Adding an equivalence relation to the interval logic $AB\overline{B}$

complexity and expressiveness

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Outline

- First-Order logic and equivalence relations
- The interval temporal logic $AB\overline{B}\sim$
  - Syntax and semantics of $AB\overline{B}\sim$
  - A geometrical interpretation of $AB\overline{B}\sim$ models
- Decidability and complexity of $AB\overline{B}\sim$ over finite linear orders
- Undecidability of $AB\overline{B}\sim$ over $\mathbb{N}$
- ωS-regular languages
  - Encoding ωS-regular languages in $AB\overline{B}\sim$
- Conclusions

Logics with equivalence relations

In last years, various (un)decidability results about extensions of the 2-variable fragment of first-order logic with one or more equivalence relations have been given in the literature. We mention two of them:

$FO^2[<, \sim, +1]$ (over finite linear orders)

M. Bojanczyk, C. David, A. Muscholl, T. Schwentick, and L. Segoufin, "Two-variable logic on data words" , ACM Transactions on Computational Logic, vol. 12, no. 4, p. 27, 2011

$FO^2[\sim_1, \sim_2]$

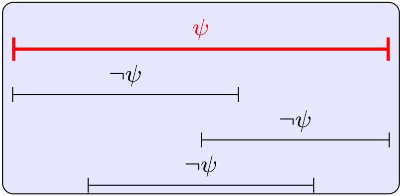
E. Kieronski and L. Tendera, "On finite satisfiability of two-variable first-order logic with equivalence relations", in *Proc. of LICS*. IEEE Computer Society, 2009, pp. 123–132.

E. Kieronski, J. Michaliszyn, I. Pratt-Hartmann, and L. Tendera, "Two-Variable First-Order Logic with Equivalence Closure", in *Proc. of LICS*. IEEE Computer Society, 2012, pp. 431–440.

We study the effects of the addition of an equivalence relation to an expressive enough and robustly decidable interval temporal logic.

The distinctive features of interval temporal logics

Truth of formulae is defined over intervals (not points).



Interval temporal logics are very expressive (compared to point-based temporal logics).
In particular, formulas of interval logics express properties of pairs of time points rather than of single time points, and are evaluated as sets of such pairs, i.e., as binary relations.

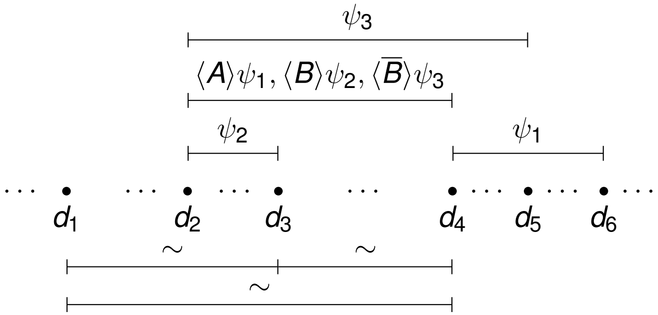
Thus, in general there is no reduction of the satisfiability/validity in interval logics to monadic second-order logic, and therefore Rabin's theorem is not applicable here.

The logic $AB\overline{B}\sim$ over (prefixes of) $\mathbb{N}$

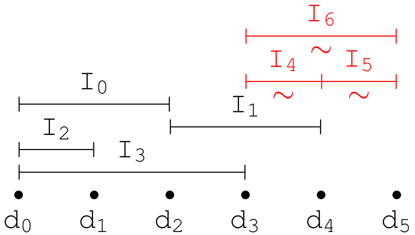
The logic $AB\overline{B}$ of Allen's relations *meets*, *begins* and *begin by* extended with an equivalence relation $\sim$

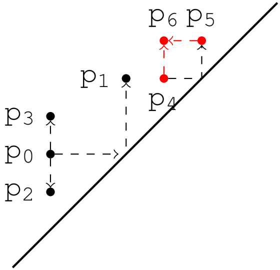
Syntax: $\varphi ::= p \in \mathcal{AP} \mid \sim \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid \langle A\rangle\psi \mid \langle B\rangle\psi \mid \langle \overline{B}\rangle\psi$

Semantics:



A geometrical interpretation of $AB\bar{B} \sim$ models



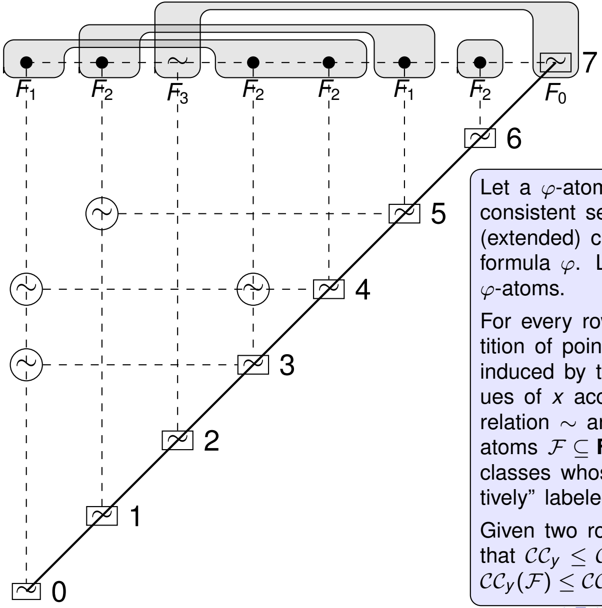


A geometrical interpretation of interval structures: *compass structures* – intervals are mapped into points of the second octant of the Euclidean plane and interval relations are mapped into suitable spatial relations between pairs of points

Proposition

An $AB\bar{B} \sim$ formula φ is satisfied by some interval model iff it is featured by some labeled compass structure.

The basic notion of row class counter



$\mathcal{CC}_7(\{F_1, F_2\}) = 2$
$\mathcal{CC}_7(\{F_3, F_0\}) = 1$
$\mathcal{CC}_7(\{F_2\}) = 1$

Let a φ-atom be a maximal locally consistent set of formulas from the (extended) closure of an $AB\overline{B} \sim$ formula φ. Let **F** be the set of all φ-atoms.

For every row y, $\mathcal{CC}_y$ takes the partition of points (x, y) belonging to y induced by the partition of the values of x according the equivalence relation $\sim$ and, for each set of φ-atoms $\mathcal{F} \subseteq \mathbf{F}$, it counts the number of classes whose elements are "collectively" labeled by $\mathcal{F}$.

Given two rows $y < y'$, we say that $\mathcal{CC}_y \leq \mathcal{CC}_{y'}$ iff for each $\mathcal{F}$ in **F**, $\mathcal{CC}_y(\mathcal{F}) \leq \mathcal{CC}_{y'}(\mathcal{F})$.

Decidability of $AB\bar{B}\sim$ over finite linear orders

Lemma (1)

For every $AB\bar{B}\sim$-formula φ and every finite labeled compass structure $\mathcal{G} = (\mathbb{P}(N), \sim, A, B, \bar{B}, \mathcal{L})$ for φ (if any), the partial order $\leq$ over row class counters $\mathcal{CC}_y$ of $\mathcal{G}$ is a well quasi-ordering.

Lemma (2)

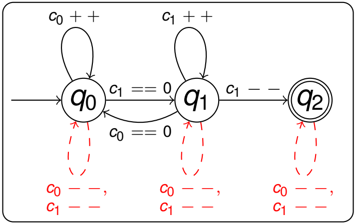
Let $\mathcal{G} = (\mathbb{P}(N), \sim, A, B, \bar{B}, \mathcal{L})$ be a finite labeled compass structure for an $AB\bar{B}\sim$-formula φ. If there exist $y < y' \leq N$, with $\mathcal{CC}_y \leq \mathcal{CC}_{y'}$, then there exists a finite labeled compass structure $\mathcal{G}' = (\mathbb{P}(N'), \sim', A, B, \bar{B}, \mathcal{L}')$ for φ with $N' = N - (y' - y)$.

Theorem

The satisfiability problem for $AB\bar{B}\sim$ over finite linear orders is decidable.

Reductions from reachability problems for LMMs

Lossy Minsky Machine (LMM)



0-0 Reachability is decidable and it is non-primitive hard. There exists a polynomial time reduction from **0-0** Reachability to finite satisfiability for $AB\bar{B} \sim$ formulas.

0-n Reachability is undecidable. There exists a polynomial time reduction from **0-n** Reachability to satisfiability of $AB\bar{B} \sim$ formulas over $\mathbb{N}$.

0-0 Reachability
Input: a lossy counter machine $M = (Q, k, \delta)$ and two states $q_0, q_f \in Q$;
Output: YES, if there exists a computation of M from configuration $(q_0, 0^k)$ to configuration $(q_f, 0^k)$;
NO otherwise.

0-n Reachability
Input: a lossy counter machine $M = (Q, k, \delta)$ and two states $q_0, q_f \in Q$;
Output: YES, if, for every k dimensional vector $\bar{z} \in \mathbb{N}^k$, there exists a computation in M from configuration $(q_0, 0^k)$ to configuration$(q_f, \bar{z})$;
NO otherwise.

lossy computation

$(q_0, [0, 0])$
$\downarrow$
$(q_1, [0, 0])$
$\downarrow$
$(q_1, [0, 1])$
$\downarrow$
$(q_0, [0, 1])$
$\downarrow$
$(q_0, [1, 1])$
$\downarrow$
$(q_0, [2, 1])$
$\downarrow$
$(q_0, [2, 0])$
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without lossyness we would be stuck here

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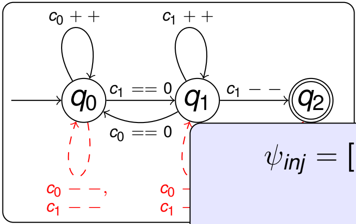

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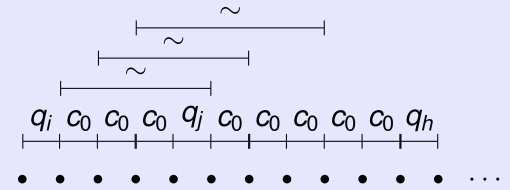
$$\psi_{inj} = [B][A]\neg\psi_{\exists Q} \rightarrow \neg \sim$$



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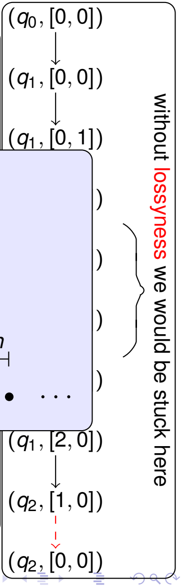
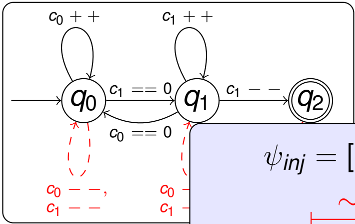


Reductions from reachability problems for LMMs

Lossy Minsky Machine (LMM)



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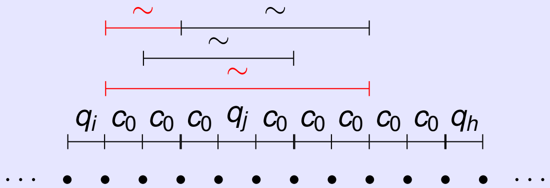

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$\downarrow$
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without lossyness we would be stuck here

Expressiveness

$AB\overline{B} \sim$ is expressive enough to provide a logical characterization of strictly unbounded ω-languages.

M. Bojańczyk and T. Colcombet. ω-regular expressions with bounds. In *LICS*. IEEE Computer Society, 2006, pp. 285–296.

Strictly unbounded ω-languages

Strictly unbounded ω-languages are obtained from ω-regular languages by adding a variant of Kleene star $(.)^*$, denoted by $(.)^S$, to be used in the scope of the ω-constructor $(.)^\omega$.

The exponent S allows one to constrain the number of iterated concatenations of the regular language R in the expression R^S to tend to infinity.

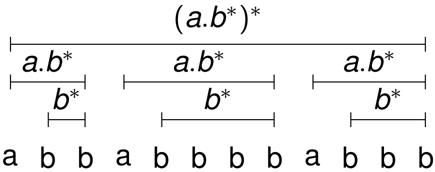
More precisely, for any natural number k, R^S constrains the number of ω-iterations in which R is repeated exactly k times to be finite.

Regular and ω-Regular Expression

$$e ::= a \in \Sigma \mid e_1 . e_2 \mid e_1 + e_2 \mid e^*$$

$$o ::= e.o \mid o_1 + o_2 \mid e^\omega$$

$$e = (a.b^*)^*$$



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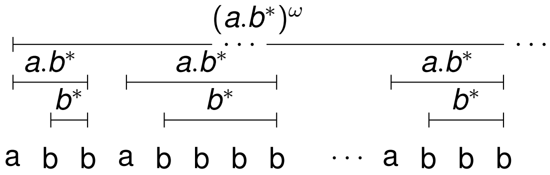

ωS-regular expressions

$$e ::= a \in \Sigma \mid e_1 . e_2 \mid e_1 + e_2 \mid e^* \mid e^S$$

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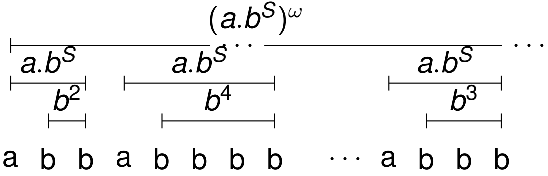
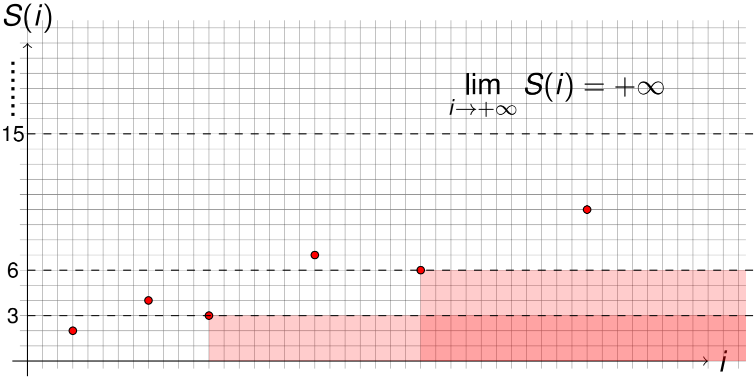
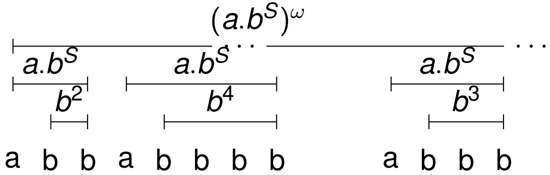


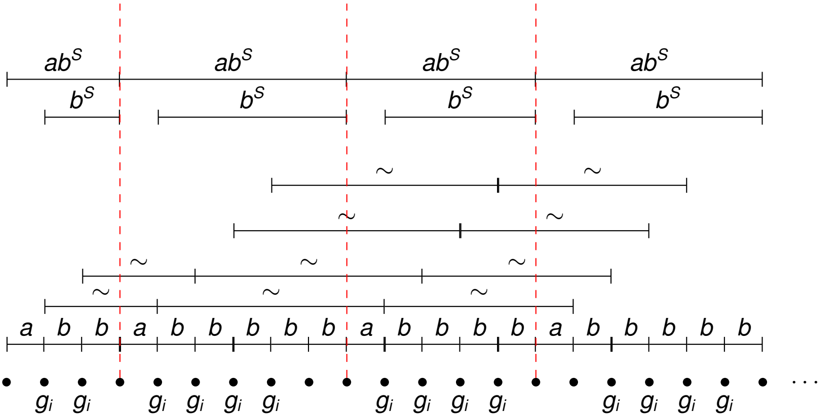
ωS-regular expressions





Encoding ωS-regular languages in $AB\overline{B}\sim$ over $\mathbb{N}$

$$e = (e^k)^\omega \text{ , where } e^k = ab^S$$



$$[G](\sim\rightarrow \langle B\rangle \bigvee_{e_k \in Sub_\omega(e)} \langle A\rangle expr_k)$$

Conclusions

In the present work:

- we studied the extension of the interval temporal logic $AB\overline{B}$, interpreted over finite linear orders and the linear order of natural numbers, with an equivalence relation $\sim$;
- we showed how ωS-regular languages can be encoded in $AB\overline{B}\sim$ (in a previous paper, we established a similar result for $A\overline{A}B\overline{B}$ and ωB-regular languages), thus establishing a promising connection between extensions of ω-languages and interval temporal logics.

Future work:

- to study the expressiveness of the extension of other interval temporal logics with an equivalence relation;
- to identify the interval temporal logic counterpart of ωBS-regular languages (a natural candidate is $A\overline{A}B\overline{B}\sim$).