Intersection Types: an Introduction

AAL’02, Canberra, December 2, 2002

Plan of the talk

- simple types
- intersection types
- properties of λ-terms
- type preorder
- filter models
- Stone duality

Implicational Propositional Logics

$$(ax) \quad \Gamma, \quad \sigma \vdash \quad \sigma$$

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$$(\to E) \qquad \frac{\Gamma \vdash \quad \sigma \to \tau \quad \Gamma \vdash \quad \sigma}{\Gamma \vdash \quad \tau}$$

Implicational Propositional Logics

$$(ax) \quad \Gamma, \sigma \vdash \sigma$$

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$$(\to I) \quad \frac{\Gamma, \sigma \vdash \tau}{\Gamma \vdash \sigma \to \tau}$$

Functional Interpretation of Implication

$\sigma \rightarrow \tau$ is the set of functions which
applied to an argument belonging to σ
give a result belonging to τ

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even: $number \rightarrow bool$

Lambda Notation

If M is an expression (possibly containing the variable x)
then $\lambda x.M$ represents a function
which applied to an argument N gives $M[N/x]$

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$\lambda x.x^2$ is the square function

$$(\lambda x.x^2)2 \longrightarrow 4$$

Lambda Calculus

variables: $x, y, \ldots$

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application: $M N$

Lambda Calculus

variables: $x, y, \ldots$

application: MN

abstraction: $\lambda x.M$

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(β-rule) $(\lambda x.M)N \longrightarrow M[N/x]$

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$$(\lambda xy.xyy)(\lambda z.z) \longrightarrow \lambda y.(\lambda z.z)yy$$

Lambda Calculus

variables: $x, y, \ldots$

application: MN

abstraction: $\lambda x.M$

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$$(\lambda y.yy)(\lambda y.yy) \longrightarrow (\lambda y.yy)(\lambda y.yy)$$

Lambda Calculus

variables: $x, y, \ldots$
application: MN
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(β-rule) $(\lambda x.M)N \longrightarrow M[N/x]$

$$(\lambda xy.xyy)(\lambda z.z) \longrightarrow \lambda y.(\lambda z.z)yy \longrightarrow \lambda y.yy$$

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Implicational Propositional Logics

$$(ax) \quad \Gamma, \quad \sigma \vdash \quad \sigma$$

$$(\to E) \qquad \frac{\Gamma \vdash \quad \sigma \to \tau \quad \Gamma \vdash \quad \sigma}{\Gamma \vdash \quad \tau}$$

$$(\to I) \qquad \frac{\Gamma, \quad \sigma \vdash \quad \tau}{\Gamma \vdash \quad \sigma \to \tau}$$

Simple types

$$(ax) \qquad \Gamma, x : \sigma \vdash x : \sigma$$

$$(\to E) \qquad \frac{\Gamma \vdash \quad \sigma \to \tau \quad \Gamma \vdash \quad \sigma}{\Gamma \vdash \quad \tau}$$

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$$(\rightarrow E) \qquad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

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Simple types

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$$(\rightarrow I) \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \rightarrow \tau}$$

$$\frac{z : \sigma \vdash z : \sigma}{\vdash \lambda z.z : \sigma \to \sigma} \,(\to I)$$

$$\frac{z : \sigma \vdash z : \sigma}{\vdash \lambda z.z : \sigma \rightarrow \sigma}\ (\rightarrow I)$$

$$\frac{\Gamma \vdash x : \sigma \rightarrow \sigma \rightarrow \tau \quad \Gamma \vdash y : \sigma}{\Gamma \vdash xy : \sigma \rightarrow \tau}\ (\rightarrow E)$$

$$\Gamma = \{x : \sigma \rightarrow \sigma \rightarrow \tau, y : \sigma\}$$

$$\frac{z : \sigma \vdash z : \sigma}{\vdash \lambda z.z : \sigma \to \sigma}\,(\to I)$$

$$\frac{\dfrac{\Gamma \vdash x : \sigma \to \sigma \to \tau \quad \Gamma \vdash y : \sigma}{\Gamma \vdash xy : \sigma \to \tau}\,(\to E) \qquad \Gamma \vdash y : \sigma}{\Gamma \vdash xyy : \tau}\,(\to E)$$

$$\Gamma = \{x : \sigma \to \sigma \to \tau, y : \sigma\}$$

$$\dfrac{z : \sigma \vdash z : \sigma}{\vdash \lambda z.z : \sigma \to \sigma}\,(\to I)$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash x : \sigma \to \sigma \to \tau \quad \Gamma \vdash y : \sigma}{\Gamma \vdash xy : \sigma \to \tau}\,(\to E) \qquad \Gamma \vdash y : \sigma}{\Gamma \vdash xyy : \tau}\,(\to E)}{x : \sigma \to \sigma \to \tau \vdash \lambda y.xyy : \sigma \to \tau}\,(\to I)$$

$$\Gamma = \{x : \sigma \to \sigma \to \tau, y : \sigma\}$$

$$\dfrac{z : \sigma \vdash z : \sigma}{\vdash \lambda z.z : \sigma \to \sigma}\,(\to I)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash x : \sigma \to \sigma \to \tau \quad \Gamma \vdash y : \sigma}{\Gamma \vdash xy : \sigma \to \tau}\,(\to E) \qquad \Gamma \vdash y : \sigma}{\Gamma \vdash xyy : \tau}\,(\to E)}{x : \sigma \to \sigma \to \tau \vdash \lambda y.xyy : \sigma \to \tau}\,(\to I)}{\vdash \lambda xy.xyy : (\sigma \to \sigma \to \tau) \to \sigma \to \tau}\,(\to I)$$

$$\Gamma = \{x : \sigma \to \sigma \to \tau, y : \sigma\}$$

$$\frac{z : \sigma \vdash z : \sigma}{\vdash \lambda z.z : \sigma \to \sigma}\,(\to I)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \vdash x : \sigma \to \sigma \to \tau \quad \Gamma \vdash y : \sigma}{\Gamma \vdash xy : \sigma \to \tau}\,(\to E) \qquad \Gamma \vdash y : \sigma}{\Gamma \vdash xyy : \tau}\,(\to E)}{x : \sigma \to \sigma \to \tau \vdash \lambda y.xyy : \sigma \to \tau}\,(\to I)}{\vdash \lambda xy.xyy : (\sigma \to \sigma \to \tau) \to \sigma \to \tau}\,(\to I)$$

$$\Gamma = \{x : \sigma \to \sigma \to \tau, y : \sigma\}$$

$\lambda y.yy$ cannot be typed

Simple types are preserved by reduction

$$\dfrac{\dfrac{\begin{array}{c}[x:\sigma] \quad \cdots \quad [x:\sigma]\\ \boxed{M^{\tau}}\\ M:\tau\end{array}}{\lambda x.M:\sigma\longrightarrow\tau}\,(\longrightarrow I) \qquad \begin{array}{c}\boxed{N^{\sigma}}\\ N:\sigma\end{array}}{(\lambda x.M)N:\tau}\,(\longrightarrow E)$$

Simple types are preserved by reduction

$$\dfrac{\dfrac{\begin{array}{c}[x:\sigma] \quad \cdots \quad [x:\sigma] \\ \boxed{M^{\tau}} \\ M:\tau\end{array}}{\lambda x.M : \sigma\to\tau}\,(\to I) \qquad \begin{array}{c}\boxed{N^{\sigma}}\\ N:\sigma\end{array}}{(\lambda x.M)N : \tau}\,(\to E) \qquad \xRightarrow{\beta\text{-red}} \qquad M[N/x]$$

Simple types are preserved by reduction

$$\dfrac{\dfrac{\begin{array}{c}[x\colon\sigma]\quad\cdots\quad[x\colon\sigma]\\ \boxed{M^{\tau}}\\ M\colon\tau\end{array}}{\lambda x.M\colon\sigma\to\tau}(\to I)\qquad \begin{array}{c}\boxed{N^{\sigma}}\\ N\colon\sigma\end{array}}{(\lambda x.M)N\colon\tau}(\to E)\qquad \overset{\beta\text{-red}}{\Longrightarrow}\qquad \begin{array}{c}\boxed{N^{\sigma}}\quad\vdots\quad\boxed{N^{\sigma}}\\ N\colon\sigma\quad\cdots\quad N\colon\sigma\\ \\ M[N/x]\end{array}$$

Simple types are preserved by reduction

$$
\dfrac{\dfrac{\begin{array}{c}[x\colon \sigma] \quad \cdots \quad [x\colon \sigma] \\ \boxed{\quad M^{\tau} \quad} \\ M\colon \tau\end{array}}{\lambda x.M\colon \sigma \to \tau}\,(\to I) \qquad \begin{array}{c}\boxed{N^{\sigma}} \\ N\colon \sigma\end{array}}{(\lambda x.M)N\colon \tau}\,(\to E)
\quad \xRightarrow{\beta\text{-red}} \quad
\begin{array}{c}\boxed{N^{\sigma}} \quad \vdots \quad \boxed{N^{\sigma}} \\ N\colon \sigma \quad \cdots \quad N\colon \sigma \\ \boxed{\quad M[N/x]^{\tau} \quad} \\ M[N/x]\colon \tau\end{array}
$$

Simple types are NOT preserved by expansion

Exactly 1 occurrence of N

$$\boxed{N^{\sigma}}$$

$$N : \sigma$$

$$\boxed{M[N/x]^{\tau}}$$

$$M[N/x] : \tau$$

Simple types are NOT preserved by expansion

Exactly 1 occurrence of N

$$\boxed{N^{\sigma}}$$

$$N : \sigma$$

$$\boxed{M[N/x]^{\tau}} \qquad \overset{\beta\text{-exp}}{\Longrightarrow}$$

$$M[N/x] : \tau \qquad\qquad (\lambda x.M)N$$

Simple types are NOT preserved by expansion

Exactly 1 occurrence of N

$$[x\colon \sigma]$$

$$\boxed{M^{\tau}}$$

$$M\colon \tau$$

$$\boxed{N^{\sigma}}$$

$$N\colon \sigma$$

$$\boxed{M[N/x]^{\tau}}$$

$$M[N/x]\colon \tau \quad \xRightarrow{\beta\text{-exp}} \quad (\lambda x.M)N$$

Simple types are NOT preserved by expansion

Exactly 1 occurrence of N

$$\begin{array}{c} \boxed{N^{\sigma}} \\ N : \sigma \\ \boxed{M[N/x]^{\tau}} \\ M[N/x] : \tau \end{array} \quad \xRightarrow{\beta\text{-exp}} \quad \begin{array}{c} [x : \sigma] \\ \boxed{M^{\tau}} \\ \dfrac{M : \tau}{\lambda x.M : \sigma \to \tau} \; (\to I) \\ (\lambda x.M)N \end{array}$$

Simple types are NOT preserved by expansion

Exactly 1 occurrence of N

$$
\begin{array}{c}
\boxed{N^{\sigma}} \\
N : \sigma \\
\boxed{M[N/x]^{\tau}} \\
M[N/x] : \tau
\end{array}
\quad \overset{\beta\text{-exp}}{\Longrightarrow} \quad
\dfrac{\dfrac{\begin{array}{c}[x : \sigma] \\ \boxed{M^{\tau}} \\ M : \tau\end{array}}{\lambda x.M : \sigma \to \tau}\,(\to I) \qquad \begin{array}{c}\boxed{N^{\sigma}} \\ N : \sigma\end{array}}{(\lambda x.M)N : \tau}\,(\to E)
$$

Simple types are NOT preserved by expansion

No occurrences of N

$$\boxed{M[N/x]^{\tau}} \qquad \overset{\beta\text{-exp}}{\Longrightarrow} \qquad \dfrac{\boxed{M^{\tau}}}{\dfrac{M:\tau}{\lambda x.M:\sigma\to\tau}\,(\to I) \qquad N:?}$$

$$M[N/x]:\tau \qquad\qquad (\lambda x.M)N$$

Simple types are NOT preserved by expansion

Two or more occurrences of N

$$\boxed{N^{\sigma_1}} \qquad \boxed{N^{\sigma_2}}$$

$$N: \sigma_1 \qquad N: \sigma_2$$

$$\boxed{M[N/x]^{\tau}} \qquad \stackrel{\beta\text{-exp}}{\Longrightarrow} \qquad [x: ?] \quad N: ?$$

$$M[N/x]: \tau \qquad\qquad (\lambda x.M)N$$

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Simple types are NOT preserved by expansion

No occurrences of N

$$\boxed{M[N/x]^{\tau}} \qquad \overset{\beta\text{-exp}}{\Longrightarrow} \qquad \dfrac{\dfrac{\boxed{M^{\tau}}}{M : \tau}}{\lambda x.M : \sigma \to \tau}\,(\to I) \qquad N : ?$$

$$M[N/x] : \tau \qquad\qquad (\lambda x.M)N$$

No occurrences of N

Universal type Ω

$$\boxed{M[N/x]^{\tau}} \atop M[N/x] : \tau \qquad \xRightarrow{\beta\text{-exp}} \qquad \dfrac{\dfrac{\dfrac{[x:\Omega]}{\boxed{M^{\tau}}}}{\dfrac{M:\tau}{\lambda x.M : \Omega \to \tau}\,(\to I)} \qquad N:\Omega}{(\lambda x.M)N : \tau}\,(\to E)$$

Simple types are NOT preserved by expansion

Two or more occurrences of N

$$\boxed{N^{\sigma_1}} \qquad \boxed{N^{\sigma_2}}$$

$$N : \sigma_1 \qquad N : \sigma_2$$

$$\boxed{M[N/x]^{\tau}} \qquad \xrightarrow{\beta\text{-exp}} \qquad [x : ?] \quad N : ?$$

$$M[N/x] : \tau \qquad\qquad (\lambda x.M)N$$

Two or more occurrences of N

Type intersection $\cap$

$$\begin{array}{c} \boxed{N^{\sigma_1}} \quad \boxed{N^{\sigma_2}} \\ N:\sigma_1 \quad N:\sigma_2 \\ \boxed{M[N/x]^{\tau}} \\ M[N/x]:\tau \end{array} \quad \overset{\beta\text{-exp}}{\Longrightarrow} \quad \dfrac{\dfrac{\begin{array}{c} \dfrac{[x:\sigma_1\cap\sigma_2]}{x:\sigma_1}\,(\cap E) \quad \dfrac{[x:\sigma_1\cap\sigma_2]}{x:\sigma_2}\,(\cap E) \\ \boxed{M^{\tau}} \\ M:\tau \end{array}}{(\lambda x.M):\sigma_1\cap\sigma_2\to\tau}\,(\to I) \qquad \dfrac{\begin{array}{c} \boxed{N^{\sigma_1}} \quad \boxed{N^{\sigma_2}} \\ N:\sigma_1 \quad N:\sigma_2 \end{array}}{N:\sigma_1\cap\sigma_2}\,(\cap I)}{(\lambda x.M)N:\tau}\,(\to E)$$

Intersection types

$$(ax) \qquad \Gamma, x : \sigma \vdash x : \sigma$$

$$(\to E) \qquad \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

$$(\to I) \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}$$

Intersection types

$$(\Omega) \qquad \Gamma \vdash M : \Omega$$

Intersection types

$$(\Omega) \qquad \Gamma \vdash M : \Omega$$

$$(\cap I) \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau}$$

Intersection types

$$(\Omega) \qquad \Gamma \vdash M : \Omega$$

$$(\cap I) \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau}$$

$$(\cap E) \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \sigma} \qquad\qquad (\cap E) \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \tau}$$

Intersection types

$$(\Omega) \qquad \Gamma \vdash M : \Omega$$

$$(\cap I) \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau}$$

$$(\cap E) \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \sigma} \qquad\qquad (\cap E) \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \tau}$$

$$\frac{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon (\sigma \to \tau)\cap\sigma}{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon \sigma \to \tau}\,(\cap E) \qquad \frac{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon (\sigma \to \tau)\cap\sigma}{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon \sigma}\,(\cap E)$$

$$
\dfrac{\dfrac{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon (\sigma \to \tau)\cap\sigma}{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon \sigma \to \tau}\,(\cap E) \qquad \dfrac{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon (\sigma \to \tau)\cap\sigma}{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon \sigma}\,(\cap E)}{x\colon (\sigma \to \tau)\cap\sigma \vdash xx\colon \tau}\,(\to E)
$$

$$
\dfrac{\dfrac{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon (\sigma \to \tau)\cap\sigma}{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon \sigma \to \tau}\,(\cap E) \qquad \dfrac{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon (\sigma \to \tau)\cap\sigma}{x\colon (\sigma \to \tau)\cap\sigma \vdash x\colon \sigma}\,(\cap E)}{\dfrac{x\colon (\sigma \to \tau)\cap\sigma \vdash xx\colon \tau}{\vdash \lambda x.xx\colon (\sigma \to \tau)\cap\sigma \to \tau}\,(\to I)}\,(\to E)
$$

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Normalization properties

$$(\lambda xy.xyy)(\lambda z.z) \longrightarrow \lambda y.(\lambda z.z)yy \longrightarrow \lambda y.yy$$

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$M \in \mathcal{N}$ iff $M \twoheadrightarrow_\beta$ a normal form

$(\lambda xy.xyy)(\lambda z.z) \in \mathcal{N}$ $\qquad$ $\lambda x.x((\lambda y.yy)(\lambda y.yy)) \notin \mathcal{N}$

Normalization properties

$$(\lambda xy.xyy)(\lambda z.z) \longrightarrow \lambda y.(\lambda z.z)yy \longrightarrow \lambda y.yy$$

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$M \in \mathcal{N}$ iff $M \twoheadrightarrow_\beta$ a normal form

$$(\lambda xy.xyy)(\lambda z.z) \in \mathcal{N} \qquad \lambda x.x((\lambda y.yy)(\lambda y.yy)) \notin \mathcal{N}$$

$M \in \mathcal{HN}$ iff $M \twoheadrightarrow_\beta \lambda \vec{x}.y\vec{N}$

$$\lambda x.x((\lambda y.yy)(\lambda y.yy)) \in \mathcal{HN} \qquad \lambda x.(\lambda y.yy)(\lambda y.yy) \notin \mathcal{HN}$$

Normalization properties

$$(\lambda xy.xyy)(\lambda z.z) \longrightarrow \lambda y.(\lambda z.z)yy \longrightarrow \lambda y.yy$$

$$(\lambda y.yy)(\lambda y.yy) \longrightarrow (\lambda y.yy)(\lambda y.yy) \longrightarrow \ldots$$

$M \in \mathcal{N}$ iff $M \twoheadrightarrow_\beta$ a normal form

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$M \in \mathcal{HN}$ iff $M \twoheadrightarrow_\beta \lambda \vec{x}.y\vec{N}$

$$\lambda x.x((\lambda y.yy)(\lambda y.yy)) \in \mathcal{HN} \qquad \lambda x.(\lambda y.yy)(\lambda y.yy) \notin \mathcal{HN}$$

$M \in \mathcal{WN}$ iff $M \twoheadrightarrow_\beta \lambda x.N$ or $M \twoheadrightarrow_\beta x\vec{N}$

$$\lambda x.(\lambda y.yy)(\lambda y.yy) \in \mathcal{WN} \qquad (\lambda y.yy)(\lambda y.yy) \notin \mathcal{WN}$$

Normalization properties

$$(\lambda xy.xyy)(\lambda z.z) \longrightarrow \lambda y.(\lambda z.z)yy \longrightarrow \lambda y.yy$$

$$(\lambda y.yy)(\lambda y.yy) \longrightarrow (\lambda y.yy)(\lambda y.yy) \longrightarrow \dots$$

$M \in \mathcal{N}$ iff $M \twoheadrightarrow_\beta$ a normal form

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$M \in \mathcal{HN}$ iff $M \twoheadrightarrow_\beta \lambda \vec{x}.y\vec{N}$

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$M \in \mathcal{WN}$ iff $M \twoheadrightarrow_\beta \lambda x.N$ or $M \twoheadrightarrow_\beta x\vec{N}$

$$\lambda x.(\lambda y.yy)(\lambda y.yy) \in \mathcal{WN} \qquad (\lambda y.yy)(\lambda y.yy) \notin \mathcal{WN}$$

Characterization of $\mathcal{N}$ by types

$M \in \mathcal{N}$ iff $\Gamma \vdash M : \sigma$ for some Γ, σ not containing Ω

$$
\dfrac{
  \dfrac{x{:}\,(\sigma \to \tau)\cap\sigma \vdash x{:}\,(\sigma \to \tau)\cap\sigma}{x{:}\,(\sigma \to \tau)\cap\sigma \vdash x{:}\,\sigma \to \tau}\,(\cap E)
  \quad
  \dfrac{x{:}\,(\sigma \to \tau)\cap\sigma \vdash x{:}\,(\sigma \to \tau)\cap\sigma}{x{:}\,(\sigma \to \tau)\cap\sigma \vdash x{:}\,\sigma}\,(\cap E)
}{
  \dfrac{x{:}\,(\sigma \to \tau)\cap\sigma \vdash xx{:}\,\tau}{\vdash \lambda x.xx{:}\,(\sigma \to \tau)\cap\sigma \to \tau}\,(\to I)
}\,(\to E)
$$

Characterization of $\mathcal{HN}$ by types

$M \in \mathcal{HN}$ iff $\Gamma \vdash M : \sigma$ for some Γ, σ not containing Ω at top level

$$
\cfrac{\cfrac{x{:}\,\Omega \to \tau \vdash x{:}\,\Omega \to \tau \quad x{:}\,\Omega \to \tau \vdash (\lambda y.yy)(\lambda y.yy){:}\,\Omega}{x{:}\,\Omega \to \tau \vdash x((\lambda y.yy)(\lambda y.yy)){:}\,\tau}\,(\to E)}{\vdash \lambda x.x((\lambda y.yy)(\lambda y.yy)){:}\,(\Omega \to \tau) \to \tau}\,(\to I)
$$

Characterization of $\mathcal{WN}$ by types

$$M \in \mathcal{WN} \text{ iff } \Gamma \vdash M : \Omega \rightarrow \Omega$$

$$\frac{x : \Omega \vdash (\lambda y.yy)(\lambda y.yy) : \Omega}{\vdash \lambda x.(\lambda y.yy)(\lambda y.yy) : \Omega \rightarrow \Omega} (\rightarrow I)$$

Intersection types characterize also

strongly normalizable terms

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strongly normalizable terms

closable terms

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strongly normalizable terms

closable terms

terms of the **I**-calculus

Intersection types characterize also

strongly normalizable terms

closable terms

terms of the **I**-calculus

persistently normalizable terms

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. . .

Intersection types characterize also

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. . .

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Preorder on intersection types

$\sigma \leq \sigma \cap \sigma$

Preorder on intersection types

$\sigma \leq \sigma \cap \sigma$

$\sigma \cap \tau \leq \sigma, \ \sigma \cap \tau \leq \tau$

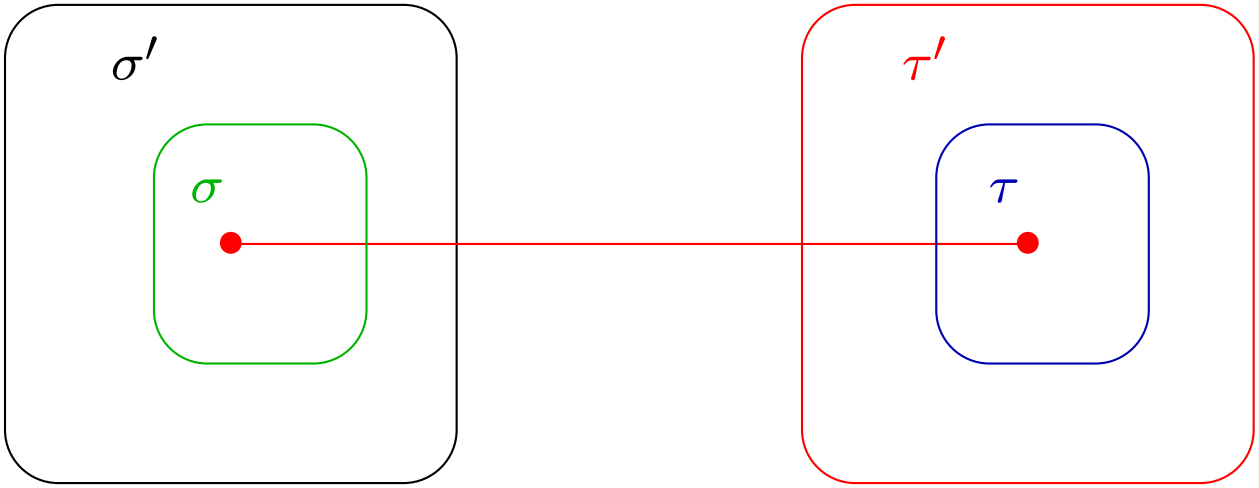
Preorder on intersection types

$\sigma \leq \sigma \cap \sigma$

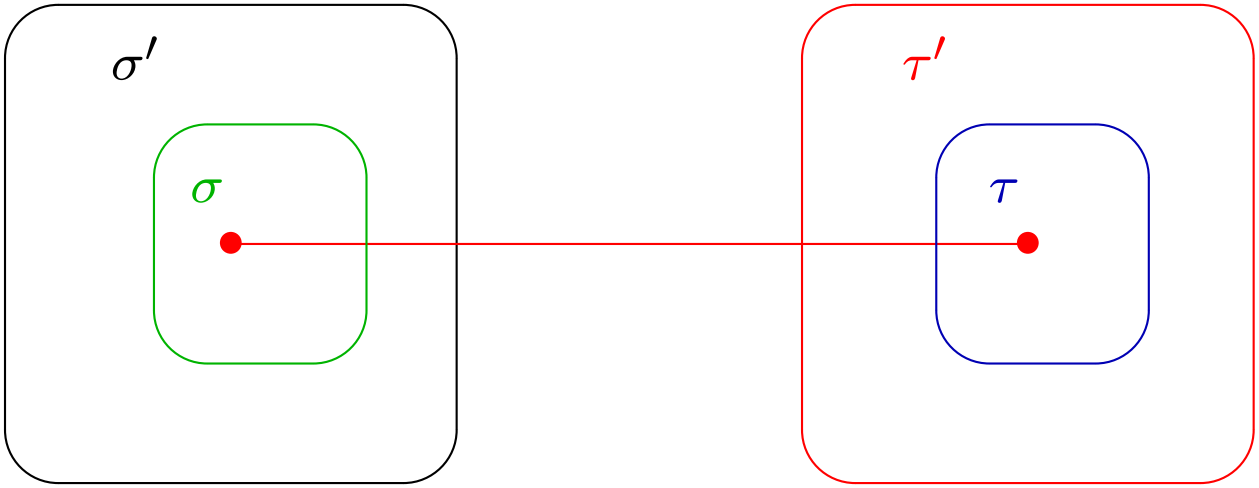
$\sigma \cap \tau \leq \sigma, \; \sigma \cap \tau \leq \tau$

$\sigma \leq \sigma', \; \tau \leq \tau' \Rightarrow \sigma \cap \tau \leq \sigma' \cap \tau'$

Preorder on intersection types

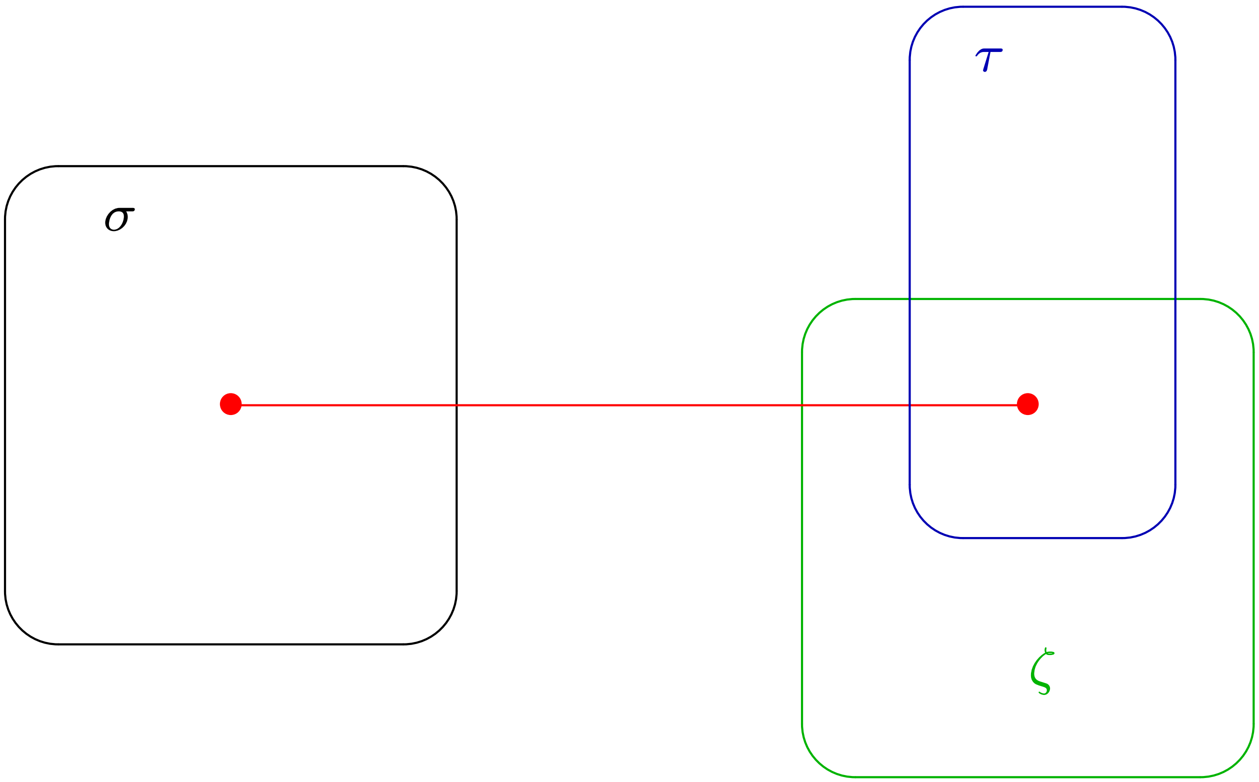


Preorder on intersection types

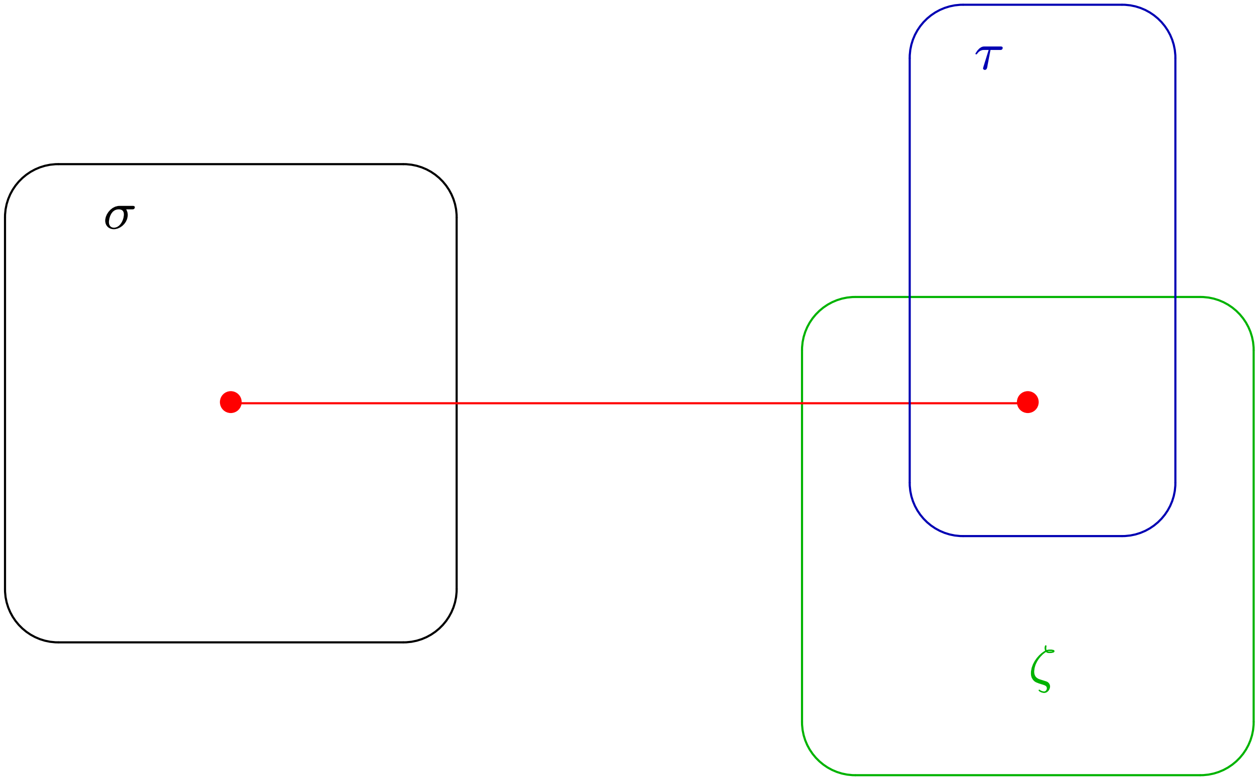


$$\sigma \leq \sigma',\ \tau \leq \tau' \Rightarrow \sigma' \rightarrow \tau \leq \sigma \rightarrow \tau'$$

Preorder on intersection types

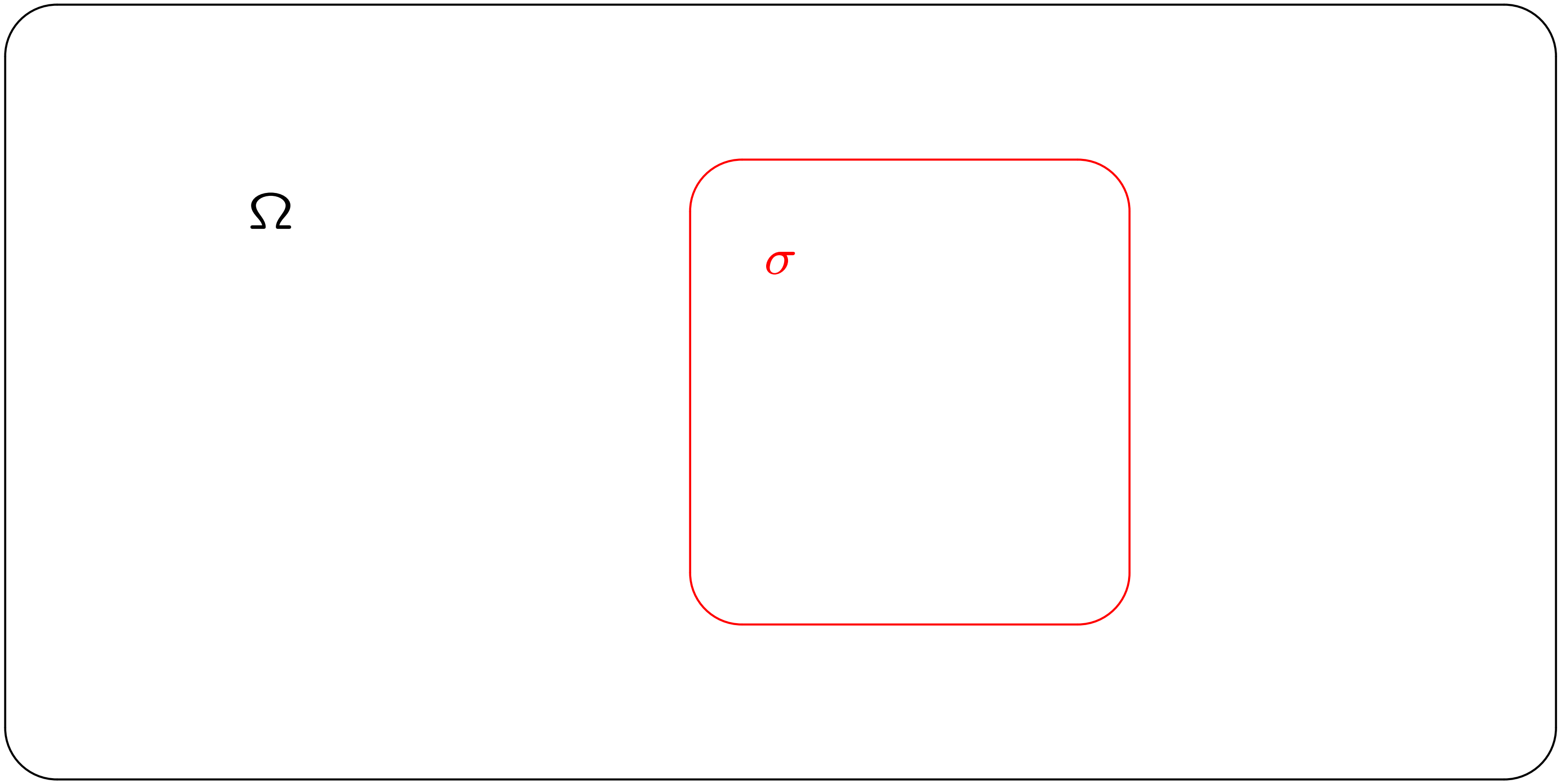


Preorder on intersection types

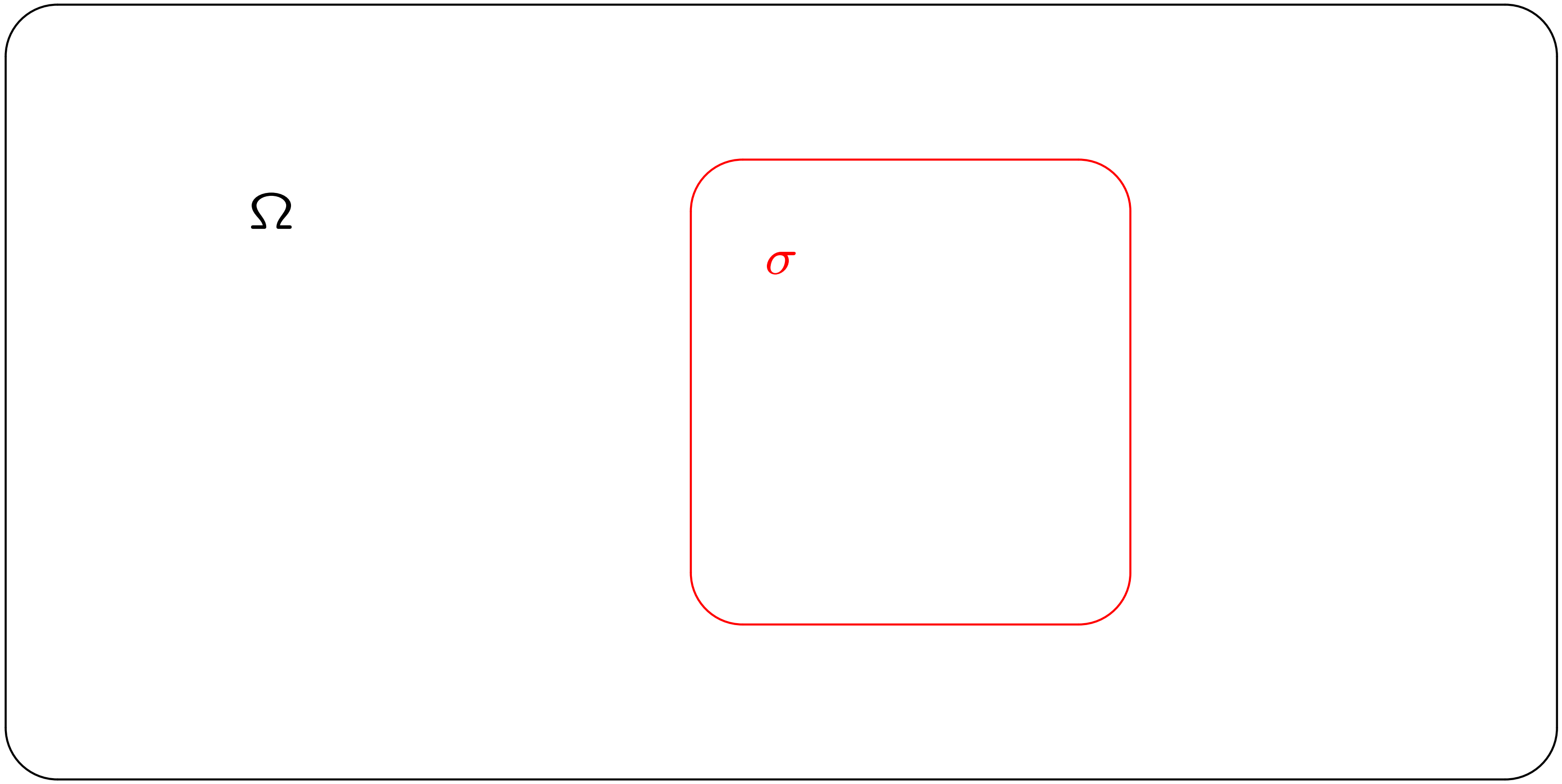


$$(\sigma \to \tau) \cap (\sigma \to \zeta) \leq \sigma \to \tau \cap \zeta$$

Preorder on intersection types

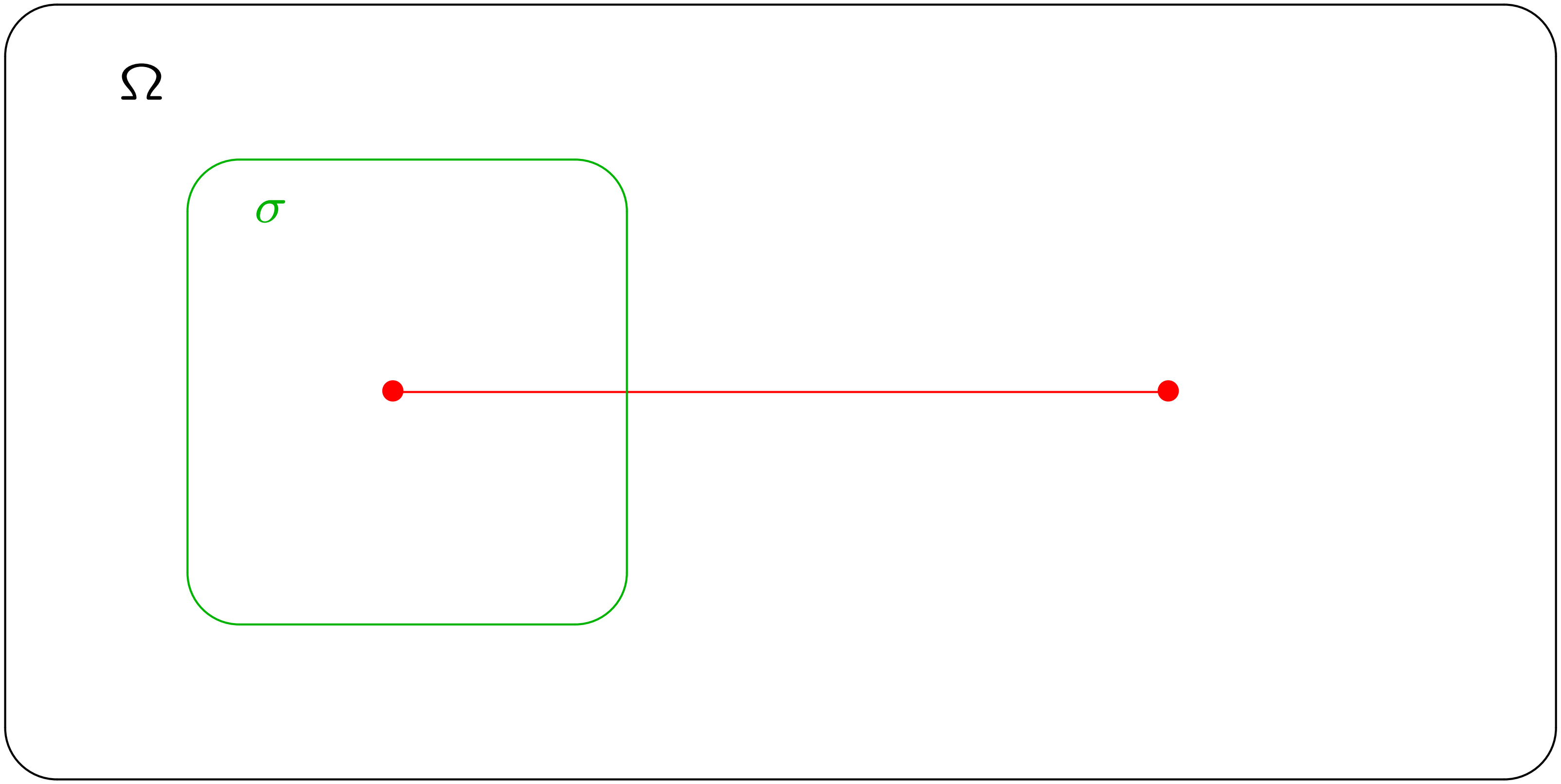


Preorder on intersection types

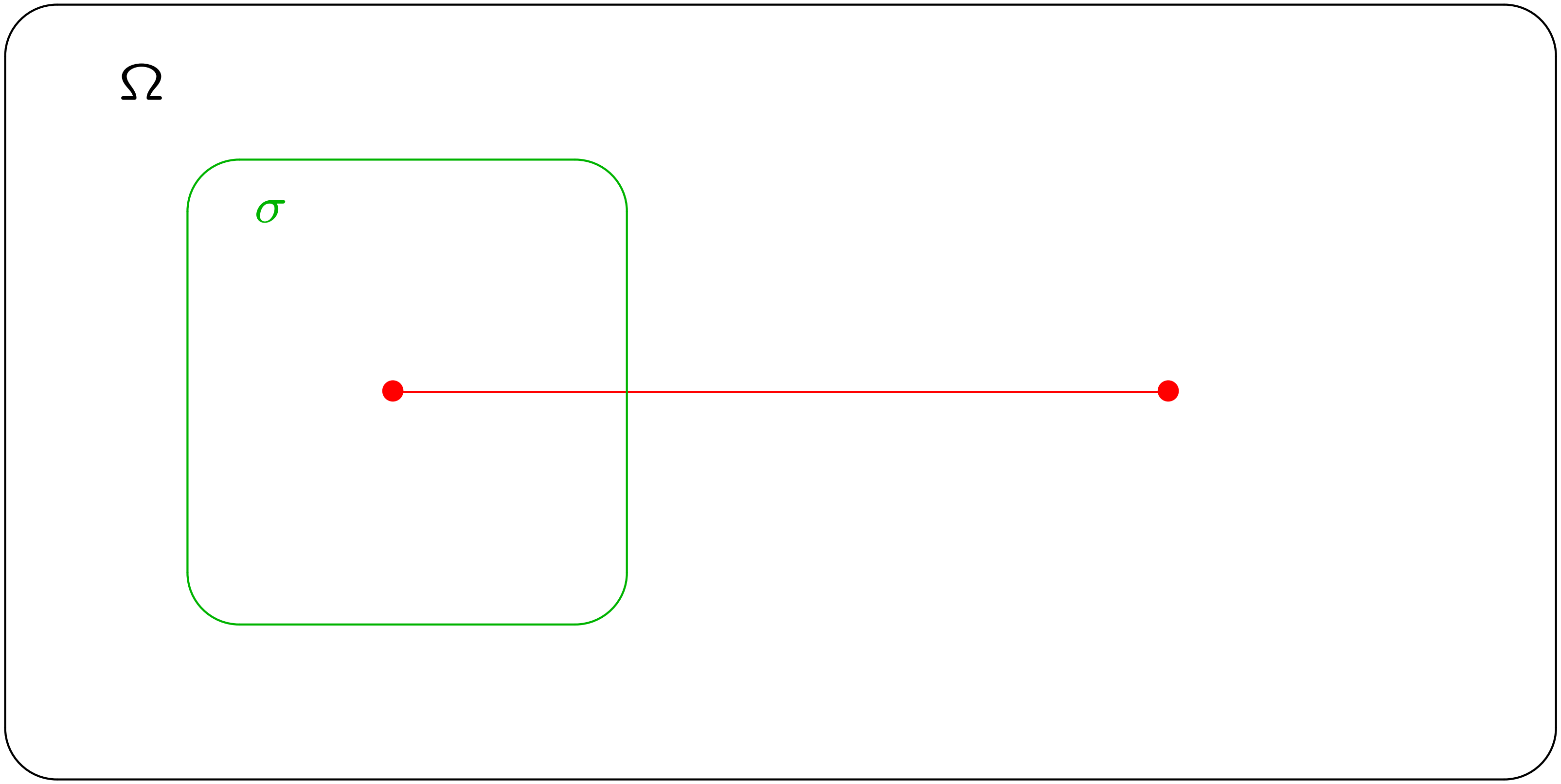


$$\sigma \leq \Omega$$

Preorder on intersection types



Preorder on intersection types



$$\sigma \rightarrow \Omega \leq \Omega \rightarrow \Omega$$

Preorder on intersection types

$\sigma \leq \sigma \cap \sigma$

$\sigma \cap \tau \leq \sigma, \ \sigma \cap \tau \leq \tau$

$\sigma \leq \sigma', \ \tau \leq \tau' \Rightarrow \sigma \cap \tau \leq \sigma' \cap \tau'$

$\sigma \leq \sigma', \ \tau \leq \tau' \Rightarrow \sigma' \to \tau \leq \sigma \to \tau'$

$(\sigma \to \tau) \cap (\sigma \to \zeta) \leq \sigma \to \tau \cap \zeta$

$\sigma \leq \Omega \quad \sigma \to \Omega \leq \Omega \to \Omega$

$\sigma \leq \sigma$

$\sigma \leq \tau, \ \tau \leq \zeta \ \Rightarrow \sigma \leq \zeta$

deleting Ω and replacing $\rightarrow$ to $\leq$

$\sigma \leq \sigma \cap \sigma$	$\sigma \cap \tau \leq \sigma,\ \sigma \cap \tau \leq \tau$
$\sigma \leq \sigma',\ \tau \leq \tau' \Rightarrow \sigma \cap \tau \leq \sigma' \cap \tau'$	$\sigma \leq \sigma',\ \tau \leq \tau' \Rightarrow \sigma' \rightarrow \tau \leq \sigma \rightarrow \tau'$
$(\sigma \rightarrow \tau) \cap (\sigma \rightarrow \zeta) \leq \sigma \rightarrow \tau \cap \zeta$	$\sigma \leq \Omega \quad \sigma \rightarrow \Omega \leq \Omega \rightarrow \Omega$
$\sigma \leq \sigma$	$\sigma \leq \tau,\ \tau \leq \zeta \ \Rightarrow \sigma \leq \zeta$

deleting Ω and replacing $\rightarrow$ to $\leq$

$\sigma \rightarrow \sigma \cap \sigma$ $\quad$ $\sigma \cap \tau \rightarrow \sigma, \ \sigma \cap \tau \rightarrow \tau$

$\sigma \rightarrow \sigma', \ \tau \rightarrow \tau' \Rightarrow \sigma \cap \sigma' \rightarrow \tau \cap \tau'$ $\quad$ $\sigma \rightarrow \sigma', \ \tau \rightarrow \tau' \Rightarrow (\sigma' \rightarrow \tau) \rightarrow \sigma \rightarrow \tau'$

$(\sigma \rightarrow \tau) \cap (\sigma \rightarrow \zeta) \rightarrow \sigma \rightarrow \tau \cap \zeta$

$\sigma \rightarrow \sigma$ $\quad$ $\sigma \rightarrow \tau, \ \tau \rightarrow \zeta \ \Rightarrow \sigma \rightarrow \zeta$

the minimal relevant logic B_+

Subsumption rule

$$(\leq) \qquad \frac{\Gamma \vdash M : \sigma \quad \sigma \leq \tau}{\Gamma \vdash M : \tau}$$

$$\dfrac{\dfrac{x\colon \sigma\to\zeta \vdash x\colon \sigma\to\zeta}{x\colon \sigma\to\zeta \vdash x\colon \sigma\cap\tau\to\zeta}\,(\leq)}{\vdash \lambda x.x\colon (\sigma\to\zeta)\to\sigma\cap\tau\to\zeta}\,(\to I)$$

Plan of the talk

- simple types
- intersection types
- properties of λ-terms
- type preorder
- filter models
- Stone duality

The set $\mathcal{F}$ of filters

A filter is a set X of intersection types such that:

- $\Omega \in X$
- if $\sigma \leq \tau$ and $\sigma \in X$, then $\tau \in X$
- if $\sigma, \tau \in X$, then $\sigma \cap \tau \in X$

$\mathcal{F}$ is the set of filters

$\uparrow X$ is the filter generated by X

$\uparrow \sigma$ is $\uparrow \{\sigma\}$

$\langle \mathcal{F}, \subseteq \rangle$ is an ω-algebraic complete lattice

$$\begin{tikzcd}
& \mathcal{T} \arrow[d, dotted] & \\
& \uparrow (X \cup Y) \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& X \cap Y \arrow[d, dotted] & \\
& \uparrow \Omega &
\end{tikzcd}$$

$\langle \mathcal{F}, \subseteq \rangle$ is a λ-model (filter model)

For any lambda term M and environment $\rho : \mathtt{var} \to \mathcal{F}$

$$[\![M]\!]^{\mathcal{F}}_{\rho} = \{\tau \in \mathcal{T} \mid \exists \Gamma \models \rho.\ \Gamma \vdash M : \tau\}$$

where $\Gamma \models \rho$ if and only if $(x : \sigma) \in \Gamma$ implies $\sigma \in \rho(x)$.

If $\Gamma \vdash M : \tau$ and $M =_{\beta} N$, then $\Gamma \vdash N : \tau$

with suitable type preorders we can obtain filter models isomorphic to

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Scott inverse limit models;

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Scott P_ω model;

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$\ldots$

Plan of the talk

- simple types
- intersection types
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- Stone duality

we stared from types and arrived to models: what is the framework?

Stone dualities

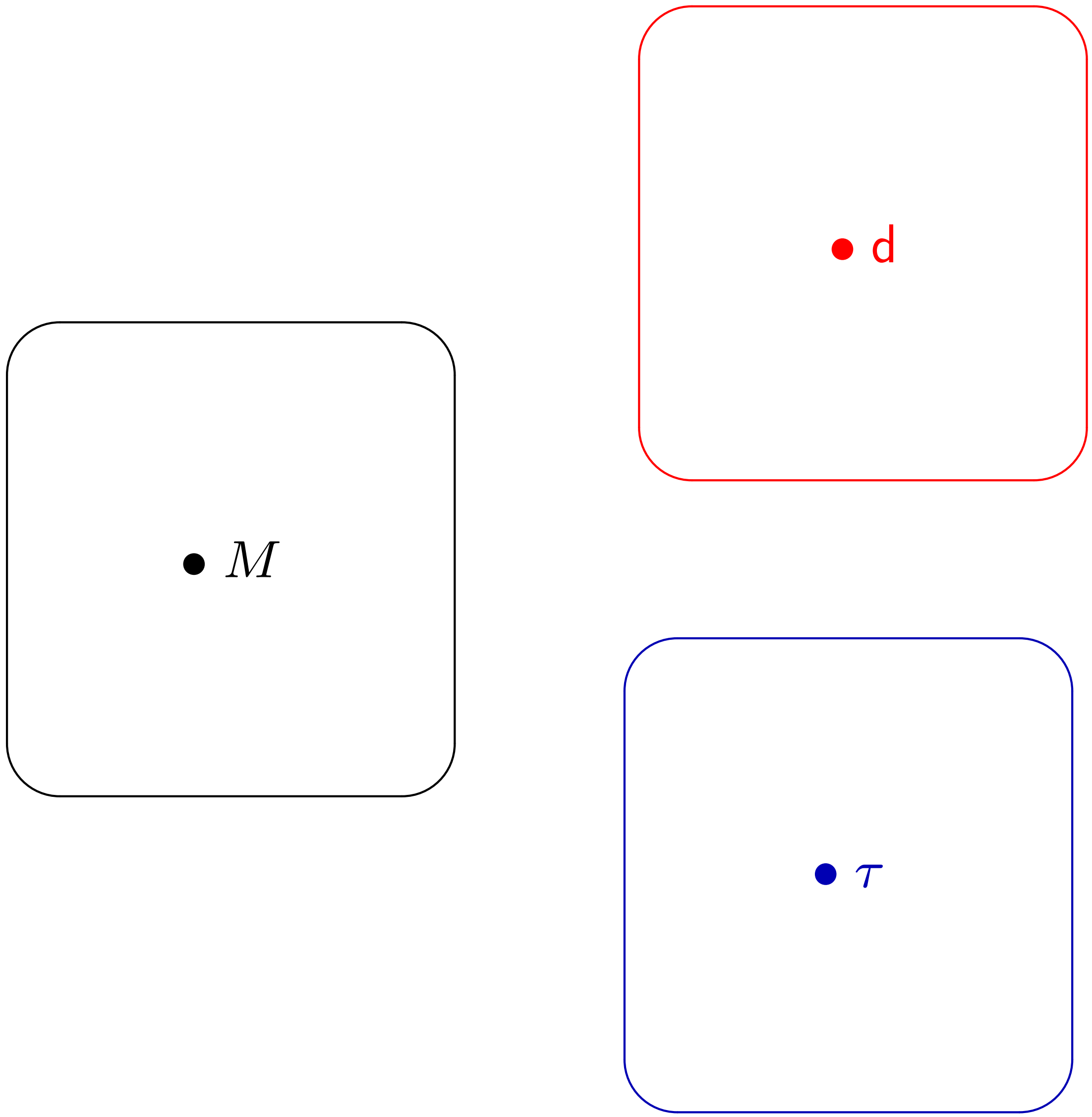
topological spaces as partial orders

Stone spaces as Boolean algebras
(Stone, 36)

Scott domains as information systems
(Scott, 82)

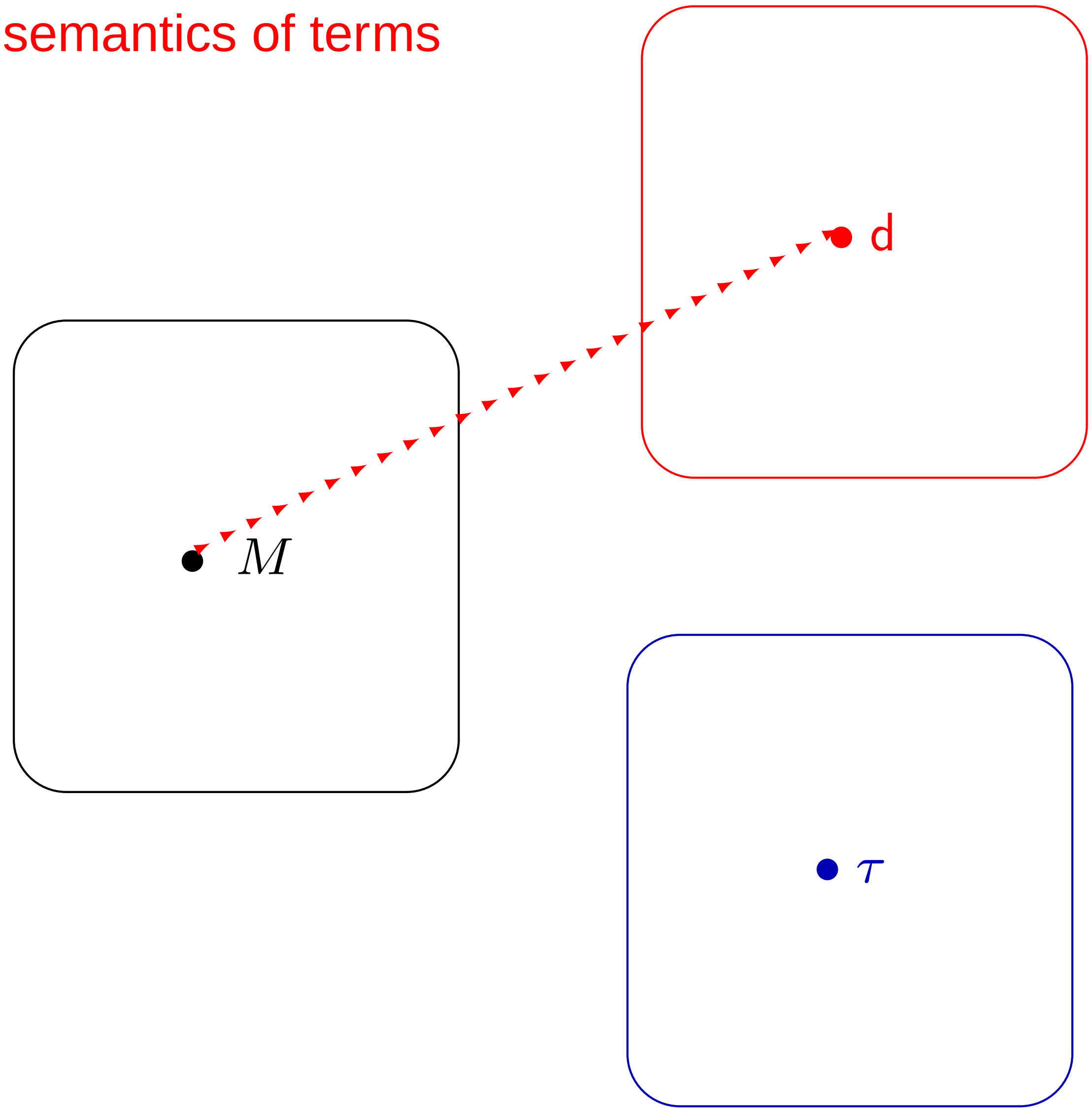
ω-algebraic complete lattices as intersection type theories
(Coppo et al., 84)

SFP domains as pre-locales
(Abramsky, 91)

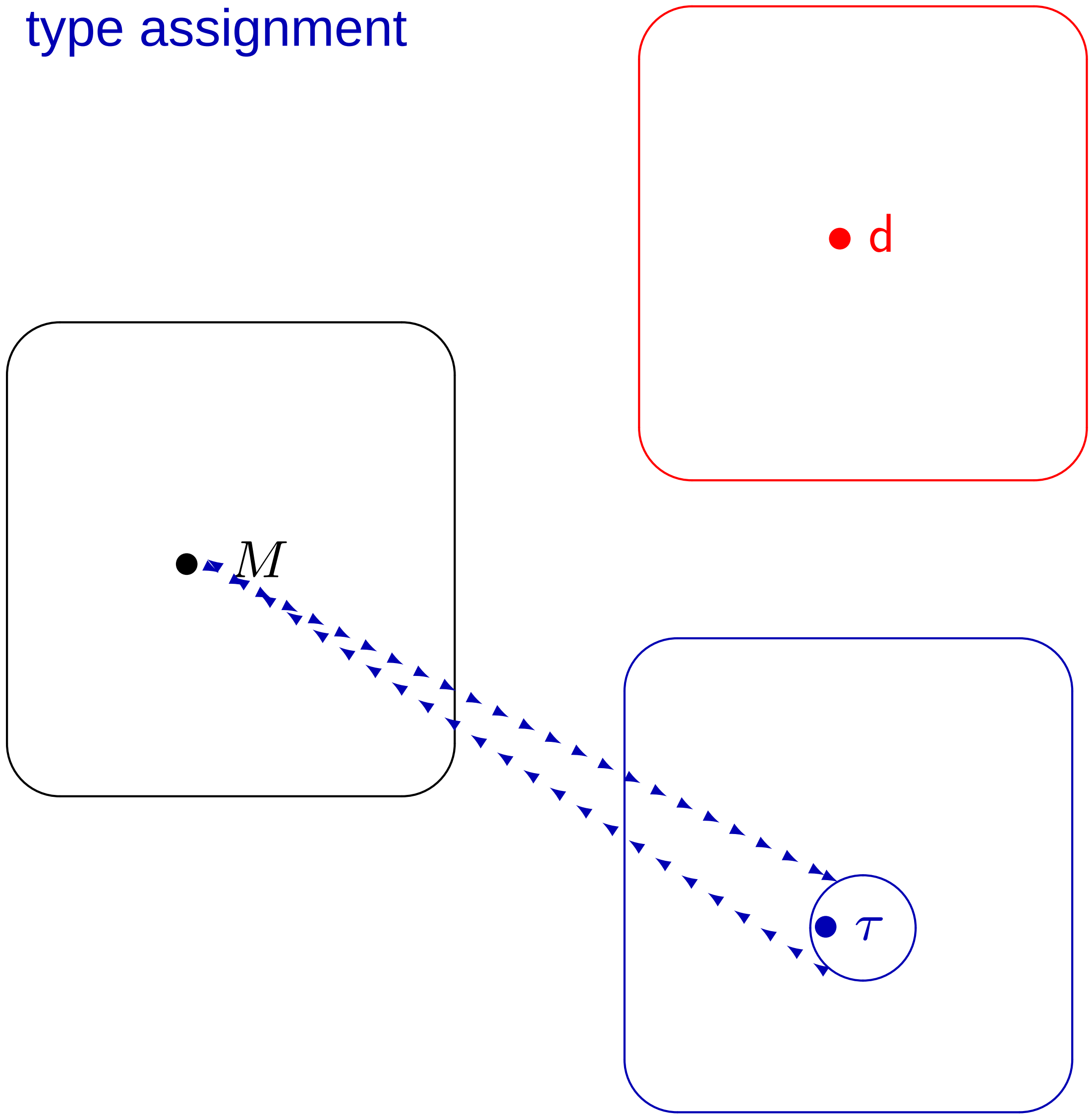


M
d
τ

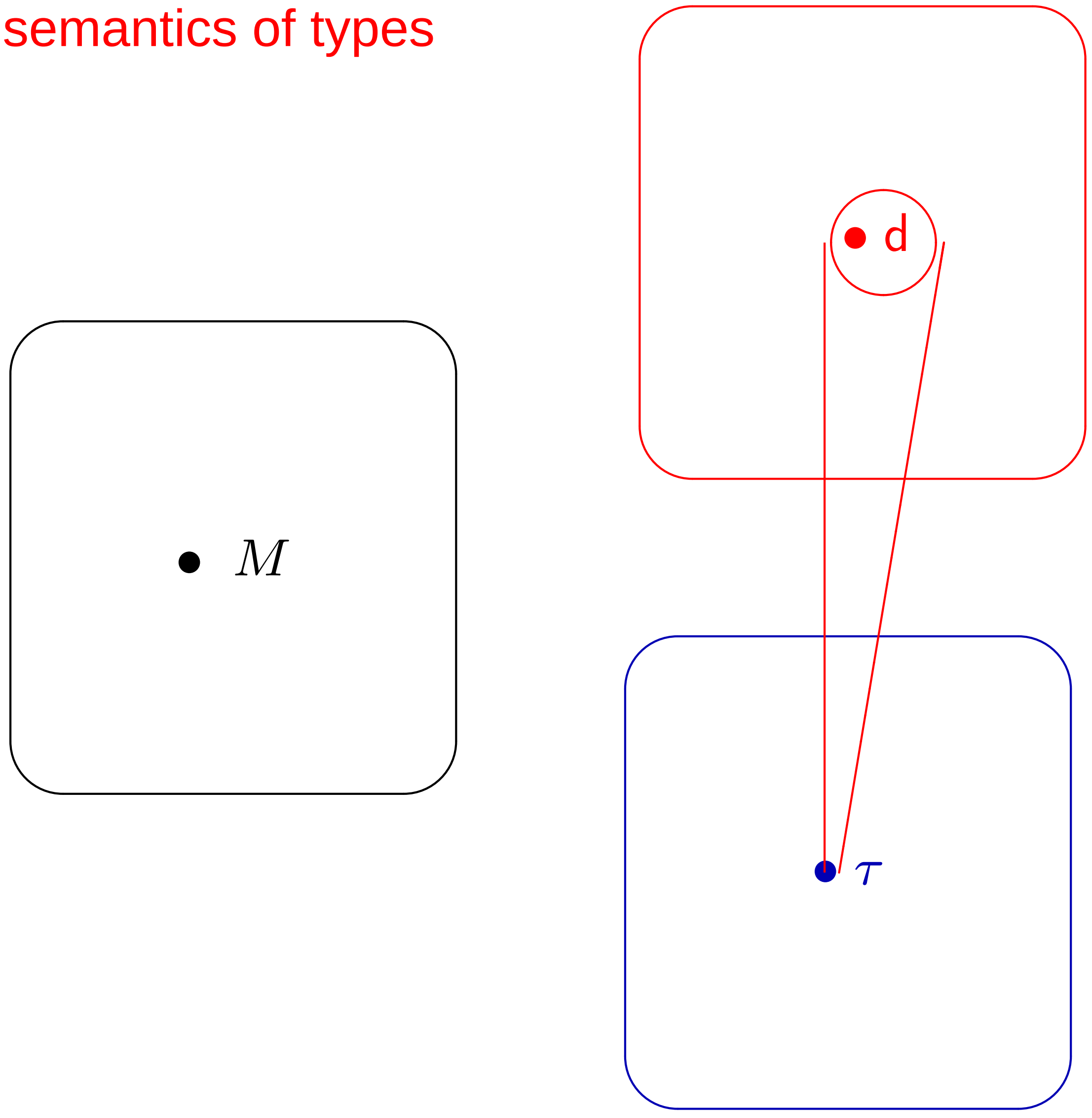
semantics of terms



type assignment



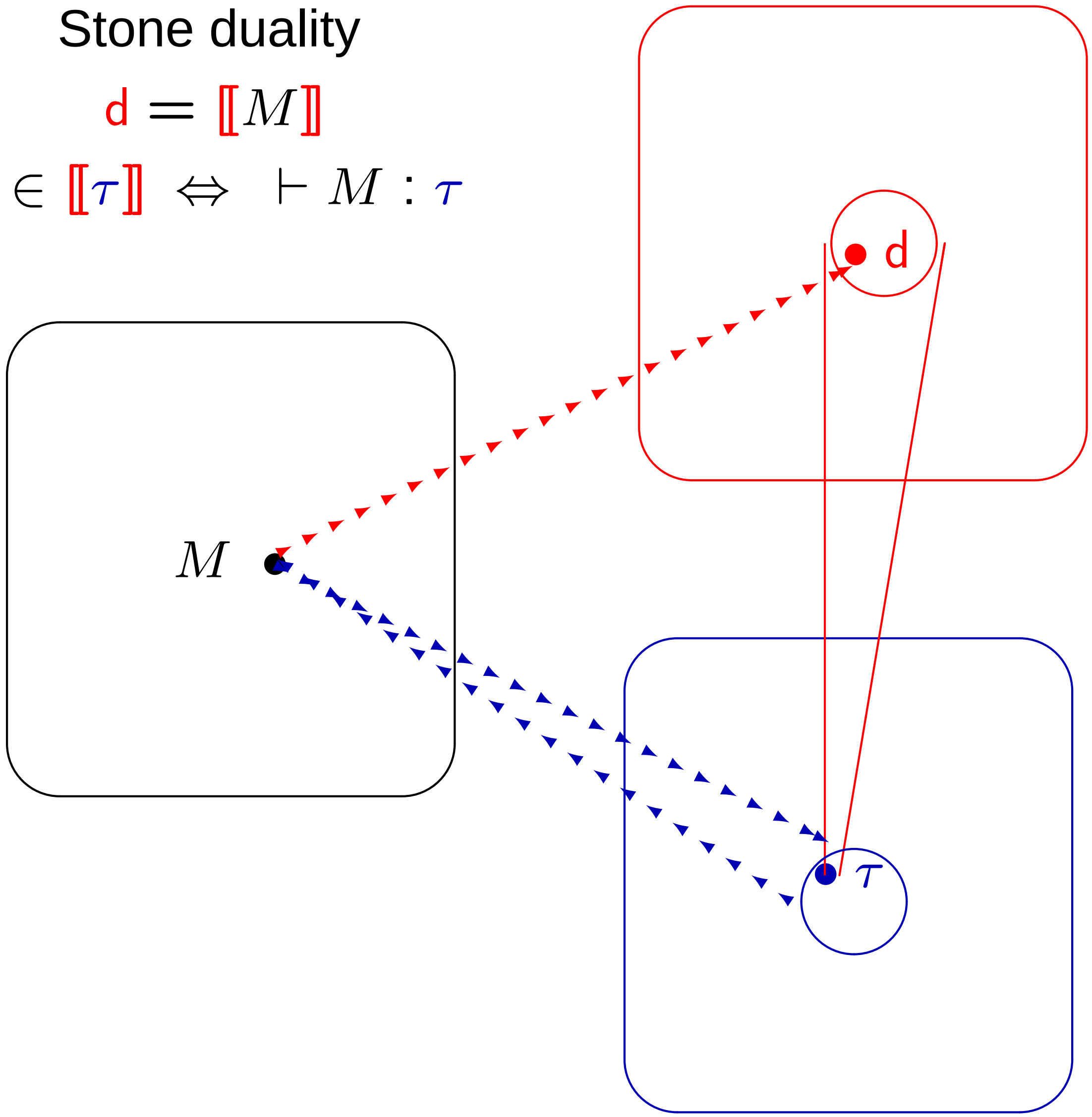
semantics of types



Stone duality

$$\mathsf{d} = [\![M]\!]$$

$$\mathsf{d} \in [\![\tau]\!] \iff \vdash M : \tau$$



to sum up

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intersection types are a bridge between logic and λ-models

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thank you for your attention