Ellipsoidal Techniques for Reachability Analysis of Discrete-Time Linear Systems

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***Abstract*—This paper describes the computation of reach sets for discrete-time linear control systems with time-varying coefficients and ellipsoidal bounds on the controls and initial conditions. The algorithms construct external and internal ellipsoidal approximations that touch the reach set boundary from outside and from inside. Recurrence relations describe the time evolution of these approximations. An essential part of the paper deals with singular discrete-time linear systems.**

***Index Terms*—Ellipsoidal methods, reach sets, regularization, singular discrete-time systems.**

I. Introduction

WE EXTEND the use of ellipsoidal methods for continuous-time linear systems developed in [1] to compute the reach set for discrete-time linear systems. We also deal with systems with a singular state transition matrix and show how to approximate the reach set with any given accuracy using regularization and provide the means for picking the appropriate regularization parameters that guarantee this accuracy over a specified finite time interval. We make no controllability assumptions regarding the system. Thus, the suggested regularization technique also extends results of [1] to uncontrollable systems.

In the case of singular state transition matrix, or if the system is not controllable at every time step, the exact representation of the reach set on every time step by external ellipsoidal approximation is not possible. However, the exact representation is possible for the regularized reach set that overapproximates the actual one, and whose boundary can be made arbitrarily close to the boundary of the actual one over any given finite time interval. Using internal ellipsoidal approximations, on the other hand, it is always possible to exactly represent the actual reach set. However, the concept of "good curves" that was introduced in [1] and without which the computation process becomes heavy and complicated, can only be applied for systems with nonsingular state transition matrix. Therefore, in order to make use of 'good curves' in the singular case, we perform the internal approximation of the regularized reach set.

The paper is organized as follows. Section II reviews related work on reachability. In Section III, basic definitions and results from [1] are translated to the discrete-time case. In Section IV, the recurrence relation for the external approximating ellipsoids is derived. Here it is assumed that the system has no singularities and is controllable in one time step. In Sections V and VI, we consider systems with no controllability assumptions and systems with singular state transition matrix. The regularization technique is proposed that guarantees the overapproximation of the reach set of the original system with given ε-accuracy. Section VII contains a simple example whose purpose is to illustrate how regularization of a singular discrete-time linear system works. In Section VIII, the recurrence relation for the internal ellipsoidal approximation is derived. For systems with singular state transition matrix, they require regularization. Section IX collects some conclusions.

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II. Related Work

A common step in methods for verification and controller synthesis is the calculation of reach sets. Consider the discrete-time linear system with time-varying coefficients

$$x[k+1] = A[k]x[k] + B[k]u[k], \qquad k \geq k_0 \tag{1}$$
$$x[k_0] = x^0 \tag{2}$$

in which $x \in \mathbf{R}^n$ is the state and $u \in \mathbf{R}^m$ is the control. The state transition matrix is

$$\Phi(k+1, k_0) = A[k]\Phi(k, k_0), \qquad k \geq k_0, \quad \Phi(k,k) = I$$

which for time-invariant systems simplifies as

$$\Phi(k, k_0) = A^{k-k_0}.$$

The control $u = u[k]$ and the initial condition $x[k_0]$ are restricted to convex, compact sets: $u[k] \in \mathcal{P}[k], x[k_0] \in \mathcal{X}_0$.

Definition 2.1: The reach set $\mathcal{X}(k, k_0, x^0)$ at time $k > k_0$ from the initial position (k_0, x^0) is the set of all states $x[k]$ reachable at time k by the system (1) with $x[k_0] = x^0$ through all possible controls.

The reach set $\mathcal{X}(k, k_0, \mathcal{X}_0)$ is

$$\mathcal{X}(k, k_0, \mathcal{X}_0) = \bigcup\{\mathcal{X}(k, k_0, x^0) \mid x^0 \in \mathcal{X}_0)\}.$$

Some important facts about reach sets follow.

Lemma 2.1: The set-valued map $\mathcal{X}(k, k_0, \mathcal{X}_0)$ satisfies the semigroup property

$$\mathcal{X}(k, k_0, \mathcal{X}_0) = \mathcal{X}(k, i, \mathcal{X}(i, k_0, \mathcal{X}_0)), \qquad k_0 \leq i \leq k. \tag{3}$$

The reach set $\mathcal{X}(k,k_0,\mathcal{X}_0)$ can be expressed directly as

$$\mathcal{X}(k,k_0,\mathcal{X}_0) = \Phi(k,k_0)\mathcal{X}_0 + \sum_{i=k_0}^{k-1}\Phi(k,i+1)B[i]\mathcal{P}[i], \qquad k \geq k_0 \quad (4)$$

or recursively as

$$\begin{aligned}&\mathcal{X}(k+1,k_0,\mathcal{X}_0)\\ &\quad = A[k]\mathcal{X}(k,k_0,\mathcal{X}_0) + B[k]\mathcal{P}[k], \qquad k \geq k_0 \quad (5)\\ &\quad \mathcal{X}(k_0,k_0,\mathcal{X}_0) = \mathcal{X}_0.\end{aligned}$$

Operation "$+$" in (4) and (5) is the geometric or Minkowski sum: for sets V and $W, V+W = \{v+w \mid v \in V, w \in W\}$.

Reachability analysis is concerned with the computation of the reach set $\mathcal{X}(k,k_0,\mathcal{X}_0)$ in a way that can effectively meet requests like the following.

1) Graphically display the projection of the reach set along any specified two-dimensional subspace.
2) For a given convex target set M and time K, determine whether for some $k \leq K, \mathcal{X}(k,k_0,\mathcal{X}_0)$ intersects M.
3) For specified $x^K \in \mathcal{X}(K,k_0,\mathcal{X}_0)$, find an initial condition $x^0 \in \mathcal{X}_0$ and control $u = u[k] \in \mathcal{P}[k], k_0 \leq k < K$, that steers the state from x^0 to x^K.

Because the set of initial condition $\mathcal{X}_0$ and the control set $\mathcal{P}[k]$ are convex, so is the reach set $\mathcal{X}(k,k_0,\mathcal{X}_0)$. Hence, reachability analysis reduces to effective manipulation with convex sets, performing set-valued operations such as unions, intersections, geometric sums and differences. Two basic objects are used as convex approximations: various kinds of (convex) polytopes, e.g., general polytopes, zonotopes, parallelotopes, rectangular polytopes; and ellipsoids. Polytopes can give arbitrarily close approximations to any convex set, but the number of vertices can grow prohibitively large and, as shown in [2], the computation of a polytope by its convex hull becomes intractable for large number of vertices in high dimensions.

Reachability analysis for general polytopes is implemented in the Multi-Parametric Toolbox (MPT) for Matlab [3], [4]. The reach set at every time step is computed as the geometric sum of two polytopes (5). The procedure consists in finding the vertices of the resulting polytope and calculating its convex hull. The convex hull algorithm employed by MPT is based on the double description method [5] and implemented in the CDD/CDD+ package [6]. Its complexity is V^n, where V is the number of vertices and n is the state space dimension. Hence, the use of MPT is practical for low dimensional systems. However, even in low dimensional systems the number of vertices in the reach set polytope can grow very large with the number of time steps. For example, consider the system

$$x_{k+1} = Ax_k + u_k$$

with $A = \begin{bmatrix}\cos 1 & -\sin 1\\ \sin 1 & \cos 1\end{bmatrix}, u_k \in \{u \in \mathbf{R}^2 \mid \|u\|_\infty \leq 1\}$, and $x_0 \in \{x \in \mathbf{R}^2 \mid \|x\|_\infty \leq 1\}$. Starting with a rectangular initial set, the number of vertices of the reach set polytope is $4k+4$ at the kth step.

The method of zonotopes for external approximation of reach sets is presented in [7]. A zonotope is a special class of polytopes (see [8]) of the form

$$Z = \left\{ x \in \mathbf{R}^n \;\middle|\; x = c + \sum_{i=1}^{p}\alpha_i g_i, -1 \leq \alpha_i \leq 1 \right\}$$

wherein c and $g_1,\ldots,g_p$ are vectors in $\mathbf{R}^n$. Thus, a zonotope Z is compactly represented by its center c and generator vectors $g_1,\ldots,g_p$. The value p/n is called the order of the zonotope. The difficulty is that with every time step the order of the approximating zonotope increases by p/n. This difficulty can be averted by limiting the number of generator vectors, and overapproximating zonotopes whose number of generator vectors exceeds this limit by lower order zonotopes. The zonotope method of [7] provides neither a notion of tightness of approximation nor an estimate of the error resulting from the zonotope order reduction. Further, the method does not indicate any way of control synthesis, i.e., finding a sequence of controls that drives the system to the desired terminal set or point.

CheckMate [9] is a Matlab toolbox that can evaluate specifications for trajectories starting from the set of initial (continuous) states corresponding to the parameter values at the vertices of the parameter set. This provides preliminary insight into whether the specifications will be true for all parameter values. The method of oriented rectangluar polytopes for external approximation of reach sets is introduced in [10]. The basic idea is to construct an oriented rectangular hull of the reach set for every time step, whose orientation is determined by the singular value decomposition of the sample covariance matrix for the states reachable from the vertices of the initial polytope. The limitation of CheckMate and the method of oriented rectangles is that only autonomous (i.e., uncontrolled) systems are allowed, and only an external approximation of the reach set is provided.

Requiem [11] is a *Mathematica* notebook which, given a linear system, the set of initial conditions and control bounds, symbolically computes the exact reach set, using the experimental quantifier elimination package. Quantifier elimination is the removal of all quantifiers (the universal quantifier $\forall$ and the existential quantifier $\exists$) from a quantified system. Each quantified formula is substituted with quantifier-free expression with operations $+, \times, =$, and $<$. For example, consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

with $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}0\\ 1\end{bmatrix}$. For initial conditions $x_0 \in \{x \in \mathbf{R}^2 \mid \|x\|_\infty \leq 1\}$ and controls $u_k \in \{u \in \mathbf{R} \mid -1 \leq u \leq 1\}$, the reach set for $k \geq 0$ is given by the quantified formula

$$\left\{ x \in \mathbf{R}^2 \mid \exists x_0, \exists k \geq 0, \exists u_i, 0 \leq i \leq k : \quad x = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1}Bu_i \right\}$$

which is equivalent to the quantifier-free expression

$$-1 \leq [1 \quad 0]x \leq 1 \wedge -1 \leq [0 \quad 1]x \leq 1.$$

It is proved in [12] that if A is constant and nilpotent or is diagonalizable with rational real or purely imaginary eigenvalues, the quantifier elimination package returns a quantifier free formula describing the reach set.

The reach set approximation via parallelotopes [13] employs the idea of parametrization described in [1] for ellipsoids. The reach set is represented as the intersection of tight external, and the union of tight internal, parallelotopes. The evolution equations for the centers and orientation matrices of both external and internal parallelotopes are provided. This method also finds controls that can drive the system to the boundary points of the reach set, similarly to [14] and [1]. It works for general linear systems. The computation to solve the evolution equations for tight approximating parallelotopes, however, is more involved than the one for ellipsoids, and in the case of discrete-time system this method does not deal with singular state transition matrices.

We note that the level set method [15], [16] deals with general nonlinear controlled systems and gives exact representation of their reach sets, but requires solving the HJB equation and finding the set of states that belong to sub-zero level set of the value function. The method [16] is impractical for systems of dimension higher than three.

In this paper, we describe ellipsoidal techniques that extend the results of [1]. The ellipsoidal calculus provides the following benefits.

- Approximating the reach set of an n-dimensional system by L ellipsoids over k time steps requires $k[L(8n^3+4n^2+2n)+2n^2]$ scalar multiplications.
- It is possible to exactly represent the reach set of linear system through both external and internal ellipsoids.
- It is possible to single out individual external and internal approximating ellipsoids that are optimal to some given criterion (e.g., trace, volume, diameter), or combination of such criteria.
- We obtain simple analytical expressions for control sequences that steer the state to a desired target.
- The Ellipsoidal Toolbox [17] implements the reach set computations described here.

III. Reachability Problem

The control $u = u[k]$ is restricted to the nondegenerate ellipsoid

$$\begin{aligned} u[k] \in \mathcal{P}[k] &= \mathcal{E}(p[k], P[k]) \\ &= \{u[k] \,|\, \langle (u[k]-p[k]), P^{-1}[k](u[k]-p[k])\rangle \le 1\} \end{aligned} \tag{6}$$

with $p[k] \in \mathbf{R}^m$ being the center of the ellipsoid and $P[k] \in \mathbf{R}^{m\times m}$ its shape matrix. The initial condition is restricted to the nondegenerate ellipsoid $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$. The support function of the ellipsoid $\mathcal{E}(p[k], P[k])$ is

$$\begin{aligned} &\rho(l \,|\, \mathcal{E}(p[k], P[k])) \\ &\quad = \max\{\langle l, x\rangle \,|\, x \in \mathcal{E}(p[k], P[k])\} \\ &\quad = \langle l, p[k]\rangle + \langle l, P[k]l\rangle^{1/2}. \end{aligned} \tag{7}$$

Lemma 3.1: The reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ can be expressed as

$$\begin{aligned} \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0)) &= q[k] + \Phi(k, k_0)\mathcal{E}(0, X_0) \\ &\quad + \sum_{i=k_0}^{k-1} \Phi(k, i+1)\mathcal{E}(0, B[i]P[i]B^T[i]) \end{aligned} \tag{8}$$

in which

$$q[k] = \Phi(k, k_0)x_0 + \sum_{i=k_0}^{k-1} \Phi(k, i+1)B[i]p[i]. \tag{9}$$

Using (8) and (7), we obtain the next result.

Lemma 3.2: The support function of the reach set is

$$\begin{aligned} &\rho(l \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))) \\ &\quad = \langle l, q[k]\rangle + \langle l, \Phi(k, k_0)X_0\Phi^T(k, k_0)l\rangle^{1/2} \\ &\qquad + \sum_{i=k_0}^{k-1} \langle l \\ &\qquad \Phi(k, i+1)B[i]P[i]B^T[i]\Phi^T(k, i+1)l\rangle^{1/2}. \end{aligned} \tag{10}$$

Representation (8) leads to the next fact.

Lemma 3.3: The reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ is a convex compact set in $\mathbf{R}^n$.

If the matrices $A[i], k_0 \le i < k$, are nonsingular, the ellipsoid $\Phi(k, k_0)\mathcal{E}(0, X_0)$ in sum (8) is nondegenerate, and so the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ contains an open set of $\mathbf{R}^n$. Its boundary points have an important characterization: y is a boundary point of $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ only if there exists a support vector $l_y \ne 0$ such that

$$\begin{aligned} \langle l_y, y\rangle &= \rho(l_y \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))) \\ &= \max\{\langle l_y, x\rangle \,|\, x \in \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))\}. \end{aligned} \tag{11}$$

The control $u_y[i], k_0 \le i < k$, and the initial state $x[k_0] = x_{0y} \in \mathcal{E}(x_0, X_0)$, which drive the system (1) from the state $x[k_0] = x_{0y}$ to a boundary point $x[k] = y$ is determined from the following theorem.

Theorem 3.1: Suppose $A[i], k_0 \le i < k$, are nonsingular, and $x[k] = y$ is a boundary point of $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$. Then the control $u_y[i], k_0 \le i < k$, and the initial state $x[k_0] = x_{0y}$, which yield the unique trajectory $x_y[i]$ that reaches $y = x_y[k]$ from $x_{0y} = x[k_0]$, and ensures (11), satisfy the following maximum principle for control:

$$\begin{aligned} &\langle l_y, \Phi(k, i+1)B[i]u_y[i]\rangle \\ &\quad = \max\{\langle l_y, \Phi(k, i+1)B[i]u\rangle \,| \\ &\qquad u \in \Phi(k, i+1)\mathcal{E}(p[i], P[i])\} \\ &\quad = \langle l_y, \Phi(k, i+1)B[i]p[i]\rangle \\ &\qquad + \langle l_y, \Phi(k, i+1)B[i]P[i]B^T[i]\Phi^T \\ &\qquad \times (k, i+1)l_y\rangle^{1/2} \end{aligned} \tag{12}$$

for $k_0 \le i < k$, and the maximum condition for the initial state

$$\begin{aligned} &\langle l_y, \Phi(k, k_0)x_{0y}\rangle \\ &\quad = \max\{\langle l_y, x\rangle \,|\, x \in \Phi(k, k_0)\mathcal{E}(x_0, X_0)\} \\ &\quad = \langle l_y, \Phi(k, k_0)x_0\rangle \\ &\qquad + \langle l_y, \Phi(k, k_0)X_0\Phi^T(k, k_0)l_y\rangle^{1/2}. \end{aligned} \tag{13}$$

Here, l_y is the support vector for the set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ at the point y that satisfies (11).

The proof of this theorem is basically contained in the expressions (12) and (13) themselves.

Remark: To simplify the notation, we will denote $R[i] = B[i]P[i]B^T[i]$ and from now on use $R[i]$ instead of $B[i]P[i]B^T[i]$ where possible.

IV. External Ellipsoidal Approximations

Relation (8) shows that the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ is a sum of $k - k_0 + 1$ ellipsoids. We overapproximate the reach set by a single ellipsoid. Here, we consider the situation when matrices $A[i]$ are nonsingular. The case with singular $A[i]$ is treated in Section VI. In the beginning, we will also assume that matrices $R[i]$ are nonsingular.

Remark: $R[i]$ are nonsingular if the matrices $B[i] \in \mathbf{R}^{n\times m}$ have rank $n\,(n \leq m)$.

We will look for external approximating ellipsoids for the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ in the class of ellipsoids $\mathcal{E}(q[k], Q[k])$ that satisfy the following two relations:

$$q[k] = A[k-1]q[k-1] + B[k-1]p[k-1] \qquad (14)$$

with $q[k_0] = x_0$, and

$$\begin{aligned} Q[k] = &\left(\sum_{i=k_0}^{k-1} s_i[k] + s_0[k]\right) \\ &\times \left(\sum_{i=k_0}^{k-1} s_i^{-1}[k]\Phi(k, i+1)R[i]\Phi^T(k, i+1)\right. \\ &\left. + s_0^{-1}[k]\Phi(k, k_0)X_0\Phi^T(k, k_0)\right) \end{aligned} \qquad (15)$$

with $Q[k_0] = X_0$, for arbitrary $s_0[k] > 0$, $s_i[k] > 0$ for $k > k_0$ and $k_0 \leq i < k$.

Using (10), it can be easily checked that

$$\rho(l \,|\, \mathcal{E}(q[k], Q[k])) \geq \rho(l \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0)))$$

for all directions l. Hence

$$\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0)) \subseteq \mathcal{E}(q[k], Q[k]).$$

Definition 4.1: We say that ellipsoid $\mathcal{E}(q[k], Q[k])$, defined by (14) and (15), is a tight external approximation of the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ if there exists direction l such that

$$\rho(\pm l \,|\, \mathcal{E}(q[k], Q[k])) = \rho(\pm l \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))).$$

A tight external ellipsoid defined by direction l, touches the boundary of the reach set at point x_l such that

$$\begin{aligned} \rho(l \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))) &= \langle l, x_l\rangle \\ &= \rho(l \,|\, \mathcal{E}(q[k], Q[k])) = \langle l, q[k]\rangle + \langle l, Q[k]l\rangle^{1/2}. \end{aligned}$$

This condition will be fulfilled if we choose

$$s_i[k] = \langle l, \Phi(k, i+1)R[i]\Phi^T(k, i+1)l\rangle^{1/2} \qquad (16)$$

and

$$s_0[k] = \langle l, \Phi(k, k_0)X_0\Phi^T(k, k_0)l\rangle^{1/2}. \qquad (17)$$

To check it, we substitute (16) and (17) into (15), calculate the support function of the resulting ellipsoid and arrive at (10). As we can see, for every direction l there exists a tight external ellipsoidal approximation of the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$, whose shape matrix is calculated by (15).

For all $0 \leq i < k$, the parameters $s_i[k]$ in (17) and (16) depend on k. As a result, the sum (15) needs to be recomputed for every k with new values of $s_0[k]$ and $s_i[k]$. To get rid of this dependency on k, and thus greatly reduce the computation effort, we introduce the notion of "good curves." These are trajectories $l[k]$ that satisfy the homogeneous adjoint equation

$$l[k] = A^T[k]l[k+1] \quad l(k_0) = l_0. \qquad (18)$$

Vector $l[k]$ can be expressed as

$$l[k] = \Phi^T(k_0, k)l_0. \qquad (19)$$

With $l[k]$ defined in (19), relations (16) and (17) simplify into

$$s_i = \langle l_0, \Phi(k_0, i+1)R[i]\Phi(k_0, i+1)l_0\rangle^{1/2} \qquad (20)$$

and

$$s_0 = \langle l_0, X_0 l_0\rangle^{1/2}. \qquad (21)$$

Now, s_i and s_0 no longer depend on k. When substituted into (15), they give us a recurrence relation for matrix $Q[k]$

$$\begin{aligned} Q[k+1] = (1 + \pi[k])A[k]Q[k]A^T[k]& \\ +(1 + \pi^{-1}[k])R[k]& \end{aligned} \qquad (22)$$

with $Q[k_0] = X_0$, in which $\pi[k]$ is defined by

$$\begin{aligned} \pi[k] &= \frac{s_k}{\sum_{i=k_0}^{k-1} s_i + s_0} \\ &= \frac{\langle l_0, \Phi(k_0, k+1)R[k]\Phi^T(k_0, k+1)l_0\rangle^{1/2}}{\langle l_0, \Phi(k_0, k)Q[k]\Phi^T(k_0, k)l_0\rangle^{1/2}}. \end{aligned} \qquad (23)$$

The expression for the center of the approximating ellipsoid is given by

$$q[k+1] = A[k]q[k] + B[k]p[k] \qquad (24)$$

with $q[k_0] = x_0$. In both (22) and (24) $k \geq k_0$. We state this result as a theorem.

Theorem 4.1: Let $l[k]$ satisfy the (19). Then the ellipsoid $\mathcal{E}(q[k], Q[k])$ given by (22) and (24), is such that for all $k \geq k_0$ the reach set

$$\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0)) \subseteq \mathcal{E}(q[k], Q[k])$$

and

$$\rho(l[k] \,|\, \mathcal{E}(q[k], Q[k])) = \rho(l[k] \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))).$$

The supporting hyperplane for the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ generated by the vector $l[k]$ is also a supporting hyperplane for $\mathcal{E}(q[k], Q[k])$ and touches it at the same point $x_l[k]$, given by

$$x_l[k] = q[k] + \frac{Q[k]l[k]}{\langle l[k], Q[k]l[k]\rangle^{1/2}}. \tag{25}$$

Since for every boundary point $x_l[k]$ of the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ there exists direction $l[k]$ and corresponding tight external ellipsoid $\mathcal{E}(q[k], Q[k])$, the following result is established.

Theorem 4.2: The reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ can be expressed as

$$\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0)) = \bigcap_{l[k]} \mathcal{E}(q[k], Q[k]). \tag{26}$$

This theorem says that the exact reach set of the system can be expressed as an intersection of infinite number of external approximating ellipsoids. In practice, we only deal with finite number of ellipsoids, each of which is defined by parameter $l[k]$ that is related to l_0 by (19). The greater the number of different values of l_0, for which the approximating ellipsoids are computed, the better is the approximation

$$\bigcap_{l[k]} \mathcal{E}(q[k], Q[k]).$$

The choice of l_0 is arbitrary. It can be random, or such that vectors $l[k]$, for given k, satisfy certain criteria - for instance, are orthogonal to each other. The question about how the accuracy of approximation depends on the number of different values of l_0 and the choice of these values, remains open.

Remark: For n-dimensional system, every step of computation of external ellipsoidal approximation requires $2n^2$ scalar multiplications for the center of ellipsoid and $(8n^3 + 4n^2 + 2n)$ scalar multiplications for its shape matrix. Since the trajectory of the center is the same for different values of parameter l_0, computing L ellipsoidal approximations for k time steps requires

$$k(L(8n^3 + 4n^2 + 2n) + 2n^2)$$

scalar multiplications. So, the complexity of computation grows linearly with the number of time steps and number of approximations, and polynomially, as $O(n^3)$, with the state space dimension. Ellipsoidal approximations for each of the values of l_0 are independent of each other and can be computed in parallel. The number of these approximations, L, is fixed and does not depend on k. Reach set computation via polytopes, on the other hand, may encounter the increase of the number of vertices with each time step, which results in the fact that complexity of one time step computation grows with k. Zonotope method, although more effective than general polytope algorithm, still has this problem when the representation of zonotope describing the reach set grows with the number of time steps.

If $l[k]$ is a "good curve," it specifies the external ellipsoid $\mathcal{E}(q[k], Q[k])$, with $Q[k]$ and $q[k]$ coming from (22) and (24), which touches the reach set at some point $x_l[k]$. In order for the points $x_l[\,\cdot\,]$ to form the system trajectory on $k_0, \ldots, k$, that is, for $x_l[k]$ to satisfy

$$x_l[k] = \Phi(k, k_0)x^0 + \sum_{i=k_0}^{k-1} \Phi(k, i+1)B[i]u[i]$$

the corresponding control $u[i] \in \mathcal{E}(p[i], P[i])$ and the initial condition $x^0 \in \mathcal{E}(x_0, X_0)$ are derived from the maximum principle for control (12) and maximum condition (13). They are specified by

$$u[i] = p[i] + \frac{P[i]B^T[i]\Phi^T(k_0, i+1)l_0}{\langle l_0, \Phi(k_0, i+1)R[i]\Phi^T(k_0, i+1)l_0\rangle^{1/2}} \tag{27}$$

and

$$x_l^0 = x_0 + \frac{X_0 l_0}{\langle l_0, X_0 l_0\rangle^{1/2}}. \tag{28}$$

Thus, if the system starts at x_l^0 at time step k_0, the control $u[i]$ will bring it at time step k to the boundary point of the reach set $x_l[k]$, at which the reach set is touched by the external approximating ellipsoid $\mathcal{E}(q[k], Q[k])$ defined by the direction $l[k]$.

By scaling the shape matrices of controls $P[i]$ and the initial conditions X_0 in (27) and (28) with some coefficient $\nu, 0 \le \nu \le 1$, the system can be driven to an internal point of the reach set that belongs to the line segment connecting $x_l[k]$ and $q[k] : x = \nu x_l[k] + (1-\nu)q[k]$. For L ellipsoidal approximations, we know L boundary points of the reach set at time step k, $\{x_l[k]\}_j^L$. Any point that belongs to the convex hull of $q[k]$ and $\{x_l[k]\}_j^L$ can be reached at time step k by taking the corresponding convex combinations of controls (27) and the initial conditions (28).

V. Singular R

We start by stating a simple fact that will be useful in our proofs.

Lemma 5.1: Let $a \ge b > 0$. Then, from $a^2 - b^2 < \varepsilon^2$ for some $\varepsilon > 0$, it follows that $a - b < \varepsilon$.

In case the matrix $R[i]$ is singular for some i, the recurrence relations (22) are not valid because we cannot guarantee that parameter s_i defined in (20) is strictly positive. Thus, we need to replace $R[i]$ with a suitable nonsingular matrix. Define $R_\alpha[i]$

$$R_\alpha[i] = R[i] + \alpha^2 I, \qquad k_0 \le i < k \tag{29}$$

for $\alpha > 0$. $R_\alpha[i]$ is positive definite and $R_\alpha[i] \to R[i]$ as $\alpha \to 0$. Corresponding to $R_\alpha[i]$ we define the "regularized" reach set $\mathcal{X}_\alpha(k, k_0, \mathcal{E}(x_0, X_0))$

$$\mathcal{X}_\alpha(k, k_0, \mathcal{E}(x_0, X_0)) = q[k] + \Phi(k, k_0)\mathcal{E}(0, X_0) + \sum_{i=k_0}^{k-1} \Phi(k, i+1)\mathcal{E}(0, R_\alpha[i]) \tag{30}$$

with support function

$$\begin{aligned}\rho(l \mid \mathcal{X}_\alpha(k, k_0, \mathcal{E}(x_0, X_0))) &= \langle l, q[k]\rangle + \langle l, \Phi(k, k_0)X_0\Phi^T(k, k_0)l\rangle^{1/2} \\ &\quad + \sum_{i=k_0}^{k-1} \langle l, \Phi(k, i+1)R_\alpha[i]\Phi^T(k, i+1)l\rangle^{1/2}\end{aligned} \tag{31}$$

with $q[k]$ determined from (9). The difference between the support functions of the regularized reach set $\mathcal{X}_\alpha(k,k_0,\mathcal{E}(x_0,X_0))$ and the actual reach set $\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0))$ is

$$\begin{aligned}\rho(l\,|\,&\mathcal{X}_\alpha(k,k_0,\mathcal{E}(x_0,X_0)))\\ &-\rho(l\,|\,\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0)))\\ &=\sum_{i=k_0}^{k-1}(\langle l,\Phi(k,i+1)R_\alpha[i]\Phi^T(k,i+1)l\rangle^{1/2}\\ &\quad-\langle l,\Phi(k,i+1)R[i]\Phi^T(k,i+1)l\rangle^{1/2}).\end{aligned}$$

Let us estimate the ith element of this sum,

$$\begin{aligned}\langle l,\Phi(k,i+1)&R_\alpha[i]\Phi^T(k,i+1)l\rangle^{1/2}\\ &-\langle l,\Phi(k,i+1)R[i]\Phi^T(k,i+1)l\rangle^{1/2}\end{aligned}$$

for $k_0\le i<k$. Let $\overline{\sigma}(A[i])$ be the largest singular value of matrix $A[i]$, and denote

$$\sigma_{\max}=\max_{k_0\le i<k}\{\overline{\sigma}(A[i])\}. \tag{32}$$

Then, we can write

$$\begin{aligned}\langle l,\Phi(k,i+1)&R_\alpha[i]\Phi^T(k,i+1)l\rangle\\ &-\langle l,\Phi(k,i+1)R[i]\Phi^T(k,i+1)l\rangle\\ &=\alpha^2\langle l,\Phi(k,i+1)\Phi^T(k,i+1)l\rangle\\ &\le\sigma_{\max}^{2(k-i-1)}\alpha^2.\end{aligned}$$

Thus, by Lemma 5.1, we obtain

$$\begin{aligned}0\le\langle l,\Phi(k,i+1)&R_\alpha[i]\Phi^T(k,i+1)l\rangle^{1/2}\\ &-\langle l,\Phi(k,i+1)R[i]\Phi^T(k,i+1)l\rangle^{1/2}\le\sigma_{\max}^{k-i-1}\alpha.\end{aligned}$$

We now state the main result of this section.

Theorem 5.1: For any $\varepsilon>0$ and any $k\ge k_0$ there exists $\alpha>0$, such that

$$\begin{aligned}0\le\rho(l\,|\,&\mathcal{X}_\alpha(k,k_0,\mathcal{E}(x_0,X_0)))\\ &-\rho(l\,|\,\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0)))<\varepsilon\end{aligned}$$

for all vectors l, $\langle l,l\rangle=1$. More precisely, α can be chosen as

$$\alpha=\frac{\varepsilon}{2\left(\sigma_{\max}^{k-k_0-1}+\cdots+\sigma_{\max}^2+\sigma_{\max}+1\right)} \tag{33}$$

where $\sigma_{\max}$ is defined in (32).

Remark: In (32) we could have taken i strictly greater than k_0. We leave this definition as is, however, because it will be used in the next section in its present form.

Substituting R with R_α in (20) and (23), we obtain recurrence relations for $Q_\alpha[k]$, the shape matrix of the tight external ellipsoid of the regularized reach set $\mathcal{X}_\alpha(k,k_0,\mathcal{E}(x_0,X_0))$. The recurrence relation for the center of this ellipsoid (24) remains untouched. So, the relation

$$\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0))=\bigcap_{l[k]}\mathcal{E}(q[k],Q_\alpha[k]) \tag{34}$$

holds with ε accuracy in the Hausdorff metric. Expression (27) for the control is similarly modified by replacing R with R_α.

Remark: In the continuous-time case described in [1], the use of "good curves" is allowed under the condition

$$\int_{t'}^{t''}\langle l,\Phi(t'',s)B(s)P(s)B^T(s)\Phi^T(t'',s)l\rangle^{1/2}ds>0$$

for all $l\neq 0$ and any $t',t'',t''>t'$, which holds if the system is completely controllable. Instead of the controllability assumption, the regularization

$$B(t)P(t)B^T(t)+\alpha^2I$$

can be used. Choosing

$$\alpha=\frac{\varepsilon}{(t-t_0)\max_{t_0\le s\le t}\|\Phi(t,t_0)\|_2}$$

for given $\varepsilon>0$ would ensure ε-accuracy in approximation of the original reach set by the regularized one in the time interval between t_0 and t.

VI. SINGULAR A

When $A[i]$ is singular for some $i,k_0\le i<k$, relation (19) is invalid because the inverse for the state transition matrix $\Phi(k,k_0)$ does not exist; hence, we cannot use the expression $\Phi(k_0,k)$; and the subsequent formulas (20) and (22) do not make sense for singular $A[i]$. The way around it is to regularize matrix $A[i]$. Let the singular value decomposition of $A[i]$ be

$$A[i]=U_i\Sigma_iV_i$$

and define the new matrix $A_\delta[i]$ by

$$A_\delta[i]=U_i(\Sigma_i+\delta I)V_i=A[i]+\delta U_iV_i \tag{35}$$

for $k_0\le i>k$, where $\delta>0$. As one can see, $A_\delta[i]$ is nonsingular for $\delta>0$, and $A_\delta[i]\to A[i]$ as $\delta\to 0$. Corresponding to the matrix $A_\delta[i]$ there is an invertible state transition matrix $\Phi_\delta(k,k_0)$.

Remark: Here, actually, δ may depend on i, and $\Phi_\delta(k,k_0)$ is

$$\Phi_\delta(k,k_0)=A_{\delta_{k-1}}[k-1]A_{\delta_{k-2}}[k-2]\ldots A_{\delta_{k_0}}[k_0].$$

We drop the subindexes, however, to simplify the notation.

Let H be any symmetric positive–definite matrix. We would like to find such $\delta > 0$ that for given $\varepsilon > 0$

$$-\varepsilon^2 < \langle l, A_\delta[i]HA_\delta^T[i]l\rangle - \langle l, A[i]HA^T[i]l\rangle < \varepsilon^2$$

for all vectors $l, \langle l, l\rangle = 1$, and any i, then by Lemma 5.1

$$-\varepsilon < \langle l, A_\delta[i]HA_\delta^T[i]l\rangle^{1/2} - \langle l, A[i]HA^T[i]l\rangle^{1/2} < \varepsilon.$$

For that, we write

$$\begin{aligned} &A_\delta[i]HA_\delta^T[i] - A[i]HA^T[i] \\ &\quad = \delta(A[i]HV_i^TU_i^T + U_iV_iHA^T[i]) + \delta^2 U_iV_iHV_i^TU_i^T. \end{aligned}$$

If we denote the largest singular value of H as h, and the largest singular value of the symmetric matrix $A[i]HV_i^TU_i^T + U_iV_iHA^T[i]$ as

$$\overline{\lambda} = \overline{\lambda}(A[i]HV_i^TU_i^T + U_iV_iHA^T[i])$$

then δ must be chosen so that it satifies the following inequality:

$$h\delta^2 + \overline{\lambda}\delta < \varepsilon^2$$

which is achieved by picking δ from the interval

$$0 < \delta < \frac{-\overline{\lambda} + \sqrt{\overline{\lambda}^2 + 4h\varepsilon^2}}{2h}. \tag{36}$$

Thus, we arrive at the following lemma.

Lemma 6.1: Let $H = H^T > 0$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$-\varepsilon < \langle l, A_\delta[i]HA_\delta^T[i]l\rangle^{1/2} - \langle l, A[i]HA^T[i]l\rangle^{1/2} < \varepsilon$$

for all vectors $l, \langle l, l\rangle = 1$. More precisely, δ can be chosen as

$$\delta = \frac{-\overline{\lambda} + \sqrt{\overline{\lambda}^2 + 4h\varepsilon^2}}{2h+1} \tag{37}$$

where $\overline{\lambda}$ is the largest singular value of the matrix $A[i]HV_i^TU_i^T + U_iV_iHA^T[i]$.

Remark: The choice of δ in (37) is not unique, but is such that it satisfies (36).

We need another auxiliary result, which will be later used in the proof. Given symmetric positive–semidefinite matrix R, and R_α being defined as in (29), what should be the value of α to guarantee

$$\varepsilon \leq \rho(l \,|\, \mathcal{E}(0, R_\alpha)) - \rho(l \,|\, \mathcal{E}(0, R)) \leq \alpha \tag{38}$$

for some $\varepsilon > 0$ and all directions $l, \langle l, l\rangle = 1$. The right-hand side of this inequality holds for any $\alpha > 0$-it follows directly from lemma 5.1. Let us address the left side of (38). Denote the largest singular value of R as $\overline{\mu}$. Then,

$$\langle l, Rl\rangle = \mu \leq \overline{\mu}.$$

By choosing $\alpha = \sqrt{\varepsilon^2 + 2\varepsilon\sqrt{\overline{\mu}}}$ we get

$$\begin{aligned} &\rho(l \,|\, \mathcal{E}(0, R_\alpha)) - \rho(l \,|\, \mathcal{E}(0, R)) \\ &\quad = (\langle l, Rl\rangle + \alpha^2)^{1/2} - \langle l, Rl\rangle^{1/2} \\ &\quad = (\mu + 2\varepsilon\sqrt{\overline{\mu}} + \varepsilon^2)^{1/2} - \sqrt{\mu} \\ &\quad \geq ((\sqrt{\mu} + \varepsilon)^2)^{1/2} - \sqrt{\mu} = \varepsilon \end{aligned}$$

So, we can state the following lemma.

Lemma 6.2: Let $R = R^T \geq 0$, and R_α is defined as in (29). Then, for any $\varepsilon > 0$ and α chosen as

$$\alpha = \sqrt{\varepsilon^2 + 2\varepsilon\sqrt{\overline{\mu}}}$$

where $\overline{\mu}$ is the largest singular value of R, inequality (38) holds.

We define the regularized reach set $\mathcal{X}_{\delta,\alpha}(k, k_0, \mathcal{E}(x_0, X_0))$ by

$$\begin{aligned} \mathcal{X}_{\delta,\alpha}(k, k_0, \mathcal{E}(x_0, X_0)) &= q[k] + \Phi_\delta(k, k_0)\mathcal{E}(0, X_0) \\ &\quad + \sum_{i=k_0}^{k-1} \Phi_\delta(k, i+1)\mathcal{E}(0, R_\alpha[i]). \end{aligned} \tag{39}$$

Its support function is

$$\begin{aligned} &\rho(l \,|\, \mathcal{X}_{\delta,\alpha}(k, k_0, \mathcal{E}(x_0, X_0))) \\ &\quad = \langle l, q[k]\rangle + \langle l, \Phi_\delta(k, k_0)X_0\Phi_\delta^T(k, k_0)l\rangle^{1/2} \\ &\qquad + \sum_{i=k_0}^{k-1} \langle l, \Phi_\delta(k, i+1)R_\alpha[i]\Phi_\delta^T(k, i+1)l\rangle^{1/2} \end{aligned} \tag{40}$$

where $R_\alpha[i]$ is defined in (29) and $q[k]$ comes from (9) as before.

Our goal is to make the regularized reach set arbitrary close to the actual reach set for every $k \geq k_0$, by the appropriate choice of parameters $\delta_i, k_0 \leq i < k$ and α. In other words, for any $\varepsilon > 0$ we should find $\delta_i > 0, k_0 \leq i < k$, and $\alpha > 0$, such that

$$\begin{aligned} 0 \leq \rho(l \,|\, &\mathcal{X}_{\delta,\alpha}(k, k_0, \mathcal{E}(x_0, X_0))) \\ &- \rho(l \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))) < \varepsilon \end{aligned}$$

for all $k \geq k_0$, and all vectors l. We now show how to construct such α and the sequence δ_i. Let us estimate the difference of the support functions of the regularized reach set and the actual reach set

$$\begin{aligned} &\rho(l \,|\, \mathcal{X}_{\delta,\alpha}(k, k_0, \mathcal{E}(x_0, X_0))) \\ &\quad - \rho(l \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))) \\ &= \rho(l \,|\, A_\delta[k-1]\mathcal{X}_{\delta,\alpha}(k-1, k_0, \mathcal{E}(x_0, X_0))) \\ &\quad - \rho(l \,|\, A[k-1]\mathcal{X}_{\delta,\alpha}(k-1, k_0, \mathcal{E}(x_0, X_0))) \\ &\quad + \rho(l \,|\, A[k-1]\mathcal{X}_{\delta,\alpha}(k-1, k_0, \mathcal{E}(x_0, X_0))) \\ &\quad - \rho(l \,|\, A[k-1]\mathcal{X}(k-1, k_0, \mathcal{E}(x_0, X_0))) \\ &\quad + \rho(l \,|\, \mathcal{E}(0, R_\alpha[k-1])) \\ &\quad - \rho(l \,|\, \mathcal{E}(0, R[k-1])). \end{aligned} \tag{41}$$

As we can see, this difference breaks up into three components. First, we analyze

$$\begin{aligned}\rho(l\,|\,A_\delta[k-1]\mathcal{X}_{\delta,\alpha}(k-1,k_0,\mathcal{E}(x_0,X_0)))\\ -\rho(l\,|\,A[k-1]\mathcal{X}_{\delta,\alpha}(k-1,k_0,\mathcal{E}(x_0,X_0)))\\ =\sum_{i=k_0}^{k-2}\big(\langle l,A_\delta[k-1]\Phi_\delta(k-1,i+1)R_\alpha[i]\\ \times\Phi_\delta^T(k-1,i+1)A_\delta^T[k-1]l\rangle^{1/2}\\ -\langle l,A[k-1]\Phi_\delta(k-1,i+1)R_\alpha[i]\\ \times\Phi_\delta^T(k-1,i+1)A^T[k-1]l\rangle^{1/2}\big)\\ +\langle l,A_\delta[k-1]\Phi_\delta(k-1,k_0)X_0\\ \times\Phi_\delta^T(k-1,k_0)A_\delta^T[k-1]l\rangle^{1/2}\\ -\langle l,A[k-1]\Phi_\delta(k-1,k_0)X_0\\ \times\Phi_\delta^T(k-1,k_0)A^T[k-1]l\rangle^{1/2}.\end{aligned}$$

Matrices $\Phi_\delta(k-1,i+1)R_\alpha[i]\Phi_\delta^T(k-1,i+1), k_0\leq i<k-1$, and $\Phi_\delta(k-1,k_0)X_0\Phi_\delta^T(k-1,k_0)$ are symmetric positive definite. We use Lemma 6.1: For any $\varepsilon_1>0$ and $k,k\geq k_0$, there exists δ_{k-1} such that

$$\begin{aligned}-\varepsilon_1<\rho(l\,|\,A_\delta[k-1]\mathcal{X}_{\delta,\alpha}(k-1,k_0,\mathcal{E}(x_0,X_0)))\\ -\rho(l\,|\,A[k-1]\mathcal{X}_\delta(k-1,k_0,\mathcal{E}(x_0,X_0)))<\varepsilon_1.\end{aligned}$$

If we denote the maximum of the largest singular values of matrices

$$\Phi_\delta(k-1,i+1)R_\alpha[i]\Phi_\delta^T(k-1,i+1)$$

for $k_0\leq i<k-1$, and

$$\Phi_\delta(k-1,k_0)X_0\Phi_\delta^T(k-1,k_0)$$

by $\overline{h}_{k-1}$, and the maximum of the largest singular values of the matrices

$$\begin{aligned}A[k-1]\Phi_\delta(k-1,i+1)R_\alpha[i]\\ \times\Phi_\delta^T(k-1,i+1)V_{k-1}^TU_{k-1}^T\\ +U_{k-1}V_{k-1}\Phi_\delta(k-1,i+1)R_\alpha[i]\\ \times\Phi_\delta^T(k-1,i+1)A^T[k-1]\end{aligned}$$

for $k_0\leq i<k-1$, and

$$\begin{aligned}A[k-1]\Phi_\delta(k-1,k_0)X_0\Phi_\delta^T(k-1,k_0)V_{k-1}^TU_{k-1}^T\\ +U_{k-1}V_{k-1}\Phi_\delta(k-1,k_0)X_0\Phi_\delta^T(k-1,k_0)A^T[k-1]\end{aligned}$$

by $\overline{\lambda}_{k-1}$, we have

$$\delta_{k-1}=\frac{-\overline{\lambda}_{k-1}+\sqrt{\overline{\lambda}_{k-1}^2+4\overline{h}_{k-1}\left(\frac{\varepsilon_1}{k-k_0}\right)^2}}{2\overline{h}_{k-1}+1}. \tag{42}$$

Remark: If we could find upper bounds

$$\overline{h}\geq\overline{h}_i\quad\overline{\lambda}\geq\overline{\lambda}_i$$

for all $i,k_0\leq i<k,\delta$ would not depend on i:

$$\begin{aligned}\delta_{k_0}=\cdots=\delta_{k-1}=\delta\\ =\frac{-\overline{\lambda}+\sqrt{\overline{\lambda}^2+4\overline{h}\left(\frac{\varepsilon_1}{k-k_0}\right)^2}}{2\overline{h}+1}.\end{aligned} \tag{43}$$

We now consider the third component of the difference (41). Let $\overline{\mu}(R[i])$ be the largest singular value of the matrix $R[i]$, and define

$$\mu_{\max}=\max_{k_0\leq i<k}\{\overline{\mu}(R[i])\}. \tag{44}$$

By Lemma 6.2, choosing α as

$$\alpha=\sqrt{\varepsilon_1^2+2\varepsilon_1\sqrt{\mu_{\max}}} \tag{45}$$

we can guarantee

$$\varepsilon_1\leq\rho(l\,|\,\mathcal{E}(0,R_\alpha[i]))-\rho(l\,|\,\mathcal{E}(0,R[i]))\leq\alpha$$

for $k_0\leq i<k$. We are finally ready to estimate the whole difference (41) together with its second component. We start with $k=k_0+1$. Since $\mathcal{X}_{\delta,\alpha}(k_0,k_0,\mathcal{E}(x_0,X_0))=\mathcal{X}(k_0,k_0,\mathcal{E}(x_0,X_0))=\mathcal{E}(x_0,X_0)$, we can find δ_{k_0} from (42) such that

$$\begin{aligned}0\leq\rho(l\,|\,\mathcal{X}_{\delta,\alpha}(k_0+1,k_0,\mathcal{E}(x_0,X_0)))\\ -\rho(l\,|\,\mathcal{X}(k_0+1,k_0,\mathcal{E}(x_0,X_0)))\leq\varepsilon_1+\alpha\end{aligned}$$

because the second component of the difference (41) is 0 for $k=k_0+1$. From here, we get the estimate of the second component used in the next time step

$$\begin{aligned}0\leq\rho(l\,|\,A[k_0+1]\mathcal{X}_{\delta,\alpha}(k_0+1,k_0,\mathcal{E}(x_0,X_0)))\\ -\rho(l\,|\,A[k_0+1]\mathcal{X}(k_0+1,k_0,\mathcal{E}(x_0,X_0)))\\ \leq\sigma_{\max}(\varepsilon_1+\alpha)\end{aligned}$$

where $\sigma_{\max}$ is defined in (32). Again, there exists δ_{k_0+1} such that for $i=k_0+2$ we have

$$\begin{aligned}0\leq\rho(l\,|\,\mathcal{X}_{\delta,\alpha}(k_0+2,k_0,\mathcal{E}(x_0,X_0)))\\ -\rho(l\,|\,\mathcal{X}(k_0+2,k_0,\mathcal{E}(x_0,X_0)))\\ \leq\sigma_{\max}(\varepsilon_1+\alpha)+\varepsilon_1+\alpha.\end{aligned}$$

Continuing this process, for $i=k$ we arrive at

$$\begin{aligned}0\leq\rho(l\,|\,\mathcal{X}_{\delta,\alpha}(k,k_0,\mathcal{E}(x_0,X_0)))\\ -\rho(l\,|\,\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0)))\\ \leq(\sigma_{\max}^{k-k_0-1}+\sigma_{\max}^{k-k_0-2}+\cdots+\sigma_{\max}+1)(\varepsilon_1+\alpha).\end{aligned}$$

Theorem 6.1: For any $\varepsilon>0$ and $k\geq k_0$ there exist sequence $\{\delta_i>0\},k_0\leq i<k$, and parameter $\alpha>0$ such that

$$\begin{aligned}0\leq\rho(l\,|\,\mathcal{X}_{\delta,\alpha}(k,k_0,\mathcal{E}(x_0,X_0)))\\ -\rho(l\,|\,\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0)))<\varepsilon\end{aligned}$$

for all vectors $l, \langle l, l\rangle = 1$. Values for δ_i can be found from (42), α is defined in (45), with ε_1 determined from the inequality

$$\varepsilon_1 + \sqrt{\varepsilon_1^2 + 2\varepsilon_1\sqrt{\mu_{\max}}} < \frac{\varepsilon}{\sigma_{\max}^{k-k_0-1} + \sigma_{\max}^{k-k_0-2} + \cdots + \sigma_{\max} + 1} \quad (46)$$

where $\sigma_{\max}$ and $\mu_{\max}$ come from (32) and (44) respectively.

Replacing Φ with Φ_δ and R with R_α in (19), (20) and (23), we obtain a recurrence relation for the shape matrix of the tight external approximating ellipsoid of the regularized reach set:

$$Q_{\delta,\alpha}[k+1] = (1+\pi_{\delta,\alpha}[k])A_\delta[k]Q_{\delta,\alpha}[k]A_\delta^T[k] + \left(1+\pi_{\delta,\alpha}^{-1}[k]\right)R_\alpha[k] \quad (47)$$

with $Q_{\delta,\alpha}[k_0] = X_0$, where

$$\pi_{\delta,\alpha}[k] = \frac{\left\langle l_0, \Phi_\delta(k_0,k+1)R_\alpha[k]\Phi_\delta^T(k_0,k+1)l_0\right\rangle^{1/2}}{\left\langle l_0, \Phi_\delta(k_0,k)Q_{\delta,\alpha}[k]\Phi_\delta^T(k_0,k)l_0\right\rangle^{1/2}}. \quad (48)$$

The expression (26) is accordingly replaced with

$$\mathcal{X}_{\delta,\alpha}(k,k_0,\mathcal{E}(x_0,X_0)) = \bigcap_{l_\delta[k]} \mathcal{E}(q[k], Q_{\delta,\alpha}[k]) \quad (49)$$

where $l_\delta[k]$ satisfies

$$l_\delta[k] = \Phi_\delta^T(k_0,k)l_0 \quad (50)$$

and $q[k]$ remains the same as before, coming from (24). Thus, by appropriately choosing the parameters $\delta_i, k_0 \leq i < k$, and α, we can externally approximate the actual reach set $\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0)), k \geq k_0$, with ε-accuracy in Hausdorff metric for any given $\varepsilon > 0$. The corresponding control and the initial condition, which drive the system along the good trajectory are given by (27) and (28), with Φ substituted by Φ_δ and R by R_α. Formula (28) remains unchanged, and (27) is modified as

$$u_{\delta,\alpha}[i] = p[i] + \frac{P[i]B^T[i]\Phi_\delta^T(k_0,i+1)l_0}{\left\langle l_0, \Phi_\delta(k_0,i+1)R_\alpha[i]\Phi_\delta^T(k_0,i+1)l_0\right\rangle^{1/2}}. \quad (51)$$

Remark: As one can see from (42) and (46), for small ε, δ should be one order smaller. It results in matrices $\Phi_\delta(k,k_0)$, although nonsingular, being still ill-conditioned. Computing the inverse for ill-conditioned matrix is error-prone. Therefore, we suggest the following simple trick. First of all, instead of computing the inverse of $\Phi_\delta(k,k_0)$ directly, obtain it as

$$\Phi_\delta(k_0,k) = A_\delta^{-1}[k_0]\dots A_\delta^{-1}[k-1]$$

because each individual A_δ is better conditioned than their product. Second, compute the approximate inverse of A_δ by any usual method (e.g., Gauss–Jordan, LDU, or QR) and denote it $A_\delta^\dagger$, then

$$A_\delta^{-1} = (A_\delta^\dagger A_\delta)^{-1}A_\delta^\dagger.$$

Notice that $(A_\delta^\dagger A_\delta)$ is well-conditioned matrix—its condition number is close to 1 and, hence, is properly invertible by standard inversion methods. For more information about matrix inversion, we refer the reader to [18], [19].

One other way to use the result of this section, is to fix δ, and based on that, compute ε_1, α, and ε for given k, treating ε as overapproximation error bound.

VII. Example: Regularized Reach Set

Consider the system

$$\begin{aligned} x_1[k+1] &= x_2[k] \\ x_2[k+1] &= u[k] \end{aligned} \quad (52)$$

for $k \geq 0$, where $|u[k]| \leq \mu$ for all k, and $x[0] \in \mathcal{E}(0,I)$. The matrices $R[i]$ are now constant and have the form

$$R[i] = R = \begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix}.$$

Since $A^k = 0, k > 1$, the reach set $\mathcal{X}(k,0,\mathcal{E}(0,I))$ is a sum of two degenerate ellipsoids

$$\mathcal{X}(k,0,\mathcal{E}(0,I)) = \mathcal{E}(0,ARA^T) + \mathcal{E}(0,R)$$

with

$$ARA^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}.$$

As seen in Fig. 1(a), the reach set $\mathcal{X}(k,0,\mathcal{E}(0,I))$ is actually a square with side 2μ and centered at the origin.

The singular value decomposition of A is

$$A = U\Sigma V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus, by (35), A_δ can be expressed as

$$A_\delta = A + \delta UV = \begin{bmatrix} 0 & 1+\delta \\ -\delta & 0 \end{bmatrix}.$$

The expression for R_α comes from (29)

$$R_\alpha = R + \alpha^2 I = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \mu+\alpha^2 \end{bmatrix}.$$

The choice of α and δ depends on the desired accuracy ε and the number of steps k (in our example $k_0 = 0$) we want this accuracy to be maintained. So, α is determined from (45), wherein $\mu_{\max} = \mu$. δ is given by (43), wherein $\overline{\lambda} = 2\mu$ and $\overline{h} = 1$. In both formulas ε_1 should satisfy condition (46). For example, for $\mu = 1$, the choice of $\varepsilon_1 = 10^{-10}$ guarantees accuracy $\varepsilon \leq 0.01$ of the exact reach set approximation after 100 steps. Fig. 1(b) shows the regularized reach set at time step $k = 10$. Here, we

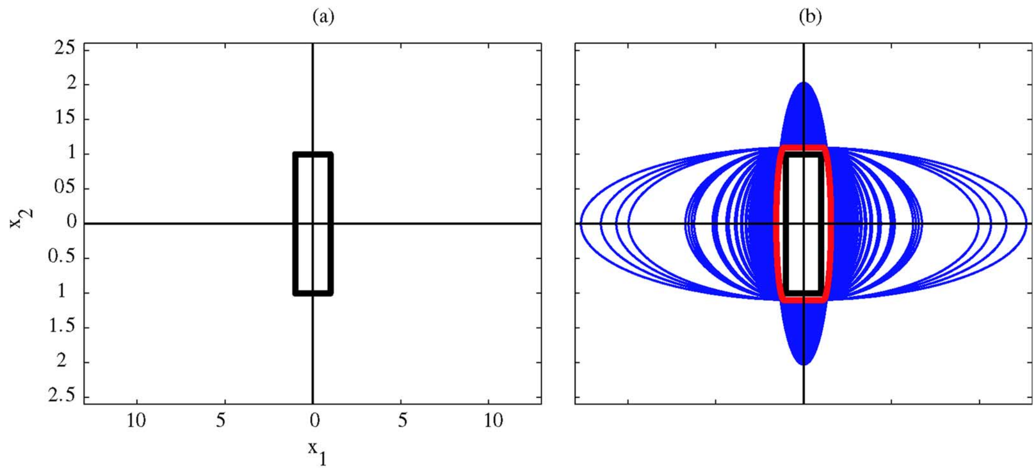

Fig. 1. (a) Reach set $\mathcal{X}(k,0,\mathcal{E}(0,I))$ of the system (52) for $k > 1 (\mu = 1)$. (b) Regularized reach set $\mathcal{X}_{\delta,\alpha}$ at $k = 10$ with $\varepsilon_1 = 0.1$.

picked $\varepsilon_1 = 0.1$, calculated the corresponding $\delta = 0.000025$ and $\alpha = 0.45$ and used the recurrence relation (47) to calculate the shape matrix of the external ellipsoid for different direction vectors l_0.

VIII. Internal Ellipsoidal Approximations

Consider the sum of $k+1$ ellipsoids $\mathcal{E}(0,H_i), 0 \le i \le k$ with positive semidefinite H_i. An internal ellipsoidal approximation of this sum is given by

$$\mathcal{E}(0,Q) \subseteq \sum_{i=0}^{k} \mathcal{E}(0,H_i)$$

with

$$Q = Q[k] = \left(\sum_{i=0}^{k} S_i H_i^{1/2}\right)^T \left(\sum_{i=0}^{k} S_i H_i^{1/2}\right) \tag{53}$$

and any orthogonal matrices $S_i, 0 \le i \le k, S_i^T S_i = I$ (see [1]).

$Q[k]$ satisfies the recurrence relation

$$\begin{aligned} Q[k+1] &= \left(\sum_{i=0}^{k+1} S_i H_i^{1/2}\right)^T \left(\sum_{i=0}^{k+1} S_i H_i^{1/2}\right) \\ &= \left(\sum_{i=0}^{k} S_i H_i^{1/2} + S_{k+1} H_{k+1}^{1/2}\right)^T \\ &\quad \times \left(\sum_{i=0}^{k} S_i H_i^{1/2} + S_{k+1} H_{k+1}^{1/2}\right) \\ &= Q[k] + H_{k+1} \\ &\quad + \left(\sum_{i=0}^{k} S_i H_i^{1/2}\right)^T S_{k+1} H_{k+1}^{1/2} \\ &\quad + \left(S_{k+1} H_{k+1}^{1/2}\right)^T \left(\sum_{i=0}^{k} S_i H_i^{1/2}\right) \end{aligned} \tag{54}$$

with $Q[0] = H_0$.

The following tightness condition is true (see [1]).

Theorem 8.1: Let H_0 be positive definite. Then, the inequality

$$\begin{aligned} \langle l, Q[k] l \rangle &= \left\langle l, \left(\sum_{i=0}^{k} S_i H_i^{1/2}\right)^T \left(\sum_{i=0}^{k} S_i H_i^{1/2}\right) l \right\rangle \\ &\le \left(\sum_{i=0}^{k} \langle l, H_i l \rangle^{1/2}\right)^2 \end{aligned} \tag{55}$$

which holds for any $l \in \mathbf{R}^n$ and any array of orthogonal matrices $S_i, 0 \le i \le k$, turns into equality for a given $l \in \mathbf{R}^n$ iff for all H_i there exist numbers η_i such that

$$S_i H_i^{1/2} l = \eta_i S_0 H_0^{1/2} l, \qquad 0 \le i \le k. \tag{56}$$

Now, let us return to (1) with nonsingular matrices $A[i], k_0 \le i < k$, and its reach set at time step k, $\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0))$ given by (8). As we can see, the reach set is a sum of $k+1$ ellipsoids. If we define $Q[k]$ as

$$\begin{aligned} Q[k] = &\left(\Phi(k,k_0) X_0^{1/2} S_0^T \right. \\ &\left. + \sum_{i=k_0}^{k-1} \Phi(k,i+1) R^{1/2}[i] S_i^T[k]\right) \\ &\times \left(S_0 X_0^{1/2} \Phi^T(k,k_0) \right. \\ &\left. + \sum_{i=k_0}^{k-1} S_i[k] R^{1/2}[i] \Phi^T(k,i+1)\right) \end{aligned} \tag{57}$$

with $Q[k_0] = X_0$, and in which S_0 and $S_i[k], k_0 \le i < k$, are any orthogonal matrices

$$\begin{aligned} &\mathcal{E}(q[k],Q[k]) \\ &\quad \subseteq \mathcal{E}(\Phi(k,k_0)x_0, \Phi(k,k_0) X_0 \Phi^T(k,k_0)) \\ &\qquad + \mathcal{E}\left(\sum_{i=k_0}^{k-1} \Phi(k,i+1) B[i] p[i], \right. \\ &\qquad \left. \sum_{i=k_0}^{k-1} \Phi(k,i+1) R[i] \Phi^T(k,i+1)\right) \\ &\quad = \mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0)) \end{aligned} \tag{58}$$

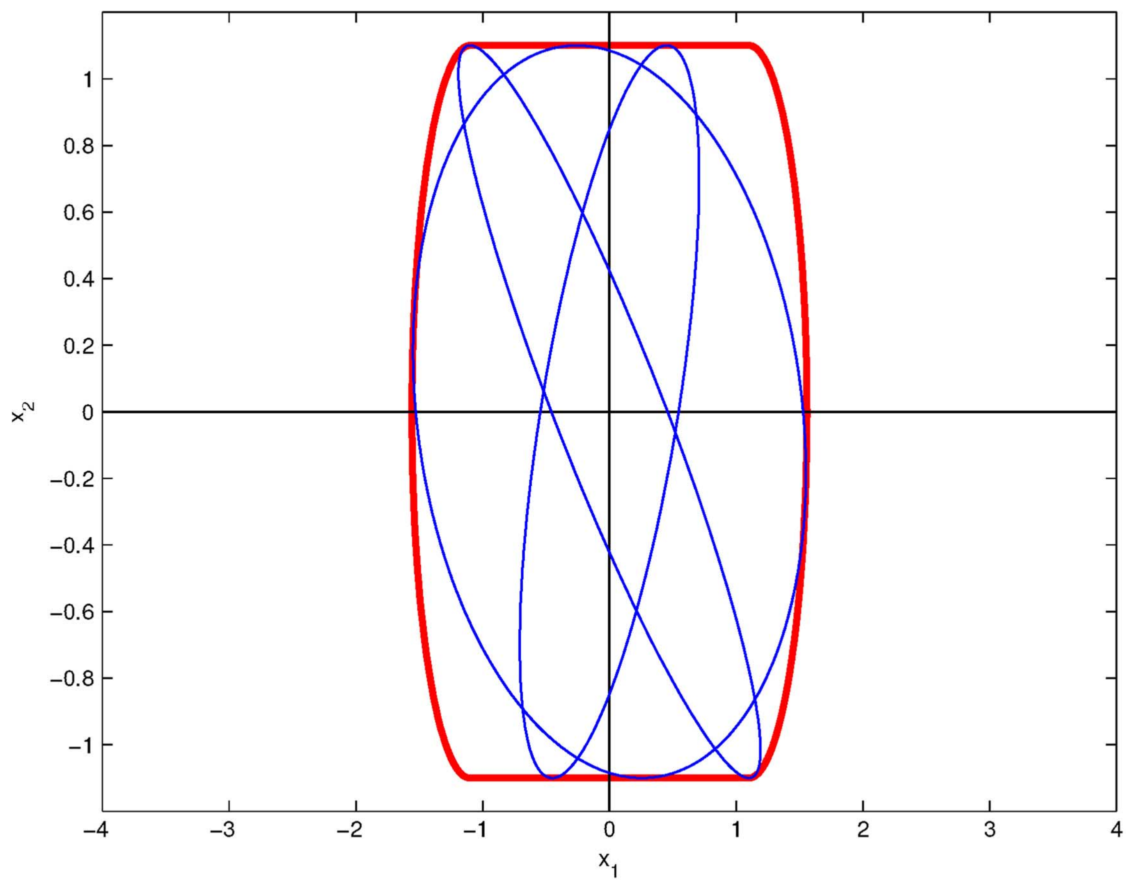

Fig. 2. Regularized reach set $\mathcal{X}_{\delta,\alpha}$ at $k = 10$ and its internal approximating ellipsoids.

with $q[k]$ coming from (9). So, the ellipsoid $\mathcal{E}(q[k], Q[k])$ lies inside the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$.

Definition 8.1: Ellipsoid $\mathcal{E}(q[k], Q[k]) \subseteq \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ is a tight internal approximation of the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ if there exists direction l such that

$$\rho(\pm l \,|\, \mathcal{E}(q[k], Q[k])) = \rho(\pm l \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))).$$

In order for $\mathcal{E}(q[k], Q[k])$ to be a tight internal approximation of the reach set for a given direction l, we use Theorem 8.1 and come up with the following tightness condition that can be checked by direct calculation.

Theorem 8.2: For a given time step $k \geq k_0$ and given direction l

$$\rho(l \,|\, \mathcal{E}(q[k], Q[k])) = \rho(l \,|\, \mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0)))$$

iff S_0 and $S_i[k], k_0 \leq i < k$, satisfy the relation

$$S_i[k]R^{1/2}[i]\Phi^T(k, i+1)l = \eta_i[k]S_0X_0^{1/2}\Phi^T(k, k_0)l \quad (59)$$

with

$$\eta_i[k] = \frac{\langle l, \Phi(k, i+1)R[i]\Phi^T(k, i+1)l\rangle^{1/2}}{\langle l, \Phi(k, k_0)X_0\Phi^T(k, k_0)l\rangle^{1/2}}. \quad (60)$$

Let $l[k]$ satisfy the "good curve" relation (19). In this case, expressions (59) and (60) simplify into

$$S_iR^{1/2}[i]\Phi^T(k_0, i+1)l_0 = \eta_iS_0X_0^{1/2}l_0 \quad (61)$$

and

$$\eta_i = \frac{\langle l_0, \Phi(k_0, i+1)R[i]\Phi^T(k_0, i+1)l_0\rangle^{1/2}}{\langle l_0, X_0l_0\rangle^{1/2}}. \quad (62)$$

Now, η_i and $S_i, k_0 \leq i < k$ no longer depend on k. We are ready to write the recurrence relation for the shape matrix $Q[k]$ of the internal ellipsoid $\mathcal{E}(q[k], Q[k])$ that touches the reach set $\mathcal{X}(k, k_0, \mathcal{E}(x_0, X_0))$ at points of the good trajectory specified by (19). Using definition (57) and relation (54) we get

$$\begin{aligned} Q[k+1] = {} & A[k]Q[k]A^T[k] + R[k] \\ & + A[k]Q^{1/2}[k]S_kR^{1/2}[k] \\ & + R^{1/2}[k]S_k^TQ^{1/2}[k]A^T[k] \end{aligned} \quad (63)$$

with $Q[k_0] = X_0$. The orthogonal matrices $S_0, S_k, k \geq k_0$, are related by (61) and (62). The recurrence relation for the center of the internal approximating ellipsoid $q[k]$ is the same as for external and is given by (24).

Theorem 8.3: Let $l[k]$ satisfy the (19), then the ellipsoid $\mathcal{E}(q[k],Q[k])$, where $Q[k]$ and $q[k]$ are the solutions of (63) and (24), is such that for all $k \geq k_0$, the following is true:

$$\mathcal{E}(q[k],Q[k]) \subseteq \mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0))$$
$$\text{and}$$
$$\rho(l[k] \,|\, \mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0))) = \rho(l[k] \,|\, \mathcal{E}(q[k],Q[k])).$$

The control that drives the system along the good trajectory is indicated in (27) for those vectors l_0, for which

$$\langle l_0, \Phi(k_0,i+1)R[i]\Phi^T(k_0,i+1)l_0\rangle \neq 0, \qquad k_0 \leq i < k$$

and the initial condition in (28).

For every boundary point of the reach set $\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0))$ there exists direction $l[k]$ and corresponding tight internal ellipsoid $\mathcal{E}(q[k],Q[k])$. Thus, we arrive at the result similar to Theorem 4.2.

Theorem 8.4: The reach set $\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0))$ can be expressed as

$$\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0)) = \bigcup_{l[k]} \mathcal{E}(q[k],Q[k]). \tag{64}$$

Remark: The results of this section are also valid for singular matrices $R[i]$, and no special regularization as in the case of external approximation is needed.

Until now, in discussing internal ellipsoidal approximations, we assumed $A[i]$ to be nonsingular. In case of singular $A[i]$, the aforementioned recurrence relations lose their sense because matrix $\Phi(k_0,k)$ does not exist. It is still possible to build internal approximating ellipsoids using (53) even for the singular case, since the reach set of a linear discrete-time system for every time step k is nothing else but a finite sum of ellipsoids. However, the recurrence relations in the case of singular matrix A cease to exist, making the calculation of tight internal approximating ellipsoids a rather involved procedure because for every k the matrices S_i need to be recomputed. Therefore, we suggest the following way out for singular case. Instead of the actual reach set $\mathcal{X}(k,k_0,\mathcal{E}(x_0,X_0))$ we consider the regularized reach set $\mathcal{X}_{\delta,\alpha}(k,k_0,\mathcal{E}(x_0,X_0))$ defined in (39) and constructed according to Theorem 6.1. Matrices $A_\delta[i]$ are nonsingular, so are $R_\alpha[i]$, and the relations (61)–(63) hold. Recall the example (52) from Section VII. Regularization does not depend on the approximation, and parameters δ and α need to be computed once for all the external and internal approximations, which was already done previously. Fig. 2 shows the regularized reach set at $k = 10$ and its internal ellipsoidal approximations for three different initial directions, $l_0 = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}(\sqrt{3})/(2)\\-(1)/(2)\end{pmatrix}$, and $\begin{pmatrix}(1)/(2)\\(\sqrt{3})/(2)\end{pmatrix}$.

IX. Conclusion

This paper extends the results of [1] to linear discrete-time systems. The emphasis is made on the situation specific to the discrete-time case, which has no continuous-time analog, when the state transition matrix $\Phi(k,k_0)$ is singular. We also do not assume controllability of the system. It was shown that in these situations it is possible to approximate the actual reach set with any given ε-accuracy by regularizing matrices $B[i]P[i]B^T[i]$ and $A[i]$. Theorem 6.1 provides the means of picking the appropriate regularization parameters. Internal ellipsoidal approximations work well even with singular matrices $B[i]P[i]B^T[i]$ and $\Phi(k,k_0)$. However, when $\Phi(k,k_0)$ is singular, in order to use the concept of "good curves" and the recurrence relations (61)–(63), we still need to perform the regularization.

What the regularization does, it bloats the reach set so as to make it strictly convex. In such a way, for every boundary point of the regularized reach set there exists an external ellipsoid that touches the reach set at that point.

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