Presburger Vector Addition Systems

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***Abstract*—The reachability problem for Vector Addition Systems (VAS) is a central problem of net theory. The problem is known to be decidable by inductive invariants definable in the Presburger arithmetic. When the reachability set is definable in the Presburger arithmetic, the existence of such an inductive invariant is immediate. However, in this case, the computation of a Presburger formula denoting the reachability set is an open problem. In this paper we close this problem by proving that if the reachability set of a VAS is definable in the Presburger arithmetic, then the VAS is flatable, i.e. its reachability set can be obtained by runs labeled by words in a bounded language. As a direct consequence, classical algorithms based on acceleration techniques effectively compute a formula in the Presburger arithmetic denoting the reachability set.**

I. Introduction

Vector Addition Systems (VAS) or equivalently Petri Nets are one of the most popular formal methods for the representation and the analysis of parallel processes [1]. The reachability problem is central since many computational problems (even outside the realm of parallel processes) reduce to this problem. Sacerdote and Tenney provided in [2] a partial proof of decidability of this problem. The proof was completed in 1981 by Mayr [3] and simplified by Kosaraju [4] from [2], [3]. Ten years later [5], Lambert provided a further simplified version based on [4]. This last proof still remains difficult and the upper-bound complexity of the corresponding algorithm is just known to be non-primitive recursive. Nowadays, the exact complexity of the reachability problem for VAS is still an open-question. Even the existence of an elementary upper-bound complexity is open.

Recently, in [6], we proved that even if reachability sets of VAS are not definable in the Presburger arithmetic $\mathrm{FO}(\mathbb{Z}, +, \leq)$, they are *almost semilinear*, a class of sets that extends the class of Presburger sets inspired by the *semilinear sets* [7]. An application of this result was provided; we proved that a final configuration is not reachable from an initial one if and only if there exists a forward inductive invariant definable in the Presburger arithmetic that contains the initial configuration but not the final one. Since we can decide if a Presburger formula denotes a forward inductive invariant, we deduce that there exist checkable certificates of non-reachability in the Presburger arithmetic. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by enumerating finite sequences of actions and a second one that tries to prove the non-reachability by enumerating Presburger formulas. Such an algorithm always terminates in theory but in practice an enumeration does not provide an efficient way for deciding the reachability problem. In particular the problem of deciding *efficiently* the reachability problem is still an *open question*.

When the reachability set is definable in the Presburger arithmetic, the existence of checkable certificates of non-reachability in the Presburger arithmetic is immediate since the reachability set is a forward inductive invariant (in fact the most precise one). The problem of deciding if the reachability set of a VAS is definable in the Presburger arithmetic was studied twenty years ago independently by Dirk Hauschildt during his PhD [8] and and Jean-Luc Lambert. Unfortunately, the work of Lambert was never published, and experts think that the proof of Hauschildt is incomplete. Moreover, from these two works, it is difficult to deduce a simple algorithm for computing a Presburger formula denoting the reachability set when such a formula exists.

For the class of *flatable* vector addition systems, such a computation can be performed with *accelerations techniques*. Let us recall that a VAS is said to be *flatable* if there exists a language included in $w_1^* \dots w_m^*$ for some words $w_1, \dots, w_m$ such that that every reachable configuration is reachable by a run labeled by a word in this language (such a language is said to be *bounded* [9]). *Acceleration techniques* provide a framework for deciding reachability properties that works well in practice but without termination guaranty in theory. Intuitively, acceleration techniques consist in computing with some symbolic representations transitive closures of sequences of actions. For vector addition systems, the Presburger arithmetic is known to be expressive enough for this computation. As a direct consequence, when the reachability set of a vector addition system is computable with acceleration techniques, this set is necessarily definable in the Presburger arithmetic. In [10], we proved that a VAS is flatable if, and only if, its reachability set is computable by acceleration.

Recently, we proved that many classes of VAS with known Presburger reachability sets are flatable [10] and we conjectured that VAS with reachability sets definable in the Presburger arithmetic are flatable. In this paper, we prove this conjecture. As a direct consequence, classical acceleration techniques always terminate on the computation of Presburger formulas denoting reachability sets of VAS when such a formula exists.

Outline In section III we introduce the acceleration framework and the notion of flatable subreachability sets and flatable subreachability relations. We also recall why Presburger formulas denoting reachability sets of flatable vector addition systems are computable with acceleration techniques. In section IV we recall the definition of well-preorders, the Dickson's lemma and the Higman's lemma. In Section V we recall some classical elements of linear algebra and we introduce the central notion of *smooth periodic sets* defined as follows.

A *periodic set* is a set $\mathbf{P} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{P}$ and $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$. Let us recall [7] that a set $\mathbf{X} \subseteq \mathbb{Z}^d$ is definable in the Presburger arithmetic if and only if it is a finite union of *linear sets*, sets of the form $\mathbf{b} + \mathbf{P}$ where $\mathbf{b} \in \mathbb{N}^d$ and $\mathbf{P}$ is a periodic set of the form $\mathbf{P} = \mathbb{N}\mathbf{p}_1 + \cdots + \mathbb{N}\mathbf{p}_k$ for some vectors $\mathbf{p}_1, \ldots, \mathbf{p}_k \in \mathbb{Z}^d$. A limit of a periodic set $\mathbf{P} \subseteq \mathbb{Q}^d$ is a vector $\mathbf{v} \in \mathbb{Q}^d$ such that there exist $\mathbf{p} \in \mathbf{P}$ and $n \in \mathbb{N}_{>0}$ satisfying $\mathbf{p} + n\mathbb{N}\mathbf{v} \subseteq \mathbf{P}$. The set of limits is denoted by $\lim(\mathbf{P})$. A periodic set is said to be *smooth* if $\lim(\mathbf{P})$ is definable in the decidable logic $\mathrm{FO}(\mathbb{Q}, +, \leq)$ and if for every sequence $(\mathbf{p}_n)_{n \in \mathbb{N}}$ of vectors $\mathbf{p}_n \in \mathbf{P}$ there exists an infinite set $N \subseteq \mathbb{N}$ such that $\mathbf{p}_m - \mathbf{p}_n \in \lim(\mathbf{P})$ for every $n \leq m$ in N.

In Section VI we recall the well-order over the runs first introduced in [11] central in the analysis of vector addition systems. Sections VII and VIII provide independent results that are used in Section IX to prove that reachability sets of vector additions systems intersected with Presburger sets are finite unions of sets $\mathbf{b} + \mathbf{P}$ where $\mathbf{b} \in \mathbb{N}^d$ and $\mathbf{P} \subseteq \mathbb{N}^d$ is a smooth periodic set such that for every linear set $\mathbf{Y} \subseteq \mathbf{b} + \mathbf{P}$ there exists $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{p} + \mathbf{Y}$ is a flatable subreachability set (intuitively a subset of the reachability set computable by acceleration). The last sections show that this decomposition of the reachability set is sufficient for proving that if the reachability set of a VAS is definable in the Presburger arithmetic then it is flatable. Due to space limitation, most mathematical results are only proved in appendix.

II. Vectors And Numbers

We denote by $\mathbb{N}, \mathbb{N}_{>0}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}_{\geq 0}, \mathbb{Q}_{>0}$ the set of *natural numbers*, *positive integers*, *integers*, *rational numbers*, *non negative rational numbers*, and *positive rational numbers*. *Vectors* and *sets of vectors* are denoted in bold face. The ith *component* of a vector $\mathbf{v} \in \mathbb{Q}^d$ is denoted by $\mathbf{v}(i)$. We introduce $||\mathbf{v}||_\infty = \max_{1 \leq i \leq d} |\mathbf{v}(i)|$ where $|\mathbf{v}(i)|$ is the *absolute value* of $\mathbf{v}(i)$. A set $\mathbf{B} \subseteq \mathbb{Q}^d$ is said to be *bounded* if there exists $m \in \mathbb{Q}_{\geq 0}$ such that $||\mathbf{b}||_\infty \leq m$ for every $\mathbf{b} \in \mathbf{B}$. The addition function $+$ is also extended component-wise over $\mathbb{Q}^d$.

The *dot product* of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$ is the rational number $\sum_{i=1}^{d} \mathbf{x}(i)\mathbf{y}(i)$ denoted by $\mathbf{x} \cdot \mathbf{y}$. A *linear form* is a totally-defined function $f : \mathbb{Q}^d \to \mathbb{Q}$ such that there exists $\mathbf{h} \in \mathbb{Q}^d$ satisfying $f(\mathbf{x}) = \mathbf{h} \cdot \mathbf{x}$ for every $\mathbf{x} \in \mathbb{Q}^d$. A *linear function* is a totally-defined function $f : \mathbb{Q}^d \to \mathbb{Q}^p$ such that $f = (f_1, \ldots, f_p)$ where $f_j : \mathbb{Q}^d \to \mathbb{Q}$ is a *linear form*.

Given two sets $\mathbf{V}_1, \mathbf{V}_2 \subseteq \mathbb{Q}^d$ we denote by $\mathbf{V}_1 + \mathbf{V}_2$ the set $\{\mathbf{v}_1 + \mathbf{v}_2 \mid (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2\}$, and we denote by $\mathbf{V}_1 - \mathbf{V}_2$ the set $\{\mathbf{v}_1 - \mathbf{v}_2 \mid (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2\}$. In the same way given $T \subseteq \mathbb{Q}$ and $\mathbf{V} \subseteq \mathbb{Q}^d$ we let $T\mathbf{V} = \{t\mathbf{v} \mid (t, \mathbf{v}) \in T \times \mathbf{V}\}$. We also denote by $\mathbf{v}_1 + \mathbf{V}_2$ and $\mathbf{V}_1 + \mathbf{v}_2$ the sets $\{\mathbf{v}_1\} + \mathbf{V}_2$ and $\mathbf{V}_1 + \{\mathbf{v}_2\}$, and we denote by $t\mathbf{V}$ and $T\mathbf{v}$ the sets $\{t\}\mathbf{V}$ and $T\{\mathbf{v}\}$. In the sequel, an empty sum of sets included in $\mathbb{Q}^d$ denotes the set reduced to the zero vector $\{\mathbf{0}\}$.

III. Flatable Vector Addition Systems

A *Vector Addition System (VAS)* is a pair $(\mathbf{c}_{\text{init}}, \mathbf{A})$ where $\mathbf{c}_{\text{init}} \in \mathbb{N}^d$ is an *initial configuration* and $\mathbf{A} \subseteq \mathbb{Z}^d$ is a finite set of *actions*.

The semantics of vector addition systems is obtained as follows. A vector $\mathbf{c} \in \mathbb{N}^d$ is called a *configuration*. We introduce the labeled relation $\to$ defined by $\mathbf{x} \xrightarrow{\mathbf{a}} \mathbf{y}$ if $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$ are configurations, $\mathbf{a} \in \mathbf{A}$ is an action, and $\mathbf{y} = \mathbf{x} + \mathbf{a}$. As expected, a *run* is a non-empty word $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$ of configurations $\mathbf{c}_j \in \mathbb{N}^d$ such that $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$ is a vector in $\mathbf{A}$. The word $w = \mathbf{a}_1 \ldots \mathbf{a}_k$ is called the *label* of ρ. The configurations $\mathbf{c}_0$ and $\mathbf{c}_k$ are respectively called the *source* and the *target* and they are denoted by $\mathrm{src}(\rho)$ and $\mathrm{tgt}(\rho)$. We also denote by $\mathrm{dir}(\rho)$ the couple $(\mathrm{src}(\rho), \mathrm{tgt}(\rho))$ called the *direction* of ρ. The relation $\to$ is extended over the words $w = \mathbf{a}_1 \ldots \mathbf{a}_k$ of actions $\mathbf{a}_j \in \mathbf{A}$ by $\mathbf{x} \xrightarrow{w} \mathbf{y}$ if there exists a run from $\mathbf{x}$ to $\mathbf{y}$ labeled by w. Given a language $W \subseteq \mathbf{A}^*$, we denote by $\xrightarrow{W}$ the relation $\bigcup_{w \in W} \xrightarrow{w}$. The relation $\xrightarrow{\mathbf{A}^*}$ is called the reachability relation and it is denoted by $\xrightarrow{*}$. A *subreachability relation* is a relation included in $\xrightarrow{*}$.

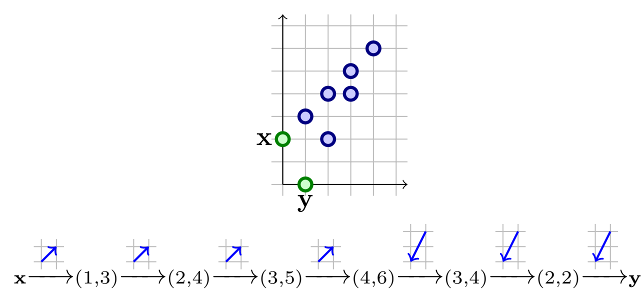

Figure 1. The run ρ labeled by $(1,1)^4(-1,-2)^3$ with $\mathrm{dir}(\rho) = (\mathbf{x}, \mathbf{y})$.

Given a configuration $\mathbf{c} \in \mathbb{N}^d$ and a language $W \subseteq \mathbf{A}^*$ we denote by $\mathrm{post}(\mathbf{c}, W)$ the set of configurations $\mathbf{y} \in \mathbb{N}^d$ such that $\mathbf{c} \xrightarrow{W} \mathbf{y}$. Given a set of configurations $\mathbf{C} \subseteq \mathbb{N}^d$ and a language $W \subseteq (\mathbb{Z}^d)^*$ we denote by $\mathrm{post}(\mathbf{C}, W)$ the set of configurations $\bigcup_{\mathbf{c} \in \mathbf{C}} \mathrm{post}(\mathbf{c}, W)$. The set $\mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*)$ is called the *reachability set*. A subset of this set if called a *subreachability set*.

Flatability properties are defined thanks to *bounded languages*. A language $W \subseteq \mathbf{A}^*$ is said to be *bounded* if there exists a finite sequence $w_1, \ldots w_m$ of words $w_j \in \mathbf{A}^*$ such that $W \subseteq w_1^* \ldots w_m^*$. Let us recall that bounded languages are stable by concatenation, union, intersection, and subset. A subreachability relation is said to be *flatable* if it is included in $\xrightarrow{W}$ where $W \subseteq \mathbf{A}^*$ is a bounded language. A subreachability set is said to be *flatable* if it is included in $\mathrm{post}(\mathbf{c}_{\text{init}}, W)$ where $W \subseteq \mathbf{A}^*$ is a bounded language.

Definition III.1. *A VAS is said to be* flatable *if its reachability set is flatable. A VAS is said to be* Presburger *if its reachability set is definable in the Presburger arithmetic.*

In this paper we show that the class of Presburger VAS coincides with the class of flatable VAS. In the remainder of this section we recall elements of acceleration techniques that explain why flatable VAS are Presburger. We also explain why a Presburger formula denoting the reachability set is effectively computable in this case.

The *displacement* of a word $w = \mathbf{a}_1 \dots \mathbf{a}_k$ of actions $\mathbf{a}_j \in \mathbf{A}$ is the vector $\Delta(w) = \sum_{j=1}^{k} \mathbf{a}_j$. Observe that $\mathbf{x} \xrightarrow{w} \mathbf{y}$ implies $\mathbf{x} + \Delta(w) = \mathbf{y}$ but the converse is not true in general. The converse property can be obtained by associating to every word $w = \mathbf{a}_1 \dots \mathbf{a}_k$ the configuration $\mathbf{c}_w$ defined for every $i \in \{1, \dots, d\}$ by:

$$\mathbf{c}_w(i) = \max\{-(\mathbf{a}_1 + \dots + \mathbf{a}_j)(i) \mid 0 \leq j \leq k\}$$

The following lemma shows that $\mathbf{c}_w$ is the minimal for $\leq$ configuration from which there exists a run labeled by w.

Lemma III.2. *There exists a run from a configuration $\mathbf{x} \in \mathbb{N}^d$ labeled by a word $w \in \mathbf{A}^*$ if, and only if, $\mathbf{x} \geq \mathbf{c}_w$.*

Proof: We assume that $w = \mathbf{a}_1 \dots \mathbf{a}_k$ where $\mathbf{a}_j \in \mathbf{A}$. Assume first that there exists a run $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ labeled by w from $\mathbf{c}_0 = \mathbf{x}$. Since $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$ we deduce that $\mathbf{c}_j = \mathbf{x} + \mathbf{a}_1 + \dots + \mathbf{a}_j$. Since $\mathbf{c}_j \geq \mathbf{0}$ we get $\mathbf{x} \geq -(\mathbf{a}_1 + \dots + \mathbf{a}_j)$. We have proved that $\mathbf{x} \geq \mathbf{c}_w$. Conversely, let us assume that $\mathbf{x} \geq \mathbf{c}_w$ and let us prove that there exists a run from $\mathbf{x}$ labeled by w. We introduce the vectors $\mathbf{c}_j = \mathbf{x} + \mathbf{a}_1 + \dots + \mathbf{a}_j$. Since $\mathbf{x} \geq \mathbf{c}_w$ we deduce that $\mathbf{c}_j \in \mathbb{N}^d$. Therefore $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$ is a run. Just observe that $\mathbf{c}_0 = \mathbf{x}$ and ρ is labeled by w. ∎

The following lemma shows that the set of triples $(\mathbf{x}, n, \mathbf{y}) \in \mathbb{N}^d \times \mathbb{N} \times \mathbb{N}^d$ such that $\mathbf{x} \xrightarrow{w^n} \mathbf{y}$ is effectively definable in the Presburger arithmetic. In particular with an existential quantification of the variable n, we deduce that the relation $\xrightarrow{w^*}$ is effectively definable in the Presburger arithmetic. Hence if a set of configurations $\mathbf{C} \subseteq \mathbb{N}^d$ is denoted by a Presburger formula then for every word $w \in \mathbf{A}^*$ we can effectively compute a Presburger formula denoting $\mathrm{post}(\mathbf{C}, w^*)$.

Lemma III.3. *A pair $(\mathbf{x}, \mathbf{y}) \in \mathbb{N}^d \times \mathbb{N}^d$ of configurations satisfies $\mathbf{x} \xrightarrow{w^n} \mathbf{y}$ where $w \in \mathbf{A}^*$ and $n \in \mathbb{N}_{>0}$ if and only if:*

$$\mathbf{x} \geq \mathbf{c}_w \quad \wedge \quad \mathbf{x} + n\Delta(w) = \mathbf{y} \quad \wedge \quad \mathbf{y} - \Delta(w) \geq \mathbf{c}_w$$

Proof: Assume first that we have a run $\mathbf{x} \xrightarrow{w^n} \mathbf{y}$. Since $n \geq 1$, a prefix and a suffix of this run show that $\mathbf{x} \xrightarrow{w} \mathbf{x} + \Delta(w)$ and $\mathbf{y} - \Delta(w) \xrightarrow{w} \mathbf{y}$. Lemma III.2 shows that $\mathbf{x} \geq \mathbf{c}_w$ and $\mathbf{y} - \Delta(w) \geq \mathbf{c}_w$. Moreover, since $\mathbf{x} + n\Delta(w) = \mathbf{y}$ we have proved one way of the lemma. For the other way, let us assume that $\mathbf{x} \geq \mathbf{c}_w$, $\mathbf{x} + n\Delta(w) = \mathbf{y}$, and $\mathbf{y} - \Delta(w) \geq \mathbf{c}_w$. We introduce the sequence $\mathbf{c}_0, \dots, \mathbf{c}_n$ defined by $\mathbf{c}_j = \mathbf{x} + j\Delta(w)$. Let us prove that $\mathbf{c}_{j-1} \geq \mathbf{c}_w$ for every $1 \leq j \leq n$. Let $i \in \{1, \dots, d\}$. If $\Delta(w)(i) \geq 0$ then $\mathbf{c}_{j-1}(i) \geq \mathbf{x}(i) \geq \mathbf{c}_w(i)$. Next, assume that $\Delta(w)(i) < 0$. In this case, since $\mathbf{x} + n\Delta(w) = \mathbf{y}$ we deduce that $\mathbf{c}_{j-1} = \mathbf{y} - \Delta(w) + (n - j)(-\Delta(w))$. Thus $\mathbf{c}_{j-1}(i) \geq \mathbf{y}(i) - \Delta(w)(i) \geq \mathbf{c}_w(i)$. We have proved that $\mathbf{c}_{j-1} \geq \mathbf{c}_w$. Lemma III.2 shows that $\mathbf{c}_{j-1} \xrightarrow{w} \mathbf{c}_j$. We have proved that $\mathbf{c}_0 \xrightarrow{w^n} \mathbf{c}_n$. Since $\mathbf{c}_0 = \mathbf{x}$ and $\mathbf{c}_n = \mathbf{y}$ we have proved the other way. ∎

We deduce the following theorem also proved in [12] in a more general context. This theorem shows that we can effectively compute a Presburger formula denoting the reachability set of flatable VAS.

Theorem III.4 ([12]). *There exists an algorithm computing for any flatable VAS $(\mathbf{c}_{init}, \mathbf{A})$ a sequence $w_1, \dots, w_m \in \mathbf{A}^*$ such that:*

$$\mathrm{post}(\mathbf{c}_{init}, \mathbf{A}^*) = \mathrm{post}(\mathbf{c}_{init}, w_1^* \dots w_m^*)$$

Proof: Let us consider an algorithm that takes as input a VAS $(\mathbf{c}_{\text{init}}, \mathbf{A})$ and it computes inductively a sequence $(w_m)_{m \geq 1}$ of words $w_m \in \mathbf{A}^*$ such that every finite sequence $(\sigma_j)_{1 \leq j \leq n}$ of words $\sigma_j \in \mathbf{A}^*$ is a sub-sequence. Note that such an algorithm exists. From this sequence, another algorithm computes inductively Presburger formulas denoting sets of configurations $\mathbf{C}_m \subseteq \mathbb{N}^d$ satisfying $\mathbf{C}_0 = \{\mathbf{c}_{\text{init}}\}$ and $\mathbf{C}_m = \mathrm{post}(\mathbf{C}_{m-1}, w_m^*)$ for every $m \in \mathbb{N}_{>0}$. The algorithm stops and it returns $w_1, \dots, w_m$ when $\mathrm{post}(\mathbf{C}_m, \mathbf{A}) \subseteq \mathbf{C}_m$. Note that such a test is implementable since $\mathbf{C}_m$ is denoted by a Presburger formula and the Presburger arithmetic is a decidable logic. When the algorithm stops the set $\mathbf{C}_m$ is included in the reachability set and it satisfies $\mathrm{post}(\mathbf{C}_m, \mathbf{A}) \subseteq \mathbf{C}_m$. We deduce that $\mathbf{C}_m$ is equal to the reachability set. In particular the reachability set if equal to $\mathrm{post}(\mathbf{c}_{\text{init}}, w_1^* \dots w_m^*)$ and the algorithm is correct.

For the termination, since the VAS is flatable, there exists a bounded language $W \subseteq \mathbf{A}^*$ such that the reachability set is included in $\mathrm{post}(\mathbf{c}_{\text{init}}, W)$. As W is bounded, there exists a finite sequence $\sigma_1, \dots, \sigma_n \in \mathbf{A}^*$ such that $W \subseteq \sigma_1^* \dots \sigma_n^*$. There exists $m \in \mathbb{N}$ such that this sequence is a sub-sequence of $w_1, \dots, w_m$. Let us observe that $W \subseteq \sigma_1^* \dots \sigma_n^* \subseteq w_1^* \dots w_m^*$. From the following inclusions we deduce that $\mathbf{C}_m$ is equal to the reachability set:

$$\begin{aligned}
\mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*) &\subseteq \mathrm{post}(\mathbf{c}_{\text{init}}, W) \\
&\subseteq \mathrm{post}(\mathbf{c}_{\text{init}}, w_1^* \dots w_m^*) \\
&= \mathbf{C}_m \\
&\subseteq \mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*)
\end{aligned}$$

In particular $\mathrm{post}(\mathbf{C}_m, \mathbf{A}) \subseteq \mathbf{C}_m$ and the algorithm terminates before the mth iteration. ∎

Corollary III.5. *Reachability sets of flatable VAS are effectively definable in the Presburger arithmetic.*

In the remainder of this paper, we proved that Presburger VAS are flatable. As a direct consequence a Presburger formula denoting the reachability set of a Presburger VAS is effectively computable using classical acceleration techniques.

IV. Well-Preorders

A relation R over a set S is a subset $R \subseteq S \times S$. The composition of two relations R_1, R_2 over S is the relation over S denoted by $R_1 \circ R_2$ and defined as the set $\bigcup_{i \in S}\{(s,t) \in S \times S \mid (s,i) \in R_1 \ \wedge \ (i,t) \in R_2\}$. A relation R over S is said to be *reflexive* if $(s,s) \in R$ for every $s \in S$, *transitive* if $R \circ R \subseteq R$, *antisymmetric* if $(s,t),(t,s) \in R$ implies $s = t$, a *preorder* if R is reflexive and transitive, and an *order* if R is an antisymmetric preorder. The composition of R by itself n times where $n \in \mathbb{N}_{>0}$ is denoted by R^n. The transitive closure of a relation R is the relation $\bigcup_{n \geq 1} R^n$ denoted by R^+.

A preorder $\sqsubseteq$ over a set S is said to be *well* if for every sequence $(s_n)_{n \in \mathbb{N}}$ of elements $s_n \in S$ there exist an infinite set $N \subseteq \mathbb{N}$ such that $s_n \sqsubseteq s_m$ for every $n \leq m$ in N. Observe that $(\mathbb{N}, \leq)$ is a well-ordered set whereas $(\mathbb{Z}, \leq)$ is not well-ordered. As another example, the pigeonhole principle shows that a set S is well-ordered by the equality relation if, and only if, S is finite. Well-preorders can be easily defined thanks to *Dickson's lemma* and *Higman's lemma* as follows.

Dickson's lemma: Dickson's lemma shows that the cartesian product of two well-preordered sets is well-preordered. More formally, given two preordered sets $(S_1, \sqsubseteq_1)$ and $(S_2, \sqsubseteq_2)$ we denote by $\sqsubseteq_1 \times \sqsubseteq_2$ the preorder defined component-wise over the cartesian product $S_1 \times S_2$ by $(s_1, s_2) \sqsubseteq_1 \times \sqsubseteq_2 (s_1', s_2')$ if $s_1 \sqsubseteq_1 s_1'$ and $s_2 \sqsubseteq_2 s_2'$. *Dickson's lemma* says that $(S_1 \times S_2, \sqsubseteq_1 \times \sqsubseteq_2)$ is well-preordered for every well-preordered sets $(S_1, \sqsubseteq_1)$ and $(S_2, \sqsubseteq_2)$. As a direct application, the set $\mathbb{N}^d$ equipped with the component-wise extension of $\leq$ is well-ordered.

Higman's lemma: Higman's lemma shows that words over well-preordered alphabets can be well-preordered. More formally, given a preordered set $(S, \sqsubseteq)$, we introduce the set S^* of words over S equipped with the preorder $\sqsubseteq^*$ defined by $w \sqsubseteq^* w'$ if w and w' can be decomposed into $w = s_1 \ldots s_k$ and $w' \in S^* s_1' S^* \ldots s_k' S^*$ where $s_j \sqsubseteq s_j'$ are in S for every $j \in \{1, \ldots, k\}$. *Higman's lemma* says that $(S^*, \sqsubseteq^*)$ is well-preordered for every well-preordered set $(S, \sqsubseteq)$. As a classical application, the set of words over a finite alphabet S is well-ordered by the sub-word relation $=^*$.

V. Vector Spaces, Conic Sets, Periodic Sets, And Lattices

In this section we recall some elements of linear algebra. We also introduce the central notions of *definable conic sets* and *smooth periodic sets*.

A *vector space* is a set $\mathbf{V} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{V}$, $\mathbf{V} + \mathbf{V} \subseteq \mathbf{V}$, and $\mathbb{Q}\mathbf{V} \subseteq \mathbf{V}$. The following set is a vector space called the *vector space generated* by $\mathbf{X} \subseteq \mathbb{Q}^d$:

$$\left\{ \sum_{j=1}^{k} \lambda_j \mathbf{x}_j \mid k \in \mathbb{N} \text{ and } (\lambda_j, \mathbf{x}_j) \in \mathbb{Q} \times \mathbf{X} \right\}$$

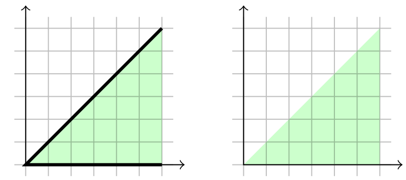

Figure 2. The finitely generated conic set $\mathbb{Q}_{\geq 0}(1,1) + \mathbb{Q}_{\geq 0}(1,0)$ and the definable conic set $\{(0,0)\} \cup \{(c_1, c_2) \in \mathbb{Q}_{>0}^2 \mid c_2 < c_1\}$

This vector space is the minimal for the inclusion among the vector spaces that contain $\mathbf{X}$. Let us recall that every vector space $\mathbf{V}$ is generated by a finite set. The *rank* $\mathrm{rank}(\mathbf{V})$ of a vector space $\mathbf{V}$ is the minimal natural number $r \in \mathbb{N}$ such that there exists a finite set $\mathbf{B}$ with r vectors that generates $\mathbf{V}$. Let us recall that $\mathrm{rank}(\mathbf{V}) \leq \mathrm{rank}(\mathbf{W})$ for every pair of vector spaces $\mathbf{V} \subseteq \mathbf{W}$. Moreover, if $\mathbf{V}$ is strictly included in $\mathbf{W}$ then $\mathrm{rank}(\mathbf{V}) < \mathrm{rank}(\mathbf{W})$. Vectors spaces are geometrically characterized as follows:

Lemma V.1 ([13]). *A set $\mathbf{V} \subseteq \mathbb{Q}^d$ is a vector space if and only if there exists a finite set $\mathbf{H} \subseteq \mathbb{Q}^d$ such that:*

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbb{Q}^d \mid \bigwedge_{\mathbf{h} \in \mathbf{H}} \mathbf{h} \cdot \mathbf{v} = 0 \right\}$$

A *conic set* is a set $\mathbf{C} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{C}$, $\mathbf{C} + \mathbf{C} \subseteq \mathbf{C}$ and $\mathbb{Q}_{\geq 0}\mathbf{C} \subseteq \mathbf{C}$. The following set is a conic set called the *conic set generated* by $\mathbf{X} \subseteq \mathbb{Q}^d$:

$$\left\{ \sum_{j=1}^{k} \lambda_j \mathbf{x}_j \mid k \in \mathbb{N} \text{ and } (\lambda_j, \mathbf{x}_j) \in \mathbb{Q}_{\geq 0} \times \mathbf{X} \right\}$$

This conic set is the minimal for the inclusion among the conic sets that contain $\mathbf{X}$. Contrary to the vector spaces, some conic sets are not finitely generated. Fig. 2 depicts examples of finitely generated conic sets and (non finitely generated) conic sets. Finitely generated conic sets are geometrically characterized by the following lemma.

Lemma V.2 ([13]). *A set $\mathbf{C} \subseteq \mathbb{Q}^d$ is a finitely generated conic set if and only if there exists a finite set $\mathbf{H} \subseteq \mathbb{Q}^d$ such that:*

$$\mathbf{C} = \left\{ \mathbf{c} \in \mathbb{Q}^d \mid \bigwedge_{\mathbf{h} \in \mathbf{H}} \mathbf{h} \cdot \mathbf{c} \geq 0 \right\}$$

Definition V.3. *A conic set is said to be* definable *if it can be denoted by a formula in* FO $(\mathbb{Q}, +, \leq)$.

A *periodic set* is a set $\mathbf{P} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{P}$, and $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$. The following set is a periodic set called the *periodic set generated* by $\mathbf{X} \subseteq \mathbb{Q}^d$:

$$\left\{ \sum_{j=1}^{k} n_j \mathbf{x}_j \mid k \in \mathbb{N} \text{ and } (n_j, \mathbf{x}_j) \in \mathbb{N} \times \mathbf{X} \right\}$$

This periodic set is the minimal for the inclusion among the periodic sets that contains $\mathbf{X}$. Observe that the conic set $\mathbf{C}$ generated by a periodic set $\mathbf{P}$ is $\mathbf{C} = \mathbb{Q}_{\geq 0}\mathbf{P}$. The finitely generated periodic sets are characterized as follows. Given a periodic set $\mathbf{P}$ we denote by $\leq_{\mathbf{P}}$ the preorder over $\mathbf{P}$ defined by $\mathbf{p} \leq_{\mathbf{P}} \mathbf{q}$ if $\mathbf{q} \in \mathbf{p} + \mathbf{P}$. A periodic set $\mathbf{P} \subseteq \mathbb{Q}^d$ is said to be *discrete* if there exists $n \in \mathbb{N}_{>0}$ such that $\mathbf{P} \subseteq \frac{1}{n}\mathbb{Z}^d$. Observe that finitely generated periodic sets are discrete. The following lemma characterizes the discrete periodic sets that are finitely generated. The proof is given in appendix.

Lemma V.4. *Let $\mathbf{P}$ be discrete periodic set. The following conditions are equivalent:*

- *$\mathbf{P}$ is finitely generated as a periodic set.*
- *$(\mathbf{P}, \leq_{\mathbf{P}})$ is well-preordered.*
- *$\mathbb{Q}_{\geq 0}\mathbf{P}$ is finitely generated as a conic set.*

Remark V.5. *A set $\mathbf{X} \subseteq \mathbb{Z}^d$ is definable in the Presburger arithmetic $\mathrm{FO}(\mathbb{Z}, +, \leq)$ if, and only if, it is a finite union of* linear set *$\mathbf{b} + \mathbf{P}$ where $\mathbf{b} \in \mathbb{Z}^d$ and $\mathbf{P} \subseteq \mathbb{Z}^d$ is a finitely generated periodic set [7].*

A *limit* of a periodic set $\mathbf{P} \subseteq \mathbb{Q}^d$ is a vector $\mathbf{v} \in \mathbb{Q}^d$ such that there exists $\mathbf{p} \in \mathbf{P}$ and $n \in \mathbb{N}_{>0}$ satisfying $\mathbf{p} + n\mathbb{N}\mathbf{v} \subseteq \mathbf{P}$. The set of limits of $\mathbf{P}$ is denoted by $\lim(\mathbf{P})$.

Lemma V.6. $\lim(\mathbf{P})$ *is a conic set.*

Proof: Let $\mathbf{C} = \lim(\mathbf{P})$. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{C}$. There exist $\mathbf{p}_1, \mathbf{p}_2 \in \mathbf{P}$ and $n_1, n_2 \in \mathbb{N}_{>0}$ such that $\mathbf{p}_1 + n_1\mathbb{N}\mathbf{v}_1$ and $\mathbf{p}_2 + n_2\mathbb{N}\mathbf{v}_2$ are included in $\mathbf{P}$. Let $n = n_1 n_2$. Since $n\mathbb{N}$ is included in $n_1\mathbb{N}$ and $n_2\mathbb{N}$ we deduce that $\mathbf{p}_1 + n\mathbb{N}\mathbf{v}_1$ and $\mathbf{p}_2 + n\mathbb{N}\mathbf{v}_2$ are included in $\mathbf{P}$. As $\mathbf{P}$ is periodic we deduce that $\mathbf{p} + n\mathbb{N}\mathbf{v} \subseteq \mathbf{P}$ where $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. As $\mathbf{p} \in \mathbf{P}$ we get $\mathbf{v} \in \mathbf{C}$. We deduce that $\mathbf{C} + \mathbf{C} \subseteq \mathbf{C}$. Since $\mathbf{0} \in \mathbf{C}$ and $\mathbb{Q}_{\geq 0}\mathbf{C} \subseteq \mathbf{C}$ are immediate, we have proved that $\mathbf{C}$ is a conic set. ∎

A *periodic set* $\mathbf{P}$ is said to be *well-limit* if for every sequence $(\mathbf{p}_n)_{n\in\mathbb{N}}$ of vectors $\mathbf{p}_n \in \mathbf{P}$ there exists an infinite set $N \subseteq \mathbb{N}$ such that $\mathbf{p}_m - \mathbf{p}_n \in \lim(\mathbf{P})$ for every $n \leq m$ in N. The periodic set $\mathbf{P}$ is said to be *smooth* if $\lim(\mathbf{P})$ is a definable conic set and $\mathbf{P}$ is well-limit.

Example V.7. *Let us consider the periodic set $\mathbf{P} \subseteq \mathbb{N}^2$ generated by $(0,1)$ and the pairs $(2^m, 1)$ where $m \in \mathbb{N}$. The limit of $\mathbf{P}$ is the definable conic set $\mathbf{C} = \{(0,0)\} \cup (\mathbb{Q}_{\geq 0} \times \mathbb{Q}_{>0})$. Note that $\mathbf{P}$ is not well-limit since the sequence $(\mathbf{p}_n)_{n\in\mathbb{N}}$ defined by $\mathbf{p}_n = (2^n, 1)$ is such that $\mathbf{p}_m - \mathbf{p}_n = (2^m - 2^n, 0) \notin \mathbf{C}$ for every $n < m$.*

A *lattice* is a set $\mathbf{L} \subseteq \mathbb{Q}^d$ such that $\mathbf{0} \in \mathbf{L}$, $\mathbf{L} + \mathbf{L} \subseteq \mathbf{L}$ and $-\mathbf{L} \subseteq \mathbf{L}$. The following set is a lattice called the *lattice generated* by $\mathbf{X} \subseteq \mathbb{Q}^d$:

$$\left\{ \sum_{j=1}^{k} z_j \mathbf{x}_j \mid k \in \mathbb{N} \text{ and } (z_j, \mathbf{x}_j) \in \mathbb{Z} \times \mathbf{X} \right\}$$

This lattice is the minimal for the inclusion among the lattices that contain $\mathbf{X}$. Observe that the conic set generated by a lattice $\mathbf{L}$ is equal to the vector space $\mathbf{V} = \mathbb{Q}_{\geq 0}\mathbf{L}$. Since vector spaces are finitely generated, the previous Lemma V.4 shows that discrete lattices are finitely generated.

Remark V.8. *The following inclusions hold:*

$$\begin{array}{ccccccc}
\textit{conic sets} & \subseteq & \textit{periodic sets} & \supseteq & \textit{discrete periodic sets} & \supseteq & \textit{finitely gen. periodic sets} \\
\cup\!| & & \cup\!| & & \cup\!| & & \cup\!| \\
\textit{vector spaces} & \subseteq & \textit{lattices} & \supseteq & \textit{discrete lattices} & = & \textit{finitely gen. lattices}
\end{array}$$

VI. Well-Order Over The Runs

We define a well-order over the runs as follows. We introduce the relation $\trianglelefteq$ over the runs defined by $\rho \trianglelefteq \rho'$ if ρ is a run of the form $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$ where $\mathbf{c}_j \in \mathbb{N}^d$ and if there exists a sequence $(\mathbf{v}_j)_{0 \leq j \leq k+1}$ of vectors $\mathbf{v}_j \in \mathbb{N}^d$ such that ρ' is a run of the form $\rho' = \rho_0 \ldots \rho_k$ where ρ_j is a run from $\mathbf{c}_j + \mathbf{v}_j$ to $\mathbf{c}_j + \mathbf{v}_{j+1}$.

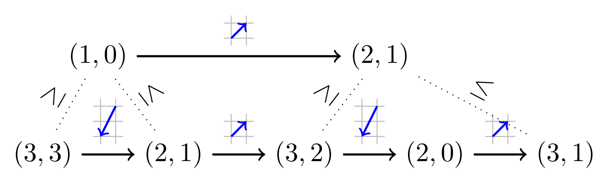

Figure 3. $(1,0)(2,1) \quad \trianglelefteq \quad (3,3)(2,1)(3,2)(2,0)(3,1)$

Example VI.1. *This example is depicted on Figure 3. Let $\rho = (1,0)(2,1)$ and observe that $\rho \trianglelefteq \rho_1\rho_2$ where $\rho_1 = (3,3)(2,1)$ and $\rho_2 = (3,2)(2,0)(3,1)$.*

Let us recall the following lemma based on the Higman's Lemma.

Lemma VI.2 ([11], [14])**.** *The relation $\trianglelefteq$ is a well-order.*

Lemma VI.3. *For every runs $\rho \trianglelefteq \rho'$, the pair $(\mathbf{e}, \mathbf{f}) = \mathrm{dir}(\rho') - \mathrm{dir}(\rho)$ satisfies $\mathrm{dir}(\rho) + \mathbb{N}(\mathbf{e}, \mathbf{f})$ is a flatable subreachability relation.*

Proof: Assume that $\rho \trianglelefteq \rho'$. In this case $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$ where $\mathbf{c}_j \in \mathbb{N}^d$ and there exists a sequence $\mathbf{v}_0, \ldots, \mathbf{v}_{k+1} \in \mathbb{N}^d$ such that $\rho' = \rho_0 \ldots \rho_k$ where ρ_j is a run from $\mathbf{c}_j + \mathbf{v}_j$ to $\mathbf{c}_j + \mathbf{v}_{j+1}$ labeled by a word σ_j. We introduce the actions $\mathbf{a}_1, \ldots, \mathbf{a}_k$ defined by $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$. By monotony we deduce that for every $r \in \mathbb{N}$ we have a run from $\mathbf{c}_j + r\mathbf{v}_j$ to $\mathbf{c}_j + r\mathbf{v}_{j+1}$ labeled by σ_j^r. We also have $\mathbf{c}_j + r\mathbf{v}_{j+1} \xrightarrow{\mathbf{a}_j} \mathbf{c}_{j+1} + r\mathbf{v}_{j+1}$. We obtain from these runs, a run ρ_r from $\mathbf{c}_0 + r\mathbf{v}_0$ to $\mathbf{c}_k + r\mathbf{v}_{k+1}$ labeled by $\sigma_0^r \mathbf{a}_1 \sigma_1^r \ldots \mathbf{a}_k \sigma_k^r$. Since $(\mathbf{e}, \mathbf{f}) = \mathrm{dir}(\rho') - \mathrm{dir}(\rho)$ is the pair $(\mathbf{v}_0, \mathbf{v}_{k+1})$ we deduce that $\mathrm{dir}(\rho) + \mathbb{N}(\mathbf{e}, \mathbf{f})$ is included in $\xrightarrow{W}$ where $W = \sigma_0^* \mathbf{a}_1 \sigma_1^* \ldots \mathbf{a}_k \sigma_k^*$. ∎

Based on the definition of the well-order $\trianglelefteq$, we introduce the *transformer relations with capacity* $\mathbf{c} \in \mathbb{N}^d$ as the relation

$\overset{\mathbf{c}}{\curvearrowright}$ over $\mathbb{N}^d$ defined by $\mathbf{x} \overset{\mathbf{c}}{\curvearrowright} \mathbf{y}$ if there exists a run from $\mathbf{c}+\mathbf{x}$ to $\mathbf{c}+\mathbf{y}$. By monotony, let us observe that $\overset{\mathbf{c}}{\curvearrowright}$ is a periodic relation.

Remark VI.4. *In [14], the conic set $\mathbb{Q}_{\geq 0} \overset{\mathbf{c}}{\curvearrowright}$ is shown to be definable.*

VII. Reflexive Definable Conic Relations

The class of finite unions of reflexive definable conic relations over $\mathbb{Q}_{\geq 0}^d$ are clearly stable by composition, sum, intersection, and union. In the appendix we prove the following theorem:

Theorem VII.1. *Transitive closures of finite unions of reflexive definable conic relations over $\mathbb{Q}_{\geq 0}^d$ are reflexive definable conic relations.*

Example VII.2. *Let us consider the reflexive definable conic relation $R = \{(x, x') \in \mathbb{Q}_{\geq 0}^2 \mid x \leq x' \leq 2x\}$. Observe that R^n where $n \geq 1$ is the reflexive definable conic relation $\{(x, x') \in \mathbb{Q}_{\geq 0}^2 \mid x \leq x' \leq 2^n x\}$. Thus $R^+ = \{(0,0)\} \cup \{(x, x') \mid 0 < x \leq x'\}$. Observe that R^n is strictly included in R^+ for every $n \geq 1$. Hence R^+ cannot be computed with a finite Kleene iteration $R^1 \cup \ldots \cup R^n$.*

VIII. Transformer Relations

In this section, we prove the following theorem. All other results are not used in the sequel.

Theorem VIII.1. *For every capacity $\mathbf{c} \in \mathbb{N}^d$ and for every periodic relation P included in $\overset{\mathbf{c}}{\curvearrowright}$, there exists a definable conic relation $R \subseteq \mathbb{Q}_{\geq 0}^d \times \mathbb{Q}_{\geq 0}^d$ such that $\lim(P) \subseteq R$ and such that for every $(\mathbf{e}, \mathbf{f}) \in \bar{R}$ there exists $(\mathbf{x}, \mathbf{y}) \in P$ and $n \in \mathbb{N}_{>0}$ such that*

$$(\mathbf{c}, \mathbf{c}) + (\mathbf{x}, \mathbf{y}) + n\mathbb{N}(\mathbf{e}, \mathbf{f})$$

is a flatable subreachability relation.

Theorem VIII.1 is obtained by following the approach introduced in [14]. Note that even if some lemmas are very similar to the ones given in that paper, proofs must be adapted to our context. In the remainder of the section, γ denotes a triple $(\mathbf{c}, P)$ where $\mathbf{c} \in \mathbb{N}^d$ is a capacity, and $P \subseteq \overset{\mathbf{c}}{\curvearrowright}$ is a periodic relation. We introduce the set Ω_γ of runs ρ such that $\mathrm{dir}(\rho) \in (\mathbf{c}, \mathbf{c}) + P$. Note that Ω_γ is non empty since it contains the run reduced to the single configuration $\mathbf{c}$. We denote by $\mathbf{Q}_\gamma$ the set of configurations $\mathbf{q} \in \mathbb{N}^d$ such that there exists a run $\rho \in \Omega_\gamma$ in which $\mathbf{q}$ occurs. We denote by I_γ the set of indexes $i \in \{1, \ldots, d\}$ such that $\{\mathbf{q}(i) \mid \mathbf{q} \in \mathbf{Q}_\gamma\}$ is finite. We consider the projection function $\pi_\gamma : \mathbf{Q}_\gamma \to \mathbb{N}^{I_\gamma}$ defined by $\pi_\gamma(\mathbf{q})(i) = \mathbf{q}(i)$. We introduce the finite set of *states* $S_\gamma = \pi_\gamma(\mathbf{Q}_\gamma)$ and the set T_γ of *transitions* $(\pi_\gamma(\mathbf{q}), \mathbf{q}' - \mathbf{q}, \pi_\gamma(\mathbf{q}'))$ where $\mathbf{q}\mathbf{q}'$ is a factor of a run in Ω_γ. We introduce $s_\gamma = \pi_\gamma(\mathbf{c})$. Since $T_\gamma \subseteq S_\gamma \times \mathbf{A} \times S_\gamma$ we deduce that T_γ is finite. We introduce the graph $G_\gamma = (S_\gamma, T_\gamma)$.

An *intraproduction* for γ is a vector $\mathbf{h} \in \mathbb{N}^d$ such that $\mathbf{c} + \mathbf{h} \in \mathbf{Q}_\gamma$. We denote by $\mathbf{H}_\gamma$ the set of intraproduction for γ. Since $\mathbf{c} \in \mathbf{Q}_\gamma$, the following Lemma VIII.2 shows that this set is periodic. In particular for every $\mathbf{h} \in \mathbf{H}_\gamma$, from $\mathbf{c} + \mathbb{N}\mathbf{h} \subseteq \mathbf{Q}_\gamma$ we deduce that $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$.

Lemma VIII.2. *We have $\mathbf{Q}_\gamma + \mathbf{H}_\gamma \subseteq \mathbf{Q}_\gamma$.*

Proof: Let $\mathbf{q} \in \mathbf{Q}_\gamma$ and $\mathbf{h} \in \mathbf{H}_\gamma$. As $\mathbf{q} \in \mathbf{Q}_\gamma$, there exist $(\mathbf{x}, \mathbf{y}) \in P$ and words $u, v \in \mathbf{A}^*$ such that $\mathbf{c}+\mathbf{x} \xrightarrow{u} \mathbf{q} \xrightarrow{v} \mathbf{c}+\mathbf{y}$. Since $\mathbf{h} \in \mathbf{H}_\gamma$ there exist $(\mathbf{x}', \mathbf{y}') \in P$ and words $u', v' \in \mathbf{A}^*$ such that $\mathbf{c}+\mathbf{x}'+n'\mathbf{e} \xrightarrow{u'} \mathbf{c}+\mathbf{h} \xrightarrow{v'} \mathbf{c}+\mathbf{y}'+n'\mathbf{f}$. By monotony, we have $\mathbf{c} + (\mathbf{x} + \mathbf{x}') \xrightarrow{u'u} \mathbf{q} + \mathbf{h} \xrightarrow{vv'} \mathbf{c} + (\mathbf{y} + \mathbf{y}')$. As P is periodic, we deduce that $\mathbf{q} + \mathbf{h} \in \mathbf{Q}_\gamma$. ■

Corollary VIII.3. *We have $\pi_\gamma(\mathrm{src}(\rho)) = s_\gamma = \pi_\gamma(\mathrm{tgt}(\rho))$ for every run $\rho \in \Omega_\gamma$.*

Proof: Since $\rho \in \Omega_\gamma$ there exists $(\mathbf{x}, \mathbf{y}) \in P$ such that ρ is a run from $\mathbf{c} + \mathbf{x}$ to $\mathbf{c} + \mathbf{y}$. In particular $\mathbf{x}$ and $\mathbf{y}$ are two intraproductions for γ. We deduce that $\mathbf{x}(i) = 0 = \mathbf{y}(i)$ for every $i \in I_\gamma$. Hence $\pi_\gamma(\mathrm{src}(\rho)) = \pi_\gamma(\mathbf{c}) = \pi_\gamma(\mathrm{tgt}(\rho))$. ■

A *path* in G_γ is a word $p = (s_0, \mathbf{a}_1, s_1) \ldots (s_{k-1}, \mathbf{a}_k, s_k)$ of transitions $(s_{j-1}, \mathbf{a}_j, s_j)$ in T_γ. Such a path is called a path from s_0 to s_k labeled by $w = \mathbf{a}_1 \ldots \mathbf{a}_k$. When $s_0 = s_k$ the path is called a *cycle*. The previous corollary shows that every run $\rho = \mathbf{c}_0 \ldots \mathbf{c}_k$ in Ω_γ labeled by a word $w = \mathbf{a}_1 \ldots \mathbf{a}_k$ provides the cycle $t_1 \ldots t_k$ in G_γ on s_γ labeled by w where $t_j = (\pi_\gamma(\mathbf{c}_{j-1}), \mathbf{a}_j, \pi_\gamma(\mathbf{c}_j))$. We deduce that G_γ is strongly connected.

Lemma VIII.4. *For every $\mathbf{q} \leq \mathbf{q}'$ in $\mathbf{Q}_\gamma$ there exists an intraproduction $\mathbf{h} \in \mathbf{H}_\gamma$ such that $\mathbf{q}' \leq \mathbf{q} + \mathbf{h}$.*

Proof: As $\mathbf{q}, \mathbf{q}' \in \mathbf{Q}_\gamma$ there exist $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in P$, and there exist $u, v, u', v' \in \mathbf{A}^*$ such that:

$$\mathbf{c}+\mathbf{x} \xrightarrow{u} \mathbf{q} \xrightarrow{v} \mathbf{c}+\mathbf{y} \qquad \text{and} \qquad \mathbf{c}+\mathbf{x}' \xrightarrow{u'} \mathbf{q}' \xrightarrow{v'} \mathbf{c}+\mathbf{y}'$$

Let us introduce $\mathbf{z} = \mathbf{q}' - \mathbf{q}$. By monotony:

$$\begin{aligned} \mathbf{c} + \mathbf{x} + \mathbf{x}' &\xrightarrow{u'} \mathbf{q}' + \mathbf{x} \\ \mathbf{q} + \mathbf{z} + \mathbf{x} &\xrightarrow{v} \mathbf{c} + \mathbf{y} + \mathbf{z} + \mathbf{x} \\ \mathbf{c} + \mathbf{x} + \mathbf{z} + \mathbf{y} &\xrightarrow{u} \mathbf{q} + \mathbf{z} + \mathbf{y} \\ \mathbf{q}' + \mathbf{y} &\xrightarrow{v'} \mathbf{c} + \mathbf{y} + \mathbf{y}' \end{aligned}$$

Since $\mathbf{q}' + \mathbf{x} = \mathbf{q} + \mathbf{z} + \mathbf{x}$ and $\mathbf{q} + \mathbf{z} + \mathbf{y} = \mathbf{q}' + \mathbf{y}$, we have proved that $\mathbf{c} + \mathbf{x} + \mathbf{x}' \xrightarrow{u'v} \mathbf{c} + \mathbf{h} \xrightarrow{uv'} \mathbf{c} + \mathbf{y} + \mathbf{y}'$ with $\mathbf{h} = \mathbf{z} + \mathbf{x}$. Thus $\mathbf{h}$ is an intraproduction. Observe that $\mathbf{q} + \mathbf{h} = \mathbf{q}' + \mathbf{x} + \mathbf{y} \geq \mathbf{q}'$. ■

Lemma VIII.5. *There exist intraproductions $\mathbf{h} \in \mathbf{H}_\gamma$ such that $I_\gamma = \{i \mid \mathbf{h}(i) = 0\}$.*

Proof: Let $i \notin I_\gamma$. There exists a sequence $(\mathbf{q}_k)_{k \in \mathbb{N}}$ of configurations $\mathbf{q}_k \in \mathbf{Q}_\gamma$ such that $(\mathbf{q}_k(i))_{k \in \mathbb{N}}$ is strictly increasing. Since $(\mathbb{N}^d, \leq)$ is well-ordered there exists $k < k'$ such that $\mathbf{q}_k \leq \mathbf{q}_{k'}$. Lemma VIII.4 shows that there exists an intraproduction $\mathbf{h}_i$ for γ such that $\mathbf{q}_{k'} \leq \mathbf{q}_k + \mathbf{h}_i$. In particular $\mathbf{h}_i(i) > 0$. As the set of intraproductions $\mathbf{H}_\gamma$ is periodic we deduce that $\mathbf{h} = \sum_{i \notin I} \mathbf{h}_i$ is an intraproduction for γ. By construction we have $\mathbf{h}(i) > 0$ for every $i \notin I_\gamma$.

Since $\mathbf{h} \in \mathbf{H}_\gamma$ we deduce that $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$. Therefore $I_\gamma = \{i \mid \mathbf{h}(i) = 0\}$. ∎

Given $s \in S_\gamma$ we introduce the relation $R_{\gamma,s}$ of couples $(\mathbf{e}, \mathbf{f}) \in \mathbb{Q}^d_{\geq 0} \times \mathbb{Q}^d_{\geq 0}$ such that $\mathbf{f} - \mathbf{e} \in \mathbb{Q}_{\geq 0}\Delta(\sigma)$ where σ is the label of a cycle on s in G_γ. Observe that $R_{\gamma,s}$ is a reflexive definable conic relation. From Theorem VII.1 we deduce that the transitive closure $R_\gamma = (\bigcup_{s \in S_\gamma} R_{\gamma,s})^+$ is a reflexive definable conic relation.

Lemma VIII.6. *For every $s_1, \ldots, s_k \in S_\gamma$ there exists $(\mathbf{x}, \mathbf{y}) \in P$ and $\mathbf{q}_1, \ldots, \mathbf{q}_k \in \mathbf{Q}_\gamma$ such that $s_j = \pi_\gamma(\mathbf{q}_j)$ for every $1 \leq j \leq k$ and such that:*

$$\mathbf{c} + \mathbf{x} \xrightarrow{*} \mathbf{q}_1 \cdots \xrightarrow{*} \mathbf{q}_k \xrightarrow{*} \mathbf{c} + \mathbf{y}$$

Proof: Since $s_j \in S_\gamma$ there exists $\mathbf{p}_j \in \mathbf{Q}_\gamma$ and $(\mathbf{x}_j, \mathbf{y}_j) \in P$ such that $\mathbf{c}+\mathbf{x}_j \xrightarrow{*} \mathbf{p}_j \xrightarrow{*} \mathbf{c}+\mathbf{y}_j$. Let us introduce $(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^k (\mathbf{x}_j, \mathbf{y}_j)$. Since P is periodic this pair is in P. Let us introduce $\mathbf{h}_j = \mathbf{y}_1 + \cdots + \mathbf{y}_{j-1} + \mathbf{x}_j + \cdots + \mathbf{x}_k$. By monotony, since $\mathbf{c} + \mathbf{x}_j \xrightarrow{*} \mathbf{p}_j \xrightarrow{*} \mathbf{c} + \mathbf{y}_j$, we deduce that $\mathbf{c} + \mathbf{h}_j \xrightarrow{*} \mathbf{q}_j \xrightarrow{*} \mathbf{c} + \mathbf{h}_{j+1}$ where $\mathbf{q}_j = \mathbf{p}_j + (\mathbf{h}_j - \mathbf{x}_j)$. Since $\mathbf{h}_j - \mathbf{x}_j$ is a sum of intraproductions, we deduce that $\mathbf{h}_j - \mathbf{x}_j$ is an intraproduction. In particular $\pi_\gamma(\mathbf{q}_j) = \pi_\gamma(\mathbf{p}_j) = s_j$. We have proved the lemma. ∎

Lemma VIII.7. *For every $(\mathbf{e}, \mathbf{f}) \in R_\gamma$ there exists $(\mathbf{x}, \mathbf{y}) \in P$ and $n \in \mathbb{N}_{>0}$ such that:*

$$(\mathbf{c}, \mathbf{c}) + (\mathbf{x}, \mathbf{y}) + n\mathbb{N}(\mathbf{e}, \mathbf{f})$$

is a flatable subreachability relation.

Proof: Let us consider $(\mathbf{e}, \mathbf{f}) \in R_\gamma$. There exists a non-empty sequence $s_1, \ldots, s_k$ of states $s_j \in S_\gamma$ such that $(\mathbf{e}, \mathbf{f}) \in R_{\gamma,s_1} \circ \cdots \circ R_{\gamma,s_k}$. We introduce s_0, s_{k+1} equal to s_γ. Let us consider the sequence $(\mathbf{v}_j)_{0 \leq j \leq k}$ such that $\mathbf{v}_0 = \mathbf{e}$, $\mathbf{v}_k = \mathbf{f}$ and such that $(\mathbf{v}_{j-1}, \mathbf{v}_j) \in R_{\gamma,s_j}$ for every $j \in \{1, \ldots, k\}$. By definition of R_{γ,s_j}, there exists $\lambda_j \in \mathbb{Q}_{\geq 0}$ and a cycle in G_γ on s_j labeled by a word σ_j such that $\mathbf{v}_j - \mathbf{v}_{j-1} = \lambda_j \Delta(\sigma_j)$. By multiplying $(\mathbf{e}, \mathbf{f})$ by a positive natural number, we can assume without loss of generality that $\lambda_j \in \mathbb{N}$ for every $j \in \{1, \ldots, k\}$, and $\mathbf{v}_j \in \mathbb{N}^d$ for every $j \in \{0, \ldots, k\}$. Moreover, by replacing σ_j by $\sigma_j^{\lambda_j}$ we can assume that $\mathbf{v}_j - \mathbf{v}_{j-1} = \Delta(\sigma_j)$.

Lemma VIII.6 shows that there exists $(\mathbf{x}, \mathbf{y}) \in P$ and words $w_0, \ldots, w_k \in \mathbf{A}^*$ $\mathbf{q}_1, \ldots, \mathbf{q}_k \in \mathbf{Q}_\gamma$ such that $s_j = \pi_\gamma(\mathbf{q}_j)$ for every $1 \leq j \leq k$ and such that:

$$\mathbf{c} + \mathbf{x} \xrightarrow{w_0} \mathbf{q}_1 \cdots \xrightarrow{w_{k-1}} \mathbf{q}_k \xrightarrow{w_k} \mathbf{c} + \mathbf{y}$$

Note that $w = w_0\sigma_1 w_1 \ldots \sigma_k w_k$ is the label of a cycle on s_γ. Lemma VIII.5 shows that there exist intraproductions $\mathbf{h} \in \mathbf{H}_\gamma$ such that $I_\gamma = \{i \mid \mathbf{h}(i) = 0\}$. Since the set of intraproductions is periodic, by multiplying $\mathbf{h}$ by a large positive natural number we can assume without loss of generality that there exists a run from $\mathbf{c}+\mathbf{h}$ labeled by w. As $\mathbf{h}$ is an intraproduction there exist $(\mathbf{x}', \mathbf{y}') \in P$ and $u, v \in \mathbf{A}^*$ such that $\mathbf{c}+\mathbf{x}' \xrightarrow{u} \mathbf{c}+\mathbf{h} \xrightarrow{v} \mathbf{c}+\mathbf{y}'$. By monotony, we deduce that for every $r \in \mathbb{N}$ we have:

$$\mathbf{c} + \mathbf{x} + \mathbf{x}' + r\mathbf{e} \xrightarrow{uw_0\sigma_1^r w_1 \ldots \sigma_k^r w_k v} \mathbf{c} + \mathbf{y} + \mathbf{y}' + r\mathbf{f}$$

Since P is periodic we deduce that $(\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}') \in P$. We have proved the lemma with the bounded language $W = uw_0\sigma_1^* w_1 \ldots \sigma_k^* w_k v$. ∎

Lemma VIII.8. *States in S_γ are incomparable.*

Proof: Let us consider $s \leq s'$ in S_γ. There exists $\mathbf{q}, \mathbf{q}' \in \mathbf{Q}_\gamma$ such that $s = \pi_\gamma(\mathbf{q})$ and $s' = \pi_\gamma(\mathbf{q}')$. Lemma VIII.5 shows that there exists an intraproduction $\mathbf{h}' \in \mathbf{H}_\gamma$ such that $I_\gamma = \{i \mid \mathbf{h}'(i) = 0\}$. By replacing $\mathbf{h}'$ by a vector in $\mathbb{N}_{>0}\mathbf{h}'$ we can assume without loss of generality that $\mathbf{q}(i) \leq \mathbf{q}'(i) + \mathbf{h}'(i)$ for every $i \notin I_\gamma$. As $\mathbf{q}(i) = s(i) \leq s'(i) = \mathbf{q}'(i) = \mathbf{q}'(i) + \mathbf{h}'(i)$ for every $i \in I_\gamma$ we deduce that $\mathbf{q} \leq \mathbf{q}' + \mathbf{h}'$. Lemma VIII.2 shows that $\mathbf{q}' + \mathbf{h}' \in \mathbf{Q}_\gamma$. Lemma VIII.4 shows that there exists an intraproduction $\mathbf{h} \in \mathbf{H}_\gamma$ such that $\mathbf{q}'+\mathbf{h}' \leq \mathbf{q}+\mathbf{h}$. As $\mathbf{h} \in \mathbf{H}_\gamma$ we deduce that $\mathbf{h}(i) = 0$ for every $i \in I_\gamma$. In particular $\mathbf{q}'(i) \leq \mathbf{q}(i)$ for every $i \in I_\gamma$. Hence $s' \leq s$ and we get $s = s'$. ∎

Lemma VIII.9. *We have $\lim(P) \subseteq R_\gamma$.*

Proof: Let $(\mathbf{e}, \mathbf{f}) \in \lim(P)$. By multiplying this pair by a positive integer, we can assume that there exists $(\mathbf{x}, \mathbf{y}) \in P$ such that $(\mathbf{x}, \mathbf{y}) + \mathbb{N}(\mathbf{e}, \mathbf{f}) \subseteq \mathbf{P}$. Thus for every $n \in \mathbb{N}$ there exists a run ρ_n labeled by a word in $\mathbf{A}^*$ such that $\mathrm{dir}(\rho_n) = (\mathbf{c}, \mathbf{c}) + (\mathbf{x}, \mathbf{y}) + n(\mathbf{e}, \mathbf{f})$. Lemma VI.2 shows that there exists $n < m$ such that $\rho_n \trianglelefteq \rho_m$. Assume that ρ_n is the run $\mathbf{c}_0 \ldots \mathbf{c}_k$ where $\mathbf{c}_j \in \mathbb{N}^d$. There exists a sequence $\mathbf{v}_0, \ldots, \mathbf{v}_{k+1} \in \mathbb{N}^d$ such that $\rho_m = \rho'_0 \ldots \rho'_k$ where ρ'_j is a run from $\mathbf{c}_j + \mathbf{v}_j$ to $\mathbf{c}_j + \mathbf{v}_{j+1}$ labeled by a word σ_j. Observe that $s_j = \pi_\gamma(\mathbf{c}_j)$ is in S_γ. Since $s_j \leq \pi_\gamma(\mathbf{c}_j + \mathbf{v}_j)$, Lemma VIII.8 shows that $s_j = \pi_\gamma(\mathbf{c}_j + \mathbf{v}_j)$. Since $s_j \leq \pi_\gamma(\mathbf{c}_j + \mathbf{v}_{j+1})$, we also deduce that $s_j = \pi_\gamma(\mathbf{c}_j + \mathbf{v}_{j+1})$. Thus σ_j is the label of a cycle on s_j in G_γ. We deduce that $(\mathbf{v}_j, \mathbf{v}_{j+1}) \in R_{\gamma,s_j}$. Thus $(\mathbf{v}_0, \mathbf{v}_{k+1}) \in R_\gamma$. Since this pair is equal to $(\mathbf{e}, \mathbf{f})$, we are done. ∎

We have proved Theorem VIII.1.

IX. Reachability Decomposition

In this section, we prove the following theorem. All other results are not used in the sequel.

Theorem IX.1. *For every Presburger set $\mathbf{X} \subseteq \mathbb{N}^d$, the set $\mathrm{post}(\mathbf{c}_{init}, \mathbf{A}^*) \cap \mathbf{X}$ is a finite union of sets $\mathbf{b}+\mathbf{P}$ where $\mathbf{b} \in \mathbb{N}^d$ and $\mathbf{P} \subseteq \mathbb{N}^d$ is a smooth periodic set such that for every linear set $\mathbf{Y} \subseteq \mathbf{b}+\mathbf{P}$ there exists $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{p}+\mathbf{Y}$ is flatable.*

The proof of the previous theorem is based on the following simple lemma.

Lemma IX.2. *For every relations $R_1, R_2 \subseteq \mathbb{N}^d \times \mathbb{N}^d$ and for every capacity $\mathbf{c} \in \mathbb{N}^d$ such that $(\mathbf{c}, \mathbf{c}) + R_1$ and $(\mathbf{c}, \mathbf{c}) + R_2$ are flatable subreachability relations, then $(\mathbf{c}, \mathbf{c}) + R_1 + R_2$ is a flatable subreachability relation.*

Proof: There exist bounded languages $W_1, W_2 \subseteq \mathbf{A}^*$ such that $(\mathbf{c}, \mathbf{c}) + R_1$ and $(\mathbf{c}, \mathbf{c}) + R_2$ are included respectively in $\xrightarrow{W_1}$ and $\xrightarrow{W_2}$. By monotony, we deduce that $(\mathbf{c}, \mathbf{c}) + R_1 + R_2$ is included in $\xrightarrow{W_1 W_2}$. ∎

Since Presburger sets are finite union of linear sets, we can assume that $\mathbf{X}$ is a linear set in the previous Theorem IX.1. Hence, we can assume that there exists a configuration $\mathbf{x} \in \mathbb{N}^d$ and a finitely generated periodic set $\mathbf{M} \subseteq \mathbb{N}^d$ such that $\mathbf{X} = \mathbf{x} + \mathbf{M}$. We introduce the set Ω of runs ρ from the initial configuration $\mathbf{c}_{\text{init}}$ to a configuration in $\mathbf{X}$. Lemma VI.2 shows that $\trianglelefteq$ is a well-order over Ω and Lemma V.4 shows that $\leq_{\mathbf{M}}$ is a well-order over $\mathbf{M}$. We deduce that Ω is well-ordered by the relation $\sqsubseteq$ defined by $\rho \sqsubseteq \rho'$ if $\rho \trianglelefteq \rho'$ and $\mathrm{tgt}(\rho) - \mathbf{x} \leq_{\mathbf{M}} \mathrm{tgt}(\rho') - \mathbf{x}$. In particular $\Omega_0 = \min_{\sqsubseteq}(\Omega)$ is a finite set. Let us observe that we have the following equality:

$$\mathbf{X} = \bigcup_{\rho \in \Omega_0} \mathrm{tgt}(\rho) + \mathbf{M}_\rho$$

Where $\mathbf{M}_\rho$ is the following periodic set:

$$\mathbf{M}_\rho = \{\mathbf{m} \in \mathbf{M} \mid \mathbf{0} \overset{\mathbf{c}_0}{\curvearrowright} \circ \cdots \circ \overset{\mathbf{c}_k}{\curvearrowright} \mathbf{m}\}$$

So, the proof of Theorem IX.1 reduces to show that $\mathbf{M}_\rho$ is a smooth periodic set such that for every $\mathbf{y} \in \mathbb{N}^d$ and for every finitely generated periodic set $\mathbf{Q} \subseteq \mathbb{N}^d$ such that $\mathbf{y} + \mathbf{Q} \subseteq \mathrm{tgt}(\rho) + \mathbf{M}_\rho$, there exists $\mathbf{m} \in \mathbf{M}_\rho$ such that $\mathbf{y} + \mathbf{m} + \mathbf{Q}$ is flatable.

In the sequel ρ is a run in Ω of the form $\rho = \mathbf{c}_0 \dots \mathbf{c}_k$. We introduce the periodic set P of tuples $(\mathbf{x}_0, \dots, \mathbf{x}_{k+1}) \in (\mathbb{N}^d)^{k+2}$ such that $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{x}_{k+1} \in \mathbf{M}$ and $\mathbf{x}_j \overset{\mathbf{c}_j}{\curvearrowright} \mathbf{x}_{j+1}$ for every j. We consider the projection function $\pi_j : (\mathbb{N}^d)^{k+2} \to \mathbb{N}^d \times \mathbb{N}^d$ defined by $\pi_j(\mathbf{x}_0, \dots, \mathbf{x}_{k+1}) = (\mathbf{x}_j, \mathbf{x}_{j+1})$. We also introduce the periodic set $P_j = \pi_j(P)$. Theorem VIII.1 shows that there exists a definable conic relation $R_j \subseteq \mathbb{Q}_{\geq 0}^d \times \mathbb{Q}_{\geq 0}^d$ such that $\lim(P_j) \subseteq R_j$ and such that for every $r_j \in R_j$, there exists $p_j \in P$ and $n_j \in \mathbb{N}_{>0}$ such that $(\mathbf{c}_j, \mathbf{c}_j) + \pi_j(p_j) + n_j \mathbb{N} r_j$ is a flatable subreachability relation.

We introduce the following definable conic set:

$$\mathbf{C} = \{\mathbf{c} \in \mathbb{Q}_{\geq 0}^d \mid \mathbf{0} \ R_0 \circ \cdots \circ R_k \ \mathbf{c}\}$$

Lemma IX.3. *The periodic set $\mathbf{M}_\rho$ is well-limit and its limit is included in $\mathbf{C} \cap \mathbb{Q}_{\geq 0} M$.*

Proof: Let us consider a sequence $(\mathbf{m}_n)_{n \in \mathbb{N}}$ of vectors $\mathbf{m}_n \in \mathbf{M}_{\rho,\mathbf{A}}$. For every n, there exists a sequence $(\mathbf{x}_{0,n}, \dots, \mathbf{x}_{k+1,n})$ in P such that $\mathbf{x}_{k+1,n} = \mathbf{m}_n$. So, there exists a run $\rho_{j,n}$ from $\mathbf{c}_j + \mathbf{x}_{j,n}$ to $\mathbf{c}_j + \mathbf{x}_{j+1,n}$ labeled by a word in $\mathbf{A}^*$. Lemma VI.2 shows that $\trianglelefteq$ is a well-order over the runs and Lemma V.4 shows that $\leq_{\mathbf{M}}$ is a well-order over $\mathbf{M}$. We deduce that there exists an infinite set $N \subseteq \mathbb{N}$ such that $\rho_{j,n} \trianglelefteq \rho_{j,m}$ and $\mathbf{m}_n \leq_{\mathbf{M}} \mathbf{m}_m$ for every $n \leq m$ in N and for every $0 \leq j \leq k$. Lemma VI.3 shows that for every $r \in \mathbb{N}$ there exists a run labeled by $\mathbf{A}^*$ with a direction equals to $\mathrm{dir}(\rho_{j,n}) + r(\mathrm{dir}(\rho_{j,m}) - \mathrm{dir}(\rho_{j,n}))$. Let us introduce $\mathbf{z}_{j,r} = \mathbf{x}_{j,n} + r(\mathbf{x}_{j,m} - \mathbf{x}_{j,n})$ and observe that the previous direction is equal to $(\mathbf{c}_j, \mathbf{c}_j) + (\mathbf{z}_{j,r}, \mathbf{z}_{j+1,r})$. Thus $\mathbf{z}_{j,r} \overset{\mathbf{c}_j}{\curvearrowright} \mathbf{z}_{j+1,r}$. Since $\mathbf{z}_{0,r} = \mathbf{0}$ and $\mathbf{z}_{k+1,r} = \mathbf{m}_n + r(\mathbf{m}_m - \mathbf{m}_n) \in \mathbf{M}$ from $\mathbf{m}_n \leq_{\mathbf{M}} \mathbf{m}_m$, we deduce that $(\mathbf{z}_{0,r}, \dots, \mathbf{z}_{k+1,r}) \in P$. Thus $\mathbf{m}_n + r(\mathbf{m}_m - \mathbf{m}_n) \in \mathbf{M}_\rho$. We deduce that $\mathbf{m}_m - \mathbf{m}_n \in \lim(\mathbf{M}_\rho)$. Therefore $\mathbf{M}_\rho$ is well-limit periodic.

Now, let us consider $\mathbf{v} \in \lim(\mathbf{M}_\rho)$. By multiplying this vector by a positive integer, we can assume that that there exists $\mathbf{m} \in \mathbf{M}$ such that $\mathbf{m}_n = \mathbf{m} + n\mathbf{v}$ is in $\mathbf{M}_\rho$ for every $n \in \mathbb{N}$. We can then apply the previous paragraph on this sequence. Let $n < m$ in $\mathbb{N}$. Since $(\mathbf{z}_{0,r}, \dots, \mathbf{z}_{k+1,r}) \in P$ we deduce that $(\mathbf{z}_{j,r}, \mathbf{z}_{j,r+1}) \in P_j$. Thus $(\mathbf{x}_{j,n}, \mathbf{x}_{j+1,n}) + \mathbb{N}((\mathbf{x}_{j,m}, \mathbf{x}_{j+1,m}) - (\mathbf{x}_{j,n}, \mathbf{x}_{j+1,n}))$ is included in $\mathbf{P}_j$ and we deduce that $(\mathbf{x}_{j,m}, \mathbf{x}_{j+1,m}) - (\mathbf{x}_{j,n}, \mathbf{x}_{j+1,n}) \in \lim(P_j)$. Hence $(\mathbf{x}_{j,m}, \mathbf{x}_{j+1,m}) - (\mathbf{x}_{j,n}, \mathbf{x}_{j+1,n}) \in R_j$. We deduce that $(\mathbf{x}_{0,m} - \mathbf{x}_{0,n}, \mathbf{x}_{k+1,m} - \mathbf{x}_{k+1,n}) \in R_0 \circ \cdots \circ R_k$. From $\mathbf{x}_{0,m} - \mathbf{x}_{0,n} = \mathbf{0}$ and $\mathbf{x}_{k+1,m} - \mathbf{x}_{k+1,n} = \mathbf{m}_m - \mathbf{m}_n = (m-n)\mathbf{v}$, we deduce that $\mathbf{v} \in \mathbf{C}$. Moreover, from $\mathbf{m}_n \leq_{\mathbf{M}} \mathbf{m}_n$ we get $(m-n)\mathbf{v} \in \mathbf{M}$. We have proved that $\mathbf{v} \in \mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{M}$. ∎

Lemma IX.4. *For every $\mathbf{v} \in \mathbf{C}$, there exist relations $\tilde{R}_0, \dots, \tilde{R}_k \subseteq \mathbb{N}^d \times \mathbb{N}^d$ such that $(\mathbf{c}_j, \mathbf{c}_j) + \tilde{R}_j$ is a flatable subreachability relation, $\mathbf{m} \in \mathbf{M}$, and $n \in \mathbb{N}_{>0}$ such that for every $r \in \mathbb{N}$:*

$$\mathbf{0} \ \tilde{R}_0 \circ \cdots \circ \tilde{R}_k \ \mathbf{m} + rn\mathbf{v}$$

Proof: Let us consider $\mathbf{v} \in \mathbf{C}$. There exists a sequence $(\mathbf{v}_0, \dots, \mathbf{v}_{k+1}) \in (\mathbb{Q}_{\geq 0}^d)^{k+1}$ such that $\mathbf{v}_0 = \mathbf{0}$, $\mathbf{v}_{k+1} = \mathbf{v}$ and $(\mathbf{v}_j, \mathbf{v}_{j+1}) \in R_j$ for every j. There exist $n_j \in \mathbb{N}_{>0}$, $p_j \in P$, such that $(\mathbf{c}_j, \mathbf{c}_j) + \pi_j(p_j) + n_j \mathbb{N}(\mathbf{v}_j, \mathbf{v}_{j+1})$ is a flatable subreachability relation. Let $n = \prod_{j=0}^k n_j$. Since $n\mathbb{N} \subseteq n_j\mathbb{N}$ we deduce that $(\mathbf{c}_j, \mathbf{c}_j) + \pi_j(p_j) + n\mathbb{N}(\mathbf{v}_j, \mathbf{v}_{j+1})$ is a flatable subreachability relation. Let us consider $p = \sum_{j=1}^k p_j$. Note that $p - p_j \in P$ and in particular $(c_j, c_j) + \pi_j(p - p_j)$ is in the reachability relation. Lemma IX.2 shows that $(c_j, c_j) + \tilde{R}_j$ is a flatable subreachability relation where $\tilde{R}_j = \pi_j(p) + n\mathbb{N}(\mathbf{v}_j, \mathbf{v}_{j+1})$. Assume that $p = (\mathbf{x}_0, \dots, \mathbf{x}_{k+1})$. We have proved that for every $r \in \mathbb{N}$ we have $\mathbf{x}_j + nr\mathbf{v}_j \ \tilde{R}_j \ \mathbf{x}_{j+1} + nr\mathbf{v}_{j+1}$. Since $p \in P$ we deduce that $\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{m} = \mathbf{x}_{k+1}$ is a vector in $\mathbf{M}$. Since $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{v}_{k+1} = \mathbf{v}$, we have proved the lemma. ∎

The previous Lemma IX.4 shows that $\mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{M}$ is included in $\lim(\mathbf{M}_\rho)$. Hence, with Lemma IX.3 we deduce that $\lim(\mathbf{M}_\rho)$ is equal to the definable conic set $\mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{M}$.

Lemma IX.5. *For every $\mathbf{y} \in \mathbb{N}^d$ and for every finitely generated periodic set $\mathbf{Q} \subseteq \mathbb{N}^d$ such that $\mathbf{y} + \mathbf{Q} \subseteq \mathrm{tgt}(\rho) + \mathbf{M}_\rho$, there exists $\mathbf{m} \in \mathbf{M}_\rho$ such that $\mathbf{y} + \mathbf{m} + \mathbf{Q}$ is flatable.*

Proof: Since $\mathbf{Q}$ is finitely generated, there exists a finite set $\mathbf{V} \subseteq \mathbf{Q}$ that generates $\mathbf{Q}$. Observe that $\mathbf{x} - \mathrm{tgt}(\rho) + \mathbb{N}\mathbf{v} \subseteq \mathbf{M}_\rho$ for every $\mathbf{v} \in \mathbf{V}$. Thus $\mathbf{v} \in \lim(\mathbf{M}_\rho)$. As $\lim(\mathbf{M}_\rho) \subseteq \mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{M}$, we deduce that there exist relations $\tilde{R}_{0,\mathbf{v}}, \dots, \tilde{R}_{k,\mathbf{v}} \subseteq \mathbb{N}^d \times \mathbb{N}^d$ such that $(\mathbf{c}_j, \mathbf{c}_j) + \tilde{R}_{j,\mathbf{v}}$ is a flatable subreachability relation, $\mathbf{m}_\mathbf{v} \in \mathbf{M}$, and $n_\mathbf{v} \in \mathbb{N}_{>0}$ such that for every $r \in \mathbb{N}$:

$$\mathbf{0} \ \tilde{R}_{0,\mathbf{v}} \circ \cdots \circ \tilde{R}_{k,\mathbf{v}} \ \mathbf{m}_\mathbf{v} + rn_\mathbf{v}\mathbf{v}$$

Let us consider $n = \prod_{\mathbf{v} \in \mathbf{V}} n_\mathbf{v}$, $\mathbf{m} = \sum_{\mathbf{v} \in \mathbf{V}} \mathbf{m}_\mathbf{v}$ and $\tilde{R}_j = \sum_{\mathbf{v} \in \mathbf{V}} \tilde{R}_{j,\mathbf{v}}$. Lemma IX.2 shows $(\mathbf{c}_j, \mathbf{c}_j) + \tilde{R}_j$ is a flatable subreachability relation. Moreover, since $\mathbf{Q}$ is generated by $\mathbf{V}$ we deduce that for every $\mathbf{q} \in \mathbf{Q}$ we have:

$$\mathbf{0} \ \tilde{R}_0 \circ \cdots \circ \tilde{R}_k \ \mathbf{m} + n\mathbf{q}$$

Now, let us consider the set $\mathbf{Z} = \sum_{\mathbf{v} \in \mathbf{V}} \{0, \ldots, n-1\}\mathbf{v}$. Observe that $\mathbf{Z}$ is finite and since $\mathbf{Z} \subseteq \mathbf{M}_\rho$, we deduce that for every $\mathbf{z} \in \mathbf{M}_\rho$, there exists $p_{\mathbf{z}} = (\mathbf{x}_{0,\mathbf{z}}, \ldots, \mathbf{x}_{k+1,\mathbf{z}}) \in P$ such that $\mathbf{x}_{k+1,\mathbf{z}} = \mathbf{z}$. Let us consider the relation $\tilde{R}'_j = \bigcup_{\mathbf{z} \in \mathbf{Z}} (\tilde{R}_j + \pi_j(p_{\mathbf{z}}))$. Lemma IX.2 shows that $(\mathbf{c}_j, \mathbf{c}_j) + \tilde{R}'_j$ is flatable. Since $\mathbf{Q} = \mathbf{Z} + n\mathbf{Q}$ we deduce that for every $\mathbf{q} \in \mathbf{Q}$ we have:

$$\mathbf{0} \ \tilde{R}'_0 \circ \cdots \circ \tilde{R}'_k \ \mathbf{m} + \mathbf{q}$$

Finally, since $\mathbf{y} - \mathrm{tgt}(\rho) \in \mathbf{M}_\rho$ we deduce that there exists $p = (\mathbf{x}_0, \ldots, \mathbf{x}_{k+1})$ in P such that $\mathbf{x}_{k+1} = \mathbf{y} - \mathrm{tgt}(\rho)$. Lemma IX.2 shows that $\tilde{R}''_j = \tilde{R}'_j + \pi_j(p)$ is such that $(\mathbf{c}_j, \mathbf{c}_j) + \tilde{R}''_j$ is flatable. Hence, this relation is included in $\xrightarrow{W_j}$ where $W_j \subseteq \mathbf{A}^*$ is a bounded language.

Let us introduce the actions $\mathbf{a}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$ and the bounded language $W = W_0 \mathbf{a}_1 W_1 \ldots \mathbf{a}_k W_k$. We have proved that $\mathrm{post}(\mathbf{c}_{\text{init}}, W)$ contains $\mathbf{y} + \mathbf{m} + \mathbf{Q}$. Thus this set is flatable. ■

We have proved Theorem IX.1.

X. Dimension

The *dimension* of a set $\mathbf{X} \subseteq \mathbb{Q}^d$ is the minimal integer $r \in \{-1, \ldots, d\}$ such that $\mathbf{X} \subseteq \bigcup_{j=1}^k \mathbf{B}_j + \mathbf{V}_j$ where $\mathbf{B}_j$ is a bounded subset of $\mathbb{Q}^d$ and $\mathbf{V}_j \subseteq \mathbb{Q}^d$ is a vector space satisfying $\mathrm{rank}(\mathbf{V}_j) \leq r$ for every j. We denote by $\dim(\mathbf{X})$ the dimension of $\mathbf{X}$. Observe that $\dim(\mathbf{v} + \mathbf{X}) = \dim(\mathbf{X})$ for every $\mathbf{X} \subseteq \mathbb{Q}^d$ and for every $\mathbf{v} \in \mathbb{Q}^d$. Observe that $\dim(\mathbf{X}) = -1$ if and only if $\mathbf{X}$ is empty. Note that $\dim(\mathbf{X} \cup \mathbf{Y}) = \max\{\dim(\mathbf{X}), \dim(\mathbf{Y})\}$ for every subsets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{Q}^d$.

Example X.1. $\dim(\mathbb{N}) = 1$, $\dim(\mathbb{Q}) = 1$, $\dim(\mathbb{N}(1,0) + \mathbb{N}(1,1)) = 2$, $\dim(\mathbb{N}(1,0) \cup \mathbb{N}(1,1)) = 1$.

The dimension of a periodic set is obtained as follows.

Lemma X.2. *We have* $\dim(\mathbf{P}) = \mathrm{rank}(\mathbf{V})$ *for every periodic set* $\mathbf{P}$ *where* $\mathbf{V}$ *is the vector space generated by* $\mathbf{P}$.

XI. Equivalent Sets

Given a natural number $r \in \{0, \ldots, d\}$, we introduce the equivalence relation $\equiv_r$ over the subsets of $\mathbb{Q}^d$ by $\mathbf{X} \equiv_r \mathbf{Y}$ if $\dim(\mathbf{X} \Delta \mathbf{Y}) < r$. Note that $\equiv_r$ is distributive over $\cup$ and $\cap$.

Lemma XI.1. *Let* $\mathbf{V}$ *be a vector space and* $r = \mathrm{rank}(\mathbf{V})$. *For every* $\mathbf{h} \in \mathbb{Q}^d$ *such that* $\mathbf{h} \cdot \mathbf{v} \neq 0$ *for at least one* $\mathbf{v} \in \mathbf{V}$, *for every* $c \in \mathbb{Q}$ *and for every* $\# \in \{>, \geq\}$, *we have:*

$$\{\mathbf{x} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{x} \geq 0\} \equiv_r \{\mathbf{x} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{x} \# c\}$$

Proof: Let us introduce a vector $\mathbf{v} \in \mathbf{V}$ such that $\mathbf{h} \cdot \mathbf{v} \neq 0$. By replacing $\mathbf{v}$ by $-\mathbf{v}$ we can assume that $\mathbf{h} \cdot \mathbf{v} > 0$. We introduce the set $\mathbf{B} = \{\lambda \mathbf{v} \mid |\lambda| \leq \frac{|c|}{\mathbf{h} \cdot \mathbf{v}}\}$ and the vector space $\mathbf{W} = \{\mathbf{w} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{w} = 0\}$. Since $\mathbf{W}$ is included in $\mathbf{V} \backslash \{\mathbf{v}\}$ we deduce that $\mathrm{rank}(\mathbf{W}) < \mathrm{rank}(\mathbf{V}) = r$. Let us prove that the symmetrical difference of $\{\mathbf{x} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{x} \geq 0\}$ and $\{\mathbf{x} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{x} \# c\}$ is included in $\mathbf{B} + \mathbf{W}$. Let $\mathbf{x}$ be a vector in this difference. Then $\mathbf{x} \in \mathbf{V}$ and either $\mathbf{h} \cdot \mathbf{x} \geq 0$ and $\mathbf{h} \cdot \mathbf{x} \leq c$ or we have $\mathbf{h} \cdot \mathbf{x} < 0$ and $\mathbf{h} \cdot \mathbf{x} \geq c$. In any case we deduce that $-|c| \leq \mathbf{h} \cdot \mathbf{x} \leq |c|$. Let us consider $\lambda = \frac{\mathbf{h} \cdot \mathbf{x}}{\mathbf{h} \cdot \mathbf{v}}$. Note that $\mathbf{b} = \lambda \mathbf{v}$ is a vector in $\mathbf{B}$ and $\mathbf{w} = \mathbf{x} - \lambda \mathbf{v}$ is a vector in $\mathbf{W}$. Thus $\mathbf{x} \in \mathbf{B} + \mathbf{W}$. ■

We deduce the following two corollaries:

Corollary XI.2. *Let* $\mathbf{V}$ *be a vector space and* $r = \mathrm{rank}(\mathbf{V})$. *For every* $\mathbf{X} \subseteq \mathbf{V}$ *definable in* $\mathrm{FO}(\mathbb{Q}, +, \leq, 0, 1)$ *and for every* $\mathbf{v} \in \mathbf{V}$ *we have* $\mathbf{X} \equiv_r \mathbf{v} + \mathbf{X}$.

Proof: Since $\mathrm{FO}(\mathbb{Q}, +, \leq, 0, 1)$ admits quantifier elimination we deduce that $\mathbf{X}$ is a Boolean (union and intersection) combination of sets of the form $\mathbf{S} = \{\mathbf{s} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{s} \# c\}$ where $\mathbf{h} \in \mathbb{Q}^d$, $\# \in \{>, \geq\}$, and $c \in \mathbb{Q}$. Note that $\mathbf{v} + \mathbf{S} = \{\mathbf{s} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{s} \# \mathbf{h} \cdot \mathbf{v}\}$. In particular if $\mathbf{h} \cdot \mathbf{v} = 0$ then $\mathbf{S} = \mathbf{S} + \mathbf{v}$ and if $\mathbf{h} \cdot \mathbf{v} \neq 0$ Lemma XI.1 shows that $\mathbf{S} \equiv_r \mathbf{v} + \mathbf{S}$. We deduce that $\mathbf{S} \equiv_r \mathbf{v} + \mathbf{S}$ is both case. Since $\equiv_r$ is distributive over $\cup$ and $\cap$ we get the corollary. ■

Corollary XI.3. *Let* $\mathbf{P} \subseteq \mathbb{Z}^d$ *be a finitely generated periodic set,* $\mathbf{L} = \mathbf{P} - \mathbf{P}$ *the lattice generated by* $\mathbf{P}$, *and* $\mathbf{C} = \mathbb{Q}_{\geq 0}\mathbf{P}$ *be the conic set generated by* $\mathbf{P}$. *For every* $\mathbf{x} \in \mathbf{L}$ *we have* $\mathbf{x} + \mathbf{P} \ \equiv_r \ \mathbf{L} \cap \mathbf{C}$ *where* $r = \dim(\mathbf{P})$.

Proof: Since $\mathbf{P}$ is finitely generated, there exists $\mathbf{p}_1, \ldots, \mathbf{p}_k \in \mathbf{P}$ such that $\mathbf{P} = \mathbb{N}\mathbf{p}_1 + \cdots + \mathbb{N}\mathbf{p}_k$. We introduce the set $\mathbf{B}$ of vectors $\mathbf{b} \in \mathbf{L}$ such that $\mathbf{b} \in [0,1]\mathbf{p}_1 + \cdots + [0,1]\mathbf{p}_k$. Note that $\mathbf{B}$ is a bounded finite subset of $\mathbb{Z}^d$. Thus $\mathbf{B}$ is finite. Since $\mathbf{B} \subseteq \mathbf{P} - \mathbf{P}$ we deduce that there exists $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{p} + \mathbf{b} \in \mathbf{P}$ for every $\mathbf{b} \in \mathbf{B}$. Let us prove that $\mathbf{p} + (\mathbf{L} \cap \mathbf{C}) \subseteq \mathbf{P}$. Let us consider $\mathbf{v} \in \mathbf{L} \cap \mathbf{C}$. There exists a sequence $\mu_1, \ldots, \mu_k \in \mathbb{Q}_{\geq 0}$ such that $\mathbf{v} = \mu_1 \mathbf{p}_1 + \cdots + \mu_k \mathbf{v}_k$. Let $n_j \in \mathbb{N}$ such that $\mu_j - n_j \in [0,1]$ and let $\mathbf{q} = n_1 \mathbf{p}_1 + \cdots + n_k \mathbf{p}_k$. Note that $\mathbf{q} \in \mathbf{P}$ and $\mathbf{v} - \mathbf{q} \in \mathbf{B}$. Thus $\mathbf{p} + \mathbf{v} - \mathbf{q} \in \mathbf{P}$. In particular $\mathbf{p} + \mathbf{v} \in \mathbf{P}$ and we have proved the inclusion $\mathbf{p} + (\mathbf{L} \cap \mathbf{C}) \subseteq \mathbf{P}$. Since $\mathbf{p} \in \mathbf{L}$ we get $\mathbf{p} + \mathbf{L} = \mathbf{L}$. Thus $\mathbf{p} + (\mathbf{L} \cap \mathbf{C}) = \mathbf{L} \cap (\mathbf{p} + \mathbf{C})$. Corollary XI.2 shows that $\mathbf{C} \equiv_r \mathbf{p} + \mathbf{C}$. Since $\equiv_r$ is distributive over the intersection, we get $\mathbf{L} \cap (\mathbf{p} + \mathbf{C}) \equiv_r \mathbf{L} \cap \mathbf{C}$. Moreover, from $\mathbf{L} \cap (\mathbf{p} + \mathbf{C}) \subseteq \mathbf{P} \subseteq \mathbf{L} \cap \mathbf{C}$ we deduce that $\mathbf{P} \equiv_r \mathbf{L} \cap \mathbf{C}$. Note that for every $\mathbf{x} \in \mathbf{L}$ we have $-\mathbf{x} + (\mathbf{L} \cap \mathbf{C}) = \mathbf{L} \cap (-\mathbf{x} + \mathbf{C}) \equiv_r \mathbf{L} \cap \mathbf{C}$ thanks to corollary XI.2. We have proved that $\mathbf{x} + \mathbf{P} \ \equiv_r \ \mathbf{L} \cap \mathbf{C}$ for every $\mathbf{x} \in \mathbf{L}$. ■

XII. Equivalent Presburger Sets

In appendix we prove the following Theorem XII.1.

Theorem XII.1. *Let* $\mathbf{X} = \bigcup_{j=1}^k \mathbf{b}_j + \mathbf{P}_j$ *where* $\mathbf{b}_j \in \mathbb{Z}^d$ *and* $\mathbf{P}_j \subseteq \mathbb{Z}^d$ *is a smooth periodic set. We assume that* $\mathbf{X}$ *is non empty and we introduce* $r = \dim(\mathbf{X})$. *If* $\mathbf{X}$ *is equivalent for* $\equiv_r$ *to a Presburger set then there exists a sequence* $(\mathbf{Y}_j)_{1 \leq j \leq k}$ *of linear sets* $\mathbf{Y}_j \subseteq \mathbf{b}_j + \mathbf{P}_j$ *such that* $\mathbf{X} \equiv_r \bigcup_{j=1}^k \mathbf{p}_j + \mathbf{Y}_j$ *for every sequence* $(\mathbf{p}_j)_{1 \leq j \leq k}$ *of vectors* $\mathbf{p}_j \in \mathbf{P}_j$.

XIII. Presburger Reachability Sets

In this section we prove that Presburger subreachability sets are flatable. As a direct consequence, we deduce that Presburger VAS are flatable.

Lemma XIII.1. *Presburger subreachability sets are flatable.*

Proof: We prove by induction over $r \in \{-1, \ldots, d\}$ that Presburger subreachability sets $\mathbf{X}$ with $\dim(\mathbf{X}) \leq r$ are flatable. Note that if $\dim(\mathbf{X}) = -1$ then $\mathbf{X}$ is empty and the proof is immediate. Let us assume that the lemma is proved in dimension $r \in \{-1, \ldots, d\}$ and let us consider a Presburger subreachability set $\mathbf{X} \subseteq \mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*)$ such that $\dim(\mathbf{X}) = r+1$. In particular $\mathbf{X}$ is non empty. Theorem IX.1 shows $\mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*) \cap \mathbf{X}$ is a finite union of sets $\bigcup_{j=1}^{k} \mathbf{b}_j + \mathbf{P}_j$ where $\mathbf{b}_j \in \mathbb{N}^d$ and $\mathbf{P}_j \subseteq \mathbb{N}^d$ is a smooth periodic set such that for every linear set $\mathbf{Y}_j \subseteq \mathbf{b}_j + \mathbf{P}_j$ there exists $\mathbf{p}_j \in \mathbf{P}_j$ such that $\mathbf{p}_j + \mathbf{Y}_j$ is flatable.

Since $\mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*) \cap \mathbf{X}$ is equal to $\mathbf{X}$ which is a Presburger set, Theorem XII.1 shows that there exists a sequence $(\mathbf{Y}_j)_{1 \leq j \leq k}$ of linear sets $\mathbf{Y}_j \subseteq \mathbf{b}_j + \mathbf{P}_j$ such that $\mathbf{X} \equiv_r \bigcup_{j=1}^{k} \mathbf{p}_j + \mathbf{Y}_j$ for every sequence $(\mathbf{p}_j)_{1 \leq j \leq k}$ of vectors $\mathbf{p}_j \in \mathbf{P}_j$.

Let us consider a sequence $(\mathbf{p}_j)_{1 \leq j \leq k}$ of vectors $\mathbf{p}_j \in \mathbf{P}_j$ such that $\mathbf{p}_j + \mathbf{Y}_j$ is flatable. We deduce that $\mathbf{Y} = \bigcup_{j=1}^{k} \mathbf{p}_j + \mathbf{P}_j$ is flatable. Since $\mathbf{X} \equiv_r \mathbf{Y}$ we deduce that $\dim(\mathbf{X} \backslash \mathbf{Y}) < r$. Since $\mathbf{X} \backslash \mathbf{Y}$ is a Presburger subreachability set, by induction, this set is flatable. From $\mathbf{X} \subseteq (\mathbf{X} \backslash \mathbf{Y}) \cup \mathbf{Y}$, we deduce that $\mathbf{X}$ is flatable. We have proved the rank $r+1$. ■

Theorem XIII.2. *The class of flatable VAS coincides with the class of Presburger VAS.*

Proof: Assume first that the VAS is Presburger. Then $\mathbf{X} = \mathrm{post}(\mathbf{c}_{\text{init}}, \mathbf{A}^*)$ is a Presburger set. The previous lemma shows that $\mathbf{X}$ is flatable. Hence the VAS is flatable. Conversely, if the VAS is flatable, Theorem III.4 shows that the VAS is Presburger. ■

Corollary XIII.3. *Presburger subreachability relations are flatable.*

Proof: Let $\mathbf{A} \subseteq \mathbb{Z}^d$ be a finite set of actions. We consider the VAS $((\mathbf{0}, \mathbf{0}), A')$ in dimension $2d$ where A' is the set $\{\mathbf{0}\} \times \mathbf{A}$ and the vectors $(\mathbf{u}_i, \mathbf{u}_i)$ where $\mathbf{u}_i \in \mathbb{Z}^d$ satisfies $\mathbf{u}_i(j) = 0$ if $j \neq i$ and $\mathbf{u}_i(i) = 1$. Observe that the reachability set of this VAS is $\xrightarrow{\mathbf{A}^*}$. Hence, if a subreachability relation R of $\xrightarrow{\mathbf{A}^*}$ is Presburger, we deduce that there exists a bounded language $W' \subseteq (A')^*$ such that $R \subseteq \mathrm{post}((\mathbf{0}, \mathbf{0}), W')$. Let us consider the word morphism $\phi : (A')^* \to A^*$ defined by $\phi(\mathbf{0}, \mathbf{a}) = \mathbf{a}$ and $\phi(\mathbf{u}_i, \mathbf{u}_i) = \varepsilon$. Observe that $W = \phi(W')$ is a bounded language and $\mathrm{post}((\mathbf{0}, \mathbf{0}), W')$ is included in $\xrightarrow{W}$. We deduce that R is flatable. ■

XIV. Conclusion

We have proved that acceleration techniques are complete for the computation of Presburger formulas denoting the reachability sets of Presburger vector addition systems. Since there exist vector addition systems with finite reachability sets of Ackermann cardinals [15], acceleration-based algorithms have an Ackermann lower bound of complexity. In the future, we are interested in improving acceleration techniques to avoid this bound thanks to over-approximation techniques. More generally, we are interested in characterizing vector addition systems with reachability sets not definable in the Presburger arithmetic. These vector addition systems are interesting since we know that there exist inductive invariants definable in the Presburger arithmetic obtained by over-approximating the reachability set. The main objective is an algorithm for deciding the general reachability problem for vector addition systems based on accelerations and on-demand over-approximations that works well in practice.

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Appendix A
Proof of Lemma V.4

Lemma V.4. *Let* $\mathbf{P}$ *be discrete periodic set. The following conditions are equivalent:*

- $\mathbf{P}$ *is finitely generated as a periodic set.*
- $(\mathbf{P}, \leq_{\mathbf{P}})$ *is well-preordered.*
- $\mathbb{Q}_{\geq 0}\mathbf{P}$ *is finitely generated as a conic set.*

Proof: Let us consider a discrete periodic set $\mathbf{P} \subseteq \mathbb{Q}^d$. By replacing $\mathbf{P}$ by $n\mathbf{P}$ for some $n \in \mathbb{N}_{>0}$ we can assume in the sequel that $\mathbf{P} \subseteq \mathbb{Z}^d$.

Assume first that $(\mathbf{P}, \leq_{\mathbf{P}})$ is well-preordered and let us prove that $\mathbf{P}$ is finitely generated as a periodic set. We introduce the relation $\sqsubseteq$ over $\mathbf{P}$ defined by $\mathbf{p} \sqsubseteq \mathbf{q}$ if $\mathbf{p} \leq_{\mathbf{P}} \mathbf{q}$ and if $|\mathbf{p}(i)| \leq |\mathbf{q}(i)|$ and $\mathbf{p}(i)\mathbf{q}(i) \geq 0$ for every $i \in \{1, \dots, d\}$. Since $\leq$ is a well-order over $\mathbb{N}^d$ we deduce that $\sqsubseteq$ is a well-order over $\mathbf{P}$. The set $\mathbf{M}$ of minimal elements of $\mathbf{P}\backslash\{\mathbf{0}\}$ for this order is finite. We denote by $\mathbf{Q}$ be the periodic set generated by $\mathbf{M}$. Observe that $\mathbf{Q} \subseteq \mathbf{P}$. Assume by contradiction that $\mathbf{P}\backslash\mathbf{Q}$ is non empty and let us consider an element $\mathbf{p}$ in this set minimal for $\sqsubseteq$. Since $\mathbf{0} \in \mathbf{Q}$ we deduce that $\mathbf{p} \in \mathbf{P}\backslash\{\mathbf{0}\}$. Thus there exists $\mathbf{m} \in \mathbf{M}$ such that $\mathbf{m} \sqsubseteq \mathbf{p}$. Let $\mathbf{q} = \mathbf{p} - \mathbf{m}$. Since $\mathbf{m} \leq_{\mathbf{P}} \mathbf{p}$ we get $\mathbf{q} \in \mathbf{P}$. Moreover, $\mathbf{q} \sqsubseteq \mathbf{p}$. Thus, if $\mathbf{q} \notin \mathbf{Q}$, by minimality of $\mathbf{p}$ we get $\mathbf{q} = \mathbf{p}$ and $\mathbf{m} = \mathbf{0}$ which is impossible since $\mathbf{M} \subseteq \mathbf{P}\backslash\{\mathbf{0}\}$. Thus $\mathbf{q} \in \mathbf{Q}$. From $\mathbf{p} = \mathbf{q} + \mathbf{m}$ we get $\mathbf{p} \in \mathbf{Q}$ and we get a contradiction. Thus $\mathbf{P}\backslash\mathbf{Q}$ is empty and we get $\mathbf{P} = \mathbf{Q}$. In particular $\mathbf{P}$ is finitely generated as a periodic set.

Now, assume that $\mathbf{P}$ is finitely generated as a periodic set and let us prove that $\mathbf{C} = \mathbb{Q}_{\geq 0}\mathbf{P}$ is finitely generated as a conic set. We have $\mathbf{P} = \mathbb{N}\mathbf{p}_1 + \cdots + \mathbb{N}\mathbf{p}_k$ for some vectors $\mathbf{p}_1, \dots, \mathbf{p}_k \in \mathbf{P}$. In particular $\mathbf{C} = \mathbb{Q}_{\geq 0}\mathbf{p}_1 + \cdots + \mathbb{Q}_{\geq 0}\mathbf{p}_k$ and we deduce that $\mathbf{C}$ is finitely generated as a conic set.

Finally assume that $\mathbf{C} = \mathbb{Q}_{\geq 0}\mathbf{P}$ is finitely generated as a conic set and let us prove that $(\mathbf{P}, \leq_{\mathbf{P}})$ is well-preordered. There exists some vectors $\mathbf{q}_1, \dots, \mathbf{q}_k \in \mathbf{C}$ such that $\mathbf{C} = \mathbb{Q}_{\geq 0}\mathbf{q}_1 + \cdots + \mathbb{Q}_{\geq 0}\mathbf{q}_k$. Since $\mathbf{C} = \mathbb{Q}_{\geq 0}\mathbf{P}$ by multiplying vectors $\mathbf{q}_j$ by a positive natural number, we can assume that $\mathbf{q}_j \in \mathbf{P}$. We denote by $\mathbf{Q}$ the periodic set generated by $\mathbf{q}_1, \dots, \mathbf{q}_k$. Let us introduce the following set:

$$\mathbf{B} = (\mathbf{P} - \mathbf{P}) \cap ([0, 1]\mathbf{p}_1 + \cdots + [0, 1]\mathbf{p}_k)$$

Note that $\mathbf{B}$ is bounded and vectors in this set are in $\mathbb{Z}^d$. Thus $\mathbf{B}$ is finite. Let us prove that $\mathbf{P} \subseteq \mathbf{B} + \mathbf{Q}$. Note that for every $\mathbf{p} \in \mathbf{P}$ from $\mathbf{P} \subseteq \mathbf{C}$, we deduce that there exists $\lambda_1, \dots, \lambda_k \in \mathbb{Q}_{\geq 0}$ such that $\mathbf{p} = \sum_{j=1}^{k} \lambda_j \mathbf{q}_j$. There exists $\mu_j \in [0, 1]$ and $n_j \in \mathbb{N}$ such that $\lambda_j = \mu_j + n_j$. In particular $\mathbf{p} = \mathbf{b} + \mathbf{q}$ where $\mathbf{b} = \sum_{j=1}^{k} \mu_j \mathbf{q}_j$ and $\mathbf{q} = \sum_{j=1}^{k} n_j \mathbf{q}_j$. Note that $\mathbf{q} \in \mathbf{Q}$ and $\mathbf{b} \in \mathbf{B}$.

Now, let us consider an infinite sequence $(\mathbf{p}_n)_{n \in \mathbb{N}}$ of vectors in $\mathbf{P}$. For every $n \in \mathbb{N}$ there exists $\mathbf{b}_n \in \mathbf{B}$ and $\mathbf{q}_n \in \mathbf{Q}$ such that $\mathbf{p}_n = \mathbf{b}_n + \mathbf{q}_n$. Since $(\mathbf{B}, =)$ and $(\mathbf{Q}, \leq_{\mathbf{Q}})$ are two well-ordered sets, Dickson's Lemma show that there exists an infinite set $N \subseteq \mathbb{N}$ such that $\mathbf{b}_n = \mathbf{b}_m$ and $\mathbf{q}_n \leq_{\mathbf{Q}} \mathbf{q}_m$ for every $n \leq m$ in N. Thus $\mathbf{p}_n \leq_{\mathbf{P}} \mathbf{p}_m$ for every $n \leq m$ in N. We have proved that $(\mathbf{P}, \leq_{\mathbf{P}})$ is well-preordered. ∎

Appendix B
Proof of Theorem VII.1

In this section we prove the following theorem.

Theorem VII.1. *Transitive closures of finite unions of reflexive definable conic relations over $\mathbb{Q}^d_{\geq 0}$ are reflexive definable conic relations.*

The following lemma shows that the transitive closure of $R_1 \cup \ldots \cup R_k$ where R_j is a definable conic relation for every j is equal to the transitive closure of the reflexive definable conic relation $R = R_1 \circ \cdots \circ R_k$. That means Theorem VII.1 reduces to show that the class of reflexive definable conic relation over $\mathbb{Q}^d_{\geq 0}$ is stable by transitive closure.

Lemma B.1. *For every reflexive conic relations $R_1, \ldots, R_k$ over $\mathbb{Q}^d_{\geq 0}$, we have:*

$$R_1 \cup \ldots \cup R_k \quad \subseteq \quad R_1 + \cdots + R_k \quad \subseteq \quad R_1 \circ \cdots \circ R_k$$

Proof: Since $(\mathbf{0}, \mathbf{0}) \in R_j$ for every j we deduce that $R_1 \cup \ldots \cup R_k \subseteq R_1 + \cdots + R_k$. Let us consider a sequence $(\mathbf{x}_j, \mathbf{y}_j)_{1 \leq j \leq k}$ of couples $(\mathbf{x}_j, \mathbf{y}_j) \in R_j$. We introduce $\mathbf{z}_j = \mathbf{y}_1 + \cdots + \mathbf{y}_j + \mathbf{x}_{j+1} + \cdots + \mathbf{x}_k$. Let $j \in \{1, \ldots k\}$. Since $\mathbf{z}_{j-1} - \mathbf{x}_j \in \mathbb{Q}^d_{\geq 0}$ and R_j is a reflexive relation we get $(\mathbf{z}_{j-1} - \mathbf{x}_j, \mathbf{z}_{j-1} - \mathbf{x}_j) \in R_j$. Moreover, as $(\mathbf{x}_j, \mathbf{y}_j) \in R_j$ and R_j is conic we get $(\mathbf{z}_{j-1} - \mathbf{x}_j, \mathbf{z}_{j-1} - \mathbf{x}_j) + (\mathbf{x}_j, \mathbf{y}_j) \in R_j$. This couple is equal to $(\mathbf{z}_{j-1}, \mathbf{z}_j)$. We have proved that $(\mathbf{z}_0, \mathbf{z}_k) \in R_1 \circ \cdots \circ R_k$. Now just observe that $(\mathbf{z}_0, \mathbf{z}_k) = \sum_{j=1}^{k} (\mathbf{x}_j, \mathbf{y}_j)$. ∎

Transitive closures of reflexive conic relations can be characterized as follows. We introduce the function $\nabla : \mathbb{Q}^d_{\geq 0} \times \mathbb{Q}^d_{\geq 0} \rightarrow \mathbb{Q}^d$ defined by $\nabla(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \mathbf{x}$. Given a set $I \subseteq \{1, \ldots, d\}$ we introduce $\mathbb{Q}^d_I = \{\mathbf{x} \in \mathbb{Q}^d_{\geq 0} \mid \mathbf{x}(i) > 0 \iff i \in I\}$ and the function $\nabla_I : \mathbb{Q}^d_{\geq 0} \times \mathbb{Q}^d_{\geq 0} \rightarrow \mathbb{Q}^d$ partially defined over $\mathbb{Q}^d_I \times \mathbb{Q}^d_I$ by $\nabla_I(r) = \nabla(r)$ for every $r \in \mathbb{Q}^d_I \times \mathbb{Q}^d_I$.

Lemma B.2. *We have $\nabla_I^{-1}(\nabla_I(R)) \subseteq R^+$ for every $I \subseteq \{1, \ldots, d\}$ and for every reflexive conic relation R over $\mathbb{Q}^d_{\geq 0}$.*

Proof: Let $(\mathbf{x}, \mathbf{y}) \in \nabla_I^{-1}(\nabla_I(R))$. Then $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d_I$. We introduce the vector $\mathbf{z} \in \mathbb{Q}^d_I$ defined by $\mathbf{z}(i) = \min\{\mathbf{x}(i), \mathbf{y}(i)\}$. We also introduce $\mathbf{v} = \mathbf{y} - \mathbf{x}$. Since $\mathbf{v} \in \nabla_I(R)$, there exists $(\mathbf{a}, \mathbf{b}) \in (\mathbb{Q}^d_I \times \mathbb{Q}^d_I) \cap R$ such that $\mathbf{v} = \mathbf{b} - \mathbf{a}$. Since $\mathbf{z}, \mathbf{a} \in \mathbb{Q}^d_I$ there exists $n \in \mathbb{N}_{>0}$ such that $\frac{1}{n}\mathbf{a} \leq \mathbf{z}$. Hence there exists $\mathbf{e} \in \mathbb{Q}^d_{\geq 0}$ such that $\mathbf{z} = \mathbf{e} + \frac{1}{n}\mathbf{a}$. As R is reflexive we get $(\mathbf{e}, \mathbf{e}) \in R$ and since R is conic we have $(\mathbf{e}, \mathbf{e}) + \frac{1}{n}(\mathbf{a}, \mathbf{b}) \in R$. This couple is equal to $(\mathbf{z}, \mathbf{z} + \frac{1}{n}\mathbf{v})$. Let $k \in \{0, \ldots, n\}$ and let us prove that $\mathbf{e}_k = \mathbf{x} + \frac{k}{n}\mathbf{v} - \mathbf{z}$ is in $\mathbb{Q}^d_{\geq 0}$. Let $i \in \{1, \ldots, d\}$. If $\mathbf{v}(i) \geq 0$ then $\mathbf{e}_k(i) \geq \mathbf{x}(i) - \mathbf{z}(i) \geq 0$. If $\mathbf{v}(i) \leq 0$ since $\mathbf{e}_k = \mathbf{y} - \frac{n-k}{n}\mathbf{v} - \mathbf{z}$ we deduce that $\mathbf{e}_k(i) = \mathbf{y}(i) - \frac{n-k}{n}\mathbf{v}(i) - \mathbf{z}(i) \geq \mathbf{y}(i) - \mathbf{z}(i) \geq 0$. Thus $\mathbf{e}_k \in \mathbb{Q}^d_{\geq 0}$. Since R is reflexive we get $(\mathbf{e}_k, \mathbf{e}_k) \in R$. As R is conic we deduce that $(\mathbf{e}_k, \mathbf{e}_k) + (\mathbf{z}, \mathbf{z} + \frac{1}{n}\mathbf{v})$ is in R. Since this couple is equal to $(\mathbf{x} + \frac{k}{n}\mathbf{v}, \mathbf{x} + \frac{k+1}{n}\mathbf{v})$ we have proved that $(\mathbf{x}, \mathbf{y}) \in R^n$. ∎

Lemma B.3. *Let R be reflexive conic relation over $\mathbb{Q}^d$, let $\mathbf{v}_0, \ldots, \mathbf{v}_k \in \mathbb{Q}^d$ such that $\mathbf{v}_0 \ R \ \mathbf{v}_1 \ \cdots \ R \ \mathbf{v}_k$ and let $\mu_1, \ldots, \mu_k \in \mathbb{Q}_{\geq 0}$ such that the following vector $\mathbf{x}_j$ is in $\mathbb{Q}^d_{\geq 0}$ for every $1 \leq j \leq k$:*

$$\mathbf{x}_j = (\mu_0 - \mu_1)\mathbf{v}_0 + \cdots + (\mu_j - \mu_{j+1})\mathbf{v}_j$$

where $\mu_0 = 1$ and $\mu_{k+1} = 0$. Then $\mathbf{v}_0 \ R^n \ \mathbf{x}_k$ where n is the cardinal of $\{j \in \{1, \ldots, k\} \mid \mu_j > 0\}$.

Proof: Let us consider the vector $\mathbf{z}_j = \mathbf{x}_j + \mu_{j+1}\mathbf{v}_j$. As R is reflexive, we deduce that $(\mathbf{x}_{j-1}, \mathbf{x}_{j-1}) \in R$. Since R is conic, we get $(\mathbf{x}_{j-1}, \mathbf{x}_{j-1}) + \mu_j(\mathbf{v}_{j-1}, \mathbf{v}_j) \in R$. This pair is equal to $(\mathbf{z}_{j-1}, \mathbf{z}_j)$. Thus $(\mathbf{z}_{j-1}, \mathbf{z}_j) \in R$. Since $\mathbf{z}_{j-1} = \mathbf{z}_j$ if $\mu_j = 0$ we deduce that $\mathbf{z}_0 \ R^n \ \mathbf{z}_k$. Observe that $\mathbf{z}_0 = \mathbf{x}_0 + \mu_1\mathbf{v}_0 = \mu_0\mathbf{v}_0 = \mathbf{v}_0$ and $\mathbf{z}_k = \mathbf{x}_k + \mu_{k+1}\mathbf{v}_k = \mathbf{x}_k$. ∎

Lemma B.4. *Let $\mathbf{v}_0, \ldots, \mathbf{v}_k \in \mathbb{Q}^d_{\geq 0}$ and let us consider the sets $I_j = \{i \in \{1, \ldots, k\} \mid \mathbf{v}_0(i) > 0 \vee \ldots \vee \mathbf{v}_j(i) > 0\}$. There exist non-negative rational numbers $\mu_1, \ldots, \mu_k \geq 0$ such that $\mu_j = 0$ if $I_j = I_{j-1}$ and such that for every $0 \leq j \leq k$:*

$$(\mu_0 - \mu_1)\mathbf{v}_0 + \cdots + (\mu_j - \mu_{j+1})\mathbf{v}_j \in \mathbb{Q}^d_{I_j}$$

where $\mu_0 = 1$ and $\mu_{k+1} = 0$.

Proof: The lemma is immediate with $k = 0$. Assume the lemma proved for k and let us consider a sequence $\mathbf{v}_0, \ldots, \mathbf{v}_{k+1} \in \mathbb{Q}^d_{\geq 0}$ and let us introduce a sequence $\mu_1, \ldots, \mu_k \geq 0$ such that $\mu_j = 0$ if $I_j = I_{j-1}$ and such that:

$$(\mu_0 - \mu_1)\mathbf{v}_0 + \cdots + (\mu_j - \mu_{j+1})\mathbf{v}_j \in \mathbb{Q}^d_{I_j}$$

where $\mu_0 = 1$ and $\mu_{k+1} = 0$. Let us consider $\mathbf{x} = (\mu_0 - \mu_1)\mathbf{v}_0 + \cdots + (\mu_k - \mu_{k+1})\mathbf{v}_k$. Note that if $I_{k+1} = I_k$, by considering $\mu_{k+2} = 0$ we are done. So, let us assume that $I_{k+1} \neq I_k$. Since $\mathbf{x} \in \mathbb{Q}^d_{I_k}$ there exists $\epsilon > 0$ such that $\mathbf{x}(i) > \epsilon\mathbf{v}_k(i)$ for every $i \in I_k$. Let us consider the sequence $(\mu'_0, \ldots, \mu'_{k+2}) = (\mu_0, \ldots, \mu_k, \epsilon, 0)$. Observe that $(\mu'_0 - \mu'_1)\mathbf{v}_0 + \cdots + (\mu'_j - \mu'_{j+1})\mathbf{v}_j \in \mathbb{Q}^d_{I_j}$ for every $1 \leq j \leq k+1$. We have proved the lemma by induction. ∎

Corollary B.5. *Let R be reflexive conic relation over $\mathbb{Q}^d$, let $\mathbf{v}_0, \ldots, \mathbf{v}_k \in \mathbb{Q}^d$ such that $\mathbf{v}_0 \ R \ \mathbf{v}_1 \ \cdots \ R \ \mathbf{v}_k$, and let $I = \{i \mid \bigvee_{j=0}^{k} \mathbf{v}_j(i) > 0\}$. There exist non-negative rational numbers $\mu_1, \ldots, \mu_k \geq 0$ such that the following vector $\mathbf{e}$ is in $\mathbb{Q}^d_I$ and such that $\mathbf{v}_0 \ R^d \ \mathbf{e}$:*

$$\mathbf{e} = \mathbf{v}_0 + \sum_{j=1}^{k} \mu_j(\mathbf{v}_j - \mathbf{v}_{j-1})$$

Proof: Let us consider the sets $I_j = \{i \in \{1, \ldots, k\} \mid \mathbf{v}_0(i) > 0 \vee \ldots \vee \mathbf{v}_j(i) > 0\}$. Lemma B.4 shows that there exist non-negative rational numbers $\mu_1, \ldots, \mu_k \geq 0$ such that $\mu_j = 0$ if $I_j = I_{j-1}$ and such that the following vector $\mathbf{x}_j$ is in $\mathbb{Q}^d_{I_j}$ for every $0 \leq j \leq k$:

$$\mathbf{x}_j = (\mu_0 - \mu_1)\mathbf{v}_0 + \cdots + (\mu_j - \mu_{j+1})\mathbf{v}_j$$

where $\mu_0 = 1$ and $\mu_{k+1} = 0$. Lemma B.3 shows that $\mathbf{v}_0 \ R^n \ \mathbf{x}_k$ where n is the cardinal of $\{j \in \{1, \ldots, k\} \mid \mu_j > 0\}$. Since

$n \leq d$ and R is reflexive, we deduce that $R^n \subseteq R^d$. Note that $\mathbf{e} = \mathbf{x}_k$ is in $\mathbb{Q}^d_{I_k}$. Since $I_k = I$, we are done. ■

Lemma B.6. *For every reflexive conic relation R over $\mathbb{Q}^d_{\geq 0}$ we have:*

$$R^+ = R^d \circ \left(\sum_{I \subseteq \{1,\dots,d\}} \nabla_I^{-1}(\nabla_I(R)) \right) \circ R^d$$

Proof: From Lemma B.2 we deduce that $\nabla_I^{-1}(\nabla_I(R)) \subseteq R^+$ for every $I \subseteq \{1, \dots, d\}$. With Lemma B.1 we deduce that $\sum_{I \subseteq \{1,\dots,d\}} \nabla_I^{-1}(\nabla_I(R))$ is included in R^+. We have proved the inclusion $\supseteq$. Let us now prove the inclusion $\subseteq$.

Let us consider $(\mathbf{x}, \mathbf{y}) \in R^+$. There exists a sequence $(\mathbf{v}_j)_{0 \leq j \leq k}$ with $k \geq 1$ of vectors $\mathbf{v}_j \in \mathbb{Q}^d_{\geq 0}$ such that $\mathbf{v}_0 = \mathbf{x}$, $\mathbf{v}_k = \mathbf{y}$ and $(\mathbf{v}_{j-1}, \mathbf{v}_j) \in R$ for every $j \in \{1, \dots, k\}$. We introduce the set $I = \{i \mid \mathbf{v}_0(i) > 0 \vee \ldots \vee \mathbf{v}_k(i) > 0\}$.

Corollary B.5 shows that there exist $\mu_1, \dots, \mu_k \geq 0$ such that $\mathbf{x} \ R^d \ \mathbf{e}$ where $\mathbf{e} = \mathbf{x} + \sum_{j=1}^k \mu_j(\mathbf{v}_j - \mathbf{v}_{j-1})$ is a vector in $\mathbb{Q}^d_I$. The inverse of R and Corollary B.5 show that there exist $\mu'_1, \dots, \mu'_k \geq 0$ such that $\mathbf{f} \ R^d \ \mathbf{y}$ where $\mathbf{f} = \mathbf{y} + \sum_{j=1}^k \mu'_j(\mathbf{v}_{j-1} - \mathbf{v}_j)$ is a vector in $\mathbb{Q}^d_I$.

Let us consider $\mu \geq 0$ such that $\mu - \mu_j - \mu'_j \geq 0$ for every j. Let $\mathbf{a} = \sum_{j=1}^k (1 + \mu - \mu_j - \mu'_j)(\mathbf{v}_j - \mathbf{v}_{j-1})$ and let us prove that $\mathbf{a} \in \nabla_I(R)$. Let us introduce the vector $\mathbf{e} \in \mathbb{Q}^d_I$ defined by $\mathbf{e}(i) = 1$ if $i \in I$ and $\mathbf{e}(i) = 0$ otherwise. Since R is reflexive we get $(\mathbf{e}, \mathbf{e}) \in R$ and since R is conic then $r_j = (\mathbf{e} + \mathbf{v}_{j-1}, \mathbf{e} + \mathbf{v}_j)$ is in R. Observe that $\mathbf{e} + \mathbf{v}_{j-1}$ and $\mathbf{e} + \mathbf{v}_j$ are both in $\mathbb{Q}^d_I$. We deduce that $\nabla(r_j) \in \nabla_I(R)$. Then $\mathbf{v}_j - \mathbf{v}_{j-1} \in \nabla_I(R)$. Since $\nabla_I(R)$ is a conic set we deduce that $\mathbf{a} \in \nabla_I(R)$.

We have:

$$\begin{aligned}
&(\mathbf{f} + \mu\mathbf{y}) - (\mathbf{e} + \mu\mathbf{x}) \\
&= (1+\mu)\mathbf{y} - (1+\mu)\mathbf{x} - \sum_{j=1}^k (\mu_j + \mu'_j)(\mathbf{v}_j - \mathbf{v}_{j-1}) \\
&= (1+\mu)\sum_{j=1}^k (\mathbf{v}_j - \mathbf{v}_{j-1}) - \sum_{j=1}^k (\mu_j + \mu'_j)(\mathbf{v}_j - \mathbf{v}_{j-1}) \\
&= \sum_{j=1}^k (1 + \mu - \mu_j - \mu'_j)(\mathbf{v}_j - \mathbf{v}_{j-1}) \\
&= \mathbf{a}
\end{aligned}$$

As $\mathbf{a} \in \nabla_I(R)$ and $\mathbf{e} + \mu\mathbf{x}, \mathbf{f} + \mu\mathbf{y} \in \mathbb{Q}^d_I$ we deduce that $\mathbf{e} + \mu\mathbf{x} \ \nabla_I^{-1}(\nabla_I(R)) \ \mathbf{f} + \mu\mathbf{y}$. From $(\mathbf{x}, \mathbf{e}) \in R^d$ and $(\mathbf{x}, \mathbf{x}) \in R^d$ and since R^d is conic, we deduce that $(1+\mu)\mathbf{x} \ R^d \ \mathbf{e} + \mu\mathbf{x}$. Symmetrically we get $\mathbf{f} + \mu\mathbf{y} \ R^d (1+\mu)\mathbf{f}$. We have proved that the relation $R^d \circ \nabla_I^{-1}(\nabla_I(R)) \circ R^d$ contains $(1 + \mu)(\mathbf{x}, \mathbf{y})$. Since this relation is conic we deduce that it contains $(\mathbf{x}, \mathbf{y})$. ■

We deduce the proof of Theorem VII.1.

Appendix C
Proof of Lemma X.2

Lemma C.1. *Let $\mathbf{P} \subseteq \mathbb{Q}^d$ be a periodic set included in $\bigcup_{j=1}^{k} \mathbf{B}_j + \mathbf{V}_j$ where $k \in \mathbb{N}_{>0}$, $\mathbf{B}_j \subseteq \mathbb{Q}^d$ is a bounded set and $\mathbf{V}_j \subseteq \mathbb{Q}^d$ is a vector space. There exists $j \in \{1, \dots, k\}$ such that $\mathbf{P} \subseteq \mathbf{V}_j \subseteq \mathbf{B}_j + \mathbf{V}_j$.*

Proof: Let us first prove by induction over $k \in \mathbb{N}_{>0}$ that for every periodic set $\mathbf{P} \subseteq \mathbb{Q}^d$ included in $\bigcup_{j=1}^{k} \mathbf{V}_j$ where $\mathbf{V}_j \subseteq \mathbb{Q}^d$ is a vector space, there exists $j \in \{1, \dots, k\}$ such that $\mathbf{P} \subseteq \mathbf{V}_j$. The rank $k = 1$ is immediate. Let us prove the rank $k+1$ and assume that $\mathbf{P}$ is included in $\bigcup_{j=1}^{k+1} \mathbf{V}_j$. If $\mathbf{P} \subseteq \mathbf{V}_{k+1}$ the induction is proved. So we can assume that there exists $\mathbf{p} \in \mathbf{P} \backslash \mathbf{V}_{k+1}$. Let $\mathbf{x} \in \mathbf{P}$. Since $n\mathbf{p} + \mathbf{x} \in \mathbf{P}$ for every $n \in \mathbb{N}$ there exists $j \in \{1, \dots, k+1\}$ such that $n\mathbf{p} + \mathbf{x} \in \mathbf{V}_j$. As $\{1, \dots, k+1\}$ is finite, there exists j in this set and $n < n'$ such that $n\mathbf{p} + \mathbf{x}$ and $n'\mathbf{p} + \mathbf{x}$ are both in $\mathbf{V}_j$. In particular the difference of this two vectors is in $\mathbf{V}_j$. Since this difference is $(n' - n)\mathbf{p}$ and $\mathbf{p} \not\in \mathbf{V}_{k+1}$ we get $j \in \{1, \dots, k\}$. Observe that $n(n'\mathbf{p} + \mathbf{x}) - n'(n\mathbf{p} + \mathbf{x})$ is the difference of two vectors in $\mathbf{V}_j$. Thus this vector is in $\mathbf{V}_j$ and we deduce that $\mathbf{x} \in \mathbf{V}_j$. We have shown that $\mathbf{P} \subseteq \bigcup_{j=1}^{k} \mathbf{V}_j$. By induction there exists $j \in \{1, \dots, k\}$ such that $\mathbf{P} \subseteq \mathbf{V}_j$. We have proved the induction.

Finally, let $\mathbf{P} \subseteq \mathbb{Q}^d$ be a periodic set included in $\bigcup_{j=1}^{k} \mathbf{B}_j + \mathbf{V}_j$ where $k \in \mathbb{N}_{>0}$, $\mathbf{B}_j \subseteq \mathbb{Q}^d$ is a bounded set and $\mathbf{V}_j \subseteq \mathbb{Q}^d$ is a vector space. Let us consider the set J of $j \in \{1, \dots, k\}$ such that $\mathbf{V}_j \subseteq \mathbf{B}_j + \mathbf{V}_j$. Let us prove that $\mathbf{P} \subseteq \bigcup_{j \in J} \mathbf{V}_j$. Let us consider $\mathbf{p} \in \mathbf{P}$. Since $n\mathbf{p} \in \mathbf{P}$ for every $n \in \mathbb{N}$, the pigeon-hole principle shows that there exists $j \in \{1, \dots, k\}$ and an infinite set $N \subseteq \mathbb{N}$ such that $n\mathbf{p} \in \mathbf{B}_j + \mathbf{V}_j$ for every $n \in N$. We deduce that for every $n \in N$ there exists $\mathbf{b}_n \in \mathbf{B}_j$ such that $n\mathbf{p} - \mathbf{b}_n \in \mathbf{V}_j$. Lemma V.1 shows that there exists a finite set $\mathbf{H} \subseteq \mathbb{Q}^d$ such that $\mathbf{V}_j = \{\mathbf{v} \in \mathbb{Q}^d \mid \bigwedge_{\mathbf{h} \in \mathbf{H}} \mathbf{h} \cdot \mathbf{v} = 0\}$. Let $\mathbf{h} \in \mathbf{H}$. Since $n\mathbf{p} - \mathbf{b}_n \in \mathbf{V}_j$ we get $n\mathbf{h} \cdot \mathbf{p} = \mathbf{h} \cdot \mathbf{b}_n$ for every $n \in N$. Since $\mathbf{B}_j$ is bounded, there exists $c \in \mathbb{Q}_{\geq 0}$ such that $|\mathbf{h} \cdot \mathbf{b}_n| \leq c$ for every $n \in N$. Thus $\mathbf{h} \cdot \mathbf{p} = 0$ and we have proved that $\mathbf{p} \in \mathbf{V}_j$. From $n\mathbf{p} + \mathbf{b}_n \in \mathbf{V}_j$ and $\mathbf{p} \in \mathbf{V}_j$ we deduce that $\mathbf{b}_n \in \mathbf{V}_j$. Thus $\mathbf{V}_j = \mathbf{b}_n + \mathbf{V}_j \subseteq \mathbf{B}_j + \mathbf{V}_j$ and we have proved that $j \in J$. We deduce that $\mathbf{P}$ is included in $\bigcup_{j \in J} \mathbf{V}_j$. From the previous paragraph, there exists $j \in J$ such that $\mathbf{P} \subseteq \mathbf{V}_j$. ■

Lemma X.2. *We have $\dim(\mathbf{P}) = \mathrm{rank}(\mathbf{V})$ for every periodic set $\mathbf{P}$ where $\mathbf{V}$ is the vector space generated by $\mathbf{P}$.*

Proof: Since $\mathbf{P} \subseteq \mathbf{V}$ we deduce that $\dim(\mathbf{P}) \leq \mathrm{rank}(\mathbf{V})$. For the converse inequality, there exist $k \in \mathbb{N}$, $(\mathbf{B}_j)_{1 \leq j \leq k}$ a sequence of bounded subsets $\mathbf{B}_j \subseteq \mathbb{Q}^d$ and a sequence $\mathbf{V}_j \subseteq \mathbb{Q}^d$ of vector spaces such that $\mathbf{P} \subseteq \bigcup_{j=1}^{k} \mathbf{b}_j + \mathbf{V}_j$ and such that $\mathrm{rank}(\mathbf{V}_j) \leq \dim(\mathbf{P})$ for every j. Since $\mathbf{P}$ is non empty we deduce that $k \in \mathbb{N}_{>0}$. Lemma C.1 proves that there exists $j \in \{1, \dots, k\}$ such that $\mathbf{P} \subseteq \mathbf{V}_j$. By minimality of the vector space generated by $\mathbf{P}$ we get $\mathbf{V} \subseteq \mathbf{V}_j$. Hence $\mathrm{rank}(\mathbf{V}) \leq \mathrm{rank}(\mathbf{V}_j)$. From $\mathrm{rank}(\mathbf{V}_j) \leq \dim(\mathbf{P})$ we get $\mathrm{rank}(\mathbf{V}) \leq \dim(\mathbf{P})$. ■

APPENDIX D
COMPLETE EXTRACTIONS

Let $\mathcal{K}$ be a finite class of definable conic sets of $\mathbb{Q}^d$. We denote by $\Sigma(\mathcal{K})$ the set $\bigcup_{\mathbf{K}\in\mathcal{K}} \mathbf{K}$. An *extraction* of $\mathcal{K}$ is a finite class $\mathcal{C}$ of finitely generated conic sets of $\mathbb{Q}^d$ such that for every $\mathbf{C} \in \mathcal{C}$ there exists $\mathbf{K} \in \mathcal{K}$ such that $\mathbf{C} \subseteq \mathbf{K}$. An *extraction* $\mathcal{C}$ of $\mathcal{K}$ is said to be *complete* if $\Sigma(\mathcal{C}) = \Sigma(\mathcal{K})$.

Example D.1. *Let us consider the class $\mathcal{K} = \{\mathbf{K}_1, \mathbf{K}_2\}$ with $\mathbf{K}_1 = \{\mathbf{0}\} \cup (\mathbb{Q} \times \mathbb{Q}_{>0})$ and $\mathbf{K}_2 = \mathbb{Q} \times \mathbb{Q}_{\leq 0}$. Observe that $\Sigma(\mathcal{K})$ is equal to $\mathbb{Q}^2$. We show that there does not exist a complete extraction of $\mathcal{K}$ as follow. We first consider a finitely generated conic set $\mathbf{C}$ included in $\mathbf{K}_1$. Such a conic set is generated by a finite set of vectors in $\mathbf{K}_1 \backslash \{\mathbf{0}\} = \mathbb{Q} \times \mathbb{Q}_{>0}$. So there exists $\epsilon \in \mathbb{Q}_{>0}$ such that $\mathbf{C} \subseteq \mathbb{Q}_{\geq 0}(1, \epsilon) + \mathbb{Q}_{\geq 0}(-1, \epsilon)$. Now, let us consider an extraction $\mathcal{C}$ of $\mathcal{K}$. We have proved that there exists $\epsilon \in \mathbb{Q}_{>0}$ such that $\Sigma(\mathcal{C}) \subseteq (\mathbb{Q}_{\geq 0}(1, \epsilon) + \mathbb{Q}_{\geq 0}(-1, \epsilon)) \cup (\mathbb{Q} \times \mathbb{Q}_{\leq 0})$ which is strictly included in $\mathbb{Q}^2$ (for instance $(1, \frac{\epsilon}{2})$ is not in this set).*

In this section finite classes $\mathcal{K}$ of definable conic sets of $\mathbb{Q}^d$ having a complete extraction are topologically characterized thanks to the *overlapping property*[1]. The class $\mathcal{K}$ is said to have the *overlapping property* if for every $\mathbf{K} \in \mathcal{K}$ and for every finite sequence $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of vectors $\mathbf{v}_j \in \mathbb{Q}^d$ satisfying $\mathbb{Q}_{>0}\mathbf{v}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_k \subseteq \mathbf{K}$ there exists $\mathbf{K}' \in \mathcal{K}$ such that $\mathbf{K}' \cap (\mathbb{Q}_{>0}\mathbf{v}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_j)$ is non empty for every $j \in \{1, \ldots, k\}$. We are going to prove the following result:

Theorem D.2. *A finite class $\mathcal{K}$ of definable conic sets of $\mathbb{Q}^d$ has the overlapping property if and only if it has the complete extraction property.*

Example D.3. *Let us come back to the class $\mathcal{K} = \{\mathbf{K}_1, \mathbf{K}_2\}$ with $\mathbf{K}_1 = \{\mathbf{0}\} \cup (\mathbb{Q} \times \mathbb{Q}_{>0})$ and $\mathbf{K}_2 = \mathbb{Q} \times \mathbb{Q}_{\leq 0}$ introduced in Example D.1. We show that $\mathcal{K}$ does not satisfy the overlapping property by considering the sequence $\mathbf{v}_1, \mathbf{v}_2$ defined by $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (1, 1)$. Now, just observe that $\mathbb{Q}_{>0}\mathbf{v}_1 + \mathbb{Q}_{>0}\mathbf{v}_2 \subseteq \mathbf{K}_1$ but $\mathbf{K}_1 \cap (\mathbb{Q}_{>0}\mathbf{v}_1)$ and $\mathbf{K}_2 \cap (\mathbb{Q}_{>0}\mathbf{v}_1 + \mathbb{Q}_{>0}\mathbf{v}_2)$ are empty.*

We observe that if a finite class $\mathcal{K}$ of definable conic sets of $\mathbb{Q}^d$ has a complete extraction $\mathcal{C}$, then for every $\mathbf{K} \in \mathcal{K}$ and for every sequence $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of vectors $\mathbf{v}_j \in \mathbb{Q}^d$ such that $\mathbb{Q}_{>0}\mathbf{v}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_k \subseteq \mathbf{K}$, from $\mathbf{K} \subseteq \bigcup_{\mathbf{C}\in\mathcal{C}} \mathbf{C}$, the following lemma shows that there exists $\mathbf{C} \in \mathcal{C}$ such that $\mathbf{C} \cap (\mathbb{Q}_{>0}\mathbf{v}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_j) \neq \emptyset$ for every $j \in \{1, \ldots, k\}$. Since $\mathcal{C}$ is an extraction of $\mathcal{K}$ we deduce that there exists $\mathbf{K}' \in \mathcal{K}$ such that $\mathbf{C} \subseteq \mathbf{K}'$. Therefore $\mathbf{K}' \cap (\mathbb{Q}_{>0}\mathbf{v}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_j) \neq \emptyset$ for every $j \in \{1, \ldots, k\}$. We have proved that $\mathcal{K}$ has the overlapping property.

Lemma D.4. *For every sequence $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of vectors $\mathbf{v}_j \in \mathbb{Q}^d$ and for every finite class $\mathcal{C}$ of finitely generated conic sets of $\mathbb{Q}^d$ such that $\mathbb{Q}_{>0}\mathbf{v}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_k \subseteq \bigcup_{\mathbf{C}\in\mathcal{C}} \mathbf{C}$, there exists $\mathbf{C} \in \mathcal{C}$ such that $\mathbf{C} \cap (\mathbb{Q}_{>0}\mathbf{v}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_j) \neq \emptyset$ for every $j \in \{1, \ldots, k\}$.*

Proof: We prove the lemma by induction over $k \in \mathbb{N}_{>0}$. The rank $k = 1$ is immediate since from $\mathbb{Q}_{>0}\mathbf{v}_1 \subseteq \bigcup_{\mathbf{C}\in\mathcal{C}} \mathbf{C}$ we deduce that there exists $\mathbf{C} \in \mathcal{C}$ such that $\mathbf{C} \cap (\mathbb{Q}_{>0}\mathbf{v}_1)$ is non empty. Let us assume the induction proved for a rank $k \in \mathbb{N}_{>0}$ and let us consider a sequence $\mathbf{v}_0, \ldots, \mathbf{v}_k$ of vectors in $\mathbb{Q}^d$ and a finite class $\mathcal{C}$ of finitely generated conic sets of $\mathbb{Q}^d$ such that $\mathbb{Q}_{>0}\mathbf{v}_0 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_k \subseteq \bigcup_{\mathbf{C}\in\mathcal{C}} \mathbf{C}$. We introduce the finite class $\mathcal{C}_0 = \{\mathbf{C} \in \mathcal{C} \mid \mathbf{v}_0 \in \mathbf{C}\}$. We are going to prove that there exists a sequence $(\lambda_j)_{1\leq j\leq k}$ of rational numbers $\lambda_j \in \mathbb{Q}_{>0}$ such that $\mathbb{Q}_{>0}(\mathbf{v}_1 + \lambda_1\mathbf{v}_0) + \cdots + \mathbb{Q}_{>0}(\mathbf{v}_k + \lambda_k\mathbf{v}_0) \subseteq \bigcup_{\mathbf{C}\in\mathcal{C}_0} \mathbf{C}$.

Since every $\mathbf{C} \in \mathcal{C}$ is a finitely generated conic set, Lemma V.2 shows that there exists a finite set $\mathbf{H}_\mathbf{C} \subseteq \mathbb{Q}^d$ such that:

$$\mathbf{C} = \bigcap_{\mathbf{h}\in\mathbf{H}_\mathbf{C}} \{\mathbf{v} \in \mathbb{Q}^d \mid \mathbf{h} \cdot \mathbf{v} \geq 0\}$$

We introduce the set $\mathbf{H} = \bigcup_{\mathbf{C}\in\mathcal{C}} \mathbf{H}_\mathbf{C}$ and the set $\mathbf{H}_0 = \{\mathbf{h} \in \mathbf{H} \mid \mathbf{h} \cdot \mathbf{v}_0 > 0\}$.

We build up a sequence $(\lambda_j)_{1\leq j\leq k}$ of rational numbers $\lambda_j \in \mathbb{Q}_{>0}$ such that $\mathbf{h} \cdot (\mathbf{v}_j + \lambda_j\mathbf{v}_0) \geq 0$ for every $\mathbf{h} \in \mathbf{H}_0$ as follows. Let $\mathbf{h} \in \mathbf{H}_0$ and $j \in \{1, \ldots, k\}$. Since $\mathbf{h} \cdot \mathbf{v}_0 > 0$ we deduce that there exists $\lambda_{\mathbf{h},j} \in \mathbb{Q}_{\geq 0}$ such that $\mathbf{h} \cdot (\mathbf{v}_j + \lambda_{\mathbf{h},j}\mathbf{v}_0) \geq 0$. We introduce a rational number $\lambda_j \in \mathbb{Q}_{>0}$ such that $\lambda_j \geq \lambda_{\mathbf{h},j}$ for every $\mathbf{h} \in \mathbf{H}_0$. By construction observe that $\mathbf{h} \cdot (\mathbf{v}_j + \lambda_j\mathbf{v}_0) \geq 0$ for every $\mathbf{h} \in \mathbf{H}_0$ and for every $j \in \{1, \ldots, k\}$.

We introduce the sequence $(\mathbf{w}_j)_{1\leq j\leq k}$ of vectors $\mathbf{w}_j = \mathbf{v}_j + \lambda_j\mathbf{v}_0$. Now, let us consider $\mathbf{x} \in \mathbb{Q}_{>0}\mathbf{w}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{w}_k$ and let us prove that $\mathbf{x} \in \bigcup_{\mathbf{C}\in\mathcal{C}_0} \mathbf{C}$. Observe that for every $n \in \mathbb{N}$ we have $n\mathbf{v}_0 + \mathbf{x} \in \mathbb{Q}_{>0}\mathbf{v}_0 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_k \subseteq \bigcup_{\mathbf{C}\in\mathcal{C}} \mathbf{C}$. Hence there exists $\mathbf{C}_n \in \mathcal{C}$ such that $n\mathbf{v}_0 + \mathbf{x} \in \mathbf{C}_n$. Since $\mathcal{C}$ is finite, there exists $\mathbf{C} \in \mathcal{C}$ such that $\mathbf{C}_n = \mathbf{C}$ for an infinite number of $n \in \mathbb{N}$. Let $\mathbf{h} \in \mathbf{H}_\mathbf{C}$. Since $n\mathbf{v}_0 + \mathbf{x} \in \mathbf{C}$ we get $n\mathbf{h} \cdot \mathbf{v}_0 + \mathbf{h} \cdot \mathbf{x} \geq 0$. As this inequality holds for an infinite number of $n \in \mathbb{N}$ we deduce that $\mathbf{h} \cdot \mathbf{v}_0 \geq 0$. In particular $\mathbf{v}_0 \in \mathbf{C}$ and we deduce that $\mathbf{C} \in \mathcal{C}_0$. Note that if $\mathbf{h} \cdot \mathbf{v}_0 = 0$ then $\mathbf{h} \cdot \mathbf{x} \geq 0$. Otherwise, if $\mathbf{h} \cdot \mathbf{v}_0 > 0$ then $\mathbf{h} \in \mathbf{H}_0$. In this case $\mathbf{h} \cdot \mathbf{w}_j \geq 0$ for every j. From $\mathbf{x} \in \mathbb{Q}_{>0}\mathbf{w}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{w}_k$ we get $\mathbf{h} \cdot \mathbf{x} \geq 0$. We have proved that $\mathbf{h} \cdot \mathbf{x} \geq 0$ for every $\mathbf{h} \in \mathbf{H}_\mathbf{C}$. Therefore $\mathbf{x} \in \mathbf{C}$ and we have proved the inclusion $\mathbb{Q}_{>0}\mathbf{w}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{w}_k \subseteq \bigcup_{\mathbf{C}\in\mathcal{C}_0} \mathbf{C}$.

By induction, there exists $\mathbf{C} \in \mathcal{C}_0$ such that $\mathbf{C} \cap (\mathbb{Q}_{>0}\mathbf{w}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{w}_j)$ is non empty for every $j \in \{1, \ldots, k\}$. Since $\mathbb{Q}_{>0}\mathbf{w}_1 + \cdots + \mathbb{Q}_{>0}\mathbf{w}_k \subseteq \mathbb{Q}_{>0}\mathbf{v}_0 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_j$ we deduce that $\mathbf{C} \cap (\mathbb{Q}_{>0}\mathbf{v}_0 + \cdots + \mathbb{Q}_{>0}\mathbf{v}_j)$ is non empty for every $j \in \{1, \ldots, k\}$. As $\mathbf{C} \cap (\mathbb{Q}_{>0}\mathbf{v}_0)$ contains $\mathbf{v}_0$, this set is also non empty. Therefore, we have proved the induction at rank $k + 1$. ∎

Given a finitely generated conic set $\mathbf{C} \subseteq \mathbb{Q}^d$ and a finite class $\mathcal{K}$ of definable conic sets, we denote by $\mathbf{C} \cap \mathcal{K}$ the finite class $\{\mathbf{C} \cap \mathbf{K} \mid \mathbf{K} \in \mathcal{K}\}$.

[1]The term "overlapping" comes from a topological property introduced by Lambert in an unpublished work similar to the one we consider in this paper.

Lemma D.5. *For every finite class $\mathcal{K}$ of definable conic sets of $\mathbb{Q}^d$ with the overlapping property and for every finitely generated conic set $\mathbf{C} \subseteq \mathbb{Q}^d$, the class $\mathbf{C} \cap \mathcal{K}$ has the overlapping property.*

Proof: Let us consider $\mathbf{K} \in \mathcal{K}$ and a sequence $\mathbf{c}_1, \dots, \mathbf{c}_k$ of vector $\mathbf{c}_j \in \mathbb{Q}^d$ such that $\mathbb{Q}_{>0}\mathbf{c}_1 + \dots + \mathbb{Q}_{>0}\mathbf{c}_k \subseteq \mathbf{C} \cap \mathbf{K}$. Since $\mathcal{K}$ has the overlapping property, there exists $\mathbf{K}' \in \mathcal{K}$ such that $\mathbf{K}' \cap (\mathbb{Q}_{>0}\mathbf{c}_1 + \dots + \mathbb{Q}_{>0}\mathbf{c}_j)$ is non empty for every $j \in \{1, \dots, k\}$. As $\mathbf{C}$ is a finitely generated conic set, Lemma V.2 shows that there exists a finite set $\mathbf{H} \subseteq \mathbb{Q}^d$ such that:

$$\mathbf{C} = \{\mathbf{c} \in \mathbb{Q}^d \mid \bigwedge_{\mathbf{h} \in \mathbf{H}} \mathbf{h} \cdot \mathbf{v} \geq 0\}$$

Let $\mathbf{c} = \sum_{j=1}^{k} \mathbf{c}_j$. As $\mathbb{Q}_{>0}\mathbf{c}_1 + \dots + \mathbb{Q}_{>0}\mathbf{c}_k \subseteq \mathbf{C}$ we deduce that $\mathbf{c} + \mathbb{Q}_{\geq 0}\mathbf{c}_j \subseteq \mathbf{C}$. In particular $\mathbf{h} \cdot \mathbf{c} + \lambda \mathbf{h} \cdot \mathbf{c}_j \geq 0$ for every $\lambda \in \mathbb{Q}_{\geq 0}$. Thus $\mathbf{h} \cdot \mathbf{c}_j \geq 0$. We deduce that $\mathbf{c}_j \in \mathbf{C}$. Hence $\mathbb{Q}_{>0}\mathbf{c}_1 + \dots + \mathbb{Q}_{>0}\mathbf{c}_j \subseteq \mathbf{C}$ for every $j \in \{1, \dots, k\}$. In particular $\mathbf{C} \cap \mathbf{K}' \cap (\mathbb{Q}_{>0}\mathbf{c}_1 + \dots + \mathbb{Q}_{>0}\mathbf{c}_j)$ is non empty for every $j \in \{1, \dots, k\}$. We have proved that the class $\mathbf{C} \cap \mathcal{K}$ has the overlapping property. ∎

Lemma D.6. *Let $\mathcal{K}$ be a finite class of definable conic sets of $\mathbb{Q}^d$ with the overlapping property then $\mathcal{K}$ has the complete extraction property.*

Proof: We prove by induction over $r \in \mathbb{N}$ that for every vector space $\mathbf{V} \subseteq \mathbb{Q}^d$ with $\mathrm{rank}(\mathbf{V}) \leq r$ and for every finite class $\mathcal{K}$ of definable conic subsets of $\mathbf{V}$, if $\mathcal{K}$ has the overlapping property then it has the complete extraction property. The rank $r = 0$ is immediate since in this case $\mathbf{V} = \{\mathbf{0}\}$. So, let us assume the induction proved for a rank $r \in \mathbb{N}$ and let us consider a vector space $\mathbf{V} \subseteq \mathbb{Q}^d$ with $\mathrm{rank}(\mathbf{V}) \leq r + 1$ and a finite class $\mathcal{K}$ of definable conic subsets of $\mathbf{V}$. We assume that $\mathcal{K}$ has the overlapping property.

Since $\mathcal{K}$ is a finite class of sets definable in FO $(\mathbb{Q}, +, \leq, 0)$, and this logic admits a quantifier elimination algorithm, we deduce that there exists a finite set $\mathbf{H} \subseteq \mathbb{Q}^d$ such that every $\mathbf{K} \in \mathcal{K}$ is the set of vectors $\mathbf{v} \in \mathbf{V}$ satisfying a boolean combination of constraints of the form $\mathbf{h} \cdot \mathbf{x} \# 0$ where $\# \in \{<, \leq, \geq, >\}$. Note that if a vector $\mathbf{h} \in \mathbf{H}$ satisfies $\mathbf{h} \cdot \mathbf{v} = 0$ for every $\mathbf{v} \in \mathbf{V}$ then the constraints $\mathbf{h} \cdot \mathbf{x} \# 0$ is useless. So, we can assume without loss of generality that for every $\mathbf{h} \in \mathbf{H}$ there exists $\mathbf{v} \in \mathbf{V}$ such that $\mathbf{h} \cdot \mathbf{v} \neq 0$.

Let us consider for every $s : \mathbf{H} \to \{-1, 1\}$ the finitely generated conic set $\mathbf{C}_s = \{\mathbf{v} \in \mathbf{V} \mid s(\mathbf{h})\mathbf{h} \cdot \mathbf{v} \geq 0\}$. Since $\mathcal{K}$ has the overlapping property, LemmaD.5 shows that $\mathcal{K}_s = \mathbf{C}_s \cap \mathcal{K}$ has the overlapping property. From $\mathbf{V} = \bigcup_s \mathbf{C}_s$ we deduce that $\Sigma(\mathcal{K}) = \bigcup_s \Sigma(\mathcal{K}_s)$. So, it is sufficient to prove that $\mathcal{K}_s$ has the complete extraction property. By replacing $\mathcal{K}$ by $\mathcal{K}_s$ and $\mathbf{H}$ by $\{s(\mathbf{h})\mathbf{h} \mid \mathbf{h} \in \mathbf{H}\}$, we can assume without loss of generality that $\mathbf{h} \cdot \mathbf{v} \geq 0$ for every $\mathbf{v} \in \Sigma(\mathcal{K})$.

We introduce the following finitely generated conic set $\mathbf{C}$ and the following set $\mathbf{X}$:

$$\mathbf{C} = \bigcap_{\mathbf{h} \in \mathbf{H}_0} \{\mathbf{c} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{c} \geq 0\}$$

$$\mathbf{X} = \bigcap_{\mathbf{h} \in \mathbf{H}_0} \{\mathbf{x} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{x} > 0\}$$

We also introduce for every $\mathbf{h} \in \mathbf{H}$ the vector space $\mathbf{V}_\mathbf{h} = \{\mathbf{v} \in \mathbf{V} \mid \mathbf{h} \cdot \mathbf{v} = 0\}$. Since for every $\mathbf{h} \in \mathbf{H}$ there exists a vector $\mathbf{v} \in \mathbf{V}$ such that $\mathbf{h} \cdot \mathbf{v} \neq 0$ we deduce that $\mathbf{V}_\mathbf{h}$ is strictly included in $\mathbf{V}$ and in particular $\mathrm{rank}(\mathbf{V}_\mathbf{h}) \leq r$. Lemma D.5 shows that $\mathbf{V}_\mathbf{h} \cap \mathcal{K}$ has the overlapping property and by induction we deduce that this class has the complete extraction property. We introduce the set $\mathcal{K}' = \{\mathbf{K} \in \mathcal{K} \mid \mathbf{K} \cap \mathbf{X} \neq \emptyset\}$. Since $\mathbf{C} \backslash \mathbf{X}$ is included in $\bigcup_{\mathbf{h} \in \mathbf{H}} \mathbf{V}_\mathbf{h}$ we deduce that $\Sigma(\mathcal{K})$ is equal to the union of the sets $\Sigma(\mathbf{V}_\mathbf{h} \cap \mathcal{K})$ indexed by $\mathbf{h} \in \mathbf{H}$ and $\Sigma(\mathcal{K}')$. Therefore, in order to prove that $\mathcal{K}$ has the complete extraction property it is sufficient to prove that $\mathcal{K}'$ has the complete extraction property is immediate.

Let us prove that $\mathbf{X} \subseteq \mathbf{K}$ for every $\mathbf{K} \in \mathcal{K}'$. Recall that $\mathbf{K}$ is the set of vectors $\mathbf{v} \in \mathbf{V}$ satisfying a boolean combination of constraints of the form $\mathbf{h} \cdot \mathbf{x} \# 0$ where $\# \in \{<, \leq, \geq, >\}$. As $\mathbf{K} \cap \mathbf{X}$ is non empty we deduce that this boolean combination is true when the predicates $\mathbf{h} \cdot \mathbf{x} \# 0$ with $\# \in \{\geq, >\}$ are evaluated to true. We deduce that $\mathbf{X} \subseteq \mathbf{K}$.

Let us prove that $\mathcal{K}'$ has the overlapping property. Let us consider $\mathbf{K} \in \mathcal{K}'$ and a sequence $\mathbf{v}_1, \dots, \mathbf{v}_k$ of vectors in $\mathbb{Q}^d$ such that $\mathbb{Q}_{>0}\mathbf{v}_1 + \dots + \mathbb{Q}_{>0}\mathbf{v}_k \subseteq \mathbf{K}$. Since $\mathbf{K} \cap \mathbf{X}$ is non empty, there exists a vector $\mathbf{x}$ in this intersection. As $\mathbf{X} \subseteq \mathbf{K}$ we deduce that $\mathbb{Q}_{>0}\mathbf{v}_1 + \dots + \mathbb{Q}_{>0}\mathbf{v}_k + \mathbb{Q}_{>0}\mathbf{x} \subseteq \mathbf{X} \subseteq \mathbf{K}$. As $\mathcal{K}$ has the overlapping property we deduce that there exists $\mathbf{K}' \in \mathcal{K}$ such that $\mathbf{K}' \cap (\mathbb{Q}_{>0}\mathbf{v}_1 + \dots + \mathbb{Q}_{>0}\mathbf{v}_k + \mathbb{Q}_{>0}\mathbf{x})$ is non empty and such that $\mathbf{K}' \cap (\mathbb{Q}_{>0}\mathbf{v}_1 + \dots + \mathbb{Q}_{>0}\mathbf{v}_j)$ is non empty for every $j \in \{1, \dots, k\}$. Since $\mathbb{Q}_{>0}\mathbf{v}_1 + \dots + \mathbb{Q}_{>0}\mathbf{v}_k + \mathbb{Q}_{>0}\mathbf{x}_s \subseteq \mathbf{X}_s$ we deduce that $\mathbf{K}' \in \mathcal{K}'$. Therefore $\mathcal{K}'$ has the overlapping property.

Note that if $\mathbf{X}$ is empty then $\mathcal{K}'$ has a complete extraction. So, we can assume that $\mathbf{X}$ is non empty. We fix $\mathbf{x} \in \mathbf{X}$. Lemma D.5 shows that $\mathbf{V}_\mathbf{h} \cap \mathcal{K}'$ has the overlapping property for every $\mathbf{h} \in \mathbf{H}$. By induction we deduce that this class has the complete extraction property. We denote by $\mathcal{C}_\mathbf{h}$ a complete extraction of $\mathbf{V}_\mathbf{h} \cap \mathcal{K}'$ and we consider the following class $\mathcal{C}$:

$$\mathcal{C} = \{\mathbf{C} + \mathbb{Q}_{\geq 0}\mathbf{x} \mid \mathbf{C} \in \bigcup_{\mathbf{h} \in \mathbf{H}} \mathcal{C}_\mathbf{h}\}$$

Let us first prove that $\mathcal{C}$ is an extraction of $\mathcal{K}'$. Let $\mathbf{h} \in \mathbf{H}$ and $\mathbf{C} \in \mathcal{C}_\mathbf{h}$. Since $\mathcal{C}_\mathbf{h}$ is an extraction of $\mathbf{V}_\mathbf{h} \cap \mathcal{K}_s$ we deduce that there exists $\mathbf{K} \in \mathcal{K}'$ such that $\mathbf{C} \subseteq \mathbf{V}_\mathbf{h} \cap \mathbf{K}$. Let $\lambda \in \mathbb{Q}_{>0}$ and observe that $\mathbf{C} + \lambda\mathbf{x} \subseteq \mathbf{X} \subseteq \mathbf{K}$. Hence $\mathbf{C} + \mathbb{Q}_{\geq 0}\mathbf{x} \subseteq \mathbf{K}$. We have proved that $\mathcal{C}$ is an extraction of $\mathcal{K}'$.

Let us prove that the completeness of the extraction $\mathcal{C}$ of $\mathcal{K}'$. We consider $\mathbf{y} \in \Sigma(\mathcal{K}')$. Since $\mathbf{x} \in \mathbf{X}$ we deduce that $\mathbf{h} \cdot \mathbf{x} > 0$ and since $\mathbf{y} \in \mathbf{C}$ we get $\mathbf{h} \cdot \mathbf{y} \geq 0$. Let us introduce $\lambda = \min_{\mathbf{h} \in \mathbf{H}} \frac{\mathbf{h} \cdot \mathbf{y}}{\mathbf{h} \cdot \mathbf{x}}$ and observe that $\mathbf{c} = \mathbf{y} - \lambda \mathbf{x}_s$ satisfies $\mathbf{h} \cdot \mathbf{c} \geq 0$ for every $\mathbf{h} \in \mathbf{H}$. Hence $\mathbf{c} \in \mathbf{C}$. In particular $\mathbb{Q}_{>0}\mathbf{c} + \mathbb{Q}_{>0}\mathbf{x} \subseteq \mathbf{X}$. Let $\mathbf{K} \in \mathcal{K}'$. Since $\mathbf{X} \subseteq \mathbf{K}$, we get $\mathbb{Q}_{>0}\mathbf{c} + \mathbb{Q}_{>0}\mathbf{x} \subseteq \mathbf{K}$. As $\mathcal{K}'$ has the overlapping property we deduce that there exists $\mathbf{K}' \in \mathcal{K}$ such that $\mathbf{K}' \cap (\mathbb{Q}_{>0}\mathbf{c})$ is non empty. Hence there exists $\mu \in \mathbb{Q}_{>0}$ such that $\mu \mathbf{c} \in \mathbf{K}'$. Since $\mathbf{K}'$ is a conic set we deduce that $\frac{1}{\mu}(\mu \mathbf{c}) \in \mathbf{K}'$. Therefore $\mathbf{c} \in \Sigma(\mathcal{K}')$. Moreover by definition of λ we deduce that there exists $\mathbf{h} \in \mathbf{H}$ such that $\mathbf{c} \in \mathbf{V_h}$. We deduce that $\mathbf{c} \in \Sigma(\mathbf{V_h} \cap \mathcal{K}_s)$. Therefore, there exists $\mathbf{C} \in \mathcal{C}_\mathbf{h}$ such that $\mathbf{c} \in \mathbf{C}$. We have proved that $\mathbf{y} \in \Sigma(\mathcal{C})$. Therefore $\mathcal{C}$ is a complete extraction of $\mathcal{K}'$.

The induction is proved. ■

We have proved Theorem D.2.

Appendix E
Proof of Theorem XII.1

In this section, we prove the following theorem.

Theorem XII.1. *Let $\mathbf{X} = \bigcup_{j=1}^{k} \mathbf{b}_j + \mathbf{P}_j$ where $\mathbf{b}_j \in \mathbb{Z}^d$ and $\mathbf{P}_j \subseteq \mathbb{Z}^d$ is a smooth periodic set. We assume that $\mathbf{X}$ is non empty and we introduce $r = \dim(\mathbf{X})$. If $\mathbf{X}$ is equivalent for $\equiv_r$ to a Presburger set then there exists a sequence $(\mathbf{Y}_j)_{1 \leq j \leq k}$ of linear sets $\mathbf{Y}_j \subseteq \mathbf{b}_j + \mathbf{P}_j$ such that $\mathbf{X} \equiv_r \bigcup_{j=1}^{k} \mathbf{p}_j + \mathbf{Y}_j$ for every sequence $(\mathbf{p}_j)_{1 \leq j \leq k}$ of vectors $\mathbf{p}_j \in \mathbf{P}_j$.*

We first prove the following three lemmas.

Lemma E.1. *For every periodic set $\mathbf{P} \subseteq \mathbb{Q}^d$ and for every vector $\mathbf{v} \in \mathbb{Q}^d$, we have $\mathbf{v} \in (\mathbf{P} - \mathbf{P}) \cap \lim(\mathbf{P})$ if, and only if there exists $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{p} + \mathbb{N}\mathbf{v} \subseteq \mathbf{P}$.*

Proof: Let $\mathbf{v} \in \mathbb{Q}^d$ and assume first that $\mathbf{p} + \mathbb{N}\mathbf{v} \subseteq \mathbf{P}$ for some $\mathbf{p} \in \mathbf{P}$. In this case $\mathbf{v} \in \lim(\mathbf{P})$ and from $\mathbf{v} = (\mathbf{p} + \mathbf{v}) - \mathbf{p}$ we deduce that $\mathbf{v} \in \mathbf{P} - \mathbf{P}$. Thus $\mathbf{v} \in (\mathbf{P} - \mathbf{P}) \cap \lim(\mathbf{P})$. Conversely, let us consider $\mathbf{v} \in (\mathbf{P} - \mathbf{P}) \cap \lim(\mathbf{P})$. There exists $\mathbf{p}_+, \mathbf{p}_-$ such that $\mathbf{v} = \mathbf{p}_+ - \mathbf{p}_-$. Moreover there exists $\mathbf{q} \in \mathbf{P}$ and $n \in \mathbb{N}_{>0}$ such that $\mathbf{q} + n\mathbb{N}\mathbf{v} \subseteq \mathbf{P}$. Let us consider $\mathbf{p} = \mathbf{q} + n\mathbf{v} + (n-1)\mathbf{p}_-$ and let us prove that $\mathbf{p} + \mathbb{N}\mathbf{v} \subseteq \mathbf{P}$. Let us consider $k \in \mathbb{N}$. The Euclidean divisor of k by n shows that there exists $q \in \mathbb{N}$ and $r \in \{0, \dots, n-1\}$ such that $k = qn + r$. Note that $r\mathbf{p}_- + r\mathbf{v} = r\mathbf{p}_+$. Thus $(n-1)\mathbf{p}_- + r\mathbf{v} = r\mathbf{p}_+ + (n-1-r)\mathbf{p}_- \in \mathbf{P}$. We deduce that $\mathbf{p} + k\mathbf{v} = (\mathbf{q} + n(q+1)\mathbf{v}) + ((n-1)\mathbf{p}_- + r\mathbf{v}) \in \mathbf{P}$. We have proved that $\mathbf{p} + \mathbb{N}\mathbf{v} \subseteq \mathbf{P}$. In particular $\mathbf{p} \in \mathbf{P}$. ■

Lemma E.2. *Let $\mathbf{P}$ be a periodic set included in a Presburger set $\mathbf{S} \subseteq \mathbb{Z}^d$. We have:*

$$\dim((\mathbf{P} - \mathbf{P}) \cap \lim(\mathbf{P}) \backslash \mathbf{S}) < \dim(\mathbf{S})$$

Proof: Let $\mathbf{V}$ be the vector space generated by $\mathbf{P}$. Lemma X.2 shows that $\dim(\mathbf{P}) = \operatorname{rank}(\mathbf{V})$. By replacing $\mathbf{S}$ by $\mathbf{S} \cap \mathbf{V}$ we can assume without loss of generality that $\mathbf{S} \subseteq \mathbf{V}$. Since the Presburger arithmetic admits a quantifier elimination algorithm, a quantifier free formula in disjunctive normal form shows that $\mathbf{S}$ can be decomposed into a finite union $\bigcup_{j=1}^{k} (\mathbf{R}_j \cap \mathbf{X}_j)$ where $\mathbf{R}_j$ is the set of vectors $\mathbf{z} \in \mathbb{Z}^d \cap \mathbf{V}$ satisfying a conjunction of formulas of the form $\mathbf{h} \cdot \mathbf{z} \in c + m\mathbb{Z}$ with $\mathbf{h} \in \mathbb{Z}^d$, $c \in \mathbb{Z}$ and $m \in \mathbb{N}_{>0}$, and where $\mathbf{X}_j$ is a subset of $\mathbf{V}$ such that there exists a finite set $A_j \subseteq \mathbb{Q}^d \times \{>, \geq\} \times \mathbb{Q}$ such that:

$$\mathbf{X}_j = \{\mathbf{v} \in \mathbf{V} \mid \bigwedge_{(\mathbf{h}, \#, c) \in A_j} \mathbf{h} \cdot \mathbf{v} \# c\}$$

We can assume that for every $(\mathbf{h}, \#, c) \in A_j$ there exists a vector $\mathbf{v} \in \mathbf{V}$ such that $\mathbf{h} \cdot \mathbf{v} \neq 0$ since otherwise the constraint $\mathbf{h} \cdot \mathbf{v} \# c$ reduces to $0 \# c$. Without loss of generality we can also assume that $\mathbf{R}_j$ is non empty. Let $\mathbf{r}_j \in \mathbf{R}_j$ and observe that $\mathbf{L}_j = \mathbf{R}_j - \mathbf{r}_j$ is a lattice that generates $\mathbf{V}$ since for every $\mathbf{v} \in \mathbf{V}$ there exists $m \in \mathbb{N}_{>0}$ such that $n\mathbf{v} \in \mathbf{L}_j$. We introduce the lattice $\mathbf{L} = \bigcap_{j=1}^{k} \mathbf{L}_j$. By considering a product of the natural numbers $m \in \mathbb{N}_{>0}$ (one for each j), we deduce that for every $\mathbf{v} \in \mathbf{V}$ there exists $m \in \mathbb{N}_{>0}$ such that $m\mathbf{v} \in \mathbf{L}$.

Let $\mathbf{v} \in (\mathbf{P} - \mathbf{P}) \cap \lim(\mathbf{P})$. Lemma E.1 shows that there exists $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{p} + \mathbb{N}\mathbf{v} \subseteq \mathbf{P}$. By replacing $\mathbf{p}$ by a vector in $\mathbb{N}_{>0}\mathbf{p}$ we can assume that $\mathbf{p} \in \mathbf{L}$. Since $\mathbf{v} \in \mathbf{P} - \mathbf{P} \subseteq \mathbf{V}$, we deduce that there exists $m \in \mathbb{N}_{>0}$ such that $m\mathbf{v} \in \mathbf{L}$. Since $\mathbf{p} + \mathbf{v} + m\mathbb{N}\mathbf{v} \subseteq \mathbf{P} \subseteq \mathbf{S}$, there exists $j \in \{1, \dots, k\}$ and an infinite subset $N \subseteq \mathbb{N}$ such that $\mathbf{p} + \mathbf{v} + mN\mathbf{v} \subseteq \mathbf{R}_j \cap \mathbf{X}_j$. Let $n \in N$ and observe that $\mathbf{p} + \mathbf{v} + mn\mathbf{v} \in \mathbf{r}_j + \mathbf{L}_j$ and from $\mathbf{p}, nm\mathbf{v} \in \mathbf{L} \subseteq \mathbf{L}_j$ we get $\mathbf{v} \in \mathbf{r}_j + \mathbf{L}_j = \mathbf{R}_j$. Moreover, since N is infinite and $\mathbf{p} + \mathbf{v} + mN\mathbf{v} \subseteq \mathbf{X}_j$ we deduce that $\mathbf{v}$ is in the following set $\tilde{\mathbf{X}}_j$:

$$\tilde{\mathbf{X}}_j = \{\mathbf{v} \in \mathbf{V} \mid \bigwedge_{(\mathbf{h}, \#, c) \in A_j} \mathbf{h} \cdot \mathbf{v} \geq 0\}$$

We have proved that $(\mathbf{P} - \mathbf{P}) \cap \lim(\mathbf{P}) \subseteq \tilde{\mathbf{S}}$ where $\tilde{\mathbf{S}} = \bigcup_{j=1}^{k} (\mathbf{R}_j \cap \tilde{\mathbf{X}}_j)$. Thus $\dim((\mathbf{P} - \mathbf{P}) \cap \lim(\mathbf{P}) \backslash \mathbf{S}) \leq \dim(\tilde{\mathbf{S}} \backslash \mathbf{S})$. From Lemma XI.1 since $\equiv_r$ is distributive over $\cup$ and $\cap$, we get $\mathbf{S} \equiv_r \tilde{\mathbf{S}}$. Thus $\dim((\mathbf{P} - \mathbf{P}) \cap \lim(\mathbf{P}) \backslash \mathbf{S}) < r$. ■

Lemma E.3. *Let $(\mathbf{m}_r)_{1 \leq r \leq n}$ be a sequence of vectors $\mathbf{m}_r \in \mathbb{Z}^d$ and let $\mathbf{M}_r = \mathbb{N}\mathbf{m}_1 + \dots + \mathbb{N}\mathbf{m}_r$ for every $r \in \{1, \dots, n\}$. If $\mathbf{M}_n$ is included in $\bigcup_{j=1}^{k} \mathbf{b}_j + \mathbf{P}_j$ where $\mathbf{b}_j \in \mathbb{Z}^d$ and $\mathbf{P}_j \subseteq \mathbb{Z}^d$ is a well-limit periodic set then there exists $j \in \{1, \dots, k\}$ such that $\mathbf{M}_n \cap (\mathbf{b}_j + \mathbf{P}_j)$ is non empty and such that $(\mathbf{m}_r + \mathbf{M}_r) \cap \lim(\mathbf{P}_j)$ is non empty for every $r \in \{1, \dots, n\}$.*

Proof: We prove the lemma by induction over $n \in \mathbb{N}$. The rank $n = 0$ is immediate. Assume the rank $n \in \mathbb{N}$ proved and let us consider a sequence $(\mathbf{m}_r)_{1 \leq r \leq n+1}$ of vectors $\mathbf{m}_r \in \mathbb{Z}^d$ and let $\mathbf{M}_r = \mathbb{N}\mathbf{m}_1 + \dots + \mathbb{N}\mathbf{m}_r$ for every $r \in \{1, \dots, n+1\}$. Assume $\mathbf{M}_{n+1}$ is included in $\bigcup_{j=1}^{k} \mathbf{b}_j + \mathbf{P}_j$ where $\mathbf{b}_j \in \mathbb{Z}^d$ and $\mathbf{P}_j \subseteq \mathbb{Z}^d$ is a well-limit periodic set. Let $t \in \mathbb{N}$ and observe that $\mathbf{M}_n \subseteq \bigcup_{j=1}^{k} \mathbf{b}_j - t\mathbf{m}_{n+1} + \mathbf{P}_j$. By induction there exists $j \in \{1, \dots, k\}$ such that $\mathbf{M}_n \cap (\mathbf{b}_j - t\mathbf{m}_{n+1} + \mathbf{P}_j)$ is non empty and such that $(\mathbf{m}_r + \mathbf{M}_r) \cap \lim(\mathbf{P}_j)$ is non empty for every $r \in \{1, \dots, n\}$. We deduce that there exists $j \in \{1, \dots, k\}$ and an infinite subset $T \subseteq \mathbb{N}$ such that $(\mathbf{m}_r + \mathbf{M}_r) \cap \lim(\mathbf{P}_j)$ is non empty for every $r \in \{1, \dots, n\}$ and such that $\mathbf{M}_n \cap (\mathbf{b}_j - t\mathbf{m}_{n+1} + \mathbf{P}_j)$ is non empty for every $t \in T$. Since $\mathbf{M}_n + t\mathbf{m}_{n+1} \subseteq \mathbf{M}_{n+1}$ we deduce that $\mathbf{M}_{n+1} \cap (\mathbf{b}_j + \mathbf{P}_j)$ is non empty. For every $t \in T$ there exists $\mathbf{k}_t \in \mathbf{M}_n$ such that $\mathbf{v}_t = \mathbf{k}_t - \mathbf{b}_j + t\mathbf{m}_{n+1}$ in in $\mathbf{P}_j$. As $\mathbf{M}_n$ is finitely generated and $\mathbf{P}_j$ is a well-limit periodic set, we deduce that there exists $t < t'$ such that $\mathbf{k}_{t'} - \mathbf{k}_t \in \mathbf{M}_n$ and $\mathbf{v}_{t'} - \mathbf{v}_t \in \lim(\mathbf{P}_j)$. Observe that this last vector is equal to $\mathbf{k}_{t'} - \mathbf{k}_t + (t' - t)\mathbf{m}_{n+1}$ which is in $\mathbf{m}_{n+1} + \mathbf{M}_{n+1}$. So we have proved the rank $n+1$. Therefore, the lemma is proved by induction. ■

Now, let us prove Theorem XII.1. We consider a non-empty set $\mathbf{X} = \bigcup_{j=1}^{k} \mathbf{b}_j + \mathbf{P}_j$ where $\mathbf{b}_j \in \mathbb{Z}^d$ and $\mathbf{P}_j \subseteq \mathbb{Z}^d$ is a smooth periodic set. We introduce the definable conic set $\mathbf{K}_j = \lim(\mathbf{P}_j)$. We denote by r the dimension of $\mathbf{X}$. Note that $k > 0$ and $r \in \{1, \dots, d\}$ since $\mathbf{X}$ is non empty. We introduce the lattices $\mathbf{L}_j = \mathbf{P}_j - \mathbf{P}_j$. We denote by $\mathbf{V}_j$ the vector space generated by $\mathbf{P}_j$. We introduce the set $J = \{j \in \{1, \dots, k\} \mid$

$\mathrm{rank}(\mathbf{V}_j) = r\}$, the class $\mathcal{V} = \{\mathbf{V}_j \mid j \in J\}$. For every vector space $\mathbf{V} \in \mathcal{V}$ we introduce the set $J_{\mathbf{V}} = \{j \in J \mid \mathbf{V} = \mathbf{V}_j\}$, the lattice $\mathbf{L}_{\mathbf{V}} = \bigcap_{j \in J_{\mathbf{V}}} \mathbf{L}_j$. For every $\mathbf{V} \in \mathcal{V}$ and for every $\mathbf{z} \in \mathbb{Z}^d$ we introduce the set $J_{\mathbf{V},\mathbf{z}} = \{j \in J_{\mathbf{V}} \mid \mathbf{z} \in \mathbf{b}_j + \mathbf{L}_j\}$ and the finite class $\mathcal{K}_{\mathbf{V},\mathbf{z}} = \{\mathbf{K}_j \mid j \in J_{\mathbf{V},\mathbf{z}}\}$.

Lemma E.4. *For every $\mathbf{V} \in \mathcal{V}$ and for every $\mathbf{z} \in \mathbb{Z}^d$, we have:*

$$\mathbf{L}_{\mathbf{V}} \cap (\mathbf{X} - \mathbf{z}) \equiv_r \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + \mathbf{P}_j$$

Proof: Let us consider $j \in \{1, \dots, k\}$ such that the dimension of the intersection $\mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{P}_j)$ is greater or equal to r and let us prove that $j \in J_{\mathbf{V},\mathbf{z}}$. In that case, this intersection is non empty and thus it contains a vector $\mathbf{x}$. We deduce that the intersection is included in $\mathbf{x} + (\mathbf{V} \cap \mathbf{V}_j)$. Hence $\mathrm{rank}(\mathbf{V} \cap \mathbf{V}_j) \geq r$. From $\mathbf{V} \cap \mathbf{V}_j \subseteq \mathbf{V}$ and $\mathrm{rank}(\mathbf{V}) = r$ we get $\mathbf{V} \cap \mathbf{V}_j = \mathbf{V}$. Hence $\mathbf{V} \subseteq \mathbf{V}_j$. As $\mathrm{rank}(\mathbf{V}) = r$ and $\mathrm{rank}(\mathbf{V}_j) \leq r$ we deduce that $\mathbf{V} = \mathbf{V}_j$. Thus $j \in J_{\mathbf{V}}$. Moreover, since $\mathbf{x} \in \mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{P}_j)$ we deduce that $\mathbf{b}_j - \mathbf{z} \in \mathbf{x} + \mathbf{L}_j \subseteq \mathbf{L}_j$ as $\mathbf{x} \in \mathbf{L}_{\mathbf{V}} \subseteq \mathbf{L}_j$. Thus $j \in J_{\mathbf{V},\mathbf{z}}$. We deduce the relations:

$$\begin{aligned}
\mathbf{L}_{\mathbf{V}} \cap (\mathbf{X} - \mathbf{z}) &= \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j=1}^{k} (\mathbf{b}_j - \mathbf{z} + \mathbf{P}_j) \\
&= \bigcup_{j=1}^{k} \mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{P}_j) \\
&\equiv_r \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{P}_j) \\
&= \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} (\mathbf{b}_j - \mathbf{z} + \mathbf{P}_j)
\end{aligned}$$

We have proved the lemma. ■

Lemma E.5. *If there exists a Presburger set $\mathbf{S} \subseteq \mathbb{Z}^d$ such that $\mathbf{X} \equiv_r \mathbf{S}$ then for every $\mathbf{V} \in \mathcal{V}$ and for every $\mathbf{z} \in \mathbb{Z}^d$ we have:*

$$\mathbf{L}_{\mathbf{V}} \cap \Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}}) \equiv_r \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + \mathbf{P}_j$$

Proof: Assume that there exists a Presburger set $\mathbf{S} \subseteq \mathbb{Z}^d$ such that $\mathbf{X} \equiv_r \mathbf{S}$. Lemma E.4 shows the following relation:

$$\mathbf{L}_{\mathbf{V}} \cap (\mathbf{S} - \mathbf{z}) \equiv_r \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + \mathbf{P}_j$$

Hence, there exists a Presburger set $\mathbf{D} \subseteq \mathbb{Z}^d$ such that $\dim(\mathbf{D}) < r$ and such that the Presburger set $\mathbf{R} = \mathbf{L}_{\mathbf{V}} \cap (\mathbf{S} - \mathbf{z})$ satisfies $\mathbf{R} \backslash \mathbf{D} \subseteq \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + \mathbf{P}_j \subseteq \mathbf{D} \cup \mathbf{R}$. Lemma E.2 shows that for every $j \in J_{\mathbf{V},\mathbf{z}}$ there exists a Presburger set $\mathbf{D}_j \subseteq \mathbb{Z}^d$ such that $\dim(\mathbf{D}_j) < \dim(\mathbf{P}_j)$ and such that $\mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j) \subseteq \mathbf{D}_j \cup \mathbf{D} \cup \mathbf{R}$. Therefore $\mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j)$ is included in the union of $\mathbf{Z} = \mathbf{D} \cup \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{D}_j$ and $\mathbf{R}$. We get the following inclusions:

$$\mathbf{R} \backslash \mathbf{Z} \subseteq \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j) \subseteq \mathbf{R} \cup \mathbf{Z}$$

Since $\dim(\mathbf{Z}) < r$ we deduce the following relation:

$$\mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j) \equiv_r \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + \mathbf{P}_j$$

Finally observe that for every $j \in J_{\mathbf{V},\mathbf{z}}$, since $\mathbf{b}_j - \mathbf{z} \in \mathbf{L}_j$, we have $\mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j) = \mathbf{L}_j \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{K}_j)$. Corollary XI.2 shows that $\mathbf{L}_j \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{K}_j) \equiv_r \mathbf{L}_j \cap \mathbf{K}_j$. ■

Lemma E.6. *Let $\mathbf{V} \in \mathcal{V}$ and $\mathbf{z} \in \mathbb{Z}^d$ such that:*

$$\mathbf{L}_{\mathbf{V}} \cap \Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}}) \equiv_r \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + \mathbf{P}_j$$

Then the class $\Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}})$ has the overlapping property.

Proof: There exists a Presburger set $\mathbf{D} \subseteq \mathbb{Z}^d$ such that $\dim(\mathbf{D}) < r$ and such that the Presburger set $\mathbf{S} = \mathbf{L}_{\mathbf{V}} \cap \Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}})$ and the set $\mathbf{R} = \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + \mathbf{P}_j$ satisfy $\mathbf{R} \backslash \mathbf{D} \subseteq \mathbf{S} \subseteq \mathbf{R} \cup \mathbf{D}$.

Let us consider $j_0 \in J_{\mathbf{V},\mathbf{z}}$ and a sequence $\mathbf{v}_1, \dots, \mathbf{v}_n$ of vectors $\mathbf{v}_n \in \mathbb{Q}^d$ such that $\mathbb{Q}_{>0}\mathbf{v}_1 + \dots + \mathbb{Q}_{>0}\mathbf{v}_n \subseteq \mathbf{K}_{j_0}$ and let us prove that there exists $\mathbf{K} \in \mathcal{K}_{\mathbf{V},\mathbf{z}}$ such that $\mathbf{K} \cap (\mathbb{Q}_{>0}\mathbf{v}_1 + \dots + \mathbb{Q}_{>0}\mathbf{v}_r) \neq \emptyset$ for every $r \in \{1, \dots, n\}$. By extending the sequence we can assume that $\mathbf{v}_1, \dots, \mathbf{v}_n$ generates $\mathbf{V}$. Moreover, by replacing vectors $\mathbf{v}_r$ by vectors in $\mathbb{N}_{>0}\mathbf{v}_r$ we can assume without loss of generality that $\mathbf{v}_r \in \mathbf{L}_{\mathbf{V}}$. Therefore $\mathbf{v} = \mathbf{v}_1 + \dots + \mathbf{v}_n$ satisfies $\mathbf{v} + \mathbb{N}\mathbf{v}_1 + \dots + \mathbb{N}\mathbf{v}_n \subseteq \mathbf{L}_{\mathbf{V}} \cap \mathbf{K}_{j_0} \subseteq \mathbf{S} \subseteq \mathbf{R} \cup \mathbf{D}$. By decomposing $\mathbf{D}$ into linear sets, since $\mathbf{v}_1, \dots, \mathbf{v}_n$ generates $\mathbf{V}$, Lemma E.3 shows that there exists $j \in J_{\mathbf{V},\mathbf{z}}$ such that $(\mathbf{v} + \mathbb{N}\mathbf{v}_1 + \dots + \mathbb{N}\mathbf{v}_n) \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{P}_j) \neq \emptyset$ and such that $(\mathbb{N}_{>0}\mathbf{v}_1 + \dots + \mathbb{N}_{>0}\mathbf{v}_r) \cap \mathbf{K}_j \neq \emptyset$ for every $r \in \{1, \dots, n\}$. We have proved that $\mathcal{K}_{\mathbf{V},\mathbf{z}}$ has the overlapping property. ■

Lemma E.7. *For every $\mathbf{V} \in \mathcal{V}$ and for every $\mathbf{z} \in \mathbb{Z}^d$ we have:*

$$\mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j) \equiv_r \mathbf{L}_{\mathbf{V}} \cap \Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}})$$

Proof: We observe that for every $j \in J_{\mathbf{V}}$, the intersection $\mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{L}_j)$ is equal to $\mathbf{L}_{\mathbf{V}}$. We deduce the following equality:

$$\begin{aligned}
&\mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j) \\
&= \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j))
\end{aligned}$$

Since $\mathbf{b}_j - \mathbf{z} \in \mathbf{L}_j$ we get $\mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j) = \mathbf{L}_j \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{K}_j)$. Corollary XI.1 shows that $\mathbf{L}_j \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{K}_j) \equiv_r \mathbf{L}_j \cap \mathbf{K}_j$. Hence $\mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j)) \equiv_r \mathbf{L}_{\mathbf{V}} \cap \mathbf{K}_j$. We have proved:

$$\begin{aligned}
&\bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j)) \\
&\equiv_r \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{L}_{\mathbf{V}} \cap \mathbf{K}_j = \mathbf{L}_{\mathbf{V}} \cap \Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}})
\end{aligned}$$

We deduce the lemma. ■

Lemma E.8. *If $\mathcal{K}_{\mathbf{V},\mathbf{z}}$ has a complete extraction for every $\mathbf{V} \in \mathcal{V}$ and for every $\mathbf{z} \in \mathbb{Z}^d$ then there exists a finite sequence $(\mathbf{C}_j)_{1 \leq j \leq k}$ of finitely generated conic sets $\mathbf{C}_j \subseteq \mathbf{K}_j$ such that $\bigcup_{j=1}^k \mathbf{b}_j + \mathbf{L}_j \cap \mathbf{K}_j \equiv \bigcup_{j=1}^k \mathbf{b}_j + \mathbf{L}_j \cap \mathbf{C}_j$.*

Proof: Let $\mathbf{V} \in \mathcal{V}$ and $\mathbf{z} \in \mathbb{Z}^d$. From Lemma E.7 we deduce the following relation:

$$\mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j) \equiv_r \mathbf{L}_{\mathbf{V}} \cap \Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}})$$

Since $\mathcal{K}_{\mathbf{V},\mathbf{z}}$ has a complete extraction, there exists a sequence $(\mathbf{C}_{\mathbf{V},\mathbf{z},j})_{j \in J_{\mathbf{V},\mathbf{z}}}$ of finitely generated conic sets $\mathbf{C}_{\mathbf{V},\mathbf{z},j} \subseteq \mathbf{K}_j$ such that $\bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{C}_{\mathbf{V},\mathbf{z},j} = \Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}})$. Since for every $j \in J_{\mathbf{V},\mathbf{z}}$ we have $\mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{L}_j) = \mathbf{L}_{\mathbf{V}}$, we deduce:

$$\begin{aligned}\mathbf{L}_{\mathbf{V}} \cap \Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}}) &= \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{L}_{\mathbf{V}} \cap \mathbf{C}_{\mathbf{V},\mathbf{z},j} \\ &= \mathbf{L}_{\mathbf{V}} \cap (\bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} (\mathbf{b}_j - \mathbf{z} + \mathbf{L}_j) \cap \mathbf{C}_{\mathbf{V},\mathbf{z},j})\end{aligned}$$

Lemma XI.1 shows that $(\mathbf{b}_j - \mathbf{z} + \mathbf{L}_j) \cap \mathbf{C}_{\mathbf{V},\mathbf{z},j} \equiv_r \mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{C}_{\mathbf{V},\mathbf{z},j})$ for every $j \in J_{\mathbf{V},\mathbf{z}}$. We have proved:

$$\mathbf{L}_{\mathbf{V}} \cap \Sigma(\mathcal{K}_{\mathbf{V},\mathbf{z}}) \equiv_r \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{C}_{\mathbf{V},\mathbf{z},j})$$

Let us introduce a finite set $\mathbf{Z}_{\mathbf{V}} \subseteq \mathbb{Z}^d$ such that $\sum_{j \in J_{\mathbf{V}}} \mathbf{b}_j + \mathbf{L}_j = \mathbf{Z}_{\mathbf{V}} + \mathbf{L}_{\mathbf{V}}$. We consider the sequence $(\mathbf{C}_j)_{1 \leq j \leq k}$ of finitely generated conic sets defined by $\mathbf{C}_j = \{\mathbf{0}\}$ if $j \notin J$ and defined for every $j \in J$ by:

$$\mathbf{C}_j = \sum_{\mathbf{z} \in \mathbf{Z}_{\mathbf{V}_j}} \mathbf{C}_{\mathbf{V}_j,\mathbf{z},j}$$

Observe that $\mathbf{C}_j \subseteq \mathbf{K}_j$ for every $j \in \{1, \dots, k\}$. In particular $\mathbf{L}_j \cap \mathbf{C}_j \subseteq \mathbf{L}_j \cap \mathbf{K}_j$. We consider the sequence $(\mathbf{M}_j)_{1 \leq j \leq k}$ of sets $\mathbf{M}_j = \mathbf{L}_j \cap \mathbf{C}_j$. Since $\mathbf{L}_{\mathbf{V}} \cap (\mathbf{b}_j - \mathbf{z} + \mathbf{M}_j)$ is empty for every $j \in J_{\mathbf{V}} \backslash J_{\mathbf{V},\mathbf{z}}$ we deduce:

$$\begin{aligned}&\mathbf{L}_{\mathbf{V}} \cap (\bigcup_{j \in J_{\mathbf{V}}} \mathbf{b}_j - \mathbf{z} + (\mathbf{L}_j \cap \mathbf{K}_j)) \\ &\equiv_r \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V},\mathbf{z}}} \mathbf{b}_j - \mathbf{z} + \mathbf{M}_j \\ &= \mathbf{L}_{\mathbf{V}} \cap \bigcup_{j \in J_{\mathbf{V}}} \mathbf{b}_j - \mathbf{z} + \mathbf{M}_j\end{aligned}$$

Since $\sum_{j \in J_{\mathbf{V}}} \mathbf{b}_j + \mathbf{L}_j = \mathbf{Z}_{\mathbf{V}} + \mathbf{L}_{\mathbf{V}}$, we deduce the relation $\bigcup_{j \in J_{\mathbf{V}}} \mathbf{b}_j + (\mathbf{L}_j \cap \mathbf{K}_j) \equiv_r \bigcup_{j \in J_{\mathbf{V}}} \mathbf{b}_j + \mathbf{M}_j$. Therefore $\bigcup_{j \in J} \mathbf{b}_j + (\mathbf{L}_j \cap \mathbf{K}_j) \equiv_r \bigcup_{j \in J} \mathbf{b}_j + \mathbf{M}_j$. Since $\dim(\mathbf{b}_j + (\mathbf{L}_j \cap \mathbf{K}_j)) < r$ and $\dim(\mathbf{b}_j + \mathbf{M}_j) < r$ for every $j \in \{1, \dots, k\} \backslash J$ we deduce that $\mathbf{b}_j + (\mathbf{L}_j \cap \mathbf{K}_j) \equiv_r \emptyset$ and $\mathbf{b}_j + \mathbf{M}_j \equiv_r \emptyset$. We have proved the following relation:

$$\bigcup_{j=1}^k \mathbf{b}_j + (\mathbf{L}_j \cap \mathbf{K}_j) \equiv_r \bigcup_{j=1}^k \mathbf{b}_j + \mathbf{M}_j$$

In the previous relation, the relation $\equiv_r$ can be replaced by $\equiv$ since the set $\mathbf{Y} = \bigcup_{j=1}^k \mathbf{b}_j + (\mathbf{L}_j \cap \mathbf{K}_j)$ satisfies $\dim(\mathbf{Y}) = r$. ■

Now let us prove Theorem XII.1. Assume that $\mathbf{X}$ is equivalent for $\equiv_r$ to a Presburger set. We deduce that there exists a finite sequence $(\mathbf{C}_j)_{1 \leq j \leq k}$ of finitely generated conic sets $\mathbf{C}_j \subseteq \mathbf{K}_j$ such that $\mathbf{X} \equiv_r \bigcup_{j=1}^k \mathbf{b}_j + (\mathbf{L}_j \cap \mathbf{C}_j)$.

Let us introduce the periodic set $\mathbf{Q}_j = \mathbf{L}_j \cap \mathbf{C}_j$. Since the conic set generated by $\mathbf{Q}_j$ is $\mathbf{C}_j$ which is finitely generated, Lemma V.4 shows that $\mathbf{Q}_j$ is finitely generated. Lemma E.1 shows that for every $\mathbf{v} \in \mathbf{Q}_j$ there exists $\mathbf{p} \in \mathbf{P}_j$ such that $\mathbf{p} + \mathbb{N}\mathbf{v} \subseteq \mathbf{P}_j$. Since $\mathbf{Q}_j$ is finitely generated, there exists $\mathbf{y}_j \in \mathbf{b}_j + \mathbf{P}_j$ such that the linear set $\mathbf{Y}_j = \mathbf{y}_j + \mathbf{Q}_j$ is included in $\mathbf{b}_j + \mathbf{P}_j$.

Now, let us consider a sequence $(\mathbf{p}_j)_{1 \leq j \leq k}$ of vectors $\mathbf{p}_j \in \mathbf{P}_j$. Note that $\mathbf{p}_j + \mathbf{Y}_j = \mathbf{y}_j + (\mathbf{L}_j \cap (\mathbf{p}_j + \mathbf{C}_j))$. Note that the vector space $\mathbf{W}_j$ generated by $\mathbf{C}_j$ is included in $\mathbf{V}_j$. If the inclusion is strict then $\mathrm{rank}(\mathbf{W}_j) < r$ and we get $\mathbf{p}_j + \mathbf{Y}_j \equiv_r \emptyset \equiv_r \mathbf{b}_j + \mathbf{Q}_j$. Otherwise, if $\mathbf{W}_j = \mathbf{V}_j$ then Corollary XI.2 shows that $\mathbf{p}_j + (\mathbf{y}_j - \mathbf{b}_j) + \mathbf{C}_j \equiv_r \mathbf{C}_j$. Thus $\mathbf{p}_j + \mathbf{Y}_j \equiv_r \mathbf{b}_j + \mathbf{Q}_j$. We have proved that $\mathbf{X} \equiv_r \bigcup_{j=1}^k \mathbf{p}_j + \mathbf{Y}_j$.