Elimination for Systems of Algebraic Differential Equations
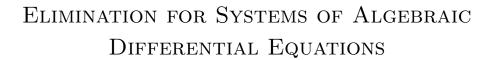
Richard Gustavson
The Graduate Center, City University of New York

Elimination for Systems of Algebraic Differential Equations

by

Richard Gustavson

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2017

Elimination for systems of algebraic differential equations
by
Richard Gustavson

This manuscript has been read and accepted by the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

(required signature)

——————————————   ——————————————————————

Date   Alexey Ovchinnikov

                                                       Chair of Examining Committee

(required signature)

——————————————   ——————————————————————

Date   Ara Basmajian

                                                       Executive Officer

Alexey Ovchinnikov

——————————————————————

Richard Churchill

——————————————————————

Benjamin Steinberg

——————————————————————

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Elimination for systems of algebraic differential equations

by

Richard Gustavson

Adviser: Alexey Ovchinnikov

We develop new upper bounds for several effective differential elimination techniques for systems of algebraic ordinary and partial differential equations. Differential elimination, also known as decoupling, is the process of eliminating a fixed subset of unknown functions from a system of differential equations in order to obtain differential algebraic consequences of the original system that do not depend on that fixed subset of unknowns. A special case of differential elimination, which we study extensively, is the question of consistency, that is, if the given system of differential equations has a solution. We first look solely at the "algebraic data" of the system of differential equations through the theory of differential kernels to provide a new upper bound for proving the consistency of the system. We then prove a new upper bound for the effective differential Nullstellensatz, which determines a sufficient number of times to differentiate the original system in order to prove its inconsistency. Finally, we study the Rosenfeld-Gröbner algorithm, which approaches differential elimination by decomposing the given system of differential equations into simpler systems. We analyze the complexity of the Rosenfeld-Gröbner algorithm by computing an upper bound for the orders of the derivatives in all intermediate steps and in the output of the algorithm.

Acknowledgments

First, I thank my advisor, Alexey Ovchinnikov, for introducing me to the field of differential algebra, for all of the advice and suggestions he has given me throughout the years, and for always supporting my research career to the fullest. I would not be the mathematician I am today without his mentorship.

I thank my collaborators, Marina Kondratieva, Omar León Sánchez, and Gleb Pogudin, for working with me, sharing their insights as we made progress on our projects, and without whose efforts this dissertation would not have been possible.

Thank you Richard Churchill and Benjamin Steinberg for being on my dissertation committee and providing useful comments and feedback as I have worked on my research. Thank you to Phyllis Cassidy, William Sit, Alice Medvedev, Raymond Hoobler, Joel Nagloo, and all of the other members of the Kolchin Seminar in Differential Algebra for providing such inspiring discourse week in and week out and for teaching me so much about the subject.

I spent more time than I know at the Graduate Center with my academic siblings, Eli Amzallag, Carlos Arreche, Mengxiao Sun, and Peter Thompson. Thank you all for working through challenging material with me, the time we spent at conferences, and for keeping me sane these last six years. A special thank you to Eli, for all of the math we did together and for all of the talks about every subject imaginable under the sun.

I presented results appearing in this dissertation at the following conferences: International Congress on Industrial and Applied Mathematics (ICIAM) in Beijing, China (August

Kramer-Miller, Aradhana Kumari, Ivan Levcovitz, Ni Lu, Zachary McGuirk, Kaethe Minden, Byungdo Park, Sunil Philip, Rachel Popp, Edger Sterjo, Alex Taam, Warren Tai, Kameryn Williams, Tian An Wong, and Gabriel Zapata.

I thank my parents, who raised me, put me through school, and taught me to follow my passions, even if it means they won't understand a single word of this dissertation. Thank you to my brothers Henry and Ethan for always making sure I remember that there is more to life than math. Finally, I thank my wife, Melanie. She has been by my side every day for the past ten-plus years. She has pushed me when I needed encouragement, lifted me up when I was down, and stood by my side no matter the hardship. I could not have asked for a better life partner.

The material presented in Section 2.2 and Chapter 3 appeared previously in modified form in the following:

- R. Gustavson and O. León Sánchez. Effective bounds for the consistency of differential equations. Preprint, 2016. URL: `http://arxiv.org/abs/1601.02995`.

- R. Gustavson and O. León Sánchez. A new bound for the existence of differential field extensions. In *Mathematical Aspects of Computer and Information Sciences*, number 9582 in Lecture Notes in Computer Science, pages 358-361, Switzerland, 2016. Springer. URL: `http://dx.doi.org/10.1007/978-3-319-32859-1_30`.

The material appearing in Chapter 4 appeared previously in the following:

- R. Gustavson, M. Kondratieva, and A. Ovchinnikov. New effective differential Nullstellensatz. *Advances in Mathematics*, 290: 1138-1158, 2016. URL: `http://dx.doi.org/10.1016/j.aim.2015.12.021`.

The material presented in Section 2.1 and Chapter 5 appeared previously in modified form in the following:

- R. Gustavson, A. Ovchinnikov, and G. Pogudin. New order bounds in differential elimination algorithms. Preprint, 2016. URL: `https://arxiv.org/abs/1602.00246`.

- R. Gustavson, A. Ovchinnikov, and G. Pogudin. Bounds for orders of derivatives in differential elimination algorithms. In *ISSAC '16: Proceedings of the 2016 ACM International Symposium on Symbolic and Algebraic Computation*, pages 239-246, New York, NY, USA, 2016. ACM Press. URL: `http://dx.doi.org/10.1145/2930889.2930922`.

Contents

List of Tables

Chapter 1

Introduction

It is a fundamental problem to determine whether a system $F = 0$, $F = f_1, \ldots, f_r$, of polynomial partial differential equations (PDEs) with coefficients in a differential field $\mathcal{K}$ is consistent, that is, whether it has a solution in a differential field containing $\mathcal{K}$. This is a special case of differential elimination, which is the process of eliminating a given subset of differential indeterminates from the system $F = 0$ in order to produce differential algebraic consequences of the system that do not depend on the chosen subset of differential indeterminates. This dissertation is mainly focused on the question of consistency. We approach this problem in three different ways. First, we look at the system $F = 0$ as a purely algebraic system using differential kernels. Second, we build off this theory using the differential Nullstellensatz. Third, we decompose the system $F = 0$ into simpler systems using the Rosenfeld-Gröbner algorithm. In each case, we seek effective methods for solving the stated problem. That is, we look for upper bounds for various properties involved, which allows for the creation of algorithms. This chapter begins by providing a brief overview of each method and its history, before describing our results.

Consider the following system of polynomial PDEs

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 1 \end{cases} \tag{1.0.1}$$

with differential indeterminates u, v and $\partial_1 = \partial/\partial_x$, $\partial_2 = \partial/\partial_y$. The corresponding system of polynomial equations is

$$\begin{cases} z_1 + z_2 = 0 \\ z_3 - z_4 = 0 \\ (z_5 + z_6)^2 + (z_7 + z_8)^2 = 1 \end{cases}$$

which is consistent (e.g., take $z_1 = \ldots = z_7 = 0$ and $z_8 = 1$). On the other hand, system (1.0.1) is inconsistent. Indeed, applying ∂_1 and ∂_2 to the first and second equations in (1.0.1), consider the extended system

$$\begin{cases} u_x + v_y = 0 \\ u_y - v_x = 0 \\ u_{xx} + v_{xy} = 0 \\ u_{yy} - v_{xy} = 0 \\ u_{xy} + v_{yy} = 0 \\ u_{xy} - v_{xx} = 0 \\ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 = 1. \end{cases}$$

It now remains to substitute the sum of the third and fourth equations and the difference of the fifth and sixth equations into the last equation to obtain $0 = 1$. The equivalent

polynomial system is

$$
\begin{cases}
z_1 + z_2 = 0 \\
z_3 - z_4 = 0 \\
z_5 + z_9 = 0 \\
z_6 - z_9 = 0 \\
z_{10} + z_8 = 0 \\
z_{10} - z_7 = 0 \\
(z_5 + z_6)^2 + (z_7 + z_8)^2 = 1
\end{cases}
$$

which is inconsistent by the above reasoning. In this particular example, by differentiating the first two equations of (1.0.1) one time, we discover that the corresponding polynomial system is inconsistent. This example illustrates the essence of the first two methods we will study, differential kernels and the effective differential Nullstellensatz.

The method of differential kernels is aimed to study the set of algebraic solutions of a given system of algebraic differential equations (viewed as a purely algebraic system), and then determine if an algebraic solution can be used to construct a differential solution. This construction is not always possible, as evidenced by the above example. More precisely, a *differential kernel* is a field extension of the ground differential field $(\mathcal{K}, \partial_1, \ldots, \partial_m)$ obtained by adjoining a solution of the associated algebraic system such that this solution serves as a means to "prolong" the derivations from $\mathcal{K}$. Kernels were first studied in the context of functional equations [1, 7]. Differential kernels in a single derivation were studied by Cohn [8] and Lando [30]. We consider differential kernels with an arbitrary number of commuting derivations. A differential kernel is said to have a *regular realization* if there is a differential field extension of $\mathcal{K}$ containing the differential kernel and such that the generators of the kernel form the sequence of derivatives of the generators of order zero. A differential kernel has a regular realization if and only if the chosen solution of the associated algebraic system can be prolonged to yield a differential solution to the original system of differential equations.

Thus, the problem of determining the consistency of a given system of differential equations is equivalent to the problem of determining the existence of regular realizations of a given differential kernel. In a single derivation, every differential kernel has a regular realization [30, Proposition 3]. However, this is no longer the case with more than one derivation.

The first analysis of differential kernels with several commuting derivations appears in the work of Pierce [35], using different terminology (there a differential kernel is referred to as a field extension satisfying the *differential condition*). In that paper it is shown that if a differential kernel has a *prolongation* of a certain length (that is, we can extend the derivations from the algebraic solution some finite number of times), then it has a regular realization. We note here that even if a differential kernel has a proper prolongation, this is no guarantee that a regular realization will exist. We denote by $T_{h,m}^n$ the smallest prolongation length that guarantees the existence of a regular realization of any differential kernel of length h in n differential indeterminates (dependent variables) over any differential field of characteristic zero with m commuting derivation operators $\partial_1, \ldots, \partial_m$ (that is, with m independent variables). Note that this number only depends on the data (h, m, n). A recursive construction of an upper bound for $T_{h,m}^n$ was provided in [31]; unfortunately, this upper bound is unwieldy from a computational standpoint even when $m = 2$ or 3.

Another method of studying consistency is the differential Nullstellensatz, which states that the consistency of the differential algebraic system $F = 0$ is equivalent to showing that the equation $1 = 0$ is not a differential-algebraic consequence of the system $F = 0$. Algebraically, the latter says that 1 does not belong to the differential ideal generated by F in the ring of differential polynomials. The differential Nullstellensatz was first proved by Ritt [36] for the field of meromorphic functions, and then for arbitrary differential fields by Kolchin [27]. The differential Nullstellensatz does not tell us how many derivatives is sufficient to apply to the system $F = 0$ in order to determine whether $1 = 0$ is a differential-algebraic consequence of it. The solution to this problem is the effective differential Nullstellensatz.

Let $F = 0$ be a system of polynomial PDEs in n differential indeterminates and m commuting derivation operators $\partial_1, \ldots, \partial_m$, of total order h and degree d, with coefficients in a differential field $\mathcal{K}$ of characteristic zero. For every non-negative integer b, let $F^{(b)} = 0$ be the set of differential equations obtained from the system $F = 0$ by differentiating each equation in it b times with respect to any combination of $\partial_1, \ldots, \partial_m$. An upper bound for the effective differential Nullstellensatz is a function $b(m, n, h, d)$ such that, for all such F, the system $F = 0$ is inconsistent if and only if the system of polynomial equations in $F^{(b(m,n,h,d))}$ is inconsistent. By the usual Hilbert's Nullstellensatz, the latter is equivalent to

$$1 \in \left(F^{(b(m,n,h,d))} \right),$$

the ideal generated by $F^{(b(m,n,h,d))}$.

The effective differential Nullstellensatz was first addressed in [39], without providing a complete solution. In the ordinary case ($m = 1$), the first bound, which was triple-exponential in n and polynomial in d, appeared in [16]. The first general formula for the upper bound and first series of examples for the lower bound in the case of m derivations appeared in [15]. That formula is expressed in terms of the Ackermann function and is primitive recursive but not elementary recursive in n, h, d for each fixed m and is not primitive recursive in m. A model-theoretic treatment was given in [22]. In the case of constant coefficients and $m = 1$, an important breakthrough was made in [9], where a double-exponential bound in n was given. The Ackermannian nature of the general bound, however, made it not computationally viable even for $m = 2$ or 3.

Decomposition algorithms take a different approach to the problem of differential elimination by decomposing a system of differential equations into an intersection of systems, each with specific properties that can be more easily studied. The Rosenfeld-Gröbner algorithm is a fundamental decomposition algorithm which allows us to study both the problems of

consistency and differential elimination. This algorithm, which first appeared in [2, 3], takes as its input a finite set F of differential polynomials and outputs a representation of the radical differential ideal generated by F as a finite intersection of regular differential ideals. The algorithm can test membership in a radical differential ideal, and, in conjunction with the differential Nullstellensatz, can test the consistency of a system of polynomial differential equations. The Rosenfeld-Gröbner algorithm has an advantage over other decomposition algorithms, such as the Ritt-Kolchin algorithm [27, 37], since the Rosenfeld-Gröbner algorithm does not depend on the factorization problem [27, §IV.9], which is too complex to implement in computer algebra systems.

The Rosenfeld-Gröbner algorithm has been implemented in MAPLE as a part of the DifferentialAlgebra package. In order to determine the complexity of the algorithm, we need to find an upper bound on the orders of derivatives that appear in all intermediate steps and in the output of the algorithm. The first step in answering this question was completed in [14], in which an upper bound in the case of a single derivation and any ranking on the set of derivatives was found. If there are n unknown functions and the order of the original system is h, it was shown that an upper bound on the orders of the output of the Rosenfeld-Gröbner algorithm is $h(n-1)!$. Nothing was shown, however, for the case of multiple derivations.

In this dissertation, we provide new and improved upper bounds for all of these methods. In Chapter 2 we introduce the concept of differential kernels and prove some basic results about them. Chapter 3 is focused on providing an improved upper bound for $T_{h,m}^n$, the smallest prolongation length of a differential kernel that guarantees the existence of a regular realization. The material in these chapters originally appeared in [17, 18]. This new upper bound is given in Theorem 3.1.4 by the number $C_{h,m}^n$, which we introduce in Section 3.1. In further sections we show that there is a recursive algorithm that computes the value of the integer $C_{h,m}^n$. This is a nontrivial task, as we develop a series of new combinatorial results in order to complete the proof. In Section 3.2, we prove the main combinatorial result of

the chapter, Theorem 3.2.7. This theorem is a strengthening of Macaulay's theorem on the growth of the Hilbert-Samuel function when applied to certain sequences called *connected antichain sequences* of $\mathbb{Z}_{\geqslant 0}^m$. We then use a consequence of this combinatorial result, namely Corollary 3.2.9, in Section 3.3 to show that the integer $C_{h,m}^n$ can be expressed in terms of the maximal length of certain antichain sequences (see Theorem 3.3.9). At this point, we use the results from [31, §3] to derive an algorithm that computes the number $C_{h,m}^n$. This new upper bound $C_{h,m}^n$ of $T_{h,m}^n$ allows us to produce specific, computationally viable upper bounds for a small numbers of derivations (for example, one, two, or three derivations), which the previously known bound does not produce. In Section 3.4 we provide some concrete computations to show how our new upper bound compares with what was previously known. For instance, our bound produces

$$T_{h,2}^n \leq 2^n h \quad \text{and} \quad T_{h,3}^1 \leq 3(2^h - 1),$$

which, surprisingly, was not known previously.

In Chapter 4, we improve the upper bound for the effective differential Nullstellensatz. The material in this chapter originally appeared in [19]. Our main result, Theorem 4.2.1, provides a uniform upper bound on the number of differentiations needed for all systems of polynomial PDEs with the number of derivations, indeterminates, total order, and total degree bounded by m, n, h, and d, respectively. This bound outperforms the previously known general upper bound [15]. Our result reduces the problem to the polynomial effective Nullstellensatz, which has been very well studied, with many sharp results available [5, 10, 25, 28, 29]. On the other hand, note that our problem is substantially more difficult than this problem, because the polynomial effective Nullstellensatz corresponds (see Theorem 4.4.3) to the effective differential Nullstellensatz restricted to systems of linear ($d = 1$) PDEs in one indeterminate ($n = 1$) with constant coefficients, and we do not make these restrictions.

We go beyond the recent result of [9] and use the methods of differential kernels for fields with several commuting derivations [12, 35] to obtain a new upper bound for the most general case in terms of $T_{h,m}^n$: the coefficients do not have to be constant and we allow any number of derivations m. For any m, our bound is polynomial in d. For $m = 1, 2, 3$, a more concrete analysis of the bound is given in Section 4.3 using the bounds on $T_{h,m}^n$ given in Chapter 3, which shows that our bound is elementary recursive in these cases. In particular, for $m = 1$, it is double-exponential in n and h and is polynomial in d, as in [9], but does not require constant coefficients. For $m = 2$ and $n = 1$, it is triple-exponential in h. Our Examples 4.4.2 and 4.4.6 show lower bounds that are polynomial in h and d and exponential in mn.

In Chapter 5, we find an upper bound for the orders of derivatives that appear in all intermediate steps and in the output of the Rosenfeld-Gröbner algorithm in the case of an arbitrary number of commuting derivations and a weighted ranking on the derivatives. The material in this chapter originally appeared in [20, 21]. We show in Theorem 5.3.4 that an upper bound for the weights of derivatives in the intermediate steps and in the output of the Rosenfeld-Gröbner algorithm is given by hf_{L+1}, where h is the weight of our input system of differential equations, $\{f_0, f_1, f_2, \dots\}$ is the Fibonacci sequence $\{0, 1, 1, 2, 3, 5, \dots\}$, and L is the maximal possible length of a certain antichain sequence (that depends solely on h, the number m of derivations, and the number n of differential indeterminates). By choosing a specific weight, we are able to produce an upper bound for the orders of the derivatives in the output of the Rosenfeld-Gröbner algorithm. For $m = 2$, we refine this upper bound in a new way (see Corollary 5.3.5) by showing that the weights of the derivatives are bounded above by a sequence defined similarly to the Fibonacci sequence but with a slower growth rate.

The upper bound for $T_{h,m}^n$ (and thus for the effective differential Nullstellensatz) and for the orders of the output of the Rosenfeld-Gröbner algorithm depend on the lengths of

certain antichain sequences of the set $\mathbb{Z}_{\geqslant 0}^m \times \{1, \ldots, n\}$ with the partial order $\leqslant$ defined by $((a_1, \ldots, a_m), i) \leqslant ((b_1, \ldots, b_m), j)$ if and only if $i = j$ and $a_k \leqslant b_k$ for all $1 \leqslant k \leqslant m$. An analysis of the lengths of these antichain sequences began in [35] and continued in [12]. In [31] a recursive function is given that calculates the maximal length of an antichain sequence in $(\mathbb{Z}_{\geqslant 0}^m \times \{1, \ldots, n\}, \leqslant)$ with degree growth bounded by a given function f. This breakthrough allowed for many of the calculations of specific upper bounds found in this dissertation.

There are other similarities between the nature of the three upper bounds that we have obtained in addition to their dependence on lengths of antichain sequences, as well as differences. The bound on the effective differential Nullstellensatz relies on the bound for $T_{h,m}^n$, but also on the classical polynomial Nullstellensatz; as a result, the upper bound for the effective differential Nullstellensatz is larger than the other upper bounds, and also depends on the *degree* of the original system of differential equations (which is not the case for the other upper bounds). One final similarity among all of the bounds is that (with some exceptions) they are not sharp, that is, the current upper bounds do not equal the lower bounds (in the case of $T_{h,m}^n$, a general form for the lower bound is not known). Recently, improvements have been made to the upper bound for the effective differential Nullstellensatz in [34] using triangular sets; also appearing there is the first upper bound for the problem of effective differential elimination. It is an ongoing project to continue to improve both the upper and lower bounds for all of these quantities.

Chapter 2

Differential Kernels

2.1 Background on differential algebra

In this section, we present background material from differential algebra that is pertinent to the dissertation. For a more in-depth discussion, we refer the reader to [24, 26, 27, 37].

A *differential ring* is a commutative ring R with a collection of m commuting derivations $\Delta = \{\partial_1, \ldots, \partial_m\}$ on R. An ideal I of a differential ring is a *differential ideal* if $\delta a \in I$ for all $a \in I$, $\delta \in \Delta$. For a set $A \subseteq R$, let (A), $\sqrt{(A)}$, $[A]$, and $\{A\}$ denote the smallest ideal, radical ideal, differential ideal, and radical differential ideal containing A, respectively. If $\mathbb{Q} \subseteq R$, then $\{A\} = \sqrt{[A]}$.

Remark 2.1.1. In this dissertation, as usual, we also use the braces $\{a_1, a_2, \ldots\}$ to denote the set containing the elements $a_1, a_2, \ldots$. Even though this notation conflicts with the above notation for radical differential ideals (used here for historical reasons), it will be clear from the context which of the two objects we mean in each particular situation.

In this dissertation, $\mathcal{K}$ is a differential field of characteristic zero with m commuting

derivations $\Delta = \{\partial_1, \ldots, \partial_m\}$. The set of derivative operators is denoted by

$$\Theta := \left\{\partial_1^{i_1} \ldots \partial_m^{i_m} : i_j \in \mathbb{Z}_{\geqslant 0}, 1 \leqslant j \leqslant m\right\}.$$

For $Y = \{y_1, \ldots, y_n\}$ a set of n *differential indeterminates*, the set of *derivatives* of Y is

$$\Theta Y := \{\theta y : \theta \in \Theta, y \in Y\}.$$

Then the ring of *differential polynomials* over $\mathcal{K}$ is defined to be

$$\mathcal{K}\{Y\} = \mathcal{K}\{y_1, \ldots, y_n\} := \mathcal{K}[\theta y : \theta y \in \Theta Y].$$

We can naturally extend the derivations $\partial_1, \ldots, \partial_m$ to the ring $\mathcal{K}\{Y\}$ by defining

$$\partial_j \left(\partial_1^{i_1} \ldots \partial_m^{i_m} y_k\right) := \partial_1^{i_1} \ldots \partial_j^{i_j+1} \ldots \partial_m^{i_m} y_k.$$

For any $\theta = \partial_1^{i_1} \ldots \partial_m^{i_m} \in \Theta$, we define the *order* of θ to be

$$\mathrm{ord}(\theta) := i_1 + \ldots + i_m.$$

For any derivative $u = \theta y \in \Theta Y$, we define

$$\mathrm{ord}(u) := \mathrm{ord}(\theta).$$

For a differential polynomial $f \in \mathcal{K}\{Y\} \setminus \mathcal{K}$, we define the order of f to be the maximum order of all derivatives that appear in f. For any finite set $A \subseteq \mathcal{K}\{Y\} \setminus \mathcal{K}$, we set

$$\mathcal{H}(A) := \max\{\mathrm{ord}(f) : f \in A\}. \tag{2.1.1}$$

For any $\theta = \partial_1^{i_1} \ldots \partial_m^{i_m}$ and positive integers $c_1, \ldots, c_m \in \mathbb{Z}_{>0}$, we define the *weight* of θ to be

$$w(\theta) = w\left(\partial_1^{i_1} \ldots \partial_m^{i_m}\right) := c_1 i_1 + \ldots + c_m i_m.$$

Note that if all of the $c_i = 1$, then $w(\theta) = \mathrm{ord}(\theta)$ for all $\theta \in \Theta$. For a derivative $u = \theta y \in \Theta Y$, we define the weight of u to be $w(u) := w(\theta)$. For any differential polynomial $f \in \mathcal{K}\{Y\} \setminus \mathcal{K}$, we define the weight of f, $w(f)$, to be the maximum weight of all derivatives that appear in f. For any finite set $A \subseteq \mathcal{K}\{Y\} \setminus \mathcal{K}$, we set

$$\mathcal{W}(A) := \max\{w(f) : f \in A\}. \tag{2.1.2}$$

A *ranking* on the set ΘY is a total order $<$ satisfying the following two additional properties: for all $u, v \in \Theta Y$ and all $\theta \in \Theta$, $\theta \neq \mathrm{id}$,

$$u < \theta u \quad \text{and} \quad u < v \implies \theta u < \theta v.$$

A ranking $<$ is called an *orderly ranking* if for all $u, v \in \Theta Y$,

$$\mathrm{ord}(u) < \mathrm{ord}(v) \implies u < v.$$

Given a weight w, a ranking $<$ on ΘY is called a *weighted ranking* if for all $u, v \in \Theta Y$,

$$w(u) < w(v) \implies u < v.$$

Remark 2.1.2. Note that if $w\left(\partial_1^{i_1} \ldots \partial_m^{i_m}\right) = i_1 + \ldots + i_m$ (that is, $w(\theta) = \mathrm{ord}(\theta)$), then a weighted ranking $<$ on ΘY is in fact an orderly ranking.

2.2 Differential kernels

Fix a positive integer n. We are interested in field extensions of $\mathcal{K}$ whose generators over $\mathcal{K}$ are indexed by elements of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$, where $\mathfrak{n} = \{1, \ldots, n\}$. To do so, we introduce the following terminology: Given an element $\xi = (u_1, \ldots, u_m) \in \mathbb{Z}_{\geqslant 0}^m$, we define the *degree* of ξ to be

$$\deg(\xi) = u_1 + \ldots + u_m.$$

If $\alpha = (\xi, i) \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$, we set $\deg(\alpha) = \deg(\xi)$. For any $h \in \mathbb{Z}_{\geqslant 0}$, we let

$$\Gamma(h) = \{\alpha \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n} : \deg(\alpha) \leqslant h\}.$$

We will consider two different orders $\leqslant$ and $\trianglelefteq$ on $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$. Given two elements $\alpha = (\xi, i)$ and $\beta = (\tau, j)$ of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$, we set $\alpha \leqslant \beta$ if and only if $i = j$ and $\xi \leqslant \tau$ in the product order of $\mathbb{Z}_{\geqslant 0}^m$ (recall the product order on $\mathbb{Z}_{\geqslant 0}^m$ says that $(a_1, \ldots, a_m) \leqslant (b_1, \ldots, b_m)$ if and only if $a_i \leqslant b_i$ for all $1 \leqslant i \leqslant m$). On the other hand, if $\xi = (u_1, \ldots, u_m)$ and $\tau = (v_1, \ldots, v_m)$, we set $(\xi, i) \trianglelefteq (\tau, j)$ if and only if

$$(\deg(\xi), i, u_1, \ldots, u_m) \text{ is less than or equal to } (\deg(\tau), j, v_1, \ldots, v_m)$$

in the (left) lexicographic order. Note that if $x = (x_1, \ldots, x_n)$ are differential indeterminates and we identify $\alpha = (\xi, i)$ with $\partial^\xi x_i := \partial_1^{u_1} \ldots \partial_m^{u_m} x_i$, then $\leqslant$ induces an order on the set of algebraic indeterminates $\{\partial^\xi x_i : (\xi, i) \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}\}$ given by $\partial^\xi x_i \leqslant \partial^\tau x_j$ if and only if $\partial^\tau x_j$ is a derivative of $\partial^\xi x_i$ (in particular this implies that $i = j$). On the other hand, the ordering $\trianglelefteq$ induces the canonical orderly ranking on the set of algebraic indeterminates.

Recall that an *antichain* in a partially ordered set $(P, <)$ is a collection of elements of P that are all pairwise incomparable with respect to the partial order. An *antichain sequence* in $(P, <)$ is a sequence $(p_1, p_2, p_3, \ldots)$ of elements of P such that the set containing these

elements is an antichain. We will work mostly with antichain sequences in the partially ordered set $(\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}, \leqslant)$, although occasionally we will look at antichain sequences in other partially ordered sets. By Dickson's lemma every antichain sequence $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ in $(\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}, \leqslant)$ must be finite.

We will look at field extensions of $\mathcal{K}$ of the form

$$\mathcal{L} := \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(h)) \tag{2.2.1}$$

for some fixed $h \in \mathbb{Z}_{\geqslant 0}$, although occasionally we will have to consider extensions of the form $\mathcal{K}(a_i^\xi : (\xi, i) \lhd (\tau, k))$ for some fixed $(\tau, k) \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$. Here we use a_i^ξ as a way to index the generators of $\mathcal{L}$ over $\mathcal{K}$. The element $(\tau, j) \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ is said to be a *leader* of $\mathcal{L}$ if a_j^τ is algebraic over $\mathcal{K}(a_i^\xi : (\xi, i) \lhd (\tau, j))$, and a leader (τ, j) is a *minimal leader* of $\mathcal{L}$ if there is no leader (ξ, i) with $(\xi, i) < (\tau, j)$. The set of minimal leaders of $\mathcal{L}$ forms an antichain of $(\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}, \leqslant)$. We note that the notions of leader and minimal leader make sense even when we allow $h = \infty$.

We now define the main object of study for this section and Chapter 3.

Definition 2.2.1. The field extension $\mathcal{L}$, as in (2.2.1), is said to be a *differential kernel* over $\mathcal{K}$ if there exist derivations

$$D_k : \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(h-1)) \to \mathcal{L}$$

extending ∂_k for $1 \leqslant k \leqslant m$ such that $D_k a_i^\xi = a_i^{\xi + \mathbf{k}}$ for all $(\xi, i) \in \Gamma(h-1)$, where $\mathbf{k} \in \mathbb{Z}_{\geqslant 0}^m$ is the m-tuple with a one in the k-th component and zeros elsewhere. The number h is called the *length* of the differential kernel. If $\mathcal{L}$ has the form $\mathcal{K}(a_i^\xi : (\xi, i) \lhd (\tau, j))$ for some fixed

$(\tau, j) \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$, we say that $\mathcal{L}$ is a differential kernel over $\mathcal{K}$ if there exist derivations

$$D_k : \mathcal{K}(a_i^\xi : (\xi + \mathbf{k}, i) \lhd (\tau, j)) \to \mathcal{L}$$

extending ∂_k for $1 \leqslant k \leqslant m$ such that $D_k a_i^\xi = a_i^{\xi + \mathbf{k}}$ whenever $(\xi + \mathbf{k}, i) \lhd (\tau, j)$.

Unless stated otherwise every differential kernel $\mathcal{L}$ will have the form (2.2.1).

Definition 2.2.2. A *prolongation* of a differential kernel $(\mathcal{L}, D_1, \ldots, D_m)$ of *length* $s \geqslant h$ is a differential kernel $\mathcal{L}' = \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(s))$ over $\mathcal{K}$ with derivations $D_1', \ldots, D_m'$ such that $\mathcal{L}'$ is a field extension of $\mathcal{L}$ and D_k' extends D_k for $1 \leqslant k \leqslant m$. The prolongation $\mathcal{L}'$ of $\mathcal{L}$ is called *generic* if the set of minimal leaders of $\mathcal{L}$ and $\mathcal{L}'$ coincide.

In the ordinary case, $m = 1$, every differential kernel of length h has a prolongation of length $h + 1$ (in fact a generic one) [30, Proposition 1]. However, for $m > 1$, prolongations need not exist.

Example 2.2.3. Working with $m = 2$ and $n = 1$, set $\mathcal{K} = \mathbb{Q}$ and $\mathcal{L} = \mathbb{Q}(t, t, 1)$ where t is transcendental over $\mathbb{Q}$. Here we are setting

$$a^{(0,0)} = t, \quad a^{(1,0)} = t, \quad \text{and} \quad a^{(0,1)} = 1.$$

The field $\mathcal{L}$ equipped with derivations D_1 and D_2 such that $D_1(t) = t$ and $D_2(t) = 1$ is a differential kernel over $\mathbb{Q}$ of length 1; however, it does not have a prolongation of length 2. Indeed, if $\mathcal{L}$ had a prolongation

$$\mathcal{L}' = \mathbb{Q}(a^\xi : \deg(\xi) \leqslant 2)$$

with derivations D_1' and D_2', then we would get the contradiction

$$0 = D_1'(1) = D_1'a^{(0,1)} = a^{(1,1)} = D_2'a^{(1,0)} = D_2'(t) = 1.$$

Definition 2.2.4. A differential kernel $\mathcal{L}' = \mathcal{K}(b_i^\xi : (\xi, i) \in \Gamma(h))$ is said to be a *specialization* (over $\mathcal{K}$) of the differential kernel $\mathcal{L}$ if the tuple $(b_i^\xi : (\xi, i) \in \Gamma(h))$ is a specialization of $(a_i^\xi : (\xi, i) \in \Gamma(h))$ over $\mathcal{K}$ in the algebraic sense, that is, there is a $\mathcal{K}$-algebra homomorphism

$$\phi : \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(h)) \to \mathcal{K}(b_i^\xi : (\xi, i) \in \Gamma(h))$$

that maps $a_i^\xi \mapsto b_i^\xi$. The specialization is said to be *generic* if ϕ is an isomorphism.

Lemma 2.2.5. *Suppose $\mathcal{L}'$ is a generic prolongation of $\mathcal{L}$ of length s. If $\bar{\mathcal{L}}$ is another prolongation of $\mathcal{L}$ of length s, then $\bar{\mathcal{L}}$ is a specialization of $\mathcal{L}'$.*

Proof. Let $\mathcal{L}' = \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(s))$ with derivations $D_1', \ldots, D_m'$, and $\bar{\mathcal{L}} = \mathcal{K}(b_i^\xi : (\xi, i) \in \Gamma(s))$ with derivations $\bar{D}_1, \ldots, \bar{D}_m$. Since $\bar{\mathcal{L}}$ is a prolongation of $\mathcal{L}$, we have that $b_i^\xi = a_i^\xi$ for all $(\xi, i) \in \Gamma(h)$. For convenience of notation we let

$$\mathcal{L}'_{\unlhd(\tau,j)} := \mathcal{K}(a_i^\xi : (\xi, i) \unlhd (\tau, j)) \text{ and } \mathcal{L}'_{\lhd(\tau,j)} := \mathcal{K}(a_i^\xi : (\xi, i) \lhd (\tau, j)),$$

when $h \leqslant \deg(\tau) \leqslant s$. Note that

$$\mathcal{L}'_{\unlhd(h\mathbf{1},n)} = \mathcal{L} \text{ and } \mathcal{L}'_{\unlhd(s\mathbf{1},n)} = \mathcal{L}'.$$

Similar notation, and remarks, apply to $\bar{\mathcal{L}}_{\unlhd(\tau,j)}$ and $\bar{\mathcal{L}}_{\lhd(\tau,j)}$.

We prove the lemma by constructing the desired $\mathcal{K}$-algebra homomorphism

$$\phi : \mathcal{L}'_{\unlhd(\tau,j)} \to \bar{\mathcal{L}}_{\unlhd(\tau,j)}$$

recursively where $(h\mathbf{1}, n) \trianglelefteq (\tau, j) \trianglelefteq (s\mathbf{1}, n)$. The base case, $(\tau, j) = (h\mathbf{1}, n)$, is trivial since then

$$\mathcal{L}'_{\trianglelefteq(h\mathbf{1},n)} = \mathcal{L} = \bar{\mathcal{L}}_{\trianglelefteq(h\mathbf{1},n)}.$$

Now assume $(h\mathbf{1}, n) \triangleleft (\tau, j) \trianglelefteq (s\mathbf{1}, n)$ and that we have a $\mathcal{K}$-algebra homomorphism ϕ' : $\mathcal{L}'_{\triangleleft(\tau,j)} \to \bar{\mathcal{L}}_{\triangleleft(\tau,j)}$ mapping $a_i^\xi \mapsto b_i^\xi$ for $(\xi, i) \triangleleft (\tau, j)$. If (τ, j) is not a leader of $\mathcal{L}'$, then a_j^τ is transcendental over $\mathcal{L}'_{\triangleleft(\tau,j)}$, and so ϕ' extends to the desired $\mathcal{K}$-algebra homomorphism $\phi : \mathcal{L}'_{\trianglelefteq(\tau,j)} \to \bar{\mathcal{L}}_{\trianglelefteq(\tau,j)}$.

Hence, it remains to show the case when (τ, j) is a leader of $\mathcal{L}'$. In this case, since $\mathcal{L}'$ is a generic prolongation of $\mathcal{L}$, (τ, j) is a nonminimal leader of $\mathcal{L}'$, and moreover $a_j^\tau = (D')^\zeta a_j^\eta$ for some minimal leader (η, j) of $\mathcal{L}$ and nonzero $\zeta \in \mathbb{Z}_{\geqslant 0}^m$. Let f be the minimal polynomial of $a_j^\eta \in \mathcal{L}$ over $\mathcal{K}(a_i^\xi : (\xi, i) \triangleleft (\eta, j))$. The standard argument (in characteristic zero) to compute the derivative of an algebraic element in terms of its minimal polynomial yields a polynomial g over $\mathcal{L}'_{\triangleleft(\tau,j)}$ and a positive integer l such that

$$a_j^\tau = (D')^\zeta a_j^\eta = \frac{g(a_j^\eta)}{(f'(a_j^\eta))^l} \in \mathcal{L}'_{\triangleleft(\tau,j)}.$$

Similarly, there is a polynomial $\tilde{g}$ over $\bar{\mathcal{L}}_{\triangleleft(\tau,j)}$ such that

$$b_j^\tau = \bar{D}^\zeta a_j^\eta = \frac{\tilde{g}(a_j^\eta)}{(f'(a_j^\eta))^l} \in \bar{\mathcal{L}}_{\triangleleft(\tau,j)},$$

and, moreover, one such $\tilde{g}$ is obtained by applying ϕ' to the coefficients of g. This latter observation, together with the two equalities above, imply that $\mathcal{L}'_{\trianglelefteq(\tau,j)} = \mathcal{L}'_{\triangleleft(\tau,j)}$ and that $\phi'(a_j^\tau) = b_j^\tau$. Hence, in the case when a_j^τ is a leader, setting $\phi := \phi'$ yields the desired $\mathcal{K}$-algebra homomorphism. $\qquad\square$

Definition 2.2.6. An n-tuple $g = (g_1, \ldots, g_n)$ contained in a differential field extension $(\mathcal{M}, \partial'_1, \ldots, \partial'_m)$ of $(\mathcal{K}, \Delta)$ is said to be a *regular realization* of the differential kernel $\mathcal{L}$ if the tuple

$$((\partial')^\xi g_i : (\xi, i) \in \Gamma(h))$$

is a generic specialization of $(a_i^\xi : (\xi, i) \in \Gamma(h))$ over $\mathcal{K}$. The tuple g is said to be a *principal realization* of $\mathcal{L}$ if there exists a sequence of differential kernels $\mathcal{L} = \mathcal{L}_0, \mathcal{L}_1, \ldots$, each a generic prolongation of the preceding, such that g is a regular realization of each $\mathcal{L}_i$.

Remark 2.2.7. Note that the differential kernel $\mathcal{L}$ has a regular realization if and only if there exists a differential field extension $(\mathcal{M}, \partial'_1, \ldots, \partial'_m)$ of $(\mathcal{K}, \Delta)$ such that $\mathcal{L}$ is a subfield of $\mathcal{M}$ and $\partial'_k a_i^\xi = a_i^{\xi+\mathbf{k}}$ for all $(\xi, i) \in \Gamma(h-1)$ and $1 \leqslant k \leqslant m$. In this case, $g := (a_1^\mathbf{0}, \ldots, a_n^\mathbf{0})$ is a regular realization of $\mathcal{L}$, and g will be a principal realization of $\mathcal{L}$ if and only if the minimal leaders of $\mathcal{L}$ and $\mathcal{K}\langle g \rangle$ coincide (here $\mathcal{K}\langle g \rangle \subseteq \mathcal{L}$ denotes the smallest differential field that contains both $\mathcal{K}$ and g).

Lemma 2.2.8. *If f is a principal realization and g is a regular realization of the differential kernel $\mathcal{L}$, then g is a differential specialization of f.*

Proof. Since f is a principal realization of $\mathcal{L}$, there exists a differential field extension $(\mathcal{M}, \partial'_1, \ldots, \partial'_m)$ of $\mathcal{K}$ containing $\mathcal{L} = \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(h))$ such that $\partial'_k a_i^\xi = a_i^{\xi+\mathbf{k}}$ for all $(\xi, i) \in \Gamma(h-1)$ and $\mathcal{K}\langle a_1^\mathbf{0}, \ldots, a_n^\mathbf{0} \rangle$ has the same minimal leaders as $\mathcal{L}$. Similarly, since g is a regular realization of $\mathcal{L}$, there is a differential field extension $(\mathcal{N}, \bar{\partial}_1, \ldots, \bar{\partial}_m)$ of $\mathcal{K}$ containing $\mathcal{L}$ such that $\bar{\partial}_k a_i^\xi = a_i^{\xi+\mathbf{k}}$.

Now, for each $s \geqslant h$, the differential kernel given by

$$\mathcal{L}'_s := \mathcal{K}((\partial')^\eta a_i^\mathbf{0} : (\eta, i) \in \Gamma(s))$$

is a generic prolongation of $\mathcal{L}$, and the one given by

$$\bar{\mathcal{L}}_s := \mathcal{K}(\bar{\partial}^\eta a_i^{\mathbf{0}} : (\eta, i) \in \Gamma(s))$$

is a prolongation of $\mathcal{L}$. By Lemma 2.2.5, $\bar{\mathcal{L}}_s$ is a specialization of $\mathcal{L}_s'$. Since this holds for all $s \geqslant h$, the desired differential specialization is obtained by taking the union of this chain. $\qquad\square$

Remark 2.2.9. One can similarly define prolongations, and regular and principal realizations, if the differential kernel is of the form $\mathcal{K}(a_i^\xi : (\xi, i) \vartriangleleft (\tau, j))$ for some fixed $(\tau, j) \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$. In addition, Lemmas 2.2.5 and 2.2.8 also hold in this case, with the same proofs.

In the ordinary case, $m = 1$, every differential kernel has a regular realization (in fact a principal one) [30, Proposition 3]. However, if $m > 1$, regular realizations do not always exist. Moreover, as the following example shows, there are differential kernels of length h with a prolongation of length $2h - 1$ but with no regular realization.

Example 2.2.10. Working with $m = 2$ and $n = 1$, set $\mathcal{K} = \mathbb{Q}$. Let

$$\mathcal{L} = \mathbb{Q}(a^{(i,j)} : i + j \leqslant h)$$

where the $a^{(i,j)}$'s are all algebraically independent over $\mathbb{Q}$ except for the algebraic relations $a^{(0,h)} = a^{(0,h-1)}$ and $a^{(h,0)} = (a^{(0,h-1)})^2$. Set $t := a^{(0,h-1)}$, so $a^{(0,h)} = t$ and $a^{(h,0)} = t^2$. The field $\mathcal{L}$ is a differential kernel over $\mathbb{Q}$ of length h. Moreover, it has a (generic) prolongation of length $2h - 1$. However, it does not have a prolongation of length $2h$ (and consequently no regular realization of $\mathcal{L}$ exists). Indeed, if $\mathcal{L}$ had a prolongation

$$\mathcal{L}' = \mathbb{Q}(a^{(i,j)} : i + j \leqslant 2h)$$

with derivations D_1' and D_2', then, as $D_2'(t) = t$, we would have

$$D_1' a^{(i,h)} = a^{(i+1,h-1)}$$

for $0 \leqslant i \leqslant h - 1$, and

$$a^{(h,j)} = (D_2')^j(t^2) = 2^j t^2$$

for $1 \leqslant j \leqslant h - 1$. In particular,

$$D_1' a^{(h-1,h)} = a^{(h,h-1)} = 2^{h-1} t^2 \quad \text{and} \quad D_2' a^{(h,h-1)} = D_2'(2^{h-1} t^2) = 2^h t^2.$$

This would yield the contradiction

$$2^{h-1} t^2 = D_1' a^{(h-1,h)} = a^{(h,h)} = D_2' a^{(h,h-1)} = 2^h t^2.$$

Nonetheless, there are conditions on the minimal leaders of a differential kernel that guarantee the existence of a regular realization. In [35], Pierce proved results of this type using different terminology: In his paper differential kernels are referred to as fields satisfying the *differential condition*, and a regular realization of $\mathcal{L}$ is referred to as the existence of a differential field extension of $\mathcal{K}$ *compatible* with $\mathcal{L}$. Using the terminology of differential kernels [35, Theorem 4.3] translates to:

Theorem 2.2.11. *Let* $\mathcal{L} = \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(h))$ *be a differential kernel over* $\mathcal{K}$ *for some even integer* $h > 0$. *Suppose further that*

(†) *for every minimal leader* (ξ, i) *of* $\mathcal{L}$ *we have that* $\deg(\xi) \leqslant \frac{h}{2}$.

Then the differential kernel $\mathcal{L}$ *has a regular realization.*

Note that a differential kernel $\mathcal{L}$ has a regular realization if and only if it has prolongations of any length. Thus the natural question to ask is: Is the existence of a regular realization guaranteed as long as one can find prolongations of a certain (finite) length? And if so, how can one compute this length, and what is its complexity? To answer these questions we will need the following terminology. Given an increasing function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$, we say that f *bounds the degree growth* of a sequence $\alpha_1, \ldots, \alpha_k$ of elements of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ if $\deg(\alpha_i) \leqslant f(i)$, for $i = 1, \ldots, k$. We let $\mathfrak{L}_{f,m}^n$ be the maximal length of an antichain sequence of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ of degree growth bounded by f. The existence of the number $\mathfrak{L}_{f,m}^n$ follows from generalizations of Dickson's lemma [11]. Recently, in [31], an algorithm that computes the exact value of $\mathfrak{L}_{f,m}^n$ was established (in fact, an antichain sequence of degree growth bounded by f of maximal length was built).

The following is a consequence of Theorem 2.2.11 (for details see the proof of [35, Theorem 4.10] or the discussion after Fact 3.6 of [12]).

Theorem 2.2.12. *Suppose $\mathcal{L} = \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(h))$ is a differential kernel over $\mathcal{K}$. Let $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ be defined as $f(i) = 2^i h$. If $\mathcal{L}$ has a prolongation of length $2^{\mathfrak{L}_{f,m}^n + 1} h$, then $\mathcal{L}$ has a regular realization.*

The above theorem motivates the following definition:

Definition 2.2.13. Given integers $m, n > 0$ and $h \geqslant 0$, we let $T_{h,m}^n$ be the smallest integer $\geqslant h$ with the following property: For any differential field $(\mathcal{K}, \partial_1, \ldots, \partial_m)$ of characteristic zero with m commuting derivations and any differential kernel $\mathcal{L}$ over $\mathcal{K}$ of length h, if $\mathcal{L}$ has a prolongation of length $T_{h,m}^n$, then $\mathcal{L}$ has a regular realization.

Theorem 2.2.12 shows that

$$T_{h,m}^n \leqslant 2^{\mathfrak{L}_{f,m}^n + 1} h \quad \text{where } f(i) = 2^i h.$$

This upper bound of $T_{h,m}^n$ is not sharp. For instance, [30, Proposition 3] shows that $T_{h,1}^n = h$, while $2^{\mathcal{L}_{f,1}^n + 1}h = 2^{n+1}h$. Also, by Proposition 3.1.6(3) below we have that $T_{h,2}^1 = 2h$, while $2^{\mathcal{L}_{f,2}^1 + 1}h = 2^{2h+2}h$. In general, for $m > 1$, a formula that computes the value of $T_{h,m}^n$ has not yet been found, and thus establishing computationally practical upper bounds is an important problem.

Chapter 3

Realizations of Differential Kernels

In Chapter 2, it was shown that the previously known upper bounds for the number of prolongations of a differential kernel needed to guarantee the existence of a regular realization, $T_{h,m}^n$, were not computationally viable even for small numbers of derivations. In this chapter a new, recursive upper bound for $T_{h,m}^n$ is given that can be computed in many realistic cases. In Section 3.1, the new upper bound, $C_{h,m}^n$, is defined from a theoretical standpoint and is shown in Theorem 3.1.4 to be an upper bound for $T_{h,m}^n$. In Section 3.2, the main combinatorial result is proven (Theorem 3.2.7). Then, in Section 3.3, a recursive algorithm to construct $C_{h,m}^n$ is given (Theorem 3.3.9). Finally, in Section 3.4, some specific values of $C_{h,m}^n$ are discussed in relation to previously known upper bounds for $T_{h,m}^n$.

3.1   On the existence of principal realizations

In this section we give an improvement of Theorems 2.2.11 and 2.2.12. This will come from replacing condition (†) by a *weaker* condition that guarantees the existence of a principal realization of a given differential kernel. We use the notation of Chapter 2; in particular, $(\mathcal{K}, \partial_1, \ldots, \partial_m)$ is a differential field of characteristic zero with m commuting derivations.

Given two elements $\eta = (u_1, \ldots, u_m)$ and $\tau = (v_1, \ldots, v_m)$ in $\mathbb{Z}_{\geqslant 0}^m$, we let

$$\mathrm{LUB}(\eta, \tau) = (\max(u_1, v_1), \ldots, \max(u_m, v_m))$$

be the *least upper bound* of η and τ with respect to the order $\leqslant$. Given an antichain sequence $\bar{\alpha}$ of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ we let

$$\gamma(\bar{\alpha}) = \{(\mathrm{LUB}(\eta, \tau), i) : \eta \neq \tau \text{ with } (\eta, i), (\tau, i) \in \bar{\alpha} \text{ for some } i\}.$$

Clearly, if for some integer $h \geqslant 0$ we have $\bar{\alpha} \subseteq \Gamma(h)$, then $\gamma(\bar{\alpha}) \subseteq \Gamma(2h)$. For a field extension of $\mathcal{K}$ of the form $\mathcal{L} = \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(h))$, we let $\gamma(\mathcal{L})$ denote $\gamma(\bar{\alpha})$ where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is the antichain sequence consisting of all minimal leaders of $\mathcal{L}$ ordered increasingly with respect to $\trianglelefteq$. Note that

$$\gamma(\mathcal{L}) \subseteq \Gamma(2h).$$

Theorem 3.1.1. *Let $\mathcal{L} = \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(h))$ be a differential kernel over $\mathcal{K}$. Suppose further that*

(♯) *For every $(\tau, l) \in \gamma(\mathcal{L}) \setminus \Gamma(h)$ and $1 \leqslant i < j \leqslant m$ such that $(\tau - \mathbf{i}, l)$ and $(\tau - \mathbf{j}, l)$ are leaders, there exists a sequence of minimal leaders $(\eta_1, l), \ldots, (\eta_s, l)$ such that $\eta_\ell \leqslant \tau - \mathbf{k}_\ell$, with $k_1 = i$, $k_s = j$ and some $k_2, \ldots, k_{s-1}$, and*

$$\deg(\mathrm{LUB}(\eta_\ell, \eta_{\ell+1})) \leqslant h \quad \text{for } \ell = 1, \ldots, s-1. \tag{3.1.1}$$

Then the differential kernel $\mathcal{L}$ has a principal realization.

Remark 3.1.2.

1. One can check that condition (†) of Theorem 2.2.11 implies condition (♯). On the other hand, if $m = 2$, $n = 1$, $h = 4$, and the only minimal leader of $\mathcal{L}$ is $(3, 0)$, then condition (†) does *not* hold; however, condition (♯) holds trivially. Thus, indeed (♯) is a *weaker* condition on the minimal leaders.

2. The converse of Theorem 3.1.1 does not generally hold (i.e., (♯) is not a necessary condition for the existence of principal realizations). For instance, if $m = 2$, $n = 1$, $h = 2$, and $a^{(1,0)} = a^{(0,1)} = 0$, then $\mathcal{L}$ has a principal realization but (♯) does not hold.

Proof. We construct the principal realization recursively. Let $(\tau, l) \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ with $\deg(\tau) > h$. We want to specify a value for a_l^τ. We assume that we have defined all a_i^ξ, where $(\xi, i) \lhd (\tau, l)$, such that the field extension

$$\mathcal{K}(a_i^\xi : (\xi, i) \lhd (\tau, l))$$

is a generic prolongation of $\mathcal{L}$.

If $(\tau - \mathbf{i}, l)$ is not a leader for all $1 \leqslant i \leqslant m$, then set a_l^τ to be transcendental over $\mathcal{K}(a_i^\xi : (\xi, i) \lhd (\tau, l))$ and define $D_i a_l^{\tau - \mathbf{i}} := a_l^\tau$. If there is an i such that $(\tau - \mathbf{i}, l)$ is a leader, then the algebraicity of $a_l^{\tau - \mathbf{i}}$ over $\mathcal{K}(a_i^\xi : (\xi, i) \lhd (\tau - \mathbf{i}, l))$ determines the value of a_l^τ; more precisely, the minimal polynomial of $a_l^{\tau - \mathbf{i}}$ determines the value $D_i a_l^{\tau - \mathbf{i}}$, and then we must set $a_l^\tau := D_i a_l^{\tau - \mathbf{i}}$. All we need to check is that if there is another j such that $(\tau - \mathbf{j}, l)$ is a leader, then the value $D_j a_l^{\tau - \mathbf{j}}$ (determined by the minimal polynomial of $a_l^{\tau - \mathbf{j}}$) is equal to $D_i a_l^{\tau - \mathbf{i}}$.

We now check that indeed $D_i a_l^{\tau - \mathbf{i}} = D_j a_l^{\tau - \mathbf{j}}$. First assume $(\tau, l) \in \gamma(\mathcal{L})$ (the other case will be considered below). Condition (♯) guarantees the existence of a sequence of minimal leaders $(\eta_1, l), \ldots, (\eta_s, l)$ such that $\eta_\ell \leqslant \tau - \mathbf{k}_\ell$, with $k_1 = i$, $k_s = j$ and some $k_2, \ldots, k_{s-1}$, and satisfying (3.1.1).

Claim. For every $1 \leqslant \ell \leqslant s - 1$, we have $D_{k_\ell} a_l^{\tau - \mathbf{k}_\ell} = D_{k_{\ell+1}} a_l^{\tau - \mathbf{k}_{\ell+1}}$.

Proof of Claim. If $k_\ell = k_{\ell+1}$, then the statement holds trivially. Let $k_\ell \neq k_{\ell+1}$ and $\pi = \text{LUB}(\eta_\ell, \eta_{\ell+1})$. By (3.1.1), we have $\deg(\pi) \leqslant h < \deg(\tau)$. In particular, there is $1 \leqslant k \leqslant m$ such that $\eta_\ell(k) \leqslant \pi(k) < \tau(k)$, where $\xi(k)$ denotes the k-entry of ξ. Since $k_\ell \neq k_{\ell+1}$, either $k \neq k_\ell$ or $k \neq k_{\ell+1}$; without loss of generality, we assume that $k \neq k_\ell$. We now prove that $(\tau - \mathbf{k}_\ell - \mathbf{k}, l)$ is a leader. Since $\eta_\ell \leqslant \tau - \mathbf{k}_\ell$, $\eta_\ell(k) < \tau(k)$, and $k \neq k_\ell$, we get that $\eta_\ell \leqslant \tau - \mathbf{k}_\ell - \mathbf{k}$. So, since (η_ℓ, l) is a (minimal) leader, $(\tau - \mathbf{k}_\ell - \mathbf{k}, l)$ is also a leader. This implies that the derivations D_{k_ℓ} and D_k commute on $a_l^{\tau - \mathbf{k}_\ell - \mathbf{k}}$ (see [35, Lemma 4.2]), and so

$$D_{k_\ell} a_l^{\tau - \mathbf{k}_\ell} = D_{k_\ell} D_k a_l^{\tau - \mathbf{k}_\ell - \mathbf{k}} = D_k D_{k_\ell} a_l^{\tau - \mathbf{k}_\ell - \mathbf{k}} = D_k a_l^{\tau - \mathbf{k}}.$$

If $k_{\ell+1} = k$ the result follows from the above equalities. If $k_{\ell+1} \neq k$, we can proceed as before (using the same k) to show that $(\tau - \mathbf{k}_{\ell+1} - \mathbf{k}, l)$ is leader, and thus obtain

$$D_{k_{\ell+1}} a_l^{\tau - \mathbf{k}_{\ell+1}} = D_k a_l^{\tau - \mathbf{k}}.$$

This proves the claim.

It now follows from the claim, since $k_1 = i$ and $k_s = j$, that $D_i a_l^{\tau - \mathbf{i}} = D_j a_l^{\tau - \mathbf{j}}$, as desired. Now, for the case when $(\tau, l) \notin \gamma(\mathcal{L})$. Let (η_1, l) and (η_2, l) be any pair of minimal leaders such that $\eta_1 \leqslant \tau - \mathbf{i}$ and $\eta_2 \leqslant \tau - \mathbf{j}$. By definition of $\gamma(\mathcal{L})$, we have that $\deg(\text{LUB}(\eta_1, \eta_2)) < \deg(\tau)$. One can now proceed as in the proof of the claim, with $\pi = \text{LUB}(\eta_1, \eta_2)$, to show that $D_i a_l^{\tau - \mathbf{i}} = D_j a_l^{\tau - \mathbf{j}}$.

One continues this recursive construction with the tuple succeeding τ (in the $\lhd$ order). Note, in each step of this construction, we do not add new minimal leaders, so the prolongations we obtain at each step still satisfy condition ($\sharp$) and are generic. By the genericity of each prolongation, this construction yields the desired principal realization of $\mathcal{L}$. $\square$

Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be an antichain sequence of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$. For each integer $h \geqslant 0$, let

$$\Gamma_{\bar{\alpha}}(h) = \{\alpha \in \bar{\alpha} : \alpha \in \Gamma(h)\}.$$

We define $D_{h,\bar{\alpha}}$ as the smallest integer $p \geqslant h$ with the following property:

($\sharp'$) For every $(\tau, l) \in \gamma(\Gamma_{\bar{\alpha}}(p)) \setminus \Gamma(p)$ and $1 \leqslant i < j \leqslant m$ such that $(\tau - \mathbf{i}, l) \geqslant \beta_1$ and

$(\tau - \mathbf{j}, l) \geqslant \beta_2$ for some $\beta_1, \beta_2 \in \Gamma_{\bar{\alpha}}(p)$, there exists a sequence $(\eta_1, l), \ldots, (\eta_s, l)$ in $\Gamma_{\bar{\alpha}}(p)$

such that $\eta_\ell \leqslant \tau - \mathbf{k}_\ell$, with $k_1 = i$, $k_s = j$ and some $k_2, \ldots, k_{s-1}$, and

$$\deg(\mathrm{LUB}(\eta_\ell, \eta_{\ell+1})) \leqslant p \quad \text{for } \ell = 1, \ldots, s - 1. \tag{3.1.2}$$

Note that if $r \geqslant h$ is such that $\bar{\alpha} \subseteq \Gamma(r)$, then $D_{h,\bar{\alpha}} \leqslant 2r$.

Remark 3.1.3. Note that, given $h \geqslant 0$ and an antichain sequence $\bar{\alpha}$ of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$, $D_{s,\bar{\alpha}} = D_{h,\bar{\alpha}}$ for any $h \leqslant s \leqslant D_{h,\bar{\alpha}}$.

Finally, we set

$$C_{h,m}^n := \max\{D_{h,\bar{\alpha}} : \bar{\alpha} \text{ is an antichain sequence of } \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}\}.$$

In Section 3.3 we will see that in fact $C_{h,m}^n < \infty$.

We can now prove the main result of this section.

Theorem 3.1.4. *Let h be a nonnegative integer. Suppose $\mathcal{L} = \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(h))$ is a differential kernel over $\mathcal{K}$. If $\mathcal{L}$ has a prolongation of length $C_{h,m}^n$, then there is some $h \leqslant r \leqslant C_{h,m}^n$ such that the differential kernel $\mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(r))$ has a principal realization. In particular, $\mathcal{L}$ has a regular realization and so*

$$T_{h,m}^n \leqslant C_{h,m}^n.$$

Proof. Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be the antichain sequence of minimal leaders of the prolongation

$$\mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(C_{h,m}^n)).$$

By definition of $D_{h,\bar{\alpha}}$ (see property ($\sharp'$) above), if we set $r := D_{h,\bar{\alpha}}$, then r has the following three properties:

(i) $r \geqslant h$

(ii) $C_{h,m}^n \geqslant r$, so $\mathcal{L}' := \mathcal{K}(a_i^\xi : (\xi, i) \in \Gamma(r))$ is a differential kernel over $\mathcal{K}$

(iii) Since $\Gamma_{\bar{\alpha}}(r)$ is equal to the set of minimal leaders of $\mathcal{L}'$, we have that for every $(\tau, l) \in \gamma(\mathcal{L}') \setminus \Gamma(r)$ and $1 \leqslant i < j \leqslant m$ such that $\tau - \mathbf{i}, \tau - \mathbf{j}$ are leaders of $\mathcal{L}'$, there exists a sequence $(\eta_1, l), \ldots, (\eta_s, l)$ of minimal leaders of $\mathcal{L}'$ such that $\eta_\ell \leqslant \tau - \mathbf{k}_\ell$, with $k_1 = i$, $k_s = j$ and some $k_2, \ldots, k_{s-1}$, and

$$\deg(\mathrm{LUB}(\eta_\ell, \eta_{\ell+1})) \leqslant r \quad \text{for } \ell = 1, \ldots s - 1.$$

Property (iii) is precisely saying that $\mathcal{L}'$ satisfies condition ($\sharp$) of Theorem 3.1.1. Thus, properties (ii) and (iii), together with Theorem 3.1.1, yield a principal realization of $\mathcal{L}'$. Finally, property (i) implies this principal realization of $\mathcal{L}'$ is a regular realization of $\mathcal{L}$. □

Remark 3.1.5. So far, to the author's knowledge, there are *no* known cases where $T_{h,m}^n < C_{h,m}^n$. It is thus an interesting problem to determine whether or not these two numbers are equal. Such open questions on the optimality of $C_{h,m}^n$ are part of an ongoing project.

In Sections 3.2 and 3.3 we work towards building a recursive algorithm that computes the value of $C_{h,m}^n$. For now, we prove some basic cases.

Proposition 3.1.6.

1. $C_{0,m}^n = 0$.

2. *For any $h > 0$, $C_{h,1}^n = h$.*

3. *For any $h > 0$, $C_{h,2}^1 = 2h$. Consequently, by Example 2.2.10, $T_{h,2}^1 = 2h$.*

Proof.

(1) This is clear.

(2) For any antichain sequence $\bar{\alpha}$ of $\mathbb{Z}_{\geqslant 0} \times \mathfrak{n}$, condition ($\sharp'$) above is trivially satisfied for any integer $p \geqslant 0$ since in this case $\gamma(\Gamma_{\bar{\alpha}}(p)) = \emptyset$. Hence, $D_{h,\bar{\alpha}} = h$, and so $C_{h,1}^n = h$.

(3) First, to see that $C_{h,2}^1 \geqslant 2h$, consider the antichain sequence $\bar{\alpha} = ((h,0),(0,h))$ of $\mathbb{Z}_{\geqslant 0}^2$. Since $\gamma(\bar{\alpha}) = \{\text{LUB}((h,0),(0,h))\} = \{(h,h)\}$, the integer $2h$ satisfies condition ($\sharp'$), and it is indeed the smallest one as $\bar{\alpha}$ consists of exactly two elements. Hence, $D_{h,\bar{\alpha}} = 2h$ and so $C_{h,2}^1 \geqslant 2h$.

Now we prove $C_{h,2}^1 \leqslant 2h$. Towards a contradiction assume there is an antichain sequence $\bar{\alpha}$ of $\mathbb{Z}_{\geqslant 0}^2$ such that $D_{h,\bar{\alpha}} > 2h$. First, let us recall a basic fact about blocks of $\mathbb{Z}_{\geqslant 0}^2$. Recall that a *block* of $\mathbb{Z}_{\geqslant 0}^2$ is a subset of the form

$$\{(u_1,u_2),(u_1+1,u_2-1),\ldots,(u_1+c,u_2-c)\}$$

for some $u_1,u_2,c \in \mathbb{Z}_{\geqslant 0}$. Suppose B is a set of elements of $\mathbb{Z}_{\geqslant 0}^2$ all of degree $d \geqslant 0$, and let B' be those elements of degree $d+1$ which are $\geqslant$ some element in B. One can check that $|B'| \geqslant |B| + 1$ and $|B'| = |B| + 1$ if and only if B is a block.

Now, for each integer $i \geqslant 0$, we let

$$\mathfrak{M}_{\bar{\alpha}}(i) = |\{\xi \in \mathbb{Z}_{\geqslant 0}^2 : \deg(\xi) = i \text{ and } \xi \geqslant \tau \text{ for some } \tau \in \bar{\alpha}\}|.$$

Note that $|\mathfrak{M}_{\bar{\alpha}}(h)| \geqslant 2$. Indeed, if this were not the case the integer h would satisfy condition ($\sharp'$) and so $D_{h,\bar{\alpha}}$ would equal h, contradicting the fact that $D_{h,\bar{\alpha}} > 2h$. We now claim that $|\mathfrak{M}_{\bar{\alpha}}(i+1)| \geqslant |\mathfrak{M}_{\bar{\alpha}}(i)| + 2$ for $h \leqslant i < 2h$. If this were not the case, then, as we are working in $\mathbb{Z}_{\geqslant 0}^2$, $|\mathfrak{M}_{\bar{\alpha}}(i+1)| = |\mathfrak{M}_{\bar{\alpha}}(i)| + 1$. However, as we pointed out above, the latter could only happen if $\mathfrak{M}_{\bar{\alpha}}(i)$ is a block and $\bar{\alpha}$ has no elements of degree $i+1$. But this would imply that the integer $i + 1 \leqslant 2h$ satisfies condition ($\sharp'$), contradicting again the fact that $D_{h,\bar{\alpha}} > 2h$. Putting the previous inequalities together we get $\mathfrak{M}_{\bar{\alpha}}(i) \geqslant 2(i+1-h)$ for $h \leqslant i \leqslant 2h$. In particular, $\mathfrak{M}_{\bar{\alpha}}(2h) \geqslant 2h + 2$. However, this is impossible since the number of elements of degree $2h$ of $\mathbb{Z}_{\geqslant 0}^2$ is $2h + 1$, and so we have reached the desired contradiction. $\qquad \square$

3.2 On Macaulay's theorem

In this section we prove a key result on the Hilbert-Samuel function that will be used to derive Corollary 3.2.9 below. This will then be used in Section 3.3 to provide an algorithm that computes the value of $C_{h,m}^n$.

Recall, from previous sections, that we denote $\mathbb{Z}_{\geqslant 0}^m$ equipped with the product order by $(\mathbb{Z}_{\geqslant 0}^m, \leqslant)$, and we denote $\mathbb{Z}_{\geqslant 0}^m$ equipped with the (left) degree-lexicographic order by $(\mathbb{Z}_{\geqslant 0}^m, \trianglelefteq)$. We start by recalling some basic notions (for details see [6, Chap.4, §2]). A subset M of $\mathbb{Z}_{\geqslant 0}^m$ is said to be *compressed* if whenever $\xi, \eta \in \mathbb{Z}_{\geqslant 0}^m$ and $\deg(\xi) = \deg(\eta)$ we have

$$(\xi \in M \text{ and } \xi \triangleleft \eta) \implies (\eta \geqslant \zeta \text{ for some } \zeta \in M).$$

If d is a positive integer, M is said to be a *d-segment* of $\mathbb{Z}_{\geqslant 0}^m$ if all the elements of M have degree d and, given $\xi, \eta \in M$ with $\xi \triangleleft \eta$, if $\xi \in M$ then $\eta \in M$. We note that if M is compressed and

$$N := \{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = d \text{ and } \xi \geqslant \zeta \text{ for some } \zeta \in M\},$$

then N is a d-segment of $\mathbb{Z}_{\geqslant 0}^m$.

Given positive integers a and d, one can write a uniquely in the form

$$a = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \ldots + \binom{k_j}{j}, \tag{3.2.1}$$

where $k_d > k_{d-1} > \ldots > k_j \geqslant j \geqslant 1$ for some j (see [6, Lemma 4.2.6]). One refers to (3.2.1) as the *d-binomial representation* of a. Now define

$$a^{\langle d \rangle} := \binom{k_d + 1}{d + 1} + \binom{k_{d-1} + 1}{d} + \ldots + \binom{k_j + 1}{j + 1},$$

and $0^{\langle d \rangle} := 0$.

We now recall the Hilbert-Samuel function. Given any subset M of $\mathbb{Z}_{\geqslant 0}^m$, the *Hilbert-Samuel function* $H_M : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ is defined as

$$H_M(d) = |\{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = d \text{ and } \xi \not\geqslant \eta \text{ for all } \eta \in M\}|.$$

Macaulay's theorem on the Hilbert-Samuel function states the following (for a proof see Corollary 4.2.9 and Theorem 4.2.10(c) from [6]).

Theorem 3.2.1. *For any subset M of $\mathbb{Z}_{\geqslant 0}^m$, and $d > 0$, we have that*

$$H_M(d + 1) \leqslant H_M(d)^{\langle d \rangle}.$$

Moreover, if M is compressed and $M \subseteq \Gamma(d)$ (i.e., $\deg(\xi) \leqslant d$ for all $\xi \in M$), then

$$H_M(d + 1) = H_M(d)^{\langle d \rangle}.$$

We will also make use of the function S_M which is complementary to the Hilbert-Samuel function; that is, for any subset M of $\mathbb{Z}_{\geqslant 0}^m$, $S_M : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ is given by

$$S_M(d) = |\{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = d \text{ and } \xi \geqslant \eta \text{ for some } \eta \in M\}|.$$

Note that

$$S_M(d) + H_M(d) = |\{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = d\}| = \binom{m-1+d}{d}. \tag{3.2.2}$$

For any $M \subseteq \mathbb{Z}_{\geqslant 0}^m$, we define $(1, \ldots, m) \cdot M$ to be the set containing all m-tuples of the form $(u_1, \ldots, u_j + 1, \ldots, u_m)$ with $(u_1, \ldots, u_m) \in M$ and $j = 1, \ldots, m$. More generally, for a sequence of integers $1 \leqslant i_1 < \ldots < i_s \leqslant m$, we let $(i_1, \ldots, i_s) \cdot M$ be the set $(u_1, \ldots, u_{i_j} + 1, \ldots, u_m)$ with $(u_1, \ldots, u_m) \in M$ and $j = 1, \ldots, s$. We now recall *Macaulay's function $a^{(m)}$*. For integers $a \geqslant 0$ and $d > 0$, with $a \leqslant |\{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = d\}|$, we let

$$a^{(m)} := |(1, \ldots, m) \cdot N_{a,d}| = S_{N_{a,d}}(d+1), \tag{3.2.3}$$

where $N_{a,d}$ is the subset of $\mathbb{Z}_{\geqslant 0}^m$ consisting of the a largest elements of $\Gamma(d)$ with respect to $\vartriangleleft$. Note that, by our assumption on a and d, the set $N_{a,d}$ is a d-segment of $\mathbb{Z}_{\geqslant 0}^m$ (as defined above); in particular, it is compressed. To justify our notation in (3.2.3), we must show that the value $a^{(m)}$ is independent of d. To that end, let $d' = d + p$, for a positive integer p. Clearly, $N_{a,d'} = (1)^p \cdot N_{a,d}$, where the latter denotes the set of $(u_1 + p, \ldots, u_m)$ with $(u_1, \ldots, u_m) \in N_{a,d}$. Then we have

$$(1, \ldots, m) \cdot N_{a,d'} = (1)^p \cdot ((1, \ldots, m) \cdot N_{a,d}),$$

and hence $|(1, \ldots, m) \cdot N_{a,d'}| = |(1, \ldots, m) \cdot N_{a,d}|$, as desired.

As a consequence of the moreover clause of Macaulay's theorem (Theorem 3.2.1), for

integers $b \geqslant 0$ and $d > 0$, with $b \leqslant |\{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = d\}| = \binom{m-1+d}{d}$, we have that

$$b^{\langle d \rangle} = |\{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = d+1 \text{ and } \xi \notin (1, \dots, m) \cdot N_{a,d}\}| = H_{N_{a,d}}(d+1),$$

where $a := \binom{m-1+d}{d} - b$. This implies that $b^{\langle d \rangle} = \binom{m+d}{d+1} - a^{(m)}$; in particular, for any M we

have

$$H_M(d)^{\langle d \rangle} = \binom{m+d}{d+1} - S_M(d)^{(m)}. \tag{3.2.4}$$

Thus, with the above notation, Theorem 3.2.1 can be reformulated as

Corollary 3.2.2. *For any subset M of $\mathbb{Z}_{\geqslant 0}^m$, and $d > 0$, we have that*

$$S_M(d+1) \geqslant S_M(d)^{(m)}.$$

Moreover, if M is compressed and $M \subseteq \Gamma(d)$, then

$$S_M(d+1) = S_M(d)^{(m)}.$$

Remark 3.2.3. The formulation of this corollary is quite similar to how Macaulay originally

presented his theorem in the 1920s (see [32] or [41]).

Proof. By (3.2.2), (3.2.4) and Theorem 3.2.1, we have

$$\begin{aligned}
S_M(d+1) &= \binom{m+d}{d+1} - H_M(d+1) \\
&\geqslant \binom{m+d}{d+1} - H_M(d)^{\langle d \rangle} \\
&= S_M(d)^{(m)}.
\end{aligned}$$

For the moreover clause one simply replaces the above inequality by equality.   □

We now fix some notation that will be used in the proof of Theorem 3.2.7 below.

Definition 3.2.4. Let d be a nonnegative integer and M a subset of $\mathbb{Z}_{\geqslant 0}^m$. Given $\tau \in \mathbb{Z}_{\geqslant 0}^m$ of $\deg(\tau) > d + 1$, and distinct $\xi, \zeta \in M \cap \Gamma(d)$ both $< \tau$ (recall that $<$ denotes the product order of $\mathbb{Z}_{\geqslant 0}^m$), we say that ξ and ζ are $\tau_{d,M}$-*connected* if there is a sequence of elements $\eta_1, \ldots, \eta_s$ of $M \cap \Gamma(d)$ all $< \tau$ with $\eta_1 = \xi$, $\eta_s = \zeta$, and such that for all $1 \leqslant i \leqslant s - 1$,

$$\deg(\mathrm{LUB}(\eta_i, \eta_{i+1})) \leqslant d + 1.$$

Given an integer $d \geqslant 0$, consider the following condition on $M \subseteq \mathbb{Z}_{\geqslant 0}^m$:

(∗) There are two distinct elements $\xi, \zeta \in M \cap \Gamma(d)$ such that for every sequence $\eta_1, \ldots, \eta_s$ of elements of $M \cap \Gamma(d)$ all $< \mathrm{LUB}(\xi, \zeta)$ with $\eta_1 = \xi$ and $\eta_s = \zeta$, there exists $1 \leqslant i \leqslant s - 1$ such that $\deg(\mathrm{LUB}(\eta_i, \eta_{i+1})) > d + 1$.

Remark 3.2.5. Suppose M satisfies condition (∗) for a fixed d. Then, for any pair of distinct elements $\xi, \zeta \in M \cap \Gamma(d)$ given as in condition (∗), we have that $\deg(\mathrm{LUB}(\xi, \zeta)) > d + 1$. Hence, $M \cap \Gamma(d)$ contains two distinct elements ξ and ζ that are not $\mathrm{LUB}(\xi, \zeta)_{d,M}$-connected. Moreover, such a pair (ξ, ζ) can be chosen with the following additional property: for any two distinct elements $\eta, \pi \in M \cap \Gamma(d)$ both $< \mathrm{LUB}(\xi, \zeta)$, either η and π are $\mathrm{LUB}(\xi, \zeta)_{d,M}$-connected, or $\mathrm{LUB}(\eta, \pi) = \mathrm{LUB}(\xi, \zeta)$. To see this, suppose there exist distinct $\xi', \zeta' \in M \cap \Gamma(d)$ both $< \mathrm{LUB}(\xi, \zeta)$ that are not $\mathrm{LUB}(\xi, \zeta)_{d,M}$-connected but $\mathrm{LUB}(\xi', \zeta') \neq \mathrm{LUB}(\xi, \zeta)$. Then $\mathrm{LUB}(\xi', \zeta') < \mathrm{LUB}(\xi, \zeta)$. In this case, we replace the pair (ξ, ζ) with the pair (ξ', ζ'). This process will eventually produce the desired pair (after finitely many steps, since at each step the degree of $\mathrm{LUB}(\xi, \zeta)$ decreases).

We will need the following technical result of Sperner on the Macaulay function (see [41, §3, p.196]).

Lemma 3.2.6. *Let A, B, C be nonnegative integers. If $A > 0$, $A = B + C$, and $C^{(m-1)} < A^{(m)} - A$, then*

$$B^{(m)} + C^{(m-1)} \geqslant A^{(m)}.$$

We are finally ready to prove the main theorem of this section, which can be regarded as the key result of this chapter.

Theorem 3.2.7. *Let $d > 0$ be an integer and M a subset of $\mathbb{Z}_{\geqslant 0}^m$. If M satisfies condition $(*)$ above, then we have the following strict inequality*

$$H_M(d+1) < H_M(d)^{\langle d \rangle}.$$

Proof. We first make some simplifications. By definition of the Hilbert-Samuel function,

$$H_M(d) = H_N(d) \text{ and } H_M(d+1) \leqslant H_N(d+1),$$

where $N = \{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = d \text{ and } \xi \geqslant \eta \text{ for some } \eta \in M\}$, and so it would suffice to prove the theorem for N. Hence, we assume that all the elements of M have degree d.

Note that the desired inequality is equivalent to

$$S_M(d+1) > S_M(d)^{(m)}. \tag{3.2.5}$$

Indeed, if (3.2.5) holds, by (3.2.2) and (3.2.4), we would have

$$H_M(d+1) = \binom{m+d}{d+1} - S_M(d+1) < \binom{m+d}{d+1} - S_M(d)^{(m)} = H_M(d)^{\langle d \rangle}.$$

Thus, it suffices to prove (3.2.5). Note that, by our assumption that all the elements of M have degree d, we have $|M| = S_M(d)$ and $|(1, \ldots, m) \cdot M| = S_M(d+1)$.

Now let (ξ, ζ) be a pair of distinct elements of M as in Remark 3.2.5 and set

$$\tau := \mathrm{LUB}(\xi, \zeta) = (v_1, \ldots, v_m);$$

that is, $\xi = (a_1, \ldots, a_m)$ and $\zeta = (b_1, \ldots, b_m)$ are elements of M that are not $\tau_{d,M}$-connected, and for any two distinct elements $\eta, \pi \in M$ both $< \tau$, either η and π are $\tau_{d,M}$-connected or $\mathrm{LUB}(\eta, \pi) = \tau$. We assume that $a_1 < b_1$; if not simply permute the variables.

Let $A := S_M(d) = |M|$ and $F := S_M(d+1) = |(1, \ldots, m) \cdot M|$. Thus, we must show that

$$F > A^{(m)}.$$

We prove the result by induction on the size of A. Since M has at least two elements, the base case is $A = 2$ and so $M = \{\xi, \zeta\}$. In this case, $A^{(m)} = 2m - 1$, and saying that ξ and ζ are not $\tau_{d,M}$-connected is equivalent to saying that $\deg(\tau) > d + 1$. But then we cannot have $\xi + \mathbf{i} = \zeta + \mathbf{j}$ for any $1 \leqslant i, j \leqslant m$, so $F = 2m > A^{(m)}$.

Now we prove the induction step, and so assume $A \geqslant 3$. Let $(u_1, \ldots, u_m)$ be the least element of M with respect to the (left) degree-lexicographical order $\trianglelefteq$. We can then write

$$M = M_0 \cup M_1,$$

where M_0 consists of all $(t_1, \ldots, t_m) \in M$ with $t_1 > u_1$, and M_1 consists of all $(t_1, \ldots, t_m) \in M$ with $t_1 = u_1$. Note that $M_0 \cap M_1 = \emptyset$. We then have the following inclusions:

$$(1) \cdot M \cup (2, \ldots, m) \cdot M_1 \subseteq (1, \ldots, m) \cdot M, \tag{3.2.6}$$

$$(1, \ldots, m) \cdot M_0 \cup (2, \ldots, m) \cdot M_1 \subseteq (1, \ldots, m) \cdot M. \tag{3.2.7}$$

In addition we have that

$$(1) \cdot M \cap (2, \ldots, m) \cdot M_1 = \emptyset \text{ and } (1, \ldots, m) \cdot M_0 \cap (2, \ldots, m) \cdot M_1 = \emptyset.$$

We now prove that, under our assumptions, the inclusion (3.2.6) is strict. First note that all tuples $\pi = (c_1, \ldots, c_m) \in M$ such that $\pi < \tau$ and $c_1 = a_1 < b_1$ are $\tau_{d,M}$-connected to ξ, otherwise this would contradict the choice of τ as $\mathrm{LUB}(\xi, \pi) \neq \tau$. Let a be the smallest integer with $a > a_1$ and such that there is $\pi = (c_1, \ldots, c_m) \in M$ with $\pi < \tau$, not $\tau_{d,M}$-connected to ξ, and $c_1 = a$. Note that $a \leqslant b_1$. Also, note that there is $1 < i \leqslant m$ such that $c_i < v_i$ (if not, we would have $\pi > \xi$). Set

$$\rho = (c_1 - 1, c_2, \ldots, c_i + 1, \ldots, c_m).$$

Then, ρ is not in M. Indeed, it it were, π would be $\tau_{d,M}$-connected to ξ. This shows that $\pi + \mathbf{i} \in (1, \ldots, m) \cdot M$ but $\pi + \mathbf{i} \notin (1) \cdot M \cup (2, \ldots, m) \cdot M_1$, as desired.

Now we prove that if M_0 does not satisfy condition $(*)$, then containment (3.2.7) is strict. In this case, we must have $\xi \in M_1$. Also, note that for every $1 < i \leqslant m$ such that $a_i > 0$, if we set

$$\nu = (a_1 + 1, a_2, \ldots, a_i - 1, \ldots, a_m),$$

then $\nu < \tau$ but it cannot be in M_0. Indeed, if it were, then ν and ζ would witness that M_0 satisfies condition $(*)$ since ξ and ν are $\tau_{d,M}$-connected. This shows that $\xi + \mathbf{1} \in (1, \ldots, m) \cdot M$ but $\xi + \mathbf{1} \notin (1, \ldots, m) \cdot M_0 \cup (2, \ldots, m) \cdot M_1$, as desired.

Let $B = |M_0|$ and let $C = |M_1|$. Since we have shown that inclusion (3.2.6) is strict, an application of Corollary 3.2.2 yields

$$F > A + C^{(m-1)}.$$

On the other hand, if inclusion (3.2.7) is strict, another application of Corollary 3.2.2 yields

$$F > B^{(m)} + C^{(m-1)}.$$

Finally, if (3.2.7) is an equality, then we have shown that M_0 must satisfy $(*)$. Since $B < A$ (as $M_0 \subsetneq M$), by induction we have that in this case $|(1, \ldots, m) \cdot B| > B^{(m)}$, and so

$$F > B^{(m)} + C^{(m-1)}.$$

Therefore, we always have that

$$F > A + C^{(m-1)} \quad \text{and} \quad F > B^{(m)} + C^{(m-1)}. \tag{3.2.8}$$

If $C^{(m-1)} \geqslant A^{(m)} - A$, then it follows from the first inequality of (3.2.8) that $F > A^{(m)}$. For the remaining case $C < A^{(m)} - A$, since $A \geqslant 3$ and $A = B + C$, Lemma 3.2.6 yields $B^{(m)} + C^{(m-1)} \geqslant A^{(m)}$. It now follows from the second inequality of (3.2.8) that $F > A^{(m)}$. This concludes the proof. $\qquad\square$

Remark 3.2.8.

1. Theorem 3.2.7 seems to be of independent interest. It states that a necessary condition for the Hilbert-Samuel function of M to have maximal growth at $d + 1$ is that every pair ξ, ζ of distinct elements of $M \cap \Gamma(d)$ is $\mathrm{LUB}(\xi, \zeta)_{d,M}$-connected.

2. For $m = 2$, the converse of Theorem 3.2.7 holds. Indeed, if M is a subset of $\mathbb{Z}_{\geqslant 0}^2$ all of whose elements have degree d, then $H_M(d+1) = H_M(d)^{\langle d \rangle}$ if and only if M is a block (i.e., M is of the form $\{(u_1, u_2), (u_1 + 1, u_2 - 1), \ldots, (u_1 + c, u_2 - c)\}$ for some $u_1, u_2, c \in \mathbb{Z}_{\geqslant 0}$), and if M is a block then M does not satisfy condition $(*)$. On the other hand, when $m \geqslant 3$, the converse of Theorem 3.2.7 does not generally hold. For a (counter-)

example, consider the case $m = 3$ and $M = \{\xi \in \mathbb{Z}_{\geqslant 0}^3 : \deg(\xi) = 2\} \setminus \{(1, 1, 0)\}$. One can easily check that this M does not satisfy condition $(*)$; however,

$$H_M(3) = 0 < 1 = H_M(2)^{\langle 2 \rangle}.$$

To finish this section, we want to connect the previous discussion to our work with antichains from previous sections. Given an antichain sequence $\bar{\xi} = (\xi_1, \ldots, \xi_k)$ of $\mathbb{Z}_{\geqslant 0}^m$, for each $i \geqslant 0$ the Hilbert-Samuel function $H_{\bar{\xi}}^i : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ is defined as

$$H_{\bar{\xi}}^i(d) = |\{\eta \in \mathbb{Z}_{\geqslant 0}^m : \deg(\eta) = d \text{ and } \eta \not\geqslant \xi_j \text{ for all } j \leqslant i \text{ for which } \xi_j \text{ is defined}\}|.$$

If for each $i \geqslant 0$ we let

$$M_i = \{\eta \in \mathbb{Z}_{\geqslant 0}^m : \deg(\eta) = d \text{ and } \eta \geqslant \xi_j \text{ for some } \xi_j \text{ with } j \leqslant i\},$$

we see then that $H_{M_i}(d) = H_{\bar{\xi}}^i(d)$. Hence, a direct consequence of Theorem 3.2.7 is the following:

Corollary 3.2.9. *Let $d > 1$ be an integer and $\bar{\xi} = (\xi_1, \ldots, \xi_k)$ an antichain sequence of $\mathbb{Z}_{\geqslant 0}^m$. If $H_{\bar{\xi}}^k(d) = H_{\bar{\xi}}^k(d-1)^{\langle d-1 \rangle}$, then for each pair $\xi_i \neq \xi_j$, both having degree at most $d-1$, there exists a sequence $\eta_1, \ldots, \eta_s$ of distinct elements of $\Gamma_{\bar{\xi}}(d-1) = \bar{\xi} \cap \Gamma(d-1)$ all $< \mathrm{LUB}(\xi_i, \xi_j)$ such that $\eta_1 = \xi_i$, $\eta_s = \xi_j$, and*

$$\deg(\mathrm{LUB}(\eta_\ell, \eta_{\ell+1})) \leqslant d, \quad \text{for all } \ell = 1, \ldots, s-1.$$

3.3 An algorithm to compute $C_{h,m}^n$

In this section we prove that there is a recursive algorithm that computes $C_{h,m}^n$. We first deal with the case $n = 1$ (Theorem 3.3.5), and then we prove that for $n > 1$ the value is obtained by compositions in the "h" input (Theorem 3.3.9).

We will make use of the following combinatorial lemma (for a proof see [31, Lemma 3.12]).

Lemma 3.3.1. *Suppose $a_1, \ldots, a_t$ and $b_1, \ldots, b_s$ are sequences of nonnegative integers such that $b_1 = \ldots = b_{s-1} \geqslant b_s$ and $b_1 \geqslant a_i$ for all $i \leqslant t$. If $a_1 + \ldots + a_t \leqslant b_1 + \ldots + b_s$, then, for every integer $d > 0$, we have that*

$$a_1^{\langle d \rangle} + \ldots + a_t^{\langle d \rangle} \leqslant b_1^{\langle d \rangle} + \ldots + b_s^{\langle d \rangle}.$$

Recall from Section 2.2 that for any increasing function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ we say f bounds the degree growth of a sequence $\alpha_1, \ldots, \alpha_k$ of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ if $\deg(\alpha_i) \leqslant f(i)$ for all $i = 1, \ldots, k$. Also, $\mathfrak{L}_{f,m}^n$ denotes the maximal length of an antichain sequence of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ of degree growth bounded by f. In [31] an algorithm computing the value of $\mathfrak{L}_{f,m}^n$ was established and an antichain sequence of maximal length was built. We discuss this in more detail below.

3.3.1 The case $n = 1$

Throughout this subsection we let $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ be the increasing function defined as $g(1) = h$ and $g(i) = i + h - 2$ for $i \geqslant 2$. We will prove that

$$C_{h,m}^1 = \mathfrak{L}_{g,m}^1 + h - 1.$$

In Proposition 3.1.6 we already proved that this equality holds in the case $h = 0$ or $m = 1$. We now assume $h \geqslant 1$ and $m \geqslant 2$. Note that in this case we have $\mathfrak{L}_{g,m}^1 \geqslant 2$, and so the

above equality is equivalent to

$$C_{h,m}^1 = g(\mathfrak{L}_{g,m}^1) + 1. \tag{3.3.1}$$

Let $\bar{\mu} = (\mu_1, \ldots, \mu_L)$ be the antichain sequence defined recursively as follows:

$$\mu_1 = \max_{\trianglelefteq}\{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = g(1)\},$$

and, as long as it is possible,

$$\mu_i = \max_{\trianglelefteq}\{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = g(i) \text{ and } \xi \not\trianglerighteq \mu_1, \ldots, \mu_{i-1}\}.$$

In [31, §3.2], it is shown that $\bar{\mu}$ is a compressed antichain sequence of $\mathbb{Z}_{\geqslant 0}^m$ having length $L = \mathfrak{L}_{g,m}^1$ (i.e., $\bar{\mu}$ is of maximal length among antichain sequences of $\mathbb{Z}_{\geqslant 0}^m$ with degree growth bounded by g). It is also observed that $H_{\bar{\mu}}^L(\deg(\mu_L)) = H_{\bar{\mu}}^L(g(L)) = 0$, where recall that $H_{\bar{\mu}}^i$ denotes the Hilbert-Samuel function of $\bar{\mu}$, that is, for $i, d \geqslant 0$,

$$H_{\bar{\mu}}^i(d) = |\{\xi \in \mathbb{Z}_{\geqslant 0}^m : \deg(\xi) = d \text{ and } \xi \not\trianglerighteq \mu_j \text{ for all } j \leqslant i \text{ for which } \mu_j \text{ is defined}\}|.$$

The antichain sequence $\bar{\mu}$ can be more explicitly constructed as follows:

(i) if $\mu_i = (u_1, \ldots, u_s, 0, \ldots, 0, u_m)$ with $s < m-1$ and $u_s > 0$, then

$$\mu_{i+1} = (u_1, \ldots, u_s - 1, g(i+1) - g(i) + u_m + 1, 0, \ldots, 0)$$

(ii) if $\mu_i = (u_1, \ldots, u_{m-1}, u_m)$ with $u_{m-1} > 0$, then

$$\mu_{i+1} = (u_1, \ldots, u_{m-1} - 1, g(i+1) - g(i) + u_m + 1).$$

Define the function $\Psi_{g,m} : \mathbb{Z}_{>0} \times \mathbb{Z}_{\geqslant 0}^m \to \mathbb{Z}_{\geqslant 0}$ by the following relations:

$$
\begin{cases}
\Psi_{g,m}(i, (0, \ldots, 0, u_m)) = i \\[2mm]
\Psi_{g,m}(i-1, (u_1, \ldots, u_s, 0, \ldots, 0, u_m)) \\[1mm]
\qquad = \Psi_{g,m}(i, (u_1, \ldots, u_s - 1, g(i) - g(i-1) + u_m + 1, 0, \ldots, 0)), \quad s < m-1, u_s > 0 \\[2mm]
\Psi_{g,m}(i-1, (u_1, \ldots, u_m)) \\[1mm]
\qquad = \Psi_{g,m}(i, (u_1, \ldots, u_{m-1} - 1, g(i) - g(i-1) + u_m + 1)), \qquad u_{m-1} > 0.
\end{cases}
$$

$$(3.3.2)$$

Then by the recursive construction of $\bar{\mu}$, one obtains

Proposition 3.3.2 ([31, Corollary 3.10])**.** *With the above terminology,*

$$\mathfrak{L}_{g,m}^1 = \Psi_{g,m}(1, (g(1), 0 \ldots, 0)).$$

For example, when $m = 2$, the sequence $\bar{\mu}$ is given by

$$\mu_1 = (h, 0), \ \mu_2 = (h-1, 1), \ \mu_3 = (h-2, 3), \ \mu_4 = (h-3, 5), \ldots, \ \mu_{h+1} = (0, 2h-1),$$

and so $L = \mathfrak{L}_{g,2}^1 = h + 1$.

By the above discussion, it suffices to establish (3.3.1) to prove that there is a recursive algorithm that computes the value of $C_{h,m}^1$. We first prove that $C_{h,m}^1 \geqslant g(\mathfrak{L}_{g,m}^1) + 1$.

Proposition 3.3.3. *With $\bar{\mu}$ as above, we have $D_{h,\bar{\mu}} = g(\mathfrak{L}_{g,m}^1) + 1$ (see Section 3.1 for the definition of $D_{h,\bar{\mu}}$). In particular, $C_{h,m}^1 \geqslant g(\mathfrak{L}_{g,m}^1) + 1$.*

Proof. For each $i = 1, \ldots, L$, we let $\bar{\xi}_i$ be the antichain sequence $(\mu_1, \ldots, \mu_i)$. Recall that $L = \mathfrak{L}^1_{g,m}$. It suffices to prove

$$D_{h,\bar{\xi}_i} = \deg(\mu_i) + 1 \quad \text{for } i = 2, \ldots, L. \tag{3.3.3}$$

Indeed, if (3.3.3) holds, then taking $i = L$ yields $D_{h,\bar{\mu}} = \deg(\mu_L) + 1 = g(\mathfrak{L}^1_{g,m}) + 1$.

We now prove (3.3.3) by induction on i. We actually prove a little bit more: in addition to (3.3.3), we prove that for each pair of distinct elements $\mu_q, \mu_t \in \bar{\xi}_i$

$$\text{there are } \eta_1, \ldots, \eta_s \in \bar{\xi}_i \text{ all } < \text{LUB}(\mu_q, \mu_t) \text{ such that } \eta_1 = \mu_q, \eta_s = \mu_t$$

$$\text{and } \deg(\text{LUB}(\eta_\ell, \eta_{\ell+1})) \leqslant \deg(\mu_i) + 1 \text{ for } \ell = 1, \ldots, s - 1. \tag{3.3.4}$$

For the base case $i = 2$, note that

$$\bar{\xi}_2 = (\mu_1, \mu_2) = ((h, 0, \ldots, 0), (h - 1, 1, 0, \ldots, 0)),$$

so $\gamma(\bar{\xi}_2) = \{\text{LUB}(\mu_1, \mu_2)\} = \{(h, 1, 0 \ldots, 0)\}$. Since $\deg((h, 1, 0, \ldots, 0)) = h + 1$, we see that $D_{h,\bar{\xi}_2} = h + 1 = \deg(\mu_2) + 1$. To show condition (3.3.4) we simply take $\eta_1 = \mu_1$ and $\eta_2 = \mu_2$.

For the induction step we fix $3 \leqslant i \leqslant L$, and assume $D_{h,\bar{\xi}_{i-1}} = \deg(\mu_{i-1}) + 1$ and that condition (3.3.4) holds for $i - 1$. Since $\bar{\xi}_i$ is the concatenation of $\bar{\xi}_{i-1}$ and μ_i (with $\deg(\mu_i) = \deg(\mu_{i-1}) + 1 = D_{h,\bar{\xi}_{i-1}}$), we have that $D_{h,\bar{\xi}_i} \geqslant D_{r,\bar{\xi}_{i-1}} = \deg(\mu_i)$. It remains to show that $D_{h,\bar{\xi}_i} \neq \deg(\mu_i)$, that the integer $\deg(\mu_i) + 1$ satisfies condition ($\sharp'$) of Section 3.1, and that condition (3.3.4) holds. To do this we will prove that for any $q < i$ there exists $t < i$ such that

$$\mu_t < \text{LUB}(\mu_q, \mu_i) \text{ and } \deg(\text{LUB}(\mu_t, \mu_i)) = \deg(\mu_i) + 1, \tag{3.3.5}$$

and this will complete the proof. Indeed, suppose (3.3.5) holds, and set $\zeta = \mathrm{LUB}(\mu_t, \mu_i) \in \gamma(\bar{\xi}_i)$, where this t is the one associated to $q = 1$. Then, there exists $1 \leqslant k \leqslant m$ such that $\mu_i = \zeta - \mathbf{k}$, and so there cannot be $p < i$ such that $\mu_p \leqslant \zeta - \mathbf{k}$. Thus, this ζ witnesses the fact that $D_{h,\xi_i} \neq \deg(\mu_i)$. On the other hand, observe that if condition (3.3.4) holds then the integer $\deg(\mu_i) + 1$ satisfies condition ($\sharp'$). Thus, it would be enough to prove condition (3.3.4). To do this, let $\mu_p \rhd \mu_q$ be a pair of elements of $\bar{\xi}_i$. If $\mu_p, \mu_q \in \bar{\xi}_{i-1}$, then, by induction, there is a sequence with the desired properties. So now suppose $p = i$. By (3.3.5), there is $\mu_t \in \bar{\xi}_{i-1}$ such that $\mu_t < \mathrm{LUB}(\mu_p, \mu_q)$ and $\deg(\mathrm{LUB}(\mu_t, \mu_p)) \leqslant \deg(\mu_i) + 1$. Hence, in this case, the desired sequence can be obtained by starting with $\eta_1 = \mu_p$, $\eta_2 = \mu_t$, and then continuing with an appropriate sequence going from μ_t to μ_q (which exists by induction).

Finally, we prove (3.3.5). To do this, let $q < i$ and consider the two possible shapes that μ_i can take according to the construction of $\bar{\mu}$ above:

<u>Case 1.</u> Suppose $\mu_{i-1} = (u_1, \ldots, u_{m-1}, u_m)$ with $u_{m-1} > 0$. Then, by construction of $\bar{\mu}$,

$$\mu_i = (u_1, \ldots, u_{m-1} - 1, a), \quad \text{where } a = g(i) - u_1 - \ldots - u_{m-1} + 1.$$

Let $\mu_q = (v_1, \ldots, v_m)$ and $1 \leqslant l \leqslant m$ be the smallest integer such that the L-entry of μ_q is strictly larger than the L-entry of μ_i. Note that we must have $l < m$. Indeed, since $q < i$, the L-entry is the first entry (from left to right) where μ_q and μ_i differ. By construction of $\bar{\mu}$, we can find $t < i$ such that μ_t has the form $(u_1, \ldots, u_{l-1}, w_l, \ldots, w_m)$ with w_l equal to $1 + $ (the L-entry of μ_i), and w_p less than or equal to the p-entry of μ_i for $l < p \leqslant m$. Then $\mu_t < \mathrm{LUB}(\mu_q, \mu_i)$ and

$$\deg(\mathrm{LUB}(\mu_t, \mu_i)) = \deg(\mu_i) + 1.$$

<u>Case 2.</u> Suppose $\mu_{i-1} = (u_1, \ldots, u_s, 0, \ldots, 0, u_m)$ with $s < m - 1$ and $u_s > 0$. Then, by construction of $\bar{\mu}$,

$$\mu_i = (u_1, \ldots, u_s - 1, a, 0, \ldots, 0), \quad \text{where } a = g(i) - u_1 - \cdots - u_s + 1.$$

Let $\mu_q = (v_1, \ldots, v_m)$ and $1 \leqslant l \leqslant m$ be the smallest integer such that the L-entry of μ_q is strictly larger than the L-entry of μ_i. The same reasoning as in Case 1 yields that $l \leqslant s$. Again by construction of $\bar{\mu}$, we can find $t < i$ such that μ_t has the form $(u_1, \ldots, u_{l-1}, w_l, \ldots, w_s, w_{s+1}, 0, \ldots, 0)$ with w_l equal to $1 + (\text{the } L\text{-entry of } \mu_i)$, and w_p less than or equal to the p-entry of μ_i for $l < p \leqslant s + 1$. Then $\mu_t < \text{LUB}(\mu_q, \mu_i)$ and

$$\deg(\text{LUB}(\mu_t, \mu_i)) = \deg(\mu_i) + 1. \quad \square$$

It remains to show that $C^1_{h,m} \leqslant g(\mathfrak{L}^1_{g,m}) + 1$. To do this, suppose there is an antichain sequence $\bar{\xi} = (\xi_1, \ldots, \xi_M)$ of $\mathbb{Z}^m_{\geqslant 0}$ such that $D_{h,\bar{\xi}} \geqslant g(\mathfrak{L}^1_{g,m}) + 1$. We must show that then $D_{h,\bar{\xi}} \leqslant g(\mathfrak{L}^1_{g,m}) + 1$.

The following result gives the relationship between the Hilbert-Samuel functions of $\bar{\mu}$ and $\bar{\xi}$. This is where Corollary 3.2.9 is used.

Theorem 3.3.4. *With $\bar{\mu}$ and $\bar{\xi}$ as above, we have that*

$$H^i_{\bar{\xi}}(d) \leqslant H^i_{\bar{\mu}}(d)$$

for all $i, d \geqslant 0$. Consequently, $D_{h,\bar{\xi}} \leqslant g(\mathfrak{L}^1_{g,m}) + 1$.

Proof. We proceed by induction on i. For the base case $i = 0$, we have

$$H^0_{\bar{\xi}}(d) = \binom{m - 1 + d}{d} = H^0_{\bar{\mu}}(d),$$

which is the number of m-tuples of degree d.

Now for the induction step $i + 1$. Note that, since $D_{h,\bar{\xi}} \geqslant g(\mathcal{L}^1_{g,m}) + 1$, the sequence $\bar{\xi}$ contains at least two elements of degree at most h. It follows then that $H^1_{\bar{\xi}}(d) \leqslant H^1_{\bar{\mu}}(d)$ and $H^2_{\bar{\xi}}(d) \leqslant H^2_{\bar{\mu}}(d)$ for all $d \geqslant 0$. Thus, we assume that $i \geqslant 2$. We have that for $d < \deg(\mu_{i+1})$,

$$H^{i+1}_{\bar{\xi}}(d) \leqslant H^i_{\bar{\xi}}(d) \leqslant H^i_{\bar{\mu}}(d) = H^{i+1}_{\bar{\mu}}(d).$$

Now consider the case when $d = \deg(\mu_{i+1})$ (note that $d > 1$ since $h > 0$ and $i \geqslant 2$).

Claim. Either $H^{i+1}_{\bar{\xi}}(d) < H^i_{\bar{\xi}}(d)$ or $H^i_{\bar{\xi}}(d) < H^i_{\bar{\mu}}(d)$.

Proof of Claim. Towards a contradiction suppose

$$H^{i+1}_{\bar{\xi}}(d) = H^i_{\bar{\xi}}(d) = H^i_{\bar{\mu}}(d). \tag{3.3.6}$$

By the induction hypothesis, Lemma 3.3.1, and Macaulay's theorem (Theorem 3.2.1),

$$H^i_{\bar{\xi}}(d-1)^{\langle d-1 \rangle} \leqslant H^i_{\bar{\mu}}(d-1)^{\langle d-1 \rangle} = H^i_{\bar{\mu}}(d) = H^i_{\bar{\xi}}(d).$$

By Macaulay's theorem, this inequality implies that $H^i_{\bar{\xi}}(d) = H^i_{\bar{\xi}}(d-1)^{\langle d-1 \rangle}$. This equality, together with $H^{i+1}_{\bar{\xi}}(d) = H^i_{\bar{\xi}}(d)$, implies that $\deg(\xi_j) \neq d$ for all $j \leqslant i+1$ for which ξ_j is defined. This fact and Corollary 3.2.9 imply that

$$D_{h,\bar{\xi}} \leqslant d = \deg(\mu_{i+1}) < D_{h,\bar{\mu}}.$$

But this contradicts our assumption on $D_{h,\bar{\xi}}$, and so we have proven the claim.

Hence, either $H^{i+1}_{\bar{\xi}}(d) < H^i_{\bar{\xi}}(d)$ or $H^i_{\bar{\xi}}(d) < H^i_{\bar{\mu}}(d)$. Induction yields then that $H^{i+1}_{\bar{\xi}}(d) < H^i_{\bar{\mu}}(d)$, which implies that

$$H^{i+1}_{\bar{\xi}}(d) \leqslant H^i_{\bar{\mu}}(d) - 1 = H^{i+1}_{\bar{\mu}}(d),$$

as desired.

Now let $d \geqslant \deg(\mu_{i+1})$. By Macaulay's theorem

$$H^{i+1}_{\bar{\xi}}(d+1) \leqslant H^{i+1}_{\bar{\xi}}(d)^{\langle d \rangle}, \tag{3.3.7}$$

and

$$H^{i+1}_{\bar{\mu}}(d+1) = H^{i+1}_{\bar{\mu}}(d)^{\langle d \rangle}. \tag{3.3.8}$$

It then follows, by induction on $d \geqslant \deg(\mu_{i+1})$ and Lemma 3.3.1, that

$$H^{i+1}_{\bar{\xi}}(d)^{\langle d \rangle} \leqslant H^{i+1}_{\bar{\mu}}(d)^{\langle d \rangle}. \tag{3.3.9}$$

Thus, putting (3.3.7), (3.3.8), and (3.3.9) together, we get

$$H^{i+1}_{\bar{\xi}}(d+1) \leqslant H^{i+1}_{\bar{\mu}}(d+1),$$

and the result follows.

For the "consequently" clause, note that setting $i = L$ (recall $L = \mathfrak{L}^1_{g,m}$) and $d = \deg(\mu_L)$ yields

$$H^L_{\bar{\xi}}(\deg(\mu_L)) \leqslant H^L_{\bar{\mu}}(\deg(\mu_L)) = 0.$$

Thus, for every $\eta \in \mathbb{Z}^m_{\geqslant 0}$ with $\deg(\eta) = \deg(\mu_L)$ we have that $\eta \geqslant \xi_j$ for some $\xi_j \in \bar{\xi}$. This implies that $D_{h,\bar{\xi}} \leqslant \deg(\mu_L) + 1 = g(\mathfrak{L}^1_{g,m}) + 1$. $\qquad \square$

Recall the Ackermann function $A : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$, defined as

$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } x > 0 \text{ and } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{if } x, y > 0. \end{cases}$$

We can now conclude:

Theorem 3.3.5. *For all $h \geqslant 0$ we have*

$$C_{h,m}^1 = \mathfrak{L}_{g,m}^1 + h - 1.$$

In particular, if $h \geqslant 1$ then

$$C_{h,m}^1 = A(m - 1, C_{h-1,m}^1) \tag{3.3.10}$$

and

$$C_{h,m}^1 \leqslant A(m, h - 1) - 1,$$

and if $h \geqslant 2$ then

$$A(m, h - 2) \leqslant C_{h,m}^1$$

where A denotes the Ackermann function.

Proof. By the discussion above, all that is left to prove is the "in particular" clause. In [40, Proposition 1.1] it is shown that if $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ is a function of the form $f(i) = s + i - 1$, for some integer $s \geqslant 1$, then $\mathfrak{L}_{f,m}^1 = A(m, s - 1) - s$. Now, by Proposition 3.1.6, $C_{h,1}^1 = h$; on the other hand, $A(0, C_{h-1,1}^1) = C_{h-1,m}^1 + 1 = h$, so (3.3.10) holds when $m = 1$. Assume

$m > 1$. Observe that the antichain sequence $\bar{\mu}$ defined above has the form

$$(h, 0, \ldots, 0), (h - 1, 1, 0, \ldots, 0), (h - 1, 0, 2, 0, \ldots, 0), \ldots, (1, 0, \ldots, 0, C^1_{h-1,m} - 1),$$

$$(0, C^1_{h-1,m} + 1, 0, \ldots, 0), (0, C^1_{h-1,m}, 2, 0, \ldots, 0), \ldots, (0, 0, C^1_{h,m} - 1).$$

Again by [40, Proposition 1.1], the length of the sequence in the second line equals $A(m - 1, C^1_{h-1,m}) - C^1_{h-1,m} - 1$. Hence, the degree of the last tuple of the sequence equals $A(m - 1, C^1_{h-1,m}) - 1$. Consequently, $C^1_{h,m} = A(m - 1, C^1_{h-1,m})$, as desired.

Now consider the function $r : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ given by $r(i) = h + i - 1$. Then $g(i) \leqslant r(i)$ for all i, and so $\mathfrak{L}^1_{g,m} \leqslant \mathfrak{L}^1_{r,m} = A(m, h - 1) - h$. Hence,

$$C^1_{h,m} = \mathfrak{L}^1_{g,m} + h - 1 \leqslant A(m, h - 1) - 1.$$

For the second inequality consider the function $t(i) = h + i - 2$. Then $t(i) \leqslant g(i)$ for all i, and so $\mathfrak{L}^1_{g,m} \geqslant \mathfrak{L}^1_{t,m} = A(m, h - 2) - h + 1$. Hence,

$$C^1_{h,m} = \mathfrak{L}^1_{g,m} + h - 1 \geqslant A(m, h - 2). \quad \square$$

3.3.2 The case $n > 1$.

We now extend the results of the previous subsection to arbitrary $n \geqslant 1$. Let $h_1 := h$ and $g_1 : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ be defined as $g_1(1) = h$ and $g_1(i) = i + h - 2$ for $i \geqslant 2$. For $n > 1$, we define h_n and $g_n : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ recursively by

$$h_n := \mathfrak{L}^{n-1}_{g_{n-1},m} + h - (n - 1)$$

and

$$
g_n(i) = \begin{cases} g_{n-1}(i) & \text{if } i \leqslant \mathfrak{L}^{n-1}_{g_{n-1},m} \\ h_n & \text{if } i = \mathfrak{L}^{n-1}_{g_{n-1},m} + 1 \\ i + h_n - \mathfrak{L}^{n-1}_{g_{n-1},m} - 2 & \text{if } i \geqslant \mathfrak{L}^{n-1}_{g_{n-1},m} + 2 \end{cases}
$$

Note that $h_2 = \mathfrak{L}^1_{g_1,m} + h - 1 = C^1_{h,m}$ (by Theorem 3.3.5).

We will prove that

$$
C^n_{h,m} = \mathfrak{L}^n_{g_n,m} + h - n. \tag{3.3.11}
$$

This will imply that

$$
C^n_{h,m} = C^1_{C^{n-1}_{h,m},m} \quad \text{for } n \geqslant 2.
$$

In Proposition 3.1.6 we proved that (3.3.11) holds in the case $h = 0$ or $m = 1$. We now assume $h \geqslant 1$ and $m \geqslant 2$. In this case $\mathfrak{L}^n_{g_n,m} \geqslant \mathfrak{L}^{n-1}_{g_{n-1},m} + 2$, and so by definition of h_n and g_n we get

$$
g_n(\mathfrak{L}^n_{g_n,m}) + 1 = \mathfrak{L}^n_{g_n,m} + h_n - \mathfrak{L}^{n-1}_{g_{n-1},m} - 1 = \mathfrak{L}^n_{g_n,m} + h - n.
$$

Thus, to prove (3.3.11) it suffices to prove

$$
C^n_{h,m} = g_n(\mathfrak{L}^n_{g_n,m}) + 1. \tag{3.3.12}
$$

Let $\bar{\mu} = (\mu_1, \ldots, \mu_L)$ be the antichain sequence in $\mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ defined recursively as follows:

$$
\mu_1 = \max_{\trianglelefteq}\{\alpha \in \mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n} : \deg(\alpha) = g_n(1)\},
$$

and, as long as it is possible,

$$
\mu_i = \max_{\trianglelefteq}\{\alpha \in \mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n} : \deg(\alpha) = g_n(i) \text{ and } \alpha \not\trianglerighteq \mu_1, \ldots, \mu_{i-1}\}.
$$

In [31, §3.3], it is shown that $\bar{\mu}$ is an antichain sequence of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ having length $L = \mathfrak{L}_{g_n,m}^n$ (i.e., $\bar{\mu}$ is of maximal length among antichain sequences of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ with degree growth bounded by g_n). It is also observed that

$$H_{\bar{\mu}}^L(\deg(\mu_L)) = H_{\bar{\mu}}^L(g(L)) = 0,$$

where $H_{\bar{\mu}}^i$ denotes the Hilbert-Samuel function of $\bar{\mu}$, that is, for $i, d \geqslant 0$,

$$H_{\bar{\mu}}^i(d) = |\{\alpha \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n} : \deg(\alpha) = d \text{ and } \alpha \not\geqslant \mu_j \text{ for all } j \leqslant i \text{ for which } \mu_j \text{ is defined}\}|.$$

The antichain sequence $\bar{\mu}$ can be more explicitly constructed as follows:

Let $\bar{\mu}^{(1)}$ be the antichain sequence of maximal length with degree growth bounded by $f_1(i) := g_1(i)$ constructed in Section 3.3.1 inside of $\mathbb{Z}_{\geqslant 0}^m \times \{n\}$ (i.e., the n-th copy of $\mathbb{Z}_{\geqslant 0}^m$ in $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$). Let L_1 be the length of $\bar{\mu}^{(1)}$; in other words $L_1 = \mathfrak{L}_{f_1,m}^1$. Thus, $\bar{\mu}^{(1)}$ is of the form

$$((\mu_1^{(1)}, n), \dots, (\mu_{L_1}^{(1)}, n)).$$

Similarly, let $\bar{\mu}^{(2)}$ be the antichain sequence of maximal length with degree growth bounded by $f_2(i) := g_2(i + L_1)$ inside of $\mathbb{Z}_{\geqslant 0}^m \times \{n - 1\}$, and let L_2 be the length of $\bar{\mu}^{(2)}$ (that is, $L_2 = \mathfrak{L}_{f_2,m}^1$). Then,

$$\bar{\mu}^{(2)} = ((\mu_1^{(2)}, n - 1), \dots, (\mu_{L_2}^{(2)}, n - 1)).$$

Continuing in this fashion, we build $\bar{\mu}^{(j)}$ for $j = 3, \dots n$ as the antichain sequence of maximal length with degree growth bounded by

$$f_j(i) = g_j(i + L_1 + \dots + L_{j-1})$$

inside of $\mathbb{Z}_{\geqslant 0}^m \times \{n - j + 1\}$, and let L_j be the length of $\bar{\mu}^{(j)}$ (that is, $L_j = \mathfrak{L}_{f_j,m}^1$). Then the

sequence $\bar{\mu}$ is the concatenation of $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(n)}$; in particular, we get the following:

Proposition 3.3.6 ([31, Proposition 3.13])**.** *With the above terminology,*

$$\mathfrak{L}^n_{g_n,m} = \mathfrak{L}^1_{f_1,m} + \ldots + \mathfrak{L}^1_{f_n,m}.$$

Note that this implies that

$$\mathfrak{L}^n_{g_n,m} = \mathfrak{L}^{n-1}_{g_{n-1},m} + \mathfrak{L}^1_{f_n,m}. \tag{3.3.13}$$

From this construction of $\bar{\mu}$, one obtains the following recursive formula

$$\mathfrak{L}^n_{g_n,m} = \Psi_{f_1,m}(1, (f_1(1), 0, \ldots, 0)) + \ldots + \Psi_{f_n,m}(1, (f_n(1), 0, \ldots, 0)),$$

where $\Psi_{f_j,m}$ is defined as in (3.3.2) with f_j in place of g.

We now prove (3.3.12). First, we show that $C^n_{h,m} \geqslant g(\mathfrak{L}^n_{g_n,m}) + 1$.

Lemma 3.3.7. *With $\bar{\mu}$ as above, we have $D_{h,\bar{\mu}} = g_n(\mathfrak{L}^n_{g_n,m}) + 1$. In particular, $C^n_{h,m} \geqslant g_n(\mathfrak{L}^n_{g_n,m}) + 1$.*

Proof. We proceed by induction on n. The case $n = 1$ is given in Proposition 3.3.3. Assume it holds for $n - 1$. Then $D_{h,\bar{\mu}'} = g_{n-1}(\mathfrak{L}^{n-1}_{g_{n-1},m}) + 1 = h_n$, where $\bar{\mu}'$ is the concatenation of $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(n-1)}$. Since $\bar{\mu}$ is the concatenation of $\bar{\mu}'$ and $\bar{\mu}^{(n)}$, we have that $D_{h,\bar{\mu}} \geqslant D_{h,\bar{\mu}'}$. Thus, by Remark 3.1.3, $D_{h,\bar{\mu}} = D_{h_n,\bar{\mu}}$. It follows that $D_{h,\bar{\mu}} = D_{h_n,\bar{\mu}^{(n)}}$. Since $\deg\left(\mu_1^{(n)}\right) = h_n$, Proposition 3.3.3 (applied with $\bar{\mu}^{(n)}$ and h_n in place of $\bar{\mu}$ and h, respectively) yields

$$D_{h_n,\bar{\mu}^{(n)}} = \deg\left(\mu^{(n)}_{L_n}\right) + 1 = g_n(\mathfrak{L}^n_{g_n,m}) + 1,$$

as desired. $\square$

We now prove that $C_{h,m}^n \leqslant g_n(\mathfrak{L}_{g_n,m}^n) + 1$. To do this, suppose there is an antichain sequence $\bar{\alpha} = (\alpha_1, \ldots, \alpha_M)$ of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ such that $D_{h,\bar{\alpha}} \geqslant g_n(\mathfrak{L}_{g_n,m}^n) + 1$. We must show that then $D_{h,\bar{\alpha}} \leqslant g_n(\mathfrak{L}_{g_n,m}^n) + 1$.

We now establish the relationship between the Hilbert-Samuel functions of $\bar{\mu}$ and $\bar{\alpha}$. This is where we use the full potential of Lemma 3.3.1.

Theorem 3.3.8. *With $\bar{\mu}$ and $\bar{\alpha}$ as above, we have that*

$$H_{\bar{\alpha}}^i(d) \leqslant H_{\bar{\mu}}^i(d)$$

for all $i, d \geqslant 0$. Consequently, $D_{h,\bar{\alpha}} \leqslant g_n(\mathfrak{L}_{g_n,m}^n) + 1$.

Proof. First we make some observations. For any antichain sequence $\bar{\beta}$ of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ and each $1 \leqslant j \leqslant n$, we let $H_{\bar{\beta}}^{i,j}$ be the Hilbert-Samuel function of the subsequence of $\bar{\beta}$ consisting of its elements inside of $\mathbb{Z}_{\geqslant 0} \times \{n - j + 1\}$ (i.e., the $(n - j + 1)$-th copy of $\mathbb{Z}_{\geqslant 0}^m$ in $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$). Then

$$H_{\bar{\beta}}^i(d) = H_{\bar{\beta}}^{i,1}(d) + \ldots + H_{\bar{\beta}}^{i,n}(d). \tag{3.3.14}$$

By the construction of $\bar{\mu}$, we have that

$$H_{\bar{\mu}}^{i,j}(d) = H_{\bar{\mu}^{(j)}}^i(d).$$

Thus, if $\mathfrak{L}_{g_j,m}^j < i \leqslant \mathfrak{L}_{g_{j+1},m}^{j+1}$, for some $0 \leqslant j < n$, then for $d \geqslant \deg\left(\mu_{L_j}^{(j)}\right)$ we have

$$0 = H_{\bar{\mu}}^{i,0}(d) = \ldots = H_{\bar{\mu}}^{i,j}(d) \leqslant H_{\bar{\mu}}^{i,j+1}(d) \leqslant H_{\bar{\mu}}^{i,j+2}(d) = \ldots = H_{\bar{\mu}}^{i,n}(d), \tag{3.3.15}$$

where the last terms all equal $\binom{m-1+d}{d}$, the number of m-tuples of degree d. For the case $j = 0$, we are setting $\mathfrak{L}_{g_0,m}^0 = 0$, $\mu_{L_0}^{(0)} = (0, \ldots, 0)$, and $H_{\bar{\mu}}^{i,0}(d) = 0$.

We now go back to the proof of the theorem. We proceed by induction on i. For the

base case $i = 0$, we have

$$H^0_{\bar{\alpha}}(d) = n \cdot \binom{m-1+d}{d} = H^0_{\bar{\mu}}(d).$$

Now assume the inequality holds for $i \geqslant 0$. We prove it for $i + 1$. Note that, since $D_{h,\bar{\alpha}} \geqslant g_n(\mathfrak{L}^n_{g_n,m}) + 1$, the sequence $\bar{\alpha}$ contains at least two elements of degree at most h. It follows then that $H^1_{\bar{\alpha}}(d) \leqslant H^1_{\bar{\mu}}(d)$ and $H^2_{\bar{\alpha}}(d) \leqslant H^2_{\bar{\mu}}(d)$ for all $d \geqslant 0$. Thus, we assume $i \geqslant 2$.

We have that for $d < \deg(\mu_{i+1})$

$$H^{i+1}_{\bar{\alpha}}(d) \leqslant H^i_{\bar{\alpha}}(d) \leqslant H^i_{\bar{\mu}}(d) = H^{i+1}_{\bar{\mu}}(d).$$

Now consider the case $d = \deg(\mu_{i+1})$ (note that $d > 1$ since $h > 0$ and $i \geqslant 2$). Let $0 \leqslant j < n$ be such that $\mathfrak{L}^j_{g_j,m} < i + 1 \leqslant \mathfrak{L}^{j+1}_{g_{j+1},m}$. Note that then $d \geqslant \deg\left(\mu^{(j)}_{L_j}\right)$.

Claim. Either $H^{i+1}_{\bar{\alpha}}(d) < H^i_{\bar{\alpha}}(d)$ or $H^i_{\bar{\alpha}}(d) < H^i_{\bar{\mu}}(d)$.

Proof of Claim. Towards a contradiction suppose

$$H^{i+1}_{\bar{\alpha}}(d) = H^i_{\bar{\alpha}}(d) = H^i_{\bar{\mu}}(d). \tag{3.3.16}$$

By the induction hypothesis, Lemma 3.3.1 (which can be applied by (3.3.15)), and Macaulay's theorem (Theorem 3.2.1), we get

$$\sum_{k=1}^n H^{i,k}_{\bar{\alpha}}(d-1)^{\langle d-1 \rangle} \leqslant \sum_{k=1}^n H^{i,k}_{\bar{\mu}}(d-1)^{\langle d-1 \rangle} = H^i_{\bar{\mu}}(d) = H^i_{\bar{\alpha}}(d).$$

This inequality and Macaulay's theorem imply that

$$H^{i,k}_{\bar{\alpha}}(d) = H^{i,k}_{\bar{\alpha}}(d-1)^{\langle d-1 \rangle}$$

for $k = 1, \ldots, n$. These equalities, together with $H_{\bar{\alpha}}^{i+1}(d) = H_{\bar{\alpha}}^{i}(d)$, imply that $\deg(\alpha_s) \neq d$ for all $s \leqslant i + 1$ for which α_s is defined. This fact and Corollary 3.2.9 imply that

$$D_{h,\bar{\alpha}} \leqslant d = \deg(\mu_{i+1}) < D_{h,\bar{\mu}}.$$

However, this contradicts our assumption on $D_{h,\bar{\alpha}}$, and so we have proven the claim.

Hence, either

$$H_{\bar{\alpha}}^{i+1}(d) < H_{\bar{\alpha}}^{i}(d) \quad \text{or} \quad H_{\bar{\alpha}}^{i}(d) < H_{\bar{\mu}}^{i}(d).$$

Induction yields then that $H_{\bar{\alpha}}^{i+1}(d) < H_{\bar{\mu}}^{i}(d)$, which implies that

$$H_{\bar{\alpha}}^{i+1}(d) \leqslant H_{\bar{\mu}}^{i}(d) - 1 = H_{\bar{\mu}}^{i+1}(d),$$

as desired.

Now let $d \geqslant \deg(\mu_{i+1})$ (note that then $d \geqslant \deg\left(\mu_{L_j}^{(j)}\right)$). By Macaulay's theorem,

$$H_{\bar{\alpha}}^{i+1,k}(d + 1) \leqslant H_{\bar{\alpha}}^{i+1,k}(d)^{\langle d \rangle} \tag{3.3.17}$$

and

$$H_{\bar{\mu}}^{i+1,k}(d + 1) = H_{\bar{\mu}}^{i+1,k}(d)^{\langle d \rangle} \tag{3.3.18}$$

for $k = 1, \ldots, n$. It then follows, by induction on $d \geqslant \deg(\mu_{i+1})$ and using Lemma 3.3.1, that

$$H_{\bar{\alpha}}^{i+1,1}(d)^{\langle d \rangle} + \ldots + H_{\bar{\alpha}}^{i+1,n}(d)^{\langle d \rangle} \leqslant H_{\bar{\mu}}^{i+1,1}(d)^{\langle d \rangle} + \ldots + H_{\bar{\mu}}^{i+1,n}(d)^{\langle d \rangle}. \tag{3.3.19}$$

Thus, putting (3.3.14), (3.3.17), (3.3.18), and (3.3.19) together, we conclude

$$H_{\bar{\alpha}}^{i+1}(d + 1) \leqslant H_{\bar{\mu}}^{i+1}(d + 1),$$

and the result follows.

For the "consequently" clause, note that setting $i = L$ (recall $L = \mathfrak{L}^n_{g_n,m}$) and $d = \deg(\mu_L)$ yields

$$H^L_{\bar{\alpha}}(\deg(\mu_L)) \leqslant H^L_{\bar{\mu}}(\deg(\mu_L)) = 0.$$

Thus, for every $\beta \in \mathbb{Z}^m_{\geqslant 0} \times \mathfrak{n}$ with $\deg(\beta) = \deg(\mu_L)$ we have that $\beta \geqslant \alpha_j$ for some $\alpha_j \in \bar{\alpha}$. This implies that $D_{h,\bar{\alpha}} \leqslant \deg(\mu_L) + 1 = g_n(\mathfrak{L}^n_{g_n,m}) + 1$. $\qquad\square$

We can now conclude:

Theorem 3.3.9. *For all $h \geqslant 0$, we have*

$$C^n_{h,m} = \mathfrak{L}^n_{g_n,m} + h - n.$$

Consequently,

$$C^n_{h,m} = C^1_{C^{n-1}_{h,m},m} \quad \text{for } n \geqslant 2.$$

Proof. By the discussion above, all that is left to show is the "consequently" clause. Note that $f_n(1) = C^{n-1}_{h,m}$ and $f(i) = i + C^{n-1}_{h,m} - 2$ for $i \geqslant 2$. Thus, by Theorem 3.3.5, $C^1_{C^{n-1}_{h,m},m} = \mathfrak{L}^1_{f_n,m} + C^{n-1}_{h,m} - 1$. By (3.3.13), we thus have

$$C^n_{h,m} = \mathfrak{L}^n_{g_n,m} + h - n = \mathfrak{L}^1_{f_n,m} + \mathfrak{L}^{n-1}_{g_{n-1},m} + h - n = \mathfrak{L}^1_{f_n,m} + C^{n-1}_{h,m} - 1 = C^1_{C^{n-1}_{h,m},m}. \quad\square$$

Define the function $A_n : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ by

$$A_n(x,y) = \begin{cases} A(x, y-1) - 1 & \text{if } n = 1 \\ A(x, A_{n-1}(x,y) - 1) - 1 & \text{if } n > 1 \end{cases}$$

where A denotes the Ackermann function. We then have the following:

Corollary 3.3.10. *For all $h \geqslant 1$, we have*

$$C_{h,m}^n \leqslant A_n(m, h).$$

Additionally, if $h \geqslant 2$, then

$$A_n(m, h-1) + 1 \leqslant C_{h,m}^n.$$

Proof. We prove both inequalities by induction on n. The base case $n = 1$ is given by Theorem 3.3.5. Now suppose both inequalities are true for $n - 1$. Then, by induction and Theorems 3.3.5 and 3.3.9, we get

$$C_{h,m}^n = C_{C_{h,m}^{n-1},m}^1 \leqslant A(m, C_{h,m}^{n-1} - 1) - 1 \leqslant A(m, A_{n-1}(m, h) - 1) - 1 = A_n(m, h)$$

and, if $h \geqslant 2$,

$$C_{h,m}^n = C_{C_{h,m}^{n-1},m}^1 \geqslant A(m, C_{h,m}^{n-1} - 2) \geqslant A(m, A_{n-1}(m, h-1) - 1) = A_n(m, h-1) + 1. \quad \square$$

3.4   Specific values

In this section we provide some specific values for $C_{h,m}^n$ and compare this to what was previously known in terms of upper bounds for $T_{h,m}^n$.

1. For $m = 1$, by Proposition 3.1.6(2), $C_{h,1}^n = h$ for all n and h. Since every differential kernel with a single derivation has a regular realization (see [30, Proposition 3]), this is the exact value of $T_{h,1}^n$.

2. For $m = 2$, the previous bound yields

$$T_{h,2}^n \leqslant 2^{b_n + 1} h,$$

where b_n is given recursively by $b_0 = 0$ and $b_{i+1} = 2^{b_i+1}h + b_i + 1$; see [31, §3]. In particular,

$$T^1_{h,2} \leqslant 2^{2h+2}h \text{ and } T^2_{h,2} \leqslant 2^{2^{2h+2}h+2h+3}h.$$

On the other hand, we claim that our new bound yields

$$T^n_{h,2} \leqslant 2^n h,$$

which is a new and practical result. To see this, first observe that for the case when $n = 1$, by (3.3.10) of Theorem 3.3.5, $C^1_{h,2} = A(1, C^1_{h-1,2})$ for all h, so by induction on h, assuming $C^1_{h-1,2} = 2(h-1)$, we have

$$C^1_{h,2} = A(1, C^1_{h-1,m}) = A(1, 2h - 2) = 2h.$$

Note that in Proposition 3.1.6(3) we also proved that $C^1_{h,2} = 2h$ directly using the definition of $C^n_{h,m}$. For the case of $m > 2$, we use Theorem 3.3.9 and induction on n; the base case is $n = 1$, and so

$$C^n_{h,2} = C^1_{C^{n-1}_{h,2},2} = C^1_{2^{n-1}h,2} = 2(2^{n-1}h) = 2^n h.$$

3. For $m = 3$, up until now it was only known that

$$T^1_{1,3} \leqslant 2^{71} \text{ and } T^1_{2,3} \leqslant 2^{2^{2^{2^{520}+520}+2^{520}+521}};$$

see [31, Example 3.15]. We claim that our bound yields

$$T^1_{h,3} \leqslant 3(2^h - 1).$$

We show this by induction on h. The case $h = 0$ is given in Proposition 3.1.6(1).

Assume $C^1_{h-1,3} = 3(2^{h-1} - 1)$. Then, by (3.3.10) of Theorem 3.3.5,

$$C^1_{h,3} = A(2, C^1_{h-1,3}) = A(2, 3(2^{h-1} - 1)) = 2(3(2^{h-1} - 1)) + 3 = 3(2^h - 1).$$

4. So far no feasible upper bound was known for $m \geqslant 4$. Using (3.3.10) of Theorem 3.3.5, our bound yields

$$T^1_{1,4} \leqslant C^1_{1,4} = 5, \quad T^1_{1,5} \leqslant C^1_{1,5} = 13, \quad \text{and} \ \ T^1_{1,6} \leqslant C^1_{1,6} = 65533.$$

5. More generally (for arbitrary m), in [31], it was shown that

$$T^n_{h,m} < \begin{cases} 2A(m+3, 4h-1) & \text{when } n = 1 \\ \frac{2}{n} A(m+5, 4nh-1) & \text{when } n > 1. \end{cases} \tag{3.4.1}$$

Recall $A : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ denotes the Ackermann function:

$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } x > 0 \text{ and } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{if } x, y > 0. \end{cases}$$

The Ackermann function is known to have extremely large growth, especially in the first input. For example, $A(1, y) = y + 2$, $A(2, y) = 2y + 3$, $A(3, y) = 2^{y+3} - 3$, and

$$A(4, y) = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}_{y+3} - 3.$$

Thus, the upper bounds (3.4.1) are not computationally feasible, since the first input

is $m + 3$ when $n = 1$, and $m + 5$ when $n > 1$. On the other hand, by Corollary 3.3.10, our bound implies that

$$T^n_{h,m} \leqslant A_n(m, h),$$

where $A_n : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{> 0} \to \mathbb{Z}_{\geqslant 0}$ is an iterated Ackermann function given by

$$A_n(x, y) = \begin{cases} A(x, y - 1) - 1 & \text{if } n = 1 \\ A(x, A_{n-1}(x, y) - 1) - 1 & \text{if } n > 1. \end{cases}$$

This new upper bound is easier to work with, especially for small inputs. For example, $A_n(3, h)$ is a tower of exponentials in h, where the height of the tower is equal to n.

Chapter 4

Effective Differential Nullstellensatz

This chapter is focused on providing a new upper bound for the effective differential Null-stellensatz. It begins by showing several auxiliary results in Section 4.1; the key lemma in this section, Lemma 4.1.3, uses the value $T_{h,m}^n$ which was bounded in Chapter 3. The main result, Theorem 4.2.1, is contained in Section 4.2. This is continued with an analysis of our estimate for particular numbers of derivations in Section 4.3. In Section 4.4, a series of examples showing a new lower bound is given.

4.1 Preparation

As above, let $(\mathcal{K}, \Delta)$ be a differential field of characteristic zero with m commuting deriva-tions. In this chapter we study the effective differential Nullstellensatz. The *weak form of the differential Nullstellensatz* states that, for all $F \subseteq \mathcal{K}\{y_1, \ldots, y_n\}$, $1 \notin [F]$ if and only if, for all differentially closed fields $\mathcal{L} \supseteq \mathcal{K}$, there exists $(a_1, \ldots, a_n) \in \mathcal{L}^n$ such that, for all $f \in F$, $f(a_1, \ldots, a_n) = 0$. The *strong form of the differential Nullstellensatz* states that for all $F \subseteq \mathcal{K}\{y_1, \ldots, y_n\}$ and $g \in \mathcal{K}\{y_1, \ldots, y_n\}$, $g \in \{F\}$ if and only if, for all differentially closed fields $\mathcal{L} \supseteq \mathcal{K}$ and all $(a_1, \ldots, a_n) \in \mathcal{L}^n$ such that, for all $f \in F$, $f(a_1, \ldots, a_n) = 0$, we

have $g(a_1, \ldots, a_n) = 0$.

In this chapter we will set $R := \mathcal{K}\{y_1, \ldots, y_n\}$ to be the ring of differential polynomials on n differential indeterminates $y_1, \ldots, y_n$, and for all $h \geqslant 0$, we set

$$R_h := \mathcal{K}\big[\theta y_i : 1 \leqslant i \leqslant n, \theta \in \Theta, \mathrm{ord}(\theta) \leqslant h\big],$$

that is, R_h is the set of all differential polynomials of order at most h. In this chapter, we work with an arbitrary orderly ranking $<$ on R (note that $<$ is different than the partial order defined on $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ from before). Let $l \geqslant 0$ and $J \subset R_l$ be an ideal. For each $k \in \mathbb{Z}_{\geqslant 0}$, let $J^{(k)}$ be the ideal of the ring R_{l+k} generated by the derivatives of the elements of J up to order k (cf. [33]), that is,

$$J^{(k)} = \big(\theta g : g \in J, \, \mathrm{ord}(\theta) \leqslant k\big).$$

For $D \in \Theta$, let $J^{(D)}$ be the ideal of $R_{l+\mathrm{ord}(D)}$ generated by the derivatives of the elements of J not exceeding D in the given orderly ranking $<$, that is,

$$J^{(D)} = \big(\theta g : g \in J, \, \theta \leqslant D\big).$$

For every ideal J of the ring R_l, we let

$$J' = \sqrt{(\theta J : \mathrm{ord}\,\theta \leqslant 1)} \cap R_l.$$

We also let

$$\alpha_l = \binom{l+m}{m}.$$

Note that

$$\dim_{\mathcal{K}}(R_l) = n\alpha_l.$$

Lemma 4.1.1. *Let $J \subset R_l$ be an ideal, $p > 0$, and $(J')^p \subseteq J^{(1)}$. Then, for all $k \geqslant 0$,*

$$\sqrt{J'^{(k)}} \subseteq \sqrt{J^{(kp+1)}}.$$

Proof. Fix an orderly ranking on the ring of Δ-polynomials $\mathcal{K}\{y\}$. Let $D \in \Theta$, $p \in \mathbb{Z}_{\geqslant 0}$. Then there exist $\beta_l \in \mathcal{K}\{y\}$ and an element $c \in \mathbb{Q}$ such that

$$D^p\big(y^p\big) = c(D(y))^p + \sum_{\theta(l)y < Dy} \beta_l \theta(l) y. \tag{4.1.1}$$

Indeed, let $D = \partial_1^{i_1} \ldots \partial_m^{i_m} \in \Theta$ of order r. By the Leibniz rule, for every weight- and degree-homogeneous differential polynomial z, the differential polynomial ∂z is homogeneous of degree equal to $\deg z$ and of weight with respect to ∂ equal to that of z plus one. Hence,

$$D^p\big(y^p\big) = \sum_{\sum_{k=1}^p l_1^k = pi_1, \ldots, \sum_{k=1}^p l_m^k = pi_m} c_? \partial_1^{l_1^1} \ldots \partial_m^{l_m^1} y \cdot \ldots \cdot \partial_1^{l_1^p} \ldots \partial_m^{l_m^p} y, \tag{4.1.2}$$

where $c_?$ are some elements of $\mathcal{K}$. Consider a monomial in the right-hand side of (4.1.2). Suppose that it is of order greater than r in every differential indeterminate that appears in it. Then, for each k, $1 \leqslant k \leqslant p$, we have

$$l_1^k + \ldots + l_m^k > r.$$

Adding p inequalities, we obtain

$$pr = p(i_1 + \ldots + i_m) = \sum_{k=1}^p \sum_{t=1}^m l_t^k > pr,$$

which is a contradiction. Therefore, for each monomial in the right-hand side of (4.1.2), one

of the factors has order $\leqslant r$, and we have:

$$D^p\left(y^p\right) = \sum_{\substack{\sum_{k=1}^p l_1^k = pi_1, \ldots, \sum_{k=1}^p l_m^k = pi_m \\ \sum_{k=1}^m l_k^1 \geqslant r, \ldots, \sum_{k=1}^m l_k^p \geqslant r}} c_? \partial_1^{l_1^1} \ldots \partial_m^{l_m^1} y \cdot \ldots \cdot \partial_1^{l_1^p} \ldots \partial_m^{l_m^p} y + \sum_{l < r} \beta_l \theta(l) y, \qquad (4.1.3)$$

If, in a monomial from the first sum in (4.1.3), at least one of the factors had order greater than r, then, as in the above, by adding p inequalities, we would arrive at a contradiction. Thus, we obtain:

$$D^p\left(y^p\right) = \sum_{\substack{\sum_{k=1}^p l_1^k = pi_1, \ldots, \sum_{k=1}^p l_m^k = pi_m \\ \sum_{k=1}^m l_k^1 = r, \ldots, \sum_{k=1}^m l_k^p = r}} c_? \partial_1^{l_1^1} \ldots \partial_m^{l_m^1} y \cdot \ldots \cdot \partial_1^{l_1^p} \ldots \partial_m^{l_m^p} y + \sum_{l < r} \beta_l \theta(l) y, \qquad (4.1.4)$$

Let the orderly ranking on ΘY be such that $\partial_1 > \ldots > \partial_m$ and, in the first sum in (4.1.4), for one of the factors, we have $l_1^k > i_1$ for all k, $1 \leqslant k \leqslant p$. Adding these p inequalities, we obtain

$$pi_1 = \sum_{k=1}^p l_1^k > pi_1,$$

which gives a contradiction. Thus,

$$D^p\left(y^p\right) = \sum_{\substack{\sum_{k=1}^p l_1^k = pi_1, \ldots, \sum_{k=1}^p l_m^k = pi_m \\ \sum_{k=1}^m l_k^1 = r, \ldots, \sum_{k=1}^m l_k^p = r \\ l_1^1 = i_1, \ldots, l_1^p = i_1}} c_? \partial_1^{l_1^1} \ldots \partial_m^{l_m^1} y \cdot \ldots \cdot \partial_1^{l_1^p} \ldots \partial_m^{l_m^p} y + \sum_{\theta(l) y < Dy} \beta_l \theta(l) y,$$

As before, note that, for each monomial from the first sum, one cannot have $l_2^k > i_2$ for all k, $1 \leqslant k \leqslant p$. Thus in this sum, we are left with just the monomials of order $\leqslant r$ of the form

$$\partial_1^{i_1} \partial_2^{i_2} \partial_3^{l_3^1} \ldots \partial_m^{l_m^1} y \cdot \partial_1^{i_1} \partial_2^{i_2} \partial_3^{l_3^p} \ldots \partial_m^{l_m^p} y,$$

moving the rest of the monomials to the other sum. We now see that, distributing all

monomials between these two sums accordingly, we obtain that the first sum contains only one summand and, therefore, obtain (4.1.1).

We will prove the statement of the lemma now. By induction on k, we will show that, for all $D \in \Theta$ of order k,

$$\sqrt{J'(D)} \subseteq \sqrt{J^{(kp+1)}}.$$

By the definition of J', there exists $p \geqslant 1$ such that, for every $j' \in J'$,

$$j'^p = \sum_i \theta_i j_i, \quad j_i \in J, \quad \mathrm{ord}(\theta_i) \leqslant 1. \tag{4.1.5}$$

The base case is $k = 0$, so $D \in \Theta$ is of order 0. By the definition of J',

$$\sqrt{J'} = J' \subseteq \sqrt{J^{(1)}},$$

and the statement holds. Now let $k := \mathrm{ord}(D) > 0$ and suppose that, for all $D' < D$, we have

$$\sqrt{J'(D')} \subseteq \sqrt{J^{(kp+1)}}.$$

By (4.1.1) for $y = j'$, (4.1.5) implies

$$c\big(D(j')\big)^p + \sum_{D' < D} \beta_{D'} D' j' = D^p \left(\sum_i \theta_i j_i \right). \tag{4.1.6}$$

Since

$$D^p \left(\sum_i \theta_i j_i \right) \in J^{(kp+1)}$$

and, by the inductive hypothesis, for all $D' < D$, $D'j' \in \sqrt{J^{(kp+1)}}$ in (4.1.6), we have

$$\sqrt{J'(D)} \subseteq \sqrt{J^{(kp+1)}}. \quad \square$$

Lemma 4.1.2. *Let $s \geqslant 0$,*

$$I_0 = (F_0) \subseteq I_1 = (F_1) \subseteq \ldots \subseteq I_s = (F_s) \subseteq R_l$$

be ideals of R_l, $p_j \geqslant 0$, $0 \leqslant j \leqslant s$, and, for all i, $1 \leqslant i \leqslant s$,

$$I_i = \left(I_{i-1} \right)' \quad and \quad I_i^{p_i} \subseteq I_{i-1}^{(1)}.$$

Then, for all $q \in \mathbb{Z}_{\geqslant 0}$, there exists k such that

$$I_s^{(q)} \subseteq \sqrt{(F_0)^{(k)}} \quad and \quad k \leqslant 1 + p_1 + p_1 \cdot p_2 + \ldots + q \cdot p_1 \cdot \ldots \cdot p_s.$$

Proof. Let $g \in I_s^{(q)}$. Then $g \in I_{s-1}'^{(q)}$. Set $J = I_{s-1}$. Then

$$J^{(1)} = \left(F_{s-1}, \partial F_{s-1} : \partial \in \Delta \right), \quad J' = \sqrt{J^{(1)}} \cap R_l.$$

Applying Lemma 4.1.1 with $k = q$, we obtain

$$g \in \sqrt{I_{s-1}^{(qp_s+1)}}.$$

Again, by Lemma 4.1.1 with $k = qp_s + 1$ and $J = I_{s-1}$, we have

$$g \in \sqrt{I_{s-2}^{(p_{s-1}(qp_s+1)+1)}}.$$

Arguing similarly, we obtain

$$g \in \sqrt{I_0^{(1+\ldots+(1+(1+p_sq)p_{s-1})\ldots p_1)}} \subseteq \sqrt{(F_0)^{(1+\ldots+(1+(1+qp_s)p_{s-1})\ldots p_1)}}. \quad \square$$

Recall from Definition 2.2.13 that $T_{h,m}^n$ is the smallest integer $\geqslant h$ such that for any differential field $(\mathcal{K}, \Delta)$ of characteristic zero with m commuting derivations and any differential kernel $\mathcal{L}$ over $\mathcal{K}$ of length h, if $\mathcal{L}$ has a prolongation of length $T_{h,m}^n$, then $\mathcal{L}$ has a regular realization. Note that this means that $T_{h,m}^n$ is the smallest integer $\geqslant h$ such that if $\mathcal{K}\left(a_i^\theta : 1 \leqslant i \leqslant n,\ \mathrm{ord}(\theta) \leqslant T_{h,m}^n\right)$ is a differential kernel over $\mathcal{K}$, then there is a differential field extension $(\mathcal{M}, \partial_1', \ldots, \partial_m')$ of $(\mathcal{K}, \Delta)$ containing $\mathcal{K}\left(a_i^\theta : 1 \leqslant i \leqslant n,\ \mathrm{ord}(\theta) \leqslant h\right)$ such that for all $\mathcal{K}$, $1 \leqslant k \leqslant m$,

$$\partial_k' a_i^\theta = a_i^{\partial_k \cdot \theta}$$

whenever $\mathrm{ord}(\theta) \leqslant h-1$. In Chapter 3 we proved an upper bound for $T_{h,m}^n$ and an algorithm to compute that upper bound; see Theorem 3.1.4 and Theorem 3.3.9.

For all $F \subseteq R_h$, we let

$$I = \sqrt{(F)}, \quad T = T_{h,m}^n \,(\text{for } m > 1), \quad T = h+1 \,(\text{for } m = 1), \quad I_0 = \sqrt{I^{(T)}} \cap R_{T-1} \quad (4.1.7)$$

and

$$I_k = \sqrt{\left(g, \partial g : g \in I_{k-1}, \partial \in \Delta\right)} \cap R_{T-1} = \sqrt{I_{k-1}^{(1)}} \cap R_{T-1}. \qquad (4.1.8)$$

Lemma 4.1.3 (cf. [12, Proposition 4.1]). *If $1 \notin [F]$, then, for all $k \geqslant 1$ such that $I_k \neq R_{T-1}$,*

$$\dim(I_{k-1}) > \dim(I_k).$$

Proof. Suppose that

$$\dim(I_k) = \dim(I_{k-1}) \qquad (4.1.9)$$

for some $k \geqslant 1$. Fix such k. Since $I_{k-1} \subseteq I_k$, by (4.1.9), there exists a minimal prime component of I_k that is a minimal prime component of I_{k-1}. Pick such a component and

denote it by Q. Let P be a prime component of $\sqrt{I_{k-1}^{(1)}} \subseteq R_T$ such that

$$Q = P \cap R_{T-1}, \tag{4.1.10}$$

which exists by [4, Proposition 16, Section 2, Chapter II]. Let

$$R_T/P = \mathcal{K}\left[a_i^\theta : 1 \leqslant i \leqslant n, \ \mathrm{ord}(\theta) \leqslant T\right].$$

Then, by (4.1.10),

$$R_{T-1}/Q = \mathcal{K}[b], \quad b := \left(a_i^\theta : 1 \leqslant i \leqslant n, \ \mathrm{ord}(\theta) < T\right).$$

We will show that the field

$$\mathcal{L} = \mathcal{K}\left(a_i^\theta : 1 \leqslant i \leqslant n, \ \mathrm{ord}(\theta) \leqslant T\right)$$

is a differential kernel over $\mathcal{K}$. For this, it is sufficient to show that, if

$$f \in \mathcal{K}\left[x_i^\theta : 1 \leqslant i \leqslant n, \ \mathrm{ord}(\theta) \leqslant T - 1\right] \quad \text{and} \quad f(b) = 0,$$

then

$$\sum_{\substack{1 \leqslant i \leqslant n \\ \mathrm{ord}(\theta) \leqslant T-1}} \frac{\partial f}{\partial x_i^\theta}(b) a_i^{\partial_k \cdot \theta} + f^{\partial_k}(b) = 0 \tag{4.1.11}$$

(here f^{∂_k} is the polynomial obtained from f by applying ∂_k to its coefficients). Note that $Q^{(1)} \subseteq P$. Indeed, let J be the intersection of all minimal prime components of I_{k-1} not equal to Q, $h \in Q$, $\partial \in \Delta$, and $g \in J \setminus Q$ be such that $hg \in I_{k-1}$. By [26, Lemma 1.3, Chapter I],

$$g \cdot \partial h \in \sqrt{I_{k-1}^{(1)}} \subseteq P.$$

Since $g \notin Q$ and $g \in R_{T-1}$, $g \notin P$. Hence, $\partial h \in P$.

Finally, since $f(b) = 0$, $f \in Q$. Hence, $\partial f \in Q^{(1)} \subset P$, which implies (4.1.11). By the choice of T and Theorem 2.2.12, if $\mathcal{L}$ is a differential kernel over $\mathcal{K}$, $\text{Quot}(R/\{Q\})$ is a non-trivial extension of the differential field $(\mathcal{K}, \Delta)$, which contradicts $1 \in [F] \subseteq [I_k]$. $\qquad \square$

4.2 Upper bound

We can now state the main result of this chapter. This is the effective weak differential Nullstellensatz.

Theorem 4.2.1. *Let $h, D \geqslant 0$, $F \subset R_h$, $\deg(F) \leqslant D$. Then $1 \in [F]$ if and only if there exists $k \geqslant 0$ such that*

$$k \leqslant (n\alpha_{T-1}D)^{2^{O\left(n^3\alpha_T^3\right)}} \quad \text{and} \quad 1 \in (F)^{(k)},$$

where $\alpha_T = \binom{T+m}{m}$ and T is any function of m, n, and h for which the statement of Lemma 4.1.3 holds, for instance, $T = T_{h,m}^n$, defined as above.

Remark 4.2.2. If the statement of Lemma 4.1.3 is improved by finding a function that grows slower than $T_{h,m}^n$ such that the conclusion of the lemma still holds, one will not have to reprove Theorem 4.2.1 to have the correspondingly improved bound.

Proof. If $1 \in (F)^{(k)}$, then $1 \in [F]$ by definition. We will now show the reverse implication. Let $s = \dim(Z(F))$ and also

$$a := n\alpha_{T-1}, \quad b := n\alpha_T, \quad c := O\left(n^2\alpha_{T-1}^2\right), \tag{4.2.1}$$

where the assignment of c in the above is simply a way of shortening formulas below and is to be treated as just a replacement of the O-expression by the symbol c. Then, by [23,

Proposition 2.3],

$$\deg(Z(F)) = \deg(Z(I)) \leqslant D^{n\alpha_h}.$$

Hence, by [9, Proposition 4], the ideal I (as well as the ideal $I^{(T)}$, see (4.1.8)) can be generated by polynomials of degree at most

$$\left(n\alpha_h D^{n\alpha_h}\right)^{2^{O(sn\alpha_h)}} \leqslant (n\alpha_h)^{2^{O\left(n^2\alpha_h^2\right)}} D^{n\alpha_h \cdot 2^{O\left(n^2\alpha_h^2\right)}} = (n\alpha_h D)^{2^{O\left(n^2\alpha_h^2\right)}} =: d_F.$$

Then

$$\deg(Z(I_0)) \leqslant d_F^{n\alpha_T} = (n\alpha_h D)^{b2^{O\left(n^2\alpha_h^2\right)}} =: D_0$$

and the ideal I_0 can be generated by polynomials of degrees at most

$$(aD_0)^{2^c} = (a)^{2^c} (n\alpha_h D)^{b2^{O\left(n^2\alpha_h^2\right)+c}} =: d_0.$$

Moreover, by [25, Theorem 1.3],

$$\sqrt{I^{(T)}}^{p_0} \subseteq I^{(T)}, \quad p_0 := d_F^b.$$

Hence,

$$I_0^{p_0} \subseteq I^{(T)} \cap R_{T-1}.$$

Continuing this way, we obtain that

$$\deg(Z(I_{i+1})) \leqslant d_i^{n\alpha_T} =: D_{i+1}$$

and the ideal I_{i+1} can be generated by polynomials of degrees at most

$$d_{i+1} := a^{2^c}(d_i)^{b2^c} = a^{2^c}\left(a^{2^c}d_{i-1}^{b2^c}\right)^{b2^c} = a^{2^c+b2^{2c}}d_{i-1}^{b^2 2^{2c}}$$
$$= a^{2^c+b2^{2c}}\left(a^{2^c}d_{i-2}^{b2^c}\right)^{b^2 2^{2c}} = a^{2^c+b2^{2c}+b^2 2^{3c}}d_{i-2}^{b^3 2^{3c}}.$$

Therefore,

$$d_{i+1} = a^{2^c\sum\limits_{j=0}^{q}(b2^c)^j}d_{i-q}^{(b2^c)^{q+1}} = a^{2^c\sum\limits_{j=0}^{i}(b2^c)^j}d_0^{(b2^c)^{i+1}}$$
$$= a^{2^c\frac{(b2^c)^{i+1}-1}{b2^c-1}}d_0^{(b2^c)^{i+1}} \leqslant \left(a^{2^c}d_0\right)^{(b2^c)^{i+1}}$$

and

$$D_{i+1} \leqslant \left(a^{2^c}d_0\right)^{b(b2^c)^i}.$$

Again, by [25, Theorem 1.3],

$$\sqrt{I_i^{(1)}}^{p_{i+1}} \subseteq I_i^{(1)}, \quad p_{i+1} := \left(a^{2^c}d_0\right)^{b(b2^c)^i} \geqslant d_i^b, \quad i \geqslant 0.$$

Hence,

$$I_{i+1}^{p_{i+1}} \subseteq I_i^{(1)} \cap R_{T-1}.$$

By Lemma 4.1.3, since $\dim(R_{T-1}) = a$, $1 \in I_a$. By Lemma 4.1.2 applied to (4.1.8),

$$1 \in I_0^{(1+p_1+p_1\cdot p_2+...+p_1\cdot...\cdot p_a)}.$$

Again by Lemma 4.1.2, for all $q \geqslant 0$,

$$I_0^{(q)} \subseteq \sqrt{I^{(T+1+qp_0)}}.$$

Hence,

$$1 \in I^{(T+1+p_0(1+p_1+p_1 \cdot p_2+...+p_1 \cdot ... \cdot p_a))}.$$

By Lemma 4.1.1 applied to (4.1.7), we obtain

$$1 \in (F)^{(1+p_0(T+1+p_0(1+p_1+p_1 \cdot p_2+...+p_1 \cdot ... \cdot p_a)))}$$

$$= (F)^{(1+p_0(T+1+...(1+(1+p_a)p_{a-1})...p_0))} = (F)^{(p_0(T+p_0 \cdot ... \cdot p_a))}, \qquad (4.2.2)$$

with the latter equality following from the definition of c via the O-symbol. Note that

$$p_0^2 p_1 \cdot ... \cdot p_a = d_F^{2b} \left(a^{2^c} d_0\right)^{b \sum_{j=0}^{a-1} (b2^c)^j} = d_F^{2b} \left(a^{2^c} d_0\right)^{b\left(\frac{(b2^c)^a - 1}{b2^c - 1} - 1\right)}$$

$$\leqslant d_F^{2b} \left(a^{2^c} d_0\right)^{b(b2^c)^a} = d_F^{2b} \left(a^2 d_F^b\right)^{(b2^c)^{a+1}}$$

$$\leqslant (ad_F)^{b\left((b2^c)^{a+1}+2\right)} = (ad_F)^{2^{cb}} = a^{2^{cb}} (n\alpha_h D)^{2^{O\left(n^2 \alpha_h^2\right)+cb}}$$

$$= a^{2^{cb}} (n\alpha_h D)^{2^{cb}} = (aD)^{2^{cb}},$$

(the equalities hold by the O-definition of c and because $\alpha_{T-1} \geqslant 1$ if $h \geqslant 1$ and $k = a = 0$ if $h = 0$) and the result follows by substituting (4.2.1) in the above and using (4.2.2). □

Using the Rabinowitz trick, we obtain the effective strong differential Nullstellensatz.

Corollary 4.2.3. *(cf. [9, Corollary 21]) Let $h, D \geqslant 0$, $F \subset R_h$, $f \in R_h$, and*

$$\max\{\deg(f), \deg(F)\} \leqslant D.$$

Then $f \in \{F\}$ if and only if there exists $k \geqslant 0$ such that

$$k \leqslant (n\alpha_{T'-1} D)^{2^{O\left(n^3 \alpha_{T'}^3\right)}} \quad and \quad f \in \sqrt{(F)^{(k)}},$$

where $T' := T_{h,m}^{n+1}$.

Proof. If $f \in \sqrt{(F)^{(k)}}$, then $f \in \{F\}$ by definition. Let $f \in \{F\}$. Then $1 \in [1 - tf, F] \subseteq \mathcal{K}\{y_1 \ldots, y_n, t\}$. By Theorem 4.2.1,

$$1 \in \left((1 - tf)^{(k)}, F^{(k)} \right),$$

for which we used the properties of O to go down from $D + 1$ (which appears because $\deg(tf) = \deg(f) + 1$) to D and from $n + 1$ to n outside of T'. As usual, by substituting $1/f$ into t and clearing out the denominators, we obtain the result. $\qquad\square$

Remark 4.2.4. Note that, for $m \geqslant 2$, $T \neq O(T')$ (see Section 4.3), and so we do not replace T' by T in the corollary. However, for $m = 1$, we simply have $T' = T = h + 1$.

4.3 Concrete values of the number of derivations

Recall that, if $m = 1$, then $T = h + 1$. Then the bound from Theorem 4.2.1 is

$$(n(h + 1)D)^{2^{O\left(n^3 (h+2)^3 \right)}}$$

and is better than the bound from [9, Corollary 19], because our result holds for non-constant coefficients.

If $m = 2$, by Theorem 3.3.9, $T = T(n, h) \leqslant 2^n h$. Therefore, in this case, the bound from Theorem 4.2.1 is

$$\left(n2^{n-1} h(2^n h + 1)D \right)^{2^{O\left(n^3 2^{6n} h^6 \right)}}.$$

By Section 3.3.1, when there is only one differential indeterminate ($n = 1$), then $T = T(m, h) \leqslant A(m, h - 1) - 1$. Hence, for a small number of derivations, the value of T is quite manageable. For example, by Section 3.4(3), if $m = 3$ and $n = 1$, then $T = T(h) \leqslant 3(2^h - 1)$. As a result, in this case, the bound from Theorem 4.2.1 is triple-exponential in h.

For comparison, note that the bound from [15, Theorem 1], $A(m + 8, \max(n, h, d))$, has a substantially higher growth rate, as, for example, $A(3, x)$ is exponential in x and $A(4, x)$ is a tower of exponentials of length $x + 3$, and the minimal possible value here, $A(9, 1)$, is out of reach for any existing computer even to output.

4.4 Lower bound

The examples in [15] show that the lower bound for the effective differential Nullstellensatz is exponential in the number of variables and the number of derivations and polynomial in the degree of the system. We expand on these results, first by observing how the order of the system affects the lower bound.

Example 4.4.1. Consider the system

$$F = \left\{ y_1^d, y_1 - y_2^d, \ldots, y_{n-1} - y_n^d, 1 - y_n^{(h)} \right\} \subseteq \mathcal{K}\{y_1, \ldots, y_n\} =: R$$

with one derivation. A particular and essential case of this, $h = 1$, was considered in an unpublished manuscript by York Kitajima, and the argument in the present example is based on Kitajima's argument and extends it, with extra subtleties. Recall that for $s \geqslant 2$ and $m, m_1, \ldots, m_s \in \mathbb{Z}_{\geqslant 0}$ with $m_1 + \ldots + m_s = m$, the multinomial coefficient is

$$\binom{m}{m_1, \ldots, m_s} = \frac{m!}{m_1! \cdot \ldots \cdot m_s!}.$$

For $l \geqslant 1$, denote by M_l the multinomial coefficient

$$M_l = \binom{d^l h}{d^{l-1}h, \ldots, d^{l-1}h},$$

where this multinomial coefficient contains d terms. We claim that $(F)^{(j)} \subseteq I_j$ where

$$I_0 = \left(y_1, y_2, \ldots, y_{n-1}, y_n, 1 - y_n^{(h)}\right)$$

$$I_j = \left(I_{j-1}, y_1^{(j)}, y_2^{(j)}, \ldots, y_{n-1}^{(j)}, y_n^{(j)}, y_n^{(h+j)}\right) \qquad\qquad 1 \leqslant j \leqslant h-1$$

$$I_j = \left(I_{j-1}, y_1^{(j)}, \ldots, y_{n-i}^{(j)} - \prod_{l=1}^{i} M_l^{d^{i-l}}, y_{n-i+1}^{(j)}, \ldots, y_{n-1}^{(j)}, y_n^{(h+j)}\right) \quad j = d^i h, 1 \leqslant i \leqslant n-1$$

$$I_j = \left(I_{j-1}, y_1^{(j)}, \ldots, y_{n-1}^{(j)}, y_n^{(h+j)}\right) \qquad\qquad \text{otherwise, } j \leqslant d^n h - 1.$$

Indeed, we can show this by induction on j. The base case $j = 0$ is clear. Now assume $(F)^{(k)} \subseteq I_k$ for all $k < j$. By induction, we only need to show the inclusion of unmixed monomials, i.e. powers of a single derivative of a y_i. The generalized Leibniz rule says that for all $s \geqslant 1$, $m \geqslant 0$, and $f_1, \ldots, f_s \in R$,

$$\left(\prod_{r=1}^{s} f_r\right)^{(m)} = \sum_{m_1 + \ldots + m_s = m} \binom{m}{m_1, \ldots, m_s} \prod_{r=1}^{s} f_r^{(m_r)}.$$

In our system F, we have $s = d$. Unmixed monomials thus occur when $m_1 = \ldots = m_d = m/d$. When $j \neq d^i h$, $y_i^{(j/d)} \in I_j$ for all i, $1 \leqslant i \leqslant n$, so there is nothing to prove. The case we must consider is when $j = d^i h$, in which case each m_α in the multinomial coefficient is $d^{i-1}h$ and $y_{n-i}^{(d^i h)} \notin I_{d^i h}$.

It remains to show that $\left(y_{n-i} - y_{n-i+1}^d\right)^{(d^i h)} \in I_{d^i h}$, since $y_{n-i+1}^{(d^{i-1}h)} \notin I_{d^{i-1}h}$ by construction. Observe that

$$\left(y_{n-i} - y_{n-i+1}^d\right)^{(d^i h)} = y_{n-i}^{(d^i h)} - M_i \left(y_{n-i+1}^{(d^{i-1}h)}\right)^d + g,$$

where $g \in \mathcal{K}\{y\}$ contains no unmixed monomials, and so is in $I_{d^i h}$. Thus, it suffices to show

$$y_{n-i}^{(d^i h)} - M_i \left(y_{n-i+1}^{(d^{i-1}h)}\right)^d \in I_{d^i h}.$$

By construction,

$$y_{n-i+1}^{(d^{i-1}h)} - \prod_{l=1}^{i-1} M_l^{d^{i-l-1}} \in I_{d^{i-1}h} \subseteq I_{d^ih}.$$

We can thus write

$$y_{n-i}^{(d^ih)} - M_i \left(y_{n-i+1}^{(d^{i-1}h)} \right)^d =$$

$$\left(y_{n-i}^{(d^ih)} - \prod_{l=1}^{i} M_l^{d^{i-l}} \right) - M_i \sum_{\alpha=0}^{d-1} \left[\left(\prod_{l=1}^{i-1} M_l^{d^{i-l-1}} \right)^{\alpha} \left(y_{n-i+1}^{(d^{i-1}h)} \right)^{d-1-\alpha} \right] \left(y_{n-i+1}^{(d^{i-1}h)} - \prod_{l=1}^{i-1} M_l^{d^{i-l-1}} \right)$$

in terms of elements of I_{d^ih}, completing the induction step and proving that $F^{(j)} \subseteq I_j$ for all j, $1 \leqslant j \leqslant d^nh - 1$.

Since $1 \notin I_j$ for all j, $0 \leqslant j \leqslant d^nh - 1$, then $1 \notin (F)^{(d^nh-1)}$. Observe that

$$\left(y_n^{d^n} \right)^{(d^nh)} = \left(\left(y_n^{d^n} \right)^{(h)} \right)^{((d^n-1)h)}$$

$$= \left(\left(\left(\sum_{n_{1,1}+...+n_{1,d^n}=h} \binom{h}{n_{1,1}, \ldots, n_{1,d^n}} \prod_{i=1}^{d^n} y_n^{(n_{1,i})} \right)^{(h)} \right)^{((d^n-2)h)} \right.$$

$$= \left(\left(\sum_{n_{1,1}+...+n_{1,d^n}=h} \sum_{n_{2,1}+...+n_{2,d^n}=h} \binom{h}{n_{1,1}, \ldots, n_{1,d^n}} \binom{h}{n_{2,1}, \ldots, n_{2,d^n}} \prod_{i=1}^{d^n} y_n^{(n_{1,i}+n_{2,i})} \right)^{(h)} \right)^{((d^n-3)h)}$$

$$= \ldots = \sum_{n_{1,1}+...+n_{1,d^n}=h} \ldots \sum_{n_{d^n,1}+...+n_{d^n,d^n}=h} \left(\prod_{j=1}^{d^n} \binom{h}{n_{j,1}, \ldots, n_{j,d^n}} \prod_{i=1}^{d^n} y_n^{(n_{1,i}+...+n_{d^n,i})} \right).$$

Since $y_n^{(h)} \equiv 1$ modulo the system F, then $y_n^{(l)} \equiv 0$ for all $l > h$, so the only non-zero terms in this sum will be powers of $y_n^{(h)}$. We thus have that, modulo F, $\left(y_n^{d^n} \right)^{(d^nh)} \equiv 1$, so $1 \in (F)^{(d^nh)}$ and $1 \notin (F)^{(d^nh-1)}$.

Example 4.4.2. Consider the following collections of differential polynomials in $\mathcal{K}\{y_1, \ldots, y_n\}$ with derivatives $\Delta = \{\partial_1, \ldots, \partial_m\}$, with $d, h \geqslant 1$:

$$G_1 = \left\{ (\partial_1 y_1)^d, \partial_1 y_1 - (\partial_2 y_1)^d, \ldots, \partial_{m-1} y_1 - (\partial_m y_1)^d \right\}$$

$$G_i = \left\{ \partial_m y_{i-1} - (\partial_1 y_i)^d, \partial_1 y_i - (\partial_2 y_i)^d, \ldots, \partial_{m-1} y_i - (\partial_m y_i)^d \right\} \quad 2 \leqslant i \leqslant n-1$$

$$G_n = \left\{ \partial_m y_{n-1} - (\partial_1 y_n)^d, \partial_1 y_n - (\partial_2 y_n)^d, \ldots, \partial_{m-1} y_n - (\partial_m y_n)^d, 1 - \partial_m^{h+1} y_n \right\}.$$

Similar to what is done in [15], if we replace F in the previous example by $G = \bigcup_{i=1}^n G_i$, then the elements of G will need to be differentiated a minimum of $d^{mn}h$ times in order to reduce the system to 1, so $1 \in (G)^{(d^{mn}h)}$ and $1 \notin (G)^{(d^{mn}h-1)}$.

In these examples, the lower bound for having $f \in (G)^{(k)}$ is exponential in the number of derivations and number of variables and linear in the order of the system. The systems of partial differential equations presented in these examples are non-linear. The existence of a lower bound for linear systems that is double-exponential in the number of derivations m is shown in [38]. It is currently unknown how to combine this result with the non-linear examples presented here to produce a lower bound that more closely resembles the current upper bound.

We now present an alternative approach, using the lower bound on the effective polynomial Nullstellensatz, to construct an example of a linear system $G \subseteq \mathcal{K}\{y_1, \ldots, y_n\}$ with $f \in (G)^{(k)}$ but $f \notin (G)^{(k-1)}$, where $\mathcal{K}$ is exponential in the number of derivations and the number of variables and polynomial in the order of the system. We believe that this approach can be extended to the case of non-linear systems to produce better lower bounds.

We use a system of polynomials to construct a system of differential polynomials. We begin with polynomials in $\mathcal{K}[X_1, \ldots, X_m]$ and construct differential polynomials in $\mathcal{K}\{y\}$ with derivations $\Delta = \{\partial_1, \ldots, \partial_m\}$, where $\mathcal{K}$ is constant with respect to each ∂_i. Given $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_{\geqslant 0}^m$, denote $X^\alpha = X_1^{\alpha_1} \cdot \ldots \cdot X_m^{\alpha_m}$ and $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_m^{\alpha_m}$.

Suppose we have $f_1, \ldots, f_r \in \mathcal{K}[X_1, \ldots, X_m]$. For each i, $1 \leqslant i \leqslant r$, there exist $\alpha_{i,1}, \ldots, \alpha_{i,N_i} \in \mathbb{Z}_{\geqslant 0}^m$ and $c_{i,1}, \ldots, c_{i,N_i} \in \mathcal{K}$ such that

$$f_i = \sum_{j=1}^{N_i} c_{i,j} X^{\alpha_{i,j}}.$$

We then define $\tilde{f}_i \in \mathcal{K}\{y\}$ to be

$$\tilde{f}_i = \sum_{j=1}^{N_i} c_{i,j} \partial^{\alpha_{i,j}} y.$$

Similarly, given $f = \sum_{j=1}^{N} s_j X^{\gamma_j} \in \mathcal{K}[X_1, \ldots, X_m]$, we can define $\tilde{f} = \sum_{j=1}^{N} s_j \partial^{\gamma_j} y \in \mathcal{K}\{y\}$. Consider the system $G = \{\tilde{f}_1, \ldots, \tilde{f}_r\}$.

Theorem 4.4.3. *Let $f, f_1, \ldots, f_r \in \mathcal{K}[X_1, \ldots, X_m]$, $\tilde{f}, \tilde{f}_1, \ldots, \tilde{f}_r \in \mathcal{K}\{y\}$, and $G \subseteq \mathcal{K}\{y\}$ be defined as above. Suppose $f \in (f_1, \ldots, f_r)$ and let k be the lower bound for the degree of the coefficients of the f_i in any possible representation of f. Then $\tilde{f} \in (G)^{(k)}$ but $\tilde{f} \notin (G)^{(k-1)}$.*

Proof. Suppose $f \in (f_1, \ldots, f_r)$, so there exist $g_1, \ldots, g_r \in \mathcal{K}[X_1, \ldots, X_m]$ such that $f = g_1 f_1 + \ldots + g_r f_r$. As with each f_i, there exist $\beta_{i,j} \in \mathbb{Z}_{\geqslant 0}^m$ and $d_{i,j} \in \mathcal{K}$, $j = 1, \ldots, M_i$, such that we can write each g_i as

$$g_i = \sum_{j=1}^{M_i} d_{i,j} X^{\beta_{i,j}}.$$

It is then easy to see that

$$\sum_{j=1}^{M_1} d_{1,j} \partial^{\beta_{1,j}} \left(\tilde{f}_1 \right) + \ldots + \sum_{j=1}^{M_r} d_{r,j} \partial^{\beta_{r,j}} \left(\tilde{f}_r \right) = \sum_{j=1}^{N} s_j \partial^{\gamma_j} y = \tilde{f}. \tag{4.4.1}$$

Since $G = \{\tilde{f}_1, \ldots, \tilde{f}_r\}$, we thus have that $\tilde{f} \in [G]$, and since the maximum degree of each g_i is k, the maximum order of each $\partial^{\beta_{i,j}}$ is also k, so $\tilde{f} \in (G)^{(k)}$.

It remains to show that $\tilde{f} \notin (G)^{(k-1)}$. Suppose for a contradiction we have $\tilde{f} \in (G)^{(l)}$ for

some $l < k$, so we can write

$$\tilde{f} = \sum_{j=1}^{K_1} \alpha_{1,j}(y)\partial^{\sigma_{1,j}}\left(\tilde{f}_1\right) + \ldots + \sum_{j=1}^{K_r} \alpha_{r,j}(y)\partial^{\sigma_{r,j}}\left(\tilde{f}_r\right) \qquad (4.4.2)$$

where the $\alpha_{i,j} \in \mathcal{K}\{y\}$ and $\operatorname{ord}\partial^{\sigma_{i,j}} \leqslant l < k$.

To complete the proof, we need the following fact about systems of homogeneous degree 1 polynomials. Suppose $p, p_1, \ldots, p_s \in \mathcal{K}[X_1, \ldots, X_n]$ are homogeneous degree 1 polynomials. If there exist $q_1, \ldots, q_s \in \mathcal{K}[X_1, \ldots, X_n]$ such that $p = q_1 p_1 + \ldots + q_s p_s$, then we can in fact assume that all of the q_i are constant. Indeed, write $p = a_1 X_1 + \ldots + a_n X_n$. Assume without loss of generality that $a_n \neq 0$. Since $p = q_1 p_1 + \ldots + q_s p_s$, then

$$X_n = q_0 + \frac{q_1}{a_n} p_1 + \ldots + \frac{q_s}{a_n} p_s, \quad q_0 := -\frac{a_1}{a_n} X_1 - \ldots - \frac{a_{n-1}}{a_n} X_{n-1}.$$

Thus, it suffices to prove the result when $p = X_n$.

For this, order the variables so that $X_1 > \ldots > X_n$. Applying Gauss-Jordan elimination to the system $\{p_i = 0\}$, we obtain a new system $\{p'_i = 0\}$ that is in reduced row echelon form. Moreover, every p'_i is a linear combination of $p_1, \ldots, p_r$ (with coefficients in $\mathcal{K}$) and vice versa. There are two cases to consider. If X_n is a leading variable in $\{p'_i = 0\}$, then because of the ordering on the X_i, we must have in fact that X_n is one of the p'_i, so the proof is complete. Therefore, suppose X_n is not a leading variable of $\{p'_i = 0\}$. By assumption, $X_n \in (p_1, \ldots, p_s) = (p'_1, \ldots, p'_s)$. This implies that for every solution $(\alpha_1, \ldots, \alpha_n)$ of the system $\{p_i = 0\}$ (or equivalently $\{p'_i = 0\}$), $\alpha_n = 0$. Thus, X_n cannot be a free variable of $\{p'_i = 0\}$, since there is a solution of the system $\{p'_i = 0\}$ for every possible value of any free variable (provided that a solution exists, which in this case is true, given by $(0, \ldots, 0)$).

Now, since the $\partial^{\gamma_j} y$ and $\partial^{\sigma_{i,j}}(\tilde{f}_i)$ in (4.4.2) are all homogeneous of degree 1, by the above

discussion, we can assume that the $\alpha_{i,j}$ are all constants $b_{i,j} \in \mathcal{K}$, so we obtain

$$\tilde{f} = \sum_{j=1}^{K_1} b_{1,j} \partial^{\sigma_{1,j}} \left(\tilde{f}_1 \right) + \ldots + \sum_{j=1}^{K_r} b_{r,j} \partial^{\sigma_{r,j}} \left(\tilde{f}_r \right). \tag{4.4.3}$$

Let

$$h_i = \sum_{j=1}^{K_i} b_{i,j} X^{\sigma_{i,j}}.$$

Based on our construction of (4.4.1) we can go backwards and deduce, using (4.4.3), that $f = h_1 f_1 + \ldots + h_r f_r$. Since we know that $\mathrm{ord}(\partial^{\sigma_{i,j}}) \leqslant l$, this means that $\deg(h_i) \leqslant l$, $1 \leqslant i \leqslant r$, contradicting the fact that the maximum degree must be at least $k > l$. □

Remark 4.4.4. If $f = 1$ in Theorem 4.4.3, then $\tilde{f} = y$. Thus, by considering the system $G_1 = \{G, 1 - ty\} \subseteq K\{t, y\}$, we have $1 \in (G_1)^{(k)}$ and $1 \notin (G_1)^{(k-1)}$.

Example 4.4.5. For $m \geqslant 2$, $h \geqslant 1$, consider the following system of polynomial equations in $\mathcal{K}[X_1, \ldots, X_m]$; cf. [5, page 578]:

$$f_1 = X_1^h, f_2 = X_1 - X_2^h, \ldots, f_{m-1} = X_{m-2} - X_{m-1}^h, f_m = 1 - X_{m-1} X_m^{h-1}.$$

It is shown that $1 \in (f_1, \ldots, f_m)$ and if $1 = g_1 f_1 + \ldots + g_m f_m$, then

$$\deg(g_1) \geqslant h^m - h^{m-1} = h^{m-1}(h - 1).$$

Thus, if k is the maximum degree of the g_i (that is smallest possible over the collection of all g_i so that $1 = \sum g_i f_i$), we must have that $k \geqslant h^{m-1}(h - 1)$.

Let us use this polynomial system to create a system of differential polynomial in $\mathcal{K}\{y\}$ with derivations $\Delta = \{\partial_1, \ldots, \partial_m\}$. Let G be the system in $\mathcal{K}\{y\}$ given by

$$\tilde{f}_1 = \partial_1^h y, \tilde{f}_2 = \partial_1 y - \partial_2^h y, \ldots, \tilde{f}_{m-1} = \partial_{m-2} y - \partial_{m-1}^h y, \tilde{f}_m = y - \partial_{m-1} \partial_m^{h-1} y. \tag{4.4.4}$$

By the above discussion, we have $y \in (G)^{(k)}$ where $k \geqslant h^{m-1}(h-1)$ and $y \notin (G)^{\left(h^{m-1}(h-1)-1\right)}$. We have thus constructed a linear system G in which the number of derivations of the elements of G needed is exponential in the number of derivatives and polynomial in the order of the system.

We can construct an explicit linear combination of the $\tilde{f}_i$s and their derivatives equaling y that requires exactly $h^{m-1}(h-1)$ derivations of $\tilde{f}_1$. Explicit g_is are constructed in [5] such that $1 = g_1 f_1 + \ldots + g_m f_m$ and $\deg(g_1) = h^{m-1}(h-1)$ by observing that, setting $D = h^{m-1}(h-1)$,

$$X_m^D\left(X_1^h\right) - \sum_{i=2}^{m-1} X_m^D\left(X_{i-1}^{h^{i-1}} - \left(X_i^h\right)^{h^{i-1}}\right) + \left(1 - \left(X_{m-1}X_m^{h-1}\right)^{h^{m-1}}\right) = 1. \qquad (4.4.5)$$

Thus, if we set

$$g_1 = X_m^D$$

$$g_i = X_m^D\left(\sum_{j=0}^{h^{i-1}-1} X_{i-1}^{h^{i-1}-1-j}\left(X_i^h\right)^j\right) \qquad 2 \leqslant i \leqslant m-1$$

$$g_m = \sum_{j=0}^{h^{m-1}-1}\left(X_{m-1}X_m^{h-1}\right)^j,$$

then using (4.4.5), we have $1 = g_1 f_1 + \ldots + g_m f_m$.

We can use these g_is to find the desired linear combination of the $\tilde{f}_i$s and their derivatives. Using the corresponding identities in $\mathcal{K}[X_1, \ldots, X_m]$, we obtain that

$$\partial_{i-1}^{h^{i-1}} y - \partial_i^{h^i} y = \sum_{j=0}^{h^{i-1}-1} \partial_{i-1}^{h^{i-1}-j-1}\partial_i^{hj}\left(f_i\right) \qquad 2 \leqslant i \leqslant m-1$$

$$y - \partial_{m-1}^{h^{m-1}}\partial_m^{h^{m-1}(h-1)} y = \sum_{j=0}^{h^{m-1}-1} \partial_{m-1}^j \partial_m^{j(h-1)}\left(f_m\right).$$

Thus, setting $D = h^{m-1}(h-1)$, we can directly adapt (4.4.5) to see that

$$\partial_m^D \left(\partial_1^h y\right) - \sum_{i=2}^{m-1} \partial_m^D \left(\partial_{i-1}^{h^{i-1}} y - \partial_i^{h^i} y\right) + \left(y - \partial_{m-1}^{h^{m-1}} \partial_m^{h^{m-1}(h-1)} y\right) = y. \tag{4.4.6}$$

This gives us a linear combination of the $\tilde{f}_i$s and their derivatives that requires exactly $h^{m-1}(h-1)$ derivations of $\tilde{f}_1$, which we know is minimal by the polynomial case.

Example 4.4.6. We can use (4.4.6) to generalize this result to the case of multiple variables. We define a system in $\mathcal{K}\{y_1, \ldots, y_n\}$ with derivatives $\Delta = \{\partial_1, \ldots, \partial_m\}$. Let $m \geqslant 2$, $h \geqslant 1$. For $n = 1$, we have (4.4.4). For $n \geqslant 2$, consider the collection of differential polynomials:

$$G_1 = \left\{\partial_1^h y_1, \partial_1 y_1 - \partial_2^h y_1, \partial_2 y_1 - \partial_3^h y_1, \ldots, \partial_{m-2} y_1 - \partial_{m-1}^h y_1\right\}$$

$$G_i = \left\{\partial_{m-1} y_{i-1} - \partial_1^h y_i, \partial_1 y_i - \partial_2^h y_i, \ldots, \partial_{m-2} y_i - \partial_{m-1}^h y_i\right\} \quad 2 \leqslant i \leqslant n-1$$

$$G_n = \left\{\partial_{m-1} y_{n-1} - \partial_1^h y_n, \partial_1 y_n - \partial_2^h y_n, \ldots, \partial_{m-2} y_n - \partial_{m-1}^h y_n, y_n - \partial_{m-1} \partial_m^{h-1} y_n\right\}.$$

Then let $G = \bigcup_{i=1}^n G_i$. We claim that $y_n \in (G)^{(k)}$ where $k \geqslant h^{n(m-1)}(h-1)$ and

$$y_n \notin (G)^{\left(h^{n(m-1)}(h-1)-1\right)}. \tag{4.4.7}$$

We can write a system as in (4.4.6) to produce y_n in terms of the elements of G and their derivatives needing exactly $h^{n(m-1)}(h-1)$ derivations of $\partial_1^h y_1$. Let $E = h^{n(m-1)}(h-1)$. Then

$$\partial_m^E \left(\partial_1^h y_1\right) - \sum_{j=1}^n \sum_{i=2}^{m-1} \partial_m^E \left(\partial_{i-1}^{h^{(j-1)(m-1)+i-1}} y_j - \partial_i^{h^{(j-1)(m-1)+i}} y_j\right)$$

$$- \sum_{j=1}^{n-1} \partial_m^E \left(\partial_{m-1}^{h^{j(m-1)}} y_j - \partial_1^{h^{j(m-1)+1}} y_{j+1}\right) + \left(y_n - \partial_{m-1}^{h^{n(m-1)}} \partial_m^{h^{n(m-1)}(h-1)} y_n\right) = y_n.$$

By the same minimality argument used in Example 4.4.5, we must differentiate $\partial_1^h y_1$ at least E times. This shows (4.4.7) and there is a $k \geqslant E$ with $y_n \in (G)^{(k)}$.

Chapter 5

Rosenfeld-Gröbner Algorithm

In this chapter an upper bound for the orders of the derivatives in all intermediate steps and in the output of the Rosenfeld-Gröbner decomposition algorithm is found. The Rosenfeld-Gröbner algorithm approaches differential elimination by decomposing a differential ideal into an intersection of simpler ideals. Background information on differential rankings, differential remainders, and regular differential systems is given in Section 5.1. The Rosenfeld-Gröbner algorithm and basic applications are stated in Section 5.2. The main result, Theorem 5.3.4, is given in Section 5.3. We analyze the upper bound for specific inputs in Section 5.4, and address the shape of the lower bound in Section 5.5.

5.1 Differential rankings

This section contains background material on differential rankings that we will need when discussing the Rosenfeld-Gröbner algorithm. Recall that a *ranking* on the set ΘY is a total order $<$ satisfying the following two additional properties: for all $u, v \in \Theta Y$ and all $\theta \in \Theta$, $\theta \neq \mathrm{id}$,

$$u < \theta u \quad \text{and} \quad u < v \implies \theta u < \theta v.$$

From now on, we fix a *weighted ranking* $<$ on ΘY, that is we fix a weight $w\left(\partial_1^{i_1} \ldots \partial_m^{i_m}\right) = c_1 i_1 + \ldots + c_m i_m$ and a ranking $<$ such that if $u, v \in \Theta Y$ and $w(u) < w(v)$, then $u < v$. Note that as in Chapter 4, $<$ is different than the partial order defined on $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ used in Chapters 2 and 3.

Definition 5.1.1. Let $f \in \mathcal{K}\{Y\} \setminus \mathcal{K}$.

- The derivative $u \in \Theta Y$ of highest rank appearing in f is called the *leader* of f, denoted $\text{lead}(f)$.

- If we write f as a univariate polynomial in $\text{lead}(f)$, the leading coefficient is called the *initial* of f, denoted $\text{init}(f)$.

- If we apply any derivative $\delta \in \Delta$ to f, the leader of δf is $\delta(\text{lead}(f))$, and the initial of δf is called the *separant* of f, denoted $\text{sep}(f)$.

Note that the term "leader" in this context is different than how it was used in Chapters 2 and 3.

Given a set $A \subseteq \mathcal{K}\{Y\} \setminus \mathcal{K}$, we will denote the set of leaders of A by L_A, the set of initials of A by I_A, and the set of separants of A by S_A; we then let $H_A = I_A \cup S_A$ be the set of initials and separants of A.

For a derivative $u \in \Theta Y$, we let $(\Theta Y)_{<u}$ (respectively, $(\Theta Y)_{\leqslant u}$) be the collection of all derivatives $v \in \Theta Y$ with $v < u$ (respectively, $v \leqslant u$). For any derivative $u \in \Theta Y$, we let $A_{<u}$ (respectively, $A_{\leqslant u}$) be the elements of A with leader $< u$ (respectively, $\leqslant u$), that is,

$$A_{<u} := A \cap \mathcal{K}[(\Theta Y)_{<u}] \quad \text{and} \quad A_{\leqslant u} := A \cap \mathcal{K}[(\Theta Y)_{\leqslant u}].$$

We can similarly define $(\Theta A)_{<u}$ and $(\Theta A)_{\leqslant u}$, where

$$\Theta A := \{\theta f : \theta \in \Theta, f \in A\}.$$

Given $f \in \mathcal{K}\{Y\} \setminus \mathcal{K}$ such that $\deg_{\mathrm{lead}(f)}(f) = d$, we define the *rank* of f to be

$$\mathrm{rank}(f) := \mathrm{lead}(f)^d.$$

The weighted ranking $<$ on ΘY determines a pre-order (that is, a relation satisfying all of the properties of an order, except for the property that $a \leqslant b$ and $b \leqslant a$ imply that $a = b$) on $\mathcal{K}\{Y\} \setminus \mathcal{K}$, as follows. Given $f_1, f_2 \in \mathcal{K}\{Y\} \setminus \mathcal{K}$, we say that

$$\mathrm{rank}(f_1) < \mathrm{rank}(f_2)$$

if $\mathrm{lead}(f_1) < \mathrm{lead}(f_2)$ or if $\mathrm{lead}(f_1) = \mathrm{lead}(f_2)$ and $\deg_{\mathrm{lead}(f_1)}(f_1) < \deg_{\mathrm{lead}(f_2)}(f_2)$.

Definition 5.1.2. A differential polynomial f is *partially reduced* with respect to another differential polynomial g if no proper derivative of $\mathrm{lead}(g)$ appears in f, and f is *reduced* with respect to g if, in addition,

$$\deg_{\mathrm{lead}(g)}(f) < \deg_{\mathrm{lead}(g)}(g).$$

A differential polynomial is then (partially) reduced with respect to a set $A \subseteq \mathcal{K}\{Y\} \setminus \mathcal{K}$ if it is (partially) reduced with respect to every element of A.

Definition 5.1.3. For a set $A \subseteq \mathcal{K}\{Y\} \setminus \mathcal{K}$, we say that A is:

- *autoreduced* if every element of A is reduced with respect to every other element.

- *weak d-triangular* if the set of leaders L_A is autoreduced.

- *d-triangular* if A is weak d-triangular and every element of A is partially reduced with respect to every other element.

Note that every autoreduced set is d-triangular. Every weak d-triangular set (and thus every d-triangular and autoreduced set) is finite [24, Proposition 3.9]. Since the set of leaders of a weak d-triangular set A is autoreduced, distinct elements of A must have distinct leaders. If $u \in \Theta Y$ is the leader of some element of a weak d-triangular set A, we let A_u denote this element.

We define a pre-order on the collection of all weak d-triangular sets, which we also call *rank*, as follows. Given two weak d-triangular sets $A = \{A_1, \ldots, A_r\}$ and $B = \{B_1, \ldots, B_s\}$, in each case arranged in increasing rank, we say that $\text{rank}(A) < \text{rank}(B)$ if either:

- there exists a $k \leqslant \min(r, s)$ such that $\text{rank}(A_i) = \text{rank}(B_i)$ for all $1 \leqslant i < k$ and $\text{rank}(A_k) < \text{rank}(B_k)$, or

- $r > s$ and $\text{rank}(A_i) = \text{rank}(B_i)$ for all $1 \leqslant i \leqslant s$.

We also say that $\text{rank}(A) = \text{rank}(B)$ if $r = s$ and $\text{rank}(A_i) = \text{rank}(B_i)$ for all $1 \leqslant i \leqslant r$.

We can restrict this ranking to the collection of all d-triangular sets or the collection of all autoreduced sets.

Definition 5.1.4. A *characteristic set* of a differential ideal I is an autoreduced set $C \subseteq I$ of minimal rank among all autoreduced subsets of I.

Given a finite set $S \subseteq \mathcal{K}\{Y\}$, let S^∞ denote the multiplicative set containing 1 and generated by S. For an ideal $I \subseteq \mathcal{K}\{Y\}$, we define the *saturated ideal* to be

$$I : S^\infty := \{a \in \mathcal{K}\{Y\} : \exists s \in S^\infty \text{ with } sa \in I\}.$$

If I is a differential ideal, then $I : S^\infty$ is also a differential ideal [27, Section I.2].

Definition 5.1.5. For a differential polynomial $f \in \mathcal{K}\{Y\}$ and a weak d-triangular set $A \subseteq \mathcal{K}\{Y\}$, a *differential partial remainder* f_1 and a *differential remainder* f_2 of f with respect to A are differential polynomials such that there exist $s \in S_A^\infty$, $h \in H_A^\infty$ such that $sf \equiv f_1 \mod [A]$ and $hf \equiv f_2 \mod [A]$, with f_1 partially reduced with respect to A and f_2 reduced with respect to A.

We denote a differential partial remainder of f with respect to A by pd-red(f, A) and a differential remainder of f with respect to A by d-red(f, A). There are algorithms to compute pd-red(f, A) and d-red(f, A) for any f and A [24, Algorithms 3.12 and 3.13]. These algorithms have the property that

$$\mathrm{rank}(\text{pd-red}(f, A)),\ \mathrm{rank}(\text{d-red}(f, A)) \leqslant \mathrm{rank}(f);$$

since we have a weighted ranking, this implies that

$$w(\text{pd-red}(f, A)),\ w(\text{d-red}(f, A)) \leqslant w(f).$$

Two derivatives $u, v \in \Theta Y$ are said to have a *common derivative* if there exist $\phi, \psi \in \Theta$ such that $\phi u = \psi v$. Note this is the case precisely when $u = \theta_1 y$ and $v = \theta_2 y$ for some $y \in Y$ and $\theta_1, \theta_2 \in \Theta$. If $u = \partial_1^{i_1} \ldots \partial_m^{i_m} y$ and $v = \partial_1^{j_1} \ldots \partial_m^{j_m} y$ for some $y \in Y$, we define the *least common derivative* of u and v, denoted $\mathrm{lcd}(u, v)$, to be

$$\mathrm{lcd}(u, v) = \partial_1^{\max(i_1, j_1)} \ldots \partial_m^{\max(i_m, j_m)} y.$$

Definition 5.1.6. For $f, g \in \mathcal{K}\{Y\} \setminus \mathcal{K}$, we define the Δ-*polynomial* of f and g, denoted $\Delta(f, g)$, as follows. If $\mathrm{lead}(f)$ and $\mathrm{lead}(g)$ have no common derivatives, set $\Delta(f, g) = 0$.

Otherwise, let $\phi, \psi \in \Theta$ be such that

$$\mathrm{lcd}(\mathrm{lead}(f), \mathrm{lead}(g)) = \phi(\mathrm{lead}(f)) = \psi(\mathrm{lead}(g)),$$

and define

$$\Delta(f, g) := \mathrm{sep}(g)\phi(f) - \mathrm{sep}(f)\psi(g).$$

Definition 5.1.7. A pair (A, H) is called a *regular differential system* if:

- A is a d-triangular set

- H is a set of differential polynomials that are all partially reduced with respect to A

- $S_A \subseteq H^\infty$

- for all $f, g \in A$, $\Delta(f, g) \in ((\Theta A)_{<u}) : H^\infty$, where $u = \mathrm{lcd}(\mathrm{lead}(f), \mathrm{lead}(g))$.

Any ideal of the form $[A] : H^\infty$, where (A, H) is a regular differential system, is called a *regular differential ideal*. Every regular differential ideal is a radical differential ideal [24, Theorem 4.12]. Given a radical differential ideal $I \subseteq \mathcal{K}\{Y\}$, a *regular decomposition* of I is a finite collection of regular differential systems $\{(A_1, H_1), \ldots, (A_r, H_r)\}$ such that

$$I = \bigcap_{i=1}^{r} [A_i] : H_i^\infty.$$

As we will see, due to the Rosenfeld-Gröbner algorithm, every radical differential ideal in $\mathcal{K}\{Y\}$ has a regular decomposition.

A d-triangular set C is called a *differential regular chain* if it is a characteristic set of $[C] : H_C^\infty$; in this case, we call $[C] : H_C^\infty$ a *characterizable differential ideal*. A *characteristic decomposition* of a radical differential ideal $I \subseteq \mathcal{K}\{Y\}$ is a representation of I as an intersection of characterizable differential ideals. As we will recall in Section 5.2, every radical differential ideal also has a characteristic decomposition.

5.2 Statement of the algorithm

Below we reproduce the Rosenfeld-Gröbner algorithm from [24, Section 6]. This algorithm relies on two others, called auto-partial-reduce and update, which we also include. We include these two auxiliary algorithms because, in Section 5.3, we will study their effect on the growth of the weights of derivatives in Rosenfeld-Gröbner.

Rosenfeld-Gröbner takes as its input two finite subsets $F, K \in \mathcal{K}\{Y\}$ and outputs a finite set $\mathcal{A}$ of regular differential systems such that

$$\{F\} : K^\infty = \bigcap_{(A,H)\in\mathcal{A}} [A] : H^\infty, \tag{5.2.1}$$

where $\mathcal{A} = \emptyset$ if $1 \in \{F\} : K^\infty$.

If we have a decomposition of $\{F\} : K^\infty$ as in (5.2.1), we can compute, using only algebraic operations, a decomposition of the form

$$\{F\} : K^\infty = \bigcap_{C\in\mathcal{C}} [C] : H_C^\infty, \tag{5.2.2}$$

where $\mathcal{C}$ is finite and each $C \in \mathcal{C}$ is a differential regular chain [24, Algorithms 7.1 and 7.2]. This means that an upper bound on $\bigcup_{(A,H)\in\mathcal{A}} \mathcal{W}(A \cup H)$ from (5.2.1) will also be an upper bound on $\bigcup_{C\in\mathcal{C}} \mathcal{W}(C)$ from (5.2.2).

Rosenfeld-Gröbner has many immediate applications. For example, if $K = \{1\}$, then $\{F\} : K^\infty = \{F\}$, so in this case, Rosenfeld-Gröbner computes a regular decomposition of $\{F\}$, which then also gives us a characteristic decomposition of $\{F\}$ by the discussion in the previous paragraph.

Recall from Chapter 4 that the *weak differential Nullstellensatz* says that a system of polynomial differential equations $F = 0$ is consistent (that is, has a solution in some differential field extension of $\mathcal{K}$) if and only if $1 \notin [F]$ [27, Section IV.2]. Thus, since

Rosenfeld-Gröbner$(F, K) = \emptyset$ if and only if $1 \in \{F\} : K^\infty$, we see that $F = 0$ is consistent if and only if Rosenfeld-Gröbner$(F, \{1\}) \neq \emptyset$.

More generally, Rosenfeld-Gröbner and its extension for computing a characteristic decomposition of a radical differential ideal allow us to test for membership in a radical differential ideal, as follows. Suppose we have computed a characteristic decomposition

$$\{F\} = \bigcap_{C \in \mathcal{C}} [C] : H_C^\infty.$$

Now, a differential polynomial $f \in \mathcal{K}\{Y\}$ is contained in $\{F\}$ if and only if $f \in [C] : H_C^\infty$ for all $C \in \mathcal{C}$; this latter case is true if and only if d-red$(f, C) = 0$, which can be tested using [24, Algorithm 3.13].

Rosenfeld-Gröbner, auto-partial-reduce, and update rely on the following tuples of differential polynomials:

Definition 5.2.1. A *Rosenfeld-Gröbner quadruple* (or *RG-quadruple*) is a 4-tuple (G, D, A, H) of finite subsets of $\mathcal{K}\{Y\}$ such that:

- A is a weak d-triangular set, $H_A \subseteq H$, D is a set of Δ-polynomials, and

- for all $f, g \in A$, either $\Delta(f, g) = 0$ or $\Delta(f, g) \in D$ or

$$\Delta(f, g) \in (\Theta(A \cup G)_{<u}) : H_u^\infty,$$

  where $u = \text{lcd}(\text{lead}(f), \text{lead}(g))$ and $H_u = H_{A_{<u}} \cup (H \setminus H_A) \cap \mathcal{K}[(\Theta Y)_{<u}]$.

Remark 5.2.2. The RG-quadruple that is output by update satisfies additional properties that we do not list, as they are not important for our analysis. For more information, we refer the reader to [24, Algorithm 6.10]

Algorithm: Rosenfeld-Gröbner, [24, Algorithm 6.11]

Data: F, K finite subsets of $\mathcal{K}\{Y\}$

Result: A set $\mathcal{A}$ of regular differential systems such that:

- $\mathcal{A}$ is empty if it has been detected that $1 \in \{F\} : K^\infty$

- $\{F\} : K^\infty = \bigcap\limits_{(A,H) \in \mathcal{A}} [A] : H^\infty$ otherwise

$\mathcal{S} := \{(F, \emptyset, \emptyset, K)\}$;
$\mathcal{A} := \emptyset$;
while $\mathcal{S} \neq \emptyset$ **do**
    $(G, D, A, H) :=$ an element of $\mathcal{S}$;
    $\bar{\mathcal{S}} = \mathcal{S} \setminus (G, D, A, H)$;
    if $G \cup D = \emptyset$ **then**
        $\mathcal{A} := \mathcal{A} \cup \mathsf{auto\text{-}partial\text{-}reduce}(A, H)$;
    else
        $p :=$ an element of $G \cup D$;
        $\bar{G}, \bar{D} := G \setminus \{p\}, D \setminus \{p\}$;
        $\bar{p} := \mathrm{d\text{-}red}(p, A)$;
        if $\bar{p} = 0$ **then**
            $\bar{\mathcal{S}} := \bar{\mathcal{S}} \cup \{(\bar{G}, \bar{D}, A, H)\}$;
        else
            if $\bar{p} \notin \mathcal{K}$ **then**
                $\bar{p}_i := \bar{p} - \mathrm{init}(\bar{p}) \, \mathrm{rank}(\bar{p})$ $\bar{p}_s := \deg_{\mathrm{lead}(\bar{p})}(\bar{p}) \bar{p} - \mathrm{lead}(\bar{p}) \, \mathrm{sep}(\bar{p})$;
                $\bar{\mathcal{S}} := \bar{\mathcal{S}} \cup \{\mathsf{update}(\bar{G}, \bar{D}, A, H, \bar{p}), (G \cup \{\bar{p}_s, \mathrm{sep}(\bar{p})\}, \bar{D}, A, H \cup \{\mathrm{init}(\bar{p})\}), (\bar{G} \cup \{\bar{p}_i, \mathrm{init}(\bar{p})\}, \bar{D}, A, H)\}$;
            end
        end
    end
    $\mathcal{S} := \bar{\mathcal{S}}$;
end
return $\mathcal{A}$;

Algorithm: auto-partial-reduce, [24, Algorithm 6.8]

Data: Two finite subsets A, H of $\mathcal{K}\{Y\}$ such that $(\emptyset, \emptyset, A, H)$ is an RG-quadruple

Result:

- The empty set if it is detected that $1 \in [A] : H^\infty$

- Otherwise, a set with a single regular differential system (B, K) with $L_A = L_B$, $H_B \subseteq K$, and $[A] : H^\infty = [B] : K^\infty$

$B := \emptyset$;
for $u \in L_A$ *increasingly* **do**
$\quad$ $b := \text{pd-red}(A_u, B)$;
$\quad$ **if** $\text{rank}(b) = \text{rank}(A_u)$ **then**
$\quad\quad$ $B := B \cup \{b\}$;
$\quad$ **else**
$\quad\quad$ **return** $(\emptyset)$;
$\quad$ **end**
end
$K := H_B \cup \{\text{pd-red}(p, B) : p \in H \setminus H_A\}$;
if $0 \in K$ **then**
$\quad$ **return** $(\emptyset)$;
else
$\quad$ **return** $\{(B, K)\}$;
end

Algorithm: update [24, Algorithm 6.10]

Data:

- A 4-tuple (G, D, A, H) of finite subsets of $\mathcal{K}\{Y\}$

- A differential polynomial p reduced with respect to A such that $(G \cup \{p\}, D, A, H)$ is an RG-quadruple

Result: A new RG-quadruple $(\bar{G}, \bar{D}, \bar{A}, \bar{H})$
$u := \text{lead}(p)$;
$G_A := \{a \in A \mid \text{lead}(a) \in \Theta u\}$;
$\bar{A} := A \setminus G_A$;
$\bar{G} := G \cup G_A$;
$\bar{D} := D \cup \{\Delta(p, a) \mid a \in \bar{A}\} \setminus \{0\}$;
$\bar{H} := H \cup \{\text{sep}(p), \text{init}(p)\}$;
return $(\bar{G}, \bar{D}, \bar{A} \cup \{p\}, \bar{H})$;

5.3   Order upper bound

Given finite subsets $F, K \subseteq \mathcal{K}\{Y\}$, let $h = \mathcal{W}(F \cup K)$. Our goal is to find an upper bound for

$$\mathcal{W}\left(\bigcup_{(A,H) \in \mathcal{A}} (A \cup H) \right),$$

where $\mathcal{A} = \mathsf{Rosenfeld\text{-}Gr\ddot{o}bner}(F, K)$, in terms of h, m (the number of derivations), and n (the number of differential indeterminates). By then choosing a specific weight, we can find an upper bound for $\mathcal{H}\left(\bigcup_{(A,H) \in \mathcal{A}}(A \cup H) \right)$ in terms of m, n, and $\mathcal{H}(F \cup K)$ (see (2.1.1) and (2.1.2) for the definitions of $\mathcal{H}$ and $\mathcal{W}$, respectively).

We approach this problem as follows. Every $(A, H) \in \mathcal{A}$ is formed by applying auto-partial-reduce to a 4-tuple $(\emptyset, \emptyset, A', H') \in \mathcal{S}$. Thus, it suffices:

- to bound how auto-partial-reduce increases the weight of a collection of differential polynomials (it turns out to not increase the weight), and

- to bound $\mathcal{W}(G \cup D \cup A \cup H)$ for all (G, D, A, H) added to $\mathcal{S}$ throughout the course of Rosenfeld-Gröbner.

We accomplish the latter by determining when the weight of a tuple (G, D, A, H) added to $\mathcal{S}$ is larger than the weights of the previous elements of $\mathcal{S}$ and bounding $\mathcal{W}(G \cup D \cup A \cup H)$ in this instance, and then bounding the number of times we can add such elements to $\mathcal{S}$.

There is a sequence $\{(G_i, D_i, A_i, H_i)\}_{i=0}^{N}$ corresponding to each regular differential system (A, H) in the output of Rosenfeld-Gröbner, where $N = N_{(A,H)}$, such that the tuple $(G_{i+1}, D_{i+1}, A_{i+1}, H_{i+1})$ is obtained from the tuple (G_i, D_i, A_i, H_i) during the while loop, $(G_0, D_0, A_0, H_0) = (F, \emptyset, \emptyset, K)$, and $(A, H) = \mathsf{auto\text{-}partial\text{-}reduce}(A_N, H_N)$.

We begin with an auxiliary result, which is an analogue of the *first* property from [15, Section 5.1].

Lemma 5.3.1. *For every* $f \in A_i$ *and* $i < j$, *there exists* $g \in A_j$ *such that* $\mathrm{lead}(f) \in \Theta \mathrm{lead}(g)$. *In particular, if* p *is reduced with respect to* A_j, *then* p *is reduced with respect to* A_i *for all* $i < j$.

Proof. It is sufficient to consider the case $j = i+1$. If $(G_{i+1}, D_{i+1}, A_{i+1}, H_{i+1})$ was obtained from (G_i, D_i, A_i, H_i) without applying update, then $A_i = A_{i+1}$. Otherwise, either $f \in A_i \backslash G_{A_i}$ (we use the notation from update), or $f \in G_{A_i}$. In the former case, $f \in A_{i+1}$ as well, so we can set $g = f$. In the latter case, $\mathrm{lead}(f) \in \Theta \mathrm{lead}(p)$, so we can set $g = p$. $\square$

We define a partial order $\preccurlyeq$ on the set of derivatives ΘY as follows. For $u, v \in \Theta Y$, we say that $u \preccurlyeq v$ if there exists $\theta \in \Theta$ such that $\theta u = v$. Note that this implies that u and v are both derivatives of the same $y \in Y$.

Given a sequence $\{(G_i, D_i, A_i, H_i)\}_{i=0}^{N}$ as above (where $N = N_{(A,H)}$ for some regular differential system (A, H) in the output of Rosenfeld-Gröbner), we will construct an antichain sequence $S = \{s_1, s_2, \ldots\} \subseteq \Theta Y$ inductively going along the sequence $\{(G_i, D_i, A_i, H_i)\}$. Suppose $S_{j-1} = \{s_1, \ldots, s_{j-1}\}$ has been constructed after considering

$$(G_0, D_0, A_0, H_0), \ldots, (G_{i-1}, D_{i-1}, A_{i-1}, H_{i-1}),$$

where $S_0 = \emptyset$. A 4-tuple (G_i, D_i, A_i, H_i) can be obtained from $(G_{i-1}, D_{i-1}, A_{i-1}, H_{i-1})$ in two ways:

1. We did not perform update. In this case, we do not append a new element to S.

2. We performed update with respect to a differential polynomial $\bar{p}$. If there exists $s_k \in S_{j-1}$ such that $\mathrm{lead}(\bar{p}) \preccurlyeq s_k$, we do not append a new element to S_{j-1}. Otherwise, let $s_j = \mathrm{lead}(\bar{p})$ and define $S_j = \{s_1, \ldots, s_j\}$. In the latter case, we set $k_j = i$. We also set $k_0 = 0$.

Theorem 5.3.2. *The sequence $\{s_j\}$ is an antichain sequence in ΘY and, for all $j \geqslant 1$,*

$$w(s_j) \leqslant h f_j,$$

where $\{f_j\}$ is the Fibonacci sequence.

For $m = 2$, we provide a refined version of Theorem 5.3.2. Let $\{f(n,h)_k\}$ be the sequence:

$$\begin{cases} f(n,h)_0 = 0, \ \ f(n,h)_1 = f(n,h)_2 = h \\[2mm] f(n,h)_k = f(n,h)_{k-1} + f(n,h)_{k-2} & \text{for } k \leqslant n+1 \\[2mm] f(n,h)_k = f(n,h)_{k-1} + f(n,h)_{k-2} - 1 & \text{for } k > n+1. \end{cases} \qquad (5.3.1)$$

Proposition 5.3.3. *For $m = 2$ the sequence $\{s_j\}$ satisfies, for all $j \geqslant 1$,*

$$w(s_j) \leqslant f(n,h)_j.$$

We will prove Proposition 5.3.3 while proving Theorem 5.3.2, highlighting the case $m = 2$.

Proof. Let $i < j$. Assume that $s_j \succcurlyeq s_i$. Then, p is not reduced with respect to A_{k_i}, which contradicts Lemma 5.3.1. On the other hand, the case $s_j \preccurlyeq s_i$ is impossible by the construction of the sequence, so $\{s_j\}$ is an antichain sequence.

Let L denote the length of the sequence $\{s_j\}$. We denote the maximal $j \in \mathbb{Z}_{\geqslant 0}$ such that $k_j \leqslant i$ by anti-k_i. For all $i \geqslant 0$, let us set $j = \text{anti-}k_i$ and prove by induction on i that

1. $\mathcal{W}\left(\bigcup\limits_{t=0}^{i}(G_t \cup D_t \cup H_t)\right) \leqslant h f_{j+1}$

2. $\mathcal{W}\left(\bigcup\limits_{t=0}^{i} A_t\right) \leqslant h f_j$

3. For all distinct elements of $\bigcup\limits_{t=0}^{i} A_t$, the weights of the least common derivatives of their leaders do not exceed $h f_{j+1}$.

If $m = 2$, let $F_0 = 0$, $F_1 = F_2 = h$. We will show that there exists a sequence $\{F_r\}$ such that

- for all $r \geqslant 1$, $w(s_r) \leqslant F_r$ and

- $F_r = F_{r-1} + F_{r-2} - 1$ for all $r \geqslant 3$ except at most $n - 1$ of them, for which $F_r = F_{r-1} + F_{r-2}$. In the latter case, we will say that r is *a jump index*. Note that 2 is not a jump index by the definition, although $F_2 = F_1 + F_0$.

For each such sequence, the induction hypothesis will be the following:

1. $\mathcal{W}\left(\bigcup_{t=0}^{i}(G_t \cup D_t \cup H_t)\right) \leqslant F_{j+1}$ for $j < L$ and $\mathcal{W}\left(\bigcup_{t=0}^{i}(G_t \cup D_t \cup H_t)\right) \leqslant F_{L+1} + 1$ for $j = L$

2. $\mathcal{W}\left(\bigcup_{t=0}^{i} A_t\right) \leqslant F_j$

3. For all distinct elements of $\bigcup_{t=0}^{i} A_t$, the weights of the least common derivatives of their leaders do not exceed F_{j+1} for $j < L$ and $F_{L+1} + 1$ for $j = L$

4. If, in either of (1) or (3), the equality holds in the case $j = L$, then, for every q, $1 \leqslant q \leqslant n$, the sequence $\{s_r\}$ contains $\partial_1^{a_q} y_q$ and $\partial_2^{b_q} y_q$ for some a_q and b_q.

In the base case $i = 0 = k_0$, we have

$$\mathcal{W}(G_0 \cup D_0 \cup H_0) = h = h f_1 \ (F_1 \text{ in the case m} = 2)$$

and

$$\mathcal{W}(A_0) = \mathcal{W}(\emptyset) = 0 = h f_0 \ (F_0 \text{ in the case m} = 2).$$

There are two distinct cases for $i + 1$:

1. Case $i + 1 < k_{j+1}$ (so anti-$k_{i+1} = j$). Then, $(G_{i+1}, D_{i+1}, A_{i+1}, H_{i+1})$ was obtained from (G_i, D_i, A_i, H_i) in one of the following ways:

(a) We did not perform update. In this case, $A_{i+1} = A_i$ and

$$\mathcal{W}(G_{i+1} \cup D_{i+1} \cup H_{i+1}) \leqslant \mathcal{W}(G_i \cup D_i \cup H_i).$$

(b) We performed update with respect to a differential polynomial p such that $\mathrm{lead}(f) \in \Theta \, \mathrm{lead}(p)$ for some $f \in \bigcup_{t=0}^{i} A_t$. In this case,

$$\mathcal{W}(A_{i+1}) \leqslant \mathcal{W}\left(\bigcup_{t=0}^{i} A_t\right).$$

Then, for all $g \in A_t$ $(t \leqslant i)$,

$$w(\Delta(p, g)) \leqslant w(\mathrm{lcd}(\mathrm{lead}(g), \mathrm{lead}(f))),$$

which is bounded by hf_{j+1} (by F_{j+1} or $F_{L+1} + 1$ in the case $m = 2$) due to the third inductive hypothesis. Since $D_{i+1} \setminus D_i$ consists of some of these polynomials, $G_{i+1} \setminus G_i \subseteq A_i$, and $H_{i+1} \setminus H_i = \{\mathrm{sep}(p), \mathrm{init}(p)\}$, then

$$\mathcal{W}(G_{i+1} \cup D_{i+1} \cup H_{i+1}) \leqslant \mathcal{W}(G_i \cup D_i \cup H_i).$$

2. Case $i + 1 = k_{j+1}$ (so now anti-$k_{i+1} = j + 1$). We performed update with respect to a differential polynomial p, which is a result of reduction of some $\tilde{p} \in G_i \cup D_i$ with respect to A_i. Then

$$\mathcal{W}(A_{i+1}) \leqslant \max(\mathcal{W}(A_i), w(p)) \leqslant hf_{j+1}.$$

Moreover, for every $g \in \bigcup_{t=0}^{i} A_t$,

$$w(\mathrm{lcd}(\mathrm{lead}(g), \mathrm{lead}(p))) \leqslant hf_j + hf_{j+1} = hf_{j+2}. \tag{5.3.2}$$

Since $D_{i+1} \setminus D_i$ consists of some of these polynomials, $G_{i+1} \setminus G_i \subseteq A_i$, and $H_{i+1} \setminus H_i = \{\mathrm{sep}(p), \mathrm{init}(p)\}$, we have

$$\mathcal{W}(G_{i+1} \cup D_{i+1} \cup H_{i+1}) \leqslant \max(\mathcal{W}(G_i \cup D_i \cup H_i), hf_{j+2}) = hf_{j+2}.$$

In the case $m = 2$, instead of (5.3.2), we obtain

$$w(\mathrm{lcd}(\mathrm{lead}(g), \mathrm{lead}(p))) \leqslant w(\mathrm{lead}(p)) + w(\mathrm{lead}(g)) \tag{5.3.3}$$

If (5.3.3) is strict, we have

$$w(\mathrm{lcd}(\mathrm{lead}(g), \mathrm{lead}(p))) \leqslant w(\mathrm{lead}(p)) + w(\mathrm{lead}(g)) - 1 \leqslant F_j + F_{j+1} - 1 = F_{j+2},$$

and $j + 2$ is not a jump index. Otherwise, (5.3.3) turns out to be an equality. In this case, the only possibility is $\mathrm{lead}(p) = \partial_1^a y_r$ and $\mathrm{lead}(g) = \partial_2^b y_r$ (or vice versa) for some r. Note that, for every r, such a situation occurs at most once. Consider the following two cases:

(a) For every q, $1 \leqslant q \leqslant n$, the sequence $s_1, \ldots, s_{j+1}$ already contains $\partial_1^{a_q} y_q$ and $\partial_2^{b_q} y_q$ for some a_q and b_q. In this case, $s_1, \ldots, s_{j+1}$ already form an antichain sequence that cannot be extended further, so $j + 1 = L$. We set $F_{L+1} = F_L + F_{L-1} - 1$, so we can bound the right-hand side of (5.3.3) from above by $F_{L+1} + 1$.

(b) Otherwise, we just set $F_{j+2} = F_{j+1} + F_j$, so $j + 2$ is a jump index, and we still have less than n of them.

Since $w(s_j) \leqslant \mathcal{W}(A_{k_j}) \leqslant hf_j$, this completes the proof of Theorem 5.3.2.

In order to complete the proof of Proposition 5.3.3, it is sufficient to show that, for every such sequence $\{F_j\}$, for all j, $f(n, h)_j \geqslant F_j$. Let $i_1, \ldots, i_{n-1}$ denote the jump indices of $\{F_j\}$. Note that $\{f(n, h)_j\}$ is uniquely defined as a sequence of the same type as $\{F_j\}$ with jump indices $3, \ldots, n + 1$. It is sufficient to prove that, after decreasing any jump index of $\{F_j\}$ by one, we obtain a sequence which is not smaller than $\{F_j\}$. Then, since we will obtain $\{f(n, h)_j\}$ after some number of such operations and the jump indices of $\{f(n, h)_j\}$ cannot be further decreased, we will have that $\{f(n, h)_j\}$ is the largest such sequence. The claim is true since, before decreasing i_j, the sequence was of the form

$$\ldots, F_{i_j-2}, \quad F_{i_j-1} = F_{i_j-3} + F_{i_j-2} - 1, \quad F_{i_j} = F_{i_j-1} + F_{i_j-2} = F_{i_j-3} + 2F_{i_j-2} - 1, \ldots$$

but, after decreasing i_j by one, it will be of the form

$$\ldots, F_{i_j-2}, \quad F_{i_j-1} = F_{i_j-3} + F_{i_j-2}, \quad F_{i_j} = F_{i_j-1} + F_{i_j-2} - 1 = F_{i_j-3} + 2F_{i_j-2} - 1, \ldots$$

Since the rest of terms obey the same recurrence for both sequences, the latter is not smaller than the former. $\square$

Recall that the *degree* of an element $((i_1, \ldots, i_m), k) \in \mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ is defined to be $i_1 + \ldots + i_m$. Given a weight $w\left(\partial_1^{i_1} \ldots \partial_m^{i_m}\right) = c_1 i_1 + \ldots + c_m i_m$ on Θ, define a map from the set of derivatives ΘY to the set $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ by

$$\partial_1^{i_1} \ldots \partial_m^{i_m} y_k \mapsto ((c_1 i_1, \ldots, c_m i_m), k).$$

Note the degree of the image of θy in $\mathbb{Z}_{\geqslant 0} \times \mathfrak{n}$ is equal to the weight of θy in ΘY.

Under this map, the partial order $\preccurlyeq$ on ΘY determines the partial order $\leqslant$ on $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ given in Section 2.2 by

$$((i_1, \ldots, i_m), k) \leqslant ((j_1, \ldots, j_m), l) \iff k = l \text{ and } i_r \leqslant j_r \text{ for all } r, \ 1 \leqslant r \leqslant m.$$

Thus, every antichain sequence of ΘY determines an antichain sequence of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$. Since every antichain sequence of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ is finite, so is every antichain sequence of ΘY.

For an increasing function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$, let $\mathfrak{L}_{f,m}^n$ be the maximal length of an antichain sequence of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ with degree growth bounded by f, which exists as described in Chapter 3.

Theorem 5.3.4. *Let $F, K \subseteq \mathcal{K}\{Y\}$ be finite subsets with $h = \mathcal{W}(F \cup K)$, $L = \mathfrak{L}_{f,m}^n$, and $\mathcal{A} = \mathsf{Rosenfeld\text{-}Gr\ddot{o}bner}(F, K)$, where $f(i) = hf_i$ with $\{f_i\}$ the Fibonacci sequence. Then*

$$\mathcal{W}\left(\bigcup_{(A,H) \in \mathcal{A}} (A \cup H) \right) \leqslant hf_{L+1}.$$

Proof. Since $w(\mathrm{pd\text{-}red}(p, B)) \leqslant w(p)$ for any $p \in \mathcal{K}\{Y\}$ and weak d-triangular set B, we have $\mathcal{W}(B \cup K) \leqslant \mathcal{W}(A \cup H)$, where $\{(B, K)\} = \mathsf{auto\text{-}partial\text{-}reduce}(A, H)$. Hence, it suffices to bound $\mathcal{W}(G \cup D \cup A \cup H)$ whenever the tuple (G, D, A, H) is added to $\mathcal{S}$ in $\mathsf{Rosenfeld\text{-}Gr\ddot{o}bner}$.

By Theorem 5.3.2 and the correspondence between antichain sequences of ΘY and $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$, we obtain an antichain sequence of $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ of degree growth bounded by $f(i)$, so the length of this sequence (and thus the sequence from Theorem 5.3.2) is at most L.

In the proof of Theorem 5.3.2, it is shown that for all $i \leqslant N$, for $j := \mathrm{anti\text{-}}k_i$, we have

$$\mathcal{W}\left(\bigcup_{t=1}^{i} (G_t \cup D_t \cup A_t \cup H_t) \right) \leqslant hf_{j+1}.$$

Since the largest possible j is the length of the antichain sequence (and this j is equal to anti-k_N), for every (G_i, D_i, A_i, H_i), we have

$$\mathcal{W}(G_i \cup D_i \cup A_i \cup H_i) \leqslant h f_{L+1}.$$

Since every (G, D, A, H) added to $\mathcal{S}$ equals (G_i, D_i, A_i, H_i) for some i, this ends the proof. $\square$

Corollary 5.3.5. *Let $m = 2$, $F, K \subseteq \mathcal{K}\{Y\}$ be finite subsets with $h = \mathcal{W}(F \cup K)$, $L = \mathfrak{L}^n_{f,2}$, and $\mathcal{A} = $ Rosenfeld-Gröbner(F, K), where $f(i) = f(n, h)_i$ with $\{f(n, h)_i\}$ given by (5.3.1). Then*

$$\mathcal{W}\left(\bigcup_{(A,H)\in\mathcal{A}} (A \cup H) \right) \leqslant f(n, h)_{L+1}.$$

Proof. Replacing $h f_i$ with $f(n, h)_i$ everywhere in the proof of Theorem 5.3.4, we obtain an argument that is valid in all cases except for the case in which, for every q, $1 \leqslant q \leqslant n$, the antichain sequence $\{s_j\}$ contains $\partial_1^{a_q} y_q$ and $\partial_2^{b_q} y_q$ for some a_q and b_q. In this case, we still have $\mathcal{W}(A_i) \leqslant f(n, h)_L$ for all i. We will prove that $\mathcal{W}(H_i) \leqslant f(n, h)_{L+1}$ for all i. For $i < k_L$, this inequality follows from the proof of Theorem 5.3.2. For $i \geqslant k_L$, every h added to H_i is reduced with respect to A_i (see Rosenfeld-Gröbner). The definition of k_j implies that the set of leaders of A_{k_j} contains s_j. While performing update for A_i, every leader s of A_i either survives or is replaced with $\tilde{s}$ such that s is a derivative of $\tilde{s}$. Hence, for all $i \geqslant k_j$, the set of leaders of A_i contains either s_j or $\tilde{s}$ such that s_j is a derivative of $\tilde{s}$. Thus, since h is reduced with respect to A_i for $i \geqslant k_L$, for every variable $\partial_1^a \partial_2^b y_q$ occurring in h, we have $a < a_q$ and $b < b_q$. Thus,

$$w(h) \leqslant \max_{1 \leqslant q \leqslant n} \left(w\left(\partial_1^{a_q-1} y_q\right) + w\left(\partial_2^{b_q-1} y_q\right) \right) \leqslant f(n, h)_L + f(n, h)_{L-1} - 2 < f(n, h)_{L+1}. \quad \square$$

We can use Theorem 5.3.4 and Corollary 5.3.5 to bound the orders of the output Rosenfeld-Gröbner. Let $F, K \subseteq \mathcal{K}\{Y\}$ be two finite subsets, and define a weight w on Θ such that

$$\mathcal{W}(F \cup K) = \mathcal{H}(F \cup K). \tag{5.3.4}$$

This can always be done by letting $w(\theta) = \mathrm{ord}(\theta)$ for all derivatives θ, but there are sometimes other weights that lead to equation (5.3.4) being satisfied.

Example 5.3.6. We provide examples of differential polynomials f that arise as part of systems of PDEs for which it is possible to construct a nontrivial weight w such that $w(f) = \mathrm{ord}(f)$. We note that we are not applying Rosenfeld-Gröbner to these examples; we simply present them to demonstrate that there are nontrivial weights satisfying equation (5.3.4).

1. Consider the heat equation

$$u_t - \alpha \cdot (u_{xx} + u_{yy}) = 0, \quad f(u) := \partial_t u - \alpha \cdot (\partial_x^2 u + \partial_y^2 u) \in \mathcal{K}\{u\},$$

   where $u(x, y, t)$ is the unknown, α is a positive constant, and $\mathcal{K}\{u\}$ has derivations $\{\partial_x, \partial_y, \partial_t\}$. If we define a weight w on Θ by

$$w\left(\partial_x^i \partial_y^j \partial_t^k\right) = i + j + 2k,$$

   then $w(f) = 2 = \mathrm{ord}(f)$.

2. Consider the K-dV equation

$$\phi_t + \phi_{xxx} + 6\phi\phi_x = 0, \quad f(\phi) := \partial_t \phi + \partial_x^3 \phi + 6\phi\partial_x\phi \in \mathcal{K}\{\phi\},$$

where $\phi(x,t)$ is the unknown and $\mathcal{K}\{\phi\}$ has derivations $\{\partial_x, \partial_t\}$. Define a weight w on Θ by

$$w\left(\partial_x^i \partial_t^j\right) = i + 3j,$$

so that $w(f) = 3 = \mathrm{ord}(f)$.

Using Theorem 5.3.4, Corollary 5.3.5, and (5.3.4), we obtain the following order bound for the output of Rosenfeld-Gröbner:

Corollary 5.3.7. *Let* $F, K \subseteq \mathcal{K}\{Y\}$ *be finite subsets with* $h = \mathcal{H}(F \cup K)$, $L = \mathcal{L}_{f,m}^n$, $\mathcal{A} =$ Rosenfeld-Gröbner(F, K), *where* $f(i) = f(n,h)_i$ *with* $\{f(n,h)_i\}$ *the sequence given by* (5.3.1) *if* $m = 2$ *and* $f(i) = hf_i$ *with* $\{f_i\}$ *the Fibonacci sequence if* $m > 2$. *Let* $w\left(\partial_1^{i_1} \ldots \partial_m^{i_m}\right) = c_1 i_1 + \ldots + c_m i_m$ *be a weight defined on* ΘY *such that* $\mathcal{W}(F \cup K) = \mathcal{H}(F \cup K)$. *Then, for all* $g \in \mathcal{A}$,

$$\mathrm{ord}(g, \partial_i) \leqslant \begin{cases} \frac{f(n,h)_{L+1}}{c_i} & \textit{if } m = 2 \\[2mm] \frac{hf_{L+1}}{c_i} & \textit{if } m > 2. \end{cases}$$

5.4 Specific values

In order to apply the results of the previous section, we need to be able to effectively compute $\mathcal{L}_{f,m}^n$. [35] only proved the existence of this number, without an analysis of how to construct it. [12] constructed an upper bound for $m = 1, 2$. The first analysis for the case of arbitrary m appears in [31]. This was studied extensively in Chapter 3; we repeat the main points.

For an increasing function $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$, recall the definition of $\Psi_{f,m} \colon \mathbb{Z}_{>0} \times \mathbb{Z}_{\geqslant 0}^m \to \mathbb{Z}_{\geqslant 0}$

given in (3.3.2):

$$
\begin{cases}
\Psi_{f,m}(i, (0, \ldots, 0, u_m)) = i \\[2mm]
\Psi_{f,m}(i-1, (u_1, \ldots, u_r, 0, \ldots, 0, u_m)) \\[2mm]
\quad = \Psi_{f,m}(i, (u_1, \ldots, u_r - 1, f(i) - f(i-1) + u_m + 1, 0, \ldots, 0)), \quad r < m - 1, u_r > 0 \\[2mm]
\Psi_{f,m}(i-1, (u_1, \ldots, u_m)) \\[2mm]
\quad = \Psi_{f,m}(i, (u_1, \ldots, u_{m-1} - 1, f(i) - f(i-1) + u_m + 1)), \qquad u_{m-1} > 0.
\end{cases}
$$

By Proposition 3.3.2, we know that the maximal length of an antichain sequence in $\mathbb{Z}_{\geqslant 0}^m$ with degree growth bounded by f does not exceed

$$
\Psi_{f,m}(1, (f(1), 0, \ldots, 0)).
$$

Let us also define the sequence $\psi_0, \psi_1, \ldots$ by the relations $\psi_0 = 0$ and

$$
\psi_{i+1} = \Psi_{f_i, m}(1, (f_i(1), 0, \ldots, 0)) + \psi_i, \quad f_i(x) := f(x + \psi_i).
$$

Then Proposition 3.3.6 implies the following:

Proposition 5.4.1 ([31, Corollary 3.14]). *The maximal length of an antichain sequence in* $\mathbb{Z}_{\geqslant 0}^m \times \mathfrak{n}$ *with degree growth bounded by f does not exceed ψ_n.*

Now, let us apply this technique to the functions $f_1(i) = f(n, h)_i$ and $f_2(i) = h f_i$. Then, by Theorem 5.3.4 and Corollary 5.3.5, an upper bound on the weights of the output of Rosenfeld-Gröbner will be $f_1(\mathfrak{L}_{f_1,2}^n + 1)$ if $m = 2$ and $f_2(\mathfrak{L}_{f_2,m}^n + 1)$ if $m > 2$. In general, we do not have formulas for $\mathfrak{L}_{f_1,2}^n$ and $\mathfrak{L}_{f_2,m}^n$ for arbitrary h, m, n that improves the one given in Proposition 5.4.1; however, we can compute $\mathfrak{L}_{f_1,2}^n$ and $\mathfrak{L}_{f_2,m}^n$ for some specific values of h, m, n.

If $\mathcal{W}(F \cup K) = \mathcal{H}(F \cup K) = h$, we can use Corollary 5.3.7 to produce perhaps sharper bounds for the order of the elements of Rosenfeld-Gröbner(F, K) with respect to particular derivations. In the examples that follow, we calculate upper bounds for $\mathrm{ord}(g, \partial_1)$ for $g \in$ Rosenfeld-Gröbner(F, K), where $w\left(\partial_1^{i_1} \ldots \partial_m^{i_m}\right) = c_1 i_1 + \ldots + c_m i_m$ in the case in which $c_1 = 2$ and the case in which $c_1 = 3$. We note that in the tables that follow, "N/A" appears whenever we cannot have the given initial order h with given c_i as part of the weight function.

1. Assume that $n = 1$ and $m = 2$. Then the maximal length of an antichain sequence does not exceed $h + 1$. In this case, the weights of the resulting polynomials are bounded by $f(1, h)_{h+2}$, which results in Table 5.1.

Table 5.1: Case $m = 2$ and $n = 1$

h	1	2	3	4	5	6	7	8	9	10
$f(1, h)_{h+2}$	1	4	11	25	55	106	205	386	713	1297
$\mathrm{ord}(g, \partial_1)$, $c_1 = 2$	N/A	2	5	12	26	53	102	193	356	648
$\mathrm{ord}(g, \partial_1)$, $c_1 = 3$	N/A	N/A	3	8	17	35	68	128	237	432

2. Assume that $m = 2$ and n is arbitrary. Then the maximal length of an antichain sequence does not exceed b_n, where b_n satisfies $b_1 = h+1$ and $b_{n+1} = f(n, h)_{b_n+1}+b_n+1$, which results in Table 5.2.

Table 5.2: Case $m = 2$ and n arbitrary

n	h	b_n	$f(n, h)_{b_n+1}$	$\mathrm{ord}(g, \partial_1)$, $c_1 = 2$	$\mathrm{ord}(g, \partial_1)$, $c_1 = 3$
2	1	5	4	N/A	N/A
2	2	9	77	38	N/A
2	3	18	9,960	4,980	3,320
2	4	34	$31,206,974$	$15,603,487$	$10,402,324$
3	1	11	90	N/A	N/A

3. Assume that $m = 3$ and $n = 1$. We can construct the maximal length antichain sequence of $\mathbb{Z}_{\geqslant 0}^3$ using the methods of [31] and the function $f(i) = hf_i$, resulting in the following sequence:

$$(h, 0, 0), (h - 1, 1, 0), (h - 1, 0, h + 1), (h - 2, 2h + 2, 0), \ldots,$$

$$(h - 2, 0, hf_{2h+6} - (h - 2)), \ldots, (h - i, hf_{c_{i-1}+1} - (h - i), 0), \ldots,$$

$$(h - i, 0, hf_{c_i} - (h - i)), \ldots, (0, hf_{c_{h-1}+1}, 0), \ldots, (0, 0, hf_{c_h}),$$

where the sequence c_i is given by $c_0 = 1$ and for $1 \leqslant i \leqslant h$,

$$c_i = c_{i-1} + 1 + hf_{c_{i-1}+1} - (h - i).$$

As a result, we see that the maximal length of an antichain sequence is equal to c_h.

Table 5.3 shows some maximal lengths $\mathfrak{L}_{f,m}^n$ and weights $f(\mathfrak{L}_{f,m}^n + 1)$, where $f(i) = hf_i$, for $m = 3$, 4, and 5.

Table 5.3: Case m, n arbitrary

m	n	h	length	weight
3	1	1	3	3
3	1	2	10	178
3	1	3	712	$\leqslant 10^{150}$
3	2	1	$433,494,480$	$\leqslant 10^{90,594,990}$
4	1	1	5	8
5	1	1	20	10,946

5.5   Order lower bound

This section gives a lower bound for the orders of the output of Rosenfeld-Gröbner, coming from the lower bound for degrees of elements of a Gröbner basis from [43]. To be specific, we show that for m, h sufficiently large, there is a set of r differential polynomials $F \subseteq \mathcal{K}\{y\}$ of order at most h, where $\mathcal{K}$ is equipped with m derivations, $r \sim m/2$, and $\mathcal{K}$ is constant with respect to all of the derivations, such that if $\mathcal{A} = \mathsf{Rosenfeld\text{-}Gröbner}(F, \{1\})$, then

$$\mathcal{H}\left(\bigcup_{(A,H)\in\mathcal{A}} (A \cup H) \right) \geqslant h^{2^r}. \tag{5.5.1}$$

The arguments presented here are standard, and we include them for completeness. We first note the following standard fact about differential ideals generated by linear differential polynomials.

Proposition 5.5.1. *Suppose $F, K \subseteq \mathcal{K}\{Y\}$ are composed of linear differential polynomials. Then the output of $\mathsf{Rosenfeld\text{-}Gröbner}(F, K)$ is either empty or consists of a single regular differential system (A, H) with A and H both composed of linear differential polynomials.*

Suppose now we apply Rosenfeld-Gröbner to $(F, \{1\})$, where F consists of linear differential polynomials, in order to obtain a regular decomposition of $\{F\}$. Since every element of F is linear, $[F]$ is a prime differential ideal, so by Proposition 5.5.1, we have

$$[F] = \{F\} = [A] : H^\infty$$

for some regular differential system (A, H), with A and H both composed of linear differential polynomials. Since every element of A is linear, after performing scalar multiplications and addition, A can be transformed to an autoreduced set $\bar{A}$ without affecting the leaders and orders of elements of A. Since (A, H) is a regular differential system, $\bar{A}$ is a characteristic set

of $[F]$. So, it suffices to find a lower bound on the orders of elements of linear characteristic sets in $\mathcal{K}\{Y\}$.

There is a well-studied one-to-one correspondence between polynomials in $\mathcal{K}[X_1, \ldots, X_m]$ and homogeneous linear differential polynomials in $\mathcal{K}\{y\}$ with m derivations and $\mathcal{K}$ a field of constants, as seen in Section 4.4:

$$\sum c_{i_1,\ldots,i_m} X_1^{i_1} \cdot \ldots \cdot X_m^{i_m} \leftrightarrow \sum c_{i_1,\ldots,i_m} \partial_1^{i_1} \ldots \partial_m^{i_m} y. \tag{5.5.2}$$

Any orderly ranking on Θy then determines a graded monomial order on $\mathcal{K}[X_1, \ldots, X_m]$.

Given a polynomial $f \in \mathcal{K}[X_1, \ldots, X_m]$, let $\tilde{f} \in \mathcal{K}\{y\}$ be its corresponding differential polynomial under (5.5.2). By the discussion above, if we have a collection of polynomials $f_1, \ldots, f_r \in \mathcal{K}[X_1, \ldots, X_m]$, we can construct a characteristic set $C = \{C_1, \ldots, C_s\}$ of $[\tilde{f}_1, \ldots, \tilde{f}_r] \subseteq \mathcal{K}\{y\}$ consisting of homogeneous linear differential polynomials, and so each $C_i \in \mathcal{K}\{y\}$ is in fact equal to $\tilde{g}_i$ for some $g_i \in \mathcal{K}[X_1, \ldots, X_m]$.

Proposition 5.5.2 (cf. [42, page 352],[13]). *With the notation above,*

$$\{g_1, \ldots, g_s\} \subseteq \mathcal{K}[X_1, \ldots, X_m]$$

is a Gröbner basis of the ideal $I = (f_1, \ldots, f_r)$.

By Proposition 5.5.2, we can find a lower bound for the orders of the output of Rosenfeld-Gröbner via a lower bound for the degrees of elements of a Gröbner basis, as shown:

Example 5.5.3. This example demonstrates the lower bound (5.5.1) for the orders of the output of Rosenfeld-Gröbner. In [43, Section 8], for m, h sufficiently large, a collection of m algebraic polynomials $f_1, \ldots, f_r$ of degree at most h in m algebraic indeterminates, with $r \sim m/2$, is constructed such that any Gröbner basis of $(f_1, \ldots, f_r)$ with respect to a graded monomial order has an element of degree at least h^{2^r}.

As a result of the previous discussion, we have a collection of differential polynomials $F = \tilde{f}_1, \ldots, \tilde{f}_r \in \mathcal{K}\{y\}$ of order h with m derivations such that any linear characteristic set of $[\tilde{f}_1, \ldots, \tilde{f}_r]$ will contain a differential polynomial of order at least h^{2^r}. Since in this case $\{(A, H)\} = \mathsf{Rosenfeld\text{-}Gröbner}(F, \{1\})$ can be transformed into a linear characteristic set without affecting the orders of the elements, this means that

$$\mathcal{H}(A \cup H) \geqslant h^{2^r}.$$

Bibliography

[1] I. Bentsen. The existence of solutions of abstract partial difference polynomials. *Transactions of the American Mathematical Society*, 158:373–397, 1971. URL `https://doi.org/10.1090/S0002-9947-1971-0279078-0`.

[2] F. Boulier, D. Lazard, F. Ollivier, and M. Petitot. Representation for the radical of a finitely generated differential ideal. In *ISSAC'95: Proceedings of the 1995 international symposium on Symbolic and algebraic computation*, pages 158–166, New York, NY, USA, 1995. ACM Press. URL `http://dx.doi.org/10.1145/220346.220367`.

[3] F. Boulier, D. Lazard, F. Ollivier, and M. Petitot. Computing representations for radicals of finitely generated differential ideals. *Applicable Algebra in Engineering, Communication and Computing*, 20(1):73–121, 2009. URL `http://dx.doi.org/10.1007/s00200-009-0091-7`.

[4] N. Bourbaki. *Algèbre commutative. Chapitres 1 à 4.* Éléments de Mathématique. Springer, 2006. URL `http://dx.doi.org/10.1007/978-3-540-33976-2`.

[5] W. D. Brownawell. Bounds for the degrees in the Nullstellensatz. *Annals of Mathematics*, 126(3):577–591, 1987. URL `http://dx.doi.org/10.2307/1971361`.

[6] W. Bruns and J. Herzog. *Cohen-Macaulay rings.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1993.

[7] R. M. Cohn. *Difference algebra.* Interscience Publishers, John Wiley & Sons, New York-London-Sydeny, 1965.

[8] R. M. Cohn. Specializations of differential kernels and the Ritt problem. *Journal of Algebra*, 61:256–268, 1979. URL `http://dx.doi.org/doi:10.1016/0021-8693(79)90316-8`.

[9] L. D'Alfonso, G. Jeronimo, and P. Solernó. Effective differential Nullstellensatz for ordinary DAE systems with constant coefficients. *Journal of Complexity*, 30(5):588–603, 2014. URL `http://dx.doi.org/10.1016/j.jco.2014.01.001`.

[10] T. Dubé. A combinatorial proof of the effective Nullstellensatz. *Journal of Symbolic Computation*, 15(3):277–296, 1993. URL `http://dx.doi.org/10.1006/jsco.1993.1020`.

[11] D. Figueira, S. Figueira, S. Schmitz, and P. Schnoebelen. Ackermannian and primitive-recursive bounds with Dickson's lemma. In *Logic in Computer Science (LICS), 26th Annual IEEE Symposium*, pages 269–278, 2011. URL `http://dx.doi.org/10.1109/LICS.2011.39`.

[12] J. Freitag and O. León Sánchez. Effective uniform bounds in partial differential fields. *Advances in Mathematics*, 288:308–336, 2016. URL `http://dx.doi.org/10.1016/j.aim.2015.10.013`.

[13] V. Gerdt. Involutive algorithms for computing Gröbner bases. In *Computational commutative and non-commutative algebraic geometry*, volume 196 of *NATO Science Series III: Computer and Systems Sciences*. IOS, Amsterdam, 2005. URL `http://arxiv.org/abs/math/0501111`.

[14] O. Golubitsky, M. Kondratieva, M. Moreno Maza, and A. Ovchinnikov. A bound for Rosenfeld-Gröbner algorithm. *Journal of Symbolic Computation*, 43(8):582–610, 2008. URL `http://dx.doi.org/10.1016/j.jsc.2007.12.002`.

[15] O. Golubitsky, M. Kondratieva, A. Ovchinnikov, and A. Szanto. A bound for orders in differential Nullstellensatz. *Journal of Algebra*, 322(11):3852–3877, 2009. URL `http://dx.doi.org/10.1016/j.jalgebra.2009.05.032`.

[16] D. Grigoriev. Complexity of quantifier elimination in the theory of ordinary differential equations. *Lecture Notes Computer Science*, 378:11–25, 1989. URL `http://dx.doi.org/10.1007/3-540-51517-8_81`.

[17] R. Gustavson and O. León Sánchez. Effective bounds for the consistency of differential equations. Preprint, 2016. URL `http://arxiv.org/abs/1601.02995`.

[18] R. Gustavson and O. León Sánchez. A new bound for the existence of differential field extensions. In *Mathematical Aspects of Computer and Information Sciences*, number 9582 in Lecture Notes in Computer Science, pages 358–361, Switzerland, 2016. Springer. URL `http://dx.doi.org/10.1007/978-3-319-32859-1_30`.

[19] R. Gustavson, M. Kondratieva, and A. Ovchinnikov. New effective differential Nullstellensatz. *Advances in Mathematics*, 290:1138–1158, 2016. URL `http://dx.doi.org/10.1016/j.aim.2015.12.021`.

[20] R. Gustavson, A. Ovchinnikov, and G. Pogudin. Bounds for orders of derivatives in differential elimination algorithms. In *ISSAC '16: Proceedings of the 2016 ACM International Symposium on Symbolic and Algebraic Computation*, pages 239–246, New York, NY, USA, 2016. ACM Press. URL `http://dx.doi.org/10.1145/2930889.2930922`.

[21] R. Gustavson, A. Ovchinnikov, and G. Pogudin. New order bounds in differential elimination algorithms. Preprint, 2016. URL `https://arxiv.org/abs/1602.00246`.

[22] M. Harrison-Trainor, J. Klys, and R. Moosa. Nonstandard methods for bounds in differential polynomial rings. *Journal of Algebra*, 360:71–86, 2012. URL `http://dx.doi.org/10.1016/j.jalgebra.2012.03.013`.

[23] J. Heintz and C.-P. Schnorr. Testing polynomials which are easy to compute. In *Logic and algorithmic (Zurich, 1980)*, volume 30 of *Monograph. Enseign. Math.*, pages 237–254. Univ. Genève, Geneva, 1982.

[24] E. Hubert. Notes on triangular sets and triangulation-decomposition algorithms II: Differential systems. In U. Langer and F. Winkler, editors, *Symbolic and Numerical Scientific Computations*, volume 2630 of *Lecture Notes in Computer Science*, pages 40–87. Springer, 2003. URL `http://dx.doi.org/10.1007/3-540-45084-X_2`.

[25] Z. Jelonek. On the effective Nullstellensatz. *Inventiones Mathematicae*, 162(1):1–17, 2005. URL `http://dx.doi.org/10.1007/s00222-004-0434-8`.

[26] I. Kaplansky. *An Introduction to Differential Algebra*. Hermann, Paris, 1957.

[27] E. R. Kolchin. *Differential algebra and algebraic groups*. Academic Press, New York, 1973. Pure and Applied Mathematics, Vol. 54.

[28] J. Kollár. Sharp effective Nullstellensatz. *Journal of the Americal Mathematical Society*, 1(4):963–975, 1988. URL `http://dx.doi.org/10.1090/S0894-0347-1988-0944576-7`.

[29] T. Krick, L. M. Pardo, and M. Sombra. Sharp estimates for the arithmetic Nullstellensatz. *Duke Mathematical Journal*, 109(3):521–598, 2001. URL `http://dx.doi.org/10.1215/S0012-7094-01-10934-4`.

[30] B. A. Lando. Jacobi's bound for the order of systems of first order differential equations. *Trans. Amer. Math. Soc.*, 152:119–135, 1970. ISSN 0002-9947. URL `http://dx.doi.org/10.2307/1995642`.

[31] O. León Sánchez and A. Ovchinnikov. On bounds for the effective differential Nullstellensatz. *Journal of Algebra*, 449:1–21, 2016. URL `http://dx.doi.org/10.1016/j.jalgebra.2015.10.009`.

[32] F. S. Macaulay. Some properties of enumeration in the theory of modular systems. *Proceedings of the London Mathematical Society*, 26(2):531–555, 1927.

[33] R. Moosa and T. Scanlon. Jet and prolongation spaces. *Journal of the Institute of Mathematics of Jussieu*, 9(2):391–430, 2010. URL `http://dx.doi.org/10.1017/S1474748010000010`.

[34] A. Ovchinnikov, G. Pogudin, and N. T. Vo. Effective differential elimination and Nullstellensatz. Preprint, 2016. URL `https://arxiv.org/abs/1610.04022`.

[35] D. Pierce. Fields with several commuting derivations. *Journal of Symbolic Logic*, 79(1): 1–19, 2014. URL http://dx.doi.org/10.1017/jsl.2013.19.

[36] J. Ritt. *Differential Equations from the Algebraic Standpoint*. American Mathematical Society Colloquium Publications, New York, 1932.

[37] J. Ritt. *Differential Algebra*. American Mathematical Society, 1950.

[38] B. Sadik. The complexity of formal resolution of linear partial differential equations. *Lecture Notes in Computer Science*, 948:408–414, 1995. URL http://dx.doi.org/10. 1007/3-540-60114-7_31.

[39] A. Seidenberg. An elimination theory for differential algebra. *University of California publications in Mathematics*, III(2):31–66, 1956.

[40] G. M. Socías. An ackermannian polynomial ideal. In *Volume 539 of Lecture Notes in Computer Science*, pages 269–280. Springer, 1991. URL http://dx.doi.org/10.1007/ 3-540-54522-0_116.

[41] E. Sperner. On a combinatorial theorem of macaulay and its applications to the theory of polynomial ideals. *ACM Communications in Computer Algebra*, 42(4):192–201, 2008. URL http://dx.doi.org/10.1145/1504341.1504343.

[42] W. Wu. On the construction of Groebner basis of a polynomial ideal based on Riquier-Janet theory. In D. Wang and Z. Zheng, editors, *Differential Equations with Symbolic Computation*, Trends in Mathematics, pages 351–368. Birkhäuser Verlag, 2005. URL http://dx.doi.org/10.1007/3-7643-7429-2_20.

[43] C. Yap. A new lower bound construction for commutative Thue systems with applications. *Journal of Symbolic Computation*, 12:1–27, 1991. URL http://dx.doi.org/10. 1016/S0747-7171(08)80138-1.