for Noetherian spaces.

**Well-quasi-orderings.** Let  $(X, \leq)$  be a set endowed with a quasi-order. A sequence  $(x_n)_n \in X^{\mathbb{N}}$  is *good* whenever there exists  $i < j$  such that  $x_i \leq x_j$ . A quasi-ordered set  $(X, \leq)$  is a *well-quasi-ordered* if every sequence is good. By calling a sequence bad whenever it is not good, well-quasi-orderings are equivalently defined as having no infinite bad sequences. Instances of well-quasi-orderings include  $(\mathbb{N}, \leq)$ ,  $(\Sigma, =)$  when  $\Sigma$  is a finite set, and totally

ordered well-founded sets. New well-quasi-orderings can be constructed using the following *algebra* [8]: finite disjoint sums  $\Sigma_{i=1}^n D_i$ , finite products  $\Pi_{i=1}^n D_i$ , finite words  $D^*$  with the subword embedding, finite multisets  $D^\circledast$  with the multiset embedding, finite sets  $\mathcal{P}_f(D)$  with the Hoare quasi-ordering, finite trees  $T(D)$  with the tree embedding relation.

Notice that for every choice of unary constructor  $F$  in the above algebra of wqos, one can see  $F(D)$  as building specific subsets of finite relational structures coloured using elements of  $D$ , ordered using the **induced substructure relation** allowing to increase colours. In the concrete case of finite words, we colour a linear finite ordering with elements of  $D$  and a word  $w$  *embeds* into  $w'$  whenever there exists a strictly increasing map  $h: |w| \rightarrow |w'|$  such that  $w_i \leq w_{h(i)}$  for  $1 \leq i \leq |w|$ . We say that  $u \leq_* v$  whenever  $u$  embeds into  $v$ . Similarly, a *tree embedding* is between  $t$  and  $t'$  is a map  $h$  from nodes of  $t$  to nodes of  $t'$  respecting the ancestor relation and increasing the colours of the nodes, and  $t \leq_{\text{tree}} t'$  whenever  $t$  embeds in  $t'$ . Proofs that these constructions preserves well-quasi-orderings typically rely on so called *minimal bad sequence arguments* [11, 14, 8], which have been generalised in several categorical settings [10, 3].

**Noetherian spaces.** A **topological space** is the pair  $(X, \tau)$  where  $\tau \subseteq P(X)$ ,  $\tau$  is stable under finite intersections, and  $\tau$  is stable under arbitrary unions. Before presenting the topological counterpart to well-quasi-orderings, let us provide a small dictionary from topology to orders. Given a quasi-ordered set  $(X, \leq)$ , one can build the **Alexandroff topology**  $\tau_\leq$  over  $X$  by collecting all subsets of  $X$  that are *upwards-closed*, that is, sets  $U \subseteq X$  such that  $\forall x \in U, \forall y \in X, x \leq y \implies y \in U$ . Conversely, given a topology  $\tau$ , one can build the **specialisation pre-order**  $\leq_\tau$  defined by  $x \leq_\tau y$  if  $\forall U \in \tau, x \in U \implies y \in U$ .

► **Remark 1.1.** The specialisation pre-order of the Alexandroff topology of a quasi-order  $\leq$  is the quasi-order itself.

A space  $X$  endowed with an Alexandroff topology comes from a well-quasi-ordering if and only if **every subset  $F$  of  $X$  is compact**, i.e., from every family  $(U_i)_{i \in I}$  of open sets such that  $F \subseteq \bigcup_{i \in I} U_i$ , one can extract a finite subset  $J \subseteq_f I$  such that  $F \subseteq \bigcup_{i \in J} U_i$ . Dropping the condition that  $X$  is Alexandroff, we define a topological space  $(X, \tau)$  to be a **Noetherian space** whenever **every subset of  $X$  is compact**.

▷ **Claim 1.2** ([5, Proposition 9.7.18]). **The product of two Noetherian spaces is Noetherian, and the join of two Noetherian topologies over the same set  $X$  is Noetherian.**

► **Remark 1.3.** A space  $(X, \tau)$  is Noetherian if and only if for every increasing sequence of open subsets  $(U_i)_{i \in \mathbb{N}}$ , there exists  $j \in \mathbb{N}$  such that  $\bigcup_{i \in \mathbb{N}} U_i = \bigcup_{i \leq j} U_i$ .

Several topologies can share the same specialisation pre-order  $\leq$ , among those, the Alexandroff topology is the finest. Generalising our study to non-Alexandroff topologies allows to escape roadblocks in the theory of well-quasi-orderings. For instance, **the powerset of a Noetherian space is Noetherian** [4], so are the  $\omega$ -length words [6], or even transfinite ordinal words [9] over a Noetherian space; results which are known to fail in the case of wqos. Moreover, Noetherian spaces allow considering spaces that are better described through an algebraic or topological point of view. A classical example following this pattern is the complex plane  $\mathbb{C}$  endowed with the Zariski topology, where the closed sets are the algebraic sets  $V(S) \triangleq \{z \in \mathbb{C} \mid \forall f \in S, f(z) = 0\}$  where  $S$  is a subset of  $C[X]$ .

## 1.1 Contributions of this paper

The main contribution of this paper is to provide an abstract topological version of the *minimal bad sequence argument* of well-quasi-orderings. A prototype of this argument can be

found in the study of the “topological Higman Lemma” [5]. The technical tool allowing this topological minimal bad sequence argument is the notion of **topology expander** over a set  $X$ , which we define as a function mapping topologies over  $X$  to topologies over  $X$  and satisfying mild extra properties. Our main result is Corollary 2.9, stating that **the least fixed point of every topology expander is a Noetherian topology**. This allows us to **inductively build Noetherian topologies, bypassing the well known fact that Noetherian spaces are not closed under limits or co-limits in  $\mathbf{Top}$**  [e.g. Exercise 9.7.22 of 5].

The ability to separate the construction of the space from the construction of the topology might be perceived as a weakness of the theory, when it is in fact a strength of our approach. We illustrate this by studying a topology over words of ordinal length, a research area already present in Goubault-Larrecq [6] and Goubault-Larrecq, Halfon, and Lopez [9]. To demonstrate the power of our framework, we introduce a natural topology over ordinal branching trees (with finite depth) and prove that it is Noetherian (Definition 3.9 and Theorem 3.10).

Finally, we can leverage Corollary 2.9 to build topologies over initial algebras of some well-behaved endofunctors of  $\mathbf{Top}$ , following the path taken by Hasegawa [10] for well-quasi-orders. Furthermore, we prove in Theorem 4.10 that this divisibility topology over an initial algebra coincides with the Alexandroff topology of the divisibility preorder defined by Hasegawa [10] in the case of a wqo. We believe that this answers a crucial question arising in the field of well-quasi-orders and Noetherian spaces, by **substituting to a unicity argument —there is no finest topology that is Noetherian— a canonicity argument —there is a uniform proof on inductive constructions—**.

## 2 Refinements of Noetherian topologies

Let us fix a set  $X$  equipped the *trivial topology*  $\tau_{\text{triv}} \triangleq \{\emptyset, X\}$ . This space is Noetherian because there are finitely many open sets. The approach taken in this paper is to iteratively refine this topology while keeping it Noetherian, and ultimately prove that *limit* of this construction remains Noetherian.

To capture the refinement of our topology  $\tau$ , we consider a *refinement function*  $F$  mapping topologies over  $X$  to topologies over  $X$ . Moreover, we assume that  $F(\tau)$  is Noetherian whenever  $\tau$  is, and that  $F(\tau) \subseteq F(\tau')$  when  $\tau \subseteq \tau'$ . This unrestricted setting already allows us, thanks to Tarski’s fixed point theorem, to consider the least fixed point of  $F$ , obtained by transfinitely iterating  $F$  from the trivial topology.

Let us briefly demonstrate that this general setting allows us to obtain non-Noetherian topologies as fixed points. For that we use the special case of  $\Sigma \triangleq \{a, b\}$  with the discrete topology, and study the set  $X \triangleq \Sigma^*$  of finite words over  $\Sigma$ .

► **Definition 2.1.** *The prefix topology over  $\Sigma^*$  is generated by the following open sets:  $U_1 \dots U_n \Sigma^*$  where  $U_i \subseteq \Sigma$ . We have written  $UV \triangleq \{uv \mid u \in U, v \in V\}$ .*

► **Lemma 2.2.** *The prefix topology over  $\Sigma^*$  is not Noetherian.*

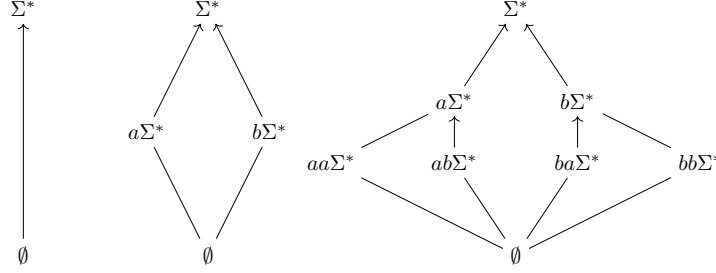
**Proof.** Let  $U_i \triangleq \{a^i b w \mid w \in \Sigma^*\}$  for  $i \geq 0$ . The union  $\bigcup_{i \geq 0} U_i$  is infinite and strictly increasing. ◀

► **Lemma 2.3.** *The prefix topology over  $(\Sigma^*, \tau)$  is the least fixed point of  $F_{\text{pref}}(\tau) \triangleq \langle UV \mid U \subseteq \Sigma, V \in \tau \rangle$ . Moreover,  $F_{\text{pref}}$  is monotone and preserves Noetherian spaces.*

**Proof.** It is clear that  $F_{\text{pref}}$  is monotone and that the prefix topology is the least fixed point of  $F_{\text{pref}}$ . Consider a subset  $E \subseteq \Sigma^*$  and let us prove that  $E$  is compact in  $F_{\text{pref}}(\tau)$ . Thanks to Alexander’s subbase lemma, it suffices to consider open covers using subbasic open sets. Let

us write such an open cover  $E \subseteq \bigcup_{i \in I} U_i V_i$ . Since  $\tau \times \tau$  is Noetherian Claim 1.2. Hence, there exists a finite set  $J \subseteq I$  such that  $\bigcup_{i \in J} U_i \times V_i = \bigcup_{i \in I} U_i \times V_i$ . However, this immediately implies that  $E \subseteq \bigcup_{i \in J} U_i V_i$ , and we successfully extracted a finite subcover of  $E$ . ◀

We illustrate the computation of the least fixed point of  $F_{\text{pref}}$  in Figure 1. As depicted, the union  $\bigcup_{i \in \mathbb{N}} a^i b \Sigma^*$  is an infinite strictly increasing sequence of opens, hence the least fixed point of  $F_{\text{pref}}$  is not a Noetherian topology.



■ **Figure 1** Iterating  $F_{\text{pref}}$  over  $\Sigma^*$ . On the left the trivial topology  $\tau_{\text{triv}}$ , followed by  $F_{\text{pref}}$ , and on the right  $F_{\text{pref}}$ .<sup>2</sup>

## 2.1 Topology expanders

The least fixed point of  $F_{\text{pref}}$  is not Noetherian because  $F_{\text{pref}}$  lacks uniformity. Let us consider  $V \triangleq a\Sigma^*$ , which happens to be a clopen (closed and open) subset of  $(\Sigma^*, F_{\text{pref}}(\tau_{\text{triv}}))$ . When endowing  $V$  with the topology induced by  $\Sigma^*$ , we obtain a space  $(V, \tau_{\text{triv}})$ .

As one cannot directly apply  $F_{\text{pref}}$  over  $(V, \tau_{\text{triv}})$ , we use the fact that  $V$  is a closed subset of  $\Sigma^*$  to define the subset restriction of a topology to  $V$ , the “topology without point” counterpart to induced topologies.

► **Definition 2.4** (Subset restriction). *Let  $(X, \tau)$  be a topological space and  $H$  be a closed subset of  $X$ . Define the subset restriction  $\tau|_H$  to be the topology generated by the opens  $U \cup H^c$  where  $U$  ranges in  $\tau$ .*

The subset restriction  $\tau|_H$  mimics the behaviour of the topology  $\tau \cap H$  induced by  $\tau$  over  $H$ , and is a sub-topology of  $\tau$ . In order to build intuition, let consider the special case of an Alexandroff topology over  $X$  and compute the specialisation preorder of  $\tau|_H$ , where  $H$  is a downwards closed set.

► **Lemma 2.5** (Specialisation preorder and restrictions). *Let  $\tau = \tau_{\leq}$  over a set  $X$ , and  $x, y \in X$ . Then,  $x \leq_{\tau|_H} y$  if and only if  $x \leq_{\tau} y \in H$  or  $y \notin H$ .*

**Proof.** Let us write  $\uparrow F$  for the set of points that are above  $F$  for  $\leq$ , and  $\uparrow x$  as a shorthand notation for  $\uparrow \{x\}$ , the set of points above  $x$ . Let us now unpack the definition of  $x \leq_{\tau|_H} y$ .

$$\begin{aligned} x \leq_{\tau|_H} y &\iff \forall U \in \tau|_H, x \in U \Rightarrow y \in U \\ &\iff \forall U \in \tau, x \in U \cup H^c \Rightarrow y \in U \cup H^c \end{aligned}$$

Let  $x \leq_{\tau|_H} y$ . Then, for every open set  $U \in \tau$ , if  $x \in U \cup H^c$ , then  $y \in U \cup H^c$ . Assume that  $y \in H$ , then  $x \leq_{\tau} y$ . Otherwise,  $y \notin H$  and we concluded.

Conversely, assume that  $x \leq_\tau y$  with  $y \in H$ . Then for every open  $U \in \tau$ ,  $U \cup H^c$  is open in  $\tau$  and therefore  $x \in U \cup H^c \implies y \in U \cup H^c$ . As a consequence,  $x \leq_{\tau|H} y$ . If  $y \notin H$ , then the implication still holds vacuously.  $\blacktriangleleft$

Concretely, the restriction to a downwards closed subset  $H$  of a quasi order  $X$  can be defined directly as merging every element in  $H^c$  to a single point on top of  $H$ .

Leveraging this analogue to the “induced topology” over  $V$ , it is an easy check that  $F_{\text{pref}}(\tau_{\text{triv}})|V$  is exactly  $\{\emptyset, a\Sigma^*, \Sigma^*\}$ . Contrarily to  $F_{\text{pref}}, F_{\text{pref}}(\cdot)|V$  builds a Noetherian topology when iterated transfinitely, as its least fixed point consists precisely of  $\{\emptyset, a^i\Sigma^*, \Sigma^*\}$ . When applying  $F_{\text{pref}}$  to  $V$ , one cannot obtain the open set  $ab\Sigma^*$ . The absence of the open set  $ab\Sigma^*$  in this iteration follows from the fact that  $ab\Sigma^*$  only appears in  $\Sigma^*$  as the result of an interaction between opens in  $a\Sigma^*$  and opens in  $b\Sigma^*$ , two disjoint closed sets. We rule out this behaviour in the following definition of a topology expander.

► **Definition 2.6** (Topology Expander). *A topology expander is a refinement function  $F$  that satisfies the following properties:*

1.  $F$  respects Noetherian spaces: if  $\tau$  is Noetherian, so is  $F(\tau)$ ;
2.  $F$  is monotone:  $\tau \subseteq \tau' \implies F(\tau) \subseteq F(\tau')$ ;
3.  $F$  respects subsets: for every Noetherian topology  $\tau$  satisfying  $\tau \subseteq F(\tau)$ , for all closed set  $H$  in  $\tau$ ,  $F(\tau)|H \subseteq F(\tau|H)|H$ .

As proven at the beginning of the section,  $F_{\text{pref}}$  fails to be a topology expander. Notice that the inclusion  $F(\tau)|H \supseteq F(\tau|H)|H$  holds trivially because  $\tau|H \subseteq \tau$  and  $F$  is monotone, hence Item 3 of Definition 2.6 can be replaced by  $F(\tau)|H = F(\tau|H)|H$ .

## 2.2 Iterating Expanders

Our goal is now to prove that the definition of topology expanders allows iterating them transfinitely in a sense that we precise hereafter. This will allow us to reach a least fixed point using transfinite induction and prove that this topology is still Noetherian.

► **Definition 2.7** ( $F^\alpha$ -topology). *Given a topological space  $(X, \tau)$  a topology expander  $F$ , and a limit ordinal  $\alpha$ , the limit topology  $F^\alpha(\tau)$  is defined as the join over all topologies  $F^\beta(\tau)$  for  $\beta < \alpha$ .*

We now devote the remaining of this section to prove our main theorem, which immediately implies that least fixed points of topology expanders are Noetherian.

► **Theorem 2.8.** *Let  $\alpha$  be an ordinal and  $F$  be a topology expander. If  $F^\beta(\tau)$  is Noetherian for all  $\beta < \alpha$ , and  $\tau \subseteq F(\tau)$ , then  $F^\alpha(\tau)$  is Noetherian.*

► **Corollary 2.9** (Noetherian preservation). *Let  $X$  be a set and  $F$  be a topology expander for  $X$ . The least fixed point of  $F$  is a Noetherian topology over  $X$ .*

### 2.2.1 The topological minimal bad sequence argument

Before introducing the so-called topological minimal bad sequence argument, let us first put a well-founded partial ordering over the open sets in a limit topology  $F^\alpha(\tau)$ , where  $\tau \subseteq F(\tau)$  is a Noetherian topology. To an open  $U \in F^\alpha(\tau)$ , we associate a depth  $\text{depth}(U)$ , defined as the smallest ordinal  $\beta \leq \alpha$  such that  $U \in F^\beta(\tau)$ . We then define  $U \preceq V$  to hold whenever  $\text{depth}(U) \leq \text{depth}(V)$ , and  $U \triangleleft V$  whenever  $\text{depth}(U) < \text{depth}(V)$ . It is an easy check that this is a well-founded partial order over  $F^\alpha(\tau)$ . The first step towards proving that  $F^\alpha(\tau)$  is Noetherian restricting our study to opens of depth strictly less than  $\alpha$  itself.

► **Fact 2.9.1** (Shallow sub-basis). *Let  $\alpha$  be a limit ordinal. The topology  $F^\alpha(\tau)$  has a subbasis of elements of depth strictly below  $\alpha$ .*

We restate hereafter the topological bad sequence argument designed by Goubault-Larrecq [5, Lemma 9.7.31] in the proof of the Topological Kruskal Theorem, adapted to our ordering of subbasic open sets. Let  $(X, \tau)$  be a topological space. A sequence  $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$  of open sets is *good* if there exists  $i \in \mathbb{N}$  such that  $U_i \subseteq \bigcup_{j < i} U_j$ . A sequence that is not good is called *bad*. These properties are tailored to mimic the notion of good sequences and bad sequences in well-quasi-orderings, as shown in the following lemma.

► **Lemma 2.10** (Minimal bad sequence [5, Lemma 9.7.15]). *Let  $\alpha$  be a limit ordinal. Assume that  $F^\alpha(\tau)$  is not a Noetherian space. There exists a bad sequence  $\mathcal{U}$  of opens in  $F^\alpha(\tau)$  of depth less than  $\alpha$  such that every strictly lexicographically smaller sequences of opens for  $\trianglelefteq$  is a good sequence.*

One key observation is that in a limit topology, minimal bad sequences are not allowed to use opens of arbitrary depth: they are either 0 or a successor ordinal.

► **Lemma 2.11** (Depth of minimal bad sequences). *Let  $\alpha$  be a limit ordinal,  $\tau$  a topology and  $F$  a topology expander such that  $F^\beta(\tau)$  is Noetherian for all  $\beta < \alpha$ . Assume that  $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$  is a minimal bad sequence of  $F^\alpha(\tau)$ , for all  $i \in \mathbb{N}$ ,  $\text{depth}(U_i)$  is either 0 or a successor ordinal.*

**Proof.** Assume by contradiction that there exists  $i \in \mathbb{N}$  such that  $\text{depth}(U_i)$  is a limit ordinal. This proves that  $U_i$  is obtained as a union of opens in  $F^\beta(\tau)$  for  $\beta < d_i$ . Since  $F^{d_i}(\tau)$  is Noetherian, this union is finite and of depth strictly below  $d_i$  which is absurd. ◀

Notice that if  $\mathcal{U}$  is a minimal bad sequence in  $(X, F^\alpha(\tau))$ , then  $U_i \not\subseteq \bigcup_{j < i} U_j \triangleq V_i$ , i.e.,  $U_i \cap V_i^c \neq \emptyset$ . We can now use our subset restriction operator to devise a minimal topology associated to this minimal bad sequence. The topology  $\text{Down}(U_i)$  is obtained using all opens strictly below  $U_i$  for  $\triangleleft$ . In particular, if  $\text{depth}(U_i) = \gamma + 1$  then  $\text{Down}(U_i)$  equals  $F^\gamma(\tau)$ . Noticing that  $H_i \triangleq V_i^c$  is a closed set in  $F^\alpha(\tau)$ , we can build a restricted topology  $\text{Down}(U_i)|_{H_i}$ .

► **Lemma 2.12** (Minimal topology). *Given a minimal bad sequence  $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ , the topology  $\mathcal{U}(F^\alpha(\tau))$ , generated by  $\bigcup_{i \in \mathbb{N}} \text{Down}(U_i)|_{H_i}$ , is Noetherian.*

**Proof.** Assume by contradiction that  $\mathcal{U}(F^\alpha(\tau))$  is not Noetherian. Thanks to Goubault-Larrecq [5, Lemma 9.7.15] there exists a bad sequence  $\mathcal{W} \triangleq (W_i)_{i \in \mathbb{N}}$  of subbasic elements of  $\mathcal{U}(F^\alpha(\tau))$ . By definition,  $W_i$  is in some  $\text{Down}(U_j)|_{H_j}$ . Let us write  $\rho: \mathbb{N} \rightarrow \mathbb{N}$  the mapping such that  $W_i \in \text{Down}(U_{\rho(i)})|_{H_{\rho(i)}}$ . In practice, this amounts to the existence of an open  $T_{\rho(i)} \triangleleft U_{\rho(i)}$  such that  $W_i = T_{\rho(i)} \cup V_{\rho(i)}$ . Without loss of generality we assume that  $\rho$  is increasing.

Let us build the sequence  $\mathcal{Y}$  defined by  $Y_i \triangleq U_i$  if  $i < \rho(0)$  and  $Y_i \triangleq T_{\rho(i)}$  otherwise. This is a sequence of open sets in  $F^\alpha(\tau)$  that is lexicographically smaller than  $\mathcal{U}$ , hence  $\mathcal{Y}$  is a good sequence. There exists  $i \in \mathbb{N}$  such that  $Y_i \subseteq \bigcup_{j < i} Y_j$ .

- If  $i < \rho(0)$ , then  $U_i \subseteq \bigcup_{j < i} U_j$  contradicting that  $\mathcal{U}$  is bad.
- If  $i \geq \rho(0)$ , let us write  $n \triangleq i - \rho(0)$  and remark that  $Y_i = T_{\rho(n)} \subseteq \bigcup_{j < \rho(0)} U_j \cup \bigcup_{j < n} T_{\rho(j)}$ . By taking on both sides the union with  $V_{\rho(i)}$ , we obtain  $W_i \subseteq \bigcup_{j < i} W_j$ , contradicting the fact that  $\mathcal{W}$  is a bad sequence.

This is absurd. ◀

### 2.2.2 Proof of Theorem 2.8

We are now ready to leverage our knowledge of minimal topologies associated to minimal bad sequences to carry on the proof of our main theorem.

► **Theorem 2.8.** *Let  $\alpha$  be an ordinal and  $F$  be a topology expander. If  $F^\beta(\tau)$  is Noetherian for all  $\beta < \alpha$ , and  $\tau \subseteq F(\tau)$ , then  $F^\alpha(\tau)$  is Noetherian.*

**Proof.** If  $\alpha$  is a successor ordinal, then  $\alpha = \beta + 1$  and  $F^\alpha(\tau) = F(F^\beta(\tau))$ . Because  $F$  respects Noetherian topologies, we immediately conclude that  $F^\alpha(\tau)$  is Noetherian. We are therefore only interested in the case where  $\alpha$  is a limit ordinal.

Assume by contradiction that  $F^\alpha(\tau)$  is not Noetherian, using Lemma 2.10 there exists a minimal bad sequence  $\mathcal{U} \triangleq (U_i)_{i \in \mathbb{N}}$ . Let us write  $d_i \triangleq \text{depth}(U_i) < \alpha$ . Thanks to Lemma 2.11  $d_i$  is either 0 or a successor ordinal.

▷ **Claim 2.13.** Because  $F^\beta(\tau)$  is Noetherian for  $\beta < \alpha$ , there are finitely many opens  $U_i$  at depth  $\beta$  for every ordinal  $\beta < \alpha$ .

**Proof.** Assume by contradiction that there exists infinitely many opens  $(U_i)_{i \in J}$  at depth  $\beta$ . Then the sequence lies in  $F^\beta(\tau)$ , which is Noetherian. As a consequence, there exists  $j \in J$  such that  $U_j \subseteq \bigcup_{i < j} U_i$ , which contradicts the fact that  $\mathcal{U}$  is bad. ◁

Furthermore, the sequence of depths of opens must be increasing, otherwise  $\mathcal{U}$  would not be lexicographically minimal. We can therefore construct a strictly increasing map  $\rho: \mathbb{N} \rightarrow \mathbb{N}$  such that  $0 < \text{depth}(U_{\rho(j)})$  and  $\text{depth}(U_i) < \text{depth}(U_{\rho(j)})$  whenever  $0 \leq i < \rho(j)$ .

Let us consider some  $i = \rho(n)$  for some  $n \in \mathbb{N}$ . Let us write  $V_i \triangleq \bigcup_{j < i} U_j$ ,  $H_i \triangleq X \setminus V_i$ . The set  $V_i$  is open in  $\text{Down}(U_i)$  by construction of  $\rho$ , hence  $H_i$  is closed in  $\text{Down}(U_i)$ . As  $F$  is a topology expander, we derive the following inclusions:

$$\begin{aligned} F(\text{Down}(U_i))|H_i &\subseteq F(\text{Down}(U_i)|H_i)|H_i \\ &\subseteq F(\mathcal{U}(F^\alpha(\tau)))|H_i \end{aligned}$$

Recall that  $U_i \in F(\text{Down}(U_i))$ . As a consequence,  $U_i \cup V_i = W_i \cup V_i$  where  $W_i$  is open in  $F(\mathcal{U}(F^\alpha(\tau)))$ . Thanks to Lemma 2.12, and preservation of Noetherian topologies through topology expanders, the latter is a Noetherian topology and  $(W_{\rho(i)})_{i \in \mathbb{N}}$  is good. This provides a  $i \in \mathbb{N}$  such that  $W_{\rho(i)} \subseteq \bigcup_{\rho(j) < \rho(i)} W_{\rho(j)}$ . In particular,

$$\begin{aligned} U_{\rho(i)} \cup V_{\rho(i)} &= W_{\rho(i)} \cup V_{\rho(i)} \subseteq \bigcup_{\rho(j) < \rho(i)} W_{\rho(j)} \cup V_{\rho(i)} \subseteq \bigcup_{\rho(j) < \rho(i)} W_{\rho(j)} \cup V_{\rho(j)} \\ &\subseteq \bigcup_{\rho(j) < \rho(i)} U_{\rho(j)} \cup V_{\rho(j)} \subseteq \bigcup_{j < \rho(i)} U_j = V_{\rho(i)} \end{aligned}$$

This proves that  $U_{\rho(i)} \subseteq V_{\rho(i)}$ , i.e. that  $U_{\rho(i)} \subseteq \bigcup_{j < \rho(i)} U_j$ . This contradicts the fact that  $\mathcal{U}$  is bad. ◁

We have effectively proven that being well-behaved with respect to closed subspaces is enough to consider least fixed points of refinement functions. This behaviour should become clearer in the upcoming sections, where we illustrate how this property can be ensured both in the case of Noetherian spaces and wqos.



### 3 Applications of Topology Expanders

We now briefly explore topologies that can be built using Corollary 2.9. As a first example, we will derive classical results from the theory of Noetherian spaces such as the *topological Higman lemma* [5, Theorem 9.7.33], and the *topological Kruskal theorem* [5, Theorem 9.7.46]. We will then discuss whether Corollary 2.9 can be applied to more recent developments on Noetherian topologies for words of ordinal length [6, 9].

► **Definition 3.1** (Regular subword topology [5, Definition 9.7.26]). *Given a topological space  $(\Sigma, \tau)$ , the space  $\Sigma^*$  of finite words over  $\Sigma$  can be endowed with the regular subword topology, generated by the open sets  $[U_1, \dots, U_n]$  when  $U_i \in \tau$ , where  $[U_1, \dots, U_n]$  is a shorthand notation for the set  $\Sigma^*U_1\Sigma^* \dots \Sigma^*U_n\Sigma^*$ .*

We first demonstrate in detail how to recover the topological Higman lemma by simply unrolling the definition of the regular subword topology as a fixed point, with the slight technicality that we have to consider upwards closures with respect to  $\leq_*$  via  $\uparrow_{\leq_*} U \triangleq \{w \in \Sigma^* \mid \exists v \in U. v \leq_* w\}$ . This is needed to guarantee that we build a topology expander.

► **Lemma 3.2.** *Given a Noetherian space  $(\Sigma, \theta)$ , the regular subword topology is the least fixed point of the topology topology expander  $F_{\text{words}}$  that maps  $\tau$  to the topology generated by the following sets:*

- $\uparrow_{\leq_*} UV$  for  $U, V$  opens in  $\tau$ ;
- $\uparrow_{\leq_*} W$ , for  $W$  open in  $\theta$ .

**Proof.** It is obvious that  $F_{\text{words}}$  is monotone, let us focus on the two other requirements for  $F_{\text{words}}$  to be a topology expander.

▷ **Claim 3.3.** If  $\tau$  is Noetherian, then  $F_{\text{words}}(\tau)$  is Noetherian.

**Proof.** Because the join of two Noetherian topologies is Noetherian (Claim 1.2), it suffices to prove that the topology generated by the sets  $\uparrow_{\leq_*} UV$  ( $U, V$  open in  $\tau$ ) and the topology generated by the sets  $\uparrow_{\leq_*} W$  ( $W$  open in  $\theta$ ) are Noetherian.

Let  $\uparrow_{\leq_*} U_i V_i$  be a sequence of opens. Because Noetherian topologies are stable under product (Claim 1.2), the sequence  $U_i \times V_i$  stabilises in finite time, hence so does the sequence  $\uparrow_{\leq_*} U_i V_i$ .

Let  $\uparrow_{\leq_*} W_i$  be a sequence of opens. Because  $\theta$  is Noetherian, the sequence  $W_i$  stabilises in finite time, hence so does  $\uparrow_{\leq_*} W_i$ . ◁

▷ **Claim 3.4.** If  $\tau$  is Noetherian,  $\tau \subseteq F_{\text{words}}(\tau)$  and  $H$  is closed in  $\tau$ , then  $F_{\text{words}}(\tau)|H \subseteq F_{\text{words}}(\tau|H)|H$ .

**Proof.** Notice that if  $H$  is closed in  $\tau$ , as  $\tau \subseteq F_{\text{words}}(\tau)$ ,  $H$  is downwards closed for  $\leq_*$ . As a consequence,  $(\uparrow_{\leq_*} UV) \cap H = (\uparrow_{\leq_*} (U \cap H)(V \cap H)) \cap H$ . Similarly,  $(\uparrow_{\leq_*} W) \cap H = (\uparrow_{\leq_*} (W \cap H)) \cap H$ . Hence, we have proven that  $F_{\text{words}}(\tau)|H \subseteq F_{\text{words}}(\tau|H)|H$ . ◁

It remains to show that the least fixed point of  $F_{\text{words}}$  is the regular subword topology. One inclusion is obtained by the fact that the opens in the regular subword topology are stable under  $F_{\text{words}}$ . For the other inclusion, a straightforward induction on  $n$  shows that  $[U_1, \dots, U_n]$  is open in the least fixed point of  $F_{\text{words}}$ . ◀

The exact same proof scheme applies for the topological Kruskal theorem, for which the opens are also given using an inductive definition.



► **Definition 3.5** (Tree topology [5, Definition 9.7.39]). *Given a topological space  $(\Sigma, \tau)$ , the space  $\mathsf{T}(\Sigma)$  of finite trees over  $\Sigma$  can be endowed with the tree topology, generated by the sets  $\diamond U \langle V \rangle$  where  $U$  is an open set of  $\Sigma$ ,  $V$  is an open set of  $\Sigma^*$  in its regular subword topology, and  $t \in \diamond U \langle V \rangle$  whenever there exists a subtree  $t'$  of  $t$  whose root is labelled by an element of  $U$  and whose list of children belong to  $V$ .*

► **Lemma 3.6.** *Given a Noetherian space  $(\Sigma, \theta)$ , the tree topology subword topology is the least fixed point of the following topology expander  $\mathsf{F}_{\text{tree}}$  over  $\mathsf{T}(\Sigma)$  that maps a topology  $\tau$  to the topology generated by the sets  $\uparrow_{\leq_{\text{tree}}} U \langle V \rangle$ , for  $U$  open in  $\theta$ ,  $V$  open in  $\Sigma^*$ , where  $U \langle V \rangle$  is the set of trees whose root is labelled by  $U$  and list of children is in  $V$ .*

It should not be surprising that both the topological Higman lemma and the topological Kruskal theorem fit in the framework of topology expanders, as those rely on the same key argument: the topological minimal bad sequence. Now, we will proceed to extend the use of topology expander to spaces for which the original proof did not use a minimal bad sequence argument, obtaining simpler and shorter proofs.

Our main example will be the space  $\Sigma^{<\alpha}$  of words of ordinal length less than  $\alpha$ , where  $\alpha$  is a fixed ordinal. Since  $\leq_*$  is in general not a wqo on  $\Sigma^{<\alpha}$  when  $\leq$  is wqo on  $\Sigma$ , this also provides an example of a topological minimal bad sequence argument that has no counterpart in the realm of wqos.

► **Definition 3.7** (Regular Ordinal Subword Topology [9]). *Let  $(\Sigma, \theta)$  be a topological space. The regular ordinal subword topology over  $\Sigma^{<\alpha}$  is the topology generated by the closed sets  $F_1^{<\beta_1} \dots F_n^{<\beta_n}$ , for  $n \in \mathbb{N}$ , for  $F_i$  closed in  $\theta$ , and where  $F^{<\beta}$  is the set of words of length less than  $\beta$  with all of their letters in  $F$ .*

The regular ordinal subword topology is Noetherian [9], but the proof is quite technical and relies on the in-depth study of the possible inclusions between the subbasic closed sets. We will prove that one can devise a topological expander computing a finer topology as its least fixed point, which provides a shorter proof of this result. We do now know yet if the two topologies coincide.

Given a Noetherian space  $X$ , the ordinal rank of the lattice of closed sets of  $X$  is called the *stature* of  $X$  [7] and generalises the homonymous notion in the case of well-quasi-orderings [1]. As opposed to the original proof [9, Proposition 33], our new approach does not yield any bounds on the stature of  $\Sigma^{<\alpha}$ .

► **Lemma 3.8.** *Given a Noetherian space  $(\Sigma, \theta)$ , and an ordinal  $\alpha$ , the regular ordinal subword topology over  $\Sigma^{<\alpha}$  is coarser than the least fixed point of the topology topology expander  $\mathsf{F}_{\alpha\text{-words}}$  that maps  $\tau$  to the topology generated by the following sets:*

- $\uparrow_{\leq_*} UV$  for  $U, V$  opens in  $\tau$ ;
- $\uparrow_{\leq_*} \beta \triangleright U$ , for  $U$  open in  $\tau$ ,  $\beta \leq \alpha$ , where  $w \in \beta \triangleright U$  if and only if  $w_{>\gamma} \in U$  for all  $0 \leq \gamma < \beta$ ;
- $\uparrow_{\leq_*} W$ , for  $W$  open in  $\theta$ .

As an example of a new Noetherian topology derived using Corollary 2.9, we will consider  $\alpha$ -branching trees  $\mathsf{T}^{<\alpha}(\Sigma)$ , i.e., the least fixed point of  $X \mapsto 1 + \Sigma \times X^{<\alpha}$  where  $\alpha$  is a given ordinal. Let us write  $\mathsf{T}^{<\alpha}(\Sigma)$  for the set

► **Definition 3.9.** *Let  $(\Sigma, \theta)$  be a Noetherian space. The regular ordinal tree topology over  $\alpha$ -branching trees is the least fixed point of  $\mathsf{F}_{\alpha\text{-trees}}$ , mapping a topology  $\tau$  to the topology generated by the sets  $\uparrow_{\leq_{\text{tree}}} U \langle V \rangle$ , where  $U \in \theta$ ,  $V$  is open in  $(\mathsf{T}^{<\alpha}(\Sigma))^{<\alpha}$  with the regular ordinal subword topology, and  $U \langle V \rangle$  is the set of trees whose root is labelled by an element of  $U$  and list of children belongs to  $V$ .*

► **Theorem 3.10.** *The  $\alpha$ -branching trees endowed with the regular ordinal tree topology forms a Noetherian space.*

**Proof.** It suffices to prove that  $F_{\alpha\text{-trees}}$  is a topology expander. It is clear that  $F_{\alpha\text{-trees}}$  is monotone, and a closed set of  $F_{\alpha\text{-trees}}(\tau)$  is always downwards closed for  $\leq_{\text{tree}}$ . As a consequence, if  $\tau \subseteq F_{\alpha\text{-trees}}(\tau)$  and  $H$  is closed in  $\tau$ ,  $t \in V \triangleq (\uparrow_{\leq_{\text{tree}}} U(V)) \cap H$  if and only if  $t \in H$  and every children of  $t$  belongs to  $H$ . Therefore,  $(\uparrow_{\leq_{\text{tree}}} U(V)) \cap H = (\uparrow_{\leq_{\text{tree}}} U(V \cap H^{<\alpha})) \cap H$ . Notice that  $H^{<\alpha} \cap V$  is an open of the regular ordinal subword topology over  $\tau|H$ . As a consequence,  $V \cap H \in F_{\alpha\text{-trees}}(\tau|H)|H$ .

Let us now check that  $F_{\alpha\text{-trees}}$  preserves Noetherian topologies. Let  $W_i \triangleq \uparrow_{\leq_{\text{tree}}} U_i(V_i)$  be a  $\mathbb{N}$ -indexed sequence of open sets in  $F_{\alpha\text{-trees}}(\tau)$  where  $\tau$  is Noetherian. The product of the topology  $\theta$  and the regular ordinal subword topology over  $\tau$  is Noetherian thanks to Claim 1.2 and Lemma 3.8. Hence, there exists a  $i \in \mathbb{N}$  such that  $U_i \times V_i \subseteq \bigcup_{j < i} U_j \times V_j$ . As a consequence,  $W_i \subseteq \bigcup_{j < i} W_j$ . We have proven that  $F_{\alpha\text{-trees}}(\tau)$  is Noetherian. ◀

## 4 Consequences on inductive definitions

The goal of this section is to provide a systematic way to define a topology expander over a space defined inductively. For instance,  $\Sigma^*$  is the least fixed point of  $X \mapsto 1 + \Sigma \times X$ , and  $T(\Sigma)$  is the least fixed point of  $X \mapsto 1 + \Sigma \times X^*$ . As noticed by Hasegawa [10] and Freund [3], usual orderings on words and trees can be derived from their least fixed point definitions.

This inductive definition is better expressed in a categorical setting. In this paper only three categories will appear, the category **Set** of sets and functions, the category **Top** of topological spaces and continuous maps, and the category **Ord** of quasi-ordered spaces and monotone maps. Using this language, a unary constructor  $G$  in the algebra of wqos defines an *endofunctor* from objects of the category **Ord** to objects of the category **Ord** preserving well-quasi-orderings.

In our study of Noetherian spaces (resp. well-quasi-orderings), we will often see constructors  $G'$  as first building a new set of structures, and then adapting the topology (resp. ordering) to this new set. In categorical terms, we are interested in endofunctors  $G'$  that are U-lifts of endofunctors on **Set**. In particular, this implies the existence of an endofunctor  $G$  of **Set** and a map  $H$  such that  $G'((X, \tau)) = (G(X), H(\tau))$ .

### 4.1 Divisibility Topologies over Analytic Functors

We will avoid as much as possible the use of complex machinery related to analytic functors, and use as a definition an equivalent characterisation given by Hasegawa [10, Theorem 1.6]. For an introduction to analytic functors and combinatorial species, we redirect the reader to Joyal [12].

► **Definition 4.1** (Category of elements). *Given  $G$  an endofunctor of **Set**, the category of elements  $\text{el}(G)$  has as objects pairs  $(E, a)$  with  $a \in G(E)$ , and as morphisms between  $(E, a)$  and  $(E', a')$  maps  $f: E \rightarrow E'$  such that  $G_f(a) = a'$ .*

As an intuition to the unfamiliar reader, an element  $(E, a)$  in  $\text{el}(G)$  is a witness that  $a$  can be produced through  $G$  by using elements of  $E$ . Morphisms of elements are witnessing how relations between elements of  $G(E)$  and  $G(E')$  arise from relations between  $E$  and  $E'$ . It is quite natural to define the notion of a “smallest” set of elements  $E$  such that  $a$  can be found in  $G(E)$ . This is achieved by considering so-called transitive objects.

► **Definition 4.2** (Transitive object). *A transitive object in a category  $\mathcal{C}$  is an object  $X$  satisfying the following two conditions for every object  $A$  of  $\mathcal{C}$ :*

- $\text{Hom}(X, A)$  is non-empty;
- The right action of  $\text{Aut}(X)$  on  $\text{Hom}(X, A)$  by composition is transitive.

Given an object  $A$  in a category  $\mathcal{C}$ , one can build the slice category  $\mathcal{C}/A$  whose objects are elements of  $\text{Hom}(B, A)$  when  $B$  ranges over objects of  $\mathcal{C}$  and morphisms between  $c_1 \in \text{Hom}(B_1, A)$  and  $c_2 \in \text{Hom}(B_2, A)$  are maps  $f: B_1 \rightarrow B_2$  such that  $c_2 \circ f = c_1$ . This notion of slice category can be combined with the one of transitive object to build so-called “weak normal forms”.

► **Definition 4.3** (Weak Normal Form). *A weak normal form of an object  $A$  in a category  $\mathcal{C}$  is a transitive object in  $\mathcal{C}/A$ .*

A category  $\mathcal{C}$  has the *weak normal form property* whenever every object  $A$  has a weak normal form. We are now ready to formulate a definition of analytic functors through the existence of weak normal forms for objects in their category of elements.

► **Definition 4.4** (Analytic functor). *An endofunctor  $G$  of  $\text{Set}$  is an analytic functor whenever its category of elements  $\text{el}(G)$  has the weak normal form property. Moreover;  $X$  is a finite set for every weak normal form  $f \in \text{Hom}((X, x), (Y, y))$  in  $\text{el}(G)/(Y, y)$ .*

► **Example 4.5.** The functor mapping  $X$  to  $X^*$  is analytic, and the weak normal form of a word  $(X^*, w)$  is  $(\text{letters}(w), w)$  together with the canonical injection from  $\text{letters}(w)$  to  $X$ . In this specific case, the weak normal forms are in fact initial objects.

► **Example 4.6.** The functor mapping  $X$  to  $X^{<\alpha}$  is not analytic when  $\alpha \geq \omega$ , because of the restriction that weak normal forms are defined using finite sets.

Let us now explain how these weak normal forms can be used to define a support associated to the analytic functor. Given an analytic functor  $G$  and an element  $(X, x)$  in  $\text{el}(G)$ , there exists a weak normal form  $f \in \text{Hom}((Y, y), (X, x))$  in the slice category  $\text{el}(G)/(X, x)$ . By definition,  $f: Y \rightarrow X$  and  $G_f(y) = x$ . We define  $f(Y)$  as the support of  $x$  in  $X$ .

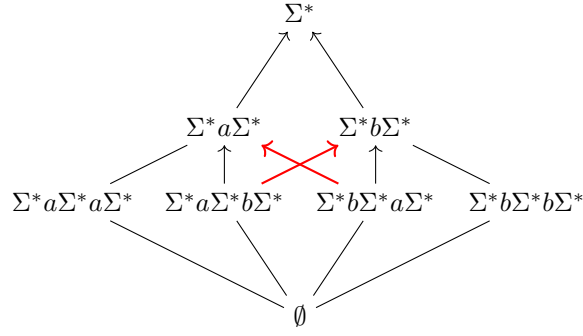
In turn, this construction of support allows building a substructure ordering on initial algebras  $(\mu G, \delta)$  of  $G$ : an element  $a \in \mu G$  is a children of an element  $b \in \mu G$  whenever  $a = b$  or  $a \in \text{supp}(\delta^{-1}(b))$ . The transitive closure of the children relation is a quasi-order called the *substructure ordering* on  $\mu G$  written  $\sqsubseteq$ .

► **Example 4.7** (Embedding of finite words). The substructure ordering on  $\mu G$  for  $G(X) \triangleq 1 + \Sigma \times X$  is the suffix ordering of words.

As analytic functors induce a quasi-ordering on their initial algebras, it is natural to import this quasi-ordering when dealing with lifts of analytic functors in the category  $\text{Ord}$ . This follows the construction of Hasegawa [10, Definition 2.7], although this substructure ordering is implicitly built. Given a topology  $\tau$  on  $\mu G$ , one can build open sets as  $\uparrow_{\sqsubseteq} U$  for  $U \in \tau$ . Open sets of this new topology are automatically upwards closed for  $\sqsubseteq$ .

► **Definition 4.8** (Divisibility Topology). *Let  $G': \text{Top} \rightarrow \text{Top}$  be a lifting of an analytic functor  $G$ , and  $(\mu G, \delta)$  an initial algebra of  $G$ . Moreover, we suppose that  $G'$  preserves inclusions. The divisibility topology on  $\mu G$  is the least fixed point of  $F_\diamond(\tau) \triangleq \{\uparrow_{\sqsubseteq} \delta(U) \mid U \text{ open in } G'(\mu G, \tau)\}$ .*

► **Example 4.9.** The divisibility map  $F_\diamond$  corrects the behaviour of  $F_{\text{pref}}$  over  $\Sigma^*$  by noticing that  $a \sqsubseteq ba$  and  $b \sqsubseteq ab$ . The differences can be seen on Figure 1 and Figure 2.



■ **Figure 2** The topology  $F_{\Diamond}^2(\tau_{\text{triv}})$ , with bold red arrows for the inclusions that were not present in  $F_{\text{pref}}^2(\tau_{\text{triv}})$ .

► **Theorem 4.10.** *The divisibility topology is Noetherian.*

**Proof.** We prove that  $F_{\Diamond}$  is a topology expander and conclude thanks to Corollary 2.9.

1. Let us prove that  $F_{\Diamond}$  sends Noetherian topologies to Noetherian topologies. This is because it is the upwards closure of the image of a Noetherian topology through  $\delta$ .
2. Let us show that  $F_{\Diamond}$  is monotone.  
Consider  $\tau \subseteq \tau'$  two topologies on  $\mu G$ . Let us write  $X \triangleq (\mu G, \tau)$  and  $Y \triangleq (\mu G, \tau')$ . By definition of the inclusion of topologies, exists an embedding  $\iota: X \rightarrow Y$  in  $\mathbf{Top}$  whose underlying function is the identity on  $\mu G$ . Because  $G'$  preserves embeddings,  $G'_\iota$  is an embedding from  $G'(X)$  to  $G'(Y)$ , that is, an embedding from  $(G'(\mu G), G'(\tau))$  to  $(G'(\mu G), G'(\tau'))$ . Moreover,  $UG'_\iota = G_{U\iota} = G_{\text{Id}_{\mu G}} = \text{Id}_{\mu G}$ . As a consequence,  $G'(\tau) \subseteq G'(\tau')$  and  $F_{\Diamond}(\tau) \subseteq F_{\Diamond}(\tau')$ .
3. Let us consider a Noetherian topology  $\tau$  such that  $\tau \subseteq F_{\Diamond}(\tau)$ ,  $H$  closed in  $\tau$ , and prove that  $F_{\Diamond}(\tau)|H \subseteq F_{\Diamond}(\tau|H)|H$ . Because  $G$  is an analytic functor, we can assume without loss of generality that  $G(H) \subseteq G(\mu G)$ .

▷ **Claim 4.11.**  $\delta^{-1}(H) \subseteq G(H)$

**Proof.** Let  $t \in H$ , because  $H$  is downwards closed for  $\sqsubseteq$ , for every  $u \in \text{supp}(\delta^{-1}(t))$ ,  $u \in H$ . As a consequence,  $\text{supp}(\delta^{-1}(t)) \subseteq H$ , and this means that  $\delta^{-1}(t) \in G(H)$ . ◀

Let  $U = \uparrow_{\sqsubseteq} \delta(V)$  be an open set of  $F_{\Diamond}(\tau)$ . Notice that  $H$  is a closed subset of  $F_{\Diamond}(\tau)$  because  $\tau \subseteq F_{\Diamond}(\tau)$ . Therefore,

$$\begin{aligned} U \cup H^c &= (\uparrow_{\sqsubseteq} \delta(V)) \cup H^c = \uparrow_{\sqsubseteq} (\delta(V) \cup H^c) = \uparrow_{\sqsubseteq} \delta(V) \cup \delta(G(H)^c) \cup H^c \\ &= \uparrow_{\sqsubseteq} \delta(V \cup G(H)^c) \cup H^c \end{aligned}$$

To conclude that  $U \cup H^c$  is open in  $F_{\Diamond}(\tau|H)|H$  it suffices to show that  $V \cup G(H)^c$  can be rewritten as  $W \cup G(H)^c$  where  $W$  is open in  $G'(\mu G, \tau|H)$ . Let us consider two maps  $e_1: (H, \tau_H) \rightarrow (\mu G, \tau)$ , and  $e_2: (H, \tau_H) \rightarrow (\mu G, \tau|H)$ . These two maps are embeddings, hence preserved by  $G'$ . As a consequence,  $V \cap G(H) = (G'_{e_1})^{-1}(V)$ , which is open. Because  $G'_{e_2}$  is an embedding, there exists a  $W$  open in  $G'(\mu G, \tau|H)$  such that  $(G'_{e_2})^{-1}(W) = V \cap G(H)$ .

We have proven that  $W \cup G(H)^c = V \cup G(H)^c$  with  $W$  open in  $G'(\mu G, \tau|H)$  ◀

We can apply Theorem 4.10 to the sets of finite words and finite trees, to recover the regular subword topology and the tree topology that were obtained in an ad-hoc fashion in Section 3.

► **Lemma 4.12.** *The regular subword topology over  $\Sigma^*$ , is the divisibility topology associated to the analytic functor  $X \mapsto 1 + \Sigma \times X$ .*

► **Lemma 4.13.** *The regular tree topology over  $T(\Sigma)$ , is the divisibility topology associated to the analytic functor  $X \mapsto 1 + X \times \Sigma^*$ .*

We believe that Lemmas 4.12 and 4.13 are strong indicators that the topologies introduced prior to this work were the right generalisations of Higman's word embedding and Kruskal's tree embedding in a topological setting. This partially addresses the canonicity issue of the aforementioned topologies.

► **Remark 4.14.** Beware that Theorem 4.10 cannot be applied to words of ordinal length as it would require to design  $\Sigma^{<\alpha}$  as the initial algebra of an analytic functor. Similarly, Theorem 4.10 cannot be applied to  $\alpha$ -branching trees (Definition 3.9). In both cases, one can directly apply Corollary 2.9 to conclude that the topology is Noetherian (see. Lemma 3.8 and Theorem 3.10).

## 4.2 Divisibility Preorders

In the case of finite words and finite trees, it has been proven that the specialisation preorder of the corresponding topology coincides respectively with the Alexandroff topologies of the word embedding and tree embedding [5, Exercise 9.7.30, Exercise 9.7.44]. We will proceed to generalise this result to every divisibility topology by relating it to the divisibility preorder introduced by Hasegawa.

Let us recall briefly how the divisibility preorder is built by Hasegawa [10, Definition 2.7]. Given an analytic functor  $G$  and its lift  $G^O$  to quasi-orderings respecting embeddings and wqos, let us build a family  $A_i$  of quasi-orders and  $e_i: A_i \rightarrow A_{i+1}$  of embeddings as follows:

- $A_0 = \emptyset$ ,  $A_1 = G^O(A_0)$  and  $e_0$  is the empty map.
- $e_{n+1} = G_{e_n}^O$  and  $A_{n+1}$  has as carrier set  $G(A_n)$  and preordering the transitive closure of the union of the two following relations: The one is the quasi-order  $G^O(A_n)$ , and the other is the collection of  $b \triangleleft a$  for each weak normal form  $(X, z) \rightarrow^f (A_n, a)$  in  $\text{el}(G)$  and each  $b$  in the image of  $X \rightarrow^f A_n \rightarrow^{e_n} A_{n+1}$ .

The *divisibility ordering*  $\preceq$  is the  $\omega$ -inductive limit in the category  $\text{Ord}$  of the diagram  $A_0 \rightarrow^{e_0} A_1 \rightarrow^{e_1} \dots$ . As remarked by Hasegawa, the maps  $e_n$  are injective order embeddings, and so are the morphisms  $c_n: A_n \rightarrow \mu G$  of the colimiting cone [10, Lemma 2.8]. Without loss of generality, we can assume that  $A_0 \subseteq A_1 \dots$  and that the colimit  $\mu G$  is the union of the sets  $A_i$  for  $0 \leq i < \omega$ . In particular, the map  $\delta$  is the identity map in this setting.

► **Lemma 4.15.** *We have  $(\preceq)^* = \preceq$ .*

► **Corollary 4.16** (Topology inclusion). *The Alexandroff topology of the divisibility preorder contains the divisibility topology.*

**Proof.** It suffices to prove that  $F_\Diamond(\tau_\preceq) \subseteq \tau_\preceq$ . Let us consider an open set  $V$  of  $F_\Diamond(\tau_\preceq)$  of the form  $\uparrow_{\sqsubseteq} \delta(U)$ , where  $U$  is open in  $\tau_\preceq$ . In particular,  $U = \uparrow_{\preceq} U$ . Notice that  $\uparrow_{\sqsubseteq} \uparrow_{\preceq} U = U$  because of Lemma 4.15. We have proven that  $V \in \tau_\preceq$ . ◀

► **Lemma 4.17.** *For all  $n \in \mathbb{N}$ ,  $\tau_{\preceq_n} \subseteq F_\Diamond(\tau_{\preceq_n})$ , where  $\preceq_n = \preceq|_{A_n}$  (as in Lemma 2.5).*

► **Corollary 4.18.**  $\tau_{\leq}$  is contained in the divisibility topology.

We are now ready to state our correctness theorem, i.e., that the divisibility topology is a generalisation to the topological setting of the divisibility preorder from Hasegawa.

► **Theorem 4.19.** Let  $G'$  be the lift of an analytic functor respecting Alexandroff topologies, Noetherian spaces, and topology embeddings. The divisibility topology of  $\mu G$  is the Alexandroff topology of the divisibility preorder of  $\mu G$ , which is a well-quasi-ordering.

## 5 Concluding Remarks

We have provided an abstract version of the minimal bad sequence lemma in the case of Noetherian spaces, allowing to prove that least fixed points of topology expanders are Noetherian topologies. We illustrate the strength of this result by recovering known Noetherian topologies as least fixed points, even in the case where the space itself is not inductively defined. In the cases where the spaces are defined as initial algebras of analytic functors, we uniformly defined a divisibility topology, which strictly generalises the divisibility preorder of Hasegawa to topological spaces.

Surprisingly, we could not provide examples where the least fixed point of the topology expander is obtained in more than  $\omega$  steps, even when dealing with ordinal indexed structures such as  $\alpha$ -words and  $\alpha$ -trees of Definitions 3.7 and 3.9.

## Acknowledgements

I thank Jean Goubault-Larrecq and Sylvain Schmitz for their help and support in writing this paper. I thank Simon Halfon for his help on transfinite words.

## References

- 1 Andreas Blass and Yuri Gurevich. Program termination and well partial orderings. *ACM Trans. Comput. Logic*, 9(3), jun 2008. ISSN 1529-3785. doi:10.1145/1352582.1352586. URL <https://doi.org/10.1145/1352582.1352586>.
- 2 Jean Daligault, Michael Rao, and Stéphan Thomassé. Well-quasi-order of relabel functions. *Order*, 27(3):301–315, Nov 2010. ISSN 1572-9273. doi:10.1007/s11083-010-9174-0. URL <https://doi.org/10.1007/s11083-010-9174-0>.
- 3 Anton Freund. From kruskal’s theorem to friedman’s gap condition. *Mathematical Structures in Computer Science*, 30(8):952–975, 2020.
- 4 Jean Goubault-Larrecq. **On Noetherian spaces**. In *Proceedings of LICS’07*, pages 453–462, 2007. doi:10.1109/LICS.2007.34.
- 5 Jean Goubault-Larrecq. **Non-Hausdorff Topology and Domain Theory**, volume 22 of *New Mathematical Monographs*. Cambridge University Press, 2013.
- 6 Jean Goubault-Larrecq. Infinitary noetherian constructions i. infinite words, 2021.
- 7 Jean Goubault-Larrecq and Bastien Laboureix. Statures and sobriification ranks of noetherian spaces, 2021. URL <https://arxiv.org/abs/2112.06828>.
- 8 Jean Goubault-Larrecq, Monika Seisenberger, Victor L. Selivanov, and Andreas Weiermann. Well quasi-orders in computer science (dagstuhl seminar 16031). *Dagstuhl Reports*, 6(1):69–98, 2016. doi:10.4230/DagRep.6.1.69. URL <https://doi.org/10.4230/DagRep.6.1.69>.
- 9 Jean Goubault-Larrecq, Simon Halfon, and Aliaume Lopez. Infinitary noetherian constructions ii. transfinite words and the regular subword topology, 2022.

- 10 Ryu Hasegawa. Two applications of analytic functors. *Theoretical Computer Science*, 272(1):113–175, 2002. ISSN 0304-3975. doi:[https://doi.org/10.1016/S0304-3975\(00\)00349-2](https://doi.org/10.1016/S0304-3975(00)00349-2). URL <https://www.sciencedirect.com/science/article/pii/S0304397500003492>. Theories of Types and Proofs 1997.
- 11 Graham Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, 3(1):326–336, 1952. doi:10.1112/plms/s3-2.1.326.
- 12 André Joyal. Foncteurs analytiques et espèces de structures. In Gilbert Labelle and Pierre Leroux, editors, *Combinatoire énumérative*, pages 126–159, Berlin, Heidelberg, 1986. Springer Berlin Heidelberg. ISBN 978-3-540-47402-9.
- 13 Igor Kríž and Robin Thomas. On well-quasi-ordering finite structures with labels. *Graphs and Combinatorics*, 6(1):41–49, 1990.
- 14 Joseph B. Kruskal. The theory of well-quasi-ordering: A frequently discovered concept. *Journal of Combinatorial Theory, Series A*, 13(3):297–305, 1972. doi:10.1016/0097-3165(72)90063-5.
- 15 Eric C Milner. Basic wqo-and bqo-theory. In *Graphs and order*, pages 487–502. Springer, 1985.
- 16 Maurice Pouzet. Un bel ordre d’abritement et ses rapports avec les bornes d’une multirelation. *CR Acad. Sci. Paris Sér. AB*, 274:A1677–A1680, 1972.
- 17 R. Rado. Partial well-ordering of sets of vectors. *Mathematika*, 1(2):89–95, 1954. doi:10.1112/S0025579300000565.

## A

 Appendix to “Applications of Topology Expanders” (Section 3)

► **Lemma 3.6.** *Given a Noetherian space  $(\Sigma, \theta)$ , the tree topology subword topology is the least fixed point of the following topology expander  $F_{\text{tree}}$  over  $\mathcal{T}(\Sigma)$  that maps a topology  $\tau$  to the topology generated by the sets  $\uparrow_{\leq \text{tree}} U\langle V \rangle$ , for  $U$  open in  $\theta$ ,  $V$  open in  $\Sigma^*$ , where  $U\langle V \rangle$  is the set of trees whose root is labelled by  $U$  and list of children is in  $V$ .*

**Proof.** The proof follows the same pattern as for the regular subword topology. The only technical part is to notice that a downwards closed set  $H$  for  $\leq_{\text{tree}}$  satisfies  $(\uparrow_{\leq \text{tree}} U\langle V \rangle) \cap H = (\uparrow_{\leq \text{tree}} U\langle [V_1 \cap H, \dots, V_n \cap H] \rangle) \cap H$ , whenever  $V = [V_1, \dots, V_n]$ . ◀

► **Lemma 3.8.** *Given a Noetherian space  $(\Sigma, \theta)$ , and an ordinal  $\alpha$ , the regular ordinal subword topology over  $\Sigma^{<\alpha}$  is coarser than the least fixed point of the topology topology expander  $F_{\alpha\text{-words}}$  that maps  $\tau$  to the topology generated by the following sets:*

- $\uparrow_{\leq_*} UV$  for  $U, V$  opens in  $\tau$ ;
- $\uparrow_{\leq_*} \beta \triangleright U$ , for  $U$  open in  $\tau$ ,  $\beta \leq \alpha$ , where  $w \in \beta \triangleright U$  if and only if  $w_{>\gamma} \in U$  for all  $0 \leq \gamma < \beta$ ;
- $\uparrow_{\leq_*} W$ , for  $W$  open in  $\theta$ .

**Proof.** It is obvious that  $F_{\alpha\text{-words}}$  is monotone. Moreover, closed sets  $H$  in  $F_{\alpha\text{-words}}(\tau)$  are downwards closed with respect to  $\leq_*$ . As a consequence,  $(\uparrow_{\leq_*} UV) \cap H = (\uparrow_{\leq_*} (U \cap H)(V \cap H)) \cap H$ ,  $(\uparrow_{\leq_*} W) \cap H = (\uparrow_{\leq_*} (W \cap H)) \cap H$ , and  $(\uparrow_{\leq_*} \beta \triangleright U) \cap H = (\uparrow_{\leq_*} \beta \triangleright (U \cap H)) \cap H$ . Hence,  $F_{\alpha\text{-words}}$  respects subsets. To conclude that  $F_{\alpha\text{-words}}$  is a topology expander, it remains to prove that it preserves Noetherian topologies.

▷ **Claim A.1.** Let  $\tau$  be a Noetherian topology,  $F_{\alpha\text{-words}}(\tau)$  is Noetherian.



**Proof.** As a consequence of Lemma 3.2, the topology generated by the sets  $\uparrow_{\leq_*} UV$ , and  $\uparrow_{\leq_*} W$  is Noetherian. Therefore, it suffices to check that the topology generated by the sets  $\uparrow_{\leq_*} \beta \triangleright U$  is Noetherian to conclude that  $F_{\alpha\text{-words}}(\tau)$  is too.

For that, consider a bad sequence  $\beta_i \triangleright U_i$  of open sets, indexed by  $\mathbb{N}$ . Because for all  $i$ ,  $\beta_i < \alpha + 1$ , we can extract our sequence so that  $\beta_i \leq \beta_j$  when  $i \leq j$ . The extracted sequence is still bad. Because  $\tau$  is Noetherian, there exists  $i \in \mathbb{N}$  such that  $U_i \subseteq \bigcup_{j < i} U_j$ . Let us now conclude that  $\beta_i \triangleright U_i \subseteq \bigcup_{j < i} \beta_j \triangleright U_j$ , which is in contradiction with the fact that the sequence is bad.

Let  $w \in \beta_i \triangleright U_i$ , and assume by contradiction that for all  $j < i$ , there exists a  $\gamma_j < \beta_j \leq \beta_i$  such that  $w_{>\gamma_j} \notin U_j$ . Let  $\gamma \triangleq \max_{j < i} \gamma_j < \beta_i$ . The word  $w_{>\gamma}$  does not belong to  $U_j$  for  $j < i$ , because  $U_j$  is upwards closed for  $\leq_*$ . As a consequence,  $w_{>\gamma} \notin \bigcup_{j < i} U_j$ . However,  $w_{>\gamma} \in U_i$ , which is absurd.  $\blacktriangleleft$

We now have to check that every open set in the regular ordinal subword topology is open in the least fixed point of  $F_{\alpha\text{-words}}$ . We prove by induction over  $n$  that a product  $F_1^{<\beta_1} \dots F_n^{<\beta_n}$  has a complement that is open.

**Empty product** this is the whole space.

$P \triangleq F^{<\beta} P'$  By induction hypothesis,  $P'^c$  is an open  $U$  in the least fixed point topology. Let us prove that  $P^c = A \cup B$ , where  $A \triangleq \uparrow_{\leq_*} \{av \mid u \notin F \wedge av \in U\}$ , and  $B \triangleq \uparrow_{\leq_*} (\beta \triangleright U)$ .

$\triangleright$  Claim A.2.  $P^c \subseteq A \cup B$ .

Proof. Let  $w \notin P$  and distinguish two cases.

- Either there exists a smallest  $\gamma < \beta$  such that  $w_\gamma \notin F$ . In which case  $w = w_{<\gamma} w_\gamma w_{>\gamma}$ . Since  $\gamma < \beta$ ,  $w_{<\gamma} \in F^{<\beta}$ , hence  $w_{>\gamma} \in U$  because  $w \notin P$ . As a consequence,  $w \in A$ .
- Or  $w_\gamma \in F$  for every  $\gamma < \beta$ . However, this proves that  $w_{>\gamma} \in U$  for every  $\gamma < \beta$ , which means that  $w \in B$ .  $\triangleleft$

$\triangleright$  Claim A.3.  $A \subseteq P^c$ .

Proof. Because  $P$  is downwards closed for  $\leq_*$ , it suffices to check that every word  $av$  with  $a \notin F$  and  $av \in U$  lies in  $P^c$ .

Assume by contradiction that  $av \in P$ , then  $av = u_1 u_2$  with  $u_1 \in F^{<\beta}$  and  $u_2 \in P'$ . Because  $a \notin F$ , this proves that  $u_1$  is the empty word, and that  $u_2 = w \in P'$ . This is absurd because  $w \in U = (P')^c$ .  $\triangleleft$

$\triangleright$  Claim A.4.  $B \subseteq P^c$ .

Proof. Because  $P$  is downwards closed for  $\leq_*$  it suffices to check that every word  $w \in \beta \triangleright U$  lies in  $P^c$ .

Assume by contradiction that such a word  $w$  is in  $P$ . One can write  $w = uv$  with  $u \in F^{<\beta}$  and  $v \in P'$ . However,  $|u| = \gamma < \beta$ , and  $\gamma + 1 < \beta$  because  $\beta$  is a limit ordinal. Therefore,  $v = w_{>\gamma} \in U = (P')^c$  which is absurd.  $\triangleleft$

$\triangleright$  Claim A.5.  $A$  and  $B$  are open in the least fixed point of  $F_{\alpha\text{-words}}$ .

Proof. The set  $B$  is open because  $U$  is open. Let us prove by induction that whenever  $U$  is open and  $F$  is closed in  $\theta$ , the set  $F \rtimes U \triangleq \uparrow_{\leq*} \{av \mid a \notin F, av \in U\}$  is open. It is easy to check that  $F \rtimes (\uparrow_{\leq*} W) = \uparrow_{\leq*} (W \cap F^c) \cup \uparrow_{\leq*} F^c W$ . Moreover,  $F \rtimes (\uparrow_{\leq*} UV) = \uparrow_{\leq*} (F \rtimes U)V$ . Finally, for  $\beta \geq 1$ ,  $F \rtimes (\uparrow_{\leq*} \beta \triangleright U) = \uparrow_{\leq*} F^c(\beta' \triangleright U)$  with  $\beta' = \beta$  if  $\beta$  is limit, and  $\beta' = \gamma$  if  $\beta = \gamma + 1$ .  $\triangleleft$

We have proven that  $P^c$  is open.  $\triangleleft$

## B Appendix to “Consequences on inductive definitions” (Section 4)

► **Definition B.1** (Lift). *An endofunctor  $G'$  of  $\mathbf{Top}$  is a lift of an endofunctor  $G$  of  $\mathbf{Set}$  if the following diagram commutes, where  $U$  is the forgetful functor*

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow{G'} & \mathbf{Top} \\ \downarrow U & & \downarrow U \\ \mathbf{Set} & \xrightarrow{G} & \mathbf{Set} \end{array}$$

► **Lemma 4.12.** *The regular subword topology over  $\Sigma^*$ , is the divisibility topology associated to the analytic functor  $X \mapsto 1 + \Sigma \times X$ .*

**Proof.** It suffices to remark that the functions  $F_{\diamond}$  and  $F_{\text{words}}$  have the same least fixed point, and conclude using Lemma 3.2.  $\triangleleft$

► **Lemma 4.13.** *The regular tree topology over  $T(\Sigma)$ , is the divisibility topology associated to the analytic functor  $X \mapsto 1 + X \times \Sigma^*$ .*

**Proof.** It suffices to remark that the functions  $F_{\diamond}$  and  $F_{\text{tree}}$  have the same least fixed point, and conclude using Lemma 3.6.  $\triangleleft$

► **Lemma B.2.**  *$a \triangleleft b$  in  $A_{n+1}$  if and only if  $a \in \text{supp}(\delta^{-1}(b))$ .*

**Proof.** Assume that  $a \triangleleft b$ , then  $b \in G(A_n)$  and there exists a weak normal form  $(X, z) \rightarrow^f (A_n, b)$  such that  $a \in f(X)$ . As  $(A_n, b) \rightarrow^{\iota} (\mu G, b)$ ,  $(X, z) \rightarrow^{f \circ \iota} (\mu G, b)$  is also a weak normal form [10, Lemma 1.5]. As a consequence,  $a \in \iota(f(X))$  and  $a \in \text{supp}(\delta^{-1}(b))$ .

Assume that  $a \sqsubset b$ , there exists a weak normal form  $(X, z) \rightarrow^f (\mu G, b)$  such that  $a \in f(X)$ . As  $b \in G(A_n)$  for some  $n \in \mathbb{N}$ , this means that  $(A_n, b) \rightarrow^{\iota} (\mu G, b)$  is an element of the slice category, hence that there exists  $g$  such that  $(X, z) \rightarrow^g (A_n, b)$  is a weak normal form of  $b$  and  $\iota \circ g = f$ . In particular,  $a \in g(X)$ , hence  $a \in \text{supp}(\delta^{-1}(b))$ .  $\triangleleft$

A direct consequence is that our substructure relation captures the height of the sets  $A_n$  in the following sense:

► **Fact B.2.1.** *If  $a \sqsubset b$  and  $b \in A_{n+1}$  then  $a \in A_n$ .*

► **Fact B.2.2.** *For all  $n \in \mathbb{N}$ ,  $A_n$  is a downwards closed subset of  $A_{n+1}$ .*

Now, it is an easy check that the divisibility preorder on  $\mu G$  is compatible with substructures as this is true for the sets  $A_n$ .

► **Lemma 4.15.** *We have  $(\preceq \sqsubseteq)^* = \preceq$ .*

**Proof.** By induction we prove it on  $A_n$  using the fact that  $a \sqsubset b$  and  $b \in A_{n+1}$  implies  $a \in A_n$ .  $\triangleleft$

► **Lemma B.3** (Compatibility with substructures). *For all  $n \in \mathbb{N}$ ,*

$$(\sqsubseteq_{\leq_{G^O(A_n)}})^* \sqsubset = \leq_{G^O(A_n)} \sqsubset$$

*Note that this equality is only over elements of  $A_{n+1}$ .*

**Proof.** Let  $n \in \mathbb{N}$ . Only one inclusion is non trivial. We know that  $\leq_{A_{n+1}} = (\leq_{G^O(A_n)} \sqsubseteq)^*$ . As the maps  $e_n$  is an order embedding, for every  $a, b \in A_n$ ,  $a \leq_{A_{n+1}} b$  implies  $a \leq_{A_n} b$ . In particular,  $\leq_{A_{n+1}} \sqsubset = \leq_{A_n} \sqsubset$ . As  $e_n$  is monotone from  $A_n$  to  $G^O(A_n)$ ,  $x \leq_{A_n} y$  implies  $x \leq_{G^O(A_n)} y$  and therefore  $(\leq_{A_{n+1}} \sqsubset) \subseteq (\leq_{G^O(A_n)} \sqsubset)$ . ◀

► **Corollary B.4.** *For all  $n \in \mathbb{N}$ ,  $\leq_{G^O(A_n)} \sqsubseteq = \leq_{A_{n+1}}$*

► **Lemma 4.17.** *For all  $n \in \mathbb{N}$ ,  $\tau_{\preceq_n} \subseteq F_\Diamond(\tau_{\preceq_n})$ , where  $\preceq_n = \preceq|_{A_n}$  (as in Lemma 2.5).*

**Proof.** Let  $x \in \mu G$  and consider  $U = \uparrow_{\preceq_n} x$ , which is open in  $\tau_{\preceq_n}$ . Let us write  $V = \uparrow_{\preceq_{n+1}} \{y \mid x \preceq_n y\}$ . It is clear that  $U = V$ , let us now prove that  $V$  is open in  $F_\Diamond(\tau_{\preceq_n})$ .

Thanks to Corollary B.4,  $V = \uparrow_{\sqsubseteq} \uparrow_{F(\preceq_n)} \{y \mid x \preceq_n y\}$ . Moreover,  $\uparrow_{G^O(\preceq_n)} \{y \mid x \preceq_n y\}$  is open in  $G'(\mu G, \tau_{\preceq_n})$ . As a consequence, we have proven that  $V$  is open in  $F_\Diamond(\tau_{\preceq_n})$ . ◀