Approximations and complex multiplication according to Ramanujan

by

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Introduction

This talk revolves around two focuses: complex multiplications (for elliptic curves and Abelian varieties) in connection with algebraic period relations, and (diophantine) approximations to numbers related to these periods. Our starting point is Ramanujan's works [1], [2] on approximations to π via the theory of modular and hypergeometric functions. We describe in chapter 1 Ramanujan's original quadratic period--quasiperiod relations for elliptic curves with complex multiplication and their applications to representations of fractions of π and other logarithms in terms of rapidly convergent nearly integral (hypergeometric) series. These representations serve as a basis of our investigation of diophantine approximations to π and other related numbers. In Chapter 2 we look at period relations for arbitrary CM-varieties following Shimura and Deligne. Our main interest lies with modular (Shimura) curves arising from arithmetic Fuchsian groups acting on H. From these we choose arithmetic triangular groups, where period relations can be expressed in the form of hypergeometric function identities. Particular attention is devoted to two (commensurable) triangle groups, (0,3;2,6,6) and (0,3;2,4,6), arising from the quaternion algebra over $\mathbb{Q}$ with the discriminant $D = 2\cdot 3$. We also touch upon the algebraic independence problem for periods and quasiperiods of general CM-varieties and particularly CM-curves associated with the triangle groups (hypergeometric curves as we call them). The diophantine approximation problem for numbers connected with periods, particularly for multiples of π, is analyzed using Padé approximations to power series representing these numbers. We give a brief review of Padé approximations, their effective construction, and problems of analytic and arithmetic (p-adic) convergence of Padé approximants. Padé approximations to nearly integral power series (G-functions) are used in connection with Ramanujan-like representations of $1/\pi$ and other similar period constants. We discuss measures of irrationality for algebraic multiples of π and related numbers that follow from Padé approximation methods.

The problem of uniformization by nonarithmetic subgroups is discussed in connection with the Whittaker conjecture [11] on an explicit expression for accessory parameters in the (Schottkey-type) uniformization of hyperelliptic Riemann surfaces of genus $g \geq 2$ by Fuchsian groups. On the basis of numerical computations of monodromy groups of linear differential equations, we concluded that the conjecture [11] is generically incorrect. Moreover, it appears that accessory parameters in the uniformization problem of Riemann surfaces defined over $\bar{\mathbb{Q}}$ are nonalgebraic with the exception of uniformization by arithmetic subgroups and of cases when the differential equations are reduced to hypergeometric ones (the monodromy group is connected to one of the triangle groups). We briefly describe numerical and theoretical results on the transcendence of elements of the monodromy groups of linear differential equations over $\bar{\mathbb{Q}}(x)$.

We conclude the paper with a discussion of numerical approximations to solutions of algebraic and differential equations. We present generalizations of our previous results [12] on the complexity of approximations to solutions of linear differential equations. We describe a new, "bit-burst" method of evaluation of solutions of linear differential equations everywhere on their Riemann surface. In the worst case, an evaluation with n bits of precision at an n-bit point requires $O(M_{bit}(n)\log^3 n)$ boolean (bit) operations, where $M_{bit}(n)$ is the boolean complexity of an n-bit multiplication. For functions satisfying additional arithmetic conditions (e.g. E-functions and G-functions) the factor $\log^3 n$ could be further decreased to $\log^2 n$ or, even, $\log n$. We also describe the natural parallelizations of the presented algorithms that are well suited for practical implementation.

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1. Complex multiplications and Ramanujan's period relations

Most of the material in this talk evolves around mathematics closely associated with one of the earliest Ramanujan papers "Modular equations and approximations to π" [1] published in 1914, which according to Hardy [2] "is mainly Indian work, done before he came to England." In that or another way the same kind of mathematics appears in later Ramanujan research, including his notebooks. Hardy interpreted many of Ramanujan's results and identities as connected mainly with "complex multiplication", and Ramanujan's interest in resolving modular equations in explicit radicals was later picked up in Watson's outstanding series [3] of "Singular Moduli" papers. Singular moduli themselves, and general modular equations relating automorphic functions with respect to congruence subgroup of a full modular group are traditional subjects of late $\overline{\underline{\text{XIX}}}$ century mathematics, whose importance is clearly realized in modern number theory and algebraic geometry, particularly in diophantine geometry in connection with arithmetic theory of elliptic curves and rational points on them. Also modular equations turned up as a convenient tool of fast operational complexity algorithms of computation of π and of values of elementary functions (see corresponding chapters in Borweins' book [5]).

Instead of complex multiplication as merely a subject of "singular moduli" of elliptic functions we will touch upon the complex multiplication in a slightly more general context: from the point of view of nontrivial endomorphisms of certain classes of Jacobians of particular algebraic curves. (We are not going to discuss a variety of complex "complex multiplication" subjects on L-functions and Abelian varieties, though we'll have to borrow particular consequences of vast theories developed in general by Shimura, Deligne and others.)

The choice of curves is clearly determined by Ramanujan's interest: these are curves with 4 critical points, whose Abelian periods are expressed via hypergeometric integrals (for simplicity one can call those curves hypergeometric ones), see [10].

Transforming one of the 4 critical points into infinity and normalizing, one recovers the Gauss hypergeometric function integrals representative of these periods. We display these well-known expressions:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-xz)^{-a}dx,$$

$(c > b > 0)$, with the expansion near $z = 0$:

$$F(a,b;c;z) = 1 + \frac{a\cdot b}{c\cdot 1} \cdot z + \frac{a(a+1)\cdot b(b+1)}{c\cdot(c+1)\cdot 1\cdot 2} \cdot z^2 + \dots .$$

Our primary interest in complex multiplication is not arithmetical but transcendental; rather than to study the number-theoretical functions associated with complex multiplication invariants, we want to know the basic facts about the values of these invariants: are these numbers transcendental (over $\mathbb{Z}$)? If algebraic relations do exist, over what fields of definition do they exist? These basic questions of transcendence, algebraic independence and linear independence are the subject of diophantine approximations. With these questions come their qualitative counterparts: when numbers are irrational (transcendental), how well are they approximated by rationals? Can these best approximations be determined effectively and/or explicitly? (Usually one asks in this context: can one determine the continued fraction expansion of the number?)

The class of numbers we are interested in is generated over $\bar{\mathbb{Q}}$ by periods and quasiperiods of Abelian varieties, i.e. by integrals

$$\int_\gamma \omega \quad \text{and} \quad \int_\gamma \eta$$

for differentials ω and η of the first and the second kind, respectively, from $H^1_{DR}(A)$, and $\gamma \in H_1(A,\mathbb{Z})$, for an Abelian variety A defined over $\bar{\mathbb{Q}}$.

In particular, when A is a Jacobian $J(\Gamma)$ of a nonsingular curve Γ over $\bar{\mathbb{Q}}$, we are looking at periods and quasiperiods forming the full Riemann matrix of Γ-total of $2g \times 2g$ elements, where g denotes the genus of Γ.

In this context, complex multiplications, understood as nontrivial endomorphisms of A, are usually expressed as nontrivial algebraic relations among the elements of Riemann's original $2g \times g$ pure period matrix π of A. These "purely period relations" are well known, and are mainly algebraic in nature, and in one-dimensional case ($g = 1$), give pairs of periods whose ratio is a "singular module". An interesting thing discovered by Ramanujan in this classical (even in his time) field was the existence of new quasiperiods relations. In the Weierstrass-like notation, commonly accepted nowdays, period and quasiperiod relations in the elliptic curve case can be described as follows.

One starts with an elliptic curve over $\bar{\mathbb{Q}}$ with a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ ($g_2, g_3 \in \bar{\mathbb{Q}}$), having the fundamental periods ω_1, ω_2 (with $\mathrm{Im}\frac{\omega_2}{\omega_1} > 0$) and the corresponding quasi-periods η_1, η_2. The only relation between ω_i and η_j that always holds is the Legendre identity:

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i.$$

Thus π belongs to the field generated by periods and quasiperiods over $\bar{\mathbb{Q}}$. In the complex multiplication case $\tau = \frac{\omega_2}{\omega_1}$ is an imaginary quadratic number.

Whenever $\tau \in \mathbb{Q}(\sqrt{-d})$ for $d > 0$, invariants g_2 and g_3 can be chosen from the Hilbert class field of $\mathbb{Q}(\sqrt{-d})$, and this field is the minimal extension with this property.

A priori complex multiplication means only a single relation between ω_i.

It seems that until Ramanujan's paper nobody explicitly stated the existence of the second relation between periods and quasiperiods. This relation is the following one:

Whenever τ is a quadratic number: $A\tau^2 + B\tau + C = 0$, the four numbers: $\omega_1, \omega_2, \eta_1, \eta_2$ are linearly dependent over $\bar{\mathbb{Q}}$ only on two of them.

Explicitly:

$$\begin{aligned} &\omega_2 = \tau\omega_1, \\ &A\tau\eta_2 - C\eta_1 + \alpha\omega_1 = 0 \end{aligned} \tag{1.1}$$

for $\alpha \in \bar{\mathbb{Q}}$ $(\alpha \in \mathbb{Q}(\tau, g_2, g_3))$.

The relations (1.1) are not entirely original; Legendre's investigation of the lemmiscate case provides with (1.1) in two cases, where τ is equivalent to i or to ρ under $SL_2(\mathbb{Z})$; moreover, these cases were clearly known to Euler, who evaluated the appropriate complete elliptic integrals. However, those two particular cases are "wrong ones": in these cases the importance of the relation (1.1) is lost because $\alpha = 0$, and it is hard to understand the need for its appearance. In a few other special singular moduli cases, (1.1) appears in the classical literature, see [4].

Of course, Ramanujan did not use the Weierstrass equations and preferred the Legendre ones, where one can see the hypergeometric functions instantly.

In order to pass to Legendre notations [5], one puts $4x^3 - g_2x - g_3 = 4(x-e_1)(x-e_2)(x-e_3)$, and looks at the modular function

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}$$

(an automorphic function $k^2 = k^2(\tau)$ with respect to the principal congruence subgroup $\Gamma(2)$ of $\Gamma(1) = SL_2(\mathbb{Z})$ in the variable τ in $H = \{\tilde{z}: \operatorname{Im} \tilde{z} > 0\}$). Then the periods and quasiperiods are expressed through the complete elliptic integrals of the first and second kind:

$$K(k) = \frac{\pi}{2} \cdot F(\frac{1}{2}, \frac{1}{2}; 1; k^2),$$

$$E(k) = \frac{\pi}{2} \cdot F(-\frac{1}{2}, \frac{1}{2}; 1; k^2).$$

We also look at $K(k')$, $E(k')$ for $k'^2 = 1 - k^2$; then ω_1, η_1 are expressed in terms of $K = K(k)$, $E = E(k)$, while ω_2, η_2 are expressed in terms of iK' and iE':

$$\omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \quad \omega_2 = \frac{iK'}{\sqrt{e_1 - e_3}};$$

$$\eta_1 = \sqrt{e_1 - e_3} \cdot E - e_1 \omega_1, \quad \eta_2 = -\sqrt{e_1 - e_3}\, iE' - e_3 \omega_2.$$

Invariant α in (1.1)--a nontrivial part of the Ramanujan quasiperiod relation--is easily recognized as one of the simplest values of "nonholomorphic Eisenstein series". Weil's treatise [6] creates a clear impression that this quantity and its algebraicity had been known to Kronecker (or even Eisenstein). It seems to us that though similar and more general functions were carefully examined, this particular connection had been reconstructed by Weil, and cannot be easily separated from his own work on period relations. The "nonholomorphic" Eisenstein series are too important to be ignored.

The usual Eisenstein series associated with the lattice $\mathcal{L}$ of periods: $\mathcal{L} = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ are

$$G_k(\omega_1,\omega_2) = \sum_{\substack{\omega\in\mathcal{L}\\ \omega\neq 0}} \omega^{-k} \quad \text{for} \quad k = 4,6,\ldots .$$

The corresponding normalized inhomogeneous series $E_k(\tau)$ are defined as

$$G_k(\omega_1,\omega_2) = \left(\frac{2\pi i}{\omega_2}\right)^k \cdot \frac{-B_k}{k!} \cdot E_k(\tau),$$

or

$$E_k(\tau) = 1 - \frac{2k}{B_k} \cdot \sum_{n=1}^{\infty} \sigma_{k-1}(n) \cdot q^n$$

for $\sigma_{k-1}(n) = \Sigma_{d|n} d^{k-1}$, and $q = e^{2\pi i\tau}$.

These q-series were subject of a variety of Ramanujan's studies [1-2] with his preferred notations a P,Q,R for $E_k(\tau)$ with $k = 2,4,6$, respectively.

For $k = 2$ the proper definition of $G_k(\omega_1,\omega_2)$ is a nonholomorphic one arising from

$$H(s,z) = \sum_{\omega\in\mathcal{L}} (\bar{z}+\bar{\omega})\,|z+\omega|^{-2s}$$

as:

$$G_2(\omega_1,\omega_2) = \lim_{s\to\infty^+} \sum_{\omega\in\mathcal{L}}{}^{\!*}\ \omega^{-2}\cdot|\omega|^{-2s}.$$

In the $E_k(\tau)$ notations, the quasiperiod relation is expressed by means of the function

$$s_2(\tau) \stackrel{\text{def}}{=} \frac{E_4(\tau)}{E_6(\tau)} \cdot \left(E_2(\tau) - \frac{3}{\pi \operatorname{Im}(\tau)}\right), \tag{1.2}$$

which is invariant under the action of $\Gamma(1)$ but nonholomorphic. It is this object that was studied by Ramanujan in connection with α in (1.1). Ramanujan actually proved in [1] that this function admits algebraic values whenever τ is imaginary quadratic. Moreover, Ramanujan [1] presented a variety of algebraic expressions for this function, differentiating modular equations.

His work, or Weil's, shows that the function in (1.2) has values from the Hilbert class field $\mathbb{Q}(\tau, j(\tau))$ of $\mathbb{Q}(\tau)$ for quadratic τ. The relation of (1.2) to α is simple: for $\beta = s_2(\tau)$ from (1.2), $\alpha = (B + 2A\tau)\beta \cdot g_3/g_2$.

Amazingly, Mordell in his notes [1] on Ramanujan paper missed the true importance of (1.1) or (1.2), stating merely "... Ramanujan's method of obtaining purely algebraical approximations appears to be new." These relations were rediscovered in the 70's (among the rediscovers was Siegel [7]), see particularly [9], and stimulated search for multidimensional generalizations of the period relations promoted by Weil [8]. We'll talk about generalizations of elliptic period relations later, but meanwhile let us look on relations (1.1) once more. One can combine (1.1) with the Legendre relation to arrive to a phenomenally looking "quadratic relation" derived by Ramanujan, that expresses π in terms of squares of ω_1 and η_1 only (no ω_2 and η_2!) All this is interesting, as an algebraic identity, but Ramanujan transforms these quadratic relations into rapidly convergent generalized hypergeometric representation of simple algebraic multiples of $1/\pi$. To do this he needed not only modular functions but also hypergeometric function identities.

We reproduce first Ramanujan's own favorite [1], which was used by Gosper in 1985 in his $17.5 \cdot 10^6$ decimal digit computation of π :

$$\frac{9801}{2\sqrt{2}\pi} = \sum_{n=0}^{\infty} \{1103 + 26390n\} \frac{(4n)!}{n!^4 \cdot (4 \cdot 99)^{4n}} \qquad (44).$$

(Numeration here is temporarily borrowed from [1].)

The reason for this pretty representation of $1/\pi$ lies in the representation of $(K(k)/\pi)^2$ as a ${}_3F_2$-hypergeometric function. Apparently there are four classes of such representations all of which were determined by Ramanujan: these are <u>four</u> distinct classes of ${}_3F_2$-representation of $1/\pi$, all based on special cases of Clausen identity (and all presented by Ramanujan [1]):

$$F(a,b;a+b+\tfrac{1}{2};z)^2 = {}_3F_2\left(\begin{matrix}2a,a+b,2b\\ a+b+\frac{1}{2},2a+2b\end{matrix} \,\middle|\, z\right). \qquad (1.3)$$

Unfortunately, the Clausen identity is a unique one--no other nontrivial relation between parameters makes a product of hypergeometric functions a generalized hypergeometric function.

We display the basis for the Ramanujan's series representation for $1/\pi$. We'll discuss them later in connection with arithmetic triangle groups. Meanwhile, what is good in these identities for diophantine approximations? First of all, Ramanujan approximations to π are indeed remarkably fast numerical schemes of evaluation of π. Unlike some other numerical schemes these are series schemes, that can be accelerated into Padé approximation schemes. These Padé approximations schemes are better numerically, but more important, they are nontrivial arithmetically good rational approximations to algebraic multiples of π, that provide nontrivial measures of diophantine approximations.

(Deviating momentarily, we want to compare Padé approximations vs. power series approximations in numerical evaluation of functions and constants. Remarkably, asymptotically there is no significant difference between Boolean complexities of evaluation of Padé approximations to solutions of linear differential equations and of power series approximations within a given precision. Unfortunately, asymptotically there is no gain in the degree of approximations either; moreover there is a significant difference in storage requirements. Padé approximations require more storage. Even in cases when explicit Padé approximations are known, gains of using them can be visible only in about hundreds of digits of precision; not below or above. That is why unless special circumstances call for (like uniform approximations with a minimal storage in ordinary precision range), Padé approximations and continued fraction expansion techniques should not be used for numerical evaluation.

However, in diophantine approximations we have no choice. Only Padé approximations and a vast army of their generalizations are capable of approximating functions and constants and tell something of their arithmetic nature, of their irrationalities and transcendences, measures of approximation, etc.).

All Ramanujan's quadratic period relations (four types) can be deduced from one series by modular transformations, and we choose the series as the one associated with the modular invariant $J = J(\tau)$. In the place of Ramanujan's nonholomorphic function we take, as above in (1.2):

$$s_2(\tau) = \frac{E_4(\tau)}{E_6(\tau)}\left(E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}\right).$$

Then the Clausen identity gives the following ${}_3F_2$-representation for an algebraic multiple of $1/\pi$:

$$\sum_{n=0}^{\infty}\left\{\frac{1}{6}(1-s_2(\tau))+n\right\}\cdot\frac{(6n)!}{(3n)!n!^3}\cdot\frac{1}{J(\tau)^n}$$
$$= \frac{(-J(\tau))^{1/2}}{\pi}\cdot\frac{1}{(d(1728-J(\tau)))^{1/2}}. \tag{1.4}$$

Here if $\tau = (1+\sqrt{-d})/2$. If $h(-d) = 1$ the second factor in the right hand side is a rational number. The largest one class discriminat $-d = -163$ gives the most rapidly convergent series (though coefficients are slightly strange):

$$\sum_{n=0}^{\infty}\{c_1+n\}\cdot\frac{(6n)!}{(3n)!n!^3}\,\frac{(-1)^n}{(640,320)^n}$$
$$= \frac{(640,320)^{3/2}}{163\cdot 8\cdot 27\cdot 7\cdot 11\cdot 19\cdot 127}\cdot\frac{1}{\pi}. \tag{1.5}$$

Here

$$c_1 = \frac{13,591,409}{163 \cdot 2 \cdot 9 \cdot 7 \cdot 11 \cdot 19 \cdot 127}$$

(and, of course, $J(\frac{1+\sqrt{-163}}{2}) = -(640,320)^3$).

Ramanujan [1] provides instead of this a variety of other formulas connected mainly with the three other triangle groups commensurable with $\Gamma(1)$.

Four classes of ${}_3F_2$ representations of algebraic multiples of $1/\pi$ correspond to four ${}_3F_2$ hypergeometric functions (that are squares of ${}_2F_1$ representations of complete elliptic integrals via the Clausen identity). These are

$$ {}_3F_2\left({1/2\,,\ 1/6\,,\ 5/6 \atop 1\,,\ 1}\Big|x\right) = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!n!^3}\left(\frac{x}{12^3}\right)^n \tag{1.6}$$

$$ {}_3F_2\left({1/4\,,\ 3/4\,,\ 1/2 \atop 1\,,\ 1}\Big|x\right) = \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4}\left(\frac{x}{4^4}\right)^n \tag{1.7}$$

$$ {}_3F_2\left({1/2\,,\ 1/2\,,\ 1/2 \atop 1\,,\ 1}\Big|x\right) = \sum_{n=0}^{\infty} \frac{(2n)!^3}{n!^6}\left(\frac{x}{2^6}\right)^n \tag{1.8}$$

$$ {}_3F_2\left({1/3\,,\ 2/3\,,\ 1/2 \atop 1\,,\ 1}\Big|x\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\cdot\frac{(2n)!}{n!^2}\left(\frac{x}{3^3.2^2}\right)^n. \tag{1.9}$$

The first function is the one arising in (1.4) with $x = 12^3/J(\tau)$. Other three functions correspond to modular transformations of $J(\tau)$. This means that appropriate $x = x(\tau)$ is a modular function of higher level (e.g. in (1.8), $x = 4k^2(1-k^2)$ for $k^2 = k^2(\tau)$), and series (1.7)-(1.9) for the same τ have slower convergence rates than the series in (1.6).

Representations similar to (1.5) can be derived for any of the series (1.6)-(1.9) for any singular moduli $\tau \in \mathbb{Q}(\sqrt{-d})$

and for any class number $h(-d)$, thus extending Ramanujan list [1] ad infinum. There is a simple recipe to generate these new identities, instead of elaborate procedure proposed in [1] (see also [5]) based on differentiating of modular equations. To derive these identities one needs the explicit expressions of $x_j = x(\tau_j)$ with the representatives τ_j in H of algebraically conjugate values of automorphic function $x = x(\tau)$ for $\tau \in \mathbb{Q}(\sqrt{-d})$ (say $\tau = \sqrt{-d}$ or $\tau = \frac{1+\sqrt{-d}}{2}$). E.g. for $x(\tau) = 12^3/J(\tau)$, τ_j: $j = 1,\ldots,h(-d)$ corresponds to the class number of $\mathbb{Q}(\sqrt{-d})$. The necessary values of $s_2(\tau_j)$ are easy to compute from q-series representation of $E_k(\tau)$, if to use the formula (1.2). These computations can be carried out in bounded precision, because, as we know, $s_2(\tau_j)$ lies in the Hilbert class field of $\mathbb{Q}(\sqrt{-d})$ and because, whenever $J(\tau_j)$ is algebraically conjugate to $J(\tau_i)$, numbers $s_2(\tau_j)$ and $s_2(\tau_i)$ are also algebraically conjugate. This allows us to express $s_2(\tau)$ in terms of $J(\tau)$ and $\sqrt{-d}$ explicitly using only finite precision approximations to all $s_2(\tau_j)$. This way one obtains rapidly convergent ${}_3F_2$ representations of algebraic multiples of π by nearly integral power series. When $h(-d) > 1$, these series contain nonrational numbers, making the series (1.5) the fastest convergent series representing a multiple of $1/\pi$, and having rational number entries only.

Even before Ramanujan's remarkable approximations to π, singular moduli evaluations were used to approximate multiples of π by logarithms of algebraic numbers (usually the values of modular invariants). One of the first series of such approximations belongs to Hermite [13]. Of course, by now it is reproduced in hundreds of papers and we have to give a customary example. One is looking here at the expansion of the modular invariant near infinity:

$$J(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots$$

for $q = e^{2\pi i\tau}$. The elementary theory of complex multiplication shows that for $\tau = (1+\sqrt{-d})/2$, $q^{-1} = e^{-\pi\sqrt{d}}$ is very close to an algebraic integer $J(\tau)$ of degree $h(-d)$. Usual examples (see description in [14]) involve the largest one class discriminant -163, $d = 163$, when:

$$e^{\pi\sqrt{163}} = -262537412640768743.9999999999992\ldots\ .$$

There is a variety of these and similar approximations of π by logarithms of other classical automorphic functions. One of the most popular, studied by Shanks et. al. [15], has a simple q-expansion:

For $f = f_1(\sqrt{-d})^{-24} = (k/4k'^2)^2$ at $\tau = \sqrt{-d}$,

$$\log f + \sqrt{d}\cdot\pi = 24 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} q^k(1-q^k)^{-1}.$$

It is also known that the right side can be expanded in powers of f:

$$\log f + \sqrt{d}\cdot\pi = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a_n}{n}\cdot f^n.$$

Here a_n are integers. Shanks [15] looks at specialization of this formula for -d with class groups of special structure for relatively large d. These approximations are not technically approximations to π, but rather to a linear form in π and in another logarithm. All of them are natural consequences of Schwarz theory and the representation of the function inverse to the automorphic one (say $J(\tau)$) as a ratio of two solutions of a hypergeometric equation. One such formula is

$$\pi i\cdot\tau = \ln(k^2) - \ln(16) + \frac{G(\frac{1}{2},\frac{1}{2};1;k^2)}{F(\frac{1}{2},\frac{1}{2};1;k^2)}, \tag{1.10}$$

and another (our favorite) is Fricke's [16]

$$2\pi i\cdot\tau = \ln J + \frac{G(\frac{1}{12},\frac{5}{12};1;\frac{12^3}{J})}{F(\frac{1}{12},\frac{5}{12};1;\frac{12^3}{J})}. \tag{1.11}$$

Here $G(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}\cdot\{\sum_{j=0}^{n-1}(\frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{c+j})\}$ is the hypergeometric function (of the second kind) in the exceptional case, when there are logarithmic terms.

Perhaps the most popular approximations to linear forms in π and in another logarithm are associated with Stark-Baker solution to one-class and two-class problems (cf. [17]). Stark's approach [18] is based on Kronecker's limit formula, and in a way, similar to approximations given above, one represents

$$L(1,\chi)\cdot L(1,\chi\chi') - \frac{\pi^2}{6}\prod_{p|k}(1-\frac{1}{p^2})\sum_f \frac{\chi(a)}{a}$$

for $\chi(\cdot) = (\frac{k}{\cdot})$, $\chi'(\cdot) = (\frac{-d}{\cdot})$, $k > 0$ as a rapidly convergent

$q^{1/k}$-series (here $f = ax^2 + bxy + cy^2$ runs through a complete set of unequivalent quadratic forms with the discriminant $-d$). Using Dirichlet's class number formula one obtains an exceptional approximation (as above) to the linear form in two logarithms:

$$|h(-kd)\cdot\log \varepsilon_{\sqrt{k}} - q\pi\sqrt{d}| < e^{-\pi O(\sqrt{d}/k)}$$

for an arbitrary fundamental unit $\varepsilon_{\sqrt{k}}$ of $\mathbb{Q}(\sqrt{k})$ and for one-class discriminant $-d$. These remarkable linear forms were generalized by Stark to three-term linear forms in the class number two case.

(While these unusually good approximations can be used in the class number problem--approximations are so good that they are impossible for large d--these linear forms cannot be used for the analysis of arithmetic properties of the individual logarithms, like π, entering the linear form. Moreover, as the class number grows, the number of the terms in the linear form grows.)

Important developments initiated by Ramanujan in his truly algebraic approximations to $1/\pi$ can be extended to the analysis of linear forms in logarithms presented above. In fact, each term in these linear forms can be separately represented by a rapidly convergent series in $1/J$ with nearly integral coefficients.

For this one takes an automorphic function $\varphi(\tau)$ with respect to one of the congruence subgroups of $\Gamma(1)$ and expand functions like $F(\frac{1}{12},\frac{5}{12};1;12^3/J)$, $G(\frac{1}{12},\frac{5}{12};1,;12^3/J)$ in powers of $\varphi(\tau)$ instead of $J(\tau)$. Whenever $\varphi(\tau)$ is automorphic with respect to a classical triangle group, we arrive to the corresponding usual hypergeometric functions.

Other logarithms, like π, can be represented as values of convergent series satisfying Fuchsian linear differential equations. This is particularly clear for $\log \varepsilon_{\sqrt{k}}$ of a fundamental unit $\varepsilon_{\sqrt{k}}$ of a real quadratic field $\mathbb{Q}(\sqrt{k})$. To represent this number as a convergent series (in, say, $1/J(\tau)$) one uses Kronecker's limit formula expressing this logarithm $\log \varepsilon_{\sqrt{k}}$ in terms of products of values of Dedekind's Δ-function ("Jugandtraum", see [6]). Such an expression of $\log \varepsilon_{\sqrt{k}}$ in terms of power series in $1/J(\tau)$ for $\tau = (1+\sqrt{-d})/2$ for any $d \equiv 3(8)$, depends, unfortunately, on k, because k is related to the level of the appropriate modular form $\varphi = \varphi_k(\tau)$.

For $k = 5$ Siegel [19] made an explicit computation that expresses $J(\tau)$ in terms of the resolvent $\varphi_5(\tau)$ of 5-th degree modular equation known from the classical theory of 5-th degree equations. His relations [19] were:

$$(\varphi-\varepsilon^3)((\varphi-\varepsilon)(\varphi^2+\varepsilon^{-1}\varphi+\varepsilon^{-2}))^3 + (\varphi/\sqrt{5})^5 J = 0$$

and

$$\varphi(\tau)(=\varphi_5(\tau)) = \varepsilon^{h(-5p)/2}$$

for $\tau = (1+\sqrt{-p})/2$ and $\varepsilon = \varepsilon_{\sqrt{5}}$. Here one has $p \equiv 3(5)$; if $p \equiv 2(5)$ and replaces ε by ε^{-1} in the expression of $J = J(\tau)$.

This, in combination with Ramanujan's approximation to π, allows one to express $\log \varepsilon$ (its multiple) as a convergent series in $1/J$ (or in $1/\omega$).

References

[1] S. Ramanujan, Collected Papers, Cambridge, 1927, 23-39.

[2] G.H. Hardy, Ramanujan, Cambridge, 1940.

[3] G.N. Watson, Some singular moduli (I);(II);(III);(IV); Quart. J. Math. Oxford, 3 (1932), 81-98; 189-212; Proc. London Math. Soc. 40 (1936), 83-142; Acta Arithmetica, 1 (1936), 284-323.

[4] J.M. Borwein, P.B. Borwein, Pi and the AGM, Wiley, 1987.

[5] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, 4 ed., Cambridge, 1927.

[6] A. Weil, Elliptic Functions According to Eisenstein and Kronecker, Springer, 1976.

[7] C.L. Siegel, Bestimmung der elliptischen Modulfunktionen durch eine Transformations gleichung, Abh. Math. Sem. Univ. Hamburg, 27 (1964), 32-38.

[8] A. Weil, Sur les Périodes des Intégrales Abéliennes, Comm. Pure Appl. Math., 29 (1976), 813-819.

[9] D. Masser, Elliptic Functions and Transcendence, Lecture Notes Math., v. 437, Springer, 1975.

[10] S. Lefschetz, On certain numerical invariants of algebraic varieties with application to Abelian varieties, Trans. Amer. Math. Soc., 22 (1921), 327-482.

[11] E.T. Whittaker, On hyperlemniscate functions, a family of automorphic functions, J. London Math. Soc., 4 (1929), 274-278.

[12] D.V. Chudnovsky, G.V. Chudnovsky, Computer assisted number theory, Lecture Notes Math., Springer, 1240, 1987, 1-68.

[13] Ch. Hermite, Sur la Théorie des Équations Modulaires, C.R. Acad. Sci. Paris., 48 (1859), 940-1079-1097; 49 (1859), 16-110-141.

[14] H.M. Stark, Class-numbers of complex quadratic fields, Lecture Notes Math., Springer, v. 320, 1973, 153-174.

[15] D. Shanks, Dihedral quartic approximation and series for π, J. Number Theory, 14 (1982), 397-423.

[16] R. Fricke, Die Elliptischen Funktionen und ihre Anwendungen, v. 1, Teubner, 1916.

[17] A. Baker, Transcendental Number Theory, Cambridge, 1979.

[18] H.M. Stark, A transcendence theorem for class number problems, I; II; Ann. Math. 94 (1971), 153-173; 96(1972), 174-209.

[19] C.L. Siegel, Zum Beweise des Starkschen Satzes, Invent. Math., 5 (1968), 180-191.

[20] G. Shimura, Automorphic forms and the periods of Abelian varieties, J. Math. Soc. Japan, 31 (1979), 561-59.

[21] G. Shimura, The arithmetic of certain zeta functions and automorphic forms on orthogonal groups, Ann. of Math., 111 (1980), 313-375.

[22] P. Deligne, Valeurs de fonctions L et périodes d'integrales, Proc. Symp. Pure Math., v. 33, Part 2, Amer. Math. Soc., Providence, R.I., 313-346.

[23] P. Deligne, Cycles de Hodge absolus et périodes des intégrales des variétés abéliennes, Soc. Math. de France, Memoire, N° 2, 1980, 23-33.

[24] G. Shimura, Introduction to the Arithmetic Theory of Automorphic Forms, Princeton, University Press, 1971.

[25] G. Shimura, Y. Taniyama, Complex Multiplication of Abelian Varieties and Its Applications to Number Theory, Publications of the Mathematical Society of Japan, N° 6, 1961.

[26] G.V. Chudnovsky, Algebraic independence of values of exponential and elliptic functions. Proceedings of the International Congress of Mathematicians, Helsinki, 1979, Acad. Sci. Tennice, Helsinki, 1980, v. 1, 339-350.

[27] B.H. Gross, N. Koblitz, Gauss sums and the p-adic Γ-function, Ann. of Math., 109 (1979), 569-581.

[28] R. Fricke, F. Klein, Vorlesungen über die Theorie der Automorphen Functionen, bd. 2, Tenbner, 1926.

[29] R. Morris, On the automorphic functions of the group $(0,3;\ell_1,\ell_2,\ell_3)$, Trans. Amer. Math. Soc., 7 (1906), 425-448.

[30] K. Takeuchi, Arithmetic triangle groups, J. Math. Soc. Japan, 29 (1977), 91-106.

[31] H.P.F. Swinnerton-Dyer, Arithmetic groups in Discrete Groups and Automorphic Functions, Academic Press, 1977, 377-401.

[32] J.I. Hutchinson, On the automorphic functions of the group (0,3;2,6,6), Trans. Amer. Math. Soc., 5 (1904), 447-460.

[33] D.V. Chudnovsky, G.V. Chudnovsky, Note on Eisenstein's system of differential equations, in Classical and Quantum Models and Arithmetic Problems, M. Dekker, 1984, 99-116.

[34] D.V. Chudnovsky, G.V. Chudnovsky, The use of computer algebra for diophantine and differential equations, in Computer Algebra as a Tool for Research in Mathematics and Physics, Proceeding of the International Conference, 1984, New York University, M. Dekker, 1987.

[35] D.V. Chudnovsky, G.V. Chudnovsky, Applications of Padé approximations to diophantine inequalities in values of G-functions, Lecture Notes Math., v. 1135, Springer, 1985, 9-50.

[36] K. Mahler, Perfect systems, Compositio Math., 19 (1968), 95-166.

[37] D.V. Chudnovsky, G.V. Chudnovsky, Applications of Padé approximations to the Grothendieck conjecture on linear differential equations, Lecture Notes Math., v. 1135, Springer, 1985, 51-100.

[38] C.L. Siegel, Über einige Anwendungen diophantischer Approximationen, Abh. Preuss. Akad. Wiss. Phys. Math. Kl. 1, 1929.

[39] G.V. Chudnovsky, Contributions to the Theory of Transcendental Numbers, Mathematical Surveys and Monographs, v. 19, Amer. Math. Soc., Providence, R.I., 1984.

[40] G.V. Chudnovsky, Padé approximation and the Riemann monodromy problem, in Bifurcation Phenomena in Mathematical Physics and Related Topics, D. Reidel, Boston, 1980, 448-510.

[41] G.V. Chudnovsky, Rational and Padé approximation to solutions of linear differential equations and the monodromy theory. Lecture Notes Physics, v. 126, Springer, 1980, 136-169.

[42] G.V. Chudnovsky, Padé approximations to the generalized hypergeometric functions. I., J. Math. Pures et Appliques, Paris, 58 (1979), 445-476.

[43] G.V. Chudnovsky, On the method of Thue-Siegel, Ann. of Math., 117 (1983), 325-382.

[44] C.L. Siegel, Transcendental Numbers, Princeton University Press, 1949.

[45] G.V. Chudnovsky, Rational approximations to linear forms of exponentials and binomials, Proc. Nat'l Acad. Sci. U.S.A., 80 (1983), 3138-3141.

[46] D.V. Chudnovsky, G.V. Chudnovsky, A random walk in higher arithmetic, Adv. Appl. Math., 7 (1986), 101-122.

[47] H. Poincare, Sur les groupes des équations lineaires, Acta. Math., 4 (1884) 201-312.

[48] D.A. Hejhel, Monodromy groups and Poincaré series, Bull. Amer. Math. Soc., 84 (1978), 339-376.

[49] E.T. Whittaker, On the connexion of algebraic functions with automorphic functions, Phil. Trans., 122 A(1898), 1-32.

[50] R.A. Rankin, The differential equations associated with the uniformization of certain algebraic curves, Proc. Roy. Soc. Edinburgh, 65 (1958) 35-62.

[51] A. Schonhage, V. Strassen, Schnelle Multiplikation grosser Zahlen, Computing, 7 (1971), 281-292.

[52] D.V. Chudnovsky, G.V. Chudnovsky, Algebraic complexities and algebraic curves over finite fields, Proc. Natl. Acad. Sci. USA, 84 (1987), 1739-1743.

[53] A. Schonhage, Equation solving in terms of computational complexity, Proc. International Congress of Mathematicians, Berkeley, 1986.

[54] R.P. Brent, Multiple-precision zero-finding methods and the complexity of elementary function evaluation, in Analytic Computational Complexity, J.F. Traub, Ed., Academic Press, 1975, 151-176.

[55] R.P. Brent, The complexity of multiple-precision arithmetic, in Complexity of Computational Problem Solving, R.S. Anderson and R.P. Brent, Eds. Univ. of Queensland Press, Brisbane, Australia, 1975, 126-165.

[56] D.V. Chudnovsky, G.V. Chudnovsky, On expansions of algebraic functions in power and Puiseux series, I; II; J. of Complexity, 2 (1986), 271-294; 3 (1987), 1-25.

[57] O. Perron, Die Lehre von den Kettenbrüchen, Teubner, 1929.

[58] D. Bini, V. Pan, Polynomial division and its computational complexity, J. of Complexity, 2 (1986), 179-203.

[59] D. A. Hejhal, A classical approach to a well-known spectral correspondence on quaternion groups, Lecture Notes Math., v. 1135, Springer, 1985, 127-196.