Iterated regret minimization: A new solution concept ☆

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A B S T R A C T

For some well-known games, such as the Traveler's Dilemma or the Centipede Game, traditional game-theoretic solution concepts—most notably Nash equilibrium—predict outcomes that are not consistent with empirical observations. We introduce a new solution concept, *iterated regret minimization*, that exhibits the same qualitative behavior as that observed in experiments in many games of interest, including Traveler's Dilemma, the Centipede Game, Nash bargaining, and Bertrand competition. As the name suggests, iterated regret minimization involves the iterated deletion of strategies that do not minimize regret.

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1. Introduction

Perhaps the most common solution concept considered in game theory is Nash equilibrium. Other solution concepts have been considered, such as *sequential equilibrium* (Kreps and Wilson, 1982), *perfect equilibrium* (Selten, 1975), and *rationalizability* (Bernheim, 1984; Pearce, 1984); see Osborne and Rubinstein (1994) for an overview. There are a number of well-known games where none of them seems appropriate.

Consider the well-known *Traveler's Dilemma* (Basu, 1994, 2007). Suppose that two travelers have identical luggage, for which they both paid the same price. Their luggage is damaged (in an identical way) by an airline. The airline offers to recompense them for their luggage. They may ask for any dollar amount between $2 and $100. There is only one catch. If they ask for the same amount, then that is what they will both receive. However, if they ask for different amounts—say one asks for m and the other for m', with $m < m'$—then whoever asks for m (the lower amount) will get $(m + p)$, while the other traveler will get $(m − p)$, where p can be viewed as a reward for the person who asked for the lower amount, and a penalty for the person who asked for the higher amount.

It seems at first blush that both travelers should ask for $100, the maximum amount, for then they will both get that. However, as long as $p > 1$, one of them might then realize that he is actually better off asking for $99 if the other traveler asks for $100, since he then gets $(99 + p)$. In fact, $99 *weakly dominates* $100, in that no matter what Traveler 1 asks for, Traveler 2 is always at least as well off asking for $99 than $100, and in one case (if Traveler 2 asks for $100) Traveler 1 is strictly better off asking for $99. Thus, it seems we can eliminate 100 as an amount to ask for. However, if we eliminate 100, a similar argument shows that 98 weakly dominates 99. And once we eliminate 99, then 97 weakly dominates 98. Continuing this argument, both travelers end up asking for $2! In fact, it is easy to see that $(2, 2)$ is the

only Nash equilibrium. With any other pair of requests, at least one of the travelers would want to change his request if he knew what the other traveler was asking. Since $(2, 2)$ is the only Nash equilibrium, it is also the only sequential and perfect equilibrium. Moreover, it is the only rationalizable strategy profile.

This seems like a strange result. It seems that no reasonable person—even a game theorist!—would ever play 2. Indeed, when the Traveler's Dilemma was empirically tested among game theorists (with $p = 2$) they typically did not play anywhere close to 2. Becker et al. (2005) asked members of the Game Theory Society (presumably, all experts in game theory) to submit a strategy for the game. Fifty-one of them did so. Of the 45 that submitted pure strategies, 31 submitted a strategy of 96 or higher, and 38 submitted a strategy of 90 or higher; only 3 submitted the "recommended" strategy of 2. The strategy that performed best (in pairwise matchups against all submitted strategies) was 97, which had an average payoff of $85.09. The worst average payoff went to those who played 2; it was only $3.92.

Another sequence of experiments by Capra et al. (1999) showed, among other things, that this result was quite sensitive to the choice of p. For low values of p, people tended to play high values, and keep playing them when the game was repeated. By way of contrast, for high values of p, people started much lower, and converged to playing 2 after a few rounds of repeated play. The standard solution concepts (Nash equilibrium, rationalizability, etc.) are all insensitive to the choice of p; for example, $(2, 2)$ is the only Nash equilibrium for all choices of $p > 1$.

We introduce a new solution concept, *iterated regret minimization*, which has the same qualitative behavior as that observed in the experiments, not just in Traveler's Dilemma, but in many other games that have proved problematic for Nash equilibrium, including the Centipede Game, Nash bargaining, and Bertrand competition. For many of these games, alternative explanations of observed behavior exist. For instance, the famous "Gang of Four" analysis (Kreps et al., 1982) provides an explanation for cooperative behavior by assuming that a small fraction of player types are "irrational" and are restricted to playing some specific strategy (say, announcing 100 in the Traveler's Dilemma). While this explanation certainly is plausible, iterated regret minimization provides a conceptually different explanation for cooperative behavior, without requiring *ad hoc* assumptions (such as specifying the strategy employed by irrational players in the context of the gang of four analysis). Of course, iterated regret minimization is not without its own limitations. As we show, there are natural games where Nash equilibrium provides more reasonable predictions than regret minimization.

We now describe our new solution concept. As the name suggests, iterated regret minimization involves the iterated deletion of strategies that do not minimize regret. It is a solution concept that leads to quite different predictions than Nash equilibrium, while being grounded on regret, a decision rule that has a long history in decision theory (Niehans, 1948; Savage, 1951) and has a great deal of support from the psychology literature (see, for example, Grosskopf et al., 2006; Ritov, 1996).

Roughly speaking, the idea of regret in decision theory is that an agent chooses an action a that minimizes regret across states, where the regret of action a in a state s is the difference between the agent's utility when he performs a in a state s and when he performs the act that gives the highest utility in state s. To apply regret in a strategic-form game, we take the states to be the other players' strategy choices. Iterated regret minimization takes this idea one step further: we see what inferences we can draw starting with minimal beliefs about the other players' strategies, using only the fact that the other players are regret minimizer.

However, there are some subtleties involved trying to define a solution concept this way. We cannot use the standard (informal) assumption that the strategy profile in equilibrium is common knowledge, or that the use of regret minimization is common knowledge. For example, minimizing regret with respect to a fixed strategy profile is equivalent to maximizing expected utility, and we are quickly led back to Nash equilibrium. We nevertheless provide an epistemic characterization for iterated regret minimization. Unlike epistemic characterizations of many other solution concepts, our epistemic characterization of iterated regret minimization does *not* involve common knowledge or common belief of rationality. Indeed, rather than believing, for example, that everyone believes that everyone believes … (k times) that everyone is rational, an agent will view this event as very unlikely. Moreover, the larger k is, the more unlikely the event is.

The rest of this paper is organized as follows. Section 2 contains preliminaries. Section 3 is the heart of the paper: we first define iterated regret minimization in strategic games, provide an epistemic characterization of it, and then show how iterated regret minimization works in numerous standard examples from the game theory literature, including Traveler's Dilemma, Prisoner's Dilemma, the Centipede Game, Bertrand competition, and Nash bargaining, both with pure strategies and mixed strategies. In the interests of full disclosure, we also show some games where iterated regret minimization provides problematic predictions. In Section 4, we briefly consider regret minimization in Bayesian games. We discuss related work in Section 5, and conclude in Section 6. Proofs are relegated to Appendix B.

2. Preliminaries

As usual, we refer to a vector $\vec{x}$ consisting of a value x_j for each player j, as a *profile*, and let $\vec{x}_{-i}$ denote the collection of all values x_j for $j \neq i$.

A strategic game is characterized by a tuple $([n], A, \vec{u})$, where $[n] = \{1, \ldots, n\}$ is the set of players, $A = A_1 \times \cdots \times A_n$ is a set of action profiles, and $\vec{u}$ is the profile of utility functions, where $u_i(\vec{a})$ is player i's utility or payoff if the action profile $\vec{a}$ is played. A (*mixed*) *strategy* for player i is a probability distribution $\sigma_i \in \Delta(A_i)$ (where, as usual, we denote by $\Delta(X)$ the set of distributions on the set X). Let $\Sigma_i = \Delta(A_i)$ denote the mixed strategies for player i in game G, and let $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ denote the set of mixed strategy profiles. To simplify notation, we abuse notation slightly and let $a_i \in A_i$ also denote the

mixed strategy that puts probability 1 on action a_i. If σ is a strategy for player i then $\sigma(a)$ denotes the probability given to action a by strategy σ. Given a strategy profile $\vec{\sigma}$, player i's expected utility if $\vec{\sigma}$ is played, denoted $U_i(\vec{\sigma})$, is $\mathbf{E}_{\Pr}[u_i^{\vec{\sigma}}]$, where the expectation is taken with respect to the probability Pr induced by $\vec{\sigma}$ (where the players are assumed to choose their actions independently).

3. Iterated regret minimization in strategic games

We start by providing an informal discussion of iterated regret minimization in strategic games, and applying it to the Traveler's Dilemma; we then give a more formal treatment.

Nash equilibrium implicitly assumes that the players know what strategy the other players are using. Such knowledge seems unreasonable, especially in one-shot games. Regret minimization is one way of trying to capture the intuition that a player wants to do well no matter what the other players do.

The idea of minimizing regret was introduced (independently) in decision theory by Savage (1951) and Niehans (1948). To explain how we use it in a game-theoretic context, we first review how it works in a single-agent decision problem. Suppose that an agent chooses an act from a set A of acts. The agent is uncertain as to the true state of the world; there is a set S of possible states. Associated with each state $s \in S$ and act $a \in A$ is the utility $u(a, s)$ of performing act a if s is the true state of the world. For simplicity, we take S and A to be finite here. The idea behind the *minimax-regret* rule is to hedge the agent's bets, by doing reasonably well no matter what the actual state is. For each state s, let $u^*(s)$ be the best outcome in state s; that is, $u^*(s) = \max_{a \in A} u(a, s)$. The *regret* of a in state s, denoted $regret_u(a, s)$, is $u^*(s) - u(a, s)$; that is, the regret of a in s is the difference between the utility of the best possible outcome in s and the utility of performing act a in s. Let $regret_u(a) = \max_{s \in S} regret_u(a, s)$. For example, if $regret_u(a) = 2$, then in each state s, the utility of performing a in s is guaranteed to be within 2 of the utility of any act the agent could choose, even if she knew that the actual state was s. The minimax-regret decision rule orders acts by their regret; the "best" act is the one that minimizes regret. Intuitively, this rule is trying to minimize the regret that an agent would feel if she discovered what the situation actually was: the "I wish I had chosen a' instead of a" feeling.

Despite having been used in decision making for over 50 years, up until recently, there was little work on applying regret minimization in the context of game theory. We discuss other work on applying regret minimization to game theory in Section 5; here, we describe our own approach. We start by explaining it in the context of the Traveler's Dilemma and restrict our attention to pure strategies. We take the acts for one player to be that player's pure strategy choices and take the states to be the other player's pure strategy choices. Each act–state pair is then just a strategy profile; the utility of the act–state pair for player i is just the payoff to player i of the strategy profile. Intuitively, each agent is uncertain about what the other agent will do, and tries to choose an act that minimizes his regret, given that uncertainty.

It is easy to see that, if the penalty/reward p is such that $2 \leqslant p \leqslant 48$, then the acts that minimize regret are the ones in the interval $[100 - 2p, 100]$; the regret for all these acts is $2p - 1$. For if player 1 asks for an amount $m \in [100 - 2p, 100]$ and player 2 asks for an amount $m' < m$ (which he can do), then the payoff to player 1 is $m' - p$, compared to the payoff of $m' + p - 1$ (or just m' if $m' = 2$) that is achieved with the best response; thus, the regret is $2p - 1$ in this case. If, instead, player 2 asks for $m' \geqslant m$, then player 1 gets a payoff of $m + p$, and the best possible payoff in the game is $99 + p$, so his regret is at most $99 - m \leqslant 2p - 1$. On the other hand, if player 1 chooses $m < 100 - 2p$, then his regret will be $99 - m > 2p - 1$ if player 2 plays 100. If $49 \leqslant p \leqslant 96$, then these arguments show that the unique act that minimizes regret is asking for \$3; when $p = 97$, both \$3 and \$2 minimize regret, and when $p \geqslant 98$ only \$2 minimizes regret.

Suppose that $2 \leqslant p \leqslant 48$. Applying regret minimization once suggests using a strategy in the interval $[100 - 2p, 100]$. But we can iterate this process. If we assume that both players use a strategy in this interval, then the strategy that minimizes regret is that of asking for \$$(100 - 2p + 1)$. A straightforward check shows that this has regret $2p - 2$; all other strategies have regret $2p - 1$.

Note that, in the case that $p = 2$, the actions in $[96, 100]$ are the ones that minimize regret. Recall that 31 of the 45 members of the Game Theory Society who submitted a strategy submitted one in the interval $[96, 100]$; thus, their actions certainly seem to be compatible with regret minimization. In this case, iterating regret minimization singles out the strategy of asking for \$97, which was found to be the best strategy by Becker et al. (2005). Given that a large subset of the members of the Game Theory Society were in fact choosing an action that minimized regret, it is not surprising that iterated regret minimization should lead to a strategy that does well. As p increases, all the acts in the interval $[100 - 2p, 100]$ survive one round of regret minimization. We would expect that if players are in fact minimizing regret, then after a number of rounds, players would start minimizing regret with respect to strategies in $[100 - 2p, 100]$ rather than all the strategies in $[2, 100]$, and thus we should see the actions played decreasing as a function of p. This matches, at a qualitative level, the findings of Capra et al. (1999).

3.1. Deletion operators and iterated regret minimization

Iterated regret minimization proceeds much like other notions of iterated deletion. To put it in context, we first abstract the notion of iterated deletion.

Let $G = ([n], A, \vec{u})$ be a strategic game. We define iterated regret minimization in a way that makes it clear how it relates to other solution concepts based on iterated deletion. A *deletion operator* $\mathcal{D}$ maps sets $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ of strategy profiles

in G to sets of strategy profiles such that $\mathcal{D}(\mathcal{S}) \subseteq \mathcal{S}$. We require that $\mathcal{D}(\mathcal{S}) = \mathcal{D}_1(\mathcal{S}) \times \cdots \times \mathcal{D}_n(\mathcal{S})$, where $\mathcal{D}_i$ maps sets of strategy profiles to strategies for player i. Intuitively, $\mathcal{D}_i(\mathcal{S})$ is the set of strategies for player i that survive deletion, starting with $\mathcal{S}$. Note that the set of strategies that survive deletion may depend on the set that we start with. Iterated deletion then amounts to applying the $\mathcal{D}$ operator repeatedly, starting with an appropriate initial set $\mathcal{S}_0$ of strategies, where $\mathcal{S}_0$ is typically either the set of pure strategy profiles (i.e., action profiles) in G or the set of mixed strategy profiles in G.

Definition 3.1. Given a deletion operator $\mathcal{D}$ and an initial set $\mathcal{S}_0$ of strategies, the set of strategy profiles that survive iterated deletion with respect to $\mathcal{D}$ and $\mathcal{S}_0$ is

$$\mathcal{D}^\infty(\mathcal{S}_0) = \bigcap_{k>0} \mathcal{D}^k(\mathcal{S}_0)$$

(where $\mathcal{D}^1(\mathcal{S}) = \mathcal{D}(\mathcal{S})$ and $\mathcal{D}^{k+1}(\mathcal{S}) = \mathcal{D}(\mathcal{D}^k(\mathcal{S}))$). Similarly, the set of strategy profiles for player i that survive iterated deletion with respect to $\mathcal{D}$ and $\mathcal{S}_0$ is $\mathcal{D}_i^\infty(\mathcal{S}_0) = \bigcap_{k>0} \mathcal{D}_i^k(\mathcal{S}_0)$, where $\mathcal{D}_i^1 = \mathcal{D}_i$ and $\mathcal{D}_i^{k+i} = \mathcal{D}_i \circ \mathcal{D}^k$.

We can now define the deletion operator $\mathcal{RM}$ appropriate for regret minimization in strategic games (we deal with Bayesian games in Section 4). Intuitively, $\mathcal{RM}_i(\mathcal{S})$ consists of all the strategies in $\mathcal{S}_i$ that minimize regret, given that the other players are using a strategy in $\mathcal{S}_{-i}$. In more detail, we proceed as follows. Suppose that $G = ([n], A, \vec{u})$ and that $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \subseteq A$, the set of pure strategy profiles (i.e., action profiles). For $\vec{a}_{-i} \in \mathcal{S}_{-i}$, let $u_i^{\mathcal{S}_i}(\vec{a}_{-i}) = \max_{a_i \in \mathcal{S}_i} u_i(a_i, \vec{a}_{-i})$. Thus, $u_i^{\mathcal{S}_i}(\vec{a}_{-i})$ is the best outcome for i given that the remaining players play $\vec{a}_{-i}$ and that i can select actions only in $\mathcal{S}_i$. For $a_i \in \mathcal{S}_i$ and $\vec{a}_{-i} \in \mathcal{S}_{-i}$, let the *regret* of a_i for player i given $\vec{a}_{-i}$ relative to $\mathcal{S}_i$, denoted $regret_i^{\mathcal{S}_i}(a_i \mid \vec{a}_{-i})$, be $u_i^{\mathcal{S}_i}(\vec{a}_{-i}) - u_i(a_i, \vec{a}_{-i})$. Let $regret_i^{\mathcal{S}}(a_i) = \max_{\vec{a}_{-i} \in \mathcal{S}_{-i}} regret^{\mathcal{S}_i}(a_i \mid \vec{a}_{-i})$ denote the maximum regret of a_i for player i (given that the other players' actions are chosen from $\mathcal{S}_{-i}$). Let $minregret_i^{\mathcal{S}} = \min_{a_i \in \mathcal{S}_i} regret_i^{\mathcal{S}_{-i}}(a_i)$ be the minimum regret for player i relative to $\mathcal{S}$. Finally, let

$$\mathcal{RM}_i(\mathcal{S}) = \left\{ a_i \in \mathcal{S}_i \colon regret_i^{\mathcal{S}}(a_i) = minregret_i^{\mathcal{S}} \right\}.$$

Thus, $\mathcal{RM}_i(\mathcal{S})$ consists of the set of actions that achieve the minimal regret with respect to $\mathcal{S}$. Clearly $\mathcal{RM}_i(\mathcal{S}) \subseteq \mathcal{S}$. Let $\mathcal{RM}(\mathcal{S}) = \mathcal{RM}_1(\mathcal{S}) \times \cdots \times \mathcal{RM}_n(\mathcal{S})$. Note that because $\mathcal{RM}^k(\mathcal{S})$ is a cross-product of the form $\mathcal{RM}_1^k(\mathcal{S}) \times \cdots \times \mathcal{RM}_n^k(\mathcal{S})$ for all k, then so is $\mathcal{RM}^\infty(\mathcal{S}) = \bigcap_k \mathcal{RM}^k(\mathcal{S})$. Thus, it makes sense to write $\mathcal{RM}_i^\infty(\mathcal{S})$. Moreover, it is easy to see that $\mathcal{RM}_i^\infty(\mathcal{S}) = \bigcap_k \mathcal{RM}_i^k(\mathcal{S})$.

If $\mathcal{S}$ consists of mixed strategies, then the construction of $\mathcal{RM}(\mathcal{S})$ is the same, except that the expected utility operator U_i is used rather than u_i in defining $regret_i$. To ensure that this is well defined, we need to argue that there is in fact a strategy s_i for player i that maximizes $regret_i$ and one that minimizes $minregret_i$. This follows from the compactness of the sets of which the max and min are taken, and the continuity of the functions being maximized and minimized.

Definition 3.2. Given a strategic game $G = ([n], A, \vec{u})$, $\mathcal{RM}_i^\infty(A)$ is the set of (pure) strategies for player i that survive iterated regret minimization with respect to pure strategies in G. Similarly, $\mathcal{RM}_i^\infty(\Sigma)$ is the set of (mixed) strategies for player i that survive iterated regret minimization with respect to mixed strategies in G.

The following theorem, whose proof is in Appendix B, shows that iterated regret minimization is a reasonable concept in that, for all games G, $\mathcal{RM}^\infty(A)$ and $\mathcal{RM}^\infty(\Sigma)$ are nonempty fixed points of the deletion process, that is, $\mathcal{RM}(\mathcal{RM}^\infty(A)) = \mathcal{RM}^\infty(A)$ and $\mathcal{RM}(\mathcal{RM}^\infty(\Sigma)) = \mathcal{RM}^\infty(\Sigma)$; the deletion process converges at $\mathcal{RM}^\infty$. (Our proof actually shows that for any nonempty closed set $\mathcal{S}$ of strategies, the set $\mathcal{RM}^\infty(\mathcal{S})$ is nonempty and is a fixed point of the deletion process.)

Theorem 3.3. *Let $G = ([n], A, \vec{u})$ be a strategic game. If $\mathcal{S}$ is a closed, nonempty set of strategies of the form $\mathcal{S}_1 \times \cdots \times \mathcal{S}_n$, then $\mathcal{RM}^\infty(\mathcal{S})$ is nonempty and $\mathcal{RM}(\mathcal{RM}^\infty(\mathcal{S})) = \mathcal{RM}^\infty(\mathcal{S})$.*

Unlike solution concepts that implicitly assume that agents know other agents' strategies, in a strategy profile that survives iterated regret minimization, a player is not making a best response to the strategies used by the other players since, intuitively, he does not know what these strategies are. As a result, a player chooses a strategy that ensures that he does reasonably well compared to the best he could have done, no matter what the other players do. We shall see the impact of this in the examples of Section 3.4.

We remark that Renou and Schlag (2010) (independently) also considered regret in games. They provide a different solution concept for strategic games based on regret minimization; among other things, their approach deals with mixed strategies in a different way than we do. See Section 5 and Appendix C for detailed comparisons.

3.2. Comparison to other solution concepts involving iterated deletion

Iterated deletion has been applied in other solution concepts. We mention three here. Given a set $\mathcal{S}$ of strategies, a strategy $\sigma \in \mathcal{S}_i$ is *weakly dominated* by $\tau \in \mathcal{S}_i$ with respect to $\mathcal{S}$ if, for some strategy $\vec{\sigma}_{-i} \in \mathcal{S}_{-i}$, we have $U_i(\sigma, \vec{\sigma}_{-i}) < U_i(\tau, \vec{\sigma}_{-i})$ and, for all strategies $\vec{\sigma}'_{-i} \in \mathcal{S}_{-i}$, we have $U_i(\sigma, \vec{\sigma}'_{-i}) \leqslant U_i(\tau, \vec{\sigma}'_{-i})$. Similarly, σ is *strongly dominated* by τ with respect to $\mathcal{S}$ if $U_i(\sigma, \vec{\sigma}'_{-i}) < U_i(\tau, \vec{\sigma}'_{-i})$ for all strategies $\vec{\sigma}'_{-i} \in \mathcal{S}_{-i}$. Thus, if σ is weakly dominated by τ with respect to $\mathcal{S}_{-i}$, then i always does at least as well with τ as with σ, and sometimes does better (given that we restrict to strategies in $\mathcal{S}_{-i}$); if σ is strongly dominated by τ, then player i always does better with τ as with σ. Let $\mathcal{WD}_i(\mathcal{S})$ (resp., $\mathcal{SD}_i(\mathcal{S})$) consist of all strategies $\sigma_i \in \mathcal{S}_i$ that are not weakly (resp., strongly) dominated by some strategy in $\mathcal{S}_i$ with respect to $\mathcal{S}$. We can then define the *pure strategies that survive iterated weak* (*resp., strong*) *deletion with respect to pure strategies* as $\mathcal{WD}(A)$ (resp., $\mathcal{SD}(A)$). And again, we can start with Σ to get corresponding notions for mixed strategies.

As is well known (Pearce, 1984), the *rationalizable* strategies can also be considered as the outcome of an iterated deletion process, which results in the same set of strategy profiles as iterated strong deletion with respect to mixed strategies. Intuitively, a pure strategy for player i is rationalizable if it is a best response to some beliefs that player i may have about the pure strategies that other players are following. Given a set $\mathcal{S}$ of pure strategy profiles, $\sigma \in \mathcal{S}_i$ is *justifiable* if there is some distribution μ on the strategies in $\mathcal{S}_{-i}$ such that σ is a best response to the resulting mixed strategy. Intuitively, μ describes player i's beliefs about the likelihood that other players are following various strategies; thus, a strategy σ for i is justifiable if there are beliefs that i could have to which σ is a best response. Let $\mathcal{J}_i(\mathcal{S})$ consist of all strategies for player i that are justifiable with respect to $\mathcal{S}$. A pure strategy σ for player i is *rationalizable* if $\sigma \in \mathcal{J}_i^\infty(A)$.[1]

Like iterated deletion of weakly dominated strategies, but unlike iterated deletion of strongly dominated strategies (and thus also rationalizability), the order of deletion matters for regret. If we delete only some of the strategies that can be deleted at a given step, we may end up with a different outcome; see Example 3.16.

3.3. An epistemic characterization of iterated regret minimization

Traditional solution concepts typically assume common knowledge of rationality, or at least a high degree of mutual knowledge of rationality. For example, it is well known that rationalizability can be characterized in terms of common knowledge of rationality (Tan and Werlang, 1988), where a player is rational if he has some beliefs according to which what he does is a best response in terms of maximizing expected utility; Aumann and Brandenburger (1995) show that mutual knowledge of rationality (together with mutual knowledge of the payoffs and the players' beliefs) suffices for Nash equilibrium (where, again, rationality means playing a utility-maximizing best response); and Brandenburger et al. (2008) show that iterated deletion of weakly dominated strategies requires sufficiently high mutual assumption of rationality, where "assumption" is a variant of "knowledge", and "rationality" means "does not play a weakly dominated strategy". But if we make this assumption (and identify rationality with minimizing regret), we seem to run into a serious problem with iterated regret minimization, which is well illustrated by the Traveler's Dilemma. As we observed earlier, the strategy profile $(97, 97)$ is the only one that survives iterated regret minimization when $p = 2$. However, if agent 1 knows that player 2 is playing 97, then he should play 96, not 97! That is, among all strategies, 97 is certainly not the strategy that minimizes regret with respect to $\{97\}$.

Some of these difficulties also arise when dealing with iterated deletion of weakly dominated strategies. The justification for deleting a weakly dominated strategy is the existence of other strategies. But this justification may disappear in later deletions. As Mas-Colell et al. (1995, p. 240) put in their textbook when discussing iterated deletion of weakly dominated strategies:

> The argument for deletion of a weakly dominated strategy for player i is that he contemplates the possibility that every strategy combination of his rivals occurs with positive probability. However, this hypothesis clashes with the logic of iterated deletion, which assumes, precisely, that eliminated strategies are not expected to occur.

Brandenburger et al. (2008) resolve this paradox in the context of iterated deletion of weakly dominated strategies by assuming that strategies were not really eliminated. Rather, they assumed that strategies that are weakly dominated occur with infinitesimal (but nonzero) probability. This is formally modeled in a framework where uncertainty is captured using a lexicographic probability system (LPS) (Blume et al., 1991), whose support consists of all types. (Recall that an LPS is a sequence $(\mu_0, \mu_1, \ldots)$ of probability measures, in this case on type profiles, where μ_1 represents events that have infinitesimal probability relative to μ_0, μ_1 represents events that have infinitesimal probability relative to μ_1, and so on. Thus, a probability of $(1/2, 1/3, 1/4)$ can be identified with a nonstandard probability of $1/2 + \epsilon/3 + \epsilon^2/4$, where ϵ is an infinitesimal.) In this framework, they show that iterated deletion of weakly dominated strategies corresponds to sufficiently high mutual assumption of rationality.

Unfortunately, this approach does not seem to help in the context of iterated regret minimization. Assigning deleted strategies infinitesimal probability will not make 97 a best response to a set of strategies where 97 is given very high

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[1] The notion of rationalizability is typically applied to pure strategies, although the definitions can be easily extended to deal with mixed strategies.

probability. We deal with this problem by essentially reversing the approach taken by Brandenburger, Friedenberg, and Keisler. Rather than assuming common knowledge of rationality, we assign successively lower probability to higher orders of rationality. The idea is that now, with overwhelming probability, no assumptions are made about the other players; with probability ϵ, they are assumed to be rational, with probability ϵ^2, the other players are assumed to be rational and to believe that they are playing rational players, and so on. (Of course, "rationality" is interpreted here as minimizing expected regret.) Thus, players proceed lexicographically. Their first priority is to minimize regret with respect to all strategies; their next priority is to minimize regret with respect to strategies that a rational player would use; and so on. For example, in Traveler's Dilemma, players do not play 96 because they are still assuming that, with overwhelming likelihood, the other player is playing an arbitrary strategy (not 97); 97 is slightly better than the other strategies that minimize regret given the slight likelihood that the other player is minimizing regret.

This approach is consistent with the spirit of Camerer et al.'s (2004) *cognitive hierarchy* model, where the fraction of people with kth-order beliefs declines as a function of k, although not as quickly as this informal discussion suggests.

Since regret minimization is non-probabilistic, the formal model of a lexicographic belief is a countable sequence $(\mathcal{S}^0, \mathcal{S}^1, \ldots)$ of sets of strategy profiles. The strategy profiles in $\mathcal{S}^0$ represent the players' primary beliefs, the strategy profiles in $\mathcal{S}^1$ are the players' secondary beliefs, and so on. (We can think of $\mathcal{S}^k$ as the support of the measure μ_k in an LPS.[2]) We call $\mathcal{S}^i$ the *level-i belief* of the lexicographic belief $(\mathcal{S}^0, \mathcal{S}^1, \ldots)$.

Given such lexicographic beliefs, what strategy should a rational player i choose? Clearly the most important thing is to minimize regret with respect to his primary beliefs, $\mathcal{S}^0_{-i}$. But among strategies that minimize regret with respect to $\mathcal{S}^0_{-i}$, the best are those strategies that also minimize regret with respect to $\mathcal{S}^1_{-i}$; similarly, among strategies that minimize regret with respect to each of $\mathcal{S}^1_{-i}, \ldots, \mathcal{S}^{k-1}_{-i}$, the best are those that also minimize regret with respect to $\mathcal{S}^k_{-i}$. In other words, level 0 beliefs are used to identify an initial set of strategies that minimize regret, level 1 beliefs are used to "break ties" among those, and so on. Formally, a strategy σ for player i is *rational with respect to a lexicographic sequence* $(\mathcal{S}^0, \mathcal{S}^1, \ldots)$ if there exists a sequence $(\mathcal{T}^0, \mathcal{T}^1, \ldots)$ of strategy profiles such that $\mathcal{T}^0_i$ consists of all strategies τ such that τ_i minimizes regret with respect to $\mathcal{S}^0_{-i}$ for all players i; and $\mathcal{T}^k$ for $k > 0$ is defined inductively to consist of all strategies $\tau \in \mathcal{T}^{k-1}$ such that τ_i has the least regret with respect to $\mathcal{S}^k_{-i}$ among all strategies in $\mathcal{T}^{k-1}_i$; and $\sigma \in \bigcap_{k=0}^{\infty} \mathcal{T}^k_i$.[3] Of course, this definition makes perfect sense if the lexicographic sequence is finite and has the form $(\mathcal{S}^0, \ldots, \mathcal{S}^k)$; in that case we consider $(\mathcal{T}^0, \ldots, \mathcal{T}^k)$. Such a sequence $(\mathcal{S}^0, \ldots, \mathcal{S}^k)$ is called a $(k+1)$*st-order lexicographic belief*. It easily follows that $\emptyset \neq \cdots \subseteq \mathcal{T}^k \subseteq \mathcal{T}^{k-1} \subseteq \cdots \subseteq \mathcal{T}^0$, so that a strategy that is rational with respect to $(\mathcal{S}^0, \mathcal{S}^1, \ldots)$ is also rational with respect to each of the finite prefixes $(\mathcal{S}^0, \mathcal{S}^1, \ldots)$.

Up to now, we have not imposed any constraints on *justifiability* of beliefs. We provide a recursive definition of justifiability. A (kth-order) lexicographic belief $(\mathcal{S}^j)_{j \in I}$ is *justifiable* if, for each $j \in I$, the level-j belief $\mathcal{S}_j$ is *level-j justifiable*, where level-j justifiability is defined as follows.

- To capture the intuition that players' primary beliefs are such that they make no assumptions about the other players, we say that a belief $\mathcal{S}^0_i$ is *level-0 justifiable* if it is the full set of strategies $\mathcal{S}_i$ available to player i.[4]
- To capture the intuition that players' level-k belief is that the other players are $(k-1)$st-order rational, we say that a belief, $\mathcal{S}^k_i$, is *level-k justifiable* if there exists some justifiable kth-order belief $(\mathcal{S}'^0, \mathcal{S}'^1, \ldots, \mathcal{S}'^{k-1})$ such that $\mathcal{S}^k_i$ is the set of rational strategies for player i with respect to $(\mathcal{S}'^0_{-i}, \mathcal{S}'^1_{-i}, \ldots, \mathcal{S}'^{k-1}_{-i})$.

This notion of justifiability captures the intuition that, with probability ϵ^k, each player j_k believes that each other player j_{k-1} is rational with respect to a kth-order belief and believes that, with probability ϵ^{k-1}, each other player j_{k-2} is rational with respect to a $(k-1)$st-order belief and believes that, with probability $\epsilon^{k-2}, \ldots$, and with probability ϵ believes that each other player j_1 is rational with respect to a first-order belief and believes that each other player j_0 is playing an arbitrary strategy in $\mathcal{S}^0$. (As usual, "rationality" here means "minimizes regret with respect to his beliefs".)

Given these definition, we have the following theorem.

Theorem 3.4. *Let $G = ([n], A, \vec{u})$ be a strategic game and let $\mathcal{S}$ be either A or Σ (i.e., either all pure strategy profiles or all mixed strategy profiles). Then for each $k \in N$ there exists a unique level-k justifiable belief $\tilde{\mathcal{S}}^k = \mathcal{RM}^{k-1}(\mathcal{S})$. Furthermore, $\mathcal{RM}^{\infty}(\mathcal{S})$ is the set of rational strategies with respect to the belief $(\tilde{\mathcal{S}}^0, \tilde{\mathcal{S}}^1, \ldots)$ and $\mathcal{RM}^k(\mathcal{S})$ is the set of rational strategies with respect to the belief $(\tilde{\mathcal{S}}^0, \tilde{\mathcal{S}}^1, \ldots, \tilde{\mathcal{S}}^k)$.*

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[2] Like LPS's, this model implicitly assumes, among other things, that players i and j have the same beliefs about players $j' \notin \{i, j\}$. To get an epistemic characterization of a more general setting, where players' initial beliefs about other players strategies are not commonly known, we need a slightly more general model of beliefs, where each player has his or her own lexicographic sequence; see Section 3.5.

[3] A straightforward argument by induction shows that $\mathcal{T}^k$ is nonempty and compact, so that there will be a strategy τ_i that has the least regret with respect to $\mathcal{S}^k_{-i}$ among all strategies in $\mathcal{T}^{k-1}_i$.

[4] As we discuss in Section 3.5, we can also consider a more general model where players have prior beliefs; in such a setting, $\mathcal{S}^0_i$ need not be the full set of strategies.

Proof. By definition there is a unique level-0 justifiable belief $\tilde{\mathcal{S}}^0 = \mathcal{S}$. It inductively follows that there exists a unique level-k justifiable belief $\tilde{\mathcal{S}}^k = \mathcal{RM}(\mathcal{S}^{k-1}) = \mathcal{RM}^{k-1}(\mathcal{S})$. The theorem then follows from the definition of rationality with respect to a lexicographic belief. $\quad\square$

Note that the sets $(\tilde{\mathcal{S}}^1, \tilde{\mathcal{S}}^2, \ldots)$ are just the sets $(\mathcal{T}^0, \mathcal{T}^1, \ldots)$ given in the definition of rationality.

In Appendix A, we provide an alternative epistemic characterization of iterated regret minimization in terms of Kripke structures. The idea is to say that, at level 0, each player has no idea what strategy the other players are using; at level 1, the player knows that the other players are rational, but that is all he knows; at level 2, the player knows that other players are rational and believe, at level 1, that other players are rational, but that is all he knows; and so on. This is made precise using ideas from Halpern and Pass (2009).

3.4. Examples

We now consider the outcome of iterated regret minimization in a number of standard games, showing how it compares to the strategies recommended by other solution concepts. We start by considering what happens if we restrict to pure strategies, and then consider mixed strategies. As we show, for many games of interest, regret does a better job of predicting what people do than other solution concepts (although we also give some games where Nash equilibrium does an arguably better job than regret). While there have been many explanations for peoples' behavior in various games, we are not aware of any single explanation that deals with such a wide variety of games.

3.4.1. Pure strategies

Example 3.5. *Traveler's Dilemma:* If $G = ([n], A, \vec{u})$ is the Traveler's Dilemma, then using the arguments sketched in the introduction, we get that $\mathcal{RM}_i^\infty(A) = \mathcal{RM}_i^2(A) = \{100 - 2p + 1\}$ if $p \leqslant 48$. As we mentioned, $(2, 2)$ is the only action profile (and also the only mixed strategy profile) that is rationalizable (resp., survives iterated deletion of weakly dominated strategies, is a Nash equilibrium). Moreover, if we allow an action to be deleted if it is strongly dominated by a mixed strategy, then $(2, 2)$ is the only action profile that survives iterated deletion of strongly dominated strategies. (This is not surprising, since it is well known (Pearce, 1984) that a strategy survives iterated deletion of strongly dominated strategies iff it is rationalizable.) Thus, iterated regret minimization is quite different from all these other approaches in the Traveler's Dilemma, and gives results that are in line with experimental observations.

Interestingly, although $(2, 2)$ is the only Nash equilibrium, it is a rather fragile equilibrium. If, rather than assuming common belief in rationality, we assume only common p-belief (Monderer and Samet, 1989) for even a relatively small p such as 0.02 (where common p-belief of a statement ϕ holds if everyone believes with probability $1 - p$ that everyone believes with probability $1 - p \ldots$ that ϕ holds), then almost every bid can be justified. But this leaves us with the problem of choosing what will be played; iterated regret minimization provides some insight.

Example 3.6. *Centipede Game:* Another well-known game for which traditional solution concepts provide an answer that is not consistent with empirical observations is the *Centipede Game* (Rosenthal, 1982). In the Centipede Game, two players play for a fixed number k of rounds (known at the outset). They move in turns; the first player moves in all odd-numbered rounds, while the second player moves in even-numbered rounds. At her move, a player can either stop the game, or continue playing (except at the very last step, when a player can only stop the game). For instance, if $k = 3$, in the first round, player 1 may either continue or stop, in the second round, player 2 may either continue or stop, and in the final, third round, player 1's only option is to stop. For all t, player 1 prefers the stopping outcome in round $2t + 1$ (when she moves) to the stopping outcome in round $2t + 2$; similarly, for all t, player 2 prefers stopping in round $2t$ (when he moves) to player 1 stopping in round $2t + 1$. However, for all t, stopping in round $t + 2$ leads to a better outcome for both players than stopping in round t.

Consider two versions of the Centipede Game. The first has *exponential* payoffs. In this case, the utility of stopping at odd-numbered rounds t is $(2^t + 1, 2^{t-1})$, while the utility of stopping at even-numbered rounds is $(2^{t-1}, 2^t + 1)$. Thus, if player 1 stops at round 1, player 1 gets 3 and player 2 gets 1; if player 2 stops at round 4, then player 1 gets 8 and player 2 gets 17; if player 2 stops at round 20, the both players get over 500,000. In the version with *linear payoffs with punishment* $p > 1$, if t is odd, the payoff is $(t, t - p)$, while if t is even, the payoff is $(t - p, t)$.

The game can be described as a strategic game where A_i is the set of strategies for player i in the extensive-form game. It is straightforward to show (by backwards induction) that the only strategy profiles that survive iterated deletion of weakly dominated strategies are ones where player 1 stops at the first move and player 2 stops at the second move. Moreover, in all Nash equilibria, the first player stops at the first move. By way of contrast, a large number of strategies are rationalizable. Indeed, it is not hard to show that all pure strategies where player 1 quits at some point (including the one where he continues until to his last move, and then quits) are rationalizable. For example, it is rationalizable for player 2 to continue until the end (since he can believe that, with probability 1, player 1 will stop at the first move, so that his strategy is irrelevant). Given that this is rationalizable, it is also rationalizable for player 1 to continue to his last move and then

quit. Thus, Nash equilibrium seems to eliminate too many strategies, while rationalizability eliminates too few.[5] In empirical tests, subjects usually cooperate for a certain number of rounds, although it is rare for them to cooperate throughout the whole game (McKelvey and Palfrey, 1992; Nagel and Tang, 1998). As we now show, using iterated regret minimization, with linear payoffs, we also get cooperation for a number of rounds (which depends on the penalty); with exponential payoffs, we get cooperation up to the end of the game. Adding some minimal prior beliefs (see Section 3.5) leads to cooperation almost to the end of the game, but not all the way to the end. Our results suggest some further experimental work, with regard to the sensitivity of the game to the payoffs and prior beliefs.

Before going on, note that, technically, a strategy in the Centipede Game must specify what a player does whenever he is called upon to move, including cases where he is called upon to move after he has already stopped the game. Thus, if t is odd and $t + 2 < k$, there is more than one strategy where player 1 stops at round t. For example, there is one where player 1 also stops at round $t + 2$, and another where he continues at round $t + 2$. However, all the strategies where player 1 first stops at round t are payoff equivalent for player 1 (and, in particular, are equivalent with respect to regret minimization). We use $[\![t]\!]$ to denote the set of strategies where player 1 stops at round t, and similarly for player 2; additionally, we let $[\![k + 1]\!]$ denote the strategy for the player moving in round $k - 1$ of always continuing. It is easy to see that in the k-round Centipede Game with exponential payoffs, the unique strategy that minimizes regret is to stop at the last possible round. On the other hand, with linear payoffs and punishment p, the situation is somewhat similar to the Traveler's Dilemma. All strategies (actions) for either player that stop at or after stage $k - p + 1$ have regret $p - 1$, which is minimal, but what happens with iterated deletion depends on whether k and p are even or odd. For example, if k and p are both even, then $\mathcal{RM}_1(A) = \{[\![k - p + 1]\!], [\![k - p + 3]\!], \ldots, [\![k + 1]\!]\}$ and $\mathcal{RM}_2(A) = \{[\![k - p + 2]\!], [\![k - p + 4]\!], \ldots, [\![k]\!]\}$. Relative to $\mathcal{RM}(A)$, the strategies in $[\![k - p + 1]\!]$ have regret $p - 2$; the remaining strategies have regret $p - 1$. Thus, $\mathcal{RM}_1^2(A) = \mathcal{RM}_1^\infty(A) = \{[\![k - p + 1]\!]\}$. Similarly, $\mathcal{RM}_2^2(A) = \mathcal{RM}_2^\infty(A) = \{[\![k - p + 2]\!]\}$.

If, on the other hand, both k and p are odd, $\mathcal{RM}_1(A) = \{[\![k - p + 1]\!], [\![k - p + 3]\!], \ldots, [\![k]\!]\}$ and $\mathcal{RM}_2(A) = \{[\![k - p]\!], [\![k - p + 2]\!], \ldots, [\![k + 1]\!]\}$. But, here, iteration does not remove any strategies (since $k - p + 1$ still has regret $p - 1$ for player 1, and $k - p$ still has regret $p - 1$ for player 2). Thus, $\mathcal{RM}_1^\infty(A) = \mathcal{RM}_1(A)$ and $\mathcal{RM}_2^\infty(A) = \mathcal{RM}_2(A)$. Note that none of player 2's strategies are removed; in particular the strategy $[\![k + 1]\!]$ of always continuing survives iterated regret minimization, although it is strictly dominated by stopping in round $k - 1$ (since player 1 must stop in round k). As mentioned above, if we consider iterated regret minimization with prior beliefs (see Section 3.5), we may exclude all strictly dominated strategies from the initial strategy space; doing this yields similar results but now $[\![k + 1]\!]$ is excluded for player 2.

Example 3.7. *Matching pennies:* Suppose that $A_1 = A_2 = \{a, b\}$, and consider the matching pennies game, with payoffs as given in the table below:

	a	b
a	$(80, 40)$	$(40, 80)$
b	$(40, 80)$	$(80, 40)$

Since the players have opposite interest there are no pure strategy Nash equilibria. Randomizing with equal probability over both actions is the only Nash equilibria; this is consistent with experimental results (see e.g., Goeree and Holt, 2001). With regret minimization, both actions have identical regret for both players, thus using regret minimization (with respect to pure strategies) both actions are viable.

Consider a variant of this game (called the *asymmetric matching pennies*, Goeree et al., 2000), where $u(a, a) = (320, 40)$.

	a	b
a	$(320, 40)$	$(40, 80)$
b	$(40, 80)$	$(80, 40)$

Here, the unique Nash equilibrium is one where player 1 still randomizes between a and b with equal probability, but player 2 picks b with probability 0.875 (Goeree et al., 2000). Experimental results by Goeree and Holt (2001) show quite different results: player 1 chooses a with probability 0.96; player 2, on the other hand, is consistent with the Nash equilibrium and chooses b with probability 0.86. In other words, players most of the time end up with the outcome (a, b). With iterated regret minimization, we get a qualitatively similar result. It is easy to see that in the first round of deletion, a minimizes the regret for player 1, whereas both a and b minimize the regret for player 2; thus $\mathcal{RM}_1^1(A) = a$ and $\mathcal{RM}_2^1(A) = a, b$. In the second round of the iteration, b is the only action that minimizes regret for player 2. Thus, $\mathcal{RM}^2(A) = \mathcal{RM}^\infty = (a, b)$; that is, (a, b) is the only strategy profile that survives iterated deletion.

Note that we would get exactly the same prediction even if we change $u(a, a)$ from $(320, 40)$ to $(80 + \epsilon, 40)$, where $\epsilon > 0$. This seems like an unnecessarily strong (and arguably unreasonable) prediction. More intuitive predictions are obtained once we consider iterated regret minimization with mixed strategies; see Example 3.21.

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[5] It is also well known that if we consider Pearce (1984) notion of *extensive-form rationalizability*, then only the backward induction outcome is rationalizable. That is, extensive-form rationalizability eliminates even more strategies than Nash equilibrium.

Example 3.8. *Coordination games:* Suppose that $A_1 = A_2 = \{a, b\}$, and $u(a, a) = (k_1, k_2)$, $u(b, b) = (1, 1)$, $u(a, b) = u(b, a) = (0, 0)$, where $k_1, k_2 > 0$, as shown in the table below:

	a	b
a	(k_1, k_2)	$(0, 0)$
b	$(0, 0)$	$(1, 1)$

Both (a, a) and (b, b) are Nash equilibria, but (a, a) *Pareto dominates* (b, b) if $k_1, k_2 > 1$: both players are better off with the equilibrium (a, a) than with (b, b). With regret minimization, we do not have to appeal to Pareto dominance if we stick to pure strategies. It is easy to see that if $k_1, k_2 > 1$, $\mathcal{RM}_1^1(A) = \mathcal{RM}_2^1(A) = \mathcal{RM}_1^\infty(A) = \mathcal{RM}_2^\infty(A) = \{a\}$ (yielding regret 1). Similarly, if $k_1, k_2 < 1$, then $\mathcal{RM}_1^\infty(A) = \mathcal{RM}_2^\infty(A) = \{b\}$, and if $k_1 = k_2 = 1$, $\mathcal{RM}_1^1(A) = \mathcal{RM}_2^1(A) = \mathcal{RM}_1^\infty(A) = \mathcal{RM}_2^\infty(A) = \{a, b\}$.

Finally, if $k_1 > 1$, and $k_2 < 1$, then the unique profile that minimizes regret is (a, b), which results in both players getting a payoff of 0. While this is unfortunate, in the absence of any coordinating device, this may well be the outcome that is played between two players meeting for the first time. On the other hand, if k_1 is huge, say 200, whereas k_2 is almost 1, the outcome (a, b) might seem unreasonable. Intuitively, by inspecting the game, player 2 should conclude that player 1 will play a with high probability. Roughly speaking, the game itself provides player 2 with a prior belief about player 1. We consider regret minimization with such prior beliefs in Section 3.5.

Example 3.9. *Bertrand competition:* Bertrand competition is a 2-player game where the players can be viewed as firms producing a homogeneous good. There is demand for 100 units of the good at any price up to \$200. (Unlike standard descriptions of Bertrand competition, for simplicity, we assume that prices are in whole dollar amounts, rather than varying continuously.) If both firms charge the same price, each will sell 50 units of the good. Otherwise, the firm that charges the lower price will sell 100 units at that price. Each firm has a cost of production of 0, as long as it sells for a positive price, it makes a profit. It is easy to see that the only Nash equilibria of this game are $(0, 0)$, $(1, 1)$, $(0, 1)$, $(1, 0)$, and $(2, 2)$. But it does not seem so reasonable that firms playing this game only once will charge \$2 or less, when they could charge up to \$200. And indeed, experimental evidence shows that people in general choose significantly higher prices (Dufwenberg and Gneezy, 2000).

Now consider regret. Suppose that firm 1 charges $n \geqslant 1$. If firm 2 charges $m > 1$, then the best response is for firm 1 to charge $m - 1$. If $m > n$, then firm 1's regret is $(m - 1 - n)100$; if $m = n > 1$, firm 1's regret is $(n/2 - 1)100$; if $m = n = 1$, firm 1's regret is 0; and if $m < n$, firm 1's regret is $(m - 1)100$. If $m = 1$, firm 1's best response is to charge 1, and the regret is 0 if $n = 1$ and 100 if $n > 1$. It follows that firm 1's regret is $\max((199 - n)100, (n - 2)100)$. Clearly if $n = 0$, firm 1's regret is 199×100 (if firm 2 charges 200). Thus, firm 1 minimizes regret by playing 100 or 101, and similarly for firm 2. A second round of regret minimization, with respect to $\{100, 101\}$, leads to 100 as the unique strategy that results from iterated regret minimization.

Example 3.10. *The Nash bargaining game (Nash, 1950):* In this 2-player game, each player must choose an integer between 0 and 100. If player 1 chooses x and player 2 chooses y and $x + y \leqslant 100$, then player 1 receives x and player 2 receives y; otherwise, both players receive 0. All strategy profiles of the form $(x, 100 - x)$ are Nash equilibria. The problem is deciding which of these equilibria should be played. Nash (1950) suggested a number of axioms, for which it followed that $(50, 50)$ was the unique acceptable strategy profile.

Using iterated regret minimization leads to the same result, without requiring additional axioms. Using arguments much like those used in the case of Bertrand competition, it is easy to see that the regret of playing x is $\max(100 - x, x - 1)$. If the other player plays $y \leqslant 100 - x$, then the best response is $100 - y$, and the regret is $100 - y - x$. Clearly, the greatest regret is $100 - x$, when $y = 0$. On the other hand, if the other player plays $y > 100 - x$, then the best response is $100 - y$, so the regret is $100 - y$. The greatest regret comes if $y = 100 - x + 1$, in which case the regret is $x - 1$. It follows that regret is minimized by playing either 50 or 51. Iterating regret minimization with respect to $\{50, 51\}$ leaves us with 50. Thus, $(50, 50)$ is the only strategy profile that survives iterated regret minimization.

We have implicitly assumed here that the utility of a payoff of x is just x. If, more generally, it is $u(x)$, and u is an increasing function, then the same argument shows that the regret is $\max(u(100) - u(x), u(x - 1) - u(0))$. Again, there will be either one value for which the regret is maximized (as there would have been above if we had taken the total to be 99 instead of 100) or two consecutive values. A second round of regret minimization will lead to a single value; that is, again there will be a single strategy profile of the form (x, x) that survives iterated regret minimization. However, x may be such that $2x < 100$ or $2x > 100$. This can be viewed as a consequence of the fact that, in a strategy profile that survives iterated regret minimization, a player is not making a best response to what the other players are doing.

We next show that iterated regret and Nash equilibrium agree on Prisoner's Dilemma. This follows from a more general observation, that iterated regret always recommends a dominant action. A *dominant action* a for player i is one such that $u_i(a, \vec{b}_{-i}) \geqslant u_i(a', \vec{b}_{-i})$ for all $a' \in A_i$ and $\vec{b} \in A$. We can similarly define a dominant (mixed) strategy. It is easy to see that dominant actions survive iterated deletion of weakly and of strongly dominated actions with respect to A, and are rationalizable. Indeed, if there is a dominant action, the only actions that survive one round of iterated deletion of weakly

dominated strategies are dominant actions. Similar observations hold in the case of mixed strategies. The next result shows that iterated regret minimization acts like iterated deletion of weakly dominated strategies in the presence of dominant actions and strategies.

Proposition 3.11. *Let $G = ([n], A, \vec{u})$ be a strategic game. If player i has a dominant action a_i, then*

(a) $\mathcal{RM}_i(A) = \mathcal{RM}_i^{\infty}(A)$;
(b) $\mathcal{RM}_i(A)$ *consists of the dominant actions in G.*

Proof. Every action that is dominant has regret 0 (which is minimal); furthermore, only dominant actions have regret 0. It follows that $\mathcal{RM}_i(A)$ consists of the dominant actions for i (if such actions exist). Since none of these actions will be removed in later deletions, it follows $\mathcal{RM}_i(A) = \mathcal{RM}_i^{\infty}(A)$. □

Example 3.12. *(Repeated) Prisoner's Dilemma:* Recall that in Prisoner's Dilemma, players can either cooperate (c) or defect (d). The payoffs are $u(d,d) = (u_1, u_1)$, $u(c,c) = (u_2, u_2)$, $u(d,c) = (u_3, 0)$, $u(c,d) = (0, u_3)$, where $0 < u_1 < u_2 < u_3$ and $u_2 > u_3/2$ (so that alternating between (c,d) and (d,c) is not as good as always cooperating). It is well known (and easy to check) that d is the only dominant action for both 1 and 2, so it follows by Proposition 3.11 that traditional solutions concepts coincide with iterated regret minimization for this game.

Things get more interesting if we consider repeated Prisoner's Dilemma. We show that if both players use iterated regret deletion, they will defect in every round, both in finitely and infinitely repeated Prisoner's Dilemma.

First consider Prisoner's Dilemma repeated n times. Let s_{ad} be the strategy where player 1 always defects, and let S consist of all pure strategies in n-round Prisoner's Dilemma.

Claim 3.13. *In n-round repeated Prisoner's Dilemma, $regret_1^S(s_{ad}) = (n-1)(u_3 - u_2) + \max(-u_1, u_2 - u_3)$. Moreover, if s is a strategy for player 1 where he plays c before seeing player 2 play c (i.e., where player 1 either starts out playing c or plays c at the kth for $k > 1$ move after seeing player 2 play d for the first $k-1$ moves), then $regret_1^S(s) > (n-1)(u_3 - u_2) + \max(-u_1, u_2 - u_3)$.*

It follows from Claim 3.13 that the only strategies that remain after one round of deletion are strategies that start out defecting, and continue to defect as long as the other player defects. If the players both play such a strategy, they both defect at every round. Thus, all these strategies that survive one round of deletion survive iterated deletion. It follows that with iterated regret minimization, we observe defection in every round of finitely repeated Prisoner's Dilemma. Essentially the same argument shows that this is true in infinitely repeated Prisoner's Dilemma (where payoffs are discounted by δ, for $0 < \delta < 1$). By way of contrast, while always defecting is the only Nash equilibrium in finitely repeated Prisoner's Dilemma, the Folk Theorem shows that for all p with $0 \leqslant p \leqslant 1$, if δ is sufficiently close to 1, there is an equilibrium in which p is the fraction of times that both players cooperate. Thus, with Nash equilibrium, there is a discontinuity between the behavior in finitely and infinitely repeated Prisoner's Dilemma that does not occur with regret minimization. Nevertheless, intuition suggests that there should be a way to justify cooperation using regret minimization, just as in the case of the Centipede Game. This is indeed the case, as we show in Section 3.5.

Example 3.14. *Hawk–Dove*[6]*:* In this game, $A_1 = A_2 = \{d, h\}$; a player can choose to be a dove (d) or a hawk (h). The payoffs are $u(d,d) = (b,b)$, $u(d,h) = (a,c)$, $u(h,d) = (c,a)$, $u(h,h) = (0,0)$, where $0 < a < b < c$ and $c < a + b$. The game has two Nash equilibria: (h,d) and (d,h). But d is the only action that minimizes regret (yielding regret $c - b$). Thus, $\mathcal{RM}(A) = \mathcal{RM}^{\infty}(A) = (d,d)$.

Example 3.15. The game described below shows that the strategies that survive iterated deletion of strongly dominated strategies by pure strategies can be disjoint from those that survive iterated regret minimization.

	x	y
a	(0, 100)	(0, 0)
b	(1, 0)	(1, 1)

First consider iterated deletion of strongly dominated strategies. For player 1, b strongly dominates a. Once a is deleted, y strongly dominates x for player 2. Thus, iterated deletion of strongly dominated strategies leads to the unique strategy profile (b, y). Now consider regret minimization. The regret of x is less than that of y, while the regret of b is less than that of a. Thus, iterated regret minimization leads to (b, x).

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[6] This game is sometimes known as *Chicken*.

Example 3.16. The following example shows that order of deletion matters. Consider a 2-player game where $A_1 = \{a_1, a_2, a_3\}$, $A_2 = \{b_1, b_2, b_3\}$, no matter what the action profile, player 2's payoff is 1 (so that the game is effectively a one-player game), and player 1's payoffs are given in the table below:

	b_1	b_2	b_3
a_1	5	5	−1
a_2	2	2	0
a_3	−1	−10	3

It is easy to see that a_1 has regret 4, a_2 has regret 3, and a_3 has regret 15. Thus, a_2 is the action that minimizes regret. However, if we first delete only a_3, then among a_1 and a_2, a_1 has regret 1 and a_2 has regret 3, so a_2 is deleted, leaving a_1. This example also illustrates what has been perceived as a problem with regret. Suppose that the agent has only actions a_1 and a_2 available. Then the action that minimizes regret, as we have seen, is a_1. But now suppose that we add a_3 to his set of available options, an action that is dominated (as far as regret goes) by both a_1 and a_2. Although a_3 is dominated, as we have seen, it affects the regret of both a_1 and a_2 (and makes a_2 the action that minimizes regret). The fact that regret does not satisfy independence of "irrelevant" alternatives has been viewed as a problem. We do not view it as a problem; indeed, the psychology literature shows that people in fact do not view actions such as a_3 as irrelevant.[7]

Note that in the examples above, the deletion process converges after (at most) two steps. We can construct examples of games where we need $\max(|A_1| - 1, \ldots, |A_n| - 1)$ deletion steps. The following example shows this in the case that $n = 2$ and $|A_1| = |A_2|$, and arguably illustrates some additional problems with iterated regret minimization.

Example 3.17. Consider a symmetric 2-player game with $A_1 = A_2 = \{a_1, \ldots, a_n\}$. If both players play a_k, then the payoff for each one is k. For $k > 1$, if one player plays a_k and the other plays a_{k-1}, then the player who plays a_k gets -2, and the other gets 0. In all other cases, both players get payoff of 0.

	a_1	a_2	…	a_{n-1}	a_n
a_1	$(1, 1)$	$(0, -2)$	…	$(0, 0)$	$(0, 0)$
a_2	$(-2, 0)$	$(2, 2)$	…	$(0, 0)$	$(0, 0)$
…			…		
a_{n-1}	$(0, 0)$	$(0, 0)$	…	$(n-1, n-1)$	$(0, -2)$
a_n	$(0, 0)$	$(0, 0)$	…	$(-2, 0)$	(n, n)

The strategy a_n has regret $n+1$ (if, for example, player 1 plays a_n and player 2 plays a_{n-1}, player 1 could have gotten $n+1$ more by playing a_{n-1}); all other actions have regret n. Thus, only a_n is eliminated in the first round. Similar considerations show that we eliminate a_{n-1} in the next round, then a_{n-2}, and so on. Thus, a_1 is the only pure strategy that survives iterated regret minimization.

Note that (a_k, a_k) is a Nash equilibrium for all k. Thus, the strategy that survives iterated regret minimization is the Nash equilibrium that is Pareto dominated by all other Nash equilibria. We get a similar result if we modify the payoffs so that if both players play a_k, then they both get 2^k, while if one player plays a_k and the other plays a_{k-1}, then the one that plays a_k gets $-2 - 2^{k-1}$, and the other gets 0. Suppose that $n = 20$ in the latter game. Would players really accept a payoff of 2 when they could get a payoff of over 1,000,000 if they could coordinate on a_{20}?

The following variant of a *generalized coordination game* demonstrates the same effect even without iteration.

	a	b
a	$(1, 1)$	$(0, -10)$
b	$(-10, 0)$	$(10, 10)$

Clearly (b, b) is the Pareto optimal Nash equilibrium, but playing b has regret 11, whereas a has regret only 10; thus (a, a) is the only profile that minimizes regret. Note, however, that (a, a) is the *risk dominant* Nash equilibrium. (Recall that in a *generalized coordination game*—a 2-player, 2-action game where $u_1(a, a) > u_1(b, a)$, $u_1(b, b) > u_1(a, b)$, $u_2(a, a) > u_2(a, b)$, and $u_2(b, b) > u_2(b, a)$—the Nash equilibrium (a, a) is risk dominant if the product of the "deviation losses" for (b, b) (i.e., $(u_1(b, a) - u_1(b, b))(u_2(a, b) - u_2(b, b))$) is higher than the product of the deviation losses for (a, a).) In the game above, the product of the deviation losses for (b, b) is $100 = (0 - 10)(0 - 10)$, while the product of the deviation losses for (a, a) is $121 = (-10 - 1)(-10 - 1)$; thus, (a, a) is risk dominant. In fact, in every generalized coordination game, the product of the deviation losses for (x, x) is $regret_1(x)regret_2(x)$, so if the game is symmetric (i.e., if $u_1(x, y) = u_2(y, x)$, which implies that $regret_1(x) = regret_2(x)$), the risk dominant Nash equilibrium is the only action profile that minimizes regret. (It is easy to see that this is no longer the case if the game is asymmetric.)

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[7] Stoye (2007) gives an axiomatization for regret minimization in decision theory that involves independence of *never-optimal* alternatives, where an alternative is never-optimal if it is not optimal in any state. In our context, this means that the action is not optimal no matter what the other players do. The action a_3 is optimal if player 2 plays b_3, so it is not never-optimal.

3.4.2. Mixed strategies

Applying regret in the presence of mixed strategies can lead to quite different results than if we consider only pure strategies. We first generalize Proposition 3.11, to show that if there is a dominant strategy that happens to be pure (as is the case, for example, in Prisoner's Dilemma), nothing changes in the presence of mixed strategies. But in general, things can change significantly.

Proposition 3.18. *Let $G = ([n], A, \vec{u})$ be a strategic game. If player i has a dominant action a_i, then $\mathcal{RM}_i(\Sigma) = \mathcal{RM}_i^{\infty}(\Sigma) = \Delta(\mathcal{RM}_i(A))$.*

Proof. The argument is similar to the proof of Proposition 3.11 and is left to the reader. $\square$

To understand what happens in general, we first show that we need to consider regret relative to only pure strategies when minimizing regret at the first step. (The proof is relegated to Appendix B.)

Proposition 3.19. *Let $G = ([n], A, \vec{u})$ be a strategic game and let σ_i be a mixed strategy for player i. Then $regret_i^{\Sigma}(\sigma_i) = \max_{\vec{a}_{-i} \in A_{-i}} regret^{\mathcal{S}_i}(\sigma_i \mid \vec{a}_{-i})$.*

Example 3.20. *Roshambo (rock–paper–scissors):* In the rock–paper–scissors game, $A_1 = A_2 = \{r, s, p\}$; rock (r) beats scissors (s), scissors beats paper (p), and paper beats rock; $u(a, b)$ is $(2, 0)$ if a beats b, $(0, 2)$ if b beats a, and $(1, 1)$ if $a = b$. If we stick with pure strategies, by symmetry, we have $\mathcal{RM}_1(A) = \mathcal{RM}_2(A) = \mathcal{RM}_1^{\infty}(A) = \mathcal{RM}_2^{\infty}(A) = \{r, s, p\}$. If we move to mixed strategies, it follows by Proposition 3.19 that picking r, s, and p each with probability $1/3$ is the only strategy that minimizes regret. (This is also the only Nash equilibrium.)

Example 3.21. *Matching pennies with mixed strategies:* Consider again the matching pennies game, where $A_1 = A_2 = \{a, b\}$, and $u(a, a) = u(b, b) = (80, 40)$, $u(a, b) = u(b, a) = (40, 80)$. Recall that $\mathcal{RM}_1^{\infty}(A) = \mathcal{RM}_2^{\infty}(A) = \{(a, b)\}$. Consider a mixed strategy that puts weight p on a. By Proposition 3.19, the regret of this strategy is $\max(40(1 - p), 40p)$, which is minimized when $p = \frac{1}{2}$ (yielding regret 20). Thus, randomizing with equal probability over a, b is the only strategy that minimizes regret; it is also the only Nash equilibrium. But now consider the asymmetric matching pennies game, where $u(a, a) = (320, 40)$. Recall that $\mathcal{RM}^{\infty}(A) = (a, b)$. Since the utilities have not changed for player 2, it is still the case that $1/2a + 1/2b$ is the only strategy that minimizes regret for player 2. On the other hand, by Proposition 3.19, the regret of the strategy for player 1 that puts weight p on a is $\max(280(1 - p), 40p)$, which is minimized when $p = 0.875$. Thus, $\mathcal{RM}^{\infty}(\Sigma) = (0.875a + 0.225b, 0.5a + 0.5b)$.

Example 3.22. *Coordination games with mixed strategies:* Consider again the coordination game where $A_1 = A_2 = \{a, b\}$ and $u(a, a) = (k, k)$, $u(b, b) = (1, 1)$, $u(a, b) = u(b, a) = 0$. Recall that if $k > 1$, then $\mathcal{RM}_1^{\infty}(A) = \mathcal{RM}_2^{\infty}(A) = \{(a)\}$, while if $k = 1$, then $\mathcal{RM}_1^{\infty}(A) = \mathcal{RM}_2^{\infty}(A) = \{a, b\}$. Things change if we consider mixed strategies. Consider a mixed strategy that puts weight p on b. By Proposition 3.19, the regret of this strategy is $\max(kp, 1 - p)$ which is minimized when $p = \frac{1}{k+1}$ (yielding regret $\frac{k}{k+1}$). Thus, mixed strategies that minimize regret can put positive weight on actions that have sub-optimal regret.

Example 3.23. *Traveler's Dilemma with mixed strategies:* Consider the case when $p = 2$. As we saw earlier, each of the choices 96–100 has regret 3 relative to other pure strategies. It turns out that there are mixed strategies that have regret less than 3. Consider the mixed strategy that puts probability $1/2$ on 100, $1/4$ on 99, and decreases exponentially, putting probability $1/2^{98}$ on both 3 and 2. Call this mixed strategy σ. Let Σ consist of all the mixed strategies for Traveler's Dilemma.

Claim 3.24. $regret_1^{\Sigma}(\sigma) < 3$.

The proof of Claim 3.24 shows that $regret_1^{\Sigma}(\sigma)$ is not much less than 3; it is roughly $3 \times (1 - 1/2^{101-k})$. Nor is σ the strategy that minimizes regret. For example, it follows from the proof of Claim 3.24 that we can do better by using a strategy that puts probability $1/2$ on 98 and decreases exponentially from there. While we have not computed the exact strategy that minimizes (or strategies that minimize) regret—the computation is nontrivial and does not add much insight—we can make two useful observations:

- The mixed strategy that minimizes regret places probability at most $3/(99 - k)$ on the pure strategies k or less. For suppose player places probability α on the pure strategies k or less. If player 2 plays 100, player 1 could have gotten 101 by playing 99, and gets at most $k + 2$ by playing k or less. Thus, the regret is at least $(99 - k)\alpha$, which is at least 3 if $\alpha \geqslant 3/(99 - k)$. Thus, for example, the probability that 90 or less is played is at most $1/3$.
- The strategy that minimizes regret has regret greater than 2.9. To see this, note that the strategy can put probability at most $3/97$ on 2 and at most $3/96$ on 3. This means that the regret relative to 3 is at least

$$(1 - 3/96 - 3/97)3 + 3/96 = 3 - 6/96 - 3/97 > 2.9.$$

The fact that it is hard to compute the exact strategy that minimizes regret suggests that people are unlikely to be using it. On the other hand, it is easy to compute that the optimal strategy puts high weight on actions in the high 90's. In retrospect, it is also not surprising that one can come close to minimizing regret by putting some weight on (almost) all actions. This was also the case in Example 3.22; as we observed there, we can sometimes do better by putting some weight even on actions that do not minimize regret.[8] Interestingly, the distribution of strategies observed by Becker et al. (2005) is qualitatively similar to the distribution induced by a mixed strategy that is close to optimal. If everyone in the population was playing a mixed strategy that was close to optimal in terms of minimizing regret, we would expect to see something close to the observed distribution.

The phenomena observed in the previous two examples apply to all the other examples considered in Section 3.4.1. For example, in the Bertrand competition, while the pure strategies of least regret (100 and 101) have regret 9,900, there are mixed strategies with regret less than 7,900 (e.g., by putting probability 1/5 on each of 80, 100, 120, 140, and 160). We can do somewhat better than this, but not much. Moreover, we believe that in both Traveler's Dilemma and in Bertrand competition there is a unique mixed strategy that minimizes regret, so that one round of deletion will suffice. This is not true in general, as the following example shows.

Example 3.25. Consider a 2-player symmetric game where $A_1 = A_2 = \{a_{mk}: m = 1, \ldots, n, \ k = 1, 2\}$. Define

$$u_1(a_{ij}, a_{kl}) = \begin{cases} -3^{\max(i,k)} & \text{if } i \neq k, \\ 0 & \text{if } i = k, \ j = l, \\ -3^{i+1} & \text{if } i = k, \ j \neq l. \end{cases}$$

Let Σ consists of all mixed strategies in this game. We claim that, for every strategy σ for player 1, $regret_1^{\Sigma}(\sigma) \geqslant 3^n$, and similarly for player 2. To see this, consider a mixed strategy of the form $\sum_{ij} p_{ij} a_{ij}$. The best response to a_{nj} is a_{nj}, which gives a payoff of 0. Thus, the regret of this strategy relative to a_{n1} is $3^n(\sum_{ij, i \neq n} p_{ij} + 3p_{n2})$. Similarly, the regret relative to a_{n3} is $3^n(\sum_{ij, i \neq n} p_{ij} + 3p_{n1})$. Thus, the sum of the regrets relative to a_{n1} and a_{n2} is $3^n(2 + p_{n1} + p_{n2})$. It follows that the regret relative to one of a_{n1} and a_{n2} is at least 3^n. It also easily follows that every convex combination of strategies a_{ij} with $i < n$ has regret exactly 3^n (the regret relative to a_{n1} is 3^n, and the regret relative to every other strategy is no worse). Moreover, every strategy that puts positive weight on a_{n1} or a_{n2} has regret greater than 3^n. Thus, at the first step, we eliminate all and only strategies that put positive weight on a_{n1} or a_{n2}. An easy induction shows that at the kth step we eliminate all and only strategies that put positive weight on $a_{(n-k+1)1}$ or $a_{(n-k+1)2}$. After $n - 1$ steps of iterated deletion, the only strategies that are convex combinations of a_{11} and a_{12}. One more step of deletion leaves us $(1/2)a_{11} + (1/2)a_{12}$.[9]

In the case of pure strategies, it is immediate that there cannot be more than $|A_1| + \cdots + |A_n|$ rounds of deletion, although we do not have an example that requires more than $\max(|A_1|, \ldots, |A_n|)$ rounds. Example 3.25 shows that $\max(|A_1|, \ldots, |A_n|)$ may be required with mixed strategies, but all that follows from Theorem 3.3 is that the deletion process converges after at most countably many steps. We conjecture that, in fact, the process converges after at most $\max(|A_1|, \ldots, |A_n|)$ steps, both with pure and mixed strategies, but we have not proved this.

3.5. Iterated regret minimization with prior beliefs

We have assumed that we start the deletion process with all pure (resp., mixed) strategy profiles. Moreover, we have assumed that, at all stages in the deletion process (and, in particular, at the beginning), the set of strategies that the agents consider possible is the same for all agents. This does not seem reasonable after a game has been played repeatedly, and players have had a chance to observe the strategies used by other players. Even if a game is played for the first time between players that have never met, as we argue below, the structure of the game may "naturally" signal something about the what strategies the players consider "reasonable".

To model these beliefs, we allow each agent i to start the deletion process with a strict subset Σ^i of the set of all profiles. Intuitively, the strategies in Σ_j^i are the only strategies that i is considering for j. For $j \neq i$, it is perhaps most natural to think of Σ_j^i as representing i's beliefs about what strategies j will use; however, it may also make sense to interpret Σ_j^i as a representative set of j's strategies from i's point of view, or as the only ones that i is considering because it is too complicated to consider them all (see below). For $i = j$, Σ_i^i is the set of strategies that i is still considering using; thus, i essentially ignores all strategies other than those in Σ_i^i. When we do regret minimization with respect to a single set $\mathcal{S}$

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[8] It is not necessarily the case that the support of the optimal strategy consists of all actions, or even all undominated actions. For example, consider a coordination game with three actions a, b, and c, where $u(a, a) = u(b, b) = k$, $u(c, c) = 1$, and $u(x, y) = 0$ if $x \neq y$. If $k > 2$, then the strategy that minimizes regret places probability 1/2 on each of a and b, and probability 0 on c.

[9] In this example, at every step but the last step, the set of strategies that remain consist of all convex combinations of a subset of pure strategies. But this is not necessarily the case. If we replace 3^k by 2^k in all the utilities above, then we do not eliminate all strategies that put positive weight on a_{n1} or a_{n2}; in particular, we do not eliminate strategies that put the *same* weight on a_{n1} and a_{n2} (i.e., where $p_{n1} = p_{n2}$).

of strategy profiles (as we do in the definition of iterated regret minimization), we are implicitly assuming that the players have common beliefs.

The changes required to deal with this generalization are straightforward: each agent simply applies the standard regret minimization operator to his set of strategy profiles. More formally, the generalized regret minimization $\mathcal{RM}'$ takes as an argument a tuple $(\Pi_1, \ldots, \Pi_n)$ of strategy profiles and returns such a tuple; we define $\mathcal{RM}'(\Pi_1, \ldots, \Pi_n) = (\mathcal{RM}(\Pi_1), \ldots, \mathcal{RM}(\Pi_n))$.[10]

Example 3.26. *Repeated Prisoner's Dilemma with prior beliefs:* The role of prior beliefs is particularly well illustrated in repeated Prisoner's Dilemma. In the proof of Claim 3.13, to show that the regret of a strategy like tit for tat is greater than $(n-1)(u_3 - u_2) + \max(-u_1, u_2 - u_3)$, it is necessary to consider a strategy where player 2 starts out by defecting, and then cooperates as long as player 1 defects. This seems like an extremely unreasonable strategy for player 2 to use! Given that there are $2^{2^n - 1}$ pure strategies for each player in n-round Prisoner's Dilemma, and computing the regret of each one can be rather complicated, it is reasonable for the players to focus on a much more limited set of strategies. Suppose that each player believes that the other player is using a strategy where he plays tit for tat for k rounds, and then defects from then on. Call this strategy s_k. (So, in particular, $s_0 = s_{ad}$ and s_n is tit for tat.) Let S_i^* consist of all the strategies s_k for player i; let S_{2-i}^+ be any set of strategies for player $2 - i$ that includes S_{2-i}^*. It is easy to see that the best response to s_0 is s_0, and the best response to s_k for $k > 1$ is s_{k-1} (i.e., you are best off defecting just before the other player starts to defect). Thus,

$$regret_i^{S_i^+ \times S_{2-i}^*}(s_k \mid s_l) = \begin{cases} (l - k - 1)(u_2 - u_1) & \text{if } k < l, \\ u_3 + u_1 - 2u_2 & \text{if } k = l > 0, \\ u_3 + u_1 - u_2 & \text{if } k > l > 0, \\ u_1 & \text{if } k > l = 0, \\ 0 & \text{if } k = l = 0. \end{cases}$$

It follows that

$$regret_i^{S_i^+ \times S_{2-i}^*}(s_k) = \begin{cases} \max((n - k - 1)(u_2 - u_1), u_3 + u_1 - u_2) & \text{if } k \geqslant 2, \\ \max((n - 1)(u_2 - u_1), u_3 + u_1 - 2u_2, u_1) & \text{if } k = 1, \\ n(u_2 - u_1) & \text{if } k = 0. \end{cases}$$

Intuitively, if player 1 plays s_k and player 2 is playing a strategy in S_2^*, then player 1's regret is maximized if player 2 plays either s_n (in which case 1 would have been better off by continuing to cooperate longer) or if player 2 plays s_{k-1} (assuming that $k > 1$), in which case 1 would have been better off by defecting earlier. Thus, the strategy that minimizes player 1's regret is either s_{n-1}, s_1, or s_0. (This is true whatever strategies player 1 is considering for himself, as long as it includes these strategies.) If n is sufficiently large, then it will be s_{n-1}. This seems intuitively reasonable. In the long run, a long stretch of cooperation pays off, and minimizes regret. Moreover, it is not hard to show that allowing mixtures over $s_0, \ldots, s_n$ makes no difference; for large n, s_{n-1} is still the unique strategy that minimizes regret.

To summarize, if each player i believes that the other player $2 - i$ is playing a strategy in S_{2-i}^*—a reasonable set of strategies to consider—then we get a strategy that looks much more like what people do in practice.

As shown in Example 3.26, starting the deletion process with a subset of the set of all strategy profiles provides an explanation of observed behavior in the repeated Prisoner's Dilemma. But to make this approach useful, we need to motivate where these sets are coming from. One approach would be to simply require that the sets are exogenously given; that is, it is up to the modeler to restrict the set of strategies to those that he believes are "psychologically viable" for the players. While this is admittedly somewhat *ad hoc*, Example 3.26 shows that we can, in some cases, at least provide plausible motivation for our choices. Another approach would be to provide some systematic way of determining the subsets of strategies to use given just the traditional description of the game. For instance, it might seem natural to exclude all strongly, or even weakly, dominated strategies. For example, if we exclude dominated strategies, then we get cooperation in the Centipede Game up to the second-last round, which is somewhat closer to the experimental results.

Thinking in terms of beliefs makes it easy to relate iterated regret to other notions of equilibrium. Suppose that there exists a strategy profile $\vec{\sigma}$ such that player i's beliefs have the form $\Sigma_i \times \{\sigma_{-i}\}$. That is, player i believes that each of the other players are playing their component of $\vec{\sigma}$, and there are no constraints on his choice of strategy. Then it is easy to see that the strategies that minimize player i's regret with respect to these beliefs are just the best responses to σ_{-i}. In particular, $(\vec{\sigma}, \ldots, \vec{\sigma}) \in \mathcal{RM}'(\Sigma_1 \times \sigma_{-1}, \ldots, \Sigma_n \times \sigma_{-n})$ if and only if $\vec{\sigma}$ is a Nash equilibrium since minimizing regret with respect to a single strategy profile is equivalent to maximizing expected utility with respect to the strategy profile. The key point here is that if the agent's beliefs are represented by a "small" set, then the agent makes a best response in the standard sense by minimizing regret; minimizing regret with respect to a "large" belief set looks more like traditional regret minimization.

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[10] As we hinted in Section 3.3, an epistemic justification of this more general notion of regret minimization would require a more general notion of lexicographic beliefs, where each player has a separate sequence of beliefs.

4. Iterated regret minimization in Bayesian games

Up to now we have considered strategic games. We now show how to extend iterated regret minimization to Bayesian games. This will allow us to make some general conceptual comments on dealing with probabilistic vs. non-probabilistic events, and to be able to compare our work better to other work in the literature.

Formally, a Bayesian game is tuple $([n], A, \vec{u}, T, \pi)$, where, as before, $[n]$ is the set of players, A is the set of action profiles, $\vec{u}$ is the profile of utility functions, $T = T_1 \times \cdots \times T_n$ is the set of type profiles (where T_i represents the set of possible types for player i), and π is a probability measure on T. A player's utility can depend, not just on the action profile, but on the type profile. Thus, $u_i : A \times T \to \mathbb{R}$. For simplicity, we assume that $\pi(t_i) > 0$ for all types $t_i \in T_i$ and $i = 1, \ldots, n$ (where t_i is an abbreviation of $\{\vec{t}' : t_i' = t_i\}$).

A strategy for player i in a Bayesian game is a function from player i's type to an action in A_i; that is, what a player does will in general depend on his type. For a pure strategy profile $\vec{\sigma}$, player i's expected utility is

$$U_i(\vec{\sigma}) = \sum_{\vec{t} \in T} \pi(\vec{t}) u_i\big(\sigma_1(t_1), \ldots, \sigma_n(t_n)\big).$$

Player i's expected utility with a mixed strategy profile $\vec{\sigma}$ is the expectation with respect to the probability on pure strategy profiles induced by $\vec{\sigma}$. Given these definitions, a Nash equilibrium in a Bayesian game is defined in the same way as a Nash equilibrium in a strategic game.

There are some subtleties involved in doing iterated deletion in Bayesian games. Roughly speaking, we need to relativize all the previous definitions so that they take types into account. We give the definitions for pure strategies; the modifications to deal with mixed strategies are straightforward and left to the reader.

As before, suppose that $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$. Moreover, suppose that, for each player i, $\mathcal{S}_i$ is also a cross-product; that is, for each type $t \in T_i$, there exists a set of actions $A(t) \subseteq A_i$ such that $\mathcal{S}_i$ consists of all strategies σ such that $\sigma(t) \in A(t)$ for all $t \in T_i$. For $\vec{a}_{-i} \in \mathcal{S}_{-i}$ and $\vec{t} \in T$, let $u_i^{\mathcal{S}_i}(\vec{a}_{-i}, \vec{t}) = \max_{a_i \in \mathcal{S}} u_i(a_i, \vec{a}_{-i}, \vec{t})$. For $a_i \in \mathcal{S}_i$, $\vec{a}_{-i} \in \mathcal{S}_{-i}$, and $\vec{t} \in T$, the *regret* of a_i for player i given $\vec{a}_{-i}$ and $\vec{t}$, relative to $\mathcal{S}_i$, denoted $regret_i^{\mathcal{S}_i}(a_i \mid \vec{a}_{-i}, \vec{t})$, is $u_i^{\mathcal{S}_i}(\vec{a}_{-i}, \vec{t}) - u_i(a_i, \vec{a}_{-i}, \vec{t})$. Let $regret_i^{\mathcal{S}}(a_i \mid \vec{t}) = \max_{\vec{a}_{-i} \in \mathcal{S}_{-i}(\vec{t}_{-i})} regret^{\mathcal{S}_i}(a_i \mid \vec{a}_{-i}, \vec{t})$ denote the maximum regret of player i given $\vec{t}$. The *expected regret of a_i given t_i and $\mathcal{S}_{-i}$* is $E[regret_i^{\mathcal{S}_i}(a_i \mid t_i)] = \sum_{\vec{t} \in T} \pi(\vec{t} \mid t_i) regret_i^{\mathcal{S}_i}(a_i \mid \vec{t})$. Let $minregret^{\mathcal{S}_i}(t_i) = \min_{a_i \in \mathcal{S}_i(t_i)} E[regret_i^{\mathcal{S}}(a_i \mid t_i)]$. We delete all those strategies that do not give an action that minimizes expected regret for each type. Thus, let $\mathcal{RM}_i(\mathcal{S}_i) = \{\sigma \in \mathcal{S}_i : regret_i^{\mathcal{S}_i}(\sigma(t_i) \mid t_i) = minregret^{\mathcal{S}_i}(t_i)\}$, and define $\mathcal{RM}(\mathcal{S}) = \mathcal{RM}_1(\mathcal{S}_1) \times \cdots \times \mathcal{RM}_n(\mathcal{S}_n)$. Having defined the deletion operator, we can apply iterated deletion as before.

These definitions can be viewed as capturing a general philosophy: when we have probabilities, we take expectation; when we do not have probabilities, we use regret. Thus, we take expectation over type profiles, since we have a probability over types; we take regret with respect to action profiles, because we do not have a probability on action profiles.

Example 4.1. *First-price auction:* In a first-price auction, the player with the highest bid wins the auction, but pays his actual bid. Assume, for simplicity, that bids are natural numbers, that the lower-numbered player wins the auction in the event of a tie, that all valuations are even, and that the product is sold only if some player bids above 0. If a player's valuation is v, then bidding v' has regret $\max(v' - 1, v - v' - 1)$. To see this, consider player i. Suppose that the highest bid of the other agents is v'', and that the highest-numbered agent that bids v'' is agent j. If $v'' < v'$ or $v'' = v'$ and $i < j$, then i wins the bid. He may have done better by bidding lower, but the lowest he can bid and still win is 1, so his maximum regret in this case is $v' - 1$ (which occurs, for example, if $v'' = 0$). On the other hand, if $v'' > v'$ or $v'' = v$ and $j < i$, then i does not win the auction. He feels regret if he could have won the auction and still paid at most v. To win, if $j < i$, he must bid $v'' + 1$, in which case his regret is $v - v'' - 1$. In this case, v'' can be as small as v', so his regret is at most $v - v' - 1$. If $j > i$, then he must only bid v'' to win, so his regret is $v - v''$, but $v'' \geqslant v' + 1$, so his regret is again at most $v - v' - 1$. It follows that bidding $v' = v/2$ is the unique action that minimizes regret (yielding a regret of $v/2 - 1$).

Note that in this example, the bid for player i that minimizes his regret does not depend on his prior distribution over the valuations (i.e., types) of the other players. This is so since no matter what prior player i has on the valuation v_j of a player j, he still considers it possible that j bids an arbitrary value b_j; in particular, it might very well be that $b_j > v_j$. If we restrict the prior beliefs of players (as in Section 3.5) to include only bids $b_j \leqslant v_j$ (as bids $b > v_j$ are weakly dominated by bidding v_j), the analysis no longer holds; the bid for player i that minimizes his regret now depends on his prior.

5. Related work

While the notion of regret has been well studied in the context of decision theory (see Hayashi, 2008b; Ritov, 1996, and the references therein to get sense of the range of the work), there has been surprisingly little work on applying regret to game theory. Linhart and Radner (1989) applied regret to a bargaining problem, and showed that it leads to solutions that seem much more reasonable than Nash equilibrium. In the words of Linhart and Radner, "in the application to sealed-bid bargaining, the instruction 'Bid so as to minimize your maximum regret' has several advantages over the instruction 'Bid the best response to your opponent's bid'".

More recently, Ozbay and Ozbay (2007) applied regret to explain the puzzle of overbidding in auctions. One issue of relevance in their analysis was the extent to which bidders learn after the bidding process how well they would have done had they bid something else. Our definition of regret does not take this into account, but work by Ritov (1996) shows that uncertainty resolution affects how much regret people actually feel, and the anticipation of uncertainty resolution affects decisions.

The explanatory power of regret suggests that there should be some reasonable solution concepts based on regret. There has been some work on this in the computer science literature. Hyafil and Boutilier (2004) consider *pre-Bayesian* games, where each agent has a type and a player's utility depends on both the action profile and the type profile, just as in a Bayesian game, but now there is no probability on types.[11] The solution concept they use is a hybrid of Nash equilibrium and regret. Roughly speaking, they take regret with respect to the types of the other players, but use Nash equilibrium with respect to the strategies of other players. That is, they define $regret_i^{S_i}(a_i \mid \vec{a}_{-i}, \vec{t})$ as we do (taking S_i to consist of all strategies for player i), but then define $regret^{S_i}(a_i \mid \vec{a}_{-i})$ by minimizing over all $\vec{t}_{-i}$. They then define a profile $\vec{\sigma}$ to be a *minimax-regret equilibrium* if, for all type profiles $\vec{t}$, no agent can decrease his regret by changing his action. For strategic games, where there are no types (i.e., $|T| = 1$), their solution concept collapses to Nash equilibrium. Thus, their definitions differ from ours in that they take regret with respect to types, not with respect to the strategies of other players as we do, and they do not iterate the regret operation.

Even closer to our work is a recent paper by Renou and Schlag (2010) (their work was done independently of ours). Just as we do, they focus on strategic games. Their motivation for considering regret, and the way they do it in the case of pure strategies, is identical to ours (although they do not iterate the deletion process). Renou and Schlag call $\vec{\sigma}$ a *minimax-regret equilibrium relative to* $(\Pi_1, \ldots, \Pi_n)$, where Π_i is a closed, convex set of strategies for player i (which can be viewed as player i's prior beliefs, as in Section 3.5), if σ_i minimizes regret with respect to S. They are particularly interested in strategy profiles $\vec{\sigma}$ that minimize regret for each agent with respect to all the strategy profiles in an ϵ neighborhood of $\vec{\sigma}$. Roughly speaking, this can be understood informally as saying that rather than the strategies being used in equilibrium being commonly known, they are commonly approximately known; in equilibrium, players are best responding (in the sense of minimizing regret), given these beliefs.

Although Renou and Schlag define regret for pure strategies, this is only a tool for dealing with mixed strategies; they do not consider the regret of a pure strategy with respect to a set of pure strategies, as we do. In particular, they have no analogue to our analysis of pure strategies in Section 3.4.1. Their definition of regret for mixed strategies is in general somewhat different from ours; see Appendix C for a detailed comparison. Nevertheless, as we show in Appendix C, despite these differences, the approaches often give the same results (and are certainly inspired by similar intuitions).

Finally, Selten et al. (2005) use regret in a repeated game to define what they call in *impulse balance point*. Roughly speaking, they argue that the experience of regret causes a player in a game to shift his actions in the direction of lowering regret; the impulse balance point is the distribution of play where the impulses in all directions are at a balance. Experimental work (Avrahami et al., 2005; Avrahami and Kareev, in press) supports the role of such "impulsive" behavior. Since impulse balance theory is very much grounded in repeated games, whereas we are looking at strategic games, the predictions of the two approaches cannot be compared. It would certainly be of interest to extend our approach to repeated games (where there would presumably be a change in players' beliefs over time).

6. Discussion

The need to find solution concepts that reflect more accurately how people actually play games has long been recognized. This is a particularly important issue because the gap between "descriptive" and "normative" is particularly small in game theory. An action is normatively the "right" thing to do only if it is the right thing to do with respect to how others actually play the game; thus, a good descriptive theory is an essential element of a good normative theory.

There are many examples in the game-theory literature where Nash equilibrium and its refinements do not describe what people do. We have introduced a new solution concept, iterated regret minimization, that, at least in some games, seems to capture better what people are doing than more standard solution concepts.

Of course, we are far from the first to try to find ways of explaining the outcomes of the games we have considered. Kreps et al. (1982) show that even if there is a small probability of there being a player who plays tit for tat, then cooperation until almost the end of the game is a sequential equilibrium in Prisoner's Dilemma. The outcomes of games like the Traveler's Dilemma and the Centipede Game have sometimes been explained by assuming that a certain fraction of agents will be "altruistic", and play the helpful action (e.g., playing 100 in Traveler's Dilemma or cooperating in the Centipede Game) (cf., Capra et al., 1999). There seems to be some empirical truth to this assumption; for example, 10 of 45 game theorists that submitted pure strategies in the experiments of Becker et al. (2005) submitted 100. With an assumption of altruism, then the strategies of many of the remaining players can be explained as best responses to their (essentially accurate) beliefs.

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[11] Hyafil and Boutilier actually consider a slightly less general setting, where the utility for player i depends only on player i's type, not the whole type profile. Modifying their definitions to deal with the more general setting is straightforward.

Altruism may indeed be part of an accurate descriptive theory, but to use it, we first need to decide what the "right" action is, and also the likelihood that agents are altruistic. Moreover, there are many games (such as coordination games) where there is no obvious "altruistic" action. Similarly, while players of repeated Prisoner's Dilemma may be reacting the way they do because they believe that one of them might be a tit for tat player, it is not clear how to use this insight to explain behavior in other games. Iterated regret minimization provides a different descriptive explanation, one which applies to all games. Given the importance of regret as a motivating factor demonstrated in the psychology literature, there is some reason to believe that it provides some descriptive accuracy. Moreover, regret has some normative import as well, since it can be viewed as describing rational play, in a precise sense.

It seems particularly appealing when considering inexperienced but intelligent players that play a game for the first time. In this setting, it seems unreasonable to assume that players know what strategies other players are using (as is implicitly assumed in Nash equilibrium).

We emphasize that although there exist alternative explanations of the observed behavior in the games considered here, these explanations in general depend on the specific game under consideration. In contrast, iterated regret minimization provides a single, simple, explanation that—at least at a qualitative level—is consistent with observed behavior in all of them.

While we have illustrated some of the properties of iterated regret minimization, we view this paper as more of a "proof of concept". There are clearly many issues we have left open. We mention a few of the issues we are currently exploring here.

- As we observed in Section 3.5, some behavior is well explained by assuming that agents start the regret minimization procedure with a subset of the set of all strategy profiles, which can be thought of as representing the strategy profiles that the agent is considering. But we need better motivation for where this set is coming from.
- We have considered "greedy" deletion, where all strategies that do not minimize regret are deleted at each step. We could instead delete only a subset of such strategies at each step of deletion. As we observed in Example 3.16, the final set of strategies can depend on the order of deletion. Getting an understanding of how robust the deletion process is would be of interest.
- We have focused on normal-form games and Bayesian games. It would also be interesting to extend regret minimization to extensive-form games. A host of new issues arise here, particularly because, as is well known, regret minimization is not time consistent (see Hayashi, 2008a, for some discussion of the relevant issues).
- A natural next step would be to apply our solution concepts to mechanism design. As we mentioned earlier, there has already been some work on applying regret (Linhart and Radner, 1989; Ozbay and Ozbay, 2007), but there seems to be scope for much more.

One last point is worth mentioning. There are times when regret seems too weak a solution concept. Suppose that we modify the Traveler's Dilemma so as to allow a new action, call it P (for *punish*), where both players get 2 if they both play P, but if one player plays P and the other plays an action other than P, then the player who plays P gets 2, and the other player gets -95. Playing P has regret 97, just as playing 2 does (if I play P and the other player plays 100, I wish I had played 99), the existence of P means that every action has regret 97. Moreover, if the -95 is changed to -96, playing P is the only strategy that minimizes regret. We do not expect that people would actually play P in this situation (although (P, P) is in fact a Nash equilibrium).

Ignoring the question of whether it is rational to play P for a moment, note that there is a huge penalty for taking P into account when doing a regret calculation. There are strategies that minimize regret with respect to all strategies other than p, and have a regret of 3 (namely, the strategies in $[96, 100]$). If we think of minimizing regret as somewhat akin to buying insurance, insuring against all risks comes at a "cost" of 94 compared to the cost of insuring against almost all risks (that is, all but the risk of the opponent playing P). Traveler's Dilemma is a game where regret is attractive precisely because we can minimize regret so "cheaply"; the regret of the strategy that minimizes regret is low. In the modified version, minimizing regret is expensive. It would be interesting to attempt to define a solution concept based on regret that somehow takes these ideas into account.

Appendix A. An epistemic characterization using Kripke structures

Let $G = ([n], A, \vec{u})$ be a strategic game. We start by considering iterated regret minimization with pure strategy profiles. A (*lexicographic belief*) *Kripke structure* for G has the form $(W, \mathbf{s}, \mathcal{B}_1, \ldots, \mathcal{B}_n)$, where $\mathbf{s}$ associates with each world $w \in W$ a pure strategy profile $\mathbf{s}(w)$ in the game Γ, and $\mathcal{B}_i$ associates with each world w a sequence $(W_0, W_1, \ldots)$ of sets of worlds. For convenience, we use $\mathcal{B}_i^k(w)$ to denote the set W_k in the sequence, so $\mathcal{B}_i(w) = (\mathcal{B}_i^0(w), \mathcal{B}_i^1(w), \ldots)$. Intuitively, $\mathcal{B}_i^0(w)$ consists of the worlds that player i considers most likely at w, the worlds in $\mathcal{B}_i^1(w)$ are less likely, and so on. Thus, the sequence $\mathcal{B}_i(w)$ models player i's beliefs at w. We assume that at each world in $\bigcup_{j=0}^{\infty} \mathcal{B}_i(w)$ player i uses the same strategy and has the same beliefs as he does at w. That is, if $w' \in \bigcup_{j=0}^{\infty} \mathcal{B}_i(w)$, then $\mathbf{s}_i(w) = \mathbf{s}_i(w')$, where $\mathbf{s}_i(w)$ denotes player i's strategy in the strategy profile $\mathbf{s}(w)$, and $\mathcal{B}_i(w) = \mathcal{B}_i(w')$. This captures the intuition that a player knows his own strategy and beliefs.

Given a set W' of worlds, let $\mathbf{s}(W')$ denote the set of strategy profiles associated with the worlds in W'; that is, $\mathbf{s}(W') = \{\mathbf{s}(w'): w' \in W'\}$. Note that while the notion of a lexicographic belief sequence defined in Section 3.3 considers a countable sequence $(\mathcal{S}_0, \mathcal{S}_1, \ldots)$ of strategies, a belief sequence here considers a sequence $\mathcal{B}_i(w)$ of sets of worlds. Player i is *rational in world w* if player i's strategy in w is rational with respect to $\langle \mathbf{s}(\mathcal{B}_i^0(w)), \mathbf{s}(\mathcal{B}_i^1(w)), \ldots \rangle$, where rationality with respect to lexicographic sequences is defined as in Section 3.3.

To capture the type of reasoning we want to do in these Kripke structures, we consider a propositional language similar in spirit to that considered in Halpern and Pass (2009). We start with the special proposition *true*, a primitive proposition $play_i(a)$ for each player i and action $a \in A_i$ (which intuitively says that player i is playing action a), a primitive proposition RAT_i for each player i (which, intuitively, says that player i is rational), and close off under conjunction, negation, and the family of modal operators O_i^k. Roughly speaking, $O_i^k \phi$ says that "all player i knows at level k is ϕ". We take $play_{-i}(\vec{a}_{-i})$ to be an abbreviation for $\bigwedge_{j \neq i} play_j(a_j)$, and $play(\vec{a})$ to be an abbreviation for $\bigwedge_j play_j(a_j)$.

A formula is either true or false at a world in a Kripke structure. As usual, we write $(M, w) \models \phi$ if ϕ is true at world w in Kripke structure M. The semantics for formulas other that $O_i^k \phi$ is given as follows.

- $(M, w) \models true$ (so *true* is vacuously true).
- $(M, w) \models play_i(a)$ if $\mathbf{s}_i(w) = a$.
- $(M, w) \models RAT_i$ if $\mathbf{s}_i(w)$ is a best response to the strategy sequence $\langle \mathbf{s}(\mathcal{B}_i^0(w)), \mathbf{s}(\mathcal{B}_i^1(w)), \ldots \rangle$.
- $(M, w) \models \neg\phi$ if $(M, \omega) \not\models \phi$.
- $(M, w) \models \phi \wedge \phi'$ iff $(M, \omega) \models \phi$ and $(M, \omega) \models \phi'$.

The "all agent i knows" operator was first considered by Levesque (1990) and later by Halpern and Lakemeyer (2001). While our definition is similar in spirit to these earlier definitions, it is more syntactic, and follows the lines of the definition used in Halpern and Pass (2009). Intuitively, "all agent i knows is ϕ" if agent i knows ϕ and considers possible every formula consistent with ϕ. The phrase "every formula" makes it clear that the meaning of "all i knows" is dependent on what formulas we consider. For simplicity here, we just consider formulas of the form $play_{-i}(\sigma_{-i})$ for $j \neq i$, so "all i knows is ϕ" says that i considers possible all strategy profiles that other players could use, consistent with ϕ.[12] Thus,

- $(M, w) \models O_i^k \phi$ if $(M, w') \models \phi$ for all $w' \in \mathcal{B}_i^k(w)$ and for all strategies $a_{-i} \in A_{-i}$, if there exists some structure $M^\dagger$ and world $w^\dagger$ in $M^\dagger$ such that $(M^\dagger, w^\dagger) \models \phi \wedge play_{-i}(a_{-i})$, then there exists a world $w'' \in \mathcal{B}^k(w)$ such that $(M, w'') \models play_{-i}(a_{-i})$.

Thus, this formula says that if it is consistent with ϕ that the players other than i play a_{-i}, then player i must consider it possible at the kth level that the players play a_{-i}.

Define the formulas ND_i^k inductively, by taking $ND_i^0 = true$ and

$$ND_i^{k+1} = RAT_i \wedge \bigwedge_{\ell=0}^{k} O_i^\ell \left(\bigwedge_{j \neq i} ND_j^\ell \right).$$

This formula captures the intuitions that we outlined in the main text. Player i's top-level belief is that the other players could be using an arbitrary strategy; his first-level belief is that the other players are rational, but that they believe that other players could be using an arbitrary strategy; his second-level belief is that the other players are rational and believe that other players are rational and believe that other players are using an arbitrary strategy; and so on.

The following theorem shows that $\bigwedge_i ND_i^k$ completely characterizes k rounds of iterated regret minimization.

Theorem A.1. *The following are equivalent*:

- *The action profile $\vec{a}$ survives k rounds of iterated regret minimization (i.e., $\vec{a} \in \mathcal{RM}^k(A)$).*
- *There exist a structure M and world w in M such that $(M, w) \models play(\vec{a}) \wedge (\bigwedge_i ND_i^k)$.*

Proof. We proceed by induction on k. The result is trivially true if $k = 0$. Suppose the result is true for k; we prove it for $k + 1$. Suppose that $(M, w) \models play(\vec{a}) \wedge (\bigwedge_i ND_i^k)$. It is almost immediate from the induction hypothesis that $\mathbf{s}_{-i}(\mathcal{B}_i^\ell) = \mathcal{RM}_{-i}^\ell(A)$ for $\ell = 0, \ldots, k$ and every player i. Since $(M, w) \models play_i(a_i) \wedge RAT_i$, it follows that $a_i \in \mathcal{RM}_i^{k+1}(A)$. Thus, $\vec{a} \in \mathcal{RM}^{k+1}(A)$.

Now suppose that $\vec{a} \in \mathcal{RM}^{k+1}(A)$. We construct a finite structure M^{k+1} as follows. Let W consist of all the worlds of the form $(\vec{a}, \ell, i)$, where $\vec{a} \in A$, $\ell \in \{0, \ldots, k+1\}$, and i is a player. Define $\mathbf{s}(\vec{a}, \ell, i) = \vec{a}$. Finally, define

$$\mathcal{B}_j^h((\vec{a}, i, \ell)) = \begin{cases} \{(\vec{a}', j, \ell - 1) \colon \vec{a}'_{-j} \in \mathcal{RM}^h_{-j}(A), \ a_j = a'_j\} & \text{if } 0 \leqslant h \leqslant \ell, \ j \neq i, \ \ell > 0; \\ \{(\vec{a}', j, 0) \colon \vec{a}'_{-j} \in A_{-j}, \ a_j = a'_j\} & \text{if } h = \ell = 0, \ j \neq i; \\ \{(\vec{a}', i, \ell) \colon \vec{a}'_{-i} \in \mathcal{RM}^h_{-i}(A), \ a_i = a'_i\} & \text{if } 0 \leqslant h \leqslant \ell, \ j = i; \\ \mathcal{B}_j^\ell((\vec{a}, i, \ell)) & \text{if } h > \ell. \end{cases}$$

It is easy to see that if $a_i = a'_i$, then $\mathcal{B}_i^h(\vec{a}, i, \ell) = \mathcal{B}_i^h(\vec{a}', i, \ell)$, and if $j \neq i$ and $a_j = a'_j$, then $\mathcal{B}_j^h(\vec{a}, i, \ell) = \mathcal{B}_j^h(\vec{a}', j, \ell - 1)$. It follows that, for all worlds $w, w' \in M^{k+1}$, if $w' \in \bigcup_{j=0}^{\infty} \mathcal{B}_i(w)$, then $\mathbf{s}_i(w) = \mathbf{s}_i(w')$ and $\mathcal{B}_i(w) = \mathcal{B}_i(w')$. Thus, M^{k+1} is indeed a lexicographic belief Kripke structure. Now a straightforward induction on h shows that if $\vec{a} \in \mathcal{RM}^h(A)$ and $h \leqslant \ell$, then $(M, (\vec{a}, \ell, j)) \models \bigwedge_i ND_i^h$. The desired result follows. $\quad\square$

We have now provided an epistemic characterization for iterated regret minimization with pure strategies. The characterization for mixed strategies is identical, except that now, in a lexicographic belief Kripke structure, the strategy $\mathbf{s}(w)$ associated with a world w may be a mixed strategy.

Appendix B. Proofs

We provide proofs of all the results not proved in the main text here. We repeat the statements for the convenience of the reader.

Theorem 3.3: *Let* $G = ([n], A, \vec{u})$ *be a strategic game. If* $\mathcal{S}$ *is a closed, nonempty set of strategies of the form* $\mathcal{S}_1 \times \cdots \times \mathcal{S}_n$, *then* $\mathcal{RM}^{\infty}(\mathcal{S})$ *is nonempty and* $\mathcal{RM}(\mathcal{RM}^{\infty}(\mathcal{S})) = \mathcal{RM}^{\infty}(\mathcal{S})$.

Proof. We start with the case that $\mathcal{S}$ consists of pure strategies, since it is so simple. Since $\mathcal{D}(\mathcal{S}) \subseteq \mathcal{S} \subseteq A$ for any deletion operator $\mathcal{D}$, and A is finite, we must have $\mathcal{RM}^{\infty}(\mathcal{S}) = \mathcal{RM}^k(\mathcal{S})$ for some finite k. Moreover, we have $\mathcal{RM}(\mathcal{RM}^{\infty}(\mathcal{S})) = \mathcal{RM}^{\infty}(\mathcal{S})$.

To deal with the general case, we must work a little harder. We first need the following lemma.

Lemma B.1. *Let* $\mathcal{S}$ *be a nonempty closed set of strategies. Then* $\mathcal{RM}(\mathcal{S})$ *is closed and nonempty.*

Proof. We start by showing that $regret_i^{\mathcal{S}_i}$ is continuous. First note that $regret_i^{\mathcal{S}_i}(a_i \mid \vec{a}_{-i})$ is a continuous function of $\vec{a}_i$. By the closedness (and hence compactness) of $\mathcal{S}$ it follows that $regret_i^{\mathcal{S}}(a_i) = \max_{\vec{a}_{-i} \in \mathcal{S}_{-i}} regret_i^{\mathcal{S}_i}(a_i \mid \vec{a}_{-i})$ is well defined (even though it involves a max). To see that $regret_i^{\mathcal{S}}$ is continuous, suppose not. This means that there exist some a, δ such that for all n, there exists an a_n within $1/n$ of a such that $|regret_i^{\mathcal{S}}(a_n) - regret_i^{\mathcal{S}}(a)| > \delta$. By compactness of $\mathcal{S}$, it follows by the Bolzano–Weierstrass theorem (Rudin, 1976) that there exists a convergent subsequence $(a_{n_m}, regret_i(a_{n_m}))$ that converges to (a, b). We have $|b - regret_i(a)| \geqslant \delta$, which is a contradiction.

Now, to see that $\mathcal{RM}(\mathcal{S})$ is nonempty, it suffices to observe that, because $\mathcal{S}$ is compact, and $regret_i^{\mathcal{S}}$ is continuous, for each player i, there must be some strategy σ_i such that $regret_i^{\mathcal{S}}(\sigma_i) = minregret_i^{\mathcal{S}}$. Thus, $\sigma_i \in \mathcal{RM}_i(\mathcal{S})$.

To show that $\mathcal{RM}(\mathcal{S})$ is closed, suppose that $\langle \sigma^m \rangle_{m=1,2,3,\ldots}$ is a sequence of mixed strategy profiles in $\mathcal{RM}(\mathcal{S})$ converging to σ (in the sense that the probability placed by σ^m on a pure strategy profile converges to the probability placed by σ on that strategy profile) and, by way of contradiction, that $\sigma \notin \mathcal{RM}(\mathcal{S})$. Thus, for some player i, $\sigma_i \notin \mathcal{RM}_i(\mathcal{S})$. Note that, since $\mathcal{RM}(\mathcal{S}) \subseteq \mathcal{S}$, the sequence $\langle \sigma_i^m \rangle_{m=1,2,3,\ldots}$ is in $\mathcal{S}$; since $\mathcal{S}$ is closed, $\sigma \in \mathcal{S}$. Let $minregret_i^{\mathcal{S}} = b$. Since $\sigma_i^m \in \mathcal{RM}_i(\mathcal{S})$, we must have that $regret_i^{\mathcal{S}}(\sigma_i^m) = b$ for all m. Since $\sigma_i \notin \mathcal{RM}_i(\mathcal{S})$, there must exist some strategy profile $\vec{\tau} \in \mathcal{S}$ such that $U_i(\vec{\tau}) - U_i(\sigma_i, \tau_{-i}) = b' > b$. But by the continuity of utility, $\lim_m U_i(\sigma_i^m, \tau) = b'$. This contradicts the assumption that $regret_i^{\mathcal{S}}(\sigma_i^m) = b$ for all m. Thus, $\mathcal{RM}(\mathcal{S})$ must be closed. $\quad\square$

Returning to the proof of Proposition 3.3, note that since $\mathcal{S}$ is closed and nonempty, it follows by Lemma B.1 that $\mathcal{RM}^k(\mathcal{S})$ is a closed nonempty set for all k. Additionally, note that $\mathcal{RM}^k(\mathcal{S})$ can be viewed as a subset of the compact set $[0, 1]^{|A|}$ (since a probability distribution on a finite set X can be identified with a tuple of numbers in $[0, 1]^{|X|}$); it follows that $\mathcal{RM}^k(\mathcal{S})$ is also bounded, and thus compact. Finally, note that the set $\{\mathcal{RM}^k(\mathcal{S}) \colon k = 1, 2, 3, \ldots\}$ has the *finite intersection property*: the intersection of any finite collection of its elements is nonempty (since it is equal to the smallest element). The compactness of $\mathcal{S}$ now guarantees that the intersection of all the sets in a collection of closed subsets of $\mathcal{S}$ with the finite intersection property is nonempty (Munkres, 2000). In particular, it follows that $\mathcal{RM}^{\infty}(\mathcal{S})$ is nonempty.

To see that $\mathcal{RM}^{\infty}(\mathcal{S})$ is a fixed point of the deletion process, suppose, by way of contradiction, that $\sigma_i \in \mathcal{RM}_i^{\infty}(\mathcal{S}) - \mathcal{RM}_i(\mathcal{RM}^{\infty}(\mathcal{S}))$. Let $minregret_i^{\mathcal{RM}^{\infty}(\mathcal{S})} = b$ and choose $\sigma'_i \in \mathcal{RM}_i^{\infty}(\mathcal{S})$ such that $regret_i^{\mathcal{RM}^{\infty}(\mathcal{S})}(\sigma'_i) = b$. Since $\sigma_i \notin$

$\mathcal{RM}(\mathcal{RM}^\infty(\mathcal{S}))$, it must be the case that $regret_i^{\mathcal{RM}^\infty(\mathcal{S})}(\sigma_i') = b' > b$. By assumption, $\sigma_i \in \mathcal{RM}^\infty(\mathcal{S})$, so $\sigma_i \in \mathcal{RM}^k(\mathcal{S})$ for all k; moreover, $regret_i^{\mathcal{RM}^k(\mathcal{S})}(\sigma_i) \geqslant b'$. Since $\sigma_i \in \mathcal{RM}^{k+1}(\mathcal{S})$, it follows that $minregret_i^{\mathcal{RM}^k(\mathcal{S})} \geqslant b'$. This means that there exists a strategy profile $\vec{\tau}^k \in \mathcal{RM}^k$ such that $U_i(\vec{\tau}^k) - U_i(\sigma_i', \vec{\tau}^k_{-i}) \geqslant b'$. By the Bolzano–Weierstrass theorem, the sequence of strategies $\langle \vec{\tau}^k \rangle_{k=1,2,\dots}$ has a convergent subsequence $\langle \vec{\tau}^{k_j} \rangle_{j=1,2,\dots}$ that converges to some strategy profile $\vec{\tau}$. Since $\tau^k \in \mathcal{RM}^m(\mathcal{S})$ for all $k \geqslant m$, it must be the case that, except for possibly a finite initial segment, this convergent subsequence is in $\mathcal{RM}^m(\mathcal{S})$. Since $\mathcal{RM}^m(\mathcal{S})$ is closed, $\vec{\tau}$, the limit of the convergent subsequence, is in $\mathcal{RM}^m(\mathcal{S})$ for all $m \geqslant 1$. Thus, $\vec{\tau} \in \mathcal{RM}^\infty(\mathcal{S})$. Now a simple continuity argument shows that $U_i(\vec{\tau}) - U_i(\sigma_i', \vec{\tau}_{-i}) \geqslant b' > b$, a contradiction. $\quad\square$

Claim 3.13: $regret_1^S(s_{ad}) = (n-1)(u_3 - u_2) + \max(-u_1, u_2 - u_3)$. *Moreover, if s is a strategy for player 1 where he plays c before seeing player 2 play c (i.e., where player 1 either starts out playing c or plays c at the kth for $k > 1$ move after seeing player 2 play d for the first $k - 1$ moves), then $regret_1^S(s) > (n-1)(u_3 - u_2) + \max(-u_1, u_2 - u_3)$.*

Proof. Let s_c be the strategy where player 2 starts out playing d and then plays c to the end of the game if player 1 plays c, and plays d to the end of the game if player 1 plays d. We have $regret_1^{S_1}(s_{ad} \mid s_c) = (n-1)(u_3 - u_1) - u_1$: player 1 gets nu_1 with (s_{ad}, s_c), and could have gotten $(n-1)u_3$ if he had cooperated on the first move and then always defected.

Let s_c' be the strategy where player 2 starts out playing c and then plays c to the end of the game if player 1 plays c, and plays d to the end of the game if player 1 plays d. It is easy to see that $regret_1^{S_1}(s_{ad} \mid s_c') = (n-1)(u_3 - u_1) + (u_2 - u_3)$. Thus, $regret_1^S(s_{ad}) \geqslant (n-1)(u_3 - u_2) + \max(-u_1, u_2 - u_3)$. We now show that $regret_1^S(s_{ad}) = (n-1)(u_3 - u_2) + \max(-u_1, u_2 - u_3)$. For suppose that the regret is maximized if player 2 plays some strategy s, and player 1's best response to s is s'. Consider the first place where the play of (s', s) differs from that of (s_{ad}, s). This must happen *after* a move where player 1 plays c with s'. For as long as player 1 plays d with s', player 2 cannot distinguish s' from s_{ad}, and so does the same thing in response. So suppose that player 1 plays c at move k with s'. If player 2 plays c at step k, player 1 gets a payoff of u_3 with (s_{ad}, s) at step k and a payoff of u_2 with (s', s). Thus, player 1's total payoff with (s_{ad}, s) is at least $(n-1)u_1 + u_3$, while his payoff with (s_{ad}, s) is at most $(n-1)u_3 + u_2$; thus, his regret is at most $(n-1)(u_3 - u_1) + (u_2 - u_3)$. On the other hand, if player 2 plays d with s at step k, then player 1's payoff at step k with (s', s) is 0, while his payoff at step k with (s_{ad}, s) is u_1. Thus, his regret is at most $(n-1)(u_3 - u_1) - u_1$. (In both cases, the regret can be that high only if $k = 1$.)

We next show that if s is a strategy for player 1 where he plays c before seeing player 2 play c, then $regret_1^S(s) > (n-1)(u_3 - u_2) + \max(-u_1, u_2 - u_3)$. Suppose that k is the first move where player 1 plays c despite not having seen c before. If $k = 1$ (so that player 1 cooperates on the first move), let s_d be the strategy where player 2 plays d for the first move, then plays c to the end of the game if player 1 has played d for the first move, and otherwise plays d to the end of the game. It is easy to see that then $regret_1^{S_1}(s \mid s_d) = (n-1)(u_3 - u_1) + u_1$. On the other hand, if $k > 1$, then the regret $regret_1^{S_1}(s \mid s_c) \geqslant (n-1)(u_3 - u_1)$. Thus, $regret_1^S(s) > (n-1)(u_3 - u_2) + \max(-u_1, u_2 - u_3)$. $\quad\square$

Proposition 3.19: *Let $G = ([n], A, \vec{u})$ be a strategic game and let σ_i be a mixed strategy for player i. Then $regret_i^\Sigma(\sigma_i) = \max_{\vec{a}_{-i} \in A_{-i}} regret_i^{\Sigma_i}(\sigma_i \mid \vec{a}_{-i})$.*

Proof. Note that, for all strategies profiles $\vec{\sigma}_{-i}$, there exists some strategy σ_i^* such that $U_i^{\Sigma_i}(\vec{\sigma}_{-i}) = \max_{\sigma_i' \in \Sigma_i} U_i(\sigma_i', \vec{\sigma}_{-i}) = U_i(\sigma_i^*, \vec{\sigma}_{-i})$. It follows that

$$
\begin{aligned}
U_i^{\Sigma_i}(\vec{\sigma}_{-i}) &= U_i(\sigma_i^*, \vec{\sigma}_{-i}) \\
&= \sum_{\vec{a}_{-i} \in A_{-i}} \vec{\sigma}_{-i}(\vec{a}_{-i}) U_i(\sigma_i^*, \vec{a}_{-i}) \\
&\leqslant \sum_{\vec{a}_{-i} \in A_{-i}} \vec{\sigma}_{-i}(\vec{a}_{-i}) U_i^{\Sigma_i}(\vec{a}_{-i}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
regret_i^{\Sigma_i}(\sigma_i \mid \vec{\sigma}_{-i}) &= U_i^{\Sigma_i}(\vec{\sigma}_{-i}) - U_i(\sigma_i, \vec{\sigma}_{-i}) \\
&= U_i^{\Sigma_i}(\vec{\sigma}_{-i}) - \sum_{\vec{a}_{-i} \in A_{-i}} \vec{\sigma}_{-i}(\vec{a}_{-i}) U_i(\sigma_i, \vec{a}_{-i}) \\
&\leqslant \sum_{\vec{a}_{-i} \in A_{-i}} \vec{\sigma}_{-i}(\vec{a}_{-i}) U_i^{\Sigma_i}(\vec{a}_{-i}) - \sum_{\vec{a}_{-i} \in A_{-i}} \vec{\sigma}_{-i}(\vec{a}_{-i}) U_i(\sigma_i, \vec{a}_{-i}) \\
&= \sum_{\vec{a}_{-i} \in A_{-i}} \vec{\sigma}_{-i}(\vec{a}_{-i}) \, regret_i^{\Sigma_i}(\sigma_i \mid \vec{a}_{-i}) \\
&\leqslant \max_{\vec{a}_{-i} \in A_{-i}} regret_i^{\Sigma_i}(\sigma_i \mid \vec{a}_{-i}).
\end{aligned}
$$

It follows that

$$regret_i^\Sigma(\sigma_i) = \max_{\vec{\sigma}_{-i} \in \Sigma_{-i}} regret_i^{\Sigma_i}(\sigma_i \mid \vec{\sigma}_{-i}) = \max_{\vec{a}_{-i} \in A_{-i}} regret_i^{\Sigma_i}(\sigma_i \mid \vec{a}_{-i}). \qquad \square$$

Claim 3.24: $regret_1^\Sigma(\sigma) < 3$.

Proof. By Proposition 3.19, to compute $regret_1^\Sigma(\sigma)$, it suffices to compute $regret_1^{\Sigma_1}(\sigma \mid a)$ for each action a of player 2. If player 1 plays σ and player 2 plays 100, then the best response for player 1 is 99, giving him a payoff of 101. The payoff with σ is

$$100 \times \frac{1}{2} + 101 \times \frac{1}{4} + 100 \times \frac{1}{8} + \cdots + 5 \times \frac{1}{2^{98}} + 4 \times \frac{1}{2^{98}}.$$

To evaluate the expression, observe that we get an equivalent expression by adding $2 \times \frac{1}{2} - 1$. Thus,

$$100 \times \frac{1}{2} + 101 \times \frac{1}{4} + 100 \times \frac{1}{8} + \cdots + 5 \times \frac{1}{2^{98}} + 4 \times \frac{1}{2^{98}}$$

$$= 102 \times \frac{1}{2} + 101 \times \frac{1}{4} + 100 \times \frac{1}{8} + \cdots + 5 \times \frac{1}{2^{98}} + 4 \times \frac{1}{2^{98}} - 1$$

$$= 4 \times \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{98}} + \frac{1}{2^{98}} \right) + 1 \times \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{98}} \right)$$

$$\quad + 1 \times \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{97}} \right) + \cdots + \frac{1}{2} - 1$$

$$= 4 + \left(1 - \frac{1}{2^{98}} \right) + \left(1 - \frac{1}{2^{97}} \right) + \cdots + \left(1 - \frac{1}{2} \right) - 1$$

$$= 102 - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{98}} \right) - 1$$

$$= 100 + \frac{1}{2^{98}},$$

so the regret in this case is less than 1. Similarly, if player 2 plays k with $3 \leqslant k \leqslant 99$, the best response for player 1 is $k-1$, which would give player 1 a payoff of $k+1$, while the payoff from σ is

$$(k-2)\left(\frac{1}{2} + \cdots + \frac{1}{2^{100-k}} \right) + k \times \frac{1}{2^{101-k}} + (k+1) \times \frac{1}{2^{102-k}} + k \times \frac{1}{2^{103-k}}$$

$$\quad + (k-1) \times \frac{1}{2^{104-k}} + (k-2) \times \frac{1}{2^{105-k}} + \cdots + 5 \times \frac{1}{2^{98}} + 4 \times \frac{1}{2^{98}}.$$

Thus, player 1's regret if player 2 plays k is

$$3 \times \left(\frac{1}{2} + \cdots + \frac{1}{2^{100-k}} + \frac{1}{2^{105-k}} \right) + 2 \times \frac{1}{2^{104-k}} + 1 \times \left(\frac{1}{2^{101-k}} + \frac{1}{2^{103-k}} \right)$$

$$\quad + \frac{1}{2^{106-k}} \left(4 + 5 \times \frac{1}{2} + 6 \times \frac{1}{4} + \cdots + (k-4) \times \frac{1}{2^{k-8}} + (k-3) \times \frac{1}{2^{k-8}} \right)$$

$$< 3 \times \left(\frac{1}{2} + \cdots + \frac{1}{2^{100-k}} + \frac{1}{2^{105-k}} \right) + 2 \times \frac{1}{2^{104-k}} + 1 \times \left(\frac{1}{2^{101-k}} + \frac{1}{2^{103-k}} \right) + 10 \times \frac{1}{2^{106-k}}$$

$$< 3 \times \left(\frac{1}{2} + \cdots + \frac{1}{2^{103-k}} \right)$$

$$= 3 \times \left(1 - \frac{1}{2^{103-k}} \right)$$

$$< 3.$$

Finally, if player 2 plays 2, the best response for player 1 is 2, which would give player 1 a payoff of 2. Since player 1's payoff with σ is nonnegative, his regret in this case is less than 2. $\quad \square$

Appendix C. A comparison to the Renou–Schlag approach

As we said, the Renou–Schlag definition of regret in the case of mixed strategies is somewhat different from ours. Our definition of $regret_i^{S_i}(\sigma_i \mid \vec{\sigma}_{-i})$ does not depend on whether $\vec{\sigma}$ consists of pure strategies or mixed strategies (except that expected utility must be used in the case of mixed strategies, rather than utility). By way of contrast, Renou and Schlag (2010) define

$$regret_i'(\sigma_i \mid \vec{\sigma}_{-i}) = \sum_{a_i \in A_i,\, a_{-i} \in A_{-i}} \sigma_i(a) \vec{\sigma}_{-i}(\vec{a}_{-i})\, regret_i^{A_i}(a_i \mid \vec{a}_{-i}),$$

where, as before, $regret_i^{A_i}(a_i \mid \vec{a}_{-i})$ denotes the regret of player i relative to the actions A_i. That is, $regret_i'(\sigma_i \mid \vec{\sigma}_{-i})$ is calculated much like the expected utility of $\vec{\sigma}$ to agent i, in terms of the appropriate convex combination regrets for pure strategies. Note that $regret_i'$ is independent of any set S_i of strategies.

While we view players as actually choosing mixed strategies, intuitively, Renou and Schlag view players as really choosing only pure strategies. A mixed strategy is taken to represent the beliefs of the players regarding other players' strategies. With this viewpoint, to calculate the regret of an action a relative to a mixed strategy $\vec{\sigma}_{-i}$ for the other players, the player should calculate his regret relative to every pure strategy for the other players, and multiply that by the likelihood of that strategy occurring, according to his beliefs (i.e., according to $\vec{\sigma}_{-i}$). Thus, with this viewpoint, we should have

$$regret_i'(\sigma_i \mid \vec{\sigma}_{-i}) = \sum_{a_{-i} \in A_{-i}} \vec{\sigma}_{-i}(\vec{a}_{-i})\, regret_i'(\sigma_i \mid \vec{a}_{-i}).$$

The fact that $regret'(\sigma_i \mid \vec{a}_{-i})$ should be $\sum_{a_i \in A} \sigma_i(a_i)\, regret_i^{A_i}(a_i \mid \vec{a}_{-i})$ is noncontroversial. As the following lemma shows, both approaches agree on the regret relative to a pure strategy, although they disagree in general.

Lemma C.1. *If $\vec{\sigma}$ is a mixed strategy and $\vec{a}$ is a pure strategy, then $regret_i'(\sigma_i \mid \vec{a}_{-i}) = regret_i^{\Sigma_i}(\sigma_i \mid \vec{a}_{-i})$, but in general, $regret_i'(\sigma_i \mid \vec{\sigma}_{-i}) \neq regret_i^{\Sigma_i}(\sigma_i \mid \vec{\sigma}_{-i})$.*

Proof. For the first claim, it suffices to show that $regret_i^{\Sigma_i}(\sigma_i \mid \vec{a}_{-i}) = \sum_{a_i \in A} \sigma_i(a_i)\, regret^{A_i}(a_i \mid \vec{a}_{-i})$. It is easy to see that there is a pure strategy for player i that is a best response to $\vec{a}_{-i}$; call it a^*. Thus,

$$
\begin{aligned}
regret_i^{\Sigma_i}(\sigma_i \mid \vec{a}_{-i}) &= u_i(a^*, \vec{a}_{-i}) - u_i(\sigma_i, \vec{a}_{-i}) \\
&= u_i(a^*, \vec{a}_{-i}) - \sum_{a_i \in A_i} \sigma_i(a_i) u_i(a_i, \vec{a}_{-i}) \\
&= \sum_{a_i \in A_i} \sigma_i(a_i)\big(u_i(a^*, \vec{a}_{-i}) - u_i(a_i, \vec{a}_{-i})\big) \quad \left[\text{since } u_i(a^*, \vec{a}_{-i}) = \sum_{a_i \in A_i} \sigma_i(a_i) u_i(a^*, \vec{a}_{-i})\right] \\
&= \sum_{a_i \in A_i} \sigma_i(a_i)\, regret_i^{A_i}(a_i \mid \vec{a}_{-i}).
\end{aligned}
$$

For the second part, consider the following game, where $A_1 = A_2 = \{a, b\}$, and player 1's payoffs are given by the following table:

	a	b
a	3	0
b	0	2

Clearly $regret_1^{\Sigma_1}(a \mid a) = regret_1^{\Sigma_1}(b \mid b) = 0$, $regret_1^{\Sigma_1}(a \mid b) = 2$, and $regret_1^{\Sigma_1}(b \mid a) = 3$. Let σ be the strategy $(1/2)a + (1/2)b$. Then an easy computation shows that $regret'(a \mid \sigma) = 1$, while $regret_1^{\Sigma_1}(a \mid \sigma) = 0$ (since a is the best response to σ). $\quad\square$

In light of Lemma C.1, it is perhaps not surprising that the two approaches rank strategies differently. What is perhaps surprising is that, as we now show, if we are considering the strategy that minimizes regret with respect to all strategies, then it does not matter which approach we take.

Proposition C.2. *Let $G = ([n], A, \vec{u})$ be a strategic game and let σ_i be a mixed strategy for player i. Then $regret_i^{\Sigma}(\sigma_i) = \max_{\sigma_{-i} \in \Sigma_{-i}} regret_i'(\sigma_i \mid \sigma_{-i})$.*

Proof. By Proposition 3.19, $regret_i^{\Sigma_i}(\sigma_i) = \max_{\vec{a}_{-i} \in A_{-i}} regret_i^{\Sigma_i}(\sigma_i \mid \vec{a}_{-i})$. It also easily follows from the definition of $regret_i'$ that $\max_{\sigma_{-i} \in \Sigma_{-i}} regret_i'(\sigma_i \mid \sigma_{-i}) = \max_{\vec{a}_{-i} \in A_{-i}} regret_i'(\sigma_i \mid \vec{a}_{-i})$. Thus, it suffices to show that $regret_i^{\Sigma_i}(\sigma_i \mid \vec{a}_{-i}) = regret_i'(\sigma_i \mid \vec{a}_{-i})$ for all $\vec{a}_{-i} \in A_{-i}$; this follows from Lemma C.1. $\quad\square$

Proposition C.2 shows that, for many of the examples in Section 3.4.2, it would not matter whether we had used *regret′* instead of *regret*. On the other hand, the difference between the two approaches becomes more significant if we consider prior beliefs. Just as we do, Renou and Schlag consider regret minimization not just with respect to all strategies, but with respect to a set of strategies (representing a player's beliefs). When a player is considering his best response, he always considers all possible strategies for himself. Thus, in our language, Renou and Schlag consider generalized regret minimization with respect to profiles of sets of strategy profiles of the form $(\Sigma_1 \times \Pi_{-1}, \ldots, \Sigma_n \times \Pi_{-n})$; that is, each agent i puts no restriction on his own strategies. (They also require that Π_{-i} be a closed, convex set of strategies.) In general, *regret* and *regret′* differ with prior beliefs, as the following example shows.

Example C.3. Consider the game from Lemma C.1. Let σ_α denote the strategy $\alpha a + (1 - \alpha)b$; let Π_2 be the (closed, convex) set of strategies for player 2 of the form σ_α, where $\alpha \in [1/5, 3/5]$. It is easy to check that

$$regret_1^{\Sigma_1}(\sigma_\alpha \mid \sigma_{\alpha'}) = \begin{cases} 3\alpha' - (3\alpha\alpha' + 2(1-\alpha)(1-\alpha')) & \text{if } \alpha' \geqslant 2/5, \\ 2(1-\alpha') - (3\alpha\alpha' + 2(1-\alpha)(1-\alpha')) & \text{if } \alpha' < 2/5, \end{cases}$$

since a is the best response to $\sigma_{\alpha'}$ if $\alpha' \geqslant 2/5$, and otherwise b is the best response. Simplifying this expression, we get

$$regret_1^{\Sigma_1}(\sigma_\alpha \mid \sigma_{\alpha'}) = \begin{cases} 5(1-\alpha)(\alpha' - 2/5) & \text{if } \alpha' \geqslant 2/5, \\ 5\alpha(2/5 - \alpha') & \text{if } \alpha' < 2/5. \end{cases}$$

If player 1 believes that player 2 is playing a strategy in Π_2 (i.e., $\alpha' \in [1/5, 3/5]$), then the strategy that minimizes $regret_1^{\Sigma_1}$ is $\sigma_{1/2}$. The strategy $\sigma_{1/2}$ has regret 1/2; any other strategy will have greater regret relative to either $\sigma_{1/5}$ or $\sigma_{3/5}$. (Note that without this restriction on α', $\sigma_{3/5}$ would have minimized $regret_1$.) By way of contrast,

$$regret_1'(\sigma_\alpha \mid \sigma_{\alpha'}) = 2\alpha(1 - \alpha') + 3(1 - \alpha)\alpha',$$

and the strategy that minimizes $regret_1'$ is $\sigma_{3/5}$. This guarantees that $regret_1'$ is 6/5, no matter what player 2 does; on the other hand, if player 1 chooses σ_α and $\alpha > 3/5$, then $regret_1'(\sigma_\alpha \mid \sigma_{1/5}) > 6/5$, while if $\alpha < 3/5$, then $regret_1'(\sigma_\alpha \mid \sigma_{3/5}) > 6/5$.

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