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The generating function of Kreweras walks with interacting boundaries is not algebraic

Alin Bostan*[1], Manuel Kauers†[2] and Thibaut Verron‡[2]

[1]*Inria, Université Paris-Saclay, 1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France*
[2]*Institute for Algebra, Johannes Kepler University, Linz, Austria*

Abstract. Beaton, Owczarek and Xu (2019) studied generating functions of Kreweras walks and of reverse Kreweras walks in the quarter plane, with interacting boundaries. They proved that for the reverse Kreweras step set, the generating function is always algebraic, and for the Kreweras step set, the generating function is always D-finite. However, apart from the particular case where the interactions are symmetric in x and y, they left open the question of whether the latter one is algebraic. Using computer algebra tools, we confirm their intuition that the generating function of Kreweras walks is not algebraic, apart from the particular case already identified.

Abstract. Beaton, Owczarek et Xu (2019) ont étudié les séries génératrices des marches avec interactions de type Kreweras et Kreweras inversé dans le quart de plan. Pour le modèle de Kreweras renversé, ils ont prouvé que la série génératrice est toujours algébrique, et pour le modèle de Kreweras, que la série génératrice est toujours D-finie. Cependant, mis à part le cas particulier où les interactions sont symétriques en x et y, ils ont laissé ouverte la question de savoir si cette dernière est algébrique. En utilisant du calcul formel, nous confirmons leur intuition que la série génératrice des marches de Kreweras n'est jamais algébrique, mis à part le cas particulier déjà identifié.

Keywords: Enumerative combinatorics, generating functions, lattice paths, Kreweras walks, kernel method, computer algebra, creative telescoping, automated guessing, algebraic functions, D-finite functions, hypergeometric functions.

1 Introduction

It is always interesting to know whether a generating function is D-finite, because D-finiteness gives easy access to a lot of useful information about the series. It is also interesting to know whether a D-finite series is algebraic, because algebraicity gives access to even more useful information or makes more efficient algorithms applicable.

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†manuel.kauers@jku.at; supported by the Austrian FWF grants F5004 and P31571-N32.
‡thibaut.verron@jku.at; supported by the Austrian FWF grants F5004 and P31571-N32.

Every algebraic series is D-finite but not vice versa, and it is a notoriously difficult problem to decide for a given D-finite power series whether it is algebraic or not [16, §4(g)]. There exists an algorithm for deciding whether a given linear differential equation has only algebraic solutions [15]. This algorithm can be generalized in order to compute, for a given linear differential operator L, another differential operator L^{alg}, whose solution space is spanned by the algebraic solutions of L [14]. The operator L^{alg} can then be used to decide whether a specific solution y of $L(y) = 0$ is algebraic or transcendental. However, the algorithm for computing L^{alg} is very expensive, and to our knowledge it was never implemented.

A popular and simple check is to inspect the asymptotic behaviour of the coefficient sequence: if it is not of the form $c\phi^n n^\alpha$ with $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$, then the series is transcendental [6]. However, this condition is only necessary but not sufficient. The purpose of this paper is to highlight a less popular condition which is also necessary but not sufficient, and which can be tried in many cases where the asymptotic test fails. The method consists in finding (using computer algebra) a closed form representation of the D-finite function in question and to prove (also using computer algebra) that the function has a logarithmic singularity.

Our method is an illustration of the *guess-and-prove* paradigm, which is classically used to prove algebraicity [3]: one guesses an algebraic equation, then post-certifies it. Transcendence is a more difficult task, as one needs to prove that no algebraic equation exists. However, if one can still guess a differential equation, and solve it in explicit form, then the explicit solution can lead to transcendence proofs. This is the methodology promoted here. In order to facilitate its application to other examples, we include a detailed description of the required computer algebra calculations for Maple and Mathematica[1].

As a concrete example, we consider a power series that appears in a recent study of restricted lattice walk models with interacting boundaries. A model is determined by a step set $S \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ and consists of walks in the quarter plane $\mathbb{N}^2$ starting at $(0, 0)$. For each step set S, Beaton et al. [1, 2] are interested in the generating functions $Q(a, b, c, x, y, t) \in \mathbb{Q}[[a, b, c, x, y, t]]$, where $[a^h b^v c^u x^i y^j t^n]Q$ is the number of walks of length n starting at $(0, 0)$ ending at (i, j), with h visits of the horizontal axis, v visits of the vertical axis, and u visits of the origin. Among other things, they show that the generating function is algebraic for the step set $\{\swarrow, \rightarrow, \uparrow\}$ (known as reverse Kreweras), and that the generating function is D-finite for the step set $\{\nearrow, \leftarrow, \downarrow\}$ (known as Kreweras). They conjecture that this latter series is not algebraic, and this is what we will prove here. More precisely, we will show the following:

Theorem 1. *Let $Q(a, b, c; x, y; t) \in \mathbb{Q}[a, b, c][[x, y, t]]$ be the generating function counting interactive Kreweras walks restricted to the quarter plane. The generating function $Q(a, b, c; x, y; t)$ is not algebraic over $\mathbb{Q}(a, b, c, x, y, t)$.*

[1] Also available online: `http://www.algebra.uni-linz.ac.at/research/kreweras-interacting`

Furthermore, for values $a, b, c \in \mathbb{Q}$ with $c \neq 0$, the generating function $Q(a, b, c; x, y; t) \in \mathbb{Q}[[x, y, t]]$ is algebraic over $\mathbb{Q}(x, y, t)$ if and only if $a = b$.

2 Notations and the kernel equation

We first recall some notations used in this paper. Whenever possible, we follow the notations used in [2]. Let R be an integral domain with fraction field K. We denote:

- $R[t]$ the ring of polynomials in t with coefficients in R;
- $K(t)$ the field of rational functions in t with coefficients in K, which is the fraction field of $R[t]$;
- $R[t, 1/t]$ the ring of Laurent polynomials in t with coefficients in R;
- $R[[t]]$ the ring of formal power series in t with coefficients in R.

Given $f(t) \in K((t))$, we denote by $[t^n]f$ the coefficient of t^n in $f(t)$, so that $f(t) = \sum_{n\in\mathbb{Z}}([t^n]f)t^n$. We denote by $[t^>]f$ the sum of the terms of f with positive exponents, that is, $[t^>]f = \sum_{n\in\mathbb{Z}_{>0}}([t^n]f)t^n$.

We denote by $R[t]\langle\partial_t\rangle$ the Ore algebra of differential operators in t with polynomial coefficients. It is a non-commutative ring, and it has a left-action on the rings above, given by $\partial_t(f) = \frac{\partial f}{\partial t}$.

Those definitions can be iterated to extend them to multiple variables, and we group together the brackets when applicable: for example $R[[x, y]]$ is the ring of formal power series in x, y with coefficients in R, and given $f \in R[[x, y]]$, we denote by $[x^> y^0]f$ the sum of terms of positive degree in x and degree 0 in y in f.

For $n, k, l, h, v, u \in \mathbb{Z}$, we denote by $q_{h,v,u;k,l;n}$ the number of walks of length n which:

- start at $(0, 0)$;
- end at (k, l);
- never leave the upper-right quadrant $\{(x, y) \in \mathbb{Z}^2 : x \geq 0, y \geq 0\}$;
- visit the horizontal boundary (excluding the origin) $\{(x, y) \in \mathbb{Z}^2 : x > 0, y = 0\}$ exactly h times;
- visit the horizontal vertical boundary (excluding the origin) $\{(x, y) \in \mathbb{Z}^2 : x = 0, y > 0\}$ exactly v times;
- visit the origin u times (not counting the starting point).

The associated generating function $Q(a, b, c; x, y; t)$ is defined as

$$Q(a, b, c; x, y; t) = \sum_n t^n \sum_{k,l} x^k y^l \sum_{h,v,u} q_{h,v,u;k,l;n} a^h b^v c^u.$$

Note that, since there are only finitely many walks of a given length, for each n, the two innermost sums define a polynomial. Hence $Q(a, b, c; x, y; t)$ lives in $\mathbb{Q}[a, b, c, x, y][[t]] \subset \mathbb{Q}[a, b, c][[x, y, t]]$. For shortness, we shall write $Q(x, y) := Q(a, b, c; x, y; t)$. In particular, $Q(0, 0)$ is the generating function counting interacting walks ending at $(0, 0)$, $Q(x, 0)$ is the generating function counting interacting walks ending on the horizontal axis and $Q(0, y)$ is the generating function counting interacting walks ending on the vertical axis.

Finally, we denote by $Q_{i,j} = Q_{i,j}(a, b, c; t) := [x^i y^j]Q(x, y) \in \mathbb{Q}[a, b, c][[t]]$ the generating function counting interacting walks ending at point (i, j). The coefficient of t^n in $Q_{i,j}$ is a polynomial in a, b, c, and its coefficient for the monomial $a^h b^v c^u$ is exactly $q_{h,v,u;k,l;n}$.

The elements a, b, c are called weights associated respectively to the horizontal boundary (excluding the origin), the vertical boundary (excluding the origin) and the origin.

Given a step set $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, the step generator S is

$$S(x, y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j \in \mathbb{Q}[x, 1/x, y, 1/y].$$

We denote by

- $A(x, y) = \sum_{(i,-1)\in\mathcal{S}} x^i y^{-1}$ the step generator for the steps going southwards;
- $B(x, y) = \sum_{(-1,j)\in\mathcal{S}} x^{-1} y^j$ the step generator for the steps going westwards;
- $G(x, y) = x^{-1} y^{-1}$ if $(-1, -1) \in \mathcal{S}$ and 0 otherwise, the step generator for the steps going south-westwards.

The *kernel* of the step set is $K(x, y) = 1 - tS(x, y) \in \mathbb{Q}[t, x, 1/x, y, 1/y]$.

The kernel equation is a functional equation satisfied by the generating function counting walks restricted to the quarter plane.

Theorem 2 ([2, Theorem 1]). *For a lattice walk restricted to the quarter-plane, starting at the origin, with weights a (resp. b) associated with vertices on the x-axis excluding the origin (resp. the y axis excluding the origin), and weight c associated with the origin, the generating function $Q(x, y)$ satisfies the following functional equation*

$$\begin{aligned} K(x, y)Q(x, y) = \frac{1}{c} + \frac{1}{a}\left(a - 1 - taA(x, y)\right) Q(x, 0) + \frac{1}{b}\left(b - 1 - tbB(x, y)\right) Q(0, y) \\ + \left(\frac{1}{abc}(ac + bc - ab - abc) + tG(x, y)\right) Q(0, 0). \end{aligned} \tag{2.1}$$

3 Main result and the power series Θ

We consider specifically the Kreweras step set $\mathcal{S} = \{(1,1), (-1,0), (0,-1)\}$. By exhaustive enumeration, the generating function $Q(a,b,c;x,y;t) \in \mathbb{Q}[a,b,c,x,y][[t]]$ starts

$$1 + xy\,t + \left(x^2y^2 + ax + by\right) t^2 + \left(x^3y^3 + (a+1)x^2y + (b+1)xy^2 + ac + bc\right) t^3 + \cdots.$$

For instance, at length 3, the walk $(0,0) \to (1,1) \to (2,2) \to (3,3)$ corresponds to the term $a^0b^0c^0x^3y^3t^3 = x^3y^3t^3$, as it does not touch any of the axes after leaving the origin, while the walk $(0,0) \to (1,1) \to (1,0) \to (0,0)$, corresponds to $a^1b^0c^1t^3x^0y^0 = act^3$, as after leaving the origin it touches the positive horizontal axis once, it returns to the origin once, but does not touch the positive vertical axis.

The main result of this paper is that Q is not algebraic (Theorem 1). In order to prove it, we define

$$\Theta = [x^{>}y^0]\left(\frac{(x-y)(x^2y-1)(xy^2-1)}{xyK(x,y)}\right), \tag{3.1}$$

where $K(x,y) = 1 - t\left(xy + x^{-1} + y^{-1}\right)$. We will prove that Θ is not algebraic. The connection between Θ and Q comes from the following lemma.

Lemma 3 ([2, Lemma 10])**.** *There exist Laurent polynomials $\beta, \beta_{x,0}, \beta_{0,x}, \beta_{0,0}, \beta_{1,0}, \beta_{2,0}, \beta_{3,0} \in \mathbb{Q}[t, 1/t, x, 1/x]$, such that*

$$\begin{aligned}\beta + \beta_{x,0}Q(x,0) + \beta_{0,x}Q(0,x) + \beta_{0,0}Q(0,0) + \beta_{1,0}Q_{1,0} + \beta_{2,0}Q_{2,0} + \beta_{3,0}Q_{3,0}\\ = t^3\frac{a-b}{c}(ab - (ab - ac - bc + abc)Q(0,0))\Theta.\end{aligned} \tag{3.2}$$

Proof. It is a straightforward transposition of [2, Lemma 10], by observing that with the notations therein, $\theta = t^3\frac{a-b}{c}ab\Theta$ and $\theta_{0,0} = t^3\frac{a-b}{c}(ab - ac - bc + abc)Q(0,0)\Theta$. □

4 Transcendence of Θ

Recall that for $\alpha, \beta, \gamma \in \mathbb{Q}$ and $-\gamma \notin \mathbb{N}$, the Gaussian hypergeometric series ${}_2F_1\left({\alpha,\beta \atop \gamma}; t\right)$ is defined as

$${}_2F_1\left({\alpha,\beta \atop \gamma}; t\right) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{t^n}{n!} \in \mathbb{Q}[[t]],$$

where $(u)_n$ denotes the Pochhammer symbol $(u)_n = u(u+1)\cdots(u+n-1)$ for $n \in \mathbb{N}$. It satisfies the differential equation $\left(t^2 - t\right) y''(t) + ((\alpha+\beta+1)t - \gamma)\, y'(t) + \alpha\beta\, y(t) = 0$.

Theorem 4. *The power series Θ defined in (3.1) admits the following closed form representation:*

$$\Theta(t;x) = A_1(t;x) + A_2(t;x)\int_0^t A_3(s;x)T(s;x)ds,$$

where

$$A_0 = \sqrt{1 - \frac{2t}{x} - (4x^3-1)\frac{t^2}{x^2}},$$

$$A_1 = \frac{1}{6xt^3} - \frac{x^3-1}{2x^2t^2} + \frac{2-3x^3}{6x^3t} + \frac{tx^3+2t-x}{6t^3x^2}A_0,$$

$$A_2 = \frac{x^2(x-tx^3-2t)}{3t^3}A_0,$$

$$A_3 = \frac{1}{(tx^3+2t-x)^2(4t^2x^3-(x-t)^2)A_0},$$

$$T = (3t-x)x\,{}_2F_1\left(\begin{matrix}-1/3,-2/3\\1\end{matrix};27t^3\right) + 4t(2tx^3+t-x)\,{}_2F_1\left(\begin{matrix}-1/3,1/3\\2\end{matrix};27t^3\right).$$

The power series A_0, A_1, A_2 and A_3 are algebraic, and the power series T is transcendental. In particular, Θ is transcendental.

Proof. We prove that Θ is equal to $C := A_1 + A_2 \int A_3 T$, in four steps:

1. use creative telescoping [5, 11] to obtain a differential operator $L_{ct} \in \mathbb{Q}(x,t)\langle\partial_t\rangle$ annihilating Θ;
2. verify that L_{ct} annihilates C;
3. find $r \in \mathbb{N}$ such that if $s \in \mathbb{Q}(x)[[t]]$ is a power series solution of L_{ct} and $s = 0 \mod t^r$, then $s = 0$;
4. verify that Θ and C, both power series solution of L_{ct}, are equal modulo t^r and so they are actually equal.

For the first step, define

$$\Theta_0(t;x,y) = \frac{(x-y)(x^2y-1)(xy^2-1)}{xyK(x,y)} \in \mathbb{Q}(x,y,t)$$

so that $\Theta = [x^>y^0]\Theta_0$. In order to bring the problem to a form suitable to creative telescoping algorithms, we encode the coefficient extractions as residues. Extracting the constant coefficient in y is immediate: for any $F \in \mathbb{Q}[x,1/x,y,1/y][[t]]$, as by definition,

$$[y^0]F(t;x,y) = \mathrm{Res}_{y=0}\left(\frac{F(t;x,y)}{y}\right) \in \mathbb{Q}[x,1/x][[t]].$$

For extracting the positive part, we follow [4, Theorem 3]: for any $F \in \mathbb{Q}[x,1/x][[t]]$,

$$[x^>]F(t;x) = \mathrm{Res}_{z=0}\left[\frac{1}{z}F(t;z)\frac{\frac{x}{z}}{1-\frac{x}{z}}\right] \in \mathbb{Q}[x][[t]].$$

So composing the two, we get

$$\Theta(t;x) = \operatorname{Res}_{z=0} \operatorname{Res}_{y=0} \left[\frac{1}{yz} \Theta_0(t;z,y) \frac{\frac{x}{z}}{1-\frac{x}{z}} \right]. \tag{4.1}$$

An annihilator for this can now be computed using creative telescoping, for example, using the Mathematica package `HolonomicFunctions` [10], with the following:

```
(* In Mathematica *)
<< "HolonomicFunctions.m"
Theta0 = (x-y)*(x^2*y-1)*(x*y^2-1)/(x*y*(1-t*(1/x+1/y+x*y)))
Theta0z = Theta0 /. x -> z
Lct = First[First[CreativeTelescoping[
        First[CreativeTelescoping[Theta0z/z/y * (x/z)/(1-x/z),
                                  Der[y], {Der[z],Der[t]}]],
        Der[z], {Der[t]}]]]
```

This yields an operator $L_{ct} \in \mathbb{Q}(x,t)\langle \partial_t \rangle$ of order 6, which annihilates Θ.

Checking that L_{ct} annihilates C is a straightforward computation with a computer algebra software. For instance, in Maple, the following command evaluates to 0:

```
# In Maple
with(DEtools);
simplify(eval(diffop2de(Lct, [Dt,t], y(t)), y(t) = C));
```

For the last step, we need to look at a basis of power series solutions of L_{ct}. Computer algebra software can again be used to compute (truncations of) elements in such a basis. For instance, this can be done with the following lines in Maple.

```
# In Maple
Order := 8;
sols := formal_sol(Lct,[Dt,t]);

# Keep only the power series solutions
sols := select(s -> type(series(s,t=0),'taylor'), sols);
```

The output shows that the set of power series solutions is a $\mathbb{Q}(x)$-vector space of dimension 2 spanned, after a change of basis bringing the first terms to echelon form, by

$$s_0 = 1 + xt^2 + t^3 + O(t^4),$$
$$s_1 = t + \frac{1-x^3}{x} t^2 + \frac{1}{x^2} t^3 + O(t^4).$$

So a power series solution of L_{ct} is entirely determined by its coefficients of degree 0 and 1, and in particular knowing a power series modulo t^2 is enough.

Finally, checking that the first two coefficients of C and Θ are equal is again a straightforward computation. For instance, again using Maple:

```
# In Maple
map(normal,series(C,t,5));
series(Theta,t,2);
```

returns the same result $-x^2 + O(t^2)$. This allows to conclude that $\Theta = C$.

For the second statement of the theorem, note that A_0, A_1, A_2 and A_3 are algebraic by closure properties of algebraic functions. The proof of the fact that T is transcendental combines human observation and computer algebra. First, observe that if $T(t;x)$ was algebraic, then by closure properties so would be $T(t;3t)$, which has only one hypergeometric term, and thus $H(t) = {}_2F_1\left({-1/3,1/3 \atop 2}; t\right)$ would be algebraic. But it is straightforward to verify that this cannot be the case, either by a lookup in Schwarz's classification of algebraic ${}_2F_1$'s [13], or simply by observing that the minimal-order linear differential equation $\left(9t^2 - 9t\right) H''(t) + (9t - 18) H'(t) - H(t) = 0$ of H has solutions which cannot be algebraic because one of them has logarithms in its local expansion at 0.

Finally, it follows that Θ is also transcendental, again by closure properties: if Θ was algebraic, so would be $\int_0^t A_3(s;x)T(s;x)ds$, and so would be its derivative A_3T, and so would be T. $\square$

We now give some more explanation on the process we followed to find the closed form proved above. The first step is to compute a small-order differential operator annihilating Θ. It is possible, given the data of the series coefficients, to guess such an operator L_g of order 4 (hence smaller in order than L_{ct}), by using for instance the guesser [9]:

```
(* In Mathematica, continuation of the previous calculations *)
<< "Guess.m"
Theta = Expand[x*Expand[1/x CoefficientList[Series[Theta0,{t,0,70}],t]]
        /. (x^i_ /; i<0) -> 0 /. (y^i_ /; i!=0) -> 0];
Lg = GuessMinDE[Theta,TT[t]]
```

The output is an operator $L_g \in \mathbb{Q}(x,t)\langle\partial_t\rangle$ which is likely to annihilate Θ. We can increase our trust in this operator by verifying that L_g right-divides L_{ct}:

```
(* In Mathematica, continuation of the previous calculations *)
OreReduce[Lct,{ToOrePolynomial[Lg,TT[t]]}]
```

This line returns the right-remainder of L_{ct} modulo L_g, which is 0 as expected.

At this point, we could also compute the quotient, and, examining its solutions similarly to what was done in the proof of Theorem 4, prove that L_g annihilates Θ. But this is not necessary: the constructed closed form will (by design) be annihilated by L_g, so the fact that L_g is an annihilator of Θ is a consequence of the theorem.

As a next step, we compute a closed form solution C of L_g. The starting point is to decompose L_g as the least common left multiple (LCLM) of two operators of smaller order [7]:

```
# In Maple
L1, L3 := op(DFactorLCLM(Lg, [Dt,t]));
```

The output is a pair of two operators L_1 of order 1, and L_3 of order 3, such that $L_g = \mathrm{LCLM}(L_1, L_3)$. Equivalently, in terms of solution spaces, this means that a basis of solutions of $L_g(y) = 0$ is obtained by the union of the bases of L_1 and L_3, respectively. The operator L_1 admits a simple solution; this can be seen using the Maple command

```
dsolve(diffop2de(L1, [Dt,t], y(t)), y(t));
```

which outputs

$$\frac{3\,x^3}{t} + \frac{3\,x^4}{t^2} - \frac{2}{t} + \frac{3\,x}{t^2} - \frac{x^2}{t^3}.$$

It remains to treat the operator L_3. The starting point is to decompose it as the product of two operators of smaller order [8]:

```
# In Maple
fac := DFactor(L3, [Dt,t]);
```

The output is a pair `fac` $= [L_2, S_1]$ of two operators of order 2, respectively 1, such that $L_3 = L_2\, S_1$. Now the differential equation $L_3(z) = 0$ is equivalent to $L_2(y) = 0$ and $S_1(z) = y$. Hence, it remains to solve $L_2(y) = 0$. This can be done by using the algorithm in [12] and its Maple implementation provided by the authors[2]. Using the command `hypergeomdeg3`, one gets a solution in terms of hypergeometric ${}_2F_1$ functions:

```
SOL:=x/t^3/(t*x^3+2*t-x)/(4*t^2*x^3-t^2+2*t*x-x^2)*
        ((x-3*t)*x*hypergeom([-1/3, -2/3],[1],27*t^3)
            -4*t*(2*t*x^3+t-x)*hypergeom([-1/3, 1/3],[2],27*t^3));
```

One can check that this is indeed a solution of L_2; indeed, the simplification command

```
simplify(eval(diffop2de(fac[1], [Dt,t], y(t)), y(t) = SOL));
```

return 0. Moreover, one can show that this solution coincides (locally at $t = 0$) with the unique power series solution of L_2. Finally, the solution of $L_3(z) = 0$ can be found using

```
simplify(dsolve( diffop2de(fac[2], [Dt,t], z(t) ) = SOL, z(t)));
```

[2] https://www.math.fsu.edu/~vkunwar/hypergeomdeg3/hypergeomdeg3

which yields

$$\frac{tx^3+2t-x}{t^3}\sqrt{(4x^3-1)\,t^2+2tx-x^2}\left(\int\frac{\mathrm{SOL}\cdot t^3}{tx^3+2t-x}\frac{1}{\sqrt{(4x^3-1)\,t^2+2tx-x^2}}\,dt+c\right),$$

where SOL is the hypergeometric expression found above and $c = c(x)$ is a constant function in t, that is found by fitting initial terms of the power series expansions. Putting pieces together yields the expression in the statement of Theorem 4.

Note that the method sketched above is rigorous in the sense that the closed form solution of L_{ct} found in this way is correct by construction. The alternative correctness argument in the proof of the theorem is independent of how the closed form was found.

5 Transcendence of Q

Theorem 5. *Assume that $a \neq b$ and $c \neq 0$. In particular, this is the case if a, b, c are variables in the polynomial ring $\mathbb{Q}[a, b, c]$. Then the power series $Q(x, y)$, $Q(x, 0)$ and $Q(0, y)$ are transcendental over $\mathbb{Q}(a, b, c, x, y, t)$.*

Proof. First note that the algebraicity of the three series is equivalent: if $Q(x, y)$ is algebraic, then so are its specializations $Q(0, y)$ and $Q(x, 0)$; and conversely, if, say, $Q(0, y)$ is algebraic, then by symmetry of the step set so is $Q(x, 0)$, and by the kernel equation, so is $Q(x, y)$.

To reach a contradiction, assume that $Q(x, y)$ is algebraic. Then, by taking the derivative along x and taking the value at $x = y = 0$, the power series $Q_{1,0}$ is also algebraic. Repeating the same process, $Q_{2,0}$ and $Q_{3,0}$ are algebraic. Recall that $Q(0, 0)$ is algebraic [2, Corollary 3]. So all in all, the left-hand side L of Equation (3.2) is algebraic.

If $(a-b)\Big(ab-(ab-ac-bc+abc)Q(0,0)\Big) \neq 0$, this would imply that

$$\Theta = \frac{c\,L}{(a-b)\Big(ab-(ab-ac-bc+abc)Q(0,0)\Big)t^3}$$

is also algebraic, which is a contradiction with Theorem 4.

Thus, $(a-b)\big(ab-(ab-ac-bc+abc)Q(0,0)\big) = 0$. By assumption, $a \neq b$, so the second factor has to be zero. Since $Q(0,0) = 1 + (a+b)ct^3 + \cdots$, extracting coefficients of t^0 and t^3 in this second factor yields $abc = ac + bc$ and $0 = ab(a+b)c$. Since $c \neq 0$, these relations imply $ab = a + b$ and $0 = ab(a + b)$, thus $ab = a + b = 0$, and finally $a = b = 0$, which contradicts the assumption $a \neq b$. $\square$

6 Particular cases and additional remarks

If $a = b$, as observed in [2, Section 5.5], the right-hand side of Equation (3.2) vanishes, and then the series $Q(x,y)$ is algebraic. If $c = 0$, then in particular $Q(0,0) = 1$, and both sides of Equations (3.2) and (2.1) (after clearing out the denominator c) vanish. We do not know if the power series $Q(x,y)$ is algebraic or even D-finite in that case. With sample values of a, b, x, y and $c = 0$, we were not able to guess any algebraic, differential or recurrence relation with the first 10 000 coefficients in t of the series $Q(a,b,c;x,y;t)$.

The generating function $Q(1,1)$, which counts interacting walks regardless of their ending point, is also of interest, besides $Q(0,0)$ and $Q(x,y)$. Experimentally, this generating function appears to be algebraic: for hundreds of thousands of triples $(a,b,c) \in \mathbb{F}_{45007}^3$, with $\mathbb{F}_{45007}$ the finite field with 45007 elements, we could guess a polynomial $P_{a,b,c}(t,u) \in \mathbb{F}_{45007}[t,u]$ of degree 92 in t and degree 24 in u with $P_{a,b,c}(t, Q(a,b,c;1,1;t)) = 0 \bmod t^{2350}$. The next step would be to *lift* it to a polynomial $P(a,b,c;t,u) \in \mathbb{Q}[a,b,c,t,u]$ and to *prove* that $P(a,b,c;t,Q(a,b,c;1,1;t)) = 0$. In principle, this is doable using (a variant of) the approach in [3].

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