DECIDABILITY OF MONADIC THEORIES

A. L. Semenov
Academy of Sciences
Moscow

On the geographical map of logical theories we now recognize an immense ocean of undecidable and sparse islands of decidability. Monadic decidable theories constitute an important archipelago on the map. Büchi's introduction of automata technique opened a new epoch in the exploration of monadic theories. Rabin's discovery of decidability for monadic theory of the tree can be compared by its importance for mathematical logic only with Tarski theorem on decidability of the real field.

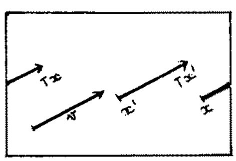
The purpose of this paper is to survay some results and methods obtained in Moscow during last five years as a further developemnt of Büchi's and Rabin's achievements. In the first part we consider some extensions of Büchi's result - namely decidability of monadic theories of structures $\langle N; \leqslant, f \rangle$, where f is a unary function. In the second part we discuss some ideas of a new proof for Rabin's theorem and extensions of this theorem. In the third part we mention some results concerning weak monadic theories. Monadic theory of a structure $\mathcal{S}$ will be denoted by $\mathcal{MTS}$. N is the set of all non-negative integers, ω-word is an infinite sequence of symbols from a finite alphabet; its limit is the set of symbols which have an infinite number of occurences in it.

I. MONADIC THEORIES OF UNARY FUNCTIONS ON N .

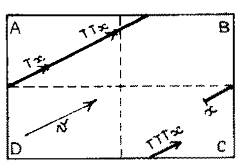
Let f be a fixed $N \longrightarrow N$ function with a finite range. We can try to express different properties of f using the monadic language of the structure $\langle N; \leqslant \rangle$ and ask if there is an algorithm to decide the resulting theory $\mathcal{MT}f \rightleftharpoons \mathcal{MT}\langle N; \leqslant, f \rangle$. In <u>this</u> part we will use abbreviation $\mathcal{MT}$ for $\mathcal{MT}\langle N; \leqslant \rangle$ (another notation is S1S). The first results on these theories were obtained by C. Elgot and M. Rabin in their paper [ER]. Their idea was very simple and powerful. Any sentence Φ in the language of $\mathcal{MT}f$ can be treated as a formula in the lantuage of $\mathcal{MT}$ with a single functional variable f. Using Büchi's method we can construct an automaton which accepts ω-word representing f if and only if the sentence Φ is true. As was proved by Elgot and Rabin the acceptance condition can be effectively verified for f being characteristic functions of sets of squares, factorials and so on. Siefkes in [S1] studied the class of functions, for which the decidability can be proved by Elgot - Rabin method. In [S2] he settled a question on the decidability of $\mathcal{MT}f$, where

$$f(x) = \text{sign} \sin x .$$

For this function nor positive answer can be given by Elgot - Rabin - Siefkes method, neither negative can be given by theorems by Siefkes and Thomas (see below). The study of the function sign sin x became a starting point of our investigation. This function is an example of an almost periodic function of symbolic dynamics. Symbolic dynamics studies the behaviour of dynamical systems by using corresponding discrete systems, elements of which are ω-words (or symbolic sequences) (see [H]). We will illustrate the idea by one example of connection between continuous and symbolic dynamic systems. Let us take as our dynamic system a torus and a transformation of it, say translation $T : x \mapsto x + v$.



The trajectory of x will be $x, Tx, TTx, \ldots$. Then we cut the torus into four parts and consider symbolic trajectory - the sequence



of parts in which x, Tx, ... lie. For x it will be CAAC... . The key-point is that the symbolic trajectory describes the trajectory well enough. It is clear now how such functions as sign sin x appear in symbolic dynamics. Different other examples of sequences, such as Thue, Morse or Keane examples are almost periodic (see [J1]). Now we recall this notion. We formulate it in terms of ω-words. A function f (an ω-word) is almost periodic (a.p.) if for any word u there exists an integer Δ such that either

1) all occurences of u in f are contained in the initial segment of f of length Δ, or

2) the word u occurs in every segment of f having length Δ. Actually particular examples of a.p. functions constructed in symbolic dynamics are effectively almost periodic, that is an integer Δ can be found by u algorithmically and we can decide, which alternative 1) or 2) is true. The function sign sin x is effectively a.p. as well as other sequences mentioned above.

The following question was the origin of the general study of decidability criterion for $\mathcal{MT}$f : Do the finite-automata transformations preserve almost periodicity? The answer is - "Yes" and in fact leads to the characterisation of decidability of $\mathcal{MT}$f for almost periodic f.

<u>Decidability theorem (almost periodic case)</u>. For any a.p. f $\mathcal{MT}$ f is dècidable $\Longleftrightarrow$ f is computable and effectively almost periodic.

So $\mathcal{MT}$ sign sin x as well as monadic theories of many other particular functions are decidable. The theorem is a special case of the general decidability theorem we shall consider now.

The notion of congruence $\mathcal{C}$ (with finite set of $\mathcal{C}$-classes, we call it simply congruence) on a free monoid Σ^* was used in many works on automata on ω-words (see [B1], [T2]). Let $\mathcal{C}$ be such a congruence. A <u>$\mathcal{C}$-index</u> is a nonempty word over the alphabet of $\mathcal{C}$-classes. The length of it does not exceed $|\mathcal{C}|$ - the number of $\mathcal{C}$-classes. For every $\mathcal{C}$-index $\varkappa$ we define its value $\langle\!\langle \varkappa \rangle\!\rangle$ by induction. If K is a $\mathcal{C}$-class, $\langle\!\langle K \rangle\!\rangle \rightleftharpoons \Sigma \cap K$; $w \in \langle\!\langle \varkappa K \rangle\!\rangle \Longleftrightarrow w \in K$ & there exist prefix and suffix of w belonging to $\langle\!\langle \varkappa \rangle\!\rangle$ & each subword of w, which belongs to $\langle\!\langle \varkappa \rangle\!\rangle$, is its prefix or suffix.

The idea here is to find a repetition structure in an ω-word modulo $\mathcal{C}$. Firstly, we recognize neighborous letters which are $\mathcal{C}$-congruent, then - $\mathcal{C}$-congruent segments having appeared in our search for letters, and so on.

<u>Decidability theorem</u>. The decision problem for $\mathcal{MT}\langle N; \leqslant, W\rangle$ is equivalent to the problem: to compute f and

for every congruence $\mathcal{C}$ to find the setof all $\mathcal{C}$-classes for which no element of them occurs in some end of W &

to find such an end.

The condition of the decidability given by the theorem is easy to be verified in the cases in which decidability was proved by the use of Elgot - Rabin - Siefkes theorem. We have mentioned yet one consequence of the decidability theorem - a.p. case. Another consequence can be trivially obtained from the theorem by substituting all regular sets to indices.

<u>Corollary</u>. For any W the decision problem for $\mathcal{MT}$W is equivalent to the problem of both :

1) to calculate W

2) to decide for every regular set A if there is an end of W which does not contain elements of A as subwords and if so - to find such an end.

<u>Z-case</u>. In the symbolic dynamics and other applications it is important to consider Z-word - a bisequence of letters from a finite alphabeth Σ . The set of all such words is denoted by Σ^Z. Let $w \in \Sigma^Z$. If we are interested in the decidability of $\mathcal{MT}W = \mathcal{MT}\langle Z; \leqslant, W\rangle$, two main cases arise. In accordance with symbolic dynamics we call a Z-word recurrent if no word has the first or the last occurence in it.

<u>Theorem on zero undefinability</u>. For any $W \in \Sigma^Z$ the following conditions are equivalent :

1) 0 is not definable in $\mathcal{MT}$W

2) W is recurrent.

It is clear that in the nonrecurrent case the decidability of $\mathcal{MT}$W and the decidability of the pair of theories obtained by cutting W into two ω-words are mutually reducible. In the recurrent case the situation is more simple and in some sense more exotic.

<u>Decidability theorem (recurrent Z-case)</u>. For any recurrent Z-word W the decision problem for $\mathcal{MT}$W is algorithmically equivalent to the problem: for every congruence $\mathcal{C}$ and $\mathcal{C}$-index $\varkappa$ recognize if there is an occurence of some element from $\langle\langle \varkappa \rangle\rangle$ in W .

We see that for recurrent W monadic theory $\mathcal{MT}$W is completely defined by the set of all subwords of W . In particular, all W for which this set is Σ^* have the same $\mathcal{MT}$W. From the theorem we get the decidability of it. In symbolic dynamic such Z-words are called transitive; they constitute a subset in Σ^Z of measure 1 . This leads us to some consequencesfor symbolic dynamics. One of them is the following :

<u>Corollary</u> (Gleason - Welch, see [H], sect. 11). For a given endomorphism f of the shift dynamical system, each recurrent transitive point x has the same cardinality of $f^{-1}(x)$.

We formulate the <u>uniformisation problem</u> for the theory in the following way. For any formula $\phi(X, Y)$ in the language of $\mathcal{MT}$ construct a transducer T , which for given input X produces some output Y , for which $\phi(X, Y)$ is true if such Y exists, and stops if there is no such Y .

It is well known that if transducer is understood as finite-state transducer, the uniformisation is not possible in all cases. It can be effectively constructed if it does exist. (See [R2] for historical references.) Transducer which perform uniformisation in all cases must have an unbounded memory and must get some information about infinite behaviour of ω-word.

We introduce the notion of <u>minimal transducers</u> which form (in some informal sense) minimal extension of the class of finite state transducers.

Let us fix a family of finite sets: input alphabet Σ , output alphabet Ω , two sets of marks - Π-marks (they form the set Π) and Γ-marks (they form the set Γ). Minimal transducers have input tape, output tape, control unit (a finite deterministic automaton) and memory. A transducer activity at any moment is determined by the part of input tape which it has not read, control state and memory state. State transition is determined (as in finite automata) by the previous state and input symbol. Memory state is defined by two marking maps which assign values to marks. The first is a total map $p : \Pi \longrightarrow \Omega^*$, the second is partial, $g : \Gamma \longrightarrow \Omega^*$. All elements from the image of g have equal lengths. Every control state has its printing mark - an element of Π and marking move - a pair of partial maps P, G :

$$P : \Pi \longrightarrow \Gamma, \quad G : \Gamma \longrightarrow \Gamma \times \Omega .$$

Each step begins with reading an input symbol, the control state changes, and then three actions are peformed: Γ-transition , printing, Π-transition. The marking changes in all these actions and each action uses the marking which appears after the previous action.

<u>Γ-transition</u>. For every $\gamma \in \Gamma$, if $G(\gamma)$ is undefined, then $g(\gamma)$ becomes also undefined. If $G(\gamma) = \langle \delta, \sigma \rangle$, then $g(\gamma) \longleftarrow g(\delta)$ ($\longleftarrow$ is the assignement). So, the value of γ becomes equal to the previous value of some Γ-mark extended by one symbol.

Printing. The value of printing mark is printed. Let it be x. Then we divide (if possible) the value of each mark by x from left. If it is impossible for a Π-mark, its value becomes Λ, if it is impossible for some Γ-mark, its value becomes undefined.

Π-transition. For every Π-mark ρ, if $P(\rho)$ is undefined, then the value of ρ remains unchanged, if it is defined, then $P(\rho) \longleftarrow g(P(\rho))$.

To end the description we note that the work of a transducer begins in some initial control state with some initial marking. It can be proved that if a minimal transducer is total, then it has a finite delay.

Theorem (definability of minimal transducers). Given a minimal transducer one can construct a formula $\phi(X, Y)$ in the monadic language of $\mathcal{MT}$ which defines the same function on ω-words as the transducer.

The technique developed for proving the decidability theorem gives us tools for the uniformisation. We state a uniformisation theorem in a simplified style. Given a family of regular sets R and an ω-word W we define W_R as any ω-word, obtained from W by marking two symbols of W by benchmarks. The part of W following the first benchmark contains the occurences of the elements of only those members from R the elements of which occur in each end of W; between the first and the second marks you can meet elements of all those members.

A uniformisation theorem. Given a formula $\phi(X, Y)$ in the language of $\mathcal{MT}$ one can construct a minimal transducer T and a finite family of regular sets R such that T transforms any W_R into some U, for which $\phi(W, U)$ holds and it falls into some rejecting state if no such U exists.

For every W all images of W obtained by minimal transducers constitute a model of $\mathcal{MT}W$. For almost periodic W these images are almost periodic. All a.p. ω-words form a model for $\mathcal{MT}$.

Unbounded functions. Till now we were interested in functions with finite range only. The major reason for it is that usually the theory $\mathcal{MT}f$ is undecidable for f with infinite range. Actually, as follows from a theorem of Thomas [T1], for every strictly increasing f which is not definable in $\mathcal{MT}$ the theory $\mathcal{MT}f$ is undecidable. It is easy to prove that every strictly increasing function definable in $\mathcal{MT}$ coincides, for a sufficiently large x, with $x + c$. As it was mentioned in [ER], the theory $\mathcal{M}\mathcal{T}f$ is decidable if f is definable in some decidable theory $\mathcal{M}\mathcal{T}g$ where g has a finite range. A question naturally arises : is there any decidable $\mathcal{M}\mathcal{T}f$ where f is definable in no theory of the form $\mathcal{M}\mathcal{T}g$, g having a finite range ? Such f does exist (see [Se2]).

II. RABIN THEOREM. A NEW PROOF AND GENERALIZATION.

In his report on the International congress of Mathematicians in Nice, 1970, Rabin stated as number 1 the following problem :

"Find a simpler proof for theorem 2(ii), possibly avoiding the transfinite induction used in [2]".(Theorem 2(ii) is the so called "Complementation lemma" and [2] is Rabin's publication in Transaction AMS - our reference [R1].)

In 1979 a new proof of Rabin's theorem was given by An. A. Muchnik - a student at Moscow University. In May, 1980 the (unpublished) paper of Muchnik was approved as his magister thesis (see [M1]). For technical reasons it is not published yet, a very meagre compensation was lectures on it in Moscow University (A. L. Semenov 1979/80, A. G. Dragalin, A. Shen' 1980/81). In 1981 (see [GH]) Rabin's problem was solved also by Yu. Gurevich and L. Harrington. Their solution is very close to Muchnik's. There are some differences in general idea, the Gurevich - Harrington proof is more "game-oriented". Other differences will be briefly discussed later. Gurevich - Harrington method also gives a new proof for the Stupp's generalization of Rabin's theorem (see [Sh]). The same is true for Muchnik's proof, theorem 2 below is a further generalization of Stupp's theorem. This and some other results were obtained thanks to stimulating discussions with P. Schupp.

Muchnik used finite automata in his proof. His automata are a generalization of Rabin's. The most important distinction is the introduction of deadends, it will be discussed later. In the simplest case of automata without input the notion of automaton in fact coincides with the notion of game of a special kind. This case corresponds to the problem of emptiness part in Rabin's proof. So we begin with games.

<u>Games</u>. Our games have two players : White and Black. White positions (W-positions) are elements of some set Π, Black positions (B-positions) are subsets of Π. The <u>rule</u> of a game is a set R of pairs of the form $\langle p, P\rangle$, where p is a W-position, P is a B-position. <u>Party</u> is a sequence

$$p_0, P_0, p_1, P_1, \dots$$

where $p_0, p_1, \dots$ are W-positions, $P_0, P_1, \dots$ are B-positions and for every i

$$\langle p_i, P_i \rangle \in R$$

$$p_{i+1} \in P_i .$$

So, White plays by choosing an element from R and giving a position to Black. Black plays by choosing an element of his position and giving it to White. Each party is either white party or black party, not both. Finite party is a finite sequence defined in a similar way; it is not a party.

White strategy (W-strategy) is a map φ from finite parties $p_0, P_0, \dots, p_n$ into B-positions P_n for which $\langle p_n, P_n \rangle \in R$. White wins in the game with an initial position p_0 if every party

$$p_0, P_0, \dots$$

where

$$P_i = \varphi(p_0, P_0, \dots, p_i), \quad i = 0, 1, \dots$$

is a white party; similarly for black.

Now we have to introduce a particular sort of games for which the division of parties into white and black will be of a special kind. Let Π have the form $\Pi = F \times S$, where F is a nonempty field of the game, S is a nonempty finite set of states of the game; let σ be a projection : $\sigma\langle x, s \rangle = s$. The notation of a party $p_0, P_0, \dots$ is an ω-word

$$\sigma(p_0) \ \sigma(p_1) \ \sigma(p_2) \ \dots .$$

All subsets of S are divided into white and black. A party is white if the limit of its notation is white, otherwise it is black. Now we will consider the monadic language of the monadic structure $\langle \Pi ; R, \sigma \rangle$. The atomic formulas of this language are of the form $x \in X$, $R(x, X)$, $\sigma(x) = s$, where x is an individual variable, X is a set variable, $s \in S$. So the language is determined by S.

Theorem 1. (Winner definability). There is an algorithm which for every set of states S and a family of white subsets of it constructs a monadic formula $\phi(x)$ such that for every game with this set of states and an arbitrary set of white states and an arbitrary field and rule it holds :

White wins in the game with initial position $p_0 \Longleftrightarrow \phi(p_0)$ is true.

An important and obvious consequence of this theorem is the following. Let us have a monadic structure $\langle F, \mathcal{K} \rangle$. Sometimes we can define a game on it by introducing a set S , a family of formulas $R_{s,t}(x, X)$ of the monadic language of signature π , where $R_{s,t}(x, X) \Longleftrightarrow R(\langle x, s\rangle , \langle X, t\rangle)$, a family of white subsets of S and an initial state $s_0 \in S$. We shall suppose that each of $R_{s,t}$ is defined by a fixed formula in the monadic language of $\langle F, \pi\rangle$.

<u>Corollary</u>. There is an algorithm which for every set of states S , family of formulas $R_{s,t}$, family $L \subseteq 2^S$ and a state s_0 constructs a formula ϕ in the monadic language of signature π such that for arbitrary structure $\langle F, \pi\rangle$ we have :

White wins in the game with the field F , the set of states S , the rule defined by $R_{s,t}$, the family of white subsets L , and the initial position $\langle x, s_0\rangle \Longleftrightarrow \phi(x)$ is true.

So, if a game is definable in the monadic language of some monadic structure, then the set of winning positions for it is definable in the same language.

Now we pass to the Muchnik's generalization of Rabin's theorem. It is also a generalization of a Stupp's theorem. The method of the proof for this theorem and Rabin theorem is the same and it uses no transfinite induction.

Let $\mathcal{S} = \langle A; \ldots\rangle$ be a monadic structure. We construct some new monadic structure <u>Tree</u>$(\mathcal{S})$ - the tree structure over $\mathcal{S}$. Its elements are finite sequences of elements of $\mathcal{S}$. The signature of it contains the relation $x \prec y$ - " x is a proper initial segment of y " . Then, for each formula $(\bar{x}, \bar{X})$ of the monadic language of $\mathcal{S}$ the signature contains the symbol $\hat{\varphi}$ with the interpretation

$$\hat{\varphi}(\bar{y}, \bar{Y}) \Longleftrightarrow \exists z \in A^*, \bar{x}, \bar{X} \ (\ \bar{y} = z\bar{x} \quad \bar{Y} = z\bar{X} \quad \varphi(\bar{x}, \bar{X}) \ ).$$

Here, $\bar{y}, \bar{Y} \ldots$ are vectors over the elements of A^*, 2^{A^*}, etc. Finally, for each formula $\varphi(t, \bar{x}, \bar{X})$ we place to the signature of <u>Tree</u>$(\mathcal{S})$ the symbol $\hat{\hat{\varphi}}$ with the interpretation

$$\hat{\hat{\varphi}}(\bar{y}, \bar{Y}) \Longleftrightarrow \exists z \in A^*, t \in A, \bar{x}, \bar{X} \ (\ \bar{y} = zt\bar{x} \ \& \ \bar{Y} = zt\bar{X} \ \& \ \varphi(t, \bar{x}, \bar{X}) \ ).$$

<u>Reduction theorem for tree structures</u>. There is an algorithm to construct for every sentence in the monadic language of <u>Tree</u>$(\mathcal{S})$ an equivalent sentence in the monadic language of $\mathcal{S}$.

If we do not use $\hat{\hat{\varphi}}$ in the construction of <u>Tree</u>$(\mathcal{S})$ and suppose that $\mathcal{S}$ is an elementary structure, we obtain the theorem of Stupp (see [Sh] , [GH]). As a special case for $\mathcal{S}$ of cardinality two we get Rabin's theorem.

<u>About the proof</u>. Firstly the proper notion of automaton is introduced and the theorem on automata definability for definable subsets of <u>Tree</u>($\mathcal{S}$) is proved. Secondly, it is noted that the automaton constructed for a closed formula is a game in essence. So we can use theorem 1 to obtain a sentence in the monadic language of $\mathcal{S}$ which is true iff the automaton accepts the tree, that is equivalent to the truthness of the given sentence in the monadic language of <u>Tree</u>($\mathcal{S}$).

The proof uses the same construction as in theorem 1. Naturally the most difficult part is the construction of an automaton, especially for the negation of a formula.

<u>The notion of automaton</u>. An automaton has a finite set of states S, and an input alphabet Σ (usually $\Sigma = \{0, 1\}^n$ for some n). The set of positions is $\Pi \rightleftharpoons A \times S$. Transition rule is a family of definable in $\mathcal{MTS}$ relations $R_\alpha \subseteq \Pi \times 2^\Pi$. Initial condition is a family of $\mathcal{MTS}$-definable predicates $I_\alpha \subseteq \Pi$. Some subsets of S are white. A run ϑ of the automaton on a marked tree (that is a map $\mu : A^* \longrightarrow \Sigma$) is a set of pairs $\langle x, \langle s_0, s_1\rangle\rangle$, where $x \in A^+$; for $x \in A$ $\langle x, s_1\rangle \in I_{\mu(x)}$ and, for every $x \in A^+$, $a \in A$ if $\langle xa, \langle s_0, s_1\rangle\rangle \in \vartheta$ then

$$\langle\langle a, s_1\rangle , \{\langle b, s_1'\rangle \mid b \in A \;\&\; \langle xab, \langle s_1, s_1'\rangle\rangle \in \vartheta\}\rangle \in R_{\mu(x)} .$$

Now we shall formulate the acceptance condition for the automaton. Let μ be a marked tree. Then it is accepted iff there is an accepting run ϑ of the automaton on it. The run is accepting if for any path $x_0 = a_0$, $x_1 = x_0a_1$, ..., $x_{i+1} = x_ia_i$, $a_i \in A$ and any sequence of states $s_0, s_1, \ldots$, where $\langle s_i, s_{i+1}\rangle \in \vartheta(x_i)$, the limit of the sequence is white. So the "actual state" of automaton in the vertex x_i is s_{i+1} and we use pairs for the connection of times.

As we see, for a fixed run an element of tree does not determine a pair of the automaton states in it. But actually every automaton can be effectively transformed into one for which this kind of determinism takes place. Let us call these automata standard. (This is a straightforward generalization of Rabin's automata.)

The proof of automata definability now proceeds in the usual way. We construct automata defining signature relations and then prove closeness under $\vee$, $\exists$, $\neg$ for standard automata. The main point again is a complementation lemma. For its proof the notion of automata with deadends is introduced. A set of deadends plays in the description of automata the role similar to that of states, but no transition from deadends is permitted. Deadends are also used as marks of the vertices in the tree. Then, if the automaton with deadends arrives to some vertex marked by a deadend then the acceptance condition demands that the automaton must be in a deadend (not in state), and the deadend must coincide with the mark in the vertex. The complementation lemma is now proved using the induction by number of states (the number of deadends is irrelevant). The steps are similar to the proof of the first theorem. No transfinite induction is used! The states of automaton for the complement are the permutations of the given automaton $\mathfrak{A}$. (This method adds three levels of exponentation at each quantifier change in the decision procedure.) In constructing its rule we use the following operation acting on permutations. We suppose that all states of $\mathfrak{A}$ are cyclically ordered. Let w be a permutation and s - a state. If $w = usv$ and $v's$ is the reordering of sv in the cyclical order, then s transforms w into $uv's$. This method of storing the information on the history is used in constructing accepting runs for automaton accepting the complement. The technical improvements we have discussed very briefly lead us to the short and transparent proof of Rabin's theorem using only a few simple ideas. Gurevich and Harrington's proof is, as we have said, very close to Muchnik's as can be understood from [GH], but there are no mentions about such constructions as deadends and methods for storing a history.

<u>Uniformisation</u>. As it was conjectured by Siefkes in [S3], the definable uniformisation for S2S (the tree structure over the structure with two elements) is impossible. That conjecture was proved by Gurevich & Harrington in a strong form of nonuniformisability of some relation in $2^{\{0,1\}^*} \times \{0,1\}^*$. The relation is the natural $\in$-relation. So the choice function is indefinable in S2S. On the other hand, any relation in $\{0,1\}^* \times 2^{\{0,1\}^*}$ (and, consequently in $\{0,1\}^* \times \{0,1\}^*$) has a definable uniformisation. It was also proved by Muchnik.

III. WEAK MONADIC AND ELEMENTARY THEORIES.

There is a natural correspondence between finite subsets of N and N based on finary coding (see [B2]). This correspondence was used in [B], [ER] for treating the weak monadic theory of $\langle N; \leq \rangle$ as elementary. The addition of natural numbers and predicate Pw_2 "to be a power of two" are definable in this theory. Actually the set of relations definable in this theory coincides with the set of relations definable by finite automata using binary encoding of integers. As it was conjectured in [McN] and proved in [Se1] (and also can easily be obtained from results of [Se2]), there are some relations definable by finite automata but not definable in elementary theory of $\langle N; +, Pw_2 \rangle$. But we can use another unary predicate instead of Pw_2 to characterize the definability.

<u>Theorem</u>. Let P_0 be a set of all natural numbers with binary codings from the set

$$\{000, 001\}^* \ 010 \ \{000,001\}^* \cup \{000,001\}^*.$$

Then a relation is definable by a finite automaton iff it is definable in elementary theory of $\langle N; +, P_0 \rangle$.

The decidability theorem for elementary theories of structures $\langle N; \leq, W \rangle$, where W is an ω-word, and other results from the first part of this paper can be proved in a modified form. The modification is made by restricting the set of congruences to special congruences only. We call a congruence special if it corresponds to the homomorphism on a finite monoid with no subgroups. The connection of this notion with elementary theories was found by Schützenberger (see also [M2] for a simple proof of Schützenberger's theorem). Some other questions on decidability of the elementary theories of $\langle N; +, f \rangle$, $\langle N; \leq, f \rangle$ were discussed in [Se1], [Se2]. In particular, in [Se2] a predicate P was constructed for which elementary theory of $\langle N; +, P \rangle$ is undecidable but all relations definable in it are decidable. (This answers a question of Büchi from [B1].)

Elgot and Rabin asked whether there is a maximal decidable theory, e.g. a decidable theory which becomes undecidable if we add to it some predicate which is not definable in it. As Soprunov proved, there is no maximal decidable weak monadic theory.

REFERENCES

[B1] Büchi J.R.: On a decision method in restricted second order arithmetic. Proc. 1960 Intern. Congr. for Logic, Methodol. and Philos. of Sci., Stanford, 1962, pp. 1 - 11.

[B2] Büchi J.R.: Weak second-order arithmetic and finite automata. Z.Math. Logik und Grundl. Math. 6(1960), N° 1, pp. 66 - 92.

[ER] Elgot C.C., Rabin M.O.: Decidability and undecidability of extentions of second (first) order theory of (generalized) successor. J. Symbolic Logic 31(1966), N° 2, pp. 169 - 181.

[GH] Gurevich Yu., Harrington L.: Trees, automata and games. Proc. ACM STOC , May 1982, pp. 60 - 65.

[H] Hedlund G.A.: Endomorphisms and automorphisms of the shift dynamical systems. Math. syst. theory 3(1971), N° 4, pp. 320 - 375.

[J] Jacobs K.: Maschinenerzeugte 0-1-Folgen, Selecta Mathematica, II, Springer-Verlag, 1970.

[M1] Мучник Ан. А. [Muchnik An.A.]: Игры на бесконечных деревьях и автоматы с тупиками. Новое доказательство разрешимости монадической теории нескольких следований. [Games on infinite trees and automata with deadends. A new proof of the decidability for the monadic theory of several successors.] Semiotica i Informatica 1984 (to appear) .

[M2] Мучник Ан.А. [Muchnik An.A.] : Новое доказательство теоремы Шютценберже о моноидах без нетривиальных подгрупп. [A new proof of Schützenberger's theorem on monoids without nontrivial subgroups.] Matematicheskaia logika, mat. lingvistika i teoria algoritmov. Kalinin 1983, pp. 65 - 68 .

[McN] McNaughton R.: Rev. of [B2]. J. Symb. Logic 28(1983) N°2, pp. 100 - 102.

[R1] Rabin M.O.: Decidability of second-order theories and automata on infinite trees. Transactions AMS 141(1969), pp. 1 - 35.

[R2] Rabin M.O.: Automata on infinite objects and Church's problem. AMS, Providence, 1972.

[S1] Siefkes D.: Decidable extentions of monadic second order successor arithmetic. Automatentheorie un Formale Sprachen, Mannheim,1969, pp. 441 - 472.

[S2] Siefkes D.: Undecidable extentions of monadic second order successor arithmetic. Z. Math. Logik und Grundlagen der Mathematik 17(1971), N° 5, pp. 383 - 394.

[S3] Siefkes D.: The recursive sets in certain monadic second order fragments of arithmetic. Arch. Math. Logik, 17(1975), N° 1-2, pp. 71 - 80.

[Se1] Семенов А.Л. [Semenov A.L.] : О некоторых расширениях арифметики сложения натуральных чисел. [On certain extensions of the arithmetic of addition of natural numbers.] Iz. Akad. Nauk SSSR 43(1979), N° 5, pp. 1175 - 1195; Math. USSR - Izv. 15(1980), N° 2, pp. 401 - 418.

[Se2] Семенов А.Л. [Semenov A.L.]: Логические теории одноместных функций на натуральном ряде. [Logical theories of unary functions on natural numbers.] Izv. Akad. Nauk SSSR 47(1983),

N^o 3, pp. 623 - 658.

[Sh] Shelah S.: The monadic theory of order. Annals of Math. 102 (1975), pp. 379 - 419.

[T1] Thomas W.: A note on undecidable extentions of monadic second order arithmetic. Z. Math. Logik und Grundlagen Math. 17 (1975), N^o 1-2, pp. 43 - 44.

[T2] Thomas W.: A combinatorial approach to the theory of ω-automata. Information and Control 48(1981), pp. 261 - 283.