Brzozowski's algorithm (co)algebraically.

Helle Hvid Hansen and Alexandra Silva

Radboud Universiteit Nijmegen and CWI

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Motivation

- duality between reachability and observability (Arbib and Manes 1975): beautiful, not very well-known.

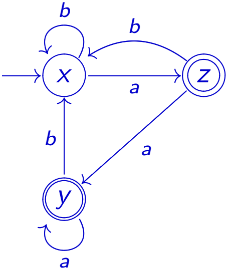
  Bidoit&Hennicker&Kurz. On the duality between observability and reachability (2001)

- combined use of algebra and coalgebra.

- our understanding of automata is still very limited;
  cf. recent research: universal automata, àtomata, weighted automata (Sakarovitch, Brzozowski, . . .)

- joint work with Bonchi, Bonsangue, Rutten (Dexter's festschrift 2012) and Hansen, Panangaden and Bezhanishvilli, Kozen, Kupke.

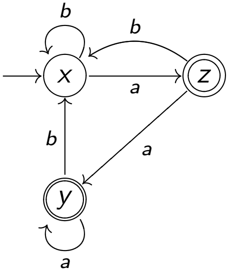
Brzozowski algorithm (by example)



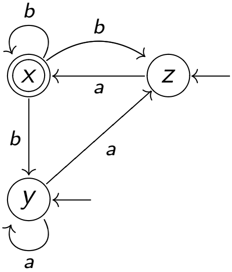
- initial state: x • final states: y and z
- $L(x) = \{a, b\}^* a$
- X is reachable but not minimal: $L(y) = \varepsilon + \{a, b\}^* a = L(z)$

Reversing the automaton: $rev(X)$

$X =$

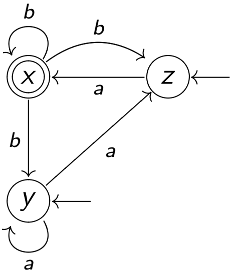


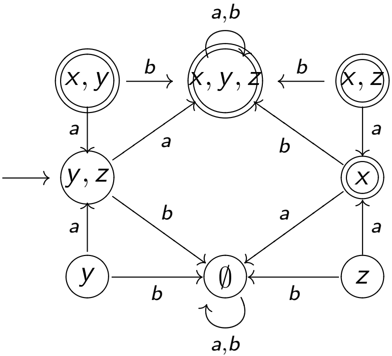
$rev(X) =$



- transitions are reversed
- initial states $\Leftrightarrow$ final states
- $rev(X)$ is non-deterministic

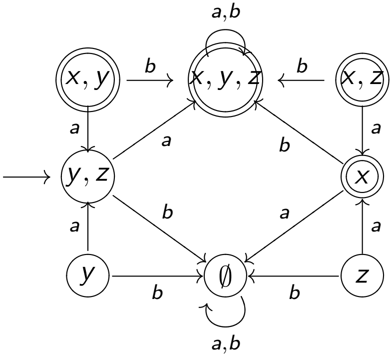
Making it deterministic again: $det(rev(X))$





- new state space: $2^X = \{V \mid V \subseteq \{x, y, z\}\}$
- $V \xrightarrow{a} W \qquad W = \{w \mid v \xrightarrow{a} w, v \in V\}$
- initial state: $\{y, z\}$
- final states: all V with $x \in V$

The automaton $det(rev(X))$ . . .



- . . . accepts the reverse of the language accepted by X:

$$L(\,det(rev(X))\,) \;=\; a\,\{a,b\}^* \;=\; reverse(\,L(X)\,)$$

- . . . and is observable!

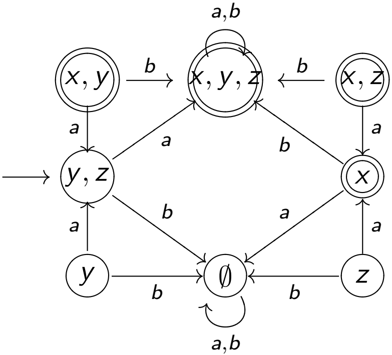
Today's Theorem

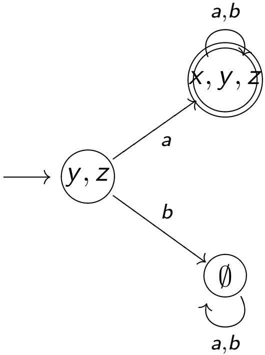
If: a deterministic automaton X is *reachable* and accepts $L(X)$

then: $det(rev(X))$ is *minimal* and

$$L(\,det(rev(X))\,) \;=\; reverse(\,L(X)\,)$$

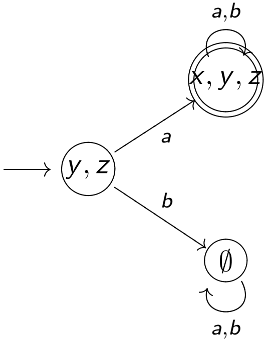
Taking the reachable part of *det*(*rev*(*X*))

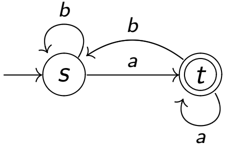




- *reach*(*det*(*rev*(*X*))) is reachable (by construction)

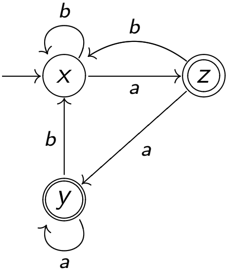
Repeating everything, now for $reach(det(rev(X)))$

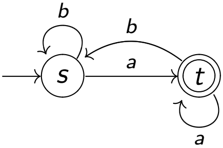




- . . . gives us $reach(det(rev(reach(det(rev(X))))))$
- which is (reachable and) minimal and accepts $\{a, b\}^* a$.

All in all: Brzozowski's algorithm





- X is reachable and accepts $\{a, b\}^* \, a$
- $reach(det(rev(reach(det(rev(X))))))$ also accepts $\{a, b\}^* \, a$
- . . . and is minimal!!

Goal of the day

- Correctness of Brzozowski's algorithm (co)algebraically
- Generalizations to other types of automata

Deterministic Automata are Algebras and Coalgebras

$(1 = \{0\})$ $\qquad$ $(2 = \{0, 1\})$

$$\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t"] & \\
& X^A &
\end{tikzcd}$$

$$\begin{tikzcd}
1 + A \times X \arrow[d, "{[i,t]}"] \\
X
\end{tikzcd}
\qquad
\dfrac{\dfrac{X \to X^A}{A \times X \to X}}{A \to (X \to X)}
\qquad
\begin{tikzcd}
X \arrow[d, "{\langle f,t \rangle}"] \\
2 \times X^A
\end{tikzcd}$$

initial state, transitions $1 + A \times (-)$-algebra	transitions are both algebra and coalgebra	output, transitions $2 \times (-)^A$-coalgebra

Initial Algebras and Final Coalgebras

$$\begin{tikzcd}
1 \arrow[d, "\varepsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dotted, "r"] \arrow[d, "\alpha"'] & X \arrow[ur, "f"] \arrow[r, dotted, "o"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\varepsilon?"'] \arrow[d, "\zeta"] \\
(A^*)^A \arrow[r, dotted, "r^A"] & X^A \arrow[r, dotted, "o^A"] & (2^{A^*})^A
\end{tikzcd}$$

For all $a \in A, w \in A^*$:

$$\begin{array}{rcll}
\alpha(w)(a) & = & wa & \text{(append } a\text{)} \\
\zeta(S)(a) & = & \{w \in A^* \mid aw \in S\} = a^{-1}S & \text{(left } a\text{-derivative)} \\
\\
r(w) & = & t(i)(w) & \text{(state reached on input } w\text{)} \\
o(x) & = & \{w \in A^* \mid f(t(x)(w)) = 1\} & \text{(language acepted by } x\text{)}
\end{array}$$

Reachability, Observability, Minimality

$$
\begin{tikzcd}
1 \arrow[d, "\varepsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dotted, "r"] \arrow[d, "\alpha"'] & X \arrow[r, dotted, "o"] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\varepsilon?"'] \arrow[d, "\zeta"] \\
(A^*)^A \arrow[r, dotted, "r^A"] & X^A \arrow[r, dotted, "o^A"] & (2^{A^*})^A
\end{tikzcd}
$$

Def. (Arbib & Manes)

Automaton $\langle X, t, i, f \rangle$ is ...

- reachable if <u>r is surjective</u> (no algebraic redundancy).
- observable if <u>o is injective</u> (no coalgebraic redundancy).
- minimal if it is *reachable and observable*.

(Contravariant) Powerset construction

$$2^{(-)} : \quad \begin{array}{c} V \\ {\scriptstyle g}\big\downarrow \\ W \end{array} \quad \mapsto \quad \begin{array}{c} 2^V \\ \big\uparrow{\scriptstyle 2^g} \\ 2^W \end{array}$$

where $2^V = \{S \mid S \subseteq V\}$ and, for all $S \subseteq W$,

$$2^g(S) = g^{-1}(S) \quad (= \{v \in V \mid g(v) \in S\})$$

- Note: if g is *surjective*, then 2^g is *injective*.

Reversing an Automaton

- $2^{(-)}$ reverses transitions and determinises:

$$\begin{array}{c|c} X & X \times A \\ {\scriptstyle t}\downarrow & \downarrow \\ X^A & X \end{array} \quad \xmapsto{2^{(-)}} \quad \begin{array}{c|c} 2^{X\times A} & (2^X)^A \\ \uparrow & \uparrow{\scriptstyle 2^t} \\ 2^X & 2^X \end{array}$$

Reversed transitions: $S \xrightarrow{a} t_a^{-1}(S)$ (a-predecessors of S)

- initial becomes final:

$$i\colon 1 \to X \quad \longmapsto \quad 2^i\colon 2^X \to 2^1 = 2$$

In reversed automaton: S is final iff $i \in S$.

- final becomes initial:

$$f\colon X \to 2 = 2^1 \quad \longmapsto \quad f\colon 1 \to 2^X$$

In reversed automaton: initial state is set of final states f.

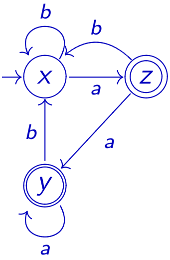
Reversing the entire automaton

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t"] & \\
& X^A &
\end{tikzcd}
\qquad \xmapsto{\;2^{(-)}\;} \qquad
\begin{tikzcd}
1 \arrow[dr, "f"] & & 2 \\
& 2^X \arrow[ur, "2^i"] \arrow[d, "2^t"] & \\
& (2^X)^A &
\end{tikzcd}
$$

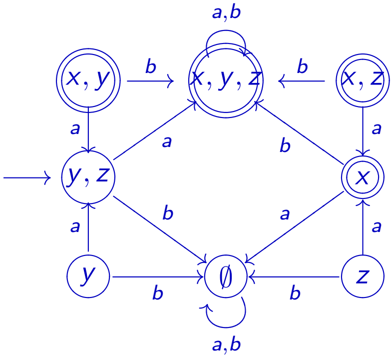
- Initial and final are exchanged . . .
- transitions are reversed . . .
- and the result is again deterministic!

Our previous example

$X =$



$2^X =$



- Note that X has been reversed and determinized:

$$2^X = det(rev(X))$$

Proving today's Theorem

If: a deterministic automaton X is *reachable* and accepts $L(X)$

then: 2^X $(=\ det(rev(X)))$ is *observable* and

$$L(2^X) \ = \ reverse(L(X))$$

Proof: by reversing $A^* \xrightarrow{r} X$

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & \\
A^* \arrow[r, "r"'] \arrow[d, "\alpha"'] & X \arrow[d, "t"] \\
(A^*)^A \arrow[r] & X^A
\end{tikzcd}
\qquad \xmapsto{2^{(-)}} \qquad
\begin{tikzcd}
 & 2 \\
2^X \arrow[ur, "2^i"] \arrow[r, "2^r"'] \arrow[d, "2^t"'] & 2^{A^*} \arrow[u, "2^\epsilon"'] \arrow[d, "2^\alpha"] \\
(2^X)^A \arrow[r] & (2^{A^*})^A
\end{tikzcd}
$$

- X becomes 2^X
- initial automaton A^* becomes (almost) final automaton 2^{A^*}
- r is *surjective* $\Rightarrow$ 2^r is *injective*

Reachable becomes observable

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & \\
A^* \arrow[r, "r"'] \arrow[d, "\alpha"'] & X \arrow[d, "t"] \\
(A^*)^A \arrow[r] & X^A
\end{tikzcd}
\quad \xmapsto{2^{(-)}} \quad
\begin{tikzcd}
 & 2 & \\
2^X \arrow[ur, "2^i"] \arrow[r, "2^r"'] \arrow[d, "2^t"'] & 2^{A^*} \arrow[u, "2^\epsilon"'] \arrow[r, dashed, "rev"'] \arrow[d, "2^\alpha"] & 2^{A^*} \arrow[ul, "\epsilon?"'] \arrow[d, "\zeta"] \\
(2^X)^A \arrow[r] & (2^{A^*})^A \arrow[r, dashed] & (2^{A^*})^A
\end{tikzcd}
$$

- If r is *surjective* then (2^r and hence) $rev \circ 2^r$ is *injective*.
- That is, 2^X is observable.

Summarizing

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[ur, "f"'] \arrow[d, "t"'] & \\
(A^*)^A \arrow[r, dashed] & X^A &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[dr, "f"] & & 2 \\
& 2^X \arrow[ur, "2^i"] \arrow[r, dashed, "rev \circ 2^r"'] \arrow[d, "2^t"'] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\zeta"] \\
& (2^X)^A \arrow[r, dashed] & (2^{A^*})^A
\end{tikzcd}
$$

- If: X is reachable, i.e., r is surjective

  then: $rev \circ 2^r$ is injective, i.e., 2^X is observable.
- And: $rev(2^r(f)) = rev(o(i))$, i.e., $L(\,2^X\,) \;=\; reverse(\,L(X)\,)$

Corollary: Brzozowski's algorithm

- X becomes 2^X, accepting *reverse*$(L(X))$
- take reachable part: $Y = \mathit{reachable}(2^X)$
- Y becomes 2^Y, which is minimal and accepts

$$\mathit{reverse}(\mathit{reverse}(L(X))) = L(X)$$

Generalizations

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\zeta"] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & B \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & B^{A^*} \arrow[u] \arrow[d] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (B^{A^*})^A
\end{tikzcd}
$$

- A Brzozowski minimization algorithm for *Moore* automata.

$$B^X = \{\phi \mid \phi\colon X \to B\} \qquad B^f(\phi) = \phi \circ f$$

Brzozowski for Weighted Automata

Weighted Automata

$$\begin{tikzcd}
1 \arrow[dr, "i"] & & \mathbb{S} \\
& X \arrow[ur, "f"] \arrow[d, "t"] & \\
& V(X)^A &
\end{tikzcd}$$

Weighted languages

$$\begin{tikzcd}
& \mathbb{S} \\
\mathbb{S}^{A^*} \arrow[ur, "o"] \arrow[d, "d"] & \\
(\mathbb{S}^{A^*})^A &
\end{tikzcd}$$

Brzozowski for Weighted Automata

Weighted Automata

$$\begin{tikzcd}
1 \arrow[dr, "i"] & & \mathbb{S} \\
& X \arrow[ur, "f"] \arrow[r, "\eta"] \arrow[d, "t"] & V(X) \arrow[u, "f^\sharp"] \arrow[dl, "t^\sharp"] \\
& V(X)^A &
\end{tikzcd}$$

Weighted languages

$$\begin{tikzcd}
& \mathbb{S} \\
\mathbb{S}^{A^*} \arrow[ur, "o"] \arrow[d, "d"] & \\
(\mathbb{S}^{A^*})^A &
\end{tikzcd}$$

Brzozowski for Weighted Automata

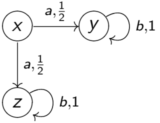
Weighted Automata **Weighted languages**

$$\begin{tikzcd}
1 \arrow[dr, "i"] & & \mathbb{S} \arrow[rr, dashed] & & \mathbb{S} \\
& X \arrow[ur, "f"] \arrow[r, "\eta"] \arrow[d, "t"] & V(X) \arrow[u, "f^\sharp"] \arrow[dl, "t^\sharp"] \arrow[r, dashed] & \mathbb{S}^{A^*} \arrow[ur, "o"] \arrow[d, "d"] & \\
& V(X)^A \arrow[rr, dashed] & & (\mathbb{S}^{A^*})^A &
\end{tikzcd}$$

Brzozowski for weighted languages: given a weighted automaton we want a **canonical** representative of the image in the final coalgebra – Moore automaton.

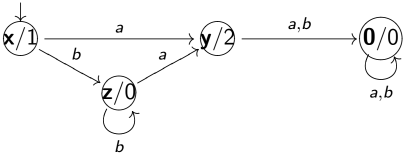
Brzozowski for weighted automata

Weighted automaton which recognizes $\sigma\colon A^* \to \mathbb{S}$:



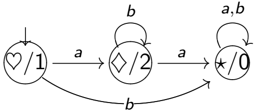
$ab^* \mapsto 2$

"Reverse and determinize" (Worthington):



$b^*a \mapsto 2$

"Reverse and determinize" for Moore automata (using B^-):



$ab^* \mapsto 2$

Example

$$X = \{x, y, z\},\ i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ f = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$$

$$\begin{tikzcd} x \arrow[r, "a{,}\frac{1}{2}"] \arrow[d, "a{,}\frac{1}{2}"] & y \arrow[loop right, "b{,}1"] \\ z \arrow[loop right, "b{,}1"] & \end{tikzcd}$$

$$t_a = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \qquad t_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$ab \mapsto \begin{pmatrix} 1 & 2 & 2 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2$$

$$L(x) = \{\varepsilon \mapsto 1, ab^* \mapsto 2\}$$

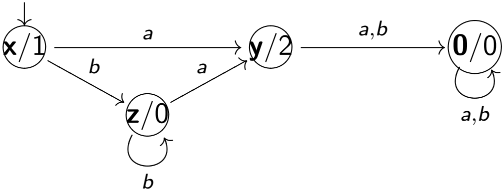
Reversing the automaton (Worthinghton)

Moore automaton that recognizes the reverse weighted language. Initial vector: $f^T = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix}^T$, final vector: $i^T = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ and transition function is **transposed**.

$$t_a^T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t_b^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t^T \colon V(X) \to V(X)^A$$

Reachable automaton from $\mathbf{x} = f^T$:



$$L(\mathbf{x}) = \{\varepsilon \mapsto 1; b^* a \mapsto 2\} = rev(L(x))$$

Part II: Brzozowski's algorithm via adjunctions

Part II: Brzozowski's algorithm via adjunctions

Motivation: Gain deeper understanding of

- the construction/algorithm,
- relation to similar constructions,
- uniform proofs.

Overview:

- Categories of automata.
- Adjunction of automata via reversal.
- Brzozowski, functorially.
- Generalisations to Moore and weighted automata.
- Generalised dual adjunction.
- Related work.

Categories of Automata

Aut $=$ category of all deterministic automata, and automaton morphisms:

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t"] & \\
& X^A &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[d, "i"] \arrow[dr, "j"] & 2 \\
X \arrow[ur, "f"] \arrow[r, "h"] \arrow[d, "t"] & Y \arrow[u, "g"] \arrow[d, "s"] \\
X^A \arrow[r, "h^A"] & Y^A
\end{tikzcd}
$$

Note:

- Automaton morphisms preserve language.
- No initial object, no final object in Aut.

$\mathrm{Aut}(L)$ = subcategory of Aut of automata accepting L.

Initial and final objects regained:

$$\begin{tikzcd}
1 \arrow[d, "\varepsilon"'] \arrow[dr, "i"] \arrow[drr, bend left, "L"] & & 2 \\
A^* \arrow[d, "\alpha"'] \arrow[r, dotted, "r"] \arrow[urr, bend left, "\chi_L"] & X \arrow[d, "t"] \arrow[r, dotted, "o"] \arrow[ur, "f"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
(A^*)^A \arrow[r, dotted, "r^A"] & X^A \arrow[r, dotted, "o^A"] & (2^{A^*})^A
\end{tikzcd}$$

Automaton $\langle X, t, i, f \rangle$ in Aut(L) is ...

- reachable if initial morphism r is surjective.
- observable if final morphism o is injective.

Myhill-Nerode via Aut(L)

$$\begin{tikzcd}
1 \arrow[d, "\varepsilon"'] \arrow[dr, bend left, "L"] & 2 \\
A^* \arrow[ur, bend left, "\chi_L"] \arrow[r, dotted, "r=o"] \arrow[d, "\alpha"'] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\zeta"] \\
(A^*)^A \arrow[r, dotted, "r^A=o^A"] & (2^{A^*})^A
\end{tikzcd}$$

Characterisation:

L regular

iff index($\equiv_L$) is finite

iff left-quotients(L) is finite

- $o(w) = \{u \in A^* \mid wu \in L\} = w^{-1}L$
- $\ker(o)$ is Myhill-Nerode-equivalence:
  $w \equiv_L v \quad \text{iff} \quad \forall u \in A^* : wu \in L \iff vu \in L$
- $\mathrm{img}(o)$ is set of left-quotients of L.
- $|\mathrm{img}(o)| = \mathrm{index}(\equiv_L)$

Adjoint Automata: Main tools

- Adjunction of state spaces:

$$
\begin{tikzcd}
\mathsf{Set} \arrow[r, bend left, "2^{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathrm{op}} \arrow[l, bend left, "2^{(-)^{\mathrm{op}}}"]
\end{tikzcd}
\qquad\qquad
\frac{X \to 2^Y \quad \text{in Set}}{2^X \to Y \quad \text{in Set}^{\mathrm{op}}}
$$

- Exponential transpose:

$$
\frac{g\colon X \to 2^Y \quad \text{in Set}}{\theta(g)\colon Y \to 2^X \quad \text{in Set}}
$$

- Transpose lemma:

$$
\theta(X \xrightarrow{h} Y \xrightarrow{f} 2^Z) \quad = \quad Z \xrightarrow{\theta(f)} 2^Y \xrightarrow{2^h} 2^X
$$

Reversing an Automaton

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t=\langle t_a \rangle_{a\in A}"] & \\
& X^A &
\end{tikzcd}
\qquad \xmapsto{\overline{2}} \qquad
\begin{tikzcd}
1 \arrow[dr, "\theta(f)"] & & 2 \\
& 2^X \arrow[ur, "2^i"] \arrow[d, "2^t=\langle 2^{t_a} \rangle_{a\in A}"] & \\
& (2^X)^A &
\end{tikzcd}
$$

$$
\mathcal{X} : \qquad 1 \xrightarrow{i} X \xrightarrow{t_{a_1}} X \quad \cdots \quad X \xrightarrow{t_{a_n}} X \xrightarrow{f} 2
$$

$$
\overline{2}(\mathcal{X}) : \qquad 2 \xleftarrow{2^i} 2^X \xleftarrow{2^{t_{a_1}}} 2^X \quad \cdots \quad 2^X \xleftarrow{2^{t_{a_n}}} 2^X \xleftarrow{\theta(f)} 1
$$

Reversal is Functorial

Theorem:

- $L(\overline{2}(\mathcal{X})) = \mathit{rev}(L(\mathcal{X}))$.
- Reversing is functor $\overline{2}\colon \mathsf{Aut} \to \mathsf{Aut}^{\mathrm{op}}$.
- Reversing is functor $\overline{2}\colon \mathsf{Aut}(L) \to \mathsf{Aut}(\mathit{rev}(L))^{\mathrm{op}}$.

Adjunction of Automata

Theorem: Reversal lifts dual adjunction on Set to dual adjunction of automata:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{Aut}(L) \arrow[r, bend left, "\overline{2}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathsf{Aut}(\mathit{rev}(L))^{\mathrm{op}} \arrow[l, bend left, "\overline{2}^{\mathrm{op}}"] \arrow[d] \\
\mathsf{Set} \arrow[r, bend left, "2"] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathrm{op}} \arrow[l, bend left, "2^{\mathrm{op}}"]
\end{tikzcd}$$

Corollary (duality): Let $\mathcal{A}$ be initial object in Aut(L), $\mathcal{Z}$ the final object in Aut($rev(L)$), and let $\mathcal{X}$ be an automaton in Aut(L).

$$\begin{array}{ccc}
r\colon \mathcal{A} \twoheadrightarrow \mathcal{X} & \xmapsto{\overline{2}} & o\colon \overline{2}(\mathcal{X}) \rightarrowtail \overline{2}(\mathcal{A}) = \mathcal{Z} \\
\mathcal{X} \text{ reachable} & \Longrightarrow & \overline{2}(\mathcal{X}) \text{ observable}
\end{array}$$

Brzozowski's algorithm, functorially

- Let: $\mathrm{rAut}(L) = $ reachable automata accepting L,
  $\mathrm{oAut}(L) = $ observable automata accepting L,
  $\mathrm{mAut}(L) = $ minimal automata accepting L.
- Reachability is functor $R\colon \mathrm{Aut}(L) \to \mathrm{rAut}(L)$ (coreflector).
  Restricts to $R\colon \mathrm{oAut}(L) \to \mathrm{mAut}(L)$.
- Brzozowski's algorithm is $R \circ \overline{2}^{\mathrm{op}} \circ R^{\mathrm{op}} \circ \overline{2}$:

$$\begin{tikzcd}
\mathrm{Aut}(L) \arrow[r, "\overline{2}"] \arrow[d, dotted, "O"'] & \mathrm{Aut}(\mathit{rev}(L))^{\mathrm{op}} \arrow[d, "R^{\mathrm{op}}"] \\
\mathrm{oAut}(L) \arrow[d, "R"'] & \mathrm{rAut}(\mathit{rev}(L))^{\mathrm{op}} \arrow[l, "\overline{2}^{\mathrm{op}}"'] \\
\mathrm{mAut}(L) &
\end{tikzcd}$$

Brzozowski for Moore Automata, revisited

Adjunction of Moore Automata:

$$\begin{tikzcd}
\mathsf{Moore}(L) \arrow[rr, bend left, "\overline{B}^{(-)}"] \arrow[rr, phantom, "\perp"] \arrow[d] & & \mathsf{Moore}(rev(L))^{\mathrm{op}} \arrow[ll, bend left, "\overline{B}^{(-)\mathrm{op}}"] \arrow[d] \\
\mathsf{Set} \arrow[rr, bend left, "B^{(-)}"] \arrow[rr, phantom, "\perp"] & & \mathsf{Set}^{\mathrm{op}} \arrow[ll, bend left, "B^{(-)\mathrm{op}}"]
\end{tikzcd}$$

- $L \colon A^* \to B$ (B-weighted language), $rev(L)(w) = L(w^R)$.
- Reversal functor $B^{(-)} = \mathsf{Set}(-, B)$.
- Brzozowski minimization, functorially ✓

Brzozowski for Weighted Automata

Weighted Automaton in Set

$$\begin{tikzcd} 1 \arrow[dr, "i"] & & \mathbb{S} \\ & X \arrow[ur, "f"] \arrow[d, "t"] & \\ & V(X)^A & \end{tikzcd}$$

Moore Automaton in SMod

$$\begin{tikzcd} 1 \arrow[dr, "\eta \circ i"] & & \mathbb{S} \\ & V(X) \arrow[ur, "f^\sharp"] \arrow[d, "t^\sharp"] & \\ & V(X)^A & \end{tikzcd}$$

- $\mathbb{S}$ is a commutative semiring $(S, +, \cdot, 0, 1)$.
- SMod $=$ $\mathbb{S}$-semimodules and $\mathbb{S}$-linear maps
- $V(X) = \{s_1 x_1 + \ldots + s_n x_n \mid s_i \in \mathbb{S}, x_i \in X\}$ (free on X)

Brzozowski for Weighted Automata, revisited

Adjunction of Moore Automata over SMod:

$$\begin{tikzcd}
\mathrm{Moore}(L) \arrow[r, bend left, "\overline{\mathbb{S}}^{(-)}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathrm{Moore}(rev(L))^{\mathrm{op}} \arrow[l, bend left, "\overline{\mathbb{S}}^{(-)^{\mathrm{op}}}"] \arrow[d] \\
\mathrm{SMod} \arrow[r, bend left, "\mathbb{S}^{(-)}"] \arrow[r, phantom, "\perp"] & \mathrm{SMod}^{\mathrm{op}} \arrow[l, bend left, "\mathbb{S}^{(-)^{\mathrm{op}}}"]
\end{tikzcd}$$

- $L \colon A^* \to \mathbb{S}$ (formal power series), $rev(L)(w) = L(w^R)$.
- Reversal functor: $(-)^* = \mathbb{S}^{(-)} = \mathrm{SMod}(-, \mathbb{S})$ (dual space)
- Note: $V(X)^* = V(X^*)$ for finite X.

Brzozowski for WAut via Brzozowski for Moore

$$
\begin{tikzcd}
\mathsf{WAut}_{\mathsf{Set}}(L) \arrow[d, "(-)^\sharp"] \arrow[dr, bend left] & \\
\mathsf{Moore}_{\mathsf{SMod}}(L) \arrow[r, "\mathbb{S}^{(-)}"] & \mathsf{Moore}_{\mathsf{SMod}}(\mathit{rev}(L))^{\mathrm{op}} \arrow[d, "\mathsf{forget}"] \\
& \mathsf{Moore}_{\mathsf{Set}}(\mathit{rev}(L))^{\mathrm{op}} \arrow[d, "R^{\mathrm{op}}"] \\
\mathsf{oMoore}_{\mathsf{Set}}(L) \arrow[d, "R"] & \mathsf{rMoore}_{\mathsf{Set}}(\mathit{rev}(L))^{\mathrm{op}} \arrow[l, "B^{(-)^{\mathrm{op}}}"'] \\
\mathsf{mMoore}_{\mathsf{Set}}(L) & \text{(possibly infinite!)}
\end{tikzcd}
$$

Brzozowski for WAut in SMod

$$
\begin{tikzcd}
\mathsf{WAut}(L) \text{ over } \mathsf{Set} \arrow[d, "(-)^\sharp"] \arrow[dr, bend left] & \\
\mathsf{Moore}_{\mathsf{SMod}}(L) \arrow[r, "\overline{\mathbb{S}}"] & \mathsf{Moore}_{\mathsf{SMod}}(rev(L))^{\mathrm{op}} \arrow[d, "Q^{\mathrm{op}}"] \\
\mathsf{oMoore}_{\mathsf{SMod}}(L) \arrow[d, "Q"'] & \mathsf{rMoore}_{\mathsf{SMod}}(rev(L))^{\mathrm{op}} \arrow[l, "\overline{\mathbb{S}}^{\mathrm{op}}"'] \\
\mathsf{mMoore}_{\mathsf{SMod}}(L) & \text{(finitely generated state space)}
\end{tikzcd}
$$

Note:

- reachability "illegal operation" in $\mathsf{Moore}_{\mathsf{SMod}}(L)$.
- $\mathcal{A} \twoheadrightarrow Q(\mathcal{X})$ (image/quotient of initial object?).
- $Q(\mathcal{X})$ finitely generated if $\mathbb{S}$ Noetherian.

Generalised Duality

Assume $\mathbf{C}, \mathbf{D}$ have products.

$$
\begin{tikzcd}
\mathbf{C} \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] & \mathrm{Moore}_{\mathbf{C}}(L) \arrow[r, dotted, bend left, "\overline{C_0}^{(-)}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathrm{Moore}_{\mathbf{D}}(rev(L))^{\mathrm{op}} \arrow[l, dotted, bend left, "\overline{D_0}^{(-)}"] \arrow[d] & \mathbf{D} \arrow[d, bend left, "U'"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Set} \arrow[u, bend left, "\Phi"] & \mathbf{C} \arrow[r, bend left, "C_0^{(-)}"] \arrow[r, phantom, "\perp"] & \mathbf{D}^{\mathrm{op}} \arrow[l, bend left, "D_0^{(-)}"] & \mathrm{Set} \arrow[u, bend left, "\Phi'"]
\end{tikzcd}
$$

Moore automaton over $\mathbf{C}$:

state space:	$C \in \mathbf{C}$
initial state:	$i: 1 \to UC$
transitions:	$t_a: C \to C,\ a \in A$
output:	$f: C \to D_0^{\Phi' 1}$

Example: Quantum Automata

Cf. Bart Jacobs, Frank Roumen.

$$\begin{tikzcd}
\mathrm{Moore}(L) \arrow[rr, bend left, "\overline{[0,1]}^{(-)}"] \arrow[d] & \perp & \mathrm{Moore}(\mathit{rev}(L))^{\mathrm{op}} \arrow[ll, bend left, "\overline{[0,1]}^{(-)}"] \arrow[d] \\
\mathrm{Conv} \arrow[rr, bend left, "{[0,1]}^{(-)}"] & \perp & \mathrm{EMod}^{\mathrm{op}} \arrow[ll, bend left, "{[0,1]}^{(-)}"]
\end{tikzcd}$$

State	$\leftrightsquigarrow$	Observations
$\mathcal{DM}(\mathcal{H})$		$\mathcal{E}\mathit{ff}(\mathcal{H})$
Schrödinger		Heisenberg

Brzozowski minimization? Finitely generated resulting automaton?

Related Work

- Bezhanishvili, Kupke, Panangaden (WoLLIC 2012): minimisation via dual equivalence coalgebra-algebra (deterministic, linear weighted, belief automata).

$$\begin{tikzcd} \mathrm{Coalg}(T) \arrow[r, bend left] \arrow[r, phantom, "\cong"] \arrow[d] & \mathrm{Alg}(L)^{\mathrm{op}} \arrow[l, bend left] \arrow[d] \\ \mathsf{C} \arrow[r, bend left] \arrow[r, phantom, "\cong"] & \mathsf{D}^{\mathrm{op}} \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{PODFA} \arrow[r, bend left] \arrow[r, phantom, "\cong"] \arrow[d] & \mathrm{FBAO}^{\mathrm{op}} \arrow[l, bend left] \arrow[d] \\ \mathrm{FinSet} \arrow[r, bend left] \arrow[r, phantom, "\cong"] & \mathrm{FinBA}^{\mathrm{op}} \arrow[l, bend left] \end{tikzcd}$$

(Both left and right adjoint must preserve epis.)

- Arbib, Manes, Gehrke, Pin, König, Hülsbusch, Milius, Adamek, Myers, Worthington,...

Conclusion

Summary:

- Brzozowski algorithm via dual adjunction of automata.
- Duality: state and observations.
- Generalisations: given Moore/nondeterm/weighted automaton accepting L, construct minimal Moore automaton accepting L (language equivalence!).
- Future work: other automaton types (probabilistic, multi-sorted, ...), combination with generalised powerset construction, algebraic-coalgebraic automata theory.

Message:

- duality $\rightsquigarrow$ algorithms.
- categories $\rightsquigarrow$ generalisations, clarification.