Mean-Payoff Games on Timed Automata

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Abstract

Mean-payoff games on timed automata are played on the infinite weighted graph of configurations of priced timed automata between two players—Player Min and Player Max—by moving a token along the states of the graph to form an infinite run. The goal of Player Min is to minimize the limit average weight of the run, while the goal of the Player Max is the opposite. Brenguier, Cassez, and Raskin recently studied a variation of these games and showed that mean-payoff games are undecidable for timed automata with five or more clocks. We refine this result by proving the undecidability of mean-payoff games with three clocks. On a positive side, we show the decidability of mean-payoff games on one-clock timed automata with binary price-rates. A key contribution of this paper is the application of dynamic programming based proof techniques applied in the context of average reward optimization on an uncountable state and action space.

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1 Introduction

The classical mean-payoff games [24, 13, 15, 4] are two-player zero-sum games that are played on weighted finite graphs, where two players—Max and Min—take turn to move a token along the edges of the graph to jointly construct an infinite play. The objectives of the players Max and Min are to respectively maximize and minimize the limit average reward associated with the play. Mean-payoff games are well-studied in the context of optimal controller synthesis in the framework of Ramadge-Wonham [22], where the goal of the game is to find a control strategy that maximises the average reward earned during the evolution of the system. Mean-payoff games enjoy a special status in verification, since μ -calculus model checking and parity games can be reduced in polynomial-time to solving mean-payoff games. Mean-payoff objectives can also be considered as quantitative extensions [16] of classical Büchi objectives, where we are interested in the limit-average share of occurrences of accepting states rather than merely in whether or not infinitely many accepting states occur. For a broader discussion on quantitative verification, in general, and the transition from the classical qualitative to the modern quantitative interpretation of deterministic Büchi automata, we refer the reader to Henzinger's excellent survey [16].

We study mean-payoff games played on an infinite configuration graph of timed automata. Asarin and Maler [3] were the first to study games on timed automata and they gave an algorithm to solve timed games with reachability time objective. Their work was later generalized and improved upon by Alur et al. [1] and Bouyer et al. [8]. Bouyer et al. [7, 5] also studied the more difficult average payoffs, but only in the context of scheduling, which in game-theoretic terminology corresponds to 1-player games. However, they left the problem of proving decidability of 2-player average reward games on priced timed automata open. Jurdziński and Trivedi [19] proved the decidability of the special case of average time games

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where all locations have unit costs. More recently, mean-payoff games on timed automata have been studied by Brenguier, Cassez and Raskin [10] where they consider average payoff per time-unit. Using the undecidability of energy games [9], they showed undecidability of mean-payoff games on weighted timed games with five or more clocks. They also gave a semi-algorithm to solve cycle-forming games on timed automata and characterized the conditions under which a solution of these games gives a solution for mean-payoff games.

On the positive side, we characterize general conditions under which dynamic programming based techniques can be used to solve the mean-payoff games on timed automata. As a proof-of-concept, we consider one-clock binary-priced timed games, and prove the decidability of mean-payoff games for this subclass. Our decidability result can be considered as the average-payoff analog of the decidability result by Brihaye et al. [11] for reachability-price games on timed automata. We strengthen the known undecidability results for mean-payoff games on timed automata in three ways: i) we show that the mean-payoff games over priced timed games is undecidable for timed games with only three clocks; ii) secondly, we show that undecidability can be achieved with binary price-rates; and finally, iii) our undecidability results are applicable for problems where the average payoff is considered per move as well as for problems when it is defined per time-unit.

Howard [17, 21] introduced gain and bias optimality equations to characterize optimal average on one-player finite game arenas. Gain and bias optimality equations based characterization has been extended to two-player game arenas [14] as well as many subclasses of uncountable state and action spaces [12, 6]. The work of Bouyer et al. [6] is perhaps the closest to our approach—they extended optimality equations approach to solve games on hybrid automata with certain strong reset assumption that requires all continuous variables to be reset at each transition, which in the case of timed automata is akin to requiring all clocks to be reset at each transition. To the best of our knowledge, the exact decidability for timed games does not immediately follow from any previously known results.

Howard's Optimality equations requires two variable per state: the gain of the state and the bias of the state. Informally speaking, the gain of a state corresponds to the optimal mean-payoff for games starting from that state, while the bias corresponds to the limit of transient sum of step-wise deviations from the optimal average. Hence, intuitively at a given point in a game, both players would prefer to first optimize the gain, and then choose to optimize bias among choices with equal gains. We give general conditions under which a solution of gain-bias equations for a finitary abstraction of timed games can provide a solution of gain-bias equations for the original timed game. For this purpose, we exploit a region-graph like abstraction of timed automata [18] called the boundary region abstraction (BRA). Our key contribution is the theorem that states that every solution of gain-bias optimality equations for boundary region abstraction carries over to the original timed game, as long as for every region, the gain values are constant and the bias values are affine.

The paper is organized in the following manner. In Section 2 we describe mean-payoff games and introduce the notions of gain and bias optimality equations. This section also introduces mean-payoff games over timed automata and states the key results of the paper. Section 3 introduces the boundary region abstraction for timed automata and characterizes the conditions under which the solution of a game played over the boundary region abstraction can be lifted to a solution of mean payoff game over priced timed automata. In Section 4 we present the strategy improvement algorithm to solve optimality equations for mean-payoff games played over boundary region abstraction and connect them to solution of optimality equations over corresponding timed automata. Finally, Section 5 sketches the undecidability of mean-payoff games for binary-priced timed automata with three clocks.

2 Mean-Payoff Games on Timed Automata

We begin this section by introducing mean-payoff games on graphs with uncountably infinite vertices and edges, and show how, and under what conditions, gain-bias optimality equations characterize the value of mean-payoff games. We then set-up mean-payoff games for timed automata and state our key contributions.

2.1 Mean-Payoff Games

▶ Definition 1 (Turn-Based Game Arena). A game arena Γ is a tuple $(S, S_{\text{Min}}, S_{\text{Max}}, A, T, \pi)$ where S is a (potentially uncountable) set of states partitioned between sets S_{Min} and S_{Max} of states controlled by Player Min and Player Max, respectively; A is a (potentially uncountable) set of actions; $T: S \times A \to S$ is a partial function called transition function; and $\pi: S \times A \to \mathbb{R}$ is a partial function called price function.

We say that a game arena is *finite* if both S and A are finite. For any state $s \in S$, we let A(s) denote the set of actions available in s, i.e., the actions $a \in A$ for which T(s,a) and $\pi(s,a)$ are defined. A transition of a game arena is a tuple $(s,a,s') \in S \times A \times S$ such that s' = T(s,a) and we write $s \xrightarrow{a} s'$. A finite play starting at a state s_0 is a sequence of transitions $\langle s_0, a_1, s_1, a_2, \ldots, s_n \rangle \in S \times (A \times S)^*$ such that for all $0 \le i < n$ we have that $s_i \xrightarrow{a_{i+1}} s_{i+1}$ is a transition. For a finite play $\rho = \langle s_0, a_1, \ldots, s_n \rangle$ we write $\text{Last}(\rho)$ for the final state of ρ , here $\text{Last}(\rho) = s_n$. The concept of an infinite play $\langle s_0, a_1, s_1, \ldots \rangle$ is defined in an analogous way. We write Runs(s) and $\text{Runs}_{\text{fin}}(s)$ for the set of plays and the set of finite plays starting at $s \in S$ respectively.

A strategy of Player Min is a function $\mu: \operatorname{Runs_{fin}} \to A$ such that $\mu(\rho) \in A(\operatorname{Last}(\rho))$ for all finite plays $\rho \in \operatorname{Runs_{fin}}$, i.e. for any finite play, a strategy of Min returns an action available to Min in the last state of the play. A strategy χ of Max is defined analogously and we let $\Sigma_{\operatorname{Min}}$ and $\Sigma_{\operatorname{Max}}$ denote the sets of strategies of Min and Max, respectively. A strategy σ is positional if $\operatorname{Last}(\rho) = \operatorname{Last}(\rho')$ implies $\sigma(\rho) = \sigma(\rho')$ for all $\rho, \rho' \in \operatorname{Runs_{fin}}$. This allows us to represent a positional strategy as a function in $[S \to A]$. Let Π_{Min} and Π_{Max} denote the set of positional strategies of Min and Max, respectively. For any state s and strategy pair $(\mu, \chi) \in \Sigma_{\operatorname{Min}} \times \Sigma_{\operatorname{Max}}$, let $\operatorname{Run}(s, \mu, \chi)$ denote the unique infinite play $\langle s_0, a_1, s_1, \ldots \rangle$ in which Min and Max play according to μ and χ , respectively, i.e. for all $i \geqslant 0$ we have that $s_i \in S_{\operatorname{Min}}$ implies $a_{i+1} = \mu(\langle s_0, a_1, \ldots, s_i \rangle)$ and $s_i \in S_{\operatorname{Max}}$ implies $a_{i+1} = \chi(\langle s_0, a_1, \ldots, s_i \rangle)$.

In a mean-payoff game on a game arena, players Min and Max move a token along the transitions indefinitely thus forming an infinite play $\rho = \langle s_0, a_1, s_1, \ldots \rangle$ in the game graph. The goal of player Min is to minimize $\mathcal{A}_{\mathrm{Min}}(\rho) = \limsup_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} \pi(s_i, a_{i+1})$ and the goal of player Max is to maximize $\mathcal{A}_{\mathrm{Max}}(\rho) = \liminf_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} \pi(s_i, a_{i+1})$. The upper value $\mathrm{Val}^*(s)$ and the lower value $\mathrm{Val}_*(s)$ of a state $s \in S$ are defined as:

$$\operatorname{Val}^*(s) = \inf_{\mu \in \Sigma_{\operatorname{Min}}} \sup_{\chi \in \Sigma_{\operatorname{Max}}} \mathcal{A}_{\operatorname{Min}}(\operatorname{Run}(s,\mu,\chi)) \text{ and } \operatorname{Val}_*(s) = \sup_{\chi \in \Sigma_{\operatorname{Max}}} \inf_{\mu \in \Sigma_{\operatorname{Min}}} \mathcal{A}_{\operatorname{Max}}(\operatorname{Run}(s,\mu,\chi))$$

respectively. It is always the case that $\operatorname{Val}_*(s) \leqslant \operatorname{Val}^*(s)$. A mean-payoff game is called determined if for every state $s \in S$ we have that $\operatorname{Val}_*(s) = \operatorname{Val}^*(s)$. Then, we write $\operatorname{Val}(s)$ for this number and we call it the value of the mean-payoff game at state s. We say that a game is positionally-determined if for every $\varepsilon > 0$ we have strategies $\mu_{\varepsilon} \in \Pi_{\operatorname{Min}}$ and $\chi_{\varepsilon} \in \Pi_{\operatorname{Max}}$ such that for every initial state $s \in S$, we have that

$$\operatorname{Val}_*(s) - \varepsilon \leqslant \inf_{\mu' \in \Sigma_{\operatorname{Min}}} \mathcal{A}_{\operatorname{Max}}(\operatorname{Run}(s, \mu', \chi_{\varepsilon})) \text{ and } \operatorname{Val}^*(s) + \varepsilon \geqslant \sup_{\chi' \in \Sigma_{\operatorname{Max}}} \mathcal{A}_{\operatorname{Min}}(\operatorname{Run}(s, \mu_{\varepsilon}, \chi')).$$

For a given ε we call each such strategy an ε -optimal strategy for the respective player.

Given two functions $G: S \to \mathbb{R}$ (gain) and $B: S \to \mathbb{R}$ (bias), we say that (G, B) is a solution to the optimality equations for mean-payoff game on $\Gamma = (S, S_{\text{Min}}, S_{\text{Max}}, A, T, \pi)$, denoted $(G, B) \models \text{Opt}(\Gamma)$ if

$$\begin{split} G(s) &= \begin{cases} \sup_{a \in A(s)} \{G(s') \, : \, s \xrightarrow{a} s'\} & \text{if } s \in S_{\text{Max}} \\ \inf_{a \in A(s)} \{G(s') \, : \, s \xrightarrow{a} s'\} & \text{if } s \in S_{\text{Min}}. \end{cases} \\ B(s) &= \begin{cases} \sup_{a \in A(s)} \{\pi(s, a) - G(s) + B(s') \, : \, s \xrightarrow{a} s' \text{ and } G(s) = G(s')\} & \text{if } s \in S_{\text{Max}} \\ \inf_{a \in A(s)} \{\pi(s, a) - G(s) + B(s') \, : \, s \xrightarrow{a} s' \text{ and } G(s) = G(s')\} & \text{if } s \in S_{\text{Min}}. \end{cases} \end{split}$$

We prove the following theorem connecting a solution of the optimality equations with mean-payoff games. We exploit this theorem to solve mean-payoff games on timed automata.

▶ Theorem 2. If there exists a function $G: S \to \mathbb{R}$ with finite image and a function $B: S \to \mathbb{R}$ with bounded image such that $(G, B) \models Opt(\Gamma)$ then for every state $s \in S$, we have that G(s) = Val(s) and for every $\varepsilon > 0$ both players have positional ε -optimal strategies.

Proof. Assume that we are given the functions $G: S \to \mathbb{R}$ with finite image and $B: S \to \mathbb{R}$ with bounded image such that $(G, B) \models \operatorname{Opt}(\Gamma)$. In order to prove the result we show, for every $\varepsilon > 0$, the existence of positional strategies μ_{ε} and χ_{ε} such that

$$G(s) - \varepsilon \leqslant \inf_{\mu' \in \Sigma_{\text{Min}}} \mathcal{A}_{\text{Max}}(\text{Run}(s, \mu', \chi_{\varepsilon})) \text{ and } G(s) + \varepsilon \geqslant \sup_{\chi' \in \Sigma_{\text{Max}}} \mathcal{A}_{\text{Min}}(\text{Run}(s, \mu_{\varepsilon}, \chi')).$$

The proof is in two parts.

■ Given $\varepsilon > 0$ we compute the positional strategy $\mu_{\varepsilon} \in \Pi_{\text{Min}}$ satisfying the following conditions: $\mu_{\varepsilon}(s) = a$ if

$$G(s) = G(s') \tag{1}$$

$$B(s) \geqslant \pi(s,a) - G(s) + B(s') - \varepsilon,$$
 (2)

where $s \xrightarrow{a} s'$. Notice that it is always possible to find such strategy since (G, B) satisfies optimality equations and G is finite image.

Now consider an arbitrary strategy $\chi \in \Sigma_{\text{Max}}$ and consider the run $\text{Run}(s, \mu_{\varepsilon}, \chi) = \langle s_0, a_1, s_1, \ldots, s_n, \ldots \rangle$. Notice that for every $i \geq 0$ we have that $G(s_i) \geq G(s_{i+1})$ if $s_i \in S_{\text{Max}}$ and $G(s_i) = G(s_{i+1})$ if $s_i \in S_{\text{Min}}$. Hence $G(s_0), G(s_1), \ldots$ is a non-increasing sequence. Since G is finite image, the sequence eventually becomes constant. Assume that for $i \geq N$ we have that $G(s_i) = g$. Now notice that for all $i \geq N$ we have that $B(s_i) \geq \pi(s_i, a_{i+1}) - g + B(s_{i+1})$ if $s_i \in S_{\text{Max}}$ and $B(s_i) \geq \pi(s_i, a_{i+1}) - g + B(s_{i+1}) - \varepsilon$ if $s_i \in S_{\text{Min}}$. Summing these equations sidewise form i = N to N + k we have that $B(s_N) \geq \sum_{i=N}^{N+k} \pi(s_i, a_{i+1}) - (k+1) \cdot g + B(s_{N+k+1}) - (k+1) \cdot \varepsilon$. Rearranging, we get

$$g \geqslant \frac{1}{k+1} \sum_{i=N}^{N+k} \pi(s_i, a_{i+1}) + \frac{1}{k+1} (B(s_{N+k+1}) - B(s_N)) - \varepsilon.$$

Hence
$$g \geqslant \limsup_{k \to \infty} \frac{1}{k+1} \sum_{i=N}^{N+k} \pi(s_i, a_{i+1}) + \limsup_{k \to \infty} \frac{1}{k+1} (B(s_{N+k+1}) - B(s_N)) - \varepsilon$$

$$= \limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \pi(s_i, a_{i+1}) - \varepsilon$$

Hence $G(s) + \varepsilon \geqslant \mathcal{A}_{Min}(Run(s, \mu_{\varepsilon}, \chi)).$

Since χ is an arbitrary strategy in Σ_{Max} , we have $G(s) + \varepsilon \geqslant \sup_{\chi' \in \Sigma_{\text{Max}}} \mathcal{A}_{\text{Min}}(\text{Run}(s, \mu_{\varepsilon}, \chi'))$.

This part is analogous to the first part of the proof and is omitted.

The proof is now complete.

2.2 Timed Automata

Priced Timed Game Arenas (PTGAs) extend classical timed automata [2] with a partition of the actions between two players Min and Max. Before we present the syntax and semantics of PTGAs, we need to introduce the concept of clock variables and related notions.

Clocks. Let \mathcal{X} be a finite set of clocks. A clock valuation on \mathcal{X} is a function $\nu: \mathcal{X} \to \mathbb{R}_{\geqslant 0}$ and we write $V(\mathcal{X})$ (or just V when \mathcal{X} is clear from the context) for the set of clock valuations. Abusing notation, we also treat a valuation ν as a point in $(\mathbb{R}_{\geqslant 0})^{|\mathcal{X}|}$. Let $\mathbf{0}$ denote the clock valuation that assigns 0 to all clocks. If $\nu \in V$ and $t \in \mathbb{R}_{\geqslant 0}$ then we write $\nu + t$ for the clock valuation defined by $(\nu + t)(c) = \nu(c) + t$ for all $c \in \mathcal{X}$. For $C \subseteq \mathcal{X}$, we write ν for the valuation where ν equals 0 if $c \in C$ and ν otherwise. For $\mathcal{X} \subseteq V(\mathcal{X})$, we write $\overline{\mathcal{X}}$ for the smallest closed set in V containing X. Although clocks are usually allowed to take arbitrary non-negative values, for notational convenience we assume that there is a $K \in \mathbb{N}$ such that for every $c \in \mathcal{X}$ we have $\nu(c) \leqslant K$.

Clock Constraints. A clock constraint over \mathcal{X} with upper bound $K \in \mathbb{N}$ is a conjunction of simple constraints of the form $c \bowtie i$ or $c-c' \bowtie i$, where $c, c' \in \mathcal{X}$, $i \in \mathbb{N}$, $i \leqslant K$, and $\bowtie \in \{<,>,=,\leqslant,\geqslant\}$. For $\nu \in V(\mathcal{X})$ and $K \in \mathbb{N}$, let $\mathrm{CC}(\nu,K)$ be the set of clock constraints with upper bound K which hold in ν , i.e. those constraints that resolve to true after substituting each occurrence of a clock x with $\nu(x)$.

Regions and Zones. Every clock region is an equivalence class of the indistinguishability-by-clock-constraints relation. For a given set of clocks \mathcal{X} and upper bound $K \in \mathbb{N}$ on clock constraints, a clock region is a maximal set $\zeta \subseteq V(\mathcal{X})$ such that $CC(\nu, K) = CC(\nu', K)$ for all $\nu, \nu' \in \zeta$. For the set of clocks \mathcal{X} and upper bound K we write $\mathcal{R}(\mathcal{X}, K)$ for the corresponding finite set of clock regions. We write $[\nu]$ for the clock region of ν . A clock zone is a convex set of clock valuations that satisfies constraints of the form $\gamma ::= c_1 \bowtie k \mid c_1 - c_2 \bowtie k \mid \gamma \wedge \gamma$, $k \in \mathbb{N}, c_1, c_2 \in \mathcal{X}$ and $\bowtie \in \{\leq, <, =, >, \geq\}$. We write $\mathcal{Z}(\mathcal{X}, K)$ for the set of clock zones over the set of clocks \mathcal{X} and upper bound K. When \mathcal{X} and K are clear from the context we write \mathcal{R} and \mathcal{Z} for the set of regions and zones. In this paper we fix a positive integer K, and work with K-bounded clocks and clock constraints.

2.3 Priced Timed Game Arena: Syntax and Semantics

▶ **Definition 3.** A priced timed game arena is a tuple $T=(L_{\min}, L_{\max}, Act, \mathcal{X}, Inv, E, \rho, \delta, p)$ where L_{\min} and L_{\max} are sets of *locations* controlled by Player Min and Player Max and we write $L = L_{\min} \cup L_{\max}$; Act is a finite set of *actions*; \mathcal{X} is a finite set of *clocks*; $Inv: L \to \mathcal{Z}$ is an *invariant condition*; $E: L \times Act \to \mathcal{Z}$ is an *action enabledness function*; $\rho: Act \to 2^C$ is a *clock reset function*; $\delta: L \times Act \to L$ is a *transition function*; and $p: L \cup L \times Act \to \mathbb{R}$ is a *price information function*. A PTGA is binary-priced when $p(\ell) \in \{0, 1\}$ for all $\ell \in L$.

When we consider a PTGA as an input of an algorithm, its size is understood as the sum of the sizes of encodings of L, \mathcal{X} , Inv, Act, E, ρ , δ and p. We draw the states of Min players as circles, while states of Max player as boxes.

Let $T = (L_{\min}, L_{\max}, Act, \mathcal{X}, Inv, E, \rho, \delta, p)$ be a PTGA. A configuration of a PTGA is a pair (ℓ, ν) , where ℓ is a location and ν a clock valuation such that $\nu \in Inv(\ell)$. For any $t \in \mathbb{R}_{\geq 0}$, we let $(\ell, \nu) + t$ equal the configuration $(\ell, \nu + t)$. In a configuration (ℓ, ν) , a timed action (time-action pair) (t, a) is available if and only if the invariant condition $Inv(\ell)$ is continuously satisfied while t time units elapse, and a is enabled (i.e. the enabling condition $E(\ell, a)$ is satisfied) after t time units have elapsed. Furthermore, if the timed action (t, a) is performed, then the next configuration is determined by the transition relation δ and the reset function ρ , i.e. the clocks in $\rho(a)$ are reset and we move to the location $\delta(\ell, a)$.

A game on a PTGA starts in an initial configuration $(\ell, \nu) \in L \times V$ and players Min and Max construct an infinite play by taking turns to choose available timed actions (t, a) whenever the current location is controlled by them and the price $p(\ell) \cdot t + p(\ell, a)$ is paid to the Max by player Min. Formally, PTGA semantics is given as a game arena.

- ▶ **Definition 4** (PTGA Semantics). Let $T = (L_{Min}, L_{Max}, Act, \mathcal{X}, Inv, E, \rho, \delta, p)$ be a PTGA. The semantics of T is given by game arena $[\![T]\!] = (S, S_{Min}, S_{Max}, A, T, \pi)$ where
- $S \subseteq L \times V$ is the set of states such that $(\ell, \nu) \in S$ if and only if $\nu \in Inv(\ell)$;
- $(\ell, \nu) \in S_{\text{Min}}$ (or $(\ell, \nu) \in S_{\text{Max}}$) if $(\ell, \nu) \in S$ and $\ell \in L_{\text{Min}}$ (or $\ell \in L_{\text{Max}}$, respectively).
- $A = Act \times \mathbb{R}_{\geq 0}$ is the set of *timed actions*;
- $T: S \times Act \to S$ is the transition function such that for $(\ell, \nu) \in S$ and $(a, t) \in Act$, we have $T((\ell, \nu), (a, t)) = (\ell', \nu')$ if and only if
- $\nu + t' \in Inv(\ell) \text{ for all } t' \in [0, t]; \ \nu + t \in E(\ell, a); \ (\ell', \nu') \in S, \ \delta(\ell, a) = \ell', \ (\nu + t)_{\rho(a)} = \nu'.$
- $\pi: S \times Act \to \mathbb{R}$ is the reward function where $\pi((\ell, \nu), (a, t)) = p(\ell, a) + p(\ell) \cdot t$.

We are interested in the mean-payoff decision problem for timed automata T that asks to decide whether the value of the mean-payoff game for a given state is below a given budget. For a PTGA T and budget $r \in \mathbb{R}$, we write $\mathsf{MPG}(\mathsf{T},r)$ for the r-mean payoff decision problem that asks whether value of the game at the state $(\ell,\mathbf{0})$ is smaller than r. The following theorem summarizes the key contribution of this paper.

▶ **Theorem 5.** The decision problem MPG(T,r) for binary-priced timed automata T is undecidable for automata with three clocks, and decidable for automata with one clock.

3 Boundary Region Graph Abstraction

In this section we introduce an abstraction of priced timed games called the boundary region abstraction (that generalizes classical corner-point abstraction [7]), and characterize conditions under which a solution of optimality equations for the boundary region abstraction can be lifted to a solution of optimality equations for timed automata. Observe that in order to keep our result as general as possible, we present the abstraction and corresponding results for timed automata with an arbitrary number of clocks. In the following section, we show that the required conditions hold for the case of one-clock binary-priced timed automata.

Timed Successor Regions. Recall that \mathcal{R} is the set of clock regions. For $\zeta, \zeta' \in \mathcal{R}$, we say that ζ' is in the future of ζ , denoted $\zeta \stackrel{*}{\to} \zeta'$, if there exist $\nu \in \zeta$, $\nu' \in \zeta'$ and $t \in \mathbb{R}_{\geq 0}$ such that $\nu' = \nu + t$ and say ζ' is the *time successor* of ζ if $\nu + t' \in \zeta \cup \zeta'$ for all $t' \leq t$ and write $\zeta \to \zeta'$, or equivalently $\zeta' \leftarrow \zeta$, to denote this fact. For regions $\zeta, \zeta' \in \mathcal{R}$ such that $\zeta \stackrel{*}{\to} \zeta'$ we write $[\zeta, \zeta']$ for the zone $\bigcup \{\zeta'' \mid \zeta \stackrel{*}{\to} \zeta'' \wedge \zeta'' \stackrel{*}{\to} \zeta'\}$.

Thin and Thick Regions. We say that a region ζ is thin if $[\nu] \neq [\nu + \varepsilon]$ for every $\nu \in \zeta$ and $\varepsilon > 0$ and thick otherwise. We write \mathcal{R}_{Thin} and \mathcal{R}_{Thick} for the sets of thin and thick regions, respectively. Observe that if $\zeta \in \mathcal{R}_{Thick}$ then, for any $\nu \in \zeta$, there exists $\varepsilon > 0$, such that $[\nu] = [\nu + \varepsilon]$ and the time successor of a thin region is thick, and vice versa.

Intuition for the Boundary Region Graph (BRG). Recall that K is an upper bound on clock values and let $\llbracket K \rrbracket_{\mathbb{N}} = \{0,1,\ldots,K\}$. For any $\nu \in V$, $b \in \llbracket K \rrbracket_{\mathbb{N}}$ and $c \in \mathcal{X}$, we define $\mathsf{time}(\nu,(b,c)) \stackrel{\text{def}}{=} b - \nu(c)$ if $\nu(c) \leqslant b$, and $\mathsf{time}(\nu,(b,c)) \stackrel{\text{def}}{=} 0$ if $\nu(c) \gt b$. Intuitively, $\mathsf{time}(\nu,(b,c))$ returns the amount of time that must elapse in ν before the clock c reaches the integer value b. Observe that, for any $\zeta' \in \mathcal{R}_{\mathsf{Thin}}$, there exists $b \in \llbracket K \rrbracket_{\mathbb{N}}$ and $c \in \mathcal{X}$, such that $\nu \in \zeta$ implies $(\nu + (b - \nu(c)) \in \zeta'$ for all $\zeta \in \mathcal{R}$ in the past of ζ' and write $\zeta \to_{b,c} \zeta'$. The boundary region abstraction is motivated by the following. Consider $a \in \mathsf{Act}$, (ℓ, ν) and $\zeta \stackrel{*}{\to} \zeta'$ such that $\nu \in \zeta$, $[\zeta, \zeta'] \subseteq \mathit{Inv}(\ell)$ and $\nu' \in E(\ell, a)$. (For illustration, see Figure 2 in Appendix).

- If $\zeta' \in \mathcal{R}_{\text{Thick}}$, then there are infinitely many $t \in \mathbb{R}_{\geqslant 0}$ such that $\nu+t \in \zeta'$. However, amongst all such t's, for one of the boundaries of ζ' , the closer $\nu+t$ is to this boundary, the 'better' the timed action (t,a) becomes for a player's objective. However, since ζ' is a thick region, the set $\{t \in \mathbb{R}_{\geqslant 0} \mid \nu+t \in \zeta'\}$ is an open interval, and hence does not contain its boundary values. Let the closest boundary of ζ' from ν be defined by the hyperplane $c = b_{\text{sup}}$. b_{inf} , $b_{\text{sup}} \in \mathbb{N}$ are such that $b_{\text{inf}} \nu(c)$ ($b_{\text{sup}} \nu(c)$) is the infimum (supremum) of the time spent to reach the lower (upper) boundary of region ζ' . Let the zones that correspond to these boundaries be denoted by ζ'_{inf} and ζ'_{sup} respectively. Then $\zeta \to_{b_{\text{inf}},c} \zeta'_{\text{inf}} \to \zeta'$ and $\zeta \to_{b_{\text{sup}},c} \zeta'_{\text{sup}} \leftarrow \zeta'$. In the boundary region abstraction we include these 'best' timed actions through (b_{inf} , c, a, ζ') and (b_{sup} , c, a, a, a.
- If $\zeta' \in \mathcal{R}_{Thin}$, then there exists a unique $t \in \mathbb{R}_{\geq 0}$ such that $\nu+t \in \zeta'$. Moreover since ζ' is a thin region, there exists a clock $c \in C$ and a number $b \in \mathbb{N}$ such that $\zeta \to_{b,c} \zeta'$ and $t = b \nu(c)$. In the boundary region abstraction we summarise this 'best' timed action from region ζ via region ζ' through the action (b, c, a, ζ') .

Based on this intuition above the boundary region abstraction (BRA) is defined as follows.

- ▶ **Definition 6.** For a priced timed game arena $\mathsf{T} = (L_{\mathrm{Min}}, L_{\mathrm{Max}}, Act, \mathcal{X}, Inv, E, \rho, \delta, p)$ the boundary region abstraction of T is given by the game arena $\widehat{\mathsf{T}} = (\widehat{S}, \widehat{S}_{\mathrm{Min}}, \widehat{S}_{\mathrm{Max}}, \widehat{A}, \widehat{T}, \widehat{\pi})$
- $\widehat{S} \subseteq L \times V \times \mathcal{R}$ is the set of states such that $(\ell, \nu, \zeta) \in \widehat{S}$ if and only if $\zeta \subseteq Inv(\ell)$ and $\nu \in \overline{\zeta}$ (recall that $\overline{\zeta}$ denotes the closure of ζ);
- $= (\ell, \nu, \zeta) \in \widehat{S}_{\text{Min}} \text{ (or } (\ell, \nu, \zeta) \in \widehat{S}_{\text{Max}}) \text{ if } (\ell, \nu, \zeta) \in \widehat{S} \text{ and } \ell \in L_{\text{Min}} \text{ (or } \ell \in L_{\text{Max}}, \text{ resp.)}.$
- $\widehat{A} = (\llbracket K \rrbracket_{\mathbb{N}} \times \mathcal{X} \times Act \times \mathcal{R})$ is the set of actions;
- For \hat{s} =(ℓ, ν, ζ)∈ \hat{S} and α =($b_{\alpha}, c_{\alpha}, a_{\alpha}, \zeta_{\alpha}$)∈ \hat{A} , function $\hat{T}(\hat{s}, \alpha)$ is defined if $[\zeta, \zeta_{\alpha}]$ ⊆ $Inv(\ell)$ and $\zeta_{\alpha} \subseteq E(\ell, a_{\alpha})$ and it equals (ℓ', ν', ζ') ∈ \hat{S} where $\delta(\ell, a_{\alpha}) = \ell', \nu_{\alpha}[C:=0] = \nu'$ and $\zeta_{\alpha}[C:=0] = \zeta'$ with $\nu_{\alpha} = \nu + time(\nu, (b_{\alpha}, c_{\alpha}))$ and one of the following conditions holds: $\zeta \to_{b_{\alpha}, c_{\alpha}} \zeta_{\alpha}$; $\zeta \to_{b_{\alpha}, c_{\alpha}} \zeta_{inf} \to \zeta_{\alpha}$ for some $\zeta_{inf} \in \mathcal{R}$; $\zeta \to_{b_{\alpha}, c_{\alpha}} \zeta_{sup} \leftarrow \zeta_{\alpha}$ for some $\zeta_{sup} \in \mathcal{R}$;
- for $(\ell, \nu, \zeta) \in \widehat{S}$ and $(b_{\alpha}, c_{\alpha}, a_{\alpha}, \zeta_{\alpha}) \in \widehat{A}$ the reward function $\widehat{\pi}$ is given by: $\widehat{\pi}((\ell, \nu, \zeta), (b_{\alpha}, c_{\alpha}, a_{\alpha}, \zeta_{\alpha})) = p(\ell, a_{\alpha}) + p(\ell) \cdot (b_{\alpha} \nu(c_{\alpha}))$

Although the boundary region abstraction is not a finite game arena, every state has only finitely many time-successor (the boundaries of the regions) and for a fixed initial state we can restrict attention to a finite game arena due to the following observation.

▶ **Lemma 7** ([23].). Let T be a priced timed game arena and \widehat{T} the corresponding BRA. For any state of \widehat{T} , its reachable sub-graph is finite and can be constructed in time exponential in the size of T when T has more than one clock. For one clock T, the reachable sub-graph of \widehat{T} can be constructed in time polynomial in the size of T. Moreover, the reachable sub-graph from the initial location and clock valuation is precisely the corner-point abstraction.

3.1 Reduction to Boundary Region Abstraction

In what follows, unless specified otherwise, we fix a PTGA $\mathsf{T} = (L_{\mathrm{Min}}, L_{\mathrm{Max}}, Act, \mathcal{X}, Inv, E, \rho, \delta, p)$ with semantics $[\![\mathsf{T}]\!] = (S, S_{\mathrm{Min}}, S_{\mathrm{Max}}, A, T, \pi)$ and BRA $\widehat{\mathsf{T}} = (\widehat{S}, \widehat{S}_{\mathrm{Min}}, \widehat{S}_{\mathrm{Max}}, \widehat{A}, \widehat{T}, \widehat{\pi})$. Let $G: \widehat{S} \to \mathbb{R}$ and $B: \widehat{S} \to \mathbb{R}$ be such that $(G, B) \models \mathrm{Opt}(\widehat{\mathsf{T}})$, i.e. for every $\widehat{s} \in \widehat{S}$ we have that

$$G(\hat{s}) = \begin{cases} \max_{\alpha \in \widehat{A}(\hat{s})} \{G(\hat{s}') : \hat{s} \xrightarrow{\alpha} \hat{s}'\} & \text{if } \hat{s} \in \widehat{S}_{\text{Max}} \\ \min_{\alpha \in \widehat{A}(\hat{s})} \{G(\hat{s}') : \hat{s} \xrightarrow{\alpha} \hat{s}'\} & \text{if } \hat{s} \in \widehat{S}_{\text{Min}}. \end{cases}$$

$$B(\hat{s}) = \begin{cases} \max_{\alpha \in \widehat{A}(\hat{s})} \{\pi(\hat{s}, \alpha) - G(\hat{s}) + B(\hat{s}') : \hat{s} \xrightarrow{\alpha} \hat{s}' \text{ and } G(\hat{s}) = G(\hat{s}')\} & \text{if } \hat{s} \in \widehat{S}_{\text{Max}} \\ \min_{\alpha \in \widehat{A}(\hat{s})} \{\pi(\hat{s}, \alpha) - G(\hat{s}) + B(\hat{s}') : \hat{s} \xrightarrow{\alpha} \hat{s}' \text{ and } G(\hat{s}) = G(\hat{s}')\} & \text{if } \hat{s} \in \widehat{S}_{\text{Min}}. \end{cases}$$

For a function $F:\widehat{S}\to\mathbb{R}$ we define a function $F^{\boxplus}:S\to\mathbb{R}$ as $(\ell,\nu)\mapsto F(\ell,\nu,[\nu])$. In this section we show under what conditions we can lift a solution (G,B) of optimality equations of BRA to $(G^{\boxplus},B^{\boxplus})$ for priced timed game arena. Given a set of valuations $X\subseteq V$, a function $f:X\to\mathbb{R}_{\geqslant 0}$ is affine if for any valuations $\nu_x,\nu_y\in X$ we have that for all $\lambda\in[0,1], f(\lambda\nu_x+(1-\lambda)\nu_y)=\lambda f(\nu_x)+(1-\lambda)f(\nu_y)$. We say that a function $f:\widehat{S}\to\mathbb{R}_{\geqslant 0}$ is regionally affine if $f(\ell,\cdot,\zeta)$ is affine over a region for all $\ell\in L$ and $\zeta\in\mathbb{R}$, and f is regionally constant if $f(\ell,\cdot,\zeta)$ is constant over a region for all $\ell\in L$ and $\zeta\in\mathbb{R}$. Some properties of affine functions that are useful in the proof of the key lemma are given in Lemma 8.

- ▶ Lemma 8. Let $X \subseteq V$ and $Y \subseteq \mathbb{R}_{\geqslant 0}$ be convex sets. Let $f: X \to \mathbb{R}$ and $w: X \times Y \to \mathbb{R}$ be affine functions. Then for $C \subseteq \mathcal{X}$ we have that $\phi_C(\nu,t) = w(\nu,t) + f((\nu+t)[C:=0])$ is also an affine function, and $\inf_{t_1 < t < t_2} \phi_C(\nu,t) = \min\{\overline{\phi}_C(\nu,t_1), \overline{\phi}_C(\nu,t_2)\}$ and $\sup_{t_1 < t < t_2} \phi_C(\nu,t) = \max\{\overline{\phi}_C(\nu,t_1), \overline{\phi}_C(\nu,t_2)\}$, $\overline{\phi}$ is the unique continuous closure of ϕ .
- ▶ Theorem 9. Let $G: \widehat{S} \to \mathbb{R}$ and $B: \widehat{S} \to \mathbb{R}$ are such that $(G, B) \models Opt(\widehat{\mathsf{T}})$ and G is regionally constant and B is regionally affine, then $(G^{\boxplus}, B^{\boxplus}) \models Opt(\mathsf{T})$.

Proof. We need to show that $(G^{\boxplus}, B^{\boxplus}) \models \operatorname{Opt}(\mathsf{T})$, i.e. for every

$$G^{\boxplus}(s) = \begin{cases} \sup_{(t,a) \in A(s)} \{G^{\boxplus}(s') : s \xrightarrow{(t,a)} s'\} & \text{if } s \in S_{\text{Max}} \\ \inf_{(t,a) \in A(s)} \{G^{\boxplus}(s') : s \xrightarrow{(t,a)} s'\} & \text{if } s \in S_{\text{Min}}. \end{cases}$$

$$B^{\boxplus}(s) = \begin{cases} \sup_{(t,a) \in A(s)} \{\pi(s,(t,a)) - G^{\boxplus}(s) + B^{\boxplus}(s') : s \xrightarrow{(t,a)} s' \text{ and } G^{\boxplus}(s) = G^{\boxplus}(s')\} & \text{if } s \in S_{\text{Max}} \\ \inf_{(t,a) \in A(s)} \{\pi(s,(t,a)) - G^{\boxplus}(s) + B^{\boxplus}(s') : s \xrightarrow{(t,a)} s' \text{ and } G^{\boxplus}(s) = G^{\boxplus}(s')\} & \text{if } s \in S_{\text{Min}}. \end{cases}$$
Consider the case where $s = (\ell, \tau) \in S_{\text{Min}}$ and consider the right side of the gain equations.

Consider the case when $s = (\ell, \nu) \in S_{\text{Min}}$ and consider the right side of the gain equations.

$$\inf_{\substack{(t,a)\in A(s)}} \{G^{\boxplus}(s') : s \xrightarrow{(t,a)} s'\}$$

$$= \min_{\substack{\zeta'':[\nu]\to^*\zeta''\\[\zeta,\zeta'']\in Inv(\ell)}} \min_{\substack{a\in Act\\\nu+t\in\zeta''}} \{G(\delta(\ell,a),(\nu+t)[\rho(a){:=}0],[(\nu+t)][\rho(a){:=}0])\}$$

$$= \min_{\substack{\alpha\in\widehat{A}(\ell,\nu,[\nu])}} \{G(\ell',\nu',\zeta') : (\ell,\nu,\zeta) \xrightarrow{\alpha} (\ell',\nu',\zeta')\} = G(\ell,\nu,[\nu]) = G^{\boxplus}(\ell,\nu).$$

The first equality holds since $(G,B) \models \operatorname{Opt}(\widehat{\mathsf{T}})$. The second equality follows since G is regionally constant and hence it suffices to consider the delay $\operatorname{time}(\nu,(b,c))$ that corresponds to either left or right boundary of the region ζ'' , i.e. for fixed ν,ζ'' and $a\in Act$ we have that $\inf_{\substack{t:\\\nu+t\in\zeta''\\\ell'=0}} \{G(\ell',(\nu+t)[\rho(a){:=}0],\zeta')\} = G(\ell',\nu_{\alpha}[C{:=}0],\zeta') \text{ where } \nu_{\alpha}=\nu+\operatorname{time}(\nu,(b_{\alpha},c_{\alpha})),$ $\zeta''[C{:=}0]=\zeta'$ with $\zeta\to_{b_{\alpha},c_{\alpha}}\zeta''$ if ζ'' is thin, and $\zeta\to_{b_{\alpha},c_{\alpha}}\zeta_{\inf}\to\zeta''$ for some $\zeta_{\inf}\in\mathcal{R}$ if ζ'' is thick. Similarly, for the bias equations, we need to show:

$$\begin{split} &\inf_{\substack{t:\\ \nu+t\in\zeta''}} \left\{\pi((\ell,\nu),(t,a)) - G(\ell,\nu) + B(\ell',(\nu+t)[\rho(a){:=}0],\zeta')\right\} \\ &= & \pi((\ell,\nu,[\nu]),(\mathsf{time}(\nu,(b_\alpha,c_\alpha)))) - G(\ell,\nu,[\nu]) + B(\ell',\nu_\alpha[C{:=}0],\zeta') \end{split}$$

where $\nu_{\alpha} = \nu + \operatorname{time}(\nu, (b_{\alpha}, c_{\alpha}))$, $\zeta''[C := 0] = \zeta'$ with $\zeta \to_{b_{\alpha}, c_{\alpha}} \zeta''$ if ζ'' is thin; and $\zeta \to_{b_{\alpha}, c_{\alpha}} \zeta_{\inf} \to \zeta''$ for some $\zeta_{\inf} \in \mathcal{R}$ or $\zeta \to_{b_{\alpha}, c_{\alpha}} \zeta_{\sup} \to \zeta''$ for some $\zeta_{\sup} \in \mathcal{R}$ if ζ'' is thick. Given B is regionally affine (and hence linear in t) and the price function is linear in t, the whole expression $\pi((\ell, \nu), (t, a)) - G(\ell, \nu) + B(\ell', (\nu + t)[\rho(a) := 0], \zeta')$ is linear in t and from Lemma 8 it attains its infimum or supremum on either boundary of the region.

4 Decidability for One Clock Binary-priced PTGA

Given the undecidability with 3 or more clocks, we focus on one clock PTGA. We provide a strategy improvement algorithm to compute a solution $G:\widehat{S}\to\mathbb{R}$ and $B:\widehat{S}\to\mathbb{R}$ of the optimality equations, i.e. $(G,B)\models \mathrm{Opt}(\widehat{\mathsf{T}})$ for the BRA $\widehat{\mathsf{T}}=(\widehat{S},\widehat{S}_{\mathrm{Min}},\widehat{S}_{\mathrm{Max}},\widehat{A},\widehat{T},\widehat{\pi})$ of one-clock binary-priced PTGAs with certain "integral payoff" restriction. Further, we show that for one clock binary-priced integral-payoff PTGA, the solution of optimality equations of corresponding BRG is such that the gains are regionally constant and biases are regionally affine. Hence by Theorem 9, the algorithm can be applied to solve mean-payoff games for one-clock binary-priced integral-payoff PTGAs. We also show how to lift the integral-payoff restriction to recover decidability for one-clock binary-priced PTGA.

Regionally constant positional strategies. Standard strategy improvement algorithms iterate over a finite set of strategies such that the value of the subgame at each iteration gets strictly improved. However, since there are infinitely many positional strategies in a boundary region abstraction, we focus on "regionally constant" positional strategies (RCPSs). We say that a positional strategy $\mu:\widehat{S}\to\widehat{A}$ of player Min is regionally-constant if for all $(\ell,\nu,\zeta), (\ell,\nu',\zeta)\in\widehat{S}_{\mathrm{Min}}$ we have that $[\nu]=[\nu']$ implies that $\mu(\ell,\nu,\zeta)=\mu(\ell,\nu',\zeta)$. We similarly define RCPSs for player Max. In other words, in an RCPS a player chooses same boundary action for every valuation of a region—as a side-result we show that optimal strategies for both players have this form. Observe that there are finitely many RCPSs for both players. We write $\widehat{\Pi}_{\mathrm{Min}}$ and $\widehat{\Pi}_{\mathrm{Max}}$ for the set of RCPSs for player Min and player Max, respectively. For a BRA $\widehat{\mathsf{T}},\ \chi\in\widehat{\Pi}_{\mathrm{Max}}$, and $\mu\in\widehat{\Pi}_{\mathrm{Min}}$ we write $\widehat{\mathsf{T}}(\chi)$ and $\widehat{\mathsf{T}}(\mu)$ for the "one-player" game on the sub-graph of BRAs where the strategies of player Max and Min have been fixed to RCPSs χ and μ , respectively. Similarly we define the zero-player game $\widehat{\mathsf{T}}(\mu,\chi)$ where strategies of both players are fixed to RCPSs μ and χ .

Let $\widehat{\mathsf{T}}(\chi,\mu)$ be a zero-player game on the subgraph where strategies of player Max (and Min) is fixed to RCPSs χ (and μ). Observe that for $\widehat{\mathsf{T}}(\mu,\chi)$ the unique runs originating from states $\hat{s}_0 = (\ell,\nu,\zeta)$ and $\hat{s}'_0 = (\ell,\nu',\zeta)$ with $[\nu] = [\nu']$ follow the same "lasso" after one step, i.e. the unique runs $\hat{s}_0 \xrightarrow{\alpha_1} \hat{s}_1 \cdots \hat{s}_k (\xrightarrow{\alpha_{k+1}} \cdots \hat{s}_{k+N-1} \xrightarrow{\alpha_{k+N}} \hat{s}_k)^*$ and $\hat{s}'_0 \xrightarrow{\alpha_1} \hat{s}'_1 \cdots \hat{s}'_k (\xrightarrow{\alpha_{k+1}} \cdots \hat{s}'_{k+N-1} \xrightarrow{\alpha_{k+N}} \hat{s}'_k)^*$ are such that for $\hat{s}_i = (\ell_i, \nu_i, \zeta_i)$ and $\hat{s}'_i = (\ell'_i, \nu'_i, \zeta'_i)$ we have that $\ell_i = \ell'_i, \zeta_i = \zeta'_i$ and $\nu_i = \nu'_i$ for all $i \in [1, k+N-1]$. This is so because for one-clock timed automata the successors of the states $\hat{s}_0 = (\ell, \nu, \zeta)$ and $\hat{s}'_0 = (\ell, \nu', \zeta)$ for action $\alpha_1 = (b, c, a, \zeta')$ is the same (ℓ'', ν'', ζ'') where $\nu''(c) = \nu(c) + (b - \nu(c)) = b = \nu(c) + (b - \nu(c))$ if $c \notin \rho(a)$ and $\nu''(c) = 0$ otherwise. Consider the optimality equations (Section C.3) for the lasso. Observe that the gain for the states $\hat{s}_0, \ldots, \hat{s}_{k+N-1}$ is the same, and let's call it g. If we add the bias equations side-wise for the cycle, we get $g = \frac{1}{N} \sum_{i=0}^{N-1} \pi(\hat{s}_{k+i}, \alpha_{k+i+1})$. It follows from the previous observation that the gains are regionally constant.

Integral Payoff PTGA. The gain in a zero-player game, $\widehat{\mathsf{T}}(\chi,\mu)$, although regionally-constant, may not be a whole number. We say that a PTGA is integral-payoff if for every pair $(\mu,\chi) \in \widehat{\Pi}_{\text{Min}} \times \widehat{\Pi}_{\text{Max}}$ of RCPSs the gain as defined above is a whole number. Observe that the denominator in the gains correspond to the number of edges in a simple cycle of the BRA $\widehat{\mathsf{T}}$. If there are N simple cycles in the region graph of length n_1, n_2, \ldots, n_N , then let \mathcal{L} be the least-common multiple of n_1, n_2, \ldots, n_N . We multiply constants appearing in the guards and invariants of the timed automata by \mathcal{L} . It is easy to observe that mean-payoff of any state in the original PTGA T is the mean-payoff in Υ_T divided by \mathcal{L} . For notational convenience, we assume that the given PTGA is an integral-payoff PTGA and hence for RCPS strategy profile (μ,χ) the gain is regionally constant and integral.

Algorithm 1: ComputeValueZeroPlayer(T, μ, χ)

```
1 Consider \widehat{\mathsf{T}}(\mu,\chi) as a (single successor) weighted graph \mathcal{G} = (V,E,w) where

\quad = V = L \times \mathcal{R} \times \mathcal{R} (with an order \preceq) and E \subseteq V \times \widehat{A} \times V

\quad = (v_1,\alpha,v_2) \in E \text{ if } v_1 = (\ell_1,\zeta_1,\zeta_1'), v_2 = (\ell_2,\zeta_2,\zeta_2'), \text{ and } \mu(\ell_1,\nu_1,\zeta_1') = \alpha \text{ (or } \chi(\ell_1,\nu_1,\zeta_1') = \alpha) \text{ for all } \nu_1 \in \zeta_1 \text{ and } (\ell_1,\nu_1,\zeta_1') \xrightarrow{\alpha} (\ell_1,\nu_2,\zeta_2') \text{ for some } \nu_2 \in \zeta_2.

\quad = w(v_1,\alpha,v_2) \text{ is the expression } \nu \mapsto b_\alpha - \nu(c_\alpha);

for every cycle C of \mathcal{G} do

\quad \text{Let Reach}(C) \text{ be set of vertices that reach } C;

Let \gamma be the average weight of the cycle (w is constant on cycles);

For every vertex V in Reach(C) set G(V) = \gamma and B(V) = \bot;

For the smallest \preceq-vertex V_* in C. Set B(V_*) = 0;

while there is V \in Reach(C) with B(V) = \bot do

\quad \text{Let } (V',\alpha,V'') \in E \text{ with } B(V'') \neq \bot;

\quad B(V') := \nu \mapsto (w(V',\alpha,V'')(\nu) - G + B(V''));

return (G,B);
```

4.1 Strategy Improvement Algorithm for Binary-Priced PTGA

Let T be a one-clock integral-payoff binary-priced PTGA T and $\widehat{\mathsf{T}}$ be its boundary region graph. For a given RCPS profile $(\mu,\chi)\in\widehat{\Pi}_{\mathrm{Min}}\times\widehat{\Pi}_{\mathrm{Max}}$, Algorithm 1 computes the solution for the optimality equations $\mathrm{Opt}(\mathsf{T}(\mu,\chi))$. This algorithm considers $\widehat{\mathsf{T}}(\mu,\chi)$ as a graph whose vertices are "regions" $(\ell,[\nu],\zeta)$ corresponding to state $(\ell,\nu,\zeta)\in\widehat{S}$ of the boundary region graph, edges are boundary actions between them determined by the regionally constant strategy profile, and weight of an edge is the time function associated with the boundary action. Observe that every cycle in this graph will have constant weight on the edges since taking boundary actions in a loop will require going from an integral valuation to another integral valuation, and the average cost of such a cycle can be easily computed.

Also observe that, not unlike standard convention [21], our algorithm chooses a vertex in a cycle arbitrarily and fixes the bias of all of the states in that vertex to 0. This is possible since optimality equations over a cycle are underdetermined, and we exploit this flexibility to achieve solution to biases in a particularly "simple" structure. We say that a function $f: \widehat{S} \to \mathbb{R}_{\geqslant 0}$ is regionally simple [3] if for all $\ell \in L$, $\zeta, \zeta' \in \mathcal{R}$ either i) there exists a $d \in \mathbb{N}$ such that $f(\ell, \nu, \zeta') = d$ for all $\nu \in \zeta$; or ii) there exists $d \in \mathbb{N}$ and $c \in \mathcal{X}$ such that $f(\ell, \nu, \zeta') = d - \nu(c)$ for all $\nu \in \zeta$. Key properties of regionally simple functions (Lemma 20 in Appendix C.2) include that they are also regionally affine, closed under minimum and maximum, and if $B: \widehat{S} \to \mathbb{R}$ be a regionally simple function and $G: \widehat{s} \to \mathbb{N}$ be a regionally constant function, then $\widehat{s} \mapsto \pi(\widehat{s}, \alpha) - G(\widehat{s}) + B(\widehat{s}')$ where $\widehat{s} \stackrel{\triangle}{\to} \widehat{s}'$ is a regionally simple function. Using these properties and induction on the distance to \preceq -minimal element in the reachable cycle, we prove the correctness and following property of Algorithm 1.

▶ **Lemma 10.** Algorithm 1 computes solution of optimality equations $(G, B) \models Opt(\widehat{\mathsf{T}}(\mu, \chi))$ for $\mu \in \widehat{\Pi}_{Min}$ and $\chi \in \widehat{\Pi}_{Max}$. Moreover, G is regionally constant and B is regionally simple.

The strategy improvement algorithm to solve optimality equations is given as Algorithm 2. It begins by choosing an arbitrary regionally constant positional strategy χ' and at every iteration of the loop (2–11) the algorithm computes (5–9) the value (G,B) of the current RCPS χ and based on the value, the function IMPROVEMAXSTRATEGY returns an improved strategy by picking boundary action that lexicographically maximizes gain and bias respect-

Algorithm 2: ComputeValueTwoPlayer(T)

```
1 Choose an arbitrary regionally constant positional strategy \chi' \in \Pi_{\text{Max}};
 2 repeat
 3
        Choose an arbitrary regionally constant positional strategy \mu' \in \Pi_{\text{Min}};
 4
        repeat
 5
            \mu := \mu';
 6
            (G, B) := \text{ComputeValueZeroPlayer}(\mathsf{T}, \mu, \chi);
            \mu' := \text{ImproveMinStrategy}(\mathsf{T}, \mu, G, B) ;
        until \mu = \mu';
        \chi' := \text{ImproveMaxStrategy}(\mathsf{T}, \chi, G, B) ;
10
11 until \chi = \chi';
12 return (G,B);
```

ing the policy that switches a decision only for a strict improvement. We formally define the function ImproveMaxStrategy as follows: for $\chi \in \widehat{\Sigma}_{\text{Max}}$, $G : \widehat{S} \to \mathbb{R}$, and $B : \widehat{S} \to \mathbb{R}$ we let strategy ImproveMaxStrategy(T, χ , G, B) be such that for all $\widehat{s} \in \widehat{S}_{\text{Max}}$ we have

$$\text{ImproveMaxStrategy}(\mathsf{T},\chi,G,B)(\hat{s}) = \begin{cases} \chi(\hat{s}) & \text{if } \chi(\hat{s}) \in M^*(\hat{s},G,B) \\ \text{Choose}(M^*(\hat{s},G,B)) & \text{Otherwise}. \end{cases}$$

where $M^*(\hat{s}, G, B) = \underset{\alpha \in \widehat{A}}{\operatorname{argmax}^{\operatorname{lex}}} \{ (G(\hat{s}'), \pi(\hat{s}, \alpha) - G(\hat{s}) + B(\hat{s}')) : \hat{s} \xrightarrow{\alpha} \hat{s}' \}$ and Choose picks an arbitrary element from a set. IMPROVEMAXSTRATEGY satisfies the following.

▶ Lemma 11. If $\chi \in \widehat{\Pi}_{Max}$, G is regionally constant, and B is regionally simple, then function IMPROVEMAXSTRATEGY(T, χ , G, B) returns a regionally constant positional strategy.

The lines (5–9) compute the value of the strategy χ of Player Max via a strategy improvement algorithm. This sub-algorithm works by starting with an arbitrary strategy of Player Min and computing the value (G, B) of the zero-player PTGA $\widehat{\mathsf{T}}(\mu, \chi)$. Based on the value, the function IMPROVEMINSTRATEGY returns an improved strategy of Min. The function IMPROVEMINSTRATEGY is defined as a dual of the function IMPROVEMAXSTRATEGY where χ is replaced by μ and argmax by argmin. IMPROVEMINSTRATEGY satisfies the following.

▶ Lemma 12. If $\mu \in \widehat{\Pi}_{Min}$, G is regionally constant, and B is regionally simple, then function IMPROVEMINSTRATEGY(T, χ, G, B) returns a regionally constant positional strategy.

It follows from Lemma 11 and Lemma 12 that at every iteration of the strategy improvement the strategies μ and χ are RCPSs. Together with finiteness of the set of RCPSs and strict improvement at every step (Lemma 21 and 22), we get following result.

▶ **Theorem 13.** Algorithm 2 computes solution of optimality equations $(G, B) \models Opt(\widehat{\mathsf{T}})$ for integral payoff PTGA T. Moreover, G is regionally constant and B is regionally affine.

This theorem—together with Theorem 9 and Theorem 2—gives a proof of decidability for mean-payoff games for integral-payoff binary-priced one-clock timed automata.

5 Undecidability Results

▶ **Theorem 14.** The mean-payoff problem $\mathsf{MPG}(\mathsf{T},r)$ is undecidable for PTGA T with 3 clocks having location-wise price-rates $\pi(\ell) \in \{0,1,-1\}$ for all $\ell \in L$ and r=0. Moreover, it is undecidable for binary-priced T with 3 clocks and r>=0.

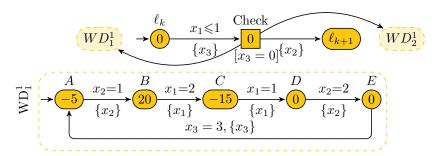


Figure 1 Simulation to decrement counter C_1 , mean cost is ε for error ε . The widget WD₂¹ has exactly the same structure and guards on all transitions as WD₁¹, but the price signs are reversed.

Proof. We first show the undecidability result of the mean-payoff problem $\mathsf{MPG}(\mathsf{T},0)$ with location prices $\{1,0,-1\}$ and no edge prices. We prove the result by reducing the non-halting problem of 2 counter machines. Our reduction uses a PTGA with 3 clocks x_1,x_2,x_3 , location prices $\{1,0,-1\}$, and no edge prices. Each counter machine instruction (increment, decrement, zero check) is specified using a PTGA module. The main invariant in our reduction is that on entry into any module, we have $x_1 = \frac{1}{5^{c_1}7^{c_2}}, x_2 = 0$ and $x_3 = 0$, where c_1, c_2 are the values of counters C_1, C_2 . We outline the construction for the decrement instruction of counter C_1 in Figure 1. For conciseness, we present here modules using arbitrary location prices. However, we can redraw these with extra locations and edges using only the location prices from $\{1,0,-1\}$ as shown for WD_1^1 in Figure 5 in Appendix.

The role of the Min player is to faithfully simulate the two counter machine, by choosing appropriate delays to adjust the clocks to reflect changes in counter values. Player Max will have the opportunity to verify that player Min did not cheat while simulating the machine.

We enter location ℓ_k with $x_1 = \frac{1}{5^{c_1}7^{c_2}}, x_2 = 0$ and $x_3 = 0$. Lets denote by x_{old} the value $\frac{1}{5^{c_1}7^{c_2}}$. To correctly decrement C_1 , player Min should choose a delay of $4x_{old}$ at location ℓ_k . At location Check, there is no time elapse and player Max has three possibilities: (i) to go to ℓ_{k+1} and continue the simulation, or (ii) to enter the widget WD₁, or (iii) to enter the widget WD₂. If player Min makes an error, and delays $4x_{old} + \varepsilon$ or $4x_{old} - \varepsilon$ at ℓ_k ($\varepsilon > 0$), then player Max can enter one of the widgets and punish player Min. Player Max enters widget WD₁ if the error made by player Min is of the form $4x_{old} + \varepsilon$ at ℓ_k and enters widget WD₂ if the error made by player Min is of the form $4x_{old} - \varepsilon$ at ℓ_k .

Let us examine the widget WD₁¹. When we enter WD₁¹ for the first time, we have $x_1 = x_{old} + 4x_{old} + \varepsilon$, $x_2 = 4x_{old} + \varepsilon$ and $x_3 = 0$. In WD₁¹, the cost of going once from location A to E is 5ε . Also, when we get back to A after going through the loop once, the clock values with which we entered WD₁¹ are restored; thus, each time, we come back to A, we restore the starting values with which we enter WD₁¹. The third clock is really useful for this purpose only. It can be seen that the mean cost of transiting from A to A through E is ε . In a similar way, it can be checked that the mean cost of transiting from A to A through E in widget WD₂¹ is ε when player Min chooses a delay $4x_{old} - \varepsilon$ at ℓ_k . Thus, if player Min makes a simulation error, player Max can always choose to goto one of the widgets, and ensure that the mean pay-off is not ≤ 0 . Note that when $\varepsilon = 0$, then player Min will achieve his objective: the mean pay-off will be 0. Details of other gadgets are in Appendix D.1.

In the Appendix D.2, we show how this undecidability results extends (with the same parameters) if one defines mean payoff per time unit instead of per step. This way of averaging across time spent was considered in [10], where the authors show the undecidability of MPG(T,0) with 5 clocks. We improve this result to show undecidability already in 3 clocks.

References

- 1 R. Alur, M. Bernadsky, and P. Madhusudan. Optimal reachability for weighted timed games. In *Proc. of ICALP*, pages 122–133. Springer, 2004.
- 2 R. Alur and D. Dill. A theory of timed automata. *Theoretical Computer Science*, 126(2):183–235, 1994.
- 3 E. Asarin and O. Maler. As soon as possible: Time optimal control for timed automata. In F. W. Vaandrager and J. H. van Schuppen, editors, *Proc. of HSCC*, pages 19–30, 1999.
- **4** H. Björklund, S. Sandberg, and S. Vorobyov. A combinatorial strongly subexponential strategy improvement algorithm for mean payoff games. In *Proc. of MFCS*, pages 673–685, 2004.
- **5** P. Bouyer. Weighted timed automata: Model-checking and games. In *Proc. of MFPS*, volume 158, pages 3–17, 2006.
- 6 P. Bouyer, T. Brihaye, M. Jurdzinski, R. Lazic, and M. Rutkowski. Average-price and reachability-price games on hybrid automata with strong resets. In *FORMATS*, volume 5215 of *LNCS*, pages 63–77, 2008.
- 7 P. Bouyer, E. Brinksma, and K. G. Larsen. Staying alive as cheaply as possible. In *Proc.* of HSCC, volume 2993 of LNCS, pages 203–218. Springer, 2004.
- 8 P. Bouyer, F. Cassez, E. Fleury, and K. G. Larsen. Optimal strategies in priced timed game automata. In *Proc. of FSTTCS*, volume 3328 of *LNCS*, pages 148–160. Springer, 2004.
- **9** P. Bouyer, K. G. Larsen, and N. Markey. Lower-bound constrained runs in weighted timed automata. In *Proc. of QEST*, pages 128–137, 2012.
- 10 R. Brenguier, F. Cassez, and J. F. Raskin. Energy and mean-payoff timed games. In Proc. of HSCC, pages 283–292, 2014.
- 11 T. Brihaye, G. Geeraerts, S. N. Krishna, L. Manasa, B. Monmege, and A. Trivedi. Adding negative prices to priced timed games. In *Proc. of CONCUR*, pages 560–575, 2014.
- 12 E. Dynkin and A. Yushkevich. Controlled Markov Processes. Springer, 1979.
- A. Ehrenfeucht and A. Mycielski. Positional strategies for mean payoff games. *International Journal of Game Theory*, 8:109–113, 1979.
- 14 J. Filar and K. Vrieze. Competitive Markov Decision Processes. Springer, 1997.
- 15 V. A. Gurvich, A. V. Karzanov, and L. G. Khachiyan. Cyclic games and an algorithm to find minimax cycle means in directed graphs. *USSR Computational Mathematics and Mathematical Physics*, 28:85–91, 1988.
- 16 Thomas A. Henzinger. Quantitative reactive modeling and verification. Computer Science
 Research and Development, 28(4):331–344, 2013.
- 17 R. A. Howard. Dynamic Programming and Markov Processes. MIT Press, 1960.
- 18 M. Jurdziński and A. Trivedi. Reachability-time games on timed automata. In Proc. of ICALP, pages 838–849. Springer, 2007.
- 19 M. Jurdziński and A. Trivedi. Average-time games. In R. Hariharan, M. Mukund, and V. Vinay, editors, *Proc. of FSTTCS*, Dagstuhl Seminar Proceedings, 2008.
- 20 Marvin L. Minsky. Computation: finite and infinite machines. Prentice-Hall, Inc., 1967.
- 21 M. L. Puterman. Markov Decision Processes: Disc. Stoc. Dynamic Prog. Wiley, 1994.
- 22 P. J. Ramadge and W. M. Wonham. The control of discrete event systems. In *IEEE*, volume 77, pages 81–98, 1989.
- 23 A. Trivedi. Competitive Optimisation on Timed Automata. PhD thesis, Department of Computer Science, The University of Warwick, 2009.
- 24 U. Zwick and M. Paterson. The complexity of mean payoff games on graphs. Theoretical Computer Science, 158:343–359, 1996.

Appendix

A Supplementary material to Section 2

A.1 Strategy improvement algorithm for Finite Game Arenas

Let Γ be a finite game arena. For technical convenience let us fix an arbitrary but fixed linear order $\leq \subseteq S^2$ on the set of states S. For a positional strategy $\chi \in \Pi_{\text{Max}}$ we write Γ_{χ} for the subgame of Γ where the outgoing transitions from the states controlled by Player Max have been restricted to the ones allowed by χ . We similarly define Γ_{μ} and $\Gamma_{\chi}\mu$.

Strategy Improvement Algorithm for Finite Game Arenas. The strategy improvement algorithm to compute s solution of optimality equations works as follows.

- 1. Fix an arbitrary positional strategy $\chi: S_{\text{Max}} \to A$ for player Max.
- 2. (Best counter-strategy against χ .) Compute the best counter-strategy μ for player Min against the strategy χ by performing the following steps.
 - a. (Minimize gain.) For every state $s \in S$, let G(s) be the value of the minimum average weight of a cycle reachable from the state s in the strategy subgame Γ_{χ} ; let C(s) denote such a cycle reachable from the state s. Let m_s be the \preceq -smallest state on the cycle C(s). Let W_s be the set of states s', such that $m_{s'} = m_s$, i.e., the set of states which have the same reachable minimum average weight cycle in the graph Γ_{χ} as the state s.
 - b. (Minimize bias.) For every state $s \in S$, let B(s) be the weight of the shortest path from the state s to the state m_s in the subgraph of Γ_{χ} induced by the set of states W_s , and where G(s) was subtracted from the price of every transition in the subgraph Γ_{χ} . For every state $s \in S_{\text{Min}}$, set $\mu(s)$ to be the state $s' \in V$, such that $s \xrightarrow{a} s'$ for some $a, s' \in W_s$ (and hence G(s) = G(s')), and $B(s) = (\pi(s, a) G(s)) + B(s')$.
 - Observe that the functions G and B thus obtained satisfy the optimality equations for the subgame Γ_{χ} . However, these function may not satisfy the optimality equations for the original game Γ . The next step "locally" changes the strategy χ , intuitively in order to make progress towards computing the optimality equations.
- 3. (Local improvement of strategy χ .) For every state $s \in S_{\text{Max}}$, set $\chi(s)$ to be a successor s' of the state s, which first maximizes G(s') and then maximizes B(s'). In other words, $\chi(s)$ is the successor s' of the state s which maximizes (G(s'), B(s')) according to the lexicographical ordering on pairs, where we use the usual ordering on the reals in both coordinates. Importantly, if the current χ -successor of the state s is already maximum in the above lexicographic sense, $\chi(s)$ remains unchanged, even if there are other successors s' of s with the same pair of values (G(s'), B(s')) as the state $\chi(s)$. This assumption is important for finite termination of the strategy improvement algorithm.
- **4.** If the local improvement of the strategy χ in the previous step resulted in a change of χ in at least one state then go back to step 2. Otherwise stop.

We establish the following two fundamental properties of the iterative scheme of strategy improvement described above. The first observation that a locally optimal strategy yields a solution to optimality equations is straightforward to check.

▶ **Lemma 15** (OE Solution from a locally optimal strategy). If the algorithm stops then the tuple (G, B) computed in the last iteration is a solution to optimality equations.

Next, we show that in every non-terminating iteration of the algorithm, the pair (G, B) consisting of the gain function G and the bias function B, that are uniquely determined from the current pair of strategies χ and μ , strictly increases according to a certain linear ordering

as a result of the local improvement. This implies finite termination of the algorithm, since there are only finitely many positional strategies. Thus, together with Lemma 15, we get the existence of an OE Solution, which establishes positional determinacy of mean-payoff games on finite game arenas.

- ▶ **Theorem 16** (Strict global improvement from myopic improvement). Let μ (μ') be the best counter-strategy for player Min against a strategy χ (χ') for player Max, and let G (G') and B (B') be as computed in step 2 of an iteration of the algorithm starting from the strategy χ (χ' .) If the strategy χ' is a non-trivial local improvement of the strategy χ , as computed in step 3 of the algorithm, then for every state $s \in S$, the following hold.
- 1. We have $G'(s) \geqslant G(s)$.
- **2.** If G'(s) = G(s) then $B'(s) \ge B(s)$.
- 3. If $s \in S_{Max}$ and $\chi'(s) \neq \chi(s)$ then either G'(s) > G(s), or G'(s) = G(s) and B'(s) > B(s).

Proof. In order to verify property 1 it suffices to show that the average weight of every cycle reachable from a state s in the strategy subgraph $\Gamma_{\chi'}$ is no smaller than the smallest average weight of a cycle reachable from the state s in the strategy subgraph Γ_{χ} . First, observe that for every transition (s, a, s') in the subgraph of the graph $\Gamma_{\chi'}$, we have the inequality $G(s) \leq G(s')$. It implies in $\Gamma_{\chi'}$ we have that G(s) is smaller than average of the cheapest reachable cycle. On the other hand G'(s) is the average of the cheapest cycle in $\Gamma_{\chi'}$. It follows that $G(s) \leq G(s')$.

Now we argue that the properties 2 and 3 hold. From the assumption that G'(s) = G(s) it follows that the paths from the state s in graphs $\Gamma \chi \mu$ and $\Gamma_{\chi' \mu'}$ lead to the same cycle. We need to prove that $B'(s) \geq B(s)$ and that $\chi'(s) \neq \chi(s)$ implies B'(s) > B(s).

First, observe that for every transition (s,a,s') in the subgraph of the graph Γ_{χ} induced by the set of states W_s , we have the inequality $B(s) \leq (\pi(s,a) - G(s)) + B(s')$; it follows by the construction of B as the weights of shortest paths to the state s (step 2(b)). Moreover, by the definition of the myopic improvement of the strategy χ (step 3), for every transition (s,a,s') in the subgraph of the graph $\Gamma_{\chi'}$ induced by the set of states W'_s , we have $B(s) \leq (\pi(s,a) - G(s)) + B(s')$.

Now let $\langle s=s_0,a_1,s_1,\ldots,s_p=m_v\rangle$ be a path from the state s to the state m_s in the subgraph of the graph $\Gamma_{\chi'}$ induced by the set of states W'_s . Then adding the p inequalities $B(s_i) \leqslant (\pi(s_i,a_{i+1})-G(s_i))+B(s_{i+1})$, for $i=0,1,\ldots,p-1$, we get that $B(s) \leqslant \sum_{i=0}^{p-1} (\pi(s_i,a_{i+1})-G(s))$. This, however, implies that $B(s) \leqslant B'(s)$, since if the previous inequality holds for all the paths from s to m_s in the appropriate subgraph of $\Gamma_{\chi'}$, then it also holds for the shortest such. This establishes property 2. Property 3 now follows from the strictness of the inequality $B(s_0) < (\pi(s_0,a_1)-G(s_0))+B(s_1)$ if we assume that $\chi'(s) \neq \chi(s)$. Note that this is when the assumption at the end of step 3 is necessary to avoid looping without strict improvement of neither the gain nor the bias function from one iteration of the algorithm to another.

B Boundary Region Abstraction: Illustration

A PTGA is shown at the top of Figure 2. A sub-graph of BRA reachable from $(\ell_0, (0.3, 0.1), 0 < y < x < 1)$ is shown below the PTGA in the same figure. The names of the regions correspond to the regions depicted in the bottom right corner. Edges are labelled (a, c, b, ζ) and the intuitive meaning is to wait until clock c reaches the value b in the boundary of the region ζ . Considering the region ζ_1 , we see that it is determined by the constraints $(1 < x < 2) \land (0 < y < 1) \land (y < x - 1)$. The bold numbers on edges correspond to the time delay be-

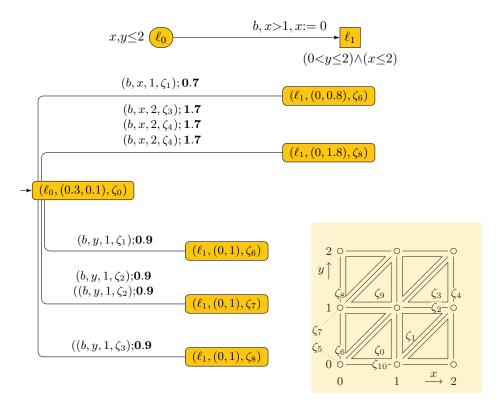


Figure 2 Sub-graph of the boundary region abstraction for the PTGA with the region names as depicted in the bottom right corner.

fore the action labelling the edge is taken. Figure 2 includes the actions available in the initial state and one of the action pairs that are available in the state $(\ell_1, (0, 1), (x=0) \land (1 < y < 2))$.

C Proofs from Section 4

C.1 Examples of PTGAs with non-affine Bias Functions

- ▶ Example 17. Consider the timed game shown in Figure 3. All the locations here belong to the player Min. There are two cycles in which the gains or the average weight of the cycles is $\frac{1}{2}$. If corresponding to two or more strategies, the gains are the same, then the strategy that minimizes the bias is chosen by the player Min. The bias B(s) of a player Min state $s = (\ell, \nu)$ where ℓ is a location of the priced timed game and ν is a clock valuation is given by $\inf_{(t,a)\in A(s)} \{\pi(s,(t,a)) G(s) + B(s') : s \xrightarrow{(t,a)} s' \text{ and } G(s) = G(s')\}$ if $s \in S_{\text{Min}}$. The inf is replaced by a sup for a player Max state. G(s) denotes the gain of state s. Considering the bias of the states (ℓ_2, ν) and (ℓ_5, ν) to be 0, the bias of a state (ℓ_1, ν) turns out to be $\frac{1}{2} \nu_x$ when the edge $(\ell_1 \to \ell_2)$ is considered and the bias of (ℓ_1, ν) is the constant function -1 when the edge $(\ell_1 \to \ell_4)$ is considered. Here ν_x denotes the value of clock x. For $x > \frac{3}{2}$, the value of $\frac{1}{2} \nu_x$ is smaller than -1 and hence the edge $(\ell_1 \to \ell_4)$ is chosen while the edge $(\ell_1 \to \ell_2)$ is chosen when $x \leq \frac{3}{2}$. Thus the bias is not regionally affine. This can be attributed to the fact that the gain is not integral.
- ▶ **Example 18.** Let us consider another example of one-clock priced timed games with three price rates: 0, 1 and -1 shown in Figure 4. The value of the game is $\frac{1}{4}$ which is obtained by

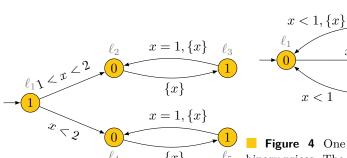


Figure 3 One clock PTGA with binary prices may not have regionally simple bias.

Figure 4 One clock PTGA with non-binary prices. The ε -optimal strategies of Min and Max players are not regionally positional boundary strategies and the value of the game is $\frac{1}{4}$.

the following sequence of moves by both players. From the initial state $(\ell_1, x = 0)$, player Min waits at ℓ_1 for 0.5 time units and then moves the token to location ℓ_2 . player Max moves the token to ℓ_4 immediately and subsequently the token reaches ℓ_1 with the value of clock $x = 1 - \delta$, where δ is an infintesimal positive quantity. The token is forwarded to ℓ_2 now instantaneously or after an infintesimal delay so that the value of clock x is still less than 1. player Max now forwards the token to ℓ_3 when the value of clock x becomes 1. At location ℓ_3 , $1 - \delta$ amount of time is elapsed. In the next move of player Max, the token reaches the initial state $(\ell_1, x = 0)$. The ε -optimal strategies of both Min and player Max are not regionally constant boundary strategies.

▶ Lemma 19. In one clock binary non-integral payoff PTGA the bias may not be regionally affine. The same is true for for the BRA of the PTGA.

Proof. Consider the example in Figure 3. There are two loops, one consisting of locations ℓ_2 and ℓ_3 and the other one with locations ℓ_4 and ℓ_5 . For the states (ℓ_2, ν) and (ℓ_3, ν) where ν is any valuation of clock x, the gain g equals $\frac{1}{2}$. For the states (ℓ_4, ν) and (ℓ_5, ν) such that $\nu_x \leq 1$ also have their gain g equal to $\frac{1}{2}$. Hence the corresponding PTGA is a non-integral payoff one. Note that all the states in the non-integral payoff PTGA in Figure 3 belong to the player Min. Since the gains corresponding to both the loops are the same, the optimal strategy of the player Min from a state (ℓ_1, ν) is determined by the bias of a successor state.

We now show that the bias of a state (ℓ_1, ν) may not be regionally affine. For computing the bias in the loop consisting of the locations ℓ_2 and ℓ_3 , let us consider that $B(\ell_2, \nu) = 0$. Thus considering the successor state of a state (ℓ_1, ν) to be the state $(\ell_2, \nu + t)$, we have

$$\begin{split} B(\ell_1,\nu) &= \inf_{t:1<\nu+t<2} \ t-g+B(\ell_2,\nu+t) \\ &= \inf_{t:1-\nu< t<2-\nu} \ t-\frac{1}{2}+0 = 1-\nu-\frac{1}{2} = \frac{1}{2}-\nu. \end{split}$$

In the loop consisting of locations ℓ_4 and ℓ_5 , let $B(\ell_5, \nu) = 0$. Now

$$B(\ell_4, \nu) = \inf_t t - g + B(\ell_5, 0) = 0 - \frac{1}{2} + 0 = -\frac{1}{2}.$$

Thus considering the successor state of a state (ℓ_1, ν) to be the state $(\ell_4, \nu + t)$, we have

$$B(\ell_1, \nu) = \inf_{t:0 \le t < 2} t - g + B(\ell_4, \nu + t) = 0 - \frac{1}{2} - \frac{1}{2} = -1.$$

Hence the bias at (ℓ_1, ν) is given by $\min(\frac{1}{2} - \nu_x, -1)$. This gives us that the edge $\ell_1 \to \ell_2$ is chosen when $x \leq \frac{3}{2}$, while the edge $\ell_1 \to \ell_4$ is chosen when $x > \frac{3}{2}$. Hence the bias of location ℓ_1 is not regionally affine in non-integral payoff PTGA. It is easy to see that the lemma also holds for the BRA on the PTGA.

C.2 Regionally Simple Functions: Properties

Simple functions were introduced by Asarin and Maler [3] in the context of reachability timed games for timed automata.

- ▶ **Lemma 20** (Simple Functions and Their Properties). A function $f: V \to \mathbb{R}_{\geq 0}$ is simple if \blacksquare either, there exists $d \in \mathbb{N}$ such that for all $\nu \in V$ we have $f(\nu) = d$;
- or, there exists $d \in \mathbb{N}$ and $c \in \mathcal{X}$ such that for all $\nu \in V$ we have $f(\nu) = d \nu(c)$.

We say that a function $f: \widehat{S} \to \mathbb{R}_{\geqslant 0}$ is regionally simple if for all $\ell \in L$, $\zeta, \zeta' \in \mathcal{R}$ either i) there exists a $d \in \mathbb{N}$ such that $f(\ell, \nu, \zeta') = d$ for all $\nu \in \zeta$; or ii) there exists $d \in \mathbb{N}$ and $c \in \mathcal{X}$ such that $f(\ell, \nu, \zeta') = d - \nu(c)$ for all $\nu \in \zeta$. The simple functions have the following properties.

- 1. Every simple function is an affine function.
- 2. Every regionally simple function is a regionally affine function.
- **3.** For $\ell \in L$, $\zeta, \zeta' \in \mathcal{R}$ let $D_{\ell,\zeta,\zeta'} = \{(\ell,\nu,\zeta') : [\nu] = \zeta\}$. For simple functions $f,g: D_{\ell,\zeta,\zeta'} \to \mathbb{R}$ we have that
 - $= \min\{f, g\}$ is either f or g.
 - $= \max\{f, g\}$ is either f or g.
- 4. Minimum and maximum of a finite set of regionally simple functions is regionally simple.
- **5.** Let $B: \widehat{S} \to \mathbb{R}$ be a regionally simple function and $G: \widehat{s} \to \mathbb{N}$ be a regionally constant function. Let $B_{\alpha}^{\oplus}: \widehat{S} \to \mathbb{R}$ as $\widehat{s} \mapsto \pi(\widehat{s}, \alpha) G(\widehat{s}) + B(\widehat{s}')$ where $\widehat{s} \xrightarrow{\alpha} \widehat{s}'$. We have that B_{α}^{\oplus} is a regionally simple function.

Proof. The proof is in many parts.

- 1. Trivial.
- 2. Trivial.
- **3.** Let $f, g: D_{\ell,\zeta,\zeta'} \to \mathbb{R}$. There are four cases to consider.
 - For all $\nu \in \zeta$ we have $f(\nu) = d_1$ and $g(\nu) = d_2$. If $d_1 < d_2$ the min $\{f, g\} = f$ else min $\{f, g\} = g$. Similarly if $d_1 < d_2$ then max $\{f, g\} = g$ else min $\{f, g\} = f$.
 - For all $\nu \in \zeta$ we have $f(\nu) = d_1$ and $g(\nu) = d_2 \nu(c_2)$. Notice that if for some valuation $d_1 < d_2 \nu(c_2)$ then for all valuations $d_1 < d_2 \nu'(c_2)$ since the difference between various valuations in a region is strictly less than 1. The lemma follows for this case from this observation.
 - For all $\nu \in \zeta$ we have $f(\nu) = d_1 \nu(c_1)$ and $g(\nu) = d_2$. This case is analogous to previous case and omitted.
 - For all $\nu \in \zeta$ we have $f(\nu) = d_1 \nu(c_1)$ and $g(\nu) = d_2 \nu(c_2)$. If for some region $d_1 \nu(c_1) < d_2 \nu(c_2)$ then for all valuations this inequality will follow since difference between various values of clocks from different valuations in a region is strictly smaller than 1.
- **4.** Similar to property 3.
- **5.** Let $B: \widehat{S} \to \mathbb{R}$ be a regionally simple function and $G: \widehat{s} \to \mathbb{N}$ be a regionally constant function. Let $B_{\alpha}^{\oplus}: \widehat{S} \to \mathbb{R}$ as $\widehat{s} \mapsto \pi(\widehat{s}, \alpha) G(\widehat{s}) + B(\widehat{s}')$ where $\widehat{s} \xrightarrow{\alpha} \widehat{s}'$. Let $\widehat{s} = (\ell, \nu, \zeta)$ and $\widehat{s}' = (\ell', \nu', \zeta')$. Let $g \in \mathbb{N}$ be such that $G(\ell, \nu, \zeta) = g$. There are eight cases to consider.

- $B(\ell', \cdot, \zeta')$ is $\nu \mapsto d$, $p(\ell) = 0$ and $c_{\alpha} \notin \rho(a_{\alpha})$. In this case $B(\ell, \cdot, \zeta) = p(\ell, a_{\alpha}) g + d$ (an integer) is a simple function.
- $B(\ell', \cdot, \zeta')$ is $\nu \mapsto d$, $p(\ell) = 0$ and $c_{\alpha} \in \rho(a_{\alpha})$ In this case $B(\ell, \cdot, \zeta) = p(\ell, a_{\alpha}) g + d$ (an integer) is a simple function.
- $B(\ell', \cdot, \zeta')$ is $\nu \mapsto d$, $p(\ell) = 1$ and $c_{\alpha} \notin \rho(a_{\alpha})$ In this case $B(\ell, \cdot, \zeta) = (b_{\alpha} \nu(c_{\alpha})) + p(\ell, a_{\alpha}) g + d = K \nu(c_{\alpha})$ is a simple function.
- $B(\ell', \cdot, \zeta')$ is $\nu \mapsto d$, $p(\ell) = 1$ and $c_{\alpha} \in \rho(a_{\alpha})$ In this case $B(\ell, \cdot, \zeta) = (b_{\alpha} \nu(c_{\alpha})) + p(\ell, a_{\alpha}) g + d = K \nu(c_{\alpha})$ is a simple function.
- $B(\ell', \cdot, \zeta')$ is $\nu \mapsto d \nu(c)$, $p(\ell) = 0$ and $c_{\alpha} \notin \rho(a_{\alpha})$ In this case $B(\ell, \cdot, \zeta) = p(\ell, a_{\alpha}) g + d (\nu(c) + b_{\alpha} \nu(c_{\alpha})) = p(\ell, a_{\alpha}) g + d b_{\alpha}$ (an integer) is a simple function. We have $\nu(c) = \nu(c_{\alpha})$ as there in only one clock.
- $B(\ell', \cdot, \zeta')$ is $\nu \mapsto d \nu(c)$, $p(\ell) = 0$ and $c_{\alpha} \in \rho(a_{\alpha})$ In this case $B(\ell, \cdot, \zeta) = p(\ell, a_{\alpha}) g + d$ (an integer) is a simple function.
- $B(\ell', \cdot, \zeta')$ is $\nu \mapsto d \nu(c)$, $p(\ell) = 1$ and $c_{\alpha} \notin \rho(a_{\alpha})$ In this case $B(\ell, \cdot, \zeta) = (b_{\alpha} \nu(c_{\alpha})) + p(\ell, a_{\alpha}) g + d (\nu(c) + b_{\alpha} \nu(c_{\alpha})) = (b_{\alpha} \nu(c_{\alpha})) + p(\ell, a_{\alpha}) g + d b_{\alpha} = K \nu(c_{\alpha})$ is a simple function.
- $B(\ell', \cdot, \zeta')$ is $\nu \mapsto d \nu(c)$, $p(\ell) = 1$ and $c_{\alpha} \in \rho(a_{\alpha})$ In this case $B(\ell, \cdot, \zeta) = (b_{\alpha} \nu(c_{\alpha})) + p(\ell, a_{\alpha}) g + d = K \nu(c_{\alpha})$ is a simple function.

The proof is now complete.

C.3 Proof of Lemma 10

Let $\widehat{\mathsf{T}}(\chi,\mu)$ be a zero-player game on the subgraph where strategies of player Max (and Min) is fixed to RCPSs χ (and μ). We sketch an algorithm ComputeValueZeroPlayer(T,μ,χ) which returns the solution of optimality equations $\mathrm{Opt}(\mathsf{T}(\mu,\chi))$. Observe that for $\mathsf{T}(\mu,\chi)$ the unique runs originating from states $\hat{s}_0 = (\ell,\nu,\zeta)$ and $\hat{s}'_0 = (\ell,\nu',\zeta)$ with $[\nu] = [\nu']$ follow the same "lasso" after one step, i.e. the unique runs

$$\hat{s}_0 \xrightarrow{\alpha_1} \hat{s}_1 \cdots \hat{s}_k \left(\xrightarrow{\alpha_{k+1}} \cdots \hat{s}_{k+N-1} \xrightarrow{\alpha_{k+N}} \hat{s}_k \right)^*$$
 and $\hat{s}'_0 \xrightarrow{\alpha_1} \hat{s}'_1 \cdots \hat{s}'_k \left(\xrightarrow{\alpha_{k+1}} \cdots \hat{s}'_{k+N-1} \xrightarrow{\alpha_{k+N}} \hat{s}'_k \right)^*$

are such that for $\hat{s}_i = (\ell_i, \nu_i, \zeta_i)$ and $\hat{s}'_i = (\ell'_i, \nu'_i, \zeta'_i)$ we have that $\ell_i = \ell'_i$, $\zeta_i = \zeta'_i$ and $\nu_i = \nu'_i$ for all $i \in [1, k + N - 1]$. This is so because for one-clock timed automata the successors of the states $\hat{s}_0 = (\ell, \nu, \zeta)$ and $\hat{s}'_0 = (\ell, \nu', \zeta)$ for action $\alpha_1 = (b, c, a, \zeta')$ is the same (ℓ'', ν'', ζ'') where $\nu''(c) = \nu(c) + (b - \nu(c)) = b = \nu(c) + (b - \nu(c))$ if $c \notin \rho(a)$ and $\nu''(c) = 0$ otherwise. Consider the optimality equations for the lasso.

$$G(\hat{s}_{0}) = G(\hat{s}_{1}) = \cdots = G(\hat{s}_{k}) = \cdots G(\hat{s}_{k+N-1}) = g \text{ say}$$

$$B(\hat{s}_{0}) = \pi(\hat{s}_{0}, \alpha_{1}) - g + B(\hat{s}_{1})$$

$$B(\hat{s}_{1}) = \pi(\hat{s}_{1}, \alpha_{2}) - g + B(\hat{s}_{2})$$

$$\cdots \qquad \cdots$$

$$B(\hat{s}_{k}) = \pi(\hat{s}_{k}, \alpha_{k+1}) - g + B(\hat{s}_{k+1})$$

$$B(\hat{s}_{k+1}) = \pi(\hat{s}_{k+1}, \alpha_{k+1}) - g + B(\hat{s}_{k+3})$$

$$\cdots \qquad \cdots$$

$$B(\hat{s}_{k+N-1}) = \pi(\hat{s}_{k+N-1}, \alpha_{k+N}) - g + B(\hat{s}_{k})$$
(3)

We show this via an induction on the distance from the region whose biases we fixed earlier to 0. Let $\hat{s}_0 \xrightarrow{\alpha_1} \hat{s}_1 \xrightarrow{\alpha_2} \cdots \hat{s}_k$ be the run such that $\hat{s}_i = (\ell_i, \nu_i, \zeta_i)$ and for all states

 $B(\ell_k, \cdot, \zeta_k) = 0$. We prove via induction that $B(\ell_i, \cdot, \zeta_i)$ is a simple function. The base case follows easily since $B(\ell_k, \cdot, \zeta_k) = 0$ which is a simple function. Assume that $B(\ell_i, \cdot, \zeta_i)$ is simple we show that $B(\ell_{i-1}, \cdot, \zeta_{i-1})$ defined as

$$\nu \mapsto p(\ell_{i-1}) \cdot (b_{\alpha} - \nu(c_{\alpha})) + p(\ell_{i-1}, a_{\alpha}) - g + B(\ell_i, \nu_{\alpha}, \zeta_i)$$

is also simple where $\alpha_i = (b_\alpha, c_\alpha, a_\alpha, \zeta_\alpha)$. This follows form property 5 of Lemma 20.

C.4 Proof of Lemma 11 and Lemma 12

Proof. To prove this lemma it suffices to show that the function

$$M^*(\cdot, G, B) : \hat{s} \mapsto \underset{\alpha \in \widehat{A}}{\operatorname{argmax}}^{\operatorname{lex}} \{ (G(\hat{s}'), \pi(\hat{s}, \alpha) - G(\hat{s}) + B(\hat{s}')) : \hat{s} \xrightarrow{\alpha} \hat{s}' \}$$

is regionally constant. We already have that G is regionally constant. From the property 5 of Lemma 20 it follows that $\hat{s} \mapsto \pi(\hat{s}, \alpha) - G(\hat{s}) + B(\hat{s}')$ is regionally simple. Note that since maximum (Property 3 of Lemma 20 of any finite set of simple functions over a region is one of the functions from the set, it follows that the set $M^*(\cdot, G, B)$ is regionally constant. The proof for the Lemma 12 is similar and hence omitted.

C.5 Proof of Theorem 13

The following two Lemmas together with finiteness of regionally constant positional strategies give the proof of Theorem 13. The proofs of these lemma are similar to Lemma 16 and hence omitted.

- ▶ Lemma 21. Let $(G, B) = Opt(\mathsf{T}(\chi))$. If $\chi' = \mathsf{IMPROVEMAXSTRATEGY}(\mathsf{T}, \chi, G, B)$ then $(G'B') = Opt(\mathsf{T}(\chi'))$ is such that $(G', B') \ge (G, B)$ and if $\chi' \ne \chi$ then (G', B') > (G, B).
- ▶ Lemma 22. Let $(G, B) = Opt(\mathsf{T}(\chi, \mu))$. If $\mu' = \mathsf{IMPROVEMINSTRATEGY}(\mathsf{T}, \mu, G, B)$ then $(G'B') = Opt(\mathsf{T}(\chi, \mu'))$ is such that $(G', B') \leq (G, B)$ and if $\mu' \neq \mu$ then (G', B') < (G, B).

D Proofs from Section 5

We prove our undecidability result using a reduction from two-counter machine. A two-counter machine M is a tuple (L,C) where $L=\{\ell_0,\ell_1,\ldots,\ell_n\}$ is the set of instructions—including a distinguished terminal instruction ℓ_n called HALT—and $C=\{c_1,c_2\}$ is the set of two *counters*. The instructions L are one of the following types:

- 1. (increment c) $\ell_i : c := c + 1$; goto ℓ_k ,
- **2.** (decrement c) $\ell_i : c := c 1$; goto ℓ_k ,
- **3.** (zero-check c) ℓ_i : if (c>0) then goto ℓ_k else goto ℓ_m ,
- **4.** (Halt) ℓ_n : HALT.

where $c \in C$, $\ell_i, \ell_k, \ell_m \in L$. A configuration of a two-counter machine is a tuple (l, c, d) where $l \in L$ is an instruction, and c, d are natural numbers that specify the value of counters c_1 and c_2 , respectively. The initial configuration is $(\ell_0, 0, 0)$. A run of a two-counter machine is a (finite or infinite) sequence of configurations $\langle k_0, k_1, \ldots \rangle$ where k_0 is the initial configuration, and the relation between subsequent configurations is governed by transitions between respective instructions. The run is a finite sequence if and only if the last configuration is the terminal instruction ℓ_n . Note that a two-counter machine has exactly one run starting from the initial configuration. The halting problem for a two-counter machine asks whether its unique run ends at the terminal instruction ℓ_n . It is well known ([20]) that the halting problem for two-counter machines is undecidable.

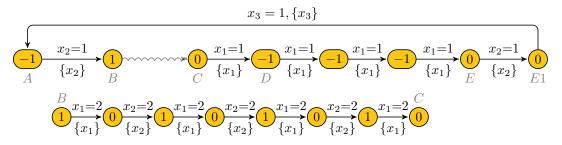


Figure 5 WD_1^1 redrawn with location prices from $\{1,0,-1\}$. Every location has a self loop with the guard $x_2, x_3 = 1$, reset x_2, x_3 , which is not shown here for conciseness. The curly edge from B to C is shown below. The mean-payoff incurred in one transit from A to A via E is $\frac{\epsilon}{14}$. If Min makes no error, this is 0.

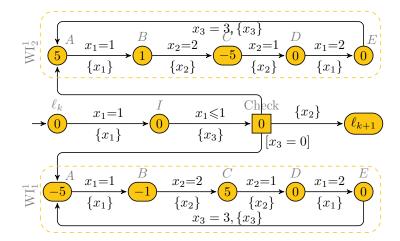


Figure 6 Simulation to increment counter C_1 , mean cost is ε for error ε . As seen in Figure 5, the widgets WI₂ and WI₁ can be redrawn with location prices in $\{0, 1, -1\}$.

D.1 Proofs for undecidability of mean-payoff (per-transition) games

Simulation of increment instruction.: The module to increment C_1 is given in Fig. 6. Again, we start at ℓ_k with $x_1 = \frac{1}{5^{c_1}7^{c_2}}, x_2 = 0$ and $x_3 = 0$. Let $\frac{1}{5^{c_1}7^{c_2}}$ be called as x_{old} . A time of $1 - x_{old}$ is spent at ℓ_k . Let the time spent at I be denoted x_{new} . To correctly increment counter 1, x_{new} must be $\frac{x_{old}}{5}$. No time is spent at Check. Player Max can either continue simulation of the next instruction, or can enter one of the widgets WI₁, WI₂ to verify if x_{new} is indeed $\frac{x_{old}}{5}$.

If player Min makes an error by elapsing a time $\frac{x_{old}}{5} + \varepsilon$, then player Max chooses the widget WI₁¹. It can be seen that the cost of going from A to E once in WI₁¹ is 5ε . The mean cost incurred by Min in going from E to E once, when Max chooses WI₁¹ is hence E. Similarly, if Min makes an error by elapsing a time $\frac{x_{old}}{5} - \varepsilon$, then player Max chooses the widget WI₂¹. The mean cost incurred in one transit from E to E through E in WI₂¹ is again E. If Min makes no simulation error, then the mean cost incurred is E.

Simulation of Zero-check.: Fig. 7 shows the module for zero-check instruction for counter C_2 . ℓ_k is a no time elapse location, from where, player Min chooses one of the locations $\operatorname{Check}_{c_2=0}$ or $\operatorname{Check}_{c_2\neq 0}$. Both these are player Max locations, and player Max can either continue the simulation, or can go to the check widgets $W_2^{=0}$ or $W_2^{\neq 0}$ to verify the correctness of player Min's choice. The widgets $W_2^{=0}$ and $W_2^{\neq 0}$ are given in Fig. 8 and Fig. 9 respectively.

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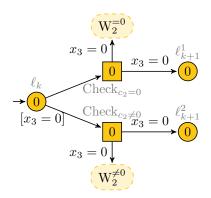


Figure 7 Widget WZ₂ simulating zero-check for C_2

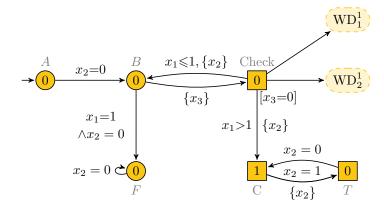


Figure 8 Widget W₂⁼⁰

- Consider the case when player Min guessed that C_2 is zero, and entered the location Check $_{c_2=0}$ in Fig. 7. Let us assume that player Max verifies player Min's guess by entering $W_2^{=0}$ (Fig. 8). No time is spent in the initial location A of $W_2^{=0}$. We are therefore at B with $x_1 = \frac{1}{5^{c_1}T^{c_2}} = x_{old}$ and $x_2, x_3 = 0$. In case $c_1 = c_2 = 0$, we can directly go to the F state, and keep looping there forever, incurring mean cost 0. If that is not the case, player Min has to prove his claim right, by multiplying x_1 with 5 repeatedly, till x_1 becomes 1; clearly, this is possible iff $c_2 = 0$. The loop between B and Check precisely does this: each time player Min spends a time x_{delay} in B, player Max can verify that $x_{delay} = 4x_{old}$ by going to WD_1^1 or WD_2^1 or come back to B. No time is elapsed in Check. Finally, if $x_1 = 1$, we can go to F, and player Min achieves his objective. However, if C_2 was non-zero, then x_1 will never reach 1 after repeatedly multiplying x_1 with 5; in this case, at some point, the edge from Check to C will be enabled. In this case, the infinite loop between C and T, will lead to a mean cost greater than 0.
- Consider now the case when player Min guessed that C_2 is non-zero, and hence entered the location $\operatorname{Check}_{c_2\neq 0}$ in Fig. 7. Let us assume now that player Max enters $W_2^{\neq 0}$ (Fig. 9) to verify player Min's guess. Similar to $W_2^{=0}$, no time is spent at location A of $W_2^{\neq 0}$, and the clock values at B are $x_1 = \frac{1}{5^{c_1}7^{c_2}} = x_{old}$ and $x_2, x_3 = 0$. If $c_1 = c_2 = 0$, then $x_1 = 1$, in which case, the location D is reached, from where, the loop between D, T is

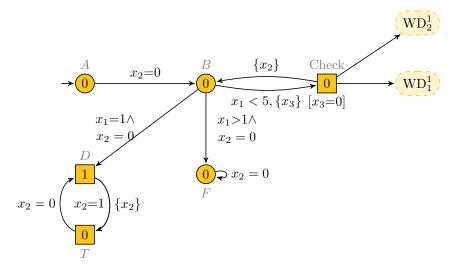


Figure 9 Widget $W_2^{\neq 0}$

taken incurring a mean cost greater than 0.

If that is not the case, player Min, to prove his claim, repeatedly multiplies x_1 by 5 using the loop between B and Check. x_1 becomes 1 iff $c_2 = 0$. Once x_1 becomes 1, the edge from B to D will be enabled. In this case, the infinite loop between D and T, will lead to a mean cost greater than 0. Note that once x_1 becomes 1, player Min can also wait in B and transit to Check. However, due to the guard on the edge from B to Check, the delay at B will be less than $4x_{old}$, (when $x_1 = 1$, $x_1 = x_{old} = \frac{1}{5^{c_1}7^{c_2}} = 1$, $c_1 = c_2 = 0$) say $4x_{old} - \varepsilon$ in which case too player Max can go to WD $_1^1$ or WD $_2^1$ and the mean cost will be ε . If C_2 was non-zero, then x_1 will never reach 1 after repeatedly multiplying x_1 with 5; in this case, at some point, x_1 will be greater than 1 and the edge from B to F will be enabled and player Min can achieve his objective by moving to F.

Correctness of the construction. On entry into the location ℓ_n (for HALT instruction), we reset clock x_1 to 0; from ℓ_n , we go to a state F with price 1, with a self loop that checks $x_1 = 1$, and resets x_1 .

- 1. Assume that the two counter machine halts. If player Min simulates all the instructions correctly, he will incur a mean cost > 0, by either reaching the F after ℓ_n : when player Min does not cheat, player Max has no incentive to enter any of the check widgets, he will just let the computation continue, till the Halt location is reached. This will incur a mean cost > 0. If player Min makes an error in his computation, player Max can always enter an appropriate widget, making the mean cost > 0. In summary, if the two counter machine halts, then player Min has no strategy to achieve his goal (mean pay off ≤ 0).
- 2. Assume that the two counter machine does not halt.
 - If player Min simulates all the instructions correctly, and if player Max never enters a check widget, then player Min incurs cost 0, since all locations in the main modules have price 0. Even if player Max enters some widget, the mean cost of player Min is still 0, since no errors were made by Min. However, if player Min makes an error, player Max can enter a check widget, ensuring that the mean cost is > 0. Thus, if the two counter machine does not halt, player Min has a strategy (by making no errors) to achieve mean cost 0.

In summary, if the two counter machine does not halt, player Min has a strategy to

achieve his goal, and vice-versa.

Thus, player Min incurs a mean cost ≤ 0 iff he chooses the strategy of faithfully simulating the two counter machine, when the machine does not halt. When the machine halts, the mean cost incurred by player Min is more than 0 irrespective of whether he makes a simulation error or not.

MPG(T, r) problem, r > 0

Now we argue that the $\mathsf{MPG}(\mathsf{T},r)$ problem is undecidable for PTGAs with $\geqslant 3$ clocks, having only binary location prices and no edge prices, for $r=\frac{1}{3}$. We note that in the previous undecidability result, all the modules apart from the WD_1^1,WD_2^1,WI_1^1 and WI_2^1 use binary location prices. Thus we now give here these modules with only binary prices.

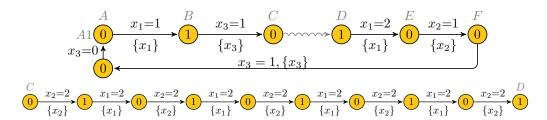


Figure 10 WD_1^1 here has locations with binary price rates. All locations have a self loop $x_3=1?$, reset x_3 . This module is chosen by the Max player when the error made by Min player during the decrement operation is ε . The curly edge from C to D is shown by the path below. The mean-payoff incurred in one transit from A to A via F is $\frac{1}{3} + \frac{\varepsilon}{15}$. If Min makes no error, the cost is $\frac{1}{3}$.

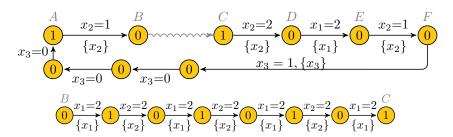


Figure 11 WD_2^1 here has locations with binary price rates. All locations have a self loop $x_3 = 1$?, reset x_3 . This module is chosen by the Max player when the error made by Min player during the decrement operation is $-\varepsilon$. The curly edge from C to D is shown by the path below. The mean-payoff incurred in one transit from A to A via F is $\frac{\epsilon}{3} + \frac{\epsilon}{15}$. If Min makes no error, it is $\frac{1}{3}$.

Fig. 10 shows the module WD_1^1 while Fig. 11 shows the module WD_2^1 with location prices 0 and 1. The modules WI_1^1 and WI_2^1 can also be similarly made with only binary prices. The module WD_1^1 is chosen by the Max player when the error made by the Min player while simulating the decrement counter C_1 operation is ε . The module WD_2^1 is chosen by the Max player when the error made by the Min player is $-\varepsilon$. When no error is made by the Min player, the mean cost WD_1^1 is $\frac{1}{3}$ while the cost in the presence of an error is $\frac{1}{3} + \frac{\varepsilon}{15}$.

The modules for zerocheck (Figure 7) can be used as they are. In case an error is made in the guess, the meanpayoff incurred by shuttling between locations C, T (Figure 8) or D, T (Figure 9) is $\frac{1}{2} > \frac{1}{3}$.

D.2 Proofs for undecidability of mean-payoff (per-time-unit) games

Mean-payoff game (per time unit) has been studied in [10]. The mean-payoff (per time unit) of a play is defined as the long-run average of cost per time unit. Formally, the mean payoff of a play $\operatorname{Run}(s,\mu.\chi) = \sigma = (\ell_0,\nu_0) \xrightarrow{(d_0,a_0)} (\ell_1,\nu_1) \xrightarrow{(d_1,a_1)} \cdots$ starting from a state s in which players Min and Max play according to μ and χ respectively is

$$\lim_{n \to \infty} \frac{\sum_{i=0}^{n} (p(\ell_i, a) + p(\ell_1) \cdot t_i)}{\sum_{i=0}^{n} t_i}.$$

We refine the undecidability result in [10] by showing the following result.

▶ **Theorem 23.** The mean-payoff problem $\mathsf{MPG}(\mathsf{T},r)$ is undecidable for PTGA T with 3 clocks having location-wise price-rates $\pi(\ell) \in \{0,1,-1\}$ for all $\ell \in L$ and r=0. Moreover, it is undecidable for binary-priced T with 3 clocks and r>0.

Similar to the proof of Theorem 14, this proof also involves reduction from the non-halting problem of two counter machines.

Proof. We first show the undecidability of $\mathsf{MPG}(\mathsf{T},0)$ with location prices $\{1,0,-1\}$ and no edge prices. We prove the result by reducing the non-halting problem of 2 counter machines. Given a two counter machine \mathcal{M} , we construct a PTGA Γ with 3 clocks x_1, x_2, x_3 , and arbitrary location prices, but no edge prices, and show that player Min has a winning strategy iff \mathcal{M} does not halt.

We specify a module for each instruction of the two counter machine. On entry into a module, we have $x_1 = \frac{1}{5^{c_1}7^{c_2}}$, $x_2 = 0$ and $x_3 = 0$, where c_1, c_2 are the values of counters C_1, C_2 . We construct the PTGA Γ whose building blocks are the modules for instructions. The role of player Min is to faithfully simulate the two counter machine, by choosing appropriate delays to adjust the clocks to reflect changes in counter values. Player Max will have the opportunity to verify that player Min did not cheat while simulating the machine. We shall now present modules for increment, decrement and zero check instructions. For conciseness of the figures, we present here modules using arbitrary prices. However, we can redraw these with extra locations and edges using only the location prices from $\{1,0,-1\}$, as was done in the case of Theorem 14.

Simulation of decrement instruction. The module to simulate the decrement of counter C_1 is given in Figure 12. We enter location ℓ_k with $x_1 = \frac{1}{5^{c_1}7^{c_2}}$, $x_2 = 0$ and $x_3 = 0$. Lets denote by x_{old} the value $\frac{1}{5^{c_1}7^{c_2}}$. To correctly decrement C_1 , player Min should choose a delay of $4x_{old}$ at location ℓ_k . At location Check, there is no time elapse. Player Max has three possibilities: (i) to go to ℓ_{k+1} , or (ii) to enter the widget WD₁¹, or (iii) to enter the widget WD₂¹. If player Min makes an error, and delays $4x_{old} + \varepsilon$ or $4x_{old} - \varepsilon$ at ℓ_k ($\varepsilon > 0$), then player Max can enter one of the widgets and punish player Min. Player Max enters widget WD₁¹ if the error made by player Min is of the form $4x_{old} + \varepsilon$ at ℓ_k and enters widget WD₂¹ if the error made by player Min is of the form $4x_{old} - \varepsilon$ at ℓ_k . Let us examine the widget WD₁¹.

When we enter WD₁¹ for the first time, we have $x_1 = x_{old} + 4x_{old} + \varepsilon$, $x_2 = 4x_{old} + \varepsilon$ and $x_3 = 0$. In WD₁¹, the cost of going once from location A to E is 5ε . Also, when we get back to A after going through the loop once, the clock values with which we entered WD₁¹ are restored; thus, each time, we come back to A, we restore the starting values with which we enter WD₁¹. The third clock is really useful for this purpose only. The total time elapsed in 3 time units; at A, the value of clock x_3 is 0 and the transition from E to A is enabled when $x_3 = 3$. Hence the mean cost of transiting from A to A through E is $\frac{5\varepsilon}{3}$. In a similar

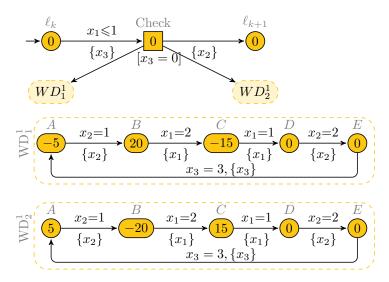


Figure 12 Simulation to decrement counter C_1 , mean cost is $\frac{5\varepsilon}{3}$ for error ε

way, it can be checked that the mean cost of transiting from A to A through E in widget WD_2^1 is $\frac{5\varepsilon}{3}$ when player Min chooses a delay $4x_{old} - \varepsilon$ at ℓ_k . Thus, if player Min makes a simulation error, player Max can always choose to go to one of the widgets, and ensure that the mean pay-off is $not \leq 0$. Note however that when $\varepsilon = 0$, then player Min will always achieve his objective: the mean pay-off will be 0.

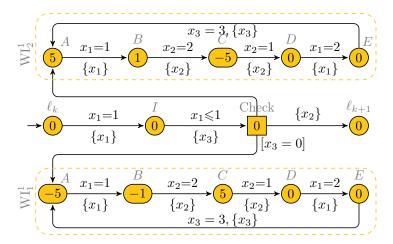


Figure 13 Simulation to increment counter C_1 , mean cost is $\frac{5\varepsilon}{3}$ for error ε

Simulation of increment instruction.: The module to increment C_1 is given in Fig. 13. Again, we start at ℓ_k with $x_1 = \frac{1}{5^{c_1}7^{c_2}}$, $x_2 = 0$ and $x_3 = 0$. Let $\frac{1}{5^{c_1}7^{c_2}}$ be called as x_{old} . A time of $1 - x_{old}$ is spent at ℓ_k . Let the time spent at I be denoted x_{new} . To correctly increment counter 1, x_{new} must be $\frac{x_{old}}{5}$. No time is spent at Check. Player Max can either continue simulation of the next instruction, or can enter one of the widgets WI_1^1, WI_2^1 to verify if x_{new} is indeed $\frac{x_{old}}{5}$.

If player Min makes an error by elapsing a time $\frac{x_{old}}{5} + \varepsilon$, then player Max chooses the widget WI₁¹. It can be seen that the cost of going from A to E once in WI₁¹ is 5ε and the time spent in moving from A to A through E is 3 time units. Hence the mean cost incurred

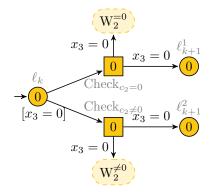


Figure 14 Widget WZ₂ simulating zero-check for C_2

by Min in going from A to A through E once, when Max chooses WI_1^1 is hence $\frac{5\varepsilon}{3}$. Similarly, if Min makes an error by elapsing a time $\frac{x_{old}}{5} - \varepsilon$, then player Max chooses the widget WI_2^1 . The mean cost incurred in one transit from A to A through E in WI_2^1 is again $\frac{5\varepsilon}{3}$. If Min makes no simulation error, then the mean cost incurred is 0.

Simulation of Zero-check.: Figure 14 shows the module for zero-check instruction for counter C_2 . ℓ_k is a no time elapse location, from where player Min chooses one of the locations $\operatorname{Check}_{c_2=0}$ or $\operatorname{Check}_{c_2\neq 0}$. Both these are player Max locations, and player Max can either continue the simulation, or can go to the check widgets $W_2^{=0}$ or $W_2^{\neq 0}$ to verify the correctness of player Min's choice. The widgets $W_2^{=0}$ and $W_2^{\neq 0}$ are given in Figure 15 and Figure 16 respectively.

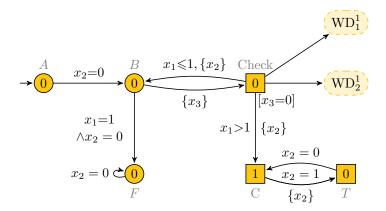


Figure 15 Widget $W_2^{=0}$

Consider the case when player Min guessed that C_2 is zero, and entered the location $\operatorname{Check}_{c_2=0}$ in Figure 14. Let us assume that player Max verifies player Min's guess by entering $W_2^{=0}$ (Figure 15). No time is spent in the initial location A of $W_2^{=0}$. We are therefore at B with $x_1 = \frac{1}{5^{c_1}7^{c_2}} = x_{old}$ and $x_2, x_3 = 0$. In case $c_1 = c_2 = 0$, we can directly go to the F state, and keep looping there forever, incurring mean cost 0. If that is not the case, player Min has to prove his claim right, by multiplying x_1 with 5 repeatedly, till x_1 becomes 1; clearly, this is possible iff $c_2 = 0$. The loop between B and Check precisely does this: each time player Min spends a time x_{delay} in B, player Max can verify that $x_{delay} = 4x_{old}$ by going to WD_1^1 or WD_2^1 or come back to B. No time is elapsed in Check. Finally, if $x_1 = 1$, we can go to F, and player Min achieves

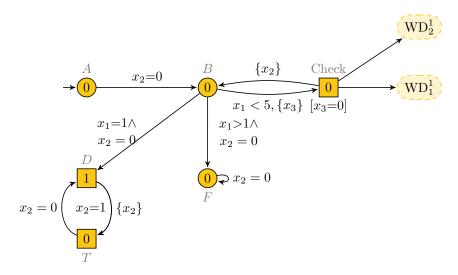


Figure 16 Widget $W_2^{\neq 0}$

his objective. However, if C_2 was non-zero, then x_1 will never reach 1 after repeatedly multiplying x_1 with 5; in this case, at some point, the edge from Check to C will be enabled. In this case, the infinite loop between C and T, will lead to a mean cost of 1.

Consider now the case when player Min guessed that C_2 is non-zero, and hence entered the location $\operatorname{Check}_{c_2\neq 0}$ in Fig. 14. Let us assume now that player Max enters $W_2^{\neq 0}$ (Fig. 16) to verify player Min's guess. Similar to $W_2^{=0}$, no time is spent at location A of $W_2^{\neq 0}$, and the clock values at B are $x_1 = \frac{1}{5^{c_1}7^{c_2}} = x_{old}$ and $x_2, x_3 = 0$. If $c_1 = c_2 = 0$, then $x_1 = 1$, in which case, the location D is reached, from where, the loop between D, T is taken incurring a mean cost of 1. If that is not the case, player Min repeatedly multiplies x_1 by 5 using the loop between B and C heck. x_1 becomes 1 iff $c_2 = 0$. Once x_1 becomes 1, the edge from B to D will be enabled. In this case, the infinite loop between D and T, will lead to a mean cost of 1. Note that if once x_1 becomes 1, player Min could also wait in B and transit to C heck. However, due to the guard on the edge from B to C heck, the delay at B will be less than $4x_{old}$, where $x_{old} = 1$. Say the delay is $4x_{old} - \varepsilon$ in which case too player Max can go to WD_1^1 or WD_2^1 and the mean cost will be greater than 0. If C_2 was non-zero, then x_1 will never reach 1 after repeatedly multiplying x_1 with 5; in this case, at some point, x_1 will be greater than 1 and the edge from B to F will be enabled and player Min can achieve its objective by moving to F.

Correctness of the construction. On entry into the location ℓ_n (for HALT instruction), we reset clock x_1 to 0; from ℓ_n , we go to a state F with price 1, with a self loop that checks $x_1 = 1$, and resets x_1 .

- 1. Assume that the two counter machine halts. If player Min simulates all the instructions correctly, he will incur a mean cost > 0, by either reaching the F after ℓ_n : when player Min does not cheat, player Max has no incentive to enter any of the check widgets, he will just let the computation continue, till the Halt location is reached. This will incur a mean cost > 0. If player Min makes an error in his computation, player Max can always enter an appropriate widget, making the mean cost > 0. In summary, if the two counter machine halts, then player Min has no strategy to achieve his goal (mean pay off ≤ 0).
- 2. Assume that the two counter machine does not halt.
 - If player Min simulates all the instructions correctly, and if player Max never enters

a check widget, then player Min incurs cost 0, since all locations in the main modules (modules simulating decrement counters, increment counters and zero check) have price 0. Even if player Max enters some widget, the mean cost of player Min is still 0, since no errors were made by Min. However, if player Min makes an error, player Max can enter a check widget, ensuring that the mean cost is > 0. Thus, if the two counter machine does not halt, player Min has a strategy (by making no errors) to achieve mean cost 0.

In summary, if the two counter machine does not halt, player Min has a strategy to achieve his goal, and vice-versa.

Thus, player Min incurs a mean $\cos t \le 0$ iff he chooses the strategy of faithfully simulating the two counter machine, when the machine does not halt. When the machine halts, the mean cost incurred by player Min is more than 0 irrespective of whether he makes a simulation error or not.

Now we argue that the $\mathsf{MPG}(\mathsf{T},r)$ (r>0) problem is undecidable for PTGAs with $\geqslant 3$ clocks, having only binary location prices and no edge prices. We note that in the previous undecidability result, all the modules apart from the WD_1^1, WD_2^1, WI_1^1 and WI_2^1 use binary location prices. As in Figure 11 and Figure 10, these modules can be constructed easily. The module WD_1^1 is chosen by the Max player when the error made by the Min player while simulating the decrement counter C_1 operation is ε . The module WD_2^1 is chosen by the Max player when the error made by the Min player is $-\varepsilon$.