LECTURE NOTES ON COMPLEXITY OF QUANTIFIER ELIMINATION OVER THE REALS

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Introduction

These are lecture notes for a course I gave in mid-1990s for MSc students at the University of Bath. It presents an algorithm with singly exponential complexity for the existential theory of the reals, in the spirit of J. Renegar [7]. Some ideas are borrowed from [6] and [5]. The aim was to convey the main underlying ideas, so many of the proofs and finer details of algorithms are either missing or just sketched. Full proofs and algorithms can be found in the aforementioned papers or, in a somewhat different form, in the fundamental monograph [1].

I changed nothing in the original notes except adding references, bibliography, and correcting obvious typos.

1. Semialgebraic sets

1.1. **Objectives of the course.** We are going to study algorithms for solving systems of polynomial equations and inequalities. The emphasis is on the *complexity*, that is, on the running time of algorithms. Elementary procedures, like Gröbner basis or cylindrical decomposition, have high complexity. Effective algorithms require more advanced mathematical tools.

Computational problems we will be dealing with go beyond just solving systems. The most general problem we are going to consider is *quantifier elimination in the first order theory of the reals*.

1.2. **Formulas and semialgebraic sets.** Let $\mathbb{R}$ denote the field of all real numbers.

Suppose that we are interested in deciding whether or not a system of inequalities

$$f_1(X_1, \ldots, X_n) \geq 0, \ldots, f_k(X_1, \ldots, X_n) \geq 0$$

is consistent, that is, has a solution in $\mathbb{R}^n$. Here $f_i \in \mathbb{R}[X_1, \ldots, X_n]$, $1 \leq i \leq k$ are polynomials with real coefficients in variables $X_1, \ldots, X_n$.

We can express the statement that the system is consistent by writing

$$(1.1) \qquad \exists X_1 \exists X_2 \cdots \exists X_n \left(f_1(X_1, \ldots, X_n) \geq 0, \ldots, f_k(X_1, \ldots, X_n) \geq 0 \right).$$

This statement is either true (if there is a real solution to the system) or false.

We can generalize formula (1.1) in two natural ways. Firstly, we can consider formulae in which not all the variables are bound by quantifier $\exists$.

Example 1.1.

$$(1.2) \qquad \exists X_{n_1} \exists X_{n_1+1} \cdots \exists X_n \left(f_1(X_1, \ldots, X_n) \geq 0, \ldots, f_k(X_1, \ldots, X_n) \geq 0 \right)$$

for a certain n_1, $1 < n_1 \leq n$.

In (1.2) variables

$$X_{n_1}, \ldots, X_n$$

are called *bound*, while the remaining variables

$$X_1, \ldots, X_{n_1-1}$$

are *free*.

Secondly, we can consider formulae with both types of quantifiers, $\forall$ as well as $\exists$.

Example 1.2. A formula with the prefix

$$\forall X_{n_1} \forall X_{n_1+1} \exists X_{n_1+2} \forall X_{n_1+3} \cdots$$

has free variables $X_1, \ldots, X_{n_1-1}$ and both types of quantifiers. Formulae of such kind are called *formulae of the first order theory of $\mathbb{R}$ (or of the reals)*.

Formulae of the first order theory of the reals can be used to express a great variety of statements, properties, problems about real numbers, and therefore about the geometry of $\mathbb{R}^n$.

Formulae without free variables express statements (true or false), while the ones with free variables describe *properties*.

A set of points $(x_1, \ldots, x_{n_1-1}) \in \mathbb{R}^{n_1-1}$, which after substituting their coordinates instead of free variables in a formula give a true statement, is called *semialgebraic*.

1.3. **Tarski-Seidenberg Theorem.** The most important fact about real formulae and semialgebraic sets is the following

Theorem 1.3 (Tarski-Seidenberg). *Any semialgebraic set can be described as a union of sets, each of which is a set of all solutions of a system of polynomial inequalities.*

In other words, a set defined by a formula with quantifiers can be defined by a formula without quantifiers. The process of passing from one representation to another is called *quantifier elimination*.

Remark 1.4. All the above definitions can be modified to suit the field $\mathbb{C}$ of complex numbers rather than the reals. Inequalities $f > 0$ should then be replaced by $f \neq 0$. Subsets of $\mathbb{C}^n$ defined by formulae with quantifiers are called "constructible". Analogy of the Tarski-Seidenberg Theorem is also true.

1.4. **Formal input.** The general problem we are concerned with in this course is quantifier elimination. The input for this problem is of the kind

$$(Q_1 X^{(1)}) \cdots (Q_q X^{(q)}) P(Z, X^{(1)}, \ldots, X^{(q)}).$$

Here:

- $Q_1, \ldots, Q_q$ are quantifiers $\exists$ or $\forall$;
- $X^{(i)} = (X_1^{(i)}, \ldots, X_{n_i}^{(i)})$ — bound variables;
- $Z = (Z_1, \ldots, Z_{n_0})$ — free variables;
- P — Boolean combination of k "atomic" formulae of the kind $f > 0$ or $f = 0$, and for every f the degree $\deg(f) < d$.

Set $n = n_0 + n_1 + \cdots + n_q$.

2. COMPLEXITY

2.1. Complexity for systems of inequalities. Model of computation: Turing machine (or RAM — random access machine).

Consider the system of inequalities

$$(2.1) \qquad f_1 \geq 0, \ldots, f_k \geq 0$$

with $f_i \in \mathbb{Z}[X_1, \ldots X_n]$ (as usual, $\mathbb{Z}$ stands for the ring of all integers), $\deg(f_i) < d$, maxima of bit lengths of coefficients of f_i, $l(f_i) < M$.

We consider first the problem of deciding consistency of system (2.1).

There are some deep reasons to look for an upper complexity bounds of the kind

$$(kd)^E M^{O(1)}.$$

The first algorithm for the problem was proposed by A.Tarski in late 40s [8]. In his bound E can't be estimated from above by any tower of exponents. Such bounds are called "non-elementary" and indicate that the corresponding algorithm is absolutely impractical.

In mid-1970s G. Collins [3] and, independently, H.R. Wüthrich [9] published an algorithm, called "Cylindrical Algebraic Decomposition", for which $E = 2^{O(n)}$. Note that the Gröbner basis algorithm for the similar problem over complex numbers has essentially the same complexity.

Our aim is to construct an algorithm with $E = O(n)$. Thus, the complexity bound we are going to get is $(kd)^{O(n)} M^{O(1)}$.

2.2. Complexity for quantifier elimination. For quantifier elimination problem the upper complexity bound has the same form as for deciding consistency of systems:

$$(kd)^E M^{O(1)}.$$

For Cylindrical Algebraic Decomposition (CAD) the value of E is again equal to $2^{O(n)}$.

Further in this course we are going to describe, with some detail, an algorithm with

$$E = \prod_{0 \leq i \leq q} O(n_i) < (cn)^q$$

for a certain constant c.

2.3. "Real numbers" model. Along with the Turing machine (bit) model of computation, one can consider another model in which the elementary step is an arithmetic operation or comparison operation over real numbers. This model is more abstract in the sense that no constructive representation of real numbers is assumed, the latter are viewed merely as symbols. Real numbers model, therefore, can't serve as a foundation of algorithm's theory in classical sense, however a formal theory of computation based on this model exists (Blum-Shub-Smale theory).

Complexity results for real numbers model are usually stronger than similar results for bit model. One reason is that computations in bit model can take advantage of a clear connection between the size of the input and metric characteristics of definable semialgebraic sets, which allows sometimes to use rapidly converging methods of numerical analysis. Another technical advantage of bit model is a possibility to use effective polynomial factorization as a subroutine which is not always possible for polynomials with abstract coefficients.

On the weak side, real number model algorithm, being applied to polynomials with essentially constructive (e.g., integer) coefficients, can't control the growth of intermediate data. Thus, it is theoretically possible that the bit complexity of an algorithm is significantly worse than it's real number complexity. In practice, however this almost never happens.

Tarski's algorithm works in real number model as well as for Turing machines. Collins' CAD in original version can't handle symbolic coefficients, but probably could be redesigned for real number model.

In this lecture course we follow the approach of James Renegar. Our algorithms will work equally well in both models.

The complexity of algorithms in real number model will be of the form $(kd)^E$ with $E = O(n)$ for deciding consistency problem and $E = \prod_{1 \le i \le q} O(n_i)$ for quantifier elimination problem.

2.4. Lower bounds.

2.4. **Lower bounds.** Observe that the number of the monomials (terms of the kind $a_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n}$) in a polynomial in n variables of the degree d can be equal to

$$\binom{d+n}{n} = d^{O(n)},$$

so the complexity $(kd)^{O(n)}$ for deciding consistency is quite close to the "best possible" (for *dense* representation of polynomials, i.e., if we agree to write zero coefficients). The best possible would be $kd^{O(n)}$, but the existence of an algorithm with such complexity is a known open problem.

For quantifier elimination problem there is a result of Davenport and Heintz [4] proving that polynomials occuring in quantifier-free formula *after* quantifier elimination can have the degrees not less than d^{n^q}. It follows that Renegar's algorithm for quantifier elimination is "close to the best possible".

3. Univariate case: Sturm's theorem

3.1. Computational problems for univariate polynomials.

3.1. **Computational problems for univariate polynomials.** The idea behind almost all methods for solving the problems under consideration, is to reduce the problems to the univariate case. All straightforward methods, like CAD, do that, our effective algorithms will use the same trick.

Strangely enough, we shall not consider the problem that seems the most natural here: of *finding* solutions of univariate equations or inequalities. This may mean producing a rational approximation with a given accuracy to a root. This problem belongs, in fact, to numerical analysis in the classical sense and is rather far from the methods of this course. However, it's worth mentioning that such an approximation indeed can be produced in time polynomial in the length of the description of input polynomials and accuracy bound.

So, if we don't "compute" roots (in the above sense) what is left algorithmically? Over $\mathbb{C}$ (complex numbers) there is really not much to do since, according to Fundamental Theorem of Algebra, any polynomial f has exactly $\deg_X(f)$ roots, and if we want to "define" a root by an irreducible over rational numbers polynomial, we simply have to factorize f. Also, if we have a system of equations $f_1 = \cdots = f_k = 0$ we can decide its consistency by computing GCD (greatest common divisor) over rational numbers of $f_1, \ldots, f_k$.

In real case, already deciding consistency of $f = 0$ is a nontrivial task.

3.2. **Sturm's Theorem: formulation.** Let $f \in \mathbb{R}[X]$, an interval $[a, b] \subset \mathbb{R}$, the degree $\deg(f) < d$.

Let f' be the derivative of f.

Set

$h_1 = f;$

$h_2 = f';$

h_{i+1} — the reminder of division of h_{i-1} by h_i, taken with the opposite sign: $h_{i-1} = q_i h_i - h_{i+1}$, $\deg(h_{i+1}) < \deg(h_i)$. Thus, $h_{i+1} = q_i h_i - h_{i-1}$.

Continue the process of producing polynomials h_i while the division is possible (i.e., until the reminder becomes a constant). If the last (constant) reminder $h_t \neq 0$ (in this case f and f' are relatively prime), then divide all h_i's by h_t. The resulting sequence of polynomials $f_1 = f/h_t, f_2 = f'/h_t, f_3 = h_3/h_t, \ldots, f_{t-1} = h_{t-1}/h_t, f_t = 1$ is called *Sturm's sequence*. If $h_t = 0$, then $f_1 = f/h_{t-1}, f_2 = f'/h_{t-1}, \ldots, f_{t-2} = h_{t-2}/h_{t-1}, 1$ is the *Sturm sequence*.

For any $x \in \mathbb{R}$ let $V(x)$ be the number of *sign changes* in the sequence of values of the Sturm's sequence: $f_1(x), f_2(x), \ldots$ Formally, $V(x)$ is the number of pairs (i, j) of indices in Sturm's sequence such that $i < j$, $f_i(x)f_j(x) < 0$ and $f_k(x) = 0$ if $i < k < j$.

Theorem 3.1 (Sturm's Theorem). *The number of distinct real roots of f in $[a, b]$ is $V(a) - V(b)$.*

Proof. Can be easily found in literature. $\square$

3.3. **Sturm's Theorem: discussion.** Sturm's Theorem gives and algorithm for deciding whether f has a real root. Indeed, from the formulation follows how to decide this for an interval $[a, b]$. It's well known that *all* real roots of a univariate polynomial $a_d X^d + \cdots + a_1 X + a_0$ belong to the interval $[-R, R]$ with

$$R = 1 + \max_{0 \leq i \leq d-1} |a_i/a_d|.$$

So, it is enough to compute $V(-R) - V(R)$. If this difference is > 0 then there is a real root.

A simpler way is to pass from intervals to whole $\mathbb{R}$ is to notice that for any polynomial $f = a_d X^d + \cdots + a_1 X + a_0$ the sign of $f(\infty)$ (i.e., sign of f at any sufficiently large point $x \in \mathbb{R}$) coincides with the sign of a_d (or, equivalently, with the sign of $a_d X^d$ at 1), while the sign of $f(-\infty)$ coincides with the sign of $a_d X^d$ at -1.

Observe also that Sturm's Theorem provides conditions in terms of polynomial equations and inequalities on coefficients of f, which are true if and only if f has a root in $\mathbb{R}$.

4. Univariate case: Ben-Or – Kozen – Reif algorithm

4.1. **Towards generalization of Sturm's Theorem.** Our nearest goal is to generalize Sturm's Theorem to the case of several univariate polynomials.

Let $f_1, \ldots, f_k \in \mathbb{R}[X]$ with degrees $\deg(f_i) < d$ for all i, $1 \leq i \leq k$.

Definition 4.1. Consistent sign assignment for $f_1, \ldots, f_k$ is a string

$$\sigma = (\sigma_1, \ldots, \sigma_k),$$

where $\sigma_i \in \{>, <, =\}$, such that the system $f_1 \sigma_1 0, \ldots, f_k \sigma_k 0$ has a root in $\mathbb{R}$.

We want to generalize Sturm's Theorem and construct an algorithm for listing all consistent sign assignments. Moreover, we want to get polynomial conditions on coefficients, guaranteeing the existence of a common root.

The following algorithm belongs to M. Ben-Or, D. Kozen, and J. Reif [2].

4.2. **Preparation of the input.** First we "prepare" the input family so that it would be more simple.

(1)   Make all f_i's *squarefree* and *relatively prime*. It is sufficient to compute greatest common divisor (GCD) g_i of f_i and its derivative f_i' for all $1 \le i \le k$ (with the help of Euclidean algorithm), and then to divide each g_i by GCD g of the family $g_1, \ldots, g_k$. Now we can reconstruct all consistent sign assignments of $f_1, \ldots, f_k$ from all consistent sign assignments of $g, g_1, \ldots, g_k$.

Throughout this portion of notes we assume that all f_i's are *squarefree* and *relatively prime*. It follows that all the sets of all roots of polynomials f_i are disjoint, thus *any* consistent sign assignment for the family has *at most one* "=" sign.

(2)   Add a new member f in the family $f_1, \ldots, f_k$, so that any consistent sign assignment for $f, f_1, \ldots, f_k$ has *exactly* one "=" sign, and each consistent sign assignment for $f_1, \ldots, f_k$ can be obtained from a consistent sign assignment for $f, f_1, \ldots, f_k$. Let the product $f_1 \cdots f_k = a_{kd} X^{kd} + \cdots + a_0$. Define

$$R = 1 + \max_{0 \le j < kd} \left\{ \frac{a_j}{a_{kd}} \right\}.$$

Then we can take

$$f = \frac{d(f_1 \cdots f_k)}{dX}(X - R)(X + R).$$

Now the problem can be reformulated as follows: list all consistent assignments of signs $<$ and $>$ for $f_1, \ldots, f_k$ at roots of f.

4.3. **Cases $k = 0$ and $k = 1$.** For $k = 0$ use Sturm's sequence obtained by Euclidean algorithm.

Denote $V(-\infty) - V(\infty) = S(f, f')$ (recall that we start with Euclidean division of f by f').

In case $k = 1$ we have a family of two polynomials f, f_1.

Denote:

$$C_1 = \{f = 0 \ \& \ f_1 > 0\};$$
$$\overline{C_1} = \{f = 0 \ \& \ f_1 < 0\}.$$

For a finite set A, let $|A|$ be the number of elements in A. Obviously, $S(f, f') = |C_1| + |\overline{C_1}|$.

The following trick is very important for generalization of Sturm's Theorem. Construct Sturm's sequence of remainders starting with dividing f by $f' f_1$ (rather than simply by f' as in original Sturm's sequence). Denote the difference of numbers of sign changes at $-\infty$ and at ∞ by $S(f, f' f_1)$.

Lemma 4.2. $S(f, f' f_1) = |C_1| - |\overline{C_1}|$.

Proof. Trivial modification of a proof of original Sturm's Theorem. We don't consider the proof in this course.   □

Original Sturm and this Lemma imply that the following system of linear equations with the unknowns $|C_1|, |\overline{C_1}|$ is true:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |C_1| \\ |\overline{C_1}| \end{pmatrix} = \begin{pmatrix} S(f, f') \\ S(f, f'f_1) \end{pmatrix}.$$

Denote the (2×2)-matrix of this system by A_1.

The value $|C_1|$ determines whether $\{f = 0 \ \& \ f_1 > 0\}$ is consistent. The value $|\overline{C_1}|$ determines whether $\{f = 0 \ \& \ f_1 < 0\}$ is consistent. Hence, after the linear system is solved all consistent sign assignments will be listed.

4.4. **Case $k = 2$.** In this case we come to the following system of linear equations:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} |C_1 \cap C_2| \\ |\overline{C_1} \cap C_2| \\ |C_1 \cap \overline{C_2}| \\ |\overline{C_1} \cap \overline{C_2}| \end{pmatrix} = \begin{pmatrix} S(f, f') \\ S(f, f'f_1) \\ S(f, f'f_2) \\ S(f, f'f_1f_2) \end{pmatrix},$$

where

$$C_2 = \{f = 0 \ \& \ f_2 > 0\}, \quad \overline{C_2} = \{f = 0 \ \& \ f_2 < 0\}.$$

Denote the matrix of this system by A_2.

Observe, that if this system of linear equations is true, then all consistent sign assignments for f, f_1, f_2 will be found.

Checking that the system is indeed true, is straightforward:

(1) $|C_1 \cap C_2| + |C_1 \cap \overline{C_2}|$ is the number of all roots of f with $f_1 > 0$, while $|\overline{C_1} \cap C_2| + |\overline{C_1} \cap \overline{C_2}|$ is the number of all roots of f with $f_1 < 0$, so the left-hand part of the first equation is the whole number of roots of f;

(2) $|C_1 \cap C_2| + |C_1 \cap \overline{C_2}|$ is the number of all roots of f with $f_1 > 0$, while $|\overline{C_1} \cap C_2| + |\overline{C_1} \cap \overline{C_2}|$ is the number of all roots of f with $f_1 < 0$, so the left-hand part of the second equation equals to the difference between these two quantities, which is $S(f, f'f_1)$ by the Lemma 4.2;

(3) similar to (2);

(4) $|C_1 \cap C_2| + |\overline{C_1} \cap \overline{C_2}|$ is the number of all roots of f with f_1 and f_2 having the same sign (i.e., $f_1 f_2 > 0$), while $|\overline{C_1} \cap C_2| + |C_1 \cap \overline{C_2}|$ is the number of all roots of f with f_1 and f_2 having distinct signs (i.e., $f_1 f_2 < 0$), so the left-hand part of the last equation is $S(f, f'f_1f_2)$ by the Lemma 4.2 (applied to $f_1 f_2$ instead of f_1).

4.5. **Tensor product of matrices.** Observe that $A_2 = A_1 \otimes A_1$, where $\otimes$ is the operation of *tensor product*.

Definition 4.3. Let $A = \|a_{ij}\|_{1 \leq i \leq m; 1 \leq j \leq n}$ and $B = \|b_{ij}\|_{1 \leq i \leq m'; 1 \leq j \leq n'}$ be two matrices. Then the tensor product of A by B is the $(mm' \times nn')$-matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \cdots & \cdots & \cdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix},$$

where

$$a_{ij}B = \begin{pmatrix} a_{ij}b_{11} & \cdots & a_{ij}b_{1n'} \\ \cdots & \cdots & \cdots \\ a_{ij}b_{m'1} & \cdots & a_{ij}b_{m'n'} \end{pmatrix}.$$

4.6. **Case of arbitrary** k**.** Let us consider now the case of a family $f, f_1, \ldots, f_k$ for arbitrary k. Define the matrix $A_k = A_1 \otimes A_1 \otimes \cdots \otimes A_1$ (k times). Let C be the vector of all possible elements of the form $|D_1 \cap \cdots \cap D_k|$ where $D_i \in \{C_i, \overline{C_i}\}$, $1 \leq i \leq k$; and S be the vector of all possible elements of the form $S(f, f' f_{j_1} \cdots f_{j_t})$ for all $(j_1, \ldots, j_t) \subset \{1, \ldots, k\}$. Herewith, the component of C having "bared" sets $\overline{C_{j_1}}, \ldots, \overline{C_{j_t}}$ corresponds to the component $S(f, f' f_{j_1} \cdots f_{j_t})$ of S.

Lemma 4.4. $A_k C = S$

Proof.   Streightforward observation (similar to particular case of $k = 2$ in Section 4.4). $\qquad\square$

Square matrix A_k is nonsingular. Its unique solution describes all consistent sign assignments for $f, f_1, \ldots, f_k$. Solving the system by any of standard methods (e.g., Gaussian elimination) we list all consistent sign assignments. Let us estimate the complexity of this algorithm.

From definition of tensor product it follows that the order of A_k is $2^k \times 2^k$. Vectors C and S have obviously 2^k components each. Each component of the vector S can be computed (in real numbers model) by Euclidean division in time polynomial in kd, and we need to find 2^k components. In bit model, supposing that the bit sizes of integer coefficients of polynomials $f, f_1, \ldots, f_k$ are less than M, vector S can be computed with complexity polynomial in $M 2^k d$. The complexity of Gaussian algorithm is polynomial in the order of the matrix, i.e., in 2^k (in real numbers model) and $M 2^k$ (in bit model). Thus, the total running time of finding all consistent sign assignments via system $A_k C = S$ is polynomial in $2^k d$ (real numbers) or in $M 2^k d$ (bit).

We want, however, to obtain an algorithm whose running time is polynomial in k, d (real numbers) and in M, k, d (bit).

4.7. **Divide-and-conquer algorithm.** A key observation to get a better algorithm is that since $\deg(f_i) < d$ and $\deg(f) < kd + 2$, the number of roots of f is less than $kd + 2$, and therefore, the total number of all consistent sign assignments is less than $kd + 2$.

It follows that among the components of the solution of the system $A_k C = S$ there are less than $kd + 2$ nonzero ones. Thus, $A_k C = S$ is equivalent to a much smaller (for large k) $\big((kd + 2) \times (kd + 2)\big)$-linear system. Now the problem is how to construct that smaller system effectively.

We shall apply the usual divide-and-conquer recursion on k (supposing for simplicity of notations that k is even).

Suppose that we can do all the computation in the case of $k/2$. Namely, assume that we already have the solutions for two problems for two families of polynomials:

$$f, f_1, \ldots, f_{k/2} \quad \text{and} \quad f, f_{k/2+1}, \ldots, f_k.$$

This means, we have constructed:

(1)   two square nonsingular matrices A and A' with elements from $\{-1, 1\}$, of orders $m \times m$ and $n \times n$ respectively, $m, n < kd + 2$;

(2)   two vectors S and S' of sizes m and n respectively with elements of the forms

$$S(f, f' \prod I), \, I \subset \{f_1, \ldots, f_{k/2}\}$$

and

$$S(f, f' \prod J), \ J \subset \{f_{k/2+1}, \ldots, f_k\}$$

respectively, such that the systems $AC = S$, $A'C' = S'$ have solutions C, C' respectively with *nonzero coordinates only* of the forms

$$\Big| \bigcap_{q \in I} C_q \cap \bigcap_{q \in \{f_1, \ldots, f_{k/2}\} \backslash I} \overline{C_q} \Big|, \ I \subset \{f_1, \ldots, f_{k/2}\},$$

and

$$\Big| \bigcap_{q \in J} C_q \cap \bigcap_{q \in \{f_{k/2+1}, \ldots, f_k\} \backslash J} \overline{C_q} \Big|, \ J \subset \{f_{k/2+1}, \ldots, f_k\}$$

respectively;

(3) solutions C and C' which correspond to all consistent sign assignments for $f_1, \ldots, f_{k/2}$ and $f_{k/2+1}, \ldots, f_k$ respectively.

Now let us construct a new linear system. As a matrix take $A \otimes A'$, which is a square nonsingular $(mn \times mn)$-matrix.

Two vectors S, S' are combined in a new column vector SS' consisting of m column vectors placed end-to-end, each of length n.

The (i, j)th component of SS' is

$$S(f, f' \prod(I \triangle J)),$$

where $S(f, f' \prod I)$ is the ith component of S, $S(f, f' \prod J)$ is the jth component of S', and $\triangle$ denotes the *symmetric difference* of two sets of polynomials:

$$I \triangle J = (I \cup J) \setminus (I \cap J).$$

Finally, two vectors C and C' are combined into a new column vector CC'.

Components of CC' are indexed by pairs (i, j). The (i, j)th component of CC' is $|E \cap E'|$ where $|E|$ is the ith component of C and $|E'|$ is jth component of C'.

Lemma 4.5. $(A \otimes A')(CC') = SS'$.

We don't consider a proof of Lemma 4.5 in the course.

Observe that CC' describes all consistent sign assignments for $f, f_1, \ldots, f_k$. However, unlike vectors C and C', the vector CC' *may* have zero components. As was already noted, the new system has the order $mn \times mn$. It may happen that $mn > kd + 2$. In this case CC' indeed contains zero components, not less than $mn - (kd + 2)$.

By solving system, we find CC', and after that *reduce* the system $(A \otimes A')(CC') = SS'$. Namely, we delete all zero components of CC' and the corresponding columns of $A \otimes A'$ to obtain a rectangular system of order at most $mn \times (kd+2)$. After that, choose a basis among the rows of the matrix and form a new $\big((kd+2) \times (kd+2)\big)$-matrix. Delete all components of SS' *not* corresponding to basis rows.

The resulting system is the output of the recursive step of the algorithm.

4.8. **Complexity of Ben-Or–Kozen–Reif algorithm.** It's easy to see that the algorithm has the running time polynomial in k, d (and M, in bit model). Indeed, denoting the complexity function by $T(k)$, we get a functional inequality

$$T(k) \le 2T(k/2) + p(kd),$$

for a certain polynomial p, whose solution $T(k) \le O\big(p(kd)\big)$.

5. Systems of polynomial inequalities

5.1. **Deciding consistency of systems of polynomial inequalities.** We address the basic problem of deciding consistency of systems of inequalities

$$(5.1) \qquad\qquad f_1 \geq 0, \ldots, f_{k_1} \geq 0, f_{k_1+1} > 0, \ldots, f_k > 0.$$

Here polynomials $f_i \in \mathbb{Z}[X_1, \ldots, X_n]$ (for bit model) or $f_i \in \mathbb{R}[X_1, \ldots, X_n]$ (for real numbers model). A version of the algorithm presented below belongs to James Renegar.

It is convenient to deal with systems of nonstrict inequalities only. Introduce a new variable Y and replace strict inequalities in (5.1) by

$$Y f_{k_1+1} \geq 1, \ldots, Y f_k \geq 1, Y \geq 0.$$

Obviously, (5.1) is consistent if and only if the new system is consistent. Changing the notations, we shall consider, in what follows, the problem of consistency of a system of nonstrict inequalities

$$(5.2) \qquad\qquad f_1 \geq 0, \ldots, f_k \geq 0.$$

Because it is easy to check whether or not the origin $(0, \ldots, 0)$ is a solution of (5.2), we shall assume that the origin does not satisfy (5.2).

Finally, we suppose that the set $\{(5.2)\}$ is bounded, if nonempty. The non-bounded case can be reduced to a bounded by intersecting $\{(5.2)\}$ with a ball in $\mathbb{R}^n$ of a sufficiently large radius, we are not considering the details of this construction in the course.

5.2. **Towards reduction to a system of equations.** Let the degrees $\deg(f_i) < d$ for all $1 \leq i \leq k$, and D be the minimal even integer not less than $kd+1$. Introduce a new variable ε, and consider the polynomial

$$g(\varepsilon) = \prod_{1 \leq i \leq k} (f_i + \varepsilon) - \varepsilon^{k+1} \sum_{1 \leq j \leq n} X_j^D.$$

Lemma 5.1. *If (5.2) is consistent, then there is a solution of (5.2) which is a limit of a root of*

$$(5.3) \qquad\qquad \frac{\partial g(\varepsilon)}{\partial X_1} = \cdots = \frac{\partial g(\varepsilon)}{\partial X_n} = 0$$

as $\varepsilon \longrightarrow 0$.

Proof. Let $x \in \{(5.2)\}$. Then for all sufficiently small positive values of ε

$$\varepsilon^{k+1} \sum_j x_j^D < \prod_i (f_i(x) + \varepsilon),$$

hence $g(\varepsilon)(x) > 0$. Fix for a time being one of these values of ε.

Let C be any connected component of $\{g(\varepsilon) > 0\}$ such that $x \in C$. Note that at all points $v \in C$ the sign of $f_i(v) + \varepsilon$ is the same for each $i \in \{1, \ldots, k\}$, hence $f_i(v) + \varepsilon > 0$.

Observe that C is bounded. Indeed, suppose C is *not bounded*. Then, there exists a point $y \in C \setminus \{(5.2)\}$ at an arbitrarily large distance from $\{(5.2)\}$. In particular, for such y there is at least one $i \in \{1, \ldots, k\}$, with $f_i(y) < 0$, having a large distance $\mathrm{dist}(y, \{f_i(y) = 0\})$. It follows from basic facts of analytic geometry (Łojasiewicz

inequality), that for $\text{dist}(y, \{f_i(y) = 0\})$ sufficiently larger than ε, we have $|f_i(y)| > \varepsilon$, i.e., $f_i(y) + \varepsilon < 0$. Since C is connected, there is a connected (semialgebraic) curve $\Gamma \subset C$ containing x and y. Because $f_i(x) + \varepsilon > 0$ and $f_i(y) + \varepsilon < 0$, there exists a point $z \in \Gamma$ with $f_i(z) + \varepsilon = 0$. We get $g(\varepsilon)(z) = -\varepsilon^{k+1} \sum_j z_j^D \leq 0$, which contradicts $z \in \{g(\varepsilon) > 0\}$.

At every point w in the *boundary* of C polynomial $g(\varepsilon)(w) = 0$. It follows that $g(\varepsilon)$ attains a *local maximum* at a point $v \in C$. It is well known that any such point v is a root of the system of equations

$$\frac{\partial g(\varepsilon)}{\partial X_1}(v) = \cdots = \frac{\partial g(\varepsilon)}{\partial X_n}(v) = 0.$$

Also, since $v \in C$, we have

$$f_1(v) + \varepsilon > 0, \ldots, f_k(v) + \varepsilon > 0.$$

Passing to limit in the latter system of inequalities, we get

$$f_1(v') \geq 0, \ldots, f_k(v') \geq 0,$$

where $v \longrightarrow v'$ as $\varepsilon \longrightarrow 0$.

The lemma is proved. $\square$

5.3. **Homogeneous polynomials and projective spaces.** Any polynomial is an expression of the form

$$\sum_{(i_1,\ldots,i_n)} a_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n}.$$

A polynomial is called *homogeneous polynomial* or *form* if $i_1 + \cdots + i_n$ is a non-zero constant for all $(i_1, \ldots, i_n)$ with $a_{i_1,\ldots,i_n} \neq 0$.

Examples.

(i)   Homogeneous polynomial: $X^2Y^3 + X^5 + XY^4$;

(ii) Homogeneous polynomial: $X^2 - Y^2$;

(iii) Nonhomogeneous polynomial: $X^2 - 1$.

Observe that zero is always a root of a homogeneous polynomial. Also, if $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ is a root of a homogeneous polynomial $h \in \mathbb{C}[X_1, \ldots, X_n]$, then for any $a \in \mathbb{C}$ the point $ax = (ax_1, \ldots, ax_n)$ is again a root of h. For a fixed $x \neq 0$ the set of all points of the form ax, $a \in \mathbb{C}$ is a straight line passing through the origin. So, if $x \neq 0$ is a root of h, then the line passing through x and the origin consists of roots of h. Thus, we can say that the *line itself is a root* of h.

This motivates the idea to consider (in homogeneous case) a space of *straight lines*, passing through the origin, instead of the usual space of *points*. It is called an $(n-1)$-*dimensional projective space* and is denoted by $\mathbb{P}^{n-1}(\mathbb{C})$. An element of $\mathbb{P}^{n-1}(\mathbb{C})$, which is a line passing through $x = (x_1, x_2, \ldots, x_n) \neq (0, \ldots, 0)$, is denoted by $(x_1 : x_2 : \cdots : x_n)$ (i.e., we use colon instead of comma). Of course, for any $0 \neq a \in \mathbb{C}$ elements of $\mathbb{P}^{n-1}(\mathbb{C})$:

$$(x_1 : x_2 : \cdots : x_n) \quad \text{and} \quad (ax_1 : ax_2 : \cdots : ax_n)$$

coincide. The usual space of points $\mathbb{C}^n$ is called *affine* if there is a possibility of a confusion.

Observe that our construction is leading to the projective space of the dimension $n-1$, which is quite natural as can be seen from the example of $\mathbb{P}^1(\mathbb{C})$. The latter is the space of all straight lines on the plane P of (complex) coordinates X, Y, passing through the origin. All elements of the projective space $\mathbb{P}^1(\mathbb{C})$, except one, correspond bijectively to all points of the line $\{X = 1\}$. Namely, if $(x : y) \in \mathbb{P}^1(\mathbb{C})$ and $x \neq 0$, then $(x : y) = \big(1 : (y/x)\big) \in \mathbb{P}^1(\mathbb{C})$, and *the point* of P $\big(1, (y/x)\big) \in \{X = 1\}$. The only element, for which this correspondence fails, is $(0 : y)$ (for all $y \in \mathbb{C}$ this is the same element, called *point at infinity*). Thus, we have a parametrization of $\mathbb{P}^1(\mathbb{C})$ minus point at infinity by points of the line. This justifiers the definition of the dimension of $\mathbb{P}^1(\mathbb{C})$ as "one".

An arbitrary polynomial $h \in \mathbb{C}[X_1, \ldots, X_n]$ can be naturally homogenized. Namely, let $\deg_{X_1,\ldots,X_n}(h) = d$. Introduce a new variable X_0 and consider the homogeneous polynomial $\mathrm{hom}(h) = X_0^d h(X_1/X_0, \ldots, X_n/X_0)$. If $(x_1, \ldots, x_n)$ is a root of h, then $(1 : x_1 : \cdots : x_n)$ is a root of $\mathrm{hom}(h)$. On the other hand, if $(y_0 : y_1 : \cdots : y_n)$ is a root of $\mathrm{hom}(h)$ and $y_0 \neq 0$, then $(y_1/y_0, \ldots, y_n/y_0)$ is a root of h. So, there is a bijective correspondence between roots of h and roots of of $\mathrm{hom}(h)$ with $y_0 \neq 0$. However, $\mathrm{hom}(h)$ can have a root of the form $(0 : z_1 : \cdots : z_n)$ (then, of course, at least one of $z_1, \ldots, z_n$ is different from 0). This root does not correspond to any root of h (in the sense described above), it's called "root at infinity".

Example.

$h = X_1 - X_2 + 1$;
$\mathrm{hom}(h) = X_1 - X_2 + X_0$.
$(0 : 1 : 1) = (0 : a : a)$ for any $0 \neq a \in \mathbf{C}$, is the only root at infinity of $\mathrm{hom}(h)$.

The described above relation between the roots of a polynomial and it's homogenization can be extended to roots of *systems of polynomial equations*. A system can have *no roots* (i.e., be inconsistent), while its homogenization can have roots.

Example.

$X_1 - X_2 = X_1 - X_2 + 1 = 0$;
homogenization: $X_1 - X_2 = X_1 - X_2 + X_0 = 0$.
The first system is inconsistent, while the homogenized one has the unique root at infinity $(0 : 1 : 1)$.

We conclude that the homogenized system has not less roots than the original system. This implies, for instance, that if the homogenized system has *at most finite number of roots*, then the original system has at most finite number of roots as well. We shall use this remark further on.

The reason for considering homogeneous equations in projective space instead of arbitrary ones in affine space, is that the projective case is much richer in useful properties.

5.4. u-**resultant.** Consider a system of n *homogeneous* polynomial equations in variables $X_0, X_1, \ldots, X_n$:

$$(5.4) \qquad\qquad\qquad h_1 = \cdots = h_n = 0.$$

Let $\deg(h_i) < d$ for each $i \in \{1, \ldots, n\}$.

For some deep algebraic reasons, which we are not discussing in this course, the system (5.4) is *always consistent* in $\mathbb{P}^n(\mathbb{C})$. It can have either finite or infinite number of roots.

Introduce new variables $U_0, U_1, \ldots, U_n$.

Lemma 5.2. *There exists a homogeneous polynomial* $R_{h_1, \ldots, h_n} \in \mathbb{C}[U_0, \ldots, U_n]$ *called u-resultant, such that* $R_{h_1, \ldots, h_n} \equiv 0$ *(is identically zero) if and only if (5.4) has infinite number of roots. Moreover, if* $R_{h_1, \ldots, h_n} \not\equiv 0$ *(and, thus (5.4) has finite number r of roots), then it can be decomposed into linear factors:*

$$R_{h_1, \ldots, h_n} = \prod_{1 \leq i \leq r} (\chi_0^{(i)} U_0 + \chi_1^{(i)} U_1 + \cdots + \chi_n^{(i)} U_n)$$

where $(\chi_0^{(i)} : \cdots : \chi_n^{(i)})$, $1 \leq i \leq r$ *are all the roots of (5.4) in* $\mathbb{P}^n(\mathbb{C})$. *The degree of R is* $d^{O(n)}$.

We are not considering the full proof of this important lemma in this course. At certain point our algorithm will need to *compute* the u-resultant. At that point we shall give a sketch of the procedure together with some hints on a proof of Lemma 5.2.

5.5. **System (5.3) has a finite number of roots.**

Lemma 5.3. *For all sufficiently small positive values of ε, if (5.2) is consistent, then (5.3) has a finite number of roots.*

Proof. We already know (Lemma 5.1) that if (5.2) is consistent, then (5.3) has at least one root. We also know (end of Section 5.3) that it's sufficient to prove that the homogenization of (5.3) has finite number of roots in the projective space $\mathbb{P}^n(\mathbb{C})$.

Let $G_i(\varepsilon)$ be the homogenization of $\partial g(\varepsilon)/\partial X_i$. We shall now prove that

$$(5.5) \qquad\qquad G_1(\varepsilon) = \cdots = G_n(\varepsilon) = 0$$

has finite number of roots in $\mathbb{P}^n(\mathbb{C})$ for all sufficiently small $\varepsilon > 0$.

Homogeneous polynomial $G_i(\varepsilon)$ is of the form:

$$G_i(\varepsilon) = F(\varepsilon) - \varepsilon^{k+1} D X_i^{D-1},$$

where $F(\varepsilon)$ is a homogeneous polynomial (w.r.t. variables $X_0, \ldots, X_n$).

The degree of $F(\varepsilon)$ (w.r.t. ε) is k. Thus, for any fixed values of $X_0, \ldots, X_n$, we have $F(\varepsilon)/\varepsilon^{k+1} \longrightarrow 0$ as $\varepsilon \longrightarrow \infty$.

Observe that for $\varepsilon \neq 0$ the system (5.5) has the same set of roots as

$$(5.6) \qquad\qquad \frac{G_1(\varepsilon)}{\varepsilon^{k+1}} = \cdots = \frac{G_n(\varepsilon)}{\varepsilon^{k+1}} = 0.$$

The u-resultant of (5.6) is either identically zero (as a polynomial in $U_0, \ldots, U_n$) for *all* values of ε, or is identically zero for *at most finite number* of values of ε.

The first alternative does not occur, because for *large* positive values of ε, the system (5.6) tends to

$$(5.7) \qquad\qquad X_1^{D-1} = \cdots = X_n^{D-1} = 0,$$

so the u-resultant of (5.6) tends to u-resultant of (5.7). But (5.7) has exactly one root in $\mathbb{P}^n(\mathbb{C})$ (namely, $(1 : 0 : \cdots : 0)$), so u-resultant for (5.7), and hence for (5.6) is not identically zero for large values of ε.

It follows that the u-resultant is not identically zero also for all sufficiently *small* positive values of ε, so for these values (5.5) has a finite number of roots.   □

5.6. **General scheme of the algorithm.** Recall that, according to Lemma 5.1, if (5.2) is consistent, then there is a solution to (5.2) which is a limit of a root of (5.3) as $\varepsilon \longrightarrow 0$.

On the other hand, for all sufficiently small positive values of ε (5.3) has a finite number of roots, due to Lemma 5.3. Moreover, the homogenization (5.5) of (5.3) has a finite number of roots. Now Lemma 5.2 implies that every root of (5.5) is a coefficient vector in a divisor of u-resultant $R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)}$.

If we were able to factorize $R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)}$, over $\mathbb{C}$, then the process of deciding the consistency of (5.2) could be roughly as follows: construct the u-resultant for (5.5), factorize, then for each coefficient vector $(\chi_0^{(i)}, \ldots, \chi_n^{(i)})$ of each factor of the u-resultant find its component-wise limit $(\eta_0^{(i)}, \ldots, \eta_n^{(i)})$, and check whether the point

$$(\eta_1^{(i)}/\eta_0^{(i)}, \ldots, \eta_n^{(i)}/\eta_0^{(i)})$$

satisfies the system (5.2). If yes, then (5.2) is consistent, else (5.2) is inconsistent.

Unfortunately, we are unable to factorize a polynomial in general case. Even in special cases, when effective multivariate factorization algorithms are known, they are very involved.

Therefore our scheme will be different. After constructing the u-resultant, we shall find another polynomial in $U_0, \ldots, U_n$, whose factors correspond to limits of factors of the u-resultant.

Instead of factoring this new polynomial, we shall use the fact that the set of its roots in $\mathbb{C}^{n+1}$ is a union of a finite number of *hyperplanes*, passing through the origin. The coefficient vectors defining these hyperplanes are exactly the coefficient vectors of factors and can be found by computing the gradients of the polynomial at the appropriate points.

5.7. **Passing to limit in the u-resultant.** Recall that $R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)}$ is a polynomial in the coefficients of homogeneous polynomials $G_i(\varepsilon)$. In particular, it is a polynomial in ε.

Represent $R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)}$ in the form:

$$R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)} = R_m \varepsilon^m + \sum_{j>m} R_j \varepsilon^j,$$

where R_m is not identically zero in $U_0, \ldots, U_n$, i.e., m is the *lowerst* degree of a term in ε of $R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)}$.

Lemma 5.4.
$$R_m = \prod_{1 \leq i \leq r} (\eta_0^{(i)} U_0 + \cdots + \eta_n^{(i)} U_n),$$

where $(\eta_0^{(i)}, \ldots, \eta_n^{(i)})$ is the limit of $(\chi_0^{(i)}, \ldots, \chi_n^{(i)})$ as $\varepsilon \longrightarrow 0$ (some, but not all, of the $\eta^{(i)}$-vectors can be zero).

Proof. Consider
$$\frac{R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)}}{\varepsilon^m} = R_m + \sum_{j>m} R_j \varepsilon^{j-m}.$$

As ε tends to zero, the polynomial $R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)}/\varepsilon^m$ tends to the polynomial R_m. Hence the roots (in the projective space $\mathbb{P}^n(\mathbb{C})$ with projective coordinates $U_0,\ldots,U_n$) of $R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)}$ tend to roots of R_m, as $\varepsilon \longrightarrow 0$.

Because $R_{G_1(\varepsilon),\ldots,G_n(\varepsilon)}$ factors linearly, the set of all roots of this polynomial is a collection of r hyperplanes $\left\{ \chi_0^{(i)} U_0 + \cdots + \chi_n^{(i)} U_n = 0 \right\}$. Each hyperplane is uniquely described by its normal vector $(\chi_0^{(i)},\ldots,\chi_n^{(i)})$.

If follows that the set of all roots of R_m is a collection of *at most* r hyperplanes, each of which is uniquely described by its normal vector

$$\left(\lim_{\varepsilon \longrightarrow 0}(\chi_0^{(i)}),\ldots, \lim_{\varepsilon \longrightarrow 0}(\chi_n^{(i)}) \right).$$

Denoting

$$\eta_j^{(i)} = \lim_{\varepsilon \longrightarrow 0}(\chi_j^{(i)})$$

for $1 \le i \le r$; $0 \le j \le n$, we get:

$$R_m = \prod_{1 \le i \le r} (\eta_0^{(i)} U_0 + \cdots + \eta_n^{(i)} U_n).$$

$\square$

Corollary 5.5. *If (5.2) is consistent, then a solution of (5.2) is*

$$(\eta_1^{(i)}/\eta_0^{(i)},\ldots,\eta_n^{(i)}/\eta_0^{(i)}),$$

where $\eta_0^{(i)} U_0 + \cdots + \eta_n^{(i)} U_n$ is a divisor of R_m.

5.8. **The algorithm modulo u-resultant construction.** We noticed in the proof of the Lemma 5.4 that factors of R_m correspond to hyperplanes in $(n+1)$-dimensional affine space of coordinates $U_0,\ldots,U_n$. Namely, $\eta_0^{(i)} U_0 + \cdots + \eta_n^{(i)} U_n = 0$, $1 \le i \le r$ are equations for the hyperplanes. Then $(\eta_0^{(i)},\ldots,\eta_n^{(i)})$ are vectors orthogonal to the corresponding hyperplanes. Obviously, the orthogonal vectors are collinear to gradients, i.e., vectors of the kind

$$\left(\frac{\partial R_m}{\partial U_0}(x),\ldots, \frac{\partial R_m}{\partial U_n}(x) \right),$$

for x taken on all hyperplanes (provided that these gradients are nonzero, see Section 5.9).

We reduced the consistency problem for (5.2) to the following: compute gradients of the polynomial R_m at the points $x^{(1)},\ldots,x^{(r)}$, where $x^{(i)}$ belongs to the hyperplane $\eta_0^{(i)} U_0 + \cdots + \eta_n^{(i)} U_n = 0$, and does not belong to any other hyperplane. We shall address the problem of constructing R_m in Section 5.11 below. Now let us describe the rest of the algorithm.

We shall obtain points $x^{(1)},\ldots,x^{(r)}$ as the points of intersections of a "generic" straight line in $\mathbb{R}^{n+1}$ with $\{R_m = 0\}$. Intuitively, *almost every* straight line intersects each member of our family of hyperplanes at exactly one point, and all points are distinct.

Suppose that we have found the generic line in a parametric form $\alpha t + \beta$, where t is a new variable, and $\alpha, \beta \in \mathbb{R}^{n+1}$. Thus, a point on the line has coordinates $\alpha_0 t' + \beta_0,\ldots,\alpha_n t' + \beta_n$ for a value $t' \in \mathbb{R}$ of the parameter t. The roots of

$$R_m(\alpha t + \beta) = 0$$

(w.r.t. the variable t) correspond to points of intersection of the line with the hyperplanes. Now Corollary 5.5 implies that if (5.2) is consistent, then for a root t' of $R_m(\alpha t + \beta) = 0$ the point

$$\left(\frac{\partial R_m/\partial U_1}{\partial R_m/\partial U_0}(\alpha t' + \beta), \ldots, \frac{\partial R_m/\partial U_n}{\partial R_m/\partial U_0}(\alpha t' + \beta) \right)$$

is a solution to (5.2).

Denoting

$$\frac{\partial R_m/\partial U_j}{\partial R_m/\partial U_0}(\alpha t + \beta) = y_j(\alpha t + \beta), \ 1 \le j \le n,$$

we get:

The system (5.2) is consistent if and only if there exists $t' \in \mathbf{R}$ such that

$$R_m(\alpha t' + \beta) = 0 \quad \text{and}$$

(5.8)
$$f_i\big(y_1(\alpha t' + \beta), \ldots, y_n(\alpha t' + \beta)\big) \ge 0$$
$$\text{for all } 1 \le i \le k.$$

The system (5.8) is a system of polynomial inequalities in *one* variable. We can check its consistency by the algorithm of Ben-Or, Kozen and Reif.

5.9. **Case of a zero gradient.** In Section 5.8 we used the gradients of R_m at points x of intersection of $\{R_m = 0\}$ with a generic line, i.e., vectors of the kind

$$\left(\frac{\partial R_m}{\partial U_0}(x), \ldots, \frac{\partial R_m}{\partial U_n}(x) \right),$$

silently assuming that these vectors are nonzero. This is a simplification of the actual procedure, since the gradient *may be* zero at a chosen point. This happens exactly when a factor $\eta_0^{(i)} U_0 + \cdots + \eta_n^{(i)} U_n$ occurs in R_m with a power greater than 1. In terms of $R_m(\alpha t + \beta)$ it means that a root t', corresponding to an intersection of the straight line with $\{R_m = 0\}$, is *multiple*.

Lemma 5.6.
$$\eta_j^{(i)} = \frac{\partial^p}{\partial s \partial^{p-1} t} R_m(\alpha t + s e_j + \beta) \Big|_{t=t', s=0}$$
for a certain p, $1 \le p \le r$, where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 occurs at jth position).

Proof. By direct computation. □

5.10. **Computing a generic straight line in $\mathbb{R}^{n+1}$.** Let

$$S_1 = \{(1, i, i^2, \ldots, i^n) : \quad 1 \le i \le nr + 1\};$$
$$S_2 = \{(1, i, i^2, \ldots, i^n) : \quad 1 \le i \le nr(r-1)/2 + 1\}.$$

Lemma 5.7. *There exist $\alpha \in S_1$ and $\beta \in S_2$ such that:*

(1) $\{\alpha t + \beta\}$ intersects every $\{\eta_0^{(i)} U_0 + \cdots + \eta_n^{(i)} U_n = 0\}$ at exactly one point, say, $\alpha t^{(i)} + \beta$;

(2) if $t^{(i)} = t^{(j)}$ for $i \ne j$ then

$$\frac{\eta_l^{(i)}}{\eta_0^{(i)}} = \frac{\eta_l^{(j)}}{\eta_0^{(j)}}$$

for all $1 \leq l \leq n$.

Proof. Each subset of $n+1$ vectors from S_1 or from S_2 is linearly independent (they form a famous *Vandermonde matrix*). Then at least one $\alpha \in S_1$ does not belong to

$$\{R_m = 0\} = \bigcup_{1 \leq i \leq r} \{\eta_0^{(i)} U_0 + \cdots + \eta_n^{(i)} U_n\}.$$

Indeed, let *all* points $\alpha^{(i)} = (1, i, i^2, \ldots, i^n) \in S_1$ belong to $\{R_m = 0\}$. Then, denoting $\eta^{(i)} = (\eta_0^{(i)}, \ldots, \eta_n^{(i)})$, we get:

$$\alpha^{(1)} \eta^{(1)} = 0 \quad \text{or} \quad \alpha^{(1)} \eta^{(2)} = 0 \quad \text{or} \quad \cdots \quad \text{or} \quad \alpha^{(1)} \eta^{(r)} = 0$$

$$\text{and}$$

$$\alpha^{(2)} \eta^{(1)} = 0 \quad \text{or} \quad \alpha^{(2)} \eta^{(2)} = 0 \quad \text{or} \quad \cdots \quad \text{or} \quad \alpha^{(2)} \eta^{(r)} = 0$$

$$\text{and}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\alpha^{(nr+1)} \eta^{(1)} = 0 \quad \text{or} \quad \alpha^{(nr+1)} \eta^{(2)} = 0 \quad \text{or} \quad \cdots \quad \text{or} \quad \alpha^{(nr+1)} \eta^{(r)} = 0.$$

It follows that there exist i_0 and pair-wise distinct $j_1, \ldots, j_{n+1}$ such that

$$\alpha^{(j_1)} \eta^{(i_0)} = 0 \quad \text{and} \quad \alpha^{(j_2)} \eta^{(i_0)} = 0 \quad \text{and} \quad \ldots \quad \text{and} \quad \alpha^{(j_{n+1})} \eta^{(i_0)} = 0,$$

i.e., vectors $\alpha^{(j_1)}, \ldots, \alpha^{(j_{n+1})}$ are linearly dependent. We have got a contradiction.

It follows that for any β, (1) is satisfied. Now let us choose β so that (2) would be satisfied.

Let $H_1 \subset \mathbb{R}^{n+1}$ be the set of all points contained in two distinct hyperplanes. For any two distinct hyperplanes P_1, P_2 the dimension $\dim(P_1 \cap P_2) = n - 1$ (recall that each hyperplane passes through the origin). The number of all pairs of hyperplanes is $\binom{r}{2} = r(r-1)/2$. Hence H_1 is contained in a union of $r(r-1)/2$ planes of the dimension $n - 1$. It follows that the set

$$H_2 = \{\beta \in \mathbb{R}^{n+1} : \quad \alpha t + \beta \in H_1 \text{ for some } t\}$$

is contained in a union of $r(r-1)/2$ planes of the dimension n (i.e., hyperplanes). Reasoning as before, we get that there exists $\beta \in S_2$ such that $\beta \notin H_2$. $\square$

5.11. Computing u-resultant.

In this section we shall discuss briefly the problem of computing the u-resultant. It's more convenient to do that for a general system of homogeneous equations (rather than for the concrete system (5.5). The construction will be "approximately correct", that is, we shall ignore some technical details.

We start with a system of homogeneous equations

$$(5.9) \qquad\qquad\qquad\qquad g_1 = \cdots = g_n = 0$$

in variables $X_0, X_1, \ldots, X_n$. Let $\deg(g_i) = d_i$ and $D = \sum_{1 \leq i \leq n} d_i$. Introduce new variables $U_0, U_1, \ldots, U_n$ and construct a new polynomial

$$g_0 \equiv U_0 X_0 + U_1 X_1 + \cdots + U_n X_n.$$

The degree $\deg(g_0) = d_0 = 1$.

Observe that each homogeneous polynomial h can be uniquely defined by a vector of its coefficients (that is, we must fix a certain order of all possible monomials and take coefficients in that order). Thus, we can view the set of all polynomials of degree less or equal to d as a vector space of the dimension $\dim = \binom{n+d}{n}$. Identically zero polynomial plays a role of the zero in that space.

Let $\mathcal{H}_i$ be the vector space of all homogeneous polynomials of degree $D - d_i$ $(0 \leq i \leq n)$, and $\mathcal{H}$ be the vector space of all polynomials of the degree D.

Consider the direct sum $\mathcal{H}_0 \times \cdots \times \mathcal{H}_n$ which is again the a vector space. Its elements are vectors of spaces $\mathcal{H}_i$, joined end-to-end. Observe that

$$\dim(\mathcal{H}_0 \times \cdots \times \mathcal{H}_n) = \sum_i \binom{n + D - d_i}{n}; \quad \dim(\mathcal{H}) = \binom{n + D}{n}.$$

Define the linear map from $\mathcal{H}_0 \times \cdots \times \mathcal{H}_n$ to $\mathcal{H}$ by the following formula:

$$(h_0, h_1, \ldots, h_n) \longrightarrow g_0 h_0 + \cdots + g_n h_n.$$

The matrix M of this map is of the order

$$\binom{n + D}{n} \times \sum_i \binom{n + D - d_i}{n}.$$

Lemma 5.8. *(1)   The system (5.9) has a finite number of roots if and only if at least one of the $\binom{n+D}{n} \times \binom{n+D}{n}$ minors of M is nonsingular;*

(2)   If M' is a maximal nonsingular minor of M, then

$$\det(M') = \prod_i (\xi_0^{(i)} U_0 + \cdots + \xi_n^{(i)} U_n),$$

where $(\xi_0^{(i)} : \cdots : \xi_n^{(i)})$ are all the roots of the system (5.9).

Hint for the proof. Let (5.9) have finite number of roots and let $(\xi_0^{(i)} : \cdots : \xi_n^{(i)})$ be one of them. Then, by the definition of the map M, for any vector $h_0, h_1, \ldots h_n$ the value of the homogeneous polynomial $M(h_0, \ldots, h_n)$ at the point $(\xi_0^{(i)} : \cdots : \xi_n^{(i)})$ is zero. Hence for the specializations of $U_0, \ldots, U_n$ for which $\xi_0^{(i)} U_0 + \cdots + \xi_n^{(i)} U_n = 0$, the map *is not an isomorphism.* Indeed, for this specialization every image under the map is a homogeneous polynomial having $(\xi_0^{(i)} : \cdots : \xi_n^{(i)})$ as a root. Not every polynomial of the degree D has this property.

It follows that all maximal minors of M (for the specialization of $U_0, \ldots, U_n$) vanish, in particular, $\det(M')$ vanishes. In other words: polynomial $\det(M')$ vanishes on the hyperplane $\{\xi_0^{(i)} U_0 + \cdots + \xi_n^{(i)} U_n = 0\}$. It follows that $\xi_0^{(i)} U_0 + \cdots + \xi_n^{(i)} U_n$ divides $\det(M')$. $\square$

5.12. **Complexity of deciding consistency.** A straightforward observation shows that the complexity of the described algorithm is $(kd)^{O(n)} M^{O(1)}$ for the Turing machine model and $(kd)^{O(n)}$ for the real number model. Here the term kd appears as the degree of polynomial $g(\varepsilon)$ in Section 5.2.

6. Quantifier elimination

Here we briefly sketch main ideas behind the quantifier elimination algorithm. To make formulas simpler we will work in the real number model.

6.1. **Formulation of the problem.** We first consider the problem of elimination one block of existential quantifiers, i.e., representing a set

$$\{\exists X_{n_1} \exists X_{n_1+1} \cdots \exists X_n \left(f_1(X_1, \ldots, X_n) \geq 0, \ldots, f_k(X_1, \ldots, X_n) \geq 0\right)\} \subset \mathbb{R}^{n_1-1}$$

by a Boolean combination of polynomial equations and inequalities.

Changing, for convenience, the notations we write:

$$X = (X_{n_1}, \ldots, X_n), \quad Y = (X_1, \ldots, X_{n_1-1}),$$

$$F(X_1, \ldots, X_n) = f_1(X_1, \ldots, X_n) \geq 0, \ldots, f_k(X_1, \ldots, X_n) \geq 0,$$

thus the input formula becomes $\exists X \; F(Y)$.

6.2. **Main idea.** Consider free variables Y as *parameters* and try to execute the decidability algorithm for a parametric formula (i.e., with variable coefficients).

Decidability algorithm from Section 5 uses only arithmetic operations $\pm, \times, \div$, and comparisons over real numbers. Arithmetic can be easily performed parametrically (it is arithmetic of polynomials), while comparisons require branching.

The parametric algorithm can be represented by *algebraic decision tree*.

The tree is described as follows. Root of the tree is associated with an arithmetic operation on some variables in the variable vector Y. Proceeding by induction, at each vertex of the tree, except leaves, an arithmetic operation is performed between two elements of the set consisting of variables in Y and the results of operations performed along the branch leading from the root to this vertex. As a result, this vertex is associated with a polynomial in variables in Y, which is a composition of arithmetic operations performed along the branch.

Upon a *specialization* of Y, the polynomial at each vertex is evaluated, and the value is compared to zero. Depending on the result of comparison, one of the three branches is chosen, thereby determining the next vertex, where the next polynomial evaluation will be made, and so on, until a leaf is reached.

Each leaf is assigned the value **True** or **False**. All specializations of Y, arriving at a leaf marked **True**, correspond to closed formulas that are true. Similar with **False**.

6.3. **Elimination algorithm.**
- Use the algorithm from Section 5 parametrically to obtain a decision tree with the variable input Y.
- Determine all branches from the root to leaves marked **True**.
- Take conjunction of the inequalities along each such branch:
$$\bigwedge_{i \in \text{branch}} h_i(Y) \sigma_i 0,$$
where $\sigma_i \in \{<, >, =\}$.
-
$$\exists X \; F(X, Y) \Leftrightarrow \bigvee_{\text{branches}} \bigwedge_{i \in \text{branch}} h_i(Y) \sigma_i 0$$

6.4. **Complexity of elimination algorithm.** The complexity of the consistency algorithm from Section 5 coincides with *height* of the tree (the length of the longest branch from the root to a leaf). From the first glance, this leads to a bad complexity for elimination algorithm, because there are too many branches (hence, leaves) in the tree. Indeed, the height equals to complexity of consistency, which is $(kd)^{O(n)}$. It follows that combinatorially there may be as much as $3^{(kd)^{O(n)}}$ leaves (recall that we take the union of leaves marked **True** in the elimination algorithm). However, we will now show that most branches are not followed under any specialization of Y, i.e., most leaves correspond to Boolean formulas defining $\emptyset$.

Let us first generalize Definition 4.1 to the case of many variables.

Let $f_1, \ldots, f_k \in \mathbb{R}[X_1, \ldots, X_n]$ with degrees $\deg(f_i) < d$ for all i, $1 \le i \le k$.

Definition 6.1. Consistent sign assignment for $f_1, \ldots, f_k$ is a string

$$\sigma = (\sigma_1, \ldots, \sigma_k),$$

where $\sigma_i \in \{>, <, =\}$, such that the system $f_1 \sigma_1 0, \ldots, f_k \sigma_k 0$ has a root in $\mathbb{R}^n$.

Lemma 6.2. *The number of distinct consistent sign assignments for $f_1, \ldots, f_k$ is at most $(kd)^{O(n)}$, where $\deg(f_i) < d$ for all $i \in \{1, \ldots, k\}$.*

Proof. Choose in every $\{f_1 \sigma_1 0, \ldots, f_k \sigma_k 0\}$ one arbitrary point. There exists a positive $\varepsilon \in \mathbb{R}$ such that for every chosen point x and every $i \in \{1, \ldots, k\}$ the inequality $f_i(x) > 0$ implies $f_i(x) > \varepsilon$, and $f_i(x) < 0$ implies $f_i(x) < -\varepsilon$. It is easy to prove that the number of distinct consistent sign assignments does not exceed the number of connected components of the semialgebraic set

$$S = \left\{ h = \prod_{1 \le i \le k} (f_i + \varepsilon)^2 (f_i - \varepsilon)^2 > 0 \right\}.$$

In turn, according to the famous Thom-Milnor bound on homologies of semialgebraic sets, this number does not exceed $(kd)^{O(n)}$.

Alternatively, for a small enough positive $\delta < \varepsilon$, the number of connected components of S does not exceed the number of connected components of $\{h = \delta\}$. A careful examination of the consistency algorithm from Section 5 shows that, being applied to the equality $h = \delta$, it will produce among all of its roots

$$(\eta_1^{(i)}/\eta_0^{(i)}, \ldots, \eta_n^{(i)}/\eta_0^{(i)})$$

a representative on each connected component of $\{h = \delta\}$. The upper bound on the number of roots follows from the complexity of the consistency algorithm. $\qquad\square$

Carefully examining the decidability algorithm we can see that the number of distinct polynomials h_i in the tree is essentially the same as in one branch, i.e., $(kd)^{O(n)}$. From the complexity estimate we know that their degrees are $(kd)^{O(n)}$. Then, by Lemma 6.2, the number of distinct non-empty sets associated with leaves is $(kd)^{O(n^2)}$. We conclude that the complexity of $\exists$-elimination is $(sd)^{O(n^2)}$.

6.5. **Universal quantifier and quantifier alternation.** To eliminate the universal quantifier in $\forall X \ F(X, Y)$ the algorithm first re-writes this formula in an equivalent form $\neg \exists X \neg F(X, Y)$, then eliminates $\exists$ in $\exists X \neg F(X, Y)$, as above, and finally re-writes the resulting Boolean combination of equations and inequalities without the negation symbol.

Finally, consider the general case of a formula

$$(Q_1 X^{(1)}) \cdots (Q_q X^{(q)}) P(Y, X^{(1)}, \ldots, X^{(q)}),$$

where

- $Q_1, \ldots, Q_q$ are quantifiers $\exists$ or $\forall$;
- $X^{(i)} = (X_1^{(i)}, \ldots, X_{n_i}^{(i)})$ — bound variables;
- $Y = (Y_1, \ldots, Y_{n_0})$ — free variables;
- P — Boolean combination of k formulas of the kind $f > 0$ or $f = 0$, and for every f the degree $\deg(f) < d$.

Set $n = n_0 + n_1 + \cdots + n_q$.

The algorithm begins with eliminating the most "internal" quantifier Q_q in the formula

$$(Q_q X^{(q)}) P(Y, X^{(1)}, \ldots, X^{(q)}),$$

obtaining a Boolean combination $P'(Y, X^{(1)}, \ldots, X^{(q-1)})$, and then continues recursively, eliminating quantifiers one by one, inside out.

Clearly, the complexity bound of this procedure is $(kd)^{n^{O(q)}}$

Using finer technical tools one can construct quantifier elimination algorithm having the complexity

$$(kd)^{\prod_{0 \leq i \leq q} O(n_i)}.$$

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