Nonlinear Systems: A Polynomial Approach

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Abstract. The modern development of nonlinear control theory is related mainly to the use of algebraic methods which show great applicability to solve a number of nonlinear control problems. Recently, the power of such methods were extended by introducing the concept of transfer functions of nonlinear systems. Such a concept represents a generalization, for the transfer functions of nonlinear systems have many analogical properties like those of linear systems. In this chapter some basic properties of the transfer function formalism of nonlinear systems are briefly discussed and references to possible applications are given.

Keywords: nonlinear systems, algebraic approach, polynomial approach, transfer functions.

1 Introduction

Although the Laplace and Z transforms of nonlinear differential and respectively difference equations are not defined transfer functions of nonlinear continuous-, discrete-time and time-delay systems were developed recently. For continuous-time case it was given in [7,10,24], for discrete-time case in [12,14] and for time-delay systems in [8,9]. Such a formalism is equivalent to that of [5] for linear time-varying systems and allow us to associate to a nonlinear system the tangent (or variational) linear system, see for instance [6], over Kähler differentials [17] except that now the time-varying coefficients of the polynomials are not necessarily independent [18]. The nonlinear transfer function formalism is, in principle, similar to the linear theory, except that the polynomial description relates now the differentials of system inputs and outputs, and the resulting polynomial rings are non-commutative. Such a formalism has been already employed in [22] to investigate some structural properties of nonlinear systems, in [15] to study the nonlinear model matching, in [11] to study the observer design and in [13] to study the realization problem of nonlinear systems. In this chapter we discuss some of basic properties of transfer functions of nonlinear systems with a possible application to a few control problems, namely nonlinear model matching [15,16]. To make the text easy-to-follow, many technicalities are rather avoided and the reader is referred to the works listed above.

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2 Algebraic Background

We will use the algebraic formalism of [3,4,26] which employ the concept of differential one-forms to study nonlinear continuous-, discrete-time and, respectively, time-delay systems and that of [7,8,14] which introduces the transfer functions of such systems. For the sake of simplicity, here our attention is restricted to the case of SISO systems.

To make notations compact we for any variable $\xi(t)$ write only ξ. In case of differential equations, e.g. continuous-time nonlinear systems, we use the symbols $\dot{\xi}$, $\ddot{\xi}$ and $\xi^{(k)}$ to denote the first, second and k-th time derivation of $\xi(t)$ respectively. Similarly, for discrete-time case we use ξ^+, ξ^{++} and $\xi^{[k]}$ to denote the first, second and, respectively, k-th forward time shift. Finally, for nonlinear time-delay systems we mix up the two previous notations. In particular, the time delays are denoted as backward time shifts ξ^-, ξ^{--} and $\xi^{[-k]}$ respectively.

Using the introduced notation, nonlinear continuous-time systems are objects of the form

$$y^{(n)} = \varphi(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(m)}) \tag{1}$$

discrete-time systems are objects of the form

$$y^{[n]} = \varphi(y, y^+, \ldots, y^{[n-1]}, u, u^+, \ldots, u^{[m]}) \tag{2}$$

and finally nonlinear time-delay systems are objects of the form

$$y^{(n)} = \varphi(\{y^{[-k]}, \dot{y}^{[-k]}, \ldots, y^{(n-1)[-k]}, u^{[-k]}, \dot{u}^{[-k]}, \ldots, u^{(m)[-k]}; k \geq 0\}) \tag{3}$$

In all φ is assumed to be an element of the field of meromorphic functions $\mathcal{K}$ and $y, u \in \mathbf{R}$ and $m < n$.

Remark 1. In the case of systems without delays even if one starts with the usual state-space representation it is, at least locally, always possible to eliminate the state variables to get an input-output equation of the form (1) or (2) respectively, see for instance [3]. In the time-delay case the state elimination algorithm [1] might, however, results in an input-output equation representing a system of a neutral type. For the sake of simplicity, here we consider only systems that admit input-output equations of the form (3), see also [16].

We define three separate left Ore rings (algebras) $\mathcal{K}[s]$, $\mathcal{K}[\delta]$ and $\mathcal{K}[\delta, s]$ of polynomials over $\mathcal{K}$ with the usual addition and the (non-commutative) multiplications given by the commutation rules:

$$sa = as + \dot{a} \qquad \text{for } \mathcal{K}[s] \tag{4}$$

$$\delta a = a^+\delta \qquad \text{for } \mathcal{K}[\delta] \tag{5}$$

$$\begin{aligned} sa &= as + \dot{a} \\ \delta a &= a^-\delta \\ \delta s &= s\delta \end{aligned} \qquad \text{for } \mathcal{K}[\delta, s] \tag{6}$$

In all $a \in \mathcal{K}$. The rings $\mathcal{K}[s]$, $\mathcal{K}[\delta]$ and $\mathcal{K}[\delta, s]$ thus represent the rings of linear ordinary differential, shift and, respectively, differential time-delay operators that act over the vector space of one-forms $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$ in the following ways:

$$
\begin{aligned}
* : \mathcal{K}[s] \times \mathcal{E} \to \mathcal{E};\; & \left(\sum_i a_i s^i\right) * v = \sum_i a_i v^{(i)} \\
* : \mathcal{K}[\delta] \times \mathcal{E} \to \mathcal{E};\; & \left(\sum_i a_i \delta^i\right) * v = \sum_i a_i v^{[i]} \\
* : \mathcal{K}[\delta, s] \times \mathcal{E} \to \mathcal{E};\; & \left(\sum_{i,j} a_{ij} \delta^j s^i\right) * v = \sum_{i,j} a_{ij} v^{(i)[-j]}
\end{aligned}
$$

for any $v \in \mathcal{E}$. For the sake of simplicity the symbols $*$ are usually dropped.

Note that the commutation rules (4), (5) and, respectively, (6) actually represent the rules for differentiating, shifting and differential time-delaying respectively.

Note also that we use the same symbol δ to denote both a forward shift operator in the discrete-time case and a delay operator in the time-delay case, as it is a convention in both. Whether we think of δ as a forward shift operator (5) or a delay operator (6) will be clear from the context and the class of systems we work with.

Lemma 1 (Ore condition). *For all non-zero $a, b \in \mathcal{K}[s]$ ($\mathcal{K}[\delta]$ or, respectively, $\mathcal{K}[\delta, s]$), there exist non-zero $a_1, b_1 \in \mathcal{K}[s]$ ($\mathcal{K}[\delta]$ or, respectively, $\mathcal{K}[\delta, s]$) such that $a_1 b = b_1 a$.*

Thus, the ring $\mathcal{K}[s]$ ($\mathcal{K}[\delta]$ or, respectively, $\mathcal{K}[\delta, s]$) can be embedded to the non-commutative quotient field $\mathcal{K}\langle s\rangle$ ($\mathcal{K}\langle\delta\rangle$ or, respectively, $\mathcal{K}\langle\delta, s\rangle$) by defining quotients [21] as

$$\frac{a}{b} = b^{-1} \cdot a$$

The addition and multiplication are defined as

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{\beta_2 a_1 + \beta_1 a_2}{\beta_2 b_1}$$

where $\beta_2 b_1 = \beta_1 b_2$ by Ore condition and

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{\alpha_1 a_2}{\beta_2 b_1} \tag{7}$$

where $\beta_2 a_1 = \alpha_1 b_2$ again by Ore condition.

Due to the non-commutative multiplication (4) ((5) or, respectively, (6)) they, of course, differ from the usual rules. In particular, in case of the multiplication (7) we, in general, cannot simply multiply numerators and denominators, nor cancel them in a usual manner. We neither can commute them as the multiplication of quotients is non-commutative as well.

Example 1 ([15]). Consider two quotients

$$\frac{1}{s-y}, \frac{1}{s}$$

from $\mathcal{K}\langle s\rangle$. Then

$$\frac{1}{s-y} + \frac{1}{s} = \frac{2s - y - 2\dot{y}/y}{s^2 - (y + \dot{y}/y)s}$$

and

$$\frac{1}{s-y} \cdot \frac{1}{s} = \frac{1}{s^2 - ys - \dot{y}} \neq \frac{1}{s} \cdot \frac{1}{s-y} = \frac{1}{s^2 - ys}$$

Once the fraction of two skew polynomials is defined we can introduce the transfer function of the nonlinear systems (1), (2) and (3) respectively as elements $F(s) \in \mathcal{K}\langle s\rangle$, $F(\delta) \in \mathcal{K}\langle\delta\rangle$ and, respectively, $F(\delta, s) \in \mathcal{K}\langle\delta, s\rangle$ such that $\mathrm{d}y = F(s)\mathrm{d}u$, $\mathrm{d}y = F(\delta)\mathrm{d}u$ and $\mathrm{d}y = F(\delta, s)\mathrm{d}u$ respectively.
For instance, after differentiating (3) we get

$$\mathrm{d}y^{(n)} - \sum_{i,j} \frac{\partial\varphi}{\partial y^{(i)[-j]}}\mathrm{d}y^{(i)[-j]} = \sum_{i,j} \frac{\partial\varphi}{\partial u^{(i)[-j]}}\mathrm{d}u^{(i)[-j]}$$

or alternatively

$$a(\delta, s)\mathrm{d}y = b(\delta, s)\mathrm{d}u$$

where $a(\delta, s) = s^n - \sum_{i,j} \frac{\partial\varphi}{\partial y^{(i)[-j]}}\delta^j s^i$ and $b(\delta, s) = \sum_{i,j} \frac{\partial\varphi}{\partial u^{(i)[-j]}}\delta^j s^i$ are in $\mathcal{K}[\delta, s]$.
Then

$$F(\delta, s) = \frac{b(\delta, s)}{a(\delta, s)}$$

Example 2. Consider the nonlinear time-delay system $\ddot{y} = \dot{y}^- u^-$. Then

$$\begin{aligned} \mathrm{d}\ddot{y} &= u^- \mathrm{d}\dot{y}^- + \dot{y}^- \mathrm{d}u^- \\ s^2\mathrm{d}y &= u^-\delta s\mathrm{d}y + \dot{y}^-\delta\mathrm{d}u \end{aligned}$$

and the transfer function

$$F(\delta, s) = \frac{\dot{y}^-\delta}{s^2 - u^-\delta s}$$

Notice that here s and δ stand for differential and, respectively, time-delay operator (6).

3 Properties of Transfer Functions of Nonlinear Systems

Transfer functions of nonlinear systems have many properties we expect from transfer functions. They are invariant with respect to state-transformations. They provide input-output description and are related to the accessibility and observability of a nonlinear system. Finally, they also allow us to use the transfer function algebra when combining systems in series, parallel and feedback connection.

3.1 Invariance of Transfer Functions

Consider for instance the discrete-time case [14] and a system described by the following state-space representation

$$\begin{aligned} x^+ &= f(x,u) \\ y &= g(x) \end{aligned} \tag{8}$$

where the entries of f and g are from the field of meromorphic functions $\mathcal{K}$ and $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ and $y \in \mathbf{R}$.

The transfer function can be computed as

$$F(\delta) = C(\delta I - A)^{-1}B \tag{9}$$

where $A = (\partial f/\partial x)$, $B = (\partial f/\partial u)$ and $C = (\partial g/\partial x)$.

Note that the entries of $(\delta I - A)$ are (non-commutative) skew polynomials and thus the inversion is not trivial, see [7], and leads to solving linear equations over non-commutative fields [20].

Proposition 1 ([14]). *Transfer function (9) of nonlinear discrete-time system (8) is invariant with respect to the state transformation $\xi = \phi(x)$.*

Proof. For any state transformation $\xi = \phi(x)$ one has $\operatorname{rank}_{\mathcal{K}} T = n$ where $T = (\partial\phi/\partial x)$. Since $\mathrm{d}\xi = T\mathrm{d}x$, in the new coordinates we have

$$\begin{aligned} \mathrm{d}\xi^+ &= T^+AT^{-1}\mathrm{d}\xi + T^+B\mathrm{d}u \\ \mathrm{d}y &= CT^{-1}\mathrm{d}\xi \end{aligned}$$

where T^+ means δ is applied entrywise to T. Thus, the transfer function reads

$$F(\delta) = CT^{-1}(\delta I - T^+AT^{-1})^{-1}T^+B = C({T^+}^{-1}\delta T - A)^{-1}B + D$$

After applying the commutation rule $\delta T = T^+\delta$ we get $F(\delta) = C(\delta I - A)^{-1}B$.

Example 3. Consider the system

$$\begin{aligned} x^+ &= \frac{e^u}{x} \\ y &= \ln x \end{aligned}$$

The input-output map of the system is linear $y^+ = -y + u$. Note that $A = \partial f/\partial x = -e^u/x^2$, $B = \partial f/\partial u = e^u/x$ and $C = \partial g/\partial x = 1/x$ and following the non-commutative multiplication rules (5) and (7) the transfer function can be computed as

$$\begin{aligned} F(\delta) = C(\delta I - A)^{-1}B &= \frac{1}{x} \cdot \frac{1}{\delta + e^u/x^2} \cdot \frac{e^u}{x} = \frac{1}{\delta x + e^u/x} \cdot \frac{e^u}{x} = \\ &= \frac{1}{e^u/x\delta + e^u/x} \cdot \frac{e^u}{x} = \frac{e^u}{e^u\delta + e^u} = \frac{1}{\delta + 1} \end{aligned}$$

For continuous-time and time-delay counterparts see [7] and [8] respectively.

Remark 2. The invariance of transfer functions can be viewed as a consequence of the change of basis formula of the pseudo-linear map [2] and also a consequence of the fact that the tangent linear systems of two equivalent nonlinear systems are almost identical [6] with that sense that certain associated left modules are isomorphic.

3.2 Accessibility and Observability

The transfer functions as those of linear systems are strongly related to the notions of accessibility (controllability) and observability of nonlinear systems.

For instance for continuous-time case we can use the concept of autonomous elements introduced in [3] which in terms of the polynomial approach leads to the presence or non-presence of common left factors of polynomials derived from input-output equation of the system [25].

Proposition 2. *Let $F(s) = \frac{b(s)}{a(s)}$ be the transfer function of the nonlinear system (1). Then the system is accessible if and only if the polynomials $a(s)$ and $b(s)$ have no common left factors.*

Also the observability condition can be stated in terms of polynomials.

Proposition 3 ([22]). *Let $F(s) = \frac{b(s)}{a(s)}$ be the transfer function of the nonlinear system*

$$\dot{x} = f(x, u)$$
$$y = g(x)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ and $y \in \mathbf{R}$. Then the nonlinear system is observable if and only if

$$\deg a(s) = n$$

Proof (Sketch). If the system is not observable one obtains, by eliminating the state variables, an i/o equation $y^{(r)} = \varphi(y, \dot{y}, \ldots, y^{(r-1)}, u, \dot{u}, \ldots, u^{(r-1)})$ where $r < n$ and then $\deg a(s) = r$.

For discrete-time and time-delay counterparts see [14] and [8,16] respectively.

Remark 3. In the module approach the presence of common left factors of the polynomials is equivalent to that whether the module associated to the system is torsion free or not, see for instance [23].

3.3 Transfer Function Algebra and Model Matching

When the nonlinear systems are being combined in series, parallel and feedback connection we can even use the algebra of transfer functions of nonlinear systems [7,14,8]. Following example demonstrates how to handle a series connection of two nonlinear time-delay systems.

Example 4 ([8]). Consider the two systems $\dot{y}_A = y_A u_A^-$ and $y_B = \ln u_B$ with the transfer functions

$$F_A(\delta, s) = \frac{y_A \delta}{s - u_A^-} \qquad F_B(\delta, s) = \frac{1}{u_B}$$

The systems are combined together in a series connection. For the connection A→B, when $u_B = y_A$, the resulting transfer function is

$$F(\delta, s) = F_B(\delta, s) F_A(\delta, s) = \frac{1}{u_B} \cdot \frac{y_A \delta}{s - u_A^-} = \frac{y_A \delta}{y_A s + \dot{y}_A - u_A^- y_A} = \frac{\delta}{s}$$

Hence, the combination A→B is linear from an input-output point of view which is easy to check $\dot{y}_B = u_A^-$. However, when the systems are connected as B→A, that is $u_A = y_B$, the result is different

$$F(\delta, s) = F_A(\delta, s) F_B(\delta, s) = \frac{y_A \delta}{s - u_A^-} \cdot \frac{1}{u_B} = \frac{y_A \delta}{u_B^- s - u_B^- \ln u_B^-}$$

This time, it does not yield a linear system. Note that here we used the multiplication rules (6) and (7).

The transfer function algebra can be, to advantage, employed in the nonlinear model matching problem, following the same ideas as in the linear case. The previous example serves as a stepping stone, and a typical control problem can be formulated as for given model $G(s)$ and a system $F(s)$ find a compensator $R(s)$ such that $G(s) = F(s) \cdot R(s)$. Then clearly, solution to the problem consists of computing $R(s) = F^{-1}(s) \cdot G(s)$. Preliminary problem statements and solutions to nonlinear model matching following the above mentioned ideas were given in [15,16] and have contact points to that of [19] for linear time-varying systems except that now the time-varying coefficients of the polynomials are not necessarily independent [18] and the resulting input-output differential forms of compensators might not be integrable. Other applications of the transfer function formalism of nonlinear systems can be found in [11,13].

References

1. Anguelova, M., Wennberg, B.: State elimination and identifiability of the delay parameter for nonlinear time-delay systems. Automatica 44, 1373–1378 (2008)
2. Bronstein, M., Petkovšek, M.: An introduction to pseudo-linear algebra. Theoretical Computer Science 157, 3–33 (1996)
3. Conte, G., Moog, C.H., Perdon, A.M.: Algebraic Methods for Nonlinear Control Systems, 2nd edn. Theory and Applications. Springer, London (2007)
4. Aranda-Bricaire, E., Kotta, Ü., Moog, C.H.: Linearization of discrete-time systems. SIAM J. Cont. Opt. 34, 1999–2023 (1996)
5. Fliess, M.: Une interprétation algébrique de la transformation de Laplace et des matrices de transfert. Linear Algebra and its Applications 203, 429–442 (1994)
6. Fliess, M., Lévine, J., Martin, P., Rouchon, P.: Flatness and defect of non-linear systems: introductory theory and examples. Int. J. Control 61, 1327–1361 (1995)

7. Halás, M.: An algebraic framework generalizing the concept of transfer functions to nonlinear systems. Automatica 44, 1181–1190 (2008)
8. Halás, M.: Nonlinear time-delay systems: a polynomial approach using Ore algebras. In: Loiseau, J.J., et al. (eds.) Topics in Time-Delay Systems. LNCIS, vol. 388, pp. 109–119. Springer, Heidelberg (2009)
9. Halás, M.: Ore algebras: a polynomial approach to nonlinear time-delay systems. In: 9th IFAC Workshop Time-Delay Systems, Nantes, France (2007)
10. Halás, M., Huba, M.: Symbolic computation for nonlinear systems using quotients over skew polynomial ring. In: 14th Mediterranean Conference on Control and Automation, Ancona, Italy (2006)
11. Halás, M., Kotta, Ü.: A polynomial approach to the synthesis of observers for nonlinear systems. In: 47th Conference on Decision and Control, Cancun, Mexico (2008)
12. Halás, M., Kotta, Ü.: Extension of the concept of transfer function to discrete-time nonlinear control systems. In: European Control Conference, Kos, Greece (2007)
13. Halás, M., Kotta, Ü.: Realization problem of SISO nonlinear systems: a transfer function approach. In: 7th IEEE International Conference on Control & Automation, Christchurch, New Zealand (2009)
14. Halás, M., Kotta, Ü.: Transfer Functions of Discrete-time Nonlinear Control Systems, Proc. Estonian Acad. Sci. Phys. Math. 56, 322–335 (2007)
15. Halás, M., Kotta, Ü., Moog, C.H.: Transfer function approach to the model matching problem of nonlinear systems. In: 17th IFAC World Congress, Seoul, Korea (2008)
16. Halás, M., Moog, C.: A Polynomial Solution to the Model Matching Problem of Nonlinear Time-delay Systems. In: 10th European Control Conference, Budapest, Hungary (2009)
17. Johnson, J.: Kähler differentials and differential algebra. Annals of Mathematics 89, 92–98 (1969)
18. Li, Z., Ondera, M., Wang, H.: Simplifying skew fractions modulo differential and difference relations. In: International Symposium on Symbolic and Algebraic Computation ISSAC, Linz, Austria (2008)
19. Marinescu, B., Bourlès, H.: The exact model-matching problem for linear time-varying systems: an algebraic approach. IEEE Transactions on Automatic Control 48, 166–169 (2003)
20. Ore, O.: Linear equations in non-commutative fields. Annals of Mathematics 32, 463–477 (1931)
21. Ore, O.: Theory of non-commutative polynomials. Annals of Mathematics 34, 480–508 (1933)
22. Perdon, A.M., Moog, C.H., Conte, G.: The pole-zero structure of nonlinear control systems. In: 7th IFAC Symposium NOLCOS, Pretoria, South Africa (2007)
23. Pommaret, J.F., Quadrat, A.: Localization and parametrization of linear multidimensional control systems. Systems and control letters 37, 247–260 (1999)
24. Zheng, Y., Cao, L.: Transfer function description for nonlinear systems. Journal of East China Normal University (Natural Science) 2, 15–26 (1995)
25. Zheng, Y., Willems, J., Zhang, C.: A polynomial approach to nonlinear system controllability. IEEE Transactions on Automatic Control 46, 1782–1788 (2001)
26. Xia, X., Márquez-Martínez, L.A., Zagalak, P., Moog, C.H.: Analysis of nonlinear time-delay systems using modules over non-commutative rings. Automatica 38, 1549–1555 (2002)