The Power of Priority Channel Systems

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Outline

priority channel systems a model of computation

priority embedding a well quasi ordering

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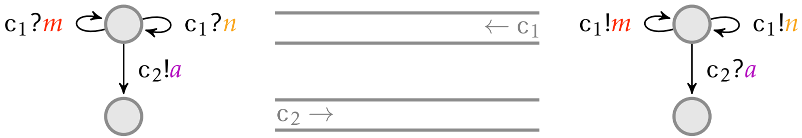

- modeling network communications
- Turing-powerful: Σ^0_1-complete

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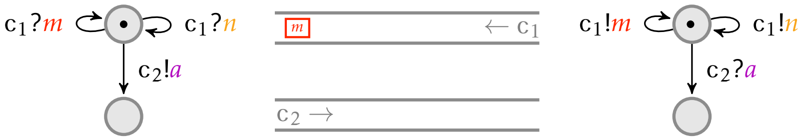

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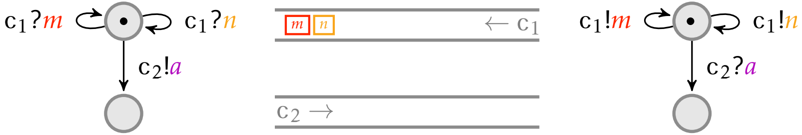

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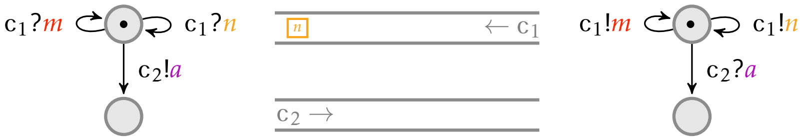

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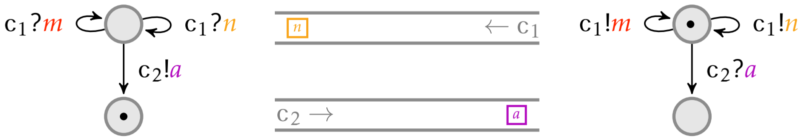

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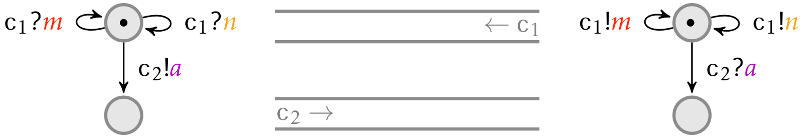
Lossy Channel Systems (LCSs)

(Abdulla and Jonsson, 1996; Cécé et al., 1996)



- apply losing rewriting rules to channel contents:

$$m \rightarrow_* \varepsilon \quad m \in M$$

- modeling imperfect communications, e.g. packet dropping policies against congestions
- decidable: $\mathbf{F}_{\omega^\omega}$-complete (Chambart and Schnoebelen, 2008)

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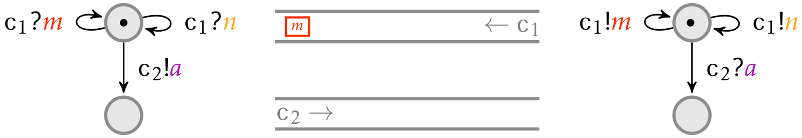

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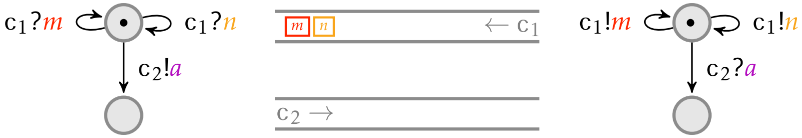

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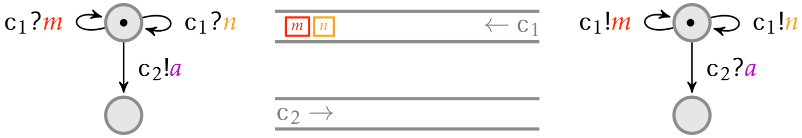

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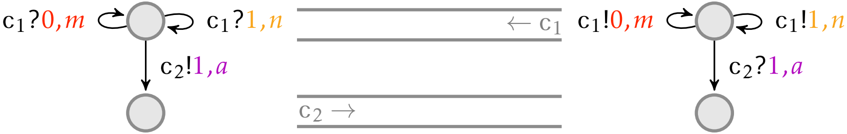

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$$(a,m)(b,n) \to_{\#} (b,n) \quad a \leqslant b \in \mathbb{N},\, m,n \in M$$

- modeling communications with QoS, e.g. differentiated services (RFC2475)

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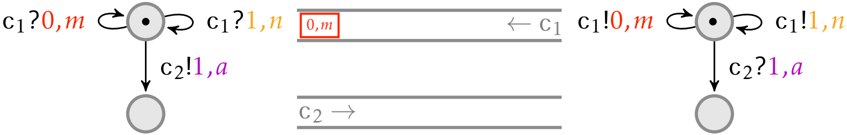

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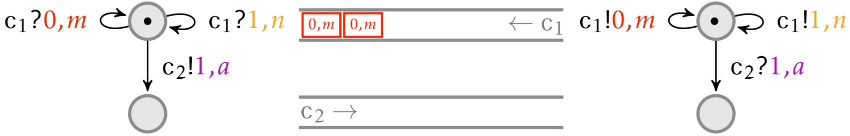

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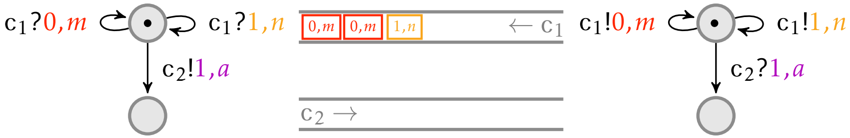

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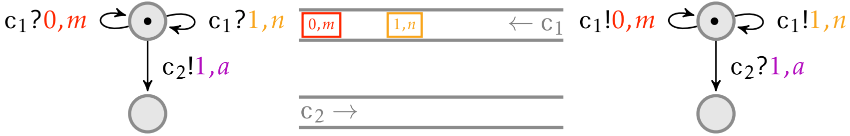

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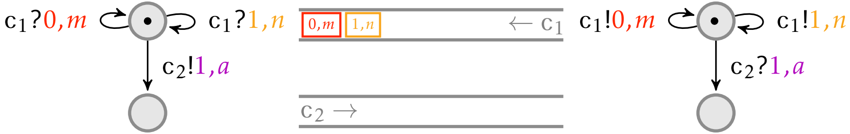

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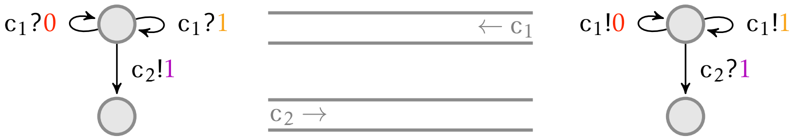

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Remark: Alternative Models

strict superseding Turing-powerful

ordered channels with rules

$$a\,b \rightarrow_s b \quad a < b \in \mathbb{N}$$

overtaking Turing-powerful

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Losing as an Embedding

- losing rules define a quasi-ordering $\xleftarrow{*}_{*}$ over M^*
- can be restated as substring embedding:

$$x \sqsubseteq_* y \overset{\text{def}}{\Leftrightarrow} x = m_1 \cdots m_\ell, y = z_1 m_1 z_2 \cdots z_\ell m_\ell z_{\ell+1}, \forall i. z_i \in M^*$$

- examples:
  - $201 \sqsubseteq_* 22011$
  - $120 \sqsubseteq_* 10210$
  - $\forall y \in M^*. \varepsilon \sqsubseteq_* y$

SUPERSEDING AS AN EMBEDDING

- if $d \in \mathbb{N}$, write $\Sigma_d \stackrel{\text{def}}{=} \{0, \ldots, d\}$
- superseding rules define a quasi-ordering $\stackrel{*}{\leftarrow}_{\#}$ over Σ_d^*
- can be restated as priority embedding:

$$x \sqsubseteq_{\mathrm{p}} y \stackrel{\text{def}}{\Leftrightarrow} x = a_1 \cdots a_\ell, y = z_1 a_1 z_2 \cdots z_\ell a_\ell, \forall i. z_i \in \Sigma_{a_i}^*$$

- examples:
  - $201 \sqsubseteq_{\mathrm{p}} 22011$
  - $120 \not\sqsubseteq_{\mathrm{p}} 10210$
  - $\varepsilon \sqsubseteq_{\mathrm{p}} y$ iff $y = \varepsilon$

Priority Embedding is Well

c.f. related orderings of Schütte and Simpson (1985)

Definition (wqo)

A quasi-order $(A, \leqslant_A)$ is well $\overset{\text{def}}{\Leftrightarrow}$ in any infinite sequence $x_0, x_1, \ldots$ over A, there exist $i < j$ s.t. $x_i \leqslant_A x_j$.

Theorem

$(\Sigma_d^*, \sqsubseteq_p)$ *is a wqo.*

- proof by induction over d
- nested applications of Higman's Lemma

PCSs are Well-Structured

(Abdulla et al., 2000; Finkel and Schnoebelen, 2001)

For a PCS with state set Q and m channels:

transition system $(Q \times (\Sigma_d^*)^m, \rightarrow)$ with superseding steps or perfect steps

wqo $(Q \times (\Sigma_d^*)^m, \sqsubseteq_\mathrm{p})$ by Dickson's Lemma

monotonicity $\forall (p,\bar{x}),(q,\bar{x}'),(p,\bar{y}) \in Q \times (\Sigma_d^*)^m$,
if $(p,\bar{x}) \rightarrow (q,\bar{x}')$ and $\bar{x} \sqsubseteq_\mathrm{p} \bar{y}$,
then $\exists \bar{y}' \in (\Sigma_d^*)^m, \bar{y} \sqsubseteq_\mathrm{p} \bar{y}$ and $(p,\bar{y}) \rightarrow (q,\bar{y}')$.

Generic Algorithms

for Reachability, Inevitability, Simulation w. a finite-state system, etc.

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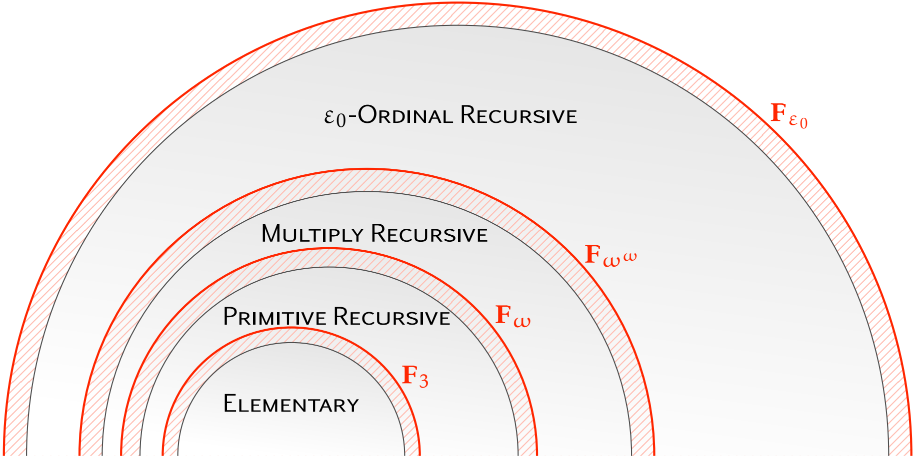
Fast-Growing Complexity Classes

(Schmitz and Schnoebelen, 2012)

Ordinal-indexed complexity hierarchy inside R:



Complexity of PCS Problems

Theorem

Reachability and Termination in PCSs are $\mathbf{F}_{\varepsilon_0}$-complete.

upper bound using length function theorems for applications of Higman's Lemma (Schmitz and Schnoebelen, 2011)

lower bound reduction from acceptance of a Turing machine working in $H^{\varepsilon_0}(n)$ space

Lower Bound: Hardy Functions

fundamental sequences $(\lambda(x))_x$
for limit ordinals λ in $\varepsilon_0 + 1$: $\lambda(x) < \lambda$
with $\lim_{x\to\omega} \lambda(x) = \lambda$

Example

$$\omega(x) = x + 1\,,$$
$$\omega^{\omega\cdot 2}(x) = \omega^{\omega+x+1}\,,$$
$$(\varepsilon_0)(x) = \Omega_{x+1} \stackrel{\text{def}}{=} \left.\omega^{\omega^{\cdots^{\omega}}}\right\}x+1 \text{ stacked } \omega\text{'s}$$

Lower Bound: Hardy Functions

Hardy functions $(H^\alpha)_{\alpha \leqslant \varepsilon_0}$

$$H^0(x) \stackrel{\text{def}}{=} x\,, \quad H^{\alpha+1}(x) \stackrel{\text{def}}{=} H^\alpha(x+1)\,, \quad H^\lambda(x) \stackrel{\text{def}}{=} H^{\lambda(x)}(x)\,.$$

Example

$$H^n(x) = x + n\,,$$
$$H^\omega(x) = 2x + 1\,,$$
$$H^{\omega^2}(x) = 2^{x+1}(x+1) - 1\,,$$

H^{ω^3} non elementary,

H^{ω^ω} Ackermannian,

H^{ε_0} not provably total in Peano arithmetic

Lower Bound: Hardy Computations

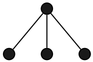
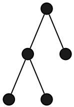
rewrite system over $(\varepsilon_0+1)\times\omega$:

$$\alpha+1, x \xrightarrow{H} \alpha, x+1$$
$$\lambda, x \xrightarrow{H} \lambda(x), x$$

computations $\alpha_0,x_0 \xrightarrow{H} \alpha_1,x_1 \xrightarrow{H} \cdots \xrightarrow{H} \alpha_n,x_n$

- preserve $H^{\alpha_i}(x_i)$
- in particular if $\alpha_n = 0$ then $x_n = H^{\alpha_0}(x_0)$

Lower Bound: Encoding Ordinals

$\alpha \in \Omega_{d+1}$	3	$\omega^2 + 1$
$t(\alpha) \in T_{d+1}$		
$s_d(\alpha) \in \Sigma_d^*$	222	1122

Proposition (Robustness)

If $s_d(\alpha) \sqsubseteq_p s_d(\beta)$, *then* $\forall x, H^\alpha(x) \leqslant H^\beta(x)$.

Lower Bound: Weak Hardy Computations

Implement Hardy steps $\alpha, n \to \beta, m$ as a PCS:

- work on string encodings: $s_d(\alpha), n \xrightarrow{H}_{\#} s_d(\beta'), m'$
- weak: $s_d(\beta') \sqsubseteq_{\mathrm{p}} s_d(\beta)$ and $m' \leqslant m$, but the perfect behaviour is possible
- also for inverse steps $s_d(\beta), m \xrightarrow{H^{-1}}_{\#} s_d(\alpha), n'$ with $s_d(\alpha') \sqsubseteq_{\mathrm{p}} s_d(\alpha)$ and $n' \leqslant n$

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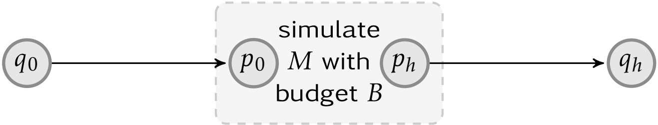
Lower Bound: Wrapping Up

reduction from a Turing machine M working in space $H^{\varepsilon_0}(n) = H^{\Omega_d}(n)$ for $d = n+1$.



$\Omega_d, n \xrightarrow{H}_{\#} \cdots \xrightarrow{H}_{\#} 0, B \qquad 0, B' \xrightarrow{H^{-1}}_{\#} \cdots \xrightarrow{H^{-1}}_{\#} \alpha, n'$

- robustness: $H^{\Omega_d}(n) \geqslant B \geqslant B' \geqslant H^{\alpha}(n')$
- coverability: $\alpha = \Omega_d \wedge n = n'$
- implies perfect simulation

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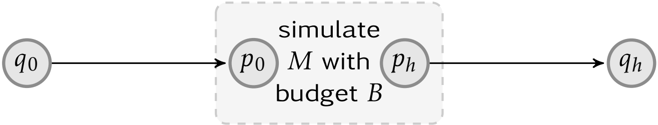

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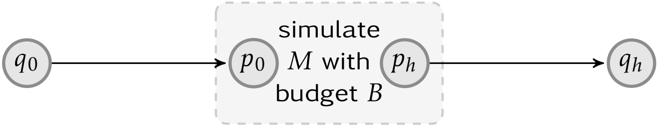

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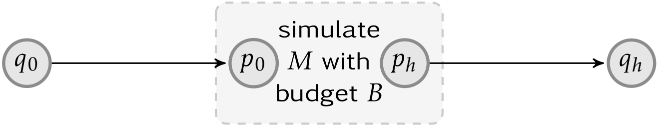

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Concluding Remarks

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ordering priority embedding

Perspectives

verifying PCSs regular model checking and acceleration

using PCSs reducing problems about other models (e.g. manipulating bounded depth trees and graphs)

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Fundamental Sequences

Definition (Fundamental Sequences)

For limit ordinals in $\varepsilon_0 + 1$:

$$
\begin{aligned}
(\gamma + \omega^{\beta+1})(x) &\stackrel{\text{def}}{=} \gamma + \omega^{\beta} \cdot (x+1) \\
(\gamma + \omega^{\lambda})(x) &\stackrel{\text{def}}{=} \gamma + \omega^{\lambda(x)} \\
(\varepsilon_0)(x) &\stackrel{\text{def}}{=} \Omega_{x+1} \stackrel{\text{def}}{=} \omega^{\omega^{\cdots^{\omega}}} \Big\}\, x+1 \text{ stacked } \omega\text{'s}
\end{aligned}
$$

Encodings

$s_d\colon T_{d+1} \to \Sigma_d^*$ by induction on d:

$$s_d(\bullet(t_1 \cdots t_n)) \stackrel{\text{def}}{=} \begin{cases} \varepsilon & \text{if } n = 0, \\ s_{d-1}(t_1)d \cdots s_{d-1}(t_n)d & \text{otherwise.} \end{cases}$$

$s_d\colon \Omega_{d+1} \to \Sigma_d^*$ by induction on d:

$$s_d\left(\sum_{i=1}^{n} \gamma_i\right) \stackrel{\text{def}}{=} s_d(\gamma_1) \cdots s_d(\gamma_n)\,, \quad s_d(\omega^\alpha) \stackrel{\text{def}}{=} s_{d-1}(\alpha)\, d\,.$$