A HIERARCHY OF REGULAR SEQUENCE SETS

Klaus Wagner
Sektion Mathematik der Friedrich-Schiller-Universität Jena
Jena, DDR

In the paper STAIGER/WAGNER [1] several topological classes of regular sequence sets are characterized both by suitable notions of acceptance for finite automata and also without the notion of automaton, namely the classes of regular open, closed, $G_\delta - F_\sigma$, G_δ and F_σ-sets, respectively. As an information about further investigations in this field, in the present paper a survey about a hierarchy of reducibility degress of regular sequence sets generated by m-reducing with finite automata is given. These degrees can be characterized as complexity classes with respect to suitable complexity measures of finite automata.

Let a FDSA (FDAA) $\mathfrak{X}$ be an initial finite deterministic synchronous (asynchronous) automaton with input and output alphabet X. $\Phi_{\mathfrak{X}}$ denotes the deterministic synchronous (asynchronous) sequential operator generated by $\mathfrak{X}$ and X^ω denotes the set of all infinite sequences of type ω over the alphabet X.

Definition: $A \subseteq X^\omega$ is said to be DS- (DA-) reducible to $B \subseteq X^\omega$, which is written $A \leq_{DS} B$ ($A \leq_{DA} B$), iff there exists a FDSA (FDAA) $\mathfrak{X}$ such that

$$\forall \xi (\xi \in X^\omega \rightarrow (\xi \in A \leftrightarrow \Phi_{\mathfrak{X}}(\xi) \in B))$$

Furthermore, $A \equiv_{DS} B$ iff $A \leq_{DS} B$ and $B \leq_{DS} A$, and $A \equiv_{DA} B$ iff $A \leq_{DA} B$ and $B \leq_{DA} A$. As usual we transfer the relation $\leq_{DS}$ ($\leq_{DA}$) to the DS- (DA-) equivalence classes, which are called DS- (DA-) degrees.

Let a FDA $\mathfrak{M} = [X, Z, f, z_0, \mathfrak{Z}]$ be an initial finite deterministic automaton with the input alphabet X, the set of states Z, the transition function f, the initial state z_0, and the system $\mathfrak{Z} \subseteq \mathfrak{P}(Z)$ of final sets. Then $T(\mathfrak{M}) =_{df} \{\xi ; U(\Phi_{\mathfrak{M}}(\xi)) \in \mathfrak{Z}\}$ is called the set of all sequences accepted by the FDA $\mathfrak{M}$ in the sense of MÜLLER [2], where we define $U(\eta) =_{df} \{z; \text{card}\{n; \eta(n) = z\} = \aleph_0\}$.
Now we define the complexity measures $m_1(\mathfrak{M})$, $m_2(\mathfrak{M})$, $m_3(\mathfrak{M})$ and

$m_4(\mathfrak{A})$ which are characteristic for the automaton $\mathfrak{A}$.

Definition:

1. $K_1^1(\mathfrak{A}) =_{df} \{Z';\quad T([X,Z,f,z_0,\{Z'\}]) \neq \emptyset \wedge Z' \notin \mathfrak{F}\}$,

   $K_1^2(\mathfrak{A}) =_{df} \{Z';\quad T([X,Z,f,z_0,\{Z'\}]) \neq \emptyset \wedge Z' \in \mathfrak{F}\}$,

   $K_{2m}^1(\mathfrak{A}) =_{df} \{Z'; Z' \in K_1^2(\mathfrak{A}) \wedge \exists Z''(Z'' \in K_{2m-1}^1(\mathfrak{A}) \wedge Z'' \subset Z')\}$ for $m \geq 1$,

   $K_{2m}^2(\mathfrak{A}) =_{df} \{Z'; Z' \in K_1^1(\mathfrak{A}) \wedge \exists Z''(Z'' \in K_{2m-1}^2(\mathfrak{A}) \wedge Z'' \subset Z')\}$ for $m \geq 1$,

   $K_{2m+1}^1(\mathfrak{A}) =_{df} \{Z'; Z' \in K_1^1(\mathfrak{A}) \wedge \exists Z''(Z'' \in K_{2m}^1(\mathfrak{A}) \wedge Z'' \subset Z')\}$ for $m \geq 1$,

   $K_{2m+1}^2(\mathfrak{A}) =_{df} \{Z'; Z' \in K_1^2(\mathfrak{A}) \wedge \exists Z''(Z'' \in K_{2m}^2(\mathfrak{A}) \wedge Z'' \subset Z')\}$ for $m \geq 1$,

   $m_1(\mathfrak{A}) =_{df} \max(\{0\} \cup \{m;\ K_m^1(\mathfrak{A}) \neq \emptyset\})$ and

   $m_2(\mathfrak{A}) =_{df} \max(\{0\} \cup \{m;\ K_m^2(\mathfrak{A}) \neq \emptyset\})$.

2. For $Z_1, Z_2 \subseteq Z$ we define

   $Z_1 \overset{\mathfrak{A}}{\vdash} Z_2 =_{df} \exists z_1 \exists z_2 \exists w(z_1 \in Z_1 \wedge z_2 \in Z_2 \wedge w \in X^* \wedge f(z_1,w) = z_2)$.

   Let $m =_{df} \max\{m_1(\mathfrak{A}), m_2(\mathfrak{A})\}$.

   $K_1^3(\mathfrak{A}) =_{df} K_m^1(\mathfrak{A})$,

   $K_1^4(\mathfrak{A}) =_{df} K_m^2(\mathfrak{A})$,

   $K_{2n}^3(\mathfrak{A}) =_{df} \{Z'; Z' \in K_1^4(\mathfrak{A}) \wedge \exists Z''(Z'' \in K_{2n-1}^3(\mathfrak{A}) \wedge Z'' \overset{\mathfrak{A}}{\vdash} Z')\}$ for $n \geq 1$,

   $K_{2n}^4(\mathfrak{A}) =_{df} \{Z'; Z' \in K_1^3(\mathfrak{A}) \wedge \exists Z''(Z'' \in K_{2n-1}^4(\mathfrak{A}) \wedge Z'' \overset{\mathfrak{A}}{\vdash} Z')\}$ for $n \geq 1$,

   $K_{2n+1}^3(\mathfrak{A}) =_{df} \{Z'; Z' \in K_1^3(\mathfrak{A}) \wedge \exists Z''(Z'' \in K_{2n}^3(\mathfrak{A}) \wedge Z'' \overset{\mathfrak{A}}{\vdash} Z')\}$ for $n \geq 1$,

   $K_{2n+1}^4(\mathfrak{A}) =_{df} \{Z'; Z' \in K_1^4(\mathfrak{A}) \wedge \exists Z''(Z'' \in K_{2n}^4(\mathfrak{A}) \wedge Z'' \overset{\mathfrak{A}}{\vdash} Z')\}$ for $n \geq 1$,

   $m_3(\mathfrak{A}) =_{df} \max(\{0\} \cup \{n;\ K_n^3(\mathfrak{A}) \neq \emptyset\})$ and

   $m_4(\mathfrak{A}) =_{df} \max(\{0\} \cup \{n;\ K_n^4(\mathfrak{A}) \neq \emptyset\})$.

Now we can define the classes C_m^n, D_m^n and E_m^n which are very important for our further investigations.

Definition: Let be $m \geq 1$ and $n \geq 1$.

$$C_m^n =_{df} \{T(\mathfrak{A});\ \max\{m_1(\mathfrak{A}), m_2(\mathfrak{A})\} = m \wedge m_3(\mathfrak{A}) = n \wedge m_4(\mathfrak{A}) < n\}$$
$$D_m^n =_{df} \{T(\mathfrak{A});\ \max\{m_1(\mathfrak{A}), m_2(\mathfrak{A})\} = m \wedge m_3(\mathfrak{A}) < n \wedge m_4(\mathfrak{A}) = n\}$$
$$E_m^n =_{df} \{T(\mathfrak{A});\ \max\{m_1(\mathfrak{A}), m_2(\mathfrak{A})\} = m \wedge m_3(\mathfrak{A}) = m_4(\mathfrak{A}) = n\}$$

Theorem 1: 1. C_m^n and D_m^n are DS-degrees as well as DA-degrees.

2. E_m^n is a union of DS-degrees as well as DA-degrees.

Let G^R, F^R, G^R_δ and F^R_σ denote the classes of regular open, closed, G_δ -, F_σ -sets respectively.

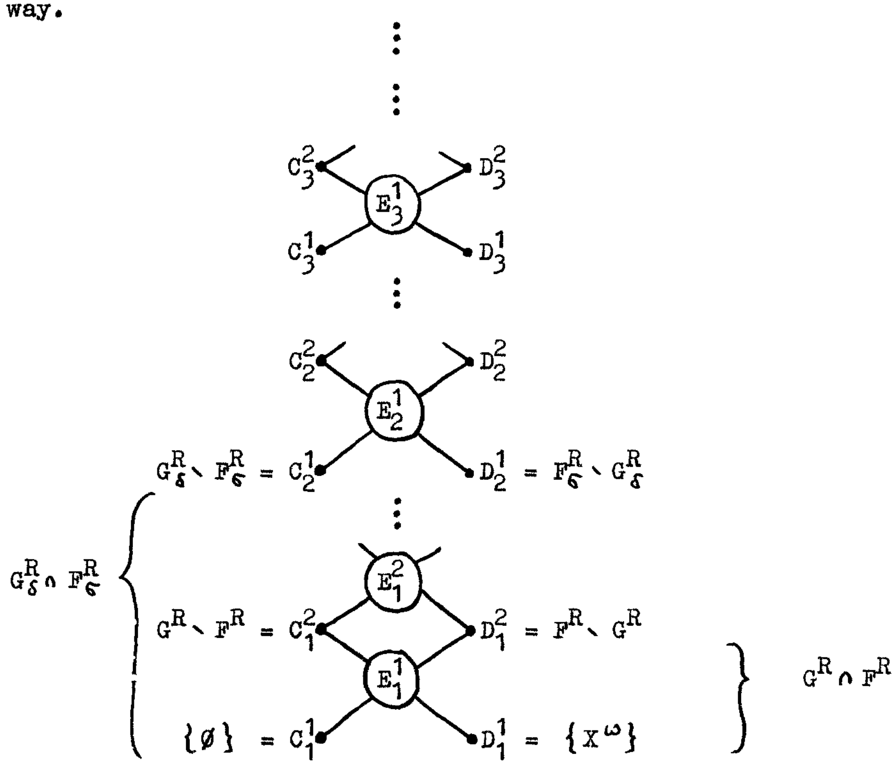
<u>Theorem 2</u>: 1. $G^R \cap F^R = C^1_1 \cup D^1_1 \cup E^1_1$.

2. $G^R \setminus F^R = C^2_1$ and $F^R \setminus G^R = D^2_1$.

3. $G^R_\delta \cap F^R_\sigma = \bigcup_{n=1}^{\infty} (C^n_1 \cup D^n_1 \cup E^n_1)$.

4. $G^R_\delta \setminus F^R_\sigma = C^1_2$ and $F^R_\sigma \setminus G^R_\delta = D^1_2$.

<u>Theorem 3</u>: The coarse structure of the partially orderd set of all DS-degrees as well as DA-degrees can be represented in the following way.



And now we have to investigate the structure of these DS-degrees and DA-degrees, respectively, which are in E^n_m . For this aim the notion of the derivation of an automaton is important.

<u>Definition</u>: Let $\max \{ m_1(\mathfrak{A}), m_2(\mathfrak{A}) \} = m$ and $\max \{ m_3(\mathfrak{A}), m_4(\mathfrak{A}) \} = n$. Then the automaton $\partial\mathfrak{A} = [X, \partial Z, \partial f, \partial z_o, \partial\mathfrak{Z}]$ is said to be the <u>first derivation of the automaton</u> $\mathfrak{A} = [X, Z, f, z_o, \mathfrak{Z}]$

where we define

$$\partial Z =_{df} \begin{cases} (\partial_1 Z \cap \partial_2 Z) \cup \{s_1, s_2\}, & \text{if } \partial_1 Z \cap \partial_2 Z \neq \emptyset \\ \{s_1\}, & \text{if } \partial_2 Z = \emptyset \\ \{s_2\}, & \text{if } \partial_1 Z = \emptyset, \end{cases}$$

where $s_1, s_2 \notin Z \cup \bigcup \mathfrak{Z}$,

$$\partial_1 Z =_{df} \{z;\ z \in Z \wedge \exists Z_1 \ldots \exists Z_n (Z_1 \in K_1^3(\mathfrak{R}) \wedge \ldots \wedge Z_n \in K_n^3(\mathfrak{R}) \wedge$$
$$\wedge \{z_o\} \vdash^{\mathfrak{R}} \{z\} \vdash^{\mathfrak{R}} Z_1 \vdash^{\mathfrak{R}} \ldots \vdash^{\mathfrak{R}} Z_n)\} \text{ and}$$

$$\partial_2 Z =_{df} \{z;\ z \in Z \wedge \exists Z_1 \ldots \exists Z_n (Z_1 \in K_1^4(\mathfrak{R}) \wedge \ldots \wedge Z_n \in K_n^4(\mathfrak{R}) \wedge$$
$$\wedge \{z_o\} \vdash^{\mathfrak{R}} \{z\} \vdash^{\mathfrak{R}} Z_1 \vdash^{\mathfrak{R}} \ldots \vdash^{\mathfrak{R}} Z_n)\},$$

$$\partial z_o =_{df} \begin{cases} z_o, & \text{if } \partial_1 Z \cap \partial_2 Z \neq \emptyset \\ s_1, & \text{if } \partial_2 Z = \emptyset \\ s_2, & \text{if } \partial_1 Z = \emptyset, \end{cases}$$

$$\partial f(z,x) =_{df} \begin{cases} f(z,x), & \text{if } z \in \partial_1 Z \cap \partial_2 Z \text{ and } f(z,x) \in \partial_1 Z \cap \partial_2 Z \\ s_1, & \text{if } z \in \partial_1 Z \cap \partial_2 Z \text{ and } f(z,x) \in \partial_1 Z \setminus \partial_2 Z \\ s_2, & \text{if } z \in \partial_1 Z \cap \partial_2 Z \text{ and } f(z,x) \notin \partial_1 Z \\ s_1, & \text{if } z = s_1 \\ s_2, & \text{if } z = s_2 \end{cases}$$

and $\partial \mathfrak{Z} =_{df} \mathfrak{Z} \cup \{\{s_2\}\}$.

<u>Theorem 4</u>: 1. $T(\mathfrak{R}) \in C_m^n$ implies $T(\partial\mathfrak{R}) \in C_1^1$.

2. $T(\mathfrak{R}) \in D_m^n$ implies $T(\partial\mathfrak{R}) \in D_1^1$.

3. $T(\mathfrak{R}) \in E_m^n$ implies $T(\partial\mathfrak{R}) \in \bigcup_{\substack{\mu=1,\ldots,m-1 \\ \nu=1,2,\ldots}} (C_\mu^\nu \cup D_\mu^\nu \cup E_\mu^\nu)$ for $m \geq 2$.

4. $T(\mathfrak{R}) \in E_1^n$ implies $T(\partial\mathfrak{R}) \in E_1^1$.

<u>Theorem 5</u>: Let be $T(\mathfrak{R})$, $T(\mathfrak{R}') \in E_m^n$

1. $T(\mathfrak{R}) \leq_{DS} T(\mathfrak{R}') \Longleftrightarrow T(\partial\mathfrak{R}) \leq_{DS} T(\partial\mathfrak{R}')$.

2. $T(\mathfrak{R}) \leq_{DA} T(\mathfrak{R}') \Longleftrightarrow T(\partial\mathfrak{R}) \leq_{DA} T(\partial\mathfrak{R}')$.

Thus the structure of these DS-degrees (DA-degrees) which are in E_m^n $(m \geq 2)$ resembles the structure of all DS-degrees (DA-degrees) which are in the classes C_μ^ν , D_μ^ν and E_μ^ν with $\mu < m$. Further the structure of all DS-degrees (DA-degrees) which are in E_1^n resembles the structure of all DS-degrees (DA-degrees) which are in E_1^1. In this manner we can inductively get clarity about the structure of the partially ordered set of all regular DS-degrees (DA-degrees) if we know the structure of all DS-degrees (DA-degrees) which are in E_1^1. For the investigation of this last question we define

Definition: $m_5(\mathfrak{A}) =_{df} \max \{ |w| \; ; \; f(z_o,w) \in \partial_1 Z \cap \partial_2 Z \}$

$E_k =_{df} \{ \; T(\mathfrak{A}); \; m_5(\mathfrak{A}) = k \} \cap E_1^1$

Theorem 6:

1. $E_1^1 = \bigcup_{k=o}^{\infty} E_k$.
2. E_k is a DS-degree.
3. $E_{k_1} \leq_{DS} E_{k_2} \Longleftrightarrow k_1 \leq k_2$
4. E_1^1 is a DA-degree.

This completes our knowledge of the structure of all regular DS-degrees (DA-degrees) with respect to the partial ordering $\leq_{DS}$ ($\leq_{DA}$).

References.

1. Staiger, L. and Wagner, K., Automatentheoretische und automatenfreie Charakterisierungen topologischer Klassen regulärer Folgenmengen. Elektronische Informationsverarbeitung und Kybernetik, EIK 10 (1974) 7, 379-392.

2. Müller, D.E., Infinite sequences and finite machines. AIEE Proc. Fourth Annual Symp. Switching Circuit Theory and Logical Design, 3-16.