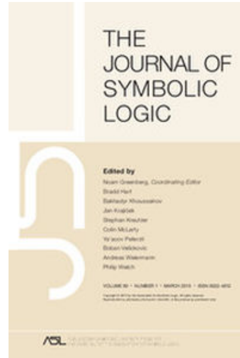

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FLUTED FORMULAS AND THE LIMITS OF DECIDABILITY

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Abstract. In the predicate calculus, variables provide a flexible indexing service which selects the actual arguments to a predicate letter from among possible arguments that precede the predicate letter (in the parse of the formula). In the process of selection, the possible arguments can be permuted, repeated (used more than once), and skipped. If this service is withheld, so that arguments must be the immediately preceding ones, taken in the order in which they occur, the formula is said to be *fluted*. Quine showed that if a fluted formula contains only homogeneous conjunction (conjoins only subformulas of equal arity), then the satisfiability of the formula is decidable. It remained an open question whether the satisfiability of a fluted formula without this restriction is decidable. This paper answers that question.

§1. Introduction. In 1960, in "Variables explained away" [12], Quine presented his Predicate Functor Logic (PFL), a system equivalent to predicate logic, but without variables. Quine sought to explicate the notion of *variable* by carefully delineating the roles that variables play in predicate logic. He did this by introducing predicate functors that provided the various services normally provided by variables.

Quine returned to PFL in a number of his papers and books in the following years (e.g., [14, 15, 16]). The set of predicate functors varied in different versions of PFL. One could try to make do with as few as possible, or try to make the functors individually as simple as possible. A set that achieves the latter goal is the following.

$\exists$ (crop), $\neg$, $\wedge$, inv, Inv, pad, ref.

This set falls naturally into two subsets:

(i) the alethic functors, $\exists$, $\neg$, $\wedge$; and
(ii) the combinatory functors, inv, Inv, pad, ref.

The formulas (or schemas) that can be formed using only predicate letters and the alethic functors were named *fluted* formulas by Quine. In 1969 in "On the limits of decision" [13], Quine showed that if the fluted formulas are restricted to conjoin only subformulas of the same arity (called *homogeneous* conjunction), then their satisfiability is decidable. However the method used (an extension of the method used by Herbrand to show monadic logic decidable) breaks down when the restriction on conjunction is relaxed (Noah [9]). It remained an open question whether satisfiability of unrestricted fluted formulas is decidable.

This paper answers the latter question in the affirmative.

§2. Preliminaries. This paper assumes the usual definition of the pure predicate calculus. Typically the set of predicate symbols will be those that occur in some given finite set of formulas or *premises*. The finite set of predicate symbols will be

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referred to as the *lexicon*. Let L be a lexicon and $R \in L$. Then $ar(R)$ denotes the arity of R. Define $ar(L) := max\{ar(R) : R \in L\}$.

A standard result from predicate calculus is the following.

THEOREM 1. (The Principle of Monotonicity) *Let θ be a subformula, not in the scope of $\neg$, that occurs as a conjunct in formula ϕ. Then ϕ' can be inferred from ϕ, where ϕ' is obtained from ϕ by deleting θ.*

PROOF. See Andrews [1], Theorem 2105, Substitutivity of Implication. Note that the empty conjunction is defined to be equivalent to $\top$ (verum).

An *interpretation* $\mathcal{I}$ of a lexicon L consists of a set $\mathcal{D}$, the *domain* of $\mathcal{I}$, and a mapping that assigns to each $R \in L$ a subset of $\mathcal{D}^{ar(R)}$. The notions of satisfaction and truth are the standard ones. If ϕ is a formula over L with free variables among $\{x_1, \ldots, x_k\}$, and ϕ is satisfied in $\mathcal{I}$ by the assignment to variables $\{x_i \mapsto a_i\}_{1 \leq i \leq k}$, we write $a_1 \cdots a_k \models \phi$. If ϕ is a sentence and ϕ is true in $\mathcal{I}$, we write $\mathcal{I} \models \phi$.

§3. **Fluted formulas.** Let L be a finite set of predicate symbols. Let $X_m := \{x_1, \ldots, x_m\}$ be a set of m variables where $m \geq 0$. An *atomic fluted formula* of L over X_m is $Rx_{m-n+1} \cdots x_m$, where $R \in L$ and $ar(R) = n \leq m$. The set of all atomic fluted formulas of L over X_m will be denoted $Af_L(X_m)$. Define $Af_L(X_0) := \{\top\}$.

A *fluted formula* of L over X_m is defined inductively.

(i) An atomic fluted formula of L over X_m is a fluted formula of L over X_m.

(ii) If ϕ is a fluted formula of L over X_{m+1}, then $\exists x_{m+1}\phi$ and $\forall x_{m+1}\phi$ are fluted formulas of L over X_m.

(iii) If ϕ and ψ are fluted formulas of L over X_m, then $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, and $\neg\phi$ are fluted formulas of L over X_m.

This definition can be generalized as follows. Call the fluted formulas just defined *standard* fluted formulas. Now any formula that is alpha-equivalent to a standard fluted formula is defined to be a fluted formula. Two formulas are alpha-equivalent if they are alphabetic variants of one another, i.e., differ only in an inessential renaming of variables (see Enderton [3], pp. 118-120 for a precise definition).

The fluted formulas of L form a proper subset of the formulas of the pure predicate calculus with predicate symbols L. The semantics of the fluted formulas of L coincides with the usual semantics of the pure predicate calculus. In connection with standard fluted formulas, $abc \cdots \models \phi$ will always mean that ϕ is satisfied (in the interpretation given by the context) by the assignment to variables $\{x_1 \mapsto a, x_2 \mapsto b, x_3 \mapsto c, \ldots\}$.

It might be noted in passing that in the predicate calculus restricted to fluted formulas, it is possible to dispense with variables entirely, since the arity and position of a predicate symbol completely determine the sequence of variables that follow the predicate symbol. However, variables will be retained to make the presentation more familiar and more explicit.

§4. **Fluted constituents.** A conjunction in which for each $\rho \in Af_L(X_m)$ either ρ or $\neg\rho$ (but not both) occurs as a conjunct will be called a *minimal conjunction over* $Af_L(X_m)$ (because it is an atom in the Boolean lattice generated by $Af_L(X_m)$). The arity $ar(\theta)$ of a minimal conjunction is defined to be the maximum of the arities of the predicate symbols occurring in θ. The set of minimal conjunctions over $Af_L(X_m)$ will be denoted $\Delta Af_L(X_m)$ (cf. Rantala [17]). Note that if $\Delta Af_L(X_m) = \{\theta_1, \dots, \theta_l\}$, and ϕ is any quantifier-free formula over $Af_L(X_m)$, then

(i) $\neg(\theta_i \wedge \theta_j)$ for $i \neq j$,
(ii) $\theta_1 \vee \cdots \vee \theta_l$, and
(iii) either $\theta_i \to \phi$, or $\theta_i \to \neg\phi$, for $1 \leq i \leq l$,

are tautologies (see [17]).

Let $\mathbf{P}$ be the positive integers, and $\mathbf{P}^*$ the set of finite strings over $\mathbf{P}$. String concatenation is denoted by juxtaposition. The empty string is ε. If $i_1, \dots, i_n \in \mathbf{P}$, and $\alpha = i_1 \cdots i_n$, then for $k \leq n$, $(k : \alpha) := i_1 \cdots i_k$ is the k-prefix of α.

A subset $\mathcal{T} \subseteq \mathbf{P}^*$ is a *tree domain* if

(i) $\varepsilon \in \mathcal{T}$, and
(ii) if $\alpha i \in \mathcal{T}$, where $\alpha \in \mathbf{P}^*$ and $i \in \mathbf{P}$, then
  (a) $\alpha j \in \mathcal{T}$ for $0 < j < i$, and
  (b) $\alpha \in \mathcal{T}$.

Define the *height of* $\alpha \in \mathcal{T}$, $h(\alpha) :=$ the length of string α. For all $\alpha, \beta \in \mathbf{P}^*, i \in \mathbf{P}$, if $\alpha i \beta \in \mathcal{T}$ then $\alpha i \beta$ is a *descendant of* α and α is an *ancestor of* $\alpha i \beta$. Moreover, αi is an *immediate descendant of* α and α is an *immediate ancestor of* αi. Define $w(\alpha) :=$ the number of immediate descendants of α. Thus $\alpha 1, \alpha 2, \dots, \alpha w(\alpha)$ are the immediate descendants of α. If $w(\alpha) = 0$, then α is *terminal* in $\mathcal{T}$. If all terminal elements of $\mathcal{T}$ have the same height, then $\mathcal{T}$ is *balanced.* In this case, $h(\mathcal{T}) := h(\alpha)$, where α is any terminal element in $\mathcal{T}$. If $0 < h(\alpha) < h(\mathcal{T})$, then α is *internal* in $\mathcal{T}$.

An element α together with all of its descendants is defined to be the *subtree rooted on* α, and is denoted $(\alpha]$. An element α together with all of its immediate descendants will be called the *elementary subtree rooted on* α. An element α together with all of its ancestors is defined to be the *path from* ε *to* α, and is denoted $[\alpha)$.

Tree domain is an algebraic representation of tree. It is not a new notion. For example, it can be found in Arbib et al. [2], in Gallier [4], and in Huet [8]. The notation $(\alpha]$ and $[\alpha)$ also is not new. $(\alpha]$ has been used to denote the lower interval (lower closure, principal ideal) of the element α in a poset, and $[\alpha)$ has been used to denote the dual (e.g., see Grätzer [5]). Note that the notation makes the usual assumption that trees grow downward.

Let $\mathcal{T}$ be a balanced tree domain. A *labeled tree domain* $\mathcal{T}_L$ is defined to be $\mathcal{T}$ with a formula $\theta_\alpha \in \Delta Af_L(X_{h(\alpha)})$ associated with each $\alpha \in \mathcal{T}$. The labeled subtree of $\mathcal{T}_L$ rooted on α will be denoted $(\theta_\alpha]$. The labeled path in $\mathcal{T}_L$ from ε to α will be denoted $[\theta_\alpha)$. The subtree $(\theta_\alpha]$ is given the following interpretation.

(i) If α is terminal, then $(\theta_\alpha]$ denotes θ_α.
(ii) If α is nonterminal with height k, then $(\theta_\alpha]$ denotes $\theta_\alpha \wedge \exists x_{k+1}(\theta_{\alpha 1}] \wedge \cdots \wedge \exists x_{k+1}(\theta_{\alpha w(\alpha)}] \wedge \forall x_{k+1}((\theta_{\alpha 1}] \vee \cdots \vee (\theta_{\alpha w(\alpha)}])$.

The formula denoted by $(\theta_\alpha]$ is a *fluted constituent of* L *of height* $h(\mathcal{T}) - h(\alpha)$ *over the variables* $X_{h(\alpha)}$. If $h(\alpha) = 0$, the formula denoted by $(\theta_\alpha]$ is a *constituent sentence.*

The path $[\theta_\alpha)$ denotes $\theta_\varepsilon \wedge \theta_{1:\alpha} \wedge \theta_{2:\alpha} \wedge \cdots \wedge \theta_\alpha$. If $\theta_\varepsilon = \neg\top$, then $\mathcal{T}_L$ is *trivial.* If $\mathcal{T}_L$ is nontrivial, θ_ε can usually be elided. Notice that for paths of nonzero length, $[\theta_\alpha)$ is not a fluted formula, but rather a conjunction of fluted formulas, each over a different set of variables. Nonetheless, it will be possible, and convenient, to consider paths together with fluted formulas.

In the remainder of this paper, all tree domains will be nontrivial labeled balanced tree domains. Moreover, $(\theta_\alpha]$ and $[\theta_\alpha)$ will not be distinguished from the formulas they denote.

Let $\alpha \in \mathscr{T}$ and $\theta_\alpha \in \Delta Af_L(X_{h(\alpha)})$. Define $g(\alpha) := max(1, 1 + h(\alpha) - ar(L))$. Then the variables occurring in θ_α are precisely $x_{g(\alpha)}, \dots, x_{h(\alpha)}$.

If ϕ is a constituent or path, then define:

(i) $\phi^{[-k]}$ is ϕ with the last k variables eliminated;

(ii) $\phi_{[-k]}$ is ϕ with the first k variables eliminated.

Here elimination of a variable is accomplished by removing all atomic formulas in which that variable occurs, as well as the quantifier, if any, associated with that variable.

If ϕ is a fluted formula (including tree and path), containing only occurrences of variables $x_l, \dots, x_k$ in that order, then $\phi^\dagger := \phi\{x_l \mapsto x_1, \dots, x_k \mapsto x_{k-l+1}\}$ is the *standardization* of ϕ.

Fluted constituents are Hintikka constituents of the second kind ([17]) restricted to fluted formulas. The proofs of the main results in [17] are indifferent to the precise nature of the atomic formulas. The proofs go through unchanged if atomic fluted formulas replace atomic formulas of the pure predicate calculus. Therefore, the main results for Hintikka constituents hold for fluted constituents. The following theorems extend the results for atomic constituents given at the beginning of this section to constituents in general.

THEOREM 2. (The Fundamental Property of Constituents) (i) *If ϕ and ψ are fluted constituents of L of height k over the variables X_l, and $\phi \neq \psi$, then $\phi \wedge \psi$ is inconsistent.*

(ii) *The disjunction of all fluted constituents of L of height k over the variables X_l is logically valid.*

PROOF. See [17], Theorem 3.10.

THEOREM 3. *Let ϕ be a standard fluted formula of L containing variables X_m, where variables $X_k \subseteq X_m$ are free. Then ϕ is logically equivalent to a disjunction of fluted constituents of height $m - k$ over X_k.*

PROOF. See [17], Theorem 4.1.

Notice that as formulas, paths of the same height, like constituents of the same height, are either identical or inconsistent. Therefore, if $\alpha, \beta \in \mathscr{T}$ at the same height, and $a_1 \cdots a_k \models [\theta_\alpha)$ and $a_1 \cdots a_k \models [\theta_\beta)$, then $[\theta_\alpha) = [\theta_\beta)$.

§5. Trivial inconsistency. If ϕ is a constituent sentence, then $\phi \rightarrow \phi^{[-k]}$ and $\phi \rightarrow \phi_{[-k]}$ by the Principle of Monotonicity. Hence $\phi \rightarrow (\phi^{[-k]} \wedge \phi_{[-k]})$. Moreover, $\phi^{[-k]}$ and $\phi_{[-k]}$ are constituent sentences of the same height. It follows from the Fundamental Property of Constituents that either $\phi^{[-k]}$ and $\phi_{[-k]}$ are identical (up to possible repetition of constituents, order of conjunction and disjunction, and alpha-equivalence), or ϕ is inconsistent. In the latter case, ϕ is said to be *trivially inconsistent* (cf. Hintikka [6, 7]).

A convenient definition of trivial inconsistency for purposes of this paper is the following. A constituent $\mathscr{T}_L$ is not trivially inconsistent iff for any $\alpha \in \mathscr{T}$, for all k such that $0 \leq k \leq h(\alpha)$, there exists $\gamma \in \mathscr{T}$ such that

(i) $[\theta_\gamma) = ([\theta_\alpha)_{[-k]})^\dagger$, and
(ii) $\{(\theta_{\gamma j}]^{[-k]} : 1 \leq j \leq w(\gamma)\} = \{((\theta_{\alpha j}]_{[-k]})^\dagger : 1 \leq j \leq w(\alpha)\}$.

An equivalent, and simpler, definition is obtained by restricting k to 1. A constituent $\mathscr{T}_L$ is not trivially inconsistent iff for any $\alpha \in \mathscr{T}$, there exists $\gamma \in \mathscr{T}$ such that

(i) $[\theta_\gamma) = ([\theta_\alpha)_{[-1]})^\dagger$, and
(ii) $\{(\theta_{\gamma j}]^{[-1]} : 1 \leq j \leq w(\gamma)\} = \{((\theta_{\alpha j}]_{[-1]})^\dagger : 1 \leq j \leq w(\alpha)\}$.

Obviously, the first implies the second. To see the converse, inductively assume that for any $\alpha \in \mathscr{T}$, there exists $\gamma \in \mathscr{T}$ such that

(i) $[\theta_\gamma) = ([\theta_\alpha)_{[-k+1]})^\dagger$, and
(ii) $\{(\theta_{\gamma j}]^{[-k+1]} : 1 \leq j \leq w(\gamma)\} = \{((\theta_{\alpha j}]_{[-k+1]})^\dagger : 1 \leq j \leq w(\alpha)\}$.

By the second definition, there exists $\delta \in \mathscr{T}$ such that

(i) $[\theta_\delta) = ([\theta_\gamma)_{[-1]})^\dagger$, and
(ii) $\{(\theta_{\delta j}]^{[-1]} : 1 \leq j \leq w(\delta)\} = \{((\theta_{\gamma j}]_{[-1]})^\dagger : 1 \leq j \leq w(\gamma)\}$.

Hence

(i) $[\theta_\delta) = ([\theta_\gamma)_{[-1]})^\dagger = ((([\theta_\alpha)_{[-k+1]})^\dagger)_{[-1]})^\dagger$, and
(ii) $\{((\theta_{\delta j}]^{[-1]})^{[-k+1]} : 1 \leq j \leq w(\delta)\} = \{(((\theta_{\gamma j}]_{[-1]})^\dagger)^{[-k+1]} : 1 \leq j \leq w(\gamma)\} = \{(((((\theta_{\alpha j}]_{[-k+1]})^\dagger)_{[-1]})^\dagger : 1 \leq j \leq w(\alpha)\}$,

which yields the desired result.

It is a corollary of the first definition that if a constituent $\mathscr{T}_L$ is not trivially inconsistent, then for any nonterminal $\alpha \in \mathscr{T}$, for all k such that $0 \leq k \leq h(\alpha)$, there exists $\gamma \in \mathscr{T}$, such that

(i) $[\theta_\gamma) = ([\theta_\alpha)_{[-k]})^\dagger$, and
(ii) $\{[\theta_{\gamma j}) : 1 \leq j \leq w(\gamma)\} = \{([\theta_{\alpha j})_{[-k]})^\dagger : 1 \leq j \leq w(\alpha)\}$.

Note that γ is not necessarily unique. But $[\theta_\gamma)$, as a formula, is unique.

Thus constituents that are not trivially inconsistent exhibit a pattern. This pattern might be described by saying that elementary subtrees at any given level are embedded at higher levels throughout the constituent according to the rule given by the above equations.

§6. Simple fluted constituents. A fluted constituent sentence $\mathscr{T}_L$ is *simple* if for all $\alpha \in \mathscr{T}$,

(i) $ar(\theta_\alpha) = h(\alpha)$, and
(ii) $1 \leq i < j \leq w(\alpha)$ implies $\theta_{\alpha i} \neq \theta_{\alpha j}$.

A constituent that fails to satisfy (ii) will be said to have *occurrences of equal siblings*.

A simple constituent sentence possesses a regularity that eliminates the need for consideration of a number of special cases when reasoning about it. If $\mathscr{T}_L$ is a simple constituent sentence, then it follows easily that

(i) for all $\alpha \in \mathcal{T} : g(\alpha) = 1$;
(ii) $ar(L) \geq h(\mathcal{T})$;
(iii) the siblings in any elementary subtree are pairwise inconsistent;
(iv) no two distinct paths denote the same formula; therefore, any two distinct paths of the same length from the same element are inconsistent;
(v) If $\mathcal{T}_L$ is not trivially inconsistent, then for all $\alpha \in \mathcal{T}$ and for all k such that $0 < k \leq h(\alpha)$, there exists a unique $\gamma \in \mathcal{T}$, such that
$[\theta_\gamma) = ([\theta_\alpha)_{[-k]})^\dagger$, and
$\{[\theta_{\gamma j}) : 1 \leq j \leq w(\gamma)\} = \{([\theta_{\alpha j})_{[-k]})^\dagger : 1 \leq j \leq w(\alpha)\}$.

The next objective is to show that it is possible to restrict our attention to simple constituent sentences.

Lemma 4. *Let $\mathcal{T}_L$ be a fluted constituent sentence. Then there exists a fluted constituent sentence $\mathcal{T}'_{L'}$, such that*
(i) $L \subseteq L'$
(ii) *for all* $\alpha \in \mathcal{T}' : ar(\theta_\alpha) = h(\alpha)$
(iii) $\mathcal{T}'_{L'}$ *is trivially inconsistent iff* $\mathcal{T}_L$ *is*
(iv) $\mathcal{T}'_{L'} \rightarrow \mathcal{T}_L$

Proof. The proof is by induction on the number of $\beta \in \mathcal{T}$ such that $ar(\theta_\beta) < h(\beta)$. The basis is vacuous. For the induction step, let $ar(\theta_\beta) < h(\beta)$, and let β have minimal height among such elements. Since $h(\beta)$ is minimal, $ar(\theta_\beta) = h(\beta) - 1$. Let Q be a new predicate symbol of arity $h(\beta)$, and define $L' := L \cup \{Q\}$. $\mathcal{T}'_{L'}$ is obtained from $\mathcal{T}_L$ as follows.

If $h(\alpha) < h(\beta)$, then θ_α is unchanged.

If $h(\alpha) \geq h(\beta)$, then substitute $\theta_\alpha \wedge Qx_p \cdots x_q$ for θ_α, where $p = h(\alpha) - h(\beta) + 1$ and $q = h(\alpha)$.

Now it is obvious that $\mathcal{T}'_{L'}$ is trivially inconsistent iff $\mathcal{T}_L$ is. Moreover, by the Principle of Monotonicity, $\mathcal{T}'_{L'} \rightarrow \mathcal{T}_L$. This completes the proof.

If $\mathcal{T}_L$ is viewed as a formula over the lexicon L', then $\mathcal{T}'_{L'}$ is a constituent of $\mathcal{T}_L$. If $\mathcal{T}'_{L'}$ is consistent, Q will be interpreted as the universal predicate of arity $h(\beta)$.

Lemma 5. *Let $\mathcal{T}_L$ be a fluted constituent sentence that is not trivially inconsistent, and that has occurrences of equal siblings. Let m be the minimum height of such occurrences. Then there exists a fluted constituent sentence $\mathcal{T}'_{L'}$, such that*
(i) $L \subseteq L'$
(ii) *the number of occurrences of equal siblings at height m in $\mathcal{T}'_{L'}$ is less than the number of occurrences of equal siblings at height m in $\mathcal{T}_L$*
(iii) $\mathcal{T}'_{L'}$ *is not trivially inconsistent*
(iv) $\mathcal{T}'_{L'} \rightarrow \mathcal{T}_L$

Proof. Without loss of generality, it can be assumed that for all $\alpha \in \mathcal{T}$: $1 \leq i < j \leq w(\alpha)$ implies $(\theta_{\alpha i}] \neq (\theta_{\alpha j}]$; that is, $\mathcal{T}_L$ contains no occurrences of repeated constituents. This assumption also will be in force throughout the construction described below. Moreover, in view of Lemma 4, it can be assumed that for all $\alpha \in \mathcal{T}$: $ar(\theta_\alpha) = h(\alpha)$. Let $h(\mathcal{T}_L) = h$.

Let $\beta \in \mathcal{T}$ be an element at height m such that $1 \leq i < j \leq w(\beta)$ and $\theta_{\beta i} = \theta_{\beta j}$. To simplify notation, suppose that $\theta_{\beta 1} = \theta_{\beta 2} = \cdots = \theta_{\beta l}$, where $l \leq w(\beta)$. Let $Q_1, \ldots, Q_r$ be new predicate symbols of arity $h(\beta) + 1$, where $2^{r-1} < l \leq 2^r$, and define $L' := L \cup \{Q_1, \ldots, Q_r\}$. Let $\rho_1, \ldots, \rho_l$ be any distinct minimal conjunctions over $\{Q_1, \ldots, Q_r\}$. If $\rho = \sigma_1 \wedge \cdots \wedge \sigma_r$, where for $1 \leq i \leq r$, $\sigma_i = Q_i$ or $\sigma_i = \neg Q_i$, then let $\rho x_p \cdots x_q$ abbreviate $\sigma_1 x_p \cdots x_q \wedge \cdots \wedge \sigma_r x_p \cdots x_q$. If ϕ is any formula over the lexicon L', then define $\phi^\flat$ to be ϕ with all occurrences of $Q_1, \ldots, Q_r$ deleted.

The construction of $\mathcal{T}'_{L'}$ proceeds in the order of height k. $\mathcal{T}^{(k)}_{L'}$ will be the result corresponding to height k. $\mathcal{T}^{(h)}_{L'}$ will be the result at the conclusion of the construction. The proof that $\mathcal{T}^{(h)}_{L'}$ satisfies the lemma is by induction. The induction hypothesis is

(i) $\mathcal{T}^{(k)}_{L'}$ is not trivially inconsistent, up to height k. That is, in $\mathcal{T}^{(k)}_{L'}$: for each $\alpha \in \mathcal{T}^{(k)}$ such that $h(\alpha) \leq k$: there exist $\gamma \in \mathcal{T}^{(k)}$ such that $h(\gamma) = h(\alpha) - 1$, and

$[\theta_\gamma) = ([\theta_\alpha)_{[-1]})^\dagger$ and

$\{((\theta_{\gamma j}]^{[-1]})^{\flat k} : 1 \leq j \leq w(\gamma)\} = \{((\theta_{\alpha j}]_{[-1]})^\dagger : 1 \leq j \leq w(\alpha)\}$,

where $((\theta_{\gamma j}]^{[-1]})^{\flat k}$ is $(\theta_{\gamma j}]^{[-1]}$ with $\theta^\flat_\delta$ substituted for θ_δ when $h(\delta) \geq k$.

(ii) $(\mathcal{T}^{(k)}_{L'})^\flat = \mathcal{T}_L$.

In $\mathcal{T}^{(h)}_{L'}$, $((\theta_{\gamma j}]^{[-1]})^{\flat h} = (\theta_{\gamma j}]^{[-1]}$, and so $\mathcal{T}^{(h)}_{L'}$ is not trivially inconsistent. Moreover, since $(\mathcal{T}^{(k)}_{L'})^\flat$ is obtained from $\mathcal{T}^{(k)}_{L'}$ by deleting conjuncts of the form $\rho x_p \cdots x_q$, then by the Principle of Monotonicity, it follows from (ii) that $\mathcal{T}^{(k)}_{L'} \rightarrow \mathcal{T}_L$.

For the basis step, let $k = h(\beta) + 1$. Then $\mathcal{T}^{(k)}_{L'}$ is obtained from $\mathcal{T}_L$ as follows.

For $1 \leq i \leq l$, substitute $\theta_{\beta i} \wedge \rho_i x_1 \cdots x_k$ for $\theta_{\beta i}$.

For $l < i \leq w(\beta)$, substitute $\theta_{\beta i} \wedge \rho_1 x_1 \cdots x_k$ for $\theta_{\beta i}$.

For all other elements αi at height k, substitute $\theta_{\alpha i} \wedge \rho_1 x_1 \cdots x_k$ for $\theta_{\alpha i}$.

The basis step introduces a partition of $[\theta_{\beta 1})$ (which by assumption is equal to $[\theta_{\beta 2}), \ldots$, and $[\theta_{\beta l})$), making $[\theta_{\beta 1} \wedge \rho_1 x_1 \cdots x_k), \ldots, [\theta_{\beta l} \wedge \rho_l x_1 \cdots x_k)$ distinct in $\mathcal{T}^{(k)}_{L'}$. (i) of the induction hypothesis holds since $\mathcal{T}_L$ is not trivially inconsistent. Obviously, $(\mathcal{T}^{(k)}_{L'})^\flat = \mathcal{T}_L$. Therefore the induction hypothesis holds for the basis step.

For the induction step, let $h(\beta) + 1 < k \leq h$. The induction step modifies the tree $\mathcal{T}^{(k-1)}_{L'}$ to yield a tree $\mathcal{T}^{(k)}_{L'}$ that is not trivially inconsistent, up to height k. The construction considers in turn each $\alpha \in \mathcal{T}^{(k-1)}$ such that $h(\alpha) = k - 1$ as follows. By the induction hypothesis, there exist $\gamma \in \mathcal{T}^{(k-1)}$ such that $h(\gamma) = k - 2$, and

$[\theta_\gamma) = ([\theta_\alpha)_{[-1]})^\dagger$ and

$\{((\theta_{\gamma j}]^{[-1]})^{\flat(k-1)} : 1 \leq j \leq w(\gamma)\} = \{((\theta_{\gamma j}]^{[-1]})^\flat : 1 \leq j \leq w(\gamma)\} = \{((\theta_{\alpha j}]_{[-1]})^\dagger : 1 \leq j \leq w(\alpha)\}$.

Define $(\alpha i, \gamma j) := 1$ if $((\theta_{\gamma j}]^{[-1]})^\flat = ((\theta_{\alpha i}]_{[-1]})^\dagger$. Otherwise $(\alpha i, \gamma j) := 0$. Replace the subtrees $\{(\theta_{\alpha i}] : 1 \leq i \leq w(\alpha)\}$ with the subtrees $\{(\theta_{\alpha i} \wedge \rho x_p \cdots x_k] : (\alpha i, \gamma j) = 1\}$, where $\theta_{\gamma j} = \theta^\flat_{\gamma j} \wedge \rho x_{p-1} \cdots x_{k-1}$ and $(\theta_{\alpha i} \wedge \rho x_p \cdots x_k]$ is obtained from $(\theta_{\alpha i}]$ by substitution of $\theta_{\alpha i} \wedge \rho x_p \cdots x_k$ for $\theta_{\alpha i}$. Since the number of subtrees lying above

α may increase in number as a result of this replacement, it may be necessary to reindex the tree domain.

When all $\alpha \in \mathcal{T}^{(k-1)}$ at height $k-1$ have been considered, the result is $\mathcal{T}_{L'}^{(k)}$. Now (i) of the induction hypothesis holds for $\mathcal{T}_{L'}^{(k)}$. Moreover, $(\mathcal{T}_{L'}^{(k)})^\flat = (\mathcal{T}_{L'}^{(k-1)})^\flat = \mathcal{T}_L$. Therefore the induction hypothesis holds for the induction step.

Finally, define $\mathcal{T}_{L'}' := \mathcal{T}_{L'}^{(h)}$. This completes the proof.

If $\mathcal{T}_L$ is viewed as a formula over the lexicon L', then $\mathcal{T}_{L'}'$ is a constituent of $\mathcal{T}_L$. In an interpretation of $\mathcal{T}_{L'}'$, the $\rho_1, \ldots, \rho_l$ will be interpreted as subsets of $\mathcal{D}^{h(\beta)+1}$ that separate the subset that interprets $\theta_{\beta 1}$ into l disjoint parts such that each part satisfies one of the existential claims on $\theta_{\beta 1}$. Such separation is always possible since $(\theta_{\beta 1}], \ldots, (\theta_{\beta l}]$ are distinct constituents, and so pairwise not simultaneously satisfiable. That is, for any assignment to the free variables of these constituents, no element of the domain can bear witness for more than one of them.

Together these lemmas yield the following theorem.

THEOREM 6. *Let $\mathcal{T}_L$ be a fluted constituent sentence that is not trivially inconsistent. Then there exists a simple fluted constituent sentence $\mathcal{T}_{L'}'$, such that*

(i) $L \subseteq L'$

(ii) $\mathcal{T}_{L'}'$ *is not trivially inconsistent*

(iii) $\mathcal{T}_{L'}' \to \mathcal{T}_L$

PROOF. As before, it can be assumed without loss of generality that $\mathcal{T}_L$ contains no occurrences of repeated constituents. Also, in view of Lemma 4, it can be assumed that for all $\alpha \in \mathcal{T}$: $ar(\theta_\alpha) = h(\alpha)$.

$\mathcal{T}_{L'}'$ is constructed inductively. The construction begins with $\mathcal{T}_L$. Inductively, suppose that n steps have been performed, and that m is the minimum height at which there are occurrences of equal siblings. The claim made for the construction is: after step $(n+1)$, there are fewer occurrences of equal siblings at height m than after step n. Each step employs the construction of Lemma 5, which reduces the number of occurrences of equal siblings at the minimum height of such occurrences. When this number reaches zero, it increases the minimum height of such occurrences. While some steps may increase the total number of occurrences of equal siblings, the construction acts to restrict these occurrences to greater and greater heights, until they only occur at height h, where they are finally eliminated entirely by the assumption that $\mathcal{T}_{L'}'$ contains no occurrences of repeated constituents. This completes the proof of the theorem.

§7. Satisfiability of fluted constituents. By Theorem 3, every fluted formula can be expressed as a disjunction of fluted constituents of sufficient depth of the lexicon of that formula. Therefore, the question of satisfiability of a fluted formula reduces to the question of satisfiability of a fluted constituent. The following theorem, which provides a decision procedure for the latter question, is the main result of the paper.

THEOREM 7. *A fluted constituent sentence is unsatisfiable iff it is trivially inconsistent.*

PROOF. The 'if' direction is obvious. The 'only-if' direction is proved in its contrapositive form. Let $\mathcal{T}_L$ be a fluted constituent sentence of height h that is not trivially inconsistent. In view of Theorem 6, it can be assumed without loss of generality that $\mathscr{T}_L$ is simple. It will be shown that $\mathscr{T}_L$ is satisfiable in an interpretation $\mathscr{I}$ with domain

$$\mathscr{D} := \{a_\alpha : (\alpha \in \mathscr{T}) \wedge (\alpha \neq \varepsilon)\}.$$

It suffices to interpret the $\theta_\alpha \in \mathscr{T}_L$, since, if done consistently, this fixes a unique interpretation of the elements of L. Two claims will be proved.

Claim 1. An interpretation $\mathscr{I}$ of L can be constructed with the property that if α is nonterminal at height k, and $a_{\beta_1} \cdots a_{\beta_k} \models [\theta_\alpha)$, then

(i) for $1 \leq j \leq w(\alpha) : \exists a_\beta \in \mathscr{D} : a_{\beta_1} \cdots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$, and

(ii) $\forall a_\beta \in \mathscr{D} : a_{\beta_1} \cdots a_{\beta_k} a_\beta \models [\theta_{\alpha 1}) \vee \cdots \vee [\theta_{\alpha w(\alpha)})$.

Claim 2. In an interpretation $\mathscr{I}$ of L with the property of Claim 1, if $h(\alpha) = k$ and $a_{\beta_1} \cdots a_{\beta_k} \models [\theta_\alpha)$, then $a_{\beta_1} \cdots a_{\beta_k} \models (\theta_\alpha]$.

The theorem follows from these claims since, letting $\mathscr{I}$ be the interpretation of Claim 1, we have $\mathscr{I} \models [\theta_\varepsilon)$, because $\mathscr{T}_L$ is nontrivial, and so by Claim 2, $\mathscr{I} \models (\theta_\varepsilon]$, i.e., $\mathscr{I} \models \mathscr{T}_L$. Proofs of the claims are now given.

Proof of Claim 1. For the proof of this claim, it will be helpful to invoke geometric intuition by viewing $[\theta_\alpha)$, where $h(\alpha) = k$, as a subspace in the k-dimensional space with coordinate axes $x_1, x_2, \ldots, x_k$. On this view, the tuple $(a_{\beta_1}, \ldots, a_{\beta_k})$, which will be written $a_{\beta_1} \cdots a_{\beta_k}$, is a point in the k-dimensional space. The statement $a_{\beta_1} \cdots a_{\beta_k} \in [\theta_\alpha)$ is defined to be equivalent to $a_{\beta_1} \cdots a_{\beta_k} \models [\theta_\alpha)$. $a_{1:\alpha} \cdots a_\alpha$ also is a point in the k-dimensional space. Points of this latter form, as well as points of the form $a_{i:\alpha} \cdots a_\alpha$ $(1 \leq i \leq k)$, will be called *standard points*. In the usual way, a subspace of k-dimensional space becomes a subspace of $(k+1)$-dimensional space by cylindrification or ringing along the $(k+1)$-st coordinate.

The mapping of $\mathscr{I}$ is defined in three parts. Each part is ordered by height. The first part of the mapping is given as follows. For each $\alpha \in \mathscr{T}$, define

$$a_{1:\alpha} \cdots a_\alpha \models \theta_\alpha,$$

or equivalently,

$$a_{1:\alpha} \cdots a_\alpha \models [\theta_\alpha).$$

This is immediately followed by the definition

$$\text{for } 1 < i \leq h(\alpha) : a_{i:\alpha} \cdots a_\alpha \models \theta_\gamma, \text{ where } [\theta_\gamma) = ([\theta_\alpha)_{[-i+1]})^\dagger.$$

This part of the definition is proper, since for each α and for $1 \leq i \leq h(\alpha)$, before the definition is made, the point $a_{i:\alpha} \cdots a_\alpha$ is *uncommitted*, i.e., has not been defined to satisfy any θ_γ. This concludes the first part of the mapping. Following this part, every standard point is committed.

The second part of the mapping is defined, ordered by height, as follows. Let $h(\alpha) = k > 0$ and consider $\theta_{\alpha j}$, where $1 \leq j \leq w(\alpha)$. From the first part of the mapping, $a_{m:\delta} \cdots a_\delta a_{\delta n} \models [\theta_{\alpha j})$. We extend the interpretation of the $\theta_{\alpha j}$ as follows. For each $\beta \in \mathscr{T}$, if

(i) $a_{m:\delta} \cdots a_\delta a_\beta \models [\theta_{\alpha j})_{[-1]}$, and

(ii) it is not the case that $1 \leq l \leq w(\alpha)$ and $a_{m:\delta} \cdots a_\delta a_\beta \models \theta_{\alpha l}$,

then define $a_{m:\delta} \cdots a_\delta a_\beta \models [\theta_{\alpha j})$. As before, this is immediately followed by the definition

$$\text{for } m < i \leq h(\alpha) : a_{i:\delta} \cdots a_\delta a_\beta \models \theta_\gamma, \text{ where } [\theta_\gamma) = ([\theta_{\alpha j})_{[-i+m]})^\dagger.$$

If (ii) fails, then the extension under consideration either has already been made, or cannot be made without introducing inconsistency. If (ii) is satisfied, the points $a_{i:\delta} \cdots a_\delta a_\beta$, for $m \leq i \leq h(\alpha)$, are uncommitted. Hence the definition is proper. Note that this extension may not be unique, since it may depend on the order in which the j are considered. (The order need not be the same for each β.) This concludes the second part of the mapping. Now every point of the form $a_{i:\alpha} \cdots a_\alpha a_\beta$ is committed, where α is nonterminal and $1 \leq i \leq h(\alpha)$.

The intent of the first two parts of the mapping is to ensure that at every point $a_{i:\alpha} \cdots a_\alpha$, if $a_{i:\alpha} \cdots a_\alpha \models [\theta_\gamma)$, then: (i) for every $[\theta_{\gamma j})$ there is some a_β such that $a_{i:\alpha} \cdots a_\alpha a_\beta \models [\theta_{\gamma j})$, and (ii) for every a_β, there is some $[\theta_{\gamma j})$ such that $a_{i:\alpha} \cdots a_\alpha a_\beta \models [\theta_{\gamma j})$.

The third and final part of the mapping is now defined, ordered by height. Let $h(\alpha) = k > 1$ and consider $\theta_{\alpha j}$, where $1 \leq j \leq w(\alpha)$. The interpretation of the $\theta_{\alpha j}$ is extended as follows. Let $a_{\beta_1} \cdots a_{\beta_k}$ be a nonstandard point. For each $\beta \in \mathscr{T}$, if

(i) $a_{\beta_1} \cdots a_{\beta_k} \models [\theta_\alpha)$, and

(ii) $a_{1:\alpha} \cdots a_\alpha a_\beta \models [\theta_{\alpha j})$,

then define $a_{\beta_1} \cdots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$. If (i) is satisfied, the point $a_{\beta_1} \cdots a_{\beta_k}$ belongs only to $[\theta_\alpha)$, since $\mathscr{T}_L$ is simple. Therefore, the point $a_{\beta_1} \cdots a_{\beta_k} a_\beta$ is uncommitted. Hence the definition is proper. The intent is that the definition inherit from the first and second parts the property that for every $[\theta_{\alpha j})$ there is some a_β such that $a_{\beta_1} \cdots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$, and also that for every a_β, there is some $[\theta_{\alpha j})$ such that $a_{\beta_1} \cdots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$. Following this final part, every point is committed. This concludes the definition of the mapping.

It remains to show that this interpretation has the property claimed for it. The inductive proof is subdivided into three cases.

Case 1. $a_{\beta_1} \cdots a_{\beta_k} = a_{1:\alpha} \cdots a_\alpha$. By the first part of the definition, for $1 \leq j \leq w(\alpha)$, $a_{1:\alpha} \cdots a_\alpha a_{\alpha j} \models \theta_{\alpha j}$. Therefore, item (i) of Claim 1 holds. By the second part of the definition and induction hypothesis, for all $\beta \in \mathscr{T}$, there is some j such that $a_{1:\alpha} \cdots a_\alpha a_\beta \models \theta_{\alpha j}$. Therefore, item (ii) of Claim 1 holds.

Case 2. $a_{\beta_1} \cdots a_{\beta_k} = a_{i:\delta} \cdots a_\delta$ for some $\delta \in \mathscr{T}$. From Case 1, Claim 1 holds for $a_{1:\delta} \cdots a_\delta \models \theta_\delta$. Since $\mathscr{T}_L$ is not trivially inconsistent, $\exists \gamma \in \mathscr{T}$, such that

(i) $[\theta_\gamma) = ([\theta_\delta)_{[-i+1]})^\dagger$, and

(ii) $\{[\theta_{\gamma j}) : 1 \leq j \leq w(\gamma)\} = \{([\theta_{\delta j})_{[-i+1]})^\dagger : 1 \leq j \leq w(\delta)\}$.

Since $\mathscr{T}_L$ is simple, γ is unique. Hence $\gamma = \alpha$. By the first and second parts of the definition, therefore, Claim 1 holds for $a_{i:\delta} \cdots a_\delta \models \theta_\alpha$ also.

Case 3. $a_{\beta_1} \cdots a_{\beta_k}$ is nonstandard. Claim 1 follows from the third part of the definition.

In every case, then, Claim 1 holds. This concludes the proof of Claim 1.

Proof of Claim 2. This proof is by induction on the depth $d = h - k$, where k is the height of $\alpha \in \mathscr{T}$. The induction hypothesis is that Claim 2 holds for all elements with depth $< d$.

For the basis step, $d = 0$, θ_α is at height h. Here $(\theta_\alpha] = \theta_\alpha$ by definition, and so the induction hypothesis is trivially true.

For the induction step, $d > 0$, θ_α is at height $k = h - d$. Suppose $a_{\beta_1} \cdots a_{\beta_k} \models [\theta_\alpha)$. Since $\mathscr{I}$ is assumed to have the property of Claim 1,

(i) for $1 \leq j \leq w(\alpha) : \exists a_\beta \in \mathscr{D} : a_{\beta_1} \cdots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$, and

(ii) $\forall a_\beta \in \mathscr{D} : a_{\beta_1} \cdots a_{\beta_k} a_\beta \models [\theta_{\alpha 1}) \vee \cdots \vee [\theta_{\alpha w(\alpha)})$.
By the induction hypothesis, if $a_{\beta_1} \cdots a_{\beta_k} a_\beta \models [\theta_{\alpha j})$, then $a_{\beta_1} \cdots a_{\beta_k} a_\beta \models (\theta_{\alpha j}]$. Therefore,

(i) for $1 \leq j \leq w(\alpha) : \exists a_\beta \in \mathscr{D} : a_{\beta_1} \cdots a_{\beta_k} a_\beta \models (\theta_{\alpha j}]$, and
(ii) $\forall a_\beta \in \mathscr{D} : a_{\beta_1} \cdots a_{\beta_k} a_\beta \models (\theta_{\alpha 1}] \vee \cdots \vee (\theta_{\alpha w(\alpha)}]$.

Thus $a_{\beta_1} \cdots a_{\beta_k} \models (\theta_\alpha]$, and the induction hypothesis holds at height k. This concludes the proof of Claim 2, and of the theorem.

As is the case with constituents in general, the number of fluted constituents grows rapidly with height. Suppose $card(L) = n$. Then the number of fluted constituents of height 0, is $N(0) = 2^n$, the number of minimal conjunctions. Inductively, the number of fluted constituents of height k is $N(k) = N(0)(2^{N(k-1)} - 1)$. Each constituent is the result of putting a new root element over some nonempty subset of the constituents of height $k - 1$.

The number of elements in a fluted constituent (i.e., tree domain) of height 0 is $M(0) = 1$. The maximum number of elements in a fluted constituent of height k is $M(k) = 1 + N(k-1)M(k-1)$, i.e., the root element plus the maximum number of elements in each of the maximum number of subtrees descending from the root element. $M(h)$ provides an upper bound on the number of elements in the tree domain representing a fluted constituent of height h. This yields the following corollary to Theorem 7.

COROLLARY 8. *If a fluted constituent of L of height h is satisfiable, it is satisfiable in a finite domain, whose cardinality is bounded above by $M(h)$.*

If ϕ is a fluted formula, Theorem 3 states that ϕ is equivalent to the disjunction of its constituents. Moreover, the proof of Theorem 3 provides an effective method of transforming ϕ into the disjunction of its constituents. Obviously ϕ is satisfiable iff one of its constituents is satisfiable. Theorem 7 states that a constituent is satisfiable iff it is not trivially inconsistent. Trivial inconsistency can be decided by a finite number of tests on the syntax of the constituent. Theorems 3 and 7 therefore yield the following conclusion.

THEOREM 9. *The satisfiability of a fluted formula is decidable.*

§8. Discussion. Theorem 9 locates the boundary between decidable and undecidable logic more precisely than heretofore, putting fluted logic on the same side as monadic logic and homogeneous fluted logic. Quine's conjecture that PFL (and general quantification theory) gets its 'escape velocity' from the combinatory functors is given further support.

Fluted logic may also have an importance beyond its relation to the limits of decidability. It seems to be related to natural language in a way that sheds light on natural language reasoning. It was noted (in Section 3) that variables play no essential role in fluted formulas, since fluted formulas are deprived of the services of the combinatory functors.

Natural language also does not employ variables. It is true that when intersentence linking is required, anaphoric pronouns are used, but these cannot be considered simply as variables (see Purdy [11] and references cited there). The observation that natural language is variable-free has inspired a number of variable-free formal languages, whose syntax is designed to closely parallel that of natural language (e.g., Suppes [20], Sommers [18], Purdy [10]). However, to match the expressive power of predicate calculus, they incorporate devices equivalent to the combinatory functors of PFL, and thereby deviate from natural language.

But perhaps predicate calculus is not an appropriate model for natural language reasoning. Much of natural language reasoning is conducted within the constraints of fluted logic. Many examples can be found in [18]. Even the infamous Schubert's Steamroller (Stickel [19]) can be stated in fluted formulas. The most complex premise is:

> Every animal either likes to eat all plants or all animals much smaller than itself that like to eat some plants.

This can be rendered by the fluted sentence:

$$(1)\quad \forall x_1(Ax_1 \rightarrow$$
$$(\forall x_2(Px_2 \rightarrow Ex_1x_2) \vee \forall x_2((Ax_2 \wedge Mx_1x_2 \wedge \exists x_3(Px_3 \wedge Ex_2x_3)) \rightarrow Ex_1x_2)))$$

Thus it appears that fluted logic is a more appropriate model for natural language reasoning. If this intuition is correct, it is no coincidence that fluted logic falls close to or at the boundary of decidability. This suggests the interesting but as yet unanswered question whether there exists a usefully efficient decision procedure for satisfiability of fluted formulas.

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