Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 158 (2015) 213–231
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Notes

On the nonemptiness of the α-core of discontinuous games: Transferable and nontransferable utilities ☆

Metin Uyanık

Department of Economics, The Johns Hopkins University, MD 21218, United States

Received 4 November 2014; final version received 18 February 2015; accepted 14 April 2015

Available online 27 April 2015

Abstract

The nonemptiness of the α-core of games with continuous payoff functions was proved by Scarf (1971) for nontransferable utilities and by Zhao (1999) for transferable utilities. In this paper we present generalizations of their results to games with possibly discontinuous payoff functions. Our handling of discontinuity is based on Reny's (1999) better-reply-security concept. We present examples to show that our generalizations are nonvacuous.

JEL classification: C62; D50; D82; G13

Keywords: NTU α-core; TU α-core; Discontinuous games

1. Introduction

Two major solution concepts for normal form games are the Nash equilibrium and the core. A Nash equilibrium is a noncooperative solution in which the joint interest of groups of players is not explicitly considered, whereas the core is a cooperative solution involving the behavior

☆ The results reported here constitute part of the author's Ph.D. dissertation being written under the supervision of Professors Hülya Eraslan and M. Ali Khan. I am grateful to them and to Richard P. McLean for their careful reading of the preliminary drafts. I also thank Ying Chen, Idione Meneghel, Burçin Kısacıkoğlu, Sezer Yaşar and the seminar participants at the University of Queensland and at the Sunday Theory Workshops at Johns Hopkins University for stimulating discussion. Finally, I am grateful to an anonymous referee of this journal for his/her report and for bringing Zhao (1999) to my attention.

E-mail address: muyanik1@jhu.edu.

http://dx.doi.org/10.1016/j.jet.2015.04.005
0022-0531/

of coalitions of players. Motivated by economic problems that are suitably modeled by games with discontinuous payoff functions, Nash's existence result has been considerably extended following the seminal works of Dasgupta and Maskin (1986) and Reny (1999).[1] The existence of a cooperative solution for such games has yet to be provided. This note fulfills this objective.

An action profile belongs to the core of a game if no group of players has an incentive to form a coalition in which each of its members are made better-off, i.e. the action profile cannot be *blocked* by any coalition. In a (normal form) game, the actions of the complementary coalition affect the payoff of the members of a coalition, and therefore the definition of "blocking" hinges crucially on what the complementary coalition does. Among various blocking concepts defined in the literature, the *α-core* due to Aumann (1961) has attracted a significant attention.[2] In this paper we study the existence of this cooperative solution for games with possibly discontinuous payoff functions without transferable utilities (α-core) and with transferable utilities (α^T-core).

An action profile is in the α-core of a game if no coalition has an alternative action which makes all of its members better off, independently of the actions of the complementary coalition. Hence it is a pessimistic solution concept regarding the actions of the complementary coalition. And a pair of an action profile and a payoff profile is in the α^T-core of a game if the action profile maximizes the grand coalition's aggregate payoff, and no coalition has an alternative action which guarantees a higher aggregate payoff, independently of the actions of the complementary coalition. Therefore, the α-core allows the coordination of the actions among the members of the coalitions, whereas the α^T-core allows payoff transfers within the coalitions in addition to the coordination of actions. Scarf (1971) and Zhao (1999) proved the following theorems.

Theorem *(Scarf). Let $G = (X_i, u_i)_{i\in N}$ be a game such that for each player i,*

(i) *X_i is a nonempty, compact and convex subset of a finite dimensional Euclidean space,*
(ii) *u_i is a quasiconcave and continuous function on $X = \prod_{i\in N} X_i$.*

Then G has a nonempty α-core.

Theorem *(Zhao). Let $G = (X_i, u_i)_{i\in N}$ be a game such that for each player i,*

(i) *X_i is a nonempty, compact and convex subset of a finite dimensional Euclidean space,*
(ii) *u_i is a concave and continuous function on $X = \prod_{i\in N} X_i$,*
(iii) *G is weakly separable.*

Then G has a nonempty α^T-core.

In this paper we generalize these results to games where the continuity assumptions are weakened. And in line with the recent literature on discontinuous games pioneered by Reny (1999),

[1] See Carmona (2011) for a symposium on the recent developments in the discontinuous games literature.

[2] The *α-core,* along with the *strong Nash equilibrium* and the *β-core,* are the standard cooperative solution concepts for normal form games, see, for example, Ray and Vohra (1997). These cooperative solution concepts differ on the definition of blocking. The α-core requires a blocking coalition to select a specific strategy independently of the complementary coalition's choice, the β-core allows a blocking coalition to vary its blocking strategy as a function of the complementary coalition's choice, and the strong Nash equilibrium requires the complementary coalition to stick on its choice. See Ichiishi (1993, Section 2.3, p. 36) for details.

the notions of *coalitionally* C*-secure*, *coalitionally* C^T*-secure* and *coalitionally* C_N^T*-secure* games are presented, and the existence of an imputation in the α-core and α^T-core are shown for them.

The paper is organized as follows. Section 2 defines the basic concepts and states the results, Section 3 illustrates examples, Section 4 provides proofs of the results, and Section 5 concludes.

2. The model and the results

A (*normal form*) *game* is a list $G = (X_i, u_i)_{i \in N}$ where

(i) $N = \{1, \ldots, n\}$ is the finite set of players,
(ii) X_i is the nonempty set of actions of player $i \in N$,
(iii) $u_i : X \to \mathbb{R}$ is the utility function of player $i \in N$ defined on $X = \prod_{i \in N} X_i$.

A *quasiconcave* (*concave*) *game* is a game $G = (X_i, u_i)_{i \in N}$ such that for each player i, X_i is a nonempty, compact and convex subset of a finite dimensional Euclidean space and u_i is a quasiconcave (concave) function on X.

In his pioneering work, Reny (1999) proved the existence of a Nash equilibrium of normal form games satisfying the following weak continuity assumption.

Definition 1. A game G is *better-reply-secure*[3] (*BRS*) if at each $x \in X$ that is not a Nash equilibrium of G, there exist $y^x \in X$, $\delta^x > 0$, and an open neighborhood U^x of x such that for each $x' \in U^x$ there exists $i \in N$ such that for each $z \in U^x$, $u_i(y_i^x, z_{-i}) > u_i(x') + \delta^x$.

McLennan et al. (2011) provided the following continuity concept which is weaker than *BRS*, and used it to prove the existence of a (pure strategy) Nash equilibrium. Barelli and Meneghel (2013) extended this existence result by further weakening the continuity assumption.

Definition 2. A (quasiconcave) game G is C*-secure* if at each $x \in X$ that is not a Nash equilibrium of G, there exist $v_i^x \in \mathbb{R}$ and $y_i^x \in X_i$ for each $i \in N$ and an open neighborhood U^x of x such that

(i) $u_i(y_i^x, z_{-i}) \geq v_i^x$ for each $i \in N$ and each $z \in U^x$,
(ii) for each $x' \in U^x$ there exists $i \in N$ such that $u_i(x') < v_i^x$.

Remark 1. If an action profile x is not a Nash equilibrium, then by definition at least one player deviates. The C-security imposes the following structure on the individual deviations: (i) an open neighborhood of x contains no Nash equilibrium, i.e. at each point on the neighborhood at least one player deviates, (ii) the identity of the deviant is allowed to vary, but the deviation for any deviant must be fixed on the neighborhood, and (iii) each deviant's deviation should be robust against all other players' trembles.

A *coalition* is an element S in $\mathcal{N} = 2^N \backslash \emptyset$. Let $\mathring{\mathcal{N}} = \mathcal{N} \backslash N$. The set of actions available to a coalition S is denoted as $X_S = \prod_{i \in S} X_i$, and the vector of utility functions of coalition S

[3] This definition is equivalent to but different from the original definition of better-reply-security. See footnote 7 in Reny (2013) and B-security concept in McLennan et al. (2011, Definition 2.4, p. 1646).

as $u_S = (u_i)_{i \in S}$.[4] For each coalition S, $-S$ denotes the complementary coalition $N \backslash S$. Next, we define equilibrium and continuity concepts for normal form games where each coalition is allowed to coordinate actions among its members, but payoff transfers are not allowed.

Definition 3. Let G be a game. A coalition S *α-blocks* an action profile $x \in X$ if $\exists x'_S \in X_S$ such that $u_S(x'_S, z_{-S}) \gg u_S(x)$[5] for each $z_{-S} \in X_{-S}$. An action profile $x^* \in X$ is in the *α-core* of G if it is not α-blocked by any coalition.

This equilibrium definition differs in two major aspects from the Nash equilibrium. First, an arbitrary coalition is permitted to modify its action profile. Second, the complementary coalition is permitted a subsequent modification of its action profile. Now we define a continuity concept analogous to C-security (Definition 2 above) which involves the behavior of the coalitions as follows.

Definition 4. A game G is *coalitionally C-secure* if at each $x \in X$ that is not in the α-core of G, there exist $v^x_S \in \mathbb{R}^{|S|}$ and $y^x_S \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood U^x of x such that

(i) $u_S(y^x_S, z_{-S}) \geq v^x_S$ for each $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$,
(ii) for each $x' \in U^x$ there exists $S \in \mathcal{N}$ such that $u_S(x') \ll v^x_S$.

Remark 2. Note that coalitional C-security is conceptually analogous to the notion of C-security (see Remark 1). First, the role of individual players is taken by coalitions. Second, we impose restrictions on *coalitional blockings* instead of *individual deviations.* The main difference is that we assume $u_S(y^x_S, z_{-S}) \geq v^x_S$ for each $S \in \mathcal{N}$ and $z_{-S} \in X_{-S}$, instead of each $z \in U^x$. In other words, we assume each coalition can guarantee the payoff level independent of all actions of the complementary coalition, instead of the complementary coalition's tremble. Although this is a strong assumption, it is consistent with the definition of blocking and Reny's insight. The C-security assumes a structure on *individual deviations,* and coalitional C-security on *coalitional blockings*, and the definition of blocking, unlike that of deviation, already incorporates *all* actions of the complementary coalition. Therefore, there is no inclusion relation between the coalitional C-security and C-security concepts. Example 1 below illustrates this relation.

Now we define equilibrium and continuity concepts for normal form games where coalitions are allowed to both coordinate their actions and reallocate their aggregate payoffs, i.e. payoff transfers are allowed.

Definition 5. Let G be a game. A coalition S *α^T-blocks* a payoff profile $v \in \mathbb{R}^n$ if $\exists x'_S \in X_S$ such that $\sum_{i \in S} u_i(x'_S, z_{-S}) > \sum_{i \in S} v_i$ for each $z_{-S} \in X_{-S}$. A pair of an action profile and a payoff profile $(x^*, v^*) \in X \times \mathbb{R}^n$ with $\sum_{i \in N} u_i(x^*) = \sum_{i \in N} v^*_i$ is in the *α^T-core* of G if v^* is not α^T-blocked by any coalition.

[4] By abusing the notation, we drop the subscript N for the grand coalition, and when there it is clear from the context, we use i and $\{i\}$ interchangeably for the singleton coalition $S = \{i\}$.

[5] We use usual vector comparison symbols: $x \geq y$ represents $x^k \geq y^k$ for each index k; $x > y$ represents $x^k \geq y^k$ for each index k and inequality is strict at least for one k; $x \gg y$ represents $x^k > y^k$ for each index k. Also, $\subset$ represents subset and $\subsetneq$ proper subset.

Note that due to payoff transfers, this equilibrium definition requires two variables, an action profile x^* and a payoff profile v^*, whereas the former is enough to define the α-core. Now we define a class of games which plays an essential role for the nonemptiness of the α^T-core.

Definition 6. A game G is *bounded* if u_i is bounded for each player i.

For a bounded game G, define

$$\mathcal{X} = \{(x,v) \in X \times \mathbb{R}^n \mid \sum_{i \in N} v_i = \sum_{i \in N} u_i(x),\ \inf_{x \in X} u_i(x) \le v_i\ \forall i \in N\}.$$

$\mathcal{X}$ is the set of all attainable pairs of action profiles and payoff profiles that are individually rational. Since any payoff profile which assigns player i a payoff below $\inf_{x\in X} u_i(x)$ is blocked by player i, hence $\mathcal{X}$ contains the α^T-core of G. If X is compact then it is clear that the closure of $\mathcal{X}$, denoted by $\bar{\mathcal{X}}$, is compact. Now we define a continuity concept analogous to C-security (Definition 2) for TU games as follows.

Definition 7. A bounded game G is *coalitionally C^T-secure* if at each $(x,v) \in \bar{\mathcal{X}}$ that is not in the α^T-core of G, there exist $w_S^{x,v} \in \mathbb{R}$ and $y_S^{x,v} \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood $U^{x,v}$ of (x,v) such that

(i) $\sum_{i\in S} u_i(y_S^{x,v}, z_{-S}) \ge w_S^{x,v}$ for each $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$,
(ii) for each $(x',v') \in U^{x,v}$ there exists $S \in \mathcal{N}$ such that $\sum_{i\in S} v_i' < w_S^{x,v}$.

Note that since α^T-core consists of a pair of an action profile and a payoff profile, the coalitional C^T-security is defined on the set of pairs. Moreover, a point (x^*, v^*) in the α^T-core must satisfy the following two conditions: (i) $\sum_{i\in N} v_i^* = \sum_{i \in N} u_i(x^*) = \max_{x\in X} \sum_{i\in N} u_i(x)$, and (ii) v^* is not α^T-blocked by any coalition. In order to guarantee the existence of a solution to the maximization problem (i), we define coalitional C^T-security at points in $\bar{\mathcal{X}}$, not only in $\mathcal{X}$.[6] Although coalitional C^T-security is the natural analogue of C-security, some important games, such as Bertrand duopoly game with different marginal costs, have a nonempty α^T-core and are not coalitionally C^T-secure, see Example 1 below. Therefore, we introduce a separate continuity concept which we term coalitional C_N^T-security as follows. For a given game $G = (X_i, u_i)_{i\in N}$, let $G_N = (X, \bar{u})$ be the induced one player where $X = \prod_{i\in N} X_i$ and $\bar{u}(x) = \sum_{i\in N} u_i(x)$. If G_N is C-secure, or equivalently coalitionally C-secure which are identical for one player games, then Theorem 1 implies there exists a maximizer $\bar{x} \in X$ of $\bar{u}$. (And our proof technique provides an alternative proof of the existence of a maximal element based on Scarf's (1967) theorem.) Since any point in the α^T-core must satisfy condition (i) described above, focusing only on the redistributions of the grand coalition's maximum aggregate payoff is enough to show the nonemptiness of the α^T-core.[7] In particular, define

$$\mathcal{V} = \{v \in \mathbb{R}^n \mid \sum_{i\in N} v_i = \sum_{i\in N} u_i(\bar{x}),\ \inf_{x\in X} u_i(x) \le v_i\ \forall i \in N\}.$$

[6] This property of coalitional C^T-security is similar to the *BRS* notion of Reny (1999) which takes the closure of the graph of the game into account.

[7] A one player game may be helpful in illustrating the underlying structure. Consider the following example. $X = [0, 1]$ and $u(0) = 1$, $u(1) = 0$, and $u(x) = x$ for all $x \in (0, 1)$. This 1-player game is not coalitionally C^T-secure since 1 is not maximizer and the game is not coalitionally C^T-secure at $(x, v) = (1, 1) \in \bar{\mathcal{X}}$. However, it will be clear below that the game is coalitionally C_N^T-secure.

Now imposing a structure on the deviations at each point in $\mathcal{V}$ will guarantee the nonemptiness of the α^T-core. Note that by construction, $\mathcal{V}$ is compact.

Definition 8. A bounded game G is *coalitionally C_N^T-secure* if the induced one player game G_N is C-secure, and at each $v \in \mathcal{V}$ such that $(\bar{x}, v)$ is not in the α^T-core of G, there exist $w_S^v \in \mathbb{R}$ and $y_S^v \in X_S$ for each $S \in \mathring{\mathcal{N}}$ and an open neighborhood U^v of v such that

(i) $\sum_{i \in S} u_i(y_S^v, z_{-S}) \geq w_S^v$ for each $S \in \mathring{\mathcal{N}}$ and each $z_{-S} \in X_{-S}$,
(ii) for each $v' \in U^v$ there exists $S \in \mathring{\mathcal{N}}$ such that $\sum_{i \in S} v_i' < w_S^v$.

Note that if $\bar{u}$ is an upper semicontinuous function, then G_N is C-secure.[8] But the converse of this claim is not true, see the example presented in footnote 7. Note that although the coalitional C_N^T-security imposes restriction only on $\{\bar{x}\} \times \mathcal{V} \subset \bar{\mathcal{X}}$, it explicitly requires the grand coalition to have a well-behaved blocking behavior at each points it blocks that is not imposed by the coalitional C^T-security. Under certain conditions, coalitional C_N^T-security is weaker than coalitional C^T-security. This relation is summarized in Claims 1 and 2 at the end of this section. Moreover, there is no inclusion relation between coalitional C-security and coalitional C^T-security as well as coalitional C_N^T-security. These relations are illustrated in Examples 2 and 3 below.

Definition 9. A coalitionally C^T-secure game G is *quasiseparable* if for each $S \in \mathcal{N} \backslash N$ and $y_S \in \{y_S^{x,v} | (x, v) \in \bar{\mathcal{X}}$ is not in the α^T-core of $G\}$,

$$\inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(y_S, z_{-S}) = \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(y_S, z_{-S}).$$

Definition 10. A coalitionally C_N^T-secure game G is *quasiseparable* if for each $S \in \mathcal{N} \backslash N$ and $y_S \in \{y_S^v | v \in \mathcal{V}$ such that $(\bar{x}, v)$ is not in the α^T-core of $G\}$,

$$\inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(y_S, z_{-S}) = \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(y_S, z_{-S}).$$

In Zhao (1999), a game is called *weakly separable* if for each $S \in \mathring{\mathcal{N}}$ and $i \in S$,

$$u_i(x_S^*, x_{-S}^*(x_S^*)) = \min_{z_{-S} \in X_{-S}} u_i(x_S^*, z_{-S})$$

where $(x_S^*, x_{-S}^*(x_S^*)) \in X$ is a solution to the problem $\max_{z_S \in X_S} \min_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(z_S, z_{-S})$. Note that every weakly separable game is quasiseparable. To see this, if the game is weakly separable, then the maxmin problem has a solution for each coalition. Hence at each $(x, v) \in \bar{\mathcal{X}}$ and $S \neq N$, we can set $y_S^{x,v} = x_S^*$. But, even for continuous payoff functions, quasiseparability is weaker than weak separability since it imposes a restriction on the aggregate payoff functions of coalitions, not on each member of the coalitions' payoff functions.

Moreover, Zhao (1999) briefly noted in footnote 4 that if the maxmin problem does not have a solution, one can define the induced *TU* game by simply replacing maxmin with supinf. This

[8] In order to see this, choose $x \in X$ such that there exists $y^x \in X$ with $\bar{u}(y^x) > \bar{u}(x)$. Then upper semicontinuity implies that for each $\varepsilon \in (0, \bar{u}(y^x) - \bar{u}(x))$, there exists an open neighborhood U^x of x such that $\bar{u}(x') \leq \bar{u}(x) + \varepsilon$ for all $x' \in \bar{U}^x$. Setting $v_1^x = \bar{u}(y^x)$ implies G_N is C-secure.

line of argumentation requires the boundedness of the payoff functions (otherwise one has to take the complications of the extended real line into consideration) and can only be used to show the nonemptiness of the epsilon α^T-core. In order to see this, first redefine the weak separability as follows. A game G is weakly separable if for each $S \neq \mathcal{N}$,

$$\sup_{z_S \in X_S} \inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(z_S, z_{-S}) = \sum_{i \in S} \sup_{z_S \in X_S} \inf_{z_{-S} \in X_{-S}} u_i(z_S, z_{-S}).$$

And consider the following trivial two-player game. $X_1 = X_2 = [0, 1]$, $u_1(x_1, x_2) = x_1$ for all $x_1 \in [0, 1)$ and $x_2 \in X_2$, and $u_1(1, x_2) = 0$. And $u_2(x_1, x_2) = u_1(x_2, x_1)$. It is clear that the maximum of the aggregate utility does not exist, i.e. the grand coalition always blocks, hence the α^T-core of the game is empty. Note that every two-player game trivially satisfies weak separability.

Now we are ready to state our results.

Theorem 1. *Every coalitionally C-secure, quasiconcave game has a nonempty α-core.*

Theorem 2. *Every coalitionally C^T-secure, concave,*[9] *bounded, quasiseparable game has a nonempty α^T-core.*

Theorem 3. *Every coalitionally C^T_N-secure, concave, bounded, quasiseparable game has a nonempty α^T-core.*

As is by now well-understood in the context of NTU games, the special case of a 2-player game is rather important in that the 2-player set-up allows one to dispense with a linear space structure on both the actions sets and the payoff functions, see Ichiishi (1993, Remark 2.3.2, p. 37). Proposition 1 shows that this carries over *verbatim* to the discontinuous setting as is formalized and presented here. Proposition 2 shows that the analogous result holds for TU games. But more to the point the 2-player game results serve as an important backdrop to the examples we presented that validate our results as meaningful and useful generalizations of Scarf's and Zhao's results. First, define a *compact game* as a game $G = (X_i, u_i)_{i \in N}$ such that for each player i, X_i is a nonempty and compact subset of a finite dimensional Euclidean space.

Proposition 1. *Every coalitionally C-secure, compact 2-player game has a nonempty α-core.*

Proposition 2. *Every coalitionally C^T_N-secure, compact, bounded 2-player game has a nonempty α^T-core.*

Now we provide two results related to the relationship between coalitional C^T-security and coalitional C^T_N-security.

[9] It is well known that every real-valued concave function on a Euclidean space is continuous on its domain's relative interior. And it is easy to define a concave function that is discontinuous at every point of the relative boundary of its domain. If the domain is contained in $\mathbb{R}$, it is easy to see that such functions are lower semicontinuous at the relative boundary of their domain. However this result is not necessarily true for the higher dimensional Euclidean spaces, see Ernst (2013, Theorem 2.4, p. 3672). Therefore, the minimization problems defined above may not have a solution, and hence our setup with *infimum* is crucial.

Claim 1. *Every bounded and coalitionally C^T-secure 2-player game is coalitionally C_N^T-secure. A bounded and coalitionally C_N^T-secure 2-player game is not necessarily coalitionally C^T-secure.*

Claim 2. *Let G be a bounded and coalitionally C^T-secure game such that $\bar{u} = \sum_{i \in N} u_i$ has a maximizer. Then G is coalitionally C_N^T-secure. A bounded and coalitionally C_N^T-secure game is not necessarily coalitionally C^T-secure.*

Note that Claim 2 shows that coalitional C_N^T-security is a necessary and sufficient condition for the existence of a maximizer of $\bar{u}$ for bounded and coalitionally C^T-secure games with compact action sets.

3. Examples

The first example illustrates a game which does not have a (pure strategy) Nash equilibrium, but has a nonempty α-core and α^T-core. The game satisfies all the assumptions of Propositions 1 and 2, particularly coalitional C-security and coalitional C_N^T-security. However, it neither satisfies C-security nor coalitional C^T-security.

Example 1. Consider the following Bertrand duopoly game. Each firm's action set is $P_i = [0, 10]$. The market demand function $D : [0, 10] \to \mathbb{R}$ is defined as

$$D(p) = \max\{4 - p, 0\}.$$

And the profit functions of the firms are defined as

$$\pi_1(p) = \begin{cases} p_1 D(p_1) & \text{if } p_1 < p_2, \\ p_1 D(p_1)/2 & \text{if } p_1 = p_2, \\ 0 & \text{otherwise,} \end{cases} \qquad \pi_2(p) = \begin{cases} p_2 D(p_2) - 1 & \text{if } p_2 < p_1, \\ p_2 D(p_2)/2 - 1 & \text{if } p_1 = p_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that this game does not have a Nash equilibrium.[10] Also, it is not hard to check that the game is not C-secure, but is coalitionally C-secure. For example, $p = (0, 0)$ is not in the α-core. Consider an ε-neighborhood of p with $\varepsilon = 0.1$. Setting $\{0, 0, (1, 1); 0, 0, (2, 0.5)\}$ as the collection of actions and utility levels for coalitions $\{1\}$, $\{2\}$ and $\{1, 2\}$, respectively, proves the coalitional C-security at p. The price pair $(2, 4)$ is in the α-core of this game since firm 1 gets the monopoly profit, hence it cannot be part of any blocking coalition, and firm 2 cannot block since it cannot guarantee a profit above 0 for itself. Note that $(2, 2, 4, 0) \in \bar{\mathcal{X}}$ is not in the α^T-core of G since $\sum_{i \in N} u_i(2, 2) = 3 < 4$. But the game is not coalitionally C^T-secure at $(2, 2, 4, 0)$ since no coalition can block it. However, it is not hard to check that the game is coalitionally C_N^T secure and the α^T-core is nonempty.

Although the discontinuity of the payoff functions prevents us from applying the theorem of Scarf presented in the introduction, the induced non-transferable utility game satisfies the assumptions of Scarf's Theorem presented in Appendix A, see Fig. 1(a). Therefore, the nonemptiness of the α-core is guaranteed without referring to our result. A more severe problem due to discontinuity is the violation of the closedness of the set of attainable utilities of the coalitions, especially the grand coalition. The following example is more interesting from this perspective.

[10] Note that in this paper we consider only the existence of Nash equilibrium in pure-strategies. It is known that this game has a Nash equilibrium in mixed-strategies, see Blume (2003).

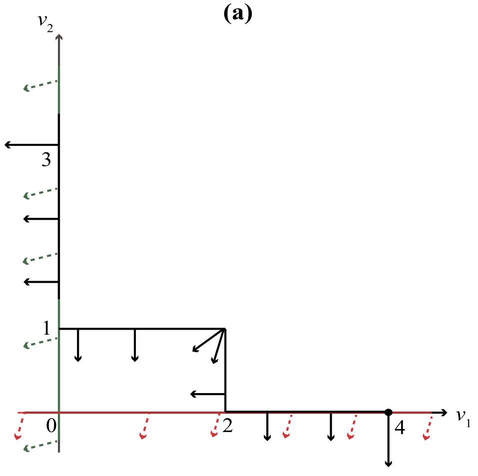

Fig. 1. The induced *NTU* games in Examples 1 and 2 are illustrated in panels (a) and (b), respectively. The boundary of the set of attainable utilities for coalition $\{1\}$ is illustrated by the green color, for coalition $\{2\}$ by the red color and for coalition $\{1, 2\}$ by the black color. The arrows show which side of the boundary can be attained by the coalitions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example 2. Consider a two player game where the action set of each player is $X_i = [0, 2]$. And, the payoff function $u_i : X = X_1 \times X_2 \to \mathbb{R}$ of player $i = 1, 2$ is defined as

$$u_1(x_1, x_2) = \begin{cases} x_2 & \text{if } x_1 + x_2 < 2, \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad u_2(x_1, x_2) = \frac{1}{2}u_1(x_2, x_1).$$

It is easy to check that the action profile $(1, 1)$ is in the α-core of the game. Moreover, the set of attainable utilities for the grand coalition is not closed, see Fig. 1, panel (b). Also, it is not hard to check that this game is coalitionally C-secure. Lastly, it is easy to see that this game has an empty α^T-core, and is not both coalitionally C^T-secure and coalitionally C^T_N-secure.

The following example illustrates a game which has a nonempty α^T-core and an empty α-core. The game is coalitionally C^T-secure but not coalitionally C-secure.

Example 3. Consider a two player game where the action set of each player is $X_i = [0, 2]$ (Fig. 2). And, the payoff function $u_i : X = X_1 \times X_2 \to \mathbb{R}$ of player $i = 1, 2$ is defined as

$$u_1(x_1, x_2) = \begin{cases} x_2 & \text{if } x_1 \leq 1.5, x_1 + x_2 < 2, \\ x_2 & \text{if } x_1 > 1.5, x_1 + x_2 \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} \frac{1}{2}x_1 & \text{if } x_1 \leq 1.5, x_1 + x_2 < 2, \\ \frac{1}{2}x_1 & \text{if } x_1 > 1.5, x_1 + x_2 \leq 2, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Note that this game is not coalitionally C-secure and has an empty α-core. However, it is bounded and coalitionally C^T-secure, and hence has a nonempty α^T-core.

In the context of the industrial organization theory, our results would have implications for both the formation and the outcome of the grand cartel (covert collusion) and monopoly merger (overt collusion). In line with this observation, we close this section by providing two results

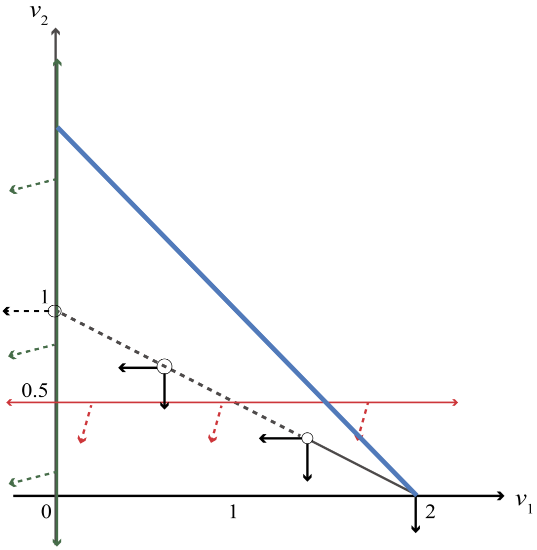

Fig. 2. The induced *NTU* and *TU* games in Example 3 are illustrated. The boundary of the set of attainable utilities for coalition $\{1\}$ is illustrated by the green color, for coalition $\{2\}$ by the red color and for coalition $\{1, 2\}$ by the black color. The arrows show which side of the boundary can be attained by the coalitions. The blue line describes the set of redistributions of the grand coalition's maximum aggregate payoff. The game is coalitionally C^T-secure but not coalitionally C-secure. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

on Cournot oligopoly models for which we considerably relax quasiconcavity and concavity assumptions and allow discontinuity in payoffs. A stronger version of the first one is introduced in Ichiishi (1993, Example 2A.11, p. 64) and the second in Zhao (1999, Theorem 2, p. 29). A Cournot oligopoly game is a game $G = (X_i, \pi_i)_{i \in N}$ where $N = \{1, \dots, n\}$ is the set of firms, $X_i = [0, \bar{y}_i]$ is the production set of firm i where $\bar{y}_i > 0$ denotes firm i's capacity constraint, and $\pi_i : X \to \mathbb{R}$ is the profit function of firm i which is defined as

$$\pi_i(x) = f_i(x_i, \sum_{j \in N \setminus \{i\}} x_j) = p(x_i + \sum_{j \in N \setminus \{i\}} x_j)x_i - c_i(x_i),$$

where $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ is firm i's cost function, $p : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse demand function and $f_i : \mathbb{R}^2_+ \to \mathbb{R}$.

Claim 3. *A coalitionally C-secure Cournot oligopoly game has a nonempty α-core if f_i is quasiconcave for each firm i.*

Claim 4. *A coalitionally C^T-secure (coalitionally C^T_N-secure), bounded Cournot oligopoly game has a nonempty α^T-core if p is decreasing and f_i is concave for each firm i.*

4. Proofs of the results

The proofs of Theorems 1–3 require a delicate construction. In order to give a general overview of the proofs, we first provide a *heuristic outline* of the proof of Theorem 1. Whereas the details of the proofs of Theorems 2 and 3 are different, the construction in the proof is quite similar. Our proof of Theorem 1 is by contradiction. We assume the α-core of the game is empty. By using the compactness of the action sets and the coalitional C-security of the game, we obtain a finite selection of points. And by using these finite selections, and quasiconcavity of the payoff functions we construct an *NTU* game which satisfies the assumptions of Scarf's Theorem. The nonemptiness of the core of this *NTU* game furnishes us a contradiction with the coalitional C-security assumption. Both Scarf's and our proof are *algorithmic*. He used an algorithm to find the core, we uses an algorithm to obtain a contradiction which is in line with Reny's insight. Note that in the proof of Theorem 1, we use Scarf's Theorem provided in Appendix A as Scarf (1971) did. But our application of Scarf's Theorem is very different. In Scarf's proof, continuity of the payoff functions and compactness of the action sets trivially implies that the induced *NTU* game satisfies assumptions (i)–(iii) of Scarf's Theorem. The nontrivial part of his proof is to show that quasiconcavity of the payoff functions implies the balancedness of the induced *NTU* game. Hence, Scarf's proof is a direct proof. In our problem, due to discontinuity in the payoff functions, the induced *NTU* game does not necessarily satisfy the closedness assumption, hence we cannot directly use Scarf's Theorem.

Proof of Theorem 1. Assume G has an empty α-core. Then, since G is coalitionally C-secure, for each $x \in X$, there exist $v_S^x \in \mathbb{R}^{|S|}$ and $y_S^x \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood U^x of x such that $u_S(y_S^x, z_{-S}) \geq v_S^x$ for each coalition $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$, and for each $z \in U^x$ there exists a coalition $S \in \mathcal{N}$ such that $u_S(z) \ll v_S^x$. The family $\{U^x | \, x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering $\{U^{x_k} | \, k = 1, \dots, m\}$. Let $U^k = U^{x_k}$, and for all $S \in \mathcal{N}$, $v_S^k = v_S^{x_k}$ and $y_S^k = y_S^{x_k}$ for all $k \in K = \{1, \dots, m\}$.

Now define an *NTU* game[11] $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^n$ as follows. For all $S \in \mathcal{N} \backslash N$,

$$V(S) = \bigcup_{k \in K} \{v \in \mathbb{R}^n | \, v_S \leq v_S^k\},$$

$$V(N) = \left(\bigcup_{k \in K} \{v \in \mathbb{R}^n | \, v \leq u(y_N^k)\} \right) \bigcup \left(\bigcup_{l \in L, \mathbf{k} \in K^{|\mathcal{B}^l|}} \{v \in \mathbb{R}^n | \, v \leq u(x^{l,\mathbf{k}})\} \right),$$

where $L, \mathcal{B}^l$ and $x^{l,\mathbf{k}} \in X$ are defined as follows. Let $\mathcal{T} = \{\mathcal{B}^l\}_{l \in L}$ be the set of all balanced collections of coalitions. Since $\mathcal{N}$ is finite, the collection of all balanced coalitions is finite and we denote this collection as $\{\mathcal{B}^l\}_{l \in L}$. For each balanced collection $\mathcal{B}^l$, let $\lambda^l = \{\lambda_S^l\}_{S \in \mathcal{B}^l}$ be the balancing weights (if the balancing weights of a balanced collection of coalitions are not unique, pick and fix an arbitrary one). Now, for each $l \in L$ and $\mathbf{k} \in K^{|\mathcal{B}^l|}$, define $x^{l,\mathbf{k}} \in X$ as

$$x_i^{l,\mathbf{k}} = \sum_{S \in \mathcal{B}^l : i \in S} \lambda_S^l y_{S,i}^{k_S} \quad \text{for all } i \in N,$$

where $y_{S,i}^{k_S}$ is the action of player $i \in S$ in the joint action $y_S^{k_S}$ of coalition S. Since each $x_i^{l,\mathbf{k}}$ is a convex combination of points in X_i, and X_i is convex, $x_i^{l,\mathbf{k}} \in X_i$.

By construction, the *NTU* game V is balanced. To see this, pick a balanced collection of coalitions $\mathcal{B} \in \mathcal{T}$ with the balancing weights λ, and let $v \in \bigcap_{S \in \mathcal{B}} V(S)$. If $N \in \mathcal{B}$, then $v \in V(N)$

[11] See Appendix A for the definition and properties of *NTU* games, and *TU* games as well which are used in the proofs of the results.

trivially holds. Otherwise, by construction of V, for each $S \in \mathcal{B}$ there exists $k_S \in K$ such that $v_S \leq v_S^{k_S} \leq u_S(y_S^{k_S}, z_{-S})$ for all $z_{-S} \in X_{-S}$. Define $x \in X$ as

$$x_i = \sum_{S \in \mathcal{B}: i \in S} \lambda_S y_{S,i}^{k_S} \quad \text{for all } i \in N.$$

By construction of V, $u(x) \in V(N)$. Therefore, showing

$$v_i \leq u_i(x) \quad \text{for each } i \in N$$

implies V is balanced. At this level of generality, it is sufficient to demonstrate that $v_1 \leq u_1(x)$, since by a suitable renaming of players any particular player can be made the first. We now adapt the proof technique in Scarf (1971) to our framework. Define $y^S \in X$ for each $S \in \mathcal{B}$ containing player 1 as follows. If $i \in S$, then

$$y_i^S = y_{S,i}^{k_S}.$$

If $i \notin S$, then

$$y_i^S = \frac{\sum \lambda_E y_{E,i}^{k_E}}{\sum \lambda_E},$$

where in both the numerator and the denominator the summation is taken over all $E \in \mathcal{B}$ which contain player i but not player 1. From Scarf (1971, p. 179)

$$x = \sum_{S \in \mathcal{B}: 1 \in S} \lambda_S y^S.$$

For each coalition $S \in \mathcal{B}$ containing player 1, we have defined an action profile such that each player i in S use the $y_{S,i}^{k_S}$, and each player i not in S use a specific strategy y_i^S. But since $y_S^{k_S}$ guarantees player 1 a utility of at least v_1 regardless of the strategy choices of the players not in S, we see that $u_1(y^S) \geq v_1$, for all $S \in \mathcal{B}$ which contain player 1. And, the quasiconcavity of u_1 implies

$$v_1 \leq u_1(x).$$

Therefore, V is balanced.

It is clear that conditions (i)–(ii) of Scarf's Theorem provided in Appendix A are satisfied. And since for each coalition S, the set $V(S)$ is constructed by using finitely many points, condition (iii) of Scarf's Theorem is satisfied. Hence, V has a nonempty core, i.e. there exists $v^* \in V(N)$ such that $v^* \notin \operatorname{int} V(S)$ for all $S \in \mathcal{N}$. Since $v^* \in V(N)$, by construction there exists $x^* \in X$ such that $v^* \leq u(x^*)$. Also, since G is coalitionally C-secure and $x^* \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $u_S(x^*) \ll v_S^k$. Since $\{v_S^k\} \times \mathbb{R}^{|-S|} \subset V(S)$, $v^* \in \operatorname{int} V(S)$. This furnishes us a contradiction. $\square$

Proof of Theorem 2. Assume G has an empty α^T-core. Then, since G is coalitionally C^T-secure, for each $(x, v) \in \bar{\mathcal{X}}$, there exist $w_S^{x,v} \in \mathbb{R}$ and $y_S^{x,v} \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood $U^{x,v}$ of (x, v) such that $\sum_{i \in S} u_i(y_S^{x,v}, z_{-S}) \geq w_S^{x,v}$ for each coalition $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$, and for each $(x', v') \in U^{x,v}$ there exists a coalition $S \in \mathcal{N}$ such that $\sum_{i \in S} v_i' < w_S^{x,v}$. The family $\{U^{x,v} | (x, v) \in \bar{\mathcal{X}}\}$ is an open covering of $\bar{\mathcal{X}}$ which, by compactness of $\bar{\mathcal{X}}$, contains a finite subcovering $\{U^{x_k,v_k} | k = 1, \ldots, m\}$. Let $U^k = U^{x_k,v_k}$, and for all $S \in \mathcal{N}$, $w_S^k = w_S^{x_k,v_k}$ and $y_S^k = y_S^{x_k,v_k}$ for all $k \in K = \{1, \ldots, m\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $S \in \mathcal{N} \backslash N$,

$$W(S) = \max_{k \in K} w_S^k,$$

$$W(N) = \max \left\{ \max_{k \in K} \sum_{i \in N} u_i(y_N^k), \max_{l \in L, \mathbf{k} \in K^{|\mathcal{B}^l|}} \sum_{i \in N} u_i(x^{l,\mathbf{k}}) \right\},$$

where $L, \mathcal{B}^l$ and $x^{l,\mathbf{k}} \in X$ are defined as follows. Let $\mathcal{T} = \{\mathcal{B}^l\}_{l \in L}$ be the set of all minimally balanced collections of coalitions. Since $\mathcal{N}$ is finite, the collection of all minimally balanced coalitions is finite and we denote this collection as $\{\mathcal{B}^l\}_{l \in L}$. For each minimal balanced collection $\mathcal{B}^l$, let $\lambda^l = \{\lambda_S^l\}_{S \in \mathcal{B}^l}$ be the balancing weights (note that the balancing weights of a minimal balanced collection of coalitions are always unique). Now, for each $l \in L$ and $\mathbf{k} \in K^{|\mathcal{B}^l|}$, define $x^{l,\mathbf{k}} \in X$ as

$$x_i^{l,\mathbf{k}} = \sum_{S \in \mathcal{B}^l : i \in S} \lambda_S^l y_{S,i}^{k_S} \quad \text{for all } i \in N,$$

where $y_{S,i}^{k_S}$ is the action of player $i \in S$ in the joint action $y_S^{k_S}$ of coalition S. Since each $x_i^{l,\mathbf{k}}$ is a convex combination of points in X_i, and X_i is convex, $x_i^{l,\mathbf{k}} \in X_i$.

We shall show for each minimally balanced collection of coalitions $\mathcal{B} \in \mathcal{T}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Pick $\mathcal{B} \in \mathcal{T}$ with the balancing weights λ. If $\mathcal{B} = \{N\}$, then $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$ trivially holds. Otherwise, since $\mathcal{B}$ is a minimal balanced collection of coalition, it does not contain N. By construction of W, for each $S \in \mathcal{B}$ there exists $k_S \in K$ such that $W(S) = w_S^{k_S} \leq \sum_{i \in S} u_i(y_S^{k_S}, z_{-S})$ for all $z_{-S} \in X_{-S}$. Define $x \in X$ as

$$x_i = \sum_{S \in \mathcal{B} : i \in S} \lambda_S y_{S,i}^{k_S} \quad \text{for all } i \in N.$$

By construction of W, $\sum_{i \in N} u_i(x) \leq W(N)$. Therefore, showing

$$\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq \sum_{i \in N} u_i(x)$$

will be sufficient. First, from the construction of W and quasiseparability,

$$\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq \sum_{S \in \mathcal{B}} \lambda_S \inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(y_S^{k_S}, z_{-S}) = \sum_{i \in N} \sum_{S \in \mathcal{B} : i \in S} \lambda_S \inf_{z_{-S} \in X_{-S}} u_i(y_S^{k_S}, z_{-S}).$$

Hence, showing the following inequality implies the desired result

$$\sum_{S \in \mathcal{B} : i \in S} \lambda_S \inf_{z_{-S} \in X_{-S}} u_i(y_S^{k_S}, z_{-S}) \leq u_i(x) \text{ for each } i \in N.$$

At this level of generality, it is sufficient to demonstrate that the above inequality holds for player 1, since by a suitable renaming of players any particular player can be made the first. We now define $y^S \in X$ for each $S \in \mathcal{B}$ containing player 1 as follows. If $i \in S$, then

$$y_i^S = y_{S,i}^{k_S}.$$

If $i \notin S$, then

$$y_i^S = \frac{\sum \lambda_E y_{E,i}^{k_E}}{\sum \lambda_E},$$

where in both the numerator and the denominator the summation is taken over all $E \in \mathcal{B}$ which contain player i but not player 1. From Scarf (1971, p. 179)

$$x = \sum_{S \in \mathcal{B}: 1 \in S} \lambda_S y^S.$$

Pick a coalition $S \in \mathcal{B}$ containing player 1. Then by construction of y^S,

$$\inf_{z_{-S} \in X_{-S}} u_1(y_S^{k_S}, z_{-S}) \leq u_1(y^S).$$

Therefore, from the concavity of u_1,

$$\sum_{S \in \mathcal{B}: 1 \in S} \lambda_S \inf_{z_{-S} \in X_{-S}} u_1(y_S^{k_S}, z_{-S}) \leq \sum_{S \in \mathcal{B}: 1 \in S} \lambda_S u_1(y^S) \leq u_1(x).$$

Therefore, since $\mathcal{B}$ is arbitrarily chosen, for each $\mathcal{B} \in \mathcal{T}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$.

Bondareva–Shapley Theorem provided in Appendix A implies W has a nonempty core, i.e. there exists $v^* \in \mathbb{R}^n$ such that $\sum_{i \in N} v_i^* = W(N)$ and $\sum_{i \in S} v_i^* \geq W(S)$ for all $S \in \mathcal{N}$. By construction of $W(N)$, there exists $x^* \in X$ such that $\sum_{i \in N} v_i^* = \sum_{i \in N} u_i(x^*)$. In particular, $x^* = y_N^{k'}$, or $x^{l,\mathbf{k}}$ for some $k' \in K$, $l \in L$, $\mathbf{k} \in K^{|\mathcal{B}^l|}$. Hence, $(x^*, v^*) \in \bar{\mathcal{X}}$. Since G is coalitionally C^T-secure and $(x^*, v^*) \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

Proof of Theorem 3. Since G_N is C-secure, there exists $\bar{x} \in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Hence, $\mathcal{V}$ is a well-defined compact set. Now assume G has an empty α^T-core. Then, since G is coalitionally C_N^T-secure, for each $v \in \mathcal{V}$, there exist $w_S^v \in \mathbb{R}$ and $y_S^v \in X_S$ for each $S \in \mathring{\mathcal{N}}$ and an open neighborhood U^v of v such that $\sum_{i \in S} u_i(y_S^v, z_{-S}) \geq w_S^v$ for each coalition $S \in \mathring{\mathcal{N}}$ and each $z_{-S} \in X_{-S}$, and for each $z \in U^v$ there exists a coalition $S \in \mathring{\mathcal{N}}$ such that $\sum_{i \in S} v_i' < w_S^v$. The family $\{U^v | \, v \in \mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k} | \, k = 1, \ldots, m\}$. Let $U^v = U^{v_k}$, and for all $S \in \mathring{\mathcal{N}}$, $w_S^k = w_S^{v_k}$ and $y_S^k = y_S^{v_k}$ for all $k \in K = \{1, \ldots, m\}$.

Now define a *TU* game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $S \in \mathcal{N} \backslash N$,

$$W(S) = \max_{k \in K} w_S^k, \quad \text{and } W(N) = \bar{w} = \max_{x \in X} \sum_{i \in N} u_i(x).$$

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Pick a minimal balanced collection of coalitions $\mathcal{B}$ with the balancing weights $\lambda = \{\lambda_S\}_{S \in \mathcal{B}}$ (note that the balancing weights of a minimal balanced collection of coalitions are always unique). If $\mathcal{B} = \{N\}$, then $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$ trivially holds. Otherwise, since $\mathcal{B}$ is a minimal balanced collection of coalition, it does not contain N. We shall show that there exists $x' \in X$ such that

$$\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq \sum_{i \in N} u_i(x') \leq \bar{w} = W(N),$$

where the last inequality follows from the definition of $\bar{w}$. By construction of W, for each $S \in \mathcal{B}$ there exists $k_S \in K$ such that $W(S) = w_S^{k_S} \leq \sum_{i \in S} u_i(y_S^{k_S}, z_{-S})$ for all $z_{-S} \in X_{-S}$. Define $x \in X$ as

$$x_i = \sum_{S \in \mathcal{B}: i \in S} \lambda_S y_{S,i}^{k_S} \quad \text{for all } i \in N.$$

By construction of W, $\sum_{i\in N} u_i(x) \leq W(N) = \bar{w}$. Therefore, showing

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{i\in N} u_i(x)$$

will be sufficient. The remainder of the proof now follows the argument of Theorem 2 verbatim. In particular, we conclude that $\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq W(N)$ for each $\mathcal{B} \in \mathcal{T}$.

Bondareva–Shapley Theorem provided in Appendix A implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i\in N} v_i^* \leq W(N)$ and $\sum_{i\in S} v_i^* \geq W(S)$ for all $S \in \mathcal{N}$. Since G is coalitionally C_N^T-secure and $v^* \in U^k$ for some $k \in K$, there exists $S \in \mathring{\mathcal{N}}$ such that $\sum_{i\in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

Proof of Proposition 1. Assume G has an empty α-core. Then, since G is coalitionally C-secure, for each $x \in X$, there exist $v_S^x \in \mathbb{R}^{|S|}$ and $y_S^x \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood U^x of x such that $u_S(y_S^x, z_{-S}) \geq v_S^x$ for each coalition $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$, and for each $x' \in U^x$ there exists a coalition $S \in \mathcal{N}$ such that $u_S(x') \ll v_S^x$. The family $\{U^x \mid x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering $\{U^{x_k} \mid k = 1, \dots, m\}$. Let $U^k = U^{x_k}$, and for all $S \in \mathcal{N}$, $v_S^k = v_S^{x_k}$ and $y_S^k = y_S^{x_k}$ for all $k \in K = \{1, \dots, m\}$.

Now define an *NTU* game $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^2$ as follows. For each $i \in N$,

$$V(\{i\}) = \bigcup_{k\in K} \{v \in \mathbb{R}^2 \mid v_i \leq v_i^k\},$$
$$V(N) = \bigcup_{k,k'\in K} \left(\{v \in \mathbb{R}^2 \mid v \leq u(y_N^k)\} \cup \{v \in \mathbb{R}^2 \mid v \leq u(y_1^k, y_2^{k'})\}\right).$$

By construction, the *NTU* game V is balanced. To see this, note that there are three balanced collections of coalitions for this game of which two contain N. For these collections, there is nothing to prove. The only balanced collection of coalitions which does not include N is $\mathcal{B} = \{\{1\}, \{2\}\}$. Pick $v \in V(S)$ for all $S \in \mathcal{B}$. Then, there exists $k, k' \in K$ such that[12] $v_1 \leq v_1^k$ and $v_2 \leq v_2^{k'}$. By construction, $v_1^k \leq u_1(y_1^k, z_2)$ for all $z_2 \in X_2$ and $v_2^k \leq u_2(z_1, y_2^{k'})$ for all $z_1 \in X_1$. Therefore $(v_1^k, v_2^{k'}) \leq u(y_1^k, y_2^{k'})$, and hence $v \in V(N)$. Therefore V is balanced.

It is clear that conditions (i)–(ii) of Scarf's Theorem provided in Appendix A are satisfied. And since for each coalition S, the set $V(S)$ is constructed by using finitely many points, condition (iii) of Scarf's Theorem is satisfied. Hence, V has a nonempty core, i.e. there exists $v^* \in V(N)$ such that $v^* \notin \operatorname{int} V(S)$ for all $S \in \mathcal{N}$. Since $v^* \in V(N)$, by construction there exists $x^* \in X$ such that $v^* \leq u(x^*)$. Also, since G is coalitionally C-secure and $x^* \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $u_S(x^*) \ll v_S^k$. Since $\{v_S^k\} \times \mathbb{R}^{|-S|} \subset V(S)$, $v^* \in \operatorname{int} V(S)$. This furnishes us a contradiction. □

Proof of Proposition 2. Since G_N is C-secure, there exists $\bar{x} \in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Hence, $\mathcal{V}$ is a well-defined compact set. Now assume G has an empty α^T-core. Then, since G is coalitionally C_N^T-secure, for each $v \in \mathcal{V}$, there exist $w_i^v \in \mathbb{R}$ and $y_i^v \in X_S$ for each $i \in N$ and an open neighborhood U^v of v such that

[12] Recall that we abuse the notation here, and refer to the singleton coalitions without using curly brackets, see footnotes 4 and 5 for details about the notation.

$u_i(y_i^v, z_j) \geq w_i^v$ for each player $i \neq j$ and each $z_j \in X_j$, and for each $z \in U^v$ there exists a player $i \in N$ such that $v_i' < w_i^v$. The family $\{U^v | \ v \in \mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k} | \ k = 1, \dots, m\}$. Let $U^v = U^{v_k}$, and for all $i \in N$, $w_i^k = w_i^{v_k}$ and $y_i^k = y_i^{v_k}$ for all $k \in K = \{1, \dots, m\}$.

Now define a *TU* game $W : \mathcal{N} \to \mathbb{R}$ as follows. For each $i \in N$,

$$W(\{i\}) = \max_{k \in K} w_i^k, \quad \text{and } W(N) = \bar{w} = \max_{x \in X} \sum_{i \in N} u_i(x).$$

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Note that there are only two minimal balanced collections of coalitions: $\mathcal{B} = \{N\}$, or $\{\{1\}, \{2\}\}$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Let $\mathcal{B} = \{\{1\}, \{2\}\}$. By construction, there exists $k, k' \in K$ such that $W(\{1\}) = w_1^k$ and $W(\{2\}) = w_2^{k'}$. By construction, $w_1^k \leq u_1(y_1^k, z_2)$ for all $z_2 \in X_2$ and $w_2^k \leq u_2(z_1, y_2^{k'})$ for all $z_1 \in X_1$. Therefore, $(w_1^k, w_2^{k'}) \leq u(y_1^k, y_2^{k'})$ and $\sum_{i \in N} u_i(y_1^k, y_2^{k'}) \leq \bar{w}$ imply $W(\{1\}) + W(\{2\}) \leq W(N)$.

Bondareva–Shapley Theorem provided in Appendix A implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i \in N} v_i^* = W(N)$ and $v_i^* \geq W(\{i\})$ for all $i \in N$. Since G is coalitionally C_N^T-secure and $v^* \in U^k$ for some $k \in K$, there exists $i \in N$ such that $v_i^* < w_i^k$. By construction, $w_i^k \leq W(\{i\})$. This furnishes us a contradiction. □

Proof of Claim 1. Let G be a bounded and coalitionally C^T-secure 2-player game. Showing the induced one player game G_N is C-secure is enough to prove that G is coalitionally C_N^T-secure. Now pick $x \in X$ that is not a maximizer of $\bar{u} = u_1 + u_2$. Then for each $v \in \mathbb{R}^2$ such that $(x, v) \in \bar{\mathcal{X}}$ cannot be in the α^T-core of G. Then there exist $w_S^{x,v} \in \mathbb{R}$ and $y_S^{x,v} \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood $U^{x,v}$ of (x, v) such that $\sum_{i \in S} u_i(y_S^{x,v}, z_{-S}) \geq w_S^{x,v}$ for each coalition $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$, and for each $(x', v') \in U^{x,v}$ there exists a coalition $S \in \mathcal{N}$ such that $\sum_{i \in S} v_i' < w_S^{x,v}$. Let U^x be the projection of $U^{x,v}$ on X. Since the projection is an open map, U^x is an open neighborhood of x. Define y^x as the maximizer of $\bar{u}$ over the set $\{(y_1^{x,v}, y_2^{x,v}), y_N^{x,v}\}$. Pick $x' \in U^x$. Then $(x', v') \in U^{x,v}$ for each $v' \in \mathcal{V}^{x'} = \{v'' \in \mathbb{R}^2 | v_1'' + v_2'' = \bar{u}(x'), \ \inf_{x'' \in X} \ u_i(x'') \leq v_i'' \ \forall i \in N\}$. If for some $v' \in \mathcal{V}^{x'}$, $v_1' + v_2' < w_N^{x,w}$ then $\bar{u}(x') < w_N^{x,w} \leq \bar{u}(y^x)$. Otherwise, for each $v' \in \mathcal{V}^{x'}$, either $v_1' < w_1^{x,v}$ or $v_2' < w_2^{x,v}$. If for some player $i \in N$, $v_i' < w_i^{x,v}$ for each $v' \in \mathcal{V}^{x'}$, then $\bar{u}(x') < w_i^{x,v} \leq \bar{u}(y^x)$. Otherwise, let $v_i' = w_i^{x,v}$ for some $v' \in \mathcal{V}^{x'}$. Then $w_j^{x,v} > \bar{u}(x') - v_i'$ for player $j \neq i$. Hence $\bar{u}(x') = v_1' + v_2' < w_1^{x,v} + w_2^{x,v} \leq \bar{u}(y^x)$. Therefore, for each $x'' \in U^x$, $\bar{u}(x'') < w^x = \bar{u}(y^x)$.

Example 1 illustrates a bounded and coalitionally C_N^T-secure 2-player game that is not coalitionally C^T-secure. □

Proof of Claim 2. Let G be a bounded and coalitionally C^T-secure game such that $\bar{u}$ has a maximizer $\bar{x} \in X$, and let $\bar{w} = \bar{u}(\bar{x})$. Note that since G is coalitionally C^T-secure, showing the induced one player game $G_N = (X, \bar{u})$ is C-secure is equivalent to showing G is coalitionally C_N^T-secure. Now assume G is not C-secure. Then, there exists $x \in X$ such that $\bar{u}(x) < \bar{w}$ and for each open neighborhood U^x of x there exists $x' \in X$ such that $\bar{u}(x') \geq \bar{u}(y)$ for each $y \in X$. Therefore there exists a sequence $x^m \in X$ such that $x^m \to x$ and for each $m \in \mathbb{N}$, $\bar{u}(x^m) \geq \bar{u}(y)$ for each $y \in X$. Hence $\bar{u}(x^m) \to \bar{w}$. But since x is not a maximizer of $\bar{u}$, G cannot be coalitionally C^T-secure.

Now we illustrate a bounded and coalitionally C_N^T-secure game that is not C^T-secure. Let $X_i = [0, 1]$, $u_i(x_i, x_{-i}) = x_i$ for $x_i \in (0, 1)$, $u_i(0, x_{-i}) = 1$ and $u_i(1, x_{-i}) = 0$ for each $i \in N$ and

$x_{-i} \in X_{-i}$. It is clear that $\{((0,\dots,0),(1,\dots,1))\}$ is the α^T-core of this game. For each $S \in \mathcal{N}$ and $x \in X \backslash \{(0,\dots,0)\}$, set $w_S^v = |S|$, y_S^x is the vector of zeros in $\mathbb{R}^{|S|}$, and the ε-neighborhood of x by choosing $\varepsilon = (1-u_i(x))/4$ for some $i \in N$. Hence this game is coalitionally C_N^T-secure. However, since $((1,\dots,1),(1,\dots,1))$ is in $\bar{\bar{\mathcal{X}}}$, this game is not coalitionally C^T-secure. □

Proof of Claim 3. It is enough to show that π_i is quasiconcave on X if and only if f_i is quasiconcave in the two argument for each $i \in N$. Pick $i \in N, x, y \in X$ and $\delta \in (0,1)$. Then, $\pi_i(\delta x + (1-\delta)y) \geq \min\{\pi_i(x), \pi_i(y)\}$ if and only if $p(\delta x_i + (1-\delta)y_i + \delta \sum_{j \in N\backslash\{i\}} x_j + (1-\delta)\sum_{j \in N\backslash\{i\}} x_j)(\delta x_i + (1-\delta)y_i) - c_i(\delta x_i + (1-\delta)y_i) \geq \min\{p(x_i + \sum_{j \in N\backslash\{i\}} x_j)x_i - c_i(x_i),\ p(y_i + \sum_{j \in N\backslash\{i\}} y_j)y_i - c_i(y_i)\}$ if and only if $f_i(\delta x + (1-\delta)y) \geq \min\{f_i(x), f_i(y)\}$. □

Proof of Claim 4. We shall first show that π_i is concave on X if and only if f_i is concave in the two argument for each $i \in N$. Pick $i \in N, x, y \in X$ and $\delta \in (0,1)$. Then, $\pi_i(\delta x + (1-\delta)y) \geq \delta\pi_i(x) + (1-\delta)\pi_i(y)$ if and only if $p(\delta x_i + (1-\delta)y_i + \delta \sum_{j \in N\backslash\{i\}} x_j + (1-\delta)\sum_{j \in N\backslash\{i\}} x_j)(\delta x_i + (1-\delta)y_i) - c_i(\delta x_i + (1-\delta)y_i) \geq \delta(p(x_i + \sum_{j \in N\backslash\{i\}} x_j)x_i - c_i(x_i)) + (1-\delta)(p(y_i + \sum_{j \in N\backslash\{i\}} y_j)y_i - c_i(y_i))\}$ if and only if $f_i(\delta x + (1-\delta)y) \geq \delta f_i(x) + (1-\delta) f_i(y)$. Next, since p is a decreasing function, for each $S \neq N$, $\min_{z_{-S} \in X_{-S}} \sum_{i \in S} \pi_i(x_S, z_{-S})$ is well defined for each $x_S \in X_S$ and $\bar{y}_T$ is a minimizer for both this problem and for $\sum_{i \in S} \min_{z_{-S} \in X_{-S}} \pi_i(x_S, z_{-S})$. Therefore, the game is quasiseparable. □

5. Concluding remarks

This paper provides sufficient conditions for the nonemptiness of the TU and NTU α-cores of games with possibly discontinuous payoff functions. We end this paper with three remarks. First, although the α-core is widely applied cooperative solution concept for normal form games, a number of different solution concepts, such as β-core, strong equilibrium and hybrid solution of Zhao (1999) are also of interest for analyzing specific problems. It will be of interest to discuss the existence of such solutions for discontinuous games.

Second, a generalization of Scarf (1971) to games with nonordered preferences is provided by Kajii (1992), and a generalization to nonordered and discontinuous preferences by Martins-da Rocha and Yannelis (2011). Although their models capture nonordered preferences on infinite dimensional spaces, our coalitional-security condition is substantially weaker than the continuity assumption they imposed when the set of actions are finite dimensional and the preferences are represented by payoff functions. They worked with correspondences $P_S : X \twoheadrightarrow X_S$ which map each action of the grand coalition to the set of blocking actions of coalition S. Martins-da-Rocha and Yannelis assumed that the correspondence P_S has open fibers on X which implies if a coalition S blocks an action profile x by using action y_S, then it blocks all actions around an open neighborhood of x by using y_S. Whereas, coalitional-security implies each point in some neighborhood of x is blocked by some coalition by using a fixed action profile. Hence, the blocking coalition may alternate for different points in the neighborhood. It is possible to generalize the state of the art result of Martins-da Rocha and Yannelis (2011) in this line of literature by weakening the open fibers assumption and using the concepts and methods defined in Uyanık (2014). Moreover, the nonemptiness of the α^T-core has not been studied even for nonordered and continuous preferences.

Third, the unbounded payoff functions, especially the logarithmic functions, are essential for many economic models. Our results in this paper require compactness of the action sets and boundedness of the payoff functions. It may be of interest to provide existence results for games with unbounded payoff functions and noncompact action sets.

Appendix A

A *nontransferable utility (NTU) game* is a nonempty-valued correspondence $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^n$, where $N = \{1, \dots, n\}$ is the set of players and $\mathcal{N} = 2^{N \setminus \emptyset}$ the set of coalitions. The *core* of an *NTU* game V is defined as $\text{Core}(V) = V(N) \setminus \left(\bigcup_{S \in \mathcal{N}} \text{int}\, V(S)\right)$, where $\text{int}\, V(S)$ is the (topological) interior of the set $V(S)$. A collection of coalitions $\mathcal{B} \subset 2^{\mathcal{N}}$ is *balanced* if for each coalition S, there exists a nonnegative scalar λ_S with $\lambda_S = 0$ if $S \notin \mathcal{B}$ such that for each $i \in N$, $\sum_{S:i \in S} \lambda_S = 1$. An *NTU* game V is *balanced* if for all balanced collections of coalitions $\mathcal{B}$, $\bigcap_{S \in \mathcal{B}} V(S) \subset V(N)$. Now, we state the beautiful theorem of Scarf (1967) which is used to prove our results.

Theorem *(Scarf). A balanced NTU game V has a nonempty core if for each coalition S,*

(i) *$V(S)$ is closed,*
(ii) *$v' \in \mathbb{R}^n$, $v \in V(S)$ and $v'_S \leq v_S$ imply $v' \in V(S)$,*
(iii) *there exists $M_S \in \mathbb{R}^{|S|}$ such that $v \in V(S)$ implies $v_S \leq M_S$.*

A *transferable utility (TU) game* is a function $W : \mathcal{N} \to \mathbb{R}$. The *core* of a *TU* game W is defined as $\text{Core}^T(W) = \{v \in \mathbb{R}^n | \sum_{i \in N} v_i \leq W(N) \text{ and } \sum_{i \in S} v_i \geq W(S)\ \forall S \in \mathcal{N}\}$. A balanced collection of coalitions $\mathcal{B} \subset 2^{\mathcal{N}}$ is *minimal* if it does not have a balanced proper subcollection. Note that the balancing weights λ of every minimally balanced collection of coalitions are unique. We end this paper by stating the influential theorem of Bondareva (1962) and Shapley (1967) which is used to prove our results.

Theorem *(Bondareva–Shapley). A TU game has a nonempty core if and only if for every minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$.*

References

Aumann, R.J., 1961. The core of a cooperative game without side payments. Trans. Am. Math. Soc. 98, 539–552.
Barelli, P., Meneghel, I., 2013. A note on the equilibrium existence problem in discontinuous games. Econometrica 81, 813–824.
Blume, A., 2003. Bertrand without fudge. Econ. Lett. 78, 167–168.
Bondareva, O., 1962. The theory of the core in an n-person game. Vestn. Leningr. Univ. 13, 141–142.
Carmona, G., 2011. Symposium on: existence of Nash equilibria in discontinuous games. Econ. Theory 48, 1–4.
Dasgupta, P., Maskin, E., 1986. The existence of equilibrium in discontinuous economic games, I: theory, II: applications. Rev. Econ. Stud. 53, 1–41.
Ernst, E., 2013. A converse of the Gale–Klee–Rockafellar theorem: continuity of convex functions at the boundary of their domains. Proc. Am. Math. Soc. 141, 3665–3672.
Ichiishi, T., 1993. The Cooperative Nature of the Firm. Cambridge University Press, New York.
Kajii, A., 1992. A generalization of Scarf's theorem: an α-core existence theorem without transitivity or completeness. J. Econ. Theory 56, 194–205.
Martins-da-Rocha, V.F., Yannelis, N., 2011. Nonemptiness of the alpha core. Working paper. Manchester School of Social Sciences, University of Manchester.
McLennan, A., Monteiro, P.K., Tourky, R., 2011. Games with discontinuous payoffs: a strengthening of Reny's existence theorem. Econometrica 79, 1643–1664.
Ray, D., Vohra, R., 1997. Equilibrium binding agreements. J. Econ. Theory 73, 30–78.

Reny, P.J., 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica 67, 1029–1056.
Reny, P.J., 2013. Nash equilibrium in discontinuous games. BFI working paper series, 2013-004.
Scarf, H.E., 1967. The core of an N person game. Econometrica 35, 50–69.
Scarf, H.E., 1971. On the existence of a cooperative solution for a general class of n-person games. J. Econ. Theory 3, 169–181.
Shapley, L.S., 1967. On balanced sets and cores. Nav. Res. Logist. Q. 14, 453–460.
Uyanık, M., 2014. Maximality in games and economies: a unified treatment. Working paper. Johns Hopkins University.
Zhao, J., 1999. The existence of TU α-core in normal form games. Int. J. Game Theory 28, 25–34.