

Darboux evaluations of algebraic Gauss hypergeometric functions

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Abstract

Algebraic Gauss hypergeometric functions can be expressed explicitly in several ways. One attractive way is to pull-back their hypergeometric equations (with a finite monodromy) to Fuchsian equations with a finite cyclic monodromy, and express the algebraic solutions as radical functions on the covering curve. This article presents these pull-back transformations of minimal degree for the hypergeometric equations with the tetrahedral, octahedral or icosahedral projective monodromy. The minimal degree is 4, 6 or 12, respectively. The covering curves are called *Darboux curves*, and they have genus 0 or (for some icosahedral Schwarz types) genus 1.

1 Introduction

Algebraic Gauss hypergeometric functions were studied by many authors; see for example [Sch72, Fuc75, Bri77, Kle78, Pep81, Bou98, Kat72, BD79, SU93, vdPU98]. This purpose of this article is to derive and present satisfying explicit forms for these functions.

The Gauss hypergeometric equation is

$$X(1-X) \frac{d^2 Y(X)}{dX^2} + (c - (a+b+1)X) \frac{dY(X)}{dX} - a b Y(X) = 0. \quad (1.1)$$

This is a Fuchsian equation on \mathbb{P}_X^1 with 3 regular singular points $X = 0, 1, \infty$. The local exponent differences are: $1 - c$ at $X = 0$; $c - a - b$ at $X = 1$; and $a - b$ at $X = \infty$. Let $E(e_1, e_2, e_3)$ denote a hypergeometric equation with the local exponent differences e_1, e_2, e_3 at the 3 singular points. The order and multiplication by -1 of the exponent differences is disregarded in this notation.

As well known [Sch72, BD79, vdPU98], hypergeometric equation (1.1) has a basis of algebraic solutions if and only if its monodromy group is finite. The following hypergeometric equations (and their fractional-linear transformations; see Appendix 5.4) have this property and are called *standard hypergeometric equations* with algebraic solutions:

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- $E(1, 1/n, 1/n)$, where n is a positive integer. The hypergeometric equation degenerates to a Fuchsian equation with two singular points. Its monodromy group is a cyclic group with n elements.
- $E(1/2, 1/2, 1/n)$, where n is an integer, $n \geq 2$. The projective monodromy group of this equation is the dihedral group with $2n$ elements.
- $E(1/2, 1/3, 1/3)$. The projective monodromy is the tetrahedral group, isomorphic to A_4 .
- $E(1/2, 1/3, 1/4)$. The projective monodromy is the octahedral group, isomorphic to S_4 .
- $E(1/2, 1/3, 1/5)$. The projective monodromy is the icosahedral group, isomorphic to A_5 .

Recall that the projective monodromy group is the monodromy group (or the Galois group, if solutions are algebraic) of y_1/y_2 , where y_1, y_2 is a basis of solutions of the differential equation. The quotient y_1/y_2 is known as a *Schwarz map*. In general, the two-dimensional monodromy representation of second order Fuchsian equations is a cyclic extension of the projective monodromy.

The celebrated theorem of Klein [Kle77] states that if a second order linear homogeneous differential equation only has algebraic solutions, then that equation is a *pull-back transformation* of a standard hypergeometric equation from the list above. Explicitly, if the Fuchsian equation has coefficients in $\mathbb{C}(X)$, the pull-back transformation changes the variable Z in (1.1) to a rational function $\varphi(X)$. In geometric terms, we have a finite covering $\varphi : \mathbb{P}_X^1 \rightarrow \mathbb{P}_Z^1$ between two projective lines, and we *pull-back* the standard hypergeometric equation from \mathbb{P}_Z^1 onto \mathbb{P}_X^1 .

The possible finite projective monodromies for the second order Fuchsian equations are the same as listed above: a cyclic, a dihedral, or the tetrahedral, octahedral or icosahedral groups. The standard hypergeometric equation in Klein's theorem can be chosen to have the same projective monodromy. In particular, the theorem of Klein implies that if a hypergeometric equation only has algebraic solutions, then it is a pull-back transformation of a listed standard hypergeometric equation with the same projective monodromy.

Hypergeometric equations with finite monodromies were first classified by Schwarz in [Sch72]. Disregarding hypergeometric equations with a cyclic monodromy, Schwarz gave a list of 15 types of these hypergeometric equations. One type consists of hypergeometric equations with a dihedral projective monodromy. The other types are represented by the following hypergeometric equations:

- $E(1/2, 1/3, 1/3)$, $E(1/3, 1/3, 2/3)$. The projective monodromy is tetrahedral.
- $E(1/2, 1/3, 1/4)$, $E(2/3, 1/4, 1/4)$. The projective monodromy is octahedral.
- $E(1/2, 1/3, 1/5)$, $E(1/2, 1/3, 2/5)$, $E(1/2, 1/5, 2/5)$, $E(1/3, 1/3, 2/5)$, $E(1/3, 2/3, 1/5)$, $E(2/3, 1/5, 1/5)$, $E(1/3, 2/5, 3/5)$, $E(1/3, 1/5, 3/5)$, $E(1/5, 1/5, 4/5)$, $E(2/5, 2/5, 2/5)$. The projective monodromy is icosahedral.

We refer to Schwarz type of hypergeometric equations with algebraic solutions by the triple of the exponent differences e_1, e_2, e_3 of these representative equations. (Usually, the Schwarz

type is denoted by a roman numeral from I to XV.) We refer to the listed 14 hypergeometric equations as *main representatives* of the Schwarz types. Hypergeometric equations of the same Schwarz type are characterized by the property that their hypergeometric solutions are *contiguous* to hypergeometric solutions of the main representative (see Appendix 5.6).

Algebraic solutions of differential equations can be represented in several ways. For example, by minimal polynomial equations they satisfy [SU93], or (if the Galois group is solvable) by nested radical expressions. In [BvHW03], [vHW05] [Ber04, Chapter 1] an algorithm is developed to represent algebraic solutions of second order linear differential equations using Klein's theorem. The representation form is

$$\theta(x) H(\varphi(x)), \tag{1.2}$$

where $\varphi(x)$ is a rational function, $\theta(x)$ is a radical function that define Klein's pull-back transformation, and $H(z)$ is a solution of a corresponding standard hypergeometric equation. The degree of $\varphi(x)$ is equal to the order of the projective monodromy group, that is, 12, 24 or 60 for the three most interesting monodromies.

We propose to pull-back a hypergeometric equations with a finite monodromy to Fuchsian equations with a cyclic monodromy group. Then some hypergeometric solutions are transformed to rather simple radical functions. We call a pull-back covering $\phi : D \rightarrow \mathbb{P}^1$ of this kind a *Darboux covering*. The covering curve D is called a *Darboux curve*. Identification of algebraic Gauss hypergeometric functions with radical functions on an algebraic curve offers satisfying geometric intuition, especially when the Darboux covering has low degree. We use the term *Darboux evaluation* to refer to the aspired identification of hypergeometric functions with radical functions.

The Darboux coverings for hypergeometric equations with finite dihedral projective monodromies have degree 2. They are simply $Z = x^2$ if half-integer exponent differences are at $Z = 0$ and $Z = \infty$. In [Vid11], the pulled-back dihedral hypergeometric functions are expressed in terms of power functions and terminating Appell's (double hypergeometric) F_2 or F_3 sums.

Theory of Darboux coverings is developed in §3. It turns out that Darboux coverings for hypergeometric equations of the same Schwarz type are identical. Therefore we have finitely many different Darboux coverings and Darboux curves. We compute all Darboux coverings of minimal degree, which turns out to be 4, 6 or 12 for the tetrahedral, octahedral and icosahedral types, respectively. The corresponding Darboux curves have genus 0 or (for some icosahedral types) genus 1. For each Schwarz type, we use Darboux coverings of minimal degree and compute Darboux evaluations for two hypergeometric solutions of the main representative equation. The evaluations are presented in §2. Using these formulas, differentiation and contiguous relations, one can compute Darboux evaluations for two different hypergeometric solutions of any hypergeometric equation with the tetrahedral, octahedral or icosahedral projective monodromy.

Section 4 presents the method used to compute Darboux evaluations. Most attention is paid to describing complicated computations on genus 1 Darboux curves. The computer algebra system *Maple* was used in the computations. Appendix §5 gives short introductions to pull-back transformations, Fuchsian equations, contiguous relations and other relevant topics.

2 Hypergeometric formulas

This section presents the main explicit results: Darboux evaluations for two hypergeometric functions of each Schwarz type with the tetrahedral, octahedral and icosahedral projective monodromies. Each pair of hypergeometric evaluations forms a solution basis of the main representative equation of the respective Schwarz type, up to a power factor to the second evaluation. Darboux evaluations for other hypergeometric functions of the same Schwarz type can be computed by using contiguous relations if another evaluation in the same contiguous orbit is known. Another contiguous evaluation can be computed by differentiation.

We use Darboux coverings of minimal possible degree, which is 4, 6 and 12 for the three considered monodromies, respectively, as shown in Lemma 3.3. For the tetrahedral and octahedral Schwarz types, Darboux coverings are evident from the arguments of the hypergeometric functions. Darboux coverings for icosahedral Schwarz types are given in (2.9), (2.14), (2.18), (2.24), (2.30), (2.34). Some of these Darboux coverings are valid for two different Schwarz types. For seven icosahedral types, the Darboux curve has genus 1 rather than 0. Weierstrass equations $\xi^2 = x(1 + \alpha x + \beta x^2)$, with $\alpha, \beta \in \mathbb{C}$, for these curves are given in (2.17), (2.23), (2.29), (2.33). The Darboux evaluations hold locally around $x = 0$ or, if the Darboux curve has genus 1, around the point $(x, \xi) = (0, 0)$. A simple way to check each evaluation is to expand both sides in power series around $x = 0$. If the Darboux curve has genus 1, one has to replace ξ by the respective $\sqrt{x} \sqrt{1 + \alpha x + \beta x^2}$, and expand the power series in \sqrt{x} .

2.1 Tetrahedral hypergeometric equations

Here are two hypergeometric evaluations for solutions of $E(1/2, 1/3, 1/3)$:

$${}_2F_1\left(\begin{matrix} 1/4, -1/12 \\ 2/3 \end{matrix} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) = (1-2x)^{-1/4}, \quad (2.1)$$

$${}_2F_1\left(\begin{matrix} 1/4, 7/12 \\ 4/3 \end{matrix} \middle| \frac{x(x+4)^3}{4(2x-1)^3}\right) = \frac{1}{1 + \frac{1}{4}x} (1-2x)^{3/4}. \quad (2.2)$$

If we multiply the second formula by $x^{1/3}(x+4)/(2x-1)$, the two evaluated functions form a basis of solutions of the same hypergeometric equation. Here are two hypergeometric evaluations for solutions of $E(1/3, 1/3, 2/3)$, of the other tetrahedral Schwarz type:

$${}_2F_1\left(\begin{matrix} 1/2, -1/6 \\ 2/3 \end{matrix} \middle| \frac{x(x+2)^3}{(2x+1)^3}\right) = (1+2x)^{-1/2}, \quad (2.3)$$

$${}_2F_1\left(\begin{matrix} 1/6, 5/6 \\ 4/3 \end{matrix} \middle| \frac{x(x+2)^3}{(2x+1)^3}\right) = \frac{1}{1 + \frac{1}{2}x} (1+2x)^{1/2} (1+x)^{1/3}. \quad (2.4)$$

2.2 Octahedral hypergeometric equations

Here are two representative solutions of $E(1/2, 1/3, 1/4)$:

$${}_2F_1\left(\begin{matrix} 7/24, -1/24 \\ 3/4 \end{matrix} \middle| \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = (1+14x+x^2)^{-1/8}, \quad (2.5)$$

$${}_2F_1\left(\begin{matrix} 5/24, 13/24 \\ 5/4 \end{matrix} \middle| \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1}{1-x} (1+14x+x^2)^{5/8}. \quad (2.6)$$

And here are two representative solutions of $E(1/4, 1/4, 2/3)$:

$${}_2F_1\left(\begin{matrix} 7/12, -1/12 \\ 3/4 \end{matrix} \middle| \frac{27x(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{(1+\frac{1}{2}x)^{1/4}}{(1+4x+x^2)^{1/4}}, \quad (2.7)$$

$${}_2F_1\left(\begin{matrix} 1/6, 5/6 \\ 5/4 \end{matrix} \middle| \frac{27x(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{(1+2x)^{1/4} (1+4x+x^2)^{1/2}}{1+x}. \quad (2.8)$$

2.3 Icosahedral hypergeometric equations

The Darboux covering for hypergeometric equations of the Schwarz types $(1/2, 1/3, 1/5)$ and $(1/2, 1/3, 2/5)$ is

$$\varphi_1(x) = \frac{1728x(x^2-11x-1)^5}{(x^4+228x^3+494x^2-228x+1)^3} \quad (2.9)$$

Representative evaluations for solutions of $E(1/2, 1/3, 1/5)$ are:

$${}_2F_1\left(\begin{matrix} 19/60, -1/60 \\ 4/5 \end{matrix} \middle| \varphi_1(x)\right) = (1-228x+494x^2+228x^3+x^4)^{-1/20}, \quad (2.10)$$

$${}_2F_1\left(\begin{matrix} 11/60, 31/60 \\ 6/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{(1-228x+494x^2+228x^3+x^4)^{11/20}}{1+11x-x^2}. \quad (2.11)$$

Similarly, basic evaluations for solutions of $E(1/2, 1/3, 2/5)$ are:

$${}_2F_1\left(\begin{matrix} 13/60, -7/60 \\ 3/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{1-7x}{(1-228x+494x^2+228x^3+x^4)^{7/20}}, \quad (2.12)$$

$${}_2F_1\left(\begin{matrix} 17/60, 37/60 \\ 7/5 \end{matrix} \middle| \varphi_1(x)\right) = \frac{(1+\frac{1}{7}x)(1-228x+494x^2+228x^3+x^4)^{17/20}}{(1+11x-x^2)^2}. \quad (2.13)$$

The Darboux covering for Schwarz type $(1/2, 1/5, 2/5)$ is the following:

$$\varphi_2(x) = \frac{64x(x^2-x-1)^5}{(x^2-1)(x^2+4x-1)^5}. \quad (2.14)$$

Representative evaluations for solutions of $E(1/2, 1/5, 2/5)$ are:

$${}_2F_1\left(\begin{matrix} 7/20, -1/20 \\ 4/5 \end{matrix} \middle| \varphi_2(x)\right) = \frac{(1+x)^{7/20}}{(1-x)^{1/20}(1-4x-x^2)^{1/4}}, \quad (2.15)$$

$${}_2F_1\left(\begin{matrix} 3/20, 11/20 \\ 6/5 \end{matrix} \middle| \varphi_2(x)\right) = \frac{(1+x)^{3/20}(1-x)^{11/20}(1-4x-x^2)^{3/4}}{1+x-x^2}. \quad (2.16)$$

Darboux curves for other icosahedral Schwarz types have genus 1. The Darboux curve for hypergeometric equations of the Schwarz types $(1/3, 1/3, 2/5)$ and $(1/3, 2/3, 1/5)$ is given by the equation

$$C_3 : \quad \xi^2 = x(1+33x-9x^2). \quad (2.17)$$

The Darboux covering is

$$\varphi_3(x, \xi) = \frac{144 \xi (1 + 33x - 9x^2)^2 (1 - 9\xi + 54x)}{(1 + 21\xi - 117x + 9x\xi - 234x^2)^3} \quad (2.18)$$

Here are basic evaluations for solutions of $E(1/3, 1/3, 2/5)$:

$${}_2F_1\left(\begin{matrix} 3/10, -1/30 \\ 3/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1 - 9\xi + 54x)^{1/30}}{(1 + 21\xi - 117x + 9x\xi - 234x^2)^{1/10}}, \quad (2.19)$$

$${}_2F_1\left(\begin{matrix} 7/10, 11/30 \\ 7/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1 + 21\xi - 117x + 9x\xi - 234x^2)^{11/10}}{(1 - 9\xi + 54x)^{11/30} (1 + 33x - 9x^2)}. \quad (2.20)$$

And here are basic evaluations for solutions of $E(1/3, 2/3, 1/5)$:

$${}_2F_1\left(\begin{matrix} 17/30, -1/10 \\ 4/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1 - 9\xi + 54x)^{13/30}}{(1 + 21\xi - 117x + 9x\xi - 234x^2)^{3/10}}, \quad (2.21)$$

$${}_2F_1\left(\begin{matrix} 1/10, 23/30 \\ 6/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \frac{(1 + 21\xi - 117x + 9x\xi - 234x^2)^{3/10} (1 - 9\xi + 54x)^{7/30} (\xi + 5x)}{\xi (1 + 9x)}. \quad (2.22)$$

The Darboux curve for the Schwarz types $(2/3, 1/5, 1/5)$ and $(1/3, 2/5, 3/5)$ is given by

$$C_4 : \quad \xi^2 = x (1 + 5x - 5x^2). \quad (2.23)$$

The Darboux covering is

$$\varphi_4(x, \xi) = \frac{432 x (1 - \frac{7}{5}\xi - 9x - x^2)^5 (1 + 50x - 125\xi^2 + 450x\xi - 500x^2)}{(5\xi + 57x) (1 + \frac{18}{5}\xi - 16x + x^2)^5 (1 + 50x - 125\xi^2 - 450x\xi - 500x^2)}. \quad (2.24)$$

Basic evaluations for solutions of $E(2/3, 1/5, 1/5)$ are:

$${}_2F_1\left(\begin{matrix} 1/6, -1/30 \\ 4/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = \frac{(1 - \frac{3}{5}\xi - \frac{34}{5}x)^{1/6}}{(1 + 3\xi - 20x)^{1/6} (1 + 50x - 125\xi^2 - 450x\xi - 500x^2)^{1/30}}, \quad (2.25)$$

$${}_2F_1\left(\begin{matrix} 1/6, 11/30 \\ 6/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = (1 + 50x - 125\xi^2 - 450x\xi - 500x^2)^{11/30} \times \frac{(1 + 3\xi - 20x)^{5/6} (1 - \frac{3}{5}\xi - \frac{34}{5}x)^{1/6} (1 + \frac{21}{4}\xi + \frac{41}{4}x)}{(1 - 9x) (1 - \frac{7}{4}\xi - \frac{15}{2}x) (1 + 5\xi + 10x)}. \quad (2.26)$$

Basic evaluations for solutions of $E(1/3, 2/5, 3/5)$ are:

$${}_2F_1\left(\begin{matrix} 13/30, -1/6 \\ 3/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = \frac{(1 + 50x - 125\xi^2 - 450x\xi - 500x^2)^{13/30} (1 - 3\xi + 2x)}{(1 + 3\xi - 20x)^{5/6} (1 - \frac{3}{5}\xi - \frac{34}{5}x)^{1/6} (1 + 5\xi + 10x)}, \quad (2.27)$$

$${}_2F_1\left(\begin{matrix} 5/6, 7/30 \\ 7/5 \end{matrix} \middle| \varphi_4(x, \xi)\right) = (1 + 50x - 125\xi^2 - 450x\xi - 500x^2)^{7/30} \times \frac{(1 + \frac{18}{5}\xi - 16x + x^2)^{7/6} (1 + \frac{1}{25}\xi)^{5/6} (1 + 5\xi + 10x)}{(1 - \frac{7}{5}\xi - 9x - x^2)^2 (1 - 5x)^{7/6}}. \quad (2.28)$$

The Darboux curve for the Schwarz type $(1/3, 1/5, 3/5)$ is given by

$$C_5 : \quad \xi^2 = x(1+x)(1+16x). \quad (2.29)$$

The Darboux covering is

$$\varphi_5(x, \xi) = -\frac{54(\xi+5x)^3(1-2\xi+6x)^5}{(1-16x^2)(\xi-5x)^2(1-2\xi-14x)^5}. \quad (2.30)$$

Representative evaluations for solutions of $E(1/3, 1/5, 3/5)$ are:

$${}_2F_1\left(\begin{matrix} 8/15, -1/15 \\ 4/5 \end{matrix} \middle| \varphi_5(x, \xi)\right) = \frac{(1+4x)^{8/15}(\xi+5x)^{1/6}x^{1/15}}{(1-2\xi-14x)^{1/3}(\xi-3x)^{3/10}}, \quad (2.31)$$

$${}_2F_1\left(\begin{matrix} 2/15, 11/15 \\ 6/5 \end{matrix} \middle| \varphi_5(x, \xi)\right) = \frac{(1-\xi+x)(1-2\xi-14x)^{2/3}(\xi+5x)^{1/6}(\xi-3x)^{13/10}}{(1+\xi+x)(1-2\xi+6x)(1+4x)^{13/15}x^{11/15}}. \quad (2.32)$$

The Darboux curve for the Schwarz types $(1/5, 1/5, 4/5)$ and $(2/5, 2/5, 2/5)$ is given by

$$C_6 : \quad \xi^2 = x(1+x-x^2). \quad (2.33)$$

The Darboux covering is

$$\varphi_6(x, \xi) = \frac{16\xi(1+x-x^2)^2(1-\xi)^2}{(1+\xi+2x)(1+\xi-2x)^5}. \quad (2.34)$$

Basic evaluations for solutions of $E(1/5, 1/5, 4/5)$ are:

$${}_2F_1\left(\begin{matrix} 7/10, -1/10 \\ 4/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1-\xi+2x)^{1/15}(1-\xi)^{3/5}}{(1+\xi+2x)^{7/30}\sqrt{1+\xi-2x}}, \quad (2.35)$$

$${}_2F_1\left(\begin{matrix} 1/10, 9/10 \\ 6/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(\xi+2x+x^2)(1+\xi)^{1/10}(1-\xi)^{3/10}}{\xi(1-\xi+2x)^{1/30}(1+\xi+2x)^{2/15}\sqrt{1+\xi-2x}}. \quad (2.36)$$

Basic evaluations for solutions of $E(2/5, 2/5, 2/5)$ are:

$${}_2F_1\left(\begin{matrix} 3/10, -1/10 \\ 3/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1-\xi+2x)^{2/15}(1+\xi+2x)^{1/30}(1-\xi)^{1/5}}{\sqrt{1+\xi-2x}}, \quad (2.37)$$

$${}_2F_1\left(\begin{matrix} 3/10, 7/10 \\ 7/5 \end{matrix} \middle| \varphi_6(x, \xi)\right) = \frac{(1+\xi+2x)^{7/30}(1+\xi)^{1/5}(1+\xi-2x)^{3/2}}{(1-\xi+2x)^{1/15}(1-\xi)^{2/5}(1+x-x^2)}. \quad (2.38)$$

3 Darboux curves

In the context of integration theory of vector fields [Dar78], [Oll01], one considers a polynomial vector field on \mathbb{C}^2 given by a derivation $\mathcal{L} = f(x, y)\partial/\partial x + g(x, y)\partial/\partial y$ with $f, g \in \mathbb{C}[x, y]$. A polynomial $p(x, y)$ is called a *Darboux polynomial* for the vector field (or the derivation \mathcal{L}) if $p(x, y)$ divides $\mathcal{L}p(x, y)$ in the ring $\mathbb{C}[x, y]$. An algebraic curve defined as the zero set of a Darboux polynomial is called a *Darboux curve*. A Darboux curve is infinitesimally invariant

under the vector field. Hence an alternative term is *invariant algebraic curve*, as in [CLPZ02] for example.

In differential Galois theory we have the following definition [Wei95], [Sin92]. Let K be a differential field, and let $R = K[y_1, \dots, y_n]$ be a differential ring. Let \mathcal{D} denote the derivation on R , and suppose that it extends the derivation on K . Then $p \in R$ is a *Darboux polynomial* for \mathcal{D} if p divides $\mathcal{D}p$ in R . For example, consider differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = 0, \quad \text{with } a_0, a_1, \dots, a_{n-1} \in K. \quad (3.1)$$

Let \mathcal{D} be the derivation on $K[y, y', \dots, y^{(n-1)}]$ defined by $\mathcal{D}y = y'$, $\mathcal{D}y' = y''$, \dots , $\mathcal{D}y^{(n-2)} = y^{(n-1)}$, $\mathcal{D}y^{(n-1)} = -a_{n-1}y^{(n-1)} - \dots - a_1y' - a_0y$. Darboux polynomials for this derivation correspond to semi-invariants of the differential Galois group for (3.1); see [Wei95, Theorem 38]. If the order n of (3.1) is equal to 2, then one considers Darboux polynomials in $K[u]$ for the derivation \mathcal{D} defined by $\mathcal{D}u = -u^2 - a_1u - a_0$. In [Wei94, UW96] these Darboux polynomials are called *special polynomials*.

As we see, the terms “Darboux polynomials”, “Darboux curves” are not consistently used though there is a common theme. In the context of algebraic solutions of differential equations, we wish to use the term “Darboux curves” to offer a geometric guidance. We offer the following definition of Darboux curves and Darboux coverings.

Definition 3.1 Let C denote an algebraic curve (see Appendix 5.1). We suppose that the function field $\mathbb{C}(C)$ is a differential field. Consider differential equation (3.1) on \mathbb{P}^1 , assuming $K = \mathbb{C}(\mathbb{P}^1)$. We say that a finite covering $\phi : C \rightarrow \mathbb{P}^1$ is a *Darboux covering* for (3.1), if a pull-back of (3.1) with respect to ϕ has a solution Y such that:

- (i) Its logarithmic derivative $u = Y'/Y$ is in $\mathbb{C}(C)$;
- (ii) The algebraic degree of u over K is precisely the degree of ϕ .

In the described situation, C is called a *Darboux curve* for (3.1).

To see connection with previous definitions, let us assume that the order n of (3.1) is equal to 2. We take the differential field $K = \mathbb{C}(z)$ with $z' = 1$. Suppose that we have a Darboux covering $\phi : C \rightarrow \mathbb{P}^1$ of degree d , and let u be a required logarithmic derivative in $\mathbb{C}(C)$. Then we have the following facts:

- The logarithmic derivative u is an algebraic solution of the associated Riccati equation $u' + u^2 + a_1u + a_0 = 0$. Let $P(u) = 0$ denote the minimal monic polynomial equation defining u over K . The polynomial P has degree d , and it is a defining equation for the Darboux curve C . Therefore $\mathbb{C}(C) = \mathbb{C}(z, u)$.
- The polynomial P is a Darboux polynomial for the derivation $\mathcal{D}u = -u^2 - a_1u - a_0$ in the definition of differential Galois theory; see [UW96, Lemma 2.4].
- The expression $y^{n+1}P(y'/y)$ is a homogeneous polynomial in $K[y, y']$. It is a Darboux polynomial for the corresponding derivation on $K[y, y']$.
- Assume that $a_1, a_0 \in \mathbb{C}[z]$, and consider the vector field $\partial/\partial z - (u^2 + a_1u + a_0)\partial/\partial u$. Then P is a Darboux polynomial according the first definition above.

Besides, we have the following facts.

- Suppose that $P = u^m + \sum_{j=0}^{d-1} b_j u^j$. Then $-b_{d-1}$ is the logarithmic derivative of an exponential solution of the d -th symmetric tensor power of differential equation (3.1); see [UW96, Theorem 2.1] or [SU93, §3.2]. All other coefficients b_j are determined by b_{d-1} and the differential equation [UW96, §2].
- The mentioned exponential solution (which can be expressed as $\exp \int -b_{d-1}$) is a degree d semi-invariant of the differential Galois group [UW96, §2].
- Inside a Picard-Vessiot extension for (3.1), the field $\mathbb{C}(C)$ is fixed by a 1-reducible subgroup of the differential Galois group for (3.1) of finite index [SU93, Lemma 3.1].
- If the differential Galois group of (3.1) is finite, the 1-reducible subgroups are cyclic subgroups [UW96, Lemma 1.5].

As we see, Darboux coverings correspond to algebraic Riccati solutions and semi-invariants of the differential Galois group.

We defined Darboux curves and coverings for general linear differential equations. They can be computed from minimal polynomials for Riccati solutions or by direct solution of pull-backed equations. Pulled-back equations on Darboux curves have cyclic monodromy groups, hence their solutions can be expressed in terms of radical functions. Solutions of the original equation can be presented as those radical solutions of the transformed equation, keeping in mind the inverse substitution defined by the Darboux covering. We refer to these radical expressions as *Darboux evaluations* of solutions of the original equation. They could be satisfying for many purposes. For convenience, Darboux curves of genus 0 can be parametrized, and Weierstrass forms of genus 1 Darboux used. Thanks to contiguous relations, Darboux evaluations for algebraic hypergeometric functions can be derived using only a few representative Darboux evaluations for each Schwarz type.

Darboux evaluations for solutions of hypergeometric equations with dihedral projective monodromies is recognized in [Vid11] or [Erd53, 2.5.5]. The Darboux coverings in the dihedral case are simple quadratic coverings, such as $\varphi(x) = x^2$. The Darboux curves and coverings for hypergeometric equations were introduced earlier in [Vid99, §4.2]. The Darboux coverings of minimal degree for all tetrahedral, octahedral types and for the icosahedral types $(1/2, 1/3, 1/5)$, $(1/2, 1/3, 2/5)$, $(1/2, 1/5, 2/5)$, $(1/3, 1/3, 2/5)$, $(1/3, 1/3, 2/5)$ were computed in that PhD thesis.

3.1 Basic properties of Darboux curves

The following lemmas identify Darboux coverings for hypergeometric equations of the same Schwarz type, and imply that there are only finitely many different Darboux coverings for all hypergeometric equations. These are key attractive features for using Darboux pull-backs.

Lemma 3.2 *Suppose that $\phi : D \rightarrow \mathbb{P}^1$ is a Darboux covering for a hypergeometric equation E_1 with a tetrahedral, octahedral or icosahedral monodromy, and E_2 is hypergeometric equation of the same Schwarz type. Then $\phi : D \rightarrow \mathbb{P}^1$ is a Darboux covering for E_2 as well.*

Proof. Let y_1 be a hypergeometric solution of E_1 . Since E_1 and E_2 have the same Schwarz type, there is a hypergeometric solution y_2 which is contiguous to y_1 . Recall that y'_1 is contiguous to y_1 as well. Therefore there is a contiguous relation $y_2 = ay_1 + by'_1$, with $a, b \in \mathbb{C}(z)$. There is also a contiguity relation $y'_2 = cy_1 + dy'_1$, with $c, d \in \mathbb{C}(z)$. Let u_1 denote the Riccati solution y'_1/y_1 for E_1 , and let u_2 denote the Riccati solution y'_2/y_2 for E_2 . Then $u_2 = (c + du_1)/(a + bu_1)$. This implies that the function fields $\mathbb{C}(z, u_1)$ and $\mathbb{C}(z, u_2)$ are isomorphic. \square

The following lemma categorizes all possible Darboux coverings for hypergeometric equations with the considered monodromies.

Lemma 3.3 *Suppose that the projective monodromy G of a hypergeometric equation (1.1) is either tetrahedral A_4 , or octahedral S_4 , or icosahedral A_5 .*

- *If $G \cong A_4$ (tetrahedral group), the Darboux coverings have degree 4, 6 or 12.*
- *If $G \cong S_4$ (octahedral group), the Darboux coverings have degree 6, 8, 12 or 24.*
- *If $G \cong A_5$ (icosahedral group), the Darboux coverings have degree 12, 20, 30 or 60.*
- *The Darboux coverings of each degree are unique up to automorphisms of the Darboux curve.*

Proof. As noted, Darboux curves correspond to algebraic Riccati solutions. According to [UW96, Corollary 1.7], the given numbers are possible degrees of minimal polynomials for solutions of the Riccati equation associated to the hypergeometric equations with considered monodromy groups. For each projective monodromy group, Riccati solutions of the maximal degree are not unique, but they define the same function field (because of irreducibility of the monodromy) and hence the same Darboux curve. Irreducible polynomials for Riccati solutions of other (non-maximal) degrees are unique except that there are two irreducible polynomials of degree 4 in the tetrahedral case. To show that those two degree 4 polynomials define isomorphic Darboux curves, we first use Lemma 3.2 to reduce consideration to the two equations $E(1/2, 1/3, 1/3)$ and $E(1/3, 1/3, 2/3)$, representing the two distinct tetrahedral Schwarz types. The two points with the exponent differences $1/3$ of either of these equations are interchanged by a fractional-linear transformation (say, $X \mapsto 1 - X$), and the two minimal polynomials differ by the same transformation. \square

The uniqueness claim does not mean that coverings for pull-backs of considered hypergeometric equations to Fuchsian equations with the isomorphic cyclic monodromies have to be isomorphic. In fact, Litcanu [Lit04, §2] observed that there are two pull-back transformations of $E(1/2, 1/3, 1/4)$ to a Fuchsian equation with the $\mathbb{Z}/2\mathbb{Z}$ monodromy. One pull-back covering is the degree 12 Darboux covering

$$\frac{27(x-1)^4(x^2+6x+1)^4}{(x^2-10x+1)^3(3x^2+2x+3)^3}, \quad (3.2)$$

and the other is $108x^2(x^2-1)^4/(x^4+14x^2+1)^3$. The latter covering is a composition of the degree 6 covering for octahedral equations (for a pull-back to $\mathbb{Z}/4\mathbb{Z}$) and a quadratic covering.

It is not a Darboux covering, because the respective Riccati solution is still of degree 6. The two coverings of degree 12 have different branching patterns.

3.2 Darboux coverings for standard hypergeometric equations

Here are the main facts about Darboux coverings for standard hypergeometric equations.

Lemma 3.4 *Let H denote a standard hypergeometric equation with tetrahedral, octahedral or icosahedral differential Galois group G . Let $\varphi : D \rightarrow \mathbb{P}^1$ be a Darboux covering for H of degree m . Then:*

- (i) *The Darboux curve D has genus zero.*
- (ii) *Let $\tilde{\varphi} : \tilde{D} \rightarrow \mathbb{P}^1$ be a Darboux covering for H of the maximal degree $|G|$. Then we have a factorization $\tilde{\varphi} = \gamma \circ \varphi$, where the covering $\gamma : \tilde{D} \rightarrow D$ is given (up to fractional-linear transformations of D and \tilde{D}) by $x \mapsto x^{|G|/m}$.*
- (iii) *Let $X \in \mathbb{P}^1$ be a regular singular point of the hypergeometric equation. Assume that the local exponent difference at X has denominator k . Then there are $\lfloor m/k \rfloor$ points above X with branching index k , and other points above X are unramified.*

Proof. First we prove this theorem for the Darboux covering $\tilde{\varphi}$ of maximal degree $m = |G|$. The first statement follows classically from [Kle84]. In particular, there is an action of G on $\tilde{D} \cong \mathbb{P}^1$, and the projection $\mathbb{P}^1 \rightarrow \mathbb{P}^1/G \cong \mathbb{P}^1$ is precisely $\tilde{\varphi}$. This map is also the inverse of a Schwarz map for the hypergeometric equation. At each fiber of $\tilde{\varphi}$ the points have the same branching index, since $\mathbb{C}(\tilde{D}) \supset \mathbb{C}[z]$ is a Galois extension. This extension is also the Piccard-Vessiot extension for H , so a suitable pull-back of H with respect to $\tilde{\varphi}$ has trivial Galois group. Hence all exponent differences of any pull-back of H with respect to $\tilde{\varphi}$ are integers. As a consequence, in the situation of part (iii) the branching indices should be integer multiples of k . Since $g(\tilde{D}) = 0$, Hurwitz theorem leaves only one possibility which is described in part (iii). Part (ii) is trivial in the considered case.

Now we consider a general Darboux covering $\varphi : D \rightarrow \mathbb{P}^1$. Let u be a corresponding Riccati solution of degree m , so that $\mathbb{C}(D) \cong \mathbb{C}(z, u)$. Let $K \supset \mathbb{C}(z)$ denote the the Piccard-Vessiot extension of (1.1). We have $K \cong \mathbb{C}(\tilde{D})$ as just above. Consider the tower of field extensions $K \supset \mathbb{C}(D) \supset \mathbb{C}(z)$. Here $K \supset \mathbb{C}(D)$ is the Piccard-Vessiot extension for the differential equation $y' = uy$, so its Galois group must be a cyclic subgroup of G of index m . The existence of the corresponding covering $\gamma : \tilde{D} \rightarrow D$ implies (i). Hurwitz theorem implies that there are exactly two branching points of γ . This implies (ii). We see that the equation $y' = uy$ has exactly two singular points, so in the situation of part (iii) branching indices are equal to either k or 1, and that there in total only two unramified points above the three regular singular points of (1.1). This information gives part (iii). \square

The Darboux coverings for the standard hypergeometric equations can be computed from scratch, by using the branching pattern prescribed by part (iii) of Lemma 3.4, and Algorithm 1 in [Vid05, §3] for example. On the other hand, the coefficients in their rational expressions

are familiar from the (semi)-invariants of the classical action of A_4 , S_4 , A_5 on $\mathbb{C}[x]$ (or on homogeneous polynomials in $\mathbb{C}[x, y]$); see [Kle84].

Explicit expressions for the Darboux coverings of the minimal degree 4, 6 or 12 are evident in formulas (2.1), (2.5), (2.9), respectively. The covering of degree 6 for the standard tetrahedral equation is given by

$$X \mapsto \frac{(x^2 - 6x - 3)^3}{(x^2 + 6x - 3)^3}. \quad (3.3)$$

For the standard octahedral equation, the Darboux covering of degree 8 is given by

$$X \mapsto \frac{(x^2 + 20x - 8)^4}{256x(x+1)^3(x-8)^3}, \quad (3.4)$$

while the covering of degree 12 is given in (3.2). The Darboux coverings of degree 20 and 30 for the standard icosahedral equation are given by the rational functions

$$\frac{64(x^4 + 55x^3 - 165x^2 - 275x + 25)^5}{125x(x^2 + 5x + 40)^3(x^2 - 40x - 5)^3(8x^2 - 5x + 5)^3},$$

$$\frac{27(x^2 + 2x + 5)^5(x^4 + 20x^3 - 210x^2 + 100x + 25)^5}{(3x^2 - 10x + 15)^3(x^4 + 70x^2 + 25)^3(x^4 - 60x^3 - 370x^2 - 300x + 25)^3}. \quad (3.5)$$

The Darboux coverings of the maximal degree $D \in \{12, 24, 60\}$ can be obtained by composing an indicated Darboux covering of smaller degree m with the cyclic covering $x \mapsto x^{D/m}$. Different rational expressions for the maximal Darboux coverings are related to each other by Möbius transformations, though the coefficients are cumbersome algebraic numbers. For example, to transform $x^3(x^3 + 4)^3/4(2x^3 - 1)^3$ to the composition of (3.3) with $x \mapsto x^2$, a Möbius transformation is

$$x \mapsto \sqrt[3]{5 - 3\sqrt{3}} \frac{x + \sqrt{3 + 2\sqrt{3}}}{x - \sqrt{-3 + 2\sqrt{3}}}.$$

3.3 Darboux curves for other Schwarz types

The following lemma allows us to compute all necessary Darboux coverings once Darboux coverings for standard hypergeometric equations (from §3.2) are known. For this purpose we usually take E_1 to be a main representative equation of a chosen Schwarz type, and we usually take E_0 to be the corresponding standard equation.

Lemma 3.5 *Let E_0 , E_1 denote two linear differential equations. Suppose that E_1 is a pull-back of E_0 with respect to a covering $\psi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Suppose that $\phi_0 : D_0 \rightarrow \mathbb{P}^1$ is a Darboux covering for E_0 . Then the fiber product D_1 of $\psi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and $\phi_0 : D_0 \rightarrow \mathbb{P}^1$ is a Darboux curve for E_1 , and the projection $\phi_1 : D_1 \rightarrow \mathbb{P}^1$ (to the source of ψ) is a Darboux covering for E_1 of the same degree as ϕ_0 .*

Proof. Let z denote a projective parameter for the \mathbb{P}^1 below, and let x denote a projective parameter of the other \mathbb{P}^1 , so that the covering ψ corresponds to the extension $\mathbb{C}(x) \supset \mathbb{C}(z)$.

Denominators k, ℓ, m	Schwarz type	Klein degree	$g(D)$, when $\deg \gamma : \tilde{D} \rightarrow D$ is equal to			
			m	ℓ	k	1
2, 3, 3	(1/2, 1/3, 1/3)	1	0	0	0	0
	(1/3, 1/3, 2/3)	2	0	0	1	1
2, 3, 4	(1/2, 1/3, 1/4)	1	0	0	0	0
	(2/3, 1/4, 1/4)	2	0	1	2	3
2, 3, 5	(1/2, 1/3, 1/5)	1	0	0	0	0
	(1/2, 1/3, 2/5)	7	0	0	0	0
	(1/2, 1/5, 2/5)	3	0	2	2	4
	(1/3, 1/3, 2/5)	2	1	1	3	5
	(1/3, 2/3, 1/5)	6	1	1	3	5
	(2/3, 1/5, 1/5)	2	1	3	5	9
	(1/3, 2/5, 3/5)	10	1	3	5	9
	(1/3, 1/5, 3/5)	4	1	3	5	9
	(1/5, 1/5, 4/5)	6	1	5	7	13
	(2/5, 2/5, 2/5)	6	1	5	7	13

Table 1: Genus of Darboux curves

Suppose that u_0 be a Riccati solution for E_0 which determines D_0 , so that $\mathbb{C}(D_0) \cong \mathbb{C}(z, u_0)$. Then there is a solution y for E_0 which satisfies $y' = u_0 y$. The logarithmic derivative of the pull-back of y with respect to (5.3) is $u_0 \psi' + \theta'/\theta$. It is a Riccati solution for E_1 , and it lies in $\mathbb{C}(x, u_0) \cong \mathbb{C}(x) \otimes_{\mathbb{C}(z)} \mathbb{C}(x, u_0)$ which is the function field for the fiber product of ψ and ϕ_0 . Since the degree of the projection to \mathbb{P}^1 is equal to the degree of ψ , the claim follows. \square

The simplest coverings for Klein pull-backs ψ from standard hypergeometric equations to other main representatives were first computed in [Sch72]. These Klein coverings are familiar from classical transformations of hypergeometric series (see for instance [AAR99] or [Vid09, §5]), except the ones for the Schwarz types $(1/2, 1/3, 2/5)$, $(1/3, 2/3, 1/5)$, $(1/3, 2/5, 3/5)$. Here are rational functions that define Klein pull-backs for their main representatives:

$$\frac{x^2(189 - 64x)^5}{(3584x^2 + 2457x - 2916)^3}, \quad \frac{4x(25x - 9)^5}{27(x - 1)(125x + 3)^3}, \quad \frac{3125x^2(x - 1)^3(5x + 27)^5}{4(625x^3 - 2875x^2 + 675x - 729)^3}. \quad (3.6)$$

Table 1 shows the genus of all Darboux curves; this information can be computed using Lemma 3.5, part (iii) of Lemma 3.4, remarks in Appendixes 5.7 and 5.2. The third column shows the degree of Klein morphism for the main representatives of each Schwarz type. We explicitly compute and utilize Darboux coverings of minimal degree.

3.4 Hypergeometric functions on Darboux curves

The purpose of this subsection is to describe technical features of identifying and representing radical functions. First we describe radical solutions of a Fuchsian equation with cyclic monodromy.

Lemma 3.6 *Let C be an algebraic curve. Consider a second order Fuchsian differential equation (3.1) on C . Suppose that its differential Galois group G is finite and cyclic. Suppose that there are singular points with non-integer local exponent differences. Then:*

- (i) *There exist exactly two independent radical solutions f_1, f_2 .*
- (ii) *For any regular singular point $P \in C$ of (3.1) where the exponent difference is not an integer, the local exponents of f_1 and f_2 are different.*

Proof. Let $K_{PV} \supset \mathbb{C}(C)$ be the Piccard-Vessiot extension of the differential equation (3.1), and let V denote its space of solutions in K_{PV} . The monodromy group G acts on V by scalar multiplication, because otherwise the quotient of two independent solutions would be in $\mathbb{C}(C)$. Hence the representation of G on V splits into two irreducible representations. Let $f_1, f_2 \in V$ be generators of the two G -invariant subspaces. If $P \in C$ is a point where the exponent difference is not an integer, f_1, f_2 are in different spaces invariant under the local monodromy group. Hence they have well-defined local exponents, which should be different. \square

Hypergeometric functions which pull-back to radical solutions under a Darboux covering are characterized as follows.

Lemma 3.7 *Let H denote a hypergeometric equation. Suppose that its monodromy G is tetrahedral, octahedral or icosahedral. Let $\phi : D \rightarrow \mathbb{P}^1$ denote a Darboux morphism for H , of degree d . Let $\theta(z) {}_2F_1\left(\begin{smallmatrix} A, B \\ C \end{smallmatrix} \middle| f(z)\right)$ denote a hypergeometric solution of H , with $f(z)$ a fractional-linear function, and $\theta(z)$ a radical factor. Assume that the denominator of the lower parameter C is equal to $|G|/d$. Then the hypergeometric function is pull-backed to a radical function. Conversely, each radical solution of the pull-backed equation represents (up to a scalar multiple) a hypergeometric equation with the assumed property.*

Proof. Set $m = |G|/d$. Suppose that the pull-backed equation is normalized so that its monodromy is the cyclic group of order m . Then all non-integer local exponents of the pull-backed equation have denominator m . We may assume that $f(z) = z$. There is a point P above $z = 0$ such that the exponent difference δ for the pull-backed equation at P is non-integer. The denominator of δ is m . Let λ_1, λ_2 denote the local exponents at P , and let t denote a local parameter at P . For each local exponent λ_j there is a unique power series solution of the form $t^{\lambda_j}(1 + \alpha_1 t + \alpha_2 t + \dots)$. By part (ii) of Lemma 3.6, both power series represent radical functions.

The hypergeometric function is pull-backed, up to a constant multiple, to one of the two mentioned power series. Hence the pull-back is a radical function. Conversely, push-forwards of the two power series must be hypergeometric series. \square

Lemma 3.8 *Let F_1, F_2 denote two contiguous algebraic (but not rational) Gauss hypergeometric functions. Let D denote their common Darboux curve. Then the pull-back of F_1/F_2 is a rational function on D .*

Proof. The logarithmic derivative F'_1/F_1 is a rational function on D . Up to a factor in $C(D)$, F'_1 is Gauss hypergeometric function contiguous to F_1 . The contiguous relation between F_1, F'_1 and F_2 has coefficients in $\mathbb{C}(x)$, hence the claim follows. \square

If we take any pair of contiguous evaluations in §2, the quotient of the right-hand sides is a rational (rather than radical) function on the Darboux curve, as suggested by Lemma 3.8. This allows us to express other contiguous evaluations conveniently as product of a fixed radical function and some rational function on the Darboux curve.

4 Computation of Darboux evaluations

This is an outline of actual computations that led us to the list of Darboux evaluations in §2. If the Darboux curve has genus 0, computations are quite straightforward. Subsections 4.2 and 4.3 are devoted to difficulties of expressing rational and radical functions on genus 1 Darboux curves.

4.1 Computation of Darboux curves and coverings

Darboux coverings for the three standard Schwarz types are considered in §3.2. To compute other Darboux coverings and curves, Lemma 3.5 is used. Let us fix an Schwarz type which is not standard, and let E_1 denote its representative hypergeometric equation as listed in §1. Suppose that $\varphi : \mathbb{P}^1_z \mapsto \mathbb{P}^1_Z$ is a Darboux covering (of minimal degree) for the corresponding standard hypergeometric equation. Suppose that $\psi : \mathbb{P}^1_X \mapsto \mathbb{P}^1_Z$ is a Klein covering for E_1 . Then the Darboux curve is a fiber product of φ and ψ . An equation for it is given by the equation $\psi(X) = \phi(z)$; see §5.7.

If the Darboux curve has genus 0, then a parameterization of it immediately gives the Darboux covering. As an example, consider the icosahedral type $(1/2, 1/3, 2/5)$. We apply Lemma 3.5 with $E_0 = E(1/5, 1/2, 1/3)$ and $E_1 = E(2/5, 1/2, 1/3)$. By formulas (2.9) and (3.6) we get the following equation for the Darboux curve:

$$\frac{X^2(189 - 64X)^5}{(3584X^2 + 2457X - 2916)^3} = \frac{1728z(z^2 - 11z - 1)^5}{(z^4 + 228z^3 + 494z^2 - 228z + 1)^3}. \quad (4.1)$$

This is a rational curve. It can be parameterized by standard algorithms and computer algebra systems such as **Maple**. Here is a parameterization:

$$X = \frac{1728x(x^2 - 11x - 1)^5}{(x^4 + 228x^3 + 494x^2 - 228x + 1)^3}, \quad z = \frac{(7x - 1)^5}{x^2(x + 7)^5}. \quad (4.2)$$

(Recall that parameterizations are unique up to fractional-linear transformations on \mathbb{P}^1_x .) The parametric expression for X gives the Darboux covering for $E(2/5, 1/2, 1/3)$. We recognize that this is the same Darboux covering (2.9) as for the Schwarz type $(1/2, 1/3, 1/5)$.

With a computer algebra package and standard algorithms [vH94], [Kov86] at hand, it is straightforward to pull-back a hypergeometric equation to a Darboux curve of genus 0, and solve the pull-backed differential equation. Lemma 3.7 characterizes hypergeometric functions which have to be identified with radical solutions. Since there are only 2 radical solutions by Lemma 3.6, computer algebra systems should return them. (Otherwise we may consider a simplified version of the procedure in §4.3.)

If the Darboux curve D has genus 1, we first wish to compute a convenient Weierstrass model from the equation $\psi(X) = \varphi(z)$. With such a model at hand, we identify D with the elliptic curve (D, \mathcal{O}) , where \mathcal{O} denotes the point at infinity. We always seek to have the point $(x, \xi) = (0, 0)$ on D above $X = 0$, so to allow easy power series verification of the evaluations in (2.19)–(2.38).

The Darboux covering is given by the X -component of an isomorphism between the elliptic curve and the model $\psi(X) = \varphi(z)$. We wish to express the covering function in such a way that multiplicities of its zeroes and poles would be well visible. This is not a straightforward problem; it is discussed in §4.2. Pull-backs of hypergeometric equations onto elliptic curves and finding radical solutions of those pull-backs are discussed in §4.3. In the rest of this subsection, we derive the four elliptic curves C_3, C_4, C_5, C_6 , introduced in (2.17), (2.23), (2.29), (2.33) as Darboux curves for some icosahedral Schwarz types.

For the Schwarz type $(1/3, 1/3, 2/5)$, we use Lemma 3.5 with $E_0 = E(1/5, 1/2, 1/3)$ and $E_1 = E(2/5, 1/3, 1/3)$. This gives the equation $X^2/4(X-1) = \varphi_1(z)$ for the Darboux curve. After applying the fractional-linear transformation $F \mapsto F/(F-1)$ to both sides we get:

$$\frac{X^2}{(X-2)^2} = -\frac{1728 z (z^2 - 11z - 1)^5}{(z^2 + 1)^2 (z^4 - 522z^3 - 10006z^2 + 522z + 1)^2}. \quad (4.3)$$

We collect full squares onto the left-hand side and observe that the Darboux curve is isomorphic to the genus 1 curve

$$\tilde{\xi}^2 = -1728 z (z^2 - 11z - 1). \quad (4.4)$$

This curve is isomorphic to C_3 via the isomorphism $(z, \tilde{\xi}) \mapsto (3x, 72\xi)$. The Darboux covering is given by the X -component of the isomorphism between C_3 and (4.3). We have

$$\frac{X}{X-2} = \frac{72 \xi (9x^2 - 33x - 1)^2}{(9x^2 + 1)(81x^4 - 14094x^3 - 90054x^2 + 1566x + 1)},$$

so

$$\varphi_3(x, \xi) = 2 / \left(1 + \frac{(9x^2 + 1)(81x^4 - 14094x^3 - 90054x^2 + 1566x + 1)}{72 \xi (9x^2 - 33x - 1)^2} \right). \quad (4.5)$$

Expression (2.18) is derived by methods of §4.2.

For the Schwarz type $(1/3, 2/3, 1/5)$, we use Lemma 3.5 with $E_0 = E(2/5, 1/2, 1/3)$ and $E_1 = E(4/5, 1/3, 1/3)$. Then we get the same equation $X^2/4(X-1) = \varphi_1(z)$. Hence the Darboux curve is the same as for the type $(1/3, 1/3, 2/5)$.

For the Schwarz type $(2/3, 1/5, 1/5)$, we use Lemma 3.5 with $E_0 = E(1/5, 1/2, 1/3)$ and $E_1 = E(1/5, 1/5, 2/3)$. This gives the equation $4X(1-X) = \varphi_1(z)$ for the Darboux curve. After applying the fractional-linear transformation $F \mapsto 1-F$ to both sides we get:

$$(1-2X)^2 = \frac{(z^2 + 1)^2 (z^4 - 522z^3 - 10006z^2 + 522z + 1)^2}{(z^4 + 228z^3 + 494z^2 - 228z + 1)^3}. \quad (4.6)$$

We collect full squares to the left-hand side and observe that the Darboux curve is isomorphic to the genus 1 curve

$$\tilde{\xi}^2 = z^4 + 228z^3 + 494z^2 - 228z + 1. \quad (4.7)$$

This curve is isomorphic to C_4 by the isomorphism

$$(z, \tilde{\xi}) \mapsto \left(\frac{57x - 5\xi}{x + 25}, \frac{25(\xi^2 - 570\xi - 380x^2 + 248x + 25)}{(x + 25)^2} \right). \quad (4.8)$$

Like with $\varphi_3(x, \xi)$, we identify $\varphi_4(x, \xi)$ with the X -component of the isomorphism between C_4 and (4.6), and apply methods of Lemma 4.2 to get expression (2.24).

For the Schwarz type $(1/3, 2/5, 3/5)$, we use Lemma 3.5 with $E_0 = E(2/5, 1/2, 1/3)$ and $E_1 = E(2/5, 2/5, 2/3)$. Then we get the same equation $4X(1 - X) = \varphi_1(z)$. Hence the Darboux curve is the same as for the type $(2/3, 1/5, 1/5)$.

For the Schwarz type $(1/3, 1/5, 3/5)$, we use Lemma 3.5 with $E_0 = E(1/5, 1/2, 1/3)$ and $E_1 = E(1/5, 1/3, 3/5)$. This gives the equation $-64X/(X - 1)(9X - 1)^3 = \varphi_1(z)$ for the Darboux curve. This curve is isomorphic to C_5 , though it is not straightforward to compute a handy isomorphism with current computer algebra packages. The package `algcures` of **Maple 9.0** can be used to obtain a Weierstrass form and an isomorphism. The isomorphism ought to be simplified using methods of §4.2. Eventually, we obtain an isomorphism given by $z = -2(\xi - 3x)^2/(\xi + 3x)(4x + 1)$ and $X = \varphi_5(x, \xi)$ as in (2.30).

For the Schwarz type $(1/5, 1/5, 4/5)$, we use Lemma 3.5 with $E_0 = E(1/5, 1/2, 2/5)$ and $E_1 = E(1/5, 1/5, 4/5)$. This gives the equation $4X(1 - X) = \varphi_2(z)$ for the Darboux curve. After applying the fractional-linear transformation $F \mapsto 1 - F$ to both sides we get

$$(1 - 2X)^2 = -\frac{64x(x^2 - x - 1)^5}{(x^2 + 1)^2(x^4 - 22x^3 - 6x^2 + 22x + 1)^2}. \quad (4.9)$$

We collect full squares to the left-hand side and easily observe that the Darboux curve is isomorphic C_6 .

For the Schwarz type $(2/5, 2/5, 2/5)$, we use Lemma 3.5 with $E_0 = E(2/5, 1/2, 1/5)$ and $E_1 = E(2/5, 2/5, 2/5)$. Then we get the same equation $4X(1 - X) = \varphi_2(z)$. Hence the Darboux curve is the same as for the type $(1/5, 1/5, 4/5)$.

Notice that the elliptic curves C_3, C_4, C_5, C_6 are defined over \mathbb{Q} . It is useful to know rational points on them. Table 2 gives this arithmetic information [Sil86]. It was computed using **Maple** package `Apeps` [Con]. Recall that by \mathcal{O} we denote the point at infinity. As we see, only the curve C_4 has infinitely many rational points. In Table 2 we introduce the notation $\mathcal{O}^*, A_n, \tilde{A}_n, A_n^*, \tilde{A}_n^*$ (with positive $n \in \mathbb{Z}$) for the rational points on C_4 .

4.2 Representing functions on genus 1 curves

Here we consider the problem of representation of rational functions on elliptic curves. Foremost, we use techniques of this subsection to compute expressions (2.18), (2.24), (2.30), (2.34) for Darboux coverings from genus 1 Darboux curves. Subsection 4.3 extends these techniques for computation of expressions on the right-hand sides of (2.19)–(2.22), (2.25)–(2.28), (2.31)–(2.32), (2.35)–(2.38).

Elliptic curve	Mordell-Weil group	Rational points
C_3	$\mathbb{Z}/6\mathbb{Z}$	$\mathcal{O}, (0, 0), (-\frac{1}{9}, \frac{5}{9}), (-\frac{1}{9}, -\frac{5}{9}), (1, -5), (1, 5)$
C_4	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathcal{O}, \mathcal{O}^* = (0, 0), A_n = n(\frac{1}{5}, \frac{3}{5}), \tilde{A}_n = -n(\frac{1}{5}, \frac{3}{5}),$ $A_n^* = n(\frac{1}{5}, \frac{3}{5}) + \mathcal{O}^*, \tilde{A}_n^* = -n(\frac{1}{5}, \frac{3}{5}) + \mathcal{O}^*$
C_5	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathcal{O}, (0, 0), (-\frac{1}{4}, -\frac{3}{4}), (-\frac{1}{4}, -\frac{3}{4}),$ $(-1, 0), (-\frac{1}{16}, 0), (\frac{1}{4}, -\frac{5}{4}), (\frac{1}{4}, \frac{5}{4})$
C_6	$\mathbb{Z}/6\mathbb{Z}$	$\mathcal{O}, (0, 0), (-1, 1), (-1, -1), (1, -1), (1, 1)$

Table 2: Rational points on elliptic curves

A canonical way to represent a rational functions F on a (hyper)elliptic curve $\xi^2 = G(x)$ (with $G(x) \in \mathbb{C}[x]$) is the sum $f_1(x) + \xi f_2(x)$, with $f_1(x), f_2(x) \in \mathbb{C}(x)$. This representation suits well algebraic computations, but it gives little geometric information about the function. For example, the principal divisor for a function can be much simpler than the degree of $f_1(x)$ and $f_2(x)$ may suggest. We would like to have a compact expression that reflects well multiplicities in the principal divisor. We do not give strict definitions or algorithms for an alternative representation. Rather, we give tables of principal divisors on the elliptic curves C_3, C_4, C_5, C_6 , and propose to combine those principal divisors to make the divisor for F . The corresponding multiplicative expression in $\mathbb{C}[x, \xi]$ -polynomials from our tables will give, up to a constant multiple, a compact expression for F that we seek. We need to compute only finitely many rational and radical functions on elliptic curves, and our tables give enough information for these purposes.

Concretely, we start with Darboux covering $\varphi_3(x, \xi)$. Its principal divisor can be computed from (4.5) to be the following:

$$(0, 0) + \mathcal{O} + 5 \left(\frac{11+5\sqrt{5}}{6}, 0 \right) + 5 \left(\frac{11-5\sqrt{5}}{6}, 0 \right) - 3R_1 - 3R_2 - 3R_3 - 3R_4. \quad (4.10)$$

Here the four points R_1, R_2, R_3, R_4 are defined by the equations

$$81x^4 + 6156x^3 + 4446x^2 - 684x + 1 = 0, \quad 150\xi = 27x^3 + 1989x^2 + 741x - 7. \quad (4.11)$$

Table 3 gives a list of principal divisors on C_3 . For $i \in \{1, 2, 3, 4\}$, by \tilde{R}_i we denote the inverse of R_i in the group structure of C_3 . Divisor (4.10) can be rewritten as follows:

$$\begin{aligned} & \left\{ (0, 0) + \left(\frac{11+5\sqrt{5}}{6}, 0 \right) + \left(\frac{11-5\sqrt{5}}{6}, 0 \right) - 3\mathcal{O} \right\} + 2 \left\{ 2 \left(\frac{11+5\sqrt{5}}{6}, 0 \right) + 2 \left(\frac{11-5\sqrt{5}}{6}, 0 \right) - 4\mathcal{O} \right\} \\ & + \left\{ 3 \left(-\frac{1}{9}, -\frac{5}{9} \right) - 3\mathcal{O} \right\} - 3 \left\{ R_1 + R_2 + R_3 + R_4 + \left(-\frac{1}{9}, -\frac{5}{9} \right) - 5\mathcal{O} \right\}. \end{aligned}$$

Observe that each divisor in curled brackets is present in Table 3. We can immediately build the corresponding multiplicative combination of the functions $\xi, 1 + 33x - 9x^2, 1 - 9\xi + 54x, 1 + 21\xi - 117x + 9x\xi - 234x^2$. Up to undetermined constant multiple, the multiplicative

Function	Divisor
ξ	$(0, 0) + \left(\frac{11+5\sqrt{5}}{6}, 0\right) + \left(\frac{11-5\sqrt{5}}{6}, 0\right) - 3\mathcal{O}$
$1 + 33x - 9x^2$	$2\left(\frac{11+5\sqrt{5}}{6}, 0\right) + 2\left(\frac{11-5\sqrt{5}}{6}, 0\right) - 4\mathcal{O}$
$1 - 9\xi + 54x$	$3\left(-\frac{1}{9}, -\frac{5}{9}\right) - 3\mathcal{O}$
$1 + 9\xi + 54x$	$3\left(-\frac{1}{9}, \frac{5}{9}\right) - 3\mathcal{O}$
$1 + 9x$	$\left(-\frac{1}{9}, -\frac{5}{9}\right) + \left(-\frac{1}{9}, \frac{5}{9}\right) - 2\mathcal{O}$
$\xi + 5x$	$(0, 0) + \left(-\frac{1}{9}, \frac{5}{9}\right) + (1, -5) - 3\mathcal{O}$
$1 + 21\xi - 117x + 9x\xi - 234x^2$	$R_1 + R_2 + R_3 + R_4 + \left(-\frac{1}{9}, -\frac{5}{9}\right) - 5\mathcal{O}$
$1 - 21\xi - 117x - 9x\xi - 234x^2$	$\tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 + \tilde{R}_4 + \left(-\frac{1}{9}, \frac{5}{9}\right) - 5\mathcal{O}$

Table 3: Principal divisors on C_3

expression is (2.18). The constant multiple can be determined by evaluating the multiplicative expression and (4.5) a convenient point, say $(-\frac{1}{9}, \frac{5}{9})$.

As an extra exercise, one may consider the function $1 - \varphi_3$. Its divisor can be computed from (2.18) or (4.5) to be

$$3\tilde{R}_1 + 3\tilde{R}_2 + 3\tilde{R}_3 + 3\tilde{R}_4 - 3R_1 - 3R_2 - 3R_3 - 3R_4. \quad (4.12)$$

A straightforward combinatorial work suggests the expression

$$1 - \varphi_3(x, \xi) = \frac{(1 - 21\xi - 117x - 9x\xi - 234x^2)^3 (1 - 9\xi + 54x)}{(1 + 21\xi - 117x + 9x\xi - 234x^2)^3 (1 + 9\xi + 54x)}. \quad (4.13)$$

Our proposal boils down in building a sufficient table of principal divisors, and combining the known principal divisors to arrive at the principal divisor of a target function. In practise, both things are done in parallel. We start with the functions we wish to express, and compute their divisors. We look at \mathbb{Q} -rational points that occur, and use knowledge of the Mordell-Weil group (see Table 2) to foresee and compute suitable $\mathbb{C}[x, \xi]$ -polynomials that vanish only on rational points. Then we distinguish \mathbb{Q} -irreducible divisor components of higher degree. For each such divisor component Γ , we use Gröbner bases to find $\mathbb{C}[x, \xi]$ -polynomials of minimal degree that vanish on Γ with sufficient multiplicities. We choose those polynomials whose divisors enlarge Γ minimally or least awkwardly. We look at additional components that occur (usually they are rational points); if they are new, we introduce new polynomials that could compensate the additional components.

For example, consider the \mathbb{Q} -irreducible component $R_1 + R_2 + R_3 + R_4$ defined by (4.11). A Gröbner basis gives the following quadratic polynomials that vanish on it:

$$9x\xi - 234x^2 + 21\xi - 117x + 1, \quad 3\xi^2 - 2088x^2 + 150\xi - 744x + 7. \quad (4.14)$$

Other quadratic polynomials are obtained by linear combination. We have chosen the first polynomial in (4.14), and consequently we had to compensate its additional component

$(-\frac{1}{9}, -\frac{5}{9})$. A reasonable alternative is the quadratic polynomial $\xi^2 - 21x\xi - 150x^2 + \xi + 25x$, whose divisor is $R_1 + R_2 + R_3 + R_4 + (0, 0) + (1, -5) - 6\mathcal{O}$.

Next consider computation of expression (2.24) for the Darboux covering $\varphi_4(x, \xi)$. This is the most complicated case, so our description of the computational method reaches deeper refinement level. Let us introduce the functions N and L on the rational points of C_4 :

$$\begin{aligned} N(A_n) = n, \quad N(A_n^*) = n, \quad N(\tilde{A}_n) = -n, \quad N(\tilde{A}_n^*) = -n, \quad N(\mathcal{O}^*) = 0, \\ L(A_n) = 0, \quad L(A_n^*) = 1, \quad L(\tilde{A}_n) = 0, \quad L(\tilde{A}_n^*) = 1, \quad L(\mathcal{O}^*) = 1. \end{aligned} \quad (4.15)$$

Principal divisors of $\mathbb{C}[x, \xi]$ -polynomials have the form $\sum_{j=1}^n S_j - n\mathcal{O}$. If all points S_j are rational, by Lemma 5.1 we must have

$$\sum_{j=1}^n N(S_j) = 0 \quad \text{and} \quad \sum_{j=1}^n L(S_j) \quad \text{even.} \quad (4.16)$$

A preliminary expression for $\varphi_4(x, \xi)$ can be computed by composing the obvious isomorphism between the curves in (4.6) and (4.7) with isomorphism (4.8). The principal divisor of $\varphi_4(x, \xi)$ is:

$$(0, 0) + \mathcal{O} + 5P_1 + 5P_2 - \left(\frac{1}{125}, \frac{57}{625}\right) - (-25, 285) - 5Q_1 - 5Q_2, \quad (4.17)$$

where

$$P_{1,2} = \left(-\frac{3}{2} \pm \frac{7}{2\sqrt{5}}, 7 \mp 3\sqrt{5}\right), \quad Q_{1,2} = \left(-4 \pm \frac{9}{\sqrt{5}}, -27 \pm 12\sqrt{5}\right).$$

In the notation of Table 2, we have $(\frac{1}{125}, \frac{57}{625}) = A_5$ and $(-25, 285) = A_5^*$.

It seems convenient to consider the lines through P_1, P_2 and through Q_1, Q_2 . Their equations are $7\xi = 4 - 30x$ and $3\xi = 20x - 1$, respectively. The third points on these two lines are, respectively, \tilde{A}_6^* and \tilde{A}_4^* . But after adding or subtracting extra divisor terms $5\tilde{A}_6^*, 5\tilde{A}_4^*$ in (4.17), it is very cumbersome to compensate them due to (4.16). The $\mathbb{C}[x, \xi]$ polynomials for compensating principal divisors are expected to have very large coefficients.

Rather than introducing the sub-expression $P_1 + P_2 + \tilde{A}_6^* - 3\mathcal{O}$ in (4.17), we may try to work with the principal divisor $P_1 + P_2 + \tilde{A}_4^* + \tilde{A}_2 - 4\mathcal{O}$. Then \tilde{A}_4^* is compensated automatically. Keeping in mind that the divisors $P_1 + P_2$ and $Q_1 + Q_2$ are equivalent (respectively) to A_6^* and A_4^* for the purposes of restrictions (4.16), we work out the following expressions of (4.17) as sums of principal divisors:

$$\begin{aligned} & 5 \left(P_1 + P_2 + \tilde{A}_4^* + \tilde{A}_2 - 4\mathcal{O} \right) + \left(A_5 + \tilde{A}_5^* + \mathcal{O}^* - 3\mathcal{O} \right) \\ & - 5 \left(Q_1 + Q_2 + \tilde{A}_4^* - 3\mathcal{O} \right) - \left(5\tilde{A}_2 + 2A_5 - 7\mathcal{O} \right) - \left(A_5^* + \tilde{A}_5^* - 2\mathcal{O} \right), \end{aligned}$$

and

$$\begin{aligned} & 5 \left(P_1 + P_2 + \tilde{A}_4^* + \tilde{A}_2 - 4\mathcal{O} \right) + \left(A_{10} + \tilde{A}_{10}^* + \mathcal{O}^* - 3\mathcal{O} \right) \\ & - 5 \left(Q_1 + Q_2 + \tilde{A}_4^* - 3\mathcal{O} \right) - \left(5\tilde{A}_2 + A_{10} - 6\mathcal{O} \right) - \left(A_5 + A_5^* + \tilde{A}_{10}^* - 3\mathcal{O} \right). \end{aligned}$$

Function	Divisor	Function	Divisor
x	$2\mathcal{O}^* - 2\mathcal{O}$	$1 - 15x - 5x^2$	$P_1 + P_2 + \tilde{P}_1 + \tilde{P}_2 - 4\mathcal{O}$
$1 - 5x$	$A_1 + \tilde{A}_1 - 2\mathcal{O}$	$1 - 40x - 5x^2$	$Q_1 + Q_2 + \tilde{Q}_1 + \tilde{Q}_2 - 4\mathcal{O}$
$25 + x$	$A_5^* + \tilde{A}_5^* - 2\mathcal{O}$	$5 - 7\xi - 45x - 5x^2$	$P_1 + P_2 + \tilde{A}_5^* + \tilde{A}_1 - 4\mathcal{O}$
$1 - 125x$	$A_5 + \tilde{A}_5 - 2\mathcal{O}$	$5 + 18\xi - 80x + 5x^2$	$Q_1 + Q_2 + \tilde{A}_5^* + A_1 - 4\mathcal{O}$
$5\xi + 57x$	$A_5^* + \tilde{A}_5 + \mathcal{O}^* - 3\mathcal{O}$	$1 - 7y + 15x + 15x^2$	$P_1 + P_2 + \tilde{A}_4^* + \tilde{A}_2 - 4\mathcal{O}$
$-5\xi + 57x$	$\tilde{A}_5^* + A_5 + \mathcal{O}^* - 3\mathcal{O}$	$4 - 7\xi - 30x$	$P_1 + P_2 + \tilde{A}_6^* - 3\mathcal{O}$
$1 + 5\xi + 10x$	$2\tilde{A}_1 + A_2 - 3\mathcal{O}$	$1 + 3\xi - 20x$	$Q_1 + Q_2 + \tilde{A}_4^* - 3\mathcal{O}$
$1 - 3\xi + 2x$	$A_2 + A_2^* + \tilde{A}_4^* - 3\mathcal{O}$	$1 - 8\xi + 22x - 15x^2$	$2A_1 + A_2^* + \tilde{A}_4^* - 4\mathcal{O}$
$4 + 21\xi + 41x$	$A_2 + A_4^* + \tilde{A}_6^* - 3\mathcal{O}$	$4 - 35\xi - 101x$	$A_1^* + A_5 + \tilde{A}_6^* - 3\mathcal{O}$
$5 - 3\xi - 34x$	$A_5^* + \tilde{A}_4^* + \tilde{A}_1 - 3\mathcal{O}$	$20 - 7\xi - 79x$	$A_1 + A_5^* + \tilde{A}_6^* - 3\mathcal{O}$

Function	Divisor
$1 + 50x - 125\xi^2 + 450x\xi - 500x^2$	$5A_1 + \tilde{A}_5 - 6\mathcal{O}$
$1 + 50x - 125\xi^2 - 450x\xi - 500x^2$	$5\tilde{A}_1 + A_5 - 6\mathcal{O}$
$25 - 570\xi + 248x + \xi^2 - 380x^2$	$R_1 + R_2 + R_3 + R_4 + 2\tilde{A}_5^* - 6\mathcal{O}$
$4 + 95\xi + 83x + 21\xi^2 - 475x\xi + 40x^2$	$R_1 + R_2 + R_3 + R_4 + \tilde{A}_6^* + \tilde{A}_4^* - 6\mathcal{O}$

Table 4: Principal divisors on C_4

Function	Divisor
$\xi + 5x$	$2\left(\frac{1}{4}, -\frac{5}{4}\right) + (0, 0) - 3\mathcal{O}$
$\xi - 5x$	$2\left(\frac{1}{4}, \frac{5}{4}\right) + (0, 0) - 3\mathcal{O}$
$\xi + 3x$	$2\left(-\frac{1}{4}, \frac{3}{4}\right) + (0, 0) - 3\mathcal{O}$
$\xi - 3x$	$2\left(-\frac{1}{4}, -\frac{3}{4}\right) + (0, 0) - 3\mathcal{O}$
$1 + \xi + x$	$\left(-\frac{1}{4}, -\frac{3}{4}\right) + \left(\frac{1}{4}, -\frac{5}{4}\right) + (-1, 0)$
$1 - 2\xi + 6x$	$P_1 + P_2 + \left(\frac{1}{4}, \frac{5}{4}\right) - 3\mathcal{O}$
$1 + 12x + 16x^2$	$P_1 + P_2 + \tilde{P}_1 + \tilde{P}_2 - 4\mathcal{O}$
$1 - 2\xi - 14x$	$Q_1 + Q_2 + \left(\frac{1}{4}, -\frac{5}{4}\right) - 3\mathcal{O}$
$1 - 28x + 16x^2$	$Q_1 + Q_2 + \tilde{Q}_1 + \tilde{Q}_2 - 4\mathcal{O}$
$1 + 8\xi - 28x + 8x\xi - 104x^2$	$R_1 + R_2 + R_3 + R_4 + \left(\frac{1}{4}, \frac{5}{4}\right) - 5\mathcal{O}$

Table 5: Divisors on C_5

Now we can build a table of $\mathbb{C}[x, \xi]$ -polynomials of the involved principal divisors. Eventually, the two decompositions of (4.17) give the following expressions for $\varphi_4(x, \xi)$:

$$\frac{8208(1-7\xi+15x+15x^2)^5(-5\xi+57x)}{(1+3\xi-20x)^5(59375x^2\xi+12350x\xi+475\xi-166250x^3-49875x^2-3800x-19)(x+25)},$$

and, respectively,

$$\frac{432(1-7\xi+15x+15x^2)^5(49495\xi+292441x)}{(1+3\xi-20x)^5(2375\xi^2+7400x\xi+1150\xi-42000x^2-10200x-521)(2605\xi+29678x-475)}.$$

Compensation of $5\tilde{A}_2$ look quite awkward in both formulas.

As the last attempt, we introduce $P_1 + P_2 + \tilde{A}_5^* + \tilde{A}_1 - 4\mathcal{O}$ and $Q_1 + Q_2 + \tilde{A}_5^* + A_1 - 4\mathcal{O}$ in (4.17), forgetting divisors of the linear polynomials $7\xi + 30x - 4$ and $3\xi - 20x + 1$. A natural effort to compensate $5\tilde{A}_1$ and $5A_1$ leads to the following expression decomposition of (4.17):

$$\begin{aligned} & 5\left(P_1 + P_2 + \tilde{A}_5^* + \tilde{A}_1 - 4\mathcal{O}\right) + \left(5A_1 + \tilde{A}_5 - 6\mathcal{O}\right) + (2\mathcal{O}^* - 2\mathcal{O}) \\ & - 5\left(Q_1 + Q_2 + \tilde{A}_5^* + A_1 - 4\mathcal{O}\right) - \left(5\tilde{A}_1 + A_5 - 6\mathcal{O}\right) - \left(A_5^* + \tilde{A}_5 + \mathcal{O}^* - 3\mathcal{O}\right). \end{aligned}$$

This expression gives formula (2.24). The most convenient principal divisors are listed in Table 4. The points R_1, R_2, R_3, R_4 on C_4 are the points in the fiber $z = 1$ of $\varphi_4(x, \xi)$.

The principal divisor for $\varphi_5(x, \xi)$ on C_5 turns out to be

$$(0, 0) + \mathcal{O} + 5P_1 + 5P_2 - \left(-\frac{1}{4}, \frac{3}{4}\right) - \left(-\frac{1}{4}, -\frac{3}{4}\right) - 5Q_1 - 5Q_2, \quad (4.18)$$

where

$$P_{1,2} = \left(\frac{-3 \pm \sqrt{5}}{8}, \frac{-5 \pm 3\sqrt{5}}{8}\right), \quad Q_{1,2} = \left(\frac{7 \pm 3\sqrt{5}}{8}, \frac{-45 \mp 21\sqrt{5}}{8}\right).$$

As mentioned in §4.1, computation of C_5 and a preliminary expression for $\varphi_5(x, \xi)$ was not straightforward. A natural effort leads to the following expression decomposition of (4.18) into principal ideals:

$$\begin{aligned} & 3\left(2\left(\frac{1}{4}, -\frac{5}{4}\right) + (0, 0) - 3\mathcal{O}\right) + 5\left(P_1 + P_2 + \left(\frac{1}{4}, \frac{5}{4}\right) - 3\mathcal{O}\right) \\ & - \left(\left(\frac{1}{4}, \frac{5}{4}\right) + \left(\frac{1}{4}, -\frac{5}{4}\right) + \left(-\frac{1}{4}, \frac{3}{4}\right) + \left(-\frac{1}{4}, -\frac{3}{4}\right) - 4\mathcal{O}\right) \\ & - 2\left(2\left(\frac{1}{4}, \frac{5}{4}\right) + (0, 0) - 3\mathcal{O}\right) - 5\left(Q_1 + Q_2 + \left(\frac{1}{4}, -\frac{5}{4}\right) - 3\mathcal{O}\right). \end{aligned} \quad (4.19)$$

This expression gives formula (2.30). The most convenient principal divisors are listed in Table 5. The points R_1, R_2, R_3, R_4 on C_5 are the points in the fiber $z = 1$ of $\varphi_5(x, \xi)$.

The principal divisor for $\varphi_6(x, \xi)$ on C_5 turns out to be

$$(0, 0) + \mathcal{O} + 5P_1 + 5P_2 - (1, 1) - (-1, 1) - 5Q_1 - 5Q_2. \quad (4.20)$$

where

$$P_{1,2} = \left(\frac{1 \pm \sqrt{5}}{2}, 0\right), \quad Q_{1,2} = (-2 \pm \sqrt{5}, -5 \pm 2\sqrt{5}).$$

Function	Divisor	Function	Divisor
ξ	$(0, 0) + P_1 + P_2 - 3\mathcal{O}$	$1 + x - x^2$	$2P_1 + 2P_2 - 4\mathcal{O}$
$1 - \xi$	$(-1, 1) + 2(1, 1) - 3\mathcal{O}$	$1 - 4x - x^2$	$Q_1 + Q_2 + R_1 + R_2 - 4\mathcal{O}$
$1 + \xi$	$(-1, -1) + 2(1, -1) - 3\mathcal{O}$	$\xi + 2x + x^2$	$(0, 0) + Q_1 + Q_2 - 3\mathcal{O}$
$1 + \xi + 2x$	$3(-1, 1) - 3\mathcal{O}$	$1 + \xi - 2x$	$Q_1 + Q_2 + (1, 1) - 3\mathcal{O}$
$1 - \xi + 2x$	$3(-1, -1) - 3\mathcal{O}$	$1 - \xi - 2x$	$R_1 + R_2 + (1, -1) - 3\mathcal{O}$

Table 6: Divisors on C_6

We naturally arrive at the following expression for (4.20):

$$\begin{aligned} & \left(P_1 + P_2 + (0, 0) - \mathcal{O} \right) + 2 \left(2P_1 + 2P_2 - 4\mathcal{O} \right) + 2 \left(2(1, 1) + (1, -1) - 2\mathcal{O} \right) \\ & - \left(3(-1, 1) - 3\mathcal{O} \right) - 5 \left(Q_1 + Q_2 + (1, 1) - 3\mathcal{O} \right). \end{aligned}$$

This expression gives formula (2.34). The most convenient principal divisors are listed in Table 6. The points R_1, R_2 on C_6 are in the fiber $z = 1$ of $\varphi_6(x, \xi)$.

4.3 Computation of hypergeometric evaluations

In principle, evaluations (2.19)–(2.22), (2.25)–(2.28), (2.31)–(2.32), (2.35)–(2.38) are computed by pulling-back their hypergeometric equations onto C_3, C_4, C_5, C_6 , respectively, and finding radical solutions of the pull-backed differential equations. Their divisors will have coefficients in \mathbb{Q} rather than \mathbb{Z} , as described in Appendix 5.1. But standard computer algebra systems do not handle differential equations on higher genus curves.

The pull-backed equations are cumbersome as we will see. On the other hand, their singular points and exponent differences are easy to see, as explained in Appendix 5.5. The local exponents tell us possible coefficients in the principal divisors of the radical solutions. Possible principal divisors are restricted by Lemma 5.2 and Lemma 3.6. For simplest hypergeometric equations, we may end up with just 2 possible principal divisors for radical solutions. Then we can find and check those solutions without computing the pull-back equation explicitly. Additional contiguous evaluations can be obtained by differentiating known solutions and contiguous relations, while respecting Lemma 3.8 and avoiding explicit computation with pull-back equation again.

Generally, we may have several candidates for radical solutions, which we must check by substituting into the pull-back equation. If we have one undetermined simple zero for a radical solution, its location can be restricted by the arithmetical argument that the principal divisor should be invariant under the Galois action of $\overline{\mathbb{Q}}$.

We start with computation of evaluations (2.19)–(2.20). We use Riemann notation (5.6)

and consider the icosahedral hypergeometric equation with the solution space

$$P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & -1/30 \\ 2/5 & 1/3 & 3/10 \end{array} \begin{array}{c} z \\ \\ \end{array} \right\}.$$

Its pull-back $z \mapsto \varphi_3(x, \xi)$, $y(z) \mapsto Y(\varphi_3(x, \xi))$ onto C_3 has the following regular singular points and local exponents:

$$\left\{ \begin{array}{cccccc} (0,0) & \mathcal{O} & \left(\frac{11+5\sqrt{5}}{6}, 0\right) & \left(\frac{11-5\sqrt{5}}{6}, 0\right) & R_1 & R_2 & R_3 & R_4 \\ 0 & 0 & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ \frac{2}{5} & \frac{2}{5} & 2 & 2 & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} \end{array} \right\}$$

Condition (i) of Lemma 5.2 leaves only two candidates for the divisors of radical solutions:

$$\frac{2}{5} \mathcal{O} - \frac{1}{10} R_1 - \frac{1}{10} R_2 - \frac{1}{10} R_3 - \frac{1}{10} R_4, \quad \frac{2}{5} (0,0) - \frac{1}{10} R_1 - \frac{1}{10} R_2 - \frac{1}{10} R_3 - \frac{1}{10} R_4. \quad (4.21)$$

They must represent divisors of the two solutions of Lemma (3.6) and, by identification of local solutions, the hypergeometric series (2.19) and (2.20). Since the point $x = 0$ corresponds to $(0,0) \in C_3$, the first divisor correspond to the series in (2.19). The right-hand side of that formula is easy to construct from the divisor by using Table 3 and the known value of left-hand side at $x = 0$. Similarly, the second divisor in (4.21) implies the identity

$$\varphi_3(x, \xi)^{2/5} {}_2F_1 \left(\begin{array}{c} 7/10, 11/30 \\ 7/5 \end{array} \middle| \varphi_3(x, \xi) \right) = \frac{x^{1/5} (1 - 9\xi + 54x)^{1/30}}{(1 + 21\xi - 117x + 9x\xi - 234x^2)^{1/10}}.$$

This gives formula (2.20). We derived these identities without even computing the pull-back differential equation on C_3 .

Next we consider computation of evaluations (2.21)–(2.22). The pull-back $z \mapsto \varphi_3(x, \xi)$, $y(z) \mapsto Y(\varphi_3(x, \xi))$ of the icosahedral hypergeometric equation with the solution space

$$P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & -1/10 \\ 1/5 & 1/3 & 17/30 \end{array} \begin{array}{c} z \\ \\ \end{array} \right\}.$$

is the following differential equation (with coefficients in a convenient form):

$$\begin{aligned} Y'' + \frac{3(3+47\xi+1974x-2051\xi^2+2676x\xi+54348x^2+33\xi^3-1002x\xi^2-1548x^2\xi)}{10\xi^2(1+21\xi-117x+9x\xi-234x^2)} Y' \\ + \frac{51\xi(1-9\xi+54x)(1-21\xi-117x-9x\xi-234x^2)}{25x^2(1+9x)(1+21\xi-117x+9x\xi-234x^2)^2} Y = 0. \end{aligned} \quad (4.22)$$

Like in the previous case above, we know singularities and local exponents of this equation without cumbersome computations. Here they are:

$$\left\{ \begin{array}{cccccc} (0,0) & \mathcal{O} & R_1 & R_2 & R_3 & R_4 \\ 0 & 0 & -\frac{3}{10} & -\frac{3}{10} & -\frac{3}{10} & -\frac{3}{10} \\ \frac{1}{5} & \frac{1}{5} & \frac{17}{10} & \frac{17}{10} & \frac{17}{10} & \frac{17}{10} \end{array} \right\}$$

Condition (i) of Lemma 5.2 gives the following candidates for the divisors of radical solutions:

$$\frac{1}{5}\mathcal{O} + X - \frac{3}{10}R_1 - \frac{3}{10}R_2 - \frac{3}{10}R_3 - \frac{3}{10}R_4, \quad \frac{1}{5}(0,0) + Y - \frac{3}{10}R_1 - \frac{3}{10}R_2 - \frac{3}{10}R_3 - \frac{3}{10}R_4.$$

Here X and Y are some regular points of (4.22). By condition (ii) and the additional statement of the same lemma, these should be torsion points on C_3 defined over \mathbb{Q} . The possibilities for X and Y are: $(-\frac{1}{9}, -\frac{5}{9})$, $(-\frac{1}{9}, \frac{5}{9})$, $(1, -5)$, $(1, 5)$. This gives 8 possible divisors of a radical solution. For each possibility, one has to construct a radical function (in any form) with that divisor, and to check whether it is a solution of (4.22). Alternatively, candidate solutions can be expanded in power series around $(x, \xi) = (0, 0)$ and compared with the hypergeometric series in (2.21) and (2.22). Then one does not have to know explicit equation (4.22), but has to find enough power series terms of *all* candidate solutions (so that the right candidates could be selected). It turns out that actual solutions have $X = (-\frac{1}{9}, -\frac{5}{9})$ and $Y = (1, -5)$. These two solutions and evaluations (2.21), (2.22) are expressed in a convenient form by using methods of §4.2.

Next we consider evaluation of formulas (2.25)–(2.26). A pull-back of $E(1/5, 1/5, 2/3)$ has the following singularities and local exponents:

$$\left\{ \begin{array}{cccccccccc} \mathcal{O}^* & \mathcal{O} & R_1 & R_2 & R_3 & R_4 & A_5 & A_5^* & Q_1 & Q_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{30} & -\frac{1}{30} & -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{5} & \frac{1}{5} & 2 & 2 & 2 & 2 & \frac{1}{6} & \frac{1}{6} & \frac{5}{6} & \frac{5}{6} \end{array} \right\}$$

Condition (i) of Lemma 5.2 gives the following candidates for the divisors on C_4 of radical solutions:

$$\begin{aligned} & -\frac{1}{6}Q_1 - \frac{1}{6}Q_2 + \frac{1}{6}A_5 + \frac{1}{6}A_5^*, & \frac{1}{5}\mathcal{O} + \frac{1}{5}(0,0) - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 - \frac{1}{30}A_5 - \frac{1}{30}A_5^*, \\ & \frac{1}{5}\mathcal{O} - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 + \frac{1}{6}A_5 - \frac{1}{30}A_5^*, & \frac{1}{5}(0,0) - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 - \frac{1}{30}A_5 + \frac{1}{6}A_5^*, \\ & \frac{1}{5}\mathcal{O} - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 - \frac{1}{30}A_5 + \frac{1}{6}A_5^*, & \frac{1}{5}(0,0) - \frac{1}{6}Q_1 - \frac{1}{6}Q_2 + \frac{1}{6}A_5 - \frac{1}{30}A_5^*. \end{aligned}$$

The candidate divisors are grouped into possible pairs for a basis of actual solutions, following Lemma 3.6. The divisors in the first pair do not satisfy condition (ii) of Lemma 5.2. To decide the right pair, one may take one divisor from each of the one two pairs, construct an expression for a corresponding radical function, and compare its power series around $(0, 0)$. It turns out, the last pair is the right one. Once we have the right divisors for (2.25) and (2.26), we can proceed similarly as in the pervious cases. Application of the methods of §4.2 may require some combinatorial creativeness. For example, here is a convenient splitting of the right divisor for (2.25):

$$\frac{1}{6}(\tilde{A}_4^* + A_5^* + \tilde{A}_1 - 3\mathcal{O}) - \frac{1}{6}(Q_1 + Q_2 + \tilde{A}_4^* - 3\mathcal{O}) - \frac{1}{30}(5\tilde{A}_1 + A_5 - 6\mathcal{O}).$$

Now we consider evaluation of formulas (2.27)–(2.28). We comment only the most complicated step of choosing the divisors of the actual solutions of the pull-back of $E(2/5, 3/5, 1/3)$. The singularities and local exponents are the following:

$$\left\{ \begin{array}{cccccccc} \mathcal{O}^* & \mathcal{O} & P_1 & P_2 & A_5 & A_5^* & Q_1 & Q_2 \\ 0 & 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} & -\frac{5}{6} & -\frac{5}{6} \\ \frac{2}{5} & \frac{2}{5} & 2 & 2 & \frac{13}{30} & \frac{13}{30} & \frac{13}{6} & \frac{13}{6} \end{array} \right\}$$

Condition (i) of Lemma 5.2 gives the following candidates for the divisors of radical solutions:

$$\begin{aligned}
& -\frac{1}{6}A_5 - \frac{1}{6}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2 + 2P_1, & -\frac{1}{6}A_5 - \frac{1}{6}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2 + 2P_2, \\
& -\frac{1}{6}A_5 - \frac{1}{6}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2 + X + Y, & X + Y \in \{A_5, A_5^*\} \\
& \frac{2}{5}\mathcal{O} + \frac{2}{5}\mathcal{O}^* + \frac{13}{30}A_5 + \frac{13}{30}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2, \\
& \frac{2}{5}\mathcal{O} + \frac{13}{30}A_5 - \frac{1}{6}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2 + X, & X \in \{A_2, A_2^*\} \\
& \frac{2}{5}\mathcal{O} - \frac{1}{6}A_5 + \frac{13}{30}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2 + X, & X \in \{A_2, A_2^*\}, \\
& \frac{2}{5}\mathcal{O}^* + \frac{13}{30}A_5 - \frac{1}{6}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2 + X, & X \in \{A_2, A_2^*\} \\
& \frac{2}{5}\mathcal{O}^* - \frac{1}{6}A_5 + \frac{13}{30}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2 + X, & X \in \{A_2, A_2^*\}.
\end{aligned}$$

Here the restrictions on the additional points X, Y follow from condition (ii) and the additional statement of Lemma 5.2. The first three possibilities can be paired only with the fourth divisor as (divisors of) functions f_1, f_2 of Lemma 3.6. To refute them, one has to check only that a function with the fourth divisor (as the principal divisor) is not a solution of the pull-back equation. Other possibilities have to be paired and checked as we did for equation (4.22). The right divisors are these:

$$\frac{2}{5}\mathcal{O} + \frac{13}{30}A_5 - \frac{1}{6}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2 + A_2^*, \quad \frac{2}{5}\mathcal{O}^* - \frac{1}{6}A_5 + \frac{13}{30}A_5^* - \frac{5}{6}Q_1 - \frac{5}{6}Q_2 + A_2.$$

Now we can compute (2.27), (2.28) like in the previous cases, etc.

Computation of evaluations (2.31)–(2.32), (2.35)–(2.38) is similar and not more complicated. First we find singularities and local exponents of the pull-backed equation (of a corresponding main hypergeometric equation); make a list of possible divisors for radical solutions of the pull-back; use Lemma 3.6 to make a short divisor list for necessary check, find radical functions for the candidate divisors from the short list; compare their power series around $(0, 0)$ with the expansions of the hypergeometric series; take the right divisors and find a convenient expression for their functions (using methods of §4.2).

Differentiation and computation of contiguous evaluations to (2.19)–(2.38) quickly leads to large expressions. For example, an expression contiguous to (2.19) is

$$\begin{aligned}
& {}_2F_1\left(\begin{matrix} 3/10, 29/30 \\ 3/5 \end{matrix} \middle| \varphi_3(x, \xi)\right) = \\
& \frac{(1+21\xi-117x+9x\xi-234x^2)^{9/10}(1+9x)^2(1+198x-99x^2)}{(1-9\xi+54x)^{29/30}(1-21\xi-117x-9x\xi-234x^2)^2}. \quad (4.23)
\end{aligned}$$

The divisor for this function on C_3 is

$$\frac{9}{10}R_1 + \frac{9}{10}R_2 + \frac{9}{10}R_3 + \frac{9}{10}R_4 + S_1 + S_2 + \tilde{S}_1 + \tilde{S}_2 + \frac{2}{5}\mathcal{O} - 2\tilde{R}_1 - 2\tilde{R}_2 - 2\tilde{R}_3 - 2\tilde{R}_4,$$

where $S_1, S_2, \tilde{S}_1, \tilde{S}_2$ are the points with the x -coordinate equal to $1 \pm 10/3\sqrt{11}$. For least painful computation of contiguous expressions, keep in mind that the quotient of contiguous hypergeometric functions is a rational function on a Darboux curve, by Lemma 3.8. In particular, the above expression should be viewed as $G(x, \xi) \times {}_2F_1\left(\begin{matrix} 3/10, -1/30 \\ 3/5 \end{matrix} \middle| \varphi_3\right)$ with $G(x, \xi) \in \mathbb{C}(C_3)$. To compute $G(x, \xi)$, only divisors with integer (rather than \mathbb{Q}) coefficients have to be considered as in §4.2.

5 Appendix

Here we recall definitions and facts which are important to us. This material is widely known, but quite rarely presented in a way which is most convenient for our purposes. We concentrate the details that we use. For similar introductions, we refer to [vdW02], [Beu07], [Ber04].

5.1 Algebraic curves

For general theory of algebraic curves we refer to [Ful69] or to [Sha74]. We assume algebraic curves to be reduced, irreducible, smooth, complete (or projective), defined over \mathbb{C} . In particular, the projective line \mathbb{P}^1 is $\mathbb{C} \cup \{\infty\}$ set-theoretically. (All curves in this paper are defined over \mathbb{Q} , but we do not consider arithmetic properties here.)

Let C denote an algebraic curve. We denote the field of rational functions on C by $\mathbb{C}(C)$. It can be generated by 2 functions, since C is birationally isomorphic to a (possibly singular) curve in \mathbb{P}^2 . The function field $\mathbb{C}(\mathbb{P}^1)$ can be generated by 1 function; such a generator is called a *rational parameter*.

If $P \in C$ and $f \in \mathbb{C}(C)$, then $\text{ord}_P(f)$ denotes the *valuation* of f at P . If negative, this is the order of a pole of f at P ; otherwise this is the vanishing order of f at P . A *local parameter* at P is a function $t_P \in \mathbb{C}(C)$ such that $\text{ord}_P(t_P) = 1$. For example, if C is the projective line, then $x - \alpha$ is a local parameter at the point $x = \alpha$, and $1/x$ is a local parameter at $x = \infty$.

A *divisor* on C is a finite formal sum $\sum_{P \in C} a_P P$, with $a_P \in \mathbb{Z}$. The *degree* of such a divisor is the integer $\sum_{P \in C} a_P$. The divisors form a commutative group under addition. For a function $f \in \mathbb{C}(C)$ we have its *principal divisor* $\sum_{P \in C} \text{ord}_P(f) P$, which has degree 0. Principal divisors form a subgroup of degree 0 divisors. The quotient of these two groups is a *Picard group* of C ; it is denoted by $\text{Pic}(C)$. For example, $\text{Pic}(\mathbb{P}^1)$ is the trivial group because all degree zero divisors on \mathbb{P}^1 are principal.

Explicit curves in this paper have either genus 0 (i.e., isomorphic to \mathbb{P}^1) or genus 1. Let E denote a curve of genus 1. It can be represented in a *Weierstrass form* $\xi^2 = G_3(x)$, where $G_3(x)$ is a cubic polynomial in $\mathbb{C}[x]$. The point at infinity in this model by \mathcal{O} . The Picard group of E is isomorphic (set-theoretically) to E itself. As usual, we identify a point $P \in E$ with the element of $\text{Pic}(E)$ represented by the divisor $P - \mathcal{O}$. Then the additive group law on E can be given by the known chord-and-tangent method. In particular, if three points of E lie on one line of \mathbb{P}^2 , they add up to the neutral element \mathcal{O} . The curve E with this group law is an *elliptic curve* (E, \mathcal{O}) . Recall that a *torsion point* on E is a point of finite order.

Lemma 5.1 *Let (E, \mathcal{O}) denote an elliptic curve, and let $T = \sum_{P \in E} a_P P$ be a divisor on E . Then T is a principal divisor if and only if $\sum_{P \in E} a_P P = \mathcal{O}$ in the additive group of (E, \mathcal{O}) .*

Proof. Follows from the specified identification of (E, \mathcal{O}) with $\text{Pic}(E)$. \square

We also consider *radical functions* on C , that is, products of \mathbb{Q} -powers of functions from $\mathbb{C}(C)$. These are multi-valued functions, but their branching points are poles or zeroes with finitely many complex branches coming together. Valuations of those functions are well defined at any point, and have values in \mathbb{Q} at the branching points. Accordingly, we consider their principal divisors $\sum_{P \in C} a_P P$ with coefficients $a_P \in \mathbb{Q}$.

Lemma 5.2 *Let (E, \mathcal{O}) denote an elliptic curve, and let $T = \sum_{P \in E} a_P P$ be a divisor with coefficients in \mathbb{Q} . Then T is the principal divisor for a radical function if and only if the following conditions hold:*

- (i) *The degree $\sum_{P \in E} a_P$ is zero.*
- (ii) *Let $\sum_{P \in E} \tilde{a}_P P$ be an integer multiple of T such that all coefficients \tilde{a}_P are integers. Then, in the additive group of E , the point $\sum_{P \in E} \tilde{a}_P P$ must be a torsion point.*

Proof. Under these conditions, an integer factor nT of T would be a principle divisor with integer coefficients. Then T is a divisor of $G^{1/n}$ for some $G \in \mathbb{C}(E)$.

On the other hand, if T is a divisor of a radical function f , then an integer power f^n is a rational function. The divisor nT sums up to \mathcal{O} by Lemma 5.1. Other integer factors of T with integral coefficients may sum up to a torsion point. \square

5.2 Finite coverings and pull-back transformations

Consider a finite covering $\phi : C \rightarrow D$ from C to other algebraic curve D . It induces an algebraic field extension $\mathbb{C}(C) \supset \mathbb{C}(D)$. We denote the degree of ϕ by $\deg \phi$. The genus $g(C)$ and $g(D)$ of both curves and branching data are related by the Hurwitz formula:

$$2g(C) - 2 = (2g(D) - 2) \deg \phi + \sum_{P \in C} (r_P - 1). \quad (5.1)$$

Here r_P is the branching order at P . It is equal to $\text{ord}_P(t_{\phi(P)} \circ \phi)$.

Now we convene what we mean by a pull-back of hypergeometric equation (1.1) with respect to a finite covering. Let C denote an algebraic curve. Suppose that the function field $\mathbb{C}(C)$ of C is generated by functions x, ξ . If C is a rational curve, we may assume that ξ is not used and x is a rational parameter of C .

Consider a finite covering $\phi : C \rightarrow \mathbb{P}^1$. Let z denote a rational parameter for \mathbb{P}^1 . Then a pull-back of (1.1) with respect to ϕ is a differential equation is defined by transformation:

$$z \mapsto \phi(x, \xi), \quad y(z) \mapsto Y(x, \xi) = \theta(x, \xi) y(\phi(x, \xi)). \quad (5.2)$$

Here $\theta(x, \xi)$ is a radical function. Note that such a function has the property that its logarithmic derivative $\theta'(x, \xi)/\theta(x, \xi)$ is in $\mathbb{C}(C)$. We use the derivation on $\mathbb{C}(C)$ that extends the usual derivative on $\mathbb{C}(x)$. If $C \cong \mathbb{P}^1$, then transformation (5.2) is the following:

$$z \mapsto \phi(x), \quad y(z) \mapsto Y(x) = \theta(x) y(\phi(x)). \quad (5.3)$$

5.3 Differential Galois theory

A *differential field* K is a field with a derivation, i.e., a map $D : K \rightarrow K$ which satisfies $D(a + b) = D(a) + D(b)$ and the Leibnitz rule $D(ab) = aD(b) + bD(a)$. One usually denotes $D(a)$ by a' . An *extension* of the differential field K is a differential field L which contains K and whose derivation extends the derivation of K . The basic example of a differential field

is the field $\mathbb{C}(z)$ of rational functions on \mathbb{P}^1 with the usual derivation. Other example is the field $\mathbb{C}(C)$ of rational functions on an algebraic curve. To give a derivation on $\mathbb{C}(C)$ one may consider a finite covering $\phi : C \rightarrow \mathbb{P}^1$ and the corresponding unique extension of the usual derivation of $\mathbb{C}(z)$.

Fix a differential field K and consider a linear homogeneous differential equation (3.1). Solutions in any extension of K form a linear space over the *constant field* $\{a \in K \mid a' = 0\}$. The dimension of the solution space is at most n . A *Piccard-Vessiot extension* $K_{PV} \supset K$ for (3.1) is, roughly speaking, a minimal extension of differential fields, such that the solutions of (3.1) in K_{PV} form a linear space of dimension n . The *differential Galois group* G of (3.1) is the group of autocoverings of K that fix the elements of K_{PV} . The action of G on the n -dimensional space of solutions in K_{PV} gives a faithful n -dimensional representation of G . Therefore the differential Galois group G is usually considered as an algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$.

In §3, we utilize the *Riccati equation* associated to (3.1). Solutions for the Riccati equation are precisely the logarithmic derivatives y'/y of solutions for (3.1). Explicitly, the Riccati equation for (3.1) with $n = 2$ is $u' + u^2 + a_1 u + a_0 = 0$. Rational or algebraic solutions of the Riccati equation are important in finding “closed form” solutions of the original equation (3.1), see [Kov86]. We refer to algebraic solutions of the Riccati equation in our working definition of Darboux curves.

Suppose that hypergeometric equation (1.1) has a finite monodromy G . Then the differential Galois group is isomorphic to G . (More generally, the differential Galois group of a Fuchsian equation is isomorphic to the Zariski closure of a representation of the monodromy group.) The Piccard-Vessiot extension $K_{PV} \supset \mathbb{C}(z)$ is a finite Galois extension, the usual Galois group is isomorphic to G as well. If $y(z) \in K_{PV}$ is a solution of (1.1), then $K_{PV} = \mathbb{C}(z, y)$.

In most papers on differential Galois theory, second order differential equations are normalized to the form $y(z)'' = r(z)y(z)$, with $r(z) \in \mathbb{C}(z)$. Hypergeometric equation (1.1) can be normalized by the transformation

$$y(z) \mapsto z^{(e_0-1)/2} (1-z)^{(e_1-1)/2} y(z). \quad (5.4)$$

The normalized equation is:

$$\frac{d^2 y(z)}{dz^2} = \left(\frac{e_1^2 - 1}{4(z-1)^2} + \frac{e_0^2 - 1}{4z^2} + \frac{1 + e_\infty^2 - e_0^2 - e_1^2}{4z(z-1)} \right) y(z). \quad (5.5)$$

If the monodromy G of a hypergeometric equation is isomorphic to A_4, S_4 or A_5 , then the differential Galois group of the normalized equation (5.5) is $G \times \{1, -1\}$. This does not change facts that are important to us. Algebraic degree of Riccati solutions for (1.1) is the same as of Riccati solutions for (5.5).

5.4 Hypergeometric equations

The singularities and local exponents of the hypergeometric equation (1.1) are conveniently revealed in Riemann's P -notation:

$$P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{array} \middle| z \right\}. \quad (5.6)$$

As we see, the first row indicates the regular singular points, and the other rows contain the local exponents and the variable z . The exponent differences obviously are $e_0 = 1 - c$, $e_1 = c - a - b$ and $e_\infty = a - b$. These linear expressions in a, b, c can be inverted, so that an ordered sequence of exponent differences determines the hypergeometric equation. If we permute the exponent differences e_0, e_1, e_∞ or multiply some of them by -1 , we get hypergeometric equations related by well-known fractional-linear transformations [AAR99]. In general, there are 24 hypergeometric equations related in this way, and they share the same (up to radical factors and fractional-linear change of the independent variable) 24 hypergeometric Kummer's solutions.

A general basis of solutions for (1.1) is

$${}_2F_1 \left(\begin{array}{c} a, b \\ c \end{array} \middle| z \right), \quad z^{1-c} {}_2F_1 \left(\begin{array}{c} a+1-c, b+1-c \\ 2-c \end{array} \middle| z \right), \quad (5.7)$$

where ${}_2F_1 \left(\begin{array}{c} a, b \\ c \end{array} \middle| z \right) := 1 + \frac{ab}{c \cdot 1!} z + \frac{a(a+1)b(b+1)}{c(c+1) \cdot 2!} z^2 + \dots$ is the Gauss hypergeometric series.

5.5 Fuchsian equations

All differential equations that we explicitly consider are *Fuchsian equations*. These equations have only regular singular points. For equation (3.1) this means the following: if $K = \mathbb{C}(C)$ for an algebraic curve C , then for any point $P \in C$ and for $i = 1, \dots, n$ we must have $\text{ord}_P(a_i) \geq (n - i)(\text{ord}_P(t_P') - 1)$, where t_P is a local parameter at P . Local exponents at P can be defined as follows: substitute $y = t_P^\mu$ into the Fuchsian equation and consider the terms to the power $\mu + n((\text{ord}_P(t_P') - 1))$ of t_P as an equation in μ ; the roots of that equation are precisely the local exponents. The local exponents at regular points are equal to $0, 1, \dots, n - 1$.

In general, singularities and local exponents do not determine a Fuchsian equation uniquely. Hence we cannot always use the P -notation for general Fuchsian equations. However, in §4.3 we write down arrays of singularities and local exponents similar to (5.6).

5.6 Contiguous relations of Gauss hypergeometric functions

Two Gauss hypergeometric functions are called *contiguous* (or *associated* in [Erd53]) if they have the same argument z and their parameters a, b and c differ respectively by integers. As is known [AAR99, §2.5], for any three contiguous ${}_2F_1$ functions there is a *contiguous relation*, which is a linear relation between the three functions where the coefficients are rational functions in the parameters a, b, c and the argument z . A straightforward (though

not efficient) method to compute a contiguous expression for ${}_2F_1\left(\begin{smallmatrix} a+k, b+\ell \\ c+m \end{smallmatrix} \middle| z\right)$ in terms of ${}_2F_1\left(\begin{smallmatrix} a+1, b \\ c \end{smallmatrix} \middle| z\right)$ and ${}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix} \middle| z\right)$ is the following. By using the contiguous relations

$$b {}_2F_1\left(\begin{smallmatrix} a, b+1 \\ c \end{smallmatrix} \middle| z\right) = (b-a) {}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix} \middle| z\right) + a {}_2F_1\left(\begin{smallmatrix} a+1, b \\ c \end{smallmatrix} \middle| z\right), \quad (5.8)$$

$$(c-1) {}_2F_1\left(\begin{smallmatrix} a, b \\ c-1 \end{smallmatrix} \middle| z\right) = (c-a-1) {}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix} \middle| z\right) + a {}_2F_1\left(\begin{smallmatrix} a+1, b \\ c \end{smallmatrix} \middle| z\right), \quad (5.9)$$

one eliminates the shifts in b and c , and then by using the contiguous relation

$$a(1-z) {}_2F_1\left(\begin{smallmatrix} a+1, b \\ c \end{smallmatrix} \middle| z\right) = (2a-c-az+bz) {}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix} \middle| z\right) + (c-a) {}_2F_1\left(\begin{smallmatrix} a-1, b \\ c \end{smallmatrix} \middle| z\right) \quad (5.10)$$

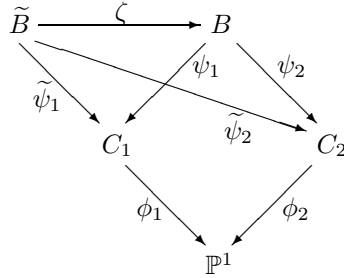
one gets an expression with two contiguous terms. Effective computation of contiguous relations is considered in [Vid03]. They can be computed in $O(\log(\max(k, l, m)))$ steps, but complexity of expressions in each such step grows exponentially, and the output is $O(\max(k, l, m))$.

One can rewrite contiguity conditions in terms of local exponent differences at $z = 0, 1, \infty$ for the hypergeometric equation, since the parameters a, b, c determine the exponent differences and vice versa (if the sign of exponent differences is taken into account). The main hypergeometric solutions (5.7) of two hypergeometric equations (1.1) are contiguous if for each $X \in \{0, 1, \infty\}$ the difference of signed exponent differences at X of the two equations is an integer, and the sum of the three integer differences is even. Two hypergeometric equations have solutions contiguous to each other (or equivalently, they have the same Schwarz type) if one can choose a permutation of exponent differences and their sign in such a way that the just described situation occurs.

For example, the parameters e_0, e_1, e_∞ of hypergeometric equations of the Schwarz type $(1/3, 1/3, 2/3)$ can be characterized as follows: they are rational numbers, their denominators are equal to 3, and the sum of their numerators is even.

5.7 Fiber products of curves

Let C_1 and C_2 denote two curves over \mathbb{C} . Let $\phi_1 : C_1 \rightarrow \mathbb{P}^1$ and $\phi_2 : C_2 \rightarrow \mathbb{P}^1$ be two finite coverings of degree m and n respectively. The fiber product of $\phi_1 : C_1 \rightarrow \mathbb{P}^1$ and $\phi_2 : C_2 \rightarrow \mathbb{P}^1$ is a curve B with two coverings $\psi_1 : B \rightarrow C_1$ and $\psi_2 : B \rightarrow C_2$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$, and for any other curve \tilde{B} with coverings $\tilde{\psi}_1 : \tilde{B} \rightarrow C_1$ and $\tilde{\psi}_2 : \tilde{B} \rightarrow C_2$ satisfying $\phi_1 \circ \tilde{\psi}_1 = \phi_2 \circ \tilde{\psi}_2$ there is a unique covering $\xi : \tilde{B} \rightarrow B$ such that $\tilde{\psi}_1 = \psi_1 \circ \xi$ and $\tilde{\psi}_2 = \psi_2 \circ \xi$. Then the following diagram commutes:



We have $\deg \psi_1 = \deg \phi_2$ and $\deg \psi_2 = \deg \phi_1$. On the level for function fields, we have $\mathbb{C}(B) = \mathbb{C}(C_1) \otimes_{\mathbb{C}(\mathbb{P}^1)} \mathbb{C}(C_2)$.

A birational model for B is the curve on $C_1 \times C_2$ of those points $(X_1, X_2) \in C_1 \times C_2$ which satisfy $\varphi_1(X_1) = \varphi_2(X_2)$. This is a singular model in general. A singular point corresponds to a pair $(X_1, X_2) \in B$ such that X_1 and X_2 have branching indices $r_1 > 1$, $r_2 > 1$ respectively (with respect to ϕ_1 and ϕ_2). Such a singularity is of type $x_1^{r_1} - x_2^{r_2}$; by resolving it we get $\gcd(r_1, r_2)$ points that correspond to (X_1, X_2) on a non-singular model for B . This information allows us to compute the branching data for the projections $\psi : B \rightarrow C_1$ and $\psi : B \rightarrow C_2$ and the genus of B . For example, if $X_1 \in C_1$ has branching index r_1 (with respect to ϕ_1), and the branching data of ϕ_2 above $\phi_1(X_1)$ is $a_1 + \dots + a_k$, then the branching data for ψ_1 above X_1 is the following: $\gcd(a_1, r_1) * \frac{\text{lcm}(a_1, r_1)}{r_1} + \dots + \gcd(a_k, r_1) * \frac{\text{lcm}(a_k, r_1)}{r_1}$.

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