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A QUASI-ORDER ON CONTINUOUS FUNCTIONS

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Abstract. We define a quasi-order on Borel functions from a zero-dimensional Polish space into another that both refines the order induced by the Baire hierarchy of functions and generalises the embeddability order on Borel sets. We study the properties of this quasi-order on continuous functions, and we prove that the closed subsets of a zero-dimensional Polish space are well-quasi-ordered by bi-continuous embeddability.

§1. Introduction.

A *quasi-order* (qo) on a set Q is a reflexive and transitive relation $\leq_Q$ on Q. The qo $\leq_Q$ is a well-quasi-order (wqo) if it contains no infinite descending chain and no infinite antichain. It is very natural to ask whether a qo that is not of a known order type is in fact a wqo or not. The first famous example of such a question is Fraïssé's conjecture (see [2]), stating that countable linear orders with order embedding is a wqo. It was proved to be true by Laver in [6], using the stronger notion of better quasi order (bqo), developed by Nash-Williams in [12]. In the study of partial Borel functions between 0-dimensional Polish spaces there are three classical quasi-orders.

We restrict our attention to a canonical Polish 0-dimensional space: the Baire space ω^ω, or $\mathscr{N}$. The first qo is the Wadge order on subsets: given $A, B \subseteq \mathscr{N}$ we write $A \leq_W B$ if there is a continuous function f such that (s.t.) $A = f^{-1}(B)$. This is a wqo on Borel subsets of $\mathscr{N}$ inducing a hierarchy that refines greatly the Borel hierarchy of sets. Remark that $\leq_W$ has a simple functional expression: $A \leq_W B$ if there is a continuous σ s.t. $\mathbf{1}_A = \mathbf{1}_B \circ \sigma$, where $\mathbf{1}_X$ denotes the characteristic function of X with image in ω^ω, so $\mathbf{1}_X(x) = 1^\omega$ when x is in X and 0^ω otherwise. For results on the Wadge hierarchy of sets, see [16] and [1].

The second equally classical qo on subsets of $\mathscr{N}$ is the qo based on continuous embeddability. We say that $A \subseteq \mathscr{N}$ *continuously embeds* into $B \subseteq \mathscr{N}$ if there is a continuous injection from A to B. Since we consider partial functions, continuity of a function means, without further notice, with respect to the topology induced on its domain. Here we consider the following refinement. We say that A *bi-continuously embeds* in B if there are two continuous functions $f\colon A \to B$ and $g\colon B \to A$ s.t. $g \circ f = \mathrm{Id}_A$, the identity function on A.

The third qo is a refinement of the Baire order on functions that is induced on Borel functions by the Borel rank. Given a Borel function $f\colon \mathscr{N} \to \mathscr{N}$ we can define

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the degree function $d_f\colon \omega_1 \to \omega_1$ as follows: if $\alpha \in \omega_1$, $d_f(\alpha)$ is the smallest ordinal β s.t. $f^{-1}(\Sigma^0_\alpha) \subseteq \Sigma^0_\beta$. We then say that $f \leq_B g$ if for all $\alpha \in \omega_1$ $d_f(\alpha) \leq d_g(\alpha)$.

The qo on functions from $\mathcal{N}$ into itself we are interested in, is a generalisation of those three qos. Given two partial functions $f\colon \mathcal{N} \to \mathcal{N}$ and $g\colon \mathcal{N} \to \mathcal{N}$, we define $f \leq g$ and we say that g *reduces* f if and only if (iff) there are two continuous functions σ and τ such that $\operatorname{dom} \sigma = \operatorname{dom} f$ holds, σ ranges into the domain of g, $\operatorname{dom} \tau = \operatorname{im} g\sigma$ holds, τ ranges onto $\operatorname{im} f$, and when f is defined, $f = \tau \circ g \circ \sigma$. We denote $f \equiv g$ when both $f \leq g$ and $g \leq f$. Here $\operatorname{dom} f$ (resp. $\operatorname{im} f$) is the domain (resp. the range) of f.

This qo was first introduced by Hertling and Weihrauch, see [4] or [3], in order to study computable degeneracies of functions from the Cantor space into ω.

Given any two functions f and g from $\mathcal{N}$ into itself we have:

FACT.
- *The relation $\leq$ is a* qo.
- *If $A \leq_W B$ then $\mathbf{1}_A \leq \mathbf{1}_B$ and if $\mathbf{1}_A \leq \mathbf{1}_B$ then $A \leq_W B$ or $A^c \leq_W B$.*
- *For any subset $A \subseteq \mathcal{N}$ we have that $\mathbf{1}_A \equiv \mathbf{1}_{A^c}$, where $A^c = \mathcal{N} \setminus A$.*
- $\mathrm{Id}_A \leq \mathrm{Id}_B$ *iff A bi-continuously embeds into B.*
- *If $f \leq g$ then $f \leq_B g$, i.e., $\leq$ refines $\leq_B$.*

Several other qos have been considered on functions, but for any of them either the preceding fact is wrong or it is certainly not a wqo. First, the Wadge order can be generalised to functions in a brutal way: for f, g in $\mathcal{N}^{\mathcal{N}}$, say $f \leq_W g$ iff there is a continuous σ s.t. $f = g\sigma$, but then there already exist some uncountable $\leq_W$-antichains among the constant functions.

Hertling studied $\leq_W$ on functions with discrete image in [3], along with a second qo on functions, introduced by Weihrauch in [17]: say $f \leq_2 g$ iff there exist two continuous functions σ and τ s.t. for all $x \in \operatorname{dom} f$ we have $f(x) = \tau(x, g\sigma(x))$. We did not choose to study this qo for it does not refine $\leq_B$ (for instance all continuous functions are $\leq_2$-equivalent).

We discarded, for the same reason, the qo $f \sqsubseteq g$ if there are continuous σ, τ s.t. $\tau \circ f = g \circ \sigma$ inspired by Solecki [14]. It does not refine $\leq_B$ either, because if f is any discontinuous Borel isomorphism with continuous inverse the relation $f \sqsubseteq \mathrm{Id}_{\mathcal{N}}$ holds.

In this paper, we begin the study of $\leq$ on Borel functions by focusing on the very special case of the continuous functions with closed domain. We first show that among such functions, the ones with uncountable range are all equivalent with respect to $\leq$. Since the range of a Borel function with Borel domain has the perfect set property, it is either countable or of cardinality $2^{\aleph_0}$. Hence we study more specifically the functions with countable image. We let both C denote the set of continuous functions from $\mathcal{N}$ into itself, with closed domain and countable range, and C_∞ denote the set of all continuous functions with closed domain.

In the sequel we define what we call the Cantor–Bendixson rank $\mathrm{CB}(f)$ of a function f in C. It coincides with the usual Cantor–Bendixson rank on closed sets, in the sense that given F closed, $\mathrm{CB}(\mathrm{Id}_F) = \mathrm{CB}(F)$. This rank stratifies C in sets C_α of all functions in C of Cantor–Bendixson rank α, for α countable.

Let $\mathsf{C}^\star$ be the subset of C of all functions with compact domain. The first result we get is the following:

THEOREM 1.1. *The relation $\leq$ is a well-order on $\mathsf{C}^\star/\equiv$ of order type ω_1.*

Using $\mathsf{C}^{\star}$ as a leverage point in C we then obtain a criterion for any subset of $(\mathsf{C}_{\infty}, \leq)$ to be a bqo.

THEOREM 1.2. *Let Q be a subset of C_{∞}. If for all $\alpha \in \omega_1$ $(Q \cap \mathsf{C}_{\alpha}, \leq)$ is a* bqo *then $(Q, \leq)$ is a* bqo.

Notice that we do not know yet if C_{∞} is a bqo or not, but even if turns out not to be, this criterion gives a tool to find a subclass of C_{∞} that is bqo.

In particular, we can apply Theorem 1.2 to the subclass of identity functions and obtain the following result.

THEOREM 1.3. *The set $\Pi^0_1(\mathscr{N})$ of closed subsets of $\mathscr{N}$, quasi ordered by bi-continuous embedding, is a* bqo.

Notice that this result can also be obtained as a corollary of Laver's result on labelled trees in [7], so that we provide here an alternative proof.

§2. The rank of a function.

We remind that given any finite sequences of integers $u, v \in \omega^{<\omega}$, $u \perp v$ stands for $u \nsubseteq v$ and $v \nsubseteq u$, $[u] = \{x \in \mathscr{N} : u \subset x\}$, and $\{[u] : u \in \omega^{<\omega}\}$ is a basis of clopen sets for the product topology on $\mathscr{N}$. We let ε be the empty sequence in $\omega^{<\omega}$. For all classical results in descriptive set theory we refer the reader to [5, 11].

We first prove that every continuous function with uncountable image is equivalent to the identity, so that we can concentrate on the sole case of the continuous functions with countable image. In fact we prove a slightly stronger result:

PROPOSITION 2.1. *Let $f : \mathscr{N} \to \mathscr{N}$ be any Borel function with uncountable image. Then $\mathrm{Id}_{\mathscr{N}} \leq f$.*

As a consequence, if f and g belong to C_{∞} and satisfy $|\mathsf{im}\, f| = |\mathsf{im}\, g| = 2^{\aleph_0}$, then $f \equiv g$ holds.

PROOF. The last remark comes from the fact that $f \leq \mathrm{Id}_{\mathscr{N}}$ holds for any continuous f, so we prove that $\mathrm{Id}_{\mathscr{N}} \leq f$ holds for every Borel function f with uncountable image.

We let $f : \mathscr{N} \to \mathscr{N}$ be such a function. We consider the relation $x E_f y$ iff $f(x) = f(y)$. By Silver's theorem on Borel equivalence relations we get a compact $K_0 \subseteq \mathsf{dom}\, f$, with $K_0 \cong 2^{\omega}$ the Cantor set, on which the function f is one-to-one.

Then, as any Borel function, $f|_{K_0}$ is continuous on a dense Π^0_2-set A (see [5, Theorem 8.38]). Since A is Π^0_2 it is Polish, and since it is dense it contains a compact perfect set, so we can take another compact set $K \subseteq A$ still homeomorphic to 2^{ω}, along with continuous embeddings $\sigma : \mathscr{N} \to K$ and $\tau_0 : \mathsf{im}\, \sigma \to \mathscr{N}$ s.t. $\tau_0 \circ \sigma = \mathrm{Id}_{\mathscr{N}}$ holds. The map $f|_K$ is one-to-one and continuous, hence since K is compact, there is a continuous function τ_1 s.t. $\tau_1 \circ f = \mathrm{Id}_K$. We set $\tau = \tau_0 \circ \tau_1$, so to obtain finally $\mathrm{Id}_{\mathscr{N}} = \tau \circ f \circ \sigma$. ⊣

We now consider only the continuous functions with countable image. We want to generalise the Cantor–Bendixson rank to such functions. The naive way to proceed would be to take the rank of the image. But, although locally constant functions are all equivalent for the qo we defined, the rank of the image of a locally constant function can be of any countable ordinal.

We have to be more precise in the definition of the rank and we must take into account the local behaviour of the function. We thus make the following definition.

Definitions.

- Given a function $f \in \mathsf{C}$ and $x \in \mathsf{dom}\ f$, we say that x is *f-isolated* if $f^{-1}(\{f(x)\})$ is a neighbourhood of x, i.e., f is locally constant on x.
- Given $A \subseteq \mathsf{dom}\ f$, we write $I(f, A)$ for the set of all $f|_A$-isolated points. As $f \in \mathsf{C}$, we have $I(f, A) = \bigcup\{A \cap [u]\colon f([u]) \cap A \text{ is a singleton}\}$ which makes this set open and dense in A.
- We inductively define:

$$\mathrm{CB}_0(f) = \mathsf{dom}\ f,$$
$$\mathrm{CB}_{\alpha+1}(f) = \mathrm{CB}_\alpha(f) \setminus I(f, \mathrm{CB}_\alpha(f)),$$
$$\mathrm{CB}_\lambda(f) = \bigcap_{\alpha\in\lambda} \mathrm{CB}_\alpha(f) \text{ for } \lambda \text{ limit.}$$

- We set $\mathrm{CB}(f)$ for the least ordinal α s.t. $\mathrm{CB}_\alpha(f) = \mathrm{CB}_{\alpha+1}(f)$. Since f is in C, one sees that $\alpha = \mathrm{CB}(f)$ is countable and $\mathrm{CB}_\alpha(f) = \emptyset$, we call α the *Cantor–Bendixson rank* of f. Finally, given any countable ordinal α, we write C_α for the set of all functions of rank α in C.

We also denote $\mathrm{CB}(F)$ the classical Cantor–Bendixson rank of any closed set F. Since x is isolated in F iff it is Id_F-isolated, we have indeed that $\mathrm{CB}(F) = \mathrm{CB}(\mathrm{Id}_F)$ holds.

We give some properties of the Cantor–Bendixson rank of a function, among which a necessary condition for the reduction of two functions:

Proposition 2.2. *Let f and g be two functions in C such that f is reduced by g, σ and τ the two continuous functions witnessing the reduction, i.e., $f = \tau \circ g \circ \sigma$ holds. Then we have the following properties*:

1. *If N (resp. M) is a subset of* $\mathsf{dom}\ f$ *(resp.* $\mathsf{dom}\ g$*), and $\sigma(N) \subseteq M$ holds, then $f|_N \leq g|_M$.*
2. *For every x in* $\mathsf{dom}\ f$, *when $\sigma(x)$ is g-isolated, x is f-isolated.*
3. *For all $\alpha < \omega_1$, $\sigma(\mathrm{CB}_\alpha(f)) \subseteq \mathrm{CB}_\alpha(g)$.*
4. $\mathrm{CB}(f) \leq \mathrm{CB}(g)$.

Proof. 1. Since $f = \tau g \sigma$ holds, so does $f|_N = (\tau g \sigma)|_N$ but $\sigma(N) \subseteq M$ implies $(\tau g \sigma)|_N = \tau(g|_M)(\sigma|_N)$ and hence $f|_N \leq g|_M$.

2. Suppose $x' = \sigma(x)$ is g-isolated, for some $x \in \mathsf{dom}\ f$. Then g is constant on an open set $U \ni x'$, but σ is continuous and $f = \tau g \sigma$, so f is constant on the open set $\sigma^{-1}(U) \ni x$, hence x is f-isolated.

3. Let us prove this point by induction on α.

For $\alpha = 0$ we have that $\sigma(\mathsf{dom}\ f) \subseteq \mathsf{dom}\ g$ holds since $f = \tau g \sigma$ does.

Suppose $\sigma(\mathrm{CB}_\alpha(f)) \subseteq \mathrm{CB}_\alpha(g)$. Set $F_\alpha = \mathrm{CB}_\alpha(f)$ and $G_\alpha = \mathrm{CB}_\alpha(g)$, $f' = f|_{F_\alpha}$ and $g' = g|_{G_\alpha}$. Apply the first point and get $f' \leq g'$, then apply the second and get $\sigma(\mathrm{CB}_1(f')) \subseteq \mathrm{CB}_1(g')$. Notice now that $\mathrm{CB}_1(f') = F_\alpha \setminus I(f', F_\alpha)$ and since $f' = f|_{F_\alpha}$ we have $I(f', F_\alpha) = I(f, F_\alpha)$ so $\mathrm{CB}_1(f')$ is in fact $\mathrm{CB}_{\alpha+1}(f)$. Similarly, $\mathrm{CB}_1(g') = \mathrm{CB}_{\alpha+1}(g)$ hence we finally have that $\sigma(\mathrm{CB}_{\alpha+1}(f)) \subseteq \mathrm{CB}_{\alpha+1}(g)$ holds.

Suppose α is limit and for all $\beta < \alpha$ $\sigma(\mathrm{CB}_\beta(f)) \subseteq \mathrm{CB}_\beta(g)$ holds. Then

$$\begin{aligned}\sigma(\mathrm{CB}_\alpha(f)) &= \sigma\Big(\bigcap_{\beta<\alpha} \mathrm{CB}_\beta(f)\Big)\\ &\subseteq \bigcap_{\beta<\alpha} \sigma(\mathrm{CB}_\beta(f))\\ &\subseteq \bigcap_{\beta<\alpha} \mathrm{CB}_\beta(g)\\ &= \mathrm{CB}_\alpha(g)\end{aligned}$$

and this concludes this point.

4. Let α be the Cantor–Bendixson rank of g. By definition $\mathrm{CB}_\alpha(g) = \emptyset$, and by the previous point $\mathrm{CB}_\alpha(f) \subseteq \mathrm{CB}_\alpha(g)$ so $\mathrm{CB}_\alpha(f)$ is empty. Since $\mathrm{CB}(f)$ is the least such ordinal, we have $\mathrm{CB}(f) \leq \alpha$. ⊣

Corollary 2.3. *For f in C and A a closed subset of $\mathsf{dom}\ f$, we have for all α countable that $\mathrm{CB}_\alpha(f|_A)$ is a subset of $\mathrm{CB}_\alpha(f)$ and so $\mathrm{CB}(f|_A) \leq \mathrm{CB}(f)$ holds. Moreover, when A is clopen we have $\mathrm{CB}_\alpha(f|_A) = \mathrm{CB}_\alpha(f) \cap A$ for all countable α.*

Proof. For the first sentence, remark that $f|_A = \tau f \sigma$ with $\sigma = \mathrm{Id}_A$ and $\tau = \mathrm{Id}_{f(A)}$, and apply points 3 and 4 of Proposition 2.2.

For the second part, we claim first that when A is clopen, an element x of A is f-isolated if and only if it is $f|_A$-isolated. The direct implication is given by point 2 of 2.2, let us prove the other one. If x is $f|_A$-isolated, there is an open neighborhood U of x in A such that $f|_A$ is constant on U. But since A is clopen, $A \cap U$ is an open neighborhood of x in $\mathsf{dom}\ f$ and f is constant on it, which means that x is f-isolated.

It remains to prove the equality. We proceed by induction on α, and follow the same pattern as in point 3 of 2.2.

For $\alpha = 0$ we have that $\mathsf{dom}\ (f|_A) = A = \mathsf{dom}\ f \cap A$ holds.

Suppose $\mathrm{CB}_\alpha(f|_A) = \mathrm{CB}_\alpha(f) \cap A$. Set $f' = f|_{\mathrm{CB}_\alpha(f)}$. Apply the claim to f' and get $I(f', A) = I(f') \cap A$. We hence have that $\mathrm{CB}_{\alpha+1}(f|_A) = \mathrm{CB}_{\alpha+1}(f) \cap A$ holds.

Suppose α is limit and for all $\beta < \alpha$ $\mathrm{CB}_\beta(f|_A) = \mathrm{CB}_\beta(f) \cap A$ holds. Then

$$\mathrm{CB}_\alpha(f|_A) = \bigcap_{\beta<\alpha} \mathrm{CB}_\beta(f|_A) = \bigcap_{\beta<\alpha} (\mathrm{CB}_\beta(f) \cap A) = A \cap \bigcap_{\beta<\alpha} \mathrm{CB}_\beta(f) = \mathrm{CB}_\alpha(f) \cap A$$

and this concludes the corollary. ⊣

We notice now that, given any $f \in \mathsf{C}$, the decreasing sequence $(\mathrm{CB}_\alpha(f))_{\alpha<\mathrm{CB}(f)}$ has a least element iff $\mathrm{CB}(f)$ is a successor, and in this case f is locally constant on this set. We now isolate the special class of the functions that are constant on this least closed set. We show that the more general functions can be decomposed into such simpler ones.

We say that $f \in \mathsf{C}$ is *simple* if $\mathrm{CB}(f) = \alpha + 1$ and $f|_{\mathrm{CB}_\alpha(f)}$ is constant, of range $\{y_f\}$.

Lemma 2.4. (*Decomposition Lemma*) *Given $f \in \mathsf{C}$, there is a countable partition in clopen sets $(A_i)_{i\in I}$ of $\mathsf{dom}\ f$ s.t. for all $i \in I$, $f|_{A_i}$ is simple, and if $\mathrm{CB}(f)$ is a successor ordinal, then $\mathrm{CB}(f|_{A_i}) = \mathrm{CB}(f)$ holds, otherwise we have $\mathrm{CB}(f|_{A_i}) < \mathrm{CB}(f)$.*

Proof. We proceed by induction on $\alpha = \mathrm{CB}(f)$. If $\mathrm{CB}(f) = 0$ then since $f \in \mathsf{C}$, $f = \emptyset$ holds.

Assume α is limit. Since $\mathrm{CB}_\alpha(f) = \emptyset$, we have $\mathsf{dom}\ f = \bigcup_{\beta\in\alpha} \mathrm{CB}_\beta(f)^c$, but $\mathrm{CB}_\beta(f)^c$ is open for all $\beta < \alpha$, and $\boldsymbol{\Sigma}^0_1$ has the generalised reduction property, so there is a countable partition $(A_i)_{i\in\omega}$ of $\mathsf{dom}\ f$ in clopen sets with the property that for all $i \in \omega$ there is some $\beta_i < \alpha$ such that $A_i \subseteq \mathrm{CB}_{\beta_i}(f)^c$ holds. Since $\mathrm{CB}(f|_{A_i}) < \alpha$ holds, we can apply the induction hypothesis to each $f|_{A_i}$ to obtain a countable partition $(A_{i,j})_{j\in N_i}$ of A_i in clopen sets of A_i such that $\mathrm{CB}(f_{A_{i,j}}) \leq \mathrm{CB}(f_{A_i})$ holds and $f|_{A_{i,j}}$ is simple. Since A_i is clopen for all $i \in I$, and $\mathrm{CB}(f|_{A_i}) < \mathrm{CB}(f)$ we have that for all $(i,j) \in \prod_{i\in I} N_i$, $A_{i,j}$ is clopen and $\mathrm{CB}(f_{A_{i,j}}) < \mathrm{CB}(f)$ holds, and $\{A_{i,j} : (i,j) \in \prod_{i\in I} N_i\}$ is the partition we were looking for.

Assume now that there is $\beta < \omega_1$ with $\alpha = \beta + 1$. This means that $f|_{\mathrm{CB}_\beta(f)}$ is locally constant, so $\{f^{-1}(\{y\}) : y \in \mathsf{im}\ f|_{\mathrm{CB}_\beta(f)}\}$ is an open countable partition of $\mathrm{CB}_\beta(f)$. It is induced by a countable family $(A_i)_{i\in I}$ of open sets of $\mathsf{dom}\ f$ such that every $\{A_i \cap \mathrm{CB}_\beta(f) : i \in I\}$ is a partition of $\mathrm{CB}_\beta(f)$ and $f|_{\mathrm{CB}_\beta(f)\cap A_i}$ is a constant function.

Hence the family $\{(\mathrm{CB}_\beta(f)^c \cup A_i) : i \in I\}$ is a countable open covering of $\mathsf{dom}\ f$. Use the generalised reduction property once more to obtain for all $i \in I$ an open subset A'_i of $\mathrm{CB}_\beta(f)^c \cup A_i$. Since $(A'_i)_{i\in I}$ is an open partition of $\mathsf{dom}\ f$, every A'_i is clopen. But $(A_i)_{i\in i}$ partitions $\mathrm{CB}_\beta(f)$ and for all i, $A_i \cap \mathrm{CB}_\beta(f)$ is non empty, so $A'_i \cap \mathrm{CB}_\beta(f)$ is also non empty. Since A'_i is clopen, we have by Corollary 2.3 that $\mathrm{CB}(f|_{A'_i}) \geq \beta + 1$. As it can not be above also by Corollary 2.3, $\mathrm{CB}(f|_{A'_i}) = \alpha$ holds. Finally every $f|_{A'_i}$ is simple because by definition $f|_{\mathrm{CB}_\beta(f)\cap A'_i}$ is constant. The partition we were looking for is $(A'_i)_{i\in I}$. ⊣

§3. **Operations on functions.** The Decomposition Lemma implies in particular that an identity function is the "direct sum" of simple identity functions. This is a well-known fact in the analysis of closed sets. In fact all closed sets can be decomposed using two types of summing operations, and those turn out to have some equivalent operations on functions.

Precisely, let $(A_i)_{i\in I}$ be a family of subsets of $\mathscr{N}$ with $I \in \omega + 1$ (I is an initial segment of ω). Define:

$$\bigoplus_{i\in I} A_i = \bigcup \{i^\frown x : i \in I, x \in A_i\},$$
$$\lim_{i\in I} A_i = \{0^\omega\} \cup \bigcup \{(0^i)^\frown 1^\frown x : i \in I, x \in A_i\}.$$

If $(f_i\colon A_i \to B_i)_{i\in I}$ is a family of functions, then the equivalent operations on functions are:

$$\begin{aligned} \bigoplus_{i\in I} f_i \colon \bigoplus_{i\in I} A_i &\longrightarrow \bigoplus_{i\in I} B_i \\ i^\frown x &\longmapsto i^\frown f_i(x), \end{aligned}$$

$$\lim_{i\in I} f_i\colon \lim_{i\in I} A_i \longrightarrow \lim_{i\in I} B_i$$
$$x \longmapsto \begin{cases} (0^i)^\frown 1^\frown f_i(x') & \text{if } x = (0^i)^\frown 1^\frown x', \\ 0^\omega & \text{otherwise.} \end{cases}$$

If $A_i = A$ holds for all $i \in I$, we set $\bigoplus_{i\in I} A_i = I \cdot A$ and when moreover $I = \omega$ we set $\lim_{i\in\omega} A_i = \lim A$, if $f_i = f$ for all $i \in I$ put $\bigoplus_{i\in I} f_i = I \cdot f$ and if moreover I is ω, put $\lim_{i\in\omega} f_i = \lim f$.

We notice now that if we see the identity as an operator between sets and functions then it commutes with those operations. Indeed both $\bigoplus_{i\in I} \mathrm{Id}_{A_i} = \mathrm{Id}_{\bigoplus_{i\in I} A_i}$ and $\lim_{i\in I} \mathrm{Id}_{A_i} = \mathrm{Id}_{\lim_{i\in I} A_i}$ hold. Now we say that a closed set F is *simple* if Id_F is simple, the well-known fact in the analysis of closed sets we were refering to may now be stated as follows.

FACT. *Let F be a countable closed set of rank $\alpha + 1$.*
There are $I \in \omega + 1$ and $F_i \subseteq F$ for $i \in I$ simple closed sets also of rank $\alpha + 1$ s.t. $F \cong \bigoplus F_i$.
Moreover, when F is simple, there are $F_i \subseteq F$ for $i \in \omega$ closed sets with $\mathrm{CB}(F_i) \leq \alpha$ s.t. $F \cong \lim F_i$.

PROOF. The first fact comes from the application of the Decomposition Lemma to Id_F, for the second one, we set $y_F = y_{\mathrm{Id}_F}$ and $F_i = F \cap [y_F|_i] \setminus [y_F|_{i+1}]$. ⊣

Furthermore, these operations preserve the following properties.

PROPOSITION 3.1. 1. *If $(A_i)_{i\in I}$ is a family of closed sets, then $\bigoplus_{i\in I} A_i$ and $\lim_{i\in I} A_i$ are also closed. When all A_i's are compact, so is $\lim_{i\in I} A_i$.*
2. *If for all $i \in I$, f_i is continuous (resp. one-to-one, onto, an embedding, an homeomorphism)*[1] *then so are $\bigoplus_{i\in I} f_i$ and $\lim_{i\in I} f_i$*
3. *If $(f_i)_{i\in I}$ is a family of functions in C, then we have*
$$\mathrm{CB}\Big(\bigoplus f_i\Big) = \sup\{\mathrm{CB}(f_i)\colon i \in I\}.$$
4. *If $(f_i)_{i\in\omega}$ is a family of functions in C and $(\mathrm{CB}(f_i))_{i\in\omega}$ is monotone,*
$$\mathrm{CB}(\lim_{i\in I} f_i) = \sup\{\mathrm{CB}(f_i)\colon i \in I\} + 1$$
holds.

PROOF. 1. For the first point, notice that $(\bigoplus A_i)^c = \bigoplus A_i^c$ and $(\lim A_i)^c = \lim A_i^c \setminus \{0^\omega\}$ which are open when the A_i's are closed.

For the second point, we let $(U_n)_{n\in\omega}$ be an open covering of $\lim A_i$. There is an integer n_0 s.t. $0^\omega \in U_{n_0}$, and $i_0 \in \omega$ s.t. $[0^{i_0}] \subseteq U_{n_0}$. But $A = \bigcup_{i<i_0} (0^i)^\frown 1^\frown A_i$ is compact as a finite union of compact sets, so there is an integer n_1 s.t. $(U_n)_{n<n_1}$ covers A, hence $\{U_{n_0}\} \cup \{U_n\colon n < n_1\}$ forms a finite covering of $\lim A_i$.

2. Notice that we have both
$$\Big(\bigoplus_{i\in I} f_i\Big)^{-1}(A) = \bigoplus_{i\in I} f_i^{-1}(A)$$
and $(\lim_{i\in I} f_i)^{-1}(A) = \lim_{i\in I} f_i^{-1}(A)$.

[1]That is a continuous bijection which inverse is also continuous.

3. Set $f = \bigoplus_{i \in I} f_i$. For all $i \in I$, by definition of $\bigoplus$, we have $f|_{[(i)]} \equiv f_i$, so by Proposition 2.2 $\mathrm{CB}\left(f|_{[(i)]}\right) = \mathrm{CB}(f_i)$ holds for all $i \in I$ and by Corollary 2.3 so does $\mathrm{CB}(f) \geq \sup\{\mathrm{CB}(f_i) \colon i \in I\}$.

But $[(i)]$ is a clopen set for all i, so by Corollary 2.3 again, if α stands for $\sup\{\mathrm{CB}(f_i) \colon i \in I\}$, we have

$$\mathrm{CB}_\alpha(f) = \bigcup_{i \in I} \mathrm{CB}_\alpha(f) \cap [(i)] = \bigcup_{i \in I} \mathrm{CB}_\alpha(f|_{[(i)]}) = \emptyset,$$

so $\alpha = \mathrm{CB}(f)$ holds, and it concludes this point.

4. Set $f = \lim_{i \in \omega} f_i$, $\alpha_i = \mathrm{CB}(f_i)$ and $\alpha = \sup\{\mathrm{CB}(f_i) \colon i \in I\}$. By definition of lim, $f|_{[(0^i)^\frown 1]} \equiv f_i$ for all integers i so $\mathrm{CB}(f|_{[(0^i)^\frown 1]}) = \alpha_i$ by Corollary 2.3.

Fix $\beta < \alpha$. The sequence $(\alpha_i)_{i \in \omega}$ is monotone, so there is an integer n s.t. $\alpha_i > \beta$ for all integers $i \geq n$. The set $[(0^i)^\frown 1]$ is clopen so by Corollary 2.3, $\mathrm{CB}_\beta(f) \cap [(0^i)^\frown 1]$ is non empty for all $i \geq n$. Hence by definition of lim, the point 0^ω is not $f|_{\mathrm{CB}_\beta(f)}$-isolated, and so 0^ω is in $\mathrm{CB}_\alpha(f)$.

But $\alpha = \sup\{\alpha_i \colon i \in \omega\}$, so for all integers i $\mathrm{CB}_\alpha(f) \cap [(0^i)^\frown 1]$ is empty, and finally 0^ω is the unique point of $\mathrm{CB}_\alpha(f)$. Hence $\mathrm{CB}(f) = \alpha + 1$ and we obtain the result. ⊣

There is a canonical way to associate a sequence of functions of strictly smaller rank to a simple function $f \in \mathsf{C}$. We just consider a partition of $\mathscr{N} \setminus \{y_f\}$ in clopen sets, and take the induced functions on the inverse image of each piece of this partition.

Precisely, for every integer i we define $f^{(i)}$ to be $f|_{A_i}$, where we take $A_i = f^{-1}([y_f|_i] \setminus [y_f|_{i+1}])$. Notice that $\mathrm{CB}(f^{(i)}) < \mathrm{CB}(f)$ for all $i \in \omega$, and $\mathsf{dom}\, f^{(i)} \cap \mathsf{dom}\, f^{(j)} = \emptyset$ holds whenever $i \neq j$.

We first state a sufficient condition for a function defined on a partition to be continuous.

Claim 3.2. *Given any function $f \colon \omega^\omega \to \omega^\omega$, any family $(A_i)_{i \in \omega}$ of pairwise disjoint clopen sets and $F = (\bigcup_{i \in \omega} A_i)^c$. Assume that*

- *f is continuous both on every A_i and on F,*
- *every sequence $(x_n)_{n \in \omega}$ which meets infinitely many A_i's*[2] *and converges to x in F satisfies that the sequence $(f(x_n))_{n \in \omega}$ converges to $f(x)$.*

Then f is continuous.

Proof. Let $(x_n)_{n \in \omega}$ be a sequence converging to x. Either x is in A_i for some integer i, then since A_i is open the sequence is cofinitely in it, and $(f(x_n))_{n \in \omega}$ converges indeed to $f(x)$ because f is continuous on A_i.

Or x is in F, then since every A_i is closed, no subsequence of $(x_n)_n$ is entirely in one A_i. Hence $(x_n)_n$ is either cofinitely in F or meets infinitely many A_i's. In both case $(f(x_n))_n$ converges to $f(x)$. Finally f is continuous. ⊣

Lemma 3.3. 1. *If $f \in \mathsf{C}$, and $(A_i)_{i \in I}$ is any partition of $\mathsf{dom}\, f$ into clopen sets, then $f \leq \bigoplus_{i \in I} f|_{A_i}$.*

2. *If f is simple, then $\bigoplus_{i \in \omega} f^{(i)} \leq f \leq \lim_{i \in \omega} f^{(i)}$ holds.*

[2] Meaning that $\{i \in \omega \colon \exists n \in \omega (x_n \in A_i)\}$ is infinite.

3. *If* $(f_i)_{i\in\omega}$ *and* $(g_i)_{i\in\omega}$ *are two sequences in* C *along with an increasing sequence of integers* $(k_i)_{i\in\omega}$ *s.t.* $f_i \leq g_{k_i}$ *then both* $\bigoplus_{i\in\omega} f_i \leq \bigoplus_{i\in\omega} g_i$ *and* $\lim_{i\in\omega} f_i \leq \lim_{i\in\omega} g_i$ *hold.*

PROOF. 1. Put $\sigma_0\colon \mathsf{dom}\ f \to \mathscr{N},\ x \mapsto i^\frown x$, with i the unique integer s.t. $x \in A_i$, then $\tau_0\colon \mathsf{im}\ (\bigoplus f|_{A_i}) \to \mathscr{N},\ i^\frown f(x) \mapsto f(x)$. The functions σ_0 and τ_0 are continuous, and $f = \tau_0(\bigoplus f|_{A_i})\sigma_0$ holds.

2. Let $\sigma_1\colon \mathsf{dom}\ \bigoplus f^{(i)} \to \mathscr{N},\ i^\frown x \mapsto x$ and $\tau_1\colon (\mathsf{im}\ f)\backslash\{y_f\} \to \mathsf{im}\ (\bigoplus f^{(i)})$, $f(x) \mapsto i^\frown f(x)$ with i the unique integer s.t. $f(x) \in A_i$ (it exists because $f(x) \neq y_f$). The functions σ_1 and τ_1 are continuous, and $\bigoplus f^{(i)} = \tau_1 f \sigma_1$ holds.

For the second reduction consider

$$\sigma_2\colon \mathsf{dom}\ f \longrightarrow \lim_{i\in\omega} \mathsf{dom}\ f^{(i)}$$
$$x \longmapsto \begin{cases} (0^i)^\frown 1^\frown x & \text{if } x \in \mathsf{dom}\ f^{(i)} \text{ for some } i \in \omega, \\ 0^\omega & \text{otherwise} \end{cases}$$

and

$$\tau_2\colon \lim_{i\in\omega} \mathsf{im}\ f^{(i)} \longrightarrow \mathsf{im}\ f$$
$$y \longmapsto \begin{cases} y' & \text{if } y = (0^i)^\frown 1^\frown y', \\ y_f & \text{otherwise.} \end{cases}$$

The construction provides $f = \tau_2(\lim f^{(i)})\sigma_2$. Apply Claim 3.2 to prove that σ_2 and τ_2 are continuous.

3. Set first $f_i\colon A_i \to B_i,\ g_i\colon C_i \to D_i$, and define an increasing sequence of integers $(k_i)_{i\in\omega}$, and two sequences of continuous functions (δ_i) and (ν_i) s.t. for all $i \in \omega$, $f_i = \nu_i g_{k_i} \delta_i$. Define then

$$\sigma_3\colon \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} C_i \quad \text{and} \quad \tau_3\colon \bigoplus_{i\in I} D_i \to \bigoplus_{i\in I} B_i$$
$$i^\frown x \mapsto k_i^\frown \delta_i(x) \qquad\qquad k_i^\frown x \mapsto i^\frown \nu_i(x),$$

Note that τ_3 is only partial. However, both σ_3 and τ_3 are continuous, and $\bigoplus f_i = \tau_3(\bigoplus g_i)\sigma_3$ holds.

Finally, consider

$$\sigma_4\colon \mathsf{dom}\ \lim f_i \longrightarrow \mathsf{dom}\ \lim g_i$$
$$x \longmapsto \begin{cases} (0^{k_i})^\frown 1^\frown \delta_i(x') & \text{if } x = (0^i)^\frown 1^\frown x', \\ 0^\omega & \text{otherwise} \end{cases}$$

and

$$\tau_4\colon \lim \mathsf{im}\ g_i \longrightarrow \lim \mathsf{im}\ f_i$$
$$y \longmapsto \begin{cases} (0^i)^\frown 1^\frown \nu_i(y') & \text{if } y = (0^{k_i})^\frown 1^\frown y', \\ y & \text{otherwise.} \end{cases}$$

These functions are continuous by Claim 3.2, and $\lim f_i = \tau_4(\lim g_i)\sigma_4$. $\dashv$

§4. Functions with compact domain. In this section we focus on $\mathsf{C}^\star$: the set of continuous functions with compact domain, and prove that this set is well-ordered by $\leq$, of order type $\omega_1 + 1$. The following proposition shows furthermore that $\mathsf{C}^\star$ is useful in the study of C, because the functions with compact domain appear to be a supporting pillar for C, as they are "elementary sub-functions" of every other function.

PROPOSITION 4.1. *We let $f \in \mathsf{C}_{\alpha+1}$. There is a compact subset K of* $\mathsf{dom}\ f$ *s.t. $f|_K$ is injective and $f|_K \in \mathsf{C}_{\alpha+1}$ holds.*

PROOF. We can suppose w.l.o.g., that f is simple, and we then proceed by induction on α.

If $\alpha = 0$, the function f is constant. We set $K = \{x\}$, with $x \in \mathsf{dom}\ f$.

If $\alpha \neq 0$, let $x \in \mathrm{CB}_\alpha(f)$, we set $A_i = [x|_i]\backslash[x|_{i+1}]$ and $f_i = f|_{A_i}$.

Then, when α is successor, say $K_i \subseteq \mathsf{dom}\ f_i$ is either the compact set given by induction hypothesis if $\mathrm{CB}(f_i) = \alpha$ and $\emptyset$ otherwise. We set $K = \{x\} \cup \bigcup K_i \cong \lim K_i$ and we are done.

Finally, if α is limit, for all $i \in \omega$ we choose a successor ordinal $\beta_i \leq \mathrm{CB}(f_i)$ s.t. $\sup\{\beta_i : i \in \omega\} = \alpha$. We use the induction hypothesis to find $K_i \subseteq \mathsf{dom}\ (f_i)$ and set $K = \{x\} \cup \bigcup K_i \cong \lim K_i$. ⊣

We now define an ordinal which is invariant for the relation $\leq$.

We let $f \in \mathsf{C}$, and set $N_f = |f(\mathrm{CB}_\alpha(f))|$ if $\mathrm{CB}(f) = \alpha + 1$, $N_f = 0$ otherwise. We set $tp(f) = (\mathrm{CB}(f), N_f)$. We call it the *type* of f, and for F a closed set we write $tp(F) = tp(\mathrm{Id}_F)$.

FACT. *The type along with the lexicographic ordering $\leq_{lex}$ is an ordinal invariant for $\leq$ on C, meaning that for all f, g in C, $f \leq g$ implies $tp(f) \leq_{lex} tp(g)$.*

PROOF. Assume that $f \leq g$ holds, and fix two continuous functions σ and τ such that $f = \tau g \sigma$ holds.

By Proposition 2.2, since $f \leq g$ holds, so does $\mathrm{CB}(f) \leq \mathrm{CB}(g)$. If $\mathrm{CB}(f) < \mathrm{CB}(g)$ then by definition of the type $tp(f) \leq_{lex} tp(g)$ and we are done. Suppose now that $\mathrm{CB}(f) = \mathrm{CB}(g) = \alpha$ for some $\alpha < \omega_1$, and prove by induction on α that $N_f \leq N_g$ holds.

If α is null or limit then $N_f = N_g = 0$ holds because f and g are in C.

Suppose α is successor and denote β its predecessor. Since $f = \tau g \sigma$ holds, by point 3 of Proposition 2.2, $\sigma(\mathrm{CB}_\beta(f))$ is a subset of $\mathrm{CB}_\beta(g)$. Hence τ is surjective from $\mathsf{im}\ g|_{\mathrm{CB}_\beta(g)}$ onto $\mathsf{im}\ f|_{\mathrm{CB}_\beta(f)}$, so we have $N_f \leq N_g$ and $tp(f) \leq tp(g)$ holds. ⊣

The main result in this section is the following one, stating that the type is a complete invariant for functions with compact domain. In particular, Theorem 1.1 is a direct consequence of it.

THEOREM 4.2. *We let f and g be two functions in C^*. Then the following holds*:

$$tp(f) \leq tp(g) \textit{ iff } f \leq g.$$

We need first to prove a lemma.

LEMMA 4.3. 1. *$N_f < \omega$ and there is a sequence $(f_i)_{i<N_f}$ of simple functions s.t. $f \equiv \bigoplus f_i$ and for all $i < N_f$ $\mathrm{CB}(f_i) = \mathrm{CB}(f)$,*
2. *$tp(f) = tp(\mathsf{im}\ f)$,*
3. *if K and L are two compact sets s.t. $tp(K) \leq tp(L)$, then $\mathrm{Id}_K \leq \mathrm{Id}_L$ holds.*

PROOF. 1. Since $\mathsf{dom}\ f$ is compact, and $\{\mathrm{CB}_\beta(f) : \beta < \mathrm{CB}(f)\}$ is a strictly decreasing sequence of closed sets and $\mathsf{dom}\ f = \mathrm{CB}_0(f)$, we have $\mathrm{CB}(f) = \alpha + 1$ for some $\alpha < \omega_1$. So $f|_{\mathrm{CB}_\alpha(f)}$ is a locally constant function on a compact set, hence it is of finite range. The partition induced by the Decomposition Lemma is then indeed finite, which allows us to conclude.

2. Following the first point we can assume that f is simple. Since $\mathsf{im}\ f$ is compact, and we have $f \leq \mathrm{Id}_{\mathsf{im}\ f}$, thus we also have $\mathrm{CB}(f) \leq \mathrm{CB}(\mathsf{im}\ f)$, so it only remains to prove $\mathrm{CB}(f) \geq \mathrm{CB}(\mathsf{im}\ f)$. We proceed by induction on $\mathrm{CB}(f)$.

We let K_i and L_i for all $i \in \omega$ be closed (hence compact) subsets of $\mathsf{dom}\ f$ and $\mathsf{im}\ f$ respectively, s.t. K_i is the domain of $f^{(i)}$ and L_i its image. By definition of $f^{(i)}$ (see Lemma 3.3), we have $\mathrm{CB}(f^{(i)}) = \beta_i < \mathrm{CB}(f)$ hence by induction hypothesis $\mathrm{CB}(L_i) \leq \beta_i$ holds. But we also have

$$\mathrm{CB}(\mathsf{im}\ f) = (\sup \mathrm{CB}(L_i)) + 1 \leq (\sup \beta_i) + 1 = \mathrm{CB}(f),$$

indeed $\mathsf{im}\ f \cong \lim L_i$ holds.

3. We can assume w.l.o.g., that K and L are simple and that $\mathrm{CB}(K) \leq \mathrm{CB}(L)$. We proceed by induction on $\mathrm{CB}(L)$ and write $K \cong \lim K_i$, $L \cong \lim L_i$ with $\sup(\mathrm{CB}(K_i)) + 1 = \mathrm{CB}(K)$ and $\sup(\mathrm{CB}(L_i)) + 1 = \mathrm{CB}(L)$. There is an increasing sequence of integers $(n_i)_{i\in\omega}$ s.t. by the induction hypothesis $\mathrm{Id}_{K_i} \leq \mathrm{Id}_{L_{n_i}}$ holds. We conclude using Lemma 3.3. ⊣

PROOF (OF THEOREM 4.2). ($\Leftarrow$) We already noticed that the type is an invariant. ($\Rightarrow$) By Proposition 4.1 and Lemma 4.3 there is a compact $K \subseteq \mathsf{dom}\ g$ s.t. $g|_K$ is one-to-one and $tp(g|_K) = tp(g)$ holds. Then $g|_K$ is continuous from K onto $\mathsf{im}\ g|_K$ so that we have

$$tp(K) \geq tp(\mathsf{im}\ g|_K) = tp(g|_K) = tp(g) \geq tp(f) = tp(\mathsf{im}\ f),$$

thus by Lemma 4.3 there are two continuous functions σ_0 and τ_0 s.t. $\mathsf{im}\ \sigma_0 = \mathsf{dom}\ \tau_0 = K$ and $\mathrm{Id}_{\mathsf{im}\ f} = \tau_0\sigma_0$. But g is one-to-one on $K = \mathsf{im}\ \sigma_0$ which is compact, so there exists a continuous function τ_1 s.t. $\tau_1 g|_K = \mathrm{Id}_K$. Set finally $\sigma = \sigma_0 f$ and $\tau = \tau_0\tau_1$, we have $f = \tau g \sigma$. ⊣

Towards the definition of a complete function for each class, we first define the canonical compact sets. For all limit ordinal α, we fix a sequence $(\beta_n)_{n\in\omega}$ s.t. for all $n \in \omega$, β_n is a successor ordinal, and $\sup\{\beta_n : n \in \omega\} = \alpha$. By induction on α we define a set $K_{\alpha+1}$ for any $\alpha < \omega_1$: $K_1 = \{0^\omega\}$; $K_{\alpha+2} = \lim K_{\alpha+1}$; and $K_{\alpha+1} = \lim_{n\in\omega} K_{\beta_n}$ if α is limit.

Using Proposition 3.1, an induction on α shows that, for all $\alpha < \omega_1$, $K_{\alpha+1}$ is a simple compact countable set of rank $\alpha + 1$.

COROLLARY 4.4. *Let* $f \in \mathsf{C}$,

1. *if* $\mathsf{dom}\ f$ *is compact then* $f \equiv \mathrm{Id}_{\mathsf{im}\ f}$,
2. *if* $\mathsf{dom}\ f$ *is compact and* $tp(f) = (\alpha, N)$ *then* $f \equiv N \cdot \mathrm{Id}_{K_\alpha} = \mathrm{Id}_{N\cdot K_\alpha}$,
3. *if* $\mathrm{CB}(f) \geq \alpha + 1$ *then* $\mathrm{Id}_{K_{\alpha+1}} \leq f$.

PROOF. 1. Use Theorem 4.2, noticing that $tp(f) = tp(\mathsf{im}\ f) = tp(\mathrm{Id}_{\mathsf{im}\ f})$.

2. Use Theorem 4.2, noticing that $tp(N \cdot \mathrm{Id}_{K_\alpha}) = (\alpha, N) = tp(f)$.

3. By Proposition 4.1 there is a compact $K \subseteq \mathsf{dom}\ f$ s.t.

$$\mathrm{CB}(f|_K) = \mathrm{CB}(f) \geq \alpha + 1,$$

hence $tp(f|_K) \geq tp(\mathrm{Id}_{K_{\alpha+1}})$ so $f \geq \mathrm{Id}_{K_{\alpha+1}}$ by theorem 4.2. ⊣

§5. **Better-quasi-ordering.** We now have all we need to prove Theorems 1.2 and 1.3, but we first recall some of the background, and some of the results about wqos and bqos.

We let $\leq_Q$ be a qo on a set Q. A sequence (q_n) of elements of Q is *bad* iff for all n, m in ω s.t. $n < m$ we have $q_n \not\leq q_m$. A qo is a wqo iff there is no bad sequences. We use the stronger notion of better-quasi-order.

We let $[\omega]^\omega$ denote the set of infinite subsets of ω with its canonical topology (induced by the topology on 2^ω considering that $[\omega]^\omega \subset \mathscr{P}(\omega) = 2^\omega$), and for $X \in [\omega]^\omega$, we let $[X]^\omega$ be the set of infinite subsets of X. If Q is a set, a *Q-array* is a function $f\colon [\omega]^\omega \to Q$, with domain $[X_0]^\omega$ for some $X_0 \in [\omega]^\omega$, with a countable image and s.t. $f^{-1}(\{y\})$ is Borel for all $y \in \mathsf{im}\, f$.

Now if $(Q, \leq_Q)$ is a qo, a Q-array f is *good* if $\exists X \in \mathsf{dom}\, f\,(f(X) \leq_Q f(X^+))$, where $X^+ = X \setminus \{\min(X)\}$. Otherwise f is *bad*.

A qo $(Q, \leq_Q)$ is a *better-quasi-order* (bqo) if every Q-array is good. Every bad sequence $(q_n)_{n\in\omega}$ in Q induces a bad Q-array defined by $f(X) = q_{\min X}$, so that every bqo is indeed a wqo.

We let Q and P be two qos, and $\varphi\colon Q \to P$ be an order-preserving function. If Q is a bqo then so is $\varphi(Q)$.

Suppose $(I \leq_I)$ and $(Q_i, \leq_i)$ for $i \in I$ are qos, then the sum $\sum_{i\in I} Q_i = \{(i, q)\colon i \in I, q \in Q_i\}$ equipped with the lexicographic order is a qo, and a bqo sum of bqos is still a bqo.

Given a qo $(Q, \leq_Q)$, we say that a Q-sequence is a function s from an ordinal α to Q. Set $lg(s) = \alpha$ the length of s, and $\tilde{Q}$ for the class of all Q-sequences. It is a qo for the following relation: for s, t in $\tilde{Q}$ we say $s \leq_{\tilde{Q}} t$ if there is a strictly increasing map $\varphi\colon lg(s) \to lg(t)$ s.t. for all $x < lg(s)$, $s(x) \leq t(\varphi(x))$.

We also consider $S_\omega(Q)$ the set of strictly increasing sequences of length ω in $\tilde{Q}$, quasi-ordered by $\leq_{\tilde{Q}}$.

For more results about bqos, see [8, 9, 15]. Notice that Simpson gives in [9] for the first time the definition of a bqo that we use. The original definition of Nash-Williams in [12] is still considered mainly for the reverse mathematics of bqo, see for instance [10]. We remind the following classical theorems about bqos:

THEOREM 5.1.

- (Nash-Williams) *If Q is* bqo *then so is $\tilde{Q}$.*
- (Pouzet, Sauer in [13]) *A* qo $(Q, \leq_Q)$ *is a* bqo *iff it is* wqo *and* $(S_\omega(Q), \leq)$ *is a* bqo.

In order to prove Theorem 1.2 we need to describe precisely the structure of the C_α layers. Namely, we want to prove that there is an order-preserving function from a bqo sum of the C_αs onto C. This statement, along with Proposition 2.1, will imply Theorem 1.2.

We first define the qo we use for the sum. Given an ordinal α, denote $(\lambda_\alpha, n_\alpha)$ the unique pair of ordinals s.t. λ_α is limit, $n_\alpha \in \omega$ and $\alpha = \lambda_\alpha + n_\alpha$.

Given two countable ordinals α and β we write $\alpha <^\bullet \beta$ iff:

$$\lambda_\alpha < \lambda_\beta \text{ or}$$
$$\lambda_\alpha = \lambda_\beta \text{ and } 2n_\alpha < n_\beta.$$

Set now $\alpha \leq^{\bullet} \beta$ if $\alpha <^{\bullet} \beta$ or $\alpha = \beta$, set also $\omega_1^{\bullet} = (\omega_1, \leq^{\bullet})$, and define

$$\begin{aligned}\pi\colon \sum_{\alpha \in \omega_1^{\bullet}} \mathsf{C}_\alpha &\longrightarrow \mathsf{C}\\ (\alpha, f) &\longmapsto f.\end{aligned}$$

The function π is the one we were searching. Theorem 1.2 comes from the following two results. The first one implies that π is order-preserving.

THEOREM 5.2. *For any countable ordinal* α, *we have*:

1. *If* $f \in \mathsf{C}_\alpha$ *and* $\mathrm{CB}(g) \geq \alpha + n_\alpha + 1$ *then* $f \leq g$.
2. *If* α *is limit then for all* f *and* g *in* C_α $f \equiv g$.

We first need the following technical claim.

CLAIM. *If* $\mathrm{CB}(f) = \alpha$ *is a limit ordinal then there is an antichain* $(v_i)_{i\in\omega} \subset \omega^{<\omega}$ *s.t.*

$$\sup\{\mathrm{CB}(f|_{f^{-1}([v_i])})\colon i \in \omega\} = \alpha.$$

PROOF. Set $\rho(v) = \mathrm{CB}(f|_{f^{-1}([v])})$ and $T = \{v \in \omega^{<\omega}\colon \rho(v) = \alpha\}$. Notice that T is indeed a tree, for $v \subseteq v'$ implies $\rho(v) \geq \rho(v')$. There are two possibilities.

The first one is that $[T]$ (the set of all infinite branches of T) is finite. If $\gamma = \sup\{\rho(v)\colon v \notin T\}$ then $\gamma = \alpha$, otherwise since in this case $f|_{f^{-1}([T])}$ is locally constant, $\alpha = \gamma + 1$ which contradicts the limitness of α. Just pick a convenient antichain in $\omega^{<\omega} \setminus T$ to conclude.

Suppose now as a second possibility that $[T]$ is infinite, then it is possible to find an infinite antichain in T, which also concludes. ⊣

PROOF (OF THEOREM 5.2). If $\alpha = 0$, $f = \emptyset$ so we assume $\alpha \neq 0$. We prove both points by the same induction on α.

- *First case*: α *limit*, $n_\alpha = 0$.

We first show the second point. In order to do this we only need to prove that, given any $f \in \mathsf{C}_\alpha$, we have $\bigoplus f|_{A_i} \leq f$, where $(A_i)_{i\in\omega}$ is the partition induced by the Decomposition Lemma.

Following Lemma 3.3 it means indeed that $\bigoplus f|_{A_i} \equiv f$. Thus, given f and g in C_α, with $(A_i)_{i\in\omega}$ and $(B_i)_{i\in\omega}$ the respective partitions from the Decomposition Lemma, $\beta_i < \alpha$ and $\gamma_i < \alpha$ the respective ranks of $f|_{A_i} = f_i$ and $g|_{B_i} = g_i$ we have $f \equiv \bigoplus f_i$ and $g \equiv \bigoplus g_i$ and, by Proposition 3.1,

$$\sup\{\beta_i\colon i \in \omega\} = \sup\{\gamma_i\colon i \in \omega\} = \alpha.$$

Define then an increasing sequence of integers $(k_i)_{i\in\omega}$ s.t. $\beta_i + n_{\beta_i} + 1 < \gamma_{k_i}$. By induction hypothesis $f_i \leq g_{k_i}$, so using Lemma 3.3 again we get $\bigoplus f_i \leq \bigoplus g_i$ which means that the relation $f \leq g$ is satisfied.

Since this proof is symmetric in f and g we get the result.

(Claim $\Rightarrow \bigoplus f_i \leq f$): Use the claim to pick for all $i \in \omega$ some $v_{n_i} \in \omega^{<\omega}$, s.t. $\mathrm{CB}(f|_{f^{-1}([v_{n_i}])}) \geq \beta_i + n_{\beta_i} + 1$ holds. By induction hypothesis $f_i \leq f|_{f^{-1}([v_{n_i}])}$, conclude using Lemma 3.3.

Let us now prove the first point. Consider $g \in \mathsf{C}_{\alpha+1}$, following the Decomposition Lemma we can suppose w.l.o.g. that g is simple. Then, by Lemma 3.3 $\bigoplus_{i\in\omega} g^{(i)} \leq$

g, and by Proposition 3.1 $\mathrm{CB}(\bigoplus_{i\in\omega} g^{(i)}) = \alpha$ so what precedes implies $f \equiv \bigoplus_{i\in\omega} g^{(i)} \leq g$ holds.

- *Second case*: *α successor*, $n_\alpha = n+1$.

By Corollary 4.4 we need to prove that for any $f \in \mathsf{C}_\alpha$, $f \leq \mathrm{Id}_{K_{\alpha+n+2}}$. Let $(A_i)_{i\in I}$ be the partition given by the Decomposition Lemma, and $f_i = f|_{A_i}$. Lemma 3.3 guarantees

$$f \leq \bigoplus_{i\in I} f_i \leq \bigoplus_{i\in I} \lim_{j\in\omega} f_i^{(j)}.$$

But $\mathrm{CB}(f_i^{(j)}) \leq \lambda_\alpha + n$ holds, thus by induction hypothesis $f_i^{(j)} \leq \mathrm{Id}_{K_{\alpha+n}}$. Hence by Lemma 3.3,

$$f \leq \bigoplus_{i\in I} \lim \mathrm{Id}_{K_{\alpha+n}} \leq \bigoplus_{i\in I} \mathrm{Id}_{\lim K_{\alpha+n}} \leq \bigoplus_{i\in I} \mathrm{Id}_{K_{\alpha+n+1}} \leq \mathrm{Id}_{K_{\alpha+n+2}}.$$

This concludes the second case, and finishes the proof. ⊣

Here is the final step to prove theorem 1.2:

Proposition 5.3. *The* qo $(\omega_1, \leq^\bullet)$ *is a* bqo.

Proof. Notice that

$$\begin{aligned}\phi\colon \omega_1^\bullet &\longrightarrow \sum \{(\omega, \leq^\bullet)\colon \lambda \in \omega_1, \lambda \text{ limit}\}\\ \alpha &\longmapsto (\lambda_\alpha, n_\alpha)\end{aligned}$$

is an order isomorphism by definition of $\leq^\bullet$, so we only need to show that $(\omega, \leq^\bullet)$ is a bqo. Indeed $\omega_1^\bullet$ will then be a well-ordered sum of bqo and therefore a bqo itself. For that purpose we use Pouzet's characterisation of bqos.

Now, given (a_n) a sequence of integers, there are two possibilities. Either $\{a_n\colon n \in \omega\}$ is finite (Case 1), or it is infinite (Case 2).

In Case 1: there are two integers n, m such that $n < m$ and $a_n = a_m$ hence (a_n) is not bad.

In Case 2: let n_0 be the smallest integer s.t. a_{n_0} is the minimum of $\{a_n\colon n \in \omega\}$, there is $m > n_0$ s.t. $2a_{n_0} < a_m$ hence (a_n) is not bad either.

So $(\omega, \leq^\bullet)$ is a wqo. But case (2) implies that any two $\leq^\bullet$-increasing sequences are $\leq^\bullet_\omega$-equivalents, so by Theorem 5.1 $(\omega, \leq^\bullet)$ is a bqo. ⊣

Proof (of Theorem 1.2). Let Q be any subset of C_∞. Suppose that for every $\alpha < \omega_1$, $Q \cap \mathsf{C}_\alpha$ is a bqo. By Theorem 5.2, the projection $\pi\colon \sum_{\alpha\in\omega_1^\bullet} Q \cap \mathsf{C}_\alpha \to Q \cap \mathsf{C}$ is an order preserving function from $\sum_{\alpha\in\omega_1^\bullet} Q \cap \mathsf{C}_\alpha$ which is a bqo sum (by the previous proposition) of bqos, to $Q \cap \mathsf{C}$. Hence $Q \cap \mathsf{C}$ is a bqo, and since $Q \setminus \mathsf{C}$ is either empty or a class of functions that are all equivalent by Proposition 2.1, and hence a bqo, we conclude that Q itself is a bqo as union of two bqos. ⊣

Suppose now one wants to show that a specific Q is indeed a bqo. By Theorem 1.2 it is enough to prove inductively that $Q_\alpha = Q \cap \bigcup_{\beta\leq\alpha} \mathsf{C}_\beta$ is a bqo for all $\alpha \in \omega_1$, and by Theorem 5.2 it is enough to prove it for $\alpha + 1$ supposing that is holds for α.

In the case of Q being the set of all functions in C_∞ which are equivalent to the identity on some closed subset of $\mathcal{N}$, this successor step works out.

As the map

$$\begin{aligned}\mathrm{Id}\colon \mathbf{\Pi}^0_1 &\longrightarrow Q\\ F &\longmapsto \mathrm{Id}_F\end{aligned}$$

is an embedding of the class of $\mathbf{\Pi}^0_1$-sets, quasi-ordered by bi-continuous embeddability, into Q, we have Theorem 1.3 as a corollary of the following.

THEOREM 5.4. *For all* $\alpha \in \omega_1$, Q_α *is a* bqo *implies* $Q_{\alpha+1}$ *is a* bqo.

PROOF. Set $Q^s_{\alpha+1} = \{f \in Q_{\alpha+1} : tp(f) = (\alpha+1, 1)\}$.

For $F \subseteq \mathcal{N}$ s.t. $\mathrm{Id}_F \in Q^s_{\alpha+1}$ define $F^{(i)} = \mathsf{dom}\,\mathrm{Id}^{(i)}_F$. As $F \cong \lim F^{(i)}$, we have

$$\mathrm{Id}_F \leq \mathrm{Id}_G \text{ iff } \mathrm{Id}_{\lim F^{(i)}} \leq \mathrm{Id}_{\lim G^{(i)}} \text{ iff } \lim \mathrm{Id}_{F^{(i)}} \leq \lim \mathrm{Id}_{G^{(i)}}$$

but by Lemma 3.3, $(\mathrm{Id}_{F^{(i)}})_{i\in I} \leq_{\tilde{Q}_\alpha} (\mathrm{Id}_{G^{(i)}})_{i\in I}$ implies $\lim \mathrm{Id}_{F^{(i)}} \leq \lim \mathrm{Id}_{G^{(i)}}$. This means that lim is an order-preserving function from Q^ω_α, the set of all infinite sequences of elements of Q_α, onto $Q^s_{\alpha+1}$. Hence if Q_α is a bqo, by Nash-Williams' theorem so is Q^ω_α and as a consequence $Q^s_{\alpha+1}$ is also a bqo.

Given now $F \in Q_{\alpha+1}$ we have by the Decomposition Lemma that $F \equiv \bigoplus F_i$ with F_i simple so by Lemma 3.3 again, there is an order-preserving function from $(Q^s_{\alpha+1})^\omega$ onto $Q_{\alpha+1}$. The same use of Nash-Williams' theorem as before gives us that $Q_{\alpha+1}$ is a bqo, and this concludes. $\dashv$

Concluding remarks. In this paper we essentially proved that a very peculiar fragment of C, which is itself a fragment of all Borel functions between Polish spaces, is a bqo with respect to $\leq$. We gave furthermore a criterion for the question we obviously need to ask:

QUESTION 5.5. *Is* $(\mathsf{C}, \leq)$ *a* bqo?

We conjecture that the answer is positive. We are far less positive concerning the next natural question.

QUESTION 5.6. *Is the set of all Borel functions between* 0*-dimensional Polish spaces, equipped with* $\leq$, *a* bqo?

Towards a positive answer to this general question, we ask whether the structure of $\equiv$-classes of Borel functions can be described by only looking at specific functions. Generalising the dichotomy principle generated by Proposition 2.1 as indicated below would be a major step in this direction.

QUESTION 5.7. *Given any Borel function between* 0*-dimensional Polish spaces, is there an open set* U *with* $\mathsf{im}\ f|_U$ *countable and* $f|_{U^c}$ $\equiv$*-equivalent to some Borel isomorphism*?

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