Arch. Math. Logic (1996) 35: 191–208

Archive for
Mathematical
Logic

A complete many-valued logic with product-conjunction

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Received July 27, 1995 / Revised November 13, 1995

Abstract. A simple complete axiomatic system is presented for the many-valued propositional logic based on the conjunction interpreted as product, the coresponding implication (Goguen's implication) and the corresponding negation (Gödel's negation). Algebraic proof methods are used. The meaning for fuzzy logic (in the narrow sense) is shortly discussed.

1 Introduction

Many-valued logics have become a subject of increased interest as logics of vagueness, i.e. fuzzy logics; intermediate truth values are understood as degrees of truth of fuzzy propositions (see [25]). Many-valued semantics underlying fuzzy logic works with various binary operations on the unit interval $[0,1]$ generalizing the classical boolean truth functions on $\{0,1\}$. In particular, the so-called *t-norms* and their duals *t-conorms* ([22, 23, 1, 18]) have become popular to model conjunction and disjunction-like operations with fuzzy sets. A (continuous) t-norm is a commutative and associative binary operation in $[0,1]$, non-decreasing in both variables and having 1 and 0 as neutral and absorbent elements respectively.

In this paper we investigate some logics whose set of truth values is the real interval $[0,1]$ and we concentrate our attention to logics having a conjunction whose truth function $t(x,y)$ is a t-norm, and having a corresponding residuated implication (or, as Pavelka [19] observes, the conjunction and the implication form an adjoint couple); i.e., if $i(x,y)$ is the truth function of the implication then

$$z \leq i(x,y) \text{ iff } t(x,z) \leq y.$$

There are three main examples: (1) *Łukasiewicz's logic* [16] with the conjunction $x\&y = \max(0, x+y-1)$ and implication $x \rightarrow_{\mathrm{L}} y = \min(1, 1-x+y)$,

Mathematics Subject Classification: 03B50, 03B52
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(2) *Gödel's logic* [6] with the conjunction $x \wedge y = \min(x, y)$ and the implication $x \rightarrow_G y = 1$ for $x \leq y$ and $x \rightarrow_G y = y$ otherwise, and finally (3) *product logic* with the conjunction $x \odot y = x.y$ and implication $x \rightarrow_P y = 1$ for $x \leq y$ and $x \rightarrow_P y = y/x$ otherwise. We also have truth constants 0, 1 (absolute falsity and absolute truth). Each of the above implications $\rightarrow$ defines its negation as $\neg x = x \rightarrow 0$; for Łukasiewicz's logic we get $\neg x = 1 - x$, for Gödel's logic and product logic we get Gödel's negation $\neg 0 = 1$, $\neg x = 0$ for $x > 0$.

Note that each (continuous) t-norm is a "mixture" of the three above t-norms. Namely, Ling has proved [15] that any archimedean t-norm, i.e. a t-norm having 0 and 1 as the only idempotent elements, is isomorphic either to the t-norm *product* $t(x, y) = x.y$ or to the *Łukasiewicz*'s t-norm $t(x, y) = max(0, x + y - 1)$, and that any other continous t-norm is either the t-norm *minimum* $t(x, y) = min(x, y)$, or is an ordinal sum of the t-norm *min* and archimedean t-norms (see e.g. [15] or [18] for the exact formulation.)

In Łukasiewicz's logic, implication and 0 can be taken as primitives and other connectives (including Łukasiewicz's conjunction and the minimum conjunction as well as their de Morgan duals - disjunctions $\min(1, x + y)$, $\max(x, y)$) are definable. Łukasiewicz's propositional logic has an elegant axiomatization by four schemes, which was shown complete with respect to 1-tautologies (formulas having identically the value 1) by Rose and Rosser [20] (see [7] for a full proof and very detailed information on Łukasiewicz's and Gödel's logic). As was shown by Scarpelini [21], the corresponding predicate calculus is not recursively axiomatizable, i.e. the set of its 1-tautologies is not recursively enumerable. On the other hand, both Łukasiewicz's propositional and predicate logic have a "graded" version, originally formulated and investigated by Pavelka [19] and Novák [17]; for very simplified versions see [10], [11].

Gödel's logic also has, besides implication and 0, the disjunction with maximum as truth function as a primitive connective; Gödel's propositional logic is completely axiomatized by the axioms of intuitionistic propositional logic plus the linearity axiom $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$; see [7] for a completeness proof. The corresponding Gödel's predicate logic has a finitary (recursive) axiomatization and hence the set of its 1-tautologies is recursively enumerable (see[24][1]). An analogon of Pavelka's graded logic is not possible since Gödel's implication is not continuous (see [19]).

The third logic, based on the product conjunction, seems not to be studied in a similar extent as the two logics above. In this paper we show that the propositional logic based on product and the corresponding implication is completely axiomatized by a finite set of axiom schemes. It is important that this logic contains both minimum and maximum as defined connectives. (Note that minimum is definable in each t-norm based logic, cf. [4].) In proving the completeness we apply the same method as Gottwald for Łukasiewicz's logic [7], i.e. we algebraize the logic, introducing a notion of product algebras. We show that one of them is the unit interval [0,1] with truth functions of connectives as operations, and that

[1] Surprisingly, Takeuti and Titani do not refer to Gödel and their axiomatization allows simplifications following from the completeness of the mentioned axiom system of propositional logic

another example is the algebra of classes of provably equivalent formulas. Then we show that:

(i) if φ is provable and we understand φ as a term in the language of product algebras then $\varphi = 1$ is valid in all product algebras,

(ii) each product algebra is a subalgebra of the direct product of some linearly ordered product algebras,

(iii) if an identity $\varphi = 1$ is valid in the algebra [0,1] then it is valid in all linearly ordered algebras, and hence, by (ii), in all algebras, in particular in the algebra of classes of formulas, which means, by (i), that φ is a provable formula.

At the end we present some open problems.

2 Product logic: basic definitions

Formulas are built from propositional variables and the truth constant 0, using connectives $\odot, \rightarrow$ (both binary). The *semantics* is as above, i.e. the truth functions are defined as follows, for any $x, y \in [0,1]$,

$$\begin{array}{rcl} x \odot y & = & x.y \\ x \rightarrow y & = & \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise} \end{cases} \end{array}$$

for any $x, y \in [0,1]$. We take the freedom of denoting the truth function of a connective by the same symbol as the connective itself. From these primitive connectives, one can define the constant 1 and the four derived connectives given below:

$$\begin{array}{rcl} 1 & \text{is} & 0 \rightarrow 0, \\ \neg\varphi & \text{is} & \varphi \rightarrow 0, \\ \varphi \wedge \psi & \text{is} & \varphi \odot (\varphi \rightarrow \psi) \\ \varphi \vee \psi & \text{is} & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\ \varphi \leftrightarrow \psi & \text{is} & (\varphi \rightarrow \psi) \odot (\psi \rightarrow \varphi) \end{array}$$

The corresponding truth functions for these connectives are:

$$\begin{array}{rcl} \neg x & = & \begin{cases} 1, \text{ if } x = 0 \\ 0, \text{ otherwise} \end{cases} \\ x \wedge y & = & \min(x, y), \\ x \vee y & = & \max(x, y) \\ x \leftrightarrow y & = & \min(x \rightarrow y, y \rightarrow x) = (x \rightarrow y).(y \rightarrow x) \end{array}$$

(Trivial checking.)

Lemma 1 *The following formulas are 1-tautologies:*

for $\rightarrow$:

(A1)	$\varphi \rightarrow (\psi \rightarrow \varphi)$	*(adding assumptions)*
(A2)	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$	*(transitivity)*
(A3)	$0 \rightarrow \varphi$	*(extremal)*

for $\rightarrow, \odot$:

(A4)	$(\varphi \odot \psi) \rightarrow (\psi \odot \varphi)$	*(commutativity)*
(A5)	$(\varphi \odot (\psi \odot \chi)) \rightarrow ((\varphi \odot \psi) \odot \chi)$	*(associativity)*
	$((\varphi \odot \psi) \odot \chi) \rightarrow (\varphi \odot (\psi \odot \chi))$	*(associativity)*
(A6)	$((\varphi \odot \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$	*(residuation)*
	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \odot \psi) \rightarrow \chi)$	*(residuation)*
(A7)	$(\varphi \rightarrow \psi) \rightarrow ((\varphi \odot \chi) \rightarrow (\psi \odot \chi))$	*(monotonicity)*
(A8)	$\neg\neg\chi \rightarrow ((\varphi \odot \chi \rightarrow \psi \odot \chi) \rightarrow (\varphi \rightarrow \psi))$	*(cancellation)*

for $\wedge, \vee, \rightarrow$:

(A9)	$(\chi \rightarrow \varphi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\chi \rightarrow (\varphi \wedge \psi)))$	*(implied conjunction)*
(A10)	$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$	*(implying disjunction)*
(A11)	$(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$	*(pre-linearity)*
(A12)	$(\varphi \wedge \neg\varphi) \rightarrow 0$	*(contradiction)*

Definition 1 (product logic ΠL) We take all formulas of the form (A1)-(A12) for axioms and allow modus ponens as the only deduction rule. The notion of proof is as in classical logic, but relative to our set of axioms. This logic will be called ΠL (product logic).

Soundness of ΠL with respect to the above semantics is clear: it is very easy to check that any truth-evaluation respecting the above truth-functions evaluates each axiom to 1, and that modus ponens preserves 1-tautologies.

For the sake of comparative purposes, it is interesting to test the validity of the set of axioms of ΠL in the Gödel's and Łukasiewicz's logics when we interpret $\odot$ by the *minimum* and *Łukasiewicz's* t-norms, respectively. Under these assumptions, it turns out that all the axioms of ΠL are 1-tautologies of both Gödel's and Łukasiewicz's logics except for the *cancellation* axiom (A8), which fails in both logics, and for the *contradiction* axiom (A12) which fails in Łukasiewicz's logic. Finally, notice that the negation in ΠL is exactly the same as in Gödel's logic.

As usual in developing any formal logical system, first we have to show that various important formulas are provable in our logic. The provabilities below are needed to show besides other things that the algebra of classes of logically equivalent formulas has desired properties.

Lemma 2 *The following formulas related to implication are provable in ΠL:*

(1)	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$	*(exchange)*
	$(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$	
(2)	$\varphi \rightarrow 1$	*(extremal)*

Proof. (1) From (A2) we get $\vdash ((\psi \odot \varphi) \rightarrow (\varphi \odot \psi)) \rightarrow (((\varphi \odot \psi) \rightarrow \chi) \rightarrow ((\psi \odot \varphi) \rightarrow \chi))$. Now using (A4) and modus ponens we get $\vdash ((\varphi \odot \psi) \rightarrow \chi) \rightarrow ((\psi \odot \varphi) \rightarrow \chi)$. Finally (1) comes from (A6), applying transitivity (A2) twice. As a consequence $\vdash (\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$ easily follows from (A2) and (1).

(2) By (A1) we have $\vdash 0 \rightarrow (\varphi \rightarrow 0)$ and by (1) we have $\vdash \varphi \rightarrow (0 \rightarrow 0)$. □

Lemma 3 *The following formulas relating product conjunction and implication are provable in ΠL:*

(3) $\varphi \odot \psi \to \varphi$, $\psi \odot \varphi \to \varphi$; $\varphi \to \varphi$,

(4) $\varphi \to (\psi \to (\varphi \odot \psi))$

(5) $\varphi \odot 0 \to 0$, $\varphi \to (1 \odot \varphi)$, *(extremals)*

(6) $(\varphi \to \psi) \to ((\chi \odot \varphi) \to (\chi \odot \psi))$ *(monotonicity)*

(7) $(\varphi \to \psi) \to ((\chi \to \gamma) \to ((\varphi \odot \chi) \to (\psi \odot \gamma)))$ *(monotonicity)*

Proof. (3) $\vdash \varphi \to (\psi \to \varphi)$ by (A1), thus $\vdash (\varphi \odot \psi) \to \varphi$ by (A6); furthermore, $\vdash (\psi \odot \varphi) \to \varphi$ by (A4). From this, $\vdash \psi \to (\varphi \to \varphi)$ by (A6), thus $\vdash (\varphi \to \varphi)$ (substitute an axiom for ψ).

(4) $\vdash \varphi \to (\psi \to (\varphi \odot \psi))$ follows from $(\varphi \odot \psi) \to (\varphi \odot \psi)$ by (A6).

(5) $\vdash (\varphi \odot 0) \to 0$ follows from $\vdash (\varphi \odot \psi) \to \psi$. The proof of $\vdash \varphi \to (1 \odot \varphi)$ is as follows. $\vdash (1 \to (\varphi \to (1 \odot \varphi))$ comes from (4), and $\vdash 1$ obviously holds by (A3), and thus (5) is obtained by modus ponens.

(6) follows from (A7) and (A4) by multiple use of transitivity (A2) (and Lemma 2(1)).

(7) Notice that

$$\vdash (\chi \to \gamma) \to ((\psi \odot \chi) \to (\psi \odot \gamma)) \tag{a}$$

by (6),

$$\vdash (\varphi \to \psi) \to ((\varphi \odot \chi) \to (\psi \odot \chi)), \tag{b}$$

by (A7),

$$\vdash ((\varphi \to \psi) \odot (\varphi \odot \chi)) \to (\psi \odot \chi)), \tag{c}$$

by (A6) from (b),

$$\vdash (\psi \odot \chi) \to ((\chi \to \gamma) \to (\psi \odot \gamma)), \tag{d}$$

by (1), from (a),

$$\vdash ((\varphi \to \psi) \odot (\varphi \odot \chi)) \to ((\chi \to \gamma) \to (\psi \odot \gamma)), \tag{e}$$

by transitivity from (c), (d), and finally

$$\vdash (\varphi \to \psi) \to ((\chi \to \gamma) \to ((\varphi \odot \chi) \to (\psi \odot \gamma)))$$

by (A6), (1) and transitivity. □

As easy consequences from this lemma we get the list of properties of the equivalence connective given in the following corollary. Notice that $\varphi \leftrightarrow \psi$ could be equivalently defined as $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

Corollary 1 *The following formulas concerning the equivalence connective are provable formulas in ΠL:*

(8) $(\varphi \leftrightarrow \psi) \to (\varphi \to \psi)$, $(\varphi \leftrightarrow \psi) \to (\psi \to \varphi)$,
$(\varphi \to \psi) \to ((\psi \to \varphi) \to (\varphi \leftrightarrow \psi))$

(9) $\varphi \leftrightarrow \varphi$ *(reflexivity)*
$(\varphi \leftrightarrow \psi) \to (\psi \leftrightarrow \varphi)$ *(symmetry)*
$(\varphi \leftrightarrow \psi) \to ((\psi \leftrightarrow \chi) \to (\varphi \leftrightarrow \chi))$ *(transitivity)*

Notice that the theorems in (9) show that the relation defined on the set of formulas by "φ is logically equivalent to ψ iff ΠL proves $\varphi \leftrightarrow \psi$" is indeed an equivalence relation. Moreover, the next proposition shows that this relation is indeed a congruence with respect to implication and product conjunction, and consequently, w.r.t. any other connective definabe from them as well.

Proposition 1 *The following substitution principles of provable equivalents hold in ΠL:*

(10) $(\varphi \leftrightarrow \psi) \rightarrow ((\varphi \rightarrow \chi) \leftrightarrow (\psi \rightarrow \chi))$
(11) $(\varphi \leftrightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \leftrightarrow (\chi \rightarrow \psi))$
(12) $(\varphi \leftrightarrow \psi) \rightarrow ((\varphi \odot \chi) \leftrightarrow (\psi \odot \chi))$
(13) $(\varphi \leftrightarrow \psi) \rightarrow ((\chi \odot \varphi) \leftrightarrow (\chi \odot \psi))$

Proof. All proofs follow from the definition of $\leftrightarrow$ using results of Lemmas 2 and 3. □

Although ΠL does not have proofs by cases in general (since tertium non datur is not provable), it *does* admit proof by cases if the cases are $\varphi \rightarrow \psi$, $\psi \rightarrow \varphi$:

Corollary 2 *If* $\vdash (\varphi \rightarrow \psi) \rightarrow \chi$ *and* $\vdash (\psi \rightarrow \varphi) \rightarrow \chi$ *then* $\vdash \chi$.

Proof. This is a consequence of (A11) and (A10): from

$$((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow (((\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)) \rightarrow \chi)$$

we get χ by triple use of modus ponens. □

Note that any pair of formulas whose disjunction is provable may be used instead of $\varphi \rightarrow \psi$ and $\psi \rightarrow \varphi$, e.g. the pair $\neg\phi$, $\neg\neg\phi$.

In this paper we do not investigate provability in theories, but the definition would be obvious. Thus notice only in passing that the *deduction theorem* fails in ΠL since the formula $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ is not a 1-tautology in ΠL. However, formula (7) in Lemma 3 allows for a variant of the deduction theorem, analogous to that of Łukasiewicz's logic, which reads as follows:

Let T be a set of formulas. Then, $T \cup \{\varphi\} \vdash \psi$ iff there exists a natural n such that $T \vdash \varphi^n \rightarrow \psi$, where φ^n is $\varphi \odot \ldots \odot \varphi, n$ times.

Next group of provable formulas will be needed in the next section to show the lattice structure of the algebra of classes of equivalent provable formulas of our logic. Besides, the basic relationships of $\wedge$ and $\vee$ connectives with implication are also shown.

Lemma 4 *The following formulas are provable in ΠL:*

(14) $(\varphi \wedge \psi) \rightarrow \varphi$, $(\varphi \wedge \psi) \rightarrow \psi$
(15) $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$, $(\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$ *(commutativity)*
(16) $\varphi \rightarrow (\varphi \vee \psi)$, $\psi \rightarrow (\varphi \vee \psi)$
(17) $(\varphi \odot \psi) \rightarrow (\varphi \wedge \psi)$
(18) $(\varphi \wedge (\psi \wedge \chi)) \leftrightarrow ((\varphi \wedge \psi) \wedge \chi)$ *(associativity - $\wedge$)*
$(\varphi \vee (\psi \vee \chi)) \leftrightarrow ((\varphi \vee \psi) \vee \chi)$ *(associativity - $\vee$)*
(19) $\varphi \leftrightarrow (\varphi \wedge \varphi)$, $(\varphi \vee \varphi) \leftrightarrow \varphi$ *(idempotence)*
(20) $\varphi \leftrightarrow ((\varphi \wedge \psi) \vee \varphi)$, $\varphi \leftrightarrow ((\varphi \vee \psi) \wedge \varphi)$ *(absorption)*

Proof. (14) $\vdash (\varphi \wedge \psi) \to \varphi$ is evident from the definition of $\wedge$; we show $\vdash (\varphi \wedge \psi) \to \psi$, i.e. $\vdash (\varphi \odot (\varphi \to \psi)) \to \psi$, i.e. $\vdash ((\varphi \to \psi) \odot \varphi) \to \psi$, i.e. $\vdash (\varphi \to \psi) \to (\varphi \to \psi)$, which is evident from (3).

(15) Use $\vdash ((\varphi \wedge \psi) \to \psi) \to (((\varphi \wedge \psi) \to \varphi) \to ((\varphi \wedge \psi) \to (\psi \wedge \varphi)))$ (i.e. (A9)) and modus ponens. $\vdash (\varphi \vee \psi) \to (\psi \vee \varphi)$ follows from the definition of $\vee$ and from the commutativity of $\wedge$ just proved.

(16) We show $\vdash \varphi \to ((\varphi \to \psi) \to \psi)$ (see (14)) and $\vdash \varphi \to ((\psi \to \varphi) \to \varphi)$ (from (A1)); thus $\vdash \varphi \to (\varphi \vee \psi)$ by (A9) and the definition of $\vee$. $\vdash \varphi \to (\psi \vee \varphi)$ follows from $\vdash (\varphi \vee \psi) \to (\psi \vee \varphi)$.

(17) $\vdash \varphi \odot \psi \to \varphi \wedge \psi$ follows from (3) and (A9).

(18) Show $\vdash (\varphi \wedge (\psi \wedge \chi)) \to \varphi$, $\vdash (\varphi \wedge (\psi \wedge \chi)) \to \psi$, $\vdash (\varphi \wedge (\psi \wedge \chi)) \to \chi$ (using (14)); thus $\vdash (\varphi \wedge (\psi \wedge \chi)) \to (\varphi \wedge \psi)$ and $\vdash (\varphi \wedge (\psi \wedge \chi)) \to ((\varphi \wedge \psi) \wedge \chi)$ by double use of (A9). The proof of the associativity for $\vee$ is analogous.

(19) Use $\vdash (\varphi \to \varphi) \to ((\varphi \to \varphi) \to (\varphi \to (\varphi \wedge \varphi)))$, i.e. (A9), and (3). For $\vee$ analogously using (A10).

(20) $\vdash \varphi \to ((\varphi \wedge \psi) \vee \varphi)$ follows from (16); to show the converse implication, use $\vdash \varphi \to \varphi$, $\vdash (\varphi \wedge \psi) \to \varphi$ and (A10). The second formula is proved analogously. □

Next corollary shows that the lattice order is actually determined by the implication (21) and, therefore, meet ($\wedge$) and join ($\vee$) are compatible with this order (22).

Corollary 3 *Next formulas are theorems of ΠL:*

(21) $(\varphi \to \psi) \leftrightarrow (\varphi \leftrightarrow (\varphi \wedge \psi))$

$(\varphi \to \psi) \leftrightarrow (\psi \leftrightarrow (\varphi \vee \psi))$

(22) $(\varphi \to \psi) \to ((\varphi \wedge \chi) \to (\psi \wedge \chi))$

$(\varphi \to \psi) \to ((\varphi \vee \chi) \to (\psi \vee \chi))$

Proof. (21) It suffices to show $\vdash (\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$; but by (A9) we have $\vdash (\varphi \to \varphi) \to ((\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi)))$.The proof of the second formula is analogous.

(22) We have $\vdash (\varphi \to \psi) \to ((\varphi \wedge \chi) \to \psi)$, $\vdash (\varphi \wedge \chi) \to \chi$, further
$\vdash ((\varphi \wedge \chi) \to \psi) \to (((\varphi \wedge \chi) \to \chi) \to ((\varphi \wedge \chi) \to (\psi \wedge \chi)))$, thus
$\vdash (\varphi \to \psi) \to (((\varphi \wedge \chi) \to \chi) \to ((\varphi \wedge \chi) \to (\psi \wedge \chi)))$, thus
$\vdash ((\varphi \wedge \chi) \to \chi) \to ((\varphi \to \psi) \to ((\varphi \wedge \chi) \to (\psi \wedge \chi)))$, hence the first formula of (22) is provable. The second formula is proved dually. □

Lemma 5 *ΠL proves the following formulas about the truth constants* 0 *and* 1, *and properties of the negation:*

(23) $((\varphi \odot \psi) \to 0) \to ((\varphi \wedge \psi) \to 0)$
$(\varphi \odot \varphi \to 0) \to (\varphi \to 0)$

(24) $\neg 0 \leftrightarrow 1;\ \neg 1 \leftrightarrow 0$

(25) $(\varphi \to \neg\varphi) \to \neg\varphi$

(26) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

(27) $\varphi \to \neg\neg\varphi$

(28) $\neg\neg\neg\varphi \to \neg\varphi$

(29) $\neg\varphi \vee \neg\neg\varphi.$

Proof. (23) We want to prove $((\varphi \odot \psi) \to 0) \to ((\varphi \wedge \psi) \to 0)$, i.e. $(\varphi \to (\psi \to 0)) \to ((\varphi \wedge \psi) \to 0)$, i.e. $(\varphi \to \neg\psi) \to ((\varphi \wedge \psi) \to 0)$, thus finally $((\varphi \to \neg\psi) \odot (\varphi \wedge \psi)) \to 0$. The following chains of implications are provable.
$((\varphi \to \neg\psi) \odot (\varphi \wedge \psi)) \to [(\varphi \to \neg\psi) \odot \varphi] \to \neg\psi$,
$((\varphi \to \neg\psi) \odot (\varphi \wedge \psi)) \to [(\varphi \to \neg\psi) \odot \psi] \to \psi$, thus
$((\varphi \to \neg\psi) \odot (\varphi \wedge \psi)) \to [\psi \wedge \neg\psi] \to 0$.

(24) We have $\vdash \neg 0 \to 1$ by (A3); conversely, $\neg 0$ is $0 \to 0$, hence $\vdash \neg 0$, thus $\vdash 1 \to \neg 0$. Furthermore $\vdash 0 \to \neg 1$ by (A3); conversely, we succesively have $\vdash \neg 1 \to (1 \odot \neg 1)$ by (5), $\vdash (1 \odot \neg 1) \to (1 \wedge \neg 1)$ by (16) and $\vdash (1 \wedge \neg 1) \to 0$ by (A12). Therefore by (A2) we finally get $\vdash \neg 1 \to 0$.

(25) It follows from $\vdash ((\varphi \odot \varphi) \to 0) \to (\varphi \to 0)$, i.e. $\vdash (\varphi \to (\varphi \to 0)) \to (\varphi \to 0)$, which is $\vdash (\varphi \to \neg\varphi) \to \neg\varphi$.

(26) It follows from transitivity: $\vdash (\varphi \to \psi) \to ((\psi \to 0) \to (\varphi \to 0))$.

(27) It follows from $\vdash (1 \to \varphi) \to (\neg\varphi \to 0)$

(28) This follows from (26) and (27).

(29) Observe the following provable chains of implications:
$(\neg\varphi \to \neg\neg\varphi) \to \neg\neg\varphi$ by (25),
$\neg\neg\varphi \to (\neg\varphi \vee \neg\neg\varphi)$ by (16),
and
$(\neg\neg\varphi \to \neg\varphi) \to (\neg\neg\varphi \to \neg\neg\neg\varphi)$ by (26),
$(\neg\neg\varphi \to \neg\neg\neg\varphi) \to \neg\neg\neg\varphi$ by (25),
$\neg\neg\neg\varphi \to \neg\varphi$ by (28),
$\neg\varphi \to (\neg\varphi \vee \neg\neg\varphi)$ by (16).
Thus $\vdash \neg\varphi \vee \neg\neg\varphi$ using the above chains of implications, (A10) and (A11). □

Next we show some decomposition rules of implications when their antecedent and consequent subformulae contain lattice connectives.

Lemma 6 *ΠL proves the following formulas on lattice connectives and implication :*

(30) $((\varphi \wedge \psi) \to \chi) \leftrightarrow ((\varphi \to \chi) \vee (\psi \to \chi))$;
$((\varphi \vee \psi) \to \chi) \leftrightarrow ((\varphi \to \chi) \wedge (\psi \to \chi))$

(31) $(\varphi \to (\psi \wedge \chi)) \leftrightarrow ((\varphi \to \psi) \wedge (\varphi \to \chi))$;
$(\varphi \to (\psi \vee \chi)) \leftrightarrow ((\varphi \to \psi) \vee (\varphi \to \chi))$

Proof. (30) We only prove the first formula. It suffices to show $\vdash ((\varphi \wedge \psi) \to \chi) \to ((\varphi \to \chi) \vee (\psi \to \chi))$, since the other direction is straightforward by (A10). We prove it by cases $\varphi \to \psi$, $\psi \to \varphi$. The following implications are provable:
$(\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$
$(\varphi \to (\varphi \wedge \psi)) \to (((\varphi \wedge \psi) \to \chi) \to (\varphi \to \chi))$,
$(\varphi \to \psi) \to (((\varphi \wedge \psi) \to \chi) \to (\varphi \to \chi))$, and thus
$(\varphi \to \psi) \to (((\varphi \wedge \psi) \to \chi) \to ((\varphi \to \chi) \vee (\psi \to \chi)))$
Analogously, one proves $(\psi \to \varphi) \to (((\varphi \wedge \psi) \to \chi) \to ((\varphi \to \chi) \vee (\psi \to \chi)))$.

(31) The proofs are very similar to (30). In one direction it is easy from (A9) and (A10) respectively. The converse implications are proved by cases $\chi \to \psi$, $\psi \to \chi$. □

The distributivity of the lattice connectives and the distributivity of product conjunction with respect to them are shown next ((33) and (32) respectively).

Lemma 7 *ΠL proves the next distributive laws:*
(32) $\varphi \odot (\psi \vee \chi) \leftrightarrow (\varphi \odot \psi) \vee (\varphi \odot \chi)$
$\varphi \odot (\psi \wedge \chi) \leftrightarrow (\varphi \odot \psi) \wedge (\varphi \odot \chi)$
(33) $(\varphi \wedge (\psi \vee \chi)) \leftrightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
$(\varphi \vee (\psi \wedge \chi)) \leftrightarrow ((\varphi \vee \psi) \wedge (\varphi \vee \chi))$

Proof. (32) (a) Let us first prove the first formula.
$\vdash ((\varphi \odot \chi) \vee (\psi \odot \chi)) \to ((\varphi \vee \psi) \odot \chi)$ using (A7), (A10) and (4); we prove the converse implication. This means to prove

$$\vdash ((\varphi \vee \psi) \odot \chi) \to [((\varphi \odot \chi) \to (\psi \odot \chi)) \to (\psi \odot \chi)]$$

and the same formula with φ, ψ in [...] exchanged (by (A9) and the definition of $\vee$). After obvious transformations we have to prove

$$\vdash ((\varphi \odot \chi) \to (\psi \odot \chi)) \to [((\varphi \vee \psi) \odot \chi) \to (\psi \odot \chi)].$$

Denote this formula by (*). We use a proof by cases $\neg\chi, \neg\neg\chi$.
$\vdash \neg\chi \to (\chi \to 0) \to [((\varphi \vee \psi) \odot \chi) \to 0] \to [((\varphi \vee \psi) \odot \chi) \to (\psi \odot \chi)] \to (*)$, thus $\vdash \neg\chi \to (*)$. On the other hand,
$\vdash \neg\neg\chi \to [((\varphi \odot \chi) \to (\psi \odot \chi)) \to (\varphi \to \psi)] \to [((\varphi \odot \chi) \to (\psi \odot \chi)) \to ((\varphi \vee \psi) \leftrightarrow \psi)] \to (*)$;
thus $\vdash (\neg\chi \vee \neg\neg\chi) \to (*)$ and hence $\vdash (*)$.

(b) Now let us prove the second formula.
$\vdash (\varphi \wedge \psi) \odot \chi \to ((\varphi \odot \chi) \wedge (\psi \odot \chi))$ by (3) and (A9); we prove the converse, i.e. $\vdash ((\varphi \odot \chi) \wedge (\psi \odot \chi)) \to (\varphi \wedge \psi) \odot \chi$, i.e.
$\vdash [(\varphi \odot \chi) \odot ((\varphi \odot \chi) \to \psi \odot \chi)] \to ((\varphi \wedge \psi) \odot \chi)$, or
$\vdash ((\varphi \odot \chi) \to (\psi \odot \chi)) \to [(\varphi \odot \chi) \to ((\varphi \wedge \psi) \odot \chi)]$.
This is proved by cases $\neg\chi$, $\neg\neg\chi$ as in the first formula.

(33) It suffices to show $\vdash (\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$ and $\vdash ((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \to (\varphi \vee (\psi \wedge \chi))$. We prove $\vdash (\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$

by first considering the definition of $\wedge$ and then by applying succesively the distributivity of $\rightarrow$ and $\odot$ w.r.t. the disjunction $\vee$, namely (32) and (31). To prove now $\vdash ((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \rightarrow (\varphi \vee (\psi \wedge \chi))$ it suffices to notice that $((\varphi \vee \psi) \wedge (\varphi \vee \chi))$ is equivalent to the conjunction $A \wedge B \wedge C \wedge D$ while $(\varphi \vee (\psi \wedge \chi))$ is equivalent to the conjunction $(A \vee X) \wedge B \wedge (C \vee Y) \wedge D$, where: A is $((\varphi \rightarrow \psi) \rightarrow \psi)$, B is $((\psi \rightarrow \varphi) \rightarrow \varphi)$, C is $((\varphi \rightarrow \chi) \rightarrow \chi)$, D is $((\chi \rightarrow \varphi) \rightarrow \varphi)$, X is $((\varphi \rightarrow \chi) \rightarrow \psi)$, and Y is $((\varphi \rightarrow \psi) \rightarrow \chi)$. □

Finally, we show the provability of some special formulas resulting from the iterated use of the product conjunction.

Lemma 8 *ΠL proves:*

(34) $(\varphi \vee \psi) \odot (\varphi \vee \psi) \rightarrow ((\varphi \odot \varphi) \vee (\psi \odot \psi))$
$(\varphi \wedge \psi) \odot (\varphi \wedge \psi) \rightarrow ((\varphi \odot \varphi) \wedge (\psi \odot \psi))$

(35) $(\varphi \rightarrow \psi)^n \vee (\psi \rightarrow \varphi)^n$, *for each* n,
where α^n *is* $\alpha \odot \ldots \odot \alpha, n$ *times.*

Proof. (34) We write φ^2 for $\varphi \odot \varphi$ etc.; we have to prove $(\varphi \vee \psi)^2 \leftrightarrow (\varphi^2 \vee \psi^2)$. Now, by (32), $\vdash (\varphi \vee \psi)^2 \leftrightarrow (\varphi^2 \vee \psi^2 \vee \varphi \odot \psi)$, thus it suffices to prove $\vdash (\varphi \odot \psi) \rightarrow (\varphi^2 \vee \psi^2)$ to get $(\varphi^2 \vee \psi^2 \vee \varphi \odot \psi) \leftrightarrow \varphi^2 \vee \psi^2$.
Prove by cases $(\varphi \rightarrow \psi)$, $(\psi \rightarrow \varphi)$:
$\vdash (\varphi \rightarrow \psi) \rightarrow (\varphi \odot \psi \rightarrow \psi^2)$, thus $\vdash (\varphi \rightarrow \psi) \rightarrow (\varphi \odot \psi \rightarrow \varphi^2 \vee \psi^2)$, and dually, $\vdash (\psi \rightarrow \varphi) \rightarrow (\varphi \odot \psi \rightarrow \varphi^2 \vee \psi^2)$, which gives the result.

(35) First show $\vdash \varphi \rightarrow (\varphi \rightarrow (\varphi \odot \varphi))$ (using residuation, adding assumption); thus, $\vdash [(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)]^2$. By (34) we get $(\varphi \rightarrow \psi)^2 \vee (\psi \rightarrow \varphi)^2$. Iterate and use $\vdash ((\varphi \rightarrow \psi)^{n+1} \vee (\psi \rightarrow \varphi)^{n+1}) \rightarrow ((\varphi \rightarrow \psi)^n \vee (\psi \rightarrow \varphi)^n)$, which comes easily from $\vdash \varphi^{n+1} \rightarrow \varphi^n$ and (22), used doubly. □

3 Product algebras and completeness

To prove completeness for our product logic *ΠL*, we next define what we call *product algebras* and show that they play an analogous role for *ΠL* as *MV-Algebras* do for the infinitely many-valued Łukasiewicz's logic (see for instance [7]). Namely, one can prove that the quotient algebra of classes of equivalent *ΠL*-formulas is a product algebra. Moreover, the unit interval equipped with the truth functions of product logic is a special linearly ordered product algebra because every valid identity there can be shown to be also valid in every linearly ordered product algebra. Then completeness for 1-tautologies comes from the fact that each product algebra is a subdirect product of linearly ordered product algebras.

Definition 2 A *product algebra* is an algebra $\mathcal{A} = \langle A, \odot, \rightarrow, 0, 1 \rangle$ such that, defining

$$\begin{aligned} x \wedge y &= x \odot (x \rightarrow y) \\ x \vee y &= ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x) \\ \neg x &= x \rightarrow 0 \end{aligned}$$

the following conditions are satisfied:

- $\mathscr{A} = \langle A, \odot, \rightarrow, \wedge, \vee, 0, 1\rangle$ is a *residuated lattice* (see [19][2]), i.e.
  (i) $\langle A, \wedge, \vee, 0, 1\rangle$ is a lattice ($x \leq y \equiv x \wedge y = x \equiv x \vee y = y$), 1 and 0 are the top and bottom elements respectively,
  (ii) $\langle \odot, \rightarrow\rangle$ is an *adjoint couple* on A,that is:
    1. $\odot$ and $\rightarrow$ are binary operations on A,
    2. $\odot$ is non-decreasing in both variables,
    3. $\rightarrow$ is non-increasing in the first variable and non-decreasing in the second one, and
    4. the adjointness condition $x \leq (y \rightarrow z)$ iff $(x \odot y) \leq z$ holds.

  (iii) $\langle A, \odot, 1\rangle$ is a commutative monoid, that is, $\odot$ is associative and commutative, and $1 \odot x = x$
- $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (pre-linearity)
- $\neg\neg z \odot (x \odot z \rightarrow y \odot z) \leq (x \rightarrow y)$ (pre-cancellation)
- $x \wedge \neg x = 0$ (bottom)
- $x \odot (y \vee z) \leq (x \odot y) \vee (x \odot z)$, (distributive laws)
  $x \odot (y \wedge z) \geq (x \odot y) \wedge (x \odot z)$

Again, it is easy to notice that it is the pre-cancellation condition that makes product algebras different from the algebras corresponding to Gödel's logic (i.e. Heyting algebras satisfying the pre-linearity condition [3]) even if we replace $\odot$ by $\wedge$ throughout in the definition of a product algebra. On the other hand, product algebras also differ from MV-algebras in that condition, but also in the "bottom" condition.

Lemma 9 *Any product algebra satisfies the following further properties:*
(i) $x \leq y$ *iff* $(x \rightarrow y) = 1$
(ii) $x \leq y$ *implies* $x = y \odot (y \rightarrow x)$
(iii) $\neg 0 = 1$, $\neg 1 = 0$
Furthermore, if the algebra is linearly ordered it also satisfies:
(iv) $\neg x = 0$, *for* $x > 0$
(v) *if* $z \neq 0$ *then* $x \odot z = y \odot z$ *iff* $x = y$
(vi) *if* $z \neq 0$ *then* $x \odot z < y \odot z$ *iff* $x < y$
($x < y$ *means* $x \leq y$ *and* $x \neq y$)

Proof. (i) It is true in each residuated lattice. (ii) It follows from the definition of $\wedge$ from $\odot, \rightarrow$.

(iii) We prove $\neg 0 = 1$. Note $\neg 0 = 0 \rightarrow 0$ by definition and $1 \odot 0 = 0$ by Definition 2 (iii); thus $1 \leq 0 \rightarrow 0$ by (i) of the present lemma, but since 1 is the top element we get $0 \rightarrow 0 = 1$, thus $\neg 0 = 1$.

[2] Residuated lattices in relation to many valued logics are studied in depth in the recent paper[14]. Höhle's paper unfortunately came to our possession only after the completion of the present paper. Some parts of the subsequent proofs could be shortened by referring to [14]; but to keep the paper more self-contained we have decided to keep full proofs here

[3] Called Heyting chains in [7] and relative Stone algebras in [2]

Now consider $\neg 1$; by *(bottom)* in Definition 2, $1 \wedge \neg 1 = 0$ but since 1 is the top we have $1 \wedge x = x$ for all x, thus $\neg 1 = 0$.

(iv) Here $\wedge$ is minimum and $x \wedge \neg x = 0$ and thus, if $x > 0$ then $\neg x = 0$.

(v) If $z \neq 0$ then $\neg\neg z = 1$, thus if $x \odot z \leq y \odot z$ then $(x \odot z \rightarrow y \odot z) = 1$, thus $x \rightarrow y = 1$, $x \leq y$. This gives also (vi). $\square$

Remark. Observe that the above definition of product algebras, even if cumbersome, can be written as a sequence of (universally quantified) Horn clauses. Therefore the class of product algebras is a quasivariety and thus it is closed under direct products and subalgebras (see e.g. [8]).

We have the trivial one-element product algebra with $0 = 1$. Besides this algebra, the only finite linearly ordered product algebra is the two elements boolean algebra $\{0, 1\}$. This is an easy consequence of the fact that in any finite chain the above cancellation property forbids the identification of meet and product. Moreover, any boolean algebra is a product algebra, taking product as meet, since it is a subdirect product of copies of $\{0, 1\}$.

For our purposes, the most interesting and notorious examples of product algebras are given in the following lemma.

Lemma 10 *(i) The unit interval with truth functions is a linearly ordered product algebra.*
(ii) The algebra of classes of equivalent formulas in ΠL is a product algebra (not linearly ordered.)

Proof. (i) It is straightforward to check that $\langle [0,1], \odot, \rightarrow, \wedge, \vee, 0, 1 \rangle$ is a residuated lattice with the usual order, where $\odot$ is product, $\rightarrow$ is its corresponding residuated implication function, $x \wedge y$ is $\min(x, y)$ and $x \vee y$ is $\max(x, y)$, and that the rest of conditions also hold.
(ii) This follows from the set of the formulas shown to be provable in ΠL in the lemmas and corollaries of the previous section. We patiently verify all items from the definition of a product algebra. Namely:

- Corollary 1 and Proposition 1 show, among other things, that the quotient algebra $\mathscr{PL}$ of classes of provably equivalent formulas is properly defined, i.e. $\mathscr{PL}/\!\sim\, = \langle \Pi L/\!\sim, \odot, \rightarrow, [0], [1] \rangle$, where $\Pi L/\!\sim\, = \{[\varphi] \mid \varphi \in \Pi L\}$, where $[\varphi] = \{\psi \mid \psi \in \Pi L$ and $\Pi L \vdash \varphi \leftrightarrow \psi\}$, and the operations (we denote them with the same connective symbols) are definable in the usual way, i.e.
  $[\varphi] \odot [\psi] = [\varphi \odot \psi]$
  $[\varphi] \rightarrow [\psi] = [\varphi \rightarrow \psi]$
  This is a sound definition since evidently $[\varphi] = [\psi]$ iff $\vdash \varphi \leftrightarrow \psi$, thus $[\varphi] = [\psi]$ implies $[\varphi \odot \chi] = [\psi \odot \chi]$, $[\varphi \rightarrow \chi] = [\psi \rightarrow \chi]$, $[\chi \rightarrow \varphi] = [\chi \rightarrow \psi]$ by Proposition 1. Hence the algebra is well defined and we get also $[\varphi] \vee [\psi] = [\varphi \vee \psi]$ etc.

- Moreover, we have that $\vdash \varphi \to (1 \to \varphi)$ from Axiom (A1), and thus $\vdash 1 \to \varphi$ if φ is provable. This, together with (2), shows us that the class of the truth constant 1 is the top element of the algebra, and any provable formula in ΠL is in that class. In turn, axiom (A3) shows that the class of the truth constant 0 is the bottom element of the algebra.
- Lema 4 and Corollary 3 show that $\langle \Pi L/\!\sim, \wedge, \vee, [0], [1] \rangle$ is a lattice where the order induced is that defined by $\to$, i.e. $[\varphi] \leq [\psi]$ iff $[\varphi \to \psi] = [1]$ iff $\vdash \varphi \to \psi$: indeed, by definition, $[\varphi] \leq [\psi]$ means $[\varphi \wedge \psi] = [\varphi]$, i.e. $\vdash (\varphi \wedge \psi) \leftrightarrow \varphi$, thus $\vdash \varphi \to \psi$ by Lemma 4(14). Conversely, if $\vdash \varphi \to \psi$ then $\vdash \varphi \leftrightarrow (\varphi \wedge \psi)$ by Corollary 3 (21).
- Axiom (A2), together with (1), (A7), (6) and axioms (A6), prove that $\langle \odot, \to \rangle$ is an adjoint couple in $\Pi L/\!\sim$.
- Axioms (A4) and (A5), together with (3) and (5) show that $\langle \Pi L/\!\sim, \odot, [1] \rangle$ is a commutative monoid.
- Finally the conditions of pre-linearity, pre-cancellation, bottom and the distributive laws are guaranteed by axioms (A11), (A8), (A12) and (32) respectively. □

Next lemma shows that every provable formula in ΠL is interpreted in any product algebra as its top element.

Lemma 11 *For each formula φ, understand each propositional variable p_i as an object variable; you get a term of the language of product algebras. If φ is provable then the identity $\varphi = 1$ holds in each product algebra.*

Proof. Verify that $\varphi = 1$ is a true identity of product algebras for each axiom (A1) - (A12) and observe that in each product algebra if $x = 1$ and $x \to y = 1$, i.e. $1 \leq x \to y$, thus $x = 1 \odot x \leq y$, then $y = 1$; thus modus ponens preserves 1-tautologicity of our product algebras.

We only verify (A6), (A9), (A10). Now $(x \odot y) \to z \leq x \to (y \to z)$ iff $x \odot ((x \odot y) \to z) \leq y \to z$ iff $x \odot y \odot ((x \odot y) \to z) \leq z$, and the last inequality is true $(a \odot (a \to b) \leq b)$. Conversely, to show $x \to (y \to z) \leq (x \odot y) \to z$ show

$$x \odot y \odot (x \to (y \to z)) = y \odot x \odot (x \to (y \to z)) \leq y \odot (y \to z) \leq z.$$

This proves (A6).

We verify (A10), i.e. $(x \to z) \odot (y \to z) \leq (x \vee y) \to z$. Compute: $(x \vee y) \odot ((x \to z) \odot (y \to z)) = (x \odot (x \to z) \odot (y \to z)) \vee (y \odot (y \to z) \odot (x \to z)) \leq z \odot (y \to z) \vee z \odot (x \to z) \leq z \vee z = z$. (distributivity used).

Finally, we prove (A9), i.e. $(z \to x) \odot (z \to y) \leq z \to (x \wedge y)$.
$z \odot (z \to x) \odot (z \to y) \leq x \odot (z \to y) \leq x$, analogously $z \odot (z \to x) \odot (z \to y) \leq y$, thus $z \odot (z \to x) \odot (z \to y) \leq x \wedge y$. Note that the distributivity for $\odot, \wedge$ appears to be redundant. □

Observe that the result of lemma 11 and (ii) of lemma 10 prove that ΠL is a complete axiomatization of 1-tautologies over product algebras, in the sense that theorems of ΠL coincide with the formulas φ such that the identity $\varphi = 1$ holds in each product algebra.

Next step is to prove that each product algebra is a subdirect product of linearly ordered product algebras.

Definition 3 Let $\mathscr{A}$ be a product algebra. A *filter* is a set $F \subseteq A$ such that, for each $a, b \in A$:

$$a \in F \text{ and } b \in F \text{ implies } a \odot b \in F$$
$$a \in F \text{ and } a \leq b \text{ implies } b \in F.$$

Furthermore, F is an *ultrafilter* iff for each $c, d \in A$,

$$(c \to d) \in F \text{ or } (d \to c) \in F.$$

Lemma 12 *Let $\mathscr{A}$ be a product algebra and let F be a filter. Define the corresponding equivalence*

$$a \sim_F b \text{ iff } (a \to b) \in F \text{ and } (b \to a) \in F,$$

i.e. $a \sim_F b$ iff $(a \leftrightarrow b) \in F$. Then:
(1) $\sim_F$ is a congruence, and the corresponding quotient algebra $\mathscr{A}/{\sim_F}$ is a product algebra too.
(2) $\mathscr{A}/{\sim_F}$ is linearly ordered iff F is an ultrafilter.

Proof. The proof is rather standard; we present some details. Let $[x]$ denote the class $\{y | x \sim y\}$. Evidently, $\sim$ is a congruence; thus the factor algebra is a lattice with the top element $[1]$ and the bottom element $[0]$ with respect to $\wedge, \vee$ and the corresponding ordering; in addition, it is a commutative semigroup with the unit element $[1]$ for $\odot$. We show the following:

(a) $[x] \leq [y]$ iff $[x \to y] = [1]$. Indeed, if $[x] \leq [y]$ then $[x \wedge y] = [x]$, thus $(x \to (x \wedge y)) \in F$, but $(x \to (x \wedge y)) \sim (x \to y)$, thus $x \to y \in F$ and $[x \to y] = 1$. Similarly for the converse.

(b) Adjointness: $[z] \leq [x \to y]$ iff $z \to (x \to y) \in F$ iff $((z \odot x) \to y) \in F$ iff $[z \odot x] \leq [y]$.

(c) Monotonicity. Recall that $x \to y \leq (x \odot z \to y \odot z)$, thus $(x \to y) \in F$ implies $(x \odot z \to y \odot z) \in F$, thus $[x] \leq [y]$ implies $[x \odot z] \leq [y \odot z]$. Other monotonicities are proved in the same way.

The remaining conditions are equalities and therefore preserved by factorizations. This completes the proof. □

Theorem 1 *Let $\mathscr{A}$ be a product algebra and $a \in A$, $a \neq 1$. Then there is an ultrafilter F on A not containing a.*

Proof. Start with $F_0 = \{1\}$ $(a \notin F_0)$ and successively process all pairs $(c \to d), (d \to c)$; if F is a filter not containing $(c \to d), (d \to c)$ then create F_1, F_2 as the smallest filters, extensions of F, containing $(c \to d)$ and $(d \to c)$ respectively, i.e. $F_1 = \{u \mid (\exists v \in F)(\exists n \text{ natural })(v \odot (c \to d)^n \leq u)\}$, and similarly for F_2 and $(d \to c)$. Clearly, if a filter contains F as a subset and contains $c \to d$ as an element then it contains each $v \odot (c \to d)^n$ since it is closed under $\odot$; on the other hand, the set F_1 as defined here is a filter since if $v_1, v_2 \in F$ then

$v_1 \odot (c \to d)^n \odot v_2 \odot (c \to d)^m = (v_1 \odot v_2) \odot (c \to d)^{n+m}$ and $v_1 \odot v_2 \in F$. We show next that either $a \notin F_1$ or $a \notin F_2$. If $a \in F_1$ and $a \in F_2$ then, for some $v \in F$ and n natural, $v \odot (c \to d)^n \leq a$ and $v \odot (d \to c)^n \leq a$, thus $a \geq (v \odot (c \to d)^n) \vee (v \odot (d \to c)^n) = v \odot ((c \to d)^n \vee ((d \to c)^n) = v \odot 1 = v$, thus $a \in F$, contradiction. □

Corollary 4 *Each product algebra is a subdirect product of linearly ordered product algebras.*

Proof. From last theorem, it is easy to check that, for any algebra $\mathscr{A}$, $\cap\{F \mid F$ is an ultrafilter of $\mathscr{A}\} = \{1\}$. Therefore, the intersection of the corresponding congruences $\sim_F$ is just the minimum congruence, that is, the identity. The corollary then comes from Lemma 7 (2) by applying the standard result about subdirect products of algebras (see e.g. [3]) saying that if the intersection of a family of congruences of some algebra A is the minimum congruence, A is a subdirect product of the corresponding quotient algebras. □

Finally, before proving completeness of ΠL we need some results relating linearly ordered product algebras and ordered Abelian groups.

Theorem 2 *Let $\mathscr{A} = \langle A, \odot, \to, 0_A, 1_A \rangle$ be a linearly ordered product algebra. Then there is a linearly ordered Abelian group $\mathscr{G} = (G, +_G, o_G, \leq_G)$ and an isomorphism ι of the non-positive part $N = \{g \in G, g \leq_G o_G\}$ and $A - \{0_A\}$, i.e. a one-to-one mapping from N to $A - \{0_A\}$ such that*

$$\iota(o_G) = 1_A$$
$$\iota(g +_G h) = \iota(g) \odot \iota(h)$$
$$g \leq_G h \text{ iff } \iota(g) \leq \iota(h)$$

Proof. For basics on linearly ordered (l.o.) groups and semigroups the classical reference is [5]; for the reader's convenience we recall the definitions needed. An algebra $\mathscr{G} = (G, +_G, \leq_G)$ is a l.o. commutative semigroup if $\leq_G$ is a linear order on G, $+_G$ is a binary commutative and associative operation on G and the following monotonicity holds: $x \leq_G y$ implies $x +_G z \leq_G y +_G z$ for each $x, y, z \in G$. If in addition $(G, +_G)$ is a group, i.e. has a neutral element o_G and each element x has its inverse $-_G x$ such that $x +_G (-_G x) = o_G$ then $\mathscr{G}$ is a l.o. Abelian group. $\mathscr{G}$ satisfies *subtraction* if for each $x \leq_G y$ from G, there is a unique $z \in G$ such that $x +_G z = y$. Each l.o. Abelian group satisfies subtraction; and if $\mathscr{G}$ is a l.o. commutative semigroup with subtraction and having a least element o_G which is at the same time a neutral element ($o_G +_G x = x$ for each x) then there is a l.o. Abelian group $\hat{\mathscr{G}}$ such that $\mathscr{G}$ is the subsemigroup of $\hat{\mathscr{G}}$ consisting of all non-negative elements of $\hat{\mathscr{G}}$. See [5] p. 155 (or observe that the construction of $\hat{\mathscr{G}}$ is the same as the construction of integers from natural numbers).

Now observe that if $\mathscr{A}$ is a l.o. product algebra then $\mathscr{A} - \{0_A\}$ is a l.o. commutative semigroup having the greatest element 1_A which is neutral; moreover, $\mathscr{A} - \{0_A\}$ satisfies dual subtraction by Lemma 9(ii) and (v): $y \odot z = x$ has a solution whenever $x \leq y$, the solution being $y \rightarrow x$. Thus $\mathscr{A} - \{0_A\}$ is (up to an isomorphism ι) the set of all non-positive elements of a l.o. Abelian group as desired. Note that for the embedding ι we have

$$\iota(g +_{\mathrm{G}} (-_{\mathrm{G}} h)) = \iota(h) \rightarrow \iota(g)$$

for $g \leq_{\mathrm{G}} h$. □

Theorem 3 *If an identity $\tau = \sigma$, in the language of product algebras, is valid in the unit interval algebra then it is valid in all linearly ordered product algebras.*

Proof. Assume there is a linearly ordered product algebra $\mathscr{A} = \langle A, \odot_A, \rightarrow_A, 0_A, 1_A \rangle$, with order $\leq_A$, in which the identity $\tau = \sigma$ is not valid, i.e. there is a tuple $\mathbf{a} = (a_1, \ldots, a_n)$ of elements of A such that $\tau(\mathbf{a}) \neq \sigma(\mathbf{a})$ in $\mathscr{A}$. Let D_1 be the finite set of values of all subterms of τ, σ (including τ, σ) for the arguments $\mathbf{a}$; assume $0_A, 1_A \in D_1$ and set $D = D_1 - \{0_A\}$. We use the following theorem of the theory of ordered Abelian groups (proved by Gurevich and Kokorin [9]; a full proof is found both in [12] Lemma 7.3.20 and in [13]):

If $\mathscr{G} = (G, \odot_{\mathrm{G}}, \leq_{\mathrm{G}})$ is a l.o. Abelian group and D is a finite subset of G then there is a finite subset E of the multiplicative group $((0, +\infty), ., 1)$ of positive reals, with the usual order $\leq$, and an isomorphism κ of D onto E, i.e. for $x, y, z \in D$ we have

$$x \odot_{\mathrm{G}} y = z \textit{ iff } \kappa(x).\kappa(y) = \kappa(z),$$
$$x \leq_{\mathrm{G}} y \textit{ iff } \kappa(x) \leq \kappa(y).$$

Needless to say, the whole group $\mathscr{G}$ may be not embeddable into $(0, +\infty)$; but each *finite* subset is. In fact, the usual formulation uses the additive l.o. group of all reals or the so-called PROSPECTOR group instead of the multiplicative group of positive reals, but these l.o. groups are obviously isomorphic.

Now take a $\mathscr{G}$ whose set of non-positive elements is (isomorphic to) our $\mathscr{A} - \{0_A\}$. Take our D and find the corresponding $E \subset (0, +\infty)$. Since $\kappa(1_A) = 1$ we get $E \subset (0, 1]$. Put $E_1 = E \cup \{0\}$ and extend κ defining $\kappa(0_A) = 0$. We claim that for each $x, y, z \in D_1$,

$$x \odot_A y = z \text{ iff } \kappa(x).\kappa(y) = \kappa(z),$$
$$x \rightarrow_A y = z \text{ iff } \kappa(x) \rightarrow \kappa(y) = \kappa(z),$$
$$x \leq_A y \text{ iff } \kappa(x) \leq \kappa(y),$$

where $\rightarrow$ stands for the truth-function corresponding to the product implication. This is clear for $x, y, z >_A 0_A$ from the isomorphism property of κ; indeed, for $x \leq_A y$ we have $x \rightarrow_A y = z$ iff $z = 1_A$ iff $\kappa(z) = 1$ iff $\kappa(x) \rightarrow \kappa(y) = \kappa(z)$; for $x >_A y$, $x \rightarrow_A y = z$ iff $y = x \odot_A z$ iff $\kappa(y) = \kappa(x).\kappa(z)$ iff $\kappa(x) \rightarrow \kappa(y) = \kappa(z)$. (Use Lemma 9 (iii) and (v).)

To check the property for the case that some of x, y, z are zeros discuss the cases $0_A \rightarrow_A y = z$, $x \rightarrow_A 0_A = z$, $x \rightarrow y = 0_A$ and similarly for $\odot_A$. (For example, take $x \rightarrow_A y = 0_A$: it follows in each product algebra that $x >_A y$ and $y = 0_A$, that $x \rightarrow_A y = 0_A$ is preserved.)

Summarizing, we have succeeded to construct an injection κ of D_1 onto an $E_1 \subset [0, 1]$ which is an isomorphism w.r.t. $\mathscr{A}$ and the product algebra $[0, 1]$ in the above sense (for $x, y, z \in D$, κ preserves $x \odot_A y = z$, $x \rightarrow_A y = z$, $x \leq_A y$, $x = 0_A$, $x = 1_A$); thus for $b_i = \kappa(a_i)$, the tuple $b_1, \dots b_n$ fails to satisfy $\tau = \sigma$ in $[0, 1]$; $\tau = \sigma$ is not valid in the latter algebra. This completes the proof. □

Theorem 4 (Completeness) *The following are mutually equivalent:*
(i) φ is a 1-tautology;
(ii) the identity $\varphi = 1$ is valid in each product algebra;
(iii) φ is provable in ΠL.

Proof. If φ is a 1-tautology then the corresponding identity $\varphi = 1$ is valid in $[0, 1]$, hence in all linearly ordered product algebras (due to theorem 3), hence in *all* product algebras (due to Corollary 1), hence, in particular, in the algebra of classes of equivalent formulas of ΠL , hence $\Pi L \vdash \varphi \leftrightarrow 1$, hence $\Pi L \vdash \varphi$. □

4 Concluding remarks

The next problem is: what happens if we add the "classical" fuzzy negation $1 - x$ and thus the MYCIN disjunction $x + y - xy$? Note that a completeness theorem for a fuzzy logic containing all these, but with respect to an infinitary system, is contained in [24]. Another problem reads: how far can we develop a "graded" variant of product logic, similar to Pavelka's logic, cf. [19, 10] ? Complete analogy is impossible since our implication is not continuous in [0, 1] (see [19]), but a partial analogy seems possible. This is a subject of current research.

Acknowledgement. The authors are indebted to Pere Garcia for his comments on earlier drafts of this paper, and to an anonymous referee for his (her) useful remarks that have lead to improvements in the paper. This research has been partially supported by the COPERNICUS project MUM (10053) from the European Union. P. Hajek also recognizes partial support by the grant No. 130108 of the Academy of Sciences of the Czech Republic.

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