Ontology Focusing: Knowledge-enriched Databases on Demand

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ABSTRACT

We propose a novel framework to facilitate the on-demand design of data-centric systems by exploiting domain knowledge from an existing ontology. Its key ingredient is a process that we call focusing, which allows to obtain a schema for a (possibly knowledgeenriched) database semi-automatically, given an ontology and a specification of the scope of the desired system. We formalize the inputs and outputs of focusing, and identify relevant computational problems: finding a schema via focusing, testing its consistency, and answering queries in the knowledge-enriched databases it produces. These definitions are fully independent from the ontology language. We then instantiate the framework using selected description logics as ontology languages, and popular classes of queries for specifying the scope of the system. For several representative combinations, we study the decidability and complexity of the identified computational problems. As a by-product, we isolate (and solve) variants of classical decision problems in description logics, that are interesting in their own right.

CCS CONCEPTS

• Information systems → Database design & models; • Computing methodologies → Description logics;

KEYWORDS

Ontologies; Description logics; Database Schemas; Open and Closed World Assumption

1 INTRODUCTION

In the design of data-centric systems, coming up with the right data organization (in terms of database schemas, integrity constraints, conceptual models, etc.) is of paramount importance. If well-chosen, it can make the implementation of the remaining functionality more evident, as it binds the developers to one shared and unambiguous view of the data to be managed by the target system. Unfortunately, coming up with the right data organization remains challenging and time-consuming, despite the many techniques and tools that are available to aid the design of data-centric systems. In addition, modern systems face further challenges, like incompleteness of information, or the need for interoperability with multiple other systems [1].

We propose a novel way to exploit domain knowledge captured in ontologies in the design of data-centric systems. Ontologies, understood here as logical theories expressing domain knowledge, provide a shared understanding of the domain to different users and applications; justified by expected reusability, considerable resources have been invested in constructing high-quality ontologies for many domains [21]. In data management they have already proved to be a powerful tool. Successful applications include data integration and querying incomplete data sources, where they are used on-line during the system operation to infer additional facts from incomplete data, and to provide a unified view of heterogenous data sources [6, 33, 44]. Here, we would like to use an existing ontology to produce, quickly and with moderate effort, datacentric systems on demand. To achieve this goal, two key challenges need to be overcome.

Specificity. Ontologies are typically broad, containing many terms irrelevant for the intended application. We need methods to restrict their scope to obtain more manageable conceptualizations.

Data completeness. Ontologies have an *open-world* semantics that treats data as incomplete, while every meaningful data-centric application will call for some completeness assumptions on (parts of) the data.

As a response to these challenges, we propose a process called *focusing* that allows us to trim away irrelevant information, and establish completeness assumptions. The goal of this process is to find so-called *focusing solutions*. Syntactically, a focusing solution provides a schema for a database and specifies how its instances are enriched with the knowledge from the ontology, by prescribing which 'parts' of the ontology are relevant and which are complete. Semantically, focusing solutions define a set of *intended models* for these knowledge-enriched databases: those that give the expected answers for the relevant queries.

Our main contribution is formalizing focusing solutions. The notion is independent from the ontology language and gives several options for specifying the scope of the system. We identify key computational problems relevant for obtaining and using focusing solutions. As an advanced proof of concept, we instantiate our general notions by considering a few choices of ontology languages and scope specifications. For these combinations, we study the decidability and complexity of the introduced computational problems. As a by-product, we isolate (and solve) variants of classical reasoning tasks that seem to be interesting in their own right.

2 GENERAL FRAMEWORK

We now discuss the key features and design choices in the framework we propose. We begin from a motivating use case.

2.1 Emergency response in a smart city

City authorities rely on an ontology to quickly build situation-specific applications supporting response to emergency events. The ontology contains, among others: (1) Data about the city, e.g., districts, population, facilities such as hospitals, schools, and sport centers, together with associated details like their capacity, size, and existing services; (2) The city's risk assessment data, detailing possible emergencies; (3) Knowledge about public health emergencies, e.g., types of emergencies include disease outbreaks, radiation emergencies, and disasters and weather emergencies; the latter include extreme heat, floods, hurricanes, and wild fires; (4) Emergency response knowledge, like types of responders (paramedics, firefighters, police officers, etc.); (5) General knowledge about relevant topics like weather, buildings, or cities.

Having such an ontology for this purpose is possible and realistic. In fact, many cities now compile and store data concerning (1) and (2), and there are vast repositories of online resources and readily available ontologies for (3)–(5); for example, see [32]. Moreover, maintaining such an ontology can be part of the routine preparation measures carried out in emergency response departments when there are no emergencies.

When a disaster happens, an automated focusing engine is fed with relevant parameters, to obtain focusing solutions that can help quickly build tools specific to this kind of disaster. For example, in many emergencies, so-called 'community assessment' questionnaires are used to gather critical public health data such as availability of drinking water and electricity in households. Existing guidelines suggest to first design the database that will be used to gather and analyze the data, and to guide the questionnaire design with it. Using an ontology like the one we have described, one could instead propose the database tables automatically. In the event of a disaster like a flood, one could quickly create an application for storing, for example, the current alert level in the different districts of the city and the complete list of operating shelters together with currently available resources, while disregarding any knowledge in the ontology related to other possible disasters, such as wild fires or extreme heat.

2.2 What should focusing be?

The starting point is an ontology, expressed as a theory in some logical formalism over a relational signature. Based on some specification of the intended scope of the system, we need to decide which predicates are to be supported by the system and how the database will interact with the ontology at run time. Let us assume that we only allow ourselves two make two decisions about each predicate: whether it will evolve during the run of the system and so should be stored in the database (dynamic vs static predicates), and whether it should be viewed as a complete representation of the data it represents (closed vs open predicates). The focusing solutions are designed to support these decisions, as well as the specification of the scope of the system. We now informally describe the four components of focusing solutions.

A bare-bones focusing solution is a **database schema**, collecting all dynamic predicates. Such system allows storing relevant

data and updating it as the reality evolves, but neither gives any completeness guarantees, nor improves the specificity.

We address the **data completeness** issue by declaring some predicates as *closed*. Semantically, these declarations trim the set of represented models, keeping only those that agree with the current database instance on the closed predicates. In particular, if we decide to close all dynamic predicates, we end up with a standard database, with the ontology acting as a set of integrity constraints.

The actual intention behind declaring a predicate as closed, is to commit the system to keep the stored content of the predicate identical to its real-world interpretation. Thus, closed dynamic predicates should be used to represent changing aspects of reality that are fully observable to the system. In our motivating example, these could include the precise list of districts for which an evacuation was ordered, the open shelters, assignments of personal to tasks, shelters, etc. Some of these aspects may be fully observable because they are actually controlled by the system (maybe the evacuation orders are issued by the system itself), for others we might rely on updates from some other trusted system.

We address the **specificity** issue by declaring some predicates as *fixed*. When we fix a predicate, we require its extension to be determined by the ontology alone. That is, in an intended model, a fixed predicate will contain a tuple of constants only if the ontology alone entails the corresponding atom. When we fix a predicate, it effectively becomes static and closed: even if it is stored in the database, it has only one allowed extension.

If a predicate is *not* populated by the ontology (which is quite common since most ontologies focus on *terminological* rather than *assertional* knowledge), fixing it enforces its extension to be empty. By fixing irrelevant predicates, we avoid reasoning about them and make the ontology more specific for the current situation. This reflects the intuition that the more specific our knowledge of the situation is, the more inferences we can draw from the ontology about our situation.

Fixed predicates will typically include aspects of reality that are captured in the ontology but irrelevant to our application, like the specifics of other types of emergencies not related to the current one. We can also fix predicates that are relevant, but correspond to immutable aspects of reality, and their extension is uniquely (and accurately) determined by the ontology. For instance, all the districts of a city, or all hospitals in a district.

A fixed predicate can in principle act as an additional integrity constraint. This happens if declaring the predicate as fixed discards all intended models for some instance, thus making the instance *inconsistent*. This is undesired, and will be explicitly forbidden in the definition of focusing solution.

Our focusing solutions already provide a database schema and an interface to the ontology that addresses the specificity and data completeness issues, but so far we have been assuming that the designer makes all the choices, as appropriate for the intended scope of the system. In order to support these choices using automated reasoning, or at least check that they are correct, we need the designer to provide a formal **specification of the scope**. We propose to specify the scope in terms of queries that will be posed when the system is running. We shall have three families of such queries.

Determined queries are the ones for which we want a guarantee that the answers entailed by the data and the ontology are complete with respect to all possible models. Equivalently, the answers do not depend on the concrete model, as long as it is compatible with the database contents and the ontology. Declaring a query as determined should be viewed as a demand of the designer: this query needs to be determined, how do we guarantee this?

Fixed queries generalize fixed predicates: complete answers to these queries are entailed by the ontology alone. Declaring a query as fixed is a decision rather than a demand: We freeze the answers as the ones entailed by the ontology alone, and models yielding more answers should be discarded. The more assumptions of this kind, the easier it is to make other queries determined.

Closed queries generalize closed (dynamic) predicates. For these queries, we are making the assumption that complete answers can be obtained from the data alone, by directly evaluating the query. Declaring a query as closed is a promise made by the designer: I am prepared to maintain the data in such a way that this query is closed. Again, the more assumptions of this kind, the easier it is to make other queries determined.

Note that, in fact, all three families of queries can be viewed as completeness assertions: closed queries talk about completeness of the data, fixed queries talk about completeness of the ontology, and determined queries talk about completeness of the combination of both. Allowing fixed and closed queries, rather than just predicates, gives a bit more flexibility to the designer. For example, it might not be reasonable to assume complete knowledge about all buildings in the city, or all hospitals in the country, but maintaining up-to-date information about these buildings in the city that are hospitals is a perfectly reasonable requirement.

The focusing problem is to find a focusing solution that guarantees that certain queries are determined, assuming that certain other queries are closed or fixed. These solutions are in general not unique and there are many trade-offs involved. For example, the more predicates are closed, the easier it is for a query to be determined, but we must pay the maintenance costs for each predicate we close. It seems desirable to have as few closed predicates as possible, provided we can guarantee that the suitable queries are determined. On the other hand, if suitable queries are already determined, it may be desirable to fix as many predicates as is possible without making any instances inconsistent. In the following subsection we formalize the outcome of this discussion.

What is a focusing solution? 2.3

We shall keep our notions independent from the specific formalisms used to express ontologies and queries.

We assume an infinite set Const of constants and an infinite set Rel of relation symbols. Each relation symbol $r \in \text{Rel}$ has a nonnegative integer arity, denoted by arity(r). An atom is an expression of the form $r(c_1, \ldots, c_n)$, where $r \in \text{Rel}, \{c_1, \ldots, c_n\} \subseteq \text{Const}$, and n = arity(r). A (database) instance I is any set of atoms. A signature Σ is any set of relation symbols. An instance I is over a signature Σ , if $r(\vec{t}) \in I$ implies $r \in \Sigma$. The active domain of I, denoted by adom(I), is the set of all constants in the atoms of I.

We assume an infinite set T of *theories*. Each theory $\varphi \in T$ is associated with a set of instances that are called *models* of φ . We write $I \models \varphi$ if I is a model of φ .

We assume an infinite set Q of *queries*. Each query $q \in Q$ has a non-negative integer arity, denoted arity(q). Each query $q \in Q$ is associated with a function $\llbracket \cdot \rrbracket_q$ that maps every instance I to an *n*-ary relation $[\![I]\!]_q \subseteq \mathbf{Const}^n$, where n = arity(q). Queries of arity 0 are called Boolean; their associated functions map instances to subsets of $Const^0 = \{\varepsilon\}$, where ε is the empty tuple. A Boolean query *holds* in an instance I, if $[\![I]\!]_q = \{\varepsilon\}$. We need the notion of *certain answers*. Given an *n*-ary query $q \in \mathbb{Q}$, a theory φ and an instance \mathcal{I} , we let $\llbracket \varphi, \mathcal{I} \rrbracket_q$ denote the set of n-tuples $\vec{u} \in \mathbf{Const}^n$ satisfying the following implication: if $\mathcal{J} \models \varphi$ and $I \subseteq \mathcal{J}$, then $\vec{u} \in [\![\mathcal{J}]\!]_q$.

A focusing configuration is a database schema together with three sets of queries representing completeness assertions about the data and about the theory, and determinacy assertions.

Definition 1 (Focusing configuration). A (focusing) configuration is a tuple $\mathcal{F} = (\Sigma, Q_{CL}, Q_{FIX}, Q_{DET})$, where $\Sigma \subseteq \mathbf{Rel}$ is a signature, and Q_{CL} , Q_{FIX} , $Q_{DET} \subseteq \mathbf{Q}$ are sets of queries. An instance I is legal for \mathcal{F} in case it is over Σ .

Let us explain how the four ingredients of a focusing configuration $\mathcal{F} = (\Sigma, Q_{\mathsf{CL}}, Q_{\mathsf{FIX}}, Q_{\mathsf{DET}})$ work. First, the signature Σ is a database schema that determines database instances of interest: an instance is *legal* for \mathcal{F} it is over the signature Σ .

The queries from Q_{CL} specify completeness assertions about data, effectively restricting the set of models represented by a legal instance as follows.

Definition 2 (Query-based Completeness). Given a theory φ , an instance I, and a set Q of queries, we let $CL(\varphi, I, Q)$ be the set of all instances $\mathcal J$ such that

- (1) $I \subseteq \mathcal{J}$,
- (2) $\mathcal{J} \models \varphi$, and (3) $\llbracket I \rrbracket_q = \llbracket \mathcal{J} \rrbracket_q$ for all $q \in Q$.

Intuitively, $CL(\varphi, \mathcal{I}, Q)$ contains exactly the models of φ and \mathcal{I} that provide no new information about the queries in Q compared to the information given by \mathcal{I} alone.

The queries in Q_{FIX} specify completeness assertions about the theory, further restricting the set of represented models.

Definition 3 (Query-based Fixing). For a theory φ , an instance I, and a set Q of queries, we let $FIX(\varphi, I, Q)$ be the set of all instances J such that

- (1) $I \subseteq \mathcal{J}$,
- (2) $\mathcal{J} \models \varphi$, and
- (3) $\llbracket \mathcal{J} \rrbracket_q = \llbracket \varphi, \emptyset \rrbracket_q \text{ for all } q \in Q.$

Intuitively, $FIX(\varphi, I, Q)$ contains exactly the models of φ and Ithat provide no new information about the queries in Q compared to the information given by φ alone.

Thus, a configuration $\mathcal{F} = (\Sigma, Q_{CL}, Q_{FIX}, Q_{DET})$ tells us to consider only database instances I over Σ as legal. For each such instance I, we are to assume that the information retrieved from I alone by the queries in Q_{CL} is complete, and we are to restrict our attention to models where the answers to queries in Q_{FIX} are

frozen to the facts inferred from the initial theory φ (without any data). These are the intended models represented by a concrete legal database instance I.

Definition 4 (Intended models). For a theory φ , a configuration $\mathcal{F} = (\Sigma, Q_{CL}, Q_{FIX}, Q_{DET})$, and an instance I over Σ , let

$$\mathsf{MOD}(\varphi,\mathcal{F},\mathcal{I}) = \mathsf{CL}(\varphi,\mathcal{I},Q_\mathsf{CL}) \cap \mathsf{FIX}(\varphi,\mathcal{I},Q_\mathsf{FIX}) \,.$$

We are now ready to provide the definition of focusing, which makes the role of Q_{DET} precise: we shall demand that freezing answers to the queries from Q_{FIX} does not affect consistency of instances and that the answers to the queries in Q_{DET} coincide over all intended models.

Definition 5 (Focusing). A focusing solution for a theory φ is a configuration $\mathcal{F} = (\Sigma, Q_{\text{CL}}, Q_{\text{FIX}}, Q_{\text{DET}})$ such that the next two conditions are obeyed for all instances I over Σ :

- (1) if $CL(\varphi, I, Q_{CL}) \neq \emptyset$, then $MOD(\varphi, \mathcal{F}, I) \neq \emptyset$;
- (2) for all $\mathcal{J}_1, \mathcal{J}_2 \in \mathsf{MOD}(\varphi, \mathcal{F}, \mathcal{I})$, and all $q \in Q_{\mathsf{DET}}$, we have $[\![\mathcal{J}_1]\!]_q = [\![\mathcal{J}_2]\!]_q$.

Let us see how this definition captures the scenario discussed in the previous subsection. Suppose that the designer specifies a set Q_{DET} determined queries that need to be guaranteed as well as sets Q_{CL}^u of queries that are promised to be closed, and Q_{FIX}^u of queries are chosen to be interpreted as fixed. We now want to find a correct focusing solution of the form

$$\mathcal{F} = (\Sigma, Q_{CL}, Q_{FIX}, Q_{DET})$$

with $Q_{\mathsf{CL}}^u \subseteq Q_{\mathsf{CL}}$ and $Q_{\mathsf{FIX}}^u \subseteq Q_{\mathsf{FIX}}$. There may be many such focusing solutions, and they are all good in the sense that they guarantee correct answers to Q_{DET} . This leaves some space to accommodate additional preferences of the designer; we discuss it in the following subsection.

2.4 Getting and using focusing solutions

We shall now identify key reasoning problems crucial in obtaining and using focusing solutions. For each problem the input includes a focusing configuration \mathcal{F} ; some problems additionally input theories, database instances, and queries. To be able to speak about concrete formalisms, we parameterize the problems by query and ontology languages. We write T: \mathcal{L}_{TH} to indicate that the language used for expressing theories is $\mathcal{L}_{TH} \subseteq T$. Similarly, we use C: \mathcal{L}_{CL} , F: \mathcal{L}_{FIX} , D: \mathcal{L}_{DET} , and Q: \mathcal{L}_{Q} for \mathcal{L}_{CL} , \mathcal{L}_{FIX} , \mathcal{L}_{DET} , $\mathcal{L}_{Q} \subseteq Q$.

The main problem is recognizing focusing solutions among focusing configurations.

```
FOCUS(T: \mathcal{L}_{TH}, C: \mathcal{L}_{CL}, F: \mathcal{L}_{FIX}, D: \mathcal{L}_{DET})

Input: A pair (\varphi, \mathcal{F}) with \mathcal{F} = (\Sigma, Q_{CL}, Q_{FIX}, Q_{DET}), and

- \varphi \in \mathcal{L}_{TH},

- \Sigma \subseteq \text{Rel},

- Q_{CL} \subseteq \mathcal{L}_Q, Q_{FIX} \subseteq \mathcal{L}_{FIX}, Q_{DET} \subseteq \mathcal{L}_{DET}.

Question: Is \mathcal{F} a focusing solution to \varphi?
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Thus, in the focusing problem a candidate focusing configuration is given, and the task is to decide if it is a focusing solution. In the scenario discussed previously, the input consists of closed queries $Q^u_{\rm CL}$, fixed queries $Q^u_{\rm FIX}$, and determined queries $Q^{\rm DET}_{\rm CL}$, and

the output is a focusing solution $(\Sigma, Q_{CL}, Q_{FIX}, Q_{DET})$ with $Q_{CL}^u \subseteq$ Q_{CL} and $Q_{FIX}^u \subseteq Q_{FIX}$. If we consider for Q_{CL} and Q_{FIX} a query language that gives us a finite number of candidates, like atomic queries (which covers the basic scenario with closed and fixed predicates), the recognition problem can be used directly in the search for such a solution by applying exhaustive search. This search can be guided by some preferences of the designer. For example, a basic strategy could be to minimize the set of closed queries, and then maximize the set of fixed ones. More sophisticated strategies could involve a specified order in which the designer prefers to close predicates, reflecting, for instance, the cost of maintaining them (size statistics, availability, acquisition costs, etc). One could also consider semi-automated approaches, like a dialog approach where successive solutions are proposed to the designer, who adjusts the specification, or even the ontology, and accepts some suggested choices while rejecting others, thus converging to a satisfactory focusing solution. We leave investigating such strategies to future research.

An additional criterion that might be useful in the search for suitable focusing solutions is the existence of consistent database instances. This is embodied in the following decision problem. Here $\mathcal P$ stands for a collection of parameters.

```
EMPTINESS(\mathcal{P})

Input: A triple (\varphi, \mathcal{F}), where (\varphi, \mathcal{F}) \in FOCUS(\mathcal{P})

Question: Is MOD(\varphi, \mathcal{F}, I) = \emptyset for each I legal for \mathcal{F}?
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Obviously a single consistent database instance does not make a focusing solution very useful, but the criterion can help eliminate some utterly useless solutions.

Assuming that a satisfactory focusing solution is found, how do we use it? Two reasoning tasks are crucial in the operation of the system resulting from focusing. First, whenever a tuple is inserted, deleted, or updated, the system needs to check that the resulting database instance is still consistent. This makes the feasibility of the following problem essential.

```
CONSISTENCY(\mathcal{P})

Input: A triple (\varphi, \mathcal{F}, I), where

- I is legal for \mathcal{F},

- (\varphi, \mathcal{F}) \in FOCUS(\mathcal{P}).

Question: MOD(\varphi, \mathcal{F}, I) \neq \emptyset?
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Finally, the system is there to be queried. This makes the following entailment problem relevant.

```
ENTAILMENT(Q: \mathcal{L}_{\mathbf{Q}}, \mathcal{P})

Input: A tuple (\varphi, \mathcal{F}, I, q), where

- I is legal for \mathcal{F},

- q \in \mathcal{L}_{\mathbf{Q}} is Boolean,

- (\varphi, \mathcal{F}) \in \mathsf{FOCUS}(\mathcal{P}).

Question: Is q true in all \mathcal{J} \in \mathsf{MOD}(\varphi, \mathcal{F}, I)?
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Note that CONSISTENCY is a special case of non-ENTAILMENT, but not conversely, so complexity of the two problems might differ.

3 CONCRETE PROBLEMS

To illustrate some concrete settings that may be useful, in this section we instantiate the reasoning problems with some selected languages. We discuss how the problems can be solved in these concrete cases, and provide complexity results for them.

3.1 Ontology and Query Languages

Here we recall the concrete formalisms for expressing theories and queries that we study. As ontology languages we look at description logics (DLs), and as query languages we consider instance queries, atomic queries, and conjunctive queries (CQs).

Description Logics (DLs). DLs is a family of logics, specifically tailored for writing ontologies [4]. Most DLs are fragments of first-order logic, which allow only unary and binary relation symbols. This and other restrictions allow DLs to be equipped with a special syntax, which allows to write formulas in a more concise way. We now introduce the main DL of this paper, called ALCHOIF.

Let $N_C \subseteq \text{Rel}$ be a countably infinite set of unary relation symbols, called *concept names*, and let $N_R \subseteq \text{Rel}$ be a countably infinite set of binary relation symbols, called *role names*. If $r \in N_R$, then rand the expression r^- are roles (r^- is also called the *inverse of r*). For a role r^- , we let $(r^-)^- = r$. We let $N_R^+ = N_R \cup \{r^- \mid r \in N_R\}$. The set of $\mathcal{ALCHOIF}$ concepts is defined inductively as follows: (a) every concept name $A \in N_C$ is a concept; (b) the expressions \top and \bot are concepts; (c) the expression $\{c\}$, where $c \in \mathbf{Const}$, is also a concept (called *nominal*); (d) if *C*, *D* are concepts, and *p* is a role, then $\neg C$, $C \sqcap D$, $C \sqcup D$, $\exists p.C$ and $\forall p.C$ are also concepts. A *concept inclusion* is an expression of the form $C \sqsubseteq D$, where C, Dare concepts. A role inclusion is an expression of the form $r \sqsubseteq p$, where r, p are roles. A functionality assertion is an expression of the form func(p), where p is a role. An $(\mathcal{ALCHOIF})$ ontology O is a finite set of concept inclusions, role inclusions, and functionality assertions. The DL that is obtained by disallowing functionality assertions, role inclusions, or nominals is indicated by, respectively, dropping 'F', 'H', or 'O' from its name. An ontology is in the DL $\mathcal{EL}I_{\perp}$, if it contains only concept inclusions, and they are built using only the constructs \sqcap and \exists (i.e., nominals, \sqcup , and ∀, as well as role inclusions and functionality assertions, are forbidden). We use $N_C(O)$ and $N_R(O)$ to denote the sets of concept names and role names that appear in O, respectively. We let $N_{R}^{+}(O) = N_{R}(O) \cup \{r^{-} \mid r \in N_{R}(O)\}.$

The semantics to ontologies is given using instances, defined in Section 2.3. Assume an instance I and an ontology O. We define a function \cdot^I that maps every concept C to a set $C^I \subseteq \mathbf{Const}$, and every role p to a relation $p^I \subseteq \mathbf{Const} \times \mathbf{Const}$. For a concept name A, and a role name r, we let $A^I = \{c \mid A(c) \in I\}$ and $r^I = \{(c,d) \mid r(c,d) \in I\}$. The function \cdot^I is then extended to the

remaining concepts and roles as follows:

$$\begin{split} & \top^{I} = \operatorname{adom}(I), \qquad \bot^{I} = \emptyset \,, \qquad \{c\}^{I} = \{c\} \,, \\ & (C \sqcup D)^{I} = C^{I} \cup D^{I} \,, \qquad (C \sqcap D)^{I} = C^{I} \cap D^{I} \,, \\ & (\neg C)^{I} = \operatorname{adom}(I) \setminus C^{I} \,, \\ & (\exists p.C)^{I} = \{c \mid \exists d : (c,d) \in p^{I} \text{ and } d \in C^{I} \} \,, \\ & (\forall p.C)^{I} = \{c \mid \forall d : (c,d) \in p^{I} \text{ implies } d \in C^{I} \} \,, \\ & (r^{-})^{I} = \{(c,d) \mid (d,c) \in r^{I} \} \,. \end{split}$$

We say I is a model of O, if the following are satisfied: (1) $\alpha^I \subseteq \beta^I$ for all concept and role inclusions $\alpha \sqsubseteq \beta \in O$, and (2) $(c,d_1) \in p^I$ and $(c,d_2) \in p^I$ imply $d_1 = d_2$, for all $c \in Const$ and func $(p) \in O$. We note that by defining the semantics using database instances, instead of general first-order structures (or interpretations, as they are called in DLs), we are effectively interpreting ontologies under the Standard Name Assumption (SNA). That is, the domain of interpretation is always the set Const, and the interpretation of constants is given by the identity function. However, the active domain of the instance, as defined in Section 2.3, may well be a proper subset of Const.

In order to simplify presentation, when providing upper bounds we can concentrate wlog. on ontologies in *normal form*, which is defined as following. A *simple concept B* is any concept in $Const \cup \{\top, \bot\} \cup \{\{c\} \mid c \in Const\}$. We use $N_{C}^{+}(O)$ to denote the set of simple concepts that appear in O. An ontology O is in *normal form* if all its statements are of one of the following forms:

$$B_1 \sqcap \ldots \sqcap B_{k-1} \sqsubseteq B_k \sqcup \ldots \sqcup B_m$$
,
 $B_1 \sqsubseteq \exists r.B_2$, $B_1 \sqsubseteq \forall r.B_2$, $r \sqsubseteq s$, func (r) .

where B_1, \ldots, B_m are simple concepts, k > 1, $m \ge k$, and r, s are roles. It is well known that any ontology O can be transformed into an ontology O' in normal form such that O and O' have the same models up to the signature of O.

We will also study DLs whose definition is based on the above normal form. In particular, $DL\text{-}Lite^{\mathcal{HOF}}_{\mathrm{Bool}}$ ontologies are ontologies in normal form that additionally satisfy the following:

- (i) for all $B_1 \sqsubseteq \exists r.B_2$, we have $B_2 = \top$,
- (ii) for all $B_1 \sqsubseteq \forall r.B_2$, we have $B_1 = \top$, and
- (iii) functional roles have no subroles: func(s) $\in O$ and $r \sqsubseteq s \in O$ imply r = s.

The DL DL- $Lite^{\mathcal{HF}}$ is obtained by applying the above restrictions, but additionally prohibiting nominals, as well as \sqcup and \sqcap ; that is, we only have concept inclusions of the form $B_1 \sqsubseteq B_2$ and the two forms (i) and (ii) shown above. Observe that

$$\mathit{DL\text{-}Lite}^{\mathcal{HF}} \subseteq \mathit{DL\text{-}Lite}^{\mathcal{HOF}}_{Bool} \subseteq \mathcal{ALCHOIF} \,.$$

Queries. We let CQ be the class of conjunctive queries (primitive positive first-order formulas) over the signature $N_C \cup N_R$, with the usual semantics. Occasionally we talk about $\mathcal{U}CQ$, the class of unions of conjunctive queries (UCQs), corresponding to positive existential first-order formulas. We also consider the class $\mathcal{A}Q \subseteq CQ$ of atomic queries, and the class $IQ \subseteq \mathcal{A}Q$ of instance queries, that is, atomic queries over concepts. If $Q \subseteq \mathcal{A}Q$, we can view Q as a set of predicates.

3.2 Recognizing focusing solutions

In this section we deal with recognizing focusing solutions for \mathcal{ALCHOI} ontologies. The two conditions in the definition of focusing solutions (Definition 5) are closely related to natural variants of two classical problems in description logics.

The first condition of Definition 5 boils down to a variant of the classical query entailment problem that only considers models where selected predicates have finite extensions (the remaining predicates may have a finite or an infinite extension).

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\begin{aligned} & \text{MIXED-OMQA}(\text{T:}\ \mathcal{L}_{\text{TH}}, \text{Q:}\ \mathcal{L}_{\text{Q}}) \\ & & \text{Input:} \quad \text{A tuple}\ (\varphi, \Sigma, I, q) \text{ where} \\ & & - \varphi \in \mathcal{L}_{\text{TH}}, \\ & & - \Sigma \subseteq \text{Rel}, \\ & & - I \text{ is an instance}, \\ & & - q \in \mathcal{L}_{\text{Q}} \text{ is Boolean}. \\ & & Question: \quad \text{Is } q \text{ satisfied in every model } \mathcal{J} \text{ of } \varphi \text{ such that} \\ & & - I \subseteq \mathcal{J}, \text{ and} \\ & & - R^{\mathcal{J}} \text{ is finite for all } R \in \Sigma? \end{aligned}
```

This mixed variant generalizes the usual finite and unrestricted variants of OMQA. Recall that a logic has finite controllability if, for each theory expressed in this logic, non-entailed queries have finite counter-models. For logics enjoying finite controllability, the three variants of OMQA coincide. This is the case for \mathcal{ALCHOI} , which makes our problem 2ExpTIME-complete [11].

In the second condition of Definition 5, the computational crux of the matter is the following problem.

```
NULLABILITY(T: \mathcal{L}_{TH}, C: \mathcal{L}_{CL}, Q: \mathcal{L}_{Q})

Input: A tuple (\varphi, \Sigma, Q_{CL}, q) where

- \varphi \in \mathcal{L}_{TH},
- Q_{CL} \subseteq \mathcal{L}_{CL},
- q \in \mathcal{L}_{Q}.
Question: Do all I over \Sigma with CL(\varphi, I, Q_{CL}) \neq \emptyset admit
\mathcal{J} \in CL(\varphi, I, Q_{CL}) \text{ with } \|\mathcal{J}\|_{q} = \emptyset?
```

The nullability problem is closely related to the query emptiness problem, which is known to be NExpTime-complete for atomic queries, \mathcal{ALCI} ontologies, and no closed predicates [3]. We show that nullability of atomic queries for \mathcal{ALCHOI} with closed instance queries (i.e., closed concepts) is in CONExpTime^{NP}. Allowing closed roles rather quickly leads to undecidability, but for sufficiently restricted DLs we regain decidability.

Theorem 1. The following hold:

- (1) NULLABILITY(T: ALCHOI, C: IQ, Q: AQ) is in CONExpTime^{NP};
- (2) NULLABILITY(T: \mathcal{ELI}_{\perp} , C: \mathcal{AQ} , Q: \mathcal{AQ}) is undecidable;
- (3) NULLABILITY(T: DL- $Lite_{Bool}^{O}$, C: $\mathcal{A}Q$, Q: $\mathcal{A}Q$) is in $CONExpTIME^{NP}$.

PROOF. Given an \mathcal{ALCHOI} ontology $O, \Sigma \subseteq N_{\mathbb{C}}(O) \cup N_{\mathbb{R}}(O)$, $Q_{\mathbb{CL}} \subseteq IQ$, and $q \in \mathcal{AQ}$, we need to check that for each I over Σ with non-empty $\mathsf{CL}(O,I,Q_{\mathbb{CL}})$ there exists $\mathcal{J} \in \mathsf{CL}(O,I,Q_{\mathbb{CL}})$

with $[\![\mathcal{J}]\!]_q = \emptyset$. We first show that if this is not the case, then there is a witnessing instance I' over Σ with $|\mathsf{adom}(I')| \le 2^{|\mathsf{N}_{\mathsf{C}}^+(O)|}$.

In what follows, a type T is any subset of $N_C^+(O)$ such that $T \in T$ and $L \notin T$. For an instance \mathcal{J} and $c \in \mathbf{Const}$, the *type of c* in \mathcal{J} is the set of concepts $B \in N_C^+(O)$ such that $c \in B^{\mathcal{J}}$.

Let I be an instance over Σ such that $\mathrm{CL}(O,I,Q_{\mathrm{CL}}) \neq \emptyset$ and $[\![\mathcal{J}]\!]_q \neq \emptyset$ for all $\mathcal{J} \in \mathrm{CL}(O,Q_{\mathrm{CL}})$. Let $\mathcal{J} \in \mathrm{CL}(O,I,Q_{\mathrm{CL}})$. Let I' be obtained from I by identifying constants that have the same type in \mathcal{J} ; when some constants are identified, all edges incident with them become incident with the resulting constant; nominals remain themselves.

It is easy to see that $CL(O, I', Q_{CL}) \neq \emptyset$. A suitable witness \mathcal{J}' is obtained from \mathcal{J} by identifying constants in adom(I) that have the same type in \mathcal{J} . By construction, \mathcal{J}' is an extension of I' that is a model of O and preserves closed concepts.

Suppose that $[\![\mathcal{M}']\!]_q = \emptyset$ for some $\mathcal{M}' \in \operatorname{CL}(O, I', Q_{\operatorname{CL}})$. We shall turn \mathcal{M}' into $\mathcal{M} \in \operatorname{CL}(O, I, Q_{\operatorname{CL}})$ such that $[\![\mathcal{M}]\!]_q = \emptyset$. Let us replace the copy of I' contained in \mathcal{M}' by a copy of I. That is, $\operatorname{adom}(\mathcal{M}) = (\operatorname{adom}(\mathcal{M}') \setminus \operatorname{adom}(I')) \uplus \operatorname{adom}(I)$. We now lift the interpretation of concepts and roles from \mathcal{M}' . For $c \in \operatorname{adom}(I)$, let c' be the unique constant in $\operatorname{adom}(I')$ that has the same type in \mathcal{J} as c. For $c \in \operatorname{adom}(\mathcal{M}') \setminus \operatorname{adom}(I')$, let c' = c. We let $c \in \mathcal{A}^{\mathcal{M}}$ iff $c' \in \mathcal{A}^{\mathcal{M}'}$ for $A \in \operatorname{N}_{\mathbb{C}}(O)$ and $(c,d) \in r^{\mathcal{M}}$ iff $(c',d') \in \mathcal{A}^{\mathcal{M}'}$ for $r \in \operatorname{N}_{\mathbb{R}}(O)$. By construction, \mathcal{M} is an extension of I that is a model of O, preserves closed concepts, and realizes the same types. It follows that $\mathcal{M} \in \operatorname{CL}(O, I, Q_{\operatorname{CL}})$ and $[\![\mathcal{M}]\!]_q = \emptyset$.

We can now describe the algorithm. We guess universally an instance I over Σ with $|adom(I)| \le 2^{|N_C^+(O)|}$. We should accept if either $CL(O, I, Q_{CL}) = \emptyset$ or $CL(O \cup \{\alpha_q\}, I, Q_{CL}) \neq \emptyset$, where $\alpha_{A(x)}$ is $A \sqsubseteq \bot$ and $\alpha_{r(x,y)}$ is $\exists r. \top \sqsubseteq \bot$. Both these checks amount to deciding if there is a model $\mathcal J$ of a given ontology O' that extends a given instance I while preserving a given set of closed concepts. For \mathcal{ALCHOI} ontologies this problem is in NP in terms of data complexity [28]. More precisely, it can be solved by a nondeterministic algorithm with running time $poly(|\mathcal{I}|) \cdot 2^{poly(|\mathcal{O}'|)}$, which is sufficient to guarantee the CONEXPTIME NP upper bound. One way to do this is as follows. Guess the restriction \mathcal{J}_0 of \mathcal{J} to adom(I). Use type elimination to compute realizable types. Start from the set of types that are realized in \mathcal{J}_0 or do not contain closed concepts. Iteratively remove types that cannot have their existential restrictions satisfied using available types; for types realized in \mathcal{J}_0 ignore existential restrictions that are already fulfilled in \mathcal{J}_0 . When the set of available types stabilizes, accept iff it contains each type realized in \mathcal{J}_0 .

Undecidability is obtained via a reduction from the halting problem for deterministic Turing machines, relying on the ability enforce that the counter witness to nullability is a grid, and the upper bound for the third problem is obtained by a a polynomial reduction to the first problem (see Appendix for details).

Let us now see that the two introduced problems indeed help solve the focusing problem, and that lower bounds and undecidability propagate to focusing as well. THEOREM 2. The problems

FOCUS(T: $\mathcal{A}\mathcal{L}CHOI$, C: IQ, F: $\mathcal{A}Q$, D: CQ), FOCUS(T: $\mathcal{A}\mathcal{L}CHOI$, C: $\mathcal{A}Q$, F: \emptyset , D: CQ), FOCUS(T: $\mathcal{A}\mathcal{L}CO$, C: \emptyset , F: \emptyset , D: CQ)

are 2ExpTime-complete, and

FOCUS(T:
$$\mathcal{ELI}_{\perp}$$
, C: \mathcal{AQ} , F: \mathcal{AQ} , D: \emptyset)

is undecidable.

PROOF. We begin with the upper bound for the first problem. Let O be an \mathcal{ALCHOI} ontology and let $\mathcal{F} = (\Sigma, Q_{\text{CL}}, Q_{\text{FIX}}, Q_{\text{DET}})$ with $\Sigma \subseteq \mathsf{N}_{\mathsf{C}}(O) \cup \mathsf{N}_{\mathsf{R}}(O), Q_{\mathsf{CL}} \subseteq IQ, Q_{\mathsf{FIX}} \subseteq \mathcal{A}Q$, and $Q_{\mathsf{DET}} \subseteq CQ$. As a first step, we reduce to the case with only one fixed query q_{FIX} , that additionally satisfies $[\![O,\emptyset]\!]_{q_{\mathsf{FIX}}} = \emptyset$. Let O' be the following extension of O. For each $q \in Q_{\mathsf{FIX}}$, introduce a fresh concept name A_q . If q = A(x) and $[\![O,\emptyset]\!]_q = \{(a_1),\ldots,(a_k)\}$, axiomatize A_q as

$$A_q \equiv A \sqcap \neg \{a_1, \ldots, a_k\}.$$

If q = r(x, y) and $[O, \emptyset]_q = \bigcup_{i=1}^k \{(a_i, b_{i,1}), \dots, (a_i, b_{i,j_i})\}$, axiomatize A_q as

$$A_q \equiv (\neg \{a_1, \dots, a_k\} \sqcap \exists r. \top) \sqcup \bigsqcup_i \{a_i\} \sqcap \exists r. \neg \{b_{i,1}, \dots, b_{i,j_k}\}.$$

Finally, add yet another fresh concept name B, axiomatized as

$$B \equiv \bigsqcup_{q} A_{q} .$$

Let $q_{\mathsf{F}|\mathsf{X}} = B(x)$. By construction, \mathcal{F} is a focusing solution for O iff $\mathcal{F}' = (\Sigma, Q_{\mathsf{CL}}, \{q_{\mathsf{F}|\mathsf{X}}\}, Q_{\mathsf{DET}})$ is a focusing solution for O'. Notice that O' has polynomial size, but computing it involves finding $[\![O,\emptyset]\!]_q$ for each $q \in Q_{\mathsf{F}|\mathsf{X}}$. As we have already mentioned, for \mathcal{FLCHOI} this is in $2\mathsf{ExpTime}$ [11], so we are within the intended complexity bounds.

Thus, without loss of generality we can assume that our input is O and $\mathcal{F} = (\Sigma, Q_{CL}, \{B(x)\}, Q_{DET})$ such that $B \in N_C(O)$ and $[\![O,\emptyset]\!]_{B(x)} = \emptyset$. For such inputs, the first condition of Definition 5 is an instance of NULLABILITY (T: \mathcal{ALCHOI} , C: \mathcal{AQ} , Q: \mathcal{AQ}), which we shall prove to be in CONEXPTIME $^{NP} \subseteq 2EXPTIME$ (Theorem 1). For the second condition of Definition 5, we first observe that we can eliminate the single fixed query B(x) without affecting the set of intended models: it suffices to add the concept inclusion $B \sqsubseteq \bot$ to O. Then, to reduce the second condition to mixed query answering, we first introduce fresh duplicates A' and r' for all $A \in N_{\mathbb{C}}(O)$ and $r \in N_R(O)$ that do not occur in Q_{CL} . Next, we construct O' and q' for all $q \in Q_{DET}$ by replacing original predicates in Q and q with their duplicates. It is straightforward to see that $[\![\mathcal{J}_1]\!]_q = [\![\mathcal{J}_2]\!]_q$ for all I and $\mathcal{J}_1, \mathcal{J}_2 \in \mathsf{MOD}(O, \mathcal{F}, I)$ iff $q \subseteq q'$ over all models of $O \cup O'$ with finite predicates Q_{CL} . Query containment can be reduced to query answering in the usual way, by incorporating the query q into the ontology using nominals; the presence of finiteness assumptions (in both problems) does not affect the construction. This way we have reduced the second condition to $|Q_{DET}|$ polynomial-size instances of mixed query answering. As we have argued, the latter is in 2ExpTime for \mathcal{ALCHOI} , which gives the desired upper bound for the focusing problem.

To obtain the upper bound for the second problem, note that in the absence of fixed predicates the first condition of Definition 5 trivializes and the second condition can be checked exactly like before, but without the first preprocessing step.

For the third problem the upper bound follows directly from either of the previous cases. To show the lower bound, we reduce from the standard unrestricted query answering problem for \mathcal{ALCO} [31]. Let O be an \mathcal{ALCO} ontology and let $q \in CQ$. Let O' be the ontology obtained from O by introducing a fresh concept name A, axiomatized with $\{c\} \subseteq A$ and $A \subseteq \forall r.A$ for all $\{c\} \in \mathbb{N}_{\mathbb{C}}^+(O)$, $r \in \mathbb{N}_{\mathbb{R}}^+(O)$, and replacing each concept inclusion $C \subseteq D$ with $C \sqcap A \subseteq D$. Each model I of O can be turned into a model of O' by letting $A^I = \operatorname{adom}(I)$; each model I' of O' can be turned into a model of O by restricting it to $A^{I'}$. Consequently, $[\![O,\emptyset]\!]_q = [\![O',\emptyset]\!]_q$, but there is a model I' of O' that satisfies I0, because outside of I1 no restrictions are imposed by I2. It follows that I3 is a focusing solution for I3 if I4 entails I5.

For the last claim, note that if $[\![\mathcal{O},\emptyset]\!]_q=\emptyset$, then the following are equivalent:

- (O, Σ, Q_{CL}, q) is a positive instance of NULLABILITY;
- $(O, (\Sigma, Q_{CL}, \{q\}, \emptyset))$ is a positive instance of FOCUS.

Because the reduction in the proof of Theorem 1 (2) gives a query that has the above property, we can conclude that this variant of the focusing problem is undecidable.

3.3 Entailment

The goal of this section is to prove the following theorem.

THEOREM 3. The problem

ENTAILMENT(T:
$$\mathcal{ALCHOI}$$
, C: CQ , F: \emptyset , Q: CQ)

is 2ExpTime-complete in combined complexity, and coNP-complete in data complexity.

Let O be an \mathcal{ALCHOI} KB, Q_{CL} a set of CQs, q a Boolean CQ, and I an instance. Unless specified otherwise, we will always consider Σ_O -instances where Σ_O is the signature of O, i.e., the set of concept and role names occurring in O. Our goal is to decide if for every instance $\mathcal{J} \in \mathsf{CL}(O,I,Q_{\text{CL}}), \mathcal{J} \models q$. This is clearly equivalent to the problem of finding a counter model for q: an instance $\mathcal{J} \in \mathsf{CL}(O,I,Q_{\text{CL}})$ such that $\mathcal{J} \nvDash q$. Moreover, it suffices to consider counter models that are almost forests. An instance \mathcal{J} is a tree extension of I if $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1$ such that $\mathrm{adom}(\mathcal{J}_0) = \mathrm{adom}(I)$ and \mathcal{J}_1 is a collection of trees of bounded degree in which elements of $\mathrm{adom}(I)$ occur only in leaves (and never in roots, even if they are also leaves). Note that the partition of \mathcal{J} into \mathcal{J}_0 and \mathcal{J}_1 is unique. We call \mathcal{J}_1 the forest of \mathcal{J} .

LEMMA 1. The following are equivalent:

- (1) There exists $\mathcal{J} \in CL(O, I, Q_{CL})$ such that $\mathcal{J} \nvDash q$;
- (2) There exists $\mathcal{T} \in \mathsf{CL}(O, I, Q_{\mathsf{CL}})$ such that $\mathcal{T} \nvDash q$ and \mathcal{T} is a tree extension of I.

Next we observe that $[\![\mathcal{J}]\!]_{q'} = [\![\mathcal{I}]\!]_{q'}$ for every query $q' \in Q_{\text{CL}}$ can be reformulated as a non-entailment problem of a UCQ capturing 'bad matches' of queries in Q_{CL} . Intuitively, a match π of a query $q'(\vec{x})$ in Q_{CL} is bad if $\pi(\vec{x}) \notin [\![\mathcal{I}]\!]_{q'}$, or if it maps an answer variable to some d not in $adom(\mathcal{I})$. To this aim, we assume

below that we have two concept names A_0 and \bar{A}_0 such that for each instance \mathcal{J} in $CL(O, I, Q_{CL})$ we have $A_0^{\mathcal{J}} = \operatorname{adom}(I)$ and $\bar{A}_0^{\mathcal{J}} = \operatorname{adom}(\mathcal{J}) \setminus \operatorname{adom}(I)$.

Lemma 2. For every $q(\vec{x}) = \exists \vec{y} \, \varphi(\vec{x}, \vec{y})$ in Q_{CL} , let

$$\widehat{q} = \bigvee_{\overrightarrow{a} \notin \llbracket T \rrbracket_q} \exists \overrightarrow{y} \, \varphi(\overrightarrow{a}, \overrightarrow{y}) \ \lor \bigvee_{x \in \overrightarrow{x}} \exists \overrightarrow{x} \, \exists \overrightarrow{y} \, A_0(x) \land \varphi(\overrightarrow{x}, \overrightarrow{y}) \ \ and \ \ \widehat{Q} = \bigvee_{q \in Q_{\mathsf{CL}}} \widehat{q} \, .$$

Then
$$\mathcal{J} \in \mathsf{CL}(O, I, Q_{\mathsf{CL}})$$
 iff $\mathcal{J} \models (O, I)$ and $\mathcal{J} \nvDash \widehat{Q}$.

Putting all pieces together, the problem thus reduces to deciding the existence of a tree-like model of O that extends I and is a counter model for the UCQ $\widehat{Q} \vee q$.

Using an approach similar to that used e.g. in [15, 28] we will start by establishing the coNP upper bound in data complexity by decomposing counter models and then use a guess and check algorithm for finding such decompositions.

Given an instance $\mathcal J$ and an element $d\in \mathbf{Const}$, the O-type of d in $\mathcal J$ is defined as $\operatorname{tp}_{\mathcal J}(d)=\left\{A\in\operatorname{N}_{\mathbb C}(O)\mid d\in A^{\mathcal J}\right\}$. A (realizable) unary type τ for O is a subset of $\operatorname{N}_{\mathbb C}(O)$ such that there is a model $\mathcal J$ of O and $d\in \mathbf{Const}$ with $\tau=\operatorname{tp}_{\mathcal J}(d)$. Further, for unary types τ,τ' and role r we write $\tau\leadsto_r\tau'$ if there is a model $\mathcal J$ of O and $d,e\in \mathbf{Const}$ such that $(d,e)\in r^{\mathcal J}$, $\operatorname{tp}_{\mathcal J}(d)=\tau$ and $\operatorname{tp}_{\mathcal J}(e)=\tau'$. We now extend the notion of types to capture small substructures of tree extensions of $\mathcal I$.

DEFINITION 6. Let $n \geq 1$. A n-type \mathcal{M} for (O, I) is a finite tree extension of I such that

- the forest of M is a single tree of out-degree at most |O| and depth at most n;
- (2) for each role r, and all $(d, d') \in r^{\mathcal{M}}$, $\operatorname{tp}_{\mathcal{M}}(d) \rightsquigarrow_r \operatorname{tp}_{\mathcal{M}}(d')$;
- (3) for each $d \in A^{\mathcal{M}} \setminus \operatorname{adom}(I)$ at depth at most n-1 and each $A \sqsubseteq \exists r.B$ in O there exists $e \in B^{\mathcal{M}}$ such that $(d, e) \in r^{\mathcal{M}}$;
- (4) $r^{\overline{M}} \subseteq s^{\overline{M}}$ for all $r \sqsubseteq s$ in O.

We write $root(\mathcal{M})$ for the root of the unique tree in the forest of \mathcal{M} .

For an element $d \in \operatorname{adom}(\mathcal{M})$, we use $\mathcal{M}(d)_k$ to denote the subinstance of \mathcal{M} induced by $\operatorname{adom}(I)$ and the subtree of depth at most k of the only tree in the forest of \mathcal{M} , rooted at d. We write $\mathcal{M}(d)_k \simeq \mathcal{M}'(e)_k$ if there is an isomorphism g from $\mathcal{M}(d)_k$ to $\mathcal{M}'(e)_k$ such that g(d) = e.

Definition 7. A set Γ of n-types for (O, I) is coherent if the following are satisfied:

- (1) for all $\mathcal{M}, \mathcal{M}' \in \Gamma$, $adom(M) \cap adom(M') = adom(I)$ and $\mathcal{M}|_{adom(I)} = \mathcal{M}'|_{adom(I)}$;
- (2) for every $c \in A^{\mathcal{M}} \cap \operatorname{adom}(I)$ and $A \sqsubseteq \exists r.B$ in O there exists $\mathcal{M} \in \Gamma$ such that there is an r-edge from c to $\operatorname{root}(\mathcal{M})$ and $\operatorname{root}(\mathcal{M}) \in B^{\mathcal{M}}$;
- (3) for every $\mathcal{M} \in \Gamma$ and every successor d of $root(\mathcal{M})$, there is $\mathcal{M}' \in \Gamma$ such that $\mathcal{M}(d)_{n-1} \simeq \mathcal{M}'(e)_{n-1}$ with $e = root(\mathcal{M}')$.

LEMMA 3 ([15, 28]). Let Q be union of Boolean CQs, each using at most n variables. Then $(O, I) \models Q$ iff for each coherent set Γ of n-types for (O, I) it holds that $\bigcup_{M \in \Gamma} M \models Q$.

Thus, we can test whether $(O, I) \models Q$, by universally guessing a set of coherent types Γ and verifying that $\bigcup_{M \in \Gamma} M \nvDash Q$. The

size of an n-type for (O, I) is at most $|O|^n + |I|$. Because all n-types in a coherent set coincide over $\operatorname{adom}(I)$, up to isomorphism there is at most $(2+|I|)^{|O|^{\operatorname{poly}(n)}}$ different n-types. Hence, we can impose the same bound on the size the sets Γ guessed in the algorithm. Further, checking whether a coherent set of n-types satisfies a union Q of Boolean CQs, each using at most n variables can be done in time $\operatorname{poly}(|Q|,|I|^n)$. We apply this algorithm to $Q=\widehat{Q}$. We can take for n the maximal number of variables in queries from Q_{CL} . It then follows that the size of $\operatorname{each}\widehat{q}$ is $O(|q|\cdot|\operatorname{adom}(I)|^n+|q|\cdot n)$, and \widehat{Q} is the union of $|Q_{\operatorname{CL}}|$ such queries. This gives the coNP upper bound for the data complexity of the entailment problem. The lower bound follows from IQ entailment in a sublogic of $\operatorname{\mathcal{ALCHOI}}$ [40].

To establish the 2ExpTime-completeness in combined complexity, observe that the algorithm in [11] for deciding entailment of a UCQ Q in \mathcal{ALCHOI} runs in time $2^{(|O|+|I|+|Q|)^{\operatorname{poly}(n)}}$, where n is the maximal number of variables in any CQ constituting Q. The lower bound follows from CQ entailment in \mathcal{ALCI} [25].

3.4 Emptiness and Mixed Satisfiability

In this section we present a collection of complexity results for the EMPTINESS problem. As a technical tool, we introduce another closely related problem, called mixed satisfiability, or MIXED-SAT. Similarly to MIXED-OMQA, MIXED-SAT corresponds to a stronger version of unrestricted satisfiability of a theory, but a relaxed version of finite satisfiability: we are looking for models of the input theory in which all predicates from a given set have finite extensions (the remaining predicates may have a finite or an infinite extension). This problem will not only make the characterization of EMPTINESS clearer, but it is interesting in its own right. For instance, given a theory φ and a set Σ of predicate symbols, we can use MIXED-SAT to check whether there exists a finite database D over Σ such that D can be extended to a model of φ while preserving D, i.e., by finding a finite or an infinite extension for all predicates of φ that do not appear in Σ . Formally, MIXED-SAT is defined as follows.

```
\mathsf{MIXED}\text{-}\mathsf{SAT}(\mathcal{L}_\mathsf{TH})
```

Input: A pair (φ, Σ) with $\varphi \in \mathcal{L}_{TH}$ and $\Sigma \subseteq Rel$.

Question: Does exist a model \mathcal{J} of φ such that $R^{\mathcal{J}}$ is finite

for all $R \in \Sigma$?

We let MIXED-UNSAT(\mathcal{L}_{TH}) be the complement of the decision problem MIXED-SAT(\mathcal{L}_{TH}).

We next observe that if we consider only atomic closed queries (closed predicates) and we do not require predicate fixing, then MIXED-UNSAT and EMPTINESS collapse into one problem.

Proposition 4. For each \mathcal{L}_{TH} there exist polynomial reductions

- (1) from EMPTINESS(T: \mathcal{L}_{TH} , C: $\mathcal{A}Q$, D: Q, F: \emptyset) to MIXED-UNSAT(\mathcal{L}_{TH}), and
- (2) from MIXED-UNSAT(\mathcal{L}_{TH}) to EMPTINESS(T: \mathcal{L}_{TH} , C: $\mathcal{A}Q$, D: \emptyset , F: \emptyset).

Recall that a fragment \mathcal{L}_{TH} of first-order logic has the *finite model property (FMP)*, if any satisfiable theory $\varphi \in \mathcal{L}_{TH}$ has a finite model. Clearly, for any \mathcal{L}_{TH} with the FMP, MIXED-SAT(\mathcal{L}_{TH})

corresponds to ordinary satisfiability in \mathcal{L}_{TH} , i.e. (φ, Σ) is a positive instance of MIXED-SAT($\mathcal{L}_{\mathsf{TH}}$) iff φ is satisfiable. Here, we look at some fragments that do not have the FMP: we concentrate on mixed satisfiability in DLs that support inverse roles, and simultaneously allow to express functionality of roles. These features cause the loss of the FMP, and make mixed satisfiability non-trivial.

Example 1. Consider the following ontology O:

$$A \sqsubseteq \exists r.B,$$
 $B \sqsubseteq \exists r.A,$ $\exists p^- \sqsubseteq \exists p,$
 $A \sqsubseteq \exists p \sqcap \neg \exists .p^-,$ $\{d\} \sqsubseteq A,$ func $(p^-).$

Let c_1, c_2, c_3, \ldots be an enumeration of Const with $c_1 = d$. It is easy to see that $(O, \{A, B, r\})$ is a positive instance of MIXED-SAT for $\mathcal{ALCHOIF}$, which is witnessed by a model I with $A^{I} = \{c_1\}$, $B^{I} = \{c_{2}\}, r^{I} = \{(c_{1}, c_{2})\}, and p^{I} = \{(c_{i}, c_{i+1}) \mid i \in \mathbb{N}\}. Note that$ in such I, only p has an infinite extension. Observe that O forces p to be infinite, i.e. (O, Σ) is a negative instance of MIXED-SAT for $\mathcal{ALCHOIF}$ for any Σ that contains p.

We next study mixed satisfiability for the DL $\mathcal{ALCHOIF}$ and its sublogic \mathcal{ALCHIF} , and show that the problem for these DLs is complete for NExpTime and ExpTime, respectively. We then argue that for some important DLs of the DL-Lite family the complexity can be lowered significantly.

We first observe that, in all DLs considered here, role names can be eliminated in polynomial time from Σ in instances (O, Σ) while preserving mixed satisfiability. To see this, suppose (O, Σ) is an instance of mixed satisfiability with *O* in one of the above DLs, and suppose Σ' is the set of role names in Σ . We let

$$O' = O \cup \left\{ \top \sqsubseteq \forall r.A, \ \top \sqsubseteq \forall r^-.A \mid r \in \Sigma' \right\} \,,$$

where A is a fresh concept name that does not occur in O. Intuitively, we use A to collect the domain and the co-domain of every role name in Σ . It is easy to see that (O, Σ) is a positive instance of mixed satisfiability iff $(O, \{A\} \cup \Sigma \setminus \Sigma')$ is a positive instance of mixed satisfiability.

Keeping in mind the above observation, our goal next is to provide an algorithm to decide, given a pair (O, Σ) , where O is an ontology expressed in $\mathcal{ALCHOIF}$ and Σ is a set of *concept names*, whether there exists a model I of O such that A^{I} is finite for all $A \in \Sigma$. Our algorithm for $\mathcal{ALCHOIF}$, as well as the algorithms for the remaining DLs, build on the previously developed methods for finite model reasoning in DLs. Specifically, there exist algorithms that can solve MIXED-SAT($\mathcal{ALCHOIF}$) and MIXED-SAT($\mathcal{ALCHOVEF}$) wf formally define mosaics, which are finite representations for instances (O, Σ) , where Σ must be the set of all concept and roles names in O. For us the most relevant are [26, 34, 35], which present algorithms based on reductions to integer programming. Our algorithm and the presentation can be described as follows. We define the notions of a *tile T* for O and a *mosaic N* for (O, Σ) . Note that DLs considered here support only unary and binary relations, and thus models here can be naturally seen as directed graphs with labels on nodes (capturing the content of concept names), and labels on edges (describing the extensions of role names). Intuitively, a tile *T* is a (finite and small) description of a single domain element together with its (relevant) neighborhood in a model of O. A tile can be seen as a building block for constructing models of O. A mosaic N is a multiset of tiles that has to satisfy certain

coherence conditions. The conditions ensure that a desired model of O can be assembled by instantiating tiles in N (according to the tile multiplicities given by N). In the final step, we show that deciding the existence of a mosaic for (O, Σ) can be reduced to a variant of integer programming problem, which, as we argue, can be solved algorithmically and yields worst-case optimal upper bounds for $\mathcal{ALCHOIF}$ and \mathcal{ALCHIF} . We concentrate here on ontologies in normal form.

We start with the formal definition of tiles.

DEFINITION 8 (TILES). Let O be an $\mathcal{ALCHOIF}$ ontology in normal form. A type T (for O) is any subset of $N_C^+(O)$ such that $\top \in T$ and $\bot \notin T$. We use Types(O) to denote the set of types for O. A tile τ (for O) is any tuple $\tau = (T, \rho)$, where $T \in \text{Types}(O)$, ρ is a set of pairs (R, T') with $R \subseteq N_p^+(O)$ and $T' \in Types(O)$, and such that the following conditions are satisfied:

- (1) $|\rho| \le |O|$;
- (2) If $B_1 \sqcap \ldots \sqcap B_{k-1} \sqsubseteq B_k \sqcup \ldots \sqcup B_m \in O$ and $\{B_1,\ldots,B_{k-1}\}\subseteq T$, then $\{B_k,\ldots,B_m\}\cap T\neq\emptyset$;
- (3) If $A \sqsubseteq \exists r.B \in O$ and $A \in T$, then there is $(R, T') \in \rho$ such that $r \in R$ and $B \in T'$;
- (4) For all $(R, T') \in \rho$, the following hold:
 - (a) If $A \subseteq \forall r.B \in O$, $A \in T$ and $r \in R$, then $B \in T'$;
 - (b) If $A \subseteq \forall r.B \in O$, $A \in T'$ and $r^- \in R$, then $B \in T$;
 - (c) If $r \sqsubseteq s \in O$ and $r \in R$, then $s \in R$;
 - (d) If $r \sqsubseteq s \in O$ and $r^- \in R$, then $s^- \in R$.
- (5) If func $(r) \in O$, then $|\{(R, T) \in \rho \mid r \in R\}| \le 1$.

We use Tiles(O) to denote the set of all tiles for O.

Intuitively, a tile $\tau = (T, \rho)$ for O describes an element e that participates in basic concepts given by T, and whose neighborhood is described (at least partially) by ρ . Each $(R, T') \in \rho$ can be seen as a labeled edge outgoing from e. The sets T and ρ form a configuration that is consistent with the statements in O (from the 'perspective' of e): e participates in suitable simple concepts (Condition (2)), there is a proper witness edge for all existential inclusions $A \sqsubseteq \exists r.B$ that are 'triggered' by e (Condition (3)), all edges outgoing from *e* satisfy all universal inclusions $A \sqsubseteq \forall r.B$ and all role inclusions (Condition (4)), and the edges described by ρ are compatible with the functionality assertions in O (Condition (5)).

tions of desired models. Intuitively, a mosaic tells us how many instances of different tiles we need in order to construct a desired model. Since in our setting some tiles might have to be instantiated infinitely many times, we need a value to talk about the cardinality of a countable infinite set. In dealing with this, we closely follow [34], which shows how an algorithm for (fully) finite satisfiability in C^2 can be turned into an algorithm for (fully) unrestricted satis fiability. We let $\mathbb{N}^* = \mathbb{N} \cup \{\aleph_0\}$. Since mosaics will involve linear inequalities, the usual ordering <, and the arithmetic operations + and \cdot on $\mathbb N$ need to be extended to accommodate the new element \aleph_0 . In particular, $n < \aleph_0$ for all $n \in \mathbb{N}$. We let $\aleph_0 \cdot \aleph_0 = \aleph_0 + \aleph_0 = \aleph_0$. We let $\aleph_0 + n = n + \aleph_0 = \aleph_0$ for all $n \in \mathbb{N}$. We let $\aleph_0 \cdot n = n \cdot \aleph_0 = \aleph_0$ for all $n \in \mathbb{N}$ with n > 0. Finally, we let $\aleph_0 \cdot 0 = 0 \cdot \aleph_0 = 0$.

DEFINITION 9 (Mosaics). Assume an ontology O, and a set Σ of concept names. Given a set R of roles, we let $R^- = \{r^- \mid r \in R\}$. A mosaic for (O, Σ) is a function N: Tiles $(O) \to \mathbb{N}^*$ that satisfies the following conditions:

(1) For every nominal $\{c\} \in N_C^+(O)$ we have:

$$\sum_{\substack{(T,\rho)\in\mathsf{Tiles}(O)\\\{c\}\in T}} N((T,\rho))=1\,;$$

(2) The following inequation is satisfied:

$$\sum_{\tau \in \mathsf{Tiles}(O)} N(\tau) \geq 1\,;$$

- (3) For every $A \in \Sigma$ and every tile $(T, \rho) \in \mathsf{Tiles}(O)$ with $A \in T$, we have $N((T, \rho)) \neq \aleph_0$;
- (4) For all $(T, \rho) \in \text{Tiles}(O)$ and $(R, T') \in \rho$ the following implication holds: if $N((T, \rho)) > 0$, then there is some ρ' such that $(T', \rho') \in \text{Tiles}(O)$ and $N((T', \rho')) > 0$;
- (5) For every pair $T, T' \in \mathsf{Types}(O)$ and every $R \subseteq \mathsf{N}^+_\mathsf{R}(O)$ with $\mathsf{func}(r^-) \in O$, the following inequation is satisfied:

$$\sum_{\substack{(T,\rho)\in\mathsf{Tiles}(O)\,\wedge\\(R,T')\in\rho\\}} N((T,\rho)) \leq \sum_{\substack{(T',\rho')\in\mathsf{Tiles}(O)\,\wedge\\(R^-,T)\in\rho'}} N((T',\rho'))\;.$$

The conditions in Definition 9 are geared to ensure that a mosaic *N* for (O, Σ) witnesses the existence of a model *I* for *O* where each concept name from Σ has a finite extension. Condition (1) tells that in a candidate for I there is exactly one element that satisfies a nominal $\{c\}$ (which must be the constant c itself). Condition (2) tells that at least one tile needs to be realized in a model of O. Condition (3) ensures that tiles (T, ρ) with $\Sigma \cap T \neq \emptyset$ are allowed to be instantiated only finitely many times. Condition (4) ensures that for tile (T, ρ) that will be instantiated at some element e, we can also instantiate tiles to provide neighbors for e as prescribed by ρ . Finally, to understand the Condition 5, let us take a pair $T, T' \in \mathsf{Types}(O)$ and a set $R \subseteq \mathsf{N}^+_\mathsf{R}(O)$, and let us view a candidate for I as labeled graph. Suppose the graph has n nodes that are labeled with T, and having an R-labeled edge to a node labeled with T'. If func $(r^-) \in O$, then clearly there must exists at least n nodes that are labeled with T', and having an R^- -labeled edge to a node labeled with *T*.

The conditions placed on mosaics correctly characterize mixed satisfiability in $\mathcal{ALCHOIF}$ (see Appendix for a proof).

THEOREM 5. Assume an $\mathcal{ALCHOIF}$ ontology O, and a set $\Sigma \subseteq N_C$. Then the following statements are equivalent:

- (1) O has a model I such A^{I} is finite for all $A \in \Sigma$.
- (2) There exists a mosaic N for (O, Σ) .

Due to Theorem 5, mixed satisfiability in $\mathcal{ALCHOIF}$ reduces to deciding the existence of a mosaic. To reason about mosaics, we employ (an extension of) integer programming. In particular, we consider linear inequations of the form

$$a_1 \cdot x_1 + \dots + a_n \cdot x_n + c \le b_1 \cdot y_1 + \dots + b_m \cdot y_m \tag{1}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m$ are positive integers, c is a (possibly negative) integer, and $x_1, \ldots, x_n, y_1, \ldots, y_m$ are variables. In case

 $c \geq 0$, the inequation (1) is a called *positive*. An *ordinary inequation system* is a pair (V, \mathcal{E}) , where \mathcal{E} is a set of linear inequations of the form (1), and V is the set of variables that appear in \mathcal{E} . The notion of a *solution* $S: V \to \mathbb{N}$ for (V, \mathcal{E}) is the standard one. The definition of solutions *over* \mathbb{N} can be lifted to solutions *over* \mathbb{N}^* in the obvious way, i.e., we will consider solutions to (V, \mathcal{E}) that are functions of the form $S: V \to \mathbb{N}^*$. An *enriched inequation system* is a tuple (V, \mathcal{E}, F, I) , were (V, \mathcal{E}) is an ordinary inequation system, $F \subseteq V$, and I is a set of *implications*, which are expressions of the form

$$y_1 + \cdots + y_m > 0 \Rightarrow x_1 + \cdots + x_n > 0$$

with $y_1, \ldots, y_k, x_1, \ldots, x_n \in V$. A solution S (over \mathbb{N} or over \mathbb{N}^*) to (V, \mathcal{E}, F, I) is a solution to (V, \mathcal{E}) that additionally satisfies the following conditions:

- (1) $S(x) \neq \aleph_0$ for all $x \in F$;
- (2) S(x) > 0 implies $S(x_1) + \cdots + S(x_n) > 0$ for all $x > 0 \Rightarrow x_1 + \cdots + x_n > 0$ in I.

The following result on the complexity of reasoning with enriched inequation systems can be shown (see Appendix).

THEOREM 6. Deciding the existence of a solution for an enriched inequation system $\mathcal{H} = (V, \mathcal{E}, F, I)$ is feasible in non-deterministic polynomial time in the size of \mathcal{H} . If \mathcal{E} contains only positive inequations, the problem is solvable in polynomial time in the size of \mathcal{H} .

Assume an $\mathcal{ALCHOIF}$ ontology O, and a set $\Sigma \subseteq \mathbb{N}_{\mathbb{C}}$. Due to Theorem 5, checking the existence of a model I of O where A^{Σ} is finite for all $A \in \Sigma$ reduces to checking the existence of a mosaic for (O, Σ) . It is easy to see that the latter can be reduced to checking the existence of a solution to an exponentially sized enriched inequation system $(V^*, \mathcal{E}^*, F^*, I^*)$. Intuitively, V^* is obtained by taking a variable x_{τ} for every $\tau \in \mathsf{Tiles}(O)$. The inequations in \mathcal{E}^* are obtained directly from the expressions in points (1), (2) and (5) of Definition 9, by replacing $N(\tau)$ with the variable x_{τ} for all $\tau \in \mathsf{Tiles}(O)$. The set I^* is obtained directly from point (4). That is, for all $(T, \rho) \in \mathsf{Tiles}(O)$ and $(R, T') \in \rho$, I^* contains

$$x_{(T,\rho)} > 0 \Rightarrow x_{\tau^1} + \cdots + x_{\tau^m} > 0$$
,

where $\{\tau^1, \dots, \tau^m\} = \{(T', \rho') \in \mathsf{Tiles}(O) \mid \text{ for some } \rho'\}$. Finally, $F^* = \{x_\tau \mid \tau = (T, \rho) \in \mathsf{Tiles}(O) \land \Sigma \cap T \neq \emptyset\}$. We observe here that in the case O is an \mathcal{ALCHIF} , we can ignore point (1) in the construction of \mathcal{E}^* , which then results in a set that contains only positive inequations.

As the above encoding requires exponential time, the complexity results in Theorem 6 lead to the following complexity bounds for $\mathcal{ALCHOIF}$ and \mathcal{ALCHIF} .

THEOREM 7. The following hold:

- (1) MIXED-SAT(T: $\mathcal{ALCHOIF}$) is NExpTime-complete.
- (2) MIXED-SAT(T: \mathcal{ALCHIF}) is ExpTime-complete.
- (3) EMPTINESS(T: $\mathcal{ALCHOIF}$, C: \mathcal{AQ} , D: Q, F: \emptyset) is conexptime-complete.
- (4) EMPTINESS(T: \mathcal{ALCHIF} , C: \mathcal{AQ} , D: Q, F: \emptyset) is ExpTime-complete.

The lower bounds for the above problems are inherited from general satisfiability in $\mathcal{ALCHOIF}$ [43] and \mathcal{ALCHIF} [41].

We now turn our attention of the *DL-Lite* family of DLs, and, in particular, to the DLs $DL-Lite^{\mathcal{H}O\mathcal{F}}_{\mathrm{Bool}}$ and $DL-Lite^{\mathcal{H}\mathcal{F}}_{\mathrm{H}}$, which are important members of this family without the FMP. Their syntactic restrictions open the way to different algorithms and allow us to obtain further complexity results.

THEOREM 8. The following are true:

- (1) MIXED-SAT(T: DL-Lite $^{\mathcal{HOF}}_{\mathrm{Bool}}$) is NP-complete. (2) MIXED-SAT(T: DL-Lite $^{\mathcal{HF}}$) is in PTIME.
- (3) EMPTINESS(T: DL- $Lite_{Bool}^{\mathcal{HOF}}$, C: \mathcal{AQ} , D: Q, F: \emptyset) is coNP-complete.
- (4) EMPTINESS(T: DL- $Lite^{\mathcal{HF}}$, C: $\mathcal{A}Q$, D: \mathbb{Q} , F: \emptyset) is PTIME-complete.

The upper bounds for T: $DL\text{-}Lite_{Bool}^{\mathcal{HOF}}$ above are obtained by modifying our reduction to (enriched) integer programming, while the lower bound is inherited from satisfiability in propositional logic. The upper bound for DL- $Lite^{\mathcal{HF}}$ can be obtained by applying the cycle reversion technique [38]. In particular, using polynomial time computation mixed satisfiability in DL-Lite $^{\mathcal{HF}}$ can be reduced to ordinary satisfiability, which is known to be tractable.

The above results on the EMPTINESS problem deal with the case where $F:\emptyset$. We can additionally deal with the case $F:\mathcal{A}Q$ in DLs that allow to freeze the interpretation of concept and role names to some given extensions, enabling us to reduce the case where $Q_{\mathsf{FIX}} \neq \emptyset$ to the setting where $Q_{\mathsf{FIX}} = \emptyset$. This can be done in $\mathcal{ALCHOIF}$ as shown in the proof of Theorem 2, and the same can be done in $DL\text{-}Lite_{Bool}^{\mathcal{HOF}}$. From this we can infer the following

THEOREM 9. The following hold:

- (1) EMPTINESS(T: $\mathcal{ALCHOIF}$, C: \mathcal{AQ} , D: Q, F: \mathcal{AQ}) is in PTIME^{NEXPTIME}.
- (2) EMPTINESS(T: DL-Lite $_{\text{Bool}}^{\mathcal{HOF}}$, C: \mathcal{AQ} , D: Q, F: \mathcal{AQ}) is in PTIME^{NP}

4 RELATED WORK

Module Extraction. Focusing as defined in this paper is related to module extraction, which has been studied extensively in DLs as a tool to facilitate the development and use of very large ontologies. Various notions of modularity and algorithms for module extraction have been proposed in the literature [13, 19, 23, 24, 37, 42]. Important technical tools in modularity are various algorithms related to conservative extensions, inseparability, uniform interpolation, and forgetting, which have been studied in DLs but are also classic problems in other areas of logic (see, e.g., [3, 9, 29]). Roughly speaking, a fragment O_1 of an ontology O_2 is called a *module* if the terms defined in O_1 have precisely the meaning they have in the larger ontology O_2 . More precisely, one usually requires that O_1 and O_2 agree on the entailment of inclusions over a given signature Σ . In this way, an application whose scope is limited to the entities in Σ can safely use O_1 instead of the full O_2 . The difference between the mentioned works and our work is that focusing-unlike module extraction-will in general change the meaning of terms.

To see this, consider a (rather trivial) disaster management ontology $O = \{ \text{Disaster} \equiv \text{Flood} \sqcup \text{Drought} \}$. Suppose we want to focus on floods, and thus we naturally expect a focusing configuration to lead to the entailment of Disaster ≡ Flood. This can be achieved via a focusing solution $\mathcal{F} = (\{Flood\}, \{Flood\}, \{Drought\}, \emptyset)$. Since the desired entailment does not hold in O, module extraction cannot help us in this case. As a method that does change the meaning of terms, we mention [12], where the authors consider the problem of selecting n inclusions from an input ontology that preserve as much entailments as possible.

Closed Predicates. Combining closed-world and open-world reasoning is a recognized challenge both in database and AI research, and has received significant attention in the literature. The use of closed predicates is a particular way to overcome the challenge [17, 27, 28]. Note that Definition 2 generalizes the idea of closed predicates in DLs. Closing extensions of predicates is similar in spirit to circumscription [30], but instead of minimizing the inference of new tuples in selected predicates, such inferences are prohibited altogether. Circumscription has been studied both for expressive and lightweight DLs [7, 8]. Note that closed predicates, or other kinds of statements to assert information completeness have also been studied in databases (see, e.g., [2, 5, 16]). In particular, [16] studies completeness assertions made using queries, and explores reasoning about databases and queries in the presence of such assertions.

Query Emptiness. The nullability problem studied in Section 3.2 is closely related to the query emptiness problem studied in [3], and the so-called schema-level positive query implication ($\exists PQI$) problem studied in [5]. For Boolean queries, the 3 problems effectively collapse (here and in the mentioned papers, the problems are studied for different languages). For non-Boolean queries, there is a slight divergence. Consider the ontology $O = \{\{c\} \subseteq \exists r.A\}$, the instance query A(x), and suppose the signature of legal databases is $\Sigma = \emptyset$. In the sense of [3, 5], this yields a positive instance, i.e., there is a database (the empty database) in which the certain answer to A(x) is empty. In our setting, this is a negative instance of the nullability problem (the extension of A always has an element, still we cannot identify it via a constant). Note that the undecidability result for nullability with \mathcal{ELI}_{\perp} ontologies already holds for simple Boolean CQs of the form $\exists x A(x)$ (see Appendix). This contributes to the study initiated in [5], which showed undecidability results for disjunctive linear tuple generating dependencies (TGDs) and linear TGDs with constants. Since \mathcal{ELI}_{\perp} can be translated into guarded TGDs without constants (and using only predicates of arity at most two) but extended with constraints, this provides a new class of constraints for which $\exists PQI$ is undecidable.

Finite Model Reasoning. Reasoning about finite models of DL ontologies has received some (albeit limited) attention [10, 18, 22, 26, 38, 39]. Since DLs are closely related to the fragments C^1 and C^2 of first-order logic with counting quantifiers, the work by Pratt-Hartmann on the complexity of finite and unrestricted satisfiability in these logics is particularly relevant [34, 36]. The inequations we use in Section 3.4 are inspired by [26], but we use some tricks from [34] (in addition to our own) to deal with mixed satisfiability.

5 DISCUSSION

We have introduced *focusing*, which makes it possible to reuse the knowledge in an ontology as a basis for the on-demand design of data-centric applications. The selected complexity results we have provided are not meant to paint a complexity landscape, or to identify the best formalisms to be used for focusing. Rather, they constitute a preliminary study of the limits and possibilities of the focusing framework, and many questions remain to be answered.

We have briefly mentioned as an open issue the design of strategies to find preferred focusing solutions, which also calls for suitable ways to capture preferences of the designer. There are also many challenges related to reasoning about the queries in focusing specifications: e.g. to derive additional queries that may be closed or fixed, or to use the closed and fixed assumptions for query optimization.

The specification of queries that are complete or fixed is a way to add knowledge to the ontology, and this knowledge could be leveraged in many ways. For example, it is well known that CQs with negation are undecidable in the presence of ontologies [20], but over our enriched ontologies, non-trivial queries with negation admit algorithms. This applies, e.g., to CQs with negation where each variable that occurs in a negative atom also occurs in a positive atom over a closed predicate. Moreover, one could imagine a scenario where one may use queries with negation to specify the scope of the desired system, and the focusing engine could take that into account and search for focusing solutions where suitable predicates are closed as to guarantee decidability of query answering.

We would also like to understand whether in some cases, fixing predicates has a significant effect on the complexity of reasoning, for example, if the more specific ontology that is implicitly built by fixing is in a logic with lower complexity (this could happen, for example, if we fixed a predicate from every disjunction, making the specific part of the ontology effectively Horn). Are there ways to decide whether this is the case, and if so, can this be effectively leveraged by algorithms?

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Ontology Focusing: Knowledge-enriched Databases on Demand

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A MISSING PROOFS OF SECTION 3.2

We now argue that allowing roles names to be used to specify completeness queries leads to undecidability of the nullability problem. This holds for the DL \mathcal{ELI}_{\perp} , which is a well-known *Horn* DL.

THEOREM 10. NULLABILITY(T: \mathcal{ELI}_{\perp} , C: \mathcal{AQ} , Q: IQ) is undecidable. Moreover, NULLABILITY(T: \mathcal{ELI}_{\perp} , C: \mathcal{AQ} , Q: Q) is undecidable even for Boolean Qs of the form $\exists x.A(x)$, where A is a concept name.

PROOF. We provide a reduction from the halting problem for (deterministic) Turing Machines (TMs). Assume a TM

$$M = (\Gamma, Q, q_0, q_{ves}, q_{no}, \delta)$$

with the alphabet $\Gamma = \{0,1\}$, the state set Q, the initial state $q_0 \in Q$, the accepting state $q_{yes} \in Q$, the rejecting state $q_{no} \in Q$, and the transition function $\delta: Q \setminus \{q_{yes}, q_{no}\} \times \Gamma \cup \{b\} \to Q \times \Gamma \cup \{b\} \times \{+1, -1\}$. The problem is to decide whether M terminates on the empty word ϵ ; that is, starting from the one-way infinite tape that contains in every cell only the blank symbol b. We assume that the machine never comes back to the initial state q_0 , and never attempts to move to the left of the initial position of the head. We will now construct an instance $(O, \Sigma, Q_{\text{CL}}, q)$ of NULLABILITY(T: \mathcal{ELI}_{\perp} , C: \mathcal{AQ} , Q: \mathcal{IQ}) such that $(O, \Sigma, Q_{\text{CL}}, q)$ is a negative instance iff the machine M terminates on ϵ . In the construction of O, we use the following symbols to encode runs of TMs:

- We use role names up and right to encode a grid on which the computation of M can be represented.
- We use the concept name Head to indicate the position of the read-write head of *M*.
- We use the concept name NoHeadRight and NoHeadLeft to indicate that the read-write head is not here or to the right (resp. left) of
 a given tape cell.
- We use the concept name Q_q for every state $q \in Q$.
- We use the concept names S_0 , S_1 , S_b to indicate the content of a tape cell.
- We use the concept names X, Y_1, Y_2, Z to perform a trick, which will become clear later.
- We use an auxiliary role name *r*.

Our next goal is to obtain the ontology O. Its construction is split into two parts. In the first part, we simply define inclusions that simulate the computation of M on a grid given by the roles up and right. The (extension of the) role right corresponds to the tape of M, while the (extension of the) role up corresponds to time. To make the presentation easier, we allow infinite instances over Q_{CL} ; modifying the below reduction to fit precisely the definition of nullability (which requires finite instances) is not too dificult. The following inclusions of O are self-explanatory (assuming that up and right form a grid, which is the tricky part here):

- (A) $\top \sqsubseteq \exists r.(Q_{q_0} \sqcap \mathsf{Head})$
- (B) $Q_q \sqsubseteq (\exists \mathsf{up}.\top) \sqcap (\exists \mathsf{right}.\top)$ for all $q \in Q \setminus \{q_{\mathsf{ves}}, q_{\mathsf{no}}\}$
- (C) $\exists right^-.Head \sqsubseteq NoHeadRight$
- (D) $\exists right.Head \sqsubseteq NoHeadLeft$
- (E) NoHeadRight \sqsubseteq NoHead
- (F) NoHeadLeft

 □ NoHead
- (G) NoHead \sqcap Head $\sqsubseteq \bot$
- (H) ∃right⁻.NoHeadRight

 □ NoHeadRight
- (I) ∃right.NoHeadRight

 NoHeadLeft
- (J) $\exists \text{right.} Q_q \sqsubseteq Q_q \text{ for all } q \in Q$
- (K) $\exists \text{right}^-.Q_q \sqsubseteq Q_q \text{ for all } q \in Q$
- (L) $Q_{q_0} \sqsubseteq S_b$
- (M) $Q_q \sqcap Q_s \sqsubseteq \bot$ for all $q, s \in Q$
- (N) $\exists up^-.(S_{\sigma} \sqcap NoHead) \sqsubseteq S_{\sigma} \text{ for all } \sigma \in \{0, 1, b\}$
- (O) $\exists \mathsf{up}^-.(S_\sigma\sqcap Q_q\sqcap\mathsf{Head})\sqsubseteq S_{\sigma'}\sqcap Q_{q'}$ for all $(q,\sigma)\in Q\setminus \{q_{yes},q_{no}\}\times\Gamma\cup\{b\}$ such that $\delta(q,\sigma)=(q',\sigma',D)$
- $(P) \ \exists \mathsf{right}^-. (\exists \mathsf{up}^-. (S_\sigma \sqcap Q_q \sqcap \mathsf{Head})) \sqsubseteq \mathsf{Head} \ \mathsf{for} \ \mathsf{all} \ (q,\sigma) \in Q \setminus \{q_{\mathit{yes}}, q_{\mathit{no}}\} \times \Gamma \cup \{b\} \ \mathsf{such} \ \mathsf{that} \ \delta(q,\sigma) = (q',\sigma',+1)$
- (Q) $\exists \mathsf{right}.(\exists \mathsf{up}^-.(S_\sigma\sqcap Q_q\sqcap \mathsf{Head})) \sqsubseteq \mathsf{Head} \text{ for all } (q,\sigma) \in Q \setminus \{q_{\mathit{yes}},q_{\mathit{no}}\} \times \Gamma \cup \{b\} \text{ such that } \delta(q,\sigma) = (q',\sigma',-1) = (q',\sigma',$

We let Q_{CL} be the set of concepts and roles mentioned in the above inclusions, except r. Consider an instance I over Q_{CL} . We say I is grid-like in case $\{up(c,d), right(d,e)\} \subseteq I$ and $\{right(c,d'), up(d',e')\} \subseteq I$ imply e = e'. It is easy to see that the above encoding is correct on grid-like instances: (a) if M halts, then there is a grid-like instance I over Q_{CL} such that I satisfies every inclusion above, and (b) if there exists a grid-like instance I over Q_{CL} that satisfies all the above inclusions, then M halts. We need a tool to ensure that our instances are grid-like. For this we use further inclusions, which will involve predicates that are not part of Q_{CL} . These predicates are the concept names X, Y_1, Y_2, Z , and the role name r. We add the following inclusions to O:

- (1) $\top \sqsubseteq \exists r.(X \sqcap (\exists up. \exists right. Y_1) \sqcap (\exists right. \exists up. Y_2))$
- (2) $Y_1 \sqcap Y_2 \sqsubseteq Z$

The inclusions above can be explained as follows. We use X, Y_1 , Y_2 as "flags". The inclusion (1) says that (i) X needs to be arbitrarily placed somewhere in the candidate to be a grid-like structure, (ii) Y_1 needs to placed at some up-right neighbor of (the placement of) X, and (iii) Y_2 needs to placed at some right-up neighbor of (the placement of) X. The inclusion (2) just says that if a location is flagged with Y_1 and Y_2 , then also Z is placed there.

Finally, the query *q* is q = Z(x), or alternatively $q = \exists x. Z(x)$.

It is not difficult to see that the above is a correct reduction. Indeed, if M halts, then we can easily translate a terminating run of M into a grid-like instance I over Q_{CL} such that $Z^{\mathcal{J}} \neq \emptyset$ for all models \mathcal{J} of O that agree with I on the extensions of the predicates in Q_{CL} . For the other direction, assume we have an instance I over Q_{CL} that forces the extension of Z to be non-empty in all relevant models of O. This implies that I is grid-like, and thus a terminating run of M can be extracted. To see this a bit more formally, suppose that I is not grid-like. Then there exists some $\{\text{up}(c,d), \text{right}(d,e), \text{right}(c,d'), \text{up}(d',e')\} \subseteq I$ such that $e \neq e'$. Extend I to a model of I by adding I and additionally I (and additionally I) for all constants I). One arrives at a contradiction: the extension of I in this instance is empty. I

Next, we show that more radical restriction of the logic can restore the decidability of the nullability problem with closed roles.

Theorem 11. NULLABILITY(T: DL- $Lite_{Bool}^{O}$, C: \mathcal{AQ} , Q: \mathcal{AQ}) is in $coNExpTime^{NP}$.

PROOF. We give a polynomial reduction to NULLABILITY(T: \mathcal{ALCOI} , C: IQ, Q: $\mathcal{A}Q$). Consider a DL- $Lite_{\mathrm{Bool}}^O$ ontology O, a signature $\Sigma \subseteq \mathsf{N}_{\mathrm{C}}(O) \cup \mathsf{N}_{\mathrm{R}}(O)$, a set of queries $Q_{\mathrm{CL}} \subseteq \mathcal{A}Q$, and a query $q \in \mathcal{A}Q$. Let Σ_{R} be the set of roles r such that $r \in Q_{\mathrm{CL}}$ or $r^- \in Q_{\mathrm{CL}}$. For each $r \in \Sigma_{\mathrm{R}}$, introduce a fresh concept name A_r axiomatized as $A_r \equiv \exists r. \top$; that is, A_r is intended to contain origins of all r-edges. We also include axiom $A_r \sqsubseteq \exists r. A_{r^-}$, which says that each origin of r has an origin of r^- among its r-successors. The resulting ontology O' is not in DL- $Lite_{\mathrm{Bool}}^O$, but it is in \mathcal{ALCOI} . Let Q'_{CL} be obtained by removing from Q_{CL} all role queries, and adding all queries of the form $A_r(x)$ for $r \in \Sigma_{\mathrm{R}}$. Let $\Sigma' = \Sigma \cup \{A_r \mid r \in \Sigma \cap \Sigma_{\mathrm{R}}\}$. We claim that q is nullable wrt. O, Σ , and Q_{CL} iff it is nullable wrt. O', Σ' , and Q'_{CL} .

For any instance \mathcal{M} over the signature of O, let $\widehat{\mathcal{M}}$ be obtained by extending \mathcal{M} minimally so that $(A_r)^{\widehat{\mathcal{M}}} = (\exists r. \top)^{\mathcal{M}}$. Note that if $\mathcal{M} \models O$, then $\widehat{\mathcal{M}} \models O'$.

Suppose there is I over Σ such that $\operatorname{CL}(I,O,Q_{\operatorname{CL}}) \neq \emptyset$ and $[\![\mathcal{J}]\!]_q \neq \emptyset$ for all $\mathcal{J} \in \operatorname{CL}(I,O,Q_{\operatorname{CL}})$. We claim that $I' = \widehat{I}$ is a counterwitness for the nullability of q wrt. O', Σ' , and Q'_{CL} . If $\mathcal{J} \in \operatorname{CL}(I,O,Q_{\operatorname{CL}})$ then $\widehat{\mathcal{J}} \in \operatorname{CL}(I',O',Q'_{\operatorname{CL}})$, so $\operatorname{CL}(I',O',Q'_{\operatorname{CL}}) \neq \emptyset$. Suppose $[\![\mathcal{M}']\!]_q = \emptyset$ for some $\mathcal{M}' \in \operatorname{CL}(I',O',Q'_{\operatorname{CL}})$. Because each A_r is closed, only elements that had an outgoing r-edge in \widehat{I} will have new outgoing r-edges in \mathcal{M}' . Hence, after removing all new r-edges, \mathcal{M}' is still a model of O'. Consequently, we can assume wlog. that there are no new r-edges in \mathcal{M}' for all $r \in \Sigma_R$; that is, $r^{\mathcal{M}'} = r^I$ for all $r \in \Sigma_R$. Let \mathcal{M} be the restriction of \mathcal{M}' to the signature of O. Clearly $\mathcal{M} \in \operatorname{CL}(O,I,Q_{\operatorname{CL}})$ and $[\![\mathcal{M}]\!]_q = \emptyset$.

Conversely, suppose there is I' over Σ' such that $\mathrm{CL}(O',I',Q'_{\mathsf{CL}}) \neq \emptyset$ and $[\![\mathcal{J}']\!]_q \neq \emptyset$ for all $\mathcal{J}' \in \mathrm{CL}(O',I',Q'_{\mathsf{CL}})$. Take some $\mathcal{J}' \in \mathrm{CL}(O',I',Q'_{\mathsf{CL}})$. Let I'' be obtained from I' by including all r-edges present in \mathcal{J}' for all $r \in \Sigma_R \cap \Sigma'$. Note that

$$\mathcal{J}' \in \mathsf{CL}(\mathcal{O}', \mathcal{I}'', \mathcal{Q}'_{\mathsf{Cl}}) \subseteq \mathsf{CL}(\mathcal{O}', \mathcal{I}', \mathcal{Q}'_{\mathsf{Cl}}).$$

Hence, wlog. we can assume that all these edges are already present in \mathcal{J}' ; that is, $r^{\mathcal{I}'} = r^{\mathcal{J}'}$ for all $r \in \Sigma_{\mathbb{R}} \cap \Sigma'$. Let I and \mathcal{J} be the restrictions to the signature of O of the instances I' and \mathcal{J}' . It holds that $\mathcal{J} \in \mathsf{CL}(O, I, Q_{\mathsf{CL}})$. Suppose that there exists $\mathcal{M} \in \mathsf{CL}(O, I, Q_{\mathsf{CL}})$ with $[\![\mathcal{M}]\!]_q = \emptyset$. Let $\mathcal{M}' = \widehat{\mathcal{M}}$. We have $\mathcal{M}' \in \mathsf{CL}(O', I', Q'_{\mathsf{CL}})$ and $[\![\mathcal{M}']\!]_q = \emptyset$.

B MISSING PROOFS OF SECTION 3.3

PROOF OF LEMMA 1. The second condition trivially implies the first one. For the converse implication, assume there is some counter model \mathcal{J} for q. Using a standard unraveling technique, we will inductively construct from \mathcal{J} a tree extension \mathcal{T} of I together with a homomorphism g from \mathcal{T} to \mathcal{J} , and show that $\mathcal{T} \in CL(\mathcal{O}, \mathcal{I}, \mathcal{Q}_{CL})$. Let \mathcal{T}_0 be the subinstance of \mathcal{J} induced by $\operatorname{adom}(I)$ and let g_0 be the identity homomorphism from \mathcal{T}_0 to \mathcal{J} . Assume \mathcal{T}_i and g_i are defined. To define \mathcal{T}_{i+1} and g_{i+1} , for each element $d \in \operatorname{adom}(\mathcal{T}_i)$ and each axiom $K \sqsubseteq \exists r.B$ that is not yet satisfied in d, consider an element $e \in B^{\mathcal{J}}$ such that $(g_i(d), e) \in r^{\mathcal{J}}$. If $e \in \operatorname{adom}(I)$ then \mathcal{T}_{i+1} extends \mathcal{T}_i by adding an s-edge from d to e for every super role s of e. Otherwise, we add a fresh copy e' of e and an e-edge from e0 to e1, for every super role e2 of e3, and extend e3 to e4 to e6. Because e7 is a tree-extension of e7 such that e7. Because e7 is and e7 maps

homomorphically to \mathcal{J} we have $[\![T]\!]_{q'}\subseteq [\![T]\!]_{q'}$, for every $q'\in Q_{\mathsf{CL}}$. Because $[\![\mathcal{J}]\!]_{q'}=[\![T]\!]_{q'}$, it follows that $[\![T]\!]_{q'}=[\![T]\!]_{q'}$. An analogous argument proves that $\mathcal{T}\not\models q$.

MISSING PROOFS OF SECTION 3.4

PROOF OF PROPOSITION 4. Assume $(\varphi, \mathcal{F}) \in \mathsf{FOCUS}(\mathsf{T}: \mathcal{L}_\mathsf{TH}, \mathsf{C}: \mathcal{A}Q, \mathsf{D}: \mathsf{Q})$, where $\mathcal{F} = (\Sigma, Q_\mathsf{CL}, Q_\mathsf{DET}, \emptyset)$. Then (φ, \mathcal{F}) is a positive instance of EMPTINESS(T: $\mathcal{L}_\mathsf{TH}, \mathsf{C}: \mathcal{A}Q, \mathsf{D}: \mathsf{Q})$ iff (φ, Q_CL) is a positive instance of MIXED-UNSAT(T: $\mathcal{L}_\mathsf{TH})$.

Assume an instance (φ, Σ) of MIXED-UNSAT(T: \mathcal{L}_{TH}). Then $(\varphi, (\Sigma, \Sigma, \emptyset, \emptyset))$ is trivially a positive instance of FOCUS(T: \mathcal{L}_{TH} , C: \mathcal{AQ}), and further (φ, Σ) is a positive instance of MIXED-UNSAT(T: \mathcal{L}_{TH}) iff $(\varphi, (\Sigma, \Sigma, \emptyset, \emptyset))$ is a positive instance of EMPTINESS(T: \mathcal{L}_{TH} , C: \mathcal{AQ}). \square

PROOF OF THEOREM 5. (from (1) to (2)) Assume a model I of O where all the concept names in Σ have a finite extension. We can assume that adom(I) is finite or countably infinite. We define a mosaic N for (O, Σ) . For an element $e \in adom(I)$, let

$$ut(e) = \{ \top \} \cup \{ B \in N_C^+(O) \mid e \in B^I \}.$$

For a pair of elements $e, e' \in adom(I)$, let

$$bt(e, e') = \{r \in N_{R}^{+}(O) \mid (e, e') \in r^{I}\}.$$

Assume an element $e \in \text{adom}(I)$. Since I is a model of O, for all $\alpha = A \sqsubseteq \exists r.B \in O$ with $A \in ut(e)$, there exists an element e_{α} such that $(e, e_{\alpha}) \in r^{I}$ and $e_{\alpha} \in B^{I}$. Let $e_{\alpha_{1}}, \ldots, e_{\alpha_{n}}$ be the set such elements for each axiom α_{i} of the form $A \sqsubseteq \exists r.B$ occurring in O. Let F be the set of elements $e' \in \text{adom}(I)$ such that $(e, e') \in r^{I}$ for some func $(r) \in O$. Let $S = \{e_{\alpha_{1}}, \ldots, e_{\alpha_{n}}\} \cup F$. We next define the tile extracted for the element e as follows:

$$tile(e) = (ut(e), \{(bt(e, e'), ut(e')) \mid e' \in S\}).$$

Note that for a given element e, there can be several ways to extract a tile tile(e), because $e_{\alpha_1}, \ldots, e_{\alpha_n}$ can be chosen in multiple ways. It is not difficult to check that tile(e) is indeed a proper tile. Let N be the function such that, for each $\tau \in Tiles(O)$,

$$N(\tau) = |\{e \in \operatorname{adom}(\mathcal{I}) \mid \tau = tile(e)\}|.$$

Note that $N(\tau) = \aleph_0$ in case $\{e \in \text{adom}(I) \mid \tau = tile(e)\}$ is infinite. It is not difficult to check that N is a proper mosaic.

(from (2) to (1)) Assume there exists a mosaic N for (O, Σ) . We show how to construct a model I of O such that all predicates from Σ have a finite extension. Let $\Delta = \{(\tau, i) \in \mathsf{Tiles}(O) \times \mathbb{N} \mid 0 < i \leq N(\tau)\}$. For each $(\tau, i) \in \Delta$ we take a designated constant, which in a slight abuse of notation, we denote simply as (τ, i) .

For simple concepts $B \in N_C^+(O) \setminus \{T, \bot\}$, we let $B^T = \{((T, \rho), i) \in \Delta \mid B \in T\}$. It now remains to define the extensions of roles. First, we deal with roles r such that both $\operatorname{func}(r) \in O$ and $\operatorname{func}(r^-) \in O$, and for this we employ the inequation (5) in two directions. Intuitively, the role r needs to connect an element e with some "unary type" T to some element e' with some unary type T' via some "binary type" T that contains T. For each such triple T, T', T, due to the equation (5), there is a bijection T between the sets T and T and T where

$$A = \{((T, \rho), i) \in \Delta \mid (R, T') \in \rho\}$$
 and $B = \{((T', \rho'), i) \in \Delta \mid (R^-, T) \in \rho'\}$.

For every $e \in A$, let $(e, f(e)) \in s^T$ iff $s \in R$. Now we consider each role r such that $func(r^-) \in O$ but $func(r) \notin O$. Again, the role r connects an element e with some unary type T to some element e' with some unary type T' via some binary type R that contains r. For each such triple T, T', R, due to the equation (5), there is an injective function f from the set A to the set B, where

$$A = \left\{ ((T, \rho), i) \in \Delta \mid (R, T') \in \rho \right\} \qquad \text{and} \qquad B = \left\{ ((T', \rho'), i) \in \Delta \mid (R^-, T) \in \rho' \right\}.$$

For every $e \in A$, let $(e, f(e)) \in s^T$ iff $s \in R$. Remains to define the remaining role extensions. In particular, pick a pair $T, T' \in \mathsf{Types}(O)$ and a set $R \subseteq \mathsf{N}^+_\mathsf{R}(O)$ that does not contain any functional or inverse functional roles of O. Let

$$A = \{ ((T, \rho), i) \in \Delta \mid (R, T') \in \rho \}.$$

If $A = \emptyset$, then do nothing. If $A \neq \emptyset$, then we know by condition (4) that this is some ρ' such that $e' = ((T', \rho), 1) \in \Delta$. For every $e \in A$, we let $(e, e') \in s^I$ iff $s \in R$. This finishes the construction of I. It is not too difficult to check that I is indeed a model of O such that all predicates from Σ have a finite extension.

PROOF OF THEOREM 6. Given the above reduction, in order to prove the NEXPTIME upper bound, it remains to argue that checking the existence of a solution S to any (V, \mathcal{E}, F, I) is feasible in non-deterministic polynomial time. For the EXPTIME upper bound, it suffices to show that the existence of S can be decided in polynomial time in case \mathcal{E} contains only positive inequations. Consider a system

$$\mathcal{H}^* = (V^*, \mathcal{E}^*, V^*, I^*).$$

Checking the existence of a solution to \mathcal{H}^* is solvable in non-deterministic polynomial time, because in non-deterministic polynomial time we can build an ordinary inequation system $(V^*, \mathcal{E}^* \cup \hat{\mathcal{E}})$ such that \mathcal{H}^* has a solution iff $(V^*, \mathcal{E}^* \cup \hat{\mathcal{E}})$ has a solution. The inequations $\hat{\mathcal{E}}$ are obtained from the implications $y_1 + \cdots + y_m > 0 \Rightarrow x_1 + \cdots + x_n > 0$ in I^* by non-deterministically adding one of the inequations $y_1 + \cdots + y_m = 0$, $x_1 > 0$, ..., $x_n > 0$ to $\hat{\mathcal{E}}$. If \mathcal{E}^* is a set of positive inequations, then checking the existence of a solution to \mathcal{H}^* is feasible in polynomial time.

We next argue that a given (V, \mathcal{E}, F, I) can be transformed in polynomial time into an enriched inequation system $(V', \mathcal{E}', V', I')$, while preserving the existence of a solution. The resulting system still contains implications, but its solutions must be over \mathbb{N} . We let V' be obtained by adding to V a fresh variable x^{∞} for every $x \in V$. The inequations in \mathcal{E}' are obtained from the inequations in \mathcal{E} by replacing every summand $b_i \cdot y_i$ that occurs on the right-hand-side of " \leq " of an inequation by the summand $b_i \cdot y_i + y_i^{\infty}$. Second, for every $x \in F$, we add $x^{\infty} = 0$. For example, an inequation $2 \cdot x \leq 2 \cdot y + z$ is replaced by $2 \cdot x \leq 2 \cdot y + y^{\infty} + z + z^{\infty}$. The set I' is obtained from I in two steps. First, we replace in every implication every summand $b_i \cdot y_i$ by the summand $b_i \cdot y_i + y_i^{\infty}$. Second, we add to I' the implication

$$x_1^{\infty} + \dots + x_n^{\infty} > 0 \Rightarrow y_1^{\infty} + \dots + y_m^{\infty} > 0 \tag{2}$$

for every inequation of the form (1) in \mathcal{E} . It is not difficult to see that a solution S to \mathcal{H} can transformed into a solution S' for \mathcal{H}' . Let S be a solution to \mathcal{H} , and let $B = 1 + c + \sum_{x \in F} c \cdot S(x)$, where c is the maximal value among the constants that appear in \mathcal{E} . We let S' be defined as follows:

$$S'(x) = \begin{cases} S(x), & x \in V \land S(x) \in \mathbb{N} \\ 0, & x \in V \land S(x) = \aleph_0 \\ 0, & x = y^{\infty} \land S(y) \in \mathbb{N} \\ B, & x = y^{\infty} \land S(y) = \aleph_0 \end{cases}$$
(3)

It is easy to see that S' is a solution to \mathcal{H}' . For the other direction, assume S' is an arbitrary solution to \mathcal{H}' . We define $S:V'\to\mathbb{N}^*$ as follows:

$$S(x) = \begin{cases} S'(x), & S'(x^{\infty}) = 0\\ \aleph_0, & S'(x^{\infty}) > 0 \end{cases}$$

$$\tag{4}$$

It is not difficult to see that that S is a solution to \mathcal{H} .

Mixed Satisfiability in $\textit{DL-Lite}^{\mathcal{HOF}}_{\text{Bool}}$

Let us assume a $DL\text{-}Lite^{\mathcal{H}O\mathcal{F}}_{\mathrm{Bool}}$ ontology O, and let Σ be a set of concept names we want to keep finite. We use \sqsubseteq_O^* to denote the reflexive transitive closure of $\{(r,s),(r^-,s^-)\mid r\sqsubseteq s\in O\}$. A set T of simple concepts is an r-sink (w.r.t. O) if $A\in T$ for all $\top\sqsubseteq \forall s.A\in O$ such that $r\sqsubseteq_O^*$ s. A tile for O is a pair $\tau=(T,R)$, where T is a set of simple concepts, and $R\subseteq N^+_R(O)$ such that

- (1) for all $B_1 \sqcap B_2 \sqsubseteq B_3 \sqcup B_4 \in O$, $\{B_1, B_2\} \subseteq T$ implies $\{B_3, B_4\} \cap T \neq \emptyset$;
- (2) for all $A \sqsubseteq \exists r. \top \in O, A \in T$ implies $r \in R$
- (3) T is an r-sink for all $r^- \in R$

We use Tiles(O) to denote the set of all tiles for O. A tile (T, R) is an r-sink, if T is an r-sink. We let root((T, R)) = T. We now want to use mosaics to check the existence of a model for O that keeps finite the extensions of the predicates in Σ .

DEFINITION 10. A mosaic for (O, Σ) is a function $N : \mathsf{Tiles}(O) \to \mathbb{N} \cup \{\aleph_0\}$ satisfying the following:

- (1) There is a tile $\tau \in \text{Tiles}(O)$ such that $N(\tau) > 0$.
- (2) For every nominal $\{o\}$ that appears in O, there is a tile (T,R) such that $\{o\} \in T$ and N((T,R)) > 0.
- (3) For every pair of tiles $\tau_1, \tau_2 \in \mathsf{Tiles}(O)$ and every nominal $\{o\}$ that appears in $O, \{o\} \in \mathsf{root}(\tau_1) \cap \mathsf{root}(\tau_2), N(\tau_1) > 0$ and $N(\tau_2) > 0$ imply $\tau_1 = \tau_2$ and $N(\tau_1) = 1$.
- (4) For every $A \in \Sigma$ and every tile $\tau \in \text{Tiles}(O)$ with $A \in \text{root}(\tau)$, we have $N(\tau) \neq \aleph_0$.
- (5) For all tiles $\tau = (T, R)$ and all $r \in R$, if $N(\tau) > 0$, then there exists a tile τ' such that $N(\tau') > 0$ and τ' is an r-sink.
- (6) For every r with func(r), $func(r^{-}) \in O$, we have

$$\sum_{\substack{(T,R)\in\mathsf{Tilles}(O)\,\wedge\\r\in R}} N((T,R)) \quad = \quad \sum_{\substack{(T,R)\in\mathsf{Tilles}(O)\,\wedge\\r^-\in R}} N((T,R))\;.$$

(7) For every r with $func(r^-) \in O$ but $func(r) \notin O$, we have

$$\sum_{\substack{(T,R)\in\mathsf{Tiles}(O)\,\wedge\\r\in R}} N((T,R)) \leq \sum_{\substack{(T,R)\in\mathsf{Tiles}(O)\,\wedge\\T \ is \ an \ r-sink}} N((T,R)) \ .$$

Mosaics as above correctly characterize mixed satisfiability in $\textit{DL-Lite}^{\mathcal{HOF}}_{\text{Bool}}$:

Theorem 12. Let O be a DL-Lite $^{\mathcal{HOF}}_{\mathrm{Bool}}$ ontology and let $\Sigma\subseteq N_{\mathbb{C}}(O)$. Then the following are equivalent:

- (A) O has a model I such A^{I} is finite for each $A \in \Sigma$.
- (B) There exists a mosaic N for (O, Σ) as in Definition 10.

PROOF. The proof is analogous to the proof of Theorem 5.

The argument for the following theorem is very similar to the argument for Theorem 7, where a NEXPTIME upper bound was shown. The key difference here is that the inequation systems that result from Definition 10 have only polynomially many inequations in the size of the input ontology. The only challenge to obtain an NP upper bound is the fact that individual inequations might have exponentially many variables. However, using the result in [36] (which is in essence due to [14]), we know that we can concentrate on solutions in which only polynomially many variables take non-zero values.

Theorem 13. It is an NP-complete problem to decide, given (O, Σ) with O in DL-Lite ${}^{\mathcal{HOF}}_{Bool}$, whether O has a model I such A^I is finite for all predicates $A \in \Sigma$.

PROOF. To obtain the upper-bound, we use a procedure that non-deterministically generates a polynomially sized integer inequation system, which then can be checked for the existence of a solution in non-deterministic polynomial time. The generation of the inequations has 3 steps, which overcome 3 challenges.

- (A) This step deals with the conditional inequalities in point (5) of Definition 10. The "conditionals" can be eliminated by guessing a set *G* of precisely the roles for which a sink tile will exists. Given a guess *G*, we take the following inequations:
 - for all $r \in G$,

$$\sum_{\substack{\tau \in \mathsf{Tiles}(O) \, \land \\ \tau \text{ is an } r\text{-sink}}} x_{\tau} > 0$$

- for all $r \notin G$.

$$\sum_{\substack{\tau = (T, R) \in \mathsf{Tiles}(O) \, \land}} x_{\tau} = 0$$

- (B) The above already leads us to an almost ordinary set \mathcal{E} of integer inequations. We need to deal with the special value \aleph_0 . We argue that by using a small guess we can obtain an inequation system over pure integers. Note that we have a small number of equations: we have at most $n = 3 \times 2 \times |\mathsf{N}_R(O)|$ inequations (we count that the equation from point (7) as two inequations). We guess a set X of variables from the equations such that $|X| \le n$ (we in fact need less than that), and for each $x_\tau \in X$ we have that the root of τ does not have a predicate from Σ . Intuitively, we assume that the variables from X will have the value \aleph_0 . Based the guessed X, we can reduce our inequation system. If a variable from X occurs on the right-hand-side of an inequation, then we can drop the inequation. If a variable from X occurs on the left-hand-side of an inequation, but there is no variable in X that occurs on the right-hand-side, then we know that the guess of X was wrong (there exist no mosaic under such guess of X). We have that the original system is satisfiable with the special value \aleph_0 iff there exists X as above such that the reduced system has a plain integer solution.
- (C) After guessing the small *G* and *X*, we can concentrate on an ordinary inequation system, but the problem is that it has exponentially many variables. However, due to Lemma 3 in [36], the inequation system has a solution iff it has a solution where only a small number (polynomial in the number of inequations) of variables have non-zero values. That means we can guess those variables, and consider an inequation system over those variables only, which is small as desired.

The lower bound is inherited from propositional logic.

Mixed Satisfiability in Horn DLs with functionality

By generalizing the cycle reversion technique for finite model reasoning in Horn description logics [22, 38], we will reduce the problem of mixed satisfiability to (unrestricted) satisfiability. This yields a series of results for Horn DLs with functionality: $DL\text{-}Lite_{core}^{\mathcal{F}}$, $DL\text{-}Lite_{Horn}^{\mathcal{F}}$, and Horn- \mathcal{ALCIF} , subsuming \mathcal{ELI}_{\perp} with functionality. Since Horn- \mathcal{ALCIF} subsumes all the above mentioned DLs, we will present the approach assuming a Horn- \mathcal{ALCIF} ontology O.

We start by recalling how the cycle reversion technique works in the case of finite model reasoning in Horn- \mathcal{ALCIF} .

DEFINITION 11. A finmed cycle in O is a sequence $K_1, r_1, \ldots, r_{n-1}, K_n$, with $n \ge 1, K_1, \ldots, K_n$ conjunctions of concept names, and r_0, \ldots, r_n roles such that:

- $-K_1 = K_n,$
- $K_i \sqsubseteq_O \exists r_i.K_{i+1}$, for $1 \le i < n-1$, and
- $K_{i+1} \sqsubseteq_{O} (\leq 1 r_{i}^{-} A)$, for some $A \in K_{i}$,

where $K \sqsubseteq_O C$ is a short-hand for $O \models K \sqsubseteq C$, for some concept C.

Further, we say that a finmod cycle $K_1, r_1, \ldots, r_{n-1}, K_n$ in O is reversed if $O \models K_{i+1} \sqsubseteq \exists r_i^-.K_i$ and $O \models K_i \sqsubseteq (\leq 1 \ r_i \ K_{i+1})$.

Ontology Focusing: Knowledge-enriched Databases on Demand

We then have the announced result.

Theorem 14 ([22]). Let O be a Horn- \mathcal{ALCIF} ontology. An ontology $\widehat{O}\supseteq O$ can be constructed such that

- the size of \widehat{O} is bounded by poly(|O|),
- all finmod cycles in O are reversed in \widehat{O} , and
- O is finitely satisfiable iff \widehat{O} is satisfiable.

So, in a nutshell, \widehat{O} above is obtained by adding axioms to O that ensure that all the finmod cycles in O are reversed.

Now, we proceed to extend this result for deciding mixed satisfiability. We start by introducing some additional notions. Let $\Sigma \subseteq N_C(O)$, we say that a model I of O is a Σ -finite model of O if A^I is finite for every $A \in \Sigma$, and that a type τ is finitely realized in I if the set of elements in I realizing τ is finite. The first step will then be to compute the set of all types that are finitely realized in some model Σ -finite model. We will show that to compute this set is enough to compute a set of conjunctions that determine such types. Then, as a second step, by reversing all cycles in O in which those "partially specified" types participate, we will obtain the desired reduction from mixed satisfiability to satisfiability.

In what follows, we will deliberately confuse conjunctions of concept names and sets of concept names in what follows.

DEFINITION 12. Let Σ^* be the smallest set of conjunctions occurring in O such that:

- $K \in \Sigma^*$, for every $K \sqsubseteq_O A \in O$ such that $A \in \Sigma$;
- $K \in \Sigma^*$, for every $K' \in \Sigma^*$ such that $K \sqsubseteq_O \exists r.K'$ and $A \sqsubseteq_O (\leq 1 \ r^- \ K)$.

The following lemma establishes that Σ^* precisely determines all the types that have to be finitely realized in a Σ -finite model of O.

Lemma 4. Let τ be a realizable O-type τ . τ is finitely realized in every model Σ finite model I of O iff $K \subseteq \tau$ for some $K \in \Sigma^*$.

PROOF. The direction (\Leftarrow) follows from the definition of Σ^* and the semantics of Horn- \mathcal{ALCIF} . For the direction (\Rightarrow), we show the contrapositive. So, we assume that there is no $K \subseteq \tau$ as in the statement of the lemma. Let I be a Σ -finite model such that τ is finitely realized in I. Then, we can construct a Σ -finite model \mathcal{J} from I such that τ is not finitely realized by creating countably many copies d_1, d_2, \ldots of some fixed $d \in \operatorname{adom}(I)$ with $\operatorname{tp}_I(d) = \tau$. The construction of \mathcal{J} relies on an special kind of unraveling, inductively defined as follows. The initial instance \mathcal{J}_0 contains a single 'distinguished' element d_0 such that $\operatorname{tp}_{\mathcal{J}_0}(d_0) = \tau$ and a copy of every element f in I such that $K \subseteq \operatorname{tp}_I(f)$ for some $K \in \Sigma^*$ with their unary types set as in I. We will denote this set of copies as Δ_F . For the inductive step, start by setting $\mathcal{J}_{i+1} = \mathcal{J}_i$ and then extend \mathcal{J}_{i+1} as follows. Include a fresh copy d_{i+1} of d in \mathcal{J}_{i+1} , and for every element e in \mathcal{J}_i such that its type contains some K, $K \sqsubseteq \exists r.A \in O$ and $e \notin (\exists r.A)^{\mathcal{J}_i}$ proceed as follows. Let e' be an element in I witnessing this requirement (such e' exists since I is a model of O). Then, if there is a copy of e' in Δ_F , add an r-edge from e to that copy. Otherwise, add a fresh copy of e' to \mathcal{J}_{i+1} and an r-edge from e to that copy.

It can be readily checked that the interpretation obtained in the limit is a model of O. To see that it is a Σ-finite model, it suffices to observe that (1) Δ_F is finite, and (2) the types of the fresh witnesses (which are copied from the original Σ-finite model I) added in the inductive steps do not contain any concept from Σ , as otherwise they would be in Δ_F .

Before describing the generalized cycle reversion, we make a note on the complexity of computing Σ^* in all the DLs considered.

Proposition 15. Σ^* can be computed

- in NLogSpace for DL-Lite $_{core}^{\mathcal{F}}$,
- in PTime for DL-Lite $_{Horn}^{\mathcal{F}}$
- in ExpTime for Horn- \mathcal{ALCIF} .

PROOF. The complexity bounds follow from the definition of Σ^* and the known complexity of deciding \sqsubseteq_Q for each of the DLs.

With the computation of Σ^* in place, we can now define what we will call Σ^* -cycle reversion. Firs, we need a straightforward generalization of finmed cycles.

DEFINITION 13. Let O be a Horn- \mathcal{ALCIF} ontology, and Σ^* a set of conjunctions occurring in O. A Σ^* -cycle in O is a sequence

$$K_1, r_1, \ldots, K_{n-1}, r_n, K_n$$
 such that

- $K_1 \in \Sigma^*$,
- $-\ K_1=K_n,$
- $K_i \sqsubseteq_O \exists r_i.K_{i+1}$, for $1 \le i < n-1$, and
- $K_{i+1} \sqsubseteq_O (\leq 1 r_i^- A)$, for some $A \in K_i$.

R1
$$\overline{K \sqcap A \sqsubseteq A}$$
 R2 $\overline{K \sqsubseteq A_i} \qquad R4$ $\overline{K \sqsubseteq A_i} \qquad R5$ $\overline{K \sqsubseteq A_i} \qquad R6$ $\overline{K \sqsubseteq A_i} \qquad \overline{K \sqsubseteq$

Figure 1: Inference Rules

For our purpose then it will be enough to reverse all Σ^* -cycles in O with Σ^* as in Definition 12. As for finmod cycles in Horn- \mathcal{ALCIF} , we can achieve this using a consequence-driven calculus [22] (see figure 1 for reference), by adding a precondition in rule R9 that $K_1 \in \Sigma^*$. The extended ontology \widehat{O} in which all Σ^* -cycles are reversed can be obtained after an exponential number of rule applications. The following lemma provides the reduction for Horn- \mathcal{ALCIF} .

Lemma 5. Let $\widehat{O} \supseteq O$ be an ontology in which all Σ^* -cycles in O are reversed. Then \widehat{O} is satisfiable iff O has a Σ -finite model.

PROOF. The (\Leftarrow) direction follows since $O \subseteq \widehat{O}$ and every every rule is sound w.r.t. the Σ -finite model semantics.

For the (\Rightarrow) direction, let I be a model of O that realizes at least one type containing some conjunction in Σ^* (otherwise the statement holds trivially). One can construct a finite interpretation \mathcal{J} using the special unraveling in the proof of Theorem 3 in [22], by considering only the set of types realized in I, that contain some conjunction from Σ^* .

In order to extend $\mathcal J$ to a Σ -model of O it remains to add missing witnesses for existential restrictions in O, this can be done as follows. For every element d in $\mathcal J$ whose type contains some K, such that $K \sqsubseteq \exists r.A \in O$ and $d \notin (\exists r.A)^{\mathcal J_i}$, let e be an element in I witnessing this requirement, which exists since I is a model of \widehat{O} . If the type of e' contains some conjunction from Σ^* , then then add an r-edge from e to some element d' in $\mathcal J$ with $\operatorname{tp}_{\mathcal J}(d') = \operatorname{tp}_{I}(e')$. This element exists by construction and moreover, adding this r-edge does not violates functionality.

Otherwise, if the type of e' does not contain any conjunction from Σ^* , add a fresh element of d' to $\mathcal J$ and an r-edge from e to d' and set tp $\mathcal J$ (d') = tp $\mathcal J$ (e').

We then obtain the following result, where the upper bound follows from Lemma 5 above and the fact (unrestricted) satisfiability in Horn- \mathcal{HLCIF} is Exptime-complete. The lower bound follows from the complexity of satisfiability in \mathcal{ELI}_{\perp} .

THEOREM 16. Mixed satisfiability in Horn- \mathcal{ALCIF} is ExpTime-complete

To obtain the complexity bounds for the other considered DLs, first note that Definition 13 applies to all of them. Then, note that for instance, in DL- $Lite_{core}^{\mathcal{F}}$ conjunction considered for cycles are either empty i.e., \top or have a single concept name, and functionality assertions func(r) are equivalent to having the axiom $\top \sqsubseteq (\le 1 \ r \ \top)$. Finally, note that for DL- $Lite_{core}$ ontologies, which are not able to express conjunction on the left side of axioms Σ^* can be regarded simply as a superset of Σ .

Next, we assume that the unary type of an element in a model O indicates whether it belongs to the domain and range of a given role r. This consideration is without loss of generality as every ontology O can be transformed into such an equisatisfiable ontology, by adding for every role r name (and its inverse r^-) occurring in O the following axioms $A_r \equiv \exists r. \top$ and $A_{r^-} \equiv \exists r^-. \top$. Now, identifying a Σ^* -cycle in DL- $Lite^{\mathcal{F}}_{Horn}$ can be done by representing the ontology as a directed graph in which the nodes are conjunctions of concept names occurring in the ontology, and where there is an edge between K, K' if $K \sqsubseteq_O \exists r$ and $\exists r^- \sqsubseteq_O K'$ with func $(r) \in O$.

Ontology Focusing: Knowledge-enriched Databases on Demand

In the case of $DL\text{-}Lite_{core}^{\mathcal{F}}$ the graph is simpler. The nodes are the concept names representing the domain and range of roles in the ontology, with an edge between A_r and A_{S^-} if $A_r \subseteq_O \exists s^-$ and $\operatorname{func}(s^-) \in O$.

Now, note that deciding \sqsubseteq_O can be done in polynomial time on the size of the ontology for DL- $Lite_{Horn}^{\mathcal{F}}$ and in NLogSpace for DL- $Lite_{core}^{\mathcal{F}}$. Further, deciding whether an edge in a directed graph belongs to a cycle in LogSpace. We then obtain the following result.

Proposition 17. An ontology $\widehat{O} \supseteq O$ in which all Σ^* -cycles are reversed can be constructed

- in PTIME, for a DL-Lite thorn ontology O, and
 in NL for a DL-Lite ontology O.

Proposition 17 and Lemma 5 yield the desired result:

Corollary 1. Mixed satisfiability is PTIME-complete for DL-Lite $_{horn}^{\mathcal{F}}$, and complete for NL for DL-Lite $_{core}^{\mathcal{F}}$.