FINITE INVESTIGATIONS OF TRANSFINITE DERIVATIONS*

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ω-derivations of arithmetic formulas are analyzed. A primitive recursive normalization operator E is constructed in the first part. It cut-eliminates not only recursively described derivations (i.e., well-founded proof-figures) but also arbitrary (not necessarily well-founded) proof-figures constructed from an axiom by derivation rules. This permits us to apply E in the theory of models. Its application in the theory of proofs is based on the formalizability of the fundamental properties of E in a primitive recursive arithmetic. Cut-eliminability in the Heyting arithmetic $HA^{\omega}+AC$ with the axiom of choice of all finite types is proved in the second part. The formulation allowing cut-elimination uses terms associated with the derivations by a method due to Carry, Howard, Girard, and Martin-Löf. These terms are included in the very formulation of the rules. The conservativity of $HA^{\omega}+AC$ over HA is obtained as one of the corollaries.

Introduction

The main objects of our investigation are derivations in Gentzen systems containing an infinite induction rule (infinite $\forall$-rule, ω-rule, Carnap rule):

$$\frac{\Gamma\rightarrow A[0]\ldots\Gamma\rightarrow A[N]\ldots}{\Gamma\rightarrow\forall xA}\left(\rightarrow\forall_{\infty}\right). \tag{1}$$

Systems of such kind were first considered (as far as the author is aware) by P. S. Novikov and P. Lorenzen as long ago as the end of the Thirties. An elegant formulation, allowing numerous applications in the theory of proofs, was presented and investigated by Schütte [1].

A majority of applications of Gentzen systems depend on the proof of admissibility (or eliminability) of the cut rule, which we find convenient to formulate thus:

$$\frac{\Gamma,\Sigma\rightarrow C\quad C,\Gamma,\Pi\rightarrow B}{\Gamma,\Sigma,\Pi\rightarrow B}. \tag{2}$$

The majority of proofs existing in the literature (see [1], for instance, or the Appendix to the Russian translation of [2]) for cut-eliminability in infinite systems are generalizations of the Gentzen's original proof for predicate calculus. They are carried out by induction on the maximal complexity of the cut formulas; moreover, the induction transition is proved by induction on the length of the transformed derivation. In view of the presence of the rule $\rightarrow\forall_{\infty}$ this length, in general, is an infinite ordinal, so that the latter induction is

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transfinite. In the situation (most often examined) when the derivations being studied are described by general recursive functions, the resulting cut-free derivation is constructed either with the aid of the theorem on recursions or by recursion over the length of the given derivation. In both cases the resulting cut-elimination operator proves to be only partially recursive: it operates correctly on infinite proof-figures which are derivations; in return it may not yield any results on the remaining proof-figures. In connection with this the problem arises of constructing an everywhere-defined cut-elimination operator. Let us show, restricting ourselves for simplicity to arithmetic systems, how to construct a primitive recursive cut-elimination operator. The paper's first part is devoted to this. (To avoid misunderstanding we remark that we have in mind operators obtained by a finite number of substitutions and primitive recursions and, by the same token, given on arbitrary functions and not just on the Gödel numbers of general recursive or primitive recursive functions. In particular, such an operator is automatically continuous.) It turns out that we can succeed in investigating by finite means the fundamental properties of the operator E constructed; more precisely, within the framework of primitive recursive arithmetic with free functional variables. This, in its own turn, enables us to give finite proofs of a number of statements which till now have been proved only by transfinite induction. A typical example is the equivalence of the usual formalization of arithmetic and the cut-free formalization from [3]. It is precisely such kind of application in the theory of proofs that served as an impetus for the construction of E. The finiteness of E makes it a convenient tool for the investigation of derivations in the usual systems of formal arithmetic. In this way we have succeeded, for instance, in establishing the equivalence of various theorems of effective existence [4] and also in giving a proof of the normalizability theorem for finite arithmetic [5], which the author feels is the simplest one of the known proofs.

The applications of E in the theory of models arise, in particular, from the fact that E is defined on proof-figures constructed in accord with the rules of the system being examined, but not being derivations because of the presence of infinite branches (i.e., not being well-founded). Proof-figures of such kind appear when considering models of limited complexity [6]. It turns out that E processes well-founded ones, while too trivial infinite branches do not occur in non-well-founded proof-figures. This enables us to estimate the models' complexity.

The construction of operator E became possible as a result of using the repetition rule $\mathfrak{S} \vdash \mathfrak{S}\,(\mathcal{R}ep)$, which can be eliminated in the classical case.

A comparision with the well-known constructions from recursion theory (like the theorem on the primitive recursive enumerability of a nonempty enumerable set) suggests the following scheme. We fix some standard partially recursive cut-elimination operator E^{p} (described in [1], for instance). Now, in order to compute the value of E at some point we shall build over this point applications of $\mathcal{R}ep$ until the value of E^{p} is computed. Such an approach, however, would make it difficult to obtain the applications described above (and mentioned below); actually, we shall use a more economical scheme based on an analysis of the working of operator E^{p}.

The construction of operator E enables us to establish that certain initial data, necessary for the working of the standard operator E^{p}, are in reality not needed for solving some of the problems to be examined. The operator E^{p} uses not only the very given derivation with a cut but also the degree (the maximal complexity of the cuts) of the given derivation and its ordinal length (actually, instead of the length we could everywhere speak of the upper bound of the length). For the construction of the resulting derivation the operator E uses only the original derivation itself. The remaining parameters (the degree and the length) are used only for the computation of the length of the resulting derivation. (Moreover, as A. G. Dragalin showed, E^{p} can be modeified in such a way that it too will not require superfluous initial data on its own domain.)

In order to construct, as far as possible, a description of the working of operator E, we shall, as a preliminary, describe in detail and analyze the working of the standard operator E^{p}. In all the remaining analyses (besides a few particularly specified cases) we shall stay within the framework of a conservative extension of primitive recursive arithmetic in whose language we shall allow any arithmetic formulas (including those containing quantifiers), but we shall allow only quantifier-free induction predicates in the induction postulate. An equivalent formulation is: the induction axiom is excluded entirely, but defining relations for the bounded μ-operator are added on. Among the logical rules there are all the rules for quantifiers in the usual formulation. It is precisely this system that we shall denote as PRA (primitive recursive arithmetic). We remark that the derivability in PRA of a theorem of the form

$$\forall x\, R(x) \rightarrow \forall y\, S(y) \tag{3}$$

with primitive recursive R and S follows from the derivability in PRA (and, hence, in the usual quantifier-free formulation of primitive recursive arithmetic) of a finite strengthening of theorem (3): $\forall x_{<\varphi(y)} R(x) \rightarrow S(y)$ for some primitive recursive φ .

This remark proves to be essential for what follows in view of the fact that the majority of the syntactical statements of interest to us are of form (3).

In the paper's second part we examine extensions of Heyting arithmetic containing some form or other of the axiom of choice (AC)

$$\forall \alpha \exists \beta A \rightarrow \exists \gamma \forall \alpha A_{\beta}[\gamma(\alpha)]. \tag{4}$$

Normalizability proofs for such systems were unknown up to this point. The weakest of the extensions to be considered, $HA+AC^{0\to 0}$, has the same language as HA (Heyting arithmetic) (with free variables for unary functions) and in it instead of (2) there is the rule

$$\frac{\Gamma \rightarrow \forall x \exists y A \quad \forall x A_{y}[f(x)], \Gamma \rightarrow C}{\Gamma \rightarrow C}, \tag{5}$$

where f does not enter into A, Γ, and C. This formulation is equivalent to a special case of scheme (4), in which α and β are variables of type 0 and γ is a variable of type 1.

The strongest of the extensions to be considered, $HA^{\omega}+AC$, is an arithmetic with quantifiers and with an axiom of choice with respect to variables of all finite types. The construction, for the systems being examined, of formulations admitting of normalization (i.e., of cut-elimination) is made difficult by the fact that a normal proof of a formula of the form $\exists\alpha\forall x A[x,\alpha(x)]$ must end with an indication of α, whereas its natural mathematical proof most often will end with the formula $\forall x\exists y A[x,y]$ and an application of the axiom of choice. The introduction of constants for all β-recursive functions with $\beta<\varepsilon_0$ or for Gödel primitive recursive functionals also does not allow us to give explicitly all the functions α needed. A formulation admitting of normalization is constructed by using terms which are associated with the proofs by the Carry—Howard—Girard—Martin-Löf method (cf. [4]).

For $HA+AC^{0\to 0}$ we can succeed in obtaining the properties of being a corollary by passing to a formulation with a (constructively applicable) ε-symbol. In the case of $HA^{\omega}+AC$ this property, generally speaking, does not hold even in derivations which we call normal, but it can be obtained in the derivations of HA-formulas with the aid of additional transformations. By the same token we obtain a proof of the theorem announced by Goodman on the conservativity of $HA^{\omega}+AC$ over HA.

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PART 1. FIRST-ORDER ARITHMETIC SYSTEMS

1. Formal Heyting Arithmetic System

The terms of a HA system are constructed from 0 and from individual variables with the aid of unary free functional variables, of the binary functionals symbols + and ·, and of the unary functional symbol S. The terms $0, S0, SS0, \ldots$ are the digits. Simple terms are terms of the form C', where C is a variable, as well as terms of the form $f(m)$, $m+n$, $m\cdot n$, where f is a functional variable and m and n are variables or digits. Atomic formulas are equalities of terms, while arbitrary formulas are constructed in the usual way from atomic ones with the aid of the connectives $\&, \supset, \neg, \forall, \exists$. A disjunction is introduced as an abbreviation:

$$A\vee B \leftrightharpoons \exists x((x=0\supset A)\,\&\,(x\neq 0\supset B)).$$

Sequences are expressions of the form $\Gamma\to A$, where Γ is a list (possibly, empty) of formulas and A is a formula.

Our axiomatics for HA will differ from the axiomatics for the intuitionistic arithmetic system from [2], mainly, in the intensive use of the convolution axiom $\exists x(t=x)$. Its postulates are propositional rules of the Gentzen formulation of intuitionistic predicate calculus with the retention of the principal antecedent formulas, as well as $\to\forall$ and $\exists\to$. A cut will be applied in the form

$$\frac{\Gamma,\Sigma\to A \quad A,\Gamma,\Pi\to B}{\Gamma,\Sigma,\Pi\to B}\,(C).$$

We shall apply rules without accounting for the order of the formulas within the antecedents. (Thus, we are freed from rearrangement rules. The rule for the cancellation of repetitions turns out to be admissible because of the preservation of principal formulas, while the rule for refinement, because of the fact that the axioms will be invariant relative to it.)

The rules $\forall\to, \to\exists$ differ from the usual versions only in that the term m being substituted can be only a variable or a digit,

$$\frac{\Gamma\to A[m]}{\Gamma\to\exists x A}(\to\exists) \qquad \frac{A[m],\forall x A,\Gamma\to B}{\forall x A,\Gamma\to B}(\forall\to).$$

Among the postulates for the equalities, besides the axiom $a=a$; $t=a,\ E[t],\ \Gamma\to E[a]$; $t=a,\ E[a],\ \Gamma\to E[t]$ $(\mathcal{E}q)$, where a is a variable and t is a simple term or a variable, there also occurs the convolution rule

$$\frac{t=b,\Gamma\to A}{\Gamma\to A}\,(Comp),$$

where b is a variable not occurring in t,Γ,A; t is a simple term.

The arithmetic postulates are the defining equalities for $+$ and $\cdot$, the axioms S are

$$\Gamma,\ sa=sb\to a=b \qquad \Gamma,\ sa=0\to B,$$

and the induction rule is

$$\frac{\Gamma\to A[0] \quad \Gamma,A[b]\to A[sb]}{\Gamma\to\forall x A}\,(JR).$$

Let us prove the equivalence of the HA system described and the system from [2] which we shall denote by G. First of all, all the postulates of HA are derivable in G; in particular, $Comp$ is justified with the aid of the derivable formula $\exists x(t=x)$.

The derivability in HA of the postulates G is proved in several stages. The sign $\vdash$ in this section denotes derivability in the HA formulation being examined.

<u>LEMMA 1.1.</u> (a) $S\vdash S_h[N]$; (b) $\vdash a=b\to b=a$; (c) $\vdash a=b,\ b=c\to a=c$.

(a) is proved in standard fashion with the aid of $\to\supset, \to\forall$, $\forall\to$. (b) is obtained from $a=a$ with the aid of $\mathcal{E}q$. (c) is a special case of $\mathcal{E}q$.

LEMMA 1.2. (a) $\vdash t=b, A[t]\to A[b]$; $\vdash t=b, A[b]\to A[t]$, t is a variable or a simple term and A is an arbitrary formula;

(b) $\vdash \tau=b, A[\tau]\to A[b]$ $\quad \vdash \tau=b, A[b]\to A[t]$;

(c) $\vdash \forall x A\to A[\tau]$ $\quad \vdash A[\tau]\to\exists x A$;

(d) $S\vdash S_b[\tau]$, where τ is an arbitrary term and S is an arbitrary sequent.

Proof. (a) Induction on the length of A. (b) Induction on the length of τ. *Comp* is applied in the induction transition. (c) From (b) with the aid of $\to\exists$, $\forall\to$, *Comp*. (d) Analogously to (a) in Lemma 1.1.

Items (c) and (d) of Lemma 1.2 permit us to derive all the postulates of the positive predicate calculus. Having substituted the formula $A[0]\,\&\,\forall x(A\supset A[sx])$ in the place of Γ in the induction rule, it is easy to derive the induction axiom. By the same token, we have proved the equivalence of G and HA.

2. Language and Postulates of an Infinite (Semiformal) System

With the aim of simplifying the formal details a large part of the results will be stated and proved for the arithmetic $\{\forall,\neg\}$-language with free functional variables and intuitionistic rules. Almost all of these results admit of generalization to wider languages (including a second-order language) and to classical systems. These generalization will be considered in § 8.

The choice of the $\{\forall,\neg\}$-language and the intuitionistic rule is due, on the one hand, to the fact that this combination is especially convenient for presenting the traditional normalization procedures, and on the other, to the fact that it suffices for (the immersion in it) classical arithmetic. Indeed, by leading an arbitrary arithmetic formula to a prenex form without implications and with negations only for equalities, by elimination negations by the formula

$$a\neq b\leftrightarrow\exists z(a^2+b^2=sz+2ab),$$

and by carrying forward the emergence of $\exists z$, we can eliminate $\&$ and $\vee$ by the formulas

$$a=a_1\,\&\,b=b_1\leftrightarrow a^2+a_1^2+b^2+b_1^2=2aa_1+2bb_1$$

$$a=a_1\vee b=b_1\leftrightarrow ab+a_1b_1=ab_1+a_1b.$$

After this $\exists$ is replaced by $\neg\forall\neg$. As is well known, intuitionistic rules are sufficient for the resultant fragment of classical arithmetic.

The free functional variables are included for the purpose of illustrating the treatment of a decidable equality.

Terms are constructed from 0 and from the object variables with the aid of $s,+,\cdot$ and unary functional variables. Formulas are constructed from equalities with the aid of $\forall$ and $\neg$. A formula is taken to be closed if it does not contain free object variables.

Rules of the Infinite System.

$$\frac{\Gamma\to A[0]\ldots\Gamma\to A[N]\ldots}{\Gamma\to\forall x A}(\to\forall_\infty)\qquad\frac{A[N],\forall x A,\Gamma\to B}{\forall x A,\Gamma\to B}(\forall\to)$$

$$\frac{\neg A,\Gamma\to A}{\neg A,\Gamma\to B}(\neg\to)\qquad\frac{A,\Gamma\to 0=1}{\Gamma\to\neg A}\ (\to\neg)$$

$$\frac{f(M)=0,\Gamma\to B\ldots f(M)=N,\Gamma\to B\ldots}{\Gamma\to B}\qquad\frac{\Gamma\to C}{\Gamma\to C}(Rep)$$

$$\frac{\Gamma,\Sigma\to A\quad A,\Gamma,\Pi\to B}{\Gamma,\Sigma,\Pi\to B}(C). \tag{1}$$

Axioms of the Infinite System.

$$F,\Gamma\to B\qquad\Gamma\to T\ , \tag{2}$$

where F is a false equality (not containing variables) and T is a true equality;

$$f(M)=N,\ E[f(M)],\Gamma\to E[N];\ f(M)=N,\ E[N],\Gamma\to E[f(M)]\ , \tag{3}$$

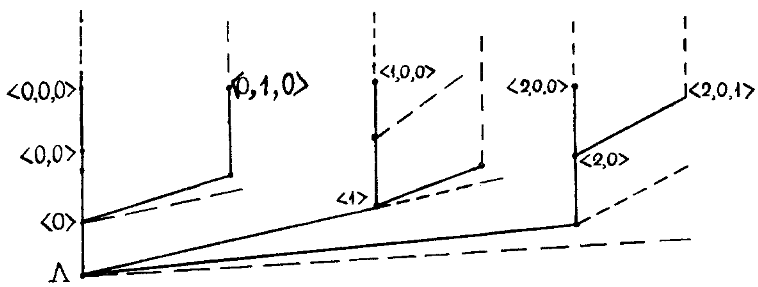
and the sequents obtained from the ones indicated by replacing $f(M)=N$ by $M=f(N)$.

In addition, all sequents derivable from the ones indicated by a finite number of cuts are reckoned as axioms (all the conclusions of such a "complex" axiom must be indicated in its analysis).

The order of the antecedent formulas is considered immaterial.

The definition of a derivation in the infinite systems will consist of two parts: local correctness (the proof is constructed from an axiom by the derivation rules) and well-foundedness, viz., a condition which is formulated from another, but each time in such a way that bar-induction with respect to the proof can be justified by admissible metamathematical means: if something is true for an axiom and is inherited under applications of the rules, then it is true also for any provable sequence.

Infinite derivations will be represented by functions given on the universal flow



i.e., on the (tree-shape ordered) set of all finite sequences of positive integers. We shall use standard functions and relations for such sequences: $(a)_i$ is the i-th term of sequences a ; $lh(a)$ is the length of a (more precisely, $lh(\Lambda)=0$; $lh(\langle a_0,\ldots,a_n\rangle)=n$); $a\subset a'$, where a' is a continuation of a ; $a*i$ is the continuation of a by the number i , i.e., $\langle a_0,\ldots, a_n\rangle * i = \langle a_0,\ldots,a_n,i\rangle$; $a*a'$ is the continuation of a by the sequence a'.

Function f is a proof-figure locally correct relative to the infinite system being examined (abbreviated $LC(f)$) if with each node a of the universal flow it associates either the empty symbol $\varnothing$, indicating that this node already does not belong to the proof, or information (a certain number coding it) including some sequent S_a (located at node a) and an analysis $\mathcal{A}_a$, containing complete information on that application of the postulated, by which S_a is obtained; moreover, the following conditions are fulfilled:

(a) $S_\Lambda \neq \varnothing$ and all analyses are correct: axioms are located at nodes described as axioms; a sequent located at node a, described as the conclusion of a rule, is connected with the sequents located at the immediately preceding nodes, precisely as indicated in the rule. Here, for all rules except Rep the first premise is situated at the node $a*0$, then the second (if it exists) at node $a*1$, etc. In the case of Rep a number $\mathcal{A}_a$ is indicated in analysis i_a, such that the premise is situated at node $a*i_a$.

(b) If the rule indicated in $\mathcal{A}_a$ has a finite number of premises, then the nodes $a*j$ at which there are no premises are empty, i.e., $f(a*j)=\varnothing$.

(c) If $f(a)=\varnothing$ or S_a is an axiom according to $\mathcal{A}_a$, then $f(a*j)=\varnothing$ for any j.

Remark. In the case of a nonempty node a the information $f(a)$ can include, besides S_a and $\mathcal{A}_a$ other objects as well, for example, ordinals or derivations in HA (see below). The condition relative to i_a was introduced in order to obtain in § 5 the elementary operator of lowering the degree of a cut. Up to this point we can take it that $i_a=0$, i.e., a repetition is treated in exactly the same way as the other rules. In what follows it will be established as well that a primitive recursive transformation can be added on to achieve that $i_a=0$ for any a.

For the standard Gödel numbering the condition $LC(f)$ is equivalent to a formula of the form $\forall x\, G(f,x)=0$, where G is an elementary functional.

Kinship relations between occurrences of formulas are defined in a natural way in a locally correct proof-figure. An occurrence of a rule in a premise, arising from an occurrence in Γ, is the ancestor of the same occurrence in Γ of the conclusion (which is reckoned as the descendent of the occurrence in the premise); it is analogous for the succeedent C and the recurrent principal formulas of the antecedent rules. Only the side formulas in the premises ($A, B, A[N]$, $t=N$ for $Comp$) and the formula A in rule (C) have no descendents in the conclusion. The kinship relations are transitive by definition.

It is very well known that the definition of a derivation as a locally correct well-founded proof-figure would yield too broad (for our purposes) a class of provable formulas: all true formulas in the $\{\forall, \neg\}$-language being examined. However, there is a trivial method for obtaining a system equivalent to the original formal system: it is sufficient to equip the infinite derivations of our interest with the information that they were obtained from derivations in the formal system.

If H is the notation for some formal system, then $LCH(f)$ means the $LC(f)$ holds and some derivation d_a of sequent S_a in system H is contained, together with other information at each nonempty node a of proof-figure f.

In connection with the question of cut-eliminability from the proof-figures being examined it is natural to be interested in the complexity of the cuts encountered. Knowledge of this quantity permits us, as we see later on, to give a more exact estimate of the ordinal length (or, as is often said, the height) of the resulting cut-free proof-figure. In the first part we shall use the traditional measure of complexity, viz., the maximal complexity of the cut formulas. In the second part, for systems with the axiom of choice we shall use the complexity measures, stemming from the Hilbert school, traditional for systems with an ε-symbol.

Definition. $LC^+(f)$ means that $LC(f)$ holds and $f(\Lambda)$ contains a positive integer, denoted $deg(f)$ and bounding from above the complexities of all cuts from f.

The complexity of the equality $t=s$ equals 0; the complexity of the formula δA, where $\delta = \neg, \forall x$, is greater by one than the complexity of formula A.

The already mentioned ordinal length of a derivation is given usually with the aid of some system ON of ordinal notation. In order to ensure the formalizability* of all the considerations in PRA, when the ON system is mentioned it is assumed that the ordinal functions and relations needed are given by Kalmár-elementary functions and relations and that their needed properties are provable in PRA. Examples of ON systems of such kind can be drawn from [1].

It can prove to be inconvenient to associate with a derivation its exact ordinal length, and sometimes even impossible. Therefore, in the theory of proofs we use a method which Howard called the ambiguous association of ordinals.

Definition. $WF_{ON}(f)$ signifies that each nonempty node of proof-figure f contains, together with other information, also the ordinal α_a belonging to ON; moreover, the ordinals grow under motion downward with respect to the tree: $\alpha_a > \alpha_{a*i}$.

We remark that the requirement of growth of the ordinals refers to rule Rep. α_a is the upper bound of the length (or the height) of the tree terminating at node a. The ordinal $h(f)$ of proof-figure f is the ordinal α_Λ of its lower node Λ.

3. Connection of an Infinite System with a Normal One

As is well known, formal arithmetic systems are immersed in infinite systems by means of the replacements of $\to\forall$ and $Comp$ by their "infinite" forms and of the induction rule

$$\frac{\Gamma\to A[0] \quad A[b],\Gamma\to A[sb]}{\Gamma\to\forall x A}$$

*If we do not strive to do this essential constraints may not be imposed on ON and, without materially changing the cut-elimination operator, we can use the postulates for ON only in the part touching on order preservation.

by the rule $\to\forall_\infty$ and the cut

$$\dfrac{\Gamma\to A[0]\quad \dfrac{\Gamma\to A[0]\;\; A[0],\Gamma\to A[1]}{\Gamma\to A[1]}\;\ldots\quad \dfrac{\begin{array}{c}\Gamma\to A[0]\\ \vdots\\ \Gamma\to A[N]\end{array}\;\; A[N],\Gamma\to A[sN]}{\Gamma\to A[sN]}\;\ldots}{\Gamma\to\forall x A} \tag{1}$$

Let us describe the immersion in detail so as to illustrate in this simple situation the means which will often be used in what follows and let there exist an elementary operator Φ, yielding, from each HA-derivation d of a closed sequent S not containing connectives other than $\forall, \neg$, a derivation $\Phi(d)$ of the same sequent in HA_∞ of length $<\omega^2$.

THEOREM 3.1. An elementary operator Φ exists such that

$$\mathrm{PRA} \vdash Prov(d,S) \to (LC^+H+WF_{\mathcal{E}_0})(\Phi(d))\;\&\; h(\Phi(d))<\omega\cdot(n_d+1),$$

where $\mathcal{E}_0$ denotes a standard system of notations or ordinals less than ε_0 and n_d denotes the maximum number of applications of induction in one branch of derivation d.

Proof. It can be reckoned that the given derivation d does not contain rules for connectives other than $\forall, \neg$. Let us denote the desired derivation $\Phi(d)$ by f. The contents $f(a)$ of node a are determined by the contents of some node a^* (the preimage of a) from d. This refers to the case when a turns out to be one of the nodes of proof-figure (1). In this case the preimage of the node at which the sequent $\Gamma\to A[0]$ is located will be the node from d, at which the left premise of the corresponding induction is located. The preimage of $\Gamma\to A[M]$ with $M>0$ will be the conclusion of the induction; the preimage $A[M]$ of $\Gamma\to A[sM]$ will be the right premise of the induction. The rule Rep will not be applied. More formally, let us determine by recursion over a the node a^* of derivation d, and the substitution ξ_a of positive integers for certain free variables. It is assumed that d has been written in the form of a tree and possesses the property of purity of variables: the characteristic variable b of rule $\to\forall$ or JR is encountered really only over the corresponding premise of the rule. A superscript d indicates that the object being considered belongs to the given derivation d.

At the node a of the new derivation we place the sequent $S^d_{a^*}\xi_a$ if a is nonempty in d, is not the conclusion of an induction in d, or is such a conclusion but ξ_a does not contain substitutions in the place of the characteristic variable b of this induction. Otherwise, at a we place the sequent $(\Gamma\to A_x[b])\xi_a$, where $\Gamma\to\forall x A$ is the conclusion of the induction being examined. The derivation d_a of sequent S_a in the original system is obtained in the usual manner from the part of derivation d, located over a.

Thus, we assume: $\Lambda^*=\Lambda$; ξ_Λ is empty. Now suppose that a^* and ξ_a have been determined and that a^* is nonempty in d. We consider a number of cases depending on the contents of a^* in d.

1. $\mathfrak{a}^*$ is an axiom. We assume $\mathcal{A}^{f}_{a}$ = axiom.

2. $\mathfrak{a}^*$ is a conclusion of a negation rule. Then

$$(\mathfrak{a}*0)^* = \mathfrak{a}^* * 0,\ \xi_{a*0} = \xi_a,\ \mathcal{A}^f_a = \mathcal{A}^d_{a^*}.$$

3. $\mathfrak{a}^*$ is a conclusion of $\forall\rightarrow$. Then $(\mathfrak{a}*0) = \mathfrak{a}^* * 0$, $\xi_{a*0} = \xi_a \cup \mathbb{0}$, where $\mathbb{0}$ denotes the substitution of zeros in the place of the variables occurring in the premise but not in the $\mathcal{A}^f_a = \mathcal{A}^d_{a^*}$ $(=\forall\rightarrow)$.

3.† $\mathfrak{a}^*$ is a conclusion of $\rightarrow\forall$ or *Comp*. Then $(\mathfrak{a}*i)^* = \mathfrak{a}^* * 0$, $\xi_{a*i} = \xi_a \cup (b/i)$, where b is the characteristic variable of this rule, and $\mathcal{A}^f_a = \mathcal{A}^d_{a^*}$.

4. $\mathfrak{a}^*$ is a conclusion of induction.

4.1. ξ_a does not contain substitutions for the characteristic variable b. Then $\mathcal{A}^f_a = \rightarrow\forall$; $\xi_{a*i} = \xi_a \cup (b/i)$;

$$(\mathfrak{a}*i)^* = \text{ if } i=0 \text{ then } \mathfrak{a}^* * 0 \text{ else } \mathfrak{a}^*$$

4.2. $\xi_a = \xi' \cup (b/M)$. Then $\mathcal{A}^f_a = (\text{cut})$,

$$\xi_{a*0} = \xi_{a*1} = \xi' \cup (b/M-1);\ (\mathfrak{a}*1)^* = \mathfrak{a}^* * 1$$

$$(\mathfrak{a}*0)^* = \text{ if } M=1 \text{ then } \mathfrak{a}^* * 0 \text{ else } \mathfrak{a}^*.$$

It remains to place the ordinals. For each node $\mathfrak{a}$ of derivation f we denote by n_a the maximum number of inductions encountered in one branch of that part of derivation d that is terminated by $\mathfrak{a}^*$, and by m_a the distance from $\mathfrak{a}^*$ to the closest upward conclusion of the induction.

In cases 1, 2, 3, and 4.1 we set $\alpha_a = \omega\cdot n_a + m_a$.

In case 4.2, $\alpha_a = \omega\cdot(n_a - 1) + M$.

Now it is easy to verify that the proof-figure obtained indeed is a proof. The primitive recursion defining f is reduced to an elementary operator since all the $\mathfrak{a}^*$ are bounded by the maximal node encountered in d. The theorem is proved.

4. Reduction of a Cut. Standard Normalization Operator

As in the majority of the cut-elimination procedures available in the literature, our procedure will consist of a series of standard formal transformations, viz., reductions decreasing the complexity of the cut. The individual reductions will almost coincide with the reduction from [1]. The difference will consist, firstly, in the use of repetition rules for

†Case numbered 3 appears twice in the original. I have not renumbered these because the author refers to Case 3 below and I am unable to tell which of them he means — Translator's note.

preserving the structure of the derivation and, secondly, in the use of the quantities i_a to make the transformation of the infinite derivation Kalmar-elementary. All transformations will be defined for arbitrary (in general, non-well-founded) locally correct proof-figures. In addition, it will be proved that well-founded proof-figures go into well-founded ones and that additional information (ordinals, for example) can be transferred it were given.

In a standard cut-elimination scheme there is first determined the reduction operator $\mathcal{R}_0$ applied to the derivations, terminated by a cut, and splitting this cut into finer ones (or obliterating it if the cut formula is an equality) without disturbing the remaining cuts. Next, starting from $\mathcal{R}_0$ there is defined the operator $\mathcal{R}_1$ of lowering by one the power of the cut. $\mathcal{R}_1$ simply applies $\mathcal{R}_0$ to cuts of maximal power, beginning with the uppermost one. In other words, $\mathcal{R}_1$ is determined by bar-recursion on f or by ordinal recursion on the length of f. However, we can treat the resulting recursive equalities as equations on $\mathcal{R}_1$ still to be solved. Precisely such an approach is applied when we examine proofs described by general recursive functions (or, more commonly, when we wish to prove that $\mathcal{R}_1$ is a partially recursive operator defined on all derivations). In this case the equations being analyzed are solved with the aid of a recursion theorem, which yields $\mathcal{R}_1$ in the form of a partially recursive operator on arbitrary LC^+ proof-figures. Next, with the aid of bar-induction or of transfinite induction on the length it is proved that $\mathcal{R}_1$ is defined on all derivations and from them yields derivations once again. In this section we describe $\mathcal{R}_0$ and in the next one we show how to find the solution of the equations for $\mathcal{R}_1$ in the form of an elementary operator.

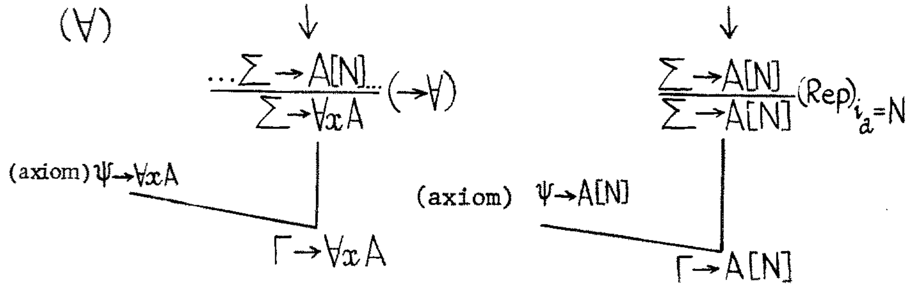
Reductions use the invertibility of the succeedent rules, i.e., the possibility of rearranging any LC proof-figure f terminating in (a sequent which can be) a conclusion of such a rule, in the LC proof-figure g ending in its premise; moreover, the length does not increase (if WF_{ON} is fulfilled). At the same time is proved the admissibility of curtailing the repetitions. We reproduce the corresponding transformations in order to facilitate the understanding of what is to follow.

LEMMA 4.1. An elementary operator exists which from each LC proof-figure f ending in a sequent from the left column forms a certain LC proof-figure g ending in a sequent from the same row of the right column. Here

$$WF_{ON}(f) \to WF_{ON}(g) \,\&\, h(f) = h(g)$$

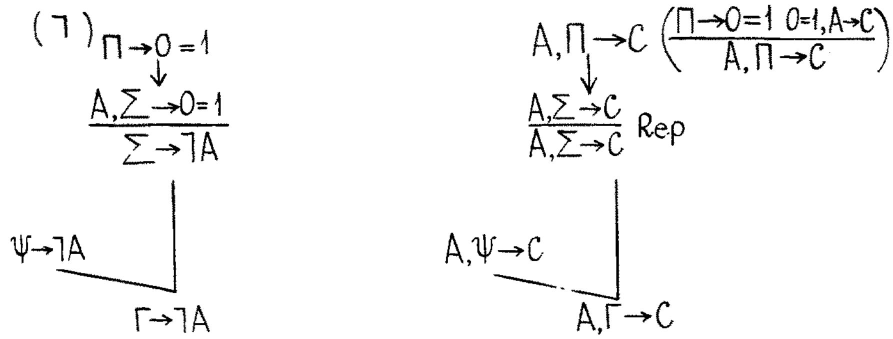
$$\begin{array}{lll} (\forall) & \Gamma \to \forall x A & \Gamma \to A[N] \\ (\neg) & \Gamma \to \neg A & A, \Gamma \to C \\ (\mathrm{Contr}) & A, A, \Gamma \to C & A, \Gamma \to C \end{array}$$

Proof. The given proof-figure f is shown on the left and the new proof-figure g on the right.



All ancestors $\forall x A$ are replaced by $A[N]$. The corresponding $\to\forall$ translate into Rep, whose premises are shifted to the right by N positions in order to ensure that the resulting operator is elementary. More precisely, for each node a of the new proof-figure we could determine its preimage a^* in the old proof-figure. Here it could turn out that (for nonempty a) the lengths of a and a^* coincide and that the maximum of the members of a^* does not exceed the maximum of the members of a. Therefore, the recursion on the length of a, defining a^* in terms of a, proves to be bounded.

The ordinals of the sequents (as in the two remaining subcases) are not changed.



All ancestors $\neg A$ and formulas $0=1$ which pass into this $\neg A$ are replaced by C. A is added on in the antecedents. The cut indicated within the parentheses is included in the analysis of the axiom $A,\Pi\to C$, but not in the proof-figure $\mathcal{g}$ itself, and, therefore, does not lead to a growth in the length.

(Contr) We delete the superfluous antecedent occurrence of A and of all its ancestors. If A is not an equality, then all the postulates are retained. In the single nontrivial case when A is $\varphi(N)=N$, the axiom $\varphi(N)=N,\varphi(N)=N,\Gamma\to\varphi(\varphi(N))=N$ passes into the axiom $\varphi(N)=N,\Gamma\to\varphi(\varphi(N))=N$, which is obtained with the aid of a cut with respect to $N=N$. The lemma is proved.

Let us describe operator R_0. If the given proof-figure f does not end with a sequent, we set $[R_0 f]=f$. If f does end with a sequent, then $[R_0 f]$ is determined by analyzing the cases in dependence on the form of the cut formula. If f has been equipped with ordinals, then $h([R_0 f])=\beta+h(f)$, where β is the ordinal of the left premise of the cut. In each of the cases to be examined the given proof-figure f is shown at the top and the new proof-figure $[R_0 f]$ at the bottom. The transformation of the ordinals is indicated within parentheses.

$$
\begin{array}{ll}
(\beta_2)\ \dfrac{\Sigma \rightarrow A[N]}{(\beta_1)\ \Sigma \rightarrow \forall x A} & \dfrac{A[N], \forall x A, \Sigma \rightarrow \mathfrak{D} \quad (\alpha_3)}{\forall x A, \Sigma \rightarrow \mathfrak{D} \quad (\alpha_2)} \\
(\beta_3)\ \Psi \rightarrow \forall x A & \qquad \forall x A, \Omega \rightarrow F\ (\alpha_4)
\end{array}
$$

$$
(\beta)\ \frac{\Gamma, \Pi \rightarrow \forall x A \qquad \forall x A, \Gamma, \Phi \rightarrow C \quad (\alpha_1)}{\Gamma, \Pi, \Phi \rightarrow C \quad (\alpha)}
$$

New proof-figure:

$$
\begin{array}{l}
(\beta_2)\ \dfrac{\Sigma \rightarrow A[N]}{(\beta_1)\ \Sigma \rightarrow A[N]} \\
(\beta_3)\ \Psi \rightarrow A[N]
\end{array}
$$

$$
(\beta)\ \frac{\Gamma, \Pi \rightarrow A[N] \qquad A[N], [\Gamma, \Pi, \Sigma] \rightarrow \mathfrak{D} \quad (\beta + \alpha_3)}{[\Gamma, \Pi, \Sigma] \rightarrow \mathfrak{D} \quad (\beta + \alpha_2)}
$$

$$
[\Gamma \Pi, \Omega] \rightarrow F \ (\beta + \alpha_4)
$$

$$
(\mathrm{Rep})\ \frac{[\Gamma, \Pi, \Gamma, \Phi] \rightarrow C \quad (\beta + \alpha_1)}{\Gamma, \Pi, \Phi \rightarrow C \quad (\beta + \alpha)}
$$

All ancestors of the antecedent $\forall x A$ are replaced by $\Gamma \Pi$ with an identification in the resulting sequents of the members of list Γ having common ancestors. β is added on from the left to the ordinals of the sequents containing antecedent $\forall x A$. The corresponding $\forall \rightarrow$ are replaced by sequents the derivations of whose left premises are obtained by the lemma on invertibility.

The $\neg$-case is analyzed analogously.

$$
\begin{array}{ll}
\dfrac{A, \Sigma \rightarrow 0 = 1 \ (\beta_2)}{\Sigma \rightarrow \neg A \quad (\beta_1)} & \dfrac{\neg A, \Sigma \rightarrow A \quad (\alpha_3)}{\neg A, \Sigma \rightarrow \mathfrak{D} \quad (\alpha_2)} \\
\qquad \Psi \rightarrow \neg A (\beta_3) & \qquad \neg A, \Omega \rightarrow F \ (\alpha_4)
\end{array}
$$

$$
(\beta)\ \frac{\Gamma, \Pi \rightarrow \neg A \qquad \neg A, \Gamma, \Phi \rightarrow C \quad (\alpha_1)}{\Gamma, \Pi, \Phi \rightarrow C \quad (\alpha)}
$$

New proof-figure:

$$
\begin{array}{l}
\dfrac{A, \Sigma \rightarrow \mathfrak{D} \quad (\beta_2)}{A, \Sigma \rightarrow \mathfrak{D} \quad (\beta_1)} \\
\qquad A, \Psi \rightarrow \mathfrak{D} \ (\beta_3)
\end{array}
$$

$$
\begin{array}{c}
\downarrow \\
(\beta+\alpha_3)\ \dfrac{[\Gamma,\Pi,\Sigma]\to A \quad A,\Gamma,\Pi\to\mathfrak{D}\ (\beta)}{[\Gamma,\Pi,\Sigma]\to\mathfrak{D}\quad(\beta+\alpha_2)} \\
\vdots \\
[\Gamma,\Pi,\Omega]\to F\quad(\beta+\alpha_4) \\
\vdots \\
\dfrac{[\Gamma,\Pi,\Gamma,\Phi]\to C\quad(\beta+\alpha_1)}{\Gamma,\Pi,\Phi\to C\quad(\beta+\alpha)}
\end{array}
$$

Finally, if the cut formula is an equality, we separate the working of R_0 into two stages: at first the cut is raised up to an axiom with respect to the right branch; next, with respect to the left. The given proof-figure

$$
\begin{array}{cccc}
(0) & \Omega\to E & E,\Psi\to F & (0) \\
 & \downarrow & \downarrow & \\
(\gamma) & \Xi\to E & E,\Theta\to G & (\alpha_2) \\
 & \downarrow & \downarrow & \\
(\beta) & \multicolumn{2}{c}{\dfrac{\Gamma,\Pi\to E \qquad E,\Gamma,\Phi\to C}{\Gamma,\Pi,\Phi\to C}} & \begin{array}{c}(\alpha_1)\\(\alpha)\end{array}
\end{array}
$$

is transformed thus:

$$
\begin{array}{ll}
[\Omega,\Psi]\to F & (0)\ \left(\dfrac{\Omega\to E \quad E,\Psi\to F}{[\Omega,\Psi]\to F}\right) \\
\downarrow & \\
[\Xi,\Psi]\to F & (\gamma) \\
\downarrow & \\
[\Gamma,\Pi,\Psi]\to F & (\beta+0) \\
\downarrow & \\
[\Gamma,\Pi,\Theta]\to G & (\beta+\alpha_2) \\
\downarrow & \\
\dfrac{[\Gamma,\Pi,\Gamma,\Phi]\to C}{\Gamma,\Pi,\Phi\to C} & \begin{array}{l}(\beta+\alpha_1)\\(\beta+\alpha)\end{array}
\end{array}
$$

The upper sequents of the new proof-figure are obtained from the upper sequents of the old by means of cuts. β is added on from the left to the ordinals of the sequents containing ancestors of antecedent E. The ordinals of sequents not containing such ancestors are not changed.

The lower part of the new proof-figure (from $[\Gamma,\Pi,\Psi]\to F$ to the end) is obtained from the right part of the old by replacing the ancestors of antecedent E by $\Gamma\Pi$. The upper part is obtained from the left part of the old proof-figure by replacing succeedent E by F and adding Ψ into the antecedent.

We have completed the description of operator R_0.

Remark 1. It is possible to consider a cut with respect to equalities, more like the standard procedure consisting of the deletion of the whole left part (and the replacement of E by $\Gamma\Pi$ in the right part), if E is a true formula, and the deletion of the whole right part (with an analogous replacement), if E is a false formula. (We recall that the standard procedure is applied in the absence of functional variables.) Under such deletions the length of the proof, in general, does not increase. For simplicity let us consider how to write the case when E has the form $f(M)=N$. We continue the given derivation by the rule $Comp$ for $f(M)$. Over the N-th premise of this $Comp$ our cut can be replaced by Rep, while over the remaining premises on the left there turn out to be the axioms

$$f(M)=N,\ f(M)=K, \Gamma, \Phi \to C \qquad N \neq K,$$

which are eliminated by transformation R_0 (without an increase of the ordinal length).

Remark 2. By adding on the rules for equality

$$\frac{t=N, \Gamma[N] \to E[N]}{t=N, \Gamma[t] \to E[t]} \qquad \frac{N=t, \Gamma[N] \to E[N]}{N=t, \Gamma[t] \to E[N]}$$

we can eliminate a cut also from (the definition of) an axiom.

Let us show that operator R_0 is elementary. We consider only the $\forall$-case; the rest are easier. We define the preimages a^* of the nodes a of the new proof-figure $[R_0 f]$ in the old proof-figure f and we show how the contents of node a is computed from the contents of a^*. For the cases when a^* contains ancestors of a succeedent cut formula of the form $\forall x A$ we define the index J_a of node a. J_a is the number N with which we need to apply the inversion lemma. The preimage and the index follow under all passages from bottom to top (compare with cases 2 and 3 of Theorem 3.1) besides the new sequents. In this last case, i.e., when a^* is a conclusion of rule $\forall\to$ belonging to the right out formula, we set

$$(a*1)^* = a^* * 0; \quad (a*0)^* = *0,$$

i.e., $(a*0)^*$ is the left premise of the lower cut.

$J_{a*0}=N$, where N is the number being substituted in the $\forall\to$ being considered, located at node a^*. The sequent $S_a^{[R_0 f]}$ placed at node a of the new derivation $[R_0 f]$ coincides with $S_{a^*}^f$ if $S_{a^*}^f$ does not contain ancestors of the cut formulas and is obtained by the replacement of the succeedent $\forall x A$ by $A[J_a]$ if $S_{a^*}^f$ contains the succeedent ancestor. Finally, if $S_{a^*}^f$ contains an antecedent ancestor, then $S_a^{[R_0 f]}$ is obtained from it by replacing this ancestor by $\Gamma\Pi$. Since R_0 is defined by the correspondence $a \to a^*$ and is given by a bounded recursion, R_0 is elementary, which is what we required.

If now $\mathcal{R}_1$ is determined as indicated at the beginning of this section, then for well-founded proof-figure f satisfying the condition LC^+ there holds $LC^+[\mathcal{R}_1 f]$ & $deg\ [\mathcal{R}_1 f] \leq deg\ (f) \dot{-} 1$. The standard operator $\mathcal{R}$ of complete elimination of cuts from derivations f, satisfying $LC^+[f]$ is determined by recursion on $deg(f)$: $[\mathcal{R}f]=[\mathcal{R}_1\ldots[\mathcal{R}_1 f]\ldots]$, where $\mathcal{R}$ is applied $1+deg(f)$ times.

When $h[\mathcal{R}f]$ is a limit ordinal the ordinal length $h(f)$ is estimated in terms of $h(f)$ and $deg(f)$ in the well known way:

$$h[\mathcal{R}f]=2^{\cdot^{\cdot^{2^{h(f)}}}} \qquad deg(f) \text{ times.}$$

This follows from the relation $h[\mathcal{R}_1 f]\leqslant 2^{h(f)}$ which is justified with the aid of the equations for $\mathcal{R}_1$ and of the bound for R_0. The bounding by limit $h(f)$ is due to the technical peculiarities of our R_0 and will be made clear below. We recall that its fulfillment can always be achieved by replacing $h(f)$ by $h(f)+\omega$.

5. Elementary Operator of Lowering the Degree

The natural domain of operator $\mathcal{R}_1$ consists of LC^+-derivations since we need to know the maximal complexity of the cut in order to apply it. The domain is extended onto LC-derivations if all cuts are reduced or (as we shall do) all cuts of nonzero complexity (nontrivial cuts) are reduced. Therefore, let us consider the equations on operator R_1:

$$[R_1 f]=\begin{cases} \dfrac{[R_1 f^+]}{f(\Lambda)}, & f \text{ does not end with a nontrivial cut,} \\[2ex] \left[R_0 \dfrac{[R_1 f^+]}{f(\Lambda)}\right], & f \text{ ends with a nontrivial cut,} \end{cases}$$

where f^+ denotes the result of deleting the lower node from f.

These are precisely the equations we shall solve. It turns out that the requirement of continuity of R_1 (each finite stump of proof-figure $[R_1 f]$, resulting after the transformation is determined by some finite stump of proof-figure f) together with Eqs. (1) practically compels the primitive recursive definition of R_1. Indeed, comparing the proof-figures f and $[R_1 f]$, we note that the contents of node a of proof-figure $[R_1 f]$ are determined mainly by the contents of some node a^* of proof-figure f. We state below certain properties of the correspondence $a\to a^*$. It turns out that these properties yield the primitive recursive definition needed. The recursion is found to be bounded by virtue of our definition of i_a in the lemma on invertibility, so that as a result we obtain an elementary operator.

We remark that equality in (1) refers only to sequents and analyses and not to additional information. If we denote the right-hand side by g, then

$$S_a^{[R_1 f]}=S_a^{g} \text{ and } \mathcal{A}_a^{[R_1 f]}=\mathcal{A}_a^{g}$$

for any node a. The question on equipping with ordinals will be considered later.

An important role in the description of operator R_1 will be played by the ancestors of the principal formulas of the nontrivial cuts, which we call **C**-formulas. Antecedent C-formulas are called $(C\to)$-formulas and succeedent C-formulas are called $(\to C)$-formulas. The applications of the rules whose principal formulas are C-formulas are analogously classified into C-rules, $(C\to)$-rules, and $(\to C)$-rules.

Thus, let us describe the properties of the correspondence $a \to a^*$ between the nodes of the given proof-figure f and the resulting proof-figure g.

I. $\Lambda^* = \Lambda$

II. If $g(a) \neq \emptyset$, then $f(a^*) = \emptyset$ and one (and only one) of the following conditions 1–2.3 is fulfilled:

1. $\mathcal{A}^g_a = \mathcal{A}^f_{a^*}$, where $\mathcal{A}^f_{a^*} \neq (C)$, the rule situated at nodes a (of proof-figure g) and a^* (of proof-figure f) have like principal and side formulas, and $(a*i)^* = a^* * i$;

2. $\mathcal{A}^g_a \neq \mathcal{A}^f_{a^*}$ and one of the following conditions is fulfilled:

2.1. $\mathcal{A}^f_{a^*}$ is a $(C\to)$-rule, $\mathcal{A}^g_a$ is a cut, the principal formula of cut $\mathcal{A}^g_a$ coincides with the side formula of $\mathcal{A}^f_{a^*}$ and (denoting by s the node of proof-figure f, at which is situated the conclusion of the cut from which the $(C\to)$-rule being considered originates);

2.1.1. If $\mathcal{A}^f_{a^*} = \forall\to$, then $(a*0)^* = s$; $(a*1)^* = a^* * 0$;

2.1.2. If $\mathcal{A}^f_{a^*} = \neg\to$, then the premises are represented by:

$$(a*0)^* = a^* * 0;\ (a*1)^* = s;$$

2.2. $\mathcal{A}^f_{a^*}$ is a cut, $\mathcal{A}^g_a = \mathrm{Rep}$; $(a*0)^* = a^* * 1$, $i_a = 0$;

2.3 $\mathcal{A}^f_{a^*}$ is a $(\to C)$-rule, $\mathcal{A}^g_a = \mathrm{Rep}$; $i_a = 0$ if $\mathcal{A}^f_{a^*} = \to\neg$; $i_a = N$ if $\mathcal{A}^f_{a^*} = \to\forall$, while a is found in ' ' over a cut by formula $A[N]$; $(a*i_a)^* = a^* * i_a$ (see below for a more exact statement).

In order to convert I and II into a complete definition of a^* and $\mathcal{A}^g_a$ by primitive recursion on the length of a, it is sufficient to introduce the additional quantity N_a, determining the N from item 2.3. This quantity will carry more information, by defining the formula $\mathfrak{D}$ which we need to substitute into the succeedent when "inverting" rule $\to\neg$.

$$\text{III. } N_\Lambda = \emptyset;$$

$$N_{a*i} = \begin{cases} N_a, & \mathcal{A}^f_{a^*} \text{ is Rep, Comp, } (C) \text{ or } (\forall\to) \text{ but is not a } (C\to)\text{-rule, or} \\ & (i=1 \text{ and } \mathcal{A}^f_{a^*} \text{ is a } (C\to)\text{-rule}); \\ N, & \mathcal{A}^f_{a^*} = (\forall\to) \text{ is a } (C\to)\text{-rule and } N \text{ is the number to be} \\ & \text{substituted in this rule;} \\ \mathfrak{D}, & \forall a\ \mathcal{A}^f_{a^*} = (\neg\to), \text{ is a } (C\to)\text{-rule and } \mathfrak{D} \text{ is the succeedent} \\ & \text{of its conclusion;} \\ \emptyset & \text{otherwise.} \end{cases}$$

In order to have the right to take it that all the proof-figures being considered are locally correct, we construct an operator transforming any proof-figure into a locally correct one and not changing the initial segments of the proof-figures which are already locally correct. Let $LC_{\leq b}(f)$ denote a formula obtained from $LC(f)$ by replacing the quantifier by the quantifier $\forall a_{\leq b}$; slightly more explicit is: for each application of a rule situated in a branch going from Λ and b it is required that its premise, situated in this branch, be connected with the conclusion in precisely the way indicated in the analysis and that the conclusion can in principle be obtained by this rule. If analysis $\mathcal{A}_b$ indicates that S_b is an axiom, then S_b must be an axiom.

LEMMA 5.1. An elementary operator $\mathcal{L}$ exists such that in PRA we can prove:

$$LC[\mathcal{L}f];\ LC_{\leq b}(f) \rightarrow [\mathcal{L}f](b) = f(b).$$

Proof. Operator $\mathcal{L}$ will change the contents of all nodes for which it is already clear that LC is violated at them or below. If, for example, S_Λ is not a sequent, then a sequent fixed earlier is placed in Λ with analysis Rep; it is placed in this analysis at all nodes of form $<0,\ldots,0>$, while $\emptyset$ is placed at the remaining nodes. We act analogously each time that some analysis proves to be noncorrect (when first we look from bottom to top). For example, if sequent S_a cannot generally be obtained by the rule indicated, then at the nodes $<a,0,\ldots,0>$ we place S_a with analysis Rep, while we place the symbol $\emptyset$ at the remaining nodes $b \supset a$. However, if it can be so obtained, but at any node at which something else stands where there should be a premise, then the needed premise is placed at this node and, once again, Rep over it. The lemma is proved.

Let us define R_1. Assuming that the given proof-figure f is locally correct and denoting $[R_1 f]$ by g, we define a^* , $\mathcal{A}^g_a$, and N_a by recursions I-III. It remains to define S^g_a. If there were a single nontrivial cut in f, the definition would be relatively simple: S^g_a is obtained from $S^f_{a^*}$ by excluding the (only one, if it is there at all) $(C\rightarrow)$-formula by replacing it by the antecedent of the corresponding cut and by "inverting" the (only one, if it is there) $(\rightarrow C)$-formula, i.e., by replacing $\forall x A$ by $A[N_a]$ or replacing $\neg A$ by N_a with the adding on of A in the antecedent. In the general case the procedure of excluding the $(C\rightarrow)$-formula is iterated. For each application L of a nontrivial cut in f there is defined a formula list L^* which will be substituted into the antecedent in the place of the $(C\rightarrow)$-formulas connected with this cut. Such a substitution is called the exclusion of these $(C\rightarrow)$-formulas. We write L in the form

$$\frac{\Gamma,\Sigma\rightarrow A \quad A,\Gamma,\Pi\rightarrow B}{\Gamma,\Sigma,\Pi\rightarrow B}.$$

If Σ does not contain $(C\rightarrow)$-formulas (in particular, if there are no nontrivial cuts below), then $L^* = \Gamma\Sigma$. If, however, Σ does contain $(C\rightarrow)$-formulas of cuts $L_1,\ldots,L_k$, then L^* is the result of their exclusion from $\Gamma\Sigma$.

Now, writing the result of the exclusion of all $(C\rightarrow)$-formulas from $S^f_{a^*}$ in the form $\Delta\rightarrow B$, we set

$$S_a^g = \begin{cases} \Delta \to B, & \text{if } B \text{ is not a}(\to C)\text{-formula in } S_{a^*}^f \\ A, \Delta \to N_a, & \text{if } B = \neg A \text{ is a}(\to\neg)\text{-formula} \\ \Delta \to A[N_a], & \text{if } B = \forall x A \text{ is a } (\to\forall)\text{-formula.} \end{cases}$$

The definition of R_1 is complete. The theorem presented below, summarizing the fundamental properties of this operator, can be proved by induction on the length of node a of the resulting proof-figure $[R_1 f]$. The main ideas of this proof have been essentially expressed above and we shall not carry it through in detail.

In item (c), where we have mentioned the modulus of continuity for R_1 we have used the notation:

$$b \leqslant a \leftrightharpoons lh(b) = lh(a) \,\&\, \forall i_{\leqslant lh(b)} (b)_i \leqslant max\{1, \max_j \{(a)_j\}\}.$$

Item (d) ensures the boundedness of recursions I-III. It is precisely for its fulfillment that we chose the definition of $a * \iota_a$ as the successor of a in Paragraph 2.3.

Items (e) and (f), which follow (d), will be used to estimate the convergence of the cut-elimination process in the cases when the given proof-figure is not equipped with a finite limit on the degrees of the cut.

The symbol $\tilde{\tau}_a^f$ in item (e) denotes the maximum of the degrees of the cuts located at the nodes C of proof-figure f, such that $C \leqslant a$ if there are cuts below a; denotes 0 otherwise; δ_a^f denotes the maximum distance from a up to the conclusion of the nontrivial cut situated below a .

We remark that since induction is applied to single-quantifier predicates, the proof on the whole is provable in PRA (with free functional variables; see item (b) of the theorem).

<u>THEOREM 5.2.</u> (a) Operator R_1 satisfies relations (1); (b) $LC([R_1 f])$;

$$(c)\ \forall b \leqslant a^* f(b) = \varphi(b) \to [R_1 f](a) = [R_1 \varphi](a)$$

$$(d)\ a^* \leqslant a;\ (e)\ \tilde{\tau}_a^f \neq 0 \to \tilde{\tau}_a^{[R_1 f]} < \tilde{\tau}_a^f;$$

$$(f)\ \delta_a^f \neq 0 \to \delta_a^{[R_1 f]} < \tilde{\tau}_a^f;$$

6. Preservation of Well-Foundedness and of Additional Information

Within the framework of classical or specifically intuitionistic abstractions the preservation of well-foundedness follows from item (a) of Theorem 5.2.

<u>COROLLARY 6.1.</u>

$$LC(f) \,\&\, \forall \alpha \exists x f(\bar{\alpha} x) = \varnothing \to \forall \alpha \exists x [R_1 f](\bar{\alpha} x) = \varnothing.$$

<u>Proof.</u> Apply induction on f (bar-induction) to establish the relation

$$f(a) \neq \varnothing \to \forall \alpha \exists x [R_1(\lambda b f(a * b))](\bar{\alpha} x) = \varnothing.$$

Completely analogously it can be established that R_1 takes derivations again into derivations if the derivations have been introduced by a generalized inductive definition.

Indeed, the recursion defining R_1 yields the possibility of obtaining information on the infinite branches of proof-figure f if we are given such information for $[R_1 f]$. A study of this kind of questions proves useful in model theory [6]. Let us introduce some auxiliary concepts.

For a nonempty node $a = b * N$ of the universal flow the immediate predecessor b of this node will be denoted a^-. A segment of the universal flow is a finite sequence $\Lambda = a_0$, $a_1, \ldots, a_n$ of nodes, such that $\forall j_{<n}\, a^-_{j+1} = a_j$. Two nodes are congruent if one of them is a continuation of the other. A cut C (with a conclusion at a node) separates node a from node b if a is found over the left premise or coincides with this premise and b contains an ancestor of the antecedent formula of cut C.

The following statement is provable in PRA.

LEMMA 6.2. If $LC(f) \& g = [R_1 f] \& g(a) \neq \emptyset$, then:

(a) $a^{*-} \neq a^{-*} \to$ (a^* is the left premise of the cut in f, separating a^* from a^{-*});

(b) $\forall c_{\subset a^*} \exists b_{\subset a}\, b^* = c$;

(c) the image $\{a^*_0, \ldots, a^*_n\}$ of segment $\{a_0, \ldots, a_n\}$ is such that $g(a_n) \neq \emptyset$ is a subtree with branches really only at conclusions of cuts; here, if a^*_i is the left premise of cut C, then

$$n \geq j > i \,\&\, (a^*_j \text{ is congruent with } C) \to a^*_j \supset a^*_i \tag{1}$$

(i.e., all points following from a^*_i either fall above a^*_i with respect to the left side of the cut or turn cut to be "on one side" but not over the right premise).

(d) $i \neq j \to a^*_i \neq a^*_j$;

(e) $\exists i_{\leq n}\, lh(a^*_i) \geq [\log_2(n+1)]$ (the image of a sufficiently long segment contains a sufficiently long segment);

(f) $\forall \theta \varphi_1 \varphi_2 \gamma_1 \gamma_2 [\forall i \{g(\bar\theta i) \neq \emptyset \,\&\, \bar\varphi_1(i) = (\bar\theta(\gamma_1(i)))^* \,\&\, \bar\varphi_2(i) = (\bar\theta(\gamma_2(i)))^*\} \to \forall j\, \varphi_1(j) = \varphi_2(j)]$ (the image of an infinite branch θ contains no more than one infinite branch).

Proof. (a) and (b) by induction on the length of a. (c) Let $I = \{a_0, \ldots, a_n\}$ be a segment and $g(a_n) \neq \emptyset$. By virtue of item (b), $I^* = \{a^*_0, \ldots, a^*_n\}$ is a subtree of the universal flow. Let I^* have a branching at point a^*_i : $a^{*-}_k = a^*_i, a^{*-}_l = a^*_i$, $a^*_k \neq a^*_l$. Then one of the numbers $k-1$ or $l-1$ (say, $k-1$) differs from i. Then, by virtue of (a), a^*_k is the left premise of a cut in f. Let us consider the minimal pair (i, j) for which (1) is violated even though a^*_i is the left premise of cut C. By virtue of the minimality, (1) is fulfilled with $j-1$ instead of j. Therefore, $a^{*-}_j \neq a^*_{j-1}$. By virtue of (a) it follows hence that a^*_j

is the left premise of cut C_1 separating a_j^* from a_{j-1}^*. If C_1 is found to be below C then (1) is violated for the pair $(j,j-1)$ and for cut C_1, while if C_1 is above C or coincides with it, then (1) is violated for the pair $(i,j-1)$. (d) follows from (c). (e) follows from (c) and (d). (f) is obtained from (c) by induction on j; here condition (1) is essential. The lemma is proved.

With the aid of König's Lemma 6.2 yields classically the following statement strengthening Corollary 6.1.

THEOREM 6.3. If $LC(f)$ and if φ is an infinite branch of tree $[R_1 f]$, then in f there is an infinite branch of class $\Delta_2^0(\varphi)$.

Indeed, the image of every infinite branch φ from g is a binary tree containing arbitrarily long segments and, therefore, contains an infinite branch (the only one, by virtue of Lemma 6.2 (f). Indeed, when seeking this infinite branch we have to make a choice only with respect to the predicate $\exists i\,((\bar{\varphi}(i))^*=a)$.

The next statement shows that we can achieve $\forall a(i_a=0)$, completely preserving the structure of the given proof-figure.

THEOREM 6.4. Primitive recursive operators F and t exist such that (in the notation $g \leftrightharpoons [Ff],\ a^*=[tf](a)$) in PRA we can prove: if $LC(f)$, then $LC(g)\,\&\,\forall a(i_a^g=0\,\&\,lh(a)=lh(a^*)\,\&\, g(a)\neq\emptyset \leftrightarrow f(a^*)\neq\emptyset)\,\&\,\forall a\,(g(a)$ coincides with $f(a^*)$ except that $i_a^g=0)$.

In order to achieve $i_a=0$, we shift to the left by i_a all premises of those Rep for whose conclusions $i_a\neq 0$, and everything that lies above. The preimage a^* of node a of the "shifted" derivation is determined by the following primitive recursion (and we do not see how it can be replaced by a Kalmár-elementary operation):

$$\Lambda^*=\Lambda$$

$$(a*n)^*=\begin{cases} a^**n, & \text{if } a \text{ is not a conclusion of Rep} \\ a^**(n+i_{a^*}^f) & \text{otherwise} \end{cases}$$

All the information, except i_a is carried over into a from a^*.

Since R_1 lowers by one the degrees of all nontrivial cuts, it is trivially continued up to an operator R_1^+ such that

$$LC^+(f)\to LC^+[R_1^+f]\ \&\ deg[R_1^+f]=deg(f)\dot{-}1.$$

Let us continue R_1 onto derivations enriched by ordinals. The small difference from the standard consideration is caused by the fact that our definition of operator R_1 does differ somewhat from the standard one (in which repetitions are not introduced). Because of this, instead of the standard exponent satisfying the condition $\beta<\alpha\to 2^\beta+2^\beta\leqslant 2^\alpha$ we require a function satisfying an analogous strict inequality. We set: $2_+^0=1$; $2_+^\alpha=2^\alpha$ if α is a limit ordinal; $2_+^{\alpha+1}=2_+^\alpha+2_+^\alpha+1$. The last relation shows that $\beta<\alpha\to 2_+^\beta+2_+^\beta<2_+^\alpha$. It is easily verified that $2_+^\alpha=2^{\alpha+1}-1$ for finite α, and $2_+^\alpha=2^\alpha+N$ for $\alpha=\alpha_l+N$, where α_l is a limit ordinal.

We can now continue R_1 up to operator R_1^{ON} preserving the property WF_{ON}. For this it is sufficient to determine the ordinals α_a^g for the nodes a of the proof-figure $g=[R_1 f]$. For each node a of proof-figure $[R_1 f]$ we write out, beginning with the lowest one, all the cuts $L_1,\ldots,L_k$ whose conclusions are located below a^* in the original proof-figure f and whose right premises are congruent (i.e., are located below or coincide) with a^*. Let $\beta_1,\ldots,\beta_k$ be the ordinals of the left premises of cuts $L_1,\ldots,L_k$; $\alpha^* \leftrightharpoons \alpha_{a^*}^f$. We set

$$\alpha_a^g = 2_+^{\beta_1}+\ldots+2_+^{\beta_k}+2_+^{\alpha^*} \tag{2}$$

and we verify the condition of the increase of ordinals by examining the same cases as in recursion II of § 5.

1 and 2.3. In these cases the numbers $\beta_1,\ldots,\beta_k$ are one and the same for the premises and the conclusions and it is enough to refer to the analogous condition for f.

2.1.

$$\alpha_a^g = 2_+^{\beta_1}+\ldots+2_+^{\beta_k}+2_+^{\alpha^*};\ \alpha_{a*0}^g = 2_+^{\beta_1}+\ldots+2_+^{\beta_i}\ (i\leq k)$$
$$\alpha_{a*1}^g = 2_+^{\beta_1}+\ldots+2_+^{\beta_k}+2^{\alpha_{a^*1}^f}.$$

(These relations have been written out for the $(\forall\rightarrow)$-case. In the $(\neg\rightarrow)$-case it is necessary to transpose α_{a*0}^g and α_{a*1}^g.) Now, obviously, $\alpha_{a*0}^g < \alpha_a^g$, while $\alpha_{a*1}^g < \alpha_a^g$ follows from the analogous condition for f.

2.2. $\alpha_a^g = 2_+^{\beta_1}+\ldots+2_+^{\beta_{k-1}}+2_+^{\alpha^*}$; $\alpha_{a*0}^g = 2_+^{\beta_1}+\ldots+2_+^{\beta_{k-1}}+2_+^{\beta_k}+2_+^{\alpha_{a^*1}^f}$, where β_k is the ordinal of the left premise a^*0 of the cut with a conclusion at a^*. Setting $\beta=max(\beta_k,\alpha_{a^*1}^f)$ we have, by the condition for f, $\beta<\alpha^*$, whence $\alpha_{a*0}^g<\alpha_a^g$.

It remains to note for what follows that $h[R_1^{ON}f]=2_+^{h(f)}$.

The continuation of R_1 up to the operator R_1^H preserves the property LCH (the presence at each node of an H-derivation corresponding to a sequent is obtained more simply). The derivation d_a^g is obtained from $d_{a^*}^f$ analogously to how S_a^g is obtained from $S_{a^*}^f$. Here cuts are applied for the exclusion of the $(C\rightarrow)$-formulas and for the modeling of the inversion lemma. The details (if needed) can be found in [5].

7. Complete Elimination of Cut and Repetition Rules

By R we denote the iteration of operator R_1

$$[R0f]=f,\quad [R(k+1)f]=[R_1[Rkf]]$$

and by E the result of a "sufficiently long" application of this operator (cf. Theorem 5.2 (e); (f)):

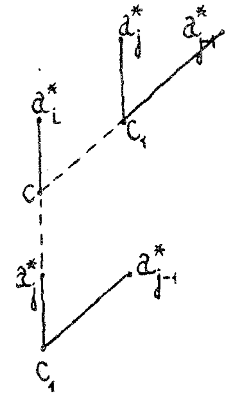


Fig. 1

$$[Ef](a) = [R(\min\{\tilde{\tau}^f_a, \delta^f_a\})f]\ (a).$$

<u>THEOREM 7.1.</u> If $LC(f)$ then

$(a)_R$ $\quad LC[Rkf] \mathbin{\&} \tilde{\tau}^{[Rkf]}_a \leqslant \tilde{\tau}^f_a \dot{-} k$,

$(b)_R$

$$[Rkf] = \begin{cases} \dfrac{[Rkf^+]}{f(\Lambda)}, & f \text{ does not end in a nontrivial cut,} \\[2ex] [R(k-1)[R_1 f]], & f \text{ does end in a nontrivial cut,} \end{cases}$$

$(a)_E$ $\quad LC[Ef] \mathbin{\&} \delta^{[Ef]}_a = 0$,

$(b)_E$

$$[Ef] = \begin{cases} \dfrac{[Ef^+]}{f(\Lambda)}, & \text{if } f \text{ does not end in a nontrivial cut,} \\[2ex] \left[Rk \dfrac{[Ef^+]}{f(\Lambda)}\right], & \text{if } f \text{ does end in a power of a cut, } k>0; \end{cases}$$

$(C)_E$ $\quad [Ef]$ contains cuts only by equalities,

$(C)_{RE}$ $\quad LC^+(f) \to [Ef] = [R(\deg(f))f]$

<u>Remark to Theorem 7.1 $(C)_E$.</u> Applying operator R_0, we can eliminate cuts also by equalities. Here the modification mentioned after the description of R_0 enables us not to increase the ordinal length.

Information on the relation of infinite branches in f and $[Rkf]$ can be extracted from Theorems 6.1 and 6.2; however, we have not yet succeeded in obtaining analogous information for E. Regarding the preservation of well-foundedness, it is obtained by bar-induction from Theorem 7.1 $(a)_E$.

It is not difficult to continue R and E up to operators R^H and E^H preserving condition H: it is sufficient to determine them, starting from R_1^H in just the same way that R and E were determined, starting from R_1.

Let us estimate the variation of the ordinal length under an application of R_k (i.e., $\lambda f[Rkf]$) and of E. From the estimates for R_1^{ON} we once again obtain

$$WF_{ON}(f) \to WF_{ON}[R^{ON}kf] \,\&\, h[R^{ON}kf] \leqslant 2_+^{\cdot^{\cdot^{2_+^{h}(f)}}} \quad k \text{ times}$$

where R^{ON} is an iteration of operator R_1^{ON}. When $k = deg(f)$ we obtain an "almost standard" estimate of the length. If, however, the maximum complexity of the cuts is not bounded (or not known), the estimate changes. During the working of the standard operator when lowering the degree by γ the length increases from α to $\varphi_\gamma(\alpha)$, where the φ_β are the normal (i.e., continuous and strictly increasing) solutions of the functional equation

$$\varphi_\beta(\varphi_\gamma(\alpha)) = \varphi_{\beta+\gamma}(\alpha) \tag{1}$$

with initial conditions $\varphi_0(\alpha) = \alpha$, $\varphi_1(\alpha) = 2^\alpha$ (see [11]). For $\beta > 0$ they satisfy the relation

$$\varphi_\beta(\alpha) \cdot 2 \leqslant \varphi_\beta(\alpha+1). \tag{2}$$

For operator E^{ON} the ordinal lengths vary with respect to the type of estimates for R_1^{ON}: functions which we denote by φ_β are changed instead of the functions φ_β^+. These are the normal solutions of Eq. (1) with the initial conditions $\varphi_0^+(\alpha) = \alpha$, $\varphi_1^+(\alpha) = 2_+^\alpha$. For $\beta > 0$ they satisfy the relation

$$\varphi_\beta^+(\alpha) \cdot 2 < \varphi_\beta^+(\alpha+1) \tag{3}$$

for $k < \omega$, $\varphi_k^+(\alpha)$ is the k-th iteration of function 2_+^α, while $\varphi_\omega^+(\alpha)$ coincides with $\varphi_\omega(\alpha)$ and equals $\varepsilon_{\alpha-1}$, viz., the ε-number $(\alpha-1)$-th is order.

Let us show how to continue E up to operator E^{ON} satisfying the conditions

$$(LC + WF_{ON})(f) \to [E^{ON}f] = [Ef] \,\&\, h[E^{ON}f] = \varphi_\omega(h(f)).$$

For a given proof-figure f such that $(LC + WF_{ON})(f)$ and for a given node a (of proof-figure $g = [E^{ON}f]$) we write out, beginning with the uppermost, all the conclusions of the cuts

$$C_1, \ldots, C_\ell \tag{4}$$

in proof-figure f, which "enter into the working" when computing $g(a)$. By f_0 we denote the proof-figure which is obtained from f by replacing all ordinals α by $\varphi_\omega(\alpha)$. Suppose that we have already determined the proof-figure f_i satisfying the condition $LC + WF_{ON}$ and differing from f really only in the parts located above $C_1, \ldots, C_i$. By f_{i+1} we denote the result of replacing in f_i the part θ_{i+1} terminating in cut C_{i+1} by $[R^{ON}k_{i+1}\theta_{i+1}]$, where k_{i+1} is the degree of cut c_{i+1}. To verify $WF_{ON}(f_{i+1})$ it is sufficient to note that the ordinal $\varphi^{+}_{k_{i+1}}(\varphi_\omega(\alpha))$ of the lower sequent of proof-figure $[R^{ON}k_{i+1}\,\theta_{i+1}]$ equals the ordinal $\varphi_\omega(\alpha)$ of the lower sequent of proof-figure θ_{i+1}: indeed,

$$\varphi^{+}_{k_{i+1}}(\varphi_\omega(\alpha)) = \varphi_{k_{i+1}}(\varphi_\omega(\alpha)) = \varphi_{k_{i+1}+\omega}(\alpha) = \varphi_\omega(\alpha).$$

We now set $\alpha^{g}_{a} = \alpha^{f\ell}_{a}$. We make use of the fact that $f_\ell(a) = [Ef](a)$. The proof of this assertion includes the establishment of the invariance of the result of normalization relative to the order of reductions (strong normalizability). After strong normalizability has been established, we can justify the correctness of our assignment of the ordinals, i.e., their increase when moving downward in proof-figure g. Here we take into account that list (4) for node $a * i$ coincides with the analogous list for node a except in the case when the preimage a^* of node a in the original proof-figure f is a conclusion of a nontrivial cut. In this latter case we once again use the invariance of the values of function φ_ω relative to φ^{+}_{k}:

$$\varphi^{+}_{k}(\varphi_\omega(\alpha)) = \varphi_\omega(\alpha).$$

As a result we obtain a derivation satisfying condition WF_{ON}.

Let us take up the exclusion of Rep. Once more it is clear that applications of Rep can be replaced by applications of Comp. This latter rule is necessary for the derivations of sequents containing free functional variables (for example, $\neg\forall x f(0) \neq x$), but it is desirable to exclude from it the derivations of sequents not containing free variables.

The sequent $T_1,\dots,T_n \to F$ is said to be trivially false if $T_1,\dots,T_n$ are true constant equalities and F is a false constant equality. It is clear that in a locally correct tree without cuts or Comp, the last sequent of which is trivially false, we must apply Rep. It turns out that this is a natural obstacle to the exclusion of Rep. Before we state the corresponding theorem we remark that if a (trivially) false sequent is encountered in some locally correct tree, then the lower sequent of this tree is false.

THEOREM 7.2. An elementary operator Φ exists such that if f is a locally correct proof-figure without a cut with lower sequent S, then $[\Phi f]$ is a locally correct proof-figure without a cut with the same lower sequent. If S contains free functional variables, then $[\Phi f]$ does not contain Rep. If S does not contain free variables, then Rep is applied in $[\Phi f]$ really only to trivially false sequents. An elementary operator exists, constructing from each infinite branch of $[\Phi f]$ an infinite branch in f. Φ can be continued up to an operator Φ^+ preserving property WF_{ON}; moreover, $h[\Phi^+ f] = h(f)$.

Proof. We shall take it that the lower sequent $\Gamma \to A$ of the proof-figure f being examined does not contain functional variables and that f itself does not contain Comp. We consider the cases possible.

1. Γ contains the member $\forall x B$. Then Rep can be replaced by $\forall\to$, since all sequents of f contain $\forall x B$.

2. Γ contains the member $\neg B$ and A is a false equality. Then Rep is replaced by $\neg\rightarrow$.

3. Γ contains a false equality or A is a true equality. Then $\Gamma\rightarrow A$ is an axiom.

4. $\Gamma\rightarrow A$ is a false sequent. Then $\mathfrak{f}$ is simply a (infinite) chain of Rep, satisfying all the conditions necessary.

5. Γ consists of true equalities and A has the form $\forall x_1\ldots\forall x_k B$ $(k\geqslant 0)$, where B starts with a negation or is an equality. Applying the transformation from the lemma on invertibility, we reduce this case to the preceding one

$$\frac{\Gamma, C[N_1,\ldots,N_k]\rightarrow 0=1}{\Gamma\rightarrow\forall x_1\ldots\forall x_k\neg C} \quad \text{or} \quad \frac{\Gamma\rightarrow E[N_1,\ldots,N_k]}{\Gamma\rightarrow\forall x_1\ldots\forall x_k E}.$$

The theorem is proved.

8. Generalizations and Applications

All the results in the preceding sections, excepting Theorem 7.2, generalize (together with the proofs) to wider systems. First of all, this is HA with a full complement of connectives. The corresponding infinite system contains, besides the postulates from § 2 and the standard Gentzen rules $\rightarrow\&\rightarrow, \rightarrow\supset\rightarrow, \rightarrow\exists$, also the infinite rule

$$\frac{\ldots A[N], \Gamma\rightarrow C\ldots}{\exists x A, \Gamma\rightarrow C}\,(\exists\rightarrow).$$

Theorem 7.2 is already false in the presence of $\exists$. Indeed, if we could effectively restrict the applications of Rep to only false sequents, then we could effectively find an element in each nonempty enumerable set. It is sufficient to consider trees $\mathfrak{f}$ consisting of a single branch: downward we find $\exists x R(x)$, while at the $(N+1)$-th stage we find $\exists x Rx$ or $R(\mu x_{\leqslant N} R(x))$ depending on whether $\exists x_{\leqslant N} R(x)$ or not.

We note further that an analog of Theorem 7.2 is not obtained even if we pass to the many-succeedent version. We recall that in such a version the succeedent rules $\rightarrow\&, \rightarrow\exists$ are formulated in the same way as in the classical case, but the rules $\rightarrow\supset, \rightarrow\forall$ have single-succeedent premises. For example,

$$(\rightarrow\exists)\ \frac{\Gamma\rightarrow\Delta, \exists x A, A[N]}{\Gamma\rightarrow\Delta, \exists x A}, \qquad \frac{A, \Gamma\rightarrow B}{\Gamma\rightarrow\Delta, A\supset B}\,(\rightarrow\supset).$$

In order to obtain a counterexample to an analog of Theorem 7.2, it is sufficient to consider trees $\mathfrak{f}$ in which downward are found sequents of the form

$$\rightarrow 0=0\supset\exists x R(N,x),\ 0=0\supset\exists x S(N,x),$$

where $\exists x R(a,x), \exists x S(a,x)$ give a pair of disjunctive effectively nonseparable sets.

In return, all the remaining results of §§ 5-7 carry over to the many-succeedent version.

The increase in the supply of functional symbols does not cause any additional difficulties. For example, we can add on constants for all provable-recursive functions with corresponding defining equalities as axioms.

Intuitionistic arithmetic systems of higher types with weak axioms of the existence of functions are analyzed in just the same way. An example is the elementary analysis, an extension of HA in which are allowed quantifiers on the functional variables and are postulated the properties of pairing functions, as well as closedness relative to the operations of substitution and primitive recursion. The corresponding infinite formulation contains infinite rules for quantifiers of type 0 and the usual rules for quantifiers of type 1, as well as defining equalities for the operators of substitution and primitive recursion.

The adding on of constants for substitutions and recursions of all finite types requires no changes.

The classical arithmetic PA could be treated as a subsystem of Heyting arithmetic; however, for some applications it is convenient to consider it separately. The many-succeedent sequents $A_1,\dots,A_n \to B_1,\dots,B_m$ are used and the cut takes the form

$$\frac{\Gamma,\Sigma \to \Delta,\Omega,C \qquad C,\Gamma,\Pi \to \Delta,\Phi}{\Gamma,\Sigma,\Pi \to \Delta,\Omega,\Phi}.$$

The only change in the description of operator R_0 is: the ancestors of the antecedent (for the $\{\forall, \neg\}$-language) C being excluded are replaced by the sequent $\Gamma,\Sigma \to \Delta,\Omega$, i.e., Γ and Σ are assigned to the antecedents and Δ and Ω to the succeedents. All the remaining operators are determined, starting from the new R_0, in the same way as before. An analog of Theorem 7.2 is valid here.

Into consideration we can enter as well formal arithmetic system with the axiom of transfinite induction up to some ordinal α. Such an axiom

$$\forall y(\forall x_{\prec y} A \supset A[y]) \supset \forall x A,$$

where $\prec$ is an ordering of the positive integers by the type of α, is replaced analogously by the usual induction

$$\cfrac{\cfrac{\cfrac{\begin{array}{c} K \prec 0, \Gamma \to A[K] \\ \dots \\ M \prec N \quad \Gamma \to A[N] \dots \end{array}}{\Gamma \to \forall x_{\prec N} A} \qquad \forall x_{\prec N} A, \Gamma \to A[N]}{\dots \Gamma \to A[N] \dots}}{\Gamma \to \forall x A}.$$

The derivation d in the system H being examined leads to a derivation having a length less than $\alpha \cdot N + \omega (< \alpha \cdot \omega)$, where N is the maximum number of inductions in one branch of derivation d.

A branching analysis of transfinite level also can be considered within the framework of the general scheme. Here we shall not bother with the details of assigning ordinals for this case. We merely remark that the same scheme is used as when constructing the operator E^{ON} and the same bound as in [1] is used.

Operator E can be extended as well to systems of nonpredicate analysis (second-order arithmetic). However, in this case the proof of the well-foundedness of proof-figure $[\mathsf{E}f]$ even when is obtained directly from a finite derivation, cannot set within the framework of PRA with the aid of some "good" ON system. The well-foundedness of $[\mathsf{E}f]$ can be proved by the Girard–Tait–Martin-Löf method of "computability predicates."

We consider a natural formulation. In order to avoid the introduction of unnatural rules corresponding to axioms for equality we shall take it that there are no functional variables in the language.

For example, the rules of the $NHA_{\neg\forall\infty}$ system will be $\rightarrow\forall_\infty, \rightarrow\neg$, as in HA_∞ (they are denoted also by $\forall^+, \neg^+$ and called introduction rules),

$$\frac{\Gamma\rightarrow\forall x A}{\Gamma\rightarrow A[N]}(\forall^-) \qquad \frac{\Gamma\rightarrow A \quad \Sigma\rightarrow\neg A}{\Gamma,\Sigma\rightarrow 0=1}(\neg^-) \text{ removal rules,}$$

$$\frac{\Gamma,\Gamma,\Sigma\rightarrow A}{\Gamma,\Sigma\rightarrow A}(\mathrm{Contr}) \qquad \frac{\Gamma\rightarrow F}{\Gamma\rightarrow A}(y) \quad (F \text{ is a false equality}).$$

The axioms are: $A,\Gamma\rightarrow A$; $F,\Gamma\rightarrow A$; $\Gamma\rightarrow T$, where F is a false equality and T is a true equality.

The principal formula of a removal rule is said to be maximal if some ancestor of it in the given proof-figure is obtained by an introduction rule or by y. We cannot require, as was done in the case of predicate calculus, that the principal removal formula itself was obtained by an introduction rule or by y, since we have Contr.

A locally correct (relative to the new rules) proof-figure is called normal if there are no maximal formulas in it. The analog NR_0 of operator R_0 with whose aid we obtain the analogs of the results in §§5, 6, and 7, has been described in [5]. The question of normalization in classical systems of natural type with a full complement of connectives (having, as noted, interesting applications; see [6]) will be considered elsewhere.

Among the possible applications of the results obtained we note, for instance, the proof in PRA of the equivalence of the usual formalization of Heyting or of classical arithmetic and the "cut-free" formalization Z' from [3] with the additional scheme of induction on each ordinal $\alpha < \varepsilon_0$. The proof given in [3] used induction on ε_0.

Another application is the proof (in particular, for a system with free functional variables) of Parson's theorem on the fact that PRA is closed relative to the rule $JR\Pi_2^0$ of induction with respect to Π_2^0-formulas:

$$\frac{\forall x\,\exists y\, A[0,x,y] \quad \forall x\,\exists y\, A[b,x,y]\rightarrow\forall x\,\exists y\, A[sb,x,y]}{\forall z\,\forall x\,\exists y\, A[z,x,y]}.$$

Let us outline how to extend the proof in [4] of the equivalence of different theorems of effective existence onto the Gödel interpreation.* The transformations $\mathfrak{T}$ of realiza-

*Translator's Note: This is also sometimes called the Dialectica interpretation.

bility type, considered in [4], associate with each derivation sequents $A_1,\dots,A_n \to B$ one term (containing, in general, the variables $x^{A_1},\dots,x^{A_n}$) such that there holds

$$x^{A_1} \circledR A_1, \dots, x^{A_n} \circledR A_n \to t^B \circledR B.$$

The Gödel interpretation $\textcircled{g}$ (under our approach) associates with such a derivation a system consisting of $n+1$ terms $t_1^{A_1},\dots,t_n^{A_n}, t^B$, such that there holds

$$\mathcal{A}_1[x_1,t_1^{A_1}],\dots,\mathcal{A}_n[x_n,t_n^{A_n}] \to \mathcal{B}[t^B,y],$$

where $\mathcal{A}_1,\dots,\mathcal{A}_n,\mathcal{B}$ are constructed as if by means of the application of the Gödel interpretation to the implication $A_1 \& \dots \& A_n \supset B$. More precisely, for each rule there is indicated a canonic method of translating the system of terms for the premise into a system of terms for the conclusion. For example, for the conjunction introduction rule $B, C \vdash B \& C$ the terms $t_i^{A_i}$ from the antecedent are preserved, while the new succeedent term is a pair of the old ones. For our present purposes it is convenient to use a formulation in the form of a natural calculus. Several essential difficulties arise when considering the rule for the shortening of repetitions. Let us consider a typical application of this rule and the proof-figure into which it passes after the substitution of the Gödel terms:

$$\frac{A,A,C\to B}{A,C\to B} \quad \frac{\mathcal{A}[x_1,t_1],\mathcal{A}[x_2,t_2],\mathcal{C}[x_3,t_3]\to\mathcal{B}[t_4,y]}{\mathcal{A}[x,\tau],\mathcal{C}[x_3,t_3^*]\to\mathcal{B}[t_4^*,y]}.$$

Here the asterisk denotes the substitution of x_1 for x_2 and τ is determined (by an identification of the variables x_1 and x_2 and) by an analysis of the cases:

$$\tau = \mathbf{if}\ \neg\mathcal{A}[x_1,t_1^*]\ \mathbf{then}\ t_1^*\ \mathbf{else}\ t_2^*.$$

However, under such a method of assigning terms the connections between the terms assigned to the premises and the conclusions of the rules prove to be noninvariant relative to normalization. Therefore, we shall take it that τ is correctly connected with t_1 and t_2 also when the analysis of the cases takes place within τ, i.e., if t_1 and t_2 have, respectively, the forms $t_a[\tau_1]$ and $t_a[\tau_2]$, with one and the same t and $\tau = t_a^*[\tau']$, where

$$\tau' = \mathbf{if}\ \neg\mathcal{A}[x_1,t_a^*[\tau_1^*]]\ \text{then}\ \tau_1^*\ \text{else}\ \tau_2^*.$$

Under such a definition all the results of [4], together with the proofs, re retained.

PART II. SYSTEMS WITH THE AXIOM OF CHOICE

9. $HA + AC^{0\to 0}$ and the ε-Symbol

As was noted in the Introduction the $HA + AC^{0\to 0}$ system is obtained by adding on to HA choice rules of type 0:

$$\frac{\Gamma\to\forall x\,\exists y\,A \qquad \forall x\,A_y[f(x)],\Gamma\to C}{\Gamma\to C}, \tag{1}$$

where f does not enter into A, Γ , and C. From (1) we derive the corresponding "many-place"* principle:

$$\frac{\Gamma \to \forall x_1 \ldots \forall x_n \exists y A \qquad \forall x_1 \ldots \forall x_n A_y[f(<x_1, \ldots, x_n>)], \Gamma \to C}{\Gamma \to C}.$$

To prove the admissibility of $AC^{0\to 0}$ in HA we prove cut-eliminability, for which, in its own turn, we consider the HA^{ε} system whose language is obtained by adding on to the language of the HA system a new rule for the formation of terms: if A is a formula, then $\varepsilon x A$ is a term.

Actually, the concept of a derivation will be formulated in such a way that the ε-type [7] of the terms encountered represent unary functions or constants.

The occurrence V of term t in the term $\varepsilon x A$ is subordinate to term $\varepsilon x A$ if t contains freely the variable x which is connected with the outermost εx. The occurrence V of term t in the term τ inhabits τ if it is not subordinate to any subterm of τ.

In addition to the terms of form c', $f(m)$, $m+n$, $m \cdot n$, we consider simple terms, as well as terms of the form $\varepsilon x A$ inhabited really only by variables and digits.

All postulates of the HA system, except $\exists\to$, are modified in connection with the new language. In particular, the application of the rule Comp can have the form $\varepsilon x A = b, \Gamma \to C \vdash \Gamma \to C$. On the axioms for equality

$$(t=c), E[t], \Gamma \to E[c], \qquad (t=c), E[c], \Gamma \to E[t]$$

(and analogously for $(c=t)$), besides the old restriction "c is a variable or a digit and t is a simple term," imposed in HA, we require that t satisfy the same condition as c if the replacement takes place inside the domain of action of ε .

The rule $\exists\to$ is replaced by the rule

$$\frac{A[\varepsilon x A], \exists x A, \Gamma \to C}{\exists x A, \Gamma \to C} (\to \exists_{\varepsilon}).$$

Finally, the concept of proof is modified with respect to the type of calculi with partially defined functions. We introduce an auxiliary definition.

The formula $\forall x \exists z A$ (respectively, the formula $\exists z A$) is called the basis of any term of the form $\varepsilon z A_x[b]$ (respectively, the term $\varepsilon z A$), where b is a digit or a variable, free for x, in $\exists z A$. The basis is minimal if it has the form $\exists z A$ or it has the form $\forall x \exists z A$ and b is not a digit. By definition, only a formula of the form $\forall x \exists z A$ can be a basis of the occurrence of the term $\varepsilon z A_x[b]$ in which the variable b is connected.

*Translator's Note: Since the attributes "unary," "binary," "ternary," etc. are so often used instead of "one-place," "two-place," "three-place," respectively, I should like to suggest the use here of a newly coined word "multi-ary" instead of "many-place;" note that "n-ary" is already in common use for "n-place."

We stress that b cannot be a complex term: $\forall x \exists z \varphi(x)=z$ is not a basis for $\varepsilon z(\varphi(\varepsilon y(0=y))=z)$, although it is such for $\varepsilon z \varphi(0)=z$.

A derivation in HA^{ε} is a proof-figure constructed by starting from an axiom in accordance with rules; moreover, for each occurrence of an ε-term t in some sequent there is a basis of t in the antecedent of this sequent.

THEOREM 9.1. HA^{ε} is an extension of $HA + AC^{0\to 0}$.

Proof. Let us establish the admissibility of rule (1). Let HA^{ε}-derivations of the premises of rule (1) be given. We can take it that f does not enter into the derivation of the left premise. Using the well-known method for the exclusion of complex terms with the aid of equality and quantifiers

$$E[f(t)] \leftrightarrow \forall u(u=t \to E[f(u)])$$

we achieve that t is a variable or a positive integer in all terms $f(t)$ occurring in the derivation of the right premise. Now, replacing $f(t)$ by $\varepsilon y A_x[t]$ and adding $\forall x \exists y A$ to the antecedents of all sequents, we obtain the HA^{ε}-derivation of the sequent $\forall x \exists y A, \forall x A_y[\varepsilon x A], \Gamma \to C$. In order to derive $\Gamma \to C$ it remains only to apply a cut with sequents $\forall x \exists y A \to \forall x A_y[\varepsilon x A]$ and $\Gamma \to \forall x \exists y A$. The conservativity of HA^{ε} relative to $HA+AC^{0\to 0}$ will follow from the conservativity relative to HA.

To establish the normalization theorem for HA^{ε} we consider the corresponding infinite $HA^{\varepsilon}_{\infty}$ system. (We note in advance that the proof of normalizability proceeds analogously to the case of predicate calculus with ε-symbol; moreover, some simplifications are introduced, connected with the use of equality.)

The postulates of the $HA^{\varepsilon}_{\infty}$ is obtained from the postulates of HA^{ε} by a well-known scheme: the induction rule is deleted, $\to\forall$ and Comp are turned into infinite rules, and the repetition rule (Rep) is added on. We remark that $\exists_{\varepsilon}\to$ remains single-premise.

All results of the substitution of digits in the place of variables in the axioms of HA^{ε} (with one restriction; see below) are reckoned to be axioms. Only closed sequents are examined. This latter fact permits us to consider that in the axioms for equality

$$t=N, E[t] \to E[N] \text{ , etc.} \tag{3}$$

the replacement of t by N does not take place inside the ε-symbol. Indeed, by virtue of our restrictions, t is a positive integer in this case. Therefore, in case $t \equiv N$ sequent (3) has the form $T, E \to E$, while if t is different from N, (3) is obtained from other axioms in view of the falsity of the antecedent's first member.

Derivations in $HA^{\varepsilon}_{\infty}$ are defined as those locally correct proof-figures f in which all occurrences of ε-terms have bases, the nodes are equipped with ordinals $<\varepsilon_0$, increasing from the rules premises to the conclusions, and the lower node Λ contains a list

τ of formulas, such that for any formula C of a cut from f we have $C' \in \tau$, such that $C = C'_{x_1 \dots x_k}[N_1, \dots, N_k]$ for the digits $N_1, \dots, N_k$. This list τ will be denoted $deg(f)$. Thus, derivations are proof-formulas satisfying WF_{ON} (for our concrete ON system) and a strengthening of condition LC^+.

A term S is a special case of term τ if there exists a substitution ξ of positive integers in the place of the variables occurring freely in τ, such that $S \equiv \tau\xi$.

The following statement on substitution in an individual application of a rule can be proved by looking over all rules.

LEMMA 9.2. (a) A given application L of a postulate of the $HA^{\varepsilon}_{\infty}$ system remains an application of the same postulate after the substitution of the positive integer N in the place of all occurrences of the closed ε-term S, if the following conditions are fulfilled:

(1) if L is a quantifier rule introducing the formula QxA, then S is not a special case of some term of A, freely containing x;

(2) if $L = Comp$, then S is different from the principal term of this Comp;

(3) if L is $\exists_{\varepsilon}\rightarrow$, then S is different from the ε-term introduced.

(b) if L is Comp with a principal term S, then the substitution of N in the place of S (and the deletion of false premises) takes L into Rep; if L is $\exists_{\varepsilon}\rightarrow$ with a principal formula $\exists xA$, S is εxA, and $A[N]$ is the antecedent member of the sequent into which the conclusion of L after the substitution leads, then L leads to a curtailment of repetitions.

The invertibility of rules, except $\exists_{\varepsilon}\rightarrow$, and the admissibility of the curtailment of repetitions can be proved in exactly the same way as in § 4. Only the invertibility of $\exists_{\varepsilon}\rightarrow$ requires a separate proof. We introduce an auxiliary definition.

A derivation d in $HA^{\varepsilon}_{\infty}$ is said to be reduced if: (R1) the principal formula A of any $\exists\rightarrow$ originates from the lowest antecedent A of d;

(R2) below the line of the cut with respect to formula A there are no antecedent occurrences of formulas $\forall x_1 \dots \forall x_k \mathcal{A}$, such that $A = \mathcal{A}_{x_1 \dots x_k}[N_1, \dots, N_k]$;

(R3) all bases are minimal.

LEMMA 9.3 (On Reduction). From every $HA^{\varepsilon}_{\infty}$-derivation of length α we can construct a reduced $HA^{\varepsilon}_{\infty}$-derivation of the same degree and of length $<\alpha+\omega$ with the same last sequent.

Indeed, the fulfillment of condition R1 can be achieved by replacing the analysis, and the fulfillment of condition R2, by replacing the cut

$$\frac{\Gamma, \Sigma \rightarrow A \qquad \forall x_1 \dots \forall x_k \mathcal{A}, A, \Gamma, \Pi \rightarrow B}{\forall x_1 \dots \forall x_k \mathcal{A}, \Gamma, \Sigma, \Pi \rightarrow B} \tag{4}$$

by a series of $\forall\rightarrow$ (if $k>0$) and by a curtailment of repetitions. The fulfillment of R3 is achieved analogously.

Let us prove the invertibility of $\exists_\varepsilon\rightarrow$.

LEMMA 9.4. Let there be given a reduced derivation of the sequent

$$\exists_y A, \Gamma \rightarrow C \tag{5}$$

of degree τ and length α, where Γ, C; and τ do not contain ε-terms of which $\varepsilon y A$ is a special case. Then a derivation of sequent $A[N], \Gamma \rightarrow C$ exists, having the same degree and length.

Proof. We take it that the given derivation of sequent (5) is reduced and that $\exists y A$ does not enter into Γ. We replace all ancestors of $\exists y A$ by $A[N]$ and all occurrences of $\varepsilon y A$ by N. For each application of the postulate, except, possibly, $\exists_\varepsilon\rightarrow$ and Comp, the conditions of item (a) of Lemma 9.3 are fulfilled. Spoiled $\exists_\varepsilon\rightarrow$ and Comp are modeled as indicated in item (b) of the same lemma.

Let A^- denote the list of direct subformulas of formula A. In other words $A^- = \{A\}$ if A is atomic; $A^- = \{A_1, A_2\}$ if $A = A_1 \supset A_2$ or $A = A_1 \& A_2$; $A^- = \{B\}$ if $A = QxB$, where $Q = \forall, \exists$.

Let us recall some definitions from [8]. The length of a formula or a term is the number of occurrences of predicate and functional symbols (besides the occurrences of S in the digits), of logical connectives (including quantifiers), and of the ε-symbol. The skeleton of a formula or term A is the result of replacing by 0 all those (maximal) occurrences of subterms which do not contain occurrences of the variables connected in A. The index $ind(A)$ of a formula or term A is the length of the skeleton. The maximal member of a list of formulas is the formula containing logical connectives, having the maximal index among all the members of the list, and having the maximal length among all the members of maximal index.

LEMMA 9.5 (On Lowering Cut Degree). If A is a maximal member of list $\{A\} \cup \tau$, then from every reduced derivation of degree $\{A\} \cup \tau$ and length α we can construct a derivation of the same sequent, having the length 2^{α}_{+} and the degree $A^- \cup \tau'$, where τ' is obtained from τ by the replacement of certain occurrences of ε-terms by variables.

The proof is carried out by looking over the cases in dependence on the principal connective of formula A. The $\&$-and $\supset$-cases are examined in the standard manner. The peculiarity of the $\exists$-case is the verification of the conditions in Lemma 9.4 and the (possible) replacement of τ by τ'. Let $A = \exists y \mathcal{A}$.

By replacing, if necessary, the positive integers by variables in A and in certain members of list τ and by deleting from τ the members set down as false, we achieve that any concretization (the result of substituting positive integers in the place of the free occurrences of variables) of any member of list τ is different from any concretization of formula A. In those members of list τ that contain "free" occurrences of terms of the form $\varepsilon y \mathcal{A}[b_1, \ldots, b_k]$, we replace these "free" occurrences by new variables and we denote the result of the replacement by τ'.

Using the reducibility of the given derivation, we verify that the conditions in Lemma 9.4 are fulfilled for the right premise $\exists y\mathcal{A}, \Gamma \to C$ of any A-cut. If this premise were to contain $\varepsilon y\mathcal{A}$, then condition R2 would be violated for the given cut. For the same reason Γ and C cannot contain terms of which a special case is $\varepsilon y\mathcal{A}$. Finally, if τ were to contain such terms, then A would not be maximal. We now proceed with the standard consideration of the $\exists$-case.

The $\forall$-case. We only need to convince ourselves that the subdivision of the antecedent formula of cut $\forall x\mathcal{A}$ into formulas $\mathcal{A}[N]$ does not spoil the bases of the ε-terms. Let us assume that the basis of the ε-term t is violated. Then above the cut being examined there is another one from whose principal formula originates t. Together with the minmality of the bases, this contradicts the maximality of A. The lemma is proved.

The next statement is now trivial.

THEOREM 9.6. From every $HA^{\varepsilon}_{\infty}$-derivation we can construct a derivation of the same cut-free sequent.

Let us prove the conservativity of HA^{ε} over HA.

THEOREM 9.7. From every HA^{ε}-derivation of a sequent not containing ε we can construct its derivation in HA.

Proof. By virtue of the fact that the proof of Theorem 9.6 is formalizable in PRA we can recken that we are given a cut-free derivation in $HA^{\varepsilon}_{\infty}$ having a length $<\varepsilon_0$. By virtue of the admissibility in HA of induction up to any concrete ordinal $<\varepsilon_0$ it is sufficient to prove the presence of a cut-free derivation of length $<\varepsilon_0$ in HA_{∞}. Indeed, having the latter, we can prove the sequent, we are interested in, of induction on the length. Everything leads to the following lemma.

LEMMA 9.8. Let a cut-free $HA^{\varepsilon}_{\infty}$-derivation of length α be given and, moreover, let its last sequent not contain ε. Then we can construct a cut-free HA_{∞}-derivation S having a length $<\alpha+\omega$.

Proof. Increasing the length by less than ω, we can achieve that below any sequent containing ε there would exist $\exists\to$ with a principal formula $\exists yA$ for the maximal (occurrences of) terms εyA. The rearrangement needed is:

$$\begin{array}{ccc} \overset{\downarrow}{\Sigma} \to B & \qquad & A[\varepsilon yA], \overset{\downarrow}{\Sigma} \to B \\ \downarrow & & \downarrow \\ \exists yA, \Gamma \to C & & \dfrac{A[\varepsilon yA], \exists yA, \Gamma \to C}{\exists yA, \Gamma \to C}. \end{array}$$

We now replace $\exists\to$ by its HA-formula, using Lemma 2.4, and we obtain a derivation not containing ε.

10. $HA^{\omega}+AC$

Finite types are generated from type 0 by the scheme: if σ and τ are types, then

$(\sigma\to\tau)$ is a finite type (an operation reworking objects of type σ into objects of type τ). The types of many-place operations are introduced as the abbreviations

$$(\sigma_1,\dots,\sigma_n\to\tau)\leftrightharpoons(\sigma_1\to\dots\to(\sigma_n\to\tau)\dots).$$

For each finite type σ there exist variables $x^\sigma, y^\sigma, \dots$ of type σ.

Terms of finite types are determined by the usual rules, starting from variables and constants. To us 0 (of type 0), S (of type $0\to0$), and + and $\cdot$ (of type $0,0\to0$) will be constants. Equalities of terms of like type are the atomic formulas of the language of HA^ω while the remaining formulas are constructed from the atomic ones with the aid of $\&, \supset$ and of the quantifiers $\forall$ and $\exists$ with respect to the variables of all the finite types.

The postulates of the HA^ω system are, essentially, the postulates of the Gentzen variant of the HA system, modified in connection with the multigrade language. In particular, the rule Comp and the quantifier rules are extended to all types. Here, in Comp the term t must have the form $\varphi(c)$, where φ and c are variables or constants, while in the rules $\to\exists, \forall\to$ it is permitted to substitute only variables or constants.

The main subject of the subsequent discussion will be the scheme

$$\forall\alpha^\sigma\exists\beta^\tau A\to\exists\beta^{\sigma\to\tau}\forall\alpha^\sigma A_\beta[(\beta\alpha)]. \qquad (AC^{\sigma\to\tau}).$$

The $HA^\omega+AC$ system is the result of adding on to HA^ω the scheme $AC^{\sigma\to\tau}$ for all finite types σ and τ.

The expressive power of the $HA^\omega+AC$ system is sufficiently great: even the constant + and $\cdot$ with their own defining equalities are expressed in terms of S. Indeed, in $HA^\omega+AC$ without these constants we can prove

$$\exists\alpha\forall x\forall y[\alpha(x,0)=x\,\&\,\alpha(x,Sy)=S(\alpha(x,y))]$$

and the analogous formula defining $\cdot$ (starting from +). The proof is constructed along the pattern of Appendix 4 in [7].

Using the rule Comp, we can derive the formula $\exists\beta(\beta=t)$ for β and t belonging to one type, which by virtue of AC yields the λ-operator: $\exists\gamma\forall\alpha(\gamma(\alpha)=t)$.

We can now justify the definition by looking over the cases

$$\textbf{if } t=0 \textbf{ then } \tau_1 \textbf{ else } \tau_2,$$

where t is a term of type 0. For this it is enough to lower all types to zero (by writing out to the end all the variables of the types needed), to apply the well-known construction for type 0 (using the function $\lambda x(1\dot{-}x)$ whose existence follows from AC), and then to apply the λ-operator.

After this the existence of the iteration operator for each finite type can be proved without difficulty by the Dedekind method:

$$\forall\alpha\forall\beta\exists\gamma\forall x_{<a}(\gamma(0)=\alpha\ \&\ \gamma(Sx)=\beta(x,\gamma(x)))$$

is established by induction on a and, next, a diagonal function is obtained after an application of AC.

Thus, the constants are obtained for all primitive recursive functionals of the finite types. They suffice for the "realization" of the quantifiers $\exists$ in the theorems $\exists\beta A$. However, in theorems of the form $B\rightarrow\exists\beta A$ the realizations $\exists\beta$ depend, in general, on the hypothetical justification of formula B. It turns out that a normalization in $HA^{\omega}+AC$ becomes possible if we add on the tools for the explicit expression of such a dependency. We apply the technique using formulas as types [9]. Variables of type A can be interpreted as variables for constructions justifying A. Under such an interpretation the items of the definition presented below become clear. For example, the basis of formula $A\&B$ is the pair $\langle\tau,q\rangle$, where τ is a basis of A and q is a basis of B.

Let us define a simultaneous induction of a type, of the terms of corresponding types, and of formulas. The notation $t\in A$ signifies that t is a term of type A.

1. Finite types are types. Variables and constants of type σ belong to type σ.

2. If $\tau\in\sigma$, $q\in(\sigma\rightarrow\tau)$, then $(\tau q)\in\tau$.

3. $0\in 0, S\in(0\rightarrow 0); +, \cdot$ are terms of type $(0,0\rightarrow 0)$.

4. If σ is a finite type and $\tau,q\in\sigma$, then $\tau=q$ is a (atomic) formula. σ is called the type of this formula

5. The set of formulas is closed relative to the connectives $\&$ and $\supset$ and to the quantifiers with respect to the variables of the finite types.

6. Every formula is a type.

7. $\tau\in B\rightarrow\lambda a^{A}\tau\in(A\supset B)$, where a^{A} is a variable of type A.

8. $\tau\in A\rightarrow\lambda\alpha\tau_{b}[\alpha]\in\forall\alpha A$, if b is a variable of the same finite type as α, not occurring in the superscripts of the variables from τ.

9. $\tau\in A, q\in B\rightarrow\langle\tau,q\rangle\in A\& B$.

10. $\tau\in A_{\alpha}[q]\rightarrow\langle\tau,q\rangle\in\exists\alpha A$.

11. $0^{\sigma}\in E$, where E is an atomic formula of type σ.

12. $\tau\in A\& B\rightarrow(\tau)_{0}\in A\ \&\ (\tau)_{1}\in B$.

13. $\tau\in A,\ q\in A\supset B\rightarrow(\tau q)\in B$.

14. $\tau\in\forall\alpha^{\sigma} A;\ q\in\sigma\rightarrow(\tau q)\in A[q]$.

15. $\tau\in\exists\alpha^{\sigma}A\rightarrow(\tau)_{0}\in A[(\tau)_{1}]\ \&\ (\tau)_{1}\in\sigma$.

16. $\rho_{A}\in(A[0]\supset(\forall x(A\supset A[Sx])\supset\forall xA))$. (It is implied that ρ_{A} is not containing parts of formula A.

Remark. Item 13 and item 14 (if we write $\forall_\alpha^\sigma A$ as $\sigma \to A$ and $A[q]$ as $((\forall_\alpha A)q)$) can be looked upon as special cases of item 2.

Conversions (rules for computing terms):

1. Defining equalities for + and ·.

2. $((\lambda_\alpha \tau)t) = \tau_\alpha[t]$. 3. $(\langle\tau,q\rangle)_0=\tau$; $(\langle\tau,q\rangle)_1=q$.

4. $(\rho\tau q 0)=\tau$; $(\rho\tau q(St))=(qt(\rho\tau q t))$.

A sequent is an expression $\Gamma\to A$, where Γ is a list of formulas and equalities of terms of arbitrary types and A is a formula; moreover, if the variable a^B occurs in a sequent, then B is a member of list Γ.

Rules of the infinite $(HA^\omega+AC)_\infty$ will be applied not to sequents but to more complex objects: sequents $\Gamma\to A$, equipped with a term τ of type A. It is implied that τ contains a variable with superscript B only when B is a member of Γ. Furthermore, each variable a^B with superscript B is associated with some occurrence of B in Γ; moreover, different occurrences are associated with different variables. In the rules cited below $\supset\to Ch$ and $(C)q$ denote terms associated with the left premise.

Simple terms are the variables of all finite types >0, the positive integers, and also the terms of form $\lambda_\alpha(q\alpha)$ where $q\in\forall_\alpha\exists\beta A$.

Semisimple terms are terms of the form (τq) where τ and q are simple terms, as well as terms of the form $(q)_1$ where $q\in\exists\beta A$.

<u>Postulates of the Infinite $(HA^\omega+AC)_\infty$ System and the Association of Terms.</u> The letter c denotes an arbitrary simple term, t denotes a semisimple term; τ and q are terms of suitable types:

$$\frac{\Gamma\to A \quad \Gamma\to B}{\Gamma\to A\& B}\quad \frac{\tau\ \ q}{\langle\tau,q\rangle}\qquad\qquad \frac{A,\Gamma\to B}{\Gamma\to A\supset B}\quad\frac{\tau}{\lambda a^A\tau}$$

$$\frac{\Gamma\to A[c]}{\Gamma\to\exists_\alpha A}\quad\frac{\tau}{\langle\tau,c\rangle}\qquad\qquad \sigma\neq 0\ \ \frac{\Gamma\to A[b]}{\Gamma\to\forall_\alpha^\sigma A}\quad\frac{\tau}{\lambda_\alpha\tau_b[\alpha]}$$

$$\frac{\ldots\Gamma\to A[N]\ldots}{\Gamma\to\forall x A}\quad\frac{\ldots(\tau N)\ldots}{\tau}$$

$$\frac{a^A=(a^{A\&B})_0, A, A\&B,\Gamma\to C}{A\&B,\Gamma\to C}\quad\frac{\tau}{\tau_{a^A}[(a^{A\&B})_0]}\qquad \frac{a^B=(a^{A\&B})_1, B, A\&B,\Gamma\to C}{A\&B,\Gamma\to C}\quad\frac{\tau}{\tau_{a^B}[(a^{A\&B})_1]}$$

$$\frac{A\supset B,\Gamma\to A \quad a^B=(a^{A\supset B}q), B, A\supset B,\Gamma\to C}{A\supset B,\Gamma\to C}\qquad\frac{q\quad\tau}{\tau_{a^B}[(a^{A\supset B}q)]}$$

$$\frac{a^A=(a^\forall c), A[c], \forall_\alpha A,\Gamma\to C}{\forall_\alpha A,\Gamma\to C}\quad\frac{\tau}{\tau_{a^A}[(a^\forall c)]}\qquad \begin{array}{l} a^A \leftrightharpoons a^{A[c]} \\ a^\forall \leftrightharpoons a^{\forall_\alpha A}\end{array}$$

$$\frac{a^{A}=(a^{\exists})_0,\ b=(a^{\exists})_1,\ A[b],\exists\alpha A,\Gamma\to C}{\exists\alpha A,\Gamma\to C}\quad\frac{\tau}{\tau_{a^A,b}[(a^{\exists})_0,(a^{\exists})_1]}\quad \sigma>0,\ a^{A}\leftrightharpoons a^{A[b]},\ a^{\exists}\leftrightharpoons a^{\exists\alpha A}$$

$$\frac{\ldots a^{A[N]}=(a^{\exists})_0,\ N=(a^{\exists})_1,\ A[N],\exists x A,\Gamma\to C\ldots}{\exists x A,\Gamma\to C}\quad\frac{\tau_{a^{A[b]},b}[a^{A[N]},N]}{\tau_{a^{A[b]},b}[(a^{\exists})_0,(a^{\exists})_1]}$$

$$(\mathrm{Comp}_\sigma)\quad\frac{t=b,\ \Gamma\to C}{\Gamma\to C}\quad\frac{\tau}{\tau_b[t_\sigma]}\qquad \sigma\neq 0,\ \sigma \text{ is a finite type}$$

$$(\mathrm{Comp}_0)\quad\frac{\ldots t=N,\ \Gamma\to C\ldots}{\Gamma\to C}\quad\frac{\tau_b[N]}{\tau_b[t]}$$

$$(\mathrm{Ch}_\forall)\quad\frac{\Gamma\to\forall\alpha\exists\beta A\quad a^{\forall}=\lambda\alpha(q\alpha)_0,\forall\alpha A\beta[(q\alpha)_1],\Gamma\to C}{\Gamma\to C}\quad\frac{q\qquad\tau}{\tau_{a^\forall}[\lambda\alpha(q\alpha)_0]}$$

$$(\mathrm{Ch}_\exists)\quad\frac{\Gamma\to\exists\beta A\quad a^{A}=(q)_0,A[(q)_1],\Gamma\to C}{\Gamma\to C}\quad\frac{q\qquad\tau}{\tau_{a^A}[(q)_0]}$$

$$(\mathrm{Rep})\quad\frac{\Gamma\to C}{\Gamma\to C}\ \frac{\tau}{\tau'}\ \Gamma\models\tau=\tau'\qquad (C)\quad\frac{\Gamma\to C\quad a^{C}=q,C,\Gamma\to A}{\Gamma\to A}\quad\frac{q\ \ \tau}{\tau_{a^C}[q]}$$

$$(\mathrm{Rep})_\exists\quad\frac{\Gamma\to\exists\beta A\ \ \Gamma,\Pi\to C}{\Gamma,\Pi\to C}\qquad(\mathrm{Rep})_\forall\quad\frac{\Gamma\to\forall\alpha\exists\beta A\quad\Gamma\to C}{\Gamma\to C}\quad\frac{\tau}{\tau}$$

<u>Axioms</u>. Conversions and axioms for equality with an arbitrary antecedent:

$$\Gamma\to E;\ \Gamma\to q^{\sigma}=q^{\sigma}\ \ (\sigma\neq 0);\ \ t=c,E[t],\Gamma\to[E[c];\ t=c,E[c],\Gamma\to E(t)$$

and analogously for $c=t$. Restrictions: c is a simple term and t is a semisimple term. In addition, as axioms we take the sequents $F,\Gamma\to A$, $\Gamma\to T$, where F is a false numerical equality and T is a true numerical equality, as well as all sequents that can be obtained from the cut axioms. With an axiom with succeedent $\tau=q$, where τ and q are terms of type σ, we associate the term $\mathbb{O}^{\sigma}$.

The notation $\Gamma\models\tau=q$ signifies that the sequent $\Gamma\to\tau=q$ is an axiom.

We say that a term t of the form $\lambda\alpha(q\alpha)_1$ (respectively, of the form $(q)_1$) participates directly in the application L of a rule if one of the following conditions is fulfilled: (1) L is $\to\exists$ or $\forall\to$ and t is the c of the rule; (2) L is Comp with the principal term $(t\tau)$ (respectively, t).

The base of variable a^{A} of type A in a given sequent is the antecedent member of A associated with this variable. The base of term t of the form $\lambda\alpha(q\alpha)_1$, where $q\in\forall\alpha\exists\beta A$, is the left premise of the rule $Ch_\forall$ or $Rep_\forall$, having the form $\Gamma\to\forall\alpha\exists\beta A;\, q$. The base of term $(q)_1$, where $q\in\exists\beta A$ and is not a variable, is defined analogously, with reference to $Ch_\exists$ $Rep_\exists$.

A derivation in $(HA^\omega+AC)_\infty$ is a locally correct proof-figure f equipped with ordinals $<\varepsilon_0$ and a list τ_f consisting of the three parts C_f, Ch_f, and Rep_f, where C_f (respectively, Ch_f and Rep_f) contains, to within substitutions of simple terms for variables, all cut formulas other than equalities (respectively, all principle formulas of rules Ch and Rep_Q) from f, and the base of t exists below L for any term t participating directly in the application L of some rule.

When considering parts g of a given derivation f, consisting of all sequents located over some node a (including a) it is implied that $\tau_g=\tau_f$.

It is easy to see that $(HA^\omega+AC)_\infty$ is an extension of $HA^\omega+AC$: the rearrangement of a finite derivation into an infinite one is carried out by a standard scheme. The axiom of choice is replaced by the rule of choice and induction is replaced by cuts with a subsequent Rep (for the matching of equipping terms).

We prove the normalization theorem for $(HA^\omega+AC)_\infty$ by the same scheme as for $HA+AC^{0\to0}$. Here, however, the normalization will be carried out in two stages. At first we shall take the steps not requiring the distinguishing of the levels of types (the levels correspond to the magnitudes of the indices from § 9), and only then the steps requiring such distinguishment.

<u>LEMMA 10.1 (On Invertibility)</u>. From every derivation f equipped with a sequent from the left column we can construct a derivation g equipped with a sequent from the right column, such that $h(f)=h(g)$ and τ_g differs from τ_f really only by the substitution of an appropriate term in the place of the occurrences of certain variables.

$$\Gamma\to\forall\alpha A;\, q \qquad\qquad \Gamma\to A[C];\ (q,c)$$

$$\Gamma\to A\& B;q \qquad\qquad \Gamma\to A;(q)_0\quad \Gamma\to B;(q)_1$$

$$\Gamma\to A\supset B;\, q \qquad\qquad A,\Gamma\to B;\ (q\,a^A)$$

$$a^\exists=u,\exists\alpha A,\Gamma\to C;\, q \qquad\qquad \langle a^{A[C]},c\rangle=u, A[C],\Gamma\to C;\, q_{a^\exists}[\langle a^{A[C]},c\rangle]$$

u,Γ, and C do not contain $a^\exists$.

The proof is carried out by a standard scheme; however, we need to observe the equipping terms. In the cases of $\forall,\&,\supset$ we make, over all equipping terms of sequents containing the formula "to be inverted" in the succeedent, the same transformation (writing out to the end) as over the lower term. Therefore, the equipping can be spoiled only for Rep which replaces the succeedent rule introducing the formula to be inverted. However, the conversions help here.

Let us consider the $\exists$-case. For simplicity of writing let the type of the variable introduced be different from 0; $a_i \leftrightharpoons a^{A[b_i]}$; $\tilde{a} \leftrightharpoons \langle a^{A[c]}, c\rangle$.

The old derivation:

$$a_m = \ldots = (a^{\exists})_0,\ b_m = \ldots = (a^{\exists})_1,\ \Phi \to F$$

$$\frac{(a^{\exists})_1 = b,\ \Omega \to G}{\Omega \to G} \qquad \frac{t}{t_b[(a^{\exists})_1]}$$

$$\frac{a_{n+1} = \ldots = a_1 = (a^{\exists})_0,\ b_{n+1} = \ldots = b_1 = (a^{\exists})_1,\ A[b_1], \ldots, A[b_{n+1}], \exists\alpha A, \Gamma, \Sigma \to \mathfrak{D} \qquad \tau}{a_n = \ldots = a_1 = (a^{\exists})_0,\ b_n = \ldots = b_1 = (a^{\exists})_1,\ A[b_1], \ldots, A[b_n], \exists\alpha A, \Gamma, \Sigma \to \mathfrak{D} \qquad \tau_{a_{n+1}, b_{n+1}}[(a^{\exists})_0, (a^{\exists})_1]}$$

$$\downarrow$$

$$a^{\exists} = u, \exists\alpha A, \Gamma \to C;\ q.$$

The new derivation:

$$\Phi' \to F' \qquad \left(\frac{\to a^{A[c]} = (\tilde{a})_0 \qquad \to c = (\tilde{a})_1 \qquad a^{A[c]} = (\tilde{a})_0, c = (\tilde{a})_1, \Phi' \to F'}{\Phi' \to F'}\right)$$

$$\frac{\Omega' \to G'}{\Omega' \to G'} \qquad \frac{t'_b[c]}{t'_b[(\tilde{a})_1]}$$

$$\frac{A[c], \Gamma, \Sigma' \to \mathfrak{D}'}{A[c], \Gamma, \Sigma' \to \mathfrak{D}} \qquad \frac{\tau_{a_{n+1}, b_{n+1}, \ldots, a_1, b_1}[(\tilde{a})_0, (\tilde{a})_1, \ldots, (\tilde{a})_0, \ldots, (\tilde{a})_1]}{\tau_{a_{n+1}, b_{n+1}, \ldots, a_1, b_1}[(\tilde{a})_0, (\tilde{a})_1, \ldots, (\tilde{a})_0, \ldots, (\tilde{a})_1]}$$

$$\downarrow$$

$$\tilde{a} = u, A[c], \Gamma \to C;\quad q_{a^{\exists}}[\tilde{a}].$$

All ancestors of the formula $\exists\alpha A$ and the formula $A[b_i]$ (the side formulas corresponding to the applications of $\exists\to$) are deleted. We act in the same way with all ancestors of the additional side formulas of these $\exists\to$, as well as of the side formulas of Comp, in which $(a^{\exists})_1$ participates directly. We write out $A[c]$ to the end in all the antecedents. We replace all occurrences of variables b_i by c, all occurrences of a_i by $a^{A[c]}$ and all occurrences of $a^{\exists}$ by $\tilde{a}$. The deleted occurrences of the additional logical formulas are restored in the axioms with the aid of cuts. The left premises of the cuts are conversion axioms. For Rep, replacing Comp, the condition of the correctness of the equipping is fulfilled.

If α is a variable of type 0, then c is some digit and in each of the applications of $\exists\to$, Comp being considered we need to select the premise with number c.

One of the fundamental conditions for the correctness of the subsequent transformations is given by the following statement on the invariability of the equipping.

LEMMA 10.2. Let the equipped sequent $\Xi, \Omega \to C;\tau$ be found over the equipped sequent

$\Xi \to C; q$ in some $(HA^{\omega}+AC)_{\infty}$ -derivation and, moreover, let the succeedent of the lower sequent originate from the succeedent of the upper one. Then $\Xi, \Omega \models \tau = q$.

Proof. It is carried out by induction on the number of applications of the rules leading from τ to q. We note that if on the way there are no and Rep, then we can strengthen the statement up to $\Omega \models \tau = q$.

In the theorem on lowering the degree, presented below, we shall distinguish the cases when the degree of a cut is lowered and when the degree of Ch is lowered. A° denotes $\{A\}$ if A starts with $\forall\alpha\exists\beta$ or $\exists\beta$. In the remaining cases A° is empty. $A^{\exists}$ denotes $\exists_{\beta}B$ if $A = \forall\alpha\exists_{\beta}B$ and is empty in the remaining cases.

THEOREM 10.3 (On Lowering Cut Degree). From every derivation f of length α and of degree (C_f, Ch_f, Rep_f) we can construct a derivation g of the same sequent, of length 2^{α}_{+} and of degree

$$(\bar{A} \cup C_f \setminus A),\ \overset{\exists}{A} \cup (Ch_f \setminus A),\ \overset{\circ}{A} \cup Rep_f).$$

Proof. We consider only the case when f ends in a cut or in Ch with respect to formula A and in f there are no other applications of these rules with respect to A. In other words, we determine the analog of operator R_0 from § 4. The analog we need of operator R_1 is determined in the same way as in § 5.

1. The $\forall$ -case.

1.1. Cut. Here we go through the standard consideration.

The old derivation:

$$\begin{array}{c} \overset{\forall}{a} = q,\ \Phi \to E \\ \downarrow \\ \dfrac{\overset{A}{a} = (\overset{\forall}{a}c),\ \overset{\forall}{a} = q,\ A[c],\ \forall\alpha A, \Omega \to \mathfrak{D}}{\overset{\forall}{a} = q,\ \forall\alpha A,\ \Omega \to \mathfrak{D}} \\ \downarrow \\ \dfrac{\Gamma \to \forall\alpha A \quad \overset{\forall}{a} = q, \forall\alpha A. \Gamma \to C}{\Gamma \to C} \end{array} \qquad \begin{array}{c} \dfrac{u}{u_{a^A}[(a^{\forall}c)]} \\ \\ \dfrac{q \quad \tau}{\tau_{a^{\forall}}[q]}. \end{array}$$

The new derivation:

$$\begin{array}{c} \Phi' \to E' \\ \downarrow \\ \downarrow \\ \dfrac{\Omega \to A[c] \quad \overset{\forall}{a} = (sc), A[c], \Omega' \to \mathfrak{D}'}{\Omega' \to \mathfrak{D}'} \\ \downarrow \end{array} \qquad \dfrac{(qc) \quad u_{a^{\forall}}[q]}{u_{a^A, a^{\forall}}[(qc), q]}$$

$$\frac{\Gamma\to C}{\Gamma\to C} \qquad \frac{\tau_{a^\forall}[q]}{\tau_{a^\forall}[q]}.$$

The ancestor $\overset{\forall}{a}=q,\forall\alpha A$ is destroyed, after which $\overset{\forall}{a}$ is replaced by q. The equipping by terms in the new cut proves to be correct. The axioms are restored with the aid of cuts.

1.2. $Ch_\forall$. The old derivation $(q_0 \leftrightharpoons \lambda\alpha(q\alpha)_0)$:

$$\overset{\forall}{a}=q_0,\ \overset{A}{a}=(\overset{\forall}{a}c),\ \Phi\to E$$

$$\downarrow$$

$$\frac{\overset{A}{a}=(\overset{\forall}{a}c),\ A_{\alpha\beta}[c,(qc)_1],\ \overset{\forall}{a}=q_0,\ \forall\alpha A_\beta[(q\alpha)_1],\ \Omega\to\mathfrak{D}}{\overset{\forall}{a}=q_0,\ \forall\alpha A_\beta[(q\alpha)_1],\ \Omega\to\mathfrak{D}} \qquad \frac{u}{u_{a^A}[(\overset{\forall}{a}c)]}$$

$$\downarrow$$

$$\frac{\Gamma\to\forall\alpha\exists\beta A \qquad \overset{\forall}{a}=q_0,\ \forall\alpha A_\beta[(q\alpha)_1],\ \Gamma\to C}{\Gamma\to C}\qquad \frac{q\quad \tau}{\tau_{a^\forall}[q_0]}.$$

The new derivation:

$$\overset{A}{a}=(qc)_0,\ \Phi'\to E' \qquad \left(\frac{\overset{A}{a}=(qc)_0\to\overset{A}{a}=(q_0c)\quad \overset{A}{a}=(q_0c),\ \Phi'\to E'}{\overset{A}{a}=(qc)_0,\ \Phi'\to E''}\right)$$

$$\searrow$$

$$Ch_\exists\,\frac{\Omega'\to\exists\beta A_\alpha[c]\qquad \overset{A}{a}=(qc)_0,\ A_{\alpha\beta}[c,(qc)_1],\ \Omega'\to\mathfrak{D}'}{\Omega'\to\mathfrak{D}'}\qquad\frac{(qc)\quad u_{a^\forall}[q_0]}{u_{a^A,a^\forall}[(q_0c),q_0]}$$

$$\downarrow$$

$$\frac{\Gamma\to\forall\alpha\exists\beta A\qquad\Gamma\to C}{\Gamma\to C}\qquad\frac{\tau_{a^\forall}[q_0]}{\tau_{a^\forall}[q_0]}.$$

All $\overset{\forall}{a}$ are replaced by q_0. After this the ancestors of formulas $\forall\alpha A_\beta[(q\alpha)_1]$ are replaced by $\overset{A}{a}=(qc)_0$ and $\forall\to$ by $Ch_\exists$. The axioms spoiled under the latter replacement are restored with the aid of cuts and conversions.

2. The $\exists$-case.

2.1. Cut. The rearrangement is carried out in two stages. At first a proof-figure is obtained, not being, in general, a derivation.

The old derivation:

$$\downarrow$$

$$\frac{\Omega\to A[c]}{\Omega\to\exists\alpha A}\qquad\frac{\tau}{\langle\tau,c\rangle}$$

$$\frac{\begin{matrix}\downarrow \\ \Gamma\to\exists\alpha A\end{matrix} \qquad \begin{matrix}\downarrow \\ a^{\exists}=\tau,\ \exists\alpha A,\ \Gamma\to C\end{matrix}}{\Gamma\to C} \qquad \frac{\tau \quad q}{q_{a^{\exists}}[\tau]}.$$

The new proof-figure:

$$\begin{matrix} \langle a^{A}, c\rangle=\tau, \Phi\to E \\ \downarrow \\ \dfrac{\Omega\to A[c] \qquad \langle a^{A},c\rangle=\tau, A[c], \Omega\to C}{\Omega\to C} \qquad \dfrac{\tau \quad q_{a^{\exists}}[\langle a^{A},c\rangle]}{q_{a^{\exists}}[\langle\tau,c\rangle]} \\ \downarrow \\ \dfrac{\Gamma\to C}{\Gamma\to C} \qquad \dfrac{q_{a^{\exists}}[\tau]}{q_{a^{\exists}}[\tau]}. \end{matrix}$$

The new proof-figure is obtained by a standard transformation: the ancestors of $\exists\alpha A$ are replaced by C and Γ is added to the antecedents. The applications of $\to\exists$ are replaced by "almost cuts" with the application of the invertibility lemma. (The modifier "almost" is introduced because the additional side formula $\langle a^{A}, c\rangle=\tau$ does not have the needed form $a^{A}=\tau$.) The equipping terms in the new proof-figure are obtained by substituting the old terms in q in the place of $a^{\exists}$. We now replace (the ancestors of the explicitly mentioned occurrences of the formulas) $\langle a^{A}, c\rangle=\tau$ by $a^{A}=\tau$. The axioms spoiled by this are restored with the aid of Lemma 2.4.

$$\cfrac{\cfrac{a^{A}=\tau, \Phi\to\langle\tau,c\rangle=\tau \qquad a^{A}=\tau, \langle\tau,c\rangle=\tau, \Phi\to\langle a^{A},c\rangle=\tau}{a^{A}=\tau, \Phi\to\langle a^{A},c\rangle=\tau} \qquad \langle a^{A},c\rangle=\tau, \Phi\to E}{a^{A}=\tau, \Phi\to E}$$

2.2. $Ch_{\exists}$. The old derivation:

$$\begin{matrix} (\tau)_1=b,\ a^{A[(\tau)_1]}=(\tau)_0,\ \Phi\to E \\ \downarrow \\ \dfrac{(\tau)_1=b,\ a^{A[(\tau)_1]}=(\tau)_0,\ \Xi\to H}{a^{A[(\tau)_1]}=(\tau)_0,\ \Xi\to H} \qquad \dfrac{v}{v_b[(\tau)_1]} \\ \downarrow \\ \dfrac{\tau}{\langle\tau,c\rangle}\ \dfrac{\Omega\to A[c]}{\Omega\to\exists\alpha A} \qquad a^{A[(\tau)_1]}=(\tau)_0,\ A[(\tau)_1],\ \Theta,\ B[(\tau)_1],\ldots,C[(\tau)_1]\to G \quad u \\ \downarrow \qquad\qquad\qquad \downarrow \\ \dfrac{\Gamma\to\exists\alpha A \qquad a^{A[(\tau)_1]}=(\tau)_0,\ A[(\tau)_1],\ \Gamma\to\mathfrak{D}}{\Gamma\to\mathfrak{D}} \qquad \dfrac{\tau \quad q}{q_{a^{A[(\tau)_1]}}[(\tau)_0]}. \end{matrix}$$

In the derivation of the right premise of the cut we pick out all antecedent members observable up to (explicitly indicated occurrences of) $A[(\tau)_1]$ and in them we replace $(\tau)_1$

by c. In the equipping terms and the additional side formulas we replace $a^{B[(\tau)_1]},\dots,$ $a^{C[(\tau)_1]}$ by $a^{B[c]},\dots a^{C[c]}$. After this we construct a new derivation as in the case of the $\exists$-cut:

$$\frac{\Phi''\to(\tau)_0=\tau \qquad \Phi'',(\tau)_0=\tau,\ a^{A[c]}=\tau\to a^{A[c]}=(\tau)_0 \qquad a^{A[c]}=(\tau)_0, c=(\tau)_1,\Phi''\to E''}{(\tau)_1=c,\ a^{A[c]}=\tau,\Phi''\to E''}$$

$$\frac{\Phi'\to(\tau)_1=c \qquad (\tau)_1=c,\ a^{A[c]}=\tau,\Phi'\to E'}{a^{A[c]}=\tau,\ \Phi'\to E'}$$

$$\downarrow$$

$$\frac{a^{A[c]}=\tau,\Xi'\to H'}{a^{A[c]}=\tau,\Xi'\to H'} \qquad \frac{v_b[c]}{v_b[(\tau)_1]}$$

$$\downarrow$$

$$a^{A[c]}=\tau, A[c],\Theta', B[c],\dots,C[c]\to G' \qquad u_{A[(\tau)_1]}[a^{A[c]}\dots]$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{\Omega\to A[c] \qquad a^{A[c]}=\tau, A[c],\Omega\to \mathfrak{D}}{\mathrm{Rep}\ \dfrac{\Omega\to\mathfrak{D}}{\Omega\to\mathfrak{D}}} \qquad \frac{q[a^{A[c]}]}{\dfrac{q[\tau]}{q[(\langle\tau,c\rangle)_0]}}$$

$$\downarrow$$

$$\mathrm{Rep}_{\exists}\frac{\Gamma\to\exists\alpha A \qquad \Gamma\to\mathfrak{D}.}{\Gamma\to\mathfrak{D}} \qquad \frac{q[(\tau)_0]}{q[(\tau)_0]}.$$

All ancestors of $a^{A[(\tau)_1]}=(\tau)_0$ are replaced by $a^{A[c]}=\tau$. All ancestors of the side formulas of Comp for $(\tau)_1$ over the right premise of the cut are deleted and the corresponding variables b are replaced by c. The formulas $B[(\tau)_1]$, including those in the superscript of the variables $a^{B[(\tau)_1]}$, are replaced by $B[c]$. All applications of the rules are preserved or lead to Rep. Let us clarify the transformation of the axiom. The passage from $\Phi\to E$ to $\Phi'\to E'$ is put together from two stages: at first $a^{B[(\tau)_1]}$ and b are substituted for all occurrences of the variables $a^{B[c]}$ and c, which yields $\Phi''\to E''$. Next, c is substituted for certain occurrences of $(\tau)_1$ (precisely, the occurrences in formulas $B[(\tau)_1]$). The first substitution leads the axiom written on the top of the old proof once more into an axiom (written in the upper right corner of the new proof). The second substitution is justified with the aid of cuts with respect to the axioms for equality and, next, a cut with respect to $(\tau)_1=c$ is used, whose left premise is derivable by virtue of Lemma 2.4.

3. The $\&$-case. It is analyzed analogously to the $\forall$-case.

4. The $\supset$-case. The old derivation:

$$
\begin{array}{c}
a^{\supset}=\tau,\ a^{B}=(a^{\supset}v),\ \Phi\to E\\
\downarrow\\
\dfrac{a^{\supset}=\tau, A\supset B, \Omega\to A \quad a^{B}=(a^{\supset}v), B, a^{\supset}=\tau, A\supset B, \Omega\to\mathfrak{D}}{a^{\supset}=\tau, A\supset B, \Omega\to\mathfrak{D}} \quad \dfrac{v \qquad u}{u_{a^{B}}[(a^{\supset}v)]}\\
\downarrow\\
\dfrac{\Gamma\to A\supset B \quad a^{\supset}=\tau, A\supset B, \Gamma\to C}{\Gamma\to C} \quad \dfrac{\tau \quad q}{q_{a^{\supset}}[\tau]}.
\end{array}
$$

Let a prime denote the substitution of τ in the place of $a^{\supset}$. Then the new derivation has the form:

$$
\begin{array}{c}
a^{B}=(\tau v'),\ \Phi'\to E'\\
\downarrow\\
\dfrac{v'(\tau a^{A})}{(\tau v')} \quad \dfrac{\dfrac{\Omega'\to A \quad a^{A}=v', A, \Omega'\to B}{\Omega'\to B} \qquad a^{B}=\tau v', B, \Omega'\to\mathfrak{D}'}{\Omega'\to\mathfrak{D}'} \quad \dfrac{u'}{u'_{a^{B}}[(\tau v')]}\\
\downarrow\\
\dfrac{\Gamma\to C}{\Gamma\to C} \quad \dfrac{q'}{q'}.
\end{array}
$$

The proof is complete.

The normalization theorem is obtained without difficulty from Theorem 10.3.

THEOREM 10.4. From every $(HA+AC)_{\infty}$-derivation we can construct a derivation of the same last sequent not containing applications of a cut or of the rule of choice.

Let us take up a further normalization of derivations. We shall try to compute the values of all terms q justified with the aid of the rules Rep_{Q} by applying the inversion operation to $Rep_{\forall}$ (which permits us to take the step of computing the terms $(\lambda\alpha(q,\alpha)_{1}, C)$ by replacing them by $(q,C)_{1})$ and, next, by raising with respect to the derivation obtained the sequent $\exists\beta A$ up to the rule introducing $\exists\beta$. We are already not interested in everything that is above the conclusion of this rule: we need only the term C of this rule.

Definition. A node a of derivation f is called unessential if a is empty or if below a there exists a conclusion b of rule $\to\exists$, and further below, the left premise b_1 of rule Rep_{Q}; moreover, the succeedent of b_1 originates from the succeedent of b and either coincides with it or is obtained by a weighting of $\forall\alpha$.

The essential part of a derivation consists of all its nodes that are not unessential.

A derivation is said to be computed if it does not contain cuts or Ch, and in its essential part there are neither applications of $Rep_{\exists}$ nor applications of Comp, in which terms of type $\exists\alpha A$ are computed.

LEMMA 10.5. From every derivation f we can construct a computed derivation with the same last sequent.

Proof. We can take it that C_f and Ch_f are empty. We shall "compute" f by lowering the levels of the terms which violate the condition of being computed (the level of a term is the number of layers of arrows in its type). At first we annihilate the "bad" Comp for the terms justified by Rep_Q by choosing among the members not already looked over those which are justified by terms of maximal level and by examining two cases, depending on the form of the justification.

1. $\forall\alpha\exists\beta A$. We replace $(\lambda\alpha(S\alpha)_1, C)$ by $(SC)_1$ in the corresponding Comp and we turn them into $Rep_\exists$ by adding on the proof of the sequent with the succeedent $\exists\beta A_\alpha[C]$, obtained by the invertibility lemma.

2. $\exists\beta A$. We eliminate the $Rep_\exists$ being examined in exactly the same way as a cut, by raising upward to the corresponding $\to\exists$.

3. Comp, computing $(a^3)_1$. We act with the corresponding antecedent member $\exists\beta A$ as in the invertibility lemma.

Let us now prove the conservativity relative to HA.

THEOREM 10.6. From every derivation in $HA^\omega+AC$ of the sequent $\Gamma\to A$, where Γ and A are closed formulas of the HA system, we can construct the derivation in HA of this sequent.

Proof. By virtue of the preceding it is sufficient to show that a computed derivation f of the sequent $\Gamma\to A$ can be rearranged into a HA_∞-derivation. Only variables of type 0 are introduced by virtue of $\forall\to$ and $\to\exists$ having the property of being subformulas; therefore, only digits are substituted into them. We replace all free variables of type $\sigma>0$ by 0^σ; after which we delete the side formulas of all Comp (and in Comp of type 0 we restrict ourselves to the first premise).

We obtain a proof-figure which differs from the HA_∞-derivation only in the presence of additional logical side formulas and in the fact that the conversions mentioned earlier, as well as the additional conversion

$$(0^{\sigma\to\tau}t)=0^\tau$$

are applied when justifying the axiom.

By virtue of Lemma 9.8 we need only prove that after the deletion of the additional side formulas the axioms go into axioms. We can take it that the axioms have the form

$$M_1=N_1,\ldots,M_k=N_k,\ E,\Gamma\to M=N, \tag{1}$$

where E is a complete list of additional side formulas, $M_1,N_1,\ldots,M,N$ are positive integers, and the remaining members are not equalities. It remains to prove that M coincides with N under the assumption that M_i coincides with N_i for all $i\,(i=1,\ldots,k)$. The additional equalities occurring in the antecedent have one of the following forms:

$$a^{A[N]} = (a^{\exists x A})_0\,,\quad N = (a^{\exists x A})_1$$
$$a^{B} = (a^{C \supset B} \tau)$$
$$a^{D} = (a^{D\& E})_0\,;\quad a^{E} = (a^{D \& E})_1 \qquad (2)$$
$$a^{A[N]} = (a^{\forall x A} N)$$

where the rows of (2) correspond one-to-one to the applications of rules in the branch from the axiom being examined to the lowest sequent. We identify all occurrences of the formulas in this branch, being the ancestors of each other. Then each member of the antecedent in (1) is either a member of the antecedent of a lower sequent or an antecedent side formula of some application of the rule. We say that antecedent members (i.e., occurrences in an antecedent) are isomorphic if they coincide or are side formulas of the applications of one and the same rule with isomorphic principal formulas (here the $\& \to$ separating, respectively, the first and second members are reckoned to be distinct; the same thing refers to application of $\forall \to$, in which different numbers are substituted; however, all applications of $\exists \to$ with isomorphic principal formulas are identified). We say that the branch being examined (or, what is the same, the system of equalities (2)) is consistent if in the applications of $\exists \to$ with isomorphic principal formulas there are substituted (in the branch given) like positive integers. Now we show how to solve a consistent system of equalities (2), i.e., for each logical variable a^G we find a term t_{a^G} such that the substitution of t_{a^G} for a^G turns all equalities into identities or conversions. Here the isomorphic variables (i.e., the variables corresponding to isomorphic occurrences of formulas) will be identified. In particular, like terms will correspond to them. Instead of t_{a^G} we shall write t_G.

If G is an elementary formulas, then $t_G = 0$.

If $G = \exists x A$, then $t_G = \langle t_{a^{A[N]}}, N \rangle$ for that (single) N for which the equalities $(a^G)_0 = a^{A[N]}$ and $(a^G)_1 = N$ are contained among equalities (2). (It is precisely here that the consistency is used.) If there are no such equalities, then $t_G = a^G$.

If $G = \forall x A$, then

$$t_G = \lambda x(\textbf{if } x = N_1 \textbf{ then } t_{A[N_1]} \textbf{ else} \ldots \textbf{if } x = N_k \textbf{ then } t_{A[N_k]} \textbf{ else } a^G),$$

where $A[N_1], \ldots, A[N_k]$ is a complete list of (occurrences of) formulas for which the equality $a^{A[N_i]} = (a^G N_i)$ exists.

If $G = A_0 \& A_1$, then $t_G = \langle t_{A_0}, t_{A_1} \rangle$ if there are both the equalities $a^{A_i} = (a^G)_i$, $i = 0, 1$. If the i-th one of them is not there, we replace t_{A_i} by a^{A_i}.

If $G = A \supset B$, then $t_G = \lambda x^A t_B$ for (any of the isomorphic occurrences of) B, for which the actuality $a^B = (a^G x)$ exists. If there are none such, then $t_G = a^G$. Here we took advantage of the fact that the t_B for isomorphic B coincide.

It is easy to see that all equalities (2) pass into conversions after the substitution of t_G for a^G, so that $\rightarrow M = N$ is derivable with the aid of the axiom for equality, of conversions, and of cuts. If it were that $M \neq N$, then certain terms would have different normal forms, which contradicts the Church—Rosser theorem. Hence $M = N$.

Now, from all sequents of the given derivation we delete all the additional side formulas; after which we replace by Rep all the "degenerate" applications of $\exists \rightarrow$, i.e., the applications below which there exists an application of $\exists \rightarrow$ to an isomorphic formula. This transformation removes all inconsistent branches, so that by virtue of the preceding, the axioms go into axioms of HA. Q.E.D.

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