INFINITE TYPES, INFINITE DATA, INFINITE INTERACTION

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ABSTRACT. We describe a way to represent computable functions between coinductive types as particular transducers in type theory. This generalizes earlier work on functions between streams by P. Hancock to a much richer class of coinductive types. Those transducers can be defined in dependent type theory *without* any notion of equality but require inductive-recursive definitions. Most of the properties of these constructions only rely on a mild notion of equality (intensional equality) and can thus be formalized in the dependently typed language Agda.

INTRODUCTION

This papers gives a type theoretic representation theorem for continuous functions between a wide class of spaces of infinite values. By infinite, we understand a value having a coinductive type. Continuity means that finite information about a result of the function requires only finite information about its argument. Because of that, it is a necessary condition for computability.

The simplest coinductive space is the Cantor space of infinite boolean sequences. It corresponds to the coinductive type $\nu_X(X + X)$ and its topology is well known. Programs that implement continuous functions from the Cantor space to a discrete space D can be represented by finite binary trees in which leaves are labelled by values in D. We extend this to a class of functors going beyond $X \mapsto X + X$ on the category Set by considering so-called (finitary) polynomial functors on Set^I for some index set I. The final coalgebra νP of such a functor P always exists and may be constructed as the inverse limit of: $1 \leftarrow P(\mathbf{1}) \leftarrow P^2(\mathbf{1}) \leftarrow \cdots$. Those final coalgebras have a natural topology, and when the functor F is *finitary* (commutes with filtered colimits), the topology enjoys a close connection with the intuitive notion of "finite amount of information" about potentially infinite values. However, representing such topologies inside a formalized system such as

dependent type theory is far from trivial because their definition relies heavily on delicate topics like equality.

Our main result pertains to the question of how continuous functions between these natural classes of spaces can be represented in dependent type theory. It turns out that any such "implementation" can itself be put into the form of a potentially infinite data-structure, inhabiting a final coalgebra for an appropriate functor, albeit one which is in most cases no longer finitary. This settles a conjecture of P. Hancock about representability of continuous between "dependent streams" [13].

We obtain this result via a more general construction, without any cardinality restrictions on the initial functors. One can still topologise the final coalgebras, though the topology that arises from the inverse chain construction no longer enjoys much connection with any intuition of finite information, and there are (classically) continuous functions that cannot be implemented by programs.

1. Preliminaries I. Streams and Trees in Point Set Topology

1.1. **Streams.** Given a set X endowed with the discrete topology, the set of *streams* over X, written $\mathtt{stream}(X)$, is defined as the infinite product $\prod_{i \geq 0} X$. The product topology is generated from the basic open sets $\prod_{i \geq 0} U_i$ where finitely many U_is are of the form $\{x_i\}$ for some $x_i \in X$ and the other U_is are equal to X. This topological space is usually called the Cantor space (when X is finite) or the Baire space (when X is countably infinite). Continuity for functions between streams amounts to the following:

Lemma 1.1. *A function $f : \mathtt{stream}(X) \to \mathtt{stream}(Y)$ is continuous if and only if, for each stream s in $\mathtt{stream}(X)$, each projection $f(s)_k$ of $f(s)$ depends on at most a finite prefix of s.*

Writing $s_{\restriction n}$ for the restriction of stream s to its finite prefix of length n, the condition is equivalent to

$$\forall s \in \mathtt{stream}(X), \forall k \geq 0, \exists n \geq 0, \forall t \in \mathtt{stream}(X), s_{\restriction n} = t_{\restriction n} \Rightarrow f(s)_k = f(t)_k \,. \qquad (*)$$

Before proving lemma 1.1, let's look at a preliminary result:

Lemma 1.2. *For any subset $V \subseteq \mathtt{stream}(X)$, we have: V is open iff*

$$\forall s \in V, \exists n \geq 0, \forall t \in \mathtt{stream}(X), s_{\restriction n} = t_{\restriction n} \Rightarrow t \in V \,.$$

Proof. The $\Rightarrow$ direction is immediate: an open set is a union of basic open sets, which satisfy the condition. (Recall that a basic open set is of the form $\prod_{i \geq 0} U_i$, where each U_i is X, except for finitely many that are singleton sets.)

For the $\Leftarrow$ direction, we define, for each $s \in V$, the set $V_s = \{t \mid s_{\restriction n_s} = t_{\restriction n_s}\}$, where n_s is the integer coming from condition. We have $V = \bigcup_{s \in V} V_s$. $\qquad \square$

Proof of lemma 1.1. Suppose the function $f : \mathtt{stream}(X) \to \mathtt{stream}(Y)$ satisfies condition $(*)$. To show that f is continuous, it is enough to show that the inverse image of any basic open set is an open set. Because the inverse image commutes with intersections, it is sufficient to look at *pre* basic open sets of the form $V_{k,y} = \{s \mid s_k = y\}$.

To show that $f^{-1}(V_{k,y})$ is open, we use lemma 1.2 and show that $s \in f^{-1}(V_{k,y})$ implies

$$\forall s \in f^{-1}(V_{k,y}) \exists n \geq 0, \forall t \in \mathtt{stream}(X), s_{\restriction n} = t_{\restriction n} \Rightarrow t \in f^{-1}(V_{k,y}) \,.$$

Because $s \in f^{-1}(V_{k,y})$ is $f(s)_k = y$, this is implied by condition $(*)$.

For the converse, suppose $f : \mathtt{stream}(X) \to \mathtt{stream}(Y)$ is continuous. We want to show that it satisfies condition $(*)$. Let $s \in \mathtt{stream}(X)$ and $k \geq 0$. The set $\{t \mid f(t)_k = f(s)_k\}$ is open and because f is continuous, its inverse image also is open. By lemma 1.2, we now that there is some n such that $s_{\restriction n} = t_{\restriction n} \Rightarrow f(t)_k = f(s)_k$. This finishes the proof. $\qquad\square$

Because of this, *constructive* functions between streams are usually held to be continuous: we expect them to arise as continuous functions with the additional properties that:

- finding the finite prefix needed to compute a chosen element of $f(s)$ is computable, and
- finding the value of the element of $f(s)$ from that finite prefix is computable.

Note that the discrete space X may be generalized to a family $(X_i)_{i \geq 0}$ that needs not be constant. More interestingly, we can allow the set X_i (giving the set of possible values for the ith element of a stream) to depend on the ith prefix of the stream. We can in this way obtain for example the space of increasing streams of natural numbers:

- the set X_0 doesn't depend on anything and is defined as $\mathbb{N}$,
- the set X_1 depends on the value of $x_0 \in X_0$: $X_{1,x_0} = \{k \in \mathbb{N}, x_0 \leq k\}$,
- the set X_2 depends on $x_1 \in X_{1,x_0}$, etc.

The set of increasing streams isn't naturally a product space but is a subspace of $\mathtt{stream}(\mathbb{N})$. The topology is the expected one, and continuous functions are still characterized by lemma 1.1.

1.2. **Infinite Trees, Natural Topology.** The natural topology for sets of *infinite trees* is less well known than the Cantor and Baire topologies. The simplest kind of infinite tree, the infinite *binary tree* has a root, and two distinguished "branches" going from that root to two "nodes". Each of these two nodes also has two branches, etc. An infinite binary tree *over* X is a way to label each node of the infinite binary tree with an element of X. If we write $\mathbb{B}$ for the set $\{0, 1\}$, each node of the infinite binary tree can be identified by a list of elements of $\mathbb{B}$: this list simply represents the branch leading to this node from the root. The set of infinite binary trees over X, written $\mathtt{tree}_{\mathbb{B}}(X)$, can thus be defined as

$$\mathtt{tree}_{\mathbb{B}}(X) = X \times (\mathbb{B} \to X) \times (\mathbb{B}^2 \to X) \times \cdots \times (\mathbb{B}^i \to X) \times \ldots$$

where each term gives the ith "layer" of the tree as a function from finite branches of length i to X. We can rewrite this as

$$\mathtt{tree}_{\mathbb{B}}(X) \quad = \quad \prod_{i \geq 0} \left(\mathbb{B}^i \to X \right) \quad = \quad \prod_{i \geq 0} \left(\prod_{t \in \mathbb{B}^i} X \right).$$

By replacing the set $\mathbb{B}$ by some other set B, we obtain the ternary trees over X or countably-branching trees over X, etc. Streams themselves are recovered by taking $B = \{\star\}$. If both B and X are endowed with the discrete topology, we obtain a natural topology on $\mathtt{tree}_B(X)$. Note that when B is infinite, the spaces $B^i \to X$ aren't discrete anymore. Nevertheless, we have:

Lemma 1.3. *Let A, B, X and Y be discrete spaces; a function $f : \mathtt{tree}_A(X) \to \mathtt{tree}_B(Y)$ is continuous iff for every $t \in \mathtt{tree}_A(X)$, the value at each node of $f(t)$ only depends on a finite subtree[1] of t.*

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[1] A *subtree* is a set of nodes that contains the root of the tree and is closed by the ancestor relation.

Proof. The proof of this lemma is exactly the same as the proof of lemma 1.1, except that we replace the natural number n in $t_{\restriction n}$ (for some $t \in \mathtt{tree}_B(Y)$) by a *finite subtree*. The only remark is that basic open sets of $\mathtt{tree}_B(Y)$ are of the form $\prod_{i \geq 0} \prod_{t \in B^i} X_{i,t}$ where all sets $X_{i,t}$ are equal to X, except for finitely many that are singleton of the form $\{x_{i,t}\}$ for some $x_{i,t} \in X$. $\qquad\square$

It is again possible to devise more general notions of trees by allowing the set X at a node in the tree to depend on the values of its location as a path from the root. The resulting space is endowed with the subspace topology and lemma 1.3 still holds. We will later generalize this notion further by allowing the branching of a node (given by the set B) to depend itself on the value stored at the node. With this generalisation we can model very general objects, such as infinite automata that issue commands and change to a new state (choose a branch) based on the responses.

1.3. **Infinite Trees, Wild Topology.** The topology we naturally get in this paper corresponds to a different topology on trees. When looking at

$$\mathtt{tree}_B(X) \quad = \quad \prod_{i \geq 0} \left(\prod_{t \in B^i} X \right) ,$$

we can endow the inner product space with the "box" topology, where basic opens are given by arbitrary products of open sets. Because B and X are discrete sets, this amounts to giving the discrete topology to each layers $B^i \to X$. Instead of being generated by "finite subtrees", open sets are generated by "subtrees with finite depth".

Lemma 1.4. *Let A, B, X and Y be discrete spaces; a function $f : \mathtt{tree}_A(X) \to \mathtt{tree}_B(Y)$ is continuous for the wild topology iff for every $t \in \mathtt{tree}_A(X)$, and $k \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that the nodes of $f(t)$ at depth less than k depend only on the nodes of t at depth less than n.*

Formally, this looks very similar to condition $*$ on page 2:

$$\forall t \in \mathtt{tree}_A(X), \forall k \geq 0, \exists n \geq 0, \forall t' \in \mathtt{tree}_A(X), t_{\restriction k} = t'_{\restriction k} \Rightarrow f(t)_n = f(t')_n , \qquad (**)$$

where $t_{\restriction k}$ is the complete subtree of t up-to depth k. Intuitively, this topology considers infinite trees as streams of their layers, where layers are discrete.

When A and B are finite, the two notions of continuity (lemma 1.3 and 1.4) coincide, but when this is not the case, we cannot compare continuous functions for the two topologies.

- Consider $f : \mathtt{stream}(\mathbb{N}) \to \mathtt{tree}_{\mathbb{N}}(\mathbb{N})$ sending the stream s to the tree $f(s)$ where the node indexed by $(i_0, \ldots, i_n)$ is $s_{i_0} + s_{i_1} + \ldots + s_{i_n}$. This is certainly continuous for the natural topology. However, because the first layer of the output is infinite, we cannot bound the number of layers (elements) of the input stream s that are needed to construct it: this function isn't continuous for the wild topology.

- Consider $g : \mathtt{tree}_{\mathbb{N}}(\mathbb{B}) \to \mathtt{stream}(\mathbb{B})$ where the ith element of $g(t)$ is the maximum of the complete ith layer of t ($\mathbb{B}$ being the complete lattice with two elements). This function is continuous in the wild sense, but because we need to know the whole ith layer of the input to get the ith value of the output, this function isn't continuous for the natural topology (and certainly not computable).

2. Preliminaries II. Martin Löf Type Theory

2.1. **Basic Features.** We work in a meta theory that is in essence Martin-Löf's dependent type theory [19] with two additional features:
- coinductive types,
- inductive-recursive definitions.

All the constructions described in the paper have been defined using the dependently typed functional programming language Agda [26].

A Note about Equality. This paper is concerned with constructions in "pure" type theory, i.e. dependent type theory without identity. Those constructions enjoy many interesting properties, but proving them requires some notion of equality. Equality in Martin-Löf type theory is a complex subject about which whole books have been written [27]. We try to be mostly agnostic about the flavor of equality we are using and only rely on the "simplest" one: intentional equality, written $a \equiv_T b$, or simply $a \equiv b$. This makes it possible to use vanilla Agda for checking proofs.[2]

We annotate the proofs in the paper with
- $[Agda\checkmark]$: for those that have been formalized using Agda,
- $[Agda\text{✗}]$: for those which have only partly been formalized, typically because the proof is too complex to write in Agda.

Because we want to explain Agda code only slightly less than you want to read about Agda code, the formalized proofs are either omitted from the paper, or explained informally.

We sometimes need to assume equality for functions is extensional, i.e. that $f \equiv g$ iff $f\,a \equiv_B g\,a$ for all $a : A$. Those proofs are clearly identified.

Notation. The notation for dependent types is standard. Here is a summary:
- We write $\Gamma \vdash B$ to mean that B is a well-formed type in context Γ. We write $A = B$ to express that A and B are the same by definition.
- We write $\Gamma \vdash b : B$ to mean that b is an element of type B in context Γ. The context is often left implicit and we usually write $b : B$. We write $a = b : B$ to express that a and b are definitionally equal in type B. When the type B can easily be deduced from the context, we will usually write just $a = b$.
- If B is a type and C is a type depending on $x : B$, i.e. , $x : B \vdash C$, we write
  - $(\Sigma\, x : B)\, C$ for the dependent sum. Its canonical elements are pairs $\langle b, c \rangle$ with $b : B$ and $c : C[x/b]$,
  - $(\Pi\, x : B)\, C$ for the dependent product. Its canonical elements are functions $(\lambda\, x : B)\, u$ where $x : B \vdash u : C$.

  When the type C is constant, we abbreviate those by $B \times C$ and $B \to C$.
- The usual amenities are present: the natural numbers, W-types and so on.

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[2]All the Agda code was checked using `Agda 2.5.4.2` with the flag `--without-K`. The Agda code is available at `http://www.lama.univ-smb.fr/~hyvernat/Files/Infinite/agda.tgz`, with the file `PAPER.agda` referencing all the formalized results from the paper. For those without a working Agda installation, the code is also browsable directly from `http://www.lama.univ-smb.fr/~hyvernat/Files/Infinite/browse/PAPER.html`.

We use a universe Set of "small types" containing $\mathbf{0}$ (with no element), $\mathbf{1}$ (with a single element $\star$) and $\mathbf{2}$ (with two elements 0 and 1). Moreover, the dependent sums and products are reflected in this universe, and we use the same notation $(\Pi\, b : B)\, C$ and $(\Pi\, b : B)\, C$ whenever $B : \mathsf{Set}$ and $b : B \vdash C : \mathsf{Set}$. We assume that this universe is closed under many inductive-recursive and coinductive definitions which will be treated below.

We are not always consistent with notation for application of functions and usually write
- $f\, x$ when the result is an element of a small type (an element of Set),
- $A(i)$ when the result is itself a small type (and thus, not an element of Set).

Predicates and Families. The Curry-Howard isomorphism makes the type Set into a universe of propositions.

Definition 2.1. If $A : \mathsf{Set}$, the collection of *predicates* on A is defined as
$$\mathsf{Pow}(A) \quad = \quad A \to \mathsf{Set}\,.$$
We introduce the following notations [24]: if X and Y are predicates on A,
- "$a \,\epsilon\, X$" is "$X(a)$",
- "$X \subset Y$" is "$(\Pi\, a : A)\, (a \,\epsilon\, X) \to (a \,\epsilon\, Y)$,
- "$X \between Y$" is "$(\Sigma\, a : A)\, (a \,\epsilon\, X) \times (a \,\epsilon\, Y)$",
- "$X \cap Y$" is "$(\lambda\, a : A)\, (a \,\epsilon\, X) \times (a \in Y)$",
- "$X \cup Y$" is "$(\lambda\, a : A)\, (a \,\epsilon\, X) + (a \in Y)$".

The intuition is that a predicate on A is just a subset of A in some constructive and predicative[3] set theory. We will also need the following definition.

Definition 2.2. A family of C is given by a set I together with a function from I to C. In other words,
$$\mathsf{Fam}(C) \quad = \quad (\Sigma\, I : \mathsf{Set})\, I \to C\,.$$

2.2. **Inductive-recursive Definitions.** Inductive definitions are a way to define (weak) initial algebras for endofunctors on Set. A typical example is defining $\mathtt{list}(X)$ as the least fixed point of $Y \mapsto \mathbf{1} + X \times Y$. In their simplest form, inductive-recursive definitions [9] give a way to define (weak) initial algebras for endofunctors on $\mathsf{Fam}(C)$.[4] It means we can define, at the same time:
- a inductive set U, called the *index set*,
- and a recursive function $f : U \to C$.

Of course, the set U and the function f may be mutually dependent.

The type C may be large or small. Taking $C = \mathbf{1}$, we recover usual inductive types as the index part of such a definition, the recursive function f always being trivial. In non-degenerate cases, the inductive clauses defining U are expressed using the simultaneously defined function f. Here is a traditional example with $C = \mathsf{Set}$: complete binary trees, defined one layer at a time. Here is the definition, first using Agda syntax:

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[3]if $A : \mathsf{Set}$, $\mathsf{Pow}(A)$ isn't of type Set

[4]To ensure the definition is healthy, the endofunctor has to be expressible according to a certain coding scheme [10].

```
mutual
  data Tree : Set where
    Empty : Tree
    AddLayer : (t : Tree) → (branches t × Bool → Nat) → Tree
  branches : Tree → Set
  branches Empty = One
  branches (AddLayer t l) = (branches t) × Bool
```
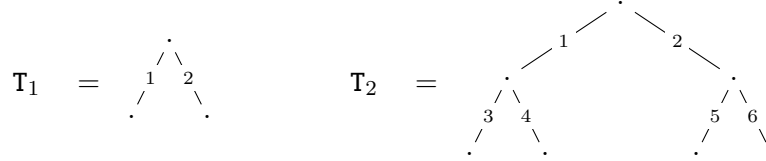
While `Empty` corresponds to the empty tree, the definitions

```
T₁ = AddLayer Empty (λ b → if b.2 then 1 else 2)
T₂ = AddLayer T₁ (λ b →
                    if   b.1.2 && not b.2 then 3
                    elif b.1.2 && not b.2 then 4
                    elif not b.1.2 && b.2 then 5
                    else                       6)
```

correspond to the trees

The corresponding functor takes the family $\langle T : \mathsf{Set}, b : T \to \mathsf{Set} \rangle$ to the family

- index set: $T' = \mathbf{1} + \big(\Sigma\, t : T\big)\, \big((b\, t \times \mathbf{2}) \to \mathbb{N}\big)$,
- recursive function b' defined with
  - $b'\, \star = \mathbf{1}$, where $\star$ is the only element of $\mathbf{1}$,
  - $b'\, \langle t, l \rangle = (b\, t) \times \mathbf{2}$.

We will, in section 5.1, encounter a similar situation in which the type C will be $\mathsf{Fam}(I)$, i.e. we will need to take the least fixed point of a functor from $\mathsf{Fam}\big(\mathsf{Fam}(I)\big)$ to itself.

Another typical example involves defining a universe U of types closed under dependent function space: U needs to contains inductive elements of the form $\big(\Pi\, A : U\big)\, (B : \mathsf{fam}(A))$, but $\mathsf{fam}(A)$ is defined as $\mathsf{El}(A) \to U$ and makes use of the decoding function $\mathsf{El} : U \to \mathsf{Set}$.

2.3. **Greatest Fixed Points.** In its simplest form a coinductive definition introduces some $\nu_F : \mathsf{Set}$ together with a weakly terminal coalgebra $c : \nu_F \to F(\nu_F)$ for a sufficiently healthy[5] functor $F : \mathsf{Set} \to \mathsf{Set}$. For example, given a set A, the functor $F(X) = A \times X$ is certainly healthy and the set ν_F corresponds to the set of streams over A. The coalgebra c "decomposes" a stream $[a_0, a_1, \ldots]$ into the pair $\langle a_0, [a_1, a_2, \ldots] \rangle$ of its head and its tail. Because the coalgebra is weakly terminal, for any other such coalgebra $x : X \to F(X)$ there is a map $X \to \nu_F$. The corresponding typing rules are given by

$$\frac{\sigma : X \to F(X)}{\nu_{\mathrm{intro}}\, \sigma : X \to \nu_F} \quad \text{and} \quad \frac{}{\nu_{\mathrm{elim}} : \nu_F \to F(\nu_F)}$$

and the computation rule is

$$\nu_{\mathrm{elim}}\, (\nu_{\mathrm{intro}}\, \sigma\, x) \quad = \quad F_{\nu_{\mathrm{intro}}\, \sigma}\, (\sigma\, x).$$

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[5]for our purposes, "sufficiently healthy" amounts to "polynomial"

Such coinductive definitions can be extended to families of sets: given an index set I, we introduce weakly terminal coalgebras for sufficiently healthy functor acting on $\mathsf{Pow}(I)$. The typing rules are extended as follows:

$$\frac{\sigma : X \subset F(X)}{\nu_{\mathrm{intro}}\,\sigma : X \subset \nu_F} \qquad \text{and} \qquad \frac{}{\nu_{\mathrm{elim}} : \nu_F \subset F(\nu_F)}$$

together with the computation rule

$$\nu_{\mathrm{elim}}\,(\nu_{\mathrm{intro}}\,\sigma\,i\,x) \quad = \quad F_{\nu_{\mathrm{intro}}\,\sigma}\,i\,(\sigma\,i\,x).$$

Note that it doesn't seem possible to guarantee the existence of strict terminal coalgebras [20] without extending considerably the type theory.[6]

Bisimulations. Equality in type theory is a delicate subject, and it is even more so in the presence of coinductive types. The usual (extensional or intensional) equality is easily defined and shown to enjoy most of the expected properties. However, it isn't powerful enough to deal with infinite objects. Two infinite objects are usually considered "equal" when they are *bisimilar.* Semantically, bisimilarity has a simple definition [2, 25, 3]. In the internal language, two coalgebras are bisimilar if their decompositions are equal, coinductively.

Definition 2.3. Given a locally cartesian closed category $\mathbb{C}$ with an endofunctor F and two coalgebra $c_i : T_i \to F(T_i)$ $(i = 1, 2)$, a *bisimulation* between T_1 and T_2 is given by a span $T_1 \leftarrow R \to T_2$ with a coalgebra structure such that the following diagram commutes.

$$
\begin{array}{ccccc}
T_1 & \xleftarrow{\;r_1\;} & R & \xrightarrow{\;r_2\;} & T_2 \\
{\scriptstyle c_1}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle c_2} \\
F(T_2) & \xleftarrow{\;F_{r_1}\;} & F(R) & \xrightarrow{\;F_{r_2}\;} & F(T_2)
\end{array}
$$

In particular, the identity span $T \leftarrow T \to T$ is always a bisimulation and the converse of a bisimulation between T_1 and T_2 is a bisimulation between T_2 and T_1.

Functions between coinductive types ought to be congruences for bisimilarity:

Definition 2.4. If $f_i : C_i \to D_i$ $(i = 1, 2)$ are morphisms from coalgebras $c_i : C_i \to F(C_i)$ to coalgebras $d_i : D_i \to F(D_i)$, we say that f_1 and f_2 are equal *up to bisimulation,* written $f_1 \approx f_2$, if for every bisimulation $C_1 \leftarrow R \to C_2$, there is a bisimulation $D_1 \leftarrow S \to D_2$

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[6]It is apparently possible to do so in univalent type theory [3], where coinductive type can be defined from inductive ones!

and a morphism $h : R \to S$ making the following diagram commute.

$$
\begin{array}{ccccc}
F(C_1) & \xleftarrow{\ F_{r_1}\ } & F(R) & \xrightarrow{\ F_{r_2}\ } & F(C_2) \\
\uparrow{\scriptstyle c_1} & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle c_1} \\
C_1 & \xleftarrow{\ r_1\ } & R & \xrightarrow{\ r_2\ } & C_2 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f_2} \\
D_1 & \xleftarrow{\ s_1\ } & S & \xrightarrow{\ s_2\ } & D_2 \\
\downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle d_2} \\
F(D_1) & \xleftarrow{\ F_{s_1}\ } & F(S) & \xrightarrow{\ F_{s_2}\ } & F(D_2)
\end{array}
$$

Translated in the internal language, $f_1 \approx f_2$ means that if x and y are bisimilar, then $f_1\, x$ and $f_2\, y$ are also bisimilar. It is not difficult to show that

- $\approx$ is reflexive and symmetric,
- $\approx$ is compositional: if $f_1 \approx f_2$ and $g_1 \approx g_2$, then $f_1 g_1 \approx f_2 g_2$ (if the composition makes sense).

Without additional properties (which holds in our context), $\approx$ is however not transitive.

Coinductive types are interpreted by *weakly* terminal coalgebras. There can be, in principle, several non-isomorphic weakly terminal coalgebras. We however have the following.

Lemma 2.5. *Let T_1 and T_2 be weakly terminal coalgebra for the endofunctor F,*

- *if $m : T_1 \to T_2$ is a mediating morphisms, then $m \approx \mathrm{id}$,*
- *if $f : T_1 \to T_2$ and $g : T_2 \to T_1$ are mediating morphisms, then $gf \approx \mathrm{id}_{T_1}$ and $fg \approx \mathrm{id}_{T_2}$.*

Proof. Consider the following diagram:

$$
\begin{array}{ccccc}
F(T_2) & \xleftarrow{\ F_{r_2}\ } & F(R) & \xrightarrow{\ F_{r_1}\ } & F(T_1) \\
\uparrow{\scriptstyle t_2} & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle t_1} \\
T_2 & \xleftarrow{\ r_2\ } & R & \xrightarrow{\ r_1\ } & T_1 \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle m} \\
T_2 & \xleftarrow{\ r_2\ } & R & \xrightarrow{\ mr_1\ } & T_2 \\
\downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle t_2} \\
F(T_2) & \xleftarrow{\ F_{r_2}\ } & F(R) & \xrightarrow{\ F_{mr_1}\ } & F(T_2)
\end{array}
$$

The only thing needed to make it commutative is that the bottom right square is commutative. This follows from the fact that m is a mediating morphism between the weakly terminal

coalgebras:

$$
\begin{array}{ccccc}
R & \xrightarrow{\;r_1\;} & T_1 & \xrightarrow{\;m\;} & T_2 \\
{\scriptstyle r}\Big\downarrow & & {\scriptstyle t_1}\Big\downarrow & & {\scriptstyle t_2}\Big\downarrow \\
F(R) & \xrightarrow[F(r_1)]{} & F(T_1) & \xrightarrow[F(m)]{} & F(T_2)
\end{array}
$$

The second point is a direct consequence of the first one. $\qquad\square$

Assumption 2.6. *We assume our type theory is "compatible" with bisimilarity, in the sense that any definable function f between* datatypes *satisfies $f \approx f$.*

Since it is possible to construct a dependent type theory where bisimilarity *is* intensional equality [3, 21], this assumption is reasonable. In our case, it allows to simplify some of the (pen and paper) proofs while not needing the extension of our type theory.

Note however that the hypothesis that the functions are between *datatypes* (i.e. not involving equality) is crucial: the whole problem is precisely that we don't assume bisimilarity is the same as (intensional) equality.

3. Indexed Containers

Indexed containers [5] were first considered (implicitly) in type theory 30 years ago by K. Petersson and D. Synek [22]. Depending on the context and the authors, they are also called *interaction systems* [14] or *polynomial diagrams* [12, 15]

Definition 3.1. For $I : \mathsf{Set}$, an indexed container over I is a triple $w = \langle A, D, n \rangle$ where

- $i : I \vdash A(i) : \mathsf{Set}$,
- $i : I, a : A \vdash D(i, a) : \mathsf{Set}$,
- $i : I, a : A, d : D \vdash n(i, a, d) : I$.[7]

In Agda, the definition looks like

```
record IS (I : Set) where
  field
    A : I → Set
    D : (i : I) → A i → Set
    n : (i : I) → (a : A i) → D i c → I
```

A useful intuition is that I is a set of states and that $\langle A, D, n \rangle$ is a game:

- $A(i)$ is the set of *moves* (or *actions* or *commands*) in state i
- $D(i, a)$ is the set of *counter-moves* (or *reactions* or *responses*) after move a in state i,
- $n(i, a, d) : I$ is the *new state* after move a and counter-move d have been played. When no confusion arises, we usually write $i[a/d]$ for $n(i, a, d)$.

Each indexed container gives rise to a monotone operator on predicates over I:

Definition 3.2. If $w = \langle A, D, n \rangle$ is an indexed container over $I : \mathsf{Set}$, the *extension of w* is the operator: $[\![w]\!] : \mathsf{Pow}(I) \to \mathsf{Pow}(I)$ where

$$
i \,\epsilon\, [\![w]\!](X) \quad = \quad \big(\Sigma\, a : A(i)\big) \ \big(\Pi\, d : D(i, a)\big) \ i[a/d] \,\epsilon\, X \, .
$$

---

[7]An abstract, but equivalent, way of defining indexed containers over I is as functions $I \to \mathsf{Fam}\big(\mathsf{Fam}(I)\big)$.

Lemma 3.3. *The operator $[\![w]\!]$ is monotonic, i.e., the following type is inhabited:*

$$X \subset Y \quad \to \quad [\![w]\!](X) \subset [\![w]\!](Y)$$

for every predicates X and Y of the appropriate type.

Proof. [Agda✓] This is direct: given $i : X \subset Y$ and $\langle a, f \rangle : [\![w]\!](X)$, we just need to "carry" i through $[\![w]\!]$ and return $\langle a, i \circ f \rangle$. □

Indexed containers form the objects of several categories of interest [14, 12, 5, 15], but that will only play a very minor role here.

3.1. **Composition.** Extensions of indexed containers can be composed as functions. There is a corresponding operation on the indexed containers.

Definition 3.4. If $w_1 = \langle A_1, D_1, n_1 \rangle$ and $w_2 = \langle A_2, D_2, n_2 \rangle$ are two indexed containers on I, the *composition* of w_1 and w_2 is the indexed container $w_2 \circ w_1 = \langle A, D, n \rangle$ where
- $A(i) = \big(\Sigma\, a_1 : A_1(i)\big)\ \big(\Pi\, d_1 : D_1(i, a_1)\big)\ A_2\big(i_1[a_1/d_1]\big) = i \,\epsilon\, [\![w_1]\!](A_2)$,
- $D\big(i, \langle a_1, f \rangle\big) = \big(\Sigma\, d_1 : D_1(i, a_1)\big)\ D_2\big(i[a_1/d_1], f\, d_1\big)$,
- $n\big(i, \langle a_1, f \rangle, \langle d_1, d_2 \rangle\big) = i[a_1/d_1][f\, d_1/d_2]$.

Lemma 3.5. *For every indexed containers w_1 and w_2 and predicate X, we have*

$$[\![w_2]\!] \circ [\![w_1]\!](X) == [\![w_2 \circ w_1]\!](X)$$

where $Y == Z$ is an abbreviation for $(X \subset Y) \times (Y \subset X)$.

If function extensionality holds, the pair of functions are inverse to each other.

Proof. [Agda✓] The main point is that the intensional axiom of choice

$$\big(\Pi\, d : D\big)\ \big(\Sigma\, a : A(d)\big)\ \varphi(d, a) \quad \leftrightarrow \quad \big(\Sigma\, f : \big(\Pi\, d : D\big)\ A(d)\big)\ \big(\Pi\, d : D\big)\ \varphi(d, f\, d)$$

is provable in Martin-Löf type theory. We then have

$$
\begin{aligned}
i \,\epsilon\, [\![w_2]\!] \circ [\![w_1]\!](X) \quad = \quad & \big(\Sigma\, a_1 : A_1(i)\big)\ \big(\Pi\, d_1 : D_1(i, a_1)\big)\ \big(\Sigma\, a_2 : A_2(i[a_1/d_1])\big) \\
& \big(\Pi\, d_2 : D_2(i[a_1/d_1], a_2)\big)\ i[a_1/d_1][a_2/d_2] \,\epsilon\, X \\
\text{(axiom of choice)} \quad \leftrightarrow \quad & \big(\Sigma\, a_1\big)\ \big(\Sigma\, f : \big(\Pi\, d_1 : D_1(i, a_1)\big)\ A_2(i[a_1/d_1])\big)\ \big(\Pi\, d_1\big) \\
& \big(\Pi\, d_2\big)\ i[a_1/d_1][f\, d_1/d_2] \,\epsilon\, X \\
\leftrightarrow \quad & \big(\Sigma\, \langle a_1, f \rangle\big)\ \big(\Pi\, \langle d_1, d_2 \rangle\big)\ i[a_1/d_1][f\, d_1/d_2] \,\epsilon\, X \\
= \quad & i \,\epsilon\, [\![w_2 \circ w_1]\!](X).
\end{aligned}
$$

□

3.2. **Duality.**

Definition 3.6. If $w = \langle A, D, n \rangle$ is an indexed container over I, we write $w^\perp$ for the indexed container $\langle A^\perp, D^\perp, n^\perp \rangle$ where
- $A^\perp(i) = \big(\Pi\, a : A(i)\big)\ D(i, a)$,
- $D^\perp(i, \_) = A(i)$, (note that it does not depend on the value of $f : A^\perp(i)$)
- $i[f/a] = i[a/f\, a]$.

Lemma 3.7. *For every indexed container* $w = \langle A, D, n \rangle$, *the following type is inhabited:*

$$i \; \epsilon \; \left[\!\left[w^\perp \right]\!\right] (X) \quad \longleftrightarrow \quad \left(\Pi\, a : A(i) \right) \; \left(\Sigma\, d : D(i, a) \right) \; i[a/d] \; \epsilon \; X \,.$$

With function extensionality, this is an isomorphism

Sketch of proof. $[Agda\checkmark]$ Just like lemma 3.5, the proof relies on the intensional axiom of choice, which shows that

$$\left(\Sigma\, f : A^\perp(i) \right) \; \left(\Pi\, a : D^\perp(i, f) \right) \; \varphi(a, f\, a) \leftrightarrow \left(\Pi\, a : A(i) \right) \; \left(\Sigma\, d : D(i, a) \right) \; \varphi(a, d) \,.$$

$\square$

3.3. Free Monad. If $w = \langle A, D, n \rangle$ is an indexed container on I, we can consider the free monad generated by $[\![w]\!]$. N. Gambino and M. Hyland proved that the free monad F_w generated by some $[\![w]\!]$ (a dependent polynomial functors) is of the form $[\![w^*]\!]$ for some indexed container w^* [11]. It is characterized by the fact that $[\![w^*]\!] (X)$ is the least fixed point of $Y \mapsto X \cup [\![w]\!] (Y)$. In other words, we are looking for an indexed container w^* satisfying
- $X \cup [\![w]\!] \left([\![w^*]\!] (X) \right) \subset [\![w^*]\!] (X)$
- $X \cup [\![w]\!] (Y) \subset Y \rightarrow [\![w^*]\!] (X) \subset Y$.

Informally, $i \; \epsilon \; [\![w]\!]^* (X)$ iff $i \; \epsilon \; X \cup w\big(X \cup w(X \cup w(X \cup \dots))\big)$. Expending the definition of $[\![w]\!]$, this means that

$$\left(\Sigma\, a_1 \right) \; \left(\Pi\, d_1 \right) \; \left(\Sigma\, a_2 \right) \; \left(\Pi\, d_2 \right) \quad \dots \quad i[a_1/d_1][a_2/d_2] \cdots \epsilon \; X \,.$$

This pseudo formula depicts a well-founded tree (with branching on $d_i : D(\_,\_)$) in which branches are finite: either they end at a Πd_i because the corresponding domain $D(\_,\_)$ is empty, or they end in a state $i[a_1/d_1][a_2/d_2]\dots$ which belongs to X. The length of the sequence $a_1/d_1, a_2/d_2, \dots$ is finite but may depend on what moves / counter-moves are chosen as the sequence grows longer. We can define w^* inductively.

Definition 3.8. Define $w^* = \langle A^*, D^*, n^* \rangle$ over I as:
- $A^* : \mathsf{Pow}(I)$ is a weak initial algebra for the endofunctor $X \mapsto (\lambda\, i) \big(\mathbf{1} + [\![w]\!] (X)(i) \big)$ on $\mathsf{Pow}(I)$. Concretely, there are two constructors for $A^*(i)$:

$$\frac{}{\mathtt{Leaf} : A^*(i)} \quad \text{and} \quad \frac{a : A(i) \qquad k : \big(\Pi\, d : D(i, a) \big) \; A^*(n\, i\, a\, d)}{\mathtt{Node}(a, k) : A^*(i)} \,.$$

  Thus, an element of $A^*(i)$ is a well-founded tree where each internal node is labeled with an elements $a : A(i)$ and the branching is given by the set $D(i, a)$.
- The components D^* and n^* are defined by recursion:
  - in the case of a $\mathtt{Leaf}$:
    $$\begin{aligned} D^*\big(i, \mathtt{Leaf}(i)\big) &= \mathbf{1} : \mathsf{Set} \\ n^*\big(i, \mathtt{Leaf}(i), \star\big) &= i : I \end{aligned}$$
  - in the case of a $\mathtt{Node}$:
    $$\begin{aligned} D^*\big(i, \mathtt{Node}(a, k)\big) &= \big(\Sigma\, d : D(i, a) \big) \; D^*\big(i[a/d], f\, d\big) : \mathsf{Set} \\ n^*\big(i, \mathtt{Node}(a, k), \langle d, d' \rangle\big) &= n^*\big(i[a/d], k\, d, d'\big) : I \,. \end{aligned}$$

  The corresponding Agda definition is

```
module FreeMonad (I : Set) (w : IS I) where
  open IS w

  data A* : I → Set where
    Leaf : (i : I) → A* i
    Node : (i : I) → (a : A i) → (f : (d : (D i a)) → A* (n a d)) → A* i

  D* : (i : I) → (t : A* i) → Set
  D* i Leaf = One
  D* i (Node a f) = Σ (D i a) (λ d → D* (n a d) (f d))

  n* : (i : I) → (t : A* i) → (b : D* i t) → I
  n* i Leaf ⋆ = i
  n* i (Node a f) ( d , ds ) = n* (f d) ds
```

In the presence of extensional equality, this particular inductive definition can be encoded using standard W-types [12]. As it is given, it avoids using equality but needs either a universe, or degenerate[8] induction-recursion on $\mathsf{Fam}(I)$. This construction does indeed correspond to the free monad described by N. Gambino and M. Hyland:

Lemma 3.9. *If w is an interaction system on I : Set, we have*
(1) *for all X : $\mathsf{Pow}(I)$, $X \cup [\![w]\!]\left([\![w^*]\!](X)\right) \subset [\![w^*]\!](X)$*
(2) *for all X, Y : $\mathsf{Pow}(I)$, $X \cup [\![w]\!](Y) \subset Y \to [\![w^*]\!](X) \subset Y$.*

Proof. $[Agda\checkmark]$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3.4. **Greatest Fixed Points.** Agda has some support for infinite values via the $\infty\_$ type constructor making a type "lazy", i.e. stopping computation. Using this and the operators $\natural : A \to \infty A$ (to freeze a value) and $\flat : \infty A \to A$ (to unfreeze it), it is possible to define the type $\nu_{[\![w]\!]}$ (which we'll write ν_w) for any interaction system w. The termination checker used in Agda [1] also checks productivity of recursive definition, but since it is not clear that this is sound when inductive and coinductive types are mixed [4, 17] we will only use the introduction and elimination rules in our developments:

$$\frac{\sigma : X \subset [\![w]\!](X)}{\nu_{\mathrm{intro}}\,\sigma : X \subset \nu_w} \quad \text{and} \quad \frac{}{\nu_{\mathrm{elim}} : \nu_w \subset [\![w]\!](\nu_w)}$$

which are definable in Agda.

Elements of ν_w are formed by coalgebras for $[\![w]\!]$, and any element of $i \,\epsilon\, \nu_w$ can be "unfolded" into an element of $i \,\epsilon\, [\![w]\!](\nu_w)$ i.e. into an element of

$$\left(\Sigma\, a : A(i)\right)\,\left(\Pi\, d : D(i, a)\right)\, i[a/d] \,\epsilon\, \nu_w\,.$$

We can repeat this unfolding and informally decompose an element of $i \,\epsilon\, \nu_w$ into an infinite "lazy" process of the form

$$\left(\Sigma\, a_1 : A(i)\right)\,\left(\Pi\, d_1 : D(i, a_1)\right)\,\left(\Sigma\, a_2 : A(i[a_1/d_1])\right)\,\left(\Pi\, d_2 : D(i[a_1/d_1], a_2)\right)\,\cdots$$

We therefore picture an element of $i \,\epsilon\, \nu_w$ as an infinite tree (which needs not be well-founded). Each node of such a tree has an implicit state in I, and the root has state i. If the state of a

---

[8]in the sense that the inductive set $A^*(i)$ doesn't depend on the recursive functions $D^*(i, \_)$ and $n^*(i, \_, \_)$.

node is j, then the node contains an element a of $A(j)$, and the branching of that node is given by $D(j, a)$. Note that some finite branches may be inextensible when they end at a node of sort j with label a for which $D(j, a)$ is empty.

Examples. We will be particularly interested in fixed points of the form $\nu_{w^\perp}$. Because of lemma 3.7, an element of $i \,\epsilon\, \nu_{w^\perp}$ unfolds to a potentially infinite object of the form

$$\left(\Pi\, a_1 : A(i)\right)\ \left(\Sigma\, d_1 : D(i, a_1)\right)\ \left(\Pi\, a_2 : A(i[a_1/d_1])\right)\ \left(\Sigma\, d_2 : D(i[a_1/d_1], a_2)\right)\ \cdots$$

For such types, the branching comes from $A(\_)$ and the labels from $D(\_, \_)$. Here are some example of the kind of objects we get.
(1) Streams on T are isomorphic to "$\star \,\epsilon\, \nu_{w^\perp}$" where $I = \mathbf{1} = \{\star\}$ and $w = \langle A, D, n \rangle$ with
  - $A(\star) = \mathbf{1}$,
  - $D(\star, \star) = X$,
  - $n(\star, \star, x) = \star$.
(2) Increasing streams of natural numbers are isomorphic to "$0 \,\epsilon\, \nu_{w^\perp}$" where $I = \mathbb{N}$ and $w = \langle A, D, n \rangle$ with
  - $A(i) = \mathbf{1}$,
  - $D(i, \star) = \left(\Sigma\, j : \mathbb{N}\right)\ (i < j)$,
  - $n(i, \star, \langle j, p \rangle) = j$.
(3) Infinite, finitely branching trees labeled by X are isomorphic to $\star \,\epsilon\, \nu_{w^\perp}$ where $I = \mathbf{1}$ and $w = \langle A, D, n \rangle$ to be
  - $A(\star) = \left(\Sigma\, k : \mathbb{N}\right)\ N(k)$, where $N(k)$ is the set with exactly k elements,
  - $D(\star, \langle k, i \rangle) = X$,
  - $n(\star, k, x) = \star$.
(4) In general, non-losing strategies for the second player from state $i : I$ in game w are given by $i \,\epsilon\, \nu_{w^\perp}$.

Bisimulations. The appropriate equivalence relation on coinductive types is *bisimilarity*. Translating the categorical notion from page 8 for the type ν_w,[9] we get that $T_1 : i_0 \,\epsilon\, \nu_w$ is bisimilar to $T_2 : i_0 \,\epsilon\, \nu_w$ if:
(1) there is an I indexed family $R_i : (i \,\epsilon\, \nu_w) \times (i \,\epsilon\, \nu_w) \to \mathsf{Set}$ (an "indexed relation") s.t.
(2) $R_{i_0}\langle T_1, T_2 \rangle$ is inhabited,
(3) whenever $R_i \langle T_1, T_2 \rangle$ is inhabited, we have
  - $a_1 \equiv a_2$,
  - for every $d_1 : D(i, a_1)$, the elements $f_1\, i[a_1/d_1]$ and $f_2\, i[a_2/d_2]$ are related by R, where $\langle a_1, f_1 \rangle$ [resp. $\langle a_2, f_2 \rangle$] comes from the coalgebra structure $\nu_w \subset [\![w]\!]\,\nu_w$ applied to T_1 [resp. T_2].

Expressing this formally is quite verbose as values need to be transported along equalities to have an appropriate type. For example, having $a_1 \equiv a_2 \in A(i)$ only entails that $D(i, a_1)$ and $D(i, a_2)$ are isomorphic, and thus, $d_1 : D(i, a_1)$ is not strictly speaking an element of $D(i, a_2)$!

We noted on page 9 that categorically speaking, without hypothesis on the functor F, $\approx$ was not necessarily transitive. We have

---

[9]Note that we only define bisimilarity for elements of $i \,\epsilon\, \nu_w$, and not for arbitrary $[\![w]\!]$-coalgebras.

Lemma 3.10. *If w is an interaction system over I, then $\approx$ is an equivalence relation on any $i \,\epsilon\, \nu_w$:*

- *for any $T : i \,\epsilon\, \nu_w$, there is an element in $T \approx T$,*
- *for any $T_1, T_2 : i \,\epsilon\, \nu_w$, there is a function in $(T_1 \approx T_2) \to (T_2 \approx T_1)$,*
- *for any $T_1, T_2, T_3 : i \,\epsilon\, \nu_w$, there is a function in $(T_1 \approx T_2) \to (T_2 \approx T_3) \to (T_1 \approx T_3)$.*

Proof. [*Agda*✓] The result is intuitively obvious but while reflexivity is easy, proving transitivity (and to a lesser extend symmetry) in Agda is surprisingly tedious. Explaining the formal proof is probably pointless as it mostly consists of transporting elements along equalities back and forth. □

We will keep some of the bisimilarity proofs in the meta theory in order to simplify the arguments. The only consequence of the assumption from page 10 that we'll need is the following.

Lemma 3.11. *Suppose that $f, g : i_1 \,\epsilon\, \nu_{w_1} \to i_2 \,\epsilon\, \nu_{w_2}$ are definable in type theory, and that neither w_1 nor w_2 involve the identity type, then, to prove that $f \approx g$,[10] it is enough to show that $f\,T \approx g\,T$ for any $T : i_1 \,\epsilon\, \nu_{w_1}$.*

Proof. If $S \approx T$, we have $f\,S \approx f\,T \approx g\,T$ where the first bisimulation comes from the assumption $f \approx f$ from page 10, and the second bisimulation comes from the hypothesis of the lemma. □

This makes proving $f \approx g$ simpler as we can replace the hypothesis $T_1 \approx T_2$ by the stronger $T_1 \equiv T_2$.

Weakly Terminal Coalgebras. We will have to show that some sets are isomorphic "up to bisimilarity". To do that, we'll use lemma 2.5 by showing that the two sets are weakly terminal coalgebras for the same functor $[\![w]\!]$. (One of the sets will always be ν_w, making half of this automatic.)

To show what $T : \mathsf{Pow}(I)$ is a weakly terminal coalgebra for $[\![w]\!]$, we have to define, mimicking the typing rules for coinductive types:

- $\mathtt{elim} : T \subset [\![w]\!]\,(T)$,
- $\mathtt{intro} : X \subset [\![w]\!]\,(X) \to X \subset T$,
- $\mathtt{comp}_{X,c,x,i} : \mathtt{elim}(\mathtt{intro}\,c\,i\,x) \equiv [\![w]\!]_{\mathtt{intro}\,c}\,i\,(c\,i\,x)$ whenever
  - $X : \mathsf{Pow}(I)$, $c : X \subset [\![w]\!]\,(X)$, $i : I$ and $x : i \,\epsilon\, X$,
  - $[\![w]\!]_{\mathtt{intro}\,c} : [\![w]\!]\,(X) \subset [\![w]\!]\,(T)$ comes from lemma 3.3.

By lemma 2.5, we have

Corollary 3.12. *If C is a weakly terminal coalgebra for $[\![w]\!]$, then there are functions $f : \nu_w \subset C$ and $g : C \subset \nu_w$ such that*

$$f\,i\,(g\,i\,T) \approx T$$

for any $T : i \,\epsilon\, \nu_w$.

Proof. [*Agda*✓] This is the second point of lemma 2.5, and it has been formalized in Agda. □

---

[10]i.e. $S \approx T \to f\,S \approx g\,T$

4. Simulations and Evaluation

4.1. **Functions on Streams.** Continuous function from $\mathtt{stream}(A)$ to $\mathtt{stream}(B)$ can be described by infinite, A-branching "decision trees" with two kinds of nodes: ϵ and $\mathtt{output}_b$ with $b \in B$. The idea is that $f(s) = [b_1, b_2, \dots]$ if and only if the infinite branch corresponding to s contains, in order, the nodes $\mathtt{output}_{b_1}$, $\mathtt{output}_{b_2}$, etc. For that to be well defined, we need to guarantee that there are no infinite path in the tree without any $\mathtt{output}$ node.

Theorem 4.1 ([13]). *The set of continuous functions from $\mathtt{stream}(A)$ to $\mathtt{stream}(B)$ is isomorphic to the set $\nu X.\mu Y.(B \times X) + (A \to Y)$.*

The nested fixed points guarantee that along a branch, there can only be finitely many consecutive non-$\mathtt{output}$ nodes:

- $\mu Y.(B \times X) + (A \to Y)$: well-founded A-branching trees with leafs in $B \times X$, i.e. consisting of an $\mathtt{output}_b$ node and an element of X,
- $\nu X. \cdots$ the element in X at each leaf for the well-founded trees is another such well-founded tree, ad infinitum.

We can evaluate each such tree into a continuous function, a process colloquially referred to as "stream eating": we consume the elements of the stream to follow a branch in the infinite tree, and output b (an element of the result) whenever we find an $\mathtt{output}_b$ node.[11]

Our aim is to extend theorem 4.1 to coinductive types of the form $i \epsilon \nu_{w^\perp}$. The problem is doing so in a type-theoretic manner and even the case of dependent streams (where the type of an element may depend on the value of the previous element) is not trivial. Retrospectively, the difficulty was that there are two generalizations of streams:

- adding states: we consider dependent streams instead of streams;
- adding branching: we consider trees instead of streams.

Both cases required the introduction of states *and* branching, making those seemingly simpler cases as hard as the general one.

4.2. **Linear Simulations as Transducers.** We are going to define a notion of "transducer" that can explore some branch of its input and produce some output along the way. Such notions have already been considered as natural notions of morphisms between dependent containers: linear simulations [15] and general simulations [14]. These notions generalize morphism as representations of pointwise inclusions $[\![w_1]\!] \subset [\![w_2]\!]$ (called cartesian strong natural transformations), which only make sense for containers with the same index set [12, 11, 5].

A transducer from type $i_1 \epsilon \nu_{w_1^\perp}$ to type $i_2 \epsilon \nu_{w_2^\perp}$ works as follows: given an argument (input) in $i_1 \epsilon \nu_{w_1^\perp}$, morally of the form

$$\left(\Pi\, a_0\right)\ \left(\Sigma\, d_0\right)\ \left(\Pi\, a_1\right)\ \left(\Sigma\, d_1\right)\ \dots$$

it must produce (output) an object of the form

$$\left(\Pi\, b_0\right)\ \left(\Sigma\, e_0\right)\ \left(\Pi\, b_1\right)\ \left(\Sigma\, e_1\right)\ \dots$$

In other words, the transducer

---

[11]Constructing the tree from a function isn't constructive without access to a function describing the modulus of continuity of the function, i.e. the size of the prefix we need to inspect to get a given finite prefix of the result.

- consumes b (they are given by the environment when constructing the result) and d (they may be produced internally by the input),
- produces e (as part of the output) and a (to be used internally by feeding them to the input).

Graphically, it can suggestively represented as
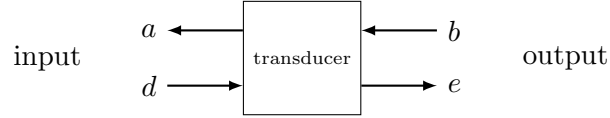
A very simple kind of transducer works as follows:
(1) when given some b_0,
(2) it produces an a_0 and feeds it to it argument,
(3) it receives a d_0 from its argument,
(4) and produces an e_0 for its result.
(5) It starts again at step (1) possibly in a different internal state.

This intuition is captured by the following definition.

Definition 4.2. Let $w_1 = \langle A_1, D_1, n_1 \rangle$ and $w_2 = \langle A_2, D_2, n_2 \rangle$ be indexed containers over I_1 and I_2 and let $R : \mathsf{Pow}(I_1 \times I_2)$ be a relation between states. We say that R is a *linear simulation* from w_1 to w_2 if it comes with a proof:

$$\rho \quad : \quad \big(\Pi\, i_1 : I_1\big)\ \big(\Pi\, i_2 : I_2\big)\ R(i_1, i_2) \quad \to \quad \begin{aligned}&\big(\Pi\, a_2 : A_2(i_2)\big)\\&\big(\Sigma\, a_1 : A_1(i_1)\big)\\&\big(\Pi\, d_1 : D_1(i_1, a_1)\big)\\&\big(\Sigma\, d_2 : D_2(i_2, a_2)\big)\\&\quad R\big(i_1[a_1/d_1], i_1[a_2/d_2]\big)\,.\end{aligned}$$

We write $(R, \rho) : w_1 \multimap w_2$, but usually leave the ρ implicit and write $R : w_1 \multimap w_2$.

To justify the fact that this can serve as a transducer, we need to "evaluate" a simulation on elements of $i \in \nu_{w_1^\perp}$.

Lemma 4.3. *Let $w_1 = \langle A_1, D_1, n_1 \rangle$ and $w_2 = \langle A_2, D_2, n_2 \rangle$ be indexed containers over I_1 and I_2, and $(R, \rho) : w_1 \multimap w_2$. We have a function*

$$\mathrm{eval}_R : \big(\Pi\, i_1 : I_1\big)\ \big(\Pi\, i_2 : I_2\big)\ R(i_1, i_2) \to (i_1 \in \nu_{w_1^\perp}) \to (i_2 \in \nu_{w_2^\perp})\,.$$

Proof. [Agda✓] This amounts to unfolding the simulation as a linear transducer. The main point in the Agda proof is to show that the predicate $\nu_{w_1^\perp} \mathbin{\lozenge} R^\sim(i_2)$ is a coalgebra for $[\![w_2^\perp]\!]$. □

The next lemma shows that this notion of simulation gives an appropriate notion of morphism between indexed containers.

Lemma 4.4. *We have:*
- *the identity type on I is a linear simulation from any w over I to itself,*
- *if R is a linear simulation from w_1 to w_2, and if S is a linear simulation from w_2 to w_3, then $S \circ R$ is a linear simulation from w_1 to w_3, where $S \circ R$ is the relational composition of R and S:*

$$(S \circ R)(i_1, i_3) \quad = \quad \big(\Sigma\, i_2 : I_2\big)\ R(i_1, i_2) \times S(i_2, i_3)$$

Proof. [*Agda*✓] That the identity type is a linear simulation is straightforward. Composing simulation amounts to extracting the functions $a_1 \mapsto a_2$ and $d_2 \mapsto d_1$ from the simulations, and composing them. □

Note that composition of simulations is only associative *up to associativity of relational composition* (pullback of spans) so that a quotient is needed to turn indexed containers into a category [15]. What is nice is that composition of simulation corresponds to composition of their evaluations, up to bisimilarity.

Lemma 4.5. *If w_1, w_2 and w_3 are interaction systems on I_1, I_2 and I_3, and if $R : w_1 \multimap w_2$ and $S : w_2 \multimap w_3$, then we have*

$$\mathrm{eval}_{S \circ R}\, i_1\, i_3\, \langle i_2, \langle r, s \rangle \rangle \quad \approx \quad \mathrm{eval}_S\, i_2\, i_3\, s \circ \mathrm{eval}_R\, i_1\, i_2\, r$$

where

- $i_1 : I_1$, $i_2 : I_2$, $i_3 : I_3$,
- $r : R(i_1, i_2)$ and $s : S(i_2, i_3)$,
- *and thus,* $\langle i_2, \langle r, s \rangle \rangle : (S \circ R)(i_1, i_3)$.

Recall that for functions, $f \approx g$ means that for every input T (here of type $i_1 \in \nu_{w_1^\perp}$), we have "$f\,T \approx g\,T$, i.e. $f\,T$ is bisimilar to $g\,T$".

Proof. [*Agda*✓] This is one instance where the direct, type theoretic proof of bisimilarity is possible, and not (too) tedious. With the transducer intuition in mind, this result is natural: starting from some $a_3 : A_1(i_3)$, we can either

- transform it to $a_2 : A_2(i_2)$ (with the simulation from w_2 to w_3) and then to $a_1 : A_1(i_1)$ (with the simulation from w_1 to w_2),
- or transform it directly to $a_1 : A(i_1)$ (with the composition of the two simulations).

Because composition is precisely defined by composing the functions making the simulations, the two transformations are obviously equal. (The Agda proof is messier than that but amounts to the same thing.)

Note that because of lemma 3.11, the Agda proof only needs to show that $(\mathrm{eval}_S \circ \mathrm{eval}_R)\,T$ is bisimilar to $\mathrm{eval}_{S \circ R}\,T$. □

4.3. General Simulations.

As far as representing functions from $i_1 \in \nu_{w_1^\perp}$ to $i_2 \in \nu_{w_2^\perp}$, linear simulation aren't very powerful. For streams, the first n elements of the result may depend at most on the first n elements of the input! Here is a typical continuous function that cannot be represented by a linear simulation. Given a stream s of natural numbers, look at the head of s:

- if it is 0, output 0 and start again with the tail of the stream,
- if it is $n > 0$, remove the next n element of the stream, output there sum, and start again.

For example, on the stream $[0, 1, 2, 3, 4, 5, 6, \ldots]$, the function outputs

$$\left[0,\, 2,\, \overbrace{4 + 5 + 6}^{3 \text{ elements}},\, \overbrace{8 + 9 + \cdots + 14}^{7 \text{ elements}},\, \overbrace{16 + 17 + \cdots + 31}^{15 \text{ elements}},\, \ldots\right] = [0, 2, 15, 77, 376, \ldots]$$
$$\underbrace{}_{=15} \quad \underbrace{}_{=77} \quad \underbrace{}_{=376}$$

We can generalize transducers by allowing them to work in the following manner.

(1) When given some b_0,
(2) they produce an a_0 and feed it to their argument,

(3) they receive a d_0 from their argument,

(4) and *either go back to step* (2) *or* produce an e_0 (output) for their result,

(5) start again at step (1)...

In other words, steps (2) and (3) can occur several times in a row. We can even allow the transducer to go directly from step (1) to step (4), bypassing steps (2) and (3) entirely. Of course steps (3) and (4) should never happen infinitely many times consecutively.

Definition 4.6. Let w_1 and w_2 be indexed containers over I_1 and I_2, let $R : \mathsf{Pow}(I_1 \times I_2)$ be a relation between states; we say that R is a *general simulation from w_1 to w_2* if it is a linear simulation from w_1^* to w_2.

In other words, $\langle R, \rho \rangle$ is a general simulation from $\langle A_1, D_1, n_1 \rangle$ to $\langle A_2, D_2, n_2 \rangle$ if

$$\rho : \big(\Pi\, i_1 : I_1\big)\ \big(\Pi\, i_2 : I_2\big)\ R(i_1, i_2)\ \rightarrow\ \begin{array}{l} \big(\Pi\, a_2 : A_2(i_2)\big) \\ \big(\Sigma\, \alpha_1 : A_1^*(i_1)\big) \\ \big(\Pi\, \delta_1 : D_1^*(i_1, \alpha_1)\big) \\ \big(\Sigma\, d_2 : D_2(i_2, a_2)\big) \\ \quad R\big(i_1[\alpha_1/\delta_1], i_2[a_2/d_2]\big)\,. \end{array}$$

Thanks to lemma 4.3, such a simulation automatically gives rise to a function from $\nu_{w_1^{*\perp}}$ to $\nu_{w_2^\perp}$. Fortunately, $\nu_{w_1^{*\perp}}$ is, up to bisimulation, isomorphic to $\nu_{w_1^\perp}$.

Lemma 4.7. $\nu_{w^{*\perp}}$ *is a weakly terminal coalgebra for* $[\![w^\perp]\!]$.

Proof. [*Agda*✓] From an element of $i\,\epsilon\,\nu_{w^{*\perp}}$ we can use the elimination rule and extract a member of $\big(\Pi\,\alpha : A^*(i)\big)\ \big(\Sigma\,\delta : D^*(i, \alpha)\big)\ i[\alpha/\delta]\nu_{w^{*\perp}}$. Given some $a : A(i)$, we instantiate α to $\mathtt{Node}(a, (\lambda\, d : D(i, a))\,\mathtt{Leaf}) : A^*(i)$ (a single a, followed by nothing), and its responses are just responses to a. This produces an element of $\big(\Pi\, a : A(i)\big)\ \big(\Sigma\, d : D(i, a)\big)\ i[a/d]\,\epsilon\,\nu_{w^{*\perp}}$, i.e. , an element of $i\,\epsilon\,[\![w^\perp]\!]\,(\nu_{w^{*\perp}})$. We've just shown that $\nu_{w^{*\perp}} \subset [\![w^\perp]\!]\,\nu_{w^{*\perp}}$. We refer to the Agda code for the rest of the proof. $\square$

Corollary 4.8. $\nu_{w^{*\perp}}$ *and* $\nu_{w^\perp}$ *are isomorphic up to bisimulation:*

$$\nu_{w^{*\perp}} \xleftrightarrow{\ \approx\ } \nu_{w^\perp}\,.$$

Proof. This is a direct consequence of lemma 2.5. $\square$

By composing the function $\nu_{w^\perp} \subset \nu_{w^{*\perp}}$ with the evaluation of linear simulations (lemma 4.3), we get an evaluation function for general simulations.

Corollary 4.9. *Let* $w_1 = \langle A_1, D_1, n_1 \rangle$ *and* $w_2 = \langle A_2, D_2, n_2 \rangle$ *be indexed containers over I and I_2, and* $(R, \rho) : w^* \multimap w_2$. *We have a function*

$$\mathrm{eval}_R^* : \big(\Pi\, i_1 : I_1\big)\ \big(\Pi\, i_2 : I_2\big)\ R(i_1, i_2) \rightarrow (i_1\,\epsilon\,\nu_{w_1^\perp}) \rightarrow (i_2\,\epsilon\,\nu_{w_2^\perp})\,.$$

Sidenote on formal topology. When evaluating a general simulation $R : w_1^* \multimap w_2$ directly (i.e. not relying on corollary 4.8), we need to compute an element of $i_2\,\epsilon\,\nu_{w_2^\perp}$ from

- a state $i_1 : I_1$ and a state $i_2 : I_2$,
- an element $r : R(i_1, i_2)$,
- an element $T_1 : i_1\,\epsilon\,\nu_{w_1^\perp}$.

We then need to find, for each branch $a_2 : A_2(i_2)$, a corresponding $d_2 : D_2(i_2, a_2)$ and a way to continue the computation. Given a_2, by the general simulation property, we can compute an element of

$$\left(\Sigma\,\alpha_1 : A_1^*(i_1)\right)\ \left(\Pi\,\delta_1 : D_1^*(i_1, a_1)\right)\ \left(\Sigma\,d_2 : D_2(i_2, a_2)\right)\ R\big(i_1[\alpha_1/\delta_1], i_2[a_2/d_2]\big)$$

which can be rewritten as

$$i_1 \,\epsilon\, [\![w_1^*]\!] \left((\lambda\,i : I_1)\ \left(\Sigma\,d_2 : D_2(i_2, a_2)\right)\ R\big(i, i_2[a_2/d_2]\big) \right).$$

The crucial first step is matching the well-founded tree α_1 with the infinite tree T_1. This can be done with

Lemma 4.10. *Let* $I : \mathsf{Set}$*, and* w *an indexed container over* I*, and* X *a predicate over* I*. The type* $w^*(X) \between \nu_{w^\perp} \to X \between \nu_{w^\perp}$ *is inhabited.*

Proof. [Agda✓] By matching dual quantifiers coming from $i \,\epsilon\, w^*(X)$ and $i \,\epsilon\, \nu_{w^\perp}$, we get an alternating sequence of moves a_i and counter moves d_i reaching a final state i_f in X, together with a infinite tree in $i_f \,\epsilon\, \nu_{w^\perp}$. More formally, suppose $w^*(X) \between \nu_{w^\perp}$ is inhabited, i.e., that we have $i : I$, $\langle \alpha, f \rangle : i \,\epsilon\, w^*(X)$ and a (non-well-founded) tree T in $i \,\epsilon\, \nu_{w^\perp}$. We examine α:

- $\alpha = \mathtt{Leaf}(i)$: we have $f \star : i \,\epsilon\, X$, in which case $U \between \nu_{w^\perp}$ is inhabited.
- $\alpha = \mathtt{Node}(a, k)$: where $k : \big(\Pi\,d\big)\ i[a/d] \,\epsilon\, w^*(X)$. We can apply ν_{elim} to $T : i \,\epsilon\, \nu_{w^\perp}$ to obtain a function in $\big(\Pi\,a\big)\ \big(\Sigma\,d\big)\ i[a/d] \,\epsilon\, \nu_{w^\perp}$. Applying that function to $a : A(i)$, we get $d : D(i, a)$ s.t.
  - $i[a/d] \,\epsilon\, \nu_{w^\perp}$,
  - $k\,d : i[a/d] \,\epsilon\, w^*(X)$.
  This pair inhabits $w^*(X) \between \nu_{w^\perp}$, and by induction hypothesis, yields an inhabitant of $U \between \nu_{w^\perp}$

$$\square$$

This formula is at the heart of formal topology, where it is called "monotonicity" [8]. There, $i \,\epsilon\, [\![w^*]\!]\,(U)$ is read "the basic open i is covered by U" (written $i \lhd U$), and $i \,\epsilon\, \nu_{w^\perp}$ is read "the basic open i contains a point" and is written $i \,\epsilon\, \mathsf{Pos}$.

Applied to the present situation with $X = (\lambda\,i : I_1)\ \big(\Sigma\,d_2 : D_2(i_2, a_2)\big)\ R\big(i, i_2[a_2/d_2]\big)$, we get an element of $X \between \nu_{w_2^\perp}$, which is precisely given by

- a state $i_1 : I_1$,
- a pair $\langle d_2, r \rangle$ with $d_2 : D_2(i_2, a_2)$ and $r : R(i_1, i_2[a_2/d_2])$,
- an element $T_1 : \nu_{w_1^\perp}$.

In other words, we get the sought after d_2 (giving the first element of the a_2 branch), together with enough data to continue the computation.

4.4. **The Free Monad Construction as a Comonad.** Composition of general simulations doesn't follow directly from composition of linear simulations because there is a mismatch on the middle interaction system: composing $R : w_1^* \multimap w_2$ and $S : w_2^* \multimap w_3$ to get a simulation from w_1^* to w_3 isn't obvious. Fortunately, the operation $w \mapsto w^*$ lifts to a comonad, and the composition corresponds to composition in its (co)Kleisli category.

Proposition 4.11. *Let $\mathbb{C}$ be a locally cartesian closed category, the operation $P \mapsto P^*$ lifts to a monad on the category of polynomial functors over I with cartesian natural transformations between them.*

Proof. [*Agda✗*] The operation $P \mapsto P^*$ goes from $\mathsf{End}(\mathbb{C}/I)$, the category of polynomial endofunctors on $\mathbb{C}/I$ to $\mathsf{Mnd}(\mathbb{C}/I)$, the category of (polynomial) monads over $\mathbb{C}/I$ [11, 12]. We write $\mathsf{Nat}(F, G)$ for natural transformations from F to G, and $\mathsf{Nat}_{\mathsf{Mnd}}(F, G)$ for those transformations that respect the monad structures of F and G. Writing $F\text{-}\mathsf{alg}$ for the category of F-algebras, and $F\text{-}\mathsf{Alg}$ (when F is a monad) for the category of F-algebras that respect the monad operations, we have

$$
\begin{aligned}
\mathsf{Nat}_{\mathsf{Mnd}}(P^*, M) &\cong M\text{-}\mathsf{Alg} \longrightarrow_{\mathbb{C}} P^*\text{-}\mathsf{Alg} &&\text{[6, proposition (5.3)]} \\
&\cong M\text{-}\mathsf{Alg} \longrightarrow_{\mathbb{C}} P\text{-}\mathsf{alg} &&\text{[11, proposition 17]} \\
&\cong \mathsf{Nat}(P, M) &&\text{[6, proposition (5.2)]}
\end{aligned}
$$

This shows that $\_^*$ is left adjoint to the forgetful functor $\mathcal{U} : \mathsf{Mnd}(\mathbb{C}/I) \to \mathsf{End}(\mathbb{C}/I)$, and makes the composition $\mathcal{U}\_^*$ a monad. It only remains to show that the monad operations are strong cartesian transformations.

Since strong natural transformations from $[\![w_1]\!]$ to $[\![w_2]\!]$ correspond exactly to linear simulations $(\equiv, \rho) : w_2 \multimap w_1$ [12, 16], it is enough to define the monad operations as simulations:

- $(\equiv, \varepsilon_w) : w^* \multimap w$
- and $(\equiv, \delta_w) : w^* \multimap w^{**}$

Those constructions are relatively straightforward in Agda: the first operation corresponds to embedding a single action $a : A(i)$ into $A^*(i)$ as $\mathtt{Node}(a, (\lambda\, d : D(i, a))\, \mathtt{Leaf})$. A direct definition of δ_w is done by induction and can be found in the Agda code. Semantically speaking, it can be derived from lemma 3.9:

$$
\begin{aligned}
& [\![w]\!]\, [\![w^*]\!]\, (X) \subset [\![w^*]\!]\, (X) &&\text{(first point of lemma 3.9)} \\
\to\ & [\![w^*]\!]\, (X) \cup [\![w]\!]\, [\![w^*]\!]\, (X) \subset [\![w^*]\!]\, (X) &&\text{(because } [\![w^*]\!]\, (X) \subset [\![w^*]\!]\, (X)) \\
\to\ & [\![w^*]\!]\, [\![w^*]\!]\, (X) \subset [\![w^*]\!]\, (X) &&\text{(second point of lemma 3.9)} \\
\to\ & X \cup [\![w^*]\!]\, [\![w^*]\!]\, (X) \subset [\![w^*]\!]\, (X) &&\text{(because } X \subset [\![w^*]\!]\, (X)) \\
\to\ & [\![w^{**}]\!]\, (X) \subset [\![w^*]\!]\, (X) &&\text{(second point of lemma 3.9)}
\end{aligned}
$$

$\square$

What makes this lemma interesting is that the constructions themselves are easy to define in Agda without identity types. A purely type theoretic proof that the constructions satisfies the monad laws couldn't be completed in Agda, because the overhead of reasoning with equality on dependent structures is *very* tedious. The categorical proof guarantees that it holds in all models for extensional type theory which is good enough for us.

Because of the reversal (strong natural transformation from w_1 to w_2 are equivalent to identity linear simulations from w_2 to w_1), this translates to "$w \mapsto w^*$ lifts to a comonad in the category of interaction systems with linear simulations".

Corollary 4.12. *The operation $w \mapsto w^*$ lifts to a comonad in the category of interaction systems over I and linear simulations.*

4.5. **Composition of General Simulations.** Recall that a comonad may be given in triple form with the following data:

- a natural transformation $\varepsilon_w : w^* \multimap w$,
- a "cobinding" operation taking $R : w_1^* \multimap w_2$ to $R^\sharp : w_1^* \multimap w_2^*$ (defined as $R^* \circ \delta_{w_1}$),

satisfying the following laws:

(1) $\varepsilon_w^\sharp = \mathrm{id}_{w^*}$,
(2) $\varepsilon_{w_2} \circ R^\sharp = R$ for $R : w_1^* \multimap w_2$,
(3) $(S \circ R^\sharp)^\sharp = S^\sharp \circ R^\sharp$ for $R : w_1 \multimap w_2^*$ and $S : w_2^* \multimap w_3^*$.

We can now define

Definition 4.13. If $R : w_1^* \multimap w_2$ and $S : w_2^* \multimap w_3$, define $S \bullet R$ with $S \circ R^\sharp$.

The comonad laws are then enough to prove that composition of general simulations corresponds to composition of their evaluations.

Proposition 4.14. *Let w_1, w_2, w_3 be interaction systems on I_1, I_2 and I_3, with $R : w_1^* \multimap w_2$ and $S : w_2^* \multimap w_3$, then we have*

$$(\mathrm{eval}_S^* \, i_2 \, i_3 \, s) \circ (\mathrm{eval}_R^* \, i_1 \, i_2 \, t) \quad \approx \quad \mathrm{eval}_{S \bullet R}^* \, i_1 \, i_3 \, \langle i_2, \langle s, t \rangle \rangle$$

where

- $i_1 : I_1$, $i_2 : I_2$, $i_3 : I_3$,
- $s : S(i_2, i_3)$ *and* $t : T(i_2, i_3)$,
- *and thus,* $\langle i_2, \langle s, t \rangle \rangle : (T \bullet S)(i_1, i_3)$.

Proof. [Agda✗] Because $\nu_{w_2^* \perp}$ is a weakly terminal algebra for $[\![w_2^\perp]\!]$, by lemma 2.5, we have a pair of morphisms $f_2 : i_2 \,\epsilon\, \nu_{w_2^\perp} \to i_2 \,\epsilon\, \nu_{w_2^* \perp}$ and $g_2 : i_2 \,\epsilon\, \nu_{w_2^* \perp} \to i_2 \,\epsilon\, \nu_{w_2^\perp}$ such that $f_2 g_2 \approx \mathrm{id}$ and $g_2 f_2 \approx \mathrm{id}$.

Expanding the definitions (and omitting all non-essential parameters, except for the first and last lines), we have:

$$i_1 \,\epsilon\, \nu_{w_1^\perp} \xrightarrow{\mathrm{eval}_{S \bullet R}^* \, i_1 \, i_3 \, \langle i_2 \langle r, s \rangle \rangle} i_3 \,\epsilon\, \nu_{w_3^\perp}$$

$$= \nu_{w_1^\perp} \xrightarrow{\mathrm{eval}_{S \bullet R}^*} \nu_{w_3^\perp}$$

$$(\text{def of eval}^*) \quad = \quad \nu_{w_1^\perp} \xrightarrow{f_1} \nu_{w_1^* \perp} \xrightarrow{\mathrm{eval}_{S \bullet R}} \nu_{w_3^\perp}$$

$$(\text{def of } \bullet) \quad = \quad \nu_{w_1^\perp} \xrightarrow{f_1} \nu_{w_1^* \perp} \xrightarrow{\mathrm{eval}_{S \circ R^\sharp}} \nu_{w_3^\perp}$$

$$(\text{lemma 3.5}) \quad = \quad \nu_{w_1^\perp} \xrightarrow{f_1} \nu_{w_1^* \perp} \xrightarrow{\mathrm{eval}_{R^\sharp}} \nu_{w_2^* \perp} \xrightarrow{\mathrm{eval}_S} \nu_{w_3^\perp}$$

$$= \quad \nu_{w_1^\perp} \xrightarrow{f_1} \nu_{w_1^* \perp} \xrightarrow{\mathrm{eval}_{R^\sharp}} \nu_{w_2^* \perp} \xrightarrow{\mathrm{id}} \nu_{w_2^* \perp} \xrightarrow{\mathrm{eval}_S} \nu_{w_3^\perp}$$

$$(\text{remark above}) \quad \approx \quad \nu_{w_1^\perp} \xrightarrow{f_1} \nu_{w_1^* \perp} \xrightarrow{\mathrm{eval}_{R^\sharp}} \nu_{w_2^* \perp} \xrightarrow{g_2} \nu_{w_2^\perp} \xrightarrow{f_2} \nu_{w_2^* \perp} \xrightarrow{\mathrm{eval}_S} \nu_{w_3^\perp}$$

$$(\text{remark below}) \quad = \quad \nu_{w_1^\perp} \xrightarrow{f_1} \nu_{w_1^* \perp} \xrightarrow{\mathrm{eval}_{R^\sharp}} \nu_{w_2^* \perp} \xrightarrow{\mathrm{eval}_{\varepsilon w_2}} \nu_{w_2^\perp} \xrightarrow{f_2} \nu_{w_2^* \perp} \xrightarrow{\mathrm{eval}_S} \nu_{w_3^\perp} \quad (*)$$

$$(\text{comonad law}) \quad = \quad \nu_{w_1^\perp} \xrightarrow{f_1} \nu_{w_1^* \perp} \xrightarrow{\mathrm{eval}_R} \nu_{w_2^\perp} \xrightarrow{f_2} \nu_{w_2^* \perp} \xrightarrow{\mathrm{eval}_S} \nu_{w_3^\perp}$$

$$(\text{def of eval}^*) \quad = \quad \nu_{w_1^\perp} \xrightarrow{\mathrm{eval}_R^*} \nu_{w_2^\perp} \xrightarrow{\mathrm{eval}_S^*} \nu_{w_3^\perp}$$

$$= \quad i_1 \,\epsilon\, \nu_{w_1^\perp} \xrightarrow{\mathrm{eval}_R^* \, i_1 \, i_2 \, r} i_2 \,\epsilon\, \nu_{w_2^\perp} \xrightarrow{\mathrm{eval}_S^* \, i_2 \, i_3 \, s}{}_3 \epsilon\, \nu_{w_3^\perp}$$

The only missing part $(*)$ is showing that $\mathrm{eval}_{\varepsilon_{w_2}} \approx g_2$ where $g_2 : i_2 \, \epsilon \, \nu_{w_2^{*\perp}} \to i_2 \, \epsilon \, \nu_{w_2^\perp}$ is the mediating morphism coming from the $\llbracket w_2^\perp \rrbracket$ coalgebra structure of $i_2 \, \epsilon \, \nu_{w_2^{*\perp}}$. This was formally proved in Agda. (The reason is essentially that both morphisms are, up-to bisimilarity, the identity.) $\qquad\square$

Note that this proof is hybrid with a formal part proved in Agda, and a pen and paper proof.

5. Layering and Infinite Trees

General simulations allow to represent all computable (continuous) functions on streams: for a given piece of the output, we only need to look at finitely many elements of the stream. A general simulation does that by asking as many elements as it needs. However, for branching structures, general simulations are not enough: general simulations explore their arguments along a single branch $a_1/d_1, a_2/d_2, \ldots$. For example, the function summing each layer of a binary tree to form a stream is not representable by a general simulation:
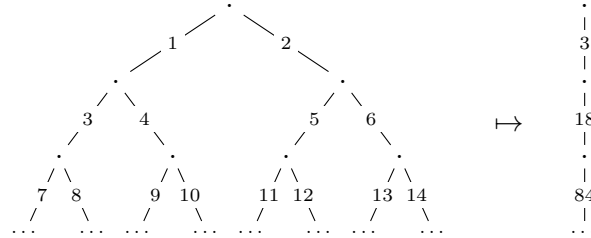
We want a notion of "backtracking" transducer that can explore several branches. Describing such a transducer is difficult in type theory if we try to refrain from using equality.

5.1. **Layering.** To give a simulation access to several branches, we are going to replace a branching structure by the stream of its "layers". Then, a simulation will be able to access as many layers as it needs to get information about as many branches as it needs.

Definition 5.1. Given an indexed container $w = \langle A, D, n \rangle$ over $I : \mathsf{Set}$ we define an indexed container $w^\sharp = (A^\sharp, D^\sharp, n^\sharp)$ on I by induction-recursion on $\mathsf{Fam}(I)$.

- given $i : I$, the index set $A^\sharp(i) : \mathsf{Set}$ is defined with

$$\frac{}{\mathtt{Leaf} : A^\sharp(i)} \qquad \frac{\alpha : A^\sharp(i) \qquad l : \left(\Pi \beta : D^\sharp(i, \alpha)\right) A(n^\sharp \, i \, \alpha \, \beta)}{(\alpha \triangleleft l) : A^\sharp(i)} \, ;$$

- for $\alpha : A^\sharp(i)$, the family $\left\langle D^\sharp(i, \alpha) : \mathsf{Set}, n^\sharp \, i \, \alpha : D^\sharp(i, \alpha) \to I \right\rangle$ is defined with
  - $D^\sharp(i, \star) = \mathbf{1}$ and $D^\sharp(i, \alpha \triangleleft l) = \left(\Sigma \beta : D^\sharp(i, \alpha)\right) D\left(n^\sharp \, i \, \alpha \, \beta, l \, \beta\right)$,
  - $n^\sharp \, i \, \star \, \star = i$ and $n^\sharp \, i \, (\alpha \triangleleft l) \, \langle \beta, d \rangle = n \, (n^\sharp \, i \, \alpha \, l) \, (l \, \beta) \, d$.

The Agda definition looks like

```
mutual
  data A# : I → Set where
    Leaf :  i : I  → A# i
    _◁_ : i : I → (t : A# i) → ((b : D# i t) → A (n# t b)) → A# i

  D# : (i : I) → A# i → Set
```

```
D# i Leaf = One
D# i (t ◁ l) = Σ (D# i t) (λ ds → D (n# t ds) (l ds))

n# : i : I → (t : A# i) → D# i t → I
n# i Leaf * = i
n# i (t ◁ l) ( ds , d ) = n n# i t ds (l ds) d
```

This definition generalizes the example of complete binary trees as an inductive recursive definition from page 7 to arbitrary dependent A-labelled and D-branching trees. Given such a tree $\alpha : A^\sharp(i)$, $D^\sharp(i, \alpha)$ is the set of its terminal leaves. A new layer assigns a new element $a : A(\dots)$ at each leaf, and $\alpha \triangleleft l$ is the new, increased tree.

Of particular interest is the interaction system $w^{\perp\sharp}$. An element of $A^{\perp\sharp}(i)$ is a complete tree of finite depth where branching occurs at $A(\_)$ and labels are chosen in $D(\_,\_)$. In particular, an element of $D^{\perp\sharp}(i, \alpha)$ is a finite sequence of actions.

We can now construct a "non-branching" structure from an arbitrary interaction system:

Definition 5.2. Given w an interaction system on I and a fixed $i : I$, we define a new interaction system $\langle\!\langle w, i \rangle\!\rangle$:
- states are $A^\sharp(i)$,
- actions in state $\alpha : A^\sharp(i)$ are given by $\big(\Pi\,\beta : D^\sharp(i, \alpha)\big)\ A(n^\sharp\,i\,\alpha\,\beta)$, i.e., "layers" on top of α,
- responses are trivial: there is only ever one possible response: $\star$,
- the next state after action l in state α (and response $\star$) is $\alpha \triangleleft l$.

States of this new interaction system record a complete tree of finite depth. Moreover, since it has trivial responses, an element of $\texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w,i \rangle\!\rangle}$ is not very different from a (dependent) stream of layers, and so, from an infinite tree in $i\ \epsilon\ \nu_w$. This is generalized and formalized in the next lemmas.

Lemma 5.3. *The predicate* $\big(\Pi\,\beta : D^\sharp(i, \alpha)\big)\ (n^\sharp\,i\,\alpha\,\beta)\ \epsilon\ \nu_w$,[12] *depending on $\alpha : A^\sharp(i)$, is a weakly terminal coalgebra for $[\![\langle\!\langle w, i \rangle\!\rangle]\!]$.*

Proof. $[Agda\checkmark]$ The proof corresponds to the above remark that an infinite tree is equivalently given by the infinite stream of its layers. It has been formalized in Agda. $\qquad\square$

Corollary 5.4. *Given $\alpha : A^\sharp(i)$, there are functions*

$$\Big(\Pi\,\beta : D^\sharp(i, \alpha)\Big)\ (n^\sharp\,i\,\alpha\,\beta)\ \epsilon\ \nu_w \quad\overset{\approx}{\longleftrightarrow}\quad \alpha\ \epsilon\ \nu_{\langle\!\langle w,i \rangle\!\rangle}$$

that are, up to bisimulation, inverse to each other. In particular, there are functions

$$f : i\ \epsilon\ \nu_w \to \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w,i \rangle\!\rangle} \quad and \quad g : \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w,i \rangle\!\rangle} \to i\ \epsilon\ \nu_w$$

such that $fg \approx \mathrm{id}$ and $gf \approx \mathrm{id}$.

Proof. This is a direct consequence of lemma 2.5. $\qquad\square$

Interaction system with trivial actions or reactions are special in the sense that duality is essentially idempotent, in the sense that w and $w^{\perp\perp}$ are isomorphic:
- there are bijections f_i between $A(i)$ and $A^{\perp\perp}(i)$,

---

[12]An element of $\big(\Pi\,\beta : D^\sharp(i, \alpha)\big)\ (n^\sharp\,i\,\alpha\,\beta)\ \epsilon\ \nu_w$ is a way to extend the complete tree α of finite depth to a full infinite complete tree by appending an infinite tree at each leaf.

- there is are bijections $g_{i,a}$ between $D(i,a)$ and $D^{\perp\perp}(i, f_i\, a)$,
- the next state functions are compatible with them: $n(i,a,d) \equiv n^{\perp\perp}(i, f_i\, a, g_{i,a}\, d)$.

Rather than developing this notion, we will only state and prove the only consequence we'll need.

Lemma 5.5. *Suppose w has trivial (singleton) reactions, then ν_w is a weakly terminal coalgebra for $[\![w^{\perp\perp}]\!]$.*

Proof. $[Agda\checkmark]$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 5.6. *There are function*

$$\varphi : \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w\rangle\!\rangle, i} \to \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w, i\rangle\!\rangle^{\perp\perp}} \quad and \quad \psi : \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w, i\rangle\!\rangle^{\perp\perp}} \to \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w\rangle\!\rangle, i}$$

such that $\varphi\psi \approx \mathrm{id}$ and $\psi\varphi \approx \mathrm{id}$.

5.2. **Continuous Functions.** We now have everything we need.

Definition 5.7. A layered simulation from w_1 to w_2 at states $i_1 : I_1$ and $i_2 : I_2$ is a simulation from $\langle\!\langle w_1^\perp, i_1\rangle\!\rangle^\perp$ to $\langle\!\langle w_2^\perp, i_2\rangle\!\rangle^\perp$. A general layered simulation is a simulation from $\langle\!\langle w_1^\perp, i_1\rangle\!\rangle^{\perp*}$ to $\langle\!\langle w_2^\perp, i_2\rangle\!\rangle^\perp$.

Lemma 5.8. *For every general layered simulation $R : \langle\!\langle w_1^\perp, i_1\rangle\!\rangle^{\perp*} \multimap \langle\!\langle w_2^\perp, i_2\rangle\!\rangle^\perp$ there is an evaluation function*

$$\mathrm{eval}_R : R(\texttt{Leaf}, \texttt{Leaf}) \to i_1\ \epsilon\ \nu_{w_1^\perp} \to i_2\ \epsilon\ \nu_{w_2^\perp}$$

Proof. We have

$$
\begin{array}{lll}
i_1\ \epsilon\ \nu_{w_1^\perp} & \xrightarrow{\quad f_1\quad} & \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w_1^\perp, i_1\rangle\!\rangle} \quad\text{(corollary 5.4)} \\[4pt]
& \xrightarrow{\quad\varphi\quad} & \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w_2^\perp, i_2\rangle\!\rangle^{\perp\perp}} \quad\text{(corollary 5.6)} \\[4pt]
& \xrightarrow{\mathrm{eval}_R^*\,\texttt{Leaf}\,\texttt{Leaf}\,r} & \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w_2^\perp, i_2\rangle\!\rangle^{\perp\perp}} \quad\text{(corollary 4.9)} \\[4pt]
& \xrightarrow{\quad\psi\quad} & \texttt{Leaf}\ \epsilon\ \nu_{\langle\!\langle w_2^\perp, i_2\rangle\!\rangle} \quad\text{(corollary 5.6)} \\[4pt]
& \xrightarrow{\quad g_2\quad} & i_2\ \epsilon\ \nu_{w_2^\perp} \quad\text{(corollary 5.4)}
\end{array}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 5.9. *If composition of general layered simulations is defined as general composition of layered simulations, then evaluation of a composition is bisimilar to the composition of their evaluations.*

Proof. The composition of evaluations gives

$$i_1\ \epsilon\ \nu_{w_1^\perp} \xrightarrow{f_1} \_ \xrightarrow{\varphi_1} \_ \xrightarrow{\mathrm{eval}_R^*} \_ \xrightarrow{\psi_2} \_ \xrightarrow{g_2} \_ \xrightarrow{f_2} \_ \xrightarrow{\varphi_2} \_ \xrightarrow{\mathrm{eval}_S^*} \_ \xrightarrow{\psi_3} \_ \xrightarrow{g_3} i_3\ \epsilon\ \nu_{w_3^\perp}\ .$$

Since $f_2 g_2 \approx \mathrm{id}$ (corollary 5.4) and $\varphi_2\psi_2 \approx \mathrm{id}$ (corollary 5.6), this whole composition is bisimilar to

$$i_1\ \epsilon\ \nu_{w_1^\perp} \xrightarrow{f_1} \_ \xrightarrow{\varphi_1} \_ \xrightarrow{\mathrm{eval}_R^*} \_ \xrightarrow{\mathrm{eval}_S^*} \_ \xrightarrow{\psi_3} \_ \xrightarrow{g_3} i_3\ \epsilon\ \nu_{w_3^\perp}$$

and thus, by proposition 4.14, to

$$i_1\ \epsilon\ \nu_{w_1^\perp} \xrightarrow{f_1} \_ \xrightarrow{\varphi_1} \_ \xrightarrow{\mathrm{eval}_{S\bullet R}^*} \_ \xrightarrow{\psi_3} \_ \xrightarrow{g_3} i_3\ \epsilon\ \nu_{w_3^\perp}\ .$$

This corresponds to evaluation of $S\bullet R : \langle\!\langle w_1^\perp, i_1\rangle\!\rangle^{\perp*} \multimap \langle\!\langle w_2^\perp, i_2\rangle\!\rangle^\perp$ as defined in lemma 5.8. $\quad\square$

Concluding Remarks

Internal Simulations. It is possible to internalize the notion of linear simulation by defining an interaction system $w_1 \multimap w_2$ on $I_1 \times I_2$ satisfying "R is a linear simulation from w_1 to w_2 iff $R \subset [\![w_1 \multimap w_2]\!](R)$" [14, 15].

Definition 5.10. If w_1 and w_2 are interaction systems on I_1 and I_2, then the interaction system $w_1 \multimap w_2$ on $I_1 \times I_2$ is defined by

- $A\big(\langle i_1, i_2 \rangle\big) = \big(\Sigma\, f : A_2(i_2) \to A_1(i_1)\big)\ \big(\Pi\, a_2 : A_2(i_2)\big)\ D_1\big(i_1, f(a_2)\big) \to D_2(i_2, a_2),$

- $D\big(\langle i_1, i_2 \rangle, \langle f, \varphi \rangle\big) = \big(\Sigma\, a_2 : A_2(i_2)\big)\ D_1\big(i_1, f(a_2)\big),$

- $n\, \langle i_1, i_2 \rangle\, \langle f, \varphi \rangle\, \langle a_2, d_1 \rangle = \big\langle i_2[a_2/\varphi(a_2)(d_3)], i_3[f(a_2)/d_3] \big\rangle.$

The resulting structure is nicer in the opposite category because then, $w_1 \multimap w_2$ generalizes duality (definition 3.6) in the sense that $w^{\perp}$ is the same as $w \multimap \perp$, where $\perp$ is the trivial interaction system (on $I = \{\star\}$, with a single action and a single reaction) and is right adjoint to the pointwise cartesian product of interaction systems. Linear simulations thus give rise to a symmetric monoidal closed category.

That a simulation $R : w_1 \multimap w_2$ is nothing more than a coalgebra for $[\![w_1 \multimap w_2]\!]$ means that, up to bisimilarity, it can be seen as elements of $\nu_{w_1 \multimap w_2}$. However, even if w_1 and w_2 are finitary, w_1^* or $\langle\!\langle w_1^{\perp} \rangle\!\rangle^*$ aren't. We cannot really represent higher order continuous functions in this way.

Thoughts about Completeness. Stating and proving formally completeness of this representation is yet to be done. It is, in a way, intuitively obvious but with several subtleties.

- Semantically, when interpreting all the constructions in the category of sets and functions,[13] every function that is continuous for the "wild" topology from section 1.3 is representable as a general layered simulation. This would be an analogous to theorem 4.1 and the proof would go as follows
  - we generalize theorem 4.1 to dependent streams (i.e. consider interaction systems with trivial actions) and show that in this case, any continuous function from $\nu_{w_1^{\perp}}(i_1)$ to $\nu_{w_2^{\perp}}(i_2)$ is represented by an element of $\nu_{w_1^* \multimap w_2}(i_1, i_2)$.
  - we use corollary 5.6 showing that any $\nu_{w^{\perp}}(i)$ (without restriction) is isomorphic to some $\nu_{w'^{\perp}}(j)$ where w' has trivial actions.
- In particular, every function continuous for the natural topology (section 1.2) between greatest fixed points of finitary interaction systems[14] is thus representable as a general layered simulation.
- If only the codomain is finitary, all continuous functions between fixed points of non-finitary interaction are representable by simulations of the form $\langle\!\langle w_1^{\perp}, i_1 \rangle\!\rangle^{\perp *} \multimap w_2$. This is for example the case of the naturally continuous but non-wildly continuous function from page 4. However, we don't know how to compose such simulations.

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[13]Proving such theorems in Agda isn't possible, as their can be non-computable continuous functions. We would need at least to add hypothesis about computability of modulus of continuity and other similar hypothesis.

[14]An interaction systems is finitary if its sets of actions are finite.

Notes about Formalization. Some proofs haven't been formalized in Agda, most notably:

- the proof of lemma 3.11 or of assumption 2.6,
- the proof of corollary 4.12.

We think the second holds in intensional type theory with function extensionality but were unable to complete the proof. As it stands, we only know it holds semantically by categorical reasoning. (It thus holds in extensional type theory.)

The situation is subtle with lemma 3.11. Is is possible it can be bypassed entirely. A direct proof of corollary 5.4 was in fact checked in Agda (with the `--with-K` flag), but its complexity convinced us to base similar proofs on lemma 3.11. If no other way is known, it might mean we need to extend our type theory to the whole of cubical type theory, where assumption 2.6 holds in a strong sense (bisimilarity is the same as path equality).

Considering the degree of obfuscation for many of the formal proofs, it is tempting to investigate extensions of type theory to simplify reasoning. Currently, the best candidate seems to be cubical type theory [7] where we can use a kind of "heterogeneous equality" and have equalities "over" equalities [18]. A relation

$$R : \left(\Sigma\, i_1 : I \right)\ \left(\Sigma\, i_2 : I \right)\ \left(\Sigma\, \rho_i : i_1 \equiv i_2 \right) (i_1 \,\epsilon\, \nu_{w_1}) \times (i_2 \,\epsilon\, \nu_{w_2})$$

would then be a bisimulation if

$$R\big(\langle i_1, i_2, \rho_i, T_1, T_2 \rangle\big) \qquad \rightarrow \qquad \begin{aligned} & \left(\Sigma\, \rho_a : a_1 \equiv_{\rho_i} a_2 \right) \\ & \left(\Pi\, d_1 : D(i, a_1) \right)\ \left(\Pi\, d_2 : D(i, a_2) \right)\ \left(\Pi\, \rho_d : d_1 \equiv_{\rho_a} d_2 \right) \\ & R(\langle i_1[a_1/d_1], i_2[a_2/d_2], \cdots, T_1', T_2' \rangle) \end{aligned}$$

where a_1 and T_1' come from unfolding T_1 (and similarly for a_2, T_2'), $\equiv_\rho$ denotes the equality type *over equality ρ*, and the "$\cdots$" denotes the equality between $i_1[a_1/d_1]$ and $i_2[a_2/d_2]$ coming from ρ_i, ρ_a and ρ_d.

This is left for future work.

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