arXiv:0706.0248v2 [math.GR] 17 Jun 2007

APERIODIC POINTLIKES AND BEYOND

KARSTEN HENCKELL, JOHN RHODES, AND BENJAMIN STEINBERG

Dedicated to the memory of Bret Tilson

Abstract. We prove that if π is a recursive set of primes, then pointlike sets are decidable for the pseudovariety of semigroups whose subgroups are π-groups. In particular, when π is the empty set, we obtain Henckell's decidability of aperiodic pointlikes. Our proof, restricted to the case of aperiodic semigroups, is simpler than the original proof.

1. Introduction

In [?] the first author showed that aperiodic pointlikes are computable; the companion result for groups was proved by Ash [?]. Recently there has been renewed interest in the decidability of aperiodic pointlikes: the third author used it to compute certain joins [?, ?]; the authors have recently used it to study aperiodic idempotent pointlikes and stable pairs [?, ?, ?]. As a consequence, the Mal'cev product $\mathbf{V} \textcircled{m} \mathbf{A}$ is always decidable if $\mathbf{V}$ is decidable and the semidirect product $\mathbf{V} * \mathbf{A}$ is decidable so long as $\mathbf{V}$ is local and decidable. The original proof of the decidability of aperiodic pointlikes in [?] is quite long. The key complication is that the aperiodic semigroup used to compute the pointlikes is given in terms of generators of a transformation semigroup. To prove the semigroup is aperiodic, the first author used a complicated Zeiger coding of the Rhodes expansion to show that these generators live inside a wreath product of aperiodic semigroups, and hence generate an aperiodic semigroup. An alternate approach was given by the first author in [?] involving a simpler coding into a wreath product.

We prove here a considerable generalization of this result. If π is a set of primes, let $\mathbf{G}_\pi$ denote the pseudovariety of groups with order divisible only by primes in π, that is, the pseudovariety of π-groups. Then $\overline{\mathbf{G}}_\pi$ denotes the pseudovariety of semigroups whose subgroups are π-groups. For instance, when $\pi = \emptyset$, then $\overline{\mathbf{G}}_\pi$ is the pseudovariety of aperiodic semigroups; if $\pi = \{p\}$, then $\overline{\mathbf{G}}_\pi$ is the pseudovariety of semigroups whose subgroups are p-groups. Notice that $\overline{\mathbf{G}}_\pi$ has decidable membership if and only if π is recursive. We prove in this case that $\overline{\mathbf{G}}_\pi$ has decidable pointlikes. Our

Date: June 2, 2007.
1991 *Mathematics Subject Classification.* 20M07.
The third author was supported in part by NSERC.

construction is inspired by Henckell's proof [?], but we sidestep the Zeiger coding by working directly with Ł-chains.

The paper is organized as follows. Given a finite semigroup T, we first introduce a certain computable semigroup $CP_\pi(T)$ of $\overline{\mathbf{G}}_\pi$-pointlikes. Then we discuss Schützenberger groups and the notion of a π'-free element. In the following section, we show how to associate a finite semigroup $S^\pi \in \overline{\mathbf{G}}_\pi$ to any finite semigroup S. By working with an arbitrary semigroup and axiomatizing the essential properties of Henckell's original construction, we manage to simplify Henckell's proof scheme. We draw inspiration from the Grigorchuk school's theory of self-similar (or automaton) groups [**?**, **?**]. The subsequent section shows how to construct a relational morphism from T to $CP_\pi(T)^\pi$ such that the inverse image of each element belongs to $CP_\pi(T)$. The construction assumes the existence of a blowup operator on $CP_\pi(T)$, the existence of which is established in the final section. This last bit again simplifies the corresponding construction in [?].

2. Pointlikes

As usual, if S is a finite semigroup, then s^ω denotes as usual the unique idempotent power of s. The notation S^I stands for S with an adjoined identity I. The reader is referred to [**?**,**?**,**?**,**?**,**?**] for background and undefined terminology concerning finite semigroups.

Definition 2.1 ($\mathbf{V}$-pointlikes)**.** Let $\mathbf{V}$ be a pseudovariety of semigroups and T a semigroup. A subset $Z \subseteq T$ is said to be $\mathbf{V}$-*pointlike* if, for all relational morphisms $\varphi : T \to S$ with $S \in \mathbf{V}$, there exists $s \in S$ such that $Z \subseteq s\varphi^{-1}$.

The collection $\mathsf{PL}_{\mathbf{V}}(T)$ of $\mathbf{V}$-pointlikes of T is a subsemigroup of the power set $P(T)$, containing the singletons, and which is a downset for the order $\subseteq$. One says that $\mathbf{V}$ has decidable pointlikes if one can effectively compute $\mathsf{PL}_{\mathbf{V}}(T)$ from the multiplication table of T. See [**?**, ?, ?, ?, **?**, ?] for more on pointlikes. If $Z \in P(T)$, let us define $Z^{\omega+*} = Z^\omega \bigcup_{n\geq 1} Z^n$. Since products distribute over union in $P(T)$, it follows easily that

$$ZZ^{\omega+*} = Z^{\omega+*} = Z^{\omega+*}Z. \tag{2.1}$$

One deduces immediately from (2.1) that $Z^{\omega+*}$ is an idempotent. Observe that if Z is a group element, then $Z^{\omega+*} = \bigcup_{n\geq 1} Z^n$.

Let π be a set of primes; then π' denotes the set of primes not belonging to π. Denote by $\mathbf{G}_\pi$ the pseudovariety of π-groups, that is, groups whose orders only involve primes from π. Let $\overline{\mathbf{G}}_\pi$ be the pseudovariety of semigroups whose subgroups are π-groups. As mentioned in the introduction, if $\pi = \emptyset$, then $\mathbf{G}_\pi$ is the trivial pseudovariety and $\overline{\mathbf{G}}_\pi$ is the pseudovariety of aperiodic semigroups; if $\pi = \{p\}$, then $\mathbf{G}_\pi$ is the pseudovariety of p-groups. Notice that the membership problems for π, $\mathbf{G}_\pi$ and $\overline{\mathbf{G}}_\pi$ are equivalent.

The following proposition shows that the semigroup of $\overline{\mathbf{G}}_\pi$-pointlikes is closed under unioning up cyclic π'-subgroups. If π is empty, this means one can union up any cyclic subgroup, as was observed by Henckell [?]. In fact, $\mathsf{PL}_{\mathbf{A}}(T)$ is closed under the operation $Z \mapsto Z^{\omega+*}$.

Proposition 2.2 (Cyclic amalgamation)**.** *Let π be a set of primes and T a finite semigroup. Suppose $Z \in \mathsf{PL}_{\overline{\mathbf{G}}_\pi}(T)$ generates a cyclic π'-group. Then $Z^{\omega+*} \in \mathsf{PL}_{\overline{\mathbf{G}}_\pi}(T)$.*

Proof. Suppose the group element Z has order k. Let $\varphi : T \to S$ be a relational morphism with $S \in \overline{\mathbf{G}}_\pi$. Choose $s \in S$ with $Z \subseteq s\varphi^{-1}$. Then $Z = Z^\omega Z \subseteq s^\omega s\varphi^{-1}$. So without loss of generality, we may assume that s is a group element. The order n of s must be prime to k, so we can find an positive integer m with $mn \equiv 1 \bmod k$. Then $Z = Z^{mn} \subseteq s^{mn}\varphi^{-1} = s^\omega\varphi^{-1}$. Thus $Z^r \subseteq s^\omega\varphi^{-1}$ for all $r > 0$ and so $Z^{\omega+*} \subseteq s^\omega\varphi^{-1}$. We conclude $Z^{\omega+*} \in \mathsf{PL}_{\overline{\mathbf{G}}_\pi}(T)$. $\square$

This paper is devoted to proving the following generalization of Henckell's theorem describing the **A**-pointlike sets [?].

Theorem 2.3. *Let π be a set of primes and T be a finite semigroup. Denote by $CP_\pi(T)$ the smallest subsemigroup of $P(T)$ containing the singletons and closed under $Z \mapsto Z^{\omega+*}$ whenever Z generates a cyclic π'-group. Then $\mathsf{PL}_{\overline{\mathbf{G}}_\pi}(T)$ consists of all $X \in P(T)$ with $X \subseteq Y$ some $Y \in CP_\pi(T)$.*

Proposition 2.2 shows that $CP_\pi(T) \subseteq \mathsf{PL}_{\overline{\mathbf{G}}_\pi}(T)$ and hence each of the subsets described in Theorem 2.3 is indeed $\overline{\mathbf{G}}_\pi$-pointlike. The hard part of the result is proving the converse.

Corollary 2.4. *Let π be a recursive set of prime numbers. Then $\overline{\mathbf{G}}_\pi$-pointlikes are decidable.*

In [?, ?] it is shown that if $\mathbf{V}$ is a pseudovariety such that $\mathbf{A} \circledm \mathbf{V} = \mathbf{V}$ and $\mathbf{V}$-pointlikes are decidable, then the $\mathbf{V}$-idempotent pointlikes are decidable. This in particular applies to pseudovarieties of the form $\overline{\mathbf{G}}_\pi$.

Corollary 2.5. *Let π be a recursive set of prime numbers. Then $\overline{\mathbf{G}}_\pi$-idempotent pointlikes are decidable and hence the Mal'cev product $\mathbf{V} \circledm \overline{\mathbf{G}}_\pi$ is decidable whenever $\mathbf{V}$ is decidable.*

3. Schützenberger groups and π'-free elements

Fix a semigroup S. Let H be an $\mathscr{H}$-class of S and set

$$\mathrm{St}_R(H) = \{s \in S^I \mid Hs \subseteq H\}.$$

The faithful quotient transformation monoid, denoted $(H, \Gamma_R(H))$, is a transitive regular permutation group called the *Schützenberger group* of H [?, ?]. If H is a maximal subgroup, then $\Gamma_R(H) = H$. In general, one can always find a subgroup $\widetilde{\Gamma}_R(H) \subseteq \mathrm{St}_R(H)$ acting transitively on H with faithful quotient $\Gamma_R(H)$ [?]. The following proposition is well-known.

Proposition 3.1. *Let H and H' be Ł-equivalent $\mathscr{H}$-classes of S. Then* $\mathrm{St}_R(H) = \mathrm{St}_R(H')$ *and one can take* $\widetilde{\Gamma}_R(H) = \widetilde{\Gamma}_R(H')$. *Moreover, the kernels of the natural maps* $\widetilde{\Gamma}_R(H) \to \Gamma_R(H)$ *and* $\widetilde{\Gamma}_R(H) \to \Gamma_R(H')$ *coincide. In particular,* $\Gamma_R(H) \cong \Gamma_R(H')$.

Proof. Suppose that $H = H_a$, $H' = H_b$ and $ya = b$ with $y \in S^I$. By Green's lemma, $yH_a = H_b$. The equality $\mathrm{St}_R(H) = \mathrm{St}_R(H')$ is classical [?, ?]. Now $H_a = a\widetilde{\Gamma}_R(H)$, so $b\widetilde{\Gamma}_R(H) = ya\widetilde{\Gamma}_R(H) = yH_a = H_b$. Thus $\widetilde{\Gamma}_R(H)$ is a group in $\mathrm{St}_R(H')$ acting transitively on H. By regularity of the action we can take $\widetilde{\Gamma}_R(H') = \widetilde{\Gamma}_R(H)$. The statement about kernels follows since the right stabilizer of any two Ł-equivalent elements of a semigroup coincide. $\square$

Similarly, there is a left Schützenberger group $(\Gamma_L(H), H)$ and a subgroup $\widetilde{\Gamma}_L(H)$ of the left stabilizer of H mapping onto $\Gamma_L(H)$. The groups $\Gamma_L(H)$ and $\Gamma_R(H)$ are isomorphic. In fact, if $h_0 \in H$ is a fixed base point and $g \in \Gamma_R(H)$, then the map γ sending g to the unique $g\gamma \in \Gamma_L(H)$ with $g\gamma h_0 = h_0 g$ is an anti-isomorphism. In particular, using Proposition 3.1 and its dual, we see that the Schützenberger group depends up to isomorphism only on the $\mathscr{J}$-class. See [?, ?] for details. The following proposition describes when an element belongs to $\mathrm{St}_R(H)$, and hence represents an element of $\Gamma_R(H)$.

Proposition 3.2. *Let H be an $\mathscr{H}$-class of S. Then $s \in S^I$ belongs to* $\mathrm{St}_R(H)$ *if and only if, for some $h \in H$, $hs \in H$.*

Proof. Necessity is clear. Suppose $hs \in H$ and $h' \in H$. Then we have $h's$ŁhsŁh'. Therefore, $h's \mathrel{\mathscr{J}} h'$ and so $h's \mathrel{\mathscr{R}} h'$. This shows $h's \in H$. $\square$

We now introduce the important notion of π'-freeness.

Definition 3.3 (π'-free)**.** Let π be a set of primes. A $\mathscr{J}$-class (respectively, Ł-, $\mathscr{R}$-class) of S is called *π'-free* if its Schutzenberger group is a π-group. Likewise, an element of a π'-free $\mathscr{J}$-class is called π'-free.

We shall need the following well-known and easy to prove lemma.

Lemma 3.4. *Let $\varphi : G \to H$ be an onto group homomorphism and let $h \in H$ have prime order p. There there is an element $g \in G$ of prime power order p^n with $g\varphi = h$.*

4. A $\overline{\mathbf{G}}_\pi$-variant of the Rhodes expansion

Our goal in this section is to associate a finite semigroup $S^\pi \in \overline{\mathbf{G}}_\pi$ to each finite semigroup S. The case of $CP_\pi(T)$ will yield a semigroup in $\overline{\mathbf{G}}_\pi$ and a relational morphism that establishes Theorem 2.3.

Fix a finite semigroup S for this section. Elements of the free monoid S^* will be written as strings $\vec{x} = (x_n, x_{n-1}, \ldots, x_1)$. The empty string is denoted ε. We omit parentheses for strings of length 1. If $n \geq \ell$, define

$$(x_n, x_{n-1}, \ldots, x_1)\alpha_\ell = (x_\ell, x_{\ell-1}, \ldots, x_1)$$

and $(x_n, \ldots, x_1)\tau_\ell = x_\ell$. We identify $\vec{x}\alpha_1$ with $\vec{x}\tau_1$, the first letter of $\vec{x}$. By convention $\vec{x}\alpha_0 = \varepsilon$. Set $(x_n, \ldots, x_1)\omega = x_n$. We use $\vec{b} \cdot \vec{a}$ for the concatenation of $\vec{b}$ and $\vec{a}$. As the notation suggests, we read strings from right to left.

If P is a pre-ordered set, then a *flag* of elements of P is a strict chain $p_n < p_{n-1} < \cdots < p_1$. We also allow an empty flag. Denote by $\mathscr{F}(S)$ the set of flags for the Ł-order on S. Of course, $\mathscr{F}(S)$ is a finite set. A typical flag $s_n <_{Ł} s_{n-1} <_{Ł} \cdots <_{Ł} s_1$ shall be denoted $(s_n, s_{n-1}, \cdots, s_1)$. We shall also consider the set $\overline{\mathscr{F}}(S)$ of Ł-chains, that is, all strings $(s_n, s_{n-1}, \ldots, s_1) \in S^*$ (including the empty string) such that $s_{i+1} \leq_{Ł} s_i$ for all i. Of course $\mathscr{F}(S) \subseteq \overline{\mathscr{F}}(S) \subseteq S^*$. A string $(s_n, \ldots, s_1)$ is termed π'*-free* if each s_i is π'-free (see Definition 3.3).

We use $\mathscr{F}_\pi(S)$ and $\overline{\mathscr{F}}_\pi(S)$ to denote the respective subsets of $\mathscr{F}(S)$ and $\overline{\mathscr{F}}(S)$ consisting of π'-free strings. There is a natural retraction from $\overline{\mathscr{F}}(S)$ to $\mathscr{F}(S)$ (mapping $\overline{\mathscr{F}}_\pi(S)$ onto $\mathscr{F}_\pi(S)$), which we proceed to define. Define an elementary reduction to be a rule of the from $(s', s) \to s'$ where $s'Ł s$. Elementary reductions are length-decreasing. It is well known and easy to prove that the elementary reductions form a confluent rewriting system and so each element $\vec{x} \in \overline{\mathscr{F}}(S)$ can be reduced to a unique flag $\vec{x}\rho \in \mathscr{F}(S)$, called its *reduction* [**?**]. Clearly the reduction map is a retract and takes $\overline{\mathscr{F}}_\pi(S)$ to $\mathscr{F}_\pi(S)$. The Rhodes expansion [?] defines a multiplication on $\mathscr{F}(S)$ using the reduction map. Our constructions are motivated by properties of the Rhodes expansion, but we shall not need this expansion per se. Some key properties of the reduction map, which are immediate from the definition, are recorded in the following lemma.

Lemma 4.1. *The reduction map ρ enjoys the following properties:*

(1) *For $\vec{x} \in \overline{\mathscr{F}}(S)$, $\vec{x}\rho\omega = \vec{x}\omega$;*
(2) *Let $\vec{b}, \vec{a}, \vec{b} \cdot \vec{a} \in \overline{\mathscr{F}}(S)$ and suppose $\vec{a}\rho = (a_\ell, a_{\ell-1}, \ldots, a_1)$. Then one has $(\vec{b} \cdot \vec{a})\rho\alpha_\ell = (x_\ell, a_{\ell-1}, \ldots, a_1)$ where $x_\ell Ł a_\ell$.*

Let us now turn to defining an auxiliary semigroup that will play a role in the proof.

Definition 4.2 ($\check{S}$)**.** Denote by $\check{S}$ the monoid of all functions $f : S \to S$ such that

(1) $sf \leq_{\mathscr{R}} s$ for all $s \in S$;
(2) f preserves Ł, i.e. $sŁs'$ implies $sfŁs'f$;
(3) There exists $s_f \in S^I$ such that $sf \mathrel{\mathscr{R}} s$ implies $sf = ss_f$.

Notice that the natural action of S^I on the right of S belongs to $\check{S}$.

Proposition 4.3. *The set $\check{S}$ is a monoid.*

Proof. Clearly $\check{S}$ contains the identity. The set of functions satisfying the second item is obviously closed under composition. Suppose $f, g \in S$. Then

$sfg \leq_{\mathscr{R}} sf \leq_{\mathscr{R}} s$. Moreover, if $s \mathrel{\mathscr{R}} sfg$, all the inequalities are equalities and so $sfg = sfs_g = ss_fs_g$. In particular, we can take $s_{fg} = s_fs_g$. □

Let us write $\check{S}^\infty$ for the action monoid of the infinite wreath product $\wr^\infty(S, \check{S})$ of right transformation monoids $(S, \check{S})$. There is a natural action of $\check{S}^\infty$ on S^* by length-preserving, sequential functions via the projections $\wr^\infty(S, \check{S}) \to \wr^n(S, \check{S})$; to obtain the action on a word of length n, project first to $\wr^n(S, \check{S})$ and then act. If $F \in \check{S}^\infty$ and $\vec{a} \in S^*$, then there is a unique element ${}_{\vec{a}}F \in \check{S}^\infty$ such that $(\vec{b} \cdot \vec{a})F = \vec{b}_{\vec{a}}F \cdot \vec{a}F$ for all $\vec{b} \in S^*$; for example, ${}_\varepsilon F = F$. Also if $F \in \check{S}^\infty$, then there is also a unique element $\sigma_F \in \check{S}$ such that, for $s \in S$, the equality $sF = s\sigma_F$ holds. In particular,

$$(s_n, \ldots, s_1)F = (s_n, \ldots, s_2)_{s_1}F \cdot s_1\sigma_F. \tag{4.1}$$

So σ_F describes the action on the first letter, and must belong to $\check{S}$, while ${}_{s_1}F \in \check{S}^\infty$ is how F acts on the rest of a string starting with s_1. In fact, (4.1) can serve as a recursive definition of what it means to belong to $\check{S}^\infty$ (c.f. [?, ?]). Now in our situation, by definition of $\check{S}$, there is an element $s_{\sigma_F} \in S^I$ so that $s_1F = s_1\sigma_F \mathrel{\mathscr{R}} s_1$ implies $s_1F = s_1s_{\sigma_F}$.

Let us consider some examples to illustrate this formalism for infinite iterated wreath products, in particular the wreath recursion (4.1). For simplicity, we work with $\wr^\infty(\{0,1\}, T_2)$ where T_2 is the full transformation semigroup on two letters. Notice that the iterated wreath product $\wr^\infty(\{0,1\}, T_2)$ is isomorphic to the semidirect product $(\wr^\infty(\{0,1\}, T_2))^2 \rtimes T_2$. As a first example example, consider the 2-adic odometer, which adds one to the 2-adic expansion of an integer (where the least significant bit is the first one read from *right to left*) [?, ?].

Example 4.4 (Odometer). Let A be the 2-adic odometer considered above, acting on $\{0,1\}^*$, and let I be the identity function on $\{0,1\}^*$. If a 2-adic integer has 0 as its least significant bit, we change the 0 to a 1 and then continue with the identity map the rest of the way; if the least significant bit is 1, we change it to 0 and we add 1 to what remains (i.e. perform a carry). So in terms of the wreath recursion (4.1), $\sigma_A = (01)$ and ${}_0A = I$, ${}_1A = A$. If we identify $\wr^\infty(\{0,1\}, T_2)$ with $\wr^\infty(\{0,1\}, T_2)^2 \rtimes T_2$, then $A = ((I, A), (01))$.

Next we consider the two sections to the unilateral shift.

Example 4.5 (Shift). Consider the functions F, G on $\{0,1\}^*$ that send $x_nx_{n-1}\cdots x_1$ to, respectively, $x_{n-1}x_{n-2}\cdots x_10$ and $x_{n-1}x_{n-2}\cdots x_11$. Both of these functions act by remembering the first letter, then resetting it to a predetermined symbol, and then resetting the second letter to the first and so on and so forth. Formally, the wreath recursion (4.1) is given by $\sigma_F = \overline{0}$, $\sigma_G = \overline{1}$ (where $\overline{x}$ is the constant map to x) and ${}_0F = F = {}_0G$, ${}_1F = G = {}_1G$. Identifying $\wr^\infty(\{0,1\}, T_2)$ with $\wr^\infty(\{0,1\}, T_2)^2 \rtimes T_2$, we have $F = ((F, G), \overline{0})$ and $G = ((F, G), \overline{1})$. So, for example, the wreath recursion

$$(x_nx_{n-1}\cdots x_21)F = (x_nx_{n-1}\cdots x_2)G \cdot 0 = x_{n-1}x_{n-2}\cdots x_210$$

holds. Notice that on infinite bit strings, F and G are the two sections to the unilateral shift that erases the first letter.

From these examples, the reader should instantly see the connection between iterated wreath products and sequential functions [?, **?**, ?, ?].

A subsemigroup T of $\check{S}^\infty$ is called *self-similar* if, for all $F \in T$ and $\vec{a} \in S^*$, one has ${}_{\vec{a}}F \in T$; so $\check{S}^\infty$ itself is self-similar. It is actually enough that, for each letter $s \in S$, one has ${}_sF \in T$. For instance, the group generated by the 2-adic odometer A is self-similar in $\wr^\infty(\{0,1\}, T_2)$ since ${}_0A = I$, ${}_1A = A$. Similarly the semigroup generated by the two sections F, G to the shift is self-similar since ${}_0F = F = {}_0G$, ${}_1F = G = {}_1G$. This viewpoint on infinite wreath products is due to Grigorchuk and Nekrashevych [?, ?].

Definition 4.6 ($\check{S}_0^\infty$)**.** Denote by $\check{S}_0^\infty$ the collection of all transformations $F \in \check{S}^\infty$ such that whenever $(x_n, x_{n-1}, \dots, x_1)F = (y_n, y_{n-1}, \dots, y_1)$ with $x_{n-1} \mathrel{\mathscr{R}} y_{n-1}$ and $x_n \mathrel{\mathscr{R}} y_n$, there exists $s \in S^I$ with $x_{n-1}s = y_{n-1}$ and $x_ns = y_n$.

The element s can depend on the string $(x_n, \dots, x_1)$.

Proposition 4.7. *$\check{S}_0^\infty$ is a self-similar submonoid of $\check{S}^\infty$.*

Proof. Clearly it contains the identity. Suppose $F, G \in \check{S}_0^\infty$ and

$$(x_n, x_{n-1}, \dots, x_1)FG = (z_n, z_{n-1}, \dots, z_1)$$

with $x_{n-1} \mathrel{\mathscr{R}} z_{n-1}$ and $x_n \mathrel{\mathscr{R}} z_n$. Suppose that $(x_n, x_{n-1}, \dots, x_1)F = (y_n, y_{n-1}, \dots, y_1)$. Then, for $i = n-1, n$, we have $x_i \geq_{\mathscr{R}} y_i \geq_{\mathscr{R}} z_i \mathrel{\mathscr{R}} x_i$. Thus $x_i \mathrel{\mathscr{R}} y_i$ and $y_i \mathrel{\mathscr{R}} z_i$, $i = n-1, n$. By assumption, there exists $s, t \in S$ with $x_is = y_i$ and $y_it = z_i$, $i = n-1, n$. Then $x_ist = z_i$, for $i = n-1, n$. Hence $\check{S}_0^\infty$ is submonoid of $\check{S}^\infty$. Self-similarity is immediate from the equation $(x_n, \dots, x_1)_{\vec{a}}F \cdot \vec{a}F = ((x_n, \dots, x_1) \cdot \vec{a})F$ and the definition of $\check{S}_0^\infty$. $\square$

If $s \in S$, define the diagonal operator $\Delta_s : S^* \to S^*$ by

$$(x_n, x_{n-1}, \dots, x_1)\Delta_s = (x_ns, x_{n-1}s, \dots, x_1s). \tag{4.2}$$

It is immediate $\Delta_s \in \check{S}_0^\infty$. The next lemma expresses the so-called Zeiger property of $\check{S}_0^\infty$.

Lemma 4.8 (Zeiger Property)**.** *Suppose $\vec{x} = (x_n, \dots, x_1) \in \overline{\mathscr{F}}(S)$ and $F \in \check{S}_0^\infty$ is such that $\vec{x}F = (y_n, y_{n-1}, \dots, y_1)$ with $x_{n-1} = y_{n-1}$ and $x_n \mathrel{\mathscr{R}} y_n$. Then $x_n = y_n$.*

Proof. By definition of $\check{S}_0^\infty$, there exists $s \in S^I$ such that $x_{n-1}s = y_{n-1} = x_{n-1}$ and $x_ns = y_n$. Since $x_n \leq_{\mathrm{L}} x_{n-1}$, we can write $x_n = ux_{n-1}$ with $u \in S^I$. Then $y_n = x_ns = ux_{n-1}s = ux_{n-1} = x_n$, as required. $\square$

We now define an important transformation semigroup on $\overline{\mathscr{F}}(S)$.

Definition 4.9 ($\mathscr{C}$)**.** Let $\mathscr{C}$ consist of all transformations $f : \overline{\mathscr{F}}(S) \to \overline{\mathscr{F}}(S)$ such that there exists $(\widehat{f}, \overline{f}) \in \check{S}_0^\infty \times (\overline{\mathscr{F}}(S) \setminus \{\varepsilon\})$ with $\vec{x}f = \vec{x}\widehat{f} \cdot \overline{f}$.

For instance, if $s \in S$, then one readily checks that (Δ_s, s) defines an element of $\mathscr{C}$ via the formula:

$$(x_n, x_{n-1}, \ldots, x_1)(\Delta_s, s) = (x_n s, x_{n-1}s, \ldots, x_1 s, s).$$

Such elements correspond to generators of the Rhodes expansion [?]. Notice that in order for $(\widehat{f}, \overline{f})$ to define an element of $\mathscr{C}$ one must have $x\widehat{f} \leq_{\mathrm{L}} \overline{f}\omega$ for every $x \in S$. It is essential that $\overline{f}$ is not empty in the definition of $\mathscr{C}$.

Proposition 4.10. *The set $\mathscr{C}$ is a semigroup.*

Proof. Let $f, g \in \mathscr{C}$. We claim $\widehat{fg} = \widehat{f}_{\overline{f}}\widehat{g}$ and $\overline{fg} = (\overline{f})\widehat{g} \cdot \overline{g}$. Indeed,

$$\vec{x}fg = (\vec{x}\widehat{f} \cdot \overline{f})g = (\vec{x}\widehat{f} \cdot \overline{f})\widehat{g} \cdot \overline{g} = \vec{x}\widehat{f}_{\overline{f}}\widehat{g} \cdot (\overline{f})\widehat{g} \cdot \overline{g}.$$

As $\check{S}_0^\infty$ is self-similar, ${}_{\overline{f}}\widehat{g} \in \check{S}_0^\infty$ and so $fg \in \mathscr{C}$. □

An immediate consequence of the definition is:

Lemma 4.11. *If $f \in \mathscr{C}$ and $\vec{a}, \vec{b}, \vec{b} \cdot \vec{a} \in \overline{\mathscr{F}}(S)$, then $(\vec{b} \cdot \vec{a})f = \vec{b}_{\vec{a}}\widehat{f} \cdot \vec{a}f$.*

Proof. Indeed, a straightforward computation yields

$$(\vec{b} \cdot \vec{a})f = (\vec{b} \cdot \vec{a})\widehat{f} \cdot \overline{f} = \vec{b}_{\vec{a}}\widehat{f} \cdot (\vec{a}\widehat{f} \cdot \overline{f}) = \vec{b}_{\vec{a}}\widehat{f} \cdot \vec{a}f,$$

proving the lemma. □

Lemma 4.11 shows that $\mathscr{C}$ behaves very much like an iterated wreath product, a property that we shall exploit repeatedly. In fact, an element of $\mathscr{C}$ is like an asynchronous transducer that outputs $\overline{f}$ with empty input and then computes synchronously. We are almost prepared to define our semigroup in $\overline{\mathbf{G}}_\pi$. Recall that ρ denotes the reduction map.

Definition 4.12 ($\mathscr{C}^\pi$)**.** Let $\mathscr{C}^\pi$ be the subset of $\mathscr{C}$ consisting of all transformations $f \in \mathscr{C}$ such that:

(1) $\overline{\mathscr{F}}_\pi(S)f \subseteq \overline{\mathscr{F}}_\pi(S)$;
(2) $\rho f \rho = f\rho$.

Proposition 4.13. *The set $\mathscr{C}^\pi$ is a semigroup.*

Proof. Closure of the first item under composition is clear. The computation

$$\rho fg\rho = \rho f(\rho g\rho) = (\rho f\rho)g\rho = f\rho g\rho = fg\rho \tag{4.3}$$

completes the proof that $\mathscr{C}^\pi$ is a semigroup. □

Notice that (4.3) allows us to define an action of $\mathscr{C}^\pi$ on $\mathscr{F}_\pi(S)$ by $\vec{x} \mapsto \vec{x}f\rho$ for $f \in \mathscr{C}^\pi$. Let us denote the resulting faithful transformation semigroup by $(\mathscr{F}_\pi(S), S^\pi)$. Observe that $\overline{\mathscr{F}}_\pi(S)$ appears in Definition 4.12, while in the definition of S^π we use $\mathscr{F}_\pi(S)$. Since $\mathscr{F}_\pi(S)$ is finite, so is S^π. Our goal is to prove $S^\pi \in \overline{\mathbf{G}}_\pi$. We do this by showing that if $\vec{x} \in \mathscr{F}_\pi(S)$, $p \in \pi'$ and $f \in S^\pi$ with $\vec{x}f^p = \vec{x}$, then $\vec{x}f = \vec{x}$. This implies that S^π has no cyclic subgroup of prime order belonging to π' and hence $S^\pi \in \overline{\mathbf{G}}_\pi$. First we make a simple observation.

Lemma 4.14. *Let $f \in \mathscr{C}$ and $\varepsilon \neq \vec{x} \in \overline{\mathscr{F}}(S)$. Then*

$$\vec{x}f\rho\omega = \vec{x}f\omega = \vec{x}\widehat{f}\omega \leq_{\mathscr{R}} \vec{x}\omega.$$

Proof. By definition of $\mathscr{C}$, we have $\vec{x}f\omega = (\vec{x}\widehat{f} \cdot \overline{f})\omega = \vec{x}\widehat{f}\omega$. Since $\widehat{f} \in \check{S}_0^{\infty} \subseteq \check{S}^{\infty}$, we conclude $\vec{x}\widehat{f}\omega \leq_{\mathscr{R}} \vec{x}\omega$. The lemma then follows from Lemma 4.1. $\square$

Notice if $f \in \mathscr{C}^{\pi}$, then $\vec{x}f\rho \neq \varepsilon$, for all $\vec{x} \in \overline{\mathscr{F}}(S)$. Indeed, $\overline{f} \neq \varepsilon$ implies $\vec{x}f\rho = (\vec{x}\widehat{f} \cdot \overline{f})\rho \neq \varepsilon$. Finally, we turn to the main result of this section.

Theorem 4.15. *Let S be a finite semigroup. Then $S^{\pi} \in \overline{\mathbf{G}}_{\pi}$.*

Proof. As $\rho : \mathscr{C}^{\pi} \to S^{\pi}$ is a homomorphism by (4.3), it suffices to prove that if $\vec{x} \in \mathscr{F}_{\pi}(S)$, $p \in \pi'$ and $f \in \mathscr{C}^{\pi}$ are such that $\vec{x}f^{p}\rho = \vec{x}$, then $\vec{x}f\rho = \vec{x}$. Set $\vec{x}_i = \vec{x}f^{i}\rho$; so $\vec{x}_0 = \vec{x}_p = \vec{x}$. As $f \in \mathscr{C}^{\pi}$, the above discussion shows $\vec{x}_i \neq \varepsilon$ for all i. Suppose that $\vec{x} = (x_n, x_{n-1}, \ldots, x_1)$. We prove the following critical claim by induction on ℓ.

Claim 1. *For each $1 \leq i \leq p$ and each $0 \leq \ell \leq n$, we have $|\vec{x}_i| \geq \ell$ and $\vec{x}_i\alpha_\ell = \vec{x}\alpha_\ell = (x_\ell, x_{\ell-1}, \ldots, x_1)$.*

Let us show how the claim implies the theorem. As $\vec{x}_1 = \vec{x}f\rho$, it suffices from the claim to show that $|\vec{x}_1| = |\vec{x}|$. But Lemma 4.14 implies that

$$\vec{x}\omega \geq_{\mathscr{R}} \vec{x}_1\omega \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} \vec{x}_{p-1}\omega \geq_{\mathscr{R}} \vec{x}_p\omega = \vec{x}\omega \tag{4.4}$$

and so $x_n = \vec{x}_1\tau_n \geq_{\mathrm{L}} \vec{x}_1\omega \mathrel{\mathscr{R}} \vec{x}\omega = x_n$. Therefore, $\vec{x}_1\omega \mathrel{\mathrm{L}} x_n$ and hence, since $\vec{x}_1$ is a flag, we conclude $|\vec{x}_1| = n$. This shows that $\vec{x} = \vec{x}f\rho$ and completes the proof of Theorem 4.15 once we establish the claim.

The claim is trivial for $\ell = 0$. Assume the claim is true for $0 \leq \ell \leq n-1$. We prove it for $\ell + 1$. Let $\vec{a} = \vec{x}\alpha_\ell = (x_\ell, x_{\ell-1}, \ldots, x_1)$ and set $\vec{x} = \vec{b} \cdot \vec{a}$; if $\ell = 0$, then $\vec{a} = \varepsilon$ and $\vec{b} = \vec{x}$. Then $\vec{x}_i = \vec{b}_i \cdot \vec{a}$ for some $\vec{b}_i \in \mathscr{F}_{\pi}(S)$, each $1 \leq i \leq p$, by the inductive hypothesis. First we show $|\vec{x}_i| \geq \ell + 1$. Indeed, (4.4) implies $\vec{x}_i\omega \mathrel{\mathscr{R}} x_n <_{\mathrm{L}} x_\ell$ since $\vec{x}$ is a flag. We conclude $\vec{b}_i \neq \varepsilon$, that is $|\vec{x}_i| \geq \ell + 1$, for all i.

Let us set $\vec{z} = \vec{a}f\rho$. First observe that $|\vec{z}| \geq \ell$ and $\vec{z}\alpha_\ell = \vec{a}$. Indeed,

$$\vec{b}_1 \cdot \vec{a} = \vec{x}_1 = \vec{x}f\rho = (\vec{b} \cdot \vec{a})f\rho = (\vec{b}_{\vec{a}}\widehat{f} \cdot \vec{a}f)\rho = (\vec{b}_{\vec{a}}\widehat{f} \cdot \vec{z})\rho$$

by Lemma 4.11 and the confluence of reduction. As each entry of $\vec{b}_{\vec{a}}\widehat{f}$ is $<_{\mathscr{J}}$-below x_ℓ (since $\vec{x}$ is a flag and ${}_{\vec{a}}\widehat{f} \in \check{S}_0^{\infty}$), we must have $\vec{z}\alpha_\ell = \vec{a}$. There are three cases.

Case 1. Suppose that $|\vec{z}| \geq \ell + 2$. Then an application of Lemma 4.11 and the second item of Lemma 4.1 shows that, for any $\vec{y} \in \mathscr{F}_{\pi}(S)$,

$$(\vec{y} \cdot \vec{a})f\rho\alpha_{\ell+1} = (\vec{y}_{\vec{a}}\widehat{f} \cdot \vec{a}f)\rho\alpha_{\ell+1} = (\vec{y}_{\vec{a}}\widehat{f} \cdot \vec{z})\rho\alpha_{\ell+1} = \vec{z}\alpha_{\ell+1}.$$

In particular, $(\vec{x}_i)\alpha_{\ell+1}$ is independent of i and so equals $(\vec{x})\alpha_{\ell+1}$, as desired.

Case 2. Suppose that $|\vec{z}| = \ell+1$. Our goal is to show that $\vec{b}_i\alpha_1 = x_{\ell+1}$ for all $1 \leq i \leq p$. Set $s = \vec{z}\omega$ for convenience. Then we have by Lemma 4.11 and confluence of the reduction map

$$\vec{b}_i \cdot \vec{a} = \vec{x}_i = \vec{x}_{i-1}f\rho = (\vec{b}_{i-1}\cdot\vec{a})f\rho = ((\vec{b}_{i-1})_{\vec{a}}\widehat{f}\cdot\vec{a}f)\rho = ((\vec{b}_{i-1})_{\vec{a}}\widehat{f}\cdot\vec{z})\rho.$$

So the second item of Lemma 4.1 allows us to deduce $\vec{b}_i\alpha_1 \mathrel{Ł} \vec{z}\omega = s$. On the other hand, Lemma 4.1 implies $s = \vec{z}\omega = \vec{a}f\rho\omega = \vec{a}f\omega$ and so, since $\vec{x}_if = (\vec{b}_i)_{\vec{a}}\widehat{f}\cdot\vec{a}f \in \overline{\mathscr{F}}(S)$, we must have

$$(\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \leq_{Ł} \vec{a}f\omega = s \mathrel{Ł} \vec{b}_i\alpha_1, \text{ for all } i. \tag{4.5}$$

Subcase 1. Suppose $(\vec{b}_j\alpha_1)_{\vec{a}}\widehat{f} \mathrel{\mathscr{R}} \vec{b}_j\alpha_1$ for some $1 \leq j \leq p$. By definition of $\check{S}$ and of the wreath product there is an element $u \in S^I$ so that if $y \in S$ and $y_{\vec{a}}\widehat{f} \mathrel{\mathscr{R}} y$, then $y_{\vec{a}}\widehat{f} = yu$. With this notation, we are assuming $\vec{b}_j\alpha_1 u \mathrel{\mathscr{R}} \vec{b}_j\alpha_1$. By (4.5), $\vec{b}_j\alpha_1 u = (\vec{b}_j\alpha_1)_{\vec{a}}\widehat{f} \leq_{Ł} \vec{b}_j\alpha_1$, whence $\vec{b}_j\alpha_1 u \mathrel{\mathscr{H}} \vec{b}_j\alpha_1$. In particular, u represents an element of $\Gamma_R(H_{\vec{b}_j\alpha_1})$ by Proposition 3.2. Notice in the case of aperiodic pointlikes, this already yields $(\vec{b}_j\alpha_1)_{\vec{a}}\widehat{f} = \vec{b}_j\alpha_1 u = \vec{b}_j\alpha_1$ and so the following subclaim is essentially trivial in the aperiodic case.

Subclaim 1. *For $i \geq j$, we have $(\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \mathrel{\mathscr{R}} \vec{b}_i\alpha_1$ and $\vec{b}_i\alpha_1 = \vec{b}_j\alpha_1 u^{i-j}$.*

We prove the subclaim by induction, the case $i = j$ being by assumption. Assume it is true for i. Then $(\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \mathrel{\mathscr{R}} \vec{b}_i\alpha_1$ implies $(\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} = \vec{b}_i\alpha_1 u$. Since $(\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \leq_{Ł} \vec{b}_i\alpha_1$ by (4.5) and $(\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \mathrel{\mathscr{R}} \vec{b}_i\alpha_1$ by the induction hypothesis, we conclude $\vec{b}_i\alpha_1 u = (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \mathrel{Ł} \vec{b}_i\alpha_1 \mathrel{Ł} s$, where the last Ł-equivalence uses (4.5). As $\vec{b}_i$ is a flag, $\vec{b}_i\tau_2 <_{Ł} \vec{b}_i\alpha_1$ and so $(\vec{b}_i)_{\vec{a}}\widehat{f}\tau_2 <_{\mathscr{J}} \vec{b}_i\alpha_1$. Therefore, $(\vec{b}_i)_{\vec{a}}\widehat{f}\tau_2 <_{\mathscr{J}} (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f}$. Recalling $\vec{z}$ is a flag of length $\ell+1$, reduction is confluent and $(\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \mathrel{Ł} s = \vec{z}\omega$, we obtain

$$\vec{b}_{i+1}\alpha_1 = ((\vec{b}_i)_{\vec{a}}\widehat{f}\cdot\vec{a}f)\rho\tau_{\ell+1} = ((\vec{b}_i)_{\vec{a}}\widehat{f}\cdot\vec{z})\rho\tau_{\ell+1} = (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} = \vec{b}_i\alpha_1 u \tag{4.6}$$

Since $\vec{b}_{i+1}\alpha_1 = \vec{b}_i\alpha_1 u \mathrel{Ł} \vec{b}_i\alpha_1$, the second item of Definition 4.2 implies $(\vec{b}_{i+1}\alpha_1)_{\vec{a}}\widehat{f} \mathrel{Ł} (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} = \vec{b}_{i+1}\alpha_1$, where the equality uses (4.6). Since, $(\vec{b}_{i+1}\alpha_1)_{\vec{a}}\widehat{f} \leq_{\mathscr{R}} \vec{b}_{i+1}\alpha_1$, we conclude $(\vec{b}_{i+1}\alpha_1)_{\vec{a}}\widehat{f} \mathrel{\mathscr{R}} \vec{b}_{i+1}\alpha_1$, as was required for the subclaim. Also, by induction and (4.6)

$$\vec{b}_{i+1}\alpha_1 = \vec{b}_i\alpha_1 u = \vec{b}_j\alpha_1 u^{i-j}u = \vec{b}_j\alpha_1 u^{i+1-j},$$

where the first equality uses (4.6). This proves Subclaim 1.

Since $\vec{b}_j = \vec{b}_{j+p}$, Subclaim 1 implies $\vec{b}_j\alpha_1 = \vec{b}_{j+p}\alpha_1 = \vec{b}_j\alpha_1 u^p$. Since u represents an element of $\Gamma_R(H_{\vec{b}_j\alpha_1})$ and the Schützenberger group is a regular permutation group, it follows u^p represents the identity. But $\vec{b}_j\alpha_1$ is π'-free and $p \in \pi'$. We conclude that u represents the identity of $\Gamma_R(H_{\vec{b}_j\alpha_1})$ and so $\vec{b}_j\alpha_1 u = \vec{b}_j\alpha_1$. Applying the subclaim, it follows $\vec{b}_j\alpha_1 = \vec{b}_i\alpha_1$ for all $i \geq j$. Since the sequence $\vec{b}_i$ is periodic with period p, we conclude $\vec{b}_i\alpha_1$ is independent of i and in particular coincides with $\vec{b}_p\alpha_1 = x_{\ell+1}$, as required.

Subcase 2. Suppose that $(\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} <_{\mathscr{R}} \vec{b}_i\alpha_1$ for all $1 \le i \le p$. Then since $\vec{x}_i f = (\vec{b}_i)_{\vec{a}}\widehat{f} \cdot \vec{a}f \in \overline{\mathscr{F}}_\pi(S)$, we deduce $(\vec{b}_i)_{\vec{a}}\widehat{f} \in \overline{\mathscr{F}}_\pi(S)$. Therefore, every entry of $(\vec{b}_i)_{\vec{a}}\widehat{f}$ is $<_{\mathscr{J}}$-below $\vec{b}_i\alpha_1 Łs = \vec{z}\omega$ (see (4.5)). Since $|\vec{z}| = \ell + 1$,

$$\vec{b}_{i+1}\alpha_1 = \vec{x}_i f\rho\tau_{\ell+1} = ((\vec{b}_i)_{\vec{a}}\widehat{f} \cdot \vec{a}f)\rho\tau_{\ell+1} = ((\vec{b}_i)_{\vec{a}}\widehat{f} \cdot \vec{z})\rho\tau_{\ell+1} = \vec{z}\omega$$

In particular, $\vec{b}_i\alpha_1$ is independent of i, and so taking $i = p$ shows that $\vec{b}_i\alpha_1 = \vec{b}\alpha_1 = x_{\ell+1}$, as desired.

Case 3. We now arrive at the final case: when $|\vec{z}| = \ell$, i.e. $\vec{z} = \vec{a}$. This case does not arise when $\ell = 0$ since $\varepsilon f\rho = \overline{f}\rho \neq \varepsilon$. So assume from now on $\ell \ge 1$. This is the only case that makes use of the definition of $\check{S}_0^\infty$. Observe

$$x_\ell = \vec{a}\omega = \vec{z}\omega = \vec{a}f\rho\omega = \vec{a}f\omega = (\vec{a}\widehat{f} \cdot \overline{f})\omega = \vec{a}\widehat{f}\omega. \tag{4.7}$$

Since $\vec{x}_i = \vec{b}_i \cdot \vec{a}$ is a flag, we have the important formula

$$\vec{b}_{i+1}\alpha_1 = \vec{x}_i f\rho\tau_{\ell+1} = ((\vec{b}_i)_{\vec{a}}\widehat{f} \cdot \vec{a}f)\rho\tau_{\ell+1} = (\vec{b}_i)_{\vec{a}}\widehat{f}\rho\alpha_1 Ł (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \tag{4.8}$$

where the last equality uses $|\vec{a}f\rho| = |\vec{z}| = \ell$, while the Ł-equivalence comes from the second item of Lemma 4.1. The following subclaim will be used to seal the rest of the proof.

Subclaim 2. *There do not exist $m > i \ge 0$ such that $\vec{b}_m\alpha_1 <_{\mathscr{J}} \vec{b}_i\alpha_1$.*

Indeed, suppose $\vec{b}_m\alpha_1 <_{\mathscr{J}} \vec{b}_i\alpha_1$. Then by (4.8), we have

$$\vec{b}_{m+1}\alpha_1 Ł (\vec{b}_m\alpha_1)_{\vec{a}}\widehat{f} \le_{\mathscr{R}} \vec{b}_m\alpha_1 <_{\mathscr{J}} \vec{b}_i\alpha_1.$$

Continuing, we see that $\vec{b}_n\alpha_1 <_{\mathscr{J}} \vec{b}_i\alpha_1$ for all $n \ge m$. But the sequence $\vec{b}_n$ is periodic with period p, so choosing an appropriate n yields $\vec{b}_i <_{\mathscr{J}} \vec{b}_i$. This contradiction establishes Subclaim 2.

Now we are in a position to prove that the Ł in (4.8) is really an equality:

$$\vec{b}_{i+1}\alpha_1 = (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f}. \tag{4.9}$$

Indeed, by (4.8) $\vec{b}_{i+1}\alpha_1 Ł (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \le_{\mathscr{R}} \vec{b}_i\alpha_1$. Subclaim 2 then implies

$$(\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \mathrel{\mathscr{R}} \vec{b}_i\alpha_1 >_{\mathscr{J}} \vec{b}_i\tau_2 \ge_{\mathscr{R}} (\vec{b}_i)_{\vec{a}}\widehat{f}\tau_2 \tag{4.10}$$

since $\vec{b}_i$ is a flag. From (4.8) and (4.10) it follows that indeed

$$\vec{b}_{i+1}\alpha_1 = (\vec{b}_i)_{\vec{a}}\widehat{f}\rho\alpha_1 = (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f}.$$

Let us now prove by induction on $0 \le i \le p-1$ that $\vec{b}_i\alpha_1 = x_{\ell+1}$ (where $\vec{b}_0 = \vec{b}$). The case $i = 0$ is trivial. Suppose that the statement is true for i with $0 \le i \le p-2$. Then (4.9) and induction implies

$$\vec{b}_{i+1}\alpha_1 = (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \le_{\mathscr{R}} \vec{b}_i\alpha_1 = x_{\ell+1}.$$

Since $\vec{b}_{i+1}\alpha_1 \not<_{\mathscr{J}} \vec{b}_i\alpha_1$ (by Subclaim 2), we must in fact have $\vec{b}_{i+1}\alpha_1 \mathrel{\mathscr{R}} x_{\ell+1}$. Putting together $(\vec{b}_i\alpha_1 \cdot \vec{a})\widehat{f} = (\vec{b}_i\alpha_1)_{\vec{a}}\widehat{f} \cdot \vec{a}\widehat{f}$ with (4.7) and (4.9) yields

$$(x_{\ell+1}, x_\ell, \dots, x_1)\widehat{f} = (\vec{b}_{i+1}\alpha_1, x_\ell, y_{\ell-1}, \dots, y_1).$$

Since $x_{\ell+1} \mathscr{R} \vec{b}_{i+1}\alpha_1$, the Zeiger property (Lemma 4.8) yields $b_{i+1}\alpha_1 = x_{\ell+1}$. This completes the induction that $\vec{b}_i\alpha_1 = x_{\ell+1}$ for all i and thereby finishes the proof of Claim 1. Theorem 4.15 is now proved. □

5. Blowup operators

Fix a finite semigroup T, a set of primes π and set $S = CP_\pi(T)$. The salient idea underlying the remainder of the proof, is to construct a retraction $\widehat{B} : \overline{\mathscr{F}}(S) \to \overline{\mathscr{F}}_\pi(S)$ belonging to $\check{S}_0^\infty$. One then "conjugates" the action of the generators of the Rhodes expansion on $\mathscr{F}(S)$ by this retract to get an action on $\mathscr{F}_\pi(S)$ belonging to S^π.

Let us write $s \leq_{\mathscr{H}} t$ if both $s \leq_{\mathrm{Ł}} t$ and $s \leq_{\mathscr{R}} t$.

Definition 5.1 (Blowup operator)**.** A *preblowup operator* on S is a function $B : S \to S$ satisfying the following properties:

(1) $sB = s$ if s is π'-free;
(2) $sB <_{\mathscr{H}} s$ if s is not π'-free;
(3) $s \subseteq sB$ (the "blow up");
(4) There exists a function $m : S \to S$, written $s \mapsto m_s$, such that $sB = sm_s$ and $m_s = m_{s'}$ whenever $s\mathrm{Ł}s'$.

An idempotent preblowup operator is called a *blowup operator*.

The element m_s is called the *right multiplier* associated to s.

Lemma 5.2. *The collection of preblowup operators on S is a finite semigroup. In particular, if there are any preblowup operators on S, then there is a blowup operator on S.*

Proof. The first three conditions are obviously closed under composition. If B and B' are preblowup operators with respective right multipliers $s \mapsto m_s$ and $s \mapsto n_s$, then $sBB' = sm_s n_{sm_s}$. If $s\mathrm{Ł}s'$, then $m_s = m_{s'}$ and $sm_s\mathrm{Ł}s'm_s$. Therefore, $n_{sm_s} = n_{s'm_s} = n_{s'm_{s'}}$. This shows that BB' is a preblowup operator with $m_s n_{sm_s}$ as the right multiplier associated to s. The final statement follows from the existence of idempotents in non-empty finite semigroups. □

The next proposition collects some elementary properties of blowup operators. For the first item, the reader should consult Definition 4.2.

Proposition 5.3. *Let $B : S \to S$ be a blowup operator. Then:*

(1) *$B \in \check{S}$;*
(2) *The image of B is the set of π'-free elements of S;*
(3) *Suppose $y \leq_{\mathit{Ł}} s$. Then $y \subseteq ym_s$;*
(4) *If s is π'-free and $y \leq_{\mathit{Ł}} s$, then $y = ym_s$*

Proof. First we check $B \in \check{S}$. Since $sB = sm_s$, clearly $sB \leq_{\mathscr{R}} s$. If $s\mathrm{Ł}t$, then $m_s = m_t$ and so $sB = sm_s\mathrm{Ł}tm_s = tm_t = tB$. If $sB \mathscr{R} s$, then by the first and second items in the definition of a blowup operator, s is a π'-free and $sB = s = sI$, so we may take $s_B = I$ in the third item of Definition 4.2.

For the second item, observe that B fixes an element s if and only if s is π'-free. Since B is idempotent, it image is its fixed-point set. Turning to the third item, write $y = zs$ with $z \in S^I$. Then we have

$$y = zs \subseteq z(sB) = zsm_s = ym_s,$$

as required. For the final item, we have $s = sB = sm_s$. Since $y = zs$, some $z \in S^I$, we have $ym_s = zsm_s = zs = y$. This completes the proof. □

For the rest of this section we assume the existence of a blowup operator B on S; a construction appears in the next section. We proceed to define an "extension" $\widehat{B}$ of B to $\overline{\mathscr{F}}(S)$. Recall that Δ_s is the diagonal operator in $\check{S}^\infty$ corresponding to s (4.2).

Definition 5.4 ($\widehat{B}$)**.** Define $\widehat{B} : S^* \to S^*$ recursively by

- $\varepsilon\widehat{B} = \varepsilon$
- $(\vec{b} \cdot s)\widehat{B} = (\vec{b}\Delta_{m_s})\widehat{B} \cdot sB$.

This recursive definition is known as the Henckell formula.

Since $sB \leq_{Ł} s$, we obtain the following lemma.

Lemma 5.5. *If $\vec{x} \neq \varepsilon$, then $\vec{x}\widehat{B}\alpha_1 \leq_{Ł} \vec{x}\alpha_1$.*

We retain the notation from the previous section for the next proposition.

Proposition 5.6. *The map $\widehat{B}$ belongs to $\check{S}^\infty_0$. Moreover, $\overline{\mathscr{F}}(S)\widehat{B} = \overline{\mathscr{F}}_\pi(S)$ and $\widehat{B}|_{\overline{\mathscr{F}}(S)}$ is idempotent.*

Proof. Since $B \in \check{S}$ by Proposition 5.3 and $\Delta_{m_s} \in \check{S}^\infty$, it is immediate from the recursive definition that $\widehat{B} \in \check{S}^\infty$. Next we verify that $\overline{\mathscr{F}}(S)\widehat{B} \subseteq \overline{\mathscr{F}}_\pi(S)$ by induction on length. The base case is trivial. In general, $(\vec{b} \cdot x)\widehat{B} = (\vec{b}\Delta_{m_x})\widehat{B} \cdot xB$. Since Δ_{m_x} preserves $\overline{\mathscr{F}}(S)$, by induction $(\vec{b}\Delta_{m_x})\widehat{B} \in \overline{\mathscr{F}}_\pi(S)$. Since xB is π'-free (Proposition 5.3), if $\vec{b} = \varepsilon$ we are done. Otherwise, let x_1 be the first entry of $\vec{b}$. Then Lemma 5.5 shows that $(\vec{b}\Delta_{m_x})\widehat{B}\alpha_1 \leq_{Ł} x_1m_x$. As $xB = xm_x$ and $x_1 \leq_{Ł} x$, we see that $x_1m_x \leq_{Ł} xm_x$ and so $(\vec{b}\Delta_{m_x})\widehat{B} \cdot xB$ belongs to $\overline{\mathscr{F}}_\pi(S)$.

We show by induction on length that $\widehat{B}$ fixes $\overline{\mathscr{F}}_\pi(S)$, the case of length 0 being trivial. If $\vec{x} = \vec{b} \cdot x \in \overline{\mathscr{F}}_\pi(S)$, then $\vec{x}\widehat{B} = (\vec{b}\Delta_{m_x})\widehat{B} \cdot xB$. But Proposition 5.3, together with the fact that x is π'-free and $\vec{x}$ is an Ł-chain, implies $xB = x$ and $\vec{b}\Delta_{m_x} = \vec{b}$. So a simple induction yields $\vec{x}\widehat{B} = \vec{x}$. We conclude $\widehat{B}|_{\overline{\mathscr{F}}(S)}$ is idempotent.

Finally, we must verify $\widehat{B} \in \check{S}^\infty_0$. We proceed by induction on length, the cases of length 0 and 1 being vacuously true. Suppose $(x_2, x_1)\widehat{B} = (y_2, y_1)$ with $x_1 \mathrel{\mathscr{R}} y_1$ and $x_2 \mathrel{\mathscr{R}} y_2$. Then $y_1 = x_1B = x_1m_{x_1}$. On the other hand, $x_2 \mathrel{\mathscr{R}} y_2 = (x_2m_{x_1})B \leq_{\mathscr{R}} x_2m_{x_1} \leq_{\mathscr{R}} x_2$. Thus $x_2m_{x_1} \mathrel{\mathscr{R}} (x_2m_{x_1})B$ and so $x_2m_{x_1}$ is π'-free. Therefore $y_2 = (x_2m_{x_1})B = x_2m_{x_1}$. Thus $y_1 = x_1m_{x_1}$ and $y_2 = x_2m_{x_2}$, showing that the condition in Definition 4.6 is satisfied.

Suppose now $n > 2$ and that $(x_n, x_{n-1}, \dots, x_1)\widehat{B} = (y_n, y_{n-1}, \dots, y_1)$ with $x_i \mathscr{R}\, y_i$ for $i = n-1, n$. Then

$$(y_n, y_{n-1}, \dots, y_1) = ((x_n, x_{n-1}, \dots, x_2)\Delta_{m_{x_1}})\widehat{B} \cdot x_1 B.$$

Now $x_i \geq_{\mathscr{R}} x_i m_{x_1} \geq_{\mathscr{R}} y_i \mathscr{R}\, x_i$, for $i = n-1, n$. Therefore, $x_i m_{x_1} \mathscr{R}\, y_i$, $i = n-1, n$. Induction provides $s' \in S^I$ with $x_i m_{x_1} s' = y_i$, $i = n-1, n$. Taking $s = m_{x_1} s'$ yields $x_i s = y_i$, for $i = n-1, n$, completing the proof. $\square$

Another crucial property of $\widehat{B}$ is that it "blows up Ł-chains".

Proposition 5.7. *Let $\vec{x} = (x_n, \dots, x_1) \in \overline{\mathscr{F}}(S)$ and set $\vec{x}\widehat{B} = (y_n, \dots, y_1)$. Then $x_i \subseteq y_i$ for $i = 1, \dots, n$.*

Proof. The proof is by induction on n. For $n = 0$, the statement is vacuously true. In general, $\vec{x}\widehat{B} = (x_n m_{x_1}, \dots, x_2 m_{x_1})\widehat{B} \cdot x_1 B$. By the definition of a blowup operator $x_1 \subseteq x_1 B$. Since $\vec{x}$ is an Ł-chain, Proposition 5.3 shows $x_i \subseteq x_i m_{x_1}$ for $i = 2, \dots, n$. Induction yields $x_i m_{x_1} \subseteq y_i$ for $i = 2, \dots, n$, establishing that $x_i \subseteq y_i$. $\square$

Recall that if $s \in S$, then (Δ_s, s) denotes the element of $\mathscr{C}$ that acts by $\vec{x}(\Delta_s, s) = \vec{x}\Delta_s \cdot s$.

Proposition 5.8. *The equalities $\rho(\Delta_s, s)\rho = (\Delta_s, s)\rho$ and $\rho\widehat{B}\rho = \widehat{B}\rho$ hold.*

Proof. Consider first (Δ_s, s). Suppose $(x_n, \dots, x_1) \in \overline{\mathscr{F}}(S)$ with x_{i+1}Łx_i. Then $(x_n, \dots, x_1)(\Delta_s, s) = (x_n s, \cdots, x_1 s, s)$ has $x_{i+1}s$Ł$x_i s$. It follows that applying first the elementary reduction $(x_{i+1}, x_i) \to x_{i+1}$ and then (Δ_s, s) is the same as applying first (Δ_s, s) and then the elementary reduction $(x_{i+1}s, x_i s) \to x_{i+1}s$. We conclude $\rho(\Delta_s, s)\rho = (\Delta_s, s)\rho$.

Let us now turn to $\widehat{B}\rho$. We show by induction on i that if a string $\vec{x} = (x_n, \dots, x_1)$ with $n > i$ admits an elementary reduction $(x_{i+1}, x_i) \to x_{i+1}$ and $\vec{x}\widehat{B} = (y_n, \dots, y_1)$, then $(y_{i+1}, y_i) \to y_{i+1}$ is an elementary reduction. The base case is $i = 1$, i.e. x_1Łx_2. Then

$$(x_n, \dots, x_2, x_1)\widehat{B} = ((x_n, \dots, x_3)\Delta_{m_{x_1} m_{x_2 m_{x_1}}})\widehat{B} \cdot ((x_2 m_{x_1})B, x_1 m_{x_1})$$

By the definition of a blowup operator, x_1Łx_2 implies that $m_{x_1} = m_{x_2}$. Now $y_1 = x_1 m_{x_1}$. Since $x_2 m_{x_1} = x_2 m_{x_2} = x_2 B$ and B is idempotent, we see that $y_2 = x_2 m_{x_1}$. Now x_2Łx_1 implies $x_2 m_{x_1}$Ł$x_1 m_{x_1}$. We conclude y_2Ły_1, as required. If $i > 1$, then we use that

$$(y_n, \dots, y_1) = (x_n m_{x_1}, \dots, x_2 m_{x_1})\widehat{B} \cdot x_1 B.$$

Since $x_{i+1} m_{x_1}$Ł$x_i m_{x_1}$, the induction hypothesis gives $(y_{i+1}, y_i) \to y_{i+1}$ is an elementary reduction. This completes the induction. It is then immediate that $\rho\widehat{B}\rho = \widehat{B}\rho$. $\square$

For the next proposition, the reader is referred to Definition 4.12.

Proposition 5.9. *If $s \in S$, then $(\Delta_s, s)\widehat{B} \in \mathscr{C}^{\pi}$.*

Proof. We saw in the proof of Proposition 4.13 that the set of transformations f satisfying $\rho f \rho = f\rho$ is a semigroup. As $\widehat{B} : \overline{\mathscr{F}}(S) \to \overline{\mathscr{F}}_{\pi}(S)$ (Proposition 5.6), it suffices by Proposition 5.8, to show that $(\Delta_s, s)\widehat{B} \in \mathscr{C}$. Both (Δ_s, s) and $\widehat{B}$ leave $\overline{\mathscr{F}}(S)$ invariant. Now we obtain

$$\vec{x}(\Delta_s, s)\widehat{B} = (\vec{x}\Delta_s \cdot s)\widehat{B} = (\vec{x}\Delta_s\Delta_{m_s})\widehat{B} \cdot sB = (\vec{x}\Delta_{sm_s})\widehat{B} \cdot sB$$

and $\Delta_{sm_s}\widehat{B} \in \check{S}_0^{\infty}$ by Proposition 5.6. This establishes $(\Delta_s, s)\widehat{B} \in \mathscr{C}$, completing the proof. ☐

Remark 5.10. Let S be any finite monoid and $B : S \to S$ any idempotent operator satisfying (1), (2) and (4) of Definition 5.1. Then one can define $\widehat{B} : S^* \to S^*$ as per Definition 5.4 and Proposition 5.9 will still hold. Condition (3) is just used for Proposition 5.7 and to construct the relational morphism below.

If (X, M) and (Y, N) are faithful transformation semigroups, then a *relational morphism* $\varphi : (X, M) \to (Y, N)$ is a fully defined relation $\varphi : X \to Y$ such that, for each $m \in M$, there exists $\widetilde{m} \in N$ such that $y\varphi^{-1}m \subseteq y\widetilde{m}\varphi^{-1}$ for all $y \in Y$. If $\varphi : (X, M) \to (Y, N)$ is a relational morphism of faithful transformation semigroups, then the *companion relation* $\widetilde{\varphi} : M \to N$ is defined by $m\widetilde{\varphi} = \{n \in N \mid y\varphi^{-1}m \subseteq yn\varphi^{-1}, \forall y \in Y\}$. It is well known that $\widetilde{\varphi}$ is a relational morphism [?].

We define a relational morphism of faithful transformation semigroups $\varphi : (T^I, T) \to (\mathscr{F}_{\pi}(S), S^{\pi})$ as follows: we set $I\varphi = \varepsilon$, while, for $t \in T$, we define $t\varphi = \{\vec{x} \in \mathscr{F}_{\pi}(S) \mid t \in \vec{x}\omega\}$. Notice that φ^{-1} coincides with ω on non-empty strings.

Lemma 5.11. *The relation $\varphi : T^I \to \mathscr{F}_{\pi}(S)$ gives rise to a relational morphism $\varphi : (T^I, T) \to (\mathscr{F}_{\pi}(S), S^{\pi})$.*

Proof. Since $t \in \{t\} \subseteq \{t\}B$ and $\{t\}B \in \mathscr{F}_{\pi}(S)$, it follows that φ is fully defined. Let $t \in T$. We set $\widetilde{t} = (\Delta_{\{t\}}, \{t\})\widehat{B}\rho$. Proposition 5.9 shows $\widetilde{t} \in S^{\pi}$. We need to prove $\vec{x}\varphi^{-1}t \subseteq \vec{x}\widetilde{t}\varphi^{-1}$. If $\vec{x} = \varepsilon$, then $\vec{x}(\Delta_{\{t\}}, \{t\})\widehat{B}\rho = \{t\}B$. So $\varepsilon\varphi^{-1}t = \{I\}t = \{t\} \subseteq \{t\}B = \varepsilon\widetilde{t}\varphi^{-1}$.

If $\vec{x} \neq \varepsilon$, then we need to show $\vec{x}\omega t \subseteq \vec{x}\widetilde{t}\omega$. Abusing notation, we identify $\{t\}$ with t. Then we have $\vec{x}(\Delta_t, t)\widehat{B}\rho = \left((\vec{x}\Delta_t \cdot t)\widehat{B}\right)\rho$. Lemma 4.1 tells us $\left((\vec{x}\Delta_t \cdot t)\widehat{B}\right)\rho\omega = (\vec{x}\Delta_t \cdot t)\widehat{B}\omega$. An application of Proposition 5.7 yields

$$(\vec{x}\Delta_t \cdot t)\widehat{B}\omega \supseteq (\vec{x}\Delta_t \cdot t)\omega = \vec{x}\omega t,$$

completing the proof. ☐

Proposition 5.12. *The companion relation $\widetilde{\varphi} : T \to S^{\pi}$ satisfies the inequality $f\widetilde{\varphi}^{-1} \subseteq \varepsilon f\omega \in CP_{\pi}(T)$.*

Proof. Let $t \in f\widetilde{\varphi}^{-1}$. Then $t = It \in \varepsilon\varphi^{-1}t \subseteq \varepsilon f\varphi^{-1} = \varepsilon f\omega \in CP_{\pi}(T)$, as $\varepsilon f \neq \varepsilon$. Hence $f\widetilde{\varphi}^{-1} \subseteq \varepsilon f\omega$, as required. ☐

Corollary 5.13. *If $CP_\pi(T)$ admits a blowup operator, then Theorem 2.3 holds. That is,* $\mathsf{PL}_{\overline{\mathbf{G}}_\pi}(T) = \{X \subseteq T \mid X \subseteq Y \in CP_\pi(T)\}$.

Proof. We already know $CP_\pi(T) \subseteq \mathsf{PL}_{\overline{\mathbf{G}}_\pi}(T)$. Since S^π is π'-free by Theorem 4.15, we have that each $\overline{\mathbf{G}}_\pi$-pointlike set is contained in $f\widetilde{\varphi}^{-1}$ for some $f \in S^\pi$. An application of Proposition 5.12 then completes the proof. □

6. Construction of the blowup operator

We continue to work with our fixed finite semigroup T and to denote $CP_\pi(T)$ by S. Our task now consists of constructing a blowup operator for S. By Lemma 5.2, it suffices to construct a preblowup operator. Our approach is a variation on Henckell's [?], which leads to a shorter proof. For this purpose, we need to use Schützenberger groups. We retain the notation for Schützenberger groups introduced in Section 3.

For each non-π'-free Ł-class L, fix an $\mathscr{H}$-class H_L of L and a prime power order element $g_L \in \widetilde{\Gamma}_R(H_L)$ representing an element of $\Gamma_R(H_L)$ of prime order $p \in \pi'$ (c.f. Lemma 3.4). We are now prepared to define our preblowup operator B. If $s \in S$ is a π'-free element, define $m_s = I$. If s is not π'-free, define $m_s = g_{L_s}^{\omega+*} \in S$. Notice that $g_{L_s}^{\omega+*} = \bigcup_{n\geq 1} g_{L_s}^n \supset g_{L_s}$ since g_{L_s} is a group element. Define an operator $B : S \to S$ by $sB = sm_s$.

Proposition 6.1. *The operator B is a preblowup operator.*

Proof. If s is π'-free, then $m_s = I$ and $sB = sm_s = s$. The fourth item of Definition 5.1 is clearly satisfied by construction. We turn now to the third item. If s is π'-free, then trivially $s \subseteq sB$. If s is not π'-free, then since $g_{L_s} \in \widetilde{\Gamma}_R(H_s)$ by Proposition 3.1, we have $sg_{L_s}^\omega = s$. As $g_{L_s}^\omega \subseteq g_{L_s}^{\omega+*}$,

$$s = sg_{L_s}^\omega \subseteq sg_{L_s}^{\omega+*} = sm_s = sB,$$

as required. Finally we turn to the second item of Definition 5.1. Suppose that $s \in S$ is not π'-free. It is immediate from the definition $sB = sm_s \leq_{\mathscr{R}} s$. Let $\gamma : \Gamma_R(H_s) \to \Gamma_L(H_s)$ be the anti-isomorphism given by $sg = g\gamma s$ for $g \in \Gamma_R(H_s)$. Choose, using Lemma 3.4, an element $x \in \widetilde{\Gamma}_L(H_s)$ of order a power of p so that x maps to $g_{L_s}\gamma$ in $\Gamma_L(H_s)$ (where we view g_{L_s} as an element of $\Gamma_R(H_s)$ using Proposition 3.1 and the projection). Then we have $\bigcup_{n\geq 1} x^n = x^{\omega+*} \in S$. We calculate sB as follows:

$$sB = sg_{L_s}^{\omega+*} = s\bigcup_{n\geq 1} g_{L_s}^n = \bigcup_{n\geq 1} sg_{L_s}^n = \bigcup_{n\geq 1} (g_{L_s}\gamma)^n s = \bigcup_{n\geq 1} x^n s = x^{\omega+*}s.$$

We conclude $sB \leq_{Ł} s$ and thus $sB \leq_{\mathscr{H}} s$. To establish $sB <_{\mathscr{H}} s$, observe, using (2.1), $sBg_{L_s} = sg_{L_s}^{\omega+*}g_{L_s} = sg_{L_s}^{\omega+*} = sB$. Since g_{L_s} represents a nontrivial element of $\Gamma_R(H_s)$ (c.f. Proposition 3.1) and $(H_s, \Gamma_R(H_s))$ is a regular permutation group, we deduce that $sB \notin H_s$. This concludes the proof that $sB <_{\mathscr{H}} s$ when s is not π'-free. Therefore, B is a preblowup operator, as required. □

In light of Corollary 5.13, we have now established Theorem 2.3.

Remark 6.2. The first and second authors believe that one can make this whole approach work without blowing up null elements.

Department of Mathematics/Computer Science, New College of Florida 5800 Bay Shore Road Sarasota, Florida 34243-2109, and Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, USA, and School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, Ontario K1S 5B6, Canada

E-mail address: bsteinbg@math.carleton.ca