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On the Burnside Problem for Periodic Groups

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Introduction

The generalized Burnside problem refers to the question: Are finitely generated periodic groups finite? This was answered in the negative by Golod [1] who proved that, for each prime p, there exists a finitely generated infinite p-group. Golod's construction in not, however, direct and is based on his celebrated work with Šafarevič. Recently, Grigorčuk [2] has given a direct and elegant construction of an infinite 2-group which is generated by three elements of order 2. In this paper we give, for each odd prime p, a direct construction of an infinite p-group on two generators, each of order p. Our group is a subgroup of the automorphism group of a regular tree of degree p; and as might be expected, it is residually finite and has infinite exponent.

Preliminaries. Let p be an odd prime and let $T(0)$ be the infinite regular tree of degree p with vertex 0, so that through each vertex u of $T(0)$ there are p regular subtrees $T(u, 1), \ldots, T(u, p)$, each isomorphic to $T(0)$. For each vertex u of $T(0)$, we define an automorphism

$$t(u)\colon\ T(u) \to T(u) \tag{1}$$

by $T(u,j) \to T(u,j+1)$ for $j=1, \ldots, p-1$ and $T(u,p) \to T(u,1)$.

We note that $t(u)$ is an automorphism of order p which fixes the vertex u and cyclically permutes the vertices $u, 1, k(1), \ldots, k(l); \ldots;\ u, p, k(1), \ldots, k(l)$ for all $l \geqq 0$.

For each vertex u of $T(0)$, we define an infinite sequence $S(u)$ of vertices inductively as follows:

$$\begin{aligned} S(u) &= u, 1; u, 2; \ldots; u, p-1; S(u, p) \\ &= u, 1; u, 2; \ldots; u, p-1; u, p, 1; u, p, 2; \ldots; u, p, p-1; S(u, p, p). \end{aligned} \tag{2}$$

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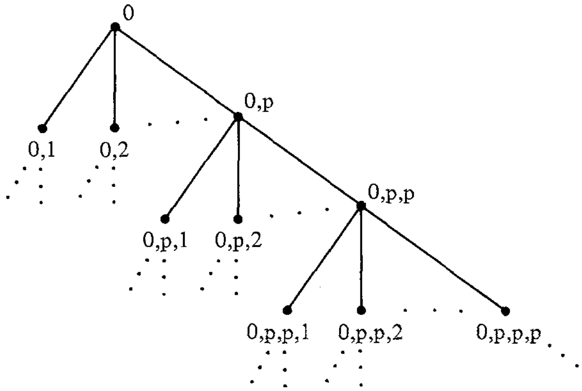

Fig. 1. The tree $T(0)$

Using the sequence $S(u)$ we define an automorphism

$$a(u)\colon\ T(u)\to T(u) \tag{3}$$

by

$$a(u)=t(u,1)t^{-1}(u,2)i(u,3)\dots i(u,p-1)a(u,p),$$

where t's are defined by (1), $i(u,j)$ is the identity automorphism of $T(u,j)$ and $a(u,p)$ is the corresponding automorphism of $T(u,p)$, i.e. $a(u,p)=t(u,p,1)t^{-1}(u,p,2)i(u,p,3)\dots i(u,p,p-1)a(u,p,p)$. Further, if $p=3$, $a(u)=t(u,1)t^{-1}(u,2)a(u,3)$. We note that $a(u)$ is an automorphism of order p which fixes each of the vertices $u; u,p; u,p,p; \dots$.

The Main Result. We are now in a position to state and prove the following theorem.

Theorem. *Let p be an odd prime and let $G(0)$ be the group generated by $t(0)$ and $a(0)$ as defined by* (1) *and* (3) *respectively. Then $G(0)$ is an infinite p-group.*

Proof. We recall from (3) that

$$\begin{aligned}a(0)&=t(0,1)t^{-1}(0,2)i(0,3)\dots i(0,p-1)a(0,p)\\&\in G(0,1)\times G(0,2)\times G(0,3)\times\dots\times G(0,p-1)\times G(0,p),\end{aligned}$$

where $G(u)$ is the group generated by $t(u)$ and $a(u)$.

For simplicity of notation we drop the inscripts and write

$$a=(t,t^{-1},i,\dots,i,a). \tag{4}$$

For each $j=0,\dots,p-1$, $t^{-j}(0)a(0)t^{j}(0)=a_j$ is also an element of $G(0,1)\times\dots\times G(0,p)$, and is obtained by a cyclic permutation of the p-tuple given by (4). Thus we have

$$\begin{aligned}a_0&=(t,t^{-1},i,\dots,i,a_0)\\a_1&=(t^{-1},i,i,\dots,a_0,t)\\&\vdots\\a_{p-1}&=(a_0,t,t^{-1},\dots,i,i).\end{aligned} \tag{5}$$

Let $H(0)$ denote the subgroup of $G(0)$ generated by $a_0, \ldots, a_{p-1}$. Then $H(0)$ is a normal subgroup of $G(0)$ of index p. Since $G(0,j)$'s are isomorphic to $G(0)$ and $H(0)$ is the subdirect product of $G(0,j)$'s, it follows that $G(0)$ is infinite. To show that $G(0)$ is a p-group, we first note that an arbitrary element $g \in G(0)$ is of the form $g = ht^j$, where $h = h(a_0, \ldots, a_{p-1})$ is a word in $a_0, \ldots, a_{p-1}$. Thus we may regard g as an element of $\langle a_0 \rangle * \ldots * \langle a_{p-1} \rangle \rtimes \langle t \rangle$. We shall prove by induction on the syllable length of g that g is a p-element. If g is of syllable length 1 then g is of order p. Let g be of syllable length $m+1 \geqq 2$ and assume that all elements of syllable length at most m are p-elements.

Case 1. $g = h(a_0, \ldots, a_{p-1}) t^{p-j}$, $j \in \{1, \ldots, p-1\}$.

The element h has syllable length $m = \lambda(0) + \ldots + \lambda(p-1)$, where $\lambda(k)$ denotes the length contribution due to a_k. Now,

$$g^p = h h^{t^j} \ldots h^{t^{(p-1)j}}$$

is an element of $H(0)$ with syllable length mp and has the property that the length contribution due to each a_k is $\lambda(0) + \ldots + \lambda(p-1) = m$. Expressing g^p as a p-tuple by (5) shows that for each j the $G(0,j)$ - component of g^p is an element of $H(0,j)$ of syllable length at most m and so is a p-element by the induction hypothesis. Thus g is a p-element.

Case 2. $g = h(a_0, \ldots, a_{p-1}) \in H(0)$.

Here h has syllable length $m+1 = \lambda(0) + \ldots + \lambda(p-1)$. Expressing h as a p-tuple by (5) shows that the $G(0,j)$ - component has syllable length $\lambda(p-j)$ or $\lambda(p-j)+1$ depending on whether or not the component is an element of $H(0,j)$. If the component has syllable length $\lambda(p-j)$ then by induction hypothesis it is a p-element (since $\lambda(p-j) \leqq m$). If the component has length $\lambda(p-j)+1$ and $\lambda(p-j)+1 \leqq m$, then again by the induction hypothesis it is a p-element. If the component has length $\lambda(p-j)+1 = m+1$, then $m=1$ and it is a p-element by Case 1 and the induction hypothesis. Thus, in turn, g is a p-element. This completes the proof that $G(0)$ is an infinite p-group.

Some Properties of the Groups $G(0)$. In this section we prove that $G(0)$ has infinite exponent and that $G(0)$ is residually finite.

Property A. $G(0)$ has infinite exponent.

Details. For $p=3$, let $c = [a_0 a_2, a_2^2 a_1 a_2^2]$ and for $p \geqq 5$, let $c = [a_{p-1}, a_0]$. It follows by (5) that for all p,

$$c = ([a, t], i, \ldots, i). \tag{6}$$

In particular, $G'(0,1)$ is the normal closure of c in H (since $G'(u)$ is the normal closure of $[a(u), t(u)]$ in $G(u)$).

Further, for $p=3$, let

$$b_1 = [a_0, t] = (t, ta, a^{-1}t)$$

and

$$b_2 = [a_0, t, t] = (a, *, *);$$

and for $p \geqq 5$, let

$$b_1 = [a_1, t] = (t, *, \ldots, *) \tag{7}$$

and

$$b_2 = [a_{p-2}, t] = (a, *, \ldots, *).$$

Thus $G'(0)$ projects onto $G(0, 1)$ for all p. Let $w(t(0), a(0))$ be any element of $G(0)$ of order p^k. In what follows, we show that $G(0)$ has another element of order at least p^{k+1}. Indeed, consider the element

$$u = (w(b_1, b_2), i, \ldots, i)t$$

which by (6) belongs to $G(0)$. By (7), it follows that $w(b_1, b_2) = (w(t, a), *, \ldots, *)$ has order at least p^k. Now,

$$u^p = (w(b_1, b_2), w(b_1, b_2), \ldots, w(b_1, b_2))$$

and the order of u is at least p^{k+1}. Thus $G(0)$ has infinite exponent.

Property B. $G(0)$ is residually finite.

Details. Let V denote the set of all vertices of the tree $T(0)$.

Then $V = \bigcup_{k=0}^{\infty} V_k$, where V_k is the set of p^k vertices at the k-th level of the tree. Every element $g \in G(0)$ induces a permutation $\Pi_k(g)$ of V_k for $k \geqq 0$. Then

$$\Pi_k\colon\ G(0) \to \operatorname{Perm}(V_k)$$

defines a natural homomorphism of $G(0)$ into the group of permutations of V_k. If $g \neq 1$ in $G(0)$ then there is a least positive k such that $\Pi_k(g)$ is not the identity element of $\operatorname{Perm}(V_k)$, so $g \notin \operatorname{Ker} \Pi_k$. Since $G(0)/\operatorname{Ker} \Pi_k$ is finite, it follows that G(0) is residually finite.

Remark. The first draft of this paper (circulated in May, 1982) did not include the Properties A and B of the present version. Roger Lyndon (written communication) has reformulated the construction of $G(0)$ in a more combinatorial setting, yielding, in particular, the solvability of the word problem for $G(0)$ (the word problem also follows from property B). We add that our emphasis in this paper is on the simplicity of our construction.

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