On the Relative Succinctness of Sentential Decision Diagrams

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Abstract

Sentential decision diagrams (SDDs) introduced by Darwiche in 2011 are a promising representation type used in knowledge compilation. The relative succinctness of representation types is an important subject in this area. The aim of the paper is to identify which kind of Boolean functions can be represented by SDDs of small size with respect to the number of variables the functions are defined on. For this reason the sets of Boolean functions representable by different representation types in polynomial size are investigated and SDDs are compared with representation types from the classical knowledge compilation map of Darwiche and Marquis. Ordered binary decision diagrams (OBDDs) which are a popular data structure for Boolean functions are one of these representation types. SDDs are more general than OBDDs by definition but only recently, a Boolean function was presented with polynomial SDD size but exponential OBDD size. This result is strengthened in several ways. The main result is a quasipolynomial simulation of SDDs by equivalent unambiguous nondeterministic OBDDs, a nondeterministic variant where there exists exactly one accepting computation for each satisfying input. As a side effect an open problem about the relative succinctness between SDDs and free binary decision diagrams (FBDDs) which are more general than OBDDs is answered.

 $\label{lem:keywords} \textbf{Keywords} \ \text{complexity theory} \cdot \text{decomposable negation normal forms} \cdot \text{knowledge compilation} \\ \cdot \ \text{ordered binary decision diagrams} \cdot \text{sentential decision diagrams} \cdot \text{storage access functions}$

1 Introduction

Knowledge compilation is an area of research with a long tradition in artificial intelligence (see, e.g., [13]). An input formula is converted into a representation of the Boolean function that the formula defines from which some tasks can (hopefully) be done efficiently. Developing their knowledge compilation map Darwiche and Marquis identified sets of useful queries and transformations in the area of knowledge compilation and compared systematically different representation types w.r.t. their succinctness and efficient support of these operations [17]. One aim of their work was to decide whether representations can be transformed into equivalent ones of another representation type at the cost of increasing the representation size at most polynomially. Here we continue this part of their work. Sentential decision diagrams, or SDDs for short, introduced by Darwiche [16] are a promising representation type for propositional knowledge bases in artificial intelligence. Our main motivation in the paper is to characterize which kind of Boolean functions can be represented by SDDs of small size.

Contribution and related work For a representation type \mathcal{M} let $\mathcal{P}(\mathcal{M})$ be the set of all Boolean functions representable by \mathcal{M} in polynomial size w.r.t. the number of Boolean variables the functions are defined on. We call $\mathcal{P}(\mathcal{M})$ a complexity class. Our aim is to characterize the complexity class $\mathcal{P}(\text{SDD})$ as precisely as possible. For the formal definitions of the following representation types see Section 2.

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If one likes to have representations of small size for Boolean functions, circuits are the most powerful model. The desire to find representation types with better algorithmic properties leads to restricted circuits. Decomposable negation normal form circuits, or DNNFs for short, introduced by Darwiche [15] are the most general one of these representation types discussed in this paper. The subcircuits leading into each \land -gate (conjunction) are defined on disjoint sets of variables. Darwiche also defined deterministic DNNFs, or d-DNNFs for short, where the subcircuits leading into each \lor -gate (disjunction) never simultaneously evaluate to the function value 1. This restriction allows polynomial-time equivalence testing [18].

In his seminal paper Bryant showed that ordered binary decision diagrams, or OBDDs for short, are well suited as data structure for Boolean functions [11]. Since some important functions have exponential OBDD size, many variants and extensions have been considered (for an extensive discussion see, e.g., the monograph of Wegener [32]). Besides nondeterministic variants and conondeterministic variants, free binary decision diagrams (FBDDs) and k-OBDDs, for constant k, have been investigated. FBDDs and k-OBDDs are by definition more general than OBDDs.

SDDs are restricted d-DNNFs more general than OBDDs. Recently, Bova provided a function in $\mathcal{P}(SDD)$ whose OBDD size is exponential [7]. This result is strengthened by our proof that there exist Boolean functions representable by SDDs of polynomial size but with exponential FBDD size (see Section 6). This result answers a question posed by Beame and Liew (see Discussion in [2]) in the affirmative whether SDDs are ever more concise than so-called decision-DNNFs which are also restricted d-DNNFs considered in database theory in the context of probabilistic databases. (See, e.g., [20] for a discussion on the importance of decision DNNFs in model counting, the problem to compute the number of satisfying assignments of a Boolean formula.) There exists a quasipolynomial simulation of decision-DNNFs by equivalent FBDDs [1]. Moreover, Beame and Liew showed that SDDs are sometimes exponentially less concise than FBDDs [2]. Therefore, we can conclude that SDDs and FBDDs are incomparable w.r.t. polynomial-size representations (see also Figure 2). In other words, $\mathcal{P}(SDD)$ is not a subset of $\mathcal{P}(FBDD)$ and vice versa. Furthermore, we prove that SDDs are even more powerful w.r.t. polynomial-size representations than k-OBDDs, where k is a constant (see Section 7). For this result we use a polynomial transformation from k-OBDDs for k into equivalent unambiguous nondeterministic OBDDs. Until now it is open whether the set of Boolean functions representable by polynomial-size unambiguous nondeterministic OBDDs, or \vee_1 -OBDDs for short, that have exactly one accepting path for every satisfying input is a subset of $\mathcal{P}(SDD)$ (see also Figure 1). One of our main results is the proof that every Boolean function f for which f and its negated function \overline{f} can be represented by polynomial-size unambiguous nondeterministic OBDDs w.r.t. the same variable ordering can also be represented by SDDs of polynomial size (see Section 3). This result is sufficient to prove that $\mathcal{P}(k\text{-OBDD}) \subset \mathcal{P}(\text{SDD})$. Adapting a result from Sauerhoff that nondeterministic OBDDs where all nondeterministic decisions are made at the beginning of the computations are less powerful w.r.t. polynomial-size representation than general nondeterministic OBDDs [26], we can strengthen our result to $\mathcal{P}(k\text{-OBDD})\subseteq \mathcal{P}(\text{SDD})$.

Razgon proved a quasipolynomial separation between decision-DNNFs and nondeterministic FBDDs, or ∨-FBDDs for short, [24]. He presented a Boolean function with polynomial decision-DNNF size but only quasipolynomial nondeterministic FBDD size. A careful inspection of his results (Theorem 2 and 3 in [24]) in combination with a result from Darwiche (Theorem 13 in [16]) also leads to a quasipolynomial separation between SDDs and nondeterministic FBDDs. Since FBDDs are more general than OBDDs this is also a quasipolynomial separation between SDDs and nondeterministic OBDDs. Recently, strengthening his result, Razgon presented a quasipolynomial separation between SDDs and a representation typ more general than nondeterministic OBDDs [25]. The second main result of our paper is the proof that SDDs can be simulated with only a quasipolynomial size increase by equivalent unambiguous nondeterministic OBDDs (see Sections 4 and 5). This simulation yields directly lower bounds on the SDD size of Boolean functions f from unambiguous nondeterministic OBDD lower bounds for f. Because of Razgon's quasipolynomial separation [25] our result is tight. For our simulation we extend ideas described independently by Beame and Liew and by Razgon for a quasipolynomial transformation from DNNFs to equivalent nondeterministic FBDDs [2, 23]. We prove that so-called structured DNNFs can be simulated by equivalent nondeterministic OBDDs with only a quasipolynomial increase in representation size. Moreover, if the structured DNNF is deterministic the result is an unambiguous nondeterministic

OBDD. Since SDDs are restricted deterministic structured DNNFs, we are done.

Figure 1 and 2 illustrate the relative succinctness of some of the representation types mentioned above. $\mathcal{P}(OBDD) \subsetneq \mathcal{P}(SDD)$ was shown in [7]. It is known that $\mathcal{P}(SDD) \not\subseteq \mathcal{P}(\vee_1 - OBDD)$ (see [25] and [16, 24]). We prove that the separation between $\mathcal{P}(SDD)$ and $\mathcal{P}(\vee_1 - OBDD)$ is only quasipolynomial. The question whether $\mathcal{P}(\vee_1 - OBDD) \subsetneq \mathcal{P}(SDD)$ is open.

 $\mathcal{P}(SDD) \not\subseteq \mathcal{P}(\lor -FBDD)$ can be proved with results in [16, 24] but the separation is only quasipolynomial. An exponential separation exists between $\mathcal{P}(SDD)$ and $\mathcal{P}(FBDD)$ and vice versa (see Section 6 and [2]).

Remarks SDDs are structured w.r.t. so-called vtrees whose leaves are labeled by Boolean variables and OBDDs respect so-called variable orderings which are lists of variables (see Section 2). Xue, Choi, and Darwiche showed a Boolean function whose SDD size w.r.t. a given vtree T is linear but whose OBDD size w.r.t. a variable ordering that corresponds to a left-right traversal of the leaves in T is exponential (Theorem 1 in [33]). Their result demonstrates that for a space-efficient simulation of SDDs by equivalent unambiguous nondeterministic OBDDs the choice of the variable ordering is not trivial. As a side effect, our quasipolynomial simulation of SDDs by equivalent unambiguous nondeterministic OBDDs presented in Section 4 and in Section 5 generates a variable ordering from a given vtree. For the SDD given in [33] it generates a variable ordering for which the represented function has polynomial OBDD size.

Only recently, Cali, Capelli, and Razgon investigated two restricted variants of decision DNNFs, so-called structured decision DNNFs and so-called decomposable \land -OBDDs which are OBDDs augmented with decomposable \land -nodes [14]. Since our quasipolynomial simulation of SDDs by equivalent unambiguous nondeterministic OBDDs generates a variable ordering from a given vtree, our constructon can be used to show that each structured decision DNNF can be seen as a decomposable \land -OBDD of the same asymptotical size. This answers the question in [14] in the affirmative whether a polynomial transformation from structured decision DNNFs to equivalent decomposable \land -OBDDs exists. Moreover, our simulation shows that every function representable by decomposable \land -OBDDs can be represented by OBDDs with only a quasipolynomial increase in representation size in general (a fact already mentioned in [21] but without proof).

Organization of the paper The rest of the paper is organized as follows. In Section 2 we recall the main definitions concerning binary decision diagrams and decomposable negation normal forms. Moreover, important Boolean functions which are discussed later on in the paper are formally defined. The next sections contain our main results. In Section 3 it is shown that every Boolean function f for which f and its negated function \overline{f} can be represented by polynomial-size unambiguous nondeterministic OBDDs w.r.t. the same variable ordering can also be represented

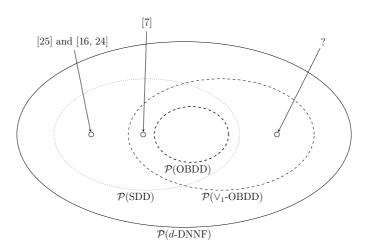


Figure 1: On the relative succinctness of SDDs and (unambiguous nondeterministic) OBDDs.

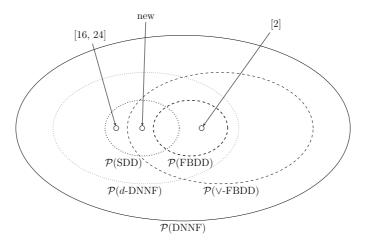


Figure 2: On the relative succinctness of SDDs and FBDDs.

by SDDs of polynomial size. Section 4 and Section 5 are devoted to the new quasipolynomial transformation from structured (deterministic) DNNFs into equivalent (unambiguous) nondeterministic OBDDs. Section 6 uses the results from Section 3 to derive small size SDDs for an important class of Boolean functions called strorage access functions. Moreover, we obtain as a corollary the result that there are functions with polynomial SDD size but exponential FBDD size. The proof that SDDs are more powerful w.r.t. polynomial-size representations than k-OBDDs for constant k, a generalization of OBDDs, is shown in Section 7. This is done by demonstrating that Boolean functions representable by k-OBDDs of polynomial size, where k is a constant, can be represented by equivalent restricted unambiguous nondeterministic OBDDs of polynomial size. Finally, we finish the paper with some open questions. For readability some tedious technical proofs are delegated into the appendix.

2 Preliminaries

In the following we assume familiarity with fundamental concepts on circuits (otherwise see, e.g., [29] and [30] for more details). In this section, we briefly recall the main notions concerning binary decision diagrams and decomposable negation normal forms, discuss the relation between ordered binary decision diagrams and sentential decision diagrams, and introduce some Boolean functions.

2.1 Binary decision diagrams

In complexity theory binary decision diagrams or in this area more often called branching programs are a well established representation type for discrete functions and the binary decision diagram size of a Boolean function is known to be a measure for the space complexity of nonuniform Turing machines and known to lie between the circuit size of the considered function and its $\{\land, \lor, \neg\}$ -formula size (see, e.g., [30, 32]).

Since binary decision diagrams are a nonuniform model of computation, usually sequences of binary decision diagrams $G = (G_n)$ representing sequences of Boolean functions $f = (f_n)$ are considered, where f_n is defined on n variables and $n \in \mathbb{N}$. In the following we simplify the notation for all nonuniform computation models because the meaning is clear from the context. Moreover, in the remaining part of the paper the size of a representation for a Boolean function refers to the number of variables the function is defined on if nothing else is explicitly mentioned.

Definition 1 (BDDs). A binary decision diagram (BDD) on a variable set $X = \{x_1, \ldots, x_n\}$ is a directed acyclic graph with one source and sinks labeled by the constants 0 and 1, respectively. Each internal node (or decision node) is labeled by a Boolean variable and has two outgoing edges, one labeled by 0 and the other by 1. A nondeterministic binary decision diagram (\vee -BDD) is

a binary decision diagram with some additional nodes called *nondeterministic nodes* (\vee -nodes) whose outgoing edges are unlabeled.

An input $b \in \{0,1\}^n$ activates all edges consistent with b, i.e., the edges labeled by b_i which leave nodes labeled by x_i (and all unlabeled edges in a nondeterministic binary decision diagram). A computation path for an input b in a BDD is a directed path of edges activated by the input b that leads from the source to a sink. A computation path for an input b that leads to the 1-sink is called accepting path for b.

Let B_n denote the set of all Boolean functions defined on n variables. A (nondeterministic) BDD represents the function $f \in B_n$ for which f(b) = 1 iff there exists an accepting path for the input b. A nondeterministic BDD is unambiguous nondeterministic, or a \vee_1 -BDD for short, iff there exists at most one accepting path for every input.

The size of a (nondeterministic) binary decision diagram G is the number of its nodes and is denoted by |G|. The (nondeterministic) binary decision diagram size of a Boolean function f is the size of a smallest BDD representing f.

Our definition of the (nondeterministic) binary decision diagram size as the number of nodes and not the number of edges is justified because both numbers are polynomially related.

In many applications, such as symbolic verification or the analysis of circuits and automata, data structures for Boolean functions are necessary that represent important functions in small size and allow the efficient execution of important operations (for the choice of these operations and a discussion see, e.g., Section 10.2 in [5] and [31]). Since satisfiability test and equality check are two important operations that are NP-hard for general BDDs, restricted variants are considered. FBDDs (with some restrictions) and k-OBDDs, where k does not depend on the number of Boolean variables the represented function is defined on, allow polynomial time algorithms for important operations. OBDDs introduced by Bryant [11] are restricted FBDDs and restricted k-OBDDs.

- **Definition 2.** (i) A free binary decision diagram (FBDD) or read-once branching program is a BDD where each directed path contains for each variable at most one node labeled by this variable. (See Figure 5 for an example of an FBDD.)
 - (ii) An ordered binary decision diagram (OBDD) is a binary decision diagram where on each directed path the node labels of the decision nodes are a subsequence of a given variable ordering $x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}$, where π is a permutation on $\{1, \ldots, n\}$. (See Figure 3 for an example of an OBDD.)
- (iii) A k-OBDD is a binary decision diagram that can be partitioned into k layers. Each layer is an OBDD (with possibly many sources) such that the edges leaving the i-th layer, $1 \le i < k$, reach only nodes of a layer j > i and the sinks. Moreover, all OBDDs respect the same variable ordering which means that on all directed paths in a layer the node labels of the decision nodes are a subsequence of a given variable ordering and this ordering is the same for all layers. (See Figure 5 for an example of a 2-OBDD.)

Nondeterministic variants of restricted BDDs can be defined similarly as for BDDs. In the rest of the paper we consider k-OBDDs, where k is a constant, if nothing else is mentioned. Since a variable ordering can be identified with the corresponding permutation, π also denotes the ordering of the variables by abuse of notation.

A 1-input or satisfying input for a function f is an assignment to the input variables whose function value is 1, in other words this assignment is mapped to 1 by f. A function is satisfiable if there exists a satisfying input for f. In the following, by abuse of notation we say that a (nondeterministic) BDD G has a 1-input or a satisfying input if G does not represent the constant 0 function.

Since OBDDs are restricted FBDDs and restricted k-OBDDs by definition, $\mathcal{P}(OBDD) \subseteq \mathcal{P}(FBDD)$ and $\mathcal{P}(OBDD) \subseteq \mathcal{P}(k\text{-}OBDD)$. Moreover, we know that $\mathcal{P}(OBDD) \subseteq \mathcal{P}(FBDD)$ and $\mathcal{P}(OBDD) \subseteq \mathcal{P}(k\text{-}OBDD)$. The hidden weighted bit function HWB_n defined below is an example of a Boolean function representable by 2-OBDDs and FBDDs of size $\mathcal{O}(n^2)$ but its OBDD size is $\Omega(2^{n/5})$ ([3] and [28]). It is well-known that the complexity classes $\mathcal{P}(FBDD)$ and $\mathcal{P}(k\text{-}OBDD)$ are incomparable which means $\mathcal{P}(FBDD) \not\subseteq \mathcal{P}(k\text{-}OBDD)$ and $\mathcal{P}(k\text{-}OBDD) \not\subseteq \mathcal{P}(FBDD)$. Moreover, there are Boolean functions representable in polynomial size by one model but only in exponential

size by the other one and vice versa (see, e.g., [32]). The same result holds for $\mathcal{P}(FBDD)$ and $\mathcal{P}(\vee_1\text{-}OBDD)$.

2.2 Decomposable negation normal forms

Many known representations of propositional knowledge bases are restricted negation normal form circuits (NNFs) and correspond to specific properties on NNFs [17]. Decomposability and determinism are two of these fundamental properties.

Definition 3 (NNFs). A negation normal form circuit on a variable set X is a Boolean circuit over fanin 2 conjunction and unbounded fanin disjunction gates, labeled by \wedge and \vee , whose inputs are labeled by literals x and \overline{x} , $x \in X$, and the Boolean constants 0 and 1. The size of an NNF C, denoted by |C|, is the number of its gates. The NNF size of a Boolean function f is the size of a smallest negation normal form circuit representing f. The Boolean function $f_C: \{0,1\}^X \to \{0,1\}$ represented by C is defined in the usual way. For an NNF C and a gate g in C the subcircuit rooted at g is denoted by C_g . An NNF is decomposable, or a DNNF for short, iff the children of each \wedge -gate are reachable from disjoint sets of input variables. A set of Boolean functions $\{f_1, \ldots, f_\ell\}$ on the same variable set is disjoint if each pair of functions $f_i, f_j, i \neq j$, is not simultaneously satisfiable. A DNNF is deterministic, or a d-DNNF for short, iff the functions computed at the children of each \vee -gate are disjoint.

Our assumption that each \land -gate has only fan-in 2 is justified because it affects the NNF size only polynomially.

Sentential decision diagrams introduced by Darwiche [16] result from so-called structured decomposability and strong determinism. They are restricted d-DNNFs and a generalization of OBDDs.

Definition 4. For a variable set X let $\bot : \{0,1\}^X \to \{0,1\}$ and $\top : \{0,1\}^X \to \{0,1\}$ denote the constant 0 function and constant 1 function, respectively. A set of Boolean functions $\{f_1,\ldots,f_\ell\}$ on the same variable set is called a *partition* iff the functions $\{f_1,\ldots,f_\ell\}$ are disjoint, none of the functions is the constant 0 function \bot , and $\bigvee_{i=1}^{\ell} f_i = \top$.

Definition 5. A vtree for a variable set X is a full, rooted binary tree whose leaves are in one-to-one correspondence with the variables in X. A sentential decision diagram C, or SDD for short, respecting a vtree T on the variable set $X = \{x_1, \ldots, x_n\}$ is defined inductively in the following way:

- C represents \perp or \top or C represents a projective function $p(X) = x_i$ or $p(X) = \overline{x_i}$, $1 \le i \le n$.
- The output gate of C is a disjunction whose inputs are wires from ∧-gates g_1, \ldots, g_ℓ , where each g_i has wires from p_i and s_i , v is an internal node in T with children v_L and v_R , $C_{p_1}, \ldots, C_{p_\ell}$ are SDDs that respect the subtree of T rooted at v_L , $C_{s_1}, \ldots, C_{s_\ell}$ are SDDs that respect the subtree of T rooted at v_R , and the functions represented by $C_{p_1}, \ldots, C_{p_\ell}$ are a partition.

Vtrees were introduced by Pipatsrisawat and Darwiche [22]. The ordering w.r.t. a vtree and the so-called partition property ensure that SDDs are decomposable and deterministic and therefore, restricted d-DNNFs. The partition property is also called strong determinism. It ensures that $\mathcal{P}(\text{SDD})$ is closed under negation which means that for each function f representable by polynomial-size SDDs also the negated function \overline{f} is in $\mathcal{P}(\text{SDD})$. To the best of our knowledge it is open whether SDDs are even more restricted in the sense of polynomial-size representations than structured d-DNNFs which are d-DNNFs respecting a vtree.

Definition 6. For a node u let vars(u) denote the set of variables that appear in a subgraph rooted at u. Let T be a vtree for the set of variables X and \mathcal{D} be a DNNF. \mathcal{D} respects the vtree T, if for every \land -node u of \mathcal{D} with children u_l, u_r , there is a node v of T with children v_l, v_r such that $vars(u_l) \subseteq vars(v_l)$ and $vars(u_r) \subseteq vars(v_r)$.

A (deterministic) DNNF that respects a given vtree T is called a (deterministic) DNNF_T. Moreover, a *structured* (deterministic) DNNF, or (deterministic) SDNNF for short, is a (deterministic) DNNF_T for an arbitrary vtree T.

Note that for each \land -node u in Definition 6 there is only one node v of T fulfilling the requirement mentioned above. We call v the decomposition node of u and d-node(u) = v.

In the rest of the paper, we look at (restricted) NNFs as classes of Boolean circuits.

2.3 On the relation between OBDDs and SDDs

A vtree is linear if for every internal node one child is a leaf. It is right-linear if for every internal node the left child is a leaf. In the following let T_{π} be a vtree whose left-right traversal of the leaves in T corresponds to the variable ordering π . OBDDs are based on the Shannon decomposition

$$f = \overline{x}_i f_{|x_i=0} \lor x_i f_{|x_i=1},$$

where $f_{|x_i=c}$ denotes the subfunction of f obtained by replacing the Boolean variable x_i by the Boolean constant c. Since the subfunctions $f_{|x_i=0}$ and $f_{|x_i=1}$ do not essentially depend on the variable x_i , i.e., there is no assignment to the remaining variables such that the function values for $x_i=0$ and $x_i=1$ differ, and the disjunction of the projective functions $p_0=\overline{x_i}$ and $p_1=x_i$ is the constant function \top but their conjunction is the function \bot , OBDDs respecting the variable ordering π can be seen as restricted SDDs w.r.t. the right-linear vtree T_{π} and vice versa (see also [16]). Figure 3 shows an OBDD for a Boolean function w.r.t. the variable ordering $\pi=a_1,a_0,x_0,x_1,x_2,x_3$, Figure 4 illustrates the corresponding right-linear vtree T_{π} and an SDD respecting T_{π} for the same Boolean function.

Structured decomposability on the notion of vtrees was originally introduced by Pipatsrisawat and Darwiche [22] but without distinction between the left and right child of a node. Xue, Choi, and Darwiche showed that switching the left and right child of a vtree node may lead to an exponential change in the size of the corresponding SDDs [33]. An SDD w.r.t. a linear vtree T_{π} can be seen as an unambiguous nondeterministic OBDD repecting π . Since it is well-known that $\mathcal{P}(\text{OBDD}) \subsetneq \mathcal{P}(\vee_1\text{-OBDD})$, it is not astonishing that swapping the children of nodes in a vtree may lead to an exponential blow-up in the representation size. We will see in Section 7 that SDDs respecting linear vtrees can represent all Boolean functions in $\mathcal{P}(k\text{-OBDD})$ in polynomial size.

2.4 Storage access functions

In the BDD literature Boolean functions modeling different aspects of storage access are well investigated. A storage access sometimes also called pointer function outputs a single bit of the input for which the address or index is also computed from the input. A very simple one is the multiplexer function MUX_n (alternative names are direct storage access function or index function) that is defined on n+k variables $a_{k-1},\ldots,a_0,x_0,\ldots,x_{n-1}$, where $n=2^k$. The function is given as $\text{MUX}_n(a,x)=x_{|a|_2}$, where $|a|_2$ is the number in $\mathbb N$ whose binary representation equals (a_{k-1},\ldots,a_0) . (See Figure 3 for an example of an OBDD representing MUX_4 .)

The following three Boolean functions are generalized storage access functions, where variables may serve as address as well as data variables. The *hidden weighted bit function* HWB_n is defined by

$$HWB_n(x_1,\ldots,x_n)=x_{\|x\|},$$

where $||x|| = x_1 + \cdots + x_n$ is the number of variables set to 1 in the input x and $x_0 := 0$ which means that the output is 0 if $x_1 + \cdots + x_n = 0$. HWB_n is an example of a function with a clear and simple structure, nevertheless the OBDD size is exponential [12]. (See Figure 5 for restricted BDDs representing the function HWB.)

The indirect storage access function ISA_n can be described in the following way. Let $n=2^k$, $k=2^\ell$, and $m=n/k=2^{k-\ell}$. ISA_n is defined on $n+k-\ell$ Boolean variables, an address vector $a=(a_{k-\ell-1},\ldots,a_0)$ and a vector $x=(x_0,\ldots,x_{n-1})$. The address vector is interpreted as the binary number with value $|a|_2$ pointing to a block $x(a)=(x_{|a|_2k},\ldots,x_{(|a|_2+1)k-1})$. Then

$$ISA_n(a,x) = x_{|x(a)|_2}$$
.

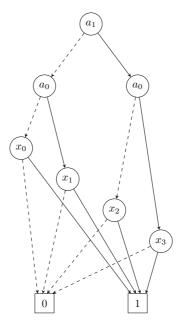


Figure 3: An OBDD for the Boolean function MUX₄ w.r.t. the variable ordering $a_1, a_0, x_0, x_1, x_2, x_3$. Dashed lines represent edges with label 0 and solid ones represent edges with label 1.

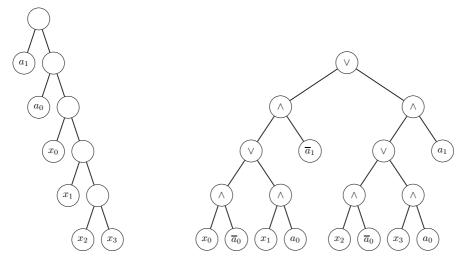


Figure 4: A right-linear vtree whose left-right traversal of the leaves corresponds to the variable ordering $a_1, a_0, x_0, x_1, x_2, x_3$ and an SDD for the Boolean function MUX₄ w.r.t. this vtree.

The function ISA_n has small size representation for BDD models like FBDDs and 2-OBDDs but its OBDD size is exponential [10]. To be more precisely its FBDD and 2-OBDD size is $\mathcal{O}(n^2)$ but its OBDD size is $\Omega(2^{\lfloor n/\log n \rfloor})$.

Another kind of storage access or pointer function is the following one. Let p be the smallest prime larger than n. The function weighted sum WS_n is defined by

$$WS_n(x_1,\ldots,x_n)=x_s,$$

where s is the sum of all ix_i in the field \mathbb{Z}_p , $1 \leq i \leq n$, if this sum is between 1 and n and 1 otherwise. The weighted sum function was introduced and analyzed by Savický and Žák [27] in order to prove a lower bound of order $2^{n-o(1)}$ on the FBDD size of a Boolean function. It is not difficult to see that the 2-OBDD size of WS_n is $\mathcal{O}(n^2)$.

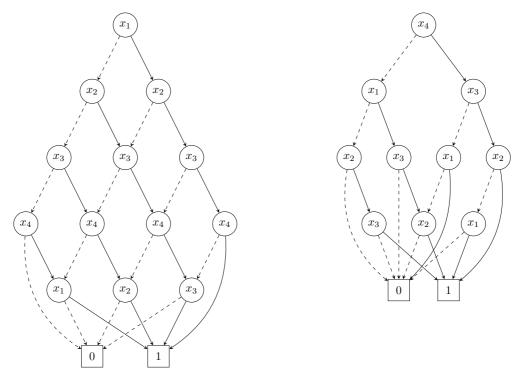


Figure 5: A 2-OBDD w.r.t. the variable ordering x_1, x_2, x_3, x_4 and an FBDD for the function HWB₄. Dashed lines represent edges with label 0 and solid ones edges with label 1. (See also [5].)

3 Simulating Unambiguous Nondeterministic OBDDs by SDDs

In this section, we will examine the relationship between unambiguous nondeterministic OBDDs and SDDs. More precisely, we will derive a way of representing a Boolean function f as an SDD provided that f and \overline{f} can both be represented by unambiguous nondeterministic OBDDs which respect a common variable ordering.

3.1 Main ideas and simulation

Let \mathcal{F}_u denote the subgraph of a given BDD \mathcal{F} rooted at node u and let f_u be the Boolean function which is represented by \mathcal{F}_u . In order to avoid corner cases, we will assume that the given unambiguous nondeterministic OBDDs are of the following form.

Definition 7. Let \mathcal{F} be an unambiguous nondeterministic OBDD. We call \mathcal{F} simple, if

- there exist no edges between \vee -nodes,
- all ∨-nodes have at least two children.
- no \vee -node is connected to a sink, and
- for each inner node u of \mathcal{F} holds that \mathcal{F}_u does not represent the constant function \top or \bot .

Observe that for each unambiguous nondeterministic OBDD that has polynomial size there exists a simple one of polynomial size representing the same function. Furthermore, we will assume w.l.o.g. that the variable ordering is given by the list of variables x_1, \ldots, x_n in the rest of this section. Next, we will present the main ideas of the simulation.

Let f and \overline{f} be Boolean functions that can be represented by unambiguous nondeterministic OBDDs \mathcal{F} and $\overline{\mathcal{F}}$, respectively. Moreover, assume \mathcal{F} and $\overline{\mathcal{F}}$ respect a common variable ordering. Darwiche already mentioned how a (deterministic) OBDD can be converted to an equivalent SDD respecting a right-linear vtree [16]. Therefore, the main question is how to deal with the occurrence

of \vee -nodes in \mathcal{F} . Let f_u be the Boolean function that is computed at an \vee -node u of \mathcal{F} . Since u can occur at any position in the given unambiguous nondeterministic OBDD \mathcal{F} , we would like to derive a way of representing f_u by an SDD. Let f_{u_1}, \ldots, f_{u_k} be the functions that are represented at the child nodes of u. Due to the assumed variable ordering, we know that the functions f_{u_1}, \ldots, f_{u_k} essentially depend on a subset of variables $Y = \{x_i, \ldots, x_n\} \subseteq X$ for $i \geq 1$. The function f_u can be represented by $f_u = (f_{u_1} \wedge \top) \vee (f_{u_2} \wedge \top) \vee \cdots \vee (f_{u_k} \wedge \top)$. However, for an SDD representing f_u in such a way it would not be guaranteed that f_{u_1}, \ldots, f_{u_k} form a partition. Hence, the main idea is to find further functions represented at inner nodes of \mathcal{F} and $\overline{\mathcal{F}}$ which essentially depend on Y and together with f_{u_1}, \ldots, f_{u_k} yield a partition.

We use the notation $f_{|x_1=c_1,\dots,x_{i-1}=c_{i-1}}$ for the subfunction that emerges of f by replacing all occurrences of x_1,\dots,x_{i-1} by constants $c_1,\dots,c_{i-1}\in\{0,1\}$. Now, observe that the subfunctions $f_{|x_1=c_1,\dots,x_{i-1}=c_{i-1}}$ and $\overline{f}_{|x_1=c_1,\dots,x_{i-1}=c_{i-1}}$ yield a partition for arbitrary assignments of the variables x_1,\dots,x_{i-1} . Fix an \vee -node u of \mathcal{F} . Define $\beta(u)$ to be the set of variable assignments over $X\setminus Y=\{x_1,\dots,x_{i-1}\}$ which can be extended by an assignment of the variables of $Y=\{x_i,\dots,x_n\}$ such that there exists an accepting path containing u for the resulting assignment in \mathcal{F} . For an arbitrary assignment $\beta\in\beta(u)$ with $\beta=(\beta_1,\dots,\beta_{i-1})\in\{0,1\}^{i-1}$ we get the relation $f_u\leq f_{|x_1=\beta_1,\dots,x_{i-1}=\beta_{i-1}}$ which means that the satisfying assignments of f_u are a subset of the satisfying assignments of $f_{|x_1=\beta_1,\dots,x_{i-1}=\beta_{i-1}}$ and $f_u\neq \bot$.

Next, we want to identify all nodes u'_1, \ldots, u'_l in \mathcal{F} for a fixed $\beta \in \beta(u)$ such that $f_{u'_j} \leq f_{|x_1=\beta_1,\ldots,x_{l-1}=\beta_{l-1}}$ and $f_{u'_j} \neq \bot$ hold. In order to get these nodes, we consider each node u' in \mathcal{F} with $\operatorname{vars}(u') \subseteq Y$ such that there is no other node u'' fulfilling $\operatorname{vars}(u') \subset \operatorname{vars}(u'') \subseteq Y$ and $\mathcal{F}_{u'}$ is a subgraph of $\mathcal{F}_{u''}$. Afterwards, for each resulting candidate u' we check whether β can be extended by an assignment of the variables of Y such that there is an accepting path in \mathcal{F} containing u'. If u' is an \lor -node, we add the children of u' instead to our set of nodes since we want to resolve \lor -nodes of \mathcal{F} .

Let $f_{u'_1}, \ldots, f_{u'_l}$ be the Boolean functions that are represented at the nodes u'_1, \ldots, u'_l in \mathcal{F} . Then, $f_{|x_1=\beta_1,\ldots,x_{i-1}=\beta_{i-1}}=f_{u_1}\vee\cdots\vee f_{u_k}\vee f_{u'_1}\vee\cdots\vee f_{u'_l}$. Analogously, we identify nodes v_1,\ldots,v_m of $\overline{\mathcal{F}}$ such that $\overline{f}_{v_j}\leq \overline{f}_{|x_1=\beta_1,\ldots,x_{i-1}=\beta_{i-1}}$ and $\overline{f}_{v_j}\neq \bot$ where the functions represented at the nodes v_1,\ldots,v_m of $\overline{\mathcal{F}}$ are denoted by $\overline{f}_{v_1},\ldots,\overline{f}_{v_m}$. Hence, we get $\overline{f}_{|x_1=\beta_1,\ldots,x_{i-1}=\beta_{i-1}}=\overline{f}_{v_1}\vee\cdots\vee\overline{f}_{v_m}$. Now, we are able to represent the function calculated at the \vee -node u as

$$f_{u} = (f_{u_{1}} \wedge \top) \vee \cdots \vee (f_{u_{k}} \wedge \top) \vee (f_{u'_{1}} \wedge \bot) \vee \cdots \vee (f_{u'_{l}} \wedge \bot) \vee \cdots \vee (\overline{f}_{v_{n}} \wedge \bot) \vee \cdots \vee (\overline{f}_{v_{m}} \wedge \bot).$$

$$(1)$$

We know that the functions $f_{u_1}, \ldots, f_{u_k}, f_{u'_1}, \ldots, f_{u'_l}, \overline{f}_{v_1}, \ldots, \overline{f}_{v_m}$ yield a partition because $f_{|x_1=\beta_1,\ldots,x_{i-1}=\beta_{i-1}}$ and $\overline{f}_{|x_1=\beta_1,\ldots,x_{i-1}=\beta_{i-1}}$ are a partition and \mathcal{F} and $\overline{\mathcal{F}}$ are unambiguous nondeterministic.

Finally, we have a look at how to construct an SDD representing f_u . Suppose there are already SDDs representing $f_{u_1}, \ldots, f_{u_k}, f_{u'_1}, \ldots, f_{u'_k}, \overline{f}_{v_1}, \ldots, \overline{f}_{v_m}$ and respecting a vtree T. Now, we construct an SDD C representing f_u composed like in Equation 1 from the given SDDs. C respects a new vtree T' which is structured in the following way. The left subtree of T' is T. The right subtree of T' is just a leaf labeled by a help variable h_{x_1,\ldots,x_n} . We need this help variable since $f_{u_1},\ldots,f_{u'_k},f_{u'_1},\ldots,f_{u'_k},\overline{f}_{v_1},\ldots,\overline{f}_{v_m}$ and \bot , \top formally have to be defined on disjoint variable sets.

If the sub-OBDDs $\mathcal{F}_{u_1}, \ldots, \mathcal{F}_{u_k}, \mathcal{F}_{u'_1}, \ldots, \mathcal{F}_{u'_l}, \overline{\mathcal{F}}_{v_1}, \ldots, \overline{\mathcal{F}}_{v_m}$ contain \vee -nodes as well, we apply the described idea recursively in order to get the needed SDDs. Observe that all functions that are represented at \vee -nodes of the mentioned sub-OBDDs essentially depend on a proper subset of variables $Y' = \{x_j, \ldots, x_n\} \subset Y$ for j > i since by assumption there are no edges between \vee -nodes. Hence, the termination of the recursion is guaranteed.

Next, we will define some notation in order to prove that the described selection of functions

always yields a partition. Afterwards, we will give the formal definition of the simulation. We start with the set $\beta(u)$.

Definition 8. Let \mathcal{F} be an unambiguous nondeterministic OBDD on the variable set $X = \{x_1, \ldots, x_n\}$ respecting the variable ordering $\pi = \text{id}$. Furthermore, let u be a node of \mathcal{F} and $Y = \{x_i, \ldots, x_n\} \subseteq X$ is chosen with the maximum value of $i \in \{1, \ldots, n\}$ fulfilling vars $(u) \subseteq Y$. Then, $\beta(u)$ is defined as the set of variable assignments over $X \setminus Y$ which can be extended by an assignment of Y such that there exists an accepting path in \mathcal{F} containing u.

The following definition helps us to identify all nodes u' of \mathcal{F} for a fixed $\beta \in \beta(u)$ at which parts of $f_{|x_1=\beta_1,...,x_{i-1}=\beta_{i-1}}$ will be computed.

Definition 9. Let \mathcal{F} be an unambiguous nondeterministic OBDD on the variable set $X = \{x_1, \ldots, x_n\}$ respecting the variable ordering $\pi = \text{id}$. In addition, let $Y = \{x_i, \ldots, x_n\} \subseteq X$ and β be a variable assignment over $X \setminus Y$. We call a node u of \mathcal{F} with vars $(u) \subseteq Y$ maximal w.r.t. Y, if there exists no other node u' in \mathcal{F} such that vars $(u) \subset \text{vars}(u') \subseteq Y$ and \mathcal{F}_u is a subgraph of $\mathcal{F}_{u'}$. Moreover, let $R(\mathcal{F}, \beta)$ be the set of all inner nodes u of \mathcal{F} such that u is maximal w.r.t. Y and β can be extended by an assignment of Y with the result that there is an accepting path for the extended assignment in \mathcal{F} containing u.

Since we want to resolve \vee -nodes of \mathcal{F} , we will replace \vee -nodes in the following way.

Definition 10. Let $R^+(\mathcal{F}, \beta)$ be the set of nodes arising from $R(\mathcal{F}, \beta)$, if every \vee -node will be replaced by its children.

The next lemma will be used in our simulation of unambiguous nondeterministic OBDDs by SDDs in order to get a partition for Boolean functions that are represented at \vee -nodes of \mathcal{F} .

Lemma 1. Let \mathcal{F} and $\overline{\mathcal{F}}$ be unambiguous nondeterministic OBDDs respecting the variable ordering π = id and representing the Boolean functions $\Phi_{\mathcal{F}}$ and $\Phi_{\overline{\mathcal{F}}}$ such that $\Phi_{\mathcal{F}} = \overline{\Phi_{\overline{\mathcal{F}}}}$. Let u be an \vee -node of \mathcal{F} and $\beta \in \beta(u)$. Furthermore, the sets $R^+(\mathcal{F},\beta) = \{u_1,\ldots,u_k\}$ and $R^+(\overline{\mathcal{F}},\beta) = \{v_1,\ldots,v_l\}$ are given. Let Φ_{u_i} and Φ_{v_j} with $i \in [k]$ and $j \in [l]$ be the functions that are represented at the nodes u_i of \mathcal{F} and v_j of $\overline{\mathcal{F}}$, respectively. Then, the set of functions $\Phi = \{\Phi_{u_1},\ldots,\Phi_{u_k},\Phi_{v_1},\ldots,\Phi_{v_l}\}$ is a partition.

Proof idea. First, we have to show that the set of functions Φ contains at least two elements. Otherwise, Φ cannot yield a partition. For this purpose, it can be shown that the children of the \vee -node u are elements of $R^+(\mathcal{F},\beta)$. Next, we have to prove that Φ fulfills all partition properties. One can show that the violation of at least one property will lead to a contradiction. The entire proof can be found in Appendix A.

Now, we give the formal definition of the simulation.

Simulation 1. Let $f \in B_n$ be a Boolean function such that f and \overline{f} can be represented by unambiguous nondeterministic OBDDs respecting the variable ordering $\pi = \text{id}$. Let \mathcal{F} and $\overline{\mathcal{F}}$ be those \vee_1 -OBDDs. We construct an SDD C representing f from the \vee_1 -OBDDs \mathcal{F} and $\overline{\mathcal{F}}$ in the following.

First, in order to define the vtree T that will be respected by C we augment $X = \{x_1, \ldots, x_n\}$ by help variables $H = \{h_{x_1, \ldots, x_n}, h_{x_2, \ldots, x_n}, \ldots, h_{x_n}\}$. We define the vtree T for the set of variables $X \cup H$ as depicted in Figure 6:

- T consists of the inner nodes $v_1, \ldots, v_n, v'_1, \ldots, v'_n$ and leaves for the variables of $X \cup H$.
- The node v_1 is the root of T.
- For all $i \in \{1, ..., n\}$: (v_i, v'_i) and $(v_i, h_{x_i,...,x_n})$ are edges of T.
- For all $i \in \{1, ..., n-1\}$: (v'_i, x_i) and (v'_i, v_{i+1}) are edges of T.
- The node v'_n is equal to the leaf labeled by x_n .

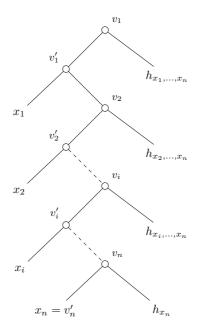


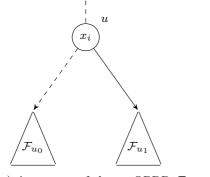
Figure 6: The vtree T for the set of variables $X \cup H$.

Let (V, E) and $(\overline{V}, \overline{E})$ be the sets of nodes and edges of the \vee_1 -OBDDs \mathcal{F} and $\overline{\mathcal{F}}$, respectively. Furthermore, let $X' \subseteq X$ be the set of variables for which there is decision node of \mathcal{F} or $\overline{\mathcal{F}}$ labeled by a variable of X'. Moreover, we have $Y = V \cup \overline{V}$ and $Z = \{ \wedge_0, \wedge_1, \emptyset \} \cup X' \cup Y$. The nodes of C are tuples $(u, v) \in Y \times Z$. We construct C respecting T by mapping nodes and edges of \mathcal{F} and $\overline{\mathcal{F}}$ to nodes and edges of C according to the following cases:

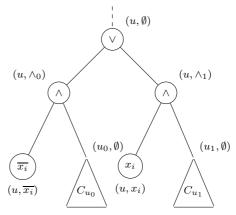
- (a) For each decision node $u \in (V \cup \overline{V})$ for a variable $x_i \in X$ which is only connected to sinks, add a decision node (u, \emptyset) to C that is labeled by a literal $\overline{x_i}$ or x_i according to the semantics of u.
- (b) For each decision node $u \in (V \cup \overline{V})$ for a variable $x_i \in X$ which is not only connected to sinks, add the \vee -node (u, \emptyset) , both \wedge -nodes (u, \wedge_0) , (u, \wedge_1) and the decision nodes $(u, \overline{x_i})$, (u, x_i) that are labeled by $\overline{x_i}$ and x_i , respectively. In addition, add the following edges to C:
 - $((u,\emptyset),(u,\wedge_0))$ and $((u,\emptyset),(u,\wedge_1)),$
 - $((u, \wedge_0), (u, \overline{x_i}))$ and $((u, \wedge_1), (u, x_i)),$
 - the 0-edge $(u, u_0) \in (E \cup \overline{E})$ is mapped to edge $((u, \wedge_0), (u_0, \emptyset))$,
 - the 1-edge $(u, u_1) \in (E \cup \overline{E})$ is mapped to edge $((u, \wedge_1), (u_1, \emptyset))$.

The case of $u \in V$ is depicted in Figure 7.

- (c) For each \vee -node $u \in (V \cup \overline{V})$, add an \vee -node (u,\emptyset) to C. Let $\beta \in \beta(u)$ be a (partial) variable assignment (uniquely chosen). If $u \in V$ holds, let $R^+ = R^+(\mathcal{F},\beta)$ and $\overline{R}^+ = R^+(\overline{\mathcal{F}},\beta)$. Otherwise, let $\overline{R}^+ = R^+(\mathcal{F},\beta)$ and $R^+ = R^+(\overline{\mathcal{F}},\beta)$. For each node $v \in (R^+ \cup \overline{R}^+)$, add an \wedge -node (u,v) to C. Moreover, add the nodes (u,\bot) and (u,\top) to C which are labeled by the constants \bot and \top , respectively. For each $v \in V$, add the following edges to C:
 - For each node $v \in R^+$ fulfilling $(u, v) \in E$ insert
 - $\begin{array}{c} \cdot \ ((u,\emptyset),(u,v)) \\ \cdot \ ((u,v),(v,\emptyset)) \\ \cdot \ ((u,v),(u,\top)) \end{array}$
 - For each node $v \in R^+$ fulfilling $(u, v) \notin E$ and each $v \in \overline{R}^+$ insert
 - $\begin{array}{c} \cdot \ ((u,\emptyset),(u,v)) \\ \cdot \ ((u,v),(v,\emptyset)) \\ \cdot \ ((u,v),(u,\bot)) \end{array}$



(a) A segment of the \vee_1 -OBDD \mathcal{F} , solid edges represent edges labeled by 1, dashed ones edges labeled by 0



(b) A segment of the constructed SDD C.

Figure 7: Case (b) in Simulation 1.

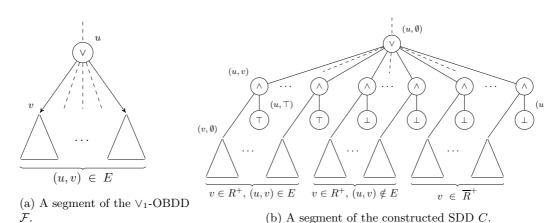


Figure 8: Case (c) in Simulation 1.

The case of $u \in V$ is depicted in Figure 8. If $u \in \overline{V}$ holds, then the edges will be inserted analogously by replacing the set of edges E by \overline{E} in the given description.

Furthermore, for each sink $u \in (V \cup \overline{V})$ we add a node (u,\emptyset) labeled by the respective constant to C. The root of C is given by $(\operatorname{root}(\mathcal{F}),\emptyset)$. Finally, we remove all nodes and edges from the resulting SDD C which cannot be reached from $\operatorname{root}(C) = (\operatorname{root}(\mathcal{F}),\emptyset)$.

In Figures 9 and 10, we give an example for the proposed simulation of unambiguous nondeterministic OBDDs by SDDs. Figure 9 depicts two unambiguous nondeterministic OBDDs \mathcal{F} and $\overline{\mathcal{F}}$ representing Boolean functions f and \overline{f} , respectively. Whereas Figure 10 shows the SDD C constructed by the simulation.

3.2 Size, correctness, and equivalence

We get a relationship between the sizes of the given unambiguous nondeterministic OBDDs and the constructed SDD by the following lemma which states that the increase in size is at most quadratic in $|\mathcal{F}| + |\overline{\mathcal{F}}|$.

Lemma 2. Let \mathcal{F} and $\overline{\mathcal{F}}$ be unambiguous nondeterministic OBDDs respecting the variable ordering $\pi = \operatorname{id}$ and representing Boolean functions $f, \overline{f} \in B_n$. Additionally, let $|\mathcal{F}| = N_1$, $|\overline{\mathcal{F}}| = N_2$, $N = N_1 + N_2$, and $X' \subseteq X$ be the set of variables for which there is decision node of \mathcal{F} or

 $\overline{\mathcal{F}}$ labeled by a variable of X'. Then, the SDD C resulting from Simulation 1 contains at most $2N^2 + 3N$ nodes.

Proof. The nodes of C are tuple $(u,v) \in Y \times Z$. By definition of Y and Z in Simulation 1 we have $Y = |\mathcal{F}| + |\overline{\mathcal{F}}| = N_1 + N_2$ and $Z = N_1 + N_2 + |X'| + 3$. Hence, C contains at most $(N_1 + N_2) \cdot (N_1 + N_2 + |X'| + 3)$ nodes. Furthermore, by assumption \mathcal{F} and $\overline{\mathcal{F}}$ contain at least one node for each variable $x \in X'$. Therefore, we also have $N_1 + N_2 \geq |X'|$. Altogether, we get the following quadratic upper bound:

$$|C| \leq (N_1 + N_2) \cdot (N_1 + N_2 + |X'| + 3)$$

$$= N \cdot (N + |X'| + 3)$$

$$\leq N \cdot (2N + 3) = 2N^2 + 3N \in \mathcal{O}(N^2).$$

Simulation 1 maps each node $u \in (V \cup \overline{V})$ to a node (u, \emptyset) of C. In order to show that C is a syntactically correct SDD computing the same function as \mathcal{F} , we will prove that each node (u, \emptyset) of C is the root of a syntactically correct SDD $C_{(u,\emptyset)}$ which computes the same function as \mathcal{F}_u or $\overline{\mathcal{F}}_u$. For this purpose, we map each node $u \in (V \cup \overline{V})$ to a node v of T such that we can show that $C_{(u,\emptyset)}$ respects subtree T_v .

Definition 11. Let T be the vtree as defined in Simulation 1 and $u \in (V \cup \overline{V})$ be an inner node of the given \vee_1 -OBDDs. We use the function *node* in order to map inner nodes of \mathcal{F} and $\overline{\mathcal{F}}$ to nodes of T in the following way:

$$\operatorname{node}(u) := \begin{cases} v_i \ , \ u \text{ is an \vee-node, } x_i \in \operatorname{vars}(u), \ \nexists x_j \in \operatorname{vars}(u) \text{ such that} \\ x_j < x_i \text{ w.r.t. } \pi. \\ v_i' \ , \ u \text{ is not an \vee-node, } x_i \in \operatorname{vars}(u), \ \nexists x_j \in \operatorname{vars}(u) \text{ such that} \\ x_i < x_i \text{ w.r.t. } \pi. \end{cases}$$

Now, we are ready to prove the stated properties of the SDDs $C_{(u,\emptyset)}$.

Lemma 3. Let \mathcal{F} and $\overline{\mathcal{F}}$ be unambiguous nondeterministic OBDDs respecting the variable ordering $\pi = \mathrm{id}$, representing Boolean functions $f, \overline{f} \in B_n$. Let C be the SDD resulting from Simulation 1. Then, each node (u,\emptyset) of C is the root of a syntactically correct SDD $C_{(u,\emptyset)}$ respecting the vtree T_v of the inner node $v = \mathrm{node}(u)$. Moreover, $C_{(u,\emptyset)}$ represents the same Boolean function as \mathcal{F}_u or $\overline{\mathcal{F}}_u$.

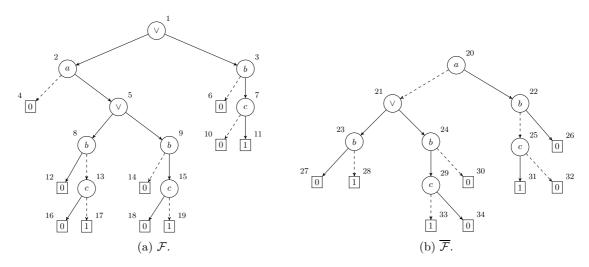


Figure 9: Unambiguous nondeterministic OBDDs \mathcal{F} and $\overline{\mathcal{F}}$ representing the Boolean functions $f(a,b,c)=(b\wedge c)\vee(a\wedge((\overline{b}\wedge\overline{c})\vee(b\wedge\overline{c})))$ and \overline{f} . Solid edges represent edges labeled by 1, dashed ones edges labeled by 0.

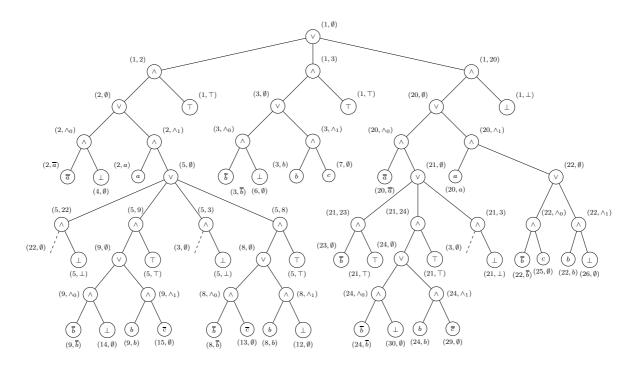


Figure 10: The SDD C which also represents f constructed by Simulation 1 with input \mathcal{F} and $\overline{\mathcal{F}}$. The dashed lines depict connections to sub-SDDs that are already shown in the diagram.

Proof idea. Consider the different cases how the node (u, \emptyset) was added to C by the given simulation. We give a proof by induction on the depth l of the subgraph $C_{(u,\emptyset)}$ of the SDD C in Appendix B.

As a consequence of Lemma 3, we know that C is a syntactically correct SDD representing the same Boolean function as \mathcal{F} .

Corollary 1. Let \mathcal{F} and $\overline{\mathcal{F}}$ be unambiguous nondeterministic OBDDs respecting the variable ordering $\pi = \mathrm{id}$, representing Boolean functions $f, \overline{f} \in B_n$. Then, C is a syntactically correct SDD respecting the vtree T as defined in the simulation. Furthermore, C represents f.

Proof. The root of C is given by the node $(\operatorname{root}(\mathcal{F}),\emptyset)$ as depicted in Simulation 1. We use Lemma 3 in order to see that $C = C_{(\operatorname{root}(\mathcal{F}),\emptyset)}$ is a syntactically correct SDD respecting the vtree T_v with $v = \operatorname{node}(\operatorname{root}(\mathcal{F}))$ and representing the same Boolean function as \mathcal{F} . Here we have $v = v_i$ or $v = v_i'$ for $i \in \{1, \ldots, n\}$. Thus, C is also respecting T.

Theorem 1. Let f be a Boolean function such that f and \overline{f} can be represented by polynomial-size unambiguous nondeterministic OBDDs respecting the same variable ordering. Then, f can also be represented by polynomial-size SDDs.

Proof. By assumption there exist polynomial-size unambiguous nondeterministic OBDDs \mathcal{F} and $\overline{\mathcal{F}}$ respecting the same variable ordering and representing f and \overline{f} , respectively. We use Simulation 1 in order to get the SDD C. On the one hand we know by Lemma 3 that C is syntactically correct and represents the same function as \mathcal{F} . On the other hand we know by Lemma 2 that the increase in size is at most quadratic in $|\mathcal{F}| + |\overline{\mathcal{F}}|$.

If we only have a representation of f as a polynomial-size unambiguous nondeterministic OBDD, we can modify Simulation 1 in order to get an equivalent structured d-DNNF representing f in polynomial size.

Corollary 2. Let f be a Boolean function representable by polynomial-size unambiguous nondeterministic OBDDs. Then, f can also be represented by structured d-DNNFs of polynomial size.

Proof idea. We can modify Simulation 1 such that in case (c) only edges to children of \vee -nodes will be added to the SDD C. For this purpose, we do not have to determine the sets R^+ and \overline{R}^+ . Furthermore, we do not need an unambiguous nondeterministic OBDD representing \overline{f} as input because we do not need a partition in order to represent Boolean functions that are computed at \vee -nodes of \mathcal{F} .

4 Simulating Structured DNNFs by Nondeterministic OBDDs

In recent works it was shown how DNNFs can be simulated by equivalent nondeterministic FBDDs with an increase in size that remains bounded by a quasipolynomial factor [2, 23]. These results were obtained by adapting a quasipolynomial simulation of decision-DNNFs by equivalent FBDDs proposed by Beame et al. [1]. In this section, we introduce another adaption in order to get a quasipolynomial simulation of structured DNNFs by equivalent nondeterministic OBDDs. Moreover, Razgon recently proved that there exists a quasipolynomial separation of SDDs (which are a subclass of d-SDNNFs) and nondeterministic OBDDs [25]. Therefore, the achieved upper bound concerning the increase in size is tight.

4.1 Recap and main ideas

At the beginning, we will briefly recap the idea of constructing a nondeterministic FBDD \mathcal{F} that computes the same Boolean function as a given DNNF $\mathcal{D}[2, 23]$. In order to construct \mathcal{F} we have to remove all \wedge -nodes of $\mathcal D$ and replace them by decision nodes. Suppose we have an \wedge -node u of $\mathcal D$ and its child nodes u_l, u_r . First, we need to find equivalent nondeterministic FBDDs \mathcal{F}_{u_l} and \mathcal{F}_{u_r} for the subgraphs \mathcal{D}_{u_l} and \mathcal{D}_{u_r} , respectively. Next up, we need to combine these nondeterministic FBDDs in order to get a larger one for the expression $\Phi_u = \Phi_{u_l} \wedge \Phi_{u_r}$. For this purpose, redirect all 1-sinks of \mathcal{F}_{u_l} to the root of \mathcal{F}_{u_r} . That way we will get the needed conjunction of the given functions. Note that we get a syntactically correct nondeterministic FBDD by this conjunction since Φ_{u_l} and Φ_{u_r} depend on disjoint sets of variables because of the decomposability of \mathcal{D} . In general this first approach fails since the node u_l can serve as input for more than one node. Then, it is not clear how to redirect the 1-sinks of \mathcal{F}_{u_l} . Therefore, we make copies of subgraphs of \mathcal{D} whenever the mentioned problem arises. Moreover, the children of ∧-nodes will be reordered to bound the blow in size. An outgoing edge of an \(\shcap-\)-node will be classified as a light edge, if the subgraph of \mathcal{D} that is connected by this edge does not contain more \wedge -nodes than the subgraph which is connected via the other edge. The latter will then be called a heavy edge. If (u, u_l) is the light edge of u, we redirect the 1-sinks of \mathcal{F}_{u_l} to the root of \mathcal{F}_{u_r} . As a consequence, each variable mentioned in \mathcal{F}_{u_l} will be queried before every other variable mentioned in \mathcal{F}_{u_r} .

For the following adaption it is crucial to observe that the order in which the functions Φ_{u_l} and Φ_{u_r} will be evaluated (and therefore the order of queried variables) essentially depends on the definition of light and heavy edges. On the one hand, we will modify the presented definition of light and heavy edges with the aid of the vtree of a given SDNNF in order to obtain a variable ordering for the constructed nondeterministic OBDD. On the other hand, this new definition of light and heavy edges also ensures that the increase in size remains bounded by a quasipolynomial factor. While the light and heavy edges of an \land -node are determined individually in the simulation of DNNFs by nondeterministic FBDDs, we will follow a more global approach using the information of a vtree to get a variable ordering.

We know that the variables which can appear in the subgraphs \mathcal{D}_{u_l} and \mathcal{D}_{u_r} of an \wedge -node u in a DNNF_T with decomposition node v are restricted to the variables mentioned in T_{v_l} and T_{v_r} , respectively. The key idea is to globally define the light and heavy edges of all \wedge -nodes of a DNNF_T which have the same decomposition node. We introduce the following quantities to formalize this approach.

Definition 12. Let T be a vtree for the set of variables X and \mathcal{D} be a DNNF_T. Furthermore, let

v be an inner node of T and v_l, v_r its children. We define the following sets and quantities:

$$A^{v} := \{u \mid u \text{ is an } \land \text{-node of } \mathcal{D}, \text{ d-node}(u) = v.\},$$
 $M^{v} := |A^{v}|, M^{v}_{l} := \sum_{w \in T_{v_{l}}} M^{w}, M^{v}_{r} := \sum_{w \in T_{v_{r}}} M^{w}.$

Our aim is to determine in a common way for all \wedge -nodes of a set A^v which subgraph can be reached via a light or heavy edge. Hereby, we achieve that all nondeterministic OBDDs representing a function $\Phi_u = \Phi_{u_l} \wedge \Phi_{u_r}$ for $u \in A^v$ will respect the same variable ordering. With an eye toward the size of the constructed nondeterministic OBDD, we will classify the edges as follows.

Definition 13. Let T be a vtree for the set of variables X and \mathcal{D} be a DNNF_T. Moreover, let u be an \land -node of \mathcal{D} with children u_l, u_r and d-node(u) = v for a node v of T. We classify the edges (u, u_l) and (u, u_r) in the following way: If $M_l^v \leq M_r^v$ holds, we call (u, u_l) a light edge and (u, u_r) a heavy edge. Otherwise, we classify the edges vice versa. We call the remainder of the edges of \mathcal{D} neutral edges.

In order to define the light and heavy edges we used the fact that given an \land -node u of \mathcal{D} with d-node(u) = v the number of \land -nodes that can occur in the subgraphs \mathcal{D}_{u_l} and \mathcal{D}_{u_r} is restricted by M_l^v and M_r^v , respectively. Thus, each time we cross a light edge on a path from the root to a leaf the number of \land -nodes that can possibly occur in the next lower subgraph will be halved. Next, we will use the quantities M_l^v and M_r^v in the same way to define a variable ordering.

Definition 14. Let \mathcal{D} be a DNNF_T and T be a vtree for the set of variables $X = \{x_1, \ldots, x_n\}$. For a pair of variables $x_i, x_j \in X$ with $i \neq j$ let v be the unique node of T with children v_l, v_r such that $x_i \in \text{vars}(v_l)$ and $x_j \in \text{vars}(v_r)$ holds. Then, we order $x_i < x_j$, if and only if $M_l^v \leq M_r^v$. Otherwise, we arrange $x_j < x_i$. We define $\pi(\mathcal{D}, T)$ to be the variable ordering induced by the previously defined relation <.

So, why do we get a variable ordering by the defined relation? Intuitively, starting from the root v of a given vtree T we order the variables that occur in T_{v_l} and T_{v_r} such that each variable of vars (v_l) precedes each variable of vars (v_r) w.r.t. to < or vice versa. Afterwards, we recursively proceed with the nodes v_l and v_r . Later on, we will formally prove that $\pi(\mathcal{D}, T)$ is the variable ordering of the constructed nondeterministic OBDD \mathcal{F} . We need the following sets in order to define the simulation.

Definition 15 ([1, 2]). Fix a DNNF_T \mathcal{D} . For a node u in \mathcal{D} and a path P from the root to u, let S(P) be the set of light edges along P and $S(u) := \{S(P) \mid P \text{ is a path from the root to } u\}$.

While we adjusted the definitions of light and heavy edges, we will use the same simulation proposed by Beame et al. in order to construct the nondeterministic OBDD [1, 2]. We will interpret a leaf of the given DNNF_T \mathcal{D} labeled by a variable $x_i \in X$ as a decision node that points to a 0-sink if $x_i = 0$ and to a 1-sink if $x_i = 1$, and vice versa for a leaf labeled by $\overline{x_i}$. Now, by the following simulation we get a nondeterministic OBDD with additional unlabeled nodes that can be removed in a further step.

Simulation 2 ([1, 2]). Let \mathcal{D} be a DNNF_T and T a vtree for the set of variables $X = \{x_1, \ldots, x_n\}$. We will construct a nondeterministic OBDD \mathcal{F} that computes the same Boolean function as \mathcal{D} . Its nodes are pairs (u, s) where u is a node of \mathcal{D} and the set of light edges s belongs to S(u). The nodes u' = (u, s) of \mathcal{F} will be labeled in the following way:

- (i) If u is a decision node for a variable $x_i \in X$ in \mathcal{D} , then u' is a decision node for the same variable in \mathcal{F} .
- (ii) If u is an \wedge -node in \mathcal{D} , then u' remains unlabeled in \mathcal{F} .
- (iii) If u is an \vee -node in \mathcal{D} , then u' is also an \vee -node in \mathcal{F} .
- (iv) If u is a 0-sink in \mathcal{D} , then u' is also a 0-sink in \mathcal{F} .
- (v) If u is a 1-sink in \mathcal{D} and $s = \emptyset$, then u' is also a 1-sink in \mathcal{F} . Otherwise, u' remains unlabeled.

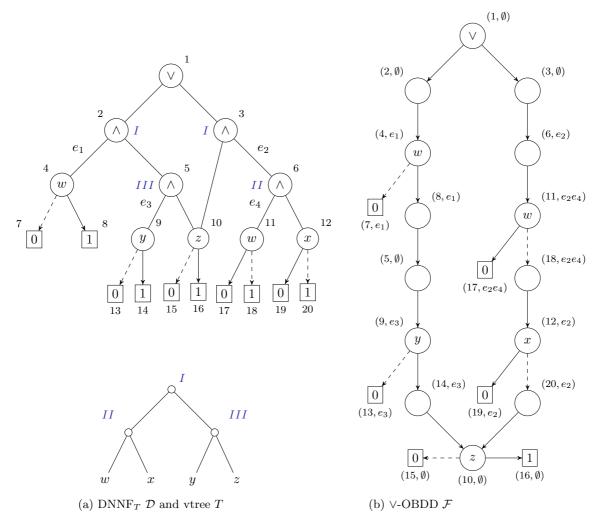


Figure 11: (a) A DNNF_T \mathcal{D} whose leaves are interpreted as decision nodes respecting the depicted vtree T for the set of variables $X = \{w, x, y, z\}$. \mathcal{D} computes the Boolean function $\Phi_{\mathcal{D}}(w, x, y, z) = wyz \vee \overline{w}\,\overline{x}z$. The light edges are marked by e_1, \ldots, e_4 and the decomposition nodes are labeled by I, II and III as in the vtree. (b) The nondeterministic OBDD \mathcal{F} resulting from the given simulation with input \mathcal{D} . The variable ordering of \mathcal{F} is given by $\pi(\mathcal{D}, T)$ resulting in the sequence w, x, y, z.

The node $(\text{root}(\mathcal{D}), \emptyset)$ is the root of \mathcal{F} . The edges in \mathcal{F} are of three types:

- 1. For each light edge e = (u, v) in \mathcal{D} and each $s \in S(u)$, add the edge $((u, s), (v, s \cup \{e\}))$ to \mathcal{F} .
- 2. For each neutral edge e = (u, v) in \mathcal{D} and each $s \in S(u)$, add the edge ((u, s), (v, s)) to \mathcal{F} .
- 3. For each heavy edge (u, v_r) with corresponding light edge $e = (u, v_l)$, each $s \in S(u)$ and each 1-sink w in \mathcal{D}_{v_l} , add the edge $((w, s \cup \{e\}), (v_r, s))$ to \mathcal{F} .

In Figure 11 we give an example for the adapted simulation. The resulting nondeterministic OBDD \mathcal{F} respects the variable ordering given by the sequence w, x, y, z. Note that we would only get a nondeterministic FBDD by the original simulation since the light edge e_2 would be classified as a heavy edge. On the one hand, there would exist a path in the resulting nondeterministic FBDD where w < z holds. On the other hand, there would also be a path where z < w holds. Hence, we cannot find a corresponding variable ordering.

4.2 Size and correctness

First, we have a look at the size of the constructed nondeterministic OBDD.

Lemma 4. Let \mathcal{D} be a DNNF_T with $M \land \text{-nodes}$, N be the total number of nodes and L the maximum number of light edges from the root to a leaf. Then, the constructed nondeterministic OBDD \mathcal{F} of Simulation 2 contains at most $N(M+1)^L \leq N \cdot 2^{\log^2(N)}$ nodes.

Proof. The upper bound of $|\mathcal{F}| \leq N(M+1)^L$ can be derived analogously to the upper bound of the simulation of DNNF by \vee -FBDDs from Beame and Liew [2]. For that to happen, one has to determine the number of nodes that are created by the simulation. Now, we have a look at the second upper bound depending only on N.

Consider a path from the root of \mathcal{D} to a leaf containing L light edges that must exist by premise. For an \wedge -node u on that path with children u_l, u_r let v be the node of T such that d-node(u) = v. Let also be v_l, v_r the children of v. By definition there exist $M^v + M_l^v + M_r^v \wedge$ -nodes having a decomposition node which is located in the subtree T_v . The subgraph \mathcal{D}_{u_l} contains at most $M_l^v \wedge$ -nodes. Assume to the contrary that there exists an \wedge -node u' in \mathcal{D}_{u_l} such that d-node(u') = v' for a node v' which is not located in T_{v_l} . Then, \mathcal{D}_{u_l} would contain at least one node labeled by a variable $x \notin \text{vars}(v_l)$ that would be a contradiction to the premise of \mathcal{D} being a DNNF $_T$. Analogously, the subgraph \mathcal{D}_{u_r} contains at most $M_r^v \wedge$ -nodes.

W.l.o.g. let (u, u_l) be the light edge of the \land -node u. Therefore, it holds that $M_l^v \leq M_r^v$. I.e., the number of \land -nodes which can be located in \mathcal{D}_{u_l} is at most half the number of \land -nodes that can possibly be located in \mathcal{D}_u . If (u, u_r) is the light edge of u, an analog result can be derived. Hence, each time we pass a light edge on the given path, the number of \land -nodes that can be located in the next lower subgraph is at least halved. Moreover, in addition to the $M \land$ -nodes there has to be at least one node labeled by a variable or literal because there must be \land -nodes which are connected to literals or variables as inputs. Altogether, we get $N > M \geq 2^L$. Now, we get the claimed upper bound by using the mentioned inequalities:

$$\begin{split} N(M+1)^L &= N \cdot 2^{\log((M+1)^L)} = N \cdot 2^{L\log(M+1)} \\ &\leq N \cdot 2^{\log(M)\log(M+1)} \\ &\leq N \cdot 2^{\log^2(N)} \,. \end{split}$$

Next up, we show an extension of Lemma 5.4 from Beame and Liew [2] which can subsequently used in order to show that the constructed nondeterministic OBDD is syntactically correct. Let $\mathcal{D}_1, \mathcal{D}_2$ be two SDNNFs. We use the notation $\mathcal{D}_1 \subset \mathcal{D}_2$ which means that \mathcal{D}_1 is a subgraph of \mathcal{D}_2 . Moreover, for two variables $x_i, x_j \in X$ we have $x_i \leq x_j$ if and only if $x_i < x_j$ w.r.t. π^* or $x_i = x_j$ holds.

Lemma 5. Let T be a vtree for the variable set $X = \{x_1, \ldots, x_n\}$, \mathcal{D} be a DNNF $_T$, and \mathcal{F} be the nondeterministic OBDD resulting from Simulation 2. Furthermore, let $\pi^* = \pi(\mathcal{D}, T)$ be the induced variable ordering. If u is a leaf in \mathcal{D} labeled by a variable $x_i \in X$ and there exists a nontrivial path (consisting of at least one edge) between (u, s) and (v, s') in \mathcal{F} , then there exists no node in \mathcal{D}_v labeled by a variable x_j fulfilling $x_j \leq x_i$ w.r.t. π^* .

Proof idea. If we assume to the contrary that there exists such a nontrivial path between (u, s) and (v, s') in \mathcal{F} and there is a node labeled by a variable $x_j \leq x_i$ in \mathcal{D}_v , we either get a violation of the decomposability of \mathcal{D} or that \mathcal{D} does not respect T which leads to a contradiction. The entire proof can be found in Appendix C.

Now, we are able to prove that the constructed nondeterministic OBDD is syntactically correct.

Lemma 6. Let T be a vtree for the set of variables $X = \{x_1, \ldots, x_n\}$, \mathcal{D} be a DNNF_T, and \mathcal{F} the nondeterministic OBDD resulting from Simulation 2. Then, \mathcal{F} is a syntactically correct nondeterministic OBDD respecting the variable ordering $\pi^* = \pi(\mathcal{D}, T)$.

Proof. We have to show that \mathcal{F} is a BDD which suffices the property that decision nodes are labeled by a subsequence of π^* on each directed path.

First, we could show with the help of Lemma 5 that \mathcal{F} is a syntactically correct nondeterministic FBDD with further unlabeled nodes. This can be done like in the proof of Lemma 5.4. from Beame and Liew [2]. Now, we only have to show that \mathcal{F} is respecting the variable ordering π^* .

Suppose there is a directed path P in \mathcal{F} such that the decision nodes appearing on P are not labeled by a subsequence of π^* . Then, there also exists a subpath of P with nodes $(u, s), \ldots, (v, s')$ fulfilling the following properties: (u, s) is a decision node labeled by a variable x_i , (v, s') is a decision node labeled by x_j with $i \neq j$, $x_j < x_i$ w.r.t. π^* . The node (u, s) is labeled by x_i in \mathcal{F} because u is a leaf in the given DNNF_T \mathcal{D} labeled by the same variable. Analogously, we know that v is a decision node labeled by x_j in \mathcal{D} . By usage of Lemma 5 we know that the subgraph \mathcal{D}_v does not contain a decision node labeled by a variable x_j such that $x_j \leq x_i$ w.r.t. π^* . Now, we have the desired contradiction because \mathcal{D}_v contains v which is labeled by x_j and $x_j < x_i$. \square

In the following we assume that \vee - and \wedge -nodes of the given DNNF do not have constants as inputs in order to simplify the proofs of correctness and completeness of the simulation. Otherwise, we could simplify a given DNNF by propagating the constants according to the semantics of \vee - and \wedge -nodes. Certificates are subgraphs of a given DNNF fulfilling the following properties.

Definition 16 ([8]). Let \mathcal{D} be a DNNF for the set of variables X. A *certificate* of \mathcal{D} is a DNNF \mathcal{C} for X with the following properties:

- (i) The DNNF \mathcal{C} is a subgraph of \mathcal{D} ($\mathcal{C} \subset \mathcal{D}$).
- (ii) The roots (output gates) of \mathcal{C} and \mathcal{D} coincide.
- (iii) If \mathcal{C} contains an \wedge -node u, \mathcal{C} also contains each child node v of u and the edge (u,v).
- (iv) If \mathcal{C} contains an \vee -node u, \mathcal{C} also contains exact one of the child nodes v of u and the edge (u, v).

Since the fanin of \wedge -nodes is restricted by 2 and because of the decomposability of \mathcal{D} a certificate can be seen as a binary tree where each leaf is labeled by a different variable of X. Now, we define 1-certificates in order to represent sets of satisfying inputs of a given DNNF.

Definition 17. A 1-certificate is a certificate with the following modifications: each leaf labeled by a literal x is a decision node labeled by x whose only outgoing edge labeled by 1 leads to the 1-sink and each leaf labeled by a literal \overline{x} is a decision node labeled by x whose only outgoing edge labeled by 0 leads to the 1-sink.

A 1-certificate represents all assignments to the input variables where the labels of outgoing edges of decision nodes are chosen as assignments for the corresponding variables. Since a 1-certificate does not have to contain a decision node for each input variable, the represented set of assignments can contain more than one element. Now, observe that according to the definition of 1-certificates each \vee - and \wedge -node will evaluate to 1 given an assignment of the defined set. Since the roots of a 1-certificate and a given DNNF coincide, this set of assignments is also satisfying for the given DNNF.

After introducing the notation of 1-certificates we are ready to show the equivalence of the Boolean functions computed by \mathcal{F} and \mathcal{D} . We will start with the correctness of the simulation, i.e., for each variable assignment b we show that $\Phi_{\mathcal{F}}[b] = 1$ implies $\Phi_{\mathcal{D}}[b] = 1$.

Lemma 7. Let \mathcal{F} be the nondeterministic OBDD resulting from Simulation 2 of a given DNNF_T \mathcal{D} . Then, for each accepting path for a (possibly partial) variable assignment b in \mathcal{F} there exists a 1-certificate of \mathcal{D} which represents b.

Proof idea. Given an accepting path for a variable assignment b in \mathcal{F} we are able to reconstruct a 1-certificate of \mathcal{D} representing the same variable assignment by inspecting Simulation 2. We give a formal proof by induction on the length l of an accepting path in \mathcal{F} in Appendix D.

Next, we will show the completeness of the given simulation, i.e., for each variable assignment b we show that $\Phi_{\mathcal{D}}[b] = 1$ implies $\Phi_{\mathcal{F}}[b] = 1$.

Lemma 8. Let \mathcal{F} be the nondeterministic OBDD resulting from Simulation 2 of a given DNNF_T \mathcal{D} . Then, for each 1-certificate of \mathcal{D} representing a (possibly partial) variable assignment b there exists an accepting path in \mathcal{F} for b.

Proof idea. Given a 1-certificate \mathcal{C} of \mathcal{D} we can decompose \mathcal{C} in order to get an accepting path in \mathcal{F} . We give a proof by induction on the depth l (longest path from the root to a leaf) of a 1-certificate of \mathcal{D} in Appendix E.

Now, we can derive the proposed equivalence of \mathcal{F} and \mathcal{D} by applying the last two lemmata.

Lemma 9. The nondeterministic OBDD \mathcal{F} computes the same Boolean function as the given DNNF_T \mathcal{D} . I.e., $\Phi_{\mathcal{F}}[b] = \Phi_{\mathcal{D}}[b]$ holds for each variable assignment b.

Altogether, we have shown that for each SDNNF there exists an equivalent nondeterministic OBDD with an increase in size that is at most quasipolynomial in $|\mathcal{D}|$. Let L, M and N be defined as in Lemma 4.

Theorem 2. For any DNNF_T \mathcal{D} there exists an equivalent nondeterministic OBDD \mathcal{F} with at most $N(M+1)^L$ nodes and \mathcal{F} can be constructed in time $\mathcal{O}(NM^L)$.

Using the described quasipolynomial simulation of SDNNF by nondeterministic OBDDs, we can derive lower bounds for SDNNFs (and also SDDs) from lower bounds for nondeterministic OBDDs.

5 Simulating (Structured) d-DNNFs

Independently, Beame and Liew and Razgon proved that DNNFs can be simulated by nondeterministic FBDDs with at most a quasipolynomial increase in size [2, 23]. In the previous section, we have adapted this construction in order to get an analogous simulation of SDNNFs by nondeterministic OBDDs. In this section, we will prove that both simulations can be used in order to simulate (structured) d-DNNFs by equivalent unambiguous nondeterministic FBDDs (OBDDs), respectively.

There are two key observations leading to the stated results. The first observation is that two different 1-certificates of a given d-DNNF \mathcal{D} do not represent a common satisfying input of \mathcal{D} .

Lemma 10. Let \mathcal{D} be a deterministic DNNF representing a Boolean function $\Phi_{\mathcal{D}}: \{0,1\}^n \to \{0,1\}$. Then, for each satisfying assignment $b \in \{0,1\}^n$ of $\Phi_{\mathcal{D}}$ there is exactly one 1-certificate of \mathcal{D} representing b.

Proof. There has to be at least one 1-certificate of \mathcal{D} representing b. Otherwise, b would not be a satisfying assignment of $\Phi_{\mathcal{D}}$. Now, suppose to the contrary there would be more 1-certificates of \mathcal{D} representing b. Let \mathcal{C}_1 and \mathcal{C}_2 be two of them. According to the definition of 1-certificates we have $\operatorname{root}(\mathcal{C}_1) = \operatorname{root}(\mathcal{C}_2) = \operatorname{root}(\mathcal{D})$. Hence, consider \mathcal{C}_1 and \mathcal{C}_2 starting from their common root. By definition of certificates we know that there has to be a common \vee -node u of \mathcal{C}_1 and \mathcal{C}_2 such that \mathcal{C}_1 only contains the left child u_l and \mathcal{C}_2 only contains the right child u_r in order that \mathcal{C}_1 and \mathcal{C}_2 differ. The subtree \mathcal{C}_{u_l} of \mathcal{C}_1 is a 1-certificate of \mathcal{D}_{u_l} representing b because otherwise \mathcal{C}_1 would be none of \mathcal{D} . Analogously, the subtree \mathcal{C}_{u_r} of \mathcal{C}_2 has to be a 1-certificate of \mathcal{D}_{u_r} . However, this implies that the Boolean functions represented by \mathcal{D}_{u_l} and \mathcal{D}_{u_r} are not disjoint since b is a satisfying assignment for both functions. This is a contradiction to the assumption of \mathcal{D} being a d-DNNF.

Now, the second observation is that the simulation from Beame and Liew (which is essentially given by Simulation 2) maps each 1-certificate of a given DNNF to a corresponding accepting path in the constructed nondeterministic FBDD.

Lemma 11. Let \mathcal{D} be a DNNF and \mathcal{F} the nondeterministic FBDD resulting from the simulation stated in [2]. Furthermore, let b be a satisfying assignment. Then, \mathcal{F} has as much accepting paths for b as \mathcal{D} has 1-certificates representing b.

Proof. Suppose to the contrary that there would exist more or less accepting paths for b in \mathcal{F} than 1-certificates of \mathcal{D} representing b.

Case 1: There are less accepting paths in \mathcal{F} than 1-certificates of \mathcal{D} . Thus, according to Lemma 8 (completeness) there exist two 1-certificates \mathcal{C}_1 and \mathcal{C}_2 of \mathcal{D} representing b which are mapped to the same accepting path P of \mathcal{F} by the given simulation. Since \mathcal{C}_1 and \mathcal{C}_2 are different 1-certificates of \mathcal{D} , one of the certificates must contain a node u which is not contained in the other certificate. Otherwise, suppose they would consist of the same set of nodes. Then, \mathcal{C}_1 and \mathcal{C}_2 had to differ in their set of edges. But, the edge set of a 1-certificate is determined by its node set according to the definition. W.l.o.g. let \mathcal{C}_1 be the certificate containing u. Now, we know that \mathcal{C}_1 was mapped to an accepting path of \mathcal{F} by the given simulation containing a node (u, s) for $s \in S(u)$. Since \mathcal{C}_2 does not contain u, \mathcal{C}_2 was mapped to an accepting path in \mathcal{F} which does not contain a node (u, s). However, this is a contradiction to the fact that \mathcal{C}_1 and \mathcal{C}_2 were both mapped to P.

Case 2: There are more accepting paths for b in \mathcal{F} than 1-certificates representing b. According to Lemma 7 (correctness) for each accepting path in \mathcal{F} there has to be a corresponding 1-certificate of \mathcal{D} . Since there are more accepting paths for b in \mathcal{F} than 1-certificates representing b, there have to be two different accepting path P_1 and P_2 which emerged from the same 1-certificate of \mathcal{D} . However, the given simulation is a function which maps nodes and edges of \mathcal{D} to nodes and edges of \mathcal{F} . Therefore, P_1 and P_2 have to be equal which leads to a contradiction.

By combining the last two lemmata we get the following result.

Proposition 1. Let \mathcal{D} be a d-DNNF and \mathcal{F} be the nondeterministic FBDD resulting from the simulation stated in [2]. Then, \mathcal{F} is an unambiguous nondeterministic FBDD.

Proof. We have to show that for each variable assignment b there exists at most one accepting path in \mathcal{F} . If b is a non-satisfying assignment, we know from the equivalence of \mathcal{D} and \mathcal{F} that there is no accepting path for b in \mathcal{F} . Now, let b be a satisfying assignment of \mathcal{D} . By Lemma 10 we know that there is exactly one 1-certificate of \mathcal{D} representing b. Furthermore, by Lemma 11 we know that there is exactly one accepting path for b in \mathcal{F} . In conclusion, for each variable assignment b there exists at most one accepting path in \mathcal{F} . Therefore, \mathcal{F} is an unambiguous nondeterministic FBDD.

Since we only changed the definition of light and heavy edges in our simulation of SDNNFs by nondeterministic OBDDs, we easily obtain the next result analogously to Lemma 11.

Lemma 12. Let \mathcal{D} be a DNNF_T and \mathcal{F} be the nondeterministic OBDD resulting from Simulation 2. Besides, let b be a satisfying assignment for the represented function. Then, there exists as many accepting paths for b in \mathcal{F} as there exists 1-certificates in \mathcal{D} representing b.

Therefore, given a d-DNNF $_T$ our simulation yields an unambiguous nondeterministic OBDD.

Proposition 2. Let \mathcal{D} be a d-DNNF_T and \mathcal{F} be the nondeterministic OBDD resulting from Simulation 2. Then, \mathcal{F} is an unambiguous nondeterministic OBDD.

6 On the SDD Size of Some Storage Access Functions

The following representations for the Boolean function HWB_n and its negation \overline{HWB}_n were presented in [3] in order to prove that generalizations of OBDDs used in applications lead to representations of small polynomial size.

$$HWB_n(x) = \bigvee_{1 \le k \le n} E_k^n(x) \wedge x_k \text{ and}$$
 (2)

$$\overline{\text{HWB}}_n(x) = \bigvee_{1 \le k \le n} (E_k^n(x) \wedge \overline{x}_k) \vee E_0^n(x), \tag{3}$$

where E_j^n , $j \in \{0, ..., n\}$, is the symmetric Boolean function on n variables computing 1 iff the number of ones in the input, that is the number of variables set to 1, is exactly j. Using equation 2 and 3 it is easy to see (and was already shown in [3]) that HWB_n and \overline{HWB}_n can be represented w.r.t. every variable ordering by unambiguous nondeterministic OBDDs of size $\mathcal{O}(n^2)$ with only one nondeterministic node at the beginning. Later on a similar construction was used in [7] in order to prove that the SDD size of the function HWB_n is polynomial.

Now, the crucial observation is that the storage access functions defined in Section 2 can all be represented in this way. The indirect storage access function is equal to

ISA_n(a, x) =
$$\bigvee_{0 \le j \le n-1} (|x(a)|_2 = j) \land x_j \text{ or}$$

$$ISA_n(a,x) = \bigvee_{\substack{1 \le i \le m-1 \\ 0 \le j \le n-1}} (|a|_2 = i) \wedge (|(x_{ik}, \dots, x_{(i+1)k-1})|_2 = j) \wedge x_j.$$

This characterization of ISA_n leads easily to a similar one for its negated function.

$$\overline{\text{ISA}}_n(a, x) = \bigvee_{\substack{1 \le i \le m-1 \\ 0 \le j \le n-1}} (|a|_2 = i) \wedge (|(x_{ik}, \dots, x_{(i+1)k-1})|_2 = j) \wedge \overline{x}_j.$$

The weighted sum function can be written as

$$WS_n(x) = \bigvee_{1 \le i \le n} ((S = i) \land x_i) \lor ((S = 0) \land x_1) \lor ((S > n) \land x_1),$$

where S is the sum of all ix_i in \mathbb{Z}_p , $1 \leq i \leq n$. The negated weighted sum function is defined in the following way.

$$\overline{\mathrm{WS}}_n(x) = \bigvee_{1 \le i \le n} ((S = i) \wedge \overline{x}_i) \vee ((S = 0) \wedge \overline{x}_1) \vee ((S > n) \wedge \overline{x}_1).$$

It is easy to see that the conjunction of a Boolean function f and a projective function both given as OBDDs can be done in time and space $\mathcal{O}(|G|)$ where G is the given OBDD representing f. W.l.o.g. let $p(X) = x_i$ be the projective function and f defined on the variable set X. Traverse the OBDD G and redirect all 0-edges leaving nodes labeled by x_i to the 0-sink. Alternatively, for all nodes v labeled by x_i all incoming edges into v are redirected to the 1-successors of v. Since v is not longer reachable afterwards, the nodes labeled by x_i can be deleted. Obviously, the size of the resulting OBDD is at most |G|. For more details see, e.g., [32].

Using the representations for HWB_n , ISA_n and WS_n mentioned above we can prove the following result as a corollary from Theorem 1.

Corollary 3. The function ISA_n can be represented by SDDs of size $\mathcal{O}(n^2)$, the functions HWB_n and WS_n by SDDs of size $\mathcal{O}(n^3)$.

Corollary 3 is an improvement on a result of Bova and Szeider that ISA_n can be represented by SDDs of size $\mathcal{O}(n^{13/5})$ [9]. Beame and Liew showed that SDDs are sometimes exponentially less concise than FBDDs [2]. For this result they analyzed Boolean functions derived from a natural class of database queries and proved that there exists a Boolean function whose FBDD size is $\mathcal{O}(m^2)$ but its SDD size is at least $2^{\sqrt{m/3}-1}$, where the number of Boolean variables the investigated function depends on is m^2+2m . Since the weighted sum function WS_n has exponential FBDD size [27], we complement Beame's and Liew's result using Corollary 3.

Corollary 4. The complexity classes $\mathcal{P}(FBDD)$ and $\mathcal{P}(SDD)$ are incomparable which means that $\mathcal{P}(FBDD) \not\subseteq \mathcal{P}(SDD)$ and vice versa.

Note that there exist Boolean functions representable by polynomial-size FBDDs but every unambiguous nondeterministic OBDD with only one nondeterministic node at the beginning has exponential size and vice versa (see, e.g., [6]). Therefore, Corollary 4 is not really astonishing.

7 On the Succinctness of SDDs and More General BDD Variants

In this section, we prove that every function representable by k-OBDDs of polynomial size, where k is a constant, can also be represented by SDDs of polynomial size. Moreover, there exist Boolean functions representable by SDDs of polynomial size whose k-OBDD size is exponential.

Theorem 3. The complexity class $\mathcal{P}(k\text{-OBDD})$ is a proper subclass of $\mathcal{P}(\text{SDD})$ which means that $\mathcal{P}(k\text{-OBDD}) \subsetneq \mathcal{P}(\text{SDD})$.

The proof of Theorem 3 is technically not too involved. We only need the following observations.

Lemma 13. Each function representable by a k-OBDD of polynomial size can be represented by an unambiguous nondeterministic OBDD of polynomial size w.r.t. the same variable ordering and with only one nondeterministic node at the beginning.

Lemma 13 can be proved by a polynomial transformation from k-OBDDs into equivalent unambiguous nondeterministic OBDDs with only one nondeterministic node at the beginning. For this we can use a construction first used in [4] and later on in [6]. For the sake of completeness we provide the proof of Lemma 13 in Appendix F.

By changing the labels of the 0- and the 1-sink a k-OBDD representing a function f can easily be transformed into a k-OBDD for the negated function \overline{f} . Therefore, for every function f representable by k-OBDDs of polynomial size also the negated function \overline{f} can be represented by k-OBDDs of polynomial size w.r.t. the same variable ordering as f. Hence, using Lemma 13 together with Theorem 1 we obtain the result $\mathcal{P}(k\text{-OBDD}) \subseteq \mathcal{P}(\text{SDD})$. Next, we prove that $\mathcal{P}(k\text{-OBDD})$ is even a proper subclass of $\mathcal{P}(\text{SDD})$.

Lemma 14. There exists Boolean functions f such that f and \overline{f} can be represented by unambiguous nondeterministic OBDDs of polynomial size w.r.t. the same variable ordering but nondeterministic OBDDs where the nondeterministic nodes are only at the beginning need exponential size for f.

Sketch of proof. Sauerhoff proved that there is a Boolean functions f representable by nondeterministic OBDDs of polynomial size but nondeterministic OBDDs for f where nondeterministic nodes are only at the beginning need exponential size [26]. A careful analysis of his proof shows that the nondeterministic OBDD for the function f which is a generalized storage access function is an unambiguous nondeterministic OBDD. Moreover, it is not too difficult but exhausting and tedious to prove that \overline{f} can also be represented by unambiguous OBDDs of polynomial size w.r.t. the same variable ordering as f.

Combining Lemma 13 and 14 with Theorem 1 we can prove Theorem 3.

Concluding Remarks

It is still open whether the complexity class $\mathcal{P}(k\text{-}\mathrm{OBDD})$, where k is a constant, is a proper subset of the complexity class that consists of all Boolean functions representable in polynomial size by unambiguous nondeterministic OBDDs with only one nondeterministic node at the beginning. Furthermore, to the best of our knowledge the question whether the complexity class that consists of all Boolean functions representable by polynomial-size unambiguous nondeterministic OBDDs is closed under negation is open. For unrestricted nondeterministic OBDDs of polynomial size the

answer is negative. Examples are all Boolean functions f for which there is an exponential gap in the so-called nondeterministic one-way communication complexity for f and \overline{f} (for communication complexity see, e.g., [19]). The existence of a Boolean function f with polynomial-size unambiguous nondeterministic OBDDs but for which \overline{f} has exponential unambiguous nondeterministic OBDD size would answer the question whether structured d-DNNFs are more powerful w.r.t. polynomial-size representations than SDDs in the affirmative.

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Appendix A: Proof of Lemma 1

Proof. First, we will show that the set Φ consists of at least two elements. For this purpose, it will be shown that the children of the \vee -node u are elements of $R^+(\mathcal{F},\beta)$. As a consequence, Φ consists of at least two elements because \mathcal{F} was assumed to be simple and therefore u has at least two children.

Let Y be the set of variables that are not assigned by β . According to the definition of $\beta(u)$, the assignment β can be extended such that there exists an accepting path for β in \mathcal{F} containing u. Suppose u were not maximal w.r.t. Y. Then, there would exist another node u' in \mathcal{F} such that $\mathrm{vars}(u) \subset \mathrm{vars}(u') \subseteq Y$ and \mathcal{F}_u is a subgraph of $\mathcal{F}_{u'}$. Let x_i be the smallest variable of $\mathrm{vars}(u)$ w.r.t. π . Then, we have $Y = \{x_i, \ldots, x_n\}$ according to the definition of $\beta(u)$. Notice that the graph $\mathcal{F}_{u'}$ must contain a variable x_j with j < i because it was assumed that there are no edges between \vee -nodes and \mathcal{F}_u is a subgraph of $\mathcal{F}_{u'}$. Hence, $\mathrm{vars}(u') \not\subseteq Y$ would hold which is a contradiction to the assumption. Therefore, $u \in R(\mathcal{F}, \beta)$ and its children are in $R^+(\mathcal{F}, \beta)$ because the node u meets both conditions of the set $R(\mathcal{F}, \beta)$.

Now, we give a proof by contradiction in order to show that Φ is a partition. Suppose to the contrary that there would be an \vee -node u of \mathcal{F} and a (partial) assignment $\beta \in \beta(u)$ such that the described set of functions Φ is not a partition. So Φ has to violate at least one of the partition properties. It will be shown that the violation of at least one partition property leads to a contradiction.

Satisfiability Suppose there would be a function $\varphi \in \Phi$ with $\varphi = \bot$. By definition of $R^+(\mathcal{F}, \beta)$ and $R^+(\overline{\mathcal{F}}, \beta)$ the nodes $u_1, \ldots, u_k, v_1, \ldots, v_l$ are no sinks. Therefore, an inner node u_i or v_j of \mathcal{F} or $\overline{\mathcal{F}}$, respectively, represents the constant function \bot . This is a contradiction to the assumption of \mathcal{F} and $\overline{\mathcal{F}}$ being simple.

Disjointness Suppose there would be functions $\varphi_1, \varphi_2 \in \Phi$ with $\varphi_1 \land \varphi_2 \neq \bot$. For this purpose, consider the following cases.

- 1. The functions φ_1, φ_2 are represented by nodes of the same \vee_1 -OBDD, i.e., either $\varphi_1 = \Phi_{u_i}$, $\varphi_2 = \Phi_{u_j}$ or $\varphi_1 = \Phi_{v_i}$, $\varphi_2 = \Phi_{v_j}$ holds for $i \neq j$. Suppose $\Phi_{u_i} \wedge \Phi_{u_j} \neq \bot$. According to the definition of $R^+(\mathcal{F},\beta)$ the assignment β can be extended (maybe differently) such that there are accepting paths for β in \mathcal{F} containing u_i and u_j . As $\Phi_{u_i} \wedge \Phi_{u_j} \neq \bot$ holds, there is an assignment β^* of Y (variables not assigned by β) such that $\Phi_{u_i}[\beta^*] = 1$ and $\Phi_{u_j}[\beta^*] = 1$. However, if we extend β by β^* then there are accepting paths for (β, β^*) in \mathcal{F} containing u_i and u_j with $i \neq j$. Because of the maximality of u_i and u_j w.r.t. Y (\mathcal{F}_{u_i} can't be a subgraph of \mathcal{F}_{u_j} or vice versa) we know that there must be two distinct accepting paths. This is a contradiction to the property of \mathcal{F} being unambiguous. If $\Phi_{v_i} \wedge \Phi_{v_j} \neq \bot$ holds, the contradiction can be derived analogously.
- 2. The functions φ_1, φ_2 are represented by nodes of \mathcal{F} and $\overline{\mathcal{F}}$, i.e., $\Phi_{u_i} \wedge \Phi_{v_j} \neq \bot$. Hence, there is an assignment β^* of Y such that $\Phi_{u_i}[\beta^*] = 1$ and $\Phi_{v_j}[\beta^*] = 1$ leading to accepting paths for β^* in the subgraphs \mathcal{F}_{u_i} and $\overline{\mathcal{F}}_{v_j}$. By definition of $R^+(\mathcal{F},\beta)$ and $R^+(\overline{\mathcal{F}},\beta)$ the assignment β can be extended such that there are accepting paths in \mathcal{F} and $\overline{\mathcal{F}}$ containing u_i and v_j , respectively. Like in the former case β can be extended by β^* such that there are accepting paths for (β,β^*) in \mathcal{F} and $\overline{\mathcal{F}}$ leading to a contradiction to $\Phi_{\mathcal{F}}=\overline{\Phi_{\overline{\mathcal{F}}}}$.

Cover Suppose $\Phi_{u_1} \vee \cdots \vee \Phi_{u_k} \vee \Phi_{v_1} \vee \cdots \vee \Phi_{v_l} \neq \top$. Then, there exists an assignment β^* of Y such that $\Phi_{u_1}[\beta^*] = \cdots = \Phi_{u_k}[\beta^*] = \Phi_{v_1}[\beta^*] = \cdots = \Phi_{v_l}[\beta^*] = 0$. Hence, there is no accepting path for β^* in $\mathcal{F}_{u_1}, \ldots, \mathcal{F}_{u_k}, \overline{\mathcal{F}}_{v_1}, \ldots, \overline{\mathcal{F}}_{v_l}$. Because every accepting path for β in \mathcal{F} and $\overline{\mathcal{F}}$ contains exactly one node from $u_1, \ldots, u_k, v_1, \ldots, v_l$, it is not possible to extend β by β^* resulting in an accepting path in \mathcal{F} or $\overline{\mathcal{F}}$. This is a contradiction to $\Phi_{\mathcal{F}} \vee \overline{\Phi_{\overline{\mathcal{F}}}} = \top$.

Now, we get the claimed lemma because the violation of at least one partition property leads to a contradiction. \Box

Appendix B: Proof of Lemma 3

Proof. We give a proof by induction on the depth l of the subgraph $C_{(u,\emptyset)}$ of the SDD C. Note that in the following proof we sometimes denote C to be the Boolean function represented at the corresponding SDD. It will be clear from the context whether the SDD or the represented function is meant.

Base case (l=0): Since the depth of the subgraph $C_{(u,\emptyset)}$ is zero, it only consists of the node (u,\emptyset) . Therefore, (u,\emptyset) was added to C because of rule (a) from Simulation 1. Otherwise, in case (b) or (c) the node (u,\emptyset) would be connected to other nodes by outgoing edges resulting in an increase of depth.

First, we will show that $C_{(u,\emptyset)}$ is a syntactically correct SDD. According to rule (a) of Simulation 1 the node (u,\emptyset) was added to C because of a decision node $u \in (V \cup \overline{V})$ for a variable $x_i \in X$ that is connected only to sinks. In this particular case (u,\emptyset) was labeled by a literal x_i or $\overline{x_i}$ depending on the semantics of the decision node u. Then, we know that node u and u is an SDD representing a projective function as in the base case of Definition 5 respecting vtree u is since it contains a leaf labeled by the variable u. It is evident from rule (a) that u is represents the same function as the node u of u of u or u is a syntactically correct SDD. According to rule (a) of Simulation 1 the node u of u is a syntactically correct SDD. According to rule (a) of Simulation 1 the node u of u is a syntactically correct SDD. According to rule (a) of Simulation 1 the node u of Simulation 1 the node u of u is a syntactically correct SDD. According to u is a syntactical u in u is a syntactical u in u in

Induction hypothesis: Each subgraph $C_{(u,\emptyset)}$ of C with depth of at most l is a syntactically correct SDD respecting vtree T_v with v = node(u). Moreover, it represents the same Boolean function as the node u of \mathcal{F} or $\overline{\mathcal{F}}$.

Inductive step $(l \to l+1, l \geq 0)$: In this particular case (u, \emptyset) of C was added because of rule (b) or (c). Otherwise, the depth of $C_{(u,\emptyset)}$ would be zero as mentioned in the base case. Subsequently, we will have a look at both cases.

Case 1: The node (u, \emptyset) was added to C due to rule (b) because of the decision node $u \in (V \cup \overline{V})$ for a variable $x_i \in X$. Then, (u, \emptyset) is an \vee -node which is connected to the \wedge -nodes (u, \wedge_0) and (u, \wedge_1) . The node (u, \wedge_0) is connected to the node $(u, \overline{x_i})$ labeled by $\overline{x_i}$ and (u, \wedge_1) is connected to (u, x_i) labeled by x_i . Let (u, u_0) and (u, u_1) be the outgoing 0- and 1-edges of u, respectively. Then, C also contains the edges $((u, \wedge_0), (u_0, \emptyset))$ and $((u, \wedge_1), (u_1, \emptyset))$. Since we have this setup of nodes and edges, $C_{(u,\emptyset)}$ is an inductively defined SDD constructed by smaller SDDs (see Definition 5). Next, we will show that $C_{(u,\emptyset)}$ is a syntactically correct SDD respecting the vtree $T_{v_i'}$ with $v_i' = \text{node}(u)$. For this purpose, we show that the smaller SDDs are syntactically correct and that they represent Boolean functions which form a partition.

 $C_{p_1}=C_{(u,\overline{x_i})}$ and $C_{p_2}=C_{(u,x_i)}$ are SDDs representing a projective function and they consist of a single node labeled by $\overline{x_i}$ or x_i , respectively. According to the construction of T in Simulation 1 the left subtree of $T_{v_i'}$ is a leaf labeled by x_i . Hence, C_{p_1} and C_{p_2} are SDDs respecting this left subtree. $C_{s_1}=C_{(u_0,\emptyset)}$ and $C_{s_2}=C_{(u_1,\emptyset)}$ are subgraphs of C with a depth of at most l-1 since $C_{(u,\emptyset)}$ is a subgraph with depth of at most l+1 and (u,\emptyset) is connected to the nodes (u_0,\emptyset) , (u_1,\emptyset) by paths of length two. By induction hypothesis $C_{(u_0,\emptyset)}$ and $C_{(u_1,\emptyset)}$ are syntactically correct SDDs respecting vtrees $T_{\text{node}(u_0)}$ and $T_{\text{node}(u_1)}$, respectively. Since there are edges (u,u_0) and (u,u_1) in $\mathcal F$ or $\overline{\mathcal F}$ and the variable ordering is given by x_1,\ldots,x_n , we know that $\text{node}(u_0)=v_j$ or $\text{node}(u_0)=v_j'$ holds for j>i. Otherwise, the variable ordering of $\mathcal F$ or $\overline{\mathcal F}$ would be violated. Analogously, we can derive $\text{node}(u_1)=v_h$ or $\text{node}(u_1)=v_h'$ for h>i. Therefore, both SDDs respect the right subtree $T_{v_{i+1}}$. Moreover, we know that the set of functions $\{C_{p_1},C_{p_2}\}$ yield a partition since the following conditions are satisfied:

$$-C_{p_1} = C_{(u,\overline{x_i})} = \overline{x_i} \neq \bot, C_{p_2} = C_{(u,x_i)} = x_i \neq \bot,$$

$$-C_{p_1} \wedge C_{p_2} = \overline{x_i} \wedge x_i = \bot, \text{ and}$$

$$-C_{p_1} \vee C_{p_2} = \overline{x_i} \vee x_i = \top.$$
(satisfiability)
(disjointness)
(cover)

Now, we want to show the equivalence of the represented functions. According to rule (b) of the simulation we have $C_{(u,\emptyset)} = \overline{x_i}C_{(u_0,\emptyset)} \vee x_iC_{(u_1,\emptyset)}$. W.l.o.g. let $u \in V$. Since u is a decision node for the variable x_i , we know that $\Phi_{\mathcal{F}_u} = \overline{x_i}\Phi_{\mathcal{F}_{u_0}} \vee x_i\Phi_{\mathcal{F}_{u_1}}$ because of the Shannon decomposition rule. By induction hypothesis we get $C_{(u_0,\emptyset)} = \Phi_{\mathcal{F}_{u_0}}$ and $C_{(u_1,\emptyset)} = \Phi_{\mathcal{F}_{u_1}}$. Hence, $C_{(u_0,\emptyset)} = \Phi_{\mathcal{F}_u}$ holds. If $u \in \overline{V}$, we can derive the equivalence the same way.

Case 2: The node (u,\emptyset) was added to C due to rule (c) because of the \vee -node $u \in (V \cup \overline{V})$. W.l.o.g. suppose that $u \in V$ holds. According to rule (c) (u,\emptyset) is an \vee -node which is connected to an \wedge -node (u,v) for each $v \in (R^+ \cup \overline{R}^+)$. These \wedge -nodes are connected to further nodes based on rule (c). Thus, $C_{(u,\emptyset)}$ is an inductively defined SDD constructed by smaller SDDs. Next, we will show that $C_{(u,\emptyset)}$ is a syntactically correct SDD respecting the vtree T_{v_i} with $v_i = \text{node}(u)$. For this purpose, we show that the smaller SDDs are syntactically correct and that they represent Boolean functions which form a partition.

The subgraph $C_{(v,\emptyset)}$ has at most depth l-1 for each $v\in(R^+\cup\overline{R}^+)$ because by assumption $C_{(u,\emptyset)}$ is a subgraph of depth at most l+1 and (u,\emptyset) is connected to (v,\emptyset) by paths of length two. Thus, by the use of the inductive hypothesis $C_{(v,\emptyset)}$ is a syntactically correct SDD respecting the vtree $T_{\text{node}(v)}$ for each $v\in(R^+\cup\overline{R}^+)$. Since we have the edge (u,v) in $\mathcal F$ and the given variable ordering is x_1,\ldots,x_n , we know that $\text{node}(v)=v_j'$ holds for $j\geq i$ because by assumption v cannot be an \vee -node. Therefore, $C_{(v,\emptyset)}$ is an SDD respecting the vtree $T_{v_i'}$ as well. $C_{(u,\perp)}$ and $C_{(u,\top)}$ are SDDs representing \bot and \top , respectively. By definition the right subtree of T_{v_i} is a leaf labeled by the help variable h_{x_i,\ldots,x_n} . Hence, $C_{(u,\perp)}$ and $C_{(u,\top)}$ are SDDs respecting this right subtree. Furthermore, the partition properties are satisfied because the set of functions $\{C_{(v,\emptyset)}\mid v\in(R^+\cup\overline{R}^+)\}$ yield a partition: By induction hypothesis we have $C_{(v,\emptyset)}=\Phi_{\mathcal F_v}$ for each $v\in R^+$ and $C_{(v,\emptyset)}=\Phi_{\overline{\mathcal F}_v}$ for each $v\in R^+$. Thus, we know that $\{C_{(v,\emptyset)}\mid v\in(R^+\cup\overline{R}^+)\}$ is a partition using Lemma 1. Therefore, the desired properties are fulfilled:

- for each
$$v \in (R^+ \cup \overline{R}^+) : C_{(v,\emptyset)} \neq \bot$$
, (satisfiability)
- for each $v, v' \in (R^+ \cup \overline{R}^+)$ with $v \neq v' : C_{(v,\emptyset)} \wedge C_{(v',\emptyset)} = \bot$, and (disjointness)
- we have $\bigvee_{v \in (R^+ \cup \overline{R}^+)} C_{(v,\emptyset)} = \top$. (cover)

Finally, we get the equivalence of $C_{(u,\emptyset)}$ and $\Phi_{\mathcal{F}_u}$ by applying the inductive hypothesis on the representation of $C_{(v,\emptyset)}$ for each $v \in (R^+ \cup \overline{R}^+)$. Since $C_{(u,\emptyset)}$ was constructed by rule (c), $C_{(u,\emptyset)}$ represents the following Boolean function:

$$C_{(u,\emptyset)} = \bigvee_{\substack{v \in R^+, \\ (u,v) \in E}} C_{(v,\emptyset)} C_{(u,\top)} \vee \bigvee_{\substack{v \in R^+, \\ (u,v) \notin E}} C_{(v,\emptyset)} C_{(u,\bot)} \vee \bigvee_{\substack{v \in \overline{R}^+ \\ (u,v) \notin E}} C_{(v,\emptyset)} C_{(u,\bot)}$$

$$= \bigvee_{\substack{v \in R^+, \\ (u,v) \in E}} (C_{(v,\emptyset)} \wedge \top) \vee \bigvee_{\substack{v \in R^+, \\ (u,v) \notin E}} (C_{(v,\emptyset)} \wedge \bot) \vee \bigvee_{\substack{v \in \overline{R}^+ \\ (u,v) \notin E}} (C_{(v,\emptyset)} \wedge \bot)$$

$$= \bigvee_{\substack{v \in R^+, \\ (u,v) \in E}} (C_{(v,\emptyset)} \wedge \top) = \bigvee_{\substack{v \in R^+, \\ (u,v) \in E}} C_{(v,\emptyset)} \stackrel{\text{(ind.)}}{=} \bigvee_{\substack{v \in \overline{R}^+ \\ (u,v) \in E}} \Phi_{\mathcal{F}_v} = \Phi_{\mathcal{F}_u}$$

Appendix C: Proof of Lemma 5

Proof. We give a proof by contradiction adapting the proof of Lemma 5.4. from Beame and Liew [2]. If necessary, we distinguish whether i = j or $i \neq j$ holds.

Suppose to the contrary that u is a leaf of the given DNNF_T \mathcal{D} labeled by a variable $x_i \in X$ and there exists a nontrivial path between (u, s) and (v, s') in \mathcal{F} such that there exists a node in \mathcal{D}_v labeled by a variable x_j fulfilling $x_j \leq x_i$ w.r.t. π^* . We choose v such that there exists no other node v' in \mathcal{D} for which there is a path from (u, s) to (v', s'') and $\mathcal{D}_v \subset \mathcal{D}_{v'}$ holds. Therefore, we call the chosen subgraph \mathcal{D}_v to be maximal. We know that \mathcal{D}_v exists because by assumption (v, s') is a node in \mathcal{F} resulting from the node v in \mathcal{D} .

If the path from (u, s) to (v, s') only consists of one edge, then v has to be a sink in \mathcal{D} because u is a leaf node and therefore ((u, s), (v, s')) was added to \mathcal{F} because of the neutral edge (u, v) in \mathcal{D} . This leads directly to a contradiction to the assumption that \mathcal{D}_v contains a node labeled by a variable x_j . Now, consider paths from (u, s) to (v, s') in \mathcal{F} consisting of at least two edges. Especially, consider the last edge of the path:

$$(u, s), \ldots, (w, s''), (v, s').$$

Suppose that there would exist the edge (w,v) in \mathcal{D} . This would lead to a contradiction to the assumed maximality of \mathcal{D}_v because we had $\mathcal{D}_v \subset \mathcal{D}_w$ and x_j would also occur in \mathcal{D}_w . Therefore, we know that the edge between (w,s'') and (v,s') has to be of the third type and was added to \mathcal{F} because of a heavy edge in \mathcal{D} . Let z be the corresponding \wedge -node in \mathcal{D} , $e = (z, v_l)$ the light edge and $e' = (z, v_r)$ the heavy edge. Since the edge is of the third type and z is the corresponding \wedge -node, we have $v = v_r$ because the edge between (w, s'') and (v, s') was added to \mathcal{F} by mapping the heavy edge (z, v_r) . Furthermore, for that reason we have $s'' = s' \cup \{e\}$. In the following we distinguish two cases at which point the light edge e was added to the set of light edges s''. (See Figure 12 for a visualization of the two cases.)

At the beginning of the path (a) Suppose $e \in s$ holds. Hence, we know that there is a path containing the light edge e from the root of \mathcal{D} to u. There is a path from \mathcal{D} to v containing the heavy edge e' as well. Subsequently, we differentiate whether i = j or $i \neq j$ holds.

Assume we have i=j. There is a node labeled by x_i in the left subgraph \mathcal{D}_{v_l} , namely u. Additionally, by assumption the same variable $x_i=x_j$ appears in the right subgraph $\mathcal{D}_v=\mathcal{D}_{v_r}$. This is contradiction to the premise of \mathcal{D} being a DNNF_T because for the \land -node z we have: $\operatorname{vars}(v_l) \cap \operatorname{vars}(v_r) \neq \emptyset$, i.e., the decomposability is violated.

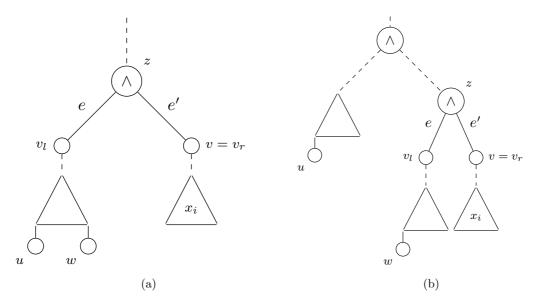


Figure 12: Subgraphs of \mathcal{D} visualizing both cases concerning the proof of Lemma 5.

Now, assume we have $i \neq j$. Let τ be the node of the vtree T such that d-node $(z) = \tau$. We can find such a node because \mathcal{D} is a DNNF_T. Let τ_l, τ_r be the children of τ . W.l.o.g. suppose $\operatorname{vars}(v_l) \subseteq \operatorname{vars}(\tau_l)$ and $\operatorname{vars}(v_r) \subseteq \operatorname{vars}(\tau_r)$. Otherwise, we could label the children of τ vice versa. Like in the preceding case we know that there is a node labeled by x_i in \mathcal{D}_{v_l} and a node labeled by x_j in \mathcal{D}_{v_r} . So, we know that $x_i \in \operatorname{vars}(\tau_l)$ and $x_j \in \operatorname{vars}(\tau_r)$. By assumption we have $x_j < x_i$ w.r.t. π^* . Therefore, it must hold that $M_r^\tau < M_l^\tau$ by definition of $\pi^* = \pi(\mathcal{D}, T)$. But now, we have a contradiction to the premise of $e = (z, v_l)$ being marked as a light edge which only holds for $M_l^\tau \leq M_r^\tau$.

During the course of the path (b) Suppose $e \notin s$ holds. Since the edge from (w, s'') to (v, s') is one of the third type, we know $e \in s''$. Hence, there must exist an edge of the first type $((z, s_1), (v_l, s_1 \cup \{e\}))$ on the path $(u, s), \ldots, (w, s''), (v, s')$. Therefore, there is also a path from (u, s) to (z, s_1) in \mathcal{F} and $\mathcal{D}_v \subset \mathcal{D}_z$ holds because of the heavy edge $e' = (z, v_r) = (z, v)$ in \mathcal{D} . The subgraph \mathcal{D}_z contains a node labeled by x_j as well because we assumed that \mathcal{D}_v contains such a node. Altogether, we get a contradiction to the maximality of \mathcal{D}_v .

Now, the claimed lemma results from the contradictions of the individual cases.

Appendix D: Proof of Lemma 7

Proof. W.l.o.g. we assume that there is no \land - or \lor -node in \mathcal{D} which uses constants as input. Otherwise, we could simplify \mathcal{D} by propagating the constant according to the semantics of \land - and \lor -nodes. We give a proof by induction on the length l (number of edges) of an accepting path and we represent a path by a list of its nodes.

Base case (l=1): Let $P=(u_1,s_1), (u_2,s_2)$ be an accepting path in \mathcal{F} for a variable assignment b. Since P is an accepting path, the node $u'_2:=(u_2,s_2)$ has to be a 1-sink of \mathcal{F} . Hence, by rule (v) of Simulation 2 the node u_2 is also a 1-sink in \mathcal{D} and $s_2=\emptyset$ holds. Furthermore, we know that u_1 cannot be an \vee - or \wedge -node because we assumed that there are no \vee - or \wedge -nodes with constant inputs. Thus, u_1 is a decision node for a variable $x_i \in X$ in \mathcal{D} and by rule (i) we know that u'_1 is a decision node for the same variable. The edge (u'_1, u'_2) was added to \mathcal{F} because of the neutral edge (u_1, u_2) in \mathcal{D} . The node u_1 has to be the root of \mathcal{D} by construction of \mathcal{F} . Thus, the decision node u_1 , the 1-sink u_2 and the edge (u_1, u_2) form a 1-certificate representing b. If the edge (u_1, u_2) in \mathcal{D} is labeled by 0, we have $b_i = 0$. Otherwise, we have $b_i = 1$.

Induction hypothesis: For each accepting path for a variable assignment b in \mathcal{F} with length at most l there is a 1-certificate of \mathcal{D} representing b.

Inductive step $(l-1 \to l, l \ge 2)$: Let $P = (u_1, s_1), \dots, (u_l, s_l), (u_{l+1}, s_{l+1})$ be an accepting path for a variable assignment b in \mathcal{F} . We do the inductive step by considering the following two cases.

<u>Case 1:</u> $u'_1 := (u_1, s_1)$ is an \vee -node of \mathcal{F} . We know that u'_1 is the root of \mathcal{F} and by construction $u'_1 = (\operatorname{root}(\mathcal{D}), \emptyset)$ holds. Furthermore, u_1 has to be an \vee -node in \mathcal{D} as well since only \vee -nodes of \mathcal{D} are mapped to \vee -nodes of \mathcal{F} by the given simulation. Therefore, the edge $((u_1, s_1), (u_2, s_2))$ was added to \mathcal{F} because of the neutral edge (u_1, u_2) in \mathcal{D} .

Consider the nondeterministic OBDD \mathcal{F}' which results from the given simulation with input \mathcal{D}_{u_2} . \mathcal{F}' corresponds to the nondeterministic OBDD with root (u_2, s_2) which arises from \mathcal{F} by removing all nodes and edges that cannot be reached from (u_2, s_2) . Now, consider the subpath $P' = (u_2, s_2), \ldots, (u_{l+1}, s_{l+1})$ of P. The subpath P' is an accepting path for b in \mathcal{F}' . Otherwise, P would be no accepting path for b in \mathcal{F} . Furthermore, P' contains an edge less than P. Thus, by the inductive hypothesis there exists a 1-certificate of \mathcal{D}_{u_2} representing b. Since u_1 is an \vee -node and the root of \mathcal{D} , we can expand the 1-certificate of \mathcal{D}_{u_2} by u_1 and the edge (u_1, u_2) in order to get a 1-certificate of \mathcal{D} .

<u>Case 2:</u> $u'_1 := (u_1, s_1)$ is not an \vee -node of \mathcal{F} . Then, u'_1 has to be an unlabeled node resulting from the \wedge -node u_1 in \mathcal{D} . Suppose to the contrary that u'_1 would be a sink. Then, P would be no computing path because P contains two sinks. Moreover, suppose u'_1 would be a decision node for a variable $x_i \in X$. Then, by rule (i) of Simulation 2 the node u_1 is also a decision node for the same variable. But now, u_1 would be a leaf in \mathcal{D} and therefore the length of the accepting path had to be 1. Finally, consider u'_1 would be an unlabeled node which was created because of a 1-sink in \mathcal{D} . Then, \mathcal{D} would only consist of this 1-sink and P had length 0.

Since u_1' is the root of \mathcal{F} , we have $u_1' = (\operatorname{root}(D), \emptyset)$. As P is an accepting path in \mathcal{F} , the node $u_{l+1}' := (u_{l+1}, s_{l+1})$ is a 1-sink. We know by rule (v) of Simulation 2 that $s_{l+1} = \emptyset$ and u_{l+1} is also a 1-sink in \mathcal{D} . The edge $((u_1, s_1), (u_2, s_2))$ was added to \mathcal{F} because of the light edge $e = (u_1, u_2)$ in \mathcal{D} since u_1 is an \wedge -node. Therefore, we have $s_2 = s_1 \cup \{e\} = \{e\}$. Since $s_{l+1} = \emptyset$ holds, there must exist an edge $((u_i, s_i), (u_{i+1}, s_{i+1}))$ in \mathcal{F} with $3 \leq i \leq l-1$ which was added because of the corresponding heavy edge $e' = (u_1, v_r)$ in \mathcal{D} . The bounds of i emerge from the first and last position of an unlabeled node (u_i, s_i) on P that is connected to (u_{i+1}, s_{i+1}) by an edge of the third type. Otherwise, we would have $e \in s_{l+1}$ resulting in (u_{l+1}, s_{l+1}) not being a 1-sink and P not being accepting. As $((u_i, s_i), (u_{i+1}, s_{i+1}))$ is an edge of the third type, we have $u_{i+1} = v_r$ and u_i is a 1-sink in \mathcal{D} .

Now, let $P' = (u_2, s_2), \ldots, (u_i, s_i)$ and $P'' = (u_{i+1}, s_{i+1}), \ldots, (u_{l+1}, s_{l+1})$ be subpaths of P such that i is chosen as described in the previous paragraph. Consider the nondeterministic OBDD \mathcal{F}' resulting from the given simulation of the left subgraph \mathcal{D}_{u_2} . Alternatively, we can get \mathcal{F}' from \mathcal{F} by removing all nodes (u, s) fulfilling $e \notin s$ and replacing unlabeled nodes without outgoing edges by 1-sinks. Moreover, consider the nondeterministic OBDD \mathcal{F}'' resulting from the given simulation of the right subgraph $\mathcal{D}_{u_{i+1}}$. We can get \mathcal{F}'' from \mathcal{F} by removing the root (u_1, s_1) and each node (u, s) for which $e \in s$ holds.

Next, we want to derive 1-certificates of \mathcal{D}_{u_2} and $\mathcal{D}_{u_{i+1}}$ representing b from the given subpaths P' and P'', respectively. The root of \mathcal{F}' is the fist node (u_2, s_2) of P'. Each edge of P' exists in \mathcal{F}' since we only removed nodes (u, s) for which $e \notin s$ holds. Furthermore, the node (u_i, s_i) is a 1-sink in \mathcal{F}' because we split up P such that $((u_i, s_i), (u_{i+1}, s_{i+1}))$ is an edge of the third type. Hence, P' is an accepting path for b in \mathcal{F}' which is shorter than P. By induction hypothesis there is a 1-certificate of \mathcal{D}_{u_2} representing b.

The root of \mathcal{F}'' is the first node (u_{i+1}, s_{i+1}) of P''. The path P'' is a proper subpath of P and has to be an accepting path for b in \mathcal{F}'' since otherwise P would be no accepting path for b in \mathcal{F} . By induction hypothesis there is a 1-certificate of $\mathcal{D}_{u_{i+1}}$ representing b.

Finally, we will combine the 1-certificates of \mathcal{D}_{u_2} and $\mathcal{D}_{u_{i+1}}$ in order to get a 1-certificate of \mathcal{D} representing b. At the beginning we observed that u_1 has to be the root of \mathcal{D} . The edge $((u_1, s_1), (u_2, s_2))$ of P was added to \mathcal{F} because of the light edge (u_1, u_2) and $((u_i, s_i), (u_{i+1}, s_{i+1}))$ was added because of the heavy edge (u_1, u_{i+1}) . Thus, the node u_1 , both edges $(u_1, u_2), (u_1, u_{i+1})$, and the 1-certificates of \mathcal{D}_{u_2} and $\mathcal{D}_{u_{i+1}}$ give a 1-certificate of \mathcal{D} representing b.

Appendix E: Proof of Lemma 8

Proof. W.l.o.g. we assume that there is no \land - or \lor -node in \mathcal{D} which uses constants as input. Otherwise, we could simplify \mathcal{D} by propagating the constant according to the semantics of \land - and \lor -nodes. Furthermore, we assume that \mathcal{D} consists not only of a sink. We give a proof by induction on the depth l (longest path from the root to a leaf) of a 1-certificate of \mathcal{D} .

Base case (l=1): Let \mathcal{C} be a 1-certificate of \mathcal{D} of depth one representing the satisfying variable assignment b. By definition of a certificate we have $\text{root}(\mathcal{C}) = \text{root}(\mathcal{D}) =: u$. The root u

has to be a decision node for a variable $x_i \in X$. Suppose to the contrary that u would be an \land -or an \lor -node. Then, the inputs of u had to be constants as \mathcal{C} is of depth one. This was precluded by assumption. Moreover, u is not a sink since we also precluded it by assumption. Therefore, \mathcal{C} consists of the root u, a 1-sink v, and an edge (u,v) which is labeled consistently with b. So, $P = (u, \emptyset), (v, \emptyset)$ is an accepting path for b in \mathcal{F} .

Induction hypothesis: For each 1-certificate of \mathcal{D} representing b with depth of at most l, there exists an accepting path for b in \mathcal{F} .

Inductive step $(l \to l+1, l \ge 1)$: Let \mathcal{C} be a 1-certificate of \mathcal{D} with depth l+1 representing the satisfying variable assignment b. Let $u := \text{root}(\mathcal{C}) = \text{root}(\mathcal{D})$. We do the inductive step by considering the following two cases.

Case 1: u is an \vee -node. By definition of certificates, \mathcal{C} contains exactly one child node of u, called v, and the edge (u,v). The subtree \mathcal{C}_v of \mathcal{C} has to be a 1-certificate of \mathcal{D}_v since \mathcal{C} would not be one of \mathcal{D} . Moreover, the depth of \mathcal{C}_v is l. By induction hypothesis there exists an accepting path for b in the nondeterministic OBDD \mathcal{F}_v which results from the given simulation by input of \mathcal{D}_v . Since we have $S(v) = \{\emptyset\}$ for v in \mathcal{D}_v and $\emptyset \in S(v)$ for v in \mathcal{D} , we know that \mathcal{F}_v is a subgraph of \mathcal{F} . Apart from the nodes and edges of \mathcal{F}_v , \mathcal{F} also contains the edge $((u,\emptyset),(v,\emptyset))$ because of the neutral edge (u,v) in \mathcal{D} . We can extend P_v to be an accepting path of \mathcal{F} by adding $((u,\emptyset),(v,\emptyset))$ as a prefix.

<u>Case 2:</u> u is not an \vee -node. The node u has to be an \wedge -node. Suppose to the contrary that u is a decision node. Then, the depth of \mathcal{C} would be 1 as in the base case. By definition of certificates, \mathcal{C} contains both children of u, called u_l and u_r . We assume that (u, u_l) is the light edge. Otherwise, we rename the child nodes of u. The subtrees \mathcal{C}_{u_l} and \mathcal{C}_{u_r} have to be 1-certificates of \mathcal{D}_{u_l} and \mathcal{D}_{u_r} , respectively, because otherwise \mathcal{C} would be no 1-certificate of \mathcal{D} . Moreover, we know that \mathcal{C}_{u_l} and \mathcal{C}_{u_r} have a depth of at most l. By induction hypothesis there are accepting paths for b in \mathcal{F}_{u_l} and \mathcal{F}_{u_r} which are nondeterministic OBDDs resulting from the simulation of \mathcal{D}_{u_l} and \mathcal{D}_{u_r} , respectively.

Let $P' = (u'_1, s'_1), \ldots, (u'_g, s'_g)$ and $P'' = (u''_1, s''_1), \ldots, (u''_h, s''_h)$ be the accepting paths for b in \mathcal{F}_{u_l} and \mathcal{F}_{u_r} , respectively. According to the simulation we know that $(u'_1, s'_1) = (\text{root}(\mathcal{D}_{u_l}), \emptyset) = (u_l, \emptyset)$ and $(u''_1, s''_1) = (\text{root}(\mathcal{D}_{u_r}), \emptyset) = (u_r, \emptyset)$. Furthermore, (u'_g, s'_g) and (u''_h, s''_h) have to be 1-sinks and $s'_g = s''_h = \emptyset$. It is our aim to identify P' and P'' in \mathcal{F} and to extend them with two further edges to an accepting path for b.

Since (u, u_r) is a heavy edge of \mathcal{D} leading to the root of \mathcal{D}_{u_r} , we have $S(u_r) = \{\emptyset\}$ in \mathcal{D} . If there was any other set of light edges in $S(u_r)$, then the decomposability property would be violated at the \land -node u: one of the light edges of a set of $S(u_r)$ has to connect a node of \mathcal{D}_{u_l} with u_r . Otherwise, \mathcal{D}_{u_r} would be cyclic. Furthermore, we have $S(u_r) = \{\emptyset\}$ in \mathcal{D}_{u_r} since u_r is the root of \mathcal{D}_{u_r} . Hence, \mathcal{F}_{u_r} is a subgraph of \mathcal{F} . Thus, \mathcal{P}'' is a path from (u_r, \emptyset) to a 1-sink in \mathcal{F} .

However, (u, u_l) is a light edge in \mathcal{D} such that $\{e\} \in S(u_l)$ holds in \mathcal{D} . Further, we also know that $S(u_l) = \{\{e\}\}$ holds in \mathcal{D} because otherwise the decomposability of \mathcal{D} would be violated. But we have $S(u_l) = \emptyset$ in \mathcal{D}_{u_l} because u_l is the root of \mathcal{D}_{u_l} . Hence, there exists an isomorphism between \mathcal{F}_{u_l} and the subgraph of \mathcal{F} which was added because of \mathcal{D}_{u_l} since ((u, s), (v, s')) is an edge of \mathcal{F}_{u_l} if and only if $((u, s \cup \{e\}), (v, s' \cup \{e\}))$ is an edge of \mathcal{F} .

Finally, consider $P = (u, \emptyset), (u'_1, s'_1 \cup \{e\}), \dots, (u'_g, s'_g \cup \{e\}), (u''_1, s''_1), \dots, (u''_h, s''_h)$. We get P by concatenating a modified version of P', P'', and two more edges. The first edge $((u, \emptyset), (u'_1, s'_1 \cup \{e\})) = ((u, \emptyset), (u_l, \{e\}))$ exists in \mathcal{F} because of the light edge (u, u_l) in \mathcal{D} . The sequence of edges $(u'_1, s'_1 \cup \{e\}), \dots, (u'_g, s'_g \cup \{e\})$ exist in \mathcal{F} since P' is an accepting path of \mathcal{F}_{u_l} and there exists the isomorphism between the nodes of \mathcal{F} and \mathcal{F}_{u_l} . Furthermore, we have $(u'_g, s'_g \cup \{e\}) = (u'_g, \{e\})$ since P' is an accepting path and therefore (u'_g, s'_g) is a 1-sink in \mathcal{F}_{u_l} with $s'_g = \emptyset$. So, the edge $((u'_g, s'_g \cup \{e\}), (u''_1, s''_1)) = ((u'_g, \{e\}), (u_r, \emptyset))$ exists because of the heavy edge (u, u_r) in \mathcal{D} . Finally, the path $(u''_1, s''_1), \dots, (u''_h, s''_h)$ ends in a 1-sink of \mathcal{F} . Hence, P is an accepting path for b in \mathcal{F} . \square

Appendix F: Proof of Lemma 13

Proof. Our aim is to prove that each function representable by a k-OBDD of polynomial size, where k is an arbitrary constant, can also be represented by an unambiguous nondeterministic OBDD of polynomial size with only one nondeterministic node at the beginning. For this reason we present a polynomial transformation from k-OBDDs into equivalent restricted unambiguous nondeterministic OBDDs. The following construction was first used in [4] proving that the satisfiability problem can be solved in polynomial time for functions represented by k-OBDDs. Later it was also used in [6] in order to prove that k-OBDDs can be polynomially transformed into OBDDs which use so-called parity nondeterminism.

Let f be the function represented by a given k-OBDD G and let k be a constant. We start with the observation that there is exactly one accepting path for each 1-input in a k-OBDD since it is a deterministic model. Now, the crucial idea is a suitable decomposition of a given k-OBDD G. For this we consider the at most $s = |G|^{k-1}$ possibilities to switch between the layers of G. The i-th auxiliary function, $1 \le i \le s$, equals 1 for the 1-inputs of f that choose the i-th possibility which means that the accepting paths for these inputs run through the layers of the given k-OBDD G in the chosen way. Such an auxiliary function can be represented by an OBDD of size $|G|^k$ by combining parts of the k-OBDD via conjunction. Here we use the fact that in a k-OBDD all layers respect the same variable ordering. (OBDDs in general do not have nice algorithmic properties. There are examples known such that g_n and h_n are two Boolean functions which have OBDDs of linear size (for different variable orderings) but $f_n = g_n \wedge h_n$ has even exponential nondeterministic FBDD size. The so-called permutation test function is an example of such a function f_n . If only OBDDs respecting the same variable ordering are considered, all important operations can be performed efficiently. For more details see, e.g., [32].)

Next, we describe these ideas more precisely. Let G_1,\ldots,G_k be the layers of G. If b is a 1-input, the accepting path for b leads through some layers $\ell(1)=1<\ell(2)<\cdots<\ell(r)\leq k$ of G, where v_1 is the source of G, $G_{\ell(i)}$ is reached at some node v_i , and from some node in $G_{\ell(r)}$ the sink labeled by 1 is reached. There are at most $|G|^{k-1}$ possibilities to choose $r,\ell(2),\ldots,\ell(r),v_2,\ldots,v_r$. For an arbitrary but fixed choice of these parameters we consider the layers $G_{\ell(1)},\ldots,G_{\ell(r)}$ and the sinks. We transform $G_{\ell(i)}$, $i\in\{1,\ldots,r\}$, into an OBDD $G'_{\ell(i)}$ with source v_i in the following way. An edge leaving $G_{\ell(i)}$ is replaced by an edge to a 1-sink if either i< r and the edge leads to v_{i+1} or i=r and the edge leads to the 1-sink. All other edges leaving a node in $G_{\ell(i)}$ are replaced by edges to the 0-sink. Now, $G'_{\ell(i)}$ consists of all nodes (and corresponding edges) reachable from v_i . The function represented by G has a 1-input iff for some $r,\ell(2),\ldots,\ell(r),v_2,\ldots,v_r$ the corresponding OBDDs $G'_{\ell(1)},\ldots,G'_{\ell(r)}$ have a common 1-input. Since all these OBDDs respect the same variable ordering, Bryant's apply algorithm [11] can be used to obtain an OBDD of size $\mathcal{O}(|G|^k)$ for the conjunction of the functions represented by $G'_{\ell(1)},\ldots,G'_{\ell(r)}$ in time $\mathcal{O}(|G|^k)$. Considering all choices of the parameters $r,\ell(2),\ldots,\ell(r),v_2,\ldots,v_r$ we obtain a unambiguous nondeterministic OBDD of size $\mathcal{O}(|G|^{2k-1})$ which has only one nondeterministic node at the beginning.