Faster Pseudopolynomial Algorithms for Mean-Payoff Games

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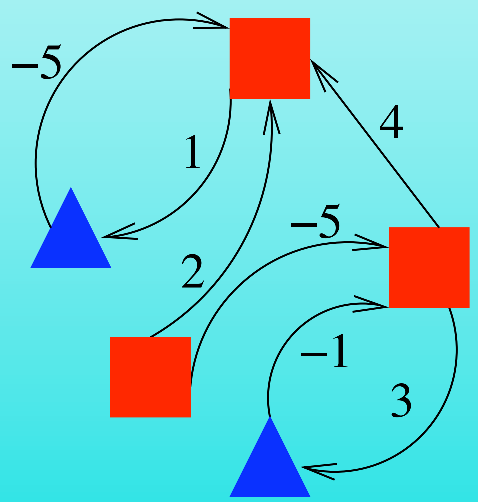
Preliminaries

Mean-Payoff Games (MPG)



- 2 players (maximizer □ vs minimizer △)
- turn based
- played on a finite graph (arena)
- infinite number of turns
- goal (for □): maximazing the long-run average weight

MPG in Formal Terms

In a MPG $\Gamma = (V, E, w : V \to \mathbb{Z}, \langle V_\Box, V_\triangle \rangle)$, player $\Box$ ($\triangle$) wants to maximize (minimize) the long-run average weight in a play (payoff).

Given a play $p = \{v_i\}_{i \in \mathbb{N}}$ in Γ, the payoff of player $\Box$ on p is:

$$\mathsf{MP}(v_0 v_1 \ldots v_n \ldots) = \liminf_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w(v_i, v_{i+1})$$

The value secured by a strategy $\sigma_\Box$: $V^* \cdot V_\Box \to V$ in vertex v is:

$$\mathsf{val}^{\sigma_\Box}(v) = \inf_{\sigma_\triangle \in \Sigma_\triangle} \mathsf{MP}(\mathsf{outcome}^\Gamma(v, \sigma_\Box, \sigma_\triangle))$$

$\sup_{\sigma_\Box \in \Sigma_\Box}(\mathsf{val}^{\sigma_\Box}(v))$ is the optimal value that player $\Box$ can secure in v

MPG are Memoryless Determined

$$\mathsf{val}^{\Gamma}(v) = \sup_{\sigma_\Box \in \Sigma_\Box} \inf_{\sigma_\triangle \in \Sigma_\triangle} \mathsf{MP}(\mathsf{outcome}^{\Gamma}(v, \sigma_\Box, \sigma_\triangle)) =$$
$$= \inf_{\sigma_\triangle \in \Sigma_\triangle} \sup_{\sigma_\Box \in \Sigma_\Box} \mathsf{MP}(\mathsf{outcome}^{\Gamma}(v, \sigma_\Box, \sigma_\Box))$$

there exist uniform memoryless strategies, $\pi_\Box : V_\Box \to V$, $\pi_\triangle : V_\triangle \to V$ such that $\mathsf{val}^{\Gamma}(v) = \mathsf{val}^{\pi_\Box}(v) = \mathsf{val}^{\pi_\triangle}(v)$

$\mathsf{val}^{\Gamma}(v)$ is said the value of the vertex v in the meanpayoff game Γ.

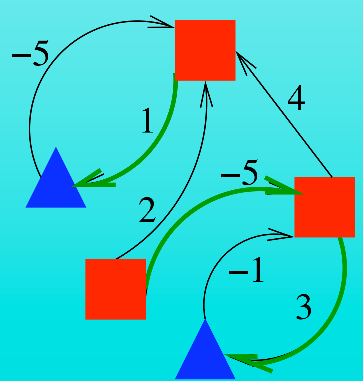
The Value in MPG

For all memoryless strategies $\pi_\square$ for player $\square$, for all $v \in V$:

$$\mathsf{val}^{\pi_\square}(v) \geq \mu$$

$$\Updownarrow$$

all cycles reachable from v in $G^{\Gamma}_{\pi_\square}$ have average weight $\geq \mu$.



$\mathsf{val}^{\Gamma}(v) = \frac{n}{d}$ such that $0 < d \leq |V|$

and $\frac{|n|}{d} \leq M$, $M = \max_{e \in E}\{|w(e)|\}$.

The Value in MPG

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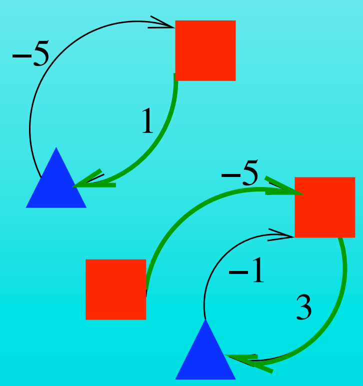


$\mathsf{val}^{\Gamma}(v) = \frac{n}{d}$ such that $0 < d \leq |V|$

and $\frac{|n|}{d} \leq M$, $M = \max_{e \in E}\{|w(e)|\}$.

MPG Problems

We consider the following four problems on MPG:

1. **Decision Problem & Strategy Synthesis** Given $v \in V$, $\mu \in \mathbb{Z}$, decide if player $\Box$ has a strategy $\pi_\Box$ to secure $\mathsf{val}^{\pi_\Box}(v) \geq \mu$.
   If yes, construct a corresponding winning strategy for player $\Box$.
2. **Threshold-partition Problem** Given $\mu \in \mathbb{Z}$, partition the set V into subsets $V_{>\mu}, V_{<\mu}, V_{=\mu}$ of vertices from which player $\Box$ can secure a payoff $> \mu$, $< \mu$, and $= \mu$, respectively.
3. **Value Problem** Compute the set of (rational) values $\{\mathsf{val}^\Gamma(v) \mid v \in V\}$
4. **Optimal Strategy Synth.** Construct an optimal strategy for player $\Box$

MPG Problems: Why They Matter?

- MPG problems have an interesting complexity status
  - MPG decision problem belongs to NP $\cap$ coNP (and even to UP $\cap$ coUP)
  - No polynomial algorithm known so far
- MPG strongly significative for theoretical and applicative aspects
  - μ-calculus model checking $\overset{\text{PTIME}}{\Longleftrightarrow}$ parity games $\overset{\text{PTIME}}{\Longrightarrow}$ MPG
  - MPG $\overset{\text{PTIME}}{\Longrightarrow}$ simple stochastic games
  - MPG $\overset{\text{PTIME}}{\Longrightarrow}$ discounted payoff games

State of the Art: An Algorithmic Statement

Consider $\Gamma = (V, E, w, \langle V_\Box, V_\triangle \rangle)$, where $w : V \to [-M, \ldots, 0, \ldots, +M]$:

U. Zwick and M. Paterson, 1996

$\Rightarrow \Theta(EV^2M)$ algorithm for the decision problem

$\Rightarrow \Theta(EV^3M)$ algorithm for the value problem

$\Rightarrow \Theta(EV^4M\log(\frac{E}{V}))$ algorithm for optimal strategy synthesis

H. Bjorklund and S. Vorobyov, 2004: Use a randomized framework

$\Rightarrow \mathcal{O}(\min(EV^2M, 2^{\mathcal{O}(\sqrt{V\log V})}))$ for the decision prob.

$\Rightarrow \mathcal{O}(\min(EV^3M(\log V + \log M), 2^{\mathcal{O}(\sqrt{V\log V})}))$ for the value prob.

Y. Lifshits and D. Pavlov, 2006

$\Rightarrow \mathcal{O}(EV2^V\log(Z))$ algorithm for the decision/value problem

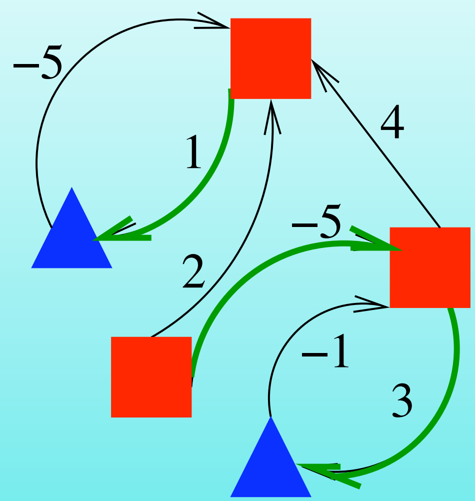
Energy Games (EG)

In an energy game $\Gamma = (V, E, w, \langle V_\Box, V_\triangle \rangle)$, the goal of player $\Box$ is building a play $p = \{v_i\}_{i \in \mathbb{N}}$ such that for some initial credit $c \in \mathbb{N}$:

$$c + \sum_{i=0}^{j} w(v_i, v_{i+1}) \geq 0) \text{ for all } j \geq 0$$

Energy games are memoryless determined, i.e. for all $v \in V$ either player $\Box$ has a winning memoryless strategy from v, or player $\triangle$ has a memoryless winning strategy from u.

Winning Strategies in EG



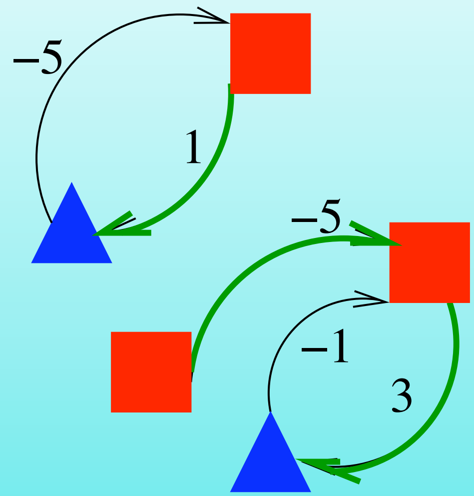
For all memoryless strategies $\pi_\Box$ for player $\Box$ in the EG Γ:

$\pi_\Box$ is winning from $v \in V$ for player $\Box$

$\Updownarrow$

all the cycles reachable from v in $G^{\Gamma}_{\pi_\Box}$ are nonnegative.

Winning Strategies in EG



For all memoryless strategies $\pi_\Box$ for player $\Box$ in the EG Γ:

$\pi_\Box$ is winning from $v \in V$ for player $\Box$

$$\Updownarrow$$

all the cycles reachable from v in $G^{\Gamma}_{\pi_\Box}$ are nonnegative.

EG Problems

We consider the following four problems on energy games:

1. **Decision Problem**. Given $v \in V$, decide if $v \in W_\Box$, i.e. if v is winning for player $\Box$.
2. **Strategy Synthesis**. Given $v \in W_\Box$ (resp. $W_\triangle$), construct a corresponding winning strategy for player $\Box$ (resp. $\triangle$) from v.
3. **Partition Problem**. Construct the sets of vertices $W_\Box, W_\triangle$ of winning vertices for the two players.
4. **Minimum Credit Problem**. For each $v \in W_\Box$ compute the minimum initial credit $c^*(v)$ such that player $\Box$ has a winning strategy from v, w.r.t. such an initial credit $c^*(v)$.

A Small Energy Progress Measure

Progress measures are functions $f : V \to \mathbb{N}$ defined on the set of vertices of a weighted graph

$$\Downarrow$$

their local consistency allows to infer global properties of the graph.

Definition [Energy Progress Measure] Let $G = \langle V, E, w \rangle$ be a weighted graph. An energy progress measure for G is a function $f : V \to \mathbb{N}$ such that for all $(v, v') \in E$:

$$f(v) \geq f(v') - w(v, v')$$

A Small Energy Progress Measure

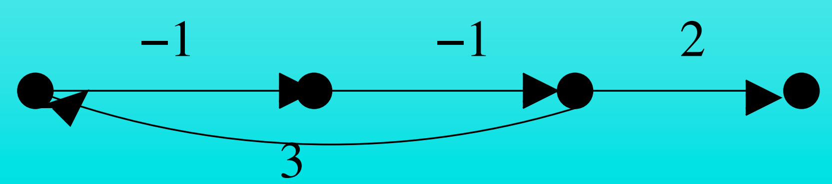
Our progress measure (PM) is referred to as energy PM since:

Let $G = (V, E, w)$ be a weighted graph. If G admits an energy progress measure, then:

$\Rightarrow$ all cycles of G are nonnegative, and

$\Rightarrow$ for all paths $(v_0, \ldots, v_n)$ in G it holds:

$$f(v_0) + \sum_{i=0}^{n-1} w(v_i, v_{i+1}) \geq 0$$



A Small Energy Progress Measure

Given $G = (V, E, w)$, where $w : V \to \{-M, \ldots, +M\}$, let

$$\mathcal{M}_G = \sum_{v \in V} \max(\{0\} \cup \{-w(v, v') \mid (v, v') \in E\})$$

Our energy progress measure is referred to as small since:

Given $G = (V, E, w)$, if all cycles of G are nonnegative, then there exists an energy progress measure $f : V \to \{0, \ldots, \mathcal{M}_G\}$ for G.

A Small Energy PM: From Graphs to Games

To extend the concept of energy PM from graphs to games, we take into account the partition of vertices between the players.

A function $f : V \to \mathcal{C}_\Gamma = \{n \in \mathbb{N} | n \leq \mathcal{M}_{G^\Gamma}\} \cup \{\top\}$ is a small energy progress measure for the game $\Gamma = (V, E, w, \langle V_\Box, V_\triangle \rangle)$ iff:

$\Rightarrow$ if $v \in V_\Box$, then $f(v) \succeq f(v') \ominus w(v, v')$ for some $(v, v') \in E$

$\Rightarrow$ if $v \in V_\triangle$, then $f(v) \succeq f(v') \ominus w(v, v')$ for all $(v, v') \in E$

- We denote by V_f the set of states $V_f = \{v \mid f(v) \neq \top\}$.
- Memoryless strategy $\pi^f_\Box$ is sayd compatible with f iff:

$$\forall v \in V_\Box . (\pi^f_\Box(v) = v' \to f(v) \succeq f(v') \ominus w(v, v'))$$

Solving the EG Problems

Lemma If $\pi_{\Box}^{f}$ is a strategy for player $\Box$ compatible with the small energy PM f for the EG Γ, then $\pi_{\Box}^{f}$ is a winning strategy for player $\Box$ from all vertices $v \in V_f$, i.e. $V_f \subseteq W_{\Box}$.

Lemma If Γ is an energy game, then Γ admits a small energy PM f with $V_f = W_{\Box}$, and such that for all $v \in W_{\Box}$, $f(v) = c^*(v)$.

Hence, determining a small energy PM on the EG Γ such that $V_f = W_{\Box}$ and for all $v \in W_{\Box}$, $f(v) = c^*(v)$, subsumes our four EG problems.

EG Algorithm: Basics

Our energy game algorithm based on the notion of small energy PM:

- Initializes the small energy PM $f : V \to \mathcal{C}_\Gamma$ to the constant function 0
- Maintain overall its execution a list L of nodes that witness a local inconsistency of f, namely:
  - $v \in L \cap V_\Box$ iff for all v' such that $(v, v') \in E$ it holds $f(v) < (v') \ominus w(v, v')$
  - $v \in L \cap V_\triangle$ iff there exists v' such that $(v, v') \in E$ and $f(v) < (v') \ominus w(v, v')$

EG Algorithm in Big Steps

The algorithm iteratively extracts a node v from L and performs:

1. Apply to f the lifting operator $\delta(f, v)$ to solve local inconsistency.
2. Insert into the list L the set of nodes witnessing a new local inconsistency, due to the increasing of $f(v)$.

Definition [Lifting Operator] Given $v \in V$, the lifting operator $\delta(\cdot, v) : [V \to \mathcal{C}_\Gamma] \to [V \to \mathcal{C}_\Gamma]$ is defined by $\delta(f, v) = g$ where:

$$g(z) = \begin{cases} f(z) & \text{if } z \neq v \\ \min\{f(v') \ominus w(v, v') \mid (v, v') \in E\} & \text{if } z = v \in V_\Box \\ \max\{f(v') \ominus w(v, v') \mid (v, v') \in E\} & \text{if } z = v \in V_\triangle \end{cases}$$

EG Algorithm: Correctness and Complexity

Correctness The energy game algorithm applied to the energy game Γ computes a small energy PM f on Γ such that:

$\Rightarrow$ if $v \in W_{\Box}$, then $f(v) = c^*(v)$, otherwise $f(v) = \top$.

To establish the complexity of our energy games algorithm note that:

- each iteration of the procedure (corresponding to a lift operation, followed by an update of the list L) costs $\mathcal{O}(post(v) + pre(v))$.
- For each $v \in V$, $f(v)$ can increase at most $\mathcal{M}_{G^\Gamma} + 1$ times.

Complexity The complexity of the energy games algorithm is $\mathcal{O}(\sum_{v \in V}(post(v) + pre(v)) \cdot \mathcal{M}_{G^\Gamma}) = \mathcal{O}(E \cdot \mathcal{M}_{G^\Gamma})$

Pseudopolynomial Upper Bounds for EG Problems

The following problems can be solved in time $\mathcal{O}(E \cdot V \cdot M)$ on the

EG $\Gamma = (V, E, w : V \to [-M, \ldots, 0, \ldots, +M], \langle V_\Box, V_\triangle \rangle)$

(1) the decision problem,

(2) the strategy synthesis problem,

(3) the partition problem, and

(4) the minimum credit problem.

Solving the MPG Problems

Given the MPG $\Gamma = (V, E, w, \langle V_\Box, V_\triangle \rangle)$, consider $\mu \in \mathbb{Z}$, and let

$$\Gamma^{-\mu} = (V, E, z - \mu, \langle V_\Box, V_\triangle \rangle).$$

Lemma If f is a small energy PM for $\Gamma^{-\mu}$ and π_f is a strategy for player $\Box$ compatible with f, then π_f applied to Γ secures player $\Box$ a payoff at least t from all $u \in V_f$.

Moreover, $\Gamma^{-\mu}$ admits a small energy PM f, such that $V_f = V_{\geq \mu}$.

Solving the MPG Decision Problem

Let $\Gamma = (V, E, w, \langle V_\Box, V_\triangle \rangle)$ be a MPG, let $\mu \in \mathbb{Z}$. The decision problem "Is $\mathsf{val}^\Gamma(v) \geq \mu$?" can be easily solved using our EG algorithm, on the ground of the previous lemma:

- If $t > M = \max_{v \in V}\{|w(v)|\}$, then no. If $-t > M$, then yes.
- Otherwise, in virtue of the previous lemma we can apply our energy games algorithm to $\Gamma^{-\mu}$ obtainin the energy PM f, such that our decision problem has a positive answer iff $f(v) \neq \bot$.
- Moreover, if $f(v) \neq \bot$, any strategy π_f compatible with f to Γ secures player $\Box$ a payoff at least μ.

Solving the MPG 3-way Partition Problem

Also the three-way partition problem can be solved using the energy games algorithm as a basic ingredient:

- Given $\Gamma' = (V, E, w - \mu, \langle V_\Box, V_\triangle \rangle)$ and $\mu \in \mathbb{Z}$, define $\Gamma' = (V, E, w - \mu, \langle V_\Box, V_\triangle \rangle), \Gamma'' = (V, E, -w + \mu, \langle V_\triangle, V_\Box \rangle)$
- Running EG algorithm on Γ' yields the partition on V into $V_{\geq \mu}, V_{<\mu}$
- Running EG algorithm on Γ'' yields the partition on V into $V_{\leq \mu}, V_{>\mu}$
- The desired three-way partition can be immediately extracted from the above two partitions.

New Pseudopolynomial Upper Bounds for MPG (I)

The following problems can be solved in $\mathcal{O}(E \cdot V \cdot M)$ on the MPG

$\Gamma = (V, E, w : V \to [-M, \ldots, 0, \ldots, +M], \langle V_{\Box}, V_{\triangle} \rangle)$

(1) the decision problem,

(2) the strategy synthesis problem,

(3) the 3-way partition problem.

New Pseudopolynomial Upper Bounds for MPG (II)

Combining a our energy games algorithm with a dichotomic search into the set of rationals:

$$S = \{\frac{p}{m} \mid 1 \leq m \leq |V| \wedge -M \leq \frac{p}{m} \leq M\}$$

we finally establish the last two new mean-payoff lower bounds:

The following problems can be solved in $\mathcal{O}(EV^2M(\log V + \log M))$ on the mean-payoff game $\Gamma = (V, E, w, \langle V_\Box, V_\triangle \rangle)$

(1) the value problem,

(2) the optimal strategy synthesis problem.

Summary of Results (I)

	Problems		
Algorithms	Decision Problem 3-Way Partition Problem	Strategy Synthesis	Remarks
This paper	$\mathcal{O}(E \cdot V \cdot W)$	$\mathcal{O}(E \cdot V \cdot W)$	Deterministic
Zwick & Paterson '96	$\Theta(E \cdot V^2 \cdot W)$	$\Theta(E \cdot V^3 \cdot W \log(\frac{E}{V}))$	Deterministic
Lifshits & Pavlov '07	$\mathcal{O}(E \cdot V \cdot 2^V)$	–	Deterministic
Bjorklund & Vorobyov '07	$\min(\mathcal{O}(E \cdot V^2 \cdot W), 2^{\mathcal{O}(\sqrt{V \cdot \log(V)})})$	$\min(\mathcal{O}(E \cdot V^2 \cdot W), 2^{\mathcal{O}(\sqrt{V \cdot \log(V)})})$	Randomized

Summary of Results (II)

	Problems	
Algorithms	Value Problem	Optimal Strategy Synthesis
This paper Daterministic	$\mathcal{O}(E \cdot V^2 \cdot W \cdot (\log(V) + \log(W)))$	$\mathcal{O}(E \cdot V^2 \cdot W \cdot (\log(V) + \log(W)))$
Zwick& Pat.'96 Deterministic	$\Theta(E \cdot V^3 \cdot W)$	$\Theta(E \cdot V^4 \cdot W \log(\frac{E}{V}))$
Lif.& Pav.'07 Deterministic	$\mathcal{O}(E \cdot V \cdot 2^V \cdot \log(W))$	$-$
Bjor.& Vor.'07 Randomized	$\min(\mathcal{O}(E \cdot V^3 \cdot W \cdot (\log(V) + \log(W))), 2^{\mathcal{O}(\sqrt{V \cdot \log(V)})})$	$\min(\mathcal{O}(E \cdot V^3 \cdot W \cdot (\log(V) + \log(W))), 2^{\mathcal{O}(\sqrt{V \cdot \log(V)})})$