A SYNTACTICAL PROOF OF LOCALITY OF DA*

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Using purely syntactical arguments, it is shown that every nontrivial pseudovariety of monoids contained in **DO** whose corresponding variety of languages is closed under unambiguous product, for instance **DA**, is local in the sense of Tilson.

1. Introduction

Pseudovarieties were introduced by Eilenberg [6] as a suitable framework for the theory of finite semigroups and monoids. Since then, a substantial body of work has evolved around the classification of pseudovarieties (see, e.g., [13, 1]). In this context, various operations on pseudovarieties have been studied and, among them, the semidirect product plays a central role in the whole theory.

Based on ideas of, among others, Krohn and Rhodes [11], Brzozowski and Simon [5], Straubing [17] and Thérien and Weiss [19], Tilson [20] developed a theory of pseudovarieties of (finite) categories and showed that there is an intimate systematic connection between the calculation of semidirect products and of certain pseudovarieties of categories associated with pseudovarieties of monoids. This idea is again explored in a forthcoming paper by P. Weil and the author [3]. Briefly, it may be stated by saying that the calculation of a semidirect product $\mathbf{V} * \mathbf{W}$ is achieved by determining when a pseudoidentity is valid in $\mathbf{W}$ (a sort of "word problem" for profinite free objects over $\mathbf{W}$) and by computing the "global" pseudovariety $g\mathbf{V}$ determined by $\mathbf{V}$ (the pseudovariety of categories generated by $\mathbf{V}$) which, in turn,

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is equivalent to the calculation of a special semidirect product, namely $\mathbf{V} * \mathbf{D}$. There is an associated "local" pseudovariety $\ell\mathbf{V}$ consisting of the finite categories whose local submonoids at each vertex lie in $\mathbf{V}$. A particularly nice situation arises when $g\mathbf{V} = \ell\mathbf{V}$, in which case $\mathbf{V}$ is said to be *local*.

Several examples of local pseudovarieties of monoids can be found in [20]. Further proofs of locality have since then been obtained by Thérien [18], Jones and Szendrei [8] and Jones and Trotter [9]. All these papers have dealt with subpseudovarieties of the pseudovariety $\mathbf{DS}$ of all finite monoids whose regular $\mathcal{D}$-classes are subsemigroups, a case in which Azevedo and the author [4, 2, 1] have shown that syntactic techniques, involving the study of profinite free objects and the algebraic-combinatorial manipulation of their elements, are particularly powerful. By making use of ideas and results of [3, 7], which extend the profinite approach to pseudovarieties of categories, this note settles, in particular, the controversy around the locality of the pseudovariety $\mathbf{DA}$, consisting of all aperiodic monoids in $\mathbf{DS}$ (see [18, 9]). More generally, denoting by $\mathbf{DO}$ the pseudovariety of all members of $\mathbf{DS}$ whose regular $\mathcal{D}$-classes are orthodox, every pseudovariety of monoids contained in $\mathbf{DO}$ whose associated variety of languages is closed under unambiguous product (cf. [14]) is local. Similar results hold for the related cases of deterministic and codeterministic products.

2. Pseudovarieties of Categories and Pseudoidentities

This section starts by reviewing some basic material on the profinite approach to categories [3, 7]. In general, for further motivation, definitions and background, see [6, 13, 1, 20].

We view a (small) category as a set $C = V \mathring{\cup} E$ with two sorts of elements, those of V being called *vertices* and those of E *edges*, which is endowed with the following partial operations:

- $\alpha, \omega : E \to V$ giving the *extremities* of an edge $\alpha(e) \xrightarrow{e} \omega(e)$; the set of all edges $v_1 \longrightarrow v_2$ is denoted by $C(v_1, v_2)$; if no further operations are given, we say that C is a *graph*;
- an associative partial *composition* $C(v_1, v_2) \times C(v_2, v_3) \to C(v_1, v_3)$;
- $1_{\_} : V \to E$ associating a *local identity* 1_v to each vertex v (with both extremities being v) which is a neutral element for composition whenever defined; in particular, $C(v, v)$ is a monoid under composition for any $v \in V$ which is called the *local submonoid* at the vertex v.

In case the set C is endowed with a topology and all the above operations are continuous, we say that C is a *topological category*. A *functor* is a function $\varphi : C_1 \to C_2$ between two categories which respects all operations and maps vertices and edges respectively to vertices and edges. Two kinds of functors $\varphi : C_1 \to C_2$ are of special interest:

- φ is a *quotient functor* if φ is onto and $\varphi|_{V_1}$ is one-to-one;
- φ is a *faithful functor* if $\varphi|_{C_1(v,w)}$ is one-to-one for each $v, w \in V_1$.

We say that a category C *divides* a category D and we write $C \prec D$ if there are a category E, a quotient functor $E \to C$ and a faithful functor $E \to D$. A *pseudovariety of categories* is a class of finite categories which is closed under finitary direct product and under taking divisors. There are two natural constructions of pseudovarieties of categories associated with pseudovarieties of monoids. For a pseudovariety $\mathbf{V}$ of monoids, the class $g\mathbf{V}$ of all categories *globally in* $\mathbf{V}$ is the class of all finite categories which divide some member of $\mathbf{V}$ (viewed as a single-vertex category); in other words, viewing the elements of $\mathbf{V}$ as single-vertex categories, $g\mathbf{V}$ is the pseudovariety of categories generated by $\mathbf{V}$. As another extreme, we define $\ell\mathbf{V}$ to be the class of all finite categories all of whose local submonoids lie in $\mathbf{V}$. Indeed, $g\mathbf{V}$ and $\ell\mathbf{V}$ are the extremes of the interval of the lattice of pseudovarieties of categories consisting of all pseudovarieties all of whose monoids lie in $\mathbf{V}$. We say that $\mathbf{V}$ is a *local pseudovariety of monoids* if $g\mathbf{V} = \ell\mathbf{V}$.

A *profinite* (*respectively pro-*$\mathbf{V}$) *category* is any projective limit of finite (resp. from $\mathbf{V}$) categories. For a finite graph A, a topological category C is said to be *A-generated* if a graph morphism $\varphi : A \to C$ is given such that $\varphi|_{V(A)}$ is one-to-one and the subcategory generated by the graph $\varphi(A)$ is dense. Given a pseudovariety $\mathbf{V}$ of categories and a finite graph A, the projective limit of all A-generated members of $\mathbf{V}$ is denoted by $\overline{\Omega}_A\mathbf{V}$; in particular, $\overline{\Omega}_A\mathbf{V}$ is also an A-generated category via a graph morphism $\iota : A \to \overline{\Omega}_A\mathbf{V}$. This is a standard construction which enjoys the following properties.

Proposition 2.1. ([3, 7]) a) *For any graph morphism $\varphi : A \to C$ where C is a pro-$\mathbf{V}$ category, there is a unique continuous functor $\hat{\varphi} : \overline{\Omega}_A\mathbf{V} \to C$ such that $\hat{\varphi} \circ \iota = \varphi$.*

b) *Each $\pi \in \overline{\Omega}_A\mathbf{V}$ is completely determined by the family $(\pi_C)_{C\in\mathbf{V}}$ where $\pi_C : \mathcal{G}(A, C) \to C$ is the function defined on the set $\mathcal{G}(A, C)$ of all graph morphisms $\varphi : A \to C$ which associates to each such φ the element $\hat{\varphi}(\pi)$ where $\hat{\varphi}$ is the unique extension of φ to $\overline{\Omega}_A\mathbf{V}$ given by (a).*

c) *The subcategory $\Omega_A\mathbf{V}$ of $\overline{\Omega}_A\mathbf{V}$ generated by A is the quotient of the free category A^* on the graph A (with $V(A^*) = V(A)$ and $A^*(v, w) = \{paths\ from\ v\ to\ w\}$) in which two coterminal edges π and ρ are identified if the category identity $\pi = \rho$ is valid in $\mathbf{V}$, i.e., $\pi'_C = \rho'_C$ for any $C \in \mathbf{V}$ where π' and ρ' are the images in $\Omega_A\mathbf{V}$ under the unique functor $A^* \to \overline{\Omega}_A\mathbf{V}$ whose restriction to A coincides with ι.*

A formal equality $\pi = \rho$ between two coterminal edges of $\overline{\Omega}_A\mathbf{V}$ is said to be a $\mathbf{V}$-*pseudoidentity over the graph A*. We say that it is *valid* in a category C from $\mathbf{V}$ if $\pi_C = \rho_C$. Intuitively, we might say that $\pi = \rho$ is valid in C if, under any morphic interpretation of the vertices and edges of A in C, the resulting edges in C for π and ρ are equal. The subclass of $\mathbf{V}$ consisting of all categories which satisfy all pseudoidentities from a given set Σ of $\mathbf{V}$-pseudoidentities is denoted by $[\![\Sigma]\!]_{\mathbf{V}}$. We may now state as follows the analogue for pseudovarieties of categories of Reiterman's theorem [15].

Theorem 2.2. ([3, 7]) *A subclass $\mathbf{W}$ of a pseudovariety $\mathbf{V}$ of categories is itself a pseudovariety of categories if and only if it is of the form $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$ for some set Σ of $\mathbf{V}$-pseudoidentities.*

We conclude this section with a simple but very useful remark. Let $\mathbf{V}$ be a pseudovariety of monoids and let A be a finite graph with set of edges $E(A)$. We denote by $\overline{\Omega}_{E(A)}\mathbf{V}$ the free pro-$\mathbf{V}$ monoid over $E(A)$, which may be described as the projective limit of all $E(A)$-generated monoids of $\mathbf{V}$. Since $\overline{\Omega}_{E(A)}\mathbf{V}$ may be viewed as a pro-$g\mathbf{V}$ (single-vertex) category, by Proposition 2.1(a) there is a unique continuous functor $\hat{\gamma} : \overline{\Omega}_A g\mathbf{V} \to \overline{\Omega}_{E(A)}\mathbf{V}$ extending the natural graph morphism $\gamma : A \to \overline{\Omega}_{E(A)}\mathbf{V}$ which identifies all vertices.

Proposition 2.3. *The functor* $\hat{\gamma} : \overline{\Omega}_A g\mathbf{V} \to \overline{\Omega}_{E(A)}\mathbf{V}$ *is faithful.*

Proof. Suppose that π and ρ are two distinct coterminal edges of $\overline{\Omega}_A g\mathbf{V}$. Since $\overline{\Omega}_A g\mathbf{V}$ is a projective limit of members of $g\mathbf{V}$, there is some $C \in g\mathbf{V}$ and some continuous functor $\psi : \overline{\Omega}_A g\mathbf{V} \to C$ such that $\psi\pi \neq \psi\rho$. By definition of $g\mathbf{V}$, there is in turn some $M \in \mathbf{V}$, some finite category D, a faithful functor $D \to M$ and a quotient functor $D \to C$.

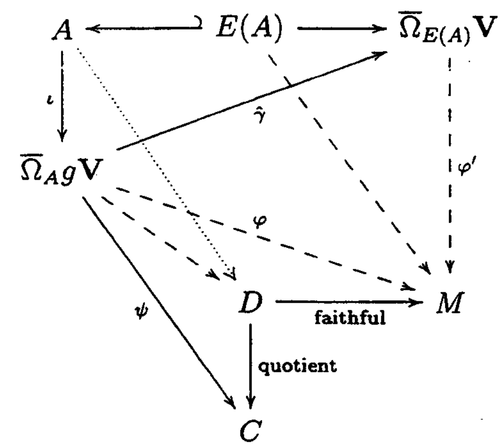

Fig. 1.

By Proposition 2.1(a), ψ lifts to a continuous functor $\overline{\Omega}_A g\mathbf{V} \to D$ and therefore yields a continuous functor $\varphi : \overline{\Omega}_A g\mathbf{V} \to M$ which, since $M \in \mathbf{V}$, induces a monoid homomorphism $\varphi' : \overline{\Omega}_{E(A)}\mathbf{V} \to M$ such that $\varphi' \circ \hat{\gamma} = \varphi$. The situation is illustrated in the above commutative diagram. Since $\psi\pi \neq \psi\rho$ and the functor $D \to M$ is faithful, it follows that $\hat{\gamma}\pi \neq \hat{\gamma}\rho$. □

3. Basic Boundary Factorizations and Unambiguous Products

Let $\mathbf{V}$ be a pseudovariety of monoids containing the pseudovariety $\mathbf{Sl}$ of all finite semilattices and let E be a finite set. Then there is a unique continuous homomorphism $c : \overline{\Omega}_E\mathbf{V} \to \mathcal{P}(E)$ into the union semilattice of all subsets of E such that $c(a) = \{a\}$ for $a \in E$; this function is called the *content function* and will be used frequently in the sequel.

Let $\pi \in \overline{\Omega}_E\mathbf{V} \setminus \{1\}$. A *basic boundary factorization* of π is a factorization of one of the following forms, where $a, b \in E$ and $\pi_1, \pi_2, \pi_3 \in \overline{\Omega}_E\mathbf{V}$:

(i) $\pi = \pi_1 a \pi_2 b \pi_3$ with $a \notin c(\pi_1)$, $b \notin c(\pi_3)$ and $c(\pi_1 a) = c(\pi) = c(b\pi_3)$;

(ii) $\pi = \pi_1 a \pi_2 b \pi_3$ with $b \notin c(\pi_1 a \pi_2)$, $a \notin c(\pi_2 b \pi_3)$ and $c(\pi_1 a \pi_2 b) = c(\pi) = c(a\pi_2 b \pi_3)$;

(iii) $\pi = \pi_1 a \pi_2$ with $a \notin c(\pi_1) = c(\pi_2)$.

If π belongs to the submonoid $\Omega_E \mathbf{V}$ of $\overline{\Omega}_E \mathbf{V}$ generated by E, then π is represented by some word w on the alphabet E so that, to obtain a basic boundary factorization of π, we just need to locate the last occurrence for the first time of a letter in w from left to right and from right to left; depending on the relative position of these two letter occurrences in w, we are led to one of the above three cases. By compactness of $\overline{\Omega}_E \mathbf{V}$, density of $\Omega_E \mathbf{V}$ in $\overline{\Omega}_E \mathbf{V}$ [1, section 3.4] and continuity of the content function, it follows that any element of $\overline{\Omega}_E \mathbf{V}$ admits a basic boundary factorization. Indeed, if $\pi \in \overline{\Omega}_E \mathbf{V}$, then there is a sequence $(w_n)_{n \geq 1}$ of words on the alphabet E whose projection in $\overline{\Omega}_E \mathbf{V}$ converges to π. Each w_n admits a basic boundary factorization and, without loss of generality we may assume that they are all of the same type and involve the same distinguished letters (a and possibly b, in the above notation). For instance, if the basic boundary factorizations $w_n = w_{1n} a w_{2n} b w_{3n}$ $(n \geq 1)$ are all of type (i) and (π_1, π_2, π_3) is an accumulation point in $(\overline{\Omega}_E \mathbf{V})^3$ of the projection of the sequence $\left((w_{1n}, w_{2n}, w_{3n})\right)_n$, then there is a factorization $\pi = \pi_1 a \pi_2 b \pi_3$ which, by continuity of c, remains a basic boundary factorization of type (i).

Given $a_1, \ldots, a_n \in E$ and $L_0, \ldots, L_n \subseteq E^*$, we say that the product $L_0 a_1 L_1 \cdots a_n L_n$ is *unambiguous* if every $w \in L_0 a_1 L_1 \cdots a_n L_n$ admits a unique factorization of the form $w = w_0 a_1 w_1 \cdots a_n w_n$ with $w_i \in L_i$ $(i = 0, \ldots, n)$. The pseudovariety of monoids $\mathbf{V}$ is said to be *closed under unambiguous product* if, for every finite set E, every $a_1, \ldots, a_n \in E$ and every $L_0, \ldots, L_n \subseteq E^*$, if each L_i is $\mathbf{V}$-recognizable (i.e., its syntactic monoid lies in $\mathbf{V}$) and the product $L_0 a_1 L_1 \cdots a_n L_n$ is unambiguous, then $L_0 a_1 L_1 \cdots a_n L_n$ is also $\mathbf{V}$-recognizable. Note that this property is equivalent to its particular case $n = 1$.

Pin, Straubing and Thérien [12, 14] have obtained a characterization of the pseudovarieties of monoids closed under unambiguous product which we proceed to describe. The pseudovariety of all finite semigroups S all of whose submonoids of the form eSe, with e an idempotent of S, are trivial is denoted by $\mathbf{LI}$; such semigroups are called *locally trivial*. Recall that, for a pseudovariety $\mathbf{W}$ of semigroups and a pseudovariety $\mathbf{V}$ of monoids, the Mal'cev product $\mathbf{W} \circledcirc \mathbf{V}$ is the pseudovariety of monoids generated by all finite monoids M such that there is a homomorphism $\varphi : M \to N$ with $N \in \mathbf{V}$ and $\varphi^{-1} e \in \mathbf{W}$ for every idempotent e of N.

Theorem 3.1. ([12, 14]) *A pseudovariety* $\mathbf{V}$ *of monoids is closed under unambiguous product if and only if it is of the form* $\mathbf{LI} \circledcirc \mathbf{V}'$ *for some pseudovariety* $\mathbf{V}'$ *of monoids.*

Examples of pseudovarieties of monoids closed under unambiguous product are now easily obtained. Consider the following pseudovarieties of monoids and semigroups, where the definitions of the latter by pseudoidentities are distinguished here by writing $[\![\ldots]\!]_{\mathbf{S}}$:

$$\begin{aligned}
\mathbf{M} &= \{\text{finite monoids}\}\\
\mathbf{DS} &= [\![\left((xy)^\omega(yx)^\omega(xy)^\omega\right)^\omega = (xy)^\omega]\!]\\
&= \{\text{finite monoids whose regular } \mathcal{D}\text{-classes are subsemigroups}\}\\
\mathbf{DO} &= [\![(xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega]\!]\\
&= \{\text{finite monoids whose regular } \mathcal{D}\text{-classes are orthodox subsemigroups}\}.
\end{aligned}$$

An elementary but rather crucial property of **DO** is the following:

if $M \in \mathbf{DO}, e, f, a, b \in M, e$ and f are idempotents, and e lies $\mathcal{J}$-below each of the elements a, b and f, then $eabe = eafbe$ (1)

(see, e.g., [2, Lemma 4.10]). More generally, if **V** is a pseudovariety of semigroups, then **DV** denotes the pseudovariety consisting of all finite monoids whose regular $\mathcal{D}$-classes are subsemigroups which belong to **V**. It is also important to note the following characteristic property of **DS** (cf. [2, Lemma 4.4]):

if $M \in \mathbf{DS}, e, a, b \in M,$ and e is an idempotent which lies $\mathcal{J}$-below each of the elements a and b, then e also lies $\mathcal{J}$-below ab. (2)

Further examples of pseudovarieties are given below:

$$\begin{aligned}
\mathbf{A} &= [\![x^{\omega+1} = x^\omega]\!] = \{\text{finite aperiodic monoids}\}\\
\mathbf{G} &= [\![x^\omega = 1]\!] = \{\text{finite groups}\}\\
\mathbf{RG} &= [\![xy^\omega = x]\!]_{\mathbf{S}} = \{\text{finite right groups}\}\\
\mathbf{LG} &= [\![x^\omega y = y]\!]_{\mathbf{S}} = \{\text{finite left groups}\}\\
\mathbf{R} &= [\![(xy)^\omega x = (xy)^\omega]\!] = \mathbf{DRG} \cap \mathbf{A} = \{\text{finite } \mathcal{R}\text{-trivial monoids}\}\\
\mathbf{L} &= [\![x(yx)^\omega = (yx)^\omega]\!] = \mathbf{DLG} \cap \mathbf{A} = \{\text{finite } \mathcal{L}\text{-trivial monoids}\}\\
\mathbf{J} &= [\![(xy)^\omega = (yx)^\omega]\!] \cap \mathbf{A} = \mathbf{R} \cap \mathbf{L} = \{\text{finite } \mathcal{J}\text{-trivial monoids}\}.
\end{aligned}$$

For a pseudovariety **H** of groups, $\overline{\mathbf{H}}$ denotes the pseudovariety of all finite monoids all of whose subgroups lie in **H**. Then, in view of Theorem 3.1, it is easy to verify that **DS**, **DO**, **H** ($\subseteq$ **G**) and $\overline{\mathbf{H}}$ are closed under unambiguous product: for each of these pseudovarieties **V**, it suffices to check that $\mathbf{LI} \circledcirc \mathbf{V} \subseteq \mathbf{V}$ by taking a homomorphism $\varphi : M \to N$ such that $N \in \mathbf{V}$ and $\varphi^{-1}e \in \mathbf{LI}$ for each idempotent e of N and showing that $M \in \mathbf{V}$. For instance, in case $\mathbf{V} = \mathbf{DO}$, if e and f are two $\mathcal{J}$-equivalent idempotents of M, then so are $\varphi(e)$ and $\varphi(f)$ and so $\varphi(e) = \varphi(efe)$ since $N \in \mathbf{DO}$. Hence efe belongs to the locally trivial semigroup $\varphi^{-1}\varphi(e)$ which implies that $efe = e\cdot efe\cdot e = e$ since e is idempotent, and this shows that $M \in \mathbf{DO}$. The other cases are similar. Since the property of being closed under unambiguous product is clearly preserved under intersection, **DA** is also closed under unambiguous product. In fact, as shown in [14], **DA** is the smallest pseudovariety closed under unambiguous product containing **Sl**. As a complement of this result, we compute some further Mal'cev products.

Proposition 3.2. *For any pseudovariety* $\mathbf{H}$ *of groups, the following equality holds:*

$$\mathbf{LI} \circledcirc (\mathbf{Sl} \vee \mathbf{H}) = \mathbf{DO} \cap \overline{\mathbf{H}}.$$

Proof. The inclusion of the Mal'cev product in $\mathbf{DO} \cap \overline{\mathbf{H}}$ was already observed above. For the reverse inclusion, let $M \in \mathbf{DO} \cap \overline{\mathbf{H}}$. Define on M a relation $\equiv$ by

$$x \equiv y \quad \text{if } x^\omega \mathrel{\mathcal{J}} y^\omega \text{ and } x^{\omega+1} = x^\omega y x^\omega.$$

Let $x, y, z \in M$ and suppose that $x \equiv y$. Then

$$(xz)^\omega = (xz)^\omega (yz)^\omega (xz)^\omega$$

by properties (1) and (2). By symmetry, it follows that $(xz)^\omega$ and $(yz)^\omega$ are $\mathcal{J}$-equivalent. On the other hand,

$$\begin{aligned}
(xz)^{\omega+1} &= (xz)^\omega x^\omega (xz)^{\omega+1} \quad \text{by (1)} \\
&= (xz)^\omega x^{\omega+1} z (xz)^\omega \\
&= (xz)^\omega x^\omega y x^\omega z (xz)^\omega \\
&= (xz)^\omega yz (xz)^\omega \quad \text{by (1) since } y, z \geq_{\mathcal{J}} (xz)^\omega.
\end{aligned}$$

Hence $xz \equiv yz$. Similarly, $zx \equiv zy$, which shows that the relation $\equiv$ is stable under multiplication. We leave it to the reader the verification that $\equiv$ is actually a congruence on M, $M/{\equiv}$ satisfies the pseudoidentities $x^\omega y = y x^\omega$ and $x^{\omega+1} = x$, every idempotent $\equiv$-class is a semigroup of $\mathbf{LI}$, and the restriction of $\equiv$ to each subgroup of M is the equality relation. Hence $M/{\equiv} \in \mathbf{Sl} \vee \mathbf{H}$ and so $M \in \mathbf{LI} \circledcirc (\mathbf{Sl} \vee \mathbf{H})$. □

As a straightforward consequence, we obtain the following corollary which gives an explicit description of the pseudovarieties to which the main results of this paper apply.

Corollary 3.3. *The subpseudovarieties of* $\mathbf{DO}$ *which are closed under unambiguous product are the pseudovarieties of groups and those of the form* $\mathbf{DO} \cap \overline{\mathbf{H}}$ *where* $\mathbf{H}$ *is a pseudovariety of groups.* □

The next result gives a useful connection between closure under unambiguous product and basic boundary factorizations.

Proposition 3.4. *Let* $\mathbf{V}$ *be a pseudovariety of monoids containing* $\mathbf{Sl}$ *and closed under unambiguous product. Then each* $\pi \in \overline{\Omega}_E \mathbf{V} \setminus \{1\}$ *has a unique basic boundary factorization. In other words, if* $\pi, \rho \in \overline{\Omega}_E \mathbf{M} \setminus \{1\}$, *the pseudoidentity* $\pi = \rho$ *is valid in* $\mathbf{V}$, *and* Φ *and* Ψ *are basic boundary factorizations of* π *and* ρ, *respectively, then the two factorizations are both of the same type ((i), (ii) or (iii)) and the equalities of factors in corresponding positions constitute pseudoidentities which are valid in* $\mathbf{V}$.

Proof. Let Φ be a basic boundary factorization of π, given by (i), (ii) or (iii). We associate with Φ a language $L(\Phi)$ by replacing each factor π_i by $[c(\pi_i)]^*$. Note that $L(\Phi)$ is an unambiguous product of letters and $\mathbf{Sl}$-recognizable languages, and therefore, $L(\Phi)$ is $\mathbf{V}$-recognizable. For $\sigma \in \overline{\Omega}_E\mathbf{M}$, we denote by $\sigma_{\mathbf{V}}$ the restriction $(\sigma_M)_{M\in\mathbf{V}}$ of the implicit operation σ to $\mathbf{V}$.

Next, let Ψ be a basic boundary factorization of ρ. Since $L(\Phi)$ and $L(\Psi)$ are $\mathbf{V}$-recognizable languages, the closures $\overline{L(\Phi)}$ and $\overline{L(\Psi)}$ of their projections in $\overline{\Omega}_E\mathbf{V}$ are open sets (cf. [1, Sec. 3.6]). Now, clearly $\pi_{\mathbf{V}} \in \overline{L(\Phi)}$ and $\rho_{\mathbf{V}} \in \overline{L(\Psi)}$ and, by hypothesis, we have $\pi_{\mathbf{V}} = \rho_{\mathbf{V}}$.

Suppose, for instance, that $\Phi : \ \pi = \pi_1 a \pi_2 b \pi_3$ is a factorization of type (i) or (ii). Since $\rho_{\mathbf{V}} \in \overline{L(\Phi)}$, there are sequences of words $(w_{ni})_n$ $(i = 1,2,3)$ such that $c(w_{ni}) = c(\pi_i)$ and, in $\overline{\Omega}_E\mathbf{V}$, $\lim_{n\to\infty} w_{n1}aw_{n2}bw_{n3} = \rho_{\mathbf{V}}$. On the other hand, since $\rho_{\mathbf{V}}$ belongs to the open set $\overline{L(\Psi)}$, it follows that, for every sufficiently large n, $w_{n1}aw_{n2}bw_{n3} \in L(\Psi)$ [1]. But basic boundary factorizations are clearly unique for words. By duality, we conclude that Φ and Ψ have the same type and $L(\Phi) = L(\Psi)$.

Consider now a pseudoidentity $\pi_i = \rho_i$ equating factors in corresponding positions of Φ and Ψ. If it fails in $\mathbf{V}$, then there is some $\mathbf{V}$-recognizable $K \subseteq E^*$ such that $\pi_{i\mathbf{V}} \in \overline{K}$ but $\rho_{i\mathbf{V}} \notin \overline{K}$. Replace in $L(\Phi)$ the factor $[c(\pi_i)]^*$ by $[c(\pi_i)]^* \cap K$ to obtain a language K' which is described as an unambiguous product of letters and $\mathbf{V}$-recognizable languages. Hence K' is $\mathbf{V}$-recognizable. Furthermore, by construction, $\pi_{\mathbf{V}} \in \overline{K'}$ and so $\rho_{\mathbf{V}} \in \overline{K'}$. Whence, say for factorizations of type (i) or (ii) and $i = 2$, if $\Phi : \ \pi = \pi_1 a \pi_2 b \pi_3$ and $\Psi : \ \rho = \rho_1 a \rho_2 b \rho_3$, then the corresponding factorization of $\rho_{\mathbf{V}}$ satisfies

$$\rho_{1\mathbf{V}} a \rho_{2\mathbf{V}} b \rho_{3\mathbf{V}} \in \overline{K'} = \overline{[c(\pi_1)]^*}\, a\, \overline{[c(\pi_2)]^* \cap K}\, b\, \overline{[c(\pi_3)]^*}.$$

Now, the subset $\overline{K'}$ of $\overline{\Omega}_E\mathbf{V}$ is open. Therefore, taking sequences of words converging to each ρ_i and arguing as above, we deduce that $\rho_{2\mathbf{V}} \in \overline{[c(\pi_2)]^* \cap K}$, in contradiction with the assumption that $\rho_{2\mathbf{V}} \notin \overline{K}$. The other cases are similar. Hence each pseudoidentity $\pi_i = \rho_i$ holds in $\mathbf{V}$. $\square$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\iota"'] & A' \arrow[d, "\text{faithful}"] \\
\overline{\Omega}_A g\mathbf{Sl} \arrow[r, "\text{faithful}"] \arrow[ur, dashed, "\varphi"] & \overline{\Omega}_{E(A)}\mathbf{Sl}
\end{tikzcd}$$

Fig. 2.

Let A be a finite graph. For a path w in A, denote by $c(w)$ the subgraph consisting of all edges and all vertices w goes through. Let A' be the category with the same set of vertices as A and with edges from a vertex v_1 to a vertex v_2 all triples of the form $(v_1, c(w), v_2)$, where w is a path in A from v_1 to v_2; the composite of two edges $(v_1, c(w), v_2)$ and $(v_2, c(w'), v_3)$ is the edge $(v_1, c(ww'), v_3)$. Then the mapping $E(A') \to \mathcal{P}\big(E(A)\big)$ associating with each edge $(v_1, c(w), v_2)$ the set $E\big(c(w)\big)$ yields, by definition of A', a faithful functor $A' \to \mathcal{P}\big(E(A)\big)$ into the monoid of subsets of

$\mathcal{P}\Big(E(A)\Big)$ under union. Hence $A' \in g\mathbf{Sl}$. From Proposition 2.3, it follows that A' is isomorphic with $\overline{\Omega}_A g\mathbf{Sl}$. Indeed, we have the commutative diagram of graph and category homomorphisms, where φ is given by Proposition 2.1(a) and $A \to A'$ is the mapping fixing vertices and sending each edge $v_1 \xrightarrow{e} v_2$ to $(v_1, c(e), v_2)$, as shown in Fig. 2. Since $\varphi|_A$ is a bijection of the generating subgraph A with a generating subgraph of A', the faithfulness of the the two homomorphisms into $\overline{\Omega}_{E(A)}\mathbf{Sl}$ and the commutativity of the diagram imply that φ is an isomorphism.

Let $\mathbf{W}$ be a pseudovariety of categories containing $g\mathbf{Sl}$. Then, by Proposition 2.1(a) there is a unique continuous quotient functor $\hat{\varphi} : \overline{\Omega}_A\mathbf{W} \to A'$ extending the natural graph morphism $\varphi : A \to A'$ defined by $\varphi(a) = \Big(\alpha(a), c(a), \omega(a)\Big)$ for $a \in E(A)$. For each edge π of $\overline{\Omega}_A\mathbf{W}$, we denote the subgraph B of A such that $\hat{\varphi}(\pi) = \Big(\alpha(\pi), B, \omega(\pi)\Big)$ by $c(\pi)$ and call it the *content* of π.

With the above notion of content for edges of $\overline{\Omega}_A\mathbf{W}$, we may define basic boundary factorizations just as at the beginning of the section in the monoid case. Again the same argument shows that every edge of $\overline{\Omega}_A\mathbf{W}$ admits some (not necessarily unique) basic boundary factorization, a fact which will be used in the next section.

4. Locality of $\mathbf{DO} \cap \overline{\mathbf{H}}$

We start with a technical lemma which will be useful in the proof of the main result.

Lemma 4.1. *Let $\mathbf{V}$ be a subpseudovariety of $\mathbf{DO}$ and suppose e is an idempotent edge of $\overline{\Omega}_A\ell\mathbf{V}$ whose content is A. Then:*

a) for every edge π of $\overline{\Omega}_A\ell\mathbf{V}$, there is some factorization $e = \sigma\pi\tau$;

b) for every loop π at $\alpha(e)$ in $\overline{\Omega}_A\ell\mathbf{V}$, π lies $\mathcal{J}$-above e in the local submonoid of $\overline{\Omega}_A\ell\mathbf{V}$ at $\alpha(e)$.

Proof. (a) By density of $\Omega_A\ell\mathbf{V}$ in $\overline{\Omega}_A\ell\mathbf{V}$ and by compactness of $\overline{\Omega}_A\ell\mathbf{V}$, it suffices to consider the case when π is given by a path w in A. The result is then proved by induction on the length of w. If $w = 1$ is the empty path, then we have the obvious factorization $e = e1e$. So, assume that the result is valid for every shorter path and let $w = va$ where a is an edge of the graph A. Then, by the induction hypothesis, there is a factorization $e = \sigma v\tau$. Since $c(e) = A$, there is some factorization $e = \sigma' a\tau'$. Now

$$e = (ee)^\omega = (\sigma' a\tau'\sigma v\tau)^\omega = \sigma' e' \cdot a\tau'\sigma v \cdot \tau\sigma' \cdot e'\tau''$$

where

$$e' = (a\tau'\sigma v \cdot \tau\sigma')^\omega \quad \text{and} \quad \tau'' = (a\tau'\sigma v \cdot \tau\sigma')^{\omega-2} a\tau'\sigma v\tau.$$

Every local submonoid of $\overline{\Omega}_A\ell\mathbf{V}$ is pro-$\mathbf{DO}$. By (1) it follows that

$$e = \sigma' e' a\tau'\sigma v \cdot e' \cdot \tau\sigma' e'\tau''$$

and so $e = \bar{\sigma} va\bar{\tau}$ for some $\bar{\sigma}$ and $\bar{\tau}$ since e' "starts" with a.

(b) This is an immediate consequence of (a). □

We may now state and prove the main result of this note. See Corollary 3.3 for its scope of application.

Theorem 4.2. *Every nontrivial pseudovariety* $\mathbf{V}$ *of monoids contained in* $\mathbf{DO}$ *which is closed under unambiguous product is local.*

Proof. Since the inclusion $g\mathbf{V} \subseteq \ell\mathbf{V}$ is always true, we show that this inclusion cannot be strict by establishing that every category pseudoidentity $\pi = \rho$ which is valid in $g\mathbf{V}$ is also valid in $\ell\mathbf{V}$. The locality of $\mathbf{V}$ follows by Theorem 2.2. If $\mathbf{V} \subseteq \mathbf{G}$ is nontrivial, then it is well-known that $\mathbf{V}$ is local (cf. [17, 20]; see also the comments at the end of this section). So, we assume that $\mathbf{V} \not\subseteq \mathbf{G}$, i.e., that $\mathbf{Sl} \subseteq \mathbf{V}$.

Let A be a finite graph and let π and ρ be two coterminal edges of $\overline{\Omega}_A\ell\mathbf{V}$ such that the pseudoidentity $\pi = \rho$ is valid in $g\mathbf{V}$. To show that the two edges π and ρ coincide, we proceed by induction on the number of edges of the graph $c(\pi) = c(\rho)$.

Consider the composite δ of the natural projection $\overline{\Omega}_A\ell\mathbf{V} \to \overline{\Omega}_A g\mathbf{V}$ (cf. Proposition 2.2(a)) with the faithful functor $\hat{\gamma}$ of Proposition 2.3:

$$\begin{tikzcd}
\overline{\Omega}_A\ell\mathbf{V} \arrow[r, "\text{projection}"] \arrow[dr, "\delta"] & \overline{\Omega}_A g\mathbf{V} \arrow[d, "\hat{\gamma}"] \\
 & \overline{\Omega}_{E(A)}\mathbf{V}
\end{tikzcd}$$

Fig. 3.

Take basic boundary factorizations of π and ρ. Since $g\mathbf{V}$ satisfies $\pi = \rho$, $\delta(\pi) = \delta(\rho)$. As δ is a functor, the basic boundary factorizations of π and ρ yield basic boundary factorizations of $\delta(\pi) = \delta(\rho)$. Since $\mathbf{V}$ is closed under unambiguous product and $\hat{\gamma}$ is faithful, it follows from Proposition 3.4 that the two factorizations of π and ρ have the same type, they break at the same edges, and the factors π_i and ρ_i in corresponding positions are such that each pseudoidentity $\pi_i = \rho_i$ is valid in $g\mathbf{V}$. By the induction hypothesis, if π_i and ρ_i involve fewer edges, then $\pi_i = \rho_i$ in $\ell\mathbf{V}$. Hence we only need to consider the case of basic boundary factorizations of type (i) and we may assume that

$$\begin{aligned}
\pi &= \pi_1^{(l)} a_1 \pi_1' b_1 \pi_1^{(r)} \\
\rho &= \pi_1^{(l)} a_1 \rho_1' b_1 \pi_1^{(r)}
\end{aligned} \tag{3}$$

where a_1 and b_1 are edges of A and $\pi_1^{(l)}$, $\pi_1^{(r)}$, π_1' and ρ_1' are edges of $\overline{\Omega}_A\ell\mathbf{V}$ such that

$$a_1 \notin c(\pi_1^{(l)}),\ b_1 \notin c(\pi_1^{(r)}),\ c(\pi) = c(\pi_1^{(l)} a_1) = c(b_1 \pi_1^{(r)}) = c(\rho),$$
and $g\mathbf{V}$ satisfies $\pi_1' = \rho_1'$.

Moreover, we may assume that $c(\pi_1') = c(\rho_1') = c(\pi)$.

Iterating the above argument, we may assume that, having constructed π_n' and ρ_n', they have basic boundary factorizations

$$\begin{aligned}\pi'_n &= \pi^{(l)}_{n+1} a_{n+1} \pi'_{n+1} b_{n+1} \pi^{(r)}_{n+1} \\ \rho'_n &= \pi^{(l)}_{n+1} a_{n+1} \rho'_{n+1} b_{n+1} \pi^{(r)}_{n+1}\end{aligned} \tag{4}$$

where a_{n+1} and b_{n+1} are edges of A and $\pi^{(l)}_{n+1}$, $\pi^{(r)}_{n+1}$, π'_{n+1} and ρ'_{n+1} are edges of $\overline{\Omega}_A \ell\mathbf{V}$ such that

$$a_{n+1} \notin c(\pi^{(l)}_{n+1}),\ b_{n+1} \notin c(\pi^{(r)}_{n+1}),\ c(\pi) = c(\pi^{(l)}_{n+1} a_{n+1}) = c(b_{n+1}\pi^{(r)}_{n+1}) = c(\rho),$$
$$g\mathbf{V} \text{ satisfies } \pi'_{n+1} = \rho'_{n+1}, \text{ and } c(\pi'_{n+1}) = c(\rho'_{n+1}) = c(\pi).$$

If this process stops at some stage, then $\pi = \rho$ in $\ell\mathbf{V}$. Otherwise, since $\overline{\Omega}_A \ell\mathbf{V}$ is compact, there is a strictly increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that each of the sequences

$$(\pi'_{n_k})_k,\ (\rho'_{n_k})_k,\ (\pi^{(l)}_1 a_1 \cdots \pi^{(l)}_{n_k} a_{n_k})_k \text{ and } (b_{n_k} \pi^{(r)}_{n_k} \cdots b_1 \pi^{(r)}_1)_k$$

converges, to the respective limits π', ρ', $\pi^{(l)}$ and $\pi^{(r)}$. By continuity of the composition operation in $\overline{\Omega}_A \ell\mathbf{V}$, in view of the formulas (3) and (4), we obtain the following factorizations:

$$\begin{aligned}\pi &= \pi^{(l)} \pi' \pi^{(r)} \\ \rho &= \pi^{(l)} \rho' \pi^{(r)}\end{aligned}$$

where the pseudoidentity $\pi' = \rho'$ is valid in $g\mathbf{V}$ and

$$c(\pi) = c(\rho) = c(\pi^{(l)}) = c(\pi^{(r)}) = c(\pi') = c(\rho').$$

As a final reduction, we note that, since $c(\pi) = c(\pi^{(l)}_n a_n) = c(b_n \pi^{(r)}_n)$ and, if necessary, by adding factors to π' and ρ', we may further assume that $\pi^{(l)}$ and $\pi^{(r)}$ are the same idempotent edge e such that $c(\pi) = c(e)$ (cf. [2]). So, we may assume that

$$\pi = e\pi' e \quad \text{and} \quad \rho = e\rho' e.$$

Now, for this simplified situation, we no longer need the hypothesis that the pseudovariety $\mathbf{V}$ be closed under unambiguous product. Since the rest of the proof yields a result which may be of independent interest, we isolate it in the following.

Proposition 4.3. *Let $\mathbf{V}$ be a pseudovariety contained in* **DO** *and containing* **Sl**, *and let e, π and ρ be edges of $\overline{\Omega}_A \ell\mathbf{V}$ with content A such that the pseudoidentity $\pi = \rho$ is valid in $g\mathbf{V}$ and e is an idempotent. Then $e\pi e = e\rho e$.*

Proof. Since $c(e) = A$, for each $v \in V(A)$ there are two edges $\alpha(e) \xrightarrow{p_v} v \xrightarrow{q_v} \alpha(e)$ such that $p_v q_v = e$ and $q_v p_v$ is an idempotent; for $v = \alpha(e)$, take $p_v = q_v = e$. Then, for each factorization $\pi = \sigma\tau$, taking $v = \omega(\sigma) = \alpha(\tau)$, we have

$$e\pi e = p_v \cdot q_v p_v \cdot q_v \sigma\tau p_v \cdot q_v p_v \cdot q_v = p_v \cdot q_v p_v \cdot q_v \sigma \cdot q_v p_v \cdot \tau p_v \cdot q_v p_v \cdot q_v = e\sigma q_v \cdot p_v \tau e$$

by Lemma 4.1(b). It follows that, for every finite path w in the graph A, denoting by $\overline{w}$ the path obtained from w by replacing each edge a by the loop $p_{\alpha(a)} a q_{\omega(a)}$

at the vertex $\alpha(e)$, then $ewe = e\overline{w}e$. Since every edge of $\overline{\Omega}_A\ell\mathbf{V}$ is the limit of a sequence of finite paths of the graph A, we deduce that, if $\sigma \mapsto \overline{\sigma}$ is the unique continuous functor $\varphi : \overline{\Omega}_A\ell\mathbf{V} \to \overline{\Omega}_A\ell\mathbf{V}$ such that all vertices are sent to $\alpha(e)$ and each edge a of the graph A is sent to $p_{\alpha(a)}aq_{\omega(a)}$, then $e\sigma e = e\overline{\sigma}e$ for every edge σ of $\overline{\Omega}_A\ell\mathbf{V}$.

Now, $e\pi e = \varphi(\pi)$, $e\rho e = \varphi(\rho)$ and the equality $\varphi(\pi) = \varphi(\rho)$ in $\ell\mathbf{V}$ holds if the local submonoid of $\overline{\Omega}_A\ell\mathbf{V}$ at the vertex $\alpha(e)$ satisfies the pseudoidentity $\pi = \rho$. Since this local submonoid is pro-$\mathbf{V}$, the latter condition is guaranteed by the hypothesis that $g\mathbf{V}$ satisfies $\pi = \rho$. This completes the proof of Proposition 4.3 and, therefore, also the proof of Theorem 4.2. □

The techniques exemplified in this note can be applied in other situations. For instance, from Proposition 4.3 it immediately follows that every pseudovariety of semilattices of groups containing $\mathbf{Sl}$ is local which is a particular case of the results of Jones and Szendrei [8]. A simplification of the same argument also yields the well-known fact that every pseudovariety of groups is local [17, 20]. We concentrated mainly on proving new results motivated by the controversy around Thérien's proof of the locality of $\mathbf{DA}$. Yet, without further ado, one can easily establish the following result by considering "deterministic" instead of unambiguous products (cf. [13]).

Theorem 4.4. *For every pseudovariety* $\mathbf{H}$ *of groups,* $\mathbf{DRG} \cap \overline{\mathbf{H}}$ *is local.*

Of course, the argument may be dualized, yielding that $\mathbf{DLG} \cap \overline{\mathbf{H}}$ is also local. In particular, $\mathbf{R}$ and $\mathbf{L}$ are local, the former being also a consequence of results of Stiffler [16] and Eilenberg [6], combined with Tilson's Delay Theorem [20]. In contrast, as proved by Knast [10], who actually calculated $g\mathbf{J}$, $\mathbf{J} = \mathbf{R} \cap \mathbf{L}$ is not local. We do not know whether $\mathbf{DG} = \mathbf{DRG} \cap \mathbf{DLG}$ is local.

References

1. J. Almeida, *Finite semigroups and universal algebra*, World Scientific, Singapore, 1995. English translation.
2. J. Almeida and A. Azevedo, *On regular implicit operations*, Portugaliae Mathematica **50** (1993), 35–61.
3. J. Almeida and P. Weil, *Profinite categories and semidirect products*, J. Pure and Applied Algebra. To appear.
4. A. Azevedo, *Operações implícitas sobre pseudovariedades de semigrupos. Aplicações*, Ph.D. thesis, Universidade do Porto, 1989.
5. J. A. Brzozowski and I. Simon, *Characterizations of locally testable events*, Discrete Math. **4** (1973), 243–271.
6. S. Eilenberg, *Automata, Languages and Machines*, vol. B, Academic Press, New York, 1976.
7. P. R. Jones, *Profinite categories, implicit operations and pseudovarieties of categories*, Int. J. Algebra and Computation. To appear.
8. P. R. Jones and M. B. Szendrei, *Local varieties of completely regular monoids*, J. of Algebra **150** (1992), 1–27.
9. P. R. Jones and P. G. Trotter, *Locality of DS and associated varieties*, J. Pure and Applied Algebra **104** (1995), 275–301.
10. R. Knast, *Some theorems on graph congruences*, RAIRO Inf. Théor. et Appl. **17** (1983), 331–342.

11. K. Krohn and J. Rhodes, *Algebraic theory of machines. I. Prime decomposition theorem for finite semigroups and machines*, Trans. Amer. Math. Soc. **116** (1965), 450–464.
12. J.-E. Pin, *Propriétés syntactiques du produit non ambigu*, vol. 85 of Lect. Notes in Comput. Sci., Berlin, 1980, Springer, 483–499.
13. J.-E. Pin, *Varieties of Formal Languages*, Plenum, London, 1986. English translation.
14. J.-E. Pin, H. Straubing, and D. Thérien, *Locally trivial categories and unambiguous concatenation*, J. Pure and Applied Algebra **52** (1988), 297–311.
15. J. Reiterman, *The Birkhoff theorem for finite algebras*, Algebra Universalis **14** (1982), 1–10.
16. P. Stiffler, *Extension of the fundamental theorem of finite semigroups*, Advances in Math. **11** (1973), 159–209.
17. H. Straubing, *Finite semigroup varieties of the form $V * D$*, J. Pure and Applied Algebra **36** (1985), 53–94.
18. D. Thérien, *Two-sided wreath product of categories*, J. Pure and Appl. Algebra **74** (1991), 307–315.
19. D. Thérien and A. Weiss, *Graph congruences and wreath products*, J. Pure and Applied Algebra **36** (1985), 205–215.
20. B. Tilson, *Categories as algebra: an essential ingredient in the theory of monoids*, J. Pure and Applied Algebra **48** (1987), 83–198.