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COMPLETENESS, INVARIANCE AND λ-DEFINABILITY

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Introduction. In [4] Gordon Plotkin considers the problem of characterizing the λ-definable functionals in full type structures. A plausible, but, as we shall see, quite false, conjecture is that a functional is λ-definable $\Leftrightarrow$ it is invariant in the sense of Fraenkel and Mostowski. A better guess might be that the set of λ-definable functionals of type σ has a uniform (in σ) definition in type theory over all full type structures (Plotkin explicitly considers a certain sort of definition by means of "logical relations"). More or less generally, one may ask if the following question is decidable.

Given: a functional F in a full type structure over a finite ground domain.

Question: is F λ-definable?

We call the statement that this question is decidable Plotkin's λ-definability conjecture. We do not know if Plotkin's conjecture is true.

In this note we consider several questions which quickly arise from Plotkin's conjecture. In §I, we ask which type structures satisfy λ-definability $=$ invariance. We construct for each consistent set of equations a model satisfying this equality. From consideration of these models we obtain a number of syntactic corollaries including a reduction of $\beta\eta$-conversion to $\beta\eta$-conversion at a single type (Theorem 3), an "ω-rule" for $\beta\eta$-conversion (Proposition 6) and consistency with $\beta\eta$-conversion (Proposition 8), definability and indefinability results for versions of Kronecker's δ (Propositions 10 and 11), and a decidability result for the problem of interdefinability among combinators. In addition, we obtain a number of useful model theoretic results including a reduction of λ-definability to relative λ-definability in hereditarily infinite full type structures (Proposition 2), and a sharpened completeness theorem for hereditarily finite full type structures (Theorem 2).

We shall, in another paper, use these models to give a (noneffective) characterization of the λ-definable functionals in arbitrary type structures as the stable solutions to systems of functional equations. Here stable solutions are, roughly, unique solutions which remain unique solutions under perturbations of the type structure by homomorphisms.

In §II we consider some consequences of Plotkin's conjecture. In particular, we show that the problem of deciding if a given hereditary finite object is the "σ-section" of a type structure is recursive and that the problem of deciding if a given statement of type theory has a hereditarily finite model is recursively enumerable,

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assuming Plotkin's conjecture. We also show that Plotkin's conjecture holds for all hereditarily finite type structures if it holds as stated.

In §III we use one of the models constructed in §I to extend the principal result of [5, Theorem 6(1)]. We also give an analogue of Scott's theorem for the untyped λ-calculus (Rice's theorem for recursive functions) [9, p. 1105] according to the analogy

$$\frac{\text{nonrecursive}}{\text{untyped } \lambda\text{-calculus}} = \frac{\text{non-(Kalmar)-elementary}}{\text{typed } \lambda\text{-calculus}}.$$

§0. Preliminaries. Λ is the set of all typed λ-terms. If X is a set of variables then Λ_X is the set of all terms with only free variables from X. $\bar{\Lambda} =_{\text{df}} \Lambda_\phi$.

For types τ, σ we write $\tau \subseteq \sigma$ if τ is a subtype of σ. The notions of positive and negative occurrence are defined as in Prawitz [8, p. 43] (read $\rightarrow$ for $\supset$). More precisely, 0 is positive in 0, and if ρ is positive (negative) in σ or negative (positive) in τ then ρ is positive (negative) in $\tau \rightarrow \sigma$. rank(σ) is defined by rank(0) = 0 and rank($\sigma \rightarrow \tau$) = max{rank(σ) + 1, rank(τ)}. A type is called binary if it has the form $\sigma_1 \rightarrow (\sigma_2 \rightarrow \sigma_3)$.

Structures and prestructures are as in Friedman [2, p. 23]. If $\mathfrak{A}$ is a prestructure then $\mathfrak{A}^\sigma$ is the set of its type σ elements and $\mathfrak{A}(\sigma) = \bigcup_{\tau\subseteq\sigma}\mathfrak{A}^\tau$. We shall always assume that prestructures consist of functionals (at higher types). If $\mathfrak{A}$ is a structure then $\mathrm{Val}_{\mathfrak{A}}$ is the unique Val satisfying (i)–(iii) of [2, p. 23] for $\mathfrak{A}$. Partial homomorphisms between prestructures are as in [2, p. 27]. The operation $(\ )^E$ and the relation $\approx$ are as in Troelstra [6, pp. 123–124]. In particular, if $X \subseteq \mathfrak{A}$ define binary relations E^ρ on $X \cap \mathfrak{A}^\rho$ as follows. $E^0(a, b) \Leftrightarrow a = b$, and $E^{\sigma\rightarrow\tau}(F, G) \Leftrightarrow$ for all $H_1H_2 \in X \cap \mathfrak{A}^\sigma$, $E^\sigma(H_1, H_2) \rightarrow E^\tau(FH_1, GH_2)$. $(X)^E = \{F \in X: E(F, F)\}$, and $\approx$ is E restricted to $(X)^E$. If $X \subseteq \mathfrak{A}$ then $\Lambda_{\mathfrak{A}}(X)$ is the set of all members of $\mathfrak{A}$ λ-definable from X. $\mathfrak{P}_\kappa$ is the full type structure over a ground domain of size κ.

If $\mathscr{E}$ is a set of equations between terms (of the same type) then $\mathscr{E} \vdash_{\beta(\eta)} M = N$ means that $M = N$ is in the closure of $\mathscr{E}$ under the inductive definition of $\beta(\eta)$-conversion (see e.g. [2, p. 24]). If $\mathscr{E} = \phi$ we write $M =_{\beta(\eta)} N$ for $\mathscr{E} \vdash_{\beta(\eta)} M = N$. The notion of the long $\beta\eta$-normal form of a term is defined as in Jensen and Pietrzykowski [7, p. 128]. Specifically, $M \in \Lambda$ is in long $\beta\eta$-normal form if $M = \lambda x_1 \cdots x_n x M_1 \cdots M_m$, where $xM_1 \cdots M_m$ has type 0 and each M_i is in long $\beta\eta$-normal form.

The language $\mathfrak{L}$ is the type theoretic language of [2, p. 30].

§I. Structures satisfying λ-definablity = invariance. If $\mathfrak{A}$ is a prestructure and π_0 is a permutation of $\mathfrak{A}^0$ then we lift π_0 to higher types by setting $\pi_{\sigma\rightarrow\tau} = \pi_\tau^{\pi_\sigma^{-1}}$ in the power group on $\mathfrak{A}^{\tau^{\mathfrak{A}^\sigma}}$, provided that $\pi''_{\sigma\rightarrow\tau}\mathfrak{A}^{\sigma\rightarrow\tau} = \mathfrak{A}^{\sigma\rightarrow\tau}$. If π_σ always exists we say π_0 fixes $\mathfrak{A}$. We say $F \in \mathfrak{A}^\sigma$ is invariant if $\pi_\sigma F = F$ for all π_0 fixing $\mathfrak{A}$.

If $\kappa \geq \aleph_0$ then the number of invariant functionals in $\mathfrak{P}_\kappa$ is uncountable. If $\kappa < \aleph_0$ then $F \in \mathfrak{P}_\kappa$ is invariant $\Leftrightarrow$ F is definable in $\mathfrak{L}$. The functional G defined by $Gxxzw = z$ and $Gxyzw = w$ if $x \neq y$ is invariant but not λ-definable. λ-definable functionals are always invariant. A structure all of whose invariant functionals are λ-definable is called a P-structure.

A set of closed equations is called consistent (in the sense of equational theories) if it has a model with a ground domain of size ≥ 2. Let $\mathscr{E}$ be a set of closed equations. Define the relation $\sim$ (between possibly open terms) to be the transitive closure of $\{(N_1, N_2)\colon (\exists M_1 = M_2 \in \mathscr{E})(\exists L \in \Lambda) LM_1M_2 =_{\beta\eta} N_1 \wedge LM_2M_1 =_{\beta\eta} N_2\}$. It is easy to verify that $\sim$ is a congruence relation on Λ with respect to application and that $\mathfrak{J}_{\mathscr{E}} =_{\mathrm{df}} \Lambda/\sim$, with $(M/\sim)(N/\sim) =_{\mathrm{df}} (MN)/\sim$, is a structure. Note that, for the assignment μ defined by $\mu(x) = x/\sim$, we have $\mathfrak{J}_{\mathscr{E}} \models M = N[\mu]$.$\Leftrightarrow$. $\vdash_{\beta\eta} M = N$.

Let π be a type preserving permutation of variables $x \overset{\pi}{\mapsto} x^\pi$. If $x_1 \cdots x_n$ are the free variables of M set $M^\pi =_{\mathrm{df}} [x_1^\pi/x_1, \ldots, x_n^\pi/x_n]M$. The map π_0 defined by $M/\sim \;\mapsto M^\pi/\sim$, for type 0 $M \in \Lambda$, is a permutation of $\mathfrak{J}^0_{\mathscr{E}}$. By induction on σ, $\pi_\sigma(M^\sigma/\sim) = (M^\sigma)^\pi/\sim$; in particular, π_0 fixes $\mathfrak{J}^0_{\mathscr{E}}$.

LEMMA 1. *Suppose $\mathscr{E}$ is consistent. If $\mathfrak{J}^\sigma_{\mathscr{E}}$ constains an invariant functional then there is a closed term of type σ.*

PROOF. Let $\sigma = \sigma_1 \to (\cdots (\sigma_n \to 0) \cdots)$, and note that there is a closed term of type $\sigma \Leftrightarrow$ for some $1 \leq i \leq n$ there is no closed term of type σ_i. Suppose the lemma fails, then there are $M_1^{\sigma_1}, \ldots, M_n^{\sigma_n} \in \bar{\Lambda}$. Let $M/\sim\; \in \mathfrak{J}^\sigma_{\mathscr{E}}$ be invariant, then the $\beta(\eta)$-normal form of $MM_1 \cdots M_n$ has the form $xN_1 \cdots N_m$ for x a free variable of M. Thus for some $y \neq x$ $\mathscr{E} \vdash_{\beta\eta} xN_1 \cdots N_m = y[y/x]N_n \cdots [y/x]N_m$, contradicting the consistency of $\mathscr{E}$.

THEOREM 1. *If $\mathscr{E}$ is consistent then $\mathfrak{J}_{\mathscr{E}}$ is a P-model.*

PROOF. Suppose M^σ is in long $\beta\eta$-normal form with free variables $x_1^{\sigma_1} \cdots x_n^{\sigma_n}$, for $n > 0$, and suppose $M/\sim$ is invariant. Let $M = \lambda y_1 \cdots y_m U^0$, and let $\lambda y_1 \cdots y_m V^0 \in \bar{\Lambda}$ have type σ. For $\sigma_i = \tau_1 \to (\cdots (\tau_k \to 0) \cdots)$ and $w_1^{\tau_1}, \ldots, w_k^{\tau_k}$ new, set $N_i^{\sigma_i} = \lambda w_1 \cdots w_k V$. We have $U \sim [N_1/x_1 \cdots N_n/x_n]U$, so $M \sim \lambda y_1 \cdots y_m[N_1/x_1, \ldots, N_n/x_n]U$.

We write $\mathfrak{J}$ for $\mathfrak{J}_\phi$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures and $h\colon \mathfrak{A} \overset{\text{onto}}{\mapsto} \mathfrak{B}$ a partial homomorphism which is 1-1 on $\mathfrak{A}^0$. Then $(\mathrm{dom}(h))^E = \mathrm{dom}(h)$ and, for $F, G \in \mathrm{dom}(h)$, $F \approx G \Leftrightarrow h(F) = h(G)$. Indeed, $h^{-1}\colon B \mapsto \mathrm{dom}(h)/\approx$ is an isomorphism. We obtain the

PROPOSITION 1. *Every structure is isomorphic to one of the form* $(\mathrm{dom}(h))^E/\approx$, *for h a partial homomorphism from some* $\mathfrak{P}_\kappa$.

We shall use $\mathfrak{J}$ immediately for the following application.

PROPOSITION 2. *For each $k \in \omega$ and $\kappa \geq \aleph_0$, for each $F \in \mathfrak{P}_\kappa$, F is λ-definable $\Leftrightarrow F$ is λ-definable from each of the k element subsets of $\mathfrak{P}^0_\kappa$.*

PROOF. Let h be a partial homomorphism from $\mathfrak{P}_\kappa$ onto $\mathfrak{J}$ and select $2k$ pairwise distinct type 0 variables $x_1 \cdots x_k y_1 \cdots y_k$.

Choose $a_i \in h^{-1}(x_i)$ and $b_i \in h^{-1}(y_i)$, and suppose $F = \mathrm{Val}_{\mathfrak{P}_\kappa}(M)a_1 \cdots a_k = \mathrm{Val}_{\mathfrak{P}_\kappa}(N)b_1 \cdots b_k$ for closed M, N. Applying h we get $Mx_1 \cdots x_k =_{\beta\eta} Ny_1 \cdots y_k$, so the $\beta\eta$-normal form of M does not contain any of the $x_1 \cdots x_k$. Let $\lambda x_1 \cdots x_k L$ be the long $\beta\eta$-normal form of M, then $F = \mathrm{Val}_{\mathfrak{P}_\kappa}(L)$.

Let $\mathscr{O}$ be the set of all variables of type with rank ≤ 1, and let $X \subseteq \mathscr{O}$ contain at least one type 0 variable. We have $\Lambda_{\mathfrak{J}}(X/\sim)^E = \Lambda_{\mathfrak{J}}(X/\sim)$, and $\mathfrak{F}_X =_{\mathrm{df}} \Lambda_{\mathfrak{J}}(X/\sim)/\approx$ is a structure with the obvious application operation. We write $M/\approx$ for $(M/\sim)/\approx$, so $\approx$ is a congruence on Λ_X with respect to application. In particular,

$$([N_1/x_1, \ldots, N_n/x_n]M)/\approx \;= ((\lambda x_1 \cdots x_n M)N_1 \cdots N_n)/\approx$$
$$= ((\lambda x_1 \cdots x_k M)/\approx)(N_1/\approx) \cdots (N_n/\approx).$$

The notion of Φ-normal form (see [1, p. 171]) is defined as follows:

(1) $\lambda x_1^{\sigma_1} \cdots x_n^{\sigma_n} x_{(i)}^0$ is in Φ-normal form.

(2) If $\sigma = \tau_1 \to (\cdots(\tau_m \to 0)\cdots)$ and for $1 \le i \le m$ $\sigma_i = \gamma_1 \to (\cdots(\gamma_k \to \tau_i)\cdots)$ then, if $M_1^{\sigma_1} \cdots M_m^{\sigma_m}$ are in Φ-normal form and $x_1^{\gamma_1}, \ldots, x_k^{\gamma_k}$, (x^σ) are not among the free variables of $M_1 \cdots M_m$ then $\lambda x_1^{\gamma_1} \cdots x_k^{\gamma_k} x^\sigma(M_1 x_1 \cdots x_k) \cdots (M_m x_1 \cdots x_k)$ is in Φ-normal form.

It is easy to prove that Φ-normal forms exist and are unique for $\beta\eta$-conversion.

We write $M \# N$ if for all $U_1 \cdots U_n V_1 \cdots V_n \in \Lambda_\emptyset$ we have $MU_1 \cdots U_n \ne_{\beta\eta} NV_1 \cdots V_n$.

LEMMA 2. *Suppose* $\sigma = \sigma_1 \to (\cdots(\sigma_n \to 0)\cdots)$ *and* $\sigma_i = \tau_1 \to (\cdots(\tau_p \to (\sigma_{i+j+1} \to (\cdots(\sigma_n \to 0)\cdots)) \to \tau))$. *Also, suppose* M^σ, $N^\sigma \in \bar{\Lambda}$, *and there are* $U_1^{\sigma_1} \cdots U_n^{\sigma_n} \in \Lambda_\emptyset$ *such that* $MU_1 \cdots U_n \ne_{\beta\eta} NU_1 \cdots U_n$. *Then there are* $V_1^{\sigma_1} \cdots V_n^{\sigma} \in \Lambda_\emptyset$ *such that* $\lambda x_1^{\tau_1} \cdots x_p^{\tau_p} V_i x_1 \cdots x_p(MV_1 \cdots V_{i+j}) \ne \lambda x_1^{\tau_1} \cdots x_p^{\tau_p} V_i x_1 \cdots x_p(NV_1 \cdots V_{i+j})$.

PROOF. W.l.o.g. we can assume $U_i = \lambda y_1 \cdots y_k U^0$, for $y_1 \cdots y_k$ not free in $U_1 \cdots U_n$. Select a new $z^{0\to(0\to0)}$, and set $V_i =_{\mathrm{df}} \lambda y_1 \cdots y_k z(y_{p+1} U_{i+j+1} \cdots U_n)U$. For $m \ne i$ set $V_m =_{\mathrm{df}} U_m$. It is easy to verify that $V_1 \cdots V_m$ have the desired properties.

PROPOSITION 3. *Suppose* $\sigma = \sigma_1 \to (\cdots(\sigma_n \to 0)\cdots)$ *and* $M^\sigma \in \bar{\Lambda}$ *is in* Φ*-normal form.*

(1) *There are* $U_1^{\sigma_1} \cdots U_n^{\sigma_n} \in \Lambda_\emptyset$ *such that, for each* Φ*-normal* $N^\sigma \in \bar{\Lambda}$ *of length* $\ne$ *length of* M, *we have* $MU_1 \cdots U_n \ne_{\beta\eta} NU_1 \cdots U_n$.

(2) *For each* Φ*-normal* $M \ne N^\sigma \in \bar{\Lambda}$ *there are* $U_1^{\sigma_1} \cdots U_n^{\sigma_n} \in \Lambda_\emptyset$ *s.t.* $MU_1 \cdots U_n \ne_{\beta\eta} NU_1 \cdots U_n$.

PROOF. (1) For each $\tau = \tau_1 \to (\cdots(\tau_m \to 0)\cdots)$ select a new z_τ of type

$$\underbrace{0 \to (\cdots}_{m}(0 \to 0)\cdots).$$

Define terms U^τ, V^τ simultaneously by $U^0 = z_0$, $V^0 = \lambda x^0 U^0$, $U^\tau = \lambda x_1^{\tau_1} \cdots x_m^{\tau_m} z_\tau(V^{\tau_1}x_1)\cdots(V^{\tau_m}x_m)$ and $V^\tau = \lambda x^\tau x U^{\tau_1} \cdots U^{\tau_m}$. It is routine to verify that $U^{\sigma_1} \cdots U^{\sigma_n}$ have the desired properties. (2) is proved by induction on the length of M. Basis: $M = \lambda x_1 \cdots x_n x_i^0$ and $N = \lambda x_1 \cdots x_n x_j N_1 \cdots N_m$ with $i \ne j$ and $0 \le m$. Choose $y^0 \ne z^0 \notin \{z_1 \cdots z_m\}$ with the type of $z_k =$ the type of N_k, and set $U_i = y$ and $U_j = \lambda z_1 \cdots z_m z$. Let the other U_l be arbitrary. Induction step: $M = \lambda x_1 \cdots x_n x_i(M_1 x_1 \ldots x_n) \cdots (M_m x_1 \cdots x_n)$ and $N = \lambda x_1 \cdots x_n x_j(N_1 x_1 \cdots x_n) \cdots (N_k x_1 \cdots x_n)$. Case: $i \ne j$. Select $\{y_1 \cdots y_m\} \not\ni y^0 \ne z^0 \notin \{z_1 \cdots z_k\}$ where $\mathrm{type}(y_l) = \mathrm{type}(M_l x_1 \cdots x_n)$ and $\mathrm{type}(z_l) = \mathrm{type}(N_l x_1 \cdots x_n)$. Let $U_i = \lambda y_1 \cdots y_m y$ and $U_j = \lambda z_1 \cdots z_k z$, and let the other U_l be arbitrary. Case: $i = j$ (so $k = m$). Since $M \ne N$, for some l $M_l \ne N_l$, so by hyp. ind. there are $U_1 \cdots U_{n+r} \in \Lambda_\emptyset$ s.t. $M_l U_1 \cdots U_{n+r} \ne_{\beta\eta} N_l U_1 \cdots U_{n+r}$ is of type 0. Thus there are $V_1 \cdots V_{n+r} \in \Lambda_\emptyset$ of corresponding types s.t. $\lambda y_1 \cdots y_{l-1} V_i y_1 \cdots y_{l-1}(M_l V_1 \cdots V_n)$ $\# \lambda y_1 \ldots y_{l-1} V_i y \cdots y_{l-1}(N_l V_1 \cdots V_n)$. In particular,

$$\begin{aligned} MV_1 \cdots V_n &=_{\beta\eta} V_i(M_1 V_1 \cdots V_n)\cdots(M_m V_1 \cdots V_n) \\ &\ne_{\beta\eta} V_i(N_1 V_1 \ldots V_n) \cdots (N_k V_1 \cdots V_n) =_{\beta\eta} NV_1 \cdots V_n. \end{aligned}$$

Let $\top =_{\mathrm{df}} (0 \to (0 \to 0)) \to (0 \to 0)$.

PROPOSITION 4. *If* rank$(\sigma) \leq 2$ *then there is an* $L^{\sigma\to\top} \in \bar{\Lambda}$ *s.t., for all* $M^\sigma, N^\sigma \in \bar{\Lambda}$, *we have* $M =_{\beta\eta} N \Leftrightarrow LM =_{\beta\eta} LN$.

PROOF. Select $z^{o\to(o\to o)}$ and x^0 and define $\mathbf{n}$ by $\mathbf{o} = x$ and $\mathbf{n} + \mathbf{1} = fx\mathbf{n}$. Suppose $\sigma = \sigma_1 \to (\cdots (\sigma_m \to o)\cdots)$, and define $M_i^{\sigma_i}$ as follows; if $\sigma_i = o$ then $M_i = \mathbf{i}$, and if

$$\sigma_i = \underbrace{0 \to (\cdots (0}_{k} \to 0)\cdots)$$

then $M_i = \lambda x_1^0 \cdots x_k^0 z\mathbf{i}\,(zx_1(\cdots (zx_{k-1}x_k)\cdots))$. Let $L^{\sigma\to\top} =_{\mathrm{df}} \lambda y\lambda zxyM_1 \cdots M_m$; it is straightforward to verify that L has the desired properties.

Let $X \subseteq \mathcal{O}$ contain at least one type o variable and at least one variable of type

$$\underbrace{0 \to (\cdots (0}_{k} \to 0)\cdots), \quad \text{for } k \geq 2.$$

We obtain the

PROPOSITION 5. *For* $M, N \in \bar{\Lambda}$, $M =_{\beta\eta} N \Leftrightarrow \mathfrak{F}_X \models M = N$.

We also obtain the following "ω-rule".

PROPOSITION 6. *For* $\sigma = \sigma_1 \to (\cdots (\sigma_n \to 0)\cdots)$ *and* $M^\sigma, N^\sigma \in \bar{\Lambda}$ *we have*

$$M =_{\beta\eta} N \Leftrightarrow \textit{for all } U_1^{\sigma_1} \ldots U_n^{\sigma_n} \in \Lambda_X\; MU_1 \cdots U_{\beta\eta} =_{\beta\eta} NU_1 \cdots U_n.$$

We now sharpen the completeness theorem of [4].

THEOREM 2. *For each* $M \in \bar{\Lambda}$ *there is an* m_M (*recursive in* M) *s.t., for all* $N \in \bar{\Lambda}$, $M =_{\beta\eta} N \Leftrightarrow \mathfrak{P}_{m_M} \models M = N$.

PROOF. Given $M^\sigma \in \bar{\Lambda}$, by Lemma 2 and Proposition 3, one can effectively find $L_1^{\sigma\to 0} \cdots L_n^{\sigma\to 0} \in \Lambda_{\mathcal{O}}$ such that if $N^\sigma \neq_{\beta\eta} M$ then for some i $L_iM \neq_{\beta\eta} L_iN$. Let $t_1 \cdots t_n \in \Lambda_{\mathcal{O}}$ be the $\beta(\eta)$-normal forms of $L_1M, \ldots, L_nM$ resp., and let $\mathscr{T}$ be the set of all type 0 terms built up from the variables in $t_1 \cdots t_n$. Let $\mathscr{E} = \{r_1 = r_2 : r_1, r_2 \in \mathscr{T}$, no r_i is a subterm of any $t_j\}$ and let $\mathscr{A}$ be the free algebra satisfying $\mathscr{E}$. The domain of $\mathscr{A}$ is some finite m_M which can be effectively caculated from M. W.l.o.g. we can assume $\mathscr{A} \subseteq \mathfrak{P}_{m_M}$; it follows easily that, for $N^\sigma \in \bar{\Lambda}$, $N \neq_{\beta\eta} M \Leftrightarrow \mathfrak{P}_{m_M} \models N \neq M$.

We also obtain the following theorem which is useful for testing structures for completeness for $\beta\eta$-conversion.

THEOREM 3. *For all* $M^\sigma, N^\sigma \in \bar{\Lambda}$, $M =_{\beta\eta} N .\Leftrightarrow.$ *for all* $L^{\sigma\to\top} \in \bar{\Lambda}$, $LM =_{\beta\eta} LN$.

If π is a type preserving permutation of $X \subseteq \mathcal{O}$ then we obtain a permutation π_0 of $\mathfrak{F}_X$, just as in the case of $\mathfrak{F}$, which fixes $\mathfrak{F}_X$.

PROPOSITION 7. $\mathfrak{F}$ *is a P-structure.*

PROOF. Let $F \in \mathfrak{F}^\sigma$ be invariant and $\sigma = \sigma_1 \to (\cdots(\sigma_n \to 0)\cdots)$. By Lemma 1 there is a closed $\lambda x_1^{\sigma_1} \cdots x_n^{\sigma_n} M$ of type σ. Let $\lambda x_1^{\sigma_1} \cdots x_n^{\sigma_n} N \in \Lambda_{\mathcal{O}}$ be such that $F = (\lambda x_1^{\sigma_1} \cdots x_n^{\sigma_n} N)/\approx$. Define a substitution Θ with domain $= \mathcal{O}$ as follows; if

$$\tau = \underbrace{0 \to (\cdots(0}_{k} \to 0)\cdots)$$

then $\Theta x^\tau = \lambda z_1^0 \cdots z_k^0 M$ for $z_1 \cdots z_k$ not free in M. Put $L = \lambda x_1 \cdots x_n \Theta N$; we claim $F = L/\approx$. Suppose $U_1^{\sigma_1} \cdots U_n^{\sigma_n} \in \Lambda_{\mathcal{O}}$, and let t be the $\beta(\eta)$-normal form of $(\lambda x_1 \cdots x_n N)U_1 \cdots U_n$. By the invariance of F, no residual of an occurrence of a member of $\mathcal{O}$ in $\lambda x_1 \cdots x_n N$ exists in t, so $t =_{\beta\eta} LU_2 \cdots U_n$. Thus $F = L/\approx$.

Let X contain only type o variables and have $|X| \geq 2$.

PROPOSITION 8. *Let* $\sigma = \sigma_1 \to (\cdots (\sigma_n \to o) \cdots)$ *and* $M^\sigma, N^\sigma \in \bar{\Lambda}$. *Then the following are equivalent.*

(1) $M = N$ *is consistent with* $\beta\eta$*-conversion.*

(2) $\mathfrak{F}_X \models M = N$.

(3) *For all* $U_1^{\sigma_1} \cdots U_n^{\sigma_n} \in \Lambda_X$, $MU_1 \cdots U_n =_{\beta\eta} NU_1 \cdots U_n$.

PROOF. Obviously (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). If $MU_1 \cdots U_n \neq_{\beta\eta} NU_1 \cdots U_n$ then there are $x^0 \neq y^0$ s.t. $M = N \vdash_{\beta\eta} x = y$, so $M = N$ is not consistent with β_η-conversion.

PROPOSITION 9. $\mathfrak{F}_X$ *is a P-structure.*

PROOF. Just as before.

When $|X| = 2$, say $X = \{u^0, v^0\}$, we write $\mathfrak{M}$ for $\mathfrak{F}_X$. We now use the structures $\mathfrak{P}_\kappa$ and $\mathfrak{M}$ to answer some questions about syntactic λ-definability.

PROPOSITION 10. *There is no* σ *and* T^σ *satisfying the following condition. For all* τ *there is an* $L^{\tau\to(\tau\to\sigma)} \in \bar{\Lambda}$ *s.t. for all* $M^\tau, N^\tau \in \bar{\Lambda}$ *we have* $M =_{\beta\eta} N \Leftrightarrow LMN =_{\beta\eta} T$.

PROOF. If such σ and T^σ exist then, by Theorem 2, $\mathfrak{P}_{m_T}$ is complete for $\beta\eta$-conversion. This contradicts Theorem 4 of [2].

Now let $\mathbf{O}$, $\underline{sg}$, e_0, $\underline{0}$, $\underline{1}$ and Σ be as in [5, §§3, 3, 4, 3, 3, and 5 resp.]. Fix σ and let $M_1/\approx, \ldots, M_n/\approx$ be an irredundant list of the elements in $\mathfrak{M}^\sigma$. Select $U_1^{\sigma\to0} \cdots U_m^{\sigma\to0} \in \Lambda_X$, for $m = \binom{n}{2}$, such that for $1 \leq i \neq j \leq n$ there is a $1 \leq k \leq m$ with $U_kM_i \neq_{\beta\eta} U_kM_j$. Let y^0, $z^{0\to0}$ be new, and set $L_k^{\sigma\to\mathbf{0}} =_{\text{df}} \lambda x^\sigma \lambda xzy([y/u, (zy)/v]U_k)x$. Define $\delta^{\sigma\to(\sigma\to\mathbf{0})} = \lambda x^\sigma y^\sigma\, \underline{sg}\, \sum_{1\leq k\leq m} e_0(L_kx)\,(L_ky)$ (this is an abuse of the notation of [5] since m may not be a power of 2 but the meaning should be clear). We have, for $M^\sigma, N^\sigma \in \bar{\Lambda}$, $\delta MN =_{\beta\eta} \underline{0} \Leftrightarrow \mathfrak{M} \models M = N$ and $\delta MN =_{\beta\eta} \underline{1} \Leftrightarrow \mathfrak{M} \models M \neq N$.

PROPOSITION 11. *Let* $\mathfrak{A}$ *be a structure and* $N_1^\tau, N_2^\tau \in \bar{\Lambda}$, *where* $N_1 = N_2$ *is not consistent with* $\beta\eta$*-conversion. The following are equivalent*:

(1) *For each* σ *there is an* $L^{\sigma\to(\sigma\to\tau)} \in \bar{\Lambda}$ *s.t., for all* $M_1^\sigma, M_2^\sigma \in \bar{\Lambda}$, $\mathfrak{A} \models M_1 = M_2 \Leftrightarrow LM_1M_2 =_{\beta\eta} N_1$ *and* $\mathfrak{A} \models M_1 \neq M_2 \Leftrightarrow LM_1M_2 =_{\beta\eta} N_2$.

(2) *For all* $M, N \in \bar{\Lambda}$ $\mathfrak{A} \models M = N .\!\Leftrightarrow\!. M = N$ *is consistent with* $\beta\eta$*-conversion.*

PROOF. Suppose (1). We have $LM_1M_2 =_{\beta\eta} N_i \Leftrightarrow \mathfrak{M} \models LM_1M_2 = N_i$, for $1 \leq i \leq 2$. If $\mathfrak{M} \models M_1 = M_2$, since $LM_1M_1 =_{\beta\eta} N_1$, then $LM_1M_2 =_{\beta\eta} N_1$. Thus $\mathfrak{M} \models M_1 = M_2 \Rightarrow \mathfrak{A} \models M_1 = M_2$. Suppose (2). Let $N_1 = \lambda x_1 \cdots x_nU^0$ and $N_2 = \lambda x_1 \cdots x_nV^0$ be in long $\beta\eta$-normal form, and set $L = \lambda x^\sigma y^\sigma \lambda x_1 \cdots x_n(\delta xy(\lambda z^0 V)U)$, for x, y, z new. (1) follows easily.

As a final application we shall prove that the following question is decidable.

Given: $M_1, \ldots, M_n, N \in \Lambda$.

Question: Does N $\beta\eta$-convert to an applicative combination (see [1, p. 160]) of $M_1 \cdots M_n$?

(This question, is, of course, undecidable for untyped, even closed, terms.)

LEMMA 3. *The following problem is decidable.*

Given: $\tau = \sigma_1 \to (\cdots (\sigma_n \to \sigma) \cdots)$ *and* $F \in \mathfrak{P}_\kappa^\tau$.

Question: Is there a proper combinator [1, p. 161] M^τ *with* λ*-prefix* $\lambda x_1^{\sigma_1} \cdots x_n^{\sigma_n}$ *s.t.* $F = \mathrm{Val}_{\mathfrak{P}_m}(M)$?

PROOF. Define the length function l on conbinations of $x_1^{\sigma_1} \cdots x_n^{\sigma_n}$ as follows: $l(x_i) = 1$ and $l(x_iU_1 \cdots U_k) = 1 + \sum_{1\leq i\leq k} l(U_i)$. Enumerate the combinations of $x_1 \cdots x_n$ in finite blocks $B_1, \ldots, B_j, \ldots$ s.t. B_j consists of all combinations of length $= j$. Set $B =_{\text{df}} \bigcup_j B_j$ and $B(r) =_{\text{df}} \bigcup_{1 \le j \le r} B_j$. For M, $N \in B$, we write $M \equiv N$ if, for all assignments μ, $\mathfrak{P}_m \models M = N[\mu]$. Since the type of any member of B is a subtype of some σ_i, $B/\equiv$ is finite. So there is a p s.t. $(\forall q \ge p)$ $(\forall M \in B_q)$ $(\exists N \in B(p))$ $M \equiv N$. Let $\sigma_i = \tau_{i1} \to (\cdots (\tau_{ik_i} \to o) \cdots)$, for $1 \le i \le n$, and set $r = \max\{k_i : 1 \le i \le n\}$. Find p smallest such that $(\forall M \in B(r \cdot p + 1))(\exists N \in B(p))$ s.t. $M \equiv N$. We claim $(\forall M \in B)(\exists N \in B(p))$ $M \equiv N$. This is proved by induction on $l(M)$. Basis: $l(M) = 1$. Obvious. Induction step: $l(M) > 1$, say $M = xM_1 \cdots M_q$. Case: for some $1 \le i \le q$, $l(M_i) > p$. By hyp. ind. applied to M_i, $(\exists N \in B(p))$ $M_i \equiv N$. Let L result from M by replacing M_i by N, then $l(L) < l(M)$ and $L \equiv M$. By hyp. ind. applied to L, we get a $T \in B(p)$ s.t. $T \equiv L \equiv M$. Case: for all $1 \le i \le q$, $l(M_i) \le p$. Since $l(M) \le r \cdot p + 1$, this case follows from the choice of p. It follows that there is a proper combinator M^τ with λ-prefix $\lambda x_1^{\sigma_1} \cdots x_n^{\sigma_n}$ s.t. $F = \mathrm{Val}_{\mathfrak{P}_m}(M) \Leftrightarrow (\exists N^\sigma \in B(p))F = \mathrm{Val}_{\mathfrak{P}_m}(\lambda x_1^{\sigma_1} \cdots x_n^{\sigma_n} N)$.

Now suppose we are given $M_1^{\sigma_1} \cdots M_n^{\sigma_n} N^\sigma$ with free variables $x_1 \cdots x_k$. We must decide if there is a proper combinator M of type $\sigma_1 \to (\cdots (\sigma_n \to \sigma) \cdots)$, with λ-prefix $\lambda y_1^{\sigma_1} \cdots y_n^{\sigma_n}$, satisfying $(\lambda x \lambda x_1 \cdots x_k x M_1 \cdots M_n)M =_{\beta\eta} \lambda x_1 \cdots x_k N$. Let $m = m_{\lambda x_1 \cdots x_k N}$ be as given by Theorem 2. The existence of such an M can be determined by searching $\mathfrak{P}_m^{\sigma_1 \to (\cdots (\sigma_n \to \sigma) \cdots)}$ and applying Lemma 3.

§II. **Some consequences of Plotkin's conjecture.** A σ-term is a term of type $\subseteq \sigma$ in long $\beta\eta$-normal form and containing only free variables of type $\subseteq \sigma$. σ-tm. is the set of σ-terms. If $\mathfrak{A}$ is a prestructure, a σ-valuation, $V_{\mathfrak{A}}$, in $\mathfrak{A}$ is a map $V_{\mathfrak{A}}: \sigma\text{-tm.} \times Asg_{\mathfrak{A}} \to \mathfrak{A}(\sigma)$ satisfying conditions (i), (ii), and (iii) in the definition of Val in [2, p. 23].

PROPOSITION 12. *For prestructures $\mathfrak{A}$, there is a structure $\mathfrak{B}$ s.t. $\mathfrak{A}(\sigma) = \mathfrak{B}(\sigma)$.$\Leftrightarrow$. there is a σ-valuation in $\mathfrak{A}$.*

PROOF. $\Rightarrow$ is obvious. Suppose $V = V_{\mathfrak{A}}$ is a σ-valuation in $\mathfrak{A}$. Let $|\mathfrak{A}^0| = \kappa$, and let h be a partial homomorphism from $\mathfrak{P}_\kappa$ onto $\mathfrak{A}$, which is a bijection of ground domains. Let M be a σ-term, ν an assignment in $\mathfrak{A}$, and μ an assignment in $\mathfrak{P}_\kappa$ satisfying $h \cdot \mu = \nu$. It is clear that $h(\mathrm{Val}_{\mathfrak{P}_\kappa}(M, \mu)) = V(M, \nu)$. Let $\mathscr{H} = \Lambda_{\mathfrak{P}_\kappa}(h^{-1''}\, \mathfrak{A}(\sigma))$. We have, for $F, G \in h^{-1''}\, \mathfrak{A}(\sigma)$, $F \approx G \Leftrightarrow h(F) = h(G)$ and $(\mathscr{H})^E = \mathscr{H}$. Set $\mathfrak{B} = \mathscr{H}/\approx$. Then h^{-1} is an embedding of $\mathfrak{A}(\sigma)$ onto $\mathfrak{B}(\sigma)$.

For prestructures $\mathfrak{A}$ with $|\mathfrak{A}^0| = \kappa$ we shall make use of the following condition: $(*)$ There is a partial homomorphism h from $\mathfrak{P}_\kappa(\sigma)$ onto $\mathfrak{A}(\sigma)$ which is a bijection of ground domains s.t. for each assignment ν in $\mathfrak{A}$, each assignment μ in $\mathfrak{P}_\kappa(\sigma)$ satisfying $h(\mu(x^\tau)) = \nu(x^\tau)$ for all x^τ with $\tau \subseteq \sigma$, and each σ-term M, $h(\mathrm{Val}_{\mathfrak{P}_\kappa}(M, \mu))$ exists.

PROPOSITION 13. $(*) \Leftrightarrow$ *there is a structure $\mathfrak{B}$ s.t. $\mathfrak{A}(\sigma) = \mathfrak{B}(\sigma)$.*

PROOF. $\Leftarrow$ follows from Lemma 17 of [2]. Suppose $(*)$. Set $V(M, \nu) =_{\text{df}} h(\mathrm{Val}_{\mathfrak{P}_\kappa}(M, \mu))$. Then V is a well-defined σ-valuation in $\mathfrak{A}$. Now apply Proposition 12.

PROPOSITION 14. *Plotkin's conjecture $\Rightarrow$ the following question is decidable.*

Given: $\mathfrak{A}(\sigma)$ of a hereditarily finite prestructure $\mathfrak{A}$.

Question: Is there a structure $\mathfrak{B}$ s.t. $\mathfrak{A}(\sigma) = \mathfrak{B}(\sigma)$?

PROOF. Given $\mathfrak{A}(\sigma)$, for $\tau \subseteq \sigma$, let $\#\tau =_{\text{df}} |\mathfrak{A}^\tau|$. Let h be a partial homomorphism from $\mathfrak{P}_{\#0}(\sigma)$ onto $\mathfrak{A}(\sigma)$ which is a bijection of ground domains. If there are M, and ν satisfying the hypotheses of $(*)$ for h but not the conclusion, then there are similar M, ν, and μ s.t. M has at most $\#\tau$ free variables of type τ. Thus we can effectively find a finite set of types $\mathscr{F}$ s.t. h satisfies $(*) \Leftrightarrow$ dom(h) is closed under all λ-definable functions of type $\in \mathscr{F}$. Since there are only finitely many h to check, we get the proposition.

The generalized Plotkin conjecture is the statement that the following question is decidable.

Given: $\mathfrak{A}(\sigma)$ for $\mathfrak{A}$ a hereditarily finite structure and $F \in \mathfrak{A}^\sigma$.

Question: Is F λ-definable?

Note that if Plotkin's conjecture is true, then we can recursively restrict the inputs for the generalized question to $\mathfrak{A}(\sigma)$ for $\mathfrak{A}$ a hereditarily finite structure (Proposition 14). Thus the generalized conjecture is perfectly general.

PROPOSITION 15. *Plotkin's conjecture $\Rightarrow$ the generalized Plotkin conjecture.*

PROOF. Given $\mathfrak{A}(\sigma)$ and $F \in \mathfrak{A}^\sigma$, F is λ-definable.$\Leftrightarrow$.there is a partial homomorphism $h\colon \mathfrak{P}_{|\mathfrak{A}^0|}(\sigma) \rightarrow^{\text{onto}} \mathfrak{A}(\sigma)$ s.t. $h^{-1}(F)$ contains a λ-definable element (Lemmas 15 and 17 of [2]).

PROPOSITION 16. *The following are equivalent.*

(1) *The following question is decidable.*

Given: $\mathfrak{A}(\sigma)$ *for* $\mathfrak{A}$ *a hereditarily finite prestructure.*

Question: *Is there a structure* $\mathfrak{B}$ *s.t.* $\mathfrak{A}(\sigma) = \mathfrak{B}(\sigma)$?

(2) *The following question is semidecidable.*

Given: *a sentence* $A \in \mathfrak{L}$.

Question: *Does* A *have a hereditarily finite model*?

PROOF. (1) $\Rightarrow$ (2) is obvious. Suppose (2). By the proof of Proposition 14, $\{\mathfrak{A}(\sigma)$: there is no structure $\mathfrak{B}$ s.t. $\mathfrak{A}(\sigma) = \mathfrak{B}(\sigma)\}$ is recursively enumerable. Given $\mathfrak{A}(\sigma)$ it is easy to construct effectively an $\mathfrak{L}$ sentence A s.t., for all structures $\mathfrak{B}$, $\mathfrak{B} \models A \Rightarrow \mathfrak{B}(\sigma)$ is isomorphic to $\mathfrak{A}(\sigma)$. Thus $\{\mathfrak{A}(\sigma)$: there is a structure $\mathfrak{B}$ s.t. $\mathfrak{A}(\sigma) = \mathfrak{B}(\sigma)\}$ is r.e.

We obtain the

THEOREM 4. *Plotkin's conjecture $\Rightarrow$ the following question is semidecidable.*

Given: *a sentence* $A \in \mathfrak{L}$.

Question: *Does* A *have a hereditarily finite model*?

We conclude with a special case of the (generally unsolvable) problem of unification, the range question.

Given: $M^{\sigma_1 \to (\cdots(\sigma_n \to \sigma)\cdots)}$ and N^σ.

Question: Are there $N_1^{\sigma_1} \cdots N_n^{\sigma_n} \in \bar{\Lambda}$ such that $MN_1 \cdots N_n =_{\beta\eta} N$?

Note that w.l.o.g. we may assume that M and N are closed. Also, the corresponding question for sequences of pairs of terms can be reduced to the range question.

THEOREM 5. *Plotkin's conjecture $\Rightarrow$ the range question is decidable.*

Proof. Assuming Plotkin's conjecture, by Theorem 2, the range question can be decided by searching through $\mathfrak{P}_{m_N}^{\sigma_1}, \ldots, \mathfrak{P}_{m_N}^{\sigma_n}$.

§III. **Complexity questions.**

LEMMA 4. *Suppose M is in long $\beta\eta$-normal form, has type with no negative occurrence of a binary type, and only free variables with type having no positive occurrence of a binary type. Then M contains at most one occurrence of a free variable of type* 0.

PROOF. Routine induction on M.

LEMMA 5. *If there is a closed term of type σ, and σ contains no positive occurrence* *of a binary type, then there is an* $M^\sigma \in \bar{\Lambda}$ *s.t., for all* $N^\sigma \in \bar{\Lambda}$, $\mathfrak{M} \models M = N$.

PROOF. We have $\sigma = \tau \to 0$ where τ contains no negative occurrence of a binary type. Let M^σ, $N^\sigma \in \bar{\Lambda}$ and $U^\tau \in \Lambda_{\{u^0, v^0\}}$. Then $MU =_{\beta\eta}$ the free variable in $U =_{\beta\eta} NU$. Thus $\mathfrak{M} \models M = N$.

LEMMA 6. *If there is a closed term of type* σ, *and* σ *contains a positive occurrence of a binary type, then there are* M^σ, $N^\sigma \in \bar{\Lambda}$ *such that* $\mathfrak{M} \models M \neq N$.

PROOF. By induction on the definition of positive and negative occurrence.

Basis: $\sigma = \sigma_1 \to (\sigma_2 \to \sigma_3)$. Let $M = \lambda x_1 \cdots x_n x_i M_1 \cdots M_m \in \bar{\Lambda}$ be in long $\beta\eta$-normal form and of type σ. Since $n \geq 2$, we can choose $1 \leq j \neq i \leq n$ and set $N = \lambda x_1 \cdots x_n x_j(\lambda \mathbf{y}_1 x_i M_1 \cdots M_m) \cdots (\lambda \mathbf{y}_k x_i M_1 \cdots M_m)$, for $\mathbf{y}_1 \cdots \mathbf{y}_k$ sequences of new variables of appropriate type. Clearly $\mathfrak{M} \models M \neq N$. Induction step: $\sigma = \sigma_1 \to 0$. Let $\sigma_1 = \tau_1 \to (\cdots(\tau_l \to 0)\cdots)$ Then some τ_i contains a positive occurrence of a binary type. Moreover, for each $1 \leq j \leq k$ there is a closed term of type τ_j, so by hyp. ind. there are $N_1^{\tau_i}$, $N_2^{\tau_i} \in \bar{\Lambda}$ s.t. $\mathfrak{M} \models N_1 \neq N_2$. Let $\lambda xx M_1 \cdots M_l \in \bar{\Lambda}$ have type σ and be in long $\beta\eta$-normal form; set $M =_{\mathrm{df}} \lambda xx M_1 \cdots M_{i-1} N_1 M_i \cdots M_l$ and $N =_{\mathrm{df}} \lambda xx M_1 \cdots M_{i-1} N_2 M_i \cdots M_l$. We have $\mathfrak{M} \models M \neq N$.

THEOREM 6. (1) *Let* $\mathscr{E}$ *be a set of equations between closed terms and* σ *a type satisfying*

(a) $M =_\beta N \Rightarrow M = N \in \mathscr{E}$,

(b) $M = N \in \mathscr{E} \Rightarrow M = N$ *is consistent with* $\beta\eta$*-conversion*,

(c) *there is a closed term of type* σ, *and*

(d) σ *contains a positive occurrence of a binary type.*

Then $\exists M^\sigma \in \bar{\Lambda}$ *such that the problem of determining for arbitrary* $N^\sigma \in \bar{\Lambda}$ *whether* $M = N \in \mathscr{E}$ *cannot be solved in elementary time on a Turing machine.*

(2) *Let* $\mathscr{S} \subseteq \bar{\Lambda}$ *satisfy*

(a) $M =_\beta N \Rightarrow (M \in \mathscr{S} \Leftrightarrow N \in \mathscr{S})$ *for* $M, N \in \bar{\Lambda}$, *and*

(b) $\mathscr{S}$ *can be decided on elementary time.*

Then for each σ, $\mathscr{S}$ *contains none or all of the closed terms of type* σ.

PROOF. (1). Let * and Ω be as in [5, §§4 and 2 resp.]. By Lemma 6 there are closed $M_0^\sigma = \lambda x_1 \cdots x_n x_i M_1 \cdots M_m$ and $N_0^\sigma = \lambda x_1 \cdots x_n x_j N_1 \cdots N_k$, in long $\beta\eta$-normal form s.t. $\mathfrak{M} \models M_0 \neq N_0$. Let $L =_{\mathrm{df}} \lambda y \lambda x_1 \cdots x_n y(\lambda z^0 x_j N_1 \cdots N_k) \cdot (x_i M_1 \cdots M_m)$ for y, z new. Then for sentences $A \in \Omega$, $A^* =_\beta \underline{0} \Leftrightarrow LA^* =_\beta M_0$ and $A^* =_\beta \underline{1} \Leftrightarrow LA^* =_\beta N_0$. By (b), not both $LA^* \neq_\beta M_0$ and $LA^* = M_0 \in \mathscr{E}$, so $LA^* =_\beta M_0 \Leftrightarrow LA^* = M_0 \in \mathscr{E}$. (1) now follows from Theorem 1 of [5]. (2) follows easily from the construction in (1).

Note that by Lemma 6, (1)(c) is essentially best possible. Also (1)(b) is quite weak; it means only that $M^\sigma = N^\sigma \in \mathscr{E} \Rightarrow$ for all $U^{\sigma \to 0} \in \Lambda_{\{u^0, v^0\}}$ $UM =_\beta UN$.

Appendix. The argument of [2] for Lemma 24 (p. 31) appears incorrect, since x and y may not have the same type. Here we repair the argument.

LEMMA. *Let* $\Phi = \exists x \exists y(s = t)$. *Then there is an existential* Ψ *with the same free variables s.t., whenever* $|B| \geq 2$, $T_B \models \Phi \leftrightarrow \Psi$.

PROOF. Let the type of $x = \sigma_1 \to (\cdots (\sigma_n \to 0) \cdots)$ and the type of $y = \tau_1 \to (\cdots (\tau_m \to 0) \cdots)$. Let $\sigma_i = \sigma_{i1} \to (\cdots (\sigma_{ik_i} \to 0) \cdots)$ and $\tau_j = \tau_{j1} \to (\cdots(\tau_{jl_j} \to 0)\cdots)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Select $\mathbf{u}_i = u_{i1}^{\sigma_{i1}} \cdots u_{ik_i}^{\sigma_{ik_i}}$ and $\mathbf{v}_j = v_{j1}^{\tau_{j1}} \cdots v_{jl_j}^{\tau_{jl_j}}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, and z^0 all new. Finally choose w

new of type $(0 \to (0 \to 0)) \to (\sigma_1 \to (\cdots (\sigma_n \to (\tau_1 \to (\cdots (\tau_m \to 0) \cdots))) \cdots))$. Let $p =_{\mathrm{df}} \lambda x_1^{\sigma_1} \cdots x_n^{\sigma_n} w0x_1 \cdots x_n(\lambda \mathbf{v}_1 z) \cdots (\lambda \mathbf{v}_m z)$ and $q = \lambda y_1^{\tau_1} \cdots y_m^{\tau_m} w1(\lambda \mathbf{u}_1 z) \cdots (\lambda \mathbf{u}_n z)y_1 \cdots y_m$ for $x_1 \cdots x_n y_1 \cdots y_m$ all new. Put $\chi =_{\mathrm{df}} s_{pq}^{xy} = t_{pq}^{sy}$. It is easy to see that $T_B \models \Phi \leftrightarrow \forall z \exists w \chi \leftrightarrow \exists z \exists w \chi$. Now use the argument in the last two sentences of Lemma 22.

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