How to decide Functionality of Compositions of Top-Down Tree Transducers

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Abstract. We prove that functionality of compositions of top-down tree transducers is decidable by reducing the problem to the functionality of one top-down tree transducer with look-ahead.

1 Introduction

Tree transducers are fundamental devices that were invented in the 1970's in the context of compilers and mathematical linguistics. Since then they have been applied in a huge variety of contexts such as, e.g., programming languages [13], security [10], or XML databases [9].

The perhaps most basic type of tree transducer is the top-down tree transducer [15,14] (for short transducer). One important decision problem for transducers concerns functionality: given a (nondeterministic) transducer, does it realize a function? This problem was shown to be decidable by Ésik [8] (even in the presence of look-ahead); note that this result also implies the decidability of equivalence of deterministic transducers [8], see also [7,11].

A natural and fundamental question is to ask whether functionality can also be decided for *compositions of transducers*. It is well known that compositions of transducers form a proper hierarchy, more precisely: compositions of n+1 transducers are strictly more expressive than compositions of n transducers [6]. Even though transducers are well studied, the question of deciding functionality for compositions of transducers has remained open. In this paper we fill this gap and show that the question can be answered affirmatively.

Deciding functionality for compositions of transducers has several applications. For instance, if an arbitrary composition of (top-down and bottom-up) tree transducers is functional, then an equivalent deterministic transducer with look-ahead can be constructed [5]. Together with our result this implies that it is decidable for such a composition whether or not it is definable by a deterministic transducer with look-ahead; note that the construction of such a single deterministic transducer improves efficiency, because it removes the need of computing intermediate results of the composition. Also other recent definability results can now be generalized to compositions: for instance, given such a composition we can now decide whether or not an equivalent linear transducer or an equivalent homomorphism exists [12] (and if so, construct it).

Let us now discuss the idea of our proof in detail. Initially, we consider a composition τ of two transducers T_1 and T_2 . Given τ , we construct a 'candidate' transducer with look-ahead M with the property that M is functional if and only if τ is functional. Our construction of M is an extension of the product construction in [2, p. 195]. The latter constructs a transducer N (without lookahead) that is obtained by translating the right-hand sides of the rules of T_1 by the transducer T_2 . It is well-known that in general, the transducer N is not equivalent to τ [2] and thus N may not be functional even though τ is. This is due to the fact that the transducer T_2 may

- copy or
- delete input subtrees.

Copying of an input tree means that the tree is translated several times and in general by different states. Deletion means that in a translation rule a particular input subtrees is not translated at all.

Imagine that T_2 copies and translates an input subtree in two different states q_1 and q_2 , so that the domains D_1 and D_2 of these states differ and moreover, T_1 nondeterministically produces outputs in the union of D_1 and D_2 . Now the problem that arises in the product construction of N is that N needs to guess the output of T_1 , however, the two states corresponding to q_1 and q_2 cannot guarantee that the same guess is used. However, the same guess may be used. This means that N (seen as a binary relation) is a superset of τ . To address this problem we show that it suffices to change T_1 so that it only outputs trees in the intersection of D_1 and D_2 . Roughly speaking this can be achieved by changing T_1 so that it runs several tree automata in parallel, in order to carry out the necessary domain checks.

Imagine now a transducer T_1 that translates two input subtrees in states q_1 and q_2 , respectively, but has no rules for state q_2 . This means that the translation of T_1 (and of τ) is empty. However, the transducer T_2 deletes the position of q_2 . This causes the translation of N to be non-empty. To address this problem we equip N with look-ahead. The look-ahead checks if the input tree is in the domains of all states of T_1 translating the current input subtree.

Finally, we are able to generalize the result to arbitrary compositions of transducers T_1, \ldots, T_n . For this, we apply the extended composition described above to the transducers T_{n-1} and T_n , giving us the transducer with look-ahead M. The look-ahead of M can be removed and incorporated into the transducer T_{n-2} using a composition result of [2]. The resulting composition of n-1 transducers is functional if and only if the original composition is.

The details of all our proofs can be found in the Appendix.

2 Top-Down Tree Transducers

For $k \in \mathbb{N}$, we denote by [k] the set $\{1, \ldots, k\}$. Let $\Sigma = \{e_1^{k_1}, \ldots, e_n^{k_n}\}$ be a ranked alphabet, where $e_j^{k_j}$ means that the symbol e_j has rank k_j . By Σ_k we denote the set of all symbols of Σ which have rank k. The set T_{Σ} of trees over

 Σ consists of all strings of the form $a(t_1, \ldots, t_k)$, where $a \in \Sigma_k$, $k \geq 0$, and $t_1, \ldots, t_k \in T_{\Sigma}$. Instead of a() we simply write a. We fix the set X of variables as $X = \{x_1, x_2, x_3, \ldots\}$.

Let B be an arbitrary set. We define $T_{\Sigma}[B] = T_{\Sigma'}$ where Σ' is obtained from Σ by $\Sigma'_0 = \Sigma_0 \cup B$ while for all k > 0, $\Sigma'_k = \Sigma_k$. In the following, let A, B be arbitrary sets. We let $A(B) = \{a(b) \mid a \in A, b \in B\}$.

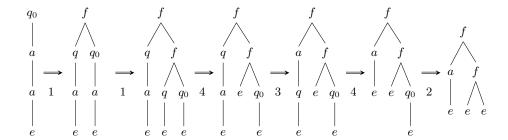
Definition 1. A top-down tree transducer T (or transducer for short) is a tuple of the form $T = (Q, \Sigma, \Delta, R, q_0)$ where Q is a finite set of states, Σ and Δ are the input and output ranked alphabets, respectively, disjoint with Q, R is a finite set of rules, and $q_0 \in Q$ is the initial state. The rules contained in R are of the form $q(a(x_1, \ldots, x_k)) \to t$, where $q \in Q$, $a \in \Sigma_k$, $k \ge 0$ and t is a tree in $T_{\Delta}[Q(X)]$.

If $q(a(x_1,...,x_k)) \to t \in R$ then we call t a right-hand side of q and a. The rules of R are used as rewrite rules in the natural way, as illustrated by the following example.

Example 1. Consider the transducer $T = (\{q_0, q\}, \Sigma, \Delta, R, q_0)$ where $\Sigma_0 = \{e\}$, $\Sigma_1 = \{a\}$, $\Delta_0 = \{e\}$, $\Delta_1 = \{a\}$ and $\Delta_2 = \{f\}$ and R consists the following rules (numbered 1 to 4):

1:
$$q_0(a(x_1)) \to f(q(x_1), q_0(x_1))$$
 2: $q_0(e) \to e$
3: $q(a(x_1)) \to a(q(x_1))$ 4: $q(e) \to e$.

On input a(a(e)), the transducer T produces the output tree f(a(e), f(e, e)) as follows



Informally, when processing a tree $s \in T_{\Sigma}$, the transducer T produces a tree t in which all proper subtrees of s occur as disjoint subtrees of t, 'ordered' by size. As the reader may realize, given an input tree s of size n, the transducer T produces an output tree that is of size $(n^2 + n)/2$. Hence, this translation has quadratic size increase, i.e., the size of the output tree is a most quadratic in size of the input tree. Note that transducers can have polynomial or exponential size increase [1].

Let $s \in T_{\Sigma}$. Then T(s) contains all trees in T_{Δ} obtainable from $q_0(s)$ by applying rules of T.

Clearly, T defines a binary relation over T_{Σ} and T_{Δ} . In the following, we denote by $\mathcal{R}(T)$ the binary relation that the transducer T defines. We say that the transducer T is functional if the relation $\mathcal{R}(T)$ is a function. Let q be a state of T. We denote by $\mathrm{dom}(q)$ the domain of q, i.e., the set of all trees $s \in T_{\Sigma}$ for which some tree $t \in T_{\Delta}$ is obtainable from q(s) by applying rules of T. We define the domain of T by $\mathrm{dom}(T) = \mathrm{dom}(q_0)$. For instance in Example 1, $\mathrm{dom}(T) = T_{\Sigma}$. However, if we remove the rule 1 for instance then the domain of T shrinks to the set $\{e\}$. We define $\mathrm{dom}(q)$, the domain of a state q of T, analogously.

A transducer $T = (Q, \Sigma, \Delta, R, q)$ is a top-down tree automaton (for short automaton) if $\Sigma = \Delta$ and all rules of T are of the form $q(a(x_1, \ldots, x_k)) \to a(q_1(x_1), \ldots, q_k(x_k))$ where $a \in \Sigma_k, k \geq 0$.

Let T_1 and T_2 be transducers. As $\mathcal{R}(T_1)$ and $\mathcal{R}(T_2)$ are relations, they can be composed. Hence,

$$\mathcal{R}(T_1) \circ \mathcal{R}(T_2) = \{(s, u) \mid \text{for some } t, (s, t) \in \mathcal{R}(T_1) \text{ and } (t, u) \in \mathcal{R}(T_2)\}.$$

If the output alphabet of T_1 and the input alphabet of T_2 coincide then the transducers T_1 and T_2 can be composed as well. The composition $T_1 \,\hat{\circ}\, T_2$ of the transducers T_1 and T_2 defines a tree translation as follows. On input s, the tree s is first translated by T_1 . Afterwards, the tree produced by T_1 is translated by T_2 which yields the output tree. Clearly, $T_1 \,\hat{\circ}\, T_2$ computes the relation $\mathcal{R}(T_1) \,\hat{\circ}\, \mathcal{R}(T_2)$. We say that the composition $T_1 \,\hat{\circ}\, T_2$ is functional if the relation $\mathcal{R}(T_1) \,\hat{\circ}\, \mathcal{R}(T_2)$ is a function.

3 Functionality of Two-Fold Compositions

In this section we show that for a composition τ of two transducers, a transducer M with look-ahead can be constructed such that M is functional if and only if τ is functional. Before formally introducing the construction for M and proving its correctness, we explain how to solve the challenges described in Section 1, i.e., we show how to handle copying and deleting rules. In the following, we call the product construction in [2, p. 195] simply the p-construction.

To see how precisely we handle copying rules, consider the transducers T_1 and T_2 . Let the transducer T_1 consist of the rules

$$q_1(a(x_1)) \to b(q_1(x_1))$$
 $q_1(e) \to e_i \mid i = 1, 2, 3$

while transducer T_2 consist of the rules

$$q_2(b(x_1)) \to f(q_2'(x_1), q_2''(x_1))$$
 $q_2'(e_j) \to e \mid j = 1, 2$
 $q_2''(e_3) \to e'$ $q_2''(e_j) \to e \mid j = 1, 2.$

The composition $\tau = T_1 \circ T_2$ defines a relation that only contains a single pair: τ only translates the tree a(e) into f(e,e). Therefore, τ is functional. For T_1 and T_2 , the p-construction yields the transducer N with the rules

$$(q_1, q_2)(a(x_1)) \to f((q_1, q_2')(x_1), (q_1, q_2'')(x_1))$$
 $(q_1, q_2')(e) \to e$
 $(q_1, q_2'')(e) \to e'$ $(q_1, q_2'')(e) \to e$.

On input a(e), the transducer N can produce either f(e,e) or f(e,e'). Therefore, N and τ are clearly not equivalent. Furthermore, the transducer N is obviously not functional even though the composition τ is.

In order to obtain a better understanding of why this phenomenon occurs, we analyze the behavior of N and τ on input a(e) in the following.

In the translation of τ , the states q_2' and q_2'' process the same tree produced by q_1 on input e due to the copying rule $q_2(b(x_1)) \to f(q_2'(x_1), q_2''(x_1))$. Furthermore, q_2' and q_2'' process a tree in $\text{dom}(q_2') \cap \text{dom}(q_2'')$. More precisely, q_2' and q_2'' both process either e_1 or e_2 .

In the translation of N on the other hand, due to the rule $(q_1, q_2)(a(x_1)) \to f((q_1, q_2')(x_1), (q_1, q_2'')(x_1))$, the states (q_1, q_2') and (q_1, q_2'') process e by 'guessing independently' from each other what q_1 might have produced on input e. In particular, the problem is that (q_1, q_2'') can apply the rule $(q_1, q_2'')(e) \to e'$ which eventually leads to the production of f(e, e'). Applying this rule means that (q_1, q_2'') guesses that e_3 is produced by q_1 . While this guess is valid, i.e., e_3 is producible by q_1 on input e, quite clearly $e_3 \notin \text{dom}(q_2')$.

In general, guesses performed by states of N cannot be 'synchronized', i.e., we cannot guarantee that states guess the same tree. Our solution to fix this issue is to restrict (q_1, q_2') and (q_1, q_2'') such that either state is only allowed to guess trees in $dom(q_2') \cap dom(q_2'')$. To understand why this approach works in general consider the following example.

Example 2. Let T_1 and T_2 be arbitrary transducers. Let $\tau = T_1 \circ T_2$ be functional. Let T_1 on input s produce either $b(t_1)$ or $b(t_2)$. Let T_2 contain the rule

$$q_2(b(x_1)) \to f(q_2^1(x_1), q_2^2(x_1))$$

where q_2 is the initial state of T_2 . The application of this rule effectively means that the states q_2^1 and q_2^2 process the same subtree produced by T_1 . Let $t_1, t_2 \in \text{dom}(q_2^1) \cap \text{dom}(q_2^2)$. Informally speaking, it does not matter whether the state q_2^1 processes t_1 or t_2 ; for either input q_2^1 produces the same output tree r and nothing else, otherwise, the functionality of τ is contradicted. The same holds for q_2^2 .

Informally, Example 2 suggests that if (q_1, q_2'') and (q_1, q_2'') only guess trees in $dom(q_2') \cap dom(q_2'')$, then it does not matter which tree exactly those states guess if the composition is functional. The final result in either case is the same. Quite clearly this is the case in our example. (In effect, q_2'' is forbidden to guess e_3 .) Thus, restricting (q_1, q_2') and (q_1, q_2'') basically achieves the same result as synchronizing their guesses if the composition is functional.

Now the question is how exactly do we restrict the states of N? Consider the states (q_1, q'_2) and (q_1, q''_2) of N in our example. The trick is to restrict q_1 such that q_1 can only produce trees in $dom(q'_2) \cap dom(q''_2)$. Thus any guess is guaranteed to be in $dom(q'_2) \cap dom(q''_2)$. In order to restrict which output trees T_1 can produce, we compose T_1 with the *domain automaton* of T_2 .

For an arbitrary transducer $T = (Q, \Sigma, \Delta, R, q)$, the domain automaton A of T is constructed analogous to the automaton in [4, Theorem 3.1]. The set of states of A is the power set of Q where $\{q\}$ is the initial state of A. The idea is that if in a translation of T on input s, the states $q_1 \ldots, q_n$ process the node v of s then $\{q_1 \ldots, q_n\}$ processes the node v of s in a computation of A. The rules of A are thus defined as follows.

Let $S = \{q_1, \ldots, q_n\}$, n > 0, and $a \in \Sigma_k$. In the following, we denote by $\mathrm{rhs}_T(q_j, a)$, where $j \in [n]$, the set of all right-hand sides of q_j and a. For all non-empty subsets $\Gamma_1 \subseteq \mathrm{rhs}_T(q_1, a), \ldots, \Gamma_n \subseteq \mathrm{rhs}_T(q_n, a)$, we define a rule

$$S(a(x_1,\ldots,x_k)) \to a(S_1(x_1),\ldots,S_k(x_k))$$

where for $i \in [k]$, S_i is defined as the set $\bigcup_{j=1}^n \Gamma_j \langle x_i \rangle$. We denote by $\Gamma_j \langle x_i \rangle$ the set of all states q' such that $q'(x_i)$ occurs in some tree γ in Γ_j ; e.g., for

$$\Gamma_i = \{a(q(x_1), q'(x_2)), \ a(a(q_1(x_1), q_2(x_2)), q_3(x_1))\},\$$

we have $\Gamma_j\langle x_1\rangle = \{q, q_1, q_3\}$ and $\Gamma_j\langle x_2\rangle = \{q', q_2\}$. We define that the state \emptyset of A realizes the identity. Hence, the rules for the state \emptyset are defined in the obvious way.

We now explain why subsets Γ_j of right-hand sides are used for the construction of rules of A. Recall that the idea is that if in a translation of T on input s, the states $q_1 \ldots, q_n$ process the node v of s then $\{q_1 \ldots, q_n\}$ processes the node v of s in a computation of A. Due to copying rules, multiple instances of a state q_1 may access v. Two instance of q_1 may process v in different manners. This necessitates the use of subsets Γ_j of right-hand sides. For a better understanding, consider the following example.

Example 3. Let $T = (\{q_0, q\}, \Sigma, \Delta, R, q_0)$ where $\Sigma_0 = \Delta_0 = \{e\}, \Sigma_1 = \Delta_1 = \{a\}$ and $\Sigma_2 = \Delta_2 = \{f\}$. The set R contains the following rules:

$$\begin{array}{lll} q_0(a(x_1)) & \to f(q_0(x_1),q_0(x_1)) & \quad q(a(x_1)) & \to e' \\ q_0(f(x_1,x_2)) \to q_0(x_1) & \quad q(f(x_1,x_2)) \to e' \\ q_0(f(x_1,x_2)) \to f(q(x_1),q(x_2)) & \quad q(e) & \to e' \\ q_0(e) & \to e. \end{array}$$

Consider the input tree s = a(f(e, e)). Clearly, on input s, the tree f(e, f(e', e')) is producible by T. In this translation, two instances of the state q_0 process the subtree f(e, e) of s, however the instances of q_0 do not process f(e, e) in the same way. The first instance of q_0 produces e on input f(e, e) while the second instance produces f(e', e'). These translations mean that the states q_0 and q process the leftmost e of s.

Consider the domain automaton A of T. By definition, A contains the rule $\{q_0\}(a(x_1)) \to a(\{q_0\}(x_1))$ which is obtained from the right-hand side of the rule $q_0(a(x_1)) \to f(q_0(x_1), q_0(x_1))$ of T. To simulate that the states q_0 and q process the leftmost e of s in the translation from s to f(e, f(e', e')), we clearly require the rule $\{q_0\}(f(x_1, x_2)) \to f(\{q_0, q\}(x_1), \{q\}(x_2))$ obtained from the right-hand sides of the rules $q_0(f(x_1, x_2) \to q_0(x_1))$ and $q_0(f(x_1, x_2) \to f(q(x_1), q(x_2)))$ of T.

For completeness, we list the remaining rules of A. The automaton A also contains the rules

For the rules of the state $\{q_0, q\}$ consider the following. The right-hand sides of rules of $\{q_0, q\}$ are identical to the right-hand sides of rules of $\{q_0\}$, i.e., the rules for $\{q_0, q\}$ are obtained by substituting $\{q_0\}$ on the left-hand-side of rules of A by $\{q_0, q\}$.

The automaton A has the following property.

Lemma 1. Let $S \neq \emptyset$ be a state of A. Then $s \in dom(S)$ if and only if $s \in \bigcap_{q \in S} dom(q)$.

Obviously, Lemma 1 implies that A recognizes the domain of T.

Using the domain automaton A of T_2 , we transform T_1 into the transducer \hat{T}_1 . Formally, the transducer \hat{T}_1 is obtained from T_1 and A using the p-construction. In our example, the transducer \hat{T}_1 obtained from T_1 and T_2 includes the following rules

$$(q_1, \{q_2\}) (a(x_1)) \to b((q_1, \{q'_2, q''_2\})(x_1))$$

 $(q_1, \{q'_2, q''_2\}) (e) \to e_j$

where j = 1, 2. The state $(q_1, \{q_2\})$ is the initial state of \hat{T}_1 . Informally, the idea is that in a translation of $\hat{\tau} = \hat{T}_1 \circ T_2$, a tree produced by a state (q, S) of \hat{T}_1 is only processed by states in S. The following result complements this idea.

Lemma 2. If the state (q, S) of \hat{T}_1 produces the tree t and $S \neq \emptyset$ then $t \in \bigcap_{q_2 \in S} dom(q_2)$.

We remark that if a state of the form (q, \emptyset) occurs then it means that in a translation of $\hat{\tau}$, no state of T_2 will process a tree produced by (q, \emptyset) . Note that as A is nondeleting and linear, \hat{T}_1 defines the same relation as $T_1 \,\hat{\circ} \, A$ [2, Th. 1]. Informally, the transducer \hat{T}_1 is a restriction of the transducer T_1 such that $\operatorname{range}(\hat{T}_1) = \operatorname{range}(T_1) \cap \operatorname{dom}(T_2)$. Therefore, the following holds.

Lemma 3.
$$\mathcal{R}(T_1) \circ \mathcal{R}(T_2) = \mathcal{R}(\hat{T}_1) \circ \mathcal{R}(T_2)$$
.

Due to Lemma 3, we focus on \hat{T}_1 instead of T_1 in the following.

Consider the transducer \hat{N} obtained from \hat{T}_1 and T_2 using the p-construction. By construction, the states of \hat{N} are of the form ((q,S),q') where (q,S) is a state of \hat{T}_1 and q' is a state of T_2 . In the following, we write (q,S,q') instead for better readability. Informally, the state (q,S,q') implies that in a translation of $\hat{\tau}$ the state q' is supposed to process a tree produced by (q,S). Because trees produced by (q,S) are only supposed to be processed by states in S, we only consider states (q,S,q') where $q' \in S$. For \hat{T}_1 and T_2 , we obtain the transducer \hat{N} with the following rules

$$\begin{array}{ll} (q_1,\{q_2\},q_2)\;(a(x_1)) \to f((q_1,S,q_2')(x_1),(q_1,S,q_2'')(x_1)) \\ (q_1,S,q_2')\;(e) &\to e \\ (q_1,S,q_2'')\;(e) &\to e \end{array}$$

where $S = \{q_2', q_2''\}$ and i = 1, 2. The initial state of \hat{N} is $(q_1, \{q_2\}, q_2)$. Obviously, \hat{N} computes the relation $\mathcal{R}(T_1) \circ \mathcal{R}(T_2)$.

In the following, we briefly explain our idea. In a translation of \hat{N} on input a(e), the subtree e is processed by (q_1, S, q_2') and (q_1, S, q_2'') . Note that in a translation of $\hat{\tau}$ the states q_2' and q_2'' would process the same tree produced by (q_1, S) on input e. Consider the state (q_1, S, q_2'') . If (q_1, S, q_2'') , when reading e, makes a valid guess, i.e., (q_1, S, q_2'') guesses a tree t that is producible by (q_1, S) on input e, then $t \in \text{dom}(q_2')$ by construction of \hat{T}_1 . Due to previous considerations (cf. Example 2), it is thus sufficient to ensure that all guesses of states of \hat{N} are valid. While obviously in the case of \hat{N} , all guesses are indeed valid, guesses of transducers obtained from the p-construction are in general not always valid; in particular if deleting rules are involved.

To be more specific, consider the following transducers T_1' and T_2' . Let T_1' contain the rules

$$q_1(a(x_1, x_2)) \to b(q_1'(x_1), q_1''(x_2), q_1'''(x_2)) \qquad q_1'(e) \to e$$

where $dom(q_1'')$ consists of all trees whose left-most leaf is labeled by e while $dom(q_1''')$ consists of all trees whose left-most leaf is labeled by c. Let T_2' contain the rules

$$q_2(b(x_1, x_2, x_3)) \to q_2(x_1)$$
 $q_2(e) \to e_j \mid j = 1, 2.$

As the translation of T_1' is empty, obviously the translation of $\tau' = T_1' \circ T_2'$ is empty as well. Thus, τ' is functional. However, the p-construction yields the transducer N' with the rules

$$(q_1, q_2)(a(x_1, x_2)) \to (q'_1, q_2)(x_1)$$
 $(q'_1, q_2)(e) \to e_j \mid j = 1, 2$

Even though $\tau' = T_1' \circ T_2'$ is functional, the transducer N' is not. More precisely, on input a(e,s), where s is an arbitrary tree, N' can produce either e_1 or e_2 while τ' would produce nothing. The reason is that in the translation of N', the tree a(e,s) is processed by the state (q_1,q_2) by applying the deleting rule $\eta = (q_1,q_2)(a(x_1,x_2)) \to (q_1',q_2)(x_1)$. Applying η means that (q_1,q_2) guesses that on input a(e,s), the state q_1 produces a tree of the form $b(t_1,t_2,t_3)$ by applying the rule $q_1(a(x_1,x_2)) \to b(q_1'(x_1),q_1''(x_2),q_1'''(x_2))$ of T_1 . However, this guess is not valid, i.e., q_1 does not produce such a tree on input a(e,s), as by definition $s \notin \text{dom}(q_1'')$ or $s \notin \text{dom}(q_1''')$. The issue is that N' itself cannot verify the validity of this guess because, due to the deleting rule η , N' does not read s.

As the reader might have guessed our idea is that the validity of each guess is verified using look-ahead. First, we need to define look-ahead.

A transducer with look-ahead (or la-transducer) M' is a transducer that is equipped with an automaton called the la-automaton. Formally, M' is a tuple $M' = (Q, \Sigma, \Delta, R, q, B)$ where Q, Σ, Δ and q are defined as for transducers and

B is the la-automaton. The rules of R are of the form $q(a(x_1:l_1,\ldots,x_k:l_k))\to t$ where for $i\in[k],\ l_i$ is a state of B. Consider the input s. The la-transducer M' processes s in two phases: First each input node of s is annotated by the states of B at its children, i.e., an input node v labeled by $a\in\Sigma_k$ is relabeled by $\langle a,l_1,\ldots,l_k\rangle$ if B arrives in the state l_i when processing the i-th subtree of v. Relabeling the node s provides M' with additional information about the subtrees of s, e.g., if the node v is relabeled by $\langle a,l_1,\ldots,l_k\rangle$ then the i-th subtree of v is a tree in $\mathrm{dom}(l_i)$. The relabeled tree is then processed by M'. To this end a rule $q(a(x_1:l_1,\ldots,x_k:l_k))\to t$ is interpreted as $q(\langle a,l_1,\ldots,l_k\rangle(x_1,\ldots,x_k))\to t$.

In our example, the idea is to equip N' with an la-automaton to verify the validity of guesses. In particular, the la-automaton is the domain automaton A' of T'_1 . Recall that a state of A' is a set consisting of states of T'_1 . To process relabeled trees the rules of N' are as follows

$$(q_1, q_2)(a(x_1 : \{q_1'\}, x_2 : \{q_1'', q_1''\})) \to (q_1', q_2)(x_1) \quad (q_1', q_2)(e) \to e_j \mid j = 1, 2$$

Consider the tree a(e,s), where s is an arbitrary tree. The idea is that if the root of a(e,s) is relabeled by $\langle a, \{q_1'\}, \{q_1'', q_1'''\} \rangle$, then due to Lemma 1, $e \in \text{dom}(q_1')$ and $s \in \text{dom}(q_1'') \cap \text{dom}(q_1'')$ and thus on input a(e,s) a tree of the form $b(t_1,t_2,t_3)$ is producible by q_1 using the rule $q_1(a(x_1,x_2)) \to b(q_1'(x_1),q_1''(x_2),q_1'''(x_2))$. Quite clearly, the root of a(e,s) is not relabeled. Thus, the translation of N' equipped with the la-automaton A' is empty as the translation of τ' is.

3.1 Construction of the LA-Transducer M

Recall that for a composition τ of two transducers T_1 and T_2 , we aim to construct an la-transducer M such that M is functional if and only if τ is functional.

In the following we show that combining the ideas presented above yields the la-transducer M. For T_1 and T_2 , we obtain M by first completing the following steps.

- 1. Construct the domain automaton A of T_2
- 2. Construct the transducer T_1 from T_1 and A using the p-construction
- 3. Construct the transducer N from T_1 and T_2 using the p-construction

We then obtain M by extending N into a transducer with look-ahead. Note that the states of N are written as (q, S, q') instead of ((q, S), q') for better readability, where (q, S) is a state of \hat{T}_1 and q' is a state of T_2 . Recall that (q, S, q') means that q' is supposed to process a tree generated by (q, S). Furthermore, recall that S is a set of states of T_2 and that the idea is that trees produced by (q, S) are only supposed to be processed by states in S. Thus, we only consider states (q, S, q') of N where $q' \in S$.

The transducer M with look-ahead is constructed as follows. The set of states of M and the initial state of M are the states of N and the initial state of N, respectively. The la-automaton of M is the domain automaton \hat{A} of \hat{T}_1 .

We now define the rules of M. First, recall that a state of \hat{A} is a set consisting of states of \hat{T}_1 . Furthermore, recall that for a set of right-hand sides Γ and a

variable x, we denote by $\Gamma\langle x\rangle$ the set of all states q such that q(x) occurs in some $\gamma\in\Gamma$. For a right-hand side γ , the set $\gamma\langle x\rangle$ is defined analogously. For all rules

$$\eta = (q, S, q')(a(x_1, \dots, x_k)) \to \gamma$$

of N we proceed as follows: If η is obtained from the rule $(q, S)(a(x_1, \ldots, x_k)) \to \xi$ of \hat{T}_1 and subsequently translating ξ by the state q' of T_2 then we define the rule

$$(q, S, q')(a(x_1: l_1, \ldots, x_k: l_k)) \rightarrow \gamma$$

for M where for $i \in [k]$, l_i is a state of \hat{A} such that $\xi\langle x_i\rangle \subseteq l_i$. Recall that relabeling a node v, that was previously labeled by a, by $\langle a, l_1, \ldots, l_k\rangle$ means that the i-th subtree of v is a tree in $\text{dom}(l_i)$. By Lemma 1, $s \in \text{dom}(l_i)$ if and only if $s \in \bigcap_{\hat{q} \in l_i} \text{dom}(\hat{q})$. Thus, if the node v of a tree s is relabeled by $\langle a, l_1, \ldots, l_k \rangle$ then it means that (q, S) can process subtree of s rooted at v using the rule $(q, S)(a(x_1, \ldots, x_k)) \to \xi$.

In the following, we present a detailed example for the construction of M for two transducers T_1 and T_2 .

Example 4. Let the transducer T_1 contain the rules

```
\begin{array}{lll} q_0(f(x_1, x_2)) \to f(q_1(x_1), q_2(x_2)) & q_0(f(x_1, x_2)) \to q_3(x_2) \\ q_2(f(x_1, x_2)) \to f(q_2(x_1), q_1(x_2)) & q_1(f(x_1, x_2)) \to f(q_1(x_1), q_1(x_2)) \\ q_2(f(x_1, x_2)) \to f'(q_2(x_1), q_1(x_2)) & q_1(f(x_1, x_2)) \to f'(q_1(x_1), q_1(x_2)) \\ q_2(e) & \to e & q_1(e) & \to e \\ q_3(d) & \to d & q_1(d) & \to d \end{array}
```

and let the initial state of T_1 be q_0 . Informally, when reading the symbol f, the states q_1 and q_2 nondeterministically decide whether or not to relabel f by f'. However, the domain of q_2 only consists of trees whose leftmost leaf is labeled by e. The state q_3 only produces the tree d on input d. Thus, the domain of T_1 only consists of trees of the form $f(s_1, s_2)$ where s_1 and s_2 are trees and either the leftmost leaf of s_2 is e or $s_2 = d$.

The initial state of the transducer T_2 is \hat{q}_0 and T_2 contains the rules

$$\begin{array}{lll} \hat{q}_{0}(f(x_{1},x_{2})) \to f(\hat{q}_{1}(x_{1}),\hat{q}_{2}(x_{1})) & & \hat{q}_{1}(f(x_{1},x_{2})) \to f(\hat{q}_{1}(x_{1}),\hat{q}_{1}(x_{2})) \\ \hat{q}_{0}(d) & \to d & & \hat{q}_{1}(f'(x_{1},x_{2})) \to f'(\hat{q}_{1}(x_{1}),\hat{q}_{2}(x_{2})) \\ \hat{q}_{2}(f(x_{1},x_{2})) \to f(\hat{q}_{2}(x_{1}),\hat{q}_{2}(x_{2})) & & \hat{q}_{1}(e) & \to e \\ \hat{q}_{2}(e) & \to e & & \hat{q}_{1}(d) & \to d \\ \hat{q}_{2}(d) & \to d. \end{array}$$

Informally, on input s, the state \hat{q}_2 produces s if the symbol f' does not occur in s; otherwise \hat{q}_2 produces no output. The state \hat{q}_1 realizes the identity. Hence, the domain of T_2 only consists of the tree d and trees $f(s_1, s_2)$ with no occurrences of f' in s_1 .

Consider the composition $\tau = T_1 \circ T_2$. On input s, the composition τ yields $f(s_1, s_1)$ if s is of the form $f(s_1, s_2)$ and the leftmost leaf of s_2 is labeled by e. If the input tree is of the form $f(s_1, d)$, the output tree d is produced. Clearly, τ is functional. We remark that both phenomena described in Section 3 occur in the

composition τ . More precisely, simply applying the p-construction to T_1 and T_2 yields a nondeterministic transducer due to 'independent guessing'. Furthermore, not checking the validity of guesses causes nondeterminism on input $f(s_1, d)$.

In the following, we show how to construct the la-automaton M from the transducers T_1 and T_2 .

Construction of the domain automaton A. We begin by constructing the domain automaton A of T_2 . The set of states of A is the power set of the set of states of T_2 and the initial state of A is $\{\hat{q}_0\}$. The rules of A are

$$\begin{cases} \hat{q}_0 \} \ (f(x_1, x_2)) \to f(S(x_1), \emptyset(x_2)) \\ \hat{q}_0 \} \ (d) & \to d \\ S \ (f(x_1, x_2)) \to f(S(x_1), S(x_2)) \\ S \ (e) & \to e \\ S \ (d) & \to d \end{cases}$$

where $S = \{\hat{q}_1, \hat{q}_2\}$. The state \emptyset realizes the identity. The rules for the state \emptyset are straight forward and hence omitted here. All remaining states, such as for instance $\{\hat{q}_0, \hat{q}_1\}$, are unreachable and hence the corresponding rules are irrelevant. Thus, we omit these rules as well. In the following, we only consider rules of states that are reachable.

Construction of the transducer $\hat{\mathbf{T}}_1$. For T_1 and A, the p-construction yields the transducer \hat{T}_1 . The transducer \hat{T}_1 contains the rules

$$\begin{aligned} &(q_0, \{\hat{q}_0\}) \; (f(x_1, x_2)) \to f((q_1, S)(x_1), q_2(x_2)) \\ &(q_0, \{\hat{q}_0\}) \; (f(x_1, x_2)) \to (q_3, \{\hat{q}_0\})(x_2) \\ &q_1 \; (f(x_1, x_2)) \to f(q_1(x_1), q_1(x_2)) \\ &q_1 \; (f(x_1, x_2)) \to f'(q_1(x_1), q_1(x_2)) \\ &q_1 \; (e) & \to e \\ &q_1 \; (d) & \to d \\ &(q_1, S) \; (f(x_1, x_2)) \to f((q_1, S)(x_1), (q_1, S)(x_2)) \\ &(q_1, S) \; (e) & \to e \\ &(q_1, S) \; (d) & \to d \\ &q_2 \; (f(x_1, x_2)) \to f(q_2(x_1), q_1(x_2)) \\ &q_2 \; (f(x_1, x_2)) \to f'(q_2(x_1), q_1(x_2)) \\ &q_2 \; (e) & \to e \\ &(q_3, \{\hat{q}_0\}) \; (d) & \to d \end{aligned}$$

and the initial state of \hat{T}_1 is $(q_0, \{\hat{q}_0\})$. For better readability, we just write q_1 and q_2 instead of (q_1, \emptyset) and (q_2, \emptyset) , respectively.

Construction of the transducer N. For \hat{T}_1 and T_2 , we construct the transducer N containing the rules

```
 \begin{array}{c} (q_0, \{\hat{q}_0\}, \hat{q}_0) \; (f(x_1, x_2)) \to f((q_1, S, \hat{q}_1)(x_1), (q_1, S, \hat{q}_2)(x_1)) \\ (q_0, \{\hat{q}_0\}, \hat{q}_0) \; (f(x_1, x_2)) \to (q_3, \{\hat{q}_0\}, \hat{q}_0)(x_2) \\ \\ (q_1, S, \hat{q}_1) \; (f(x_1, x_2)) \to f((q_1, S, \hat{q}_1)(x_1), (q_1, S, \hat{q}_1)(x_2)) \\ (q_1, S, \hat{q}_1) \; (e) & \to e \\ (q_1, S, \hat{q}_1) \; (d) & \to d \\ \\ (q_1, S, \hat{q}_2) \; (f(x_1, x_2)) \to f((q_1, S, \hat{q}_2)(x_1), (q_1, S, \hat{q}_2)(x_2)) \\ (q_1, S, \hat{q}_2) \; (e) & \to e \\ (q_1, S, \hat{q}_2) \; (d) & \to d \\ \\ (q_3, \{\hat{q}_0\}, \{\hat{q}_0\}) \; (d) & \to d \\ \end{array}
```

The initial state of N is $(q_0, \{\hat{q}_0\}, \hat{q}_0)$. Note that the states such as (q_1, S, \hat{q}_0) are not considered as \hat{q}_0 is not contained in S. We remark that though no nondeterminism is caused by 'independent guessing', N is still nondeterministic on input $f(s_1, d)$ as the validity of guesses cannot be checked. To perform validity checks for guesses, we extend N with look-ahead.

Construction of the look-ahead automaton \hat{A} . Recall that the look-ahead automaton of M is the domain automaton \hat{A} of \hat{T}_1 . The set of states of \hat{A} is the power set of the set of states of \hat{T}_1 . The initial state of \hat{A} is $\{(q_0, \{\hat{q}_0\})\}$ and \hat{A} contains the following rules.

```
 \{(q_0, \{\hat{q}_0\})\} (f(x_1, x_2)) \to f(\{(q_1, S)\}(x_1), \{q_2\})\}(x_2)) 
 \{(q_0, \{\hat{q}_0\})\} (f(x_1, x_2)) \to f(\emptyset(x_1), \{(q_3, \{\hat{q}_0\})\}(x_2)) 
 \{q_1\} (f(x_1, x_2)) \to f(\{q_1\}(x_1), \{q_1\}(x_2)) 
 \{q_1\} (d) \to d 
 \{(q_1, S)\} (f(x_1, x_2)) \to f(\{(q_1, S)(x_1)\}, \{(q_1, S)\}(x_2)) 
 \{(q_1, S)\} (e) \to e 
 \{(q_1, S)\} (d) \to d 
 \{q_2\} (f(x_1, x_2)) \to f(\{q_2\}(x_1), \{q_1\}(x_2)) 
 \{q_2\} (e) \to e 
 (q_3, \{\hat{q}_0\}, \{\hat{q}_0\}) (d) \to d
```

For better readability, we again just write q_1 and q_2 instead of (q_1, \emptyset) and (q_2, \emptyset) , respectively. We remark that, by construction of the domain automaton, \hat{A} also contains the rule

$$\{(q_0, \{\hat{q}_0\})\}(f(x_1, x_2)) \to f(\{(q_1, S)\}(x_1), \{q_2, (q_3, \{\hat{q}_0\})\}(x_2)),$$

however, since no rules are defined for the state $\{q_2, (q_3, \{\hat{q}_0\})\}\$, this rule can be omitted.

Construction of the la-transducer M. Finally, we construct the la-transducer M. The initial state of M is $(q_0, \{\hat{q}_0\}, \hat{q}_0)$ and the rules of M are

$$\begin{array}{lll} (q_0,\{\hat{q}_0\},\hat{q}_0) \; (f(x_1:\{(q_1,S)\},x_2:\{q_2\})) & \to f((q_1,S,\hat{q}_1)(x_1),(q_1,S,\hat{q}_2)(x_1)) \\ (q_0,\{\hat{q}_0\},\hat{q}_0) \; (f(x_1:\emptyset,x_2:\{q_3,\{\hat{q}_0\}\})) & \to (q_3,\{\hat{q}_0\},\hat{q}_0)(x_2) \\ & (q_1,S,\hat{q}_1) \; (f(x_1:\{(q_1,S)\},x_2:\{(q_1,S)\})) \to f((q_1,S,\hat{q}_1)(x_1),(q_1,S,\hat{q}_1)(x_2)) \\ & (q_1,S,\hat{q}_1) \; (e) & \to e \\ & (q_1,S,\hat{q}_1) \; (d) & \to d \\ & (q_1,S,\hat{q}_2) \; (f(x_1:\{(q_1,S)\},x_2:\{(q_1,S)\})) \to f((q_1,S,\hat{q}_2)(x_1),(q_1,S,\hat{q}_2)(x_2)) \\ & (q_1,S,\hat{q}_2) \; (e) & \to e \\ & (q_1,S,\hat{q}_2) \; (d) & \to d \\ & (q_3,\{\hat{q}_0\},\{\hat{q}_0\}) \; (d) & \to d \\ \end{array}$$

By construction, the transducer N contains the rule

$$\eta = (q_0, \{\hat{q}_0\}, \hat{q}_0)(f(x_1, x_2)) \to f((q_1, S, \hat{q}_1)(x_1), (q_1, S, \hat{q}_2)(x_1)).$$

This rule is obtained from the rule $(q_0, \{\hat{q}_0\})(f(x_1, x_2)) \to f((q_1, S)(x_1), q_2(x_2))$ of \hat{T}_1 .

Consider the input tree $f(s_1, s_2)$ where s_1 and s_2 are arbitrary ground trees. Clearly, translating $f(s_1, s_2)$ with N begins with the rule η . Recall that the transducer N is equipped with look-ahead in order to guarantee that guesses performed by states of N are valid. In particular, to guarantee that the guess corresponding to η is valid, we need to test whether or not $s_1 \in \text{dom}(q_1, S)$ and $s_2 \in \text{dom}(q_2)$. Therefore, M contains the rule

$$(q_0, \{\hat{q}_0\}, \hat{q}_0)(f(x_1: \{(q_1, S)\}, x_2: \{q_2\})) \to f((q_1, S, \hat{q}_1)(x_1), (q_1, S, \hat{q}_2)(x_1)).$$

Recall that if f is relabeled by $\langle f, \{q_1, S\}, \{q_2\} \rangle$ via the la-automaton \hat{A} , this means precisely that $s_1 \in \text{dom}(q_1, S)$ and $s_2 \in \text{dom}(q_2)$. We remark that by definition, M also contains rules of the form

$$(q_0, \{\hat{q}_0\}, \hat{q}_0)(f(x_1: l_1, x_2: l_2)) \to f((q_1, S, \hat{q}_1)(x_1), (q_1, S, \hat{q}_2)(x_1)),$$

where l_1 and l_2 are states of \hat{A} such that $\{q_1, S\} \subseteq l_1$ and $\{q_2\} \subseteq l_2$ and l_1 or l_2 is a proper superset. However, as none such states l_1 and l_2 are reachable by \hat{A} , we have omitted rules of this form. Other rules are omitted for the same reason. \square

3.2 Correctness of the LA-Transducer M

In the following we prove the correctness of our construction. More precisely, we prove that M is functional if and only if $T_1 \,\hat{\circ}\, T_2$ is. By Lemma 3, it is sufficient to show that M is functional if and only if $\hat{T}_1 \,\hat{\circ}\, T_2$ is.

First, we prove that the following claim: If M is functional then $\hat{T}_1 \,\hat{\circ}\, T_2$ is functional. More precisely, we show that $\mathcal{R}(\hat{T}_1) \,\circ\, \mathcal{R}(T_2) \subseteq \mathcal{R}(M)$. Obviously, this implies our claim. First of all, consider the transducers N and N' obtained from the p-construction in our examples in Section 3. Notice that the relations defined by N and N' are supersets of $\mathcal{R}(T_1) \,\circ\, \mathcal{R}(T_2)$ and $\mathcal{R}(T_1') \,\circ\, \mathcal{R}(T_2')$, respectively.

In the following, we show that this observation can be generalized. Consider arbitrary transducers T and T'. We claim that the transducer \check{N} obtained from the p-construction for T and T' always defines a superset of the composition $\mathcal{R}(T) \circ \mathcal{R}(T')$. To see that our claim holds, consider a translation of $T \circ T'$ in which the state q' of T' processes a tree t produced by the state q of T on input s. If the corresponding state (q,q') of \check{N} processes s then (q,q') can guess that q has produced t and proceed accordingly. Thus \check{N} can effectively simulate the composition $T \circ T'$.

As M is in essence obtained from the p-construction extended with look-ahead, M 'inherits' this property. Note that the addition of look-ahead does not affect this property. Therefore our claim follows.

Lemma 4. $\mathcal{R}(\hat{T}_1) \circ \mathcal{R}(T_2) \subseteq \mathcal{R}(M)$.

In fact an even stronger result holds.

Lemma 5. Let (q_1, S) be a state of \hat{T}_1 and q_2 be a state of T_2 . If on input s, (q_1, S) can produce the tree t and on input t, q_2 can produce the tree r then (q_1, S, q_2) can produce r on input s.

Consider a translation of $\hat{T}_1 \,\hat{\circ} \, T_2$ in which T_2 processes the tree t produced by T_1 on input s. We call a translation of M synchronized if the translation simulates a translation of $\hat{T}_1 \,\hat{\circ} \, T_2$, i.e., if a state (q, S, q') of M processes the subtree s' of s and the corresponding state of q' of T_2 processes the subtree t' of t and t' is produced by (q, S) on input s', then (q, S, q') guesses t'.

We now show that if $\hat{T}_1 \circ T_2$ is functional, then so is M. Before we prove our claim consider the following auxiliary results.

Lemma 6. Consider an arbitrary input tree s. Let \hat{s} be a subtree of s. Assume that in an arbitrary translation of M on input s, the state (q_1, S, q_2) processes \hat{s} . Then, a synchronized translation of M on input s exists in which the state (q_1, S, q_2) processes the subtree \hat{s} .

It is easy to see that the following result holds for arbitrary transducers.

Proposition 1. Let $\tau = T_1 \circ T_2$ where T_1 and T_2 are arbitrary transducers. Let s be a tree such that $\tau(s) = \{r\}$ is a singleton. Let t_1 and t_2 be distinct trees produced by T_1 on input s. If t_1 and t_2 are in the domain of T_2 then $T_2(t_1) = T_2(t_2) = \{r\}$.

Using Lemma 6 and Proposition 1, we now show that the following holds. Note that in the following t/v, where t is some tree and v is a node, denotes the subtree of t rooted at the node v.

Lemma 7. Consider an arbitrary input tree s. Let \hat{s} be a subtree of s. Let the state (q_1, S, q_2) process \hat{s} in a translation M on input s. If $\hat{T}_1 \hat{\circ} T_2$ is functional then (q_1, S, q_2) can only produce a single output tree on input \hat{s} .

Proof. Assume to the contrary that (q_1, S, q_2) can produce distinct trees r_1 and r_2 on input \hat{s} . For r_1 , it can be shown that a tree t_1 exists such that

- 1. on input \hat{s} , the state (q_1, S) of \hat{T}_1 produces t_1 and
- 2. on input t_1 , the state q_2 of T_2 produces r_1 .

It can be shown that a tree t_2 with the same properties exists for r_2 . Informally, this means that r_1 and r_2 are producible by (q_1, S, q_2) by simulating the 'composition of (q_1, S) and q_2 '.

Due to Lemma 6, a synchronized translation of M on input s exists in which the state (q_1, S, q_2) processes the subtree \hat{s} of s. Let g be the node at which (q_1, S, q_2) processes \hat{s} . Let $\hat{q}_1, \ldots, \hat{q}_n$ be all states of M of the form (q_1, S, q_2') , where q_2' is some state of T_2 , that occur in the synchronized translation of M and that process \hat{s} . Note that by definition $q_2' \in S$. Due to Lemmas 2 and 5, we can assume that in the synchronized translation, the states $\hat{q}_1, \ldots, \hat{q}_n$ all guess that the tree t_1 has been produced by the state (q_1, S) of \hat{T}_1 on input \hat{s} . Hence, we can assume that at the node g, the output subtree r_1 is produced. Therefore, a synchronized translation of M on input s exists, that yields an output tree \hat{r}_1 such that $\hat{r}_1/g = r_1$, where \hat{r}_1/g denotes the subtree of \hat{r}_1 rooted at the node g. Analogously, it follows that a synchronized translation of M on input s exists, that yields an output tree \hat{r}_2 such that $\hat{r}_2/g = r_2$.

As both translation are synchronized, i.e., 'simulations' of translations of $\hat{T}_1 \,\hat{\circ}\, T_2$ on input s, it follows that the trees \hat{r}_1 and \hat{r}_2 are producible by $\hat{T}_1 \,\hat{\circ}\, T_2$ on input s. Due to Proposition 1, $\hat{r}_1 = \hat{r}_2$ and therefore $r_1 = \hat{r}_1/g = \hat{r}_2/g = r_2$.

Lemma 4 implies that if M is functional then $\hat{T}_1 \circ T_2$ is functional as well. Lemma 7 implies that if $\hat{T}_1 \circ T_2$ is functional then so is M. Therefore, we deduce that due Lemmas 4 and 7 the following holds.

Corollary 1. $\hat{T}_1 \circ T_2$ is functional if and only if M is functional.

In fact, Corollary 1 together with Lemma 4 imply that $\hat{T}_1 \,\hat{\circ}\, T_2$ and M are equivalent if $\hat{T}_1 \,\hat{\circ}\, T_2$ is functional, since it can be shown that $\mathrm{dom}(\hat{T}_1 \,\hat{\circ}\, T_2) = \mathrm{dom}(M)$.

Since functionality for transducers with look-ahead is decidable [8], Corollary 1 implies that it is decidable whether or not $\hat{T}_1 \circ T_2$ is functional. Together with Lemma 3, we obtain:

Theorem 1. Let T_1 and T_2 be top-down tree transducers. It is decidable whether or not $T_1 \circ T_2$ is functional.

3.3 Functionality of Arbitrary Compositions

In this section, we show that the question whether or not an arbitrary composition is functional can be reduced to the question of whether or not a two-fold composition is functional.

Lemma 8. Let τ be a composition of transducers. Then two transducers T_1, T_2 can be constructed such that $T_1 \circ T_2$ is functional if and only if τ is functional.

Proof. Consider a composition of n transducers T'_1, \ldots, T'_n . W.l.og. assume that n > 2. For $n \le 2$, our claim follows trivially. Let τ be the composition of T'_1, \ldots, T'_n . We show that transducer $\hat{T}_1, \ldots, \hat{T}_{n-1}$ exist such that $\hat{T}_1 \hat{\circ} \cdots \hat{\circ} \hat{T}_{n-1}$ is functional if and only if τ is.

Consider an arbitrary input tree s. Let t be a tree produced by the composition $T_1' \circ \cdots \circ T_{n-2}'$ on input s. Analogously as in Proposition 1, the composition $T_{n-1}' \circ T_n'$, on input t, can only produce a single output tree if τ is functional. For the transducers T_{n-1}' and T_n' , we construct the la-transducer M according to our construction in Section 3.1. It can be shown that, the la-transducer M our construction yields has the following properties regardless of whether or not $T_{n-1} \circ T_n$ is functional

- (a) $dom(M) = dom(T_{n-1} \hat{\circ} T_n)$ and
- (b) on input t, M only produces a single output tree if and only if $T_{n-1} \circ T_n$ does

Therefore, $\tau(s)$ is a singleton if and only if $T_1' \circ \cdots \circ T_{n-2}' \circ M(s)$ is a singleton. Engelfriet has shown that every transducer with look-ahead can be decomposed to a composition of a deterministic bottom-up relabeling and a transducer (Theorem 2.6 of [4]). It is well known that (nondeterministic) relabelings are independent of whether they are defined by bottom-up transducers or by top-down transducers (Lemma 3.2 of [3]). Thus, any transducer with look-ahead can be decomposed into a composition of a nondeterministic top-down relabeling and a transducer. Let R and T be the relabeling and the transducer such that M and $R \circ T$ are equivalent. Then obviously, $\tau(s)$ is a singleton if and only if $T_1' \circ \cdots \circ T_{n-2}' \circ R \circ T(s)$ is a singleton.

Consider arbitrary transducers \bar{T}_1 and \bar{T}_2 . Baker has shown that if \bar{T}_2 is non-deleting and linear then a transducer T can be constructed such that T and $\bar{T}_1 \,\hat{\circ}\, \bar{T}_2$ are equivalent (Theorem 1 of [2]). By definition, any relabeling is non-deleting and linear. Thus, we can construct a transducer \tilde{T} such that \tilde{T} and $T'_{n-2} \,\hat{\circ}\, R$ are equivalent. Therefore, it follows that $\tau(s)$ is a singleton if and only if $T'_1 \,\hat{\circ}\, \cdots \,\hat{\circ}\, T'_{n-3} \,\hat{\circ}\, \tilde{T}\,\hat{\circ}\, T(s)$ is a singleton. This yields our claim.

Lemma 8 and Theorem 1 yield that functionality of compositions of transducers is decidable.

Engelfriet has shown that any la-transducer can be decomposed into a composition of a nondeterministic top-down relabeling and a transducer [4,3]. Recall that while la-transducers generalize transducers, bottom-up transducers and latransducers are incomparable [4]. Baker, however, has shown that the composition of n bottom-up-transducers can be realized by the composition of n+1 top-down transducers [2]. For any functional composition of transducers an equivalent deterministic la-transducer can be constructed [5]. Therefore we obtain our following main result.

Theorem 2. Functionality for arbitrary compositions of top-down and bottomup tree transducers is decidable. In the affirmative case, an equivalent deterministic top-down tree transducer with look-ahead can be constructed.

4 Conclusion

We have presented a construction of an la-transducer for a composition of transducers which is functional if and only if the composition of the transducers is functional — in which case it is equivalent to the composition. This construction is remarkable since transducers are not closed under composition in general, neither does functionality of the composition imply that each transducer occurring therein, is functional. By Engelfriet's construction in [5], our construction provides the key step to an efficient implementation (i.e., a deterministic transducer, possibly with look-ahead) for a composition of transducers — whenever possible (i.e., when their translation is functional). As an open question, it remains to see how large the resulting functional transducer necessarily must be, and whether the construction can be simplified if for instance only compositions of linear transducers are considered.

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A Appendix

In the following, we first introduce additional notation and definitions used in the proofs in the Appendix.

A.1 Definitions

Set of Nodes Let t be a tree. For t, its set V(t) of nodes is a subset of $V = \mathbb{N}^*$. More formally, $V(t) = \{\epsilon\} \cup \{iu \mid i \in [k], u \in V(t_i)\}$ where $t = a(t_1, \dots, t_k)$, $a \in \Sigma_k$, $k \geq 0$ and $t_1, \dots, t_k \in T_{\Sigma}$. For better readability we add dots between numbers. E.g. for the tree t = f(a, f(a, b)) we have $V(t) = \{\epsilon, 1, 2, 2.1, 2.2\}$. For $v \in V(t)$, t[v] is the label of v and t/v is the subtree of t rooted at v.

Substitutions Let $t_1, \ldots t_n$ be trees over Σ and v_1, \ldots, v_n be distinct nodes none of which is a prefix of the other, then we denote by $[v_i \leftarrow t_i \mid i \in [n]]$ the substitution that for each $i \in [n]$, replaces the subtree rooted at v_i with t_i .

Let t be a tree, $a \in \Sigma_0$ and \mathcal{T} be a set of trees. We denote by $t[a \leftarrow \mathcal{T}]$ the set of all trees obtained by substituting leaves labeled by a with some tree in \mathcal{T} , i.e., the set of all trees of the form $t[v \leftarrow t_v \mid v \in V(t), t[v] = a]$ where for all a-leaves $v, t_v \in \mathcal{T}$. Note that two distinct leaves labeled by a may be replaced by distinct trees in \mathcal{T} . If $\mathcal{T} = \emptyset$ then we define $t[a \leftarrow \mathcal{T}] = \emptyset$. For simplicity, we write $t[a \leftarrow t']$ if $\mathcal{T} = \{t'\}$.

Partial Trees and Semantic of a Transducer Recall that $T_{\Sigma}[B] = T_{\Sigma'}$ where Σ' is obtained from Σ by $\Sigma'_0 = \Sigma_0 \cup B$ while for all k > 0, $\Sigma'_k = \Sigma_k$. In the following, we call a tree in $T_{\Sigma}[B]$ a partial tree.

The semantic of a transducer T, defined as in Section 2, is formally defined as follows. Let $q \in Q$ and v be an arbitrary node. We denote by $[\![q]\!]_v^T$ the partial function from $T_{\Sigma}[B]$ to the power set of $T_{\Delta}[Q(V)]$ defined as follows

- for
$$s = a(s_1, \ldots, s_k), a \in \Sigma_k$$
, and $s_1, \ldots, s_k \in T_{\Sigma}[B]$,

$$[\![q]\!]_v^T(s) = \bigcup_{\xi \in \text{rhs}_T(q,a)} \xi[q(x_i) \leftarrow [\![q]\!]_{v.i}(s_i) \mid q \in Q, i \in [k]]$$

- for
$$b \in B$$
, $[q]_v^T(b) = \{q(v)\},\$

where $\operatorname{rhs}_T(q, a)$ denotes the set of all right-hand sides of q and a. The reason why input nodes of s are added to the semantic of T is that for some of our proofs we require that for states q of T it is traceable which input node q currently processes.

If clear from context which transducer is meant, we omit the superscript T and write $\llbracket q \rrbracket_v$ instead of $\llbracket q \rrbracket_v^T$. In the following, we write $\llbracket q \rrbracket$ instead of $\llbracket q \rrbracket_\epsilon$ for simplicity. We write $\llbracket q \rrbracket_v^T(s) \Rightarrow t$ if $t \in \llbracket q \rrbracket_v^T(s)$.

In the following, for trees in $T_{\Delta}[Q(X)]$ and $T_{\Delta}[Q(V)]$, we write $t\langle x \leftarrow v \rangle$ to denote the substitution $t[q(x) \leftarrow q(v) \mid q \in Q]$ for better readability where $x \in X$ and $v \in V$.

Recall that for a set Γ of right-hand sides of a transducer T, $\Gamma\langle x_i \rangle$ denotes the set of all states q of T such that $q(x_i)$ occurs in some tree γ in Γ . For a set Λ of trees in $T_{\Delta}[Q(V)]$, we define $\Lambda\langle v \rangle$ where v is some node analogously; e.g., for $\Lambda = \{f(q(v_1), f(q(v_2), q'(v_2))), f(q_1(v_1), q_2(v_2))\}$, we have $\Lambda\langle v_1 \rangle = \{q, q_1\}$ and $\Lambda\langle x_2 \rangle = \{q, q', q_2\}$.

B Properties of the Domain Automaton A

In the following we consider the *domain automaton* A introduced in Section 3 for a transducer T. In particular, we consider the properties of A. Recall that a state of A is a set consisting of states of T. In Section 3, we have claimed that if in a translation of T on input s, the states $q_1 \ldots, q_n$ process the node v of s then $\{q_1 \ldots, q_n\}$ processes the node v of s in a computation of A. We now formally prove this statement. First we prove the following auxiliary result.

Lemma 9. Let $s \in T_{\Sigma}[X]$. Let $v_1, \ldots v_n$ be the nodes of s that are labeled by a symbol in X. Let S_1 and S_2 be states of A and let for j = 1, 2,

$$[\![S_j]\!](s) \Rightarrow s[v_i \leftarrow S_i^i(v_i) \mid i \in [n]]$$

where for $i \in [n]$, S_i^i is a state of A. Then

$$[S_1 \cup S_2](s) \Rightarrow s[v_i \leftarrow S_1^i \cup S_2^i(v_i) \mid i \in [n]].$$

Proof. We prove our claim by structural induction over s. Let $s = a(s_1, \ldots, s_k)$ where $a \in \Sigma_k$, $k \ge 0$, and for $\iota \in [k]$, $s_\iota \in T_{\Sigma}[X]$. As

$$[S_1](s) \Rightarrow s[v_i \leftarrow S_1^i(v_i) \mid i \in [n]],$$

a rule $S_1(a(x_1,\ldots,x_k)) \to a(\hat{S}_1(x_1),\ldots,\hat{S}_k(x_k))$ exists such that for all $\iota \in [k]$, on input s_ι , the function $[\![\hat{S}_\iota]\!]_\iota$ yields the subtree of $s[v_i \leftarrow S_1^i(v_i) \mid i \in [n]]$ that is rooted at the node ι . More formally,

$$[\hat{S}_{\iota}]_{\iota}(s_{\iota}) \Rightarrow s_{\iota}[v' \leftarrow S_{\iota}^{i}(\iota.v') \mid v' \in V(s_{\iota}) \text{ and } \iota.v' = v_{i} \text{ where } i \in [n]],$$

which in turn implies

$$[\hat{S}_{\iota}](s_{\iota}) \Rightarrow s_{\iota}[v' \leftarrow S_{1}^{i}(v') \mid v' \in V(s_{\iota}) \text{ and } \iota.v' = v_{i} \text{ where } i \in [n]].$$

Analogously, it follows that a rule $S_2(a(x_1,\ldots,x_k)) \to a(\hat{S}'_1(x_1),\ldots,\hat{S}'_k(x_k))$ exists such that for all $\iota \in [k]$,

$$\|\hat{S}_{\iota}'\|(s_{\iota}) \Rightarrow s_{\iota}[v' \leftarrow S_{2}^{i}(v') \mid v' \in V(s_{\iota}) \text{ and } \iota.v' = v_{i} \text{ where } i \in [n]].$$

We now show that the automaton A contains the rule

$$S_1 \cup S_2(a(x_1, \dots, x_k)) \to a(\hat{S}_1 \cup \hat{S}'_1(x_1), \dots, \hat{S}_k \cup \hat{S}'_k(x_k)).$$
 (a)

By construction, the rule $S_1(a(x_1,\ldots,x_k)) \to a(\hat{S}_1(x_1),\ldots,\hat{S}_k(x_k))$ is defined only if for all $q \in S_1$ a non-empty set of right-hand sides $\Gamma_q \subseteq \operatorname{rhs}_T(q,a)$ exists such that for $\iota \in [k]$, $\hat{S}_\iota = \bigcup_{q \in S_1} \Gamma_q \langle x_\iota \rangle$.

Likewise, the rule $S_2(a(x_1,\ldots,x_k)) \to a(\hat{S}'_1(x_1),\ldots,\hat{S}'_k(x_k))$ is defined only if for all $q' \in S_2$ a non-empty set of right-hand sides $\Gamma'_{q'} \subseteq \operatorname{rhs}_T(q',a)$ exists such that for $\iota \in [k]$, $\hat{S}'_{\iota} = \bigcup_{q' \in S_2} \Gamma'_{q'} \langle x_{\iota} \rangle$. For all states $q \in S_1 \cup S_2$, we define

Clearly, the sets $\check{\Gamma}_q$ yield that the rule defined in (a) exists. Now, consider the following. As for $\iota \in [k]$,

$$[\hat{S}_{\iota}](s_{\iota}) \Rightarrow s_{\iota}[v' \leftarrow S_{1}^{i}(v') \mid v' \in V(s_{\iota}) \text{ and } \iota.v' = v_{i} \text{ where } i \in [n]]$$

and

$$[\hat{S}'_{\iota}](s_{\iota}) \Rightarrow s_{\iota}[v' \leftarrow S^{i}_{2}(v') \mid v' \in V(s_{\iota}) \text{ and } \iota.v' = v_{i} \text{ where } i \in [n]]$$

the induction hypothesis yields

$$[\hat{S}_{\iota} \cup \hat{S}'_{\iota}](s_{\iota}) \Rightarrow s_{\iota}[v' \leftarrow S_1^i \cup S_2^i(v') \mid v' \in V(s_{\iota}) \text{ and } \iota.v' = v_i \text{ where } i \in [n]]$$
 (b)

Thus, due to (a) and (b) our claim follows.

We now prove our statement.

Lemma 10. Let $s \in T_{\Sigma}[X]$. Let $v_1, \ldots v_n$ be the nodes of s that are labeled by a symbol in X. Let S be a state of A. For $q \in S$, let $[\![q]\!]^T(s) \Rightarrow t_q$. Then,

$$[S]^A(s) \Rightarrow s[v_i \leftarrow S_i(v_i) \mid i \in [n]]$$

where $S_i = \bigcup_{q \in S} t_q \langle v_i \rangle$.

Proof. Due to Lemma 9, it is sufficient to show that if $[q]^T(s) \Rightarrow t_q$, then

$$\llbracket \{q\} \rrbracket^A(s) \Rightarrow s[v_i \leftarrow S_i'(v_i) \mid i \in [n]]$$

where $S_i' = t_q \langle v_i \rangle$. We prove this claim by structural induction over s. Let $s = a(s_1, \ldots, s_k)$ where $a \in \Sigma_k$, $k \ge 0$, and for $i \in [k]$, $s_i \in T_{\Sigma}[X]$. Due to our premise, $\gamma \in \text{rhs}(q, a)$ exists such that

$$t_q \in \gamma[q'(x_i) \leftarrow \llbracket q' \rrbracket_i(s_i) \mid q' \in Q, i \in [k]]. \tag{1}$$

By definition of $A, \gamma \in \text{rhs}(q, a)$ implies that the automaton A contains the rule

$$\{q\}(a) \to a(\hat{S}_1(x_1), \dots, \hat{S}_k(x_k))$$
 (a)

where $\hat{S}_i = \gamma \langle x_i \rangle$ for $i \in [k]$.

Now consider $\gamma \in \operatorname{rhs}(q, a)$ in conjunction with the variable x_i . Let $\gamma\langle x_i \rangle = \{q_1, \ldots, q_m\}$. For $j \in [m]$, we denote by U_j the set of all nodes of γ that are labeled by $q_j(x_i)$. For all nodes $u \in U_j$, Equation 1 clearly implies $[\![q_j]\!]_i^T(s_i) \Rightarrow t_q/u$. Recall that by definition, if $\check{q}(\check{v})$ occurs in t_q/u , where \check{q} is a state of T and \check{v} is some node then \check{v} is of the form i.v'. Clearly, $[\![q_j]\!]_i^T(s_i) \Rightarrow t_q/u$ implies $[\![q_j]\!]_i^T(s_i) \Rightarrow \eta_u$ where η_u denotes the tree obtained from t_q/u by substituting occurrences of $\check{q}(i.v')$ by $\check{q}(v')$.

Recall that $\eta_u\langle v'\rangle$ denotes the set of all states q' in Q such that q'(v') occurs in η_u . In the following, let $S_{u,v'} = \eta_u\langle v'\rangle$. Due to the induction hypothesis, it follows that for all $u \in U_j$,

$$[\![\{q_j\}]\!]^A(s_i) \Rightarrow s_i[v' \leftarrow S_{u,v'}(v') \mid v' \in V(s_i), s_i[v'] \in X]. \tag{2}$$

Due to Lemma 9 and Equation 2, it follows for all $j \in [m]$ that

$$[\![\{q_j\}]\!]^A(s_i) \Rightarrow s_i[v' \leftarrow \bigcup_{u \in U_j} S_{u,v'}(v') \mid v' \in V(s_i), s_i[v'] \in X]$$
(3)

Recall that $\hat{S}_i = \gamma \langle x_i \rangle$. Thus, Lemma 9 and Equation 3 yield

$$[\![\hat{S}_i]\!]^A(s_i) \Rightarrow s_i[v' \leftarrow \bigcup_{j \in [m]} \bigcup_{u \in U_j} S_{u,v'}(v') \mid v' \in V(s_i), s_i[v'] \in X]$$
 (b)

Note that for v = i.v',

$$\bigcup_{j \in [m]} \bigcup_{u \in U_j} \eta_u \langle v' \rangle = \bigcup_{j \in [m]} \bigcup_{u \in U_j} S_{u,v'} = t \langle v \rangle.$$

Therefore it follows that (a) and (b) yield our claim.

Additionally, the domain automaton A has the following property. If in a translation of A on input s, the state S processes the node v of s then a translation of T on input s exists such that v is only processed by states in S. More formally, we prove thee following result.

Lemma 11. Let $s \in T_{\Sigma}[X]$. Let $v_1, \ldots v_n$ be the nodes of s that are labeled by a symbol in X. Let S, S_1, \ldots, S_n be states of S. Let $S = \hat{S} = \hat$

$$\hat{s} = s[v_i \leftarrow S_i(v_i) \mid i \in [n]].$$

For all $q \in S$, a tree t exists such that $[\![q]\!]^T(s) \Rightarrow t$ and for $i \in [n]$, $t\langle v_i \rangle \subseteq S_i$.

Proof. We prove our claim by structural induction. Obviously, our claim holds if $s \in X$.

In the following, let $s \notin X$. Then, a node v exist such that the subtree of s rooted at v is of the form $a(s_1, \ldots, s_k)$ where $a \in \Sigma_k$, $k \ge 0$, and $s_1, \ldots, s_k \in X$. Note that by definition v can be a leaf. Consider the tree $s' = s[v \leftarrow x_1]$. Let

 v_1', \ldots, v_m' be the nodes of s' that are labeled by a symbol in X. W.l.o.g. let

Recall that the node v is labeled by the node a in s. As $[S]^A(s) \Rightarrow \hat{s}$, it follows that a tree \tilde{s} exists such that $[S]^A(s') \Rightarrow \tilde{s}$ and the tree s can be obtained from \tilde{s} by substituting $S'_1(v)$ by $\xi \langle x_i \leftarrow v.j \mid j \in [k] \rangle$ where ξ is a right-hand side of S'_1 and a. Note that by definition, states S'_1, \ldots, S'_m of A exists such that the tree \tilde{s} is obtained from s' by relabeling the node v'_i of s' by $S'_i(v_i)$. More formally, it holds that

$$\tilde{s} = s'[v'_i \leftarrow S'_i(v'_i) \mid i \in [m]],$$

and that

$$\hat{s} = \tilde{s}[S_1'(v) \leftarrow \xi \langle x_i \leftarrow v.j \mid j \in [k] \rangle]. \tag{1}$$

By induction hypothesis, as $[S]^A(s') \Rightarrow \tilde{s}$, for all states $q \in S$, a tree t' exists such that

- 1. $[\![q]\!]^T(s') \Rightarrow t'$ and 2. for $i \in [m], t'\langle v_i' \rangle \subseteq S_i'$.

In particular, it holds that $t'\langle v_1'\rangle = t'\langle v\rangle \subseteq S_1'$.

Let $S_1' = \{q_1, \dots, q_n\}$. Recall that ξ is a right-hand side of S_1' and a. W.l.o.g. let $\xi = a(S_1(x_1), \dots, S_k(x_k))$. Then, by definition of the rules of A, it follows that for each $j \in [n]$, a tree γ_i exists such that

- (a) $\gamma_j \in \text{rhs}_T(q_j, a)$ and
- (b) for $\iota \in [k]$ it holds that $\bigcup_{i \in [n]} \gamma_i \langle x_\iota \rangle \subseteq S_\iota$.

As $t'\langle v\rangle\subseteq S_1'$, it follows that $[\![q]\!]^T(s)\Rightarrow t$ where

$$t = t'[q_j(v) \leftarrow \gamma_j \langle x_i \leftarrow v.i \mid i \in [k] \rangle \mid j \in [n]]. \tag{2}$$

Consider the node $v.\iota$ where $\iota \in [k]$. If $\check{S}(v.\iota)$ occurs in \hat{s} then it follows due to Equation 1 that $\check{S}(x_{\iota})$ occurs in ξ . Due to Equation 2 and Statement (b), it follows that $t\langle v.\iota\rangle\subseteq S_\iota$. Thus, our claim follows.

Lemmas 10 and 11 yield Lemma 1.

Properties of \hat{T}_1

In the following we consider the properties of the transducer \hat{T}_1 . Recall that \hat{T}_1 is obtained via the p-construction from the transducer T_1 and the domain automaton A of T_2 . In particular, we formally prove the statements we made about \hat{T}_1 in Section 3. First we formally prove Lemma 2, that is, we prove the following.

Lemma 12. Let (q, S) be a state of \hat{T}_1 and $S \neq \emptyset$. If the tree t over Δ is producible by (q, S) then $t \in_{q_2 \in S} \bigcap dom(q_2)$.

Proof. Let t be produced by (q, S) on input s where $s \in T_{\Sigma}$. Clearly, it is sufficient to show that $[S]^A(t) \Rightarrow t$ due to Lemma 1. We prove our claim by structural induction. Let $s = a(s_1, \ldots, s_k)$ where $a \in \Sigma_k, k \geq 0$, and $s_1, \ldots, s_k \in T_{\Sigma}$. As t is produced by (q, S) on input s, a right-hand side ξ for (q, S) and s exists such that

$$t \in \xi[(q', S')(x_i) \leftarrow \llbracket (q', S') \rrbracket_i^{\hat{T}_1}(s_i) \mid (q', S') \in \hat{Q}_1, i \in [k]]. \tag{1}$$

This means that

$$t = \xi[u \leftarrow t/u \mid u \in V(\xi), \xi[u] \text{ is of the form } (q', S')(x_i)].$$

The following statements hold:

1. By definition of \hat{T}_1 , it follows that $[S]^A(\xi) \Rightarrow \xi'$ where

$$\xi' = \xi[u \leftarrow S'(u) \mid u \in V(\xi), \xi[u] \text{ is of the form } (q', S')(x_i)].$$

2. Consider the node u. Let u be labeled by $(q', S')(x_i)$ in ξ . This means that u is labeled by S'(u) in ξ' . By Equation 1, t/u can be produced by (q', S') on input s_i .

By induction hypothesis, $[S']^A(t') \Rightarrow t'$ for all trees t' producible by (q', S'). Thus, $[S']^A(t/u) \Rightarrow t/u$. By definition $[S']^A(t/u) \Rightarrow t/u$ implies

$$[S']_u^A(t/u) \Rightarrow t/u$$

because t/u is ground.

Statements (1) and (2) yield that
$$[S]^A(t) \Rightarrow t$$
.

We now show that the converse holds as well.

Lemma 13. Let $s \in T_{\Sigma}$ and t be producible by the state q_1 of T_1 on input s. Let $S \subseteq Q_2$ such that $t \in \bigcap_{q \in S} dom(q)$. Then t be producible by the state (q_1, S) of \hat{T}_1 on input s.

Proof. We prove our claim by structural induction. Let $s = a(s_1, \ldots, s_k)$, $a \in \Sigma_k$, $k \ge 0$, and $s_1, \ldots, s_k \in T_{\Sigma}$. As t be producible by the state q_1 of T_1 on input s, $\xi \in \text{rhs}_{T_1}(q_1, a)$ exists such that

$$t \in \xi[q'(x_i) \leftarrow [\![q']\!]_i^{T_1}(s_i) \mid q' \in Q_1, i \in [k]].$$

In essence, this means that

$$t = \xi[u \leftarrow t/u \mid u \in V(\xi), \xi[u] \text{ is of the form } q'(x_i)].$$

Hence, it follows that if a node u is labeled by $q'(x_i)$ in ξ then

$$[\![q']\!]_i^{T_1}(s_i) \Rightarrow t/u. \tag{*}$$

By our premise $t \in \bigcap_{q \in S} \operatorname{dom}(q)$. Due to Lemma 1, it follows that $[S]^A(t) \Rightarrow t$. Therefore, for all leafs u of ξ that are labeled by a symbol in of the form $q_1(x_i)$, a state S_u of A exists such that

$$[S]^A(\xi) \Rightarrow \xi[u \leftarrow S_u(u) \mid u \in V(\xi), \xi[u] \in Q_1(X)]$$

and $[S_u]_u^A(t/u) \Rightarrow t/u$. By definition of \hat{T}_1 , the former implies that

$$(q_1, S)(a(x_1, \dots, x_k)) \to \xi[u \leftarrow (q', S_u)(x_i) \mid u \in V(\xi), \xi[u] = q'(x_i)]$$
 (†)

is a rule of \hat{T}_1 . The later implies $[S_u]^A(t/u) \Rightarrow t/u$ as t_u is ground. Therefore, $t/u \in \bigcap_{q \in S_u} \text{dom}(q)$ due to Lemma 1.

Consider an arbitrary node \check{u} of ξ . Let \check{u} labeled by $\check{q}(x_i)$ in ξ . Then, $t/\check{u} \in \bigcap_{q \in S_{\check{u}}} \mathrm{dom}(q)$. Furthermore, due to (*), it follows that $[\![\check{q}]\!]_i^{T_1}(s_i) \Rightarrow t/\check{u}$.

Then, the induction hypothesis yields that $[\![(\breve{q}, S_{\breve{u}})]\!]^{\hat{T}_1}(s_i) \to t/\breve{u}$. Note that $[\![(\breve{q}, S_{\breve{u}})]\!]^{\hat{T}_1}(s_i) \to t/\breve{u}$ implies $[\![(\breve{q}, S_{\breve{u}})]\!]^{\hat{T}_1}(s_i) \to t/\breve{u}$ because s_i is ground. Along with (\dagger) , this yields our claim.

Lemmas 12 and 13 allow us to prove the following statement, which implies Lemma 3.

Lemma 14. $dom(\hat{T}_1) = dom(T_1 \circ T_2)$ and for $s \in T_{\Sigma}$, $\hat{T}_1(s) = T_1(s) \cap dom(T_2)$.

Proof. First we show that $\operatorname{dom}(\hat{T}_1) = \operatorname{dom}(T_1 \,\hat{\circ}\, T_2)$. Let $s \in \operatorname{dom}(\hat{T}_1)$, i.e., a tree t over Δ exists such that $[\![q_1^0, \{q_2^0\}]\!]^{\hat{T}_1}(s) \Rightarrow t$, where $(q_1^0, \{q_2^0\})$ is the initial state of \hat{T}_1 . By construction of \hat{T}_1 , it follows that $[\![q_1^0]\!]^{T_1}(s) \Rightarrow t$ and by Lemma 12, $t \in \operatorname{dom}(q_2^0)$. Hence $s \in \operatorname{dom}(T_1 \,\hat{\circ}\, T_2)$. For the converse, let $s \in \operatorname{dom}(T_1 \,\hat{\circ}\, T_2)$. Then, a tree t over Δ exists such that $[\![q_1^0]\!]^{T_1}(s) \Rightarrow t$, where q_1^0 is the initial state of T_1 , and $t \in \operatorname{dom}(q_2)$. Hence, due to Lemma 13 it follows that $[\![q_1^0, \{q_2^0\}]\!]^{\hat{T}_1}(s) \Rightarrow t$ and thus, $s \in \operatorname{dom}(\hat{T}_1)$.

Now we show that $\hat{T}_1(s) = T_1(s) \cap \text{dom}(T_2)$. Let $[(q_1^0, \{q_2^0\})]^{\hat{T}_1}(s) \Rightarrow t$. By construction of \hat{T}_1 , $[[q_1^0]]^{T_1}(s) \Rightarrow t$ holds. By Lemma 12, $t \in \text{dom}(q_2^0)$. Therefore, our claim follows. Conversely, let $t \in T_1(s) \cap \text{dom}(T_2)$. Then, clearly $t \in \text{dom}(q_2^0)$ and $[[q_1^0]]^{T_1}(s) \Rightarrow t$. By Lemma 13, $[[(q_1^0, \{q_2^0\})]]^{\hat{T}_1}(s) \Rightarrow t$ which yields our claim.

D Correctness of the LA-Transducer M

In this section, we present the formal proof of correctness for the la-transducer M, i.e., we show that M is functional if and only if $T_1 \circ T_2$ is functional. Recall that due to Lemma 3, it is sufficient to consider $\hat{T}_1 \circ T_2$.

In the following, denote by L the set of states of the la-automaton of M. W.l.o.g. we assume that for all states l in L, $dom(l) \neq \emptyset$. In the remainder of this section, our proofs employ partial trees in $T_{\Sigma}[L]$. Consider such a tree s. Recall that in a translation of M input trees are first preprocessed by a relabeling induced by the la-automaton of M. We demand that in a translation

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of M the tree s is relabeled as follows: If the i-th child of the node v of s is labeled by $l \in L$ then we require that v be relabeled by a symbol of the form $\langle a, l_1, \ldots l_{i-1}, l, l_{i+1}, \ldots l_k \rangle$.

For instance, consider the la-automaton $B=(\{p,p'\}, \Sigma, \Sigma, R, \{p\})$ where $\Sigma=\{f^2,a^0,b^0\}$ and R contains the rules

$$\begin{array}{ll} p(f(x_1, x_2)) \to f(p(x_1), p(x_2)) & p(f(x_1, x_2)) \to f(p(x_1), p'(x_2)) \\ p'(f(x_1, x_2)) \to f(p'(x_1), p'(x_2)) & p'(f(x_1, x_2)) \to f(p'(x_1), p(x_2)) \\ p(a) \to a & p'(b) \to b. \end{array}$$

Informally, the state p checks whether or not the leftmost leaf of its input tree is a. The state p' does the same for b. Consider the tree s = f(a, f(p, b)). For s the tree $\langle f, p, p \rangle (a, \langle f, p, p' \rangle (p, b))$ is a valid relabeling. The tree $\langle f, p, p' \rangle (a, \langle f, p', p' \rangle (p, b))$ on the other hand is not.

D.1 If M is functional then $\hat{T}_1 \circ T_2$ is functional

In this section we formally prove the only-if statement of Corollary 1, i.e., we show that if M is functional then $\hat{T}_1 \,\hat{\circ}_2$ is functional. More precisely we formally show that $\mathcal{R}(\hat{T}_1) \,\hat{\circ}\, \mathcal{R}(T_2) \subseteq \mathcal{R}(M)$. Obviously this implies our result.

In the following we formally prove Lemma 5. More precisely we prove the following lemma which is a more detailed version of Lemma 5.

Lemma 15. Let (q_1, S) be a state of \hat{T}_1 and q_2 be a state of T_2 . Let $s \in T_{\Sigma}$. Consider the state (q_1, S, q_2) of M. If

$$[\![(q_1,S)]\!]^{\hat{T}_1}(s) \Rightarrow t \quad and \quad [\![q_2]\!]^{T_2}(t) \Rightarrow r$$

then $[(q_1, S, q_2)]^M(s) \Rightarrow r$.

Proof. We prove our claim by induction on the structure of s. Let $s = a(s_1, \ldots, s_k)$ where $a \in \Sigma_k$, $k \ge 0$, and for $i \in [k], s_1, \ldots, s_k \in T_{\Sigma}$. First, we prove the following claim.

Claim 16. If $[(q_1, S)]^{\hat{T}_1}(s) \Rightarrow t$ and $[q_2]^{T_2}(t) \Rightarrow r$ and $[(q_1, S, q_2)]^M(s) \Rightarrow r$, then trees ξ and ψ exist such that

$$\xi \in \operatorname{rhs}_{\hat{T}_1}((q_1, S), a)$$
 and $[q_2]^{T_2}(\xi) \Rightarrow \psi$.

Furthermore, ξ and ψ have the following properties.

- (1) Let u be a node of ξ . If a node of ψ is labeled by $q'_2(u)$ then a state (q'_1, S') of \hat{T}_1 exists such that u is labeled by $(q'_1, S')(x_\iota)$ in ξ , where $\iota \in [k]$, and $q'_2 \in S'$.
- (2) It holds that

$$t \in \xi[q(x_i) \leftarrow [q]^{T_1'}(s_i) \mid q \in Q_1', i \in [k]]$$

and

$$r \in \psi[q(u) \leftarrow \llbracket q \rrbracket^{T_2}(t/u) \mid q \in Q_2, u \in V(\xi)].$$

Proof of Claim. By definition, $[(q_1, S)]^{\hat{T}_1}(s) \Rightarrow t$ implies $[q_1]^{T_1}(s) \Rightarrow t$ and thus

$$t \in \xi'[q(x_i) \leftarrow [\![q]\!]^{T_1}(s_i) \mid q \in Q_1, i \in [k]]$$
(1)

for some $\xi' \in \operatorname{rhs}(q_1, a)$. Furthermore, by Lemma 12, $[(q_1, S)]^{\hat{T}_1}(s) \Rightarrow t$ implies that $t \in \bigcap_{q' \in S} \operatorname{dom}(q')$. In the following, let $S = \{q_2^1, \dots, q_2^n\}$. As $t \in \bigcap_{q' \in S} \operatorname{dom}(q')$ and due to Equation 1, for all $j \in [n]$, trees ψ_j and r_j exists such that

- (a) $[\![q_2^j]\!]^{T_2}(\xi') \Rightarrow \psi_j$ and
- (b) $[q_2^j]^{T_2}(t) \Rightarrow r_j$ such that

$$r_j \in \psi_j[q_2'(u) \leftarrow [\![q_2']\!]^{T_2}(t/u) \mid q_2' \in Q_2, u \in V(\xi')].$$

By our premise, the state (q_1, S, q_2) of M is defined which implies that $q_2 \in S$. W.l.o.g. let $q_2^1 = q_2$. Furthermore, as $[\![q_2]\!]^{T_2}(t) \Rightarrow r$, we can also assume that $r_1 = r$.

Recall that by definition, if a node of ψ_j is labeled by $q_2'(u)$, where $q_2' \in Q_2$ and u is a node, then the node u is labeled by some symbol in $Q_1(X)$ in ξ' . Let u_1, \ldots, u_m be the nodes of ξ' that are labeled by a symbol in $Q_1(X)$.

In the following we first prove Statement (1). Due to Lemma 10, (a) implies

$$[\![S]\!]^A(\xi') \Rightarrow \xi'[u_i \leftarrow S_i(u_i) \mid i \in [m]],$$

where A is the domain automaton of T_2 and $S_i = \bigcup_{j \in [n]} \psi_j \langle u_i \rangle$, which in turn implies that \hat{T}_1 contains the rule $(q_1, S)(a(x_1, \dots, x_k)) \to \xi$ where

$$\xi = \xi'[u_i \leftarrow (q', S_i)(x_i) \mid \xi[u_i] = q'(x_i), \iota \in [k]]$$

as T_1 is obtained from the p-construction of T_1 and A. In the following, consider the node u_i . Assume that in ψ_j a node labeled by $q_2'(u_i)$ occurs. Recall that this means that the node u_i is labeled by a symbol of the form $q'(x_i)$ in ξ' . By construction, u_i is labeled by $q'(x_i)$ in ξ' if and only if u_i is labeled by $q'(x_i)$ in ξ' . As $S_i = \bigcup_{j \in [n]} \psi_j \langle u_i \rangle$, obviously $q_2' \in S_i$.

As $q_2^1 = q_2$, Statement (1) follows with $\psi = \psi_1$. Note that clearly, for all $j \in [n]$, it holds that

$$\llbracket q_2^j \rrbracket^{T_2}(\xi') \Rightarrow \psi_j \quad \text{if and only if} \quad \llbracket q_2^j \rrbracket^{T_2}(\xi) \Rightarrow \psi_j.$$
 (2)

We now prove Statement (2). In particular, we prove the first part of Statement (2), i.e., that

$$t \in \xi[q(x_i) \leftarrow [q]^{T_1'}(s_i) \mid q \in Q_1', i \in [k]].$$

Let the node u_i be labeled by $(q'_1, S_i)(x_i)$ in ξ . Consider an arbitrary state $q'_2 \in S_i$ where $i \in [m]$. In particular, this means that $q'_2 \in \psi_j \langle u_i \rangle$ for some $j \in [n]$. In other words, a node g exists such that g is labeled by $q'_2(u_i)$ in ψ_j . Clearly, Statement (b) implies that q'_2 can produce the tree r_j/g on input t/u_i . This

statement can be generalized. More precisely, it holds that any state $q_2 \in S_i$ can produce some output tree on input t/u_i . Therefore, $t/u_i \in \bigcap_{q_2' \in S_i} \operatorname{dom}(q_2')$. Equation 1 implies that $[\![q_1']\!]^{T_1}(s_\iota) \Rightarrow t/u_i$ if $\xi'[u_i] = q_1'(x_\iota)$. Together with Lemma 13 and as $t/u_i \in \bigcap_{q_2' \in S_i} \operatorname{dom}(q_2')$, it follows that

$$\llbracket (q_1', S_i) \rrbracket^{\hat{T}_1}(s_\iota) \Rightarrow t/u_i.$$

By construction the node u_i is labeled by $(q'_1, S_i)(x_i)$ in ξ if and only if u_i is labeled by $q'(x_i)$ in ξ' . With the rule $(q_1, S)(a(x_1, \ldots, x_k)) \to \xi$ and Equation 1 it follows that the (q_1, S) can generate the tree t on input s. More precisely, it follows that

$$t \in \xi[q(x_i) \leftarrow [\![q]\!]^{\hat{T}_1}(s_i) \mid q \in Q_1', i \in [k]].$$

The second part of Statement (2) follows due to Statement (b) and Equation 2.

Let ξ and ψ be as in Claim 16. Due to Statement (1) of Claim 16, it follows that M contains the rule

$$(q_1, S, q_2)(a(x_1:l_1, \ldots, x_k:l_k)) \to \gamma$$

where γ is obtained from ψ by substituting occurrences of $q_2'(u)$ in ψ , where $q_2' \in Q_2$ and u is a leaf of ξ labeled by a symbol of the form $(q_1', S')(x_i)$, by $(q_1', S', q_2')(x_i)$. Furthermore, for $i \in [k]$, $\xi \langle x_i \rangle = l_i$.

We now show that $[(q_1, S, q_2)]^M(s) \Rightarrow r$. Recall that $s = a(s_1, \ldots, s_k)$ where

We now show that $[(q_1, S, q_2)]^M(s) \Rightarrow r$. Recall that $s = a(s_1, \ldots, s_k)$ where $a \in \Sigma_k, k \geq 0$, and for $i \in [k], s_1, \ldots, s_k \in T_{\Sigma}$. Note that as $\xi \langle x_i \rangle = l_i$ and due to Statement (2) of Claim 16 and Lemma 1, $s_i \in \text{dom}(l_i)$ for $i \in [k]$.

Consider a node g. By definition of γ , g is labeled by $(q'_1, S', q'_2)(x_i)$ in γ if and only if g is labeled by $q'_2(u)$ in ψ and the node u is labeled by $(q'_1, S')(x_i)$ in ξ . Statement (2) of Claim 16 implies that $[(q'_1, S')]^{\hat{T}_1}(s_i) \Rightarrow t/u$ and $[q'_2]^{\hat{T}_2}(t/u) \Rightarrow r/g$. Therefore, by induction hypothesis, $[(q'_1, S', q'_2)]^M(s_i) \Rightarrow r/g$. Clearly, our claim follows.

Clearly, Lemma 15 implies Lemma 4. Lemma 15 also yields the following two auxiliary results.

Lemma 17. Let (q_1, S) be a state of \hat{T}_1 and q_2 be a state of T_2 such that $q_2 \in S$. Then, for the state (q_1, S, q_2) of M, $dom((q_1, S, q_2)) = dom((q_1, S))$ holds.

Proof. Let $s = a(s_1, \ldots, s_k)$, $a \in \Sigma_k$, $k \ge 0$ and $s_1, \ldots, s_k \in T_{\Sigma}$. Let $s \in \text{dom}((q_1, S, q_2))$, i.e., $[(q_1, S, q_2)](s) \Rightarrow r$ for some tree r. Consider the first rule of M applied in this translation. Let

$$\eta = (q_1, S, q_2)(a(x_1 : l_1, \dots, x_k : l_k)) \to \gamma$$

be this rule. By construction η is obtained from a rule $(q_1, S)(a(x_1, \ldots, x_k)) \to \xi$ of \hat{T}_1 such that for $i \in [k]$, $\xi \langle x_i \rangle \subseteq l_i$. The application of η implies $s_i \in l_i$ for $i \in [k]$. This implies $s \in \text{dom}((q_1, S))$.

Conversely, let $s \in \text{dom}((q_1, S))$, i.e., $[\![(q_1, S)]\!](s) \to t$ for some tree t. Note that the state (q_1, S, q_2) of M implies $q_2 \in S$. Due to Lemma 12, it follows that $[\![q_2]\!]^{T_2}(t) \neq \emptyset$. Therefore, we deduce that due to Lemma 15, $s \in \text{dom}((q_1, S, q_2))$.

Lemma 18. Let $s \in T_{\Sigma}[L]$. Let $M(s) \Rightarrow r_M$ and let $(q_1, S, q_2)(v)$ occurs in r_M , where (q_1, S, q_2) is a state of M and v is a node of s labeled by a symbol $l \in L$. Then $dom(l) \subseteq dom((q_1, S, q_2))$.

Proof. Let the parent node of v be labeled by $a \in \Sigma_k$ where k > 0. W.l.o.g. let v be the first child of its parent node. Then, clearly the occurrence of $(q_1, S, q_2)(v)$ in r_M originates from the application of a rule $(q'_1, S', q'_2)(a(x_1 : l_1, \ldots, x_k : l_k)) \to \gamma$ of M such that $(q_1, S, q_2)(x_1)$ occurs in γ . Recall that by definition, $l_1, \ldots l_k$ are sets of states of \hat{T}_1 . By the definition of relabelings of trees in $T_{\Sigma}[L]$, the parent node of v is relabeled by a symbol of the form $\langle a, l, l'_2, \ldots l'_k \rangle$ which implies $l = l_1$.

Consider the rule $(q'_1, S', q'_2)(a(x_1 : l_1, \ldots, x_k : l_k)) \to \gamma$. Recall that by construction of M, this rule is obtained from a rule $(q'_1, S')(a(x_1, \ldots, x_k)) \to \xi$ of \hat{T}_1 such that for $i \in [k]$, $\xi\langle x_i\rangle \subseteq l_i$. Note that the occurrence of $(q_1, S, q_2)(x_1)$ in γ implies that $(q_1, S)(x_1)$ occurs in ξ . Therefore, the state (q_1, S) of \hat{T}_1 is included in l. As $s \in \text{dom}(l)$ if and only if $s \in \bigcap_{\hat{q} \in l} \text{dom}(\hat{q})$, our claim follows due to Lemma 17.

D.2 If $\hat{T}_1 \circ T_2$ is functional then M is functional.

In this section we formally prove the only-if statement of Corollary 1, i.e., we show that if $\hat{T}_1 \circ T_2$ is functional then M is functional.

First we introduce the following definition. Recall that we have introduced synchronized translations of M in Section 3.1. In the following, we extend this definition. Let $s \in T_{\Sigma}[L]$. We call the trees s, t, r and r_M synchronized if

- 1. $\hat{T}_1(s) \Rightarrow t$ and $T_2(t) \Rightarrow r$ and $M(s) \Rightarrow r_M$ and
- 2. the tree r_M is obtained from r by substituting all occurrences of $q'_2(u)$ in r by $(q'_1, S', q'_2)(v)$, where (q'_1, S') and q'_2 are states of \hat{T}_1 and T_2 , respectively, and u is a leaf of t labeled by $(q'_1, S')(v)$.

Informally, s, t, r and r_M are synchronized if on input s, M produces the tree r_M by accurately simulating $\hat{T}_1 \,\hat{\circ}\, T_2$. More precisely: Recall that when a state (q_1, S, q_2) of M processes a subtree s' of s then (q_1, S, q_2) guesses what the state (q_1, S) of \hat{T}_1 might have produced before producing output according to this guess. Informally, if all such guesses of M are correct, i.e., the states of \hat{T}_1 have indeed produced the trees M has guessed, then s, t, r and r_M are synchronized.

Before we prove a more detailed version of Lemma 6, recall that by definition, a state l in L is a set of states of \hat{T}_1 . Consider a tree $s \in T_{\Sigma}[L]$. Informally, if a symbol $l \in L$ occurs at some leaf of s then l can be considered a placeholder for some tree s' such that $s' \in \bigcap_{(q_1,S)\in l} \operatorname{dom}(q_1,S)$. We now show that the following holds.

Lemma 19. Let $s \in T_{\Sigma}[L]$. Let $M(s) \Rightarrow r_M$ and let $(q_1, S, q_2)(v)$ occur in r_M . Then trees t, r and r'_M exist such that s, t, r and r'_m are synchronized and $(q_1, S, q_2)(v)$ occurs in r'_M . Furthermore, let v' be a leaf of s that is labeled by $l \in L$. Then, it holds that if $(q'_1, S')(v')$ occurs in t then $(q'_1, S') \in l$.

Proof. We prove our claim by structural induction. First, let \bar{v} be a node of s such that the subtree of s rooted at \bar{v} is of the form $a(l_1, \ldots, l_k)$ where $a \in \Sigma_k$, $k \geq 0$, and $l_1, \ldots, l_k \in L$. Note that by definition \bar{v} can be a leaf. Then, a state $l \in L$ exists such that

$$l(a(x_1,\ldots,x_k)) \to a(l_1(x_1),\ldots,l_k(x_k))$$

is a rule of the la-automaton of M. Furthermore, as $M(s) \Rightarrow r_M$, on input $\bar{s} = s[\bar{v} \leftarrow l]$, the M produces the tree \bar{r}_M such that

$$r_M \in \bar{r}_M[q](\bar{v}) \leftarrow \llbracket q \rrbracket_{\bar{v}}^M(a(l_1,\ldots,l_k)) \mid q \text{ is a state of } M].$$

We remark that all trees in $[\![q]\!]_{\bar{v}}^M(a(l_1,\ldots,l_k))$ are of the form $\gamma\langle x_i\leftarrow\bar{v}.i\mid i\in[k]\rangle$ where $\gamma\in\operatorname{rhs}(q,a,l_1,\ldots,l_k)$. Recall that by our premise, $(q_1,S,q_2)(v)$ occurs in r_M . Then one of the following cases arises:

- (a) $(q_1, S, q_2)(v)$ does not already occur in \bar{r}_M .
- (b) $(q_1, S, q_2)(v)$ already occurs in \bar{r}_M .

First, we consider case (b). By induction hypothesis, as $M(\bar{s}) \Rightarrow \bar{r}_M$ and a node labeled by $(q_1, S, q_2)(v)$ occurs in \bar{r}_M , trees \bar{t} , \bar{r} and \bar{r}'_M exist such that \bar{s} , \bar{t} , \bar{r} and \bar{r}'_M are synchronized and $(q_1, S, q_2)(v)$ occurs in \bar{r}'_M . Furthermore, by induction hypothesis, it holds that if $(q'_1, S')(\bar{v})$ occurs in \bar{t} then $(q'_1, S') \in l$.

First, we construct the tree t. Recall that the la-automaton of M is the domain automaton of \hat{T}_1 . Therefore the existence of rule

$$l(a(x_1,\ldots,x_k)) \rightarrow a(l_1(x_1),\ldots,l_k(x_k))$$

of the la-automaton implies that for all states $(q'_1, S') \in l$, a right-hand side $\xi' \in \operatorname{rhs}_{\hat{T}_1}((q'_1, S'), a)$ exists such that $\xi'\langle x_i \rangle \subseteq l_i$ for $i \in [k]$ (*).

In the following, we define $t_{(q'_1,S')} = \xi' \langle x_i \leftarrow \bar{v}.i \mid i \in [k] \rangle$ if $(q'_1,S') \in l$. Then clearly $\hat{T}_1(s) \Rightarrow t$ where

$$t = \bar{t}[(q_1', S')(\bar{v}) \leftarrow t_{(q_1', S')} \mid (q_1', S') \in Q_1'].$$

We now show that for arbitrary nodes v' of s it holds that if v' is labeled by l in s and $(q'_1, S')(v')$ occurs in t then $(q'_1, S') \in l$. Due to (*), this holds for all nodes v' that are descendants of \bar{v} . Now assume that v' is not be a descendant of \bar{v} . Let v' be labeled by the symbol $\bar{l} \in L$ in s. Then obviously, the node v' is also labeled by \bar{l} in \bar{s} . Thus, by definition of \bar{t} , if $(q'_1, S')(v')$ occurs in \bar{t} then $(q'_1, S') \in \bar{l}$. By construction of t, $(q'_1, S')(v')$ occurs in t if and only if $(q'_1, S')(v')$ occurs in \bar{t} . This yields our claim.

We now construct r and r'_M . First recall that, by induction hypothesis, the trees \bar{s} , \bar{t} , \bar{r} and \bar{r}'_M are synchronized. Therefore, for an arbitrary node g the

following holds: g is labeled by $(q'_1, S', q'_2)(v)$ in \bar{r}_M if and only if g is labeled by $q'_2(u)$ in \bar{r} and u is a node of \bar{t} labeled by $(q'_1, S')(v)$ (†).

Now let the node g be labeled by $q_2'(u)$ in \bar{r} and let the node u be labeled by $(q_1', S')(\bar{v})$ in \bar{t} . Consider the right-hand side ξ' assigned to the state (q_1', S') in (*). Due to how rules of \hat{T}_1 are defined, it holds that

$$[\![S']\!]^A(\xi') \Rightarrow \xi'[u \leftarrow \bar{S}(u) \mid u \in V(\xi'), \xi'[u] = (\bar{q}, \bar{S})(x_i)].$$

Note that (†) implies $q_2' \in S$. This follows as the state (q_1', S', q_2') is defined. Therefore by Lemma 11, a tree ψ' exists such that

- 1. $[q_2']^{T_2}(\xi') \Rightarrow \psi'$ and
- 2. if the node u' is labeled by $(\tilde{q}, \tilde{S})(x_i)$ in ξ' then $\psi'(u') \subseteq \tilde{S}$.

The later implies that if $\tilde{q}_2(u')$ occurs in ψ' then $\tilde{q}_2 \in \tilde{S}$. Due to (*), for $i \in [k]$, it holds that $\xi'\langle x_i \rangle \subseteq l_i$. Therefore, by construction of M the rule

$$(q'_1, S', q'_2)(a(x_1: l_1, \dots, x_k: l_k)) \to \gamma'$$

is defined where γ' is obtained from ψ' by substituting occurrences of $\tilde{q}_2(u')$ in ψ by $(\tilde{q}_1, \tilde{S}, \tilde{q}_2)(x_i)$, where (\tilde{q}_1, \tilde{S}) and \tilde{q}_2 are states of \hat{T}_1 and T_2 , respectively, and u' is a leaf of ξ' labeled by a symbol of the form $(\tilde{q}_1, \tilde{S})(x_i)$.

For the node g we define $r_{T_2,g} = \psi' \langle u' \leftarrow u.u' \mid u' \in V \rangle$. Additionally, we define $r_{M,g} = \gamma' \langle x_i \leftarrow \bar{v}.i \mid i \in [k] \rangle$.

Recall that (†) holds. Then, $T_2(t) \Rightarrow r$ where

$$r = \bar{r}[g \leftarrow r_{T_2,g} \mid \bar{r}[g] = q_2'(u) \text{ and } \bar{t}[u] = (q_1', S')(\bar{v})]$$

and $M(s) \Rightarrow r_M'$ where

$$r'_{M} = \bar{r}'_{M}[g \leftarrow r_{M,g} \mid \bar{r}_{M}[g] = (q'_{1}, S', q'_{2})(\bar{v})].$$

Note that the node \bar{v} of s is relabeled by $\langle a, l_1, \ldots, l_k \rangle$ via the relabeling induced by the rule $l(a(x_1, \ldots, x_k)) \to a(l_1(x_1), \ldots, l_k(x_k))$ of the la-automaton of M. Thus, r'_M is well defined. Clearly, $\hat{T}_1(s) \Rightarrow t$ and $T_2(t) \Rightarrow r$ and $M(s) \Rightarrow r'_M$. Due to (†) and the construction of r and r'_M , it follows that the second part of the synchronized-property holds as well.

We now consider case (a). As $(q_1, S, q_2)(v)$ occurs in r_M but not in \bar{r}_M , it follows that $v = \bar{v}.i$ for some $i \in [k]$. W.l.o.g. let $v = \bar{v}.1$, i.e., v is the first child of the node \bar{v} . Furthermore, it follows that a rule

$$(\tilde{q}_1, \tilde{S}, \tilde{q}_2)(a(x_1:l_1, \ldots, x_k:l_k)) \rightarrow \tilde{\gamma}$$

exists such that $(\tilde{q}_1, \tilde{S}, \tilde{q}_2)(\bar{v})$ occurs in \bar{r}_M and $(q_1, S, q_2)(x_1)$ occurs in $\tilde{\gamma}$. Let the rule of M above be obtained from the rule $(\tilde{q}_1, \tilde{S})(a(x_1, \ldots, x_k)) \to \tilde{\xi}$ of \hat{T}_1 and subsequently translating $\tilde{\xi}$ by the state \tilde{q}_2 of T_2 . In particular, this means that a tree $\tilde{\psi}$ exists such that

- (a) $[\tilde{q}_2]^{T_2}(\tilde{\xi}) \Rightarrow \tilde{\psi}$ and
- (b) $\tilde{\gamma}$ is obtained from $\tilde{\psi}$ by substituting occurrences of $q'_2(u)$ in $\tilde{\psi}$ by $(q'_1, S', q'_2)(x_i)$, where (q'_1, S') and q'_2 are states of \hat{T}_1 and T_2 , respectively, and u is a leaf of ξ' labeled by $(q'_1, S')(x_i)$,

By induction hypothesis, as $M(\bar{s}) \Rightarrow \bar{r}_M$ and a node labeled by $(\tilde{q}_1, \tilde{S}, \tilde{q}_2)(\bar{v})$ occurs in \bar{r}_M , trees \bar{t} , \bar{r} and \bar{r}'_M exist such that \bar{s} , \bar{t} , \bar{r} and \bar{r}'_M are synchronized and $(\tilde{q}_1, \tilde{S}, \tilde{q}_2)(\bar{v})$ occurs in \bar{r}'_M .

Let the node \bar{g} be labeled by $(\tilde{q}_1, \tilde{S}, \tilde{q}_2)(\bar{v})$ in \bar{r}'_M . Due to the synchronized property, the node \bar{g} is labeled by $q_2(\bar{u})$ in \bar{r} , where \bar{u} is a node that is labeled by $(q_1, S)(\bar{v})$ in \bar{t} .

To construct the trees t, r and r'_M , we then proceed as in case (b) but set

$$\begin{aligned} &-t_{(\tilde{q}_1,\tilde{S})} = \tilde{\xi} \langle x_i \leftarrow \bar{v}.i \mid i \in [k] \rangle, \\ &-r_{T_2,\hat{g}} = \tilde{\psi} \langle u' \leftarrow \bar{u}.u' \mid u' \in V \rangle \text{ and } \\ &-r_{M,\hat{g}} = \tilde{\gamma} \langle x_i \leftarrow \bar{v}.i \mid i \in [k] \rangle. \end{aligned}$$

This yields our claim.

Lemma 19 and and Proposition 1 allow us to formally prove the following version of Lemma 7.

Lemma 20. Let $s \in T_{\Sigma}[L]$ such that only a single node v of s is labeled by a symbol in L. Let v be labeled by $l \in L$. Let $M(s) \Rightarrow r_M$ such that $(q_1, S, q_2)(v)$ occurs in r_M .

Consider the tree $\tilde{s} = s[v \leftarrow s']$ where $s' \in dom(l)$. If $T_1 \circ T_2(\tilde{s})$ is a singleton then $[(q_1, S, q_2)](s')$ is a singleton.

Proof. Note that by Lemma 18, $s' \in \text{dom}((q_1, S, q_2))$. Hence, $[(q_1, S, q_2)](s') \neq \emptyset$. Assume that $[(q_1, S, q_2)](s')$ is not a singleton, i.e., assume that distinct trees r_1, r_2 exist such that $r_1, r_2 \in [(q_1, S, q_2)](s')$.

We claim that for r_1 , a tree t_1 exists such that

- 1. on input s', the state (q_1, S) of \hat{T}_1 produces t_1 and
- 2. on input t_1 , the state q_2 of T_2 produces r_1 .

We will later prove this claim in detail. It can be shown that a tree t_2 with the same properties exists for r_2 .

Using this claim and Proposition 1, we now prove that contrary to the assumption $r_1 = r_2$.

Due to Lemma 19, as $M(s) \Rightarrow r_M$ and $(q_1, S, q_2)(v)$ occurs in r_M , it follows that trees t, r and r'_M exist such that s, t, r and r'_M are synchronized and $(q_1, S, q_2)(v)$ occurs in r'_M . Moreover, if (q', S')(v) occurs in t, where (q', S') is some state of \hat{T}_1 , then $(q', S') \in l$. Recall that by our premise, v is labeled by l in s. Therefore, dom $(l) \subseteq \text{dom}((q', S'))$ due to Lemma 1. Consequently, $s' \in \text{dom}((q', S'))$. Therefore, for all states (q', S') of \hat{T}_1 such that (q', S')(v) occurs in t, a tree t' exists such that $[(q', S')]^{\hat{T}_1}(s') \Rightarrow t'$. In the following, let

 $t\langle v \rangle = \{(q^1, S^1), \dots, (q^n, S^n)\}$ and for $j \in [n]$, let $[(q^j, S^j)]^{\hat{T}_1}(s') \Rightarrow t'_i$. Then clearly on input \tilde{s} , the transducer \hat{T}_1 can produce the tree \tilde{t} where

$$\tilde{t} = t[(q^j, S^j)(v) \leftarrow t_i' \mid j \in [n]].$$

Now consider the tree r. Let q'_2 be a state of T_2 and u be a node. By definition of t and r, if $q'_2(u)$ occurs in r, then the node u is labeled by a symbol of the form (q', S')(v) in t. Furthermore, the synchronized property implies that $q'_2 \in S'$. This follows as a state (q', S', q'_2) of M has the property that $q'_2 \in S'$. The subtree of \tilde{t} rooted at u is a tree t' such that $[(q', S')]^{\tilde{T}_1}(s') \Rightarrow t'$. By Lemma 12, it follows that $t' \in \text{dom}(q_2)$. Therefore, it follows easily that $T_2(\tilde{t}) \Rightarrow \tilde{r}$ where

$$\tilde{r} = r[q_2'(u) \leftarrow r_u \mid q_2' \in Q_2, \tilde{t}[u] = t_j' \text{ and } [\![q_2']\!]^{T_2}(t_j') \Rightarrow r_u].$$

By our premise a node g exists such that g is labeled by $(q_1, S, q_2)(v)$ in r'_{M} . As the trees s, t, r and r'_{M} are synchronized, g is labeled by $q_{2}(u)$ in r where u is a node of t such that $t[u] = (q_1, S)(v)$. Due to our claim, a tree t_1 exists such that $\llbracket (q_1, S) \rrbracket^{\hat{T}_1}(s') \Rightarrow t_1$ and and $\llbracket q_2 \rrbracket^{T_2}(t_1) \Rightarrow r_1$. W.l.o.g. we assume that $(q^1, S^1) = (q_1, S)$ and $t'_1 = t_1$. Then, it follows easily that on input \tilde{s} , the composition $\hat{T}_1 \circ T_2$ can produce a tree \tilde{r}_1 such that $\tilde{r}_1/g = r_1$. Analogously, it follows easily that on input \tilde{s} , the composition $\hat{T}_1 \circ T_2$ can produce a tree \tilde{r}_2 such that $\tilde{r}_2/g = r_2$. Due to Proposition 1, $\tilde{r}_1 = \tilde{r}_2$ and therefore

$$r_1 = \tilde{r}_1/g = \tilde{r}_2/g = r_2.$$

Now all that is left is to prove our previous claim that for r_1 and r_2 , trees t_1 and t_2 exist such that

$$- \left[\left[(q_1, S) \right]^{\hat{T}_1}(s') \Rightarrow t_1 \text{ and and } \left[q_2 \right]^{T_2}(t_1) \Rightarrow r_1$$

$$- [(q_1, S)]^{\hat{T}_1}(s') \Rightarrow t_2 \text{ and and } [q_2]^{T_2}(t_2) \Rightarrow r_2.$$

We prove our claim for r_1 . The proof for r_2 is analogous. Let $s' = a(s_1, \ldots, s_k)$ where $a \in \Sigma_k$, $k \ge 0$, and $s_1, \ldots, s_k \in T_{\Sigma}$. As r_1 is producible by (q_1, S, q_2) on input s', it follows that

$$r_1 \in \gamma[q_M(x_i) \leftarrow \llbracket q_M \rrbracket(s_i) \mid i \in [k] \text{ and } q_M \text{ is a state of } M]$$
 (2)

where $(q_1, S, q_2)(a(x_1: l_1, \ldots, x_k: l_k)) \to \gamma$ is a rule of M, l_1, \ldots, l_k are states of the la-automaton of M and for $i \in [k]$, $s_i \in \text{dom}(l_i)$.

Before we prove our claim, we prove the following result by induction on the statement of Lemma 20.

Claim 21. Let q_M be a state of M and let $q_M(x_i)$ occur γ where $i \in [k]$. Then the set $[q_M](s_i)$ is a singleton.

Proof of Claim. Before, we prove our claim consider the following. Let q'_M be a state of M. Then, by Lemma 18, $s' \in \text{dom}(q'_M)$, if $q'_M(v)$ occurs in r_M (†).

Now, we prove our claim. W.l.o.g., we consider the case i = 1. Consider the tree $\bar{s} = a(l_1, s_2, \dots, s_k)$. Due to (\dagger) , it follows that on input \bar{s} , any state q'_M

such that $q'_M(v)$ occurs in r_M can produce some partial tree, i.e., a tree with leafs with label of the form $\check{q}_M(v.1)$ where \check{q}_M is a state of M. In particular, by our premise, $(q_1, S, q_2)(v)$ occurs in r_M . As the tree r_1 is producible by (q_1, S, q_2) on input s' by applying the rule $(q_1, S, q_2)(a(x_1: l_1, \ldots, x_k: l_k)) \to \gamma$, it follows easily that on input \bar{s} , the state (q_1, S, q_2) generates a tree \hat{r} such that $q_M(v.1)$ occurs in \hat{r} if $q_M(x_1)$ occurs γ .

Thus, it follows that M on input $s[v \leftarrow \bar{s}]$ produces a tree in which $q_M(v.1)$ occurs. Clearly, the node v.1 is labeled by l_1 in \bar{s} . Note that $s_1 \in \text{dom}(l_1)$. By induction hypotheses, $\llbracket q_M \rrbracket(s_i)$ is a singleton.

We now prove our main claim.

Claim 22. A tree t_1 exists such that $[(q_1, S)]^{\hat{T}_1}(s') \Rightarrow t_1$ and and $[q_2]^{T_2}(t_1) \Rightarrow r_1$.

Proof of Claim. Let $i \in [k]$ and q_M be a state of M. Let $q_M(x_i)$ occur in γ . By Claim 21, the set $\llbracket q_M \rrbracket(s_i)$ is a singleton. Let $\llbracket q_M \rrbracket(s_i) = \{r'\}$. Let $q_M = (q'_1, S', q'_2)$ where (q'_1, S') and q'_2 are states of \hat{T}_1 and T_2 , respectively. Due to Lemma 17, $s_i \in \text{dom}(q_M)$ implies $s_i \in \text{dom}((q'_1, S'))$, i.e., $\llbracket (q'_1, S') \rrbracket^{\hat{T}_1}(s_i)$ is not empty. In the following, we show that if $\llbracket q_M \rrbracket(s_i) = \{r'\}$ then for all trees t' contained in $\llbracket (q'_1, S') \rrbracket^{\hat{T}_1}(s_i)$, it holds that $\llbracket q'_2 \rrbracket^{T_2}(t') = \{r'\}$ (*).

In the following, consider such a tree t'. By Lemma 12,

$$t' \in \bigcap_{\bar{q}_2 \in S'} \operatorname{dom}(\bar{q}_2)$$

and thus $t' \in \text{dom}(q'_2)$. Recall that by definition the state $q_M = (q'_1, S', q'_2)$ implies $q'_2 \in S'$. Therefore, the set $[\![q'_2]\!]^{T_2}(t')$ is not empty. As $[\![q_M]\!](s_i) = \{r'\}$, we deduce that $[\![q'_2]\!]^{T_2}(t') = \{r'\}$ due to Lemma 15. Therefore, (*) follows.

By definition, the rule $(q_1, S, q_2)(a(x_1 : l_1, \ldots, x_k : l_k)) \to \gamma$ of M is defined only if a rule $(q_1, S)(a(x_1, \ldots, x_k)) \to \xi$ of \hat{T}_1 and a tree ψ exist such that

- 1. $[q_2]^{T_2}(\xi) \Rightarrow \psi$
- 2. the tree γ is obtained from ψ by substituting all occurrences of $q'_2(u)$ in ψ by $(q'_1, S', q'_2)(x_i)$, where (q'_1, S') and q'_2 are states of \hat{T}_1 and T_2 , respectively, and u is a leaf of ξ labeled by $(q'_1, S')(x_i)$
- 3. for $i \in [k]$, it holds that $\xi(x_i) \subseteq l_i$.

By definition of r_1 (see Equation 2), for $i \in [k]$, it holds that $s_i \in l_i$. Therefore, it follows due to Statement 3 that if $(q'_1, S')(x_i)$ occurs in ξ then $s_i \in \text{dom}((q'_1, S'))$. Thus, $[(q_1, S)]^{\hat{T}_1}(s')) \Rightarrow t_1$ where

$$t_1 \in \xi[(q_1', S')(x_i) \leftarrow \llbracket (q_1', S') \rrbracket^{\hat{T}_1}(s_i) \mid (q_1', S') \in Q_1', i \in [k] \rrbracket$$

and $[q_2]^{T_2}(t_1) \Rightarrow r'_1$ where

$$r_1' \in \psi[q_2'(u) \leftarrow [\![q_2']\!]^{T_2}(t_1/u) \mid q_2' \in Q_2, u \in V(\xi)].$$

Note that the tree t_1/u is produced by the state (q'_1, S') on input s_i if $\xi[u] = (q'_1, S')(x_i)$. We remark that the node g is labeled by $q_2(u)$ in ψ where u is a

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$$[(q'_1, S', q'_2)]^M(s_i) = \{r'\} = [[q'_2]]^{T_2}(t_1/u).$$

Therefore, (*) yields $r_1'/g = r_1/g$. Due to the definition of γ and ψ , i.e. Statement 2, our claim follows.