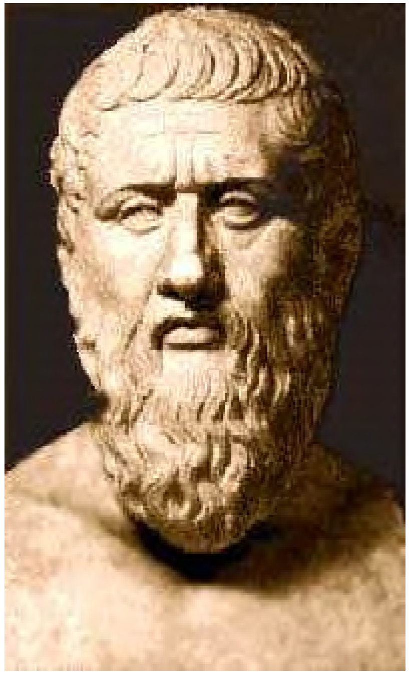
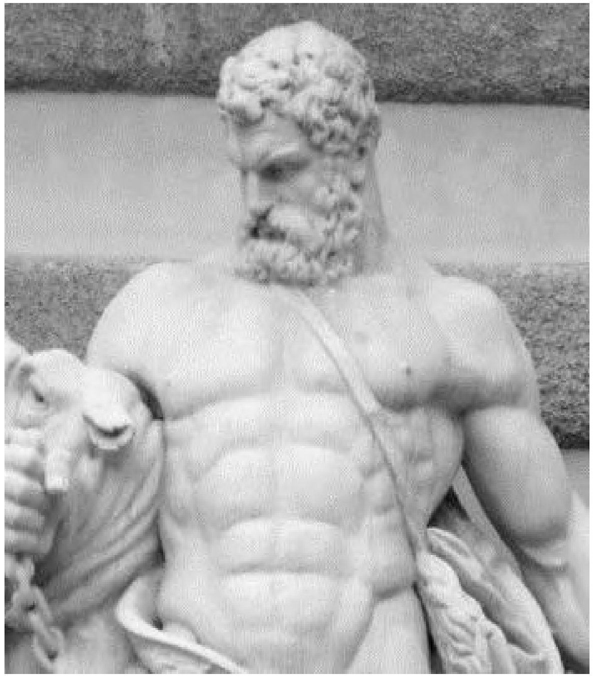
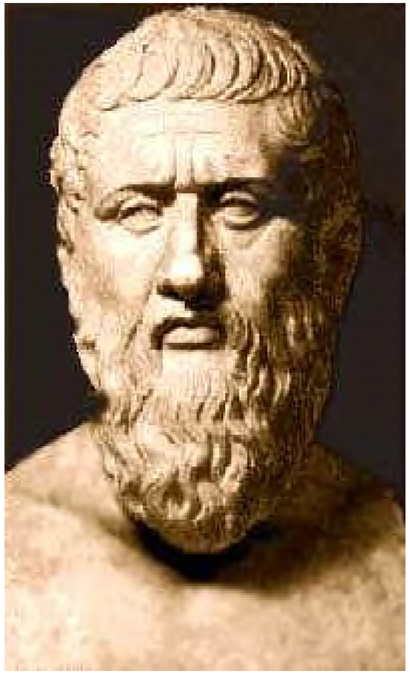
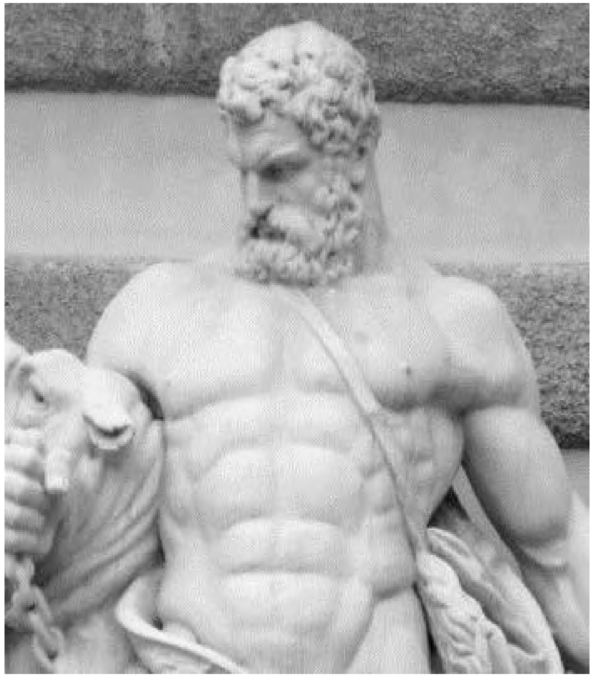
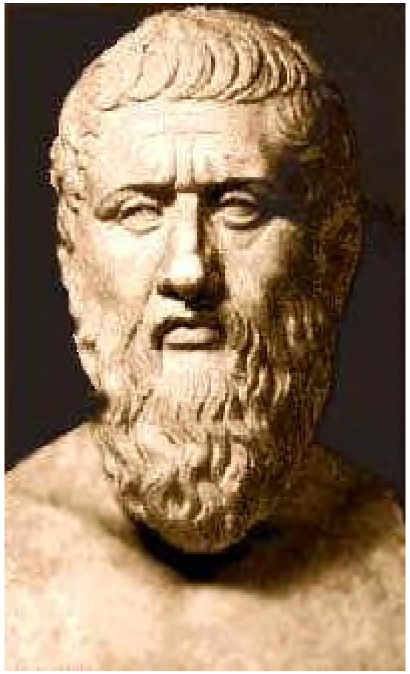
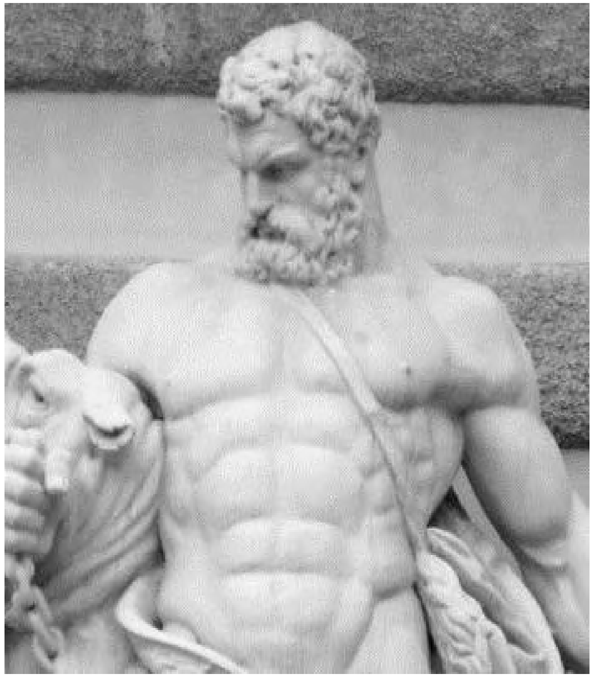
  

Downward rational termination is Ackermannian


Downward rational termination is


Downward rational termination is

Ranko Lazić
Warwick

Joël Ouaknine
Oxford

James Worrell
Oxford

Metric temporal logic [Koymans '90]

$$\varphi ::= \top \mid \bot \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid a \mid \bigcirc_I \varphi \mid \varphi_1 \, \mathcal{U}_I \, \varphi_2 \mid \varphi_1 \, \tilde{\mathcal{U}}_I \, \varphi_2$$

over finite timed words

Metric temporal logic [Koymans '90]

$$\varphi ::= \top \mid \bot \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid a \mid \bigcirc_I \varphi \mid \varphi_1 \, \mathcal{U}_I \, \varphi_2 \mid \varphi_1 \, \tilde{\mathcal{U}}_I \, \varphi_2$$

over finite timed words:

$$\begin{array}{ccccccc} a & b & b & a & c & b & c \\ 0.3 & 0.7 & 1.3 & 3.1 & 3.2 & 4.1 & 4.8 \end{array} \models \Box_{[0,\infty)} \left(a \to \Diamond_{[1,1]} b \right)$$

$$\begin{array}{ccccccc} a & b & b & a & c & b & c \\ 0.3 & 0.7 & 1.3 & 3.1 & 3.2 & 4.3 & 4.8 \end{array} \not\models \Box_{[0,\infty)} \left(a \to \Diamond_{[1,1]} b \right)$$

$$\begin{array}{ccccccc} a & b & b & a & c & b & c \\ 0.3 & 0.7 & 1.3 & 3.1 & 3.2 & 4.3 & 4.8 \end{array} \models \Box_{[0,3.1)} \left(a \to \Diamond_{[1,1]} b \right)$$

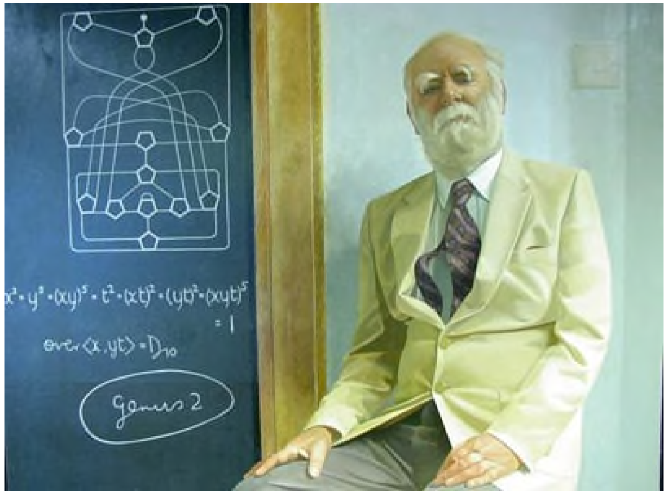
MTL
finite words

[Ouaknine & Worrell
FoSSaCS '06]

insertion channel systems
reachability

decidable [Abdulla & Jonsson
LICS '93]

not PR [Schnoebelen
IPL '02]



's

Lemma:

$\sqsubseteq$ on Σ^* is a wqo.

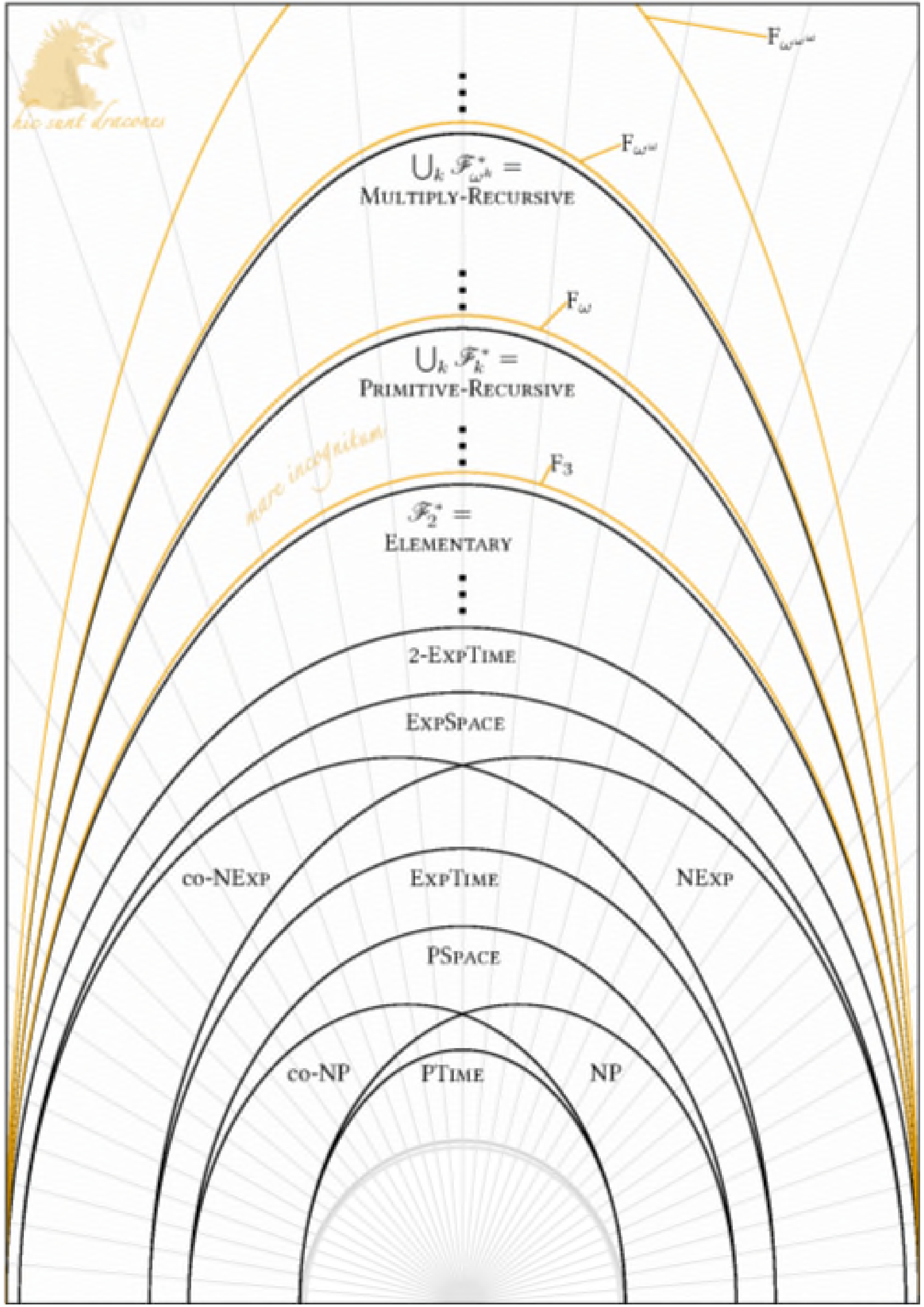
e.g. Higman $\sqsubseteq$ Highmountain

Fast growing hierarchy

$$F_0(x) = x + 1$$

$$F_{n+1}(x) = \underbrace{F_n(F_n(\dots F_n}_{x+1}(x)\dots))$$

$$F_\omega(x) = F_x(x)$$



hic sunt dracones
$F_{\omega^{\omega^\omega}}$
F_{ω^ω}
$\bigcup_k \mathscr{F}^*_{\omega^k} =$
Multiply-Recursive
F_ω
$\bigcup_k \mathscr{F}^*_k =$
Primitive-Recursive
mare incognitum
F_3
$\mathscr{F}^*_2 =$
Elementary
2-ExpTime
ExpSpace
co-NExp
ExpTime
NExp
PSpace
co-NP
PTime
NP

MTL
finite words $\leftrightarrows$ insertion channel systems reachability

[Ouaknine & Worrell FoSSaCS '06]

decidable [Abdulla & Jonsson LICS '93]

$\notin \bigcup_k \mathcal{F}_k^*$ [Schnoebelen IPL '02]

MTL
finite words $\rightleftarrows$ insertion channel systems reachability $\begin{cases} \in \mathcal{F}^*_{\omega^\omega} & \text{[Schmitz \& Schnoebelen ICALP '11]} \\ \notin \bigcup_k \mathcal{F}^*_{\omega^k} & \text{[Chambart \& Schnoebelen LICS '08]} \end{cases}$

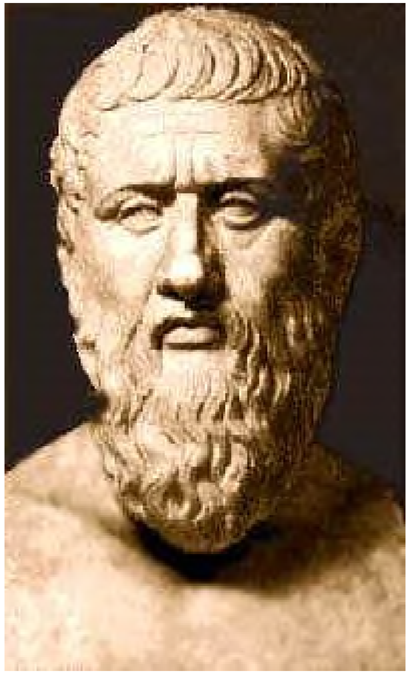
[Ouaknine & Worrell FoSSaCS '06]

Safety MTL

$$\varphi ::= \top \mid \bot \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid a \mid \bigcirc_I \varphi \mid \varphi_1 \,\mathcal{U}_F\, \varphi_2 \mid \varphi_1 \,\tilde{\mathcal{U}}_I\, \varphi_2$$

over infinite timed words

Not



!

Safety MTL

$$\varphi ::= \top \mid \bot \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid a \mid \bigcirc_I \varphi \mid \varphi_1 \,\mathcal{U}_F\, \varphi_2 \mid \varphi_1 \,\tilde{\mathcal{U}}_I\, \varphi_2$$

over infinite timed words:

$$\Diamond_{(0,\infty)} \text{thesis}$$

$$\Diamond_{(0,4]} \text{thesis}$$

MTL
finite words

[Ouaknine & Worrell
FoSSaCS '06]

insertion channel systems
reachability

$\in \mathcal{F}^*_{\omega^\omega}$ [Schmitz & Schnoebelen ICALP '11]

$\notin \bigcup_k \mathcal{F}^*_{\omega^k}$ [Chambart & Schnoebelen LICS '08]

safety MTL
infinite words

[Ouaknine & Worrell
TACAS '06]

insertion channel systems
fair termination

decidable [Ouaknine & Worrell TACAS '06]

not EL [Bouyer et al. FAC '12]

MTL

finite words

[Ouaknine & Worrell FoSSaCS '06]

insertion channel systems

reachability

$\in \mathcal{F}^{*}_{\omega^\omega}$ [Schmitz & Schnoebelen ICALP '11]

$\notin \bigcup_k \mathcal{F}^{*}_{\omega^k}$ [Chambart & Schnoebelen LICS '08]

safety MTL

infinite words

[Ouaknine & Worrell TACAS '06]

insertion channel systems

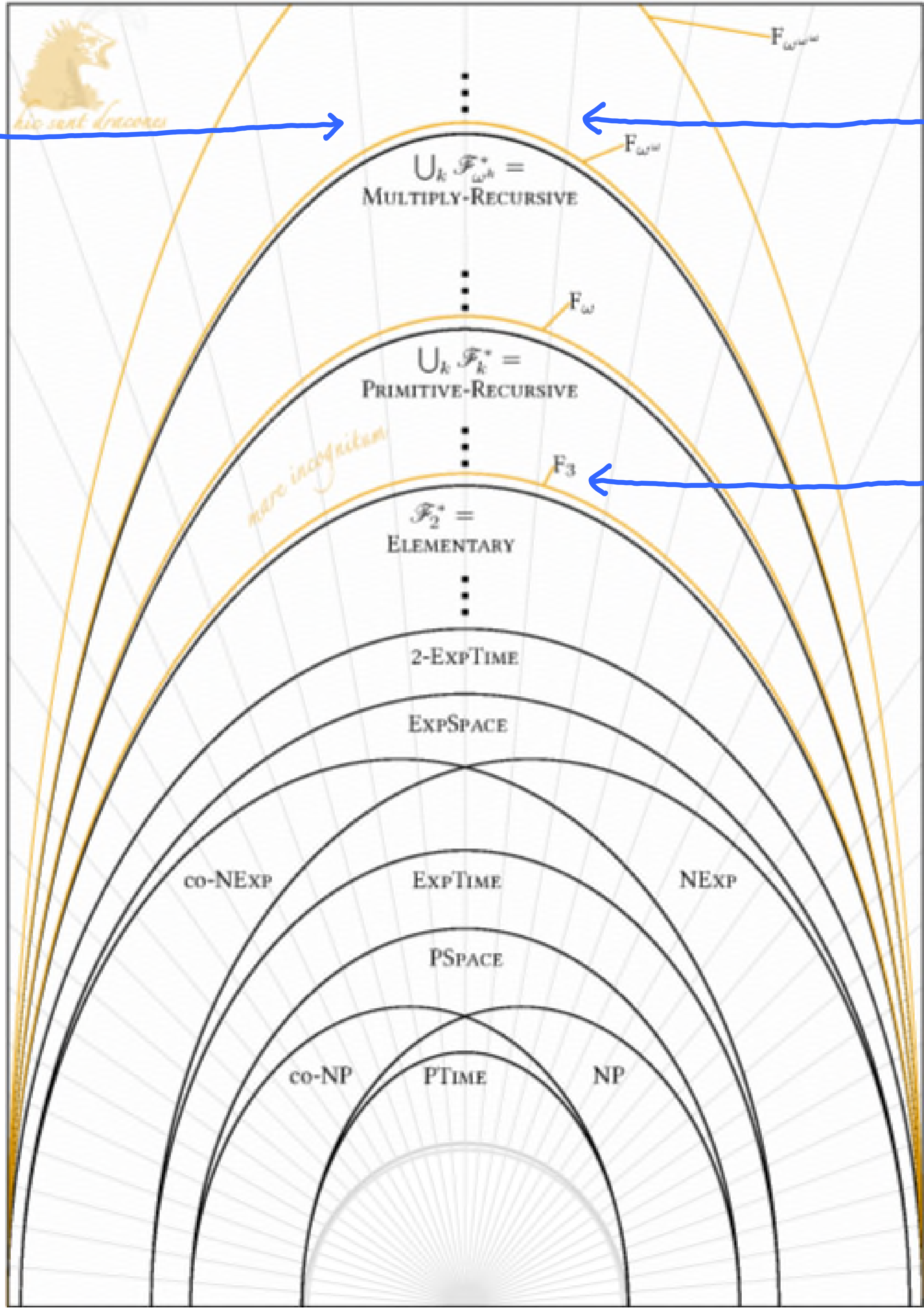
fair termination

$\in \mathcal{F}^{*}_{\omega^\omega}$ [Schmitz '12]

$\notin \mathcal{F}^{*}_{3}$ [Jenkins '12]

MTL

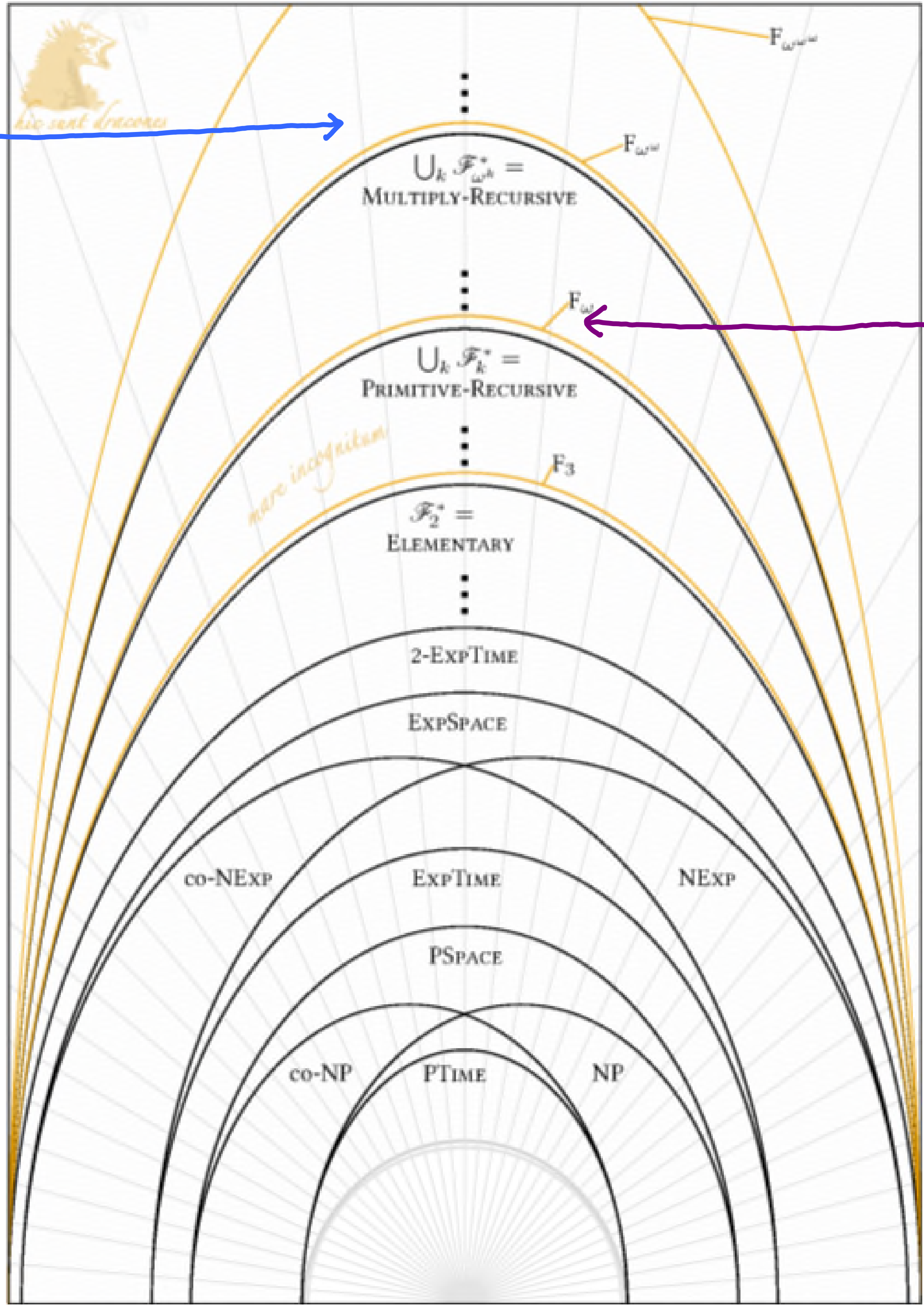
finite words



safety MTL

infinite words

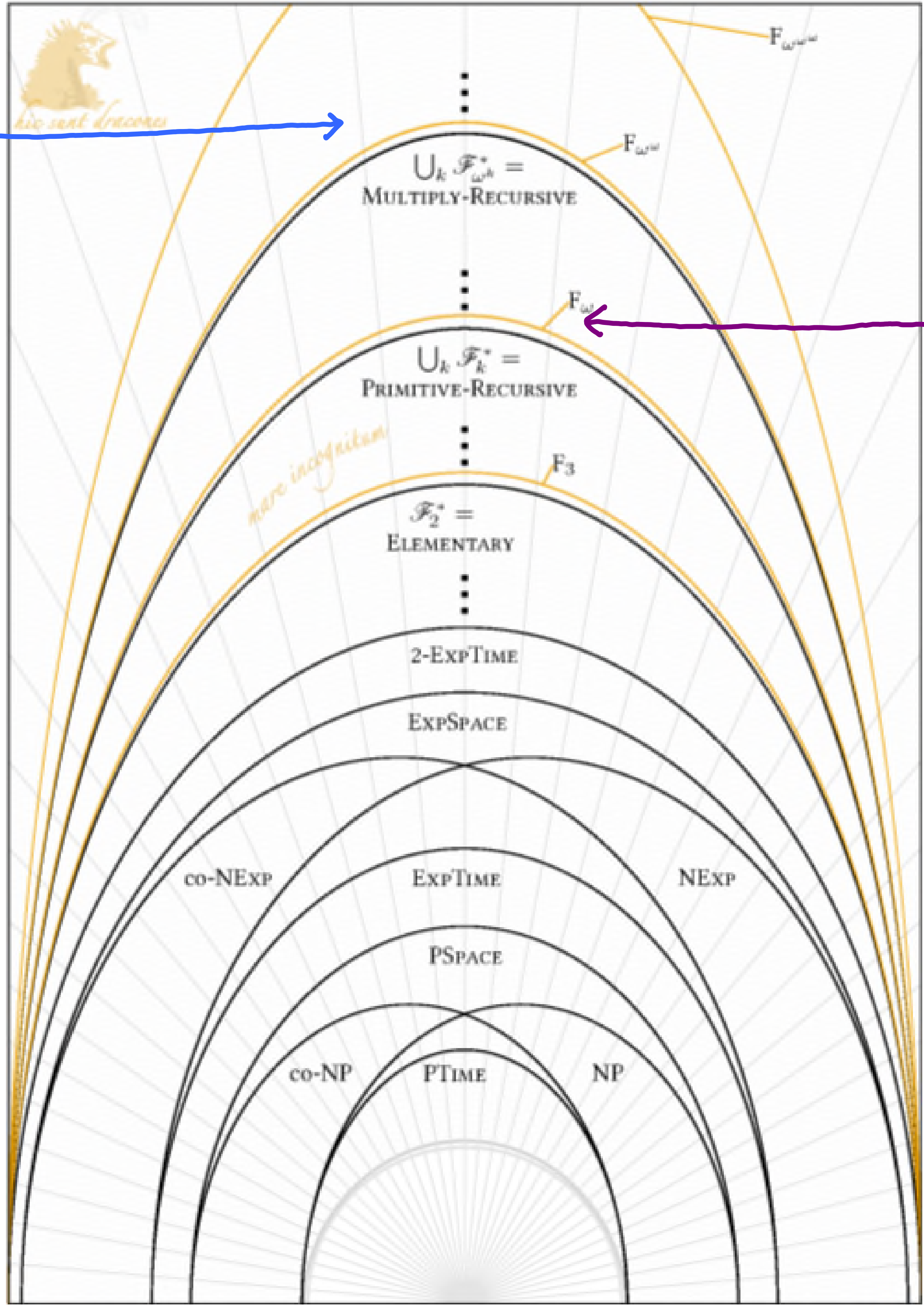
MTL
finite words



safety MTL
infinite words

MTL
finite words

'hyper-Ackermannian'



safety MTL
infinite words

'Ackermannian'


www.theoi.com

« Representations of uncomputable and uncountable sets

Lambda calculus for real analysis by Paul Taylor »

The hydra game

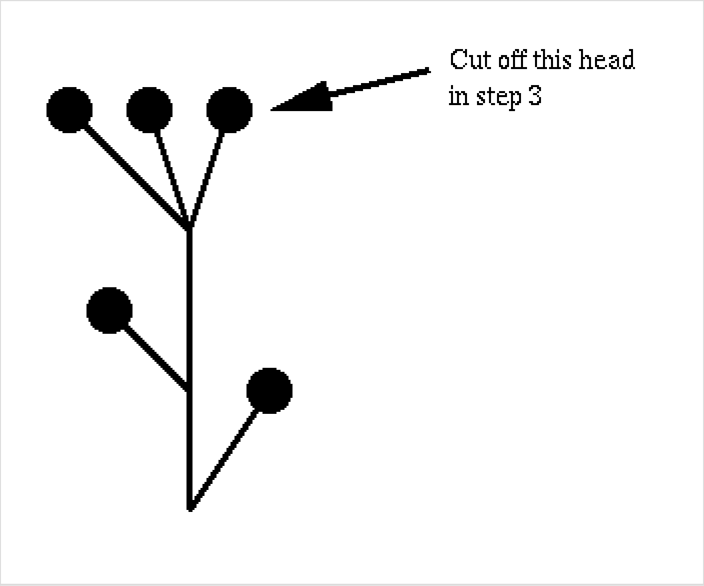
By Andrej Bauer, on February 2nd, 2008

Today I lectured about the Hydra game by Laurence Kirby and Jeff Paris (*Accessible Independence Results for Peano Arithmetic, Kirby and Paris, Bull. London Math. Soc. 1982; 14: 285-293*). For the occasion I implemented the game in Java. I am publishing the code for anyone who wants to play, or use it for teaching.

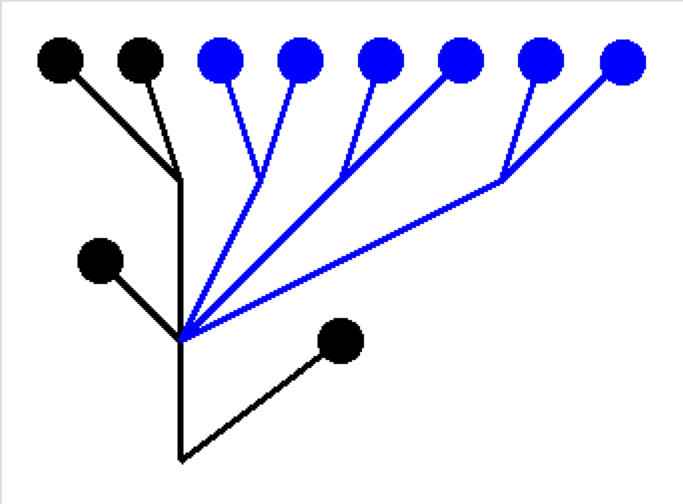
About the game

A hydra is a finite tree, with a root at the bottom. The object of the game is to cut down the hydra to its root. At each step, you can cut off one of the heads, after which the hydra grows new heads according to the following rules:

- If you cut off a head growing out of the root, the hydra does not grow any new heads.
- Suppose you cut off a head like this:



Delete the head and its neck. Descend down by 1 from the node at which the neck was attached. Look at the subtree growing from the connection through which you just descended. Pick a natural number, say 3, and grow that many copies of that subtree, like this:



My program grows I copies at step I of the game, which is one possible variant of the game. There are spoilers ahead, so before you read on you should play play the game with the Hydra applet (your browser must support Java) and try to win. Is it possible to win? How should you play to win?

Here is a surprising fact:

> **Theorem 1:** You cannot lose!

The proof uses ordinal numbers. To each hydra we assign an ordinal number:

- A head gets the number 0.
- Suppose a node S has sub-hydras $)_1, \dots,)_I$ growing from it. To each sub-hydra we assign its ordinal recursively and order the ordinals in descending order: b_1 ł b_2 ł … ł b_I. The ordinal assigned to the node S is $z^{b_1} + z^{b_2} +$ پس $+ z^{b_I}$. For example, the ordinal corresponding to the hydra from the first picture above is $z^{z^3+1} + 1$. The hydra in the second picture gets the ordinal $z^{z^2 \cdot 4+1} + 1$.
- By chopping off a head we strictly decrease the ordinal. Because there are no infinite strictly descending sequences of ordinals, the hydra will eventually die, no matter how you chop off heads.

But Theorem 1 is not the punchline. The punchline is this:

> **Theorem 2 (Kirby and Paris):** Any proof technique that proves Theorem 1 is strong enough to prove that Peano arithmetic is consistent.

's

Lemma:

$\sqsubseteq$ on Σ^* is a wqo.

e.g. Higman $\sqsubseteq$

Highmountain

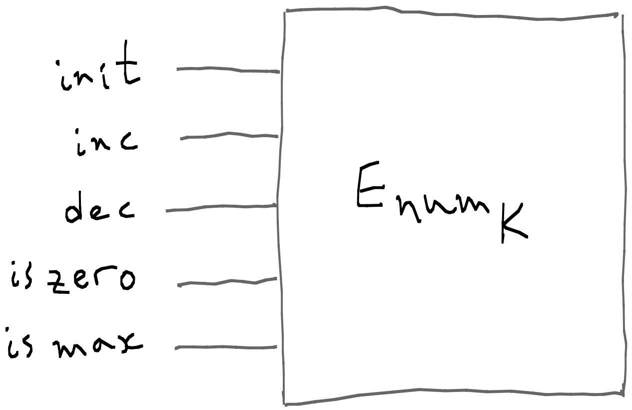
's

Lemma:

$\leq$ on $\mathbb{N}^k$ is a wqo.

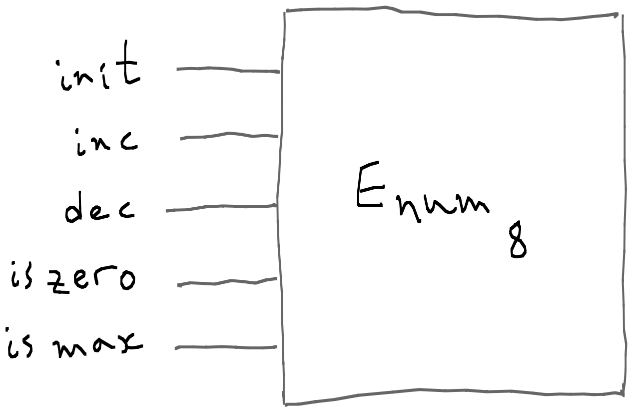
[Figueira et al. LICS'11]

Implementing counters



$$\{a_0, a_1, \ldots, a_{K-1}\}$$

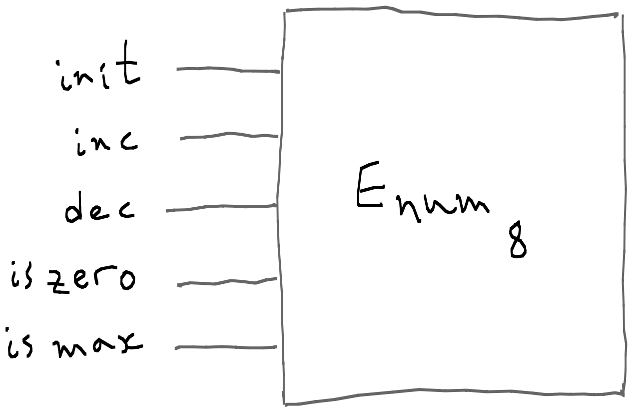
Implementing counters



$$\{a_0, a_1, \dots, a_7\}$$

a_0

Implementing counters

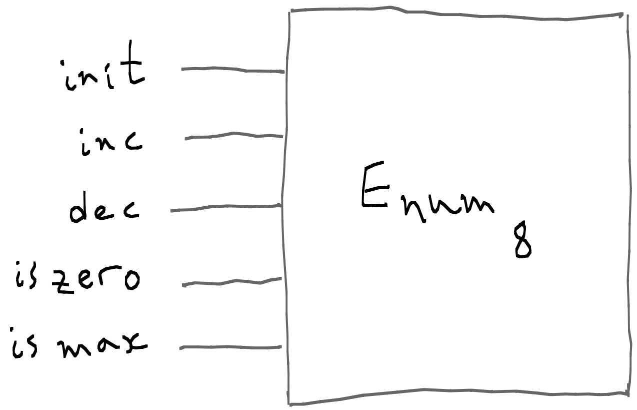


$$\{a_0, a_1, \ldots, a_7\}$$

a_0

a_1

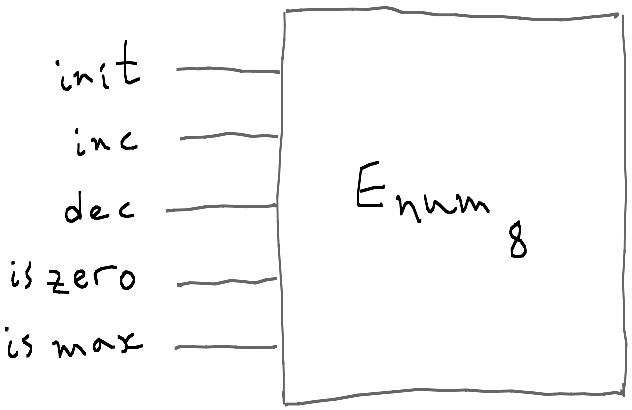
Implementing counters



$$\{a_0, a_1, \ldots, a_7\}$$

$$\begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_7 \end{array}$$

Implementing counters



$$\{a_0, a_1, \dots, a_7\}$$

a_0

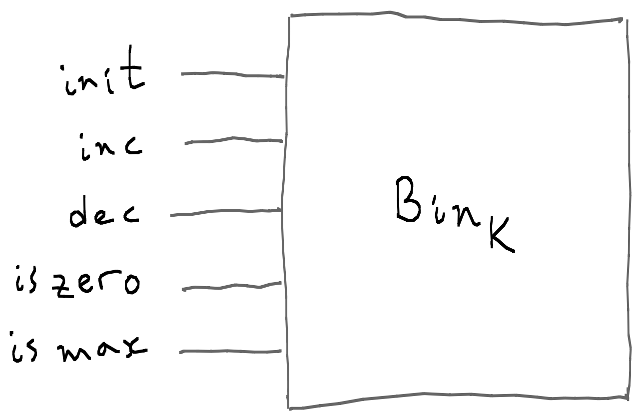
a_1

$\vdots$

a_7

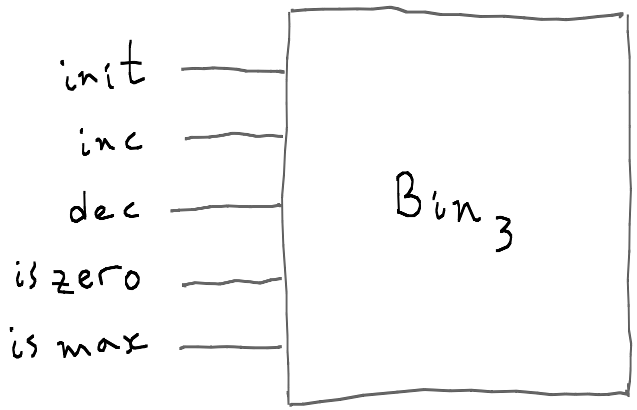
$a_3 a_7$

Implementing counters



$\{0, 1\}$

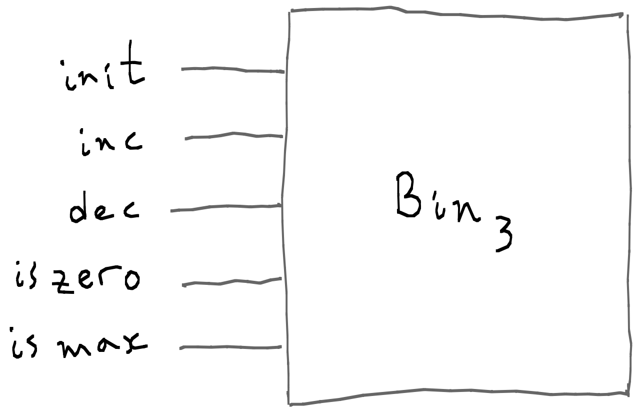
Implementing counters



$\{0, 1\}$

0 0 0

Implementing counters

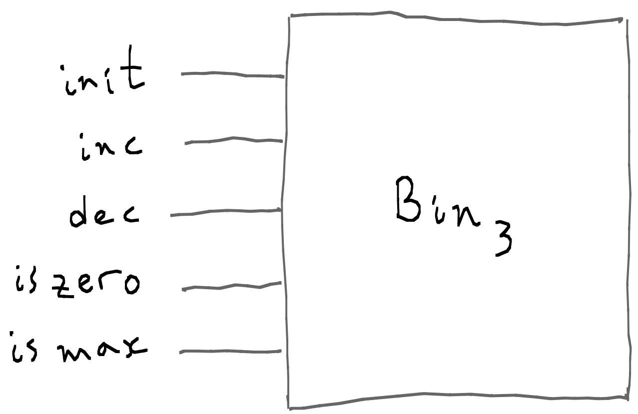


$\{0, 1\}$

0 0 0

0 0 1

Implementing counters



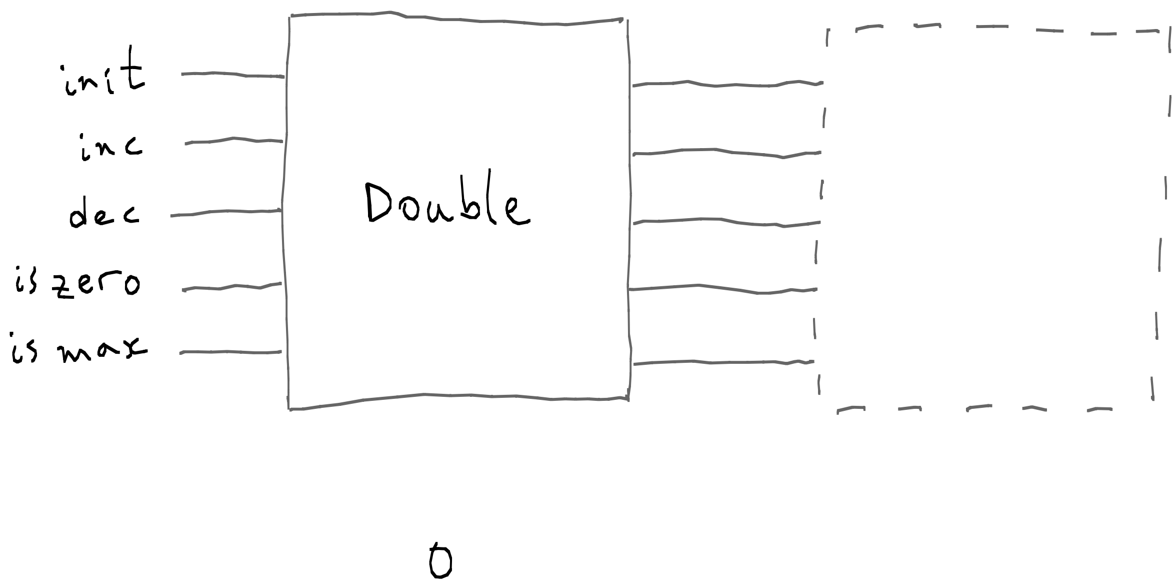
$\{0, 1\}$

0 0 0

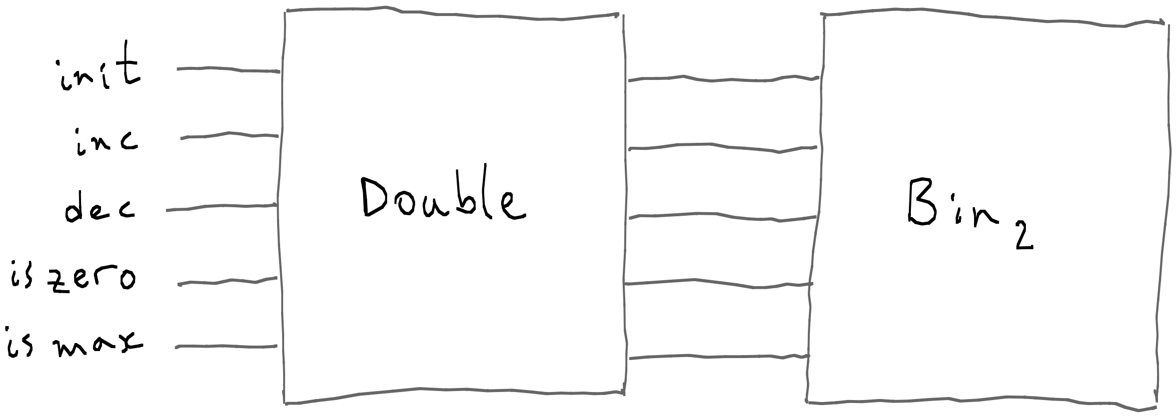
0 0 1

0 1 0 1

Implementing counters

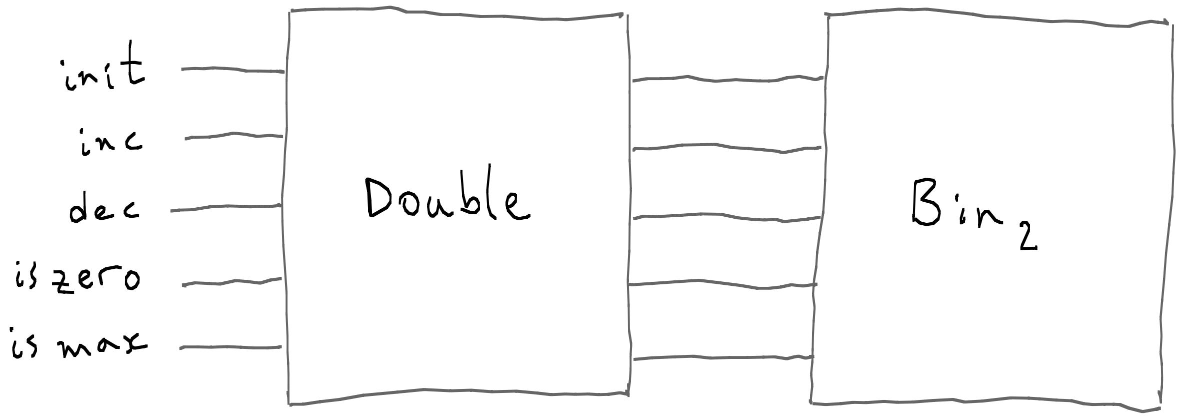


Implementing counters



0 1 0 0

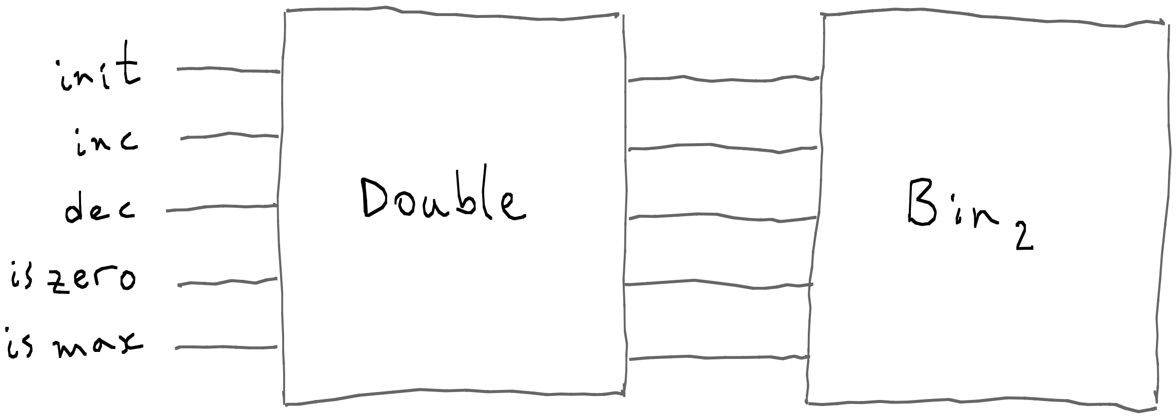
Implementing counters



0 | 0 0

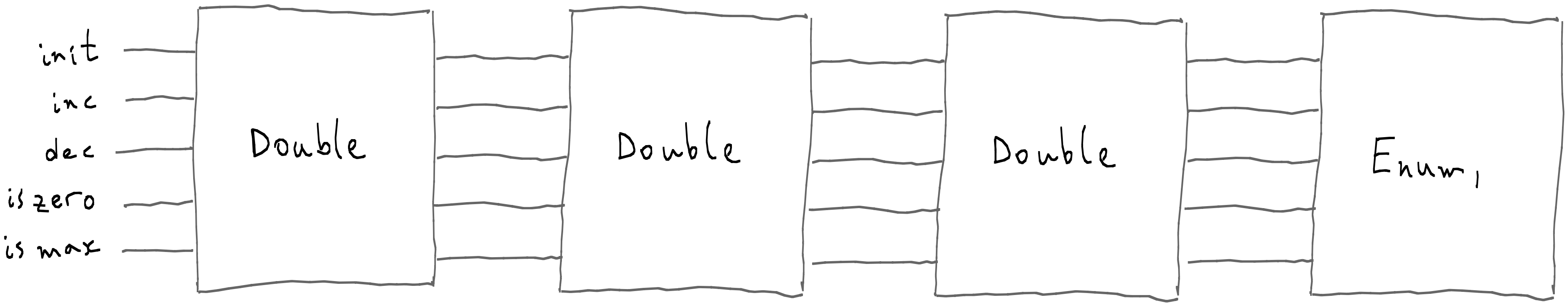
0 | 0 1

Implementing counters



0	0	0
0	0	1
	⋮	
0	1	1

Implementing counters



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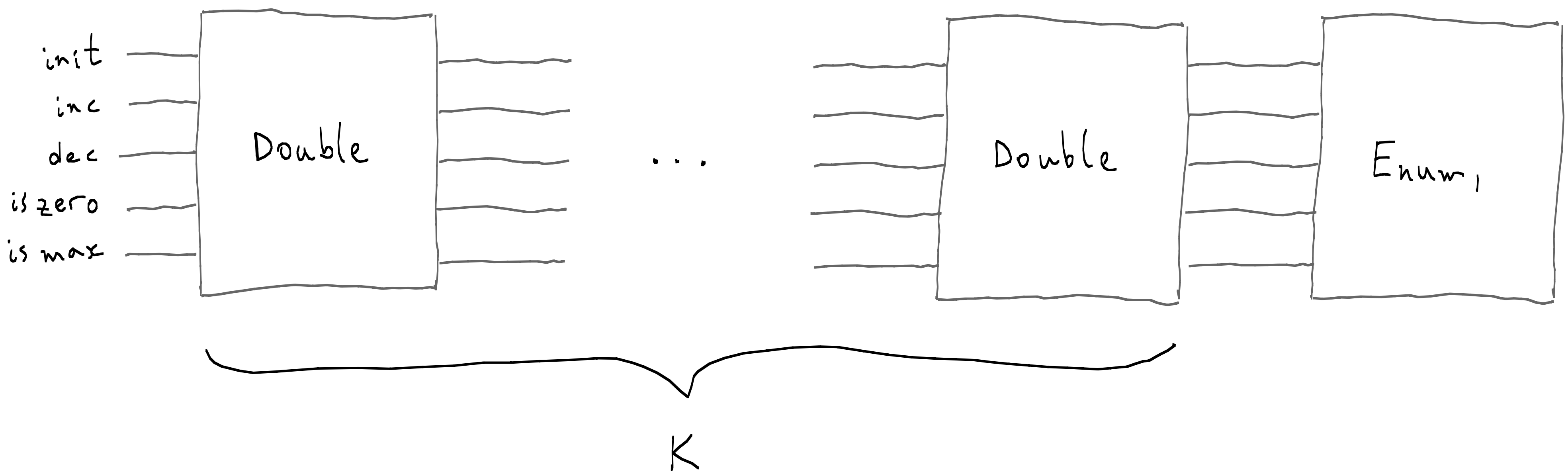

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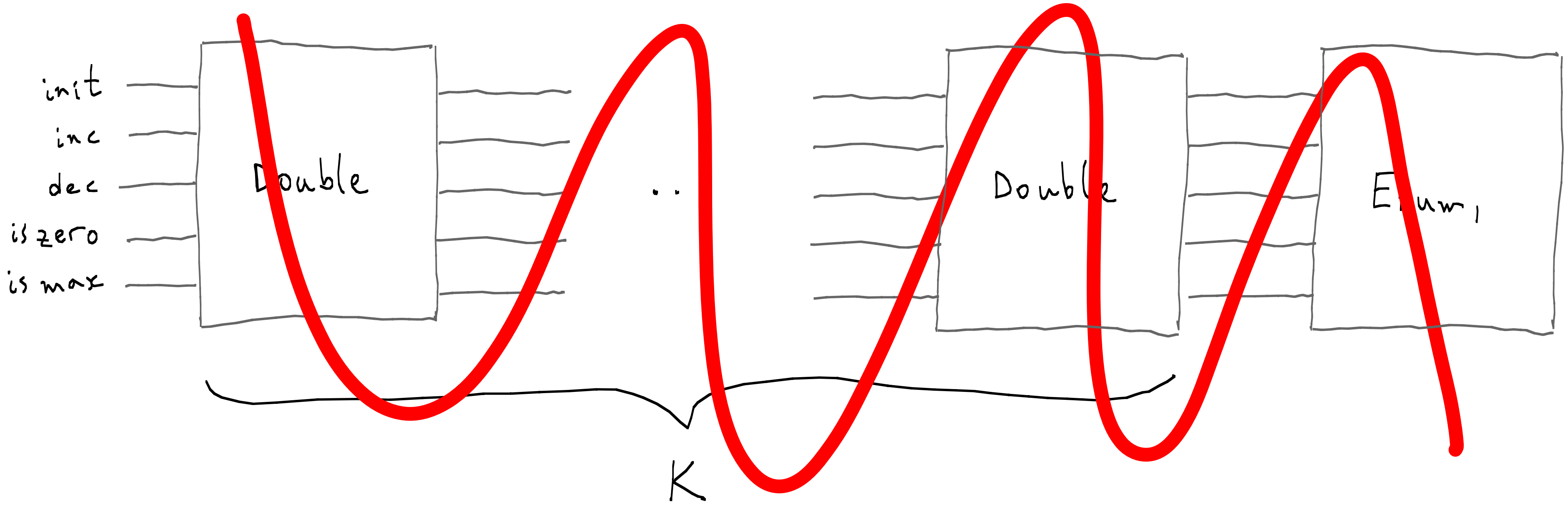
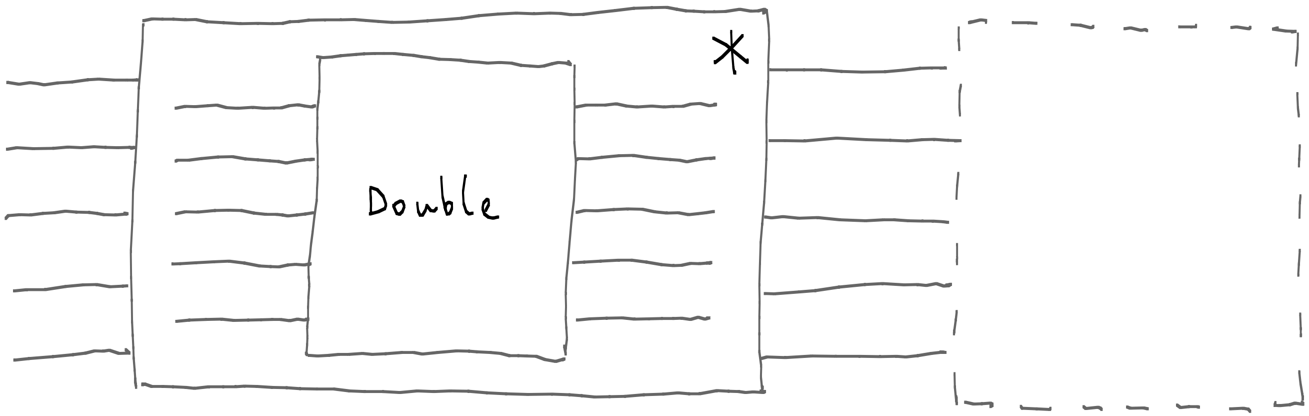
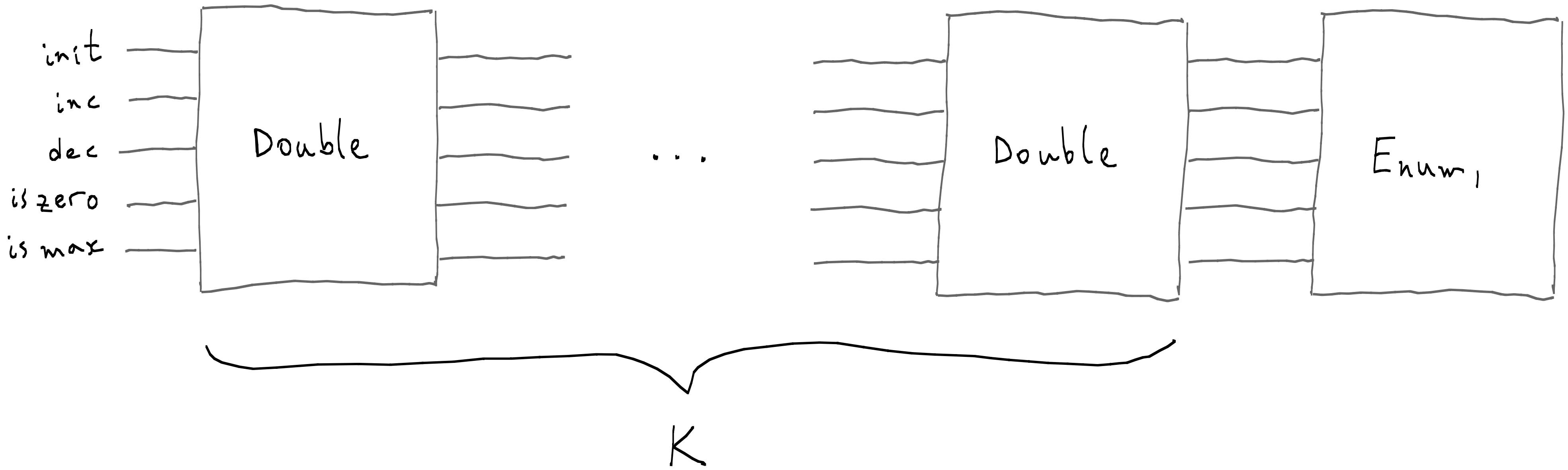
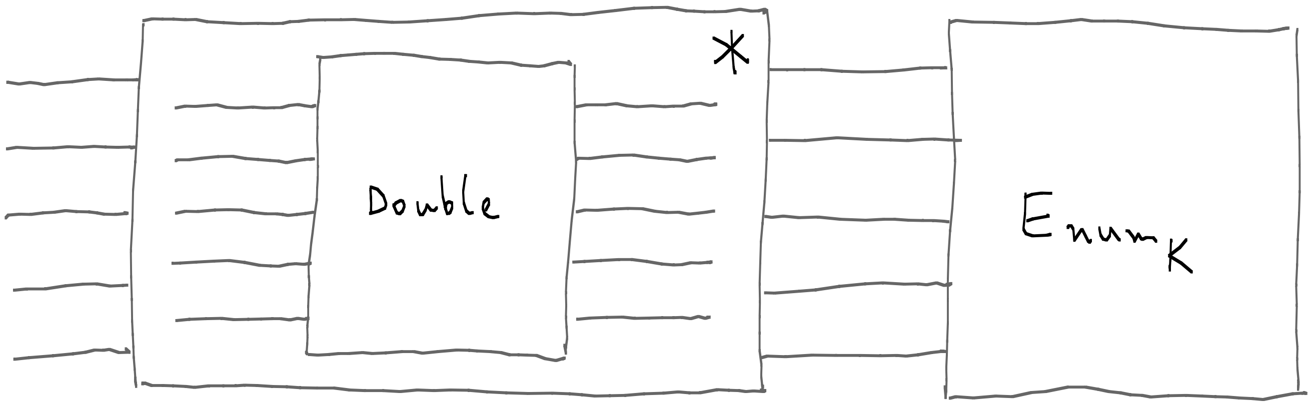




Implementing counters





Implementing counters

$$\text{Double}^*[\text{Enum}_K] \qquad\qquad 2^K$$

$$(\text{Double}^*)^*[\text{Enum}_K]$$

Implementing counters

$\mathrm{Double}^{*}[\,\mathrm{Enum}_K\,]$ $\qquad 2^K$

$(\mathrm{Double}^{*})^{*}[\,\mathrm{Enum}_K\,]$ $\qquad \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{K}$

[Bouyer et al. FAC '12]

Ackermann hierarchy

$$A_1(x) = 2x$$

$$A_{k+1}(x) = \underbrace{A_k(A_k(\dots A_k}_{x}(1)\dots))$$

Implementing counters

$\text{Double}^* [\text{Enum}_K]$ $\qquad A_2(K)$

$(\text{Double}^*)^* [\text{Enum}_K]$ $\qquad A_3(K)$

[Bouyer et al. FAC '12]

$((\text{Double}^*)^*)^* [\text{Enum}_K]$ $\qquad A_4(K)$ $\qquad$ [Jenkins '12]

Implementing counters

$\text{Double}^* [\text{Enum}_K]$ $\qquad A_2(K)$

$(\text{Double}^*)^* [\text{Enum}_K]$ $\qquad A_3(K)$

[Bouyer et al. FAC '12]

$((\text{Double}^*)^*)^* [\text{Enum}_K]$ $\qquad A_4(K)$ $\qquad$ [Jenkins '12]

$$(\cdots(\overbrace{\text{Double}^*)^* \cdots)^*}^{K-1} [\text{Enum}_K] \qquad A_K(K)$$