Morphisms of Open Games

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Abstract

We define a notion of morphisms between open games, exploiting a surprising connection between lenses in computer science and compositional game theory. This extends the (perhaps more intuitively obvious) definition of globular morphisms as mappings between strategy profiles that preserve best responses, and hence in particular preserve Nash equilibria. We construct a symmetric monoidal double category in which the horizontal 1-cells are open games, vertical 1-morphisms are lenses, and 2-cells are morphisms of open games. States (morphisms out of the monoidal unit game) in the vertical category give a flexible solution concept that includes both Nash and subgame perfect equilibria. Products in the vertical category give an external choice operator that is reminiscent of products in game semantics, and is useful in practical examples. We illustrate the above two features with a simple worked example from microeconomics, the market entry game.

Keywords: game theory, open games, lenses, category theory, double categories

1 Introduction

Open games [9,11] provide a foundation to (economic) game theory that is strongly compositional. In general open games are pieces of games, such as decisions and payoff functions, that can be composed either sequentially or in parallel. Key to achieving compositionality is the step of viewing open games as the *morphisms* of a symmetric monoidal category. (This is an instance of the more general research programme of categorical open systems [7].)

However, there are reasons that one might also wish to view open games as the objects of a category. Most obviously, we would like to characterise certain open games using universal properties, in order to reason about them in a more abstract way. In this paper we define a general notion of morphisms between open games, which we call *contravariant lens morphisms*, and explore the resulting structure and applications.

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For a pair of open games of the same type, there is an obvious definition of morphism between them, which we call a *globular morphism*: a function between the sets of strategy profiles that preserves best responses, and hence in particular takes Nash equilibria to Nash equilibria. Contravariant lens morphisms are one of several possible generalisations of this to arbitrary pairs of open games, and heavily makes use of the factorisation of open games in terms of *lenses* (a concept from database theory and functional programming, due to [8]), a connection introduced in [12].

An example of an open game is depicted as a string diagram in figure 1. (Since this string diagram has trivial boundary, it denotes an abstract scalar, or endomorphism of the monoidal unit, in the category of open games.) This represents a 2-player game in which the first player P_1 chooses x, then the second player P_2 observes $f(x)$ and chooses y, with utilities given by $q(x, y) : \mathbb{R}^2$. Such string diagrams are formalised by a coherence theorem in [12]. This open game specifies the same best response relation (and in particular the same Nash equilibria) as the usual game-theoretic analysis as an extensive-form game of imperfect information, but is built compositionally from the open games representing the atomic pieces (P_1, P_2, f, q and various structural elements) of the diagram.

We prove (theorem 6.11) that open games and contravariant lens morphisms form a *symmetric monoidal pseudo double category* [22], the expected structure of 2-cells between arbitrary morphisms in a symmetric monoidal category. A contravariant lens morphism $\alpha : \mathcal{G} \to \mathcal{H}$ between open games can be depicted as a square of the form

$$
\begin{array}{ccc}
\Phi & \xrightarrow[\Sigma(\mathcal{G})]{\mathcal{G}} & \Psi \\
{\scriptstyle \mathfrak{s}(\alpha)} \Big\uparrow & \Downarrow \Sigma(\alpha) & \Big\uparrow {\scriptstyle \mathfrak{t}(\alpha)} \\
\Phi' & \xrightarrow[\Sigma(\mathcal{H})]{\mathcal{H}} & \Psi'
\end{array}
$$

where $\Sigma(\alpha) : \Sigma(\mathcal{G}) \to \Sigma(\mathcal{H})$ is a function from strategy profiles of $\mathcal{G}$ to strategy profiles of $\mathcal{H}$, and $\mathfrak{s}(\alpha)$ and $\mathfrak{t}(\alpha)$ are lenses between the types of $\mathcal{G}$ and $\mathcal{H}$ (which represent the 'boundaries' of $\mathcal{G}, \mathcal{H}$, or the 'interfaces' between game and environment). Double categories allow coherently pasting together squares like this both horizontally and vertically, and a monoidal double category has an additional tensor operation.
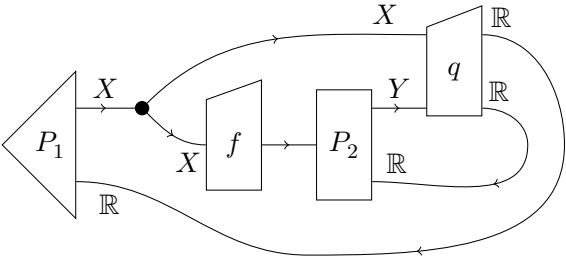

Fig. 1. Example open game

It is expected that morphisms of open games can also be represented as surface diagrams, cobordisms between the string diagrams representing the open games, and composition of morphisms as glueing surfaces. (More foundational work in 3-dimensional algebra is needed to fully justify this, however.) The long-term aim towards which we are working is to be able to reason about game-theoretic concepts using nothing but surface diagrams. Open games and globular morphisms moreover form a bicategory, but we argue by example that the more general double-categorical structure is useful.

There are many possible inequivalent definitions of morphisms between open games, and we argue in favour of contravariant lens morphisms in particular by identifying two attractive properties, as follows.

(i) *States* of open games, that is morphisms out of the monoidal unit open game, provide a fully compositional solution concept that subsumes both Nash and subgame perfect equilibria in a flexible way. States of tensor products of decisions characterise Nash equilibria of normal-form games (theorem 7.11). States of particular sequential compositions characterise Nash equilibria of extensive-form games of perfect information, and furthermore, the categorical property of $\odot$-separability of states characterises those Nash equilibria which are subgame perfect (theorem 8.2).

(ii) Categorical products of open games correspond to an *external choice* operator that is highly reminiscent of products in game semantics, and is useful in practical examples, whenever the game tree of an extensive form game is not perfectly balanced. As well as being interpretable as an external choice, products of open games share with products in game semantics the properties that strategies compose by cartesian product, and plays by disjoint union.

This paper culminates in a worked example that illustrates both of these aspects, in which we describe the market entry game, a simple but important example of game theory as applied to microeconomics. This game has 'dependent subgames', where two completely different things happen depending on an initial choice, which is typically a challenging feature to model for proposed formalisations of games. The combination of states and products of open games can be used to characterise the subgame perfect equilibria of this game in an elegant way.

An alternative definition of morphisms between open games is used in [10], in order to characterise infinitely repeated games as final coalgebras of a functor prepending one additional stage. Contravariant lens morphisms are inferior for this purpose, but on the other hand, the two attractive properties of contravariant lens morphisms identified are not shared by the morphisms of [10]. This suggests that there is no single, canonical notion of morphisms between open games, but at least two with different useful properties. It is expected, however, that all 'reasonable' definitions of morphisms between open games will form a symmetric monoidal double category, and will agree on globular morphisms (i.e. restrict to the same symmetric monoidal bicategory).

Outline. In section 2 we recall some basic definitions of game theory that we

will use. Section 3 is an introduction to open games. In section 4 we recall the categorical structure of lenses, define contravariant lens morphisms, and give some basic properties. In section 5 we prove that they form a double category, and in section 6 we extend this to a symmetric monoidal double category. Section 7 defines states (equilibria) of open games, and characterises Nash equilibria of normal form games, and section 8 extends this to Nash and subgame perfect equilibria of certain extensive form games of perfect information. Section 9 defines external choice as a product of open games using coproducts of lenses, and also incidentally discusses limits of lenses. Finally, section 10 contains a worked example, the market entry game, that illustrates the theory from the previous sections.

Omitted proofs are routine, although sometimes long.

2 Normal-form and extensive-form games

In this section we recall some basic definitions and results of game theory, which can be found in any standard textbook, such as [17].

Definition 2.1 An n-player **normal form game** consists of the following data:

- A sequence of sets $X_1, \ldots, X_n$ of **choices** for each player
- A **payoff function** $k : \prod_{i=1}^n X_i \to \mathbb{R}^n$, giving a real-valued payoff for each player given choices by each player

A (pure) **strategy** for player i in a normal form game is simply a choice $\sigma_i : X_i$, and a (pure) **strategy profile** $\sigma : \prod_{i=1}^n X_i$ is a choice of strategy for each player.

The adjective 'pure' means 'deterministic', in contrast with 'mixed' or probabilistic strategies. The equivalence of strategies and choices characterises a normal form game as both deterministic and simultaneous: in the probabilistic setting a strategy is a probability distribution over choices, and in a dynamic (sequential) game a strategy is a function from observations to choices.

Given a tuple $x : \prod_{i=1}^n X_i$, we write $x_i : X_i$ for the ith projection, $x_{-i} : \prod_{j \neq i} X_j$ for the projection onto all but the ith component, and $(x_i', x_{-i}) : \prod_{j=1}^n X_j$ for the modification of x with ith component $x_i' : X_i$. (This notation is slightly imprecise, but is both useful and standard in game theory.)

Definition 2.2 Given a normal form game $((X_i)_{i=1}^n, k)$, let $\Sigma = \prod_{i=1}^n X_i$ be its set of strategy profiles. We define the **best response relation** $\mathbf{B} \subseteq \Sigma \times \Sigma$ by $(\sigma, \sigma') \in \mathbf{B}$ iff for all players $1 \leq i \leq n$ and all $x_i : X_i$, $\left(k\left(\sigma_i', \sigma_{-i} \right) \right)_i \geq \left(k\left(x_i, \sigma_{-i} \right) \right)_i$. A pure strategy profile is called a (pure) **Nash equilibrium** if it is a fixpoint of the best response relation, that is, if $(\sigma, \sigma) \in \mathbf{B}$.

Equivalently, σ is a Nash equilibrium iff for all players $1 \leq i \leq n$ and all **unilateral deviations** $x_i' : X_i$, $(k(\sigma))_i \geq \left(k\left(x_i', \sigma_{-i} \right) \right)_i$. In words, a Nash equilibrium is a strategy profile in which no player can strictly increase their payoff by unilaterally deviating to another pure strategy. ('Unilateral' means that the strategy profiles of all other players remain fixed.)

In general Nash equilibria are not unique, and the question of how a group of players arrives at one Nash equilibrium out of many is a very subtle and difficult one that we do not address here. Pure Nash equilibria moreover need not exist at all: an example is the 2-player normal form game of *matching pennies*, which has $X_1 = X_2 = \{H, T\}$ and payoff function

$$k(x, y) = \begin{cases} (1, 0) & \text{if } x = y \\ (0, 1) & \text{if } x \neq y \end{cases}$$

(This should not be confused with mixed Nash equilibria, which famously always exist for finite games [18].)

Definition 2.3 An n-player game **extensive form game** is a tree where

- Each non-leaf node is labelled by a player, who makes the decision at that node
- Each leaf node is labelled by an n-tuple of real payoffs
- Each edge is labelled by an **action**, such that no two outgoing edges of a single node are labelled by the same action
- The nodes of the tree are partitioned into **information sets**

An information set is a nonempty set of nodes which are labelled by the same player, and have the same set of actions labelling their successors.

A game of **perfect information** is an extensive form game whose information sets are all singletons.

Small extensive-form games are usually depicted graphically as trees. We will see an example of this in section 10.

Definition 2.4 A **strategy** for a player i in an extensive-form game is a map that takes each information set x owned by player i, to a choice of action available at nodes in x. A **strategy profile** is a tuple of strategies for each player.

A strategy profile induces a **play**, which is a path from the root node to a leaf node. Plays are in bijection with leaf nodes, and so each strategy profile determines a payoff for each player.

Definition 2.5 Each n-player extensive form game induces an n-player normal form game called its **normalisation** or **strategic form** as follows. The set of choices for player i in the normalisation is defined to be the set of strategies for that player in the original game. The payoffs k are determined by the play associated to the strategy profile. A **Nash equilibrium** of an extensive form game is a strategy profile that is a Nash equilibrium of its normalisation.

Definition 2.6 A **subgame** of an extensive form game $\mathcal{G}$ is a subtree $\mathcal{H}$ with the property that if x is any node in $\mathcal{H}$, then any node in $\mathcal{G}$ in the same information set as x is also in $\mathcal{H}$. The subtree $\mathcal{H}$ inherits the structure of an extensive form game from $\mathcal{G}$.

A **subgame perfect equilibrium** of an extensive form game is a strategy profile that restricts to a Nash equilibrium on every subgame.

(This definition implies that the root node of a subgame must be in a singleton information set.) We will see an example illustrating these definitions in section 10.

The remainder of this section considers a simplified special case of perfect information games, adapted from [5,6].

Definition 2.7 An n-player **sequential game** is an extensive-form game of perfect information, whose plays all have length n, in which at level i of the tree all choices are made by player i, and in which any two nodes at the same level have the same set of actions available.

The second condition implies that the tree of a sequential game is balanced. An n-player sequential game is equivalently defined by sets $X_1, \ldots, X_n$ of actions, and a payoff function $k : \prod_{i=1}^n X_i \to \mathbb{R}^n$. Subgames are in bijection with **partial plays** $x_1, \ldots, x_{i-1}$ for $1 \leq i \leq n$. A strategy for player i is a function $\sigma_i : \prod_{j=1}^i X_j \to X_i$, and a strategy profile is a tuple $\sigma : \prod_{i=1}^n \left(\prod_{j=1}^i X_j \to X_i \right)$.

Definition 2.8 Let $(X_i)_{i=1}^n$ be a sequence of sets for $n \geq 1$. Let $1 \leq p \leq q \leq n$. For a sequence $x : \prod_{j=1}^{q-1} X_i$ and a sequence of functions $\sigma : \prod_{i=p}^n \left(\prod_{j=1}^{i-1} X_j \to X_i \right)$, we define a sequence $v_x^\sigma : \prod_{i=1}^n X_i$ extending x, called the **strategic extension** of x by σ, by the course-of-values recursion

$$(v_x^\sigma)_i = \begin{cases} x_i & \text{if } i < q \\ \sigma_i((v_x^\sigma)_1, \ldots, (v_x^\sigma)_{i-1}) & \text{if } i \geq q \end{cases}$$

With this notation, a strategy profile σ of a sequential game is a Nash equilibrium iff

$$\left(k \left(v^\sigma_{(v^\sigma)_1^{i-1}, \sigma_i'((v^\sigma)_1^{i-1})} \right) \right)_i \geq \left(k \left(v^\sigma_{(v^\sigma)_1^{i-1}, x_i} \right) \right)_i$$

for all players $1 \leq i \leq n$ and deviations $x_i : X_i$. It is a subgame-perfect equilibrium iff for all players $1 \leq i \leq n$, all partial plays $x_1, \ldots, x_{i-1}$ and all deviations $x_i : X_i$,

$$\left(k \left(v^\sigma_{x_1, \ldots, x_{i-1}} \right) \right)_i \geq \left(k \left(v^\sigma_{x_1, \ldots, x_{i-1}, x_i} \right) \right)_i$$

3   Introduction to open games

Although we refer the reader to the (mainly pedagogical) paper [9] for an introduction to open games, in this section we give a basic overview. Although this paper does contain a complete definition of the category of open games, it would take too much space to reproduce the motivation and intuition behind it in detail.

Open games are the morphisms of a symmetric monoidal category whose objects are pairs of sets. An open game $\mathcal{G} : (X, S) \to (Y, R)$ is a game-process containing some number of players (zero or more) who may partially observe the current state in X, and perform observable actions making up Y. These actions cause the players

to receive some utility from the set R, and they act *in the first place* in order to optimise the utility that they will receive. Dually, the open game specifies *coutility* which is passed backwards, to be the utility for some player in the past.

Formally, an open game $\mathcal{G} : (X, S) \to (Y, R)$ is a 4-tuple $(\Sigma_{\mathcal{G}}, \mathbf{P}_{\mathcal{G}}, \mathbf{C}_{\mathcal{G}}, \mathbf{B}_{\mathcal{G}})$, where:

- $\Sigma_{\mathcal{G}}$ is the set of *strategy profiles*
- $\mathbf{P}_{\mathcal{G}} : \Sigma_{\mathcal{G}} \times X \to Y$ is the *play function*
- $\mathbf{C}_{\mathcal{G}} : \Sigma_{\mathcal{G}} \times X \times R \to S$ is the *coplay function*
- $\mathbf{B}_{\mathcal{G}} : X \times (Y \to R) \to \mathcal{P}(\Sigma_{\mathcal{G}} \times \Sigma_{\mathcal{G}})$ is the *best response relation*

A *scalar open game* is an open game with type $(1, 1) \to (1, 1)$. It can be seen that a scalar open game $\mathcal{G}$ is defined by a set $\Sigma_{\mathcal{G}}$ and an endo-relation $\mathbf{B}_{\mathcal{G}} \subseteq \Sigma_{\mathcal{G}} \times \Sigma_{\mathcal{G}}$. A Nash equilibrium of a scalar open game is a strategy profile $\sigma : \Sigma_{\mathcal{G}}$ with the property that $(\sigma, \sigma) \in \mathbf{B}_{\mathcal{G}}$.

The idea of open games is to compositionally build best response functions for complex situations in terms of best response functions for the constituent parts. An ordinary best response relation is an endo-relation on the set of strategy profiles. These cannot be composed, but an open game specifies a best response relation *relative* to a *context* $(h, k) : X \times (Y \to R)$, and these turn out to be compositional. The context consists of a *history* $h : X$, which says what happened in the past, and a *continuation* $k : Y \to R$, which says what will happen in the future given the choices made in the present.

The other data, namely the play and coplay functions, are used when composing best responses. Of these, the play function $\mathbf{P}_{\mathcal{G}} : \Sigma_{\mathcal{G}} \times X \to Y$ is directly intuitive: it takes a strategy profile and an observation, and 'runs' the strategy profile on the observation to produce an action. The coplay function is rather less intuitive, and can be thought of as a formal trick to obtain compositionality.

One important example of an open game is a *decision* $\mathcal{D} : (X, 1) \to (Y, \mathbb{R})$, an open game that represents a single decision made by an agent who observes an element of X and then chooses an element of Y in order to maximise a real number. $\mathcal{D}$ is defined as follows:

- $\Sigma_{\mathcal{D}} = X \to Y$
- $\mathbf{P}_{\mathcal{D}}(\sigma, x) = \sigma(x)$
- $\mathbf{C}_{\mathcal{D}}(\sigma, x, r) = *$
- $\mathbf{B}_{\mathcal{D}}(h, k) = \{(\sigma, \sigma') : \Sigma_{\mathcal{G}} \times \Sigma_{\mathcal{G}} \mid \sigma'(h) \in \arg\max(k)\}$

Given open games $\mathcal{G} : (X, S) \to (Y, R)$ and $\mathcal{H} : (Y, R) \to (Z, Q)$, they can be composed to give an open game $\mathcal{H} \circ \mathcal{G} : (X, S) \to (Z, Q)$ which represents the sequential play of $\mathcal{G}$ followed by $\mathcal{H}$. Similarly, given $\mathcal{G}_1 : (X_1, S_1) \to (Y_1, R_1)$ and $\mathcal{G}_2 : (X_2, S_2) \to (Y_2, R_2)$, they can be tensored to give an open game $\mathcal{G}_1 \otimes \mathcal{G}_2 : (X_1 \times X_2, S_1 \times S_2) \to (Y_1 \times Y_2, R_1 \times R_2)$, representing $\mathcal{G}_1$ and $\mathcal{G}_2$ played simultaneously. The definitions of the operators $\circ$ and $\otimes$ are quite involved to give directly, and will be given later in this paper once we have built up more machinery. These operators make open games into the morphisms of a symmetric monoidal category.

As with morphisms in any monoidal category, open games can be denoted by *string diagrams*, a powerful but intuitive graphical syntax. The ability to define a game by its string diagram, and read information about a game's structure from its string diagram, is one of the most interesting features of the open game formalism. However in this paper we do not need string diagrams (this paper contains two string diagrams besides the illustration in the introduction, both immediately followed by the corresponding 4-tuple definition), and so we again refer the reader to [9] or [11].

4   Open games and morphisms

We briefly recall ideas from [12] on lenses. Given sets X, S, Y, R a *bimorphic lens* $\lambda : (X, S) \to (Y, R)$ consists of a *view function* $v_\lambda : X \to Y$ and an *update function* $u_\lambda : X \times R \to S$. We call a pair (X, S) a *diset*, and we refer to disets by Φ, Ψ, Θ.

The term *bimorphic lens* is proposed here by the author to distinguish it from the more familiar cases of monomorphic and polymorphic lenses. (The definition of polymorphic lens is the same, but interpreted in a polymorphic type system [19], and a monomorphic lens satisfies $X = S$ and $Y = R$.) Both monomorphic and polymorphic lenses are usually restricted to those obeying certain axioms called the *lens laws*, but these do not make sense in general for bimorphic lenses, and indeed in compositional game theory we need to consider arbitrary bimorphic lenses.

There is a category **BLens** whose objects are disets and whose morphisms are bimorphic lenses, where the composition of $\lambda : (X, S) \to (Y, R)$ and $\mu : (Y, R) \to (Z, Q)$ has has $v_{\mu \circ \lambda} = v_\mu \circ v_\lambda$ and $u_{\mu \circ \lambda}(x, q) = u_\lambda(x, u_\mu(v_\lambda(x), q))$. There is an obvious symmetric monoidal structure on **BLens** given on disets by $(X, S) \otimes (X', S') = (X \times X', S \times S')$, with monoidal unit $I = (1, 1)$. The *view functor* $\mathbb{V} : \textbf{BLens} \to \textbf{Set}$ represented by I has $\mathbb{V}(X, S) \cong X$, and the *continuation functor* $\mathbb{K} : \textbf{BLens}^{\text{op}} \to \textbf{Set}$ represented by I has $\mathbb{K}(X, S) \cong X \to S$. $\mathbb{V}$ is additionally a strong monoidal functor (for the cartesian monoidal category **Set**) and a fibration, and is the fibrewise opposite of Jacobs' simple fibration [13, section 1.3]. The *context functor* $\mathbb{C} :$ $\textbf{BLens} \times \textbf{BLens}^{\text{op}} \to \textbf{Set}$ is defined on disets by $\mathbb{C}(\Phi, \Psi) = \mathbb{V}(\Phi) \times \mathbb{K}(\Psi)$.

Equivalently, the category of lenses is an (intuitionistic) dialectica category over a trivial (1-valued) logic. This construction is implicit in de Paiva's early work such as [3], and more explicit in her later work on lineales, for example [4], although most of the discussion is about the slightly different *linear* dialectica categories.

Definition 4.1 Let Φ, Ψ be disets. An **open game** $\mathcal{G} : \Phi \longrightarrow \Psi$ consists of the following data:

- A set $\Sigma(\mathcal{G})$ of **strategy profiles**
- For every $\sigma : \Sigma(\mathcal{G})$, a lens $\mathcal{G}(\sigma) : \Phi \to \Psi$
- For every $c : \mathbb{C}(\Phi, \Psi)$, a **best response relation** $\mathbf{B}(c) \subseteq \Sigma(\mathcal{G}) \times \Sigma(\mathcal{G})$

We call a pair $c = (h, k) : \mathbb{C}(\Phi, \Psi)$ a **context** for $\mathcal{G}$, where $h : \mathbb{V}(\Phi)$ is the **history** and $k : \mathbb{K}(\Psi)$ the **continuation**. We also write $\mathfrak{s}(\mathcal{G}) = \Phi$ and $\mathfrak{t}(\mathcal{G}) = \Psi$ for the **source** and **target** of $\mathcal{G}$.

The equivalence between this definition and the more concrete definition in the previous section is easy to see. If $\Phi = (X, S)$ and $\Psi = (Y, R)$ then the family of lenses $\mathcal{G}(\sigma)$ is the same as the play and coplay functions $\Sigma(\mathcal{G}) \to (X \to Y)$, $\Sigma(\mathcal{G}) \to (X \times R \to S)$. By the isomorphisms $\mathbb{V}(\Phi) \cong X$ and $\mathbb{K}(\Psi) \cong Y \to R$, the best response relation can equivalently be written as a function $X \times (Y \to R) \to \mathcal{P}(\Sigma(\mathcal{G}) \to \Sigma(\mathcal{G}))$. The constructions on open games given in the next two sections also agree with the concrete definitions in [9].

We note that given a pair of open games $\mathcal{G}, \mathcal{G}' : \Phi \longrightarrow \Psi$ of the same type, there is an obvious way to define morphisms between them.

Definition 4.2 Let $\mathcal{G}, \mathcal{G}' : \Phi \longrightarrow \Psi$ be open games. A **globular morphism** $\alpha : \mathcal{G} \to \mathcal{G}'$ is a function $\Sigma(\alpha) : \Sigma(\mathcal{G}) \to \Sigma(\mathcal{G}')$ such that

- $\mathcal{G}(\sigma) = \mathcal{G}'(\Sigma(\alpha)(\sigma))$ for all $\sigma : \Sigma(\mathcal{G})$
- If $(\sigma, \sigma') \in \mathbf{B}_{\mathcal{G}}(c)$ then $(\Sigma(\alpha)(\sigma), \Sigma(\alpha)(\sigma')) \in \mathbf{B}_{\mathcal{G}'}(c)$

It is expected that all reasonable definitions of general morphisms between open games will agree on the globular morphisms. However it is demonstrated by example, both in this paper and in [10], that the more general morphisms are necessary. The morphisms we define in this paper are very different to those in [10], but indeed have the same globular morphisms.

The following definition is the key definition of this paper.

Definition 4.3 Let $\mathcal{G} : \Phi \longrightarrow \Psi$ and $\mathcal{G} : \Phi' \longrightarrow \Psi'$ be open games. A **contravariant lens morphism** $\alpha : \mathcal{G} \to \mathcal{G}'$ consists of the following data:

- Lenses $\mathfrak{s}(\alpha) : \Phi' \to \Phi$ and $\mathfrak{t}(\alpha) : \Psi' \to \Psi$
- A function $\Sigma(\alpha) : \Sigma(\mathcal{G}) \to \Sigma(\mathcal{G}')$

satisfying the following two axioms:

- For all $\sigma : \Sigma(\mathcal{G})$, the following diagram in **BLens** commutes:

$$
\begin{array}{ccc}
\Phi & \xrightarrow{\ \mathcal{G}(\sigma)\ } & \Psi \\
\Big\uparrow \mathfrak{s}(\alpha) & & \Big\uparrow \mathfrak{t}(\alpha) \\
\Phi' & \xrightarrow[\ \mathcal{G}'(\Sigma(\alpha)(\sigma))\ ]{} & \Psi'
\end{array}
$$

- For all contexts $(h, k) : \mathbb{C}(\Phi', \Psi)$ and all strategy profiles $\sigma, \sigma' : \Sigma(\mathcal{G})$ such that $(\sigma, \sigma') \in \mathbf{B}_{\mathcal{G}}(\mathfrak{s}(\alpha) \circ h, k)$, we have $(\Sigma(\alpha)(\sigma), \Sigma(\alpha)(\sigma')) \in \mathbf{B}_{\mathcal{G}'}(h, k \circ \mathfrak{t}(\alpha))$

We call a pair $(h, k) : \mathbb{C}(\Phi', \Psi)$ a **context** for α.

We represent a contravariant lens morphism $\alpha : \mathcal{G} \to \mathcal{G}'$ as a square

$$
\begin{array}{ccc}
\Phi & \xrightarrow[\;\Sigma(\mathcal{G})\;]{\mathcal{G}} & \Psi \\[2pt]
\mathfrak{s}(\alpha) \big\uparrow & \Downarrow \Sigma(\alpha) & \big\uparrow \mathfrak{t}(\alpha) \\[2pt]
\Phi' & \xrightarrow[\;\Sigma(\mathcal{G}')\;]{\mathcal{G}'} & \Psi'
\end{array}
$$

anticipating the double category structure.

　　Having done the work of representing open games in terms of lenses, this definition is *almost* automatic, but with one crucial twist: the lenses $\mathfrak{s}(\alpha)$ and $\mathfrak{t}(\alpha)$ go in the opposite direction to α. If they were covariant then the resulting definition would be similar to that of [10] and have many features in common, including the elegant representation of repeated games. However, we will demonstrate in this paper that the alternative definition has several attractive features. We give the definition the more specific name *contravariant lens morphism* to distinguish it from alternatives, but since this definition is the subject of this paper, we will simply refer to it as a *morphism* from now.

Proposition 4.4 *Open games and morphisms form a category* $\mathbf{Game}_1$*, with identities and composition lifted from* $\mathbf{Set}$ *and* $\mathbf{BLens}$*.*

Proposition 4.5 *Let* $\mathcal{G}, \mathcal{G}' : \Phi \longrightarrow \Psi$ *be open games. Then globular morphisms* $\mathcal{G} \to \mathcal{G}'$ *as previously defined are equivalent to morphisms* $\alpha : \mathcal{G} \to \mathcal{G}'$ *with* $\mathfrak{s}(\alpha) = \mathrm{id}_\Phi$ *and* $\mathfrak{t}(\alpha) = \mathrm{id}_\Psi$*.*

　　This amounts to viewing globes (the shape of 2-cells in a bicategory) as squares whose sides have been 'pinched together'.

Proposition 4.6 $\mathfrak{s}, \mathfrak{t} : \mathbf{Game}_1 \to \mathbf{BLens}^{\mathrm{op}}$ *and* $\Sigma : \mathbf{Game}_1 \to \mathbf{Set}$ *are functors.*

5　The double category of open games

A double category [14] is defined as an internal category object in the category of large categories and functors. (Compare to 2-categories, which are defined as categories *enriched* over categories.) Equivalently, a double category contains four sorts of things: objects, horizontal 1-cells, vertical 1-morphisms, and 2-cells. Given a pair of horizontal 1-cells $F : X \longrightarrow Y$, $G : W \longrightarrow Z$, a 2-cell $\alpha : F \to G$ consists of a pair of vertical 1-morphisms $f : X \to W$, $g : Y \to Z$ between the objects, and a square

$$
\begin{array}{ccc}
X & \xrightarrow{\;F\;} & Y \\[2pt]
f \Big\downarrow & \Downarrow \alpha & \Big\downarrow g \\[2pt]
W & \xrightarrow{\;G\;} & Z
\end{array}
$$

(Beware that there are two different conventions in use for orienting double categories. We follow the convention of [22].)

In this paper we are concerned with pseudo double categories, in which the vertical morphisms form a category, but the horizontal morphisms form a category only up to invertible 2-cells. Hence, there is a category of vertical morphisms and a bicategory of horizontal morphisms. In this paper we mostly follow the notation of [22], which gives an explicit definition of symmetric monoidal pseudo double categories. In this section we show that there is a pseudo double category whose objects are disets, horizontal 1-cells are open games, vertical 1-morphisms are reversed lenses, and 2-cells are contravariant lens morphisms.

Definition 5.1 Let Φ be a diset. We define an open game $\mathfrak{u}(\Phi) : \Phi \longrightarrow \Phi$ by $\Sigma(\mathfrak{u}(\Phi)) = 1$, $\mathfrak{u}(\Phi)(*) = \mathrm{id}_\Phi$ and $\mathbf{B}_{\mathfrak{u}(\Phi)}(c) = \{(*, *)\}$ for all contexts $c : \mathbb{C}(\Phi, \Phi)$.

Proposition 5.2 *For each lens* $\lambda : \Phi \to \Psi$ *there is a morphism of open games* $\mathfrak{u}(\lambda) : \mathfrak{u}(\Psi) \to \mathfrak{u}(\Phi)$ *defined by* $\mathfrak{s}(\mathfrak{u}(\lambda)) = \mathfrak{t}(\mathfrak{u}(\lambda)) = \lambda$ *and* $\Sigma(\mathfrak{u}(\lambda)) = \mathrm{id}_1$. *Then* $\mathfrak{u}$ *defines a functor* $\mathbf{BLens}^{\mathrm{op}} \to \mathbf{Game}_1$.

Definition 5.3 Let $\mathcal{G} : \Phi \longrightarrow \Psi$ and $\mathcal{H} : \Psi \longrightarrow \Theta$ be open games. The open game $\mathcal{H} \odot \mathcal{G} : \Phi \longrightarrow \Theta$ is defined by

- $\Sigma(\mathcal{H} \odot \mathcal{G}) = \Sigma(\mathcal{G}) \times \Sigma(\mathcal{H})$
- $(\mathcal{H} \odot \mathcal{G})(\sigma, \tau) = \mathcal{H}(\tau) \circ \mathcal{G}(\sigma)$
- $((\sigma, \tau), (\sigma', \tau')) \in \mathbf{B}_{\mathcal{H} \odot \mathcal{G}}(h, k)$ holds iff both $(\sigma, \sigma') \in \mathbf{B}_{\mathcal{G}}(h, k \circ \mathcal{H}(\tau))$ and $(\tau, \tau') \in \mathbf{B}_{\mathcal{H}}(\mathcal{G}(\sigma) \circ h, k)$

Proposition 5.4 *Let* $\Phi \xrightarrow{\mathcal{G}} \Psi \xrightarrow{\mathcal{H}} \Theta$ *and* $\Phi' \xrightarrow{\mathcal{G}'} \Psi' \xrightarrow{\mathcal{H}'} \Theta'$ *be open games, and let* $\alpha : \mathcal{G} \to \mathcal{G}'$ *and* $\beta : \mathcal{H} \to \mathcal{H}'$ *be morphisms such that* $\mathfrak{t}(\alpha) = \mathfrak{s}(\beta)$. *Then there is a morphism* $\beta \odot \alpha : \mathcal{H} \odot \mathcal{G} \to \mathcal{H}' \odot \mathcal{G}'$ *defined by* $\mathfrak{s}(\beta \odot \alpha) = \mathfrak{s}(\alpha)$, $\mathfrak{t}(\beta \odot \alpha) = \mathfrak{t}(\beta)$ *and* $\Sigma(\beta \odot \alpha) = \Sigma(\alpha) \times \Sigma(\beta)$.

Proposition 5.5 $\odot$ *defines a functor* $\mathbf{Game}_1 \times_{\mathbf{BLens}^{\mathrm{op}}} \mathbf{Game}_1 \to \mathbf{Game}_1$, *where the pullback is over*

$$\mathbf{Game}_1 \xrightarrow{\mathfrak{s}} \mathbf{BLens}^{\mathrm{op}} \xleftarrow{\mathfrak{t}} \mathbf{Game}_1$$

Proposition 5.6 *There are globular natural isomorphisms* $\mathfrak{a}_{\mathcal{I}, \mathcal{H}, \mathcal{G}} : (\mathcal{I} \odot \mathcal{H}) \odot \mathcal{G} \cong \mathcal{I} \odot (\mathcal{H} \odot \mathcal{G})$, $\mathfrak{l}_{\mathcal{G}} : \mathfrak{u}(\mathfrak{t}(\mathcal{G})) \odot \mathcal{G} \cong \mathcal{G}$ *and* $\mathfrak{r}_{\mathcal{G}} : \mathcal{G} \odot \mathfrak{u}(\mathfrak{s}(\mathcal{G})) \cong \mathcal{G}$, *each over the corresponding structure morphism of the cartesian monoidal category* $\mathbf{Set}$.

Proposition 5.7 *The above structures form a pseudo double category whose category of objects is* $\mathbf{BLens}^{\mathrm{op}}$ *and whose category of morphisms is* $\mathbf{Game}_1$.

It follows immediately that open games and globular morphisms form a bicategory. The category **Game** from [9] is precisely the corresponding 1-category of 1-cells modulo globular isomorphism.

6   Monoidal structure

Proposition 6.1 *Let* Φ_1, Φ_2 *be disets. Since* $\mathbb{V}$ *is a monoidal functor, we have projections*

$$\mathbb{V}(\Phi_1) \xleftarrow{\pi_1} \mathbb{V}(\Phi_1 \otimes \Phi_2) \xrightarrow{\pi_2} \mathbb{V}(\Phi_2)$$

Definition 6.2 Let Φ, Φ', Ψ, Ψ' be disets. We define functions

$$L : \mathrm{hom}_{\mathbf{BLens}}(\Phi', \Psi') \to (\mathbb{C}(\Phi \otimes \Phi', \Psi \otimes \Psi') \to \mathbb{C}(\Phi, \Psi))$$

$$R : \mathrm{hom}_{\mathbf{BLens}}(\Phi, \Psi) \to (\mathbb{C}(\Phi \otimes \Psi', \Psi \otimes \Psi') \to \mathbb{C}(\Phi', \Psi'))$$

by

$$L(\lambda')(h, k) = (\pi_1(h), k \circ (\Psi \otimes (\lambda' \circ \pi_2(h))) \circ r_\Psi^{-1})$$
$$R(\lambda)(h, k) = (\pi_2(h), k \circ ((\lambda \circ \pi_1(h)) \otimes \Psi') \circ l_{\Psi'}^{-1})$$

These continuations are, more explicitly,

$$\Psi \xrightarrow{r_\Psi^{-1}} \Psi \otimes I \xrightarrow{\Psi \otimes \pi_2(h)} \Psi \otimes \Phi' \xrightarrow{\Psi \otimes \lambda'} \Psi \otimes \Psi' \xrightarrow{k} I$$

$$\Psi' \xrightarrow{l_{\Psi'}^{-1}} I \otimes \Psi' \xrightarrow{\pi_1(h) \otimes \Psi'} \Phi \otimes \Psi' \xrightarrow{\lambda \otimes \Psi'} \Psi \otimes \Psi' \xrightarrow{k} I$$

Proposition 6.3 *For all lenses* $\Xi \xrightarrow{\kappa} \Phi \xrightarrow{\lambda} \Psi \xrightarrow{\mu} \Theta$ *and* $\Xi' \xrightarrow{\kappa'} \Phi' \xrightarrow{\lambda'} \Psi' \xrightarrow{\mu'} \Theta'$, *the following diagram commutes:*

$$
\begin{array}{ccccc}
& L(\mu' \circ \lambda' \circ \kappa') & & R(\mu \circ \lambda \circ \kappa) & \\
\mathbb{C}(\Xi, \Theta) & \longleftarrow & \mathbb{C}(\Xi \otimes \Xi', \Theta \otimes \Theta') & \longrightarrow & \mathbb{C}(\Xi', \Theta') \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbb{C}(\kappa, \mu) & & \mathbb{C}(\kappa \otimes \kappa', \mu \otimes \mu') & \mathbb{C}(\kappa', \mu') & \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
& L(\lambda') & & R(\lambda) & \\
\mathbb{C}(\Phi, \Psi) & \longleftarrow & \mathbb{C}(\Phi \otimes \Phi', \Psi \otimes \Psi') & \longrightarrow & \mathbb{C}(\Phi', \Psi')
\end{array}
$$

Proof. The history component is straightforward, so we check the continuation component. Let $h : \mathbb{V}(\Xi \otimes \Xi')$. By chasing a continuation $k : \mathbb{K}(\Theta \otimes \Theta')$ around the diagram
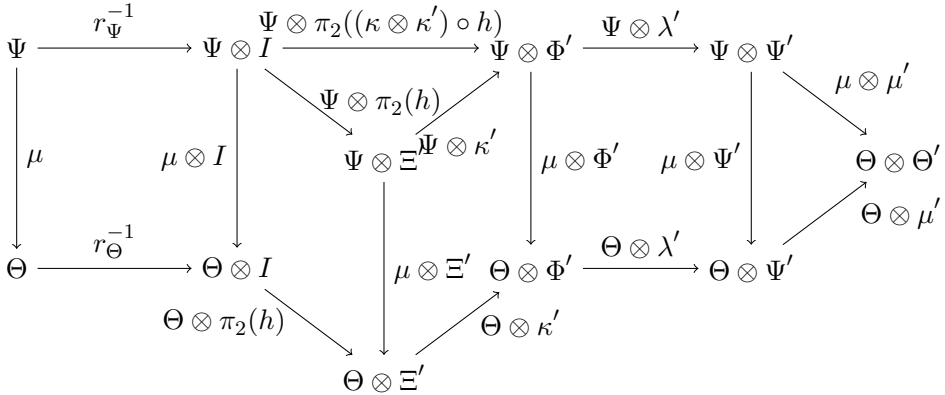
$$
\begin{array}{ccc}
\mathbb{K}(\Theta) & \longleftarrow & \mathbb{K}(\Theta \otimes \Theta') \\
\big\downarrow \mathbb{K}(\mu) & & \big\downarrow \mathbb{K}(\mu \otimes \mu') \\
\mathbb{K}(\Psi) & \longleftarrow & \mathbb{K}(\Psi \otimes \Psi')
\end{array}
$$

we see that it suffices to prove that

$$(\Theta \otimes (\mu' \circ \lambda' \circ \kappa' \circ \pi_2(h))) \circ r_\Psi^{-1} \circ \mu = (\mu \otimes \mu') \circ (\Psi \otimes (\lambda' \circ \pi_2((\kappa \otimes \kappa') \circ h))) \circ r_\Psi^{-1}$$

for commutativity of the left-hand square in the statement of the proposition. (The proof for the right-hand square is symmetric.)

Consider the diagram

Commutativity of the top triangle follows by chasing h around the right-hand square of the $\mathbb{V}$ part. The left-hand square is naturality of r^{-1}, and the other cells are functorality of $\otimes$. $\square$

Definition 6.4 Let $\mathcal{G}_1 : \Phi_1 \longrightarrow \Psi_1$ and $\mathcal{G}_2 : \Phi_2 \longrightarrow \Psi_2$ be open games. We define an open game $\mathcal{G}_1 \otimes \mathcal{G}_2 : \Phi_1 \otimes \Phi_2 \longrightarrow \Psi_1 \otimes \Psi_2$ by

- $\Sigma(\mathcal{G}_1 \otimes \mathcal{G}_2) = \Sigma(\mathcal{G}_1) \times \Sigma(\mathcal{G}_2)$
- $(\mathcal{G}_1 \otimes \mathcal{G}_2)(\sigma_1, \sigma_2) = \mathcal{G}_1(\sigma_1) \otimes \mathcal{G}_2(\sigma_2)$
- $((\sigma_1, \sigma_2), (\sigma_1', \sigma_2')) \in \mathbf{B}_{\mathcal{G}_1 \otimes \mathcal{G}_2}(c)$ iff both $(\sigma_1, \sigma_1') \in \mathbf{B}_{\mathcal{G}_1}(L(\mathcal{G}_2(\sigma_2))(c))$ and $(\sigma_2, \sigma_2') \in \mathbf{B}_{\mathcal{G}_2}(R(\mathcal{G}_1(\sigma_1))(c))$

Proposition 6.5 *Let $\alpha_1 : \mathcal{G}_1 \to \mathcal{G}_1'$ and $\alpha_2 : \mathcal{G}_2 \to \mathcal{G}_2'$ be morphisms of open games. Then there is a morphism $\alpha_1 \otimes \alpha_2 : \mathcal{G}_1 \otimes \mathcal{G}_2 \to \mathcal{G}_1' \otimes \mathcal{G}_2'$ defined by $\mathfrak{s}(\alpha_1 \otimes \alpha_2) = \mathfrak{s}(\alpha_1) \otimes \mathfrak{s}(\alpha_2)$, $\mathfrak{t}(\alpha_1 \otimes \alpha_2) = \mathfrak{t}(\alpha_1) \otimes \mathfrak{t}(\alpha_2)$, and $\Sigma(\alpha_1 \otimes \alpha_2) = \Sigma(\alpha_1) \times \Sigma(\alpha_2)$.*

Proposition 6.6 $\otimes$ *defines a functor* $\mathbf{Game}_1 \times \mathbf{Game}_1 \to \mathbf{Game}_1$.

Proposition 6.7 *There are natural isomorphisms $\alpha_{\mathcal{G}, \mathcal{H}, \mathcal{I}} : (\mathcal{G} \otimes \mathcal{H}) \otimes \mathcal{I} \cong \mathcal{G} \otimes (\mathcal{H} \otimes \mathcal{I})$, $\lambda_{\mathcal{G}} : \mathfrak{u}(I) \otimes \mathcal{G} \cong \mathcal{G}$, $\rho_{\mathcal{G}} : \mathcal{G} \otimes \mathfrak{u}(I) \cong \mathcal{G}$ and $s_{\mathcal{G}, \mathcal{H}} : \mathcal{G} \otimes \mathcal{H} \cong \mathcal{H} \otimes \mathcal{G}$, equal to the structure morphisms of the symmetric monoidal category $\mathbf{BLens}^{\mathrm{op}} \times \mathbf{Set} \times \mathbf{BLens}^{\mathrm{op}}$.*

Proposition 6.8 $\mathbf{Game}_1$ *is a symmetric monoidal category.*

Proposition 6.9 $\mathfrak{s}, \mathfrak{t} : \mathbf{Game}_1 \to \mathbf{BLens}^{\mathrm{op}}$ *and* $\Sigma : \mathbf{Game}_1 \to \mathbf{Set}$ *are strict symmetric monoidal functors.*

Proposition 6.10 *There are globular isomorphisms*

$$\mathfrak{U}_{\Phi_1, \Phi_2} : \mathfrak{u}(\Phi_1 \otimes \Phi_2) \xrightarrow{\cong} \mathfrak{u}(\Phi_1) \otimes \mathfrak{u}(\Phi_2)$$

$$\mathfrak{X}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{H}_1,\mathcal{H}_2} : (\mathcal{H}_1 \otimes \mathcal{H}_2) \odot (\mathcal{G}_1 \otimes \mathcal{G}_2) \xrightarrow{\cong} (\mathcal{H}_1 \odot \mathcal{G}_1) \otimes (\mathcal{H}_2 \odot \mathcal{G}_2)$$

Proof. We check the second of these. Take

$$\Sigma_{\mathfrak{X}} : (\Sigma(\mathcal{G}_1) \times \Sigma(\mathcal{G}_2)) \times (\Sigma(\mathcal{H}_1) \times \Sigma(\mathcal{H}_2)) \to (\Sigma(\mathcal{G}_1) \times \Sigma(\mathcal{H}_1)) \times (\Sigma(\mathcal{G}_2) \times \Sigma(\mathcal{H}_2)),$$
$$((\sigma_1, \sigma_2), (\tau_1, \tau_2)) \mapsto ((\sigma_1, \tau_1), (\sigma_2, \tau_2))$$

The first axiom follows from bifunctorality of $\otimes$ on **BLens**:

$$(\mathcal{H}_1(\tau_1) \otimes \mathcal{H}_2(\tau_2)) \circ (\mathcal{G}_1(\sigma_1) \otimes \mathcal{G}_2(\sigma_2)) = (\mathcal{H}_1(\tau_1) \circ \mathcal{G}_1(\sigma_1)) \otimes (\mathcal{H}_2(\tau_2) \circ \mathcal{G}_2(\sigma_2))$$

Write $\Phi_1 \xrightarrow{\mathcal{G}_1} \Psi_1 \xrightarrow{\mathcal{H}_1} \Theta_1$ and $\Phi_2 \xrightarrow{\mathcal{G}_2} \Psi_2 \xrightarrow{\mathcal{H}_2} \Theta_2$. For the second axiom we calculate:

$$\begin{aligned}
&(((\sigma_1, \sigma_2), (\tau_1, \tau_2)), ((\sigma_1', \sigma_2'), (\tau_1', \tau_2'))) \in \mathbf{B}_{(\mathcal{H}_1 \otimes \mathcal{H}_2) \odot (\mathcal{G}_1 \otimes \mathcal{G}_2)}(c)\\
\iff &((\sigma_1, \sigma_2), (\sigma_1', \sigma_2')) \in \mathbf{B}_{\mathcal{G}_1 \otimes \mathcal{G}_2}(\mathbb{C}(\Phi_1 \otimes \Phi_2, \mathcal{H}_1(\tau_1) \otimes \mathcal{H}_2(\tau_2))(c))\\
&\text{and } ((\tau_1, \tau_2), (\tau_1', \tau_2')) \in \mathbf{B}_{\mathcal{H}_1 \otimes \mathcal{H}_2}(\mathbb{C}(\mathcal{G}_1(\sigma_1) \otimes \mathcal{G}_2(\sigma_2), \Theta_1 \otimes \Theta_2)(c))\\
\iff &(\sigma_1, \sigma_1') \in \mathbf{B}_{\mathcal{G}_1}(L(\mathcal{G}_2(\sigma_2))(\mathbb{C}(\Phi_1 \otimes \Phi_2, \mathcal{H}_1(\tau_1) \otimes \mathcal{H}_2(\tau_2))(c)))\\
&\text{and } (\sigma_2, \sigma_2') \in \mathbf{B}_{\mathcal{G}_2}(R(\mathcal{G}_1(\sigma_1))(\mathbb{C}(\Phi_1 \otimes \Phi_2, \mathcal{H}_1(\tau_1) \otimes \mathcal{H}_2(\tau_2))(c)))\\
&\text{and } (\tau_1, \tau_1') \in \mathbf{B}_{\mathcal{H}_1}(L(\mathcal{H}_2(\tau_2))(\mathbb{C}(\mathcal{G}_1(\sigma_1) \otimes \mathcal{G}_2(\sigma_2), \Theta_1 \otimes \Theta_2)(c)))\\
&\text{and } (\tau_2, \tau_2') \in \mathbf{B}_{\mathcal{H}_2}(R(\mathcal{H}_1(\tau_1))(\mathbb{C}(\mathcal{G}_1(\sigma_1) \otimes \mathcal{G}_2(\sigma_2), \Theta_1 \otimes \Theta_2)(c)))\\
\iff &(\sigma_1, \sigma_1') \in \mathbf{B}_{\mathcal{G}_1}(\mathbb{C}(\Phi_1, \mathcal{H}_1(\tau_1))(L(\mathcal{H}_2(\tau_2) \circ \mathcal{G}_2(\sigma_2))(c)))\\
&\text{and } (\tau_1, \tau_1') \in \mathbf{B}_{\mathcal{H}_1}(\mathbb{C}(\mathcal{G}_1(\sigma_1), \Theta_1)(L(\mathcal{H}_2(\tau_2) \circ \mathcal{G}_2(\sigma_2))(c)))\\
&\text{and } (\sigma_2, \sigma_2') \in \mathbf{B}_{\mathcal{G}_2}(\mathbb{C}(\Phi_2, \mathcal{H}_2(\tau_2))(R(\mathcal{H}_1(\tau_1) \circ \mathcal{G}_1(\sigma_1))(c)))\\
&\text{and } (\tau_2, \tau_2') \in \mathbf{B}_{\mathcal{H}_2}(\mathbb{C}(\mathcal{G}_2(\sigma_2), \Theta_2)(R(\mathcal{H}_1(\tau_1) \circ \mathcal{G}_1(\sigma_1))(c)))\\
\iff &((\sigma_1, \tau_1), (\sigma_1', \tau_1')) \in \mathbf{B}_{\mathcal{H}_1 \odot \mathcal{G}_1}(L(\mathcal{H}_2(\tau_2) \circ \mathcal{G}_2(\sigma_2))(c))\\
&\text{and } ((\sigma_2, \tau_2), (\sigma_2', \tau_2')) \in \mathbf{B}_{\mathcal{H}_2 \odot \mathcal{G}_2}(R(\mathcal{H}_1(\tau_1) \circ \mathcal{G}_1(\sigma_1))(c))\\
\iff &(((\sigma_1, \tau_1), (\sigma_2, \tau_2)), ((\sigma_1', \tau_1'), (\sigma_2', \tau_2'))) \in \mathbf{B}_{(\mathcal{H}_1 \odot \mathcal{G}_1) \otimes (\mathcal{H}_2 \odot \mathcal{G}_2)}(c)
\end{aligned}$$

The relevant equalities between contexts, which all follow from proposition 6.3, are summarised in figure 2. □

Theorem 6.11 *Disets, open games and morphisms form a symmetric monoidal pseudo double category.*

Having obtained a higher-dimensional categorical language that is fully compositional and has the potential to be graphical, the next step is to formalise interesting game-theoretic concepts using this language. This is the subject of the remainder of this paper.

7 States of open games

Definition 7.1 Let $\mathcal{G} : \Phi \longrightarrow \Psi$ be an open game. A **state** of $\mathcal{G}$ is a pair (σ, k) where $\sigma : \Sigma(\mathcal{G})$ and $k : \mathbb{K}(\Psi)$ with the property that for all $h : \mathbb{V}(\Phi)$, $(\sigma, \sigma) \in \mathbf{B}_{\mathcal{G}}(h, k)$.
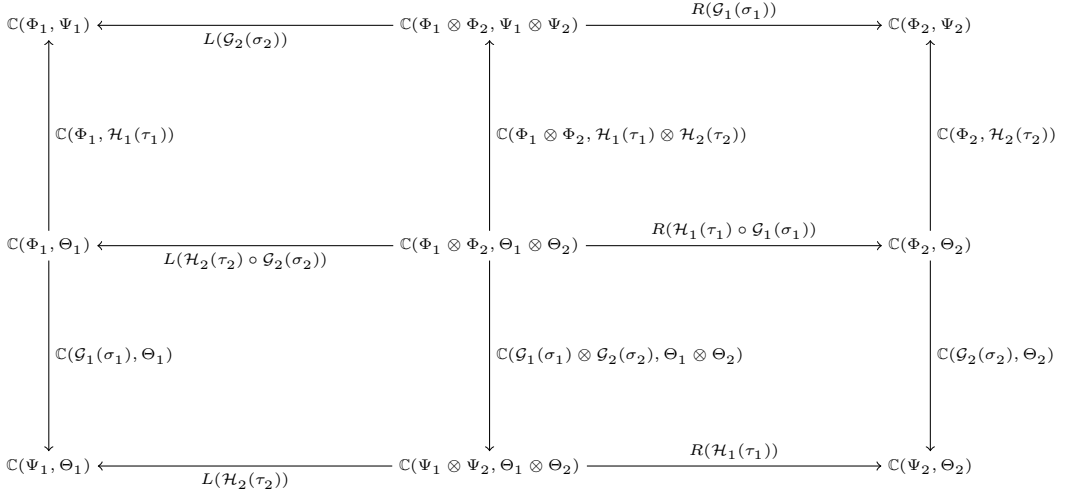
Fig. 2. Instances of proposition 6.3 used in proof of proposition 6.10

We also say that σ is a state of $\mathcal{G}$ **over** k. We write $\mathbb{S}(\mathcal{G})$ for the set of states of $\mathcal{G}$.

Informally, a state consists of a choice of continuation and a choice of strategy that is a Nash equilibrium for that continuation, for all histories. As we will see, this combines aspects of Nash and subgame perfect equilibria in a flexible way. The definition of states is a weakening of the definition of *solution* considered in [11, section 3.3], in which σ may depend on k, but not on h. The latter is more game-theoretically plausible (because many open games have interesting solutions in this stronger sense), but is incompatible with $\otimes$ and hence not fully compositional. The definition of state is intended as a compromise between game-theoretic usefulness and compositionality.

Proposition 7.2 *Let $\alpha : \mathcal{G} \to \mathcal{G}'$ be a morphism of open games, and let σ be a state of $\mathcal{G}$ over k. Then $\Sigma(\alpha)(\sigma)$ is a state of $\mathcal{G}'$ over $k \circ \mathfrak{t}(\alpha)$. Thus we have a function $\mathbb{S}(\alpha) : \mathbb{S}(\mathcal{G}) \to \mathbb{S}(\mathcal{G}')$.*

Proof. Let $h : \mathbb{V}(\mathfrak{s}(\mathcal{G}'))$. Since σ is a state of $\mathcal{G}$ over k, we have $(\sigma, \sigma) \in \mathbf{B}_{\mathcal{G}}(\mathfrak{s}(\alpha) \circ h, k)$. Since α is a morphism, it follows that $(\Sigma(\alpha)(\sigma), \Sigma(\alpha)(\sigma)) \in \mathbf{B}_{\mathcal{G}'}(h, k \circ \mathfrak{t}(\alpha))$. $\square$

Proposition 7.3 $\mathbb{S}$ *defines a functor* $\mathbf{Game}_1 \to \mathbf{Set}$.

Proposition 7.4 $\mathbb{S} \cong \hom_{\mathbf{Game}_1}(\mathfrak{u}(I), -)$.

Proof. States (σ, k) of $\mathcal{G}$ are in bijection with morphisms $\alpha : \mathfrak{u}(I) \to \mathcal{G}$, as follows. There is a unique choice $\mathfrak{s}(\alpha) = k \circ \mathcal{G}(\sigma)$ satisfying the first axiom. Since by definition $(*, *) \in \mathbf{B}_{\mathfrak{u}(I)}(c)$ for the unique $c : \mathbb{C}(I, I)$, and all contexts for α are of the form (h, id_I) for $h : \mathbb{V}(\Phi)$, the second axiom is equivalent to the condition that $(\sigma, \sigma) \in \mathbf{B}_{\mathcal{G}}(h, k)$ for all $h : \mathbb{V}(\Phi)$. $\square$

This is summarised in the diagram

$$
\begin{array}{ccc}
I & \xrightarrow{\ \mathfrak{u}(I)\ } & I \\[-2pt]
& 1 & \\
{\scriptstyle k \circ \mathcal{G}(\sigma)}\Big\uparrow & \Downarrow \sigma & \Big\uparrow{\scriptstyle k} \\
\Phi & \xrightarrow[\Sigma(\mathcal{G})]{\ \mathcal{G}\ } & \Psi
\end{array}
$$

Noting that $\mathfrak{u}(I)$ is the monoidal unit of $\mathbf{Game}_1$, states of open games are indeed states in the more general sense of monoidal categories.

It does not appear to be possible to define states in a reasonable way using only the bicategory structure. The problem is that because the double category of open games is not a framed bicategory [21], we cannot represent a non-globular 2-cell by a globular one.

By this representation, we immediately obtain definitions of $\odot$ and $\otimes$ for states, as described by the following two propositions.

Proposition 7.5 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be open games. Let σ_1 be a state of $\mathcal{G}_1$ over k_1, and let σ_2 be a state of $\mathcal{G}_2$ over k_2. Then (σ_1, σ_2) is a state of $\mathcal{G}_1 \otimes \mathcal{G}_2$ over $k_1 \otimes k_2$.*

Proposition 7.6 (Backward induction) *Let $\mathcal{G}$ and $\mathcal{H}$ be open games with $\mathfrak{t}(\mathcal{G}) = \mathfrak{s}(\mathcal{H})$. Let τ be a state of $\mathcal{H}$ over k, and let σ be a state of $\mathcal{G}$ over $k \circ \mathcal{H}(\tau)$. Then (σ, τ) is a state of $\mathcal{H} \odot \mathcal{G}$ over k.*

As its name suggests, the previous result is similar to the *backward induction* method of game theory, although in a far more general form. The intuition is that we first fix the strategy profile τ of $\mathcal{H}$, and then players in $\mathcal{G}$ reason as though players in $\mathcal{H}$ play τ.

Definition 7.7 An open game $\mathcal{G} : \Phi \longrightarrow \Psi$ is called **strategically trivial** if it satisfies the following two properties:

- $\Sigma(\mathcal{G}) \cong 1$
- $(*, *) \in \mathbf{B}_{\mathcal{G}}(c)$ for all contexts $c : \mathbb{C}(\Phi, \Psi)$

Since a strategically trivial open game $\mathcal{G} : \Phi \longrightarrow \Psi$ is defined up to globular isomorphism by its lens $\mathcal{G}(*) : \Phi \to \Psi$, the (horizontal) subcategory of strategically trivial games is isomorphic to $\mathbf{BLens}$. Given a lens $\lambda : \Phi \to \Psi$, we abuse notation and write $\lambda : \Phi \longrightarrow \Psi$ for the corresponding strategically trivial open game. In particular, for functions $f : X \to Y$ and $g : R \to S$ we have a strategically trivial open game $(f, g) : (X, S) \longrightarrow (Y, R)$, and for each set X there is a strategically trivial open game $\varepsilon : (X, X) \longrightarrow I$ corresponding to the counit lens [12] with update function $u_\varepsilon(x, *) = x$.

Proposition 7.8 *Let $\mathcal{G} : \Phi \longrightarrow \Psi$ be a strategically trivial game. Then for every continuation $k : \mathbb{K}(\Psi)$, $\mathcal{G}$ has exactly one state over k. In particular, a strategically trivial game $\mathcal{G} : \Phi \longrightarrow I$ has a unique state.*

Note that the strategically trivial games $(X, S) \longrightarrow I$ are 'internal continuations',

and are in bijection with functions $X \to S$. These include the utility functions of game theory. (A morphism to the monoidal unit in a monoidal category is sometimes called an *effect*.)

The following family of open games represents a rational agent making a choice from Y after observing an element of X. It is the main building block, besides strategically trivial open games, from which examples are built in practice.

Definition 7.9 For all sets X and Y we define an open game $\mathcal{D}_{X,Y} : (X, 1) \longrightarrow (Y, \mathbb{R})$ (where $\mathbb{R}$ is the set of real numbers, interpreted as utility) called a **decision**, as follows:

- $\Sigma(\mathcal{D}_{X,Y}) = X \to Y$
- $\mathcal{D}_{X,Y}(\sigma)$ is the unique lens $(X, 1) \to (Y, \mathbb{R})$ with $\mathbb{V}(\mathcal{D}_{X,Y}(\sigma)) = \sigma$
- $(\sigma, \sigma') \in \mathbf{B}_{\mathcal{D}_{X,Y}}(h, k)$ iff $k(\sigma'(h)) \geq k(y)$ for all $y : Y$

(Notice that this is a constant relation, which is characteristic of 1-player open games. Best response only becomes a nontrivial relation when more than one player interacts strategically.)

Proposition 7.10 *Let X, Y be sets and $k : Y \to \mathbb{R}$. Then there is a bijective correspondence between*

- *States of $\mathcal{D}_{X,Y}$ over k*
- *Functions $X \to \arg\max(k)$, where $\arg\max(k) \subseteq Y$ is the set of maximising points of k*

In particular, if Y is finite then $\mathcal{D}_{X,Y}$ has at least one state over every k.

Proof. The pair (σ, k) is a state of $\mathcal{D}_{X,Y}$ iff $k(\sigma(x)) \geq k(y)$ for all $x : X$ and $y : Y$, which is equivalent to the range of σ being $\arg\max(k)$.   □

The following theorem and its proof are essentially restatements of [11, sections 3.1.2 – 3.1.4]. This is one of the key connections between open games and classical game theory, characterising tensor products of decisions as normal form games.

Theorem 7.11 *Let $Y_1, \dots, Y_n$ be an indexed family of sets for $n \geq 1$, and let $k : \prod_{i=1}^{n} Y_i \to \mathbb{R}^n$ be a function. Then there is a bijective correspondence between*

- *States of $\bigotimes_{i=1}^{n} \mathcal{D}_{1,Y_i}$ over k*
- *Pure Nash equilibria of the n-player normal form game with outcome function k*

Proof. We can prove by induction on n that the open game

$$\bigotimes_{i=1}^{n} \mathcal{D}_{1,Y_i} : I \longrightarrow \left(\prod_{i=1}^{n} Y_i, \mathbb{R}^n \right)$$

is concretely given, up to unique natural isomorphism, as follows.

- Its set of strategy profiles is

$$\Sigma\left(\bigotimes_{i=1}^{n} \mathcal{D}_{1,Y_i}\right) = \prod_{i=1}^{n} Y_i$$

- The lens

$$\left(\bigotimes_{i=1}^{n} \mathcal{D}_{1,Y_i}\right)(\sigma) : I \to \left(\prod_{i=1}^{n} Y_i, \mathbb{R}^n\right)$$

  is the unique one with

$$\mathbb{V}\left(\left(\bigotimes_{i=1}^{n} \mathcal{D}_{1,Y_i}\right)(\sigma)\right) = \sigma$$

- The best response relation

$$(\sigma, \sigma') \in \mathbf{B}_{\bigotimes_{i=1}^{n} \mathcal{D}_{1,Y_i}}(*, k)$$

  holds iff, for all $1 \leq j \leq n$ and all $y_j : Y_j$,

$$(k(\sigma'_j, \sigma_{-j}))_j \geq (k(y_j, \sigma_{-j}))_j$$

The theorem follows from this, since the set of strategy profiles of $\bigotimes_{i=1}^{n} \mathcal{D}_{1,Y_i}$ is the set of pure strategy profiles of the normal form game, and fixpoints of $\mathbf{B}_{\bigotimes_{i=1}^{n} \mathcal{D}_{1,Y_i}}(*, k)$ are Nash equilibria by definition. □

Note that if α_1 is a state of $\mathcal{D}_{I,Y_1}$ over k_1 and α_2 is a state of $\mathcal{D}_{I,Y_2}$ over k_2 then $\alpha_1 \otimes \alpha_2$ is a state of $\mathcal{D}_{I,Y_1} \otimes \mathcal{D}_{I,Y_2}$ over $k_1 \otimes k_2$. However, the continuation $k_1 \otimes k_2$ corresponds to a function $Y_1 \times Y_2 \to \mathbb{R}^2$ that is 'separable', in the sense that $(k_1 \otimes k_2)(y_1, y_2) = (k_1(y_1), k_2(y_2))$, and hence the $\otimes$-separable states can only be equilibria of these separable games. Such games are game-theoretically trivial in the sense that there is no strategic interaction between players, and so a game degenerates into a tuple of independent maximisation problems. Only the $\otimes$-inseparable states of tensor products of decisions correspond to nontrivial Nash equilibria. More generally, computational hardness results from algorithmic game theory, such as hardness of computing mixed ϵ-Nash equilibria [20], intuitively suggest there are inescapable limits on what it will be possible to achieve by compositional methods.

8 Backward induction

The previous theorem characterised Nash equilibria of normal form games in terms of states of open games. In this section we do the same to Nash and subgame perfect equilibria of extensive form games of perfect information.

Definition 8.1 Let $n \geq 1$ and let $X_1, \ldots, X_n$ be a sequence of sets. We define an open game

$$\mathcal{D}_{X_1,\ldots,X_n}^{\Delta} : \left(\prod_{i=1}^{n-1} X_i, \mathbb{R}^{n-1}\right) \longrightarrow \left(\prod_{i=1}^{n} X_i, \mathbb{R}^n\right)$$

by the algebraic expression

$$\left(\prod_{i=1}^{n} X_i \times X_n, s_{\mathbb{R}^{n-1},\mathbb{R}^n}\right) \odot \left(\left(\left(\prod_{i=1}^{n-1} X_i, 1\right) \otimes \mathcal{D}_{\prod_{i=1}^{n-1} X_i, X_n} \otimes \left(1, \mathbb{R}^{n-1}\right)\right) \odot \left(\Delta_{\prod_{i=1}^{n-1} X_i}, \mathbb{R}^{n-1}\right)\right)$$

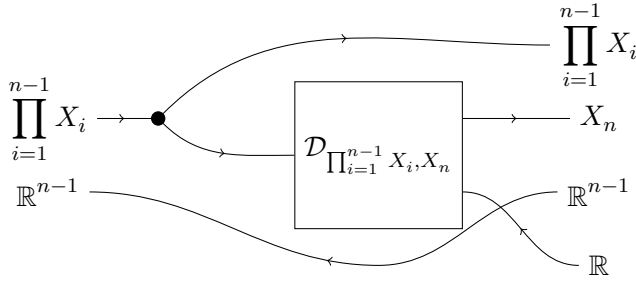This expression is visualised as a string diagram in figure 3.

Fig. 3. Definition of $\mathcal{D}_{X_1,\ldots,X_n}^{\Delta}$ in terms of $\mathcal{D}_{\prod_{i=1}^{n-1} X_i, X_n}$

We now come to one of the key results of this paper: For an extensive form game, Nash equilibria correspond to states in general, and subgame perfect equilibria correspond to $\odot$-separable states. Using this, we can use the category-theoretic notion of $\odot$-separability (which could moreover be visible in a surface diagram as a glued boundary) to *define* the game-theoretic notion of subgame perfection for general open games. We state and prove the theorem for the special case of finite sequential games [6] for simplicity.

Theorem 8.2 *Let $X_1,\ldots,X_n$ be an indexed family of sets for $n \geq 1$, and let $k : \prod_{i=1}^{n} X_i \to \mathbb{R}^n$. Then there is a bijective correspondence between:*

- *States of $\bigodot_{i=1}^{n} \mathcal{D}_{X_1,\ldots,X_i}^{\Delta}$ over k*
- *Pure Nash equilibria of the n-player sequential game with outcome function k*

Moreover there is a bijective correspondence between:

- *States of $\bigodot_{i=1}^{n} \mathcal{D}_{X_1,\ldots,X_i}^{\Delta}$ over k that are of the form $\bigodot_{i=1}^{n} \alpha_i$, where each α_i is a state of $\mathcal{D}_{X_1,\ldots,X_i}^{\Delta}$*
- *Subgame perfect equilibria of the n-player sequential game with outcome function k*

Proof. It is routine to calculate that $\mathcal{D}_{X_1,\ldots,X_n}^{\Delta}$ is concretely given as follows:

- The set of strategy profiles is

$$\Sigma\left(\mathcal{D}_{X_1,\ldots,X_n}^{\Delta}\right) = \prod_{i=1}^{n-1} X_i \to X_n$$

- For $\sigma : \prod_{i=1}^{n-1} X_i \to X_n$, the lens

$$\left(\mathcal{D}^{\Delta}_{X_1,\ldots,X_n} \right)(\sigma) : \left(\prod_{i=1}^{n-1} X_i, \mathbb{R}^{n-1} \right) \to \left(\prod_{i=1}^{n} X_i, \mathbb{R}^n \right)$$

is given by

$$v_{\left(\mathcal{D}^{\Delta}_{X_1,\ldots,X_n} \right)(\sigma)}(x) = (x, \sigma(x))$$

and

$$u_{\left(\mathcal{D}^{\Delta}_{X_1,\ldots,X_n} \right)(\sigma)}(x, r) = r_{-n}$$

- $(\sigma, \sigma') \in \mathbf{B}_{\mathcal{D}^{\Delta}_{X_1,\ldots,X_n}}(h, k)$ holds iff

$$(k(h, \sigma'(h)))_n \geq (k(h, x_n))_n$$

for all $x_n : X_n$.

For $1 \leq p \leq q \leq n$, a sequence $x : \prod_{j=1}^{q-1} X_j$ and a sequence of functions $\sigma : \prod_{i=p}^{n} \left(\prod_{j=1}^{i-1} X_j \to X_i \right)$, we define a sequence $v_x^{\sigma} : \prod_{i=1}^{n} X_i$ extending x, called the **strategic extension** of x by σ, by the course-of-values recursion

$$(v_x^{\sigma})_i = \begin{cases} x_i & \text{if } i < q \\ \sigma_i((v_x^{\sigma})_1, \ldots, (v_x^{\sigma})_{i-1}) & \text{if } i \geq q \end{cases}$$

When $p = q = 1$, and hence x is the empty sequence, we write v^{σ}. This is called the **strategic play** of σ.

Let

$$\mathcal{G}_{X_1,\ldots,X_n} = \bigodot_{i=1}^{n} \mathcal{D}^{\Delta}_{X_1,\ldots,X_i} : I \longmapsto \left(\prod_{i=1}^{n} X_i, \mathbb{R}^n \right)$$

We prove by induction on n that $\mathcal{G}_{X_1,\ldots,X_n}$ is given explicitly as follows, up to globular isomorphism.

- The set of strategy profiles is

$$\Sigma(\mathcal{G}_{X_1,\ldots,X_n}) = \prod_{i=1}^{n} \left(\prod_{j=1}^{i-1} X_j \to X_i \right)$$

- For $\sigma : \Sigma(\mathcal{G}_{X_1,\ldots,X_n})$, the lens $\mathcal{G}_{X_1,\ldots,X_n}(\sigma)$ is the unique one with

$$\mathbb{V}(\mathcal{G}_{X_1,\ldots,X_n}(\sigma))(*) = v^{\sigma}$$

- For $k : \prod_{i=1}^{n} X_i \to \mathbb{R}^n$, the best response relation

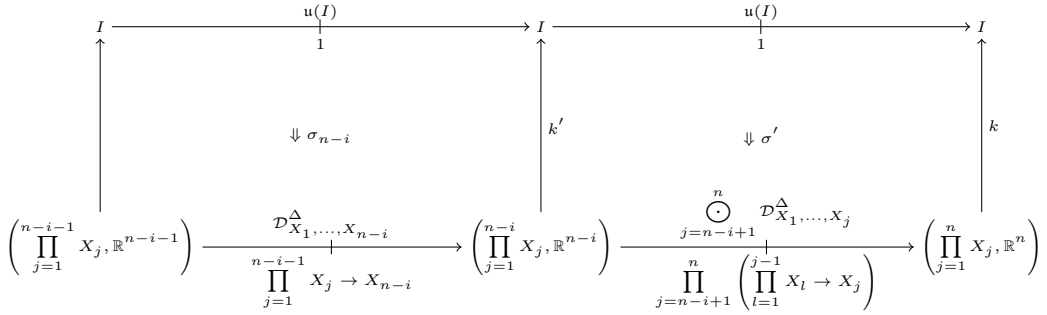$$(\sigma, \sigma') \in \mathbf{B}_{\mathcal{G}_{X_1,\ldots,X_n}}(*, k)$$

Fig. 4. Inductive step of theorem 8.2

holds iff for all $1 \leq i \leq n$ and all $x_i : X_i$,

$$\left(k \left(v^\sigma_{(v^\sigma)_1^{i-1}, \sigma_i'((v^\sigma)_1^{i-1})} \right) \right)_i \geq \left(k \left(v^\sigma_{(v^\sigma)_1^{i-1}, x_i} \right) \right)_i$$

The first part of the theorem follows from this.

Next we prove by induction on i that there is a bijective correspondence between:

- States α of

$$\bigodot_{j=n-i+1}^{n} \mathcal{D}^\Delta_{X_1,\ldots,X_j} : \left(\prod_{j=1}^{n-i} X_j, \mathbb{R}^{n-i} \right) \longrightarrow \left(\prod_{j=1}^{n} X_j, \mathbb{R}^{n} \right)$$

over $k : \prod_{j=1}^n X_j \to \mathbb{R}^n$ that are of the form

$$\alpha = \bigodot_{j=n-i+1}^{n} \alpha_j$$

where each α_j is a state of $\mathcal{D}^\Delta_{X_1,\ldots,X_j}$

- Strategy profiles

$$(\sigma)^n_{n-i+1} : \prod_{j=n-i+1}^{n} \left(\prod_{l=1}^{j-1} X_l \to X_j \right)$$

with the property that for all $j \geq n - i + 1$, all partial plays $x : \prod_{l=1}^{j-1} X_l$ and all deviations $x_j : X_j$,

$$\left(k \left(v^\sigma_{x, \sigma_j(x)} \right) \right)_j \geq \left(k \left(v^\sigma_{x, x_j} \right) \right)_j$$

The second part of the theorem follows by taking $i = n$.

For the inductive step, a $\odot$-separable state α of $\bigodot_{j=n-i}^{n} \mathcal{D}^\Delta_{X_1,\ldots,X_j}$ is of the form $\alpha = \alpha' \odot \alpha_{n-i}$, where α_{n-i} is a state of $\mathcal{D}^\Delta_{X_1,\ldots,X_{n-i}}$ and α' is a $\odot$-separable state of $\bigodot_{j=n-i+1}^{n} \mathcal{D}^\Delta_{X_1,\ldots,X_j}$. This situation is depicted in figure 4. □

In standard game theory, the *representation* of a game is typically dissociated from its *analysis*. On any game of a given class (for example extensive-form games)

one can define a variety of solution concepts (for example pure Nash equilibrium, subgame perfect equilibrium, mixed Nash equilibrium, ϵ-Nash equilibrium, sequential equilibrium). A curious fact about open games is that the representation of a model as an open game automatically 'builds in' a particular solution concept, namely pure Nash equilibrium. This is because the $\odot$ and $\otimes$ operators make essential use of Nash-like reasoning, in which the players in each component act *as though* they know which strategies will be played in the other component.

In particular, it has proven to be difficult to characterise subgame perfect equilibria, despite the fact that the *representation* of open games appears to be dynamic (that is, has a temporal component; subgame perfect equilibrium is specifically intended for dynamic games). A failed attempt was made in [11] by modifying the operator $\odot$, there called $\odot_{SP}$, but it fails to form a symmetric monoidal category with $\otimes$, and hence does not support a string diagram language. (See the conclusion section of [11].)

A direct solution is given in [10] by defining a 'subgame perfection operator' that modifies an open game's solution concept from Nash to subgame perfect equilibrium. From a practical perspective their solution is similar to the one in this paper, offering flexibility between Nash and subgame perfect equilibrium. Ours has the specific advantage that it separates the representation of a game from its analysis, as in standard game theory, although at the expense of requiring far more mathematical machinery.

9 External choice

Proposition 9.1 *For each set S, $(-, S) : \mathbf{Set} \to \mathbf{BLens}$ preserves colimits.*

Proof. For each set S, there is a functor $F_S : \mathbf{BLens} \to \mathbf{Set}$ given on disets by $F_S(Y, R) = Y \times (R \to S)$, and on lenses $\lambda : (X, R) \to (Y, Q)$ by $F_S(\lambda)(x, k) = (v_\lambda(x), k')$ where $k'(q) = k(u_\lambda(x, q))$. Then $(-, S) \dashv F_S$, since there are natural isomorphisms

$$
\begin{aligned}
\hom_{\mathbf{BLens}}((X, S), (Y, R)) &= (X \to Y) \times (X \times R \to S) \\
&\cong X \to Y \times (R \to S) \\
&= \hom_{\mathbf{Set}}(X, F_S(Y, R))
\end{aligned}
$$

Since $(-, S)$ is a left adjoint, it preserves colimits.   □

In particular, **BLens** has coproducts of the form $\coprod_{i:I}(X_i, S) = (\coprod_{i:I} X_i, S)$. (This construction is mentioned in [3, section 1.3].) $(1, 0)$ is terminal in **BLens**. Additionally $(-, -)$ is right adjoint to the universal functor $\langle \mathbb{V}, \mathbb{K} \rangle : \mathbf{BLens} \to \mathbf{Set} \times \mathbf{Set}^{\mathrm{op}}$, and hence preserves limits, but we do not use this fact.

Proposition 9.2 *The product in* $\mathbf{Game}_1$ *of a family of open games* $\mathcal{G}_i : (X_i, S) \rightarrowtail$

(Y_i, R) *is*

$$\prod_{i:I} \mathcal{G}_i : \left(\coprod_{i:I} X_i, S \right) \longrightarrow \left(\coprod_{i:I} Y_i, R \right)$$

given as follows. The set of strategy profiles is $\Sigma \left(\prod_{i:I} \mathcal{G}_i \right) = \prod_{i:I} \Sigma(\mathcal{G}_i)$*. The lenses are given by* $\left(\prod_{i:I} \mathcal{G}_i \right)(\sigma) = \coprod_{i:I} \mathcal{G}_i(\sigma_i)$*. Noting that* $\mathbb{V} \left(\coprod_{i:I} X_i, S \right) \cong \coprod_{i:I} X_i$*, the best response relation* $(\sigma, \sigma') \in \mathbf{B}_{\prod_{i:I} \mathcal{G}_i}(\iota_j(h), k)$ *holds iff* $(\sigma_j, \sigma'_j) \in \mathbf{B}_{\mathcal{G}_j}(h, k \circ \iota_j)$*.*

The product $\mathcal{G}_1 \times \mathcal{G}_2$ is an *external choice*, in which the decision of whether $\mathcal{G}_1$ or $\mathcal{G}_2$ is played is determined by the history. This is strongly reminiscent of products in categories in game semantics [1] and additive conjunction in linear logic. In particular, it is typical of products in game semantics that *strategies* compose by cartesian product, and *plays* compose by disjoint union. This analogy suggests the following correspondence between game semantics and compositional game theory: The *player* of game semantics corresponds to the $n \geq 0$ noncooperative players in an open game, and the *opponent* of game semantics corresponds to the context (h, k).

Proposition 9.3 *Let* $\mathcal{G}_i : (X_i, R) \longrightarrow (Y_i, S)$ *be a family of open games. Then a state of* $\prod_{i:I} \mathcal{G}_i$ *consists, up to isomorphism, of a state for each* $\mathcal{G}_i$*.*

Proof. Since $\mathbb{S} \cong \hom_{\mathbf{Game}_1}(\mathfrak{u}(I), -)$ it preserves limits, so

$$\mathbb{S} \left(\prod_{i:I} \mathcal{G}_i \right) \cong \prod_{i:I} \mathbb{S}(\mathcal{G}_i)$$

□

10 Example: Market entry game

We illustrate the previous three sections by considering the market entry game, a standard example of game theory in microeconomics [16, example 9.B.3]. This game is played between two players: a prospective *entrant* P into a market, and an *incumbent* Q in that market. In the first stage, P has the choice to either enter the market R, or immediately quit L. If P chooses to quit the game ends immediately and both players receive a certain utility. If P chooses to enter, in the second stage P and Q play a subgame in which simultaneously choose to either *fight F* or *accommodate A* the other (which could mean, for example, that F means setting an aggressively low price and selling at a loss, and A means setting a high price). The utilities in this subgame are an abstraction of a model of sales in a market.

The extensive form representation is depicted in figure 5. The dotted line denotes that the two connected nodes share an information set, meaning that the incumbent firm Q does not know which of the two nodes they are in; in this way, the right hand subtree represents a simultaneous game. (More precisely, the word 'simultaneous' means the choices are made independently.)

The feature of this game that is specifically awkward for approaches based on theoretical computer science is that the extensive-form tree is unbalanced. Other
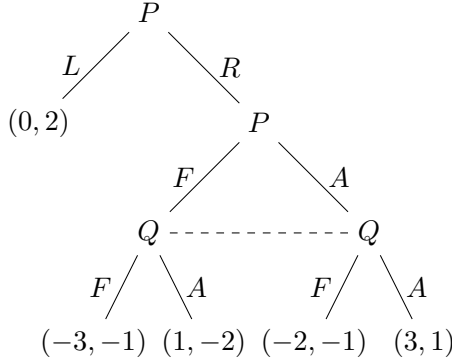
Fig. 5. Extensive form representation of the market entry game

approaches use dependent types to allow types of choices to dependent on earlier values, but can still only allow more general 'dependent subgames' such as in the market entry game using encoding tricks such as dummy moves, and using large negative utilities to rule out certain plays. (Examples of game theory developed in a dependent type system include [15,2].) An external choice operator solves the more general problem of dependent subgames in an elegant way.

We represent the market entry game as an open game as follows. The decision at the root node in figure 5 is modelled as a utility-maximising decision $\mathcal{D}_{1,1+1} : I \longrightarrow (1+1,\mathbb{R})$ where the choice $\iota_1(*)$ represents L and $\iota_2(*)$ represents R.

The left subgame is represented by $(1,c_0) : (1,\mathbb{R}) \longrightarrow I$, where $c_0 : 1 \to \mathbb{R}$ is the constant function with $c_0(*) = 0$. Notice that the utility of 2 for player Q in the left subgame has a clear economic interpretation as the profit for the incumbent firm, but it plays no role game-theoretically, and it never appears in the open game representation. (We could however add an additional, independent utility-maximising player in the left subgame with only a single choice, which has no effect on the equilibrium analysis but is more faithful to the economic situation.) As a strategically trivial effect, $(1,c_0)$ has a unique state α_L, given by the unique lenses $\mathfrak{s}(\alpha_L) : (1,\mathbb{R}) \to I$, $\mathfrak{t}(\alpha_L) : I \to I$ and function $\Sigma(\alpha_L) : 1 \to 1$.

The right subgame is an open game $\mathcal{G} : (1,\mathbb{R}) \longrightarrow I$, which can be built compositionally from a pair of utility-maximising decisions $\mathcal{D}_{1,X} : I \longrightarrow (X,\mathbb{R})$ where $X = \{F,A\}$ is the set of choices to fight or accommodate, and the utility function $q : X^2 \to \mathbb{R}^2$ given by the payoffs in the right subgame, i.e.
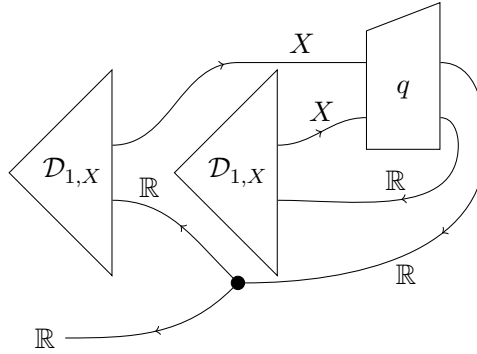
$$q(F,F) = (-3,-3) \qquad\qquad q(F,A) = (1,-2)$$
$$q(A,F) = (-2,-1) \qquad\qquad q(A,A) = (3,1)$$

This is represented by the string diagram in figure 6.

The coherence theorem proved in [12] guarantees that this diagram defines $\mathcal{G}$ up to (in general non-globular) isomorphism. Concretely, $\mathcal{G}$ is given as follows:

- $\Sigma(\mathcal{G}) = X^2$

- $\mathcal{G}(\sigma) : (1,\mathbb{R}) \to I$ is the lens with $u_{\mathcal{G}(\sigma)}(*,*) = (q(\sigma))_1$, i.e. the utility for the

Fig. 6. String diagram for the right subgame $\mathcal{G}$

entrant P given strategy profile σ for the subgame

- $(\sigma, \sigma') \in \mathbf{B}_{\mathcal{G}}(*, *)$ iff $(q(\sigma_1', \sigma_2))_1 \geq (q(\overline{\sigma_1'}, \sigma_2))_1$ and $(q(\sigma_1, \sigma_2'))_2 \geq (q(\sigma_1, \overline{\sigma_2'}))_2$, where $\overline{} : X \to X$ gives the other choice ($\overline{}$ is the non-identity automorphism of X), i.e. $\mathbf{B}_{\mathcal{G}}(*, *)$ is the best response function for the right subgame

The right subgame has a unique Nash equilibrium (A, A) in which both players accommodate. (By the comments at the end of section 7 we must necessarily discover this fact by game-theoretic methods rather than categorical methods.) Hence $\mathcal{G}$ has a unique state α_2 with $\Sigma(\alpha)(*) = (A, A)$, in which the lens $\mathfrak{s}(\alpha_2) : (1, \mathbb{R}) \to I$ has $u_{\mathfrak{s}(\alpha_2)}(*, *) = 3$. This is the payoff $(q(A, A))_1$ for the entrant P in the Nash equilibrium.

The product $(1, c_0) \times \mathcal{G} : (1 + 1, \mathbb{R}) \longrightarrow (1 + 1, 1)$ is concretely given as follows:

- $\Sigma((1, c_0) \times \mathcal{G}) \cong X^2$
- $((1, c_0) \times \mathcal{G})(\sigma)$ is the lens $(1 + 1, \mathbb{R}) \to (1 + 1, 1)$ with view function $v_{((1,c_0) \times \mathcal{G})(\sigma)} = \mathrm{id}_{1+1}$ and update function

$$u_{((1,c_0) \times \mathcal{G})(\sigma)}(h, *) = \begin{cases} 0 & \text{if } h = \iota_1(*) \\ (q(\sigma))_1 & \text{if } h = \iota_2(*) \end{cases}$$

- The best response relation $(\sigma, \sigma') \in \mathbf{B}_{(1,c_0) \times \mathcal{G}}(h, k)$ holds iff either $h = \iota_1(*)$, or $(q(\sigma_1', \sigma_2))_1 \geq (q(\overline{\sigma_1'}, \sigma_2))_1$ and $(q(\sigma_1, \sigma_2'))_2 \geq (q(\sigma_1, \overline{\sigma_2'}))_2$

Unlike the subgames $(1, c_0)$ and $\mathcal{G}$, the product $(1, c_0) \times \mathcal{G}$ cannot be seen as directly corresponding to a game in the classical sense. (It is an external choice between a forest of game trees.) By proposition 9.3, $\langle \alpha_L, \alpha_2 \rangle$ is the unique state of $(1, c_0) \times \mathcal{G}$, which is over the unique lens $\mathfrak{t}(\langle \alpha_L, \alpha_R \rangle) : (1 + 1, 1) \to I$. It has $\Sigma(\langle \alpha_L, \alpha_R \rangle)(*) = (A, A)$, and $\lambda := \mathfrak{s}(\langle \alpha_L, \alpha_R \rangle) : (1 + 1, \mathbb{R}) \to I$ is the lens with

$$u_\lambda(h, *) = \begin{cases} 0 & \text{if } h = \iota_1(*) \\ 3 & \text{if } h = \iota_2(*) \end{cases}$$

Since the previous function has a single maximising point, namely $\iota_2(*)$, by proposition 7.10 the decision $\mathcal{D}_{1,1+1}$ has a unique state α over λ. This state has $\Sigma(\alpha)(*) : 1 \to 1 + 1$ given by $\Sigma(\alpha)(*)(*) = \iota_2(*)$.

The open game $\mathcal{H} := ((1, c_0) \times \mathcal{G}) \odot \mathcal{D}_{1,1+1} : I \longrightarrow (1 + 1, 1)$ will be our representation of the original market entry game. It is concretely given, up to globular isomorphism, as follows. The set of strategy profiles is $\Sigma(\mathcal{H}) = (1 + 1) \times X^2$ consisting of a strategy for E in the first round, and a strategy for both players in the subgame in which E enters in the first round. This is the same as the set of pure strategy profiles of the original extensive-form game. (As in classical game theory, player E is required to choose a contingent strategy for the second round, even if the strategy in the first round is to quit.) $\mathcal{H}$ can be straightforwardly made into a scalar $I \longrightarrow I$ by postcomposing with the (unique) strategically trivial game $(1 + 1, 1) \longrightarrow I$.

The lens $\mathcal{H}(\sigma_1, \sigma_2, \sigma_3) : I \to (1 + 1, 1)$ has view function $v_{\mathcal{H}(\sigma_1, \sigma_2, \sigma_3)}(*) = \sigma_1$. The best response relation $\mathbf{B}_{\mathcal{H}}(*, *)$ is the same as the best response relation for the market entry game given by classical game theory. Concretely, the relation $(\sigma, \sigma') \in \mathbf{B}_{\mathcal{H}}(*, *)$ holds iff the following three conditions hold:

- If $\sigma'_1 = \iota_1(*)$ then $0 \geq (q(\sigma_2, \sigma_3))_1$, and if $\sigma'_1 = \iota_2(*)$ then $0 \leq (q(\sigma_2, \sigma_3))_1$
- If $\sigma_1 = \iota_2(*)$ then $(q(\sigma'_2, \sigma_3))_1 \geq (q(\overline{\sigma'_2}, \sigma_3))_1$
- If $\sigma_1 = \iota_2(*)$ then $(q(\sigma_2, \sigma'_3))_2 \geq (q(\sigma_2, \overline{\sigma'_3}))_2$

Even more concretely, this relation is depicted as a digraph in figure 7, where we write L and R for $\iota_1(*)$ ad $\iota_2(*)$.

By the previous reasoning $\mathcal{H}$ has a unique $\odot$-separable state, namely $\alpha = \langle \alpha_L, \alpha_R \rangle \odot \alpha_1$. This corresponds to the unique subgame perfect equilibrium of the market entry game, namely that the entrant enters the market (choice R or $\iota_2(*)$), and then both players accommodate (choice A).

$\mathcal{H}$ has two additional states that are not $\odot$-separable, corresponding to the non-subgame-perfect Nash equilibria of the market entry game (or, equivalently, to the two additional fixpoints of $\mathbf{B}_{\mathcal{H}}(*, *)$). These are the strategy profiles $(\iota_1(*), A, F)$ and $(\iota_1(*), F, F)$, recalling that $\iota_1(*)$ represents L. In both of these Nash equilibria the strategy $\sigma_3 = F$ of player Q is a *non-credible threat* (mutually harmful contingent strategy) to fight if player P chooses R in the first round. Such strategies are often considered unrealistic, and the purpose of subgame perfect equilibrium is to rule them out.
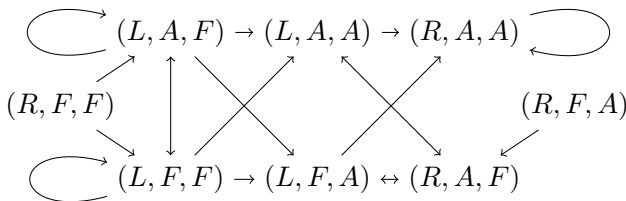
$(L, A, F) \to (L, A, A) \to (R, A, A)$

(R, F, F) (R, F, A)

$(L, F, F) \to (L, F, A) \leftrightarrow (R, A, F)$

Fig. 7. The relation $\mathbf{B}_{\mathcal{H}}(*, *)$

References

[1] Samson Abramsky and Guy McCusker. Game semantics. In *Computational logic*, volume 165 of *NATO ASI Series*. Springer, 1999. doi:10.1007/978-3-642-58622-4_1.

[2] Nicola Botta, Cezar Ionescu, and Edwin Brady. Sequential decision problems, dependently typed solutions. In *Proeedings of PLMMS13*, 2013.

[3] Valeria de Paiva. The dialectica categories. Technical report, University of Cambridge, 1991.

[4] Valeria de Paiva. Dialectica and Chu construtions: Cousins? *Theory and applications of categories*, 17(7):127–152, 2007.

[5] Martin Escardó and Paulo Oliva. Sequential games and optimal strategies. *Proceedings of the Royal Society A*, 467:1519–1545, 2011. doi:10.1098/rspa.2010.0471.

[6] Martin Escardó and Paulo Oliva. Computing Nash equilibria of unbounded games. *Proceedings of the Turing centenary conference*, 2012.

[7] Brendan Fong. *The algebra of open and interconnected systems*. PhD thesis, University of Oxford, 2016.

[8] Nate Foster, Michael Greenwald, Jonathan Moore, Benjamin Pierce, and Alan Schmitt. Combinators for bi-directional tree transformations: A linguistic approach to the view update problem. *ACM Transactions on Programming Languages and Systems*, 29(3), 2007. doi:10.1145/1232420.1232424.

[9] Neil Ghani, Jules Hedges, Viktor Winschel, and Philipp Zahn. Compositional game theory. In *Proceedings of Logic in Computer Science (LiCS) 2018*. ACM, 2018. doi:10.1145/3209108.3209165.

[10] Neil Ghani, Clemens Kupke, Alasdair Lambert, and Fredrik Nordvall Forsberg. A compositional treatment of iterated open games. To appear in *Sannellabration!*, special issue of *Theoretical computer science*. arXiv:1711.07968, 2018.

[11] Jules Hedges. *Towards compositional game theory*. PhD thesis, Queen Mary University of London, 2016.

[12] Jules Hedges. Coherence for lenses and open games. arXiv:1704.02230, 2017.

[13] Bart Jacobs. *Categorical logic and type theory*. Studies in logic and the foundations of mathematics. Elsevier, 1999.

[14] G. M. Kelly and Ross Street. Review of the elements of 2-categories. In *Category seminar*, volume 420 of *Lecture notes in mathematics*. Springer, 1974. doi:10.1007/BFb0063101.

[15] Pierre Lescanne and Matthieu Perrinel. "Backward" coinduction, Nash equilibrium and the rationality of escalation. *Acta Informatica*, 49(3):117–137, 2012. doi:10.1007/s00236-012-0153-3.

[16] Andreu Mas-Colell, Michael Whinston, and Jerry Green. *Microeconomic theory*. Oxford University Press, 1995.

[17] Michael Maschler, Eilon Solan, and Shmuel Zamir. *Game theory*. Cambridge University Press, 2013. doi:10.1017/CBO9780511794216.

[18] John Nash. Non-cooperative games. *Annals of mathematics*, 54(2):286–295, 1951.

[19] Matthew Pickering, Jeremy Gibbons, and Nicolas Wu. Profunctor optics: Modular data accessors. *The art, science and engineering of programming*, 1(2), 2017. doi:10.22152/programming-journal.org/2017/1/7.

[20] Aviad Rubinstein. Inapproximability of Nash equilibrium. In *Proceedings of the forty-seventh annual ACM symposium on Theory of computing (STOC'15)*, pages 409–418, 2015. doi:10.1145/2746539.2746578.

[21] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory and applications of categories*, 20(18):650–738, 2008.

[22] Michael Shulman. Constructing symmetric monoidal bicategories. arXiv:1004.0993, 2010.