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Spatial uniformity in diffusively-coupled systems using weighted L^2 norm contractions

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Abstract— **We present conditions that guarantee spatial uniformity in diffusively-coupled systems. Diffusive coupling is a ubiquitous form of local interaction, arising in diverse areas including multiagent coordination and pattern formation in biochemical networks. The conditions we derive make use of the Jacobian matrix and Neumann eigenvalues of elliptic operators, and generalize and unify existing theory about asymptotic convergence of trajectories of reaction-diffusion partial differential equations as well as compartmental ordinary differential equations. We present numerical tests making use of linear matrix inequalities that may be used to certify these conditions. We discuss an example pertaining to electromechanical oscillators. The paper's main contributions are unified verifiable relaxed conditions that guarantee synchrony.**

I. INTRODUCTION

Diffusively coupled models are crucial to understanding the dynamical behavior of a range of engineering and biological systems. Understanding the conditions under which a distributed system exhibits spatial uniformity is a central question in many application fields concerned with pattern formation, ranging from biology (morphogenesis developmental biology, species competition and cooperation in ecology, epidemiology) [1], [2], [3] and enzymatic reactions in chemical engineering [4] to spatio-temporal dynamics in semiconductors [5].

This paper studies reaction-diffusion partial differential equations (PDEs) of the form

$$\frac{\partial u}{\partial t}(\omega, t) = F(u(\omega, t), t) + \mathcal{L}u(\omega, t), \qquad (1)$$

where $\mathcal{L}$ denotes a diffusion operator, and compartmental systems of ordinary differential equations (ODEs):

$$\dot{u}(t) = \tilde{F}(u(t)) - \mathcal{L}^{(d)}u(t), \qquad (2)$$

where $\tilde{F}(u) = \left(F(u^1)^T, \cdots, F(u^N)^T\right)^T$ and $\mathcal{L}^{(d)}$ denotes diffusive coupling over a graph. We prove a two-part result that addresses the question of how the stability of solutions of the PDE or compartmental system of ODEs relates to stability of solutions of the underlying ordinary differential equation (ODE) $\frac{dx}{dt}(t) = F(x(t), t)$.

The first part of our result shows that when solutions of the ODE have a certain contraction property, namely $\mu_{2,Q}(J_F(u,t)) < 0$ uniformly on u and t, where $\mu_{2,Q}$ is a logarithmic norm (matrix measure) associated to a Q-weighted L^2 norm, the associated PDE, subject to no-flux (Neumann) boundary conditions, and compartmental system of ODEs, enjoy a similar property. This result complements a similar result shown in [6] which, while allowing norms L^p with p not necessarily equal to 2, had the restriction that it only applied to diagonal matrices Q and $\mathcal{L}$ was the standard Laplacian. Logarithmic norm or "contraction" approaches arose in the dynamical systems literature [7], [8], [9], and were extended and much further developed in work by Slotine e.g. [10]; see also [11] for historical comments.

The second, and complementary, part of our result shows that when $\mu_{2,Q}(J_f(u,t) - \Lambda_2) < 0$, where Λ_2 is a nonnegative diagonal matrix whose entries are the second smallest Neumann eigenvalues of the diffusion operators in (1), or respectively the second smallest eigenvalues of the diffusive coupling matrix in (2), the solutions become spatially homogeneous as $t \to \infty$. This result generalizes the previous work [12] to allow for spatially-varying diffusion, and makes a contraction principle implicitly used in [12] explicit.

We next derive convex linear matrix inequality [13] tests as in [12] that can be used to certify the conditions. Our discussion concludes with an example of synchronization in coupled ring oscillators, which have been studied in the context of cross-coupled circuits [14] and gene regulatory networks [15].

II. PRELIMINARIES

For any invertible matrix Q, and any $1 \le p \le \infty$, and continuous $u\colon \Omega \to \mathbb{R}^n$, we denote the weighted $L_{p,Q}$ norm, $\|u\|_{p,Q} = \|Qu\|_p$, where $(Qu)(\omega) = Qu(\omega)$ and $\|\cdot\|_p$ indicates the norm in $L^p(\Omega, \mathbb{R}^n)$.

Definition 1: Let $(X, \|\cdot\|_X)$ be a finite dimensional normed vector space over $\mathbb{R}$ or $\mathbb{C}$. The space $\mathcal{L}(X,X)$ of linear transformations $M\colon X \to X$ is also a normed vector space with the induced operator norm

$$\|M\|_{X\to X} = \sup_{\|x\|_X = 1} \|Mx\|_X.$$

The *logarithmic norm* $\mu_X(\cdot)$ induced by $\|\cdot\|_X$ is defined as the directional derivative of the matrix norm, that is,

$$\mu_X(M) = \lim_{h\to 0^+} \frac{1}{h}\left(\|I + hM\|_{X\to X} - 1\right),$$

where I is the identity operator on X.

The following lemma relates the logarithmic norm of a matrix to its satisfaction of a certain linear matrix inequality, and will be useful in proving our main results about spatial

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uniformity. Owing to space constraints, we omit the proof and that of most results that follow, which the interested reader may find in [16].

Lemma 1: Suppose that P is a positive definite, symmetric matrix and M is an arbitrary matrix. Then $\mu_{2,P}(M)$ is the smallest $\mu \in \mathbb{R}$ such that

$$P^2 M + M^T P^2 \leq 2\mu P^2.$$

Remark 1: Observe that for $Q > 0$,
1) $QM + M^T Q \leq \mu Q \quad \Rightarrow \quad QM + M^T Q \leq \beta I$, where $\beta = \mu\lambda$ and λ is the smallest eigenvalue of Q.
2) $QM + M^T Q \leq \beta I \quad \Rightarrow \quad QM + M^T Q \leq \gamma Q$, where $\gamma = \dfrac{\beta}{\lambda'}$ and λ' is the largest eigenvalue of Q.

III. Spatial Uniformity in Reaction-Diffusion PDEs

In this section, we study the reaction-diffusion PDE (1), subject to a Neumann boundary condition:

$$\nabla u_i \cdot \mathbf{n}(\xi, t) = 0 \quad \forall \xi \in \partial\Omega, \ \forall t \in [0, \infty). \tag{3}$$

Theorems on existence and uniqueness of solutions for PDEs such as $(1) - (3)$ can be found in standard references, e.g. [17], [18].

Assumption 1: In $(1) - (3)$ we assume:

- Ω is a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$ and outward normal $\mathbf{n}$.
- $F \colon V \times [0, \infty) \to \mathbb{R}^n$ is a (globally) Lipschitz and twice continuously differentiable vector field with respect to x, and continuous with respect to t, with components F_i:
$$F(x,t) = (F_1(x,t), \cdots, F_n(x,t))^T$$
for some functions $F_i \colon V \times [0, \infty) \to \mathbb{R}$, where V is a convex subset of $\mathbb{R}^n$.
- $\mathcal{L} = \operatorname{diag}(\mathcal{L}_1, \cdots, \mathcal{L}_n)$, and $\mathcal{L}u = (\mathcal{L}_1 u_1, \cdots, \mathcal{L}_n u_n)^T$, where for each $i = 1, \cdots, n$,
$$\mathcal{L}_i u_i = \nabla \cdot (A_i(\omega) \nabla u_i(\omega, t)), \tag{4}$$
and $A_i \colon \Omega \to \mathbb{R}^{m \times m}$ is symmetric and differentiable and there exist $\alpha_i, \beta_i > 0$ such that for all $\omega \in \Omega$ and $\zeta = (\zeta_1, \cdots, \zeta_m)^T \in \mathbb{R}^m$,
$$\alpha_i |\zeta|^2 \leq \zeta^T A_i(\omega) \zeta \leq \beta_i |\zeta|^2. \tag{5}$$

Suppose that $\mathcal{L}$ has $r \leq n$ distinct elements $\mathbf{L}_1, \cdots, \mathbf{L}_r$ (up to a scalar). Namely,

$$\begin{aligned}&\operatorname{diag}(\mathcal{L}_1, \cdots, \mathcal{L}_{n_1}, \ldots, \mathcal{L}_{n-n_r+1}, \cdots, \mathcal{L}_n) = \\ &\operatorname{diag}(d_{11}, \cdots, d_{1n_1}, \ldots, d_{r1}, \cdots, d_{rn_r}) \\ &\times \operatorname{diag}(\mathbf{L}_1, \cdots, \mathbf{L}_1, \ldots, \mathbf{L}_r, \cdots, \mathbf{L}_r),\end{aligned}$$

where $n_1 + \cdots + n_r = n$. For each $i = 1, \cdots, r$, let D_i be an $n \times n$ diagonal matrix with entries $[D_i]_{jj} = d_{ij}$, for $j = i, \cdots, n_i$ and 0 elsewhere. Also for each $i = 1, \cdots, r$, let $\mathfrak{L}_i$ be an $n \times n$ diagonal matrix with identical entries $\mathbf{L}_i$. Then $\mathcal{L}$ can be written as below,

$$\mathcal{L} = \sum_{i=1}^{r} D_i \mathfrak{L}_i. \tag{6}$$

For a fixed $i \in \{1, \cdots, n\}$, let λ_i^k be the k-th Neumann eigenvalue of the operator $-\mathcal{L}_i$ as in (4) ($\lambda_i^1 = 0$, $\lambda_i^k > 0$ for $k > 1$, and $\lambda_i^k \to \infty$ as $k \to \infty$) and e_i^k be the corresponding normalized eigenfunction:

$$\begin{aligned}\nabla \cdot \left(A_i(\omega) \nabla e_i^k(\omega)\right) &= -\lambda_i^k e_i^k(\omega), \qquad \omega \in \Omega \\ \nabla e_i^k(\xi) \cdot \mathbf{n} &= 0, \qquad \xi \in \partial\Omega\end{aligned} \tag{7}$$

Also for each $i = 1, \cdots, r$, let $\boldsymbol{\lambda}_i^k$ be the k-th Neumann eigenvalue of $-\mathbf{L}_i$. Note that

$$\Lambda_k = \sum_{i=1}^{r} \boldsymbol{\lambda}_i^k D_i, \quad \text{where} \quad \Lambda_k = \operatorname{diag}\left(\lambda_1^k, \cdots, \lambda_n^k\right). \tag{8}$$

For each $k \in \{1, 2, \cdots\}$, let E_i^k be the subspace spanned by the first k-th eigenfunctions:

$$E_i^k = \langle e_i^1, \cdots, e_i^k \rangle.$$

Now define the map $\Pi_{k,i}$ on $L^2(\Omega)$ as follows:

$$\Pi_{k,i}(v) = v - \pi_{k,i}(v),$$

where $\pi_{k,i}$ is the orthogonal projection map onto E_i^{k-1}, and we define $E_i^0 = 0$. Note that for any $i = 1, \cdots, n$,

$$\Pi_{2,i}(v) = v - \frac{1}{|\Omega|} \int_\Omega v. \tag{9}$$

For any $v = (v_1, \cdots, v_n)$, define Π_k as follows:

$$\Pi_k(v) = v - \pi_k(v),$$

where

$$\pi_k(v) = \left(\pi_{k,1}(v_1), \cdots, \pi_{k,n}(v_n)\right)^T.$$

Observe that $\pi_k(v)$ is the orthogonal projection map onto $E_1^{k-1} \times \cdots \times E_n^{k-1}$.

In [6], we proved the following lemma:

Lemma 2: Consider the PDE system $(1) - (3)$, with $\mathcal{L} = D\Delta$, where $D = \operatorname{diag}(d_1, \cdots, d_n)$. In addition suppose Assumption 1 holds. For some $1 \leq p \leq \infty$, and a positive diagonal matrix Q, let

$$\mu := \sup_{(x,t) \in V \times [0,\infty)} \mu_{p,Q}(J_F(x,t)).$$

(We are using $\mu_{p,Q}$ to denote the logarithmic norm associated to the norm $\|Qv\|_p$ in $\mathbb{R}^n$.) Then for any two solutions u and v of $(1) - (3)$, we have

$$\|u(\cdot, t) - v(\cdot, t)\|_{p,Q} \leq e^{\mu t} \|u(\cdot, 0) - v(\cdot, 0)\|_{p,Q}.$$

Before stating and proving the main result of this section, Theorem 1, we state the following lemmas.

The first lemma applies the Poincaré principle as in [19] and the orthogonality of eigenvectors of the operator $\mathcal{L}$ as in (4) to further characterize $\mathcal{L}$:

Lemma 3: Suppose $u \in L^2(\Omega)$ satisfies the Neumann boundary conditions. For any $k \in \{1, 2, \cdots\}$,

$$\left(\Pi_k(u), \mathcal{L}\Pi_k(u)\right) \leq -\left(\Pi_k(u), \Lambda_k \Pi_k(u)\right). \tag{10}$$

In addition for $k = 1, 2$ and any $n \times n$ symmetric matrix Q

with the following property:

$$QD_i + D_iQ \;>\; 0 \qquad i = 1, \cdots, r, \tag{11}$$

we have:

$$(\Pi_k(u), Q\mathcal{L}\Pi_k(u)) \;\le\; -(\Pi_k(u), Q\Lambda_k\Pi_k(u)). \tag{12}$$

The second lemma facilitates our characterization of the evolution of trajectories of (1)-(3):

Lemma 4: Let $w = u - x$, where u is a solution of $(1)-(3)$ and $x = \pi_2(u)$ or $x = v$ is another solution of $(1)-(3)$. Note that for $x = v$, $w = \Pi_1(u - v)$ and for $x = \pi_2(u)$, $w = \Pi_2(u)$. For a positive, symmetric matrix Q, let

$$\Phi(w) \;:=\; \frac{1}{2}(w, Qw).$$

Then

$$\frac{d\Phi}{dt}(w) = (w, Q(F(u,t) - F(x,t))) + (w, Q\mathcal{L}w). \tag{13}$$

We now present our main result for spatial uniformity in reaction-diffusion PDEs: the first part is a generalization of Lemma 2 to non-diagonal P for the special case of $p = 2$, and the second part is a generalization of Theorem 1 from [12] to spatially-varying diffusion.

Theorem 1: Consider the reaction-diffusion system $(1)-(3)$ and suppose Assumption 1 holds. For $k = 1, 2$, let

$$\mu_k := \sup_{(x,t)\in V\times[0,\infty)} \mu_{2,P}(J_F(x,t) - \Lambda_k),$$

for a positive symmetric matrix P such that for any $i = 1, \cdots, r$:

$$P^2D_i + D_iP^2 \;>\; 0. \tag{14}$$

Then for any two solutions, namely u and v, of $(1)-(3)$, we have:

$$\|u(\cdot,t) - v(\cdot,t)\|_{2,P} \le e^{\mu_1 t}\|u(\cdot,0) - v(\cdot,0)\|_{2,P} \tag{15}$$

$$\|\Pi_2(u(\cdot,t))\|_{2,P} \le e^{\mu_2 t}\|\Pi_2(u(\cdot,0))\|_{2,P}. \tag{16}$$

Proof: By Lemma 1,

$$Q(J_F - \Lambda_k) + (J_F - \Lambda_k)^TQ \le 2\mu_kQ, \tag{17}$$

where $Q = P^2$. Define w and $\Phi(w)$ as in Lemma 4 for $Q = P^2$. Since $\Phi(w) = \frac{1}{2}\|Pw\|_2^2$, to prove (15) and (16), it suffices to show that for $k = 1, 2$

$$\frac{d}{dt}\Phi(w) \le 2\mu_k\Phi(w).$$

Note that by Lemma 3, and the fact that $w = \Pi_1(u - v)$ or $w = \Pi_2(u)$, the second term of the right hand side of (13), $\frac{d}{dt}\Phi(w)$, satisfies:

$$(w, Q\mathcal{L}w) \le -(w, Q\Lambda_kw). \tag{18}$$

Next, by the Mean Value Theorem for integrals, and using (17), we rewrite the first term of the right hand side of (13) as follows:

$$\begin{aligned}
&(w, Q(F(u,t) - F(x,t)))\\
&= \int_\Omega w^T(\omega,t)Q(F(u(\omega,t),t) - F(x,t))\, d\omega\\
&= \int_\Omega w^T(\omega,t)Q\int_0^1 J_F(x + sw(\omega,t),t)\cdot w(\omega,t)\, ds\, d\omega\\
&= \int_0^1\int_\Omega w^T(\omega,t)QJ_F(x + sw(\omega,t),t)\cdot w(\omega,t)\, d\omega\, ds.
\end{aligned}$$

Combining the preceding equality with (18), we have:

$$\begin{aligned}
&(w, Q(F(u,t) - F(x,t))) + (w, Q\mathcal{L}w)\\
&\le \int_0^1\int_\Omega w^T(\omega,t)Q\big(J_F(x + sw(\omega,t),t) - \Lambda_k\big)\\
&\quad \times w(\omega,t)\, d\omega\, ds\\
&\le \frac{2\mu_k}{2}\int_0^1 ds\int_\Omega w^TQw\, d\omega\\
&= \frac{2\mu_k}{2}\int_\Omega w^TQw\, d\omega = 2\mu_k\Phi(w).
\end{aligned}$$

Therefore,

$$\frac{d\Phi}{dt}(w) \;\le\; 2\mu_k\Phi(w).$$

The preceding inequality implies (15) and (16) for $k = 1$ and $k = 2$ respectively. ■

Corollary 1: In Theorem 1, if $\mu_1 < 0$, then $(1)-(3)$ is contracting, meaning that solutions converge (exponentially) to each other, as $t \to +\infty$ in the weighted $L_{2,P}$ norm:

$$\|u(\cdot,t) - v(\cdot,t)\|_{2,P} \to 0 \quad \text{as } t \to \infty.$$

Corollary 2: In Theorem 1, if $\mu_2 < 0$, then solutions converge (exponentially) to uniform solutions, as $t \to +\infty$ in the weighted $L_{2,P}$ norm:

$$\|\Pi_2(u(\cdot,t))\|_{2,P} \to 0 \quad \text{as } t \to \infty.$$

IV. Spatial Uniformity in Compartmental Systems of ODEs

We next consider a compartmental ODE model where each compartment represents a spatial domain interconnected with the other compartments over an undirected graph:

$$\dot{u}(t) = \tilde{F}(u(t)) - \mathcal{L}^{(d)}u(t). \tag{19}$$

We denote the Kronecker product of two matrices A and B by $A \otimes B$.

Assumption 2: In (19), we assume:

- For a fixed convex subset of $\mathbb{R}^n$, say V, $\tilde{F}\colon V^N \to \mathbb{R}^{nN}$ is a function of the form:
$$\tilde{F}(u) = \left(F(u^1)^T, \cdots, F(u^N)^T\right)^T,$$
where $u = \left((u^1)^T, \cdots, (u^N)^T\right)^T$, with $u^i \in V$ for each i, and $F\colon V \to \mathbb{R}^n$ is a (globally) Lipschitz function.
- For any $u \in V^N$ we define $\|u\|_{p,Q}$ as follows:
$$\|u\|_{p,Q} = \left\|\left(\|Qu^1\|_p, \cdots, \|Qu^N\|_p\right)^T\right\|_p,$$

where Q is a symmetric and positive definite matrix and $1 \le p \le \infty$.

With a slight abuse of notation, we use the same symbol for a norm in $\mathbb{R}^n$:

$$\|x\|_{p,Q} := \|Qx\|_p.$$

- $u\colon [0,\infty) \to V^N$ is a continuously differentiable function.
- $\mathcal{L}^{(d)} = \sum_{i=1}^n L_i \otimes E_i$, where for any $i = 1, \cdots, n$, $L_i \in \mathbb{R}^{N\times N}$ is a symmetric positive semidefinite matrix and $L\mathbf{1}_N = 0$, where $\mathbf{1}_N = (1, \cdots, 1)^T \in \mathbf{R}^N$. The matrix L_i is the symmetric generalized graph Laplacian (see, e.g., [20]) that describes the interconnections among component subsystems. For any $i = 1, \cdots, n$, $E_i = e_i e_i^T \in \mathbb{R}^{n\times n}$ is the product of the i-th standard basis vector e_i multiplied by its transpose.

Similar to the PDE case, we assume that there exists $r \le n$ distinct matrices, $\mathbf{L}_1, \cdots, \mathbf{L}_r$ such that

$$\begin{aligned}&\operatorname{diag}(L_1, \cdots, L_{n_1}, \ldots, L_{n-n_r+1}, \cdots, L_n) = \\ &\operatorname{diag}(d_{11}, \cdots, d_{1n_1}, \ldots, d_{r1}, \cdots, d_{rn_r}) \\ &\times \operatorname{diag}(\mathbf{L}_1, \cdots, \mathbf{L}_1, \ldots, \mathbf{L}_r, \cdots, \mathbf{L}_r),\end{aligned}$$

where $n_1 + \cdots + n_r = n$. For each $i = 1, \cdots, r$, let D_i be an $n \times n$ diagonal matrix with entries $[D_i]_{jj} = d_{ij}$, for $j = i, \cdots, n_i$ and 0 elsewhere. Therefore we can write $\mathcal{L}^{(d)}$ as follows:

$$\mathcal{L}^{(d)} = \sum_{i=1}^r \mathbf{L}_i \otimes D_i. \tag{20}$$

For a fixed $i \in \{1, \cdots, n\}$, let λ_i^k be the k-th eigenvalue of the matrix L_i and e_i^k be the corresponding normalized eigenvector. Also for a fixed $i \in \{1, \cdots, r\}$, let $\boldsymbol{\lambda}_i^k$ be the k-th eigenvalue of the matrix $\mathbf{L}_i$. Note that

$$\Lambda_k = \sum_{i=1}^r \boldsymbol{\lambda}_i^k D_i, \tag{21}$$

where $\Lambda_k = \operatorname{diag}(\lambda_1^k, \cdots, \lambda_n^k)$.

We recall the Courant-Fischer minimax theorem, from [21], which characterizes the eigenvalues of a symmetric positive semidefinite matrix:

Lemma 5: Let L be a symmetric positive semidefinite matrix in $\mathbb{R}^{N\times N}$. Let $\lambda^1 \le \cdots \le \lambda^N$ be N eigenvalues with $e^1, \cdots, e^N$ corresponding normalized orthogonal eigenvectors. For any $v \in \mathbb{R}^N$, if $v^T e^j = 0$ for $1 \le j \le k-1$, then

$$v^T L v \ge \lambda^k v^T v.$$

Before presenting the main result of this section, we state the following lemma, which facilitates characterization of the evolution of trajectories of (19).

Lemma 6: Let $w := u - x$, where u is a solution of (19) and $x = \mathbf{1}_N \otimes \left(\frac{1}{N}\sum_{j=1}^N u^j\right)$ or $x = v$ is another solution of (19). For a positive, symmetric matrix Q, let

$$\Phi(w) := \frac{1}{2} w^T (I_N \otimes Q) w.$$

Then

$$\begin{aligned}\frac{d\Phi}{dt}(w) =& w^T (I_N \otimes Q)(\tilde{F}(u,t) \\ &- \tilde{F}(x,t)) - w^T (I_N \otimes Q)\mathcal{L}^{(d)} w.\end{aligned} \tag{22}$$

We now state and prove our main result for spatial uniformity in compartmental systems of ODEs.

Theorem 2: Consider the ODE system (19) and suppose Assumption 2 holds. For $k = 1, 2$, let

$$\mu_k := \sup_{(x,t)\in V\times[0,\infty)} \mu_{2,P}(J_F(x,t) - \Lambda_k),$$

for a positive symmetric matrix P such that for every $i = 1, \cdots, r$,

$$P^2 D_i + D_i P^2 > 0.$$

Then for any two solutions, namely u and v, of (19), we have:

$$\|(u-v)(t)\|_{2,P} \le e^{\mu_1 t}\|(u-v)(0)\|_{2,P}. \tag{23}$$

In addition

$$\|(u - \pi_2(u))(t)\|_{2,P} \le e^{\mu_2 t}\|(u - \pi_2(u))(0)\|_{2,P}. \tag{24}$$

Proof: By Lemma 1,

$$Q(J_F - \Lambda_k) + (J_F - \Lambda_k)^T Q \le 2\mu_k Q, \tag{25}$$

where $Q = P^2$. Define w and $\Phi(w)$ as in Lemma 6 for $Q = P^2$. Since $\Phi(w) = \frac{1}{2}\|Pw\|_2^2$, to prove (23) and (24), it suffices to show that for $k = 1, 2$

$$\frac{d}{dt}\Phi(w) \le 2\mu_k \Phi(w).$$

We rewrite the second term of the right hand side of (22) as follows. Since $Q = P^2$ and $P^2 D_i + D_i P^2 > 0$, there exists symmetric, positive definite matrices M_i such that $QD_i + D_i Q = 2M_i^T M_i$. Thus, we have:

$$\begin{aligned}&w^T (I_N \otimes Q)\mathcal{L}^{(d)} w \\ &= w^T (I_N \otimes Q)\left(\sum_{i=1}^r \mathbf{L}_i \otimes D_i\right) w \\ &= w^T \left(\sum_{i=1}^r I_N \mathbf{L}_i \otimes QD_i\right) w \\ &= \frac{1}{2}\sum_{i=1}^r w^T (\mathbf{L}_i \otimes (QD_i + D_i Q)) w \\ &= \sum_{i=1}^r w^T \left(\mathbf{L}_i \otimes M_i^T M_i\right) w \\ &= \sum_{i=1}^r w^T (I_N \otimes M_i^T)(\mathbf{L}_i \otimes I_n)(I_N \otimes M_i) w \\ &\ge \sum_{i=1}^r \boldsymbol{\lambda}_i^k ((I_N \otimes M_i) w)^T (I_N \otimes M_i) w \\ &= \sum_{i=1}^r \boldsymbol{\lambda}_i^k w^T (I_N \otimes QD_i) w = w^T (I_N \otimes Q\Lambda_k) w,\end{aligned}$$

where the final equality follows from (21. Therefore

$$-w^T(I_N \otimes Q)\mathcal{L}^{(d)}w \leq -w^T(I_N \otimes Q\Lambda_k)w. \tag{26}$$

Note that the first inequality holds for $k = 2$ by Lemma 5 and the fact that for $x = \pi_2(u)$, by definition, $w^T\mathbf{1}_{nN} = 0$ and hence $(I_N \otimes M_i)\, w\mathbf{1}_{nN} = 0$. It also holds for $k = 1$, since $\mathbf{L}_i$ and hence $\mathbf{L}_i \otimes I_n$ are positive definite, and $\boldsymbol{\lambda}_i^1 = 0$.

The remainder of the proof is analogous to that of Theorem 1. ∎

Corollary 3: In Theorem 2, if $\mu_1 < 0$, then (19) is contracting, meaning that solutions converge (exponentially) to each other, as $t \to +\infty$ in the P-weighted L_2 norm.

Corollary 4: In Theorem 2, if $\mu_2 < 0$, then solutions converge (exponentially) to uniform solutions, as $t \to +\infty$ in the P-weighted L_2 norm.

V. LMI Tests for Guaranteeing Spatial Uniformity

The next result is a modification of Theorems 3 in [12], and allows us to check the conditions in Theorems 1 and 2 through numerical tests involving linear matrix inequalities.

We define a *convex box* as:

$$\begin{aligned} &\text{box}\{M_0, M_1, \ldots, M_p\} \\ &= \{M_0 + \omega_1 M_1 + \ldots + \omega_p M_p \,|\, \omega_i \in [0,1] \\ &\qquad \text{for each } i = 1, \ldots, p\}. \end{aligned} \tag{27}$$

Theorem 3: Suppose that $J_F(x,t)$ is contained in a convex box:

$$J_F(x,t) \in \text{box}\{A_0, A_1, \ldots, A_l\} \;\; \forall\, x \in V,\; t \in [0,\infty), \tag{28}$$

where $A_1, \ldots, A_l$ are rank-one matrices that can be written as $A_i = B_i C_i^T$, with $B_i, C_i \in \mathbb{R}^n$. If there exists a scalar μ and symmetric, positive definite matrix $\mathcal{Q}$ with:

$$\mathcal{Q} = \begin{bmatrix} Q & 0 & \ldots & 0 \\ 0 & p_1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & p_l \end{bmatrix} \tag{29}$$
$$Q \in \mathbb{R}^{n\times n},\; p_i \in \mathbb{R},\; i = 1, \ldots, l,$$

satisfying:

$$\mathcal{Q}\begin{bmatrix} A_0 - \Lambda_k & B \\ C^T & -I_n \end{bmatrix} + \begin{bmatrix} A_0 - \Lambda_k & B \\ C^T & -I_n \end{bmatrix}^T \mathcal{Q} \leq \begin{bmatrix} \mu Q & 0 \\ 0 & 0 \end{bmatrix}, \tag{30}$$

with $B = [B_1 \ldots B_l]$ and $C = [C_1 \ldots C_l]$, then the upper left (symmetric, positive definite) principal submatrix Q satisfies

$$Q(J_F(x,t) - \Lambda_k) + (J_F(x,t) - \Lambda_k)^T Q \leq \mu Q; \tag{31}$$

or equivalently

$$\mu_k := \sup_{(x,t)\in V\times[0,\infty)} \mu_{2,P}(J_F(x,t) - \Lambda_k) \leq \frac{\mu}{2}, \tag{32}$$

where $P^2 = Q$.

If $l = 1$ and the image of $V \times [0,\infty)$ under J is surjective onto box$\{A_0, A_1\}$, then the converse is true.

The problem of finding the smallest μ such that there exists a matrix $\mathcal{Q}$ as in Theorem 3 is quasi-convex and is solved iteratively as a sequence of convex semidefinite programs.

Example – Ring Oscillator Circuit

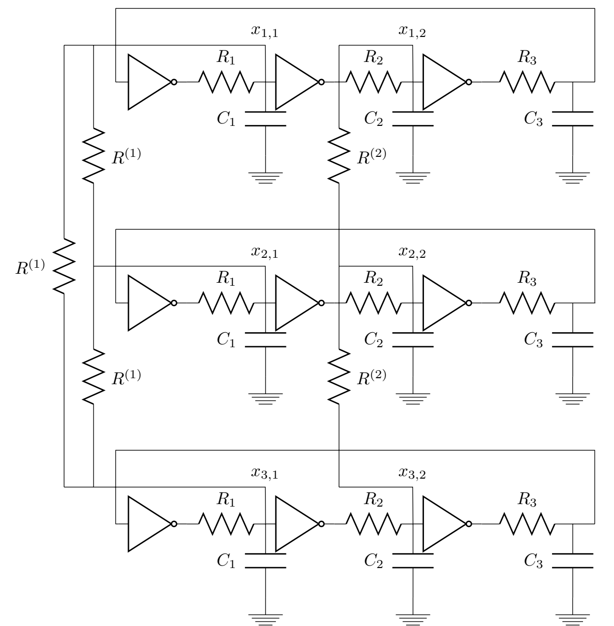

Fig. 1. An example of a network of interconnected three-stage ring oscillator circuits as in (33) coupled through nodes 1 and 2.

Consider the n-stage ring oscillator whose dynamics are given by:

$$\begin{aligned} \dot{x}_1^k &= -\eta_1 x_1^k - \alpha_1 \tanh(\beta_1 x_n^k) + w_1^k \\ &\;\vdots \\ \dot{x}_n^k &= -\eta_n x_n^k + \alpha_n \tanh(\beta_n x_{n-1}^k) + w_n^k, \end{aligned} \tag{33}$$

with coupling between corresponding nodes of each circuit. Ring oscillators have found wide application in biological oscillators such as the repressilator in [15]. The parameters $\eta_k = \frac{1}{R_k C_k}$, α_k, and β_k correspond to the gain of each inverter. The input is given by:

$$w_i^k = d_i \sum_{j\in\mathcal{N}_{k,i}} (x_i^j - x_i^k), \tag{34}$$

where $d_i = \frac{1}{R^{(i)}C_i}$ and $\mathcal{N}_{k,i}$ denotes the nodes to which node i of circuit k is connected. We wish to determine if the solution trajectories of each set of like nodes of the coupled ring oscillator circuit given by (33)-(34) synchronize, that is:

$$x_i^j - x_i^k \to 0 \text{ exponentially as } t \to \infty \tag{35}$$

for any pair $(j,k) \in \{1,\ldots,N\} \times \{1,\ldots,N\}$ and any index $i \in \{1,\ldots,n\}$.

For clarity in our discussion, we take $n = 3$ as in Figure 1. We first write the Jacobian of the system (33), where we have omitted the subscripts indicating circuit membership:

$$J(x)\big|_{x=\bar{x}} = \left[\begin{array}{ccc} -\eta_1 & 0 & \gamma_1(\bar{x}_1) \\ \gamma_2(\bar{x}_2) & -\eta_2 & 0 \\ 0 & \gamma_3(\bar{x}_3) & -\eta_3 \end{array}\right], \qquad (36)$$

with $\gamma_1(\bar{x}_1) = -\alpha_1\beta_1 \operatorname{sech}^2(\beta_1\bar{x}_3)$, $\gamma_2(\bar{x}_2) = \alpha_2\beta_2 \operatorname{sech}^2(\beta_2\bar{x}_1)$, and $\gamma_3(\bar{x}_3) = \alpha_3\beta_3 \operatorname{sech}^2(\beta_3\bar{x}_2)$. Define the matrices

$$A_0 = \left[\begin{array}{ccc} -\eta_1 & 0 & 0 \\ 0 & -\eta_2 & 0 \\ 0 & 0 & -\eta_3 \end{array}\right],\ A_1 = \left[\begin{array}{ccc} 0 & 0 & -\alpha_1\beta_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

$$A_2 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \alpha_2\beta_2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],\ A_3 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha_3\beta_3 & 0 \end{array}\right].$$

Then it follows that $J(x)$ is contained in a convex box:

$$J(x) \in \text{box}\{A_0, A_1, A_2, A_3\}. \qquad (37)$$

The problem structure can be exploited using Theorem 3 to obtain a simple analytical condition for synchronization of trajectories. In particular, the Jacobian of the ring oscillator exhibits a *cyclic* structure. The matrix M for which we seek a $\mathcal{Q}$ satisfying (30) is given by:

$$\begin{aligned} M &= \left[\begin{array}{cc} A_0 - \Lambda_2 - \frac{\mu}{2}I & B \\ C^T & -I \end{array}\right], \\ B &= \left[\begin{array}{ccc} 0 & 0 & -\alpha_1\beta_1 \\ \alpha_2\beta_2 & 0 & 0 \\ 0 & \alpha_3\beta_3 & 0 \end{array}\right], C = I_3. \end{aligned} \qquad (38)$$

Note that the matrix M exhibits a cyclic structure, and by a suitable permutation G of its rows and columns, it can be brought into a cyclic form $\tilde{M} = GMG^T$. Since $\tilde{M}$ is cyclic, it is amenable to an application of the *secant criterion* [22], which implies that the condition

$$\frac{\Pi_{i=1}^3 \alpha_i\beta_i}{\Pi_{l=1}^3(\eta_l + \lambda_l + \frac{\mu}{2})} < \sec^3\left(\frac{\pi}{3}\right) \qquad (39)$$

holds if and only if $\tilde{M}$ satisfies

$$\tilde{\mathcal{Q}}\tilde{M} + \tilde{M}^T\tilde{\mathcal{Q}} < 0 \qquad (40)$$

with negative μ, for some diagonal $\tilde{\mathcal{Q}} > 0$. Pre- and post-multiplying (40) by G^T and G, respectively, (40) is equivalent to:

$$G^T\tilde{\mathcal{Q}}GM + M^TG^T\tilde{\mathcal{Q}}G < 0. \qquad (41)$$

Thus, if $\tilde{\mathcal{Q}}$ is diagonal and satisfies (40), then $\mathcal{Q} = G^T\tilde{\mathcal{Q}}G$ is diagonal and satisfies (30). We conclude that if the secant criterion in (39) is satisfied, then by Theorem 3, we have:

$$\sup_{(x,t)\in V\times[0,\infty)} (J_F(x,t) - \Lambda_2) \ \le\ \frac{\mu}{2}.$$

Because $\mathcal{Q}$ is diagonal and positive, Q is diagonal and positive, and $QD_i + D_iQ > 0$ for each $i = 1, \cdots, r$. Therefore, since $\mu < 0$, by Corollary 4, we get:

$$x_i^j - x_i^k \to 0 \text{ exponentially as } t \to \infty \qquad (42)$$

for any pair $(j,k) \in \{1,\ldots,N\}\times\{1,\ldots,N\}$ and any index $i \in \{1, 2, 3\}$.

We note that the condition for synchrony that we have found recovers Theorem 2 in [14], which makes use of an input-output approach to synchronization [23]. We have derived the condition using Lyapunov functions in an entirely different manner from the input-output approach.

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