LINEAR EQUATIONS FOR UNORDERED DATA VECTORS.

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ABSTRACT. Following a recently considered generalisation of linear equations to unordered-data vectors and to ordered-data vectors, we perform a further generalisation to k-element-sets-of-unordered-data vectors. These generalised equations naturally appear in the analysis of vector addition systems (or Petri nets) extended so that each token carries a set of unordered data. We show that nonnegative-integer solvability of linear equations is in nondeterministic-exponential-time while integer solvability is in polynomial-time.

1. Introduction.

Diophantine linear equations. The solvability problem for systems of linear Diophantine equations is defined as follows: given a finite input set of d-dimensional integer vectors $\mathcal{I} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\} \subseteq \mathbb{Z}^d$, and a target vector $\mathbf{v} \in \mathbb{Z}^d$, we ask if there is a solution (n_1, \dots, n_m) such that

$$n_1 \cdot \mathbf{v}_1 + \ldots + n_m \cdot \mathbf{v}_m = \mathbf{v}. \tag{1.1}$$

Restricting solutions n_1, \ldots, n_m to integers \mathbb{Z} or nonnegative integers \mathbb{N} , we may speak of \mathbb{Z} -solvability or \mathbb{N} -solvability, respectively. The former problem is in \mathbf{P} - \mathbf{Time} , while the latter is equivalent to integer linear programming, a well-known \mathbf{NP} – complete problem [Kar72].

This paper is a continuation of a line of research that investigates generalisations of the solvability problem to data vectors [HLT17, HL18], i.e., on a high level of abstraction, to vectors indexed by orbit-finite sets instead of finite ones (for an introduction to orbit-finite sets, also known as sets with atoms, see [BKL11, BKLT13]). In the simplest setting, given a fixed countable infinite set \mathcal{D} of data values, a data vector is a function $\mathbf{a} : \mathcal{D} \to \mathbb{Z}^d$. Addition and scalar multiplication are defined pointwise i.e. $\mathbf{a} + \mathbf{a}'(\alpha) = \mathbf{a}(\alpha) + \mathbf{a}'(\alpha)$ for every $\alpha \in \mathcal{D}$. In the presence of data, the solvability problems are defined analogously, with the important difference that the input set \mathcal{I} of data vectors is orbit-finite, i.e., is the closure, under data

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permutations/data renaming, of a finite set of data vectors. Given such an orbit-finite set \mathcal{I} , and a target data vector \mathbf{v} , we ask if there are data vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathcal{I}$ and numbers n_1, \dots, n_m , such that the equality (1.1) is satisfied.

Example 1.1. We consider data vectors $\mathcal{D} \to \mathbb{Z}$. For every two different data values $\delta, \varepsilon \in \mathcal{D}$, let $\mathbf{v}_{\delta\varepsilon}(x) = 1$ if $x = \delta$ or $x = \varepsilon$, and $\mathbf{v}_{\delta\varepsilon}(x) = 0$ otherwise. Let $\mathcal{I} = \{\mathbf{v}_{\delta\varepsilon} \colon \mathcal{D} \to \mathbb{Z} \mid \delta, \varepsilon \in \mathcal{D}, \delta \neq \varepsilon\}$. The set \mathcal{I} is closed under data permutation. Indeed, for any data permutation (bijection) $\pi \colon \mathcal{D} \to \mathcal{D}$ and any $\mathbf{v}_{\delta\varepsilon} \in \mathcal{I}$ we have $\mathbf{v}_{\delta\varepsilon} \circ \pi = \mathbf{v}_{\pi^{-1}(\delta)\pi^{-1}(\varepsilon)}$ which is also an element of \mathcal{I} .

Finally, let $\mathbf{v}(\beta) = 2$ and $\mathbf{v}(x) = 0$ for $x \neq \beta$, where $\beta \in \mathcal{D}$ is some fixed data value. On the one hand side, this instance admits a \mathbb{Z} -solution, since \mathbf{a} is presentable as

$$\mathbf{v} = \mathbf{v}_{\beta\delta} + \mathbf{v}_{\beta\varepsilon} - \mathbf{v}_{\delta\varepsilon},$$

for any two different data values δ, ε different than β . On the other hand side, there is no \mathbb{N} -solution, as there is no similar presentation of \mathbf{v} in terms of data vectors $\mathbf{v}_{\delta\varepsilon}$ that use nonnegative coefficients. Simply, every vector that is a sum of of data vectors from the family \mathcal{I} must be strictly positive for at least two data values. Furthermore, if $\mathbf{v}(\beta) = 3$ instead of 2, then there is no \mathbb{Z} -solution, too. Indeed, notice that $\sum_{\alpha \in \mathcal{D}} \mathbf{w}(\alpha)$ is always an even number if \mathbf{w} is a sum of data vectors from the family \mathcal{I} . \square

The above simple example is covered by the theory developed in [HLT17]. Here we extend the previous result to data vectors in $\mathcal{D}^{(k)} \to \mathbb{Z}^d$, where $\mathcal{D}^{(k)}$ stands for the k elements subsets of \mathcal{D} . The complexity of analysis of data vectors in $\mathcal{D}^{(k)} \to \mathbb{Z}^d$ can already be observed for k = 2 and d = 1.

Example 1.2. Consider data vectors of a form $\mathcal{D}^{(2)} \to \mathbb{Z}$ where $\mathcal{D}^{(2)} = \{\{\delta, \varepsilon\} \colon \delta, \varepsilon \in \mathcal{D}, \ \delta \neq \varepsilon\}$. You can think of them as weighted graphs with vertices labelled with elements of \mathcal{D} . Suppose the set \mathcal{I} is a set of triangles with weights of all edges equal 1, i.e.

$$\mathbf{v}_{\gamma\delta\varepsilon}(x) = \begin{cases} 1 & \text{if } x \in \{\{\gamma, \delta\}, \{\delta, \varepsilon\}, \{\varepsilon, \gamma\}\} \\ 0 & \text{otherwise} \end{cases}$$

and $\mathcal{I} = \{ \mathbf{v}_{\gamma \delta \varepsilon} \colon \mathcal{D}^{(2)} \to \mathbb{Z} \mid \gamma, \delta, \varepsilon \in \mathcal{D}, \delta \neq \varepsilon \neq \gamma \neq \delta \}$ (see Figure 1).

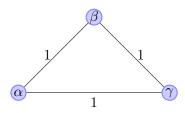


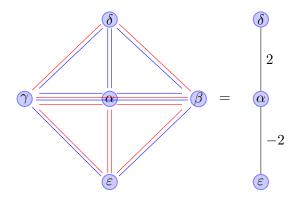
Figure 1: Graph representing the data vector $\mathbf{v}_{\alpha\beta\gamma}$.

The set \mathcal{I} is closed under data permutations, indeed for any data permutation π and any $\mathbf{v}_{\gamma\delta\varepsilon} \in \mathcal{I}$ the data vector $\mathbf{v}_{\pi^{-1}(\gamma)\pi^{-1}(\delta)\pi^{-1}(\varepsilon)} \in \mathcal{I}$. Finally, we want to know if \mathcal{I} and the following target data vector $\mathbf{v}_{\gamma\delta}$ admits a \mathbb{Z} -solution

$$\mathbf{v}_{\gamma\delta}(x) = \begin{cases} 6 & \text{if } x = \{\gamma, \delta\} \\ 0 & \text{otherwise} \end{cases}$$

 $(v_{\gamma\delta})$ is a single edge with weight 6). The answer is yes, but it is not trivial,

$$\mathbf{v}_{\gamma\delta} = (\mathbf{v}_{\beta\delta\gamma} - \mathbf{v}_{\beta\delta\alpha} + \mathbf{v}_{\delta\gamma\varepsilon} - \mathbf{v}_{\delta\varepsilon\alpha} + \mathbf{v}_{\beta\alpha\varepsilon} - \mathbf{v}_{\beta\gamma\varepsilon}) + (\mathbf{v}_{\delta\beta\gamma} - \mathbf{v}_{\gamma\beta\alpha} + \mathbf{v}_{\delta\varepsilon\gamma} - \mathbf{v}_{\gamma\varepsilon\alpha} + \mathbf{v}_{\varepsilon\beta\alpha} - \mathbf{v}_{\varepsilon\beta\delta}) + 2\mathbf{V}_{\alpha\delta\gamma}$$
The idea behind the above sum is presented in Figure 2.



First we construct a gadget $\mathbf{g}_{\delta\alpha\varepsilon}$ as presented on the left. Blue color denotes addition and red subtraction of an edge. $\mathbf{g}_{\alpha\delta\varepsilon} = \mathbf{v}_{\alpha\beta\delta} - \mathbf{v}_{\alpha\beta\varepsilon} + \mathbf{v}_{\alpha\gamma\delta} - \mathbf{v}_{\alpha\gamma\varepsilon} + \mathbf{v}_{\gamma\beta\varepsilon} - \mathbf{v}_{\gamma\beta\delta}$ Next, we use such gadgets combined with triangles in the following way $g_{\gamma\delta\alpha} + g_{\delta\gamma\alpha} + 2\mathbf{v}_{\alpha\gamma\delta} = \mathbf{v}_{\gamma\delta}$.

Figure 2: Ida behind construction in Example 1.2.

What if we change 6 to 3? The answer is postponed to Section 3. \square

1.1. Related work and our contribution. The above-discussed, simplest extension of the \mathbb{Z} -solvability problem to data vectors of the form $\mathcal{D} \to \mathbb{Z}^d$ is in **P-Time**, and the N-solvability is $\mathbf{NP}-complete$ [HLT17]. Further known results concern the more general case of ordered data domain \mathcal{D} [HL18]: while the \mathbb{Z} -solvability problem remains in **P-Time**, the complexity of the N-solvability is equivalent to the reachability problem of vector addition systems with states (VASS), or Petri nets, and hence Ackermann-complete [Ler21, CO21]. The increase of complexity caused by the order in data is thus remarkable. An example of a result that builds on top of the [HLT17] is [GSAH19]; where the continuous reachability problem for unordered data nets is shown to be in **P-Time**. The question if the continuous reachability problem is solvable for the ordered data domain remains open. It is particularly interesting due to [BFHRV10], where the coverability problem in timed data nets [AN01] is proven to be interreducible with the coverability problem in ordered data nets.

In this paper, we perform a further generalisation to k-element subsets of unordered data, i.e., consider data vectors of the form $\mathcal{D}^{(k)} \to \mathbb{Z}^d$, where $\mathcal{D}^{(k)}$ stands for the k-element subsets of \mathcal{D} . We prove two main results: first, for every fixed $k \geq 1$ the complexity of the \mathbb{Z} -solvability problem again remains polynomial. Second, we prove decidability and provide an upper **NExp-Time** complexity bound for the \mathbb{N} -solvability problem. This is done by an improvement of techniques developed in [HLT17]. Namely, we non-trivially extend theorems 11 and 15 from [HLT17].

To address \mathbb{N} -solvability, we reprove the theorem 11 from [HLT17] in a more general setting (the proof is slightly modified). Next we combined it with new idea to obtain a reduction to the (easier) \mathbb{Z} -solvability, witnessing a nondeterministic exponential blowup.

Our approach to \mathbb{Z} -solvability is an extension the theorem 15 from [HLT17]. Precisely, if we reformulate the \mathbb{Z} -solvability question in terms of (weighted) hypergraphs, then there is a natural way to lift the characterisation of \mathbb{Z} -solvability proposed in Theorem 15 [HLT17]. The main contribution of this paper is a new tool-box developed to prove the lifted theorem

15 from [HLT17]. The new characterisation is easily checkable in polynomial time, what gives us the algorithm.

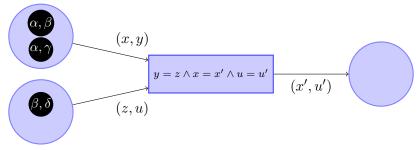
The analogous of theorems 11 and 15 from [HLT17] are not present in [HL18], thus we are pessimistic about applicability of the studied here approach in case of the ordered data domain.

As we comment in the conclusions, we believe that elaboration of the techniques of this paper allows also tackling the case of tuples of unordered data.

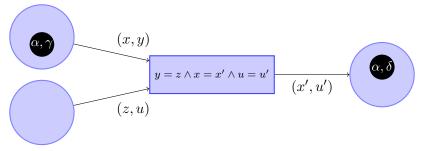
Motivation. Our motivation for this research is two-fold. From a foundational research perspective, our results are a part of a wider research program aiming at lifting computability results in finite-dimensional linear algebra to its orbit-finite-dimensional counterpart. Up to now, research is focused on understanding expressibility, but there are other natural questions about definitions of bases, dimension, linear transformations, etc. We are focused on expressibility, as it gives us a tool for the analysis of systems with data. We highlight three areas where understanding of \mathbb{N}/\mathbb{Z} -expressibility may be crucial for further development.

Unordered Data nets reachability/coverability [LNO⁺08]. The model can be seen as a special type of Coloured Petri nets [JvdAB⁺13]. Data nets are an extension of Petri nets where every token caries a tuple of data values. In addition each transition is equipped with a Boolean formula that connects data of tokens that are consumed and data of tokens that are produced (for unordered data the formula may use = and \neq). To fire a transition we take a valuation satisfying the formula and according to it we remove and produce tokens.

Example 1.3. Consider the following simple net with 3 places and one transition. The initial marking has 3 tokens each with two data values.



If we valuate $x = \alpha, y = \beta, z = \beta, u = \delta, x' = \alpha, u' = \delta$ then we may fire the transition and get the new marking.



If we chose another valuation $x = \alpha, y = \beta, z = \beta, u = \gamma, x' = \alpha, u' = \gamma$ then the formula would hold, but we would lack tokens that can be consumed so we could not fire the transition with this valuation.

The reachability and coverability questions can be formulated as usual for Petri nets. It is not hard to imagine that some workflow or a flow of data through a program can be modelled with data nets. Unfortunately, in this richer model the reachability and coverability problems are undecidable [Las16] already for k=2. For k=1, i.e. data vectors in $\mathcal{D} \to \mathbb{Z}^d$ the status of reachability is unknown and coverability is decidable but known to be Ackermann-hard [LT17]. This is not a satisfying answer for engineers and in this case we should look for over and under approximations of the reachability relation, or for some techniques that will help in the analysis of industrial cases. One of the classic overapproximations of the Petri nets reachability relation is so-called integer reachability (or Marking Equation Lemma in [DE95] lemma 2.12) where the number of tokens in some places may go negative during the run. It can be encoded as integer programming and solved in NP. Its analogue for data nets can be stated as N-solvability of an orbit finite linear equations.

Here we should mention that the integer reachability is a member of a wider family of algebraic techniques for Petri nets. We refer to [STC96] for an exhaustive overview of linear-algebraic and integer-linear-programming techniques in the analysis of Petri nets. The usefulness of these techniques is confirmed by multiple applications including, for instance, recently proposed efficient tools for the coverability problem of Petri nets [GLS16, BFHH16].

 π -calculus. Another formalism close to unordered data nets are ν -nets (unordered data nets equipped with a global freshens test). In [Ros10] Rosa-Velardo observes that they are equivalent to so-called multiset rewriting with name binding systems which, as he showed, are a formalism equivalent to π -calculus. Thus one may try to transfer algebraic techniques for data nets to π -calculus. This is a long way, but there is no possibility to start it without a good understanding of integer solutions of linear equations with data.

Here it is worth to mention that in π calculus we use constructs like $\bar{c}\langle y \rangle . P$ which send a datum y trough the channel c, thus the sending operation is parameterized with a pair of data values i.e. the name of the channel and the data value. Thus one can not expect that already existing results for data vectors $\mathcal{D} \to \mathbb{Z}^d$ [HLT17] will be sufficient and we will need at least theory for $\mathcal{D}^2 \to \mathbb{Z}^d$, for example, to count messages that are sent and received.

The encoding proposed by Rosa-Velardo motivates the development of linear algebra for even more complicated structures than pairs. The transformation from π calculus to multiset rewriting systems uses the concept of derivatives which essentially correspond to sub-processes. Then tokens are used to store data characterising whole sub-processes which leads to tokens with an arbitrary but bounded number of data.

Parikh's theorem. Finally, we may try to lift Parikh's theorem from context-free grammars and finite automata to context-free grammars with data [CK98] and register automata [KF94]. It is not clear to what extent it is possible but there are some promising results [HJLP21]. If we want to use this lifted Parikh theorem then we have to work with semilinear sets with data and be able to check things like membership or non-emptiness of the intersection. Here one more time techniques to solve systems of linear equations with data will be inevitable.

Outline. In Section 2 we introduce the setting and define the problems. Next, in Section 3 we provide the polynomial-time procedure for the \mathbb{Z} -solvability problem: the hypergraph reformulation and an effective characterisation of hypergraph solvability. In Section 4 due to pedagogical reasons we present the proof of the characterisation if data vectors are restricted to $\mathcal{D}^{(2)} \to \mathbb{Z}^d$. After this, in Sections 5, 6, 7, 8, 9, 10 we provide the full proof of the characterisation. The proof follows the same steps, as the proof of the

case $\mathcal{D}^{(2)} \to \mathbb{Z}^d$, but is much more involving on the technical level. Next, in Section 11 we present a reduction from \mathbb{N} - to \mathbb{Z} -solvability. Finally, Section 12 concludes this work.

2. Linear equations with data.

In this section, we introduce the setting of linear equations with data and formulate our results. For a gentle introduction of the setting, we start by recalling classical linear equations.

Let $\mathbb Z$ and $\mathbb N$ denote integers and nonnegative integers, respectively. Classical linear equations are of the form

$$a_1x_1 + \ldots + a_mx_m = a,$$

where $x_1 ldots x_m$ are variables (unknowns), and $a_1 ldots a_m \in \mathbb{Z}$ are integer coefficients. For a finite system \mathcal{U} of such equations over the same variables x_1, \ldots, x_m , a solution of \mathcal{U} is a vector $(n_1, \ldots, n_m) \in \mathbb{Z}^m$ such that the valuation $x_1 \mapsto n_1, \ldots, x_m \mapsto n_m$ satisfies all equations in \mathcal{U} . It is well known that integer solvability problem (\mathbb{Z} -solvability problem), i.e., the question whether \mathcal{U} has a solution $(n_1, \ldots, n_m) \in \mathbb{Z}^m$, is decidable in **P-Time**. In the sequel we are often interested in nonnegative integer solutions $(n_1, \ldots, n_m) \in \mathbb{N}^m$, but one may consider also other solution domains than \mathbb{N} . It is well known that the nonnegative-integer solvability problem (\mathbb{N} -solvability problem) of linear equations, i.e. the question whether \mathcal{U} has a nonnegative-integer solution, is NP-complete (for hardness see [Kar72]; NP-membership is a consequence of [Pot91]). The complexity remains the same for other natural variants of this problem, for instance, for inequalities instead of equations (a.k.a. integer linear programming). The \mathbb{O} -solvability problem (where $\mathbb{O} \in \{\mathbb{Z}, \mathbb{N}\}$) is equivalently formulated as follows: for a given finite set of coefficient vectors $\mathcal{I} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subseteq \mathbb{Z}^d$ and a target vector $\mathbf{v} \in \mathbb{Z}^d$ (we use bold font to distinguish vectors from other elements), check whether \mathbf{v} is an \mathbb{O} -SUMS of \mathcal{I} , i.e.,

$$\mathbf{v} \in \mathbb{O}\text{-Sums}(\mathcal{I}) = \{ n_1 \cdot \mathbf{v}_1 + \ldots + n_m \cdot \mathbf{v}_m \mid n_1, \ldots, n_m \in \mathbb{O} \}. \tag{2.1}$$

The dimension d corresponds to the number of equations in \mathcal{U} .

Data vectors. Linear equations can be naturally extended with data. In this paper, we assume that the data domain \mathcal{D} is a countable infinite set, whose elements are called data values. The bijections $\rho \colon \mathcal{D} \to \mathcal{D}$ are called data permutations. For a set \mathcal{X} and $k \in \mathbb{N}$, by $\mathcal{X}^{(k)}$ we denote the set of all k-elements subsets of \mathcal{X} (called k-sets in short). Data permutations lift naturally to k-sets of data values: $\rho(\{\alpha_1, \ldots, \alpha_k\}) = \{\rho(\alpha_1), \ldots, \rho(\alpha_k)\}$.

Fix a positive integer $k \geq 1$. A data vector is a function $\mathbf{v} \colon \mathcal{D}^{(k)} \to \mathbb{Z}^d$ such that $\mathbf{v}(x) = \mathbf{0} \in \mathbb{Z}^d$ for all but finitely many $x \in \mathcal{D}^{(k)}$. (Again, we use bold font to distinguish data vectors from other elements.) The numbers k and d we call the arity and the dimension of \mathbf{v} , respectively.

The vector addition and scalar multiplication are lifted to data vectors pointwise: $(\mathbf{v} + \mathbf{w})(x) \stackrel{\text{def}}{=} \mathbf{v}(x) + \mathbf{w}(x)$, and $(c \cdot \mathbf{v})(x) \stackrel{\text{def}}{=} c\mathbf{v}(x)$. Further for a data permutation $\pi \colon \mathcal{D} \to \mathcal{D}$ by $\mathbf{v} \circ \pi$ we mean the data vector defined as follows $(\mathbf{v} \circ \pi)(\alpha_1, \alpha_2 \dots \alpha_k) \stackrel{\text{def}}{=} \mathbf{v}(\pi(\alpha_1), \pi(\alpha_2) \dots \pi(\alpha_k))$ for all $\alpha_1, \alpha_2, \dots \alpha_k \in \mathcal{D}$. It is the natural lift of normal function composition. For a set \mathcal{I} of data vectors we define

$$PERM(\mathcal{I}) = \{ \mathbf{v} \circ \pi \mid \mathbf{v} \in \mathcal{I}, \pi \in data \text{ permutations} \}.$$

A data vector \mathbf{v} is said to be an \mathbb{O} -permutation sum of a finite set of data vectors \mathcal{I} if (we deliberately overload the symbol \mathbb{O} -Sums(\square) and use it for data vectors, while in (2.1) it is used for (plain) vectors)

$$\mathbf{v} \in \mathbb{O}\text{-Sums}(\operatorname{Perm}(\mathcal{I})) = \{n_1 \cdot \mathbf{v}_1 + \ldots + n_m \cdot \mathbf{v}_m \mid \mathbf{v}_1, \ldots, \mathbf{v}_m \in \operatorname{Perm}(\mathcal{I}), n_1, \ldots, n_m \in \mathbb{O}\}.$$
(2.2)

We investigate the following decision problems (for $\mathbb{O} \in \{\mathbb{Z}, \mathbb{N}\}$):

For complexity estimations we assume binary encoding of numbers appearing in the input to all decision problems discussed in this paper. Our main results are the following complexity bounds:

Theorem 2.1. For every fixed arity $k \in \mathbb{N}$, the \mathbb{Z} -solvability problem is in **P-Time**. (The dependency form k is exponential).

Theorem 2.2. For every fixed arity $k \in \mathbb{N}$, the \mathbb{N} -solvability problem is in **NExp-Time**.

For a special case of the solvability problems when the arity k = 1, the **P-Time** and **NP** complexity bounds, respectively, have been shown in [HLT17]. Thus, according to Theorem 2.1, in the case of \mathbb{Z} -solvability the complexity remains polynomial for every k > 1. In the case of \mathbb{N} -solvability the $\mathbb{NP} - hardness$ carries over to every k > 1, and hence a complexity gap remains open between \mathbb{NP} and $\mathbb{NExp-Time}$.

3. Proof of Theorem 2.1.

We start by reformulating the problem in terms of (undirected) weighted uniform hypergraphs (Lemma 3.2 below).

Fix a positive integer $k \geq 1$. By a k-hypergraph we mean a pair $\mathbb{H} = (V, \mu)$ where $V \subset \mathcal{D}$ is a finite set called vertices and $\mu \colon V^{(k)} \to \mathbb{Z}^d$ is a weight function (when k is not relevant we skip it and write a hypergraph). As before, the numbers k and d we call the arity and the dimension of \mathbb{H} , respectively. In the case of arity k = 2 we speak of graphs instead of hypergraphs. (Note however that the (hyper)graphs we consider are always weighted, with weights from \mathbb{Z}^d .)

Because vertices are essentially data then instead of usual u, v for vertices we will use Greek letters $\alpha, \beta, \gamma, \delta, \varepsilon \dots$

The set of hyperedges is then defined as

$$Edges(\mathbb{H}) = \{ e \in V^{(k)} \mid \mu(e) \neq \mathbf{0} \}.$$

When $\alpha \in e$ for $\alpha \in V$ and $e \in \text{Edges}(\mathbb{H})$, we say that the vertex α is *incident* with the hyperedge e. The degree of α is the number of hyperedges incident with α . Vertices of degree 0 we call *isolated*. Two hypergraphs are *isomorphic* if there is a bijection between their sets of vertices that preserves the value of the weight function. Two hypergraphs are *equivalent* if they are isomorphic after removing their isolated vertices. For a set \mathcal{H} of

hypergraphs, by $EQ(\mathcal{H})$ we denote the set of all hypergraphs equivalent to ones from \mathcal{H} . If $\mathcal{H} = \{\mathbb{H}\}$ then instead of $EQ(\{\mathbb{H}\})$ we write $EQ(\mathbb{H})$.

Scalar multiplication and addition are defined naturally for hypergraphs. First, for $c \in \mathbb{Z}$ and a hypergraph $\mathbb{H} = (V, \mu)$, let $c \cdot \mathbb{H} \stackrel{\text{def}}{=} (V, c \cdot \mu)$. Second, given two hypergraphs $\mathbb{G} = (W, \overline{\mu})$ and $\mathbb{H} = (V, \mu)$ of the same arity k and dimension d, we first add isolated vertices to both hypergraphs to make their vertex sets equal to the union $V \cup W$, thus obtaining $\mathbb{G}' = (W \cup V, \overline{\mu}')$ and $\mathbb{H}' = (W \cup V, \mu')$ with the accordingly extended weight functions $\overline{\mu}', \mu' \colon (W \cup V)^{(k)} \to \mathbb{Z}^d$, and then define $\mathbb{G} + \mathbb{H} \stackrel{\text{def}}{=} (W \cup V, \overline{\mu}' + \mu')$. Using these operations we define \mathbb{O} -sums of a family \mathcal{H} of hypergraphs of the same arity and dimension (again, we overload the symbol \mathbb{O} -SUMS(_) further and use it for hypergraphs):

$$\mathbb{O}\text{-Sums}(\mathcal{H}) = \{c_1 \cdot \mathbb{H}_1 + \ldots + c_l \cdot \mathbb{H}_l \mid c_1, \ldots, c_l \in \mathbb{O}, \mathbb{H}_1, \ldots, \mathbb{H}_l \in \mathcal{H}\}.$$

We say that a hypergraph \mathbb{H} is a \mathbb{O} -sum of \mathcal{H} up to equivalence if \mathbb{H} is a \mathbb{O} -sum of hypergraphs equivalent to elements of \mathcal{H} : $\mathbb{H} \in \mathbb{O}$ -Sums(Eq(\mathcal{H})).

Example 3.1. We illustrate the \mathbb{Z} -sums in arity k=2, i.e., using graphs. Consider the following graph \mathbb{G} consisting of 3 vertices and 2 edges:

$$\beta$$
 α γ

Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ be arbitrary vectors. Here are two examples of graphs which can be presented as a \mathbb{Z} -sum of $\{\mathbb{G}\}$ up to equivalence, using a sum of two graphs equivalent to \mathbb{G} :

and a difference of two such graphs:

We are now ready to formulate the *hypergraph* \mathbb{Z} -sum problem, to which the \mathbb{Z} -solvability is going to be reduced:

HYPERGRAPH ℤ-SUM	
INPUT:	A finite set \mathcal{H} of hypergraphs and a target hyper-
OUTPUT:	graph \mathbb{H} , all of the same arity and dimension. Is \mathbb{H} a \mathbb{Z} -sum of \mathcal{H} up to equivalence?

Lemma 3.2. The \mathbb{Z} -solvability problem reduces in logarithmic space to the hypergraph \mathbb{Z} -sum problem. The reduction preserves the arity and dimension.

Proof. The reduction encodes each data vector \mathbf{v} by a hypergraph $\mathbb{H} = (V, \mu)$, where

$$V = \{ \int \{x \in \mathcal{D}^{(k)} \mid \mathbf{v}(x) \neq \mathbf{0} \}$$

and μ is the restriction of \mathbf{v} to $V^{(k)}$. In this way a set \mathcal{I} of data vectors and a target data vector \mathbf{v} are transformed into a set of \mathcal{H} hypergraphs and a target hypergraph \mathbb{H} such that \mathbf{v} is a \mathbb{Z} -permutation sum of \mathcal{I} if, and only if \mathbb{H} is a \mathbb{Z} -sum of \mathcal{H} up to equivalence.

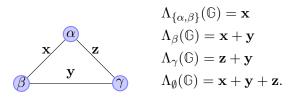
Thus, from now on, we concentrate on solving the hypergraph \mathbb{Z} -sum problem. One may ask why we perform such a reduction. The reasons are of pedagogical nature, namely graphs are more convenient for examples and proofs by pictures.

As the next step, we formulate our core technical result (Theorem 3.4). The theorem provides a local characterisation of the hypergraph \mathbb{Z} -sum problem and in consequence it enables solving the problem in polynomial time. Let $\mathbb{H} = (V, \mu)$ be a hypergraph. For a set $\mathcal{X} \subset \mathcal{D}$ we consider $\{e \in \text{Edges}(\mathbb{H}) \mid \mathcal{X} \subseteq e\}$ — the set of all hyperedges that include \mathcal{X} — and define the weight of \mathcal{X} as the sum of weights of all these edges:

$$\Lambda_{\mathcal{X}}(\mathbb{H}) \stackrel{\mathrm{def}}{=} \sum_{e \in \mathrm{EDGES}(\mathbb{H}), \mathcal{X} \subseteq e} \mu(e).$$

In particular when $\mathcal{X} \not\subseteq V$ then $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mathbf{0}$. Further, $\Lambda_{\emptyset}(\mathbb{H})$ is the sum of weights of all hyperedges of \mathbb{H} . When $\mathcal{X} = \{\alpha\}$, then $\Lambda_{\mathcal{X}}(\mathbb{H})$ is the sum of weights of all hyperedges incident with α (we write Λ_{α} instead of $\Lambda_{\{\alpha\}}$). Finally, when the cardinality of \mathcal{X} equals k i.e. $|\mathcal{X}| = k$, $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mu(\mathcal{X})$ is the weight of a hyperedge \mathcal{X} , given as a defining component of \mathbb{H} .

Example 3.3. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^d$ be arbitrary vectors. As an illustration, consider the graph \mathbb{G} on the left below, with the weights of chosen subsets of its vertices listed on the right:



Weights are important as they form a family of homomorphisms from hypergraphs to \mathbb{Z}^d . Namely, for any subset of vertices \mathcal{X} we have that

$$\Lambda_{\mathcal{X}}(\mathbb{H} + \mathbb{H}') = \Lambda_{\mathcal{X}}(\mathbb{H}) + \Lambda_{\mathcal{X}}(\mathbb{H}')$$

and

$$c\Lambda_{\mathcal{X}}(\mathbb{H}) = \Lambda_{\mathcal{X}}(c\mathbb{H})$$
, for any $c \in \mathbb{Z}$.

This allow us to design a partial test for the hypergraph \mathbb{Z} -sum problem. If there is a set of vertices \mathcal{X} such that $\Lambda_{\mathcal{X}}(\mathbb{H})$ can not be expressed as a \mathbb{Z} -sum of vectors in $\Lambda_{\mathcal{X}}(\mathrm{EQ}(\mathcal{H}))$ then $\mathbb{H} \notin \mathbb{Z}$ -SUMS(EQ(\mathcal{H})). This motivates the next definition.

We say that a k-hypergraph $\mathbb{H} = (V, \mu)$ is locally a \mathbb{Z} -sum of a family \mathcal{H} of hypergraphs if for every $\mathcal{X} \subseteq V$ of the cardinality $|\mathcal{X}| \leq k$, its weight $\Lambda_{\mathcal{X}}(\mathbb{H})$ is a \mathbb{Z} -sum of weights of $|\mathcal{X}|$ -element subsets of vertex sets of hypergraphs from \mathcal{H}

$$\Lambda_{\mathcal{X}}(\mathbb{H}) \in \mathbb{Z}\text{-Sums}(\{\Lambda_{\mathcal{X}'}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}, \mathcal{X}' \subseteq V', |\mathcal{X}'| = |\mathcal{X}|\}). \tag{3.1}$$

Note that we only consider hypergraphs $\mathbb{H}' \in \mathcal{H}$, and we do not need to consider hypergraphs $\mathbb{H}' \in E_{\mathbb{Q}}(\mathcal{H})$ equivalent to ones from \mathcal{H} as

$$\{\Lambda_{\mathcal{X}'}(\mathbb{H}')\mid \mathbb{H}'=(V',\mu')\in \mathcal{H}, \mathcal{X}'\subseteq V', |\mathcal{X}'|=|\mathcal{X}|\}=\{\Lambda_{\mathcal{X}}(\mathbb{H}')\mid \mathbb{H}'\in \mathrm{EQ}(\mathcal{H})\}.$$

Theorem 3.4. The following conditions are equivalent, for a finite set \mathcal{H} of hypergraphs and a hypergraph \mathbb{H} , all of the same arity and dimension:

(1) \mathbb{H} is a \mathbb{Z} -sum of $E_{\mathbb{Q}}(\mathcal{H})$;

(2) \mathbb{H} is locally a \mathbb{Z} -sum of \mathcal{H} .

Before we embark on proving the result (in the next section) we first discuss how it implies Theorem 2.1. Recall that the arity k is fixed. Instead of checking if $\mathbb{H} = (V, \mu)$ is a \mathbb{Z} -sum of $\mathrm{EQ}(\mathcal{H})$, the algorithm checks if \mathbb{H} is locally a \mathbb{Z} -sum of \mathcal{H} . Observe that the condition (3.1) amounts to solvability of a (classical) system of linear equations, as in (2.1). Therefore, the algorithm tests \mathbb{Z} -solvability of a system of the corresponding $d \cdot (1 + |V| + |V^{(2)}| + \dots |V^{(k)}|)$ linear equations, d for every subset $\mathcal{X} \subseteq V$ of the cardinality at most k. The number of equations is exponential in k, but due to fixing k it is polynomial in the input hypergraphs \mathbb{H} and \mathcal{H} . Thus, Theorem 2.1 is proved once we prove Theorem 3.4.

Example 3.5. Let us recall Example 1.2. For the target we have that $\Lambda_{\emptyset}(a_{\gamma\delta}) = \Lambda_{\gamma}(a_{\gamma\delta}) = \Lambda_{\delta}(a_{\gamma\delta}) = \Lambda_{\{\gamma\delta\}}(a_{\gamma\delta}) = 6$, and for the triangle we have $\Lambda_{\emptyset}(a_{\delta\gamma\varepsilon}) = 3$, $\Lambda_{x}(a_{\delta\gamma\varepsilon}) = 2$ for any $x \in \{\delta, \gamma, \varepsilon\}$, and $\Lambda_{\{x,y\}}(a_{\delta\gamma\varepsilon}) = 1$ for any $\{x,y\} \subset \{\delta, \gamma, \varepsilon\}$.

As 6=3+3 and 6=2+2+2 and 6=1+1+1+1+1+1+1 we see that the target is locally a \mathbb{Z} -sum of the triangle. Hence target is a \mathbb{Z} -sum of the triangle up to equivalence. Moreover, if we change 6 to a smaller positive number then the target graph will not be a \mathbb{Z} -sum of the triangle up to equivalence, for example 3 can not be expressed as a sum of twos.

4. Proof of Theorem 3.4 (the case for k = 2).

The implication $1 \implies 2$ is immediate. Indeed, suppose $\mathbb{H} = (V, \mu) = c_1 \cdot \mathbb{H}_1 + \ldots + c_l \cdot \mathbb{H}_l$, where $c_i \in \mathbb{Z}$ and $\mathbb{H}_i = (V_i, \mu_i) \in \text{Eq}(\mathcal{H})$. Let $\mathcal{X} \subseteq V$ be a subset of cardinality $|\mathcal{X}| \leq k$. We have

$$\Lambda_{\mathcal{X}}(\mathbb{H}) = c_1 \cdot \Lambda_{\mathcal{X}}(\mathbb{H}_1) + \ldots + c_l \cdot \Lambda_{\mathcal{X}}(\mathbb{H}_l),$$

which implies, for some hypergraphs $\mathbb{H}'_i = (V'_i, \mu'_i) \in \mathcal{H}$ equivalent to \mathbb{H}_i , and subsets $\mathcal{Y}_i \subseteq V'_i$ of cardinality $|\mathcal{Y}_i| = |\mathcal{X}|$, the following equality holds

$$\Lambda_{\mathcal{X}}(\mathbb{H}) = c_1 \cdot \Lambda_{\mathcal{Y}_1}(\mathbb{H}'_1) + \ldots + c_l \cdot \Lambda_{\mathcal{Y}_l}(\mathbb{H}'_l).$$

As \mathcal{X} was chosen arbitrarily, this shows that \mathbb{H} is locally a \mathbb{Z} -sum of \mathcal{H} .

The proof of the converse implication $2 \implies 1$ is much more involved. Here, due to pedagogical reasons, we provide a simplified version of the proof for arity k=2, i.e., for (undirected) graphs. The case for arity k=1 is considered in [HLT17]. Consequently, we speak of edges instead of hyperedges. We recall that the graphs we consider are actually \mathbb{Z}^d -weighted graphs.

Let $\mathbb{H} = (V, \mu)$ be a graph and assume that \mathbb{H} is locally a \mathbb{Z} -sum of \mathcal{H} . We are going to demonstrate that \mathbb{H} is equivalent to a \mathbb{Z} -sum of EQ(\mathcal{H}). Since we work with arity k = 2, the assumption amounts to the following conditions, for every vertex $\alpha \in V$ and every edge $e \in \text{Edges}(\mathbb{H})$:

$$\Lambda_{\emptyset}(\mathbb{H}) \in \mathbb{Z}\text{-Sums}(\{\Lambda_{\emptyset}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}\})$$

$$\tag{4.1}$$

$$\Lambda_{\alpha}(\mathbb{H}) \in \mathbb{Z}\text{-Sums}(\{\Lambda_{\alpha'}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}, \alpha' \in V'\})$$

$$\tag{4.2}$$

$$\Lambda_e(\mathbb{H}) \in \mathbb{Z}\text{-Sums}(\{\Lambda_{e'}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}, e' \in \text{Edges}(\mathbb{H}')\}).$$
 (4.3)

Claim 4.1. W.l.o.g. we may assume that $\mathbb{H} = (V, \mu)$ satisfies $\Lambda_{\emptyset}(\mathbb{H}) = \mathbf{0}$.

Proof. Indeed, due to the assumption (4.1) and due the following equality (note that the symbol \mathbb{Z} -Sums($_{-}$) applies to vectors on the left, and to graphs on the right)

$$\mathbb{Z}\text{-Sums}(\{\Lambda_{\emptyset}(\mathbb{H}') \mid \mathbb{H}' \in \mathcal{H}\}) = \{\Lambda_{\emptyset}(\mathbb{H}') \mid \mathbb{H}' \in \mathbb{Z}\text{-Sums}(\mathcal{H})\}, \tag{4.4}$$

there is a graph $\mathbb{H}' = (V', \mu') \in \mathbb{Z}\text{-Sums}(\mathcal{H})$ with $\Lambda_{\emptyset}(\mathbb{H}') = \Lambda_{\emptyset}(\mathbb{H})$. Therefore $\mathbb{H} \in \mathbb{Z}\text{-Sums}(\mathrm{Eq}(\mathcal{H}))$ if, and only if $\mathbb{H} - \mathbb{H}' \in \mathbb{Z}\text{-Sums}(\mathrm{Eq}(\mathcal{H}))$, and hence we can replace \mathbb{H} by $\mathbb{H} - \mathbb{H}'$.

We proceed in two steps. We start by defining a class of particularly simple graphs, called \mathbb{H} -simple graphs, and argue that \mathbb{H} is a \mathbb{Z} -sum of these graphs (Lemma 4.4 below). Then we prove that every \mathbb{H} -simple graph is representable as a \mathbb{Z} -sum of \mathcal{H} up to equivalence (Lemma 4.9 below). Composing the two claims we get that \mathbb{H} is a \mathbb{Z} -sum of \mathcal{H} up to equivalence.

This structure of the proof is correct due to the following simple lemma:

Lemma 4.2. For a family of k-hypergraphs \mathcal{H} and k-hypergraphs $\mathbb{H}, \mathbb{H}_1, \dots, \mathbb{H}_l$, all of the same dimension, If $\mathbb{H}_1, \dots, \mathbb{H}_l \in \mathbb{Z}$ -Sums(EQ(\mathcal{H})) and $\mathbb{H} \in \mathbb{Z}$ -Sums(EQ(\mathcal{H})).

(Lemma 4.2 is expressible, more succinctly, as

$$\mathbb{Z}$$
-Sums(Eq(\mathbb{Z} -Sums(Eq(\mathcal{H})))) = \mathbb{Z} -Sums(Eq(\mathcal{H})).)

Proof. This is simply because of three trivial facts:

- (1) $\text{Eq}(\mathbb{G}_1 + \mathbb{G}_2) \subseteq \text{Eq}(\mathbb{G}_1) + \text{Eq}(\mathbb{G}_2)$ where $\mathbb{G}_1, \mathbb{G}_2$ are any two k-hypergraphs of the same dimension, and second plus is Minkowski sum.
- (2) \mathbb{Z} -Sums(\mathcal{F}) = \mathbb{Z} -Sums(\mathbb{Z} -Sums(\mathcal{F})), for any \mathcal{F} a family of k-hypergraphs of the same dimension.
- (3) $EQ(\mathcal{F}) = EQ(EQ(\mathcal{F}))$, for any \mathcal{F} a family of k-hypergraphs of the same dimension. Now,

$$\mathbb{Z}$$
-Sums(Eq(\mathbb{Z} -Sums(Eq(\mathcal{H})))) \subset due to 1

$$\mathbb{Z}$$
-Sums(\mathbb{Z} -Sums(Eq(Eq(\mathcal{H})))) = due to 2 and 3

 \mathbb{Z} -Sums(EQ(\mathcal{H})).

The inclusion in the opposite direction is trivial.

Definition 4.3. For every vector $\mathbf{a} \in \mathbb{Z}^d$ we define an \mathbf{a} -edge simple graph $\mathbb{S}_{\mathbf{a}}^{\bullet-\bullet}$ (shown on the left) and an \mathbf{a} -vertex simple graph $\mathbb{S}_{\mathbf{a}}^{\bullet}$ (shown on the right):



We do not specify names of vertices, as anyway we will consider these graphs up to equivalence. Both types of graphs we call *simple graphs*.

Let $\mathcal{E}_{\mathbb{H}} = \mathbb{Z}\text{-Sums}(\{\Lambda_e(\mathbb{H}) \mid e \in \text{Edges}(\mathbb{H})\}) \subseteq \mathbb{Z}^d$ and $\mathcal{V}_{\mathbb{H}} = \mathbb{Z}\text{-Sums}(\{\Lambda_\alpha(\mathbb{H}) \mid \alpha \in V\}) \subseteq \mathbb{Z}^d$. Now, let $\mathcal{S}_{\mathbb{H}}^{\bullet} \stackrel{\text{def}}{=} \{\mathbb{S}_{\mathbf{a}}^{\bullet} \mid \mathbf{a} \in \mathcal{V}_{\mathbb{H}}\}$ and $\mathcal{S}_{\mathbb{H}}^{\bullet-\bullet} \stackrel{\text{def}}{=} \{\mathbb{S}_{\mathbf{a}}^{\bullet-\bullet} \mid \mathbf{a} \in \mathcal{E}_{\mathbb{H}}\}$.

Lemma 4.4. $\mathbb{H} \in \mathbb{Z}$ -Sums $(EQ(S_{\mathbb{H}}^{\bullet} \cup S_{\mathbb{H}}^{\bullet-\bullet}))$.

Proof. Suppose $\mathbb{H} = (V, \mu)$. The proof of the lemma is done in steps.

Claim 4.5. There is a $\mathbb{G} \in \mathbb{Z}$ -Sums(EQ($\mathcal{S}^{\bullet}_{\mathbb{H}}$)) such that for any vertex $\alpha \in \mathcal{D}$ holds $\Lambda_{\alpha}(\mathbb{G} + \mathbb{H}) = \mathbf{0}$.

We will use it to further simplify our problem, as $\mathbb{H} \in \mathbb{Z}$ -Sums(EQ($\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}$)) if and only if $\mathbb{G} + \mathbb{H} \in \mathbb{Z}$ -Sums(EQ($\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}$)).

Proof of the claim. Let α, α' are two vertices not in V and let $\mathcal{F} \subseteq \mathcal{S}_{\mathbb{H}}^{\bullet}$ be the family of all $\mathbb{S}_{\mathbf{a}}^{\bullet}$ simple graphs (depicted below) where $\mathbf{a} = \Lambda_{\beta}(\mathbb{H})$ for $\beta \in V$.

$$\alpha'$$
 α α β

We define $\mathbb{G} = \sum_{\mathbb{F} \in \mathcal{F}} \mathbb{F}$. As $\mathcal{F} \subset \mathrm{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet})$ the graph $\mathbb{G} \in \mathbb{Z}$ -Sums($\mathrm{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet})$). Moreover,

- (1) for every $\beta \in V$ holds $-\Lambda_{\beta}(\mathbb{H}) = \Lambda_{\beta}(\mathbb{G})$;
- (2) $\Lambda_{\alpha}(\mathbb{G}) = \mathbf{0}$ as for each $\mathbb{F} \in \mathcal{F} \Lambda_{\alpha}(\mathbb{F}) = \mathbf{0}$;
- (3) $\Lambda_{\alpha'}(\mathbb{G}) = \mathbf{0}$ as $\Lambda_{\alpha'}(\mathbb{G}) = \sum_{\beta \in V} \Lambda_{\beta}(\mathbb{H}) = 2\Lambda_{\emptyset}(\mathbb{H}) = \mathbf{0}$, where the second equality reflects the fact that every edge has two ends and the last equality is due to Claim 4.1.

As Λ_x is a homomorphism we get that $\Lambda_x(\mathbb{H} + \mathbb{G}) = \mathbf{0}$ for every $x \in V \cup \{\alpha, \alpha'\}$. The same holds for other vertices as they are isolated.

Due to the previous claim w.l.g we may assume that \mathbb{H} has following properties $\Lambda_{\beta}(\mathbb{H}) = \mathbf{0}$ for every $\beta \in V$. Here there is one issue that should be discussed. Suppose $\mathbb{H}' = \mathbb{H} + \mathbb{G}$ as in the Claim 4.5. The issue is $\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}'}^{\bullet-\bullet} = \mathcal{S}_{\mathbb{H}'}^{\bullet} \cup \mathcal{S}_{\mathbb{H}'}^{\bullet-\bullet}$ may not hold. So if we prove the lemma for \mathbb{H}' and $\mathcal{S}_{\mathbb{H}'}^{\bullet} \cup \mathcal{S}_{\mathbb{H}'}^{\bullet-\bullet}$ it does not necessarily carry on to \mathbb{H} and $\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}$. Fortunately, it is sufficient for us if \mathbb{Z} -SUMS($\mathcal{S}_{\mathbb{H}'}^{\bullet} \cup \mathcal{S}_{\mathbb{H}'}^{\bullet-\bullet}$) $\subseteq \mathbb{Z}$ -SUMS($\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}$). The last inclusion holds. Indeed, for every $\beta \in V'$ holds $\Lambda_{\beta}(\mathbb{H}')$ is a sum of weights of vertices in \mathbb{H} thus \mathbb{Z} -SUMS($\mathcal{S}_{\mathbb{H}'}^{\bullet}$) $\subseteq \mathbb{Z}$ -SUMS($\mathcal{S}_{\mathbb{H}'}^{\bullet}$). Thus, we do not loose generality due to the proposed restriction.

Claim 4.6. There is a hypergraph $\mathbb{G} \in \mathbb{Z}$ -Sums $(\mathrm{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}))$ such that $\mathbb{G} = (V', \mu'), \mathbb{H} + \mathbb{G}$ has at most 3 non-isolated vertices, and $\Lambda_{\beta}(\mathbb{H} + \mathbb{G}) = \mathbf{0}$ for ever $\beta \in V \cup V'$.

We will use it to further simplify our problem, as $\mathbb{H} \in \mathbb{Z}\text{-Sums}(\mathrm{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}))$ if and only if $\mathbb{H} + \mathbb{G} \in \mathbb{Z}\text{-Sums}(\mathrm{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}))$.

Proof of the claim. We construct \mathbb{C}' gradually as a sum $\mathbb{C}' = \sum_{i=1}^{last} \mathbb{C}'_i$ in parallel with a sequence \mathbb{H}_i where $\mathbb{H}_0 = \mathbb{H}$ and $\mathbb{H}_{i+1} = \mathbb{H}_i + \mathbb{C}'_{i+1}$. The main property of the sequence \mathbb{H}_i is that $\mathbb{H}_i > \mathbb{H}_{i+1}$ in some well founded quasi order on graphs. Let ALG be an algorithm, that takes as an input \mathbb{H}_{i-1} and produces the next $\mathbb{C}'_i \in \mathrm{EQ}(\mathcal{S}^{\bullet,\bullet})$. The precondition of the algorithm ALG is that \mathbb{H}_{i-1} has more than 3 non-isolated vertices. Thus, due to the well-foundedness of the quasi order, the sequence \mathbb{C}'_i is finite. Further, the graph \mathbb{H}_{last} has at most 3 non-isolated vertices due to the precondition of ALG. To this end, we need to define the order on graphs and provide the algorithm ALG.

Order on graphs. We assume an arbitrary total order < on vertices V. We lift the order to an order on edges $\{\alpha, \beta\}$. We define it as the lexicographic order on pairs (α, β) satisfying $\alpha > \beta$. Finally, the order is extended to a quasi-order on graphs: $\mathbb{C} < \mathbb{C}'$ iff e < e', where e and e' are the largest edges in \mathbb{C} and \mathbb{C}' , respectively.

The algorithm ALG. Suppose, (α, β) is the biggest edge in the graph \mathbb{H}_i and that \mathbb{H}_i has at least 4 non-isolated vertices. Observe that $\Lambda_{\alpha}(\mathbb{H}_i) = \mathbf{0}$. Indeed, $\Lambda_{\alpha}(\mathbb{H}_i) = \mathbf{0}$

 $\Lambda_{\alpha}(\mathbb{H} + \sum_{j=1}^{i} \mathbb{G}'_{j}) = \Lambda_{\alpha}(\mathbb{H}) + \sum_{j=1}^{i} \Lambda_{\alpha}(\mathbb{G}'_{j}) = \mathbf{0} + \sum_{j=1}^{i} \Lambda_{\alpha}(\mathbb{G}'_{j})$ but each $\mathbb{G}'_{j} \in \mathrm{EQ}(\mathcal{S}_{\mathbb{H}}^{\bullet - \bullet})$ and by the definition of edge simple graphs $\Lambda_{\varepsilon}(\mathcal{S}_{\mathbb{H}}^{\bullet - \bullet}) = \mathbf{0}$ for every vertex ε . As a consequence there must be at least one vertex $\gamma \notin \{\alpha, \beta\}$ such that (α, γ) is an edge in \mathbb{H}_{i} . As \mathbb{H}_{i} has at least 4 non-isolated vertices, there is $\delta \notin \{\gamma, \alpha, \beta\}$ a non-isolated vertex in \mathbb{H}_{i} . We define

$$\mathbb{G}_{i+1}$$
 as follows $-\mathbf{a}$ for $\mathbf{a} = \Lambda_{\{\alpha,\beta\}}(\mathbb{H}_i)$.

Observe that edges $(\alpha, \gamma), (\beta, \delta), (\gamma, \delta)$ are smaller than the edge (α, β) and $(\mathbb{H}_i + \mathbb{G}_{i+1})(\{\alpha, \beta\}) = \mathbf{0}$ so $\mathbb{H}_i + \mathbb{G}_{i+1} < \mathbb{H}_i$.

After usage of the claim we get a graph $\mathbb{H}' \stackrel{\text{def}}{=} \mathbb{H} + \sum_{i=0}^{last} \mathbb{G}'_i$. We know that $\mathbb{H} \in \mathbb{Z}\text{-Sums}(\mathrm{Eq}(\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}))$ if and only if $\mathbb{H}' \in \mathbb{Z}\text{-Sums}(\mathrm{Eq}(\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}))$. The graph \mathbb{H}' has at most 3 non-isolated nodes, $\Lambda_{\emptyset}(\mathbb{H}') = \mathbf{0}$, and $\Lambda_{\alpha}(\mathbb{H}') = \mathbf{0}$ for any vertex α .

Suppose that the set of non-isolated vertices of \mathbb{H}' is a subset of $\{\alpha, \beta, \gamma\}$. We may write the following system of equations:

$$\Lambda_{\alpha}(\mathbb{H}') = \Lambda_{\{\alpha,\beta\}}(\mathbb{H}') + \Lambda_{\{\alpha,\gamma\}}(\mathbb{H}') = \mathbf{0}
\Lambda_{\beta}(\mathbb{H}') = \Lambda_{\{\alpha,\beta\}}(\mathbb{H}') + \Lambda_{\{\beta,\gamma\}}(\mathbb{H}') = \mathbf{0}
\Lambda_{\gamma}(\mathbb{H}') = \Lambda_{\{\alpha,\gamma\}}(\mathbb{H}') + \Lambda_{\{\beta,\gamma\}}(\mathbb{H}') = \mathbf{0}$$
(4.5)

The only solution is $\Lambda_{\{\alpha,\beta\}}(\mathbb{H}') = \Lambda_{\{\beta,\gamma\}}(\mathbb{H}') = \Lambda_{\{\alpha,\gamma\}}(\mathbb{H}') = \mathbf{0}$. So \mathbb{H}' is the empty graph. In consequence, $\mathbb{H} = -\sum_{i=1}^{last} \mathbb{G}_i \in \mathbb{Z}$ -Sums $(\mathrm{EQ}(\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}))$, which completes the proof of Lemma 4.4.

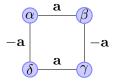
Representation of \mathbb{H} -simple graphs. As the second step we prove that every \mathbb{H} -simple graphs is representable as a \mathbb{Z} -sum of \mathcal{H} up to equivalence (Lemma 4.9). We start from two preparatory lemmas.

Lemma 4.7. Let $e \in \text{Edges}(\mathbb{G})$ for a graph $\mathbb{G} = (V, \mu)$, and let $\Lambda_e(\mathbb{G}) = \mathbf{a}$. Then $\mathbb{S}_{\mathbf{a}}^{\bullet - \bullet} \in \mathbb{Z}\text{-Sums}(\text{Eq}(\{\mathbb{G}\}))$.

Proof. Let $e = \{\alpha, \beta\} \subseteq V$, $\Lambda_e(\mathbb{G}) = \mathbf{a}$, and let $\gamma, \delta \notin V$ are two additional fresh vertices outside of V. We consider three graphs $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_{12}$ which differ from \mathbb{G} only by:

- replacing α with γ (in case of \mathbb{G}_1),
- replacing β with δ (in case of \mathbb{G}_2),
- replacing both α and β with γ and δ , respectively (in case of \mathbb{G}_{12}).

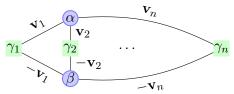
Clearly all the three graphs are equivalent to \mathbb{G} . We claim that $\mathbb{G} - \mathbb{G}_1 - \mathbb{G}_2 + \mathbb{G}_{12}$ yields:



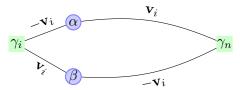
that is a graph equivalent to $\mathbb{S}_{\mathbf{a}}^{\bullet-\bullet}$. Indeed, the above graph operations cancel out all edges non-incident to the four vertices $\alpha, \beta, \gamma, \delta$, as well as all edges incident to only one of them. The two horizontal **a**-weighted edges originate from \mathbb{G} and \mathbb{G}_{12} , while the two remaining vertical ones originate from $-\mathbb{G}_1$ and $-\mathbb{G}_2$.

Lemma 4.8. Let $\alpha \in V$ be a vertex of a graph $\mathbb{G} = (V, \mu)$, and let $\Lambda_{\alpha}(\mathbb{G}) = \mathbf{a}$. Then $\mathbb{S}^{\bullet}_{\mathbf{a}} \in \mathbb{Z}$ -Sums(Eq($\{\mathbb{G}\}$)).

Proof. Let a graph \mathbb{G}' differ from \mathbb{G} only by replacing α with a fresh vertex $\beta \notin V$. The graph $\mathbb{G} - \mathbb{G}'$ has the following shape (with green square vertices representing the set $V \setminus \{\alpha\} = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$):



Relying on Lemma 4.7, we add to the graph $\mathbb{G} - \mathbb{G}'$ the following graphs equivalent to $\mathbb{S}_{\mathbf{v}_i}^{\bullet - \bullet}$, for $i = 1, 2, \ldots, n - 1$:



This results in collapsing all green vertices into one:

Since $\mathbf{a} = \mathbf{v}_1 + \ldots + \mathbf{v}_n$, the resulting graph is equivalent to $\mathbb{S}_{\mathbf{a}}^{\bullet}$, as required.

Lemma 4.9. Every \mathbb{H} -simple graph is a \mathbb{Z} -sum of \mathcal{H} up to equivalence: $\mathcal{S}_{\mathbb{H}}^{\bullet} \subseteq \mathbb{Z}$ -Sums(Eq(\mathcal{H})) and $\mathcal{S}_{\mathbb{H}}^{\bullet^{-\bullet}} \subseteq \mathbb{Z}$ -Sums(Eq(\mathcal{H})).

Proof. Let $\mathbb{H}=(V,\mu)$. We have to prove that for any $\mathbf{a}\in\mathcal{E}_{\mathbb{H}}$ holds $\mathbb{S}^{\bullet}_{\mathbf{a}}\in\mathbb{Z}$ -Sums(Eq(\mathcal{H})) and that for any $\mathbf{b}\in\mathcal{V}_{\mathbb{H}}$ holds $\mathbb{S}^{\bullet-\bullet}_{\mathbf{b}}\in\mathbb{Z}$ -Sums(Eq(\mathcal{H})). We prove only $\mathbb{S}^{\bullet}_{\mathbf{a}}\in\mathbb{Z}$ -Sums(Eq(\mathcal{H})) as the second proof is the almost same. Due to Lemma 4.2 we know that

$$\mathbb{Z}$$
-Sums(EQ(\mathbb{Z} -Sums(EQ(\mathcal{H})))) = \mathbb{Z} -Sums(EQ(\mathcal{H})).

Thus, due to Lemma 4.8 it is sufficient to prove that there is a graph $\mathbb{G}_{\mathbf{a}} \in \mathbb{Z}$ -Sums(EQ(\mathcal{H})) such that $\Lambda_{\alpha}(\mathbb{G}_{\mathbf{a}}) = \mathbf{a}$.

As $\mathbf{a} \in \mathcal{E}_{\mathbb{H}}$ we know that

$$\mathbf{a} = z_1 \cdot \mathbf{a}_1 + \ldots + z_l \cdot \mathbf{a}_l, \tag{4.6}$$

where $z_i \in \mathbb{Z}$ and \mathbf{a}_i is $\Lambda_{\beta_i}(\mathbb{H})$ for some $\beta_i \in V$. By the assumption (4.2), for every i we know

$$\mathbf{a}_i \in \mathbb{Z}\text{-Sums}(\{\Lambda_{\gamma_i}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}, \gamma_i \in V'\}).$$

we may use the equivalence relation and rewrite it as follows

$$\mathbf{a}_i \in \mathbb{Z}\text{-Sums}(\{\Lambda_{\alpha}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathrm{Eq}(\mathcal{H})\}).$$

We concretize it

$$\mathbf{a}_i = z_{i,1} \Lambda_{\alpha}(\mathbb{H}'_{i,1}) + z_{i,2} \Lambda_{\alpha}(\mathbb{H}'_{i,2}) + \ldots + z_{i,h(i)} \Lambda_{\alpha}(\mathbb{H}'_{i,h})$$
 and further

$$\mathbf{a} = \sum_{i=1}^{l} z_i \left(z_{i,1} \Lambda_{\alpha}(\mathbb{H}'_{i,1}) + z_{i,2} \Lambda_{\alpha}(\mathbb{H}'_{i,2}) + \ldots + z_{i,h(i)} \Lambda_{\alpha}(\mathbb{H}'_{i,h(i)}) \right)$$

 Λ_{α} is a homomorphism, so

$$\mathbf{a} = \Lambda_{\alpha} \left(\sum_{i=1}^{l} z_{i} \left(z_{i,1} \mathbb{H}'_{i,1} + z_{i,2} \mathbb{H}'_{i,2} + \ldots + z_{i,h(i)} \mathbb{H}'_{i,h(i)} \right) \right)$$

so we define

$$\mathbb{G}_{\mathbf{a}} \stackrel{\text{def}}{=} \sum_{i=1}^{l} z_i \left(z_{i,1} \mathbb{H}'_{i,1} + z_{i,2} \mathbb{H}'_{i,2} + \ldots + z_{i,h(i)} \mathbb{H}'_{i,h(i)} \right).$$

The construction of elements of $\mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}$ relays on Lemma 4.7 instead of Lemma 4.8.

Combining Lemmas 4.4 and 4.9 we know that \mathbb{H} is a \mathbb{Z} -sum of \mathbb{H} -simple graphs, each of which in turn is a \mathbb{Z} -sum of \mathcal{H} up to equivalence. By Lemma 4.2, \mathbb{H} is a \mathbb{Z} -sum of \mathcal{H} up to equivalence, as required. The proof of Theorem 3.4, for graphs, is thus completed.

5. Proof of Theorem 3.4 (the outline).

Now we are ready to prove the implication $2 \implies 1$ from Theorem 3.4 in full generality. The proof is split into five parts. We want to mimic the approach presented in Section 4, thus we need to generalise few things:

- (1) In Section 6 we build an algebraic background to be able to solve systems of equations that in the general case correspond to Equations 4.5.
- (2) In Section 7 we introduce simple hypergraphs that generalise graphs defined in Definition 4.3.
- (3) In Section 8 we show that our target hypergraph is a \mathbb{Z} -sum of simple hypergraphs.
- (4) In Section 9 we prove that simple hypergraphs are \mathbb{Z} -sums of $E_{\mathbb{Q}}(\mathcal{H})$.
- (5) Finally, in Section 10 we complete the proof of Theorem 3.4.

6. Reduction matrices.

In this section we introduce a notion of reduction matrices and prove a key lemma about their rank, Lemma 6.3. They are 0,1 matrices related to adjacency matrices of Kneser graphs. We recall that by x-set we mean a set with x elements.

Definition 6.1. Let $a \in \mathbb{N}$ and \mathcal{A} be a a-set. Matrix is a reduction matrix for $a \geq b \geq c$, $a, b, c \in \mathbb{N}$, denoted by $\overline{[a, b, c]}$, if

- (1) columns and rows are indexed with b-element subsets of \mathcal{A} and c-element subsets of \mathcal{A} , respectively,
- (2) $\overline{[a,b,c]}[\mathcal{C},\mathcal{B}] = 1$ if $\mathcal{C} \subseteq \mathcal{B}$ and 0 otherwise, where \mathcal{C} is an index of a row and \mathcal{B} is an index of a column.

Example 6.2. (1)
$$[4,3,1] = \begin{bmatrix} \gamma \beta \alpha & \delta \beta \alpha & \delta \gamma \alpha & \delta \gamma \beta \\ \alpha & 1 & 1 & 1 & 0 \\ \beta & 1 & 1 & 0 & 1 \\ \gamma & 1 & 0 & 1 & 1 \\ \delta & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$(2) \ \overline{[4,2,1]} = \begin{bmatrix} \beta \alpha & \gamma \alpha & \gamma \beta & \delta \alpha & \delta \beta & \delta \gamma \\ \alpha & 1 & 1 & 1 & 0 & 0 & 0 \\ \beta & 1 & 0 & 0 & 1 & 1 & 0 \\ \gamma & 0 & 1 & 0 & 1 & 0 & 1 \\ \delta & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

The below lemma expresses the important property of reduction matrices.

Lemma 6.3. Any $\overline{[2k+1,k+1,k]}$ reduction matrix has maximal rank.

The proof relies on the result from the spectral theory of Kneser graphs.

Definition 6.4. Kneser graph. Let \mathcal{A} be an a-set. The Kneser graph $K_{a,c}$ is the graph whose vertices are c-element subsets of \mathcal{A} , and where two vertices x, y are adjacent if and only if $x \cap y = \emptyset$.

The *incidence matrix* a graph (\mathcal{X}, E) is a $\mathcal{X} \times \mathcal{X}$ -matrix M such that M[x, y] = 1 if x, y are adjacent and 0 otherwise.

Theorem 6.5 ([GR01] (page. 200) Theorem 9.4.3). All eigenvalues of the incidence matrix for a Kneser graph are non-zero.

Corollary 6.6. The rank of the incidence matrix for a Kneser graph is maximal.

Proof of Lemma 6.3. We relabel columns of $\overline{[2k+1,k+1,k]}$ in the following way. If a column is labelled with the set $\mathcal{B} \subset \mathcal{A}$ we relabel it to $\mathcal{A} \setminus \mathcal{B}$. Now both rows and columns are labelled with k-subsets of \mathcal{A} . We call the relabelled matrix M'. On the one hand, observe that if a k-set \mathcal{C} is a subset of a k+1-set \mathcal{B} then $\mathcal{C} \cap (\mathcal{A} \setminus \mathcal{B}) = \emptyset$, in which case $M'[\mathcal{C}, \mathcal{A} \setminus \mathcal{B}] = 1$. On the other hand if a k-set \mathcal{C} is not a subset of a k+1-set \mathcal{B} then $\mathcal{C} \cap (\mathcal{A} \setminus \mathcal{B}) \neq \emptyset$, and $M'[\mathcal{C}, \mathcal{A} \setminus \mathcal{B}] = 0$. So we see that M' is the incidence matrix of the Kneser graph $K_{2k+1,k}$. But due to Corollary 6.6 it has maximal rank, and the same holds for M as relabelling of columns does not change the rank of the matrix.

7. SIMPLE HYPERGRAPHS.

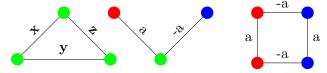
Definition 7.1. Let $0 \le m \le k$. We call a k-hypergraph $\mathbb{G} = (V, \mu)$ (m, \mathbf{a}) -simple if there exist pairwise disjoint sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that

- (1) $V = A \cup B \cup C$.
- (2) $\mathcal{A} = \{\alpha_1, \alpha_2 \dots \alpha_m\}$ and $\mathcal{B} = \{\beta_1, \beta_2 \dots \beta_m\}$ are *m*-sets.
- (3) $|\mathcal{C}| = 2(k-m) 1$ or $\mathcal{C} = \emptyset$ if m = k.
- (4) $\forall \mathcal{X} \subseteq \mathcal{A} \cup \mathcal{B}$, such that $\mathcal{X} = \{x_1, x_2 \dots x_m\}$, where $x_i \in \{\alpha_i, \beta_i\}$, the equality $\Lambda_{\mathcal{X}}(\mathbb{G}) = (-1)^{|\mathcal{B} \cap \mathcal{X}|} \cdot \mathbf{a}$ holds.
- (5) For any other $\mathcal{X} \subseteq V$ such that $|\mathcal{X}| = m$ the equality $\Lambda_{\mathcal{X}}(\mathbb{G}) = \mathbf{0}$ holds.
- (6) $\forall \mathcal{X} \subseteq V$, such that $|\mathcal{X}| < m$ the equality $\Lambda_{\mathcal{X}}(\mathbb{G}) = \mathbf{0}$ holds.

This definition looks horrible so we analyse it using examples, and explain the required properties.

(1) If m = 0 then \mathcal{A}, \mathcal{B} are empty, condition 2 states that \mathcal{C} has 3 vertices, condition 4 that $\Lambda_{\emptyset}(\mathbb{G}_0) = \mathbf{a}$, conditions 5 and 6 are empty.

Example 7.2. Figure 3: From the left to the right: a $(0, \mathbf{a})$ – simple 2-hypergraph \mathbb{G}_0 , where $\mathbf{a} = \mathbf{x} + \mathbf{y} + \mathbf{z}$; a $(1, \mathbf{a})$ – simple 2-hypergraph \mathbb{G}_1 ; a $(2, \mathbf{a})$ – simple 2-hypergraph \mathbb{G}_2 . The sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are marked with colours.



(2) If m=1 then $|\mathcal{A}|=|\mathcal{B}|=|\mathcal{C}|=1$, conditions 4 and 5 provide 3 equations

$$\mathbf{a} = \Lambda_{\bullet}(\mathbb{G}_1) = \Lambda_{\{\bullet,\bullet\}}(\mathbb{G}_1) + \Lambda_{\{\bullet,\bullet\}}(\mathbb{G}_1) \tag{7.1}$$

$$-\mathbf{a} = \Lambda_{\bullet}(\mathbb{G}_1) = \Lambda_{\{\bullet,\bullet\}}(\mathbb{G}_1) + \Lambda_{\{\bullet,\bullet\}}(\mathbb{G}_1)$$
(7.2)

$$\mathbf{0} = \Lambda_{\bullet}(\mathbb{G}_1) = \Lambda_{\{\bullet,\bullet\}}(\mathbb{G}_1) + \Lambda_{\{\bullet,\bullet\}}(\mathbb{G}_1). \tag{7.3}$$

It is not hard to derive from them that $\mathbf{a} = \Lambda_{\{\bullet,\bullet\}}(\mathbb{G}_1)$, $-\mathbf{a} = \Lambda_{\{\bullet,\bullet\}}(\mathbb{G}_1)$, and $\mathbf{0} = \Lambda_{\{\bullet,\bullet\}}(\mathbb{G}_1)$. The condition 6 states that $\Lambda_{\emptyset}(\mathbb{G}_1) = \mathbf{0}$.

(3) If m=2 then $|\mathcal{A}|=|\mathcal{B}|=2$ and $\mathcal{C}=\emptyset$. Conditions 4 and 5 define weights of all edges, the condition 4 is responsible for edges with non-zero weights and 5 for the edges with the weight **0**. The condition 6 says that the weight of \emptyset and weights of single vertices are **0**.

Now, why we characterise simple hypergraphs using such complicated conditions? The most important property of simple hypergraph is the last one. It implies the following lemma:

Lemma 7.3. Let $\mathbb{H} = (V, \mu)$ be a k-hypergraph and $\mathbb{S}^m_{\mathbf{a}}$ be an (m, \mathbf{a}) - simple k-hypergraph. Then for any $\mathcal{X} \subseteq V, |\mathcal{X}| < m$ we have $\Lambda_{\mathcal{X}}(\mathbb{H}) = \Lambda_{\mathcal{X}}(\mathbb{H} + \mathbb{S}^m_{\mathbf{a}})$.

Proof. Indeed, $\Lambda_{\mathcal{X}}$ is a homomorphism and $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}) = \mathbf{0}$ due to property 6 Definition 7.1.

It justifies the following structure of the proof of the theorem. We gradually simplify your target hypergraph by adding (i, \mathbf{a}) -simple hypergraphs for growing i. In the step i we start from a hypergraph with $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mathbf{0}$ for every $\mathcal{X} \subset V$ with at most i-1 vertices. Using (i, \mathbf{a}) -simple hypergraphs we simplify it to a hypergraph with $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mathbf{0}$ for all $\mathcal{X} \subset V$ with exactly i vertices. The property 6 guaranties that while we perform the step i, we do not ruin our work from previous steps i.e. we reach a hypergraph such $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mathbf{0}$ for every $\mathcal{X} \subset V$ with at most i vertices. Eventually, we reach a hypergraph with all the weights equal $\mathbf{0}$ i.e. the empty hypergraph.

Now conditions 4 and 5. We already mentioned that using (i, \mathbf{a}) -simple hypergraphs we want to reduce to $\mathbf{0}$ all weights of sets in $V^{(i)}$; thus it is good to keep weights on the level i as simple as possible, i.e. $\mathbf{a}, \mathbf{0}, -\mathbf{a}$.

Other conditions are a trade-off between two things:

- we wanted to have simple hypergraphs as small as possible in terms of number of vertices and number of non-zero hyperedges,
- required simple hypergraphs must be in \mathbb{Z} -Sums(EQ(\mathcal{H})).

Remark 7.4. Note that (m, \mathbf{a}) -simple hypergraph are not defined uniquely. For example $(0, \mathbf{a})$ -simple hypergraph from the example above is not fully defined. It is also not clear if simple hypergraphs exist, this is proven later (we show how to construct them).

We also introduce 2 other notions to work with families of simple hypergraphs.

Definition 7.5. Let \mathcal{G} be a family of k-hypergraphs. $\mathbb{G} = (V_{\mathbb{G}}, \mu_{\mathbb{G}})$ for every $\mathbb{G} \in \mathcal{G}$. For every $0 \leq m \leq k$ let Γ_m be a group generated by $\{\Lambda_{\mathcal{X}}(\mathbb{G}) \mid \text{ for every } \mathbb{G} \in \mathcal{G}, \mathcal{X} \in V_{\mathbb{G}}^{(i)}\}$. A family of k-hypergraphs \mathcal{S} is a *simplification of the family* \mathcal{G} if for every $0 \leq m \leq k$ and every $\mathbf{g} \in \Gamma_m$ it contains an $(m, \mathbf{g}) - simple\ k$ -hypergraph.

Definition 7.6. The family S is *simple for itself* if S is a simplification of S. A family S is *simple for* a given k-hypergraph $\mathbb{H} = (V, \Lambda_{\mathbb{H}})$ if S is a family simple for itself and is a simplification of $\{\mathbb{H}\}$.

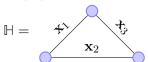
Remark 7.7. To produce a simple family, it suffices to design an algorithm alg(S) that produces a simplification of its input. As a simple family we take

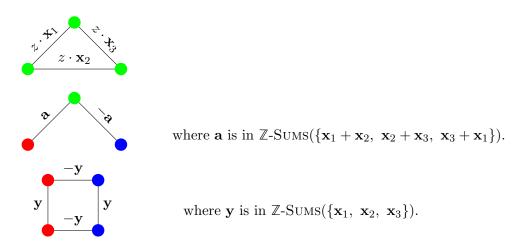
$$\bigcup_{i\in\mathbb{N}}alg^{i+1}(\{\mathbb{H}\}) \text{ where } alg^i \text{ is i-th iteration of the } alg.$$

Note that the produced family is not necessarily finite.

Example 7.8. In Figure 4 you may find an example of a simple family for a given hypergraph \mathbb{H} .

Figure 4: Figure presents an example of a simple family for the graph \mathbb{H} . $z \in \mathbb{Z}$ and the sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are marked with colours.





Lemma 7.9. Let S be a family simple for a hypergraph \mathbb{H} . Suppose $\mathbb{G} \in \mathbb{Z}$ -Sums(EQ(S)) then the family S is simple for $\mathbb{H} + \mathbb{G}$.

Proof. Let \mathcal{X} be a subset of vertices of $\mathbb{H} + \mathbb{G}$. By definition if in \mathcal{S} there are (i, \mathbf{a}) -simple and (i, \mathbf{b}) -simple hypergraphs then there is an $(i, \mathbf{a} + \mathbf{b})$ -simple hypergraph too. Thus, it is sufficient to prove that if $|\mathcal{X}| \leq k$ then in \mathcal{S} there is a $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{H} + \mathbb{G}))$ -simple hypergraph. Which is also trivial as $\Lambda_{\mathcal{X}}(\mathbb{H} + \mathbb{G}) = \Lambda_{\mathcal{X}}(\mathbb{H}) + \Lambda_{\mathcal{X}}(\mathbb{G})$ and in \mathcal{S} there are $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{H}))$ -simple and $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{G}))$ - simple hypergraphs.

8. Expressing ℍ with simple hypergraphs

Our goal in this section is to prove Theorem 8.2 which is a slightly stronger version of the the following claim.

Claim: If S is a simple family for a k-hypergraph \mathbb{H} then $\mathbb{H} \in \mathbb{Z}$ -Sums(EQ(S)).

We need a stronger version to use in the proof of existence of simple hypergraphs. The stronger version requires the notion of a support of a sum.

Definition 8.1. For a hypergraph $\mathbb{H} = (V, \mu)$ its support is V. We denote it $SUPPORT(\mathbb{H})$. Let $\mathbb{H} \in \mathbb{Z}$ -Sums(Eq(\mathcal{H})). We say that V supports \mathbb{H} for the family \mathcal{H} if there is a solution of the following equation

$$\mathbb{H} = \sum_{i>0} a_i \mathbb{H}_i$$
 where $a_i \in \mathbb{Z}, \mathbb{H}_i \in \mathrm{EQ}(\mathcal{H})$, and $\mathrm{SUPPORT}(\mathbb{H}_i) \subseteq V$ for all i .

Theorem 8.2. Let $\mathbb{H} = (V, \mu)$ be a k-hypergraph. Further, let S be a simple family for \mathbb{H} . Then, $\mathbb{H} \in \mathbb{Z}$ -Sums(EQ(S)). Moreover, if |V| > 2k - 1 then V supports \mathbb{H} for the family S.

Proof of this theorem requires a few definitions and lemmas stated below. We start with them and then we prove the theorem while proofs of lemmas are postponed.

Definition 8.3. Let $\mathbb{H} = (V, \mu)$ be a k-hypergraph. We say that it is m-isolated if for any subset $\mathcal{X} \subseteq V$ such that $|\mathcal{X}| \leq m$ holds $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mathbf{0}$.

Remark 8.4. If \mathbb{H} is a k-isolated k-hypergraph, then \mathbb{H} is equivalent to the empty hypergraph.

Definition 8.5. Let $\mathbb{H} = (V, \mu)$ be a k-hypergraph. We say that it is almost m-isolated if the following two conditions are satisfied:

- \mathbb{H} is (m-1)-isolated,
- there is $\mathcal{X} \subseteq V$ a set of vertices such that $|\mathcal{X}| \leq 2m 1$ and for any $\mathcal{Y} \in V^{(m)}$ such that $\mathcal{Y} \not\subseteq \mathcal{X}$ holds $\Lambda_{\mathcal{Y}}(\mathbb{H}) = \mathbf{0}$.

Lemma 8.6. If \mathbb{H} is a k-hypergraph that is almost m-isolated then it is m-isolated.

Lemma 8.7. Let $\mathbb{H} = (V, \mu)$ be an m-isolated k-hypergraph, and S be a simple family for \mathbb{H} . Then there is a hypergraph $\mathbb{G} \in \mathbb{Z}$ -Sums(Eq(S)), such that $\mathbb{H} + \mathbb{G}$ is almost m + 1-isolated. Moreover, if |V| > 2k - 1 then V supports \mathbb{G} for the family S.

Proof of Theorem 8.2. Without loss of generality we may assume that |V| > 2k - 1, as otherwise we extend it with a few isolated vertices.

We construct a sequence of hypergraphs $\mathbb{G}_i \in \mathbb{Z}$ -SUMS(EQ(\mathcal{S})) such that $\mathbb{H} - \sum_{i=0}^{j} \mathbb{G}_i$ is j-isolated. Also, for every $i \leq j$ the set V supports \mathbb{G}_i for the family \mathcal{S} . We define \mathbb{G} as $\sum_{i=0}^{k} \mathbb{G}_i$. Observe that $\mathbb{H} = \mathbb{G}$ as $\mathbb{H} - \sum_{i=0}^{k} \mathbb{G}_i$ is a k-isolated k-hypergraph, so is empty due to Remark 8.4. Moreover, V supports \mathbb{G} for the family \mathcal{S} as for every $i \leq k$ the set V supports \mathbb{G}_i for the family \mathcal{S} .

The construction is via induction on j. As \mathcal{S} is a simple family for \mathbb{H} then there is $\mathbb{G}_0 \in \mathrm{Eq}(\mathcal{S})$ and $\mathrm{SUPPORT}(\mathbb{G}) \subset V$ such that $\mathbb{H} - \mathbb{G}_0$ is 0-isolated. This creates the induction base.

For the inductive step we reason as follows. First we observe that due to Lemma 7.9 the family \mathcal{S} is simple for $\mathbb{H} - \sum_{i=0}^{j} \mathbb{G}_{i}$. Thus, we may use Lemma 8.7 for a hypergraph $\mathbb{H} - \sum_{i=0}^{j} \mathbb{G}_{i}$ and the family \mathcal{S} . As a consequence we get $\mathbb{G}_{j+1} \in \mathbb{Z}$ -SUMS(EQ(\mathcal{S})) such that $(\mathbb{H} - \sum_{i=0}^{j} \mathbb{G}_{i}) - \mathbb{G}_{j+1}$ is almost j+1-isolated. Moreover, V supports \mathbb{G}_{j+1} for the family \mathcal{S} . Further, Lemma 8.6 implies that $(\mathbb{H} - \sum_{i=0}^{j} \mathbb{G}_{i}) - \mathbb{G}_{j+1}$ is j+1-isolated.

Note that V supports \mathbb{G}_{j+1} for the family S thus V supports $(\mathbb{H} - \sum_{i=0}^{j+1} \mathbb{G}_i)$ for the family S, too. This ends the inductive step.

Proof of Lemma 8.6. Before we prove Lemma 8.6 we prove a simple lemma about the $\Lambda_{\mathcal{X}}$ functions. The main tool in the proof of Lemma 8.6 are reduction matrices defined in Section 6.

Lemma 8.8. Suppose $\mathbb{H} = (V, \mu)$ is a k-hypergraph, $\mathcal{X} \in V^{(m)}$ and $m \leq l \leq k$. Let $\mathcal{F} = \{ \mathcal{Y} \in V^{(l)} \mid \mathcal{X} \subseteq \mathcal{Y} \}$, be a family of l-element supersets of \mathcal{X} . Then:

$$\sum_{\mathcal{Y} \in \mathcal{F}} \Lambda_{\mathcal{Y}}(\mathbb{H}) = \binom{k-m}{l-m} \Lambda_{\mathcal{X}}(\mathbb{H}).$$

Proof. Let e be a hyperedge in \mathbb{H} such that $\mathcal{X} \subseteq e$. It suffices to prove that $\mu(e)$ appears the same number of times on the both sides of the equation.

$$\sum_{\mathcal{Y} \in \mathcal{F}} \left(\sum_{e \in V^{(k)}, \mathcal{Y} \subseteq e} \mu(e) \right) = \binom{k-m}{l-m} \sum_{e \in V^{(k)}, \mathcal{X} \subseteq e} \mu(e).$$

On the right-hand side e is added $\binom{k-m}{l-m}$ times. On the left-hand side the number of times when e is added is equal to the number of l-element supersets of $\mathcal X$ that are included in e. But this is equal to $\binom{|e|-|\mathcal X|}{l-|\mathcal X|} = \binom{k-m}{l-m}$, as required.

Proof of Lemma 8.6. Let \mathcal{X} be a set of vertices as in Definition 8.5. We only need to prove that for any $\mathcal{X}' \subseteq \mathcal{X}$ such that $|\mathcal{X}'| = m$ holds $\Lambda_{\mathcal{X}'}(\mathbb{H}) = \mathbf{0}$.

Let $\mathcal{Y}_1, \mathcal{Y}_2 \dots \mathcal{Y}_n$ be all (m-1)-subsets of \mathcal{X} and let $\mathcal{X}'_1, \mathcal{X}'_2 \dots \mathcal{X}'_{n'}$ be all m-subsets of \mathcal{X} .

Due to Lemma 8.8 we know that for any \mathcal{Y}_i the equation

$$\sum_{\mathcal{X}_j'\supset\mathcal{Y}_i}\Lambda_{\mathcal{X}_j'}(\mathbb{H})=\left((2m-1)-(m-1)\right)\cdot\Lambda_{\mathcal{Y}_j}(\mathbb{H}) \text{ holds}.$$

But \mathbb{H} is m-1 isolated so $\Lambda_{\mathcal{Y}_i}(\mathbb{H}) = \mathbf{0}$.

This system of equation may be rewritten in matrix form

$$Cu = \mathbf{0} \text{ where}$$

$$u \stackrel{\text{def}}{=} \begin{bmatrix} \Lambda_{\mathcal{X}_{1}^{\prime}}(\mathbb{H}) \\ \Lambda_{\mathcal{X}_{2}^{\prime}}(\mathbb{H}) \\ \vdots \\ \Lambda_{\mathcal{X}_{2}^{\prime}}(\mathbb{H}) \end{bmatrix}$$

and $C \stackrel{\text{def}}{=} \overline{[2m-1, m, m-1]}$ ($\overline{[\bullet, \bullet, \bullet]}$ are defined in Definition 6.1).

But according to Lemma 6.3 the rank of the matrix C is maximal, which implies $u = \mathbf{0}$ is the only solution of the system of equations. Thus $\Lambda_{\mathcal{X}'_j}(\mathbb{H}) = \mathbf{0}$ for any $j \leq n'$ and consequently \mathbb{H} is m-isolated.

Proof of Lemma 8.7. The proof of Lemma 8.7 requires some preparation.

Definition 8.9. Suppose $\mathcal{X} \subset V, |\mathcal{X}| < k$. We define $\mathbb{H}_{|-\mathcal{X}} \stackrel{\text{def}}{=} (V \setminus \mathcal{X}, \mu_{|-\mathcal{X}})$, where $\mu_{|-\mathcal{X}}(e)$ is a function from $(V - \mathcal{X})^{(k-|\mathcal{X}|)}$ to \mathbb{Z}^d and is defined as follows

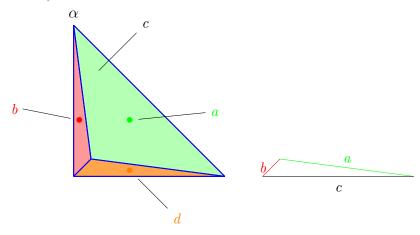
$$\mu_{|-\mathcal{X}}(e) \stackrel{\text{def}}{=} \mu(e \cup \mathcal{X}) \text{ where } e \in (V - \mathcal{X})^{(k-|\mathcal{X}|)}.$$

The above operation is called \mathcal{X} cut of \mathbb{H} . The reverse operation called enrich \mathbb{H} with \mathcal{X} is denoted by $\mathbb{H}_{|+\mathcal{X}|}$ and its effect is a minimal in the sense of inclusion $(k+|\mathcal{X}|)$ -hypergraph \mathbb{G} such that $\mathbb{G}_{|-\mathcal{X}|} = \mathbb{H}$.

If $\mathcal{X} = \{\alpha\}$ is a singleton, then we simplify the notation $\mathbb{G}_{|-\alpha} \stackrel{\text{def}}{=} \mathbb{G}_{|-\{\alpha\}}$ and $\mathbb{G}_{|+\alpha} \stackrel{\text{def}}{=} \mathbb{G}_{|+\{\alpha\}}$.

Example 8.10. See Figure 5.

Figure 5: On the left-hand side there is a 3-hypergraph $\mathbb H$ and on the right hand side there is a 2-hypergraph $\mathbb H_{|-\alpha}$. The orange bottom hyperedge disappears as it does not contain α . $\mathbb H_{|-\alpha|+\alpha}$ would look like $\mathbb H$ but without the orange hyperedge, as the enrich operation takes the minimal hypergraph among all such that cutting α returns $\mathbb H_{|-\alpha}$.



Lemma 8.11. Let $\mathbb{H} = (V, \mu)$ k-hypergraph, and $\mathcal{X} \in V^{(m)}$ for m < k. Then for any nonempty $\mathcal{Y} \subseteq V$ such that $\mathcal{Y} \cap \mathcal{X} = \emptyset$ holds $\Lambda_{\mathcal{Y} \cup \mathcal{X}}(\mathbb{H}) = \Lambda_{\mathcal{Y}}(\mathbb{H}_{|-\mathcal{X}})$. In particular, if \mathbb{H} is l-isolated and m < l then $\mathbb{H}_{|-\mathcal{X}}$ is (l-m)-isolated.

¹Inclusion of the sets of hyperedges.

Proof. Note that e and e' are parameters of the sums below.

$$\begin{split} \Lambda_{\mathcal{Y} \cup \mathcal{X}}(\mathbb{H}) &= \sum_{e \in V^{(k)}, \mathcal{Y} \cup \mathcal{X} \subseteq e} \mu(e) = \sum_{e \in V^{(k)}, \mathcal{Y} \cup \mathcal{X} \subseteq e} \Lambda_{e \setminus \mathcal{X}}(\mathbb{H}_{|-\mathcal{X}}) = \\ &\sum_{(e \setminus \mathcal{X}) \in V^{(k-|\mathcal{X}|)}, \mathcal{Y} \subseteq (e \setminus \mathcal{X})} \Lambda_{e \setminus \mathcal{X}}(\mathbb{H}_{|-\mathcal{X}}) = \end{split}$$

By e' we denote hyperedges of the hypergraph $\mathbb{H}_{|-\mathcal{X}}$.

$$\sum_{\mathcal{Y} \subset e'} \Lambda_{e'}(\mathbb{H}_{|-\mathcal{X}}) = \Lambda_{\mathcal{Y}}(\mathbb{H}_{|-\mathcal{X}}) \qquad \Box$$

Proof of Lemma 8.7. Without loss of generality, we assume that |V| > 2k - 1. Indeed, if $|V| \le 2k - 1$ then we extend $\mathbb H$ with a few isolated vertices.

In the proof, instead of constructing \mathbb{G} directly, we build a finite sequence of hypergraphs \mathbb{H}_i , such that $\mathbb{H}_0 = \mathbb{H}$, \mathbb{H}_{last} is almost l+1-isolated and each $\mathbb{H}_{i+1} = \mathbb{H}_i - \mathbb{S}^{\bullet}_{-i}$ where $\mathbb{S}^{\bullet}_{-i}$ is a $(l+1, _)$ -simple hypergraph such that $\mathbb{S}^{\bullet}_{-i} \in \mathrm{EQ}(\mathcal{S})$ and $\mathrm{SUPPORT}(\mathbb{S}^{\bullet}_{-i}) \subseteq V$. $\mathbb{S}^{\bullet}_{-i}$ is l-isolated by definition. Intuitively, we build a sequence of improvements and consequently $\mathbb{G} = -\sum_i \mathbb{S}^{\bullet}_{-i}$.

First, we define a well-founded quasi-order on hypergraphs, next we present a single improvement i.e. how to obtain \mathbb{H}_{i+1} from \mathbb{H}_i , finally we show that $\mathbb{H}_{i+1} < \mathbb{H}_i$ in the defined quasi-order and that if the hypergraph can not be improved anymore then it is almost l+1-isolated.

Order. First we impose a well founded linear order on the set of all vertices. A set of vertices \mathcal{X}_1 is smaller than a set \mathcal{X}_2 if there is a bijection $f: \mathcal{X}_1 \to \mathcal{X}_2$ such that $\alpha \leq f(\alpha)$ for every $\alpha \in \mathcal{X}_1$. Finally, a hypergraph $\mathbb{H}' = (V', \mu') \leq_{l+1} \mathbb{H}$ if for every $\mathcal{X}' \in V'^{(l+1)}$ such that $\Lambda_{\mathcal{X}'}(\mathbb{H}') \neq \mathbf{0}$ there is a set $\mathcal{X} \in V^{(l+1)}$ such that $\Lambda_{\mathcal{X}}(\mathbb{H}) \neq \mathbf{0}$ and \mathcal{X} is bigger than \mathcal{X}' .

The strict inequality $<_{l+1}$ holds if $\mathbb{H}' \leq_{l+1} \mathbb{H}$ but $\mathbb{H}' \not\geq_{l+1} \mathbb{H}$.

Assumption 8.12. From now on until the end of this proof whenever we enumerate vertices of some set $\alpha_1, \alpha_2, \ldots \alpha_n$ we assume that $\alpha_i < \alpha_j$ for i < j.

Improvement. We define the improvement for \mathbb{H}_i . Let $\mathcal{F} \subset \text{SUPPORT}(\mathbb{H}_i)^{(l+1)}$ such that $\mathcal{L} \in \mathcal{F} \Longrightarrow \Lambda_{\mathcal{L}}(\mathbb{H}_i) \neq \mathbf{0}$. Note that, if l+1=k then \mathcal{F} is the set of hyperedges of \mathbb{H}_i . Suppose $\mathcal{L} \in \mathcal{F}$ is maximal in \mathcal{F} and there is a set $\mathcal{L}' \in V^{(l+1)}$ disjoint from \mathcal{L} such that $\mathcal{L}' < \mathcal{L}$. Let $\mathcal{L} = \{\alpha_1, \alpha_2 \dots \alpha_{l+1}\}$. Let $\mathbb{S}^{\bullet}_{-i} \in \text{EQ}(\mathcal{S})$ be an $(l+1, \Lambda_{\mathcal{L}}(\mathbb{H}_i))$ -simple hypergraph supported by V such that (according to Definition 7.1) $\mathcal{A} = \mathcal{L}$ and $\mathcal{B} = \mathcal{L}' = \{f^{-1}(\alpha_1), f^{-1}(\alpha_2), \dots f^{-1}(\alpha_{l+1})\}$, where f witnesses $\mathcal{L}' < \mathcal{L}$. It exists due to Lemma 7.9. Let $\mathbb{H}_{i+1} \stackrel{\text{def}}{=} \mathbb{H}_i - \mathbb{S}^{\bullet}$.

Let $\mathbb{H}_{i+1} \stackrel{\text{def}}{=} \mathbb{H}_i - \mathbb{S}^{\bullet}_{-i}$. We claim $\mathbb{H}_{i+1} <_{l+1} \mathbb{H}_i$. Indeed, $\Lambda_{\mathcal{L}}(\mathbb{H}_{i+1}) = \mathbf{0}$ and \mathcal{L} was maximal in \mathcal{F} so it sufficient to prove that for any $\overline{\mathcal{L}} \in \text{SUPPORT}(\mathbb{S}^{\bullet}_{-i})^{(l+1)}$ such that $\overline{\mathcal{L}} \neq \mathcal{L}$ and $\Lambda_{\overline{\mathcal{L}}}(\mathbb{S}^{\bullet}_{-i}) \neq \mathbf{0}$ holds $\overline{\mathcal{L}} < \mathcal{L}$. But $\Lambda_{\overline{\mathcal{L}}}(\mathbb{S}^{\bullet}_{-i}) \neq \mathbf{0}$ if and only if $\overline{\mathcal{L}}$ is a subset of $\mathcal{L}' \cup \mathcal{L}$ that contains exactly one element from each pair of vertices $(\alpha_i, f^{-1}(\alpha_i))$ (Definition 7.1). As $\alpha_i > f^{-1}(\alpha_i)$ for every i then trivially $\mathcal{L} = \{\alpha_1, \alpha_2, \dots \alpha_{l+1}\} > \overline{\mathcal{L}}$, as required.

Reduced form. We call a hypergraph reduced if one can not improve it any further. If we consequently improve a given hypergraph then eventually we reach \mathbb{H}_{last} which is reduced. Indeed, every improvement goes down in the quasi-order on hypergraphs and the quasi-order is trivially well-founded. What remains to prove is the following claim.

Claim 8.13. A reduced hypergraph is almost l + 1-isolated.

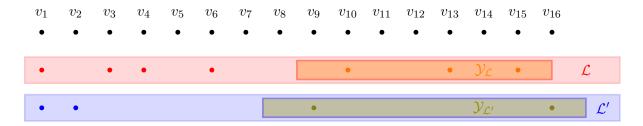


Figure 6:

Proof of the claim. First observe that \mathbb{H}_{last} is l-isolated as \mathbb{H} and each $\mathbb{S}_{-i}^{\bullet}$ are l-isolated (by Definition 7.1 every $(l+1, _)$ -simple hypergraph is l-isolated).

Observe SUPPORT(\mathbb{H}_{last}) $\subseteq V$. Suppose $V = \{\alpha_1, \alpha_2 \dots \alpha_{|V|}\}$. Let $\mathcal{F} \subseteq V^{(l+1)}$ be a family of sets of vertices such that for any $\mathcal{L} \in \mathcal{F}$ we have $\Lambda_{\mathcal{L}}(\mathbb{H}_i) \neq \mathbf{0}$ and \mathcal{L} contains a vertex with index greater than 2l+1. Our goal is to prove that \mathcal{F} is empty; this implies the claim.

We prove it via contradiction. Suppose $\mathcal{F} \neq \emptyset$. For a set $\mathcal{L} \in \mathcal{F}$, $\mathcal{L} = \{\alpha_{i_1}, \alpha_{i_2}, \dots \alpha_{i_{l+1}}\}$ we introduce $\mathcal{Y}_{\mathcal{L}} \subseteq \mathcal{L}$ the maximal (in the sense of inclusion) set of vertices $\{\alpha_{i_j}, \alpha_{i_{j+1}}, \dots, \alpha_{i_{l+1}}\}$ such that $i_j \geq 2j$, $i_{j+1} \geq 2(j+1)$, ... $i_{l+1} \geq 2(l+1)$. By definition for every $\mathcal{L} \in \mathcal{F}$ the set $\mathcal{Y}_{\mathcal{L}} \neq \emptyset$.

Let $\mathcal{L} \in \mathcal{F}$ be a set such that $\mathcal{Y}_{\mathcal{L}}$ is maximal in the following sense: for every $\overline{\mathcal{L}} \in \mathcal{F}$ holds $\mathcal{Y}_{\mathcal{L}} \not\subset \mathcal{Y}_{\overline{\mathcal{L}}}$. We show the contradiction by proving $\Lambda_{\mathcal{L}}(\mathbb{H}_{last}) = \mathbf{0}$.

Observe that $\mathcal{L} \neq \mathcal{Y}_{\mathcal{L}}$ as in this case \mathbb{H}_{last} would not be reduced.

Take, $\mathbb{G}' \stackrel{\text{def}}{=} \mathbb{H}_{last|-\mathcal{Y}_{\mathcal{L}}}$ (recall Definition 8.9). As $\mathcal{L} \neq \mathcal{Y}_{\mathcal{L}}$ we know that $\Lambda_{\mathcal{L}}(\mathbb{H}_{last}) = \Lambda_{\mathcal{L}\setminus\mathcal{Y}_{\mathcal{L}}}(\mathbb{G}')$. So, if we prove that \mathbb{G}' is $(|\mathcal{L}| - |\mathcal{Y}_{\mathcal{L}}|)$ -isolated then $\mathbf{0} = \Lambda_{\mathcal{L}\setminus\mathcal{Y}_{\mathcal{L}}}(\mathbb{G}) = \Lambda_{\mathcal{L}}(\mathbb{H}_{last})$ and we have the contradiction with the assumption that $\Lambda_{\mathcal{L}}(\mathbb{H}_{last}) \neq \mathbf{0}$.

So what remains, is to prove the following claim.

Claim 8.14. \mathbb{G}' is $(|\mathcal{L}| - |\mathcal{Y}_{\mathcal{L}}|)$ -isolated.

Track the proof on the Figure 6.

Let $l' = (|\mathcal{L}| - |\mathcal{Y}_{\mathcal{L}}|)$. It suffices to prove that \mathbb{C}' is almost l'-isolated. Indeed, then due to Lemma 8.6 \mathbb{C}' is l'-isolated.

Note that \mathbb{G}' is l'-1-isolated which is inherited from \mathbb{H}_{last} , due to Lemma 8.11. So it remains to analyse weights of l'-subsets of V. We have to prove that every $\overline{\mathcal{L}} \in V^{(l')}$ if $\Lambda_{\overline{\mathcal{L}}}(\mathbb{G}') \neq \mathbf{0}$ then it is a subset of $\{\alpha_1 \dots \alpha_{2l'-1}\}$.

Let us take $\overline{\mathcal{L}} = \{\alpha_{i'_1}, \alpha_{i'_2} \dots \alpha_{i'_{l'}}\}$, such that $i'_{l'} \geq 2l'$. We prove that $\Lambda_{\overline{\mathcal{L}}}(\mathbb{G}') = \mathbf{0}$. Observe, $\mathcal{Y}_{\mathcal{L}} \subseteq \mathcal{Y}_{\mathcal{L}} \cup \{\alpha_{i'_{l'}}\} \subseteq \mathcal{Y}_{(\overline{\mathcal{L}} \cup \mathcal{Y}_{\mathcal{L}})}$. Indeed, $\mathcal{Y}_{\mathcal{L}}$ guarantees that in $\overline{\mathcal{L}} \cup \mathcal{Y}_{\mathcal{L}}$ there is at least one element not smaller than 2(l+1), at least two elements not smaller than $2l, \dots$, at least $|\mathcal{Y}_{\mathcal{L}}|$ elements not smaller than 2(l'+1). Moreover, because of $i'_{l'} \geq 2l'$, there are at least $|\mathcal{Y}_{\mathcal{L}}| + 1$ elements not smaller than 2l'.

Due to maximality of \mathcal{L} we conclude that $\Lambda_{\mathcal{L}' \cup \mathcal{Y}_{\mathcal{L}}}(\mathbb{H}_{last}) = \mathbf{0}$, thus $\Lambda_{\mathcal{L}' \cup \mathcal{Y}_{\mathcal{L}} \setminus \mathcal{Y}_{\mathcal{L}}}(\mathbb{H}_{last|-\mathcal{Y}_{\mathcal{L}}}) = \Lambda_{\overline{\mathcal{L}}}(\mathbb{G}') = \mathbf{0}$. Therefore every $\overline{\mathcal{L}}$ an l'-set of vertices in \mathbb{G}' , for which $\Lambda_{\overline{\mathcal{L}}} \neq \mathbf{0}$, is a subset of $\{\alpha_1, \alpha_2, \dots \alpha_{2l'-1}\}$. This ends the proof that \mathbb{G}' is almost l'-isolated.

Finally, as it was written earlier, from Lemma 8.6 we derive that \mathbb{G}' is l'-isolated. \square

9. The construction of simple hypergraphs.

We start from an operator that is used in later proofs.

Definition 9.1. Suppose $\mathbb{G} = (V', \mu')$ is a k-hypergraph and $\alpha \in V', \alpha' \notin V'$ are two vertices. Let $\sigma_{\alpha} \colon V' \cup \{\alpha'\} \to V' \cup \{\alpha'\}$ such that:

$$\sigma_{\alpha}(x) \stackrel{\text{def}}{=} \begin{cases} \alpha' & \text{when } x = \alpha \\ \alpha & \text{when } x = \alpha' \\ x & \text{otherwise} \end{cases}$$

A swap of α and α' in \mathbb{G} is defined as $\tau_{\alpha'}(\mathbb{G}, \alpha) \stackrel{\text{def}}{=} \mathbb{G} \circ \sigma_{\alpha}$.

Theorem 9.2. Suppose \mathbb{G} is a k-hypergraph. Then there is \mathcal{S}' a simple family for $\{\mathbb{G}\}$. Moreover $\mathcal{S}' \subseteq \mathbb{Z}$ -SUMS $(EQ(\mathbb{G}))$.

Due to Remark 7.7 it is sufficient to propose an algorithm that for any k-hypergraph \mathbb{G} produces S a simplification of $\{\mathbb{G}\}$ such that $S \subseteq \mathbb{Z}$ -Sums(Eq(\mathbb{G})). Existence of a such algorithm is a consequence of the following lemmas.

Lemma 9.3. Let $\mathbb{G} = (V', \mu')$ be a k-hypergraph and $\mathcal{X} \in V'^{(m)}$ where $m \leq k$. Let $\mathbf{a} = \Lambda_{\mathcal{X}}(\mathbb{G})$. Then there is a (m, \mathbf{a}) - simple k-hypergraph $\mathbb{S}^m_{\mathbf{a}}$ such that $\mathbb{S}^m_{\mathbf{a}} \in \mathbb{Z}$ -SUMS(EQ(\mathbb{G})).

Lemma 9.4. For any $0 \le m \le k$ if (m, \mathbf{a}) -simple and (m, \mathbf{b}) -simple hypergraphs are elements of \mathbb{Z} -SUMS(EQ(\mathbb{G})) then $(m, \mathbf{a} + \mathbf{b})$ -simple hypergraph is also a member of \mathbb{Z} -SUMS(EQ(\mathbb{G})).

Indeed, using this two lemmas we can produce all elements in the simplification. The proof of Lemma 9.4 is easy so we start from it and then we concentrate on the much more complicated proof of Lemma 9.3.

Lemma 9.4. Let $\mathbb{S}^m_{\mathbf{a}} \in \mathbb{Z}$ -Sums(EQ(\mathbb{G})) is the (m, \mathbf{a}) -simple hypergraph and $\mathbb{S}^m_{\mathbf{b}} \in \mathbb{Z}$ -Sums(EQ(\mathbb{G})) is the (m, \mathbf{b}) -simple hypergraph. Thus we can write

$$\begin{split} \mathbb{S}^m_{\mathbf{a}} &= \sum_i a_i \mathbb{G}_i \text{ where } a_i \in \mathbb{Z} \text{ and } \mathbb{G}_i \in \mathrm{EQ}(\mathbb{G}) \\ \mathbb{S}^m_{\mathbf{b}} &= \sum_j b_j \mathbb{G}_j \text{ where } b_j \in \mathbb{Z} \text{ and } \mathbb{G}_j \in \mathrm{EQ}(\mathbb{G}) \end{split}$$

We recall the definition of simple hypergraphs; the vertices of $\mathbb{S}^m_{\mathbf{a}}$ can be split into $\mathcal{A}_{\mathbb{S}^m_{\mathbf{a}}}, \mathcal{B}_{\mathbb{S}^m_{\mathbf{a}}}, \mathcal{C}_{\mathbb{S}^m_{\mathbf{a}}}$ and similarly vertices of $\mathbb{S}^m_{\mathbf{b}}$ are in $\mathcal{A}_{\mathbb{S}^m_{\mathbf{b}}}, \mathcal{B}_{\mathbb{S}^m_{\mathbf{b}}}, \mathcal{C}_{\mathbb{S}^m_{\mathbf{b}}}$. We assume that sets $\mathcal{A}_{\mathbf{a}}$ and $\mathcal{B}_{\mathbf{a}}$ are ordered like in the definition of simple hypergraphs. Then there is a bijection π between vertices that transfers $\mathcal{A}_{\mathbb{S}^m_{\mathbf{a}}}$ to $\mathcal{A}_{\mathbb{S}^m_{\mathbf{b}}}, \mathcal{B}_{\mathbb{S}^m_{\mathbf{a}}}$ to $\mathcal{B}_{\mathbb{S}^m_{\mathbf{b}}}, \mathcal{C}_{\mathbb{S}^m_{\mathbf{a}}}$ to $\mathcal{C}_{\mathbb{S}^m_{\mathbf{b}}}$, and that preserves the orders on vertices in $\mathcal{A}_{\mathbf{a}}$ and $\mathcal{B}_{\mathbf{a}}$. Thus

$$\mathbb{S}_{\mathbf{a}}^{m} + \mathbb{S}_{\mathbf{b}}^{m} \circ \pi = \sum_{i} a_{i} \mathbb{G}_{i} + (\sum_{j} b_{j} \mathbb{G}_{j}) \circ \pi = \sum_{i} a_{i} \mathbb{G}_{i} + \sum_{j} b_{j} (\mathbb{G}_{j}) \circ \pi$$

But this mean that $\mathbb{S}^m_{\mathbf{a}} + \mathbb{S}^m_{\mathbf{a}} \circ \pi \in \mathbb{Z}$ -Sums(Eq(G)). It is not hard to see that $\mathbb{S}^m_{\mathbf{a}} + \mathbb{S}^m_{\mathbf{a}} \circ \pi$ is an $(m, \mathbf{a} + \mathbf{b})$ -simple hypergraph.

The proof of Lemma 9.3 goes via induction on k. We encapsulate the most important steps of the proof in four lemmas. Lemma 9.5 is an auxiliary lemma. Lemma 9.6 forms the induction base. Lemmas 9.7 and 9.8 cover the induction step.

Lemma 9.5. Let $\mathbb{S}^m_{\mathbf{a}} = (V, \mu)$ be an (m, \mathbf{a}) - simple k-hypergraph and let $\alpha, \alpha' \notin V$ be two vertices. Suppose $\mathbb{S}^m_{\mathbf{a}} \in \mathbb{Z}$ -Sums $(\mathrm{Eq}(\mathbb{G}_{|-\alpha}))$. Then:

- (1) $\mathbb{S}_{\mathbf{a}}^{m+1} = \mathbb{S}_{\mathbf{a}\mid+\alpha}^m \mathbb{S}_{\mathbf{a}\mid+\alpha'}^m$ is an $(m+1,\mathbf{a})$ simple (k+1)-hypergraph,
- (2) $\mathbb{S}_{\mathbf{a}}^{m+1} \in \mathbb{Z}\text{-Sums}(\mathrm{Eq}(\mathbb{G})).$

Proof. Claim 1. Let $V = \mathcal{A}' \cup \mathcal{B}' \cup \mathcal{C}'$, where $\mathcal{A}', \mathcal{B}', \mathcal{C}'$ are such as in the definition of $(m, \mathbf{a}) - simple$ hypergraph (Definition 7.1). Let $\mathcal{A}' = \{\alpha_1, \alpha_2 \dots \alpha_m\}$ and $\mathcal{B}' = \{\beta_1, \beta_2 \dots \beta_m\}$. We define $\mathcal{A} \stackrel{\text{def}}{=} \{\alpha_1, \alpha_2 \dots \alpha_m, \alpha_{m+1} = \alpha\}$ and $\mathcal{B} \stackrel{\text{def}}{=} \{\beta_1, \beta_2 \dots \beta_m, \beta_{m+1} = \alpha'\}$. To show that $\mathbb{S}^{m+1}_{\mathbf{a}}$ is $(m+1, \mathbf{a}) - simple$ we split $V \cup \{\alpha, \alpha'\}$ to \mathcal{A}, \mathcal{B} , and \mathcal{C}' and we verify the properties 1 to 7. The properties 1 to 3 are trivial. Properties 4 to 6 speak about function $\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}})$ where:

• The property 4. \mathcal{X} contains exactly one vertex from every pair α_i, β_i for $0 < i \le m+1$. We consider only one of the two cases as the second one is very similar. Suppose, $\alpha' \in \mathcal{X}$. Then $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}_{|+\alpha}}) = \mathbf{0}$ so

$$\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}}) = -\Lambda_{\mathcal{X}}(\mathbb{S}^{m}_{\mathbf{a}\mid+\alpha'}) = -\Lambda_{\mathcal{X}\backslash\{\alpha\}}(\mathbb{S}^{m}_{\mathbf{a}}).$$

Thus $\Lambda_{\mathcal{X}}(\mathbb{S}_{\mathbf{a}}^{m+1}) = -1 \cdot (-1)^{|\mathcal{X} \cap \mathcal{B}'|} \mathbf{a} = (-1)^{|\mathcal{X} \cap B|} \mathbf{a}$, as required.

- The property 5. \mathcal{X} has m+1 elements but it is not of the type considered in the property 4. In this case $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}\mid+\alpha}) = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}\mid+\alpha'}) = \mathbf{0}$, thus $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}\mid+\alpha}) = \mathbf{0}$, as required.
- The property 6. $|\mathcal{X}| \leq m$. We consider four cases: $\mathcal{X} \cap \{\alpha, \alpha'\} = \{\alpha, \alpha'\}, \mathcal{X} \cap \{\alpha, \alpha'\} = \{\alpha\}, \mathcal{X} \cap \{\alpha, \alpha'\} = \{\alpha'\}, \text{ and } \mathcal{X} \cap \{\alpha, \alpha'\} = \emptyset.$
 - In the first case, α is not a vertex of $\mathbb{S}^m_{\mathbf{a}\mid+\alpha'}$ and α' is not a vertex of $\mathbb{S}^m_{\mathbf{a}\mid+\alpha}$. So, $\mathbf{0}=\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}\mid+\alpha})=\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}\mid+\alpha'})$. Thus, $\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}})=\mathbf{0}$, as required.
 - In the second case, $\mathbf{0} = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}|_{+\alpha'})$ and $\Lambda_{\mathcal{X}\setminus\{\alpha\}}(\mathbb{S}^m_{\mathbf{a}}) = \mathbf{0}$ as $|\mathcal{X}\setminus\{\alpha\}| < m$. Thus, $\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}}) = \mathbf{0}$, as required.
 - The third case is almost the same as second.
 - In the fourth case, $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}\mid+\alpha}) = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}\mid+\alpha'})$, so $\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}}) = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}\mid+\alpha}) \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}\mid+\alpha'}) = \mathbf{0}$, as required.

Claim 2. Let $\mathbb{S}^m_{\mathbf{a}} = \sum_i a_i \mathbb{G}_i$ where $a_i \in \mathbb{Z}$, $\mathbb{G}_i \in \mathrm{EQ}(\mathbb{G}_{|-\alpha})$, and additionaly $\alpha' \notin \bigcup_i \mathrm{SUPPORT}(\mathbb{G}_i) \cup \mathrm{SUPPORT}(\mathbb{G})$. Suppose π_i are bijections that are identity on $\{\alpha, \alpha'\}$ and that witness equivalence of summands and $\mathbb{G}_{|-\alpha}$ i.e. $\mathbb{G}_i = \mathbb{G}_{|-\alpha} \circ \pi_i$. They exist as sets of vertices of $\mathbb{G}_{|-\alpha}$ and \mathbb{G}_i do not contain neither α nor α' . Due to point 1, we may write the following equation

$$\mathbb{S}_{\mathbf{a}}^{m+1} = \sum_{i} a_i (\mathbb{G}_{|-\alpha} \circ \pi_i)_{|+\alpha} - a_i (\mathbb{G}_{|-\alpha} \circ \pi_i)_{|+\alpha'}.$$

As π_i is identity on α, α' we may transform the equation.

$$\mathbb{S}_{\mathbf{a}}^{m+1} = \sum_{i} a_{i} \mathbb{G}_{|-\alpha|+\alpha} \circ \pi_{i} - a_{i} \mathbb{G}_{|-\alpha|+\alpha'} \circ \pi_{i}.$$

To prove the second claim it suffices to prove for each i

$$a_i \mathbb{G}_{|-\alpha|+\alpha} \circ \pi_i - a_i \mathbb{G}_{|-\alpha|+\alpha'} \circ \pi_i = a_i \mathbb{G} \circ \pi_i - a_i \mathbb{G}' \circ \pi_i$$

$$(9.1)$$

where $\mathbb{G}' = \tau_{\alpha'}(\mathbb{G}, \alpha)$. We simplify further

$$\mathbb{G}_{|-\alpha|+\alpha} \circ \pi_i - \mathbb{G}_{|-\alpha|+\alpha'} \circ \pi_i = \mathbb{G} \circ \pi_i - \mathbb{G}' \circ \pi_i \tag{9.2}$$

We prove Equation 9.2 by showing equality of functions $\Lambda_{\mathcal{X}}$ on both sides. So for any $\mathcal{X} \subseteq \pi_i^{-1}(\text{SUPPORT}(\mathbb{G})) \cup \text{SUPPORT}(\mathbb{G}'))$ we prove

$$\Lambda_{\mathcal{X}}\left(\mathbb{G}_{|-\alpha|+\alpha}\circ\pi_{i}-\mathbb{G}_{|-\alpha|+\alpha'}\circ\pi_{i}\right)=\Lambda_{\mathcal{X}}\left(\mathbb{G}\circ\pi_{i}-\mathbb{G}'\circ\pi_{i}\right). \tag{9.3}$$

We consider 4 cases depending on whether $\alpha, \alpha' \in \mathcal{X}$.

- If \mathcal{X} does not contain α and α' then $\Lambda_{\mathcal{X}}(\mathbb{G} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G}' \circ \pi_i)$, so the right hand side equals $\mathbf{0}$ exactly like the left-hand side of the equality.
- If \mathcal{X} contains only α then $\Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha|+\alpha} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G} \circ \pi_i)$ and $\Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha|+\alpha'} \circ \pi_i) = \mathbf{0} = \Lambda_{\mathcal{X}}(\mathbb{G}' \circ \pi_i)$, so the equality holds.
- Similarly for only α' .
- If \mathcal{X} contains both α and α' then $\mathbf{0} = \Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha|+\alpha} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha|+\alpha'} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G}' \circ \pi_i)$. So both sides of the equality are $\mathbf{0}$.

Thus the equality 9.2 holds for every \mathcal{X} .

Lemma 9.6. ² Let $\mathbb{G} = (V, \mu)$ be a 1-hypergraph. Then for any $\mathcal{X} \subseteq V$ and $|\mathcal{X}| \leq 1$ there is a $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{G}))$ – simple 1-hypergraph $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})}$, such that $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} \in \mathbb{Z}$ -Sums(Eq(\mathbb{G})).

Proof. We consider two cases (i) $|\mathcal{X}| = 1$ and (ii) $|\mathcal{X}| = 0$.

The first case. Let $\mathcal{X} = \{\alpha\}$, $\mathbf{a} = \Lambda_{\alpha}(\mathbb{G})$, and $\alpha' \notin V$. We define $\mathbb{S}_{\mathbf{a}}^{\bullet} \stackrel{\text{def}}{=} \mathbb{G} - \tau_{\alpha'}(\mathbb{G}, \alpha)$. For any vertex $\beta \neq \alpha, \alpha'$ we have $\Lambda_{\beta}(\mathbb{S}_{\mathbf{a}}^{\bullet}) = \mathbf{0}$. So $\mathbb{S}_{\mathbf{a}}^{\bullet}$ has two non-isolated vertices $\{\alpha, \alpha'\}$ and satisfies all properties of a simple 1-hypergraph.

The second case. Let $\Lambda_{\emptyset}(\mathbb{G}) = \mathbf{b}$, $\alpha' \notin V$. Further, let $\mathbb{S}^{\bullet}_{\mu(\beta)} = \mathbb{G} - \tau_{\alpha'}(\mathbb{G}, \beta)$ and for any $\beta \in V$. Then we put $\mathbb{S}^{0}_{\mathbf{b}} \stackrel{\text{def}}{=} \mathbb{G} - \left(\sum_{\beta \in V} \mathbb{S}^{\bullet}_{\mu(\beta)}\right)$. Indeed, it is a hypergraph with only one non-isolated vertex α' and $\Lambda_{\alpha'}(\mathbb{S}^{0}_{\mathbf{b}}) = -\sum_{\beta \in V} \Lambda_{\alpha'}(\mathbb{S}^{\bullet}_{\mu(\beta)}) = \Lambda_{\emptyset}(\mathbb{G})$, as required.

Lemma 9.7. Suppose that Theorem 9.2 holds if restricted to k-hypergraphs. Let $\mathbb{G} = (V, \mu)$ be a k+1-hypergraph. Then for any nonempty $\mathcal{X} \subseteq V$ and $0 < |\mathcal{X}| \le k+1$ there is a $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{G})) - simple \ k+1$ -hypergraph $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})}$, such that $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} \in \mathbb{Z}$ -Sums(Eq(\mathbb{G})).

Proof of Lemma 9.7. We show how to construct the hypergraph $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})}$. Let $\alpha \in \mathcal{X}$. Consider $\mathbb{G}_{|-\alpha}$ and a set $\mathcal{X}' = \mathcal{X} \setminus \{\alpha\}$. Observe, $\Lambda_{\mathcal{X}'}(\mathbb{G}_{|-\alpha}) = \Lambda_{\mathcal{X}}(\mathbb{G})$. So, due to the assumption we know that there is $\mathbb{S}^{|\mathcal{X}'|}_{\Lambda_{\mathcal{X}}(\mathbb{G})}$ a $(|\mathcal{X}'|, \Lambda_{\mathcal{X}}(\mathbb{G})) - simple\ k$ -hypergraph, such that $\mathbb{S}^{|\mathcal{X}'|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} \in \mathbb{Z}$ -Sums $(\mathbb{E}_{Q}(\mathbb{G}_{|-\alpha}))$. Now, due to Lemma 9.5, for some $\alpha' \notin V$, the k+1-hypergraph $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} = \mathbb{S}^{|\mathcal{X}'|}_{\Lambda_{\mathcal{X}}(\mathbb{G})_{|+\alpha}} - \mathbb{S}^{|\mathcal{X}'|}_{\Lambda_{\mathcal{X}}(\mathbb{G})_{|+\alpha'}}$ is $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{G})) - simple\ (point\ 1)$ and $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} \in \mathbb{Z}$ -Sums $(\mathbb{E}_{Q}(\mathbb{G}))$ (point 2).

Lemma 9.8. Suppose that Theorem 9.2 holds if restricted to k-hypergraphs. Let \mathbb{G} be a k+1-hypergraph and $\Lambda_{\emptyset}(\mathbb{G}) = \mathbf{a}$. Then there is a $(0,\mathbf{a})$ - simple k+1-hypergraph $\mathbb{S}^0_{\mathbf{a}}$ such that $\mathbb{S}^0_{\mathbf{a}} \in \mathbb{Z}$ -SUMS(EQ(\mathbb{G})).

Proof of Lemma 9.8. First, observe that $(0, \mathbf{a})$ -simple k + 1-hypergraph has to satisfy only two properties, it has 2k + 1 vertices and the sum of weights of all hyperedges equals \mathbf{a} .

²This lemma is reformulated Theorem 15 from [HLT17].

Thus, the lemma is a consequence of a procedure (defined below) that takes a k+1-hypergraph \mathbb{F} with l vertices and if l>2k+1 then it returns a k+1-hypergraph \mathbb{F}' such that $\Lambda_{\emptyset}(\mathbb{F}) = \Lambda_{\emptyset}(\mathbb{F}')$, $|\text{SUPPORT}(\mathbb{F}')| < l$, and $\mathbb{F}' \in \mathbb{Z}\text{-Sums}(\text{Eq}(\mathbb{F}))$. The procedure applied many times starting from \mathbb{G} produces the required simple hypergraph.

The procedure. Let $\mathbb{F} = (V, \mu)$. Suppose $\alpha \in V$. Our goal is to construct a k+1-hypergraph \mathbb{F}' such that $\text{SUPPORT}(\mathbb{F}') \subseteq \text{SUPPORT}(\mathbb{F}) \setminus \{\alpha\}$. As the first step, we construct a k+1-hypergraph $\mathbb{K} \in \mathbb{Z}$ -Sums $(\text{Eq}(\mathbb{F}))$ such that

- (1) $\Lambda_{\emptyset}(\mathbb{K}) = \mathbf{0}$.
- (2) For every \mathcal{X} , a set of vertices containing α , we have that $\Lambda_{\mathcal{X}}(\mathbb{F}) = \Lambda_{\mathcal{X}}(\mathbb{K})$.
- (3) $SUPPORT(\mathbb{K}) \subseteq SUPPORT(\mathbb{F})$.

Then, $\mathbb{F}' = \mathbb{F} - \mathbb{K}$. The second property is responsible for deleting all hyperedges containing α and the third property for not adding any new vertex.

We define \mathbb{K} as follows. Let \mathcal{S} be a simple family for $\mathbb{F}_{|-\alpha}$. By, $\hat{\mathcal{S}}$ we denote a set of all k-hypergraphs supported by $V \setminus \{\alpha\}$ and equivalent to elements of \mathcal{S} . Due to Theorem 8.2 we know that $\mathbb{F}_{|-\alpha} \in Z$ -Sums(Eq(\mathcal{S})) supported by $V \setminus \{\alpha\}$, i.e.

$$\mathbb{F}_{|-\alpha} = \sum_{i} a_i \mathbb{S}_{-i}^* \text{ where } a_i \in \mathbb{Z} \text{ and } \mathbb{S}_{-i}^* \in \hat{\mathcal{S}}.$$
 (9.4)

Thus,
$$\mathbb{F}_{|-\alpha|+\alpha} = \sum_{i} a_i \mathbb{S}^*_{-i|+\alpha}$$
. (9.5)

We define K as

$$\mathbb{K} \stackrel{\text{def}}{=} \mathbb{F}_{|-\alpha|+\alpha} - \sum_{i} a_{i} \mathbb{S}_{-i|+\alpha'_{i}}^{*} = \sum_{i} a_{i} (\mathbb{S}_{-i|+\alpha}^{*} - \mathbb{S}_{-i|+\alpha'_{i}}^{*})$$
where each $\alpha'_{i} \in V \setminus (\text{SUPPORT}(\mathbb{S}_{-i}^{*}) \cup \{\alpha\}).$

$$(9.6)$$

Observe, that $|\text{SUPPORT}(\mathbb{S}_{-i}^*) \cup \{\alpha\}| \le 2k+1 < |V| \text{ as } |V| > 2k+1.$

K satisfies the required properties:

- 0. $\mathbb{K} \in \mathbb{Z}$ -Sums(Eq(\mathbb{F})). It is sufficient to show that $\mathbb{S}^*_{-i|+\alpha} \mathbb{S}^*_{-i|+\alpha'_i} \in \mathbb{Z}$ -Sums(Eq(\mathbb{F})). Due to the assumption that Theorem 9.2 holds for k-hypergraphs we know that $\mathbb{S}^*_{-i} \in \mathbb{Z}$ -Sums(Eq($\mathbb{F}_{|-\alpha}$)). If we combine this with Lemma 9.5 (point 2) then we get that each hypergraph ($\mathbb{S}^*_{-i|+\alpha} \mathbb{S}^*_{-i|+\alpha'_i}$) $\in \mathbb{Z}$ -Sums(Eq(\mathbb{F})).
- (1) $\Lambda_{\emptyset}(\mathbb{K}) = \mathbf{0}$. Due to Lemma 9.5 (point 1) we know that $\mathbb{S}^*_{-i|+\alpha} \mathbb{S}^*_{-i|+\alpha'_i}$ is an element of a simple family for \mathbb{F} and it is $(l, \cdot) simple$ for l > 0. Thus, $\Lambda_{\emptyset}(\mathbb{S}^*_{-i|+\alpha} \mathbb{S}^*_{-i|+\alpha'_i}) = \mathbf{0}$ and $\Lambda_{\emptyset}(\mathbb{K}) = \mathbf{0}$ as \mathbb{K} is a sum of hypergraphs $\mathbb{S}^*_{-i|+\alpha} \mathbb{S}^*_{-i|+\alpha'_i}$.
- (2) $\Lambda_{\mathcal{X}}(\mathbb{K}) = \Lambda_{\mathcal{X}}(\mathbb{F})$ for any \mathcal{X} such that $\alpha \in \mathcal{X}$. Indeed,

$$\Lambda_{\mathcal{X}}(\mathbb{K}) = \sum_{i} \Lambda_{\mathcal{X}}(\mathbb{S}^*_{-i|+\alpha}) - \Lambda_{\mathcal{X}}(\mathbb{S}^*_{-i|+\alpha'_{i}}) \quad \text{(Equation 9.6)}$$

but each $\mathbb{S}^*_{-i|+\alpha'_i}$ does not contain α thus

$$\Lambda_{\mathcal{X}}(\mathbb{K}) = \sum_{i} \Lambda_{\mathcal{X}}(\mathbb{S}^*_{-i|+\alpha})) = \Lambda_{\mathcal{X}}(\mathbb{F}_{|-\alpha|+\alpha}) \quad \text{(Equation 9.5)}.$$

But, according to Definition 8.9 (of the $\cdot_{|-\alpha}$ operation) and Lemma 8.11 we know that for any set of vertices $\mathcal{Y} \subseteq V \setminus \{\alpha\}$:

$$\Lambda_{\mathcal{Y} \cup \{\alpha\}}(\mathbb{F}) = \Lambda_{\mathcal{Y}}(\mathbb{F}_{|-\alpha}) = \Lambda_{\mathcal{Y} \cup \{\alpha\}}(\mathbb{F}_{|-\alpha|+\alpha}). \tag{9.7}$$

Thus for $\mathcal{X} = \mathcal{Y} \cup \{\alpha\}$ we get $\Lambda_{\mathcal{X}}(\mathbb{K}) = \Lambda_{\mathcal{X}}(\mathbb{F})$.

(3) SUPPORT(\mathbb{K}) $\subseteq V$. Indeed, for every \mathbb{S}_{-i}^* hold SUPPORT($\mathbb{S}_{-i|+\alpha}^*$) $\subseteq V$ and SUPPORT($\mathbb{S}_{-i|+\alpha_i}^*$) $\subseteq V$.

Proof of Theorem 9.2. The proof goes via induction on k. The base for the induction is given by Lemma 9.6.

The induction step is given by Lemmas 9.7 and 9.8.

10. The proof of Theorem 3.4 itself.

Having proven theorems 9.2 and 8.2 the proof of Theorem 3.4 is simple.

Proof. From the left to the right. Observe that $\Lambda_{\mathcal{X}}$ is a homomorphism, for any set \mathcal{X} . Thus if $\mathbb{H} \in \mathbb{Z}$ -Sums(EQ(\mathcal{H})) i.e. $\mathbb{H} = \sum_i a_i \mathbb{G}_i$ where $a_i \in \mathbb{Z}$ and $\mathbb{G}_i \in \text{EQ}(\mathcal{H})$, then $\Lambda_{\mathcal{X}}(\mathbb{H}) = \sum_i a_i \Lambda_{\mathcal{X}}(\mathbb{G}_i)$, for any set $\mathcal{X} \in V$. The above holds for any set $\mathcal{X} \subseteq V$ thus \mathbb{H} is locally a \mathbb{Z} sum of \mathcal{H} .

From the right to the left. If \mathbb{H} is locally a \mathbb{Z} sum of \mathcal{H} then for any set $\mathcal{X} \subseteq V$ there is a k-hypergraph $\mathbb{G}_{\mathcal{X}}$ such that $\Lambda_{\mathcal{X}}(\mathbb{H}) = \Lambda_{\mathcal{X}}(\mathbb{G}_{\mathcal{X}})$ and $\mathbb{G}_{\mathcal{X}} \in \mathbb{Z}$ -Sums(Eq(\mathcal{H})). Due to Theorem 9.2 we conclude that there is a family simple for \mathbb{H} such that its elements are in \mathbb{Z} -Sums(Eq(\mathcal{H})). Now we can apply Theorem 8.2 together with Lemma 4.2 and conclude that $\mathbb{H} \in \mathbb{Z}$ -Sums(Eq(\mathcal{H})).

11. Proof of Theorem 2.2.

We prove Theorem 2.2 by showing that in **NExp-Time** it is possible to reduce \mathbb{N} -solvability to \mathbb{Z} -solvability. The produced instance of \mathbb{Z} -solvability is of exponential size, and can be solved in **Exp-Time** due to Theorem 2.1.

Before we start, we need to recall some facts about solution of systems of linear equations.

Hybrid linear sets.

Definition 11.1. A set of vectors is called *hybrid linear* if it is the smallest set that includes a finite set \mathcal{B} , called a base, and that is closed under the addition of elements from a finite set \mathcal{P} , called periods.

Theorem 11.2 ([Pot91]). Let M be a $d \times m$ -matrix with integer entries and $\mathbf{y} \in \mathbb{Z}^d$. The set of nonnegative integer solutions of linear equations

$$M \cdot \mathbf{x} = \mathbf{y}$$

is a hybrid linear set. The base \mathcal{B} and periods \mathcal{P} are as follows:

- Base: it is the set of minimal, in the pointwise sense, solutions $M \cdot \mathbf{x} = \mathbf{y}$.
- Periods: is the set of minimal nontrivial solutions of

$$M \cdot \mathbf{x} = \mathbf{0}.\tag{11.1}$$

In the paper [Pot91] Pottier provides bounds on the norms of \mathcal{B} and \mathcal{P} .

For a vector $\mathbf{v} \in \mathbb{Z}^m$ we introduce two norms: the infinity norm $\|\mathbf{v}\|_{\infty}$ and the norm one $\|\mathbf{v}\|_1$, defined as follows:

- $\|\mathbf{v}\|_{\infty} \stackrel{\text{def}}{=} max(\{|\mathbf{v}[i]| \mid \text{ for } 1 \leq i \leq m\}),$ $\|\mathbf{v}\|_{1} \stackrel{\text{def}}{=} \sum_{i=1}^{m} |\mathbf{v}[i]|.$

Let $\|\mathcal{H}\|_{\infty} \stackrel{\text{def}}{=} max(\{\|\mathbf{h}\|_{\infty} \mid \mathbf{h} \in \mathcal{H}\})$ and $\|\mathcal{H}\|_{1,\infty} \stackrel{\text{def}}{=} max(\{\|\mathbf{h}\|_1 \mid \mathbf{h} \in \mathcal{H}\})$. Also, for a $d \times m$ -matrix M we introduce $||M||_{1,\infty} \stackrel{\text{def}}{=} ||\mathcal{M}||_{1,\infty}$ where \mathcal{M} is the set of columns of the matrix M.

Lemma 11.3 ([Pot91]). Let $M \cdot \mathbf{x} = \mathbf{y}$ be a system of linear equations such that M is a $d \times m$ -matrix. Then the set of solutions in \mathbb{N}^m is the hybrid linear set and is described by the base \mathcal{B} and the set of periods \mathcal{P} such that:

- $\mathcal{B}, \mathcal{P} \subset \mathbb{N}^m$,
- $|\mathcal{P}| \leq d \cdot m$
- $\|\mathcal{B}\|_{\infty}$, $\|\mathcal{P}\|_{\infty} \le (\|M\|_{1,\infty} + \|\mathbf{y}\|_{\infty} + 2)^{d+m}$.

Definition 11.4. A vector $\mathbf{x} \in \mathcal{U}$ is reversible in a family of vectors \mathcal{U} if $-\mathbf{x} \in \mathbb{N}$ -Sums(\mathcal{U}). Vectors that are not reversible we call non-reversible.

Thus from Equation 11.1 and Lemma 11.3 we conclude.

Lemma 11.5. Let U_1 and U_2 be two finite sets of vectors in \mathbb{Z}^d such that every vector in \mathcal{U}_2 is non-reversible in $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$. Suppose $\mathbf{y} \in \mathbb{N}$ -Sums(\mathcal{U}).

$$\mathbf{y} = \sum_{\mathbf{v} \in \mathcal{U}_1} a_{\mathbf{v}} \cdot \mathbf{v} + \sum_{\mathbf{w} \in \mathcal{U}_2} b_{\mathbf{w}} \cdot \mathbf{w}$$
 (11.2)

where $a_{\mathbf{v}}, b_{\mathbf{w}} \in \mathbb{N}$.

Then

$$\sum_{\mathbf{w}\in\mathcal{U}_2} b_{\mathbf{w}} \le |\mathcal{U}_2| \cdot (\|\mathcal{U}\|_{1,\infty} + \|\mathbf{y}\|_{\infty} + 2)^{d+|\mathcal{U}|}$$

i.e. is bounded exponentially.

Proof. The set of solutions of Equation 11.2 is hybrid linear, and given by some $\mathcal{B}, \mathcal{P} \subset$ $\mathbb{N}^{|\mathcal{U}_1 \cup \mathcal{U}_2|}$. Every period is a solution of the equation

$$\mathbf{0} = \sum_{\mathbf{v} \in \mathcal{U}_1} a_{\mathbf{v}} \cdot \mathbf{v} + \sum_{\mathbf{v} \in \mathcal{U}_2} b_{\mathbf{w}} \cdot \mathbf{w}. \tag{11.3}$$

From the definition of reversibility we get that in any solution of Equation 11.3 all b_i are equal to 0. Thus in every solution of the Equation 11.2 the sum $\sum_{\mathbf{v} \in \mathcal{U}_2} b_{\mathbf{w}}$ is bounded by $|\mathcal{U}_2| \cdot ||\mathcal{B}||_{\infty} \leq |\mathcal{U}_2| \cdot (||\mathcal{U}||_{1,\infty} + ||\mathbf{y}||_{\infty} + 2)^{d+|\mathcal{U}|}$, where the last inequality is given by Lemma 11.3.

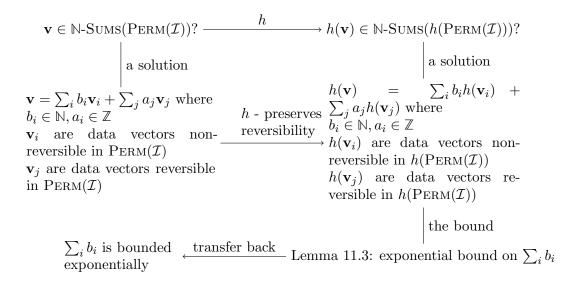


Figure 7: The diagram of the proof of Theorem 2.2.

Proof of Theorem 2.2. The idea of this reduction is as follows. We use \mathcal{I} for a finite family of data vectors in $\mathcal{D}^{(k)} \to \mathbb{Z}^d$ and \mathbf{v} for a target vector. Similarly to Definition 11.4 reversibility can be defined for data vectors. Namely, a data vector \mathbf{y} is reversible in a set of data vectors \mathcal{I} if $-\mathbf{y} \in \mathbb{N}$ -Sums(Perm(\mathcal{I})).

The schema of the proof may be tracked on the diagram in Figure 7. Next, we define a homomorphism from data vectors $\mathcal{D}^{(k)} \to \mathbb{Z}^d$ to vectors in \mathbb{Z}^d . The homomorphism is defined in such a way that it has an additional property, namely, for every data vector $\mathbf{y} \in \mathcal{I}$ its image is reversible in the image of the set \mathcal{I} if and only if \mathbf{y} is reversible in \mathcal{I} . Now, the given instance N-solvability problem for data vectors $\mathcal{D}^{(k)} \to \mathbb{Z}^d$ (called the first problem) is transformed, via the homomorphism, to the N-solvability for vectors in \mathbb{Z}^d , i.e. system of linear equations (called the second problem). The homomorphic image of any solution to the first problem is also a solution of the second problem. Due to Lemma 11.3, there is a bound on the usage of non-reversible vectors in any solution of the second problem. This bound can be then transferred back through the homomorphism. This provides us with the bound on the usage of non-reversible data vectors in any solution of the first problem. With the bound, we can guess the non-reversible part of the solution for the first problem. What remains is to verify that our guess is correct. To do this we have to show that the target minus the guessed non-reversible part can be expressed using the reversible data vectors. But this is exactly Z-solvability. Indeed, if data vectors can be reversed then we can subtract them freely.

Definition 11.6. Let $\mathbf{y} \colon \mathcal{D}^{(k)} \to \mathbb{Z}^d$ be a data vector. The *data projection* of \mathbf{y} is defined as $\mathfrak{P}(\mathbf{y}) \stackrel{\text{def}}{=} \sum_{x \in \mathcal{D}^{(k)}} \mathbf{y}(x)$. It is well defined as \mathbf{y} is almost everywhere equal to $\mathbf{0}$. For a set of data vectors \mathcal{I} we define $\mathfrak{P}(\mathcal{I}) \stackrel{\text{def}}{=} \bigcup_{\mathbf{y} \in \mathcal{I}} \{\mathfrak{P}(\mathbf{y})\}$ to be the *data projection* of \mathcal{I} . ³

Proposition 11.7. A data projection is a homomorphism from the group of data vectors with addition to \mathbb{Z}^d .

³Data projection \mathfrak{P} is the same thing as $\Lambda_{\emptyset}(\underline{\ })$ but in the hypergraphs domain.

Definition 11.8. For a data vector \mathbf{y} its support denoted $\mathrm{SUPPORT}(\mathbf{y}) \subset \mathcal{D}$ is the set $\{\alpha \mid \exists_{x \in \mathcal{D}^{(k)}} \text{ such that } \alpha \in x \text{ and } \mathbf{y}(x) \neq \mathbf{0}\}$. We say that the vector \mathbf{y} is supported by a set $V \subset \mathcal{D}$ if $\mathrm{SUPPORT}(\mathbf{y}) \subseteq V$. The definition may be lifted to the set of data vectors $\mathrm{SUPPORT}(\mathcal{I}) \stackrel{\mathrm{def}}{=} \bigcup_{\mathbf{v} \in \mathcal{I}} \mathrm{SUPPORT}(\mathbf{y})$.

Definition 11.9. Let $V \stackrel{\text{def}}{=} \text{SUPPORT}(\mathcal{I}) \cup \text{SUPPORT}(\mathbf{v})$ (as in the formulation of the N-solvability problem). By Π we denote a set of data permutations that are identity outside of V (all permutations of V). For a data vector $\mathbf{y} \in \mathcal{I} \cup \{\mathbf{v}\}$ we define the *smoothing* operator, $smooth(\mathbf{y}) \stackrel{\text{def}}{=} \sum_{\pi \in \Pi} \mathbf{y} \circ \pi$.

Lemma 11.10. Let k, V be as defined above. There is the constant c depending on k and |V|, such that for any $x \in V^{(k)}$ and any data vector $\mathbf{y} \in \mathcal{I} \cup \{\mathbf{v}\}$ the equality $smooth(\mathbf{y})(x) = \mathfrak{P}(\mathbf{y}) \cdot c$ holds.

Proof. Due to symmetry. We do not need the explicit formula.

Lemma 11.11. $\mathbf{y} \in \mathcal{I}$ is reversible in PERM(\mathcal{I}) if and only if $\mathfrak{P}(\mathbf{y})$ is reversible in $\mathfrak{P}(\mathcal{I})$.

Proof. \Longrightarrow Trivial as \mathfrak{P} is a homomorphism.

 \Leftarrow Let V and Π be as in Definition 11.9. We list elements of $\mathcal{I} = \{\mathbf{v}_1, \dots \mathbf{v}_m\}$. By M we denote the $1 \times m$ -matrix with entries in \mathbb{Z}^d such that $X[1][j] = \mathfrak{P}(\mathbf{v}_j)$.

From the assumptions we know that there is a vector $\mathbf{x} \in \mathbb{N}^m$ such that

$$M \cdot \mathbf{x} = -\mathfrak{P}(\mathbf{y}). \tag{11.4}$$

Let c be the constant from Lemma 11.10. We multiply both sides of Equation 11.4 by c getting

$$c \cdot M \cdot \mathbf{x} = -c \cdot \mathfrak{P}(\mathbf{y}). \tag{11.5}$$

But from this and Lemma 11.10 we conclude that for any $z \in V^{(k)}$

$$\left(\sum_{\mathbf{v}_i \in \mathcal{I}} (smooth(\mathbf{v}_i) \cdot \mathbf{x}[i])\right)(z) = -smooth(\mathbf{y})(z)$$

and what follows

$$\left(\sum_{\mathbf{v}_i \in \mathcal{I}} (smooth(\mathbf{v}_i) \cdot \mathbf{x}[i])\right) = -smooth(\mathbf{y}).$$
(11.6)

Using the definition of smoothing we rewrite further

$$\sum_{\mathbf{v}_i \in \mathcal{I}} (smooth(\mathbf{v}_i) \cdot \mathbf{x}[i]) + \sum_{\pi \in \Pi \setminus \{identity\}} \mathbf{y} \circ \pi = -\mathbf{y}.$$
(11.7)

As $\mathbf{y} \in \mathcal{I}$ we see that we expressed $-\mathbf{y}$ as a sum of elements in PERM(\mathcal{I}).

Corollary 11.12. Non-reversible in PERM(\mathcal{I}) elements of \mathcal{I} can be identified in P-Time (independently from d).

Proof. Let $\mathbf{y} \in \mathcal{I}$. \mathbf{y} is reversible in PERM(\mathcal{I}) if and only if there is $l \in \mathbb{N}$ such that $-l \cdot \mathfrak{P}(\mathbf{y}) \in \mathbb{N}$ -SUMS($\mathfrak{P}(\mathcal{I})$). Indeed, it is sufficient to add $(l-1)\mathfrak{P}(\mathbf{y})$. Thus the question about reversibility is equivalent to the question if $\mathfrak{P}(\mathbf{y}) \in \mathbb{Q}_+$ -SUMS($\mathfrak{P}(\mathcal{I})$), where \mathbb{Q}_+ stands for nonnegative rationals. The last question is known as the linear programming problem and is known to be solvable in **P-Time** [CLS19].

Lemma 11.13. Let \mathcal{I}_1 and \mathcal{I}_2 form the partition of the set \mathcal{I} such that \mathcal{I}_1 is the set of data vectors reversible in PERM(\mathcal{I}) and \mathcal{I}_2 is the set of data vectors non-reversible in PERM(\mathcal{I}). Suppose \mathbf{v} can be expressed as a sum

$$\mathbf{v} = \left(\sum_{\mathbf{v}_i \in PERM(\mathcal{I}_1)} a_i \mathbf{v}_i\right) + \left(\sum_{\mathbf{v}_j \in PERM(\mathcal{I}_2)} b_j \mathbf{v}_j\right) \text{ where } a_i, b_j \in \mathbb{N}.$$
 (11.8)

Then the sum $\sum_{\mathbf{v}_j \in PERM(\mathcal{I}_2)} b_j$ is bounded exponentially, precisely $\sum_{\mathbf{v}_j \in PERM(\mathcal{I}_2)} b_j \leq |\mathfrak{P}(\mathcal{I}_2)| \cdot (\|\mathfrak{P}(\mathcal{I})\|_{1,\infty} + \|\mathfrak{P}(\mathbf{v})\|_{\infty} + 2)^{d+|\mathcal{I}|}$.

Proof. The proof is based on the bound for a solution in \mathbb{N} of a system of linear equations (Lemma 11.5). By Π we denote the set of data permutations (bijections $\mathcal{D} \to \mathcal{D}$). Indeed we can apply the homomorphism \mathfrak{P} to both sides of Equation 11.8 and get

$$\mathfrak{P}(\mathbf{v}) = \mathfrak{P}\left(\sum_{\mathbf{v}_i \in \text{Perm}(\mathcal{I}_1)} a_i \mathbf{v}_i\right) + \mathfrak{P}\left(\sum_{\mathbf{v}_j \in \text{Perm}(\mathcal{I}_2)} b_j \mathbf{v}_j\right)$$

and further

$$\mathfrak{P}(\mathbf{v}) = \left(\sum_{\mathbf{v}_i \in \text{Perm}(\mathcal{I}_1)} a_i \mathfrak{P}(\mathbf{v}_i)\right) + \left(\sum_{\mathbf{v}_j \in \text{Perm}(\mathcal{I}_2)} b_j \mathfrak{P}(\mathbf{v}_j)\right)$$
(11.9)

$$\mathfrak{P}(\mathbf{v}) = \left(\sum_{\mathbf{v}_i \in \mathcal{I}_1} \mathfrak{P}(\mathbf{v}_i) \sum_{\pi \in \Pi} a_{i,\pi}\right) + \left(\sum_{\mathbf{v}_j \in \mathcal{I}_2} \mathfrak{P}(\mathbf{v}_j) \sum_{\pi \in \Pi} b_{j,\pi}\right)$$
(11.10)

But due to Lemma 11.11 we know that \mathfrak{P} preserves reversibility so the bound on the sum $\sum_{\mathbf{v}_j \in \text{Perm}(\mathcal{I}_2)} b_j = \sum_{\mathbf{v}_j \in \mathcal{I}_2} \sum_{\pi \in \Pi} b_{j,\pi}$ can be taken from Lemma 11.5, namely $\sum_{\mathbf{v}_j \in \text{Perm}(\mathcal{I}_2)} b_j \leq |\mathfrak{P}(\mathcal{I}_2)| \cdot (\|\mathfrak{P}(\mathcal{I})\|_{1,\infty} + \|\mathfrak{P}(\mathbf{v})\|_{\infty} + 2)^{d+|\mathcal{I}|}$.

Theorem 11.14. Let $\mathcal{I}_1, \mathcal{I}_2$ be a partition of \mathcal{I} into elements reversible and non-reversible in Perm(\mathcal{I}), respectively. Suppose, $s_{max} \stackrel{\text{def}}{=} max(\{|\text{SUPPORT}(\mathbf{v}_j)| \mid \mathbf{v}_j \in \mathcal{I}_2\})$. Let $\overline{V} \subset \mathcal{D}$ is a set such that $\text{SUPPORT}(\mathbf{v}) \subseteq \overline{V}$ and

$$|\overline{V}| = |\text{SUPPORT}(\mathbf{v})| + s_{max} \cdot \left(|\mathfrak{P}(\mathcal{I}_2)| \cdot (||\mathfrak{P}(\mathcal{I})||_{1,\infty} + ||\mathfrak{P}(\mathbf{v})||_{\infty} + 2)^{d+|\mathcal{I}|} \right).$$
(11.11)

Finally, let $\overline{\mathcal{I}_2} \subset \operatorname{PERM}(\mathcal{I}_2)$ is the set of all data vectors in $\operatorname{PERM}(\mathcal{I})$ that are supported by \overline{V} .

Then $\mathbf{v} \in \mathbb{N}$ -Sums(Perm(\mathcal{I})) if and only if

$$\mathbf{v} = \sum_{\mathbf{v}_i \in \mathrm{PERM}(\mathcal{I}_1)} a_i \mathbf{v}_i + \sum_{\mathbf{v}_j \in \overline{\mathcal{I}_2}} b_j \mathbf{v}_j \ \ \textit{where} \ \ a_i, b_j \in \mathbb{N}.$$

Note that the second sum is only over vectors supported by \overline{V} .

Proof. Implication from right to left is trivial. Implication from left to right is the consequence of Lemma 11.13. Suppose $\mathbf{v} = \sum_{\mathbf{v}_i \in \text{Perm}(\mathcal{I}_1)} a_i' \mathbf{v}_i + \sum_{\mathbf{v}_j \in \text{Perm}(\mathcal{I}_2)} b_j' \mathbf{v}_j$. Let \widehat{V} be the union of supports of $\mathbf{v}_j \in \text{Perm}(\mathcal{I}_2)$ that appear with nonzero coefficients in the sum above. Observe $|\widehat{V}| + |\text{SUPPORT}(\mathbf{v})| \leq |\overline{V}|$ due to Lemma 11.13 and the definition of $|\overline{V}|$. Let π be a data permutation that is the identity on $\text{SUPPORT}(\mathbf{v})$ and injects \widehat{V} into \overline{V} . Then, $\mathbf{v} = \mathbf{v} \circ \pi^{-1} = \sum_{\mathbf{v}_i \in \text{Perm}(\mathcal{I}_1)} a_i' \mathbf{v}_i \circ \pi^{-1} + \sum_{\mathbf{v}_j \in \text{Perm}(\mathcal{I}_2)} b_j' \mathbf{v}_j \circ \pi^{-1}$, as required.

Proof of Theorem 2.2. Due to Theorem 11.14, in **NExp-Time** it is possible to construct an instance of the \mathbb{Z} -solvability with data problem of exponential size, that has a solution if and only if the original problem has a solution. Precisely, using the notation from Theorem 11.14 we ask if $\mathbf{v} - \sum_{\mathbf{v}_j \in \mathcal{I}_2 \circ \Pi} b_j \mathbf{v}_j \in \mathbb{Z}$ -Sums(Perm(\mathcal{I}_1)). According to Theorem 2.1 it can be solved in **Exp-Time**. Thus the algorithm works in **NExp-Time**.

Remark 11.15. The presented reduction works in the same way for a more general class of data vectors of the form $\mathcal{D}^k \to \mathbb{Z}^d$.

12. Conclusions and future work.

We have shown **Exp-Time** and **P-Time** upper complexity bounds for \mathbb{N} - and \mathbb{Z} -solvability problems over sets (unordered tuples) of data, respectively, by (1) reducing the former problem to the latter one (with nondeterministic exponential blowup), (2) reformulating the latter problem in terms of (weighted) hypergraphs, and (3) solving the corresponding hypergraph problem by providing a local characterisation testable in polynomial time.

The characterisation of \mathbb{Z} -solvability provided by Theorem 3.4 identifies several simple to test properties that all together are equivalent to \mathbb{Z} -solvability. Each of properties is independent of others and may be used as a partial test for non-reachability in data nets. It is not clear if in every application we should use all of them.

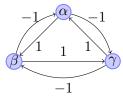
The proposed characterisation does not work for directed structures, as all weight of the following nontrivial graphs are $\mathbf{0}$.



Thus we should somehow refine the characterisation for data vectors going from k-tuples to d-dimensional vectors of integer numbers, $\mathcal{D}^k \to \mathbb{Z}^d$. For example, for directed graphs the condition stated in Equation 4.2 can be formulated as follows: For every vertex α we define $\overrightarrow{\Lambda}_{\alpha}(\mathbb{H}) = (out, in)$ where out is the sum of all edges in \mathbb{H} outgoing from α and in is the sum all edges in \mathbb{H} incoming to α . Now

$$\overrightarrow{\Lambda_{\alpha}}(\mathbb{H}) \in \mathbb{Z}\text{-Sums}(\{\overrightarrow{\Lambda_{\alpha'}}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{I}, \alpha' \in V'\})$$
(12.1)

This is still not sufficient due to the following example with all weights equal 0:



To deal with this example we need to define another invariant, namely for every pair of nodes its weight is a pair of numbers, sum of edges and the absolute value of their difference. Unfortunately, it is not clear how to generalise such invariants for hypergraphs.

Another approach to define weights for directed hypergraphs is to linearly order vertices and add only hyperedges that are oriented in the same way with respect to the order. However, in this approach the weight depends on the orientation, which does not look as a desired solution.

Although we do not have the theorem which states that the \mathbb{Z} -solvability is equivalent to some lifted version of local characterisation any of described invariants may be used as a partial test for the nonexistence of the solution. Indeed, they are homomorphisms from data vectors in $\mathcal{D}^k \to \mathbb{Z}^d$ to $\mathbb{Z}^d \cdot n$ for some $n \in \mathbb{N}$ depending on the homomorphism (for example in case of the (out, in) weight the value n = 2). Thus we identified a family of new heuristics that can be used in the analysis of data nets.

We also should mention that although in the proof of Theorem 2.1 we use data vectors with the image in \mathbb{Z}^d all proofs works for any Abelian group in which the system of equations 4.5 has a unique solution. In the general case, it corresponds to the Abelian group in which Lemma 6.3 in Section 6 holds. It speaks about maximality of the rank of some specific family of matrices.

We leave two interesting open problems. First, concerning the complexity of the N-solvability, we leave the gap between the **NP** lower bound and our **NExp-Time** upper bound. Currently, available methods seem not suitable for closing this gap. Second, the line of research we establish in this paper calls for continuation, in particular for investigation of the solvability problems over the *ordered* tuples of data. Due to Remark 11.15 we know that the problematic bit is a generalisation of the Theorem 2.1 In our opinion, the solution for this case should be possible by further development of the techniques proposed in this paper.

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References

- [AN01] Parosh Aziz Abdulla and Aletta Nylén. Timed petri nets and bqos. In José Manuel Colom and Maciej Koutny, editors, Application and Theory of Petri Nets 2001, 22nd International Conference, ICATPN 2001, Newcastle upon Tyne, UK, June 25-29, 2001, Proceedings, volume 2075 of Lecture Notes in Computer Science, pages 53-70. Springer, 2001.
- [BFHH16] Michael Blondin, Alain Finkel, Christoph Haase, and Serge Haddad. Approaching the coverability problem continuously. In Tools and Algorithms for the Construction and Analysis of Systems 22nd International Conference, TACAS 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings, pages 480–496, 2016.
- [BFHRV10] R. Bonnet, A. Finkel, Serge Haddad, and Fernando Rosa-Velardo. Comparing petri data nets and timed petri nets. 2010.
- [BKL11] Mikolaj Bojańczyk, Bartek Klin, and Sławomir Lasota. Automata with group actions. In Proceedings of the 26th Annual IEEE Symposium on Logic in Computer Science, LICS 2011, June 21-24, 2011, Toronto, Ontario, Canada, pages 355–364, Toronto, Ontario, Canada, 2011. IEEE Computer Society.
- [BKLT13] Mikolaj Bojańczyk, Bartek Klin, Sławomir Lasota, and Szymon Toruńczyk. Turing machines with atoms. In 28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013, New Orleans, LA, USA, June 25-28, 2013, pages 183-192. IEEE Computer Society, 2013.
- [CK98] Edward Y. C. Cheng and Michael Kaminski. Context-free languages over infinite alphabets. Acta Informatica, 35(3):245–267, 1998.
- [CLS19] Michael B. Cohen, Yin Tat Lee, and Zhao Song. Solving linear programs in the current matrix multiplication time. In Moses Charikar and Edith Cohen, editors, Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019, pages 938-942. ACM, 2019.

- [CO21] Wojciech Czerwinski and Lukasz Orlikowski. Reachability in vector addition systems is ackermann-complete. CoRR, abs/2104.13866, 2021.
- [DE95] Jorg Desel and Javier Esparza. Free Choice Petri Nets. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1995.
- [GLS16] Thomas Geffroy, Jérôme Leroux, and Grégoire Sutre. Occam's razor applied to the petri net coverability problem. In Reachability Problems - 10th International Workshop, RP 2016, Aalborg, Denmark, September 19-21, 2016, Proceedings, pages 77–89, 2016.
- [GR01] C. Godsil and G. Royle. Algebraic Graph Theory, volume 207 of Graduate Texts in Mathematics. volume 207 of Graduate Texts in Mathematics. Springer, 2001.
- [GSAH19] Utkarsh Gupta, Preey Shah, S. Akshay, and Piotr Hofman. Continuous reachability for unordered data petri nets is in ptime. In Mikołaj Bojańczyk and Alex Simpson, editors, Foundations of Software Science and Computation Structures, pages 260–276, Cham, 2019. Springer International Publishing.
- [HJLP21] Piotr Hofman, Marta Juzepczuk, Slawomir Lasota, and Mohnish Pattathurajan. Parikh's theorem for infinite alphabets. In 36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 July 2, 2021, pages 1–13. IEEE, 2021.
- [HL18] Piotr Hofman and Sławomir Lasota. Linear equations with ordered data. In 29th International Conference on Concurrency Theory, CONCUR 2018, September 4-7, 2018, Beijing, China, pages 24:1–24:17, 2018.
- [HLT17] Piotr Hofman, Jérôme Leroux, and Patrick Totzke. Linear combinations of unordered data vectors. In 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017, pages 1-11, 2017.
- [JvdAB+13] Kurt Jensen, Wil M. P. van der Aalst, Gianfranco Balbo, Maciej Koutny, and Karsten Wolf, editors. Transactions on Petri Nets and Other Models of Concurrency VII, volume 7480 of Lecture Notes in Computer Science. Springer, 2013.
- [Kar72] Richard M. Karp. Reducibility among combinatorial problems. In *Proceedings of a symposium* on the Complexity of Computer Computations, held March 20-22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York., pages 85–103, 1972.
- [KF94] Michael Kaminski and Nissim Francez. Finite-memory automata. *Theor. Comput. Sci.*, 134(2):329–363, 1994.
- [Las16] Sławomir Lasota. Decidability border for petri nets with data: Wqo dichotomy conjecture. In Fabrice Kordon and Daniel Moldt, editors, Application and Theory of Petri Nets and Concurrency, pages 20–36, Cham, 2016. Springer International Publishing.
- [Ler21] Jérôme Leroux. The reachability problem for petri nets is not primitive recursive. CoRR, abs/2104.12695, 2021.
- [LNO⁺08] Ranko Lazic, Thomas Christopher Newcomb, Joël Ouaknine, A. W. Roscoe, and James Worrell. Nets with tokens which carry data. Fundam. Inform., 88(3):251–274, 2008.
- [LT17] Ranko Lazic and Patrick Totzke. What makes petri nets harder to verify: Stack or data? In Thomas Gibson-Robinson, Philippa J. Hopcroft, and Ranko Lazic, editors, Concurrency, Security, and Puzzles Essays Dedicated to Andrew William Roscoe on the Occasion of His 60th Birthday, volume 10160 of Lecture Notes in Computer Science, pages 144–161. Springer, 2017.
- [Pot91] Loïc Pottier. Minimal solutions of linear diophantine systems: bounds and algorithms. In Ronald V. Book, editor, *Rewriting Techniques and Applications*, pages 162–173, Berlin, Heidelberg, 1991. Springer Berlin Heidelberg.
- [Ros10] Fernando Rosa-Velardo. Depth boundedness in multiset rewriting systems with name binding. In Antonín Kucera and Igor Potapov, editors, Reachability Problems, 4th International Workshop, RP 2010, Brno, Czech Republic, August 28-29, 2010. Proceedings, volume 6227 of Lecture Notes in Computer Science, pages 161–175, Brno, Czech Republic, 2010. Springer.
- [STC96] Manuel Silva Suárez, Enrique Teruel, and José Manuel Colom. Linear algebraic and linear programming techniques for the analysis of place or transition net systems. In Wolfgang Reisig and Grzegorz Rozenberg, editors, Lectures on Petri Nets I: Basic Models, Advances in Petri Nets, the volumes are based on the Advanced Course on Petri Nets, held in Dagstuhl, September 1996, volume 1491 of Lecture Notes in Computer Science, pages 309–373. Springer, 1996.