

ON A CONJECTURE OF MARTON

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ABSTRACT. We prove a conjecture of K. Marton, widely known as the polynomial Freiman–Ruzsa conjecture, in characteristic 2. The argument extends to odd characteristic, with details to follow in a subsequent paper.

1. INTRODUCTION

In this paper we prove a conjecture of Katalin Marton (see [8]), widely known in the literature as the *polynomial Freiman–Ruzsa conjecture* in characteristic 2. The question has many equivalent forms: see, for example, [2, Section 10] (note that the unpublished notes referenced as [30] in that paper are now available at [3]), [10] for the connection with the U^3 uniformity norm (for which it implies polynomially effective bounds), and an earlier paper of the second, third and fourth authors [4] for an entropic version.

Here is one formulation of the conjecture.

Theorem 1.1. *Suppose that $A \subset \mathbf{F}_2^n$ is a set with $|A + A| \leq K|A|$. Then A is covered by at most $2K^C$ cosets¹ of some subgroup $H \leq \mathbf{F}_2^n$ of size at most $|A|$. In fact, we can take $C = 12$.*

We prove Theorem 1.1 (and hence all equivalent formulations of the theorem) in this paper.

The first result of this general type was due to Ruzsa [8], who obtained an upper bound of $2K^2 2^{K^4}$ in place of the desired $2K^C$. It was

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¹The factor of 2 is needed to deal with the case where K is close to 1. If H_0 is a subgroup of \mathbf{F}_2^n and $A \subseteq H_0$ is a subset of size $(1 - \varepsilon)|H_0|$ for some small $\varepsilon > 0$, then in order to obtain the bound $|H| \leq |A|$, we need to pick a subgroup H of H_0 of index 2 and cover A by the two cosets of that subgroup.

Ruzsa who attributed the conjecture that Theorem 1.1 holds to Katalin Marton. Sanders [11] made a dramatic breakthrough by proving the first bounds of shape $\ll \exp(\log^{C_1} K)$, showing that $C_1 = 4 + \varepsilon$ is permissible here. A variant of Sanders' argument due to Konyagin (not published by Konyagin, but described by Sanders in [12], and see also the comments on page 8 of [4]) showed that $C_1 = 3 + \varepsilon$ is permissible.

Before the present paper, this was the best known bound.

We remark that our method does *not* in its current form give a result of Bogolyubov type, that is to say a large subspace contained in a fixed sumset such as $2A - 2A$. Konyagin's variant of Sanders' result also does not apply to this question, and so for that problem the bounds of Sanders remain the best available.

An elaboration of our arguments does give corresponding results in vector spaces over \mathbf{F}_p , p odd. This is notationally somewhat heavier and makes the underlying ideas slightly harder to appreciate so we give this general argument in a separate paper [1].

Entropy methods. Ruzsa [9] and the fourth author [14] studied entropy analogues of the notion of sumset. In a recent preprint [4], the second, third and fourth authors further developed the use of entropy techniques in additive combinatorics. Entropy methods, as well as several of the results in both [4] and [14], will be used extensively here.

For a full discussion of the relevant concepts, we refer the reader to [4] or to Appendix A. For now, we recall the notion of entropic Ruzsa distance between random variables. If X, Y are two finitely supported random variables taking values in an additive group G , then we define

$$d[X; Y] := \mathbf{H}[X' - Y'] - \frac{1}{2}\mathbf{H}[X'] - \frac{1}{2}\mathbf{H}[Y'], \quad (1.1)$$

where X', Y' are independent copies of X, Y . In particular we stress that $d[X; Y]$ depends only on the *individual distributions* of X and Y : it does not require them to be independent, or even that they are defined on a common sample space.

We refer the reader to [4, Section 1] and Appendix A for further discussion. For the moment we just mention the *Ruzsa triangle inequality* $d[X; Y] \leq d[X; Z] + d[Z; Y]$, as well as the fact that, despite $d[-; -]$ being called a distance, we have $d[X; X] = 0$ only when X is a coset of a subgroup (see Lemma 2.2 below). In this paper (since we will use no other notion of distance) we write $d[X; Y]$ instead of the more cumbersome $d_{\text{ent}}(X, Y)$ from [4]. We also switch to using square brackets, as many of our random variables X, Y are themselves expressions involving parentheses.

One consequence of the investigation in [4] was another equivalent formulation of the PFR conjecture. This is the version we will focus on in the present paper.

Theorem 1.2. *Let $G = \mathbf{F}_2^n$, and suppose that X_1^0, X_2^0 are G -valued random variables. Then there is some subgroup $H \leq G$ such that*

$$d[X_1^0; U_H] + d[X_2^0; U_H] \leq 11d[X_1^0; X_2^0],$$

where U_H denotes the uniform distribution on H . Furthermore, both $d[X_1^0; U_H]$ and $d[X_2^0; U_H]$ are at most $6d[X_1^0; X_2^0]$.

The fact that Theorem 1.2 is equivalent, up to constants, to the characteristic 2 case of Marton's original conjecture, as well as to the other combinatorial formulations of the polynomial Freiman–Ruzsa conjecture, was established in [4, Section 8]. For the convenience of the reader, we give the proof that Theorem 1.2 implies Theorem 1.1 in Appendix B.

Finally, we note that it was shown in [4, Theorem 1.11] that Theorem 1.2 (or Theorem 1.1) in characteristic 2 implies the so-called ‘weak’ polynomial Freiman–Ruzsa conjecture over \mathbf{Z} , so that is now a theorem as well.

Theorem 1.3. *Let A be a finite subset of \mathbf{Z}^D for some D , and suppose that $|A + A| \leq K|A|$. Then there is some set $A' \subseteq A$, $|A'| \geq K^{-C_1/2}|A|$, with $\dim A' \leq C_2 \log K$, for some absolute constants $C_1, C_2 > 0$.*

Indeed, if one traces through the arguments in [4] (dropping all terms involving the exponent $1 - 1/C_{\text{PFR}}$, and using the constant $C = 11$ from Theorem 1.2), one finds that one can take $C_1 = 20C = 220$ and $C_2 = \frac{40}{\log 2}$; no doubt these constants can be optimized further with additional effort. (The reason we have stated Theorem 1.3 with a $C_1/2$ in the exponent is so that the constants C_1, C_2 here are the same as the ones in [4].)

Notation. All logarithms in this paper will be natural logarithms. We will very extensively use the notation of entropy $\mathbf{H}[-]$ and mutual information $\mathbf{I}[-]$ as well as conditional variants of these concepts. Basic definitions and facts are recalled in Appendix A.

2. INDUCTION ON ENTROPY DISTANCE

Fix, for the rest of the paper, the group $G = \mathbf{F}_2^n$ and the variables X_1^0, X_2^0 appearing in Theorem 1.2. We shall think of X_1^0, X_2^0 as ‘reference variables’, and not modify them for the rest of the proof; while

they do appear in many of the expressions we will need to control, they will play a relatively minor role in the estimates. Our overall strategy can be thought of as a kind of induction on (a slight modification of) the distance $d[X_1; X_2]$, though we will ultimately phrase it as a compactness argument.

For the rest of the paper, η will denote the constant $\frac{1}{9}$. For any two G -valued random variables X_1, X_2 , we introduce the functional

$$\tau[X_1; X_2] := d[X_1; X_2] + \eta d[X_1^0; X_1] + \eta d[X_2^0; X_2]. \quad (2.1)$$

Note that this functional depends only on the distributions p_{X_1}, p_{X_2} of X_1, X_2 , and not on whether X_1, X_2 (or X_1^0, X_2^0) are independent or dependent (or even on whether they are defined on the same sample space).

Here is the main result from which we will deduce Theorem 1.2.

Proposition 2.1. *Let X_1, X_2 be two G -valued random variables with $d[X_1; X_2] > 0$. Then there are G -valued random variables X'_1, X'_2 such that*

$$\tau[X'_1; X'_2] < \tau[X_1; X_2]. \quad (2.2)$$

If we ignore the two penalty terms involving η on the right-hand side of (2.1), then Proposition 2.1 asserts that if two random variables have a positive Ruzsa distance, then we can ‘improve’ them, finding two new variables X'_1, X'_2 that are closer to each other than the original variables X_1, X_2 . To take care of the penalty terms, we will also need the new variables X'_1, X'_2 to be ‘related’ to the old variables X_1, X_2 in some sense, so that both sets of variables have a comparable Ruzsa distance to the reference variables X_1^0, X_2^0 .

In order to deduce Theorem 1.2 from Proposition 2.1 we will use the ‘100 percent’ case of our main theorem, which we note now.

Lemma 2.2. *Suppose that X_1, X_2 are G -valued random variables such that $d[X_1; X_2] = 0$. Then there exists a subgroup $H \leq G$ such that $d[X_1; U_H] = d[X_2; U_H] = 0$.*

Proof. By the triangle inequality, $d[X_1; X_1] = 0$, and so (since $G = \mathbf{F}_2^n$) we have $d[X_1; -X_1] = 0$. By [14, Theorem 1.11(i)], it follows that there exists a subgroup H with $d[X_1; U_H] = 0$. Then $d[X_2; U_H] = 0$ follows by the triangle inequality. \square

Proof of Theorem 1.2, assuming Proposition 2.1. We choose a pair of variables X_1, X_2 (or more precisely, their distributions on G) to be any

minimizer² of the functional τ . This optimization problem ranges over the square of the space of probability distributions on the finite set G . The usual topology on this space is compact, and it is straightforward that $d[-; -]$ is continuous with respect to this topology, so the minimum exists.

As (X_1, X_2) minimizes τ , by the contrapositive of Proposition 2.1 we must have $d[X_1; X_2] = 0$, and hence by Lemma 2.2 there exists a subgroup $H \leq G$ such that $d[X_1; U_H] = d[X_2; U_H] = 0$. Finally, we have

$$\begin{aligned} \eta(d[X_1^0; U_H] + d[X_2^0; U_H]) &= \eta(d[X_1^0; X_1] + d[X_2^0; X_2]) \\ &= \tau[X_1; X_2] \leq \tau[X_2^0; X_1^0] = (1 + 2\eta)d[X_1^0; X_2^0], \end{aligned}$$

and so with our choice $\eta = \frac{1}{9}$ we obtain the first claim of Theorem 1.2. Since $|d[X_1; U_H] - d[X_2; U_H]| \leq d[X_1; X_2]$ by the triangle inequality, the second statement then follows. \square

Remark. The proof of Proposition 2.1 is completely algorithmic, but our use of compactness here means that the proof of Theorem 1.2 is not. However, our argument can be modified to give a completely algorithmic proof, at the expense of slightly worse constants.

One approach is to prove Proposition 2.1 with (2.2) replaced by the stronger bound

$$d[X'_1; X'_2] \leq (1 - \eta')d[X_1; X_2] - \eta'd[X'_1; X_1] - \eta'd[X'_2; X_2], \quad (2.3)$$

for some value η' rather smaller than $\eta = \frac{1}{9}$. The proof is very similar and follows the same strategy as that outlined in Section 3. This variant is perhaps even slightly easier to follow than the current proof, since the ‘fixed’ variables X_1^0, X_2^0 no longer play a role.

Once (2.3) is established, it can be applied iteratively to give a sequence (X_1^t, X_2^t) , $t = 0, 1, \dots$ for which $d[X_1^t; X_2^t]$ decays exponentially. In particular, for some $T = O(\log d[X_1^0; X_2^0])$ we have $d[X_1^T; X_2^T] \leq \varepsilon_0$, where ε_0 is the absolute constant appearing in [4, Proposition 1.3]. By that result, X_1^T and X_2^T are both close to some variable U_H . One can then control the distance of U_H from the original variables X_1, X_2 by summing (2.3) over $0 \leq t < T$ and applying the Ruzsa triangle inequality repeatedly.

A version of this argument will be given in detail in our subsequent paper [1].

²We remark that an entropy functional minimization was previously employed by the fourth author in [13] to give an alternative proof of the Szemerédi regularity lemma.

3. PLAN OF THE REMAINING ARGUMENT

For the following preliminary discussion we write various estimates as $O(\eta k)$ for simplicity. To make the argument actually work with the parameter $\eta = \frac{1}{9}$ one of course needs to be more precise, as indeed we will be from (3.13) onwards.

To prove Proposition 2.1 we will try various choices of X'_1, X'_2 constructed using X_1, X_2 and show that at least one of them works.

To describe the choices we will try, consider a four-tuple of independent random variables $X_1, X_2, \tilde{X}_1, \tilde{X}_2$ where X_1, \tilde{X}_1 are copies of X_1 and X_2, \tilde{X}_2 are copies of X_2 ; we can in fact assume that all six variables $X_1, X_2, \tilde{X}_1, \tilde{X}_2, X_1^0, X_2^0$ are independent. Our primary choices for (X'_1, X'_2) will be sums

$$X'_1 = X_1 + \tilde{X}_2, \quad X'_2 = X_2 + \tilde{X}_1 \quad (3.1)$$

or

$$X'_1 = X_1 + \tilde{X}_1, \quad X'_2 = X_2 + \tilde{X}_2 \quad (3.2)$$

or alternatively ‘fibres’

$$X'_1 = (X_1 | X_1 + \tilde{X}_2 = g), \quad X'_2 = (X_2 | X_2 + \tilde{X}_1 = g') \quad (3.3)$$

or

$$X'_1 = (X_1 | X_1 + \tilde{X}_1 = g), \quad X'_2 = (X_2 | X_2 + \tilde{X}_2 = g') \quad (3.4)$$

for $g, g' \in G$. A result that we call the *fibring lemma* (which is [4, Proposition 1.4]), concerning the behaviour of entropy doubling under homomorphisms, may be used to relate the distances that come up when we choose the X'_i to be sums or fibres. A self-contained account of this is provided in Section 4.

The conclusion is that either one of these choices succeeds, or they all *narrowly* (by $O(\eta k)$) fail to work, and the latter can happen only if all the inequalities we used were close to equalities. Motivated by this, one can go back and analyse what happens if the inequality in the fibring lemma was almost tight. What we find in this case is that if neither choice (3.1), (3.3) works to prove (2.2) then we have an upper bound

$$I_1 := \mathbf{I}[X_1 + X_2 : \tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] = O(\eta k). \quad (3.5)$$

Informally, this says that $X_1 + X_2$ and $\tilde{X}_1 + X_2$ are almost independent conditioned on $X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2$. Similarly, if neither choice (3.2), (3.4) works to prove (2.2), we obtain a bound

$$I_2 := \mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] = O(\eta k). \quad (3.6)$$

Finally,

$$I_3 := \mathbf{I}[\tilde{X}_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] = O(\eta k), \quad (3.7)$$

since in fact we have $I_2 = I_3$, as may be seen by interchanging the names of \tilde{X}_1 and X_1 .

If none of the choices (3.1), (3.2), (3.3), (3.4) works to prove (2.2), all three estimates (3.5), (3.6) and (3.7) hold. In this case, we proceed to a part of the argument we refer to as the ‘endgame’.

Suppose first for simplicity that the mutual informations in (3.5), (3.6) and (3.7) were zero rather than merely small. It follows that for any s in the support of $X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2$, the three random variables

$$\begin{aligned} T_1 &= (X_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s), \\ T_2 &= (X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s), \\ T_3 &= (\tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s) \end{aligned} \quad (3.8)$$

are pairwise independent. We also note that $T_1 + T_2 + T_3 = 0$. However, for any trio (T_1, T_2, T_3) of pairwise independent random variables such that $T_1 + T_2 + T_3 = 0$ holds identically, a short calculation shows that

$$d[T_1; T_2] + d[T_1; T_3] + d[T_2; T_3] = 0 \quad (3.9)$$

and so, taking X'_1 and X'_2 to be some pair of T_1, T_2, T_3 , we have $d[X'_1; X'_2] = 0$. This is a very strong conclusion, and with some further estimation, it allows us to establish (2.2). (The choice of X'_1, X'_2 here is inspired, albeit rather indirectly, by a step in an argument of Katz and Koester [6].)

If the three variables in (3.8) are merely ‘almost pairwise independent’, as in (3.5) and (3.6), then things are less straightforward. Very roughly, sums of non-independent random variables are to sums of independent random variables as *partial* sumsets $A +_E B$ along a bipartite graph E are to full sumsets $A + B$. Hence, it is natural to apply a variant of the entropic Balog–Szemerédi–Gowers lemma due to the fourth author [14, Lemma 3.3] (reproduced as Lemma A.2 here) to pass from the ‘almost independent’ variables (T_1, T_2) as above, for which $\mathbf{H}[T_1 + T_2] - \frac{1}{2}\mathbf{H}[T_1] - \frac{1}{2}\mathbf{H}[T_2]$ is small (or even negative), to variables X'_1, X'_2 with $d[X'_1; X'_2] = O(\eta k)$. These variables X'_1, X'_2 , which are given to us by Lemma A.2, now establish (2.2).

Examining the proof of Lemma A.2 carefully, the final choice of X'_1 that comes out of this argument is of the form

$$(X_1 + X_2 | \tilde{X}_1 + X_2 = t, X_1 + \tilde{X}_2 = s + t) \quad (3.10)$$

or a related quantity obtained by permuting the variables, and similarly for X'_2 (using the same s, t for both X'_1 and X'_2).

Motivating examples. To understand the strategy, it is helpful to consider some cases of the form $X_1 = U_{A_1}$, $X_2 = U_{A_2}$ for various sets $A_1, A_2 \subseteq \mathbf{F}_2^n$, and to discuss which choices of X'_1, X'_2 give the desired estimate (2.2). In this discussion it is convenient to write $K := e^k$.

Consider first the case in which $A_1 = A_2 = A$ and A is a random subset of some subgroup $H \leq \mathbf{F}_2^n$ with density K^{-1} in H . Then (almost surely) $d[X_1; X_2] \approx k$. In this case the choice (3.2), that is to say $X'_1 = X_1 + \tilde{X}_1$ and $X'_2 = X_2 + \tilde{X}_2$, immediately establishes (2.2). Indeed both X'_1 and X'_2 are close to the uniform distribution U_H on H , so $d[X'_1; X'_2] \approx 0$ and $d[X'_i; X_i] \approx k/2$ for $i = 1, 2$, and (2.2) follows, with room to spare, by the triangle inequality.

Consider next the case in which $A_1 = \bigcup_{i=1}^m (x_i + H)$ and $A_2 = \bigcup_{i=1}^m (y_i + H)$, where $H < \mathbf{F}_2^n$ is a subgroup and the $x_i + H$ and $y_i + H$ are linearly independent in G/H . Setting $m := K$ gives $d[X_1; X_2] \approx k$. In this case one can obtain (2.2) with the choice (3.3), that is to say $X'_1 = (X_1 | X_1 + \tilde{X}_2 = g)$ and $X'_2 = (X_2 | X_2 + \tilde{X}_1 = g')$ for any $g, g' \in A_1 + A_2$. Indeed, X'_1 is the uniform distribution on $A_1 \cap (A_2 + g)$, which if $g \in x_i + y_j + H$ is exactly the coset $x_i + H$ of H (by linear independence). Similarly, X'_2 is uniform on a coset $y_j + H$, so $d[X'_1; X'_2] \approx 0$ and again (2.2) follows. (Using the fibres in (3.4) also gives a contradiction in this case, but the analysis is slightly more involved, since $A_1 \cap (A_1 + g)$ is typically a union of two cosets of H .)

In both of the above examples, one of the choices (3.1), (3.2), (3.3) or (3.4) already gives the estimate (2.2) and so it is not necessary to proceed to the ‘endgame’.

Here is a third example, a sort of combination of the previous two, for which the choices (3.1), (3.2), (3.3) and (3.4) all fail to work. Let A'_1, A'_2 be (independent) random subsets of density $1/m$ of the sets A_1, A_2 from the previous example, where now $m = \sqrt{K}$. One may then check that $X_1 + \tilde{X}_1, X_2 + \tilde{X}_2$ resemble the uniform distribution on the union of $\approx K/2$ cosets of H , so $d[X'_1; X'_2] \approx k - O(1)$ and we do not obtain (2.2) (if k is large) with the choice (3.2). The case of (3.1) is similar.

On the other hand, the variables $(X_1 | X_1 + \tilde{X}_2 = g), (X_2 | X_2 + \tilde{X}_1 = g')$ will be uniform on $A_1 \cap (A_2 + g)$ and $A_2 \cap (A_1 + g')$ respectively, and such sets will typically (when non-empty) resemble random subsets of a coset of H , of density $1/m^2 \approx 1/K$. For these variables we also have

$d[X'_1; X'_2] \approx k$, and so we do not obtain (2.2) with the choice (3.3). The case of (3.4) is broadly similar.

We therefore proceed to the endgame and consider the three variables in (3.8). Since $X_1 + X_2$ is roughly uniform on $B := \bigcup_{i,j} (x_i + y_j + H)$, a similar analysis to the second example shows that $(X_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s)$ is roughly uniform on $B \cap (B + s)$, which is typically a union of four cosets of H , and similarly for $(\tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s)$. Moreover, one can check that these two variables are ‘50% independent’ (in that knowing one of them narrows the choice of the other to two cosets of H).

If we then apply Lemma A.2, we find variables X'_1, X'_2 of the form given in (3.10). In other words, X'_1 is a sum of two variables of the form $(X_1 | X_1 + \tilde{X}_2 = a)$ and $(X_2 | X_2 + \tilde{X}_1 = b)$. Each will be uniform on a random subset of a coset of H , and adding them together will give something close to uniform on a coset of H . The case of X'_2 is similar, so we get $d[X'_1; X'_2] \approx 0$ and (2.2) follows. (Again, permuting the variables may give slightly different conclusions.)

Finally, we consider a further example over $G = \mathbf{Z}$ instead of \mathbf{F}_2^n . Take X_1, X_2 to have a ‘discrete Gaussian’ distribution $p(x) = C_0 e^{-x^2/2r^2}$ with some large ‘width’ r and a normalizing constant C_0 . By [14, Theorem 1.13], the distance $d[X_1; X_2] \approx \frac{1}{2} \log 2$ is roughly minimal among random variables on \mathbf{Z} with large entropy, and hence Proposition 2.1 cannot hold over \mathbf{Z} in the form stated.

Concretely, one can show that sums $X_i + \tilde{X}_j$ as in (3.1), (3.2) are (roughly) discrete Gaussians of width $\sqrt{2}r$, and fibres $(X_i | X_i + \tilde{X}_j)$ as in (3.3), (3.4) are (shifted) discrete Gaussians of width about $r/\sqrt{2}$. Hence, all these have essentially the same value of $d[-; -]$, and there is no way of ‘improving’ these variables with any of the choices considered in Sections 5 and 6. One could also see directly that the conditional near-independence statements such as

$$\mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \approx 0$$

do indeed hold for discrete Gaussians.

Even though discrete Gaussians over \mathbf{F}_2 do not exist, this example places significant constraints on what we can hope to prove in general. In particular, it shows that the endgame argument must ‘see’ the finite characteristic in an essential way.

The contrapositive formulation. The above discussion and the motivating examples which followed it were framed in terms of proving

Proposition 2.1 directly, by making various choices of X'_1 and X'_2 , and the authors found this most natural when thinking about the problem.

However, when it comes to recording the proofs, it turns out to be notationally simpler to argue in the contrapositive. Thus, suppose henceforth that we have a pair (X_1, X_2) of G -valued random variables with $d[X_1; X_2] = k$ for some k , and suppose that

$$\tau[X'_1; X'_2] \geq \tau[X_1; X_2] \quad (3.11)$$

for every pair of G -valued random variables (X'_1, X'_2) . Using the definition of τ (see (2.1)) we may rewrite (3.11) as

$$d[X'_1; X'_2] \geq k - \eta(d[X_1^0; X'_1] - d[X_1^0; X_1]) - \eta(d[X_2^0; X'_2] - d[X_2^0; X_2]). \quad (3.12)$$

As in the discussion above, the idea now is to test (3.12) with various choices of (X'_1, X'_2) generated from (X_1, X_2) , the aim being to deduce from this that $k = 0$.

Testing (3.12) with the choices (3.1), (3.3) will lead us to the conclusion that

$$I_1 := \mathbf{I}[X_1 + X_2 : \tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k \quad (3.13)$$

(that is, a more precise version of (3.5)). The details of this deduction are provided in Section 5. Testing (3.12) with the choices (3.2), (3.4) leads to

$$I_2 := \mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k + \frac{2\eta(2\eta k - I_1)}{1 - \eta} \quad (3.14)$$

(a more precise version of (3.6)), and of course the same estimate holds for I_3 , since in fact $I_2 = I_3$. The details of this deduction are provided in Section 6.

Finally, armed with the knowledge that (3.13), (3.14) hold, we proceed to the endgame, using choices of X'_1, X'_2 arising from (3.8) and entropic Balog–Szémerédi–Gowers (i.e., of forms such as (3.10)) to finally show that $k = 0$. The details are given in Section 7.

The main reason for arguing by contradiction is that it greatly simplifies our discussions of conditioned random variables. Note that (3.12) implies a conditioned version of itself: for any random variables (X'_1, Y_1) and (X'_2, Y_2) we may apply (3.12) with X'_1, X'_2 replaced by each of the

conditioned random variables $(X'_1|Y_1 = y_1), (X'_2|Y_2 = y_2)$, obtaining

$$\begin{aligned} d[(X'_1|Y_1 = y_1); (X'_2|Y_2 = y_2)] \\ \geq k - \eta(d[X_1^0; (X'_1|Y_1 = y_1)] - d[X_1^0; X_1]) \\ - \eta(d[X_2^0; (X'_2|Y_2 = y_2)] - d[X_2^0; X_2]). \end{aligned}$$

Multiplying by $p_{Y_1}(y_1)p_{Y_2}(y_2)$ and summing, we have

$$\begin{aligned} d[X'_1|Y_1; X'_2|Y_2] \geq k - \eta(d[X_1^0; X'_1|Y_1] - d[X_1^0; X_1]) \\ - \eta(d[X_2^0; X'_2|Y_2] - d[X_2^0; X_2]). \end{aligned} \quad (3.15)$$

More discussion of the notion of conditional distance may be found at (A.14).

To make the same argument in the direct (non-contrapositive) direction we would pick some particular (y_1, y_2) such that

$$\begin{aligned} d[(X'_1|Y_1 = y_1); (X'_2|Y_2 = y_2)] + \eta(d[X_1^0; (X'_1|Y_1 = y_1)] - d[X_1^0; X_1]) \\ + \eta(d[X_2^0; (X'_2|Y_2 = y_2)] - d[X_2^0; X_2]) \end{aligned}$$

is at most the (weighted) average value of the same quantity over all (y_1, y_2) . If we have to perform many such steps in sequence, this becomes notationally very taxing. However, this is purely a matter of preference: for instance, in Lemma 7.2 and its proof we instead choose to argue directly.

Remarks on odd p . We will handle the case p odd in our forthcoming paper [1]. All of the above steps go through, except for the final contradiction in the ‘endgame’ where we use an argument specific to characteristic 2. To get a contradiction in characteristic p , it is necessary to run a variant of the argument with a p -partite ‘distance’ function $D[X_1; \dots; X_p]$ replacing $d[X_1; X_2]$ as the main term in the functional to be minimized. This introduces additional notational complexity: for instance we now have an array $(X_{ij})_{i,j \in \mathbb{F}_p}$ of random variables, and the fibring lemma must be generalized to this p -partite distance and applied $p - 1$ times.

4. FIBRING LEMMA

Here we record the fibring lemma, that is to say [4, Proposition 1.4], with an explicit error term. (The latter was mentioned in [4], but only as a casual remark.)

Proposition 4.1. *Let $\pi: H \rightarrow H'$ be a homomorphism between vector spaces over \mathbf{F}_2 and let Z_1, Z_2 be H -valued random variables. Then we have*

$$d[Z_1; Z_2] \geq d[\pi(Z_1); \pi(Z_2)] + d[Z_1|\pi(Z_1); Z_2|\pi(Z_2)]$$

where the notation is as in (A.14). Moreover, if Z_1, Z_2 are taken to be independent, then the difference between the two sides is

$$\mathbf{I}[Z_1 + Z_2 : (\pi(Z_1), \pi(Z_2)) \mid \pi(Z_1 + Z_2)]. \quad (4.1)$$

Remark. Characteristic 2 is not used here in any essential way. For general abelian groups H, H' one can write $Z_1 - Z_2$ (twice) in (4.1), or else work with $D[X; Y] := \mathbf{H}[X + Y] - \frac{1}{2}\mathbf{H}[X] - \frac{1}{2}\mathbf{H}[Y]$ in place of $d[X; Y]$.

Proof. Let Z_1, Z_2 be independent throughout. We have

$$\begin{aligned} & d[Z_1|\pi(Z_1); Z_2|\pi(Z_2)] \\ &= \mathbf{H}[Z_1 + Z_2|\pi(Z_1), \pi(Z_2)] - \frac{1}{2}\mathbf{H}[Z_1|\pi(Z_1)] - \frac{1}{2}\mathbf{H}[Z_2|\pi(Z_2)] \\ &\leq \mathbf{H}[Z_1 + Z_2|\pi(Z_1 + Z_2)] - \frac{1}{2}\mathbf{H}[Z_1|\pi(Z_1)] - \frac{1}{2}\mathbf{H}[Z_2|\pi(Z_2)] \\ &= d[Z_1; Z_2] - d[\pi(Z_1); \pi(Z_2)]. \end{aligned}$$

In the middle step, we used submodularity of entropy, and in the last step we used the fact that

$$\mathbf{H}[Z_1 + Z_2|\pi(Z_1 + Z_2)] = \mathbf{H}[Z_1 + Z_2] - \mathbf{H}[\pi(Z_1 + Z_2)]$$

(since $Z_1 + Z_2$ determines $\pi(Z_1 + Z_2)$) and that

$$\mathbf{H}[Z_i|\pi(Z_i)] = \mathbf{H}[Z_i] - \mathbf{H}[\pi(Z_i)]$$

(since Z_i determines $\pi(Z_i)$). This gives the claimed inequality. The difference between the two sides is precisely

$$\mathbf{H}[Z_1 + Z_2|\pi(Z_1 + Z_2)] - \mathbf{H}[Z_1 + Z_2|\pi(Z_1), \pi(Z_2)].$$

To rewrite this in terms of (conditional) mutual information, we use the identity

$$\mathbf{H}[A|B] - \mathbf{H}[A|B, C] = \mathbf{I}[A : C|B],$$

taking $A := Z_1 + Z_2$, $B := \pi(Z_1 + Z_2)$ and $C := (\pi(Z_1), \pi(Z_2))$, and noting that in this case $\mathbf{H}[A|B, C] = \mathbf{H}[A|C]$ since C uniquely determines B . This completes the proof. \square

We extract the specific special case of this result that we will need in our arguments.

Corollary 4.2. *Let Y_1, Y_2, Y_3 and Y_4 be independent G -valued random variables. Then*

$$\begin{aligned} & d[Y_1 + Y_3; Y_2 + Y_4] + d[Y_1|Y_1 + Y_3; Y_2|Y_2 + Y_4] \\ & + \mathbf{I}[Y_1 + Y_2 : Y_2 + Y_4 | Y_1 + Y_2 + Y_3 + Y_4] = d[Y_1; Y_2] + d[Y_3; Y_4]. \end{aligned}$$

Proof. We apply Proposition 4.1 with $H := G \times G$, $H' := G$, π the addition homomorphism $\pi(x, y) := x + y$, and with the random variables $Z_1 := (Y_1, Y_3)$ and $Z_2 := (Y_2, Y_4)$. Then by independence we easily calculate

$$d[Z_1; Z_2] = d[Y_1; Y_2] + d[Y_3; Y_4]$$

while by definition

$$d[\pi(Z_1); \pi(Z_2)] = d[Y_1 + Y_3; Y_2 + Y_4].$$

Furthermore,

$$d[Z_1 | \pi(Z_1); Z_2 | \pi(Z_2)] = d[Y_1 | Y_1 + Y_3; Y_2 | Y_2 + Y_4],$$

since $Z_1 = (Y_1, Y_3)$ and Y_1 are linked by an invertible affine transformation once $\pi(Z_1) = Y_1 + Y_3$ is fixed, and similarly for Z_2 and Y_2 . Finally, we have

$$\begin{aligned} & \mathbf{I}[Z_1 + Z_2 : (\pi(Z_1), \pi(Z_2)) | \pi(Z_1) + \pi(Z_2)] \\ & = \mathbf{I}[(Y_1 + Y_2, Y_3 + Y_4) : (Y_1 + Y_3, Y_2 + Y_4) | Y_1 + Y_2 + Y_3 + Y_4] \\ & = \mathbf{I}[Y_1 + Y_2 : Y_2 + Y_4 | Y_1 + Y_2 + Y_3 + Y_4] \end{aligned}$$

where in the last line we used the fact that $(Y_1 + Y_2, Y_1 + Y_2 + Y_3 + Y_4)$ uniquely determine $Y_3 + Y_4$ and similarly $(Y_2 + Y_4, Y_1 + Y_2 + Y_3 + Y_4)$ uniquely determine $Y_1 + Y_3$. \square

5. FIRST ESTIMATE

Recall that in this and subsequent sections we are working on the assumption that (3.12) and its conditioned variant (3.15) hold, aiming to prove that $k = 0$ and hence conclude (the contrapositive of) Proposition 2.1.

Recall also that $X_1, X_2, \tilde{X}_1, \tilde{X}_2$ are independent random variables, with X_1, \tilde{X}_1 copies of X_1 and X_2, \tilde{X}_2 copies of X_2 .

In this section we establish the upper bound (3.13), which was that

$$I_1 := \mathbf{I}[X_1 + X_2 : \tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k.$$

We apply Corollary 4.2 with the choice

$$(Y_1, Y_2, Y_3, Y_4) := (X_1, X_2, \tilde{X}_2, \tilde{X}_1).$$

It gives

$$\begin{aligned} & d[X_1 + \tilde{X}_2; X_2 + \tilde{X}_1] + d[X_1|X_1 + \tilde{X}_2; X_2|X_2 + \tilde{X}_1] \\ & + \mathbf{I}[X_1 + X_2 : \tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] = 2k, \end{aligned} \quad (5.1)$$

since $d[X_1; X_2] = k$. Applying (3.12), (3.15), we have

$$\begin{aligned} d[X_1 + \tilde{X}_2; X_2 + \tilde{X}_1] & \geq k - \eta(d[X_1^0; X_1 + \tilde{X}_2] - d[X_1^0; X_1]) \\ & \quad - \eta(d[X_2^0; X_2 + \tilde{X}_1] - d[X_2^0; X_2]) \end{aligned}$$

and

$$\begin{aligned} & d[X_1|X_1 + \tilde{X}_2; X_2|X_2 + \tilde{X}_1] \\ & \geq k - \eta(d[X_1^0; X_1|X_1 + \tilde{X}_2] - d[X_1^0; X_1]) \\ & \quad - \eta(d[X_2^0; X_2|X_2 + \tilde{X}_1] - d[X_2^0; X_2]). \end{aligned} \quad (5.2)$$

It therefore suffices to prove that

$$\begin{aligned} & (d[X_1^0; X_1 + \tilde{X}_2] - d[X_1^0; X_1]) + (d[X_2^0; X_2 + \tilde{X}_1] - d[X_2^0; X_2]) \\ & + (d[X_1^0; X_1|X_1 + \tilde{X}_2] - d[X_1^0; X_1]) \\ & + (d[X_2^0; X_2|X_2 + \tilde{X}_1] - d[X_2^0; X_2]) \leq 2k. \end{aligned} \quad (5.3)$$

We pause to state some lemmas which we will use for bounding a number of similar expressions, both here and in the next two sections.

The first is a bound relating conditioned and unconditioned variants of the Ruzsa distance.

Lemma 5.1. *Suppose that (X, Z) and (Y, W) are random variables, where X, Y take values in an abelian group. Then*

$$d[X|Z; Y|W] \leq d[X; Y] + \frac{1}{2}\mathbf{I}[X : Z] + \frac{1}{2}\mathbf{I}[Y : W].$$

Proof. The definition of conditional distance is given in (A.14).

Using the alternative expression (A.15), if $(X', Z'), (Y', W')$ are independent copies of the variables $(X, Z), (Y, W)$, we have

$$\begin{aligned} d[X|Z; Y|W] & = \mathbf{H}[X' - Y'|Z', W'] - \frac{1}{2}\mathbf{H}[X'|Z'] - \frac{1}{2}\mathbf{H}[Y'|W'] \\ & \leq \mathbf{H}[X' - Y'] - \frac{1}{2}\mathbf{H}[X'|Z'] - \frac{1}{2}\mathbf{H}[Y'|W'] \\ & = d[X'; Y'] + \frac{1}{2}\mathbf{I}[X' : Z'] + \frac{1}{2}\mathbf{I}[Y' : W']. \end{aligned}$$

Here, in the middle step we used (A.5), and in the last step we used the definitions of $d[-; -]$ and $\mathbf{I}[-]$. \square

In the proof of the next lemma we will use the fact that, for any independent random variables X, Y, Z taking values in an abelian group, we have the inequality

$$\mathbf{H}[X + Y + Z] - \mathbf{H}[X + Y] \leq \mathbf{H}[Y + Z] - \mathbf{H}[Y]. \quad (5.4)$$

This is a result of Kaimanovich and Vershik [5], and can be viewed as an entropy analogue of an inequality of Plünnecke [7]. For the convenience of the reader, we give the proof in Appendix A.

Lemma 5.2. *Let X, Y, Z be random variables taking values in some abelian group, and with Y, Z independent. Then we have*

$$\begin{aligned} d[X; Y + Z] - d[X; Y] &\leq \frac{1}{2}(\mathbf{H}[Y + Z] - \mathbf{H}[Y]) \\ &= \frac{1}{2}d[Y; Z] + \frac{1}{4}\mathbf{H}[Z] - \frac{1}{4}\mathbf{H}[Y]. \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} d[X; Y|Y + Z] - d[X; Y] &\leq \frac{1}{2}(\mathbf{H}[Y + Z] - \mathbf{H}[Z]) \\ &= \frac{1}{2}d[Y; Z] + \frac{1}{4}\mathbf{H}[Y] - \frac{1}{4}\mathbf{H}[Z]. \end{aligned} \quad (5.6)$$

Proof. We first prove (5.5). We may assume (taking an independent copy) that X is independent of Y, Z . Then we have

$$\begin{aligned} d[X; Y + Z] - d[X; Y] &= \mathbf{H}[X + Y + Z] - \mathbf{H}[X + Y] - \frac{1}{2}\mathbf{H}[Y + Z] + \frac{1}{2}\mathbf{H}[Y]. \end{aligned}$$

Combining this with (5.4) gives the required bound. The second form of the result is immediate from the definition of $d[Y; Z]$, since Y, Z are independent.

Turning to (5.6), we have

$$\begin{aligned} \mathbf{I}[Y : Y + Z] &= \mathbf{H}[Y] + \mathbf{H}[Y + Z] - \mathbf{H}[Y, Y + Z] \\ &= \mathbf{H}[Y] + \mathbf{H}[Y + Z] - \mathbf{H}[Y, Z] = \mathbf{H}[Y + Z] - \mathbf{H}[Z], \end{aligned}$$

and so (5.6) is a consequence of Lemma 5.1. Once again the second form of the result is immediate from the definition of $d[Y; Z]$. \square

We return to our main task of establishing (5.3), and hence (3.13). By Lemma 5.2 (and recalling that k is defined to be $d[X_1; X_2]$) we have

$$\begin{aligned} d[X_1^0; X_1 + \tilde{X}_2] - d[X_1^0; X_1] &\leq \frac{1}{2}k + \frac{1}{4}\mathbf{H}[X_2] - \frac{1}{4}\mathbf{H}[X_1], \\ d[X_2^0; X_2 + \tilde{X}_1] - d[X_2^0; X_2] &\leq \frac{1}{2}k + \frac{1}{4}\mathbf{H}[X_1] - \frac{1}{4}\mathbf{H}[X_2], \\ d[X_1^0; X_1|X_1 + \tilde{X}_2] - d[X_1^0; X_1] &\leq \frac{1}{2}k + \frac{1}{4}\mathbf{H}[X_1] - \frac{1}{4}\mathbf{H}[X_2] \end{aligned} \quad (5.7)$$

and

$$d[X_2^0; X_2|X_2 + \tilde{X}_1] - d[X_2^0; X_2] \leq \frac{1}{2}k + \frac{1}{4}\mathbf{H}[X_2] - \frac{1}{4}\mathbf{H}[X_1]. \quad (5.8)$$

Adding all these inequalities, we obtain (5.3).

For use in the next two sections, we note that subtracting (5.2) from (5.1), and combining the resulting inequality with (5.7) and (5.8) gives the bound

$$d[X_1 + \tilde{X}_2; X_2 + \tilde{X}_1] \leq (1 + \eta)k - I_1,$$

which is equivalent to

$$\mathbf{H}[X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq \frac{1}{2}\mathbf{H}[X_1] + \frac{1}{2}\mathbf{H}[X_2] + (2 + \eta)k - I_1. \quad (5.9)$$

One could also bound the left-hand side using the Kaimanovich–Vershik inequality (5.4) (twice), but by making use of our hypothesis (3.15) as above we obtain a slightly better constant.

6. SECOND ESTIMATE

In this section we establish the upper bound (3.14), which was

$$I_2 := \mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k + \frac{2\eta(2\eta k - I_1)}{1 - \eta}.$$

We apply Corollary 4.2, but now with the choice

$$(Y_1, Y_2, Y_3, Y_4) := (X_2, X_1, \tilde{X}_2, \tilde{X}_1).$$

Now Corollary 4.2 can be rewritten as

$$\begin{aligned} & d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] + d[X_1 | X_1 + \tilde{X}_1; X_2 | X_2 + \tilde{X}_2] \\ & + \mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] = 2k, \end{aligned}$$

recalling once again that $k := d[X_1; X_2]$. From (3.12) and (3.15) as before, one has

$$\begin{aligned} d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] & \geq k - \eta(d[X_1^0; X_1 + \tilde{X}_1] - d[X_1^0; X_1]) \\ & \quad - \eta(d[X_2^0; X_2 + \tilde{X}_2] - d[X_2^0; X_2]) \end{aligned} \quad (6.1)$$

and

$$\begin{aligned} d[X_1 | X_1 + \tilde{X}_1; X_2 | X_2 + \tilde{X}_2] & \geq k - \eta(d[X_1^0; X_1 | X_1 + \tilde{X}_1] - d[X_1^0; X_1]) \\ & \quad - \eta(d[X_2^0; X_2 | X_2 + \tilde{X}_2] - d[X_2^0; X_2]). \end{aligned}$$

Now Lemma 5.2 gives

$$d[X_1^0; X_1 + \tilde{X}_1] - d[X_1^0; X_1] \leq \frac{1}{2}d[X_1; X_1], \quad (6.2)$$

$$d[X_2^0; X_2 + \tilde{X}_2] - d[X_2^0; X_2] \leq \frac{1}{2}d[X_2; X_2], \quad (6.3)$$

$$d[X_1^0; X_1 | X_1 + \tilde{X}_1] - d[X_1^0; X_1] \leq \frac{1}{2}d[X_1; X_1],$$

and

$$d[X_2^0; X_2 | X_2 + \tilde{X}_2] - d[X_2^0; X_2] \leq \frac{1}{2}d[X_2; X_2].$$

Combining all these inequalities and cancelling terms, we obtain

$$\mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq \eta(d[X_1; X_1] + d[X_2; X_2]). \quad (6.4)$$

One could bound the right-hand side by $4\eta k$ using the Ruzsa triangle inequality, but a more efficient approach is as follows. First, by combining (6.1), (6.2) and (6.3), we obtain

$$d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] \geq k - \frac{\eta}{2}(d[X_1; X_1] + d[X_2; X_2]). \quad (6.5)$$

We may also expand

$$\begin{aligned} d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] &= \mathbf{H}[X_1 + \tilde{X}_1 + X_2 + \tilde{X}_2] - \frac{1}{2}\mathbf{H}[X_1 + \tilde{X}_1] - \frac{1}{2}\mathbf{H}[X_2 + \tilde{X}_2] \\ &= \mathbf{H}[X_1 + \tilde{X}_1 + X_2 + \tilde{X}_2] - \frac{1}{2}\mathbf{H}[X_1] - \frac{1}{2}\mathbf{H}[X_2] \\ &\quad - \frac{1}{2}(d[X_1; X_1] + d[X_2; X_2]), \end{aligned}$$

and hence by (5.9)

$$d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] \leq (2 + \eta)k - \frac{1}{2}(d[X_1; X_1] + d[X_2; X_2]) - I_1.$$

Combining this bound with (6.5) we obtain

$$d[X_1; X_1] + d[X_2; X_2] \leq 2k + \frac{2(2\eta k - I_1)}{1 - \eta}.$$

Therefore by (6.4) we have the desired bound

$$\mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k + \frac{2\eta(2\eta k - I_1)}{1 - \eta}.$$

7. ENDGAME

In this section we conclude the proof of Proposition 2.1. Let us begin by recording an inequality which will be used several times in the calculations below.

Lemma 7.1. *Let X, Y, Z, Z' be random variables taking values in some abelian group, and with Y, Z, Z' independent. Then we have*

$$\begin{aligned} d[X; Y + Z | Y + Z + Z'] - d[X; Y] &\leq \frac{1}{2}(\mathbf{H}[Y + Z + Z'] + \mathbf{H}[Y + Z] - \mathbf{H}[Y] - \mathbf{H}[Z']). \end{aligned} \quad (7.1)$$

Proof. By (5.6) (with a change of variables) we have

$$d[X; Y + Z | Y + Z + Z'] - d[X; Y + Z] \leq \frac{1}{2}(\mathbf{H}[Y + Z + Z'] - \mathbf{H}[Z']).$$

Adding this to (5.5) gives the result. \square

Turning now to the main argument, let $X_1, X_2, \tilde{X}_1, \tilde{X}_2$ be as before, and introduce the random variables

$$U := X_1 + X_2, \quad V := \tilde{X}_1 + X_2, \quad W := X_1 + \tilde{X}_1$$

and

$$S := X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2.$$

From the definitions (3.5), (3.6), (3.7) of I_1, I_2, I_3 and the above notation, we see that

$$I_1 = \mathbf{I}[U : V | S], \quad I_2 = \mathbf{I}[W : U | S], \quad I_3 = \mathbf{I}[V : W | S].$$

Recall from (3.13) that $I_1 \leq 2\eta k$. From (3.14) (and since $I_2 = I_3$) we have the inequalities

$$\mathbf{I}[V : W | S], \mathbf{I}[W : U | S] \leq 2\eta k + \frac{2\eta(2\eta k - I_1)}{1 - \eta}.$$

Summing these two inequalities and the equality $\mathbf{I}[U : V | S] = I_1$ gives

$$\begin{aligned} \mathbf{I}[U : V | S] + \mathbf{I}[V : W | S] + \mathbf{I}[W : U | S] \\ \leq I_1 + 4\eta k + \frac{4\eta(2\eta k - I_1)}{1 - \eta} \\ = 6\eta k - \frac{1 - 5\eta}{1 - \eta}(2\eta k - I_1). \end{aligned} \quad (7.2)$$

We encourage the reader to read the argument assuming first that $I_1 = 2\eta k$, in which case the calculations are cleaner.

We assemble some preliminary estimates on distances. By Lemma 7.1 (taking $X = X_1^0, Y = X_1, Z = X_2$ and $Z' = \tilde{X}_1 + \tilde{X}_2$, so that $Y + Z = U$ and $Y + Z + Z' = S$) we have, noting that $\mathbf{H}[Y + Z] = \mathbf{H}[Z']$,

$$d[X_1^0; U | S] - d[X_1^0; X_1] \leq \frac{1}{2}(\mathbf{H}[S] - \mathbf{H}[X_1]).$$

Further applications of Lemma 7.1 give

$$\begin{aligned} d[X_2^0; U | S] - d[X_2^0; X_2] &\leq \frac{1}{2}(\mathbf{H}[S] - \mathbf{H}[X_2]) \\ d[X_1^0; V | S] - d[X_1^0; X_1] &\leq \frac{1}{2}(\mathbf{H}[S] - \mathbf{H}[X_1]) \\ d[X_2^0; V | S] - d[X_2^0; X_2] &\leq \frac{1}{2}(\mathbf{H}[S] - \mathbf{H}[X_2]) \end{aligned}$$

and

$$d[X_1^0; W | S] - d[X_1^0; X_1] \leq \frac{1}{2}(\mathbf{H}[S] + \mathbf{H}[W] - \mathbf{H}[X_1] - \mathbf{H}[W']),$$

where $W' := X_2 + \tilde{X}_2$. To treat $d[X_2^0; W | S]$, first note that this equals $d[X_2^0; W' | S]$, since for a fixed choice s of S we have $W' = W + s$. Now we may apply Lemma 7.1 to obtain

$$d[X_2^0; W' | S] - d[X_2^0; X_2] \leq \frac{1}{2}(\mathbf{H}[S] + \mathbf{H}[W'] - \mathbf{H}[X_2] - \mathbf{H}[W]).$$

Summing these six estimates and using (5.9), we conclude that

$$\begin{aligned} \sum_{i=1}^2 \sum_{A \in \{U, V, W\}} (\mathrm{d}[X_i^0; A|S] - \mathrm{d}[X_i^0; X_i]) \\ \leq 3\mathbf{H}[S] - \frac{3}{2}\mathbf{H}[X_1] - \frac{3}{2}\mathbf{H}[X_2] \\ \leq (6 - 3\eta)k + 3(2\eta k - I_1). \end{aligned} \quad (7.3)$$

Now we come to the key observation we will exploit, which is that

$$U + V + W = 0. \quad (7.4)$$

Here, of course, we are using the fact that we are in characteristic 2, and this time the use is critical.

To see the force of (7.2) and (7.4), we state the following general claim.

Lemma 7.2. *Let $G = \mathbf{F}_2^n$ and let (T_1, T_2, T_3) be a G^3 -valued random variable such that $T_1 + T_2 + T_3 = 0$ holds identically. Set*

$$\delta := \sum_{1 \leq i < j \leq 3} \mathbf{I}[T_i; T_j]. \quad (7.5)$$

Then there exist random variables T'_1, T'_2 such that

$$\begin{aligned} \mathrm{d}[T'_1; T'_2] + \eta(\mathrm{d}[X_1^0; T'_1] - \mathrm{d}[X_1^0; X_1]) + \eta(\mathrm{d}[X_2^0; T'_2] - \mathrm{d}[X_2^0; X_2]) \\ \leq \delta + \frac{\eta}{3} \left(\delta + \sum_{i=1}^2 \sum_{j=1}^3 (\mathrm{d}[X_i^0; T_j] - \mathrm{d}[X_i^0; X_i]) \right). \end{aligned}$$

We recommend that the reader first work through this proof in the case $\delta = 0$, when the variables T_1, T_2, T_3 are independent. Recalling (3.9), the proof below collapses to taking T'_1, T'_2 to be some two of $\{T_1, T_2, T_3\}$, without the need to apply Lemma A.2.

Proof. We apply the variant of the entropic Balog–Szemerédi–Gowers theorem stated in Lemma A.2, taking $(A, B) = (T_1, T_2)$ there. Since $T_1 + T_2 = T_3$, the conclusion is that

$$\begin{aligned} \sum_{t_3} p_{T_3}(t_3) \mathrm{d}[(T_1|T_3 = t_3); (T_2|T_3 = t_3)] \\ \leq 3\mathbf{I}[T_1 : T_2] + 2\mathbf{H}[T_3] - \mathbf{H}[T_1] - \mathbf{H}[T_2]. \end{aligned} \quad (7.6)$$

The right-hand side in (7.6) can be rearranged as

$$\begin{aligned} 2(\mathbf{H}[T_1] + \mathbf{H}[T_2] + \mathbf{H}[T_3]) - 3\mathbf{H}[T_1, T_2] \\ = 2(\mathbf{H}[T_1] + \mathbf{H}[T_2] + \mathbf{H}[T_3]) - \mathbf{H}[T_1, T_2] - \mathbf{H}[T_2, T_3] - \mathbf{H}[T_1, T_3] = \delta, \end{aligned}$$

using the fact that all three terms $\mathbf{H}[T_i, T_j]$ are equal to $\mathbf{H}[T_1, T_2, T_3]$ and hence to each other. We also have

$$\begin{aligned} & \sum_{t_3} p_{T_3}(t_3) (d[X_1^0; (T_1|T_3 = t_3)] - d[X_1^0; X_1]) \\ &= d[X_1^0; T_1|T_3] - d[X_1^0; X_1] \leq d[X_1^0; T_1] - d[X_1^0; X_1] + \tfrac{1}{2}\mathbf{I}[T_1 : T_3] \end{aligned}$$

by Lemma 5.1, and similarly

$$\begin{aligned} & \sum_{t_3} p_{T_3}(t_3) (d[X_2^0; (T_2|T_3 = t_3)] - d[X_2^0; X_2]) \\ & \leq d[X_2^0; T_2] - d[X_2^0; X_2] + \tfrac{1}{2}\mathbf{I}[T_2 : T_3]. \end{aligned}$$

Temporarily define

$$\psi[Y_1; Y_2] := d[Y_1; Y_2] + \eta(d[X_1^0; Y_1] - d[X_1^0; X_1]) + \eta(d[X_2^0; Y_2] - d[X_2^0; X_2]).$$

Putting the above observations together, we have

$$\begin{aligned} & \sum_{t_3} p_{T_3}(t_3) \psi[(T_1|T_3 = t_3); (T_2|T_3 = t_3)] \leq \delta + \eta(d[X_1^0; T_1] - d[X_1^0; X_1]) \\ & \quad + \eta(d[X_2^0; T_2] - d[X_2^0; X_2]) + \tfrac{1}{2}\eta\mathbf{I}[T_1 : T_3] + \tfrac{1}{2}\eta\mathbf{I}[T_2 : T_3]. \end{aligned}$$

Choosing some t_3 in the support of T_3 that minimizes the $\psi[-; -]$ value, and setting $T'_{1,3} := (T_1|T_3 = t_3)$, $T'_{2,3} := (T_2|T_3 = t_3)$, we have

$$\begin{aligned} \psi[T'_{1,3}; T'_{2,3}] & \leq \delta + \eta(d[X_1^0; T_1] - d[X_1^0; X_1]) + \eta(d[X_2^0; T_2] - d[X_2^0; X_2]) \\ & \quad + \tfrac{1}{2}\eta\mathbf{I}[T_1 : T_3] + \tfrac{1}{2}\eta\mathbf{I}[T_2 : T_3]. \end{aligned} \tag{7.7}$$

We now repeat this analysis for all permutations of $\{T_1, T_2, T_3\}$ to get variables $T'_{\alpha,\gamma}, T'_{\beta,\gamma}$ for $\{\alpha, \beta, \gamma\}$ ranging over all six permutations of $\{1, 2, 3\}$. Averaging the resulting inequalities (7.7), and recalling the definition (7.5) of δ , we get

$$\tfrac{1}{6} \sum_{\alpha, \beta, \gamma} \psi[T'_{\alpha,\gamma}; T'_{\beta,\gamma}] \leq \delta + \tfrac{\eta}{3} \left(\delta + \sum_{i=1}^2 \sum_{j=1}^3 (d[X_i^0; T_j] - d[X_i^0; X_i]) \right),$$

from which the result follows (taking T'_1, T'_2 to be $T'_{\alpha,\gamma}, T'_{\beta,\gamma}$ for some (α, β, γ) that gives at most the average value). \square

Applying Lemma 7.2 with any random variables (T_1, T_2, T_3) such that $T_1 + T_2 + T_3 = 0$ holds identically, and applying (3.12) with $X'_1 = T'_1$, $X'_2 = T'_2$, we deduce that

$$k \leq \delta + \tfrac{\eta}{3} \left(\delta + \sum_{i=1}^2 \sum_{j=1}^3 (d[X_i^0; T_j] - d[X_i^0; X_i]) \right).$$

Note that δ is still defined by (7.5) and thus depends on T_1, T_2, T_3 . In particular we may apply this for

$$T_1 = (U|S = s), \quad T_2 = (V|S = s), \quad T_3 = (W|S = s)$$

for s in the range of S (which is a valid choice by (7.4)) and then average over s with weights $p_S(s)$, to obtain

$$k \leq \tilde{\delta} + \frac{\eta}{3} \left(\tilde{\delta} + \sum_{i=1}^2 \sum_{A \in \{U, V, W\}} (d[X_i^0; A|S] - d[X_i^0; X_i]) \right),$$

where

$$\tilde{\delta} := \mathbf{I}[U : V|S] + \mathbf{I}[V : W|S] + \mathbf{I}[W : U|S].$$

Putting this together with (7.2) and (7.3), we conclude that

$$\begin{aligned} k &\leq \left(1 + \frac{\eta}{3}\right) \left(6\eta k - \frac{1-5\eta}{1-\eta}(2\eta k - I_1)\right) + \frac{\eta}{3} \left((6-3\eta)k + 3(2\eta k - I_1)\right) \\ &= (8\eta + \eta^2)k - \left(\frac{1-5\eta}{1-\eta} \left(1 + \frac{\eta}{3}\right) - \eta\right) (2\eta k - I_1) \\ &\leq (8\eta + \eta^2)k \end{aligned}$$

since the quantity $2\eta k - I_1$ is non-negative (by (3.13)), and its coefficient in the above expression is non-positive provided that $\eta(2\eta + 17) \leq 3$, which is certainly the case for our choice $\eta = \frac{1}{9}$ (and in fact for any $\eta \leq \frac{1}{6}$). Moreover, for³ $\eta = \frac{1}{9}$ we have $8\eta + \eta^2 < 1$. It follows that $k = 0$, as desired. The proof of Proposition 2.1 (and hence of all our results) is complete.

APPENDIX A. ENTROPY AND ADDITIVE COMBINATORICS

In this appendix we record some standard inequalities regarding Shannon entropy, as well as the standard ‘entropic Ruzsa calculus’ concerning entropies of sums of random variables.

First we remark that if X takes values in a set S then, by Jensen’s inequality,

$$\mathbf{H}[X] \leq \log |S|. \quad (\text{A.1})$$

Also, denoting by p_X the density function of X ,

$$\mathbf{H}[X] = \sum_x p_X(x) \log \frac{1}{p_X(x)} \geq \min_{x: p_X(x) > 0} \log \frac{1}{p_X(x)},$$

³In fact we can take any $\eta < \frac{1}{4+\sqrt{17}} = \frac{1}{8.1231\dots}$, and the other constants in the paper can be improved accordingly.

and therefore

$$\max_x p_X(x) \geq e^{-\mathbf{H}[X]}. \quad (\text{A.2})$$

Given a pair (X, Y) of random variables, the *conditional entropy* $\mathbf{H}[X|Y]$ is defined by the formula

$$\mathbf{H}[X|Y] := \sum_y p_Y(y) \mathbf{H}[X|Y = y]$$

where y ranges over the support of p_Y , and $X|Y = y$ denotes the random variable X conditioned on the event $Y = y$. We have the fundamental *chain rule*

$$\mathbf{H}[X, Y] = \mathbf{H}[X|Y] + \mathbf{H}[Y]. \quad (\text{A.3})$$

Here we abbreviate $\mathbf{H}[(X, Y)]$ as $\mathbf{H}[X, Y]$, and will make similar abbreviations regarding other information-theoretic quantities in this paper without further comment; for instance, $\mathbf{H}[(X, Y)|(Z, W)]$ becomes $\mathbf{H}[X, Y|Z, W]$. Note that (A.3) implies a conditional generalization

$$\mathbf{H}[X, Y|Z] = \mathbf{H}[X|Y, Z] + \mathbf{H}[Y|Z].$$

for all random variables X, Y, Z .

The *mutual information* $\mathbf{I}[X : Y]$ is defined by the formula

$$\begin{aligned} \mathbf{I}[X : Y] &= \mathbf{H}[X] + \mathbf{H}[Y] - \mathbf{H}[X, Y] \\ &= \mathbf{H}[X] - \mathbf{H}[X|Y] \\ &= \mathbf{H}[Y] - \mathbf{H}[Y|X], \end{aligned}$$

and is non-negative by a standard application of Jensen's inequality, vanishing precisely when X, Y are independent; in particular

$$\mathbf{H}[X, Y] = \mathbf{H}[X] + \mathbf{H}[Y] \quad (\text{A.4})$$

if and only if X, Y are independent, and

$$\mathbf{H}[X|Y] \leq \mathbf{H}[X] \quad (\text{A.5})$$

always.

Suppose now that (X, Y, Z) is a triple of random variables. Applying (A.5) to $(X|Z = z)$ and summing over z (weighted by $p_Z(z)$) gives

$$\mathbf{H}[X|Y, Z] \leq \mathbf{H}[X|Z], \quad (\text{A.6})$$

which is known as *submodularity*. It may equivalently be written as

$$\mathbf{H}[X, Y, Z] + \mathbf{H}[Z] \leq \mathbf{H}[X, Z] + \mathbf{H}[Y, Z]. \quad (\text{A.7})$$

The *conditional mutual information* $\mathbf{I}[X : Y|Z]$ is defined by

$$\mathbf{I}[X : Y|Z] := \sum_z p_Z(z) \mathbf{I}[(X|Z = z) : (Y|Z = z)].$$

Submodularity is equivalent to the statement that

$$\mathbf{I}[X : Y|Z] \geq 0, \quad (\text{A.8})$$

since

$$\mathbf{I}[X : Y|Z] = \mathbf{H}[X, Z] + \mathbf{H}[Y, Z] - \mathbf{H}[X, Y, Z] - \mathbf{H}[Z]. \quad (\text{A.9})$$

Equality occurs in (A.8) (and hence in (A.7)) if and only if X, Y are conditionally independent relative to Z .

G-valued random variables. We turn now to additional properties enjoyed by random variables taking values in an additive group G .

Whenever X, Y are G -valued random variables, we have that

$$\mathbf{H}[X \pm Y] \geq \mathbf{H}[X \pm Y|Y] = \mathbf{H}[X|Y] = \mathbf{H}[X] - \mathbf{I}[X : Y]$$

and similarly with the roles of X, Y reversed, thus

$$\max(\mathbf{H}[X], \mathbf{H}[Y]) - \mathbf{I}[X : Y] \leq \mathbf{H}[X \pm Y]. \quad (\text{A.10})$$

We note also the conditional variant of (A.10), namely

$$\max(\mathbf{H}[X|Z], \mathbf{H}[Y|Z]) - \mathbf{I}[X : Y|Z] \leq \mathbf{H}[X \pm Y|Z],$$

which follows from (A.10) by conditioning on $Z = z$ and summing over z (weighted by $p_Z(z)$).

A particular consequence of (A.10) is that

$$\max(\mathbf{H}[X], \mathbf{H}[Y]) \leq \mathbf{H}[X \pm Y] \quad (\text{A.11})$$

when X, Y are independent.

Continuing to suppose that X, Y are independent, recall the definition (1.1) of the Ruzsa distance $d[X; Y]$ which, since X and Y are independent, is that

$$d[X; Y] = \mathbf{H}[X - Y] - \frac{1}{2}\mathbf{H}[X] - \frac{1}{2}\mathbf{H}[Y].$$

Comparing this with (A.11) we see that

$$|\mathbf{H}[X] - \mathbf{H}[Y]| \leq 2d[X; Y]. \quad (\text{A.12})$$

We may also deduce that

$$\mathbf{H}[X - Y] - \mathbf{H}[X], \mathbf{H}[X - Y] - \mathbf{H}[Y] \leq 2d[X; Y].$$

The most important property of the Ruzsa distance is the (Ruzsa) triangle inequality

$$d[X; Y] \leq d[X; Z] + d[Z; Y].$$

This was shown in [9] and [14, (16)]; we recall a proof for completeness. This is equivalent to establishing

$$\mathbf{H}[X - Y] \leq \mathbf{H}[X - Z] + \mathbf{H}[Z - Y] - \mathbf{H}[Z] \quad (\text{A.13})$$

whenever X, Y, Z are independent. To prove this, apply (A.8) with a change of variables to get $\mathbf{I}[X - Z : Y|X - Y] \geq 0$ which, when written out in full, gives

$$\mathbf{H}[X - Z, X - Y] + \mathbf{H}[Y, X - Y] \geq \mathbf{H}[X - Z, Y, X - Y] + \mathbf{H}[X - Y].$$

Using

$$\mathbf{H}[X - Z, X - Y] = \mathbf{H}[X - Z, Y - Z] \leq \mathbf{H}[X - Z] + \mathbf{H}[Y - Z],$$

$$\mathbf{H}[Y, X - Y] = \mathbf{H}[X, Y],$$

and

$$\mathbf{H}[X - Z, Y, X - Y] = \mathbf{H}[X, Y, Z] = \mathbf{H}[X, Y] + \mathbf{H}[Z],$$

and rearranging, we indeed obtain (A.13). (As observed in [4], we do not in fact use the independence of X and Y here.)

We will also need conditional variants of the distance. If (X, Z) and (Y, W) are random variables (where X and Z are G -valued) we define

$$d[X|Z; Y|W] := \sum_{z,w} p_Z(z)p_W(w)d[(X|Z=z); (Y|W=w)]. \quad (\text{A.14})$$

Alternatively, if $(X', Z'), (Y', W')$ are independent copies of the variables $(X, Z), (Y, W)$,

$$d[X|Z; Y|W] = \mathbf{H}[X' - Y'|Z', W'] - \frac{1}{2}\mathbf{H}[X'|Z'] - \frac{1}{2}\mathbf{H}[Y'|W']. \quad (\text{A.15})$$

To conclude this appendix we give the proofs of two results from the literature which were used in the main text. The first is the inequality of Kaimanovich and Vershik, stated as (5.4) in the main text. For the original reference see [5, Proposition 1.3]; in fact, there is an inequality with more summands, which follows by induction.

Lemma A.1. *Suppose that X, Y, Z are independent random variables taking values in an abelian group. Then*

$$\mathbf{H}[X + Y + Z] - \mathbf{H}[X + Y] \leq \mathbf{H}[Y + Z] - \mathbf{H}[Y].$$

Proof. By (A.9) we have

$$\begin{aligned} \mathbf{I}[X : Z|X + Y + Z] &= \mathbf{H}[X, X + Y + Z] + \mathbf{H}[Z, X + Y + Z] \\ &\quad - \mathbf{H}[X, Z, X + Y + Z] - \mathbf{H}[X + Y + Z]. \end{aligned}$$

However, using (A.4) three times we have $\mathbf{H}[X, X + Y + Z] = \mathbf{H}[X, Y + Z] = \mathbf{H}[X] + \mathbf{H}[Y + Z]$, $\mathbf{H}[Z, X + Y + Z] = \mathbf{H}[Z, X + Y] = \mathbf{H}[Z] + \mathbf{H}[X + Y]$ and $\mathbf{H}[X, Z, X + Y + Z] = \mathbf{H}[X, Y, Z] = \mathbf{H}[X] + \mathbf{H}[Y] + \mathbf{H}[Z]$.

After a short calculation, we see that the claimed inequality is equivalent to the assertion that $\mathbf{I}[X : Z|X + Y + Z] \geq 0$, which of course is an instance of (A.8). \square

The next lemma is not quite in the literature but is very closely related to the entropic version of the Balog–Szemerédi–Gowers lemma due to the fourth author [14, Lemma 3.3]. Here we provide slightly better constants and a slightly simpler proof.

Lemma A.2. *Let (A, B) be a G^2 -valued random variable, and set $Z := A + B$. Then*

$$\sum_z p_Z(z) d[(A|Z = z); (B|Z = z)] \leq 3\mathbf{I}[A : B] + 2\mathbf{H}[Z] - \mathbf{H}[A] - \mathbf{H}[B]. \quad (\text{A.16})$$

We stress that the quantity $2\mathbf{H}[Z] - \mathbf{H}[A] - \mathbf{H}[B]$ is *not* the same as $2d[A; B]$, because (A, B) are given a joint distribution which may not be independent. In particular, $\mathbf{H}[Z] = \mathbf{H}[A + B]$ may not match the entropy of a sum of independent copies of A and B .

Proof. In the proof we will need the notion of *conditionally independent trials* of a pair of random variables (X, Y) (not necessarily independent). We say that X_1, X_2 are conditionally independent trials of X relative to Y by declaring $(X_1|Y = y)$ and $(X_2|Y = y)$ to be independent copies of $(X|Y = y)$ for all y in the range of Y . We then have

$$\mathbf{H}[(X_1|Y = y), (X_2|Y = y)] = 2\mathbf{H}[X|Y = y]$$

for all y , which upon summing over y (weighted by $p_Y(y)$) gives

$$\mathbf{H}[X_1, X_2|Y] = 2\mathbf{H}[X|Y]$$

and hence

$$\begin{aligned} \mathbf{H}[X_1, X_2, Y] &= \mathbf{H}[X_1, X_2|Y] + \mathbf{H}[Y] = 2\mathbf{H}[X|Y] + \mathbf{H}[Y] \\ &= 2\mathbf{H}[X, Y] - \mathbf{H}[Y]. \end{aligned} \quad (\text{A.17})$$

Note also that the marginal distributions of (X_1, Y) and (X_2, Y) each match the original distribution (X, Y) .

Turning to the proof of Lemma A.2 itself, let (A_1, B_1) and (A_2, B_2) be conditionally independent trials of (A, B) relative to Z , thus (A_1, B_1) and (A_2, B_2) are coupled through the random variable $A_1 + B_1 = A_2 + B_2$, which by abuse of notation we shall also call Z .

Observe that the left-hand side of (A.16) is

$$\mathbf{H}[A_1 - B_2|Z] - \frac{1}{2}\mathbf{H}[A_1|Z] - \frac{1}{2}\mathbf{H}[B_2|Z]. \quad (\text{A.18})$$

since, crucially, $(A_1|Z = z)$ and $(B_2|Z = z)$ are independent for all z .

Applying submodularity (A.7) gives

$$\begin{aligned} \mathbf{H}[A_1 - B_2] + \mathbf{H}[A_1 - B_2, A_1, B_1] \\ \leq \mathbf{H}[A_1 - B_2, A_1] + \mathbf{H}[A_1 - B_2, B_1]. \end{aligned} \quad (\text{A.19})$$

We estimate the second, third and fourth terms appearing here. First note that, by (A.17) (noting that the tuple $(A_1 - B_2, A_1, B_1)$ determines the tuple (A_1, A_2, B_1, B_2) since $A_1 + B_1 = A_2 + B_2$)

$$\mathbf{H}[A_1 - B_2, A_1, B_1] = \mathbf{H}[A_1, B_1, A_2, B_2] = 2\mathbf{H}[A, B] - \mathbf{H}[Z]. \quad (\text{A.20})$$

Next observe that

$$\mathbf{H}[A_1 - B_2, A_1] = \mathbf{H}[A_1, B_2] \leq \mathbf{H}[A] + \mathbf{H}[B]. \quad (\text{A.21})$$

Finally, we have

$$\mathbf{H}[A_1 - B_2, B_1] = \mathbf{H}[A_2 - B_1, B_1] = \mathbf{H}[A_2, B_1] \leq \mathbf{H}[A] + \mathbf{H}[B]. \quad (\text{A.22})$$

Substituting (A.20), (A.21) and (A.22) into (A.19) yields

$$\mathbf{H}[A_1 - B_2] \leq 2\mathbf{I}[A : B] + \mathbf{H}[Z]$$

and so by (A.5)

$$\mathbf{H}[A_1 - B_2|Z] \leq 2\mathbf{I}[A : B] + \mathbf{H}[Z].$$

Since

$$\begin{aligned} \mathbf{H}[A_1|Z] &= \mathbf{H}[A_1, A_1 + B_1] - \mathbf{H}[Z] \\ &= \mathbf{H}[A, B] - \mathbf{H}[Z] \\ &= \mathbf{H}[A] + \mathbf{H}[B] - \mathbf{I}[A : B] - \mathbf{H}[Z] \end{aligned}$$

and similarly for $\mathbf{H}[B_2|Z]$, we see that (A.18) is bounded by $3\mathbf{I}[A : B] + 2\mathbf{H}[Z] - \mathbf{H}[A] - \mathbf{H}[B]$ as claimed. \square

APPENDIX B. FROM ENTROPIC PFR TO COMBINATORIAL PFR

In this appendix we repeat the arguments from [4] showing that Theorem 1.2 with some constant C' in place of 11 implies Theorem 1.1 with $C = C' + 1$. In particular, with $C' = 11$ in Theorem 1.2, this gives the claimed constant $C = 12$ for Theorem 1.1.

Let A, K be as in Theorem 1.1. Let U_A be the uniform distribution on A , thus $\mathbf{H}[U_A] = \log |A|$. By (A.1) and the fact that $U_A + U_A$ is supported on $A + A$, $\mathbf{H}[U_A + U_A] \leq \log |A + A|$. The doubling condition $|A + A| \leq K|A|$ therefore gives

$$\mathbf{d}[U_A; U_A] \leq \log K.$$

By Theorem 1.2, we may thus find a subspace H of \mathbf{F}_p^n such that

$$\mathbf{d}[U_A; U_H] \leq \frac{1}{2}C' \log K. \quad (\text{B.1})$$

By (A.12) we conclude that

$$|\log |H| - \log |A|| \leq C' \log K. \quad (\text{B.2})$$

From definition of Ruzsa distance, (B.1) is equivalent to

$$\mathbf{H}[U_A - U_H] \leq \log(|A|^{1/2}|H|^{1/2}) + \frac{1}{2}C' \log K.$$

By (A.2) we conclude the existence of a point $x_0 \in \mathbf{F}_p^n$ such that

$$p_{U_A - U_H}(x_0) \geq |A|^{-1/2}|H|^{-1/2}K^{-C'/2},$$

or equivalently

$$|A \cap (H + x_0)| \geq K^{-C'/2}|A|^{1/2}|H|^{1/2}.$$

Applying the Ruzsa covering lemma [15, Lemma 2.14], we may thus cover A by at most

$$\frac{|A + (A \cap (H + x_0))|}{|A \cap (H + x_0)|} \leq \frac{K|A|}{K^{-C'/2}|A|^{1/2}|H|^{1/2}} = K^{C'/2+1} \frac{|A|^{1/2}}{|H|^{1/2}}$$

translates of

$$(A \cap (H + x_0)) - (A \cap (H + x_0)) \subseteq H.$$

If $|H| \leq |A|$ then we are already done thanks to (B.2). If $|H| > |A|$ then we can cover H by at most $2|H|/|A|$ translates of a subspace H' of H with $|H'| \leq |A|$. We can thus cover A by at most

$$2K^{C'/2+1} \frac{|H|^{1/2}}{|A|^{1/2}}$$

translates of H' , and the claim again follows from (B.2).

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