CHAPTER III

The Basis Theorem and Some Related Topics

There is no direct analog for differential polynomials of the Hilbert basis theorem for polynomials. There is, however, a weakened analog, the basis theorem of Ritt and Raudenbush. In this chapter we prove a very general version of this theorem. The Ritt–Raudenbush theorem and the known generalizations of it are corollaries of the present version.

The basis theorem and the lemma on which it is based are applied to the following varied topics: behavior of prime differential polynomial ideals under extension of the differential field of coefficients, differential fields of definition of differential polynomial ideals, universal extensions, and differential specializations.

Throughout the chapter $\mathscr{R}$ *denotes a differential ring, and* $\mathscr{F}$ *denotes a differential field for the characteristic of which we write* p *and for the field of constants of which we write* $\mathscr{C}$. *For* $\mathscr{R}$ *as well as for* $\mathscr{F}$ *we denote the set of derivation operators by* Δ, *the set of derivative operators by* Θ, *and the set of derivative operators of order less than or equal to* s *by* $\Theta(s)$. *The letters* y *and* z, *with or without subscripts, stand for differential indeterminates.*

1 Differential conservative systems

Let $\mathscr{M}$ be a differential module over the differential ring $\mathscr{R}$. By a *differential conservative system of* $\mathscr{M}$ we shall mean a conservative system of $\mathscr{M}$ (see Chapter 0, Section 7) every element of which is a differential submodule of $\mathscr{M}$.

We shall be interested exclusively in the case in which $\mathscr{M} = \mathscr{R}$. If $\mathfrak{C}$ is a differential conservative system of $\mathscr{R}$, then the elements of $\mathfrak{C}$ are differential ideals of $\mathscr{R}$; as in Chapter 0, Section 7, we call them *$\mathfrak{C}$-ideals.*

The set of all differential ideals of $\mathscr{R}$ is a differential conservative system of $\mathscr{R}$. So is the set consisting solely of the element $\mathscr{R}$.

Since the set of all perfect ideals of $\mathscr{R}$ is a conservative system of $\mathscr{R}$, it follows that the set of all perfect differential ideals of $\mathscr{R}$ is a conservative system of $\mathscr{R}$, and therefore a differential one. If $\mathfrak{k}$ is a perfect differential ideal of $\mathscr{R}$, and $s \in \mathscr{R}$, then by Chapter I, Section 2, Corollary to Lemma 1, $\mathfrak{k}:s$ is a perfect differential ideal of $\mathscr{R}$. Therefore *the set of all perfect differential ideals of $\mathscr{R}$ is a perfect differential conservative system of $\mathscr{R}$* (see Chapter 0, Section 8).

Let Σ be a subset of $\mathscr{R}$. The smallest perfect differential ideal of $\mathscr{R}$ containing Σ is called the *perfect differential ideal of $\mathscr{R}$ generated by* Σ, and is denoted by $\{\Sigma\}_{\mathscr{R}}$ or, when there is no ambiguity, by $\{\Sigma\}$. In other words, if we denote the set of all perfect differential ideals of $\mathscr{R}$ by $\mathfrak{C}$, then $\{\Sigma\} = (\Sigma)_{\mathfrak{C}}$. The set Σ is said to be a set of *perfect differential ideal generators* of $\{\Sigma\}$ or, if Σ is finite, a *perfect differential ideal basis* (or simply a *basis*) of $\{\Sigma\}$.

A description of $\{\Sigma\}$ can be given by defining recursively:

$\{\Sigma\}_1$ is the set of all $x \in \mathscr{R}$ such that $x^n \in [\Sigma]$ for some $n \in \mathbf{N}$;

$\{\Sigma\}_{k+1} = \{\{\Sigma\}_k\}_1$.

Then it is easy to see that $\{\Sigma\} = \bigcup \{\Sigma\}_k$. When $\mathscr{R}$ is an overring of $\mathbf{Q}$, the nature of $\{\Sigma\}$ is especially transparent, namely, $\{\Sigma\} = \{\Sigma\}_1$. This is an immediate consequence of Chapter I, Section 2, Lemma 2.

Let $\mathscr{A}$ be a differential algebra over $\mathscr{F}$. The set of all perfect differential ideals of $\mathscr{A}$ and the set of all $\mathscr{F}$-separable ideals of $\mathscr{A}$ are perfect conservative systems of $\mathscr{A}$, and therefore their intersection is. Thus, *the set of all $\mathscr{F}$-separable differential ideals of $\mathscr{A}$ is a perfect differential conservative system of $\mathscr{A}$.*

Let Σ be a subset of $\mathscr{A}$. The smallest $\mathscr{F}$-separable differential ideal of $\mathscr{A}$ containing Σ is called the *$\mathscr{F}$-separable differential ideal of $\mathscr{A}$ generated by* Σ, and is denoted by $\{\Sigma\}_{\mathscr{A}/\mathscr{F}}$ or, when there is no ambiguity, by $\{\Sigma\}_{/\mathscr{F}}$. Of course, when $p = 0$ then $\{\Sigma\}_{/\mathscr{F}} = \{\Sigma\}$. When $p \neq 0$ a description of $\{\Sigma\}_{/\mathscr{F}}$ can be given by defining recursively:

$\{\Sigma\}_{/\mathscr{F}}^{(1)}$ is the set of all $x \in \mathscr{A}$ for which there exists a relation $\sum {x_i}^p c_i \in [\Sigma]$ with (c_i) a family of elements of $\mathscr{C}$ linearly independent over $\mathscr{F}^p$ and with (x_i) a family of elements of $\mathscr{A}$ at least one of which equals x;

$\{\Sigma\}_{/\mathscr{F}}^{(k+1)} = \{\{\Sigma\}_{/\mathscr{F}}^{(k)}\}_{/\mathscr{F}}^{(1)}$.

Then $\{\Sigma\}_{/\mathscr{F}} = \bigcup \{\Sigma\}_{/\mathscr{F}}^{(k)}$. Indeed, consider any finite family (c_i) of elements of $\mathscr{C}$ linearly independent over $\mathscr{F}^p$. Because $\mathscr{A}/\{\Sigma\}_{/\mathscr{F}}$ is separable over $\mathscr{F}$,

(c_i) is linearly independent over $(\mathscr{A}/\{\Sigma\}_{/\mathscr{F}})^p$. Therefore if $\sum x_i{}^p c_i \in \{\Sigma\}_{/\mathscr{F}}$, then each $x_i \in \{\Sigma\}_{/\mathscr{F}}$. This shows that $\{\Sigma\}_{/\mathscr{F}}^{(1)} \subset \{\Sigma\}_{/\mathscr{F}}$ and therefore, through an easy induction argument, that the union $\mathfrak{u} = \bigcup \{\Sigma\}_{/\mathscr{F}}^{(k)}$ has the property that $\mathfrak{u} \subset \{\Sigma\}_{/\mathscr{F}}$. On the other hand, if $\sum x_i^p c_i \in \mathfrak{u}$, then $\sum x_i^p c_i \in \{\Sigma\}_{/\mathscr{F}}^{(k)}$ for some k, whence each $x_i \in \{\Sigma\}_{/\mathscr{F}}^{(k+1)} \subset \mathfrak{u}$. Thus $(\mathscr{A}/\mathfrak{u})^p$ and $\mathscr{C}$ are linearly disjoint over $\mathscr{F}^p$, so that by Chapter II, Section 2, Proposition 1, the differential algebra $\mathscr{A}/\mathfrak{u}$ over $\mathscr{F}$ is separable. This shows that the ideal $\mathfrak{u}$ is $\mathscr{F}$-separable, so that $\mathfrak{u} \supset \{\Sigma\}_{/\mathscr{F}}$.

EXERCISES

1. Let $\mathscr{A} = \mathscr{F}\{(y_i)_{i \in I}\}$ be a differential polynomial algebra over $\mathscr{F}$, and Σ be a subset of $\mathscr{A}$. Show that if each element of Σ is homogeneous then $[\Sigma]$, $\{\Sigma\}$, and $\{\Sigma\}_{/\mathscr{F}}$ are homogeneous ideals.
2. (Ritt [95, p. 146]) Let $F_1, \ldots, F_r \in \mathscr{F}\{y_1, \ldots, y_n\}$ and suppose that $\boldsymbol{\Theta}$ is independent on $\mathscr{F}$. Show that there exist $n+1$ linear combinations $L_i = \zeta_{i1} F_1 + \cdots + \zeta_{ir} F_r$ $(1 \leqslant i \leqslant n+1)$ of $F_1, \ldots, F_r$ over $\mathscr{F}$ such that $\{F_1, \ldots, F_r\} = \{L_1, \ldots, L_{n+1}\}$. (*Hint:* Set $e = \max_{1 \leqslant k \leqslant r} \operatorname{ord} F_k$ and fix $s \in \mathbf{N}$ so that $(n+1)\binom{s+m}{m} > n\binom{s+e+m}{m}$. Let $(z_{ik})_{1 \leqslant i \leqslant n+1,\, 1 \leqslant k \leqslant r}$ be a family of differential indeterminates over $\mathscr{F}\{y_1, \ldots, y_n\}$, let $M_i = z_{i1} F_1 + \cdots + z_{ir} F_r$ $(1 \leqslant i \leqslant n+1)$, and consider the ideal $\mathfrak{a}$ generated by
$$\theta M_i \qquad (\theta \in \boldsymbol{\Theta}(s), 1 \leqslant i \leqslant n+1)$$
in the polynomial algebra
$$R = \mathscr{F}[(\theta y_j)_{\theta \in \boldsymbol{\Theta}(s+e),\, 1 \leqslant j \leqslant n}, (\theta z_{ik})_{\theta \in \boldsymbol{\Theta}(s),\, 1 \leqslant i \leqslant n+1,\, 1 \leqslant k \leqslant r}]$$
over $\mathscr{F}$. Show that if $((\eta_{\theta,j}), (\zeta_{\theta,i,k}))$ is a generic zero of a prime ideal $\mathfrak{p}$ of R with $\mathfrak{a} \subset \mathfrak{p}$ and $F_1 \notin \mathfrak{p}$, then
$$\zeta_{\theta,i,1} \in \mathscr{F}((\zeta_{\theta',i,k})_{\theta' \in \boldsymbol{\Theta}(s),\, 2 \leqslant k \leqslant r})((\eta_{\theta',j})_{\theta' \in \boldsymbol{\Theta}(s+e),\, 1 \leqslant j \leqslant n})$$
for all $\theta \in \boldsymbol{\Theta}(s)$, $1 \leqslant i \leqslant n+1$, and infer that $\mathfrak{a}$ contains an element $B_{\mathfrak{p}} F_1^{d_{\mathfrak{p}}}$, where $d_{\mathfrak{p}} \in \mathbf{N}$, $B_{\mathfrak{p}} \in R$, $B_{\mathfrak{p}} \neq 0$, and $B_{\mathfrak{p}}$ is free of every θy_j. Conclude that there exist a nonzero $C \in \mathscr{F}\{(z_{ik})_{1 \leqslant i \leqslant n+1,\, 1 \leqslant k \leqslant r}\}$ and an $f \in \mathbf{N}$ such that, for every k, $C F_k{}^f \in [M_1, \ldots, M_{n+1}]$ in $\mathscr{F}\{(y_j)_{1 \leqslant j \leqslant n}, (z_{ik})_{1 \leqslant i \leqslant n+1,\, 1 \leqslant k \leqslant r}\}$, and then take elements $\zeta_{ik} \in \mathscr{F}$ such that $C((\zeta_{ik})) \neq 0$.)

2 Quasi-separable differential ideals

The purpose of this section is to prove the following lemma and its corollary.

Lemma 1 *Let $\mathscr{S} = \mathscr{R}\{y_1, \ldots, y_n\}$ be a finitely generated differential polynomial algebra over $\mathscr{R}$, and suppose given a sequential ranking of $(y_1, \ldots, y_n)$. Let $\mathfrak{p}$ be a prime differential ideal of $\mathscr{S}$ that is quasi-separable over $\mathscr{R}$, let* A *be a characteristic set of* $\mathfrak{p}$, *and let* V *denote the set of derivatives* θy_j *that are not proper derivatives of any leader* u_A *with* $A \in \mathrm{A}$. *Then there exists a finite set* $Y \subset V$ *such that every element of* $\mathfrak{p}$ *that is reduced with respect to* A *is in the ideal* $(\mathfrak{p} \cap \mathscr{R}[Y])$ *of* $\mathscr{S}$.

REMARK If $\mathscr{S}/\mathfrak{p}$ is of characteristic 0, the lemma is trivial, even when it is strengthened by omitting the requirement that the ranking be sequential and by taking Y to be the empty set. Indeed, if there existed an element of $\mathfrak{p}$ reduced with respect to A and not in $(\mathfrak{p} \cap \mathscr{R})$, then there would exist one, call it P, of minimal rank. The separant $S_P = \partial P/\partial u_P$ would be in $\mathfrak{p}$ by Chapter I, Section 10, Lemma 8. By the minimality of the rank of P then S_P would be in $(\mathfrak{p} \cap \mathscr{R})$, and by the hypothesis on the characteristic of $\mathscr{S}/\mathfrak{p}$ this would force the contradiction that $P \in (\mathfrak{p} \cap \mathscr{R})$.

Proof Assume the conclusion false. For each $s \in \mathrm{N}$ let $V(s)$ denote the set of all elements $\theta y_j \in V$ with $\operatorname{ord}\theta \leqslant s$, and let $q_s = \operatorname{Card} V(s)$. Because the conclusion is false, for each $s \in \mathrm{N}$ there exists an $s' \in \mathrm{N}$ with $s' > s$ such that some element of $\mathfrak{p} \cap \mathscr{R}[V(s')]$ is not an element of $(\mathfrak{p} \cap \mathscr{R}[V(s)])$. In other words, if $f\colon \mathscr{S} \to \mathscr{S}/\mathfrak{p}$ denotes the canonical homomorphism, then the transcendence degree of $f(\mathscr{R}[V(s')])$ over $f(\mathscr{R}[V(s)])$ is less than $q_{s'} - q_s$. It follows that there exists an infinite strictly increasing sequence of natural numbers $s, s', \ldots, s^{(v)}, \ldots$ such that the transcendence degree of $f(\mathscr{R}[V(s^{(v+1)})])$ over $f(\mathscr{R}[V(s^{(v)})])$ is less than or equal to $q_{s^{(v+1)}} - q_{s^{(v)}} - 1$. For any $h \in \mathrm{N}$ the transcendence degree of $f(\mathscr{R}[V(s^{(h)})])$ over $f(\mathscr{R})$ is then less than or equal to $q_s + \sum_{0 \leqslant v < h}(q_{s^{(v+1)}} - q_{s^{(v)}} - 1) = q_{s^{(h)}} - h$, which is less than $q_{s^{(h)}} - q_s$ provided $h > q_s$. Thus, for every $s \in \mathrm{N}$ there exists a $t \in \mathrm{N}$ with $t > s$ such that the transcendence degree of $f(\mathscr{R}[V(t)])$ over $f(\mathscr{R})$ is less than $q_t - q_s$.

Let W denote the set of all $w \in V$ such that only finitely many derivatives of w are in V. By Chapter 0, Section 17, Lemma 16, W is a finite set, so that if $s(0)$ is a large enough natural number, then $W \subset V(s(0))$. Fixing $s(0)$ large enough for this to be the case, we see from the final remark of the preceding paragraph that there exists an infinite strictly increasing sequence of natural numbers $s(0), s(1), \ldots, s(v), \ldots$ such that the transcendence degree of $f(\mathscr{R}[V(s(v+1))])$ over $f(\mathscr{R})$ is less than $q_{s(v+1)} - q_{s(v)}$. Each family $(f(v))_{v \in V(s(v+1)) - V(s(v))}$ is then algebraically dependent over $f(\mathscr{R})$. Since the sets $V(s(v+1)) - V(s(v))$ are disjoint, and $\bigcup_{v \in \mathrm{N}}(V(s(v+1)) - V(s(v))) = V - V(s(0))$, we conclude that the family $(f(v))_{v \in V - V(s(0))}$ has infinite algebraic codimension over $Q(f(\mathscr{R}))$. However, by Chapter I, Section 10, Lemma 9, this family is separably independent over $Q(f(\mathscr{R}))$. This shows that $\mathfrak{p}$ is not quasi-separable over $\mathscr{R}$.

Corollary *Let* $\mathscr{R}_0$ *be a differential subring of* $\mathscr{R}$ *over which* $\mathscr{R}$ *is finitely generated* (*as a differential ring*), *and let* $\mathfrak{p}$ *be a prime differential ideal of* $\mathscr{R}$

which is quasi-separable over $\mathscr{R}_0$. *Then there exist a finite set* $\Phi \subset \mathscr{R}$, *a finite set* $\Psi \subset \mathfrak{p}$, *and an element* $u \in \mathscr{R}$ *with* $u \notin \mathfrak{p}$, *such that* $\mathfrak{p} = ([\Psi] + (\mathfrak{p} \cap \mathscr{R}_0[\Phi])) : u^\infty$.

Proof Since $\mathscr{R}$ is finitely generated over $\mathscr{R}_0$ there exist a finitely generated differential polynomial algebra $\mathscr{S} = \mathscr{R}_0\{y_1, \ldots, y_n\}$ over $\mathscr{R}_0$ and a surjective $\mathscr{R}_0$-homomorphism $g : \mathscr{S} \to \mathscr{R}$. The inverse image $\mathfrak{q} = g^{-1}(\mathfrak{p})$ is a prime differential ideal of $\mathscr{S}$ containing the kernel of g, so that (see Chapter 0, Section 6, the Remark preceding Lemma 5) $\mathfrak{q}$ is quasi-separable over $\mathscr{R}_0$. By Lemma 1 there exist an autoreduced set $\mathrm{A} \subset \mathfrak{q}$ with $H_\mathrm{A} \notin \mathfrak{q}$ and a finite set Y of derivatives θy_j such that every element of $\mathfrak{q}$ that is reduced with respect to A is in $(\mathfrak{q} \cap \mathscr{R}_0[Y])$. Let $x \in \mathfrak{p}$. There exists an $F \in \mathfrak{p}$ with $g(F) = x$. By Chapter I, Section 9, Proposition 1, the remainder F_0 of F with respect to A is reduced with respect to A, and $F \in ((F_0) + [\mathrm{A}]) : H_\mathrm{A}^\infty$. By the above, $F_0 \in (\mathfrak{q} \cap \mathscr{R}_0[Y])$, so that $F \in ([\mathrm{A}] + (\mathfrak{q} \cap \mathscr{R}_0[Y])) : H_\mathrm{A}^\infty$. Applying g we obtain the relation $x \in ([g(\mathrm{A})] + (\mathfrak{p} \cap \mathscr{R}_0[g(Y)])) : g(H_\mathrm{A})^\infty$. Thus, the corollary holds with $\Phi = g(Y)$, $\Psi = g(\mathrm{A})$, and $u = g(H_\mathrm{A})$.

3 Differential fields of definition

A polynomial algebra $K[X] = K[(X_i)_{i \in I}]$ over a field K has, as a vector space over K, a basis consisting of the monomials in X; an ideal is a subspace. By a *field of definition* of a polynomial ideal $\mathfrak{k}$ is meant a subfield K_0 of K that is a field of definition of the subspace $\mathfrak{k}$ relative to the basis of monomials (see Chapter I, Section 5), that is, that has the property that $K \cdot (\mathfrak{k} \cap K_0[X]) = \mathfrak{k}$. It is apparent that if K_0 is field of definition of $\mathfrak{k}$, then any field of definition of $\mathfrak{k} \cap K_0[X]$ is a field of definition of $\mathfrak{k}$, and any field K_1 between K_0 and K is a field of definition of $\mathfrak{k}$ such that K_0 is a field of definition of $\mathfrak{k} \cap K_1[X]$.

If, furthermore, we denote the canonical homomorphism $K[X] \to K[X]/\mathfrak{k}$ by f, then (by Chapter 0, Section 10, Lemma 9, applied to the ideal $\mathfrak{k} \cap K_0[X]$) $f(K_0[X])$ and K are linearly disjoint over K_0. It easily follows from this that *if the ideal* $\mathfrak{k}$ *of* $K[X]$ *with field of definition* K_0 *is separable, respectively quasi-separable, respectively regular, over* K, *then* $\mathfrak{k} \cap K_0[X]$ *is separable, respectively quasi-separable, respectively regular, over* K_0.

A differential polynomial algebra $\mathscr{F}\{(y_i)_{i \in I}\}$ in a family of differential indeterminates $(y_i)_{i \in I}$ over a differential field $\mathscr{F}$ is a polynomial algebra in the family of indeterminates $(\theta y_i)_{\theta \in \Theta, i \in I}$ over the field $\mathscr{F}$, and a differential ideal is also an ideal. By a *differential field of definition* of a differential polynomial ideal $\mathfrak{k}$ over $\mathscr{F}$ we shall mean a differential subfield of $\mathscr{F}$ that is a field of definition of $\mathfrak{k}$. It is an immediate consequence of Chapter I, Section 5, Lemma 3, that *the smallest field of definition of a differential polynomial ideal over* $\mathscr{F}$ *is a differential subfield of* $\mathscr{F}$.

Proposition 1 *Let $\mathfrak{p}$ be a prime differential ideal of the finitely generated differential polynomial algebra $\mathscr{F}\{y_1,\ldots,y_n\}$ over $\mathscr{F}$, with $\mathfrak{p}$ quasi-separable over $\mathscr{F}$. Then the smallest field of definition of $\mathfrak{p}$ is a finitely generated differential field extension of the prime field.*

Proof By Section 2, the Corollary to Lemma 1, there exist an $L \in \mathscr{F}\{y_1,\ldots,y_n\}$ with $L \notin \mathfrak{p}$ and an $s \in \mathbf{N}$ such that

$$\mathfrak{p} = [\mathfrak{p} \cap \mathscr{F}[(\theta y_j)_{\theta\in\Theta(s),\,1\leqslant j\leqslant n}]]:L^\infty.$$

By Hilbert's basis theorem the polynomial ideal $\mathfrak{p} \cap \mathscr{F}[(\theta y_j)_{\theta\in\Theta(s),\,1\leqslant j\leqslant n}]$ is finitely generated. Therefore $\mathfrak{p}$ has a finite subset Φ such that $\mathfrak{p} = [\Phi]:L^\infty$. Let $\mathscr{F}_1$ be the differential field generated by the coefficients in L and in the elements of Φ, and let (φ_i) be a vector space basis of $\mathscr{F}$ over $\mathscr{F}_1$. For any $G \in \mathfrak{p}$ we may write $G = \sum G_i\varphi_i$ with each $G_i \in \mathscr{F}_1\{y_1,\ldots,y_n\}$. By the above, there is an $r \in \mathbf{N}$ such that $L^rG \in \mathscr{F}\cdot(\mathfrak{p}\cap\mathscr{F}_1\{y_1,\ldots,y_n\})$, and by Chapter 0, Section 10, Lemma 9, this implies that $L^rG_i \in \mathfrak{p}\cap\mathscr{F}_1\{y_1,\ldots,y_n\}$. Therefore each $G_i \in \mathfrak{p}\cap\mathscr{F}_1\{y_1,\ldots,y_n\}$, so that $G \in \mathscr{F}\cdot(\mathfrak{p}\cap\mathscr{F}_1\{y_1,\ldots,y_n\})$, and $\mathscr{F}_1$ is a differential field of definition. Thus, we conclude that $\mathfrak{p}$ has a finitely generated differential field of definition. Now let $\mathscr{F}_0$ be the smallest field of definition of $\mathfrak{p}$. By our earlier remarks, $\mathscr{F}_0$ is a differential field and $\mathfrak{p}\cap\mathscr{F}_0\{y_1,\ldots,y_n\}$ is a prime differential ideal quasiseparable over $\mathscr{F}_0$. Arguing for this ideal as we just did for $\mathfrak{p}$, we conclude that $\mathfrak{p}\cap\mathscr{F}_0\{y_1,\ldots,y_n\}$ has a finitely generated differential field of definition $\mathscr{F}_{01}$. However, $\mathscr{F}_{01}$ is a field of definition of $\mathfrak{p}$ and is contained in $\mathscr{F}_0$. Therefore $\mathscr{F}_0 = \mathscr{F}_{01}$.

4 The basis theorem

We are now in a position to prove one of the main results of this chapter.

Theorem 1 *Let $\mathscr{R}_0$ be a differential subring of $\mathscr{R}$ over which $\mathscr{R}$ is finitely generated (as a differential ring). Let $\mathfrak{C}$ be a perfect differential conservative system of $\mathscr{R}$. If $\mathfrak{C}|\mathscr{R}_0$ is Noetherian, and if every prime $\mathfrak{C}$-ideal is quasi-separable over $\mathscr{R}_0$, then $\mathfrak{C}$ is Noetherian.*

Proof Assume the conclusion false. By Chapter 0, Section 9, Lemma 8, there exists a maximal $\mathfrak{C}$-ideal $\mathfrak{m}$ that is not finitely $\mathfrak{C}$-generated, and $\mathfrak{m}$ is prime. By Section 2, Corollary to Lemma 1, there exist a finite $\Phi \subset \mathscr{R}$, a finite $\Psi \subset \mathfrak{m}$, and a $u \in \mathscr{R}$ with $u \notin \mathfrak{m}$, such that $u\mathfrak{m} \subset ([\Psi] + (\mathfrak{m}\cap\mathscr{R}_0[\Phi]))_{\mathfrak{C}}$. By Chapter 0, Section 9, Proposition 3 (applied to the perfect conservative system $\mathfrak{C}|\mathscr{R}_0[\Phi]$ of the ring $\mathscr{R}_0[\Phi]$), there exists a finite set $\Lambda \subset \mathfrak{m}\cap\mathscr{R}_0[\Phi]$ such that $\mathfrak{m}\cap\mathscr{R}_0[\Phi] = (\Lambda)_{\mathfrak{C}|\mathscr{R}_0[\Phi]}$, whence $\mathfrak{m}\cap\mathscr{R}_0[\Phi] \subset (\Lambda)_{\mathfrak{C}}$. Thus, $u\mathfrak{m} \subset (\Psi\cup\Lambda)_{\mathfrak{C}}$. By the maximality of $\mathfrak{m}$, $(u, \mathfrak{m})_{\mathfrak{C}}$ is finitely $\mathfrak{C}$-generated. It follows (by Chapter 0, Section 7, Lemma 6) that there exists a finite set $M \subset \mathfrak{m}$ such that $(u, \mathfrak{m})_{\mathfrak{C}} = (u, M)_{\mathfrak{C}}$. By Chapter 0, Section 8, Lemma 7, then

$$\mathfrak{m} = \mathfrak{m} \cap (u, \mathfrak{m})_{\mathfrak{C}} = \mathfrak{m} \cap (u, M)_{\mathfrak{C}} = (u\mathfrak{m}, M)_{\mathfrak{C}} \subset ((\Psi \cup \Lambda)_{\mathfrak{C}}, M)_{\mathfrak{C}}$$
$$= (\Psi \cup \Lambda \cup M)_{\mathfrak{C}},$$

so that $\mathfrak{m}$ is finitely $\mathfrak{C}$-generated. This contradiction proves the theorem.

Corollary 1 *Let $n \in \mathbf{N}$, $n \neq 0$, and consider the differential polynomial algebra $\mathcal{S} = \mathcal{R}\{y_1, \ldots, y_n\}$ over $\mathcal{R}$. A necessary and sufficient condition that the set of all perfect differential ideals of $\mathcal{S}$ be a Noetherian conservative system is that the set of all perfect differential ideals of $\mathcal{R}$ be a Noetherian conservative system and, for every prime differential ideal $\mathfrak{p}$ of $\mathcal{R}$, $Q(\mathcal{R}/\mathfrak{p})$ be differentially quasi-perfect.*

Proof If the condition is satisfied, then every prime differential ideal of $\mathcal{S}$ is quasi-separable over $\mathcal{R}$, and the theorem therefore applies to the conservative system formed by all the perfect differential ideals of $\mathcal{S}$.

Let the condition not be satisfied. If $\mathfrak{k}$ is a perfect differential ideal of $\mathcal{R}$, then $\mathcal{S}\mathfrak{k}$ is a perfect differential ideal of $\mathcal{S}$ (see Chapter 0, Section 5), and $\mathcal{S}\mathfrak{k} \cap \mathcal{R} = \mathfrak{k}$. It follows that if the set of all perfect differential ideals of $\mathcal{R}$ is not Noetherian, then neither is the set of all perfect differential ideals of $\mathcal{S}$. Therefore we may suppose that there exists a prime differential ideal $\mathfrak{p}$ of $\mathcal{R}$ such that the differential field $\mathcal{F}_0 = Q(\mathcal{R}/\mathfrak{p})$ is not differentially quasi-perfect. Now, $\mathcal{F}_0\{y_1, \ldots, y_1\}$ is the differential ring of quotients of $(\mathcal{R}/\mathfrak{p})\{y_1, \ldots, y_n\}$ over the multiplicatively stable set of nonzero elements of $\mathcal{R}/\mathfrak{p}$, and $(\mathcal{R}/\mathfrak{p})$ $\{y_1, \ldots, y_n\}$ is a homomorphic image of $\mathcal{S}$. It follows from Chapter 0, Section 9, Proposition 2 and its first corollary, that to prove that the set of all perfect differential ideals of $\mathcal{S}$ is not Noetherian, it suffices to prove the same thing for $\mathcal{F}_0\{y_1, \ldots, y_n\}$. Since $\mathcal{F}_0$ is not differentially quasi-perfect we see by Chapter II, Section 3, Proposition 5, that the characteristic p of $\mathcal{F}_0$ is not 0 and there exists an infinite sequence $c_0, c_1, \ldots, c_k, \ldots$ of constants in $\mathcal{F}_0$ such that $c_k \notin \mathcal{F}_0{}^p(c_0, \ldots, c_{k-1})$ for every k. Fixing some $\delta \in \Delta$ we see that the ideals $\mathfrak{q}_k = (y_1{}^p - c_0, (\delta y_1)^p - c_1, \ldots, (\delta^k y_1)^p - c_k)$ of $\mathcal{F}_0\{y_1, \ldots, y_n\}$ form an infinite strictly increasing sequence; each $\mathfrak{q}_k$ is obviously a differential ideal, and by Chapter 0, Section 3, Lemma 2, is prime (hence perfect).

Corollary 2 *Let $n \in \mathbf{N}$, $n \neq 0$. A necessary and sufficient condition that the set of all perfect differential ideals of the differential polynomial algebra $\mathcal{F}\{y_1, \ldots, y_n\}$ be a Noetherian conservative system, is that $\mathcal{F}$ be differentially quasi-perfect.*

Proof This is a special case of Corollary 1.

Corollary 3 *A finitely generated extension of a differentially quasi-perfect differential field is itself differentially quasi-perfect.*

Proof Let $\mathscr{F}$ be differentially quasi-perfect and $\mathscr{G} = \mathscr{F}\langle\alpha_1,\dots,\alpha_n\rangle$. Let $y_1,\dots,y_{n+1},y$ be differential indeterminates and Σ be the set of nonzero elements of $\mathscr{F}\{\alpha_1,\dots,\alpha_n\}$. There exists a surjective homomorphism $\mathscr{F}\{y_1,\dots,y_{n+1}\} \to \mathscr{F}\{\alpha_1,\dots,\alpha_n,y\}$ over $\mathscr{F}$, and if $\mathfrak{k}$ is its kernel, then $\mathscr{F}\{y_1,\dots,y_{n+1}\}/\mathfrak{k} \approx \mathscr{F}\{\alpha_1,\dots,\alpha_n,y\}$. Also, $\Sigma^{-1}\mathscr{F}\{\alpha_1,\dots,\alpha_n,y\} = \mathscr{G}\{y\}$. By Corollary 2, $\mathscr{F}\{y_1,\dots,y_{n+1}\}$ has the property that the set of all its perfect differential ideals is a Noetherian conservative system. By the above and Chapter 0, Section 9, Corollary 1 to Proposition 2, $\mathscr{G}\{y\}$ has the same property. Hence, by Corollary 2, $\mathscr{G}$ is differentially quasi-perfect.

Corollary 4 *Let $n \in \mathbf{N}$ and let $\mathfrak{p}$ be a prime differential ideal of $\mathscr{F}\{y_1,\dots,y_n\}$. If $\mathfrak{p}$ is quasi-perfect over $\mathscr{F}$, then $\mathfrak{p} = \{\Phi\}$ for some finite set $\Phi \subset \mathfrak{p}$.*

Proof By Section 3, Proposition 1, there is a finitely generated extension $\mathscr{F}_0$ of the prime field such that if we set $\mathfrak{p}_0 = \mathfrak{p} \cap \mathscr{F}_0\{y_1,\dots,y_n\}$, then $\mathfrak{p} = \mathscr{F}\mathfrak{p}_0$. By Corollary 3, $\mathscr{F}_0$ is differentially quasi-perfect; hence by Corollary 2 there is a finite set $\Phi \subset \mathfrak{p}_0$ such that $\mathfrak{p}_0 = \{\Phi\}_{\mathscr{F}_0\{y_1,\dots,y_n\}}$. Then

$$\begin{aligned}\mathfrak{p} \supset \{\Phi\}_{\mathscr{F}\{y_1,\dots,y_n\}} &\supset \mathscr{F}\cdot(\{\Phi\}_{\mathscr{F}\{y_1,\dots,y_n\}} \cap \mathscr{F}_0\{y_1,\dots,y_n\}) \\ &\supset \mathscr{F}\cdot\{\Phi\}_{\mathscr{F}_0\{y_1,\dots,y_n\}} = \mathscr{F}\mathfrak{p}_0 = \mathfrak{p},\end{aligned}$$

whence $\mathfrak{p} = \{\Phi\}_{\mathscr{F}\{y_1,\dots,y_n\}}$.

Corollary 5 *Let $n \in \mathbf{N}$. The set of all $\mathscr{F}$-separable differential ideals of $\mathscr{F}\{y_1,\dots,y_n\}$ is a Noetherian conservative system. If $\mathfrak{a}$ is any $\mathscr{F}$-separable differential ideal of $\mathscr{F}\{y_1,\dots,y_n\}$, then $\mathfrak{a} = \{\Phi\}$ for some finite set $\Phi \subset \mathfrak{a}$.*

Proof Every prime $\mathscr{F}$-separable differential ideal is quasi-separable over $\mathscr{F}$, so the first assertion follows from the theorem. Therefore $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$, where each $\mathfrak{p}_k$ is an $\mathscr{F}$-separable prime differential ideal, and by Corollary 4, $\mathfrak{p}_k = \{\Phi_k\}$ for a finite set $\Phi_k \subset \mathfrak{p}_k$. Hence by Chapter 0, Section 8, Lemma 7, $\mathfrak{a} = \{\Phi_1\} \cap \cdots \cap \{\Phi_r\} = \{\Phi_1 \cdots \Phi_r\}$.

REMARK The first result in the direction of a basis theorem was obtained by Ritt [79, 81]. Working with a differential field $\mathscr{F}$ of functions meromorphic in a region, and using the language of differential equations, he proved (a) that if Σ is a subset of $\mathscr{F}\{y_1,\dots,y_n\}$, then the system of differential equations $G = 0$ $(G \in \Sigma)$ is equivalent to (has the same solutions as) a finite subsystem, and (b) that if for an element $F \in \mathscr{F}\{y_1,\dots,y_n\}$ every solution of the above system is a solution of the differential equation $F = 0$, then some power of F is in $[\Sigma]$. (We shall take up this point of view in Chapter IV.) This led Raudenbush [73] to formalize the notion of perfect differential ideal and to prove for an abstract differential field $\mathscr{F}$ of characteristic 0 that every perfect differential ideal of $\mathscr{F}\{y_1, \dots, y_n\}$ has a basis. The same conclusion was obtained by Kolchin [37] for more general coefficient domains (including any perfect differential field, and also certain differential rings); in the same paper a counterexample was given for a nonperfect differential field $\mathscr{F}$. It was Seidenberg [108] who reestablished the Raudenbush result for an arbitrary differential field $\mathscr{F}$ by requiring that the perfect differential ideals be separable over $\mathscr{F}$, that is, who first proved the first part of Corollary 5 above.

EXERCISES

1. Let Λ_n denote the set of all $\mathscr{F}$-separable prime differential ideals of $\mathscr{F}\{y_1, \dots, y_n\}$. Show that $\operatorname{Card}\Lambda_n = \max(\aleph_0, \operatorname{Card}\mathscr{F})$. Corollary: *If $\mathscr{S}$ is a semiuniversal extension of $\mathscr{F}$* (see Chapter II, Section 2), *then there exists a family $(\mathscr{E}_\lambda)_{\lambda \in \Lambda}$ of differential subfields of $\mathscr{S}$ such that each $\mathscr{E}_\lambda$ is a finitely generated separable extension of $\mathscr{F}$, every finitely generated separable extension of $\mathscr{F}$ is $\mathscr{F}$-isomorphic to some $\mathscr{E}_\lambda$, and* $\operatorname{Card}\Lambda = \max(\aleph_0, \operatorname{Card}\mathscr{F})$.
2. Show that $\mathscr{F}$ always has a separable semiuniversal extension $\mathscr{S}$ such that $\operatorname{Card}\mathscr{S} = \max(\aleph_0, \operatorname{Card}\mathscr{F})$.

5 Differential dimension polynomials

Let $\mathfrak{p}$ be a prime differential ideal of a finitely generated differential polynomial algebra $\mathscr{F}\{y_1, \dots, y_n\}$ over $\mathscr{F}$. The canonical homomorphism of $\mathscr{F}\{y_1, \dots, y_n\}$ into $Q(\mathscr{F}\{y_1, \dots, y_n\}/\mathfrak{p})$ maps $(y_1, \dots, y_n)$ onto a family $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n)$, and maps $\mathscr{F}$ isomorphically onto a differential field that we may thus identify with $\mathscr{F}$. After this identification $Q(\mathscr{F}\{y_1, \dots, y_n\}/\mathfrak{p})$ may be written as $\mathscr{F}\langle\bar{y}\rangle = \mathscr{F}\langle\bar{y}_1, \dots, \bar{y}_n\rangle$, and the canonical homomorphism becomes the substitution of $(\bar{y}_1, \dots, \bar{y}_n)$ for $(y_1, \dots, y_n)$. The differential inseparability polynomial $\omega_{\bar{y}/\mathscr{F}}$ of $\bar{y}$ over $\mathscr{F}$ (see Chapter II, Section 12) we now call the *differential inseparability polynomial of* $\mathfrak{p}$, and we denote it by $\omega_\mathfrak{p}$. We have at our disposal in connection with $\omega_\mathfrak{p}$ all the results of Chapter II, Sections 12 and 13. In particular, $\omega_\mathfrak{p}$ is a numerical polynomial with $\deg\omega_\mathfrak{p} \leqslant m$ (the cardinal number of the set Δ of derivation operators), and if we write $\omega_\mathfrak{p} = \sum_{0 \leqslant i \leqslant m} a_i\binom{X+i}{i}$, then the coefficient a_m is the differential inseparability degree of $\mathscr{F}\langle\bar{y}\rangle$ over $\mathscr{F}$; we call this number the *differential inseparability degree of* $\mathfrak{p}$. The differential type τ of $\mathscr{F}\langle\bar{y}\rangle$ over $\mathscr{F}$ (see Chapter II, Section 13), which is defined as $\tau = \deg\omega_\mathfrak{p}$, we now call the *differential type of* $\mathfrak{p}$. The typical differential inseparability degree a_τ of

$\mathscr{F}\langle\bar{y}\rangle$ over $\mathscr{F}$ we now call the *typical differential inseparability degree of* $\mathfrak{p}$.

If $\mathfrak{p}$ is separable over $\mathscr{F}$, we also call $\omega_{\mathfrak{p}}$ the *differential dimension polynomial* of $\mathfrak{p}$, call a_m the *differential dimension* of $\mathfrak{p}$, and call a_τ the *typical differential dimension* of $\mathfrak{p}$. Since for a separable finitely generated field extension the notions of inseparability degree and transcendence degree coincide, we see by Chapter II, Section 12, Theorem 6, that if for each $s \in \mathbf{N}$ we let $\mathfrak{p}_s$ denote the polynomial ideal $\mathfrak{p} \cap \mathscr{F}[(\theta y_j)_{\theta\in\Theta(s),\,1\leqslant j\leqslant n}]$, then $\omega_{\mathfrak{p}}(s) = \dim \mathfrak{p}_s$ for all sufficiently big $s \in \mathbf{N}$.

Proposition 2 *Let $\mathfrak{p}$ and $\mathfrak{p}'$ be $\mathscr{F}$-separable prime differential ideals of a finitely generated differential polynomial algebra $\mathscr{F}\{y_1,\ldots,y_n\}$ over $\mathscr{F}$, with $\mathfrak{p} \subset \mathfrak{p}'$ and $\mathfrak{p} \neq \mathfrak{p}'$. Then $\omega_{\mathfrak{p}} > \omega_{\mathfrak{p}'}$.*

Proof For each $s \in \mathbf{N}$ let $\mathfrak{p}_s$, respectively $\mathfrak{p}_s'$, denote the prime polynomial ideal $\mathfrak{p} \cap \mathscr{F}[(\theta y_j)_{\theta\in\Theta(s),\,1\leqslant j\leqslant n}]$, respectively $\mathfrak{p}' \cap \mathscr{F}[(\theta y_j)_{\theta\in\Theta(s),\,1\leqslant j\leqslant n}]$. Since $\mathfrak{p}$ and $\mathfrak{p}'$ are separable, for all sufficiently big values of s, $\omega_{\mathfrak{p}}(s) = \dim \mathfrak{p}_s$ and $\omega_{\mathfrak{p}'}(s) = \dim \mathfrak{p}_s'$. However, for all big values of s, $\mathfrak{p}_s \subset \mathfrak{p}_s'$ and $\mathfrak{p}_s \neq \mathfrak{p}_s'$, so that (by Chapter 0, Section 11, Proposition 4) $\omega_{\mathfrak{p}}(s) > \omega_{\mathfrak{p}'}(s)$. Therefore $\omega_{\mathfrak{p}} > \omega_{\mathfrak{p}'}$.

The proposition becomes false if the hypothesis of separability is omitted (see Exercise 1 below).

EXERCISE

1. Let $p \neq 0$, let Λ be any subset of Θ, let $(c_\theta)_{\theta\in\Lambda}$ be a family of constants in $\mathscr{F}$ separably independent over $\mathscr{F}^p$, and let $\mathfrak{p}(\Lambda)$ denote the ideal $(((\theta y)^p + c_\theta)_{\theta\in\Lambda})$ of $\mathscr{F}\{y\}$. Show that $\mathfrak{p}(\Lambda)$ is a prime differential ideal and that $\omega_{\mathfrak{p}(\Lambda)} = \binom{X+m}{m}$, where $m = \operatorname{Card}\Delta$.

6 Extension of the differential field of coefficients

Let $\mathscr{G}$ be an extension of $\mathscr{F}$ and let $(y_i)_{i\in I}$ be a family of differential indeterminates over $\mathscr{G}$. We are interested in the behavior of a perfect differential ideal $\mathfrak{a}$ of $\mathscr{F}\{(y_i)_{i\in I}\}$ when $\mathscr{F}$ is extended to $\mathscr{G}$, that is, we ask about the nature of the differential ideal $\mathscr{G}\mathfrak{a}$ of $\mathscr{G}\{(y_i)_{i\in I}\}$. The question reduces, in a certain sense, to the case in which $\mathfrak{a}$ is prime. Indeed, if Π is the set of components of $\mathfrak{a}$, then (by Chapter 0, Section 8, Proposition 1) $\mathfrak{a} = \bigcap_{\mathfrak{p}\in\Pi}\mathfrak{p}$; if $F \in \mathscr{G}\{(y_i)_{i\in I}\}$ and we write $F = \sum \gamma_k F_k$, where each $F_k \in \mathscr{F}\{(y_i)_{i\in I}\}$ and (γ_k) is a basis of $\mathscr{G}$ over $\mathscr{F}$, then by Chapter 0, Section 10, Lemma 9, $F \in \mathscr{G}\mathfrak{a}$ if and only if each $F_k \in \mathfrak{a} = \bigcap_{\mathfrak{p}\in\Pi}\mathfrak{p}$, that is, each $F_k \in \mathfrak{p}$ ($\mathfrak{p} \in \Pi$), that is, $F \in \mathscr{G}\mathfrak{p}$ for every $\mathfrak{p} \in \Pi$; thus, $\mathscr{G}\mathfrak{a} = \bigcap_{\mathfrak{p}\in\Pi}\mathscr{G}\mathfrak{p}$. The situation is especially good in this respect when I is finite and $\mathfrak{a}$ is separable over $\mathscr{F}$, for then (by

Section 4, Corollary 5 to Theorem 1, and by Chapter 0, Section 9, Theorem 1) Π is finite and each $\mathfrak{p} \in \Pi$ is separable over $\mathscr{F}$.

Proposition 3 *Let $\mathfrak{p}$ be an $\mathscr{F}$-separable prime differential ideal of a finitely generated differential polynomial algebra $\mathscr{F}\{y_1, \ldots, y_n\}$ over $\mathscr{F}$, and let $\mathscr{G}$ be an extension of $\mathscr{F}$.*

(a) *$\mathscr{G}\mathfrak{p}$ is a $\mathscr{G}$-separable differential ideal of $\mathscr{G}\{y_1, \ldots, y_n\}$. If $\mathfrak{p}$ is regular over $\mathscr{F}$, then $\mathscr{G}\mathfrak{p}$ is regular over $\mathscr{G}$.*

(b) *$\mathscr{G}\mathfrak{p}$ has finitely many components, and each of them is a $\mathscr{G}$-separable prime differential ideal. If $\mathfrak{p}'$ is any one of them, then $\mathfrak{p}' \cap \mathscr{F}\{y_1, \ldots, y_n\} = \mathfrak{p}$, and $\omega_{\mathfrak{p}'} = \omega_{\mathfrak{p}}$.*

(c) *There exist, independent of $\mathscr{G}$, an irreducible polynomial P with coefficients in $\mathscr{F}$ and with some partial derivative not equal to 0, and a differential polynomial $H \in \mathscr{F}\{y_1, \ldots, y_n\}$ with $H \notin \mathfrak{p}$, such that for each extension $\mathscr{G}$ of $\mathscr{F}$, the number of components of $\mathscr{G}\mathfrak{p}$ equals the number of irreducible factors into which P splits over $\mathscr{G}$, and the sum of any two distinct components of $\mathscr{G}\mathfrak{p}$ contains H.*

Proof For each $s \in \mathbf{N}$ let

$$A_s = \mathscr{F}[(\theta y_j)_{\theta \in \Theta(s),\, 1 \leqslant j \leqslant n}], \qquad B_s = \mathscr{G}[(\theta y_j)_{\theta \in \Theta(s),\, 1 \leqslant j \leqslant n}].$$

It is obvious that $\mathscr{G}\mathfrak{p}$ is a differential ideal. Hence (a) follows from Chapter 0, Section 12, Proposition 7.

By Section 4, Corollary 5 to Theorem 1, and by Chapter 0, Section 9, Theorem 1, $\mathscr{G}\mathfrak{p}$ has finitely many components $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, these are $\mathscr{G}$-separable prime differential ideals, and $\mathscr{G}\mathfrak{p} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. It is an easy consequence of Chapter 0, Section 10, Lemma 9, that $\mathscr{G} \cdot (\mathfrak{p} \cap A_s) = (\mathscr{G}\mathfrak{p}) \cap B_s$, so that

$$\mathscr{G} \cdot (\mathfrak{p} \cap A_s) = (\mathfrak{p}_1 \cap B_s) \cap \cdots \cap (\mathfrak{p}_r \cap B_s).$$

No one of the ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ contains any other so that if s is sufficiently big, no one of the prime ideals $\mathfrak{p}_1 \cap B_s, \ldots, \mathfrak{p}_r \cap B_s$ contains another, and hence these must be the components of $\mathscr{G} \cdot (\mathfrak{p} \cap A_s)$. By Chapter 0, Section 12, Proposition 7, then $(\mathfrak{p}_i \cap B_s) \cap A_s = \mathfrak{p} \cap A_s$, whence $\mathfrak{p}_i \cap \mathscr{F}\{y_1, \ldots, y_n\} = \mathfrak{p}$, and $\dim(\mathfrak{p}_i \cap B_s) = \dim(\mathfrak{p} \cap A_s)$. For sufficiently big values of s the last equation is equivalent to the equation $\omega_{\mathfrak{p}_i}(s) = \omega_{\mathfrak{p}}(s)$, so that $\omega_{\mathfrak{p}_i} = \omega_{\mathfrak{p}}$. This proves (b).

To prove (c) let $s(\mathscr{G})$ denote the smallest natural number such that no one of the ideals $\mathfrak{p}_1 \cap B_{s(\mathscr{G})}, \ldots, \mathfrak{p}_r \cap B_{s(\mathscr{G})}$ contains any other. Then for every $s \geqslant s(\mathscr{G})$, the ideals $\mathfrak{p}_1 \cap B_s, \ldots, \mathfrak{p}_r \cap B_s$ are the components of $\mathscr{G} \cdot (\mathfrak{p} \cap A_s)$. We shall show below that $s(\mathscr{G})$ is an increasing function of $\mathscr{G}$, that is, whenever $\mathscr{G}$ and $\mathscr{H}$ are extensions of $\mathscr{F}$ with $\mathscr{G} \subset \mathscr{H}$, then $s(\mathscr{G}) \leqslant s(\mathscr{H})$. Assuming this result, let us see how we can prove (c). We may suppose that $\mathfrak{p} \neq (0)$, for otherwise (c) becomes trivial. By Chapter 0, Section 12, Proposition 7(b), for each $s \in \mathbf{N}$ there exists an irreducible polynomial P_s with coefficients in $\mathscr{F}$ and with some partial derivative not equal to 0, such that for any $\mathscr{G}$ the number of components of $\mathscr{G} \cdot (\mathfrak{p} \cap A_s)$ equals the number of irreducible factors into which P_s splits over $\mathscr{G}$. Let $\mathscr{F}_0$ denote the separable closure of $\mathscr{F}$ *in* $\mathscr{G}$, let $\mathscr{F}'$ denote a separable closure of $\mathscr{F}_0$ (and therefore of $\mathscr{F}$), and set $P = P_{s(\mathscr{F}')}$. If $\mathfrak{p}_{01}, \ldots, \mathfrak{p}_{0q}$ denote the components of $\mathscr{F}_0\,\mathfrak{p}$, then $\mathscr{G}\mathfrak{p}_{01}, \ldots, \mathscr{G}\mathfrak{p}_{0q}$ are prime (by Chapter 0, Section 12, Proposition 7(c)). As no $\mathscr{G}\mathfrak{p}_{0i}$ contains any other (by Chapter 0, Section 10, Lemma 9), they are the components of $\mathscr{G}\mathfrak{p}$. Thus, the number of components of $\mathscr{G}\mathfrak{p}$ equals that of $\mathscr{F}_0\,\mathfrak{p}$. By the result we are assuming $s(\mathscr{F}_0) \leqslant s(\mathscr{F}')$, so that the number of components of $\mathscr{F}_0\,\mathfrak{p}$ equals that of $\mathscr{F}_0 \cdot (\mathfrak{p} \cap A_{s(\mathscr{F}')})$, which by the above equals the number of irreducible factors of P over $\mathscr{F}_0$. However, by Chapter 0, Section 12, Lemma 12, this last number equals the number of irreducible factors of P over $\mathscr{G}$. Finally, by Chapter 0, Section 12, Proposition 7(b), there exists an $H \in A_{s(\mathscr{F}')}$ with $H \notin \mathfrak{p} \cap A_{s(\mathscr{F}')}$ such that H is contained in the sum of any two distinct components of $\mathscr{G} \cdot (\mathfrak{p} \cap A_{s(\mathscr{F}')})$. However, the components of $\mathscr{G} \cdot (\mathfrak{p} \cap A_{s(\mathscr{F}')})$ are the intersections with $B_{s(\mathscr{F}')}$ of the components of $\mathscr{G}\mathfrak{p}$. Thus, $H \notin \mathfrak{p}$ and H is contained in the sum of any two distinct components of $\mathscr{G}\mathfrak{p}$.

We now show that $s(\mathscr{G})$ is an increasing function of $\mathscr{G}$. Let $\mathscr{H}$ be an extension of $\mathscr{G}$ and set $C_s = \mathscr{H}[(\theta y_j)_{\theta \in \Theta(s),\, 1 \leqslant j \leqslant n}]$. Then $\mathscr{H}\mathfrak{p} = \mathscr{H} \cdot \mathscr{G}\mathfrak{p} = \mathscr{H} \cdot (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r) = \mathscr{H}\mathfrak{p}_1 \cap \cdots \cap \mathscr{H}\mathfrak{p}_r$. If $i \neq i'$, then a component $\mathfrak{q}$ of $\mathscr{H}\mathfrak{p}_i$ cannot be contained in a component $\mathfrak{q}'$ of $\mathscr{H}\mathfrak{p}_{i'}$, for otherwise we should have $\mathfrak{p}_i = \mathfrak{q} \cap \mathscr{G}\{y_1, \ldots, y_n\} \subset \mathfrak{q}' \cap \mathscr{G}\{y_1, \ldots, y_n\} = \mathfrak{p}_{i'}$. Therefore if $\mathfrak{p}_{i1}, \ldots, \mathfrak{p}_{iq_i}$ denote the components of $\mathscr{H}\mathfrak{p}_i$ $(1 \leqslant i \leqslant r)$, then the ideals $\mathfrak{p}_{ij}$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant q_i)$ are the components of $\mathscr{H}\mathfrak{p}$. If $s < s(\mathscr{G})$, there exist indices i, i' with $i \neq i'$ such that $\mathfrak{p}_i \cap B_s \subset \mathfrak{p}_{i'} \cap B_s$. For these i, i' we have

$$\begin{aligned}(\mathfrak{p}_{i1} \cap C_s) \cap \cdots \cap (\mathfrak{p}_{iq_i} \cap C_s) &= \mathscr{H}\mathfrak{p}_i \cap C_s = \mathscr{H} \cdot (\mathfrak{p}_i \cap B_s) \subset \mathscr{H} \cdot (\mathfrak{p}_{i'} \cap B_s) \\ &= \mathscr{H}\mathfrak{p}_{i'} \cap C_s \subset \mathfrak{p}_{i'1} \cap C_s,\end{aligned}$$

so that, for some j, $\mathfrak{p}_{ij} \cap C_s \subset \mathfrak{p}_{i'1} \cap C_s$ whence $s < s(\mathscr{H})$. Thus, whenever $s < s(\mathscr{G})$, then $s < s(\mathscr{H})$, so that $s(\mathscr{G}) \leqslant s(\mathscr{H})$.

Corollary *Let $(y_i)_{i \in I}$ be a family of differential indeterminates, let $(I_\lambda)_{\lambda \in \Lambda}$ be a partition of I, for each $\lambda \in \Lambda$ let $\mathfrak{p}_\lambda$ be an $\mathscr{F}$-regular differential ideal of $\mathscr{F}\{(y_i)_{i \in I_\lambda}\}$, and let $\mathfrak{r}$ be the ideal of $\mathscr{F}\{(y_i)_{i \in I}\}$ generated by $\bigcup_{\lambda \in \Lambda} \mathfrak{p}_\lambda$. Then $\mathfrak{r}$ is an $\mathscr{F}$-regular differential ideal with $\mathfrak{r} \cap \mathscr{F}\{(y_{i \in I_\lambda}\} = \mathfrak{p}_\lambda$ $(\lambda \in \Lambda)$. If I is finite, then $\omega_{\mathfrak{r}} = \sum_{\lambda \in \Lambda} \omega_{\mathfrak{p}_\lambda}$.*

Proof It is obvious that $\mathfrak{r}$ is a differential ideal. By Chapter 0, Section 12, Corollary 2 to Proposition 7, $\mathfrak{r}$ is $\mathscr{F}$-regular and $\mathfrak{r} \cap \mathscr{F}\{(y_i)_{i \in I_\lambda}\} = \mathfrak{p}_\lambda$ $(\lambda \in \Lambda)$. For the final part we may suppose that Λ consists of two elements, say the numbers 1 and 2. Let $\bar{y} = (\bar{y}_i)_{i \in I}$ denote the image of $y = (y_i)_{i \in I}$ under the canonical homomorphism $\mathscr{F}\{y\} \to \mathscr{F}\{y\}/\mathfrak{r}$, so that $\bar{y}_i \in Q(\mathscr{F}\{y\}/\mathfrak{r})$ and $\mathfrak{r}$ is the defining differential ideal of $\bar{y}$ over $\mathscr{F}$. Setting $\bar{y}' = (\bar{y}_i)_{i \in I_1}$ and $\bar{y}'' = (\bar{y}_i)_{i \in I_2}$, we see that $\mathfrak{r} \cap \mathscr{F}\{(y_i)_{i \in I_1}\} = \mathfrak{p}_1$ is the defining differential ideal of $\bar{y}'$ over $\mathscr{F}$ and $\mathfrak{p}_2$ is that of $\bar{y}''$ over $\mathscr{F}$. Also, the defining differential ideal of $\bar{y}''$ *over* $\mathscr{F}\langle\bar{y}'\rangle$ is the $\mathscr{F}\langle\bar{y}'\rangle$-regular ideal $\mathscr{F}\langle\bar{y}'\rangle\mathfrak{p}_2$. Now,

$$\begin{aligned}\operatorname{tr}\deg \mathscr{F}((\theta\bar{y}_i)_{\theta \in \Theta(s),\, i \in I})/\mathscr{F} &= \operatorname{tr}\deg \mathscr{F}((\theta\bar{y}_i)_{\theta \in \Theta(s),\, i \in I_1})/\mathscr{F} \\ &\quad + \operatorname{tr}\deg \mathscr{F}((\theta\bar{y}_i)_{\theta \in \Theta(s),\, i \in I})/\mathscr{F}((\theta\bar{y}_i)_{\theta \in \Theta(s),\, i \in I_1}).\end{aligned}$$

For big values of $s \in \mathbf{N}$ the first term of the second member here equals $\omega_{\mathfrak{p}_1}(s)$, whereas the second term is less than or equal to $\operatorname{tr}\deg \mathscr{F}((\theta\bar{y}_i)_{\theta \in \Theta(s),\, i \in I_2})/\mathscr{F} = \omega_{\mathfrak{p}_2}(s)$ and is greater than or equal to $\operatorname{tr}\deg \mathscr{F}\langle\bar{y}'\rangle((\theta\bar{y}_i)_{\theta \in \Theta(s),\, i \in I_2})/\mathscr{F}\langle\bar{y}'\rangle = \omega_{\mathscr{F}\langle\bar{y}'\rangle\mathfrak{p}_2}(s) = \omega_{\mathfrak{p}_2}(s)$. Hence $\omega_{\mathfrak{r}}(s) = \omega_{\mathfrak{p}_1}(s) + \omega_{\mathfrak{p}_2}(s)$.

7 Universal extensions

Let $\mathscr{U}$ be an extension of $\mathscr{F}$. We shall say that $\mathscr{U}$ is *universal* over $\mathscr{F}$, or that $\mathscr{U}$ is a *universal extension of* $\mathscr{F}$, if $\mathscr{U}$ is semiuniversal (see Chapter II, Section 2, especially the Corollary to Proposition 4) over every finitely generated extension of $\mathscr{F}$ in $\mathscr{U}$. By a *universal differential field* we shall mean a differential field that is universal over its prime field.

Our results on universal extensions depend on the following lemma on semiuniversal extensions.

Lemma 2 *Let $\mathscr{F}$, $\mathscr{F}'$, $\mathscr{S}'$, $\mathscr{S}$ be differential fields with $\mathscr{F} \subset \mathscr{F}' \subset \mathscr{S}' \subset \mathscr{S}$. If $\mathscr{S}'$ is semiuniversal over $\mathscr{F}'$, then $\mathscr{S}$ is semiuniversal over $\mathscr{F}$.*

Proof Let $\mathfrak{p}$ be any $\mathscr{F}$-separable prime differential ideal of $\mathscr{F}\{y_1, \ldots, y_n\}$. By Section 6, Proposition 3, $\mathscr{F}'\mathfrak{p}$ is an $\mathscr{F}'$-separable differential ideal, and has a component $\mathfrak{p}'$ that is an $\mathscr{F}'$-separable prime differential ideal with $\mathfrak{p}' \cap \mathscr{F}\{y_1, \ldots, y_n\} = \mathfrak{p}$. Because $\mathscr{S}'$ is semiuniversal over $\mathscr{F}'$, there exist elements $\eta_1, \ldots, \eta_n \in \mathscr{S}' \subset \mathscr{S}$ such that $\mathfrak{p}'$ is the defining differential ideal of $(\eta_1, \ldots, \eta_n)$ in $\mathscr{F}'\{y_1, \ldots, y_n\}$. Then $\mathfrak{p} = \mathfrak{p}' \cap \mathscr{F}\{y_1, \ldots, y_n\}$ is the defining differential ideal of $(\eta_1, \ldots, \eta_n)$ in $\mathscr{F}\{y_1, \ldots, y_n\}$. Thus, $\mathscr{S}$ is semiuniversal over $\mathscr{F}$.

Proposition 4 *Let $\mathscr{F}$, $\mathscr{F}'$, $\mathscr{U}$ be differential fields with $\mathscr{F} \subset \mathscr{F}' \subset \mathscr{U}$.*

(a) *If $\mathscr{U}$ is universal over $\mathscr{F}'$, then $\mathscr{U}$ is universal over $\mathscr{F}$.*

(b) *If $\mathscr{U}$ is universal over $\mathscr{F}$, and $\mathscr{F}'$ is finitely generated over $\mathscr{F}$, then $\mathscr{U}$ is universal over $\mathscr{F}'$.*

Proof (a) Let $\mathscr{F}_1$ be a finitely generated extension of $\mathscr{F}$ in $\mathscr{U}$. We must show that $\mathscr{U}$ is semiuniversal over $\mathscr{F}_1$. Now, $\mathscr{F}'\mathscr{F}_1$ is finitely generated over $\mathscr{F}'$. Since $\mathscr{U}$ is universal over $\mathscr{F}'$, then $\mathscr{U}$ is semiuniversal over $\mathscr{F}'\mathscr{F}_1$. By Lemma 2 then $\mathscr{U}$ is semiuniversal over $\mathscr{F}_1$.

(b) If $\mathscr{F}_1'$ is a finitely generated extension of $\mathscr{F}'$ in $\mathscr{U}$, then $\mathscr{F}_1'$ is also finitely generated over $\mathscr{F}$, so that $\mathscr{U}$ is semiuniversal over $\mathscr{F}_1'$.

We have the following existence theorem for universal extensions.

Theorem 2 *Every differential field has a separable universal extension.*

Proof Let $\mathscr{F}$ be the differential field. By Chapter II, Section 2, Corollary to Proposition 4, there exists an infinite sequence $(\mathscr{S}_k)_{k\in\mathbf{N}}$ of differential fields such that $\mathscr{S}_0 = \mathscr{F}$ and $\mathscr{S}_{k+1}$ is a separable semiuniversal extension of $\mathscr{S}_k$ $(k \in \mathbf{N})$. Then $\mathscr{U} = \bigcup_{k\in\mathbf{N}} \mathscr{S}_k$ has a unique differential field structure for which $\mathscr{U}$ is an extension of every $\mathscr{S}_k$. It is obvious that $\mathscr{U}$ is separable over $\mathscr{F}$. Let $\mathscr{F}_1$ be any finitely generated extension of $\mathscr{F}$ in $\mathscr{U}$. There exists a $k \in \mathbf{N}$ such that $\mathscr{F}_1 \subset \mathscr{S}_k$. Then $\mathscr{F}_1 \subset \mathscr{S}_k \subset \mathscr{S}_{k+1} \subset \mathscr{U}$ and $\mathscr{S}_{k+1}$ is semiuniversal over $\mathscr{S}_k$. By Lemma 2 it follows that $\mathscr{U}$ is semiuniversal over $\mathscr{F}_1$. Thus, $\mathscr{U}$ is a separable universal extension of $\mathscr{F}$.

EXERCISES

1. Let $\mathscr{U}$ be a universal extension of $\mathscr{F}$.
   (a) Show that $\mathscr{U}$ is separably closed.
   (b) Show that if $\mathscr{F}'$ is an algebraic extension of $\mathscr{F}$ in $\mathscr{U}$, then $\mathscr{U}$ is universal over $\mathscr{F}'$.
2. Let $\mathscr{U}$ be a universal extension of $\mathscr{F}$. Let $\mathscr{F}_n$ $(n \in \mathbf{N})$ and $\mathscr{G}$ be differential fields such that $\mathscr{F}_0 = \mathscr{F}$, $\mathscr{F}_{n+1}$ is a finitely generated separable extension of $\mathscr{F}_n$ $(n \in \mathbf{N})$, and $\mathscr{G} = \bigcup_{n\in\mathbf{N}} \mathscr{F}_n$. Prove that there exists an $\mathscr{F}$-homomorphism $\mathscr{G} \to \mathscr{U}$.
3. Let $\mathscr{U}$ be a universal extension of $\mathscr{F}$, let $\mathscr{H}$ be a finitely generated extension of $\mathscr{F}$ in $\mathscr{U}$, and let $\mathscr{G}$ be a finitely generated separable extension of $\mathscr{F}$. Show that there exists in $\mathscr{U}$ an extension $\mathscr{G}'$ of $\mathscr{F}$ that is $\mathscr{F}$-isomorphic to $\mathscr{G}$ such that the compositum $\mathscr{H}\mathscr{G}'$ is a finitely generated separable extension of $\mathscr{H}$. (*Hint:* Write $\mathscr{G} = \mathscr{F}\langle\eta_1, \ldots, \eta_n\rangle$, let $\mathfrak{p}$ be the defining differential ideal of $(\eta_1, \ldots, \eta_n)$ over $\mathscr{F}$, and consider $\mathscr{H}\mathfrak{p}$.)
4. Let $\mathscr{U}$ be a universal extension of $\mathscr{F}$, let $\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_n$ be differential subfields of $\mathscr{U}$ such that $\mathscr{F}_0 = \mathscr{F}$ and $\mathscr{F}_j$ is a finitely generated extension of $\mathscr{F}_{j-1}$ $(1 \leqslant j \leqslant n)$, and let $\mathscr{G}_j$ be a finitely generated separable extension of $\mathscr{F}_j$ $(0 \leqslant j \leqslant n)$. Show that there exist differential subfields $\mathscr{G}_0', \mathscr{G}_1', \ldots, \mathscr{G}_n'$ of $\mathscr{U}$ such that $\mathscr{G}_j'$ is an extension of $\mathscr{F}_j$ that is $\mathscr{F}_j$-isomorphic to $\mathscr{G}_j$

$(0 \leqslant j \leqslant n)$ and the compositum $\mathcal{G}_0'\mathcal{G}_1' \cdots \mathcal{G}_n'$ is a finitely generated separable extension of $\mathcal{F}_n$. (*Hint:* Use induction and Exercise 3.)

5. Prove: *If $\mathcal{F}$ is denumerable, then $\mathcal{F}$ has a denumerable separable universal extension $\mathcal{U}_*$ such that every universal extension of $\mathcal{F}$ contains an $\mathcal{F}$-isomorphic image of $\mathcal{U}_*$.* Outline of proof: (a) Let $\mathcal{U}$ be a universal extension of $\mathcal{F}$ (Theorem 2). Show that for every finitely generated separable extension $\mathcal{G}$ of $\mathcal{F}$ in $\mathcal{U}$ there exists an infinite sequence $(\mathcal{E}_n(\mathcal{G}))_{n \in \mathbf{N}}$ of differential subfields of $\mathcal{U}$ such that each $\mathcal{E}_n(\mathcal{G})$ is a finitely generated separable extension of $\mathcal{G}$ and every finitely generated separable extension of $\mathcal{G}$ is $\mathcal{G}$-isomorphic to some $\mathcal{E}_n(\mathcal{G})$. (*Hint:* Use Proposition 4(b), and see Section 4, Exercise 1.) (b) Show that there exists an infinite sequence $(\mathcal{F}_n)_{n \in \mathbf{N}}$ of differential subfields of $\mathcal{U}$ such that $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_{n+1}$ is a finitely generated separable extension of $\mathcal{F}_n$, and $\mathcal{F}_{n+1}$ contains an $\mathcal{F}_j$-isomorphic image of $\mathcal{E}_{n-j}(\mathcal{F}_j)$ $(0 \leqslant j \leqslant n)$. (*Hint:* Define the sequence inductively, using Exercise 3.) (c) Let $\mathcal{U}_* = \bigcup_{n \in \mathbf{N}} \mathcal{F}_n$, and show that $\mathcal{U}_*$ is denumerable, and is separable and universal over $\mathcal{F}$. (d) Show that if $\mathcal{U}'$ is any universal extension of $\mathcal{F}$, then there exists an $\mathcal{F}$-homomorphism $\mathcal{U}_* \to \mathcal{U}'$. (*Hint:* Use Exercise 2.)
6. Let p be either 0 or a prime number. Show: *There exists a denumerable universal differential field $\mathcal{U}_p$ of characteristic p such every universal differential field of characteristic p contains an isomorphic image of $\mathcal{U}_p$.* This is a special case of the result in Exercise 5.
7. Prove: *If $\mathcal{F}$ has characteristic 0, then two universal extensions of $\mathcal{F}$ always contain universal extensions of $\mathcal{F}$ that are isomorphic to each other over $\mathcal{F}$.* (*Hint:* Let $\mathcal{U}$ and $\mathcal{U}'$ be universal extensions of $\mathcal{F}$. The set of all mappings each of which is an $\mathcal{F}$-isomorphism of an extension of $\mathcal{F}$ in $\mathcal{U}$ onto an extension of $\mathcal{F}$ in $\mathcal{U}'$ can be ordered "by extension," and when so ordered has a maximal element f (Zorn's lemma). Let $\mathcal{V}$ be the domain and $\mathcal{V}'$ be the image of f, and show that $\mathcal{V}$ and $\mathcal{V}'$ are universal over $\mathcal{F}$.)

8 $\mathfrak{k}$-Coherent autoreduced sets

The purpose of the present section is to lay the groundwork for the proof in the following section of analogs for differential integral domains of some of the results on specializations described in Chapter 0, Section 14.

Throughout this section $\mathcal{S} = \mathcal{R}\{y_1, \ldots, y_n\}$ denotes a finitely generated differential polynomial algebra over $\mathcal{R}$.

Let A be an autoreduced set in $\mathcal{S}$ relative to some fixed ranking, and let $\mathfrak{k}$ be an ideal (not necessarily a differential one) of $\mathcal{S}$. We shall say that the autoreduced set A is $\mathfrak{k}$-*coherent* if the following three conditions are satisfied.

C1 *The ideal* $\mathfrak{k}$ *has a set of generators that are partially reduced with respect to* A.

C2 $[\mathfrak{k}] \subset ([\mathrm{A}]+\mathfrak{k}):H_{\mathrm{A}}^{\infty}$.

C3 *Whenever* A, $A' \in \mathrm{A}$ *and* v *is a common derivative of* u_A, $u_{A'}$, *say* $v = \theta u_A = \theta' u_{A'}$, *then* $S_{A'}\theta A - S_A \theta' A' \in ((\mathrm{A}_v)+\mathfrak{k}):H_{\mathrm{A}}^{\infty}$, *where* A_v *denotes the set of all differential polynomials* τB *with* $B \in \mathrm{A}$, $\tau \in \Theta$, *and* τu_B *of lower rank than* v.

REMARK This notion extends one previously introduced by Rosenfeld [105] for the same purpose. Limiting himself to the case in which $\mathscr{R} = \mathscr{F}$ and $p = 0$, he called an autoreduced set *coherent* when it is (0)-coherent in the present sense. For this case this more special notion suffices.

Lemma 3 *Let* $\mathfrak{p}$ *be a prime differential ideal of* $\mathscr{S}$ *that is quasi-separable over* $\mathscr{R}$, *let there be given a sequential ranking of* $(y_1, \ldots, y_n)$, *and let* A *be a characteristic set of* $\mathfrak{p}$. *Then there exists a finite set* Y *of derivatives of the* y_j, *each partially reduced with respect to* A, *such that if we set* $\mathfrak{p}_1 = \mathfrak{p} \cap \mathscr{R}[Y]$, *then* A *is* $\mathscr{S}\mathfrak{p}_1$*-coherent and* $\mathfrak{p} = ([\mathrm{A}]+\mathscr{S}\mathfrak{p}_1):H_{\mathrm{A}}^{\infty}$. *The set* Y *may be replaced by any larger finite set of derivatives of the* y_j *partially reduced with respect to* A.

REMARK If $\mathscr{S}/\mathfrak{p}$ is of characteristic 0, then the ranking need not be sequential, and we may take $Y = \varnothing$. This is evident from the proof and Section 2, the Remark following Lemma 1.

Proof For each $A \in \mathrm{A}$ we have $S_A \notin \mathfrak{p}$ (by definition of characteristic set) and $I_A \notin \mathfrak{p}$ (by Chapter I, Section 10, Lemma 8); therefore $H_{\mathrm{A}} \notin \mathfrak{p}$. By Lemma 1 we may choose a finite set Y of derivatives of the y_j, all partially reduced with respect to A, so that every element of $\mathfrak{p}$ that is reduced with respect to A is in $\mathscr{S}\mathfrak{p}_1$, where $\mathfrak{p}_1 = \mathfrak{p} \cap \mathscr{R}[Y]$; obviously any larger Y will do. Since the remainder with respect to A of any element of $\mathfrak{p}$ is reduced with respect to A (see Chapter I, Section 9, Proposition 1), we conclude that $\mathfrak{p} \subset ([\mathrm{A}]+\mathscr{S}\mathfrak{p}_1):H_{\mathrm{A}}^{\infty}$ and that the condition C3 (with $k = \mathscr{S}\mathfrak{p}_1$) is satisfield. As the inclusion $\mathfrak{p} \supset ([\mathrm{A}]+\mathscr{S}\mathfrak{p}_1):H_{\mathrm{A}}^{\infty}$ is obvious, the lemma follows.

Lemma 4 *Let* A *be a* $\mathfrak{k}$*-coherent autoreduced set in* $\mathscr{S}$, *and let* $f: \mathscr{R} \to \mathscr{R}'$ *be a differential ring homomorphism with* $H_{\mathrm{A}}{}^f \neq 0$. *Then* A^f *is a* $\mathfrak{k}^f$*-coherent autoreduced set in* $\mathscr{S}^f = f(\mathscr{R})\{y_1, \ldots, y_n\}$.

Proof Since $H_{\mathrm{A}}{}^f \neq 0$, A^f is an autoreduced set in $\mathscr{S}^f$ with $H_{\mathrm{A}^f} = H_{\mathrm{A}}{}^f$; as the homomorphism $G \mapsto G^f$ of $\mathscr{S}$ into $\mathscr{S}^f$ obviously preserves the conditions C1–C3, A^f is $\mathfrak{k}^f$-coherent.

Lemma 5 *Let* A *be a* $\mathfrak{k}$*-coherent autoreduced set in* $\mathscr{S}$, *and suppose that for each* $A \in$ A *the separant* S_A *is not a divisor of zero in* $\mathscr{S}$. *Then every element of* $([\mathrm{A}]+\mathfrak{k}):H_{\mathrm{A}}{}^{\infty}$ *that is partially reduced with respect to* A *is in* $((\mathrm{A})+\mathfrak{k}):H_{\mathrm{A}}{}^{\infty}$.

Proof Let $G \in ([\mathrm{A}]+\mathfrak{k}):H_{\mathrm{A}}{}^{\infty}$ be partially reduced with respect to A. We must show that $G \in ((\mathrm{A})+\mathfrak{k}):H_{\mathrm{A}}{}^{\infty}$. We may write

$$H_{\mathrm{A}}{}^{h}G = \sum_{1 \leqslant i \leqslant r} C_i \theta_i A_i + \sum_{1 \leqslant j \leqslant s} D_j K_j, \tag{1}$$

where $C_i \in \mathscr{S}$, $\theta_i \in \Theta$ and $\operatorname{ord}\theta_i > 0$, $A_i \in$ A, $D_j \in \mathscr{S}$, $K_j \in (\mathrm{A})+\mathfrak{k}$, and K_j is partially reduced with respect to A. If there exists for G an equation (1) with $r = 0$, then certainly $G \in ((\mathrm{A})+\mathfrak{k}):H_{\mathrm{A}}{}^{\infty}$. We assume that there does not exist for G an equation (1) with $r = 0$, and seek a contradiction.

Let v be the element of highest rank in the set consisting of $\theta_1 u_{A_1}, \ldots, \theta_r u_{A_r}$, and suppose that among all possible equations (1) for G ours is one for which v has lowest rank. Choose the notation so that $\theta_i u_{A_i}$ is lower than v for $1 \leqslant i < q$ and $\theta_i u_{A_i} = v$ for $q \leqslant i \leqslant r$. Multiplying both sides of (1) by S_{A_r} we may then write

$$\begin{aligned} S_{A_r} H_{\mathrm{A}}{}^{h} G = {} & \sum_{1 \leqslant i < q} S_{A_r} C_i \theta_i A_i + \sum_{1 \leqslant j \leqslant s} S_{A_r} D_j K_j \\ & + \sum_{q \leqslant i \leqslant r} C_i (S_{A_r} \theta_i A_i - S_{A_i} \theta_r A_r) + \sum_{q \leqslant i \leqslant r} C_i S_{A_i} \theta_r A_r. \end{aligned}$$

From this equation, condition C3, and the fact that H_{A} is a multiple of S_{A_r}, we obtain

$$H_{\mathrm{A}}{}^{k}G = \sum_{1 \leqslant i \leqslant r'} C_i' \theta_i' A_i' + \sum_{1 \leqslant j \leqslant s'} D_j' K_j' + E\theta_r A_r, \tag{2}$$

where $C_i' \in \mathscr{S}$, $\theta_i' \in \Theta$ and $\operatorname{ord}\theta_i' > 0$, $A_i' \in$ A, $\theta_i' u_{A_i'}$ is lower than v, $D_j' \in \mathscr{S}$, $K_j' \in (\mathrm{A})+\mathfrak{k}$, K_j' is partially reduced with respect to A, and $E \in \mathscr{S}$. By Chapter I, Section 8, Lemma 5, we may write $\theta_r A_r = S_{A_r} v + T$, where $T \in \mathscr{S}$ and T has lower rank than v (and therefore is free of v, as is S_{A_r}). Since S_{A_r} is not a divisor of 0, $\mathscr{S}$ may be isomorphically embedded in the ring of quotients $\Sigma^{-1}S$, where Σ is the set of all powers $S_{A_r}^{l}$ ($l \in \mathbf{N}$). Substituting $-T/S_{A_r}$ for v in (2), and then multiplying by a suitable power of H_{A}, we obtain an equation of the same form as (1) in which either r is replaced by 0 or else v is replaced by a derivative of a y_j of lower rank than v. This contradiction completes the proof.

Lemma 6 *Let* A *be a* $\mathfrak{k}$*-coherent autoreduced set in* $\mathscr{S}$, *and suppose that for each* $A \in$ A *the separant* S_A *is not a divisor of zero in* $\mathscr{S}$. *Then* $([\mathrm{A}]+\mathfrak{k}):H_{\mathrm{A}}{}^{\infty}$ *is a differential ideal of* $\mathscr{S}$, *and is prime, respectively perfect, respectively* $\mathscr{R}$*-separable if* $((\mathrm{A})+\mathfrak{k}):H_{\mathrm{A}}{}^{\infty}$ *is prime, respectively perfect, respectively* $\mathscr{R}$*-separable.*

Proof Set $\mathfrak{a} = ([A]+\mathfrak{k}):H_A^\infty$ and $\mathfrak{a}_0 = ((A)+\mathfrak{k}):H_A^\infty$. By C2, $\mathfrak{a} = ([A]+[\mathfrak{k}]):H_A^\infty$. Hence (see Chapter I, Section 2, Corollary to Lemma 1) $\mathfrak{a}$ is a differential ideal, and $\mathfrak{a}:H_A^\infty = \mathfrak{a}$.

Let $F, G \in \mathcal{S}$, $FG \in \mathfrak{a}$. Denoting the remainder with respect to A of F, respectively G, by F_0, respectively G_0, we know that $F_0 G_0 \in \mathfrak{a}$ and $F_0 G_0$ is partially reduced with respect to A, so that (by Lemma 5) $F_0 G_0 \in \mathfrak{a}_0$. Hence, if $\mathfrak{a}_0$ is prime, then F_0 or G_0 is in $\mathfrak{a}_0$, so that F or G is in $\mathfrak{a}$, and therefore $\mathfrak{a}$ is prime (an even easier argument showing that $1 \notin \mathfrak{a}$). A similar proof (starting with $F^2 \in \mathfrak{a}$ instead of $FG \in \mathfrak{a}$) shows that if $\mathfrak{a}_0$ is perfect, then so is $\mathfrak{a}$.

Suppose finally that $\mathfrak{a}_0$ is $\mathcal{R}$-separable. We must show that $\mathfrak{a}$ is $\mathcal{R}$-separable, and we may evidently suppose that $\mathfrak{a} \neq \mathcal{S}$. By what we have already proved, $\mathfrak{a}$ is perfect. If $a \in \mathcal{R}$, $a \notin \mathfrak{a}$, $B \in \mathcal{S}$, $B \notin \mathfrak{a}$, let B_0 denote the remainder of B with respect to A, so that B_0 is reduced with respect to A and $B_0 \notin \mathfrak{a}_0$. Since $\mathfrak{a}_0$ is $\mathcal{R}$-separable we infer that $aB_0 \notin \mathfrak{a}_0$ and therefore (by Lemma 5) that $aB_0 \notin \mathfrak{a}$, so that $aB \notin \mathfrak{a}$. To complete the proof we may suppose that the characteristic of $\mathcal{R}/(\mathfrak{a} \cap \mathcal{R})$ is $p \neq 0$. We must then show that $\mathcal{S}^p$ and $\mathcal{R}$ are linearly disjoint (mod $\mathfrak{a}$) over $\mathcal{R}^p$. To this end let (c_i) be a family of elements of $\mathcal{R}$ linearly dependent (mod $\mathfrak{a}$) over $\mathcal{S}^p$. Then there exist elements $D_i \in \mathcal{S}$, not all in $\mathfrak{a}$, such that $\sum D_i^p c_i \in \mathfrak{a}$. By Chapter I, Section 9, Corollary to Lemma 6, there are an exponent e and differential polynomials $E_i \in \mathcal{S}$ partially reduced with respect to A such that $H_A^e D_i \equiv E_i$ (mod $[A]$) for every i. Not every E_i is in $\mathfrak{a}$ and $\sum E_i^p c_i \in \mathfrak{a}$, so that not every E_i is in $\mathfrak{a}_0$; by Lemma 5, $\sum E_i^p c_i \in \mathfrak{a}_0$. Since $\mathfrak{a}_0$ is $\mathcal{R}$-separable this implies that there exist elements $a_i \in \mathcal{R}$ not all in $\mathfrak{a}_0$ such that $\sum a_i^p c_i \in \mathfrak{a}_0$. Hence (again by Lemma 5) the elements a_i are not all in $\mathfrak{a}$, and $\sum a_i^p c_i \in \mathfrak{a}$. This shows that $\mathcal{S}^p$ and $\mathcal{R}$ are linearly disjoint (mod $\mathfrak{a}$) over $\mathcal{R}^p$, and completes the proof.

EXERCISE

1. Let the hypothesis and notation be the same as in Lemmas 5 and 6. In addition, suppose that $\mathfrak{k}$ has a set of generators that are reduced with respect to A and that each element of A is of degree 1 in its leader.
   (a) Prove that every element of $([A]+\mathfrak{k}):H_A^\infty$ that is reduced with respect to A is in $\mathfrak{k}:H_A^\infty$.
   (b) Prove that $([A]+\mathfrak{k}):H_A^\infty$ is prime, respectively perfect, respectively $\mathcal{R}$-separable if $\mathfrak{k}:H_A^\infty$ is prime, respectively perfect, respectively $\mathcal{R}$-separable.

9 Differential specializations

We suppose in this section that $\mathcal{R}$ is a differential integral domain.

A homomorphism of $\mathcal{R}$ into a differential field $\mathcal{G}$ is called a *differential* *specialization* of $\mathscr{R}$ into $\mathscr{G}$. If $\mathscr{R}$ and $\mathscr{G}$ happen to have a common differential subring $\mathscr{R}_0$ and the homomorphism leaves invariant each element of $\mathscr{R}_0$, the differential specialization is said to be *over* $\mathscr{R}_0$.

Let $\xi = (\xi_i)_{i \in I}$ and $\xi' = (\xi_i')_{i \in I}$ be families of elements of $\mathscr{R}$ and $\mathscr{G}$, respectively. If there exists a differential specialization $f: \mathscr{R}_0\{\xi\} \to \mathscr{G}$ over $\mathscr{R}_0$ mapping ξ onto ξ' (that is, having the property that $f(\xi_i) = \xi_i'$ for every $i \in I$), we say that *ξ' is a differential specialization of ξ over $\mathscr{R}_0$*; when such an f exists it is obviously unique. A necessary and sufficient condition that ξ' be a differential specialization of ξ over $\mathscr{R}_0$ is that the defining differential ideal of ξ over $\mathscr{R}_0$ be contained in the defining differential ideal of ξ' over $\mathscr{R}_0$. Another necessary and sufficient condition is that $(\theta\xi_i')_{\theta \in \Theta, i \in I}$ be a specialization of $(\theta\xi_i)_{\theta \in \Theta, i \in I}$ over $\mathscr{R}_0$.

If ξ' is a differential specialization of ξ over $\mathscr{R}_0$ such that ξ is a differential specialization of ξ' over $\mathscr{R}_0$, we say that ξ' is a *generic* differential specialization of ξ over $\mathscr{R}_0$. This is the case if and only if there exists an $\mathscr{R}_0$-isomorphism $\mathscr{R}_0\{\xi\} \approx \mathscr{R}_0\{\xi'\}$ mapping ξ onto ξ'.

The following result is analogous to Chapter 0, Section 14, Lemma 14.

Proposition 5 *Let $\mathscr{R}$ be a differential integral domain, let $\mathscr{S} = \mathscr{R}\{y_1, \ldots, y_n\}$ be a finitely generated differential polynomial algebra over $\mathscr{R}$, let $\mathfrak{p}$ be an $\mathscr{R}$-separable prime differential ideal of $\mathscr{S}$ with $\mathfrak{p} \cap \mathscr{R} = (0)$, and let $U \in \mathscr{S}$, $U \notin \mathfrak{p}$. Then there exist a nonzero element $u \in \mathscr{R}$ and a differential polynomial $E \in \mathscr{S}$ such that, for every differential specialization $f: \mathscr{R} \to \mathscr{G}$ with $f(u) \neq 0$, $\mathfrak{p}^f:(E^f)^\infty$ is an $f(\mathscr{R})$-separable differential ideal of $\mathscr{S}^f = f(\mathscr{R})\{y_1, \ldots, y_n\}$ not containing $\alpha U^f E^f$ for any nonzero element $\alpha \in f(\mathscr{R})$.*

Proof By Section 8, Lemma 3, there exist an autoreduced set A in $\mathfrak{p}$, and a finite set Y of derivatives of the y_j partially reduced with respect to A, such that if we set $\mathfrak{p}_1 = \mathfrak{p} \cap \mathscr{R}[Y]$, then $\mathfrak{p} = ([\mathrm{A}] + \mathscr{S}\mathfrak{p}_1):H_{\mathrm{A}}^\infty$ and A is $\mathscr{S}\mathfrak{p}_1$-coherent; $\mathfrak{p}_1$ is obviously an $\mathscr{R}$-separable prime ideal of $\mathscr{R}[Y]$. Denote the remainder of U with respect to A by U_0. By the last part of Lemma 3 we may suppose that $\mathrm{A} \subset \mathscr{R}[Y]$ and $U_0 \in \mathscr{R}[Y]$, so that $H_{\mathrm{A}} U_0 \in \mathscr{R}[Y]$ and $H_{\mathrm{A}} U_0 \notin \mathfrak{p}_1$. By Chapter 0, Section 14, Lemma 14, there exist a nonzero $u \in \mathscr{R}$ and a $D \in \mathscr{R}[Y]$ such that, for every specialization $f: \mathscr{R} \to L$ with $f(u) \neq 0$, $\mathfrak{p}_1{}^f:(D^f)^\infty$ is an $f(\mathscr{R})$-separable ideal of $f(\mathscr{R})[Y]$ not containing $\alpha H_{\mathrm{A}}{}^f U_0{}^f D^f$ for any nonzero element $\alpha \in f(\mathscr{R})$.

Consider any *differential* specialization $f: \mathscr{R} \to \mathscr{G}$ with $f(u) \neq 0$. From what we have just seen, it follows that $\mathscr{S}^f \cdot (\mathfrak{p}_1{}^f:(D^f)^\infty)$ is an $f(\mathscr{R})$-separable ideal of $\mathscr{S}^f$ not containing $\alpha H_{\mathrm{A}}{}^f U_0{}^f D^f$ for any nonzero element $\alpha \in f(\mathscr{R})$. By Section 8, Lemma 4, A^f is an $\mathscr{S}^f \mathfrak{p}_1{}^f$-coherent autoreduced set in $\mathscr{S}^f$. We now prove that A^f is $\mathscr{S}^f \cdot (\mathfrak{p}_1{}^f:(D^f)^\infty)$-coherent. All that we must show for this is that $[\mathfrak{p}_1{}^f:(D^f)^\infty] \subset ([\mathrm{A}^f] + \mathscr{S}^f \cdot (\mathfrak{p}_1{}^f:(D^f)^\infty)):(H_{\mathrm{A}}{}^f)^\infty$, that is, given any $G \in \mathscr{S}$ with $G^f \in [\mathfrak{p}_1{}^f:(D^f)^\infty]$, we must show that

$$G^f \in ([\mathrm{A}^f] + \mathscr{S}^f \cdot (\mathfrak{p}_1{}^f:(D^f)^\infty)):(H_{\mathrm{A}}{}^f)^\infty.$$

Now, G^f is the sum of finitely many terms $C^f \theta L^f$ with $C \in \mathscr{S}$, $\theta \in \Theta$, $L \in \mathscr{R}[Y]$, and $L^f \in \mathfrak{p}_1{}^f:(D^f)^\infty$; the last relation here means that $(D^f)^l L^f \in \mathfrak{p}_1{}^f$ for some $l \in \mathrm{N}$ and this implies (by Chapter I, Section 2, Lemma 1) that $(D^f)^k \theta L^f \in [\mathfrak{p}_1{}^f]$ for some $k \in \mathrm{N}$. Thus, $(D^f)^k G^f \in [\mathfrak{p}_1{}^f]$. Since A^f is $\mathscr{S}^f\mathfrak{p}_1{}^f$-coherent this implies that $(D^f)^k G^f \in ([\mathrm{A}^f]+\mathscr{S}^f\mathfrak{p}_1{}^f):(H_{\mathrm{A}}{}^f)^\infty$. Denoting the remainder of G with respect to A by G_0, we easily infer that $(D^f)^k G_0{}^f \in ([\mathrm{A}^f]+\mathscr{S}^f\mathfrak{p}_1{}^f):(H_{\mathrm{A}}{}^f)^\infty$, whence (by Section 8, Lemma 5) $(D^f)^k G_0{}^f \in ((\mathrm{A}^f)+\mathscr{S}^f\mathfrak{p}_1{}^f):(H_{\mathrm{A}}{}^f)^\infty = (\mathscr{S}^f\mathfrak{p}_1{}^f):(H_{\mathrm{A}}{}^f)^\infty$. Hence

$$\begin{aligned} G_0{}^f \in ((\mathscr{S}^f\mathfrak{p}_1{}^f):(H_{\mathrm{A}}{}^f)^\infty):(D^f)^\infty &= ((\mathscr{S}^f\mathfrak{p}_1{}^f):(D^f)^\infty):(H_{\mathrm{A}}{}^f)^\infty \\ &= (\mathscr{S}^f\cdot(\mathfrak{p}_1{}^f:(D^f)^\infty)):(H_{\mathrm{A}}{}^f)^\infty, \end{aligned}$$

so that $G^f \in ([\mathrm{A}^f]+\mathscr{S}^f\cdot(\mathfrak{p}_1{}^f:(D^f)^\infty)):(H_{\mathrm{A}}{}^f)^\infty$. This shows that A^f is $\mathscr{S}^f\cdot(\mathfrak{p}_1{}^f:(D^f)^\infty)$-coherent.

By Section 8, Lemma 6, then $([\mathrm{A}^f]+\mathscr{S}^f\cdot(\mathfrak{p}_1{}^f:(D^f)^\infty)):(H_{\mathrm{A}}{}^f)^\infty$ is an $f(\mathscr{R})$-separable differential ideal of $\mathscr{S}^f$ that (by Lemma 5) does not contain $\alpha H_{\mathrm{A}}{}^f U_0{}^f D^f$ for any nonzero element $\alpha \in f(\mathscr{R})$. The same is evidently true of $([\mathrm{A}^f]+\mathscr{S}^f\cdot(\mathfrak{p}_1{}^f:(D^f)^\infty)):(H_{\mathrm{A}}{}^f D^f)^\infty$. However, $\mathfrak{p} \subset ([\mathrm{A}]+\mathscr{S}\mathfrak{p}_1):H_{\mathrm{A}}{}^\infty$, and therefore $\mathfrak{p}^f \subset ([\mathrm{A}^f]+\mathscr{S}^f\mathfrak{p}_1{}^f):(H_{\mathrm{A}}{}^f)^\infty$, whence

$$\mathfrak{p}^f:(H_{\mathrm{A}}{}^f D^f)^\infty \subset ([\mathrm{A}^f] + \mathscr{S}^f\cdot(\mathfrak{p}_1{}^f:(D^f)^\infty)):(H_{\mathrm{A}}{}^f D^f)^\infty.$$

Since the last inclusion can evidently be reversed, we see that $\mathfrak{p}^f:(H_{\mathrm{A}}{}^f D^f)^\infty = ([\mathrm{A}^f]+\mathscr{S}^f\cdot(\mathfrak{p}_1{}^f:(D^f)^\infty)):(H_{\mathrm{A}}{}^f D^f)^\infty$. Setting $E = H_{\mathrm{A}} D$, we thus see that $\mathfrak{p}^f:(E^f)^\infty$ is an $f(\mathscr{R})$-separable differential ideal of $\mathscr{S}^f$ not containing $\alpha U_0{}^f E^f$ (hence not containing $\alpha U^f E^f$) for any nonzero element $\alpha \in f(\mathscr{R})$.

From Proposition 5 we deduce the following theorem (analogous to Chapter 0, Section 14, Proposition 9(c)) on the possibility of extending differential specializations.

Theorem 3 *Let $\mathscr{R}$ be a differential integral domain, let $\mathscr{R}_0$ be a differential subring of $\mathscr{R}$ over which $\mathscr{R}$ is finitely generated and separable, and let u be a nonzero element of $\mathscr{R}$. There exists a nonzero element u_0 of $\mathscr{R}_0$ such that every differential specialization $f_0 : \mathscr{R}_0 \to \mathscr{U}$ with $\mathscr{U}$ a semiuniversal extension of $Q(f_0(\mathscr{R}_0))$ and $f_0(u_0) \neq 0$ can be extended to a differential specialization $f : \mathscr{R} \to \mathscr{U}$ with $f(\mathscr{R})$ separable over $f(\mathscr{R}_0)$ and $f(u) \neq 0$.*

Proof By hypothesis we may write $\mathscr{R} = \mathscr{R}_0\{\eta_1, \ldots, \eta_n\}$, and the defining differential ideal $\mathfrak{p}$ of $(\eta_1, \ldots, \eta_n)$ in the differential polynomial algebra $\mathscr{R}_0\{y_1, \ldots, y_n\}$ is prime and $\mathscr{R}_0$-separable with $\mathfrak{p} \cap \mathscr{R}_0 = (0)$. Also, there exists a differential polynomial $U \in \mathscr{R}_0\{y_1, \ldots, y_n\}$ with $U(\eta_1, \ldots, \eta_n) = u$, and obviously $U \notin \mathfrak{p}$. By Proposition 5 there exist a nonzero element $u_0 \in \mathscr{R}_0$ and a differential polynomial $E \in \mathscr{R}_0\{y_1, \ldots, y_n\}$ such that, for every differential specialization $f_0 : \mathscr{R}_0 \to \mathscr{U}$ with $f_0(u_0) \neq 0$, $\mathfrak{p}^{f_0}:(E^{f_0})^\infty$ is an $f_0(\mathscr{R}_0)$-separable differential ideal of $f_0(\mathscr{R}_0)\{y_1, \ldots, y_n\}$ not containing αU^{f_0} for any nonzero element $\alpha \in f_0(\mathscr{R}_0)$. The set $\mathfrak{C}$, consisting of the unit ideal and all $f_0(\mathscr{R}_0)$-separable differential ideals of $f_0(\mathscr{R}_0)\{y_1, \ldots, y_n\}$ not containing any nonzero element of $f_0(\mathscr{R}_0)$, is a perfect differential conservative system (see Chapter 0, Section 6, Lemma 5), and $\mathfrak{p}^{f_0}:(E^{f_0})^\infty \in \mathfrak{C}$. Therefore there exists a $\mathfrak{C}$-component $\mathfrak{p}'$ of $\mathfrak{p}^{f_0}:(E^{f_0})^\infty$ with $U^{f_0} \notin \mathfrak{p}'$ (see Chapter 0, Section 8, Proposition 1). If $\mathscr{U}$ is a semiuniversal extension of $Q(f_0(\mathscr{R}_0))$, then there exist elements $\eta_1', \ldots, \eta_n' \in \mathscr{U}$ such that $\mathfrak{p}'$ is the kernel of the substitution homomorphism

$$\sigma : f_0(\mathscr{R}_0)\{y_1, \ldots, y_n\} \to f_0(\mathscr{R}_0)\{\eta_1', \ldots, \eta_n'\}.$$

Denoting the homomorphism $G \mapsto G^{f_0}$ of $\mathscr{R}_0\{y_1, \ldots, y_n\}$ into $f_0(\mathscr{R}_0)\{y_1, \ldots, y_n\}$ by φ_0, we see that the composite homomorphism

$$\sigma \circ \varphi_0 : \mathscr{R}_0\{y_1, \ldots, y_n\} \to f_0(\mathscr{R}_0)\{\eta_1', \ldots, \eta_n'\}$$

has prime kernel containing $\mathfrak{p}$ but not containing U. Since the kernel of the surjective substitution homomorphism

$$\tau : \mathscr{R}_0\{y_1, \ldots, y_n\} \to \mathscr{R}_0\{\eta_1, \ldots, \eta_n\} = \mathscr{R}$$

is $\mathfrak{p}$, there must exist a homomorphism $f : \mathscr{R} \to f_0(\mathscr{R}_0)\{\eta_1', \ldots, \eta_n'\}$ such that $f \circ \tau = \sigma \circ \varphi_0$. It is now a simple matter to see that f agrees with f_0 on $\mathscr{R}_0$, $f(\mathscr{R})$ is separable over $f(\mathscr{R}_0)$, and $f(u) \neq 0$. This proves the theorem.

Remark The earliest version of Theorem 3, proved by Ritt [91], dealt with the case in which $\mathscr{R}_0$ is a finitely generated differential algebra over an ordinary differential field of functions meromorphic in a region of the complex plane. This was extended, independently and by different methods, by Seidenberg [110] and Rosenfeld [105], to the situation in which $\mathscr{R}_0$ is a finitely generated differential algebra over an arbitrary differential field of characteristic zero. The above proof of the present general theorem entails a further development of Rosenfeld's methods.

It is noteworthy that the analog for differential specializations of Chapter 0, Section 14, Proposition 9(b), is false. There exist elements η, ζ of a universal extension $\mathscr{U}$ of $\mathscr{F}$ having the following property: 0 is a differential specialization of η over $\mathscr{F}$ but there does not exist an $\alpha \in \mathscr{U}$ such that $(0, \alpha)$ is a differential specialization over $\mathscr{F}$ either of (η, ζ) or of (η, ζ^{-1}). This is more easily shown at a later stage (see Chapter IV, Section 6, Exercise 6(c)).

Another curious phenomenon is the existence of elements $\eta, \zeta \in \mathscr{U}$ such that 0 is not a differential specialization of η or ζ over $\mathscr{F}$ but is a differential specialization of $\eta\zeta$ over $\mathscr{F}$ (see Chapter IV, Section 6, Exercise 7(d)).

10 Constrained families

A family $\eta = (\eta_i)_{i \in I}$ of elements of an extension of $\mathscr{F}$ will be said to be *constrained* over $\mathscr{F}$, or to be *$\mathscr{F}$-constrained*, if η is separable over $\mathscr{F}$ (that is, $\mathscr{F}\langle\eta\rangle$ is separable over $\mathscr{F}$) and there exists a differential polynomial $B \in \mathscr{F}\{(y_i)_{i \in I}\}$ with $B(\eta) \neq 0$ such that $B(\eta') = 0$ for every *non*generic differential specialization η' of η over $\mathscr{F}$ that is separable over $\mathscr{F}$. Any such B will be called a *constraint* of η over $\mathscr{F}$. (When $\operatorname{Card} I = 1$, that is, when the constrained family η has just one coordinate η_i, we identify the family with its coordinate and call it a constrained *element*.)

If $\eta = (\eta_i)_{i \in I}$ is separably algebraic over $\mathscr{F}$ (that is, $\mathscr{F}\langle\eta\rangle$ is a separable algebraic field extension of $\mathscr{F}$), then η is constrained over $\mathscr{F}$ with constraint 1. For a familiar transcendental example see Exercise 2 below.

Proposition 6 *Let $\eta = (\eta_i)_{i \in I}$ be a family of elements of an extension of $\mathscr{F}$, with η separable over $\mathscr{F}$, and let $B \in \mathscr{F}\{(y_i)_{i \in I}\}$ be a differential polynomial such that $B(\eta) \neq 0$. There exists a differential specialization η' of η over $\mathscr{F}$ such that η' is constrained over $\mathscr{F}$ with constraint B.*

Proof By Zorn's lemma, in the set of all $\mathscr{F}$-separable prime differential ideals of $\mathscr{F}\{(y_i)_{i \in I}\}$ that contain the defining differential ideal of η over $\mathscr{F}$ but do not contain B, there is a maximal element, say $\mathfrak{p}'$. This $\mathfrak{p}'$ is the defining differential ideal over $\mathscr{F}$ of a family η' with coordinates in an extension of $\mathscr{F}$, and obviously η' is a differential specialization of η over $\mathscr{F}$ and is $\mathscr{F}$-separable, and $B(\eta') \neq 0$. If η'' is any nongeneric differential specialization of η' over $\mathscr{F}$ and is $\mathscr{F}$-separable, then the defining differential ideal of η'' in $\mathscr{F}\{(y_i)_{i \in I}\}$ is $\mathscr{F}$-separable and properly contains $\mathfrak{p}'$ and hence contains B, so that $B(\eta'') = 0$. Thus, η' is constrained over $\mathscr{F}$ with constraint B.

Proposition 7 *Let $\eta = (\eta_i)_{i \in I}$ and $\zeta = (\zeta_j)_{j \in J}$ be families of elements of an extension of $\mathscr{F}$.*

(a) *Let $\mathscr{F}\langle\eta\rangle = \mathscr{F}\langle\zeta\rangle$ and I be finite. If η is constrained over $\mathscr{F}$, then so is ζ.*

(b) *Let $\mathscr{F}\langle\eta, \zeta\rangle$ be separable over $\mathscr{F}\langle\eta\rangle$. If (η, ζ) is constrained over $\mathscr{F}$, then ζ is constrained over $\mathscr{F}\langle\eta\rangle$ and, provided J is finite, η is constrained over $\mathscr{F}$.*

(c) *Let the field of constants of $\mathscr{F}\langle\eta\rangle$ be separable over $\mathscr{F}\langle\eta\rangle^p\mathscr{C}$. If ζ is constrained over $\mathscr{F}\langle\eta\rangle$ and η is constrained over $\mathscr{F}$, then (η, ζ) is constrained over $\mathscr{F}$.*

(d) *Let I be finite. If η is constrained over $\mathcal{F}$, then the field of constants of $\mathcal{F}\langle\eta\rangle$ is separably algebraic over $\mathcal{F}\langle\eta\rangle^p\mathcal{C}$.*

Proof (a) There exist differential polynomials $M_i, N \in \mathcal{F}\{(z_j)_{j\in J}\}$ with $N(\zeta) \neq 0$ such that $\eta_i = M_i(\zeta)/N(\zeta)$. Also, η has a constraint $B \in \mathcal{F}\{(y_i)_{i\in I}\}$. For a sufficiently big $h \in \mathbf{N}$, $N^h B(M/N)$ is a differential polynomial, which we denote by C; clearly $C(\zeta) = N(\zeta)^h B(\eta) \neq 0$. Let ζ' be a differential specialization of ζ over $\mathcal{F}$ with ζ' separable over $\mathcal{F}$ and $N(\zeta')\,C(\zeta') \neq 0$. Then we may set $\eta' = (M_i(\zeta')/N(\zeta'))$. It is clear that η' is a differential specialization of η over $\mathcal{F}$ with η' separable over $\mathcal{F}$ and $B(\eta') \neq 0$. Since B is a constraint of η it follows that η' is a generic differential specialization of η over $\mathcal{F}$, and hence that ζ' is a generic differential specialization of ζ over $\mathcal{F}$. Thus, NC is a constraint of ζ over $\mathcal{F}$. Since ζ is obviously separable over $\mathcal{F}$, this shows that ζ is constrained over $\mathcal{F}$.

(b) Let $B \in \mathcal{F}\{(y_i)_{i\in I}, (z_j)_{j\in J}\}$ be a constraint of (η, ζ) over $\mathcal{F}$. If ζ' is a differential specialization of ζ over $\mathcal{F}\langle\eta\rangle$ with ζ' separable over $\mathcal{F}\langle\eta\rangle$ and $B(\eta, \zeta') \neq 0$, then (η, ζ') is a differential specialization of (η, ζ) over $\mathcal{F}$ with (η, ζ') separable over $\mathcal{F}$ and $B(\eta, \zeta') \neq 0$, so that (η, ζ') is a generic differential specialization of (η, ζ) over $\mathcal{F}$, and therefore ζ' is a generic differential specialization of ζ over $\mathcal{F}\langle\eta\rangle$. Thus, $B(\eta, z)$ is a constraint of ζ over $\mathcal{F}\langle\eta\rangle$, so that ζ is constrained over $\mathcal{F}\langle\eta\rangle$.

Now suppose that J is finite. By Section 9, Theorem 3, there exists a $U_0 \in \mathcal{F}\{(y_i)_{i\in I}\}$ with $U_0(\eta) \neq 0$ such that for every differential specialization η' of η over $\mathcal{F}$ with $U_0(\eta') \neq 0$ there is a ζ' separable over $\mathcal{F}\langle\eta'\rangle$ for which (η', ζ') is a differential specialization of (η, ζ) over $\mathcal{F}$ with $B(\eta', \zeta') \neq 0$. If η' is separable over $\mathcal{F}$, then (η', ζ') is separable over $\mathcal{F}$, and therefore (η', ζ') is a generic differential specialization of (η, ζ) over $\mathcal{F}$. Thus, U_0 is a constraint of η over $\mathcal{F}$, so that η is constrained over $\mathcal{F}$.

(c) It is obvious that (η, ζ) is separable over $\mathcal{F}$. By hypothesis, there exists a constraint $B \in \mathcal{F}\{(y_i)_{i\in I}\}$ of η over F, and there exists a $C \in \mathcal{F}\{(y_i)_{i\in I}, (z_j)_{j\in J}\}$ such that $C(\eta, (z_j)_{j\in J})$ is a constraint of ζ over $F\langle\eta\rangle$. We shall show that BC is a constraint of (η, ζ) over $\mathcal{F}$, thereby proving that (η, ζ) is constrained over $\mathcal{F}$. Indeed, let (η', ζ') be any differential specialization of (η, ζ) over $\mathcal{F}$ with $\mathcal{F}\langle\eta', \zeta'\rangle$ separable over $\mathcal{F}$ and $B(\eta')\,C(\eta', \zeta') \neq 0$. Then η' is a differential specialization of η over $\mathcal{F}$ with η' separable over $\mathcal{F}$ and $B(\eta') \neq 0$, so that η' is a generic differential specialization of η over $\mathcal{F}$. Hence, there exists an isomorphism $\mathcal{F}\langle\eta'\rangle \approx \mathcal{F}\langle\eta\rangle$ over $\mathcal{F}$ mapping η' onto η. This isomorphism can be extended to an isomorphism of $\mathcal{F}\langle\eta', \zeta'\rangle$ onto an extension of $\mathcal{F}\langle\eta\rangle$. Denoting the image of ζ' by ζ'', we see that (η, ζ'') is a generic differential specialization of (η', ζ') over $\mathcal{F}$ (so that (η, ζ'') is separable over $\mathcal{F}$), and is a differential specialization of (η, ζ) over $\mathcal{F}$, so that ζ'' is a differential specialization of ζ over $\mathcal{F}\langle\eta\rangle$ with $C(\eta, \zeta'') \neq 0$.

If we can show that $\mathscr{F}\langle\eta,\zeta''\rangle$ is separable over $\mathscr{F}\langle\eta\rangle$, this will therefore imply that ζ'' is a generic differential specialization of ζ over $\mathscr{F}\langle\eta\rangle$, hence that (η,ζ'') is a generic differential specialization of (η,ζ) over $\mathscr{F}$, and therefore that (η',ζ') is a generic differential specialization of (η,ζ) over $\mathscr{F}$, establishing the fact that BC is a constraint of (η,ζ) over $\mathscr{F}$.

If $p=0$, there is nothing to prove, so we may suppose that $p\neq 0$. Since the field $\mathscr{D}$ of constants of $\mathscr{F}\langle\eta\rangle$ obviously has the property that $\mathscr{D}^p\subset \mathscr{F}\langle\eta\rangle^p\mathscr{C}$, and since by hypothesis $\mathscr{D}$ is separable over $\mathscr{F}\langle\eta\rangle^p\mathscr{C}$, we must have $\mathscr{D}=\mathscr{F}\langle\eta\rangle^p\mathscr{C}$. By Chapter II, Section 2, part (b) of the corollary to Proposition 2, it follows that $\mathscr{F}\langle\eta,\zeta''\rangle$ is separable over $\mathscr{F}\langle\eta\rangle$.

(d) Let γ be any constant in $\mathscr{F}\langle\eta\rangle$. By part (a) of the present proposition, (γ,η) is constrained over $\mathscr{F}$. If $\gamma\in\mathscr{F}\langle\eta\rangle^p\mathscr{C}$, then certainly γ is separably algebraic over $\mathscr{F}\langle\eta\rangle^p\mathscr{C}$. Suppose then that $\gamma\notin\mathscr{F}\langle\eta\rangle^p\mathscr{C}$. By Chapter II, part (c) of the Corollary to Proposition 2, the differential field $\mathscr{F}\langle\eta\rangle = \mathscr{F}\langle\gamma,\eta\rangle$ is separable over $\mathscr{F}\langle\gamma\rangle$. By part (b) of the present proposition, then γ is constrained over $\mathscr{F}$. However, it is obvious that if a constant c is transcendental over $\mathscr{F}$, then every constant is a differential specialization of c over $\mathscr{F}$ and therefore c cannot be constrained over $\mathscr{F}$. Hence γ is algebraic over $\mathscr{F}$. Because $\mathscr{F}\langle\eta\rangle$ is separable over $\mathscr{F}$ it follows that γ is separably algebraic over $\mathscr{F}$. Since $\mathscr{F}$ and $\mathscr{C}\langle\gamma\rangle$ are linearly disjoint over $\mathscr{C}$ (see Chapter II, Section 1, Corollary 1 to Theorem 1) we conclude that γ is separably algebraic over $\mathscr{C}$, and *a fortiori* over $\mathscr{F}\langle\eta\rangle^p\mathscr{C}$.

EXERCISES

1. Let $\eta=(\eta_i)_{i\in I}$ be a family of elements of an extension of $\mathscr{F}$, let $\mathfrak{p}$ denote the defining differential ideal of η in $\mathscr{F}\{(y_i)_{i\in I}\}$, and let $\mathfrak{a}$ be the intersection of all the $\mathscr{F}$-separable prime differential ideals of $\mathscr{F}\{(y_i)_{i\in I}\}$ that properly contain $\mathfrak{p}$. Show that η is constrained over $\mathscr{F}$ if and only if $\mathfrak{p}$ is $\mathscr{F}$-separable and $\mathfrak{a}\neq\mathfrak{p}$, the set of constraints of η over $\mathscr{F}$ being $\mathfrak{a}-\mathfrak{p}$.
2. Prove that e^x is constrained, with constraint y, over the ordinary differential field of rational functions of a complex variable x (the derivation operator being d/dx).
3. Let $\mathscr{G}$ be any separable extension of $\mathscr{F}$. Show that the family $(\alpha)_{\alpha\in\mathscr{G}}$ is constrained over $\mathscr{F}$ with constraint 1.
4. Show that in Proposition 7(a), (b), and (d), the finiteness conditions cannot be omitted.
5. Let $\mathscr{F}_1$ be a separably algebraic extension of $\mathscr{F}$, let η be a finite family of elements of an extension of $\mathscr{F}_1$, and suppose that η is constrained over $\mathscr{F}_1$. Show that η is constrained over $\mathscr{F}$.