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Hadamard Products of Rational Formal Power Series

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1. INTRODUCTION

E. Borel (see [3, Theorem 7]) has shown that the Hadamard product (see Definition 2.1 below for terminology) of two rational formal power series in one variable over a field of characteristic zero is again a rational formal power series. His result can be extended to an arbitrary field. However, this result is not true in the case of several variables (see Example 3.5 below). We consider the following question:

Is the Hadamard product of two rational formal power series in several variables over a field an algebraic formal power series?

In [5] we showed that the question has a positive answer over a field of positive characteristic. Here we shall prove that the result is still true over a field of characteristic zero in the case of two variables but is not true for the case of more than two variables.

2. NOTATION AND TERMINOLOGY

Let K be a field. $K[[x_1, x_2, ..., x_k]]$ will denote the ring of formal power series in k commuting variables $x_1, x_2, ..., x_k$ with coefficients in K. We shall write $K((x_1, x_2, ..., x_k))$ for the field of fractions of $K[[x_1, x_2, ..., x_k]]$.

An element $f \in K((x_1, x_2, ..., x_k))$ is said to be an *algebraic function* over K if f is algebraic over the field of rational functions $K(x_1, x_2, ..., x_k)$. If further $f \in K[[x_1, x_2, ..., x_k]]$, then f is said to be an *algebraic series* over K.

DEFINITION 2.1. Suppose that $f, g \in K[[x_1, x_2, ..., x_k]]$, say

$$f = \sum_{\substack{n_j \geq 0 \\ j=1,2,...,k}} a_{n_1 n_2 \cdots n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \quad \text{and} \quad g = \sum_{\substack{n_j \geq 0 \\ j=1,2,...,k}} b_{n_1 n_2 \cdots n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

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where $a_{n_1 n_2 \cdots n_k}$, $b_{n_1 n_2 \cdots n_k} \in K$. *Then the Hadamard product* of f and g, which will be denoted by $f * g$, is the series which is defined by

$$f * g = \sum_{\substack{n_j \geq 0 \\ j=1,2,\ldots,k}} a_{n_1 n_2 \cdots n_k} b_{n_1 n_2 \cdots n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \in K[[x_1, x_2, \ldots, x_k]].$$

3. Preliminaries

The following result, which is well known, is due to E. Borel (see [3, Theorem 7]).

THEOREM 3.1. *Let K be a field of characteristic zero. Let $f(x) = \sum_{n \geq 0} a_n x^n$, $g(x) = \sum_{n \geq 0} b_n x^n \in K[[x]]$. If f, g are rational, then $f * g(x) = \sum_{n \geq 0} a_n b_n x^n$ is again rational.* ∎

When the field K has positive characteristic the result is still true and we shall show it below.

LEMMA 3.2. *Suppose that K is a field of characteristic $p > 0$ which is also algebraically closed. If f and g belong to $K(x)$, then $f * g \in K(x)$.*

Proof. Since K is algebraically closed we can write

$$f(x) = P(x) + \sum_{i=0}^{n} \frac{a_i}{(1 - \lambda_i x)^{\alpha_i}}$$

and

$$g(x) = Q(x) + \sum_{i=0}^{n} \frac{b_i}{(1 - \gamma_i x)^{\beta_i}},$$

where $P(x)$ and $Q(x)$ belong to $K[x]$ and a_i, b_i, λ_i, and γ_i are in K.

Since the Hadamard product operation is a K-bilinear operation it is enough to prove the lemma for the case $f_1 = 1/(1 - \lambda x)^\alpha$, $\alpha \geq 1$, and $g_1 = 1/(1 - \gamma x)^\beta$, $\beta \geq 1$, where $\lambda, \gamma \in K$.

Let α', β', k be non-negative integers such that $\alpha + \alpha' = \beta + \beta' = p^k$ so that

$$f_1 = \frac{1}{(1 - \lambda x)^\alpha} = \frac{(1 - \lambda x)^{\alpha'}}{1 - \lambda^{p^k} x^{p^k}} = (1 - \lambda x)^{\alpha'} \sum_{n \geq 0} \lambda^{np^k} x^{np^k}$$

$$g_1 = \frac{1}{(1 - \gamma x)^\beta} = \frac{(1 - \gamma x)^{\beta'}}{1 - \gamma^{p^k} x^{p^k}} = (1 - \gamma x)^{\beta'} \sum_{n \geq 0} \gamma^{np^k} x^{np^k}.$$

Since $\alpha \geqslant 1$, $\beta \geqslant 1$ we have $\alpha' < p^k$, $\beta' < p^k$ and hence

$$f_1 * g_1 = ((1-\lambda x)^{\alpha'} * (1-\gamma x)^{\beta'}) \sum_{n \geqslant 0} (\lambda\gamma x)^{np^k}$$
$$= \frac{(1-\lambda x)^{\alpha'} * (1-\gamma x)^{\beta'}}{1-(\lambda\gamma x)^{p^k}} \in K(x). \quad \blacksquare$$

LEMMA 3.3. *Suppose that K is a field and L is an extension field of K. Suppose that $f \in K[[x]]$. If $f \in L(x)$, then $f \in K(x)$.*

Proof. Suppose that $f = \sum_{n \geqslant 0} a_n x^n \in L(x) \cap K[[x]]$. Then $f = g/h$, where $g = \beta_0 + \beta_1 x + \cdots + \beta_M x^M$ and $h = \alpha_0 + \alpha_1 x + \cdots + \alpha_k x^k$, with each $\alpha_i, \beta_j \in L$ $(0 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant M)$ and $\alpha_0 \neq 0$. Hence

$$\alpha_0 a_{n+k} + \alpha_1 a_{n+k-1} + \cdots + \alpha_k a_n = 0, \qquad \forall n \in \mathbf{N} \quad \text{with } n > M. \tag{3.3.1}$$

Let B be a basis for L over K containing α_0. Define a K-linear map

$$\phi: L \to K$$

such that if $x \in B$ then

$$\phi(x) = \begin{cases} 1 & \text{if} \quad x = \alpha_0 \\ 0 & \text{otherwise.} \end{cases}$$

Applying ϕ to the relation (3.3.1) we get (since each $a_m \in K$ and $\phi(\alpha_0) = 1$)

$$a_{n+k} + \phi(\alpha_1) a_{n+k-1} + \cdots + \phi(\alpha_k) a_n = 0, \qquad \forall n > M,$$

which is a non-trivial recurrence relation over K. Hence $f \in K(x)$. $\blacksquare$

THEOREM 3.4. *Suppose that K is a field of characteristic $p > 0$. If f and $g \in K(x)$, then $f * g \in K(x)$.*

Proof. Suppose that L is an extension field of K which is algebraically closed. Then f and g belong to $L(x)$. Therefore by Lemma 3.2, $f * g \in L(x)$ and so by Lemma 3.3, $f * g \in K(x)$. $\blacksquare$

Theorems 3.1 and 3.4 do not hold in the case of several variables. For example if

$$f = \sum_{n,m \geqslant 0} \binom{n+m}{n} x^n y^m = \frac{1}{1-x-y},$$

which is rational, then (except in characteristic two)

$$f*f=\sum_{n,m\geqslant 0}\binom{n+m}{n}^2 x^n y^m=\{(1-x-y)^2-4xy\}^{-1/2},$$

which is not rational (see, for example, [4, p. 141]).

A question which arises here is

Suppose that K is a field and $f, g\in K[[x_1, x_2, ..., x_k]]$. Suppose that f, g are rational series. Is the Hadamard product $f*g$ an algebraic series?

The answer is positive when the field K has positive characteristic [see [5, Corollary 5.5]]. If $k>2$ and K has characteristic zero, then the following example shows that the answer is negative.

EXAMPLE 3.5. If

$$f=\sum_{n_1,n_2,n_3\geqslant 0}\binom{n_1+n_2+n_3}{n_1, n_2, n_3} x_1^{n_1}x_2^{n_2}x_3^{n_3}=\frac{1}{1-x_1-x_2-x_3}$$

and

$$g=\sum_{n\geqslant 0}(x_1x_2x_3)^n=\frac{1}{1-x_1x_2x_3},$$

which are rational series, then

$$f*g=\sum_{n\geqslant 0}\binom{3n}{n, n, n}t^n, \qquad \text{where} \quad t=x_1x_2x_3,$$

which is transcendental over a field of characteristic zero (see, for example, [7]). Note that, by [7], the same series is algebraic over a field of characteristic $p>0$, but of degree equal to an unbounded function of p.

The only case which is left and still seems to be unknown is the case $k=2$ and when K has characteristic zero. In this paper we intend to consider this case and prove the following theorem:

The Hadamard product of two rational series of two variables over a field of characteristic zero is an algebraic series.

First we need the following lemma:

LEMMA 3.6. *Let K be a field. Then (making appropriate natural identifications)*

(i) $K[[x, y]]=K[[x]][[y]]$,

(ii) $K[[x, y]]\subseteq K[x/y][[y]]$,

(iii) $K(x, y) = K(x)(y) \subseteq K((x, y)) \subseteq K((x))((y))$,

(iv) $K((x))[y] \subseteq K(y)((x)) \subseteq K((y))((x))$.

Proof. The proof of (i) is straightforward.

(ii) Let $f \in K[[x, y]]$, and

$$f = \sum_{n \geqslant 0} \sum_{m \geqslant 0} a_{mn} x^m y^n = \sum_{n \geqslant 0} \sum_{m \geqslant 0} a_{mn} \frac{x^m}{y^m} y^{m+n}.$$

Let $x/y = \lambda$ and $m + n = p$. Then $f = \sum_{p \geqslant 0} (\sum_{m \leqslant p} a_{m(p-m)} \lambda^m) y^p$, which is an element of $K[\lambda][[y]]$.

(iii) $K(x, y) = K(x)(y) \subseteq K((x, y))$ is trivial. For the remaining part of this case note that

$$K[[x, y]] = K[[x]][[y]] \subseteq K((x))[[y]] \subseteq K((x))((y)).$$

Thus the quotient field of $K[[x, y]]$, namely $K((x, y))$, is also contained in $K((x))((y))$.

(iv) If $f \in K((x))[y]$, then $f = \sum_{n=0}^{N} a_n(x) y^n$, where $a_n(x) = \sum_{m=-M_n}^{\infty} a_{mn} x^m$. Hence

$$\begin{aligned} f &= \sum_{n=0}^{N} \sum_{m=-M_n}^{\infty} a_{mn} x^m y^n \\ &= \sum_{\substack{m = -\mathrm{Max}\, M_i \\ 0 \leqslant i \leqslant N}}^{\infty} \left(\sum_{n=0}^{N} a_{mn} y^n \right) x^m \in K(y)((x)) \subseteq K((y))((x)). \quad \blacksquare \end{aligned}$$

4. Newton–Puiseux Theorem

The following theorem is due to Newton and Puiseux (see, for example, [6, pp. 98–102]).

THEOREM 4.1. *Let K be an algebraically closed field of characteristic zero. Then*

$$\overline{K((t))} = \bigcup_{r \geqslant 1} K((t^{1/r})),$$

where $\bar{A}$ denotes the algebraic closure of A. $\blacksquare$

We need a "Two variable version of the Newton–Puiseux Theorem." First we state the following lemma:

LEMMA 4.2. *Let K be a field of characteristic zero and let $K \subseteq L$ be a field extension. If $f(t) \in L((t))$ is algebraic over $K((t))$, then there exists a finite extension M of K with $M \subseteq L$ such that $f(t) \in M((t))$.*

Proof. See [1, Lemma 1, p. 72]. ∎

→ not necessarily algebraically closed

COROLLARY 4.3. *Let K be a field of characteristic zero. Then*

$$\overline{K((t))} = \bigcup_{L,r \geqslant 1} L((t^{1/r})),$$

where L runs through the finite extensions of K contained in $\bar{K}$.

Proof. Let $U = \bigcup_{L,r \geqslant 1} L((t^{1/r}))$, where L runs through the finite extensions of K contained in $\bar{K}$. Then $K((t)) \subseteq U$ is an algebraic extension as each $L((t^{1/r}))$ is clearly algebraic (in fact finite) over $K((t))$. On the other hand $\bigcup_{r \geqslant 1} \bar{K}((t^{1/r}))$ is an algebraically closed field by Theorem 4.1 since $\bar{K}$ is an algebraically closed field of characteristic zero. Moreover clearly

$$U \subseteq \bigcup_{r \geqslant 1} \bar{K}((t^{1/r})).$$

It remains to show that every algebraic element of $\bigcup_{r \geqslant 1} \bar{K}((t^{1/r}))$ over $K((t))$ is in fact in U. Let $f(t^{1/r}) \in \bar{K}((t^{1/r}))$ be algebraic over $K((t))$ and so over $K((t^{1/r}))$. By Lemma 4.2 there exists a finite extension L of K contained in $\bar{K}$ such that $f(t^{1/r}) \in L((t^{1/r}))$. Thus $f(t^{1/r}) \in U$ and hence

$$\overline{K((t))} = U. \quad ∎$$

COROLLARY 4.4 (A Two Variable Newton–Puiseux Theorem). *Let K be an algebraically closed field of characteristic zero. Then*

$$\overline{K((t))((u))} = \bigcup_{r,s \geqslant 1} K((t^{1/r}))((u^{1/s})).$$

Proof. By Corollary 4.3,

$$\overline{K((t))((u))} = \bigcup_{L,s \geqslant 1} L((u^{1/s})),$$

where L runs through the finite extensions of $K((t))$ contained in $\overline{K((t))}$. Since K is an algebraically closed field of characteristic zero, by Theorem 4.1 we have

$$\overline{K((t))} = \bigcup_{r \geqslant 1} K((t^{1/r})).$$

Hence

$$\overline{K((t))((u))} = \bigcup_{r,s \geq 1} K((t^{1/r}))((u^{1/s})). \blacksquare$$

LEMMA 4.5. *Let $K \subseteq L$ be a field extension. Suppose that $f(x_1, x_2, ..., x_n) \in K[[x_1, x_2, ..., x_n]]$ is algebraic over $L(x_1, x_2, ..., x_n)$ of degree N. Then f is algebraic over $K(x_1, x_2, ..., x_n)$ of degree N.*

Proof. See [5, Theorem 6.1]. $\blacksquare$

5. THE MAIN RESULT

We use an argument which is analogous to Gessel's argument in [2] to prove our main result. Note that in the proof, particular attention must be paid to the order of the variables; for example, $K((x))((y)) \neq K((y))((x))$.

THEOREM 5.1. *Let K be a field of characteristic zero. Let $R(x, y) = \sum_{m,n \geq 0} a_{mn} x^m y^n$, $S(x, y) = \sum_{m,n \geq 0} b_{mn} x^m y^n \in K[[x, y]]$. If R, S are rational series, then $R * S(t, u) = \sum_{m,n \geq 0} a_{mn} b_{mn} t^m u^n$ is an algebraic series over $K(t, u)$.*

Proof. By Lemma 4.5 we may assume that K is algebraically closed. Since $K(x, y) = K(x)(y)$ we have $R(x, y) = \sum_{n \geq 0} a_n(x) y^n$, $S(x, y) = \sum_{n \geq 0} b_n(x) y^n$ where $a_n(x) = \sum_{m \geq 0} a_{mn} x^m$, $b_n(x) = \sum_{m \geq 0} b_{mn} x^m$ are in $K(x)$. Substituting $(x, y) \to (x, u)$ in R and $(x, y) \to (t/x, u)$ in S we get

$$R(x, u) = \sum_{n \geq 0} a_n(x) u^n, \qquad S\left(\frac{t}{x}, u\right) = \sum_{n \geq 0} b_n\left(\frac{t}{x}\right) u^n.$$

Both $R(x, u)$ and $S(t/x, u)$ are power series in $K(x, t)[[u]]$ which are also contained in $K(x, t)(u)$. Applying Theorem 3.1 with K replaced by $K(x, t)$ we get

$$T(u, x, t) = R(x, u) * S\left(\frac{t}{x}, u\right) = \sum_{n \geq 0} a_n(x) b_n\left(\frac{t}{x}\right) u^n,$$

we are interested in $[x^0]T = R * S(t,u)$

which is again a rational power series in $K(x, t)[[u]] \cap K(x, t)(u)$.

Now

$$a_n(x) = \sum_{m \geq 0} a_{mn} x^m \in K((x))$$

and

$$b_n\left(\frac{t}{x}\right)=\sum_{r\geq 0} b_{rn}\frac{t^r}{x^r}\in K(x)[[t]]\subseteq K((x))[[t]].$$

Hence

$$T(u,x,t)=\sum_{n\geq 0}\sum_{r\geq 0}\sum_{m\geq 0} a_{mn}b_{rn}x^{m-r}t^r u^n\in K((x))[[t]][[u]]$$
$$=K((x))[[u]][[t]].$$

Since the coefficients of the formal power series in t are formal power series in u with coefficients in $K((x))$, bounded in the x-adic metric, we have

$$T(u,x,t)=\sum_{r\geq 0}\sum_{n\geq 0}\sum_{m\geq 0} a_{mn}b_{rn}x^{m-r}u^n t^r$$
$$=\sum_{r\geq 0}\sum_{s\geq -r}\sum_{n\geq 0} a_{(r+s)n}b_{rn}u^n x^s t^r\in K((u))((x))[[t]].$$

Since $L((t))\subseteq L((t^{1/r}))$ for $r\geq 1$, we have

$$K((u))((x))[[t]]\subseteq K((u^{1/a}))((x))((t^{1/b}))$$

for any integers $a, b\geq 1$. Let $\langle x^0\rangle T$ be the "coefficient of x^0" in $T(u,x,t)$. Then

$$\langle x^0\rangle T=\sum_{r\geq 0}\sum_{n\geq 0} a_{rn}b_{rn}u^n t^r=\sum_{m,n\geq 0} a_{mn}b_{mn}t^m u^n$$
$$=R * S(t,u)\in K[[t,u]]\subseteq K((u))((t)).$$

It remains to show that $\langle x^0\rangle T$ is algebraic over $K(t,u)$.

Since $T(u,x,t)$ is a rational series in u over $K(x,t)$ and $K(x,t)(u)=K(t,u)(x)$, there exist polynomials $P(t,u,x)$, $Q(t,u,x)\in K(t,u)[x]$ such that

$$T(u,x,t)=\frac{P(t,u,x)}{Q(t,u,x)},\qquad Q(t,u,x)\neq 0.$$

Further $K(t,u)=K(u,t)=K(u)(t)\subseteq K((u))((t))$ by Lemma 3.6(iii) and moreover

$$\overline{K((u))((t))}=\bigcup_{a,b\geq 1} K((u^{1/a}))((t^{1/b}))$$

by Corollary 4.4. Therefore, we can factorise the denominator Q into linear factors over $K((u^{1/a}))((t^{1/b}))$ for some $a, b \geqslant 1$. Hence we can expand $T = P/Q$ in partial fractions and we have

$$T(u, x, t) = \sum_{i=1}^{M} \frac{g_i(x)}{(x-\xi_i)^{c_i}},$$

where $c_i \in \mathbf{N}$, $c_i \geqslant 1$, $\xi_i \in K((u^{1/a}))((t^{1/b}))$, and

$$g_i(x) \in K((u^{1/a}))((t^{1/b}))[x] \subseteq K((u^{1/a}))(x)((t^{1/b})) \subseteq K((u^{1/a}))((x))((t^{1/b}))$$

(by Lemma 3.6(iv)).

Since ξ_i is a root of $Q \in K(u, t)[x]$, ξ_i is algebraic over $K(u, t)$ and further all the coefficients of g_i are algebraic over $K(u, t)$. Rename the ξ_i as $\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_n$ where

$$\alpha_i \in t^{1/b}K((u^{1/a}))[[t^{1/b}]], \qquad \beta_j \notin t^{1/b}K((u^{1/a}))[[t^{1/b}]],$$

so that $\beta_j^{-1} \in K((u^{1/a}))[[t^{1/b}]]$. Hence

$$T(u, x, t) = \sum_{i=1}^{m} \frac{P_i(x)}{(1-x^{-1}\alpha_i)^{c_i}} + \sum_{j=1}^{n} \frac{Q_j(x)}{(1-x\beta_j^{-1})^{d_j}} \tag{5.1.1}$$

for some positive integers c_i and d_j with

$$P_i(x) \in K((u^{1/a}))((t^{1/b}))[x, x^{-1}]$$

and

$$Q_j(x) \in K((u^{1/a}))((t^{1/b}))[x],$$

where in each case the ring is contained in $K((u^{1/a}))((x))((t^{1/b}))$ by Lemma 3.6(iv) (and P_i, Q_j still have coefficients which are algebraic over $K(u, t)$).

In order to find $\langle x^0 \rangle T$ we have to expand the terms on the right-hand side of (5.1.1) as series in $K((u^{1/a}))((x))((t^{1/b}))$. To do this, it is enough to consider the two terms $P(x)/(1-x^{-1}\alpha)^c$ and $Q(x)/(1-x\beta^{-1})^d$ for the two typical terms corresponding to the first and the second sums in (5.1.1), respectively, where

$$\alpha \in t^{1/b}K((u^{1/a}))[[t^{1/b}]], \qquad \beta \in K((u^{1/a}))((t^{1/b})) \setminus t^{1/b}K((u^{1/a}))[[t^{1/b}]],$$

$$P(x) \in K((u^{1/a}))((t^{1/b}))[x, x^{-1}]$$

and

$$Q(x) \in K((u^{1/a}))((t^{1/b}))[x].$$

Let

$$h_1 = \frac{1}{(1-x^{-1}\alpha)^c} = (1-x^{-1}\alpha)^{-c} = \sum_{n \geqslant 0} \binom{-c}{n} (-1)^n x^{-n} \alpha^n$$
$$= \sum_{n \geqslant 0} \binom{c+n-1}{n} x^{-n} \alpha^n \tag{5.1.2}$$

and $\alpha = \sum_{m=1}^{\infty} \lambda_m(u^{1/a}) t^{m/b}$, for some $\lambda_m(u^{1/a}) \in K((u^{1/a}))$. Hence

$$h_1 = \sum_{n \geqslant 0} \binom{c+n-1}{n} \left(\sum_{m=1}^{\infty} \lambda_m(u^{1/a}) t^{m/b} \right)^n x^{-n}.$$

This series is in fact a formal power series in $t^{1/b}$ as m and n are non-negative integers. Since the least exponent of t is $1/b > 0$ (when $m=1$), the coefficient of $t^{k/b}$ is a finite sum of some terms for which $n \leqslant k$. Hence

$$h_1 = \sum_{k \geqslant 0} \left(\sum_{r=0}^{k} \lambda'_{rk}(u^{1/a}) x^{-r} \right) t^{k/b},$$

for some $\lambda'_{rk}(u^{1/a}) \in K((u^{1/a}))$, which is clearly a series in $K((u^{1/a}))(x))((t^{1/b}))$ (in fact an element of $K((u^{1/a}))((x))[[t^{1/b}]]$).

Similarly, let

$$h_2 = \frac{1}{(1-x\beta^{-1})^d} = (1-x\beta^{-1})^{-d} = \sum_{n \geqslant 0} \binom{d+n-1}{n} x^n \beta^{-n}. \tag{5.1.3}$$

As $\beta^{-1} \in K((u^{1/a}))[[t^{1/b}]]$, $\beta^{-1} = \sum_{m=0}^{\infty} \gamma_m(u^{1/a}) t^{m/b}$ for some $\gamma_m(u^{1/a}) \in K((u^{1/a}))$. Hence

$$h_2 = \sum_{n \geqslant 0} \binom{d+n-1}{n} \left(\sum_{m=0}^{\infty} \gamma_m(u^{1/a}) t^{m/b} \right)^n x^n.$$

This series is obviously a formal power series in $t^{1/b}$ which is contained in $K((u^{1/a}))[[x]][[t^{1/b}]]$ and hence is a series in $K((u^{1/a}))((x))((t^{1/b}))$.

Now $P(x)$ and $Q(x)$ are polynomials in x and x^{-1} with coefficients in $K((u^{1/a}))((t^{1/b}))$ which are algebraic over $K(u, t)$ (and also α, β are algebraic over $K(u, t)$). Hence it follows that the coefficients of x^0 in $P(x)h_1$ and $Q(x)h_2$ are algebraic over $K(u, t)$ (since from (5.1.2) and (5.1.3) they are clearly finite sums of terms of the form $\delta\alpha^m$ and $\varepsilon\beta^n$ where δ, ε are algebraic over $K(u, t)$). Hence $\langle x^0 \rangle T$ is a finite sum of terms in $K((u^{1/a}))((t^{1/b}))$ which are all algebraic over $K(u, t)$ and the proof is complete. ∎

Remark 5.2. We cannot weaken the conditions of Theorem 5.1 to allow R or S to be algebraic. For example if

$$R(x, y) = \sum_{n,m \geqslant 0} \binom{n+m}{n}^2 x^n y^m = \{(1-x-y)^2 - 4xy\}^{-1/2},$$

which is algebraic (see example following Theorem 3.4) and

$$S(x, y) = \sum_{n \geqslant 0} (xy)^n = \frac{1}{1-xy},$$

which is rational, then

$$R * S = \sum_{n \geqslant 0} \binom{2n}{n}^2 t^n, \qquad \text{where } t = xy,$$

which is transcendental. (See [3, p. 298].)

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