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An asymptotic formula for solutions of linear second-order difference equations with regularly behaving coefficients

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An asymptotic formula is given for the solutions of a linear second-order difference equation $x_{n+2} - p_n x_{n+1} + q_n x_n = 0$ which holds in the case that the coefficients exhibit regular behaviour; in particular, the formula holds in the case that the coefficients are series of powers of the index n. The lowest order asymptotic behaviour is given for both x_n and $(x_{n+1} - x_n)/x_n$. Examples are given that show that the conditions on the behaviour of the coefficients cannot be weakened too much. The method presented here, which uses matrix sequences, can be extended to other cases.

Keywords: Linear difference equation; Matrix sequence; Asymptotic behaviour; Asymptotic formula

2000 Mathematics Subject Classifications: 39A11; 40A25

Lately, a number of results have appeared that are concerned with the asymptotic behaviour of the solutions $\{x_n\}$ of a linear second-order difference equation $x_{n+2} - p_n x_{n+1} + q_n x_n = 0$. The first result in the field, which has been known since a long time, is the Poincaré–Perron theorem, which essentially says that if the difference equation has a characteristic polynomial with roots of distinct moduli then for each root α there is a solution $\{x_n\}$ such that x_{n+1}/x_n converges to α (see, e.g. [8]). A matrix version of this result was treated in Ref. [7] and in Refs. [4,5]. In Refs. [4,5] it was proved that the result still holds for a root α if there are no other roots with modulus $|\alpha|$. It is known that the result does not extend in general to the case that the moduli are not all distinct, so that additional conditions must be laid on the behaviour of the coefficients. On the other extreme, the asymptotic behaviour has been studied in the case that the coefficients p_n, q_n of the difference equation converge fast ([1,2,6]). In several papers which have appeared more or less recently special cases that are intermediate between these two extremes have been treated, like in Ref. [3]. Sometimes the emphasis is on the behaviour of the quotients x_{n+1}/x_n, as in the Poincaré–Perron Theorem, sometimes on the behaviour of x_n. In this paper, we use matrix methods which we developed in Ref. [6] in order to give asymptotic formulae for both the solutions x_n and the quotients $(x_{n+1} - x_n)/x_n$ where not too severe conditions are laid on the behaviour of the coefficients. A couple of examples (notably in remark 7 and example 2) show that the conditions cannot be weakened too much. On the other hand, the method works for other cases as well (see remark 7). Our result improves on the results of Janas who studies a similar difference equation in [9].

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We prove the following result:

THEOREM 1. *Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $C_n \neq 1$ for all n, $\lim_{n\to\infty} n^{-a} C_n = C$ for some $a \in \mathbf{R}$ and $C \neq 0$. Then the difference equation*

$$x_{n+2} - 2x_{n+1} + (1 - C_n)x_n = 0 \tag{1}$$

has two linearly independent solutions $\left\{x_n^{(i)}\right\}_{n=1}^{\infty}$ $(i = 1,2)$ such that

1. *If $a > -2$ and $C_{n+1}/C_n = 1 + (a/n) + \epsilon_n$ where $\sum_{n=1}^{\infty} |\epsilon_n|$ converges,*

$$x_n^{(1)} = (1 + o(1))n^{-a/4} \prod_{k=1}^{n-1} (1 + \sqrt{C_k}), x_n^{(2)} = (1 + o(1))n^{-a/4} \prod_{k=1}^{n-1} (1 - \sqrt{C_k}) \;\; (n \to \infty)$$

and

$$x_{n+1}^{(1)} - x_n^{(1)} = (1 + o(1))\sqrt{C_n} n^{-a/4} \prod_{k=1}^{n-1} (1 + \sqrt{C_k}), x_{n+1}^{(2)} - x_n^{(2)}$$

$$= -(1 + o(1))\sqrt{C_n} n^{-a/4} \prod_{k=1}^{n-1} (1 - \sqrt{C_k}),$$

where $\sqrt{C_n}$ is defined such that $\mathrm{Re}\sqrt{C_n} > 0$ if C is not a negative number, and $\sqrt{C_n} = i\sqrt{-C_n}$ if $C < 0$. Further, in the case that $C < 0$ the additional condition that

$$\prod_{k=p}^{q} \left| \frac{1 + i\sqrt{-C_k}}{1 - i\sqrt{-C_k}} \right|$$

is bounded from above for all p, q or bounded from below for all p, q is imposed.

2. *If $a \leq -2$ and $C_n = C/n^2 + \epsilon_n/n$ for some $C \in \mathbf{C}, C \neq -1/4$ and $\sum_n |\epsilon_n|$ converges,*

$$x_n^{(1)} = (1 + o(1))n^{b_1}, \quad x_n^{(2)} = (1 + o(1))n^{b_2} \;\; (n \to \infty)$$

where b_1, b_2 are the (distinct) zeros of $X^2 - X - C$ and

$$x_{n+1}^{(1)} - x_n^{(1)} = \frac{b_1 + o(1)}{n} x_n^{(1)}, \quad x_{n+1}^{(2)} - x_n^{(2)} = \frac{b_2 + o(1)}{n} x_n^{(2)}.$$

3. *If $a = -2$ and $C_n = -1/4n^2 + \epsilon_n/n \log n$ and $\sum_n |\epsilon_n|$ converges,*

$$x_n^{(1)} = (1 + o(1))\sqrt{n}, \quad x_n^{(2)} = (1 + o(1))\sqrt{n} \log n \;(n \to \infty)$$

and

$$x_{n+1}^{(1)} - x_n^{(1)} = \frac{1}{n}\left(\frac{1}{2} + \frac{o(1)}{\log n}\right) x_n^{(1)}, \quad x_{n+1}^{(1)} - x_n^{(1)} = \frac{1}{n}\left(\frac{1}{2} + \frac{1 + o(1)}{\log n}\right) x_n^{(1)}.$$

Remarks.

1. The requirement that $C_n \neq 1$ ensures that the difference equation (1) is not degenerate.
2. A remark on notation: whenever we use the Landau symbols (like $o(1)$,$O(n^c)$) in this paper, it is always implied that $n \to \infty$. Furthermore, for solutions or coefficient sequences of difference equations (like $\{x_n\}_{n=1}^{\infty}$), we leave out the index set and simply write $\{x_n\}$.

3. If C_n is of the form $C_n = Cn^a + A_2 n^{a_2} + \ldots + A_\ell n^{a_\ell} + O(n^{a-1-\epsilon})$ with $C \neq 0, A_2, \ldots, A_\ell$ complex numbers, $a, a_2, \ldots, a_\ell$ real numbers such that $a > a_2 > \cdots > a_\ell$ and $\epsilon > 0$, then the sequence $\{C_n\}$ satisfies the conditions of Theorem 1. However, in order for the additional condition in the case $C < 0$ to hold, the last term must be $O(n^{a/2-1-\epsilon})$ instead of $O(n^{a-1-\epsilon})$ in case $a > 0$.
4. The form of equation (1) seems at first sight rather special, but in fact it is always possible to bring an arbitrary difference equation of the form

$$u_{n+2} - p_n u_{n+1} + q_n u_n = 0$$

with $p_n \neq 0$ into the form (1) with the aid of a simple transformation: if $u_n = x_n \prod_{j=1}^{n-1} r_j$, then

$$x_{n+2} - (p_n/r_{n+1})x_{n+1} + (q_n/r_n r_{n+1})x_n = 0.$$

If we choose $r_n = p_{n-1}/2$, then $\{x_n\}$ satisfies equation (1) with $C_n = 1 - (4q_n/p_n p_{n-1})$.
5. Sometimes the expressions that occur are not well defined for small values of n, i.e. when we say that $C_n = 1/n \log n$. Since it is only the asymptotic behaviour that matters, we can always redefine the values of a sequence for a finite number of indices without altering the asymptotic behaviour of the solutions. In accordance with this practice, we mostly write $\sum_n$ instead of $\sum_{n=N}^{\infty}$ and do not specify what the value of N is. This is unambiguous as long as the exact value of the sum is not needed.

We now proceed to the proof of Theorem 1. We shall make use of matrix methods. Notice that (1) is equivalent with the matrix equation

$$M_n \mathbf{y}_n = \mathbf{y}_{n+1} \tag{2}$$

where

$$M_n = \begin{pmatrix} 1 & C_n \\ 1 & 1 \end{pmatrix}$$

and

$$\mathbf{y}_n = \begin{pmatrix} x_{n+1} - x_n \\ x_n \end{pmatrix}.$$

If M_n are diagonal matrices, or if $M_n = M$ is a constant matrix, then a matrix recurrence of type (2) can be exactly solved. In general, we need approximative methods. The idea behind the proof is to find a well-behaved sequence of matrices $\{H_n\}$ such that $H_{n+1}^{-1} M_n H_n$ are diagonal matrices. If enough is known about the matrices H_n, then conclusions can be drawn on the behaviour of the solutions of (2).

In the proof, we shall construct the matrices H_n as products $G_n F_n$ of two matrices. In order to find G_n and F_n we make use of two lemmas.

Firstly, we use the following result on almost-diagonal matrix sequences:

THEOREM 2. *Let $\{A_n = \mathrm{diag}(a_1(n), a_2(n))\}$ be a sequence of complex-valued diagonal matrices such that for all p, q the products $\prod_{j=p}^{q} |a_1(n)/a_2(n)|$ are bounded either from below or from above. Further, let $\{D_n\}$ be a sequence of matrices such that $A_n + D_n$ is invertible for* *all n and* $\sum_{n=1}^{\infty}\|D_n\|/|a_j(n)|$ *converges for* $j = 1,2$. *Then, there exists a sequence of invertible matrices* $\{F_n\}$ *which converges to the identity matrix I as* $n \to \infty$ *such that*

$$F_{n+1}^{-1}(A_n + D_n)F_n = A_n.$$

Proof. This is a simplified version of Theorem 1.4 in Ref. [6] (which holds for matrices of general size and where an estimate for the convergence of the matrices is given. We shall not use this general form in this paper.) Essentially, the same result is Theorem 7.26 of Ref. [2], where the same condition on the boundedness of the products of the quotients of the diagonal elements is found (described in a slightly different fashion). □

We cannot apply Theorem 2 directly to the matrices

$$M_n = \begin{pmatrix} 1 & C_n \\ 1 & 1 \end{pmatrix}.$$

We first need to find a sequence $\{G_n\}$ such that $G_{n+1}^{-1}M_nG_n$ is almost-diagonal (in the sense of Theorem 2). We use the next lemma to find such a sequence in the case that C_n has the form stated in Theorem 1.

LEMMA 3. *Let* $G_n = \begin{pmatrix} g_n & h_n \\ 1 & 1 \end{pmatrix}$ *for certain numbers* g_n, h_n, $g_n \neq h_n$. $(n \in \mathbf{N})$ and $\mathbf{y}_n = \begin{pmatrix} x_{n+1} - x_n \\ x_n \end{pmatrix}$. Then

a.
$$(G_n^{-1}\mathbf{y}_n)_1 + (G_n^{-1}\mathbf{y}_n)_2 = x_n$$

where $()_i$ *is the ith component of the vector* $()$.

b. *If* $\{g_n\}$, $\{h_n\}$ *are distinct sequences of numbers such that* $g_{n+1} - g_n + g_ng_{n+1} - C_n = O(f_n)$ *(and similarly for* h_n*) where* $\sum_n(|f_{n-1}|)/(|g_n - h_n|)$ *converges, then* $G_{n+1}^{-1}M_nG_n = \begin{pmatrix} 1+g_n & 0 \\ 0 & 1+h_n \end{pmatrix} + D_n$ *where* $\sum_n\|D_n\|$ *converges.*

Proof.

a. This follows immediately from

$$G_n^{-1}\mathbf{y}_n = \frac{1}{g_n - h_n}\begin{pmatrix} x_{n+1} - (1+h_n)x_n \\ -x_{n+1} + (1+g_n)x_n \end{pmatrix}. \tag{3}$$

b. A straightforward calculation gives

$$\begin{aligned}&(g_{n+1} - h_{n+1})G_{n+1}^{-1}M_nG_n \\ &= \begin{pmatrix} g_n + C_n - h_{n+1}(g_n+1) & h_n + C_n - h_{n+1}(h_n+1) \\ -g_n - C_n + g_{n+1}(g_n+1) & -h_n - C_n + g_{n+1}(h_n+1) \end{pmatrix}.\end{aligned} \tag{4}$$

The off-diagonal terms of the matrix are $O(f_n)$ by assumption. The diagonal terms are then $g_n + C_n - h_{n+1}(g_n+1) = (g_{n+1} - h_{n+1})(g_n+1) + O(f_n)$ and

$-h_n - C_n + g_{n+1}(h_n + 1) = (g_{n+1} - h_{n+1})(h_n + 1) + O(f_n)$, respectively. The assertion now follows immediately. □

LEMMA 4. *For* $b \in \mathbf{R}$,

$$\prod_{k=1}^{n-1}\left(1+\frac{b}{k}\right) = (d+o(1))n^b, \quad \prod_{k=2}^{n-1}\left(1+\frac{b}{k}+\frac{c}{k\log k}\right) = (d'+o(1))n^b \log^c n \ \ (n \to \infty)$$

for some real numbers $d, d' \neq 0$ *that depend on b, c.*

Proof. Set

$$P_n = \prod_{k=1}^{n-1}\left(1+\frac{b}{k}\right) = \lambda_n n^b.$$

Then

$$\frac{\lambda_{n+1}}{\lambda_n} = \left(1+\frac{b}{n}\right)\left(\frac{n}{n+1}\right)^b = 1 + O\left(\frac{1}{n^2}\right).$$

Hence, the product

$$\frac{\lambda_n}{\lambda_1} = \prod_{k=1}^{n-1}\frac{\lambda_{k+1}}{\lambda_k}$$

converges absolutely, so that λ_n converges to some real number $d \neq 0$. The second identity follows in a similar fashion. (Notice that

$$\prod_{k=2}^{n-1}\left(1+\frac{b}{k}+\frac{c}{k\log k}\right) = \mu_n \prod_{k=2}^{n-1}\left(1+\frac{b}{k}\right)\left(1+\frac{c}{k\log k}\right) \text{ where } \lim_{n\to\infty}\mu_n = \mu \neq 0.$$

We are now ready to take on the proof of Theorem 1. □

Proof of Theorem 1. We distinguish four cases: (a) $-2 < a \leq 0$; (b) $a \leq -2, n^2 C_n \to C \neq -1/4$; (c) $a = -2, n^2 C_n \to -1/4$; (d) $a > 0$. In cases a–c we use lemma 3b to find G_n such that $G_{n+1}^{-1} M_n G_n$ is in almost-diagonal form. We shall give the argument in detail for case a. Cases b and c are similar to a, only the form of the matrices G_n differs.

a. In order to apply lemma 3b, we look for distinct sequences $\{g_n\}, \{h_n\}$ such that $g_{n+1} - g_n - C_n + g_n g_{n+1} = O(f_n)$ (similarly for h_n) such that $\sum_n(|f_{n-1}|)/(|g_n - h_n|)$ converges. Define $g_n = \sqrt{C_n} - (a/4n)(1+\sqrt{C_n})$ and $h_n = -\sqrt{C_n} - (a/4n)(1-\sqrt{C_n})$. Then for n large enough, $\sqrt{C_{n+1}} = \sqrt{C_n}(1 + a/2n + \epsilon_n)$ where ϵ_n is a generic symbol meaning that $\sum_n |\epsilon_n|$ converges. Notice that the definition of the square root guarantees that $(\sqrt{C_{n+1}}/\sqrt{C_n})$ goes to 1 (and not -1) for large n. Then

$$g_{n+1} = \sqrt{C_n} + (a/4n)\sqrt{C_n} - a/4n + \sqrt{C_n}\epsilon_n + O(1/n^2)$$

so that

$$g_{n+1} - g_n - C_n + g_n g_{n+1} = O(1/n^2) + \sqrt{C_n}\epsilon_n$$

and similarly for h_n. Then $g_n - h_n \sim 2\sqrt{C_n}$ and since both $\sum_n 1/|n^2\sqrt{C_n}|$ and $\sum_n |\epsilon_n|$ converge, we have, by lemma 3, that

$$\begin{aligned} G_{n+1}^{-1} M_n G_n &= \begin{pmatrix} 1+g_n & 0 \\ 0 & 1+h_n \end{pmatrix} + D_n \\ &= \begin{pmatrix} (1+\sqrt{C_n})(1-a/4n) & 0 \\ 0 & (1-\sqrt{C_n})(1-a/4n) \end{pmatrix} + D_n \end{aligned}$$

where $\sum_n \|D_n\|$ converges. By our definition of the square root, in the case that C is not negative, $|1+g_n|/|1+h_n| \geq 1$ for all n large enough so that the products $\prod_{n=p}^{q} |1+g_n|/|1+h_n| \geq 1$ are indeed bounded from below for all p, q, whereas in the case that C is negative, this condition has to be imposed separately. We can then apply lemma 2 to the sequence $G_{n+1}^{-1} M_n G_n$ and find that there exists a sequence $\{F_n\}$ of invertible matrices such that $F_n \to I$ as $n \to \infty$ and

$$\begin{aligned} F_{n+1}^{-1} G_{n+1}^{-1} M_n G_n F_n &= \begin{pmatrix} 1+g_n & 0 \\ 0 & 1+h_n \end{pmatrix} \\ &= \begin{pmatrix} (1+\sqrt{C_n})(1-a/4n) & 0 \\ 0 & (1-\sqrt{C_n})(1-a/4n) \end{pmatrix}. \end{aligned} \tag{5}$$

Now $F_{n+1}^{-1} G_{n+1}^{-1} M_n G_n F_n \mathbf{z}_n^{(i)} = \mathbf{z}_{n+1}^{(i)}$ where

$$\mathbf{z}_n^{(i)} = \prod_{k=1}^{n-1} (1 + (-1)^{i-1}\sqrt{C_k})(1 - a/4k)\mathbf{e}_i \quad (i = 1, 2),$$

with

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, there exist linearly independent solutions $\{\mathbf{y}_n^{(1)}\}, \{\mathbf{y}_n^{(2)}\}$ of (2) such that

$$\begin{aligned} G_n^{-1}\mathbf{y}_n^{(i)} = F_n \mathbf{z}_n^{(i)} &= \prod_{k=1}^{n-1} (1 + (-1)^{i-1}\sqrt{C_k})(1 - a/4k)(\mathbf{e}_i + o(1)) \\ &= b_n n^{-a/4} \prod_{k=1}^{n-1} (1 + (-1)^{i-1}\sqrt{C_k})(\mathbf{e}_i + o(1)) \end{aligned} \tag{6}$$

for $b_n \to b \neq 0$ (where we have used lemma 4). On the other hand, by lemma 3a we have that

$$x_n^{(i)} = \left(G_n^{-1}\mathbf{y}_n^{(i)}\right)_1 + \left(G_n^{-1}\mathbf{y}_n^{(i)}\right)_2 = b n^{-a/4} \prod_{k=1}^{n-1} (1 + (-1)^{i-1}\sqrt{C_k})(1 + o(1))$$

($i = 1,2$) are linearly independent solutions of (1). Finally, equation (3) shows that

$$\frac{\left(G_n^{-1}\mathbf{y}_n\right)_1}{\left(G_n^{-1}\mathbf{y}_n\right)_2} = \frac{\frac{x_{n+1}-x_n}{x_n} - h_n}{-\frac{x_{n+1}-x_n}{x_n} + g_n} \tag{7}$$

or, filling in the values for g_n,h_n

$$\frac{\left(G_n^{-1}\mathbf{y}_n\right)_1}{\left(G_n^{-1}\mathbf{y}_n\right)_2} = \frac{\frac{1}{\sqrt{C_n}}\frac{x_{n+1}-x_n}{x_n} + 1 + O\left(\frac{1}{n\sqrt{C_n}}\right)}{-\frac{1}{\sqrt{C_n}}\frac{x_{n+1}-x_n}{x_n} + 1 + O\left(\frac{1}{n\sqrt{C_n}}\right)}.$$

For $x_n = x_n^{(1)}$ and $x_n = x_n^{(2)}$ it follows that

$$x_{n+1}^{(i)} - x_n^{(i)} = (-1)^{i-1}\sqrt{C_n}x_n^{(i)}(1 + o(1)). \quad (i = 1, 2).$$

b. The case that $\lim_{n\to\infty} n^2 C_n = C$, $C \neq -1/4$ goes in a similar fashion. For g_n, h_n we now take $g_n = d_1/n$ and $h_n = d_2/n$, where d_1, d_2 are the zeros of $X^2 - X - C$. Then $g_{n+1} - g_n - C_n + g_n g_{n+1} = O(1/n)\epsilon_n$ (and similarly for h_n) where, as above, $\sum_n|\epsilon_n|$ converges. Analogously to (6), we have linearly independent solutions $\left\{\mathbf{y}_n^{(1)}\right\}$, $\left\{\mathbf{y}_n^{(2)}\right\}$ of (2) such that

$$G_n^{-1}\mathbf{y}_n^{(i)} = \prod_{k=1}^{n-1}(1 + d_i/n)(\mathbf{e}_i + o(1)) = b_n^{(i)} n^{d_i}(\mathbf{e}_i + o(1))$$

for $b_n^{(i)} \to b^{(i)} \neq 0$, and by lemma 4 there are solutions $\left\{x_n^{(i)}\right\}$ of (1) such that

$$x_n^{(i)} = \left(G_n^{-1}\mathbf{y}_n^{(i)}\right)_1 + \left(G_n^{-1}\mathbf{y}_n^{(i)}\right)_2 = b^{(i)} n^{d_i}(1 + o(1)) \ \ (i = 1, 2).$$

Of course, we may choose rescale $x_n^{(i)}$ such that $b^{(i)} = 1$.

Furthermore, by (7) we see that

$$\frac{\left(G_n^{-1}\mathbf{y}_n\right)_1}{\left(G_n^{-1}\mathbf{y}_n\right)_2} = \frac{n\frac{x_{n+1}-x_n}{x_n} - d_2}{-n\frac{x_{n+1}-x_n}{x_n} + d_1},$$

so that

$$x_{n+1}^{(i)} - x_n^{(i)} = \frac{d_i + o(1)}{n}x_n^{(i)}.$$

c. In the case that $\lim_{n\to\infty} n^2 C_n = C = -1/4$, the zeros d_1 and d_2 of $X^2 - X - C$ coincide, so that the sequences $\{g_n\}$ and $\{h_n\}$ of (b) are equal. We take $g_n = (1/2n)$ as in case b and define $h_n = (1/2n) + (1/n\log n)$. As in b, we have that $g_{n+1} - g_n - C_n + g_n g_{n+1} = O(1/n\log n)\epsilon_n$ (and similarly for h_n) with $\sum_n|\epsilon_n| < \infty$. Thus, by lemma 4 we have linearly independent solutions $\left\{\mathbf{y}_n^{(1)}\right\}$, $\left\{\mathbf{y}_n^{(2)}\right\}$ of (2) such that

$$G_n^{-1}\mathbf{y}_n^{(1)} = \prod_{k=1}^{n-1}(1 + 1/2k)(\mathbf{e}_1 + o(1)) = b_n^{(1)}\sqrt{n}(\mathbf{e}_1 + o(1)),$$

$$G_n^{-1}\mathbf{y}_n^{(2)} = \prod_{k=1}^{n-1}(1 + 1/2k + 1/k\log k)(\mathbf{e}_1 + o(1)) = b_n^{(2)}\sqrt{n}\,\log n(\mathbf{e}_1 + o(1)),$$

where $b_n^{(i)} \to b^{(i)} \neq 0$, and correspondingly solutions $\{x_n^{(i)}\}$ of (1) such that

$$x_n^{(1)} = (1 + o(1))\sqrt{n}, \quad x_n^{(2)} = (1 + o(1))\sqrt{n}\log n.$$

Finally, by (7), we have that

$$\frac{\left(G_n^{-1}\mathbf{y}_n\right)_1}{\left(G_n^{-1}\mathbf{y}_n\right)_2} = \frac{\log n\left(n\frac{x_{n+1}-x_n}{x_n} - 1/2\right) - 1}{\log n\left(-n\frac{x_{n+1}-x_n}{x_n} + 1/2\right)},$$

so that

$$x_{n+1}^{(1)} - x_n^{(1)} = \left(\frac{1}{2n} + \frac{o(1)}{n\log n}\right)x_n^{(1)}, \quad x_{n+1}^{(2)} - x_n^{(2)} = \left(\frac{1}{2n} + \frac{1+o(1)}{n\log n}\right)x_n^{(2)}.$$

d. In the case $a > 0$, we need two steps to bring M_n in almost-diagonal form. As above, we let $G_n = \begin{pmatrix} g_n & h_n \\ 1 & 1 \end{pmatrix}$ but we now choose $g_n = +\sqrt{C_n}, h_n = -\sqrt{C_n}$. Then by (4)

$$G_{n+1}^{-1}M_nG_n = \left(\frac{1}{2} + \frac{1}{2}\sqrt{C_n/C_{n+1}}\right)(\sqrt{C_n}+1)B_n = \left(1 - \frac{a}{4n} + \epsilon_n\right)(\sqrt{C_n}+1)B_n,$$

with

$$B_n = \begin{pmatrix} 1 & \beta_n\alpha_n \\ \beta_n & \alpha_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_n \end{pmatrix} + \frac{a}{4n}B + E_n$$

where

$$\alpha_n = \frac{1-\sqrt{C_n}}{1+\sqrt{C_n}}, \quad \beta_n = \frac{\sqrt{C_{n+1}/C_n} - 1}{\sqrt{C_{n+1}/C_n} + 1} = \frac{a}{4n} + \epsilon_n, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $\sum_n|\epsilon_n|$ and $\sum_n\|E_n\|$ converge. The matrices B_n are not in almost-diagonal form and adding a term $(a/4n)(1 \pm \sqrt{C_n})$ to g_n and h_n, as we did in case a, won't help us now. However, if we let $H_n = I + (a/8n)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then, since

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + O\left(\frac{1}{\sqrt{C_n}}\right)$$

and $\sum_n|1/n\sqrt{C_n}|$ converges, we have

$$H_{n+1}^{-1}B_nH_n = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_n \end{pmatrix} + D_n'$$

where $\sum_n\|D_n'\|$ converges. We may then apply lemma 2 to the almost-diagonal sequence $\left\{H_{n+1}^{-1}B_nH_n\right\}$ which shows that there exist a sequence of invertible matrices F_n', converging to I, such that

$$F_{n+1}^{-1}B_nF_n = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_n \end{pmatrix}$$

where we have written $F_n = H_nF_n'$. Notice that $F_n \to I$. Hence,

$$F_{n+1}^{-1}G_{n+1}^{-1}M_nG_nF_n = \left(1 - \frac{a}{4n} + \epsilon_n\right)\begin{pmatrix} 1+\sqrt{C_n} & 0 \\ 0 & 1-\sqrt{C_n} \end{pmatrix}.$$

Hence, (2) has linearly independent solutions $\left\{\mathbf{y}_n^{(i)}\right\}$ $(i = 1, 2)$ such that

$$G_n^{-1}\mathbf{y}_n^{(i)} = (1+o(1))n^{-a/4}\prod_{k=1}^{n-1}(1+(-1)^{i-1}\sqrt{C_k})F_n\mathbf{e}_i.$$

The remainder of the argument goes exactly as in case a. □

We give an example.

Example 1. If the coefficients p_n, q_n of the difference equation $x_{n+2} - p_n x_{n+1} + q_n x_n$ are rational functions of n, and $p_n \neq 0$ then with the aid of the transformation mentioned in remark 4, the difference equation can be brought into the form

$$x_{n+2} - 2x_{n+1} + (1 - Cn^a - An^{a-1} + O(n^{a-2}))x_n = 0 \tag{8}$$

with $a \in \mathbf{Z}$. All cases fall within the realm of Theorem 1. $a \leq -2$ corresponds to Theorem 1 (2,3). If $a > -2$, $C \neq 0$, then $C_{n+1}/C_n = 1 + a/n + O(1/n^2)$. If $a = -1$, then $\sqrt{C_n} = \sqrt{C}/\sqrt{n} + O(1/n\sqrt{n})$ where $\sqrt{C} = i\sqrt{-C}$ if C is real and negative. In the latter case, $(|1 - \sqrt{C_n}|)/(|1 + \sqrt{C_n}|) = 1 + O(1/n\sqrt{n})$ so that

$$\prod_{n=p}^{q} \left| \frac{1 - \sqrt{C_n}}{1 + \sqrt{C_n}} \right|$$

is bounded both from above and from below. By theorem 1, equation (8) has a basis of solutions

$$x_n^{(1)} = (1 + o(1))n^{1/4} \prod_{k=1}^{n-1} \left(1 + \frac{\sqrt{C}}{\sqrt{k}}\right), \quad x_n^{(2)} = (1 + o(1))n^{1/4} \prod_{k=1}^{n-1} \left(1 - \frac{\sqrt{C}}{\sqrt{k}}\right).$$

Expanding the products we get (in a similar fashion as in lemma 4)

$$x_n^{(1)} = b_1(1 + o(1))n^{1/4 - C/2} \cdot \mathrm{e}^{2\sqrt{Cn}}, \quad x_n^{(2)} = b_2(1 + o(1))n^{1/4 - C/2} \cdot \mathrm{e}^{-2\sqrt{Cn}}$$

where $b_i \neq 0$ $(i = 1, 2)$. If $a = 0$, then $\sqrt{C_n} = \sqrt{C} + A/(2\sqrt{C}n) + O(1/n^2)$ and equation (8) has a basis of solutions

$$x_n^+ = (1 + o(1)) \prod_{k=1}^{n-1} (1 + \sqrt{C}) \left(1 + \frac{A}{2(\sqrt{C} + C)k}\right),$$

$$x_n^- = (1 + o(1)) \prod_{k=1}^{n-1} (1 - \sqrt{C}) \left(1 + \frac{A}{2(-\sqrt{C} + C)k}\right)$$

which we can write as

$$x_n^+ = b_1(1 + o(1))(1 + \sqrt{C})^n n^{A/(2\sqrt{C}+2C)}, \quad x_n^- = b_2(1 + o(1))(1 - \sqrt{C})^n n^{A/(-2\sqrt{C}+2C)}$$

where $b_1, b_2 \neq 0$. Notice that if $C < 0$, then $\sum_k \mathrm{Im}\sqrt{-C_k}$ goes as $\sum_k \mathrm{Im}(A)/(2\sqrt{-C}k)$ which clearly is bounded either from below or from above, so that theorem 1 applies also in this case (compare remark 8). If $a = 1$, then $\sqrt{C_n} = \sqrt{Cn} + A/(2\sqrt{Cn}) + O(1/(n\sqrt{n}))$ and equation (8) has a basis of solutions

$$x_n^{\pm} = (1 + o(1))n^{-1/4} \prod_{k=1}^{n-1} (\pm\sqrt{Ck}) \left(1 + \frac{A}{2Ck} \pm \frac{1}{\sqrt{Ck}}\right),$$

which can be written as

$$x_n^+ = b_1(1 + o(1))n^{-1/4 + (A-1)/(2C)} \sqrt{(n-1)!}(\sqrt{C})^n \mathrm{e}^{2\sqrt{n/C}},$$

$$x_n^- = b_2(1 + o(1))n^{-1/4 + (A-1)/(2C)} \sqrt{(n-1)!}(-\sqrt{C})^n \mathrm{e}^{-2\sqrt{n/C}}$$

with $b_1, b_2 \neq 0$. If $a = 2$, then $\sqrt{C_n} = \sqrt{C}n + A/(2\sqrt{C}) + O(1/n)$ and equation (8) has a basis of solutions

$$x_n^{\pm} = (1 + o(1))n^{-1/2}\prod_{k=1}^{n-1}(\pm k\sqrt{C})\left(1 + \frac{A}{2kC} \pm \frac{1}{k\sqrt{C}}\right)$$

which can be written as

$$x_n^{+} = b_1(1 + o(1))(n-1)!(\sqrt{C})^n n^{1/\sqrt{C}+A/(2C)-1/2},$$

$$x_n^{-} = b_2(1 + o(1))(n-1)!(-\sqrt{C})^n n^{-1/\sqrt{C}+A/(2C)-1/2}$$

where $b_1, b_2 \neq 0$. Finally, if $a > 2$, then $\sqrt{C_n} = n^{a/2}\sqrt{C}(1 + A/(2nC) + O(1/n^2))$ so that equation (8) has a basis of solutions

$$x_n^{\pm} = (1 + o(1))n^{-a/4}\prod_{k=1}^{n-1}(\pm k^{a/2}\sqrt{C})\left(1 + \frac{A}{2kC}\right)$$

which we can write as

$$x_n^{+} = b_1(1 + o(1))(n-1)!^{a/2}(\sqrt{C})^n n^{-a/4+A/(2C)},$$

$$x_n^{+} = b_2(1 + o(1))(n-1)!^{a/2}(-\sqrt{C})^n n^{-a/4+A/(2C)}$$

where $b_1, b_2 \neq 0$.

Remarks.

6. The case that $a \leq -2$ has already been treated in Theorem 10.1 of Ref. [6]. The cases that $a = 0, C_n = C + \epsilon_n$ with $C \neq 0, 1$ and $\sum_n |\epsilon_n| < \infty$, as well as the case that $a \leq -2, C = 0$ occur in Ref. [1].
7. The method employed in the proof of Theorem 1 can of course be applied to other sequences $\{C_n\}$ than those mentioned in the statement of the theorem. E.g. if $C_n = d/n^2 + e/n^2 \log n$ with $d \neq -1/4$, then one can show that a basis of solutions of (1) is now given by $x_n^{(i)} = (1 + o(1))n^{d_i}\log^{b_i} n$ where d_1, d_2 are the zeros of $X^2 - X - d$ and $b_i = e/(2d_i - 1)$ $(i = 1, 2)$ (take $g_n = d_1/n + b_1/n \log n,\ h_n = d_2/n + b_2/n \log n$). Notice that $C_{n+1}/C_n = 1 - (2/n) - ((e/d)/n \log^2 n) + \ldots$ (where the dots stand for terms of higher order). We see that although $\sum_n (1/n \log^2 n)$ converges, the leading term in the asymptotic behaviour of the solution does depend on e as well, as distinct from the case that $a > -2$ where the summable terms ϵ_n in C_{n+1}/C_n have no bearing on the asymptotic behaviour to lowest order.
8. In the case that $n^{-a}C_n \to C$ with $C < 0$, $a > -2$, we had to impose the additional condition that the products $\prod_{k=p}^{q}(|1 + i\sqrt{-C_k}|)/(|1 - i\sqrt{-C_k}|)$ are bounded from above for all p, q or bounded from below for all p, q. This is equivalent to the condition that $\sum_{n=p}^{q} \mathrm{Im}\sqrt{-C_n}$ is bounded from above or from below for all p, q.

   One may wonder if this condition is just an artefact of the method or that it (or some similar condition) is really needed for this case. Below we give an example that shows that indeed it cannot simply be omitted. Notice that the case that $C < 0$ is in more respects different from the case that C is not a negative real number; for $C_n \in \mathbf{R}$, then all real solutions $\{x_n\}$ of (1) oscillate (i.e. $x_n x_{n+1} \leq 0$ for infinitely many n) for $C < 0$ but not for $C > 0$ (see [6]). On the other hand, for C_n real, the factors $(|1 + i\sqrt{-C_k}|)/(|1 - i\sqrt{-C_k}|)$ are all equal to 1, so that the condition is automatically satisfied.

Here follows the example:

Example 2. Let C be a negative real number, and let $r_1, r_2, \ldots$ be a sequence of numbers such that

$$\left(1 + i\sqrt{-C} - r_N\right)/\left(1 - i\sqrt{-C} + r_N\right) = \zeta_N N^{1/N^2}$$

where ζ_N is the N^2th root of unity that is closest to $(1 + i\sqrt{-C})/(1 - i\sqrt{-C})$ (if there are two possibilities, we choose any of the two). If we write for simplicity $z = 1 + i\sqrt{-C}$, then $\bar{z}\zeta_N - z = O(1/N^2)$ and

$$-r_N = \frac{\zeta_N \bar{z} N^{1/N^2} - z}{1 + \zeta_N N^{1/N^2}} = \frac{z\bar{z}}{z + \bar{z}}(N^{1/N^2} - 1) + O(1/N^2) = \frac{1 - C}{2}\frac{\log N}{N^2} + O\left(\frac{1}{N^2}\right). \quad (9)$$

In particular, $\sum_N |r_N|$ converges. Furthermore, we define a sequence of integers $\{P_n\}$ by $P_{2N} = (1/3)N(N+1)(2N+1)$ and $P_{2N-1} = P_{2N} - N^2$. Notice that $P_{2N-2} = P_{2N} - 2N^2$. We now define the numbers C_n for $n \geq 1$ in the following manner: $\sqrt{C_n} = i\sqrt{-C} - r_N$ if $P_{2N-2} < n \leq P_{2N-1}$ and $\sqrt{C_n} = i\sqrt{-C} + \overline{r_N}$ if $P_{2N-1} < n \leq P_{2N}$. Then $C_{n+1}/C_n = 1 + \epsilon_n$ with $\sum_n |\epsilon_n| < \infty$. If we set (as in case d of the proof of Theorem 1) $\alpha_n = (1 - \sqrt{C_n})/(1 + \sqrt{C_n})$, then

$$\Lambda_{2N} := \prod_{k=P_{2N-1}+1}^{P_{2N}} \alpha_k = \frac{1}{N} \quad \text{and} \quad \Lambda_{2N-1} := \prod_{k=P_{2N-2}+1}^{P_{2N-1}} \alpha_k = N,$$

so that C_n satisfies the conditions of Theorem 1 (with $a = 0, C < 0$) except for the additional condition.

Now, as in the proof of Theorem 1, we set $G_n = \begin{pmatrix} \sqrt{C_n} & -\sqrt{C_n} \\ 1 & 1 \end{pmatrix}$. Then, by (4), we have

$$G_{n+1}^{-1}\begin{pmatrix} 1 & C_n \\ 1 & 1 \end{pmatrix} G_n = (1 + \sqrt{C_n})\left(\begin{pmatrix} 1 & 0 \\ 0 & \alpha_n \end{pmatrix} + \begin{pmatrix} -\rho_n & \rho_n\alpha_n \\ \rho_n & -\rho_n\alpha_n \end{pmatrix}\right) =$$

which we write as

$$G_{n+1}^{-1}\begin{pmatrix} 1 & C_n \\ 1 & 1 \end{pmatrix} G_n = (1 + \sqrt{C_n})(\Delta_n + R_n)$$

where $\Delta_n = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_n \end{pmatrix}$ and $\rho_n = (\sqrt{C_{n+1}} - \sqrt{C_n})/2\sqrt{C_{n+1}}$. Then $\rho_k = 0$ if k is not equal to some P_N and $\rho_{P_{2N-1}} = (r_{\mathrm{N}} + \overline{r_N})/(2\sqrt{C_{P_{2N}}})$, $\rho_{P_{2N}} = (-\overline{r_N} - r_{N+1})/(2\sqrt{C_{P_{2N+1}}})$, so that, by (9),

$$2i\sqrt{-C}\rho_{P_m} = (1 - C)(-1)^{m-1}\frac{\log m}{m^2} + O\left(\frac{1}{m^2}\right). \quad (10)$$

Hence, $R_n = 0$ unless $n = P_N$ for some N and $\sum_n \|R_n\|$ converges.

We now show that there does not exist a sequence of invertible matrices $\{F_n\}$ such that $F_n \to I$ and

$$F_{n+1}^{-1}G_{n+1}^{-1}\begin{pmatrix} 1 & C_n \\ 1 & 1 \end{pmatrix} G_n F_n = \begin{pmatrix} 1+\sqrt{C_n} & 0 \\ 0 & 1-\sqrt{C_n} \end{pmatrix}$$

Notice that this implies that there is not a basis of solutions $\{x_n^{(i)}\}$ of (1) such that

$$x_n^{(i)} = (1+o(1))\prod_{k=1}^{n-1}(1+(-1)^{i-1}\sqrt{C_k})$$

and

$$x_{n+1}^{(i)} - x_n^{(i)} = (1+o(1))(-1)^{i-1}\sqrt{C_n}\prod_{k=1}^{n-1}(1+(-1)^{i-1}\sqrt{C_k}) \;\; (i=1,2).$$

Namely, if there were such a basis then, by (3) and (7) with $g_n = \sqrt{C_n} = -h_n$, we can see that there would be sequences of vectors $\left\{G_n^{-1}\mathbf{y}_n^{(i)}\right\}$ $(i=1,2)$ such that

$$\begin{pmatrix} 1 & C_n \\ 1 & 1 \end{pmatrix}\mathbf{y}_n^{(i)} = \mathbf{y}_{n+1}^{(i)}$$

and

$$\left(G_n^{-1}\mathbf{y}_n^{(i)}\right)_1 + \left(G_n^{-1}\mathbf{y}_n^{(i)}\right)_2 = (1+o(1))\prod_{k=1}^{n-1}(1+(-1)^{i-1}\sqrt{C_k}),$$

$$\left(G_n^{-1}\mathbf{y}_n^{(i)}\right)_1 / \left(G_n^{-1}\mathbf{y}_n^{(i)}\right)_2 \;\to \infty \;\text{ or }\; 0$$

for $i = 1,2$ respectively, so that there would exist a sequence of matrices $\{F_n\}$, converging to I with

$$G_n^{-1}\mathbf{y}_n^{(i)} = F_n\mathbf{e}_i\prod_{k=1}^{n-1}(1+(-1)^{i-1}\sqrt{C_k}).$$

So let us now suppose that such a sequence $\{F_n\}$ does indeed exist. Then

$$F_{n+1}^{-1}(\Delta_n + R_n)F_n = \Delta_n \tag{11}$$

for all n. Set $F_n\mathbf{e}_1 = \mathbf{y}_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}$. Thus $\lim_{n\to\infty} y_n = 1$ and $\lim_{n\to\infty} z_n = 0$. By (11), we have

$$\mathbf{y}_{n+1} = (\Delta_n + R_n)\mathbf{y}_n$$

and, in particular, $z_{n+1} = \alpha_n z_n + (\rho_n y_n - \rho_n\alpha_n z_n)$, so that

$$z_n = \alpha_{n-1}\cdot\ldots\cdot\alpha_1\left(z_1 + \sum_{k=1}^{n-1}(\alpha_k\cdot\ldots\cdot\alpha_1)^{-1}(\rho_k y_k - \rho_k\alpha_k z_k)\right).$$

In particular,

$$z_{P_N+1} = \Lambda_N\cdot\ldots\cdot\Lambda_1\left(z_1 + \sum_{k=1}^{N}(\Lambda_k\cdot\ldots\cdot\Lambda_1)^{-1}(\rho_{P_k}y_{P_k} - \rho_{P_k}\alpha_{P_k}z_{P_k})\right) \tag{12}$$

and

$$\Lambda_k \cdot \ldots \cdot \Lambda_1 = \begin{cases} (k+1)/2 & \text{for } k \text{ odd} \\ 1 & \text{for } k \text{ even} \end{cases}$$

Taking imaginary parts in (12) and using (10), together with the assumption that $z_n \longrightarrow 0$, $y_n \longrightarrow 1$, it is clear that z_n cannot converge to zero (for any choice of z_1). So we have arrived at a contradiction. This concludes example 2.

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