GLOBALLY BOUNDED SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract

The aim of this paper is to give an account of some results and conjectures involving "for almost all p" properties of power series. Our main concern is to exhibit links between three topics : automaticity, algebraicity (mod n) and D-finiteness. Diagonals of rational fractions seem to be at the heart of the problem. In the last part, we show they appear as (regular) solutions near singularity of Picard-Fuchs differential equations.

1 Definitions.

In this paper *function* will mean power series in $\mathbb{Q}[[\lambda]]$.[1]

Special functions.

- The function f will be said to be *rational* if there exists some non-zero polynomial P in $\mathbb{Q}[\lambda]$ such that Pf belongs to $\mathbb{Q}[\lambda]$.
- The function f will be said to be *algebraic* if there exists some non-zero polynomial P in $\mathbb{Q}(\lambda)[y]$ such that $P(f) = 0$.
- The function f will be said to be *D-finite* if there exists a non-zero (differential) polynomial L in $\mathbb{Q}(\lambda)[d/d\lambda]$ such that $L(f) = 0$.

Globally bounded function. Let $f = \Sigma f_n \lambda^n$ be a function. f will be said to be *globally bounded* if :

b1) for all places v of $\mathbb{Q}$ one has : $r_v(f) = \limsup |f_n|_v^{1/n} < \infty$,

b2) for almost all places v of $\mathbb{Q}$ one has : $|f_n|_v \leq 1$ for all n.

or, equivalently, if :

[1] We use $\mathbb{Q}$ as base field but, at the cost of some technical complications, everything can be generalized to number fields.

b1') f has a non-zero radius of convergence in $\mathbf{C}$,

b2') there exist α, β in $\mathbf{Q}$ such that $\alpha f(\beta\lambda)$ belongs to $\mathbf{Z}[[\lambda]]$.

Remark : Let f be a globally bounded function. For almost all primes p its coefficients f_n belong to $\mathbf{Q} \cap \mathbf{Z}_p$. Thus, there exists an integer N such that, for all h prime to N, it makes sense to reduce f modulo h.

Automatic sequences.

- A sequence $\{f_n\}$ is said to be *ultimately periodic* if there exist integers ℓ and m such that : $f_{n+\ell} = f_n$ for all integers $n \geq m$.

- A subset I of $\mathbf{N}$ is said to be *h-recognizable* if there exists some finite automaton, which reads the alphabet $\{0, 1, ..., h-1\}$, and which recognizes the word '$00...0a_s...a_0$' if and only if $\sum a_i h^i$ belongs to I.

- Let E be any finite set. A sequence $\{f_n\}$ of $E^{\mathbf{Z}}$ will be called *h-automatic* if, for all i in E, the set $\{n \in \mathbf{N} ; f_n = i\}$ is h-recognizable.

Automatic functions.

- A globally bounded function $f = \sum f_n \lambda^n$ will be said to be *totally automatic* if there exists an integer N such that, for all h prime to N, the sequence $\{f_n \bmod h\}$ of $(\mathbf{Z}/h\mathbf{Z})^{\mathbf{N}}$ is h-automatic.

- A globally bounded function $f = \sum f_n \lambda^n$ will be said to be *globally automatic* if, for almost all primes p and for all integers h, the sequence $\{f_n \bmod p^h\}$ of $(\mathbf{Z}/p^h\mathbf{Z})^{\mathbf{N}}$ is p-automatic (or equivalently p^h-automatic).

2 Rational functions.

The following proposition states obvious properties of rational functions.

PROPOSITION 1 : *Every rational function :*
a) *is D-finite,*
b) *is globally bounded,*
c) *is totally automatic.*

a) and b) are quite obvious. Let $f = \sum f_n \lambda^n$ be a rational function. There exist integers $d, a_0, \ldots, a_d$ such that, for $n > d$:

$$a_0 f_n = a_1 f_{n-1} + \cdots + a_d f_{n-d}$$

Let h be an integer prime to a_0. As the vector $v_n = \{f_{n-1}, \dots, f_{n-d}\}$ can take only h^d distinct values modulo h, there exist integers ℓ and m such that $v_{m+\ell} \equiv v_m \pmod{h}$. Thus the sequence $\{f_n \bmod h\}$ is ultimately periodic hence h-automatic. □

The converse of this proposition is not known. It is clearly false if we omit hypothesis c). In fact the rest of this paper will be devoted to study functions satisfying hypothesis a) and b). So let us begin by examining the condition c) more closely.

PROPOSITION 2 : *The globally bounded function $f = \sum f_n \lambda^n$ is totally automatic if and only if there exists an integer N such that, for all integers h prime to N, the sequence $\{f_n \bmod h\}$ is ultimately periodic.*

Only the "only if" condition is not completely obvious. Let h be an integer such that the function $f \bmod h$ is h-automatic. We can choose a prime p, not dividing h, such that the function $f \bmod ph$ is ph-automatic. Then, for all i in $\mathbf{Z}/ph\mathbf{Z}$, the set :

$$I_i = \{n \in \mathbf{N}\,; f_n \equiv i \pmod{h}\} = \bigcup_{k=0}^{\infty} \{n \in \mathbf{N}\,; f_n \equiv i + kp \pmod{ph}\}$$

is both h-recognizable and ph-recognizable. As h and ph are multiplicatively independent, we know from Cobham's theorem [9] that the sets I_i are finite unions of finite sets and of arithmetical progressions. □

This result can be expressed by means of the notion of analytic element. Let p be a prime. The function f will be said to be *p-bounded* if its *p-Gauss norm* :

$$\|f\|_p = \sup |f_n|_p$$

exists. It will be said to be a *p-analytic element* if it is the limit, for the p-Gauss norm, of a sequence of p-bounded rational functions (i.e. rational functions without pole in the disk $|\lambda| < 1$ of $\mathbf{C}_p$). It is well known that the p-analytic elements are exactly the p-bounded functions for which the sequence $\{f_n \bmod p^h\}$ is ultimately periodic for all h. Then the following result is a straightforward consequence of Chinese remainder theorem.

COROLLARY 3 : *The globally bounded function $f = \sum f_n \lambda^n$ is totally automatic if and only if it is a p-analytic element for almost all primes p.*□

If we omit hypothesis b), the converse of Proposition 1 is easily seen to be false. For instance the function defined by :

$$f_n = n!$$

satisfies conditions a), b2') and c) (hence it is a p-analytic element for almost all primes p) but is not rational because it has zero radius of convergence in $\mathbb{C}$.

Likewise the converse of Proposition 1 is false if we omit hypothesis a). For instance, if $[x]$ denotes the integral part of x, the function defined by :

$$f_n = \prod_{p\,\mathrm{prime}} p^{[\log n/\log p]}$$

satisfies conditions b) (by prime number theorem f has $\frac{1}{e}$ as radius of convergence in $\mathbb{C}$) and c) but is not rational because it is not D-finite. Indeed, suppose f is D-finite. There would then exist polynomials P_i such that, for all $n > 0$,

$$\sum_{i=0}^{\mu} P_i(n) f_{n-i} = 0$$

The set $\{p^h, p \text{ prime}, h \geq 1\}$ being of zero density, for all k one can find an integer n such that there is no p^h between n and $n+k$. Thus one has $f_{n+j} = f_n$ for $0 \leq j \leq k$. As $f_n \neq 0$, the polynomial $\sum_i P_i$ is zero at $n+\mu, \ldots, n+k$. If we choose $k > \mu + \sup deg(P_i)$, one finds that the polynomial $\sum_i P_i$, must be identically zero. This implies that the function $\frac{1}{1-x}$ is solution of the *minimal* differential equation satisfied by f. Hence $f = \frac{\lambda}{1-x}$ which is false.

It then appears that the converse of Proposition 1 is deeply connected with RUSZA's conjecture :

CONJECTURE 1 : *If the function $f = \sum f_n \lambda^n$ satisfies :*

1) *for all integers n and almost all primes p one has $f_{n+p} \equiv f_n \pmod{p}$,*

2) *the radius of convergence of f in $\mathbb{C}$ is larger than $\delta = \frac{1}{e}$,*

then f is rational and, more precisely, belongs to $\mathbb{Z}[\frac{1}{1-\lambda}]$.

[14] contains a short review on the subject and the smallest value for δ under which the conjecture is at the moment known to be true. It is also proved there that, under the hypothesis of the conjecture, the function f is D-finite.

3 Diagonals of rational fractions.

Algebraic functions : The following proposition sums up some classical properties of algebraic functions.

PROPOSITION 4 : *Every algebraic function*
a) *is D-finite,*
b) *is globally bounded,*
c) *is globally automatic.*

Property a) is well known, property b) is Eisenstein's theorem, and property c) is proved in [4].□

These properties do not characterize algebraic functions. In fact there is a wide class of functions which enjoy the same properties and which we now examine.

Diagonals : The following formula defines, for each integer $s \geq 1$, a map Δ from $\mathbb{Q}[[x_1, \ldots, x_s]]$ to $\mathbb{Q}[[\lambda]]$ called *diagonalisation* :

$$\Delta\left(\sum a_{n_1,\ldots,n_s} x_1^{n_1} \cdots x_s^{n_s}\right) = \sum a_{n,\ldots,n} \lambda^n$$

A function f will be said to be the *diagonal of a rational function* if there exist an integer s and two polynomials P and Q in $\mathbb{Q}[x_1, \ldots, x_s]$ such that :

$$\left\{ \begin{array}{l} Q(0, \ldots, 0) \neq 0 \quad {}^{2} \\ f = \Delta(P/Q) \end{array} \right.$$

We gave in [5] a complete account on the subject so we limit ourselves to a brief survey.

PROPOSITION 5 : *Every diagonal of a rational function*
a) *is D-finite,*
b) *is globally bounded,*
c) *is globally automatic.*

There exist elementary proofs of a), for instance in [13], but we can obtain more information if we use the following trick due to Deligne. We observe that the condition $f = \Delta(P/Q)$ is equivalent to the formula :

$$f = \frac{1}{(2i\pi)^{s-1}} \int_C \frac{P}{Q} \frac{dx_1 \cdots dx_{s-1}}{x_1 \cdots x_{s-1}}$$

[2] so that P/Q can be viewed as an element of $\mathbb{Q}[[x_1, \ldots, x_s]]$

where C is the "evanescent" cycle $|x_1| = \cdots = |x_{s-1}| = \varepsilon$ (ε small enough) of the variety :

$$\mathcal{X} = \begin{cases} x_1 \cdots x_s = \lambda \\ Q(x_1, \ldots, x_s) \neq 0 \end{cases}$$

Then a) is a straigthforward consequence of the finite dimensionality of the De Rham cohomology space $H^{s-1}_{DR}(\mathcal{X})$ (endowed with its Gauss-Manin connection).

Global-boundedness of the diagonal of a rational fraction is easy to verify.

The proof of D-finiteness which we just gave implies that any diagonal f of a rational fraction is solution of a Picard-Fuchs differential equation L. Now, it is known that for almost all primes p such an equation, viewed as p-adic, has a strong Frobenius structure (see for instance [11] for a definition). We also know from [6] (Theorem 7.2) that any bounded (in the p-adic unit disk) solution of L, for instance f, is an algebraic element. Hence, from the remark following ([6] Corollary 8.3), the sequence $\{f_n \bmod p^h\}$ is p-automatic and c) is proved.□

PROPOSITION 6 : *The set of diagonals of rational fractions is a subring of* $\mathbb{Q}[[\lambda]]$. *Its units consist exactly of the set of algebraic functions. It is closed under derivation, Hadamard product, Hurwitz product and algebraic change of variable.*

See [5] for the proofs and for the definition of Hadamard and Hurwitz product. By "algebraic change of variable" we mean a transformation $f \to f \circ \vartheta$ where ϑ is an algebraic function in $\lambda\mathbb{Q}[[\lambda]]$. □

The following result of Denef and Lipschitz [10] shows that nothing new can be obtained if we start from algebraic power series instead of rational ones.

PROPOSITION 7 : *Let F be an element of $Q[[x_1, \ldots, x_s]]$ which is algebraic over $\mathbb{Q}(x_1, \ldots, x_s)$. Then $\Delta(F)$ is the diagonal of a rational fraction (in $2s$ variables).* □

Examples.

Apéry's function : The following formula can be easily checked :

$$\begin{aligned} f &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \lambda^n \\ &= \Delta\left(\frac{1}{[1 - x_1x_2x_3x_4][(1-x_3)(1-x_4) - x_0(1+x_1)(1+x_2)]}\right) \end{aligned}$$

and shows Apéry's function f to be the diagonal of a rational function. Hence f is D-finite.

Hypergeometric functions : The hypergeometric functions are defined by :

$$_{h+1}F_h\,(a_0,\ldots,a_h;b_1,\ldots,b_h;\lambda)=\sum_{n=0}^{\infty}\frac{(a_0)_n\cdots(a_h)_n}{n!\,(b_1)_n\cdots(b_h)_n}\,\lambda^n$$

where the a_i's and b_i's belong to $\mathbb{Q}\setminus(-\mathbb{N})$ and $(a)_n=a(a+1)\cdots(a+n-1)$.

All these functions satisfy condition (b1). The globally bounded ones can be characterized using a computation of Dwork: let us suppose for instance that the differences a_i-b_j are not in $\mathbb{Z}$ and let N be a common denominator of the a_i's and of the b_i's. The hypergeometric function $_{h+1}F_h\,(a_0,\ldots,a_h;b_1,\ldots,b_h;\lambda)$ is globally bounded if and only if, for each δ prime to N, when running through the trigonometric circle from 1 to $e^{2i\pi}$ you always have encountered more numbers $e^{2i\pi\delta a_i}$ than numbers $e^{2i\pi\delta b_i}$ (see [7] for the general condition).

Among the globally bounded hypergeometric functions are the algebraic ones : The hypergeometric function $_{h+1}F_h\,(a_0,\ldots,a_h;b_1,\ldots,b_h;\lambda)$ is algebraic if and only if the numbers $e^{2i\pi\delta a_i}$ and $e^{2i\pi\delta b_i}$ which are distinct from 1 are intertwined on the trigonometric circle (this is a particular case of a Grothendieck's conjecture which has been proved by Katz [12] for $h=1$ and by Beukers and Heckman [2] for general h). Those algebraic hypergeometric functions are exactly the globally bounded ones for which none of the b_i's is an integer.

Using the formula :

$$\begin{aligned}&_{h+1}F_h\,(a_0,\ldots,a_h;b_1,\ldots,b_h;\lambda)*{}_{k+1}F_k\,(a'_0,\ldots,a'_k;b'_1,\ldots,b'_k;\lambda)\\ =\ &_{h+k+1}F_{h+k}\,(a_0,\ldots,a_h,a'_0,\ldots,a'_k;b_1,\ldots,b_h,1,b'_1,\ldots,b'_k;\lambda)\end{aligned}$$

and starting from algebraic hypergeometric functions, one can construct many hypergeometric functions which are the diagonals of rational fractions. The most famous example is the following :

$$\begin{aligned}{}_2F_1(\frac{1}{2},\frac{1}{2};1;\lambda) &= {}_1F_0(\frac{1}{2};\lambda)*{}_1F_0(\frac{1}{2};\lambda)\\ &= \frac{1}{\sqrt{1-\lambda}}*\frac{1}{\sqrt{1-\lambda}}\end{aligned}$$

However, not all globally bounded hypergeometric functions can be obtained this way as is shown by :

$${}_3F_2(\frac{1}{9},\frac{4}{9},\frac{5}{9};\frac{1}{3},1;\lambda)$$

It is an open question whether that globally bounded hypergeometric function is the diagonal of a rational fraction.

A function bounded for almost all p which is not the diagonal of a rational function : In [7] we proved that the function f defined by :

$$f(\lambda)=\lambda^2\,{}_3F_2(\frac{1}{3},\frac{1}{3},\frac{2}{3};\frac{5}{3},1;\lambda)$$

enjoys some curious properties :
- For each prime $p \neq 3$ it is a bounded function (i.e. for each prime p there exists a constant δ_p such that $|f_n|_p \leq \delta_p$ for all n). Hence, as explained in the proof of Proposition 5, it is an algebraic element, in other words it is globally automatic in a slightly generalized sense. Namely, for almost all primes p there exists an integer γ_p such that for any integer h, the sequence $\{p^{\gamma_p} f_n \bmod p^h\}$ is p-automatic.
- But it is not globally bounded (because for an infinity of p, $\delta_p > 1$), hence it is not the diagonal of a rational function.
- Its derivative

$$f'(\lambda) = 2\lambda \left[\sqrt[3]{(1-\lambda^3)} * \sqrt[3]{(1-\lambda^3)} \right]$$

is the diagonal of a rational fraction.

4 G-functions.

Let us first recall a theorem of ANDRE [1]. For this we need some new definitions.

Global height and radius of convergence : Let

$$\log^+(x) = \sup(0, \log(x)).$$

A function f is said to be a *G-function* if it has a non zero radius of convergence in $\mathbf{C}$ and if :

$$\sigma(f) = \limsup_{m \to \infty} \frac{1}{m} \sum_{p\,\mathrm{prime}} \max_{k \leq m} \left(\log^+ |f_k|_p\right)$$

is finite (this last condition means that the common denominator of the f_k for $0 \leq k \leq m$ has polynomial growth in m).

For v a place of $\mathbf{Q}$, recall that $r_v(f)$ denotes the radius of convergence of the function f in $\mathbf{C}_v$ (see condition b1). We define the *global radius of convergence* of f to be :

$$\rho(f) = \log^+\left(1/r_\infty(f)\right) + \sum_{p\,\mathrm{prime}} \log^+\left(1/r_p(f)\right)$$

In general there is no relation between the conditions $\rho(f) < \infty$ and $\sigma(f) < \infty$. But one of the deepest consequences of ANDRE's theorem is that these two conditions are connected in the case of D-finite functions.

Differential modules : By differential module $\mathcal{M}$ we mean a finite vector space over $\mathbf{Q}(\lambda)$ [3] endowed with an action of the derivation $\frac{d}{d\lambda}$ satisfying the

[3] More generally a free finite module over a ring closed under $\frac{d}{d\lambda}$.

classical rule :

$$\frac{d}{d\lambda}(f.m) = f'.m + f.\frac{d}{d\lambda}(m)$$

Given a basis of $\mathcal{M}$, the action of $\frac{d}{d\lambda}$ (resp. $\left(\frac{d}{d\lambda}\right)^n$) on this basis is given by a matrix G (resp. G_n). We then define :

$$\begin{aligned}
\sigma(\mathcal{M}) &= \limsup_{m\to\infty}\left(\frac{1}{m}\sum_{p\,\text{prime}}\max_{k\leq m}\left(\log^+\|\frac{G_k}{k!}\|_p\right)\right)\\
\rho(\mathcal{M}) &= \sum_{p\,\text{prime}}\limsup_{m\to\infty}\left(\frac{1}{m}\max_{k\leq m}\left(\log^+\|\frac{G_k}{k!}\|_p\right)\right)\\
&= \sum_{p\,\text{prime}}\log^+(1/\rho_p(\mathcal{M}))
\end{aligned}$$

where $\|.\|_p$ denote the p-Gauss norm (extended from polynomials to rational functions by multiplicativity and to matrices by taking the supremum on coefficients) and $\rho_p(\mathcal{M})$ the radius of convergence of solutions of $\mathcal{M}$ in the "generic disk". It is not very difficult to check that those definitions are independent of the chosen basis.

When $\sigma(\mathcal{M}) < \infty$ we will say that $\mathcal{M}$ satisfies the GALOČKIN condition and, when $\rho(\mathcal{M}) < \infty$, following BOMBIERI [3] we will say that $\mathcal{M}$ is of arithmetic type.

By a *solution of the differential module* $\mathcal{M}$ we will mean a $\mathbb{Q}(\lambda)$-linear mapping from $\mathcal{M}$ to some $\mathbb{Q}(\lambda)[\frac{d}{d\lambda}]$-module which commutes with the derivation $\frac{d}{d\lambda}$.

THEOREM 8 *([1] VI) : Let $\mathcal{M}$ be a differential module and let s be an* injective *solution from $\mathcal{M}$ to $\mathbb{Q}((\lambda))$ then the following conditions :*

i) *there exists m in $\mathcal{M}$ such that the $(\frac{d}{d\lambda})^i(m)$ generate $\mathcal{M}$ and such that $s(m)$ is a G-function,*

ii) *for all m in $\mathcal{M}$, $s(m)$ is a G-function,*

iii) *the differential module $\mathcal{M}$ satisfies the GALOČKIN condition,*

iv) *the differential module $\mathcal{M}$ is of arithmetic type,*

are equivalent and imply :

v) *for all m in $\mathcal{M}$, $\rho(s(m)) < \infty$*

i) $\Rightarrow$ ii) is easy and not deep.
ii) $\Rightarrow$ iii) is a beautiful theorem of CHUDNOVSKY [8]. It is, at the moment, the only significant known result connecting a property of a single solution with a property of the differential module.
iii) $\Leftrightarrow$ iv) is obtained by proving that the sequence

$$\frac{1}{m}\max_{k\leq m}\left(\log^+\|\frac{G_k}{k!}\|_p\right)$$

is convergent. More precisely it can be showed that

$$\rho(\mathcal{M}) \leq \sigma(\mathcal{M}) \leq \rho(\mathcal{M}) + \mu - 1$$

where μ is the dimension of $\mathcal{M}$.
iv) $\Rightarrow$ i) is obtained in two steps. It is known from BOMBIERI [3], that iv) implies the point 0 to be regular singular for $\mathcal{M}$. Hence by choosing "a good basis", one can suppose the "matrix solution near zero" to be $Y\,\lambda^C$ where the matrix Y belongs to $Gl_\mu(\mathbb{Q}((\lambda)))$ and the matrix C has rational entries and rational eigenvalues. The first and easiest step of the proof connects $\rho(\mathcal{M})$ and a similar quantity $\rho(Y)$, associated with Y. Then v) is a trivial consequence of this first point. The second step allows to go from $\rho(Y)$ to $\sigma(Y)$. It uses twice the construction of a "Frobenius antecedent" for $\mathcal{M}$.□

D-finite globally bounded functions : Let f be a D-finite globally bounded function, then it is clearly a G-function (for almost all p one has $\log^+ |f_k|_p = 0$ for all k). The (finite) $\mathbb{Q}(\lambda)$-vector space generated by f and its derivatives is a differential module $\mathcal{M}(f)$ contained in $\mathbb{Q}((\lambda))$ and the identity is an (injective !) solution from $\mathcal{M}(f)$ to $\mathbb{Q}((\lambda))$. Hence André's theorem says that $\mathcal{M}(f)$ is of arithmetic type. Thus we can "use" the following classical conjectures :

CONJECTURE 2 [BOMBIERI] *: If a differential module $\mathcal{M}$ is of arithmetic type then one has $\rho_p(\mathcal{M}) \geq 1$ for almost all p.*

CONJECTURE 3 [DWORK] *: If a differential module $\mathcal{M}$ of arithmetic type is such that $\rho_p(\mathcal{M}) \geq 1$ for almost all p then it "comes from geometry" (i.e. it can be built by mean of algebraic constructions, namely $\oplus$, $\otimes$, $\wedge$ and extensions, from cohomology spaces of algebraic varieties endowed with their Gauss-Manin connection).*

COROLLARY 9 [of preceding conjectures] *: Any D-finite globally bounded function is globally automatic.*

Let f be a D-finite globally bounded function. From the two conjectures one would obtain that f is solution of a differential equation obtained by algebraic constructions from Picard-Fuchs differential equations. The existence of a strong Frobenius structure being stable under those constructions the argument already used in Proposition 5 gives the proof.□

Remark : In fact we believe that the corollary is easier than the conjectures and should be a first step toward their proof. As $\mathcal{M}(f)$ is of arithmetic type, it is known that for a set of prime numbers of density 1 the differential module $\mathcal{M}(f)$ has a Frobenius-antecedent $\mathcal{M}(f)^\psi$. The corollary amounts to asserting that, by enlarging $\mathcal{M}(f)$ if necessary, one has $\mathcal{M}(f) = \mathcal{M}(f)^\psi$.

The analogy between Corollary 9 and Proposition 5 leads to the following conjecture :

CONJECTURE 4 : *Every D-finite globally bounded function is the diagonal of a rational function.*

Since the conjectures of Bombieri-Dwork state that the situation should occur only in the geometric case, we will examine this case more carefully.

5 The geometric situation.

Let $\overline{\mathbf{Q}}$ denote the algebraic closure of $\mathbf{Q}$ and let :

$$\iota : Z \to \mathbf{P}^1 \; ^4$$

be a quasi-projective morphism defined over $\overline{\mathbf{Q}}$, of relative dimension r, smooth over the open set $S^* = \mathbf{P}^1 \setminus \Lambda$ for a finite set Λ containing 0. We are concerned with the differential module :

$$H^r_{DR}(Z/\mathbf{P}) = \mathbf{R}^r \iota_*(\Omega^\bullet_{Z/S^*}) \otimes \overline{\mathbf{Q}}(\lambda)$$

on which the derivation $\frac{d}{d\lambda}$ acts by means of the Gauss-Manin connection. Using the existence of a lifting of Frobenius for almost all p (namely those with "good reduction"), it is easy to put a strong Frobenius structure on the differential module $H^r_{DR}(Z/\mathbf{P})$. Hence one has :

$$\rho_p(H^r_{DR}(Z/\mathbf{P})) = 1$$

for almost all p. As $\rho_p(H^r_{DR}(Z/\mathbf{P}))$ is never 0 , the module $H^r_{DR}(Z/\mathbf{P})$ is of arithmetic type. From this one can deduce :

PROPOSITION 10 [KATZ] : *The differential module $H^r_{DR}(Z/\mathbf{P})$ has at most regular singularities with rational exponents.*□

The classical theory of regular singularities allows one to construct a basis $\{e_i\}$ $(1 \le i \le \mu)$ of $H^r_{DR}(Z/\mathbf{P})$ such that :

E1) There exists a matrix G with entries in $\overline{\mathbf{Q}}[\lambda]$ such that :

$$\lambda \frac{d}{d\lambda} e_i = \sum_j G_{ij} e_j$$

[4] The whole set up can easily be generalized with a projective curve $\mathcal{C}$ instead of $\mathbf{P}^1$

E2) There exist a matrix Y in $Gl_\mu(\overline{\mathbf{Q}}[[\lambda]])$ and a matrix C with entries in $\overline{\mathbf{Q}}$ and *eigenvalues* in $\mathbf{Q}$ such that :

$$\lambda \frac{d}{d\lambda}\left(Y\,\lambda^C\right) = G\,Y\,\lambda^C$$ [5]

E3) Moreover, if we denote by α_i the eigenvalues of C, then one can impose $0 \leq \alpha_i < 1$ and $Y(0) = I$.

With the change of variable $\lambda = \xi^n$, the eigenvalues of C are multiplied by n. Thus, choosing n to be the common denominator of the eigenvalues of C, using the property 3), and possibly after a new change of basis, one can assume the matrix C to be nilpotent.

The geometric point of view : Let $S = S^* \cup \{0\}$. By using Hironaka's theorem, one sees that there exist, after enlarging Λ if necessary, a quasi-projective variety X, a proper mapping

$$\pi : X \rightarrow S$$

and a divisor $D = \bigcup D_i \subset X$ such that, if $Y = \pi^{-1}(0)$, one has :

1) $\mathcal{D} = Y \cup D$ is a divisor with normal crossing,
2) the map π and its restrictions to each intersection $D_{i_1} \cap \cdots \cap D_{i_\nu}$ are smooth over S^* and flat over S,
3) $Z|_{S^*} \simeq X - \mathcal{D}$.

Since we are looking at the situation near 0 the first step is to extend the (locally free over S^*) sheaf $\mathbf{R}^r \iota_* \Omega^\bullet_{Z/S^*}$.

Let $\Omega^\bullet_X < \mathcal{D} >$ (resp. $\Omega^\bullet_{X/S} < \mathcal{D} >$) denote the complex of differentials (resp. relative differentials) on X with at worst logarithmic poles along $\mathcal{D}$. Then Katz proved that the sheaf

$$\mathbf{R}^r \pi_* \Omega^\bullet_{X/S} < \mathcal{D} >$$

is locally free over S. In particular one has :

$$\mathbf{R}^r \pi_* \Omega^\bullet_{X/S} < \mathcal{D} > \otimes \overline{\mathbf{Q}}(\lambda) = H^r_{DR}(Z/\mathbf{P})$$

Moreover, if $\{e_i\}$ is a basis of the stalk $(\mathbf{R}^r \pi_* \Omega^\bullet_{X/S} < \mathcal{D} >)_0$ (over $\overline{\mathbf{Q}}(\lambda) \cap \overline{\mathbf{Q}}[[\lambda]]$) then the above conditions E1), E2) and E3) are satisfied.

Denote by $e_i(0)$ the image of e_i under the isomorphism :

$$\mathbf{R}^r \pi_* \Omega^\bullet_{X/S} < \mathcal{D} > \otimes (\mathcal{O}_S/\mathcal{M}_{S,0}) \cong \mathbf{H}^r(Y, \Omega^\bullet_{X/S} < \mathcal{D} > \otimes \mathcal{O}_Y)$$

[5] In this formula, λ^C is to be interpreted formally

and denote by $\mathrm{Res}_0(\nabla)$ the endomorphism of $\mathbf{H}^r(Y, \Omega^\bullet_{X/S} < \mathcal{D} > \otimes \mathcal{O}_Y)$ induced by the action of $\lambda\frac{d}{d\lambda} \otimes \mathrm{Id}$. One computes easily :

$$\mathrm{Res}_0(\nabla)\,(e_i(0)) = \sum_j C_{ij} e_j(0)$$

(in other words, the matrix of $\mathrm{Res}_0(\nabla)$ is ${}^tC$).

Now Mumford's semi-stable reduction theorem asserts that, after taking a finite covering of $S^* \cup \{0\}$ (namely $\lambda = \xi^n$), one can arrange for Y to be reduced. In this situation, $\mathrm{Res}_0(\nabla)$ and hence C are nilpotent. From now on we will assume that this reduction has been performed.

The monodromy filtration : As the matrix C is nilpotent, the matrix ξ^C is a direct sum of blocks which look like :

$$\begin{pmatrix} 1 & \log\xi & \frac{\log^2\xi}{2!} & \cdots & \frac{\log^\nu\xi}{\nu!} \\ 0 & 1 & \log\xi & \cdots & \frac{\log^{\nu-1}\xi}{(\nu-1)!} \\ & & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & \log\xi \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$$

and the columns of the matrix $Y\,\xi^C$ are clustered in a set of sequences which looks like :

$$f_0\ ,\quad f_0 \log\xi + f_1\ ,\ \ldots\ ,\quad f_0 \frac{\log^\nu \xi}{\nu!} + \cdots + f_{\nu-1}\log\xi + f_\nu$$

where the f_i's are (non zero) vectors in $\overline{\mathbf{Q}}[[\xi]]^\mu$.

The formula :

$$s_j(e_i) = \left(Y\,\xi^C\right)_{ij}$$

defines $\overline{\mathbf{Q}}(\xi)$-linear mappings s_i which make up a basis of the $\overline{\mathbf{Q}}$-space $\mathcal{S}$ of *solutions near zero* (i.e. solutions in the ring $\overline{\mathbf{Q}}((\xi))[\log\xi]$) for the module $H^r_{DR}(Z/\mathbf{P})$.

Now on the ring $\mathbf{C}((\xi))[\log\xi]$, there is a monodromy automorphism "turning once counterclockwise around 0" defined by :

$$T\,:\ \log\xi \to \log\xi + 2i\pi$$

and one can easily check that :

$$T = \exp(2i\pi N)$$

where N is the $\overline{\mathbf{Q}}((\xi))$-linear mapping defined by :

$$\begin{aligned} N(1) &= 0 \\ N(\frac{\log^{\nu}\xi}{\nu!}) &= \frac{\log^{\nu-1}\xi}{(\nu-1)!} \text{ for } \nu \geq 1 \end{aligned}$$

The monodromy automorphism T commutes with differentiation, hence acts on $\mathcal{S}\otimes\mathbf{C}$ by restriction. On this finite dimensional space, T is clearly unipotent. Then $N = \frac{1}{2i\pi}\log T$ acts on $\mathcal{S}\otimes\mathbf{C}$. Actually N acts on $\mathcal{S}$ and is represented, in the basis $\{s_j\}$, by the matrix C. Moreover N is clearly nilpotent and more precisely one has $N^{r+1}=0$.

As $\mathcal{S}$ is finite-dimensional, there is a unique increasing filtration of $\mathcal{S}$ by $\overline{\mathbf{Q}}$-spaces :

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_{2r} = \mathcal{S}$$

such that :

1. $N(M_k) \subset M_{k-2}$
2. $N(M_k) = (\mathrm{Im} N) \cap M_{k-2}$
3. N^k is an isomorphism between $Gr^M_{r+k}(\mathcal{S})$ and $Gr^M_{r-k}(\mathcal{S})$

It is called the *monodromy filtration*. For instance, if $r=2$, the following picture shows where solutions lie in accordance with the length $\nu = 0,1$ or 2 of their "block" :

$$\begin{array}{llll} M_4 & f_0\frac{\log^2\xi}{2!}+\cdots & & \\ M_3 & & g_0\log\xi+\cdots & \\ M_2 & f_0\log\xi+\cdots & & h_0 \\ M_1 & & g_0 & \\ M_0 & f_0 & & \end{array}$$

On the other hand, as $\mathrm{Res}_0(\nabla)$ is also a nilpotent endomorphism it defines along the same lines a monodromy filtration on $\mathbf{H}^r(Y, \Omega^{\bullet}_{X/S}\langle\mathcal{D}\rangle \otimes \mathcal{O}_Y)$.

Define a pairing between $\mathbf{H}^r(Y, \Omega^{\bullet}_{X/S}\langle\mathcal{D}\rangle \otimes \mathcal{O}_Y)$ and $\mathcal{S}$ by the formula :

$$< s_j, e_i(0) >= \delta_{ij}$$

so that, for any ω in $(\mathbf{R}^r\pi_*\,\Omega^{\bullet}_{X/S}\langle\mathcal{D}\rangle)_0$ and s in $\ker N$ one has

$$< s, \omega(0) >= s(\omega)(0)$$

and, for any s in $\mathcal{S}$ and ω in $\mathbf{H}^r(Y, \Omega^{\bullet}_{X/S}\langle\mathcal{D}\rangle \otimes\mathcal{O}_Y))$ one has :

$$< N(s), \omega >=< s, \mathrm{Res}_0(\nabla)\,(\omega) >$$

Finally we get :

$$M_k\mathbf{H}^r(Y, \Omega^{\bullet}_{X/S}\langle\mathcal{D}\rangle \otimes\mathcal{O}_Y) = \mathrm{Ann}\, M_{2r-k-1}\mathcal{S}$$

Construction of solutions : Our aim is to construct solutions of $\mathcal{S}$ that can be continued to 0 (i.e. containing no logarithmic term). Such solutions will lie in the kernel of N.

Denote by $\widetilde{\mathcal{D}}^{(k)}$ the disjoint union of all intersections of k irreducible components of $\mathcal{D}$. Let P be a point of $\widetilde{\mathcal{D}}^{(r+1)}$ (i.e. a point of X which lies at the intersection of $r+1$ irreducible components of the divisor $\mathcal{D}$) and let a be the embedding $\widetilde{\mathcal{D}}^{(r+1)} \to X$. From the map "taking the Poincaré residue in P" :

$$R_P : \Omega_X^{r+1} < \mathcal{D} > \to a_* \overline{\mathbf{Q}}$$

which is zero on exact forms, and the map :

$$\wedge \frac{d}{d\xi} : \Omega^r_{X/S} < \mathcal{D} > \to \Omega_X^{r+1} < \mathcal{D} >$$

we get a map of complexes :

$$R_P \circ \wedge \frac{d}{d\xi} : \Omega^\bullet_{X/S} < \mathcal{D} > \to a_* \overline{\mathbf{Q}}[-r]$$

which induces a map :

$$\mathrm{res}_P : \mathbf{R}^r \pi_* \Omega^\bullet_{X/S} < \mathcal{D} > \to (\pi \circ a)_* \overline{\mathbf{Q}}$$

PROPOSITION 11 *: Let P be a point of $\widetilde{\mathcal{D}}^{(r+1)}$. There exists a solution δ_P of the differential module $(\mathbf{R}^r \pi_* \Omega^\bullet_{X/S} < \mathcal{D} >)_0$ in the set of diagonals of rational fractions whose value in 0 is* res_P.

By assumption, there is an affine étale neighborhood U of P on which π is defined by :

$$x_0 x_1 \cdots x_s = \xi$$

and $U \cap D$ by :

$$x_{s+1} \cdots x_r = 0$$

(our assumptions imply that P belongs to Y hence $s \geq 0$).

The inclusion $U \to X$ gives a morphism of $\mathcal{O}_S$-modules with connection :

$$\theta : \mathbf{R}^r \pi_* \Omega^\bullet_{X/S} < \mathcal{D} > \to \mathbf{R}^r \pi_* \Omega^\bullet_{U/S} < \mathcal{D} >$$

Therefore it will be enough to construct a solution of the differential module $(\mathbf{R}^r \pi_* \Omega^\bullet_{U/S} < \mathcal{D} >)_0$ with the right properties.

As U is affine, one has :

$$\mathbf{R}^r \pi_* \Omega^\bullet_{U/S} < \mathcal{D} > = \pi_* \left(\Omega^r_{U/S} < \mathcal{D} > / d(\Omega^{r-1}_{U/S} < \mathcal{D} >) \right)$$

Let now ω be an element in the stalk $\left(\Omega^r_{U/S}{<}\mathcal{D}{>}\right)_P$. One has :

$$\omega = f(x_0,\ldots,x_r)\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_r}{x_r}$$

where the function f belongs to $(\mathcal{O}_U)_P$. In other words, f can be expanded in $\overline{\mathbf{Q}}[[x_0,\ldots,x_r]]$ and is algebraic over $\overline{\mathbf{Q}}(x_0,\ldots,x_r)$. From Proposition 7 we learn that the function :

$$\delta(\omega) = \Delta\left(f(x_0,\ldots,x_s,0,\ldots,0)\right)$$

is the diagonal of a rational function. This map δ will give us the expected solution.

First the map δ is actually defined on $(\mathbf{R}^r\pi_*\Omega^\bullet_{U/S}{<}\mathcal{D}{>})_0$ because $\delta(\omega)=0$ as soon as ω lies in $(d\,\Omega^{r-1}_{U/S}{<}\mathcal{D}{>})_P$. Indeed in $\Omega^\bullet_{U/S}{<}\mathcal{D}{>}$ one gets :

$$\frac{dx_0}{x_0} = -\sum_{i=1}^{s}\frac{dx_i}{x_i}\ ;\ \bigwedge_{i=0}^{s} dx_i = 0$$

and we readily compute that :

$$\delta\left(d(f\bigwedge_{i\neq k,0}\frac{dx_i}{x_i})\right) = \left\{\begin{array}{ll}\Delta(x_k f'_{x_k}|_{x_k=0}) & \text{if } k>s\\ \Delta(x_k f'_{x_k} - x_0 f'_{x_0}) & \text{if } k\le s\end{array}\right\} = 0$$

Let us now compute the action of the Gauss-Manin connection. For that purpose we may suppose the x_i's, for $i>0$, to be constant. One finds $\xi\frac{d}{d\xi} = x_0\frac{d}{dx_0}$, so that :

$$\begin{aligned}\delta(\xi\frac{d}{d\xi}\omega) &= \delta\left(x_0 f'_{x_0}(x_0,\ldots,x_r)\frac{dx_1}{x_1}\cdots\frac{dx_r}{x_r}\right)\\ &= \Delta\left(x_0 f'_{x_0}(x_0,\ldots,x_s,0,\ldots,0)\right)\\ &= \xi\frac{d}{d\xi}\Delta\left(f(x_0,\ldots,x_s,0,\ldots,0)\right) = \xi\frac{d}{d\xi}\delta(\omega)\end{aligned}$$

Finally, one checks easily that :

$$\delta(g\,\omega) = g\,\delta(\omega)$$

for any g in $\overline{\mathbf{Q}}(\xi)\cap\overline{\mathbf{Q}}[[\xi]]$.

Hence δ is a solution of $(\mathbf{R}^r\pi_*\,\Omega^\bullet_{U/S}{<}\mathcal{D}{>})_0$ and

$$\delta_P = \delta\circ\theta$$

is a solution of $(\mathbf{R}^r\pi_*\Omega^\bullet_{X/S}{<}\mathcal{D}{>})_0$.

To conclude the proof of the theorem, one has only to notice that :

$$\delta(\omega)(0) = f(0,\ldots,0) = \mathrm{res}_P(\overline{\omega})$$

as soon as $\theta(\overline{\omega}) = \omega$. □

The following formula :

$$\delta_p \otimes \mathrm{Id} \ : \ H^r_{DR}(Z/\mathbf{P}) = (\mathbf{R}^r\pi_*\Omega^\bullet_{X/S}{<}\mathcal{D}{>})_0 \otimes \overline{\mathbf{Q}}(\xi) \to \overline{\mathbf{Q}}((\xi))$$

enables to extend the solutions δ_P to elements of $\mathcal{S}$. Notice that the solutions obtained are "regular at 0" i.e. they do not involve logarithms.

The limit mixed Hodge structure : The aim of this paragraph is to prove the following theorem.

THEOREM 12 *: The space $\mathcal{E}$ of solutions of $H^r(Z/\mathbf{P})$ in the ring of diagonals of rational fractions contains $M_0(\mathcal{S})$.*

By considering the maps res_P together for the various points P of $\widetilde{\mathcal{D}}^{(r+1)}$ one obtains the "Poincaré residue map" :

$$\mathrm{res} :(\mathbf{R}^r\pi_*\Omega^\bullet_{X/S}{<}\mathcal{D}{>})_0 \ \to \ \mathbf{H}^0(\widetilde{\mathcal{D}}^{(r+1)})$$

As this map is zero on $\lambda(\mathbf{R}^r\pi_*\Omega^\bullet_{X/S}<\mathcal{D}>)_0$ one can take tensor product with $(\mathcal{O}_S/\mathcal{M}_{S,0})$ to obtain a map :

$$\rho : \mathbf{H}^r\left(Y, \Omega^\bullet_{X/S}{<}\mathcal{D}{>} \otimes \mathcal{O}_Y\right) \to \mathbf{H}^0(\widetilde{\mathcal{D}}^{(r+1)})$$

According to Proposition 11, $\mathcal{E}$ contains the subspace of $\mathcal{S}$ spanned by the solutions δ_P. Hence one has :

$$\mathrm{ann}(\mathcal{E}) \subseteq \ker\rho$$

so the theorem will follow from the following lemma :

LEMMA 13 *: $\ker\rho \subseteq M_{2r-1}\mathbf{H}^r\left(Y, \Omega^\bullet_{X/S}{<}\mathcal{D}{>} \otimes \mathcal{O}_Y\right)$.*

To prove this lemma, we need to introduce Steenbrink's theory as generalized to our situation in [15]. The basic idea is to replace the complex $\Omega^\bullet_{X/S}{<}\mathcal{D}{>} \otimes \mathcal{O}_Y$ by a quasi-isomorphic one which we will describe now.

The complex $\Omega^\bullet_X{<}\mathcal{D}{>}$ carries a filtration $W(Y)$ given by :

$$W(Y)_k\Omega^p_X{<}\mathcal{D}{>} = \Omega^k_X{<}\mathcal{D}{>} \wedge \Omega^{p-k}_X{<}D{>}$$

and one defines a double complex of Y-sheaves by :

$$\begin{aligned} &A^{p,q} = \Omega^{p+q+1}_X{<}\mathcal{D}{>} / W(Y)_q\, \Omega^{p+q+1}_X{<}\mathcal{D}{>} \\ &d' : A^{p,q} \to A^{p+1,q} \text{ the ordinary differentiation} \\ &d'' : A^{p,q} \to A^{p,q+1} \text{ cup product with } \pi^*(d\xi/\xi) \end{aligned}$$

Denote by $W(\mathcal{D})$ the filtration on $\Omega^\bullet_X{<}\mathcal{D}{>}$ defined analogously to that of $W(Y)$, and by $A^\bullet$ the single complex associated with the double complex $A^{\bullet,\bullet}$.

PROPOSITION 14 ([15]) *: Cup product with $\pi^*(d\xi/\xi)$ defines a quasi-isomorphism ϕ from $\Omega^\bullet_{X/S}<\mathcal{D}> \otimes \mathcal{O}_Y$ to $A^\bullet$.* □

On the complex $A^\bullet$ we define a new filtration :

$$M_k A^{p,q} = W(\mathcal{D})_{2q+k+1}\Omega_X^{p+q+1}<\mathcal{D}> /W(Y)_q\Omega_X^{p+q+1}<\mathcal{D}>$$

which induces, with the customary shift, the filtration :

$$M_k \mathbf{H}^r(Y, A^\bullet) = \text{image of } \mathbf{H}^r(Y, M_{r-k}A^\bullet)$$

PROPOSITION 15 [STEENBRINK] *: The isomorphism*

$$\mathbf{H}^r\left(Y, \Omega^\bullet_{X/S}<\mathcal{D}> \otimes \mathcal{O}_Y\right) \cong \mathbf{H}^r(Y, A^\bullet)$$

is compatible with the filtrations M we defined on both spaces.□

Let us consider the spectral sequence :

$$E_1^{-k,k+r} = \mathbf{H}^r(Y, Gr_k^M A^\bullet) \Rightarrow \mathbf{H}^r(Y, A^\bullet)$$

An easy computation shows :

$$\begin{aligned} Gr_k^M A^\bullet &= 0 \ \text{ for } k > r \\ Gr_r^M A^\bullet &= \oplus_{p+q=r} Gr_r A^{p,q} \\ &= Gr_r^{W(\mathcal{D})}\Omega_X^{r+1}<\mathcal{D}> [-r] \\ &= a_*\Omega^\bullet_{\widetilde{\mathcal{D}}^{(r+1)}}[-r] \end{aligned}$$

so that :

$$\begin{aligned} E_1^{p,q} &= 0 \ \text{ for } p < -r \\ E_1^{-r,2r} &= H^0(\widetilde{\mathcal{D}}^{(r+1)}) \end{aligned}$$

Hence there exists a one to one mapping :

$$\begin{aligned} Gr_r^M \mathbf{H}^r(Y, \Omega^\bullet_{X/S}<\mathcal{D}> \otimes \mathcal{O}_Y) &\cong Gr_r^M \mathbf{H}^r(Y, A^\bullet) \cong E_\infty^{-r,2r} \\ &\to E_1^{-r,2r} \cong H^0(\widetilde{\mathcal{D}}^{(r+1)}) \end{aligned}$$

That this mapping is ρ is easily seen from the construction of the spectral sequence and the definition of the quasi-isomorphism ϕ. This is enough to prove the lemma. □

Remark : The filtration W of $\Omega_X < \mathcal{D} >$, defined analogously to that of $W(Y)$ but from D, induces filtrations on $A^\bullet$, on $\mathrm{H}^r(Y, \Omega^\bullet_{X/S} < \mathcal{D} > \otimes \mathcal{O}_Y)$, then on $\mathcal{S}$ and finally on $\mathcal{E}$. We intend to study this filtration in a forthcoming paper.

THEOREM 16 *Denote by Γ the simplicial complex with one vertex for each irreducible component $\mathcal{D}_i$ of $\mathcal{D}$ and one simplex $< \mathcal{D}_{i(0)}, \ldots, \mathcal{D}_{i(k)} >$ for each connected component of $\mathcal{D}_{i(0)} \cap \ldots \cap \mathcal{D}_{i(k)}$. Then* $\dim \mathcal{E} \geq \dim H^r(\Gamma)$.

The spectral sequence $E_1^{p,q}$ degenerates at E_2. And one computes :

$$E_\infty^{-r,2r} = \ker \left(E_1^{-r,2r} \to E_1^{-r+1,2r} \cong \mathrm{H}^2(\widetilde{\mathcal{D}}^{(r)}, \Omega^\bullet_{\widetilde{\mathcal{D}}^{(r)}}) \right)$$

The proof is based on the observation that the d_1 mapping is the dual of the canonical mapping :

$$H^0(\widetilde{\mathcal{D}}^{(r)}) \to H^0(\widetilde{\mathcal{D}}^{(r+1)}) \ \square$$

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