International Game Theory Review, Vol. 12, No. 2 (2010) 115–137

DOI: 10.1142/S0219198910002556

VON NEUMANN, VILLE, AND THE MINIMAX THEOREM

HICHEM BEN-EL-MECHAIEKH

Department of Mathematics, Brock University
St. Catharines, Ontario, Canada, L2S 3A1
hmechaie@brocku.ca

ROBERT W. DIMAND

Department of Economics, Brock University
St. Catharines, Ontario, Canada, L2S 3A1
dimand@brocku.ca

Von Neumann proved the minimax theorem (existence of a saddle-point solution to 2 person, zero sum games) in 1928. While his second article on the minimax theorem, stating the proof, has long been translated from German, his first announcement of his result (communicated in French to the Academy of Sciences in Paris by Borel, who had posed the problem settled by Von Neumann's proof) is translated here for the first time. The proof presented by Von Neumann and Morgenstern (1944) is not Von Neumann's rather involved proof of 1928, but is based on what they called "The Theorem of the Alternative for Matrices" which is in essence a reformulation of an elegant and elementary result by Borel's student Jean Ville in 1938. Ville's argument was the first to bring to light the simplifying role of convexity and to highlight the connection between the existence of minimax and the solvability of systems of linear inequalities. It by-passes nontrivial topological fixed point arguments and allows the treatment of minimax by simpler geometric methods. This approach has inspired a number of seminal contributions in convex analysis including fixed point and coincidence theory for set-valued mappings. Ville's contributions are discussed briefly and von Neuman's original communication, Ville's note, and Borel's commentary on it are translated here for the first time.

Keywords: Minimax theorem; origins of game theory; John von Neumann; Jean Ville.

JEL Classifications: C7 Game Theory, B31 History of Economic Thought: Individuals

1. Introduction

The minimax theorem, proving that a zero-sum two person game must have a solution, was the starting point of the theory of strategic games as a distinct discipline. It is well known that John von Neumann (1928b) provided the first proof of the minimax theorem and that subsequently, once the economist Oskar Morgenstern persuaded him to return his attention to the subject, *The Theory of Games and Economic Behavior* (Von Neumann and Morgenstern, 1944) became the foundation work of modern game theory. However, as Von Neumann and Morgenstern clearly stated, the proof used in their 1944 book was not the original proof of

Von Neumann (1928b) nor his second proof based on the Brouwer fixed point theorem (1937). It uses a more elementary idea on the solvability of systems of linear inequalities based on the separation of convex sets originally due to Jean Ville (1938). Ville's proof of the minimax theorem was published in 1938 in an appendix to a book of notes edited by Ville himself on Emile Borel's Sorbonne lectures on the application of probability theory to games of chance and psychological games (Borel's term for games involving both chance and choice of strategy). Subsequent elementary proofs of the minimax theorem, further simplified and generalized, followed Ville's proof by way of Von Neumann and Morgenstern (see e.g. Owen (1982), 18–19, Binmore (2004), Afriat (1987), 528–541). Despite the explicit acknowledgement by Von Neumann and Morgenstern (1944), Ville (1938) has received little attention and is seldom cited in the literature. P. K. Newman, in a translator's note at the end of his translation of Ville (1946) on the Strong Axiom of Revealed Preference in the *Review of Economic Studies* in 1951–52 stated that "This is M. Ville's only paper on economics." As rightfully pointed out in Kjeldsen (2001), the connection between the minimax theorem and the solvability of systems of inequalities and concepts of convexity is far from being obvious in von Neumann's 1928 papers. It became gradually apparent to him over a period of time as he claimed in 1953. We argue here that in fact, Ville's key preparatory result for the proof of the minimax theorem is nothing else than an alternative for linear systems of inequalities, later reformulated by von Neumann and Morgenstern as an alternative for matrices. The role of convexity first acknowledged in Ville's proof prompted and stimulated a number of seminal papers and advances in convex functional analysis. For instance, a reformulation as a nonlinear alternative for quasiconcave/convex function of Ville's fundamental result on the solvability of linear systems of inequalities is the essence of Sion's generalization of von Neumann's theorem to such functions (1958) and the casting of minimax theorems as immediate — and perhaps not so important — corollaries of elegant and far reaching geometric results on fixed points, coincidences, and intersection of convex relations (e.g. Fan (1952, 1953), Browder (1968)).

Von Neumann's first publication announcing his result about saddle-point solutions to two-person, zero-sum strategic games (1928a), presented on his behalf to the Academy of Sciences by Emile Borel, as well as Ville's proof of the minimax theorem are translated here for the first time together with Borel's observations on Ville's note.

The reader is referred to Kjeldsen (2001) for a thorough and thoughtful account on the historical developments of the minimax theorem from different mathematical perspectives.

2. Von Neumann Announces the Answer to Borel's Query

Von Neumann (1928a) was presented to the Academy of Sciences in Paris by Borel at the session of May 14 (but published with the proceedings of the session of

June 18). It began by quoting Borel (1924, 1927) on the setup of a two-player game defined by a payoff matrix that states, for each combination of an action chosen by player I and an action chosen by player II, the sum (positive or negative or zero) to be paid by player II to player I, with I choosing among M possible actions (pure strategies) and II among N possible actions. Borel (1924, 101) took as his starting point the discussion by the probability theorist Joseph Bertrand (best known to economists for his hostile review of Walras and Cournot) of whether a player in baccarat should draw another card when holding a count of five. A player who always draws a card on a count of five, and a player who never draws another card on a count of five, could both be systematically beaten. Accordingly, instead of such a pure strategy, Borel proposed to solve for the optimal probability with which to draw another card in such a situation. Von Neumann identified as Borel's principal, but still open, question, how should a player play in order to insure the largest possible gain? How should player II act to minimize the amount to pay to player I, knowing that player I will act to maximize that payoff? player II will act to minimize the maximum payoff, while player I will act to maximize the minimum payoff. If the payoff that player II can guarantee is the lowest that she or he will have to make (regardless of the actions of player I) is the same as the payoff that Player I can guarantee receiving (regardless of the actions of player II), that payoff is the value of the game and the strategy combination corresponding to that payoff is the solution to the game. Borel demonstrated constructively that such a solution exists for M and N not exceeding five by solving for the minimax mixed strategies (see Dimand and Dimand (1996)). Stating that he had begun work on the problem independently before seeing Borel's writings (but certainly not before Borel began publishing on the question in 1921, when Von Neumann was a teenager), Von Neumann (1928a) announced that, in a forthcoming article in German, he had proved that such a solution would exist for any two-person, zero-sum game, without restriction of the number of pure strategies, but he did not sketch his proof in the short note communicated by Borel to the Academy. Von Neumann (1928b) then provided a proof, drawing on topology and functional calculus in ways far from reader-friendly, showing that either a minimax solution existed in general or there was a contradiction. Until he began collaborating with the economist Oskar Morgenstern in Princeton in the autumn of 1940, Von Neumann only returned to this topic on a couple of occasions: Von Neumann (1937) used a similar technique to prove existence of equilibrium in an expanding economy, and in 1937 he discussed such two person, zero sum games as rock, scissors, paper in a talk to Princeton undergraduates (Anonymous, 1937).

As Dell'Aglio (1995) emphasizes, Von Neumann (1928a, 1928b) approached the problem posed by Borel from a perspective that differed from that of Borel in one crucial aspect: Borel was concerned with psychological differences in the skillfulness of players, while Von Neumann was concerned only with the optimal strategy to be followed by any player who played correctly. Borel, commenting in on Ville (1938), recognized the importance of Von Neumann's result but predicted that, given the complexities of the psychology of players, practical application was still far in the future. Borel's next step after his 1938 volume of lectures on psychological games was an exhaustive, 424 page application of probability theory to the analysis of a single game of chance and skill, the game of bridge (Borel and Chéron, 1940), rather than an attempt at further theorizing about strategic games in general.

3. Ville's Influence on Von Neumann and Morgenstern

Von Neumann and Morgenstern (1944, 154 n1, 198 n4) cited Ville (1938) in two footnotes (a third reference, 186 n2, concerning Borel's "very instructive" considerations on poker, merely identifies Ville as a contributor to Borel, 1938). The first of these footnotes noted that the early proofs of the minimax theorem in Von Neumann (1928b) "made a rather involved use of some topology and of functional calculus" and that Von Neumann (1937), on the related problem of the existence of a solution to a system of equations of production, "contained a different proof, which was fully topological" and connected the minimax theorem to Brouwer's fixed point theorem. The sixth and seventh short paragraphs of the footnote acknowledged that "All these proofs are definitely non-elementary. The first elementary one was given by J. Ville in the collection by E. Borel and collaborators," Borel (1938). "The proof which we are going to give carries the elementarization initiated by J. Ville further, and seems to be particularly simple" (Von Neumann and Morgenstern, 1944, 154 n1). The third paragraph of other footnote (198 n4) credited Ville (1938) with an "interesting step in the direction" of extending the strategy set to choice over continuous parameters, although "The continuity assumptions made there seem, however, to be too restrictive for many applications, — in particular for the present one." Borel (1924, 1927), although quoted in Von Neumann (1928a), were not cited by Von Neumann and Morgenstern (1944).

To these terse but clear acknowledgements must be added a reminiscence by Morgenstern (1976, 811) of how, at a stage in his collaboration with Von Neumann when they were about to work on a new proof of the minimax theorem, he went for a walk "on a brilliant, snowy cold winter day" in Princeton and, going into the library of the Institute for Advanced Study to warm up, idly happened to pick up Borel (1938) and notice Ville's paper: "There, in restating Johnny's minimax theory, instead of using Brouwer's fixed point theory, he gave a more elementary proof (Johnny's two earlier proofs were definitely not elementary). I had not known of Ville's work; so I phoned Johnny to whom this was also news. We met immediately and quickly saw that the best approach was to proceed by considerations of convexity." It is in no way remarkable that John von Neumann, active in fields from quantum mechanics to computing and serving as a consultant to the Manhattan Project, had not fully kept up with Borel's work after 1928, so that Ville's advance was stumbled upon by the economist Morgenstern, who made it known to his mathematician collaborator. However, Leonard, 1992, 45, 58, emphasizes with italics that "all this went by, it appears, unbeknownst to von Neumann" and that "in

December 1941, Morgenstern accidentally discovered Borel's volume" three years after its publication. The crucial point of Morgenstern's reminiscence is that is was Ville (1938) that led Von Neumann and Morgenstern (1944) to turn to convexity for a non-topological proof of the minimax theorem (see also Leonard, 1995; Innocenti, 1995; Schmidt, 2001; and Giocoli, 2003 on influences on the development of Von Neumann's game theory).

In an overview of his scientific work, Ville (1955, 6–7) quoted (approvingly, apart from the evaluation of Borel's role) from Von Neumann (1953, 125) on the allocation of credit for developments in proving the minimax theorem: "The theorem, and its relation to the theory of convex sets were far from being obvious, witness the following facts:

(a) In 1921, and thereafter, Borel surmised the theorem to be false or possibly false.
(b) In 1928, I proved the theorem by observing its relation to the theory of fixed points and not yet to that one of convex sets.
(c) In 1935, I generalized it (for the purposes of the theory of prices and production) by an even more explicit use of the fixed-point method.
(d) It took ten years after my original proof, until J. Ville discovered, in 1938, the connection with convex sets.
(e) Even now, this connection does not tell the entire, or the simplest, story about the theorem, as the work since 1945 of S. Kakutani, J. Nash, G. Brown, and myself shows."

Ville (1955, 6) demurred from this account only by stressing that Borel, although he had not solved the question of the existence in general of a minimax solution to a two person, zero sum game, had posed the question, defined pure and mixed (randomized) strategies, and found minimax solutions to particular games. It is striking that in demurring from point (a) of Von Neumann (1953), Ville was in agreement with Von Neumann (1928a, translated in the first appendix to this paper), where Von Neumann, rather than attributing to Borel a belief that the minimax theorem was false, stated instead that Borel had formulated the hypothesis that Min Max = Max Min (that is, a saddle-point exists) for all payoff matrices, provided that the game is fair, "but he only proved to for some special cases" where each player had no more than five possible pure strategies.

4. Ville Before and After 1938

Jean-André Ville (1938), translated in the second appendix to this paper, provided a much simpler proof of the minimax theorem ("the theorem of M. von Neumann"), relying only on convex sets and H. Minkowski's concept of a separating hyperplane, and extended Von Neumann's theorem to continuous variables. Ville (1955, 6) emphasizes that it was Borel who brought to his attention the importance of Von Neumann (1928b). Drawing attention to the importance of Von Neumann's and Ville's contributions, Borel (1938, 115–17) commented on Ville's note: "Je tiens à remercier M. Jean Ville d'avoir bien voulu exposer pour les lecteurs de ce Livre, le théorème important de M. von Neumann; il a su en simplifier la démonstration et l'étendre au cas des variables continues" (Borel, 1938, 115). Leonard (1992, 46) protests that "no such credit [as in De Possel 1936] is afforded von Neumann in Borel's work of the same period ... Remarkably, however, no mention is made of what became, for de Possel at least, the 'théorème fundamental,' that of von Neumann (1928b)" (Leonard's italics), although Leonard (1992, 48) nonetheless proceeds to quote Borel (1938, 115) on Von Neumann's "important theorem."

Born in 1910, Jean Ville was, at the time of his note on Von Neumann's minimax theorem and his edition of Borel's lectures, a researcher at the CNRS, France's National Council of Scientific Research. After graduating from the Ecole Normale Supérieure, and a year of compulsory military service, Ville had spent 1933–34 at the French Institute of Berlin and 1934–35 at the University of Vienna. His first two publications (Ville, 1935a, 1935b) appeared in the proceedings of Karl Menger's celebrated Vienna mathematical colloquium, a colloquium that also published the remarkable papers on the existence of general economic equilibrium by Von Neumann (1937) and Abraham Wald (translated in Baumol and Goldfeld, 1968). It is not known whether Ville encountered Von Neumann or Von Neumann's writings in Berlin or Vienna, but Ville (1955, 2–3) makes clear the importance for him of discussion with Wald in Menger's colloquium about Richard von Mises's concept of collectives, a discussion that led Ville, upon his return to France in 1935, to propose to Maurice Fréchet martingales as a dissertation topic. Ville's doctoral dissertation (Ville, 1939) was published in a series of monographs edited by Borel (see Shafer and Vovk, 2001, Chapters 2 and 8, on Ville's work on martingales). Ville (1955, 5) proudly quoted J. L. Doob: "Although many authors had derived many martingale properties, in various forms, Ville was the first to study them systematically, and to show their wide range of applicability." Ville's interaction with Wald in Menger's colloquium and his response (and that of De Possel) to Von Neumann's minimax theorem, like Von Neumann's response to Borel's work on psychological games (communicated to the French Academy of Sciences by Borel on Von Neumann's behalf), was part of a free interchange of mathematical ideas in Continental Europe soon to be disrupted by Anschluss and war.

Ville (1955, 7) added he had returned to game theory only once after 1938 and then only incidentally, in a series of lectures at the Institut Henri Poincaré on recent developments in probability theory (Ville, 1954), in which he illustrated that while minimax was the dominant strategy to play in a strictly competitive game against another perfectly optimizing play, it need not be the best response to an imperfect player. After the Second World War, Ville (who lived until 1988) was primarily concerned with information theory and electrical problems of information transmission, publishing most often in the journal *Cables et Transmissions*, although he remained a member of the Econometric Society and of the French Society of Operational Research. While he sometimes gave series of academic lectures, his primary employment was non-academic: the title page of a set of lectures on mathematical demography Ville gave at INSEE (Ville, 1948) identifies him as engineer-in-chief of the Société Alsacienne de Constructions Mécaniques, and his curriculum vitae (circa 1956) states that he was then "technical Secretary-General" of the same company. Notwithstanding the importance of Ville (1938) in appealing to convexity for an elementary proof of the minimax theorem and of Ville (1939) in generalizing the concept of a martingale, despite the Prix Bordin des Sciences Mathématiques in 1940 and the Prix Monthyon de Statistique in 1945, Jean Ville did not pursue an academic career or take a prominent role in the postwar development of game theory.

According to Ville (1955, 11), his one postwar contribution to mathematical economics, Ville (1946) on the conditions for the existence of a utility function allowing evaluation of the change in an individual's well-being when prices and income change, was the result of his teaching assignments at the Faculty of Sciences in Lyons from 1943 to 1946, when wartime staff shortages obliged him to take on the course in financial mathematics (including index numbers) in addition to his main teaching in analytical and celestial mechanics. Ville (1946) was reviewed in *Mathematical Reviews* by Kenneth Arrow (1947) and cited both in Arrow's subsequent work and, from Arrow's review, by Paul Samuelson (1950, 358). These notices led to Newman's translation of Ville (1946) in the *Review of Economic Studies,* so that paper entered the international canon of literature on utility theory.

5. Convexity, Alternatives for Systems of Inequalities and Coincidence for Set-Valued Maps

The key result of Ville's paper, from which von Neumann's minimax theorem readily derives, is the following (Theorem in Appendix 2 below):

Theorem. *Given p linear forms of n variables*

$$f_j(x) = \sum a_j^i x_i \quad (j = 1, 2, \ldots, p; i = 1, 2, \ldots, n),$$

with the following properties:

For any non negative values assumed by the variables x_i, there exists at least one of the forms f_j that takes a non negative value at those values of the variables x_i.

Under these conditions:

There exists at least one system of non negative coefficients$X_1, X_2, \ldots, X_p$ with sum equals to 1 such that the form$\sum_{j=1}^{p} X_j f_j$ takes non negative values for every system of non negative values taken by the variables x_i.

In contemporary terms, this theorem can be rephrased as an alternative for linear systems of inequalities as follows (possibility (A) being the negation of Ville's Theorem hypotheses):

Ville's Theorem as a Linear Alternative. *Let $\mathbb{R}_+^n$ be the non-negative cone in the Euclidean space $\mathbb{R}^n$, let Δ be the standard simplex in $\mathbb{R}^p$, and let $\{f_1, \ldots, f_p\}$ be any given collection of linear forms on $\mathbb{R}^n$ Define the function φ:$\mathbb{R}_+^n \times \Delta \to \mathbb{R}$* as the convex combination $\varphi(x, X) := \sum_{j=1}^{p} X_j f_j(x), x \in \mathbb{R}_+^n, X = (X_1, \ldots, X_p) \in \Delta$.

Then the following alternative holds:

(A) *there exists* $\bar{x} \in \mathbb{R}^n_+$ *such that* $\varphi(\bar{x}, X) < 0$, *for every* $X \in \Delta$; *or*
(B) *there exists* $\bar{X} \in \Delta$ *such that* $\varphi(x, \bar{X}) \geq 0$ *for every* $x \in \mathbb{R}^n_+$.

This is in essence Von Neumann and Morgenstern *Theorem of the Alternative of Matrices* ((1944, Sec. 16.4). Their proof of the alternative is also based on the crucial idea of separating convex subsets of the Euclidean space by hyperplanes. This separation property first used by Ville for the proof of his main theorem was the key ingredient for numerous extensions of the minimax theorem after the end of World War II (e.g. extension to concave-convex function by Shiffman (1949) and H. Weyl (1950), extension to concave-convex semi-continuous functions by H. Kneser (1952), Ky Fan (1953), and C. Berge (1954)). Other extensions followed the fixed point approach of von Neumann's second proof (as in e.g. H. Nikaido (1954) with an extension to quasiconcave/convex continuous functions). A noteworthy generalization of von Neumann's minimax theorem to quasiconcave/convex and upper/lower semicontinuous is due to M. Sion (1958) where, explicitly and for the first time, the celebrated Knaster–Kuratowski–Marzukiewicz (KKM) combinatorial theorem is used. It is worth mentioning that in fact, all three fundamental results, namely the Theorem on the Separation of Convex sets, the Brouwer Fixed point Theorem, and the KKM Theorem are indeed equivalent (see e.g. Ben-El-Mechaiekh and Dimand (2007) for a proof of equivalences between some of the fundamental results connected the minimax theorem). All three have modern proofs deemed "elementary." That was certainly not the case during the first half of the twentieth century. Ville's approach was a significant improvement on von Neumann's more elaborate earlier arguments.

Kjeldsen (2001) and Guerraggio and Molho (2004) rightfully point out that the concept of quasiconcavity is in fact considered by von Neumann (1928b) in the form of his condition (K) amounting to the characterization: f is quasiconvex on a convex set X, if and only if $f(\mu x_1 + (1-\mu)x_2) \leq \max\{f(x_1), f(x_2)\}$ for all $x_1, x_2 \in X$ and all $\mu \in [0,1]$. A careful reading of von Neumann's original paper in light of Ville's approach and its generalization by von Neumann and Morgenstern, allows the reformulation of the linear alternative above as a nonlinear alternative for quasiconcave/convex and semicontinuous functions, the key step in Sion's version of the minimax theorem.

A Nonlinear Alternative for Quasiconvex/concave Functions (Ben-El-Mechaiekh *et al.*, 1982a). *Let* X *and* Y *be two convex subsets of topological vector spaces and let* $f, g : X \times Y \to \mathbb{R}$ *be two functions satisfying*:

(i) $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in X \times Y$;
(ii) $f(x, y)$ *is lower semicontinuous in* y, *for each fixed* $x \in X$ and *is quasiconcave in* x, *for each fixed* $y \in Y$;
(iii) $g(x, y)$ *is quasiconvex in* y, *for each fixed* $x \in X$ and *is upper semicontinuous in* x, *for each fixed* $y \in Y$.

If Y is compact, then for any $\lambda \in \mathbb{R}$, the following alternative holds:

(A) *there exists $\bar{x} \in X$ such that $g(\bar{x}, y) \geq \lambda$, for all $y \in Y$; or*
(B) *there exists $\bar{y} \in Y$ such that $f(x, \bar{y}) \leq \lambda$, for all $x \in X$.*

To obtain the minimax theorem, simply put $f = g$ in the alternative above and, as usually done, observe that the inequality $\alpha = \sup_X \inf_Y g(x, y) \leq \beta = \inf_Y \sup_X f(x, y)$ always holds. To prove equality, assume that $\alpha < \beta < \infty$ and let λ be an arbitrary but fixed real number strictly between α and β. By the nonlinear alternative, either there exists $\bar{y} \in Y$ such that $\tilde{f}(x, \bar{y}) \leq \lambda$, for all $x \in X$ thus $\beta \leq \lambda < \beta$ which is impossible, or there exists $\bar{x} \in X$ such that $g(\bar{x}, y) \geq \lambda$, for all $y \in Y$ thus $\alpha \geq \lambda > \alpha$, also impossible. Hence $\alpha = \beta$.

The considerations about the central role of convexity in game theory and the various mathematical concepts encountered during the investigations on the minimax theorem stimulated the developments of areas of mutlivalued analysis ((KKM theory, fixed points and coincidence for set-valued mappings, etc...) aimed at the solvability of nonlinear problems in a broad sense (variational inequalities, complementarity theory, generalized games, etc...; see e.g. Ky Fan (1984) for a comprehensive list of references). By way of illustration, the nonlinear alternative above is the analytic expression of a more elegant coincidence theorem for set-valued maps. Indeed, consider the set-valued mappings $\Phi, \Psi : X \to 2^Y$ given by, for every $x \in X$:

$$\Phi(x) := \{y \in Y; f(x, y) > \lambda\} \quad \text{and} \quad \Psi(x) := \{y \in Y; g(x, y) < \lambda\}.$$

A coincidence between Φ and Ψ, that is a pair $(x_0, y_0) \in X \times Y$ with $y_0 \in \Phi(x_0) \cap \Psi(x_0)$ is ruled out by hypothesis (i) of the nonlinear alternative. The thesis of the alternative reads: either (A): at least one value $\Psi(\bar{x})$ of Ψ is empty, or (B): at least one preimage $\Phi^{-1}(\bar{y})$ of Φ is empty. Also, hypotheses (ii)–(iii) of the nonlinear alternative in terms of Φ and Ψ read:

$$\begin{cases} \Phi \text{ has open images and convex preimages and} \\ \Psi \text{ has convex images and open preimages.} \end{cases}$$

A set-valued map is a *Ky Fan mapping* if it has non-empty convex values and open preimages (Ben-El-Mechaiekh *et al.* (1982b)). The nonlinear alternative becomes the analytical formulation of the following coincidence principle for inverse Ky Fan mappings:

Let X and Y be two convex subsets of topological vector spaces with Y compact. Two set-valued mappings $\Phi, \Psi : X \to 2^Y$ have a coincidence provided Ψ and Φ^{-1} are Ky Fan mappings.

This coincidence principle is equivalent to the Ky Fan-Browder Fixed Point Theorem (see Ben-El-Mechaiekh and Dimand (2007) for the equivalence):

Every Ky Fan mapping Φ of a convex compact subset X of a topological vector space has a fixed point $x_0 \in \Phi(x_0)$.

This elegant result is due independently to Ky Fan (1952) and Browder (1968). It embodies von Neumann's early appreciation of the importance of convexity and the power of fixed point arguments.

Recent advances in this area of mathematics are increasingly focusing on topological concepts of convexity and indicate that topology (e.g. connectedness, homology, etc...) and not linear convexity are truly the underlying mathematical reasons for the existence of solutions to nonlinear problems including minimax and equilibria for generalized games. The meaning of those abstract notions of "convexity" in problems from Economics is yet to be understood and the impact of the theoretical results still unclear.

6. Conclusion

The minimax theorem had a crucial role in the early history of game theory, because it provided a guarantee that, at least for one class of games of a kind that readers could grasp (two player, strictly competitive games such as rock, scissors, paper, matching pennies, or chess), a solution must exist. John von Neumann first provided the answer to the question posed by Emile Borel of whether this was true in general, or just for the specific cases (no more than five pure strategies) that Borel had investigated. von Neumann's first announcement of his proof, communicated in French by Borel to the Academy of Sciences in Paris, is translated in an appendix to this paper. Jean Ville's 1938 note, also translated below, greatly simplified the proof by drawing on considerations of convexity, pointing the way for further simplifications. Morgenstern's discovery of Ville (1938) in December 1941 gave a crucial impetus to his collaboration with von Neumann on what became *Theory of Games and Economic Behavior,* and opened their eyes to the relevance of convexity for the existence of solutions to strictly competitive games. Without Ville, Borel's student and associate, von Neumann and Morgenstern (1944) would have probably looked quite different, and the book's most striking definite result, that a minimax solution must exist for any two person, zero sum game, would have been presented with a proof inaccessible to many potential readers. The use of convexity in the minimax theorem encouraged the development of fixed point theory for set-valued maps and that of the KKM theory by allowing a considerable simplification of the proofs.

Acknowledgements

We thank Pierre Garrouste for biographical information about Jean Ville, and Bruna Ingrao for helpful comments. We also thanks two anonymous referees for comments that helped improve the manuscript and the translations as well as for drawing their attention to Kjeldsen (2001). We are very grateful to the Academy of Sciences of the Institute of France for permission to publish, as Appendix 1 of this paper, a translation of John von Neumann, "Sur la théorie des jeux," which appeared (communicated by Emile Borel) in the *Comptes Rendus* of the Academy, 186 (25), [1928], and to Dunod Editeur (as successor firm to Gauthier-Villars) for

permission to publish, as Appendices 2 and 3 of this paper, translations of Jean Ville's note on the minimax theorem and Emile Borel's comments on that note, both of which appeared in E. Borel *et al.*, *Traité du calcul des probabilités et de ses applications,* Tome IV, Fascicule 2 (Paris: Gauthier-Villars, 1938).

Appendix A

SESSION OF JUNE 18, 1928
COMPUTATION OF PROBABILITIES - *On the theory of games.*
Note[1] de **M. J. v. Neumann,** presented by M. Émile Borel.

I. In his book *Élements du calcul des probabilités* (Hermann, Paris, 1924, 3rd edition), as well as in numerous Notes in the *Comptes rendus,* (184, January 10, 1927, pp. 52–55: *Sur les systèmes linéaires à déterminant symétrique gauche et la théorie des jeux*), M. E. Borel treats the following problem: "Two players, I and II, play a game consisting of the choice of a number x made by I and the choice of a number y made by II; the choice of x being made amongst the numbers $1, 2, \ldots, M$ and that of y amongst the numbers $1, 2, \ldots, N$. Each one makes his choice without knowing the choice of the other, and, if I and II have chosen x, y respectively, II must pay I the sum a_{xy} ($a_{xy} \gtreqless 0$); the determinant

$$\{a_{xy}\} \quad (x = 1, 2, \ldots, M; y = 1, 2, \ldots, N)$$

is hence characteristic of the game, it is the rule of the game. What is, for I and II, the best way to play the game?" In particular, M. Borel admits that the game is fair, that is I and II have the same role; which obviously means $M = N$, $a_{xy} = -a_{yx}$ (that is the determinant $\{a_{xy}\}$ is skew-symmetric).

Having been independently working on the same (slightly more general) problem and having obtained, among other things, a result that provides an affirmative answer to the principal (and unsolved) question asked by M. Borel, I allow myself to revisit this problem. The detailed account of my results will soon appear in another publication: *Zur Theorie der Gesellschaftsspiele.*

II. Note first that the game defined earlier in terms of a determinant $\{a_{xy}\}$ might appear very special, but that by introducing the notion of the "player's tactic" (see *loc. cit.*, in M. Borel or the author) we can reduce any two-player game to this form.

A more essential remark is as follows: The question: "How should I (resp. II) play in order to insure a largest possible gain, that is to make a_{xy} as large (resp. small) as possible?" does not have a well-defined sense. Indeed, how could I (resp. II) make a_{xy} large (resp. small) given that he does not have the privilege of fixing its value — he only determines the value of x (resp. y) the row (resp. column) index in $\{a_{xy}\}$! I and II try to influence a_{xy} by determining a row (resp. column) of $\{a_{xy}\}$, while one wishes to make a_{xy} large and the other small — what is going

[1](1) Session of May 14, 1928.

to happen? The real problem is to find here an exact and plausible definition (of the goal of I, resp. II) allowing a unique solution.

III. If I chooses x, he gains at least $\text{Min}_y a_{xy}$; and he would certainly not gain any more if, II knowing his strategy, chooses the most advantageous y against this x. By suitably choosing x, this amount can at most go up to $\text{Max}_x\text{Min}_y a_{xy}$. This is what I can win in any case (independent of what II does!) — and he would not gain any more if II knows his strategy, if II is the most insightful. In the same way we also see: II can prevent I (independent of what I does!) from winning more than $\text{Min}_y\text{Max}_x a_{xy}$ — but I can gain this amount if he knows what II's strategy is, if he is the most insightful.

One can easily see that

$$\text{Max}_x\text{Min}_y a_{xy} \leq \text{Min}_y\text{Max}_x a_{xy}.$$

If the sign $=$ occurs, everything is clear : the common value gives the amount that I must and can win; II cannot prevent him from obtaining it, but he can prevent him from exceeding it. For this reason, one can easily see how I and II must play. But in general, the sign $<$ occurs as the examples

$$\begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{Bmatrix}$$

(for $\{a_{xy}\}$) show (the latter being the game "morra").

Therefore, our reasoning is in general worthless, and we must find another one.

IV. One can try the following method (see *loc. cit.*, M. Borel or the author): I must not choose one of the numbers $1, 2, \ldots, M$; he is only obliged to fix M probabilities $\alpha_1, \alpha_2, \ldots, \alpha_M$ (all ≥ 0, $\alpha_1 + \alpha_2 + \cdots + \alpha_M = 1$), and to choose $x = 1, 2, \ldots, M$ resp. with these probabilities. Similarly, II will not choose $y = 1, 2, \ldots, N$ but probabilities $\beta_1, \beta_2, \ldots, \beta_N$ (all ≥ 0, $\beta_1 + \beta_2 + \cdots + \beta_N = 1$) for these eventualities.

If I resp. II have chosen α resp. β (abbreviations for $\alpha_1, \alpha_2, \ldots, \alpha_M$ resp. $\beta_1, \beta_2, \ldots, \beta_N$), the probable value of I's gain is

$$g(\alpha\beta) = \sum_{x=1}^{M}\sum_{y=1}^{N} a_{xy}\alpha_x\beta_y.$$

[Thus $g(\alpha\beta)$ is a bilinear form in α, β]. We can repeat with $g(\alpha\beta)$ the reasoning made in paragraph III with a_{xy}. It will be of extreme importance to know if

$$\text{Max}_\alpha\text{Min}_\beta g(\alpha\beta) = \text{Min}_\beta\text{Max}_\alpha g(\alpha\beta)$$

always holds or not, the two-player game problem being solved in the first case.

V. M. Borel has formulated (*loc. cit.*) the hypothesis that $\text{Max}_\alpha\text{Min}_\beta = \text{Min}_\beta\text{Max}_\alpha$ holds for all the a_{xy} [and $g(\alpha\beta)$], provided the game is fair, that is for $a_{xy} = -a_{yx}$

[and $g(\alpha\beta) = -g(\beta\alpha)$]; but he only proved it for some special cases ($M, N \leq 5$).[2] The author was able to prove (see the Memoir cited in the *Math. Ann.*) that $\text{Max}_\alpha\text{Min}_\beta = \text{Min}_\beta\text{Max}_\alpha$ for all the systems a_{xy}, without restrictions. This theorem solves the two-player game problem in its most general form (see §II).

VI. For more than 2 players, new difficulties arise. The author was able to solve the problem of the game with 3 players.

Appendix B

On the General Theory of Games Involving Players' Skills; Note of Mr. Jean Ville

Before tackling the problem of the general theory of games involving players' skills, we shall establish a general proposition on linear forms with non negative variables which will considerably simplify the subsequent proofs.

Theorem. *Given p linear forms of n variables*

$$f_j(x) = \sum a_j^i x_i \quad (j = 1, 2, \ldots, p; i = 1, 2, \ldots, n),$$

with the following properties:

For any non negative values assumed by the variables x_i, there exists at least one of the forms f_j that takes a non negative value at those values of the variables x_i.

Under these conditions:

There exists at least one system of non negative coefficients$X_1, X_2, \ldots, X_p$ with sum equals to 1 such that the form $\sum_{j=1}^p X_j f_j$ takes non negative values for every system of non negative values taken by the variables x_i.

The theorem is easily verified in the case of a system of two forms of two variables. Let us consider the case of two forms with n variables,

$$f_1(x) = a_1^1 x_1 + a_1^2 x_2 + \cdots + a_1^n x_n,$$

$$f_2(x) = a_2^1 x_1 + a_2^2 x_2 + \cdots + a_2^n x_n,$$

satisfying the hypotheses of the theorem. Note that for any i and j, the forms $a_1^i x_i + a_1^j x_j$ and $a_2^i x_i + a_2^j x_j$ satisfy the same hypotheses. In order to convince ourselves, it suffices to let the x_k for all $k \neq i$ and $k \neq j$ vanish.

Let us consider two coefficients of the same rank, say a_1^i and a_2^i. The set of all values λ such that

$$0 \leq \lambda \leq 1,$$

$$\lambda a_1^i + (1 - \lambda) a_2^i \geq 0$$

is a closed interval which we denote by α_i.

[2](One can easily see that for $a_{xy} = -a_{yx}$, the common value of $\text{Max}_\alpha\text{Min}_\beta$ and $\text{Min}_\beta\text{Max}_\alpha$ cannot be other than 0.)

In order to prove the proposition, it is sufficient to show that there exists at least one common point to all intervals α_i. Notice first that each of these intervals is nonempty; indeed, α_i would be empty only if $a_1^i < 0$ and $a_2^i < 0$, which is incompatible with the hypothesis. Each of these intervals containing either 0 or 1, their intersection would be empty only if there are two disjoint intervals, say α_i and α_j. If this was the case, it would be impossible to find a linear combination

$$X_1(a_1^i x_i + a_1^j x_j) + X_2(a_2^i x_i + a_2^j x_j) \quad (X_1 + X_2 = 1; X_1, X_2 \geq 0)$$

with non negative coefficients. Since these two forms satisfy the conditions of the hypothesis, we can see that assuming the intervals α_i to have an empty intersection yields a contradiction

Let us now consider the general case of p forms. We will argue recursively, assuming the proof completed for $p-1$ forms, and extending it to p forms. Assume therefore that the p forms $f_1, f_2, \ldots, f_p$ satisfy the hypothesis conditions. Assume that the form f_p is not always non negative in the domain of non negative values of the variables, the proof would otherwise be completed by setting $X_p = 1$.

Let $x_1, x_2, \ldots, x_n$ be the variable appearing in the forms f_i, and let D be the domain

$$\begin{cases} x_i \geq 0 \\ f_p(x) \leq 0 \end{cases} \quad (i = 1, 2, \ldots, n).$$

The domain D is linearly convex. Hence we can define it by parametric expressions of the form

$$x_i = \sum_{j=1}^{n} b_{ij} u_j, \quad b_{ij} \geq 0 \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, s).$$

in such a way that whenever a point (u) ranges over the domain $u_j \geq 0$, $(j = 1, 2, \ldots, s)$, the point (x) describes the domain D; the correspondence being not necessarily bijective.

At every point in D, at least one of the forms $f_1, \ldots, f_{p-1}$ is non negative; thus, at the variables u_j, the forms $f_1, f_2, \ldots, f_{p-1}$ satisfy the conditions of the hypothesis, and we know how to find a combination

$$\varphi = Y_1 f_1 + \cdots + Y_{p-1} f_{p-1}, \quad Y_i \geq 0, \quad \sum Y_i = 1,$$

with non negative coefficients in u.

The form φ in the variables $x_1, x_2, \ldots, x_n$ is non negative in every point of D; thus the set of two forms f_p and φ satisfies the conditions of the hypothesis; we can therefore form a combination

$$\lambda\varphi + \mu f_p \quad (\lambda \geq 0; \mu \geq 0; \lambda + \mu = 1)$$

with non negative coefficients in $x_1, x_2, \ldots, x_n$, which is indeed of the form $\sum X_i f_i$ with $X_i \geq 0, \sum X_i = 1$.

The proposition being therefore proved, we shall state an obvious corollary:

Corollary. *Let $f_1, f_2, \ldots, f_p$ be p linear forms in n variables $x_1, x_2, \ldots, x_n$, and let φ be a linear form in the same variables. If at every point (x) with $x_i \geq 0$ $(i = 1, 2, \ldots, n)$ at least one of the forms f_i assumes a value greater than or equal to that of φ, then there exists a linear combination*

$$\psi = X_1 f_1 + X_2 f_2 + \cdots + X_p f_p, \quad X_j \geq 0 \quad (j = 1, 2, \ldots, p) \quad \sum X_j = 1,$$

such that we have $\psi \geq \varphi$ at every point(x) with $x_i \geq 0$ $(i = 1, 2, \ldots, n)$.

This corollary being established, we can proceed to the general theory of games where the players' skills intervene.

Mathematical definition of the intervention of the player's skill. Let A and B be two players in a game where both chance and the players's skills play a role. Suppose that A can choose among n distinct ways to conduct his game, say $A_1, A_2, \ldots, A_n$; and that B can choose amongst $B_1, B_2, \ldots, B_m$. For every choice made by A and by B there will correspond for B a well-defined mathematical expectation; let a_{ij} be the expectation for B given that A plays according to rule A_i and that B plays according to manner B_j.

If A and B, instead of always holding to a same manner of playing, assign to these manners probabilities chosen in advance, say

$$x_1, x_2, \ldots, x_n \quad \text{for } A, \quad x_i \geq 0, \quad \sum x_i = 1;$$
$$X_1, X_2, \ldots, X_m \quad \text{for } B, \quad X_j \geq 0, \quad \sum X_j = 1;$$

the mathematical expectation for B will be

$$G = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i X_j. \tag{B.1}$$

A having distributed its probability in a certain way, let

$$(x) = x_1, x_2, \ldots, x_n$$

be his choice. The mean gain of B will have as a maximum the expression

$$\max_X G.$$

Hence, by a suitable choice of (x), A will act in such a way that regardless of the behavior of B, the average gain G of the latter be limited by the inequality

$$G \leq \min_x \max_X G. \tag{B.2}$$

Conversely, B will be able to act in a way as to have the inequality

$$G \geq \max_X \min_x G \tag{B.3}$$

regardless of $A's$ behavior.

Theorem of Mr. Von Neumann. Mr. von Neumann has proved that the two limits appearing in (B.2) and (B.3) and between which the two players could restrict G, are equal. We shall provide an elementary proof of this proposition.

We shall set, for $j = 1, 2, \ldots, m$,

$$f_j(x) = a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{nj}x_n.$$

The system (x) being chosen, the quantity $\max_X G$ is nothing but the highest of the values

$$f_1(x), f_2(x), \ldots, f_m(x).$$

Let μ be the number $\min_x \max_X G$, and let (x^0) be a system (x) such that for $(x) = (x^0)$ one has

$$\max_X G = \mu.$$

The number μ is thus defined by the two conditions:

α. For any (x) there exists a value of j such that

$$f_j(x) \geq \mu.$$

β. There exists at least one system (x^0) such that for $(x) = (x^0)$ we have, for every value of j,

$$f_j(x^0) \leq \mu.$$

Constraining the variables $x_1, x_2, \ldots, x_n$ to being always non negative but ignoring the condition $\sum x_i = 1$, condition (α) can be expressed by:

$$(\alpha') \quad \text{For every } (x), \text{ there exists a value of } j \text{ such that}$$
$$f_j(x) \geq \mu(x_1 + x_2 + \cdots + x_n), \quad x_i \geq 0 \quad (i = 1, 2, \ldots, n).$$

By the theorem on linear forms proved earlier, there exists a system of non negative coefficients

$$X_1^0, X_2^0, \ldots, X_m^0, \quad \sum X_i^0 = 1,$$

such that the form $\sum X_j f_j$ satisfies

$$\sum X_j f_j \geq \mu(x_1 + x_2 + \cdots + x_n).$$

So, if player B chooses this system of values

$$(X^0) = (X_1^0, X_2^0, \ldots, X_m^0)$$

as probabilities for playing according to manners $B_1, B_2, \ldots, B_m$, his mathematical expectation would satisfy

$$G \geq \mu$$

regardless of how A plays; this establishes Mr. von Neumann's proposition.

Extension of the preceding theory to the continuous case. Let us now take on the case where the different playing ways available to players A and B can be considered to constitute a continuous set. We shall assume that this set is the interval $(0, 1)$. If A chooses the point with abscissa x and B the point of abscissa y, the mathematical expectation of B will be a function of x and y, say $K(x, y)$. The way A plays will be a function $F(x)$:

$$F(x) = \text{probability that } A \text{ chooses a point of abscissa} \leq x.$$

And the way B plays will be described by an analogous function $\Phi(y)$.

A and B having chosen F and Φ respectively, the mean gain of B will be

$$G = \int_0^1 \int_0^1 K(x, y) dF d\Phi.$$

We shall extend Mr. von Neumann's theorems to this case; we must show that

$$\min_F \max_\Phi G = \max_\Phi \min_F G.$$

The theorem is not necessarily true if no restriction is imposed on the function K. Consider for example the following definition:

$$\begin{aligned} K &= 0 && \text{for } x = y, \\ K &= +1 && \text{for } x = 1,\ \ y < 1 \text{ and for } x < y < 1, \\ K &= -1 && \text{for } y = 1,\ \ x < 1 \text{ and for } y < x < 1. \end{aligned}$$

For this particular function K, we show without difficulty that for any F there exists a function Φ such that $G > 1 - \varepsilon$, and for every Φ_0[3] there exists a function F such that $G > -1 + \varepsilon$. Mr. von Neumann's theorem hence fails.

We shall show that if K is a continuous function in x and y in the closed domain $0 \leq x \leq 1, 0 \leq y \leq 1$, then the theorem holds. Suppose that A has chosen a function F. The quantity $\max_\Phi G$ is nothing else but the maximum, when y varies from 0 to 1, of the function of y:

$$\int_0^1 K(x, y) dF(x).$$

This function of y being continuous, it is indeed a true maximum and not just an upper bound. Let μ be the lower bound, when F varies, of $\max_\Phi G$. This lower bound is indeed achieved for at least one form of the function F. This follows from the compactness of the set of functions F and from the fact that if a sequence of functions

$$F_1, F_2, \ldots, F_n, \ldots$$

converges to a function F_0, we have due to the continuity of K,

$$\lim_{n=\infty} \int_0^1 K(x, y) dF_n(x) = \int_0^1 K(x, y) dF_0(x)$$

uniformly with respect to y.

[3] Φ_0 should simply read Φ; a typographical error in Ville's original paper.

The number μ is hence defined by the two conditions:

α. For any $F(x)$ there exists at least one value of y such that

$$\int_0^1 K(x,y)dF(x) \geq \mu.$$

β. There exists a function $F_0(x)$ such that for any function $\Phi(y)$ we have

$$\int_0^1 K(x,y)dF_0(x)d\Phi(y) \leq \mu.$$

Let now ε be an arbitrary positive number. We can find an integer n large enough such that for

$$|x' - x''| < \frac{1}{n} \quad \text{and} \quad |y' - y''| < \frac{1}{n}$$

we have

$$|K(x', y') - K(x'', y'')| < \varepsilon. \tag{B.4}$$

This number n being chosen, we shall set

$$a_{ij} = K\left(\frac{i}{n}, \frac{j}{n}\right) \quad (i, j = 1, 2, \ldots, n).$$

Let y be a value between 0 and 1 and let $\frac{j}{n}$ be its approximation within $\frac{1}{n}$ by excess. From (B.4) we have

$$\left| \int_0^1 K(x,y)dF(x) - \int_0^1 K\left(x, \frac{j}{n}\right) dF(x) \right| < \varepsilon. \tag{B.5}$$

Consider now n non negative values with sum equals to 1

$$x_1, x_2, \ldots, x_n, \quad x_i \geq 0, \quad \sum x_i = 1. \tag{B.6}$$

If we consider the step function $F(x)$ which, for $x = \frac{1}{n}$ $(i = 1, 2, \ldots, n)$ presents a discontinuity with amplitude x_i, it follows from (α) and (B.5) that:

γ. For any system of non negative values $x_1, x_2, \ldots, x_n$ with sum equals to 1, there exists at least one value of j such that

$$\sum_{i=1}^{n} a_{ij} x_i \geq \mu - \varepsilon. \tag{B.7}$$

It follows from the theorem used in the proof of Mr. von Neumann's proposition for the discontinuous case, a system of values

$$y_1^0, y_2^0, \ldots, y_n^0; \quad y_j^0 \geq 0, \quad \sum y_j^0 = 1,$$

such that for any system of values x_i satisfying (B.6), we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i y_j^0 \geq \mu - \varepsilon. \tag{B.8}$$

Let $\Phi_0(y)$ be the step function Φ that exhibits at the points $y = \frac{j}{n}$ ($j = 1, 2, \ldots, n$) discontinuities with amplitude y_j^0. By (B.8), we have for $x = \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$

$$\int_0^1 K(x,y)d\Phi_0(y) \geq \mu - \varepsilon. \tag{B.9}$$

As we deduce from (B.4) that if $\frac{i}{n}$ is the approximated value of x from above within $\frac{1}{n}$, we have

$$\left|\int_0^1 K(x,y)d\Phi(y) - \int_0^1 K(\frac{i}{n},y)d\Phi(y)\right| < \varepsilon,$$

it follows from (B.9) that, for any x :

$$\int_0^1 K(x,y)d\Phi_0(y) \geq \mu - 2\varepsilon. \tag{B.10}$$

The number ε being arbitrarily small, we can conclude from (B.10) that

$$\max_{\Phi} \min_{F} G = \min_{F} \max_{\Phi} G,$$

which shows that for continuous K, Mr. von Neumann's theorem holds true.

Appendix C

Observations on the Preceding Note; By M. Emile Borel

I must thank M. Jean Ville for having graciously agreed to expound for this book's readers the important theorem of M. von Neumann; he was able to simplify its proof and extend it to the case of continuous variables. However, in order to prevent any misunderstanding, it seems to me indispensable to stress that the practical applications of this theorem to games of chance played effectively do not appear to materialize for a long time to come.

If, for a given game, the coefficients appearing in the equations could effectively be determined and the computations involved performed, then one would deduce a way of playing that, in absolute terms, would be the best. If, for the sake of simplicity, we limit ourselves to a non symmetric two-player game, such as "l'écarté", we would know with certainty if the advantage belongs to the player who hands the cards first or to his opponent, and we would know exactly the value of that advantage, provided both players strictly conform to the game's rules resulting from solving the equations. Of course, this would only be an average advantage, obtained after a very large number of rounds during which the random distribution of hands would have favoured one player or the other.

But it suffices to think just for a moment in order to realize that, even for a relatively simple game such as *L'Ecarté*,[4] even after believing that certain remarks

[4]A card game popular in Paris between 1825 and 1835. See the web page of the Académie des Jeux Oubliés, http://trictrac.aquitain.free.fr/

made by the best players have been taken into account in order to discard with certainty some strategies considered far from sensible, there still remains such a large number of variables that the mere task of writing the equations, let alone solving them, seems to be absolutely undoable.

It so happens that for those very ancient games practiced by many players, a phenomenon that also occurs for games that are not games of chance, such as checkers or chess, but that are also far too complex for a study of all possible combinations. Following numerous experiences involving a number of players and whose results are sometimes recorded in specialized publications, one would be able, at a given time, to conceive a tactic that could be taught to an inexperienced player exposed to the best players of that time. However, before this best tactic, result of all earlier experiences, is completely codified and becomes a teaching subject, it happens that the most skilled players that know it realize its weaknesses and discover means to exploit them by imagining little by little a new tactic, slightly different at times, or, to the contrary noticeably distant. One thus witnesses a continuous evolution in the way best players play the game, without excluding the hypothesis that a new tactic, considered rightfully as an improvement over the previous one in the sense that it insures a mean advantage for those who invented it, is nonetheless beaten if its opponents resort to an older tactic, abandoned long ago.

That is how one often goes in a circle, although in other cases, we can assert that the general theory of a given game has really progressed and that a good player nowadays is superior to the masters of a century ago. Can we hope that, by successive approximations, we would reach a result equivalent to the one provided by the resolution, impossible in practice, of the complete equations of the problem? I confess that I find it very doubtful, and besides, if that was to happen for a given game, it is almost certain that that game would be soon abandoned to the benefit of a more complicated one.

What one must retain is that in games involving a more or less free and secret choice on behalf of one of the players (a declaration, a way of discarding, etc.), any manner of playing that is too rigid has the disadvantage of informing the opponent who knows it and providing him with the possibility to counter advantageously. It is absolutely certain that the ideal manner of playing, which we do not know but try to approach, comprises rules involving probability coefficients as we have seen in the study of the most elementary psychological games. This is an inevitable complication. One has to recognize that the disadvantages for not following a rigid rule are very considerable in games, such as Bridge, where two partners must combine efforts, each one being informed by the partner at the same time and in the same way as the opponents. Thus, an attempt to mislead the opponent risks also misleading the partner. This is a difficulty of very special nature, hardly subject to computation as it is difficult to formulate in logical terms this honesty rule that partners cannot resort to a secret language, incomprehensible to their opponents. If we accept that 4 players are in turn opponents and partners according to customary rotation, it becomes even more difficult to formulate rules that are applicable to a camp, considered as the adversary of another camp. If we accept that the four players know each other well, that they have different temperaments and that each one plays his own way, roughly known to the other three, then it is evidently beneficial in order for each one of them to perfect his game to slightly modify his manner of playing depending on which partner ha has; however, these modifications will slowly become known to the other players who will modify their game accordingly.

These reflections, which could be developed a lot further, will not teach much to those who are accustomed to playing. They will perhaps explain to those who are not interested in games what pleasure and profit they provide to those who see them as entertainment; at the same time they will contribute to showing to aspiring professional players how vain is the pursuit of a perfect formula that risks to always elude us.

References

Afriat, S. N. [1987] *Logic of Choice and Economic Theory* (Oxford: Clarendon Press).

Anonymous [1937] Princeton scientist analyzes gambling: 'You can't win', *Science News Letter*, **3** (April), 216, as reprinted in Dimand and Dimand 1997, **I**.

Arrow, K. J. [1947] Review of Ville (1946), *Mathematical Reviews* 396–97.

Baumol, W. J. and Goldfeld S. [1968] *Precursors in Mathematical Economics* (London: London School of Economics and Political Science).

Ben-El-Mechaiekh, H., Deguire, P. and Granas, A. [1982a] Une alternative nonlinéaire en analye convexe et applications, *Comptes Rendus de l'Académie des Sciences de Paris* **295**, Série I, 257–259.

Ben-El-Mechaiekh, H., Deguire, P. and Granas, A. [1982b] Points fixes et coincidences pour les fonctions multivoques I (Applications de Ky Fan), *Comptes Rendus de l'Académie des Sciences de Paris* **295**, Série I, 337–340.

Ben-El-Mechaiekh, H. and Dimand, R. [2007] The von Neumann minimax theorem revisited, in *Fixed Point Theory and Applications*, Banach Center Publications **77**, 23–34.

Binmore, K. [2004] Guillermo Owen's proof of the minimax theorem, *Theory and Decision* **56**, 19–23.

Borel, E. [1921] La théorie du jeu et les équations intégrales à noyau symétrique gauche, *Comptes Rendus de l'Académie des Sciences* **173**, 1304–08, trans. L. J. Savage as "The Theory of Play and Integral Equations with Skew Symmetric Kernels," *Econometrica* **21** (1953) 97–100.

Borel, E. [1924] Sur les jeux où interviennent le hasard et l'habilité des joueurs, in E. Borel, *Théorie des probabilités,* 3rd edn. (Paris: Librairie Scientifique, J. Hermann), 204–224, trans. L. J. Savage as "On Games That Involve Chance and the Skill of Players," *Econometrica* **21** (1953) 101–15.

Borel, E. [1925–39] *Traité du Calcul des Probabilités,* 4 volumes in 18 parts, Paris: Gauthiers-Villars.

Borel, E. [1927] Sur les systèmes de formes linéaires à déterminant symétrique et la théorie générale du jeu, *Comptes Rendus de l'Académie des Sciences* **184**, 52–53, trans. L. J. Savage as "On Systems of Linear Form of Skew Symmetric Determinant and the General Theory of Play," *Econometrica* **21** (1953) 116–17.

Borel, E. *et al.* [1938] *Traité du Calcul des Probabilités* **4** (2), *Applications Aux Jeux de Hasard,* ed. J. Ville (Paris: Gauthier-Villars).

Borel, E. and Chéron, A. [1940] *Théorie Mathématique du Bridge à la Portée de Tous* (Paris: Gauthier-Villars).

Browder, F. E. [1968] The fixed point theory of multi-valued mappings in topological verctor spaces, *Math. Ann.* **177**, 238–301.

Dell'Aglio, L. [1995] Divergences in the history of mathematics: Borel, von Neumann and the genesis of game theory, *Rivista di Storia di Scienza,* 2nd series, **3**(2), 1–46.

De Possel, R. [1936] *Sur la Théorie Mathématique des Jeux de Hasard et de Réflexion* (Paris: Hermann & Cie.), as reprinted in Dimand and Dimand (1997), I.

Dimand, M. A. and Dimand, R. W., (eds.) [1997] *The Foundations of Game Theory,* 3 Vols. (Cheltenham, UK, and Brookfield, VT: Edward Elgar Publishing).

Dimand, R. W. and Dimand, M. A. [1996] From games of pure chance to strategic games: French probabilists and early game theory, in C. Schmidt, (ed.), *Uncertainty in Economic Thought* (Cheltenham, UK, and Brookfield, VT: Edward Elgar Publishing).

Fan, K. [1952] Fixed point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. USA* **38**, 121–126.

Fan, K. [1953] Minimax theorems, *Proc. Nat. Acad. Sci. USA* **39**, 42–47.

Fan, K. [1984] Some properties of convex sets related to fixed point theorems, *Math. Ann.* **266**, 519–537.

Fréchet, M. [1953a] Ėmile Borel, initiator of the theory of psychological games and its application, *Econometrica* **21**, 95–96.

Fréchet, M. [1953b] Commentary on the three notes of Ėmile Borel, *Econometrica* **21**, 118–24.

Giocoli, N. [2003] *Modeling Rational Agents* (Cheltenham, UK, and Northampton, MA: Edward Elgar Publishing).

Guerraggio, A. and Molho, E. [2004] The origins of quasi-concavity: A development between mathematics and economics, *Historia Mathematica* **31**, 62–75.

Innocenti, A. [1995] Oskar Morgenstern and the heterodox potentialities of game theory for economics, *Journal of the History of Economic Thought* **17**, 205–27.

Kjeldsen, T. H. [2001] John von Neumann's conception of the minimax theorem: A journey through different mathematical contexts, *Archive for History of Exact Sciences* **56**, 39–68.

Knaster, B., Kuratowski, C. and Mazurkiewicz, S. [1929] Ein Beweis des Fixpunktsatzes für n-Dimensionale Simplexe, *Fund. Math.* **14**, 132–138.

Kneser, H. [1952] Sur un théorème fondamental de la théorie des jeux, *C. R. Acad. Sci. Paris,* Série A **234**, 2418–2420.

Leonard, R. J. [1992] Creating a context for game theory, in E. R. Weintraub, (ed.), *Toward a History of Game Theory* (Durham, NC: Duke University Press), *Annual Supplement to History of Political Economy* **24**, 29–76.

Leonard, R. J. [1995] From parlor games to social science: Von Neumann, Morgenstern, and the creation of game theory, 1928–1944, *Journal of Economic Literature* **33**, 730–61.

Morgenstern, O. [1976] The collaboration between Oskar Morgenstern and John von Neumann on the theory of games, *Journal of Economic Literature* **14**, 805–16.

Nikaido, H. [1954] On von Neumann's minimax theorem, *Pacific J. Math* **4**, 65–72.

Owen, G. [1982] *Game Theory,* 2nd edn. (New York: Academic Press).

Samuelson, P. A. [1950] The problem of integrability in utility theory, *Economica*, n.s. **17**, 355–85.

Schmidt, C. [2001] *La Théorie des Jeux: Essai d'interprétation* (Paris: Presses Universitaires de France).

Shafer, G. and Vovk, V. [2001] *Probability and Finance: It's Only a Game*! (New York: John Wiley & Son).

Shiffman, M. [1949] On the equality min max = max min, and the theory of games, RAND Report **RM-243**.

Sion, M. [1958] On general minimax theorems, *Pacific J. Math.* **8**, 171–176.

Ville, J. [1935a] Über Kurven mit quadratischen Lange, *Ergebnisse einer Mathematischen Kolloquium Wien* **6**.

Ville, J. [1935b] Über ein Satz von O. Blumenthal, *Ergebnisse einer Mathematischen Kolloquium Wien* **6**.

Ville, J. [1938] Sur la théorie générale des jeux où intervient l'habilité des joueurs, in Borel [1938], 105–113, as reprinted in Dimand and Dimand [1997], I.

Ville, J. [1939] *Etude Critique de la Notion de Collectif* (Paris: Monographies des Probabilités).

Ville, J. [1946] Sur les conditions d'existence d'une ophélimité totale, *Annales de l'Université de Lyon, Sect. A.* **9**, 32–9, trans. P. K. Newman as "The Existence Condition of a Total Utility Function," *Review of Economic Studies* **19**(1951–52), 123–28.

Ville, J. [1948] *Leçons sur la Démographie Mathématique* (Paris: Institut National de la Statistique et des Etudes Economiques).

Ville, J. [1954] Sur quelques aspects récents de la théorie des probabilités, *Annales de l'Institut Henri Poincaré* **14**(2).

Ville, J. [1955] Notice sur les travaux scientifiques de M. Jean Ville.

Ville, J. [circa 1956] Curriculum Vitae de M. Jean-André Ville, Archives Fréchét, Université de Paris VI.

Von Neumann, J. [1928a] Sur la théorie des jeux, communicated by E. Borel, *Comptes Rendus de l'Académie des Sciences* **186**, 1689–91, as reprinted in Dimand and Dimand (1997), I.

Von Neumann, J. [1928b] Zur Theorie der Gesellschaftsspiele, *Mathematische Annalen* **100**, 295–320, trans. S. Bargmann as On the Theory of Games of Strategy, in *Contributions to the Theory of Games*, 4, (eds.) A. W. Tucker and R. D. Luce, *Annals of Mathematics Studies,* **40** (Princeton, NJ: Princeton University Press, 1959).

Von Neumann, J. [1937] Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwer'schen Fixpunktsatzes, *Ergenbisse einer Mathematischen Kolloquium Wien* **8**, 73–83, trans. G. Morton as A Model of General Economic Equilibrium, *Review of Economic Studies* **13**(1945–46), 1–9.

Von Neumann, J. [1953] Communication on the Borel Notes, *Econometrica* **21**, 124–27.

Von Neumann, J. and Morgenstern, O. [1944] *Theory of Games and Economic Behavior* (Princeton, NJ: Princeton University Press).

Weyl, H. [1950] Elementary proof of a minimax theorem due to von Neumann, in H. W. Kuhn and A. W. Tucker (eds.), *Contributions to the Theory of Games, I,* Annals of Mathematical Studies, Vol. 20, Princeton University Press, 19–25.