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On the positional determinacy of edge-labeled games

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Abstract

It is well known that games with the parity winning condition admit positional determinacy : the winner has always a *positional* (memoryless) strategy. This property continues to hold if edges rather than vertices are labeled. We show that in this latter case the converse is also true. That is, a winning condition over arbitrary set of colors admits positional determinacy in all games if and only if it can be reduced to a parity condition with some finite number of priorities.

Keywords: Parity games; Positional determinacy; Winning condition

1. Introduction

We consider games with perfect information of possibly infinite duration played by two players, Eva and Adam. The arena of a game is a graph whose vertices are positions (partitioned among the players), and edges specify possible moves, so that a play is a maximal path in the arena. If it ends in a deadlock position, the player who should move, looses. If a play is infinite, the winner is specified by a *winning criterion*. In classical setting (see, e.g. [10]), the vertices of an arena are colored in some set of colors $\mathbb{A}$ and the winning plays, of Eva say, are specified by a set $W \subseteq \mathbb{A}^\omega$. A *parity criterion* of order n involves the colors in $\{0, 1, \ldots, n\}$ (usually called *ranks*), and consists of all sequences u with $\limsup_{i\to\infty}(u_i)$ even. That is, Eva wins an infinite play if the highest[3] rank repeating infinitely often is even, otherwise Adam is the winner. A (vertex-labeled) *parity game* is a game with a parity condition of some finite order.

It is well-known that parity games admit *positional* determinacy: for any position, one of the players can win the game using a positional (memoryless, history–free) strategy.

Here the mere determinacy follows from a much more general theorem of Martin [4, 1975], which states that all games with Borel conditions are determined. Indeed, the parity conditions are on the level $\mathbf{\Delta}^0_3$ in the Borel hierarchy. Positional determinacy of these games was established in the early 1990s, independently by Emerson and Jutla [2], Mostowski [6] and McNaughton [5].

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[3] Some authors use a symmetric definition, requiring $\liminf_{i\to\infty}(u_i)$ be even, which is of course equivalent as far as the number of ranks is finite.

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Recently, Grädel and Walukiewicz [3] (see also Grädel [11]) extended the result to suitably defined parity games over *infinite* alphabet. As a special case they showed the positional determinacy of a game where Eva must force *some* rank to repeat infinitely often. It is interesting that the conditions considered by Grädel and Walukiewicz have higher Borel complexity than the parity conditions (namely $\mathbf{\Delta}_4^0$).

However, one can also find very simple winning conditions which guarantee positional determinacy, without resemblance to parity condition. For instance $(a+b)^*(ab)^\omega$, in which case Eva has to force the two labels to alternate since some moment on.

The positional determinacy is very helpful, in particular in the search for efficient algorithms for game solving. Therefore, it is natural to ask if there is any characterization of this property, e.g. in language-theoretic terms.

A starting point of the present work is an observation that if edges rather than vertices are labeled then the parity conditions (of finite order) still guarantee positional determinacy, but the other examples mentioned above loose this property. It suggests that in the edge-labeled case there is less positionally determined games. We show that indeed, in this case, the parity games of finite order are in a strong sense the *only* positionally determined games.

A similar result was previously shown by McNaughton [5] and Zielonka [10] under an extra hypothesis that the alphabet is finite [4] and the winning condition is a Muller condition. Recall that a Muller condition is given by a family $\mathcal{F} \subseteq \wp(\mathbb{A})$ and Eva wins a play if the set of colors occurring infinitely often is in $\mathcal{F}$. Strictly speaking, the setting of [5,10] was slightly different, as these authors considered arenas with partial labeling of vertices and also formulated the parity criterion slightly differently, however it is easy to see that games with labeled edges can be reduced to that case.

The result presented here is more general, as we allow an alphabet to be infinite, and do not restrict ourselves to Muller conditions. We only make a much weaker proviso that the result of a play does not depend on any initial segment of the play. This assumption permits us to disregard an initial position of the game.

2. Basic concepts

Let $\mathbb{A}$ be an alphabet of arbitrary cardinality and let $W \subseteq \mathbb{A}^\omega$. A game with the winning criterion W can be presented by a tuple

$$\langle V_{\mathrm{E}}, V_{\mathrm{A}}, \mathit{Move}, W \rangle,$$

where V_{E} and V_{A} are (disjoint) sets of positions of Eva and Adam, respectively, and $\mathit{Move} \subseteq V \times \mathbb{A} \times V$ is the relation of possible moves, with $V = V_{\mathrm{E}} \cup V_{\mathrm{A}}$. We view a triplet $(p, a, q) \in \mathit{Move}$ as an edge from p to q labeled by a. Note that there can be many edges between two vertices, labeled by different letters in $\mathbb{A}$. As we are interested in infinite plays, we will make a proviso that, for any $q \in V$, there are always some a and p such that $(q, a, p) \in \mathit{Move}$.

A play is therefore an infinite sequence of labeled edges; we represent it by a sequence

$$\pi = (p_0, a_0, p_1, a_1, p_2, \ldots),$$

where $(p_i, a_i, p_{i+1}) \in \mathit{Move}$, and we let $a_\pi = (a_0, a_1, a_2, \ldots)$ be the ω-word induced by π. Eva *wins* the play if $a_\pi \in W$, otherwise Adam is the winner.

A *strategy* for player X indicates a possible move, given an actual history of the play. We can present it as a partial mapping S from $(V \cup \mathbb{A})^*$ to Move. Whenever γ is an initial segment of some play ending in a position of X, say $p \in V_X$, the mapping S maps it to some edge $(p, a, q) \in \mathit{Move}$. A play π is consistent with S if

$$(p_i, a_i, p_{i+1}) = S(p_0, a_0, p_1, a_1, p_2, \ldots, p_i)$$

whenever $p_i \in V_X$. A strategy is *winning* for X from position p if any play starting with $p_0 = p$ and consistent with S is won by X. In this case p is a *winning position* of X (of course, p need not belong to V_X).

A strategy is *positional* if it does not depend on the whole history of the play so far, but only on the last position which is an actual position of the play. That is, a positional strategy is a (total) function $S : V_X \to \mathit{Move}$ which maps p in V_X to some labeled edge $(p, a, q) \in \mathit{Move}$. The consistency of a play π with S now amounts to the condition $(p_i, a_i, p_{i+1}) = S(p_i)$, whenever $p_i \in V_X$.

[4] Zielonka [10] removed an additional assumption of [5] that the arenas of games are finite.

We say that a game as above is *positionally determined* if, for any position p, there exists a winning positional strategy from this position for one of the players.

Definition 1. A language $W \subseteq \mathbb{A}^{\omega}$ is a generalized parity criterion (of order n) if there exists a mapping $h : \mathbb{A} \to \{0, 1, \ldots, n\}$, such that $u \in W$ iff $\limsup_{i\to\infty} (h(u_i))$ is even.

Proposition 2. *Any game with a generalized parity criterion is positionally determined.*

Proof. We will deduce it from the result in the classical, vertex-labeled case. Recall that a vertex-labeled parity game with n priorities is described by a tuple $\langle V_{\mathrm{E}}, V_{\mathrm{A}}, \mathit{Move}, \mathit{rank}\rangle$, where $\mathit{rank} : V \to \{0, 1, \ldots, n\}$. By definition, Eva wins a play $\pi = (p_0, p_1, \ldots)$ iff $\limsup_{i\to\infty} \mathit{rank}(p_i)$ is even. Now, for an edge-labeled game with generalized parity criterion as defined above, we form a vertex-labeled game by replacing each edge $(p, a, q) \in \mathit{Move}$ by two edges $(p, v_{(p,a,q)})$ and $(v_{(p,a,q)}, q)$, where $v_{(p,a,q)}$ is a new vertex, and setting $\mathit{rank}(v_{(p,a,q)}) = h(a)$ and $\mathit{rank}(p) = \mathit{rank}(q) = 0$. By positional determinacy of classical parity games (see [10] [5]), for any position p of the original game, one of the players has a positional winning strategy from this position in the modified game. Now it is easy to see that the same strategy with obvious modifications is also winning in the original game. □

Our goal here is to establish a converse to the above proposition. However, since we are interested in infinite games, we will restrict considerations to criteria which do not depend on finite initial segments of plays. [6]

Definition 3. The set $W \subseteq \mathbb{A}^{\omega}$ is *uniform* if $W = \mathbb{A}W$; in other words, for any $u \in \mathbb{A}^{\omega}$ and $v \in \mathbb{A}^{*}$, we have $u \in W \iff vu \in W$.

The main result of the paper is the following.

Theorem 4. *Suppose $W \subseteq \mathbb{A}^{\omega}$ is uniform and any game with the winning criterion W is positionally determined. Then W is a generalized parity condition of some finite order n.*

From Theorem 4 and Proposition 2 it follows immediately that a uniform set W guarantees the positional determinacy in all games if and only if it is a generalized parity condition.

We end this section by an observation that, in a positionally determined game with a uniform condition, the winning strategies can be global, i.e., independent from starting positions.

Lemma 5. *If W is uniform and a game $\langle V_{\mathrm{E}}, V_{\mathrm{A}}, \mathit{Move}, W\rangle$ is positionally determined then there are two positional strategies $S_{\mathrm{E}} : V_{\mathrm{E}} \to \mathit{Move}$ and $S_{\mathrm{A}} : V_{\mathrm{A}} \to \mathit{Move}$, for Eva and for Adam, respectively, such that, for any position p, one of them is winning from p.*

Proof. For an Eva's positional strategy S, let $w(S)$ be the set of all positions p, such that S is winning from p. By uniformity, S is also winning from any position reachable from any $p \in w(S)$, if Eva plays according to S. Consequently, if another positional strategy S' coincides with S over $w(S)$ then $w(S) \subseteq w(S')$.

Define now a preorder over positional strategies for Eva in the following way: $S_1 \sqsubseteq S_2$ if $w(S_1) \subseteq w(S_2)$ and the strategies S_1 and S_2 coincide over $w(S_1)$. For any set L of positional strategies totally ordered by $\sqsubseteq$, we can construct a positional strategy S^L which is an upper bound for L w.r.t. $\sqsubseteq$. Namely, we let $S^L(p) = S(p)$, whenever $p \in w(S)$, for some $S \in L$; elsewhere S^L is defined arbitrarily.

It follows by Zorn's lemma that there exists a positional strategy S_{E} maximal for $\sqsubseteq$. Assume now that some position p not in $w(S_{\mathrm{E}})$ is winning for Eva using some positional strategy, say S. Then the new strategy S' defined by

$$S'(q) = \begin{cases} S_{\mathrm{E}}(q) & \text{if } q \in w(S_{\mathrm{E}}), \\ S(q) & \text{otherwise} \end{cases}$$

[5] Basically, Zielonka [10] considers arenas with finite branching, but notes in a Footnote 2 that this assumption is inessential. Another proof, for arbitrary graphs, can be found in **?**.

[6] Otherwise we would have many uninteresting criteria forcing positional determinacy as, e.g., $a(a + b)^{\omega}$.

is winning for all positions in $w(S_{\mathrm{E}}) \cup \{p\}$ and do coincide with S_{E} over $w(S_{\mathrm{E}})$. This contradicts the maximality of S_{E}. Hence we have constructed a strategy which allows Eva to win from every position which was winning with some positional strategy.

Constructing by symmetry a similar strategy for Adam and applying positional determinacy yields directly the result. $\square$

3. Proof of Theorem 4

We fix an alphabet $\mathbb{A}$ and a set $W \subseteq \mathbb{A}^\omega$; we assume that W is uniform and any game with a winning criterion W is positionally determined.

For $L \subseteq \mathbb{A}^\omega$, we let L_f be the set $\{u \in \mathbb{A}^+ \mid u^\omega \in L\}$. Since this set is defined as the inverse image of L by the mapping $u \mapsto u^\omega$, for its complement $\overline{L} = \mathbb{A}^\omega - L$, we have $(\overline{L})_f = \mathbb{A}^+ - L_f$. We denote the last set by $\overline{L_f}$.

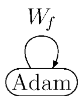
We will often use the fact that inversing the roles of Eva and Adam does not change the argument; we will refer to this as to the principle of symmetry. In particular this amounts in exchanging W with $\overline{W}$, and consequently also W_f with $\overline{W_f}$.

Lemma 6. *For any $u, v \in \mathbb{A}^+$, $uv \in W_f$ if and only if $vu \in W_f$.*

Proof. Let uv be an element of W_f. By definition of W_f we have $(uv)^\omega \in W$ or equivalently $u(vu)^\omega \in W$. By uniformity we obtain $(vu)^\omega \in W$. This proves $vu \in W_f$. The other implication follows by inverting the role of u and v. $\square$

Lemma 7. *We have ${W_f}^\omega \subseteq W$.*

Proof. Suppose to the contrary $w \in {W_f}^\omega - W$. Consider a game where all positions belong to Adam, and there is a distinguished position such that, for all $u \in W_f$, there is a loop from this position to itself labeled by u:



Then Adam wins this game by playing the successive parts of w. By assumption, Adam also wins with a positional strategy, that is by playing always the same word $u \in W_f$. Thus, $u^\omega \notin W$ and consequently $u \notin W_f$, a contradiction. $\square$

Corollary 8. *For any two words $u, v \in W_f$, $uv \in W_f$.*

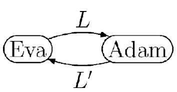
Proof. As $u \in W_f$ and $v \in W_f$, according to the Lemma 7, $(uv)^\omega \in W$ and thus $uv \in W_f$. $\square$

Lemma 9. *For any $L, L' \subseteq \mathbb{A}^+$ we have*

$$\forall v \in L'. \exists u \in L. uv \in W_f \quad \textit{iff} \quad \exists u \in L. \forall v \in L'. \ uv \in W_f \ .$$

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) Let us consider the following game $\mathcal{G}$.



Let us suppose that $\forall v \in L'.\exists u \in L. uv \in W_f$ (*). According to Lemmas 6 and 7, Eva wins the game $\mathcal{G}$ by playing after each move $v \in L'$ of Adam the corresponding $u \in L$ obtained from (*). Eva wins also with a positional strategy, which means by playing always the same word $u \in L$. Adam can then choose to play systematically any word $w \in L'$. As he always looses, we conclude that $uv \in W_f$. $\square$

Lemma 10. *For $a \in \mathbb{A}$ and $B \subseteq \mathbb{A}$ such that $aB \subseteq W_f$, there is $n > 0$ such that $a^n B^+ \subseteq W_f$.*

Proof. Let us first show (*) that for any $v \in B^+$ there is $u \in a^+$ such that $uv \in W_f$. By induction on the length of v. If $v = b$ for some $b \in B$, then the property holds with $u = a$. Let $v = bv'$ for some $b \in B$. By induction hypothesis, there exists $u' \in A^+$ such that $u'v' \in W_f$. Furthermore, by hypothesis $ab \in W_f$, hence also $ba \in W_f$, by Lemma 6. Hence, by Corollary 8, $u'v'ba \in W_f$. By Lemma 6 it follows that $(au')v \in W_f$, which is the claim for v, hence (*) is satisfied.

Now the result follows from Lemma 9 with $L = a^+$ and $L' = B^+$. $\square$

Lemma 11. *For $a \in \mathbb{A} \cap W_f$ and $B \subseteq \mathbb{A}$ such that $aB \subseteq W_f$, $aB^* \subseteq W_f$.*

Proof. Let $E = \{n \geqslant 1 \mid a^n B^+ \subseteq W_f\}$. According to Lemma 10, E is not empty.

For $n \in E$ and any $u \in B^+$, $a \in W_f$ and $a^n u \in W_f$, thus $a^{n+1}u \in W_f$ (Corollary 8). Consequently $n + 1 \in E$. According to this, there is some integer $k \geqslant 1$, such that $E = \{k + n : n < \omega\}$.

We will show that $k = 1$. Suppose to the contrary that $k > 1$ and let p be $k - 1$. We have $p \notin E$ but $2p \in E$. As $p \notin E$, there is some $u \in B^+$ such that $a^p u \in \overline{W_f}$. By Lemma 6, $ua^p \in \overline{W_f}$ also holds. Then, by Corollary 8, $a^p uua^p \in \overline{W_f}$. It follows by Lemma 6 again that $a^{2p}uu \in \overline{W_f}$ and thus $2p \notin E$, a contradiction.

Since $k = 1$, $aB^+ \subseteq W_f$. As furthermore $a \in W_f$, one obtains $aB^* \subseteq W_f$. $\square$

Let $\mathbb{A}_\mathrm{E}$ be $\{a \in \mathbb{A} \mid a \in W_f\}$ and $\mathbb{A}_\mathrm{A}$ be $\mathbb{A} - \mathbb{A}_\mathrm{E} = \{a \in \mathbb{A} \mid a \in \overline{W_f}\}$. For $a \in \mathbb{A}_\mathrm{E}$, let $g(a)$ be $\{b \in \mathbb{A}_\mathrm{A} \mid ab \in W_f\}$. For any $a, a' \in \mathbb{A}_\mathrm{E}$ one defines $a \sqsubseteq a'$ by

$$a \sqsubseteq a' \quad \text{iff} \quad g(a) \subseteq g(a').$$

We define $\sim$ the equivalence relation associated to $\sqsubseteq$, that is $\sim = \sqsubseteq \cap \sqsubseteq^{-1}$. The relation $\sqsubset$ is the strict version of $\sqsubseteq$, meaning $\sqsubset = \sqsubseteq - \sim$. Equivalently, $a \sqsubset a'$ holds for a, a' in $\mathbb{A}_\mathrm{E}$ if and only if there exists $b \in \mathbb{A}_\mathrm{A}$ such that $ab \in \overline{W_f}$ and $a'b \in W_f$.

Lemma 12. *The relation $\sqsubseteq$ is a total preorder.*

Proof. By definition, $\sqsubseteq$ is a preorder.

Let us suppose that there are two elements a and a' of $\mathbb{A}_\mathrm{E}$ incomparable for $\sqsubseteq$. This means that there is $b \in g(a) - g(a')$ and $b' \in g(a') - g(a)$.

We have $ab \in W_f$ and $a'b' \in W_f$ and thus $aba'b' \in W_f$ (Corollary 8). Similarly $b'a \in \overline{W_f}$ and $ba' \in \overline{W_f}$ and thus $aba'b' \in \overline{W_f}$ (Corollary 8 and symmetric version of Lemma 6), a contradiction. $\square$

Lemma 13. *The relation $\sim$ is of finite index.*

Proof. As the equivalences classes of $\sim$ are totally ordered by $\sqsubseteq$ (Lemma 12), it is sufficient to show that there exists no infinite strictly monotonic sequence of elements of $\mathbb{A}_\mathrm{E}$.

Let us suppose that there is an infinite sequence $a_1 \sqsubset a_2 \sqsubset \cdots$. Let b_i be for all $i \in \mathbb{N}$ such that $a_i b_i \in \overline{W_f}$ and $a_{i+1} b_i \in W_f$. From Lemma 7 and hypothesis $b_i a_{i+1} \in W_f$, we have $b_1 a_2 b_2 \cdots \in W$, and thus $a_1 b_1 a_2 \cdots \in W$ (by uniformity). However, as for all i, $a_i b_i \in \overline{W_f}$, from the symmetric version of Lemma 7, $a_1 b_1 a_2 \cdots \in \overline{W}$. There is a contradiction.

For infinite decreasing sequences the same kind of argument can be applied. $\square$

Lemma 14. *There exists an integer n and $2n$ subsets of $\mathbb{A}$ $B_1, \ldots, B_{2n}$, such that*:

(1) *the B_i's form a partition of $\mathbb{A}$,*
(2) *for any $k \in [n]$, $B_{2k} \subseteq W_f$,*
(3) *for any $k \in [n]$, $B_{2k-1} \subseteq \overline{W_f}$,*

(4) *for any* $k \in [n]$ *and* $l \leqslant 2k$, $B_{2k}B_l \subseteq W_f$,
(5) *for any* $k \in [n]$ *and* $l \leqslant 2k-1$, $B_{2k-1}B_l \subseteq \overline{W_f}$.

Proof. According to Lemma 13, there is a finite number p of equivalences classes of $\sim$, and those equivalence classes are totally ordered by $\sqsubseteq$. Let $C_1 \sqsubseteq \cdots \sqsubseteq C_p$ be those equivalence classes. We set $n = p+1$ and define $(B_i)_{i\in[2n]}$ as follows.

$$B_1 = g(C_1),$$
$$\text{for any } k \in [p], \quad B_{2k} = C_k,$$
$$\text{for any } k \in \{2, \ldots, p\}, \quad B_{2k-1} = g(C_k) - g(C_{k-1}),$$
$$B_{2p+1} = \mathbb{A}_{\mathrm{A}} - g(C_p),$$
$$B_{2p+2} = \emptyset.$$

By definition, $\{B_{2k} : k \in [p+1]\}$ is a partition of $\mathbb{A}_{\mathrm{E}}$ and $\{B_{2k-1} : k \in [p+1]\}$ is a partition of $\mathbb{A}_{\mathrm{A}}$. This establishes points 1, 2 and 3.

Ad 4. Let us consider $k \in [p]$ and $l \leqslant 2k$. If l is even, then $B_{2k} \subseteq W_f$ and $B_l \subseteq W_f$ and thus by Corollary 8, $B_{2k}B_l \subseteq W_f$. If l is odd, then by definition $B_l \subseteq g(B_{l+1})$, and thus $B_l \subseteq g(B_{2k})$ since $B_{l+1} \sqsubseteq B_{2k}$. By definition of g, we have $B_{2k}B_l \subseteq W_f$. Finally, for $k = p+1$, $B_{2k} = \emptyset$ and thus for any l, $B_{2k}B_l \subseteq W_f$.

Ad 5. Let $k \in [n]$ and $l \leqslant 2k-1$. If l is odd, then both $B_{2k-1} \subseteq \overline{W_f}$ and $B_l \subseteq \overline{W_f}$ and thus according to the symmetric version of Corollary 8, $B_{2k-1}B_l \subseteq \overline{W_f}$. Suppose l is even, hence necessarily $k > 1$, and $l/2 \leqslant k-1$. We have by definition, $B_l = C_{l/2}$, hence $B_l \sqsubseteq C_{k-1}$. Moreover, for $b \in B_{2k-1}$, $b \notin g(C_{k-1})$; hence, for any $a \in B_l$, $b \notin g(a)$. Consequently $ab \in \overline{W_f}$. □

Theorem 4 follows immediately from the following.

Lemma 15. *W is a generalized parity condition.*

Proof. Let $B_1, \ldots, B_{2n}$ be in Lemma 14. Define $h(a) = i$, whenever $a \in B_i$. We need to show that $u \in W$ iff $\limsup_{i\to\infty}(h(u_i))$ is even. Let $l = \limsup_{i\to\infty}(h(u_i))$; by uniformity we can assume that l is also maximum of $h(u_i)$. Set $A_k = \bigcup_{l=1}^{k} B_l$. Hence u can be decomposed into finite words $u = w_1w_2\ldots$, where $w_i \in B_lA_l^*$. Now, if l is even then by Lemma 14 and Corollary 8, $w_i \in W_f$. Hence, by Lemma 7, $u \in W$. If l is odd then it follows by symmetric argument that $u \in \overline{W}$. □

Finally, let us comment on the assumptions on *W* needed in Theorem 4.

It can be seen from the proof that it would be enough to require that any game with criterion *W* and *countable number of positions* is positionally determined. However we cannot restrict considerations to finite games, as the following example shows.

Let $\mathbb{A}$ be $\{a, b\}$ and ρ be the mapping from $\mathbb{A}^*$ to the set of integers defined by $\rho(\varepsilon) = 0$, $\rho(au) = \rho(u) + 1$ and $\rho(bu) = \rho(u) - 1$. Let us now consider as accepting condition the set of words $w \in \mathbb{A}^\omega$ such that $\{\rho(u) \mid u \in \mathbb{A}^*,\ u \sqsubseteq w\}$ admits an upper bound ($\sqsubseteq$ being the prefix relation). This winning condition is not a parity condition (in fact, it is not even ω-regular). However, one can show that on every *finite* game with this accepting condition, the winner has a positional strategy. In fact, this kind of accepting condition is closely related to the mean-payoff games which are positionally determined [8,1] (see also [7]).

It turns out that we cannot even restrict the assumption of Theorem 4 to games with finite out-degree. Indeed, Grädel and Walukiewicz [3] showed that the following condition over alphabet ω guarantees the positional determinacy of vertex-labeled games: Adam wins whenever some rank repeats infinitely often and the minimal such rank is odd. We claim that any edge-labeled game with this condition is positionally determined provided that each position has finite out-degree. Indeed, for any such game we can construct a vertex-labeled game similarly as in the proof of Proposition 2. For each position p replace each edge $(p, a, q) \in Move$ by two edges $(p, v_{(p,a,q)})$ and $(v_{(p,a,q)}, q)$, where $v_{(p,a,q)}$ is a new vertex, and set $rank(v_{(p,a,q)}) = a$ and $rank(p) = \max\{b : (p, b, q') \in Move\}$. Then it is easy to see that a positional winning strategy for either player can be transferred from the modified game to the original, edge-labeled game. A similar reduction is possible under the assumption of finite in-degree.

Nevertheless, we believe that there should exist some natural characterizations of the positional determinacy over finite and finitely branching games; it is the subject of further research.

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