



Countdown μ -calculus

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Abstract

We introduce the countdown μ -calculus, an extension of the modal μ -calculus with ordinal approximations of fixpoint operators. In addition to properties definable in the classical calculus, it can express (un)boundedness properties such as the existence of arbitrarily long sequences of specific actions. The standard correspondence with parity games and automata extends to suitably defined countdown games and automata. However, unlike in the classical setting, the scalar fragment is provably weaker than the full vectorial calculus and corresponds to automata satisfying a simple syntactic condition. We establish some facts, in particular decidability of the model checking problem and strictness of the hierarchy induced by the maximal allowed nesting of our new operators.

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1 Introduction

The modal μ -calculus [14] is a well-known logic for defining and verifying behavioural properties of state-and-transition systems. It extends propositional logic with basic next-step modalities and fixpoint operators to describe long-term behaviour. It is expressive enough to include other temporal logics such as CTL* as fragments, but it has good computational properties, and its simple syntax and semantics makes it a convenient formalism to study.

The μ -calculus has a straightforward inductively-defined semantics, but it is often useful to consider an alternative (but equivalent) semantics based on parity games. A formula φ together with a model \mathcal{M} define a game between two players called \forall dam and \exists ve. Positions in the game are of the form (\mathbf{m}, ψ) where \mathbf{m} is a point in \mathcal{M} and ψ is a subformula of φ , and moves are defined so that \exists ve has a winning strategy from (\mathbf{m}, φ) if and only if φ holds in \mathbf{m} . Among other advantages, the game-based semantics provides more efficient algorithms for model checking of μ -calculus formulas than an inductive computation of fixpoints [9].

The model component can be abstracted away from parity games. Indeed, a formula φ itself gives rise to an alternating parity automaton \mathcal{A}_φ that recognizes models. The behaviour of an automaton on a model is defined in terms of a parity game, states of \mathcal{A}_φ are subformulas of φ , and the transition relation is defined so that it accepts a model \mathcal{M} rooted in a point \mathbf{m} if and only if φ holds in \mathbf{m} . The advantage of this is that \mathcal{A}_φ , while conceptually closer to a parity game, is a finite structure even if it is then applied to infinite models.

The modal μ -calculus is a rather expressive formalism: it can define all bisimulation-invariant properties definable in monadic second-order logic (MSO) [13], such as “there is an infinite path of τ -labeled edges”. However, there are some properties of interest which are not definable even in MSO. Notable examples include (un)boundedness properties such as “for

every number n , there is a path with at least n consecutive τ -labeled edges". An extension of MSO called $\text{MSO} + \text{U}$, aimed at defining such properties, has been considered [6]. However, the satisfiability problem of $\text{MSO} + \text{U}$ turned out to be undecidable even for word models [4]. Since the modal μ -calculus is a fragment of MSO, it is worthwhile to extend it with a mechanism for defining (un)boundedness properties, in the hope of retaining decidability.

In this paper we propose such an extension: the *countdown μ -calculus* $\mu^\alpha\text{-ML}$. In addition to μ -calculus operators, it features countdown operators μ^α and ν^α parametrized by ordinal numbers α . Instead of least and greatest fixpoints, they define ordinal approximations of those fixpoints. Intuitively, while the meaning of classical μ -calculus formulas $\mu x.\varphi(x)$ and $\nu x.\varphi(x)$ is defined by infinite unfolding of the formula φ until a fixpoint is reached, for $\mu^\alpha x.\varphi(x)$ and $\nu^\alpha x.\varphi(x)$ the unfolding stops after α steps (which makes a difference if α is smaller than the *closure ordinal* of φ). The classical fixpoint operators are kept but renamed to μ^∞ and ν^∞ , to make clear the lack of any restrictions on the unfolding process.

An inductive definition of the semantics of countdown formulas is just as straightforward as in the classical case. With some more effort, we are able to formulate game-based semantics as well. We introduce *countdown games* and *countdown automata*, which are similar to parity games and alternating automata known from the classical setting, but are additionally equipped with counters that are decremented and reset by the two players according to specific rules. Intuitively, the counters say how many more times various ranks can be visited, in similar manner to the signatures introduced by Walukiewicz [17, Section 3]. A player responsible for decrementing a counter may lose the game if the value of that counter is zero, just as a player responsible for finding the next position in a game may lose if there is no position to go to. The key mechanism of countdown games is implicit in [11], where the authors investigate a nonstandard semantics for the scalar fragment of the μ -calculus equivalent to replacing every μ and ν by our countdown operators μ^α and ν^α , respectively. However, the authors do not abstract from formulas in their definition of games, nor consider the full vectorial calculus that corresponds to automata.

A correspondence between countdown formulas, automata and games is as tight as for the classical μ -calculus. However, complications arise: the distinction between *vectorial* and *scalar* formulas, which in the classical case disappears to a large extent due to the so-called Bekić principle, now becomes pronounced. We prove that vectorial countdown calculus is more expressive than its scalar fragment. We also prove that the countdown operator nesting hierarchy of formulas is proper.

We conjecture that the satisfiability problem is decidable for $\mu^\alpha\text{-ML}$. Unfortunately, the lack of positional determinacy in countdown games prevents us from using proof techniques known from parity automata (where one can transform an alternating automaton into a nondeterministic one that guesses the positional strategy). Nevertheless, the existence of an automata model equivalent to logic is encouraging. Apart from allowing us to solve some fragments of the logic, it implies that $\mu^\alpha\text{-ML}$ does not share some of the troublesome properties of $\text{MSO} + \text{U}$ that result in undecidability. In particular, it can be used to show that all languages definable in $\mu\text{-ML}$ have *bounded topological complexity* (i.e. at most Σ_2^1 , see [15] for an introduction to topological methods in computer science). Since $\text{MSO} + \text{U}$ defines a Σ_n^1 -complete language for every $n < \omega$ [12, Theorem 2.1], [15, Theorem 7], it follows that some $\text{MSO} + \text{U}$ -definable languages are not expressible in $\mu^\alpha\text{-ML}$ (whether $\mu^\alpha\text{-ML}$ -definability implies $\text{MSO} + \text{U}$ -definability remains an open question). Since by [8, Theorem 1.3] every logic closed under boolean combinations, projections and defining the language U from Example 4 contains $\text{MSO} + \text{U}$, this means that our calculus is *not closed under projections*. This is an arguably good news, as in the light of [3, Theorem 1.4], giving up closure under

projections is the only way to go if one wants to design a decidable extension of MSO closed under boolean operations. Decidability of the weak variant WMSO + U of MSO + U over infinite words [2] and infinite (ranked) trees [5] shows that such extensions are possible. In fact, both results are obtained by establishing a correspondence with equivalent automata models, namely deterministic max-automata [2, Theorem 1] and nested limsup automata [5, Theorem 2]. Since the existence of accepting runs for such automata can be expressed in μ^α -ML, we get that μ^α -ML contains WMSO + U on infinite words and trees. The opposite inclusion is false (due to topological reasons), at least for the trees. The relation between μ^α -ML and the ωB -, ωS - and ωBS -automata of [7] remains unclear, as these models do not admit determinization. Also, the relation between our logic and regular cost functions (see e.g. [10]) is less immediate than it could seem at first glance and requires further research.

2 Preliminaries

Fixpoints. Let Ord be the class of all ordinals, and Ord_∞ the class Ord extended with an additional element ∞ greater than all ordinals.

Knaster-Tarski theorem says that every monotonic function $F : A \rightarrow A$ on a complete lattice A has the least and the greatest fixpoint, which we denote F_μ^∞ and F_ν^∞ . Moreover:

- F_μ^∞ is the limit of the increasing sequence $F_\mu^\alpha = \bigvee_{\beta < \alpha} F(F_\mu^\beta)$
- F_ν^∞ is the limit of the decreasing sequence $F_\nu^\alpha = \bigwedge_{\beta < \alpha} F(F_\nu^\beta)$

where $\alpha \in \text{Ord}$ and \bigvee, \bigwedge are the join and meet operations in A .

Parity games. A *parity game* is played between two players \exists ve and \forall dam (or simply \exists and \forall). It consists of a set of *positions* $V = V_\exists \sqcup V_\forall$ divided between both players, an edge relation $E \subseteq V \times V$, and a labeling *rank* : $V \rightarrow \mathcal{R}$ for some finite linear order $\mathcal{R} = \mathcal{R}_\exists \sqcup \mathcal{R}_\forall$ divided between the two players.

A *play* is a sequence of positions. After a play $\pi = v_1 \dots v_n \in V^*$, the owner of v_n chooses $(v_n, v_{n+1}) \in E$ and the game moves to v_{n+1} . A player who has no legal moves loses immediately. To determine the winner of an infinite play, we look at the highest $r \in \mathcal{R}$ such that positions with rank r appear infinitely often in the play, and the owner of r loses.

A *strategy* for a player $P \in \{\exists, \forall\}$ is a partial map $\sigma : V^*V_P \rightarrow E$ that tells the player how to move. A play $v_1v_2 \dots$ is *consistent* with σ if for every n such that $v_n \in V_P$ we have $\sigma(v_1 \dots v_n) = v_{n+1}$. A strategy σ is *winning* from a position v if every play that begins in v and is consistent with σ is a win for P . A strategy is *positional* if $\sigma(\pi)$ depends only on the last position in π . Parity games are *positionally determined*: if a player has a winning strategy from v then (s)he has a winning positional strategy.

Modal μ -calculus. A model \mathcal{M} for a fixed set Act of atomic *actions* consists of a set of *points* $M \ni m, n, \dots$ together with a binary relation $\xrightarrow{\tau} \subseteq M \times M$ for every $\tau \in \text{Act}$.

Formulas of the modal μ -calculus μ -ML are given by the grammar:

$$\varphi ::= x \mid \top \mid \perp \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \mu x. \varphi \mid \nu x. \varphi \mid \langle \tau \rangle \varphi \mid [\tau] \varphi \quad (1)$$

where x ranges over a fixed infinite set Var of variables and $\tau \in \text{Act}$. Given a valuation $\text{val} : \text{Var} \rightarrow \mathcal{P}(M)$, the semantics $\llbracket \varphi \rrbracket^{\text{val}} \subseteq M$ for all formulas φ is defined inductively, with $\mu x. \varphi$ and $\nu x. \varphi$ denoting the least and greatest fixpoints, respectively, of the monotonic function $H \mapsto \llbracket \varphi \rrbracket^{\text{val}[x \mapsto H]}$ on the complete lattice $\mathcal{P}(M)$. More details can be found e.g. in [1, 16], but they can also be discerned from Section 3 below, where the semantics of countdown μ -calculus is presented in detail.

The above syntax does not include negation, but μ -calculus formulas are semantically closed under negation. For every formula φ there is a formula $\widetilde{\varphi}$ that acts as the negation of φ on every model, defined by induction in a straightforward way:

$$\widetilde{\varphi_1 \vee \varphi_2} = \widetilde{\varphi_1} \wedge \widetilde{\varphi_2}, \quad \widetilde{\langle \tau \rangle \varphi} = [\tau] \widetilde{\varphi}, \quad \widetilde{\mu x. \varphi} = \nu x. \widetilde{\varphi}, \quad \text{etc.} \quad (2)$$

Vectorial μ -calculus. A syntactically richer version of the modal μ -calculus admits mutual fixpoint definitions of multiple properties, in formulas such as $\mu_1(x_1, x_2).(\varphi_1, \varphi_2)$, where variables x_1 and x_2 may occur both in φ_1 and φ_2 . Given a valuation val as before, this formula is interpreted as the least fixpoint of the monotonic function $(H_1, H_2) \mapsto (\llbracket \varphi_1 \rrbracket^{\text{val}[x_i \mapsto H_i]}, \llbracket \varphi_2 \rrbracket^{\text{val}[x_i \mapsto H_i]})$ on the complete lattice $\mathcal{P}(M)^2$; the resulting pair of sets is then projected to the first component as dictated by the subscript in μ_1 . Tuples of any size are allowed. This *vectorial* calculus is expressively equivalent to the scalar version described before, thanks to the so-called *Bekić principle* which says that the equality:

$$\mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} \mu x_1. f_1(x_1, \mu x_2. f_2(x_1, x_2)) \\ \mu x_2. f_2(\mu x_1. f_1(x_1, x_2), x_2) \end{pmatrix} \quad (3)$$

holds for every pair of monotone operations $f_i : A_1 \times A_2 \rightarrow A_i$ on complete lattices A_1, A_2 , and similarly for the greatest fixpoint operator ν in place of μ .

3 Countdown μ -calculus

We now introduce the *countdown μ -calculus* μ^α -ML. We begin with the scalar version.

3.1 The scalar fragment

As before, fix an infinite set Var of variables and a set Act of actions. The syntax of (*scalar*) *countdown μ -calculus* is defined as follows:

$$\varphi ::= x \mid \top \mid \perp \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \mu^\alpha x. \varphi \mid \nu^\alpha x. \varphi \mid \langle \tau \rangle \varphi \mid [\tau] \varphi \quad (4)$$

for $x \in \text{Var}$, $\tau \in \text{Act}$ and $\alpha \in \text{Ord}_\infty$; the presence of ordinal numbers α is the only syntactic difference with (1). A formula with no free variables is called a *sentence*. In case $|\text{Act}| = 1$, we may skip the labels and write \Diamond and \Box instead of $\langle \tau \rangle$ and $[\tau]$. In statements that apply both to least and greatest fixpoints, we will sometimes use η^α to denote either μ^α or ν^α .

Given a model \mathcal{M} , for every valuation $\text{val} : \text{Var} \rightarrow \mathcal{P}(M)$, the *semantics* $\llbracket \varphi \rrbracket^{\text{val}} \subseteq M$ is defined inductively as follows:

$$\begin{aligned} \llbracket x \rrbracket^{\text{val}} &= \text{val}(x); \\ \llbracket \top \rrbracket^{\text{val}} &= M \quad \text{and} \quad \llbracket \perp \rrbracket^{\text{val}} = \emptyset \\ \llbracket \varphi_1 \vee \varphi_2 \rrbracket^{\text{val}} &= \llbracket \varphi_1 \rrbracket^{\text{val}} \cup \llbracket \varphi_2 \rrbracket^{\text{val}} \quad \text{and} \quad \llbracket \varphi_1 \wedge \varphi_2 \rrbracket^{\text{val}} = \llbracket \varphi_1 \rrbracket^{\text{val}} \cap \llbracket \varphi_2 \rrbracket^{\text{val}}; \\ \llbracket \langle \tau \rangle \varphi \rrbracket^{\text{val}} &= \{m \in M \mid \exists n \in \llbracket \varphi \rrbracket^{\text{val}} m \xrightarrow{\tau} n\} \quad \text{and} \quad \llbracket [\tau] \varphi \rrbracket^{\text{val}} = \{m \in M \mid \forall n \in \llbracket \varphi \rrbracket^{\text{val}} m \xrightarrow{\tau} n\}; \\ \llbracket \mu^\alpha x. \varphi \rrbracket^{\text{val}} &= F_\mu^\alpha \quad \text{and} \quad \llbracket \nu^\alpha x. \varphi \rrbracket^{\text{val}} = F_\nu^\alpha \end{aligned}$$

where in the last clause $F(H) = \llbracket \varphi \rrbracket^{\text{val}[x \mapsto H]}$. We will skip the index val if it is immaterial or clear from the context.

This obviously contains the classical μ -calculus, but is capable of capturing *boundedness* and *unboundedness* properties which are not expressible in the classical setting:

► **Example 1.** For $|\text{Act}| = 1$, consider the formula $\nu^\alpha x. \Diamond x$. In a model \mathcal{M} , for $\alpha < \omega$ the set $\llbracket \nu^\alpha x. \Diamond x \rrbracket$ consists of the points from which there is a path of length at least α . Hence, $\nu^\omega x. \Diamond x$ holds in a point if there are arbitrarily long finite paths starting from there.

3.2 The vectorial calculus

The (full) *countdown μ -calculus* is defined as for its scalar fragment, except that fixpoint operators act on tuples (vectors) of formulas rather than on single formulas.

► **Definition 2.** *The syntax of countdown μ -calculus is given as follows:*

$$\varphi ::= x \mid \top \mid \perp \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \mu_i^\alpha \bar{x}.\bar{\varphi} \mid \nu_i^\alpha \bar{x}.\bar{\varphi} \mid \langle \tau \rangle \varphi \mid [\tau] \varphi$$

for $1 \leq i \leq n < \omega$, $\bar{x} = \langle x_1, \dots, x_n \rangle \in \text{Var}^n$, $\bar{\varphi} = \langle \varphi_1, \dots, \varphi_n \rangle$ a tuple of formulas, $\tau \in \text{Act}$ and $\alpha \in \text{Ord}_\infty$.

► **Definition 3.** *The meaning $\llbracket \varphi \rrbracket^{\text{val}} \subseteq M$ of a formula φ in a model \mathcal{M} under valuation val is defined by induction the same way as for the scalar formulas except for the operators μ_i^α and ν_i^α , in which case:*

$$\llbracket \mu_i^\alpha \bar{x}.\bar{\varphi} \rrbracket^{\text{val}} = \pi_i(F_\mu^\alpha) \quad \text{and} \quad \llbracket \nu_i^\alpha \bar{x}.\bar{\varphi} \rrbracket^{\text{val}} = \pi_i(F_\nu^\alpha)$$

where the monotone map $F : (\mathcal{P}(M))^n \rightarrow (\mathcal{P}(M))^n$ is given as:

$$F(H_1, \dots, H_n) = (\llbracket \varphi_1 \rrbracket^{\text{val}'}, \dots, \llbracket \varphi_n \rrbracket^{\text{val}'})$$

for $\text{val}' = \text{val}[x_1 \mapsto H_1, \dots, x_n \mapsto H_n]$ and $\pi_i : (\mathcal{P}(M))^n \rightarrow \mathcal{P}(M)$ is the i -th projection.

Note that operators μ^∞ and ν^∞ are equivalent to μ and ν from the classical μ -calculus. Furthermore, for every ordinal α , the formula $\mu_i^{\alpha+1} \bar{x}.\bar{\psi}$ is equivalent to

$$\psi_i[x_1 \mapsto \mu_1^\alpha \bar{x}.\bar{\psi}, \dots, x_n \mapsto \mu_n^\alpha \bar{x}.\bar{\psi}]$$

and similarly for $\nu^{\alpha+1}$. As a result, without loss of generality we may assume that in countdown operators μ^α and ν^α only limit ordinals α are used.

The countdown μ -calculus is semantically closed under negation in the same way as the classical calculus, extending (2) with the straightforward $\mu_i^\alpha \bar{x}.\bar{\varphi} = \nu_i^\alpha \bar{x}.\bar{\varphi}$ and $\nu_i^\alpha \bar{x}.\bar{\varphi} = \mu_i^\alpha \bar{x}.\bar{\varphi}$.

In Section 6 we will compare the expressive power of the vectorial and scalar countdown μ -calculus in detail. For now, let us show that Bekić principle (3) fails for countdown operators:

► **Example 4.** An infinite word $W \in \Gamma^\omega$ over the alphabet $\Gamma = \{\mathbf{a}, \mathbf{b}\}$ can be seen as a model for $\text{Act} = \Gamma$ with ω as the set of points and with transition relations defined by:

$$n \xrightarrow{\tau} m \iff m = n + 1 \text{ and } W_n = \tau.$$

For every regular language $K \subseteq \Gamma^*$ and $x \in \text{Var}$, it is straightforward to define a fixpoint formula (in the classical μ -calculus, so without countdown operators) $\langle K \rangle x$ that holds in a point n , for a valuation val , if and only if there exists a word $w \in K$ and a path in W labelled with w that starts in n and ends in a point that belongs to $\text{val}(x)$. Then, the formula:

$$\varphi = \nu_1^\omega(x_1, x_2).(\langle \Gamma^* \rangle x_2, \langle \mathbf{a} \rangle x_2)$$

is true in a word W iff it contains arbitrarily long blocks of consecutive \mathbf{a} 's. To see this, observe that at the i -th step of approximation: (i) the second component (x_2) contains a point n iff the next i transitions are all labelled with \mathbf{a} , and (ii) the first component (x_1) contains a point n iff the second component contains at least one point after n .

However, the following scalar formula constructed by analogy to the Bekić principle:

$$\psi = \nu^\omega x_1.(\langle \Gamma^* \rangle (\nu^\omega x_2. \langle \mathbf{a} \rangle x_2))$$

is equivalent to $\langle \Gamma^* \rangle (\nu^\omega x_2. \langle \mathbf{a} \rangle x_2)$, and the formula under $\langle \Gamma^* \rangle$ holds in a point iff all the future transitions from that point are labelled with \mathbf{a} . Thus, ψ holds (in any point) iff the word W is of the form $\Gamma^* \mathbf{a}^\omega$, and so ψ is not equivalent to φ .

4 Countdown Games

The notion of a countdown game extends that of a parity game. As for parity games, it assumes a fixed finite linear order of ranks $\mathcal{R} = \mathcal{R}_\exists \sqcup \mathcal{R}_\forall$. In addition, we fix a subset $\mathcal{D} \subseteq \mathcal{R}$ of *nonstandard* ranks; at positions with these ranks countdowns will occur. Denote $\mathcal{D}_\exists = \mathcal{D} \cap \mathcal{R}_\exists$ and $\mathcal{D}_\forall = \mathcal{D} \cap \mathcal{R}_\forall$.

A *countdown game* consists of a set of *positions* $V = V_\exists \sqcup V_\forall$ divided between players \exists ve and \forall dam, an edge relation $E \subseteq V \times V$, a labelling $\text{rank} : V \rightarrow \mathcal{R}$, and an initial counter valuation $\text{ctr}_I : \mathcal{D} \rightarrow \text{Ord}$. Each nonstandard rank has an associated counter.

Each game *configuration* consists of a position $v \in V$ together with a counter valuation $\text{ctr} : \mathcal{D} \rightarrow \text{Ord}$. We consider *positional* and *countdown* configurations, denoted respectively $\langle v, \text{ctr} \rangle$ and $[v, \text{ctr}]$, with the following moves allowed:

- From a positional configuration $\langle v, \text{ctr} \rangle$, the owner of v chooses an edge $(v, w) \in E$ and the game proceeds from the countdown configuration $[w, \text{ctr}]$;
- From a countdown configuration $[v, \text{ctr}]$, the owner of $r = \text{rank}(v)$ chooses a counter valuation ctr' such that:
 - $\text{ctr}'(r') = \text{ctr}_I(r')$ for $r' < r$,
 - $\text{ctr}'(r) < \text{ctr}(r)$ (if r is nonstandard),
 - $\text{ctr}'(r') = \text{ctr}(r')$ for $r' > r$,

and the game proceeds from the positional configuration $\langle v, \text{ctr}' \rangle$. In words: counters for ranks lower than r are reset, the counter for r (if any) is decremented, and counters for higher ranks are left unchanged. Note that if r is standard then there is no real choice here: ctr' is determined by ctr . And if r is nonstandard then the move amounts to choosing an ordinal $\alpha < \text{ctr}(r)$.

Every play of the game alternates between positional and countdown configurations, and in each move only one component of the configuration is modified. Therefore, although a play is formally a sequence of configurations, it can be more succinctly represented as an alternating sequence of positions and counter valuations:

$$\pi = v_1 \text{ctr}_2 v_2 \text{ctr}_3 v_3 \text{ctr}_4 \dots \quad (5)$$

This has the same length as the sequence of configurations, and we will call it the length of the play. A *phase* of a game is a set of its finite plays that is convex with respect to the prefix ordering. Given a phase \mathcal{S} and a play $\pi \in \mathcal{S}$, we denote by \mathcal{S}_π the subset of \mathcal{S} consisting of all the plays having π as a prefix.

In any configuration, if the player responsible for making the next move is stuck, (s)he loses immediately. Otherwise, in an infinite play, the owner of the greatest rank appearing infinitely often loses, as in parity games. Strategies and winning strategies are defined as for classical parity games, as partial functions from finite plays to moves.

Given configuration γ , we denote the game *initialized in the configuration* γ by \mathcal{G}, γ . The default initial counter assignment is ctr_I and the default initial mode is the positional one, meaning that \mathcal{G}, v stands for $\mathcal{G}, \langle v, \text{ctr}_I \rangle$.

Note that the only way the counters may interfere with a play is when a counter has value 0 and so its owner cannot decrement it. It is therefore beneficial for a player to have greater ordinals at his/her counters.

Countdown games are not positionally determined, in the sense that the players may need to look at the counter values in order to choose a winning move (although they are *configurationally determined*, since a countdown game \mathcal{G} can be seen as a parity game with

configurations of \mathcal{G} as its positions). Later, we will show how to upgrade strategies to enforce a very limited form of counter-independence.

5 Countdown Automata

Countdown automata are a stepping stone between formulas and games. A countdown formula will define an automaton, which will then recognize a model in terms of a countdown game. Since formulas can have free variables, for technical reasons we will also consider automata with free variables. These variables resemble terminal states in that they can be targets of transitions, but no transitions originate in them, and whether they accept or not depends on an external valuation.

► **Definition 5.** A countdown automaton consists of:

- a finite set of states $Q = Q_\exists \sqcup Q_\forall$ divided between two players;
- an initial state $q_I \in Q$;
- a transition function $\delta : Q \rightarrow \mathcal{P}(Q \sqcup \text{Var}) \sqcup (\text{Act} \times (Q \sqcup \text{Var}))$ (we call the left part ϵ -transitions and the right one modal transitions);
- an assignment of ranks $\text{rank} : Q \rightarrow \mathcal{R}$ and an assignment of initial counter values $\text{ctr}_I : \mathcal{D} \rightarrow \text{Ord}$, as in a countdown game.

The language of an automaton is defined in terms of a countdown game, analogously to parity games and parity automata.

► **Definition 6.** Fix an automaton $\mathcal{A} = (Q, q_I, \delta, \text{rank}, \text{ctr}_I)$. Given a model \mathcal{M} , a valuation $\text{val} : \text{Var} \rightarrow \mathcal{P}(M)$ and a point $\mathbf{m}_I \in M$, we define the semantic game $\mathcal{G}^{\text{val}}(\mathcal{A})$ to be the countdown game $(V, E, \text{rank}', \text{ctr}_I)$ where positions are of the form $V = M \times (Q \sqcup \text{Var})$ and the edge relation E is defined as follows. In a position (\mathbf{m}, q) for $q \in Q$:

- if $\delta(q) \subseteq Q \sqcup \text{Var}$, outgoing edges (called ϵ -edges, or ϵ -moves) are $\{((\mathbf{m}, q), (\mathbf{m}, z)) \mid z \in \delta(q)\}$,
 - if $\delta(q) = (\tau, p)$, outgoing edges (modal edges, modal moves) are $\{((\mathbf{m}, q), (\mathbf{n}, p)) \mid \mathbf{m} \xrightarrow{\tau} \mathbf{n}\}$.
- There are no outgoing edges from positions (\mathbf{m}, x) for $x \in \text{Var}$.

For $q \in Q$, the owner of the position (\mathbf{m}, q) is the owner of the state q , and $\text{rank}'(\mathbf{m}, q) = \text{rank}(q)$. For $x \in \text{Var}$, the position (\mathbf{m}, x) belongs to \forall if $\mathbf{m} \in \text{val}(x)$ and to \exists otherwise. The rank $\text{rank}'(\mathbf{m}, x)$ can be set arbitrarily, as it does not affect the outcome of the game. The initial counter assignment ctr_I is kept the same.

The language $\llbracket \mathcal{A} \rrbracket^{\text{val}} \subseteq M$ of an automaton \mathcal{A} is the set of all points $\mathbf{m} \in M$ for which the configuration $\langle (\mathbf{m}, q_I), \text{ctr}_I \rangle$ in the game $\mathcal{G}^{\text{val}}(\mathcal{A})$ is winning for \exists .

It is worth to mention that although in general countdown games are not positional, one can show a much weaker but still useful fact: in the particular case of semantic games, the winning player always has a strategy that does not look at the counters in the initial *pre-modal* phase of the game (that is, *before the first modal move*). The precise statement can be found in Proposition 19 in Appendix D; its game-theoretic core is Proposition 16 in Appendix A.

The countdown calculus and countdown automata have the same expressive power, i.e. there are language-preserving translations $\varphi \mapsto \mathcal{A}_\varphi$ and $\mathcal{A} \mapsto \varphi_\mathcal{A}$ between formulas and automata. As in the classical setting, the link between formulas and automata is very useful in establishing facts about the logic. For example, one can use game semantics to show that every formula of the standard μ -ML can be transformed into an equivalent guarded one. Thanks to the equivalence between *countdown* formulas and *countdown* automata, the same is true for μ^α -ML, as stated in Proposition 18 in Appendix D.

We will now explain the translations between logic and automata in turn.

5.1 From formulas to automata – Game Semantics

Every countdown formula $\varphi \in \mu^\alpha\text{-ML}$ gives rise to a countdown automaton \mathcal{A}_φ such that $\llbracket \varphi \rrbracket^{\text{val}} = \llbracket \mathcal{A}_\varphi \rrbracket^{\text{val}}$ for every model \mathcal{M} and valuation val . Specifically, given a formula φ (with some free variables), we define an automaton $\mathcal{A}_\varphi = (Q, q_I, \delta, \text{rank}, \text{ctr}_I)$ (over the same free variables) as follows:

- $Q = \text{SubFor}(\varphi) - \text{FreeVar}(\varphi)$ is the set of all subformulas other than the free variables of φ (*without* identifying different occurrences of identical subformulas, i.e., here a subformula means a path in the syntactic tree of φ from the root of φ to the root node of the subformula). Ownership of a state in Q depends on the topmost connective, with \exists ve owning \vee and $\langle \tau \rangle$ and \forall dam owning \wedge and $[\tau]$; ownership of fixpoint subformulas, countdown subformulas and variables can be set arbitrarily as it will not matter;
- $q_I = \varphi$;
- the transition function is defined by cases:
 - $\delta(\theta_1 \vee \theta_2) = \delta(\theta_1 \wedge \theta_2) = \{\theta_1, \theta_2\}$,
 - $\delta(\langle \tau \rangle \theta) = \delta([\tau] \theta) = (\tau, \theta)$,
 - $\delta(\eta_i^\alpha \bar{x}. \bar{\theta}) = \{\theta_i\}$ (for $\eta = \mu$ or $\eta = \nu$),
 - $\delta(x) = \{\theta_i\}$, where $\eta_j^\alpha(x_1, \dots, x_n).(\theta_1, \dots, \theta_n)$ is the (unique) subformula of φ binding x with $x = x_i$.
- For the ranking function, assume that the lowest rank in \mathcal{R} is standard and call it 0 (ownership of this rank does not matter). Then let **rank** assign 0 to all subformulas of φ except for immediate subformulas of fixpoint operators. To those, assign ranks in such a way that subformulas have strictly smaller ranks than their superformulas, and for every subformula $\eta_i^\alpha \bar{x}. \bar{\varphi}$:
 - all formulas in the tuple $\bar{\varphi}$ have the same rank r ,
 - r belongs to \exists ve if $\eta = \mu$ and to \forall dam if $\eta = \nu$, and
 - if $\alpha = \infty$ then r is standard, otherwise it is nonstandard and $\text{ctr}_I(r) = \alpha$.

We denote $\mathcal{G}^{\text{val}}(\varphi) = \mathcal{G}^{\text{val}}(\mathcal{A}_\varphi)$.

► **Theorem 7 (Adequacy).** *For every model \mathcal{M} and valuation val , $\llbracket \varphi \rrbracket^{\text{val}} = \llbracket \mathcal{A}_\varphi \rrbracket^{\text{val}}$.*

Proof. As with the classical μ -calculus, the proof proceeds by induction on the complexity of the formula. The only new cases of $\mu^\alpha \bar{x}. \bar{\varphi}$ and $\nu^\alpha \bar{x}. \bar{\varphi}$ are proven by transfinite induction on α . For the details, see Appendix B. ◀

► **Example 8.** For $\text{Act} = \{\tau\}$, consider the formula $\varphi = \nu^\omega x. \Diamond x$ from Example 1. The automaton \mathcal{A}_φ has three states: $Q = \{\varphi, \Diamond x, x\}$, with φ the initial state, and the transition function comprises two deterministic ϵ -transitions and one modal transition:

$$\delta(\varphi) = \{\Diamond x\}, \quad \delta(\Diamond x) = (\tau, x), \quad \delta(x) = \{\Diamond x\}.$$

The state $\Diamond x$ is owned by \exists ve; ownership of the other two states does not matter. The automaton uses two ranks, $0 < 1$, where 0 is standard and 1 is nonstandard, assigned to states by: $\text{rank}(\varphi) = \text{rank}(x) = 0$ and $\text{rank}(\Diamond x) = 1$. Rank 1 is owned by \forall dam; ownership of rank 0 does not matter. (Note how the state $\Diamond x$ is owned by \exists ve, but its rank is owned by \forall dam). The initial counter value is $\text{ctr}_I(1) = \omega$.

Now consider any model \mathcal{M} . Since Act has only one element, \mathcal{M} is simply a directed graph. The semantic game $\mathcal{G}(\varphi)$ on \mathcal{M} (φ has no free variables, so neither has \mathcal{A}_φ and we need not consider valuations val) has positions of the form (\mathbf{m}, q) where $\mathbf{m} \in M$ and $q \in Q$, with ownership and rank inherited from q . Edges are of the form:

- $((m, \varphi), (m, \Diamond x))$ and $((m, x), (m, \varphi))$ – the ϵ -edges,
- $((m, \Diamond x), (n, x))$ such that $m \rightarrow n$ is an edge in \mathcal{M} – the modal edges.

Configurations of the game arise from positions together with counter valuations; there is only one nonstandard rank, so a counter valuation is simply an ordinal.

For a point $m \in \mathcal{M}$, the default initial configuration of the game is the positional configuration $\langle (m, \varphi), \omega \rangle$. A play that begins in this configuration proceeds as follows:

1. The first move is deterministic, to the countdown configuration $[(m, \Diamond x), \omega]$.
2. $\forall \text{dam}$, as the owner of the rank of $\Diamond x$, makes the next move: he chooses a number $k < \omega$, and the game moves to the positional configuration $\langle (m, \Diamond x), k \rangle$.
3. $\exists \text{eve}$ owns the position, so she makes the next move: she chooses a point $n \in M$ such that $m \xrightarrow{\tau} n$, and the game moves to the countdown configuration $[(n, x), k]$.
4. The rank of x is standard, so in the next move the counter does not change and the game moves to $\langle (n, x), k \rangle$. The next move is also deterministic, to the countdown configuration $[(n, \Diamond x), k]$. The game then goes back to step 2. above, with k in place of ω .

From this it is clear that $\exists \text{eve}$ wins from $\langle (m, \varphi), \omega \rangle$ if and only if \mathcal{M} has arbitrarily long paths that begin in m , as stated in Example 1.

5.2 From automata to formulas

► **Theorem 9.** *For every countdown automaton \mathcal{A} there exists a countdown formula $\varphi_{\mathcal{A}}$ s.t. $\llbracket \mathcal{A} \rrbracket^{\text{val}} = \llbracket \varphi_{\mathcal{A}} \rrbracket^{\text{val}}$ for every model \mathcal{M} and valuation val .*

Proof. See Appendix C; here we just sketch the construction of $\varphi_{\mathcal{A}}$. For an automaton $\mathcal{A} = (Q, q_I, \delta, \text{rank}, \text{ctr}_I)$, by induction on $r \in \mathcal{R}$ we build a formula $\psi_{r,q}$ for each $q \in Q$. Then we put $\varphi_{\mathcal{A}} = \psi_{r_{\max}, q_I}$. Thus for the base case of the lowest rank $r = 0$:

- if $\delta(s) = (\tau, p)$ then for $\psi_{0,s}$ we put $\langle \tau \rangle x_p$ if q belongs to $\exists \text{eve}$ and $[\tau] x_p$ if q belongs to $\forall \text{dam}$,
- if $\delta(s) \subseteq Q$ then for $\psi_{0,s}$ we put $\bigvee_{p \in \delta(s)} x_p$ if q belongs to $\exists \text{eve}$ and $\bigwedge_{p \in \delta(s)} x_p$ if q belongs to $\forall \text{dam}$.

For the inductive step, let q_1, \dots, q_d be all states in Q with rank r . For every q_i define the vectorial formula:

$$\theta_i = \eta_{q_i}^{\alpha}(x_{q_1}, \dots, x_{q_d}).(\psi_{r,q_1}, \dots, \psi_{r,q_d})$$

with $\alpha = \text{ctr}_I(r)$ and $\eta = \mu$ if r belongs to $\exists \text{eve}$ and $\eta = \nu$ if r belongs to $\forall \text{dam}$. Then put $\psi_{r+1,q} = \psi_{r,q}[x_{q_1} \mapsto \theta_1, \dots, x_{q_d} \mapsto \theta_d]$. ◀

6 Vectorial vs. scalar calculus

In this section we investigate the relation between scalar and vectorial formulas. We have already seen with Example 4 that unlike with standard fixpoints, the Bekić principle is not valid in the countdown setting. Interestingly, scalar formulas correspond to automata with a simple syntactic restriction.

► **Proposition 10.** *Scalar countdown formulas and automata where every two states have different ranks have equal expressive power.*

Proof. Inspecting the translations between formulas and automata from Sections 5.1 and 5.2, it is evident that injectively ranked automata are translated to scalar formulas, and that, although in our translation the choice of the assignment of ranks is not deterministic, every scalar formula can be translated to an injectively ranked automaton. ◀

Since the Bekić principle fails, a natural question is whether there is another way of transforming vectorial formulas to scalar form (or, equivalently, arbitrary countdown automata to injectively ranked ones). We shall give a negative answer in Theorem 11. However, before we proceed, let us analyse the following example, which shows that scalar formulas are more expressive than they may seem, covering in particular the property from Example 4.

6.1 Languages of unbounded infixes

Fix a regular language of finite words $L \subseteq \Gamma^*$. Let $\mathcal{U}(L) \subseteq \Gamma^\omega$ be the language of all infinite words that contain arbitrarily long infixes from L . For instance, the language from Example 4 is $\mathcal{U}(a^*)$. We shall now show that $\mathcal{U}(L)$ can be defined in the countdown μ -calculus, first by a vectorial formula, then by a scalar one.

Consider a finite deterministic automaton $\mathcal{A} = (Q, \delta, q_I, F)$ that recognizes L . Let $\delta^+ : \Gamma^+ \times Q \rightarrow Q$ be the unique inductive extension of the transition function $\delta : \Gamma \times Q \rightarrow Q$ to nonempty words. Define $K_{p,q} = \{w \in \Gamma^+ \mid \delta^+(w, p) = q\}$ the (regular) language of nonempty words leading from p to q in \mathcal{A} , and let $K_{p,F}$ denote the union $\bigcup_{q \in F} K_{p,q}$. By the pigeonhole principle we have $\mathcal{U}(L) = \bigcup_{q \in Q} \mathcal{U}_q(L)$, where $\mathcal{U}_q(L) \subseteq \Gamma^\omega$ consists of words such that for every $n < \omega$, w has an infix $w_n = v_I u_1 \dots u_n v_F \in L$ s.t. (i) $v_I \in K_{q_I,q}$, (ii) $u_1, \dots, u_n \in K_{q,q}$, and (iii) $v_F \in K_{q,F}$. Then $\mathcal{U}_q(L)$ can be defined by a vectorial formula:

$$\mathcal{U}_q(L) = \llbracket \nu_1^\omega(x_1, x_2).(\langle \Gamma^* K_{q_I,q} \rangle x_2, \langle K_{q,q} \rangle x_2 \wedge \langle K_{q,F} \rangle \top) \rrbracket$$

where $\langle K \rangle \psi$ is the formula as explained in Example 4. Indeed, the corresponding semantic game on a word w proceeds as follows:

1. \forall dam chooses a number $n < \omega$ as the value of his only counter,
2. \exists ve skips a prefix $v_0 v_I \in \Gamma^* K_{q_I,q}$ of w ,
3. \forall dam decrements his counter;
4. \exists ve keeps moving through $u_1, u_2, \dots \in K_{q,q}$ so that after each step, some state in F is reachable from q by some prefix of the remaining word. After each such choice of u_i \forall dam has to decrement his counter, and so \exists ve wins iff she can make at least $n - 1$ such steps. The two different stages in which \forall dam's counter is decremented reflect the two-phase dynamics of the game: first \forall dam challenges \exists ve with a number, and then \exists ve shows that she can provide an infix long enough.

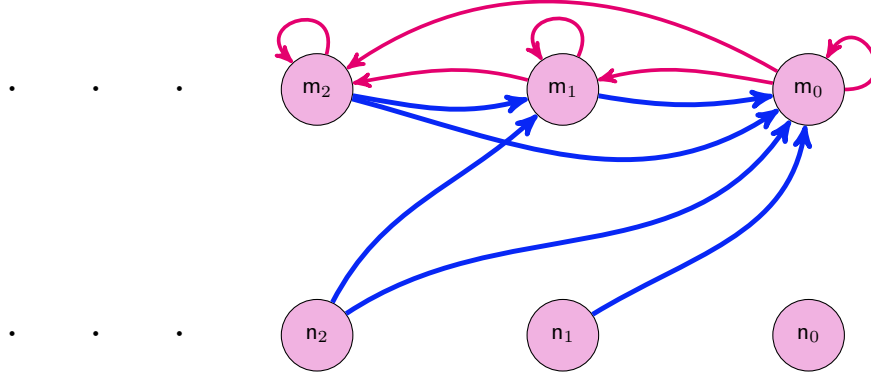
It is more tricky to define the language $\mathcal{U}_q(L)$ with a scalar formula, but it turns out to be possible. To this end, observe that without loss of generality we may restrict attention to words w such that:

1. the infixes $w_n \in L$ start arbitrarily far in w ;
2. each w_n can be decomposed as $v_I u_1 \dots u_n v_F \in L$ s.t. (i) $v_I \in K_{q_I,q}$, (ii) $u_1, \dots, u_n \in K_{q,q}$, (iii) $v_F \in K_{q,F}$, and additionally (iv) all u_i begin with the same letter $a \in \Gamma$;
3. there are at least two distinct letters $a, b \in \Gamma$ that appear infinitely often in w ;
4. the first letter of w is b .

Indeed, for (1) note that otherwise w_n start in the same position k for all n large enough. But then even the stronger property “There exists a position k such that the run of \mathcal{A} from k visits q and F infinitely often” holds, and this is easily definable by a fixpoint formula.

Item (2) follows from the pigeonhole principle and the observation that in $w_{n \times |\Gamma|} = v_I u_1 \dots u_{n \times |\Gamma|} v_F$ at least n u_i 's begin with the same letter.

For (3) observe that otherwise w has a suffix a^ω for some $a \in \Gamma$, in which case membership in $\mathcal{U}_q(L)$ is definable by a fixpoint formula. This is because an ultimately periodic word is bisimilar to a finite model, and so every monotone map reaches its fixpoints in finitely many steps, meaning that the countdown operator ν^ω is equivalent to ν^∞ .



■ **Figure 1** The model \mathcal{M} . Blue arrows represent edges labeled both with \mathbf{a} and \mathbf{b} , and pink arrows are edges labeled only with \mathbf{b} .

Finally, for (4) note that the language $\mathcal{U}_q(L)$ is closed under adding and removing finite prefixes, and so if a formula φ defines $\mathcal{U}_q(L) \cap \mathbf{b}\Gamma^\omega$, then the formula $\langle \Gamma^* \rangle \langle \mathbf{b} \rangle \top \wedge \varphi$ defines $\mathcal{U}_q(L)$.

With this in mind, define:

$$\varphi = \nu^\omega x. (\langle \mathbf{b} \rangle \top \wedge \langle \Gamma^* K_{qI,q} \rangle (\langle \mathbf{a} \rangle \top \wedge x)) \vee (\langle K_{q,q} \rangle (\langle \mathbf{a} \rangle \top \wedge x) \wedge \langle \mathbf{a} \rangle \top \wedge \langle K_{q,F} \rangle \top).$$

Note how $\langle \mathbf{b} \rangle \top \wedge x$ and $\langle \mathbf{a} \rangle \top \wedge x$ replace x_1 and x_2 from the vectorial formula. Consider the corresponding semantic game on a word w . Consider configurations of the game with the main disjunction as the formula component. Every infinite play of the game must visit such configurations infinitely often. In such a configuration, if the next letter in the model is either \mathbf{a} or \mathbf{b} then \exists ve must choose the right or left disjunct respectively. In particular, once the game reaches a configuration where $\langle \mathbf{a} \rangle \top$ holds, it must also hold every time the variable x is unraveled in the future. As a result, \exists ve wins from a configuration where $\langle \mathbf{a} \rangle \top$ holds against \forall dam's counter $n < \omega$ iff there is $u_1 \dots u_{n+1} v_F$ starting in the current position such that $u_1, \dots, u_{n+1} \in K_{q,q}$, each u_i starts with \mathbf{a} , and $v_F \in K_{q,F}$. Moreover, \exists ve wins from a position where $\langle \mathbf{b} \rangle \top$ holds, against \forall dam's $n + 1 < \omega$, iff there is $v_I \in \Gamma^* K_{qI,q}$ starting in the current position such that the next position after v_I satisfies $\langle \mathbf{a} \rangle \top$ and \exists ve wins from there against n . Putting this together, we get that \exists ve wins from a position satisfying $\langle \mathbf{b} \rangle \top$ against n iff there is $v_I u_1 \dots u_n v_F = w_n$ as in condition (2) above. Since the game starts with \forall dam choosing an arbitrary $n < \omega$, it follows that indeed φ defines $\mathcal{U}_q(L)$.

6.2 Greater expressive power of the vectorial calculus

We now show an example of a property that is definable in the vectorial countdown calculus but not in the scalar one.

Fixing $\text{Act} = \{\mathbf{a}, \mathbf{b}\}$, consider a model $\mathcal{M} = (M, \xrightarrow{\mathbf{a}}, \xrightarrow{\mathbf{b}})$ with points $M = \{m_i, n_i \mid i < \omega\}$, and with exactly the edges: $m_i \xrightarrow{\mathbf{a}} m_j$, $n_i \xrightarrow{\mathbf{a}} m_j$ and $n_i \xrightarrow{\mathbf{b}} m_j$ for all $i > j$; and $m_i \xrightarrow{\mathbf{b}} m_j$ for all i and j . Note that the relation $\xrightarrow{\mathbf{a}}$ is a subset of $\xrightarrow{\mathbf{b}}$. The model is shown in Fig. 1.

Consider the vectorial sentence $\nu_1^\omega(x_1, x_2). (\langle \mathbf{b} \rangle x_2, \langle \mathbf{a} \rangle x_2)$. This describes the property *there are arbitrarily long paths with labels in \mathbf{ba}^** , and so it is true in all points m_i and false in all points n_i . The following result immediately implies that this property cannot be defined in the scalar countdown calculus:

► **Theorem 11.** *For every scalar sentence φ , there exists $i < \omega$ s.t. $n_i \in \llbracket \varphi \rrbracket \iff m_i \in \llbracket \varphi \rrbracket$.*

Proof. The heart of the proof is Proposition 10 which says that scalar formulas correspond to injectively ranked automata. In such an automaton, whenever the counter corresponding to rank r is modified, the automaton must be in *the same state*, which allows the players to copy their strategies between different positions of the semantic game. For the details, see Appendix E. ◀

7 Strictness of the countdown nesting hierarchy

A natural question is whether greater *countdown nesting*, i.e. the maximal nesting of μ^α and ν^α operators with $\alpha \neq \infty$, results in more expressive power. We give a positive answer: under mild assumptions, the hierarchy is strict. From now on, focus on the monomodal case (i.e. $|\text{Act}| = 1$) and we assume that the only ordinal used by formulas is ω .¹

► **Theorem 12.** *For every $k < \omega$, formulas with countdown nesting $k + 1$ have strictly more expressive power than those with nesting at most k .*

In order to prove strictness, it suffices to prove it on a restricted class of models. We will show that the hierarchy is strict already on the class of transitive, linear, well-founded models – i.e. (up to isomorphism) ordinals.

More specifically, an ordinal $\kappa \in \text{Ord}$ can be seen as a model with $\alpha \rightarrow \beta$ iff $\alpha > \beta$. Since κ is an induced submodel of κ' whenever $\kappa \leq \kappa'$, we can consider a single ordinal model with κ big enough. For our purposes, the first uncountable ordinal ω_1 is sufficient.

We call a subset $S \subseteq \omega_1$ *stable above α* if either $[\alpha, \omega_1) \subseteq S$ or $[\alpha, \omega_1) \cap S = \emptyset$. A *stabilization point* of a valuation $\text{val} : \text{Var} \rightarrow \mathcal{P}(\omega_1)$ is the least $\alpha \leq \omega_1$ such that interpretations of all the variables are stable above α .

Observe that the set $[\omega^k, \omega_1) \subseteq [0, \omega_1)$ can be defined by the following sentence with countdown nesting k :

$$[\omega^k, \omega_1) = \llbracket \nu^\omega x_1 \dots \nu^\omega x_k \cdot \Diamond (\bigwedge_{i \leq k} x_i) \rrbracket. \quad (6)$$

Indeed, the semantic game can be decomposed into two alternating steps: (i) $\forall \text{dam}$ chooses a tuple of finite ordinals $(\alpha_1, \dots, \alpha_k) \in \omega^k$ and (ii) $\exists \text{eve}$ responds with a successor in the model. Since at each step $\forall \text{dam}$ has to pick a lexicographically smaller tuple (and he starts by picking any tuple) it is easy to see that he wins iff the initial point is at least ω^k . We will show that for all $k > 0$, countdown nesting k is *necessary* to define this language. The proof relies on the following lemma.

► **Lemma 13.** *For every $k < \omega$ and a formula φ with countdown nesting k , there exists an ordinal $\alpha_\varphi < \omega^{k+1}$ such that φ stabilizes α_φ above the valuation, i.e. for every valuation val stabilizing at β , $\llbracket \varphi \rrbracket^{\text{val}}$ is stable above $\beta + \alpha_\varphi$.*

Proof. See Appendix F. ◀

From this the theorem follows immediately, as the sentence φ has no free variables and thus it stabilizes at $\alpha_\varphi < \omega^{k+1}$ regardless of the valuation.

¹ This assumption could be replaced with a weaker requirement: there exists a maximal ordinal α that we are allowed to use, and α is additively indecomposable.

8 Decidability issues

We briefly discuss decidability issues in the countdown μ -calculus. Note that in a finite model every monotone map reaches its fixpoints in finitely many steps. Hence, if we replace every η^α in φ with η^∞ and denote the resulting formula by $\widehat{\varphi}$, then *in every finite model* $\llbracket \varphi \rrbracket = \llbracket \widehat{\varphi} \rrbracket$. It immediately follows that:

► **Proposition 14.** *The model checking problem for the μ^α -ML, i.e. the problem: “Given $\varphi \in \mu^\alpha$ -ML and a point m in a (finite) model \mathcal{M} , does $m \models \varphi$?” is decidable.*

Note that as a corollary we get that deciding the winner of a given (finite) countdown game \mathcal{G} is also decidable, as set of positions where \exists ve wins can be easily defined in μ^α -ML.

A more interesting problem is *satisfiability*: “Given $\varphi \in \mu^\alpha$ -ML, is there a model \mathcal{M} and a point m s.t. $m \models \varphi$?”.

► **Proposition 15.** *A formula $\varphi \in \mu^\alpha$ -ML has positive countdown if it does not use ν^α with $\alpha \neq \infty$. The satisfiability problem is decidable for such formulas.*

Proof. Observe that for φ with positive countdown, in every model we have $\llbracket \varphi \rrbracket \subseteq \llbracket \widehat{\varphi} \rrbracket$. Hence, if φ is satisfiable, then so is $\widehat{\varphi}$ – but since μ -ML has a finite model property, this means that $\widehat{\varphi}$ has a *finite* model, where $\widehat{\varphi}$ and φ are equivalent. Thus, φ is satisfiable iff $\widehat{\varphi}$ is, and the problem reduces to μ -ML satisfiability. ◀

Dualizing the above we get that the *validity* problem is decidable for formulas with *negative countdown*, i.e. with $\alpha = \infty$ for every μ^α .

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A

 Technical observations about countdown games

The following basic results about countdown games will be useful in subsequent sections.

As we mentioned, it is beneficial for each player to keep her/his counter as big as possible. More precisely, given a countdown game, define a partial order \preceq_{\exists} on its configurations: $\langle v, \text{ctr} \rangle \preceq_{\exists} \langle v, \text{ctr}' \rangle$ and $[v, \text{ctr}] \preceq_{\exists} [v, \text{ctr}']$ if and only if $\text{ctr}(r) \leq \text{ctr}'(r)$ for all $r \in \mathcal{D}_{\exists}$ and $\text{ctr}(r) \geq \text{ctr}'(r)$ for all $r \in \mathcal{D}_{\forall}$. A partial order \preceq_{\forall} is defined analogously, and it is clearly the inverse of \preceq_{\exists} , that is $\gamma \preceq_{\forall} \delta$ if and only if $\delta \preceq_{\exists} \gamma$ for all configurations γ, δ . Both orders extend to plays seen as sequences of configuration and compared pointwise. It easily follows from the definition that if $\gamma \preceq_{\exists} \delta$ and \exists ve has a move from γ to a configuration γ' , then she has a move from δ to a δ' such that $\gamma' \preceq_{\exists} \delta'$. A symmetric statement holds for \forall dam. As a result, if \exists ve has a winning strategy from γ and $\gamma \preceq_{\exists} \delta$ then she has a winning strategy from δ ; analogously for \forall dam.

Another easy observation is that if the countdown starts from limit ordinals, one may always choose values greater by a finite k than the ones given by some fixed strategy. More specifically, for a counter valuation ctr and a number $k < \omega$, define a valuation $\text{ctr} +_{\exists} k$ by:

- $(\text{ctr} +_{\exists} k)(r) = \min(\text{ctr}(r) + k, \text{ctr}_I(r))$ for $r \in \mathcal{D}_{\exists}$, and
- $(\text{ctr} +_{\exists} k)(r) = \text{ctr}(r)$ for $r \in \mathcal{D}_{\forall}$.

For a play π , let $\pi +_{\exists} k$ be the play pointwise equal to π on positions, and replacing every counter valuation ctr with $\text{ctr} +_{\exists} k$. If all the initial values ctr_I are limit ordinals, then for every strategy σ there exist a strategy k -above σ , denoted $\sigma +_{\exists} k$, such that π is a $(\sigma +_{\exists} k)$ -play iff $\pi = \pi' +_{\exists} k$ for some σ -play π' . Moreover, if σ is winning for \exists ve then so is $\sigma +_{\exists} k$. An analogous argument works for \forall dam's winning strategies.

As mentioned, countdown games are not positionally determined. Below we show a much weaker (yet still useful) property: players can win with strategies that do not depend on the counters in finite stages of the game. Consider a countdown game $(V, E, \text{rank}, \text{ctr}_I)$. For a countdown play:

$$\pi = v_1 \text{ctr}_1 v_2 \dots \text{ctr}_{n-1}, v_n, \quad \text{or} \quad \pi = v_1 \text{ctr}_1 v_2 \dots \text{ctr}_{n-1}, v_n \text{ctr}_n$$

denote by $\text{pos}(\pi)$ the sequence of consecutive positions $v_1 \dots v_n$. Given a phase \mathcal{S} of the game and a strategy σ for player P , we say that a partial function $f : V^* \rightarrow V$ guides σ in \mathcal{S} if for every $v \in V$ and σ -plays $\pi, \pi' \in \mathcal{S}$ such that v is a position chosen by P , the value $f(\text{pos}(\pi))$ is defined and equals v . We say that σ is *counter-independent in \mathcal{S}* or *\mathcal{S} -counter-independent* iff it is guided in \mathcal{S} by some partial function called the *\mathcal{S} -component* of σ and denoted $\sigma^{\mathcal{S}}$. Phase \mathcal{S} is *proper* if membership in \mathcal{S} does not depend on the counter values, meaning that for plays π, π' of the same length, $\text{pos}(\pi) = \text{pos}(\pi')$ implies $\pi \in \mathcal{S} \iff \pi' \in \mathcal{S}$.

► **Proposition 16.** *Take a countdown game $\mathcal{G} = (V, E, \text{rank}, \text{ctr}_I)$ and a proper phase \mathcal{S} of \mathcal{G} . Assume that the set $\text{pos}[\mathcal{S}] = \{\text{pos}(\pi) \mid \pi \in \mathcal{S}\}$ is finite. If \exists ve wins from configuration γ_I , then she wins with a strategy that is counter-independent in \mathcal{S} .*

Proof. The assumption on $\text{pos}[\mathcal{S}]$ implies that there exists a finite bound l_{\max} on the length of plays in \mathcal{S} . Consider a winning strategy σ for \exists ve. We show by induction on $0 \leq l \leq l_{\max}$ that:

For every σ -play π of length $|\pi| = l_{\max} - l$, there exists a winning strategy σ_{π} for \mathcal{G}, γ_I that is counter-independent in the subphase \mathcal{S}_{π} of \mathcal{S} and equal to σ on plays without a prefix from \mathcal{S}_{π} .

Once we prove the claim for $l = l_{\max}$, we obtain a strategy σ_{ϵ} counter-independent in $\mathcal{S}_{\epsilon} = \mathcal{S}$, as desired.

The base case is $l = 0$ where there is nothing to prove, as $|\pi| = l_{\max}$ implies that either $\mathcal{S}_\pi = \{\pi\}$ if $\pi \in \mathcal{S}$ or $\mathcal{S}_\pi = \emptyset$ otherwise. In both cases σ is trivially guided in \mathcal{S}_π by a partial function undefined on every argument.

For the inductive step, assume that the claim is true for l and for every σ -play π with $|\pi| = l_{\max} - l$ denote the \mathcal{S}_π -component of σ_π by $\sigma^{\mathcal{S}_\pi}$. Given a σ -play π with $|\pi| = l_{\max} - l - 1$, there are three cases to consider:

- After π it is \exists ve who makes a move. Since π is a σ -play, σ provides a move $z = \sigma(\pi)$. Since πz is also a σ -play and $|\pi z| = l_{\max} - l$, by induction hypothesis there exists a winning $\sigma_{\pi z}$ that is counter-independent in $\mathcal{S}_{\pi z}$. \exists ve can therefore win with the strategy:

$$\sigma_\pi(\rho) = \begin{cases} \sigma_{\pi z}(\rho) & \text{if } \pi z \text{ is a prefix of } \rho, \\ \sigma(\rho) & \text{otherwise.} \end{cases}$$

Unless $\pi, \pi z \in \mathcal{S}$ and $z \in V$, the strategy σ_π is guided by $\sigma^{\mathcal{S}_\pi} = \sigma^{\mathcal{S}_{\pi z}}$ in \mathcal{S}_π and otherwise it is guided by:

$$\sigma^{\mathcal{S}_\pi}(\bar{v}) = \begin{cases} z & \text{if } \bar{v} = \text{pos}(\pi), \\ \sigma^{\mathcal{S}_{\pi z}}(\bar{v}) & \text{otherwise.} \end{cases}$$

- After π \forall dam chooses a position v from a set $W \subseteq V$. For every such v , πv is a σ -play, $|\pi v| = l_{\max} - l$ and hence induction hypothesis provides $\sigma_{\pi v}$ guided by $\sigma^{\mathcal{S}_{\pi v}}$ in $\mathcal{S}_{\pi v}$. We combine strategies for all the possible choices from W :

$$\sigma_\pi(\rho) = \begin{cases} \sigma_{\pi v}(\rho) & \text{if } \pi v \text{ is a prefix of } \rho, \\ \sigma(\rho) & \text{otherwise.} \end{cases}$$

Such σ_π is guided in \mathcal{S}_π by:

$$\sigma^{\mathcal{S}_\pi}(\bar{v}) = \begin{cases} \sigma^{\mathcal{S}_{\pi v}}(\bar{v}) & \bar{v} \text{ has } \text{pos}(\pi v) \text{ as a prefix,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

- After π \forall dam updates the current counters ctr to ctr' . The only interesting case is when the current rank r is nonstandard and so ctr' is given by a choice of an ordinal $\alpha < \text{ctr}(r)$ (the case with standard r is similar to the first one). Denote such ctr' by ctr_α and πctr_α by π_α . For every $\alpha < \text{ctr}(r)$ the play π_α is consistent with σ and $|\pi_\alpha| = l_{\max} - l$, so induction hypothesis gives us σ_{π_α} guided by $\sigma^{\mathcal{S}_{\pi_\alpha}}$ in \mathcal{S}_{π_α} .

Observe that for plays π_α, π_β leading to configurations γ_α and γ_β , respectively, we have $\gamma_\alpha \succ_{\exists} \gamma_\beta$ whenever $\alpha < \beta$. It follows that if after π \forall dam chooses α , \exists ve may as well continue as if he picked β . More precisely, she may play maintaining the invariant that for the current play $\pi_\alpha \xi$ there exists a σ_β -play $\pi_\beta \xi'$ with $\pi_\beta \xi' \preceq_{\exists} \pi_\alpha \xi$. Denote such strategy by $\sigma_{[\beta/\alpha]}$.

Importantly, if $\alpha \leq \beta$ and $\sigma^{\mathcal{S}_{\pi_\beta}}$ guides σ_{π_β} in \mathcal{S}_{π_β} , then it also guides $\sigma_{[\beta/\alpha]}$ in \mathcal{S}_{π_α} . This is because whenever $\sigma_{[\beta/\alpha]}$ -plays $\pi_\alpha \xi$ and $\pi_\alpha \xi v$ belong to \mathcal{S}_{π_α} and v is chosen by \exists ve, there exists $\pi_\beta \xi' \preceq_{\exists} \pi_\alpha \xi$ such that $\pi_\beta \xi'$ and $\pi_\beta \xi' v$ are σ_{π_β} -plays. Since $\pi_\beta \xi' \preceq_{\exists} \pi_\alpha \xi$ implies $\text{pos}(\pi_\alpha \xi) = \text{pos}(\pi_\beta \xi')$, by properness of \mathcal{S} both $\pi_\beta \xi'$ and $\pi_\beta \xi' v$ belong to \mathcal{S}_{π_β} . Hence, $\sigma^{\mathcal{S}_{\pi_\beta}}(\text{pos}(\pi_\alpha \xi)) = \sigma^{\mathcal{S}_{\pi_\beta}}(\text{pos}(\pi_\beta \xi')) = v$, as desired.

There are two cases to consider, depending on whether $\text{ctr}(r)$ is a limit ordinal or not. If it is a successor ordinal then there is a maximal α that can be chosen by \forall dam. In that case, \exists ve uses the strategy:

$$\sigma_\pi(\rho) = \begin{cases} \sigma_{[\alpha/\beta]}(\rho) & \text{if } \pi_\beta \text{ is a prefix of } \rho, \\ \sigma(\rho) & \text{otherwise,} \end{cases}$$

guided in \mathcal{S}_π by $\sigma^{\mathcal{S}_\pi} = \sigma^{\mathcal{S}_{\pi_\alpha}}$.

On the other hand, if $\text{ctr}(r)$ is a limit ordinal then there is no maximal α that $\forall \text{dam}$ can choose, and for each of his choices $\exists \text{ve}$ might have used a different σ_{π_α} . However, by assumption the set of positions that appear in \mathcal{S} is finite. As a consequence, there are only finitely many possible partial functions guiding σ_{π_α} in \mathcal{S}_{π_α} and we may find $\sigma^{\mathcal{S}_\pi}$ such that σ_{π_α} is guided in \mathcal{S}_{π_α} by $\sigma^{\mathcal{S}_\pi}$ for arbitrarily big $\alpha < \text{ctr}(r)$. Define:

$$\sigma_\pi(\rho) = \begin{cases} \sigma_{[\alpha'/\alpha]}(\rho) & \text{if } \pi_\alpha \text{ is a prefix of } \rho, \\ \sigma(\rho) & \text{otherwise,} \end{cases}$$

where $\alpha' \geq \alpha$ is the least number greater than α with $\sigma^{\mathcal{S}_{\pi_\alpha}} = \sigma^{\mathcal{S}_\pi}$. By design, σ_π is guided by $\sigma^{\mathcal{S}_\pi}$ in \mathcal{S}_π . ◀

Sometimes we will want to decompose games into smaller fragments. Given a game $\mathcal{G} = (V, E, \text{rank}, \text{ctr}_I)$ and a subset $Z \subseteq V$, the *partial game* $\mathcal{G}|Z$ is played the same as \mathcal{G} , except that upon reaching a configuration with a position in Z (this is always a countdown configuration, as they are the first to be reached in any given position) the play ends with a draw, meaning that no player loses or wins. For a non-losing $\exists \text{ve}$'s strategy σ for $\mathcal{G}|Z, v$, the set of its *exit positions*, denoted $\text{exit}(\sigma) \subseteq Z$, consists of all the positions $u \in Z$ such that some σ -play ends in a countdown configuration in u .

Given another $\mathcal{G}' = (V', E', \text{rank}', \text{ctr}'_I)$ and a subset of positions $Z \subseteq V \cap V'$, we say that \mathcal{G}, v is *exit-equivalent* to \mathcal{G}', v' with respect to Z , denoted $\mathcal{G}, v \equiv_Z \mathcal{G}', v'$, if for every non-losing $\exists \text{ve}$'s strategy σ for $\mathcal{G}|Z, v$ she has a non-losing σ' for $\mathcal{G}'|Z, v'$ s.t. $\text{exit}(\sigma) = \text{exit}(\sigma')$, and symmetrically for every σ' for $\mathcal{G}'|Z, v'$. Exit-equivalence \equiv_Z is an equivalence relation between games having Z as a subset of its positions.

► **Lemma 17** (Decomposition Lemma). *Consider games \mathcal{G} and \mathcal{G}' as above such that the most important ranks $r = \max(\mathcal{R})$ and $r' = \max(\mathcal{R}')$ have the same type (meaning $r \in \mathcal{R}_\exists$ iff $r' \in \mathcal{R}'_\exists$ and either both are standard or $\text{ctr}_I(r) = \text{ctr}'_I(r')$). Assume disjoint subsets $Z_{\text{exit}}, Z_{\text{max}} \subseteq V \cap V'$ with $Z_{\text{max}} = \text{rank}^{-1}(r) = \text{rank}'^{-1}(r')$. For every $v_I \in V$ and $v'_I \in V'$:*

- $\mathcal{G}, v_I \equiv_{Z_{\text{exit}} \cup Z_{\text{max}}} \mathcal{G}', v'_I$ and:
 - $\mathcal{G}, v \equiv_{Z_{\text{exit}} \cup Z_{\text{max}}} \mathcal{G}', v$ for all $v \in Z_{\text{max}}$,
- implies $\mathcal{G}, v_I \equiv_{Z_{\text{exit}}} \mathcal{G}', v'_I$.*

Proof. Playing $\mathcal{G}|Z_{\text{exit}}, v_I$ can be decomposed into a sequence of alternating choices starting in the initial configuration $\gamma_0 = \langle v_I, \text{ctr}_I \rangle$:

1. In a configuration $\gamma_i = \langle v_i, \text{ctr}_i \rangle$ $\exists \text{ve}$ declares a fragment σ_i of her strategy that determines her moves until some countdown configuration in $Z_{\text{exit}} \cup Z_{\text{max}}$ is reached;
2. $\forall \text{dam}$ chooses a finite σ_i -play π_i leading to a configuration $[v_{i+1}, \text{ctr}]$ for some $v_{i+1} \in Z_{\text{exit}} \cup Z_{\text{max}}$;
3. If $v_{i+1} \in Z_{\text{exit}}$ the game ends in a draw, otherwise the owner of $r = \text{rank}(v_{i+1})$ updates ctr to some ctr_{i+1} , and the play proceeds from $\gamma_{i+1} = \langle v_{i+1}, \text{ctr}_{i+1} \rangle$ as in step (1) above.

For every configuration c_i as above, the fragments of strategies σ_i chosen by $\exists \text{ve}$ in (1) can be identified with strategies for $\mathcal{G}|Z_{\text{exit}} \cup Z_{\text{max}}, \langle v_i, \text{ctr}_i \rangle$. Since $\mathcal{G}|Z_{\text{exit}} \cup Z_{\text{max}}$ always stops before any position in Z_{max} is reached, the counter value for r is irrelevant *inside* $\mathcal{G}|Z_{\text{exit}} \cup Z_{\text{max}}$:

$$\mathcal{G}, \langle v_i, \text{ctr}_i \rangle \equiv_{Z_{\text{exit}} \cup Z_{\text{max}}} \mathcal{G}, \langle v_i, \text{ctr}_i[r \mapsto \beta] \rangle$$

for every $\beta \in \text{Ord}$. Moreover, ranks greater than r are never reached before the game stops and whenever the game enters a position $v \in Z_{\max}$, counters for all ranks smaller than r are reset back to their initial values from ctr_I , so each counter assignment ctr_i has to be of the form $\text{ctr}_I[r \mapsto \beta]$. Thus, the fragments of strategies in (1) are the same as strategies for $\mathcal{G}|Z_{\text{exit}} \cup Z_{\max}, \langle v_i, \text{ctr}_I \rangle$. Finally, since countdown games are configurationally determined, we may assume that in step (2) $\forall\text{dam}$ picks only a position $v_{i+1} \in Z_{\max}$ (instead of an entire play π_i ending with a countdown configuration in v_{i+1}).

Consider a game $\widehat{\mathcal{G}}$ starting in $v_0 = v_I$ and consisting of three alternating phases:

1. From $v_i \in V$, $\exists\text{ve}$ picks a non-losing strategy σ_i for $\mathcal{G}|Z_{\text{exit}} \cup Z_{\max}, v_i$;
2. $\forall\text{dam}$ chooses an exit position $v_{i+1} \in \text{exit}(\sigma_i)$, that is, a position in Z_{\max} reachable by σ_i ;
3. if r is nonstandard, its owner decrements the corresponding counter, otherwise nothing changes, and in both cases we proceed from v_{i+1} .

Formally, the game $\widehat{\mathcal{G}}$ is a countdown game with positions $\widehat{V}_{\exists} = V$ and $\widehat{V}_{\forall} = \bigcup_{v \in V} S_v$, edges $\widehat{E} = \{(v, \sigma) \mid v \in V, \sigma \in S_v\} \cup \{(\sigma, v) \mid v \in \text{exit}(\sigma)\}$ where:

$$S_v = \{\sigma \mid \sigma \text{ is a non-losing strategy for } \mathcal{G}|Z_{\text{exit}} \cup Z_{\max}, v\}$$

and $\widehat{\text{rank}}(v) = r$ and $\widehat{\text{rank}}(\sigma) = 0$ with $\widehat{\mathcal{R}} = \{0 \preceq r\}$ where r has the same type as in \mathcal{G} and 0 is an irrelevant, standard rank. It follows from the discussion above that:

$$\mathcal{G}, v \equiv_{Z_{\text{exit}}} \widehat{\mathcal{G}}, v$$

for every $v \in V$. On the other hand, the assumptions of the lemma imply that the relation:

$$\{(v_I, v'_I)\} \cup \{(v, v) \mid v \in Z_{\max}\} \cup \{(\sigma, \sigma') \in \widehat{V}_{\forall} \times \widehat{V}'_{\forall} \mid \text{exit}(\sigma) = \text{exit}(\sigma')\}$$

is a bisimulation between the arenas of $\widehat{\mathcal{G}}|Z_{\text{exit}}$ and $\widehat{\mathcal{G}}'|Z_{\text{exit}}$ that preserves order and type of ranks and ownership of positions. It follows that:

$$\mathcal{G}, v_I \equiv_{Z_{\text{exit}}} \widehat{\mathcal{G}}, v_I \equiv_{Z_{\text{exit}}} \widehat{\mathcal{G}}', v'_I \equiv_{Z_{\text{exit}}} \mathcal{G}', v'_I$$

which proves the lemma. ◀

B Proof of Theorem 7

Unfolding the definition of $\llbracket \mathcal{A}_{\varphi} \rrbracket^{\text{val}}$ from Definition 6, we prove that

$$\mathbf{m} \in \llbracket \varphi \rrbracket^{\text{val}} \iff \exists\text{ve wins } \mathcal{G}^{\text{val}}(\varphi) \text{ from } \langle (\mathbf{m}, \varphi), \text{ctr}_I \rangle \quad (7)$$

by induction on φ . The only interesting case is when $\varphi = \mu_i^{\alpha} \bar{x}. \bar{\varphi}$ for some $\bar{x} = \langle x_1, \dots, x_n \rangle$, $\bar{\varphi} = \langle \varphi_1, \dots, \varphi_n \rangle$ and $\alpha \in \text{Ord}_{\infty}$ (the case of ν^{α} in place of μ^{α} is symmetric). If $\alpha = \infty$ (i.e. μ^{α} is just a usual fixpoint operator), then the proof is essentially the same as for the classical μ -calculus [16]. For $\alpha \in \text{Ord}$, a different argument is needed. We prove (7) by induction on α :

$$\mathbf{m} \in \llbracket \mu_i^{\alpha} \bar{x}. \bar{\varphi} \rrbracket^{\text{val}} \iff \exists\text{ve wins } \mathcal{G}^{\text{val}}(\mu_i^{\alpha} \bar{x}. \bar{\varphi}) \text{ from } \langle (\mathbf{m}, \mu_i^{\alpha} \bar{x}. \bar{\varphi}), \text{ctr}_I \rangle. \quad (8)$$

To this end, denote $H_j^{\beta} = \llbracket \mu_j^{\beta} \bar{x}. \bar{\varphi} \rrbracket^{\text{val}}$ for $j \leq n$ and $\beta \in \text{Ord}$. By Definition 3 we have:

$$H_j^{\alpha} = \bigcup_{\beta < \alpha} \llbracket \varphi_j \rrbracket^{\text{val}_{\beta}}, \quad \text{where } \text{val}_{\beta} = \text{val}[x_1 \mapsto H_1^{\beta}, \dots, x_n \mapsto H_n^{\beta}].$$

By the induction hypothesis (7) applied to φ_j , $\mathbf{m} \in H_j^\alpha$ if and only if there exists a $\beta < \alpha$ such that \exists ve wins the game $\mathcal{G}^{\text{val}\beta}(\varphi_j)$ from the position $\langle \langle \mathbf{m}, \varphi_j \rangle, \text{ctr}_I \rangle$.

Consider now the game $\mathcal{G}^{\text{val}}(\mu_i^\alpha \bar{x}.\bar{\varphi})$ and its initial positional configuration $\langle \langle \mathbf{m}, \mu_i^\alpha \bar{x}.\bar{\varphi} \rangle, \text{ctr}_I \rangle$. The first move from this configuration is deterministic, to the countdown configuration $[(\mathbf{m}, \varphi_i), \text{ctr}_I]$. The next move in the game is made by \exists ve, as she is the owner of $r = \text{rank}(\varphi_i)$. Note that $\text{ctr}_I(r) = \alpha$, and r is the highest rank for which ctr_I is defined. Therefore \exists ve chooses some $\beta < \alpha$ and moves to the configuration $\langle \langle \mathbf{m}, \varphi_i \rangle, \text{ctr}_I[r \mapsto \beta] \rangle$.

The game $\mathcal{G}^{\text{val}}(\mu_i^\alpha \bar{x}.\bar{\varphi})$, played from this configuration, does not differ from $\mathcal{G}^{\text{val}\beta}(\varphi_i)$ played from $\langle \langle \mathbf{m}, \varphi_i \rangle, \text{ctr}_I \rangle$, until some variable x_j is reached. If this happens, in the former game we continue in a game isomorphic to $\mathcal{G}^{\text{val}}(\mu_j^\beta \bar{x}.\bar{\varphi})$. In the latter game, \exists ve wins if and only if the current point \mathbf{n} belongs to H_j^β . Since $\beta < \alpha$, by the induction hypothesis (8) these two conditions are equivalent. This finishes the proof.

C Proof of Theorem 9

Fix an automaton $\mathcal{A} = (Q, q_I, \delta, \text{rank}, \text{ctr}_I)$. For clarity of presentation we only consider the case when \mathcal{A} has no free variables, the general case requires no new ideas. Without losing generality assume that the highest rank r_{\max} is not assigned to any state and every other rank is assigned to at least one state.

We construct, by induction on $r \in \mathcal{R}$, a formula $\psi_{r,q}$ over the set Q treated as formal variables, with $\text{rank}(q) \geq r$ whenever q occurs free and $\text{rank}(q) < r$ if it occurs bound, and such that for every $\mathbf{m} \in \mathcal{M}$:

$$\mathcal{G}(\mathcal{A}), (\mathbf{m}, q) \equiv_{Z_r} \mathcal{G}(\psi_{r,q}), (\mathbf{m}, \psi_{r,q}) \quad (9)$$

where:

$$Z_r = \{(\mathbf{n}, q) \in M \times Q \mid r \leq \text{rank}(q)\}.$$

Note that although formally $\psi_{r,q}$ may contain free variables, the game $\mathcal{G}^{\text{val}}(\psi_{r,q})|Z_r, (\mathbf{m}, \psi_{r,q})$ always stops before any such variable is reached, so we ignore the valuation val and write $\mathcal{G}(\psi_{r,q})$.

Given (9), since no state in \mathcal{A} has the highest rank r_{\max} , the set $Z_{r_{\max}}$ is empty and so the games $\mathcal{G}(\mathcal{A}), (\mathbf{m}, q_I)$ and $\mathcal{G}(\psi_{r_{\max}, q_I}), (\mathbf{m}, \psi_{r_{\max}, q_I})$ are equivalent, which will prove the theorem.

Denote the lowest rank by 0. The set Z_0 contains all the positions of $\mathcal{G}(\mathcal{A})$, meaning that $\mathcal{G}(\mathcal{A})|Z_0$ stops immediately after the first move. Thus for the base case of (9) it is enough to put:

■ if $\delta(s) = (\tau, p)$:

$$\psi_{0,s} = \begin{cases} \langle \tau \rangle p & \text{if } q \text{ belongs to } \exists\text{ve} \\ [\tau]p & \text{if } q \text{ belongs to } \forall\text{dam} \end{cases}$$

■ if $\delta(s) \subseteq Q$:

$$\psi_{0,s} = \begin{cases} \bigvee \delta(s) & \text{if } q \text{ belongs to } \exists\text{ve} \\ \bigwedge \delta(s) & \text{if } q \text{ belongs to } \forall\text{dam}. \end{cases}$$

For the inductive step, assuming (9) for r , we will prove it for the next rank, denoted $r + 1$. Let q_1, \dots, q_d be all states in Q with rank r . For every q_i define the vectorial formula:

$$\theta_i = \eta_{q_i}^\alpha(q_1, \dots, q_d) \cdot (\psi_{r,q_1}, \dots, \psi_{r,q_d})$$

with $\alpha = \text{ctr}_I(r)$ and $\eta = \mu$ if r belongs to $\exists\text{ve}$ and $\eta = \nu$ if r belongs to $\forall\text{dam}$. Then put:

$$\psi_{r+1,q} = \psi_{r,q}[q_1 \mapsto \theta_1, \dots, q_d \mapsto \theta_d] \quad (10)$$

for every $q \in Q$. We need to prove that:

$$\mathcal{G}(\mathcal{A}), (\mathbf{m}, q) \equiv_{Z_{r+1}} \mathcal{G}(\psi_{r+1,q}), (\mathbf{m}, \psi_{r+1,q}) \quad (11)$$

for all $\mathbf{m} \in \mathcal{M}$.

The arena $V' = M \times \text{SubFor}(\psi_{r+1,q})$ of the game $\mathcal{G}(\psi_{r+1,q}) = (V', E', \text{rank}', \text{ctr}'_I)$ decomposes into $V'' = \bigcup_{1 \leq i \leq d} V_i''$ for $V_i'' = M \times \text{SubFor}(\theta_i)$ and $V^I = V' - V''$. Once a play enters V_i'' it stays there forever, since a move from V_i'' to V^I would only be possible if there was a variable free in θ_i but bound in its proper superformula (and hence also bound in $\psi_{r,q}$). However, if p is bound in $\psi_{r,q}$ then $\text{rank}(p) < r$, whereas p can be free in θ_i only if $r \leq \text{rank}(p)$. This implies that in $\psi_{r+1,q}$ the only formulas reachable from θ_i are its strict subformulas. Therefore, putting:

$$\theta = \eta_y^\alpha(y, q_1, \dots, q_d) \cdot (\psi_{r,q}, \psi_{r,q_1}, \dots, \psi_{r,q_d})$$

with y a fresh variable we obtain:

$$\mathcal{G}(\psi_{r+1,q}), (\mathbf{m}, \psi_{r+1,q}) \equiv_{Z_{r+1}} \mathcal{G}(\theta), (\mathbf{m}, \psi_{r,q})$$

because V^I corresponds to $M \times \text{SubFor}(\psi_{r,q})$ and V'' to $\bigcup_{i \leq d} M \times \text{SubFor}(\psi_{r,q_i})$ with freshness of y guaranteeing that there is no return from the second part to the first one.

Denote the rank of $\psi_{r,q_1}, \dots, \psi_{r,q_d}$ in $\mathcal{G}(\theta)$ by r' and recall that q_1, \dots, q_d all have rank 0. Consider the game $\widetilde{\mathcal{G}}(\theta)$ that is the same as $\mathcal{G}(\theta)$ except for the ranking function that swaps r' and 0, i.e. $\psi_{r,q}$ and each ψ_{r,q_i} have rank 0 and each q_i has rank r' . Since in $\mathcal{G}(\theta)$: (i) a move has $(\mathbf{n}, \psi_{r,q_i})$ as a target iff it has (\mathbf{n}, q_i) as a source and (ii) no nonempty play starting at $(\mathbf{m}, \psi_{r,q})$ reaches any $(\mathbf{n}, \psi_{r,q})$, we have:

$$\mathcal{G}(\theta), (\mathbf{m}, \psi_{r,q}) \equiv_{Z_{r+1}} \widetilde{\mathcal{G}}(\theta), (\mathbf{m}, \psi_{r,q})$$

and so for (11) it remains to prove:

$$\mathcal{G}(\mathcal{A}), (\mathbf{m}, q) \equiv_{Z_{r+1}} \widetilde{\mathcal{G}}(\theta), (\mathbf{m}, \psi_{r,q}). \quad (12)$$

Note that:

$$Z_r = Z_{r+1} \cup Y_r \quad \text{with} \quad Y_r = M \times \{q_1, \dots, q_d\}.$$

and Y_r is precisely the set of positions with rank r and r' in $\mathcal{G}(\mathcal{A})$ and $\widetilde{\mathcal{G}}(\theta)$, respectively. Since the ranks r and r' have the same type and are the most important in both games, by Lemma 17 to prove (12) it is enough to prove that:

1. $\mathcal{G}(\mathcal{A}), (\mathbf{m}, q) \equiv_{Z_r} \widetilde{\mathcal{G}}(\theta), (\mathbf{m}, \psi_{r,q})$, and
2. $\mathcal{G}(\mathcal{A}), (\mathbf{n}, q_i) \equiv_{Z_r} \widetilde{\mathcal{G}}(\theta), (\mathbf{n}, q_i)$ for all $(\mathbf{n}, q_i) \in Y_r$.

There are two cases to consider:

1. For (\mathbf{m}, q) and $(\mathbf{m}, \psi_{r+1,q})$:

$$\begin{aligned} \mathcal{G}(\mathcal{A}), (\mathbf{m}, q) &\equiv_{Z_r} \mathcal{G}(\psi_{r,q}), (\mathbf{m}, \psi_{r,q}) \\ &\equiv_{Z_r} \mathcal{G}(\theta), (\mathbf{m}, \psi_{r,q}) \\ &\equiv_{Z_r} \widetilde{\mathcal{G}}(\theta), (\mathbf{m}, \psi_{r,q}). \end{aligned}$$

The first equivalence is given by the induction hypothesis. The second one is true because the partial games $\mathcal{G}(\psi_{r,q})|Z_r, (\mathbf{m}, \psi_{r,q})$ and $\mathcal{G}(\theta)|Z_r, (\mathbf{m}, \psi_{r,q})$ are isomorphic. The third one follows from the observation that the difference between $\mathcal{G}(\theta)$ and $\widetilde{\mathcal{G}(\theta)}$ is only in ranks of positions $(\mathbf{n}, \psi_{r,q})$ and $(\mathbf{n}, \psi_{r,q_i}), (\mathbf{n}, q_i)$ for $i \leq d$ and $\mathbf{n} \in M$, but these positions cannot be reached by a nonempty play from $(\mathbf{m}, \psi_{r,q})$ before the game stops, i.e. without passing through Z_r .

2. For $(\mathbf{n}, q_i) \in Y_r$:

$$\begin{aligned} \mathcal{G}(\mathcal{A}), (\mathbf{n}, q_i) &\equiv_{Z_r} \mathcal{G}(\psi_{r,q_i}), (\mathbf{n}, \psi_{r,q_i}) \\ &\equiv_{Z_r} \mathcal{G}(\theta), (\mathbf{n}, \psi_{r,q_i}) \\ &\equiv_{Z_r} \widetilde{\mathcal{G}(\theta)}, (\mathbf{n}, \psi_{r,q_i}) \\ &\equiv_{Z_r} \widetilde{\mathcal{G}(\theta)}, (\mathbf{n}, q_i). \end{aligned}$$

The first three equivalences are true for reasons analogous to the previous case. The last one follows from the observation that in $\widetilde{\mathcal{G}(\theta)}$ the game moves deterministically from (\mathbf{n}, q_i) to $(\mathbf{n}, \psi_{r,q_i})$ and the later position has the least important, standard rank 0.

D Guarded formulas

To demonstrate usefulness of the correspondence between formulas and automata, but also for technical use in further proofs, we shall now show that without loss of generality formulas are *guarded*. We say that an automaton \mathcal{A} is guarded if it does not contain a loop without modal transitions. A formula φ is guarded if it is guarded when seen as an automaton \mathcal{A}_φ .

► **Proposition 18.** *Every countdown formula can be transformed into an equivalent guarded one.*

Proof. Note that in a countdown game, if a play moves from a position v to itself via a path without visiting ranks higher than $\text{rank}(v)$, then all the counters for lower ranks are reset and those for higher ranks remain unchanged. It follows that the resulting configuration is at least as good for the opponent P of the owner P' of $\text{rank}(v)$ as the one at the previous visit to v . Hence, P can repeat the strategy from that moment, and either eventually the game stops looping on v via lower ranks or P' loses. This means that in order to win, P' must have a strategy that avoids such loops, and therefore P' may use that strategy immediately. It follows that we obtain an equivalent game by adding the rule that whenever a play moves from any position v to itself via a path without visiting higher ranks, the owner of $\text{rank}(v)$ immediately loses.

Thanks to this, in any formula we may replace every $\eta_k^\alpha(x_1, \dots, x_n) \cdot (\psi_1, \dots, \psi_n)$ with:

$$\eta_{k,0}^\alpha(x_{i,j})_{i,j \leq n} \cdot (\psi_{i,j})_{i,j \leq n}$$

where $\psi_{i,j}$ is obtained from ψ_i by replacing

- (i) every guarded x_m with $x_{m,0}$ and
- (ii) every other x_m with \top/\perp (resp.) if $j = n$ and $\eta = \nu/\mu$, or with $x_{m,j+1}$ otherwise.

f This way, the number of unguarded unravellings of the η^α operator is counted in the index j , and the game stops whenever the play passes through the $(n+1)$ -st such unravelling (as it implies a repeated visit of a position associated with some x_m). ◀

Let us establish a few more useful facts about countdown automata (and, in light of Section 5.1, about countdown formulas) that will be useful in Section 6.

For an automaton \mathcal{A} with states Q , a valuation val and a point \mathbf{m}_I in a model \mathcal{M} , the *pre-modal phase* of the game $\mathcal{G}^{\text{val}}(\mathcal{A}), (\mathbf{m}_I, q_I)$ consists of all pre-modal plays, i.e. plays with no modal move. All the positions accessible in that phase are of the form (\mathbf{m}_I, q) for $q \in Q$ and if \mathcal{A} is guarded, then no pre-modal play is longer than $|Q|$. Hence, it follows from Proposition 16 that:

► **Proposition 19.** *In every game $\mathcal{G}^{\text{val}}(\mathcal{A}), (\mathbf{m}_I, q_I)$ for a guarded automaton \mathcal{A} , the winning player has a pre-modally counter-independent (i.e. counter-independent in the pre-modal phase) winning strategy.*

Since all the positions appearing in the pre-modal phase only have the initial point on the first coordinate, we can identify pre-modal plays π and π' starting in (\mathbf{m}, q) and (\mathbf{m}', q) for different $\mathbf{m} \neq \mathbf{m}'$ if π equals π' after swapping \mathbf{m} and \mathbf{m}' . Likewise, we simplify the pre-modal component $\sigma^I : (\{\mathbf{m}\} \times Q)^{<|Q|} \rightarrow \{\mathbf{m}\} \times Q$ guiding pre-modal σ -plays to $\sigma^I : Q^{<|Q|} \rightarrow Q$ by skipping the redundant first coordinate.

► **Proposition 20.** *Consider two points $\mathbf{m}_0, \mathbf{m}_1$ in a model \mathcal{M} , a valuation val and a guarded automaton \mathcal{A} . Assume that a player P wins the game $\mathcal{G}^{\text{val}}(\mathcal{A})$ from \mathbf{m}_0 and \mathbf{m}_1 with pre-modally counter-independent strategies σ_0 and σ_1 , respectively, both guided by the same pre-modal component σ^I . Then there are winning strategies σ'_0, σ'_1 guided by σ^I such that:*

- σ'_0 and σ'_1 behave the same in the pre-modal phase, up to swapping the points \mathbf{m}_0 and \mathbf{m}_1 , and
- for every $(\mathbf{m}_i, \text{ctr})$ reachable by a σ'_i -play, there are $(\mathbf{m}_0, \text{ctr}_0)$ and $(\mathbf{m}_1, \text{ctr}_1)$ such that each $(\mathbf{m}_j, \text{ctr}_j)$ is reachable by a σ_j -play and $\text{ctr}_i \preceq_P \text{ctr}$.

Proof. Starting in \mathbf{m}_0 or \mathbf{m}_1 , P can maintain the invariant that for the play π so far, there are π_0 and π_1 consistent with σ_0 and σ_1 respectively, such that (i) all the three plays are (point-wise) equal on P 's choices of positions and on all choices of P 's opponent, and (ii) P 's choices of counter values in π are the maximum of the corresponding choices from π_0 and π_1 . This way either P wins in the pre-modal phase, or the play reaches a modal move with counter values at least as good for P as after some σ_0 - and σ_1 -plays, respectively. P may then continue from \mathbf{m}_i with the winning strategy σ_i . ◀

► **Proposition 21.** *Consider three points $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ in a model \mathcal{M} s.t. for every $\tau \in \text{Act}$, the sets $S_1^\tau, S_2^\tau, S_3^\tau$ of their τ -successors are monotone, i.e. $S_1^\tau \subseteq S_2^\tau \subseteq S_3^\tau$; a valuation val that does not distinguish \mathbf{m}_i (i.e. $\mathbf{m}_i \in \text{val}(x) \iff \mathbf{m}_j \in \text{val}(x)$ for all $x \in \text{Var}$); and a guarded automaton \mathcal{A} . If a player P wins the semantic game $\mathcal{G}^{\text{val}}(\mathcal{A})$ from \mathbf{m}_1 and \mathbf{m}_3 using strategies σ_1, σ_3 guided by the same pre-modal component σ^I , then P also wins from \mathbf{m}_2 with a strategy σ_2 guided by σ^I .*

Proof. By Proposition 20, we may assume that σ_1 behaves the same as σ_3 in the pre-modal phase. Initially P may apply the same strategy from \mathbf{m}_2 , as the point in the model does not matter, or does not change, in the pre-modal phase. Consider any play consistent with this strategy. If P does not win already in the pre-modal phase, the play reaches a modal move, i.e. a configuration (\mathbf{m}_2, q) with $q \in Q$ such that $\delta(q) = (\tau, p)$. If the state q is owned by P then P may continue with σ_1 , and if q is owned by P 's opponent then P may continue with σ_3 . ◀

E

 Proof of Theorem 11

Observe that since scalar sentences are closed under negation, it is enough to prove that for every scalar φ there is a m_φ such that for all $i > m_\varphi$:

$$m_i \in \llbracket \varphi \rrbracket \implies n_i \in \llbracket \varphi \rrbracket \quad (\star)$$

Moreover, note that every scalar formula can be transformed into an equivalent *guarded* formula that is also scalar, by replacing every unguarded occurrence of a variable bound by μ^α (or ν^α) by \perp (or \top , respectively). Hence, it suffices to prove (\star) for guarded formulas. For the rest of the proof, we fix a guarded scalar sentence φ and denote $\mathcal{G} = \mathcal{G}(\varphi) = (V, E, \text{rank}, \text{ctr}_I)$.

Let us start with an easy fact.

► **Proposition 22.** *There exists some $N < \omega$ such that for all $N \leq i < j$:*

$$m_i \in \llbracket \varphi \rrbracket \iff m_j \in \llbracket \varphi \rrbracket$$

and if \exists ve wins the corresponding evaluation games then she does so with pre-modally counter-independent strategies σ_{m_i} and σ_{m_j} with the same pre-modal component σ^I that does not depend on i, j .

Proof. Note that the relations \xrightarrow{a} and \xrightarrow{b} are monotone, i.e. the bigger i , the more a - and b -successors m_i has. On the other hand, there are only finitely many possible pre-modal components, so by the pigeonhole principle if \exists ve wins from m_i for arbitrarily big i , some pre-modal component σ^I is used for arbitrarily big i . Thus, by Proposition 21 she can use σ^I to win for *all* i big enough. ◀

Towards (\star) , assume that $m_i \in \llbracket \varphi \rrbracket$ for all i big enough (otherwise, by Proposition 22, φ is *false* in m_i for all i big enough, which trivially implies (\star)) and denote by N the least number for which Proposition 22 holds. Then:

► **Proposition 23.** *Without loss of generality we may assume that for every $i > N$, if a σ_{m_i} -play visits a modal position for the first time and it has the shape $(m_i, \langle a \rangle \psi)$, then σ_{m_i} chooses a point m_j for some $j < N$.*

Proof. By Proposition 22, σ_{m_i} and σ_{m_N} have the same pre-modal component σ^I . Therefore, by Proposition 20, there is a strategy σ'_{m_i} winning from m_i guided by the same σ^I and only reaching pre-modal configurations at least as good for \exists ve as the ones reachable by σ_{m_N} . Then, whenever $(m_i, \langle a \rangle \psi)$ is reached in a pre-modal σ'_{m_i} -play, by the monotonicity of \xrightarrow{a} and the assumption that $N < i$, \exists ve may just continue with σ_{m_N} . Moreover, since σ_{m_N} is a legitimate strategy, it must pick a point m_j for some $j < N$, as desired. ◀

Denote by \mathcal{S} the phase of the game $\mathcal{G}(\varphi)$ that consists of plays of the shape $\pi = \xi\rho$ such that the play ξ ends with the first modal move (meaning that every proper prefix of ξ is pre-modal but ξ is not) and ρ does not visit (i) a formula beginning with $[a]$, $[b]$ or $\langle b \rangle$, nor (ii) a formula with a rank that was visited in the pre-modal phase (i.e. in a proper prefix of ξ). Note that the definition allows for empty ρ but not empty ξ . The next step is the following claim:

► **Proposition 24.** *Without loss of generality there exists a finite bound $k_{\max} < \omega$ such that no σ_{m_i} -play $\pi \in \mathcal{S}$ contains more than k_{\max} modal moves.*

Before proving the above proposition, let us demonstrate how it implies (\star) . Put:

$$m_\varphi = k_{\max} + N + 1$$

where k_{\max} is the bound from Proposition 24. We show that $n_i \in \llbracket \varphi \rrbracket$ for every $i > m_\varphi$. To this end, consider the strategy $\sigma_i = \sigma_{m_i} + \exists 1$, i.e. the strategy one above σ_{m_i} . In the pre-modal phase of the evaluation game from (n_i, φ) , use σ_i . Since m_i and n_i have the same \xrightarrow{a} -successors, if a play visits a formula beginning with $\langle a \rangle$, or $[a]$, Eve may continue with σ_i and win. The same is true for $[b]$, as every \xrightarrow{b} -successor of n_i is also a \xrightarrow{b} -successor of m_i .

The only interesting case is when a play reaches a formula that begins with $\langle b \rangle$ and σ_i chooses $m_{j'}$ for some $j' \geq i$ (if $j' < i$ then $m_{j'}$ is a \xrightarrow{b} -successor of n_i , so Eve may use σ_i). In this case, Eve may choose m_j where $j = k_{\max} + N$ and play maintaining the invariant that for the current play π , as long as it belongs to \mathcal{S} , there is a σ_i -play π' in \mathcal{S} s.t.:

1. all subformulas and ordinals are the same in π and π' ,
2. for the last points m_j and $m_{j'}$ of π and π' , respectively, we have:

$$k + N \leq j \leq j' \quad \text{and} \quad j < i$$

where k is the bound on the number of modal moves that can be made after playing π (or, equivalently, π') before leaving \mathcal{S} .

It is straightforward to maintain the invariant on ϵ -transitions and when counter values are updated.

Since m_j and $m_{j'}$ have the same \xrightarrow{b} -successors, if after the play π the game eventually moves via \xrightarrow{b} then Eve may continue as with σ_i , i.e. using the strategy $\pi\rho \mapsto \sigma_i(\pi'\rho)$. Moreover, $j \leq j'$ (guaranteed by item 2 of the invariant) and monotonicity of \xrightarrow{a} imply that if after π we encounter a formula beginning with $[a]$, Eve just uses $\pi\rho \mapsto \sigma_i(\pi'\rho)$ and win. If after π we visit a formula beginning with $\langle a \rangle$ and $\sigma_i(\pi') = m_{j''}$ for some $j'' < \omega$, then either:

1. $j'' < j$, hence $m_j \xrightarrow{a} m_{j''}$ and Eve wins using $\pi\rho \mapsto \sigma_i(\pi'\rho)$, or
2. $j \leq j''$, which combined with item 2 of the invariant gives $k + N \leq j \leq j''$. Since we have just made a modal move, we are left with at most $k - 1$ possible modal moves in \mathcal{S} , so the choice of m_{j-1} preserves the invariant, as $k - 1 + N \leq j - 1 \leq j''$ and $j - 1 < j < i$.

Since φ is guarded, the maximal number of consecutive ϵ -transitions in a play is bounded by $|\text{SubFor}(\varphi)|$. Moreover, each time a play passes through $\langle a \rangle$, the number k decreases. As a result, after at most $k_{\max} \cdot |\text{SubFor}(\varphi)|$ moves we either end the game in \mathcal{S} or leave \mathcal{S} . In the first case, thanks to item 1 of the invariant, Eve must win, for the strategy σ is winning. The second case can happen by either (i) visiting a formula that begins with $[a]$, $[b]$ or $\langle b \rangle$ (in which case Eve wins, as described above), or (ii) visiting a subformula ψ s.t. $\text{rank}(\psi)$ was visited in the pre-modal phase. But since φ is scalar (and so by Proposition 10 injectively ranked when seen as an automaton), this implies that *the same* ψ must have been visited in the pre-modal phase. Denote by π_1 the play ending with the first visit to ψ after the pre-modal phase (before the counter update, i.e. π_1 ends with a countdown configuration) and by π'_1 the corresponding σ_i -play that exists thanks to the invariant. Let π_0 and π'_0 be the prefixes of π_1 and π'_1 , respectively, ending with the first visit in ψ *after the counter update* (i.e. π_0 and π'_0 end with a positional configuration). By item 1 of the invariant, the plays π_0 and π'_0 (π_1 and π'_1) lead to the same counter assignment ctr_0 (ctr_1 , respectively).

Consider the strategy σ behaving as σ_i after π'_0 , that is $\sigma(\rho) = \sigma_i(\pi'_0\rho)$ for every ρ . Then:

$$\text{The strategy } \sigma \text{ is winning from } \langle (m_j, \psi), \text{ctr}_0 \rangle \text{ for every } N \leq j \leq i. \quad (13)$$

Indeed, for $j = i$, σ just continues a σ_i -play, and hence leads to victory. For $N \leq j < i$, we also essentially just use the same strategy σ :

1. The pre-modal phase starting from (\mathbf{m}_j, ψ) is identical as if we started from (\mathbf{m}_i, ψ) (recall that we identify pre-modal plays starting in different (\mathbf{m}_i, ψ) and (\mathbf{m}_j, ψ) if they are equal up to swapping positions (\mathbf{m}_i, θ) and (\mathbf{m}_j, θ) for all θ).
2. If after a pre-modal play ρ the game ever reaches a formula θ that begins with a modal operator, \exists ve may legally continue using σ . Indeed, since ρ is pre-modal, it does not change the point, meaning that it leads from (\mathbf{m}_j, ψ) to (\mathbf{m}_j, θ) and from (\mathbf{m}_i, ψ) to (\mathbf{m}_i, θ) . Because \mathbf{m}_i and \mathbf{m}_j have the same \xrightarrow{b} -successors, if θ begins with $\langle b \rangle$ or $[b]$, then the possible moves from the position (\mathbf{m}_j, θ) are the same as from (\mathbf{m}_i, θ) and so we can continue as if we started from (\mathbf{m}_i, θ) .

Similarly, since $j \leq i$ implies that every \xrightarrow{a} -successor of \mathbf{m}_j is an \xrightarrow{a} -successor of \mathbf{m}_i , σ can be used to win against every \forall dam's choice of an \xrightarrow{a} -successor of \mathbf{m}_j if θ begins with $[a]$. The remaining case is when θ begins with $\langle a \rangle$. By Proposition 23, if in the first modal step of a $\sigma_{\mathbf{m}_i}$ -play \exists ve has to provide an \xrightarrow{a} -successor \mathbf{m}_k of \mathbf{m}_i , then $\sigma_{\mathbf{m}_i}$ chooses some \mathbf{m}_k with $k < N$. Since $\sigma_i = \sigma_{\mathbf{m}_i} + \exists 1$, the same is true about σ_i . But since $\sigma(\rho) = \sigma_i(\pi'_0 \rho)$ and both ρ and π'_0 are pre-modal, it follows from $N \leq j$ that the choice given by σ_i is legal from \mathbf{m}_j .

This proves (13).

Note that since by definition in π_1 (and π'_1) it is the first time the game revisits a rank seen in the pre-modal phase, we have:

$$\text{ctr}_0(r') = \text{ctr}_1(r') \quad (14)$$

for every nonstandard rank $r' \geq r$. Indeed, since we are in a game for a *scalar* formula φ , every superformula θ of ψ must have been visited in the pre-modal phase. Since π_1 is a minimal play in which some rank is visited twice, guardedness of φ implies that no such θ was visited between π_0 and π_1 . This means that the only formulas that appeared between π_0 and π_1 were *strict subformulas* of ψ , and hence *all the ranks* visited between π_0 and π_1 were strictly lower than r , which implies (14).

To finish the proof of (\star) we need to show how to win once π_1 has been played and the game reached a countdown configuration $[(\mathbf{m}_j, \psi), \text{ctr}_1]$. By item 1 of the invariant, $N < j \leq i$. Consider the following cases:

- If r is standard, the counter update in $[(\mathbf{m}_j, \psi), \text{ctr}_1]$ is deterministic and by (14) leads to ctr_0 . Hence, we end up in a configuration $\langle (\mathbf{m}_j, \psi), \text{ctr}_0 \rangle$. By (13), \exists ve may use σ to win from there.
- If r is nonstandard and belongs to \forall dam, then by (14) it follows that for every ctr that \forall dam can choose in $[(\mathbf{m}_j, \psi), \text{ctr}_1]$, $\text{ctr} \preceq_{\forall} \text{ctr}_0$. Thus, again by (13), \exists ve may win using σ from every such $\langle (\mathbf{m}_j, \psi), \text{ctr} \rangle$.
- If r is nonstandard and belongs to \exists ve, the choice ctr_0 that was picked by σ_i at the end of π_0 is not legal after π_1 . However, since σ_i is one above $\sigma_{\mathbf{m}_i}$, there exist $\sigma_{\mathbf{m}_i}$ -plays π_0^- and π_1^- one below π'_0 and π'_1 , respectively, ending with configurations ctr_0^- and ctr_1^- such that $\text{ctr}_0^- + \exists 1 = \text{ctr}_0$ and $\text{ctr}_1^- + \exists 1 = \text{ctr}_1$. Define a counter assignment ctr :

$$\text{ctr}(r') = \begin{cases} \text{ctr}_1(r') & \text{if } r' > r, \\ \text{ctr}_1^-(r) & \text{if } r' = r, \\ \text{ctr}_I(r') & \text{if } r' < r. \end{cases}$$

Such ctr is a legal update from ctr_1 , because $\text{ctr}_1^- + \exists 1 = \text{ctr}_1$ implies $\text{ctr}_1^-(r) + 1 = \text{ctr}_1(r)$. Such a choice leads to the configuration $\langle (\mathbf{m}_j, \psi), \text{ctr} \rangle$, and so we show that this configuration is winning for \exists ve. We have:

$$\text{ctr}_0^- \preceq_{\exists} \text{ctr}. \quad (15)$$

Indeed, (14) implies that $\text{ctr}_0^-(r') = \text{ctr}_1^-(r')$ for all $r' \geq r$. Thus, for $r' > r$:

$$\begin{aligned} \text{ctr}_0^-(r') &= \text{ctr}_1^-(r') \leq \text{ctr}_1(r') = \text{ctr}(r') && \text{if } r' \text{ belongs to } \exists\text{ve}, \\ \text{ctr}_0^-(r') &= \text{ctr}_1^-(r') = \text{ctr}_1(r') = \text{ctr}(r') && \text{otherwise;} \end{aligned}$$

and:

$$\text{ctr}_0^-(r) = \text{ctr}_1^-(r) = \text{ctr}(r);$$

whereas for $r' < r$:

$$\text{ctr}_0^-(r') = \text{ctr}_I(r') = \text{ctr}(r').$$

Note that since $\sigma_i = \sigma_{\mathbf{m}_i} + \exists 1$ and σ is defined as $\rho \xrightarrow{\sigma} \sigma_i(\pi'_0 \rho)$, it follows that $\sigma = \sigma^- + \exists 1$ with σ^- given by $\rho \xrightarrow{\sigma^-} \sigma_{\mathbf{m}_i}(\pi'_0 \rho)$. By (13), σ wins from $\langle (\mathbf{m}_j, \psi), \text{ctr}_0 \rangle$. Consequently, $\text{ctr}_0 = \text{ctr}_0^- + \exists 1$ implies that σ^- wins from $\langle (\mathbf{m}_j, \psi), \text{ctr}_0^- \rangle$, and thanks to (15), also from $\langle (\mathbf{m}_j, \psi), \text{ctr} \rangle$.

This completes the proof of (\star) from Proposition 24.

It remains to prove Proposition 24, i.e. to refine strategies $\sigma_{\mathbf{m}_i}$ to obtain a finite bound $k_{\max} < \omega$ on the number of modal moves in a play in the phase \mathcal{S} . We will show a stronger fact: no play $\pi \in \mathcal{S}$ visits the same formula of shape $\langle \mathbf{a} \rangle \psi$ twice. Then, Proposition 24 follows with the bound $k_{\max} = |\text{SubFor}(\varphi)| + 1$, because all the other modal moves (i.e. moves corresponding to formulas beginning with $[a]$, $\langle b \rangle$ or $[b]$) end \mathcal{S} immediately.

Before we go into the somewhat technical details, let us sketch the core idea of the proof, which splits into two steps. First, we show that if instead of updating the counters during \mathcal{S} the players only decrement them once upon leaving \mathcal{S} , this does not change the winner of the game. Second, we use this equivalence to massage $\sigma_{\mathbf{m}_i}$ so that instead of performing a sequence:

$$(\mathbf{m}_j, \langle \mathbf{a} \rangle \psi) \rightarrow (\mathbf{m}_{j'}, \psi) \rightarrow \dots \rightarrow (\mathbf{m}_l, \langle \mathbf{a} \rangle \psi) \rightarrow (\mathbf{m}_{l'}, \psi) \in V^+$$

of modal moves corresponding to $\langle \mathbf{a} \rangle \psi$, \exists ve immediately goes to the *last* point $(\mathbf{m}_j, \langle \mathbf{a} \rangle \psi) \rightarrow (\mathbf{m}_{l'}, \psi)$. This is possible thanks to transitivity and well-foundedness of $\xrightarrow{\mathbf{a}}$ and avoids repetitions of $\langle \mathbf{a} \rangle \psi$.

To prove the claim, it is enough if for every *minimal* (and therefore necessarily ending with a first modal move) $\pi_I \in \mathcal{S}$ we refine $\sigma_{\mathbf{m}_i}$ to a strategy σ_{π_I} so that:

1. σ_{π_I} does not visit any $\langle \mathbf{a} \rangle \psi$ twice in any play $\rho \in \mathcal{S}_{\pi_I}$ and
2. the behaviour on all other plays is not changed, meaning that $\sigma_{\pi_I}(\rho) = \sigma_{\mathbf{m}_i}(\rho)$ for every ρ without a prefix in \mathcal{S}_{π_I} .

If we do that for every minimal $\pi_I \in \mathcal{S}$, we may combine all such refined strategies into one:

$$\sigma_{\mathcal{S}}(\rho) = \begin{cases} \sigma_{\pi_I}(\rho) & \text{if } \pi_I \text{ is the minimal prefix of } \rho \text{ from } \mathcal{S}, \\ \sigma_{\mathbf{m}_i}(\rho) & \text{otherwise (i.e. for pre-modal } \rho); \end{cases}$$

that avoids repetitions of each $\langle a \rangle \psi$ in every $\pi \in \mathcal{S}$.

Towards such a refinement of σ_{m_i} , fix a minimal $\pi_I \in \mathcal{S}$ leading to a winning countdown configuration $\gamma = [(m_z, \theta_z), \text{ctr}_z]$. Denote $v_z = (m_z, \theta_z) \in V$ and $\mathcal{Z} = \{\rho \mid \pi_I \rho \in \mathcal{S}\}$. To get our desired σ_{π_I} it suffices to construct a winning strategy for \mathcal{G}, γ that avoids repetitions of each $\langle a \rangle \psi$ in every $\pi \in \mathcal{Z}$.

Note that membership in \mathcal{Z} only depends on the underlying positions. Let $V_{\mathcal{Z}} \subseteq V$ be the set of all the positions of shape (m, ξ) with ξ either (i) beginning with $[a]$, $[b]$ or $\langle b \rangle$ or (ii) having a rank that was visited in π_I . Then $\pi \in \mathcal{Z}$ iff in π no position other than the last one belongs to $V_{\mathcal{Z}}$.

Define a *parity* game $\tilde{\mathcal{G}}$ that has arena (V, E, rank) with all the positions from $V_{\mathcal{Z}}$ turned into terminal positions immediately winning for \exists ve. To avoid confusion, we will call parity plays $\bar{v} \in V^*$ in $\tilde{\mathcal{G}}$ *paths* and reserve the term *plays* for \mathcal{G} . Observe that for every $\bar{v} \in V^*$:

$$\bar{v} \text{ is a path in } \tilde{\mathcal{G}}, v_z \iff \bar{v} = \text{pos}(\pi) \text{ for some play } \pi \in \mathcal{Z} \text{ in } \mathcal{G}, \gamma \quad (16)$$

with the left to right implication following from the fact that all the counters decremented in \mathcal{Z} have initial, and hence limit values in γ . In particular, (16) implies that every \mathcal{Z} -component of a winning strategy for \mathcal{G}, γ is a winning strategy for $\tilde{\mathcal{G}}, v_z$. This justifies the following terminology: we call a partial function $f : V^* \rightarrow V$, thought of as a candidate for a \mathcal{Z} -component of a winning strategy for \mathcal{G}, γ , a *proto-strategy* if f is a winning strategy in $\tilde{\mathcal{G}}, v_z$.

Since every modal move over \xrightarrow{b} leaves \mathcal{Z} , it follows that all the positions accessible in $\pi \in \mathcal{Z}$ are of the form (m_i, ψ) for $i \leq z$ (because \xrightarrow{a} only leads to points with a strictly smaller index) and no such position repeats in \mathcal{Z} (by guardedness of φ and acyclicity of \xrightarrow{a}). It follows that the set $\text{pos}[\mathcal{Z}]$ is finite. By (16), this means that also paths in $\tilde{\mathcal{G}}, v_z$ are all finite. Hence, if f is a proto-strategy then every maximal f -path in $\tilde{\mathcal{G}}, v_z$ ends with a position v that either (i) belongs to $V_{\mathcal{Z}}$ or (ii) is controlled by \forall dam and has no successors in both \mathcal{G} and $\tilde{\mathcal{G}}$.

We prove that for every proto-strategy f , the following are equivalent:

1. \exists ve has a winning strategy σ for \mathcal{G}, γ guided by f in \mathcal{Z} .
2. \exists ve wins in the following game:
 - (i) \forall dam picks a maximal f -path $\bar{v} \in V^*$ starting at v_z ;
 - (ii) we play a usual countdown game from γ but on arena restricted to \bar{v} (i.e. we only update the counters and deterministically follow \bar{v});
 - (iii) \exists ve wins iff the resulting configuration $[v, \text{ctr}]$, with v being the last position of \bar{v} , is winning for \exists ve in \mathcal{G} .
3. \exists ve wins in the following game:
 - (i) \forall dam picks a maximal f -path $\bar{v} \in V^*$ starting at v_z ;
 - (ii) given the set $\mathcal{D}_{\bar{v}} \subseteq \mathcal{D}$ of all the nonstandard ranks that should have non-initial value after traversing \bar{v} , the owner of each $r \in \mathcal{D}_{\bar{v}}$ (starting from more important ranks) picks a final counter value $\text{ctr}(r) < \text{ctr}_z(r)$ and we put $\text{ctr}(r) = \text{ctr}_I(r)$ for all other $r \in \mathcal{D} - \mathcal{D}_{\bar{v}}$;
 - (iii) \exists ve wins iff the resulting configuration $[v, \text{ctr}]$, with v being the last position of \bar{v} , is winning for \exists ve in \mathcal{G} .

Note that the set $\mathcal{D}_{\bar{v}}$ in (3) is uniquely determined by \bar{v} , as $r \in \mathcal{D}_{\bar{v}}$ iff r appears in \bar{v} and no higher rank appears after the last occurrence of r . However, since we are dealing with a game corresponding to a scalar formula φ , $\mathcal{D}_{\bar{v}}$ has an even more straightforward description: nonstandard r belongs to $\mathcal{D}_{\bar{v}}$ iff it is a rank of some superformula of the last formula in \bar{v} .

The implication (1) \implies (2) is straightforward. Once \forall dam picked \bar{v} , \exists ve simply uses σ until \bar{v} is traversed. By (16), until that moment the game stays in \mathcal{Z} , so the choices dictated by σ are consistent with \bar{v} , as σ is guided by f in \mathcal{Z} and \bar{v} is an f -path. Since σ is winning, the configuration reached at the end of \bar{v} must be winning.

To prove that (2) \implies (1), assume that for every maximal f -path \bar{v} starting at v_z , \exists ve has a strategy $h_{\bar{v}}$ winning in the second stage of (2). Our goal is to provide her with σ for (2). When during \mathcal{Z} she has to pick an edge, she uses $\sigma(\rho) = f(\text{pos}(\rho))$ so that σ is guided by f . For choosing ordinals, observe that the tree of all paths in $\tilde{\mathcal{G}}, v_z$ is finite, so for every countdown play ρ guided by f there are only finitely many maximal f -paths extending $\text{pos}(\rho)$. Thus, for every play ρ ending in \exists ve's choice of a counter for rank r , she takes the ordinal:

$$\max\{h_{\bar{v}}(\rho)(r) \mid \bar{v} \text{ is a maximal } f\text{-path extending } \text{pos}(\rho)\}$$

which is legal, since the longer ρ is, the fewer paths extend $\text{pos}(\rho)$. This way, she either wins before \mathcal{Z} ends, or leave it in a winning configuration, and in the later case she may continue with any winning strategy.

It remains to prove that (2) \iff (3). Note that in (2), once the path \bar{v} is chosen, the only nontrivial choice of a value for r is when its the corresponding counter has initial value and is not going to be reset further in \bar{v} .

Indeed, without lost of generality successor values are always decremented by 1 and if r is going to be reset somewhere further in \bar{v} , it suffices to pick the number k of visits in r before the closest reset. After the last reset of the counter for r , the number k of future decrements of r in \bar{v} is fixed, so in order to end the game with $\text{ctr}(r) = \alpha$ it suffices to pick the value $\alpha + k$ (and again, decrement it by 1 each time the game visits r). The above choices are legal because by definition of \mathcal{S} , r was not visited in the pre-modal phase and hence its counter has a maximal value upon entering \mathcal{S} .

Moreover, the order of these *nontrivial* choices is precisely the (decreasing) order on $\mathcal{D}_{\bar{v}}$. This establishes an equivalence between (2) and (3), therefore completing the proof of equivalence of games (1), (2) and (3).

Let σ be a winning strategy for \mathcal{G}, γ . To complete the proof of Proposition 24 it suffices to upgrade such σ so that no formula component of shape $\langle \mathbf{a} \rangle \theta$ repeats in \mathcal{Z} . Thanks to finiteness of $\text{pos}[\mathcal{Z}]$ we may apply Proposition 16 and assume that σ is guided by $\sigma^{\mathcal{Z}}$ in \mathcal{S} . As mentioned, such $\sigma^{\mathcal{Z}}$ is a legal proto-strategy. Enumerate all the subformulas $\langle \mathbf{a} \rangle \psi_1, \dots, \langle \mathbf{a} \rangle \psi_n$ of φ of shape $\langle \mathbf{a} \rangle \theta$. We construct, by induction on $i \leq n$, a sequence $f_0, \dots, f_n : V^* \rightarrow V$ of proto-strategies s.t.:

1. $f_0 = \sigma^{\mathcal{Z}}$,
2. whenever $i < j$ and $\bar{v} \in V^*$ is a maximal f_j -path, there exists a maximal f_i -path $\bar{w} \in V^*$ ending with the same position,
3. f_i avoids repetitions of $\{\psi_1, \dots, \psi_i\}$.

Assume we already have f_i and want to construct f_{i+1} . For every f_i -path $\bar{v} \in V^*$ ending in a visit in $\langle \mathbf{a} \rangle \psi_{i+1}$ consider the set:

$$\mathcal{H}_{\bar{v}} = \{\bar{w} \in V^* \mid \bar{w} \text{ is a } f_i\text{-path, has } \bar{v} \text{ as a prefix and ends with } \langle \mathbf{a} \rangle \psi_{i+1}\}$$

and fix some $(\bar{v})^\circ$ maximal in $\mathcal{H}_{\bar{v}}$ ($\mathcal{H}_{\bar{v}}$ is nonempty as it contains \bar{v} and must contain a maximal path because the length of paths is bounded).

Our new strategy f_{i+1} acts like f_i until the first visit in $\langle a \rangle \psi_{i+1}$ and then, instead of making multiple \xrightarrow{a} -moves for $\langle a \rangle \psi_{i+1}$, immediately jumps to the *last* choice from such maximal $(\bar{v})^\circ$ that extends the current path \bar{v} :

$$f_{i+1}(\bar{w}) = \begin{cases} f_i(\bar{w}) & \text{if } \bar{w} \text{ does not visit } \langle a \rangle \psi_{i+1}, \\ f_i((\bar{v})^\circ \cdot \bar{u}) & \text{otherwise, with } \bar{w} = \bar{v} \cdot \bar{u} \text{ and } \bar{v} \text{ ending in the first visit in } \langle a \rangle \psi_{i+1}. \end{cases}$$

Such f_{i+1} is a legal proto-strategy. Indeed, the new moves are allowed thanks to transitivity of \xrightarrow{a} and f_{i+1} is winning in $\tilde{\mathcal{G}}, v_z$, because positions accessible via f_{i+1} are a subset of the ones accessible by f_i . This also implies the second property, whereas the third one follows from the fact that each $(\bar{v})^\circ$ is maximal in $\mathcal{H}_{\bar{v}}$.

Since by scalarity of φ the set $\mathcal{D}_{\bar{v}}$ in the third variant of the game (3) depends only on the last formula in \bar{v} and Eve wins (3) with $f = f_0$, thanks to the second property she also wins (3) with f_n . By equivalence of (3) and (1), this means that some strategy σ_{π_I} winning from γ is guided by f_n . Moreover, the third property implies that σ_{π_I} avoids repetitions of all $\langle a \rangle \psi_1, \dots, \langle a \rangle \psi_n$ in \mathcal{Z} , thus proving Proposition 24 and completing the proof of Theorem 11.

F Proof of Lemma 13

► **Proposition 25.** *For every countdown formula φ there is a finite constant $t_\varphi < \omega$ such that for every valuation val stable above κ , in the part $[\kappa, \omega_1)$ of the model above κ , φ changes its truth value at most t_φ times.*

Proof. Since without loss of generality the formula is guarded (see Proposition 18), by Proposition 19 we may assume that in the semantic game Eve always uses a pre-modally counter-independent strategy. But the number z of possible pre-modal components for such strategies is finite, so if φ changed its value more than $t_\varphi = 2z + 2$ times above κ , there would be $\kappa \leq \alpha < \zeta < \beta$ such that Eve wins from α and β with the same pre-modal component, but loses from ζ in between, which is impossible by Proposition 21. ◀

We prove Lemma 13 by induction on the complexity of the formula φ . The base case is immediate, as for every $x \in \text{Var}$ it suffices to take $\alpha_x = 0$. For propositional connectives and modal operators we take $\alpha_{\psi_1 \vee \psi_2} = \alpha_{\psi_1 \wedge \psi_2} = \max(\alpha_{\psi_1}, \alpha_{\psi_2})$ and $\alpha_{\Diamond \psi} = \alpha_{\Box \psi} = \alpha_\psi + 1$. The remaining non-trivial cases are countdown and fixpoint operators.

■ For $\varphi = \eta_i^\omega \bar{x}.\bar{\psi}$, let $\Phi = \{\theta_1, \dots, \theta_l\}$ be the set of all maximal subformulas of $\bar{\psi}$ not using any variable x_j . For each θ , pick a fresh variable y_θ and put:

$$\psi'_j = \psi_j[\theta_1 \mapsto y_{\theta_1} \dots \theta_l \mapsto y_{\theta_l}]$$

i.e. starting from the root ψ_j , we replace every subformula θ that has no variables from \bar{x} with a fresh variable y_θ .² Observe that $\psi_j = \psi'_j[y_{\theta_1} \mapsto \theta_1 \dots y_{\theta_l} \mapsto \theta_l]$, so:

$$\eta_i^\omega \bar{x}.\bar{\psi} \equiv (\eta_i^\omega \bar{x}.\bar{\psi}') [y_{\theta_1} \mapsto \theta_1 \dots y_{\theta_l} \mapsto \theta_l].$$

Note that if φ has countdown nesting at most k , then each ψ'_j and each θ has countdown nesting less than k . Thus, by the induction hypothesis there exist $\alpha_{\psi'_j} < \omega^k$ and

² Recall that we do not identify isomorphic subformulas, and so there are no substitutions *inside* the θ 's. In particular, the order of substitutions does not matter.

$\alpha_\theta < \omega^k$ s.t. ψ'_j and θ stabilize $\alpha_{\psi'_j}$ and α_θ above the valuation, respectively. Denote $\alpha_{\overline{\psi'}} = \max\{\alpha_{\psi'_1}, \dots, \alpha_{\psi'_n}\}$.

For $m < \omega$, consider the m -th unfolding given by $\psi_j'^0 = x_j$ and $\psi_j'^{m+1} = \psi'_j[x_1 \mapsto \psi_1'^m \dots x_n \mapsto \psi_n'^m]$. It follows by a straightforward induction on m that each $\psi_j'^m$ is stable $\alpha_{\overline{\psi'}} \times m$ above the valuation. Moreover, for any valuation val we have:

$$\llbracket \mu_j^\omega \overline{x} . \overline{\psi'} \rrbracket^{\text{val}} = \bigcup_{m < \omega} \llbracket \psi_j'^m \rrbracket^{\text{val}} \quad \text{and} \quad \llbracket \nu_j^\omega \overline{x} . \overline{\psi'} \rrbracket^{\text{val}} = \bigcap_{m < \omega} \llbracket \psi_j'^m \rrbracket^{\text{val}}$$

so $\eta_j^\omega \overline{x} . \overline{\psi'}$ is stable $\alpha_{\overline{\psi'}} \times \omega$ above the valuation. Finally, we obtain that $\varphi = (\eta_j^\omega \overline{x} . \overline{\psi'})[\theta_1 \mapsto y_{\theta_1} \dots \theta_l \mapsto y_{\theta_l}]$ is stable α_φ above valuation with:

$$\alpha_\varphi = \max\{\alpha_{\theta_1} \dots \alpha_{\theta_l}\} + \alpha_{\overline{\psi'}} \times \omega.$$

Since $\alpha_{\overline{\psi'}} \times \omega < \omega^{k+1}$ and for each θ , $\alpha_\theta < \omega^k$, it follows that $\alpha_\varphi < \omega^{k+1}$.

- For $\varphi = \eta_i^\infty \overline{x} . \overline{\psi}$, note that the countdown nesting of each ψ_j is not greater than that of φ . For each $j \leq n = |\overline{x}|$, let $t_{\eta_j^\infty \overline{x} . \overline{\psi}} < \omega$ be the constant from Proposition 25 and $\alpha_{\psi_j} < \omega^{k+1}$ the constant that exists by the inductive hypothesis. Put $\alpha_{\max} = \max_{j \leq n}(\alpha_{\psi_j})$, $t_{\max} = \max_{j \leq n}(t_{\eta_j^\infty \overline{x} . \overline{\psi}})$ and $\alpha_\varphi = \alpha_{\max} \times t_{\max} \times n$. Clearly $\alpha_\varphi < \omega^{k+1}$, as $\alpha_{\max} < \omega^{k+1}$ and $t_{\max} < \omega$ – so it suffices to show that such bound works. Define a valuation:

$$\text{val}'(y) = \begin{cases} \llbracket \eta_j^\infty \overline{x} . \overline{\psi} \rrbracket^{\text{val}} & \text{if } y = x_j \\ \text{val}(y) & \text{otherwise.} \end{cases}$$

and let κ be the stabilization point of val . Note that for each $j \leq n$, $\text{val}'(x_j)$ changes value above κ at most t_{\max} times and so val' changes its value at most $t_{\max} \times n$ times above κ .

On the other hand, if val' does not change its value for at least α_{\max} steps, it remains constant forever, i.e. if for some $\kappa \leq \alpha < \omega_1$ we have that val' is constant on the interval $[\alpha, \alpha + \alpha_{\max}]$, then it is constant on the entire $[\alpha, \omega_1)$. Indeed, assuming that val' is constant on $[\alpha, \alpha + \alpha_{\max}]$, we show by induction on $\alpha_{\max} \leq \beta$ that it is constant on $[\alpha, \alpha + \beta]$. Indeed we have:

$$\text{val}'(x_j) = \llbracket \eta_j^\infty \overline{x} . \overline{\psi} \rrbracket^{\text{val}} = \llbracket \eta_j^\infty \overline{x} . \overline{\psi} \rrbracket^{\text{val}'} = \llbracket \psi_j \rrbracket^{\text{val}'}$$

and since all \overline{x} are guarded in $\overline{\psi}$, $\llbracket \psi_j \rrbracket^{\text{val}'}(\alpha + \beta)$ depends only on the values of val' strictly below $\alpha + \beta$. In particular, for every $\zeta \leq \alpha + \beta$ we have $\llbracket \psi_j \rrbracket^{\text{val}'}(\zeta) = \llbracket \psi_j \rrbracket^{\text{val}_{\alpha+\beta}}(\zeta)$ where $\text{val}_{\alpha+\beta}$ is the valuation that repeats the last value above $\alpha + \beta$:

$$\text{val}_{\alpha+\beta}(y)(\zeta) = \begin{cases} \text{val}'(y)(\zeta) & \text{if } \zeta < \alpha + \beta \\ \text{val}'(y)(\alpha + \alpha_{\max}) & \text{otherwise.} \end{cases}$$

By the inductive hypothesis, val' is constant on $[\alpha, \alpha + \beta)$, so $\text{val}_{\alpha+\beta}$ is constant on $[\alpha, \omega_1)$, i.e. it stabilizes at α . This implies that $\llbracket \psi_j \rrbracket^{\text{val}_{\alpha+\beta}}$ is stable above $\alpha + \alpha_{\psi_j}$. Since $\alpha_{\psi_j} \leq \alpha_{\max} \leq \beta$, we get that:

$$\llbracket \psi_j \rrbracket^{\text{val}'}(\alpha + \beta) = \llbracket \psi_j \rrbracket^{\text{val}_{\alpha+\beta}}(\alpha + \beta) = \llbracket \psi_j \rrbracket^{\text{val}_{\alpha+\beta}}(\alpha + \alpha_{\max}) = \llbracket \psi_j \rrbracket^{\text{val}'}(\alpha + \alpha_{\max})$$

which shows that val' is indeed constant on $[\alpha, \alpha + \beta]$.

It follows that after at most $t_{\max} \times n$ blocks, each of length at most α_{\max} , the valuation val' stabilizes.

This finishes the proof of Lemma 13 and Theorem 12.