A note about minimal non-deterministic automata

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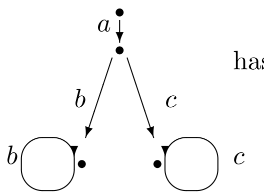
Abstract

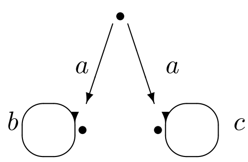
With every rational language we associate a canonical non-deterministic automaton, that subsumes all possible "minimal" automata recognizing this language.

1 Introduction

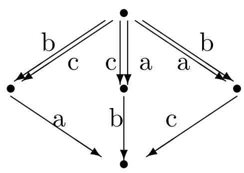
The minimal deterministic automaton associated with a rational language is not, in general, "minimal" with respect to the number of states: for instance,



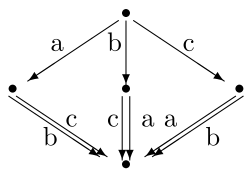
has more states than



But if we do not require automata to be deterministic, it is a problem to associate a unique minimal automaton with a given language, as shown by the following example: both automata



and



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recognize the language $\{ab, ac, bc, ba, ca, cb\}$, both are minimal with respect to the number of states, and there is no homomorphism of either automaton onto the other.

This note simply states that the problem *has* been solved: in [CC70], Christian Carrez associated, with every rational language L, a canonical non-deterministic automaton $\mathcal{C}_L$, claiming that the "truly minimal" automaton recognizing L was to be searched for among restrictions of $\mathcal{C}_L$. Indeed, this automaton $\mathcal{C}_L$ may be considered as a terminal object of the class of automata recognizing subsets of L, in the following sense:

- an automaton $\mathcal{A}$ recognizes a subset of L if, and only if, there is a homomorphism of $\mathcal{A}$ into $\mathcal{C}_L$;
- any surjective homomorphism of $\mathcal{C}_L$ onto an automaton recognizing a subset of L is an isomorphism.

2 Definitions and notations

Let Σ be a countable alphabet. An *automaton* $\mathcal{A} = (S, T, I, F)$ is defined by a set S of states, a set $T \subseteq S \times \Sigma \times S$ of transitions, and a set I (resp. F) of initial (resp. final) states.

With each state s of $\mathcal{A}$, we associate a pair of languages:

- its *history* $H_s^{\mathcal{A}}$, the set of words u such that there is a path labelled u from some initial state to s;
- its *prophecy* $P_s^{\mathcal{A}}$, the set of words u such that there is a path labelled u from s to some final state.

We require any automaton to be *trim* (or *complete*, according to the terminology of [KS]), i.e. that every state is reachable from some initial state, and that some final state is reachable from every state: thus every state has a non-empty history and a non-empty prophecy.

A *homomorphism* of an automaton $\mathcal{A}$ into an automaton $\mathcal{A}'$ is an application h of the states of $\mathcal{A}$ into the states of $\mathcal{A}'$ such that

- if s is an initial (resp. final) state of $\mathcal{A}$, $h(s)$ is an initial (resp. final) state of $\mathcal{A}'$;
- if $s \xrightarrow{a} t$ is a transition of $\mathcal{A}$, $h(s) \xrightarrow{a} h(t)$ is a transition of $\mathcal{A}'$.

Clearly, if h is a homomorphism of $\mathcal{A}$ into $\mathcal{A}'$, then for each state s of $\mathcal{A}$, $H_s^{\mathcal{A}} \subseteq H_{h(s)}^{\mathcal{A}'}$ and $P_s^{\mathcal{A}} \subseteq P_{h(s)}^{\mathcal{A}'}$. We shall say that $\mathcal{A}$ is *irreducible* if any surjective homomorphism of $\mathcal{A}$ onto an automaton recognizing the same language is an isomorphism.

3 Construction and results

Let $L \subseteq \Sigma^*$ be any language. For each subset K of Σ^*, let

$$\phi_L(K) = \{u \in \Sigma^*/uK \subseteq \text{Ł}\}$$

$$\pi_L(K) = \{u \in \Sigma^*/Ku \subseteq \text{Ł}\}.$$

We define the automata $\mathcal{C}_L$ as follows: the states of $\mathcal{C}_L$ are all $\phi_L(P)$ such that neither P nor $\phi_L(P)$ is empty. A state $\phi_L(P)$ is initial if $\phi_L(P)$ contains the empty word, and final if $\phi_L(P)$ is a subset of L. Finally, the transitions of $\mathcal{C}_L$ are all $\phi_L(P) \stackrel{a}{\to} \phi_L(P')$ such that $\phi_L(P)a \subseteq \phi_L(P')$. It is straightforward to check that each state s of $\mathcal{C}_L$ is its own history: more precisely

$$s = H_s^{\mathcal{C}_L} = \phi_L(P_s^{\mathcal{C}_L})$$

and it can be proved that, symmetrically:

$$P_s^{\mathcal{C}_L} = \pi_L(H_s^{\mathcal{C}_L})$$

We have the following results:

Proposition 1 *L is rational iff $\mathcal{C}_L$ is a finite-state automaton.*

Proof *If*: obvious. *Only if*: each $\phi_L(P)$ is the union of the equivalence classes of its elements (with respect to the syntactical congruence associated with L). $\square$

Proposition 2 *A (trim) automaton $\mathcal{A}$ recognizes a subset of L iff there is a homomorphism of $\mathcal{A}$ into $\mathcal{C}_L$.*

Proof *If*: obvious. *Only if*: for each state s of $\mathcal{A}$, let $h(s) = \phi_L(P_s^{\mathcal{A}})$; then h is a homomorphism of $\mathcal{A}$ into $\mathcal{C}_L$. $\square$

As a corollary, any irreducible automaton $\mathcal{A}$ recognizing L is isomorphic to some restriction of $\mathcal{C}_L$, and thus $\mathcal{C}_L$ contains all possible "minimal" automata recognizing L.

Proposition 3 *Any surjective homomorphism of $\mathcal{C}_L$ onto an automaton recognizing a subset of L is an isomorphism.*

Sketch of the proof If h is a surjective homomorphism of $\mathcal{C}_L$ onto $\mathcal{A}$ such that $L(\mathcal{A}) \subseteq L$, then for each state s of $\mathcal{C}_L$, we must have $H_{h(s)}^{\mathcal{A}} = H_s^{\mathcal{C}_L}$; since $s = H_s^{\mathcal{C}_L}$ it follows that h is injective, and that s is initial (resp. final) if, and only if, $h(s)$ is an initial (resp. final) state of $\mathcal{A}$. $\square$

4 Related works

In [CNP91], B. Courcelle, D. Niwinski and A. Podelski give a geometrical interpretation of the minimization: for any trim automaton $\mathcal{A}$, the set

$$\{(H_s^{\mathcal{A}}, P_s^{\mathcal{A}})/s \text{ a state of } \mathcal{A}\ \}$$

is a *rectangular decomposition* of the binary relation

$$R_L = \{(u,v) \in \Sigma^* \times \Sigma^* / uv \in L\}$$

in the sense that

$$R_L = \bigcup_{s \in S} H_s^{\mathcal{A}} \times P_s^{\mathcal{A}}$$

In a rectangular decomposition of R_L, each "rectangle" (H, P) must satisfy $H \subseteq \phi_L(P)$, and symmetrically, $P \subseteq \pi_L(H)$. Rectangular decompositions may be partially ordered: say a decomposition D' is *less than* D if every rectangle of D is included in some rectangle of D'. From this point of view, $\mathcal{C}_L$ is the automaton associated with the *least* rectangular decomposition of R_L, i.e. the set of all rectangles (H, P) such that $H = \phi_L(P)$ and $P = \pi_L(H)$. And indeed, the construction of $\mathcal{C}_L$, which depends only on L, is "symmetrical" (we might as well define each state as its own prophecy), whereas any attempt of reducing a particular automaton usually privileges either the history or the prophecy. The construction of N. Klarlund and F. B. Schneider [KS91] (aiming to characterize the language inclusion between infinite-state automata) sequentially privileges *both*. We briefly recall it: let $\mathcal{A}$ be an automaton, S its set of states. Define $\mathcal{HA}$ and $\mathcal{DA}$ as follows: their states are the non-empty subsets of S, a subset X of S is initial (resp. final) if every element of X is an initial (resp. final) state of $\mathcal{A}$, and

- $X \stackrel{a}{\to} X'$ is a transition of $\mathcal{HA}$ if for every $s \in X$, there is some $s' \in X'$ such that $s \stackrel{a}{\to} s'$ is a transition of $\mathcal{A}$;
- $X \stackrel{a}{\to} X'$ is a transition of $\mathcal{DA}$ if for every $s' \in X'$, there is some $s \in X$ such that $s \stackrel{a}{\to} s'$ is a transition of $\mathcal{A}$.

N. Klarlund and F. B. Schneider's main result is that if a trim automaton $\mathcal{A}$ recognizes a language L, then any trim automaton $\mathcal{A}'$ recognizes a subset of L if, and only if, there is a homomorphism of $\mathcal{A}'$ into $\mathcal{HDA}$. (And indeed, the construction of $\mathcal{C}_L$ is quite similar to the construction of $\mathcal{HA}$, for some deterministic trim automaton $\mathcal{A}$ recognizing L.)

References

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