



A Counterexample to a Conjecture of Scott and Suppes

Author(s): W. W. Tait

Source: *The Journal of Symbolic Logic*, Vol. 24, No. 1 (Mar., 1959), pp. 15-16

Published by: [Association for Symbolic Logic](#)

Stable URL: <http://www.jstor.org/stable/2964569>

Accessed: 12/06/2014 17:12

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
<http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



Association for Symbolic Logic is collaborating with JSTOR to digitize, preserve and extend access to *The Journal of Symbolic Logic*.

<http://www.jstor.org>

A COUNTEREXAMPLE TO A CONJECTURE OF SCOTT AND SUPPES

W. W. TAIT

In [1], it is conjectured that if S is a sentence in the first-order functional calculus with identity, and every subsystem of every finite relational system which satisfies S also satisfies S , then S is finitely equivalent to a universal sentence. (Two sentences are *finitely equivalent* if and only if they are satisfied by the same finite relational systems.) The following sentence S refutes that conjecture, and moreover S is satisfied by all finite subsystems of all (finite or infinite) relational systems which satisfy it.¹ S contains as predicate letters only the two-place predicate letters \leq , R (and the identity symbol $=$).

S is the conjunction of the following three sentences:

- $$\begin{aligned}
(1) \quad & \forall x \forall y \forall z [[x \leq y \vee y \leq x] \& [[x \leq y \& y \leq x] \supset x = y] \& \\
& \hspace{15em} [[x \leq y \& y \leq z] \supset x \leq z]], \\
(2) \quad & \forall x \forall y \forall z [Rxy \supset [z \leq x \vee y \leq z]], \\
(3) \quad & \forall x \forall y [[Rxy \& y \leq x] \supset \exists z [y \leq z < x \& \forall u [z \leq u \supset \sim Rzu]].
\end{aligned}$$

Sentence (1) asserts that \leq is a simple ordering. (2) implies that if Rxy and $x \leq y$, then either $x = y$ or y is the successor of x . Sentence (1) implies $\forall x[x \leq x]$, and this together with (3) implies

- $$(4) \quad \forall x \sim Rxx.$$

Hence (1), (2) and (3) imply that Rxy and $x \leq y$ hold only if y is the successor of x .

Let $\mathfrak{A} = \langle A, \leq, R \rangle$ be a relational system which satisfies (1), (2) and (4). Since these sentences are universal, any subsystem of \mathfrak{A} will also satisfy them. A sequence (a_1, a_2, \dots, a_n) of elements of A is called a *cycle* in \mathfrak{A} if $a_1 \leq a_2 \leq \dots \leq a_n$, $Ra_j a_{j+1}$ for $j = 1, \dots, n-1$, and $Ra_n a_1$. \mathfrak{A} satisfies (3) only if it has no cycles. If \mathfrak{A} is finite and satisfies (1), (2) and (4), then it satisfies (3) if and only if it has no cycles. If \mathfrak{B} is a subsystem of \mathfrak{A} , then every cycle of \mathfrak{B} is clearly a cycle of \mathfrak{A} . Hence, if \mathfrak{A} satisfies S , and \mathfrak{B} is a finite subsystem of \mathfrak{A} , then, since \mathfrak{B} satisfies (1), (2) and (4) and has no cycles, it satisfies (3). Thus, every finite subsystem of a relational system satisfying S also satisfies S .

For each $n > 0$, let $\mathfrak{A}_n = \langle \{0, 1, 2, \dots, n\}, \leq, R \rangle$, where \leq is the natural ordering of $\{0, 1, 2, \dots, n\}$ and Rxy means that $x \neq n$ and $y = x+1$ or

Received September 30, 1958.

¹ Both the sentence S and our argument have been simplified due to suggestions by D. Scott.

that $x = n$ and $y = 0$. Since $(0, 1, 2, \dots, n)$ is a cycle in \mathfrak{A}_n , \mathfrak{A}_n does not satisfy S , though it satisfies (1), (2) and (4). On the other hand, no proper subsystem of \mathfrak{A}_n contains cycles. Hence, every proper subsystem of \mathfrak{A}_n satisfies S . But by the remark on page 124 of [1], this proves that S is finitely equivalent to no universal sentence containing precisely n distinct variables; and this holds for every $n > 0$.

Notice that, by Theorem 1.8 of [2], a relational system \mathfrak{A} with an (infinite) subsystem \mathfrak{B} must exist such that \mathfrak{A} satisfies S and \mathfrak{B} does not. Indeed, such a system exists: Let A be the set of ordinal numbers $x \leq \omega + 1$, let \leq be the natural ordering of these ordinals, and let Rxy mean that $y = x' < \omega$ or that $y = 0$ and $x = \omega + 1$. Then $\mathfrak{A} = \langle A, \leq, R \rangle$ satisfies S , but the subsystem resulting from \mathfrak{A} by dropping the single element ω from A does not satisfy S .

REFERENCES

- [1] D. SCOTT and P. SUPPES, *Foundational aspects of theories of measure*, this JOURNAL, vol. 23 (1958), pp. 113–128.
- [2] A. TARSKI, *Contributions to the theory of models*, II, ***Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen***, ser. A, vol. 57 (1954), pp. 582–588.

STANFORD UNIVERSITY