

A NEW UPPER BOUND FOR SEPARATING WORDS

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ABSTRACT. We prove that for any distinct $x, y \in \{0, 1\}^n$, there is a deterministic finite automaton with $\tilde{O}(n^{1/3})$ states that accepts x but not y . This improves Robson’s 1989 upper bound of $\tilde{O}(n^{2/5})$.

1. INTRODUCTION

Given a positive integer n and any two distinct 0-1 strings $x, y \in \{0, 1\}^n$, let $f_n(x, y)$ denote the smallest positive integer m such that there exists a deterministic finite automaton with m states that accepts x but not y (of course, $f_n(x, y) = f_n(y, x)$). Define $f(n) := \max_{x \neq y \in \{0, 1\}^n} f_n(x, y)$. The “separating words problem” is to determine the asymptotic behavior of $f(n)$. An easy example [3] shows $f(n) = \Omega(\log n)$, which is the best lower bound known to date. Goralcik and Koubek [3] in 1986 proved an upper bound of $f(n) = o(n)$, and Robson [4] in 1989 proved an upper bound of $f(n) = O(n^{2/5} \log^{3/5} n)$. Despite much attempt, there has been no further improvement to the upper bound to date.

In this paper, we improve the upper bound on the separating words problem to $f(n) = O(n^{1/3} \log^7 n)$.

Theorem 1. *For any distinct $x, y \in \{0, 1\}^n$, there is a deterministic finite automaton with $O(n^{1/3} \log^7 n)$ states that accepts x but not y .*

We made no effort to optimize the (power of the) logarithmic term $\log^7 n$.

2. DEFINITIONS AND NOTATION

A deterministic finite automaton (DFA) M is a 4-tuple (Q, δ, q_1, F) consisting of a finite set Q , a function $\delta : Q \times \{0, 1\} \rightarrow Q$, an element $q_1 \in Q$, and a subset $F \subseteq Q$. We call elements $q \in Q$ “states”. We call q_1 the “initial state” and the elements of F the “accept states”. We say M accepts $x = x_1, \dots, x_n \in \{0, 1\}^n$ if and only if the sequence defined by $r_1 = q_1, r_{i+1} = \delta(r_i, x_i)$ for $1 \leq i \leq n$, has $r_{n+1} \in F$.

For a positive integer n , we write $[n]$ for $\{1, \dots, n\}$. We write \sim as shorthand for $= (1 + o(1))$. In our inequalities, C and c refer to (large and small, respectively) absolute constants that sometimes change from line to line. For functions f and g , we say $f = \tilde{O}(g)$ if $|f| \leq C|g| \log^C |g|$ for some absolute C . We say a set $A \subseteq [n]$ is

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d -separated if $a, a' \in A, a \neq a'$ implies $|a - a'| \geq d$. For a set $A \subseteq [n]$, a prime p , and a residue $i \in [p]_0 := \{0, \dots, p-1\}$, let $A_{i,p} = \{a \in A : a \equiv i \pmod{p}\}$.

For a string $x = x_1, \dots, x_n \in \{0, 1\}^n$ and a (sub)string $w = w_1, \dots, w_l \in \{0, 1\}^l$, let $\text{pos}_w(x) := \{j \in \{1, \dots, n-l+1\} : x_{j+k-1} = w_k \text{ for all } 1 \leq k \leq l\}$ denote the set of all (starting) positions at which w occurs as a substring in x .

3. AN EASY $\tilde{O}(n^{1/2})$ BOUND, AND MOTIVATION OF OUR ARGUMENT

In this section, we sketch an argument of an $\tilde{O}(n^{1/2})$ upper bound for the separating words problem, and then how to generalize that argument to obtain $\tilde{O}(n^{1/3})$.

For any two distinct strings $x, y \in \{0, 1\}^n$, the sets $\text{pos}_1(x)$ and $\text{pos}_1(y)$ are of course different. A natural way, therefore, to try to separate different strings x, y is to find a small prime p and a residue $i \in [p]_0$ so that $|\text{pos}_1(x)|_{i,p} \neq |\text{pos}_1(y)|_{i,p}$; if we can find such a p and i , then since there will be a prime q of size $q = O(\log n)$ with $|\text{pos}_1(x)|_{i,p} \not\equiv |\text{pos}_1(y)|_{i,p} \pmod{q}$, there will be a deterministic finite automaton with $2pq = O(p \log n)$ states that accepts one string but not the other (see Lemma 5). We are thus led to the following problem.

Problem 3.1. For given n , determine the minimum k such that for any distinct $A, B \subseteq [n]$, there is some prime $p < k$ and some $i \in [p]_0$ for which $|A_{i,p}| \neq |B_{i,p}|$.

Problem 3.1 has been considered in [5] and in [6]¹ (and possibly other places) and was essentially solved in each. We present a simple solution, also discovered in [6].

Claim 3.1. *For any distinct $A, B \subseteq [n]$, there is some prime $p = O(\sqrt{n \log n})$ and some $i \in [p]_0$ for which $|A_{i,p}| \neq |B_{i,p}|$.*

Proof. (Sketch) Fix distinct $A, B \subseteq [n]$. Suppose k is such that $|A_{i,p}| = |B_{i,p}|$ for all primes $p \leq k$ and all $i \in [p]_0$. For a prime p , let $\Phi_p(x)$ denote the p^{th} cyclotomic polynomial, of degree $p-1$. Then since $\sum_{j=1}^n 1_A(j) e^{2\pi i \frac{aj}{p}} = \sum_{j=1}^n 1_B(j) e^{2\pi i \frac{aj}{p}}$ for all $p \leq k$ and all $a \in [p]_0$, the polynomials $\Phi_p(x)$, for $p \leq k$, divide $\sum_{j=1}^n (1_A(j) - 1_B(j)) x^j =: f(x)$. Therefore, $\prod_{p \leq k} \Phi_p(x)$ divides $f(x)$. Since $A \neq B$, f is not identically 0 and thus must have degree at least $\sum_{p \leq k} (p-1) \sim \frac{1}{2} \frac{k^2}{\log k}$. Since the degree of f is obviously at most n , we must have $\frac{k^2}{\log k} \leq 3n$. The result follows. \square

By a standard pigeonhole argument (see Section 6), the bound $\tilde{O}(\sqrt{n})$ is sharp.

A natural idea to improve this $\tilde{O}(\sqrt{n})$ bound for the separating words problem is to consider the sets $\text{pos}_w(x)$ and $\text{pos}_w(y)$ for longer w . The length of w is actually not important in terms of its “cost” to the number of states needed, just as long as it is at most p , where we will be considering $|\text{pos}_w(x)|_{i,p}$ and $|\text{pos}_w(y)|_{i,p}$ (see Lemma 5). One immediate benefit of considering longer w is that the sets $\text{pos}_w(x)$ and $\text{pos}_w(y)$ are *smaller* than $\text{pos}_1(x)$ and $\text{pos}_1(y)$; indeed, for example, it can be

¹In the latter reference, they look for an *integer* $m < k$ and some $i \in [m]$ for which $|A_{i,m}| \neq |B_{i,m}|$, which is of course more economical. We decided to restrict to primes for aesthetic reasons.

shown without much difficulty that for any distinct $x, y \in \{0, 1\}^n$, there is some w of length $n^{1/3}$ such that $\text{pos}_w(x)$ and $\text{pos}_w(y)$ are distinct sets of size at most $n^{2/3}$. Thus, to get a bound of $\tilde{O}(n^{1/3})$, it suffices to show the following.

Problem 3.2. For any distinct $A, B \subseteq [n]$ of sizes $|A|, |B| \leq n^{2/3}$, there is some prime $p = \tilde{O}(n^{1/3})$ and some $i \in [p]_0$ such that $|A_{i,p}| \neq |B_{i,p}|$.

As in the proof sketch above, this problem is equivalent to a statement about a product of cyclotomic polynomials dividing a sparse polynomial of small degree (see the last page of [6]). We were not able to solve Problem 3.2. However, we make the additional observation that we can take w so that $\text{pos}_w(x)$ and $\text{pos}_w(y)$ are *well-separated* sets. Indeed, if w has length $2n^{1/3}$ and has no period of length at most $n^{1/3}$, then $\text{pos}_w(x)$ and $\text{pos}_w(y)$ are $n^{1/3}$ -separated sets. Lemmas 1 and 2 of [4] show that such w are common enough to ensure there is a choice with $\text{pos}_w(x) \neq \text{pos}_w(y)$. Our main theorem is the following².

Theorem 2. *Let A, B be distinct subsets of $[n]$ that are each $n^{1/3}$ -separated. Then there is some prime $p = \tilde{O}(n^{1/3})$ and some $i \in [p]_0$ so that $|A_{i,p}| \neq |B_{i,p}|$.*

Although Theorem 2 is also equivalent to a question about a product of cyclotomic polynomials dividing a certain type of polynomial, we were not able to make progress through number theoretic arguments. Rather, we reverse the argument of Scott [5], by noting that if there is some small m so that the m^{th} -moments of A and B differ, i.e. $\sum_{a \in A} a^m \neq \sum_{b \in B} b^m$, then there is some small p and some $i \in [p]_0$ so that $|A_{i,p}| \neq |B_{i,p}|$. The implication just written out is actually quite obvious (see the proof of Theorem 2); the implication of Scott, however, that some small p and some $i \in [p]_0$ with $|A_{i,p}| \not\equiv |B_{i,p}| \pmod{p}$ implies the existence of some small m with $\sum_{a \in A} a^m \neq \sum_{b \in B} b^m$ is less trivial, though still easy and basically just follows from the fact that $1_{x \equiv i \pmod{p}} \equiv 1 - (x - i)^{p-1} \pmod{p}$.

In any event, the benefit of considering the moments problem is that it is more susceptible to complex analytic techniques. Borwein, Erdélyi, and Kós [1] use complex analytic techniques to show that for any distinct $A, B \subseteq [n]$, there is some $m \leq C\sqrt{n}$ with $\sum_{a \in A} a^m \neq \sum_{b \in B} b^m$. They gave two proofs. One was to find a polynomial p of degree at most $C\sqrt{n}$ such that $|p(0)| > |p(1)| + \dots + |p(n)|$; the second was to show that any nonzero polynomial of degree n with coefficients bounded by 1 in absolute value must be at least $\exp(-C\sqrt{n})$ at some point close to 1. We were able to adapt this second proof to find a small m such that $\sum_{a \in A} a^m \neq \sum_{b \in B} b^m$ when A, B are well-separated sets, and thus prove Theorem 2.

The adaptations we make are quite significant. See Lemma 2 and Lemma 3.

²See page 12 for a more specific formulation.

4. PROOF OF THEOREM 2

In this section, we prove the following theorem, of which Theorem 2 will be an immediate corollary.

Theorem 3. *Let A, B be distinct subsets of $[n]$ that are each $n^{1/3}$ -separated. Then there is some non-negative integer $m = O(n^{1/3} \log^5 n)$ such that $\sum_{a \in A} a^m \neq \sum_{b \in B} b^m$.*

The idea of the proof is as follows. The sets A, B having a different small moment is equivalent to the polynomial $p(x) := \sum_{n \in A} x^n - \sum_{n \in B} x^n$ not being divisible by a large power of $x - 1$, which is roughly equivalent to $p(x)$ not being uniformly too small near $x = 1$. The latter (rough) equivalence was exploited in [1], and we follow the general proof method of Theorem 5.1 of [1] to show $p(x)$ has some not-too-small value near $x = 1$. By factoring out a large power of x from $p(x)$ and multiplying by -1 if need be, we are led to the following definition³.

Let \mathcal{P}_n denote the collection of all polynomials⁴ $p(x) = 1 - x^d + \sum_{j=n^{1/3}}^n a_j z^j \in \mathbb{C}[x]$ such that $|a_j| \leq 1$ for each j and $1 \leq d < n^{1/3}$, and all polynomials $p(x) = 1 + \sum_{j=n^{1/3}}^n a_j z^j \in \mathbb{C}[x]$ such that $|a_j| \leq 1$ for each j . We prove the following.

Proposition 4.1. *There is some absolute constant $C_1 > 0$ so that for all $n \geq 1$ and all $p \in \mathcal{P}_n$, it holds that $\max_{x \in [1-n^{-2/3}, 1]} |p(x)| \geq \exp(-C_1 n^{1/3} \log^5 n)$.*

For $a > 0$, define \tilde{E}_a to be the ellipse with foci at $1 - a$ and $1 - a + \frac{1}{4}a$ and with major axis $[1 - a - \frac{a}{32}, 1 - a + \frac{9a}{32}]$. We borrow Corollary 5.3 from [1]:

Lemma 1. *For every $n \geq 1$, $p \in \mathcal{P}_n$, and $a > 0$, we have $(\max_{z \in \tilde{E}_a} |p(z)|)^2 \leq \frac{64}{39a} \max_{x \in [1-a, 1]} |p(x)|$.*

By Lemma 1, in order to prove Proposition 4.1 it suffices to show:

Proposition 4.2. *There is an absolute constant $C > 0$ so that for every $n \geq 1$ and every $p \in \mathcal{P}_n$, it holds that $(\max_{z \in \tilde{E}_{n^{-2/3}}} |p(z)|)^2 \geq \exp(-C n^{1/3} \log^5 n)$.*

While [1] certainly uses that \tilde{E}_a is an ellipse, all we will use is about \tilde{E}_a (besides using Lemma 1 as a black box) is that the interior of \tilde{E}_a , denoted \tilde{E}_a° , contains a ball of radius $\frac{a}{10^{10}}$ centered at $1 - a$. We begin with two lemmas.

In the proof of Theorem 5.1 of [1], the authors use the function $h(z) = (1 - a)\frac{z+z^2}{2}$ for a maximum modulus principle argument to lower bound the quantity $(\max_{z \in \tilde{E}_a} |p(z)|)^2$. For $z = e^{2\pi it}$ for small t , the magnitude $|h(e^{2\pi it})|$ is quadratically in t less than 1. For our purposes, we need a linear deviation of $|h(e^{2\pi it})|$ from 1. This motivates the following lemma.

³See the comment following Theorem 4 in Section 7.

⁴Throughout the paper, we omit floor functions when they don't meaningfully affect anything.

Lemma 2. *There are absolute constants $c_4, c_5, C_6 > 0$ such that the following holds for $a > 0$ small enough. Let $\tilde{h}(z) = \sum_{j=1}^r d_j z^j$ for*

$$d_j = \frac{\lambda_a}{j^2 \log^2(j+3)}$$

and $r = a^{-1}$, where $\lambda_a \in (1, 2)$ is such that $\sum_{j=1}^r d_j = 1$. Let $h(z) = (1-a)\tilde{h}(z)$. Then $h(0) = 0$, $|h(e^{2\pi it})| \leq 1-a$ for each t , $h(e^{2\pi it}) \in \tilde{E}_a^\circ$ for $t \in [-c_4 a, c_4 a]$, and

$$|h(e^{2\pi it})| \leq 1 - c_5 \frac{|t|}{\log^2(a^{-1})}$$

for $t \in [-\frac{1}{2}, \frac{1}{2}] \setminus [-C_6 a, C_6 a]$.

Proof. Clearly $h(0) = 0$ and $|h(e^{2\pi it})| \leq 1-a$ for each t . For small t ,

$$\begin{aligned} \tilde{h}(e^{2\pi it}) &= \sum_{j=1}^r d_j e^{2\pi i t j} = \sum_{j=1}^r d_j (1 + 2\pi i t j - 2\pi^2 t^2 j^2 + O(t^3 j^3)) \\ &= 1 + 2\pi i \left(\sum_{j=1}^r j d_j \right) t - 2\pi^2 \left(\sum_{j=1}^r j^2 d_j \right) t^2 + O\left(\left(\sum_{j=1}^r j^3 d_j \right) t^3 \right). \end{aligned}$$

Note that, asymptotically as $a \rightarrow 0$,

$$\begin{aligned} \sum_{j=1}^r j d_j &= \sum_{j=1}^r \frac{\lambda_a}{j \log^2(j+3)} =: \tilde{\lambda}_a = O(1) \\ \sum_{j=1}^r j^2 d_j &= \sum_{j=1}^r \frac{\lambda_a}{\log^2(j+3)} \sim \frac{\lambda_a r}{\log^2 r} \sim \frac{\lambda_a a^{-1}}{\log^2(a^{-1})} \\ \sum_{j=1}^r j^3 d_j &= \sum_{j=1}^r \frac{\lambda_a j}{\log^2(j+3)} \sim \frac{\lambda_a r^2}{2 \log^2 r} \sim \frac{\lambda_a a^{-2}}{2 \log^2(a^{-1})}. \end{aligned}$$

Therefore, if $|t| \leq c_4 a$, we have

$$\begin{aligned} \tilde{h}(e^{2\pi it}) &= 1 + i2\pi \tilde{\lambda}_a t - 2\pi^2 (1 + o(1)) \frac{\lambda_a a^{-1}}{\log^2(a^{-1})} t^2 + O\left(\frac{\lambda_a a^{-2}}{2 \log^2(a^{-1})} t^3 \right) \\ &= 1 + o(a) + i2\pi \tilde{\lambda}_a t, \end{aligned}$$

and thus

$$|h(e^{2\pi it}) - (1-a)| = |(1-a)(o(a) + i2\pi \tilde{\lambda}_a t)| \leq \frac{a}{10^{10}},$$

provided c_4 is small enough, thereby yielding $h(e^{2\pi it}) \in \tilde{E}_a^\circ$.

We now go on to showing the last inequality in the statement of Lemma 2.

By summation by parts, for any $z \in \mathbb{C}$, we have

$$(1) \quad \sum_{j=1}^r \frac{\lambda_a z^j}{j^2 \log^2(j+3)} = \frac{\lambda_a \sum_{j=1}^r z^j}{r^2 \log^2(r+3)} + 2\lambda_a \int_1^r \frac{(\sum_{j \leq x} z^j) (\log(x+3) + \frac{x}{x+3})}{x^3 \log^3(x+3)} dx.$$

Quickly note that, for $z = 1$, (1) gives

$$(2) \quad 1 = \frac{\lambda_a}{r \log^2(r+3)} + 2\lambda_a \int_1^r \frac{\lfloor x \rfloor \left(\log(x+3) + \frac{x}{x+3} \right)}{x^3 \log^3(x+3)} dx.$$

Trivially, for any $z \in \partial\mathbb{D}$, we have

$$(3) \quad \left| \frac{\lambda_a \sum_{j=1}^r z^j}{r^2 \log^2(r+3)} \right| \leq \frac{\lambda_a}{r \log^2(r+3)}.$$

Note that, for any $x \geq 1$,

$$(4) \quad \left| \sum_{j \leq x} z^j \right| = \left| z \frac{1 - z^{\lfloor x \rfloor}}{1 - z} \right| \leq \frac{2}{|1 - z|} \leq t^{-1}$$

for all $z = e^{2\pi it}$ with $t \in (0, \frac{1}{2}]$. Take $C_6 > 3$ to be chosen later. Note $t \in (C_6 a, \frac{1}{2}]$ implies $3t^{-1} < r$. For $z = e^{2\pi it}$ with $C_6 a < t \leq \frac{1}{2}$, (4) and (2) imply

$$(5) \quad \begin{aligned} & \left| 2\lambda_a \int_1^r \frac{(\sum_{j \leq x} z^j) \left(\log(x+3) + \frac{x}{x+3} \right)}{x^3 \log^3(x+3)} dx \right| \leq \\ & 2\lambda_a \int_1^{3t^{-1}} \frac{\lfloor x \rfloor \left(\log(x+3) + \frac{x}{x+3} \right)}{x^3 \log^3(x+3)} dx + 2\lambda_a \int_{3t^{-1}}^r \frac{t^{-1} \left(\log(x+3) + \frac{x}{x+3} \right)}{x^3 \log^3(x+3)} dx \\ & = 1 - 2\lambda_a \int_{3t^{-1}}^r \frac{(\lfloor x \rfloor - t^{-1}) \cdot \left(\log(x+3) + \frac{x}{x+3} \right)}{x^3 \log^3(x+3)} dx - \frac{\lambda_a}{r \log^2(r+3)}. \end{aligned}$$

Observe $\lfloor x \rfloor - t^{-1} \geq \frac{1}{2}x$ for $x \geq 3t^{-1}$. Therefore,

$$(6) \quad \begin{aligned} 2\lambda_a \int_{3t^{-1}}^r \frac{(\lfloor x \rfloor - t^{-1}) \cdot \left(\log(x+3) + \frac{x}{x+3} \right)}{x^3 \log^3(x+3)} dx & \geq \lambda_a \int_{3t^{-1}}^r \frac{1}{x^2 \log^2(x+3)} dx \\ & \geq \frac{\lambda_a}{\log^2(r+3)} \int_{3t^{-1}}^r \frac{1}{x^2} dx \\ & = \frac{\lambda_a t}{3 \log^2(r+3)} - \frac{\lambda_a}{r \log^2(r+3)}. \end{aligned}$$

Combining (1), (3), (5), and (6), we conclude that, for any $t \in (C_6 a, \frac{1}{2}]$,

$$(7) \quad \left| \tilde{h}(e^{2\pi it}) \right| = \left| \sum_{j=1}^r \frac{\lambda_a e^{2\pi i j t}}{j^2 \log^2(j+3)} \right| \leq 1 - \frac{\lambda_a t}{3 \log^2(r+3)} + \frac{\lambda_a}{r \log^2(r+3)}.$$

Taking C_6 to be much larger than 3, (7) gives the bound

$$|\tilde{h}(e^{2\pi it})| \leq 1 - c_5 \frac{t}{\log^2(a^{-1})}$$

for $t \in (C_6 a, \frac{1}{2}]$, for suitable $c_5 > 0$. By symmetry, the proof is complete. \square

We from now on fix some $n \geq 1$ and some $p \in \mathcal{P}_n$ (defined at the beginning of the section). Let \tilde{p} be the truncation of p to terms of degree less than $n^{1/3}$; either $\tilde{p} = 1$ or $\tilde{p} = 1 - x^d$ for some $1 \leq d < n^{1/3}$. Take $a = n^{-2/3}$, and let h be as in Lemma 2. Let $m = \frac{1}{c_4 a}$. Let $J_1 = c_5^{-1} n^{-1/3} m \log^4 n$ and $J_2 = m - J_1$.

In the proof below of Proposition 4.2, we will need to upper bound the product $\prod_{j=J_1}^{J_2-1} |\tilde{p}(h(e^{2\pi i \frac{j}{m}}))|$ by $\exp(\tilde{O}(n^{1/3}))$. We must be careful in doing so, as the trivial upper bound on each term is 2 and there are approximately $n^{2/3}$ terms. However, we expect the argument of $h(e^{2\pi i \frac{j}{m}})$ to behave as if it were random, and thus we expect $|\tilde{p}(h(e^{2\pi i \frac{j}{m}}))|$ to sometimes be smaller than 1. The fact that the cancellation (between terms smaller than 1 and terms greater than 1) is nearly perfect comes from the fact that $\log |\tilde{p}(h(e^{2\pi i \frac{j}{m}}))|$ is harmonic, which we make crucial use of below.

Lemma 3. *For any $t \in [0, 1]$, we have $|\tilde{p}(h(e^{2\pi i t}))| \geq \frac{1}{2} n^{-2/3}$. For any $\delta \in [0, 1)$, we have $\prod_{j=J_1}^{J_2-1} |\tilde{p}(h(e^{2\pi i \frac{j+\delta}{m}}))| \leq \exp(C n^{1/3} \log^5 n)$ for some absolute $C > 0$.*

Proof. Clearly both inequalities hold if $\tilde{p} = 1$, so suppose $\tilde{p}(x) = 1 - x^d$ for some $1 \leq d < n^{1/3}$. For the first inequality, we use

$$|\tilde{p}(h(e^{2\pi i t}))| = |1 - h(e^{2\pi i t})^d| \geq 1 - |h(e^{2\pi i t})|^d \geq 1 - (1 - a)^d \geq \frac{1}{2} a d \geq \frac{1}{2} n^{-2/3}.$$

We now move on to the second inequality. Define $g(t) = 2 \log |\tilde{p}(h(e^{2\pi i(t+\frac{\delta}{m}})))|$. For notational ease, we assume $\delta = 0$; the argument about to come works for all $\delta \in [0, 1)$. The first inequality implies g is C^1 , so by the mean value theorem,

$$\begin{aligned} \left| \frac{1}{m} \sum_{j=J_1}^{J_2-1} g\left(\frac{j}{m}\right) - \int_{J_1/m}^{J_2/m} g(t) dt \right| &= \left| \sum_{j=J_1}^{J_2-1} \int_{j/m}^{(j+1)/m} \left(g(t) - g\left(\frac{j}{m}\right) \right) dt \right| \\ &\leq \sum_{j=J_1}^{J_2-1} \int_{j/m}^{(j+1)/m} \left(\max_{\frac{j}{m} \leq y \leq \frac{j+1}{m}} |g'(y)| \right) \frac{1}{m} dt \\ (8) \quad &\leq \frac{1}{m^2} \sum_{j=J_1}^{J_2-1} \max_{\frac{j}{m} \leq y \leq \frac{j+1}{m}} |g'(y)|. \end{aligned}$$

Since $w \mapsto \log |\tilde{p}(h(w))|$ is harmonic and $\log |\tilde{p}(h(0))| = \log |\tilde{p}(0)| = 0$, we have

$$\int_0^1 g(t) dt = 2 \int_0^1 \log |\tilde{p}(h(e^{2\pi i t}))| dt = 0,$$

and therefore

$$(9) \quad \left| \int_{J_1/m}^{J_2/m} g(t) dt \right| \leq \left| \int_0^{J_1/m} g(t) dt \right| + \left| \int_{J_2/m}^1 g(t) dt \right|.$$

Since

$$\frac{1}{2} n^{-2/3} \leq |\tilde{p}(h(e^{2\pi i t}))| \leq 1$$

for each t , we have

$$(10) \quad \left| \int_0^{J_1/m} g(t) dt \right| + \left| \int_{J_2/m}^1 g(t) dt \right| \leq 2 \left(\frac{J_1}{m} + \left(1 - \frac{J_2}{m}\right) \right) \log n \leq C \frac{\log^5 n}{n^{1/3}}.$$

By (8), (9), and (10), we have

$$\left| \frac{1}{m} \sum_{j=J_1}^{J_2-1} g\left(\frac{j}{m}\right) \right| \leq C \frac{\log^5 n}{n^{1/3}} + \frac{1}{m^2} \sum_{j=J_1}^{J_2-1} \max_{\frac{j}{m} \leq t \leq \frac{j+1}{m}} |g'(t)|.$$

Multiplying through by m , changing C slightly, and exponentiating, we obtain

$$(11) \quad \prod_{j=J_1}^{J_2-1} \left| \tilde{p}\left(h\left(e^{2\pi i \frac{j}{m}}\right)\right) \right|^2 \leq \exp \left(C n^{1/3} \log^5 n + \frac{1}{m} \sum_{j=J_1}^{J_2-1} \max_{\frac{j}{m} \leq t \leq \frac{j+1}{m}} |g'(t)| \right).$$

Note

$$g'(t_0) = \frac{\frac{\partial}{\partial t} \left[|\tilde{p}(h(e^{2\pi i t}))|^2 \right] \Big|_{t=t_0}}{|\tilde{p}(h(e^{2\pi i t_0}))|^2}.$$

We first show

$$\frac{\partial}{\partial t} \left[|\tilde{p}(h(e^{2\pi i t}))|^2 \right] \Big|_{t=t_0} \leq 100d$$

for each $t_0 \in [0, 1]$. We start by noting

$$\left| \tilde{p}(h(e^{2\pi i t})) \right|^2 = 1 + (1-a)^{2d} \left(\left| \sum_{j=1}^r d_j e^{2\pi i t j} \right|^2 \right)^d - 2 \operatorname{Re} \left[\left((1-a) \sum_{j=1}^r d_j e^{2\pi i t j} \right)^d \right].$$

Let

$$f_1(t) = (1-a)^{2d} \left(\left| \sum_{j=1}^r d_j e^{2\pi i t j} \right|^2 \right)^d.$$

Then,

$$\begin{aligned} f_1'(t) &= (1-a)^{2d} d \left(\left| \sum_{j=1}^r d_j e^{2\pi i t j} \right|^2 \right)^{d-1} \frac{\partial}{\partial t} \left[\left| \sum_{j=1}^r d_j e^{2\pi i t j} \right|^2 \right] \\ &= (1-a)^{2d} d \left(\left| \sum_{j=1}^r d_j e^{2\pi i t j} \right|^2 \right)^{d-1} \sum_{1 \leq j_1, j_2 \leq r} d_{j_1} d_{j_2} 2\pi i (j_1 - j_2) e^{2\pi i (j_1 - j_2) t}. \end{aligned}$$

Since $\sum_{j=1}^r d_j = 1$, we therefore have

$$\begin{aligned} |f'_1(t)| &\leq 2\pi d \sum_{1 \leq j_1, j_2 \leq r} \lambda_a^2 \frac{j_1 + j_2}{j_1^2 j_2^2 \log^2(j_1 + 3) \log^2(j_2 + 3)} \\ &= 4\pi d \left(\sum_{j_1=1}^r \frac{\lambda_a}{j_1 \log^2(j_1 + 3)} \right) \left(\sum_{j_2=1}^r \frac{\lambda_a}{j_2 \log^2(j_2 + 3)} \right) \\ &\leq 50d. \end{aligned}$$

Now, let

$$f_2(t) = -2 \operatorname{Re} \left[\left((1-a) \sum_{j=1}^r d_j e^{2\pi i t j} \right)^d \right]$$

and note

$$\begin{aligned} f'_2(t) &= \frac{\partial}{\partial t} \left[-2(1-a)^d \sum_{1 \leq j_1, \dots, j_d \leq r} d_{j_1} \dots d_{j_d} \cos(2\pi t(j_1 + \dots + j_d)) \right] \\ &= 4\pi(1-a)^d \sum_{1 \leq j_1, \dots, j_d \leq r} d_{j_1} \dots d_{j_d} (j_1 + \dots + j_d) \sin(2\pi t(j_1 + \dots + j_d)), \end{aligned}$$

yielding

$$\begin{aligned} |f'_2(t)| &\leq 4\pi \sum_{1 \leq j_1, \dots, j_d \leq r} \lambda_a^d \frac{j_1 + \dots + j_d}{j_1^2 \dots j_d^2 \log^2(j_1 + 3) \dots \log^2(j_d + 3)} \\ &= 4\pi d \left(\sum_{j_1=1}^r \frac{\lambda_a}{j_1 \log^2(j_1 + 3)} \right) \left(\sum_{j=1}^r \frac{\lambda_a}{j^2 \log^2(j + 3)} \right)^{d-1} \\ &\leq 50d. \end{aligned}$$

We have thus shown

$$\frac{\partial}{\partial t} \left[|\tilde{p}(h(e^{2\pi i t}))|^2 \right] \Big|_{t=t_0} \leq 100d$$

for each $t_0 \in [0, 1]$.

Recall

$$|\tilde{p}(h(e^{2\pi i t}))| = |1 - h(e^{2\pi i t})^d| \geq 1 - |h(e^{2\pi i t})|^d.$$

For $j \in [J_1, J_2] \subseteq [C_6 a m, (1 - C_6 a)m]$, we use

$$|h(e^{2\pi i \frac{j}{m}})| \leq 1 - c_5 \frac{\min(\frac{j}{m}, 1 - \frac{j}{m})}{\log^2 n}$$

to obtain

$$\frac{1}{m} \sum_{j=J_1}^{J_2-1} \max_{\frac{j}{m} \leq t \leq \frac{j+1}{m}} |g'(t)| \leq \frac{1}{m} \sum_{j=J_1}^{J_2-1} \frac{100d}{\left(1 - \left(1 - c_5 \frac{\min(\frac{j}{m}, 1 - \frac{j}{m})}{\log^2 n} \right)^d \right)^2}.$$

Up to a factor of 2, we may deal only with $j \in [J_1, \frac{m}{2}]$. Let $J_* = c_5^{-1} d^{-1} m \log^2 n$. Note that $j \leq J_*$ implies $c_5 \frac{j}{m \log^2 n} \leq d^{-1}$ and $j \geq J_*$ implies $c_5 \frac{j}{m \log^2 n} \geq d^{-1}$. Thus, using $(1-x)^d \leq 1 - \frac{1}{2}xd$ for $x \leq \frac{1}{d}$, we have

$$\begin{aligned}
\frac{1}{m} \sum_{j=J_1}^{\min(J_*, \frac{m}{2})} \frac{100d}{\left(1 - \left(1 - c_5 \frac{j}{m \log^2 n}\right)^d\right)^2} &\leq \frac{100d}{m} \sum_{j=J_1}^{\min(J_*, \frac{m}{2})} \frac{1}{\left(\frac{1}{2} c_5 \frac{j}{m \log^2 n} d\right)^2} \\
&= \frac{400m \log^4 n}{c_5^2 d} \sum_{j=J_1}^{\min(J_*, \frac{m}{2})} \frac{1}{j^2} \\
&\leq \frac{400m \log^4 n}{c_5^2 d} \frac{2}{J_1} \\
&\leq Cn^{1/3}.
\end{aligned}
\tag{12}$$

Finally, since there is some $c > 0$ such that $(1-x)^l \leq 1-c$ for all $l \in \mathbb{N}$ and $x \in [l^{-1}, 1]$, using the notation $\sum_{i=a}^b x_i = 0$ if $a > b$, we see

$$\begin{aligned}
\frac{1}{m} \sum_{j=\min(J_*, \frac{m}{2})+1}^{m/2} \frac{100d}{\left(1 - \left(1 - c_5 \frac{j}{m \log^2 n}\right)^d\right)^2} &\leq \frac{100d}{m} \sum_{j=\min(J_*, \frac{m}{2})+1}^{m/2} c^{-2} \\
&\leq Cd \\
&\leq Cn^{1/3}.
\end{aligned}
\tag{13}$$

Combining (12) and (13), we obtain

$$\frac{1}{m} \sum_{j=J_1}^{J_2-1} \max_{\frac{j}{m} \leq \frac{j+1}{m}} |g'(t)| \leq Cn^{1/3}.$$

Plugging this upper bound into (11) yields the desired result. \square

Proof of Proposition 4.2. Define $g(z) = \prod_{j=0}^{m-1} p(h(e^{2\pi i \frac{j}{m}} z))$. Fix $z \in \partial\mathbb{D}$; say $z = e^{2\pi i(\frac{j_0}{m} + \delta)}$ for some $j_0 \in \{0, \dots, m-1\}$ and $\delta \in [0, \frac{1}{m})$. For ease of notation, we assume $j_0 = 0$; the argument about to come is easily adapted to any j_0 . Then, $e^{2\pi i \frac{j}{m}} z$ is in $\{e^{2\pi i t} : -c_4 a \leq t < c_4 a\}$ if $j \in \{0, m-1\}$. Therefore, since p is analytic, the maximum modulus principle implies

$$\begin{aligned}
|g(z)| &\leq \left(\max_{w \in \tilde{E}_a^\circ} |p(w)| \right)^2 \prod_{j \notin \{0, m-1\}} |p(h(e^{2\pi i \frac{j}{m}} z))| \\
&\leq \left(\max_{w \in \tilde{E}_a} |p(w)| \right)^2 \prod_{j \notin \{0, m-1\}} |p(h(e^{2\pi i \frac{j}{m}} z))|.
\end{aligned}
\tag{14}$$

Let $I = [J_1, J_2 - 1] \cap \mathbb{Z}$. For $j \notin I$, using the bound $|p(w)| \leq \frac{1}{1-|w|}$ for each $w \in \partial\mathbb{D}$, we see

$$|p(h(e^{2\pi i \frac{j}{m}} z))| \leq \frac{1}{1 - |h(e^{2\pi i \frac{j}{m}} z)|} \leq \frac{1}{1 - (1-a)} = n^{2/3},$$

thereby obtaining

$$(15) \quad \prod_{j \notin I \cup \{0, m-1\}} |p(h(e^{2\pi i \frac{j}{m}} z))| \leq (n^{2/3})^{(J_1-1)+(m-J_2+1)} \leq (n^{2/3})^{Cn^{1/3} \log^4 n} \leq e^{Cn^{1/3} \log^5 n}.$$

Now, for $j \in I$, since

$$|h(e^{2\pi i \frac{j}{m}} z)| \leq 1 - c_5 \frac{\min(\frac{j}{m} + \delta, 1 - (\frac{j}{m} + \delta))}{\log^2 n} \leq 1 - c' n^{-1/3} \log^2 n,$$

we have

$$\left| p\left(h(e^{2\pi i \frac{j}{m}} z)\right) - \tilde{p}\left(h(e^{2\pi i \frac{j}{m}} z)\right) \right| \leq n e^{-c' \log^2 n} \leq e^{-c \log^2 n}.$$

Therefore,

$$(16) \quad \prod_{j \in I} |p(h(e^{2\pi i \frac{j}{m}} z))| \leq \prod_{j \in I} \left(|\tilde{p}(h(e^{2\pi i \frac{j}{m}} z))| + e^{-c \log^2 n} \right).$$

By both parts of Lemma 3, we obtain

$$\begin{aligned} \prod_{j \in I} \left(|\tilde{p}(h(e^{2\pi i \frac{j}{m}} z))| + e^{-c \log^2 n} \right) &= \sum_{I' \subseteq I} \left(\prod_{j \in I \setminus I'} |\tilde{p}(h(e^{2\pi i \frac{j}{m}} z))| \right) e^{-c(\log^2 n)|I'|} \\ &= \sum_{I' \subseteq I} \left(\prod_{j \in I} |\tilde{p}(h(e^{2\pi i \frac{j}{m}} z))| \right) \left(\prod_{j \in I'} |\tilde{p}(h(e^{2\pi i \frac{j}{m}} z))| \right)^{-1} e^{-c(\log^2 n)|I'|} \\ &\leq e^{Cn^{1/3} \log^5 n} \sum_{I' \subseteq I} (2n^{2/3})^{|I'|} e^{-c(\log^2 n)|I'|} \\ &\leq e^{Cn^{1/3} \log^5 n} \sum_{I' \subseteq I} e^{-c'(\log^2 n)|I'|} \\ &\leq e^{Cn^{1/3} \log^5 n} \sum_{k=0}^{|I|} \binom{|I|}{k} e^{-c'k \log^2 n} \\ (17) \quad &\leq 2e^{Cn^{1/3} \log^5 n}. \end{aligned}$$

Combining (14), (15), (16), and (17), we've shown

$$|g(z)| \leq \left(\max_{z \in \tilde{E}_a} |p(z)| \right)^2 e^{Cn^{1/3} \log^5 n}.$$

As this holds for all $z \in \partial\mathbb{D}$, we have

$$\max_{z \in \partial\mathbb{D}} |g(z)| \leq \left(\max_{z \in \tilde{E}_a} |p(z)| \right)^2 e^{Cn^{1/3} \log^5 n}.$$

To finish, note that $|g(0)| = |p(h(0))|^m = |p(0)|^m = 1$, so, as g is clearly analytic, the maximum modulus principle implies $\max_{z \in \partial \mathbb{D}} |g(z)| \geq 1$. \square

We now go on to finish the proof of Theorem 3. We will use part of Lemma 5.4 of [1], stated below.

Lemma 4. *Suppose $f(x) = \sum_{j=0}^n a_j x^j$ has $a_j \in \mathbb{C}$, $|a_j| \leq 1$ for each j . If $(x-1)^k$ divides $f(x)$, then $\max_{1-\frac{k}{9n} \leq x \leq 1} |f(x)| \leq (n+1)(\frac{e}{9})^k$.*

Proposition 4.3. *There exists an absolute constant $C > 0$ so that for all $n \geq 1$ and all $p(x) \in \mathcal{P}_n$, the polynomial $(x-1)^{\lfloor Cn^{1/3} \log^5 n \rfloor}$ does not divide $p(x)$.*

Proof. Take $C > 0$ large. Take $p(x) \in \mathcal{P}_n$. Suppose for the sake of contradiction that $(x-1)^{Cn^{1/3} \log^5 n}$ divided $p(x)$. Then, by Lemma 4 and Proposition 4.1,

$$\begin{aligned} (n+1)\left(\frac{e}{9}\right)^{Cn^{1/3} \log^5 n} &\geq \max_{x \in [1-\frac{C}{9}n^{-2/3} \log^5 n, 1]} |p(x)| \\ &\geq \max_{x \in [1-n^{-2/3} \log^5 n, 1]} |p(x)| \\ &\geq e^{-C_1 n^{1/3} \log^5 n}, \end{aligned}$$

which is a contradiction if C is large enough. \square

Proof of Theorem 3. Let $f(x) = \sum_{j=0}^n \epsilon_j x^j$, where $\epsilon_j := 1_A(j) - 1_B(j)$. Let $\tilde{f}(x) = \frac{f(x)}{x^r}$, where r is maximal with respect to $\epsilon_0, \dots, \epsilon_{r-1} = 0$. We may assume without loss of generality that $\tilde{f}(0) = 1$. Then the fact that A, B are $n^{1/3}$ -separated implies $\tilde{f}(x) \in \mathcal{P}_n$. By Proposition 4.3, $(x-1)^{Cn^{1/3} \log^5 n}$ does not divide $\tilde{f}(x)$ and thus does not divide $f(x)$. This means that there is some $k \leq Cn^{1/3} \log^5 n - 1$, $k \geq 0$, so that $f^{(k)}(1) \neq 0$. Take a minimal such k . If $k = 0$, we're of course done. Otherwise, since $f^{(m)}(1) = \sum_{j=0}^n j(j-1) \dots (j-m+1) \epsilon_j$ for $m \geq 1$, it's easy to inductively see that $\sum_{j \in A} j^m = \sum_{j \in B} j^m$ for all $0 \leq m \leq k-1$ and then $\sum_{j \in A} j^k \neq \sum_{j \in B} j^k$. \square

Theorem 2. *Let A, B be distinct subsets of $[n]$ that are each $n^{1/3}$ -separated. Then there is some prime $p \in [\frac{1}{2}C'n^{1/3} \log^6 n, C'n^{1/3} \log^6 n]$ and some $i \in [p]_0$ so that $|A_{i,p}| \neq |B_{i,p}|$. Here, $C' > 0$ is an absolute constant.*

Proof. By Theorem 3, take $m = O(n^{1/3} \log^5 n)$ such that $\sum_{a \in A} a^m \neq \sum_{b \in B} b^m$. Since $|\sum_{a \in A} a^m - \sum_{b \in B} b^m| \leq n n^m \leq \exp(O(n^{1/3} \log^6 n))$, there is some prime $p \in [\frac{1}{2}C'n^{1/3} \log^6 n, C'n^{1/3} \log^6 n]$ such that $\sum_{a \in A} a^m \not\equiv \sum_{b \in B} b^m \pmod{p}$. Noting that $\sum_{a \in A} a^m \equiv \sum_{i=0}^{p-1} |A_{i,p}| i^m \pmod{p}$ and $\sum_{b \in B} b^m \equiv \sum_{i=0}^{p-1} |B_{i,p}| i^m \pmod{p}$, we see that there is some $i \in [p]_0$ for which $|A_{i,p}| \not\equiv |B_{i,p}| \pmod{p}$. \square

5. SEPARATING WORDS WITH $O(n^{1/3} \log^7 n)$ STATES

Recall that, for a string $x = x_1, \dots, x_n \in \{0, 1\}^n$ and a (sub)string $w = w_1, \dots, w_l \in \{0, 1\}^l$, we defined $\text{pos}_w(x) = \{j \in \{1, \dots, n-l+1\} : x_{j+k-1} = w_k \text{ for all } 1 \leq k \leq l\}$.

Lemma 5. *Let m, n be positive integers, $i \in [m]$ a residue mod m , q a prime number, $a \in [q]$ a residue mod q , and $w \in \{0, 1\}^l$ a string of length $l \leq m$. Then there is a deterministic finite automaton with $2mq$ states that accepts a string $x \in \{0, 1\}^n$ if and only if $|\{j \in \text{pos}_w(x) : j \equiv i \pmod{m}\}| \equiv a \pmod{q}$.*

Proof. Write $w = w_1, \dots, w_l$. We assume $l > 1$; a minor modification to the following yields the result for $l = 1$. We interpret indices of $w \pmod{m}$, which we may, since $l \leq m$. Let the states of the DFA be $\mathbb{Z}_m \times \{0, 1\} \times \mathbb{Z}_q$. The initial state is $(1, 0, 0)$. If $j \not\equiv i \pmod{m}$ and $\epsilon \in \{0, 1\}$, set $\delta((j, 0, s), \epsilon) = (j + 1, 0, s)$. If $j \equiv i \pmod{m}$, set $\delta((j, 0, s), w_1) = (j + 1, 1, s)$ and $\delta((j, 0, s), 1 - w_1) = (j + 1, 0, s)$. If $j \not\equiv i + l - 1 \pmod{m}$, set $\delta((j, 1, s), w_{j-i+1}) = (j + 1, 1, s)$ and $\delta((j, 1, s), 1 - w_{j-i+1}) = (j + 1, 0, s)$. Finally, if $j \equiv i + l - 1 \pmod{m}$, set $\delta((j, 1, s), w_l) = (j + 1, 0, s + 1)$ and $\delta((j, 1, s), 1 - w_l) = (j + 1, 0, s)$. The accept states are $\mathbb{Z}_m \times \{0, 1\} \times \{a\}$. \square

We are now ready to prove Theorem 1, restated below.

Theorem 1. *For any distinct $x, y \in \{0, 1\}^n$, there is a deterministic finite automaton with $O(n^{1/3} \log^7 n)$ states that accepts x but not y .*

Proof. Let x_1, \dots, x_n and y_1, \dots, y_n be two distinct strings in $\{0, 1\}^n$. If $x_i \neq y_i$ for some $i < 2n^{1/3}$, then we are of course done, so we may suppose otherwise. Let $i \geq 2n^{1/3}$ be the first index with $x_i \neq y_i$. Let $w' = x_{i-2n^{1/3}+1}, \dots, x_{i-1}$ be a (sub)string of length $2n^{1/3} - 1$. By Lemma 1 and Lemma 2 of [4], there is some choice $w \in \{w'0, w'1\}$ for which $A := \text{pos}_w(x)$ is $n^{1/3}$ -separated and $B := \text{pos}_w(y)$ is $n^{1/3}$ -separated. Clearly $A \neq B$, so Theorem 2 implies there is some prime $p \in [\frac{1}{2}C'n^{1/3} \log^6 n, C'n^{1/3} \log^6 n]$ and some $i \in [p]_0$ for which $|A_{i,p}| \neq |B_{i,p}|$. Since $|A_{i,p}|$ and $|B_{i,p}|$ are at most n , there is some prime $q = O(\log n)$ for which $|A_{i,p}| \not\equiv |B_{i,p}| \pmod{q}$. Since $|w| = 2n^{1/3} \leq p$, by Lemma 5 there is a deterministic finite automaton with $2pq = O(n^{1/3} \log^7 n)$ states that accepts x but not y . \square

6. TIGHTNESS OF OUR METHODS

In this section, we prove the following, showing that our methods cannot be pushed further. We use a standard pigeonhole argument, that has been used in a variety of other papers.

Proposition 6.1. *For all n large, there are distinct $n^{1/3}$ -separated subsets A, B of $[n]$ such that $|A_{i,p}| = |B_{i,p}|$ for all $p \leq cn^{1/3} \log^{1/2} n$ and all $i \in [p]_0$.*

Proof. Let Σ denote the collection of subsets $A \subseteq [n]$ that have at most one number from each of the intervals $[1, n^{1/3}], [2n^{1/3}, 3n^{1/3}], [4n^{1/3}, 5n^{1/3}], \dots$. Note $|\Sigma| \geq (n^{1/3})^{\frac{1}{3}n^{2/3}} = e^{\frac{1}{9}n^{2/3} \log n}$. On the other hand, for any $A \subseteq [n]$, the number of

possible tuples $(|A_{i,p}|)_{\substack{p \leq k \\ i \in [p]_0}}$ is at most $\prod_{p \leq k} n^p \leq e^{\frac{k^2}{\log k} \log n}$. Taking $k = cn^{1/3} \log^{1/2} n$ yields $\frac{k^2}{\log k} \log n < \frac{1}{9} n^{2/3} \log n$, meaning there are distinct $A, B \in \Sigma$ with the same tuple, i.e. $|A_{i,p}| = |B_{i,p}|$ for all $p \leq k$ and $i \in [p]_0$. As A, B are $n^{1/3}$ -separated, the proof is complete. \square

7. FINAL REMARKS AND OPEN PROBLEMS

The proof of Theorem 2 proves the following.

Theorem 4. *Fix $\alpha \in (0, 1)$. Let A, B be distinct subsets of $[n]$ that are each n^α -separated. Then there is some prime $p = O_\alpha(n^{\frac{1-\alpha}{2}} \log^6 n)$ and some $i \in [p]_0$ so that $|A_{i,p}| \neq |B_{i,p}|$.*

The only property of n^α -separated we used is that $|a_2 - a_1| \geq n^\alpha$ and $|b_2 - b_1| \geq n^\alpha$, where a_1 and a_2 are the two smallest elements of A that are not in B , and b_1 and b_2 are the two smallest elements of B that are not in A .

The conclusion of Theorem 4 should hold if we weaken the hypothesis of A and B being n^α -separated to A and B having size at most $n^{1-\alpha}$. Taking $\alpha = \frac{1}{2}$ for concreteness and replacing n by n^2 for aesthetics, we ask the following.

Question. Let A, B be distinct subsets of $[n^2]$, each of size at most n . Must there be some prime $p = \tilde{O}(\sqrt{n})$ and some $i \in [p]_0$ so that $|A_{i,p}| \neq |B_{i,p}|$?

One may also ask the same question as above except replacing $[n^2]$ with $[n^3]$. By considering A, B that contain only elements that are multiples of all small primes, it is clear that we cannot replace $[n^2]$ or $[n^3]$ by, say, $[e^{Cn}]$.

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