

Two's Company, Three's a Crowd: Consensus-Halving for a Constant Number of Agents

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We consider the ε -CONSENSUS-HALVING problem, in which a set of heterogeneous agents aim at dividing a continuous resource into two (not necessarily contiguous) portions that all of them simultaneously consider to be of approximately the same value (up to ε). This problem was recently shown to be PPA-complete, for n agents and n cuts, even for very simple valuation functions. In a quest to understand the root of the complexity of the problem, we consider the setting where there is only a *constant* number of agents, and we consider both the *computational complexity* and the *query complexity* of the problem.

For agents with *monotone* valuation functions, we show a dichotomy: for two agents the problem is polynomial-time solvable, whereas for three or more agents it becomes PPA-complete. Similarly, we show that for two monotone agents the problem can be solved with polynomially-many queries, whereas for three or more agents, we provide exponential query complexity lower bounds. These results are enabled via an interesting connection to a *monotone Borsuk-Ulam* problem, which may be of independent interest. For agents with general valuations, we show that the problem is PPA-complete and admits exponential query complexity lower bounds, even for two agents.

CCS Concepts: • **Theory of computation** → **Problems, reductions and completeness**; *Complexity classes*; Approximation algorithms analysis.

Additional Key Words and Phrases: fair division, PPA, query complexity, generalised Robertson-Webb, monotone Borsuk-Ulam

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1 INTRODUCTION

The *Consensus-Halving* problem is a classic problem in fair division whose origins date back to the 1940s and the work of Neyman [40]. In the approximate version of the problem studied by Simmons and Su [45], coined the ε -CONSENSUS-HALVING problem, a set of n agents with different and heterogeneous valuation functions over the unit interval $[0, 1]$ aim at finding a partition of the interval into pieces labelled either “+” or “−” using at most n cuts, such that the total value of every agent for the portion labelled “+” and for the portion labelled “−” is approximately the same, up to an additive parameter ε . Very much like other well-known problems in fair division, the existence of a solution to the exact version (for $\varepsilon = 0$) can be proven via fixed-point theorems, whereas the

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approximate version ($\varepsilon > 0$) admits constructive solutions via exponential-time algorithms [45]. In fact, the Consensus-Halving problem is a continuous variant of the well-known Necklace Splitting problem [2, 32] with two thieves, and can also be seen as a generalisation of the Hobby-Rice Theorem [34].

The ε -CONSENSUS-HALVING problem received considerable attention in the literature of computer science over the past few years, as it was proven to be the first “natural” PPA-complete problem [28], i.e., a problem that does not have a polynomial-sized circuit explicitly in its definition, answering a decade-old open question [33, 41]. Additionally, Filos-Ratsikas and Goldberg [29], reduced from this problem to establish the PPA-completeness of Necklace Splitting with two thieves; these PPA-completeness results provided the first definitive evidence of intractability for these two classic problems, establishing for instance that solving them is at least as hard as finding a Nash equilibrium of a strategic game [18, 19]. Filos-Ratsikas et al. [30] improved on the results for the ε -CONSENSUS-HALVING problem, by showing that the problem remains PPA-complete, even if one restricts the attention to very small classes of agents’ valuations, namely piecewise uniform valuations with only two valuation blocks.

This latter result falls under the general umbrella of imposing restrictions on the structure of the problem, to explore if the computational hardness persists or whether we can obtain polynomial-time algorithms. Filos-Ratsikas et al. [30] applied this approach along the axis of the valuation functions, while considering a general number of agents, similarly to [28, 29]. In this paper, we take a different approach, and we restrict the number of agents to be *constant*. This is in line with most of the theoretical work on fair division, which is also concerned with solutions for a small number of agents¹ and it is also quite relevant from a practical standpoint, as fair division among a few participants is quite common. We believe that such investigations are necessary in order to truly understand the computational complexity of the problem. To this end, we state our first main question:

What is the computational complexity of ε -CONSENSUS-HALVING for a constant number of agents?

Since the number of agents is now fixed, any type of computational hardness must originate from the structure of the valuation functions. We remark that the existence results for ε -CONSENSUS-HALVING are fairly general, and in particular do not require assumptions like additivity or monotonicity of the valuation functions. For this reason, the sensible approach is to start from valuations that are as general as possible (for which hardness is easier to establish), and gradually constrain the domain to more specific classes, until eventually polynomial-time solvability becomes possible. Indeed, in a paper that is conceptually similar to ours, Deng et al. [23] studied the computational complexity of the *contiguous envy-free cake-cutting problem*² and proved that the problem is PPAD-complete, even for three agents, when agents have *ordinal preferences* over the possible pieces. These types of preferences induce no structure on the valuation functions and are therefore as general as possible. In contrast, the authors showed that for three agents and *monotone valuations*, the problem becomes polynomial-time solvable, leaving the case of four or more agents as an open problem. We adopt a similar approach in this paper for ε -CONSENSUS-HALVING, and we manage to completely settle the complexity of the problem when the agents have monotone valuations, among other results, which are highlighted in Section 1.1.

Another relevant question that has been surprisingly overlooked in the related literature is the *query complexity* of the problem. In this regime, the algorithm interacts with the agents via *queries*,

¹For example, the existence of bounded protocols for cake-cutting for more than 3 agents was not known until very recently [8, 9], but such protocols for 2 and 3 agents were known since the 1960s (see [13, 44]).

²This version of the classic envy-free cake-cutting problem requires that every agent receives a single, connected piece.

asking them to provide their values for different parts of the interval $[0, 1]$, and the complexity is measured by the number of queries required to find an ε -approximate solution. This brings us to our second main question:

What is the query complexity of ε -CONSENSUS-HALVING for a constant number of agents?

We develop appropriate machinery that allows us to answer both of our main questions at the same time. In a nutshell, for the positive results, we design algorithms that run in polynomial time and can be recast as query-based algorithms that only use a polynomial number of queries. For the negative results, we construct reductions from “hard” computational problems which allow us to simultaneously obtain computational hardness results and query complexity lower bounds.

1.1 Our Results

In this section, we list our main results regarding the computational complexity and the query complexity of the ε -CONSENSUS-HALVING problem.

Computational Complexity: We start from the *computational complexity* of the problem for a constant number of agents. We prove the following main results, parameterised by (a) the number of agents and (b) the structure of the valuation functions.

- For a *single agent* and *general valuations*, the problem is polynomial-time solvable. The same result applies to the case of any number of agents with *identical* general valuations.
- For *two or more agents* and *general valuations*, the problem is PPA-complete.
- For *two agents* and *monotone valuations*, the problem is polynomial-time solvable. This result holds even if one of the two agents has a general valuation.
- For *three or more agents* and *monotone valuations*, the problem is PPA-complete.

Finally, the ε -CONSENSUS-HALVING problem with 2 agents coincides with the well-known ε -Perfect Division problem for cake-cutting (e.g., see [14, 15]), and thus naturally our results imply that ε -Perfect Division with 2 agents with monotone valuations can be done in polynomial time, whereas it becomes PPA-complete for 2 agents with general valuations.

Before we proceed, we offer a brief discussion on the different cases that are covered by our results. The distinction on the number of agents is straightforward. For the valuation functions, we consider mainly *general valuations* and *monotone valuations*. Note that neither of these functions is additive, meaning that the value that an agent has for the union of two disjoint intervals $[a, b]$ and $[c, d]$ is not necessarily the sum of her values for the two intervals. For monotone valuations, the requirement is that for any two subsets I and I' of $[0, 1]$ such that $I \subseteq I'$, the agent values I' at least as much as I , whereas for general valuations there is no such requirement.

We remark that for agents with piecewise constant valuations (i.e., the valuations used in [28] to obtain the PPA-completeness of the problem for many agents), the problem can be solved rather straightforwardly in polynomial time for a constant number of agents, using linear programming (see the full version [6]). In terms of the classification of the complexity of the problem in order of increasing generality of the valuation functions, this observation provides the “lower bound” whereas our PPA-hardness results for monotone valuations provide the “upper bound”, see Figure 1. While the precise point of the phase transition has not yet been identified, our results make considerable progress towards this goal.

Query Complexity: Besides the computational complexity of the problem, we are also interested in its query complexity. In this setting, one can envision an algorithm which interacts with the agents via a set of queries, and aims to compute a solution to ε -CONSENSUS-HALVING using the minimum number of queries possible. In particular, a query is a question from the algorithm to

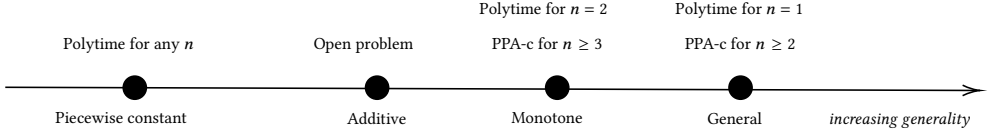


Fig. 1. A classification of ε -CONSENSUS-HALVING for a constant number n of agents, in terms of increasing generality of the valuation functions.

an agent about a subset of $[0, 1]$, who then responds with her value for that set. We provide the following results, where L denotes the Lipschitz parameter of the valuation functions:

- For a *single agent* and *general valuations*, the query complexity of the problem is $\Theta(\log \frac{L}{\varepsilon})$. The same result applies for any number of agents with *identical* general valuations.
- For $n \geq 2$ *agents* and *general valuations*, the query complexity of the problem is $\Theta\left(\left(\frac{L}{\varepsilon}\right)^{n-1}\right)$.
- For *two agents* and *monotone valuations*, the query complexity of the problem is $O(\log^2 \frac{L}{\varepsilon})$. This result holds even if one of the two agents has a general valuation.
- For $n \geq 3$ *agents* and *monotone valuations*, the query complexity of the problem is between $\Omega\left(\left(\frac{L}{\varepsilon}\right)^{n-2}\right)$ and $O\left(\left(\frac{L}{\varepsilon}\right)^{n-1}\right)$.

To put these results into context, we remark that when studying the query complexity of the problem, the input consists of the error parameter ε and the Lipschitz parameter L , given by their *binary representation*. In that sense, a $\Theta(\log^k(L/\varepsilon))$ number of queries, for some constant k , is polynomial, whereas a $\Theta(L/\varepsilon)$ number of queries is exponential. Not surprisingly, our PPA-hardness results give rise to exponential query complexity lower bounds, whereas our algorithms can be transformed into query-based algorithms of polynomial query complexity. We remark however that beyond this connection, our query complexity analysis is in fact *quantitative*, as we provide tight or almost tight bounds on the query complexity as a function of the number of agents n , for both general and monotone valuation functions.

Finally, for the case of monotone valuations, we consider a more expressive query model, which is an appropriate extension of the well-known *Robertson-Webb* query model [44, 48], the predominant query model in the literature of fair cake-cutting [13]; we refer to this extension as the *Generalised Robertson-Webb (GRW)* query model. We show that our bounds extend to this model as well, up to logarithmic factors.

1.2 Related Work

As we mentioned in the introduction, the origins of the Consensus-Halving problem can be traced back to the 1940s and the work of Neyman [40], who studied a generalisation of the problem with k labels instead of two (“+”, “−”), and proved the existence of a solution when the valuation functions are probability measures and there is no constraint on the number of cuts used to obtain the solution. The existence theorem for two labels is known as the *Hobby-Rice Theorem* [34] and has been studied rather extensively in the context of the famous *Necklace Splitting problem* [2, 4, 32]. In fact, most of the proofs of existence for Necklace Splitting (with two thieves) were established via the Consensus-Halving problem, which was at the time referred to as *Continuous Necklace Splitting* [2]. The term “Consensus-Halving” was coined by Simmons and Su [45], who studied the continuous problem in isolation, and provided a constructive proof of existence which holds for very general valuation functions, including all of the valuation functions that we consider in this paper. Interestingly, the proof of Simmons and Su [45] reduces the problem to finding edges

of complementary labels on a triangulated n -sphere, labelled as prescribed by *Tucker’s lemma*, a fundamental result in topology.

While not strictly a reduction in the sense of computational complexity, the ideas of [45] certainly paved the way for subsequent work on the problem in computer science. The first computational results were obtained by Filos-Ratsikas et al. [27], who proved that the associated computational problem, ε -CONSENSUS-HALVING, lies in PPA (adapting the constructive proof of Simmons and Su [45]) and that the problem is PPAD-hard, for n agents with piecewise constant valuation functions. Filos-Ratsikas and Goldberg [28] proved that the problem is in fact PPA-complete, establishing for the first time the existence of a “natural” problem complete for PPA, i.e., a problem that does not contain a polynomial-sized circuit explicitly in its definition, answering a long-standing open question of Papadimitriou [41]. In a follow-up paper, [29] used the PPA-completeness of Consensus-Halving to prove that the Necklace Splitting problem with two thieves is also PPA-complete. Very recently, Filos-Ratsikas et al. [30] strengthened the PPA-hardness result to the case of very simple valuation functions, namely piecewise constant valuations with at most two blocks of value. Deligkas et al. [22] studied the computational complexity of the *exact* version of the problem, and obtained among other results its membership in a newly introduced class BU (for “Borsuk-Ulam” [12]) and its computational hardness for the well-known class FIXP of Etessami and Yannakakis [25]. Very recently, Deligkas et al. [21] showed the PPA-completeness of the related Pizza Sharing problem [35], via a reduction from Consensus-Halving.

Importantly, none of the aforementioned results apply to the case of a constant number of agents, which was prior to this paper completely unexplored. Additionally, none of these works consider the query complexity of the problem. A recent work [3] studies ε -CONSENSUS-HALVING in a hybrid computational model which includes query access to the valuations, but contrary to our paper, their investigations are not targeted towards a constant number of agents, and the agents have additive valuation functions.

A relevant line of work is concerned with the query complexity of *fair cake-cutting* [13, 42], a related but markedly different fair-division problem. Conceptually closest to ours is the paper by Deng et al. [23], who study both the computational complexity and the query complexity of *contiguous envy-free cake-cutting*, for agents with either general³ or monotone valuations. For the latter case, the authors obtain a polynomial-time algorithm for three agents, and leave open the complexity of the problem for four or more agents. In our case, for ε -CONSENSUS-HALVING, we completely settle the computational complexity of the problem for agents with monotone valuations.

In the literature of fair cake-cutting, most of the related research (e.g., see [5, 8, 9, 13, 14]) has focused on the well-known *Robertson-Webb* (RW) query model, in which agents interact with the protocol via two types of queries, *evaluation queries* (EVAL) and *cut queries* (CUT). As the name suggests, this query model is due to Robertson and Webb [43, 44], but the work of Woeginger and Sgall [48] has been rather instrumental in formalising it in the form that it is being used today. Given the conceptual similarity of fair cake-cutting with Consensus-Halving, it certainly makes sense to study the latter problem under this query model as well, and in fact, the queries used by Alon and Graur [3] are essentially RW queries. As we show in Section 6, our bounds are qualitatively robust when migrating to this more expressive model, i.e., they are preserved up to logarithmic factors.

Finally, among the aforementioned works in fair cake-cutting, Brânzei and Nisan [14] study among other problems, the problem of ε -*Perfect Division*, which stipulates a partition of the cake

³More accurately, Deng et al. [23] prove their impossibility results for *ordinal preferences*, where for each possible division of the cake, the agent specifies the piece that she prefers. In particular, if one were to define valuation functions consistent with these preferences, the value of an agent for a piece would have to depend also on the way the rest of the cake is divided among the agents.

into n pieces, such that each of the n agents interprets *all* pieces to be of approximate equal value (up to ε). For the case of $n = 2$, this problem coincides with ε -CONSENSUS-HALVING, and thus one can interpret our results for $n = 2$ as extensions of the results in [14] (which are only for additive valuations) to the case of monotone valuations (for which the problem is solvable with polynomially-many queries) and to the case of general valuations (for which the problem admits exponential query complexity lower bounds).

2 PRELIMINARIES

In the ε -CONSENSUS-HALVING problem, there is a set of n agents with *valuation functions* v_i (or simply *valuations*) over the interval $[0, 1]$, and the goal is to find a partition of the interval into subintervals labelled either “+” or “−”, using at most n cuts. This partition should satisfy that for every agent i , the total value for the union of subintervals \mathcal{I}^+ labelled “+” and the total value for the union of subintervals \mathcal{I}^- labelled “−” is the same up to ε , i.e., $|v_i(\mathcal{I}^+) - v_i(\mathcal{I}^-)| \leq \varepsilon$. In this paper we will assume n to be a constant and therefore the inputs to the problem will only be ε and the valuation functions v_i .

We will be interested in fairly general valuation functions; intuitively, these will be functions mapping *measurable subsets* $A \subseteq [0, 1]$ to non-negative real numbers. Formally, let $\Lambda([0, 1])$ denote the set of Lebesgue-measurable subsets of the interval $[0, 1]$ and $\lambda : \Lambda([0, 1]) \rightarrow [0, 1]$ the Lebesgue measure. We consider valuation functions $v_i : \Lambda([0, 1]) \rightarrow \mathbb{R}_{\geq 0}$, with the interpretation that agent i has value $v_i(A)$ for the subset $A \in \Lambda([0, 1])$ of the resource. Similarly to [3, 10, 14, 15, 23], we also require that the valuation functions be Lipschitz-continuous. Following [23], a valuation function v_i is said to be Lipschitz-continuous with Lipschitz parameter $L \geq 0$, if for all $A, B \in \Lambda([0, 1])$, it holds that $|v_i(A) - v_i(B)| \leq L \cdot \lambda(A \Delta B)$. Here Δ denotes the symmetric difference, i.e., $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Valuation Classes: We will be particularly interested in the following three valuation classes, in terms of decreasing generality:

- *General valuations*, in which there is no further restriction to the functions v_i .
- *Monotone valuations*, in which $v_i(A) \leq v_i(A')$ for any two Lebesgue-measurable subsets A and A' such that $A \subseteq A'$. Intuitively, for this type of function, when comparing two sets such that one is a subset of the other, an agent cannot have a smaller value for the set that contains the other.
- *Additive valuations*, in which v_i is a function from individual intervals in $[0, 1]$ to $\mathbb{R}_{\geq 0}$ and for a set of intervals \mathcal{I} , it holds that $v_i(\mathcal{I}) = \sum_{I \in \mathcal{I}} v_i(I)$. Note that if v_i is an additive valuation function, then it is in fact a measure. We will not prove any results for this type of valuation function in this paper, but we define them for reference and comparison (e.g., see Figure 1).

Normalisation: We will also be interested in valuation functions that satisfy some standard normalisation properties. A valuation function v_i is *normalised*, if the following properties hold:

- (1) $v_i(A) \in [0, 1]$ for all $A \in \Lambda([0, 1])$,
- (2) $v_i(\emptyset) = 0$ and $v_i([0, 1]) = 1$.

In other words, we require the agents’ values to lie in $[0, 1]$ and that their value for the whole interval is normalised to 1. These are the standard assumptions in the literature of the problem for additive valuations [2], as well as in the related problem of fair cake-cutting [42]. We will only consider normalised valuation functions for our lower bounds and hardness results, whereas for the upper bounds and polynomial-time algorithms we will not impose any normalisation; this only makes both sets of results even stronger.

With regard to the valuation classes defined above, we will often be referring to their normalised versions as well, e.g., *normalised general valuations* or *normalised monotone valuations*.

Input models: Given the fairly general nature of the valuation functions, we need to specify the manner in which they will be accessed by an algorithm for ε -CONSENSUS-HALVING. Since we are interested in both the computational complexity and the query complexity of the problem, we will assume the following standard two ways of accessing these functions.

- In the **black-box model**, the valuation functions v_i can be *arbitrary functions*, and are accessed via *queries* (sometimes also referred to as *oracle calls*). A query to the function v_i inputs a Lebesgue-measurable subset A (intuitively a set of subintervals) of $[0, 1]$ and outputs $v_i(A) \in \mathbb{R}_{\geq 0}$. This input model is appropriate for studying the *query complexity* of the problem, where the complexity is measured as the number of queries to the valuation function v_i .

We will also consider the following weaker version of the black-box model, which we will use in our query complexity upper bounds, thus making them stronger: In the **weak black-box model** the input to a valuation function v_i is some set I of intervals, obtained by using at most n cuts, where n is the number of agents.

- In the **white-box model**, the valuation functions v_i are *polynomial-time algorithms*, mapping sets of intervals to non-negative rational numbers. These polynomial-time algorithms are given explicitly as part of the input, including the Lipschitz parameter L .⁴ This input model is appropriate for studying the *computational complexity* of the problem, where the complexity is measured as usual by the running time of the algorithm.

We now provide the formal definitions of the problem in the black-box and the white-box model. Note that the Lipschitz-parameter L is part of the input of the problem and thus will appear in the bounds we obtain. Some of the previous works take L to be bounded by a constant, and as a result it does not appear in their bounds.

Definition 1 (ε -CONSENSUS-HALVING (black-box model)). For any constant $n \geq 1$, the problem ε -CONSENSUS-HALVING with n agents is defined as follows:

- **Input:** $\varepsilon > 0$, the Lipschitz parameter L , *query access* to the functions v_i .
- **Output:** A partition of $[0, 1]$ into two sets of intervals I^+ and I^- such that for each agent i , it holds that $|v_i(I^+) - v_i(I^-)| \leq \varepsilon$, using at most n cuts.

Definition 2 (ε -CONSENSUS-HALVING (white-box model)). For any constant $n \geq 1$, the problem ε -CONSENSUS-HALVING with n agents is defined as follows:

- **Input:** $\varepsilon > 0$, the Lipschitz parameter L , polynomial-time algorithms v_i .
- **Output:** A partition of $[0, 1]$ into two sets of intervals I^+ and I^- such that for each agent i , it holds that $|v_i(I^+) - v_i(I^-)| \leq \varepsilon$, using at most n cuts.

Terminology: When the valuation functions are normalised, we will refer to the problem as ε -CONSENSUS-HALVING with n normalised agents. When the valuation functions are monotone, we will refer to the problem as ε -CONSENSUS-HALVING with n monotone agents. If both conditions are true, we will use the term ε -CONSENSUS-HALVING with n normalised monotone agents.

2.1 BORSUK-ULAM and TUCKER

For our PPA-hardness results and query complexity lower bounds, we will reduce from a well-known problem, the computational version of the Borsuk-Ulam Theorem [12], which states that for

⁴To avoid introducing too many technical details here, we refer the reader to [6] for the fully formal definition.

any continuous function F from S^n to \mathbb{R}^n there is a pair of antipodal points (i.e., $x, -x$) which are mapped to the same point. There are various equivalent versions of the problem (e.g., see [37]); we will provide a definition that is most appropriate for our purposes. In fact, we will include several “optional” properties of the function F in our definition, which will map to properties of the valuation functions v_i when we construct our reductions in subsequent sections. Specifically, we will impose conditions for *normalisation* and *monotonicity*, which will correspond to normalised valuation functions for our lower bounds/hardness results of Section 4, and to normalised monotone valuation functions for our lower bounds/hardness results of Section 5. Let $B^n = [-1, 1]^n$ and let $\partial(B^n)$ denote its boundary. As before, we will require that the functions we consider be Lipschitz-continuous. We say that $F : B^{n+1} \rightarrow B^n$ is Lipschitz-continuous with parameter L , if $\|F(x) - F(y)\|_\infty \leq L \cdot \|x - y\|_\infty$ for all $x, y \in B^{n+1}$, where $\|x\|_\infty = \max_i |x_i|$.

Definition 3 (nD-BORSUK-ULAM). For any constant $n \geq 1$, the problem *nD-BORSUK-ULAM* is defined as follows:

- **Input:** $\varepsilon > 0$, the Lipschitz parameter L , a function $F : B^{n+1} \rightarrow B^n$.
- **Output:** A point $x \in \partial(B^{n+1})$ such that $\|F(x) - F(-x)\|_\infty \leq \varepsilon$.
- **Optional Properties:**

Normalisation:

- $F(1, 1, \dots, 1) = (1, 1, \dots, 1)$.
- $F(-x) = -F(x)$, for all $x \in B^{n+1}$.

Monotonicity:

- If $x \leq y$, then $F(x) \leq F(y)$, for all $x, y \in B^{n+1}$, where “ \leq ” denotes coordinate-wise comparison.

In the *normalised nD-BORSUK-ULAM problem*, where F is normalised, we instead ask for a point $x \in \partial(B^{n+1})$ such that $\|F(x)\|_\infty \leq \varepsilon$. By using the fact that F is an odd function ($F(-x) = -F(x)$, for all $x \in B^{n+1}$), it is easy to see that this is equivalent to $\|F(x) - F(-x)\|_\infty \leq \varepsilon/2$. We will also use the term *normalised monotone nD-BORSUK-ULAM* to refer to the problem when both the normalisation and monotonicity properties are satisfied for F .

In the black-box version of *nD-BORSUK-ULAM*, we can query the value of the function F at any point $x \in B^{n+1}$. In the white-box version of this problem, we are given a polynomial-time algorithm that computes F . Since the number of inputs of F is fixed, we can assume that we are given an arithmetic circuit with $n + 1$ inputs and n outputs that computes F . Following the related literature [20], we will consider circuits that use the arithmetic gates $+$, $-$, \times , \max , \min , $<$ and rational constants.⁵

Another related problem that will be of interest to us is the computational version of Tucker’s Lemma [47]. Tucker’s lemma is a discrete analogue of the Borsuk-Ulam theorem, and its computational counterpart, *nD-TUCKER*, is defined below.

Definition 4 (nD-TUCKER). For any constant $n \geq 1$, the problem *nD-TUCKER* is defined as follows:

- **Input:** grid size $N \geq 2$, a labelling function $\ell : [N]^n \rightarrow \{\pm 1, \pm 2, \dots, \pm n\}$ that is antipodally anti-symmetric (i.e., for any point p on the boundary of $[N]^n$, we have $\ell(\bar{p}) = -\ell(p)$, where $\bar{p}_i = N + 1 - p_i$ for all i).
- **Output:** Two points $p, q \in [N]^n$ with $\ell(p) = -\ell(q)$ and $\|p - q\|_\infty \leq 1$.

⁵Formally speaking these circuits also need to be *well-behaved* in a certain sense (see [26]). It is easy to check that the circuits we construct in this paper all have this property.

In the black-box version of this problem, we can query the labelling function for any point $p \in [N]^n$ and retrieve its label. In the white-box version, ℓ is given in the form of a Boolean circuit with the usual gates \wedge, \vee, \neg .

In the white-box model, nD -TUCKER was recently proven to be PPA-hard for any $n \geq 2$, by Aisenberg et al. [1]; the membership of the problem in PPA was known by [41].

THEOREM 2.1 ([1, 41]). *For any constant $n \geq 2$, nD -TUCKER is PPA-complete.*

The computational class PPA was defined by Papadimitriou [41], among several subclasses of the class TFNP [38], the class of problems with a guaranteed solution which is verifiable in polynomial time. PPA is defined with respect to a graph of exponential size, which is given *implicitly* as input, via the use of a circuit that outputs the neighbours of a given vertex, and the goal is to find a vertex of odd degree, given another such vertex as input.

Given the close connection between Tucker’s Lemma and the Borsuk-Ulam Theorem, the PPA-hardness of *some appropriate computational version* of Borsuk-Ulam follows as well [1]. However, this does not apply to the version of nD -BORSUK-ULAM defined above, especially when one considers the additional properties of the function F required for normalisation and monotonicity, as discussed earlier. We will reduce from nD -TUCKER to our version of nD -BORSUK-ULAM, to obtain its PPA-hardness (even for the normalised monotone version), which will then imply the PPA-hardness of ε -CONSENSUS-HALVING, via our main reduction in Section 3.

In the black-box model, Deng et al. [24], building on the results of Chen and Deng [17], proved both query complexity lower bounds and upper bounds for nD -TUCKER.

THEOREM 2.2 ([24]). *For any constant $n \geq 2$, the query complexity of nD -TUCKER is $\Theta(N^{n-1})$.*

We remark that Deng et al. [24] use a version of nD -TUCKER that is slightly different from the one that we defined above, but their results apply to this version as well.

2.2 Efficient Black-Box Reductions

The reductions that we will construct (from nD -TUCKER to nD -BORSUK-ULAM to ε -CONSENSUS-HALVING) will be *black-box reductions*, and therefore they will also allow us to obtain query complexity lower bounds for ε -CONSENSUS-HALVING in the black-box model, given the corresponding lower bounds of Theorem 2.2. For the upper bounds, we will reduce directly from ε -CONSENSUS-HALVING to nD -TUCKER, again via a black-box reduction.

Roughly speaking,⁶ a black-box reduction from Problem A to Problem B is a procedure by which we can answer oracle calls (queries) for an instance of Problem B by using an oracle for some instance of Problem A, such that a solution to the instance of Problem B yields a solution to the instance of Problem A. For example, a black-box reduction from nD -BORSUK-ULAM to ε -CONSENSUS-HALVING is a procedure that simulates an instance for the latter problem by accessing the function F of the former problem a number of times, and such that a solution of ε -CONSENSUS-HALVING can easily be translated to a solution of nD -BORSUK-ULAM. The name “black-box” comes from the fact that this type of reduction does not need to know the structure of the functions v_i of ε -CONSENSUS-HALVING or F of nD -BORSUK-ULAM.

In order to prove lower bounds on the query complexity of some Problem B, it suffices to construct a black-box reduction from some Problem A, for which query complexity lower bounds are known; the obtained bounds will depend on the number k of oracle calls to the input of Problem A that are needed to answer an oracle call to the input of Problem B. A black-box reduction is *efficient* if k is a

⁶To keep the exposition clean, we do not provide formal definitions of these concepts here, as they are rather standard; we refer the reader to the related works of [11, 36] for more details.

constant, and therefore the query complexity lower bounds of Problems A and B are of the same asymptotic order. To obtain upper bounds on the query complexity, we can construct a reduction in the opposite direction (from Problem B to Problem A), assuming that query complexity upper bounds for Problem A are known.

Ideally, we would like to use the same reduction to also obtain computational complexity results in the white-box model. For this to be possible, the procedure described above should actually be a polynomial-time algorithm. Slightly abusing terminology, we will use the term “efficient” to describe such a reduction in the white-box model as well.

Definition 5. We say that a black-box reduction from Problem A to Problem B is *efficient* if:

- in the black-box model, it uses a constant number of queries (oracle calls) to the function (oracle) of Problem A, for each query (oracle call) to the function of Problem B;
- in the white-box model, the condition above holds, and the reduction is also a polynomial-time algorithm.

Concretely for our case, all of our reductions will be efficient black-box reductions, thus allowing us to obtain both PPA-completeness results and query complexity bounds matching those of the problems that we reduce from/to. We remark that the reductions constructed for proving the PPA-hardness of the problem in previous works (for a non-constant number of agents) [28–30] are not black-box reductions, and therefore have no implications on the query complexity of the problem.

3 BLACK-BOX REDUCTIONS TO AND FROM CONSENSUS-HALVING

In this section we develop our main machinery for proving both PPA-completeness results and query complexity upper and lower bounds for ε -CONSENSUS-HALVING. We summarise our general approach for obtaining positive and negative results below.

For our *impossibility results* (i.e., computational hardness results in the white-box model and query complexity lower bounds in the black-box model), we will construct an efficient black-box reduction from nD -BORSUK-ULAM to ε -CONSENSUS-HALVING with n agents (Proposition 3.1). This reduction will preserve the optional properties of Definition 3, meaning that if the instance of nD -BORSUK-ULAM is normalised (respectively monotone), the valuation functions of the corresponding instance of ε -CONSENSUS-HALVING will be normalised (respectively monotone) as well. This will allow us in subsequent sections to reduce the problem of proving impossibility results for ε -CONSENSUS-HALVING to proving impossibility results for the versions of nD -BORSUK-ULAM with those properties. We will obtain these latter results via reductions from nD -TUCKER, which for $n \geq 2$ is known to be PPA-hard (Theorem 2.1) and admit exponential query complexity lower bounds (Theorem 2.2).

For our *positive results* (i.e., membership in PPA in the white-box model and query complexity upper bounds in the black-box model), we will construct an efficient black-box reduction from ε -CONSENSUS-HALVING to nD -TUCKER (Proposition 3.2). We remark here that a similar reduction already exists in the related literature [31], but only applied to the case of additive valuation functions. The extension to the case of general valuations follows along the same lines, and we provide it here for completeness. We also note that some of our positive results, namely the results for one general agent and two monotone agents, will not be obtained via reductions, but rather directly via the design of polynomial-time algorithms in the white-box model or algorithms of polynomial query complexity in the black-box model.

Related to the discussion above, we have the following two propositions. Their proofs are presented in the sections below.

PROPOSITION 3.1. *There is an efficient black-box reduction from (normalised, monotone) nD -BORSUK-ULAM to (normalised, monotone) ε -CONSENSUS-HALVING.*

PROPOSITION 3.2. *There is an efficient black-box reduction from ε -CONSENSUS-HALVING to nD -TUCKER.*

3.1 Proof of Proposition 3.1

Description of the reduction. Let $n \geq 1$ be a fixed integer. Let $\varepsilon > 0$ and let $F : B^{n+1} \rightarrow B^n$ be a Lipschitz-continuous function with Lipschitz parameter L . We now construct valuation functions v_1, \dots, v_n for a Consensus-Halving instance.

Let R_1, R_2, \dots, R_{n+1} denote the partition of interval $[0, 1]$ into $n + 1$ subintervals of equal length, i.e., $R_j = [\frac{j-1}{n+1}, \frac{j}{n+1}]$ for $j \in [n + 1]$. For any $A \in \Lambda([0, 1])$, we define $x(A) \in B^{n+1} = [-1, 1]^{n+1}$ by

$$[x(A)]_j = 2(n + 1) \cdot \lambda(A \cap R_j) - 1$$

for all $j \in [n + 1]$. Recall that λ denotes the Lebesgue measure on the interval $[0, 1]$. Note that since $\lambda(A \cap R_j) \in [0, \frac{1}{n+1}]$, we indeed have $[x(A)]_j \in [-1, 1]$; see Figure 2 for a visualisation.

For $i \in [n]$, the valuation function v_i of the i th agent is defined as

$$v_i(A) = \frac{F_i(x(A)) + 1}{2}$$

for any $A \in \Lambda([0, 1])$, where $F_i : B^{n+1} \rightarrow [-1, 1]$ is the i th output of F . Note that $v_i(A) \in [0, 1]$, since $F_i(x(A)) \in [-1, 1]$.

Lipschitz-continuity. For any $A, B \in \Lambda([0, 1])$ it holds that

$$\begin{aligned} |v_i(A) - v_i(B)| &= \frac{1}{2} |F_i(x(A)) - F_i(x(B))| \leq \frac{1}{2} \|F(x(A)) - F(x(B))\|_\infty \\ &\leq \frac{L}{2} \|x(A) - x(B)\|_\infty \leq (n + 1) \cdot L \cdot \max_{j \in [n+1]} |\lambda(A \cap R_j) - \lambda(B \cap R_j)| \\ &\leq (n + 1) \cdot L \cdot \max_{j \in [n+1]} \lambda((A \Delta B) \cap R_j) \leq (n + 1) \cdot L \cdot \lambda(A \Delta B). \end{aligned}$$

Thus, v_i is Lipschitz-continuous with Lipschitz parameter $(n + 1) \cdot L$.

Correctness. Now consider an $\varepsilon/2$ -Consensus-Halving of v_1, \dots, v_n . Namely, let $\mathcal{I}^+, \mathcal{I}^-$ be a partition of $[0, 1]$ using at most n cuts, such that $|v_i(\mathcal{I}^+) - v_i(\mathcal{I}^-)| \leq \varepsilon/2$ for all $i \in [n]$. Since \mathcal{I}^+ is obtained by using at most n cuts, it follows that there exists $j \in [n + 1]$ such that R_j does not contain a cut. As a result, $\mathcal{I}^+ \cap R_j$ is either empty or equal to R_j . This implies that $\lambda(\mathcal{I}^+ \cap R_j) \in \{0, \frac{1}{n+1}\}$ and thus $[x(\mathcal{I}^+)]_j \in \{\pm 1\}$, i.e., $x(\mathcal{I}^+) \in \partial(B^{n+1})$. Furthermore, for any $j \in [n + 1]$, we have

$$\begin{aligned} [x(\mathcal{I}^+)]_j &= 2(n + 1) \cdot \lambda(\mathcal{I}^+ \cap R_j) - 1 = 2(n + 1) \cdot \left(\frac{1}{n + 1} - \lambda(\mathcal{I}^- \cap R_j) \right) - 1 \\ &= -2(n + 1) \cdot \lambda(\mathcal{I}^- \cap R_j) + 1 = -[x(\mathcal{I}^-)]_j. \end{aligned}$$

Letting $y = x(\mathcal{I}^+) \in \partial(B^{n+1})$, we have that for any $i \in [n]$

$$|F_i(y) - F_i(-y)| = |F_i(x(\mathcal{I}^+)) - F_i(x(\mathcal{I}^-))| = 2|v_i(\mathcal{I}^+) - v_i(\mathcal{I}^-)| \leq \varepsilon.$$

Thus, y is a solution to the original nD -BORSUK-ULAM instance.

White-box model. This reduction yields a polynomial-time many-one reduction from nD -BORSUK-ULAM to ε -CONSENSUS-HALVING with n agents. Thus, if we show that nD -BORSUK-ULAM is PPA-hard for some n , then we immediately obtain that ε -CONSENSUS-HALVING with n agents is also PPA-hard.

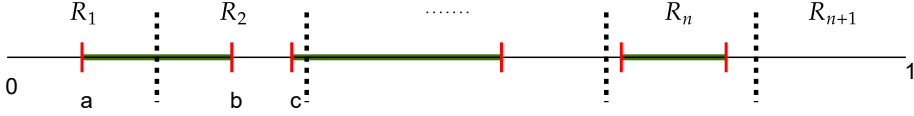


Fig. 2. The partition of $[0, 1]$ into $n+1$ subintervals of equal length and a set A , coloured by the green region, as it is defined by the red cuts. The first three cuts on the left are located at positions $a \leq b \leq c$, where $a \in R_1$ and $b, c \in R_2$. Here, since $\lambda(A \cap R_1) = 1/(n+1) - a$, we would obtain $[x(A)]_1 = 2(n+1)(1/(n+1) - a) - 1 = 1 - 2(n+1)a$. Similarly, $\lambda(A \cap R_2) = 1/(n+1) - (c - b)$, and thus $[x(A)]_2 = 1 - 2(n+1)(c - b)$.

Black-box model. It is easy to see that this is a black-box reduction. It can be formulated as follows: given access to an oracle for an instance of nD -BORSUK-ULAM with parameters (ε, L) we can simulate an oracle for an instance of ε -CONSENSUS-HALVING (with n agents) with parameters $(\varepsilon/2, (n+1)L)$ such that any solution of the latter yields a solution to the former. Furthermore, in order to answer a query to some v_i , we only need to perform a single query to F . Thus, we obtain the following query lower bound: solving an instance of ε -CONSENSUS-HALVING (with n agents) with parameters (ε, L) requires at least as many queries as solving an instance of nD -BORSUK-ULAM with parameters $(\varepsilon', L') = (2\varepsilon, \frac{L}{n+1})$. This means that if nD -BORSUK-ULAM has a query lower bound of $\Omega((\frac{L'}{\varepsilon'})^{n-1})$ for some n , then ε -CONSENSUS-HALVING (with n agents) has a query lower bound of $\Omega((\frac{L}{2\varepsilon(n+1)})^{n-1}) = \Omega((\frac{L}{\varepsilon})^{n-1})$, since n is constant.

Additional properties of the reduction. Some properties of the Borsuk-Ulam function F carry over to the valuation functions v_1, \dots, v_n . In particular, the following properties are of interest to us:

- **If F is monotone, then v_1, \dots, v_n are monotone.** Indeed, consider $A, B \in \Lambda([0, 1])$ with $A \subseteq B$. Then, it holds that $\lambda(A \cap R_j) \leq \lambda(B \cap R_j)$ for all $j \in [n+1]$, and as a result $x(A) \leq x(B)$ (coordinate-wise). By monotonicity of F , it follows that $F_i(x(A)) \leq F_i(x(B))$, and thus $v_i(A) \leq v_i(B)$ for all $i \in [n]$.
- **If F is normalised, then v_1, \dots, v_n are normalised.** As noted earlier, we already have that $v_i(A) \in [0, 1]$ for all $A \in \Lambda([0, 1])$. Thus, it remains to prove that $v_i(\emptyset) = 0$ and $v_i([0, 1]) = 1$. It is easy to see that $x([0, 1]) = (1, 1, \dots, 1)$ and thus $F(x([0, 1])) = (1, 1, \dots, 1)$ since F is normalised, which yields $v_i([0, 1]) = 1$. On the other hand, we have $x(\emptyset) = (-1, -1, \dots, -1)$ and thus $F(x(\emptyset)) = -F(-x(\emptyset)) = -F(1, 1, \dots, 1) = (-1, -1, \dots, -1)$, which yields $v_i(\emptyset) = 0$. Here we also used the fact F is an odd function, since it is normalised. In fact, since F is odd, we also obtain that $v_i(A) + v_i(A^c) = 1$ for all $A \in \Lambda([0, 1])$, where $A^c = [0, 1] \setminus A$ denotes the complement of A . This can be shown by noting that $x(A^c) = -x(A)$ (by using the same argument as for I^+ and I^- above) and then using the fact that $F(x(A^c)) = -F(x(A))$.

This means that if we are able to show (white- or black-box) hardness of nD -BORSUK-ULAM where F has additional properties, then the hardness will also hold for ε -CONSENSUS-HALVING with n agents that have the corresponding properties.

Furthermore, note that if F is a normalised nD -BORSUK-ULAM function, then an ε -approximate Consensus-Halving for v_1, \dots, v_n (i.e., $|v_i(I^+) - v_i(I^-)| \leq \varepsilon$), yields an ε -approximate solution to F , in the sense that $\|F(x(I^+))\|_\infty \leq \varepsilon$. This is due to the fact that, by definition, F is an odd function, if it is normalised.

3.2 Proof of Proposition 3.2

The reduction presented in this section is based on a proof of existence for a generalization of Consensus-Halving with general valuations given in [31]. This existence proof was also used in [31] to provide a reduction for additive valuations. It can easily be extended to work for general valuations as well. We include the full reduction here for completeness.

Description of the reduction. Consider an instance of ε -CONSENSUS-HALVING with n agents with parameters ε, L . Let v_1, \dots, v_n denote the valuations of the agents. We consider the domain K_m^n , where $K_m = \{-1, -(m-1)/m, \dots, -1/m, 0, 1/m, 2/m, \dots, (m-1)/m, 1\}$, for $m = \lceil 2nL/\varepsilon \rceil$. A point in K_m^n corresponds to a way to partition the interval $[0, 1]$ into two sets I^+, I^- using at most n cuts. A very similar encoding was also used by Meunier [39] for the Necklace Splitting problem. A point $x \in K_m^n$ corresponds to the partition $I^+(x), I^-(x)$ obtained as follows.

- (1) Provisionally put the label “+” on the whole interval $[0, 1]$
- (2) For $\ell = 1, 2, \dots, n$:
 - if $x_\ell > 0$, then put label “+” on the interval $[0, x_\ell]$;
 - if $x_\ell < 0$, then put label “−” on the interval $[0, -x_\ell]$.

Note that subsequent assignments of a label to an interval, “overwrite” previous assignments. One way of thinking about it, is that we are applying a coat of paint on the interval $[0, 1]$. Initially the whole interval is painted with colour “+”, and as the procedure is executed, various subintervals will be painted over with colour “−” or “+”. It is easy to check that the final partition into $I^+(x), I^-(x)$ that is obtained, uses at most n cuts. Furthermore, for any $x \in \partial K_m^n$, the partition $I^+(-x), I^-(-x)$ obtained from $-x$ corresponds to the partition $I^+(x), I^-(x)$ with labels “+” and “−” switched. In other words, $I^+(-x) = I^-(x)$ and $I^-(-x) = I^+(x)$. For a more formal definition of this encoding, see [31].

We define a labelling $\ell : K_m^n \rightarrow \{\pm 1, \pm 2, \dots, \pm n\}$ as follows. For any $x \in K_m^n$:

- (1) Let $i \in [n]$ be the agent that sees the largest difference between $v_i(I^+(x))$ and $v_i(I^-(x))$, i.e., $i = \arg \max_{i \in [n]} |v_i(I^+(x)) - v_i(I^-(x))|$, where we break ties by picking the smallest such i .
- (2) Pick a sign $s \in \{+, -\}$ as follows. If $v_i(I^+(x)) > v_i(I^-(x))$, then let $s = +$. If $v_i(I^+(x)) < v_i(I^-(x))$, then let $s = -$. If $v_i(I^+(x)) = v_i(I^-(x))$, then pick s such that I^s contains the left end of the interval $[0, 1]$.
- (3) Set $\ell(x) = +i$ if $s = +$, and $\ell(x) = -i$ otherwise.

With this definition, it is easy to check that $\ell(-x) = -\ell(x)$ for all $x \in \partial(K_m^n)$. By re-interpreting K_m^n as a grid $[N]^n$ with $N = 2m + 1$, we thus obtain an instance $\widehat{\ell} : [N]^n \rightarrow \{\pm 1, \pm 2, \dots, \pm n\}$ of nD -TUCKER. In particular, note that $\widehat{\ell}$ is antipodally anti-symmetric on the boundary, as required.

Correctness. Any solution to the nD -TUCKER instance $\widehat{\ell}$ yields $x, y \in K_m^n$ with $\|x - y\|_\infty \leq 1/m$ and $\ell(x) = -\ell(y)$. Without loss of generality, assume that $\ell(x) = +i$ for some $i \in [n]$. Since $\|x - y\|_\infty \leq 1/m$, we obtain that

$$\lambda(I^+(x) \Delta I^+(y)) \leq \sum_{j=1}^n |x_j - y_j| = \|x - y\|_1 \leq n\|x - y\|_\infty \leq n/m$$

and the same bound also holds for $\lambda(I^-(x) \Delta I^-(y))$. Since v_i is Lipschitz-continuous with parameter L , it follows that

$$|v_i(I^+(x)) - v_i(I^+(y))| \leq L \cdot \lambda(I^+(x) \Delta I^+(y)) \leq nL/m$$

and similarly for $|v_i(I^-(x)) - v_i(I^-(y))|$.

Since $\ell(x) = +i$, it follows that $v_i(I^+(x)) \geq v_i(I^-(x))$. For the sake of contradiction, let us assume that $v_i(I^+(x)) > v_i(I^-(x)) + \varepsilon$. Then, it follows that

$$v_i(I^+(y)) - v_i(I^-(y)) \geq v_i(I^+(x)) - v_i(I^-(x)) - 2nL/m > \varepsilon - 2nL/m \geq 0$$

since $m \geq 2nL/\varepsilon$. But this contradicts the fact that $\ell(y) = -i$. Thus, it must hold that $|v_i(I^+(x)) - v_i(I^-(x))| \leq \varepsilon$. Since $\ell(x) = +i$, it follows that for all $j \in [n]$

$$|v_j(I^+(x)) - v_j(I^-(x))| \leq |v_i(I^+(x)) - v_i(I^-(x))| \leq \varepsilon.$$

This means that $I^+(x), I^-(x)$ yields a solution to the original ε -CONSENSUS-HALVING instance.

Note that the reduction uses $N = 2m + 1 \leq 4nL/\varepsilon + 3$ for the nD -TUCKER instance. Furthermore, any query to the labelling function $\widehat{\ell}$ can be answered by performing $2n$ queries to the valuation functions v_1, \dots, v_n .

4 GENERAL VALUATIONS

We are now ready to prove our main results for the ε -CONSENSUS-HALVING problem, starting from the case of general valuations. First, for a single agent with a general valuation function, a simple binary search procedure is sufficient to solve ε -CONSENSUS-HALVING with a polynomial number of queries and in polynomial time, therefore obtaining an efficient algorithm both in the white-box and in the black-box model. We have the following theorem. The proof can be found in the full version [6].

THEOREM 4.1. *For one agent with a general valuation function (or multiple agents with identical general valuations), ε -CONSENSUS-HALVING is solvable in polynomial time and has query complexity $\Theta(\log \frac{L}{\varepsilon})$.*

We now move to our results for two or more agents with general valuations. Here we obtain a PPA-completeness result for ε -CONSENSUS-HALVING, as well as exponential bounds on the query complexity of the problem. Our results demonstrate that for general valuations, even in the case of two agents, the problem is intractable in both the black-box and the white-box model. The main technical result of the section is the following pivotal lemma, proved in the full version [6].

LEMMA 4.2. *For any constant $n \geq 1$, nD -TUCKER reduces to normalised nD -BORSUK-ULAM, via an efficient black-box reduction.*

Now we state our main theorem about the computational/query complexity of the ε -CONSENSUS-HALVING problem, as well as a corresponding theorem for nD -BORSUK-ULAM. The proofs follow from Theorems 2.1 and 2.2 characterising the complexity of nD -TUCKER and the following chain of reductions (where “ \leq ” denotes an efficient black-box reduction from the problem on the left-hand side to the problem on the right-hand side).

$$nD\text{-TUCKER} \underset{\text{Lem. 4.2}}{\leq} nD\text{-BORSUK-ULAM} \underset{\text{Prop. 3.1}}{\leq} \varepsilon\text{-CONSENSUS-HALVING} \underset{\text{Prop. 3.2}}{\leq} nD\text{-TUCKER}$$

The specific parameters that appear in the bounds below follow from the proof of Lemma 4.2.

THEOREM 4.3. *Let $n \geq 2$ be any constant. Then,*

- ε -CONSENSUS-HALVING with n normalised general agents is PPA-complete. This remains the case, even if (a) we fix $\varepsilon \in (0, 1)$, or (b) we fix $L \geq 3(n + 1)$;
- there exists a constant $c > 0$ such that for any $\varepsilon \in (0, 1)$ and any $L \geq 3(n + 1)$ with $L/\varepsilon \geq c$, the query complexity of ε -CONSENSUS-HALVING with n normalised general agents is $\Theta((L/\varepsilon)^{n-1})$.

THEOREM 4.4. *Let $n \geq 2$ be any constant. Then,*

- *normalised nD -BORSUK-ULAM is PPA-complete. This remains the case, even if (a) we fix $\varepsilon \in (0, 1)$, or (b) we fix $L \geq 3$;*
- *there exists a constant $c > 0$ such that for any $\varepsilon \in (0, 1)$ and any $L \geq 3$ with $L/\varepsilon \geq c$, the query complexity of normalised nD -BORSUK-ULAM is $\Theta((L/\varepsilon)^{n-1})$.*

In both cases, the lower bounds hold even for the normalised versions of the problems, while the upper bounds hold even for the more general, non-normalised, versions.

5 MONOTONE VALUATIONS

In this section, we present our results for agents with monotone valuations. In contrast to the results of Section 4, here we prove that for two agents with monotone valuations, the problem is solvable in polynomial time and with a polynomial number of queries, and in fact this result holds even if only one of the two agent has a monotone valuation and the other has a general valuation. For three or more agents however, the problem becomes PPA-complete once again, and we obtain a corresponding exponential lower bound on its query complexity.

5.1 An efficient algorithm for two monotone agents

We start with our efficient algorithm for the case of two agents, which is a polynomial-time algorithm in the white-box model, as well as an algorithm of polynomial query complexity in the black-box model; see Algorithm 1. The algorithm is based on a *nested binary search* procedure: we start with an interval, initially defined by a position of the two cuts that “satisfies” the agent with the monotone valuation, when one of the two cuts sits on the left endpoint 0 of the interval. Then we iteratively “shrink” this interval based on the preference of the other agent, maintaining that (a) the first agent is always “satisfied” and (b) the solution that “satisfies” the other agent is contained in the considered interval that is being shrunk. Intuitively, we are moving on the “indifference curve” of the valuation function of the agent with the monotone valuation (see the red zig-zag line in Figure 3), and we “offer” the other agent points (i.e., sets of cuts), to which the agent replies with either “+” or “−”, indicating the label of her discrepancy. Since there are “equivalent” cut positions in which the second agent responds with opposite labels (see the intersection points of the red line with the vertical axis and the horizontal axis in Figure 3), we know labels of the boundary points corresponding to this agent, and we can move in the direction towards the point with the opposite discrepancy label.

Before we proceed, we draw an interesting connection with *Austin’s moving knife procedure* [7], an Exact-Division procedure for two agents with general valuations. The procedure is based on two moving knives which one of the two agents simultaneously and continuously slides across the cake, maintaining that the corresponding cut positions ensure that she is satisfied with the partition. At some point during this process, the other agent becomes satisfied, which is guaranteed by the intermediate value theorem. Our algorithm can be interpreted as a discrete time implementation of this idea and quite interestingly, it results in a polynomial-time algorithm when one of the two agents has a monotone valuation, whereas it is computationally hard when both agents have general valuations, as shown in Section 4. On a more fundamental level, this demonstrates the intricacies of transforming moving-knife fair division protocols into discrete algorithms.

The main theorem of this section is the following. It is proved in the full version [6].

THEOREM 5.1. *For two agents with monotone valuation functions, ε -CONSENSUS-HALVING is solvable in polynomial time and has query complexity $O(\log^2 \frac{L}{\varepsilon})$, even in the weak black-box model. This result holds even if one of the two agents has a general valuation.*

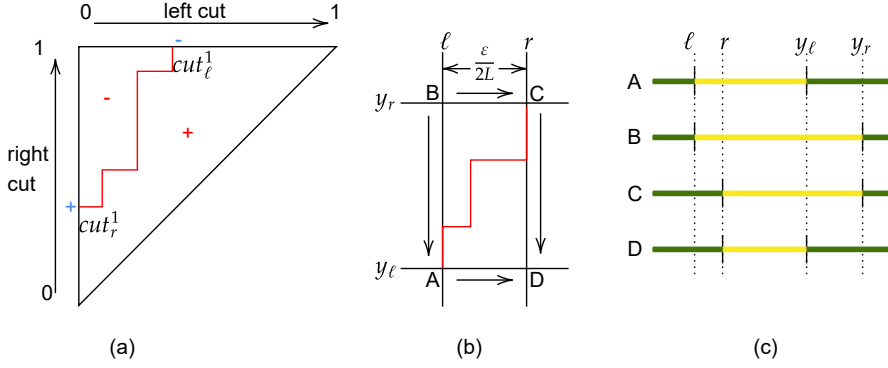


Fig. 3. Visualisation of Algorithm 1. (a) depicts our initial assumptions. The red line shows where Agent 1 is indifferent. The blue signs on $(0, \text{cut}_r^1)$ and $(\text{cut}_l^1, 1)$ show the preferences of Agent 2 under these pairs of cuts. (b) shows a possible position for the cuts $r - \ell \leq \frac{\varepsilon}{2L}$. The arrows show how the difference between the values of the positive piece and the negative piece change between the four possible combinations of pairs of cuts. (c) depicts the actual cuts on the cake: the green parts have label “+” and the yellow parts have label “-”.

ALGORITHM 1: ε -CONSENSUS-HALVING for two agents with monotone valuations

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1  Set  $\ell \leftarrow 0$  and  $r \leftarrow \text{cut}_\ell^1$ 
2  Set  $\text{cut}_\ell \leftarrow \text{cut}_\ell^1$  and  $\text{cut}_r \leftarrow 1$ 
3  while Agent 2 is not indifferent under the pair of cuts  $(\text{cut}_\ell, \text{cut}_r)$  do
4      if  $r - \ell > \frac{\varepsilon}{2L}$  then
5          Find  $y \geq \frac{\ell+r}{2}$ , such that Agent 1 is indifferent under the pair of cuts  $(\frac{\ell+r}{2}, y)$ 
6          Set  $\text{cut}_\ell \leftarrow \frac{\ell+r}{2}$  and  $\text{cut}_r \leftarrow y$ .
7          if Agent 2 prefers the positive piece under  $(\text{cut}_\ell, \text{cut}_r)$  then
8              Set  $\ell \rightarrow \frac{\ell+r}{2}$ 
9          end
10         if Agent 2 prefers the negative piece under  $(\text{cut}_\ell, \text{cut}_r)$  then
11             Set  $r \rightarrow \frac{\ell+r}{2}$ 
12         end
13     else
14         Let  $y_\ell$  and  $y_r$  be such that Agent 1 is indifferent under the pairs of cuts  $(\ell, y_\ell)$  and  $(r, y_r)$ 
15         Find  $x \in [\ell, r]$  such that Agent 1 is indifferent under the pair of cuts  $(x, \frac{y_\ell+y_r}{2})$ 
16         Set  $\text{cut}_\ell \leftarrow x$  and  $\text{cut}_r \leftarrow \frac{y_\ell+y_r}{2}$ 
17         if Agent 2 prefers the positive piece under  $(\text{cut}_\ell, \text{cut}_r)$  then
18             Set  $y_\ell \rightarrow \frac{y_\ell+y_r}{2}$ 
19         end
20         if Agent 2 prefers the negative piece under  $(\text{cut}_\ell, \text{cut}_r)$  then
21             Set  $y_r \rightarrow \frac{y_\ell+y_r}{2}$ 
22         end
23     end
24 end
    
```

5.2 Results for three or more monotone agents

We now move on to the case of three or more monotone agents, for which we manage to show that the problem becomes computationally hard and has exponential query complexity. Our results thus show a clear dichotomy on the complexity of ε -CONSENSUS-HALVING with monotone agents, between the case of two agents and the case of three or more agents.

Again we employ our general approach, but this time we need to prove computational and query-complexity hardness of the *monotone* nD -BORSUK-ULAM problem; the corresponding impossibility results for ε -CONSENSUS-HALVING with agents with monotone valuations then follow from our property-preserving reduction (Proposition 3.1). To this end, we in fact construct an efficient black-box reduction from $(n - 1)D$ -TUCKER to monotone nD -BORSUK-ULAM, i.e., we reduce from the corresponding version of nD -TUCKER of one lower dimension. In order to achieve this, we once again interpolate the $(n - 1)D$ -TUCKER instance to obtain a continuous function, but, this time, we embed it in a very specific lower dimensional subset of the nD -BORSUK-ULAM domain. We then show that the function can be extended to a monotone function on the whole domain.

The “drop in dimension” which is featured in our reduction has the following effects:

- Since $1D$ -TUCKER is solvable in polynomial-time, we can only obtain the PPA-hardness of monotone nD -BORSUK-ULAM for $n \geq 3$, and therefore the PPA-hardness of ε -CONSENSUS-HALVING for three or more monotone agents.
- The query complexity lower bounds that we “inherit” from $(n - 1)D$ -TUCKER do not *exactly* match our upper bounds, obtained via the reduction from ε -CONSENSUS-HALVING to nD -TUCKER (Proposition 3.2).

The main technical contribution of this section is the following lemma, proved in the full version [6].

LEMMA 5.2. *For any constant $n \geq 2$, $(n - 1)D$ -TUCKER reduces to normalised monotone nD -BORSUK-ULAM in polynomial time, via an efficient black-box reduction.*

Similarly to Section 4, we then obtain the following two theorems.

THEOREM 5.3. *Let $n \geq 3$ be any constant. Then,*

- ε -CONSENSUS-HALVING with n monotone agents is PPA-complete. This remains the case, even if (a) we fix $\varepsilon \in (0, 1)$, or (b) we fix $L \geq 3(n + 1)$;
- for any constant $t \in (0, 1)$, there exists a constant $c > 0$ such that for any $\varepsilon \in (0, t)$ and any $L \geq 3(n + 1)$ with $L/\varepsilon \geq c$, the query complexity of ε -CONSENSUS-HALVING with n monotone agents is between $\Omega((L/\varepsilon)^{n-2})$ and $O((L/\varepsilon)^{n-1})$.

THEOREM 5.4. *Let $n \geq 3$ be any constant. Then,*

- monotone nD -BORSUK-ULAM is PPA-complete. This remains the case, even if (a) we fix $\varepsilon \in (0, 1)$, or (b) we fix $L \geq 3$;
- for any constant $t \in (0, 1)$, there exists a constant $c > 0$ such that for any $\varepsilon \in (0, t)$ and any $L \geq 3$ with $L/\varepsilon \geq c$, the query complexity of monotone nD -BORSUK-ULAM is between $\Omega((L/\varepsilon)^{n-2})$ and $O((L/\varepsilon)^{n-1})$.

The proofs of the theorems follow from Theorems 2.1 and 2.2 and the following chain of reductions, which now crucially involve the monotone version of nD -BORSUK-ULAM and ε -CONSENSUS-HALVING.

$$(n-1)D\text{-TUCKER} \leq \underset{\text{Lem. 5.2}}{\text{monotone } nD\text{-BORSUK-ULAM}} \leq \underset{\text{Prop. 3.1}}{\text{monotone } \varepsilon\text{-CONSENSUS-HALVING}} \leq \underset{\text{Prop. 3.2}}{nD\text{-TUCKER}}$$

The specific parameters that appear in the bounds follow from the proof of Lemma 5.2. The lower bounds again hold even for the normalised versions of the problems. There is a small gap between our lower and upper bounds; we conjecture that it should be possible to prove upper bounds that match the lower bounds of Theorem 5.3, at least up to logarithmic factors, but we leave this for future work.

6 RELATIONS TO THE ROBERTSON-WEBB QUERY MODEL

The black-box query model that we used in the previous sections is the standard model used in the related literature of query complexity, where the nature of the input functions depends on the specific problems at hand. For example, for $nD\text{-BORSUK-ULAM}$ the function F inputs points on the domain and returns other points, whereas in $nD\text{-TUCKER}$ the function ℓ inputs points and outputs their labels.

At the same time, in the literature of the cake-cutting problem, the predominant query model is in fact a more expressive query model, known as the *Robertson-Webb model (RW)* [44, 48]. The RW model has been defined only for the case of *additive valuations*, and consists of the following two types of queries:

- EVAL queries, where the agent is given an interval $[a, b]$ and she returns her value for that interval, and
- CUT queries, where the agent is given a point $x \in [0, 1]$ and a real number α , and they designate the smallest interval $[x, y]$, for which their value is exactly α .

In fact, in the literature of *envy-free* cake-cutting, the query complexity in the RW model has been one of the most important open problems [13, 42], with breakthrough results coming from the literature of computer science fairly recently [8, 9]. Since $\varepsilon\text{-CONSENSUS-HALVING}$ and $\varepsilon\text{-fair}$ cake-cutting [14] are conceptually closely related, it would make sense to consider the query complexity of the former problem in the RW model as well.⁷

A potential hurdle in this investigation is that the RW model has not been defined for valuation functions beyond the additive case. To this end, we propose the following generalisation of the RW model that we call *Generalised Robertson-Webb model (GRW)*, which is appropriate for monotone valuation functions that are not necessarily additive. Intuitively, in the GRW model the agent essentially is given *sets of intervals* A rather than single intervals, and the queries are defined accordingly (see also Figure 4).

Definition 6 (Generalised Robertson-Webb (GRW) Query Model). In the GRW query model, there are two types of queries:

- EVAL queries, where agent i is given any Lebesgue-measurable subset A of $[0, 1]$ and she returns her value $v_i(A)$ for that set, and
- CUT queries, where agent i is given two disjoint Lebesgue-measurable subsets A_1 and A_2 of $[0, 1]$, an interval $I = [a, b]$, disjoint from A_1 and A_2 , and a real number $\gamma \geq 0$, and she designates some $x \in I$ such that $\frac{v_i(A_1 \cup [a, x])}{v_i(A_2 \cup [x, b])} = \gamma$, if such a point exists.

⁷The $\varepsilon\text{-CONSENSUS-HALVING}$ halving problem has in fact recently been studied under this query model as well, but in a somewhat different direction, and for agents with additive valuations [3]. Note that the authors do not refer to their query model as the RW model, but the queries that they use are essentially RW queries.



Fig. 4. Visualisation of CUT and EVAL queries. (a) The input A to an EVAL query is denoted by the green intervals. (b) The inputs A_1 and A_2 to the CUT query are denoted by the green and yellow intervals respectively, and the interval $I = [a, b]$ is denoted in blue. The agent places a cut (if possible) at a position $x \in I$ such that her value for $A_1 \cup [a, x]$ and her value for $A_2 \cup [x, b]$ are in a specified proportion.

Let us discuss why this model is the most appropriate generalisation of the RW model. First, the definition of EVAL queries is in fact the natural extension, as the agent needs to specify her value for sets of intervals; note that in the additive case, it suffices to elicit an agent's value for only single intervals, as her value for unions of intervals is then simply the sum of the elicited values. This is not the case in general for monotone valuations, and therefore we need a more expressive EVAL query. We also remark that the EVAL query is exactly the same as a query in the black-box model, as defined in Section 2, and therefore the GRW model is stronger than the black-box query model. Brânzei and Nisan [14] in fact studied the restriction of the RW model (for the cake-cutting problem and for additive valuations), for which only EVAL queries are allowed, and they coined this the *RW⁻ query model*. To put our results into context, we offer the following definition of the *GRW⁻ query model*, which is, as discussed, equivalent to the black-box query model of Section 2. By the discussion above, all of our query complexity bounds in Section 4 and Section 5 apply verbatim to the GRW⁻ query model.

Definition 7 (Generalised Robertson-Webb⁻ (GRW⁻) query model). In the GRW⁻ query model, only EVAL queries are allowed; there agent i is given a Lebesgue-measurable subset A of $[0, 1]$ and she returns her value $v_i(A)$ for that set.

While the extension of EVAL queries from the RW model to the GRW model is relatively straightforward, the generalisation of CUT queries is somewhat more intricate. Upon closer inspection of a CUT query in the (standard) RW model for additive valuations, it is clear that one can equivalently define this query as

Given an interval $I = [a, b]$ and a real number γ , place a cut at $x \in [a, b]$ such that $\frac{v_i([a, x])}{v_i([x, b])} = \gamma$.

This is because one can easily find the value of the agent for $[a, b]$ with one EVAL query, and then for any value of α used in the standard definition of a cut query, there is an appropriate value of γ in the modified definition above, which will result in exactly the same position x of the cut and vice-versa.

The simplicity of the CUT queries in the RW model is enabled by the fact that for additive valuations, the value of any agent i for an interval I does not depend on how the remainder of the interval $[0, 1]$ has been cut. This is no longer the case for monotone valuations, as now the agent needs to specify a different value for sets of intervals. We believe that our definition of the CUT query in the GRW model is the appropriate generalisation, which captures the essence of the original cut queries in RW, but also allows for enough expressiveness to make this type of query useful for monotone valuations beyond the additive case.

Finally, we remark that for general valuations (beyond monotone), any sensible definition of cut queries seems to be too strong, in the sense that it conveys unrealistically too much information (in contrast to the RW and GRW models, where the CUT queries are intuitively “shortcuts” for binary

search). For example, assume that the agent is asked to place a cut at some point x in an interval $[a, b]$, for which (a) if the cut is placed at a the agent “sees” an excess of “+” and (b) if the cut is placed at b , the agent still “sees” an excess of “+”. By the boundary conditions of the interval, there is no guarantee that a cut that “satisfies” the agent exists within that interval, and we would need to exhaustively search through the whole interval to find such a cut position, if it exists, meaning that binary search does not help us here. On the other hand, a single CUT query would either find the position or return that there is no such x within the interval.

We are now ready to state our results for the section, which we summarise in the following theorem. Qualitatively, we prove that ε -CONSENSUS-HALVING with three normalised monotone agents still has an exponential query complexity in the GRW model (with logarithmic savings compared to the black-box model), whereas for two normalised monotone agents, the problem becomes “easier” by a logarithmic factor. The proof of this theorem can be found in the full version [6].

THEOREM 6.1. *In the Generalised Robertson-Webb model:*

- ε -CONSENSUS-HALVING with $n \geq 3$ normalised monotone agents requires $\Omega\left(\frac{(L/\varepsilon)^{n-2}}{\log(L/\varepsilon)}\right)$ queries.
- ε -CONSENSUS-HALVING with $n = 2$ monotone agents can be solved with $O(\log(L/\varepsilon))$ queries.

7 CONCLUSION AND FUTURE DIRECTIONS

In this paper, we managed to completely settle the computational complexity of the ε -CONSENSUS-HALVING problem for a constant number of agents with either general or monotone valuation functions. We also studied the query complexity of the problem and we provided exponential lower bounds corresponding to our hardness results, and polynomial upper bounds corresponding to our polynomial-time algorithms. We also defined an appropriate generalisation of the Robertson-Webb query model for monotone valuations and we showed that our bounds are qualitatively robust to the added expressiveness of this model. The main open problem associated with our work is the following.

What is the computational complexity and the query complexity of ε -CONSENSUS-HALVING with a *constant* number of agents and *additive valuations*?

Our approach in this paper prescribes a formula for answering this question: One can construct a black-box reduction to this version of ε -CONSENSUS-HALVING from a computationally-hard problem like nD -TUCKER, for which we also know query complexity lower bounds, and obtain answers to both questions at the same time. Alternatively, one might be able to construct polynomial-time algorithms for solving this problem; concretely, attempting to do that for three agents with additive valuations would be the natural starting step, as this is the first case for which the problem becomes computationally hard for agents with monotone valuations.

Another line of work would be to study the query complexity of the related fair cake-cutting problem using the GRW model that we propose. In fact, while the fundamental existence results for the problem (e.g., see [46]) actually apply to quite general valuation functions, most of the work in computer science has restricted its attention to the case of additive valuations only, with a few notable exceptions [16, 23]. We believe that our work can therefore spark some interest in the study of the query complexity of fair cake-cutting with valuations beyond the additive case.

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REFERENCES

- [1] James Aisenberg, Maria Luisa Bonet, and Sam Buss. 2020. 2-D Tucker is PPA complete. *J. Comput. System Sci.* 108 (2020), 92–103. <https://doi.org/10.1016/j.jcss.2019.09.002>
- [2] Noga Alon. 1987. Splitting necklaces. *Advances in Mathematics* 63, 3 (1987), 247–253. [https://doi.org/10.1016/0001-8708\(87\)90055-7](https://doi.org/10.1016/0001-8708(87)90055-7)
- [3] Noga Alon and Andrei Graur. 2020. Efficient Splitting of Measures and Necklaces. arXiv:2006.16613
- [4] Noga Alon and Douglas B. West. 1986. The Borsuk-Ulam Theorem and Bisection of Necklaces. *Proc. Amer. Math. Soc.* 98, 4 (1986), 623–628. <https://doi.org/10.2307/2045739>
- [5] Georgios Amanatidis, George Christodoulou, John Fearnley, Evangelos Markakis, Christos-Alexandros Psomas, and Eftychia Vakaliou. 2018. An improved envy-free cake cutting protocol for four agents. In *Proceedings of the 11th International Symposium on Algorithmic Game Theory (SAGT)*. 87–99. https://doi.org/10.1007/978-3-319-99660-8_9
- [6] Aris Filos-Ratsikas, Argyrios Deligkas, and Alexandros Hollender. 2020. Two’s Company, Three’s a Crowd: Consensus-Halving for a Constant Number of Agents. arXiv:2007.15125
- [7] A. K. Austin. 1982. Sharing a cake. *The Mathematical Gazette* 66, 437 (1982), 212–215. <https://doi.org/10.2307/3616548>
- [8] Haris Aziz and Simon Mackenzie. 2016. A discrete and bounded envy-free cake cutting protocol for any number of agents. In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*. 416–427. <https://doi.org/10.1109/FOCS.2016.52>
- [9] Haris Aziz and Simon Mackenzie. 2016. A discrete and bounded envy-free cake cutting protocol for four agents. In *Proceedings of the 48th Annual ACM Symposium on Theory of Computing (STOC)*. 454–464. <https://doi.org/10.1145/2897518.2897522>
- [10] Siddharth Barman and Nidhi Rathi. 2020. Fair Cake Division Under Monotone Likelihood Ratios. In *Proceedings of the 21st ACM Conference on Economics and Computation (EC)*. 401–437. <https://doi.org/10.1145/3391403.3399512>
- [11] Paul Beame, Stephen Cook, Jeff Edmonds, Russell Impagliazzo, and Toniann Pitassi. 1998. The Relative Complexity of NP Search Problems. *J. Comput. System Sci.* 57, 1 (1998), 3–19. <https://doi.org/10.1145/225058.225147>
- [12] Karol Borsuk. 1933. Drei Sätze über die n-dimensionale euklidische Sphäre. *Fundamenta Mathematicae* 20 (1933), 177–190. <https://doi.org/10.4064/fm-20-1-177-190>
- [13] Steven J. Brams and Alan D. Taylor. 1996. *Fair Division: From cake-cutting to dispute resolution*. Cambridge University Press. <https://doi.org/10.1017/CBO9780511598975>
- [14] Simina Brânzei and Noam Nisan. 2017. The Query Complexity of Cake Cutting. arXiv:1705.02946
- [15] Simina Brânzei and Noam Nisan. 2019. Communication Complexity of Cake Cutting. In *Proceedings of the 20th ACM Conference on Economics and Computation (EC)*. 525–525. <https://doi.org/10.1145/3328526.3329644>
- [16] Ioannis Caragiannis, John K. Lai, and Ariel D. Procaccia. 2011. Towards more expressive cake cutting. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*. 127–132. <https://doi.org/10.5591/978-1-57735-516-8/IJCAI11-033>
- [17] Xi Chen and Xiaotie Deng. 2008. Matching algorithmic bounds for finding a Brouwer fixed point. *J. ACM* 55, 3 (2008), 13:1–13:26. <https://doi.org/10.1145/1379759.1379761>
- [18] Xi Chen, Xiaotie Deng, and Shang-Hua Teng. 2009. Settling the complexity of computing two-player Nash equilibria. *J. ACM* 56, 3 (2009), 14:1–14:57. <https://doi.org/10.1145/1516512.1516516>
- [19] Constantinos Daskalakis, Paul W. Goldberg, and Christos H. Papadimitriou. 2009. The complexity of computing a Nash equilibrium. *SIAM J. Comput.* 39, 1 (2009), 195–259. <https://doi.org/10.1137/070699652>
- [20] Constantinos Daskalakis and Christos H. Papadimitriou. 2011. Continuous local search. In *Proceedings of the 22nd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 790–804. <https://doi.org/10.1137/1.9781611973082.62>
- [21] Argyrios Deligkas, John Fearnley, and Themistoklis Melissourgos. 2020. Square-Cut Pizza Sharing is PPA-complete. arXiv:2012.14236
- [22] Argyrios Deligkas, John Fearnley, Themistoklis Melissourgos, and Paul G. Spirakis. 2021. Computing Exact Solutions of Consensus Halving and the Borsuk-Ulam Theorem. *J. Comput. Syst. Sci.* 117 (2021), 75–98. <https://doi.org/10.1016/j.jcss.2020.10.006>
- [23] Xiaotie Deng, Qi Qi, and Amin Saberi. 2012. Algorithmic solutions for envy-free cake cutting. *Operations Research* 60, 6 (2012), 1461–1476. <https://doi.org/10.1287/opre.1120.1116>
- [24] Xiaotie Deng, Qi Qi, Amin Saberi, and Jie Zhang. 2011. Discrete fixed points: Models, complexities, and applications. *Mathematics of Operations Research* 36, 4 (2011), 636–652. <https://doi.org/10.1287/moor.1110.0511>
- [25] Kousha Etessami and Mihalis Yannakakis. 2010. On the complexity of Nash equilibria and other fixed points. *SIAM J. Comput.* 39, 6 (2010), 2531–2597. <https://doi.org/10.1137/080720826>
- [26] John Fearnley, Paul W. Goldberg, Alexandros Hollender, and Rahul Savani. 2021. The Complexity of Gradient Descent: CLS = PPAD \cap PLS. In *Proceedings of the 53rd ACM Symposium on Theory of Computing (STOC)*. <https://doi.org/10.1145/3406325.3451052> arXiv:2011.01929

- [27] Aris Filos-Ratsikas, Søren Kristoffer Still Frederiksen, Paul W. Goldberg, and Jie Zhang. 2018. Hardness Results for Consensus-Halving. In *Proceedings of the 43rd International Symposium on Mathematical Foundations of Computer Science (MFCS)*. 24:1–24:16. <https://doi.org/10.4230/LIPIcs.MFCS.2018.24>
- [28] Aris Filos-Ratsikas and Paul W. Goldberg. 2018. Consensus Halving is PPA-complete. In *Proceedings of the 50th Annual ACM Symposium on Theory of Computing (STOC)*. 51–64. <https://doi.org/10.1145/3188745.3188880>
- [29] Aris Filos-Ratsikas and Paul W. Goldberg. 2019. The Complexity of Splitting Necklaces and Bisecting Ham Sandwiches. In *Proceedings of the 51st Annual ACM Symposium on Theory of Computing (STOC)*. 638–649. <https://doi.org/10.1145/3313276.3316334>
- [30] Aris Filos-Ratsikas, Alexandros Hollender, Katerina Sotiraki, and Manolis Zampetakis. 2020. Consensus-Halving: Does it Ever Get Easier?. In *Proceedings of the 21st ACM Conference on Economics and Computation (EC)*. 381–399. <https://doi.org/10.1145/3391403.3399527>
- [31] Aris Filos-Ratsikas, Alexandros Hollender, Katerina Sotiraki, and Manolis Zampetakis. 2021. A Topological Characterization of Modulo- p Arguments and Implications for Necklace Splitting. In *Proceedings of the 32nd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. 2615–2634. <https://doi.org/10.1137/1.9781611976465.155>
- [32] Charles H. Goldberg and Douglas B. West. 1985. Bisection of Circle Colorings. *SIAM Journal on Algebraic Discrete Methods* 6, 1 (1985), 93–106. <https://doi.org/10.1137/0606010>
- [33] Michelangelo Grigni. 2001. A Sperner lemma complete for PPA. *Inform. Process. Lett.* 77, 5–6 (2001), 255–259. [https://doi.org/10.1016/S0020-0190\(00\)00152-6](https://doi.org/10.1016/S0020-0190(00)00152-6)
- [34] Charles R. Hobby and John R. Rice. 1965. A moment problem in L_1 approximation. *Proc. Amer. Math. Soc.* 16, 4 (1965), 665–670. <https://doi.org/10.2307/2033900>
- [35] Roman N. Karasev, Edgardo Roldán-Pensado, and Pablo Soberón. 2016. Measure partitions using hyperplanes with fixed directions. *Israel Journal of Mathematics* 212, 2 (2016), 705–728. <https://doi.org/10.1007/s11856-016-1303-z>
- [36] Ilan Komargodski, Moni Naor, and Eylon Yogev. 2019. White-box vs. black-box complexity of search problems: Ramsey and graph property testing. *J. ACM* 66, 5 (2019), 1–28. <https://doi.org/10.1109/FOCS.2017.63>
- [37] Jiří Matoušek. 2008. *Using the Borsuk-Ulam theorem: lectures on topological methods in combinatorics and geometry*. Springer Science & Business Media. <https://doi.org/10.1007/978-3-540-76649-0>
- [38] Nimrod Megiddo and Christos H. Papadimitriou. 1991. On total functions, existence theorems and computational complexity. *Theoretical Computer Science* 81, 2 (1991), 317–324. [https://doi.org/10.1016/0304-3975\(91\)90200-L](https://doi.org/10.1016/0304-3975(91)90200-L)
- [39] Frédéric Meunier. 2014. Simplotop maps and necklace splitting. *Discrete Mathematics* 323, 28 (2014), 14–26. <https://doi.org/10.1016/j.disc.2014.01.008>
- [40] Jerzy Neyman. 1946. Un théorème d’existence. *CR Acad. Sci. Paris* 222 (1946), 843–845.
- [41] Christos H. Papadimitriou. 1994. On the complexity of the parity argument and other inefficient proofs of existence. *J. Comput. System Sci.* 48, 3 (1994), 498–532. [https://doi.org/10.1016/S0022-0000\(05\)80063-7](https://doi.org/10.1016/S0022-0000(05)80063-7)
- [42] Ariel D. Procaccia. 2016. Cake Cutting Algorithms. In *Handbook of Computational Social Choice*. Cambridge University Press, 311–330.
- [43] Jack M. Robertson and William A. Webb. 1995. Approximating fair division with a limited number of cuts. *Journal of Combinatorial Theory, Series A* 72, 2 (1995), 340–344. [https://doi.org/10.1016/0097-3165\(95\)90073-X](https://doi.org/10.1016/0097-3165(95)90073-X)
- [44] Jack M. Robertson and William A. Webb. 1998. *Cake-cutting algorithms: Be fair if you can*. CRC Press.
- [45] Forest W. Simmons and Francis E. Su. 2003. Consensus-Halving via theorems of Borsuk-Ulam and Tucker. *Mathematical social sciences* 45, 1 (2003), 15–25. [https://doi.org/10.1016/S0165-4896\(02\)00087-2](https://doi.org/10.1016/S0165-4896(02)00087-2)
- [46] Francis E. Su. 1999. Rental harmony: Sperner’s lemma in fair division. *The American Mathematical Monthly* 106, 10 (1999), 930–942. <https://doi.org/10.1080/00029890.1999.12005142>
- [47] Albert W. Tucker. 1945. Some Topological Properties of Disk and Sphere. In *Proceedings of the First Canadian Math. Congress, Montreal*. University of Toronto Press, 286–309.
- [48] Gerhard J. Woeginger and Jiří Sgall. 2007. On the complexity of cake cutting. *Discrete Optimization* 4, 2 (2007), 213–220. <https://doi.org/10.1016/j.disopt.2006.07.003>