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Bounds in the theory of polynomial rings over fields. A nonstandard approach

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Introduction

A familiar procedure in mathematics is to use properties of suitable *extensions* of "structures of interest" – for example, formal power series rings as extensions of polynomial rings – to obtain (or even to express) results about these structures. In connection with the example just mentioned, one often exploits the fact that localization and completion introduce flat modules, following Serre's GAGA. Most of the classical ideal theory for polynomial rings over $\mathbb{C}$ has thus been considerably simplified and generalized to suitable classes of (noetherian) rings. (This became a crucial factor in the development of abstract algebraic geometry.)

However, there still remained a number of results specific to polynomial rings over *fields*, for which only recently a similar treatment became available. In fact, in this paper we shall show that *nonstandard* extensions of polynomial rings over fields lead quite naturally to the existence of bounds of which the following is typical: *If an ideal I of $K[X_1, \dots, X_n]$, K a field, is generated by polynomials of (total) degree $\leqq d$, then its radical, its associated prime ideals, and the ideals of a suitable primary decomposition can be generated by polynomials of degree $\leqq D$, where D only depends on d and n, not on the field K or the particular ideal I.* (The *number* of associated primes can be bounded in the same way.)

The theorem just quoted and related results have first been established by means of elaborate algorithmic procedures, cf. G. Hermann's 1926 paper [5]. The many complications and case distinctions necessary in this approach led inevitably to mistakes and omissions, so A. Seidenberg [13] decided to redo this material, while using essentially similar (constructive) methods. This constructive approach gives often extra information in terms of expressions for the bounds which arise, but has also its drawbacks:

(1) The bounds obtained are usually too large to guarantee feasibility of the algorithms; 'practical' algorithms are either not known or not directly based on this work.

(2) The constructivists tend to state their theorems only for fields which are "computable" or "explicitly given", though, *suitably reformulated*, they make perfect sense and are true for arbitrary fields.

In this paper we intend to show that by concentrating on *existence proofs* for bounds, rather than on their construction, it is possible to gain a lot in efficiency of exposition.

Let us take two examples to illustrate the kind of theorems we are interested in and the method we follow to prove them.

Fix natural numbers d and n and consider polynomials $f_0, f_1, \dots, f_m \in K[X_1, \dots, X_n]$, all of (total) degree $\leqq d$, where K is a field. Then a first nontrivial fact is:

(I) *If* $f_0 \in (f_1, \dots, f_m) =$ *the ideal generated by* $f_1, \dots, f_m$, *then* $f_0 = \sum_{1 \leqq i \leqq m} h_i f_i$ *for certain* $h_i \in K[X_1, \dots, X_n]$ *whose degree is bounded by a number* A *depending only on* (d, n). (Not on K, m, or the polynomials in question.)

A second, rather deeper, theorem says:

(II) *There is a bound* $P \in \mathbb{N}$ *depending only on* (d, n) *such that if the implication*

$$gh \in (f_1, \dots, f_m) \Rightarrow g \in (f_1, \dots, f_m) \quad \text{or} \quad h \in (f_1, \dots, f_m)$$

holds for all $g, h \in K[X_1, \dots, X_n]$ *of degree* $\leqq P$, *then* $(f_1, \dots, f_m)$ *is a prime ideal or contains* 1.

Our method for proving these results may be compared to the use of power series rings $K[[X_1, \dots, X_n]]$ to prove (local) properties of polynomial rings $K[X_1, \dots, X_n]$. The role of nonstandard extensions of polynomial rings is more complicated - they are only defined over *internal* fields - but these extensions also code much more information.

More precisely, *fix a natural number n and consider the ring* $K[X_1, \dots, X_n]_{\text{int}}$ *of internal polynomials in n variables over an internal field* K. (In (0.2) we shall clarify our notion of 'internal'; for the moment the reader should just accept that the ring $K[X_1, \dots, X_n]_{\text{int}}$ contains $K[X_1, \dots, X_n]$ as a subring. An unfortunate consequence of nonstandard terminology is that internal polynomials are in general not polynomials: they can have 'infinite' degree. On the other hand, an internal field is a field in the usual sense.) We shall prove:

(I′) $K[X_1, \dots, X_n]_{\text{int}}$ *is a faithfully flat* $K[X_1, \dots, X_n]$*-module.*

(II′) *If* $\underline{p}$ *is a prime ideal of* $K[X_1, \dots, X_n]$ *then* $\underline{p} K[X_1, \dots, X_n]_{\text{int}}$ *is a prime ideal of* $K[X_1, \dots, X_n]_{\text{int}}$.

(Readers familiar with nonstandard methods will recognize that (I′) implies fact (I) above, and that (II′) is equivalent to (II).)

The typical result of our paper is that a certain part of the structure of $K[X]$ is preserved by $K[X]_{\text{int}}$, $X = (X_1, \dots, X_n)$. Two more examples of this phenomenon:

for any ideal I of $K[X]$, *K an internal field, we have:*

(III′) *I is primary in* $K[X] \Leftrightarrow IK[X]_{\text{int}}$ *is primary in* $K[X]_{\text{int}}$.

(IV′) *If I is prime then we have the equivalence:*

$$K[X]/I \text{ is integrally closed} \Leftrightarrow K[X]_{\text{int}}/IK[X]_{\text{int}} \text{ is integrally closed.}$$

Statements (III′) and (IV′) have also interesting "standard" interpretations in terms of bounds for which we refer the reader to (2.10) and (3.1).

We should mention here that the existence of bounds has an algebraic-geometric meaning, at least for *algebraically closed* K: let $f_0(C,X), \ldots, f_m(C,X) \in \mathbb{Z}[C,X]$, $C=(C_1,\ldots,C_M)$, $X=(X_1,\ldots,X_n)$. Then the set $\{c \in K^M \mid f_0(c,X) \in (f_1(c,X),\ldots,f_m(c,X))\}$ is a *constructible subset of* K^M; this follows by combining (I) with Chevalley's constructibility theorem. Similarly the existence of the bound P in (II) implies that the set $\{c \in K^M \mid (f_1(c,X),\ldots,f_m(c,X))$ is a prime ideal$\}$ is a constructible subset of K^M. As a final example, the bound implied by (IV′) leads to the constructibility of the subset $\{c \in K^M \mid (f_1(c,X),\ldots,f_m(c,X))$ is a prime ideal $\underline{p}$ of $K[X]$ such that $K[X]/\underline{p}$ is integrally closed$\}$ of K^M.

For non-algebraically closed K a similar interpretation is available if one is willing to use a notion from model theory: the sets above are *first-order definable* for general K; 'first-order definable' reduces to 'constructible' for algebraically closed K, and to 'semi-algebraic' for real closed K. (Since the bounds are independent of the field, we have in fact in each case a first-order definition uniformly for all fields; this is occasionally useful in reducing problems over fields of char. 0 to similar problems in char. $p>0$, cf. [6, (2.1)].)

For the proofs of (1.1), (2.3) and (3.9) below we borrowed the (ingenious) idea from corresponding constructive proofs, cf. [13, p. 277], [15, Th. 1], [16, p. 597], but we think the nonstandard setting brings out more clearly these ideas by removing messy complications and case distinctions.

The nonstandard proofs of (1.4), (2.5), (2.7), (2.9), and (3.11) seem completely new. We think the results in this article essentially cover the results in [13], [14] and [16] as far as existence of bounds is concerned, but we did not make a detailed comparison. Corollary (2.9)(1) seems rather more general than what can be extracted from the constructivist literature.

The extent to which the structure of $K[X]$, K an internal field, is reflected in $K[X]_{\text{int}}$ is very much an open question. For instance, for $n \geqq 3$, $m>1$ we do not know whether an ideal I of $K[X]$ such that $IK[X]_{\text{int}}$ is generated by m elements can itself be generated by m elements. See [3, (1.14)] for similar questions and further comments. In the last section we give an example of a polynomial f in $K[X_1,X_2]$, K an internal field, such that $K[X_1,X_2]/(f)$ has only trivial units (those in $K^{\cdot}$), but $K[X_1,X_2]_{\text{int}}/fK[X_1,X_2]_{\text{int}}$ has a nontrivial unit.

A remark on A. Robinson's contribution

The nonstandard approach to bounds was pioneered in Robinson [10] which influenced us considerably. The main result of [10] is a nonstandard proof of

(I) above (which is completely different from our proof in § 1). Robinson's great interest in these matters is also clear from [9, p. 503] of which two quotations: "... only a beginning has been made in the effort to replace G. Hermann's method by model theoretic arguments", and: "In this case, the existence of the required bounds results from the work of Hentzelt and Noether but again does not follow from any known model theoretic arguments".

Such model theoretic arguments were given in [2, Chap. 4], and further extensions (an simplifications) in [3] and [12]. The present paper systematizes this material and incorporates still further simplifications.

(0.1) *Conventions*

All rings are commutative with unit 1. X stands for $(X_1, \ldots, X_n)$, $n \in \mathbb{N}$. If K is a field and $f \in K[X] = K[X_1, \ldots, X_n]$ we write $\deg(f)$ for the *total* degree of f, and if $f = (f_1, \ldots, f_m) \in K[X]^m$ we let the degree of f be $\max(\deg(f_i))$. An ideal of $K[X]$ is said to be *of type d*, if it is generated by polynomials of degree $\leqq d$.

Let R be an integral domain. We call $f \in R$ *irreducible* if f is not a unit nor a product of two nonunits. We write $Ic(R)$ for the integral closure of R (inside its fraction field), and $Fr(R)$ for the fraction field of R.

(0.2) *Concerning nonstandard extensions*

We use nonstandard methods roughly as follows: we assume a structure is given which contains all (algebraic) objects, and all relations between those objects we are interested in. Then we take an *enlargement* or *nonstandard extension* of this structure; the notion of 'internal' is defined relative to this enlargement. (Readers not familiar with nonstandard methods should have no trouble in understanding the arguments in this paper after consulting Sect. 3 in [4]. A solid foundation of nonstandard methods can be found in [7].)

We write $K[X]_{\text{int}}$ for the ring of *internal* polynomials in $X = (X_1, \ldots, X_n)$ over the *internal* field K, and we write $K(X)_{\text{int}}$ for its fraction field.

$\mathbb{Z}^*$ is the set of internal integers of our enlargement, $\mathbb{N}^* = \{m \in \mathbb{Z}^* \mid m \geqq 0\}$.

In the proof of the first 'standard' theorem, (1.4), we shall indicate the structure whose enlargement should be considered in order that a previously proved nonstandard result applies. After that we shall just state without further justification the *standard translation* of the nonstandard theorems we prove, and leave to the reader the routine of checking that the translation is correct.

§ 1. Faithful flatness

(1.1) **Theorem.** *Let K be an internal field. Then $K[X]_{\text{int}}$ is a flat $K[X]$-module.*

(1.2) Before we start the proof we note that, given any ring extension $S \supset R$, the following are equivalent:

(i) S is a flat R-module.

(ii) For each homogeneous linear equation

$$f_1 Y_1 + \ldots + f_l Y_l = 0, \quad f_i \in R,$$

the solutions in S^l are S-linear combinations of solutions in R^l.

(iii) For each system of homogeneous linear equations

$$\begin{matrix} f_{11} Y_1 + \ldots + f_{1l} Y_l = 0 \\ \vdots \qquad\qquad \vdots\ \vdots \\ f_{k1} Y_1 + \ldots + f_{kl} Y_l = 0 \end{matrix} \qquad (f_{ij} \in R),$$

the solutions in S^l are S-linear combinations of solutions in R^l. For the proof, see [1, Chap. 1, § 2, n° 11] where (i)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i) is shown.

(1.3) *Proof of* (1.1). By induction on n, the number of variables, using the equivalence (i)$\Leftrightarrow$(ii) above. Assume $n>0$, and let $f_1, \ldots, f_l \in K[X]$ be given and a solution $g=(g_1, \ldots, g_l)$ in $(K[X]_{\mathrm{int}})^l$ of the equation $f_1 Y_1 + \ldots + f_l Y_l = 0$. We have to show that g is a linear combination of solutions in $K[X]^l$. Using changes of variables, etc., there is no harm in assuming that $f_1 \neq 0$, and f_1 monic in X_n. Let $d = \deg_{X_n} f_1$. Now $(-f_2, f_1, 0, \ldots, 0)$, $(-f_3, 0, f_1, 0, \ldots, 0), \ldots$, $(-f_n, 0, \ldots, 0, f_1)$ are solutions, and by subtracting suitable multiples of these solutions from g we get a solution g' with $\deg_{X_n}(g'_2), \ldots, \deg_{X_n}(g'_n)$ all less than d. Since $f_1 g'_1 + \ldots + f_l g'_l = 0$ it follows that $\deg_{X_n} g'_1$ is also finite. So g' is a solution in $K[X_1, \ldots, X_{n-1}]_{\mathrm{int}}[X_n]$. By the induction hypothesis $K[X_1, \ldots, X_{n-1}]_{\mathrm{int}}$ is a flat $K[X_1, \ldots, X_{n-1}]$-module, hence $K[X_1, \ldots, X_{n-1}]_{\mathrm{int}}[X_n]$ is a flat $K[X_1, \ldots, X_n]$-module (by preservation of flatness under scalar extension). By (i)$\Leftrightarrow$(ii) of (1.2), we conclude that g' is generated by solutions in $K[X]^l$, and therefore the original g is also generated by solutions in $K[X]^l$. □

The standard version of (1.1) is as follows.

(1.4) **Theorem.** *Given $n, d, k \in \mathbb{N}$ there is $\alpha = \alpha(n, d, k) \in \mathbb{N}$ such that for each field K and each homogeneous linear system*

$$\begin{matrix} f_{11} Y_1 + \ldots + f_{1l} Y_l = 0 \\ \vdots \qquad\qquad \vdots\ \vdots \\ f_{k1} Y_1 + \ldots + f_{kl} Y_l = 0 \end{matrix}$$

with all $f_{ij} \in K[X]$ of degree $\leqq d$ the solution submodule of $K[X]^l$ is generated by solutions $g=(g_1, \ldots, g_l)$ of degree $\leqq \alpha$ (i.e., $\deg g_i \leqq \alpha$ for $i = 1, \ldots, l$).

(1.5) *Remarks*

(1) The number α does not depend on l. For if we have a bound α for $l = l(n, d) = \dim_K \{f \in K[X] \mid \deg f \leqq d\}$, then this α will also be a bound for all other values of l. Similar arguments apply at other places in this paper.

(2) As stated in the Introduction, we shall give a detailed proof that indeed (1.4) follows from (1.1). In the remainder of this article we leave such routine arguments to the reader.

(1.6) *Proof of* (1.4). Suppose n, d, k given and there is no bound α. So for each $m \in \mathbb{N}$ there is a field K_m and a homogeneous linear system over $K_m[X]$ as above, with $l = l(n,d)$ (see (1.5), Remark (1)), that has a solution in $K_m[X]^l$ which is not generated by solutions of degree $\leqq m$. Take a structure containing all fields K_m, polynomial rings $K_m[X]$, $\mathbb{N}$, etc., and take an enlargement of this structure. By its saturation properties this enlargement contains an internal field K and polynomials $f_{ij} \in K[X]$ of degree $\leqq d$ such that the system of (1.4) has a solution in $(K[X]_{\text{int}})^l$ which is not generated by solutions in $(K[X]_{\text{int}})^l$ of finite degree, that is, it is not a $K[X]_{\text{int}}$-linear combination of solutions in $K[X]^l$. But this is in contradiction with the equivalence (i)$\Leftrightarrow$(iii) of (1.2) and the fact that $K[X]_{\text{int}}$ is a flat $K[X]$-module. $\square$

Let us mention an immediate consequence of (1.4) (or, equivalently, of (1.1)).

(1.7) **Corollary.** *Given $n, d \in \mathbb{N}$ there is $A = A(n,d) \in \mathbb{N}$ such that for each field K and for any two ideals I, J in $K[X]$ of type d the ideals $I \cap J$ and $I:J$ are of type A.*

Next we consider inhomogeneous linear equations over $K[X]$.

(1.8) **Theorem.** *Let K be an internal field. Then $K[X]_{\text{int}}$ is a faithfully flat $K[X]$-module.*

(1.9) Concerning faithful flatness, we need to know that, given any ring extension $S \supset R$ the following are equivalent, cf. [1, Chap. 1, § 3.7]:

(i) S is a faithfully flat R-module,

(ii) S is a flat R-module, and $\underline{m}S \neq S$ for each maximal ideal $\underline{m}$ of R,

(iii) S is a flat R-module, and each system

$$\begin{matrix} f_{11}Y_1 + \ldots + f_{1l}Y_l = f_1 \\ \vdots \qquad\qquad\qquad \vdots \quad \vdots \\ f_{k1}Y_1 + \ldots + f_{kl}Y_l = f_k \end{matrix} \qquad (f_{ij}, f_i \in R)$$

with a solution in S^l has a solution in R^l.

(1.10) *Proof of* (1.8). We use (i)$\Leftrightarrow$(ii) above. If $\underline{m}$ is a maximal ideal of $K[X]$ then $\underline{m}$ has a zero in a finite field extension of K, by the Nullstellensatz, and obviously this is also an (internal) zero of $\underline{m}K[X]_{\text{int}}$, and so we must have $\underline{m}K[X]_{\text{int}} \neq K[X]_{\text{int}}$. $\square$

Taking into account the equivalence (i)$\Leftrightarrow$(iii) of (1.9) we see that the standard version of (1.8) is as follows.

(1.11) **Theorem.** *Let $n, d, k \in \mathbb{N}$. There is $\beta = \beta(n,d,k) \in \mathbb{N}$ such that for every field K and every system*

$$\begin{array}{c} f_{11}Y_1+\ldots+f_{1l}Y_l=f_1 \\ \vdots \qquad\qquad \vdots\ \ \vdots \\ f_{k1}Y_1+\ldots+f_{kl}Y_l=f_k \end{array}$$

with all f_{ij}, f_i in $K[X]$ of degree $\leqq d$ the following holds: if there is a solution in $K[X]^l$ there is a solution in $K[X]^l$ of degree $\leqq \beta$.

Note that property (I) mentioned in the Introduction is a special case of this theorem.

§ 2. Prime ideals

(2.1) In this section we prefer to prove first of all the 'nonstandard' theorems, and after that to mention some of the more interesting 'standard' consequences.

Our first goal is to show that if an ideal I of $K[X]$, K an internal field, is prime, then $IK[X]_{\text{int}}$ is also prime. For the proof we need two lemmas, the first of which is due to Roquette [11, p. 255]. We repeat his proof.

(2.2) **Lemma.** *Let K be an internal field. Then $K(X)_{\text{int}}$ is a regular extension of $K(X)$.*

Proof. First we show:

(a) $K(X)$ is algebraically closed in $K(X)_{\text{int}}$. Let $z\in K(X)_{\text{int}}$ be algebraic over $K(X)$. To show that $z\in K(X)$ we may as well assume that z is integral over $K[X]$, $z\neq 0$, and since $K[X]_{\text{int}}$ is integrally closed it follows that $z\in K[X]_{\text{int}}$. Now z^{-1} is also algebraic over $K[X]$ so there is nonzero $h\in K[X]$ with $z^{-1}h\in K[X]_{\text{int}}$, so $h=zg$, where $z, g\in K[X]_{\text{int}}$. Hence $\deg h=(\deg z)+(\deg g)$, and since $\deg h$ is finite it follows that $\deg z$ must be finite, that is, $z\in K[X]$.

(b) $K(X)_{\text{int}}$ is a separable extension of $K(X)$. In the proof of this we may assume $\operatorname{char}(K)=p>0$. Now it suffices to note that if $a_1,\ldots,a_m\in K$ are p-independent in K, then $a_1,\ldots,a_m$, $X_1,\ldots,X_n$ are p-independent in $K(X)_{\text{int}}$. □

(2.3) **Lemma.** *Let K be an internal field, and $f\in K[X_1]_{\text{int}}$ be irreducible of infinite degree, and let I be an ideal of $K[X]$, $n>0$. Then the image of f in $K[X]_{\text{int}}/IK[X]_{\text{int}}$ is a non-zero-divisor.*

Proof. Suppose $g\in K[X]_{\text{int}}$ satisfies

(a) $fg=\sum h_i f_i$, $f_i\in I$, $h_i\in K[X]_{\text{int}}$, $i=1,\ldots,m$, $m\in\mathbb{N}$. It suffices to deduce from this that $g\in IK[X]_{\text{int}}$. Since f has infinite degree the canonical map

$$K[X_1]_{\text{int}}\to K[X_1]_{\text{int}}/(f)\overset{\text{def}}{=}L$$

embeds $K[X_1]$ into the internal field L, hence $K(X_1)$ is embedded into L by an embedding which we call ϕ. We consider L as an algebra over $K(X_1)$ via ϕ. Since L is a flat $K(X_1)$-module, $L[X_2,\ldots,X_n]$ is a flat $K(X_1)[X_2,\ldots,X_n]$-module. Also $L[X_2,\ldots,X_n]_{\text{int}}$ is a flat $L[X_2,\ldots,X_n]$-module, by (1.1), hence by transitivity of flatness we get:

(b) $L[X_2, \ldots, X_n]_{\text{int}}$ is a flat $K(X_1)[X_2, \ldots, X_n]$-module.

Now (a) implies that in $L[X_2, \ldots, X_n]_{\text{int}}$ we have:

(c) $\sum h_i(\phi X_1, X_2, \ldots, X_n) \cdot f_i(\phi X_1, X_2, \ldots, X_n) = 0.$

Consider the solutions $(y_1, \ldots, y_m)$ in $(K(X_1)[X_2, \ldots, X_n])^m$ of

$$Y_1 f_1 + \ldots + Y_m f_m = 0.$$

The $K(X_1)[X_2, \ldots, X_n]$-module of these solutions is generated by, say, $(h_{11}, \ldots, h_{1m}), \ldots, (h_{k1}, \ldots, h_{km})$, $h_{ij} \in K[X]$. By (b) and (c) we can write

$$h_j(\phi X_1, X_2, \ldots, X_n) = \sum_{i=1}^{k} \lambda_i h_{ij}(\phi X_1, X_2, \ldots, X_n), \quad j = 1, \ldots, m,$$

for certain λ_i in $L[X_2, \ldots, X_n]_{\text{int}}$.

Write $\lambda_i = \mu_i(\phi X_1, X_2, \ldots, X_n)$, where $\mu_i \in K[X]_{\text{int}}$. Then:

(d) $$h_j = \left(\sum_{i=1}^{k} \mu_i h_{ij}\right) + f q_j, \quad j = 1, \ldots, m, \quad q_j \in K[X]_{\text{int}}.$$

Substituting (d) in (a) and using the assumption that $(h_{i1}, \ldots, h_{im})$ is a solution of $Y_1 f_1 + \ldots + Y_m f_m = 0$ we get:

$$fg = \sum_{j=1}^{m} f q_j f_j, \quad \text{so } g = \sum q_j f_j \in IK[X]_{\text{int}}. \quad \square$$

(2.4) **Corollary.** *Let K be an internal field, I an ideal of $K[X]$. If $I: (f) = I$ for all irreducible $f \in K[X_1]$ then*

$$IK[X]_{\text{int}}: (fK[X]_{\text{int}}) = IK[X]_{\text{int}} \quad \textit{for all } f \in K[X_1]_{\text{int}} \setminus \{0\}.$$

Proof. If there is a counterexample we take one of minimal degree, say f, so there is $g \in K[X]_{\text{int}}$ with $gf \in IK[X]_{\text{int}}$ but $g \notin IK[X]_{\text{int}}$. Since f is of minimal degree f is irreducible in $K[X_1]_{\text{int}}$, and by Lemma (2.3) it cannot be of infinite degree, so $f \in K[X_1]$, but then it follows from the assumption $I: (f) = I$ and flatness that $g \in IK[X]_{\text{int}}$, contradiction. $\square$

(2.5) **Theorem.** *Let K be an internal field and I an ideal of $K[X]$.*
Then: I is prime $\Leftrightarrow IK[X]_{\text{int}}$ is prime in $K[X]_{\text{int}}$.

Proof. By faithful flatness we have $(IK[X]_{\text{int}}) \cap K[X] = I$, so ($\Leftarrow$) is obvious. For the ($\Rightarrow$)-direction we proceed by induction on n, the number of variables. If $n = 1$ then $I = (0)$ or $I = (f)$ for some irreducible $f \in K[X_1]$, and since f remains irreducible in $K[X_1]_{\text{int}}$ the result is clear in this case. Suppose now $n > 1$, and distinguish to cases:

(a) $I \cap K[X_i] \neq 0$ for each $i=1, \ldots, n$,

(b) $I \cap K[X_i]=0$ for some $i \in \{1, \ldots, n\}$.

In case (a) take $f_i \in K[X_i] \cap I$ with $\deg f_i = d_i > 0$. Then the image x_i of X_i in $K[X]/I$ is algebraic of degree $\leqq d_i$ over K, so $K[X]/I = K[x_1, \ldots, x_n]$ is a field extension of K. For the same reason the K-algebra $K[X]_{\text{int}}/IK[X]_{\text{int}}$ is generated over K by the images of the X_i (and is of finite K-dimension). So the obvious map $K[X]/I \to K[X]_{\text{int}}/IK[X]_{\text{int}}$ is surjective, and since $K[X]/I$ is a field this map is an isomorphism, hence $IK[X]_{\text{int}}$ is a prime (even a maximal) ideal of $K[X]_{\text{int}}$.

In case (b) we may suppose without loss of generality that $I \cap K[X_1]=0$. Then $IK(X_1)[X_2, \ldots, X_n]$ is a prime ideal, hence by Lemma (2.2) the ideal $IK(X_1)_{\text{int}}[X_2, \ldots, X_n]$ is prime in $K(X_1)_{\text{int}}[X_2, \ldots, X_n]$. Now we apply the induction hypothesis and get that $IK(X_1)_{\text{int}}[X_2, \ldots, X_n]_{\text{int}}$ is a prime ideal of $K(X_1)_{\text{int}}[X_2, \ldots, X_n]_{\text{int}}$, so $IK(X_1)_{\text{int}}[X_2, \ldots, X_n]_{\text{int}} \cap K[X]_{\text{int}}$ is a prime ideal of $K[X]_{\text{int}}$. It remains to show that $(IK(X_1)_{\text{int}}[X_2, \ldots, X_n]_{\text{int}}) \cap K[X]_{\text{int}} = IK[X]_{\text{int}}$. Let $g \in (IK(X_1)_{\text{int}}[X_2, \ldots, X_n]_{\text{int}}) \cap K[X]_{\text{int}}$ and take generators $f_1, \ldots, f_m$ of I. Then $g = \sum (h_i/f) \cdot f_i$ for certain $h_i \in K[X]_{\text{int}}$ and some nonzero $f \in K[X_1]_{\text{int}}$. So $fg \in IK[X]_{\text{int}}$, i.e., $g \in IK[X]_{\text{int}} : (fK[X]_{\text{int}})$. By (2.4) this gives us $g \in IK[X]_{\text{int}}$. □

(2.6) In the two corollaries which follow we assume that K is an internal field and I an ideal of $K[X]$. We also define an ideal J of $K[X]_{\text{int}}$ to be *internally radical* if J is an internal ideal such that $f^N \in J \Rightarrow f \in J$, for each $f \in K[X]_{\text{int}}$ and each $N \in \mathbb{N}^*$. Given an internal ideal J of $K[X]_{\text{int}}$ we define its *internal (nil) radical* $\sqrt[\text{int}]{J}$ as $\{f \in K[X]_{\text{int}} \mid f^N \in J \text{ for some } N \in \mathbb{N}^*\}$. This is the smallest internally radical ideal containing J.

(2.7) **Corollary.**

(i) *If* $\underline{p}_1, \ldots, \underline{p}_m$ *are the distinct minimal primes of* I *then* $\underline{p}_1 K[X]_{\text{int}}, \ldots, \underline{p}_m K[X]_{\text{int}}$ *are the distinct minimal primes of* $IK[X]_{\text{int}}$.

(ii) $\sqrt{IK[X]_{\text{int}}} = \sqrt[\text{int}]{IK[X]_{\text{int}}} = \sqrt{I} \cdot K[X]_{\text{int}}$.

Proof. By the previous theorem and faithful flatness $\underline{p}_1 K[X]_{\text{int}}, \ldots, \underline{p}_m K[X]_{\text{int}}$ are (internal) prime ideals and $\underline{p}_i K[X]_{\text{int}} \not\subset \underline{p}_j K[X]_{\text{int}}$ if $i \neq j$. Now it is a general fact that if $P_1, \ldots, P_m$ are prime ideals in a ring R such that $P_i \not\subset P_j$ $(i \neq j)$, then $P_1, \ldots, P_m$ are the minimal primes of $P_1 \cap \ldots \cap P_m$, and therefore of any ideal between $P_1 \cap \ldots \cap P_m$ and some power $(P_1 \cap \ldots \cap P_m)^M$.

Now choose $M \in \mathbb{N}$ such that $(\sqrt{I})^M = (\underline{p}_1 \cap \ldots \cap \underline{p}_m)^M \subset I \subset \underline{p}_1 \cap \ldots \cap \underline{p}_m$. By faithful flatness we get

$$\begin{aligned}(\sqrt{I})^M K[X]_{\text{int}} &= (\sqrt{I} \cdot K[X]_{\text{int}})^M = (\underline{p}_1 \cap \ldots \cap \underline{p}_m)^M K[X]_{\text{int}} \\ &= (\underline{p}_1 K[X]_{\text{int}} \cap \ldots \cap \underline{p}_m K[X]_{\text{int}})^M \subset IK[X]_{\text{int}} \\ &\subset \underline{p}_1 K[X]_{\text{int}} \cap \ldots \cap \underline{p}_m K[X]_{\text{int}}.\end{aligned}$$

Now we just apply the previous observations and we conclude that (i) and (ii) hold. □

(2.8) Before we state the next corollary we recall that a prime ideal of a ring R is called *associated with the R-module M* if it is the annihilator of an element of M. $\mathrm{Ass}_R(M)$ is the set of prime ideals associated with M. Furthermore we take the definition of primary ideal and (reduced) primary decomposition from [8, Chap. 6, §5]. We also define a proper ideal J of $K[X]_{\mathrm{int}}$ to be *internally primary* if it is internal and for all $f, g \in K[X]_{\mathrm{int}}$ with $fg \in J$, $f \notin J$, there is $N \in \mathbb{N}^*$ with $g^N \in J$. So a proper internal ideal of $K[X]_{\mathrm{int}}$ which is primary is internally primary. (The converse is not true.)

(2.9) **Corollary.**

(1) *If M is a $K[X]$-module, then*

$$\mathrm{Ass}_{K[X]_{\mathrm{int}}}(M \otimes_{K[X]} K[X]_{\mathrm{int}}) = \{\underline{p} K[X]_{\mathrm{int}} \colon \underline{p} \in \mathrm{Ass}_{K[X]} M\}.$$

(2) *I is a primary ideal of $K[X] \Leftrightarrow IK[X]_{\mathrm{int}}$ is a primary ideal of $K[X]_{\mathrm{int}}$ $\Leftrightarrow IK[X]_{\mathrm{int}}$ is an internally primary ideal of $K[X]_{\mathrm{int}}$.*

(3) *Let $I = I_1 \cap \ldots \cap I_m$ be a reduced primary decomposition of I in $K[X]$, I_k being a $\underline{p}_k$-primary ideal. Then $IK[X]_{\mathrm{int}} = I_1 K[X]_{\mathrm{int}} \cap \ldots \cap I_m K[X]_{\mathrm{int}}$ is a reduced primary decomposition of $IK[X]_{\mathrm{int}}$ in $K[X]_{\mathrm{int}}$, and $I_k K[X]_{\mathrm{int}}$ is a $\underline{p}_k K[X]_{\mathrm{int}}$-primary ideal, $k = 1, \ldots, m$.*

Proof. Statement (1) follows from [1, Chap. IV, §2.6, Th. 2] in view of (2.5) and flatness. To prove (2), suppose that I is a $\underline{p}$-primary ideal of $K[X]$. Then $\mathrm{Ass}_{K[X]} K[X]/I = \{\underline{p}\}$, and if we apply (1) to the case $M = K[X]/I$ we see that $\mathrm{Ass}(K[X]_{\mathrm{int}}/IK[X]_{\mathrm{int}}) = \{\underline{p} K[X]_{\mathrm{int}}\}$. Now $K[X]_{\mathrm{int}}$ is not noetherian but it is 'internally' noetherian, so we can still use [8, pp. 151–152] to conclude that $IK[X]_{\mathrm{int}}$ is internally primary. By (2.7)(ii) we have ${}^{\mathrm{int}}\!\sqrt{IK[X]_{\mathrm{int}}} = \sqrt{IK[X]_{\mathrm{int}}}$, so $IK[X]_{\mathrm{int}}$ is in fact a primary ideal. The other implications are trivial or follow from $IK[X]_{\mathrm{int}} \cap K[X] = I$ and (2.7)(ii). Part (3) is an immediate consequence of (2), the preceding corollary, and flatness. □

We now formulate some of the standard counterparts to the above results.

(2.10) **Theorem.** *Let $n, d \in \mathbb{N}$, $X = (X_1, \ldots, X_n)$. Then there are bounds $B = B(n, d)$ and $C = C(n, d)$ in $\mathbb{N}$ such that for each field K and each ideal I of $K[X]$ generated by polynomials of degree $\leqq d$ the following are true:*

(i) *I is prime $\Leftrightarrow 1 \notin I$, and for all $f, g \in K[X]$ of degree $\leqq B$, if $fg \in I$, then $f \in I$ or $g \in I$.*

(ii) *$\sqrt{I}$ is generated by polynomials of degree $\leqq B$, and for each $f \in K[X]$, if $f \in \sqrt{I}$, then $f^C \in I$.*

(iii) *There are at most B associated primes of I, and each of them is generated by polynomials of degree at most B.*

(iv) *I is primary $\Leftrightarrow 1 \notin I$, and for all $f, g \in K[X]$ of degree $\leqq B$, if $fg \in I$, then $f \in I$ or $g^C \in I$.*

(v) *There is a reduced primary decomposition of I consisting of at most B primary ideals, each of which is generated by polynomials of degree at most B.*

(2.11) *Remarks.*

(1) Let us indicate to which 'nonstandard' results the statements (i)–(v) correspond: (i) follows from (2.5), (ii) from (2.7)(ii) (note: it suffices that a bound C as stated exists for a set of *generators* of $\sqrt{I}$), (iii) from (2.9)(1), taking $M=K[X]/I$, (iv) from (2.9)(2), and (v) from (2.9)(3).

The verification of these implications is routine and we leave it to the reader. (Just follow the pattern of (1.6).)

(2) Using the same bound B in the statements (i)–(v) is of course not meant to suggest that the *optimal* bounds in (i)–(v) are the same. It seems that little is known about the optimal bounds.

(3) Note that (2.9)(1) is much more general than its consequence (2.10)(iii); for the case of finitely presented M one still has a straightforward standard version of (2.9)(1) but for infinitely generated M a standard counterpart seems less clear since then $M\otimes_{K[X]}K[X]_{\text{int}}$ is not naturally an internal object.

§ 3. Integral closure

Our goal in this section is to prove:

(3.1) **Theorem.** *Given n, d there are bounds $M=M(n,d)$ and $N=N(n,d)$ such that for each field k and each prime ideal I of $k[X]$ generated by polynomials of degree $\leqq d$ the $k[x]$-module $Ic(k[x])$ is generated by at most M elements of the form $f(x)/g(x)$ with $f, g\in k[X]$ of degree $\leqq N$, $x=X$* mod I.

(Geometrically: the k-irreducible variety $V\subset\mathbb{A}_n$ defined by I has a k-normalization $\bar{V}\subset\mathbb{A}_{n+M}$, the canonical map $\bar{V}\to V$ being given by the projection $\mathbb{A}_{n+M}\to\mathbb{A}_n$; the bound N leads to a bound on the degree of $\bar{V}$.)

(3.2) We shall need the following variant of Noether normalization which we state here as a property of the polynomial ring $k[X]$ over a field k. Let us call $X\leftrightarrow Y$ an *absolute change of variables* if $Y=(Y_1, \ldots, Y_n)$, each Y_i is a polynomial in X with coefficients in the prime field of k, and each X_i is a polynomial in Y with coefficients in the prime field of k. (Hence $k[X]=k[Y]$.)

(3.3) **Lemma.** *For each ideal I of $k[X]$ there is an absolute change of variables $X\leftrightarrow Y$ and an $r\leqq n$ such that $I\cap k[Y_1, \ldots, Y_r]=0$ and $k[Y]/I$ is integral over its subring $k[Y_1, \ldots, Y_r]$. If moreover I is prime and the fraction field of $k[X]/I$ is separable over k then we can arrange that $Y_1, \ldots, Y_r$ is a separating transcendence base of this fraction field over k.*

Proof. The first part of the lemma is standard, see [8, p. 260]. For the proof of the second part we may of course assume $\operatorname{char}(k)=p>0$. So let $k(x)|k$ be separable and of transcendence degree r, where $x=(x_1, \ldots, x_n)=X$ mod I. We proceed by induction on $n-r$. The case $r=n$ is trivial, so let $r<n$. After a reordering of the variables we may assume x_n separably algebraic over $k(x_1, \ldots, x_{n-1})$, cf. [8, p. 265], so $f(x_1, \ldots, x_n)=0$, $\dfrac{\partial f}{\partial X_n}(x_1, \ldots, x_n)\neq 0$ for some $f\in k[X]$. We let $x_i=y_i+x_n^{d_i}$ $(i<n)$ and $x_n=y_n$, choosing the $d_i\in\mathbb{N}$ such that $f(x)=0$ transforms into an equation $g(y)=0$ exhibiting y_n as integral over

$k[y_1, \ldots, y_{n-1}]$; from the proof in [8, p. 260] it is clear that we can choose the d_i's as powers of p, and this guarantees $\frac{\partial g}{\partial Y_n}(y) \neq 0$, hence y_n is still separably algebraic over $k(y_1, \ldots, y_{n-1})$. Now apply the induction hypothesis. □

(3.4) For our nonstandard proof of (3.1) we need 4 more lemmas. In these lemmas K is an internal field, I an ideal of $K[X]$. We put $R = K[X]/I$, $R^* = K[X]_{\text{int}}/IK[X]_{\text{int}}$. Since $IK[X]_{\text{int}} \cap K[X] = I$ (faithful flatness), we may consider R as a subring of R^*.

(3.5) **Lemma.**

(i) *R^* is a faithfully flat R-module.*

(ii) *$\underline{p} \in \operatorname{Spec}(R) \Rightarrow \underline{p}R^* \in \operatorname{Spec}(R^*)$.*

(iii) *If $\underline{p} \in \operatorname{Spec}(R)$ and $\underline{q} \subset R$ is a $\underline{p}$-primary ideal then $\underline{q}R^*$ is a $\underline{p}R^*$-primary ideal.*

Proof. The 3 statements follow immediately from (1.8), (2.5), and (2.9)(2), respectively. □

(3.6) **Lemma.** *If $r \leqq n$ and $I \cap K[X_1, \ldots, X_r] = 0$, then $IK[X]_{\text{int}} \cap K[X_1, \ldots, X_r]_{\text{int}} = 0$.*

Proof. The assumption $I \cap K[X_1, \ldots, X_r] = 0$ means that $1 \notin IK(X_1, \ldots, X_r)[X_{r+1}, \ldots, X_n]$, in other words, given any algebraically closed extension L of $K(X_1, \ldots, X_r)$ there is a point $(x_{r+1}, \ldots, x_n) \in L^{n-r}$ on which the elements of I, considered as polynomials in $(X_{r+1}, \ldots, X_n)$, vanish. Take for L the internal algebraic closure of $K(X_1, \ldots, X_r)_{\text{int}}$. Then the elements of I, as internal polynomials in $(X_{r+1}, \ldots, X_n)$ over L, vanish at some point of L^{n-r}, so

$$1 \notin IK(X_1, \ldots, X_r)_{\text{int}}[X_{r+1}, \ldots, X_n]_{\text{int}}, \quad \text{i.e.,} \quad IK[X]_{\text{int}} \cap K[X_1, \ldots, X_r]_{\text{int}} = 0. \quad \square$$

(3.7) **Lemma.** *If I is prime and $\underline{p} \in \operatorname{Spec}(R)$ is such that $R_{\underline{p}}$ is a discrete valuation ring with maximal ideal generated by π, then the internal ring $R^*_{\underline{p}^*}$, where $\underline{p}^* = \underline{p}R^*$, is a valuation ring with maximal ideal generated by π.*

Proof. Since I is prime R and R^* are domains (and we take $R_{\underline{p}}$ and $R^*_{\underline{p}^*}$ as subrings of the fraction field of R^*). From $\underline{p}R_{\underline{p}} = \pi R_{\underline{p}}$ and $\underline{p}^* = \underline{p}R^*$ it follows easily that $\underline{p}^* R^*_{\underline{p}^*} = \pi R^*_{\underline{p}^*}$. Now the internal local domain $R^*_{\underline{p}^*}$ is 'internally' noetherian with maximal ideal generated by π, so $R^*_{\underline{p}^*}$ is an 'internally discrete' valuation ring. □

(3.8) *Remark.* 'Internally discrete' means of course that the induced Krull valuation on the fraction field of R^* is internal with value group $\mathbb{Z}^*$. We shall not need this extra information.

(3.9) **Lemma.** *Suppose I is prime. Then $Ic(R^*)$ equals the internal integral closure of R^* in its fraction field, and there is nonzero $f \in R$ such that $Ic(R^*) \subset \frac{1}{f} \cdot R^*$.*

Proof. By (3.3) there is no harm in assuming $I \cap K[X_1, \ldots, X_r] = 0$ and R integral over its subring $K[X_1, \ldots, X_r]$. Put $A = K[X_1, \ldots, X_r]$, $A^* = K[X_1, \ldots, X_r]_{\text{int}}$, and let $x = (X_1, \ldots, X_r, x_{r+1}, \ldots, x_n)$ be the image of X in $R = K[X]/I$. So $R = A[x_{r+1}, \ldots, x_n]$, but also $R^* = A^*[x_{r+1}, \ldots, x_n]$ because of (3.6) and the integrality of $x_{r+1}, \ldots, x_n$ over A^*. Note that A^* is internally integrally closed. Let $\alpha \in Fr(R^*)$; α is of finite internal degree over $Fr(A^*)$ since $x_{r+1}, \ldots, x_n$ are. Hence its internal minimal polynomial $p(T) \in Fr(A^*)[T]_{\text{int}}$ is of finite degree and must equal its minimal polynomial over $Fr(A^*)$. Therefore we have the equivalences: α is internally integral over $R^* \Leftrightarrow \alpha$ is internally integral over $A^* \Leftrightarrow \alpha$ is integral over $A^* \Leftrightarrow \alpha$ is integral over R^*. This shows that $Ic(R^*) = Ic_{\text{int}}(R^*)$.

We first prove the existence of $f \in R \setminus \{0\}$ with $Ic(R^*) \subset \frac{1}{f} \cdot R^*$ for $Fr(R)|K$ separable. In that case we may assume by (3.3) that $X_1, \ldots, X_r$ is a separating transcendence base of $Fr(R)|K$. So we can take $\alpha \in R$ such that $Fr(R) = Fr(A)(\alpha)$, so $Fr(R^*) = Fr(A^*)(\alpha)$. Let $p(T)$ be the minimal polynomial of α over $Fr(A)$; so $p(T) \in A[T]$ and $p(T)$ is also the minimal polynomial of α over $Fr(A^*)$. Hence, if we put $f = \operatorname{disc}(p(T))$, we have $f \in A \setminus \{0\}$ and $Ic(R^*) \subset \frac{1}{f} \cdot A^*[\alpha] \subset \frac{1}{f} \cdot R^*$.

Let us now treat the case that $K(x)|K$ is not separable. Since the field extension $K^{p^{-\infty}}(x)|K^{p^{-\infty}}$ is separable we can take a finite extension $L \subset K^{p^{-\infty}}$ of K such that $L(x)|L$ is separable. Changing variables over K if necessary (Lemma (3.3)), we may also assume that $x = (X_1, \ldots, X_r, x_{r+1}, \ldots, x_n)$ is such that $(X_1, \ldots, X_r)$ is a separating transcendence base of $L(x)|L$ and $R_{(L)} = L[x]$ is integral over $L[X_1, \ldots, X_r]$. Let $R^*_{(L)} = L[x]_{\text{int}} = L[X_1, \ldots, X_r]_{\text{int}}[x_{r+1}, \ldots, x_n]$. The previous argument gives an $f \in R_{(L)}$, $f \neq 0$, such that $Ic(R^*_{(L)}) \subset \frac{1}{f} R^*_{(L)}$. Now some p^m-th power of f belongs to R, so we may as well take $f \in R$. Since $Ic(R^*) \subset Ic(R^*_{(L)})$ it suffices to show that for some $g \in R$

$$(*) \qquad \frac{1}{f} R^*_{(L)} \cap Fr(R^*) \subset \frac{1}{g} R^*.$$

Since L is obtained from K by successive adjunctions of p-th roots one can reduce to the case $L = K(a^{1/p})$. If $a^{1/p} \in K(x)$ then $R^*_{(L)} \subset \frac{1}{h} R^*$ with some $h \in R$, so take $g = f \cdot h$ in $(*)$. If $a^{1/p} \notin K(x)$ then $a^{1/p} \notin Fr(R^*)$, since $Fr(R^*)|K(x)$ is regular, and $(*)$ follows with $g = f$ because $1, a^{1/p}, \ldots, (a^{1/p})^{p-1}$ is a basis of the free R^*-module $R^*_{(L)}$.

(3.10) The final theorem below is the nonstandard version of (3.1). We continue to use the notations from the previous lemmas. Moreover, we put $P = \{\underline{p} \in \operatorname{Spec}(R) \mid ht(\underline{p}) = 1\}$, and we shall use the fact that if I is prime and $R = K[X]/I$ is integrally closed then $R_{\underline{p}}$ is a discrete valuation ring for each $\underline{p} \in P$ and $R = \bigcap \{R_{\underline{p}} \mid \underline{p} \in P\}$; we write $v_{\underline{p}}$ for the discrete valuation on $Fr(R)$ induced by $R_{\underline{p}}$ for $\underline{p} \in P$.

(3.11) **Theorem.** *Suppose I is prime and R is integrally closed. Then R^* is integrally closed.*

Proof. For each $\underline{p}\in P$ the ring $R^*_{\underline{p}^*}$ is a valuation ring (Lemma (3.7)), so $Ic(R^*)\subset R^*_{\underline{p}^*}$. Since also $Ic(R^*)\subset \frac{1}{f}\cdot R^*$, $f\in R\setminus\{0\}$ (Lemma (3.9)), it suffices to prove that

$$\bigcap\{R^*_{\underline{p}^*}|\underline{p}\in P\}\cap\frac{1}{f}\cdot R^*\subset R^*.$$

So let $z\in\bigcap\{R^*_{\underline{p}^*}|\underline{p}\in P\}$, $z=c/f$, $c\in R^*$. We'll show $z\in R^*$. Let $\underline{p}_1,\dots,\underline{p}_m$ be the minimal primes of fR, and put $\underline{q}_i=fR_{\underline{p}_i}\cap R$, so $\underline{q}_i$ is a $\underline{p}_i$-primary ideal of R. Since $z\in R^*_{\underline{p}_i^*}$ we can write $c/f=c_i/f_i$, $c_i\in R^*$, $f_i\in R^*\setminus\underline{p}_i^*$. From $cf_i=c_i f\in\underline{q}_i R^*$, Lemma (3.5)(iii) and the fact that R^* is internally noetherian, we get $c\in\underline{q}_i R^*$, so $c\in\underline{q}_1 R^*\cap\dots\cap\underline{q}_m R^*=(\underline{q}_1\cap\dots\cap\underline{q}_m)R^*$ (Lemma (3.5)(i)), so $c=\sum_\lambda a_\lambda b_\lambda$, $a_\lambda\in\underline{q}_1\cap\dots\cap\underline{q}_m$, $b_\lambda\in R^*$.

Now we have $v_{\underline{p}}(a_\lambda/f)\geqq 0$ for each $\underline{p}\in P$: if $\underline{p}=\underline{p}_i$ this follows from $a_\lambda\in\underline{q}_i\subset fR_{\underline{p}_i}$, otherwise we have $v_{\underline{p}}(f)=0$ and $v_{\underline{p}}(a_\lambda)\geqq 0$. So $a_\lambda/f\in\bigcap R_{\underline{p}}=R$, hence $z=\sum_\lambda\left(\frac{a_\lambda}{f}\right)\cdot b_\lambda\in R^*$. □

(3.12) The usual argument, see (1.6), gives (3.1) as a consequence of (3.11) and the first part of (3.9). Of course one also must use here the standard result that $Ic(R)$ is a finitely generated R-module, for $R=K[X]/I$, I prime, K a field.

§ 4. A counterexample

(4.1) The 'counterexample' promised at the end of the introduction stems from the fact that an elliptic curve over an algebraically closed field has torsion points of arbitrarily high order.

(4.2) **Proposition.** *There is an internal field K and a polynomial*

$$f(X_1,X_2)=X_2^2-(X_1^4+aX_1^2+bX_1+c)\in K[X_1,X_2]$$

such that $K[X_1,X_2]/(f)$ has only trivial units, i.e., units in $K^\bullet$, but $K[X_1,X_2]_{\mathrm{int}}/fK[X_1,X_2]_{\mathrm{int}}$ has a nontrivial unit. Moreover, K can be chosen algebraically closed and $X_1^4+aX_1^2+bX_1+c$ without multiple zeros.

Proof. Consider a smooth projective irreducible curve $\mathscr{C}$ of genus 1 over an algebraically closed field k of characteristic $\neq 2$. Given any two distinct points P,Q on $\mathscr{C}$ it is a routine matter to select an embedding $\mathscr{C}\hookrightarrow\mathbb{P}_2$ such that the affine part $\mathscr{C}\cap\mathbb{A}_2$ is given by an equation $X_2^2=X_1^4+aX_1^2+bX_1+c$ ($a,b,c\in k$, $X_1^4+aX_1^2+bX_1+c$ without multiple zero), while P and Q are the only two points at infinity, i.e., $\mathscr{C}\setminus\mathbb{A}_2=\{P,Q\}$. Let $k[x_1,x_2]=k[X_1,X_2]/(f)$ be the ring of regular functions on $\mathscr{C}\cap\mathbb{A}_2$, where $f=X_2^2-(X_1^4+aX_1^2+bX_1+c)$. A nontrivial unit u of $k[x_1,x_2]$ is the same as an element of $k(\mathscr{C})=k(x_1,x_2)$ whose divisor is of the form $(u)=nP-nQ=n(P-Q)$ for some $n\in\mathbb{Z}\setminus\{0\}$. Because of the isomorphism $\mathscr{C}\simeq$Divisors of degree 0/Principal divisors, of abelian groups, we see that a nontrivial unit in $k[x_1,x_2]$ exists if and only if $P-Q$ is a torsion point of $\mathscr{C}$. If we pass to an enlargement (of everything in sight), k is enlarged to an internal field K, the algebraic curve $\mathscr{C}$ is enlarged to a smooth, projective, irreducible curve $\mathscr{C}^*$ of genus 1 over K, and we can choose distinct points $P, Q \in \mathscr{C}^*$ such that $P-Q \in \mathscr{C}^*$ is an internal torsion point of infinite internal order. This means that if we select an embedding $\mathscr{C}^* \hookrightarrow \mathbb{P}_2(K)$ and an affine equation $f=0$ ($f=X_2^2-(X_1^4+aX_1^2+bX_1+c)$, $a,b,c\in K$) as before, then $K[X_1, X_2]_{\text{int}}/fK[X_1, X_2]_{\text{int}}$, the ring of internally regular functions on $\mathscr{C}^* \cap \mathbb{A}_2(K)$, has a nontrivial unit. On the other hand, since $P-Q$ is not a torsion point of $\mathscr{C}^*$, the ring $K[X_1, X_2]/(f)$ has only trivial units. $\square$

(4.3) The proposition, or rather its proof, has the following standard consequence. Let k be an algebraically closed field, $\operatorname{char}(k) \neq 2$. Then there is no fixed bound $U \in \mathbb{N}$ such that, whenever $k[X_1, X_2]/(f)$ has a nontrivial unit, $f = X_2^2-(X_1^4+aX_1^2+bX_1+c)$, $a,b,c \in k$, then it has a nontrivial unit $g+(f)$ where $g \in k[X_1, X_2]$ has degree $\leqq U$.

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