and so

$$Z^2 < 1 + \frac{2}{\mu^2}\cdot \tag{6}$$

Since X, Y, Z are positive integers, inequality (6) implies that Z can take only a finite number of values. The same holds for Y (on using (5)) and X (on using (3) or the relation $X < Y+Z$).

Since $X+Y+Z=\mu A$, A takes only a finite number of values. So do the sides a, b, c, in view of relations (2). Thus $N(\lambda)$ is finite for each positive $\lambda > 2$.

If $\lambda \leqq 2$ (i.e., $\mu \leqq 1$), we obtain the same conclusion on using the second inequality in (4) and proceeding as before.

We shall next show that $N(\lambda)=0$ for $\lambda > \sqrt{8}$ (i.e., $\mu > \sqrt{2}$). For such a value of μ we have from (6) that $Z=1$. The relation (5) then shows that $Y=1$. Since $X < Y+Z$, we now have $X=1$. Using relations (3) and (2), one obtains $\mu=3$, $A=\sqrt{3}$, $a=b=c=2$.

REMARKS. The remarks made at the beginning of this note show that $N(1)=5$ and $N(2)=1$. Our theorem shows that $N(3)=N(4)= \cdots =0$.

It would be interesting to consider a similar problem for a quadrilateral, and in general, a polygon of n sides ($n>2$).

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Material should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada.

HOW OFTEN DOES AN INTEGER OCCUR AS A BINOMIAL COEFFICIENT?

DAVID SINGMASTER, Polytechnic of the South Bank, London

Let $N(a)$ be the number of times a occurs as a binomial coefficient, $\binom{n}{k}$. We have $N(1)=\infty$, $N(2)=1$, $N(3)=N(4)=N(5)=2$, $N(6)=3$, etc. Clearly, for $a>1$, $N(a)<\infty$. Below we establish that $N(a)=O(\log a)$. We conjecture that $N(a)=O(1)$, that is, that the number of solutions of $\binom{n}{k}=a$ is bounded for $a>1$. Erdös, in a private communication, concurs in this conjecture and states that it must be very hard. In a later communication, he suggests trying to show $N(a)=O(\log \log a)$.

If we let $M(k)$ be the first integer a such that $N(a)=k$, we have: $M(1)=2$, $M(2)=3$, $M(3)=6$, $M(4)=10$, $M(6)=120$. The next values would be interesting to know.

PROPOSITION. $N(a)=O(\log a)$.

Proof: Let b be the first b such that $\binom{2b}{b}>a$. Now

$$\binom{i+j}{i}=\binom{i+j}{j}$$

is monotonically increasing in i and in j; hence

$$\binom{b+i+b+j}{b+i} \geqq \binom{b+b+j}{b} \geqq \binom{2b}{b} > a \qquad \text{for all } i, j \geqq 0.$$

Thus $\binom{i+j}{j}=a$ implies $i<b$ or $j<b$. Again by monotonicity, for each value of i (or j),

$$\binom{i+j}{j} = a$$

has at most one solution. Hence $N(a)<2b$. Now $\binom{2b}{b}\geqq 2^b$, so we have

$$a \geqq \binom{2(b-1)}{b-1} \geqq 2^{b-1};$$

hence $b\leqq \log_2 a+1$, and $N(a)\leqq 2+2\log_2 a=O(\log a)$.

Added in Proof: I have recently found that $M(8)=3003$ and this is the only solution to $N(a)\geqq 8$ with $a\leqq 2^{23}$. There are six solutions to $N(a)=6$ with $a\leqq 2^{23}$, namely: 120, 210, 1540, 7140, 11628, and 24310.

HOW IS A GRAPH'S BETTI NUMBER RELATED TO ITS GENUS?

R. A. DUKE, University of Washington

If a connected graph G has v vertices and e edges, the number $1-v+e$ is called the (1-dimensional) **Betti number** of G, denoted by $\beta(G)$. This value, which is nonnegative for connected G, was one of the first numerical characteristics of a graph studied, having been introduced by von Staudt [11] and Kirchhoff [5]. The Betti number $\beta(G)$ is also the rank of the fundamental group of G (see [9], Ch. 6) and is related to the **Euler characteristic** of G, $\chi(G)$, by the equation $\beta(G)=1-\chi(G)$.

The genus of a graph is defined in terms of its embeddings in compact, closed, orientable 2-dimensional manifolds. Each such manifold can be formed by attaching an appropriate number of "handles" to a copy of the 2-sphere. For a manifold M, the number of handles required is called the **genus** of M, denoted by $\gamma(M)$, and is related to the Euler characteristic of M by the formula $2\gamma(M)=2-\chi(M)$. Each graph can be embedded in some orientable 2-manifold by placing its vertices on the 2-sphere and attaching one handle to accommodate each edge. In general, however, such a construction yields a manifold of unneces-