Fairness and promptness in Muller formulas

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Abstract. In this paper we consider two different views of the model checking problem for the Linear Temporal Logic (LTL). On the one hand, we consider the *universal* model checking problem for LTL, where one asks that for a given system and a given formula all the runs of the system satisfy the formula. On the other hand, the *fair* model checking problem for LTL asks that for a given system and a given formula almost all the runs of the system satisfy the formula.

It was shown that these two problems have the same theoretical complexity i.e. PSPACE-complete. The question arises whether one can find a fragment of LTL for which the complexity of these two problems differs. One such fragment was identified in a previous work, namely the *Muller fragment*.

We address a similar comparison for the *prompt* fragment of LTL (pLTL). pLTL extends LTL with an additional operator, i.e. the *prompt-eventually*. This operator ensures the existence of a bound such that liveness properties are satisfied within this bound.

We show that the corresponding Muller fragment for pLTL does not enjoy the same algorithmic properties with respect to the comparison considered. We also identify a new expressive fragment for which the fair model checking is faster than the universal one.

Keywords: Model Checking · Temporal logics · Fairness.

1 Introduction

Linear Temporal Logic (LTL) allows system designers to easily describe behavioral properties of a system [17]. Its expressive power and convenience of use proved useful in many areas such as system design and verification [7,20], agent planning [4,11], knowledge representation [10], and control and synthesis [18,19]. At the heart of these applications a fundamental formal approach is always to be found, namely, the *model-checking problem* [22].

Universal and fair model-checking When trying to verify a system against a specification, one usually models the system through a formalism called *Labeled Transition System* (LTS), which can be seen as a form of graph. A run of the system is an infinite path in the LTS. The original approach to model-checking consists in verifying that all possible runs of the LTS comply with the specification. Some systems may not satisfy the specification only because of few *unlikely* runs. To avoid discarding these systems, the *fair* model checking approach gives

a formal definition of *small set* of executions, and then satisfies that the set of executions that do not satisfy the specification is indeed small. Two main notions of smallness appear in the literature. If the system is endowed with a probability distribution, then a set is small if it has probability 0. One can also define a topology on the set of executions, and then a set is small if it is of the first Baire category, or *comeager* [16]. It turns out that, for LTL specifications over *finite* LTS, these two definitions of smallness actually coincide [23]. The following example presents an intuitive difference between the *universal* and the fair *setting*.

Example 1. Consider the example of Fig. 1, this figure models a "toy protocol" that could either be in an idle state, a querying state or granting state. Assume now that the modeler wants to check that **the protocol is not always idle**. Without any fairness assumption, this system will be rejected since it could always repeat the selfloop of state a. However, as we will see later, the formalisation of the fairness assumption is able to express that the set of runs consisting of this selfloop itself is

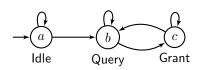


Fig. 1: A simple system.

small. Therefore, it can be ignored and we can say that the system fairly satisfies the specification.

LTL model-checking The complexity of verifying that all the runs of a system satisfy an LTL formula is known to be a PSPACE-complete problem [22]. The complexity is measured with respect to the size of the specification, an LTL formula in our case. This high complexity is due to the fact that one has to transform the specification, into one described by a non-deterministic Büchi automaton. Such an automaton is usually exponential in the size of the specification, hence the complexity blowup [1].

In order to circumvent this blow-up, a natural idea is to identify fragments of LTL with easier model checking problem. In the seminal work of Sistla and Clark [22], they identified fragments whose model checking is coNP-complete. In particular they showed this complexity for the class of formulas that only uses the temporal operator F. We also cite the fragment identified by Muscholl et al. [15]. This fragment is obtained by prohibiting the use of until operator and requiring to use only next operators indexed by a letter. They showed that the model checking problem is NP-complete. Finally we highlight the work of Emerson et al. [9] where they studied the fragment of *Muller formulas*, i.e., the fragment built upon F^{∞} , and showed that it is also coNP-complete. We will see later that this fragment enjoys a nice property.

The same complexity bound holds for the *fair* model checking problem as well [8]. Indeed, verifying that the set of executions that do not satisfy a formula is small is also a PSPACE-complete task. Again, this follows from the translation into the automata-theoretic world. However, for at least one expressive fragment of LTL, these complexities differ. Indeed for the fragment of Muller Formulas,

while the universal model checking problem is coNP-complete, the fair model checking problem can be solved by a linear time algorithm presented in [21]. This follows from the fact that Muller formulas can be checked without prior processing thus avoiding the exponential blowup.

Example 2. Consider again the example of Fig. 1. A Muller formula that one could check is of the form infinitely often a query is made and infinitely often a grant is granted. The intuition is that such a specification can be checked by only studying the strongly connected components (SCCs) of the system. Here, we see that neither the SCC $\{a\}$, nor $\{b,c\}$ satisfy the specification. It is also worth mentioning that, as we will see later, in the presence of fairness assumption, one only has to check the so-called bottom SCCs (BSCCs), in our case $\{b,c\}$.



Fig. 2: LTL vs. pLTL.

Refining LTL Consider the standard specification: All along the execution, if a message is sent, an acknowledgment will be received, that can be expressed in LTL. A system may satisfy this specification universally or fairly, even when the waiting time for the acknowledgment grows indefinitely during the execution. $Prompt\ LTL\ (pLTL)$, introduced by Kupferman et al. [14], can express the above specification in the following and more precise form: There exists a time bound t such that, along any execution, if a message is sent, an acknowledgment will be received within t seconds. Of course a natural question is whether pLTL and LTL are equivalent. In Example 3, we show a simple model that separates LTL from pLTL.

Example 3. Consider the LTS of Fig. 2, and consider the specification that asks that either A or B is seen infinitely often. Clearly enough, any infinite path in the system of Fig. 2 satisfies this LTL specification. The corresponding pLTL specification asks for the existence of a bound k such that either A or B is seen infinitely often within a window of k steps. Now, for any bound $k \geq 0$, the run $a^{k+1}b^{k+1}a^{k+1}b^{k+1}\cdots$ does not satisfy this specification.

Despite the fact that it can express more precise properties, the model checking problem for pLTL is still known to be PSPACE-complete [14]. This is achieved through an *efficient translation* into LTL. The fair model checking problem has the same complexity also: although an explicit proof is not published, a careful inspection of the proof of [14] shows that the translation into LTL formulas holds for the fair setting, and thus it is sufficient to invoke the algorithm from [8] without further blowup. **Prompt Muller Formulas** In this paper we consider the model checking problem in both the fair and universal framework in the context of pLTL, in particular of the *Muller fragment* of pLTL. As we mentioned, Muller formulas were introduced as a fragment of LTL. In this context, this fragment has a coNP-complete universal model checking problem [21]. This is already a nice property since being in this complexity class allows

one to scale better in real life application by applying symbolic approaches using SAT-solvers. We also mentioned that, for this fragment, the fair model checking problem is linear in the size of the formula.

From a practical point of view, this fragment is very handy when one wants to express asymptotic behaviors. For instance, a classical specification to seek in a reactive system is responsiveness. This specification requires from the system that queries are eventually granted, moreover, it requires that the most urgent ones are considered first. But how can one measure the urgency of a query? In LTL, one can think of two ways. The first is to consider that a query is urgent if it occurs infinitely often. The drawback with this modelization is that it is not precise enough. Indeed, a query could be asked infinitely often but with a vanishing frequency. A system could very well satisfy such property but in practice this would yield no guarantees. On the other hand, one could model the urgence as from some point on, the query is always raised until it is granted, but this is too strong now. Thanks to our fragment prompt Muller formula, one can model queries with positive frequencies. Indeed, the existence of a bound, implies that the query appears within a bounded frame of time ensuring a more robust behavior.

Prefix independent prompt Muller Formulas Finally we introduce a more permissive fragment. Along its execution, a system may violate a property during an *initial* phase, but once it enters a *steady regime* it satisfy this very property. We therefore introduce the *initialized Muller* fragment for pLTL. This fragment expresses the fact that a system should satisfy a specification in the long run, i.e., we ignore the finite initialization phase. This vision is inspired from prefix independent specifications. Such specifications are only interested in the asymptotic behavior of a system. For instance, parity, Rabin, Street, Büchi are all ω -regular specifications whose satisfaction is independent of any finite prefix [1]. Not only do these specifications seem to be more suited to real life applications, they also in general enjoy nice properties, especially in the probabilistic setting, c.f. [5,12,13]. We also mention results [3,6] where prefix independence has been considered in a setting rather close to ours, and there again they exhibited well behaved properties [2] especially when compared to the prefix dependent framework. Our fragment generalizes the concept of prefix independence for prompt Muller formulas.

Contributions and organisation Our original contributions are as follows.

We first show that universal model checking for the Muller fragment of pLTL is coNP-complete. In order to show the membership, we had to depart from the already existing reduction in the classical LTL and develop new technical tools. Our approach is based on the concept of avoiding sequences. Intuitively, these are sequences of simple cycles, each of which avoids a state formula, c.f. Definition 6. We show that for any disjunctive prompt Muller formula, the existence of a counter example in a system is equivalent to the existence of specific avoiding sequence. We also show that these sequences are small and canonical representations for counter examples. This is established in Lemma 2 and Lemma 3.

Our second contribution is to show that the complexities for universal and fair model checking for this fragment are the same. In order to obtain this, we show that universal and fair model checking for this fragment cannot be separated, c.f. Theorem 5. This latter observation can be intuitively explained by the fact that if a system is a fair model for some pLTL Muller formula, then any run, including those with measure 0, have to satisfy the formula, c.f. Lemma 6.

Our third contribution is to show that the complexity of universal and fair model checking for the prefix independent Muller Formula fragment differ. The universal model checking is again coNP-complete, but we give a *quadratic* time algorithm solving the fair model checking problem for this fragment.

2 Preliminaries

Throughout the document we will use the following notations and conventions: AP is a set of atomic propositions. For an arbitrary set E, E^* is the set of all the finite sequences over E, E^+ is the set of all the non empty finite sequences over E, E^ω is the set of all the infinite sequences of over E. When E is a finite set, |E| will denote the number of its elements.

Labelled Transition System LTS for short is a tuple $\mathcal{L} = \langle \mathsf{S}, s_{\mathsf{init}}, \mathsf{T}, \mathsf{Ibl} \colon \mathsf{S} \to 2^{\mathsf{AP}} \rangle$ such that S is a set of states, $s_{\mathsf{init}} \in \mathsf{S}$ is an initial state, $\mathsf{T} \subseteq \mathsf{S} \times \mathsf{S}$ is a set of transitions, and $\mathsf{Ibl} : \mathsf{S} \to 2^{\mathsf{AP}}$ is a labeling function. For a state $s \in \mathsf{S}$, the set of successors of s is $\mathsf{Succ}(s) = \{t \in \mathsf{S} | (s,t) \in \mathsf{T}\}$. A path in \mathcal{L} is a finite sequence of states $\pi = s_0 s_1 \cdots s_k$ of length k+1 such that $\forall 0 \leq i \leq k-1, s_{i+1} \in \mathsf{Succ}(s_i)$. We denote by $|\pi|$ the length of π , i.e. $|\pi| = k+1$. A run in \mathcal{L} is an infinite sequence of states $\rho = s_0 s_1 \cdots$ such that $\forall i \geq 0, s_{i+1} \in \mathsf{Succ}(s_i)$. Let ρ be a run and let $i \geq 0$, then $\rho[i] = s_i, \, \rho[i..] = s_i s_{i+1} \cdots$, that is the infinite suffix starting in the (i+1)th letter, and $\rho[..i]$ is the prefix up to (i+1)th position, that is $\rho[..i] = s_0 \cdots s_i$. For $i < j, \, \rho[i..j]$ is the path $s_i \cdots s_j$. We use the same notation for a path π , in this case $\pi[i..]$ will be a finite suffix.

We will also use the following notations: The set of states visited infinitely often by ρ is $\mathsf{Inf}(\rho) = \{s \in \mathsf{S} \mid \forall i \geq 0, \exists j > i, \rho[j] = s\}$. The set of all the paths (resp. runs) over $\mathcal L$ is $\mathsf{Paths}(\mathcal L)$ (resp. $\mathsf{Runs}(\mathcal L)$). The set of all initialized runs over $\mathcal L$ is $\mathsf{Runs}_{\mathsf{init}}(\mathcal L)$, that is $\{\rho \in \mathsf{Runs}(\mathcal L) \mid \rho[0] = s_{\mathsf{init}}\}$.

Linear Temporal Logic LTL for short is the set of formulas ϕ defined using the following grammar:

$$\phi ::= \alpha \mid \neg \phi \mid \phi \lor \phi \mid \mathsf{X} \ \phi \mid \phi \ \mathsf{U} \ \phi \ .$$

where α is in AP.

We also recall that LTL formulas are evaluated over runs of \mathcal{L} as follows:

$$\rho \models \alpha \text{ iff } \alpha \in \mathsf{Ibl}(\rho[0]) \ , \ \rho \models \neg \phi \text{ iff } \rho \not\models \phi \ , \ \rho \models \mathsf{X} \ \phi \text{ iff } \rho[1..] \models \phi \ ,$$

$$\rho \models \phi \lor \psi \text{ iff } \rho \models \phi \text{ or } \rho \models \psi \ , \ \rho \models \phi \ \mathsf{U} \ \psi \text{ iff } \exists i \geq 0, \ \rho[i..] \models \psi \text{ and } \forall j < i, \ \rho[j..] \models \phi \ ,$$
where $\rho \in \mathsf{Runs}(\mathcal{L}), \alpha \in \mathsf{AP}, \phi \in \mathsf{LTL}$, and $\psi \in \mathsf{LTL}$.

Given a formula $\phi \in \mathsf{LTL}$, the size $|\phi|$ is the number of operators that appear in ϕ . In the sequel, we will use the following macros:

$$\mathsf{F}\phi = \mathsf{true}\;\mathsf{U}\;\phi\;,\;\mathsf{G}\phi = \neg\mathsf{F}\neg\phi\;,\;\mathsf{F}^\infty\phi = \mathsf{GF}\phi\;.$$

The first formula is true for any run where ϕ is true in the *future*, the second formula is true for any run where ϕ always holds, and the third formula is true for any run where exists infinitely many positions where ϕ holds.

Problem 1 (Model checking). Given an LTS \mathcal{L} and an LTL formula ϕ , the model checking problem consists in checking whether $\forall \rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}), \ \rho \models \phi$. In this case, we will write $\mathcal{L} \models \phi$.

Theorem 1 ([22]). The model checking problem is PSPACE-complete.

Fairness in model checking As seen in the example of Fig. 1, the LTS is not considered a model for the formula $\mathsf{FG} \neg (\mathsf{Idle})$ due to a single run. In fact, one can say that such run is likely to occur only if the system is controlled by a demonic entity. In a less pessimistic scenario, one can imagine that the transition (a,b) in Fig. 1 should at least happen once. Under this assumption, one clearly sees that as soon as it happens, the formula $\mathsf{FG}(\neg \mathsf{Idle})$ is satisfied. This alternative vision on the behavior of the system was introduced in [21] and is called model checking under fairness assumption.

In order to impose fairness, one can use different formalisms [23]. In our case we use a fair coin. At each state, a fair coin is flipped and the succesor state is chosen accordingly. This coin flipping induces a natural probability measure over $\mathsf{Runs}_{\mathsf{init}}(\mathcal{L})$. In the fair model checking problem, one asks whether almost all the runs of an LTS satisfy a given formula ϕ . In order to build the probability measure over $\mathsf{Runs}_{\mathsf{init}}(\mathcal{L})$, we use the notion of cylinders. Let $\pi \in \mathsf{Paths}(\mathcal{L})$, the cylinder induced by π , denoted $\mathsf{Cyl}(\pi)$, is the set $\{\rho \in \mathsf{Runs}(\mathcal{L}) \mid \pi \text{ is a prefix of } \rho\}$. $\mathbb{P}_{\mathcal{L}}(\mathsf{Cyl}(\pi))$ is defined by induction over the set of all the cylinders as follows:

$$\forall s \in \mathsf{S}, \ \mathbb{P}_{\mathcal{L}}(\mathsf{Cyl}(s)) = \begin{cases} 1 \text{ if } s = s_{\mathsf{init}} \\ 0 \text{ otherwise} \end{cases}$$

Let $\pi \in \mathsf{Paths}(\mathcal{L})$ such that $|\pi| = n$ and $s \in \mathsf{S}$, then

$$\mathbb{P}_{\mathcal{L}}(\mathsf{Cyl}(\pi s)) = \begin{cases} \frac{\mathbb{P}_{\mathcal{L}}(\mathsf{Cyl}(\pi))}{|\mathsf{Succ}(\pi[n])|} \text{ if } s \in \mathsf{Succ}(\pi[n]) \\ 0 \text{ otherwise} \end{cases}$$

This measure is uniquely extended over the set $\mathsf{Runs}_\mathsf{init}$ thanks to Caratheodory's theorem.

Problem 2 (Fair model checking). Given an LTS \mathcal{L} and an LTL formula ϕ , the fair model checking problem consists in checking whether $\mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid \rho \models \phi\}) = 1$. In this case, we will write $\mathcal{L} \models_{\mathsf{AS}} \phi$.

An appealing question is then whether the latter version is any easier than the former version of the problem. It turns out that both versions are equally difficult.

Theorem 2 ([8]). The fair model checking problem is PSPACE-complete.

The Muller fragment of LTL In order to reduce the complexity of the model checking problem, one can consider subclasses of LTL formula. In order to define subclasses we use the notion of fragment. A fragment is the set of all the formulas obtained by restricting to particular temporal operators. For example, if op_1, \ldots, op_k are k temporal operators, we will denote by $\mathsf{L}(op_1, \ldots, op_k)$ the set of all the formulas where the only temporal operators allowed are op_1, \ldots, op_k . For instance, $\mathsf{L}(\mathsf{X} \ , \ \mathsf{U} \)$ coincides with LTL. Another example is $\mathsf{L}(\mathsf{F})$ which is the fragment where only F is allowed. This fragment is known as RLTL and it is interesting since its model checking problem is coNP complete [22]. In this paper our starting point will be $\mathsf{L}(\mathsf{F}^{\infty})$ whose universal model checking is coNP-complete [9], but fair model checking is linear [21].

A state formula is an LTL formula where no temporal operator appears. We also introduce the notion of positive fragment denoted $\mathsf{L}^+(op_1,\ldots,op_k)$, i.e., the set of all the formulas in $\mathsf{L}(op_1,\ldots,op_k)$ such that each atomic proposition is in the scope of some temporal operator op_1,\ldots,op_k . The positive fragment $\mathsf{L}^+(\mathsf{F}^\infty)$ is called the *Muller* fragment, and is formally defined as the smallest non empty set such that :

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 \begin{array}{l} - \ \mathrm{If} \ \phi \in \mathsf{L}(\mathsf{F}^{\infty}), \ \mathrm{then} \ \mathsf{F}^{\infty} \phi \in \mathsf{L}^{+}(\mathsf{F}^{\infty}). \\ - \ \mathrm{If} \ \phi_{1}, \phi_{2} \in \mathsf{L}^{+}(\mathsf{F}^{\infty}), \ \mathrm{then} \ \phi_{1} \vee \phi_{2}, \ \phi_{1} \wedge \phi_{2} \in \mathsf{L}^{+}(\mathsf{F}^{\infty}). \end{array}
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As explained earlier, the Muller fragment enjoys nice algorithmic properties.

Theorem 3 ([21]). The universal model checking problem for Muller formulas is coNP-complete, and the fair model checking problem can be solved in linear time.

Promptness in LTL A prompt LTL formula ϕ is defined according to the following grammar:

$$\phi ::= \alpha \mid \neg \alpha \mid \phi \lor \phi \mid \phi \land \phi \mid \mathsf{X} \phi \mid \phi \mathsf{U} \phi \mid \phi \mathsf{R} \phi \mid \mathsf{F}_{\mathsf{P}} \phi$$

The main difference lies in the addition of F_P . This operator states that ϕ has to be satisfied in a "prompt" fashion. The semantics of this operator is defined with respect to a bound $k \geq 0$. For a given k, we will write $(\rho, k) \models \mathsf{F}_\mathsf{P} \phi$ if and only if $\exists i \leq k$, $(\rho[i..], k) \models \phi$. For the other operators, the bound k is irrelevant, thus they are evaluated using the semantics of LTL. In this case we will not specify k. Another difference is that we do not allow the negation of F_P . This was already the case in [14]. Indeed, this makes more sense since the negation of the newly added operator F_P seems unnatural, as the quantification over the bound is outside the formula. Therefore, we must explicitly add the conjunction \land , as well as the release operator R the dual of the until operator U . Given

an LTS \mathcal{L} , and a pLTL formula ϕ , we say that \mathcal{L} is a model for ϕ , and write $\mathcal{L} \models \phi$ if and only if there exists a bound k such that for all $\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L})$, $(\rho, k) \models \phi$. We write $\mathcal{L} \models_{\mathsf{AS}} \phi$ if and only if there exists a bound k such $\mathbb{P}_{\mathcal{L}}(\{\rho) \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \models \phi\}) = 1$.

Problem 3 (Universal prompt model checking). Given an LTS \mathcal{L} and a pLTL formula ϕ , is it the case that $\mathcal{L} \models \phi$?

Problem 4 (Fair prompt model checking). Given an LTS \mathcal{L} and a pLTL formula ϕ , is it the case that $\mathcal{L} \models_{AS} \phi$?

From now on and when clear from the context we will refer to these two last problems as *universal model checking* and *fair model checking*.

The prompt Muller fragment The prompt Muller fragment denoted by $L(F_P^\infty)$ in the context of pLTL is obtained by the following abstract grammar:

$$\phi ::= \alpha \mid \neg \alpha \mid \phi \lor \phi \mid \phi \land \phi \mid \mathsf{F}_{\mathsf{P}}^{\infty} \phi$$

where, for a bound $k \geq 0$, $(\rho, k) \models \mathsf{F}^{\infty}_{\mathsf{P}} \phi$ if and only if $\forall i \exists j \leq k, (\rho[(i+j)..], k) \models \phi$.

Our hope as stated earlier is to show that fair model checking of this fragment is easier that its universal counterpart complexity-wise. In the sequel it will be useful to consider the so-called *positive fragment* $L^+(\mathsf{F}^{\infty}_{\mathsf{P}})$, i.e.:

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 \begin{array}{l} - \ \mathrm{If} \ \phi \in \mathsf{L}(\mathsf{F}^\infty_\mathsf{P}), \ \mathrm{then} \ \mathsf{F}^\infty \phi \in \mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P}). \\ - \ \mathrm{If} \ \phi_1, \phi_2 \in \mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P}), \ \mathrm{then} \ \phi_1 \vee \phi_2, \ \phi_1 \wedge \phi_2 \in \mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P}) \ \mathrm{and} \ \mathsf{F}^\infty_\mathsf{P} \phi_1 \in \mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P}). \end{array}
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3 Universal model checking for prompt Muller fragment

A remarkable property of the Muller fragment of LTL is that the satisfaction of a formula only depends of the asymptotic behavior of the system. Therefore, as it is shown in [21] (a corollary of a result from [9]), a system satisfies a Muller formula if and only if every strongly connected set satisfies the formula. These sets are not maximal as opposed to strongly connected components. Thus, there could exist exponentially many such sets. Roughly speaking, the algorithm guesses a strongly connected set where the formula is not satisfied in order to show that a system is not a model. The main result of this section is that the coNP complexity carries over to our setting.

Theorem 4. The universal model checking problem for $L(F_p^{\infty})$ is coNP-complete.

The proof of this theorem turns out to be more technical than what was presented in the setting of [21]. There are two main reasons why the proofs for pLTL must be different.

First, using the translation presented in [14] from pLTL to LTL builds formulas in a fragment whose model checking is already PSPACE-complete. Second,

checking only the strongly connected set is not enough in this case. In fact, it is possible for a system to satisfy a formula in all its strongly connected sets yet not be a valid model.

The rest of this section is dedicated to the proof of Theorem 4. We first present the technical tools that we will need in our algorithm. Second we establish a crucial property for the coNP membership, namely the existence of a small witness. Finally we establish the coNP membership.

3.1 Technical definitions

The goal of this section is to introduce the tools needed to prove that one can polynomially define a faulty run, and then polynomially prove that it is faulty. The first step is to limit our attention to formulas without conjunctions.

Throughout this section, we will fix a LTS $\mathcal{L} = \langle S, s_{\mathsf{init}}, \mathsf{T}, \mathsf{lbl} \colon \mathsf{S} \to 2^{\mathsf{AP}} \rangle$.

Definition 1 (Filtering). Let ϕ be a prompt Muller formula, $k \geq 0$, and ρ be some run in \mathcal{L} , assume that $(\rho, k) \not\models \phi$, then we define the filtering of ϕ denoted $\mathsf{flt}_{\rho,k}(\phi)$ as the formula obtained from ϕ as follows:

$$\begin{split} & \mathrm{flt}_{\rho,k}(\alpha) = \alpha \ , \ \mathrm{flt}_{\rho,k}(\phi_1 \vee \phi_2) = \mathrm{flt}_{\rho,k}(\phi_1) \vee \mathrm{flt}_{\rho,k}(\phi_2) \ , \ \mathrm{flt}_{\rho,k}(\mathsf{F}_\mathsf{P}^\infty \phi) = \mathsf{F}_\mathsf{P}^\infty \mathrm{flt}_{\rho,k}(\phi) \\ & \mathrm{flt}_{\rho,k}(\phi_1 \wedge \phi_2) = \begin{cases} \mathrm{flt}_{\rho,k}(\phi_1) \ if \ (\rho,k) \not\models \phi_1 \\ \mathrm{flt}_{\rho,k}(\phi_2) \ otherwise \end{cases} \end{split}$$

Lemma 1. Let ϕ be a prompt Muller formula, $k \geq 0$, and ρ some run in \mathcal{L} , then $(\rho, k) \not\models \phi$ if and only if $(\rho, k) \not\models \text{flt}_{\rho, k}(\phi)$.

The filtration of the formula is a choice over each conjunction. A formula is filtered if it is a filtration w.r.t. some run ρ and some bound k. For the sake of readability, the explicit run and bound will be omitted.

Definition 2 (Canonical form for filtered formula). Let ψ be a filtered formula. We say that ψ is in canonical form when it is written in the following form:

$$\bigvee_{i=0}^{l} a_i \vee \bigvee_{i=0}^{k} \mathsf{F}^{\infty}_{\mathsf{P}} \psi_i$$

where a_i are atomic formulas and the ψ_i are filtered formulas in canonical form.

Remark 1. Notice that if a filtered formula ψ does not start with a state formula it is always possible to write it in canonical form by adding false, i.e. $\psi \equiv \mathsf{false} \vee \bigvee_{i=0}^k \mathsf{F}_\mathsf{P}^\infty \psi_i$.

Example 4. As an ongoing example, consider the set $\{a, b, c, d\}$ of atomic variables, and let

$$\phi = \mathsf{F}^{\infty}_{\mathsf{P}} \left(a \vee \mathsf{F}^{\infty}_{\mathsf{P}} \left((b \wedge c) \vee \mathsf{F}^{\infty}_{\mathsf{P}} d \right) \vee \left(\mathsf{F}^{\infty}_{\mathsf{P}} c \wedge \mathsf{F}^{\infty}_{\mathsf{P}} d \right) \right)$$

Then, consider

$$\psi \equiv \mathsf{F}^{\infty}_{\mathsf{P}}\left(a \vee \mathsf{F}^{\infty}_{\mathsf{P}}\left(b \vee \mathsf{F}^{\infty}_{\mathsf{P}}d\right) \vee \mathsf{F}^{\infty}_{\mathsf{P}}c\right) \;\;,\;\; \psi^{*} \equiv \mathsf{false} \vee \underbrace{\mathsf{F}^{\infty}_{\mathsf{P}}\left(a \vee \underbrace{\mathsf{F}^{\infty}_{\mathsf{P}}\left(b \vee \mathsf{F}^{\infty}_{\mathsf{P}}d\right)}_{\psi_{1}} \vee \underbrace{\mathsf{F}^{\infty}_{\mathsf{P}}c}_{\psi_{2}}\right)}_{\psi}$$

On the left hand side, we have a filtration of ϕ and on the right hand side we have the same formula in canonical form. Note that ψ is indeed a filtration of ϕ for some LTS \mathcal{L} , some run ρ of \mathcal{L} , and some bound k.

In the rest of this section, we use the following notation. For any finite sequence u in S^* we denote by occ(u) the set of states visited along u. A loop is a path $\pi \in S^*$ in \mathcal{L} such that the first and last states of π are the same, that is $\pi[0] = \pi[|\pi| - 1]$. A cycle is the set of states of some loop, that is, it is a set of states $C \subseteq S$ such that there is a loop π_C that satisfies $\operatorname{occ}(\pi_C) = C$.

Definition 3 (Avoiding cycles). Let C be a cycle, and let θ be a state formula. We say that C is θ -avoiding if no state in C satisfies θ .

We now define a representation of a filtered formula as a syntax tree. This representation is useful to better visualize the formula, and the order in which sub-formulas must be falsified.

Definition 4. Let $\psi \equiv \bigvee_{i=0}^{l} a_i \vee \bigvee_{i=0}^{k} \mathsf{F}_{\mathsf{P}}^{\infty} \psi_i$ be a filtered formula in canonical form, we associate with ψ a tree $\mathsf{Tree}(\psi)$ as follows:

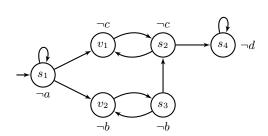
- the root t_0 of $\mathsf{Tree}(\psi)$ is labeled by $\bigvee_{i=0}^l a_i$, the children t_1, \ldots, t_k of t_0 are respectively obtained by $\mathsf{Tree}(\psi_i)$.

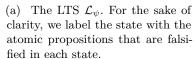
Example 5. In the continuation of Example 4, consider the LTS \mathcal{L}_{ψ} depicted in Fig. 3a. We label the state with the propositions that are not valid. For instance, in state s_1 propositions b, c, d are true. The tree associated with ψ^* is depicted in Fig. 3b.

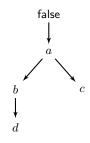
For the following definition, we denote by $\prec_{\mathcal{T}}$ the natural partial order over the nodes of a tree, i.e., for two nodes n and m, $n \prec_{\mathcal{T}} m$ if and only if n is an ancestor of m. Given a node $n \in \mathsf{Tree}(\psi)$, let θ_n be the state formula labelling n. We say that a cycle is n-avoiding, if it is θ_n -avoiding.

Definition 5 (Avoiding Sequence). Let ψ be a filtered formula in canonical form. A ψ -avoiding sequence is given by:

- a total ordering n_0, \ldots, n_l of the nodes of Tree(ψ), such that if $n_i \prec_{\mathcal{T}} n_i$, then i < j.
- a function w that associates to each node n_i a path $w(n_i) \in S^+$ such that for each $0 \le i \le l$, $occ(w(n_i))$ is a n_i -avoiding cycle.







(b) The tree associated with the formula ψ^* . According to $\mathsf{Tree}(\psi^*)$, a b-avoiding cycle should be visited before a d-avoiding cycle, but b-avoiding and c-avoiding cycles could be visited in any given order.

Fig. 3: The LTS \mathcal{L}_{ψ} and the tree associated with the formula ψ^* .

Remark 2. If i < j, and therefore $w(n_i)$ appears before $w(n_j)$ in the sequence, we will sometimes write $w(n_i) \prec_{\psi} w(n_j)$, even if \prec_{ψ} is not technically an order over words. Indeed, some words can appear multiple times, if for $i \neq j$, $w(n_i) = w(n_j)$. It is important to keep every occurrence as they correspond to different $\mathsf{F}_{\mathsf{P}}^{\mathsf{P}}$ that need to be falsified. In the following we will also say that a sequence u_0, \ldots, u_l of words S^+ is a ψ -avoiding sequence, without explicit mention of the ordering on the nodes and the function w.

Not all avoiding sequences represent a run of the system. We say that an avoiding sequence is realisable, if it is compliant with the ordering induced by the avoiding sequence.

Definition 6 (Realisable avoiding sequence). Let $U = u_0, \ldots, u_l$ be an avoiding sequence. We say that U is realisable in \mathcal{L} , if there exists a finite sequence $v_0, v_1, \ldots, v_{l+1}$ of elements in $\mathsf{Paths}(\mathcal{L})$, such that all of the following hold:

```
\begin{split} &-v_0[0] = s_{\mathsf{init}}, \\ &-\forall 1 \leq i \leq l+1, \ v_i[0] \in \mathsf{occ}(u_{i-1}), \\ &-\forall 0 \leq i \leq l, \ v_i[|v_i|-1] \in \mathsf{occ}(u_i), \\ &-v_{l+1} \in S^\omega. \end{split}
```

We call the sequence $v_0, v_1, \ldots, v_{l+1}$ a realisation witness for the pair (U, \mathcal{L}) .

We can now construct the run that will falsify the formula. Given any bound k, we need to iterate each avoiding cycle such that the resulting path is of length at least k. As each cycle has at least one element, it is enough to iterate each loop k times to guarantee this. The construction of the resulting run is formalized in the following definition.

Definition 7 (k-pumping). Let $U = u_0, \ldots, u_l$ be an avoiding sequence, and \mathcal{L} be an LTS, and $v_0, v_1, \ldots, v_{l+1}$ be a realisation witness for (U, \mathcal{L}) . A k-pumping of $(U, v_0, v_1, \ldots, v_{l+1})$ is any run of the form $v_0(u_0)^k v_1(u_1)^k \ldots v_l(u_l)^k v_{l+1}$ where k > 1.

Definition 8 (Realisation of an avoiding sequence). Let $U = u_0, \ldots, u_l$ be an avoiding sequence, and ρ be a run of some LTS \mathcal{L} . We say that ρ realizes U in \mathcal{L} if there exists a sequence of non negative integers $j_0 \leq j_1 \leq \ldots \leq j_l$ such that for any $0 \leq i \leq l, \rho[j_i] \in \mathsf{occ}(u_i)$.

Remark 3. One can easily prove that any avoiding sequence U is realized in a LTS \mathcal{L} if and only if there exists a run ρ in \mathcal{L} realizing U.

Example 6. In the continuation of Example 5, we build a ψ -avoiding sequence. We consider the following mapping,

$$a \mapsto s_1, \qquad b \mapsto v_2 s_3, \qquad c \mapsto v_1 s_2, \qquad d \mapsto s_4.$$

With the following ordering $s_1 \prec v_2 s_3 \prec v_1 s_2 \prec s_4$. One can easily check that this ordering agrees with $\mathsf{Tree}(\psi)$. Moreover, notice that each element of this sequence is an avoiding sequence for the node it is mapped to. Finally, notice that this sequence is realizable in \mathcal{L}_{ψ} . For example any run in $s_1^+(v_2 s_3)^+ s_2(v_1 s_2)^+(s_4)^{\omega}$ is a realization in \mathcal{L}_{ψ} .

Now we argue that for any k > 0 there exists a run in $s_1^+(v_2s_3)^+s_2(v_1s_2)^+(s_4)^\omega$ which is a counter example for ψ . Indeed, choose some k > 0 and consider the infinite word $\rho_k = s_1^k(v_2s_3)^ks_2(v_1s_2)^ks_4^ks_4^\omega$, then repeating s_1 k times falsifies a for k consecutive steps. Now notice that each of the k first steps also falsify the sub-formula ψ_1 . This is because from each of these positions, the cycles $\{v_2, s_3\}$, $\{v_1, s_2\}$, and $\{s_4\}$ are reached and repeated k consecutive times which falsifies in order b, c, and d. Thus the k first steps of ρ_k witness the fact that ρ_k falsifies ψ .

3.2 Small witness properties

We are now ready to state our main ingredient towards the coNP-membership, that is, the *small witness property*. This property states that for a given LTS \mathcal{L} , a filtered formula in canonical form ψ and some bound $k \geq |S| + 1$, there exists a run ρ such that $(\rho, k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi$ if and only if there exists a realisable ψ -avoiding sequence. The proof of this property relies on two lemmas. Lemma 2 establishes the left to right direction and Lemma 3 established the converse direction.

Lemma 2. Let \mathcal{L} be an LTS, ψ be a filtered formula in canonical form, and assume that there exists ρ and $k \geq |S| + 1$ such that $(\rho, k) \not\models F_{P}^{\infty} \psi$, then there exists a ψ -avoiding sequence U which is realized by ρ .

In order to prove this lemma, we need to build an avoiding sequence from a counter example. To do this, we rely on the intuition depicted in Fig. 4. In this

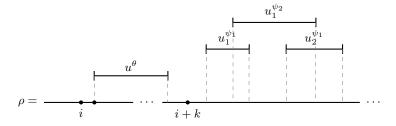


Fig. 4: Building a realisation of an avoiding sequence from a counter example.

figure, a run ρ is a counter example to a formula $\psi \equiv \mathsf{F}_\mathsf{P}^\infty(\theta \vee \mathsf{F}_\mathsf{P}^\infty \psi_1 \vee \mathsf{F}_\mathsf{P}^\infty \psi_2)$. Now because ρ is a counter example for some fixed bound k, clearly there exists a postion i such that for the next k consecutive positions, θ does not hold. From these positions one can extract a θ -avoiding cycle (u^θ in the figure). We repeat this inductively, while maintaining the order of appearance of each cycle. For instance in the figure above, ψ_1 is falsified using two cycles $u_1^{\psi_1}$ and $u_2^{\psi_1}$, but also see that ψ_2 is falsified using a cycle overlapping $u_1^{\psi_1}$ and $u_2^{\psi_1}$ but since it starts after $u_1^{\psi_1}$ the order obtained is $u_1^{\psi_1} \prec u_1^{\psi_2} \prec u_2^{\psi_1}$. The proof formalizes this intuition by extracting cycles and the induced order inductively. The full proof is in Appendix A.1.

Lemma 3. Let \mathcal{L} be an LTS, ψ be a filtered formula in canonical form, and let U be a realisable ψ -avoiding sequence in \mathcal{L} . For any k-pumping ρ_k of some realisation witness $(U, v_0, v_1, \ldots, v_{l+1})$ for (U, \mathcal{L}) , we have $(\rho_k, k) \not\models \mathsf{F}_p^{\infty} \psi$.

The proof of this lemma is less technical, it is based on the intuition that one can build counter examples by successively pumping the avoiding cycles associated with U. The proof is available in Appendix A.2.

3.3 Proof of Theorem 4

We are now ready to prove the main result of this section that is Theorem 4. To establish the membership in coNP we present a non deterministic procedure, c.f. Algorithm 1. The hardness follows easily from the results of [9]. Indeed a close inspection of the proof establishing the coNP hardness of the plain Muller fragment of LTL shows that the same encoding of 3-SAT applies in our case.

Thanks to Lemma 2 and Lemma 3, we establish the correctness of Algorithm 1.

Lemma 4. Given an LTS \mathcal{L} and a formula $\phi \in L(\mathsf{F}^{\infty}_{\mathsf{P}})$, Algorithm 1 accepts if and only if, for all k, there is a run ρ_k in Runs_{init} such that $(\rho_k, k) \not\models \phi$.

Finally Theorem 4 follows from the next lemma.

Lemma 5. The universal model checking problem for $L(F_P^{\infty})$ is in coNP.

```
Data: An LTS \mathcal{L} = \langle S, s_{\text{init}}, \mathsf{T}, \mathsf{lbl} \colon \mathsf{S} \to 2^{\mathsf{AP}} \rangle and a formula \phi \in \mathsf{L}(\mathsf{F}_\mathsf{P}^\infty)

Result: Whether \mathcal{L} \not\models \phi

Guess a filtered formula \psi \equiv \theta \lor \psi' of \phi, where \theta = \bigvee_{i=0}^l a_i and \psi' = \bigvee_{i=0}^m \mathsf{F}_\mathsf{P}^\infty \psi_i;

Guess a sequence U = u_0, \ldots, u_m of size (|S| + 1)|\psi'|;

Compute the tree \mathsf{Tree}(\psi');

Guess a mapping w between the nodes of the tree and the sequence;

Check that the initial state does not satisfy \theta;

Check that U is associated with \mathsf{Tree}(\psi');

Check that U is realisable in \mathcal{L};

Accepts the input if all checks succeeded;
```

Algorithm 1: Non deterministic algorithm for the universal model checking of $L(F_P^{\infty})$

4 Fair model checking for prompt Muller fragment

Now that we established that the universal model checking problem is coNP-complete for the prompt Muller fragment, we will address the fair version of the problem. We will actually show that assuming fairness for the behavior of the system does not simplify the problem. Formally we show that:

Theorem 5. For all LTS \mathcal{L} and formula $\phi \in L(\mathsf{F}^{\infty}_{\mathsf{P}})$, $\mathcal{L} \models \phi$ if and only if $\mathcal{L} \models_{\mathsf{AS}} \phi$.

Actually, by its own construction, any witness against the satisfaction of a formula in $L(\mathsf{F}^\infty_\mathsf{P})$ can be described by a *finite path*. One can say that the set of the counter examples of a given formula is open in the Cantor topology. Therefore, this witness has a positive probability using our measure $\mathbb{P}_\mathcal{L}$. This yields the fact that such a witness is also a witness for the fair model checking. We formalize this intuition bellow using the following lemma.

Lemma 6. Let ϕ be a formula, \mathcal{L} be an LTS and assume that there exists a run ρ such that for some k > 0 we have $(\rho, k) \not\models \phi$, then there exists a prefix w of ρ such that for any run $\rho' \in \mathsf{Cyl}(w)$ we have $(\rho', k) \not\models \phi$.

The proof consists in extracting a prefix w from a counter example ρ . We will proceed by induction over the structure of the formula. Arguably, the most technical case is the one where the formula $\phi \equiv \mathsf{F}^{\infty}_{\mathsf{P}} \psi$. To catch this intuition, consider Fig. 5 where we depict a run ρ which is a counter example to ϕ for some bound k. Using the shape of the formula, the main trick for the proof is to locate k consecutive positions where ψ does not hold. These are the positions located in Fig. 5 along ρ . From each of these positions we will extract prefixes w_i, \ldots, w_{i+k} such that they all verify the lemma, i.e., the cylinder obtained from each of them does not satisfy ϕ . To conclude the proof we extract from ρ a prefix w that contains all the prefixes w_i, \ldots, w_{i+k} and argue that this prefix w satisfies the lemma. The full proof is in Appendix B.1.

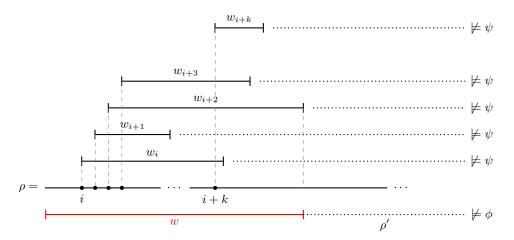


Fig. 5: Visual representation of the proof for the case $\phi = \mathsf{F}_{\mathsf{P}}^{\infty} \psi$

Proof (Proof of Theorem 5). The implication $\mathcal{L} \models \phi \implies \mathcal{L} \models_{AS} \phi$ is trivial.

The converse implication follows from Lemma 6. In order to see this, assume by contrapositive that there exists an LTS \mathcal{L} and a simple formula ϕ such that $\mathcal{L} \not\models \phi$. This means that for each $k \geq 0$, there exists a run $\rho_k \in \mathsf{Runs}_{\mathsf{init}}$ such that $(\rho_k, k) \not\models \phi$. By Lemma 6, there exists a prefix w of ρ_k such that for any run $\rho' \in \mathsf{Cyl}(w)$, $(\rho', k) \not\models \phi$. To conclude, it is sufficient to notice that w is finite thus $\mathsf{Cyl}(w)$ has positive probability, and since this holds for any $k \geq 0$, it follows that $\mathcal{L} \not\models_{\mathsf{AS}} \phi$.

5 Initialized formulas

In this section we identify a fragment for which the fair model checking problem has a lower complexity.

Definition 9. The fragment $F(L^+(F_P^\infty))$ is defined as follows: a formula $\phi \in F(L^+(F_P^\infty))$ if and only if there is a formula $\psi \in L^+(F_P^\infty)$ such that $\phi = F\psi$.

Intuitively, a formula $\mathsf{F}\psi \in \mathsf{F}(\mathsf{L}^+(\mathsf{F}_\mathsf{P}^\infty))$ means that at some point, after some initialization, the model satisfies a formula $\psi \in \mathsf{L}^+(\mathsf{F}_\mathsf{P}^\infty)$. This fragment enjoys the property of prefix independence i.e., if ρ is a run and ϕ is a formula in $\mathsf{F}(\mathsf{L}^+(\mathsf{F}_\mathsf{P}^\infty))$, then the following holds for any $k \geq 0$ and $i \geq 0$, $(\rho, k) \models \phi$ if and only if $(\rho[i..], k) \models \phi$.

5.1 Universal model checking for $F(L^+(F_P^{\infty}))$ is coNP complete

In this section we show the following Theorem:

Theorem 6. The universal model checking problem for $F(L^+(F_P^\infty))$ is coNP-complete.

This complexity result follows almost immediately from the following observation: Let \mathcal{L} be an LTS which is not a model for some formula in $F(L^+(F_P^\infty))$. This means that \mathcal{L} contains a faulty execution. We argue that this execution necessarily happens inside the same SCC, i.e., there exists an SCC that contains a counter example. We shall call this prop-



Fig. 6: $F(L^+(F_P^\infty))$ vs. $L(F_P^\infty)$.

erty, the concentration property. To get some intuition on why it is the case, consider for example the system depicted in Fig. 6, and consider the formula $F(F_P^{\infty}A \vee F_P^{\infty}B)$. Notice that this system consists of two trivial SCCs $\{a\}$, and $\{b\}$. Notice also that this system contains essentially two kinds of infinite runs; those which stay in $\{a\}$ and those which ultimately reach $\{b\}$. In the former case the formula is satisfied since A is always visited, and in the latter case the formula is satisfied because as soon as $\{b\}$ is reached, B is always visited. However if we consider the same formula for the system depicted in Fig. 2, then the formula is not satisfied anymore. The main difference between these two models is that in the first case the necessary avoiding cycles are found in different SCCs while they are in the same SCC in the second case. This observation is the core idea towards the coNP membership of the universal model checking problem for this fragment. We formalize this in Lemma 7 and prove it in Appendix C.1.

Lemma 7. Let \mathcal{L} be an LTS, $\psi \in L^+(\mathsf{F}^\infty_\mathsf{P})$ be a filtered formula in canonical form, and $k \geq |\mathsf{S}| + 1$, there exists a run ρ such that $(\rho, k) \not\models \mathsf{F}\psi$ if and only if there exists a strongly connected component C and a ψ -avoiding sequence realized in C.

To obtain the membership, we use a variation of Algorithm 1 that consists in accepting an input if and only if the guessed avoiding sequence can be realized inside the same strongly connected component. The correctness of this variation follows from Lemma 7. Notice also that this verification is not detrimental to the membership in coNP since one can decompose the system into strongly connected components in linear time. The hardness is again obtained from [9] by noticing that the same encoding once more applies to the fragment considered here. This mainly follows from the fact that the encoding of 3-SAT builds a strongly connected LTS. This yields Theorem 6. For the sake of completness the modification of Algorithm 1 is in Appendix C.1

5.2 A PTIME algorithm for the fair model checking of $F(L^+(F_P^\infty))$

The main focus of this part will be the proof of the following theorem:

Theorem 7. The fair model checking problem for $F(L^+(F_P^\infty))$ can be solved in polynomial time.

We will need the following notions:

Definition 10 (Strict predecessor). Given an LTS \mathcal{L} and a subset $U \subseteq S$ of states, the strict predecessor set of U is the set $SPred(U) = \{s \in S \mid Succ(s) \subseteq U\}$.

In other words, $\mathsf{SPred}(U)$ describes the set of states s from where it is unavoidable to hit U in one step. Consider now the operator $\mu \colon U \mapsto U \cup \mathsf{SPred}(U)$.

Definition 11 (Sure attractor). Given an LTS \mathcal{L} and a subset $U \subseteq S$ of states. We call the smallest fixed point of μ denoted by $SPred^*(U)$ the set of sure attractor.

```
Data: An LTS \mathcal{L} = \langle \mathsf{S}, s_{\mathsf{init}}, \mathsf{T}, \mathsf{Ibl} \colon \mathsf{S} \to 2^{\mathsf{AP}} \rangle and a formula \mathsf{F} \phi \in \mathsf{F} \big( \mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P}) \big)
Result: Whether \mathcal{L} \models_{\mathsf{AS}} \mathsf{F} \phi
CheckSystem(\mathcal{L}, \mathsf{F}\phi)
       Components \leftarrow The set of all the BSCCs of \mathcal{L};
       forall C \in Components do
              \mathsf{tmp} \leftarrow \mathsf{CheckBSCC}(\mathsf{C}, \mathcal{L}, \phi);
              if tmp \neq C then
                   return no;
       return yes;
CheckBSCC(C, \mathcal{L}, \phi)
      if \phi \in AP then
         return the set \{s \in \mathsf{C} \mid \phi \in \mathsf{lbl}(s)\};
       else if \phi = \psi \wedge \theta then
            return CheckBSCC(C, \mathcal{L}, \psi) \cap CheckBSCC(C, \mathcal{L}, \theta);
       else if \phi = \psi \vee \theta then
            return CheckBSCC(C, \mathcal{L}, \psi) \cup CheckBSCC(C, \mathcal{L}, \theta);
       else if \phi = \mathsf{F}^\infty_\mathsf{P} \psi then
              \mathsf{tmp} \leftarrow \mathsf{CheckBSCC}(\mathsf{C}, \mathcal{L}, \psi);
              \mathbf{if}\ \mathsf{SPred}^*(\mathsf{C},\mathsf{tmp}) = \mathsf{C}\ \mathbf{then}
                    return C;
              else
                     return Ø;
```

Algorithm 2: Polynomial time algorithm for the fair model checking of $F(L^+(F_p^\infty))$

Since formulas in $\mathsf{F}\phi \in \mathsf{F}(\mathsf{L}^+(\mathsf{F}_\mathsf{P}^\infty))$ define prefix independent sets, we can ignore any finite prefix of a run, and only check that ϕ is almost surely satisfied in each BSCC. The decomposition into BSCCs can be done using standard graph techniques. The crux resides in checking whether a BSCC B is "good", i.e., whether the formula is almost surely satisfied in B. This is done by a tableaux like approach. We recursively compute the set of states from where sub-formulas are satisfied. The key observation lies in the fact that B is "good" if and only if the formula can be satisfied from anywhere in B. This intuition is described in

Algorithm 2. Thanks to Lemma 8 and Lemma 9, Theorem 7 follows by noticing that the set of all the BSCCs can be computed in linear time.

Lemma 8. Given an LTS \mathcal{L} and a formula $F\phi \in F(L^+(F_P^\infty))$, then $\mathcal{L} \models_{AS} F\phi$ if and only if CheckSystem returns yes.

Lemma 9. Given an LTS \mathcal{L} , a BSCC B of \mathcal{L} and a formula $\mathsf{F}\phi \in \mathsf{F}(\mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P}))$, the procedure $\mathsf{CheckBSCC}(\mathsf{B},\mathcal{L},\phi)$ runs in time $O(|\phi||B|^2)$.

6 Discussion and conclusion

In this section we discuss our results, some immediate corollaries and future work. The first point we address might seem technical at a first glance, but we believe that it is worth mentioning. It concerns the maner in which the quantification on the bound is used in the evaluation of pLTL:

Strong against weak semantics Recall that pLTL formulas are evaluated with respect to a uniform bound k, i.e., for a system to be a model to some formula, all its runs are evaluated using the same bound k. Let us refer to this semantics as strong. One could want to relax this semantics and ask that a system \mathcal{L} satisfies a formula ϕ if for every run ρ there exists a bound k such that $(\rho, k) \models \phi$. Let us call this semantics weak. This raises the following questions: are both semantics equivalent? Obviously, the strong semantics always implies the weak one. However, the converse does not hold. To separate these two semantics, consider the system depicted in Fig. 6, and the formula $\phi = \mathsf{F}^{\infty}_{\mathsf{P}} \mathsf{A} \vee \mathsf{F}^{\infty}_{\mathsf{P}} \mathsf{B}$. This system is a model to the formula ψ with respect to the weak semantics but not the strong one. This raises in turn another natural question: Are these two semantics equivalent in some fragment? We answer positively to this question by noticing that they collapse for formulas in $F(L^+(F_P^{\infty}))$. This actually follows from noticing that The weak and strong semantics are equivalent for formulas in $\phi \in L(\mathsf{F}^{\infty}_{\mathsf{P}})$ if they are evaluated over a strongly connected LTS. This is formally proved in Appendix D.

Probabilistic model checking The last point we want to discuss is the complexity of quantitative verification for the fragment $F(L^+(F_P^\infty))$. Once again, our fragment behaves well, in that one can compute the satisfaction probability of a system with respect to a formula in $F(L^+(F_P^\infty))$ in polynomial time. This is formally proved in Appendix D.

Future work First, we plan on studying the complexity of the weak semantics for formulas in $F(L^+(F_P^\infty))$. Second, we plan on studying the synthesis problem induced by these fragments.

References

- Christel Baier and Joost-Pieter Katoen. Principles of model checking. MIT Press, 2008.
- 2. Thomas Brihaye, Florent Delgrange, Youssouf Oualhadj, and Mickael Randour. Life is random, time is not: Markov decision processes with window objectives. Log. Methods Comput. Sci., 16(4), 2020.
- 3. Véronique Bruyère, Quentin Hautem, and Mickael Randour. Window parity games: an alternative approach toward parity games with time bounds. In Domenico Cantone and Giorgio Delzanno, editors, Proceedings of the Seventh International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2016, Catania, Italy, 14-16 September 2016, volume 226 of EPTCS, pages 135–148, 2016.
- 4. Diego Calvanese, Giuseppe De Giacomo, and Moshe Y. Vardi. Reasoning about actions and planning in LTL action theories. In Dieter Fensel, Fausto Giunchiglia, Deborah L. McGuinness, and Mary-Anne Williams, editors, *Proceedings of the Eights International Conference on Principles and Knowledge Representation and Reasoning (KR-02), Toulouse, France, April 22-25, 2002*, pages 593–602. Morgan Kaufmann, 2002.
- Krishnendu Chatterjee. Concurrent games with tail objectives. Theor. Comput. Sci., 388(1-3):181–198, 2007.
- Krishnendu Chatterjee, Laurent Doyen, Mickael Randour, and Jean-François Raskin. Looking at mean-payoff and total-payoff through windows. *Inf. Comput.*, 242:25–52, 2015.
- Edmund M. Clarke, E. Allen Emerson, and A. Prasad Sistla. Automatic verification of finite-state concurrent systems using temporal logic specifications. ACM Trans. Program. Lang. Syst., 8(2):244–263, 1986.
- 8. Costas Courcoubetis and Mihalis Yannakakis. The complexity of probabilistic verification. J. ACM, 42(4):857–907, 1995.
- 9. E. Allen Emerson and Chin-Laung Lei. Modalities for model checking: Branching time logic strikes back. *Sci. Comput. Program.*, 8(3):275–306, 1987.
- Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. Reasoning About Knowledge. MIT Press, Cambridge, MA, USA, 2003.
- 11. Giuseppe De Giacomo and Moshe Y. Vardi. Automata-theoretic approach to planning for temporally extended goals. In Susanne Biundo and Maria Fox, editors, Recent Advances in AI Planning, 5th European Conference on Planning, ECP'99, Durham, UK, September 8-10, 1999, Proceedings, volume 1809 of Lecture Notes in Computer Science, pages 226–238. Springer, 1999.
- Hugo Gimbert and Florian Horn. Solving simple stochastic tail games. In Moses Charikar, editor, Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2010, Austin, Texas, USA, January 17-19, 2010, pages 847–862. SIAM, 2010.
- 13. Hugo Gimbert and Edon Kelmendi. Two-player perfect-information shift-invariant submixing stochastic games are half-positional. CoRR, abs/1401.6575, 2014.
- 14. Orna Kupferman, Nir Piterman, and Moshe Vardi. From liveness to promptness. Formal Methods in System Design, 34, 04 2009.
- 15. Anca Muscholl and Igor Walukiewicz. An np-complete fragment of LTL. Int. J. Found. Comput. Sci., 16(4):743–753, 2005.
- 16. J.C. Oxtoby. Measure and Category: A Survey of the Analogies Between Topological and Measure Spaces. Graduate Texts in Mathematics. Springer New York, 1971.

- 17. Amir Pnueli. The temporal logic of programs. In 18th Annual Symposium on Foundations of Computer Science (sfcs 1977), pages 46–57, 1977.
- Amir Pnueli and Roni Rosner. On the synthesis of a reactive module. In Conference Record of the Sixteenth Annual ACM Symposium on Principles of Programming Languages, Austin, Texas, USA, January 11-13, 1989, pages 179–190. ACM Press, 1989.
- 19. Amir Pnueli and Roni Rosner. On the synthesis of an asynchronous reactive module. In Giorgio Ausiello, Mariangiola Dezani-Ciancaglini, and Simona Ronchi Della Rocca, editors, Automata, Languages and Programming, 16th International Colloquium, ICALP89, Stresa, Italy, July 11-15, 1989, Proceedings, volume 372 of Lecture Notes in Computer Science, pages 652–671. Springer, 1989.
- Jean-Pierre Queille and Joseph Sifakis. Specification and verification of concurrent systems in CESAR. In Mariangiola Dezani-Ciancaglini and Ugo Montanari, editors, International Symposium on Programming, 5th Colloquium, Torino, Italy, April 6-8, 1982, Proceedings, volume 137 of Lecture Notes in Computer Science, pages 337–351. Springer, 1982.
- 21. Matthias Schmalz, Hagen Völzer, and Daniele Varacca. Model checking almost all paths can be less expensive than checking all paths. In Vikraman Arvind and Sanjiva Prasad, editors, FSTTCS 2007: Foundations of Software Technology and Theoretical Computer Science, 27th International Conference, New Delhi, India, December 12-14, 2007, Proceedings, volume 4855 of Lecture Notes in Computer Science, pages 532-543. Springer, 2007.
- 22. A. Sistla and Edmund Clarke. The complexity of propositional linear temporal logics. *J. ACM*, 32:733–749, 07 1985.
- 23. Hagen Völzer and Daniele Varacca. Defining fairness in reactive and concurrent systems. $J.\ ACM,\ 59(3):13:1-13:37,\ 2012.$

Appendix

A Proofs from Section 3

In the proof of Lemma 2 and Lemma 3, we use the notion of the rank of a formula:

Definition 12 (Rank of a filtered formula). Let ψ be a filtered formula in canonical form, the rank of ψ denoted $Rk(\psi)$ computes the nesting depth of temporalities in ψ . Formully it is obtained as follows:

$$\mathsf{Rk}(\psi) = \begin{cases} 0 \text{ if } \psi \text{ is a state formula} \\ 1 + \max\{\mathsf{Rk}(\psi_1), \dots, \mathsf{Rk}(\psi_m)\} \text{ if } \psi = \theta \vee \bigvee_{i=0}^m \mathsf{F}_\mathsf{P}^\infty \psi_i \end{cases}$$

A.1 Proof of Lemma 2

Lemma 2. Let \mathcal{L} be an LTS, ψ be a filtered formula in canonical form, and assume that there exists ρ and $k \geq |S| + 1$ such that $(\rho, k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi$, then there exists a ψ -avoiding sequence U which is realized by ρ .

Proof. Let us prove this result by induction over the rank of the formula.

If $\mathsf{Rk}(\psi) = 0$, let us write $\psi = \theta$. As $(\rho, k) \not\models \mathsf{F}_\mathsf{P}^\infty \theta$, by definition, there exists a position $j_0 \geq 0$ such that for all $j_0 \leq j \leq j_0 + k$, $\theta \not\in \mathsf{lbl}(\rho[j])$. Since $k \geq |\mathsf{S}| + 1$, $\rho[j_0..(j_0+k)]$ contains a cycle, let say $u^\theta = \rho[j_1..j_2]$, for some $j_0 \leq j_1 < j_2 \leq j_0 + k$. By construction, $\mathsf{occ}(u^\theta)$ is θ -avoiding. Since $\rho[j_1] \in \mathsf{occ}(u^\theta)$, u^θ is realized by ρ . Since $\mathsf{Tree}(\theta)$ consists of a single node labeled by θ , u^θ is a ψ -avoiding sequence realized by ρ .

If $\mathsf{Rk}(\psi) \geq 1$, ψ being filtered, let us write $\psi = \theta \vee \bigvee_{i=0}^m \mathsf{F}_\mathsf{P}^\infty \psi_i$. We build a ψ -avoiding sequence U^ψ associated with $\mathsf{Tree}(\psi)$. We build a mapping w^ψ from the nodes of $\mathsf{Tree}(\psi)$ to S^+ , and an order \prec_ψ over the image of w^ψ .

As $(\rho, k) \not\models \mathsf{F}^{\infty}_{\mathsf{P}} \psi$, by definition, exists a position $j_0 \geq 0$ such that for all $j_0 \leq j \leq j_0 + k$, $(\rho[j..], k) \not\models \theta \vee \bigvee_{i=0}^m \mathsf{F}^{\infty}_{\mathsf{P}} \psi_i$. In a similar way as before, exists a θ -avoiding cycle u^{θ} realized by ρ . The word u^{θ} will be the image of the root of $\mathsf{Tree}(\psi)$ and the minimum for the order \prec_{ψ} .

Let us write $\rho' = \rho[j_0 + k..]$. Then, $(\rho', k) \not\models \bigvee_{i=0}^m \mathsf{F}_\mathsf{P}^\infty \psi_i$. Therefore, for each $0 \le i \le m$, $(\rho', k) \not\models \mathsf{F}_\mathsf{P}^\infty \psi_i$. Moreover, for each $0 \le i \le m$, $\mathsf{Rk}(\psi_i) < \mathsf{Rk}(\psi)$. By induction hypothesis, exists for each $0 \le i \le m$ a ψ_i -avoiding sequence u^{ψ_i} realized by ρ' . Let us write $u^{\psi_i} = u^{\psi_i}_0, \ldots, u^{\psi_i}_{l_i}$ for some $l_i \ge 0$. For each node n in $\mathsf{Tree}(\psi)$, let us define $w^{\psi}(n) = u^{\theta}$ if n is the root of the tree, and $w^{\psi}(n) = w^{\psi_i}(n)$ if n is in Tree_{ψ_i} , where w^{ψ_i} is the mapping over $\mathsf{Tree}(\psi_i)$ induced by U^{ψ_i} . Note that each $w^{\psi}(n)$ is either u^{θ} or a $u^{\psi_i}_l$ for some i and l, and therefore, we use one or the other without distinction. By Definition 8, as each u^{ψ_i} for $0 \le i \le m$ is realized by ρ' , there exists a non-decreasing sequence of positions $p^{\psi_i}_0, \ldots, p^{\psi_i}_{l_i}$ such that for all $0 \le l \le l_i$, $\rho'[p^{\psi_i}_l] \in \mathsf{occ}(u^{\psi_i}_l)$. Let us construct the order $\prec_{\rho'}$ over the words $u^{\psi_i}_l$, for all $0 \le i \le m$ and $0 \le l \le l_i$ as follows:

$$- \text{ If } p_l^{\psi_i} < p_{l'}^{\psi_{i'}} \text{ then } u_l^{\psi_i} \prec_{\rho'} u_{l'}^{\psi_{i'}}.$$

$$- \text{ If } p_l^{\psi_i} = p_{l'}^{\psi_{i'}} \text{ then } \begin{cases} \text{if } i = i', \, u_l^{\psi_i} \prec_{\rho'} u_{l'}^{\psi_i} \text{ if and only if } l < l' \\ \text{otherwise the order is arbitrary} \end{cases}$$

We extend $\prec_{\rho'}$ to \prec_{ψ} by adding u^{θ} as a minimum.

We claim that \prec_{ψ} agrees with $\mathsf{Tree}(\psi)$, that is, if n and n' are two nodes in $\mathsf{Tree}(\psi)$ such that $n \prec_{\mathcal{T}} n'$, then $w^{\psi}(n) \prec_{\psi} w^{\psi}(n')$. Let $n \prec_{\mathcal{T}} n'$ be two nodes of $\mathsf{Tree}(\psi)$. Two cases can occur.

- If n is the root of Tree(ψ), then $w^{\psi}(n) = u^{\theta}$ is the minimum of \prec_{ψ} .
- If n is not the root of $\mathsf{Tree}(\psi)$, then n is a node of some $\mathsf{Tree}(\psi_i)$, for some $0 \le i \le m$. Therefore, n' is also a node of $\mathsf{Tree}(\psi_i)$, and since u^{ψ_i} is a ψ_i -avoiding sequence associated with $\mathsf{Tree}(\psi_i)$, $w^{\psi}(n) = u^{\psi_i}_l$ and $w^{\psi}(n') = u^{\psi_i}_l$ for some l < l'. Since U^{ψ_i} is realized by ρ' , by construction of $\prec_{\rho'}$,
 - either $p_l^{\psi_i} < p_{l'}^{\psi_i}$, hence $u_l^{\psi_i} \prec_{\psi} u_{l'}^{\psi_i}$,
 - or $p_l^{\psi_i} = p_{l'}^{\psi_i}$, and since $l < l', u_l^{\psi_i} \prec_{\psi} u_{l'}^{\psi_i}$

Therefore, (w^{ψ}, \prec_{ψ}) is a sequence associated with $\mathsf{Tree}(\psi)$. Let us call it $U^{\psi} = u_0^{\psi}, u_1^{\psi}, u_2^{\psi}, \dots, u_{l_{\phi}}^{\psi}$.

Now, let us prove that U^{ψ} is ψ -avoiding. By construction of U^{ψ} , $u_0^{\psi} = u^{\theta}$ is n_0 -avoiding, since n_0 is the root of $\mathsf{Tree}(\psi)$ and labeled by θ , and $\mathsf{occ}(u^{\theta})$ is θ -avoiding. For another node n, n is in $\mathsf{Tree}(\psi_i)$ for some $0 \le i \le m$. By induction hypothesis, U^{ψ_i} is ψ_i -avoiding, and therefore $\mathit{occ}(w^{\psi_i}(n))$ is n-avoiding. Since $w^{\psi}(n) = w^{\psi_i}(n)$ for each n, U^{ψ} is ψ -avoiding.

To conclude, let us prove that ρ realizes U^{ψ} . For each $1 \leq l \leq l_{\psi}$, there exists a $0 \leq i \leq m$ and a $0 \leq l' \leq l_i$ such that $u_l^{\psi} = u_{l'}^{\psi_i}$. By definition of p^{ψ_i} , $\rho'[p_{l'}^{\psi_i}] \in \operatorname{occ}(u_{l'}^{\psi_i})$ and therefore $\rho'[p_{l'}^{\psi_i}] \in \operatorname{occ}(u_l^{\psi})$. Let $p_l^{\psi} = (j_0 + k) + p_{l'}^{\psi_i}$, and $p_0^{\psi} = j_1$. Therefore, $\rho[p_0^{\psi}] = \rho[j_1] \in \operatorname{occ}[\theta]$, and for $1 \leq l \leq l_{\psi}$, $\rho[p_l^{\psi}] = \rho'[p_{l'}^{\psi_i}] \in \operatorname{occ}(u_l^{\psi})$. By construction, the sequence $p_0^{\psi}, \ldots, p_{l_{\psi}}^{\psi}$ is non-decreasing, and therefore ρ realizes U^{ψ} .

A.2 Proof of Lemma 3

Lemma 3. Let \mathcal{L} be an LTS, ψ be a filtered formula in canonical form, and let U be a realisable ψ -avoiding sequence in \mathcal{L} . For any k-pumping ρ_k of some realisation witness $(U, v_0, v_1, \ldots, v_{l+1})$ for (U, \mathcal{L}) , we have $(\rho_k, k) \not\models \mathsf{F}_p^{\infty} \psi$.

Proof. Let U be a realisable ψ -avoiding sequence, let us write $U = u_0 \dots, u_l$, and let us show by induction over the rank $\mathsf{Rk}(\psi)$ that each k-pumping ρ_k of some realisation witness $(U, v_0, v_1, \dots, v_{l+1})$ satisfies $(\rho_k, k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi$.

If $\mathsf{Rk}(\psi) = 0$, then $\psi = \theta$ is a state formula, U has a unique word u_0 , and $\mathsf{occ}(u_0)$ is θ -avoiding. Consider the k-pumping $\rho_k = v_0 u_0^k v_1$. Then, for each $0 \le j \le k$, $\rho_k[|v_0| + j] \in u_0$, therefore $\theta \notin \mathsf{lbl}(\rho_k[|v_0| + j])$. This means that $(\rho_k, k) \not\models \mathsf{F}_\mathsf{P}^{\mathsf{p}} \psi$.

If $\mathsf{Rk}(\psi) \geq 1$, since ψ is filtered and in canonical form, write $\psi \equiv \theta \vee \bigvee_{i=1}^{l'} (\mathsf{F}_\mathsf{P}^\infty \psi_i)$ where θ is a state formula, and let $j_0 = |v_0|$, and ρ_k be a k-pumping. Then, by definition, for each $0 \leq j \leq k$, $\rho_k[j_0 + j] \in u_0$, thus, $\theta \not\in \mathsf{lbl}(\rho_k[j_0 + j])$. Choose a position $0 \leq i \leq m$, we build U^{ψ_i} the ψ_i -avoiding sequence induced by U by considering the restriction of U over the subtree $\mathsf{Tree}(\psi_i)$ of $\mathsf{Tree}(\psi)$. Write $U^{\psi_i} = u_0^{\psi_i}, \ldots, u_l^{\psi_i}$ and let us define $p_0 < \ldots < p_l$ such that $u_0^{\psi_i} = u_{p_0}, \ldots, u_l^{\psi_i} = u_{p_l}$. Then

$$(U^{\psi_i}, v_1(u_1)^k v_2 \cdots v_{p_0}, v_{p_0+1}(u_{p_0+1})^k \cdots v_{p_1}, \dots, v_{p_m+1}(u_{p_m+1})^k \cdots v_{l+1})$$

is a realisation witness of $(u^{\psi_i}, \mathcal{L})$. Moreover, $\rho'_k = \rho_k[(j_0 + k|u_0|)..]$ is a k-pumping of this realisation witness.

Now by induction hypothesis, $(\rho'_k, k) \not\models \mathsf{F}^\infty_\mathsf{P} \psi_i$. Furthermore, for all positions $0 \le j \le k$, ρ'_k is a suffixe of $\rho_k[(j_0+j)..]$, thus $(\rho_k[(j_0+j)..], k) \not\models \mathsf{F}^\infty_\mathsf{P} \psi_i$. Finally, since this holds true for each $0 \le i \le m$, it follows that for all positions $0 \le j \le k$, we have

$$(\rho_k[(j_0+j)..],k) \not\models \theta \lor \bigvee_{i=1}^{l'} (\mathsf{F}_\mathsf{P}^\infty \psi_i)$$

hence, $(\rho_k, k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi$ which concludes the proof.

A.3 Proof of Lemma 4

We first prove the following lemma:

Lemma 10. Given an LTS \mathcal{L} and a filtered formula $\phi \in L^+(\mathsf{F}^{\infty}_{\mathsf{P}})$, then

$$\forall k \geq 0, \ (\rho_k, k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi \iff \forall k \geq 0, \ (\rho_k, k) \not\models \psi$$

Proof. First notice that if ψ is filtered and in $\mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P})$, then necessarily $\psi \equiv \bigvee_{i=0}^m \mathsf{F}^\infty_\mathsf{P} \psi_i$ for some m>0.

$$(\rho_{k}, k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi \implies \exists j_{0} \geq 0, \ \forall 0 \leq j \leq k, \ (\rho_{k}[(j_{0} + j)..], k) \not\models \psi$$

$$\implies \forall 0 \leq i \leq m, \ (\rho_{k}[(j_{0} + j)..], k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi_{i}$$

$$\implies \forall 0 \leq i \leq m, \ (\rho_{k}, k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi_{i}$$

$$\implies (\rho_{k}, k) \not\models \bigvee_{i=0}^{m} \mathsf{F}_{\mathsf{P}}^{\infty} \psi_{i} \equiv \psi$$

where the first and second implications are by definition of the evaluation, and the third is by property of F_P^{∞} .

To show the other direction, we show the contrapositive

$$\exists k \geq 0, \ (\rho_k, k) \models \mathsf{F}_{\mathsf{P}}^{\infty} \psi \implies \forall j_0 \geq 0, \ \exists 0 \leq j \leq k \ (\rho_k[(j_0 + j)..], k) \models \psi'$$

$$\implies \forall 0 \leq i \leq m, \ \exists 0 \leq j_i \leq k, \ (\rho_k[j_i..], k) \models \mathsf{F}_{\mathsf{P}}^{\infty} \psi_i$$

$$\implies (\rho_k, 2k) \models \mathsf{F}_{\mathsf{P}}^{\infty} \psi_i \implies (\rho_k, 2k) \models \bigvee_{i=0}^{m} \mathsf{F}_{\mathsf{P}}^{\infty} \psi_i \equiv \psi$$

where the first implication is by definition of the evaluation, the second is by setting $j_0 = 0$ and distributing over all the formulas ψ_i , and the third one holds because we add the missing prefix $\rho_k[...j_i]$.

Lemma 4. Given an LTS \mathcal{L} and a formula $\phi \in \mathsf{L}(\mathsf{F}^{\infty}_{\mathsf{P}})$, Algorithm 1 accepts if and only if, for all k, there is a run ρ_k in Runs_{init} such that $(\rho_k, k) \not\models \phi$.

Proof. We start with left to right direction. If Algorithm 1 accepts, then there exists:

- i. a filtration ψ of ϕ of the form $\psi \equiv \theta \vee \psi'$, where $\theta = \bigvee_{i=0}^{l} a_i$ and $\psi' = \bigvee_{i=0}^{m} \mathsf{F}_{\mathsf{P}}^{\infty} \psi_i$.
- -ii. a ψ' -avoiding sequence U realisable in \mathcal{L} .

Let us show that for each k, there exists a run ρ_k such that $(\rho_k, k) \not\models \phi$. Let $k \ge |S| + 1$, first, since the initial state does not satisfy $\bigvee_{i=0}^{l} a_i$, any run $\rho \in \mathsf{Runs}_{\mathsf{init}}$ is such that $(\rho, k) \not\models \theta$. Second, by Lemma 3, there exists a run $\rho_k \in \mathsf{Runs}_{\mathsf{init}}$ such that $(\rho_k, k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi'$. By Lemma 10 ρ_k is also a counter example for bound k to the formula ψ' . By Lemma 1, this implies that $(\rho_k, k) \not\models \phi$. To conclude this direction one has to notice that counter examples for the bound |S| + 1 are also counter examples for smaller bounds.

Now we show the right to left direction. Assume that for each k, there exists a run $\rho_k \in \operatorname{Runs}_{\operatorname{init}}$ such that $(\rho_k, k) \not\models \phi$. By Lemma 1 $(\rho_k, k) \not\models \operatorname{flt}_{\rho_k}(\phi)$. In order to apply Lemma 2, let $\operatorname{flt}_{\rho_k}(\phi) \equiv \theta \vee \psi'$ where $\theta = \bigvee_{i=0}^l a_i$ and $\psi' = \bigvee_{i=0}^m \mathsf{F}_{\mathsf{P}}^{\infty} \psi_i$. By Lemma 10 we have that for each $k \geq 0$, $(\rho_k, k) \not\models \mathsf{F}_{\mathsf{P}}^{\infty} \psi'$. By Lemma 2, we build a ψ' -avoiding sequence realized by ρ_k , thus realizable in \mathcal{L} , hence Algorithm 1 will guess it and the checks will be satisfied. To conclude that Algorithm 1 accepts, it remains to notice that $(\rho_k, k) \not\models \theta \vee \psi'$ implies in particular that $\rho[0] = s_{\mathsf{init}}$ does not satisfies θ .

A.4 Proof of Lemma 5

Lemma 5. The universal model checking problem for $L(F_{P}^{\infty})$ is in coNP.

Proof. Let \mathcal{L} be an LTS, and a formula $\psi \in \mathsf{L}(\mathsf{F}^\infty_\mathsf{P})$, first notice that each guess is of polynomial size. Let us show that the verification phase can be done in polynomial time.

- To check that the initial state does not satisfy $\bigvee_{i=0}^{l} a_i$, one has to check whether each $a_i \in \mathsf{lbl}(s_{\mathsf{init}})$. This is clearly done in linear time.
- To check that U is associated with $\mathsf{Tree}(\psi)$, first, we check that for each $0 \le j \le m$, $w(n_j) = u_j$. This can be done in $O(|\mathsf{S}||\psi|)$. Then, we check for each $n \prec n'$ in $\mathsf{Tree}(\psi)$ that $w(n) \prec w(n')$, i.e., if $w(n) = u_j$ and $w(n') = u_{j'}$, we check that j < j'. This can be done in $O(|\psi|^2)$.
- To check that U is ψ -avoiding, we check that for each $n \in \mathsf{Tree}(\psi)$, $\mathsf{occ}(w(n))$ is a n-avoiding cycle. As n is labeled by state formulas θ_n , we check that for each $s \in w(n)$, $\theta_n \not\in \mathsf{lbl}(s)$. This can be done in $O(|\mathsf{S}||\psi|)$.

- To check that U is realisable in \mathcal{L} , we check that $u_0[0]$ is reachable from s_{init} , and that for each $0 \leq j \leq l-1$, the first state of u_{j+1} is reachable from the last state of u_j , that is $u_{j+1}[0]$ is reachable from $u_j[|u_j^{\psi_i}|-1]$. Thus we perform m reachability test, thus it takes at most $O(|S|^2|\psi|)$.

The lemma follows.

B Proofs from Section 4

B.1 Proof of Lemma 6

Lemma 6. Let ϕ be a formula, \mathcal{L} be an LTS and assume that there exists a run ρ such that for some k > 0 we have $(\rho, k) \not\models \phi$, then there exists a prefix w of ρ such that for any run $\rho' \in \mathsf{Cyl}(w)$ we have $(\rho', k) \not\models \phi$.

Proof. We proceed by induction over the structure of the formula. Let ϕ be a formula and let ρ be a run and assume that for some k > 0 we have $(\rho, k) \not\models \phi$.

- If $\phi = \theta$ is a state formula, then $\theta \notin \mathsf{Ibl}(s_{\mathsf{init}})$. Hence any run $\rho \in \mathsf{Runs}_{\mathsf{init}}$ is such that $(\rho, k) \not\models \theta$ since $\mathsf{Runs}_{\mathsf{init}} = \mathsf{Cyl}(s_{\mathsf{init}})$.
- If $\phi = \psi_1 \wedge \psi_2$, then $(\rho, k) \not\models \psi_1 \wedge \psi_2$ if and only if $(\rho, k) \not\models \psi_1$ or $(\rho, k) \not\models \psi_2$. W.l.o.g., suppose that $(\rho, k) \not\models \psi_1$, then by induction, there exists a prefix w of ρ such that

$$\forall \rho' \in \mathsf{Cyl}(w), (\rho', k) \not\models \psi_1 \implies \forall \rho' \in \mathsf{Cyl}(w), (\rho', k) \not\models (\psi_1 \land \psi_2) \equiv \phi$$

- If $\phi = \psi_1 \vee \psi_2$, then

$$((\rho, k) \not\models \psi_1 \lor \psi_2) \iff (\rho, k) \not\models \psi_1 \text{ and } (\rho, k) \not\models \psi_2$$

By induction, there exist two prefixes w_1 and w_2 of ρ such that

$$\forall \rho' \in \mathsf{Cyl}(w_1), (\rho', k) \not\models \psi_1 \text{ and } \forall \rho' \in \mathsf{Cyl}(w_2), (\rho', k) \not\models \psi_2$$

Now, since w_1 and w_2 are prefixes of the same word, one of them is a prefix of the other, w.o.l.g., assume that w_2 is a prefix of w_1 . Then we have:

$$\mathsf{Cyl}(w_1) \subseteq \mathsf{Cyl}(w_2) \implies \forall \rho' \in \mathsf{Cyl}(w_1), (\rho', k) \not\models \psi_2$$

$$\implies \forall \rho' \in \mathsf{Cyl}(w_1), (\rho', k) \not\models \psi_1 \vee \psi_2 \equiv \phi$$

– If $\phi = \mathsf{F}^{\infty}_{\mathsf{P}} \psi$, then $(\rho, k) \not\models \mathsf{F}^{\infty}_{\mathsf{P}} \psi$. By definition, this means that there exists a position $i \geq 0$, such that for all positions $0 \leq j \leq k$, we have

$$\forall j, \text{ s.t. } 0 \leq j \leq k, \ (\rho[(i+j)..], k) \not\models \psi$$

Applying the induction hypothesis from each postion j we can extract a prefix w_j of $\rho[(i+j)...]$ such that:

$$\forall \rho_i' \in \mathsf{Cyl}(w_i) , (\rho_i', k) \not\models \psi$$

Now we will extract the prefix w, to do this, let w be the smallest prefix of ρ containing all the prefixes w_j . Formally, let $i_0 = i + \max\{0 + |w_0|, \dots, k + |w_k|\}$, and chose $w = \rho[..i_0]$. Let us show that w satisfies the claim of the lemma. Let ρ' be a run in $\mathsf{Cyl}(w)$, then

$$\forall 0 \le j \le k, \ \rho'[(i+j)..] \in \mathsf{Cyl}(w_j) \implies \forall 0 \le j \le k, \ (\rho'[(i+j)..], k) \not\models \psi$$
$$\implies (\rho', k) \not\models \mathsf{F}_{\mathsf{P}}^{\mathsf{D}} \psi$$

where the first implication is by construction of w_j and the second one by definition of $\mathsf{F}^\infty_\mathsf{P}$. This concludes the proof.

C Proofs from Section 5

C.1 Proof of Lemma 7

Lemma 7 is useful to establish the correctness of the following procedure.

```
Data: An LTS \mathcal{L} = \langle \mathsf{S}, s_{\mathsf{init}}, \mathsf{T}, \mathsf{lbl} \colon \mathsf{S} \to 2^{\mathsf{AP}} \rangle and a formula \phi \in \mathsf{L}(\mathsf{F}_\mathsf{P}^\infty)

Result: whether \mathcal{L} \not\models \phi

Guess the filtered formula \psi of \phi \in \mathsf{L}^+(\mathsf{F}_\mathsf{P}^\infty);

Guess a sequence U = u_0, \ldots, u_l of size (|S| + 1)|\psi|;

Compute the tree \mathsf{Tree}(\psi);

Guess a mapping between the nodes of the tree and the sequence;

Check that U is associated with \mathsf{Tree}(\psi);
```

Check that U is ψ -avoiding; Check that there exists an SCC C s.t. U is realisable inside C;

Algorithm 3: Non deterministic algorithm for the universal model checking of $F(L^+(F_P^\infty))$

Lemma 7. Let \mathcal{L} be an LTS, $\psi \in L^+(\mathsf{F}^\infty_\mathsf{P})$ be a filtered formula in canonical form, and $k \geq |\mathsf{S}| + 1$, there exists a run ρ such that $(\rho, k) \not\models \mathsf{F}\psi$ if and only if there exists a strongly connected component C and a ψ -avoiding sequence realized in C.

Proof. Assume that there exists a run ρ such that $(\rho, k) \not\models \mathsf{F}\psi$. The set $C = \{s \in \mathsf{S} \mid \forall i, \exists j \geq i, \rho[j] = s\}$ of infinitely often visited states in ρ belongs to some SCC C. By definition, there exists a position i_0 such that for all positions $j \geq i_0$, $\rho[j] \in C$. Since $(\rho, k) \not\models \mathsf{F}\psi$, by definition of F , for all $j \geq 0$, $(\rho[(i_0 + j)..], k) \not\models \psi$. Now since ψ is filtered and in $\mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P})$ we can apply Lemma 10. Therefore, $(\rho[i_0..], k) \not\models \mathsf{F}^\infty_\mathsf{P}\psi$. By Lemma 2 over the LTS induced by C, there exists a ψ -avoiding sequence realized in C.

Now, assume that there exists a strongly connected component C and a ψ -avoiding sequence U realized in C. By Lemma 3, exists a run ρ over C such that

 $(\rho,k) \not\models \mathsf{F}^\infty_\mathsf{P} \psi$. Since $\psi \in \mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P})$ and is filtered, Lemma 10 gives that $(\rho,k) \not\models \psi$. By Lemma 6, there exists a prefix w of ρ such that for each run $\rho' \in \mathsf{Cyl}(w)$ over C we have $(\rho',k) \not\models \psi$. Since C is strongly connected, there exists a path from w[|w|-1] the last position of w to w[0]. Let us call this path w'. Then, considere the run $\rho' = (ww')^\omega$. By construction, for each $i \geq 0$, there exists a position $j \geq i$ such that $\rho'[j..] \in \mathsf{Cyl}(w)$, therefore, $(\rho'[j..],k) \not\models \psi$. Since $\psi \in \mathsf{L}(\mathsf{F}^\infty_\mathsf{P})$, this implies that for each $j' \leq j$, $(\rho'[j'..],k) \not\models \psi$. Since this holds for infinitely many j, this means that $(\rho',k) \not\models \mathsf{F}\psi$.

C.2 Proof of Lemma 8

In order to state the proof of Lemma 8, we first need to introduce some technical properties of Algorithm 2

First we relate the set $\mathsf{SPred}^*(U)$ with the avoiding cycles. Let B be a BSCC, we denote by $S_B^\theta = \{s \in B \mid \theta \in \mathsf{Ibl}(s)\}.$

Lemma 11. Let θ be a state formula, B be a BSCC, then $\mathsf{SPred}^*(S_B^\theta) = B$ if and only if B does not contain a θ -avoiding cycle.

Proof. First, by contrapositive, assume that B contains a θ -avoiding cycle $C = \{s_0, \ldots, s_n\}$ and let us prove that $\mathsf{SPred}^*(S_B^\theta) \neq B$. By definition, for all $0 \leq i \leq n, \ \theta \not\in \mathsf{lbl}(s_i)$, and therefore $s_i \not\in S_B^\theta$. By definition of the sure attractor, it is the smallest fixed point of μ . For each $0 \leq i \leq n, \ s_i$ has a successor in C, and that successor is not in S_B^θ . Therefore, $s_i \not\in \mu(S_B^\theta)$. Since each s_i has a successor in C that is not in $\mu(S_B^\theta)$, $s_i \not\in \mu^2(S_B^\theta)$. By the same reasoning, for each $l \geq 0$ and $0 \leq i \leq n, \ s_i \not\in \mu^l(S_B^\theta)$. Therefore, $s_i \not\in \mathsf{SPred}^*(S_B^\theta)$, and $\mathsf{SPred}^*(S_B^\theta) \neq B$.

Now, by contrapositive, assume that $\mathsf{SPred}^*(s_B^\theta) \neq B$ and let us show that exists a θ -avoiding cycle. Let $s_0 \not\in \mathsf{SPred}^*(S_B^\theta) \neq B$. By definition of SPred^* , this implies that $s_0 \not\in S_B^\theta$ and, since B is a BSCC, there must exist a successor s_1 such that $s_1 \not\in \mathsf{SPred}^*(S_B^\theta)$. Repeating this argument from s_1 , there must exist $s_2 \in \mathsf{Succ}(s_1)$ such that $s_2 \not\in \mathsf{SPred}^*(S_B^\theta)$. Since B is finite, at some point, we will find two state that $s_{i_0} = s_{i_1}$ such that $C = \{s_{i_0}, \ldots, s_{i_1-1}\}$ is a cycle. Moreover for all $i_0 \leq j \leq i_1 - 1$, $s_j \notin S_B^\theta$. Therefore, C is a θ -avoiding cycle.

Lemma 12. Given an LTS \mathcal{L} , a BSCC B of \mathcal{L} and a formula $\phi \in L^+(\mathsf{F}^\infty_\mathsf{P})$, then exactly one of the followings holds:

- CheckBSCC(B, \mathcal{L}, ϕ) = B.
- CheckBSCC(B, \mathcal{L}, ϕ) = \emptyset .

Proof. Let B be a BSCC of \mathcal{L} , and $\phi \in L^+(\mathsf{F}^\infty_\mathsf{P})$. Let us prove the result by structural induction over ϕ . By definition of $L^+(\mathsf{F}^\infty_\mathsf{P})$, three cases can occur.

 $-\phi = \mathsf{F}^{\infty}_{\mathsf{P}}\psi, \ \psi \in \mathsf{L}(\mathsf{F}^{\infty}_{\mathsf{P}})$: Then, $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi)$ returns either B or \emptyset , and the result holds.

- $-\phi = \psi \lor \psi', \ \psi, \theta \in L^+(\mathsf{F}^\infty_\mathsf{P})$: By induction hypothesis, $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ is either equal to B or to \emptyset , and the same holds for $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi')$. Given that $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi) = \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi) \cup \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi')$, $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi)$ is either empty (if both $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ and $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi')$ are empty) or equal to B otherwise.
- $-\phi = \psi \wedge \psi', \ \psi, \theta \in L^+(\mathsf{F}^\infty_\mathsf{P})$: By induction hypothesis, $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ is either equal to B or to \emptyset , and the same holds for $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi')$. Given that $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi) = \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi) \cap \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi')$, $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ is either equal to B (if both $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ and $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi')$ are equal to B) or empty otherwise.

Lemma 13. Given an LTS \mathcal{L} , a BSCC B of \mathcal{L} , a formula $\phi \in L(\mathsf{F}^{\infty}_{\mathsf{P}})$, b a state in B, and $N = |\mathsf{S}| + 1$, then the following assertions are equivalent:

- $-b \in \mathsf{CheckBSCC}(\mathsf{B},\mathcal{L},\phi).$
- Any run ρ in bB^{ω} satisfies $(\rho, N) \models \phi$.

Proof. Let us show the equivalence by structural induction over ϕ .

- $-\phi = \theta \in \mathsf{AP}$: Then, $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \theta)$ if, and only if, $\theta \in \mathsf{Ibl}(b)$. As $\theta \in \mathsf{Ibl}(b)$ if and only if any run ρ in bB^ω satisfies $(\rho, N) \models \phi$, the equivalence holds.
- $-\phi = \psi \wedge \psi'$: First, assume that $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi)$. By construction, this implies that $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ and $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi')$. By induction hypothesis, any run ρ in bB^{ω} satisfies $(\rho, N) \models \psi$ and $(\rho, N) \models \psi'$, and therefore $\rho, N \models \phi$.
 - Now, assume that any run ρ in bB^{ω} satisfies $(\rho, N) \models \phi$. By definition, this means that $(\rho, N) \models \psi$ and $(\rho, N) \models \psi'$. By induction hypothesis, this implies that $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ and $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi')$. By construction, this implies that $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi)$.
- $-\phi = \psi \lor \psi'$: First, assume that $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi)$. By construction, this implies that $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ or $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi')$. W.l.o.g., assume $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$. By induction hypothesis, any run ρ in bB^{ω} satisfies $(\rho, N) \models \psi$ and therefore $(\rho, N) \models \phi$.
 - Now, assume that any run ρ in bB^{ω} satisfies $(\rho, N) \models \phi$. By definition, this means that $(\rho, N) \models \psi$ or $(\rho, N) \models \psi'$. W.l.o.g., assume that $(\rho, N) \models \psi$. By induction hypothesis, this implies that $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$. By construction, this implies that $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi)$.
- $\phi = \mathsf{F}_{\mathsf{P}}^{\infty} \psi$: First, note that $\exists b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi)$ is equivalent to $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi) = B$, and therefore to $\mathsf{SPred}^*(C, \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)) = B$. Thus, it is enough to prove that $\mathsf{SPred}^*(C, \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)) = B$ if and only if any run ρ in B^{ω} satisfies $(\rho, N) \models \phi$. Let us introduce a new label A^{ψ} . Let us label each set in $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ with A^{ψ} , and exactly those states. By induction hypothesis, for each $b \in B$, $b \in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi)$ if and only if any run ρ in bB^{ω} satisfies $(\rho, N) \models \psi$. By construction of A^{ψ} , this is equivalent to $(\rho, N) \models \theta$, where $\theta = A^{\psi}$ is a state formula. Since $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)$ is the maximum set labeled by A^{ψ} , by Lemma 11, $\mathsf{SPred}^*(C, \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \psi)) =$

B if and only if there exists no θ -avoiding cycle. By construction of A^{ψ} , there exists no θ -avoiding cycle is equivalent to there exists no ψ -avoiding cycle. Using Lemma 2 and Lemma 3, there exists no ψ -avoiding cycle if and only if for all run ρ in B^{ω} , $(\rho, N) \models \mathsf{F}_{\mathsf{P}}^{\infty} \psi$.

Lemma 8. Given an LTS \mathcal{L} and a formula $F\phi \in F(L^+(F_P^\infty))$, then $\mathcal{L} \models_{AS} F\phi$ if and only if CheckSystem returns yes.

Proof (Proof of Lemma 8). First, assume that \mathcal{L} fairly satisfies the formula, that is, exists a k such that $\mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \models \mathsf{F}\phi\}) = 1$. Consider a BSCC B of \mathcal{L} . Then, since B is reachable, $\mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid \rho \text{ ends in } B\}) \geq 0$. This implies that $\mathbb{P}_{B}(\{\rho \in B^{\omega} \mid (\rho, k) \models \mathsf{F}\phi\}) = 1$, and since B is a BSCC, $\mathbb{P}_{B}(\{\rho \in B^{\omega} \mid (\rho, k) \models \phi\}) = 1$. By lemma 6, this implies that for all $\rho \in B^{\omega}$, $(\rho, k) \models \phi$. Since this holds for any ρ starting in any state of B, by lemma 13, this implies that CheckBSCC(B, \mathcal{L}, ϕ) = B. Since this holds for any BSCC B of \mathcal{L} , by definition of CheckSystem, it returns yes.

Now, assume that \mathcal{L} does not fairly satisfy the formula, that is, for all k, $\mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \models \mathsf{F}\phi\}) < 1$. Let us fixe a k. Note that $\mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid \rho \text{ does not end in some BSCC } B\}) = 0$. Therefore exists a BSCC B_k of \mathcal{L} such that $\mathbb{P}_{B_k}(\{\rho \in B_k^\omega \mid (\rho, k) \models \mathsf{F}\phi\}) < 1$, there exists at least one $\rho_k \in B_k^\omega$ such that $(\rho_k, k) \not\models \mathsf{F}\phi$, and therefore, in particular, $(\rho_k, k) \not\models \phi$. By lemma 13, this means that $\rho[0] \not\in \mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi)$ and therefore, by lemma 12, since $\phi \in \mathsf{L}^+(\mathsf{F}_\mathsf{P}^\infty)$, $\mathsf{CheckBSCC}(\mathsf{B}, \mathcal{L}, \phi) = \emptyset$. By definition of $\mathsf{CheckSystem}$, it returns no.

C.3 Proof of Lemma 9

Lemma 9. Given an LTS \mathcal{L} , a BSCC B of \mathcal{L} and a formula $\mathsf{F}\phi \in \mathsf{F}(\mathsf{L}^+(\mathsf{F}^\infty_\mathsf{P}))$, the procedure $\mathsf{CheckBSCC}(\mathsf{B},\mathcal{L},\phi)$ runs in time $O(|\phi||B|^2)$.

Proof. We show by structural induction over ϕ that CheckBSCC(B, \mathcal{L}, ϕ) can be computed in time $|\phi||B|^2$.

- $-\phi \in \mathsf{AP}$ is a state formula. Then, $\mathsf{CheckBSCC}(\mathsf{B},\mathcal{L},\phi)$ is the set of states s such that $\phi \in \mathsf{lbl}(s)$. This can be computed in time $|B| \leq |B|^2$.
- $-\phi = \psi_1 \wedge \psi_2$. Then, CheckBSCC(B, $\mathcal{L}, \phi) = \text{CheckBSCC}(B, \mathcal{L}, \psi_1) \cap \text{CheckBSCC}(B, \mathcal{L}, \psi_2)$. By induction hypothesis, CheckBSCC(B, \mathcal{L}, ψ_1) can be computed in time $|\psi_1||B|^2$, and CheckBSCC(B, \mathcal{L}, ψ_2) can be computed in time $|\psi_2||B|^2$. Since CheckBSCC(B, \mathcal{L}, ψ_1) $\subseteq B$ and CheckBSCC(B, \mathcal{L}, ψ_2) $\subseteq B$, the intersection can be computed in time |B|. Therefore, CheckBSCC(B, \mathcal{L}, ϕ) can be computed in time $|\psi_1||B|^2 + |\psi_2||B|^2 + |B| \leq (|\psi_1| + |\psi_2| + 1)|B|^2 \leq |\phi||B|^2$.
- $\begin{array}{l} -\phi=\psi_1\vee\psi_2. \ \text{Then, CheckBSCC}(\mathsf{B},\mathcal{L},\phi)=\text{CheckBSCC}(\mathsf{B},\mathcal{L},\psi_1)\cup\text{CheckBSCC}(\mathsf{B},\mathcal{L},\psi_2). \\ \text{By induction hypothesis, CheckBSCC}(\mathsf{B},\mathcal{L},\psi_1) \ \text{can be computed in time} \\ |\psi_1||B|^2, \ \text{and CheckBSCC}(\mathsf{B},\mathcal{L},\psi_2) \ \text{can be computed in time} \ |\psi_2||B|^2. \ \text{Since} \\ \text{CheckBSCC}(\mathsf{B},\mathcal{L},\psi_1)\subseteq B \ \text{and CheckBSCC}(\mathsf{B},\mathcal{L},\psi_2)\subseteq B, \ \text{the union can be computed in time} \ |B|. \ \text{Therefore, CheckBSCC}(\mathsf{B},\mathcal{L},\phi) \ \text{can be computed in time} \ |\psi_1||B|^2+|\psi_2||B|^2+|B|\leq (|\psi_1|+|\psi_2|+1)|B|^2\leq |\phi||B|^2. \\ \end{array}$

 $-\phi = \mathsf{F}^{\infty}_{\mathsf{P}}\psi$. Then, to compute $\mathsf{CheckBSCC}(\mathsf{B},\mathcal{L},\phi)$, first, $\mathsf{CheckBSCC}(\mathsf{B},\mathcal{L},\psi)$ is computed. By induction hypothesis, this is done in time $|\psi||B|^2$. Then, SPred^* is computed. This is done in time $|B|^2$. Therefore, $\mathsf{CheckBSCC}(\mathsf{B},\mathcal{L},\phi)$ is computed in time $(|\psi|+1)|B|^2 = |\phi||B|^2$.

D Proofs from Section 6

Lemma 14. Given an LTS \mathcal{L} and a formula $\phi \in L(\mathsf{F}^{\infty}_{\mathsf{P}})$, if \mathcal{L} is strongly connected, then the weak and strong prompt model checking problems are equivalent.

Proof. The strong model checking trivially induces the weak one. We prove the converse by contrapositive. Assume that $\forall k, \ \exists \rho_k, (\rho_k, k) \not\models \phi$, and show that $\exists \rho, \ \forall k, (\rho, k) \not\models \phi$. In the following we consider $k \geq |\mathsf{S}| + 1$ since the other case follows from the case $k = |\mathsf{S}| + 1$.

Let $\psi_k = \operatorname{flt}_{\rho_k}(\phi) \equiv \theta \vee \psi_k'$, where $\theta = \bigvee_{i=0}^l a_i$ and $\psi_k' = \bigvee_{i=0}^m \mathsf{F}_{\mathsf{P}}^\infty \psi_i$. By Lemma 1, $(\rho_k, k) \not\models \psi_k$. Therefore, $\theta \not\in \operatorname{Ibl}(s_{\operatorname{init}})$ any run $\rho \in \operatorname{Runs}_{\operatorname{init}} \not\models \theta$. Moreover, $(\rho_k, k) \not\models \psi_k'$, and since $\psi_k' \in \mathsf{L}^+(\mathsf{F}_{\mathsf{P}}^\infty)$ and is filtered, by Lemma 10 we have that $(\rho_k, k) \not\models \mathsf{F}_{\mathsf{P}}^\infty \psi_k'$. By Lemma 2, there exists ψ_k' -avoiding sequence $U^k = u_0^k, \ldots, u_{l_k}^k$ for some $l_k \geq 0$, and let $v_0^k, \ldots, v_{l_{k+1}}^k$ a realisation witness for (U^k, \mathcal{L}) . We will now build a run ρ^* which is a counter example for every k. Denote w^k a path from the last element of $u_{l_k}^k$ to the first element of u_0^{k+1} . Such a path must exists since \mathcal{L} is strongly connected. Now let $\rho^* = v_0^N u_0^N v_1^N \cdots v_{l_N}^N u_{l_N}^N w^N u_0^{N+1} v_1^{N+1} \cdots$ By construction, ρ^* realizes each U^k . By Lemma 3, $(\rho^*, k) \not\models \mathsf{F}_{\mathsf{P}}^\infty \psi_k'$, and since $\psi_k' \in \mathsf{L}^+(\mathsf{F}_{\mathsf{P}}^\infty)$ is filtered, by Lemma 10, $(\rho^*, k) \not\models \psi_k'$. Since $\rho^* \in \mathsf{Runs}_{\operatorname{init}}$, $(\rho^*, k) \not\models \theta$ and $(\rho^*, k) \not\models \psi_k$ By Lemma 1, $(\rho^*, k) \not\models \phi$. The lemma follows.

Corollary 1. The strong and weak prompt model checking for $F(L^+(\mathsf{F}^\infty_\mathsf{P}))$ are equivalent.

Proof. The strong model checking trivially induces the weak one. To see why the convers holds, notice that any infinite run ends in some SCC, and since the fragment enjoys the prefix independence, one only has to study what happens in each SCC. A straight-forward application of Lemma 14 concludes the proof.

The proof that the probabilistic model checking can be computed in polynomial time comes from the following zero-one-law for the BSCCs of some LTS \mathcal{L} :

Let B be a BSCC and ϕ a formula in $L^+(\mathsf{F}^\infty_\mathsf{P})$, then either almost all the runs of B satisfy ϕ or almost no run of B does.

Now to conclude, one has to notice that the probability of reaching B can be computed in polynomial time and use the prefix independence of $F(L^+(F_P^\infty))$.

Lemma 15. Given an LTS \mathcal{L} and a formula $\phi \in L^+(\mathsf{F}^\infty_\mathsf{P})$, if \mathcal{L} is strongly connected, then either:

$$\begin{array}{l} - \ there \ exists \ k \geq 0 \ such \ that \ \mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \models \phi\}) = 1 \ or, \\ - \ for \ all \ k \geq 0, \ \mathbb{P}_{\mathcal{L}}(\{(\rho, k) \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \models \phi\}) = 0. \end{array}$$

Proof. Assume that for some $k \geq 0$,

$$\mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \models \phi\}) = \lambda \tag{1}$$

with $\lambda \in (0,1]$, and show that necessarily

$$\mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \models \phi\}) = 1$$

If $\lambda = 1$, then the result trivially follows, if not then

$$\begin{split} \mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \models \phi\}) &= \lambda \implies \mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \not\models \phi\}) = 1 - \lambda > 0 \\ &\implies \exists \rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}), \ (\rho, k) \not\models \phi \\ &\implies \exists w \text{ a prefix of } \rho, \ \forall \ \rho' \in \mathsf{Cyl}(w), \ (\rho', k) \not\models \phi \\ &\implies \mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \not\models \phi\}) = 1 \end{split}$$

The third implication is by Lemma 6, and the last implication is a consequence of the Borel Cantelli Lemma which yields that Cyl(w) is almost sure, but this contradicts Equation (1).

Corollary 2. Given an LTS \mathcal{L} and a formula $\phi \in L^+(\mathsf{F}^\infty_\mathsf{P})$, $\mathbb{P}_{\mathcal{L}}(\{\rho \in \mathsf{Runs}_{\mathsf{init}}(\mathcal{L}) \mid (\rho, k) \models \phi\})$ can be computed in polynomial time.

Proof. By Lemma 15 and prefix independence of the fragment, the probability of satisfying some formula is the same as the probability of reaching the BSCCs where this formula is satisfied. By Lemma 9 one can compute the set of BSCCs that are good for the formula in polynomial time. To conclude, recall that computing the probability of reaching a BSCC can be done in polynomial time using standard procedure from finite Markov chain literature.