A short story of the CSP dichotomy conjecture

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It has been observed long time ago that 'natural' computational problems tend to be complete in 'natural' complexity classes such as NL, P, NP, or PSPACE. Although Ladner in 1975 proved that if P$\neq$NP then there are infinitely many complexity classes between them, all the examples of such intermediate problems are based on diagonalization constructions and are very artificial. Since the seminal work by Feder and Vardi [8] this phenomenon is known as complexity dichotomy (for P and NP), see also Valiant's work [14] in the context of counting problems. Concerted efforts have been made to make this observation more precise, and since the concept of a 'natural' problem is somewhat ambiguous, a possible research direction is to pursue dichotomy results for wide classes of problems. The Constraint Satisfaction problem (CSP) is one of such classes.

Feder and Vardi in [8] observed that the CSP can be conveniently represented as the problem of deciding the existence of a homomorphism between relational structures. In particular, they emphasized the importance of nonuniform CSPs, that is, ones of the form $\mathrm{CSP}(\mathcal{H})$, which asks, given a structure $\mathcal{G}$, whether there exists a homomorphism from a structure $\mathcal{G}$ to a fixed structure $\mathcal{H}$. This problem belongs to NP in general, but can be solved in polynomial time for some structures $\mathcal{H}$. Early dichotomy results on nonuniform CSPs can be traced back to Schaefer [13] for the Generalized Satisfiability problem ($\mathcal{H}$ is 2-element), and Hell and Nesetril [9] for the H-Colouring problem ($\mathcal{H}$ is a graph). A systematic study of the complexity of nonuniform CSPs was initiated in [8], where among other advances Feder and Vardi posed the CSP Dichotomy Conjecture, which is the focus of this tutorial: For every relational structure $\mathcal{H}$ the problem $\mathrm{CSP}(\mathcal{H})$ is either solvable in polynomial time or is NP-complete.

While a wide variety of methods based on model theory, database theory, graph homomorphisms, etc. have been used to attack the dichotomy conjecture, it was the discovery of the *algebraic approach* by Jeavons et al. [11] (see also [2]) that played the decisive role in resolving the conjecture. Using the algebraic approach the dichotomy conjecture was confirmed in a number of important cases, see, e.g., [1], [3], [4], [5], [10]. A recent collection of surveys on intermediate results on the complexity of the CSP and its variants, as well as, other applications of the algebraic approach can be found in [12]. Finally, in 2017 the Bulatov and Zhuk independently confirmed the dichotomy conjecture [6], [15] for arbitrary finite relational structures. For a less technical presentation of the first of these results see [7].

In the first half of this tutorial we survey the connections of the CSP and the dichotomy conjecture with other areas of logic and computer science, and outline the history of the problem. In the second half we explain the main ideas of the first of the solution algorithms [6], [15] for nonuniform CSPs and provide some examples.

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