



Reactive Turing machines



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ABSTRACT

We propose reactive Turing machines (RTMs), extending classical Turing machines with a process-theoretical notion of interaction, and use it to define a notion of executable transition system. We show that every computable transition system with a bounded branching degree is simulated modulo divergence-preserving branching bisimilarity by an RTM, and that every effective transition system is simulated modulo the variant of branching bisimilarity that does not require divergence preservation. We conclude from these results that the parallel composition of (communicating) RTMs can be simulated by a single RTM. We prove that there exist universal RTMs modulo branching bisimilarity, but these essentially employ divergence to be able to simulate an RTM of arbitrary branching degree. We also prove that modulo divergence-preserving branching bisimilarity there are RTMs that are universal up to their own branching degree. We establish a correspondence between executability and finite definability in a simple process calculus. Finally, we establish that RTMs are at least as expressive as *persistent Turing machines*.

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1. Introduction

The Turing machine [30] is widely accepted as a computational model suitable for exploring the theoretical boundaries of computing. Motivated by the existence of universal Turing machines, many textbooks on the theory of computation present the Turing machine not just as a theoretical model to explain which functions are computable, but as an accurate conceptual model of the computer. There is, however, a well-known limitation to this view [33,19]. A Turing machine operates from the assumptions that: (1) all input it needs for the computation is available on the tape from the very beginning; (2) it performs a terminating computation; and (3) it leaves the output on the tape at the very end. That is, a Turing machine computes a function, and thus it abstracts from two key ingredients of computing: *interaction* and *non-termination*. Nowadays, most computing systems are so-called *reactive systems* [21], systems that are generally not meant to terminate and consist of computing devices that interact with each other and with their environment.

Concurrency theory emerged from the early work of Petri [26] and has now developed into a mature theory of reactive systems. We mention three of its contributions particularly relevant for our work. Firstly, it installed the notion of transition system – a generalisation of the notion of finite-state automaton from classical automata theory – as the prime mathematical model to represent discrete behaviour. Secondly, it offered the insight that language equivalence – the underlying equivalence in classical automata theory – is too coarse in a setting with interacting automata; instead one should

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consider automata up to some form of bisimilarity. Thirdly, it yielded many algebraic process calculi facilitating the formal specification and verification of reactive systems.

Several proposals have been made in the literature extending Turing machines with a notion of interaction (e.g., the *persistent* Turing machines of [18] and the *interactive* Turing machines of [23]). Since the purpose of these works is studying the effect of interaction on the expressiveness of sequential computation, interaction is added through an ad hoc input–output facility of the Turing machine. In this paper, we propose to add interaction as an orthogonal facility, in line with the way interaction is studied in concurrency theory. The result will be a semantically refined model of interactive behaviour, integrating the well-established classical theory of automata with the theory of concurrency. The advantage of the integration is that interactive behaviour can be studied both from a concurrency-theoretic perspective (e.g., up to any of the many behavioural equivalences known from the concurrency-theoretic literature, see [14]), while at the same time there is an automata-based notion of executability associated with it.

In Section 2 we propose a notion of *reactive* Turing machine (RTM), extending the classical notion of Turing machine with interaction in the style of concurrency theory. The extension consists of a facility to declare every transition to be either *observable*, by labelling it with an action symbol, or *unobservable*, by labelling it with τ . Typically, a transition labelled with an action symbol models an interaction of the RTM with its environment (or some other RTM), while a transition labelled with τ refers to an internal computation step. Thus, a conventional Turing machine can be regarded as a special kind of RTM in which all transitions are declared unobservable by labelling them with τ .

The semantic object associated with a conventional Turing machine is either the function that it computes, or the formal language that it accepts. The semantic object associated with an RTM is a behaviour, formally represented by a transition system. A function is said to be effectively computable if it can be computed by a Turing machine. By analogy, we say that a behaviour is effectively executable if it can be exhibited by an RTM. In concurrency theory, behaviours are usually considered modulo a suitable behavioural equivalence. In this paper we use (*divergence-preserving*) *branching bisimilarity* [17], which is, on the one hand, the finest behavioural equivalence in van Glabbeek's spectrum (see [14]), and, on the other hand, the coarsest behavioural equivalence that takes unobservable activity into account, is a congruence for parallel composition, and only equates processes satisfying the same CTL^*_X -formulas [16].

In Section 3 we set out to investigate the expressiveness of RTMs up to divergence-preserving branching bisimilarity. We present an example of a behaviour that is not executable up to branching bisimilarity. Then, we establish that every computable transition system with a bounded branching degree can be simulated, up to divergence-preserving branching bisimilarity, by an RTM. If the divergence-preservation requirement is dropped, then every effective transition system can be simulated. These results then allow us to conclude that the behaviour of a parallel composition of RTMs can be simulated on a single RTM.

In Section 4 we define a suitable notion of universality for RTMs and investigate the existence of universal RTMs. We find that, since bisimilarity is sensitive to branching, there are some subtleties pertaining to the branching degree bound associated with each RTM. Up to divergence-preserving branching bisimilarity, an RTM can at best simulate other RTMs with the same or a lower bound on their branching degree. If divergence-preservation is not required, however, then universal RTMs do exist.

In Section 5, we consider the correspondence between RTMs and a process calculus consisting of a few standard process-theoretic constructions. On the one hand, the process calculus provides a convenient way to specify executable behaviour; indeed, every guarded recursive specification gives rise to a computable transition system [31], which can be simulated up to branching bisimilarity with an RTM. On the other hand, we establish that every executable behaviour is, again up to divergence-preserving branching bisimilarity, finitely definable in our calculus. Recursive specifications are the concurrency-theoretic counterparts of grammars in the theory of formal languages. Thus, the result in Section 5 may be considered as the process-theoretic version of the correspondence between Turing machines and unrestricted grammars.

In Section 6, we argue that reactive Turing machines are at least as expressive as the persistent Turing machines of [18], and in Section 7 we conclude the paper with a discussion of related work and some ideas for future work.

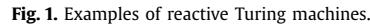
2. Reactive Turing machines

We presuppose a finite set \mathcal{A} of *action symbols* that we use to denote the observable events of a system. An unobservable event is denoted with τ , assuming that $\tau \notin \mathcal{A}$; we henceforth denote the set $\mathcal{A} \cup \{\tau\}$ by \mathcal{A}_τ . We also presuppose a finite set \mathcal{D} of *data symbols*. We add to \mathcal{D} a special symbol \square to denote a blank tape cell, assuming that $\square \notin \mathcal{D}$; we denote the set $\mathcal{D} \cup \{\square\}$ of *tape symbols* by \mathcal{D}_\square . Mostly, the precise contents of the sets \mathcal{A} and \mathcal{D} are unimportant for the developments in this paper, but it will occasionally be convenient to assume explicitly that certain special symbols are included in them.

Definition 2.1. A *reactive Turing machine* (RTM) \mathcal{M} is a quadruple $(S, \rightarrow, \uparrow, \downarrow)$ consisting of a finite set of *states* S , a distinguished *initial state* $\uparrow \in S$, a subset of *final states* $\downarrow \subseteq S$, and a $(\mathcal{D}_\square \times \mathcal{A}_\tau \times \mathcal{D}_\square \times \{L, R\})$ -labelled transition relation

$$\rightarrow \subseteq S \times \mathcal{D}_\square \times \mathcal{A}_\tau \times \mathcal{D}_\square \times \{L, R\} \times S.$$

An RTM is *deterministic* if $(s, d, a, e_1, M_1, t_1) \in \rightarrow$ and $(s, d, a, e_2, M_2, t_2) \in \rightarrow$ implies that $e_1 = e_2$, $t_1 = t_2$ and $M_1 = M_2$ for all $s, t_1, t_2 \in S$, $d, e_1, e_2 \in \mathcal{D}_\square$, $a \in \mathcal{A}_\tau$, and $M_1, M_2 \in \{L, R\}$, and, moreover, $(s, d, \tau, e_1, M_1, t_1) \in \rightarrow$ implies that there do not exist $a \neq \tau$, e_2, M_2, t_2 such that $(s, d, a, e_2, M_2, t_2) \in \rightarrow$.



With every RTM \mathcal{M} we are going to associate a transition system $\mathcal{T}(\mathcal{M})$. The states of $\mathcal{T}(\mathcal{M})$ are the configurations of the RTM, consisting of a state of the RTM, its tape contents, and the position of the read/write head on the tape. We represent the tape contents by an element of $(\mathcal{D}_{\square})^*$, replacing precisely one occurrence of a tape symbol d by a *marked* symbol \check{d} , indicating that the read/write head is on this symbol. We denote by $\check{\mathcal{D}}_{\square} = \{\check{d} \mid d \in \mathcal{D}_{\square}\}$ the set of *marked* tape symbols; a *tape instance* is a sequence $\delta \in (\mathcal{D}_{\square} \cup \check{\mathcal{D}}_{\square})^*$ such that δ contains exactly one element of $\check{\mathcal{D}}_{\square}$. Note that we do not use δ exclusively for tape instances; we also use δ for sequences over \mathcal{D} . A tape instance thus is a finite sequence of symbols that represents the contents of a two-way infinite tape. Henceforth, we do not distinguish between tape instances that are equal modulo the addition or removal of extra occurrences of the symbol \square at the left or right extremes of the sequence. That is, we do not distinguish tape instances δ_1 and δ_2 if $\square^{\omega}\delta_1\square^{\omega} = \square^{\omega}\delta_2\square^{\omega}$.

Definition 2.4. A *configuration* of an RTM $\mathcal{M} = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ is a pair (s, δ) consisting of a state $s \in \mathcal{S}$, and a tape instance δ .

Our transition system semantics defines an \mathcal{A}_τ -labelled transition relation on configurations such that an RTM-transition $s \xrightarrow{a[d/e]M} t$ corresponds with a -labelled transitions from configurations consisting of the RTM-state s and a tape instance in which some occurrence of d is marked. The transitions lead to configurations consisting of t and a tape instance in which the marked symbol d is replaced by e , and either the symbol to the left or to right of this occurrence of e is replaced by its marked version, according to whether $M = L$ or $M = R$. If e happens to be the first symbol and $M = L$, or the last symbol and $M = R$, then an additional blank symbol is appended at the left or right end of the tape instance, respectively, to model the movement of the head.

We introduce some notation to concisely denote the new placement of the tape head marker. Let δ be an element of \mathcal{D}_\square^* . Then by $\delta^<$ we denote the element of $(\mathcal{D}_\square \cup \check{\mathcal{D}}_\square)^*$ obtained by placing the tape head marker on the right-most symbol of δ if it exists, and $\check{\square}$ otherwise, i.e.,

$$\delta^< = \begin{cases} \zeta \check{d} & \text{if } \delta = \zeta d \text{ } (d \in \mathcal{D}_\square, \zeta \in \mathcal{D}_\square^*), \text{ and} \\ \check{\square} & \text{if } \delta = \varepsilon. \end{cases}$$

(We use ε to denote the empty sequence.) Similarly, by $\delta^>$ we denote the element of $(\mathcal{D}_\square \cup \check{\mathcal{D}}_\square)^*$ obtained by placing the tape head marker on the left-most symbol of δ if it exists, and $\check{\square}$ otherwise, i.e.,

$$\delta^> = \begin{cases} \check{d} \zeta & \text{if } \delta = d \zeta \text{ } (d \in \mathcal{D}_\square, \zeta \in \mathcal{D}_\square^*), \text{ and} \\ \check{\square} & \text{if } \delta = \varepsilon. \end{cases}$$

Definition 2.5. Let $\mathcal{M} = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ be an RTM. The *transition system* $\mathcal{T}(\mathcal{M})$ associated with \mathcal{M} is defined as follows:

1. its set of states is the set of all configurations of \mathcal{M} ;
2. its transition relation \rightarrow is the least relation satisfying, for all $a \in \mathcal{A}_\tau$, $d, e \in \mathcal{D}_\square$ and $\delta_L, \delta_R \in \mathcal{D}_\square^*$:

$$(s, \delta_L \check{d} \delta_R) \xrightarrow{a} (t, \delta_L^< e \delta_R) \quad \text{iff } s \xrightarrow{a[d/e]L} t, \quad \text{and}$$

$$(s, \delta_L \check{d} \delta_R) \xrightarrow{a} (t, \delta_L e^> \delta_R) \quad \text{iff } s \xrightarrow{a[d/e]R} t;$$

3. its initial state is the configuration $(\uparrow, \check{\square})$; and
4. its set of final states is the set of terminating configurations $\{(s, \delta) \mid s \downarrow\}$.

Turing introduced his machines to define the notion of *effectively computable function*. By analogy, our notion of RTM can be used to define a notion of *effectively executable behaviour*.

Definition 2.6. A transition system is *executable* if it is associated with an RTM.

Parallel composition. To illustrate how RTMs are suitable to model a form of interaction, we proceed to define on RTMs a notion of parallel composition, equipped with a simple form of communication. (We are not trying to define the most general or most suitable notion of parallel composition for RTMs here; the purpose of the notion of parallel composition defined here is just to illustrate how RTMs may run in parallel and interact.) Let \mathcal{C} be a finite set of *channels* for the communication of data symbols between one RTM and another. Intuitively, $c!d$ stands for the action of sending datum d along channel c , while $c?d$ stands for the action of receiving datum d along channel c .

First, we define a notion of parallel composition on transition systems. Let $T_1 = (\mathcal{S}_1, \rightarrow_1, \uparrow_1, \downarrow_1)$ and $T_2 = (\mathcal{S}_2, \rightarrow_2, \uparrow_2, \downarrow_2)$ be transition systems, and let $\mathcal{C}' \subseteq \mathcal{C}$. The *parallel composition* of T_1 and T_2 is the transition system $[T_1 \parallel T_2]_{\mathcal{C}'}$ = $(\mathcal{S}, \rightarrow, \uparrow, \downarrow)$, with $\mathcal{S}, \rightarrow, \uparrow$ and \downarrow defined by

1. $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$;
2. $(s_1, s_2) \xrightarrow{a} (s'_1, s'_2)$ iff $a \in \mathcal{A}_\tau - \{c!d, c?d \mid c \in \mathcal{C}', d \in \mathcal{D}_\square\}$ and either
 - (a) $s_1 \xrightarrow{a} s'_1$ and $s_2 = s'_2$, or $s_2 \xrightarrow{a} s'_2$ and $s_1 = s'_1$, or
 - (b) $a = \tau$ and either $s_1 \xrightarrow{c!d} s'_1$ and $s_2 \xrightarrow{c?d} s'_2$, or $s_1 \xrightarrow{c?d} s'_1$ and $s_2 \xrightarrow{c!d} s'_2$ for some $c \in \mathcal{C}'$ and $d \in \mathcal{D}_\square$;
3. $\uparrow = (\uparrow_1, \uparrow_2)$; and
4. $\downarrow = \{(s_1, s_2) \mid s_1 \in \downarrow_1 \text{ \& } s_2 \in \downarrow_2\}$.

Definition 2.7. Let $\mathcal{M}_1 = (\mathcal{S}_1, \rightarrow_1, \uparrow_1, \downarrow_1)$ and $\mathcal{M}_2 = (\mathcal{S}_2, \rightarrow_2, \uparrow_2, \downarrow_2)$ be RTMs, and let $\mathcal{C}' \subseteq \mathcal{C}$; by $[\mathcal{M}_1 \parallel \mathcal{M}_2]_{\mathcal{C}'}$ we denote the *parallel composition* of \mathcal{M}_1 and \mathcal{M}_2 . The transition system $\mathcal{T}([\mathcal{M}_1 \parallel \mathcal{M}_2]_{\mathcal{C}'})$ associated with the parallel composition $[\mathcal{M}_1 \parallel \mathcal{M}_2]_{\mathcal{C}'}$ of \mathcal{M}_1 and \mathcal{M}_2 is the parallel composition of the transition systems associated with \mathcal{M}_1 and \mathcal{M}_2 , i.e., $\mathcal{T}([\mathcal{M}_1 \parallel \mathcal{M}_2]_{\mathcal{C}'}) = [\mathcal{T}(\mathcal{M}_1) \parallel \mathcal{T}(\mathcal{M}_2)]_{\mathcal{C}'}$.

Example 2.8. Let \mathcal{A} be as in Example 2.2, let \mathcal{M} denote the left-hand side RTM in Fig. 1, and let \mathcal{E} denote the right-hand side RTM in Fig. 1. Then the parallel composition $[\mathcal{M} \parallel \mathcal{E}]_i$ exhibits the behaviour of outputting, along channel o , the string $11\#1111\#\dots\#1^n\#$ ($n \geq 2$, n even).

Behavioural equivalence. In automata theory, Turing machines that compute the same function or accept the same language are generally considered equivalent. In fact, functional or language equivalence is underlying many of the standard notions and results in automata theory. Perhaps most notably, a *universal* Turing machine is a Turing machine that, when started with the code of some Turing machine on its tape, simulates this machine up to functional or language equivalence. A result from concurrency theory is that functional and language equivalence are arguably too coarse for reactive systems, because they abstract from all moments of choice (see, e.g., [1]). In concurrency theory many alternative behavioural equivalences have been proposed; we refer to [14] for a classification.

The results about RTMs that are obtained in the remainder of this paper are modulo *branching bisimilarity* [17], which is the finest behavioural equivalence in van Glabbeek's linear time – branching time spectrum [14]. We consider both the divergence-insensitive and the divergence-preserving variant. (The divergence-preserving variant is called *branching bisimilarity with explicit divergence* in [17,14], but in this paper we prefer the term *divergence-preserving branching bisimilarity*.)

We proceed to define the behavioural equivalences that we employ in this paper to compare transition systems. Let \rightarrow be an \mathcal{A}_τ -labelled transition relation on a set \mathcal{S} , and let $a \in \mathcal{A}_\tau$; we write $s \xrightarrow{(a)} t$ if $s \xrightarrow{a} t$ or $a = \tau$ and $s = t$. Furthermore, we denote the transitive closure of $\xrightarrow{\tau}$ by $\xrightarrow{+}$, and we denote the reflexive-transitive closure of $\xrightarrow{\tau}$ by $\xrightarrow{*}$.

Definition 2.9. Let $T_1 = (\mathcal{S}_1, \rightarrow_1, \uparrow_1, \downarrow_1)$ and $T_2 = (\mathcal{S}_2, \rightarrow_2, \uparrow_2, \downarrow_2)$ be transition systems. A *branching bisimulation* from T_1 to T_2 is a binary relation $\mathcal{R} \subseteq \mathcal{S}_1 \times \mathcal{S}_2$ and, for all states s_1 and s_2 , $s_1 \mathcal{R} s_2$ implies

1. if $s_1 \xrightarrow{a}_1 s'_1$, then there exist $s'_2, s''_2 \in \mathcal{S}_2$ such that $s_2 \xrightarrow{*}_2 s'_2 \xrightarrow{(a)}_2 s''_2$, $s_1 \mathcal{R} s'_2$ and $s'_1 \mathcal{R} s'_2$;
2. if $s_2 \xrightarrow{a}_2 s'_2$, then there exist $s'_1, s''_1 \in \mathcal{S}_1$ such that $s_1 \xrightarrow{*}_1 s'_1 \xrightarrow{(a)}_1 s''_1$, $s'_1 \mathcal{R} s_2$ and $s'_1 \mathcal{R} s'_2$;
3. if $s_1 \downarrow_1$, then there exists s'_2 such that $s_2 \xrightarrow{*}_2 s'_2$, $s_1 \mathcal{R} s'_2$ and $s'_2 \downarrow_2$; and
4. if $s_2 \downarrow_2$, then there exists s'_1 such that $s_1 \xrightarrow{*}_1 s'_1$, $s'_1 \mathcal{R} s_2$ and $s'_1 \downarrow_1$.

The transition systems T_1 and T_2 are *branching bisimilar* (notation: $T_1 \Leftrightarrow_b T_2$) if there exists a branching bisimulation from T_1 to T_2 such that $\uparrow_1 \mathcal{R} \uparrow_2$.

A branching bisimulation \mathcal{R} from T_1 to T_2 is *divergence-preserving* if, for all states s_1 and s_2 , $s_1 \mathcal{R} s_2$ implies

5. if there exists an infinite sequence $(s_{1,i})_{i \in \mathbb{N}}$ such that $s_1 = s_{1,0}$, $s_{1,i} \xrightarrow{\tau} s_{1,i+1}$ and $s_{1,i} \mathcal{R} s_2$ for all $i \in \mathbb{N}$, then there exists a state s'_2 such that $s_2 \xrightarrow{+} s'_2$ and $s_{1,i} \mathcal{R} s'_2$ for some $i \in \mathbb{N}$; and
6. if there exists an infinite sequence $(s_{2,i})_{i \in \mathbb{N}}$ such that $s_2 = s_{2,0}$, $s_{2,i} \xrightarrow{\tau} s_{2,i+1}$ and $s_1 \mathcal{R} s_{2,i}$ for all $i \in \mathbb{N}$, then there exists a state s'_1 such that $s_1 \xrightarrow{+} s'_1$ and $s'_1 \mathcal{R} s_{2,i}$ for some $i \in \mathbb{N}$.

The transition systems T_1 and T_2 are *divergence-preserving branching bisimilar* (notation: $T_1 \Leftrightarrow_b^\Delta T_2$) if there exists a divergence-preserving branching bisimulation from T_1 to T_2 such that $\uparrow_1 \mathcal{R} \uparrow_2$.

The notions of branching bisimilarity and divergence-preserving branching bisimilarity originate with [17]. The particular divergence conditions we use to define divergence-preserving branching bisimulations here are discussed in [15], where it is also proved that divergence-preserving branching bisimilarity is an equivalence.

An unobservable transition of an RTM, i.e., a transition labelled with τ , may be thought of as an internal computation step. Divergence-preserving branching bisimilarity allows us to abstract from internal computations as long as they do not discard the option to execute a certain behaviour. The following notion is used as a technical tool in the remainder of the paper.

Definition 2.10. Given some transition system T , an *internal computation* from state s to s' is a sequence of states s_1, \dots, s_n in T such that $s = s_1 \xrightarrow{\tau} \dots \xrightarrow{\tau} s_n = s'$. An internal computation is called *deterministic* iff, for every state s_i ($1 \leq i < n$), $s_i \xrightarrow{a} s'_i$ implies $a = \tau$ and $s'_i = s_{i+1}$. If s_1, \dots, s_n is a deterministic internal computation from s to s' , then we refer to the set

$$\{s_1, \dots, s_n\}$$

as the set of *intermediate states* of the deterministic internal computation.

Proposition 2.11. Let T be a transition system and let s and t be two states in T . If there exists a deterministic internal computation from s to s' , then all its intermediate states are related by the maximal divergence-preserving branching bisimulation on T .

3. Expressiveness of RTMs

Our notion of RTMs defines the class of executable transition systems. In this section, we investigate the expressiveness of this notion up to branching bisimilarity, using the notions of effective transition system and computable transition system as a tool.

In Section 3.1, we recall the definitions of effective transition system and computable transition system, observe that executable transition systems are necessarily computable and, moreover, have a bounded branching degree. Then, we proceed to consider executable transition systems modulo (divergence-preserving) branching bisimilarity. We present an example of a (non-effective) transition system that is not executable up to branching bisimilarity. Finally, we adapt a result by Phillips [27] showing that every effective transition system is branching bisimilar to a computable transition system with branching degree at most two. Phillips' proof introduces divergence, and we present an example illustrating that this is unavoidable.

In Section 3.2, we construct, for an arbitrary boundedly branching computable transition system, an RTM that simulates the behaviour represented by the transition system up to divergence-preserving branching bisimilarity. Thus, we confirm the expressiveness of RTMs: modulo divergence-preserving branching bisimilarity, which is the finest behavioural equivalence in van Glabbeek's spectrum [14], the class of executable transition systems coincides with the class of boundedly branching computable transition systems. Moreover, in view of Phillips' result, we obtain as a corollary that every effective transition system can be simulated up to branching bisimilarity at the cost of introducing divergence.

We obtain two more interesting corollaries from the result in Section 3.2. Firstly, if a transition system is deterministic, then, by our assumption that the set \mathcal{A} of action symbols is finite, it is clearly boundedly branching; hence, every deterministic computable transition systems can be simulated, up to divergence-preserving branching bisimilarity, by a deterministic RTM. Secondly, the parallel composition of boundedly branching computable transition systems is clearly boundedly branching and computable; hence, a parallel composition of RTMs can be simulated, up to divergence-preserving branching bisimilarity, by a single RTM.

3.1. Effective and computable transition systems

Let $T = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ be a transition system; the mapping $out : \mathcal{S} \rightarrow 2^{\mathcal{A}_\tau \times \mathcal{S}}$ associates with every state its set of outgoing transitions, i.e., for all $s \in \mathcal{S}$,

$$out(s) = \{(a, t) \mid s \xrightarrow{a} t\},$$

and fin denotes the characteristic function of \downarrow . We restrict our attention in this section to *finitely branching* transition systems, i.e., transition systems for which it holds that $out(s)$ is finite for all states s . (The restriction is convenient for our definition of computable transition system below, but it is otherwise unimportant since in all our results about computable transition systems we further restrict to boundedly branching transition systems. The restriction is not necessary for the definition of effective transition system, and, in fact, our results about effective transition systems do not depend on it.)

Definition 3.1. Let $T = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ be an \mathcal{A}_τ -labelled finitely branching transition system. We say that T is *effective* if \rightarrow and \downarrow are recursively enumerable sets. We say that T is *computable* if out and fin are recursive functions.

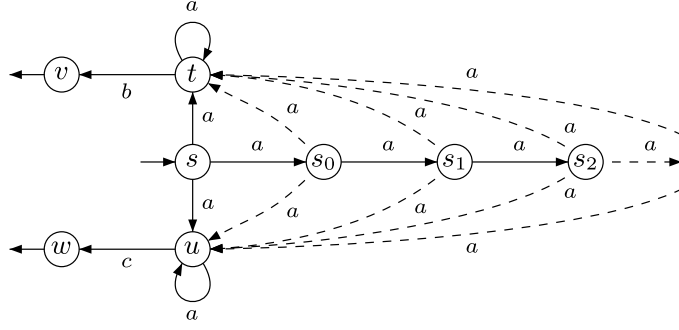
The notion of effective transition system originates with Boudol [10]. For the notion of computable transition system we have reformulated the definition in [2] to suit our needs. We temporarily step over the fact that, in order for the formal theory of recursiveness to make sense, we need suitable codings into natural numbers of the concepts involved. For now, we rely on the intuition of the reader; in Section 3.2 we return to this issue in more detail. (The reader may already want to consult [29, §1.10] for more explanations.) A transition system is effective iff there exists an algorithm that enumerates its transitions and an algorithm that enumerates its final states. Similarly, a transition system is computable iff there exists an algorithm that outputs the set of outgoing transitions of a state and also determines if it is final.

Proposition 3.2. *The transition system associated with an RTM is computable.*

Proof. We omit a formal proof, but note that Definition 2.5 describes the essence of algorithms for computing the outgoing transitions of a configuration and for determining if a configuration is final. \square

Hence, unsurprisingly, if a transition system is not computable, then it is not executable either. It is easy to define transition systems that are not computable, so there exist behaviours that are not executable. The following example takes this a little further and illustrates that there exist behaviours that are not even executable up to branching bisimilarity.

Example 3.3. (In this and later examples, we denote by φ_x the partial recursive function with index $x \in \mathbb{N}$ in some exhaustive enumeration of partial recursive functions, see, e.g., [29].) Assume that $\mathcal{A} = \{a, b, c\}$ and consider the \mathcal{A} -labelled transition system $T_0 = (\mathcal{S}_0, \rightarrow_0, \uparrow_0, \downarrow_0)$ with $\mathcal{S}_0, \rightarrow_0, \uparrow_0$ and \downarrow_0 defined by

Fig. 2. The transition system T_0 .

$$\begin{aligned}
S_0 &= \{s, t, u, v, w\} \cup \{s_x \mid x \in \mathbb{N}\}, \\
\rightarrow_0 &= \{(s, a, t), (t, a, t), (t, b, v), (s, a, u), (u, a, u), (u, c, w)\} \\
&\quad \cup \{(s, a, s_0)\} \cup \{(s_x, a, s_{x+1}) \mid x \in \mathbb{N}\} \\
&\quad \cup \{(s_x, a, t), (s_x, a, u) \mid \varphi_x \text{ is a total function}\}, \\
\uparrow_0 &= s, \quad \text{and} \\
\downarrow_0 &= \{v, w\}.
\end{aligned}$$

The transition system is depicted in Fig. 2.

To argue that T_0 is not executable up to branching bisimilarity, we proceed by contradiction. Suppose that T_0 is executable up to branching bisimilarity. Then T_0 is branching bisimilar to a computable transition system T'_0 . Then, in T'_0 , the set of states reachable by a path that contains exactly x a -transitions ($x \in \mathbb{N}$) and from which both a b - and a c -transition are still reachable, is recursively enumerable. It follows that the set of states in T'_0 branching bisimilar to s_x ($x \in \mathbb{N}$) is recursively enumerable. But then, since the problem of deciding whether from some state in T'_0 there is a path containing exactly one a -transition and one b -transition such that the a -transition precedes the b -transition, is also recursively enumerable, it follows that the problem of deciding whether φ_x is a total function must be recursively enumerable too, which it is not. We conclude that T_0 is not executable up to branching bisimilarity. Incidentally, note that the language associated with T_0 is $\{a^n b, a^n c \mid n \geq 1\}$, which is recursively enumerable (it is even context-free).

Phillips associates, in [27], with every effective transition system a *branching bisimilar* computable transition system of which, moreover, every state has a branching degree of at most 2. (Phillips actually establishes weak bisimilarity, but it is easy to see that branching bisimilarity holds.)

Definition 3.4. Let $T = (S, \rightarrow, \uparrow, \downarrow)$ be a transition system, and let B be a natural number. We say that T has a branching degree *bounded by* B if $|\text{out}(s)| \leq B$, for every state $s \in S$. We say that T is *boundedly branching* if there exists $B \in \mathbb{N}$ such that the branching degree of T is bounded by B .

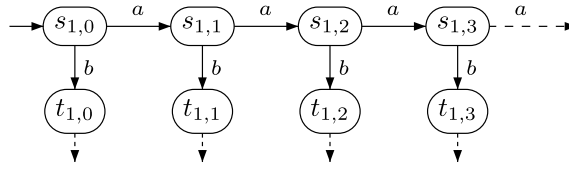
Proposition 3.5 (Phillips). *For every effective transition system T there exists a boundedly branching computable transition system T' such that $T \Leftrightarrow_b T'$.*

A crucial insight in Phillips' proof is that a divergence (i.e., an infinite sequence of τ -transitions) can be exploited to simulate a state of which the set of outgoing transitions is recursively enumerable, but not recursive. The following example, inspired by [12], shows that introducing divergence is unavoidable.

Example 3.6. Assume that $\mathcal{A} = \{a, b\}$, and consider the transition system $T_1 = (S_1, \rightarrow_1, \uparrow_1, \downarrow_1)$ with S_1 , \rightarrow_1 , \uparrow_1 and \downarrow_1 defined by

$$\begin{aligned}
S_1 &= \{s_{1,x}, t_{1,x} \mid x \in \mathbb{N}\}, \\
\rightarrow_1 &= \{(s_{1,x}, a, s_{1,x+1}) \mid x \in \mathbb{N}\} \cup \{(s_{1,x}, b, t_{1,x}) \mid x \in \mathbb{N}\}, \\
\uparrow_1 &= s_{1,0}, \quad \text{and} \\
\downarrow_1 &= \{t_{1,x} \mid \varphi_x(x) \text{ converges}\}.
\end{aligned}$$

The transition system is depicted in Fig. 3.

Fig. 3. The transition system T_1 .

Now, suppose that T_2 is a transition system such that $T_1 \leq_b^\Delta T_2$, as witnessed by some divergence-preserving branching bisimulation relation \mathcal{R} ; we argue that T_2 is not computable by deriving a contradiction from the assumption that it is.

Clearly, since T_1 does not admit infinite sequences of τ -transitions, if \mathcal{R} is divergence-preserving, then T_2 does not admit infinite sequences of τ -transitions either. It follows that if $s_1 \mathcal{R} s_2$, then there exists a state s'_2 in T_2 such that $s_2 \xrightarrow{*}_2 s'_2$, $s_1 \mathcal{R} s'_2$, and $s'_2 \not\xrightarrow{\tau}$. Moreover, since T_2 is computable and does not admit infinite sequences of consecutive τ -transitions, a state s'_2 satisfying the aforementioned properties is produced by the algorithm that, given a state of T_2 , selects an enabled τ -transition and recurses on the target of the transition until it reaches a state in which no τ -transitions are enabled.

But then we also have an algorithm that determines if $\varphi_x(x)$ converges:

1. it starts from the initial state \uparrow_2 of T_2 ;
2. it runs the algorithm to find a state without outgoing τ -transitions, and then it repeats the following steps x times:
 - (a) execute the a -transition enabled in the reached state;
 - (b) run the algorithm to find a state without outgoing τ -transitions again;
 since $\uparrow_1 \mathcal{R} \uparrow_2$, this yields a state $s_{2,x}$ in T_2 such that $s_{1,x} \mathcal{R} s_{2,x}$;
3. it executes the b -transition that must be enabled in $s_{2,x}$, followed, again, by the algorithm to find a state without outgoing τ -transitions; this yields a state $t_{2,x}$, without any outgoing transitions, such that $t_{1,x} \mathcal{R} t_{2,x}$.

From $t_{1,x} \mathcal{R} t_{2,x}$ it follows that $t_{2,x} \in \downarrow_2$ iff $\varphi_x(x)$ converges, so the problem of deciding whether $\varphi_x(x)$ converges has been reduced to the problem of deciding whether $t_{2,x} \in \downarrow_2$. Since it is undecidable if $\varphi_x(x)$ converges, it follows that \downarrow_2 is not recursive, which contradicts our assumption that T_2 is computable.

3.2. Simulation of boundedly branching computable transition systems

Let $T = (S_T, \rightarrow_T, \uparrow_T, \downarrow_T)$ be a boundedly branching computable transition system, say with branching degree bounded by B . Our goal is to construct an RTM

$$\text{Sim} = (\mathcal{S}_{\text{Sim}}, \rightarrow_{\text{Sim}}, \uparrow_{\text{Sim}}, \downarrow_{\text{Sim}}),$$

called the *simulator* for T , such that $\mathcal{T}(\text{Sim}) \leq_b^\Delta T$.

We have defined that T is computable if the associated mappings *out* and *fin* are recursive functions. As is explained in [29, §1.10], the formal theory of recursiveness can be applied to non-numerical functions (i.e., functions of which domain and range are not the set of natural numbers), through codings associating a unique natural number with every symbolic entity. In our case, we fix codings of \mathcal{A}_T and S , i.e., injections $\ulcorner \cdot \urcorner : \mathcal{A}_T \rightarrow \mathbb{N}$ and $\ulcorner \cdot \urcorner : S \rightarrow \mathbb{N}$ into the set of natural numbers \mathbb{N} . We use these codings, and standard techniques for coding and decoding tuples of natural numbers and finite sets of natural numbers¹ to define partial recursive functions *out* and *fin* on natural numbers:

- *out* : $\mathbb{N} \rightarrow \mathbb{N}$ is the partial function that, for all states s , maps $\ulcorner s \urcorner$ to the code associated with *out*(s) and is undefined for all natural numbers that are not codes of states; and
- *fin* : $\mathbb{N} \rightarrow \mathbb{N}$ is the partial function that maps $\ulcorner s \urcorner$ to *fin*(s)² and is undefined on natural numbers that are not codes of states.

For the remainder of this paper we fix an enumeration of the partial recursive functions, and we denote by $\ulcorner \text{out} \urcorner$ and $\ulcorner \text{fin} \urcorner$ indices of the partial recursive functions *out* : $\mathbb{N} \rightarrow \mathbb{N}$ and *fin* : $\mathbb{N} \rightarrow \mathbb{N}$ in this enumeration. Instead of hardcoding computations of *out* and *fin* in the RTM *Sim* to be constructed, we prefer to store their codes $\ulcorner \text{out} \urcorner$ and $\ulcorner \text{fin} \urcorner$ on the tape and interpret these codes. This is slightly more generic than necessary for simulation of the presupposed transition system T , but the genericity will pay off when we extend the simulator to obtain a universal RTM in Section 4.

We are going to define *Sim* as the union of three fragments, each with a different purpose.

1. The *initialisation fragment* prepares the tape of *Sim* for the simulation of T , writing the codes $\ulcorner \text{out} \urcorner$, $\ulcorner \text{fin} \urcorner$ and $\ulcorner \uparrow_T \urcorner$ to the tape.

¹ See, e.g., [29, §5.3 and §5.6], respectively.

² Recall that *fin* on states is a characteristic function and hence already yields a natural number.

2. In the *state fragment* the behaviour in the current state (i.e., whether it is terminating and what are its possible next transitions) is computed, and stored on the tape in coded form.
3. The *step fragment* first decodes the information on the tape about the behaviour of the current state as computed in the state fragment, moving to a special selection state of Sim that corresponds with the coded behaviour. (A crucial aspect of branching bisimilarity is that the choice for the next transition should be made in a single state. By our assumption that T is boundedly branching, we need to include only finitely many such selection states.) The aforementioned selection state has each of the possible outgoing transitions. After executing one of these transitions, the code of the target state, being the new current state, is written on the tape.

Below we present a detailed description of the construction of Sim. We first briefly discuss how we use the tape to store information regarding the current state and its behaviour. The implementation of the fragments involves several straightforward computational tasks on the contents of the tape. We do not dwell on the details of implementing these tasks; we just presuppose the existence of auxiliary deterministic Turing machines capable of carrying them out. Then, we discuss the implementation of the three fragments described above.

Tape. In the above, we have declared codes for actions and states, and for the partial recursive functions $out : \mathbb{N} \rightarrow \mathbb{N}$ and $fin : \mathbb{N} \rightarrow \mathbb{N}$. The way in which natural numbers are represented as sequences over some finite alphabet of tape symbols is largely irrelevant, but in our construction below it is sometimes convenient to have an explicit representation. In such cases, we assume that numbers are stored in unary notation using the symbol 1. That is, a natural number n is represented on the tape as the sequence 1^{n+1} of $n+1$ occurrences of the symbol 1. In addition to the symbol 1, we use the symbols \llbracket and \rrbracket to delimit the codes of out and fin that remain on the tape throughout the simulation, $|$ to separate the elements of a tuple of natural numbers, and $\#$ to separate tuples. The simulator Sim constructed below incorporates the operation of some auxiliary Turing machines that may require the use of some additional symbols; let \mathcal{D}' be the collection of all these extra symbols. Then the tape alphabet \mathcal{D} of Sim is

$$\mathcal{D} = \{1, \llbracket, \rrbracket, |, \#\} \cup \mathcal{D}'.$$

Auxiliary Turing machines. For our purposes, it is convenient to define a *deterministic Turing machine* \mathcal{M} as a quadruple $\mathcal{M} = (\mathcal{S}_{\mathcal{M}}, \rightarrow_{\mathcal{M}}, \uparrow_{\mathcal{M}}, \downarrow_{\mathcal{M}})$ with \mathcal{S} its set of *states*,

$$\rightarrow_{\mathcal{M}} \subseteq \mathcal{S}_{\mathcal{M}} \times \mathcal{D}_{\square} \times \mathcal{D}_{\square} \times \{L, R\} \times \mathcal{S}_{\mathcal{M}}$$

its *transition relation*, $\uparrow_{\mathcal{M}}$ its *initial state*, and $\downarrow_{\mathcal{M}}$ its (unique) *final state*. We assume that \mathcal{M} satisfies the following requirements:

- (i) for every pair $(s, d) \in (\mathcal{S} \setminus \{\downarrow_{\mathcal{M}}\}) \times \mathcal{D}_{\square}$ there is precisely one triple $(e, M, s') \in \mathcal{D}_{\square} \times \{L, R\} \times \mathcal{S}$ such that $(s, d, e, M, s') \in \rightarrow_{\mathcal{M}}$; and
- (ii) there do not exist $d, e \in \mathcal{D}_{\square}$, $M \in \{L, R\}$ and $s \in \mathcal{S}$ such that $(\downarrow_{\mathcal{M}}, d, e, M, s) \in \rightarrow_{\mathcal{M}}$.

Our definition of deterministic Turing machine is non-standard in assuming that whenever it halts, it does so in the special distinguished final state. This assumption is convenient when we incorporate the functionality implemented by a Turing machine in the definition of our simulator, below. To be convinced that our assumption does not limit the computational expressiveness of our notion of Turing machine, the reader may want to compare our variant with the one described in [29, pp. 13–16]. The latter does not have a distinguished halting state, but to convert it to one that satisfies our requirements, it suffices to add (in the notation of [29]) an internal state q_f and a quadruple q_i, d, d, q_f for every combination of q_i and d not already appearing as first two elements of a quadruple.

Note that a Turing machine can be viewed as an RTM without the \mathcal{A}_{τ} labels associated with transitions (see Definition 2.1) and with a singleton set of final states. Similarly as for RTMs (see Definition 2.4), a *configuration* of a Turing machine is a pair (s, δ) consisting of a state $s \in \mathcal{S}$ and a tape instance δ , and the transition relation $\rightarrow_{\mathcal{M}}$ of \mathcal{M} induces an unlabelled transition relation \longrightarrow on configurations, defined as in Definition 2.5 (simply omit all references to \mathcal{A}_{τ} and its elements).

Let δ and δ' be tape instances. By an \mathcal{M} -*computation* from δ to δ' we understand a sequence of configurations

$$(s_0, \delta_0), \dots, (s_n, \delta_n)$$

such that $s_0 = \uparrow_{\mathcal{M}}$, $\delta_0 = \delta$, $s_n = \downarrow_{\mathcal{M}}$, $\delta_n = \delta'$, and $(s_i, \delta_i) \longrightarrow (s_{i+1}, \delta_{i+1})$ for all $0 \leq i < n$. We write $\delta \mapsto_{\mathcal{M}} \delta'$ if there exists an \mathcal{M} -computation from δ to δ' .

Initialisation fragment. Note that it is straightforward to define a conventional deterministic Turing machine $\mathcal{I} = (\mathcal{S}_{\mathcal{I}}, \rightarrow_{\mathcal{I}}, \uparrow_{\mathcal{I}}, \downarrow_{\mathcal{I}})$ that, when started on an empty tape, writes the given natural numbers $\ulcorner out \urcorner$, $\ulcorner fin \urcorner$ and $\ulcorner \uparrow_T \urcorner$ on the tape in a suitable representation, yielding the tape instance

$$\llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner \uparrow_T \urcorner.$$

We use \mathcal{I} to define the *initialisation fragment* Init . The set of states of Init is defined as

$$\mathcal{S}_{\text{Init}} = \mathcal{S}_{\mathcal{I}} \setminus \downarrow_{\mathcal{I}},$$

its initial state is defined as

$$\uparrow_{\text{Init}} = \uparrow_{\mathcal{I}}; \quad \text{and}$$

its set of transitions is defined as

$$\begin{aligned} \rightarrow_{\text{Init}} = & \left\{ (in, d, \tau, e, M, in') \mid (in, d, e, M, in') \in \rightarrow_{\mathcal{I}}, in' \in \mathcal{S}_{\mathcal{I}} \setminus \downarrow_{\mathcal{I}} \right\} \\ & \cup \left\{ (in, d, \tau, e, M, \uparrow_{\text{State}}) \mid (in, d, e, M, in') \in \rightarrow_{\mathcal{I}}, in' \in \downarrow_{\mathcal{I}} \right\}. \end{aligned}$$

(Note that \uparrow_{State} is not a state in $\mathcal{S}_{\mathcal{I}}$; it is the initial state of the state fragment to be defined next.)

Fact 3.7. *The fragment Init gives rise to a deterministic internal computation from $(\uparrow_{\text{Init}}, \checkmark)$ to $(\uparrow_{\text{State}}, \llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner \uparrow_T \urcorner)$; we denote its set of intermediate states by $\text{IS}(\text{Init})$.*

State fragment. The *state fragment* State replaces the code of the current state on the tape by a sequence of codes that represents the behaviour of T in the current state. It is assumed that it starts with a tape instance of the form $\llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner s \urcorner$ for some $s \in \mathcal{S}_T$.

Recall that $\ulcorner out \urcorner$ and $\ulcorner fin \urcorner$ are indices of the partial recursive functions $out : \mathbb{N} \rightarrow \mathbb{N}$ and $fin : \mathbb{N} \rightarrow \mathbb{N}$ in some fixed enumeration of the partial recursive functions. Hence, there exists a Turing machine $\mathcal{U} = (\mathcal{S}_{\mathcal{U}}, \rightarrow_{\mathcal{U}}, \uparrow_{\mathcal{U}}, \downarrow_{\mathcal{U}})$ that interprets the codes $\ulcorner out \urcorner$ and $\ulcorner fin \urcorner$, applies the corresponding partial recursive functions out and fin to $\ulcorner s \urcorner$, and decodes the code of the finite set of pairs yielded by the function out into a list of codes of actions and target states. Without loss of generality, we may assume that \mathcal{U} , when started on a tape instance of the form

$$\llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner s \urcorner,$$

performs a terminating deterministic computation that yields the tape instance

$$\llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner fin(s) \mid \ulcorner a_1 \urcorner \mid \dots \mid \ulcorner a_k \urcorner \# \ulcorner s_1 \urcorner \mid \dots \mid \ulcorner s_k \urcorner \urcorner,$$

where $out(s) = \{(a_i, s_i) \mid 1 \leq i \leq k\}$. Note that, since the branching degree of T is bounded by B , we have that $k \leq B$. Henceforth, we refer to the sequence $fin(s), a_1, \dots, a_k$ generated and stored on the tape by \mathcal{U} as the *menu* in s .

The set of states of State is defined as

$$\mathcal{S}_{\text{State}} = \mathcal{S}_{\mathcal{U}} \setminus \downarrow_{\mathcal{U}};$$

its initial state is defined as

$$\uparrow_{\text{State}} = \uparrow_{\mathcal{U}}; \quad \text{and}$$

its set of transitions is defined as

$$\begin{aligned} \rightarrow_{\text{State}} = & \left\{ (st, d, \tau, e, M, st') \mid (st, d, e, M, st') \in \rightarrow_{\mathcal{U}}, st' \in \mathcal{S}_{\mathcal{U}} \setminus \downarrow_{\mathcal{U}} \right\} \\ & \cup \left\{ (st, d, \tau, e, M, \uparrow_{\text{Step}}) \mid (st, d, e, M, st') \in \rightarrow_{\mathcal{U}}, st' \in \downarrow_{\mathcal{U}} \right\}. \end{aligned}$$

(Again, note that \uparrow_{Step} is not a state in $\mathcal{S}_{\text{State}}$, but the initial state of the step fragment to be defined next.)

Fact 3.8. *Let $s \in \mathcal{S}_T$, let $0 \leq k \leq B$, let $a_1, \dots, a_k \in \mathcal{A}_T$ and $s_1, \dots, s_k \in \mathcal{S}_T$ such that $out(s) = \{(a_i, s_i) \mid 1 \leq i \leq k\}$. Then the fragment State gives rise to a deterministic computation from*

$$(\uparrow_{\text{State}}, \llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner s \urcorner)$$

to

$$(\uparrow_{\text{Step}}, \llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner fin(s) \mid \ulcorner a_1 \urcorner \mid \dots \mid \ulcorner a_k \urcorner \# \ulcorner s_1 \urcorner \mid \dots \mid \ulcorner s_k \urcorner \urcorner);$$

we denote its set of intermediate states by $\text{IS}(\text{State}, s)$.

Step fragment. The purpose of the *step fragment* Step is to select a transition enabled in the current state s , execute the corresponding action, and remove $\text{fin}(s)$ and all codes of actions and states from the tape, except the code of the target state of the selected transition.

The behaviour represented by the simulated transition system T when it is in state s consists of a non-deterministic choice between its k outgoing transitions $s \xrightarrow{a_1} s_1, \dots, s \xrightarrow{a_k} s_k$ and it is terminating if $\text{fin}(s) = 1$. To get a branching bisimulation between T and the transition system associated with Sim, the latter necessarily has to include a configuration offering exactly the same choice of outgoing transitions and the same termination behaviour. (It is important to note that branching bisimilarity does not, e.g., allow the choice for one of the outgoing transitions to be made by an internal computation that eliminates options one by one.) The fragment Step therefore includes one special state $sp_{\text{fin}(s), a_1, \dots, a_k}$ for every potential menu. Since $k \leq B$, the branching degree bound of T , there are $N = \sum_{k=0}^B 2 \cdot |\mathcal{A}_T|^k$ potential menus.

The functionality of the step fragment consists of two parts. The first part decodes the menu on the tape ending up in a state $sp_{\text{fin}(s), a_1, \dots, a_k}$. The second part takes care of the execution of an enabled transition and reinitialising the simulation with the target state of the executed transition as the new current state.

Let $\mathcal{D} = (\mathcal{S}_{\mathcal{D}}, \rightarrow_{\mathcal{D}}, \uparrow_{\mathcal{D}}, \downarrow_{\mathcal{D}})$ be a deterministic Turing machine with distinguished states $sp_{\text{fin}(s), a_1, \dots, a_k}$ (one for every potential menu) that, when started on a tape instance

$$\llbracket \ulcorner \text{out} \urcorner \mid \ulcorner \text{fin} \urcorner \rrbracket \text{fin}(s) \mid \ulcorner a_1 \urcorner \mid \dots \mid \ulcorner a_k \urcorner \# \ulcorner s_1 \urcorner \mid \dots \mid \ulcorner s_k \urcorner$$

performs a deterministic computation that halts in the state $sp_{\text{fin}(s), a_1, \dots, a_k}$ with tape instance $\llbracket \ulcorner \text{out} \urcorner \mid \ulcorner \text{fin} \urcorner \rrbracket \ulcorner s_1 \urcorner \mid \dots \mid \ulcorner s_k \urcorner$. Note that we can assume, without loss of generality, that

$$\downarrow_{\mathcal{D}} = \{sp_{t, a_1, \dots, a_k} \mid t \in \{0, 1\}, 0 \leq k \leq B, a_1, \dots, a_k \in \mathcal{A}_T\}.$$

The state $sp_{\text{fin}(s), a_1, \dots, a_k}$ is declared final iff $\text{fin}(s) = 1$, and it has k outgoing transitions labelled a_1, \dots, a_k , respectively. After performing the i th transition labelled with a_i , the list of codes of states $\ulcorner s_1 \urcorner \mid \dots \mid \ulcorner s_k \urcorner$ remaining on the tape should be replaced by the i th code in the list, after which the simulation returns to the state fragment. For each $1 \leq i \leq B$, let $\mathcal{R}_i = (\mathcal{S}_{\mathcal{R}_i}, \rightarrow_{\mathcal{R}_i}, \uparrow_{\mathcal{R}_i}, \downarrow_{\mathcal{R}_i})$ be a deterministic Turing machine that, when started on a tape instance of the form

$$\llbracket \ulcorner \text{out} \urcorner \mid \ulcorner \text{fin} \urcorner \rrbracket \ulcorner s_1 \urcorner \mid \dots \mid \ulcorner s_k \urcorner \quad (k \geq i)$$

halts with a tape instance

$$\llbracket \ulcorner \text{out} \urcorner \mid \ulcorner \text{fin} \urcorner \rrbracket \ulcorner s_i \urcorner.$$

The set of states of Step is defined as

$$\mathcal{S}_{\text{Step}} = \left(\mathcal{S}_{\mathcal{D}} \cup \bigcup_{i=1}^B \mathcal{S}_{\mathcal{R}_i} \right) \setminus \bigcup_{i=1}^B \downarrow_{\mathcal{R}_i};$$

its initial state is defined as

$$\uparrow_{\text{Step}} = \uparrow_{\mathcal{D}}; \quad \text{and}$$

its set of transitions is defined as

$$\begin{aligned} \rightarrow_{\text{Step}} = & \{ (sp, d, \tau, e, M, sp') \mid (sp, d, e, M, sp') \in \rightarrow_{\mathcal{D}} \} \\ & \cup \{ (sp_{t, a_1, \dots, a_k}, \llbracket, a_i, \rrbracket, R, \uparrow_{\mathcal{R}_i}) \mid t \in \{0, 1\}, a_1, \dots, a_k \in \mathcal{A}_T, k \leq B, 1 \leq i \leq k \} \\ & \cup \bigcup_{i=1}^B \{ (sp, d, \tau, e, M, sp') \mid (sp, d, e, M, sp') \in \rightarrow_{\mathcal{R}_i}, sp' \in \mathcal{S}_{\mathcal{R}_i} \setminus \downarrow_{\mathcal{R}_i} \} \\ & \cup \bigcup_{i=1}^B \{ (sp, d, \tau, e, M, \uparrow_{\text{State}}) \mid (sp, d, e, M, sp') \in \rightarrow_{\mathcal{R}_i}, sp' \in \downarrow_{\mathcal{R}_i} \}. \end{aligned}$$

Fact 3.9. Let $s \in \mathcal{S}_T$, let $0 \leq k \leq B$, let $a_1, \dots, a_k \in \mathcal{A}_T$ and $s_1, \dots, s_k \in \mathcal{S}_T$ such that $\text{out}(s) = \{(a_i, s_i) \mid 1 \leq i \leq k\}$. Then the fragment Step gives rise to the following deterministic internal computations:

(i) a deterministic internal computation from

$$(\uparrow_{\text{Step}}, \llbracket \ulcorner \text{out} \urcorner \mid \ulcorner \text{fin} \urcorner \rrbracket \text{fin}(s) \mid \ulcorner a_1 \urcorner \mid \dots \mid \ulcorner a_k \urcorner \# \ulcorner s_1 \urcorner \mid \dots \mid \ulcorner s_k \urcorner)$$

to

$$(sp_{fin(s), a_1, \dots, a_k}, \llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner s_1 \urcorner \mid \dots \mid \ulcorner s_k \urcorner \rrbracket);$$

we denote its set of intermediate states by $IS(\text{Step}, 1, s)$; and

(ii) a deterministic internal computation from

$$(\uparrow \mathcal{R}_i, \llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner s_1 \urcorner \mid \dots \mid \ulcorner s_k \urcorner \rrbracket)$$

to

$$(\uparrow \text{State}, \llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner s_i \urcorner \rrbracket);$$

we denote its set of intermediate states by $IS(\text{Step}, 2, s_i)$.

Simulator. The simulator $\text{Sim} = (\mathcal{S}_{\text{Sim}}, \rightarrow_{\text{Sim}}, \uparrow_{\text{Sim}}, \downarrow_{\text{Sim}})$ for T is defined as the union of the fragments Init , State and Step defined above: the set of states of Sim is defined as the union of the sets of states of all fragments

$$\mathcal{S}_{\text{Sim}} = \mathcal{S}_{\text{Init}} \cup \mathcal{S}_{\text{State}} \cup \mathcal{S}_{\text{Step}};$$

the transition relation of Sim is the union of the transition relations of all fragments

$$\mathcal{S}_{\text{Sim}} = \rightarrow_{\text{Init}} \cup \rightarrow_{\text{State}} \cup \rightarrow_{\text{Step}};$$

the initial state of Sim is the initial state of Init

$$\uparrow_{\text{Sim}} = \uparrow_{\text{Init}}; \quad \text{and}$$

the set of final states \downarrow_{Sim} of Sim is

$$\downarrow_{\text{Sim}} = \{sp_{1, a_1, \dots, a_k} \mid 0 \leq k \leq B \ \& \ a_1, \dots, a_k \in \mathcal{A}_\tau\}.$$

Theorem 3.10. For every boundedly branching computable transition system T there exists an RTM Sim such that $T \xrightarrow{\Delta}_{\text{b}} \mathcal{T}(\text{Sim})$.

Proof. Consider the RTM Sim of which the definition is sketched above. Referring to [Fact 3.7](#) we define the following relation:

$$\mathcal{R}_{\uparrow} = \{(\uparrow_T, t) \mid t \in IS(\text{Init})\},$$

and referring to [Facts 3.8 and 3.9](#), we define, for every $s \in \mathcal{S}_T$, the relation

$$\mathcal{R}_s = \{(s, t) \mid t \in IS(\text{State}, s) \cup IS(\text{Step}, 1, s) \cup IS(\text{Step}, 2, s)\}.$$

Then it can be verified straightforwardly that the binary relation

$$\mathcal{R} = \mathcal{R}_{\uparrow} \cup \bigcup_{s \in \mathcal{S}_T} \mathcal{R}_s$$

is a divergence-preserving branching bisimulation from T to $\mathcal{T}(\text{Sim})$. Since $(\uparrow_{\text{Sim}}, \checkmark) \in IS(\text{Init})$, it follows that $(\uparrow_T, (\uparrow_{\text{Sim}}, \checkmark)) \in \mathcal{R}$, and hence $T \xrightarrow{\Delta}_{\text{b}} \mathcal{T}(\text{Sim})$. \square

Recall that, by [Proposition 3.5](#), every effective transition system is branching bisimilar to a computable transition system with branching degree bounded by 2. According to [Theorem 3.10](#), the resulting transition can be simulated with an RTM up to divergence-preserving branching bisimilarity. We can conclude that RTMs can simulate effective transition systems up to branching bisimilarity, but, in view of [Example 3.6](#), not in a divergence-preserving manner.

Corollary 3.11. For every effective transition system T there exists a reactive Turing machine Sim such that $T \xrightarrow{\Delta}_{\text{b}} \mathcal{T}(\text{Sim})$.

Note that if T is deterministic, then $|out(s)| \leq |\mathcal{A}_\tau|$ for every state s in T , so every deterministic transition system is, in fact, boundedly branching. Furthermore, since all internal computations involved in the simulation of a boundedly branching T are deterministic, if Sim is non-deterministic, then this can only be due to a state $sp_{fin(s), a_1, \dots, a_k}$ with $a_i = a_j$ for some $1 \leq i < j \leq k$. It follows that a deterministic computable transition system can be simulated up to divergence-preserving branching bisimilarity by a deterministic RTM. The following corollary to [Theorem 3.10](#) summarises the argument.

Corollary 3.12. *For every deterministic computable transition system T there exists a deterministic RTM \mathcal{M} such that $\mathcal{T}(\mathcal{M}) \stackrel{\Delta}{\simeq}_b T$.*

Using Theorem 3.10 we can now also establish that a parallel composition of RTMs can be simulated, up to divergence-preserving branching bisimilarity, by a single RTM. To this end, note that the transition systems associated with RTMs are boundedly branching and computable. Further note that the parallel composition of boundedly branching computable transition systems is again computable. It follows that the transition system associated with a parallel composition of RTMs is boundedly branching and computable, and hence, by Theorem 3.10, there exists an RTM that simulates it up to divergence-preserving branching bisimilarity. Thus we get the following corollary.

Corollary 3.13. *For every pair of RTMs \mathcal{M}_1 and \mathcal{M}_2 and for every set of communication channels C there is an RTM \mathcal{M} such that $\mathcal{T}(\mathcal{M}) \stackrel{\Delta}{\simeq}_b \mathcal{T}(\mathcal{M}_1 \parallel \mathcal{M}_2)_C$.*

4. Universality

Recall that a *universal Turing machine* is a Turing machine that can simulate an arbitrary Turing machine on arbitrary input. When started with the code of an arbitrary Turing machine \mathcal{M} (e.g., its Gödel number, see [29]) as well as its input w on its tape, it simulates the execution of \mathcal{M} on w up to functional or language equivalence. We adapt this scheme in two ways. Firstly, we let the simulation start by inputting the description of an arbitrary RTM \mathcal{M} along some dedicated channel u , rather than assuming its presence on the tape right from the start. This is both conceptually desirable – for our aim is to give interaction a formal status – and technically necessary – for in the semantics of RTMs we have assumed that the tape is initially empty. Secondly, we require the behaviour of \mathcal{M} to be simulated up to divergence-preserving branching bisimilarity.

Thus, we arrive at the following tentative definitions. For an arbitrary RTM \mathcal{M} , denote by $\overline{\mathcal{M}}$ a deterministic RTM with no other behaviour than outputting a Gödel number $\ulcorner \mathcal{M} \urcorner$ of \mathcal{M} in an appropriate representation along channel u after which it halts in its unique final state. A *universal RTM* is then an RTM \mathcal{U} such that, for every RTM \mathcal{M} , the parallel composition $[\mathcal{U} \parallel \overline{\mathcal{M}}]_{\{u\}}$ simulates $\mathcal{T}(\mathcal{M})$.

Although such a universal RTM \mathcal{U} exists up to branching bisimilarity, as we shall see below, it does not exist up to divergence-preserving branching bisimilarity. To see this, note that the transition system associated with any particular RTM \mathcal{U} has a branching degree that is bounded by some natural number B . It can then be established, up to divergence-preserving branching bisimilarity, that \mathcal{U} can only simulate RTMs with a branching degree bounded by B .

Proposition 4.1. *There does not exist an RTM \mathcal{U} such that for all RTMs \mathcal{M} it holds that $\mathcal{T}([\mathcal{U} \parallel \overline{\mathcal{M}}]_{\{u\}}) \stackrel{\Delta}{\simeq}_b \mathcal{T}(\mathcal{M})$.*

Proof. Suppose that \mathcal{U} is an RTM such that $\mathcal{T}([\mathcal{U} \parallel \overline{\mathcal{M}}]_{\{u\}}) \stackrel{\Delta}{\simeq}_b \mathcal{T}(\mathcal{M})$ holds for every RTM \mathcal{M} . Then, by the way $\overline{\mathcal{M}}$ is defined, the branching degree of $\mathcal{T}([\mathcal{U} \parallel \overline{\mathcal{M}}]_{\{u\}}) \stackrel{\Delta}{\simeq}_b \mathcal{T}(\mathcal{M})$ is bounded by the branching degree bound on $\mathcal{T}(\mathcal{U})$, say B . Now, consider the RTM $\mathcal{M} = (\mathcal{S}_{\mathcal{M}}, \rightarrow_{\mathcal{M}}, \uparrow_{\mathcal{M}}, \downarrow_{\mathcal{M}})$ with

$$\begin{aligned} \mathcal{S}_{\mathcal{M}} &= \{\uparrow_{\mathcal{M}}, 0, \dots, B+1\}, \\ \rightarrow_{\mathcal{M}} &= \{(\uparrow_{\mathcal{M}}, \square, a, \square, R, i) \mid i = 0, \dots, B+1\} \cup \{(i+1, \square, a, \square, R, i) \mid i = 0, \dots, B\}, \quad \text{and} \\ \downarrow_{\mathcal{M}} &= \{0\}. \end{aligned}$$

Clearly, the configuration $(\uparrow_{\mathcal{M}}, \check{\square})$ in $\mathcal{T}(\mathcal{M})$ has branching degree $B+1$. Since $\mathcal{T}([\mathcal{U} \parallel \overline{\mathcal{M}}]_{\{u\}}) \stackrel{\Delta}{\simeq}_b \mathcal{T}(\mathcal{M})$, there exists a state s in $\mathcal{T}([\mathcal{U} \parallel \overline{\mathcal{M}}]_{\{u\}})$ that is related by a divergence-preserving branching bisimulation to $(\uparrow_{\mathcal{M}}, \check{\square})$. Moreover, since $(\uparrow_{\mathcal{M}}, \check{\square})$ has no outgoing τ -transitions, it follows from the definition of divergence-preserving branching bisimulation that from s we can execute at most finitely many τ -transitions to state s' without outgoing τ -transitions that must also be related to $(\uparrow_{\mathcal{M}}, \check{\square})$ by the same divergence-preserving branching bisimulation. But then s' must simulate each of the $B+1$ outgoing a -transitions to states that are pairwise not divergence-preserving branching bisimilar, and therefore it has a branching degree of $B+1$. This is a contradiction, and we conclude that the supposed RTM \mathcal{U} cannot exist. \square

If we insist on simulation up to divergence-preserving branching bisimilarity, then we need to relax the notion of universality.

Definition 4.2. Let B be a natural number. An RTM \mathcal{U}_B is *universal up to B* if for every RTM \mathcal{M} of which the associated transition system $\mathcal{T}(\mathcal{M})$ has a branching degree bounded by B it holds that $\mathcal{T}(\mathcal{M}) \stackrel{\Delta}{\simeq}_b [\mathcal{U}_B \parallel \overline{\mathcal{M}}]_{\{u\}}$.

We now present the construction of a collection of RTMs \mathcal{U}_B for all branching degree bounds B . We now benefit from our generic approach in Section 3.2: to obtain a definition of \mathcal{U}_B , it is enough to adapt the initialisation fragment of the simulator Sim.

Initialisation fragment. Recall that the initialisation fragment Init of the simulator Sim is designed to write the codes $\ulcorner out \urcorner$, $\ulcorner fin \urcorner$ and $\ulcorner \uparrow_{\mathcal{T}(\mathcal{M})} \urcorner$ for a fixed T on the tape. The initialisation fragment InitU of \mathcal{U}_B should, instead, input the Gödel number $\ulcorner \mathcal{M} \urcorner$ of an arbitrary \mathcal{M} along channel u and *compute* the codes $\ulcorner out \urcorner$, $\ulcorner fin \urcorner$ and $\ulcorner \uparrow_{\mathcal{T}(\mathcal{M})} \urcorner$ of the associated transition system $\mathcal{T}(\mathcal{M})$. We do not elaborate on the details of computing these codes from $\ulcorner \mathcal{M} \urcorner$; their existence follows from standard recursion-theoretic arguments. Here, it suffices to declare a deterministic Turing machine $\mathcal{F} = (\mathcal{S}_{\mathcal{F}}, \rightarrow_{\mathcal{F}}, \uparrow_{\mathcal{F}}, \downarrow_{\mathcal{F}})$ that, when started on a tape instance of the form

$$\llbracket \ulcorner \mathcal{M} \urcorner \rrbracket,$$

performs a terminating computation that yields the tape instance

$$\llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner \uparrow_{\mathcal{T}(\mathcal{M})} \urcorner,$$

where $\ulcorner out \urcorner$ and $\ulcorner fin \urcorner$ are indices of the partial recursive functions out and fin associated with $\mathcal{T}(\mathcal{M})$, and $\uparrow_{\mathcal{T}(\mathcal{M})}$ is the initial configuration of $\mathcal{T}(\mathcal{M})$.

Let us assume that \mathcal{M} outputs the Gödel number $\ulcorner \mathcal{M} \urcorner$ of \mathcal{M} along channel u as a sequence of $\ulcorner \mathcal{M} \urcorner + 1$ 1s delimited by \llbracket and \rrbracket . Then the initialisation fragment InitU should first receive along channel u the symbol \llbracket , then a sequence of 1s, until it receives the symbol \rrbracket , and then continue with the computation defined by the deterministic Turing machine \mathcal{F} .

The set of states of InitU is defined as

$$\mathcal{S}_{\text{InitU}} = \{in_0, in_1, in_2\} \cup (\mathcal{S}_{\mathcal{F}} \setminus \downarrow_{\mathcal{F}}),$$

its initial state is defined as

$$\uparrow_{\text{InitU}} = in_0; \quad \text{and}$$

its set of transitions is defined as

$$\begin{aligned} \rightarrow_{\text{InitU}} = & \{ (in_0, \square, u? \llbracket, \llbracket, R, in_1), (in_1, \square, u?1, 1, R, in_1), (in_1, \square, u? \rrbracket, \rrbracket, R, in_2), (in_2, \square, \tau, \square, \llbracket, L, \uparrow_{\mathcal{F}}) \} \\ & \cup \{ (in, d, \tau, e, M, \uparrow_{\mathcal{F}}) \mid (in, d, e, M, in') \in \rightarrow_{\mathcal{F}}, in' \in \mathcal{S}_{\mathcal{F}} \setminus \downarrow_{\mathcal{F}} \} \\ & \cup \{ (in, d, \tau, e, M, \uparrow_{\text{State}}) \mid (in, d, e, M, in') \in \rightarrow_{\mathcal{F}}, in' \in \downarrow_{\mathcal{F}} \}. \end{aligned}$$

Note that InitU gives rise to a deterministic internal computation only in parallel composition with an RTM $\overline{\mathcal{M}}$ that sends the Gödel number of some RTM \mathcal{M} .

Fact 4.3. Let \mathcal{M} be an arbitrary RTM, let $ci_{\overline{\mathcal{M}}}$ denote the initial configuration of $\overline{\mathcal{M}}$ and let $cf_{\overline{\mathcal{M}}}$ denote the final configuration of $\overline{\mathcal{M}}$. Then the parallel composition $[\overline{\mathcal{M}} \parallel \mathcal{U}_B]_{[u]}$ of $\overline{\mathcal{M}}$ with the fragment InitU gives rise to a deterministic internal computation from

$$(ci_{\overline{\mathcal{M}}}, (\uparrow_{\text{Init}}, \check{\square}))$$

to

$$(cf_{\overline{\mathcal{M}}}, (\uparrow_{\text{State}}, \llbracket \ulcorner out \urcorner \mid \ulcorner fin \urcorner \rrbracket \ulcorner \uparrow_T \urcorner));$$

we denote its set of intermediate states by $\text{IS}(\text{InitU})$.

A universal RTM for branching degree bound B . For a fixed branching degree bound B , we define the RTM $\mathcal{U}_B = (\mathcal{S}_{\mathcal{U}_B}, \rightarrow_{\mathcal{U}_B}, \uparrow_{\mathcal{U}_B}, \downarrow_{\mathcal{U}_B})$ as the union of the fragments InitU , State and Step defined above: the set of states of each particular \mathcal{U}_B is defined as the union of the sets of states of the fragments:

$$\mathcal{S}_{\mathcal{U}_B} = \mathcal{S}_{\text{InitU}} \cup \mathcal{S}_{\text{State}} \cup \mathcal{S}_{\text{Step}};$$

the transition relation of \mathcal{U}_B is the union of the transition relations of all fragments:

$$\mathcal{S}_{\mathcal{U}_B} = \rightarrow_{\text{InitU}} \cup \rightarrow_{\text{State}} \cup \rightarrow_{\text{Step}};$$

the initial state of \mathcal{U}_B is the initial state of InitU :

$$\uparrow_{\mathcal{U}_B} = \uparrow_{\text{InitU}}; \quad \text{and}$$

the set of final states of \mathcal{U}_B is

$$\downarrow_{\mathcal{U}_B} = \{sp_{1,a_1,\dots,a_k} \mid 0 \leq k \leq B \ \& \ a_1, \dots, a_k \in \mathcal{A}_{\tau}\}.$$

The following theorem establishes that \mathcal{U}_B is universal up to B .

Theorem 4.4. For all RTMs \mathcal{M} with a branching degree bounded by B , it holds that $\mathcal{T}(\mathcal{M}) \stackrel{\Delta}{\hookrightarrow}_b [\overline{\mathcal{M}} \parallel \mathcal{U}_B]_{\{u\}}$.

Proof. Referring to Fact 4.3 we define the following relation on configurations of the parallel composition $[\overline{\mathcal{M}} \parallel \mathcal{U}_B]_{\{u\}}$:

$$\mathcal{R}_{\uparrow} = \{(\uparrow_{\mathcal{T}(\mathcal{M})}, (t_1, t_2)) \mid t \in \text{IS}(\text{InitU})\},$$

and referring to Facts 3.8 and 3.9, we define, for every $s \in \mathcal{S}_T$, the relation

$$\mathcal{R}_s = \{(s, (cf_{\overline{\mathcal{M}}}, t)) \mid t \in \text{IS}(\text{State}, s) \cup \text{IS}(\text{Step}, 1, s) \cup \text{IS}(\text{Step}, 2, s)\}.$$

Then it is straightforward to verify that the binary relation

$$\mathcal{R} = \mathcal{R}_{\uparrow} \cup \bigcup_{s \in \mathcal{S}_T} \mathcal{R}_s$$

is a divergence-preserving branching bisimulation from $\mathcal{T}(\mathcal{M})$ to $\mathcal{T}(\mathcal{U}_B)$. Since $(ci_{\overline{\mathcal{M}}}, (\uparrow_{\mathcal{U}_B}, \check{\square})) \in \text{IS}(\text{InitU})$, it follows that $(\uparrow_{\mathcal{T}(\mathcal{M})}, (ci_{\overline{\mathcal{M}}}, (\uparrow_{\mathcal{U}_B}, \check{\square}))) \in \mathcal{R}$, and hence $\mathcal{T}(\mathcal{M}) \stackrel{\Delta}{\hookrightarrow}_b [\overline{\mathcal{M}} \parallel \mathcal{U}_B]_{\{u\}}$. \square

At the expense of introducing divergence it is possible to define a universal RTM. Recall that, by Proposition 3.5, every effective transition system is branching bisimilar to a boundedly branching transition system. The proof of this result exploits a trick, first described in [2] and adapted by Phillips in [27], to use a divergence with (infinitely many) states of at most a branching degree of 2 to simulate, up to branching bisimilarity, a state with arbitrary (even countably infinite) branching degree. The auxiliary Turing machine \mathcal{F} , used in the fragment InitU to compute the codes of *out*, *fin* and $\uparrow_{\mathcal{T}(\mathcal{M})}$ for $\mathcal{T}(\mathcal{M})$ can be adapted to deliver, instead, the codes of functions *out'*, *fin'* and \uparrow_T of a computable transition system T with branching degree bounded by 2 such that $T \stackrel{\Delta}{\hookrightarrow}_b \mathcal{T}(\mathcal{M})$. Thus, we get the following corollary to Theorem 4.4.

Corollary 4.5. There exists an RTM \mathcal{U} such that $\mathcal{T}(\mathcal{M}) \stackrel{\Delta}{\hookrightarrow}_b [\overline{\mathcal{M}} \parallel \mathcal{U}]_{\{u\}}$ for every RTM \mathcal{M} .

5. A process calculus

We have presented reactive Turing machines and studied the ensued notion of executable behaviour modulo (divergence-preserving) branching bisimilarity. In process theory, behaviour is usually specified in some process calculus. In this section, we present a simple process calculus with only standard process-theoretic notions and establish that every executable behaviour can be defined with a finite specification in our calculus up to divergence-preserving branching bisimilarity. The process calculus we define below is closest to value-passing CCS [24] for a finite set of data. It deviates from value-passing CCS in that it combines parallel composition and restriction in one construct (i.e., it includes the special form of parallel composition already presented in Section 2), omits the relabelling construction, and distinguishes successful and unsuccessful termination. Our process calculus may also be viewed as a special instance of the fragment of TCP_T , excluding sequential composition (see [11]).

Recall the finite sets \mathcal{C} of channels and \mathcal{D}_{\square} of data on which the notion of parallel composition defined in Section 2 is based. For every subset \mathcal{C}' of \mathcal{C} we define a special set of actions $\mathcal{I}_{\mathcal{C}'}$ by:

$$\mathcal{I}_{\mathcal{C}'} = \{c?d, c!d \mid d \in \mathcal{D}_{\square}, c \in \mathcal{C}'\}.$$

The actions $c?d$ and $c!d$ denote the events that a datum d is received or sent along channel c . Furthermore, let \mathcal{N} be a countably infinite set of names. The set of process expressions \mathcal{P} is generated by the following grammar ($a \in \mathcal{A}_T \cup \mathcal{I}_{\mathcal{C}}$, $N \in \mathcal{N}$, $\mathcal{C}' \subseteq \mathcal{C}$):

$$p ::= \mathbf{0} \mid \mathbf{1} \mid a.p \mid p + p \mid [p \parallel p]_{\mathcal{C}'} \mid N.$$

Let us briefly comment on the operators in this syntax. The constant $\mathbf{0}$ denotes *deadlock*, the unsuccessfully terminated process. The constant $\mathbf{1}$ denotes *skip*, the successfully terminated process. For each action $a \in \mathcal{A}_T \cup \mathcal{I}_{\mathcal{C}}$ there is a unary operator $a.$ denoting action prefix; the process denoted by $a.p$ can do an a -transition to the process denoted by p . The binary operator $+$ denotes *alternative composition* or *choice*. The binary operator $[_ \parallel _]_{\mathcal{C}'}$ denotes the special kind of *parallel composition* that we have also defined on RTMs. It enforces communication along the channels in \mathcal{C}' , and communication results in τ .

A recursive specification E is a set of equations

$$E = \{N \stackrel{\text{def}}{=} p \mid N \in \mathcal{N} \ \& \ p \in \mathcal{P}\}$$

satisfying the requirements that

- (i) for every $N \in \mathcal{N}$ it includes at most one equation with N as left-hand side, which is referred to as the *defining equation* for N ; and

Table 1

Operational rules for a recursive specification E , with $p, p', q, q' \in \mathcal{P}$, $a \in \mathcal{A}_\tau \cup \mathcal{I}_C$, $N \in \mathcal{N}$, $c \in \mathcal{C}'$, $d \in \mathcal{D}$, and $\mathcal{C}' \subseteq \mathcal{C}$.

$\frac{}{\mathbf{1} \downarrow}$		$\frac{}{a.p \xrightarrow{a} p}$	
$\frac{p \xrightarrow{a} p'}{p + q \xrightarrow{a} p'}$	$\frac{q \xrightarrow{a} q'}{p + q \xrightarrow{a} q'}$	$\frac{p \downarrow}{(p + q) \downarrow}$	$\frac{q \downarrow}{(p + q) \downarrow}$
$\frac{p \xrightarrow{a} p' \quad a \notin \mathcal{I}_{C'}}{[p \parallel q]_{C'} \xrightarrow{a} [p' \parallel q]_{C'}}$	$\frac{q \xrightarrow{a} q' \quad a \notin \mathcal{I}_{C'}}{[p \parallel q]_{C'} \xrightarrow{a} [p \parallel q']_{C'}}$	$\frac{p \downarrow \quad q \downarrow}{[p \parallel q]_{C'} \downarrow}$	
$\frac{p \xrightarrow{c?d} p' \quad q \xrightarrow{c!d} q'}{[p \parallel q]_{C'} \xrightarrow{\tau} [p' \parallel q']_{C'}}$	$\frac{p \xrightarrow{c!d} p' \quad q \xrightarrow{c?d} q'}{[p \parallel q]_{C'} \xrightarrow{\tau} [p' \parallel q']_{C'}}$		
$\frac{p \xrightarrow{a} p' \quad (N \stackrel{\text{def}}{=} p) \in E}{N \xrightarrow{a} p'}$	$\frac{p \downarrow \quad (N \stackrel{\text{def}}{=} p) \in E}{N \downarrow}$		

- (ii) if some name N occurs in the right-hand side p' of some equation $N' = p'$ in E , then E must include a defining equation for N .

Let E be a recursive specification and let p be a process expression. We say that p is *E-interpretable* if all occurrences of names in p have a defining equation in E .

We use Structural Operational Semantics [28] to associate a transition relation with process expressions: let \rightarrow be the $(\mathcal{A}_\tau \cup \mathcal{I}_C)$ -labelled transition relation induced on the set of process expressions by operational rules in Table 1. Note that the operational rules presuppose a recursive specification E .

Definition 5.1. Let E be a recursive specification and let p be an E -interpretable process expression. We define the labelled transition system

$$\mathcal{T}_E(p) = (\mathcal{S}_p, \rightarrow_p, \uparrow_p, \downarrow_p)$$

associated with p and E as follows:

1. the set of states \mathcal{S}_p consists of all process expressions reachable from p ;
2. the transition relation \rightarrow_p is the restriction to \mathcal{S}_p of the transition relation \rightarrow defined on all process expressions by the operational rules in Table 1, i.e., $\rightarrow_p = \rightarrow \cap (\mathcal{S}_p \times (\mathcal{A}_\tau \cup \mathcal{I}_C) \times \mathcal{S}_p)$.
3. the process expression p is the initial state, i.e. $\uparrow_p = p$; and
4. the set of final states consists of all process expressions $q \in \mathcal{S}_p$ such that $q \downarrow$, i.e., $\downarrow_p = \downarrow \cap \mathcal{S}_p$.

It is straightforward to associate with every RTM $\mathcal{M} = (\mathcal{S}_\mathcal{M}, \rightarrow_\mathcal{M}, \uparrow_\mathcal{M}, \downarrow_\mathcal{M})$ a (possibly infinite) recursive specification $E_\mathcal{M}^\infty$ and a process expression p such that $\mathcal{T}(\mathcal{M})$ is divergence-preserving branching bisimilar to $\mathcal{T}_{E_\mathcal{M}^\infty}(p)$:

1. Associate a distinct name M_c with every configuration c of \mathcal{M} ; and
2. let $E_\mathcal{M}$ consist of all equations

$$M_c \stackrel{\text{def}}{=} \sum_{(a, c') \in \text{out}(c)} a.M_{c'} [+ \mathbf{1}]_{\downarrow c} \quad (c \text{ a configuration of } \mathcal{M}).$$

(We use summation \sum to abbreviate an $\text{out}(c)$ -indexed alternative composition, and indicate by $[+ \mathbf{1}]_{\downarrow c}$ that the summand $\mathbf{1}$ is only included if \downarrow_c holds.)

It can be easily verified that $\mathcal{T}(\mathcal{M}) \Leftrightarrow_b^\Delta \mathcal{T}_{E_\mathcal{M}^\infty}(N_{(\uparrow_\mathcal{M}, \check{\downarrow})})$.

Our main goal in this section is to show that one does not have to resort to infinite recursive specifications: we establish that for every RTM \mathcal{M} there exists a *finite* recursive specification $E_\mathcal{M}$ and an $E_\mathcal{M}$ -interpretable process expression p such that $\mathcal{T}(\mathcal{M})$ is divergence-preserving branching bisimilar to $\mathcal{T}_{E_\mathcal{M}}(p)$. Our specification consists of two parts: a generic finite

specification of the behaviour of a tape, and a finite specification of a control process that is specific for the RTM \mathcal{M} under consideration. In the end we establish that an RTM \mathcal{M} is finitely specified by the parallel composition of its associated control with a process modelling the tape.

It is convenient to know that divergence-preserving branching bisimilarity is compatible with the notion of parallel composition in our calculus, allowing us to establish the correctness of both components separately.

Lemma 5.2. *Let E be a recursive specification and let p_1, p_2, q_1 and q_2 be E -interpretable process expressions. If $\mathcal{T}_E(p_1) \stackrel{\Delta}{\simeq}_b \mathcal{T}_E(p_2)$ and $\mathcal{T}_E(q_1) \stackrel{\Delta}{\simeq}_b \mathcal{T}_E(q_2)$, then $\mathcal{T}_E([p_1 \parallel q_1]_{C'}) \stackrel{\Delta}{\simeq}_b \mathcal{T}_E([p_2 \parallel q_2]_{C'})$.*

5.1. Tape

We want to present a finite specification of the behaviour of the tape of an RTM, but before we do so, we give a straightforward infinite specification. As an intermediate correctness result, we then establish that the behaviour defined by our finite specification is divergence-preserving branching bisimilar to the behaviour induced by our infinite specification. Recall that our definition of tape instance (see p. 145) uses a tape head marker to indicate the position of the tape head; the state of the tape is, therefore, uniquely represented by a tape instance. The behaviour of a tape in the state represented by the tape instance $\delta_L \check{d} \delta_R$ is characterised by the following equation:

$$T_{\delta_L \check{d} \delta_R} \stackrel{\text{def}}{=} r!d.T_{\delta_L \check{d} \delta_R} + \sum_{e \in \mathcal{D}_{\square}} w?e.T_{\delta_L \check{e} \delta_R} + m?L.T_{\delta_L < d \delta_R} + m?R.T_{\delta_L > \delta_R} + \mathbf{1}. \quad (1)$$

The equation expresses that a tape, when it is in the state represented by tape instance $\delta_L \check{d} \delta_R$, can either output the datum d under the head along its *read-channel* r , input a new datum e along its *write-channel* w which then replaces the datum d under the head, or receive over its *move-channel* m the instruction to move the head either one position to the left (δ_L) or one position to the right (δ_R). It is for notational convenience and not essential that we here separate the operations of reading, writing and moving, instead of combining them in a single communication. (Note that separating the operations is harmless, since our specification of the finite control specified below (see Section 5.2) will ensure that always an appropriate combination of tape operations is carried out one after the other, and the eventual parallel composition of the tape and the finite control eventually abstracts from the actual calling of tape operations.) The additional summand $\mathbf{1}$ of the right-hand side of the equation indicates that the state of the tape represented by $\delta_L \check{d} \delta_R$ is final; this is needed to ensure that the parallel composition of a tape with a finite control for an RTM is final whenever the finite control is in a final state.

We denote by E_T^∞ the recursive specification consisting of all instantiations of Eq. (1) with concrete values for d, e, δ_L and δ_R . It is easy to see that the E_T^∞ -interpretable process expression T_{\square} completely specifies all possible behaviour of the tape when it is started with an empty tape instance. It is also clear that there are infinitely many combinations of concrete values for d, e, δ_L and δ_R , so E_T^∞ is infinite.

For our finite specification of the behaviour of the tape, we make use of a seminal result in the process theoretic literature, to the effect that the behaviour of a queue can be specified in the process calculus at hand with finitely many equations. Note that the state of a queue is uniquely represented by its contents, a string δ ; we denote the behaviour of a queue with contents δ by Q_δ . Then the behaviour of a queue in all its possible states is specified by the following infinite recursive specification E_Q^∞ (with $d \in \mathcal{D}_{\square}$ and $\delta \in \mathcal{D}_{\square}^*$, and ε denoting the empty string):

$$Q_\varepsilon \stackrel{\text{def}}{=} \sum_{d \in \mathcal{D}} i?d.Q_d + \mathbf{1},$$

$$Q_{\delta d} \stackrel{\text{def}}{=} o!d.Q_\delta + \sum_{e \in \mathcal{D}} i?e.Q_{\varepsilon \delta d} + \mathbf{1}.$$

The equation for Q_ε expresses that the empty queue can only receive an arbitrary datum d along its *input-channel* i ; the equation for $Q_{\delta d}$ expresses that a queue containing at least one element d at the front of the queue may also output d along its *output-channel* o .

Bergstra and Klop [6] discovered the following intricate finite specification E_Q , consisting of six equations, completely defining the behaviour of a queue. Its correctness has been formally established by Bezem and Ponse [8]:

$$Q_p^{jk} \stackrel{\text{def}}{=} \sum_{d \in \mathcal{D}_{\square}} j?d.[Q_k^{jp} \parallel (\mathbf{1} + k!d.Q_j^{pk})]_{\{p\}} + \mathbf{1} \quad \text{for all } \{j, k, p\} = \{i, o, l\}.$$

The $\mathbf{1}$ -summands are not part of Bergstra and Klop's specification. We have added them to make sure that every state of our queue is final. It is easy to see that they have no further influence on the behaviour of the specified process. Also, it can be easily verified that the proof in [8] is still valid, and that, in fact, the proof establishes that the finite and infinite specifications of the empty queue are divergence-preserving branching bisimilar.

Theorem 5.3. $\mathcal{T}_{E_Q^\infty}(Q_\varepsilon) \stackrel{\Delta}{\simeq}_b \mathcal{T}_{E_Q}(Q_i^o)$.

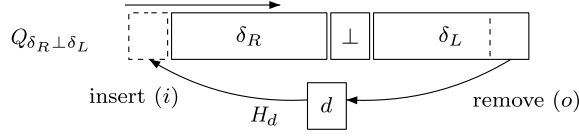


Fig. 4. Schematic representation of the tape process.

We proceed to explain how we use the queue to finitely specify a tape. Our specification consists of a *tape controller* that implements the interface of the tape with its environment and uses the queue to store information regarding the current tape instance. See Fig. 4 for an illustration of how we use the queue to store part of the tape instance $\delta_L d \delta_R$: δ_L constitutes the front of the queue δ_L , δ_R constitutes the tail of the queue, and we use a special symbol \perp to mark where δ_R ends and δ_L begins. The symbol d under the head is maintained separately by the tape controller. The four operations on the tape are implemented by the tape controller as follows:

1. If the tape controller receives the instruction to output the symbol under the head of the tape, then it suffices to send d along its read-channel. For this operation no interaction with the queue is needed.
2. If the tape controller receives the instruction to overwrite the symbol under its head with the symbol e , then it suffices to forget d and continue maintaining e . For this operation also no interaction with the queue is needed.
3. If the tape controller receives the instruction to move the head one position to the left, then this amounts to inserting the datum d at the tail of the queue and removing the datum at the front of the queue, which then becomes the new symbol maintained by the tape controller. There is one exception, though: if the symbol at the front of the queue is \perp , then the left-most position so far has been reached. The new symbol maintained by the tape controller should become \square , and the queue should be restored in its original state. This is achieved by first inserting a marker $\$$ at the tail of the queue, and then repeatedly moving symbols from the front to the tail of the queue, until the symbol $\$$ is removed from the front of the queue.
4. If the tape controller receives the instruction to move the head one position to the right, then this is implemented by first placing the marker $\$$ at the tail of the queue, and then repeatedly moving symbols from the front of the queue to the tail of the queue, until the symbol $\$$ is removed from the front of the queue. The symbol that was removed before removing $\$$ becomes the new symbol under the head.

The tape controller is defined by the finite recursive specification E_T consisting of the following equations:

$$H_d \stackrel{\text{def}}{=} r!d.H_d + \sum_{e \in \mathcal{D}_{\square}} w?e.H_e + m?L.H_d^L + m?R.H_d^R + \mathbf{1},$$

$$H_d^L \stackrel{\text{def}}{=} i!d. \left(\sum_{e \in \mathcal{D}_{\square}} o?e.H_e + o?\perp.i!\$.i!\perp.Back \right),$$

$$Back \stackrel{\text{def}}{=} \sum_{d \in \mathcal{D}_{\square}} o?d.i!d.Back + o?\$.H_{\square},$$

$$H_d^R \stackrel{\text{def}}{=} i!\$.i!d. \left(\sum_{e \in \mathcal{D}_{\square}} o?e.Fwd_e + o?\perp.Fwd_{\perp} \right),$$

$$Fwd_d \stackrel{\text{def}}{=} \sum_{e \in \mathcal{D}_{\square}} o?e.i!d.Fwd_e + o?\perp.i!d.Fwd_{\perp} + o?\$.H_d,$$

$$Fwd_{\perp} \stackrel{\text{def}}{=} \sum_{e \in \mathcal{D}_{\square}} o?e.i!\perp.Fwd_e + o?\$.i!\perp.H_{\square}.$$

To establish the correctness of our finite recursive specification of the tape controller, we first prove below that

$$\mathcal{T}_{E_T}^{\infty}(T_{\delta_L d \delta_R}) \stackrel{\Delta}{\leftrightarrow}_b \mathcal{T}_{E_T \cup E_Q^{\infty}}([H_d \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}}).$$

The following two lemmas establish that the operation of moving the tape head one position to the left, or to the right, is implemented correctly by the tape controller processes H_d^L and H_d^R .

Lemma 5.4. *Let $d \in \mathcal{D}_{\square}$ and let $\delta_L, \delta_R \in \mathcal{D}_{\square}^*$.*

- (i) *If $\delta_L = \zeta_L d_L$ for some $\zeta_L \in \mathcal{D}_{\square}^*$ and $d_L \in \mathcal{D}_{\square}$, then $[H_d^L \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}}$ has a deterministic internal computation to $[H_{d_L} \parallel Q_{d \delta_R \perp \zeta_L}]_{\{i,o\}}$; we denote its set of intermediate states by $IS(L, \check{d}_L d \delta_R)$.*

- (ii) If $\delta_L = \varepsilon$, then $[H_d^L \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}}$ has a deterministic internal computation to $[H_\square \parallel Q_{d\delta_R \perp}]_{\{i,o\}}$; we denote its set of intermediate states by $\text{IS}(L, \checkmark d\delta_R)$.

Proof. The validity of the lemma is straightforwardly proved by computing a fragment of the transition system associated with $[H_d^L \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}}$. \square

Lemma 5.5. Let $d \in \mathcal{D}_\square$ and let $\delta_L, \delta_R \in \mathcal{D}_\square^*$.

- (i) If $\delta_R = d_R \zeta_R$ for some $\zeta_R \in \mathcal{D}_\square^*$ and $d_R \in \mathcal{D}_\square$, then $[H_d^R \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}}$ has a deterministic internal computation to $[H_{d_R} \parallel Q_{\zeta_R \perp \zeta_L d}]_{\{i,o\}}$; we denote its set of intermediate states by $\text{IS}(R, \zeta_L d \checkmark \zeta_R)$.
- (ii) If $\delta_R = \varepsilon$, then $[H_d^R \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}}$ has a deterministic internal computation to $[H_\square \parallel Q_{\perp \delta_L d}]_{\{i,o\}}$; we denote its set of intermediate states by $\text{IS}(R, \delta_L d \checkmark)$.

Proof. The validity of the lemma is straightforwardly proved by computing a fragment of the transition system associated with $[H_d^R \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}}$. \square

The following theorem establishes the correct behaviour of our tape controller in a parallel composition with a queue.

Theorem 5.6. Let $d \in \mathcal{D}_\square$ and let $\delta_L, \delta_R \in \mathcal{D}_\square^*$. Then

$$\mathcal{T}_{E_T^\infty}(T_{\delta_L \checkmark \delta_R}) \stackrel{\Delta}{\hookrightarrow}_b \mathcal{T}_{E_T \cup E_Q^\infty}([H_d \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}}).$$

Proof. Referring to Lemmas 5.4 and 5.5 for the definitions of $\text{IS}(L, \delta)$ and $\text{IS}(R, \delta)$, we define the binary relation \mathcal{R} by

$$\begin{aligned} \mathcal{R} = \{ & (T_{\delta_L \checkmark \delta_R}, [H_d \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}}) \mid d \in \mathcal{D}_\square \text{ \& } \delta_L, \delta_R \in \mathcal{D}_\square^* \} \\ & \cup \{ (T_\delta, s) \mid \delta \text{ a tape instance and } s \in \text{IS}(L, \delta) \cup \text{IS}(R, \delta) \}. \end{aligned}$$

We leave it to the reader to verify that \mathcal{R} is a divergence-preserving branching bisimulation from $\mathcal{T}_{E_T^\infty}(T_{\delta_L \checkmark \delta_R})$ to $\mathcal{T}_{E_T \cup E_Q^\infty}([H_d \parallel Q_{\delta_R \perp \delta_L}]_{\{i,o\}})$. \square

Note that, as a direct consequence of Theorem 5.3, we get that

$$\mathcal{T}_{E_Q^\infty}(Q_\perp) \stackrel{\Delta}{\hookrightarrow}_b \mathcal{T}_{E_Q}([Q_o^{il} \parallel o! \perp. Q_i^{lo}]_{\{l\}}).$$

Hence, by Lemma 5.2, we can replace the infinite recursive specification of the queue in Theorem 5.6 by the finite recursive specification due to Bergstra and Klop [6], to get the following corollary.

Corollary 5.7. Let $d \in \mathcal{D}_\square$ and let $\delta_L, \delta_R \in \mathcal{D}_\square^*$. Then

$$\mathcal{T}_{E_T^\infty}(T_\square) \stackrel{\Delta}{\hookrightarrow}_b \mathcal{T}_{E_T \cup E_Q}([H_\square \parallel [Q_o^{il} \parallel o! \perp. Q_i^{lo}]_{\{l\}}]_{\{i,o\}}).$$

5.2. Finite control

It remains to associate with every RTM $\mathcal{M} = (\mathcal{S}_\mathcal{M}, \rightarrow_\mathcal{M}, \uparrow_\mathcal{M}, \downarrow_\mathcal{M})$ a finite recursive specification E_{fc} that, intuitively, implements the finite control of the RTM defined by its transition relation. For every state $s \in \mathcal{S}$ and datum $d \in \mathcal{D}_\square$, denote by $C_{s,d}$ the process that controls the behaviour of \mathcal{M} when it is in state s with d under the head. We define the behaviour of $C_{s,d}$ by the following equation ($t \in \mathcal{S}$, $a \in \mathcal{A}_\tau$, $e \in \mathcal{D}_\square$, and $M \in \{L, R\}$):

$$C_{s,d} \stackrel{\text{def}}{=} \sum_{(s,d,a,e,M,t) \in \rightarrow} \left(a.w!e.m!M. \sum_{f \in \mathcal{D}_\square} r?f.C_{t,f} \right) [+ \mathbf{1}]_s \downarrow;$$

we denote by E_{fc} the set of all instances with a concrete state $s \in \mathcal{S}$ and a concrete datum $d \in \mathcal{D}_\square$.

The following two lemmas establish that the sequence of instructions from the finite control to write e at the position of the head, move the tape head one position to the left or right, and then read the datum at the new position of the head has the desired effect.

Lemma 5.8. Let $d, e \in \mathcal{D}_\square$ and let $\delta_L, \delta_R \in \mathcal{D}_\square^*$.

(i) If $\delta_L = \zeta_L d_L$ for some $d_L \in \mathcal{D}_\square$ and $\zeta_L \in \mathcal{D}_\square^*$, then

$$\left[w!e.m!L. \sum_{f \in \mathcal{D}_\square} r?f.C_{t,f} \parallel T_{\delta_L \check{d} \delta_R} \right]_{\{r,w,m\}}$$

has a deterministic internal computation to

$$[C_{t,d_L} \parallel T_{\zeta_L \check{d}_L e \delta_R}]_{\{r,w,m\}};$$

we denote its set of intermediate states by $\text{IS}([d/e]L, t, \zeta_L \check{d}_L e \delta_R)$.

(ii) If $\delta_L = \varepsilon$, then

$$\left[w!e.m!L. \sum_{f \in \mathcal{D}_\square} r?f.C_{t,f} \parallel T_{\delta_L \check{d} \delta_R} \right]_{\{r,w,m\}}$$

has a deterministic internal computation to

$$[C_{t,\square} \parallel T_{\check{e} \delta_R}]_{\{r,w,m\}};$$

we denote its set of intermediate states by $\text{IS}([d/e]L, t, \check{e} \delta_R)$.

Lemma 5.9. Let $d, e \in \mathcal{D}_\square$ and let $\delta_L, \delta_R \in \mathcal{D}_\square^*$.

(i) If $\delta_R = d_R \zeta_R$ for some $d_R \in \mathcal{D}_\square$ and $\zeta_R \in \mathcal{D}_\square^*$, then

$$\left[w!e.m!R. \sum_{f \in \mathcal{D}_\square} r?f.C_{t,f} \parallel T_{\delta_L \check{d} \delta_R} \right]_{\{r,w,m\}}$$

has a deterministic internal computation to

$$[C_{t,d_L} \parallel T_{\delta_L e \check{d}_R \zeta_R}]_{\{r,w,m\}};$$

we denote its set of intermediate states by $\text{IS}([d/e]R, t, \delta_L e \check{d}_R \zeta_R)$.

(ii) If $\delta_R = \varepsilon$, then

$$\left[w!e.m!L. \sum_{f \in \mathcal{D}_\square} r?f.C_{t,f} \parallel T_{\delta_L \check{d} \delta_R} \right]_{\{r,w,m\}}$$

has a deterministic internal computation to

$$[C_{t,\square} \parallel T_{\delta_L e \check{\square}}]_{\{r,w,m\}};$$

we denote its set of intermediate states by $\text{IS}([d/e]R, t, \delta_L e \check{\square})$.

Theorem 5.10. $\mathcal{T}(\mathcal{M}) \xrightarrow{\Delta}_{\mathbf{b}} \mathcal{T}_{E_{\mathcal{R}} \cup E_T^\infty}([C_{\uparrow \mathcal{M}, \square} \parallel T_{\check{\square}}]_{\{r,w,m\}})$.

Proof. Referring to [Lemmas 5.8 and 5.9](#), we define a binary relation \mathcal{R} by

$$\begin{aligned} \mathcal{R} = & \{ ((s, \delta_L \check{d} \delta_R), [C_{s,d} \parallel T_{\delta_L \check{d} \delta_R}]_{\{r,w,m\}}) \mid s \in \mathcal{S}_{\mathcal{M}} \text{ \& } \delta_L, \delta_R \in \mathcal{D}_\square^* \text{ \& } d \in \mathcal{D}_\square \} \\ & \cup \{ ((t, \zeta_L \check{d}_L e \delta_R), u) \mid u \in \text{IS}([d/e]L, t, \zeta_L \check{d}_L e \delta_R) \text{ for some } d \in \mathcal{D}_\square \} \\ & \cup \{ ((t, \check{e} \delta_R), u) \mid u \in \text{IS}([d/e]L, t, \check{e} \delta_R) \text{ for some } d \in \mathcal{D}_\square \} \\ & \cup \{ ((t, \delta_L e \check{d}_R \zeta_R), u) \mid u \in \text{IS}([d/e]R, t, \delta_L e \check{d}_R \zeta_R) \text{ for some } d \in \mathcal{D}_\square \} \\ & \cup \{ ((t, \delta_L e \check{\square}), u) \mid u \in \text{IS}([d/e]R, t, \delta_L e \check{\square}) \text{ for some } d \in \mathcal{D}_\square \}. \end{aligned}$$

The relation \mathcal{R} is illustrated in [Fig. 5](#). We leave it to the reader to verify that \mathcal{R} is a divergence-preserving branching bisimulation. \square

By [Corollary 5.7](#) and [Lemma 5.2](#), we can replace the infinite recursive specification of the tape in [Theorem 5.10](#) by the finite recursive specification we found in [Section 5.1](#). We thus get the following corollary to [Theorem 5.10](#).

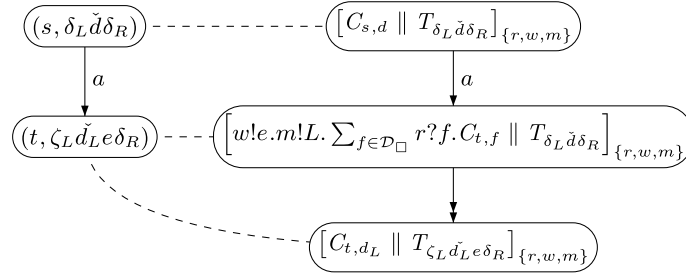


Fig. 5. Relation between an RTM transition and specification transitions.

Corollary 5.11. For every executable transition T there exists a finite recursive specification E and an E -interpretable process expression p such that $T \Leftrightarrow_{\mathcal{D}}^{\Delta} \mathcal{T}_E(p)$.

Note that if E is a finite recursive specification in our calculus, and p is an E -interpretable process expression, then $\mathcal{T}_E(p)$ is a boundedly branching computable transition system. Hence, up to divergence-preserving branching bisimilarity, we get a one-to-one correspondence between executability and finite definability in our process calculus.

Corollary 5.12. A transition system is executable modulo divergence-preserving branching bisimilarity if, and only if, it is finitely definable modulo divergence-preserving branching bisimilarity in the process calculus with deadlock, skip, action prefix, alternative composition and parallel composition with value-passing handshaking communication.

For the aforementioned corollary it is important that our calculus does not include sequential composition. If sequential composition is added to our calculus, then there exist recursive specifications with an associated transition system that is unboundedly branching (see, e.g., [3]).

6. Persistent Turing machines

In [18], the following notion of persistent Turing machine is put forward.

Definition 6.1. A *persistent Turing machine* (PTM) \mathcal{M} is a non-deterministic Turing machine with three semi-infinite tapes: a read-only input tape, a read/write work tape and a write-only output tape.

The principal semantic notion associated with PTMs in [18] is the notion of *macrostep*. Let \mathcal{D}_{\square} be the alphabet of \mathcal{M} . Then $w \xrightarrow{w_i/w_o} w'$ denotes that \mathcal{M} , when started in its initial state with its heads at the beginning of its input, work, and output tapes, containing w_i , w , and ε , respectively, has a halting computation that produces w_i , w' , and w_o as the respective contents of its input, work and output tapes. Furthermore, $w \xrightarrow{w_i/\mu} \infty$ denotes that \mathcal{M} , when started in its initial state with its heads at the beginning of its input, work and output tapes, containing w_i , w , and ε , respectively, has a diverging computation.

The macrostep semantics is used to define for every PTM a *persistent stream language* with an associated notion of equivalence, and an *interactive transition system* with associated notions of isomorphism and bisimilarity. We proceed to recall the definition of the latter.

Definition 6.2. Let \mathcal{D} be a finite set of *data symbols* (not containing the symbols $\#$ or μ). An *interactive transition system* (ITS) over \mathcal{D} is a quadruple $(\mathcal{S}, \rightarrow, \uparrow)$ consisting of a set of states \mathcal{S} containing a special state ∞ , a distinguished initial state \uparrow , and a recursively enumerable $(\mathcal{D}^* \times (\mathcal{D}^* \cup \{\mu\}))$ -labelled transition relation on states. It is assumed that all states in \mathcal{S} are reachable, and for all $s \in \mathcal{S}$ and $w_i \in \mathcal{D}^*$, whenever $s \xrightarrow{w_i/w_o} \infty$, then $w_o = \mu$, and whenever $\infty \xrightarrow{w_i/w_o} s$, then $s = \infty$ and $w_o = \mu$.

The interactive transition system associated with a PTM \mathcal{M} is defined as follows:

1. its set of states consists of all $w \in \mathcal{D}^* \cup \{\infty\}$ reachable from ε in \mathcal{M} by macrosteps, i.e., all $w \in \mathcal{D}^* \cup \{\infty\}$ such that, some $k \geq 0$, $w_{i,0}, \dots, w_{i,k} \in \mathcal{D}^*$, $w_1, \dots, w_k \in \mathcal{D}^* \cup \{\infty\}$, and $w_{o,1}, \dots, w_{o,k} \in \mathcal{D}^* \cup \{\mu\}$ such that $w_0 = w$, $w_j \xrightarrow{w_{i,j+1}/w_{o,j+1}} w_{j+1}$ and $w_k = w$.
2. its initial state is ε ; and
3. its $(\mathcal{D}^* \times (\mathcal{D}^* \cup \{\mu\}))$ -labelled transition relation is defined for all reachable $w, w' \in \mathcal{D}^* \cup \{\infty\}$, and for all $w_i \in \mathcal{D}^*$ and $w_o \in \mathcal{D}^* \cup \{\mu\}$ by $w \xrightarrow{w_i/w_o} w'$ if this is a macrostep associated with \mathcal{M} .

It is established in [18] that the above interpretation establishes a one-to-one correspondence between PTMs up to macrostep equivalence (PTMs \mathcal{M}_1 and \mathcal{M}_2 are *macrostep equivalent* if their associated ITSs are isomorphic) and ITSs up to isomorphism.

To show that PTMs can be simulated by RTMs, we associate with every ITS an effective transition system, which can then, according to Theorem 3.10, be simulated up to branching bisimilarity by an RTM. Let I be an ITS; the transition system $\mathcal{T}(I)$ associated with an ITS is defined as follows:

1. for every state s of I it includes an infinite collection of states $s_{i,w}$, one for every $w \in \mathcal{D}^*$, and transitions $s_{i,w} \xrightarrow{i?d} s_{i,wd}$ modelling that the symbol d is received along the input channel i and appended to the string w ;
2. for every state s of I it includes an infinite collection of states $s_{o,w}$, one for every $w \in \mathcal{D}^* \cup \{\mu\}$, and transitions $s_{o,dw} \xrightarrow{o!d} s_{o,w}$ and $s_{o,\mu} \xrightarrow{\tau} s_{o,\mu}$;
3. whenever I has a transition $s \xrightarrow{w_i/w_o} s'$, then $\mathcal{T}(I)$ has a transition $s_{i,w_i} \xrightarrow{i?\#} s'_{o,w_o}$ (the symbol $\#$ is used to signal the end of the input and its receipt starts the procedure to output of w_o);
4. for every state s of I , the associated transition system $\mathcal{T}(I)$ has a transition $s_{o,\varepsilon} \xrightarrow{o!\#} s_{i,\varepsilon}$ (the symbol $\#$ is used to signal the end of the output and its sending returns the transition system in input mode).

To illustrate that there is no loss of information in the encoding of ITSs into transition systems, we define, on a transition system T of which the set of actions is $\mathcal{A} = \{i?d, o!d \mid d \in \mathcal{D} \cup \{\#\}\}$, a reverse procedure. First, we associate macrosteps with sequences of transitions in T as follows:

1. for states s and s' of T and $w_i, w_o \in \mathcal{D}^*$ such that $w_i = d_0, d_1, \dots, d_k$ and $w_o = e_0, e_1, \dots, e_\ell$, let us write $s \xrightarrow{w_i/w_o} s'$ if there exist states $s_{i,0}, \dots, s_{i,k}$ and $s_{o,0}, \dots, s_{o,\ell+1}$ such that

$$s \xrightarrow{i?d_0} s_{i,0} \xrightarrow{i?d_1} \dots \xrightarrow{i?d_k} s_{i,k} \xrightarrow{i?\#} s_{o,0} \xrightarrow{o!e_0} s_{o,1} \xrightarrow{o!e_1} \dots \xrightarrow{o!e_\ell} s_{o,\ell+1} \xrightarrow{o!\#} s';$$

2. for every state s of T and $w_i \in \mathcal{D}^*$ such that $w_i = d_0, d_1, \dots, d_k$, let us write $s \xrightarrow{w_i/\mu} \infty$ if there exist states $s_{i,0}, \dots, s_{i,k}$ and an infinite sequence of states $(s_{o,\ell})_{\ell \in \mathbb{N}}$ such that

$$s \xrightarrow{i?d_0} s_{i,0} \xrightarrow{i?d_1} \dots \xrightarrow{i?d_k} s_{i,k} \xrightarrow{i?\#} s_{o,0}$$

and

$$s_{o,i} \xrightarrow{\tau} s_{o,i+1} \quad \text{for all } i \in \mathbb{N}.$$

The ITS $\mathcal{I}(T)$ associated with T has as set of states the states of T reachable by macrosteps from the initial state of T (possibly including ∞), as transitions all macrosteps between reachable states, and as initial state the initial state of T .

Theorem 6.3. *Let I be an ITS; then $\mathcal{I}(\mathcal{T}(I))$ is isomorphic to I .*

Remark 6.4. The purpose of our encoding $\mathcal{I}(_)$ on transition systems is only to illustrate that the encoding $\mathcal{T}(_)$ on ITSs is lossless; our goal has not been to define the most general encoding. Indeed, the above procedure could be generalised by allowing unobservable activity modelled as transitions labelled τ in the transition system.

Via the interpretation of PTMs as ITSs, we can associate with every PTM \mathcal{M} a transition system $\mathcal{T}(\mathcal{M})$. Clearly, the transition system $\mathcal{T}(\mathcal{M})$ associated with \mathcal{M} is effective, and hence, as a corollary to our main result (Theorem 3.10) in Section 3, there exists an RTM that simulates it up to branching bisimilarity.

Corollary 6.5. *For every PTM \mathcal{M} there exists an RTM \mathcal{M}' such that $\mathcal{T}(\mathcal{M}) \rightleftharpoons_b \mathcal{T}(\mathcal{M}')$.*

7. Concluding remarks

Our reactive Turing machines extend conventional Turing machines with a notion of interaction. Interactive computation has been studied extensively in the past two decades (see, e.g., [9,18,23]). The goal in these works is mainly to investigate to what extent interaction may have a beneficial effect on the power of sequential computation. These models essentially adopt a language- or stream-based semantics; in particular, they abstract from moments of choice in (internal) computations. Furthermore, interaction is added through special input–output facilities of the Turing machines, rather than as an orthogonal notion. We have discussed the notion of *persistent Turing machine* from [18] in Section 6; below we briefly discuss the notion of *interactive Turing machine* from [23].

Interactive Turing machines. Van Leeuwen and Wiedermann proposed *interactive Turing machines* (ITMs) in [23] (the formal details are worked out by Verbaan in [32]). An ITM is a conventional Turing machine endowed with an input port and an output port. In every step the ITM may input a symbol from some finite alphabet on its input port and outputs a symbol on its output port. ITMs are not designed to halt; they compute translations of infinite input streams to infinite output streams.

Already in [23], but more prominently in subsequent work (see, e.g., [34]), van Leeuwen and Wiedermann consider a further extension of the Turing machine paradigm, adding a notion of advice [22]. An *interactive Turing machines with advice* is an ITM that can, when needed, access some advice function that allows for inserting external information into the computation. It is established that this extension allows the modelling of non-uniform evolution. It is claimed by the authors that non-uniform evolution is essential for modelling the Internet, and that the resulting computational paradigm is more powerful than that of conventional Turing machines.

Our RTMs are not capable of modelling non-uniform evolution. We leave it as future work to consider an extension of RTMs with advice. In particular, it would be interesting to consider and extension with behavioural advice, rather than functional advice, modelling advice as an extra parallel component representing the non-uniform behaviour of the environment with which the system interacts.

Expressiveness of process calculi. The expressiveness of process calculi is an important topic in process theory. One can distinguish between *relative* expressiveness results and *absolute* expressiveness results [25]. A relative expressiveness result establishes that one calculus is at least as expressive as another calculus by exhibiting an encoding of the latter into the former; which criteria the encoding should satisfy has recently been the topic of extensive research [20,13]. An absolute expressiveness result establishes that a process calculus can or cannot express a particular process or a particular type of processes. It is then, e.g., shown that a process calculus is, or is not, Turing-powerful [11]. More refined forms of absolute expressiveness may also be exploited to compare the relative expressiveness of two process calculi, by showing that a certain type of process can be expressed in the one calculus but not in the other (see, e.g., [5,7,4]).

In [2], Baeten, Bergstra and Klop prove that computable process graphs are finitely definable in ACP_τ up to weak bisimilarity; their proof involves a finite specification of a (conventional) Turing machine. Their result was extended by Phillips in [27], who proved that all recursively enumerable process graphs are finitely definable up to weak bisimilarity. We have further extended these results by adopting a more general notion of *final state* and more refined notions of behavioural equivalence. Our proof consists of showing that RTMs can be specified in TCP_τ , a generalisation of ACP_τ . Thus, RTMs may turn out to be a useful tool in establishing the absolute expressiveness of richer process calculi. For instance, the transition system associated with a π -calculus expression is effective, so it can be simulated by an RTM, at least up to branching bisimilarity. We conjecture that the converse – every executable transition system can be specified by a π -calculus expression – is also true, but leave it for future work to work this out in detail.

Petri showed already in his thesis [26] that concurrency and interaction may serve to bridge the gap between the theoretically convenient Turing machine model of a sequential machine with unbounded memory, and the practically more realistic notion of extendable architecture of components with bounded memory. The specification we present in the proof of Corollary 5.11 is another illustration of this idea: the unbounded tape is modelled as an unbounded parallel composition. It would be interesting to further study the inherent tradeoff between unbounded parallel composition and unbounded memory in the context of RTMs, considering unbounded parallel compositions of RTMs with bounded memory.

Sequential versus parallel interactive computation. Interestingly, in the conclusions of [18] it is conjectured that parallel composition does affect the notion of interactive computability, in the sense that the parallel interactive computation is more expressive than sequential interactive computation. To verify that claim, one would need to define a notion of parallel interactive computation; it is unclear to us how this should be done in the setting of PTMs of [18]. In our setting, however, it is straightforward to define a notion of parallel composition on RTMs, and then our result at the end of Section 3 that the parallel composition of RTMs can be faithfully simulated by a single RTM shows that parallelism does not enhance the expressiveness of interactive computation as defined by the model of RTMs.

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