A GAP THEOREM FOR POWER SERIES SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS*

Dedicated to the fond memory of William Boone.

By Leonard Lipshitz and Lee A. Rubel

1. Introduction. In this paper we prove (Theorem 4.1) that a power series with sufficiently large gaps (for example Hadamard gaps) cannot satisfy an algebraic differential equation (A.D.E.). An A.D.E. is an equation of the form $P(z, \omega, \omega', \ldots, \omega^{(m)}) = 0$ where P is a polynomial over $\mathbf{C}$ in all its variables. A power series which satisfies an ADE is called differentially algebraic. We also show that many operations on power series (eg. the Hadamard product, Laplace transform, inverse Laplace transform) can take differentially algebraic power series (or functions) to power series which are not differentially algebraic.

Let $f(z) = \Sigma f_{n_k} z^{n_k}$, $f_{n_k} \in \mathbf{C}$, $f_{n_k} \neq 0$ be differentially algebraic. Maillet [MAI], Ostrowski [OST] and Popken [POP] have shown that necessarily $\limsup n_{k+1}/n_k < \infty$. This is also shown in Kolchin [KOL]. In Theorem 4.1 we prove something stronger than $\liminf n_{k+1}/n_k = 1$. This partially answers Problem 2 of [RUB]. For the precise gap condition see Section 3. We are able to exclude Hadamard gaps (i.e. $n_{k+1}/n_k \geq r > 1$) and also considerably smaller gaps such as $|n_k - k^{(\log k)^{1+\epsilon}}| < C$. (It follows from the results of Osgood [OSG] that, with an irksome side condition, one has $\limsup n_{k+1}/n_k \leq 2$.) Many examples of power series with Hadamard gaps (for example Σz^{2^k}) were shown not to be differentially algebraic by Mahler in [MAH 1]. The differentially algebraic power series with the largest gaps, that is known to us, is Jacobi's theta function $\theta(z) = 1 + 2\Sigma_{k=1}^{\infty} z^{k^2}$. For proofs that this is differentially algebraic see [JAC], [HUR] or [DRA] and [RES]. It seems to be open whether Σz^{n^3} is differentially alge-

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braic. Thus there is still a large gap between Theorem 4.1 and the known examples of differentially algebraic power series with large gaps. In Remark 4.3 we suggest a possible number-theoretic application.

In Section 2 we outline the recursion formula satisfied by the Taylor coefficients of differentially algebraic power series. (We follow [MAH 2 pp. 185-194.] This is needed in several later places. In Section 3 we discuss our gap condition and in Section 4 we prove the gap theorem.

Section 5 contains a stronger theorem on gaps in differentially algebraic power series with rational coefficients which define *entire* functions. If $f(z) = \Sigma f_k z^k$ we define the spectrum of f to be $\{k: f_k \neq 0\}$. We call a sequence $\{a_k\}$ a multiplier if whenever $\Sigma f_k z^k$ is differentially algebraic so is $\Sigma a_k f_k z^k$. In Section 6 we present some results on spectra and multipliers of differentially algebraic power series. In Section 7 we mention some open problems.

Some of the results in this paper were announced in [LIR].

2. The recursion formula. In this section we give a brief summary of the recursion formula satisfied by the coefficients of a differentially algebraic power series. We follow [Mahler, Lectures on Transcendental Numbers, pp. 186-194] where a complete exposition is given.

Let $f(z) = \Sigma_{k=0}^{\infty} f_k z^k$ be differentially algebraic. Then the transcendence degree of the field $\mathbf{C}(z, f, f', f'', \ldots)$ over $\mathbf{C}(z)$ is finite. Hence the transcendence degree of $\mathbf{C}(f, f', f'', \ldots)$ over $\mathbf{C}$ is also finite, and thus f satisfies an ADE with coefficients from $\mathbf{C}$ (instead of $\mathbf{C}[z]$.) This reduction is not essential for our proof but will simplify the notation in Section 4. Among all the ADE's with coefficients from $\mathbf{C}$, satisfied by f let $F(\omega) = F(\omega, \omega', \ldots, \omega^{(m)}) = 0$ be of lowest possible order, m, and lowest possible total degree, n. $S(\omega) = \partial F/\partial\omega^{(m)}$ is the separant of F. By our choice of F, $S(f) \neq 0$, since S has either lower order or lower degree in $\omega^{(m)}$ than F. Let

$$F(\omega) = \sum_{(\kappa)} a_{(\kappa)} \omega^{(\kappa_1)} \cdots \omega^{(\kappa_N)}$$

where each $a_{(\kappa)} \in \mathbf{C}$ and the sum is over those systems of integers $(\kappa) = (\kappa_1, \ldots, \kappa_N)$ with $0 \leq \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_N \leq m$, $N \leq n$, for which $a_{(\kappa)} \neq 0$. (Recall that m is the order of F and n the total degree of F.) The f_k then satisfy a recursion formula of the form

(2.1)

$$\alpha(k)f_k = -B(k)\sum_{(\kappa)}\sum_{[\lambda]}{}^{*} a_{(\kappa)} \frac{(\kappa_1+\lambda_1)!}{\lambda_1!} \cdots \frac{(\kappa_N+\lambda_N)!}{\lambda_N!} f_{\kappa_1+\lambda_1} \cdots f_{\kappa_N+\lambda_N}$$

where

(i) $\alpha(k)$ is a fixed nonzero polynomial in k, depending on F and f.
(ii) The first sum is over all the tuples (κ) described above.
(iii) The second sum $\Sigma^{*}_{[\lambda]}$ is over all N tuples of integers $[\lambda] = [\lambda_1, \ldots, \lambda_N]$ with $0 \leq \lambda_i$, for $i = 1, \ldots, N$ and with $\Sigma_{i=1}^{N}\lambda_i = h = k - m + s$. Here m is the order of F and $s \geq 0$ is a fixed integer depending on f and F. The * indicates that all the terms involving $f_k, f_{k+1}, \ldots, f_{k+s}$ are to be omitted.
(iv) $B(k) = 1$ if $k \geq h$

$$\frac{h!}{k!} \quad \text{if} \quad h > k.$$

The idea of the proof of Theorem 4.1 is as follows. Assume that $f(z) = \Sigma f_k z^k$ satisfies the gap condition of Section 3, so that most of the $f_k = 0$. We then show that for a large set of values k, only one of the terms $f_{\kappa_1+\lambda_1} \cdots f_{\kappa_n+\lambda_n}$ (up to a permutation) on the right hand side of 2.1 is nonzero. For these values of k (2.1) then becomes

$$\alpha(k)f_k = P(\lambda_1, \ldots, \lambda_n)f_{\kappa_1+\lambda_1} \cdots f_{\kappa_n+\lambda_n}$$

where P is a fixed polynomial. Again using the fact that most of the f_k are zero we get that $P(t_1, \ldots, t_n)$ has too many zeroes. The details of the proof are given in Sections 3 and 4.

3. The Gap Condition. In this section we introduce our gap condition and produce the "large set" of values of k mentioned in the outline at the end of Section 2. Let $\{n_k\}$ be an increasing sequence from **N**. Define

(3.1) $$\Delta_k = n_{k+1} - n_k$$

and let $d(k)$ be the largest integer $\leq k$ such that

(3.2) $$\Delta_k \geq n_{d(k)}$$

if such an integer exists, and 0 otherwise. Our condition on the sequence $\{n_k\}$ is that

$$\lim_{k\to\infty} d(k)/k = 1. \tag{3.3}$$

Note that this certainly implies that $d(k)\Delta_k \to \infty$.

Examples. (i) Hadamard Gaps. Suppose that we have $n_{k+1}/n_k > r > 1$ for all k. (i.e. that $f(z) = \Sigma a_k z^{n_k}$ has Hadamard gaps.) Then $\Delta_k \geq (r-1)n_k \geq (r-1)r^i n_{k-i}$ for $k \geq i$. Hence if we choose i such that $(r-1)r^i \geq 1$ then we certainly have $k - i \leq d(k) \leq k$, for $i \leq k$, and hence that $d(k)/k \to 1$.

(ii) Condition (3.3) is stable under bounded perturbations. Let $\{n_k\}$ satisfy condition (3.3) and let $\{n'_k\}$ satisfy $|n_k - n'_k| \leq C$. $\Delta'_k = n'_{k+1} - n'_k$. Then $|\Delta_k - \Delta'_k| \leq 2C$ and since $n_{l+1} \geq n_l + 1$ we have that $\Delta'_k \geq n'_{d(k)-3C}$ for k large enough. Hence for k large enough we have $d'(k) \geq d(k) - 3C$ and hence that $d'(k)/k \to 1$.

(iii) We give an analytic condition which implies condition (3.3) and is convenient for handling some gaps smaller than Hadamard gaps. Let $|n_k - f(k)| < C$ (C some constant) and suppose that $f(x) = e^{\rho(\log x)}$ where $\rho'(x) \uparrow \infty$ and $\rho'(x)/x \to \infty$. We shall show that n_k satisfies condition (3.3). In the light of (ii) above, it is sufficient to show that if $\Delta(x) = f(x+1) - f(x)$ and $d(x) = \sup\{z \leq x : \Delta(x) \geq f(z)\}$, then $\lim_{x\to\infty} d(x)/x = 1$. Direct calculation shows that the above conditions imply that $f'(x)$ is increasing for x large enough. Hence for large x we have that $\Delta(x) \geq f'(x)$. Hence, if we can show that for any λ with $0 < \lambda < 1$ and all x large enough, we have that $f'(x) \geq f(\lambda x)$, then we will have, for x large enough, that $d(x)/x > \lambda$ and hence that $\lim d(x)/x = 1$. Now $f'(x) \geq f(\lambda x)$ for large x is equivalent to

$$e^{\rho(\log x)} \frac{\rho'(\log x)}{x} \geq e^{\rho(\log x\lambda)} = e^{\rho(\log x + \log\lambda)}$$

for large x which is equivalent to $e^{\rho(x)}\rho'(x)e^{-x} \geq e^{\rho(x-\epsilon)}$ for large x, where $\epsilon = -\log\lambda$. This last inequality is equivalent to $\rho(x) + \log\rho'(x) - x \geq \rho(x-\epsilon)$ or $\rho(x) - \rho(x-\epsilon) \geq x - \log(\rho'(x))$. Since $\rho'(x) \to \infty$ it is sufficient to show that for x large $\rho(x) - \rho(x-\epsilon) \geq x$. Now since $\rho'(x)\uparrow$ it is sufficient to show that $\epsilon\rho'(x-\epsilon) \geq x$, or equivalently that $\rho'(x -$

$\epsilon)/(x - \epsilon) \geq 1/\epsilon(x/(x - \epsilon))$ for x large enough. This is certainly true since $\rho'(x)/x \uparrow \infty$.

Applications of this condition are, for example $\rho(x) = x^{2+\epsilon}$ for any $\epsilon > 0$ (i.e. $f(x) = e^{(\log x)^{2+\epsilon}} = x^{(\log x)^{1+\epsilon}}$) and $\rho(x) = x^2 \log x$ or $\rho(x) = x^2 \log \log x$.

(iv) An unfortunate aspect of condition (3.3) is that it is not monotone. There is a sequence $\{n_k\}$ which satisfies (3.3) but has a subsequence $\{m_k\}$ which does not satisfy (3.3). We remark however that the property of being a spectrum (see definition above) is also not monotone—i.e. there are infinite sets $X \subset Y \subset \mathbf{N}$ such that X is a spectrum but Y is not.

Next we produce the "large set" that we mentioned above. It will be the set $S_n(j)$ constructed below. $S_n(j)$ will be a set of values $k < n_j$ such that each $k \in S_n(j)$ has a unique representation as a sum $n_{i_1} + n_{i_2} + \cdots + n_{i_n}$ with $0 \leq i_1 \leq i_2 \leq \cdots \leq i_n < j$ and such that no integer k' satisfying $0 < |k - k'| < D$ (D to be specified later) has such a representation) and no integer k'' with $|k - k''| < D$ has a representation as a sum $n_{l_1} + \cdots + n_{l_N}$ with $N < n$. In other words k has a unique representation as a sum of n of the n_i with $i < j$ and no nearby integer can be represented as a sum of n or fewer of the n_i. $S_n(j)$ will be "large" in the sense that for j large enough its cardinality $|S_n(j)|$ will be $\geq (1/n!)(1 - \epsilon)j^n$ (for any $\epsilon > 0$). The construction is done in several stages, and will be much clearer if the reader first works through the simplest case $n_k = 2^k$, $d(k) = k - 1$.

Let $A \in \mathbf{N}$ (to be specified later) and define $\delta(k) = k - d(k)$. Notice that $0 \leq \delta(k) \leq k$. Let $n \in \mathbf{N}$ and let $j \in \mathbf{N}$ (thought of as large). Define

$$S_n'(j) = \{n_{i_1} + n_{i_2} + \cdots + n_{i_n} : i_{l+1} - i_l \geq \delta(i_{l+1}) + 1 \quad \text{for} \quad l = 1, \ldots, n - 1 \quad \text{and} \quad i_1 \geq A \quad \text{and} \quad j - i_n \geq \delta(j) + 1\}.$$

LEMMA 3.1. *Assume that the sequence $\{n_k\}$ satisfies condition* (3.3) *above. Let $D \in \mathbf{N}$ be fixed. If A is chosen large enough then we have that if*

$$k_1 = n_{i_1} + \cdots + n_{i_n} \quad \textit{with} \quad i_1 \geq A \quad \textit{and}$$
$$i_{l+1} - i_l \geq \delta(i_{l+1}) + 1 \quad \textit{for} \quad l = 1, \ldots, n - 1$$
$$k_2 = n_{j_1} + \cdots + n_{j_n} \quad \textit{with} \quad j_1 \geq A \quad \textit{and}$$

$$j_{l+1} - j_l \geq \delta(j_{l+1}) + 1 \quad \textit{for} \quad l = 1, \ldots, n-1$$

then either $|k_1 - k_2| > D$ *or* $i_l = j_l$ *for* $l = 1, \ldots, n$. (*In other words each element of* $S_n'(j)$ *has a unique representation in* $S_n'(j)$ *and any two distinct elements of* $S_n'(j)$ *differ by at least* $D + 1$).

Proof. Choose A large enough so that for $k > A$ we have $\Delta_k > D$ and $d(k) > 0$. Let k_1 and k_2 be given. If there is an l such that $i_l \neq j_l$, let p be the largest such and suppose that $i_p < j_p$. Then if $p = 1$ we certainly have $n_{i_1} \leq n_{i_1+1} - D \leq n_{j_1} - D$ and if $p > 1$ we have

$$\begin{aligned}
& n_{i_1} + \cdots + n_{i_p} \\
& \leq n_{i_2-\delta(i_2)-1} + n_{i_2} + \cdots + n_{i_p} && (\text{Since } i_1 \leq i_2 - \delta(i_2) - 1) \\
& = n_{d(i_2)-1} + n_{i_2} + \cdots + n_{i_p} && (\text{since } \delta(i_2) = i_2 - d(i_2)) \\
& < n_{d(i_2)} + n_{i_2} + \cdots + n_{i_p} - D && (\text{since } \Delta_k > D \quad \text{for} \quad k > A) \\
& \leq \Delta_{i_2} + n_{i_2} + \cdots + n_{i_p} - D && (\text{since } n_{d(i_2)} \leq \Delta_{i_2}) \\
& = n_{i_2+1} + n_{i_3} + \cdots + n_{i_p} - D \\
& \leq n_{d(i_3)} + n_{i_3} + \cdots + n_{i_p} - D && (\text{since } i_2 + 1 \leq d(i_3)) \\
& \leq n_{i_3+1} + n_{i_4} + \cdots + n_{i_p} && (\text{by the above argument}) \\
& \leq n_{i_p+1} - D && (\text{iterating the above}) \\
& \leq n_{j_p} - D.
\end{aligned}$$

Hence $k_2 - k_1 > D$.

Remark. Notice that the above proof also shows that if $k \in S_n'(j)$ then $k < n_j - D$.

We denote the cardinality of a set X by $|X|$.

LEMMA 3.2. *Assume that* $\{n_i\}$ *satisfies condition* (3.3). *Then for any* A *and any* $\epsilon > 0$, *we have that for all large enough* j *that*

$$|S_n'(j)| \geq \frac{1}{n!}(1-\epsilon)j^n.$$

Proof. Let $\bar{\delta}(j) = \sup\{\delta(i) : A \leq i \leq j\}$. Then $\bar{\delta}(j) = \delta(i_j)$ for some $A \leq i_j \leq j$, and $i_{j+1} \geq i_j$. If the sequence $\{i_j\}$ is bounded then certainly $\bar{\delta}(j)/j \to 0$ as $j \to \infty$. On the other hand if $i_j \to \infty$ then $\bar{\delta}(j)/j \leq \delta(i_j)/i_j \to 0$ as $j \to \infty$. Hence $\bar{\delta}(j)/j \to 0$. Let $\bar{S}_n'(j) = \{n_{i_1} + \cdots + n_{i_n} : i_{l+1} - i_l \geq \bar{\delta}(j) + 1$ for $l = 1, \ldots, n-1$ and $i_1 \geq A$ and $j - i_n \geq \bar{\delta}(j) + 1\}$. Then certainly $S_n'(j) \subset \bar{S}_n'(j)$. To count the elements of $\bar{S}_n'(j)$ think of choosing an element $n_{i_1} + \cdots + n_{i_n}$ i.e. choosing the $i_1, \ldots, i_n$, one at a time, not necessarily in order. For the first one we can choose any element l_1 with $l_1 \geq A$ and $l_1 \leq j - \bar{\delta}(j) - 1$. So there are $j(1 - A/j - (\bar{\delta}(j)+1)/j)$ possible choices. For the second choice we can choose any l_2 with $l_2 \geq A$, $l_2 \leq j - \bar{\delta}(j) - 1$ and $|l_1 - l_2| \geq \bar{\delta}(j) + 1$. Hence there are at least $j(1 - A/j - 3(\bar{\delta}(j)+1)/j)$ possible choices for the second element. Continuing in this way we see that there are

$$j^n\left(1 - \frac{A}{j} - \frac{\bar{\delta}(j)+1}{j}\right)\left(1 - \frac{A}{j} - 3\,\frac{\bar{\delta}(j)+1}{j}\right) \cdots \left(1 - \frac{A}{j} - (2n-1)\frac{\bar{\delta}(j)+1}{j}\right)$$

ways to choose $(l_1, \ldots, l_n)$ such that $|l_i - l_j| \geq \bar{\delta}(j) + 1$ for $i \neq j$ and $l_i \geq A$, $j - l_i \geq \bar{\delta}(j) + 1$. Hence there are

$$\frac{1}{n!}j^n\left(1 - \frac{A}{j} - \frac{\bar{\delta}(j)+1}{j}\right) \cdots \left(1 - \frac{A}{j} - (2n-1)\frac{\bar{\delta}(j)+1}{j}\right)$$

ways to choose $(l_1, \ldots, l_n)$ satisfying these conditions and in addition with $l_1 < l_2 < \cdots < l_n$. For A large enough we know from Lemma 3.1 that different such sequences correspond to different elements of $\bar{S}_n'(j)$. Since $A/j \to 0$ and $\bar{\delta}(j)/j \to 0$ as $j \to \infty$ the lemma now follows immediately.

Let

$$\bar{S}_n(j) = \{n_{i_1} + \cdots + n_{i_N} + \alpha : N < n,\ 1 \leq i_1 \leq \cdots \leq i_N < j \quad \text{and} \quad |\alpha| \leq D,\ \alpha \in \mathbf{Z}\}.$$

For each N there are at most j^N choices of $i_1, \ldots, i_N$ all $\leq j$. For each of these there are at most $2D + 1$ choices of α. Since N must be less than n there is a constant C_1 such that

$$|\bar{S}_n(j)| < C_1 j^{n-1}.$$

Let

$$\bar{\bar{S}}_n(j) = \{n_{i_1} + \cdots + n_{i_n} + \alpha : 0 \leq i_1 \leq \cdots \leq i_n \leq j$$

and for some l with

$$1 \leq l \leq n - 1,\ i_{l+1} - i_l \leq \delta(i_{l+1}) \quad \text{and} \quad \alpha \in Z,\ |\alpha| \leq D\}.$$

A counting argument like that in Lemma 3.2 shows that there is a constant C_2 such that

$$|\bar{\bar{S}}_n(j)| < C_2 j^{n-1}.$$

Finally let

$$S_n(j) = S_n'(j) - \{\bar{S}_n(j) \cup \bar{\bar{S}}_n(j)\}.$$

Putting together the above estimates we immediately have

LEMMA 3.3. *For any A and D and $\epsilon > 0$ if j is large enough we have that*

$$|S_n(j)| \geq \frac{1}{n!}(1 - \epsilon) j^n.$$

$S_n(j)$ is the "large set" which we mentioned above.

4. The Gap Theorem. In this section we prove

THEOREM 4.1. *Let $f(z) = \Sigma_{k=0}^{\infty} f_{n_k} z^{n_k}$ with the $f_{n_k} \in \mathbf{C}$, $f_{n_k} \neq 0$ and suppose that the sequence $\{n_k\}$ satisfies condition (3.3). Then f is not differentially algebraic.*

Proof. Suppose that f is differentially algebraic. We define $f_i = 0$ if $n_k < i < n_{k+1}$. We may suppose that f satisfies no linear ADE since if it did the f_i would satisfy a linear recursion formula of the form

$$p(k) f_k = q_1(k) f_{k-1} + q_2(k) f_{k-2} + \cdots + q_N(k) f_{k-N}$$

and it would follow immediately that for k larger than all the zeros of p in $\mathbf{N}$ that if $f_{k-1} = f_{k-2} = \cdots = f_{k-N} = 0$ then $f_k = 0$, and hence that for k large enough $\Delta_k < N$. (For details cf [STA].) Hence the f_k satisfy a recursion formula of the form (2.1) with $n \geq 2$.

Recall that the second sum on the right hand side of 2.1 is over tuples $[\lambda] = [\lambda_1, \ldots, \lambda_N]$ with $\Sigma_{i=1}^N \lambda_i = h = k - m + s$. Hence for all the terms $f_{\kappa_1+\lambda_1} \cdots f_{\kappa_N+\lambda_N}$ occurring in this sum we have $0 \leq \Sigma_{i=1}^N \kappa_i + \lambda_i - h \leq mn$. Recall that f is assumed to satisfy an ADE $F(\omega) = \Sigma_{(\kappa)} a_{(\kappa)} \omega^{(\kappa_1)} \cdots \omega^{(\kappa_N)} = 0$ of order m and total degree $n \geq 2$.
Define

$$0(\kappa) = \Sigma \kappa_i = \text{the total order of } \omega^{(\kappa_1)} \cdots \omega^{(\kappa_N)}$$

$$d(\kappa) = N = \text{the total degree of } \omega^{(\kappa_1)} \cdots \omega^{(\kappa_N)}.$$

$$b = \max\{0(\kappa) \colon d(\kappa) = n \quad \text{and} \quad a_{(\kappa)} \neq 0\}.$$

The recursion formula now becomes

(4.1)

$$\alpha(k) f_k = -B(k) \sum_{\substack{(\kappa) \\ a(\kappa)\neq 0 \\ d(\kappa)=n \\ 0(\kappa)=b}} \sum_{[\lambda]}{}^{*} a_{(\kappa)} \frac{(\kappa_1+\lambda_1)!}{\lambda_1!} \cdots \frac{(\kappa_n+\lambda_n)!}{\lambda_N!} f_{\kappa_1+\lambda_1} \cdots f_{\kappa_n+\lambda_n}$$

$$-B(k) \sum_{\substack{\text{the rest of} \\ \text{the } (\kappa) \text{ with} \\ a_{(\kappa)}\neq 0}} \sum_{[\lambda]}{}^{*} a_{(\kappa)} \frac{(\kappa_1+\lambda_1)!}{\lambda_1!} \cdots \frac{(\kappa_N+\lambda_N)!}{\lambda_N!} f_{\kappa_1+\lambda_1} \cdots f_{\kappa_N+\lambda_N}.$$

Notice that in every term in the second double sum we have either $N < n$ and $h \leq \Sigma\lambda_i + \kappa_i \leq h + mn$ or $N = n$ and $h \leq \Sigma\lambda_i + \kappa_i < h + b$. Now let $D = 2(mn + 1)$ and choose $A > m$ so that the conclusion of Lemma

3.1 is satisfied. Let $h + b \in S_n(j)$. Then $h + b$ has a unique representation as $h + b = \mu_1 + \mu_2 + \cdots + \mu_n$ with $\mu_1 < \mu_2 < \cdots < \mu_n$ and each $\mu_i \in \{n_1, \ldots, n_j\}$. Further $h + b$ is not a sum of less than n of the n_i and also no h' with $0 < |h + b - h'| < mn$ is a sum of n or fewer of the n_i's. In addition notice that $f_{\kappa_1+\lambda_1} \cdots f_{\kappa_N+\lambda_N} \neq 0$ if and only if each $\kappa_i + \lambda_i = n_q$ for some q. Hence if $h + b \in S_n(j)$ then every term $f_{\kappa_1+\lambda_1} \cdots f_{\kappa_N+\lambda_N}$ in the second double sum in 4.1 is zero and (4.1) becomes (by the uniqueness of the representation $h + b = \mu_1 + \cdots + \mu_n$, above)

(4.2)

$$\alpha(k)f_k = -B(k) \sum_{\substack{(\kappa) \\ a(\kappa)\neq 0 \\ d(\kappa)=n \\ 0(\kappa)=b}} \sum_{\substack{[\lambda] \\ \Sigma\lambda_i = h \\ [\lambda+\kappa] \text{ a} \\ \text{permutation} \\ \text{of } [\mu]}} a_{(\kappa)} \frac{(\kappa_1+\lambda_1)!}{\lambda_1!} \cdots \frac{(\kappa_n+\lambda_n)!}{\lambda_n!} f_{\mu_1} \cdots f_{\mu_n}.$$

where $[\mu] = [\mu_1, \ldots, \mu_n]$ and $[\lambda + \kappa] = [\lambda_1 + \kappa_1, \ldots, \lambda_n + \kappa_n]$. Now by our choice of A each $\mu_i > m \geq \kappa_i$ for all i. Hence (4.2) is equivalent to

(4.3)

$$\alpha(k) f_k = -B(k) \sum \sum_{\sigma \in \mathbb{S}_n} a_{(\kappa)} \frac{\mu_1!}{(\mu_1 - \kappa_{\sigma(1)})!} \cdots \frac{\mu_n!}{(\mu_n - \kappa_{\sigma(n)})!} f_{\mu_1} \cdots f_{\mu_n}$$

where the first summation is over the same set of (κ) as in (4.2) and $\mathbb{S}_n$ is the full symmetric group on $\{1, 2, \ldots, n\}$.

Now regarding the $\mu_1, \ldots, \mu_n$ as variables let

$$P_{(\kappa)\sigma}(\mu_1, \ldots, \mu_n) = \frac{\mu_1!}{(\mu_1 - \kappa_{\sigma(1)})!} \cdots \frac{\mu_n!}{(\mu_n - \kappa_{\sigma(n)})!}$$

and let

$$(4.4) \qquad P(\mu_1, \ldots, \mu_n) = \sum \sum_{\sigma \in \mathbb{S}_n} P_{(\kappa)\sigma}(\mu_1, \ldots, \mu_n)$$

where the first summation is over the same set of (κ) as in (4.2). Notice that each $P_{(\kappa)\sigma}$ has total degree b. If $(\kappa) \neq (\kappa')$ then the highest degree monomials (ie. the monomials of degree b) occurring in $P_{(\kappa)\sigma}$ and $P_{(\kappa')\sigma'}$ are different, whether $\sigma = \sigma'$ or not, since from the highest degree monomials in

$P_{(\kappa)\sigma}$ we can read off (κ). On the other hand if $\sigma \neq \sigma'$ then either $P_{(\kappa)\sigma} = P_{(\kappa)\sigma'}$ (which is the case if $\kappa_{\sigma(i)} = \kappa_{\sigma'(i)}$ for all i) or the monomials of degree b in $P_{(\kappa)\sigma}$ and $P_{(\kappa)\sigma'}$ are different. (They are $\mu_1^{\kappa_{\sigma(1)}} \cdots \mu_n^{\kappa_{\sigma(n)}}$ and $\mu_1^{\kappa_{\sigma'(1)}} \cdots \mu_n^{\kappa_{\sigma'(n)}}$ respectively). Hence in the sum on the right hand side of (4.4) there is no cancellation among monomials of degree b. Hence the polynomial $P(\mu_1, \ldots, \mu_n)$ is not identically 0.

Since $n \geq 2$ and we are assuming that $h + b \in S_n(j)$, we have that $k = h + m - s \notin S_n(j)$ (since $|h + b - k| = |b - m + s| < D$). Hence $f_k = 0$ (i.e. $k \neq n_i$ for any i). Hence for $h + b \in S_n(j)$ (4.4) becomes

$$0 = -B(k)P(\mu_1, \ldots, \mu_n) f_{\mu_1} \cdots f_{\mu_n}. \tag{4.5}$$

Since $B(k) \neq 0$ and $f_{\mu_1} \cdots f_{\mu_n} \neq 0$, we must have

$$P(\mu_1, \ldots, \mu_n) = 0 \tag{4.6}$$

where $[\mu_1, \ldots, \mu_n]$ is the unique n tuple from $\{n_1, \ldots, n_j\}$ such that $h + b = \mu_1 + \cdots + \mu_n$.

For j large enough we know that there are $>(1/n!)(1 - \epsilon)j^n$ elements in $S_n(j)$. Hence we have that $P(t_1, \ldots, t_n)$ has at least $(1/n!)(1 - \epsilon)j^n$ zeros on the set $\{n_1, \ldots, n_j\}^n$. To obtain the required contradiction all we need is the following.

LEMMA 4.2. *Let $P(t_1, \ldots, t_n)$ be a nonzero polynomial of total degree b. Then there is a constant $C(n, b)$ such that P has at most $C(n, b)j^{n-1}$ distinct zeros on the set $\{n_1, \ldots, n_j\}^n$.*

Proof. The proof is by induction on n. Let $M(n, j, b)$ be the maximum number of zeros that a polynomial in n variables of degree b can have on a set of size j. (i.e. $\{n_1, \ldots, n_j\}$). Obviously $M(1, j, b) \leq b$. Suppose that $M(n - 1, j, b) \leq C(n - 1, b)j^{n-2}$. Let $P(t_1, \ldots, t_n) = \Sigma_{i=0}^{b} Q_i(t_1, \ldots, t_{n-1})t_n^i$. If $t_1, \ldots, t_{n-1}$ are chosen so that not all the Q_i vanish, then there are at most b choices for t_n which make P vanish. On the other hand there are at most $C(n - 1, b)j^{n-2}$ choices of $t_1, \ldots, t_{n-1}$ from $\{n_1, \ldots, n_j\}$ which make all the Q_i vanish. Hence

$$\begin{aligned} M(n, j, b) &\leq j^{n-1}b + C(n - 1, b)j^{n-2} \cdot j \\ &= C(n, b)j^{n-1} \end{aligned}$$

where $C(n, b) = C(n - 1, b) + b$.

This concludes the proof of Lemma 4.2 and also Theorem 4.1.

Remark 4.3. Many of the generating functions of number theory are known to be differentially algebraic, for example the theta function $\theta(x)$, mentioned above, the discriminant function $\Delta(x)$, (cf. [RAN] and [RES]). It is well known that sums, products and compositions of differentially algebraic functions are differentially algebraic. Hence it follows from Theorem 4.1 that for any function $f(x)$, built up from these functions by addition, multiplication and composition, if f has Hadamard gaps then f is a polynomial, i.e. any relation which holds, with possible rare exceptions, among these functions holds with only finitely many exceptions. Thus for example one might consider $f(x) = P(\theta(x), \Delta(x)) = \Sigma f_n x^n$, where $P(y, z)$ is a polynomial in y and z with coefficients rational functions of x. Eg. $f(x) = (1/(1-x)^\alpha)\theta(x)^\beta - r(x)$, where $r(x)$ is a rational function. The statement that $\Sigma f_n x^n$ cannot have certain gaps can then be interpreted as a statement about the number of ways an integer n can be represented as a sum of β squares (cf. [WAL]). If we replace $\theta(x)$ in this expression by $\Delta(x)$ we get a corresponding statement about partitions. For gap theorems about automorphic forms see [KNL] and [MET].

5. A stronger theorem for entire functions. If we suppose that, in addition to being differentially algebraic, the power series converges in the whole complex plane, then a weaker gap condition will work, at least when the coefficients are rational or algebraic. Let $f(z) = \Sigma f_{n_k} z^{n_k}$ be an entire differentially algebraic power series and suppose that $0 \neq f_{n_k} \in \mathbf{Q}$. The extension to the case that the f_{n_k} are algebraic is routine and will be left to the reader (cf [MAH pp. 205-6]). As above we define $f_i = 0$ for $n_k < i < n_{k+1}$ and we call $D_l \in \mathbf{N}$ a denominator for f_l if $D_l f_l \in \mathbf{Z}$. Following exactly the same argument as in [MAH pp. 202-207], and using the fact that if $f_l = 0$ one can take $D_l = 1$ one can show that one can take

$$D_l = \prod_{\substack{L=1 \\ f_L \neq 0}}^{l} |\alpha(L)|^{[((n-1)l+1)/((n-1)L+1)]}$$

where $\alpha(L)$ is a fixed polynomial and $[r]$ is the integer part of r. Hence

$$D_l \leq \prod_{\substack{L=1 \\ f_L \neq 0}}^{l} (c' L^e)^{[((n-1)l+1)/((n-1)L+1)]}$$

for some $e \in \mathbf{N}$ and some constant c'. From this we have

$$\log D_l \leq \sum_{\substack{L=1 \\ f_L \neq 0}}^{l} \left[\frac{(n-1)l+1}{(n-1)L+1}\right] \{\log c' + e \log L\}$$

$$\leq c'' \sum_{\substack{L=1 \\ f_L \neq 0}}^{l} \frac{l}{L} \{\log c' + e \log L\}$$

for some constant c''

$$\leq cl \sum_{\substack{L=1 \\ f_L \neq 0}}^{l} \frac{1}{L} + dl \sum_{\substack{L=1 \\ f_L \neq 0}}^{l} \frac{\log L}{L}$$

for some constants c and d. Hence

$$\log D_l \leq Al \sum_{\substack{L=1 \\ f_L \neq 0}}^{l} \frac{\log L}{L} \quad \text{for some constant } A.$$

For f to be entire we must have, for every $r > 1$, that if $f_l \neq 0$ then $\log D_l \geq \log r^l = l \log r$, for l large enough. (Because $f_l \in \mathbf{Q}$ and hence $|f_l| \geq 1/D_l$ if $f_l \neq 0$). Hence we see immediately that if

$$\sum_{\substack{L=1 \\ f_L \neq 0}}^{\infty} \frac{\log L}{L}$$

is convergent then f is not entire. Hence we have shown

PROPOSITION 5.1. *If $f(z) = \Sigma_{k=0}^{\infty} f_k z^k$, $f_k \in \mathbf{Q}$ is a differentially algebraic entire function then either f is a polynomial or*

$$\sum_{\substack{L=1 \\ f_L \neq 0}}^{\infty} \frac{\log L}{L} = \infty.$$

Example 5.2. It follows from Proposition 5.1 that if $n_k \sim k^{1+\epsilon}$ (i.e. $n_k/k^{1+\epsilon} \to 1$) and $f_{n_k} \in \mathbf{Q}$, $f_{n_k} \neq 0$ and $f(z) = \Sigma f_{n_k} z^{n_k}$ is entire, then f is not differentially algebraic. This is immediate since

$$\int_a^\infty \frac{\log t^{1+\epsilon}}{t^{1+\epsilon}}\, dt < \infty.$$

This shows for example that $\Sigma_{n=0}^\infty z^{n^2}/(n^2)!$ is not differentially algebraic. We shall use this fact in the following section.

Remark 5.3. If $f(z) = \Sigma f_k z^k$ is differentially algebraic then there are constants γ_1, γ_2 such that $|f_k| \leq \gamma_1 (k!)^{\gamma_2}$ (See for example [MAH p. 200].) If in addition one knows that most of the f_k are zero one can give a smaller upper bound, by going through the argument in [MAH]. Instead of doing this we shall merely do an example, which we will need in Section 6. As in [MAH] it follows easily from the recursion formula (2.1) that

$$|f_k| \leq c_1 k^{c_2} \max_{\nu_1 + \cdots + \nu_N \leq k + c_3} |f_{\gamma_1} \cdots f_{\gamma_N}|$$

for some constants c_1, c_2, c_3. From this it follows that $\Sigma (n^2)! z^{n^2}$ is not differentially algebraic. For suppose it were. Then we would have

$$(k+1)^2! \leq c_1 k^{c_2} \max_{\nu_1 + \cdots + \nu_N \leq k + c_3} \nu_1^2! \cdots \nu_N^2!$$

$$\leq c_1{}^{c_2} k^2! (2k + c_4)!$$

for some constants c_1, c_2, c_4. But this inequality is certainly false for k large enough.

6. Spectra and Multipliers of differentially algebraic power series. Let $f(z) = \Sigma_{k=0}^\infty f_k z^k \in \mathbf{C}[[z]]$. The spectrum of f, $S_f = \{k \in \mathbf{N}: f_k \neq 0\}$. We call a sequence $\{a_k\}_{k \in \mathbf{N}}$ a multiplier if whenever $f(z) = \Sigma f_k z^k$ is differentially algebraic so is $\Sigma a_k f_k z^k$. In this section we shall prove some results on the spectra and multipliers of differentially algebraic power series.

The spectra of rational functions are well understood. If $f(z) = \Sigma f_k z^k$ is rational then we know, by the Skolem-Mahler-Lech Theorem (see [LEC]) that S_f is the union of a finite set and a finite number of arithmetic progressions. (Conversely every such set is the spectrum of some rational power series.) This result has been extended to a class of differentially algebraic power series which satisfy linear ADE's by Laohakosol [LAO]. That the spectra of arbitrary differentially algebraic power series are considerably more complicated can be seen from the Jacobi theta function $\theta(z) = 1 + 2\,\Sigma_{k=1}^{\infty} z^{k^2} (S_\theta = \{k^2 \colon k \in \mathbf{N}\})$.

A special case of the following proposition (i.e. for the spectra of solutions of *linear* algebraic differential equations) had been communicated to us by David Richman. The general case can be found in a paper by V. Laohakosol et al [LRR], and we are grateful to the authors for their permission to reproduce the proof here.

PROPOSITION 6.1. *There are only countably many spectra.*

Proof. Suppose that $f(z) = \Sigma f_k z^k$ is differentially algebraic. Then f satisfies a differential equation with coefficients in $\mathbf{Q}[a_1, \ldots, a_k, z]$ for some $a_1, \ldots, a_k \in \mathbf{C}$. Hence the transcendence degree of $\mathbf{Q}(a_1, \ldots, a_k, z, f, f', \ldots)$ over $\mathbf{Q}(a_1, \ldots, a_k, z)$ is finite and hence the transcendence degree of $\mathbf{Q}(f, f', \ldots)$ over $\mathbf{Q}$ is finite and we see that f satisfies an ADE over $\mathbf{Q}$ (i.e. of the form $P(\omega, \omega', \ldots, \omega^{(m)}) = 0$ where P is a polynomial with coefficients from $\mathbf{Q}$. If we choose $P = 0$ to be such an equation of lowest order m and lowest possible degree in $\omega^{(m)}$ then we see that the f_k satisfy a recursion relation of the form (2.1) with the $a_{(\kappa)} \in \mathbf{Q}$. If we choose $\gamma \geq$ all the integer zeros of $\alpha(k)$ (in (2.1)) then the recursion formula (2.1) determines all the f_i for $i > \gamma$ as rational functions of $f_0, \ldots, f_\gamma$. Hence we see that S_f is determined by P and the isomorphism type of $\mathbf{Q}(f_0, \ldots, f_\gamma)$ over $\mathbf{Q}$. The proposition now follows from the fact that there are only countably many such P's and only countably many such isomorphism types.

Remark. The result in [OST] that given any power series $\Sigma f_n z^n$ with infinitely many of the $f_n \neq 0$, there is a choice of $\pm$ so that $\Sigma \pm f_n z^n$ is not differentially algebraic, follows easily from Proposition 6.1 and the fact that the sum of two differentially algebraic power series is differentially algebraic.

PROPOSITION 6.2. (i) *If S_1 and S_2 are spectra so is $S_1 \cup S_2$.*

(ii) *There exist spectra S_1 and S_2 such that $S_1 \cap S_2$ is not a spectrum.*

(iii) *There is a spectrum S_3 such that $\mathbf{N} - S_3$ is not a spectrum.*

Proof. (i) Let S_i be the spectrum of f_i, $i = 1, 2$. Choose $\alpha \in \mathbf{C}$ transcendental over the subfield of $\mathbf{C}$ generated by all the coefficients of f_1 and f_2. Then the spectrum of $f_1 + \alpha f_2$ is $S_1 \cup S_2$.

(ii) Let $f_1(z) = (1/2)(\theta(z) + 1) = \Sigma_{n=0} z^{n^2}$ and $f_2(z) = zf_1(z^2) = \Sigma_{m=0}^{\infty} z^{2m^2+1}$. Then f_1 and f_2 are differentially algebraic since $\theta(z)$ is and

$$S_1 = S_{f_1} = \{n^2 \colon n \in \mathbf{N}\}$$

$$S_2 = S_{f_2} = \{2m^2 + 1 \colon m \in \mathbf{N}\}$$

$$S_1 \cap S_2 = \{n^2 \colon n \in \mathbf{N} \text{ and } \exists\, m \in \mathbf{N} \text{ such that } n^2 - 2m^2 = 1\}$$

$$= \left\{\frac{1}{4}(\epsilon_0^n - \epsilon_0^{-n})^2 \colon \epsilon_0 = 3 + 2\sqrt{2} \text{ and } n \in \mathbf{N}\right\}.$$

For this last step see any treatment of the Pell equation—for example [HAW]. Since the sequence $n_k = 1/4(\epsilon_0^k - \epsilon_0^{-k})^2 \sim 1/4\epsilon_0^{2k}$ has Hadamard gaps it follows from Theorem 4.1 that $S_1 \cap S_2$ is not a spectrum.

(iii) It is well known that

$$\theta^4(z) = 8 \sum_n \sum_{\substack{d \mid n \\ 4 \nmid d}} dz^n$$

cf. [HAW p. 314]. Notice that if $n = 2^k$ then

$$\sum_{\substack{d \mid n \\ 4 \nmid d}} d = 3$$

and if n has an odd factor then

$$\sum_{\substack{d \mid n \\ 4 \nmid d}} d > 3.$$

Hence if

$$f(z) = \theta^4(z) - \frac{24}{1 - z} = \Sigma f_n z^n \quad \text{then} \quad f_n = 0$$

if and only if n is a power of 2. Hence $\mathbf{N} - S_f = \{2^n : n \in \mathbf{N}\}$ and it follows from Theorem 4.1 that this is not a spectrum.

Remark. Using the above formula for $\theta^4(z)$ and the facts that

$$\theta^4(z) - \theta^4(-z) = 16 \sum_{n \text{ odd}} \sum_{d \mid n} dz^n$$

and that for n odd $\Sigma_{d \mid n} d = n + 1$ if and only if n is prime, it is easy to see that $\mathbf{N} - P$ is a spectrum, where P is the set of primes. We do not know if P is a spectrum.

It is known (see [STA]) that if $f(z) = \Sigma f_n z^n$ and $g(z) = \Sigma g_n z^n$ satisfy linear algebraic differential equations then so does the Hadamard product $\Sigma f_n g_n z^n$. We follow [STA] and call such functions differentially finite. Hence we see that if f and g are differentially finite then $S_f \cap S_g$ is also the spectrum of a differentially finite power series; and that the class of multipliers for differentially finite power series (i.e. the class of sequences $\{a_n\}$ such that if $\Sigma f_n z^n$ is differentially finite then so is $\Sigma a_n f_n z^n$) is just the class of sequences $\{a_n\}$ such that $\Sigma a_n z^n$ is differentially finite. These facts are in contrast to Proposition 6.1 (ii) and also Proposition 6.3.

PROPOSITION 6.3. (i) $\{1/n!\}$ *is not a multiplier.* (*Hence the class of differentially algebraic power series is not closed under Hadamard products by differentially finite power series.*)

(ii) $\{n!\}$ *is not a multiplier.*

(iii) *The sum and product of multipliers are multipliers.* $\{n^k\}_{n \in \mathbf{N}}$ *and* $\{\alpha^n\}_{n \in \mathbf{N}}$ *are multipliers.* $\{1/an+b\}_{n \in \mathbf{N}}$ *with* $a, b \in \mathbf{Q}$ *is a multiplier.*

(iv) (*Carmichael*) $\{1/(2^n - 1)\}$ *is not a multiplier.*

Proof. Σz^{n^2} is differentially algebraic. (i) We showed in Example 5.2 that $\Sigma z^{n^2}/(n^2)!$ is not differentially algebraic.

(ii) We showed that $\Sigma (n^2)! z^{n^2}$ is not differentially algebraic.

(iii) That the sum and product of multipliers are multipliers is immediate. If $f(z) = \Sigma f_n z^n$ then $\Sigma n^k f_n z^n = (z(d/dz))^k f(z)$, $\Sigma \alpha^n f_n z^n = f(\alpha z)$, and if $r \in Z$, $s \in \mathbf{N}$ then $\Sigma (1/(rn + s)) f_n z^{rn+s} = \int z^{s-1} f(z^r) dz$ is differentially algebraic and hence so is $\Sigma (1/(rn + s)) f_n z^n$. If $a, b \in \mathbf{Q}$ we can write $1/(an + b) = t/(rn + s)$ with $r \in \mathbf{N}$, $s, t \in \mathbf{Z}$ and use the fact that constants are certainly multipliers.

(iv) is proved in [CAR].

Let $f(z) = \Sigma f_k z^k$ and define $f^{\|}(z) = \Sigma |f_k| z^k$, $f^{[]}(z) = \Sigma [f_k] z^k$ and $f^{\chi}(z) = \Sigma \chi(f_k) z^k$ where $[a]$ is the greatest integer in a and $\chi(z) = 1$ if $a \neq 0$, $\chi(0) = 0$.

PROPOSITION 6.4. *There exist differentially algebraic power series $f_1(z)$, $f_2(z)$ and $f_3(z)$ such that $f_1^{\|}$, $f_2^{[]}$ and f_3^{χ} are not differentially algebraic.*

Proof.

$$f_1(z) = \frac{1}{8}\theta^4(z) - 4\frac{1}{1-z}$$

$$= \sum_n \left[\left(\sum_{\substack{d\mid n\\ 4\nmid d}} d\right) - 4\right] x^n = \Sigma\, a_n z^n$$

is differentially algebraic since $\theta(z)$ is. Then

$$f_1^{\|}(z) - f_1(z) = 2\Sigma\, z^{2^k}, \qquad \text{since} \quad \sum_{\substack{d\mid n\\ 4\nmid d}} d < 4$$

if and only if n is a power of 2. Since $\Sigma\, z^{2^k}$ is not differentially algebraic neither is $f_1^{\|}$. Let $g(z) = \Sigma_n (1/(n+1)) z^{n+1}$ and $f_2(z) = \Sigma_n (\Sigma_{k=1}^n 1/k) z^n$. Then $zf_2(z) + g(z) = f_2(z) - z$. Since g is differentially algebraic so is f_2. If $f_2^{[]} = \Sigma\, a_n z^n$ were differentially algebraic then so would $H(z) = \Sigma\,(a_{n+1} - a_n) z^n$ be. If $H(z) = \Sigma\, b_n z^n$ then the b_n are either 0 or 1 and it is easy to see that if $b_n = 1$ then $b_k = 0$ for $n < k < (3/2)n$ and hence that $H(z)$ has Hadamard gaps and thus is not differentially algebraic. Finally notice that if S is the spectrum of f then $\mathbf{N} - S$ is the spectrum of $1/(1-z) - f^{\chi}(z)$ so it follows from Proposition 6.2 (iii) that $f_3(z)$ can be differentially algebraic without f_3^{χ} being so.

To conclude this section we show

PROPOSITION 6.5. (i) *There is a differentially algebraic function $f(z)$ such that the Laplace Transform $F(s) = \mathfrak{L}\{f\}$ of f is not differentially algebraic.*

(ii) *There is a differentially algebraic function $G(s)$ such that its inverse Laplace transform $\mathfrak{L}^{-1}\{G(s)\} = g(z)$ is not differentially algebraic.*

(iii) *There are differentially algebraic functions $h(z)$ and $l(z)$ such that their convolution $h * l(z) = \int_{|t|=1} h(t) l(z/t)\,dt$ is not differentially algebraic.*

Proof. (i) $f(z) = (1/z)(1/(1 - e^{-z}) - 1/z - 1/2)$ is certainly differentially algebraic. $F(s) = \mathcal{L}\{f\}(s) = \log(e^s\Gamma(s)/2^{1/2}\pi^{1/2}s^{s-1/2})$ (see [BAT p. 262 example 8]) is not differentially algebraic because if it were $\Gamma(s)$ would also be and it is well known that $\Gamma(s)$ is not (see [HOL]).

(ii) Let $g(z) = 1 + 2\sum_{n=1}^{\infty} z^{n^2}/(n^2)!$. Then we know from Example 5.2 that $g(z)$ is not differentially algebraic. Direct calculation shows that $G(s) = \mathcal{L}\{g\}(s) = (1/s)\theta(1/s)$ which certainly is differentially algebraic.

(iii) Notice that if $F(z) = \sum a_n z^n$, $G(z) = \sum b_n z^n$ then $F*G = 1/2\pi i\int_{|w|=1} F(w)G(z/w)dw/w = \sum a_n b_n z^n$, so we see from Proposition 6.3 (i) that if $h(z) = \theta(1/2z)$ and $l(z) = e^z$ then $h*l(z) = \sum 1/n^2!(z/2)^{n^2}$ is not differentially algebraic.

7. Problems. In this section we list some open problems.

(1) Can one sharpen Theorem 4.1 so as to exclude smaller gaps? On the other hand can one find differentially algebraic power series with gaps significantly larger than those of $\theta(z)$? For example, can $\sum a_n z^{n^3}$, $a_n \neq 0$ be differentially algebraic?

(2) Can Theorem 4.1 be sharpened to show that if $\sum f_{n_k} z^{n_k}$, $f_{n_k} \neq 0$ is differentially algebraic, then lim sup $(n_k + 1)/n_k = 1$? This would for example exclude power series with blocks of nonzero coefficients and large gaps between the blocks.

(3) Can one remove the condition in Theorem 5.1 that the f_k be rational (or algebraic)? Is there such an improved gap condition for convergent (instead of entire) differentially algebraic power series?

(4) What are the multipliers for differentially algebraic power series?

(5) What are the continuous functions ϕ: $\mathbf{C} \to \mathbf{C}$ that operate on the coefficients of differentially algebraic power series, i.e. such that whenever $\sum f_n x^n$ is differentially algebraic so is $\sum \phi(f_n)x^n$? Proposition 6.4 shows that $\phi(x) = |x|$, $[x]$ and $\chi(x)$ do not operate. Proposition 6.3 shows that $\phi(x) = x^2$ does not operate (since $xy = 1/4[(x + y)^2 - (x - y)^2]$). The functions $\phi(x) = Ax + b$ and $\phi(x) = A\bar{x} + B$ (where $\bar{x}$ is the conjugate of x) certainly operate. It seems plausible to conjecture that these are the only continuous functions which operate. Note that if ϕ: $\mathbf{C} \to \mathbf{C}$ is any isomorphism, then ϕ operates.

Next we give some problems on spectra.

(6) What are the spectra of differentially algebraic power series? What are the spectra of algebraic power series? Is it true that every spectrum has rational asymptotic density? Could it be that if S is a spectrum,

then $S = M \Delta D$ where M is a finite union of arithmetic progressions, D has asymptotic density zero and Δ denotes symmetric difference?

(7) What is the class of spectra whose complements are also spectra?

(8) Is there a sequence $\{a_n\}$, with infinitely many $a_n \neq 0$, such that if S is a spectrum, then $\Sigma_{n \in S} a_n x^n$ is differentially algebraic?

(9) Does there exist a sequence $\{n_k\}$ which satisfies the gap condition (3.3) (and hence is not a spectrum) and has a subsequence $\{n_{k_l}\}$ which is a spectrum?

(10) Given a recursion formula which defines a set of positive integers, is there an algorithm for deciding whether or not it is a spectrum?

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