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Long Finite Sequences

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Let k be a positive integer. There is a longest finite sequence $x_1, ..., x_n$ in k letters in which no consecutive block $x_i, ..., x_{2i}$ is a subsequence of any later consecutive block $x_j, ..., x_{2j}$. Let $n(k)$ be this longest length. We prove that $n(1)=3$, $n(2)=11$, and $n(3)$ is incomprehensibly large. We give a lower bound for $n(3)$ in terms of the familiar Ackermann hierarchy. We also give asymptotic upper and lower bounds for $n(k)$. We view $n(3)$ as a particularly elemental description of an incomprehensibly large integer. Related problems involving binary sequences (two letters) are also addressed. We also report on some recent computer explorations of *R*. Dougherty which we use to raise the lower bound for $n(3)$.

Contents.

1. FINITENESS, AND $n(1)$, $n(2)$

We use Z for the set of all integers, Z^+ for the set of all positive integers, and N for the set of all nonnegative integers. Sequences can be either finite or infinite. For sequences x, it will be convenient to write $x[i]$ for x_i, which is the term of x with index i. Unless stated otherwise, all nonempty sequences are indexed starting with 1. Sometimes we consider sequences indexed starting at a positive integer greater than 1.

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Let $x[1], ..., x[n]$ and $y[1], ..., y[m]$ be two finite sequences, where $n, m \geqslant 0$. We use the usual notion of subsequence. Thus $x[1], ..., x[n]$ is a subsequence of $y[1], ..., y[m]$ if and only if there exist $1 \leqslant i_1 < \cdots < i_n \leqslant m$ such that for all $1 \leqslant j \leqslant n$, we have $x[j] = y[i_j]$.

We say that $x[1], ..., x[n]$ is a proper subsequence of $y[1], ..., y[m]$ if and only if $x[1], ..., x[n]$ is a subsequence of $y[1], ..., y[m]$ and $n < m$.

The focus of this paper is on finite combinatorics. But we start with the following theorem in infinitary combinatorics. It is a special case of the familiar fundamental result from wqo theory known as Higman's Lemma [Hi52]. For the sake of completeness, we give the Nash-Williams proof from [NW63] (adapted to this special case) of the second claim in Theorem 1.1. Note how remarkably nonconstructive this simplest of all proofs is.

Let $x = x[1], ..., x[n]$ be any sequence. We say that x has property * if and only if for no $i < j \leqslant n/2$ is it the case that $x[i], ..., x[2i]$ is a subsequence of $x[j], ..., x[2j]$. More generally, let $x = x[m], ..., x[n]$ be a sequence indexed from m. We say that x has property * if and only if for no $m \leqslant i < j \leqslant n/2$ is it the case that $x[i], ..., x[2i]$ is a subsequence of $x[j], ..., x[2j]$. These definitions are also made for infinite sequences by simply omitting "$\leqslant n/2$."

For any set A, let A^* be the set of all finite sequences from A (including the empty sequence).

THEOREM 1.1. *Let $k \geqslant 1$. No infinite sequence from $\{1, ..., k\}$ has property *. In fact, let $y[1], y[2], ...$ be elements of $\{1, ..., k\}^*$. Then there exists $i < j$ such that $y[i]$ is a subsequence of $y[j]$.*

Proof. To see that the second claim implies the first claim, let $x[1], x[2], ...$ be elements of $\{1, ..., k\}$. Define $y[i] = (x[i], ..., x[2i])$. According to the second claim, let $i < j$ be such that $y[i]$ is a subsequence of $y[j]$. Then $x[1], x[2], ...$ does not have property *.

Suppose the second claim is false. We say that $y[1], y[2], ...$ is bad if and only if it is a counterexample to the second claim. So there exists a bad sequence.

We now construct what Nash-Williams calls a minimal bad sequence as follows. Let $y[1]$ be an element of $[1, ..., k\}^*$ of minimal length that starts some bad sequence. Let $y[2]$ be an element of $\{1, ..., k\}^*$ of minimal length such that $y[1], y[2]$ starts some bad sequence. Let $y[3]$ be an element of $\{1, ..., k\}^*$ of minimal length such that $y[1], y[2], y[3]$ starts some bad sequence. Continue in this manner, defining $y[1], y[2], ...$ (The axiom of choice can be eliminated in an obvious way).

Now choose an infinite subsequence of the y's whose first terms are all the same (none of the y's can be empty). Call this $y[n] = z[1], z[2], z[3], ...$ Now let $z'[1], z'[2], ...$ be the result of chopping off the first terms. Then clearly $z'[1]$, $z'[2]$, ... is still bad. Also obviously $y[1]$, ..., $y[n-1]$, $z'[1]$, $z'[2]$, ... is also bad. But $z'[1]$ is shorter than $z[1] = y[n]$. This violates the definition of $y[n]$. Thus we have achieved the desired contradiction. Q.E.D.

THEOREM 1.2. *Let $k \geqslant 1$. There is a longest finite sequence from $\{1, ..., k\}$ with property* *.

Proof. Let $k \geqslant 1$, and consider the tree T of all elements of $\{1, ..., k\}^*$ which do not have property *, under extension. Then T is a finitely branching tree. If T has infinitely many nodes then T has an infinite path. (This is the fundamental König's tree lemma, or König's infinity lemma; see, e.g., [Le79, p. 298]). But this infinite path results in an infinite sequence from $\{1, ..., k\}$ without property *, contrary to Theorem 1.1. Hence T has finitely many nodes. Any node whose distance from the root of T (the empty sequence) is maximum will be a longest finite sequence from $\{1, ..., k\}$ with property *. I.e., the height of T is $n(k)$. Q.E.D.

We write $n(k)$ for the length of a longest sequence from $\{1, ..., k\}$ with property *. Obviously, $n(1) = 3$.

Consider the proof given above that $n(2)$ exists. We first give an extremely nonconstructive proof that no infinite sequence from $\{1, 2\}$ has property * (Theorem 1.1). Then we use the nonconstructive König tree lemma to conclude that $n(2)$ exists (Theorem 1.2).

But we now give a very constructive proof by actually computing $n(2)$. First observe that the eleven term sequence "12221111111" has property *. So $n(2) \geqslant 11$.

LEMMA 1.3. *Any sequence from $\{1, 2\}$ beginning with* "11," *with property* *, *must have length at most* 7.

Proof. Let $1, 1, x[3], ..., x[8]$ be from $\{1, 2\}$ and have property *. Then $x[3] = x[4] = 2$ by using $i = 1$ and $j = 2$. We have four cases:

(i) $x[5] = x[6] = 2$. Then $x[7] = x[8] = 1$ using $i = 3$ and $j = 4$. This is a contradiction using $i = 1$ and $j = 4$.

(ii) $x[5] = 2$, $x[6] = 1$. Then $x[7] = x[8] = 2$ using $i = 1$ and $j = 4$. This is a contradiction using $i = 2$ and $j = 4$.

(iii) $x[5] = 1$, $x[6] = 2$. Then $x[7] = x[8] = 2$ using $i = 1$ and $j = 4$. This is a contradiction using $i = 2$ and $j = 4$.

(iv) $x[5] = x[6] = 1$. This is a contradiction using $i = 1$ and $j = 3$.

Q.E.D.

Lemma 1.4. *Any sequence from* $\{1, 2\}$ *beginning with* "1211" *or* "1221" *with property* * *has length at most* 9.

Proof. First let $1211x[5] \cdots x[10]$ be from $\{1, 2\}$ and have property *. Then $x[5]=x[6]=1$ using $i=1$ and $j=3$. Also $x[7]=x[8]=1$ using $i=1$ and $j=4$. This is a contradiction using $i=3$ and $j=4$.

Second let $1221x[5] \cdots x[10]$ be from $\{1, 2\}$ and have property *. Then $x[5]=x[6]=1$ using $i=1$ and $j=3$. Also $x[7]=x[8]=1$ using $i=1$ and $j=4$. And $x[9]=x[10]=1$ using $i=1$ and $j=5$. This is a contradiction using $i=4$ and $j=5$. Q.E.D.

LEMMA 1.5. *Any sequence from* $\{1, 2\}$ *beginning with* "1222" *with property* * *has length at most* 11.

Proof. Let $1222x[5] \cdots x[12]$ be from $\{1, 2\}$ and have property *. Then $x[5]=x[6]=1$ using $i=2$ and $j=3$. Also $x[7]=x[8]=1$ using $i=1$ and $j=4$. And $x[9]=x[10]=1$ using $i=1$ and $j=5$. Furthermore $x[11]=x[12]=1$ using $i=1$ and $j=6$. This is a contradiction using $i=5$ and $j=6$. Q.E.D.

THEOREM 1.6. $n(2)=11$.

Proof. We have already remarked that "12221111111" has property *, and so $n(2) \geqslant 11$. Let $x[1], ..., x[12]$ be a sequence from $\{1, 2\}$ with property *. By Lemma 1.3, it cannot start with "11." By Lemmas 1.4 and 1.5, it cannot start with "1211," "1221," or "1222." It cannot start with "1212" using $i=1$ and $j=3$. Hence it cannot start with 12. By symmetry, it cannot start with "22" or "21." Hence it does not exist. Q.E.D.

Of course, we could also create a computer program to build the tree of sequences from $\{1, 2\}$ with property *. The tree would then be seen to close off at height 11 (the root is at height 0).

Since 12 is such a small number, it is feasible to use nothing but brute force by enumerating all sequences from $\{1, 2\}$ of length 12 and verifying that none of them have property * (preferably using a computer). But it is easy to imagine that in related cases of different size, the tree construction might be feasible where the brute force construction is not. See the discussion of $m(k)$ in Section 6 for a source of unexplored related problems.

As we shall see in Section 4, $n(3)$ is quite a bit larger than 11.

2. SEQUENCES OF FIXED LENGTH SEQUENCES

We now introduce (a version of) the familiar Ackermann hierarchy of functions. We define strictly increasing functions $A_k \colon Z^+ \to Z^+$, where

$k \geqslant 1$, as follows. $A_1(n) = 2n$. $A_{k+1}(n) = A_k A_k \cdots A_k(1)$, where there are n A_k's. This is iterated function application, and we have omitted parentheses.

Thus $A_2(n) = 2^n$. Also $A_3(n)$ is an exponential tower of 2's of height n.

The function $A(n) = A_n(n)$ is often called the Ackermann function. There are various minor modifications of this construction in the literature, including starting with $+1$ instead of doubling; or using a hierarchy of binary functions as Ackermann did originally, instead of a hierarchy of unary functions as we have done. These differences are inessential for our purposes and will not concern us here.

We perform a few illustrative calculations.

$A_3(1) = 2$. $A_3(2) = 4$. $A_3(3) = 16$. $A_3(4) = 2^{16} = 65{,}536$. $A_3(5) = 2^{65{,}536}$.

$A_4(1) = 2$. $A_4(2) = A_3 A_3(1) = A_3(2) = 4$. $A_4(3) = A_3 A_4(2) = A_3(4) = 2^{16} = 65{,}536$. $A_4(4) = A_3 A_4(3) = A_3(65{,}536)$, which is an exponential tower of 2's of height 65,536.

I submit that A_4 (4) is a ridiculously large number, but it is not an incomprehensibly large number. One can imagine a tower of 2's of a large height, where that height is 65,536, and 65,536 is not ridiculously large.

However, if we go much further, then a profound level of incomprehensibility emerges. The definitions are not incomprehensible, but the largeness is incomprehensible. These higher levels of largeness blur, where one is unable to sense one level of largeness from another.

For instance, $A_4(5)$ is an exponential tower of 2's of height $A_4(4)$.

It seems safe to assert that, say, $A_5(5)$ is incomprehensibly large. We propose this number as a sort of benchmark. In Section 4 we prove that $n(3) > A_7(184)$, which is considerably larger.

The following Theorem provides some useful background concerning the Ackermann hierarchy.

THEOREM 2.1. *For all $k, n \geqslant 1$, $n < A_k(n) < A_k(n+1)$. For all $k \geqslant 1$ and $n \geqslant 3$, $A_k(n) < A_{k+1}(n)$. For all $k, n \geqslant 1$, $A_k(n) \leqslant A_{k+1}(n)$. For all $k \geqslant 1$, $A_k(1) = 2$, $A_k(2) = 4$, and $A_k(3) \geqslant 2^{k+1}$. For all $k \geqslant 3$, $A_k(3) \geqslant A_{k-2}(2^k) > A_{k-2}(k-2)$. If $k \geqslant n+5$ then $A_k(3) > A_n(k)$.*

Proof. We prove by induction on k that for all n, $n < A_k(n) < A_k(n+1)$. This is clearly true if $k = 1$. Suppose this is true for $k \geqslant 1$.

First note that $A_{k+1}(n) = A_k A_k \cdots A_k(1)$, where there are n A_k's. By induction hypothesis, each application of A_k raises the argument. Hence $A_{k+1}(n) > n$.

Now $A_{k+1}(n+1) = A_k(A_{k+1}(n))$. Since A_k is strictly increasing and $n < A_{k+1}(n)$, we have $A_{k+1}(n) < A_{k+1}(n+1)$. This completes the induction.

For the second claim, we need to show that $A_k(n) < A_{k+1}(n)$, where $n \geqslant 3$. This is true for $k = 1$. Suppose this is true for all $k < m$, where $m \geqslant 2$. It suffices to show that $A_{m+1}(n) > A_m(n)$ for all $n \geqslant 3$. Fix $n \geqslant 3$.

$A_{m+1}(n) = A_m \cdots A_m(A_m A_m(1)) = A_m \cdots A_m(4)$, and $A_m(n) = A_{m-1} \cdots A_{m-1}(4)$, where there are $n-2$ A_m's and $n-2$ A_{m-1}'s. By the induction hypothesis and the first claim, we have $A_{m+1}(n) > A_m(n)$ as required.

The third claim follows from the second claim by the first two parts of the fourth claim.

For the fourth claim, $A_k(1) = 2$ is immediate, and $A_k(2) = 4$ is immediate by induction on k. We prove $A_k(3) \geqslant 2^{k+1}$ by induction on k. The cases $k = 1, 2$ are immediate. Suppose this is true for all $k < m$, where $m \geqslant 3$. $A_m(3) = A_{m-1} A_{m-1} A_{m-1}(1) = A_{m-1}(4) = A_{m-2} A_{m-1}(3) \geqslant A_{m-2}(2^m) \geqslant A_1(2^m) = 2^{m+1}$ as required.

For the fifth claim, let $k \geqslant 3$. Then $A_k(3) = A_{k-1}(4) = A_{k-2} A_{k-1}(3) \geqslant A_{k-2}(2^k) > A_{k-2}(k-2)$.

For the final claim, first note that $A_1(k-3) \geqslant k$ if $k \geqslant 6$. Therefore $A_k(3) > A_{k-2}(k-2) = A_{k-3} A_{k-2}(k-3) \geqslant A_n A_1(k-3) \geqslant A_n(k)$. Q.E.D.

Fix $k \geqslant 1$. We use the sum norm on N^k given by $|x| = x[1] + \cdots + x[k]$. We also use the partial ordering on N^k given by $x \leqslant^* y$ if and only if for all $1 \leqslant i \leqslant k$, $x[i] \leqslant y[i]$.

We define the function $f_k \colon Z^+ \to Z^+$ as follows. $f_k(p)$ is the length of the longest sequence $u[1], \ldots, u[n]$ from N^k such that

(i) each $|u[i]| \leqslant i + p - 1$;

(ii) for no $i < j$ is $u[i] \leqslant^* u[j]$.

We now prove the existence of each $f_k(p)$; i.e., that f_k does in fact have domain Z^+. We begin with the following infinitary theorem from wqo theory.

THEOREM 2.2. *Let $k \geqslant 1$ and $u[1], u[2], \ldots$ be elements of N^k. There exists $i < j$ such that $u[i] \leqslant^* u[j]$.*

Proof. Choose a subsequence whose first terms are increasing ($\leqslant$). Then choose a subsequence of that subsequence whose second terms are increasing ($\leqslant$). Continue in this way for k steps. In the last subsequence, every term is $\leqslant^*$ every later term. Q.E.D.

THEOREM 2.3. *For all $k, p \geqslant 1$, $f_k(p)$ exists.*

Proof. Fix $k, p \geqslant 1$ and form the tree T of all finite sequences from N^k obeying (i) and (ii) above such that no term is $\leqslant^*$ any later term. This is a finitely branching tree, where any infinite branch violates Theorem 2.2. Hence T has finitely many nodes. (See, e.g., [Le79, p. 298]). The height of the tree is $f_k(p)$. Q.E.D.

LEMMA 2.4. *Let* $p \geqslant 1$. $f_2(p) \geqslant 2^{p+2} - p - 3$.

Proof. Consider the sequence $(p, 0)$; $(p-1, 2), ..., (p-1, 0)$; $(p-2, 6), ..., (p-2, 0)$; $(p-3, 15), ..., (p-3, 0)$; ...; $(0, 2^{p+1}-2), ..., (0, 0)$. We have subdivided the sequence by semicolons, and the lengths of these sections are 1, 3, 7, 15,..., $2^{p+1}-1$. So (i) and (ii) are satisfied with $k=2$. The length of the sequence is $2^{p+2}-p-3$. Q.E.D.

LEMMA 2.5. *Let* $k, p \geqslant 1$. $f_{k+1}(p) > f_k \cdots f_k(2)$, *where there are* pf_k*'s.*

Proof. To obtain this lower bound on $f_{k+1}(p)$, we construct a sequence from N^{k+1} obeying (i) and (ii) with $k+1$, p, which is of length at least $f_k \cdots f_k(2)$, where there are p f_k's.

Start the sequence with $(p, 0, 0, ..., 0)$ in N^{k+1}. Now let $x[1], ..., x[n] \in N^k$ have properties (i) and (ii) for $p=2$, where $n = f(k, 2) = f_k(2)$. The next n terms are $(p-1, x[1]), (p-1, x[2]), ..., (p-1, x[n])$.

Now let $y[1], ..., y[m]$ in N^k have properties (i) and (ii) for $p = n = f_k(1)$, where $m = f(k, n) = f_k(n) = f_k\, f_k(2)$. Continue the sequence of elements of N^{k+1} with $(p-2, y[1]), ..., (p-2, y[m])$.

We can continue this process p times, where the last round of $k+1$-tuples is of the form $(0, z_1), ..., (0, z_r)$, where $r = f_k \cdots f_k(2)$, and there are p f_k's. Q.E.D.

THEOREM 2.6. *Let* $k \geqslant 2$ *and* $p \geqslant 1$. $f_k(p) \geqslant A_k(p+1)$. $f_k(1) > A_{k-1}(3)$. *For* $k \geqslant 3$, $f_k(1) > A_{k-2}(k-2)$. *The function f eventually strictly dominates every* A_n.

Proof. $f_2(p) \geqslant 2^{p+2} - p - 3 \geqslant 2^{p+1}$, which verifies the case $k=2$.

Suppose that for all $p \geqslant 1$, $f_k(p) \geqslant A_k(p+1)$, where $k \geqslant 2$. Let $p \geqslant 1$. Then $f_{k+1}(p) > f_k \cdots f_k(2) = A_k \cdots A_k(2) = A_k \cdots A_k(A_k(1)) = A_{k+1}(p+1)$, where there are p f_k's and A_k's.

For the second claim, $f_k(1) > f_{k-1}(2) \geqslant A_{k-1}(3)$ by Lemma 2.5 and the first claim. The third and fourth claims follow immediately from the second claim and Theorem 2.1. Q.E.D.

3. THE MAIN LEMMA

In this section we prove a Main Lemma concerning finite sequences from $\{2, 3\}$ which is used in Section 4 to obtain a lower bound for $n(3)$. Recall that $n(3)$ involves finite sequences from $\{1, 2, 3\}$.

Let n, m, i be positive integers, where $n < m < 2n+1$. We define $F(n, m, i)$ as follows. $F(n, m, 1) = n$, $F(n, m, 2) = m$, $F(n, m, 2i+1) = 2F(n, m, 2i-1) + 1$, $F(n, m, 2i+2) = 2F(n, m, 2i) + 1$.

Let n, m, b, k, d be positive integers such that $n < m$. We say that x is an n, m, b, k, d-sequence if and only if

(i) x is a sequence of elements from $\{2, 3\}$ indexed from n through $F(n, m, d+1) - 1$;

(ii) for all $1 \leqslant i \leqslant d$, $x[F(n, m, i)], ..., x[F(n, m, i) + b - 1] = 3$;

(iii) for all $1 \leqslant i \leqslant d$, $x[F(n, m, i) + b] = 2$;

(iv) for all $2 \leqslant i \leqslant d + 1$, $x[F(n, m, i) - 1] = 2$;

(v) for all $1 \leqslant i \leqslant d$, $x[F(n, m, i) + b], ..., x[F(n, m, i + 1) - 1]$ has exactly k 3's.

The letter "b" indicates the length of the blocks of 3's indicated in clause (ii). The letter "k" will eventually play the role of the "k" in the $f_k(p)$ of Section 2.

We introduce some useful terminology. For $1 \leqslant i \leqslant d$, we let $B_i(x)$ be the block $x[F(n, m, i)], ..., x[F(n, m, i) + b - 1]$; this is a block of 3's. For $1 \leqslant i \leqslant d$, we let $C_i(x)$ be the block $x[F(n, m, i) + b], ..., x[F(n, m, i + 1) - 1]$. Each $C_i(x)$ starts and ends with 2, and has exactly k 3's. Note that the $B_i(\mathrm{x})$ all have the same length, but the $C_i(x)$ will have differing lengths.

We will often leave off the x when we write $B_i(x)$ or $C_i(x)$. We use lth for the length of finite sequences.

Note that x is made up of the consecutive blocks $B_1, C_1, B_2, C_2, ..., B_d, C_d$.

LEMMA 3.1. *Let x be an n, m, b, k, d-sequence. Suppose m lies in the interval $((4n+1)/3, (3n+1)/2)$. Then for all $1 \leqslant i \leqslant d - 1$, $lth(C_{i+1}(x)) > lth(C_i(x)) \geqslant b + k + 2$. I.e., $F(n, m, i + 2) - F(n, m, i + 1) > F(n, m, i + 1) - F(n, m, i) \geqslant b + k + 2$. Also, for all $3 \leqslant i \leqslant d - 1$, $F(n, m, i + 1) - F(n, m, i) \geqslant 2b + 2k + 4$.*

Proof. Let $1 \leqslant i \leqslant d$. There are exactly $b + k$ 3's in the block $x[F(n, m, i)], ..., x[F(n, m, i) + b - 1], x[F(n, m, i) + b], ..., x[F(n, m, i + 1) - 1]$, according to clauses (ii) and (v). According to clauses (iii) and (iv), $x[F(n, m, i) + b] = x[F(n, m, i + 1) - 1] = 2$. Also $F(n, m, i) + b = F(n, m, i + 1) - 1$ is impossible by clause (v) and $k \geqslant 1$. Hence at least two of the terms in this block are 2. Therefore the number of terms is at least $b + k + 2$. Hence $F(n, m, i + 1) - F(n, m, i) \geqslant b + k + 2$.

It remains to show that $F(n, m, i + 2) - F(n, m, i + 1) > F(n, m, i + 1) - F(n, m, i)$. We first show this for $i = 1$. This reads: $2n + 1 - m > m - n$. I.e., $3n + 1 > 2m$, or $m < (3n + 1)/2$.

Next we show that $F(n, m, i + 2) - F(n, m, i + 1) > F(n, m, i + 1) - F(n, m, i)$ for $i = 2$. This reads: $2m + 1 - (2n + 1) > 2n + 1 - m$. I.e., $2m - 2n > 2n + 1 - m$, which is $3m > 4n + 1$, or $m > (4n + 1)/3$.

We now argue by induction. Suppose this is true strictly below $i \geqslant 3$. Now $F(n, m, i + 2) - F(n, m, i + 1) = 2(F(n, m, i) - F(n, m, i - 1))$. Also,

$F(n, m, i+1) - F(n, m, i) = 2(F(n, m, i-1) - F(n, m, i-2))$. The former is greater than the latter by the induction hypothesis.

The last claim follows since $F(n, m, i+1) = 2F(n, m, i-1) + 1$ and $F(n, m, i) = 2F(n, m, i-2) + 1$. Q.E.D.

Until Lemma 3.8, we fix x to be an n, m, b, k, d-sequence, where m lies in the interval $((4n+1)/3, (3n+1)/2)$. We will also assume that $k < b/3$.

A consecutive subsequence α of x is a sequence of the form $x[i]$, $x[i+1]$, ..., $x[j]$, $1 \leqslant i \leqslant j \leqslant F(n, m, d+1) - 1$.

We wish to consider two classes of consecutive subsequences of x. We let $lth(w)$ be the length of the finite sequence w.

The type 1 subsequences of x are the consecutive subsequences of x of the form $yB_p C_p B_{p+1} z$, where $1 \leqslant p \leqslant d-1$, y is a proper tail of C_{p-1}. Thus we allow one or both of y, z to be empty; also z can be C_{p+1} but y cannot be C_{p-1}. Of course, if $p = 1$ then y must be empty and the initial index is $F(n, m, p)$.

The type 2 subsequences of x are the consecutive subsequences of x of the form $3^r C_p B_{p+1} C_{p+1} 3^s w$, where

(i) $0 \leqslant r < b$;

(ii) $s = \min(b, 2(b-r))$;

(iii) if $s < b$ then w is empty;

(iv) if $s = b$ then w is an initial segment of C_{p+2},

where $1 \leqslant p \leqslant d-1$ and $3s$ represents s consecutive 3's. If $p = d-1$ then w must be empty.

LEMMA 3.2. *No type 1 subsequence is a type 2 subsequence. Let $n \leqslant i \leqslant F(n, m, d-1)$. Then $x[i]$, ..., $x[2i]$ is a type 1 or type 2 subsequence of x.*

Proof. For the first claim, let $yB_p C_p B_{p+1} z = 3^r C_q B_{q+1} C_{q+1} 3^s w$. We first dispense with the case $p = 1$. For here we have $B_1 C_1 B_2 z = 3^r C_q B_{q+1} C_{q+1} 3^s w$. But the left side begins with b 3's and the right side begins with r 3's, since C_q begins with 2. This is a contradiction. Thus $p \geqslant 2$.

We now consider the maximal blocks on each side of this equation consisting of at least b 3's. On the left side, there are exactly two such maximal blocks, B_p and B_{p+1}. This is because there are at most k 3's in y, C_p, z, and y ends with 2, C_p begins and ends with 2, and z begins with 2, and $k < b/3 < b$. On the right side, there are either 1 or 2 such maximal blocks, and these are B_{q+1} and possibly 3^s. This is because there are fewer than b 3's in 3^r, C_q, C_{q+1}, w, and C_q, C_{q+1} begin and end with 2, and w begins with 2. Since equality holds, there are two such maximal blocks, B_{q+1} and 3^s, and $s = b$.

Note that the number of terms strictly between these two maximal blocks is $\mathrm{lth}(C_p)$ on the left side and $\mathrm{lth}(C_{q+1})$ on the right side. Since equality holds, $\mathrm{lth}(C_p)=\mathrm{lth}(C_{q+1})$. By Lemma 3.1, the lengths of the C's are strictly increasing. Hence $p=q+1$. Also the number of terms before the first of these two maximal blocks for the left side is the same as the number of terms before the first of these two maximal blocks for the right side.

We can thus rewrite the equation in the form

$$yB_pC_pB_{p+1}z=3^rC_{p-1}B_pC_pB_{p+1}w$$

where y is a proper tail of C_{p-1}, z is an initial segment of C_{p+1}, and w is an initial segment of C_{p+1}, and where $\mathrm{lth}(y)=r+\mathrm{lth}(C_{p-1})$. But $\mathrm{lth}(y)<\mathrm{lth}(C_{p-1})$.

For the second claim, let i be as given. Then $2i\leqslant F(n,m,d+1)-1$, and so $x[i],...,x[2i]$ is a consecutive subsequence of x.

First suppose that i is at the beginning of B_p, $p\leqslant d-1$. I.e., $i=F(n,m,p)$. Then $F(n,m,p+2)=2i+1$. Hence $x[i],...,x[2i]=B_pC_pB_{p+1}C_{p+1}$, which is of type 1.

Next suppose that i is in C_p, but not at the beginning of C_p, $p\leqslant d-2$. Then $F(n,m,p)+b<i<F(n,m,p+1)$. Hence $2F(n,m,p)+2b<2i<2F(n,m,p+1)$. So $F(n,m,p+2)+b<2i<F(n,m,p+3)$. Therefore $2i$ lies in C_{p+2}. Hence $x[i],...,x[2i]$ is of the form $yB_{p+1}C_{p+1}B_{p+2}z$, where y is a proper tail of C_p, and z is an initial segment of C_{p+2}, which is of type 1.

Now suppose that i is at the beginning of C_p, $p\leqslant d-2$. Then $i=F(n,m,p)+b$. Hence $F(n,m,p+2)+b<2i=2F(n,m,p)+2b=F(n,m,p+2)+2b-1<F(n,m,p+3)$, using Lemma 3.1. Therefore $x[i],...,x[2i]$ is of the form $C_pB_{p+1}C_{p+1}B_{p+2}w$, where w is an initial segment of C_{p+2}. Also note that $B_{p+2}=3^b$, and $b=\min(b,2(b-0))$, and so $x[i],...,x[2i]$ is of type 2.

Finally suppose that i is in B_p, but not at the beginning of B_p, $p\leqslant d-2$. Then $F(n,m,p)+1\leqslant i\leqslant F(n,m,p)+b-1$. Then $F(n,m,p+2)\leqslant 2i\leqslant 2F(n,m,p)+2b-2=F(n,m,p+2)+2b-3<F(n,m,p+3)$, using Lemma 3.1. Hence $2i$ lies in B_{p+2} or C_{p+2}.

Let $r=F(n,m,p)+b-i$. Then $0\leqslant r<b$. First suppose that $r\geqslant b/2$. Then $F(n,m,p)+b-i\geqslant b/2$, and so $i\leqslant F(n,m,p)+b/2$. Hence $2i\leqslant 2F(n,m,p)+b=F(n,m,p+2)+b-1$, and so $2i$ lies in B_{p+2}. Hence $x[i],...,x[2i]$ is of the form $3^rC_pB_{p+1}C_{p+1}3^s$, where $0\leqslant s\leqslant b$. Now the position at the end of this sequence is the position of the front of B_{p+2} plus $s-1$, which is $F(n,m,p+2)+s-1=2F(n,m,p)+s=2i$. Also $i=F(n,m,p)+b-r$. Hence $2F(n,m,p)+s=2F(n,m,p)+2b-2r$. So $s=2(b-r)$. I.e., $s=\min(b,2(b-r))$, using $r\geqslant b/2$.

Now suppose that $r<b/2$. Then $F(n, m, p)+b-i<b/2$, and so $i> F(n, m, p)+b/2$. Hence $2i>2F(n, m, p)+b=F(n, m, p+2)+b-1$. Therefore $2i$ lies in C_{p+2}. Hence $x[i], ..., x[2i]$ is of the form $3^r C_p B_{p+1} C_{p+1} B_{p+2} w$, where w is an initial segment of C_{p+2}. And clearly $\min(b, 2(b-r))=b$.

Q.E.D.

Let α, β be two consecutive subsequences of x. A lifting of α into β is a strictly increasing map h: $i, ..., j\} \rightarrow \{i', ..., j'\}$, where

(i) $\alpha=x_i, ..., x_j$;

(ii) $b=x_{i'}, ..., x_{j'}$;

(iii) for all $i \leqslant p \leqslant j$, $h(p)>i$ and $x[p]=x[h(p)]$.

We say that the term $x[i]$ in α is sent to the term $x[h(i)]$ in β.

LEMMA 3.3. *Let h be a lifting from the consecutive subsequence α into the consecutive subsequence β. Then for all $m \in \mathrm{dom}(h)$, $h(m)>m$. If h sends C_p into C_q, then $p<q$ and C_p is a proper subsequence of C_q.*

Proof. Clearly by induction on m, we see that for all $i \leqslant m \leqslant j$, $h(m)>m$. Now suppose that h sends C_p into C_q. I.e., h sends the indices of the terms of C_p in x into the indices of the terms of C_q in x. Then the index in x of the first term of C_p is sent to a greater index in x, which must be the index of some term of C_q. Hence the index in x of the first term of C_p is smaller than the index in x of the first term of C_q. Therefore $p<q$. Since the lengths of the C_p are strictly increasing, we see that C_p is a proper subsequence of C_q in the usual sense. Q.E.D.

LEMMA 3.4. *Let h be a lifting from the type 1 subsequence $yB_pC_pB_{p+1}z$ into the type 1 subsequence $y'B_qC_qB_{q+1}z'$. Then C_p is a proper subsequence of C_q.*

Proof. No term of C_p is sent into y' since it has at least b 3's to its left, and $b>k$. No term of C_p is sent into z', since it has at least b 3's to its right, and $b>k$. Therefore the first and last terms of C_p, which are 2's, must be sent into C_q. I.e., C_p is sent into C_q. Now apply Lemma 3.3.

Q.E.D.

LEMMA 3.5. *Let h be a lifting from the type 1 subsequence $yB_pC_pB_{p+1}z$ into the type 2 subsequence $3^rC_qB_{q+1}C_{q+1}3^sw$. Then C_p is a proper subsequence of C_q or C_{q+1}.*

Proof. We divide the argument into cases.

Case 1. $r < b - k$. No term of C_p is sent into C_q since there are at least b 3's to its left, and $b > r + k$. No term of C_p is sent into w, since it has at least b 3's to its right, and $b > k$. Therefore the first and last terms of C_p, which are 2's, must be sent into C_{q+1}. Hence C_p is sent into C_{q+1}.

Case 2. $r \geqslant b - k$. Then $s = \min(b, 2(b-r)) \leqslant 2k < 2b/3 < b$, and hence w is empty. No term of C_p is sent into C_{q+1}, since it has at least b 3's to its right, and $b > 3k \geqslant k + s$. Therefore the first and last terms of C_p is sent into C_q. Hence C_p is sent into C_q. Q.E.D.

LEMMA 3.6. *Let h be a lifting from the type 2 subsequence $3^r C_q B_{q+1} C_{q+1} 3^s w$ into the type 1 subsequence $y B_p C_p B_{p+1} z$. Then C_{q+1} or C_q is a proper subsequence of C_p.*

Proof. We divide the argument into cases.

Case 1. $s > k$. No term of C_{q+1} is sent into z, since there are at least s 3's to its right, and $s > k$. No term of C_{q+1} is sent into y since there are at least b 3's to its left, and $b > k$. Therefore the first and last terms of C_{q+1} are sent into C_p. Hence C_{q+1} is sent into C_p.

Case 2. $s \leqslant k$. I.e., $\min(b, 2(b-r)) \leqslant k$. Hence $2(b-r) \leqslant k$. So $2r \geqslant 2b - k$, and hence $r \geqslant b - (k/2) > 3k - k/2 > k$. Also since $s < b$, w must be empty.

No term of C_q is sent into y, since it has r 3's to its left, and $r > k$. No term of C_q is sent into z, since it has at least b 3's to its right and $b > k$. Therefore the first and last terms of C_q are sent into C_p. Hence C_q is sent into C_p. Q.E.D.

LEMMA 3.7. *Let h be a lifting from the type 2 subsequence $3^r C_p B_{p+1} C_{p+1} 3^s w$ into the type 2 subsequence $3^{r'} C_q B_{q+1} C_{q+1} 3^{s'} w'$. Then either C_p is a proper subsequence of C_q or C_p is a proper subsequence of C_{q+1} or C_{p+1} is a proper subsequence of C_{q+1}.*

Proof. We divide the argument into cases.

Case 1. $s > k$. No term of C_{p+1} is sent into w' since there are at least s 3's to its right, and $s > k$. No term of C_{p+1} is sent into C_q since there are at least $r + k + b$ 3's to its left, and $r + k + b > r' + k$ (since $b > r'$). Therefore the first and last terms of C_{p+1} are sent into C_{q+1}. Hence C_{p+1} is sent into C_{q+1}.

Case 2. $s' < b$. Then w' is empty. No term of C_p is sent into C_{q+1} since there are at least $b + k + s$ 3's to its right, and $b + k + s > s' + k$. Therefore the first and last terms of C_p are sent into C_q. Hence C_p is sent into C_q.

Case 3. $s \leqslant k$ and $s' = b$. Thus $\min(b, 2(b-r)) \leqslant k$, and so $2(b-r) \leqslant k$. Hence $2b - k \leqslant 2r$, and so $r \geqslant (2b-k)/2$.

Also $\min(b, 2(b-r')) = b$. Hence $2(b-r') \geqslant b$. So $b \geqslant 2r'$. Therefore $r' \leqslant b/2$.

Note that $r \geqslant (2b-k)/2 \geqslant b/2 - k/2 + b/2 \geqslant r' + (b-k)/2 > r' + (3k-k)/2 = r' + k$.

No term of C_p is sent into C_q since there are at least r 3's to its left, and $r > r' + k$. No term of C_p is sent into w', since there are at least $b+k+s$ 3's to its right, and $b+k+s > k$. Therefore the first and last terms of C_p are sent into C_{q+1}. Hence C_p is sent into C_{q+1}. Q.E.D.

LEMMA 3.8. *Let n, m, b, k, d be positive integers such that m lies in the interval $((4n+1)/3, (3n+1)/2)$, and $k < b/3$. Let x be an n, m, b, k, d-sequence. Suppose there exists $n \leqslant i < j \leqslant F(n, m, d-1)$ such that $x[i], ..., x[2i]$ is a subsequence of $x[j], ..., x[2j]$. Then there exists $i < j \leqslant d$ such that C_i is a proper subsequence of C_j.*

Proof. Let n, m, b, k, d, x, i, j be as given. By Lemma 3.2, we see that $\alpha = x[i], ..., x[2i]$ and $\beta = x[j], ..., x[2j]$ are both consecutive subsequences of type 1 or 2. Also, let h: $\{i, ..., 2i\} \to \{j, ..., 2j\}$ be given by the subsequence relation.

We claim that h is a lifting from α into β. To see this, we argue by induction on $t = i, ..., 2i$, that $h(t) > t$. Clearly $h(i) > i$. Suppose $h(t) > t$, $i \leqslant t < 2i$. Then $h(t+1) > h(t) \geqslant t+1$, and so $h(t+1) > t+1$ as required.

We now see that exactly one of Lemmas 3.4–3.7 applies to h, α, β. Therefore we obtain i, j such that C_i is a proper subsequence of C_j. Since the lengths of the C's are strictly increasing, we also have $i < j$. Q.E.D.

We now put Lemma 3.8 in a more convenient form, eliminating the variable m.

LEMMA 3.9. *Let n, b, k, d be positive integers, where $n \geqslant 2$ and $k < b/3$. Let x be a $2n$, $3n$, b, k, d-sequence. Suppose there exists $2n \leqslant i < j \leqslant F(2n, 3n, d-1)$ such that $x[i], ..., x[2i]$ is a subsequence of $x[j], ..., x[2j]$. Then there exists $i < j \leqslant d$ such that C_i is a proper subsequence of C_j.*

Proof. By Lemma 3.8 we have only to verify $3n \in ((8n+1)/3, (6n+1)/2)$. Q.E.D.

We now refine Lemma 3.9, where we place $3^{n-1}2$ in front of the $2n$, $3n$, b, k, d-sequence.

Note that in any $2n$, $3n$, b, k, d-sequence, the length of C_1 is $n-b$, and the length of C_2 is $n-b+1$.

A strong $2n$, $3n$, b, k, d-sequence is a $2n$, $3n$, b, k, d-sequence x such that

(i) $2 \leqslant k < b/3$;

(ii) $n \geqslant 3b + 4k + 2$;

(iii) C_1 ends with 3^k22;

(iv) C_2 ends with 3^k2.

MAIN LEMMA. *Let n, b, k, d be positive integers, and let x be a strong $2n$, $3n$, b, k, d-sequence. Let $x' = 3^{n-1}2x$, where we view x' as being indexed from n. Suppose there exists $n \leqslant i < j \leqslant F(2n, 3n, d-1)$ such that $x'[i], ..., x'[2i]$ is a subsequence of $x'[j], ..., x'[2j]$. Then there exists $i < j \leqslant d$ such that $C_i(x)$ is a proper subsequence of $C_j(x)$.*

We will prove the Main Lemma according to the forms of $x'[i], ..., x'[2i]$ and $x'[j], ..., x'[2j]$, just as we proved Lemma 3.8. Obviously Lemma 3.9 takes care of $2n \leqslant i < j \leqslant F(2n, 3n, d-1)$. We need to do some extra related work in order to handle the case $n \leqslant i < 2n$, which arises because of the prefix $3^{n-1}2$.

We fix n, b, k, d, x, x' according to the hypotheses of the Main Lemma.

LEMMA 3.10. *Let $n \leqslant i < 2n$. Then the consecutive subsequence $x'[i], ..., x'[2i]$ of x' is of exactly one of the following forms:*

(I) $3^t2B_1C_1B_2z$, *where* $0 \leqslant t \leqslant n-1$, *and z is a proper initial segment of* C_2;

(II) $3^t2B_1C_13^s$, *where* $0 \leqslant t \leqslant n-1$, $0 \leqslant s < b$;

(III) 3^t2B_1z, *where* $0 \leqslant t \leqslant n-1$, *and z is a proper initial segment of* C_1;

(IV) 3^t23^s, *where* $0 \leqslant t \leqslant n$-1, *and* $1 \leqslant s < b$.

Proof. The relevant initial segment of x' is $3^{n-1}2B_1C_1B_2C_2$, where 3^{n-1} starts at position n and ends at position $2n-2$, B_1 starts at position $2n$, C_1 starts at position $2n+b$, B_2 starts at position $3n$, C_2 starts at position $3n+b$, and C_2 ends at position $4n$.

Clearly $x'[i], ..., x'[2i]$ starts somewhere in the displayed $3^{n-1}2$, and must end somewhere from the beginning of B_1 to before the next to last position in C_2. Thus $x'[i], ..., x'[2i]$ starts with 3^t, where $0 \leqslant t \leqslant n-1$. And it either ends somewhere in B_1 but not at the end of B_1 (Case IV), or ends at the end of B_1 (Case III), or ends somewhere in C_1 but not at the end of C_1 (Case III), or ends at the end of C_1 (Case II), or ends somewhere in B_2 but not at the end of B_2 (Case II), or ends at the end of B_2 (Case I), or ends somewhere in C_2 but not at the end of C_2 (Case I). Q.E.D.

We refer to these as the type I, II, III, IV subsequences of x'. Here $n \leqslant i < 2n$ is required.

LEMMA 3.11. *Let $n \leqslant i < j < 2n$ and h be a lifting from the type I subsequence $x'[i], ..., x'[2i] = 3^t 2 B_1 C_1 B_2 z$ into the type I subsequence $x'[j], ..., x'[j] = 3^{t'} 2 B_1 C_1 B_2 z'$. Then we obtain a contradiction.*

Proof. Since $i < j$, we have $t > t'$. The displayed 2 is sent into $C_1 B_2 z'$ since it has t 3's to its left and $t > t'$. But it also has $b + k + b$ 3's to its right and $b + k + b > k + b + k$. This is the desired contradiction. Q.E.D.

LEMMA 3.12. *Let $n \leqslant i < j < 2n$ and h be a lifting from the type* II *subsequence $x'[i], ..., x'[2i] = 3^t 2 B_1 C_1 3^s$ into the type* I *subsequence $x'[j], ..., x'[2j] = 3^{t'} 2 B_1 C_1 B_2 z'$. Assume $s > k$. Then we obtain a contradiction.*

Proof. No term of C_1 is sent into z', since there are at least s 3's to its right, and $s > k$. No term of C_1 is sent to the displayed 2, since there are at least $b + t$ 3's to its left, and $b + t > t'$. Therefore the first and last terms of C_1 are sent into C_1. This is a contradiction. Q.E.D.

LEMMA 3.13. *Let $n \leqslant i < j < 2n$ and h be a lifting from the type* II *subsequence $x'[i], ..., x'[2i] = 3^t 2 B_1 C_1 3^s$ into the type* I *subsequence $x'[j], ..., x'[2j] = 3^{t'} 2 B_1 C_1 B_2 z'$. Assume $s \leqslant k$. Then C_1 is a proper subsequence of C_2.*

Proof. Note that $2i = 3n + s - 1 \leqslant 3n + k - 1$. Also note that $2j \geqslant 3n + b - 1$. And $t = 2n - i - 1$, $t' = 2n - j - 1$. Therefore $t - t' = j - i \geqslant (b - k)/2 > k$, since $b > 3k$.

No term of C_1 is sent into C_1 or to the displayed 2, since there are at least $t + b$ 3's to its left and $t + b > t' + b + k$. Therefore the first and last terms of C_1 are sent into z'. Hence each term of C_1 is sent into z'. Thus C_1 is a subsequence of z'. Q.E.D.

LEMMA 3.14. *Let $n \leqslant i < j < 2n$ and h be a lifting from the type* II *subsequence $x'[i], ..., x'[2i] = 3^t 2 B_1 C_1 3^s$ into the type* II *subsequence $x'[j], ..., x'[2j] = 3^{t'} 2 B_1 C_1 3^{s'}$. Then we obtain a contradiction.*

Proof. Since $t > t'$, the displayed 2 must be sent into C_1. Therefore the first and last terms of C_1 are sent into C_1. Hence C_1 is sent into C_1, which is a contradiction. Q.E.D.

LEMMA 3.15. *Let $n \leqslant i < j < 2n$ and h be a lifting from the type* III *subsequence $x'[i], ..., x'[2i] = 3^t 2 B_1\, y$ into the type* I *subsequence $x'[j], ..., x'[2j] = 3^{t'} 2 B_1 C_1 B_2 z$. Then we obtain a contradiction.*

Proof. Since $t > t'$, the displayed 2 is sent into C_1 or z. The displayed 2 is not sent into z, since it has at least b 3's to its right, and $b > k$. Hence the displayed 2 is sent into C_1. No term of y is sent into C_1, since it has

b 3's to its left and to the right of the displayed 2, and $b>k$. Hence the first term of y (if it exists) is sent into z. So y is sent into z.

Note that $t=2n-i-1$ and $t'=2n-j-1$. Also $2i=2n+b-1+lth(y)$, and $2j \geqslant 3n+b-1+lth(z)$. Hence $2j-2i \geqslant n+lth(z)-lth(y)$. So $t-t' \geqslant (n+lth(z)-lth(y))/2$.

There are t 3's to the left of the displayed 2 in $3^t 2B_1\, y$, and at most $t'+b+k$ 3's to the left of where the displayed 2 is sent into $3^{t'}2B_1 C_1 B_2 z$. Hence $t \leqslant t'+b+k$, and so $t-t' \leqslant b+k$. Hence $(n+lth(z)-lth(y))/2 \leqslant b+k$. Therefore $n+lth(z)-lth(y) \leqslant 2b+2k$. Since $n>2b+2k$, we see that $lth(z)<lth(y)$, contradicting that y is sent into z. Q.E.D.

LEMMA 3.16. *Let* $n \leqslant i<j<2n$ *and* h *be a lifting from the type* III *subsequence* $x'[i], ..., x'[2i]=3^t 2B_1\, y$ *into the type* II *subsequence* $x'[j], ..., x'[2j]=3^{t'}2B_1 C_1 3^s$. *Then we obtain a contradiction.*

Proof. Since $t>t'$, the displayed 2 is sent into C_1. Since $b>k$, some term in B_1 is sent into 3^s. Hence y is sent into 3^s. Since y begins with 2, y is empty.

Also, since the displayed 2 is sent into C_1, we see that $b \leqslant k+s$, by looking to the right of the displayed 2. And by counting the total number of 3's, we see that $t+b \leqslant t'+b+k+s$, and so $t+b \leqslant t'+b+k+s$, and so $t \leqslant t'+k+s \leqslant t'+b+k$. Hence $t-t' \leqslant b+k$.

We have $2i=2n+b-1$ and $t=2n-i-1$. Also $2j=3n+s-1$ and $t'=2n-j-1$. Hence $t-t'=j-i=(3n+s-1)/2-(2n+b-1)/2=(n+s-b)/2$. Since $t-t' \leqslant b+k$, we have $(n+s-b)/2 \leqslant b+k$, and so $n+s-b \leqslant 2b+2k$. Hence $n \leqslant 3b+2k-s$. Since $s \geqslant b-k$, we have $n \leqslant 3b+2k-b+k=2b+3k$, which is a contradiction. Q.E.D.

LEMMA 3.17. *Let* $n \leqslant i<j<2n$ *and* h *be a lifting from the type* III *subsequence* $x'[i], ..., x'[2i]=3^t 2B_1\, y$ *into the type* III *subsequence* $x'[j], ..., x'[2j]=3^{t'}2B_1\, y'$. *Then we obtain a contradiction.*

Proof. Since $t>t'$, the displayed 2 is sent into y'. But there are at least b 3's to the right of the displayed 2, contradicting $b>k$. Q.E.D.

LEMMA 3.18. *Let* $n \leqslant i<j<2n$ *and* h *be a lifting from the type* IV *subsequence* $x'[i], ..., x'[2i]=3^t 2 3^s$ *into the type* I *subsequence* $x'[j], ..., x'[2j]=3^{t'}2B_1 C_1 B_2 z$. *Then we obtain a contradiction.*

Proof. Note that $2i=2n+s-1$ and $2j \geqslant 3n+b-1$. Also $t=2n-i-1$ and $t'=2n-j-1$. Hence $t-t'=j-i \geqslant (3n+b-1-2n-s+1)/2=(n+b-s)/2$.

Suppose the displaced 2 is sent into C_1. Then $t \leqslant t' + b + k$. Hence $(n+b-s)/2 \leqslant b+k$. So $n+b-s \geqslant 2b+2k$. Therefore $n \leqslant b+2k+s < 2b+2k$, which is a contradiction.

Suppose the displayed 2 is sent into z. By condition (iv) on x, the first 3 in C_2 occurs at position $4n-1-k$. Since $s \geqslant 1$, we see that $lth(z) \geqslant 4n-1-k-(3n+b)+1 = n-b-k$.

Note that $2j = 3n+b-1+lth(z) \geqslant 3n+b-1+n-b-k = 4n-k-1$. So $t-t' = j-i \geqslant (4n-k-1-2n-s+1)/2 = (2n-k-s)/2$.

The number of 3's in 3^t23^s is $t+s$, and the number of 3's in $3^{t'}2B_1C_1B_2z$ is at most $t'+b+k+b+k = t'+2b+2k$. Hence $t+s \leqslant t'+2b+2k$, or $t-t' \leqslant 2b+2k-s$. But $t-t' = j-i \geqslant (2n-k-s)/2$. Hence $2n-k-s \leqslant 4b+4k-2s$, and so $2n \leqslant 3b+5k-s \leqslant 3b+5k-1$. Hence $n \leqslant (3b+5k-1)/2$, which is the desired contradiction. Q.E.D.

LEMMA 3.19. *Let $n \leqslant i < j < 2n$ and h be a lifting from the type* IV *subsequence $x'[i], ..., x'[2i] = 3^t23^s$ into the type II subsequence $x'[j], ..., x'[2j] = 3^{t'}2B_1C_13^{s'}$. Then we obtain a contradiction.*

Proof. Since $t > t'$, the displayed 2 is sent into C_1. Hence $s \leqslant k+s'$, and $t \leqslant t'+b+k$. So $t-t' \leqslant b+k$, and $s-s' \leqslant k$.

Note that $2i = 2n+s-1$ and $2j = 3n+s'-1$. Also $t = 2n-1-i$ and $t' = 2n-1-j$. Now $2j-2i = n+s'-s \geqslant n-k$. But $2j-2i = 2(t-t') \leqslant 2b+2k$. Hence $n-k \leqslant 2b+2k$, and so $n \leqslant 2b+3k$, which is the desired contradiction. Q.E.D.

LEMMA 3.20. *Let $n \leqslant i < j < 2n$ and h be a lifting from the type* IV *subsequence $x'[i], ..., x'[2i] = 3^t23^s$ into the type III subsequence $x'[j], ..., x'[2j] = 3^{t'}2B_1z$. Then we obtain a contradiction.*

Proof. Since $t > t'$, the displayed 2 is sent into z. By condition (iii) on x, the first 3 in C_1 occurs at position $3n-2-k$. Since $s \geqslant 1$, we see that $lth(z) \geqslant 3n-2-k-(2n+b)+1 = n-k-b-1$.

Note that $2i = 2n+s-1$ and $2j = 2n+b-1+lth(z) \geqslant 2n+b-1+n-k-b-1 = 3n-k-2$. Also $t = 2n-i-1$ and $t' = 2n-j-1$. The number of 3's in 3^t23^s is $t+s$, and the number of 3's in $3^{t'}2B_1z$ is at most $t'+b+k$. Hence $t+s \leqslant t'+b+k$, or $t-t' \leqslant b+k-s$. But $t-t' = j-i \geqslant (3n-k-2-2n-s+1)/2 = (n-k-s-1)/2$. Hence $(n-k-s-1)/2 \leqslant b+k-s$. Therefore $n-k-s-1 \leqslant 2b+2k-2s$, and so $n \leqslant 2b+3k-s+1 \leqslant 2b+3k$, which is the desired contradiction. Q.E.D.

LEMMA 3.21. *Let $n \leqslant i < j < 2n$ and h be a lifting from the type* IV *subsequence $x'[i], ..., x'[2i] = 3^t23^s$ into the type IV subsequence $x'[j], ..., x'[2j] = 3^{t'}23^{s'}$. Then we obtain a contradiction.*

Proof. Obviously the displayed 2 is sent to the displayed 2. Hence $t \leqslant t'$, which is a contradiction. Q.E.D.

Lemmas 3.11–3.21 establish the required information concerning the case $n \leqslant i < j < 2n$. We now take up the case $n \leqslant i < 2n \leqslant j$. The first sequences will be type I–IV subsequences of x', and the second sequences will be type 1, 2 subsequences of x.

LEMMA 3.22. *Let $n \leqslant i < 2n \leqslant j \leqslant F(2n, 3n, d-1)$ and h be a lifting from the type* I *subsequence $x'[i], ..., x'[2i] = 3^t 2 B_1 C_1 B_2 z$ into the type* 1 *subsequence $x[j], ..., x[2j] = y' B_q C_q B_{q+1} z'$. Then C_1 is a proper subsequence of C_q.*

Proof. Note that $B_1 C_1 B_2 z$ is a type 1 subsequence of x. Applying Lemma 3.4, we see that C_1 is a proper subsequence of C_q. Q.E.D.

LEMMA 3.23. *Let $n \leqslant i < 2n \leqslant j \leqslant F(2n, 3n, d-1)$ and h be a lifting from the type* I *subsequence $x'[i], ..., x'[2i] = 3^t 2 B_1 C_1 B_2 z$ into the type* 2 *subsequence $x[j], ..., x[2j] = 3^r C_q B_{q+1} C_{q+1} 3^s w$. Then C_1 is a proper subsequence of C_q or C_{q+1}.*

Proof. Note that $B_1 C_1 B_2 z$ is a type 1 subsequence of x. Applying Lemma 3.5, we see that C_1 is a proper subsequence of C_q or C_{q+1}.

Q.E.D.

LEMMA 3.24. *Let $n \leqslant i < 2n \leqslant j \leqslant F(2n, 3n, d-1)$ and h be a lifting from the type* II *subsequence $x'[1], ..., x'[2i] = 3^t 2 B_1 C_1 3^s$ into the type* 1 *subsequence $x[j], ..., x[2j] = y B_p C_p B_{p+1} z$. Then C_1 is a proper subsequence of C_{p+1}.*

Proof. Note that $2i < 3n + b - 1$ and $t = 2n - 1 - i$. Hence $t > 2n - 1 - (3n + b - 1)/2 = (n - b - 1)/2 \geqslant 2k$, since $n \geqslant b + 4k + 1$. Therefore no term of C_1 is sent into $y B_p C_p$, since there are at least $t + b$ 3's to its left, and $t + b > k + b + k$. Therefore the first and last terms of C_1 are sent into z. Hence C_1 is sent into z. Q.E.D.

LEMMA 3.25. *Let $n \leqslant i < 2n \leqslant j \leqslant F(2n, 3n, d-1)$ and h be a lifting from the type* II *subsequence $x'[1], ..., x'[2i] = 3^t 2 B_1 C_1 3^s$ into the type* 2 *subsequence $x[j], ..., x[2j] = 3^{r'} C_q B_{q+1} C_{q+1} 3^{s'} w$. Then C_1 is a proper subsequence of C_{q+2}.*

Proof. As in the proof of Lemma 3.24, $t > (n - b - 1)/2 \geqslant 2k$.

First suppose $r' \leqslant b/2$. Assume that some term of C_1 is sent into $3^{r'}C_q B_{q+1} C_{q+1}$. Each term of C_1 has at least $t+b$ 3's to its left in $3^t 2B_1 C_1 3^s$, and at most $b/2+k+b+k$ 3's to its left in $3^{r'}C_q B_{q+1} C_{q+1} 3^{s'}w$. Hence $t+b \leqslant 3b/2+2k$, or $t \leqslant b/2+2k$. Hence $(n-b-1)/2 < b/2+2k$. So $n-b-1 < b+4k$, and hence $n < 2b+4k+1$, which is a contradiction.

So no term of C_1 is sent into $3^{r'}C_q B_{q+1} C_{q+1}$. Hence the first term of C_1 is sent into w. Therefore C_1 is sent into w.

Now suppose $r' > b/2$. Then $s' = \min(b, 2(b-r')) \leqslant 2(b-r') < 2(b/2) = b$, and so $s' < b$ and w is empty. Obviously the last term of C_1 is sent into $3^{r'}C_q B_{q+1} C_{q+1}$. Since there are at least $t+b+k$ 3's to the left of the last term of C_1, and $r'+k+b+k$ 3's in $3^{r'}C_q B_{q+1} C_{q+1}$, we have $t+b+k \leqslant r'+k+b+k$. Hence $t \leqslant r'+k \leqslant b+k$. So $(n-b-1)/2 \leqslant b+k$, or $n-b-1 \leqslant 2b+2k$. Hence $n \leqslant 3b+2k$, which is a contradiction. Q.E.D.

LEMMA 3.26. *Let $n \leqslant i < 2n \leqslant j \leqslant F(2n, 3n, d-1)$ and h be a lifting from the type* III *subsequence $x'[1], ..., x'[2i] = 3^t 2B_1 z$ into the type* 1 *subsequence $x[j], ..., x[2j] = yB_p C_p B_{p+1} z'$. Then we obtain a contradiction.*

Proof. Note that $2i < 3n$ and $t = 2n-1-i$. Hence $t > 2n-1-3n/2 = n/2-1$. The displayed 2 is not sent into $yB_p C_p$, since there are t 3's to its left, and $t > n/2-1 \geqslant k+b+k = b+2k$. This uses $n \geqslant 2b+4k+2$. Hence the displayed 2 is sent into z'. But this contradicts that there are at least b 3's to its right. Q.E.D.

LEMMA 3.27. *Let $n \leqslant i < 2n \leqslant j \leqslant F(2n, 3n, d-1)$ and h be a lifting from the type* III *subsequence $x'[1], ..., x'[2i] = 3^t 2B_1 z$ into the type* 2 *subsequence $x[j], ..., x[2j] = 3^{r'}C_q B_{q+1} C_{q+1} 3^{s'}w$. Then we obtain a contradiction.*

Proof. As in Lemma 3.26, $t > n/2-1$. First suppose that $r' \leqslant b/2$. The displayed 2 is not sent into $3^{r'}C_q B_{q+1} C_{q+1}$, since there are t 3's to its left, and $t > n/2-1 \geqslant b/2+k+b+k = 3b/2+2k$. This uses $n \geqslant 3b+4k+2$. Hence the displayed 2 is sent into w. But this contradicts that there are at least b 3's to its right and $b > k$.

Now suppose that $r' > b/2$. Then $s' = \min(b, 2(b-r')) = 2(b-r') < b$ and w is empty. The displayed 2 is not sent into C_q, since $t > b+k$. Hence the displayed 2 is sent into C_{q+1}. Therefore some term of B_1 is sent into $3^{s'}$. Hence z is empty. Therefore $2i = 2n+b-1$.

We can now compute $t = 2n-1-i = 2n-1-(2n+b-1)/2 = (2n-b-1)/2$. Thus there are exactly $(2n-b-1)/2 + b = (2n+b-1)/2$ 3's in $3^t 2B_1$. But there are at most $r'+k+b+k+2(b-r')$ 3's in $3^{r'}C_q B_{q+1} C_{q+1} 3^{s'}$. Hence $(2n+b-1)/2 \leqslant 2k+3b-r' \leqslant 2k+3b-b/2 = (4k+5b)/2$, and so $2n+b-1 \leqslant 4k+5b$, or $(2b+2k+1)/2$, which is a contradiction. Q.E.D.

LEMMA 3.28. *Let $n \leqslant i < 2n \leqslant j \leqslant F(2n, 3n, d-1)$ and h be a lifting from the type* IV *subsequence $x'[1], ..., x'[2i] = 3^t 2 3^s$ into the type* 1 *subsequence $x[j], ..., x[2j] = yB_p C_p B_{p+1} z$. Then we obtain a contradiction.*

Proof. Note that $2i < 2n + b - 1$ and $t = 2n - 1 - i$. Hence $t > 2n - 1 - (2n + b - 1)/2 = (2n - b - 1)/2$. Note that $(2n - b - 1)/2 \geqslant b + 2k$ because $2n - b - 1 \geqslant 2b + 4k$ follows from $n \geqslant (3b + 4k + 1)/2$. Hence $t > b + 2k$.

The displayed 2 is not sent into y, since there are t 3's to its left, and $t > k$. The displayed 2 is not sent into C_p, since there are t 3's to its left, and $t > k + b + k = b + 2k$. Hence the displayed 2 is sent into z, and so $s \leqslant k$.

Hence $2i \leqslant 2n + k$. Therefore $t = 2n - 1 - i \geqslant 2n - (2n + k)/2 - 1 = (2n - k - 2)/2$.

The number of 3's in $yB_p C_p B_1 z$ is at most $k + b + k + b + k = 3k + 2b$. Hence $(2n - k - 2)/2 \leqslant 3k + 2b$, or $2n - k - 2 \leqslant 4b + 6k$, and hence $n \leqslant (2b + 7k + 2)/2$, which is a contradiction. Q.E.D.

LEMMA 3.29. *Let $n \leqslant i < 2n \leqslant j \leqslant F(2n, 3n, d-1)$ and h be a lifting from the type IV subsequence $x'[1], ..., x'[2i] = 3^t 2 3^s$ into the type* 2 *subsequence $x[j], ..., x[2j] = 3^{r'} C_q B_{q+1} C_{q+1} 3^{s'} w$. Then we obtain a contradiction.*

Proof. As in Lemma 3.28, $t > (2n - b - 1)/2 \geqslant b + 2k$.

First suppose $r' \leqslant b/2$. The displayed 2 is not sent into $3^{r'} C_q B_{q+1} C_{q+1}$, since it has t 3's to its left, and $t > b/2 + k + b + k = (3b + 4k)/2$. This uses $n \geqslant (2b + 2k + 1)/2$. Hence the displayed 2 is sent into w. Therefore $s \leqslant k$.

Hence $2i \leqslant 2n + s \leqslant 2n + k$. Now $t = 2n - 1 - i \geqslant 2n - 1 - (2n + k)/2 = (2n - k - 2)/2$. The number of 3's in $3^{r'} C_q B_{q+1} C_{q+1} 3^{s'} w$ is at most $b/2 + k + b + k + b + k = (5b + 6k)/2$. Hence $2n - k - 2 \leqslant 5b + 6k$, or $n \leqslant (5b + 7k + 2)/2$, which is a contradiction.

Now suppose $r' > b/2$. Then $s' = 2(b - r') < b$, and w is empty. The displayed 2 is not sent into C_q since $t \geqslant b + 2k$. Hence the displayed 2 is sent into C_{q+1}. Therefore $s \leqslant k + s' \leqslant k + b - 1$.

So $2i \leqslant 2n + s \leqslant 2n + k + b - 1$. Hence $t = 2n - 1 - i \geqslant 2n - 1 - (2n + k + b - 1)/2 = (2n - k - b - 1)/2$. Now the number of 3's in $3^{r'} C_q B_{q+1} C_{q+1} 3^{s'} w$ is at most $r' + k + b + k + 2(b - r') = 3b + 2k - r' < 3b + 2k - b/2 = (5b + 4k)/2$. Therefore $2n - k - b - 1 < 5b + 4k$, and so $2n < 6b + 5k$, or $n < (6b + 5k - 1)/2$, which is a contradiction. Q.E.D.

We are now ready to complete the proof of the Main Lemma.

MAIN LEMMA. *Let n, b, k, d be positive integers, and let x be a strong $2n$, $3n$, b, k, d-sequence. Let $x' = 3^{n-1} 2x$, where we view x' as being indexed from n. Let $n \leqslant i < j \leqslant F(2n, 3n, d-1)$ be such that $x'[i], ..., x'[2i]$ is a subsequence of $x'[j], ..., x'[2j]$. Then there exists $i < j \leqslant d$ such that $C_i(x)$ is a proper subsequence of $C_j(x)$.*

Proof. Let $n \leqslant i < j \leqslant F(2n, 3n, d-1)$ be such that $x'[i], ..., x'[2i]$ is a subsequence of $x'[j], ..., x'[2j]$. Let h be a lifting from $x'[1], ..., x'[2i]$ into $x'[j], ..., x'[2j]$. First suppose that $i \geqslant 2n$. Then the result follows from Lemma 3.9.

Now suppose $n \leqslant i < j < 2n$. According to Lemma 3.10, both of these sequences are of types I, II, III, or IV. Note that the sequences in each of these four types are of strictly greater length than the later types. Hence there are $4+3+2+1=10$ cases to be considered, exactly corresponding to Lemmas 3.11–3.21. (Eleven Lemmas are used instead of ten because of Lemmas 3.12 and 3.13).

Finally suppose $n \leqslant i < 2n \leqslant j$. Then we have 8 cases according to type I, II, III, IV into type 1, 2. These cases correspond exactly to Lemmas 3.22–3.29.

Q.E.D.

4. LOWER BOUND FOR $n(3)$

We use the Main Lemma from Section 3 in order to produce a very long sequence from $\{1, 2, 3\}$ with property *.

There is a particular kind of sequence from $\{1, 2, 3\}$ that plays an important role in the lower bound for $n(3)$. We call a sequence α special if and only if

(i) α is a finite sequence from $\{1, 2, 3\}$, indexed from 1, with property *;

(ii) α is of the form $u13^{n-1}$, $n \geqslant 1$, where α is of length $2n-2$;

(iii) for all $i \leqslant n-1$, $\alpha[i], ..., \alpha[2i]$ has at least one 1.

The following Lemma shows how to use special sequences.

Lemma 4.1. *Let $n \geqslant 1$ and $\alpha = u13^{n-1}$ be special. Let b, k, d be positive integers and x be a strong $2n, 3n, b, k, d$-sequence. Suppose that there does not exist $i < j \leqslant d$ such that $C_i(x)$ is a subsequence of $C_j(x)$. Then $\alpha 2x$ has property * and is of length $\geqslant 2^{d/2}$.*

Proof. Assume n, α, u, b, k, d, x are as given. We claim that $3^{n-1}2x$ has property *, where $x' = 3^{n-1}2x$ is indexed from n. To see this, let $n \leqslant i < j \leqslant (F(2n, 3n, d+1)-1)/2 = F(2n, 3n, d-1)$, where $x'[i], ..., x'[2i]$ is a subsequence of $x'[j], ..., x'[2j]$. By the Main Lemma, there exists $i' < j' \leqslant d$ such that $C_{i'}(x)$ is a subsequence of $C_{j'}(x)$, contradicting the hypotheses.

To show that $u13^{n-1}2x$ has property *, let $i < j \leqslant (F(2n, 3n, d+1)-1)/2$. We must show that $(u13^{n-1}2x)[i], ..., (u13^{n-1}2x)[2i]$ is not a subsequence of $(u13^{n-1}2x)[j], ..., (u13^{n-1}2x)[2j]$.

Case 1. $i \leqslant n-1$. Then $(u13^{n-1}x)[i], ..., (u13^{n-1}x)[2i] = (u13^{n-1})[i], ..., (u13^{n-1})[2i]$ has at least one 1. Note that there are no 1's in $u13^{n-1}x$ past the displayed 1, which is the $(n-1)$st term. Hence if $j \geqslant n$ then we are done. Also if $j \leqslant n-1$ then we are done since $u1^{3n-1}$ has property *.

Case 2. $i > n-1$. This case is clear since $3^{n-1}2x$ has property *.
Note that $u13^{n-1}2x$ has length $F(2n, 3n, d+1) - 1 \geqslant 2^{d/2}$. Q.E.D.

LEMMA 4.2. *Let $n \geqslant 13k+5$, $k \geqslant 3$. There is a strong $2n$, $3n$, $3k+1$, k, $A_{k-1}(2n-4k-2)$-sequence x, where there does not exist $i < j \leqslant A_{k-1}(2n-4k-2)$ such that $C_i(x)$ is a subsequence of $C_j(x)$.*

Proof. Let n, k be as given. For the prospective x, these parameters already determine the endpoints of the intervals $B_i = B_i(\mathrm{x})$, $C_i = C_i(x)$ for all $i \geqslant 1$. A simple calculation shows that $B_1 = [2n, 2n+3k]$, $C_1 = [2n+3k+1, 3n-1]$, $B_2 = [3n, 3n+3k]$, $C_2 = [3n+3k+1, 4n]$, $B_3 = [4n+1, 4n+3k+1]$, $C_3 = [4n+3k+2, 6n]$, $B_4 = [6n+1, 6n+3k+1]$, etc. Also the lengths of the C's strictly increase.

By Theorem 2.6, let $y_1, y_2, ..., y_d \in N^{k-1}$, where $d = A_{k-1}(2n-4k-2)$, $|y_i| = lth(C_3) + i - k - 3 = 2n - 3k - 1 + i - k - 3 = 2n - 4k - 4 + i$, and for no $i < j \leqslant d$ is $y_i \leqslant^* y_j$.

We define a map $h\colon Z^{k-1} \to \{2, 3\}^*$ as follows. Let $z = (z_1, ..., z_{k-1})$ be given. Set $h(z) = 232^{z_1}32^{z_2} \cdots 32^{z_{k-1}}32$. Note that $z \leqslant^* z'$ if and only if $h(z)$ is a subsequence of $h(z')$. Also observe that $lth(h(z)) = |z| + k + 2$. So $lth(h(y_1)) = |y_1| + k + 2 = lth(C_3) = 2n - 3k - 1$.

Note that for all $1 \leqslant i \leqslant d$, $lth(y_i) \leqslant lth(C_{i+2})$. For each $1 \leqslant i \leqslant d$, let y'_i be the result of appending 2's at the end of y_i so that the length of y'_i is $lth(C_{i+2})$. Observe that for no $i < j \leqslant d$ is $y'_i \leqslant^* y'_j$.

We are now prepared to build the desired strong $2n$, $3n$, $3k+1$, k, $A_{k-1}(2n-4k-2)$-sequence. Note that $n \geqslant 3(3k+1) + 4k + 2 = 13k + 5$.

Set $C_1(x) = 2^{n-4k-3}3^k22$ and $C_2(x) = 2^{n-4k-1}3^k2$. For $3 \leqslant i \leqslant A_{k-1}(2n-4k-2)$, we set $C_i(x) = y'_{i-2}$. For $1 \leqslant i \leqslant d$, take $B_i(x)$ to be all 3's in the required position.

It is clear that for $3 \leqslant i < j \leqslant A_{k-1}(2n-4k-2)$, $C_i(x)$ is not a subsequence of $C_j(x)$. Now observe that $C_1(x)$, $C_2(x)$ are not subsequences of any later $C_j(x)$. This is because the 3's would have to match and $n - 4k - 3 > 1$. Finally, $C_1(x)$ is not a subsequence of $C_2(x)$. Q.E.D.

LEMMA 4.3. *Suppose there exists a special sequence of length $\geqslant 26k+8$, $k \geqslant 3$. Then $n(3) > A_{k-1}(22k+8)$.*

Proof. Let α be a special sequence of length $\geqslant 26k+8$, and write $\alpha = u13^{n-1}$. Then $n \geqslant 13k+5$. By Lemma 4.2, let x be a strong $2n$, $3n$,

$3k+1$, k, $A_{k-1}(2n-4k-2)$-sequence where there does not exist $i<j\leqslant A_{k-1}(2n-4k-2)$ such that $C_i(x)$ is a subsequence of $C_j(x)$. By Lemma 4.1, $\alpha 2x$ has property * and is of length $>A_{k-1}(2n-4k-2)$. Hence $n(3)>A_{k-1}(2n-4k-2)\geqslant A_{k-1}(22k+8)$. Q.E.D.

In order to productively apply Lemma 4.3, we need to find a long special sequence.

We do not know how to find such sequences via theoretical considerations. We have been able to construct one by hand of length 216, and verify its specialness by hand.

After this work was completed, R. Dougherty began a series of computer explorations at our suggestion. These explorations have yielded some very much longer special sequences. We report on this work in Section 6.

A nontrivial task is to verify without computer that our special sequence is indeed special. Sole brute force would require looking at $(108)(107)/2=5778$ pairs of sequences, where the lengths of the sequences range from 2 through 108, and verifying that the first of the pair is not a subsequence of the second of the pair. This is a most unpleasant task by hand.

But this task is quite manageable with the help of some simple theory which we develop now.

It is useful to work with tables associated with a sequence. Let $x[1]$, $x[2]$, ..., $x[2t]$ or $x[1]$, $x[2]$, ..., $x[2t+1]$ be a given sequence. Its associated table has the following list of lines:

1. $x[1], x[2]$
2. $x[2], x[3], x[4]$
3. $x[3], x[5], x[5], x[6]$
4. $x[4], x[5], x[6], x[7], x[8]$

...

t. $x[t], x[t+1], ..., x[2t]$.

We can now restate our condition. It is that all $x[i]$ are from $\{1, 2, 3\}$; that each line have at least one 1 (among the $x[i]$'s); and that no line be a subsequence of any later line.

It is convenient to collect blocks of like terms and write them in exponential form. Thus the entry "233331211" would be written "23^4121^2." Of course, the exponents are to be written in numerical notation. Each line in the table is to be given in this form.

It is easy to describe an efficient algorithm for determining whether one sequence put in this form is a subsequence of another sequence put in this form. This algorithm is useful both for computer implementations and for eyeballing.

Specifically, let $a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}$ and $b_1^{j_1}b_2^{j_2}\cdots b_s^{j_s}$ be given, where the a's and b's are arbitrary, adjacent a's are distinct, adjacent b's are distinct, and the i's, j's, r, s are positive integers. We start by finding (in the second sequence) the first powers of a_1 that sum to i_1. Then we find (in the second sequence) the first i_2 powers of a_2 that occur starting at a later power. And so on, until we find the first i_r powers of a_r. If this process is completed, then $a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}$ is a subsequence of $b_1^{j_1}b_2^{j_2}\cdots b_s^{j_s}$. If this process is not completed, then $a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}$ is not a subsequence of $b_1^{j_1}b_2^{j_2}\cdots b_s^{j_s}$. It is immediate that if this process is completed, then $a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}$ is a subsequence of $b_1^{j_1}b_2^{j_2}\cdots b_s^{j_s}$. Now assume $a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}$ is a subsequence of $b_1^{j_1}b_2^{j_2}\cdots b_s^{j_s}$. We can show by induction that at any stage in this process, the remaining tail of $a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}$ is a subsequence of the remaining tail of $b_1^{j_1}b_2^{j_2}\cdots b_s^{j_s}$.

It turns out to be most convenient for our immediate purposes, to give some necessary conditions for one sequence presented in this form to be a subsequence of another. We need only do this here in the case of sequences from $\{1, 3\}^*$.

Accordingly, let $a_1^{i_1}a_2^{i_2}\cdots a_r^{i_r}$ be given, where the i's and r are positive integers, and the a's lie in $\{1, 3\}$. We define the type to be the pair (r, d), where r is the number of powers (as indicated) and d is the sum of the exponents of 1. Thus the type of, say, $3^4 1^3 3^2 1 3^4$ is $(5, 4)$.

Lemma 4.4. *Let x, y be nonempty finite sequences from $\{1, 3\}$ of types (a, b) and (c, d). Suppose x is a subsequence of y. Then $a \leqslant c$ and $b \leqslant d$. Furthermore,*

(i) *if $a = c$ then x, y have the same first terms (perhaps with different powers), and any way of sending x into y must send each power in x into the corresponding power in y. As a consequence, each of the exponents in x are respectively $\leqslant$ the exponents in y (which we refer to as the exponent raising condition);*

(ii) *if $a = c$ and $b = d$, then each power of 1 in x is the same as the corresponding power of 1 in y.*

Proof. Let x, y, a, b, c, d be as given. For (i), assume $a = c$. Any two successive powers in x must be sent to distinct powers in y. Hence each power in x must be sent wholly into a power in y, for otherwise a power in y will forever be skipped over, violating $a = c$. Hence by $a = c$, each power in x is sent into the corresponding power in y. It then follows that the first terms must be the same.

Note that (ii) immediately follows from (i). Q.E.D.

Many more necessary conditions like those in Lemma 4.4 can be proved, and are generally useful. However, we will be content with using Lemma 4.4

in order to verify that our sequence of length 216 is special. When Lemma 4.4 does not apply, we bring in related considerations on an ad hoc basis. These essentially amount to uses of the algorithm presented above.

We will start with a list of finite sequences rather than with the required sequence itself. But we need to know a necessary condition for a list of finite sequences to be the table of a single sequence:

LEMMA 4.5. *Let r be a positive integer and L be a list of finite sequences. Then L is the table of a finite sequence of length $2r$ if and only if*

(i) *the first sequence of L is of length 2;*

(ii) *the last sequence of L is of length $r+1$;*

(iii) *each sequence of L is obtained from the previous sequence of L by deleting the first term and appending two additional terms.*

Furthermore, different finite sequences have different tables.

Proof. Left to the reader. Q.E.D.

We now present the table of our special sequence of length 216.

1.	12	
2.	221	
3.	2131	
4.	131^3	(3, 4)
5.	31^5	(2, 5)
6.	1^7	(1, 7)
7.	$1^6 3^2$	(2, 6)
8.	$1^5 3^2 13$	(4, 6)
9.	$1^4 3^2 1313$	(6, 6)
10.	$1^3 3^2 1313^3$	(6, 5)
11.	$1^2 3^2 1313^5$	(6, 4)
12.	$13^2 1313^7$	(6, 3)
13.	$3^2 1313^8 1$	(6, 3)
14.	$31313^8 13^2$	(7, 3)
15.	$1313^8 13^4$	(6, 3)
16.	$313^8 13^5 1$	(6, 3)
17.	$13^8 13^5 13^2$	(6, 3)
18.	$3^8 13^5 13^4$	(5, 2)
19.	$3^7 13^5 13^6$	(5, 2)

20.	$3^6 1 3^5 1 3^8$	(5, 2)
21.	$3^5 1 3^5 1 3^{10}$	(5, 2)
22.	$3^4 1 3^5 1 3^{12}$	(5, 2)
23.	$3^3 1 3^5 1 3^{14}$	(5, 2)
24.	$3^2 1 3^5 1 3^{16}$	(5, 2)
25.	$3 1 3^5 1 3^{18}$	(5, 2)
26.	$1 3^5 1 3^{20}$	(4, 2)
27.	$3^5 1 3^{20} 1^2$	(4, 3)
28.	$3^4 1 3^{20} 1^2 3^2$	(5, 3)
29.	$3^3 1 3^{20} 1^2 3^4$	(5, 3)
30.	$3^2 1 3^{20} 1^2 3^6$	(5, 3)
31.	$3 1 3^{20} 1^2 3^8$	(5, 3)
32.	$1 3^{20} 1^2 3^{10}$	(4, 3)
33.	$3^{20} 1^2 3^{12}$	(3, 2)
…		… (3,2)
53.	$1^2 3^{52}$	(2, 2)
54.	$1 3^{53} 1$	(3, 2)
55.	$3^{53} 1 3^2$	(3, 1)
…		… (3, 1)
108.	$1 3^{108}$	(2, 1)

Note that we have also presented the types of the sequences numbered 4–108. Our goal is to prove that no sequence on this list is a subsequence of any later sequence on this list. Observe by inspection that each sequence on this list has a 1.

It will be convenient to refer to the ith numbered sequence in this list as $\#i$. We say that $\#i$ is verified if and only if we have shown that $\#i$ is not a subsequence of any $\#j$, $j > i$. More specifically, in each case we assume that $\#i$ is a subsequence of $\#j$ and derive a contradiction. We must verify $\#i$ for all $1 \leqslant i \leqslant 107$.

Note that $\#1$, $\#2$, $\#3$ each have a 2, and that $\#1$ is not a subsequence of $\#2$, $\#3$, and $\#2$ is not a subsequence of $\#3$. Also note that $\#i$, $4 \leqslant i \leqslant 108$, have no 2's. Hence $\#1$, $\#2$ and $\#3$ have been verified.

We now verify $\#4$–$\#107$.

$\#4$. According to types, (first claim in Lemma 4.4), we have only to look at $\#8$–$\#11$. For $\#8$, if the 1 in $\#4$ is sent into 1^5 in $\#8$ then the 1^3 in $\#4$ is sent into the 1 in $\#8$, which is a contradiction. Hence the 1 in #4 is sent into the 1 in #8, and there is no room for the 1^3 in #4. For #9, if the 1 in #4 is sent into the 1^4 in #9, then the 1^3 in #4 is sent into the 1313 in #9, which is a contradiction. If the 1 in #4 is sent into the first 1 in #9 then the 1^3 in #4 is sent into the second 1 in #9, which is a contradiction. If the 1 in #4 is sent into the second 1 in #9 then there is no room for the 1^3. For #10, if the 1 in #4 is sent into the 1^3 in #10 then the 1^3 in #4 is sent into the 1313^3 in #10, which is a contradiction. Then we argue as for #9. For #11, if the 1 in #4 is sent into the 1^2 in #11, then the 1^3 in #4 is sent into the 1313^5 in #11, which is a contradiction. Otherwise, we argue as for #9 and #10.

#5. According to types, we have only to look at #7–#10. For #7, the 3 in #5 is sent into the 3^2 in #7, with no room for the 1^5 in #5. For #8, the 1^5 in #5 is sent into the 13 in #8, which is a contradiction. For #9, the 1^5 in #5 is sent into the 1313 in #9, which is a contradiction. For #10, the 1^5 in #5 is sent into the 1313^3 in #10, which is a contradiction.

#6. According to types, we have nothing to look at.

#7. According to types, we look at #8, #9. The last 1 in #7 is sent to the last 1 in #8 or #9. But then there is no room for the 3^2.

#8. According to types, we look at #9. The last 1 in the 1^5 in #8 is sent into the 1313 of #9. Hence the last 3 in 3^2 in #8 is sent into the last 3 in #9. But then there is no room for the 13 in #8.

#9. According to types, we have nothing to look at.

#10. According to types, we have nothing to look at.

#11. According to types, we have nothing to look at.

#12. According to types, we look at #13–#17. For #13 and #15–#17, the types are the same as the type of #12. In the case of #13, #15, #16, the first term is different from the first term of #12, violating Lemma 4.4(i). In the case of #17, the exponent raising condition in Lemma 4.4(i) is violated for the last terms. For #14, #12 is sent into a tail of: #14 with the first term deleted. By comparing the type of #12 with the type of this tail, we see that this tail is simply #14 with the first term deleted. Now #12 and #14 with the first term deleted have the same type; whereas the exponent raising condition is violated for the last terms.

#13. According to types, we look at #14–#17. For #15–#17, the types are the same as the type of #12. In the case of #15, #17, the first term differs from that of #13. For #16, the exponent raising condition fails. For #14, the result of deleting the first term in #13 is sent into the result of deleting the first two terms in #13. But the number of powers in the former is greater than the number of powers in the latter.

#14. According to types, we have nothing to look at.

#15. According to types, we look at #16, #17. For #16, the type of #16 is the same as the type of #15, and the first term of #16 is not the same as the first term of #15. For #17, the type of #17 is the same as the type of #15, but the exponent raising condition is violated at the last term.

#16. According to types, we look at #17. The type of #17 is the same as the type of #16, but the first terms are not the same.

#17. According to types, we have nothing to look at.

#18–#25. According to types, we at look #19–#25 (going forward), and #28–#31. All of these sequences have the same number of powers. For the former group, the exponent raising condition is violated at the first terms. For #18–#21 and the latter group, the exponent raising condition is violated at the first terms. For #22–#25, the exponent raising condition is violated at the last terms.

#26. According to types, we look at #27–#32. For #27, the number of powers in #26 and #27 are the same, but they do not have the same first term. For #32, the number of powers in #26 and #32 are the same, but the exponent raising condition is violated at the last terms. For #28–#31, #26 is sent into the result of deleting the first term of #28–#31. Note that the latter has the same number of powers as #26. But the exponent raising condition fails at the last terms.

#27. According to types, we look at #28–#32. For #32, the number of powers in #27 and #32 are the same, but they do not have the same first term. For numbers #28–#31, #27 is sent into #28–#31 without the last power. But the latter have the same type as #27. However, the exponent raising condition is violated at the first term.

#28–#31. According to types, we look at #29–#31 (going forward). The types are all the same, but the exponent raising condition is violated at the first term.

#32. According to types, we have nothing to look at.

#33–#52. According to types, we look at #26–#51 (going forward). The types are all the same, but the exponent raising condition is violated at the first terms.

#53. According to types, we look at #54. The second 1 in #53 is sent to the final term in #54, leaving no room for the first 3 in #53.

#54. According to types, there is nothing to look at.

#55–#107. According to types, we look at #56–#107 (going forward). The types are the same, but the exponent raising condition is violated at the first term.

Lemma 4.6. *The sequence* $\alpha =$ "$12^2 1 3 1^7 3^2 1 3 1 3^8 1 3^5 1 3^{20} 1^2 3^{53} 1 3^{108}$" *is a special sequence of length* 216.

Proof. See above. Q.E.D.

Theorem 4.7. $n(3) > A_7(184)$.

Proof. By Lemmas 4.3 and 4.6, setting $k = 8$. Q.E.D.

According to the discussion at the beginning of Section 2, we can regard $n(3)$ as incomprehensibly large. Recent computer explorations by Dougherty have demonstrated the existence of much longer special sequences. We use their existence to strengthen this lower bound for $n(3)$. See Section 6.

5. THE FUNCTION $n(k)$

In this section we give some asymptotic upper and lower bounds for the function $n(k)$. In this paper we do not consider the individual numbers $n(k)$, $k \geqslant 4$.

We also consider the related function $F: Z^+ \to Z^+$ defined as follows. $F(k)$ is the length of the longest sequence $x[1], ..., x[n]$ such that

(i) each $x[i]$ is a sequence from $\{1, ..., k\}$ of length $\leqslant i+1$;

(ii) for no $i < j$ is $x[i]$ a subsequence of $x[j]$.

Lemma 5.1. *For all* $k \geqslant 1$, $n(k) \leqslant 2F(k) + 1$.

Proof. Let $x[1], ..., x[p]$ be of longest length from $\{1, ..., k\}$ according to the definition of $n(k)$. Then p is odd. Note that $(x[1], x[2]), ..., (x[(p-1)/2], ..., x[p-1])$ have lengths $2, ..., (p+1)/2$. Hence $(p-1)/2 \leqslant F(k)$. So $p \leqslant 2F(k)+1$. Q.E.D.

Let $a_1 < a_2 < a_3 \cdots$ be defined by $a_1 = 6$, $a_2 = 9$, $a_{i+2} = 2a_i + 1$.

Lemma 5.2. *For all* $i \geqslant 1$, $a_{i+1} - a_i \geqslant i+2$. *For all* $m \geqslant 6$, *there is a unique* i *such that* $a_i, a_{i+1} \in \{m, ..., 2m\}$.

Proof. The first claim is true of $i = 1$, and since $a_3 = 13$, it is also true of $i = 2$. Now suppose $a_{i+1} - a_i \geqslant i+2$ and $a_i - a_{i-1} \geqslant i+1$, $i \geqslant 2$. Then $a_{i+2} - a_{i+1} = 2(a_i - a_{i-1}) \geqslant 2(i+1) \geqslant i+3$. In particular, the a's are strictly increasing.

For the second claim, we first prove existence. Obviously for $m = 6$ we can take $i = 1$. Suppose true for a fixed $m \geqslant 6$. Let $a_i, a_{i+1} \in \{m, ..., 2m\}$. If $a_i \neq m$ then $a_i, a_{i+1} \in \{m+1, ..., 2m+2\}$. Now assume $a_i = m$. Then $a_{i+2} = 2m+1$, and so $a_{i+1}, a_{i+2} \in \{m+1, ..., 2m+2\}$.

For uniqueness, suppose $a_i, a_{i+1}, a_j, a_{j+1} \in \{m, ..., 2m\}$, $i < j$. Then $a_i, a_{i+1}, a_{i+2} \in \{m, ..., 2m\}$, which contradicts $a_{i+2} = 2a_i + 1$. Q.E.D.

The following result is extremely crude, but suffices for our present purposes.

LEMMA 5.3. *For all $k \geqslant 1$, $n(k+7) \geqslant F(k) \geqslant (n(k)-1)/2$.*

Proof. Let $x[1], ..., x[n]$ obey (i) and (ii) above with $n = F(k)$. *Let* $x'[1], ..., x'[n]$ be sequences from $\{1, ..., k+1\}$ of lengths $a_2 - a_1 - 1$, $a_3 - a_2 - 1, ..., a_{n+1} - a_n - 1$, where $x'[i]$ is obtained from $x[i]$ by appending the requisite number of $k+1$'s. Then for no $i < j$ is $x'[i]$ a subsequence of $x'[j]$.

Now define $y[6], ..., y[a_{n+1}] \in \{1, ..., k+2\}$ as follows. Set $y[a_i]$, $1 \leqslant i \leqslant n+1$, to be $k+2$. Set each $y[a_i+1], ..., y[a_{i+1}-1]$ to be $x'[i]$.

Finally, define $y[1], ..., y[5]$ to be $k+3, ..., k+7$. Note that $a_{n+1} > F(k)$.

We have to check that $y[1], ..., y[a_{n+1}]$ has property *. Let $i < j \leqslant a_{n+1}/2$.

Case 1. $i \geqslant 6$. By Lemma 5.2, let p, q be unique such that $a_p, a_{p+1} \in \{a_i, \ldots, a_{2i}\}$ and $a_q, a_{q+1} \in \{a_j, \ldots, a_{2j}\}$. Then $y[i], \ldots, y[2i]$ and $y[j], ..., y[2j]$ both have exactly two $k+2$'s. If the former is a subsequence of the latter then a_p is sent to a_q and a_{p+1} is sent to a_{q+1}. Therefore $y[a_p+1], ..., y[a_{p+1}-1]$ is a subsequence of $y[a_q+1], ..., y[a_{q+1}-1]$. I.e., $x'[p]$ is a subsequence of $x'[q]$. Also, since $i < j$ we have $p < q$. This is a contradiction.

Case 2. $i \leqslant 5$. Then $y[i]$ does not even appear in $y[j], ..., y[2j]$.

It remains to verify that $F(k) \geqslant n(k)/3$. By Lemma 5.1, $F(k) \geqslant (n(k)-1)/2 \geqslant n(k)/3$ since $n(k) \geqslant 3$. Q.E.D.

For each $k \geqslant 1$, we define $G_k\colon Z^+ \to Z^+$ as follows. $G_k(n)$ is the length of the longest sequence $x[1], ..., x[p]$ such that

(i) each $x[i]$ is a sequence from $\{1, ..., k\}$ of length $\leqslant i+n$;

(ii) for no $i < j$ is $x[i]$ a subsequence of $x[j]$.

Let $f_1, f_2\colon Z^+ \to Z^+$. We say that f_1 dominates f_2 if and only if for all $n \in Z^+$, $f_1(n) > f_2(n)$. We say that f_1 eventually dominates f_2 if and only if for all sufficiently large n, $f_1(n) > f_2(n)$.

LEMMA 5.4. *F is strictly increasing. $G_k(n)$ is strictly increasing in each argument. F eventually dominates each G_k.*

Proof. For the first claim, let $k \geqslant 1$ and $x[1], ..., x[n]$ be of longest length according to the definition of $F(k)$. Then $x[1], ..., x[n], (k+1)$ demonstrates that $n = F(k) < F(k+1)$.

Let $x[1], ..., x[p]$ be of longest length according to the definition of $G_k(n)$. Then $x[1]\,k, ..., x[p]\,k, (k)$ demonstrates that $p = G_k(n) < G_k(n+1)$. Also $x[1], ..., x[p], (k+1)$ demonstrates that $p = G_k(n) < G_{k+1}(n)$.

For the last claim, it suffices to prove that for all $n > k \geqslant 1$, $G_k(n) < F(n)$. To see this, let $x[1], ..., x[p]$ be of longest length according to the definition of $G_k(n)$. Then $(n, 1), (n, 2), ..., (n, n), x[1], ..., x[p]$ demonstrates that $p = G_k(n) < F(n)$. Q.E.D.

We now place a norm on the ordinals $< \epsilon_0$ (actually, we will only use the norm on ordinals $< \omega^{\omega^k}$). Every $\alpha < \epsilon_0$ has a unique Cantor normal form to base ω. We define $|\alpha|$ to be the total number of occurrences of ω in the Cantor normal form of α. Note that $|\omega^\beta| = |\beta| + 1$, $|\omega| = 2$, and for $k \geqslant 0$, $|k| = k$. Clearly there are only finitely many ordinals of a given norm.

We use $\cdot$ for ordinal multiplication (and also integer multiplication).

Lemma 5.5. *For $\alpha, \beta < \epsilon_0$, $|\alpha + \beta| \leqslant |\alpha| + |\beta|$ and $|\alpha \cdot \beta| \leqslant |\alpha| \cdot |\beta|$.*

Proof. For the first claim, simply note that the sum of two normal forms is the sum of the combined components, perhaps with a rearrangement and/or cancellations. For the second claim, first observe that $|\omega^\gamma \cdot \omega^\delta| = |\omega^{\gamma+\delta}| \leqslant |\gamma| + |\delta| + 1 \leqslant (|\gamma| + 1)(|\delta| + 1) = |\omega^\gamma|\,|\omega^\delta|$. Next observe that the product of two normal forms equals a sum of the products of pairs of components, the first from the first normal form, and the second from from the second normal form, perhaps with cancellations. Even with no cancellations, the norm of the result would be at most the sum of the products of the norms of the components. By ordinary arithmetic, this is at most the product of the norms of the two normal forms. The degenerate cases where α or β is 0 can be handled separately. Q.E.D.

We write A^* for the set of all finite sequences from A, including the empty sequence.

For each $k \geqslant 1$ we define the map $g_k\colon (\omega^{\omega^{k-1}})^* \to \omega^{\omega^k}$ as follows. For all $\alpha_1, ..., \alpha_n < \omega^{\omega^{k-1}}$, $g_k(\alpha_1, ..., \alpha_n) = \omega^{\omega^{k-1} \cdot n-1} + \omega^{\omega^{k-1} n-1} \cdot \alpha_1 + \omega^{\omega^{k-1} \cdot n-2} \cdot \alpha_2 + \cdots + \alpha_n = \omega^{\omega^{k-1} \cdot n-1} \cdot (1 + \alpha_1) + \omega^{\omega^{k-1} \cdot n-2} \cdot \alpha_2 + \cdots + \alpha_n$. Set $g_k(0) = 0$.

Lemma 5.6. *For all $k \geqslant 1$, g_k is one–one onto.*

Proof. Note that the values of g_k are Cantor normal forms of the ordinals $< \omega^{\omega^k}$ to the base $\omega^{\omega^{k-1}}$. So g_k maps the 1-tuples one–one onto $[1, \omega^{\omega^{k-1}})$, the 2-tuples one–one onto $[\omega^{\omega^{k-1}}, \omega^{\omega^{k-1} \cdot 2})$, etc. Hence g_k is one–one onto. Q.E.D.

Lemma 5.7. *In the definition of $g_k(a_1, ..., a_n)$ above, the first summand has norm at least $k(n-1) + |\alpha_1|$, and the remaining summands have norms respectively at least $|\alpha_2|, ..., |\alpha_n|$.*

Proof. The second part of the claim holds because for $1 \leqslant i \leqslant n$,

(a) the infinite summands in the normal form of α_i survives the conversion into normal form of the ith summand, since the exponents in the normal form of α_i are $<\omega^{k-1}$;

(b) the finite summands ω^0 either stay intact (if $i = n$) or give rise to copies of $\omega^{\omega^{k-1} \cdot n - i}$ (if $i < n$).

For the first part of the claim, note that $\omega^{k-1} \cdot n - 1$ separately survives in the exponent, and has norm $k(n-1)$. Q.E.D.

We will need the following very weak information.

LEMMA 5.8. *Let* $k \geqslant 1$ *and* $x \in (\omega^{\omega^{k-1}})^n$. *Then* $|g_k(x)| \geqslant n - 1 + |x_1| + \cdots + |x_n|$.

Proof. This follows from Lemma 5.7 together with the observation that in the definition of $g_k(\alpha_1, ..., \alpha_n)$, every summand in the normal form of summands $1, ..., n$ is greater than every summand in the normal form of each succeeding summand. Q.E.D.

We now define maps $h_k \colon \{1, ..., k\}^* \to \omega^{\omega^{k-1}}$ as follows. Let h_1 be the length function.

Suppose h_k has been defined, $k \geqslant 1$. To define h_{k+1}, let $x \in \{1, ..., k+1\}^*$. Then x can be uniquely written as $y_1\, k+1\, y_2 \cdots k+1\, y_n$, where $n \geqslant 0$ and $y_1, ..., y_n \in \{1, ..., k\}^*$. (Some of the y's may be the empty sequence.) Define $h_{k+1}(x) = g_k(h_k(y_1), ..., h_k(y_n))$.

LEMMA 5.9. *Let* $k \geqslant 1$ *and* $x \in \{1, ..., k\}^*$. *Then* h_k *is one–one onto and* $h_k(x) \geqslant lth(x)$.

Proof. We prove by induction on $k \geqslant 1$ that h_k is one–one onto. Clearly h_1 is one–one onto. Suppose h_k is one–one onto, $k \geqslant 1$. Since g_k is one–one onto, we see that h_{k+1} is one–one onto.

We also prove the second claim by induction on $k \geqslant 1$. This is obvious for $k = 1$ since h_1 is just the length function. Suppose $h_k(x) \geqslant lth(x)$ holds for all $x \in \{1, ..., k\}^*$, where $k \geqslant 1$ is fixed. Let $x \in \{1, ..., k+1\}^*$, and write $x = y_1\, k+1\, y_2\, k+1 \cdots k+1\, y_n$. Then $h_{k+1}(x) = g_k(h_k(y_1), ..., h_k(y_n))$. By Lemma 5.8, $|h_{k+1}(x)| \geqslant n - 1 + |h_k(y_1)| + \cdots + |h_k(y_n)| \geqslant n - 1 + lth(y_1) + \cdots + lth(y_n) = lth(x)$. Q.E.D.

LEMMA 5.10. *For all* $k \geqslant 1$ *and* $x, y \in \{1, ..., k\}^*$, *if* x *is a subsequence of* y *then* $h_k(x) \leqslant h_k(y)$.

Proof. By induction on k. The case $k = 1$ is obvious. Suppose this is true for k. Let $x, y \in \{1, ..., k+1\}^*$, where x is a subsequence of y. Write

x as $z_1\, k+1\, z_2 \cdots k+1\, z_n$, and y as $w_1\, k+1\, w_2 \cdots k+1\, w_m$, where the z's and w's are from $\{1, ..., k\}^*$. Then the number of $k+1$'s in x is $\leqslant$ the number of $k+1$'s in y; i.e., $n \leqslant m$. If $n < m$, then obviously $h_{k+1}(x) < h_{k+1}(y)$. Suppose equality holds. Note that each z_i is a subsequence of w_i. Hence for all i, $h_k(z_i) \leqslant h_k(w_i)$. Therefore $h_{k+1}(x) \leqslant h_{k+1}(y)$. Q.E.D.

For each $k \geqslant 1$, we define $H_k\colon Z^+ \to Z^+$ as follows. $H_k(n)$ is the length of the longest sequence $\alpha_1 > \cdots > \alpha_n$ such that $\alpha_1 < \omega^{w^{k-1}}$ and each $|\alpha_i| \leqslant i+n$.

Lemma 5.11. *For all* $k, n \geqslant 1$, $G_k(n) \geqslant H_k(n)$.

Proof. Let $k, n \geqslant 1$. Let $\alpha_1 > \cdots > \alpha_p$ be longest as in the definition of $H_k(n)$. Consider $h_k^{-1}(\alpha_1), ..., h_k^{-1}(\alpha_n)$. No term is a subsequence of any later term because of Lemma 5.10. Also $|\alpha_i| \geqslant lth(h_k^{-1}(\alpha_i))$ by Lemma 5.9. Hence $lth(h_k^{-1}(\alpha_i)) \leqslant n+i$. Therefore $G_k(n) \geqslant p = H_k(n)$. Q.E.D.

Let $I_k(n)$ be the length of the longest sequence $\alpha_1 > \cdots > \alpha_p$ such that $\alpha_1 \leqslant \omega^k \cdot 2$ and each $|\alpha_i| \leqslant i+n$.

Lemma 5.12. *For all* $n \geqslant 1$, $I_1(n) \geqslant 3n+2 \geqslant A_1(n)$.

Proof. Consider the sequence

$$\omega + n - 1 > \cdots > \omega > 2n+1 > \cdots > 0.$$

It has length $3n+2$. Q.E.D.

Lemma 5.13. *For all* $n \geqslant 1$, $I_2(n) \geqslant 2^{n+1} \geqslant A_2(n)$.

Proof. $I_2(1) \geqslant I_1(1) \geqslant 5$. Now let $n \geqslant 2$. Consider the sequence

$$\begin{aligned}\omega^2 + n - 2 > \cdots > \omega^2 > \omega \cdot n > \omega \cdot (n-1) + 2 \\ > \cdots > \omega \cdot (n-1) > \omega \cdot (n-2) + 4 \cdots > \omega(n-2) \\ > \omega \cdot (n-3) + 8 \cdots > \omega(n-3) > \cdots > 2^n > \cdots 0\end{aligned}$$

which has length $\geqslant 2^{n+1}$. Q.E.D.

Lemma 5.14. *For all* $p \geqslant 1$ *and* $n \geqslant 2p+10$, *we have* $I_p(n) \geqslant A_p(n)$, *where* $A_p(n)$ *is the Ackermann hierarchy as defined in Section* 2.

Proof. We first claim that for all $p \geqslant 1$ and $n \geqslant 2p+6$, $(2p+2)n \leqslant 2^{n-p-2}$. We prove this by induction on $n \geqslant 2p+6$. Suppose $n = 2p+6$. Then we must verify that for all $p \geqslant 1$, $(2p+2)(2p+5) \leqslant 2^{p+4}$. This is proved by induction on $p \geqslant 1$, noting that as p increases by 1, the left side falls shy of doubling. Also note that as n increases by 1, and p is held fixed, the left side also falls short of doubling.

We now prove the Lemma by induction on $p \geqslant 2$. The basis case $p=2$ is handled by Lemma 5.12. Now fix $p \geqslant 2$ and assume that for all $n \geqslant 2p+4$, we have $I_p(n) \geqslant A_p(n)$. Let $n \geqslant 2p+6$. We wish to prove that $I_{p+1}(n) \geqslant A_{p+1}(n)$.

We first form the sequence

$$\omega^{p+1} + \omega^2 + \omega^2 > \cdots > \omega^{p+1}$$

where we have added ω^{p+1} to the left of the sequence given by Lemma 5.13 with n replaced by $n-p-3 \geqslant p+2+3+3-1$. This sequence has length at least $2^{n-p-2} \geqslant (2p+2)n$. Hence we can continue the sequence as follows.

$$\omega^{p+1} + \omega^2 + \omega^2 > \cdots > \omega^{p+1} > (\omega^p + \omega^p) \cdot n.$$

We can continue by replacing the last of the n copies of $\omega^p + \omega^p$ by a sequence for $I_p(n)$ given by the induction hypothesis of length $A_p(n)$. We can then continue by replacing the last of the remaining $n-1$ copies of $\omega^p + \omega^p$ by a sequence for $I_p(A_p(n)) \geqslant A_p A_p(n)$ of length $A_p A_p(n)$. We can continue in this way for n steps, resulting in a sequence for $I_{p+1}(n)$ of length at least $A_{p+1}(n)$. Q.E.D.

For $k, m, p \geqslant 1$, let $J_{k,m,p}: Z^+ \to Z^+$ be defined as follows. $J_{k,m,p}(n)$ is the length of the longest sequence $\alpha_1 > \cdots > \alpha_q$ such that α_1 is an ordinal $< \omega^{\omega^{k-1} \cdot m}$ and each $|\alpha_i| \leqslant A_p(i+k+m+n)$.

LEMMA 5.15. *For all $k, m, p \geqslant 1$, $J_{k,m,p}$ is eventually dominated by H_{k+1}.*

Proof. Fix k, m, p, and let n be sufficiently large. Let $\alpha_1 > \cdots > \alpha_q$ be a sequence for $J_{k,m,p}(n)$.

Consider the sequence $\omega^{p+2} \cdot \alpha_1, \omega^{p+2} \cdot \alpha_2, \ldots, \omega^{p+2} \cdot \alpha_q$. Each $|\omega^{p+2} \cdot \alpha_1| \leqslant A_{p+1}(n+i)$.

Using Lemma 5.14, choose $\omega^{p+3} > \beta_1 > \cdots > \beta_r$ such that in the sequence

$$\omega^{\omega^{k-1} \cdot m} + \omega^{p+3} + \beta_1 > \cdots > \omega^{\omega^{k-1} \cdot m} + \omega^{p+3} + \beta_r$$

the ith term has norm at most $n+i$, and $r = A_{p+1}(n+1)$.

Using Lemma 5.14, choose $\omega^{p+2} > \gamma_1 > \cdots > \gamma_s$ such that in the sequence

$$\omega^{p+2} \cdot \alpha_1 + \gamma_1 > \cdots > \omega^{p+2} \cdot \alpha_1 + \gamma_s$$

the ith term has norm at most $A_{p+1}(n+1) + i$, and $s = A_{p+1}(n+2)$.

Continue in this way, finally choosing $\omega^{p+2} > \delta_1 > \cdots > \delta_t$ such that in the sequence

$$\omega^{p+2} \cdot \alpha_q + \delta_1 > \cdots > \omega^{p+2} \cdot \alpha_q + \delta_t$$

the ith term has norm at most $A_{p+1}(n+q)+i$, and $s = A_{p+1}(n+q+1)$.

The resulting sequence demonstrates that $q = J_{k,m,p}(n) < H_{k+1}(n)$.

Q.E.D.

We want to use [Ro84], which does not use a norm on the ordinals $< \epsilon_0$, but rather a standard arithmetization of the ordinals $< \epsilon_0$ via sequence numbers, i.e., ordinal notations. This is also standard in the literature.

Let $k, n \geqslant 1$. We write $2^{[k]}(n)$ for a stack of k 2's with n on top. Thus $2^{[1]}(n) = 2^n$.

We say that $f: N^k \to N$ is elementary (or elementary recursive) if and only if for some k, it can be computed in time complexity $2^{[k]}$.

We take the approach to ordinal recursion in [FS95], which is equivalent to that in [Ro84]. Let $\alpha < \epsilon_0$, and $g, h: N^2 \to N$. We define $C(\alpha, h): N \to N$ to be the "count function" given by $C(\alpha, h)(n) = 0$ if $h(n, 0)$ is not (the notation of) an ordinal $< \alpha$; the least i such that $h(n, i) \leqslant h(n, i+1)$, where $\leqslant$ is the ordering on notations, otherwise.

Finally, define $D(\alpha, g, h)$ as the function $f: N \to N$ given by $f(n) = g(n, C(\alpha, h)(n))$. Following [FS95], the functions $D(\alpha, g, h)$, where g, h are elementary, are called the α-descent recursive functions. We also let the $<\alpha$-descent recursive functions be the union of the β-descent recursive functions for $\beta < \alpha$.

This definition can be immediately extended to functions of several variables by either adding parameters to the definition or by using an elementary pairing function on N.

The α-descent recursive functions correspond to a single step ordinal recursion on α in the sense of, say, [Ro84, p. 89]. Full ordinal recursion on α in [Ro84, p. 89], results from iterating single step ordinal recursion on α. That is, one is allowed to use functions derived by single step recursion on α, in the recursion scheme, thereby obtaining new functions, and then use these new functions, etc.

This corresponds to looking at autonomous α-descent recursion as defined in [FS95], where we close off using the binary operation $D(\alpha, g, h)$, starting with elementary g, h. (Here the unary functions produced are fed back as binary functions using an elementary pairing function.) We thus have defined what we will call here the iterated α-descent recursive functions. The iterated $<\alpha$-descent recursive functions are the union of the iterated β-descent recursive functions, for $\beta < \alpha$.

In [FS95], it is essentially shown that if $\alpha > \omega$ is closed under multiplication, then the $<\alpha$-descent recursive functions are the same as the iterated $<\alpha$-descent recursive functions, and are closed under composition. We say "essentially" because in the iteration, [FS95] allows only elementary g, thus iterating the h's only. However, by various simple devices, including Lemma 1.7 of [FS95], one easily sees that this does not make any difference.

The upshot is the following lemma.

LEMMA 5.16. *For each $k \geqslant 1$, the $<\omega^{\omega^k}$ recursive functions in the sense of* [Ro84] *are the same as the $<\omega^{\omega^k}$ descent recursive functions in the sense of* [FS95].

Proof. The details, as sketched above, are left to the reader. Q.E.D.

We now relate this to the $J_{k,m,p}$.

LEMMA 5.17. *Let $k \geqslant 1$. Every $<\omega^{w^k}$ recursive function is dominated by some $J_{k,m,p}$ at all $n \geqslant 1$.*

Proof. Let $g, h\colon N^2 \to N$ be elementary and $m \geqslant 1$. By Lemma 5.16, it suffices to show that $D(\omega^{\omega^{k-1}\cdot m}, g, h)$ is dominated by some $J_{k,m,p}$ at all $n \geqslant 1$.

Choose p such that

(i) $g(n, 0) < A_p(1+n+m)$ for all $n \geqslant 1$;

(ii) $2(h(n, i)+1) < A_p(i+n+1)$ for all $n, i \geqslant 1$;

(iii) $g(n, q) < A_p(q+n+2)$ for all $n \geqslant 1$ and $q \geqslant 0$.

The existence of p depends only on the primitive recursivity of g, h, and that every primitive recursive function is dominated by $A_p(n+2)$, for some p.

Let $n \geqslant 1$. If $h(n, 0) < \omega^{\omega^{k-1}\cdot m}$ is false then $D(\omega^{\omega^{k-1}\cdot m}, g, h)(n) = g(n, 0) < A_p(n)$.

Assume $h(n, 0) < \omega^{\omega^{k-1}\cdot m}$, and let $\omega^{\omega^{k-1}\cdot m} > h(n, 0) > h(n, 1) > \cdots > h(n,q)$ be such that $q = C(\omega^{\omega^{k-1}\cdot m}, h)$. Now consider $\omega^{\omega^{k-1}\cdot m} > \omega\cdot(h(n, 0)+1) > \omega\cdot(h(n, 1)+1) > \cdots > \omega\cdot(h(n, q)+1) > g(n, q) > g(n, q)-1 > \cdots > 0$. (The last terms from $g(n, q)$ are all finite). Using (ii) and (iii), we see that this sequence satisfies the conditions in the definition of $J_{k,m,p}(n)$. Hence $D(\omega^{\omega^{k-1}\cdot m}, g, h)(n) = g(n, q) < J_{k,m,p}(n)$ as required. Q.E.D.

THEOREM 5.18. *The functions $n(k)$ and F eventually dominate every $<\omega^{\omega^\omega}$ recursive function. For all $k \geqslant 1$, G_{k+1} eventually dominates every $<\omega^{\omega^k}$ recursive function. For all $k \geqslant 1$, G_{k+2} eventually dominates every ω^{ω^k} recursive function.*

Proof. Let g be a $<\omega^{\omega^k}$ recursive function. By Lemma 5.17, g is dominated by some $J_{k,m,p}$. By Lemma 5.15, $J_{k,m,p}$ is eventually dominated by H_{k+1}. By Lemma 5.11, $G_{k+1} \geqslant H_{k+1}$. By Lemma 5.4, F eventually dominates G_{k+1}. Hence F eventually dominates g, and G_{k+1} eventually dominates g.

Since every ω^{ω^k} recursive function is $<\omega^{\omega^{k+1}}$ recursive, the last claim follows.

Finally, to see that $n(k)$ also eventually dominates every $<\omega^{\omega^\omega}$ function, let g be $<\omega^{\omega^\omega}$ recursive. Then for all sufficiently large k, $F(k) > g(k+7)$. Hence for all sufficiently large k, $F(k-7) > g(k)$. By Lemma 5.3, for all sufficiently large k, $n(k) > g(k)$. Q.E.D.

We now use [Si88] to locate the functions F and G_k in terms of ordinal recursion.

Let $k \geqslant 1$. The tree T_k consists of all finite sequences of elements of $\{1, ..., k\}^*$ such that no term is a subsequence of any later term. Note that by Theorem 1.1 (second claim) T_k is a well founded tree, and hence has an ordinal assignment.

[Si88] investigates primitive recursive ordinal assignments for T_k.

Lemma 5.19. *There is a binary primitive recursive function B such that the following holds. For all $k \geqslant 1$, B_k is a function from the tree T_k into $\omega^{\omega^{k-1}}$ such that if s extends t in T_k then $B_k(s) < B_k(t)$.*

Proof. See [Si88, p. 971]. Q.E.D.

Theorem 5.20. *For each $k \geqslant 1$, G_{k+1} is an ω^{ω^k} recursive function. The functions $n(k)$, F, and G (as a binary function) are ω^{ω^ω} recursive functions. The functions $n(k)$ and F are strictly increasing. $G_k(n)$ is strictly increasing in each argument.*

Proof. Let $k \geqslant 1$. We use function B_{k+1} of Lemma 5.19 to give an ω^{ω^k} recursive definition of G_{k+1} by working up the tree T_{k+1}. Specifically, to compute $G_{k+1}(n)$, we do the following. For each $q \geqslant 1$, let $\alpha(q)$ be the maximum of the value of B_{k+1} at nodes in T_{k+1} of length q obeying (i) in the definition of G_{k+1}. We then find q such that $\alpha(q) = \alpha(q+1)$. Then we know that $q = G_{k+1}(n)$.

The function B provides an ω^{ω^ω} recursive definition of G by uniformly working up the trees T_k, as in the previous paragraph. By Lemma 5.1, $n(k)$, F can be defined from G by composition with an elementary function using search. Hence the functions $n(k)$, F are also ω^{ω^ω} recursive.

By Lemma 5.4, F is strictly increasing and $G_k(n)$ is strictly increasing in each argument. It remains to show that for all $k \geqslant 1$, $n(k) < n(k+1)$. Let

$x[1], \ldots, x[p]$ be according to the definition of $n(k)$. Then $x[1], \ldots, x[p]$, $k+1$ is according to the definition of $n(k+1)$. Q.E.D.

[Ro84] introduces the Hardy hierarchy (on ordinals $< \epsilon_0$) on p. 80 as follows:

$$h_0(x) = x, \qquad h_{\alpha+1}(x) = h_\alpha(x+1), \qquad h_\lambda(x) = h_{\lambda(x)}(x),$$

where $\lambda(x)$ is the xth term of the standard fundamental sequence associated with the limit ordinal $\lambda < \epsilon_0$.

Also [Ro84] defines $H_\alpha(x) = h_{\omega^\alpha}(x)$. And [Ro84, p. 81], proves the following about H:

$$H_1(x) = 2x+1, \qquad H_{\beta+1}(x) = H_\beta^{x+1}(x), \qquad H_\lambda(x) = H_{\lambda(x)}(x).$$

Here H^{x+1} is the composition of H with itself $x+1$ times.

Thus the finite levels of the H-hierarchy are (essentially) the same as the Ackermann hierarchy. This is called the "fast growing hierarchy."

From [Ro84, pp. 93, 94 (credited to "Tait, Lob, Wainer *et al.*")], we can read off the following information about the functions $n(k)$, F, and G_k. In the following, we obtain $H_{\omega^\omega+1}$ and H_{ω^k+1} instead of $H_{\omega^{\omega+1}}$ and $H_{\omega^{k+1}}$ because these functions are defined by one step ordinal recursions on ω^ω and ω^k.

Theorem 5.21. *The functions $n(k)$ and F eventually dominate all H_β, $\beta < \omega^\omega$. For all $k \geqslant 1$, G_{k+1} eventually dominates all H_β, $\beta < \omega^k$. The functions $n(k)$ and F are eventually dominated by $H_{\omega^\omega+1}$. For all $k \geqslant 1$, G_{k+1} is eventually dominated by H_{ω^k+1}.*

We will not attempt to obtain more precise information here.

[Ro84] also discusses forms of nested multiple recursion on the integers, following [Ta61].

Our favorite way of presenting nested multiple recursion on the integers is by the scheme

$$f(x_1, \ldots, x_k, y_1, \ldots, y_m) = t(f_{<x_1, \ldots, x_k}(y_1, \ldots, y_m)),$$

where

(i) $f_{<x_1, \ldots, x_k}$ is the function given by $f_{<x_1, \ldots, x_k}(z_1, \ldots, z_k, y_1, \ldots, y_m) = f(z_1, \ldots, z_k, y_1, \ldots, y_m)$ if $(z_1, \ldots, z_k) <_{lex} (x_1, \ldots, x_k)$; 0 otherwise;

(ii) t is any term involving $f_{<x_1, \ldots, x_k}$, variables $x_1, \ldots, x_k, y_1, \ldots, y_m$, the successor function, constants for integers, previously defined functions, IF THEN ELSE, with $<$, $=$ used in connection with IF THEN ELSE.

The functions generated in this way are called the nested multiply recursive functions (on the integers). This is a rather robust collection of functions on the integers, whose definition does not involve ordinal notations. It coincides with the $<\omega^{\omega^\omega}$ recursive functions, and the $<\omega^\omega$ nested recursive functions; see [Ro84, pp. 93, 94], going back to [Ta61].

COROLLARY 5.22. *The functions $n(k)$ and F eventually dominate all nested multiply recursive functions on the integers. The functions G_k are nested multiply recursive functions.*

6. RELATED PROBLEMS AND COMPUTER EXPLORATIONS

In Section 2, we introduced the functions f_k, $k \geqslant 1$, based on the partial order $\leqslant^*$ on N^k. We gave some lower bounds in Theorem 2.6 involving the Ackermann hierarchy. We now prove that each f_k is primitive recursive. We use [Si88].

Let S_k be the tree of all finite sequences of elements of N^k, where no term is $\leqslant^*$ any later term.

LEMMA 6.1. *For each $k \geqslant 1$ there is a primitive recursive function D_k such that the following holds. D_k is a function from the tree S_k into ω^k such that if s extends t in S_k then $D_k(s) < D_k(t)$.*

Proof. By [Si88, p. 970]. Q.E.D.

THEOREM 6.2. *Each f_k is primitive recursive. $f_k(n)$ is strictly increasing in each argument.*

Proof. Let $k \geqslant 1$. Then f_k can be defined by ω^k recursion using the tree S_k as follows. To compute $f_k(n)$, do the following. For each $q \geqslant 1$, let $\alpha(q)$ be the maximum value of D_k at nodes in S_k of length q representing sequences obeying (i) in the definition of f_k. Find the least q such that $\alpha(q) = \alpha(q+1)$. Then $q = f_k(n)$. As in, e.g., [Ro84], every ω^k recursive function is primitive recursive.

For the last claim, let $u[1], ..., u[n] \in N^k$ be as in the definition of $f_k(p)$. Then $u[1]\,k, ..., u[n]\,k$, (k) is as in the definition of $f_k(p+1)$, and $u[1], ..., u[n]$, $(k+1)$ is as in the definition of $f_{k+1}(p)$. Q.E.D.

We now introduce functions $M_k\colon Z^+ \to Z^+$ as follows. Let $k \geqslant 1$. $M_k(n)$ is the length of the longest sequence $x[1], ..., x[p]$ from $\{1, ..., k\}$ such that for no $n \leqslant i < j \leqslant p/2$, is $x[i], ..., x[2i]$ a subsequence of $x[j], ..., x[2j]$.

Recall the functions $G_k\colon Z^+ \to Z^+$ defined in Section 5.

LEMMA 6.3. *For all $k, n \geqslant 1$, $M_k(n) \leqslant 2G_k(n) + 2n$-1.*

Proof. Let $x[1], ..., x[p]$ be of longest length from $\{1, ..., k\}$ according to the definition of $M_k(n)$. Then p is odd, and $(x[n], ..., x[2n])$, $(x[n+1], ..., x[2n+2])$, ..., $(x[(p-1)/2], ..., x[p-1])$ have lengths $n+1, ..., (p+1)/2$. Hence $(p+1)/2 - n - 1 + 1 = (p+1)/2 - n \leqslant G_k(n)$. So $p \leqslant 2G_k(n) + 2n - 1$.

Q.E.D.

We obtain the following crude result akin to Lemma 5.3. Recall the a_i's defined just before Lemma 5.2.

LEMMA 6.4. *Let $i \geqslant 1$. $a_{i+1} - a_i \geqslant (a_i + 1)/3$. $a_{i+2} - a_{i+1} > a_{i+1} - a_i$. If $a_i, a_{i+1} \in \{3n, ..., 6n\}$ then for all $j \geqslant 1$, $a_{j+1} - a_j \geqslant n + j$.*

Proof. For the first claim, note that it is true for the basis cases $i = 1, 2$. Let $i \geqslant 2$ and suppose $a_{i+1} - a_i \geqslant (a_i + 1)/3$. Then $a_{i+2} - a_{i+1} = 2a_i + 1 - (2a_{i-1} + 1) = 2(a_i - a_{i-1}) \geqslant 2(a_{i-1} + 1)/3 = (a_{i+1} + 1)/3$.

For the second claim, note that it is true for the basis cases $i = 1, 2$. Let $i \geqslant 2$ and suppose $a_{i+2} - a_{i+1} > a_{i+1} - a_i$. Then $a_{i+3} - a_{i+2} = 2a_{i+1} - 2a_i > 2a_i - 2a_{i-1} = a_{i+2} - 2a_{i+1}$.

For the third claim, let $a_i, a_{i+1} \in \{3n, ..., 6n\}$. Then $a_{i+1} - a_i > n$ and so $a_{i+1} - a_i \geqslant n + 1$. So the claim holds for $j = 1$. It follows for $j \geqslant 1$ by the second claim. Q.E.D.

The following is a very crude result.

LEMMA 6.5. *For all $k, n \geqslant 1$, $M_{k+4}(n) \geqslant G_k(n) \geqslant (M_k(n) - 2n + 1)/2$.*

Proof. Let $x[1], ..., x[p]$ obey (i) and (ii) in the definition of $G_k(n)$ where $p = G_k(n)$. By Lemma 5.2, let t be such that $a_t, a_{t+1} \in \{3n+3, ..., 6n+6\}$. By Lemma 6.4, let $x'[1], ..., x'[p]$ be sequences from $\{1, ..., k+1\}$ of lengths $a_{t+1} - a_t - 1$, $a_{r+2} - a_{r+1} - 1$, ..., $a_{p+1} - a_p - 1$, where $x'[i]$ is obtained from $x[i]$ by appending the requisite number of $k+1$'s. Then for no $i < j$ is $x'[i]$ a subsequence of $x'[j]$.

Now define $y[a_t], y[a_t + 1], ..., y[a_{p+1}] \in \{1, ..., k+2\}$ as follows. Set $y[a_i]$, $t \leqslant i \leqslant p+1$, to be $k+2$. For $t \leqslant i \leqslant p+1$, let $y[a_i + 1], ..., y[a_{i+1} - 1]$ be $x'[i - t + 1]$. Define $y[1] = \cdots = y[a_t - 1] = 1$. Then for no $a_t \leqslant i < j \leqslant a_{p+1}/2$, is $y[i], ..., y[2i]$ a subsequence of $y[j], ..., y[2j]$. This uses Lemma 5.2.

It remains to define $y[1], ..., y[a_t - 1]$. Note that $a_t - 1 \leqslant 3n + 2$. Define $y[1] = \cdots = y[2n] = k+3$ and $y[2n+1] = \cdots = y[a_t - 1] = k+4$. Then for no $n \leqslant i < j \leqslant a_{p+1}/2$, is $y[i], ..., y[2i]$ a subsequence of $y[j], ..., y[2j]$.

Q.E.D.

THEOREM 6.6. *Let $k \geqslant 1$. M_{k+1} is an ω^{ω^k} recursive function. M_{k+5} eventually dominates every $<\omega^{\omega^k}$ recursive function. M_{k+6} eventually dominates every ω^{ω^k} recursive function. M (as a binary function) is an ω^{ω^ω} recursive function. M_{k+5} eventually dominates all H_β, $\beta < \omega^k$. M_{k+1} is eventually dominated by H_{ω^k+1}. $M_k(n)$ is strictly increasing in each argument.*

Proof. By Theorems 5.18, 5.20, 5.21, and Lemma 6.5.

For the final claim, let $x[1], ..., x[p]$ be as in the definition of $M_k(n)$, where $p = M_k(n)$. Then p is odd. We now show that $x[1], ..., x[p-1]$, $k+1$, $k+1$ is as in the definition of $M_{k+1}(n)$. Note that $p \geqslant 3$.

So see this, let $n \leqslant i < j \leqslant (p+1)/2$. Without loss of generality, we may assume $j = (p+1)/2 \geqslant 2$. I.e., we need to verify that $x[i], ..., x[2i]$ is not a subsequence of $x[j], ..., x[p-1]$, $k+1$. Suppose this is false. Then $x[i], ...,$ $x[2i]$ is a subsequence of $x[j], ..., x[p-1]$, and hence of $x[j-1], ...,$ $x[p-1]$. Therefore $i = j-1$. I.e., $x[j-1], ..., x[p-1]$ is a subsequence of $x[j], ..., x[p-1]$, $k+1$, which is impossible. Thus $M_{k+1}(n) > M_k(n)$.

Finally, to see that $M_k(n) < M_k(n+1)$, we show that $1, x[1], ..., x[p]$ is as in the definition of $M_k(n+1)$. Let $n+1 \leqslant i < j \leqslant (p+1)/2$. We need to verify that $x[i-1], ..., x[2i-1]$ is not a subsequence of $x[j-1], ...,$ $x[2j-1]$. Suppose this is false. Then $x[i-1], ..., x[2i-2]$ is a subsequence of $x[j-1], ..., x[2j-2]$, and $n \leqslant i-1 < j-1 \leqslant p/2$. This is a contradiction. Q.E.D.

The function M_2, involving two letters 1, 2, assumes special importance. In fact, we write $m(k) = M_2(k)$. By Theorem 6.6, the function m is strictly increasing.

Note that by Theorem 1.6, $m(1) = n(2) = 11$.

LEMMA 6.7. *Let $n \geqslant 13k+5$, $k \geqslant 3$. There is a sequence $3^{n-1}2y$ from $\{2, 3\}$ with property * indexed from n through $A_{k-1}(2n-4k-2)+1$.*

Proof. Let n, k be as given. By Lemma 4.2, let x be a strong $2n$, $3n$, $3k+1$, k, $A_{k-1}(2n-4k-2)$-sequence, where there does not exist $i < j \leqslant$ $A_{k-1}(2n-4k-2)$ such that $C_i(x)$ is a subsequence of $C_j(x)$. By the Main Lemma of Section 3, and the fact that the lengths of the C_i's are strictly increasing, we see that $3^{n-1}2y$ has property *, where $3^{n-1}2y$ is indexed from n, and y is the first $F(2n, 3n, A_{k-1}(2n-4k-2))$ terms of x. The result follows since $F(2n, 3n, A_{k-1}(2n-4k-2)) > A_{k-1}(2n-4k-2)$. Q.E.D.

THEOREM 6.8. *For all $k \geqslant 2$, $m(13k+5) > A_{k-1}(22k+8)$. $m(83) > A_5(140)$. The function m eventually dominates any given primitive recursive function.*

Proof. Immediate from Lemma 6.7. Q.E.D.

R. Dougherty has written and implemented software to investigate the function m. [Do98] reports the following results, where 0 and 1 are used rather than 1 and 2:

$m(1) = 11$: 01110000000

$m(2) = 31$:
0001101111110100000000000000000,
0001101111111100000000000000000

$m(3) = 199$:
0001011100011000000000010000000000000000000000011111111111111111
1111111111111111111111111111111100000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000
0110000,
0001011110011000000000010000000000000000000000011111111111111111
1111111111111111111111111111111100000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000
0110000.

[Do98] reports that the above are all of the longest sequences of the required kind, except for reversing the bits, and then changing the first $k = 0, 1, 2$ bits, respectively, and the last bit. Thus there are 4 longest sequences for $m(1)$, 16 longest sequences for $m(2)$, and 32 longest sequences for $m(3)$.

[Do98] also reports that $m(4) \geqslant 187205$, indicating that this result used man machine interaction. The far smaller sequences that were generated by the computer for $m(4)$ by brute force, were examined. The observed patterns were used to obtain an appropriate sequence of length 187,205.

[Do98] also considers the lengths of special sequences, which were used to obtain our lower bound for $n(3)$; see the beginning of Section 4, and Lemma 4.3. Let L be the longest length of a special sequence. By Lemma 4.3, L is of course much smaller than $n(3)$.

[Do98] claims that $L \leqslant m(4)$, with the help of output from the computer implementation. In addition, [Do98] reports that certain sequences for $m(4)$ can be easily modified to yield a special sequence of slightly smaller length.

In this way, [Do98] claims that $L \geqslant 187196$, using the particular sequence constructed for the result $m(4) \geqslant 187205$. Now $187188 = 26(7199) + 14$. Thus by Lemma 4.3 and $L \geqslant 187196$, we have the following improved lower bound for $n(3)$:

THEOREM 6.9. $n(3) > A_{7198}(158386)$.

Some Open Problems. What is the least k such that $m(k)$ is incomprehensibly large? E.g., $m(k) \geqslant A_5(5)$? How large is $m(4)$? How many longest sequences for $m(k)$ are there? For $m(4)$? For $n(k)$? For $n(3)$? Give upper and lower bounds for $m(k+1)$ in terms of $m(k)$. Give an upper bound for $m(k)$ in terms of the Ackermann hierarchy and k.

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