THE JOURNAL OF SYMBOLIC LOGIC
Volume 62, Number 4, Dec. 1997

THE FINITE MODEL PROPERTY FOR VARIOUS FRAGMENTS OF LINEAR LOGIC

YVES LAFONT

To show that a formula A is not provable in propositional classical logic, it suffices to exhibit a finite boolean model which does not satisfy A. A similar property holds in the intuitionistic case, with Kripke models instead of boolean models (see for instance [11]). One says that the propositional classical logic and the propositional intuitionistic logic satisfy a *finite model property*. In particular, they are decidable: there is a semi-algorithm for provability (proof search) and a semi-algorithm for non provability (model search). For that reason, a logic which is undecidable, such as first order logic, cannot satisfy a finite model property.

The case of *linear logic* is more complicated. The full propositional fragment **LL** has a complete semantics in terms of *phase spaces* [2, 3], but it is undecidable [9]. The multiplicative additive fragment **MALL** is decidable, in fact PSPACE-complete [9], but the decidability of the multiplicative exponential fragment **MELL** is still an open problem. For *affine logic*, that is, linear logic with weakening, the situation is somewhat better: the full propositional fragment **LLW** is decidable [5].

Here, we show that the finite phase semantics is complete for **MALL** and for **LLW**, but not for **MELL**. In particular, this gives a new proof of the decidability of **LLW**. The noncommutative case is mentioned, but not handled in detail.

§1. Syntax of linear logic. Roman capitals A, B stand for formulas. The connectives of propositional linear logic are:

- the *multiplicatives* $A \parr B$, $A \otimes B$, $\perp$, $\mathbf{1}$;
- the *additives* $A \,\&\, B$, $A \oplus B$, $\top$, $\mathbf{0}$;
- the *exponentials* $?A$, $!A$.

Linear negation $A^\perp$ is only given for positive atoms. It is extended to all formulas by $A^{\perp\perp} = A$ and by

$$(A \parr B)^\perp = A^\perp \otimes B^\perp,\ \perp^\perp = \mathbf{1},\ (A \,\&\, B)^\perp = A^\perp \oplus B^\perp,\ \top^\perp = \mathbf{0},\ (?A)^\perp = !A^\perp.$$

One writes $A \multimap B$ for $A^\perp \parr B$. Greek capitals Γ, Δ stand for sequents, which are multisets of formulas, so that exchange is implicit. *Identity* and *cut* are written as follows:

$$\frac{}{\vdash A, A^\perp} \qquad \frac{\vdash A, \Gamma \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$$

Received November 4, 1995; revised March 10, 1996.

The *logical rules* are:

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma} \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma} \qquad \frac{}{\vdash \mathbf{1}}$$

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma} \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \frac{}{\vdash \top, \Gamma}$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma} \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}.$$

In the last rule, $?\Gamma$ stands for a sequent of the form $?A_1, \ldots, ?A_n$. The last four rules are respectively called *dereliction*, (*logical*) *contraction*, (*logical*) *weakening*, and *promotion*.

§2. Semantics of multiplicatives and additives. If M is a commutative monoid, and $X, Y \subset M$, define

$$XY = \{xy;\ x \in X \text{ and } y \in Y\}, \qquad X \multimap Y = \{z \in M;\ xz \in Y \text{ for all } x \in X\}.$$

A *phase space* is a commutative monoid M endowed with a subset $\bot^\bullet$ of M. In that case, one writes $X^\perp$ for $X \multimap \bot^\bullet$. Clearly, $X \subset X^{\perp\perp}$ for any X, and $Y^\perp \subset X^\perp$ whenever $X \subset Y$. A *fact* is an $X \subset M$ such that $X^{\perp\perp} = X$. This amounts to say that X is of the form $Y^\perp$ for some $Y \subset M$. In particular $\bot^\bullet = \{1\}^\perp$ is a fact. Also, $X \multimap Y$ is a fact whenever Y is a fact, and any intersection of facts is a fact. In particular, $X^{\perp\perp}$ is the intersection of all facts containing X. Define $\mathbf{1}^\bullet = \bot^{\bullet\perp} = \{1\}^{\perp\perp}$, $\top^\bullet = M$, $\mathbf{0}^\bullet = \top^{\bullet\perp} = \emptyset^{\perp\perp}$, and for any facts X, Y,

$$X \parr Y = (X^\perp Y^\perp)^\perp, \quad X \otimes Y = (X^\perp \parr Y^\perp)^\perp = (XY)^{\perp\perp},$$
$$X \,\&\, Y = X \cap Y, \quad X \oplus Y = (X^\perp \,\&\, Y^\perp)^\perp = (X \cup Y)^{\perp\perp}.$$

One sees easily that $X \multimap Y = X^\perp \parr Y$ whenever Y is a fact, which justifies the notation a posteriori.

A *phase model* is given by a phase space M and an *interpretation* which maps each positive atom α to a fact $\alpha^\bullet$ of M. In that case, one defines a fact $A^\bullet$ for any formula A of **MALL** by applying the above definitions inductively, and one says that M *satisfies* A if $1 \in A^\bullet$. The *syntactical model* **MALL**$^\bullet$ is defined as follows:

- **MALL**$^\bullet$ is the free commutative monoid generated by all formulas in **MALL**. In other words, **MALL**$^\bullet$ consists of all sequents in **MALL**, the product of Γ by Δ is Γ, Δ, and the unit is the empty sequent;
- $\bot^\bullet$ is the set of all provable sequents in *cut-free* **MALL**;
- $\alpha^\bullet = \{\alpha\}^\perp$ for any positive atom α.

Note that $\{A\}^\perp$ is the set of all sequents Γ such that A, Γ is provable in cut-free **MALL**.

LEMMA 1 (Okada). *For any formula A of* **MALL**, *one has $A^\bullet \subset \{A\}^\perp$ in* **MALL**$^\bullet$. *In particular, any formula which is satisfied by* **MALL**$^\bullet$ *is provable in cut-free* **MALL**.

PROOF. By induction on formulas. If α is a positive atom, then:

- $\alpha^\bullet = \{\alpha\}^\perp$ by definition of the model;
- $\alpha^{\perp\bullet} = \{\alpha\}^{\perp\perp} \subset \{\alpha^\perp\}^\perp$ since $\alpha \in \{\alpha^\perp\}^\perp$ by the identity axiom.

If $A^\bullet \subset \{A\}^\perp$ and $B^\bullet \subset \{B\}^\perp$, then:

- $(A \mathbin{⅋} B)^\bullet = (A^{\bullet\perp}B^{\bullet\perp})^\perp \subset (\{A\}^{\perp\perp}\{B\}^{\perp\perp})^\perp = (\{A\}\{B\})^\perp \subset \{A \mathbin{⅋} B\}^\perp$ by the ⅋-rule;
- $(A \otimes B)^\bullet = (A^\bullet B^\bullet)^{\perp\perp} \subset (\{A\}^\perp\{B\}^\perp)^{\perp\perp} \subset \{A \otimes B\}^\perp$ since $\{A\}^\perp\{B\}^\perp \subset \{A \otimes B\}^\perp$ by the $\otimes$-rule;
- $(A \mathbin{\&} B)^\bullet = A^\bullet \cap B^\bullet \subset \{A\}^\perp \cap \{B\}^\perp \subset \{A \mathbin{\&} B\}^\perp$ by the &-rule;
- $(A \oplus B)^\bullet = (A^\bullet \cup B^\bullet)^{\perp\perp} \subset (\{A\}^\perp \cup \{B\}^\perp)^{\perp\perp} \subset \{A \oplus B\}^\perp$ since $\{A\}^\perp \cup \{B\}^\perp \subset \{A \oplus B\}^\perp$ by the $\oplus$-rules.

Similarly, $\perp^\bullet \subset \{\perp\}^\perp$ by the $\perp$-rule, $\mathbf{1}^\bullet = \perp^{\bullet\perp} \subset \{\mathbf{1}\}^\perp$ since $\mathbf{1} \in \perp^\bullet$ by the $\mathbf{1}$-rule, $\top^\bullet \subset \{\top\}^\perp$ by the $\top$-rule, and $\mathbf{0}^\bullet = M^\perp \subset \{\mathbf{0}\}^\perp$. ⊣

It is possible to show that the inclusion of this lemma is in fact an equality (see [2]), but the nice point about this proof is that it does not use the cut rule (see also [10]). The following theorem summarizes three properties: *soundness* of the phase semantics, *completeness* of the phase semantics, and *cut elimination*.

THEOREM 1. *For any formula A in* **MALL**, *the following statements are equivalent*:
(1) *A is provable in* **MALL**;
(2) *A is satisfied by all phase models;*
(3) *A is satisfied by* **MALL**$^\bullet$;
(4) *A is provable in cut-free* **MALL**.

PROOF. (2) implies (3) and (4) implies (1) trivially. (1) implies (2) by induction on proofs, as in [2], and (3) implies (4) by Lemma 1. ⊣

The soundness of the phase semantics has been used in [7], to show that second order multiplicative additive logic is undecidable, and subsequently in [8, 4, 6], to show that various fragments of second order linear logic are undecidable.

The proof of Theorem 1 works with smaller models than **MALL**$^\bullet$. For instance, let **MALL**$[A]$ be **MALL** restricted to the subformulas of A and let **MALL**$^\bullet[A]$ be the following phase model:

- **MALL**$^\bullet[A]$ is the free commutative monoid generated by all subformulas of A;
- $\perp^\bullet$ is the set of all provable sequents in cut-free **MALL**$[A]$;
- $\alpha^\bullet = \{\alpha\}^\perp$ if α occurs in A, and $\alpha^\bullet = \{\alpha^\perp\}^{\perp\perp}$ if $\alpha^\perp$ occurs in A, but not α.

Then A is provable in **MALL** if and only if A is satisfied by this **MALL**$^\bullet[A]$. As immediate consequences, one gets the subformula property, and the completeness of the finitely generated phase semantics.

A *logical congruence* on a phase model M is a congruence $\sim$ on M such that $\perp^\bullet$ is closed for $\sim$. This means that, if $x \in \perp^\bullet$ and $x \sim y$, then $y \in \perp^\bullet$. For instance:

- the finest logical congruence is equality;
- the coarsest logical congruence is $\equiv$, defined by $x \equiv y$ whenever $\{x\}^\perp = \{y\}^\perp$.

If $\sim$ is a logical congruence, then all facts are closed for $\sim$, so that $\sim$ induces a structure of phase model on the quotient monoid $M/{\sim}$. Let us write $\pi : M \to M/{\sim}$ for the canonical map, and $A^\bullet_\sim$ for the interpretation of a formula A in the quotient model $M/{\sim}$.

LEMMA 2. *For any formula A in* **MALL**, *one has $\pi^{-1}(A^\bullet_\sim) = A^\bullet$. In particular, $M/{\sim}$ satisfies the same formulas as M.*

PROOF. By construction, $\pi^{-1}(\bot^\bullet_\sim) = \bot^\bullet$ and $\pi^{-1}(\alpha^\bullet_\sim) = \alpha^\bullet$ for each positive atom α. Since π is a surjective morphism, the following equality holds for any $X, Y \subset M/{\sim}$:

$$\pi^{-1}(X \multimap Y) = \pi^{-1}(X) \multimap \pi^{-1}(Y).$$

The lemma follows by easy induction on A. ⊣

Let $\sqsubseteq$ be the smallest preordering on sequents which satisfies the following properties:

$$\Gamma \sqsubseteq \Gamma, \Delta \quad A, B, \Gamma \sqsubseteq A \parr B, \Gamma \quad A, \Gamma \sqsubseteq A * B, \Gamma \quad B, \Gamma \sqsubseteq A * B, \Gamma$$

where $*$ stands for $\otimes$, & or $\oplus$. It is easy to see that $\sqsubseteq$ is in fact a partial ordering, and for any formula A, there are finitely many sequents Γ such that $\Gamma \sqsubseteq A$. Let **MALL**$\langle A \rangle$ be the fragment **MALL**$[A]$ extended with an axiom $\vdash \Gamma$ for each sequent Γ such that $\Gamma \not\sqsubseteq A$, and let **MALL**$^\bullet\langle A \rangle$ be the corresponding syntactical model. The equivalence $\sim_A$ which collapses all sequents Γ such that $\Gamma \not\sqsubseteq A$ is a logical congruence on **MALL**$^\bullet\langle A \rangle$, and the quotient **MALL**$^\bullet\langle A \rangle/{\sim_A}$ is finite. The following theorem gives the completeness of the finite phase semantics for **MALL**.

THEOREM 2. *For any formula A in* **MALL**, *the following statements are equivalent:*
(1) *A is provable in* **MALL**;
(2) *A is satisfied by all finite phase models;*
(3) *A is satisfied by* **MALL**$^\bullet\langle A \rangle/{\sim_A}$;
(4) *A is satisfied by* **MALL**$^\bullet\langle A \rangle$;
(5) *A is provable in cut-free* **MALL**$\langle A \rangle$.

PROOF. (1) implies (2) by Theorem 1, and (2) implies (3) trivially. (3) implies (4) by Lemma 2, and (4) implies (5) by the analogue of Lemma 1 for **MALL**$\langle A \rangle$. Finally, (5) implies (1) because $\sqsubseteq$ has been defined in such a way that the premises of a logical rule are always smaller than its conclusion. ⊣

This proof can be adapted to the noncommutative versions of **MALL**. For the syntax and the semantics of noncommutative linear logic, we refer to [1].

§3. Semantics of exponentials. If M is a phase space, it is easy to see that $J(M) = \{x \in \mathbf{1}^\bullet;\ x \in \{x^2\}^{\bot\bot}\}$ is a submonoid of M. An *enriched phase space* is a phase space M endowed with a submonoid K of $J(M)$. This K is not required to be a fact. For instance, K may be $\{1\}$, or $J(M)$ itself, or $I(M) = \{x \in \mathbf{1}^\bullet;\ x = x^2\}$. In the latter case, one recovers Girard's phase semantics [3]. For any fact X in an enriched phase space, define

$$?X = (X^\bot \cap K)^\bot, \quad !X = (?X^\bot)^\bot = (X \cap K)^{\bot\bot}.$$

From these definitions, one deduces easily, for any facts X and $X_1, \dots, X_n$:

$$!X \subset X, \quad !X \subset !X \otimes !X, \quad !X \subset \mathbf{1}^\bullet, \quad !X_1 \cdots !X_n \subset !X \text{ whenever } !X_1 \cdots !X_n \subset X.$$

From now on, all phase models are supposed to be enriched. The syntactical model $\mathbf{LL}^\bullet$ is defined as follows:

- $\mathbf{LL}^\bullet$ is the free commutative monoid generated by all formulas in **LL**;
- $\perp^\bullet$ is the set of all provable sequents in cut-free **LL**;
- $\alpha^\bullet = \{\alpha\}^\perp$ for any positive atom α;
- K is the submonoid of $\mathbf{LL}^\bullet$ generated by all formulas of the form $?A$.

By contraction and weakening, K is indeed a submonoid of $J(M)$. Moreover, the analogue of Lemma 1 holds:

LEMMA 3. *For any formula A of* **LL**, *one has $A^\bullet \subset \{A\}^\perp$ in* $\mathbf{LL}^\bullet$. *In particular, any formula which is satisfied by* $\mathbf{LL}^\bullet$ *is provable in cut-free* **LL**.

PROOF. By induction on formulas. If $A^\bullet \subset \{A\}^\perp$, then:

- $(?A)^\bullet = (A^{\bullet\perp} \cap K)^\perp \subset (\{A\}^{\perp\perp} \cap K)^\perp \subset \{?A\}^\perp$ since $?A \in K$ and $\{A\}^\perp \subset \{?A\}^\perp$ by dereliction;
- $(!A)^\bullet = (A^\bullet \cap K)^{\perp\perp} \subset (\{A\}^\perp \cap K)^{\perp\perp} \subset \{!A\}^\perp$ since $\{A\}^\perp \cap K \subset \{!A\}^\perp$ by promotion.

The other cases are handled as in Lemma 1. ⊣

From this lemma, one deduces the analogue of Theorem 1, and in particular, the completeness of the phase semantics for **LL**. Similarly, one shows the completeness of the finitely generated phase semantics, but not the completeness of the finite phase semantics: if one extends $\sqsubseteq$ to **LL** in such a way that the premises of a logical rule are always smaller than its conclusion, then it is no more the case that there are finitely many sequents Γ such that $\Gamma \sqsubseteq A$. To show that the finite phase semantics is not complete, even for the multiplicative exponential fragment **MELL**, it suffices to consider the following formula:

$$\varphi = !a \otimes !(a \otimes b) \otimes !(a \otimes b \multimap \mathbf{1}) \multimap b = ?a^\perp \parr ?(a^\perp \parr b^\perp) \parr ?(a \otimes b \otimes \perp) \parr b,$$

where a and b are positive atoms.

THEOREM 3. *φ is satisfied by all finite phase models, but it is not provable.*

PROOF. One writes b^n for $b \otimes \cdots \otimes b$ (n times). If M is a finite phase model, there are finitely many facts in M, so that $(b^p)^\bullet = (b^q)^\bullet$ for some $p < q$. In particular, the formula $\psi = b^p \multimap b^q$ is satisfied by M, and since it is easy to see that the formula $\psi \multimap \varphi$ is provable, $\psi \multimap \varphi$ is also satisfied by M, and so is φ. To see that φ is not provable, consider the following infinite phase model:

- M is the free group generated by the symbol a. This means that any element of M can be uniquely written as a^n with $n \in \mathbb{Z}$;
- $\perp^\bullet = \{a^n;\, n \geq 0\}$, $a^\bullet = \{a\}^\perp = \{a^n;\, n \geq -1\}$, $b^\bullet = \{a^{-1}\}^\perp = \{a^n;\, n \geq 1\}$, and $K = \{1\}$.

It is easy to see that $1 \notin \varphi^\bullet = b^\bullet$, which means that φ is not satisfied. ⊣

In fact, φ belongs to the so-called *multiplicative exponential Horn fragment* of linear logic, which encodes the reachability problem for Petri nets. This means that this fragment does not satisfy the finite model property, whereas it is decidable.

If $\sim$ is a logical congruence on a phase model M, then K is not necessarily closed for $\sim$, but one can always replace K by its *closure* $\overline{K} = \{x \in M;\ x \sim y$ for some $y \in K\}$. It is indeed easy to see that, for any fact X, one has $(X \cap K)^{\perp} = (X \cap \overline{K})^{\perp}$. Moreover, $\pi(K) = \pi(\overline{K}) \subset \pi(J(M)) = J(M/{\sim})$, so that $\sim$ induces a structure of enriched phase space on $M/{\sim}$, and the analogue of Lemma 2 holds for **LL**. For instance, the smallest congruence $\sim$ such that $?A \sim ?A, ?A$ for any A is a logical congruence on $\mathbf{LL}^{\bullet}$, and K is already closed for $\sim$. In that case, $\pi(K) = I(\mathbf{LL}^{\bullet}/{\sim})$, and since $\mathbf{LL}^{\bullet}/{\sim}$ satisfies the same formulas as $\mathbf{LL}^{\bullet}$, one gets the completeness of Girard's phase semantics for **LL** [3]. Note that Girard's phase semantics is not closed under quotient by a logical congruence, because it is not true in general that $\pi(I(M)) = I(M/\sim)$.

§4. Semantics of affine logic. If M is a commutative monoid, an *ideal* of M is an $X \subset M$ such that $XM \subset X$, or equivalently, $XM = X$. Clearly, $X \multimap Y$ is an ideal whenever Y is an ideal, and any union of ideals is an ideal. Say that an ideal is *principal* if it is of the form xM for some $x \in M$, and say that it is *of finite type* if it is a finite union of principal ideals. Finally, say that M is *noetherian* if all its ideals are of finite type. We shall need the following classical result, which is also crucial in [5]:

LEMMA 4. *Any finitely generated free commutative monoid is noetherian.*

PROOF. Such a monoid is isomorphic to $\mathbb{N}^k$ with addition, and an ideal of $\mathbb{N}^k$ is a subset of $\mathbb{N}^k$ which is upwards closed for the usual ordering. $\mathbb{N}$ is clearly noetherian: its ideals are empty or principal. To see that $\mathbb{N}^2$ is noetherian, consider an ideal X of $\mathbb{N}^2$. If X is nonempty, one can choose $(p, q) \in X$, and one has

$$X = X_0 \cup \bigcup_{i<p} \{i\} \times Y_i \cup \bigcup_{j<q} Z_j \times \{j\},$$

where X_0 is the principal ideal generated by (p, q), $Y_i = \{n \in \mathbb{N};\ (i, n) \in X\}$, and $Z_j = \{n \in \mathbb{N};\ (n, j) \in X\}$. Since the Y_i and the Z_j are ideals of the noetherian monoid $\mathbb{N}$, one sees easily that X is of finite type. More generally, one proves that $\mathbb{N}^k$ is noetherian by induction on k. ⊣

An *affine phase space* is a phase space M such that $\perp^{\bullet}$ is an ideal of M, or equivalently, $\mathbf{1}^{\bullet} = M$. In that case, all facts of M are ideals, and M satisfies all formulas which are provable in **LLW**, that is, **LL** extended with the (*structural*) *weakening rule*:

$$\frac{\vdash \Gamma}{\vdash A, \Gamma}.$$

The syntactical model $\mathbf{LLW}^{\bullet}$ is an affine phase model. Again, one shows the analogue of Theorem 1, and the completeness of the finitely generated affine phase semantics. More precisely, a formula A is provable in **LLW** if and only if A is satisfied by the phase model $\mathbf{LLW}^{\bullet}[A]$ whose underlying monoid is a finitely generated free commutative monoid.

LEMMA 5. *If M is an affine phase model whose underlying monoid is a finitely generated free commutative monoid, then $M/\equiv$ is finite.*

PROOF. We can assume that the underlying monoid is $\mathbb{N}^k$ with addition. By Lemma 4,

$$\perp^\bullet = \bigcup_{i=1,\dots,n} \{x \in \mathbb{N}^k;\, x \geq u_i\}$$

for some $u_1, \dots, u_n \in \mathbb{N}^k$. Therefore, $\{x\}^\perp$ is completely determined by $x' = \inf(x, \sup(u_1, \dots, u_n))$. Since there are finitely many such x', this means that $M/\equiv$ is finite. ⊣

This lemma allows us to prove the following theorem, which gives the completeness of the finite affine phase semantics for **LLW**. In particular, **LLW** is decidable.

THEOREM 4. *For any formula A in* **LL**, *the following statements are equivalent:*
(1) *A is provable in* **LLW**;
(2) *A is satisfied by all finite affine phase models;*
(3) *A is satisfied by* $\mathbf{LLW}^\bullet[A]/\equiv$;
(4) *A is satisfied by* $\mathbf{LLW}^\bullet[A]$;
(5) *A is provable in cut-free* $\mathbf{LLW}[A]$.

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INSTITUT DE MATHÉMATIQUES DE LUMINY
UPR 9016 DU CNRS
163 AVENUE DE LUMINY, CASE 930
13288 MARSEILLE CEDEX 9, FRANCE

E-mail: lafont@iml.univ-mrs.fr