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INTENSIONAL INTERPRETATIONS OF FUNCTIONALS OF FINITE TYPE I

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§1. T_0 will denote Gödel's theory T[3] of functionals of finite type (f.t.) with intuitionistic quantification over each f.t. added. T_1 will denote T_0 together with definition by bar recursion of type o, the axiom schema of bar induction, and the schema

$$\mathbf{AC}_{00} \qquad \bigwedge x \bigvee y A(x, y) \rightarrow \bigvee \alpha \bigwedge x A(x, \alpha(x)),$$

of choice. Precise descriptions of these systems are given below in §4. The main results of this paper are interpretations of T_0 in intuitionistic arithmetic U_0 and of T_1 in intuitionistic analysis U_1. U_1 is U_0 with quantification over functionals of type (0, 0) and the axiom schemata $\mathbf{AC}_{00}$ and of bar induction. These interpretations establish that T_i is a conservative extension of U_i (for $i = 0$ and 1). If $\mathbf{AC}_{00}$ is dropped from T_1 and U_1, the interpretation of T_1 in U_1 still works, but it is open whether or not T_1 is a conservative extension of U_1. (Actually, for the conservative extension result, we need $\mathbf{AC}_{00}$ only for quantifier-free $A(x, y)$.)

The prime formulae of T_i are equations $s = t$ between terms of the same (arbitrary) f.t. The difficulty in interpreting T_i in U_i arises from the fact that $s = t \vee \neg s = t$ is an axiom of T_i, even for equations of nonnumerical type, so that $=$ cannot be interpreted simply as extensional equality. Gödel's own interpretation of $s = t$ is this: Terms are to denote *reckonable* (*berechenbaren*) functionals, where the reckonable functionals of type 0 are the natural numbers, and the reckonable functionals of type (σ, τ) are operations for which we can constructively prove that, when applied to reckonable functionals of type σ, they uniquely yield ones of type τ. $s = t$ means that s and t denote *definitionally equal* reckonable terms. Lacking a general conception of the kinds of definitions by which an operation may be introduced, the notion of definitional equality is not very clear to me. But if, as in the case of T_i, we can regard the operations ϕ as being introduced by conversion rules

$$\phi t_1 \ldots t_n \Rightarrow s(t_1, \ldots, t_n),$$

then definitional equality has a clear meaning: s and t are *definitionally equal* if they reduce to a common term by means of a sequence of applications of the conversion rules. Of course, this notion makes sense only when we have fixed a definite collection of operations with their conversion rules. If we relativize the notion of reckonability to these operations, we arrive at the notion of a *convertible* term: A term of type 0 is convertible if it reduces to a unique numeral via the conversion rules. A term r of type (σ, τ) is convertible if, for every convertible σ-term s,

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rs is a convertible τ-term. Assuming that the set of conversion rules is primitive recursive, the notion of being a convertible τ-term can be expressed in U_0 for each fixed τ. T_0 is interpreted in U_0 by interpreting the terms of type τ in T_0 as convertible τ-terms, and by interpreting $s = t$ as definitional equality (in the relativized sense). The main problem in carrying out this interpretation is to show, first, that each primitive operation of T_0 can be proved to be convertible in U_0, and secondly, that from the assumption that s and t are convertible, we can prove that they are definitionally equal or not in U_0. Actually, we cheat a little bit in this interpretation. Namely, instead of allowing as reductions arbitrary sequences of applications of the conversion rules, we make a restriction on the order in which they can be applied. This enables us to establish the above facts very simply.

The interpretation of T_1 in U_1 involves only a slight complication of the idea just described for T_0. Let 1 and 2 be abbreviations for the types (0, 0) and (1, 0), resp. The schema for bar recursion introduces functionals ϕ of certain types $(2, \tau)$, such that, for functionals ψ of type 2, $\phi\psi$ is defined by recursion on the unsecured sequences of ψ. Therefore, ψ must yield a value, not simply for all reckonable functionals of type 1 (constructive functions), but for all *infinitely proceeding sequences* of natural numbers (i.p.s.). Actually, in the absence of any thesis, such as Church's, about the totality of all constructive functions, it is hard to see how ψ could be proved to be defined for constructive functions without *a fortiori* proving it to be defined for all i.p.s. But if we relativize the notion of reckonability to the operations of T_1, this remark no longer applies: There are operations ψ which are defined for all reckonable functionals in this relativized sense, and indeed, for all recursive functions, but not for all i.p.s. Moreover, we can choose such a ψ such that $\phi\psi$ is not defined by bar recursion. So, for T_1, we somehow have to regard all i.p.s. as convertible 1-terms. In order to retain terms as concrete objects, we can do this as follows: We can consider terms containing variables of type 1. Relative to the assignment of an i.p.s. β as value of a 1-variable d, we can introduce the further conversion rule

$$dS^n \Rightarrow S^{\beta(n)},$$

where S^n denotes the numeral for n. Let α represent an assignment of values $\alpha_1, \ldots, \alpha_k$ to distinct 1-variables $d_1, \ldots, d_k$. Then an α-reduction of a term is a sequence of applications of the conversion rules, including the ones corresponding to the values α_i of d_i for $i = 1, \ldots, k$. A term of type 0 is *α-convertible* if all its variables are assigned values by α and it α-reduces to a unique numeral. A (σ, τ)-term r is α-convertible if, for every extension β of the assignment α to possibly more 1-variables and for every β-convertible σ-term s, rs is β-convertible. Note that if s is an α-convertible $(1, \tau)$-term and d is a 1-variable not assigned a value by α, then for every extension β of α to d, sd is β-convertible (and so, if $\tau = 0$, β-reduces to a unique numeral). That is, an α-convertible $(1, \tau)$-term is "defined for all i.p.s.", which is the property required (for $\tau = 0$) if bar recursion is to be valid. Terms s and t are *α-definitionally equal* if they α-reduce to the same term. Relative to an assignment of values α to the free 1-variables, the terms of T_1 will be interpreted as denoting α-convertible terms, and $s = t$ will be interpreted as α-definitional equality. Again, the crux of the interpretation is in showing that each primitive operation of

T_1 can be proved in U_1 to be α-convertible (for all α), and that from the α-convertibility of s and t we can prove in U_1 that s and t are α-definitionally equal or not. Actually, bar induction will be involved in these proofs only for showing that bar recursion is α-convertible. Otherwise, the proofs are carried out in the conservative extension of U_0 obtained by adding quantification over i.p.s. (We will, for convenience, denote this extension, itself, by U_0.)

Let T_i' be the result of restricting the prime formulae of T_i to numerical equations, and replacing the equality axioms of T_i for higher types (which can no longer be formulated) by the rule

$$\frac{sa_1 \ldots a_n = ta_1 \ldots a_n}{A(s) \rightarrow A(t)}$$

of extensionality, where the a_i are distinct new variables chosen so that the premise is a numerical equation. These systems are constructively valid, if we interpret the variables as ranging over extensional functionals, since the constant functionals of T_i are extensional. (We are using the extensional form of bar recursion introduced in Spector [9]. Of course, Brouwer's justification of recursion on the unsecured sequences of a functional ψ of type 2 [1] is intensional, in that it is based on an analysis of the possible forms of definition of ψ.) On the other hand, T_i' becomes invalid if we add nonextensional functionals such as the modulus of continuity ϕ of type (2, 1, 0) with

$$\bigwedge x(x < \phi st \rightarrow tx = ux) \rightarrow st = su,$$

where s is of type 2 and t and u of type 1. Kreisel has shown in [8] that T_0' can be interpreted in U_0 by interpreting the terms of T_0' as (extensional) effective operations in the sense of [7], and rewriting every formula about effective operations as an arithmetical formula about their Gödel numbers. This interpretation does not work for T_0, since equality between effective operations is interpreted extensionally, so that the arithmetical translation of $s = t \vee \neg s = t$ (for s and t of types $\neq 0$) is not a theorem of U_0. (Kreisel [7], [8] and Spector [9] do not distinguish between T_i and T_i', but in each case it is clear that they are referring to T_i'.) It is an open question whether a suitable notion of definitional equality can be introduced for effective operations. (This notion should be decidable, in the sense that from a proof that ϕ and ψ are effective operations, we should be able to decide $\phi = \psi$.) The question is not entirely straightforward, since a notion of definitional equality should be coupled with a redefinition of effective operations: Namely, in the condition on an effective operation ϕ that $a = b \rightarrow \phi a = \phi b$, $=$ should be interpreted as meaning definitional equality. I think that this is an interesting question, since on it depends the question of whether Gödel's notion of definitional equality is compatible with Church's thesis; but we will not discuss it here. T_1' can be interpreted in U_1 using the theory of continuous functionals of finite type [7], [10]. The details of this are given in [10]. Also, it is likely that a notion of α-effective operation (in analogy with α-convertibility) would suffice for such an interpretation. The effective operations themselves do not satisfy bar recursion, of course. In any case, these interpretations fail for T_1, since no suitable definition of definitional equality is known.

I am very grateful to William Howard, with whom I have discussed these interpretations many times, who found a serious error in an earlier version of this

paper. The error in question affected the treatment of bar recursion of type 0, and also an analogous treatment of bar recursion of finite type. The changes needed in the latter treatment are sufficiently drastic that I have decided to publish it separately in a Part II of this paper. In any case, it is distinguished from what is presented here, in that the interpretation of bar recursion of finite type is not (as far as is known) intuitionistically valid.

§2. Convertible terms of finite types. We will consider just the finite types which are inductively defined by: 0 is a f.t.; and if σ and τ are f.t., then so is (σ, τ). For $n \geq 2$, set

$$(\tau_1, \ldots, \tau_n, \tau_{n+1}) = (\tau_1, \ldots, (\tau_n, \tau_{n+1}) \ldots)$$

(association to the right). Every f.t. is uniquely of the form $(\tau_1, \ldots, \tau_n, 0)$ for some $n \geq 0$, providing we identify (0) with type 0. τ-valued functionals of n arguments of types $\tau_1, \ldots, \tau_n$ can be identified in a well-known way with functionals of type $(\tau_1, \ldots, \tau_n, \tau)$; so our restriction to types of the form 0 and (σ, τ) involves no real loss of expression. For each τ, we assume that infinitely many τ-variables are given, and we denote them by a^τ, b^τ, c^τ, a_1^τ, etc. When the type is irrelevant or has already been fixed, the type superscript may be omitted. Given a collection of constants, each of a specific f.t., the notion of a *τ-term* is inductively defined by: Each τ-variable and τ-constant (i.e., constant of type τ) is a τ-term; and if s is a (σ, τ)-term and t is a σ-term, then (st) is a τ-term. The term $(\ldots(t_1t_2) \ldots t_n)$ will be denoted by $t_1t_2 \ldots t_n$ (association to the left) for $n \geq 2$. Thus, if t is a $(\tau_1, \ldots, \tau_n, \tau)$-term and t_i a τ_i-term for $i = 1, \ldots, n$, then $tt_1 \ldots t_n$ is a τ-term. τ-terms will be denoted by r^τ, s^τ, t^τ, etc. (possibly with subscripts). Again, when no confusion will result, type superscripts may be omitted.

We will assume that the constants include the following: 0 (*zero*); S (*successor*); for each σ and τ, a constant P (*projection*) of type (σ, τ, σ); for each ρ, σ and τ, a constant K (*application*) of type $((\tau, \sigma, \rho), (\tau, \sigma), \tau, \rho)$; and for each τ, a constant R (*iteration*) of type $(\tau, (\tau, 0, \tau), 0, \tau)$. To P, K and R correspond the following *rules of conversion*:

$$Pst \Rightarrow s, \qquad Krst \Rightarrow rt(st),$$
$$Rrs0 \Rightarrow r, \qquad Rrs(St) \Rightarrow s(Rrst)t.$$

r, s and t are assumed to be of the appropriate type in each case. 0, S, projections, applications and iterations are called *impredicative primitive recursive* (i.p.r.) constants; and terms all of whose constants are i.p.r. are also called i.p.r. The term "impredicative" serves to distinguish the functionals denoted by i.p.r. terms from the primitive recursive functionals in the sense of Kleene [6], whose primitive recursive definitions involve no functionals of type higher than that of the functional being defined.

The constants ϕ which are not i.p.r. are assumed to be of type $\neq 0$ and to have associated with them exactly one conversion rule of the form

$$\phi t_1 \ldots t_n \Rightarrow s(t_1, \ldots, t_n),$$

where $n > 0$ and $s(a_1, \ldots, a_n)$ is a term built up from $a_1, \ldots, a_n$ and constants. The condition $n > 0$ is a convenience to ensure that no constant is convertible to another term. The condition that ϕ have at most one conversion rule is stronger than need be. All we really need is that conversion is *single-valued*, i.e., that there is at most one s such that $r \Rightarrow s$ for each r. For example, this is satisfied by terms $Rrst$, even though R has two conversion rules. A term r is called *directly convertible* if $r \Rightarrow s$ for some s. We assume that the set of conversions $r \Rightarrow s$ is primitive recursive under the standard Gödel numbering of the terms.

The *numerals* are: $S^0 = 0$, $S^1 = SS^0$, $S^2 = SS^1$, etc.

Let $d_1, d_2, d_3, \ldots$ be a fixed enumeration of all the 1-variables. The ith *component* α_i of an i.p.s. α is defined by

$$\alpha(x) = \prod_{i=0}^{\infty} p_i^{\alpha_i(x)}$$

where $p_0, p_1, \ldots$ is the increasing series of prime numbers. Set $|\alpha| = \alpha_0(0)$. We regard α as an assignment of values $\alpha_1, \ldots, \alpha_{|\alpha|}$ to the variables $d_1, \ldots, d_{|\alpha|}$, resp. Accordingly, for each α, there is the *α-conversion rule*

$$d_j S^n \Rightarrow_\alpha S^{\alpha_j(n)} \qquad (j \leq |\alpha|).$$

This is also single-valued for a given α. If $r \Rightarrow s$, we will also write $r \Rightarrow_\alpha s$, even though the conversion does not depend on α. r is called *directly α-convertible*, if for some s, $r \Rightarrow_\alpha s$. r is in *α-normal form* (α-n.f.) if it contains no directly α-convertible parts, i.e., no directly convertible parts and no parts $d_j S^n$ where $j \leq |\alpha|$. If $r \Rightarrow_\alpha s$, then r is uniquely of the form tu. If both t and u are in α-n.f., we write $r \Rightarrow_\alpha s$ (r is *strictly* α-convertible to s). Let $r = r_0, r_1, \ldots, r_n = s$ be a sequence of τ-terms, $n \geq 0$, such that r_{i+1} is obtained from r_i by replacing a part t of r_i by u, where $t \Rightarrow_\alpha u$ ($t \Rightarrow_\alpha u$), for $i = 0, \ldots, n-1$. Then $r_0, \ldots, r_n$ is called a (*strict*) *α-reduction* of r to s, s a (*strict*) *α-reduct* of r, and we write $r \dashv\!|_\alpha s$ ($r \dashv_\alpha s$). $r \dashv\!| s$ ($r \dashv s$) means that $r \dashv\!|_\alpha s$ ($r \dashv_\alpha s$) for some (i.e., all) α with $|\alpha| = 0$.

U_0 will denote intuitionistic arithmetic with definition of functions by primitive recursion and quantification over i.p.s. (one-place functions) added. This is clearly a conservative extension of arithmetic.

Since the conversion rules $r \Rightarrow s$ form a primitive recursive set, the property of a sequence of terms being a (strict) α-reduction is primitive recursive; and so $x \dashv\!|_\alpha y$ and $x \dashv_\alpha y$ are expressed in U_0 by purely existential formulae.

With each term t and variable b we primitive recursively associate a term $s = \lambda b \cdot t$, which contains only projections, applications, constants in t and variables in t other than b, by induction on t: Let b and t be of types σ and τ, resp. s will be of type (σ, τ). If t is a constant or variable other than b, set $s = Pt$, where P is the projection of type (τ, σ, τ). Let $t = b$, so that $\sigma = \tau$. Let P_0 and P_1 be the projections of types $\rho_0 = (\sigma, 1, \sigma)$ and $\rho_1 = (\sigma, 1)$, resp., and let K be the application of type $(\rho_0, \rho_1, \sigma, \sigma)$. Then $s = KP_0P_1$. In every other case, t is of the form t_1t_2, where we can assume that $s_i = \lambda b \cdot t_i$ is defined. Set $s = K_0 s_1 s_2$, where K_0 is the application of type $((\sigma, \tau_1), (\sigma, \tau_2), \sigma, \tau)$. (Here, τ_i is the type of t_i, so that $\tau_1 = (\tau_2, \tau)$.) Let $t(r)$ denote the result of replacing b in t by r.

I. The following is a theorem of U_0: *For all terms x, $\lambda b \cdot x$ is in $\alpha \cdot$n.f.; if y is a σ-term, then*

$$(\lambda b \cdot x) y \dashv\!| x(y),$$

and if y is a σ-term in $\alpha \cdot$n.f., then

$$(\lambda b \cdot x) y \dashv_\alpha x(y).$$

The proof in U_0 is by induction on x.

Thus, the combinators P and K of each type suffice for explicit definition (in the sense of [3, footnote 4]). In some ways (e.g., see the discussion of weak and strong definitional equality below) it would be more natural to deal with the λ-calculus (with type structure) than with combinators. The main reason for using combinators (and, presumably, the reason for their introduction originally) is that they analyze away the syntactical complications involved in changes of bound variables in the λ-operator. A similar treatment of i.p.r. functionals using combinators is given in Grzegorczyk [4].

II. The following is a theorem of U_0: *If* $r \dashv_\alpha s$, $r \dashv_\alpha t$ *and* t *is in* α-*n.f., then* $s \dashv_\alpha t$. *Consequently, if* s *is also in* α-*n.f., then* $s = t$.

The second part clearly follows from the first, since if s is in α-n.f., then $s \dashv\!|_\alpha t$ means $s = t$. Let $r \dashv_\alpha s$ and $r \dashv_\alpha t$, and assume that t is in α-n.f. We prove $s \dashv_\alpha t$ by induction on the length k of the strict α-reduction $r = s_1, \ldots, s_k = s$ of r to s, and within that, by induction on r. $k = 1$. There is nothing to prove, since $r = s$ in this case. $k = 2$. Then $r = r'(u)$ and $s = r'(v)$, where $u \Rightarrow_\alpha v$. r is uniquely of the form $r_0 r_1 \ldots r_n$, where r_0 is a constant or a variable. Since r is not in α-n.f., $n > 0$. If $r_1, \ldots, r_n$ are in α-n.f., then $u = r$, $v = s$ and every strict α-reduction of r must be of the form $r, s, \ldots$. (Here we are using the single-valuedness of the α-conversion rules.) In particular, the strict α-reduction of r to t includes a strict α-reduction of s to t. If not all of the r_i are in α-n.f., then by the definition of a strict α-reduction, $r = r_0 r_1 \ldots r_i(u) \ldots r_n$ for some $i = 1, \ldots, n$, $u \Rightarrow_\alpha v$, and $s = r_0 r_1 \ldots r_i(v) \ldots r_n$. Also, by the definition of a strict α-reduction, since t is in α-n.f. and $r \dashv_\alpha t$, we must have $r \dashv_\alpha r_0 r'_1 \ldots r'_n$, where each r'_j is in α-n.f. and $r_j \dashv_\alpha r'_j$, for $j = 1, \ldots, n$. Since the strict α-reduction of $r_i(u)$ to $r_i(v)$ is of length 2, and $r_i(u)$ is a subterm of r, $r_i(v) \dashv_\alpha r'_i$ follows by the induction hypothesis. Hence, $s = r_0 r_1 \ldots r_i(v) \ldots r_n \dashv_\alpha r_0 r'_1 \ldots r_n \dashv_\alpha t$. $k > 2$. Then there is a u with strict α-reductions of r to u and of u to s, each of length $<k$. So by the induction hypothesis, $u \dashv_\alpha t$ (from $r \dashv_\alpha u$ and $r \dashv_\alpha t$) and hence, $s \dashv_\alpha t$ (from $u \dashv_\alpha s$ and $u \dashv_\alpha t$).

This completes the proof. II remains true when $\dashv_\alpha$ is replaced by $\dashv\!|_\alpha$, but this is harder to prove.

α is said to *cover* the term t if $j \leq |\alpha|$ for all d_j in t. If $r \dashv_\beta s$ and α covers r, then it also covers s. If α covers r and r is in α-n.f., then it contains no parts $d_j S^n$ at all, and we say that it is in *normal form* (n.f.). Equivalently, r is in n.f. if and only if it is in β-n.f. for all β. $\alpha \subseteq \beta$ $(\beta \supseteq \alpha)$ means that $|\alpha| \leq |\beta|$ and that $\alpha_i = \beta_i$ for $i = 1, \ldots, |\alpha|$; i.e., that β is an extension of the assignment α of values to 1-variables. If $\alpha \subseteq \beta$, α covers r and $r \dashv_\alpha s$, then $r \dashv_\beta s$. (Without the assumption that α covers r, this holds only for $\dashv\!|$ in place of $\dashv$.)

Let $M^1_\tau(\alpha, x)$ mean that x is a τ-term containing no variables other than 1-variables, and that α covers x. $M^0_\tau(\alpha, x)$ $(\equiv M_\tau(x))$ means that x is a τ-term containing no variables. For $i = 0$ and 1, we define the predicates $C^i_\tau(\alpha, x)$ by induction on τ:

$$C^i_0(\alpha, x) \equiv M^i_0(\alpha, x) \wedge Vz(x \dashv_\alpha S^z),$$

$$C^i_{(\sigma,\tau)}(\alpha, x) \equiv \bigwedge\beta_{\supseteq\alpha}[C^i_\sigma(\beta, y) \rightarrow C^i_\tau(\beta, xy)].$$

For each τ, $C^i_\tau(\alpha, x)$ is defined in U_0; but $C^i_\tau(\alpha, x)$ as a predicate of α, x and τ

cannot be defined in U_0. (See the Remark following the proof of VI.) It is easy to see that $C^0_\tau(\alpha, x)$ is equivalent in U_0 to $C_\tau(x)$, where

$$C_0(x) \equiv M_0(x) \wedge Vz(x \dashv S^z),$$

$$C_{(\sigma,\tau)}(x) \equiv \bigwedge y[C_\sigma(y) \to C_\tau(xy)].$$

These latter predicates are arithmetical. The reason for using $C^0_\tau(\alpha, x)$ is simply so that we can deal with the two predicates C^0_τ and C^1_τ at the same time. $C^0_\tau(\alpha, x)$ arithmetizes the notion of a convertible term discussed in connection with T_0, and $C^1_\tau(\alpha, x)$ the notion of an α-convertible term, discussed in connection with T_1.

We will frequently drop the superscripts $i = 0$ and 1 on $M^i(\alpha, x)$ and $C^i_\tau(\alpha, x)$ when the discussion applies equally to the two cases. In such contexts, $C_\tau(\alpha, x)$ is expressed by saying that x is *α-convertible*, abbreviated to: *α-conv*. If x is α-conv. for some (i.e., all) α with $|\alpha| = 0$, x is called *convertible*, abbreviated to *conv*. It is easy to see that

$$C_\tau(\alpha, x) \equiv \bigwedge \beta_{\supseteq\alpha} C_\tau(\beta, x),$$

and for $\tau = (\tau_1, \ldots, \tau_n, \sigma)$,

$$C_\tau(\alpha, x) \equiv M_\tau(\alpha, x) \wedge \bigwedge \beta_{\supseteq\alpha} \bigwedge y_1 \ldots \bigwedge y_n[\bigwedge^n_{i=1} C_{\tau_i}(\beta, y_i) \to C_\sigma(\beta, xy_1 \ldots y_n)]$$

are theorems of U_0.

Set $\theta_0 = 0$ and $\theta_{(\sigma,\tau)} = \lambda b^\sigma \cdot \theta_\tau$. Then each θ_τ is in n.f.

III. For each τ, we can prove in U_0 that: *θ_τ is α-conv; and for every α-conv τ-term x, there is a y in n.f. with $x \dashv_\alpha y$*. In view of II, we can call *y the α-n.f.* of x.

The proof is by induction on τ. The proposition is evident for $\tau = 0$, since numerals are α-conv and are in α-n.f. Let $\tau = (\rho, \sigma)$ and assume the proposition for ρ and σ. Let $\alpha \subseteq \beta$ and t be a β-conv ρ-term. Then it has a β-n.f. u, and $\theta_\tau t \dashv_\beta \theta_\tau u \dashv_\beta \theta_\sigma$. θ_σ is α-conv, and so is β-conv. Hence, $\theta_\tau u$ is clearly β-conv. This proves that θ_τ is α-conv. Let r be α-conv. Since θ_ρ is too, $r\theta_\rho$ is α-conv and of type σ. So $r\theta_\rho$ has an α-n.f. u. But by the definition of a strict α-reduction, this means that $r\theta_\rho \dashv_\alpha s\theta_\rho \dashv_\alpha u$, where s is in α-n.f. and $r \dashv_\alpha s$. But since α covers s, s must be in n.f.

That the α-n.f. of an α-conv term is also α-conv follows from:

IV. We can prove in U_0 that: *If $x \dashv_\alpha y$, then $C_\tau(\alpha, x) \equiv C_\tau(\alpha, y) \wedge M(\alpha, x)$.*

Let r be a τ-term covered by α, and $r \dashv_\alpha s$. Then α covers s. If $\tau = 0$, then $r \dashv_\alpha S^m$ implies $s \dashv_\alpha S^m$ by II, and $s \dashv_\alpha S^m$ implies $r \dashv_\alpha s \dashv_\alpha S^m$. So the proposition holds for $\tau = 0$. Assume that it holds for σ and let $\tau = (\rho, \sigma)$. Let $\alpha \subseteq \beta$ and let t be β-conv. Then rt is covered by β and $rt \dashv_\beta st$. Hence, rt is β-conv just in case st is. That is, r is α-conv just in case s is.

It follows that, for $\tau = (\tau_1, \ldots, \tau_n, \sigma)$, a τ-term r is α-conv if and only if it is covered by α, and for all $\beta \supseteq \alpha$ and β-conv $t_1, \ldots, t_n$ in n.f. and of types $\tau_1, \ldots, \tau_n$, resp., $rt \ldots t_n$ is β-conv. For each τ, this equivalence can be proved in U_0.

The problem now is to define the notion of definitional equality. The simplest definition is to make s and t α-definitionally equal if they have the same α-n.f. We call this *weak α-definitional equality*. The weakness in question refers to the following phenomenon, which was pointed out by the referee of an earlier version of this paper and by William Howard: Let P be the projection of type (0, 0, 0).

Then $\phi = \lambda b^0 \cdot b$ and $\psi = \lambda b^0 \cdot Pbb$ are both in n.f., and so are not weakly α-definitionally equal. But ϕc^0 and ψc^0 have the same n.f., namely, c^0; and in the reductions to this n.f., nothing about the range of c^0 is used: it acts simply as a dummy symbol. Indeed, if we had chosen to use the λ-calculus instead of combinators, ψ would λ-convert to ϕ. So it seems reasonable, even compelling, to regard ϕ and ψ as definitionally equal. On these grounds, the appropriate notion is as follows: s^0 and t^0 are *strongly* α-definitionally equal if they are weakly so. Let s and t be (σ, τ)-terms and b a σ-variable not in s or t and $\neq d_j$ for all $j \leq |\alpha|$. Then s and t are strongly α-definitionally equal if sb and tb are. (As usual, by thinking of extensions rather than intensions, we could interchange the terms *weak* and *strong* here.) For the interpretation of T_i in U_i, either weak or strong α-definitional equality will do. We choose the former because it is simpler. Namely, we define (for $i = 0$ and 1)

$$E_\tau^i(\alpha, x, y) \equiv C_\tau^i(\alpha, x) \wedge \bigvee z[x \dashv_\alpha z \wedge y \dashv_\alpha z].$$

By IV, we can prove $E_\tau^i(\alpha, x, y) \rightarrow C_\tau^i(\alpha, y)$ in U_0. Set

$$E_\tau(x, y) \equiv C_\tau(x) \wedge \bigvee z[x \dashv z \wedge y \dashv z].$$

Then $E_\tau(x, y) \equiv E_\tau^0(\alpha, x, y)$ is a theorem of U_0.

V. The following is a theorem of U_0:

$$C_\tau(\alpha, x) \wedge C_\tau(\alpha, y) \rightarrow E_\tau(\alpha, x, y) \vee \neg E_\tau(\alpha, x, y).$$

This solves the first problem in interpreting T_i in U_i. The proof is simply this: If x and y are α-conv, then they have unique α-n.f. z and u, resp., by II and III. But, by II, $E(\alpha, x, y)$ just in case $z = u$.

The second problem in interpreting T_0 in U_0 is solved by:

VI. For each i.p.r. constant ϕ of type τ, $C_\tau(\alpha, \phi)$ is a theorem of U_0.

Case 1. $\phi = 0$. Trivial.

Case 2. $\phi = S$. If s is α-conv and in n.f., then s is a numeral S^m, and so $\phi s = S^{m+1}$, and so is α-conv. Hence, ϕ is conv.

Case 3. $\phi = P$ of type (ρ, σ, ρ). If r and s are α-conv terms of types ρ and σ, resp., in n.f., then $\phi rs \dashv_\alpha r$, and so ϕrs is α-conv. Hence, ϕ is conv.

Case 4. $\phi = K$ of type $((\pi, \sigma, \rho), (\pi, \sigma), \pi, \rho)$. If r, s and t are α-conv terms in n.f. of types (π, σ, ρ), (π, σ) and π, resp., then st is α-conv, and hence, so is $rt(st)$. Since $\phi rst \dashv_\alpha rt(st)$, ϕrst is α-conv. So ϕ is conv.

Case 5. $\phi = R$ of type $(\rho, (\rho, 0, \rho), 0, \rho)$. Let r and s be α-conv terms in n.f. of types ρ and $(\rho, 0, \rho)$, resp. Then $\phi rs0 \dashv_\alpha r$, so that $\phi rs0$ is α-conv. Assume that ϕrsS^m is α-conv. Then so is $s(\phi rsS^m)S^m$. Hence, since $\phi rsS^{m+1} \dashv_\alpha s(\phi rsS^m)S^m$, ϕrsS^{m+1} is α-conv. So by induction on m, ϕrsS^m is α-conv for all m. That is, ϕ is conv.

REMARK. By VI, any given i.p.r. constant term can be proved to be conv in U_0. But we cannot prove in U_0 that every constant i.p.r. 0-term is conv (in the sense of $C_0^0(\alpha, x)$). For if we could, the Gödel interpretation (in [3]) of this theorem of U_0 would yield an i.p.r. constant term ψ of type (0, 1), such that for every constant i.p.r. 1-term ϕ, there is a k with $\phi S^m \dashv S^n$ if and only if $\psi S^k S^m \dashv S^n$. But take $\phi = \lambda b^0 \cdot S(\psi bb)$, and a contradiction results. For the same reason, we cannot define a predicate $C(x, y)$ in U_0 such that (identifying types with their Gödel numbers), $C(S^\tau, x)$ can be proved in U_0 to satisfy the inductive definition of $C_\tau(x)$. For, if we could, we would be able to prove that every constant i.p.s. τ-term is conv. Because of this, we cannot extend the definition (in U_0) of convertibility to transfinite types.

§3. **Bar recursion of type 0.** Let θ be a closed i.p.r. (0, 1)-term with $\theta S^m S^n \dashv S^1$ if $m < n$ and $\theta S^m S^n \dashv S^0$ if $n \leq m$. Let ϕ and ψ be closed i.p.r. terms of types (1, 0, 1) and (1, 0, 0, 1), resp., such that, if s is a 1-term in n.f. then $\phi s S^m S^n \dashv_\alpha s S^n$ if if $n < m$, $\psi s S^n \dashv_\alpha 0$ if $m \leq n$, $\psi s S^m S^n S^m \dashv_\alpha S^n$, and for $k \neq m$, $\psi s S^m S^n S^k \dashv_\alpha s S^k$. We write

$$\langle s; t^0\rangle = \phi st$$

and

$$\langle s; t^0, u^0\rangle = \psi\langle s; t\rangle tu.$$

Thus, if s denotes the sequence $(n_0, n_1, \ldots)$, then $\langle s; S^m\rangle$ denotes $(n_0, n_1, \ldots, n_{m-1}, 0, 0, \ldots)$ and $\langle s; S^m, S^k\rangle$ denotes $(n_0, n_1, \ldots, n_{m-1}, k, 0, 0, \ldots)$. Let Φ_τ be a closed i.p.r. term of type $(0, \tau, (0, \tau), 0, \tau)$ such that, if r is a τ-term, s a $(0, \tau)$-term and t a 0-term, all in n.f., then

$$\Phi_\tau S^1 rst \dashv_\alpha r, \qquad \Phi_\tau S^0 rst \dashv_\alpha st.$$

For each τ, we introduce a constant B (*bar recursion*) of type $(\tau, ((0, \tau), \tau)2, 1, 0, \tau)$ with the conversion rule

$$Brstuv \Rightarrow \Phi_\tau(\theta(t\langle u; v\rangle)v)r(\lambda a^0 \cdot s(\lambda b^0 \cdot Brst\langle u; v, b\rangle(Sa)))v.$$

Recall that every term of the form $\lambda c \cdot x$ is in n.f. Hence, if $t\langle u; S^m\rangle \dashv_\alpha S^n$ and r is in n.f., then: If $n < m$, $BrstuS^m \dashv_\alpha r$; and if $m \leq n$, $BrstuS^m \dashv_\alpha s(\lambda b Brst\langle u; S^m, b\rangle S^{m+1})$. So this is clearly equivalent to the formulation of bar recursion given in Spector [9].

In this section, α-conv and conv are used in the sense of C_τ^1 always.

VII. For each bar recursion constant B, $C_\tau^1(\alpha, B)$ is a theorem of U_1.

U_1 is the system obtained from U_0 by adding the axiom schema

$$\textbf{BI} \quad \bigwedge \beta \bigvee x A(\bar{\beta}(x)) \wedge \bigwedge z(A(z) \to Q(z)) \wedge \bigwedge z(\bigwedge x Q(z^\frown x) \to Q(z)) \to Q(\langle\ \rangle)$$

of *bar induction* and the schema $\mathbf{AC}_{00}$. $A(x)$ denotes a primitive recursive predicate, $\bar{\beta}(x) = \langle\beta(0), \ldots, \beta(x-1)\rangle = \prod_{i=0}^{x-1} p_i^{\beta(i)+1}$, and if z is the sequence number $\langle z_0, \ldots, z_{p-1}\rangle$, then $z^\frown y = \langle z_0, \ldots, z_{p-1}, y\rangle$. $\langle\ \rangle = 1$ is the empty sequence number.

Let B be of type $(\tau, ((0, \tau), \tau), 2, 1, 0, \tau)$. We will prove that B is conv. Let r, s and t be α-conv terms in n.f. of types τ, $((0, \tau), \tau)$ and 2, resp. We have to show that for every m and α-conv 1-term u in n.f., $BrstuS^m$ is α-conv.

Let $j = |\alpha| + 1$, and let $A(z)$ mean that z is a sequence number $\langle z_0, \ldots, z_{p-1}\rangle$, and that td_j can be strictly α-reduced to a numeral S^n with $n < p$, using the *additional conversion rules*

$$d_j S^i \Rightarrow S^{z_i} \qquad (i < p).$$

Then $A(z)$ is primitive recursive and

$$(1) \qquad \bigwedge \beta \bigvee x A(\bar{\beta}(x))$$

follows in U_0 from $C_2^1(\alpha, t)$. Let $Q(z)$ express the proposition that: z is a sequence number $\langle z_0, \ldots, z_{p-1}\rangle$ and for all $m \geq p$ and α-conv u of type 1 in n.f., such that $uS^i \dashv_\alpha S^{z_i}$ for all $i < p$, $BrstuS^m$ is α-conv.

(2) $$A(z) \to Q(z).$$

For, assume $A(z)$, $z = \langle z_0, \ldots, z_{p-1}\rangle$, $m \geq p$ and that $uS^i \dashv_\alpha S^{z_i}$ for all $i < p$. Since $m \geq p$, it follows that $\langle u; S^m\rangle S^i \dashv_\alpha S^{z_i}$ for all $i < p$. Since $A(z)$, this means that $t\langle u; S^m\rangle \dashv_\alpha S^n$ for some $n < p \leq m$, and so $\theta(t\langle u; S^m\rangle)S^m \dashv_\alpha S^1$. Hence, $BrstuS^m \dashv_\alpha r$, and so $BrstuS^m$ is α-conv.

(3) $$\bigwedge yQ(z\char`^y) \to Q(z).$$

For, assume $Q(z\char`^y)$ for all $y$, $z = \langle z_0, \ldots, z_{p-1}\rangle, p \leq m$, and that $u$ is an $\alpha$-conv 1-term in n.f. with $uS^i \dashv_\alpha S^{z_i}$ for all $i < p$. First, assume that $m \geq p + 1$, and let $uS^p \dashv_\alpha S^y$. Then it follows from $Q(z\char`^y)$ that $BrstuS^m$ is α-conv. But if $m \ngeq p + 1$, then $m = p$. Hence, $\langle u; S^m, S^y\rangle S^i \dashv_\alpha S^{z_i}$ for all $i < p$ and $\dashv_\alpha S^y$ for $i = p$. It follows from $Q(z\char`^y)$ that $Brst\langle u; S^m, S^y\rangle S^{m+1}$ is α-conv. Since this holds for all y, $\lambda b^0(Brst\langle u; S^m, b\rangle S^{m+1})$ is α-conv and hence, so is $s(\lambda b \cdot Brst\langle u; S^m, b\rangle S^{m+1})$. But if $t\langle u; S^m\rangle \dashv_\alpha S^n$ for $n \geq m$, then $BrstuS^m$ strictly α-reduces to this term, and otherwise, it strictly α-reduces to r. So in any case, it is α-conv.

$Q(\langle\ \rangle)$ follows in U_1 from (1), (2) and (3), using **BI**. That is, if r, s, t and u are α-conv., then so is $BrstuS^m$ for all m. Hence, B is conv.

§4. **The interpretation of T_i in U_i.** The functional constants of T_0 are the i.p.r. constants. T_1 contains the bar recursion constants B in addition to the i.p.r. constants. The formulae of T_i are built up from equations $s^\tau = t^\tau$ using the propositional connectives $\neg$, $\vee$, $\wedge$, $\to$ and the quantifiers $\bigvee b^\tau$ and $\bigwedge b^\tau$. The *axioms* and *rules of inference* of T_i are

A. The axioms of intuitionistic propositional logic, together with all formulae $s = t \vee \neg s = t$.

B. Axioms of equality.

$$r = r, \qquad r = s \to (r = t \to s = t),$$
$$r = s \to rt = st, \qquad r = s \to tr = ts.$$

C. Axioms for successor.

$$\neg 0 = Sr, \qquad Sr = Ss \to r = s.$$

D. Detachment.

$$\frac{A,\ A \to B}{B}.$$

E. Mathematical induction.

$$\frac{A(0),\ A(a) \to A(Sa)}{A(r)}.$$

F. Definitional axioms. These are obtained by replacing $\Rightarrow$ in the conversion rules for the i.p.r. constants and (in the case of T_1) the bar recursion constants by $=$.

G. Quantification.

$$A(t^{\tau}) \rightarrow \bigvee a^{\tau} A(a), \qquad \bigwedge a^{\tau} A(a) \rightarrow A(t^{\tau}),$$

and the rules

$$\frac{A(a^{\tau}) \rightarrow B}{\bigvee a A(a) \rightarrow B}, \qquad \frac{B \rightarrow A(a^{\tau})}{B \rightarrow \bigwedge a A(a)}$$

where t is free for a in $A(a)$ in the axioms, and a is not free in B.

T_1 also contains the axiom schemata **BI** of bar induction and $\mathbf{AC}_{00}$ of choice.

In what follows, $a_1^{\sigma_1}, \ldots, a_k^{\sigma_k}$ will be distinct variables. $t^* = t(x_1, \ldots, x_k)$ will be the Gödel number of the term which results from replacing a_i in t by the σ_i-term with Gödel number x_i if there is such a term, and otherwise by θ_{σ_i}, for $i = 1, \ldots, k$.

VIII. For each τ-term t of T_{ι} containing no variables other than $a_1, \ldots, a_k$,

$$\bigwedge_{j=1}^{k} C^{\iota}_{\sigma_j}(\alpha, x_j) \rightarrow C^{\iota}_{\tau}(\alpha, t^*)$$

is a theorem of U_{ι}.

The proof is by induction on t. If t is a constant, then $t^* = t$, and the result follows by VI and (for $i = 1$) VII. If $t = a_i$, then $t^* = x_i$, and the result is trivial. If $t = t_1 t_2$, then $t^* = t_1^* t_2^*$ and the result follows by the inductive hypothesis, since if t_1^* and t_2^* are α-conv, so is $t_1^* t_2^*$.

Let $A = A(a_1, \ldots, a_k)$ be a formula of T_{ι} containing only $a_1^{\sigma_1}, \ldots, a_k^{\sigma_k}$ free. The formula $A^{\iota} = A^{\iota}(\alpha, x_1, \ldots, x_k)$ of U_0, containing only $\alpha, x_1, \ldots, x_k$ free, is inductively defined as follows: $(s^{\tau} = t^{\tau})^{\iota} = E^{\iota}_{\tau}(\alpha, s^*, t^*)$, $(\neg A)^{\iota} = \neg A^{\iota}$, $(A \vee B)^{\iota} = A^{\iota} \vee B^{\iota}$, $(A \wedge B)^{\iota} = A^{\iota} \wedge B^{\iota}$ and $(A \rightarrow B)^{\iota} = A^{\iota} \rightarrow B^{\iota}$. Let b^{σ} be distinct from $a_1, \ldots, a_k$, and let m be the least number $\leq |\alpha|$ such that $j \leq m$ whenever $j \leq |\alpha|$ and for some $h = 1, \ldots, k$, x_h is the Gödel number of a σ_h-term containing d_j. Let α' be defined by $\alpha'_0(x) = m$ and $\alpha'_j = \alpha_j$ for all $j > 0$. Then: $(\bigvee b^{\sigma} A(a_1, \ldots, a_k, b))^{\iota} = \bigvee \beta_{\geq \alpha'} \bigvee y[C^{\iota}_{\sigma}(\beta, y) \wedge A^{\iota}(\beta, x_1, \ldots, x_k, y)]$ and $(\bigwedge b^{\sigma} A(a_1, \ldots, a_k, b))^{\iota} = \bigwedge \beta_{\geq \alpha'} \bigwedge y[C^{\iota}_{\sigma}(\beta, y) \rightarrow A^{\iota}(\beta, x_1, \ldots, x_k, y)]$.

IX. For each $A(a_1, \ldots, a_k)$, we can prove in U_0 that *if α and β cover each term with Gödel number $x_1, \ldots, x_k$, and $\alpha_j = \beta_i$ for each d_j occurring in one of these terms, then $A^{\iota}(\alpha, x_1, \ldots, x_k) \equiv A^{\iota}(\beta, x_1, \ldots, x_k)$.*

The proof is by induction on A. If A is $(s^{\tau} = t^{\tau})$ it follows from $E^{\iota}_{\tau}(\alpha, s^*, t^*) \equiv E^{\iota}_{\tau}(\beta, s^*, t^*)$, which is evident (since α and β assign the same values to d_j in s^* or in t^*). If A is $\neg B$, $B \vee C$, $B \wedge C$ or $B \rightarrow C$, it follows from the induction hypothesis applied to B and C. If A is $\bigvee b A(a_1, \ldots, a_k, b)$ or $\bigwedge b A(a_1, \ldots, a_k, b)$, it follows from the fact that $\alpha' = \beta'$ (using the notation introduced above).

Set $F^{\iota} = \bigwedge_{j=1}^{k} C^{\iota}_{\sigma_j}(\alpha, x_j)$.

THEOREM. *If A is a theorem of T_{ι}, then $\bigwedge \alpha(F^{\iota} \rightarrow A^{\iota})$ is a theorem of U_{ι}.*

The proof is by induction on the length of a derivation of A in T_{ι}. Let $\vdash B$ mean that B is a theorem of U_{ι}, and set $A^+ = (F^{\iota} \rightarrow A^{\iota})$.

Case 1. A is an instance of **A**. Then either it is an instance of an axiom of intuitionistic propositional, in which case so is A^{ι}, or else it is of the form $s^{\tau} = t^{\tau} \vee \neg s^{\tau} = t^{\tau}$. In the first case we clearly have $\vdash A^+$, and so assume the second case. By VIII,

$$\vdash F^{\iota} \rightarrow C^{\iota}_{\tau}(\alpha, s^*) \wedge C^{\iota}_{\tau}(\alpha, t^*), \text{ and so by V, } \vdash A^+.$$

Case 2. A is an instance of **B** or **C**. Then $\vdash A^+$ follows from the derivability in U_0 of

$$C^\iota_\tau(\alpha, x) \to E^\iota_\tau(\alpha, x, x),$$

$$E^\iota_\tau(\alpha, x, y) \to [E^\iota_\tau(\alpha, x, z) \to E^\iota_\tau(\alpha, y, z)],$$

$$E^\iota_{(\sigma,\tau)}(\alpha, x, y) \wedge C^\iota_\sigma(\alpha, z) \to E^\iota_\tau(\alpha, xz, yz),$$

$$E^\iota_\sigma(\alpha, x, y) \wedge C^\iota_{(\sigma,\tau)}(\alpha, z) \to E(\alpha, zx, zy),$$

$$\neg E^\iota_0(\alpha, S^0, Sx),$$

and

$$E^\iota_0(\alpha, Sx, Sy) \to E^\iota_0(\alpha, x, y),$$

using VIII.

Using Case 2, we can show by induction on A that

X. We can prove in U_ι that

$$\bigwedge_{j=1}^{k} E^\iota_{\sigma_j}(\alpha, x_j, y_j) \to [A^\iota(\alpha, x_1, \ldots, x_k) \equiv A^\iota(\alpha, y_1, \ldots, y_k)].$$

Case 3. A is obtained by **D** from B and $B \to A$. B may contain free variables other than $a_1, \ldots, a_k$, so it is of the form $B(a_1^{\sigma_1}, \ldots, a_m^{\sigma_m})$ where $k \le m$ and $a_1, \ldots, a_m$ include all its free variables. By the induction hypothesis,

$$\vdash F^\iota \wedge \bigwedge_{j=k+1}^{m} C^\iota_{\sigma_j}(\alpha, x_j) \to B^\iota(\alpha, x_1, \ldots, x_m)$$

and

$$\vdash F^\iota \wedge \bigwedge_{j=k+1}^{m} C^\iota_{\sigma_j}(\alpha, x_j) \to [B^\iota(\alpha, x_1, \ldots, x_m) \to A^\iota].$$

But by VIII, $\bigwedge_{j=k+1}^{m} C^\iota_{\sigma_j}(\alpha, \theta_{\sigma_j})$, and so $\vdash A^+$.

Case 4. $A = B(a_1, \ldots, a_k, t) = B(t)$ is obtained by **E** from $B(0)$ and $B(b^0) \to B(Sb^0)$. Write $D(z) = B^\iota(\alpha, x_1, \ldots, x_k, S^z)$. Then $\vdash F^\iota \to D(0)$ and $\vdash F^\iota \to [D(z) \to D(z+1)]$, by the induction hypothesis. Hence, by mathematical induction in U_ι, $\vdash D(z)$. That is,

$$\vdash F^\iota \to B^\iota(\alpha, x_1, \ldots, x_k, S^z).$$

By VIII, $\vdash F^\iota \to C^\iota_0(\alpha, t^*)$, and so $\vdash F^\iota \to \bigvee z E(\alpha, t^*, S^z)$. Hence by X,

$$\vdash F^\iota \to B^\iota(\alpha, x_1, \ldots, x_k, t^*),$$

that is, $\vdash A^+$.

Case 5. A is an instance of the definitional axioms **F**. This case is trivial.

Case 6. A is $B(t^\sigma) \to \bigvee b^\sigma B(b)$, where $B(b) = B(a_1, \ldots, a_k, b)$. By VIII, $\vdash F^\iota \to C^\iota_\sigma(\alpha, t^*)$, and so

$$\vdash F^\iota \wedge B^\iota(\alpha, x_1, \ldots, x_k, t^*) \to \bigvee y[C^\iota_\sigma(\alpha, y) \to B^\iota(\alpha, x_1, \ldots, x_k, y)];$$

that is, $\vdash A^+$, since $\alpha' \subseteq \alpha$.

Case 7. A is $\bigwedge b^\sigma B(b) \to B(t)$. Like Case 6.

Case 8. A is $\bigvee b^\sigma B(b) \to D$, and is obtained from $B(b) \to D$, where b is not free in D. By the induction hypothesis.

$$\vdash \bigwedge_{j=1}^{k} C^\iota_{\sigma_j}(\beta, x_j) \wedge C^\iota_\sigma(\beta, y) \to [B^\iota(\beta, x_1, \ldots, x_k, y) \to D^\iota(\beta, x_1, \ldots, x_k)].$$

Let $\alpha' \subseteq \beta$. Then $\vdash F^i \to \bigwedge_{j=1}^k C^i_{\sigma_j}(\beta, x_j)$, and since F^i implies that α covers each x_j, $\vdash F^i \to [D^i(\alpha, x_1, \ldots, x_k) \equiv D^i(\beta, x_1, \ldots, x_k)]$, by IX. Hence,

$$\vdash F^i \wedge C^i_\sigma(\beta, y) \wedge \beta \subseteq \alpha' \to [B^i(\beta, x_1, \ldots, x_k, y) \to D^i(\alpha, x_1, \ldots, x_k)],$$

and so $\vdash A^+$.

Case 9. A is $D \to \bigwedge bB(b)$, and is obtained from $D \to B(b)$, where b is not free in D. Like Case 8.

We have completed the proof for $i = 0$. For $i = 1$, there are two further cases.

Case 10. A is an instance of **BI**. Then A is of the form $B \wedge C \wedge D \to Q(\langle\ \rangle)$, where $B = \bigwedge d^1 V b^0 G(\bar{d}b)$ with G quantifier-free, $C = \bigwedge b^0[G(b) \to Q(b)]$ and $D = \bigwedge b^0[\bigwedge c^0 Q(b^\frown c) \to Q(b)]$. Assume in U_1 that F^1, B^1, C^1 and D^1. Since $C_0(\alpha, x) \to VyE_0(\alpha, x, S^y)$, B^1 implies (writing $G^1(\beta, u)$ for $G^1(\beta, x_1, \ldots, x_k, u)$)

$$\bigwedge \beta_{\supseteq \alpha'} \bigwedge y[C^1_1(\beta, y) \to VzG^1(\beta, \bar{y}S^z)],$$

by X. Let $m = |\alpha'| + 1$. Then if $\beta \supseteq \alpha'$ and $|\beta| = m$, $\bigvee yG^1(\beta, \bar{d}_m S^y)$, since $C^1_1(\beta, d_m)$. Every such β is of the form $\alpha'^\frown\gamma$, where $\beta_1 = \alpha'_j$ for $j \leq |\alpha'|$ and $\beta_{m+1} = \gamma$. Since G^1 is primitive recursive, we can define a primitive recursive predicate $G_0(\alpha, z) = G_0(z)$ which means that z is a sequence number $\langle z_0, \ldots, z_{p-1}\rangle$ and there is a $u \leq Z$ such that, for all γ with $\bar{\gamma}(p) = z$, $G^1(\alpha'^\frown\gamma, \bar{d}_m S^u)$. Then

$$\bigwedge \gamma \bigvee xG_0(\bar{\gamma}(x)). \tag{1}$$

Let $Q_0(z)$ mean that z is a sequence number $\langle z_0, \ldots, z_{p-1}\rangle$ and that $Q^1(\alpha, x_1, \ldots, x_k, \langle S^{z_0}, \ldots, S^{z_{p-1}}\rangle)$. Then C^1 implies

$$G_0(z) \to Q_0(z), \tag{2}$$

and D^1 implies

$$\bigwedge z(\bigwedge uQ_0(z^\frown u) \to Q_0(z)). \tag{3}$$

Hence, using **BI**, $Q_0(\langle\ \rangle)$, that is, $Q(\langle\ \rangle)^1$. This proves $\vdash A^+$.

Case 11. A is an instance of $\mathbf{AC}_{00}$. That is, A is of the form

$$\bigwedge b^0 \bigvee c^0 B(b^0, c^0) \to \bigvee d^1 \bigwedge b^0 B(b^0, db).$$

Assume that F^1 and $(\bigwedge b^0 \bigvee c^0 B(b, c))^1$. Then, since $C^0_0(\alpha, x) \to \bigvee zE_0(\alpha, x, S^z)$, it follows by X that

$$\bigwedge x \bigvee yB^1(\alpha', x_1, \ldots, x_k, S^x, S^y),$$

and so by $\mathbf{AC}_{00}$, that

$$\bigvee \gamma \bigwedge xB^1(\alpha', x_1, \ldots, x_k, S^x, S^{\gamma(x)}).$$

Let $\beta = \alpha'^\frown\gamma$ and let $m = |\alpha'| + 1 = |\beta|$. Then

$$\bigwedge xB^1(\beta, x_1, \ldots, x_k, S^x, d_m S^x);$$

that is, $(\bigvee d \bigwedge bB(b, db))^1$. Thus $\vdash A^+$.

This completes the proof of the theorem.

Note that in the definition of A^0 we have

$$(s^\tau = t^\tau) \equiv E^0_\tau(s^*, t^*)$$

and

$$(\bigvee b^{\sigma} B(a_1, \ldots, a_k, b))^0 \equiv \bigvee y[C^0_{\sigma}(y) \wedge B^0(x_1, \ldots, x_k, y)],$$

$$(\bigwedge b^{\sigma} B(a_1, \ldots, a_k, b))^0 \equiv \bigwedge y[C^0_{\sigma}(y) \rightarrow B^0(x_1, \ldots, x_k, y)];$$

so that A^0 is essentially arithmetical. The proof of the theorem for this case could be given taking first order arithmetic in place of U_0. The formulation actually given was simply for the sake of preserving, as far as possible, a uniform treatment of the two interpretations.

COROLLARY. *T_ι is a conservative extension of U_ι.*

Let $A(b_1, \ldots, b_k, d_1, \ldots, d_m)$ be a formula of U_ι containing free only the 0-variables $b_1, \ldots, b_k$ and $d_1, \ldots, d_m$. Let $\beta = \langle \alpha_1, \ldots, \alpha_m \rangle$ be defined by $\beta_0(x) = m$, $\beta_j = \alpha_j$ for $j = 1, \ldots, m$, and $\beta_j(x) = 0$ for $j > m$. The corollary follows from the derivability in U_ι of

$$\bigwedge x_1 \ldots x_k \alpha_1 \ldots \alpha_m [A(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_m)$$
$$\equiv A^\iota(\langle \alpha_1, \ldots, \alpha_m \rangle, S^{x_1}, \ldots, S^{x_k}, d, \ldots, d_m)].$$

We omit the hardest part of the proof of this, namely for the case in which A is a numerical equation $s^0 = t^0$ between primitive recursive terms. It clearly suffices to prove it for the case $s = b^0$, and this is done by induction on s. If A is $\neg B$, $B \vee C$, $B \wedge C$ or $B \rightarrow C$, then the equivalence for A follows from the equivalence for B and C. Let A be $\bigvee b^0 B(b_1, \ldots, b_k, b, d_1, \ldots, d_m)$. Since $C^\iota_0(\alpha, y) \rightarrow \bigvee z E_0(\alpha, y, S^z)$, $A^\iota(\langle \alpha_1, \ldots, \alpha_m \rangle, S^{x_1}, \ldots, S^{x_k}, d_1, \ldots, d_m)$ is equivalent to $\bigvee z B^\iota(\langle \alpha_1, \ldots, \alpha_m \rangle, S^{x_1}, \ldots, S^{x_k}, S^z, d_1, \ldots, d_m)$, and so again, the equivalence for A follows from the equivalence for B. Similarly for $A = \bigwedge b^0 B(b)$. Let A be $\bigvee d B(b_1, \ldots, b_k, d_1, \ldots, d_m, d)$. By changing bound variables in B if necessary, we can suppose that d is d_{m+1}. $A^\iota(\langle \alpha_1, \ldots, \alpha_m \rangle, S^{x_1}, \ldots, S^{x_k}, d_1, \ldots, d_{m+1})$ is

$$\bigvee \beta_{\supseteq \langle \alpha_1 \ldots \alpha_m \rangle} \bigvee y[C^\iota_1(\beta, y) \wedge B^\iota(\beta, S^{x_1}, \ldots, S^{x_k}, d_1, \ldots, d_m, y)].$$

But, by $\mathbf{AC}_{00}$, if $C^1_1(\beta, y)$, then there is a γ such that for each n, $yS^n \dashv_\beta S^{\gamma(n)}$. From this it follows that $A^\iota(\langle \alpha_1, \ldots, \alpha_m \rangle, S^{x_1}, \ldots, S^{x_k}, d_1, \ldots, d_m)$ is equivalent to

$$\bigvee \gamma B^\iota(\langle \alpha_1, \ldots, \alpha_m, \gamma \rangle, S^{x_1}, \ldots, S^{x_k}, d_1, \ldots, d_{m+1}).$$

Hence, the equivalence for A follows again from the equivalence for B. Similarly for $A = \bigwedge d^1 B(d)$.

The axiom of choice is needed here because in U_ι, variables of type 1 range over i.p.s., but in the interpretation of T_ι, they range over all the functionals of type 1 built up from i.p.s. using the operations of T_ι. $\mathbf{AC}_{00}$ asserts that all such functionals are extensionally equal to i.p.s.

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