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# Splitting homotopy idempotents II

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In honor of Professor Alex Heller
on the occasion of his sixty-fifth birthday

## 1. Introduction and main results

The question studied here arose in connection with an investigation of the representability of half-exact homotopy functors [5]; specifically, (P1) in Section 2 below gives an example of a retract of a representable functor in unpointed homotopy which is not representable. The same question arises in shape theory and Dydak [1], from this starting point, arrived independently at part of our Main Theorem.

We say that an idempotent $e : A \to A$ (in any category) *splits* if there exist $g : A \to B$, $h : B \to A$ such that $A \xrightarrow{g} B \xrightarrow{h} A = e$ and $B \xrightarrow{h} A \xrightarrow{g} B = 1_B$. Brown's theorem on the representability of half-exact functors implies that all idempotents split in the homotopy category of connected, pointed CW-complexes (the maps being homotopy-classes of continuous maps).

To say that $f^2$ is homotopic to $f$ can mean one of two things: by *strict-homotopy* we mean a homotopy that preserves the base-point; by *free-homotopy* we mean a homotopy that pays no attention to the base-point. The assertion that homotopy

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* This work dates back at least 20 years (its first appearance was in a series of ETH, Zurich, Spring 1969). This particular manuscript has existed (with just a few changes) for at least ten years and for a tiresome series of reasons remained unpublished. Since it has been cited rather extensively (often just by a reference to the 'Freyd–Heller group') this seems to be an excellent occasion to get it into print. (pjf)

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idempotents split is true only for strict-homotopy—we will exhibit counterexamples for free-homotopy. (Note that if we do not insist that the space be connected we may adjoin a disjoint base-point to all spaces and translate statements about free-homotopy to statements about strict-homotopy.)

Suppose that $\mathbb{A}$ is a category in which all idempotents split and that $\equiv$ denotes a congruence on $\mathbb{A}$ (that is, $f \equiv g$ implies $hf \equiv hg$ and $fh \equiv gh$ whenever the compositions are defined). Let $\mathbb{A}/\equiv$ be the resulting quotient category (its objects are the same as those of $\mathbb{A}$, its map are $\equiv$-classes of maps in $\mathbb{A}$). We seek conditions for idempotents to split in $\mathbb{A}/\equiv$. Examples include: $\mathbb{A}$ the category of CW-complexes, $\equiv$ either strict or free homotopy; $\mathbb{A}$ the category of groups, $\equiv$ denoting conjugacy, that is, $f \equiv g$ iff there exists $\alpha$ such that $f(x) = \alpha^{-1}g(x)$ for all $x$; $\mathbb{A}$ the category of categories, $\equiv$ denoting natural equivalence of functors. We do not have an answer for all cases but there is a type of congruence for which we do, namely those that arise from certain actions of group-valued functors.

An *action of a group-valued functor*, $\pi$, on $\mathbb{A}$ is an assignment for each $f : A \to B$ in $\mathbb{A}$ and $\alpha \in \pi(B)$ a map $f^\alpha : A \to B$ subject to the conditions:

(A1) $f^1 = f$,
(A2) $(f^\alpha)^\beta = f^{\alpha\beta}$,
(A3) $A \xrightarrow{f} B \xrightarrow{g^\alpha} C = (A \xrightarrow{f} B \xrightarrow{g} C)^\alpha$,
(A4) $A \xrightarrow{f^\alpha} B \xrightarrow{g} C = (A \xrightarrow{f} B \xrightarrow{g} C)^{(\pi g)(\alpha)}$.

By a *conjugacy action* we mean an action that satisfies the further condition:

(A5) $\pi(f^\alpha) = \alpha^{-1}(\pi f)\alpha$.

The easiest and most fundamental example is the case that $\pi$ is the identity functor on the category of groups and $f^\alpha$ is defined by conjugation (as forced by A5).

Given an action of $\pi$ on $\mathbb{A}$ we obtain a congruence by defining $f \equiv g$ iff there exists $\alpha$ such that $f = g^\alpha$. ($\mathbb{A}/\equiv$ is thus an 'orbit category'.)

The motivating topological example is the case that $\mathbb{A}$ is the strict-homotopy category and $\pi$ is the fundamental-group functor in which case $\mathbb{A}/\equiv$ is the free-homotopy category. The conjugacy action of $\pi$ on $\mathbb{A}$ is obtained as follows: given $f : A \to B$ and $\alpha \in \pi(B)$ let $I$ denote the unit-interval and let $H : A \times I \to B$ be a map such that $H|_{(A \times \{0\})} = f$ and $H|_{(\{*\} \times I)}$ represents $\alpha$. Then $H|_{(A \times \{1\})}$ is unique up to strict-homotopy and we take it as the definition of $f^\alpha$. ($H$ always exists since $A \vee I$ is a retract of $A \times I$.) Note that $\mathbb{A}/\equiv$ is the category of connected pointed CW-complexes and free-homotopy classes of *pointed* maps. One could object: the free-homotopy category ought totally to ignore base-points. But it is transparent that the category whose objects are non-empty connected CW-complexes and whose maps are free-homotopy classes of maps (no base-points in sight) is equivalent (and with a hefty use of the axiom of choice, isomorphic) to $\mathbb{A}/\equiv$.

**Main Lemma.** *For any $\pi : \mathbb{A} \to \mathfrak{g}$ and conjugacy action of $\pi$ on $\mathbb{A}$ consider the commutative diagram*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\pi"] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathbb{A}/\equiv \arrow[r, "\bar{\pi}"] & \mathfrak{g}/\equiv \end{tikzcd}$$

*If idempotents split in $\mathbb{A}$ then $\bar{\pi}$ preserves and reflects the existence of splittings. That is, if $f$ is an idempotent in $\mathbb{A}/\equiv$ then $f$ splits in $\mathbb{A}/\equiv$ iff $\bar{\pi}(f)$ splits in $\mathfrak{g}/\equiv$.*

Any functor preserves splittings. The converse is proved in Section 3.

We are thus led to consider conjugacy idempotents on groups. There is a universal example, that is, there is a group $F$ together with an endomorphism $f$ and an element $a \in F$ such that $f^2 = f^\alpha$ with the property that for any other such triple $\langle G, g, \beta \rangle$ there exists a unique $h : F \to G$ such that

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d, "f"'] & G \arrow[d, "g"] \\ F \arrow[r, "h"] & G \end{tikzcd}$$

commutes and $h(\alpha) = \beta$. (The simplest construction of $F$ is as the initial algebra for the equational theory obtained by adjoining to the theory of groups a unary operation $f$, a constant $\alpha$, and two equations:

$$f(xy) = (fx)(fy)\,, \qquad f(fx) = \alpha^{-1}(fx)\alpha.)$$

**Main Theorem.** *Let $\langle F, f, \alpha \rangle$ be the universal conjugacy idempotent.*

(T1) *An arbitrary conjugacy idempotent $\langle G, g, \beta \rangle$ fails to split iff the induced map $F \to G$ is an embedding.*

(T2) *$F$ is a finitely presentable group.*

(T3) *The commutator subgroup $F'$ of $F$ is simple (and non-trivial).*

(T4) *$F'$ contains a copy of $F$.*

(T5) *$F$ is torsion-free; indeed, it is a totally ordered group.*

(T6) *$F$ contains a copy of its own infinite wreath-product.*

(T7) *Every abelian subgroup of $F$ is free abelian.*

(T8) *Every subgroup of $F$ is either finite-rank free abelian or contains an infinite-rank free abelian subgroup.*

The proof appears in Section 4.

Among the consequences:

**Main Corollary.** *Let $\mathscr{C}$ be the class of all groups that do not contain a copy of $F$.*

(C1) *If $G \in \mathscr{C}$ then every conjugacy idempotent on $G$ splits.*

(C2) *$\mathscr{C}$ is a pseudo-variety, that is, it is closed under the formation of subgroups and infinite cartesian products.*

(C3) *$\mathscr{C}$ contains all residually torsion groups.*

(C4) *$\mathscr{C}$ contains all groups of the form $\mathrm{GL}(n, K)$ where $K$ is an arbitrary field, hence it contains all groups whose linear representations are collectively faithful.*

(C5) *$\mathscr{C}$ is closed under extensions, that is, if $H \lhd G$ then $H, G/H \in \mathscr{C}$ imply $G \in \mathscr{C}$.*

(C6) *$\mathscr{C}$ contains every proper group-variety. That is, if $G$ satisfies any non-trivial equation then $G \in \mathscr{C}$.*

(C7) *$\mathscr{C}$ is closed under the formation of directed colimits.*

For the most part these are immediate consequences of the Main Theorem. C1 is just a restatement of T1. T4 says that $G \in \mathscr{C}$ iff $G$ does not contain a copy of $F'$ and since $F'$ is simple (T3) we easily obtain C2 and C5. Indeed, C5 may be improved:

(C8) *Let $G$ be a group, $\Omega$ a section of ordinal numbers, and $\{G_\alpha\}_\Omega$ a chain of subgroups that descends to $\{1\}$ such that $G_{\alpha+1} \lhd G_\alpha$ for all $\alpha$ and for all limit ordinals $\alpha$ (including 0) it is the case that $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$. Then $G_\alpha/G_{\alpha+1} \in \mathscr{C}$ all $\alpha$ implies $G \in \mathscr{C}$.*

As a sample consequence:

(C9) *$\mathscr{C}$ contains all transfinitely solvable groups.*

(C3) follows immediately from C2 and T5. It, too, may be improved.

(C10) *Let $\mathscr{G}$ be an hereditary class of groups, that is, whenever $H \to G$ is an embedding then $G \in \mathscr{G}$ implies $H \in \mathscr{G}$. If $F \not\in \mathscr{G}$ then the pseudo-variety $\hat{\mathscr{G}}$ generated by $\mathscr{G}$ is contained in $\mathscr{C}$.*

If $\mathscr{G}$ is the class of groups satisfying some property $\mathbb{P}$ then $\hat{\mathscr{G}}$ is the class of groups that are 'residually $\mathbb{P}$'. A group is in $\hat{\mathscr{G}}$ iff it is embedded in the product of all its quotients that lie in $\mathscr{G}$.

Let $\mathscr{G}$ be the class of groups with the property that each finitely presented subgroup is residually finite. T2 and T5 imply that $F \not\in \mathscr{G}$ and hence $\hat{\mathscr{G}} \subseteq \mathscr{C}$. This appears to be only a technical improvement until one recalls that all linear groups lie in $\mathscr{G}$. Thus C4.

C4 may be improved:

(C11) *$\mathscr{C}$ contains all groups that may be faithfully represented in groups of the form $\mathrm{GL}(n, R)$ where $R$ is a commutative ring (indeed, any PI ring).*

For commutative $R$ note first that the problem immediately reduces to the cases that $R$ is Noetherian (because $F$ is finitely generated). Let $\mathcal{R}$ be the radical of $R$ and let $H \subseteq \mathrm{GL}(n, R)$ be the subgroup of elements equivalent mod $\mathcal{R}$ to the identity element. Note that $H$ is the intersection of all the kernels of the maps $\mathrm{GL}(n, R) \to \mathrm{GL}(n, R/\mathcal{P})$ as $\mathcal{P}$ ranges over all prime ideals. $H$ is solvable and it thus suffices to show that any copy of $F'$ in $\mathrm{GL}(n, R)$ is contained in $H$. But we have already established that any map of the form $F' \to \mathrm{GL}(n, R) \to \mathrm{GL}(n, R/\mathcal{P})$ is trivial.

C6 is a consequence of T6. We suspect that the experts have recorded this theorem but we have not yet found a reference. We include a proof in Section 5.

The finite presentability of $F$ implies that the functor it represents, $(F, -)$, preserves directed colimits (in fact, it is equivalent to this). Any map $h : F \to G$, $G \in \mathscr{C}$, must kill $F'$ (using T3 and T4). Hence C7.

Birkhoff's theorem says that a class of groups is defined by a family of equational conditions (that is, it is a variety) iff it is a pseudo-variety closed under the formation of quotient groups. A well-known variation is that a class of groups is defined by a family of universally quantified Horn sentences iff it is a pseudo-variety closed under the formation of directed colimits. Rather than argue the general case we use, here, the fact that $F$ is generated by two elements $a,b$ subject to two relations $[b^a, ba^{-1}] = [b^{a^2}, ba^{-1}] = 1$ (as proved in Section 4).

(C12) *$G \in \mathscr{C}$ iff it satisfies the condition*

$$\forall x, y([y^x, yx^{-1}] = [y^{x^2}, yx^{-1}] = 1 \ \Rightarrow\ [x, y] = 1) .$$

## 2. Pathology

(P1) *There are free-homotopy idempotents that do not split.*

The fundamental group functor has a one-sided inverse $K(-, 1) : \mathfrak{g} \to \mathbb{A}$, the Eilenberg-Mac Lane functor. By the Main Lemma we need only verify that it carries inner automorphisms to maps that are freely homotopic to identity maps. Thus $(K(F, 1), K(f, 1))$ provides an example of an unsplit free-homotopy idempotent.

An evident question is, what other complexes share this property of carrying such an idempotent? It is known [4] that they must be infinite-dimensional.

(P2) *For any ring $K$ there exists a $K$-algebra with a conjugacy idempotent that cannot be split. If $K$ is without zero-divisors then the $K$-algebra may be chosen to be without zero-divisors. If $K$ is a division algebra then the $K$-algebra may be chosen to be a division algebra.*

Let $\mathbb{A}$ be the category of $K$-algebras, $\pi : \mathbb{A} \to \mathfrak{g}$ the 'group of units' functor, that is, $\pi(R) = R^*$, the set of units in $R$. The resulting congruence on $\mathbb{A}$ is, of course, given by conjugation. The nearest approximation to the topological $K[-, 1]$, is $K[-]$, the group-ring functor. It is not a one-sided inverse for $(-)^*$, but there is a comparison map, $G \to (K[G])^*$, and it is an embedding. T1 easily implies now that $K[F]$ is an example. If $K$ is without zero-divisors then, as for any totally ordered group (TOG), $K[F]$ is also without zero-divisors.

If $K$ is a division algebra one may obtain a new division algebra by replacing $K[F]$ with the set of functions from $F$ to $K$ not with finite support but with well-ordered support (that is, formal $K$-linear combinations of elements of $F$ for which the subset of elements that appear in any given linear combination is well-ordered under the ordering induced from $F$).

## 3. Proof of the Main Lemma

From now to the end of the paper $\mathbb{A}$ denotes a category in which all idempotents split, $\pi : \mathbb{A} \to \mathfrak{g}$ a functor with a conjugacy action on $A$.

(L1) *An idempotent in $A/\equiv$ splits iff there exists $f'$ such that $f \equiv f'$ and $(f')^2 = f'$.*

One direction is clear: given such $f'$ we may, by hypothesis, split $f'$ in $\mathbb{A}$ to obtain a splitting of $f$ in $\mathbb{A}/\equiv$.

The other direction is an easy computation. Suppose that $f$ splits in $\mathbb{A}/\equiv$, that is, there exist maps $g,h$ such that $gh \equiv f$ and $hg \equiv 1$. Let $\alpha$ be such that $hg = 1^\alpha$ and define $f'$ as $g^{\alpha^{-1}}h$. □

Henceforth we will write $\pi(f)$ as $f^*$.

(L2) *If $f^2 = f^\alpha$ and if $\alpha \in \mathrm{Image}(f^*)$ then $f$ splits in $\mathbb{A}/\equiv$.*

Let $\beta$ be such that $f^*(\beta) = \alpha^{-1}$. Then $f^\beta$ is an idempotent in $\mathbb{A}$ (because $(f^\beta)^2 = f^\beta f^\beta = (f^2)^{f^*(\beta)\beta} = (f^\alpha)^{\alpha^{-1}\beta} = f^\beta$). □

For the Main Lemma, suppose that $f^*$ splits in $\mathfrak{g}/\equiv$. By L1 there exists $\alpha$ such that $(f^*)^\alpha$ is an idempotent in $\mathfrak{g}$. A5 says, therefore, that $g = f^\alpha$ is such that $g^*$ is an idempotent in $\mathfrak{g}$. Since $g \equiv f$ it clearly suffices to split $g$ in $\mathbb{A}/\equiv$. $g^*$ obviously satisfies the hypothesis of:

(L3) *If $g^2 = g^\beta$ and if $g^*(\beta)$ is a fixed point of $g^*$ (that is, $(g^*)^2(\beta) = g^*(\beta)$) then $g$ splits in $\mathbb{A}/\equiv$.*

It clearly suffices to split $h = g^2$. But $h^2 = g^2gg = g^{\beta}gg = (g^2)^{g^*}g = g^{\beta g^*(\beta)}g = (g^2)^{g^*(\beta^2)} = h^{h^*(\beta^2)}$ and by L2, $h$ splits. $\square$

## 4. Proof of the Main Theorem

Let $F$ be the group generated by a sequence of elements $\alpha_0, \alpha_1, \alpha_2, \ldots$ subject to the relations $\alpha_j\alpha_i = \alpha_i\alpha_{j+1}$ all $0 \le i < j$. We define an endomorphism $f$ by $f(\alpha_i) = \alpha_{i+1}$. Then $f^2(\alpha_i) = \alpha_{i+2} = \alpha_{i+1}^{\alpha_0} = f^{\alpha_0}(\alpha_i)$ and $f$ is a conjugacy idempotent. Given another conjugacy idempotent $\langle G, g, \beta\rangle$ define $h : F \to G$ by $h(\alpha_i) = g^i(\beta)$. ($h$ extends from the generators to a homomorphism because $h(\alpha_i^{-1})h(\alpha_j)h(\alpha_i) = g^i(\beta^{-1})g^j(\beta)g^i(\beta) = g^i(\beta^{-1}g^{j-1}(\beta)\beta) = g^i(g^{j-i+1}(\beta)) = g^{j+1}(\beta) = \alpha_{j+1}$ for all $i \le j$); $h$ is clearly the unique map such that $h(\alpha_0) = \beta$ and $g(h(x)) = h(f(x))$ all $x$.

**Proof of T2.** $F$ is generated by $\alpha_0, \alpha_1$ because $\alpha_{i+1} = \alpha_0^{-1}\alpha_i\alpha$ for each $i > 0$ hence by iteration, $\alpha_{i+1} = \alpha_0^{-i}\alpha_1\alpha_0^i$. The relations $\alpha_j\alpha_1 = \alpha_1\alpha_{j+1}$, $j > 1$ now suffice: the case $i = 0$ is automatic from the definition of $\alpha_j$ and the case $\alpha_{j+1}\alpha_{i+1} = \alpha_{i+1}\alpha_{j+2}$ for $0 < i < j$ is obtained by $\alpha_{j+1}\alpha_{i+1} = (\alpha_{j-i+1}\alpha_1)^{\alpha_0^i} = (\alpha_1\alpha_{j-i+2})^{\alpha_0^i} = \alpha_{i+1}\alpha_{j+2}$. In fact, we need $\alpha_j\alpha_1 = \alpha_1\alpha_{j+1}$ only for $j = 2{,}3$. From the case $j = 2$ we obtain, as just seen, $\alpha_{j+1}^{-1}\alpha_{j+2}\alpha_{j+1} = \alpha_{j+3}$ for $j \ge 0$. Assume that we have $\alpha_k^{\alpha_1} = \alpha_{k+1}$ for all $1 < k < j$. Then $\alpha_j^{\alpha_1} = (\alpha_{j-2}^{-1}\alpha_{j-1}\alpha_{j-2})^{\alpha_1} = \alpha_{j-1}^{-1}\alpha_j\alpha_{j-1} = \alpha_{j+1}$. The argument requires that $1 < j - 2$, hence we must start the induction with the case $j = 3$. $\square$

The commutator form for these relations used in C11 are obtainable as follows:

$$\begin{aligned}
&\alpha_1^{-1}\alpha_{j+1}\alpha_1 = \alpha_{j+1} \\
&\Leftrightarrow \quad \alpha_1^{-1}\alpha_0^{-j}\alpha_1\alpha_0^j\alpha_1 = \alpha_0^{-j-1}\alpha_1\alpha_0^{j+1} \\
&\Leftrightarrow \quad \alpha_0^{-j}\alpha_1\alpha_0^j\alpha_1\alpha_0^{-1} = \alpha_1\alpha_0^{-j-1}\alpha_1\alpha_0^j \\
&\Leftrightarrow \quad \alpha_1^{\alpha_0^j}(\alpha_1\alpha_0^{-1}) = (\alpha_1\alpha_0^{-1})\alpha_1^{\alpha_0^j}.
\end{aligned}$$

We shall need the following technical lemma.

(L4) *Any non-trivial element of $F$ is conjugate to an element of the form $\alpha_i^n f^{i+1}(\beta)$ where $i = 0{,}1$ and $n \ne 0$.*

**Proof.** By the length of an element we mean the length of the shortest word on the $\alpha$'s needed to describe it. We may choose a conjugate of a given non-trivial element of minimal length. If we choose a minimal word on the $\alpha$'s to describe it, we may conjugate with $\alpha_0^{-1}$, if necessary, to insure that the smallest index, $i$, is either 0 or 1. If $\alpha_j^{\pm 1}\alpha_i$, $i \neq j$ appears as a sub-word it may be replaced with $\alpha_i\alpha_{j+1}^{\pm 1}$ and if $\alpha_i^{-1}\alpha_j^{\pm 1}$, $j \neq i$ appears it may be replaced with $\alpha_{j+1}^{\pm 1}\alpha_i^{-1}$. By iteration, all possible occurrences of $\alpha_i$ may be moved to the far left, all negative occurrences to the far right and the word becomes equivalent to one of the form $\alpha_i^b(\ldots)\alpha_i^{-c}$ where all indices in $(\ldots)$ are larger than $i$ and hence $(\ldots)$ describes an element of $f^{i+1}(F)$. Finally, conjugate by $\alpha_i^c$ to obtain $\alpha_i^{b-c}f^{i+1}(\beta)$. If $b-c=0$ then the original element is conjugate to $f^{i+1}(\beta)$ which is of smaller length. □

We have found two closely related permutation representations to be very useful. We began with a huge group and then cut down. Let $\Sigma$ be the group of all continuous order-preserving permutations on the real numbers. For $T \in \Sigma$ we will find it convenient to use the 'diagramatic' order to denote its action: $T$ sends a real number $x$ to $xT$. The support of $T$, $\mathrm{spt}(T)$, is its set of non-fixed points, $\{x \mid xT \neq x\}$. Let $S$ be the 'shift function': $xS = x+1$. The subgroup $\Sigma^+$ of permutations whose supports lie in the positive half of the reals is invariant under conjugation by $S$ (but not by $S^{-1}$). Define $g : \Sigma^+ \to \Sigma^+$ by $g(T) = T^S$. Let $Q \in \Sigma^+$ be anything that agrees with $S$ on $[1, \infty)$. Then $g^2 = g^Q$.

We define $Q_0$ to be the simplest such function, to wit, the piece-wise linear function which has exactly three pieces. More generally, let $Q_k$ be the function that acts trivially on $(-\infty, k]$, that linearly stretches the interval $[k, k+1]$ to the interval $[k, k+2]$ that shifts $[k+1, +\infty)$ to $k+2, +\infty)$ by adding 1. Then $Q_k^S = Q_{k+1}$ and we obtain an induced map $h : F \to \Sigma^+$ that sends $\alpha_k$ to $Q_k$. We take this as the definition of the *First Canonical Representation*.

For any $T \in \Sigma^+$ define $\mu(T)$ as $\inf(\mathrm{spt}(T))(\mu(1) = +\infty)$. It is clear that:

(L5)
$$\mu(T) < +\infty \quad \Leftrightarrow \quad T \neq 1\,.$$
$$\mu(T) < (T') \quad \Rightarrow \quad \mu(TT') = \mu(T'T) = \mu(T)\,.$$
$$\mu(g(T)) = 1 + \mu(T)\,.$$
$$\mu(g^{i+1}(T)) \geq i+1\,.$$
$$\mu(Q_i) = i\,.$$
$$\mu(T^n) = \mu(T)\,, \quad \textit{for}\ n \neq 0\,.$$

Hence an element of the form $\alpha_i^n f^{c+1}(\beta)$, $n \neq 0$ is sent to an element $T$ such that $\mu(T) = i$. L4 thus yields:

(L6) *The First Canonical Representation of F is faithful.*

**Proof of T5 (*F* is a TOG).** The last two lemmas clearly imply that $F$ is torsion-free (indeed, $\Sigma$ is torsion-free). For a total ordering define $d(\beta)$ as the right-hand derivative of $h(\beta)$ at $\mu(h(\beta))$. The set of elements $\beta$ such that $1 < d(\beta)$ is easily checked to be a normal subsemigroup with no units. As always in such a case we obtain a partial ordering by defining $\beta < \gamma$ iff $1 < d(\beta^{-1}\gamma)$. Since $d(\beta^{-1}) = (d(\beta))^{-1}$ we easily obtain that for any $\beta,\gamma$ either $\beta < \gamma$, $\gamma < \beta$ or $\beta = \gamma$. □

The faithfulness of the First Canonical Representation clearly implies:

(L7) *$f : F \to F$ is an embedding and its only fixed point is* 1.

For the record, define the set of *canonical words* as those that arise as follows: the empty word is canonical; if $w$ is canonical and if $m(w)$ denotes the lowest index appearing in $w$ then $\alpha_i^b w \alpha_i^{-c}$ is canonical if $i < m(w)$ and $b,c$ are non-negative and if either $bc \neq 0$ or $m(w) = i + 1$. Then every element is described by a unique canonical word.

(The proof of uniqueness can be proved using two lemmas stemming from the First Canonical Representation: if $\alpha_i^b w \alpha_i^{-c}$ is canonical then $i \leq \mu(\alpha_i^b w \alpha_i^{-c}) < i + 1$ and its right-hand derivative at $\mu$ is $2^{b-c}$.)

(L8) *If $\langle G, g, \beta \rangle$ is a conjugacy idempotent then $g$ splits iff $\beta$ and $g(\beta)$ commute.*

If $[\beta, g(\beta)] = 1$ then $g^2(\beta) = (g(\beta))^\beta = g(\beta)$ and L3 says that $g$ splits. For the converse, suppose $G \to H$, $H \to G$ are such that $G \to H \to G \equiv g$ and $H \to G \to H \equiv 1_H$. The only endomorphisms conjugate to the identity morphism are inner-automorphisms, in particular, they are automorphisms. Hence $H \to G$ is an embedding, $G \to H$ is onto and it becomes clear that

$$\mathrm{Image}(g) = \mathrm{Image}(g^2)\,.$$

We may compute $g^3$ in two ways:

$$g^3 = gg^2 = gg^\beta = (g^2)^\beta \quad \text{and} \quad g^3 = g^2 g = g^\beta g = (g^2)^{g(\beta)}\,.$$

Thus $\beta$ and $g(\beta)$ act, via conjugation, on $G^2(G)$ the same way. Since $g^2(G) = g(G)$ we may infer that

$$(g(\beta))(g(\beta))^{g(\beta)} = g(\beta)\,.$$

We may restate this lemma as:

(L9) *If $\langle G, g, \beta \rangle$ is a conjugacy idempotent then $g$ splits iff the kernel of the induced map $F \to G$ contains the commutator subgroup $F'$.*

The First Canonical Representation clearly shows that $F$ is not abelian, hence there does exist a non-split conjugacy idempotent.

(L10) *If* $1 \neq H \lhd F$ *and* $f(H) \subseteq H$ *then* $F' \subseteq H$.

**Proof.** The $f$-invariance of $H$ yields a diagram

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d] & F \arrow[d] \\ F/H \arrow[r, "\bar{f}"'] & F/H \end{tikzcd}$$

and $\bar{f}$ is a conjugacy idempotent. By the last lemma it suffices to prove that $\bar{f}$ splits.

By L4 we may conclude that for some $n \neq 0$ and $i = 0,1$ it is the case that $\bar{\alpha}_i^n$ is the image of $\bar{f}^{i+1}$ (where $\bar{\alpha}_i$ denotes the element in $F/H$ represented by $\alpha_i$ in $F$). We may assume that $n$ is positive. We may assume that $i < n$ (by squaring if necessary). By conjugating with $\bar{\alpha}_i$ we may insure that $\bar{\alpha}_i^n$ is in the image of $\bar{f}^n$. Let $g = \bar{f}^n$ then

$$g^2 = f\bar{f}^{2n} = (\bar{f}^{2n-1})^{\bar{\alpha}_i} = (\bar{f}^{2n-2})^{\bar{\alpha}_i^2} = \cdots = (\bar{f}^{2n-n})^{\bar{\alpha}_i^n} = g^{\bar{\alpha}_i^n}.$$

Since $\bar{\alpha}_i^n$ is in the image of $g$, L2 says that $g$ splits. But $g$ is conjugate to $\bar{f}$, hence $\bar{f}$ splits. □

(L11) *If* $1 \neq H \lhd F$ *then* $H \cap f(F) \neq 1$.

Using L4, let $\alpha_i^n f^{i+1}(\beta) \in H$ where $n \neq 0$, $i = 0,1$. If $i = 1$ we are done. For $i = 0$ we note that $\alpha_0^n f^2(\beta) = (\alpha_0^n f(\beta))^{\alpha_0}$ is also in $H$ hence $f(\beta^{-1} f(\beta)) = (\alpha_0^n f(\beta))^{-1}(\alpha_0^n f^2(\beta))$ is in $H$. By L7, $f(\beta^{-1} f(\beta)) = 1$ implies that $\beta = 1$. In that case $H$ includes $\alpha_0^n$ and therefore it includes $(\alpha_0^{-n})^{\alpha_1}\alpha_0^n = \alpha_1^{-1}\alpha_{n+1} = f(\alpha_0^{-1}\alpha_n)$.

(L12) *All non-trivial normal subgroups of F contain the commutator.*

Suppose that $1 \neq H \lhd F$. Then $f^{-1}(H)$ is normal and $f$-invariant: $f(f^{-1}(H)) \subseteq f^{-1}(H) \Leftrightarrow f^2(f^{-1}(H)) \subseteq H \Leftrightarrow (ff^{-1}(H))^{\alpha_0} \subseteq H$ and as for any subset $(ff^{-1}(H)) \subseteq H$. The last lemma says that $f^{-1}(H)$ is non-trivial thus L10 implies that $\alpha_1\alpha_2^{-1} = [\alpha_1, \alpha_0^{-1}] \in f^{-1}(H)$ hence $\alpha_2\alpha_3^{-1} = f(\alpha_1\alpha_2^{-1}) \in H$. But $\alpha_1\alpha_2^{-1} = (\alpha_2\alpha_3^{-1})^{\alpha_0^{-1}}$ is the commutator of a pair that generates $F$. Any normal subgroup that contains their commutator must contain the entire commutator. □

T1 is now an immediate consequence of L9 and L12.

Returning to $\Sigma$, the group of continuous order-preserving permutations of the reals, we note that the sequence $S, Q_1, Q_2, Q_3, \ldots$ satisfies the defining relations for $F$. The induced map $F \to \Sigma$ is the *Second Canonical Representation*. Its image is not abelian and the last lemma says, therefore, that it is faithful.

We will notationally confuse $F$ and its image under this representation. For any

$T \in F$ there exist integers $a,b$ such that $xT = x + a$ for all sufficiently small $x$ and $xT + x + b$ for all sufficiently large $x$. We use this to define $h : F \to \mathbb{Z} \times \mathbb{Z}$ ($\mathbb{Z}$ is the group of integers), to wit,

$$h(t) = \langle \lim_{x \to -\infty} (xT - x), \lim_{x \to +\infty} (xT - x) \rangle .$$

Because $F/F'$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ we conclude that the kernel of $g$ is $F'$. Thus *an element is in the commutator subgroup iff its support is bounded.*

(It is possible to characterize $F'$ as the subgroup of $\Sigma$ consisting of all piece-wise linear permutations with bounded support and with a finite number of singular points each of which is a dyadic rational and, finally, such that at all non-singular points the derivative is equal to a power of 2.)

If two elements have disjoint support they obviously commute. There is a sense in which it is correct to say that any two elements of $F'$ 'probably commute'.

**Proof of T3 ($F'$ is simple).** It suffices, given the last lemma, to show that every normal subgroup of $F'$ is normal in $F$. And for that, it suffices to show that the two conjugacy relations on $F'$—the standard one and the one induced by elements of the ambient group—coincide. We need only find for $i = 0,1$ and for any $\beta \in F'$ an element $\gamma \in F'$ such that $\beta^{\alpha_i} = \beta^{\gamma}$.

Using the notation of the Second Canonical Representation we know that for any $T \in F'$ it is the case that $[T, Q_k] = 1$ for all large $k$ (disjoint supports). Hence $T^{Q_k^{-1}Q_1} = T^{Q_1}$ and $Q_k^{-1}Q_1 \in F'$. Moreover, $T^{Q_l} = T^S$ for all small $l$ (which is allowed to be negative). Hence $T^{Q_k^{-1}Q_l} = T^S$. □

**Proof of T4 ($F$ appears in $F'$).** For all $i < j$ and all sufficiently large $n$ it is the case that $(\alpha_j\alpha_n^{-1})(\alpha_i\alpha_n^{-1}) = \alpha_j\alpha_i\alpha_{n+1}^{-1}\alpha_n^{-1} = a_i\alpha_{j+1}\alpha_{n+1}^{-1}\alpha_n^{-1} = (\alpha_i\alpha_n^{-1})(\alpha_{j+1}\alpha_n^{-1})$. Hence for all large $n$ the two elements $\alpha_a\alpha_n^{-1}$ and $\alpha_2\alpha_n^{-1}$ satisfy the two necessary defining relations for $F$. They do not commute hence we obtain a copy of $F$ (using L12). □

**Proof of T6.** In terms of the Second Canonical Representation, any finitely generated subgroup $H \subseteq F'$ has bounded support on the real line. Thus for all sufficiently large or small $n$ it is the case that $[H, H^{S^n}] = 1$. Clearly, therefore, $F'$ contains a copy of the infinite weak product $\Sigma_{\mathbb{Z}}H$ and an element $S^n$ such that conjugation by $S^n$ is the shift operation on $\Sigma_{\mathbb{Z}}H$. □

For the proofs of T7 and T8 we use the First Canonical Representation and note that for any pair of elements $\beta$ and $\gamma$, $(\mu(\beta))\gamma = \mu(\beta^{\gamma})$. If $G$ is a subgroup of $F$ and $M$ denotes the set $\{\mu(\beta) \mid \beta \in G, \beta \neq 1\}$ it follows that $M$ is invariant under the action of $G$. If $G$ is abelian, then $M$ is not only invariant but fixed by the action of $G$. In that case, we can use the right-hand derivatives, one for each element in $M$, to obtain an embedding into a cartesian power of $\mathbb{Z}$ (it is not really the derivatives but their logarithms-base-two). But any countable subgroup of such a cartesian power is known to be free abelian.

We have shown: abelian implies that $M$ is fixed and that $M$ fixed implies free-abelian. For T8 we can thus concentrate on the case that $M$ is not fixed. Without loss of generality we may assume that $M$ is bounded (if necessary use the embedding of $F$ into $F'$). Let $z$ be the least upper bound of all the non-fixed points of $M$. $z$ is easily seen to be a fixed-point for $G$. Since each non-trivial orbit is infinite we know that $M \cap [0, z]$ is infinite.

We shall find a sequence of elements $\{\beta_i\}$ such that $\mathrm{spt}(\beta_i) \cap [0, z]$ is non-empty, and such that $\mathrm{spt}(\beta_i) \cap \mathrm{spt}(\beta_j) \cap [0, z]$ is empty for $i \neq j$. The support of the commutator $[\beta_i, \beta_j]$ clearly lies to the right of $z$. But the right-hand derivative at $\mu[\beta_i, \beta_j]$ would have to be 1, hence $[\beta_i, \beta_j] = 1$.

There must be an element $\beta_1$ such that $\mu(\beta_1) < z$ but such that the left-hand derivative at $z$ is 1 (if no such element presents itself, let $\gamma$ and $\delta$ be such that $\mu(\gamma) < \mu(\delta) < z$; for suitable integers $b$ and $c$ we may take $\beta_1 = \gamma^c \delta^d$). Let $y < z$ be such that $\mathrm{spt}(\beta_1) \cap [0, z]$ is contained in $[0, y]$ and let $G_1$ be the subgroup of $G$ defined by $G_1 = \{\gamma \in G \mid \mu(\gamma) > y\}$. We may repeat this argument to find $\beta_2 \in G_1$. In this fashion we will have at each finite stage a finite sequence of elements $\beta_1, \beta_2, \ldots, \beta_n$ and a subgroup $G_n$ such that to the left of $z$ the supports of the $\beta$'s are pairwise-disjoint and disjoint from the supports of any element in $G_n$. It remains the case that $\{\mu(\gamma) \mid \gamma \in G_n\} \cap [0, z]$ is infinite. By iteration, therefore, we obtain the desired sequence.

## 5. A proof of lawlessness

Let $F$ be any group that contains a copy of its own infinite wreath product. Let $w$ be any non-trivial reduced word. We seek elements $a_0, a_1, a_2, \ldots, a_n \in F$ such that $w(a_0, a_1, \ldots, a_n) \neq 1$ (where $w(a_0, a_1, \ldots, a_n)$ indicates the result of replacing the variables of $w$ with the indicated elements and then, of course, evaluating). We will assume that the result holds for all non-trivial reduced words shorter than $w$.

If $w(x_0, 1, 1, \ldots, 1)$ is non-trivial we are done (because $F$ cannot be a torsion group). Thus the total degree of $x_0$ may be assumed to be zero, which case $w$ is a product of various $x_0$-conjugates of $x_1, x_2, \ldots, x_n$. That is, there is another word $w'(y_1, y_2, \ldots, y_m)$ such that a sequence of substitutions of the form $y_i = x_j^{x_0^k}$ transforms $w'$ into $w$. $w'$ may have more variables than $w$ but it is shorter (it is just as long as $w(1, x_1, x_2, \ldots, x_n)$). By the inductive hypothesis, there exist elements $b_1, b_2, \ldots, b_m \in F$ such that $w'(b_1, b_2, \ldots, b_m) \neq 1$.

Using the $b$'s we will construct elements $a_0, \ldots, a$ in the infinite wreath-product. We first fix some notation. The infinite wreath-product has a normal subgroup isomorphic to $\Sigma_{\mathbb{Z}} F$ and it has an element $c$ such that for all $x \in \Sigma_{\mathbb{Z}} F$ and all $i \in \mathbb{Z}$ it is the case that $p_i(x^c) = p_{i-1}(x)$. We will take $c$ as the value for $a_0$. All other $a$'s will lie in $\Sigma_{\mathbb{Z}} F$. For a given $1 \leq j \leq n$ we take $a_j$ to be an element such that $p_k(a_j) = b_i$ where $i,j,k$ are such that $y_i = x_j^{x_0^k}$ is in the above-mentioned sequence of substitutions. Then

$$p_0(w(a_0, a_1, \ldots, a_n)) = w'(b_1, \ldots, b_m) \neq 1 . \quad \square$$

*Several comments about this result and its proof*: It implies that any free group may be embedded in a cartesian power of $F$ and since powers of TOGs and TOGs, we have shown, in passing, that free groups are TOGs. The proof does not require infinite wreath-products, only arbitrarily large ones. Any variety closed under large wreath-products is therefore entire. Hence any non-trivial variety closed under semi-direct products must be entire. Any non-trivial pseudo-variety closed under extensions must contain all free groups. An example of such is the collection of $p$-nilpotent groups. Finally, note that an $mn$-fold wreath product appears as a subgroup of an $m$-fold wreath product of an $n$-fold wreath product. Thus to require arbitrarily large wreath products is to require no more than non-trivial wreath products.

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