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SOME REMARKS ON MAEHARA'S METHOD

Abstract

In proving the interpolation theorem in terms of sequent calculus, Maehara's method is usually used. This paper shows that the interpolation theorem for substructural logics obtained by deleting constants from **FL**, $\mathbf{FL_e}$, $\mathbf{FL_{ec}}$, $\mathbf{CFL_e}$ and $\mathbf{CFL_{ec}}$ can be proved by putting some restriction on partitions in Maehara's method.

1. Introduction

The following claim, called the *interpolation theorem*, is first proved by W.Craig in 1957 (see [1] and [2]):

> if $A \supset B$ is a theorem and if A and B have at least one predicate symbol in common, then there is an intermediate formula D such that both $A \supset D$ and $D \supset B$ are theorems and that all predicate symbols occurring in D also occur both in A and in B. Also, if $A \supset B$ and if A and B have no predicate symbol in common, then either $\neg A$ or B is a theorem.

After that, S.Maehara showed in 1960 that the interpolation theorem follows from the cut elimination theorem for **LK**. The method which he used in the proof is called now *Maehara's method*. According to [3], we explain Maehara's method concisely. First, let $\top$ and $\bot$ be constants and add

$$\to \top \qquad \text{and} \qquad \bot \to$$

to initial sequent. Then we generalize the interpolation theorem as follows:

> if a sequent $\Gamma \to \Delta$ is provable, and if Γ and Δ divide any two parts Γ_1, Γ_2 and Δ_1, Δ_2, respectively, then there exists a formula D such that both $\Gamma_1 \to D, \Delta_1$ and $\Gamma_2, D \to \Delta_2$ are provable and all predicate symbols occurring in D also occur both in Γ_1, Δ_1 and in Γ_2, Δ_2.

Maehara's method gives many positive results on the interpolation theorem for the logics which admit the cut elimination theorem. If we want to apply this method to the logic without constants, then we once introduce constants and eliminate them after proving the generalized interpolation theorem. This procedure is considered to simplify the proof. It is known that Maehara's method cannot be applied directly to substrucutural logics $\mathbf{FL}^-$, $\mathbf{FL}^-_{\mathbf{e}}$, $\mathbf{FL}^-_{\mathbf{ec}}$, $\mathbf{CFL}^-_{\mathbf{e}}$and $\mathbf{CFL}^-_{\mathbf{ec}}$, obtained from $\mathbf{FL}$, $\mathbf{FL}_{\mathbf{e}}$, $\mathbf{FL}_{\mathbf{ec}}$, $\mathbf{CFL}_{\mathbf{e}}$ and $\mathbf{CFL}_{\mathbf{ec}}$, respectively. As shown in this paper, the interpolation theorem for above logics can be proved by putting some restriction on partitions in Maehara's method. Then the proof is similar way to Maehara's method without the procedure introducing and eliminating constants. Further, we notice that this method gives the proof of the interpolation theorem for **LK** etc. without using constants.

2. Preliminaries

The language of our logics consists of predicate symbols and logical connectives $\neg, \supset, \wedge, \vee$ and $\circ$ (and $+$, if possible) and quantifiers $\forall$ and $\exists$. The formulas are defined in the usual way and denoted by $A, B, C, \cdots$. Further, Greek capital letters $\Gamma, \Delta, \cdots$ denote (finite, possibly empty) sequence of formulas.

A *sequent* is of the form

$$A_1, \cdots, A_m \to B_1, \cdots, B_n,$$

where $A_1, \cdots, A_m, B_1, \cdots, B_n (m, n \geq 0)$ are formulas.

The sequent calculus $\mathbf{CFL}^-_{\mathbf{ec}}$is as follows, where t and a is a term and a variable, respectively.

Initial sequent

$$A \to A$$

Contraction rules

$$\frac{\Gamma, A, A, \Sigma \to \Delta}{\Gamma, A, \Sigma \to \Delta}(c \to) \qquad \frac{\Gamma \to \Lambda, A, A, \Theta}{\Gamma \to \Lambda, A, \Theta}(\to c).$$

Exchange rules

$$\frac{\Gamma, A, B, \Sigma \to \Delta}{\Gamma, B, A, \Sigma \to \Delta}(e \to) \qquad \frac{\Gamma \to \Lambda, A, B, \Theta}{\Gamma \to \Lambda, B, A, \Theta}(\to e)$$

Cut rule

$$\frac{\Gamma \to A, \Theta \quad \Sigma, A, \Pi \to \Delta}{\Gamma, \Sigma, \Pi \to \Delta, \Theta}$$

Rules for logical connectives

$$\frac{\Gamma \to A, \Theta}{\neg A, \Gamma \to \Theta}(\neg \to) \qquad \frac{\Gamma, A \to \Theta}{\Gamma \to \neg A, \Theta}(\to \neg)$$

$$\frac{\Gamma \to A, \Theta \quad \Pi, B, \Sigma \to \Delta}{\Pi, A \supset B, \Gamma, \Sigma \to \Delta, \Theta}(\supset\to) \qquad \frac{\Gamma, A \to B, \Theta}{\Gamma \to A \supset B, \Theta}(\to\supset)$$

$$\frac{\Gamma, A, \Sigma \to \Delta}{\Gamma, A \wedge B, \Sigma \to \Delta}(\wedge \to 1) \qquad \frac{\Gamma, B, \Sigma \to \Delta}{\Gamma, A \wedge B, \Sigma \to \Delta}(\wedge \to 2)$$

$$\frac{\Gamma \to \Lambda, A, \Theta \quad \Gamma \to \Lambda, B, \Theta}{\Gamma \to \Lambda, A \wedge B, \Theta}(\to \wedge)$$

$$\frac{\Gamma, A, \Sigma \to \Delta \quad \Gamma, B, \Sigma \to \Delta}{\Gamma, A \vee B, \Sigma \to \Delta}(\vee \to)$$

$$\frac{\Gamma \to \Lambda, A, \Theta}{\Gamma \to \Lambda, A \vee B, \Theta}(\to \vee 1) \qquad \frac{\Gamma \to \Lambda, B, \Theta}{\Gamma \to \Lambda, A \vee B, \Theta}(\to \vee 2)$$

$$\frac{\Gamma, A, B, \Sigma \to \Delta}{\Gamma, A \circ B, \Sigma \to \Delta}(\circ \to) \qquad \frac{\Gamma \to A, \Lambda \quad \Sigma \to B, \Theta}{\Gamma, \Sigma \to A \circ B, \Lambda, \Theta}(\to \circ)$$

$$\frac{\Gamma, A \to \Lambda \quad \Sigma, B \to \Theta}{\Gamma, \Sigma, A + B \to \Lambda, \Theta}(+ \to) \qquad \frac{\Gamma \to \Pi, A, B, \Sigma}{\Gamma \to \Pi, A + B, \Sigma}(\to +)$$

Rules for quantifiers

$$\frac{\Gamma, A(t), \Sigma \to \Delta}{\Gamma, \forall x A(x), \Sigma \to \Delta}(\forall \to) \qquad \frac{\Gamma \to \Lambda, A(a), \Theta}{\Gamma \to \Lambda, \forall x A(x), \Theta}(\to \forall)$$

$$\frac{\Gamma, A(a), \Sigma \to \Delta}{\Gamma, \exists x A(x), \Sigma \to \Delta}(\exists \to) \qquad \frac{\Gamma \to \Lambda, A(a), \Theta}{\Gamma \to \Lambda, \exists x A(x), \Theta}(\to \exists)$$

For $(\to \forall)$ and $(\exists \to)$, there are *restrictions on variables*: the variable a must occur in neither Γ nor Δ.

$\mathbf{CFL}^-_{\mathbf{e}}$ is obtained from $\mathbf{CFL}^-_{\mathbf{ec}}$ by deleting contraction rules. $\mathbf{FL}^-_{\mathbf{ec}}$ is obtained from $\mathbf{CFL}^-_{\mathbf{ec}}$ by restricting as follows: (1) at most one formula occurs in Δ, (2) Λ and Θ must be empty, and (3) deleting $(\to c)$, $(\to e)$, $(+ \to)$ and $(\to +)$. $\mathbf{FL}^-_{\mathbf{e}}$ is obtained from $\mathbf{FL}^-_{\mathbf{ec}}$ by deleting $(c \to)$. $\mathbf{FL}^-$ is obtained from $\mathbf{FL}^-_{\mathbf{e}}$ by deleting $(e \to)$.

The definition of *proofs* and *provability* are as usual ways. In particular, a formula A is *provable* if a sequent $\to A$ is provable.

The following cut elimination theorem is well-known. For more information, see [4].

Theorem 1. *If a sequent $\Gamma \to \Delta$ is provable in $\mathbf{FL}^-$, $\mathbf{FL}^-_{\mathbf{e}}$, $\mathbf{FL}^-_{\mathbf{ec}}$, $\mathbf{CFL}^-_{\mathbf{e}}$ or $\mathbf{CFL}^-_{\mathbf{ec}}$, then it is also provable in $\mathbf{FL}^-$, $\mathbf{FL}^-_{\mathbf{e}}$, $\mathbf{FL}^-_{\mathbf{ec}}$, $\mathbf{CFL}^-_{\mathbf{e}}$ or $\mathbf{CFL}^-_{\mathbf{ec}}$ without using the cut rule.*

3. Proof of interpolation theorem

We will introduce some notions through this section. A sequent $\Gamma \to \Delta$ is *standard* if at least two formulas appears in it. For any formula A, $V(A)$ denotes the set of all predicate symbols appearing in A. When Γ is a sequence of formulas $A_1, \cdots, A_n$, $V(\Gamma)$ denotes $V(A_1) \cup \cdots \cup V(A_n)$.

First, we prove the interpolation theorem for $\mathbf{CFL}^-_{\mathbf{e}}$ and $\mathbf{CFL}^-_{\mathbf{ec}}$. To do this, we will introduce some notions. For a given (not necessary standard) sequent $\Gamma \to \Delta$, a pair $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ of the pair of (possibly empty) sequences of formulas $(\Gamma_1 : \Delta_1)$ and $(\Gamma_2 : \Delta_2)$ is called a *partition* of $\Gamma \to \Delta$, if each component of Γ appears just once either in Γ_1 or in Γ_2, and if each component of Δ appears just once either in Δ_1 or Δ_2. Furthermore, a partition $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ of $\Gamma \to \Delta$ is *conditional*, if both

Γ_1, Δ_1 and Γ_2, Δ_2 are not empty. Therefore conditional partitions (below, c-partitions) are defined only for standard sequents.

Then in order to prove the interpolation theorem for $\mathbf{CFL_e^-}$ and $\mathbf{CFL_{ec}^-}$, it is sufficient to show the following.

LEMMA 2. *Let* $\mathbf{L}$ *be* $\mathbf{CFL_e^-}$ *or* $\mathbf{CFL_{ec}^-}$. *If a standard sequent* $\Gamma \to \Delta$ *is provable in* $\mathbf{L}$ *and* $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ *is a c-partition of* $\Gamma \to \Delta$, *then there exists a formula* D *(called interpolant) such that*

(a) both $\Gamma_1 \to \Delta_1, D$ *and* $D, \Gamma_2 \to \Delta_2$ *are provable in* $\mathbf{L}$

(b) $V(D) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$.

PROOF. By Theorem 1, there is a cut-free proof P of $\Gamma \to C$. This lemma is proved by induction on the length n of P.

Here we will prove only the case that $\mathbf{L}$ is $\mathbf{CFL_e^-}$. Suppose that $n = 1$. In this case, $\Gamma \to \Delta$ must be an initial sequent $A \to A$. It is clear that an initial sequent is standard and that $V(A) \cap V(A) \neq \emptyset$. Then a c-partition is either $\langle(A :); (: A)\rangle$ or $\langle(: A); (A :)\rangle$. For the former, we may take A as an interpolant. For the latter, we may take $\neg A$ as an interpolant.

Next, suppose that $n > 1$. Let I be the last rule of inference in P. Here, we prove the problematic case that I is $(\supset\to)$. Since $\mathbf{L}$ has exchange rules, it may assume that I has the following form:

$$\frac{\Gamma \to A, \Theta \quad B, \Sigma \to \Delta}{A \supset B, \Gamma, \Sigma \to \Delta, \Theta}.$$

1) Consider a c-partition $\langle(A \supset B, \Gamma_1, \Sigma_1 : \Delta_1, \Theta_1); (\Gamma_2, \Sigma_2 : \Delta_2, \Theta_2)\rangle$ of $A \supset B, \Gamma, \Sigma \to \Delta, \Theta$. We should consider the following three cases.

1.1) Suppose that both Γ_2, Θ_2 and Σ_2, Δ_2 are not empty. Take a c-partition $\langle(\Gamma_1 : A, \Theta_1); (\Gamma_2 : \Theta_2)\rangle$ of $\Gamma \to A, \Theta$ and a c-partition $\langle(B, \Sigma_1 : \Delta_1); (\Sigma_2 : \Delta_2)\rangle$ of $B, \Sigma \to \Delta$. By the hypotheses of induction, there exist formulas D and E such that

(a) both $\Gamma_1 \to A, \Theta_1, D$ and $D, \Gamma_2 \to \Theta_2$ are provable

(b) $V(D) \subseteq V(\Gamma_1, A, \Theta_1) \cap V(\Gamma_2, \Theta_2)$

(c) both $B, \Sigma_1 \to \Delta_1, E$ and $E, \Sigma_2 \to \Delta_2$ are provable

(d) $V(E) \subseteq V(B, \Sigma_1, \Delta_1) \cap V(\Sigma_2, \Delta_2)$.

Using $(\to e), (\supset\to), (+ \to)$ and $(\to +)$, we see that

(e) both $A \supset B, \Gamma_1, \Sigma_1 \to \Delta_1, \Theta_1, D + E$ and $D + E, \Gamma_2, \Sigma_2 \to \Delta_2, \Theta_2$ are provable

(f) $V(D+E) \subseteq V(A \supset B, \Gamma_1, \Sigma_1, \Delta_1, \Lambda_1) \cap V(\Gamma_2, \Sigma_2, \Delta_2, \Lambda_2)$.

1.2) Suppose that Γ_2, Θ_2 is not empty and Σ_2, Δ_2 is empty. Then $B, \Sigma_1 \to \Delta_1$ is provable. Take a c-partition $\langle(\Gamma_1 : A, \Theta_1); (\Gamma_2 : \Theta_2)\rangle$ of $\Gamma \to A, \Theta$. By the hypothesis of induction, there exists a formula D such that

(a) both $\Gamma_1 \to A, \Theta_1, D$ and $D, \Gamma_2 \to \Theta_2$ are provable

(b) $V(D) \subseteq V(\Gamma_1, A, \Theta_1) \cap V(\Gamma_2, \Theta_2)$.

Using $(\to e)$ and $(\supset\to)$, we see that D is an interpolant.

1.3) Suppose that Γ_2, Θ_2 is empty and Σ_2, Δ_2 is not empty. Similar to (1.2).

2) Consider a c-partition $\langle(\Gamma_1, \Sigma_1 : \Delta_1, \Theta_1); (A \supset B, \Gamma_2, \Sigma_2 : \Delta_2, \Theta_2)\rangle$ of $A \supset B, \Gamma, \Sigma \to \Delta, \Theta$. Similar to (1). ■

Next, we prove the interpolation theorem for $\mathbf{FL}^-_{\mathbf{e}}$ and $\mathbf{FL}^-_{\mathbf{ec}}$. To do this, we will introduce some notions. For a given (not necessary standard) sequent $\Gamma \to C$, a pair $\langle\Gamma_1; \Gamma_2\rangle$ of (possibly empty) sequences of formulas Γ_1 and Γ_2 is called a *partition* of $\Gamma \to C$, if each component of Γ appears just once either in Γ_1 or in Γ_2. Furthermore, a partition $\langle\Gamma_1; \Gamma_2\rangle$ of $\Gamma \to C$ is *conditional*, if both Γ_1 and Γ_2, C are not empty.

Then in order to prove the interpolation theorem for $\mathbf{FL}^-_{\mathbf{e}}$ and $\mathbf{FL}^-_{\mathbf{ec}}$, it is sufficient to show the following. The proof is as in Lemma 2.

LEMMA 3. *Let* $\mathbf{L}$ *be* $\mathbf{FL}^-_{\mathbf{e}}$ *or* $\mathbf{FL}^-_{\mathbf{ec}}$. *If a standard sequent* $\Gamma \to C$ *is provable in* $\mathbf{L}$ *and* $\langle\Gamma_1; \Gamma_2\rangle$ *is a c-partition of* $\Gamma \to C$, *then there exists a formula* D *such that*

(a) both $\Gamma_1 \to D$ *and* $D, \Gamma_2 \to C$ *are provable in* $\mathbf{L}$ *and*

(b) $V(D) \subseteq V(\Gamma_1) \cap V(\Gamma_2, C)$.

Finally, we prove the interpolation theorem for $\mathbf{FL}^-$. To do this, we will introduce some notions. For a given (not necessary standard) sequent $\Gamma \to C$, we call a triple $\langle\Gamma_1; \Gamma_2; \Gamma_3\rangle$ of (possibly empty) sequences of formulas Γ_1, Γ_2 and Γ_3, *partition* of $\Gamma \to C$, if each component of Γ appears just once either in Γ_1, Γ_2 or in Γ_3 preserving the order of occurrence in Γ, i.e., $\Gamma_1, \Gamma_2, \Gamma_3$ is just Γ. Furthermore, a partition $\langle\Gamma_1; \Gamma_2; \Gamma_3\rangle$ of $\Gamma \to C$ is *conditional*, if both Γ_2 and Γ_1, Γ_3, C are not empty.

The following lemma plays an essential role in proving the interpolation theorem for $\mathbf{FL}^-$. The proof is analogue to Lemma 2.

LEMMA 4. *Let a standard sequent $\Gamma \to C$ be provable in $\mathbf{FL}^-$ and $\langle \Gamma_1; \Gamma_2; \Gamma_3 \rangle$ be a c-partition of $\Gamma \to C$. Then there exists a formula D such that*

(a) both $\Gamma_2 \to D$ and $\Gamma_1, D, \Gamma_3 \to C$ are provable in $\mathbf{FL}^-$, and

(b) $V(D) \subseteq V(\Gamma_2) \cap V(\Gamma_1, \Gamma_3, C)$.

From Lemmas 2 through 4, it is easy to see the following.

THEOREM 5. *Let $\mathbf{L}$ be $\mathbf{FL}^-$, $\mathbf{FL}^-_{\mathbf{e}}$, $\mathbf{FL}^-_{\mathbf{ec}}$, $\mathbf{CFL}^-_{\mathbf{e}}$ or $\mathbf{CFL}^-_{\mathbf{ec}}$. If $A \supset B$ is provable in $\mathbf{L}$, then there exists a formula D such that*

(a) both $A \supset D$ and $D \supset B$ are provable in $\mathbf{L}$

(b) $V(D) \subseteq V(A) \cap V(B)$.

4. Concluding remarks

By using above method, we can also prove the interpolation theorem for **LK**, **LJ** and so on. For **LK**, it is sufficient to show the following, where terminologies are as in Section 3.

Suppose that a standard sequent $\Gamma \to \Delta$ is provable in **LK** and $\langle (\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2) \rangle$ is a c-partition of $\Gamma \to \Delta$. Then either (i) or (ii) holds:

(i) if $V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2) \neq \emptyset$, there exists a formula D such that

  (a) both $\Gamma_1 \to \Delta_1, D$ and $D, \Gamma_2 \to \Delta_2$ are provable in **LK**

  (b) $V(D) \subseteq V(\Gamma_1, \Delta_1) \cap V(\Gamma_2, \Delta_2)$;

(ii) otherwise, either $\Gamma_1 \to \Delta_1$ or $\Gamma_2 \to \Delta_2$ is provable in **LK**.

For more precise proof through this paper, see [5].

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