STABLE MODELS OF TYPED λ-CALCULI

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Abstract :

Following Scott, the denotational semantics of programming languages are usually built from the notion of continuous functions. The need for restricted models has been emphasized by Plotkin and Milner, which showed thats continuous function models did not capture all operational properties of ALGOL-like sequential languages. We present new model constructions from a notion of stable function. This requires the introduction of two different orderings between stable functions which give very different cpo structures to the function spaces. We show that Milner's fully abstract model of Plotkin's PCF language only contains stable functions.

1. INTRODUCTION

The programming languages we consider are typed λ-calculi with first-order primitive functions and fixpoint operators at all types. One example is Plotkin's PCF [13], where the basic functions are the usual arithmetic functions and conditionals. In these languages, there is a clear distinction between *programs* which can be directly run by computers (for example closed terms of type integer), and *procedures* which are only intented to be part of programs. The operational semantics of a program is clearly defined by the result of its computation. The operational semantics of a procedure is harder to define, but we can easily *compare* operationally procedures by $M_1 \subset_{op} M_2$ if the result of any program containing M_1 is improved when M_1 is replaced by M_2.

On the other hand a *denotational semantics* associates values to programs and procedures in ordered denotation spaces. We can therefore also compare procedures by comparing their denotations. In particular semantics are usually obtained by *models* in which a term is interpreted as a function from environments to values. We write $M \subset_{M} M'$ if the denotation of M is less than that of M' in a model M. The problem arises of knowing whether $\subset_M$ and $\subset_{op}$ coincide in a model M, i.e if all operational properties of terms are reflected by the model. A model where $\subset_M = \subset_{op}$ is called *fully abstract* by Plotkin and Milner [13,12]. Plotkin [13] shows that the classical Scott-Milner [18,12] model of all continuous functions is *not* fully abstract for PCF, mainly because the *sequential* character of the interpretation mechanism has no counterpart in the model [2,3]. Milner [12] gives a construction of fully abstract models from syntactic material, and shows that PCF has a unique fully abstract model. However very few is known about the semantic properties of Milner's model. A fully semantic construction of it would give better understanding of the notion of sequentially computable function and functional (see also [16]) and might lead to new proofrules in LCF-like systems [10].

Hence we would like to construct models in which not all continuous functions appear but only functions which are "sequential" in some sense. Various notions of sequential functions may be found in the literature [7,12,19]. All of them are sufficient at first-order but none of them is satisfactory at higher order. However we show that model constructions are possible form the notion of *stability* [1], which is an "approximation" to sequentiality (see [3] for details). A function is stable if a *definite* information is needed from the argument in order to obtain a given approximation of the result. On a computational point of view, stability implies the existense of "optimal computations" but not the existence of "optimal computation rules" which would correspond to sequentiality (see [4,9]).

The first section is devoted to preliminaries : ordered sets, finite and λ-expressions and the syntactic ordering $<$ (Ω-match ordering of [8,20]), interpretations and the operational ordering. In the second section, we define the continuous semantics and models. We give two fully abstract model constructions. The first one is due to Milner [12] and based on a completion of the operational classes of terms. It applies to PCF. The second one uses infinite terms and the ordering $<$, and applies to the symbolic interpretation where the function symbols are left uninterpreted. The last section is devoted to stable models. We show that the first-order function domains of PCF are easily characterized by the sequentiality definitions of [7,12,19]. We give a general category - theoretic construction of models from product and exponentiation diagrams. After indicating that these diagrams do not exist with sequential functions, we show that they do exist with stable functions and obtain a first model construction. However we need to introduce a new ordering $<_s$ (the stable ordering) between functions to construct exponentiations, and the model we obtain is not order-extensional. Therefore we introduce *bidomains* which are sets ordered by two orderings : the pointwise ordering $\subset$ and the stable ordering $<_s$. Functions must be continuous w.r.t. $\subset$ and $<_s$ but stable only w.r.t. $<_s$.

By forgetting either $\subset$ or $<_s$ we obtain two semantics and two models, one of which is order-extensional. This model is still not fully abstract for PCF, but we show that Milner's fully abstract model indeed contains bidomains and stable functions. To finish, we relate the semantic orderings $\subset$ and $<_s$ to the syntactic orderings $\subset_{op}$ and $<$.

All proofs and technical details are omitted, see [3].

2. TERMS AND INTERPRETATIONS. THE OPERATIONAL ORDERING.

2.1. Complete partial orders and continuous functions.

An *ω-cpo* (or simply cpo) $\langle D,\subset,\perp\rangle$ is a partial order having a least element $\perp$ and such that any increasing ω-chain $\alpha_1,\alpha_2,\ldots,\alpha_n,\ldots$ has a least upper bound or *lub* $\bigcup_n <_n$. A *directed* set is a non empty denumerable set $\Delta \subset D$ such that $\forall\alpha_1,\alpha_2 \in \Delta, \exists\alpha_3 \in \Delta$ s.t. $\alpha_1 \subset \alpha_3$ and $\alpha_2 \subset \alpha_3$. In a cpo, any directed set has a lub $\bigcup\Delta$. A cpo is *consistently complete* if any two $\alpha,\beta \in D$ which have an upper-bound have a lub $\alpha \cup \beta$, or equivalently if any nonempty $X \subset D$ has a greatest lower bound or glb $\bigcap X$.

An element $\alpha \in D$ is *isolated* if $\alpha \subset \bigcup\Delta$ implies $\exists\,\delta \in \Delta$ s.t. $\alpha \subset \delta$ for any directed $\Delta \subset D$. A lub of isolated elements is always isolated while a glb need not be. A cpo D is *ω-algebraic* if it has denumerably many isolated elements and for every $\alpha \in D$ the set $A(\alpha) = \{\alpha' \in D \mid \alpha' \subset \alpha, \alpha' \text{ isolated}\}$ is directed and has lub α. Then isolated elements are written $e,e',\ldots$ and their set is called the *basis* $b(D)$ of D. Cartesian products of (ω-algebraic) cpos are ordered component-wise and are (ω-algebraic) cpos.

A function $h : D \to D'$ is *monotonic* if $\alpha \subset \alpha'$ implies $h(\alpha) \subset h(\alpha')$ for any $\alpha,\alpha' \in D$ and *continuous* if it is monotonic and satisfies $h(\bigcup\alpha_n) = \bigcup h(\alpha_n)$ for any ω-chain $\alpha_1,\ldots,\alpha_n,\ldots$ If D and D' are ω-algebraic then h is continuous iff

$$\forall\alpha \in D, \forall e' \subset h(\alpha), \exists e \subset \alpha \text{ s.t. } e' \subset h(e)$$

In ω-algebraic consistently complete cpos, the glb function $\bigcap$ is always continuous.

The *pointwise ordering* between functions is defined by $h \subset h'$ if $\forall\alpha\ h(\alpha) \subset h'(\alpha)$. The set $[D \to D']$ of continuous functions from D to D' ordered by the pointwise ordering is a cpo.

A *fixpoint* of $h : D \to D$ is a $d \in D$ s.t. $\alpha = h(\alpha)$. Any continuous function $h : D \to D$ has a least fixpoint $Yh = \bigcup h^n(\perp)$

If $\langle D,\subset,\perp\rangle$ is a partial order, an *ideal* $\Delta \subset D$ is a directed set such that $\forall\alpha \in \Delta, \forall\alpha' \in D, \alpha' \subset \alpha \Rightarrow \alpha' \in D$. The *completion by ideals* $\langle D^\infty,\subset,\perp\rangle$ of D is the cpo of ideals of D ordered by inclusion and with least element $\{\perp\}$. If $I : D \to D^\infty$ is defined by $I(\alpha) = \{\alpha' \mid \alpha' \subset \alpha\}$, then for any cpo D' and monotonic $h : D \to D'$ there exist a unique continuous $h^\infty : D^\infty \to D'$ such that $h = h^\infty \circ I$. (This completion is used to turn monotonic functions into continuous ones).

2.2. Types, terms and contexts.

Let $K = \{\kappa_1,\kappa_2,\ldots,\kappa_n\}$ be a set of *ground types*. The set T of *types* is generated by $K \subset T$ and $(\sigma \to \tau) \in T$ if $\sigma,\tau \in T$ if $\sigma,\tau \in T$. We write $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_n \to \kappa$ instead of $\sigma_1 \to (\sigma_2 \to (\cdots \to (\sigma_n \to \kappa)))$ and notice that every $\sigma \in T$ can be written in a unique way in this form. A *first-order type* is a type of the form $\kappa_1 \times \kappa_2 \times \cdots \times \kappa_n \to \kappa$

For each $\sigma \in T$, let $V^\sigma = \{x_1^\sigma, x_2^\sigma, \ldots\}$ be a denumerable set of *variables of type* σ, and let $V = \cup \{V^\sigma | \sigma \in T\}$. Let $F = \{f_1^{\sigma_1}, f_2^{\sigma_2}, \ldots\}$ be a set of basic function symbols of *first-order type* σ_i (see [12] for justifications). For each $\sigma \in T$, let Ω^σ be the undefined symbol of type σ and $Y^{((\sigma\to\sigma)\to\sigma)}$ be the *fixpoint combinator* of type $((\sigma\to\sigma)\to\sigma)$

The sets Λ^σ of terms of type σ are generated by :

(i) $V^\sigma \subset \Lambda^\sigma$, $f_i^{\sigma_i} \in \Lambda^{\sigma_i}$ for $f_i^{\sigma_i} \in F$, and $\Omega^\sigma \in \Lambda^\sigma$

(ii) $(Y^{((\sigma\to\sigma)\to\sigma)}\ M^{(\sigma\to\sigma)}) \in \Lambda^\sigma$ for $M^{(\sigma\to\sigma)} \in \Lambda^{(\sigma\to\sigma)}$ and $N^\sigma \in \Lambda^\sigma$

(iii) $(M^{(\sigma\to\tau)}\ N^\sigma) \in \Lambda^\tau$ for $M^{(\sigma\to\tau)} \in \Lambda^{(\sigma\to\tau)}$ and $N^\sigma \in \Lambda^\sigma$

(iv) $(\lambda x^\sigma.N^\tau) \in \Lambda^{\sigma\to\tau}$ for $x^\sigma \in V^\sigma$ and $N^\tau \in \Lambda^\tau$

We omit types whenever possible and use standard abbreviations : $MN_1N_2\ldots N_k$ or $M\vec{N}$ for $(\ldots((MN_1)N_2)\ldots N_k)$ and $\lambda x_1x_2\ldots x_m.M$ or $\lambda\vec{x}.M$ for $(\lambda x_1(\lambda x_2(\ldots(\lambda x_m\ M)\ldots))$ We denote by $M^\sigma[N^\tau/x^\tau]$ the result of the substitution of N^τ to all free occurrences of x^τ in M^σ. A term without free variables is *closed*. We also use the context notation $C[\ ]$. Context are expressions with *holes* $[\ ]_i^{\sigma_i}$ to be filled by terms of appropriate types (contexts may create bindings like in $\lambda x.[x]$).

The relations $\overset{*}{\underset{\beta}{\to}}, \overset{*}{\underset{\eta}{\to}}, \overset{*}{\underset{Y}{\to}}$ are generated by the conversion rules $(\lambda x\ M)N \underset{\beta}{\to} M[N/x]$, $\lambda x.\ Mx \underset{\eta}{\to} M$ if x is not free in M and $YM \underset{Y}{\to} M(YM)$. These relations preserve types. Expressions of the form $(\lambda x\ M)N$ are called β-*redexes* (similarly for η-*redexes* or Y-*redexes*). We write $\overset{*}{\to} = (\underset{\beta}{\to} \cup \underset{Y}{\to})^*$, since we do not consider the η-conversion rule as a computation rule but as an extensionality rule.

2.3. The example of PCF.

The arithmetic language PCF [13] has ground types $\mathbb{N}$ (integers) and $\mathbb{T}$ (booleans) and the following basic function symbols :

$\underline{0}, \underline{1}, \underline{2}, \ldots, \underline{n}$: type $\mathbb{N}$, $\underline{tt}$, $\underline{ff}$: type $\mathbb{T}$

$\underline{+1}$, $\underline{-1}$: type $\mathbb{N} \to \mathbb{N}$, $\underline{zero}$: type $\mathbb{N} \to \mathbb{T}$

$\underline{\text{Cond}\,\mathbb{N}}$: type $\mathbb{T} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $\underline{\text{Cond}\,\mathbb{T}}$: type $\mathbb{T} \times \mathbb{T} \times \mathbb{T} \to \mathbb{T}$

Plotkin [13] gives reduction rules for these function symbols. We shall instead follow Milner [12] and give *interpretations* of the symbols in section 2.5.

2.4. Infinite terms

We now construct infinite terms as in Lévy [8]. (Notice that we did not allow Y to appear as a subexpression, but only to appear in combinations YM. This will simplify the construction of infinite terms). The set $N^\sigma \subset \Lambda^\sigma$ of $\omega\beta Y$ - *normal forms* is the least set containing Ω^σ anc containing $\lambda x_1 x_2 \ldots x_k . s a_1 a_2 \ldots a_m \in \Lambda^\sigma$ if $s \in F \cup V$ and $a_1 \in N^{\sigma 1}$, $a_2 \in N^{\sigma 2}, \ldots, a_m \in N^{\sigma m}$. We use letters a, b to denote elements of $N = \cup\{N^\sigma | \sigma \in T\}$. We order N by the least ordering $<$ such that

(i) $\Omega^\sigma < a^\sigma$ for all $a^\sigma \in N^\sigma$

(ii) $\lambda x_1 x_2 \ldots x_k .\ s a_1 a_2 \ldots a_m < \lambda x_1 x_2 \ldots x_k .\ s b_1 b_2 \ldots b_m$ if $a_1 < b_1$, $a_2 < b_2, \ldots a_m < b_m$.

Then each N^σ is a partial order with least element Ω^σ. Its ideal completion $N^{\sigma\infty}$ is an ω-algebraic cpo. Elements of $N^{\sigma\infty}$ are written A^σ, $B^\sigma, \ldots$ and called *infinite terms*.

A term $M \in \Lambda$ is in *head normal form* or *hnf* [20] if it has the form $M = \lambda \vec{x}.s\vec{M}$ with $s \in F \cup V$. If M is not in hnf, it has the form $\lambda\vec{x}.R\vec{M}$ where R is an Ω or a β-redex or a Y-redex. The mapping $\omega : \Lambda^\sigma \to N^\sigma$ is defined by :

(i) $\omega(M^\sigma) = \Omega^\sigma$ if M is not in hnf

(ii) $\omega(M^\sigma) = \lambda\vec{x}.s\omega(M_1)\ \omega(M_2)\ldots\omega(M_n)$ if $M = \lambda\vec{x}.sM_1M_2\ldots M_n$

and $\omega(M)$ is called the *direct approximation* of M. If $M \overset{*}{\to} M'$, then $\omega(M) < \omega(M')$. By church-Rosser theorem, the set $A(M) = \{\omega(M') | M \overset{*}{\to} M'\}$ is directed in N and its lub in N^∞ is denoted by $\mathrm{val}(M)$ and called the *syntactic value* of M. The following result is essential.

2.4.1. Syntactic continuity theorem [8,20,22].

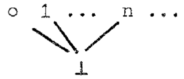
Let $C^\sigma[\ ]$ be a context with hole $[\ ]^\tau$. Then for any $M^\tau \in \Lambda^\tau$ we have :

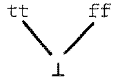
$$\forall b < \mathrm{val}(C[M])\ \ \exists a < \mathrm{val}(M)\ \text{ s.t. }\ b < \mathrm{val}(C[a]) \qquad \blacksquare$$

Only a finite approximation fo M is needed to compute a finite approximation of $C[M]$. Hence $\mathrm{val}(C[M]) = \cup\{\mathrm{val}(C[a]) | a < \mathrm{val}(M)\}$ and every context $C^\sigma[\ ]$ with hole $[\ ]^\tau$ defines a continuous mapping from $N^{\tau\infty}$ to $N^{\sigma\infty}$ by $\mathrm{val}(C[A]) = \cup\{\mathrm{val}\ C[a] | a < A\}$. Then $\mathrm{val}(C[M]) = C[\mathrm{val}(M)]$ by 2.3.1. Noticing that $\mathrm{val}(A) = A$, we shall drop out all val symbols and write $M < M'$ for $\mathrm{val}(M) < \mathrm{val}(M')$, $M < A$ for $\mathrm{val}(M) < A$ etc...

2.5. Interpretations and operational orderings.

An *interpretation* A of (K, F) is given by a cpo D_κ for each ground type $\kappa \in K$ and a continuous function $A(f) : D_{\kappa_1} \times D_{\kappa_2} \times \ldots \times D_{\kappa_n} \to D_\kappa$ for each function symbol $f \in F$ of type $\kappa_1 \times \kappa_2 \times \ldots \times \kappa_n \to \kappa$. In the example of PCF, we take :

$D_{\mathbb{N}} = N_\perp =$ 

$D_{\mathbb{T}} = T_\perp =$ 

Of course $A(\underline{n}) = n$, $A(\underline{tt}) = tt$, $A(\underline{ff}) = ff$, and $A(\underline{+1})$, $A(\underline{-1})$, $A(\underline{zero})$, $A(\underline{\text{Cond}\,\mathbb{N}})$ and $A(\underline{\text{Cond}\,\mathbb{T}})$ are the usual successor, predecessor, test to zero and conditionals.

We cannot directly use an interpretation to give a semantic value to any term, since we have no semantic domains at higher types. However we can interpret *programs*, i.e. closed terms of ground type denoted by P, P'.... Since we restricted the type of function symbols to be *first order*, it is easy to see that if P is a program then val(P) is an infinite term only containing basic function symbols and ground Ω's (no variables, no λ's, no higher type Ω's). Hence $A(\text{val}(P))$ is directly defined as in usual first-order semantics : for $a \in N^{\kappa}$ take $A(a) = \bot$ if $a = \Omega$, $A(a) = A(f)\big(A(a_1), A(a_2), \ldots, A(a_n)\big)$ if $a = f(a_1, a_2, \ldots, a_n)$, for $A \in N^{\kappa\infty}$, take $A(A) = \cup\{A(a) \mid a < A\}$.

We definie the *operational relation* $\sqsubset$op by [13,12] :

(i) $P \sqsubset_{op} P'$ if $A(\text{val}(P)) \subset_{\kappa} A(\text{val}(P'))$

(ii) $M_1^{\sigma} \sqsubset_{op} M_2^{\sigma}$ if $P[M_1^{\sigma}] \sqsubset_{op} P[M_2^{\sigma}]$ for every context $P[\ ]$ such that $P[M_1^{\sigma}]$ and $P[M_2^{\sigma}]$ are programs.

The relation $\sqsubset$op is a preorder in Λ^{σ}, and we denote by $\equiv$op the corresponding equivalence. As an example, in PCF :
$\lambda x^{\mathbb{N}}.\, x \equiv_{op} Y\,(\lambda fx.\underline{\text{Cond}\,\mathbb{N}}(\text{Zero}\ x)\ \ \underline{0}\ \ (\underline{+1}(f(\underline{-1}x))))$

Where the right part is the usual recursive definition of the identity in integers.

3. SEMANTICS AND MODELS

3.1. Syntactically continuous semantics.

A *syntactically continuous semantics* or simply a *semantics* is given by cop's $\langle E^{\sigma}, \leqslant_{\sigma}, \bot_{\sigma}\rangle$, $\sigma \in T$ and a type - preserving mapping $[\![\ ]\!] : \Lambda \to \cup E^{\sigma}$ such that

(i) $M \overset{*}{\underset{\beta}{\to}} M'$ implies $[\![M]\!] = [\![M']\!]$

(ii) $[\![M_1^{\sigma}]\!] \leqslant_{\sigma} [\![M_2^{\sigma}]\!]$ implies $\forall C^{\tau}[\ ]$, $[\![C[M_1]]\!] \leqslant_{\tau} [\![C[M]]\!]$

(iii) If $[\![M_1^{\sigma}]\!], [\![M_2^{\sigma}]\!], \ldots, [\![M_n^{\sigma}]\!], \ldots$ is an increasing chain in E^{σ} with limit $[\![M^{\sigma}]\!]$, then for any $C^{\tau}[\ ]$ the chain $[\![C[M_n]]\!]$ has limit $[\![C[M]]\!]$ in E^{τ}

(iv) $[\![\Omega^{\sigma}]\!] = \bot_{\sigma}$ and $[\![YM]\!] = \cup\, [\![M^n\ \Omega]\!]$ at all types.

3.1.1. Theorem.

If $M \underset{Y}{\to} M'$, then $[\![M]\!] = [\![M']\!]$. For all $M \in \Lambda$, $[\![M]\!] = \cup\ \{[\![a]\!] \mid a < M\}$ ∎

By the first part, $M \overset{*}{\to} M'$ implies $[\![M]\!] = [\![M']\!]$. The second part is analoguous to Wadsworth's result in D^{∞} [20], and allows to turn $[\![\ ]\!]$ into a continuous mapping from $N^{\sigma\infty}$ to E^{σ} by $[\![A]\!] = \cup\ \{[\![a]\!] \mid a < A\}$. This is consistent since $[\![M]\!] = [\![\text{val}(M)]\!]$. We call η-*semantics* a semantics such that $[\![\lambda x.Mx]\!] = [\![M]\!]$ for any M and x not free in M. A semantics is a η-semantics iff it is *term-order extensional*, i.e. satisfies $[\![Mx]\!] \leqslant [\![M'x]\!] \Rightarrow [\![M]\!] \leqslant [\![M']\!]$ if x is not free in M and M'.

The function $\text{val} : \Lambda \to N^{\infty}$ is a semantics, and is indeed an initial semantics for a suitable notion of morphism (left to the reader).

3.2. Continuous models.

We are interested here in semantics defined by models, where application and λ-abstraction are interpreted by functional application and abstraction, and where the basic function symbols are interpreted by the functions of a given interpretation A (notice that we left open the interpretation of those symbols in semantics).

A *premodel* M of Λ is a collection of cpo's $\langle D^{\sigma}_{M}, \leqslant_{\sigma}, \perp_{\sigma} \rangle$ together with continuous application mappings $. : D^{\sigma\to\tau}_{M} \times D^{\sigma}_{M} \to D^{\tau}_{M}$ which satisfy $\forall \alpha \in D^{\sigma}_{M}, \perp_{\sigma\to\tau}.\alpha = \perp_{\tau}$ and $(\forall \beta \in D^{\sigma}_{M}, \alpha.\beta = \alpha'.\beta) \Rightarrow \alpha = \alpha'$ for any $\alpha, \alpha' \in D^{\sigma\to\tau}_{M}$. Hence $D^{\sigma\to\tau}_{M}$ can be identified to a set of continuous mappings from D^{σ}_{M} to D^{τ}_{M}. A premodel is *order - extensional* if $(\forall \beta, \alpha.\beta \leqslant_{\tau} \alpha'.\beta) \Rightarrow \alpha \leqslant_{\sigma\to\tau} \alpha'$ holds for any $\alpha, \alpha' \in D^{\sigma\to\tau}_{M}$. Then the ordering in $D^{\sigma\to\tau}_{M}$ is the pointwise ordering and $\leqslant_{\sigma}$ will be simply written $\subset_{\sigma}$.

An *environnement* ρ is a type - preserving mapping $\rho : V \to \bigcup D^{\sigma}_{M}$. We order the set Env_{M} of environnements by $\rho \leqslant \rho'$ if $\rho(x^{\sigma}) \leqslant_{\sigma} \rho'(x^{\sigma})$ for all $x^{\sigma} \in V$. Then Env_{M} is a cpo with least element $\perp$ such that $\perp(x^{\sigma}) = \perp_{\sigma}$ for all σ. For $\rho \in \text{Env}_{M}$, $x^{\sigma} \in V$ and $\alpha \in D^{\sigma}_{M}$, we denote by $\rho[x \leftarrow \alpha]$ the environnement ρ' s.t. $\rho'(x) = \alpha$ and $\rho'(y) = \rho(y)$ for $y \neq x$.

Let A be an interpretation of (K,F) with ground domains D_{κ}. Then a premodel M is a *model* of (Λ, A) if it is possible to associate to any $M^{\sigma} \in \Lambda$ a continuous mapping $M[\![M]\!] : \text{Env}_{M} \to D^{\sigma}_{M}$ in such a way that the following equations hold for all $\rho \in \text{Env}_{M}$:

(i) $M[\![x]\!](\rho) = \rho(x)$

(ii) $M[\![f]\!](\rho) = A(f)$, $M[\![\Omega^{\sigma}]\!](\rho) = \perp_{\sigma}$

(iii) $M[\![MN]\!](\rho) = M[\![M]\!](\rho).\, M[\![N]\!](\rho)$

(iv) $M[\![\lambda x^{\sigma} M]\!](\rho).\alpha = M[\![M]\!](\rho[x \leftarrow \alpha])$ for all $\alpha \in D^{\sigma}_{M}$

(v) $M[\![YM]\!](\rho) = Y(M[\![M]\!](\rho))$ where Yh = least fixpoint of h.

A model is *algebraic* etc... if each D^{σ}_{M} is

It is not clear how to associate semantics to models. The semantic mapping should be $M[\![\ ]\!]$, so that the E^{σ} should be subsets of $[\text{Env}_{M} \to D^{\sigma}_{M}]$ appropriately ordered. In general the ordering $\leqslant_{\sigma\to\tau}$ of $D^{\sigma\to\tau}_{M}$ is defined in some uniform way from $\leqslant_{\sigma}$ and $\leqslant_{\tau}$, and the same way should be used to define the ordering in the E^{σ}. This will be done smoothly in the category-theoretic formalism of section 4.2. However in order - extensional models we can just take $E^{\sigma} = [\text{Env}_{M} \to D^{\sigma}_{M}]$ ordered pointwise. Then we leave to the reader to see that we indeed obtain a semantics.

A well-known example of order-extensional model is the Scott-Milner [11] model, where $D_M^{\sigma\to\tau} = [D_M^\sigma \to D_M^\tau]$ ordered pointwise.

3.3. Fully abstract models.

Let M be an order-extensional model of (Λ,A), and write $M_1^\sigma \sqsubset_M M_2^\sigma$ for $M[\![M_1^\sigma]\!] \sqsubset M[\![M_2^\sigma]\!]$, i.e. $\forall\rho \;\; M[\![M_1^\sigma]\!](\rho) \sqsubset_\sigma M[\![M_2^\sigma]\!](\rho)$. We now analyse the relations between $\sqsubset$op and $\sqsubset_M$

3.3.1. Proposition.

Let M be an order-extensional model of (Λ,A). Then $M_1 \sqsubset_M M_2$ implies $M_1 \sqsubset$op M_2. ■

The converse property is not always true. For example, Plotkin [13] shows that Scott-Milner's model is *not* fully abstract for PCF. We shall give two constructions of fully abstract models. The first one is due to Milner [12] and applies to PCF. The second one applies to the symbolic interpretation where symbols are left uninterpreted, and to which Milner's results do not apply.

3.3.2. Definition .

Let M be a model of (Λ,A). An element $\alpha \in D_M^\sigma$ is *definable* if there exists a closed M s.t. $\alpha = M[\![M]\!](\bot)$. (Notice that $M[\![M]\!](\rho)$ does not depend on ρ when M is closed). A *finite projection* $\Psi : D \to D$ is a continuous function such that $\Psi\circ\Psi = \Psi$, $\Psi(\alpha) \sqsubset \alpha$ for all $\alpha \in D$, and the set $\Psi(D)$ is finite. A *SFP-object* [14] is a cpo D which admits a chain $\Psi_1 \sqsubset \Psi_2 \sqsubset \ldots$ of finite projections with limit the identity of D. Notice that every SFP-object is ω-algebraic and that a cpo is ω-algebraic and consistently complete iff it is a SFP-object such that any two elements have a glb, see [3]. A *SFP-interpretation* is such that for any $\kappa \in K$ the cpo D_κ is a SFP-object, any isolated $e \in D_\kappa$ is definable by a closed term in N^κ, and a chain of finite projections with limit the identity of D_κ is definable by terms in $N^{\kappa\to\kappa}$ (which have the form $\lambda x^\kappa.a$ where a only contains symbols in F, ground Ω's and x^κ, so that this condition only depends on A).

3.3.3. Theorem (Milner [12]).

Every SFP-interpretation admits a fully abstract model. ■

Remark : Milner's construction [12] can be rephrased as a completion of the sets $T^\sigma = \{[M^\sigma]_{op} \mid M^\sigma \in \Lambda^\sigma\}$ of the classes of terms w.r.t. $=_{op}$ ordered by $\leqslant_{op}$. A classical ideal completion does not work since the T^σ may already contain limits which would not be preserved. Other completion [6] preserve existing lubs and turn functions which preserve existing lubs into continuous functions. The SFP - condition ensures that the application $[M].[M'] = [MM']$ does preserve existing lubs, which is not true in general.

Under additional conditions, Milner *characterizes* fully abstract models :

3.3.4. Definition.

A SFP-interpretation A is *articulate* if every D_κ is ω-algebraic and consistently complete, if the glb functions in the D_κ are definable in

$N^{\kappa\times\kappa\to\kappa}$ and *if* there exist κ_o and $tt \in D_{\kappa_o}$, $tt \neq \perp$, such that for any κ and $e \in D_\kappa$ isolated, the function $(\lambda\alpha \in D_\kappa.\ \textit{if}\ \alpha \supset e\ \textit{then}\ tt\ \textit{else}\ \perp)$ is definable in $N^{\kappa\to\kappa_o}$.

3.3.5. Theorem (Milner [12]).

If A is articulate then every order-extensional model M of (Λ,A) is ω-algebraic and consistently complete. Moreover M is fully abstract if and only if for any $\sigma \in T$ and $e \in D^\sigma_M$ isolated, e is definable, and all fully abstract models are isomorphic. ∎

Notice that PCF is articulate. Definable chains of finite projections are easily obtained from recursive definitions of the identity, like the one given in 2.4.

Let us now give the construction of a fully abstract model for the *symbolic interpretation* S, which is not SFP. The domains of S are the $N^{\kappa\infty}_c$, i.e. the sets of infinite closed trees of type κ written with symbols in F and ground Ω's. And $S(f)$ is given by $S(f)(A_1,A_2,\ldots,A_k) = f(A_1,A_2,\ldots,A_k)$. The idea of the model construction is to quotient N^∞ by η-interconvertibility. This is easily done by noticing (after Huet [23]) that η-expansion $M \to \lambda x.Mx$ always terminates in typed λ-calculi, so that we can only consider terms in maximal η-expansion.

3.3.6. Definition.

The subset N^σ_η of *terms in η-expansion* are the least sets containing

(i) Ω^σ

(ii) $\lambda x_1^{\sigma_1} x_2^{\sigma_2}\ldots x_m^{\sigma_m}.\ s\ a_1a_2\ldots a_p$

Where $\sigma = \sigma_1 \times \sigma_2 \times\cdots\times \sigma_m \to \kappa$, $s \in F \cup V$, $s\ a_1a_2\ldots a_p$ is of ground type and $a_1, a_2,\ldots a_p \in N_\eta$.

The subset $N^\sigma_{\eta c}$ of N^σ_η is the set of *closed* terms in N^σ_η, it is ordered by the restriction of $<$ and its completion by ideals is written $N^{\sigma\infty}_{\eta c}$. The *symbolic model* of (Λ,S), also denoted by S, is built by taking $D^\sigma_S = N^{\sigma\infty}_{\eta c}$, and $A.A' = [A]_1[A']_2$ using the two holes - context $[\ ]_1\ [\ ]_2$ and results of 2.3.

3.3.7. Theorem.

(Under some conditions on F, see [3]). The symbolic model is order-extensional and is a fully abstract model of (Λ,S). ∎

The syntactic model is also an initial model for a suitable notion of morphism.

4. STABLE MODELS

4.1. Sequential functions.

Let us call FA the fully abstract model of PCF which exists and is unique by Milner's results 3.3.2. and 3.3.4. This model has been constructed in a syntactic way, and we would like to analyse its semantic properties and if possible to obtain a purely semantic construction of it. Starting from the fact that isolated points must be definable (3.3.4.), we obtain a characterization of the first-order domains of FA with no great difficulty.

4.1.1. Definition.

The sense of Milner [12] or *M-sequential* if it is constant or if there exists $i \in N$, $1 \leq i \leq k$ such that $\alpha_i = \perp$ implies $h(\vec{\alpha}) = \perp$ and such that for any fixed $\alpha_i \in D_i$ the function of the remaining arguments is also M-sequential.

Notice that there is a trouble with M-sequentiality and products. Let $h : D_1 \times D_2 \to D_3$. Then h may be M-sequential if we consider it as a *one argument* function from $(D_1 \times D_2)$ to D_3, while it may not be if we consider it as a two argument function !.

4.1.2. Proposition.

Let σ be a first order type in PCF, with $\sigma = \kappa_1 \times \kappa_2 \times \cdots \times \kappa_m \to \kappa$. Let h be a continuous function of type σ. Then $h \in D^{\sigma}_{FA}$ if and only if it is M-sequential when considered as a m-any function. ■

We now see why the Scott-Milner model of all continuous functions is not fully abstract for PCF : it contains the first order "parallel or" function $por : T_\perp^2 \to T_\perp$ which is continuous but not M-sequential. This function is defined by :

$por\ (\perp,\perp) = por\ (ff,\perp) = por\ (\perp,ff) = \perp$

$por\ (tt,\perp) = por\ (tt,ff) = por\ (\perp,tt) = por\ (ff,tt) = tt$

$por\ (ff,ff) = ff$

(Plotkin [13] shows that the Scott-Milner model is indeed fully abstract if *por* is added as a basic function. However PCF with *por* is very different, and in particular its interpreter must be a parallel program).

Two other definitions of sequentiality are given by Vuillemin [19] and Kahn - Plotkin [7]. They will be referred to as V- and KP-sequentiality definitions. Both are equivalent to M-sequentiality when arguments and results are taken in flat domains, as it is the case for first-order types in PCF. However, none of M-, V-, and KP-sequentiality allow to characterize higher-type functions in PCF. (Another definition is given by Sazonov [16], but does not seem to be of the same semantic style).

4.2. Categories and Semantics.

Even if we do not succeded in construction FA in a purely semantic way, it is still interesting to construct models in which not all continuous functions appear : we can obtain more information about FA and may be new proofrules. For this purpose, it is useful to notice that the difficulty of building models can be concentrated in two categorical diagrams : product and exponentiation.

4.2.1. Definition

A *product* of two objects D, E in a category C is an object $D \times E$ with two arrows $\Pi_1 : D \times E \to D$ and $\Pi_2 : D \times E \to E$ called *projections* such that for any arrows $f : F \to D$ and $g : F \to E$ there exists a unique arrow $\langle f,g\rangle : F \to D \times E$ such that $\Pi_1 o\langle f,g\rangle = f$ and $\Pi_2 o\langle f,g\rangle = g$

$$\begin{tikzcd} & F \arrow[d, "\langle f{,}g\rangle"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & D \times E \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] & \\ D & & E \end{tikzcd}$$

Intuitively if f,g are functions then

$\langle f,g\rangle(\alpha) = \langle f(\alpha),\ g(\alpha)\rangle$.

For $f : D \to D'$ and $g : E \to E'$, we define $f \times g : D \times E \to D' \times E'$ by $f \times g = \langle fo\Pi_1,\ fo\Pi_2\rangle$ (Intuitively $f \times g(\langle\alpha\ \beta\rangle) = \langle f(\alpha),\ g(\beta)\rangle$. An *ω-product* $D_1 \times D_2 \times\cdots\times D_n \times\cdots$ is defined in the same way with projections $\Pi_1, \Pi_2, \ldots, \Pi_n, \ldots$ A category *has ω-products* if $D_1 \times D_2 \times\cdots$ exists for any objects $D_1, D_2, \ldots$

Assume a category C has products. Then an *exponentiation* $D \to E$ of D and E is an object such that there exists an arrow app : $(D \to E) \times D \to E$ such that for any object F and arrow $f : F \times D \to E$ there exists a unique arrow curry $(f) : F \to (D \to E)$ such that $f = \text{app} \circ \langle \text{curry}(f) \times i_D\rangle$ where i_D is the identity of D.

$$\begin{tikzcd} (D \to E) \times D \arrow[dr, "\text{app}"'] & & F \times D \arrow[ll, bend right, "\text{curry}(f) \times i_D"'] \arrow[dl, "f"] \\ & E & \end{tikzcd}$$

Intuitively,

$\text{curry}(f)\ (\alpha)\ (\beta) = f(\alpha,\beta)$

or $\text{curry}(f) = \lambda\alpha\cdot\lambda\beta\cdot f(\alpha,\beta)$.

A category *has exponentiation* if $D \to E$ exists for any objects D, E. Let $\text{hom}(D,E)$ be the set of arrows of D to E. A category is *cpo-enriched* [15,21] if each $\text{hom}(D,E)$ is ordered by $\leqslant_{D,E}$ and is a cpo, and if the composition o defines a continuous function from $\text{hom}(E,F) \times \text{hom}(D,E)$ to $\text{hom}(D.F)$ for any D,E,F. A *TO-category* C is a cpo-enriched category which has products and exponentiation, is such that $\langle .,.\rangle$ and curry respectively define continuous functions from $\text{hom}(F,D) \times \text{hom}(F,E)$ to $\text{hom}(F,D \times E)$ and from $\text{hom}(F \times D,E)$ to $\text{hom}(F,D \to E)$, together with the choice of one object D_C^σ for each $\sigma \in T$ in such a way that $D_C^{\sigma\to\tau}$ is an exponentiation of D_C^σ and D_C^τ.

Let $V = \{x_1^{\sigma_1}, x_2^{\sigma_2}, \ldots, x_n^{\sigma_n}, \ldots\}$ be an enumeration of the variables of Λ, and let $ENV_C = D_C^{\sigma_1} \times D_C^{\sigma_2} \times\ldots\times D_C^{\sigma_n} \times\cdots$ in a TO-category C. Let Π_{x_i} denote $\Pi_i : ENV_C \to D_C^{\sigma_i}$, and for $x^\sigma \in V$ let $S_x : ENV_C \times D_C^\sigma \to ENV_C$ be the unique arrow such that $\Pi_x o S_x = \Pi_2$ and $\Pi_y o S_x = \Pi_y o \Pi_1$ for $y \neq x$ (intuitively $S_x(\rho,\alpha) = \rho[\alpha \leftarrow x]$). Let $\langle E_C^\sigma, \leqslant_\sigma, \perp\rangle = \langle \text{hom}(ENV_C, D_C^\sigma), \leqslant_{ENV_C}, D_C^\sigma, \perp\rangle$ and let the type preserving mapping $C[\![\ ]\!] : \Lambda \to \cup E_C^\sigma$ be defined by :

(i) $C[\![x^\sigma]\!] = \Pi_x$

(ii) $C[\![MN]\!] = \mathrm{app} \circ < [\![M]\!], [\![N]\!] >$

(iii) $C[\![\lambda x M]\!] = \mathrm{curry}\ ([\![M]\!] \circ S_x)$

(iv) $C[\![\Omega]\!] = \perp$

(v) $C[\![YM]\!] = \cup\ C[\![M^n \Omega]\!]$

4.2.2. Theorem.

If C is a *TO*-category, then the E_C^σ and the mapping $C[\![\ ]\!]$ define a syntactically continuous η-semantics. ■

This theorem gives a powerful tool for building semantics, but not directly models from the D_C^σ. However in all this section we shall use it in simple cases, where the $D_C^{\sigma\to\tau}$ are cpos isomorphic to $\mathrm{hom}(D_C^\sigma, D_C^\tau)$ and where arrows are functions (see [4] for examples not in this case). Then we can impose $C[\![f]\!] = A(f)$, define $\alpha.\beta$ by $\mathrm{app}(\alpha,\beta)$ and easily recover models. What we gain is that the ordering of functions from environnements to domains is defined within the category (see 3.2.).

As easy examples, we clearly recover Scott-Milner's models by considering the category CPO-C, which has the cpo's as objects and the continuous functions as arrows (ordered pointwise). We leave to the reader to see that SFP objet - continuous functions and ω-algebraic cpos - continuous functions also form *TO*-categories (again with the pointwise ordering).

4.3. Stable functions, semantics and models.

It is easy to see that the categories of cpo's with M- or V-sequential functions can not be *TO*-categories because they have no products. The category of concrete data structures with KP-sequential functions [7] has products but not exponentiation (see [3,4]). Hence none of the know sequentiality conditions gives smooth model constructions from 4.2.2. However we shall obtain *TO*-categories by using the weaker notion of stability introduced in [1] :

4.3.1. Definition.

Let D,E be cpo's. A continuous function $h : D \to E$ is *stable* if it satisfies

$\forall\alpha \in D\ \ \forall\beta \subset h(\alpha),\ \exists\alpha' \subset \alpha$ s.t. $\beta \subset h(\alpha')$ and $(\forall\alpha'' \subset \alpha,\ \beta \subset h(\alpha'') \Rightarrow \alpha' \subset \alpha'')$

Hence h is stable iff for all α and $\beta \subset h(\alpha)$ there exists a *definite information* needed from α to reach β by applying h. This information is the least element of the set $\{\alpha' \subset \alpha | \beta \subset h(\alpha')\}$ and will be denoted by $m(h,\alpha,\beta)$. Before giving examples of stable functions, let us check that the *por* function is not stable : indeed if we take $\alpha = (tt,tt)$ and $\beta = tt$, then the set $\{\alpha' \subset \alpha | tt \subset por(\alpha')\}$ has two minimal elements $(tt,\perp)$ and $(\perp,tt)$ but no least element.

Notice that stability may be defined together with continuity when D and E are ω-algebraic, by slightly altering the ε-δ-like definition of continuity given in 2.1 :

$\forall \alpha \in D,\ \forall e' \subset h(\alpha),\ \exists$ *a least* $e \subset \alpha$ s.t. $e' \subset h(e)$.

If D and E are consistently complete, then h is stable iff $h(\cap X) = \cap\, h(X)$ holds for any upperbounded X. Write $\alpha \uparrow \beta$ if α and β are consistent. (i.e. α and β have an upper-bound). Then we can weaken the stability condition and define a function to be *consistently multiplicative* or *cm* if it satisfies $\forall \alpha, \beta \in D,\ \alpha \uparrow \beta \Rightarrow f(\alpha \cap \beta) = f(\alpha) \cap f(\beta)$. Every stable function is cm, but the converse is false in general.

Every KP-sequential function is stable, hence every function in the first-order domains of PCF is stable. However there exists a function $h : T_\perp^3 \to T_\perp$ which is stable but not KP-sequential (neither M- or V-sequential) : it is the least monotonic function such that $h(tt,ff,\perp) = h(ff,\perp,tt) = h(\perp,tt,ff) = tt$, see [1]. The context operation in N^∞ is stable, as shown by the following refinment of 2.3.1. (see [2,3]).

$\forall M, C[\ ], \forall b < C[M],\ \exists$ *a least* $a < M$ s.t. $b < C[a]$

(indeed we show in [2,3] that the context operation is KP-sequential).

Hence all functions in the symbolic model S of 3.3.6. are stable (and KP-sequential).

We define a *semantics* to be *stable* if $[\![\ \ ]\!]$ defines a stable mapping from $N^{\sigma\infty}$ to E_σ for all σ, and a *model* M to be *stable* if $M[\![M^\sigma]\!]$ is a stable mapping from ENV_M to D_M^σ for all $M^\sigma \in \Lambda$. (This is nothing but stability of the unique morphism from the initial semantics val or the initial model S). In stable semantics or models, every approximation of the value of M has minimal and optimal computations in the sense of [5,9]. Notice that Scott-Milner's model is not stable because of *por*.

We shall not study stable functions on arbitrary cpo's, but only on dI-domains, where stable finite projections exist. These domains play the same role w.r.t. stable functions as the ω-algebraic consistently complete cpo's w.r.t. continuous functions.

4.3.2. Definition.

A *dI-domain* is an ω-algebraic and consistently complete cpo D such that the two following axioms hold (see [7]) :

- *axiom I :* every isolated element dominates finitely many elements.
- *axiom d (distributivity)* :

  $\forall \alpha,\beta,\gamma \in D,\ \beta \uparrow \gamma \Rightarrow \alpha \cap (\beta \cup \gamma) = (\alpha \cap \beta) \cup (\alpha \cap \gamma)$

Notice that $N_\perp$ and $T_\perp$ are dI-domains. So are also all domains in the symbolic interpretation S.

4.3.3. Proposition.

Let D,D' be dI-domains. Then $h : D \to D'$ is stable if and only if it is cm. If $e \in D$ is isolated, then $\lambda\alpha \in D.\alpha \cap e$ is a stable finite projection on D, and if $e \uparrow e'$, then $(\lambda\alpha.\alpha \cap e) \cup (\lambda\alpha.\alpha \cap e') = \lambda\alpha.\alpha \cap (e \cup e')$. There exists a chain of stable finite projections which has limit the identity of D. ■

4.4. The *TO*-category of dI-domains and stable functions.

We now consider categories in which the arrows are stable functions $\underset{s}{\rightarrow}$ or cm functions $\underset{cm}{\rightarrow}$. Products will always be usual cartesian products and will never cause problems. Exponentiation will not be as easy : the exponentiation of $\langle D,\subset,\perp\rangle$ and $\langle D',\subset,\perp\rangle$ is *not* the set $\langle[D \underset{s}{\rightarrow} D'],\subset,\perp\rangle$ of stable functions ordered pointwise, since the application function
$app : \langle[D \underset{s}{\rightarrow} D'],\subset,\perp\rangle \times \langle D,\subset,\perp\rangle \rightarrow \langle D',\subset,\perp\rangle$ is *not* stable. Let $\mathbb{O} = \{\perp,T\}$, let h_1, h_2 : $\mathbb{O} \rightarrow \mathbb{O}$ with $h_1 = \lambda\alpha.\alpha$ and $h_2 = \lambda\alpha.T$.
Let $X = \{(h,\alpha) | h \subset h_2,\ \alpha \subset T,\ T \subset h(\alpha) = app(h,\alpha)\}$. Then X has two minimal elements (h_1,T) and $(h_2,\perp)$, but no least element, and $m(app,(h_2,T),T)$ does not exist. Also app is not cm since $app((h_1 \cap h_2), (T \cap \perp)) = \perp$ while $app(h_1,T) \cap app(h_2,\perp) = \perp$. We need to introduce other orderings between stable or cm functions.

4.4.1. Definition.

The relations $\leqslant_s$ and $\leqslant_{cm}$ in $[D \underset{s}{\rightarrow} D']$ and $[D \underset{cm}{\rightarrow} D']$ are defined by :

$h \leqslant_s h'$ if $\forall\alpha \in D,\ h(\alpha) \subset h'(\alpha)$ and $\forall\alpha \in D,\ \forall\beta \subset h(\alpha),\ m(h,\alpha,\beta) = m(h',\alpha,\beta)$

$h \leqslant_{cm} h'$ if $\forall\alpha \in D,\ h(\alpha) \subset h'(\alpha)$ and $\forall\alpha \uparrow \alpha',\ h(\alpha) \cap h'(\alpha') = h(\alpha') \cap h'(\alpha)$

Intuitively $h \leqslant_s h'$ means that not only the *values* of h approximate those of h' but also the *computations* of h approximate those of h' : namely for all α and $\beta \subset h(\alpha)$, h and h' need the *same* information about α to reach β. In general, $h \subset h'$ (pointwise) only implies $m(h',\alpha,\beta) \subset m(h,\alpha,\beta)$. In the example above, we had $h_1 \subset h_2$ but *not* $h_1 \leqslant_s h_2$ since $m(h_1,T,T) = T$ while $m(h_2,T,T) = \perp$

Let us now show that exponentiation does exist in the category *dI-s* of DI-domains with stable functions (analoguous results with cpo's and cm functions are shown in [3] but not given here).

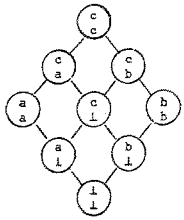
4.4.2. Theorem.

Let $\langle D,\subset,\perp\rangle$ and $\langle D',\subset,\perp\rangle$ be dI-domains. Then $\leqslant_s$ and $\leqslant_{cm}$ coincide on $[D \underset{s}{\rightarrow} D'] = [D \underset{cm}{\rightarrow} D']$. Moreover $\langle[D \underset{s}{\rightarrow} D'], \leqslant_s, \perp\rangle$ is a dI-domain and is the exponentiation of $\langle D,\subset,\perp\rangle$ and $\langle D',\subset,\perp\rangle$ in dI-s. For $h \in [D \underset{s}{\rightarrow} E]$, $e \in D$, $e' \subset h(e)$, e, e' isolated, let $e_o = m(h,e,e')$. Then

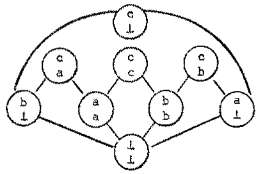
$$m(app,\ (h,e),\ e') = ((\lambda\alpha.e' \cap h(\alpha \cap e_o)),\ e_o). \quad \blacksquare$$

Notice that $\leqslant_s$ and $\subset$ give very different structures to $[D \underset{s}{\rightarrow} E]$, see diagram 1. Of course a major difference is that axiom I holds for $\leqslant_s$ but not for $\subset$.

In general a constant function $\subset$-dominates infinitely many functions.



$[D_1 \underset{s}{+} D_2], \subset$



$[D_1 \underset{s}{+} D_2], <_s$

$D_1 =$ 

$D_2 =$ 

4.5. Bidomains

Using the fact that $N_\perp$ and $T_\perp$ are dI-domains, we can easily obtain a semantics and a model of PCF by applying 4.1.1. to the category dI-s. We restricted the function spaces, but we also lost order-extensionality in the model where the orderings are $\leqslant_s$ and not $\subset$. (Although the semantics is term-order extensional, since we also order functions from environments to domains by $\leqslant_s$ and not $\subset$). Our next step will be to keep *both* orderings $\leqslant_s$ and $\subset$, and to obtain an order extensional model w.r.t. $\subset$ where the functions are restricted to be stable w.r.t. $\leqslant_s$. Hence we shall consider structures $\langle D,\subset,\leqslant,\perp\rangle$ with two different orderings on them. We shall of course need axioms relating the two orderings. However having a very long list of axioms would not even be a disadvantage : to obtain models, we simply start from $\langle D_\kappa,\subset,\subset,\perp\rangle$ at ground type, i.e. make equal the two orderings, and we obtain the higher-order domains by exponentiation. Hence any axiom which is trivial for $\leqslant \; = \; \subset$ on ground domains and carries through the exponentiation is nothing but an *additional property of the model*. For example, Kahn-Plotkin's [7] axioms C and R would carry through (but not axiom Q, see [3]).

4.5.1. Definition.

A *bidomain* is a structure $\langle D,\subset,\leqslant,\perp\rangle$ such that :

(i) $\langle D,\subset,\perp\rangle$ is an ω-algebraic and consistently complete cpo

(ii) $\langle D,\leqslant,\perp\rangle$ is a dI-domain

(iii) The identity function $\langle D,\leqslant,\perp\rangle \to \langle D,\subset,\perp\rangle$ is a morphism of ω-algebraic and consistently complete cpo's, i.e. is continuous, preserves isolated points and lubs. (Every $\leqslant$-isolated point is $\subset$-isolated, $\alpha \cup \beta = \alpha \vee \beta$ when $\alpha \uparrow_\leqslant \beta$).

(iv) the glb functions $\cap$ and $\wedge$ are continuous w.r.t. $\subset$ and $\leqslant$

(v) the relation $\leqslant$ is $\subset$-ω-inductive, i.e. if $\alpha_1,\alpha_2,\ldots,\alpha_n,\ldots$ and $\beta_1,\beta_2,\ldots,\beta_n,\ldots$ are $\subset$-increasing chains such that $\alpha_i \leqslant \beta_i$ holds for all $i \in N$, then $\cup\, \alpha_n \leqslant \cup\, \beta_n$ also holds.

Of course if $\langle D,\subset,\perp\rangle$ is a dI-domain, then $\langle D,\subset,\subset,\perp\rangle$ is a bidomain . If $\langle D,\subset,\leqslant,\perp\rangle$ is a bidomain, then $\langle D,\subset,\perp\rangle$ and $\langle D,\leqslant,\perp\rangle$ have the same basis, $\alpha \leqslant \beta$ implies $\alpha \subset \beta$, $\vee\Delta = \cup\Delta$ for any $\leqslant$-directed Δ and $\alpha \wedge \beta = \alpha \cap \beta$ if $\alpha \uparrow_\leqslant \beta$.

4.5.2. Definition.

A function $f : D \to D'$ is *bicontinuous* if it is continuous w.r.t. $\subset$ and $\leqslant$, and *stable* if it is bicontinuous and stable w.r.t. $\leqslant$. The notation $m(f,\alpha,\beta)$ is used as before, with $m(f,\alpha,\beta) = \wedge\{\alpha' \leqslant \alpha \text{ s.t. } \beta \leqslant f(\alpha)\}$. The set $[D \underset{s}{\to} D']$ of stable functions is ordered by the two orderings :

$f \subset f'$ if $\forall\alpha \in D,\ f(\alpha) \subset f'(\alpha)$

$f \leqslant_s f'$ if $\forall\alpha \in D,\ f(\alpha) \leqslant f'(\alpha)$ and $\forall\alpha \in D,\ \forall\beta \leqslant f(\alpha),\ m(f,\alpha,\beta) = m(f',\alpha,\beta)$
We can again define $f \leqslant_{cm} f'$ by

$\forall\alpha \in D,\ f(\alpha) \leqslant f'(\alpha)$ and $\forall\alpha \uparrow \beta,\ f(\alpha) \wedge f'(\beta) = f(\beta) \wedge f'(\alpha)$

and $\leqslant_s$ and $\leqslant_{cm}$ coincide on bidomains.
Let bdI-s be the category of bidomains with stable functions.

4.5.3. Theorem.

If $\langle D,\subset,\leqslant,\perp\rangle$ and $\langle D',\subset,\leqslant,\perp\rangle$ are bidomains, then $\langle [D \underset{s}{\to} D'],\subset,\leqslant_s,\perp\rangle$ is a bidomain and is the exponentiation of D and E is bdI-S. In addition the following properties hold :

- If $f \uparrow_{\leqslant_s} f'$, then $f \vee f'$ and $f \cup f'$ are determined pointwise and equal and $f \wedge f'$ and $f \cap f'$ are determined pointwise and equal.
- $f \cap f'$ is always determined pointwise (i.e $f \cap f'(\alpha) = f(\alpha) \cap f'(\alpha)$). However $f \wedge f'$ and $f \cup f'$ are not determined pointwise in general. ∎

For an example of the last property, take $h_1,\ h_2 : \mathbb{O}^2 \to \mathbb{O}$ defined by

$$h_1(\perp,\perp) = h_1(\perp,T) = h_2(\perp,\perp) = h_2(T,\perp) = \perp \qquad \text{and}$$
$$h_1(T,\perp) = h_1(T,T) = h_2(\perp,T) = h_2(T,T) = T.$$

then $h_1 \wedge h_2$ is the constant $\perp$ and $h_1 \cup h_2$ is the constant T.

Now we can use 4.2.2. to build semantics and models. For PCF, we start from $\langle N_\perp,\subset,\subset,\perp\rangle$ and $\langle T_\perp,\subset,\subset,\perp\rangle$ and build the higher-type domains by exponentiation. By forgetting either $\subset$ or $\leqslant$, we obtain *two syntactically continuous stable* η-*semantics* and *two stable models*. The model with $\subset$ is order-extensional and "strictly smaller" than Scott-Milner's one. However it is not fully abstract since we have seen that there exist stable first-order functions which are not sequential. We have no room to detail the properties of these models and the operational behaviour of $\leqslant$. Let us instead show that the fully abstract model $\mathcal{FA}$ is indeed a model of bidomains.:

4.5.4. Theorem.

Let $\mathcal{FA}$ be the fully abstract model of PCF. Then $\mathcal{FA}$ is a stable model. Moreover it is possible to define orderings $\leqslant_\sigma$ on the $D^\sigma_{\mathcal{FA}}$ in such a way that $\leqslant_{\mathbb{N}} = \subset_{\mathbb{N}}$, $\leqslant_{\mathbb{T}} = \subset_{\mathbb{T}}$, each $\langle D^{\sigma\to\tau}_{\mathcal{FA}},\subset,\leqslant_{\sigma\to\tau},\perp\rangle$ is a bidomain and is a subset of $[D^\sigma_{\mathcal{FA}} \underset{s}{\to} D^\tau_{\mathcal{FA}}]$ ordered by the restrictions of $\subset$ and $\leqslant_s$. Also $M^\sigma < M'^\sigma$ implies $\mathcal{FA}[\![M]\!] \leqslant_s \mathcal{FA}[\![M']\!]$ and $\mathcal{FA}[\![M]\!](\rho) \leqslant_\sigma \mathcal{FA}[\![M']\!](\rho)$. ∎

We conjecture the following reciprocal to the last part of 4.5.4. : For $e,e' \in D^\sigma_{\mathcal{FA}}$ isolated, $e \leqslant_\sigma e'$ holds if and only if there exist closed terms

$a, a' \in N^{\sigma}$ such that $e = FA[a](\perp)$, $e' = FA[a'](\perp)$ and $a < a'$. This would give a complete explanation of the respective role of $\subset$ and $\leqslant$: they would be *the exact semantic images of the "syntactic" orderings* $\sqsubset_{op}$ *and* $<$. (Notice that $\sqsubset_{op}$ and $<$ coincide in the symbolic model S, but differ in interpretations : in PCF we have $\underline{+1}\ \underline{0} =_{op} \underline{1}$ but not $\underline{+1}\ \underline{0} < \underline{1}$. Then we see that the stability property which exists in the syntax has a semantic counterpart with respect to the image $\leqslant$ of $<$ but *not* with respect to the image $\subset$ of $\sqsubset_{op}$). The property we conjecture is easy to prove at first-order types, but we think that a proof at all types requires the full construction of a sequential model.

CONCLUSION

The next step would be to construct sequential models which would respect the syntactic sequentiality property of [2,3] instead of the syntactic stability property of 4.3. This sequentiality property has a natural expression in Kahn-Plotkin's concrete data structures formalism [7], and work is in progress to construct *TO*-categories from concrete data structures, see [4]. The possibility of obtaining fully abstract models by quotients of the symbolic model is suggested by 3.3.6., but remains to be investigated. Finally, notice that recursive domains equations can be solved in the category bdI-s using Plotkin-Smyth constructions [15]. Hence stable models of the type-free λ-calculus analoguous to D_∞ [17,20] may be constructed. Their internal structure has not yet been investigated.

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