Reachability is Tower Complete

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Abstract

A complete characterization of the complexity of the reachability problem for vector addition system has been open for a long time. The problem is shown to be Tower complete.

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1. Introduction

Petri net theory has been studied for well over half a century [19]. As a model for concurrency and causality, Petri net model finds a wide range of applications in system specification and verification. Two equivalent formulations of Petri nets, Vector Addition System (VAS) and Vector Addition System with States (VASS), have been investigated in theoretical setting. In VASS system configurations are formulated by vectors on non-negative integers. Computation rules of systems are captured by vectors on integers. A computation is a sequence of legal transitions of configurations. Many system properties can be formalized and checked algorithmically in Petri net model [4], among which reachability has proved to be the most challenging one. A variety of problems in language, logic, concurrency can be reduced to VASS reachability problem [27]. Some famous problems are related to reachability problem of extended VASS, say branching VASS [22, 24].

Decidability result of the VASS reachability problem is among the most significant theoretical breakthroughs in computer science. The 1970's saw decidability proofs for low dimensional VASS reachability [6, 30] and an incomplete proof of the full VASS reachability problem by Sacerdote and Tenney [23]. In the decade from early 1980's to early 1990's, a complete proof of the decidability using decomposition technique was discovered by Mayr [17], and refined by Kosaraju [8] and Lambert [9]. Starting from 2010 Leroux has been studying the problem in a more logic setting [10–13].

The complexity of the reachability problem for VASS is a long standing open problem. For many years Lipton's EXPSPACE hardness result [16], announced in 1976, is the only bound we knew. In the last ten years continuing efforts have been made to attack the problem. The achievements can be classified into three categories. Firstly completeness results have been established for low dimensional VASS. Haase, Kreutzer, Ouaknine and Worrell showed that reachability of 1-dimensional VASS is NP-complete [5]. In the 2-dimensional case, Boldin, Finkel, Göller, Hasse and McKenzie proved that the problem is PSPACE complete [1] and Englert, Lazić and Totzke pointed out that the problem is NL-complete [3] if unary encoding is used. Secondly new upper bounds of the general problem have been discovered. Leroux and Schmidt have provided an F_{ω^3} upper bound in [14]. An improved F_{ω^2} upper bound is given in [28]. In the latest announcement [15], Leroux and Schmidt have proved that for $d > 2$ the d-dimensional VASS problem is in F_{d+4} and consequently that the general VASS problem is in F_{ω}. Lastly but not the least, a non-elementary lower bound has been established by Czerwińsky, Lasota, Lazić, Leroux and Mazowiecki [2]. We now know that the general VASS reachability problem is tower hard. This is a gigantic jump from the EXPSPACE hardness.

The reason that KLMST decomposition algorithm is difficult to analyze is that every decomposition step, which converts a sequence of graphs to a longer sequence of graphs, increases the size of thing exponentially in such a way that may create exponentially more decomposition steps. It was pointed out by Müller [18] that Mahr's algorithm is not even primitive recursive due to its reference to Karp and Miller's coverability trees [7]. The best upper bound that has been obtained so far is that VASS reachability is in F_{ω} [14]. The crucial observation that has led to the discovery of the primitive recursive upper bound is that a refinement of Lipton's approach [16], which has an exponential space complexity, can be exploited to reduce the dimensionality of VASS instances. Dimension reduction is part of the decomposition algorithm, although its role has not been paid enough attention it deserves. So far all the arguments for the upper bounds are combinatorial.

Building on Leroux and Schmidt's observation, a purely algebraic analysis of the decomposition technique is presented in this paper. The algebraic approach reveals a fundamental property of the decomposition procedure and the reduction procedure, they are hugely parallel. This parallelism is sufficient for us to derive an elementary bound on the height of the tower function that bounds the complexity of the decomposition algorithm. It follows immediately from this observation and the tower hardness [1] that VASS reachability is Tower complete.

The rest of the paper is organized as follows. Section 2 states the preliminaries. Section 3 discusses how linear equation systems help characterize paths in vector addition system. Section 4 does the same for local circular paths. Section 5 describes dimension reduction. Section 6 and Section 7 define reduction algorithm and decomposition algorithm respectively. Section 8 gives an account of the well known reachability checking algorithm in our setting. Section 9 establishes the tower completeness. This section is the main contribution of the paper. Section 10 makes a few comments.

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2. Preliminary

Let $\mathbb{N}$ be the set of natural numbers (nonnegative integers) and $\mathbb{Z}$ the set of integers. Let $\mathbb{V}$ denote the set of variables for nonnegative integers. We write $f, g, h, i, j, k, l, m, n, L, M, N$ for elements in $\mathbb{N}$, r, s, t for elements in $\mathbb{Z}$, and x, y, z for elements in $\mathbb{V}$. For $L \in \mathbb{N} \setminus \{0\}$ let $[L]$ be the set $\{1, \ldots, L\}$ and $[L]^+$ be $\{0\} \cup [L]$. For a finite set S let $|S|$ denote the number of element of S.

A *multi-set* $\mathcal{S}$ is a function $\mathcal{S} : S \to (\mathbb{N} \setminus \{0\})$. For each $e \in S$, the positive value $\mathcal{S}(e)$ denotes the number of occurrence of e in $\mathcal{S}$. We will apply the standard set theoretical notation to the multi-sets. We write for instance $e \in \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{S}'$, pretending that $\mathcal{S}, \mathcal{S}'$ are sets. We write $S(\mathcal{S})$ for the underlying set S.

We write $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{m}, \mathbf{n}, \mathbf{w}$ for d-dimensional vectors in $\mathbb{N}^d$, $\mathbf{r}, \mathbf{s}, \mathbf{t}$ for vectors in $\mathbb{Z}^d$, and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ for vectors in $\mathbb{V}^d$. All vectors are column vectors. For $i \in [d]$ we write for example $\mathbf{a}(i)$ for the *i-th entry* of $\mathbf{a}$. Let $\mathbf{1} = (1, \ldots, 1)^\dagger$ and $\mathbf{0} = (0, \ldots, 0)^\dagger$, where $(\_)^\dagger$ is the transposition operator. We write σ for a finite sequence of vectors and $|\sigma|$ for the length of σ. For $i \in [|\sigma|]$ we write $\sigma[i]$ for the i-th element appearing in σ.

Recall that the 1-*norm* $\|\mathbf{r}\|_1$ of $\mathbf{r}$ is $\sum_{i \in [d]} |\mathbf{r}(i)|$. The ∞-*norm* $\|\mathbf{r}\|_\infty$ of $\mathbf{r}$ is $\max_{i \in [d]} |\mathbf{r}(i)|$. The 1-*norm* $\|A\|_1$ of an integer matrix A is $\sum_{i,j} |A(i,j)|$, and the ∞-*norm* $\|A\|_\infty$ is $\max_{i,j} |A(i,j)|$.

2.1 Non-Elementary Complexity Class

VASS reachability is not elementary [2]. To characterize the problem complexity theoretically, one needs complexity classes beyond the elementary class. Schmidt introduced an ordinal indexed class of complexity classes $\mathsf{F}_3, \mathsf{F}_4, \ldots, \mathsf{F}_\omega, \ldots, \mathsf{F}_{\omega^2}, \ldots, \mathsf{F}_{\omega^\omega}, \ldots$ and showed that many problems arising in theoretical computer science are complete problems in this hierarchy [25]. In the above sequence $\mathsf{F}_3 =$ **Tower** and $\mathsf{F}_\omega =$ **Ackermann**. The class **Tower** is closed under elementary reduction and **Ackermann** is closed under primitive recursive reduction. For the purpose of this paper it suffices to say that **Tower** contains all the problems whose space complexity is bounded by tower functions of the form

$$2^{\cdot^{\cdot^{\cdot^{2^n}}}}\Big\}f(n),$$

where $f(n)$ is an elementary function.

We shall call for example a polynomial time problem a $\mathtt{poly}(n)$ time problem, and an exponential space problem a $2^{\mathtt{poly}(n)}$ space problem. Alternatively we may say that a problem can be solved in $\mathtt{poly}(n)$ time or $2^{\mathtt{poly}(n)}$ space etc.. For simplicity we shall always omit constant factors when making statements about complexity.

2.2 Integer Programming

We shall need a result in integer linear programming [29]. Let A be an $m \times k$ integer matrix and $\mathbf{x} \in \mathbb{V}^k$. The *homogeneous equation system* of A is given by the linear equation system $\mathcal{E}$ specified by

$$A\mathbf{x} = \mathbf{0}. \tag{1}$$

A nontrivial solution to (1) is some $\mathbf{a} \in \mathbb{N}^k \setminus \{\mathbf{0}\}$ such that $A\mathbf{a} = \mathbf{0}$. The set of solutions form a monoid $(\mathcal{S}, \mathbf{0}, +)$. Since the pointwise ordering $\leq$ is a well quasi order on $\mathbb{N}^k$, $\mathcal{S}$ must be generated by a finite set of nontrivial minimal solutions. This finite set is called the *Hilbert base* of $\mathcal{E}$, denoted by $\mathcal{H}(\mathcal{E})$. The following important result is proved by Pottier [20], in which r is the *rank* of A.

Lemma 1 (Pottier). *$\|\mathbf{m}\|_1 \leq (1 + k \cdot \|A\|_\infty)^r$ for every $\mathbf{m} \in \mathcal{H}(\mathcal{E})$.*

Let $\mathbf{r} \in \mathbb{Z}^k$. Nonnegative integer solutions to *equation system*

$$A\mathbf{x} = \mathbf{r} \tag{2}$$

can be derived from the Hilbert base of the homogeneous equation system $A\mathbf{x} - x'\mathbf{r} = \mathbf{0}$. Let $\mathbb{S}^{=\mathbf{r}}$ be the finite set of the minimal solutions to $A\mathbf{x} - x'\mathbf{r} = \mathbf{0}$ with $x' = 1$, and $\mathbb{S}^{=\mathbf{0}}$ be the finite set of the minimal solutions to $A\mathbf{x} - x'\mathbf{r} = \mathbf{0}$ with $x' = 0$. A solution to (2) is of the form

$$\mathbf{m} + \sum_{i \in [|\mathbb{S}^{=\mathbf{0}}|]} k_i \mathbf{m}_i,$$

where $\mathbf{m} \in \mathbb{S}^{=\mathbf{r}}$, $\mathbf{m}_i \in \mathbb{S}^{=\mathbf{0}}$ and k_i is a natural number for each $i \in [|\mathbb{S}^{=\mathbf{0}}|]$. The following is an immediate consequence of Lemma 1.

Corollary 2. *$\|\mathbf{m}\|_1 \leq (1 + k \cdot \|A\|_\infty + \|\mathbf{r}\|_\infty)^{r+1}$ for all $\mathbf{m} \in \mathbb{S}^{=\mathbf{r}} \cup \mathbb{S}^{=\mathbf{0}}$.*

The size of (1) can be defined by $mk \log(\|A\|_\infty)$, and the size of (2) by $mk \log(\|A\|_\infty) + \|\mathbf{r}\|_\infty$. The size of $(1 + k \cdot \|A\|_\infty + \|\mathbf{r}\|_\infty)^{r+1}$ is polynomial. Thus $|\mathbb{S}^{=\mathbf{r}}|$ and $|\mathbb{S}^{=\mathbf{0}}|$ are bounded by exponentials.

In polynomial space a nondeterministic algorithm can guess a solution and check if it is minimal. Hence the following.

Corollary 3. *Both $\mathbb{S}^{=\mathbf{r}}$ and $\mathbb{S}^{=\mathbf{0}}$ can be produced in $\mathtt{poly}(n)$ space.*

2.3 VASS as Labeled Graph

By a *digraph* we mean a finite directed graph in which multi-edges and self loops are admitted. A *d-dimensional vector addition system with states*, or *d-VASS*, is a labeled digraph $G = (Q, T)$ where Q is the set of vertices and T is the set of edges. Edges are labeled by elements of $\mathbb{Z}^d$, and the labels are called *displacements*. A *state* is identified to a vertex and a *transition* is identified to a labeled edge. We write o, p, q for states, ϵ and its decorated versions for edges. A transition from p to q labeled $\mathbf{t}$ is denoted by $p \xrightarrow{\mathbf{t}} q$. If ϵ denotes $p \xrightarrow{\mathbf{t}} q$, we write ϵ^{-1} for the transition $q \xrightarrow{-\mathbf{t}} p$, and we write T^{-1} for $\{\epsilon^{-1} \mid \epsilon \in T\}$. We write G^{-1} for the d-VASS (Q, T^{-1}). In the rest of the paper we refer to $G = (Q, T)$ either as a graph or as a VASS. The input size of a VASS $G = (Q, T)$ is to be understood as the length of its binary code.

A *G-path* π from p to q with labels $\sigma = \mathbf{t}_1 \ldots \mathbf{t}_k$ is a sequence of edges $p \xrightarrow{\mathbf{t}_1} p_1 \xrightarrow{\mathbf{t}_2} \ldots \xrightarrow{\mathbf{t}_{k-1}} p_{k-1} \xrightarrow{\mathbf{t}_k} p_k = q$ for some states $p_1, \ldots, p_{k-1}$, often abbreviated to $p \xrightarrow{\sigma} q$. A special instance of a G-path is a *G-edge* $p \xrightarrow{\mathbf{t}} q$, which is nothing but an edge in G. Let $\pi[i]$ denote the i-th element in the sequence π. For simplicity the G-path $p \xrightarrow{\sigma} q$ is sometimes abbreviated to σ. We shall write $p \longrightarrow^* q$ for a G-path when the labels are immaterial. A *G-cycle* θ is a G-path $p \longrightarrow^* q$ such that $p = q$. A *G-loop* $\emptyset$ is a G-cycle $p_0 \xrightarrow{\mathbf{t}_1} p_1 \xrightarrow{\mathbf{t}_2} \ldots \xrightarrow{\mathbf{t}_{k-1}} p_{k-1} \xrightarrow{\mathbf{t}_k} p_0$ such that $p_0, \ldots, p_{k-1}$ are pairwise distinct. The *underlying graph* of a G-path π, denoted by $G(\pi)$, is the subgraph of G defined by the edges appearing in π.

A Parikh image for $G = (Q, T)$ is a vector in $\mathbb{N}^T$. We will write ϕ, φ, ψ for Parikh images. The displacement $\Delta(\psi)$ is defined by $\sum_{\epsilon = (p, \mathbf{t}, q) \in T} \psi(\epsilon) \cdot \mathbf{t}$. We define the Parikh image ψ^{-1} by $\psi^{-1}(\epsilon) = \psi(\epsilon^{-1})$. A Parikh image often specifies, not necessarily uniquely, a G-path. For a G-path π we write $\Im(\pi)$ for the Parikh image defined by letting $\Im(\pi)(\epsilon)$ be the number of occurrence of ϵ in π.

Definition 4. *A* loop class *C of a VASS G is a set of G-loops. The* loop class *$\mathcal{L}(G')$ of a subgraph G' of G is the set of G-loops appearing in G'. The* loop class *$\mathcal{L}(\theta)$ of a G-cycle θ is $\mathcal{L}(G(\theta))$.*

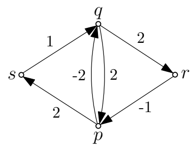
A loop class can be identified to a set of disjoint *strongly connected component* (*SCC*) of G, the *underlying graph* of the loop class. A trip made from a loop class is a trip in one of the SCCs.

The length of a loop is bounded by $|Q|$. There is some polynomial $w(n)$ such that the number of G-loops is bounded by

$$n! + \frac{n!}{2!} + \ldots + \frac{n!}{(n-1)!} < n! \cdot e = 2^{w(n)}. \tag{3}$$

Consequently the number of loop classes is bounded by $2^{2^{w(n)}}$.

Consider the 1-VASS G defined by the following labeled graph.



The G-loops are

$$\begin{aligned}\emptyset_1 &= p \to q \to r \to p,\\ \emptyset_2 &= p \to s \to q \to p,\\ \emptyset_3 &= q \to r \to p \to s \to q,\\ \emptyset_4 &= q \to p \to q.\end{aligned}$$

Let θ be the G-cycle $p \to q \to r \to p \to s \to q \to p \to q \to p$. The loop class $\mathcal{L}(\theta)$ of θ is $\{\emptyset_1, \emptyset_2, \emptyset_3, \emptyset_4\}$.

2.4 Reachability

A *located state* $p(\mathbf{s})$ is a state p at a *location* $\mathbf{s} \in \mathbb{Z}^d$. A *one-step trip* with label $\mathbf{t}$ from a located state $p(\mathbf{r})$ to a located state $q(\mathbf{s})$, is a G-edge $p \xrightarrow{\mathbf{t}} q$ rendering true the equality $\mathbf{s} = \mathbf{r} + \mathbf{t}$. Such a one-step trip is denoted by $p(\mathbf{r}) \xrightarrow{\mathbf{t}} q(\mathbf{s})$. A *trip* from $p_0(\mathbf{s}_0)$ to $p_k(\mathbf{s}_k)$ is a sequence of one-step trips between located states concatenated one after another in the following manner

$$p_0(\mathbf{s}_0) \xrightarrow{\mathbf{r}_1} p_1(\mathbf{s}_1) \xrightarrow{\mathbf{r}_2} p_2(\mathbf{s}_2) \xrightarrow{\mathbf{r}_3} \dots \xrightarrow{\mathbf{r}_{k-1}} p_{k-1}(\mathbf{s}_{k-1}) \xrightarrow{\mathbf{r}_k} p_k(\mathbf{s}_k).$$

Let τ denote the above trip. The G-path of τ, denoted by $G(\tau)$, is $p_0 \xrightarrow{\mathbf{r}_1} p_1 \xrightarrow{\mathbf{r}_2} p_2 \xrightarrow{\mathbf{r}_3} \dots \xrightarrow{\mathbf{r}_{k-1}} p_{k-1} \xrightarrow{\mathbf{r}_k} p_k$. The *displacement* of τ, denoted by $\Delta(\tau)$, is $\sum_{i\in[k]} \mathbf{r}_i$. The length of τ is denoted by $|\tau|$ and its i-th element is denoted by $\tau[i]$. We say that τ is *circular* if $p_k = p_0$. A *walk* from $p_0(\mathbf{a}_0)$ to $p_k(\mathbf{a}_k)$ is a trip

$$p_0(\mathbf{a}_0) \xrightarrow{\mathbf{r}_1} p_1(\mathbf{a}_1) \xrightarrow{\mathbf{r}_2} p_2(\mathbf{a}_2) \xrightarrow{\mathbf{r}_3} \dots \xrightarrow{\mathbf{r}_{k-1}} p_{k-1}(\mathbf{a}_{k-1}) \xrightarrow{\mathbf{r}_k} p_k(\mathbf{a}_k)$$

where $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{N}^d$. Given a VASS $G = (Q, T)$ and located states $p(\mathbf{a}), q(\mathbf{b})$, the reachability problem $\mathbb{VASS}$ asks if there is a walk from $p(\mathbf{a})$ to $q(\mathbf{b})$. If inputs are promised to be d-dimensional, the problem is denoted by $\mathbb{VASS}^d$.

2.5 Constraint Graph Sequence

Let $G = (Q, T)$ be a d-VASS. A *constraint graph (CG) of* G is a triple pCq such that $p, q \in Q$ and C is a subgraph of G. We call p the *entry state* and q the *exit state* of the CG. When C is the trivial graph with one state and no transition, the CG is essentially a state. A *constraint graph sequence (CGS)* ξ for $(\mathbf{a}, \mathbf{b}) \in \mathbb{N}^d \times \mathbb{N}^d$ is of the following form

$$\xi_0 \xrightarrow{\mathbf{t}_1} \xi_1 \xrightarrow{\mathbf{t}_2} \dots \xrightarrow{\mathbf{t}_{j-1}} \xi_{j-1} \xrightarrow{\mathbf{t}_j} \xi_j \xrightarrow{\mathbf{t}_{j+1}} \dots \xrightarrow{\mathbf{t}_k} \xi_k,$$

where $\xi_j = p_j C_j q_j$ with $C_j = (Q_j, T_j)$ is a CG of G for all $j \in [k]^+$, and $q_{j-1} \xrightarrow{\mathbf{t}_j} p_j$ for all $j \in [k]$. A diagrammatic illustration is given in Figure 1. We say that ξ is a CGS from p to q. CGSs are the prime objects of all operations of our algorithms. An instance of $\mathbb{VASS}$ consists of a VASS and a pair $p(\mathbf{a}), q(\mathbf{b})$ can be seen as a CGS for $(\mathbf{a}, \mathbf{b})$. A trip of the form

$$p_0(\mathbf{a}) \longrightarrow^* q_0(\mathbf{b}_0) \xrightarrow{\mathbf{t}_1} p_1(\mathbf{a}_1) \longrightarrow^* q_1(\mathbf{b}_1) \dots \xrightarrow{\mathbf{t}_k} p_k(\mathbf{a}_k) \longrightarrow^* q_k(\mathbf{b}),$$

where $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_0, \dots, \mathbf{b}_{k-1} \in \mathbb{N}^d$, is of *type* ξ for $(\mathbf{a}, \mathbf{b})$. A walk of type ξ for $(\mathbf{a}, \mathbf{b})$ is called a *witness* of ξ for $(\mathbf{a}, \mathbf{b})$.

Given a VASS $G = (Q, T)$ and two states $p, q \in Q$, there is a polynomial time algorithm that computes the class of the SCCs of the graph G. The SCCs are connected by the edges of G. One can nondeterministically select a CGS from p to q by linearizing the SCCs of G.

3. Algebraic Characterization

We start with an algebraic characterization of CGSs. The *characteristic system* $\mathcal{E}_\xi$ of ξ for $(\mathbf{a}, \mathbf{b})$ is an under specification of ξ by equation system. For each CG $\xi_j = p_j C_j q_j$ with $C_j = (Q_j, T_j)$, we introduce a vector of variable $\mathbf{x}_j \in \mathbb{V}^d$ for p_j and a vector of variable $\mathbf{y}_j \in \mathbb{V}^d$ for q_j. We also introduce a vector of variable $\Psi_j \in \mathbb{V}^{T_j}$ for the edges of C_j. We think of $\mathbf{x}_j$ as specifying a location for p_j, $\mathbf{y}_j$ a location for q_j, and Ψ_j a trip from $p_j(\mathbf{x}_j)$ to $q_j(\mathbf{y}_j)$ inside C_j. The equations for the initial and final locations are respectively

$$\mathbf{x}_0 = \mathbf{a}, \tag{4}$$
$$\mathbf{y}_k = \mathbf{b}. \tag{5}$$

Those concerning ξ_j are the following, where $j \in [k]^+$.

$$\sum_{\epsilon=(p,\mathbf{t},q)\in T_j} \Psi_j(\epsilon)(\mathbf{1}_q - \mathbf{1}_p) = \mathbf{1}_{q_j} - \mathbf{1}_{p_j}, \tag{6}$$
$$\mathbf{x}_j + \Delta(\Psi_j) = \mathbf{y}_j. \tag{7}$$

For each $j \in [k]$ there is also an equation connecting the exit state of ξ_{j-1} to the entry state of ξ_j:

$$\mathbf{y}_{j-1} + \mathbf{t}_j = \mathbf{x}_j. \tag{8}$$

In (6) the notation $\mathbf{1}_q$ for example is a $|Q_j|$-dimensional indicator vector whose q-th entry is 1 and all the other entries are 0. We will call equality (6) *Euler Condition*, which is a necessary and sufficient condition for the existence of a trip that enters C_j in $p_j(\mathbf{x}_j)$ and leaves C_j from $q_j(\mathbf{y}_j)$. The system $\mathcal{E}_\xi$ consists of (4) and (5), the equations for all ξ_j *a la* (6) and (7), and all the equations connecting them *a la* (8). A solution to $\mathcal{E}_\xi$ is an assignment of *nonnegative* integers to the variables that renders valid (4) through (8). The system $\mathcal{E}_\xi$ is *satisfiable* if it has a nontrivial solution. The CGS ξ is satisfiable if its characteristic system $\mathcal{E}_\xi$ is satisfiable.

Let's see an example. Consider the 1-dimensional CGS ξ for $(0, 4)$ defined in Figure 2. Let ξ_0 and ξ_1 denote respectively the left CG and the right CG. Obviously $q(4)$ can be reached from $p(0)$ in three steps. This is a trip of type $\xi_0 \xrightarrow{0} \xi_1$ for $(0, 4)$. It is constructed from the minimal solution to the characteristic equation system $\mathcal{E}_\xi$ that assigns 2 to y_{p_0} and x_{q_0}, 4 to y_q, 1 to $\Psi_0\left(p \xrightarrow{1} p_0\right)$ and $\Psi_1\left(q_0 \xrightarrow{1} q\right)$, and 0 to all the other variables. Now consider the CGS ξ for $(0, 7)$. There are many walks from $p(0)$ to $q(7)$. Four minimal solutions to the characteristic equation system $\mathcal{E}_\xi$ are defined below, where only non-zero assignments are specified.

1. $\mathbf{m}_0$ assigns 1 to $\Psi_0\left(p \xrightarrow{2} p_0\right)$, $\Psi_1\left(q_0 \xrightarrow{2} q\right)$, $\Psi_0\left(p_1 \xrightarrow{3} q_1\right)$ and $\Psi_0\left(q_1 \xrightarrow{0} p_1\right)$.
2. $\mathbf{m}_1$ assigns 5 to $\Psi_0\left(p \xrightarrow{2} p_0\right)$ and 4 to $\Psi_0\left(p_0 \xrightarrow{0} p\right)$. It also assigns 1 to $\Psi_1\left(q_0 \xrightarrow{1} p_2\right)$, $\Psi_1\left(p_2 \xrightarrow{-3} q_2\right)$, $\Psi_1\left(q_2 \xrightarrow{0} p_2\right)$ and $\Psi_1\left(p_2 \xrightarrow{-1} q\right)$.
3. $\mathbf{m}_2$ assigns 1 to $\Psi_0\left(p \xrightarrow{1} p_1\right)$, $\Psi_0\left(p_1 \xrightarrow{3} q_1\right)$, $\Psi_0\left(q_1 \xrightarrow{0} p_1\right)$ and $\Psi_0\left(p_1 \xrightarrow{-1} q_0\right)$, 2 to $\Psi_1\left(q_0 \xrightarrow{2} q\right)$, and 1 to $\Psi_1\left(q \xrightarrow{0} q_0\right)$.
4. $\mathbf{m}_3$ assigns 4 to $\Psi_0\left(p \xrightarrow{2} p_0\right)$ and 3 to $\Psi_0\left(p_0 \xrightarrow{0} p\right)$. It also assigns 1 to $\Psi_1\left(q_0 \xrightarrow{1} p_2\right)$, $\Psi_1\left(p_2 \xrightarrow{-3} q_2\right)$, $\Psi_1\left(q_2 \xrightarrow{0} p_2\right)$, $\Psi_1\left(p_2 \xrightarrow{-1} q\right)$, $\Psi_1\left(q \xrightarrow{0} q_0\right)$ and $\Psi_1\left(q_0 \xrightarrow{2} q\right)$.

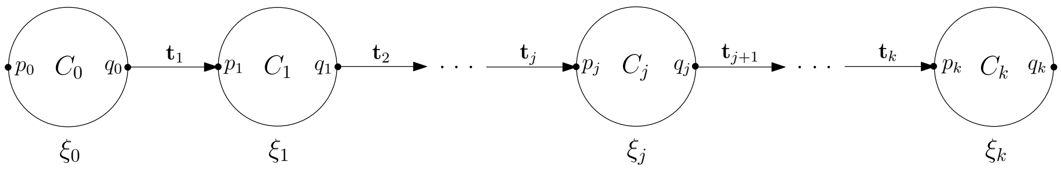


Figure 1. Constraint Graph Sequence

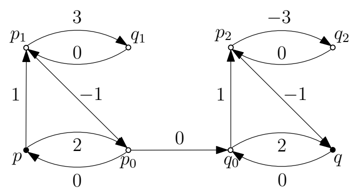


Figure 2. An Example

The solutions $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ form legitimate trips. However $\mathbf{m}_0$ does not give rise to any trip. The example tells us that while all trips of type ξ for $(\mathbf{a}, \mathbf{b})$ are solutions to $\mathcal{E}_\xi$, a solution may not define any trip. If a solution does define a trip, it is possible that no trips admitted by the solution stay inside the first quadrant. So a minimal solution may not admit any walk even if it admits a trip.

Suppose $\mathbf{s}$ is a solution to $\mathcal{E}_\xi$ and $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$. We say that $\mathbf{m}$ *supports* $\mathbf{s}$ if $\mathbf{m} \le \mathbf{s}$, where $\le$ is the pointwise ordering. The minimal solution supports a trip of ξ for $(\mathbf{a}, \mathbf{b})$ if it supports the latter seen as a solution. The difference $\mathbf{s} - \mathbf{m}$ is a solution to the *homogeneous characteristic system* $\mathcal{E}^0_\xi$ obtained from $\mathcal{E}_\xi$ by replacing the constant terms by 0. The equations are

$$\mathbf{x}_0 = \mathbf{0}, \tag{9}$$

$$\mathbf{y}_k = \mathbf{0}, \tag{10}$$

$$\sum_{\epsilon=(p,\mathbf{t},q)\in T_j} \Psi_j(\epsilon)(\mathbf{1}_q - \mathbf{1}_p) = \mathbf{0}, \tag{11}$$

$$\mathbf{x}_j + \Delta(\Psi_j) = \mathbf{y}_j, \tag{12}$$

$$\mathbf{y}_{j-1} = \mathbf{x}_j. \tag{13}$$

A solution to $\mathcal{E}^0_\xi$ is a nontrivial assignment of nonnegative integers to the variables rendering valid (9) through (13). The *Euler Condition* (11) guarantees that a solution consists of one or more circular trips inside each of the CGs $\xi_0, \ldots, \xi_k$.

In the above example the assignment that maps $\Psi_0\left(p_1 \xrightarrow{3} q_1\right)$, $\Psi_0\left(q_1 \xrightarrow{0} p_1\right)$, $\Psi_1\left(p_2 \xrightarrow{-3} q_2\right)$ and $\Psi_1\left(q_2 \xrightarrow{0} p_2\right)$ onto 1 and every other variables onto 0 is a minimal solution $\mathbf{m}_h$ to the homogeneous system $\mathcal{E}^0_\xi$, in other words $\mathbf{m}_h \in \mathcal{H}(\mathcal{E}^0_\xi)$. This solution is useless because none of $\mathbf{m}_0+g\mathbf{m}_h$, $\mathbf{m}_1+g\mathbf{m}_h$, $\mathbf{m}_2+g\mathbf{m}_h$ and $\mathbf{m}_3+g\mathbf{m}_h$ defines any trip for $g > 0$. A useful homogeneous solution is defined by the G-cycle $p \xrightarrow{1} p_1 \xrightarrow{3} q_1 \xrightarrow{0} p_1 \xrightarrow{-1} p_0 \xrightarrow{0} p$ whose displacement is 3 and the G-cycle $q_0 \xrightarrow{1} p_2 \xrightarrow{-3} q_2 \xrightarrow{0} p_2 \xrightarrow{-1} q \xrightarrow{0} q_0$ whose displacement is -3. The useless solutions suggest to introduce the following terminology.

Definition 5. *A set* $C \subseteq \mathcal{H}(\mathcal{E}^0_\xi)$ *is* connected *over* $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$ *if the summation* $\mathbf{m} + \sum C$ *defines a trip of type* ξ *for* $(\mathbf{a}, \mathbf{b})$.

When $\mathbf{m}$ is clear from context, we say that C is connected. Let $\mathcal{K}_\mathbf{m}$ denote the class of the connected subsets of $\mathcal{H}(\mathcal{E}^0_\xi)$ over $\mathbf{m}$. The largest element of $\mathcal{K}_\mathbf{m}$ is $\bigcup \mathcal{K}_\mathbf{m}$. Let $\mathbf{k_m} = \sum \bigcup \mathcal{K}_\mathbf{m}$. We call $\mathbf{k_m}$ the *principal solution to* $\mathcal{E}^0_\xi$ *over* $\mathbf{m}$. According to Pottier Lemma and its corollaries, every trip of type ξ for $(\mathbf{a}, \mathbf{b})$ supported by $\mathbf{m}$ is a solution of the form

$$\mathbf{m} + \sum_{\mathbf{c}\in C} n_\mathbf{c}\mathbf{c},$$

where $C \in \mathcal{K}_\mathbf{m}$ and $n_\mathbf{c}$ is a positive integer.

We write $\mathbf{x}^\mathbf{s}_j$ for the vector of natural number assigned to $\mathbf{x}_j$ by a solution $\mathbf{s}$ to $\mathcal{E}_\xi$ ($\mathcal{E}^0_\xi$). The notations $\mathbf{y}^\mathbf{s}_j, \Psi^\mathbf{s}_j$ are defined analogously. If $\mathbf{s} \in \mathcal{H}(\mathcal{E}^0_\xi)$, $\Psi^\mathbf{s}_j$ defines some circular trip(s) in ξ_j. If $\mathbf{s} \in \mathcal{H}(\mathcal{E}_\xi)$, $\Psi^\mathbf{s}_j$ defines a trip from the entry state p_j to the exit state q_j and possibly also some detached circular trip(s) in ξ_j. The (circular) trips defined by $\Psi^\mathbf{s}_j$ are not unique, which will not be an issue. Whenever we say a trip defined by $\Psi^\mathbf{s}_j$, we mean any trip defined by $\Psi^\mathbf{s}_j$. In fact we will see $\Psi^\mathbf{s}_j$ as a (circular) trip with possibly some additional disjoint circular trips. With these notations we introduce the notion of $\mathbf{m}$-fixed component of location.

Definition 6. *The location of the entry state* p_j*, respectively the exit state* q_j*, is* $\mathbf{m}$-unfixed at i *if* $\mathbf{x}^{\mathbf{k_m}}_j(i) > 0$*, respectively* $\mathbf{y}^{\mathbf{k_m}}_j(i) > 0$*; it is* $\mathbf{m}$-fixed at i *if* $\mathbf{x}^{\mathbf{k_m}}_j(i) = 0$*, respectively* $\mathbf{y}^{\mathbf{k_m}}_j(i) = 0$.

If the location of the entry state p_j is $\mathbf{m}$-fixed at i, then every trip of type ξ for $(\mathbf{a}, \mathbf{b})$ supported by $\mathbf{m}$ passes p_j at a location whose i-th component must be $\mathbf{x}^\mathbf{m}_j(i)$. If the location of the exit state q_j is $\mathbf{m}$-fixed at i, then every trip of type ξ for $(\mathbf{a}, \mathbf{b})$ supported by $\mathbf{m}$ passes q_j at a location whose i-th component must be $\mathbf{y}^\mathbf{m}_j(i)$. The following notations will be used.

$$\iota_\mathbf{m}(p_j) = \{i \mid \text{the location of } p_j \text{ is } \mathbf{m}\text{−fixed at } i\},$$

$$\iota_\mathbf{m}(q_j) = \{i \mid \text{the location of } q_j \text{ is } \mathbf{m}\text{−fixed at } i\}.$$

To define the notion of fixed edges we define *Hilbert solution* $\mathbf{h}$ as $\sum \mathcal{H}(\mathcal{E}^0_\xi)$. The Hilbert solution tells us a lot about the shape of the trips of type ξ for $(\mathbf{a}, \mathbf{b})$.

Definition 7. *An edge* ϵ *in the CG* ξ_j *is* unbounded *if* $\Psi^\mathbf{h}_j(\epsilon) > 0$*; it is* bounded *if* $\Psi^\mathbf{h}_j(\epsilon) = 0$.

In the example of Figure 2, all edges in the CGs are unbounded.

A trip of type ξ for $(\mathbf{a}, \mathbf{b})$ supported by some $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$ must pass a bounded edge for a fixed number of time specified by $\mathbf{m}$. All bounded edges appearing in a CG ξ_j can be removed by unfolding, producing many smaller CGs, see Section 7 for precise definition. CGs consisting of only unbounded edges of ξ_j play a key role in KLMST algorithm, hence the following definition.

Definition 8. *A CG* $\xi_j = p_jC_jq_j$ *is* unbounded *if every edge in* C_j *is unbounded. A CGS* ξ *is* unbounded *if* ξ_j *is unbounded for all* $j \in [k]^+$. *An unbounded CGS is also called an* ω-CGS.

4. Local Circular Path

We make a simple yet important observation about partially circular trips in this section. Suppose $G = (Q, T)$ is a d-dimensional VASS and $\emptyset \ne L \subsetneq [d]$. An *$L$-circular path* is a G-cycle

$$p_0 \xrightarrow{\mathbf{t}_1} p_1 \xrightarrow{\mathbf{t}_2} \ldots \xrightarrow{\mathbf{t}_{g-1}} p_{g-1} \xrightarrow{\mathbf{t}_g} p_g$$

such that $p_0 = p_g$ and $\sum_{g'\in[g]} \mathbf{t}_{g'}(l) = 0$ for all $l \in L$. We intend to understand an L-circular path in terms of those of minimal length. For that purpose let's introduce the homogeneous system $\mathcal{LC}_L$ defined by the following equations.

$$\sum_{\epsilon=(p,\mathbf{t},q)\in T} \Psi(\epsilon)(\mathbf{1}_q - \mathbf{1}_p) \quad = \quad \mathbf{0}, \tag{14}$$

$$\sum_{\epsilon=(p,\mathbf{t},q)\in T} \Psi(\epsilon)\mathbf{t}(l) \quad = \quad 0, \tag{15}$$

where (15) stands for a collection of equations indexed by $l \in L$. The equation (14) is *Euler Condition*. The system $\mathcal{LC}_L$ does not specify anything about locations. It only specifies the edges that appear in an L-circular path. However it is normally an under-specification of L-circular paths. This is because a solution to $\mathcal{LC}_L$ may consist of two G-cycles with disjoint sets of state. For example $p_0(\mathbf{r}_0) \xrightarrow{\mathbf{t}_1} \ldots \xrightarrow{\mathbf{t}_m} p_m(\mathbf{r}_m)$ and $q_0(\mathbf{s}_0) \xrightarrow{\mathbf{t}'_1} \ldots \xrightarrow{\mathbf{t}'_n} q_n(\mathbf{s}_n)$ such that $\{p_1, \ldots, p_{m-1}\} \cap \{q_1, \ldots, q_{n-1}\} = \emptyset$ and $p_1 = p_m$ and $q_1 = q_n$, and moreover $\left(\sum_{i\in[m]} \mathbf{t}_i \cup \sum_{i\in[n]} \mathbf{t}'_i\right)(l) = 0$ for all $l \in L$, may constitute a solution to $\mathcal{LC}_L$. But neither G-cycle is an L-circular path if $\left(\sum_{i\in[m]} \mathbf{t}_i\right)(l) \neq 0$ for some $l \in L$. Notice that by definition a minimal solution cannot contain two disjoint L-circular paths.

An L-circular path is *minimal* if it does not contain any proper sub-G-path that is an L-circular path. Evidently an L-circular path θ can be decomposed into a multi-set of minimal L-circular paths,

$$\theta = \sum_{\theta'\in\Theta} n_{\theta'}\theta', \tag{16}$$

where Θ is the set of the minimal L-circular paths and $n_{\theta'} \in \mathbb{N}$ for each $\theta' \in \Theta$. We are interested in representing an L-circular path by a set of minimal solutions to $\mathcal{LC}_L$. A set of L-circular paths are *connected* if their underlying graphs form a connected subgraph of G. A set of minimal solutions to $\mathcal{LC}_L$ are *connected* if the L-circular paths defined by the solutions are connected. Notice that here connectivity is equivalent to strong connectivity because we are dealing with G-cycles.

An L-circular path θ is *L-equivalent to* a set $\{\theta_1, \ldots, \theta_m\}$ of L-circular path if $\Delta(\theta)(l) = \sum_{m'\in[m]} \Delta(\theta_{m'})(l)$ for all $l \in [d] \setminus L$. It is easy to see that a minimal L-circular path θ is L-equivalent to a connected set $\{\mathbf{s}_{h'}\}_{h'\in[h]}$ of minimal solutions to $\mathcal{LC}_L$ if the following equality holds for all $l \in [d] \setminus L$,

$$\Delta(\theta)(l) = \sum_{h'\in[h]} \sum_{\epsilon=(p,\mathbf{t},q)\in T} \Psi^{\mathbf{s}_{h'}}(\epsilon)\mathbf{t}(l). \tag{17}$$

Now (16) and (17) imply that an L-circular path θ is L-equivalent to a multiset on a connected set of minimal solutions to $\mathcal{LC}_L$ in that

$$\Delta(\theta)(l) = \sum_{\mathbf{s}\in\mathbb{S}} n_{\mathbf{s}} \left(\sum_{\epsilon=(p,\mathbf{t},q)\in T} \Psi^{\mathbf{s}}(\epsilon)\mathbf{t} \right)(l) \tag{18}$$

for all $l \in [d] \setminus L$, where $\mathbb{S}$ is a connected set of minimal solutions to $\mathcal{LC}_L$ and $n_{\mathbf{s}}$ is a positive integer. The connected set $\mathbb{S}$ gives rise to a G-cycle $\dot{\theta}$ in which the G-cycle defined by each $\mathbf{s} \in \mathbb{S}$ is repeated $n_{\mathbf{s}}$ times. By Pottier Lemma the minimal solutions to $\mathcal{LC}_L$ are bounded by $2^{p(n)}$ for some polynomial $p(n)$. There are at most $2^{p(n)}$ minimal solutions. It follows that for each $l \in L$ the difference between the minimal value and the maximum value of the l-th component in the G-cycle $\dot{\theta}$ is bounded by

$$2^n \cdot 2^{p(n)} \cdot 2^{p(n)} = 2^{\ell(n)} \tag{19}$$

for a polynomial $\ell(n)$. Notice that n is the size of the VASS G.

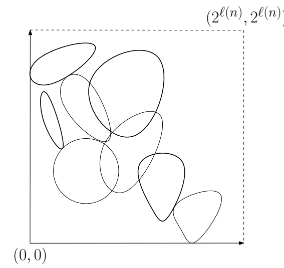
In Section 5 we will make use of circular trips whose underlying G-cycles are L-circular paths. We shall call the following trip

$$o_0(\mathbf{s}_0) \xrightarrow{\mathbf{t}_1} o_1(\mathbf{s}_1) \xrightarrow{\mathbf{t}_2} \ldots \xrightarrow{\mathbf{t}_{g-1}} o_{g-1}(\mathbf{s}_{g-1}) \xrightarrow{\mathbf{t}_g} o_g(\mathbf{s}_g) \tag{20}$$

an *L-circular trip* if $o_0 \xrightarrow{\mathbf{t}_1} o_1 \xrightarrow{\mathbf{t}_2} \ldots \xrightarrow{\mathbf{t}_{g-1}} o_{g-1} \xrightarrow{\mathbf{t}_g} o_g$ is an L-circular path. To proceed we need the following definition.

Definition 9. *Suppose $n = |G|$ and $\emptyset \subsetneq L \subsetneq [d]$. The cartesian product $\mathbb{S}_1 \times \ldots \times \mathbb{S}_d$ is* the L-space *if $\mathbb{S}_l = [0, 2^{\ell(n)}]$ for $l \in L$ and $\mathbb{S}_l = \mathbb{Z}$ for $l \notin L$. A* local space *is the L-space for some L.*

An L-circular trip in the form of (20) is *grounded* if it lies in the L-space and for each $l \in L$ there is some $g' \in [g]^+$ such that $\mathbf{s}_{g'}(l) = 0$. In the following diagram the curly cycles represent L-circular trips defined by minimal solutions and the thick curly cycles represent the ones repeated at least once. These curly cycles are connected to form a grounded L-circular trip.



The following is an immediate consequence of (18) and (19).

Proposition 10. *Every L-circular trip is L-equivalent to a grounded L-circular trip.*

5. Coverability and Pumpability

Pumpability is a fundamental idea in KLMST algorithm. A crucial observation made by Leroux and Schmidt recently is that pumpability is subsumed by coverability [15]. We give an account of this important connection between coverability and pumpability below. We start by introducing a useful terminology.

Definition 11. *Let ξ be a CGS and* $\mathbf{s}$ *be a solution to $\mathcal{E}_\xi$. A trip τ in ξ_j is* admitted by $\mathbf{s}$ *if it is a trip in the graph $\Psi^{\mathbf{s}}_j$.*

We say that a trip τ is admitted in ξ_j if it is admitted by $\mathbf{m} + \mathbf{k}_{\mathbf{m}}$, where $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$) is clear from context.

Fix an ω-CGS ξ and some $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$. Let $\xi_j = p_j C_j q_j$. A trip

$$p_0(\mathbf{s}_0) \xrightarrow{\mathbf{t}_1} p_1(\mathbf{s}_1) \xrightarrow{\mathbf{t}_2} \ldots \xrightarrow{\mathbf{t}_g} p_g(\mathbf{s}_g)$$

is a *forward* $\mathbf{m}$*-walk* in ξ_j if it is admitted by $\mathbf{m} + \mathbf{k}_{\mathbf{m}}$ and for every $g' \in [g]^+$ and every $i \in \iota_{\mathbf{m}}(p_j)$ the inequality $\mathbf{s}_{g'}(i) \geq 0$ holds. Let $\mathbf{c} \in \mathbb{N}^d$. We say that $\mathbf{c}$ is *forward* $\mathbf{m}$*-coverable in* ξ_j if there is a forward $\mathbf{m}$-walk $p_j(\mathbf{x}^{\mathbf{m}}_j) \longrightarrow^* o(\mathbf{s})$ in ξ_j such that $\mathbf{s}(i) \geq \mathbf{c}(i)$ for every $i \in \iota_{\mathbf{m}}(p_j)$. In this case we also say that $\mathbf{c}$ is covered by the forward $\mathbf{m}$-walk $p_j(\mathbf{x}^{\mathbf{m}}_j) \longrightarrow^* o(\mathbf{s})$. A trip

$$q_g(\mathbf{r}_g) \xrightarrow{\mathbf{t}_g} q_{g-1}(\mathbf{r}_{g-1}) \xrightarrow{\mathbf{t}_{g-1}} \ldots \xrightarrow{\mathbf{t}_1} q_0(\mathbf{r}_0)$$

is a *backward* $\mathbf{m}$*-walk* in ξ_j if it is admitted by $\mathbf{m} + \mathbf{k}_{\mathbf{m}}$ and for every $g' \in [g]^+$ and every $i \in \iota_{\mathbf{m}}(q_j)$ the inequality $\mathbf{r}_{g'}(i) \geq 0$ holds. We say that $\mathbf{c}$ is *backward* $\mathbf{m}$*-coverable in* ξ_j if there is a backward $\mathbf{m}$-walk $o'(\mathbf{r}) \longrightarrow^* q_j(\mathbf{y}^{\mathbf{m}}_j)$ in ξ_j such that $\mathbf{r}(i) \geq \mathbf{c}(i)$ for every $i \in \iota_{\mathbf{m}}(q_j)$. Rackoff proved in [21] that coverability is solvable in exponential space. We use a version of Rackoff Lemma given by Leroux and Schmidt [15]; and we give a sketch of their proof, which uses Rackoff's argument.

Lemma 12. *Suppose $p_j(\mathbf{x}^{\mathbf{m}}_j) = p^0(\mathbf{s}_0) \longrightarrow p^1(\mathbf{s}_1) \longrightarrow \ldots \longrightarrow p^g(\mathbf{s}_g)$ is a forward* $\mathbf{m}$*-walk. Let $c = |\iota_{\mathbf{m}}(p_j)|$ and $A > 2^{|\xi|}$. If for every $i \in \iota_{\mathbf{m}}(p_j)$, $\mathbf{s}_{g'}(i) \geq A^{1+c^c}$ for some $g' \in [g]^+$, then there is a forward* $\mathbf{m}$*-walk $p_j(\mathbf{x}^{\mathbf{m}}_j) \xrightarrow{\sigma} p^g(\mathbf{s}')$ such that $\mathbf{s}'(i) > A - 2^{|\xi|}$ for every $i \in \iota_{\mathbf{m}}(p_j)$ and $|\sigma| < A^{(c+1)^{(c+1)}}$.*

Proof. If $c = 0$, we are done; otherwise let $m \in [g-1]$ be the largest such that $\forall m' {\in} [m] \forall i {\in} \iota_{\mathbf{m}}(\mathbf{x}_j^{\mathbf{m}}).\mathbf{s}_{m'}(i) < A^{1+c^c}$. By removing sub-$\mathbf{m}$-walks of the form $p'(\mathbf{r}) \longrightarrow^* p'(\mathbf{r}')$ such that $\mathbf{r}(i) = \mathbf{r}'(i)$ for all $i \in \iota_{\mathbf{m}}(p_j)$, we get by pigeonhole principle an $\mathbf{m}$-walk $p_j(\mathbf{x}_j^{\mathbf{m}}) \longrightarrow^* p^m(\mathbf{s}_m)$ of length bounded by $|Q_j|A^{(1+c^c)c} \le |\xi_j|A^{(1+c^c)c}$. Let $c' = |I|$, in which $I = \left\{i \in \iota_{\mathbf{m}}(p_j) \mid \mathbf{s}_{m+1}(i) \ngeq A^{1+c^c}\right\}$. Now $c' < c$ by the maximality of m. By induction there is some $p^{m+1}(\mathbf{s}_{m+1}) \longrightarrow^* p^g(\mathbf{s}')$ such that its length is bounded by $A^{(c'+1)^{(c'+1)}}$ and $\mathbf{s}'(i) > A - 2^{|\xi|}$ for all $i \in I$. Then the combined sequence $p_j(\mathbf{x}_j^{\mathbf{m}}) \longrightarrow^* p^{m+1}(\mathbf{s}_{m+1}) \longrightarrow^* p^g(\mathbf{s}')$ is bounded in length by

$$|\xi_j|A^{(1+c^c)c} + A^{(c'+1)^{(c'+1)}} \le A^{1+(1+c^c)c} + A^{c^c} < A^{(c+1)^{(c+1)}}.$$

Notice that $\mathbf{s}_{m+1}(i) - 2^{|\xi_j|}A^{(c'+1)^{(c'+1)}} > (A - 2^{|\xi|})A^{c^c} > A - 2^{|\xi|} > 0$ for all $i \in \iota_{\mathbf{m}}(p_j) \setminus I$. Therefore $p_j(\mathbf{x}_j^{\mathbf{m}}) \longrightarrow^* p^{m+1}(\mathbf{s}_{m+1}) \longrightarrow^* p^g(\mathbf{s}')$ is a forward $\mathbf{m}$-walk rendering $\mathbf{s}'(i) > A - 2^{|\xi|}$ true for all $i \in \iota_{\mathbf{m}}(p_j)$. □

Pumpability is a special form of coverability. A CG ξ_j is *forward* $\mathbf{m}$*-pumpable* if $\mathbf{x}_j^{\mathbf{m}} + \mathbf{1}$ is covered in ξ_j by a forward circular $\mathbf{m}$-walk from $p_j(\mathbf{x}_j^{\mathbf{m}})$. It is *backward* $\mathbf{m}$*-pumpable* if $\mathbf{y}_j^{\mathbf{m}} + \mathbf{1}$ is covered in ξ_j by a backward circular $\mathbf{m}$-walk from $q_j(\mathbf{y}_j^{\mathbf{m}})$. A CG ξ_j is $\mathbf{m}$*-pumpable* if it is both forward $\mathbf{m}$-pumpable and backward $\mathbf{m}$-pumpable. A CGS ξ is $\mathbf{m}$*-pumpable* if ξ_j is $\mathbf{m}$-pumpable for all $j \in [k]^+$. It is *pumpable* if it is $\mathbf{m}$-pumpable for some $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$. It is *not pumpable* if it is not $\mathbf{m}$-pumpable for any $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$. The existence of a forward circular $\mathbf{m}$-walk covering $\mathbf{x}_j^{\mathbf{m}} + \mathbf{1}$ implies that for all $N > 0$ there is some $\mathbf{m}$-walk $p_j(\mathbf{x}_j^{\mathbf{m}}) \longrightarrow^* p_j(\mathbf{s}'')$ rendering $\mathbf{s}'' \ge \mathbf{x}_j^{\mathbf{m}} + N\mathbf{1}$ true. Intuitively forward $\mathbf{m}$-pumpability means that we can pump up $\mathbf{x}_j^{\mathbf{m}}$ way above $\mathbf{x}_j^{\mathbf{m}}$ on the $\mathbf{m}$-fixed entries of p_j.

If ξ_j is unbounded, pumpability checking can be reduced to coverability checking. Suppose $\mathbf{x}_j^{\mathbf{m}} + 2^{|\xi|}\mathbf{1}$ is forward $\mathbf{m}$-coverable by $p_j(\mathbf{x}_j^{\mathbf{m}}) \longrightarrow^* o(\mathbf{s})$ and there is a G-path from o to p_j in $\Psi_j^{\mathbf{m}+\mathbf{k_m}}$. Then $o(\mathbf{s}) \xrightarrow{\sigma} p_j(\mathbf{s}')$ for some $\mathbf{s}', \sigma$ such that $|\sigma| \le |Q_j| - 1$. Clearly

$$|\Delta(\sigma)| = \left|\sum_{i \in [|\sigma|]} \sigma[i]\right| \le \sum_{p' \xrightarrow{\mathbf{t}} q' \in T_j} \|\mathbf{t}\|_\infty < 2^{|\xi|}.$$

Consequently ξ_j is forward $\mathbf{m}$-pumpable. The existence of a G-path $o \longrightarrow^* p_j$ is guaranteed if ξ_j is unbounded. This is because if every edge in the connected graph $\Psi_j^{\mathbf{m}+\mathbf{k_m}}$ is unbounded, then $\Psi_j^{\mathbf{m}+\mathbf{k_m}}$ contains only one SCC. So Lemma 12 applies to pumpability checking.

Corollary 13. *A CG ξ_j is forward $\mathbf{m}$-pumpable if and only if $\mathbf{x}_j^{\mathbf{m}} + \mathbf{1}$ is covered in ξ_j by a forward circular $\mathbf{m}$-walk from $p_j(\mathbf{x}_j^{\mathbf{m}})$ whose length is bounded by $\left(\|\mathbf{x}_j^{\mathbf{m}}\|_\infty + 2^{|\xi|} + 1\right)^{(d+1)^{(d+1)}}$.*

Proof. Let $c = |\iota_{\mathbf{m}}(p_j)|$ and $A = \|\mathbf{x}_j^{\mathbf{m}}\|_\infty + 2^{|\xi|} + 1$. If ξ_j is forward $\mathbf{m}$-pumpable, then $A^{1+c^c}\mathbf{1}$ is covered in ξ_j by a forward circular $\mathbf{m}$-walk from $p_j(\mathbf{x}_j^{\mathbf{m}})$. It follows from Lemma 12 that $\mathbf{x}_j^{\mathbf{m}} + \mathbf{1}$ is covered in ξ_j by a forward circular $\mathbf{m}$-walk from $p_j(\mathbf{x}_j^{\mathbf{m}})$ whose length is bounded by $A^{(c+1)^{(c+1)}} \le A^{(d+1)^{(d+1)}}$. □

We remark that both $\|\mathbf{x}_j^{\mathbf{m}}\|_\infty$ and $2^{|\xi|}$ have exponential bound. So the size of $\left(\|\mathbf{x}_j^{\mathbf{m}}\|_\infty + 2^{|\xi|} + 1\right)^{(d+1)^{(d+1)}}$ is $2^{\mathtt{poly}(|\xi|)}$. Algorithmically the significance of the size bound is stated in the following lemma.

Lemma 14. *Suppose $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$. The $\mathbf{m}$-pumpability of an ω-CGS ξ can be checked in $2^{\wp(|\xi|)}$ space for some polynomial $\wp(n)$.*

Proof. For each $j \in [k]^+$ the algorithm guesses a vector in $\mathbb{N}^d$ whose ∞-norm is bounded by $2^{2^{\mathtt{poly}(|\xi|)}}$. It then guesses a forward $\mathbf{m}$-walk in ξ_j to the guessed vector using a counter of size $2^{\mathtt{poly}(|\xi|)}$. □

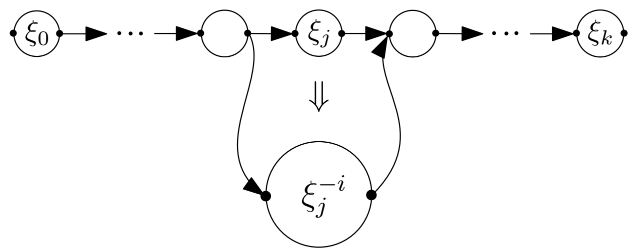


Figure 3. Reduction of ξ_j

6. Reduction

Lemma 12 is extremely useful in that it provides a spacial boundary on all the witnesses supported by $\mathbf{m}$ as long as pumpability is not available. Suppose ξ is an ω-CGS that is not $\mathbf{m}$-pumpable. By definition some $j \in [k]^+$ exists such that $\xi_j = p_j C_j q_j$ fails the $\mathbf{m}$-pumpability. If the forward $\mathbf{m}$-pumpability fails, then according to Lemma 12 every walk $p_j(\mathbf{a}_j) \longrightarrow^* q_j(\mathbf{b}_j)$ admitted by $\mathbf{m} + \mathbf{k_m}$ must fall in the region

$$\mathbb{B}_i \overset{\text{def}}{=} \underbrace{\mathbb{N} \times \ldots \times \mathbb{N}}_{i-1 \text{ times}} \times [0, B] \times \underbrace{\mathbb{N} \times \ldots \times \mathbb{N}}_{d-i \text{ times}}$$

for some $i \in [d]$, where B is defined as follows.

$$B = \max\left\{2^{\ell(|G|)}, 2^{2^{\wp(|\xi|)}}\right\},$$

where $\ell(n)$ and $\wp(n)$ are polynomials introduced in Proposition 10 and Lemma 14 respectively. If the forward $\mathbf{m}$-pumpability is available but the backward $\mathbf{m}$-pumpability is not, then every walk $q_j(\mathbf{b}_j) \longrightarrow^* p_j(\mathbf{a}_j)$ admitted by $\mathbf{m} + \mathbf{k_m}$ in C_j^{-1} must lie in $\mathbb{B}_{i'}$ for some $i' \in [d]$. It now becomes clear how to *reduce* the dimension of a non-$\mathbf{m}$-pumpable ξ_j. Let $C_j^{-i} = \left(Q_j^{-i}, T_j^{-i}\right)$, where Q_j^{-i}, T_j^{-i} are defined as follows:

$$Q_j^{-i} = \left\{(p, g) \mid p \in Q_j \text{ and } 0 \le g \le B\right\},$$

$$T_j^{-i} = \left\{(p, g) \xrightarrow{\mathbf{t}^{-i}} (q, g + \mathbf{t}(i)) \mid p \xrightarrow{\mathbf{t}} q \in T_j,\ 0 \le g, g + \mathbf{t}(i) \le B\right\}.$$

In the above definition the notation $\mathbf{t}^{-i}$ stands for the vector obtained from $\mathbf{t}$ by removing the i-th component, for example $(4, 3, 2, 1)^{-2} = (4, 2, 1)$. The definition of CG ξ_j^{-i} falls into one of four categories.

- If p_j, q_j are $\mathbf{m}$-fixed at i, then $\xi_j^{-i} = \left(p_j, \mathbf{x}_j^{\mathbf{m}}(i)\right) C_j^{-i} \left(q_j, \mathbf{y}_j^{\mathbf{m}}(i)\right)$.
- If p_j is $\mathbf{m}$-fixed at i but q_j is $\mathbf{m}$-unfixed at i, then some $b \in [0, B]$ exists such that $\xi_j^{-i} = \left(p_j, \mathbf{x}_j^{\mathbf{m}}(i)\right) C_j^{-i} \left(q_j, b\right)$.
- If q_j is $\mathbf{m}$-fixed at i but p_j is $\mathbf{m}$-unfixed at i, then some $a \in [0, B]$ exists such that $\xi_j^{-i} = \left(p_j, a\right) C_j^{-i} \left(q_j, \mathbf{y}_j^{\mathbf{m}}(i)\right)$.
- If p_j, q_j are $\mathbf{m}$-unfixed at i, then some $a, b \in [0, B]$ exist such that $\xi_j^{-i} = \left(p_j, a\right) C_j^{-i} \left(q_j, b\right)$.

In three out of the four cases our reduction algorithm has to make a guess about a and/or b. The CGS ξ *reduces to* a new CGS after substituting ξ_j^{-i} for ξ_j.

From the point of view of characteristic systems, a reduction introduces new equations of the form (4) and (5). For example if the CG ξ_j^{-i} is defined by $\left(p_j, a\right) C_j^{-i} \left(q_j, b\right)$, then $\mathbf{x}_j(i) = a$ and $\mathbf{y}_j(i) = b$ must be part of the characteristic system for ξ_j^{-i}. In the rest of the paper we shall not be bothered with this issue.

If ξ' is reduced from ξ, we get a solution to $\mathcal{E}_\xi$ respectively $\mathcal{E}_\xi^0$ from a solution to $\mathcal{E}_{\xi'}$ respectively $\mathcal{E}_{\xi'}^0$ by omitting some variables. Thus $\mathbf{m}$-fixed components remain $\mathbf{m}$-fixed after reduction.

We can now describe the reduction algorithm. We remark that since our interest is in complexity upper bound, the algorithms proposed throughout this paper are only outlined. For one thing we leave out most of the failure situations which may happen for instance when a CGS is not satisfied. Our algorithms are nondeterministic. If the input is a "yes" instance at least one execution is successful. If the input is a "no" instance, all executions fail.

Lemma 15. *A nondeterministic algorithm $\mathfrak{R}$ exists that, upon receiving a pair $(\xi, \mathbf{m})$ of an $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$ and an ω-CGS ξ that is not $\mathbf{m}$-pumpable, reduces ξ to a CGS such that if there is a witness of ξ supported by $\mathbf{m}$, there is an execution path that produces a CGS ξ' such that the same witness is supported by some $\mathbf{m}' \in \mathcal{H}(\mathcal{E}_{\xi'})$ and either ξ' is not an ω-CGS or ξ' is an $\mathbf{m}'$-pumpable ω-CGS.*

Proof. Let $\mathbf{w}$ be a witness of ξ supported by $\mathbf{m}$ and let ξ be an ω-CGS that is not $\mathbf{m}$-pumpable. The algorithm $\mathfrak{R}$ with input $(\xi, \mathbf{m})$ is defined as follows:

1. For each ξ_j that is not $\mathbf{m}$-pumpable, reduce ξ_j to a new CG ξ'_j.
2. Let ξ' be the CGS obtained after Step 1. Choose a solution $\mathbf{m}' \in \mathcal{H}(\mathcal{E}_{\xi'})$ that is supported by $\mathbf{m}$ as a solution to $\mathcal{E}_\xi$. If ξ' is not an ω-CGS, output ξ'. If ξ' is an ω-CGS that is not $\mathbf{m}'$-pumpable, invoke $\mathfrak{R}(\xi', \mathbf{m}')$; otherwise output ξ'.

The walk $\mathbf{w}$ inside $\Psi^{\mathbf{w}}_j$ tells us which execution path reduces ξ to a CGS ξ' such that $\mathbf{w}$ is also a witness of ξ'. Reduction decreases the dimension of CGs while maintaining the number of CGs. We conclude that $\mathfrak{R}$ must terminate. □

Reduction introduces constraints that may disown many trips. After reduction the new CGS may become bounded. Further round of decomposition, defined in Section 7, and reduction may reduce $(d-1)$-dimensional CGs to $(d-2)$-dimensional CGs, and so on. A 0-dimensional CG is satisfiable if there is a walk from some $p'(\mathbf{a}')$ to some $q'(\mathbf{b}')$ in some space $[0, B_1] \times \ldots \times [0, B_d]$.

Suppose ξ is a CGS of a d-dimensional VASS $G = (Q, T)$ and ξ_j is d'-dimensional for some $d' \in [d-1]$. If we see ξ_j as a d-dimensional VASS, it must be defined in an L-space for some nonempty $L \subsetneq [d]$ satisfying $|L| = d'$. We will switch freely between the d'-dimensional view and the d-dimensional view.

6.1 Loop Class After Reduction

Loop classes of circular trips in CGs of reduced dimension play a key role in our complexity analysis. In Definition 4 loop classes are introduced for subgraphs of the input graph G. We now generalize the concept to constraint graphs of any dimension. Suppose $\xi_j = p_j C_j q_j$ is defined in L-space for a nonempty $L \subsetneq [d]$. Let $c = |L|$. A transition in C_j is $(p, (a_l)_{l\in L}) \xrightarrow{\mathbf{t}^{-L}} (q, (a'_l)_{l\in L})$ obtained from a G-edge

$$p \xrightarrow{\mathbf{t}} q, \tag{21}$$

where $\mathbf{t}^{-L}$ is the projection of $\mathbf{t}$ at the index set $[d] \setminus L$, $(a_l)_{l\in L}$ for example is a c-tuple ordered by the indexes in L, and $a'_l = a_l + \mathbf{t}(l)$ for every $l \in L$. We call p the *G-state* of $(p, (a_l)_{l\in L})$ and $(a_l)_{l\in L}$ the *state vector* of $(p, (a_l)_{l\in L})$. A one-step trip from a located state $(p, (a_l)_{l\in L})(a_i)_{i\in[d]\setminus L}$ to a located state $(q, (a'_l)_{l\in L})(a'_i)_{i\in[d]\setminus L}$ labeled $\mathbf{t}^{-L}$, denoted by

$$(p, (a_l)_{l\in L})(a_i)_{i\in[d]\setminus L} \xrightarrow{\mathbf{t}^{-L}} (q, (a'_l)_{l\in L})(a'_i)_{i\in[d]\setminus L}, \tag{22}$$

is such that $a_i + \mathbf{t}(i) = a'_i$ for all $i \in [d] \setminus L$. We call (21) the *G-edge* of (22). The notation $G(C_j)$ stands for the subgraph of G constructed from the G-edges of the transitions of C_j. Similarly $G(\vartheta)$ is the subgraph of G defined by the G-edges of the transitions that appear in the circular trip ϑ. And $G(\Psi^{\mathbf{h}}_j)$ is the subgraph of G defined from the circular trips of $\Psi^{\mathbf{h}}_j$.

Definition 16. *Suppose $\xi_j = p_j C_j q_j$ and ϑ is a circular trip in C_j. Let the* loop class *$\mathcal{L}(C_j)$ be defined by the set $\mathcal{L}(G(C_j))$. Similarly let $\mathcal{L}(\vartheta) = \mathcal{L}(G(\vartheta))$ and $\mathcal{L}(\xi_j) = \mathcal{L}(G(\Psi^{\mathbf{h}}_j))$.*

It is worth remarking that loop classes are always defined on the input graph G. We will see that the loop class of a CG characterizes a fundamental invariant of VASS. The following definition should be appreciated from the point of view of this invariance.

Definition 17. *Suppose $L \subsetneq [d]$ is nonempty and C is defined on the L-space. Let D be a subgraph of C. The* loop completion *$D^{\circlearrowleft}$ of D with regards to C is the subgraph of C defined by the edge set*

$$\{\epsilon \mid \epsilon \text{ appears in } \theta,\ \theta \text{ is a path in } C,\ G(\theta) \in \mathcal{L}(D)\}.$$

The intuition behind Definition 17 is that if a G-loop based path is in D, then every path in C based on the same G-loop is in $D^{\circlearrowleft}$. It is immediate from definition that $\mathcal{L}(D^{\circlearrowleft}) = \mathcal{L}(D)$. Another way to understand Definition 17 is to think of C as the underlying graph of a VASS. The *full* sub-VASS as it were induced by the subgraph D is the VASS defined by $D^{\circlearrowleft}$.

7. Decomposition

The homogeneous solutions tell us when a CGS is decomposable. We now explain how a CGS is decomposed. If a CG is decomposable, the way it is decomposed depends on a minimal solution to $\mathcal{E}_\xi$. Suppose that ξ is a CGS and that $\xi_j = p_j C_j q_j$ is not unbounded. Let $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$ and $\mathbf{w}$ be a walk supported by $\mathbf{m}$. The *decomposition* of ξ_j is carried out in the following fashion.

1. Obtain D_j from C_j by removing all the bounded transitions.
2. Compute the SCCs of D_j.
3. Since $\mathbf{w}$ is of type ξ, it is a solution to $\mathcal{E}_\xi$. Suppose the part of the walk $\mathbf{w}$ inside C_j is of the following shape
$$p^0_j C^0_j q^0_j \xrightarrow{\mathbf{t}^1} \ldots \xrightarrow{\mathbf{t}^{k_j}} p^{k_j}_j C^{k_j}_j q^{k_j}_j,$$
where $\left\{C^0_j, \ldots, C^{k_j}_j\right\}$ is a multi-set of the SCCs of D_j, $p^0_j = p_j$, $q^{k_j}_j = q_j$, and every bounded edge ϵ in C_j appears in the above sequence for precisely $\Psi^{\mathbf{w}}_j(\epsilon)$ times. By definition $\Psi^{\mathbf{m}}_j(\epsilon) = \Psi^{\mathbf{w}}_j(\epsilon)$ for all bounded edge ϵ. The sequence is also a linearization of $\Psi^{\mathbf{m}}_j$. A new CGS is obtained by substituting for ξ_j the following CGS
$$p^0_j \left(C^0_j\right)^{\circlearrowleft} q^0_j \xrightarrow{\mathbf{t}^1} \ldots \xrightarrow{\mathbf{t}^{k_j}} p^{k_j}_j \left(C^{k_j}_j\right)^{\circlearrowleft} q^{k_j}_j, \tag{23}$$
where the loop completion $\xi^{j'}_j = \left(C^{j'}_j\right)^{\circlearrowleft}$ for $j' \in [k_j]^+$ is defined with regards to C_j and is the full sub-VASS of C_j defined by $C^{j'}_j$. A diagrammatic illustration is given in Figure 4.

Our decomposition differs from those defined in literature in that we take loop completion of SCCs. For the moment let's just say that the loop completion makes available a massively parallel decomposition strategy. By applying for all $j \in [k]^+$ the decomposition operation to ξ_j that is not unbounded, we get a new CGS ξ' that is *decomposed* from ξ. Since $\mathbf{w}$ is a solution to $\mathcal{E}_\xi$, there must be a decomposition ξ' of which the walk $\mathbf{w}$ is a witness. It follows that there is some minimal solution $\mathbf{m}'$ to $\mathcal{E}^0_{\xi'}$ that supports $\mathbf{w}$. In the above analysis we assume that we know a witness $\mathbf{w}$, and consequently $\mathbf{m}$ that supports $\mathbf{w}$. Algorithmically $\mathbf{m}$ must be guessed.

Lemma 18. *A nondeterministic algorithm $\mathfrak{D}$ exists that, upon receiving a pair $(\xi, \mathbf{m})$ of a CGS ξ and a solution $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$, decomposes ξ such that if some witness of ξ for $(\mathbf{a}, \mathbf{b})$ is supported by $\mathbf{m}$, then some execution path produces an ω-CGS ξ' such that the witness is supported by some $\mathbf{m}' \in \mathcal{H}(\mathcal{E}_{\xi'})$.*

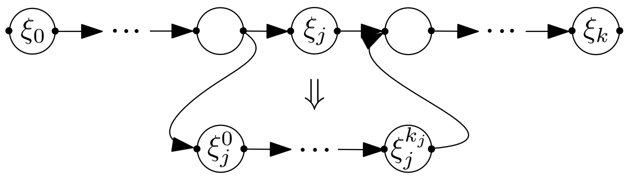


Figure 4. Decomposition of ξ_j

Proof. It follows from Pottier Lemma and its corollaries that in the input CGS ξ bounded transitions may appear at most $2^{\mathtt{poly}(n)}$ times and that the number of minimal solution to $\mathcal{E}_\xi^0$ is $2^{\mathtt{poly}(n)}$. The set $\mathcal{H}(\mathcal{E}_\xi^0)$ tells us which edges are bounded, and the minimal solution $\mathbf{m}$ tells us how many times a bounded edge should appear in a linearization *a la* (23). The nondeterministic algorithm $\mathfrak{D}(\xi, \mathbf{m})$ is defined as follows.

1. Calculate $\mathcal{H}(\mathcal{E}_\xi^0)$.
2. Construct a decomposition ξ' of ξ using $\mathcal{H}(\mathcal{E}_\xi^0)$ and $\mathbf{m}$. Abort if no decomposition exists.
3. If ξ' is an ω-CGS, output ξ'; otherwise guess an $\mathbf{m}' \in \mathcal{H}(\mathcal{E}_{\xi'})$ and then invoke $\mathfrak{D}(\xi', \mathbf{m}')$.

The construction of ξ' in Step 2 and the construction of $\mathbf{m}'$ in Step 3 are nondeterministic. One execution path must be successful in getting the right ξ' and the right $\mathbf{m}'$ in succession. Unsuccessful execution is terminated by bounding the number of recursive invocation of $\mathfrak{R}$. Such a bound will be given in Section 9. $\square$

While we are on the subject, let's point out that if ξ' is obtained from ξ by a one-step decomposition, we get a solution to $\mathcal{E}_\xi$ from a solution to $\mathcal{E}_{\xi'}$. Let $\mathbf{s}$ be a solution to $\mathcal{E}_{\xi'}$. Define an assignment to the variables of $\mathcal{E}_\xi$ in the following manner, where $j \in [k]^+$.

- If the CG ξ_j is reduced to $\xi'_{j_0} \xrightarrow{\mathbf{t}_{j_1}} \xi'_{j_1} \xrightarrow{\mathbf{t}_{j_2}} \dots \xrightarrow{\mathbf{t}_{j_g}} \xi'_{j_g}$, let
$$\Psi_j(\epsilon) \quad = \quad \begin{cases} \mathcal{T}_j(\epsilon), & \text{if } \epsilon \in \mathcal{T}_j, \\ \sum_{g' \in [g]^+} \Psi^{\mathbf{s}}_{j_{g'}}(\epsilon), & \text{otherwise,} \end{cases}$$
where $\mathcal{T}_j$ is the multi-set $\left\{ q'_{j_0} \xrightarrow{\mathbf{t}_{j_1}} p'_{j_1}, \dots, q'_{j_{g-1}} \xrightarrow{\mathbf{t}_{j_g}} p'_{j_g} \right\}$.
- If ξ_j is unaffected by the decomposition, let $\Psi_j(\epsilon) = \Psi^{\mathbf{s}}_j(\epsilon)$.

Clearly Ψ is a solution to $\mathcal{E}_\xi$. We state this fact as a corollary.

Corollary 19. *Suppose ξ' is obtained from ξ by a one-step decomposition and $\mathbf{s}$ is a solution to $\mathcal{E}_{\xi'}$. Then $\mathbf{s}$ is also a solution to $\mathcal{E}_\xi$.*

Suppose ξ' is obtained from ξ by one step decomposition and $\mathbf{s}^0$ is a solution to $\mathcal{E}_{\xi'}^0$. There must be an $\mathbf{m} \in \mathcal{H}(\mathcal{E}_{\xi'})$ and a solution $\mathbf{s}$ to $\mathcal{E}_{\xi'}$ such that $\mathbf{s}^0 = \mathbf{s} - \mathbf{m}$. By Corollary 19 both $\mathbf{s}$ and $\mathbf{m}$ are solutions to $\mathcal{E}_\xi$. Thus $\mathbf{s}^0$ is also a solution to $\mathcal{E}_\xi^0$.

Corollary 20. *Suppose ξ' is obtained from ξ by a one-step decomposition and $\mathbf{s}^0$ is a solution to $\mathcal{E}_{\xi'}^0$. Then $\mathbf{s}^0$ is a solution to $\mathcal{E}_\xi^0$.*

Referring to the linearization (23) we see that, when seeing $\mathbf{s}^0$ as a solution to $\mathcal{E}_\xi^0$, $\Psi_j^{\mathbf{s}^0}$ may consist of a set of disjoint circular trips.

8. Reachability Checking

The well known reachability algorithm is basically a repeated application of the algorithm $\mathfrak{D}$ of Lemma 18 followed by the algorithm $\mathfrak{R}$ of Lemma 15. By overlooking details, let's define the main algorithm $\mathfrak{A}$ as follows, assuming that ξ is the input CGS.

1. Guess $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$; Let $\xi' := \mathfrak{D}(\xi, \mathbf{m})$.
2. Guess $\mathbf{m}' \in \mathcal{H}(\mathcal{E}_{\xi'})$; Let $\xi := \mathfrak{R}(\xi', \mathbf{m}')$.
3. If ξ is not a pumpable ω-CGS, go to Step 1.

We will provide in Section 9 an upper bound for the number of invocations of $\mathfrak{A}$ by an amortized argument. The correctness of $\mathfrak{A}$ deserves some comment. If there is no walk from $p(\mathbf{a})$ to $q(\mathbf{b})$, $\mathfrak{A}$ would abort. Suppose there is a walk $\mathbf{w}$ from $p(\mathbf{a})$ to $q(\mathbf{b})$. The walk must be a solution to the successive characteristic systems the algorithm $\mathfrak{A}$ encounters. This is because both the algebra for the decomposition and the graph theoretical property for the reduction apply to $\mathbf{w}$. The walk $\mathbf{w}$ must be supported by a successive minimal solutions to the characteristic systems. It follows that $\mathfrak{A}$ must succeed in at least one execution, delivering a pumpable ω-CGS, which is good enough in view of the following famous proposition.

Proposition 21. *A pumpable ω-CGS has a witness.*

Proof. Suppose ξ is an $\mathbf{m}$-pumpable ω-CGS for some $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$ and τ is a trip constructed from $\mathbf{m} + \mathbf{k}_\mathbf{m}$. The trip τ is not necessarily a walk because the segment of τ inside C_j, denoted by τ_j, may go out of the first quadrant. An important observation made in previous research is that τ can be turned into a walk by adding enough loops that constitute a solution to $\mathcal{E}_\xi^0$. Let $\mathbf{g} = g\mathbf{k}_\mathbf{m}$. For all $g, n > 0$, $n\mathbf{g}$ is a solution to $\mathcal{E}_\xi^0$. By pumpability there are forward $\mathbf{m}$-walk
$$p_j(\mathbf{x}_j^\mathbf{m}) \xrightarrow{\sigma_j} p_j(\mathbf{x}_j^\mathbf{m} + \Delta(\sigma_j))$$
and backward $\mathbf{m}$-walk
$$q_j(\mathbf{y}_j^\mathbf{m} - \Delta(\varsigma_j)) \xrightarrow{\varsigma_j} q_j(\mathbf{y}_j^\mathbf{m}).$$
Let g be the minimal number rendering $\Psi_j^{\mathbf{m}+\mathbf{g}} - \Im(\sigma_j) - \Im(\varsigma_j) \geq \mathbf{0}$ true for all $j \in [k]^+$. Notice that $\Psi_j^{\mathbf{m}+\mathbf{g}} - \Im(\sigma_j) - \Im(\varsigma_j)$ satisfies the Euler Condition. It follows that there is a circular trip
$$p_j(\mathbf{x}_j^{\mathbf{m}+\mathbf{g}} + \Delta(\sigma_j)) \xrightarrow{\varpi_j} p_j(\mathbf{x}_j^{\mathbf{m}+\mathbf{g}} - \Delta(\varsigma_j)),$$
and consequently a trip
$$p_j(\mathbf{x}_j^{\mathbf{m}+\mathbf{g}}) \xrightarrow{\sigma_j}\xrightarrow{\varpi_j}\xrightarrow{\tau_j}\xrightarrow{\varsigma_j} q_j(\mathbf{y}_j^{\mathbf{m}+\mathbf{g}}). \tag{24}$$
It is possible that in (24) either σ_j or ς_j or both goes out of the first quadrant. The circular trip σ_j stays nonnegative in the $\mathbf{m}$-fixed entries. It may become negative in the $\mathbf{m}$-unfixed entries anywhere in the trip. The situation is similar for ς_j. However if g is large enough the circular trips prescribed by the homogeneous solution $\mathbf{g}$ can raise the values in the $\mathbf{m}$-unfixed entries so large that both σ_j and ς_j stay in the first quadrant. Now there is no guarantee that ϖ_j stays nonnegative in the middle of the trip in any entry. But it must end with strictly positive integers in all the $\mathbf{m}$-fixed entries of q_j because ς_j begins with these positive integers. So we can let g be large enough so that ς_j ends with strictly positive integers in all entries. That is a good news because we can let g be even larger so that τ_j stays completely in the first quadrant. It can happen though that no matter how large g is, ϖ goes into the negative in the $\mathbf{m}$-unfixed components. The way to resolve the difficulty is to repeat each of the circular trips $\sigma_j, \varpi_j, \varsigma_j$ for M times, giving rise to the following trip
$$p_j(\mathbf{x}_j^{\mathbf{m}+M\mathbf{g}}) \xrightarrow{\sigma_j^M}\xrightarrow{\varpi_j^M}\xrightarrow{\tau_j}\xrightarrow{\varsigma_j^M} q_j(\mathbf{y}_j^{\mathbf{m}+M\mathbf{g}}). \tag{25}$$
This is a walk since the values in all the entries of $\mathbf{x}_j^{\mathbf{m}+M\mathbf{g}}$ can be made as large as necessary. Finally notice that since all the components in p_0 and q_k are $\mathbf{m}$-fixed, and both ξ_0 and ξ_k are pumpable, we can pump up $\mathbf{a}$ and $\mathbf{b}$ so much that (25) is a walk for all $j \in [k]^+$. In the case that ξ_0 is a 0-dimensional CG, meaning that ξ_0 is nothing but a vertex, there must be a leftmost ξ_{j_0} that is pumpable. The same comment applies to ξ_k. $\square$

9. Tower Completeness

The complexity of $\mathfrak{A}$ is dominated by the number of decomposition step. Every decomposition step may increase the size of the current CGS exponentially. To analyze the complexity of the algorithm, one looks for a tighter bound on the number of decomposition steps the algorithm must carry out. We point out in this section that even though the number of CGs in a CGS may be astronomical, a significant proportion of them can actually be decomposed in a single decomposition step.

Let $G = (Q, T)$ be a d-dimensional VASS and ξ be a CGS of G. As it turns out, the crucial idea to analyze the complexity of decomposing ξ is a way to classify constraint graphs by G-loops. A *type* is a triple $\langle I, C, O\rangle$ where C is a loop class, $I \subseteq [d]$ and $O \subseteq [d]$. A CG $\xi_j = p_j C_j q_j$ is of *type* $\langle I, C, O\rangle$ over $\mathbf{m} \in \mathcal{H}(\mathcal{E}_\xi)$ if $\mathcal{L}(C_j) = C$, $I = \iota_{\mathbf{m}}(p_j)$ and $O = \iota_{\mathbf{m}}(q_j)$. It is of *kind* C if $\mathcal{L}(C_j) = C$.

Types help reveal some fundamental properties of $\mathfrak{A}$. Before investigating the properties, let's prove an auxiliary lemma. Suppose $\xi_j = p_j C_j q_j$ is defined in the L-space for a nonempty $L \subsetneq [d]$. A circular trip in C_j is *grounded* if it is a grounded L-circular trip.

Lemma 22. *Let ξ_j be defined in the L-space for some nonempty $L \subsetneq [d]$. For each L-circular trip ϑ satisfying $\mathcal{L}(\vartheta) \subseteq \mathcal{L}(\xi_j)$, there is a grounded circular trip admitted in ξ_j and L-equivalent to ϑ.*

Proof. Call a CG sequence $\xi^0_{j_0}, \xi^1_{j_1}, \ldots, \xi^h_{j_h}$ a *decomposition branch* for ξ_j if $\xi^h_{j_h} = \xi_j$, $\xi^0_{j_0}$ is obtained from a reduction step and, for every $h' \in [h]$, $\xi^{h'}_{j_{h'}}$ is a D-*child* of $\xi^{h'-1}_{j_{h'-1}}$, meaning that $\xi^{h'}_{j_{h'}}$ is obtained from $\xi^{h'-1}_{j_{h'-1}}$ by a decomposition step. Let $\mathbf{h}$ be the Hilbert solution to $\mathcal{E}_\xi$. It follows from Corollary 20 that $\mathbf{h}$ is a solution to $\mathcal{E}^0_{\xi^{h'}}$ for all $h' \in [h]^+$. Suppose ϑ is an L-circular trip satisfying $\mathcal{L}(\vartheta) \subseteq \mathcal{L}(\xi_j)$. By Proposition 10 the L-circular trip ϑ is equivalent to a grounded circular trip ϑ_g. Since $\mathcal{L}(\vartheta_g) = \mathcal{L}(\vartheta) \subseteq \mathcal{L}(\xi_j)$, there must be some f such that if we remove ϑ_g from $\Psi^{f\mathbf{h}}_j$, we get a number of L-circular trips, which by Proposition 10 are equivalent to some grounded circular trips $\vartheta^1, \ldots, \vartheta^m$ respectively. Since $f\mathbf{h}$ is a solution to $\mathcal{E}^0_{\xi^0}$ the circular trip(s) $\Psi^{f\mathbf{h}}_j$ in ξ^0 can be replaced by the grounded circular trips $\vartheta^1, \ldots, \vartheta^m, \vartheta_g$ because they are equivalent, producing a solution different from $f\mathbf{h}$. It follows that $\vartheta^1, \ldots, \vartheta^m, \vartheta_g$ are admitted in $\xi^0_{j_0}$. Now $f\mathbf{h}$ is also a solution to $\mathcal{E}^0_{\xi^1}$. One derives that $\vartheta^1, \ldots, \vartheta^m, \vartheta_g$ are admitted in $\xi^1_{j_1}$ as well. It follows by induction that ϑ_g is admitted in $\xi^h_{j_h} = \xi_j$. □

A CG generated during a successful execution of $\mathfrak{A}$ may be of extremely large size, but the next lemma tells us that there is an exponential bound on the length of every decomposition branch. This is a strong indication that we are on the right track.

Lemma 23 (Termination)**.** *If $\xi'_{j'} = p'_{j'} C'_{j'} q'_{j'}$ is a D-child of $\xi_j = p_j C_j q_j$, then $\mathcal{L}(C'_{j'}) \subsetneq \mathcal{L}(C_j)$.*

Proof. If ξ_j is d-dimensional, the proof is easy. Consider the case that ξ_j is defined in the L-space for a nonempty $L \subsetneq [d]$. If $\mathcal{L}(\xi_j)$ contains more than one SCC, $\mathcal{L}(C'_{j'}) \subsetneq \mathcal{L}(C_j)$ is self evident. Suppose $\mathcal{L}(\xi_j)$ forms one SCC. Assume $\mathcal{L}(\xi_j) = \mathcal{L}(C_j)$. By Lemma 22 and loop completion every circular trip in C_j is admitted in ξ_j because it is equivalent to a grounded circular trip that is admitted in ξ_j. It follows that all edges in C_j are unbounded, contradicting to the assumption that ξ_j is decomposable. So $\mathcal{L}(\xi_j) = \mathcal{L}(C_j)$ cannot be the case. We derive that $\mathcal{L}(C'_{j'}) \subseteq \mathcal{L}(\xi_j) \subsetneq \mathcal{L}(C_j)$. □

A small bound on the depth of the decomposition branches does not imply a moderate execution time if the width of the decomposition branches is not moderately bounded. An effective way to achieve moderate bound on the width is parallelization.

$$
\begin{array}{rcll}
\overline{\mathbf{x}}_h & = & \mathbf{x}^{g\mathbf{h}}_h, & \text{for } h \in [k]^+ \setminus \{j'+1, \ldots, j''\},\\
\overline{\mathbf{x}}_h & = & \mathbf{x}^{g\mathbf{h}}_h - \Delta(\Psi^{\mathbf{h}}_{j'}), & \text{for } h \in \{j'+1, \ldots, j''\},\\
\overline{\mathbf{y}}_h & = & \mathbf{y}^{g\mathbf{h}}_h, & \text{for } h \in [k]^+ \setminus \{j', \ldots, j''-1\},\\
\overline{\mathbf{y}}_h & = & \mathbf{y}^{g\mathbf{h}}_h - \Delta(\Psi^{\mathbf{h}}_{j'}), & \text{for } h \in \{j', \ldots, j''-1\},\\
\overline{\Psi}_h & = & \Psi^{g\mathbf{h}}_h, & \text{for } h \in [k]^+ \setminus \{j', j''\},\\
\overline{\Psi}_{j'} & = & \Psi^{g\mathbf{h}}_{j'} - \Psi^{\mathbf{h}}_{j'}, & \\
\overline{\Psi}_{j''} & = & \Psi^{g\mathbf{h}}_{j''} + \Psi^{\mathbf{h}}_{j'}. &
\end{array}
$$

Figure 5. Rearranging a Homogeneous Solution

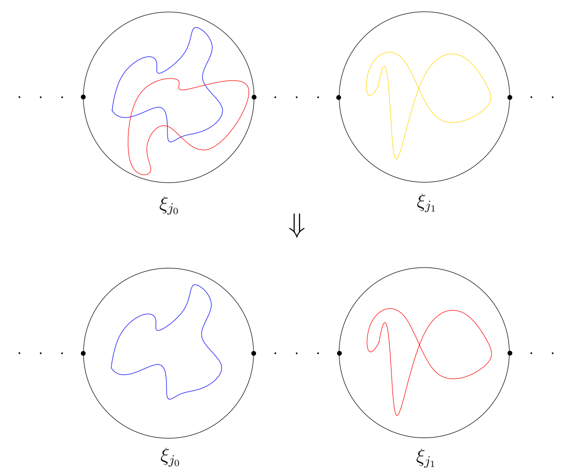


Figure 6. Transforming a Solution to a Different Solution

Lemma 24 (Parallel Decomposition)**.** *Suppose $G = (Q, T)$ is a d-dimensional VASS, ξ is a CGS for G, and $\xi_{j'}, \xi_{j''}$ are of the same type and either both are d-dimensional or both are defined in the L-space for some nonempty $L \subsetneq [d]$. Then $\xi_{j'}$ is decomposable if and only if $\xi_{j''}$ is decomposable.*

Proof. Suppose in some decomposition step $\xi_{j''} = p_{j''} C_{j''} q_{j''}$ is decomposed whereas $\xi_{j'} = p_{j'} C_{j'} q_{j'}$ is not, and $j' < j''$. Firstly we take a look at the case when both $\xi_{j'}, \xi_{j''}$ are d-dimensional CGs of type $\langle I, C, O\rangle$. Let $\mathbf{h}$ be the Hilbert solution to $\mathcal{E}^0_\xi$. By assumption some $\epsilon_1 = p \xrightarrow{\mathbf{t}} q$ in $C_{j''}$ is bounded whereas $\epsilon_0 = p \xrightarrow{\mathbf{t}} q$ in $C_{j'}$ is unbounded. Since $\mathcal{L}(C_{j'}) = \mathcal{L}(C_{j''})$, a circular path in $C_{j'}$ is a circular path in $C_{j''}$, and vice versa. Consider the solution $g\mathbf{h}$ for some positive integer g. It is not difficult to see that if g is large enough, a copy of $\Psi^{\mathbf{h}}_{j'}$ can be moved from $\xi_{j'}$ to $\xi_{j''}$, producing a solution to $\mathcal{E}^0_\xi$ different from $g\mathbf{h}$. More precisely the assignment to the variables defined in Figure 5 must be a solution to $\mathcal{E}^0_\xi$. This homogeneous solution would imply that the edge ϵ_1 is unbounded in $C_{j''}$, contradicting to the assumption. We conclude that if ϵ_1 is bounded in $C_{j''}$, ϵ_0 must be bounded in $C_{j'}$.

Suppose $\xi_{j'}, \xi_{j''}$ are of type $\langle I, C, O\rangle$ and are defined in the L-space for a nonempty $L \subsetneq [d]$. Suppose $\xi_{j'}$ is not decomposable whereas $\xi_{j''}$ is decomposable. Then there is a circular trip ϑ in $C_{j''}$ that is not admitted in $\xi_{j''}$. It follows from $L(\vartheta) \subseteq \mathcal{L}(C_{j''}) = \mathcal{L}(C_{j'})$ and Lemma 22 that ϑ is L-equivalent to a grounded circular trip ϑ_g in $C_{j'}$. The proof of Lemma 22 tells us that $\Psi^{f\mathbf{h}}_{j'}$ for some large f can be replaced by $\vartheta^1, \ldots, \vartheta^m, \vartheta_g$ for some grounded circular trips $\vartheta^1, \ldots, \vartheta^m$ admitted in $\xi_{j'}$. It is now apparent how to use the idea in the previous paragraph to show that ϑ is admitted in $\xi_{j''}$, which contradicts to the assumption. A graphical illustration is given in Figure 6, where the curly cycles are circular trips. The yellow trip ϑ in $C_{j''}$ is not admitted in $\xi_{j''}$. One may construct the red circular trip ϑ_g in $C_{j'}$ that is admitted in $\xi_{j'}$ whose overall displacement is the same as the displacement of the yellow circular trip. The red circular trip can be removed from $\xi_{j'}$ and the yellow circular trip be reinstated in $C_{j''}$, raising a contradiction. □

Lemma 24 tells us that on average a significant proportion of CGs are decomposed in one go. Parallelism is also a feature of the reduction algorithm $\mathfrak{D}$. For the sake of stating the result let's call the execution of an invocation of $\mathfrak{R}$ a *round*.

Lemma 25 (Parallel Reduction)**.** *Suppose ξ is the ω-CGS at the beginning of a round. Then the following statements are valid.*

1. *If ξ_j is not reduced in the present round, it will not be reduced in future rounds unless its type is changed.*
2. *Suppose ξ_h, ξ_j are of the same type and either both are d-dimensional or both are defined in the same L-space for some nonempty $L \subsetneq [d]$. If neither of ξ_h, ξ_j is reduced in the present round, then both are of the same type by the end of the present round.*

Proof. Suppose $\xi_j = p_j C_j q_j$ is not reduced in the round $\mathfrak{R}(\xi, \mathbf{m})$. Let ξ' be the CGS by the end of the round $\mathfrak{R}(\xi, \mathbf{m})$, and let ξ_j be denoted by $\xi'_{j'} = p'_{j'} C'_{j'} q'_{j'}$ in ξ'. The CGS ξ' must be an ω-CGS because that is the precondition for the invocation of the next round $\mathfrak{R}(\xi', \mathbf{m}')$. It follows that $\xi'_{j'}$ must be unbounded. If the type of $\xi'_{j'}$ is the same as the type of ξ_j, meaning that $\iota_{\mathbf{m}'}(p'_{j'}) = \iota_{\mathbf{m}}(p_j)$ and $\iota_{\mathbf{m}'}(q'_{j'}) = \iota_{\mathbf{m}}(q_j)$, then $\xi'_{j'}$ must be $\mathbf{m}'$-pumpable. In other words if a CG is not reduced at round $\mathfrak{R}(\xi, \mathbf{m})$ and its type has not been changed by this round, it will not be reduced at round $\mathfrak{R}(\xi', \mathbf{m}')$.

Let ξ be an ω-CGS and $\mathbf{m} \in \mathcal{H}(\mathcal{E}^0_\xi)$. After $\mathfrak{R}(\xi, \mathbf{m})$ reduces the non-$\mathbf{m}$-pumpable CGs in ξ in one go, it produces a new CGS say ξ'. If ξ' is an ω-CGS, $\mathfrak{R}(\xi', \mathbf{m}')$ is invoked in the next round for some $\mathbf{m}' \in \mathcal{H}(\mathcal{E}^0_{\xi'})$. Let's take a look at the CGs that have not been reduced in the round $\mathfrak{R}(\xi, \mathbf{m})$. Suppose $\xi_h = p_h C_h q_h$ and $\xi_j = p_j C_j q_j$ are distinct CGs in ξ of the same type and they are denoted by $\xi'_{h'} = p'_{h'} C'_{h'} q'_{h'}$ and $\xi'_{j'} = p'_{j'} C'_{j'} q'_{j'}$ respectively in ξ'. Since ξ' is an ω-CGS, both $\xi'_{h'}$ and $\xi'_{j'}$ are unbounded. Therefore

$$\mathcal{L}(\xi'_{h'}) = \mathcal{L}(C'_{h'}) = \mathcal{L}(\xi_h) = \mathcal{L}(\xi_j) = \mathcal{L}(C'_{j'}) = \mathcal{L}(\xi'_{j'}). \quad (26)$$

It also follows from the unboundedness that $\iota_{\mathbf{m}'}(p'_{j'}) \supseteq \iota_{\mathbf{m}}(p_j)$ and $\iota_{\mathbf{m}'}(p'_{h'}) \supseteq \iota_{\mathbf{m}}(p_h)$. If there exists some $i \in \iota_{\mathbf{m}'}(p'_{h'}) \setminus \iota_{\mathbf{m}}(p_h)$, then there is some solution to $\mathcal{E}^0_\xi$ that assigns a positive integer to $\mathbf{x}_h(i)$ but all solutions to $\mathcal{E}^0_{\xi'}$ assigns 0 to $\mathbf{x}_{h'}(i)$. If there is a solution to $\mathcal{E}^0_{\xi'}$ that assigns a positive integer to $\mathbf{x}_{j'}(i)$, then using the technique in the proof of Lemma 24, one routinely shows that there must be a solution to $\mathcal{E}^0_{\xi'}$ that assigns a positive integer to $\mathbf{x}_{h'}(i)$, raising a contradiction. So $\iota_{\mathbf{m}'}(p'_{j'}) = \iota_{\mathbf{m}'}(p'_{h'})$ must be valid. We have reached to the type preservation property: If a reduction produces an ω-CGS, then it turns all CGs that have not been reduced and are of the same type to CGs of the same type. □

Putting together all the results we have obtained so far in this section we are able to prove the main result of the paper.

Theorem 26. $\mathbb{VASS}$ *is Tower complete.*

Proof. In the light of Parallel Decomposition Lemma and Parallel Reduction Lemma, it makes sense to investigate the number of times $\mathfrak{D}$ and $\mathfrak{R}$ are recursively invoked in a successful run of $\mathfrak{A}$. Let's say that $\mathfrak{A}$ is executed in stages. Each stage contains one and only one invocation of $\mathfrak{D}$. Notice that the inequality in (3) implies that the number of types is bounded by $2^{2d} \cdot 2^{2^{w(n)}}$. Notice also that the number of distinct local spaces is bounded by 2^d. CGs of the same type in the same L-space are decomposed simultaneously. However CGs of the same type may emerge in different stages in the execution of $\mathfrak{A}$. Luckily the number of stages can be bounded by an elementary function. Let $S^{=l}$ denote the number of stages the CGs of kind C satisfying $|C| = l$ are produced. In other words $S^{=l}$ is the number of stages the CGs, with l size loop classes, are generated. The size of the loop class of the input VASS is, according to (3), bounded by $2^{w(n)}$. Thus

$$S^{=2^{w(n)}} \leq 1.$$

Observe that a CG of kind C can only be decomposed from a CG of a strictly bigger kind $C' \supset C$. The CGs of kind of l size are generated in different stages. But no matter in what stages they are generated, they must all be decomposed from CGs of kind of greater size. For each $l' > l$ the CGs of kind of l' size generated at some stage may split to many CGs of kind of l size. There are no more than $\binom{l'}{l} \leq \binom{2^{w(n)}}{l}$ different loop classes of l size generated from a loop class of size l'. Consequently there are no more than $2^{2d} \cdot \binom{2^{w(n)}}{l}$ different types whose loop classes are of l size generated from a type whose loop class is of size l'. The following inequalities should now be apparent, where 2^d bounds the number of L-spaces.

$$S^{=l} \leq \sum_{l'>l} 2^d \cdot 2^{2d} \cdot \binom{2^{w(n)}}{l} \cdot S^{=l'} < \left(2^{3d} \cdot 2^{2^{w(n)}} - 1\right) \cdot \sum_{l'>l} S^{=l'}. \quad (27)$$

It follows from (27) that

$$\sum_{l'>l-1} S^{=l'} = S^{=l} + \sum_{l'>l} S^{=l'} < 2^{3d} \cdot 2^{2^{w(n)}} \sum_{l'>l} S^{=l'}.$$

Therefore for some polynomial $u(n)$,

$$S^{=l} \leq \sum_{l'>l-1} S^{=l'} < \ldots < \left(2^{3d+2^{w(n)}}\right)^{2^{w(n)}} \cdot S^{=2^{w(n)}} \leq 2^{2^{u(n)}}.$$

We conclude that the maximal number of stage, and the maximal number of decomposition step as well, is bounded by

$$\sum_{l=1}^{2^{w(n)}} S^{=l} = 2^{w(n)} \cdot 2^{2^{u(n)}}. \quad (28)$$

In the execution of $\mathfrak{A}$ the subroutines $\mathfrak{D}$ and $\mathfrak{R}$ are executed in alternation. Applying Lemma 25 one sees that between two consecutive invocation of $\mathfrak{D}$, $\mathfrak{R}$ is executed recursively for at most

$$d\left(2^d \cdot 2^{2d} \cdot 2^{2^{w(n)}}\right) \quad (29)$$

times. In (29) the factor 2^d is the number of local spaces, the expression $2^{2d} \cdot 2^{2^{w(n)}}$ is the bound for the number of types, and the coefficient d is due to the fact that CG cannot be reduced consecutively more than d times without changing its type.

Combining (28) and (29) one derives that the total number of decomposition and reduction is bounded by $2^{2^{\mathtt{poly}(n)}}$. It follows that the final CGS is bounded in size by

$$2^{\cdot^{\cdot^{\cdot^{2^n}}}} \Big\} 2^{2^{\mathtt{poly}(n)}}.$$

We can modify $\mathfrak{A}$ so that both the number of execution of $\mathfrak{D}$ and the number of recursive invocation of $\mathfrak{R}$ are bounded double exponentially. That would bound the running time of all execution paths of $\mathfrak{A}$, bearing in mind that the complexity of both one step decomposition and one-step reduction is exponential space.

We have finally proved that $\mathbb{VASS} \in$ **Tower**. In view of the Tower hardness result reported in [2], we may conclude that VASS reachability is Tower complete. □

10. Conclusion

The long standing open problem, the complexity theoretical characterization of *the* reachability problem, is now settled. Not surprisingly our algorithm is based on the KLMST algorithm. The novelty of our approach is a new facility, loop completion, that speeds up decomposition and reduction. The method is based on the discovery of an invariant of circular trips, called loop class in this paper. Our study benefits a lot from a purely algebraic approach. Our presentation of the KLMST algorithm is tailored to the approach. For one thing our CGSs are graphs without any location information attached. This is different from the similar objects defined in previous studies. Our intention is to make the algebra as simple as possible. The tradeoff is that we must always talk about a CGS with one of its minimal solutions to the characteristic equation system.

One would hope that the methodology of this paper is suggestive for the studies into related issues. The proof of this paper does not give any hint about the complexity upper bound for $\mathbb{VASS}^d$. The open problem concerning the complexity of $\mathbb{VASS}^3$ is the obvious one we should attack next. Is it decidable in $2^{\mathtt{poly}(n)}$ space, as what the PSPACE completeness of $\mathbb{VASS}^2$ would make one believe [1]? Or is it already Tower hard, as the proof of this paper seems to suggest? At the moment we do not have a strong intuition about the answer. It should be mentioned that the proof of the nonelementary result [2] by Czerwinsky, Lasota, Lazic, Leroux and Mazowiecki points out that there is a constant D such that for every h the reachability problem $\mathbb{VASS}^{Dh}$ is h-EXPSPACE hard.

Tower completeness is very impractical. An avenue for further investigation is to look for practical restrictions that would significantly reduce the complexity of the problem. Hopefully there is a world of things to be discovered down the road.

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