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RATIONAL SETS IN FINITELY GENERATED NILPOTENT GROUPS

G. A. Bazhenova*

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We deal with a class of rational subsets of a group, that is, the least class of its subsets which contains all finite subsets and is closed under taking union, a product of two sets, and under generating of a submonoid by a set. It is proved that the class of rational subsets of a finitely generated nilpotent group G is a Boolean algebra iff G is Abelian-by-finite. We also study the question asking under which conditions the set of solutions for equations in groups will be rational. It is shown that the set of solutions for an arbitrary equation in one variable in a finitely generated nilpotent group of class 2 is rational. And we give an example of an equation in one variable in a free nilpotent group of nilpotency class 3 and rank 2 whose set of solutions is not rational.

INTRODUCTION

Following [1], we define a class of rational subsets of an arbitrary monoid M as the minimal class which contains all finite subsets of M and is closed under rational operations — of taking union, product, and of generating a submonoid. Choosing a group G to be a monoid M, we obtain a definition of rational subsets of G. From [2], we know of another approach which is underpinned by the idea of defining rational subsets based on the concept of a rational structure for a group. Under Sec. 4, we prove that the two definitions are sort of equivalent iff rational — in the sense of [1] — subsets of G form a Boolean algebra, that is, the class that is closed under taking union, intersection, complement, and set-theoretic difference for sets. (Since union enters the scope of rational operations, it suffices to talk about the class of rational subsets being closed under complements.)

We know from [1] that rational subsets of a free finitely generated (f.g.) monoid form a Boolean algebra. Also it is not hard to show that rational subsets of a free group of finite rank form a Boolean algebra. In the present article we prove that rational subsets of an f.g. nilpotent group G form a Boolean algebra iff G is Abelian-by-finite.

We also study into the question asking under which conditions sets of solutions of equations in f.g. nilpotent groups are not rational. An example of an equation in one variable in a free nilpotent group of rank 2 and class 3 is given for which the set of solutions is not rational (Example 1), thus solving the problem concerning the existence of such sets over f.g. nilpotent groups.

1. BASIC DEFINITIONS

Definition 1. Let M be a monoid. By induction, we define classes $\mathcal{R}_i$, $i = 0, 1, \ldots$, of subsets in M as follows.

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1. $\mathcal{R}_0$ is a class of all subsets of M.

2. If $\mathcal{R}_0, \ldots, \mathcal{R}_n$ are defined then $\mathcal{R}_{n+1}$ is a class of all sets $S \subseteq M$ such that S does not belong to any one of $\mathcal{R}_0, \ldots, \mathcal{R}_n$, but there exist sets $T_1 \in \mathcal{R}_k$, $T_2 \in \mathcal{R}_l$, $0 \leqslant k, l \leqslant n$, such that $S = T_1 \cup T_2$, or $S = T_1T_2 = \{ab \mid a \in T_1, b \in T_2\}$, or $S = T_1^* = \{1\} \cup T_1 \cup T_1T_1 \cup T_1T_1T_1 \cup \ldots$.

The union of all classes $\mathcal{R}_i$, $i = 0, 1, \ldots$, is called a class of *rational subsets* of M and is denoted by $\mathcal{R}(M)$. If the set $S \subseteq M$ belongs to $\mathcal{R}_k$, then we call the number k the *complexity* of S. The following description of rational sets in terms of finite automata is well known.

Definition 2. A *finite automaton* Γ over a monoid M is a quadruple (Q, q_0, Q_t, Ω), where Q is a finite set, called a *vertex set*, q_0 is an element of Q, called an *initial vertex*, Q_t is a subset of Q, called a *set of terminal vertices*, and Ω is a finite subset of the Cartesian product $Q \times M \times Q$, called an *edge set*.

A *successful path* π in Γ is a finite sequence of edges $\omega_1, \ldots, \omega_n$, where $\omega_i = (q_{i-1}, m_i, q_i)$, with $q_n \in Q_t$. The *label* of ω_i is an element m_i. The *label* of π is the product $m_1 \cdots m_n$. We say that Γ *accepts* $R \subseteq M$ if R is the set of labels of all successful paths in Γ.

THEOREM 1. Let M be a monoid. Then a subset of M is rational iff it is accepted by a finite automaton over M; see [1].

The following lemma is a necessary tool for proving that a set is (or is not) rational.

LEMMA 1. Let M be a monoid and $R \in \mathcal{R}(M)$. Then:

(1) either R is finite, or there exist u, v, and w in M such that $v \neq 1$, and for all integers $n \geqslant 0$, uv^nw is in R;

(2) there exist finite sets $T_0, T_1 \subseteq M$ such that $1 \notin T_1$, and all r in $R \setminus T_0$ can be represented as $r = utv$, where $t \in T_1$ and $ut^*v \subseteq R$.

Proof. Clearly, (1) is a consequence of (2). We argue for (2). Suppose that R is accepted by Γ. The number of successful paths without loops in Γ is finite. Let T_0 be the set of their labels. A set of loops without subloops is finite also. Write T_1 for the set of their labels distinct from unity. If $r \in (R \setminus T_0)$, we let $\pi = \omega_1, \ldots, \omega_n$, $\omega_i = (q_{i-1}, m_i, q_i)$ be the shortest successful path in Γ with label r. There exist indices $0 \leqslant i < j \leqslant n$ such that $q_i = q_j$ and $q_i, \ldots, q_{j-1}$ are pairwise distinct. Put $u = m_1 \cdots m_i$, $t = m_{i+1} \cdots m_j$, and $v = m_{j+1} \cdots m_n$. Note that $t \neq 1$, since otherwise our path would not be shortest. Therefore $t \in T_1$ and, moreover, $ut^*v \subseteq R$. The lemma is proved.

Let $M_1 \subseteq M_2$ be monoids, $R \subseteq M_1$. If R is a rational subset of M_1 then it is a rational subset of M_2. Generally, the converse is not true. However, if M_1 and M_2 are groups, the converse will be true, as is shown by the following:

Proposition 1. Let $H \leq G$ be groups and the set $R \subseteq H$ belong to $\mathcal{R}(G)$. Then $R \in \mathcal{R}(H)$.

Proof. By induction on the complexity of $R \in \mathcal{R}(G)$, we show that $hRg \in \mathcal{R}(H)$ for any $h, g \in G$ if $hRg \subseteq H$.

Let $R = R_1R_2$ and the statement be true for R_1 and R_2. We may assume that R_1 and R_2 are not empty. Choose $a \in R_1$ and $b \in R_2$. Let $hRg \subseteq H$. Then $hR_1bg \in \mathcal{R}(H)$ and $habg \in H$. Furthermore, $g^{-1}b^{-1}R_2g = (habg)^{-1}haR_2g \in \mathcal{R}(H)$. Hence $hRg = (hR_1bg)(g^{-1}b^{-1}R_2g) \in \mathcal{R}(H)$.

Let $R = R_1^*$ and $hRg \subseteq H$. Then $hR_1h^{-1} = hR_1g(hg)^{-1} \in \mathcal{R}(H)$. Therefore $(hR_1h^{-1})^* = hRh^{-1} \in \mathcal{R}(H)$ and $hRg = hRh^{-1}(hg) \in \mathcal{R}(H)$. The proposition is proved.

2. ABELIAN GROUPS

The group operation in Abelian groups is denoted by $+$.

THEOREM 2. Let G be an Abelian-by-finite f.g. group. Then $\mathcal{R}(G)$ is a Boolean algebra.

Proof. We first establish a number of lemmas.

LEMMA 2. Let $A = Z^n$ be a free Abelian group of finite rank n. Then any $R \in \mathcal{R}(A)$ can be represented as

$$R = \bigcup_{i=1}^{k}(a_i + M_i), \tag{1}$$

where $k \geqslant 0$, $a_i \in A$, and $M_i \subseteq A$ is an f.g. monoid, $i = 1, \ldots, k$.

Proof. Since $\{0\}$ is an f.g. monoid, all finite sets can be written in the form (1). Therefore, it suffices to show that a class of sets representable as in (1) is closed under rational operations. Clearly, it is closed under sums and unions. Let R be of the form (1). For all $1 \leqslant i \leqslant k$, the set $(a_i + M_i)^* = \{0\} \cup a_i + (a_i^* + M_i)$ assumes the form (1) since $a_i^* + M_i$ is an f.g. monoid. It follows that $R^* = \sum\limits_{i=1}^{k}(a_i + M_i)^*$ is a sum of sets of the form (1), and it also has that form (1). The lemma is proved.

Definition 3. We say that a monoid M is *free commutative* if it is trivial or is isomorphic to the monoid

$$Z_+^n = \{(z_1, \ldots, z_n) \in Z^n \mid z_1 \geqslant 0, \ldots, z_n \geqslant 0\}.$$

If $\varphi : Z_+^n \to M$ is an isomorphism then $\varphi(1, 0, \ldots, 0)$, $\varphi(0, 1, 0, \ldots, 0)$, $\ldots$, $\varphi(0, \ldots, 0, 1)$ are free generators for M.

LEMMA 3. Let $A = Z^n$ be a free Abelian group of finite rank n. Then every $R \in \mathcal{R}(A)$ can be represented as

$$R = \bigcup_{i=1}^{k}(a_i + M_i), \tag{2}$$

where $k \geqslant 0$, $a_i \in A$, and $M_i \subseteq A$ is a free commutative monoid, $i = 1, \ldots, k$.

Proof. By induction on the number of generators, we show that every f.g. monoid M has the form (2). Assume that elements $x_1, \ldots, x_s$, where $s > 1$, generate M. If x_i are not free generators then there exist tuples of nonnegative integers $n_1, \ldots, n_s, n_1', \ldots, n_s'$ such that $\sum\limits_{i=1}^{s} n_i x_i = \sum\limits_{i=1}^{s} n_i' x_i$, and for some i, we have $n_i \neq n_i'$. For $i = 1, \ldots, s$, we put $\eta_i = |n_i - n_i'|$, $\varepsilon_i = \operatorname{sgn}(n_i - n_i')$, assuming that $\operatorname{sgn}(0) = 0$. Let $y_i = \eta_i x_i$ for $\eta_i > 0$ and $y_i = x_i$ for $\eta_i = 0$. Assume $Y = \{y_1, \ldots, y_s\}$. There exists a finite set K such that $Y^* + K = M$. Hence it suffices to show that Y^* has the form (2). We know that $\sum\limits_{i=1}^{s} \varepsilon_i y_i = 0$ and that $\varepsilon_i \neq 0$ exists. In other words, we have $\sum\limits_{i \in I} y_i = \sum\limits_{i \in J} y_i$, where the sets $I, J \subseteq \{1, \ldots, s\}$ are disjoint and one of them, say I, is nonempty. Then

$$Y^* = \bigcup_{i \in I}\{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_s\}^*. \tag{3}$$

Indeed, let $u = \sum\limits_{i=1}^{s} l_i y_i$, where $l_i \in Z$ and $l_i \geqslant 0$. Put $\lambda = \min\{l_i \mid i \in I\}$ and let $\lambda = l_j$, where $j \in I$. We may assume that $l_j > 0$. Then

$$u = \sum_{i \notin I} l_i y_i + \sum_{i \in I}(l_i - \lambda) y_i + \lambda \sum_{i \in I} y_i = \sum_{i \notin I} l_i y_i + \sum_{i \in I}(l_i - \lambda) y_i + \lambda \sum_{i \in J} y_i.$$

We have arrived at (3). By induction, therefore, Y^* has the form (2). The lemma is proved.

LEMMA 4. Let $A = Z^n$, $R \in \mathcal{R}(A)$, and $K \subseteq A$ be finite. Then $(R \setminus K) \in \mathcal{R}(A)$.

Proof. Let R be a free commutative monoid and $X = \{x_1, \dots, x_s\}$ be a set of its free generators. Suppose $K = \left\{ \sum_{i=1}^{s} \lambda_i x_i \right\}$ is one-element. Then $R \setminus K = R_1 \cup R_2$, where $R_1 = \left\{ \sum_{i=1}^{s} l_i x_i \,|\, 0 \leqslant l_i \leqslant \lambda_i \right\} \setminus K$ is finite, and $R_2 = \bigcup_{i=1}^{s} ((1 + \lambda_i)x_i + R)$. The lemma is proved.

LEMMA 5. Let G be a group and $\lambda : G \to H$ an homomorphism. Then:

(1) if $R \in \mathcal{R}(G)$ then $\lambda(R) \in \mathcal{R}(H)$;

(2) if $R \in \mathcal{R}(H)$, and λ is an epimorphism with an f.g. kernel, then $\lambda^{-1}(R) \in \mathcal{R}(G)$.

Proof. (1) Is well known; see [1].

(2) Use induction on the complexity of rational sets. If $R \subseteq H$ is finite then $\lambda^{-1}(R) = \bigcup_{i=1}^{N} g_i(\ker\lambda)$, where $g_1, \dots, g_N \in G$. Since $\ker\lambda \in \mathcal{R}(G)$ as an f.g. subgroup, we have $\lambda^{-1}(R) \in \mathcal{R}(G)$. Furthermore, $\lambda^{-1}(X \cup Y) = \lambda^{-1}(X) \cup \lambda^{-1}(Y)$, $\lambda^{-1}(XY) = \lambda^{-1}(X)\lambda^{-1}(Y)$, $\lambda^{-1}(X^*) = \lambda^{-1}(X)(\lambda^{-1}(X))^* \cup (\ker\lambda)$. The lemma is proved.

LEMMA 6. Let G be a group, $H \leq G$ be a subgroup of finite index, and $D = \{d_1, \dots, d_N\} \subseteq G$ be such that $DH = G$. Then every $R \in \mathcal{R}(G)$ can be represented as $R = \bigcup_{i=1}^{N} d_i S_i$, where $S_1, \dots, S_N \in \mathcal{R}(H)$.

Proof. First we show that $R \cap H \in \mathcal{R}(G)$ for any $R \in \mathcal{R}(G)$. Let Γ be a finite automaton accepting R. Suppose that Q is its vertex set, q_0 is the initial vertex, and Q_t is the set of all terminal vertices. We construct an automaton Γ' whose vertex set is formed by the set $Q \times \{Hg \,|\, g \in G\}$ and whose edges are obtained from those in Γ using the following rule: $\omega = (q_1, g, q_2)$ "generates" all possible edges of the form $\omega' = ((q_1, Hh), g, (q_2, Hhg))$. We announce that the initial vertex is (q_0, H) and that the set of terminal vertices is $Q_t \times \{H\}$. It is easy to see that Γ' accepts $R \cap H$.

Let $R \in \mathcal{R}(G)$. Put $S_i = H \cap d_i^{-1} R$, where $i = 1, \dots, N$. Then $S_i \in \mathcal{R}(H)$ and $R = \bigcup_{i=1}^{N} d_i S_i$. The lemma is proved.

LEMMA 7. Let A be an f.g. Abelian group, $R \in \mathcal{R}(A)$, and $K \subseteq A$ be finite. Then $R \setminus K \in \mathcal{R}(A)$.

Proof. A is a direct sum of a finite Abelian group A_0 and a free Abelian group $A_1 \simeq Z^k$. By Lemma 6, $R = \bigcup_{a \in A_0} (a + R_a)$, where $R_a \in \mathcal{R}(A_1)$. The set K can be represented as $K = \bigcup_{a \in A_0} (a + K_a)$, where $K_a \subseteq A_1$ are all finite. It follows that $R \setminus K = \bigcup_{a \in A_0} (a + (R_a \setminus K_a))$. By Lemma 4, all $R_a \setminus K_a \in \mathcal{R}(A)$, and so $R \setminus K \in \mathcal{R}(A)$. The lemma is proved.

LEMMA 8. Let $A = Z^n$ be a free Abelian group of finite rank and $M \subseteq A$ be a free commutative monoid, $a \in A$. Then $A \setminus (a + M) \in \mathcal{R}(A)$.

Proof. Since $A \setminus (a + M) = (A \setminus M) + a$, we may assume that $a = 0$. Let $H = \langle M \rangle$. By Lemma 7, $((A/H) \setminus \{0\}) \in \mathcal{R}(A/H)$. By Lemma 5, $A \setminus H \in \mathcal{R}(A)$. We claim that $H \setminus M \in \mathcal{R}(A)$. Let $u_1, \dots, u_k$ be free generators for M. Then $u_1, \dots, u_k$ freely generate $H \simeq Z^k$. Furthermore, $H \setminus M = \bigcup_{i=1}^{k} (-u_i + (-u_i)^* + \sum_{j \neq i} (u_j^* + (-u_j)^*)) \in \mathcal{R}(A)$. Therefore $A \setminus M = ((A \setminus H) \cup (H \setminus M)) \in \mathcal{R}(A)$. The lemma is proved.

In view of Lemmas 3, 6, and 8, it remains to show that the intersection of two rational subsets of a free Abelian group is rational. To do this, we need to establish that sets of nonnegative solutions for systems of linear equations with integral coefficients are rational.

LEMMA 9. Let $A = Z^n$ be a free Abelian group of finite rank. Write L for an arbitrary system of linear equations in n variables with integral coefficients. Then the set S of its solutions $(x_1, \dots, x_n) \in A$ for which $x_i \geqslant 0$, $i = 1, \dots, n$, is rational in A.

The **proof** is by induction on n. The case $n = 1$ is trivial. We let $n \geqslant 2$. Assume, first, that L is uniform. Let $X = (\xi_1, \dots, \xi_n) \in S$ be nonzero. Put $N = \max_i \xi_i$. For $1 \leqslant i \leqslant n$ and $0 \leqslant T \leqslant N$, we put $R_i(T) = \{(x_1, \dots, x_n) \in S \mid x_i = T\}$. By induction, these sets are rational. We verify that

$$S = X^* + \bigcup_{i=1}^{n} \bigcup_{T=0}^{N} R_i(T). \tag{4}$$

In fact, let $Y = Y_0 = (y_1, \dots, y_n) \in S$. Put $\mu = \min_i y_i$. If μ is larger than N, we let $Y_1 = Y_0 - X$. Clearly, Y_1 belongs to S. If the minimal coordinate of the vector Y_1 is still larger than N, we put $Y_2 = Y_1 - X$ and so on and on, until we obtain a vector $Y_r = Y - rX \in S$ at least one coordinate of which does not exceed N. Since $Y_r \in \bigcup_{i=1}^{n} \bigcup_{T=0}^{N} R_i(T)$, $Y = rX + Y_r \in X^* + \bigcup_{i=1}^{n} \bigcup_{T=0}^{N} R_i(T)$. Thus (4) is proved. Hence S is rational.

Suppose that L is not uniform. We may assume that $S \neq \varnothing$. Let $X = (\xi_1, \dots, \xi_n) \in (S \setminus \{0\})$. Define N and $R_i(T)$, $i = 1, \dots, n$; $T = 0, \dots, N$, as above. For the same reasons, all $R_i(T)$ are rational. Let S' be a set of nonnegative solutions for the uniform system obtained from L by replacing all free terms by zero. By the above, S' is rational. Since

$$S = \bigcup_{i=1}^{n} \bigcup_{T=0}^{N} R_i(T) \cup (S' + X),$$

S too is rational. The lemma is proved.

LEMMA 10. Let $A = Z^n$ be a free Abelian group of finite rank, and $R_1, R_2 \in \mathcal{R}(A)$. Then $R_1 \cap R_2 \in \mathcal{R}(A)$.

Proof. By Lemma 3, it suffices to consider the case $R_i = a_i + M_i$, $i = 1, 2$, where $a_i \in A$ and $M_i \subseteq A$ are free commutative monoids. Let $u_1^i, \dots, u_{k_i}^i$ be free generators for M_i, $i = 1, 2$. Put $S = \left\{(z_1^1, \dots, z_{k_1}^1, z_1^2, \dots, z_{k_2}^2) \mid z_i^j \in Z,\ z_i^j \geqslant 0,\ a_1 + \sum_{j=1}^{k_1} z_j^1 u_j^1 = a_2 + \sum_{j=1}^{k_2} z_j^2 u_j^2\right\}$. By Lemma 9, $S \in \mathcal{R}(Z^{k_1+k_2})$. Put $\lambda : (z_1^1, \dots, z_{k_1}^1, z_1^2, \dots, z_{k_2}^2) \mapsto \sum_{j=1}^{k_1} z_j^1 u_j^1$. By Lemma 5, $\lambda(S) \in \mathcal{R}(A)$. Then $R_1 \cap R_2 = \lambda(S) + a_1 \in \mathcal{R}(A)$. The lemma is proved.

LEMMA 11. Let $A \simeq Z^n$ be a free Abelian group of finite rank, and $R \in \mathcal{R}(A)$. Then $A \setminus R \in \mathcal{R}(A)$.

The **proof** follows from Lemmas 3, 8, and 10.

We complete the proof of Theorem 2. Let $A \leq G$ be a subgroup of finite index, and $A \simeq Z^k$. Let D be a set of representatives of left cosets of G w.r.t. A. By Lemma 6, then, every $R \in \mathcal{R}(G)$ has the form $\bigcup_{d \in D} dS_d$, where $S_d \in \mathcal{R}(A)$. We have $G \setminus R = \bigcup_{d \in D} d(A \setminus S_d)$. Then $G \setminus R \in \mathcal{R}(G)$ by Lemma 11. The theorem is proved.

3. NILPOTENT GROUPS

THEOREM 3. Let G be an f.g. nilpotent group. Then $\mathcal{R}(G)$ is a Boolean algebra if and only if G is Abelian-by-finite.

Proof. The sufficiency follows from Theorem 2.

Below we need the following well-known fact.

LEMMA 12. If G is an Abelian-by-finite f.g. nilpotent group then the center of G has finite index in it.

The necessity is verified by induction on the nilpotency class. Let G be a nilpotent group of class n. Lemma 5 implies that $\mathcal{R}(G/C)$ is a Boolean algebra; here, C is the center of G. By induction, G/C is Abelian-by-finite, and then the term $\zeta_2(G)$ that succeeds C in the upper central series has finite index in G by Lemma 12. We claim that $\zeta_2(G)$ is Abelian-by-finite. It suffices to show that the order of a commutator $[x, y] = x^{-1}y^{-1}xy$ is finite for any $x, y \in \zeta_2(G)$ (then the commutant of $\zeta_2(G)$ is finite and the center of $\zeta_2(G)$ is equipped with a finite index). If the order of $[x, y]$ is infinite then x and y are free generators for a free nilpotent group N of class 2 and rank 2. Any $g \in N$ has the form

$$g = x^k y^l [x, y]^m, \tag{5}$$

where k, l, and m are integers and are defined uniquely. Write

$$R = (xy)^*([x, y]^* \cup [y, x]^*)$$

for a set of all $g \in N$ for which $k = l \geqslant 0$ in formula (5). Furthermore, let $S = x^*y^*$ be a set of all $g \in N$ for which $m = 0$, $k, l \geqslant 0$ in (5). The set $R \cap S = \{x^n y^n \mid n \geqslant 0\}$ is rational, and by Lemma 1, there then exist $u, v \neq 1$ and $w \in N$ such that $uv^*w \subseteq R \cap S$. Put $q = uvu^{-1}$ and $r = uw$. Then $q^*r \subseteq R \cap S$. Let $q = x^k y^l [x, y]^m$ and $r = x^\kappa y^\lambda [x, y]^\mu$. Then $q^n r = x^{nk+\kappa} y^{nl+\lambda} [x, y]^{nm+\mu-nl\kappa-\frac{1}{2}lkn(n-1)}$. For any $n \geqslant 0$, we obtain

$$nk + \kappa = nl + \lambda, \quad nm + \mu - nl\kappa - (1/2)lkn(n-1) = 0,$$

whence $k = l$, $kl = 0$, and $q = 1$. We have thus arrived at a contradiction, which completes the proof.

4. THE INTERPLAY BETWEEN TWO DEFINITIONS OF RATIONALITY

The definition which follows is given and studied in detail, for instance, in [2].

Definition 4. Let G be an f.g. group, Δ a finite alphabet, Δ^* a free monoid, $\lambda : \Delta^* \to G$ a surjective morphism of monoids, $L \in \mathcal{R}(\Delta^*)$, and $\lambda(L) = G$. Then the pair (Δ, λ) is called the *choice of generators* for G, and the triple (Δ, λ, L) — the *rational structure* on G. A set $R \subseteq G$ is said to be *L-rational* if $\lambda^{-1}(R) \cap L$ is rational. (This does not mean that the choice of generators is fixed; we merely mean that "information" about Δ and λ is contained in L.)

Since morphisms of monoids take rational sets to rational sets, an L-rational set will be rational for some L. On the other hand, it is not hard to give an example of a subset Z^2 which is L-rational for one L but is not L-rational for other L.

Proposition 2. Let G be an f.g. group. Then the following conditions are equivalent:

(1) $\mathcal{R}(G)$ is a Boolean algebra;

(2) each set in the class $\mathcal{R}(G)$ is L-rational for some L.

Proof. (1) $\Rightarrow$ (2). Let $R \in \mathcal{R}(G)$. Then $G \setminus R \in \mathcal{R}(G)$. Let Γ_1 and Γ_2 be finite automata over G accepting the sets R and $G \setminus R$, respectively. Let $g_1, \dots, g_N$ be the set of labels of all edges in Γ_1 and in Γ_2. Introduce a finite alphabet $\Delta = \{\delta_1, \dots, \delta_N\}$. Define a morphism $\lambda : \Delta^* \to G$ by setting $\lambda(\delta_i) = g_i$. Suppose that the finite automata Γ_1' and Γ_2' are obtained from Γ_1 and Γ_2 over Δ by replacing each label g_i by a corresponding letter δ_i. Write Λ_1 and Λ_2 for languages defined by Γ_1' and Γ_2'. Put $L = \Lambda_1 \cup \Lambda_2$. Then

L is rational, $\lambda(L) = G$, and, moreover, $\lambda^{-1}(R) \cap L = \Lambda_1$ and $\lambda^{-1}(G \setminus R) \cap L = \Lambda_2$. Hence the sets R and $G \setminus R$ are L-rational.

(2) $\Rightarrow$ (1). Let $R \in \mathcal{R}(G)$. Then R is L-rational for some L. It follows that $\Lambda = L \setminus (\lambda^{-1}(R) \cap L)$ is rational since rational subsets of an f.g. free monoid form a Boolean algebra. Hence $G \setminus R = \lambda(\Lambda)$ is rational. The proposition is proved.

6. SETS OF SOLUTIONS FOR EQUATIONS IN FINITELY GENERATED NILPOTENT GROUPS

Definition 5. An equation in one variable x with coefficients in G is an expression like

$$g_1 x^{\varepsilon_1} \dots g_n x^{\varepsilon_n} = 1, \quad \text{where } g_i \in G,\ \varepsilon_i = \pm 1\ (i = 1, \dots, n).$$

We know from [3] that a set of solutions for each equation in one variable in a free group of finite rank is rational. It is not hard to prove that a similar result will be true for f.g. Abelian groups.

Below we reason to answer the well-known question of whether sets of solutions for equations in one variable are rational in the class of f.g. nilpotent groups. Namely, we prove that any equation in one variable in an f.g. nilpotent group of class 2 has a rational set of solutions, and give an example of an equation in one variable in a free nilpotent group of class 3 and rank 2 whose set of solutions is not rational.

LEMMA 13. Let K be a group, G its subgroup, H a normal subgroup of G, $[K, G] \subseteq H$, and let the periodic part of an Abelian group G/H be finite. Assume that any equation of the form

$$a_1 x^{\varepsilon_1} a_2 x^{\varepsilon_2} \cdots a_n x^{\varepsilon_n} = 1, \tag{6}$$

where $a_i \in K$, $\varepsilon_i = \pm 1$, $i = 1, \dots, n$, x is a variable, and $s = \sum_{i=1}^{n} \varepsilon_i \neq 0$, has finitely many solutions in H. Then every equation of the form (6) has finitely many solutions in G.

Proof. Let S be the set of solutions for (6) in G. We may assume that S is not empty. Let $\xi, \eta \in S$. The element $v_\xi = a_1 \xi^{\varepsilon_1} \cdots a_n \xi^{\varepsilon_n}$ can be represented as $v_\xi = \xi^s c_\xi a_1 \cdots a_n$, where $c_\xi \in [K, G]$. Similarly, $v_\eta = a_1 \eta^{\varepsilon_1} \cdots a_n \eta^{\varepsilon_n} = \eta^s c_\eta a_1 \cdots a_n$, where $c_\eta \in [K, G]$. Since $v_\xi = 1 = v_\eta$, we have $\xi^s c_\xi = \eta^s c_\eta$ and $\eta^{-s} \xi^s = c_\eta c_\xi^{-1}$, and since $\eta^{-s} \xi^s = (\eta^{-1} \xi)^s h$ for some $h \in H$, the element $\overline{\eta^{-1} \xi}$ of the quotient group G/H has a finite order. Let $\overline{b_1}, \dots, \overline{b_m}$ be the periodic part of G/H. Then $\overline{\eta^{-1} \xi}$ is equal to some $\overline{b_i}$, and $\xi \in D = \bigcup_{i=1}^{m} \eta b_i H$. Let Q_i, $i = 1, \dots, m$, be the set of solutions for

$$a_1 (\eta b_i x)^{\varepsilon_1} \cdots a_n (\eta b_i x)^{\varepsilon_n} = 1$$

in H. By assumption, Q_i are finite. Moreover, $S = \bigcup_{i=1}^{m} \eta b_i Q_i$. The lemma is proved.

LEMMA 14. Let K be an f.g. nilpotent group. Then every equation of the form (6) has finitely many solutions in K.

Proof. Let

$$1 = K_0 \leq K_1 \leq \cdots \leq K_l = K$$

be an arbitrary central series. Using induction, we prove that (6) has finitely many solutions in K_i. If our claim is true for i, then it will be true for $i + 1$ by Lemma 13, where K_{i+1} is taken to be G and K_i is taken to be H. The lemma is proved.

Remark. It is well known that a torsion free f.g. nilpotent group K has a central series whose quotients are torsion free. For such a group, we can therefore refine Lemmas 13 and 14 as follows: every equation of the form (6) has not more than one solution.

Proposition 3. Let G be an f.g. nilpotent group of class 2. Then the set of solutions of every equation in one variable with coefficients in G is a rational subset of G.

Proof. Consider the equation

$$a_1 x^{\varepsilon_1} a_2 x^{\varepsilon_2} \cdots a_n x^{\varepsilon_n} = 1, \tag{7}$$

where $a_i \in G$, $\varepsilon_i = \pm 1$, $i = 1, \dots, n$. If $\sum_{i=1}^{n} \varepsilon_i \neq 0$, then the set of solutions for (7) is finite by Lemma 14. Otherwise, (7) can be reduced to the form $[a, x] = b$, where $a, b \in G$. Let ξ be a solution for (7). Then the set of solutions of (7) is equal to $\mathcal{Z}(a)\xi$, where $\mathcal{Z}(a)$ is the centralizer of a. Since $\mathcal{Z}(a)$ is rational as an f.g. subgroup, the proposition is proved.

Example 1. Let G be a free nilpotent group of nilpotency class 3 and rank 2. Let a and b be free generators for G. Then the set S of solutions for the equation $[x, a] = [x, b, x]$ is not rational in G.

Indeed, any element $g \in G$ can be written uniquely in the form

$$g = a^k b^l [a, b]^m [a, b, a]^r [a, b, b]^s, \; k, l, m, r, s \in Z. \tag{8}$$

It is easy to verify that the condition $g \in S$ is equivalent to

$$\begin{cases} -l = 0, \\ m = k^2, \\ -l(l-1)/2 = kl. \end{cases}$$

Let α and β be free generators for a free nilpotent group H of nilpotency class 2 and rank 2. The homomorphism $\lambda : G \to H$ specified by $\lambda(a) = \alpha$ and $\lambda(b) = \beta$ takes the set S to $S' = \{\alpha^k [\alpha, \beta]^{k^2} \mid k \in Z\}$. It suffices to prove that the set S' is not rational. Otherwise, by Lemma 1, there exist $u, v \neq 1$ and $w \in H$ such that $uv^*w \subseteq S'$. Let $u = \alpha^{n_1}\beta^{n_2}[\alpha, \beta]^{n_3}$, $v = \alpha^{m_1}\beta^{m_2}[\alpha, \beta]^{m_3}$, and $w = \alpha^{r_1}\beta^{r_2}[\alpha, \beta]^{r_3}$. The condition that $uw \in S'$ yields $n_2 + r_2 = 0$ and the condition that $uvw \in S'$ implies $n_2 + r_2 + m_2 = 0$ and $m_2 = 0$. For a natural t, we then have

$$uv^t w = \alpha^{n_1 + tm_1 + r_1} [\alpha, \beta]^{n_3 + tm_3 + r_3 - n_2 t m_1 - n_2 r_1}.$$

Since $uv^t w$ does not belong to S',

$$(n_1 + tm_1 + r_1)^2 = n_3 + tm_3 + r_3 - n_2 t m_1 - n_2 r_1. \tag{9}$$

Formula (9) is the polynomial equation in a variable t having infinitely many solutions. Therefore all coefficients of the polynomials in both parts of (9) must coincide. Hence $m_1 = 0$ and $m_3 = 0$. Then $v = 1$. We have arrived at a contradiction, which completes Example 1.

The next example shows that equations in more than one variable may have nonrational sets of solutions, even in groups of nilpotency class 2.

Example 2. Let G be a free nilpotent group of class 2 and rank 2 and let a and b be its free generators. Then the set of solutions for $[x, y] = 1$ is not rational in $G \times G$, that is, the set $S = \{(x, y) \in G \times G \mid [x, y] = 1\}$ is not rational.

In fact, let $g = a^k b^m [b,a]^l$, $h = a^\kappa b^\mu [b,a]^\lambda \in G$. The elements g and h commute iff $\begin{vmatrix} m & k \\ \mu & \kappa \end{vmatrix} = 0$. The map $G \times G \ni (g,h) \mapsto (k, m, \kappa, \mu) \in Z^4$ is an homomorphism. Therefore, it suffices to show that $A = \left\{ (k, m, \kappa, \mu) \in Z^4 \,\middle|\, \begin{vmatrix} m & k \\ \mu & \kappa \end{vmatrix} = 0 \right\}$ does not belong to $\mathcal{R}(Z^4)$. If $A \in \mathcal{R}(Z^4)$, then, by Lemma 1, there exist finite sets $D = \{d_1, \dots, d_n\}$ and $E = \{e_1, \dots, e_r\}$ in Z^4 such that $(0,0,0,0)$ is not in E, and for any vector $v \in A \setminus D$, there exist some $e_i \in E$ such that $v + Ne_i \in A$ for all natural N. We construct the vector $v = (k, m, \kappa, \mu) \in A$ which fails to satisfy the condition above. Write $e_i = (e_i^1, e_i^2, e_i^3, e_i^4)$.

(1) Let (k, m) be a vector of the form $(1, m)$, $m \geqslant 1$, which is not proportional to any nonzero vector like (e_i^1, e_i^2), (e_i^3, e_i^4).

(2) Let $R = \{\rho \mid \text{ for some } i,\ (e_i^3, e_i^4) = \rho(e_i^1, e_i^2)\}$. Choose a natural $m' \neq 0$ such that $m' \notin R$ and $(1, m, m', mm') \notin D$. Put $(\kappa, \mu) = (m', mm')$.

Let $e_j \in E$. Put

$$\Delta(N) = \begin{vmatrix} 1 + Ne_j^1 & m + Ne_j^2 \\ m' + Ne_j^3 & mm' + Ne_j^4 \end{vmatrix} = N^2 \begin{vmatrix} e_j^1 & e_j^2 \\ e_j^3 & e_j^4 \end{vmatrix} + N \begin{vmatrix} e_j^1 & e_j^2 \\ m' & mm' \end{vmatrix} - N \begin{vmatrix} e_j^3 & e_j^4 \\ 1 & m \end{vmatrix}.$$

If $\Delta(N) = 0$ for all integers $N \geqslant 1$, then the determinant $\begin{vmatrix} e_j^1 & e_j^2 \\ e_j^3 & e_j^4 \end{vmatrix}$ equals zero, its rows are proportional, and

$$0 = \begin{vmatrix} e_j^1 & e_j^2 \\ m' & mm' \end{vmatrix} - \begin{vmatrix} e_j^3 & e_j^4 \\ 1 & m \end{vmatrix} = \begin{vmatrix} m'e_j^1 & m'e_j^2 \\ 1 & m \end{vmatrix} - \begin{vmatrix} e_j^3 & e_j^4 \\ 1 & m \end{vmatrix} = \begin{vmatrix} m'e_j^1 - e_j^3 & m'e_j^2 - e_j^4 \\ 1 & m \end{vmatrix} = 0.$$

If the vector (e_j^1, e_j^2) is zero then (e_j^3, e_j^4) is nonzero and is proportional to $(1, m)$, which contradicts our choice. Therefore (e_j^1, e_j^2) is nonzero and $(e_j^3, e_j^4) = \rho(e_j^1, e_j^2)$ for some ρ. It follows that $(m' - \rho)(e_j^1, e_j^2)$ is proportional to $(1, m)$. If $m' - \rho \neq 0$, then the vectors (e_j^1, e_j^2) and $(1, m)$ are proportional, which contradicts our choice. But if $m' - \rho = 0$ then $m' \in R$, which conflicts with our choice again. Thus A is not rational, and neither is S therefore.

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