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Ramsey's theorem and König's Lemma

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Abstract We consider the relation between versions of Ramsey's Theorem and König's Infinity Lemma, in the absence of the axiom of choice.

Keywords Ramsey's theorem · König's lemma · Axiom of choice

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1 Introduction

It was remarked by Blass [1] that König's 'Infinity Lemma', the statement that an infinite tree having finite levels has an infinite branch, is a consequence of Ramsey's Theorem, in set theory without the axiom of choice. The first author enquired precisely which strength of Ramsey's Theorem was required for this. We show that Ramsey's Theorem for 2-coloured sets of pairs suffices, and since there seems to be no proof in print, we supply one here. (We note that Kleinberg [3] showed that to prove Ramsey's Theorem, some version of choice must definitely be used.)

In fact, Blass does not work directly with König's Lemma as such, but with equivalent weak forms of the axiom of choice. If we consider the following three statements:

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(1) König's Lemma,
(2) every countably infinite family of finite non-empty sets has a choice function,
(3) every countably infinite family of finite non-empty sets has an infinite subfamily with a choice function,

then Blass actually shows that (3) follows from Ramsey's Theorem, and Kleinberg's proof is based on the failure of (3), as he gives a model containing a set which is expressible as a countable union of pairs, no infinite subfamily of which has a choice function.

We show the equivalence of (1)–(3) as follows (also given in [2], for instance see the top of page 46 and the bottom of page 251).

Clearly (2) implies (3). To deduce (1) from (3), let $(T, <)$ be an infinite tree with finite levels. Its levels then form a countable family of finite sets, so by (3) there is a choice function for an infinite set of levels. If X is the set of elements so chosen then these are naturally well-ordered in type ω by which levels they lie in. The downward closure Y of X can be well-ordered since it is a subset of the image of $X \times \omega$ under the map sending each element to its ith predecessor. Then Y is an infinite tree with finite levels, so as Y can be well-ordered, the usual proof of König's Lemma provides an infinite branch. To show that (1) implies (2) let $\{X_n : n \in \omega\}$ be a countable family of finite non-empty sets. We form a tree whose nth level is the family of choice functions for $\{X_i : i < n\}$ ordered by extension. Then $(T, <)$ is an infinite tree with finite levels, and an infinite branch provides a choice function for $\{X_n : n \in \omega\}$.

Versions of König's Lemma were the subject of a paper [4] in which the degree of branching of the tree in question was constrained. These may be viewed as special cases of the axiom of dependent choices, where at each stage there are always a specified number of choices available, which is why they were written as DC_n there. We give some extensions of these results here. In particular, we remark that (from the point of view of versions of the axiom of choice), the general case of König's Lemma is equivalent to that for 'balanced trees', those in which all vertices on the same level have the same degree, and we make some further connections with the work in [4].

What seems considerably harder (and perhaps more interesting) is to discover the relations between versions of Ramsey's Theorem for varying parameters. In Lemma 2.2 we remark on the rather obvious equivalence of Ramsey's Theorem for all 'lower' parameters (the range of the map), with upper parameter fixed, and we establish that Ramsey's Theorem for a greater upper parameter implies Ramsey's Theorem for a smaller upper parameter. We have no information on the validity of the reverse implications, though in view of the heavy appeal to (full) DC in the usual inductive proof of Ramsey's Theorem, these seem unlikely to hold. König's Lemma is vacuously true for the restricted class of sets which have no countable partition, merely because any tree satisfying the hypothesis of König's Lemma is partitioned into infinitely many levels, but even for the simplest sets having no countable partition, 'amorphous' sets [5], Ramsey's Theorem is however false.

Throughout, we write $[X]^n$ for the set of n-element subsets of X, and the natural number k is identified with $\{0, 1, \ldots, k-1\}$. Thus, Ramsey's Theorem RT^n_k in its usual form, where n and k are positive integers, says that for any infinite set X and map $F : [X]^n \to k$, there is an infinite subset Y of X such that the restriction of F to $[Y]^n$ is constant (Y is then called a *homogeneous* set). Since any (well-founded) tree with an ωth level automatically has an infinite branch, namely the set of elements below some element on that level, we shall always suppose that only the finite levels of our trees are non-empty.

As is remarked in [1], if Ramsey's Theorem holds for some infinite subset of X, then it also holds for X, and since the usual proof of Ramsey's Theorem shows that it holds for ω even without choice, it follows that to violate Ramsey's Theorem we must have Dedekind finite sets present (ones which are infinite, but which have no countably infinite subset). The fact that these occur several times in the paper is therefore no surprise.

2 Deriving König's Lemma from Ramsey's theorem

Theorem 2.1 *König's Lemma follows from RT^2_2.*

Proof Let $(T, <)$ be an infinite (well-founded) tree with finite levels. Map the set of unordered pairs of members of T to $\{0, 1\}$ by F where $F\{t_1, t_2\} = 0$ if t_1 and t_2 are on the same level, 1 otherwise. By RT^2_2, there is an infinite homogeneous set X. Since all levels are finite, all elements of X lie on distinct levels. Since the set of levels of T is well-ordered (in type ω), X receives a corresponding well-ordering $\{x_n : n \in \omega\}$. Now following the outline proof of (3) $\Rightarrow$ (1) in the introduction, we find that T has an infinite branch. □

We have been unable to find upward implications between the versions of Ramsey's Theorem for pairs, triples, quadruples, ... Certainly, the usual proofs of Ramsey's Theorem use induction on this upper index, with heavy appeal to dependent choices, which we are meant to be avoiding. We can however rather easily demonstrate the equivalence of Ramsey's Theorem for different 'lower indices', needed in what follows, and also that all 'downward' implications on the upper parameter hold.

Lemma 2.2 *For each fixed n, the statements RT^n_k are equivalent for all integers $k \geq 2$.*

Proof We first remark that RT^n_2 is an immediate consequence of RT^n_k for each $k \geq 2$, since any map from $[X]^n$ to 2 also maps $[X]^n$ to k.

We next show that if $k \geq 2$, then $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^n_{k+1}$. Let $F : [X]^n \to k+1$. Let $G : [X]^n \to k$ be given by $G(A) = F(A)$ if $F(A) < k-1$, and $G(A) = k-1$ otherwise. By RT^n_k there is an infinite homogeneous set Y for G. If F takes $[Y]^n$ to some $i < k-1$, then Y is also homogeneous for F. Otherwise, $F : [Y]^n \to \{k-1, k\}$, and we may appeal to RT^n_2 to find an infinite homogeneous subset Z of Y. □

Theorem 2.3 *If $n_1 \geq n_2 \geq 1$ and $k_1, k_2 > 1$, then $RT^{n_1}_{k_1} \Rightarrow RT^{n_2}_{k_2}$.*

Proof In view of Lemma 2.2, and by using induction, it suffices to show that for each n, $\mathrm{RT}^{n+1}_{n+2} \Rightarrow \mathrm{RT}^n_2$. Suppose therefore that X is an infinite set and that $f : [X]^n \to 2$. To appeal to RT^{n+1}_{n+2} we define a map $g : [X]^{n+1} \to n+2$ by letting $g(A) = |\{B \in [A]^n : f(B) = 1\}|$ for each $n+1$-element subset A of X. Note that A has $n+1$ n-element subsets, and so $0 \leq g(A) \leq n+1$. Thus g maps $[X]^{n+1}$ into $n+2$. By RT^{n+1}_{n+2}, there is an infinite set $Y \subseteq X$ homogeneous for g. We shall show that Y is also homogeneous for f.

Let k be the constant value of g on $[Y]^{n+1}$, and let p^α be any prime power factor of $n+1$. Let Z be a subset of Y having $n+p^\alpha$ elements, and let $W = \{(u, v) : u \in [Z]^{n+1}, v \in [Z]^n, u \supset v, f(v) = 1\}$. We shall calculate $|W|$ in two different ways. Let $m = |\{v \in [Z]^n : f(v) = 1\}|$.

For each $u \in [Z]^{n+1}$, $|\{v : (u, v) \in W\}| = k$, so $|W| = \binom{n+p^\alpha}{n+1} \cdot k$.

For each $v \in [Z]^n$, $|\{u : (u, v) \in W\}| = \begin{cases} 0 & \text{if } f(v) = 0 \\ p^\alpha & \text{if } f(v) = 1 \end{cases}$,

so $|W| = p^\alpha m$.

We deduce that $\binom{n+p^\alpha}{n+1} \cdot k = p^\alpha m$. But as $\binom{n+p^\alpha}{n+1} = \binom{n+p^\alpha}{p^\alpha - 1}$ and $n+1$ is divisible by p^α, $\binom{n+p^\alpha}{n+1}$ is not divisible by p. Hence p^α divides k.

Since every prime power factor of $n+1$ divides k, we deduce that $n+1$ divides k, so $k = 0$ or $n+1$. It follows that Y is homogeneous for f. □

Corollary 2.4 *For each $n \geq 2$, König's Lemma follows from RT^n_2.*

Proof This follows from Theorems 2.1 and 2.3. □

We conclude this section by remarking that although König's Lemma is vacuously true for all sets which have no countable partition (merely because any infinite tree with finite levels automatically has such a partition), Ramsey's Theorem is unprovable even for the simplest of such sets, namely those called 'amorphous', being the ones which are infinite, but are not expressible as the disjoint union of two infinite sets [5]. For it is possible for there to be an amorphous set X which has a partition π into pairs. If we now let $F\{x, y\} = 0$ if $\{x, y\} \in \pi$, 1 otherwise, then F can have no infinite homogeneous subset, since this would have to intersect infinitely many of the members of π in a singleton, violating amorphousness, so RT^2_2 fails. This demonstrates that for a given set, Ramsey's Theorem does not follow from König's Lemma.

The stronger statement, that Ramsey's Theorem does not follow from König's Lemma, rather than just one instance of König's Lemma, is proved in [1] Theorem 1, via the equivalent statement (3) given in the introduction.

3 Versions of König's lemma

To give a more detailed analysis of versions of König's Lemma along the lines of [4], we first justify the restriction in that paper to 'balanced' trees. A tree is said to be *balanced* if all vertices on the same level have the same degree. Since all non-root elements have a unique immediate predecessor in the tree, this is the same as saying that all elements on the same level have the same number of immediate successors. The following lemma is proved without use of the axiom of choice.

Lemma 3.1 *The general case of König's Lemma is equivalent to König's Lemma just for balanced trees.*

Proof Assume König's Lemma for balanced trees, and let $(T, <)$ be an arbitrary infinite tree with finite levels. The idea is simply to 'prune' the tree until it becomes balanced. During the pruning process, we shall ensure that the elements which remain stay on the same level they were on originally.

The main step in pruning is as follows. If all elements of T on the same level have the same degree, then T is already balanced. Otherwise, there is a least n such that elements on the nth level T_n do not all have the same degree. Let k be the least degree of a vertex in T_n which has infinitely many points above it, and let T' be obtained from T by removing all elements of T_n not having degree k, and all points above them. It is clear that T' is still infinite, and since along with any point removed, all points above it are also removed, the surviving points remain on their original levels.

Now we iterate this procedure. Let $T_0 = T$, $T_{m+1} = T'_m$, and $T^* = \bigcap_{m\in\omega} T_m$. We have to see that T^* is (infinite and) balanced. Clearly the root of T lies in every T_m, and so T^* also contains the root. Since $T_0 \supseteq T_1 \supseteq T_2 \supseteq \cdots$ and the levels of all retained points are unchanged, for each n, the sequence consisting of the nth levels of the T_ms is a decreasing sequence of finite sets, so eventually stabilizes. Therefore there is some m such that all levels up to the $(n+1)$th are unchanged from that stage on. It follows that for each level up to the nth, the degrees of all points of T^* on that level are equal. Hence T^* is balanced. Furthermore, since the $(n+1)$th level of T_m is non-empty, and agrees with that of T^*, the $(n+1)$th level of T^* is also non-empty, and since n was arbitrary, T^* is infinite.

Note that the pruning process can return to previously pruned levels, since removal of points on higher levels will alter the degrees of points on that level.

By König's Lemma for balanced trees, T^* has an infinite branch, and this is also an infinite branch of T. □

This result justifies restricting consideration of versions of König's Lemma just to the following special cases. For any sequence $\sigma = (\sigma_k : k \in \omega)$ of positive integers, let DC_σ be the statement that any tree in which all elements on the kth level have exactly σ_k immediate successors, has an infinite branch. The trees referred to here are balanced, and also any balanced tree corresponds to DC_σ for some σ. The special cases considered in [4] were those where σ is constant, or generalizing this a little, for a set Z, DC_Z stood for $(\forall\sigma)((\forall k)(\sigma_k \in Z) \to \mathrm{DC}_\sigma)$. We conclude by making a few remarks about implications between the various DC_σ, without giving at all a full treatment.

Theorem 3.2 *Suppose σ and τ are infinite sequences of positive integers, such that τ is obtained from σ by one of the following methods, or a combination of all three:*

(i) *τ is a subsequence of σ,*
(ii) *for all n, τ_n divides σ_n,*
(iii) *for some sequence $0 = k_0 < k_1 < k_2 < \cdots$, for every n, $\sigma_n = \prod_{k_i \leq n < k_{i+1}} \tau_i$.*

Then $DC_\sigma \Rightarrow DC_\tau$.

Proof In each case the method is to show how to transform a balanced tree $(T_\tau, <)$ with degrees of branching given by τ to a tree $(T_\sigma, <)$ given by σ (in an 'effective' manner). One then applies DC_σ to give an infinite branch of T_σ, and this then supplies a branch of the original tree T_τ.

For (i) one adds in any 'missing' levels, with suitable duplications where required. For (ii) the tree needs to be expanded at vertices on levels n where τ_n is a proper factor of σ_n by taking σ_n/τ_n copies, and for (iii) vertices of $(T_\tau, <)$ need to be removed which lie on levels lying strictly between levels k_i and k_{i+1}. □

As far as non-implications between the DC_σ are concerned, the basic method employed in [4] is to build a Fraenkel–Mostowski model with atoms indexed by some DC_σ, using a transitive subgroup G of the group of tree automorphisms (transitivity of the group of all tree automorphisms follows since the tree is balanced), and use finite supports. This group will be a suitable iterated wreath product of finite symmetric groups. It is then automatic that DC_σ is false in the resulting model, and the main point is to show that appropriate DC_τ are true, which is done by an analysis of the subgroup structure of G.

References

1. Blass, A.: Ramsey's Theorem in the hierarchy of choice principles. J. Symbol. Logic **42**, 387–390 (1977)
2. Howard, P., Rubin, J.E.: Consequences of the axiom of choice. American Mathematical Society, Mathematical Surveys and Monographs, no. 59 (1998)
3. Kleinberg, E.M.: The independence of Ramsey's Theorem. J. Symbol. Logic **34**, 205–206 (1969)
4. Truss, J.K.: Some cases of König's Lemma. In: Marek,W., Srebrny, M., Zarach, A (eds.) Set Theory and Hierarchy Theory. Lecture Notes in Mathematics, vol. 537, pp. 273–284. Springer, Berlin Heidelberg New York (1976)
5. Truss, J.K.: The structure of amorphous sets. Ann. Pure Appl. Logic **73**, 191–233 (1995)