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Prime Ideals in Skew Polynomial Rings and Quantized Weyl Algebras*

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The concern of this paper is to investigate the structure of skew polynomial rings (Ore extensions) of the form $T = R[\theta; \sigma, \delta]$ where σ and δ are both nontrivial, and in particular to analyze the prime ideals of T. The main focus is on the case that R is commutative noetherian. In this case, the prime ideals of T are classified, polynomial identities and Artin–Rees separation in prime factor rings are investigated, and cliques of prime ideals are studied. The second layer condition is proved, as well as boundedness of uniform ranks for the prime factor rings corresponding to any clique. Further, q-skew derivations on noncommutative coefficient rings are introduced, and some preliminary results on contractions of prime ideals of T are obtained in this setting. Finally, prime ideals in quantized Weyl algebras over fields are analyzed. © 1992 Academic Press, Inc.

INTRODUCTION

The majority of previous work on skew polynomial rings $T = R[\theta; \sigma, \delta]$, with the notable exception of a paper of Irving [25], has concentrated on the two "unmixed" cases, in which either $\sigma = 1$ or $\delta = 0$. However, the recent surge of interest in quantum groups and quantized algebras (see, e.g., [14, 15, 29, 34, 38, 39]) has brought renewed interest in general skew polynomial rings, due to the fact that many of these quantized algebras and their representations can be expressed in terms of (iterated) skew polynomial rings. This development calls for a thorough study of skew polynomial rings, starting with the fundamental case in which the coefficient ring R is commutative noetherian. While Irving gave an extensive analysis of this situation, his methods did not deal with all the cases that arise. Thus our primary goal in this paper is to develop methods strong enough to handle arbitrary commutative noetherian coefficient rings. (Prime ideals in

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$R[\theta; \sigma, \delta]$ for R noncommutative noetherian are investigated in [19].) A secondary purpose is to initiate methods for studying the "q-skew derivations" that appear in the representation theory of quantized algebras, namely skew derivations (σ, δ) such that $\delta\sigma$ is a constant multiple of $\sigma\delta$.

In the case that R is commutative noetherian (and σ is an automorphism), we give a complete description of the prime ideals of T in terms of their contractions to R. Namely, an ideal I of R is the contraction of a prime of T if and only if (a) I is a σ-prime (σ, δ)-ideal of R, or (b) I is a δ-prime (σ, δ)-ideal of R and R/I has a unique associated prime ideal, or (c) I is a prime ideal of R and $\sigma(I) \neq I$.

Keeping the above assumptions on R, we then investigate the prime factor rings of T. Given primes $P > Q$ in T such that $P \cap R = Q \cap R$, we show that either T/Q is a P.I. ring or else the ideal induced from P/Q in a suitable localization of T/Q is generated by a normal element. Building on this result, we prove that for any primes $P > Q$ in T, the ideal induced from P/Q in a suitable localization of $(T/Q)[x]$ contains a nonzero AR-ideal. It follows that T satisfies the strong second layer condition. Further, we establish bounds on the uniform ranks of the prime factor rings T/P where P ranges over any clique of primes in Spec(T). Consequently, if T is an algebra over an uncountable field, then all cliques of primes in Spec(T) are classically localizable.

When a prime P of T contracts to an ideal I of R satisfying either (a) or (c) above, the structure of T/P (after localization) is already fairly accessible using results from the literature. In the remaining case, we prove (under a mild additional hypothesis) that after localizing R at the unique associated prime N of R/I, the factor ring T/IT becomes isomorphic to a full matrix ring over an ordinary differential operator ring with coefficients in the quotient field of R/N. As an application, we construct characteristic zero examples of noetherian ring extensions $R \subseteq T$ such that T is a free R-module on each side, the subring R is an R-module direct summand of T on each side, and gl.dim$(T) = 1$ while gl.dim$(R) = \infty$.

The results of the previous paragraph rely in part on our more general work on q-skew derivations over noetherian but not necessarily commutative coefficient rings R. In this context, we show in particular that if $\delta\sigma = q\sigma\delta$ for a central (σ, δ)-constant q which is not a root of unity, then for any prime P of T, the largest (σ, δ)-ideal of R contained in $P \cap R$ is necessarily σ-prime.

In the final section of the paper, we apply our results to an analysis of the prime ideals of the quantized Weyl algebras over a field k, namely the algebras $A_1(k, q) = k\{x, y\}/\langle xy - qyx - 1\rangle$ for nonzero $q \in k$. In particular, we show that whenever $q \neq 1$, the global and Krull dimensions of $A_1(k, q)$ all equal 2.

All rings in this paper are associative with unit.

1. Generalities

We recall some definitions and more or less well known facts about skew derivations and skew polynomial rings.

A *(left) skew derivation* on a ring R is a pair (σ, δ) where σ is a ring endomorphism of R and δ is a *(left) σ-derivation* on R, that is, an additive map from R to itself such that $\delta(ab) = \sigma(a)\,\delta(b) + \delta(a)b$ for all $a, b \in R$.

LEMMA 1.1. *Let (σ, δ) be a skew derivation on a ring R. Then*

$$\delta(a^m) = \sum_{i=0}^{m-1} \sigma(a)^i\,\delta(a)a^{m-1-i}$$

for all $a \in R$ and $m = 1, 2, \ldots$.

Proof. This is clear when $m = 1$. If it holds for some m, then

$$\delta(a^{m+1}) = \sigma(a)\,\delta(a^m) + \delta(a)a^m = \sum_{i=0}^{m-1} \sigma(a)^{i+1}\,\delta(a)a^{m-1-i} + \delta(a)a^m$$

$$= \sum_{i=0}^{m} \sigma(a)^i\,\delta(a)a^{m-i}. \quad \blacksquare$$

Given a ring R and maps $\sigma, \delta\colon R \to R$, consider the map $\phi\colon R \to M_2(R)$ given by the rule

$$\phi(r) = \begin{pmatrix} \sigma(r) & \delta(r) \\ 0 & r \end{pmatrix}.$$

Then ϕ is a ring homomorphism if and only if (σ, δ) is a skew derivation (see, e.g., [13, pp. 66–67]). This observation provides an easy way to construct examples and to extend skew derivations, as follows.

LEMMA 1.2. *Let (σ, δ) be a skew derivation on a ring S, and let u, v be elements in a polynomial ring $S[x]$. Then (σ, δ) extends to a skew derivation on $S[x]$ such that $\sigma(x) = u$ and $\delta(x) = v$ if and only if $u\sigma(s) = \sigma(s)u$ and $vs - \sigma(s)v = (x - u)\,\delta(s)$ for all $s \in S$.*

Proof. If there exists such an extension, then for $s \in S$ we apply σ and δ to the equation $xs = sx$ to get

$$u\sigma(s) = \sigma(s)u \qquad \text{and} \qquad u\delta(s) + vs = \sigma(s)v + \delta(s)x,$$

and from the latter equation $vs - \sigma(s)v = (x - u)\,\delta(s)$ follows.

Conversely, assume that $u\sigma(s) = \sigma(s)u$ and $vs - \sigma(s)v = (x-u)\,\delta(s)$ for all $s \in S$. It follows that the matrices

$$\phi(s) = \begin{pmatrix} \sigma(s) & \delta(s) \\ 0 & s \end{pmatrix} \qquad \text{and} \qquad y = \begin{pmatrix} u & v \\ 0 & x \end{pmatrix}$$

commute, and hence the ring homomorphism $\phi: S \to M_2(S)$ extends to a ring homomorphism

$$\phi': S[x] \to M_2(S[x])$$

such that $\phi'(x) = y$. It is clear that ϕ' satisfies a rule of the form

$$\phi'(s) = \begin{pmatrix} \sigma'(s) & \delta'(s) \\ 0 & s \end{pmatrix},$$

where σ', δ' are maps from $S[x]$ to itself extending σ, δ. Thus (σ', δ') is a skew derivation on $S[x]$, extending (σ, δ), such that $\sigma'(x) = u$ and $\delta'(x) = v$. ∎

LEMMA 1.3. *Let (σ, δ) be a skew derivation on a ring R, and let X be a right denominator set in R such that $\sigma(X) \subseteq X$. Then (σ, δ) extends uniquely to a skew derivation on RX^{-1}. Moreover,*

$$\sigma(rx^{-1}) = \sigma(r)\sigma(x)^{-1} \qquad \textit{and} \qquad \delta(rx^{-1}) = \delta(r)x^{-1} - \sigma(rx^{-1})\,\delta(x)x^{-1}$$

for all $r \in R$ and $x \in X$.

Proof. It is clear that σ induces a unique ring endomorphism on RX^{-1}. Suppose that (σ, δ) extends to a skew derivation on RX^{-1}. For $r \in R$ and $x \in X$, we can apply δ to the equation $r1^{-1} = (rx^{-1})(x1^{-1})$ to get

$$\delta(r)1^{-1} = \sigma(rx^{-1})\,\delta(x)1^{-1} + \delta(rx^{-1})(x1^{-1}),$$

whence $\delta(rx^{-1}) = \delta(r)x^{-1} - \sigma(rx^{-1})\,\delta(x)x^{-1}$. Thus if (σ, δ) extends to a skew derivation on RX^{-1}, the extension is unique and satisfies the given formulas.

Composing the natural map $M_2(R) \to M_2(RX^{-1})$ with the ring homomorphism $R \to M_2(R)$ obtained from σ and δ, we obtain a ring homomorphism $\phi: R \to M_2(RX^{-1})$ given by the rule

$$\phi(r) = \begin{pmatrix} \sigma(r)1^{-1} & \delta(r)1^{-1} \\ 0 & r1^{-1} \end{pmatrix}.$$

Since $\sigma(X) \subseteq X$, we see that $\phi(x)$ is invertible in $M_2(RX^{-1})$ for all $x \in X$; namely,

$$\phi(x)^{-1} = \begin{pmatrix} 1\sigma(x)^{-1} & -1\sigma(x)^{-1}\delta(x)x^{-1} \\ 0 & 1x^{-1} \end{pmatrix}.$$

Hence, ϕ extends (uniquely) to a ring homomorphism $\phi' \colon RX^{-1} \to M_2(RX^{-1})$ such that $\phi'(rx^{-1}) = \phi(r)\phi(x)^{-1}$ for all $r \in R$ and $x \in X$. It is clear that ϕ' satisfies a rule of the form

$$\phi'(rx^{-1}) = \begin{pmatrix} \sigma'(rx^{-1}) & \delta'(rx^{-1}) \\ 0 & rx^{-1} \end{pmatrix},$$

and therefore (σ', δ') is a skew derivation on RX^{-1} extending (σ, δ). ∎

Given a skew derivation (σ, δ) on a ring R, we denote the corresponding *skew polynomial ring* (or *Ore extension*) by $R[\theta; \sigma, \delta]$. This ring is a free left R-module with basis $1, \theta, \theta^2, \ldots$, and $\theta r = \sigma(r)\theta + \delta(r)$ for all $r \in R$. (We obtain skew polynomial rings with left-hand coefficients because we are using left skew derivations.) To abbreviate the assertion that a symbol T is to stand for the skew polynomial ring $R[\theta; \sigma, \delta]$ constructed from a ring R and a skew derivation (σ, δ) on R, we just write "let $T = R[\theta; \sigma, \delta]$."

The *degree* of a nonzero element $t \in T$ is defined in the obvious fashion. Since the standard form for elements of T is with left-hand coefficients, the *leading coefficient* of t is t_n if

$$t = t_n\theta^n + t_{n-1}\theta^{n-1} + \cdots + t_1\theta + t_0$$

with all $t_i \in R$ and $t_n \neq 0$. (If σ is an automorphism, t can also be written with right-hand coefficients, but then its θ^n-coefficient is $\sigma^{-n}(t_n)$.)

While a general formula for $\theta^n r$ (where $r \in R$ and $n \in \mathbb{N}$) is too involved to be of much use, an easy induction establishes that

$$\theta^n r = \sigma^n(r)\theta^n + r_{n-1}\theta^{n-1} + \cdots + r_1\theta + \delta^n(r)$$

for some $r_1, \ldots, r_{n-1} \in R$.

Recall that if σ is an automorphism and R is right (left) noetherian, then T is right (left) noetherian (see, e.g., [13, Sect. 12.2, Theorem 3; 21, Theorem 1.12; 37, Theorem 1.2.9]).

LEMMA 1.4. *Let $T = R[\theta; \sigma, \delta]$ where σ is an automorphism. Let X be a right denominator set in R such that $\sigma(X) = X$, and extend (σ, δ) to RX^{-1} as in Lemma 1.3. Then X is a right denominator set in T, and the identity map on RX^{-1} extends to an isomorphism of TX^{-1} onto $(RX^{-1})[\theta; \sigma, \delta]$ sending $\theta 1^{-1}$ to θ.*

Proof. Set $T^\circ = (RX^{-1})[\theta; \sigma, \delta]$, and observe that the natural map $R \to RX^{-1}$ extends to a ring homomorphism $f\colon T \to T^\circ$ such that $f(\theta) = \theta$. Note that $f(x)$ is invertible in T° for all $x \in X$.

Since $\sigma(X) = X$, it follows by an induction on degree that for any $t^\circ \in T^\circ$, there exists $x \in X$ such that $t^\circ f(x) \in f(T)$, whence $t^\circ = f(t)f(x)^{-1}$ for some $t \in T$. A similar induction shows that for any $u \in \ker(f)$, there exists $x \in X$ such that $ux = 0$.

Therefore f is a right ring of fractions for T with respect to X, whence X is a right denominator set in T and f extends to an isomorphism of TX^{-1} onto T°. ∎

Let σ be an endomorphism of a ring R. For any $a \in R$, the rule $\delta_a(r) = ar - \sigma(r)a$ defines a σ-derivation δ_a on R. Any σ-derivation of this form is *inner*; all others are *outer*.

LEMMA 1.5. *Let $T = R[\theta; \sigma, \delta]$.*

(a) *If σ is an automorphism, then $-\delta\sigma^{-1}$ is a σ^{-1}-derivation on R^{op} and the identity map on R^{op} extends to an isomorphism of $R^{op}[\theta'; \sigma^{-1}, -\delta\sigma^{-1}]$ onto T^{op} sending θ' to θ.*

(b) *Suppose that σ is an inner automorphism of R, say there is a unit $u \in R$ such that $\sigma(r) = u^{-1}ru$ for all $r \in R$. Then $u\delta$ is an ordinary derivation on R, and the identity map on R extends to an isomorphism of $R[\theta'; u\delta]$ onto T sending θ' to $u\theta$.*

(c) *Suppose that δ is an inner σ-derivation, say $\delta = \delta_a$ for some $a \in R$. Then the identity map on R extends to an isomorphism of $R[\theta'; \sigma]$ onto T sending θ' to $\theta - a$.*

Proof. (a) It is trivial to check that $-\delta\sigma^{-1}$ is a σ^{-1}-derivation on R^{op}. Observe that T is a free right R-module with basis $1, \theta, \theta^2, \ldots$, whence T^{op} is a free left R^{op}-module with basis $1, \theta, \theta^2, \ldots$. In T, we have $r\theta = \theta\sigma^{-1}(r) - \delta\sigma^{-1}(r)$ for all $r \in R$, and so in T^{op} we have $\theta r = \sigma^{-1}(r)\theta - \delta\sigma^{-1}(r)$ for all $r \in R$. Thus there exists an isomorphism of $R^{op}[\theta'; \sigma^{-1}, -\delta\sigma^{-1}]$ onto T^{op} of the form desired.

(b) Observe that $u\theta r = ru\theta + u\delta(r)$ for all $r \in R$, whence $u\delta$ is a derivation on R. Since also T is a free left R-module with basis $1, u\theta, (u\theta)^2, \ldots$, there exists an isomorphism of $R[\theta'; u\delta]$ onto T of the form desired.

(c) Observe that $(\theta - a)r = \sigma(r)(\theta - a)$ for all $r \in R$. Since also T is a free left R-module with basis 1, $\theta - a$, $(\theta - a)^2, \ldots$, there exists an isomorphism of $R[\theta'; \sigma]$ onto T of the form desired. ∎

We close this section with a final easy observation. Given $T = R[\theta; \sigma, \delta]$ and given an ideal I of R such that $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$, note that (σ, δ) induces a skew derivation on R/I, that IT is an ideal of T, and that $T/IT \cong (R/I)[\theta; \sigma, \delta]$. If σ is an automorphism, then also $IT = TI$.

2. σ-Prime, δ-Prime, and (σ, δ)-Prime Ideals

In preparation for our analysis of the types of ideals that occur when a prime ideal of a skew polynomial ring $R[\theta; \sigma, \delta]$ is contracted to the coefficient ring R, we consider σ-prime, δ-prime, and (σ, δ)-prime ideals of R. The main point of this section is to show that these types of ideals are not unrelated, at least when R is noetherian and σ is an automorphism. We prove that the prime radical of any (σ, δ)-prime ideal of R is σ-prime (Theorem 2.3), and that if R is commutative then every (σ, δ)-prime ideal is either σ-prime or δ-prime (Theorem 2.6).

Definition. Let Σ be a set of maps from a ring R to itself. A *Σ-ideal* of R is any ideal I of R such that $\sigma(I) \subseteq I$ for all $\sigma \in \Sigma$. A *Σ-prime ideal* is any proper Σ-ideal I such that whenever J, K are Σ-ideals satisfying $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$. In case 0 is a Σ-prime ideal of R, we say that R is a *Σ-prime ring*. Finally, R is *Σ-simple* provided $R \neq 0$ and 0 and R are the only Σ-ideals of R.

In the context of a ring R equipped with a skew derivation (σ, δ), we shall make use of the above definitions in the cases $\Sigma = \{\sigma\}$, $\Sigma = \{\delta\}$, or $\Sigma = \{\sigma, \delta\}$; we simplify the prefix Σ to σ, δ, or (σ, δ) in these cases. Note that if σ is an automorphism and I is a σ-ideal of R, then

$$I \subseteq \sigma^{-1}(I) \subseteq \sigma^{-2}(I) \subseteq \cdots$$

and each positive power σ^n induces an additive isomorphism of $\sigma^{-n}(I)/\sigma^{1-n}(I)$ onto $I/\sigma(I)$. Hence, if R has ACC on ideals it follows that $\sigma(I) = I$. We shall use this fact often, sometimes without explicit mention.

In [24, 25, 40], slightly stronger definitions of σ-primeness are used. However, when σ is an automorphism and R has ACC on ideals, these definitions coincide with the one given above.

Lemma 2.1. *Let σ be an automorphism of a ring R, and let I be a proper ideal of R such that $\sigma(I) = I$.*

(a) *I is σ-prime if and only if for any $a, c \in R - I$, there exist $b \in R$ and $t \in \mathbb{Z}$ such that $ab\sigma^t(c) \notin I$.*

(b) *Assume that R is right noetherian. Then I is σ-prime if and only if there exist a prime ideal P minimal over I and a positive integer n such that $\sigma^n(P)=P$ and $I=P\cap\sigma(P)\cap\cdots\cap\sigma^{n-1}(P)$. In particular, all σ-prime ideals in R are semiprime.*

Proof. (a) Assume first that I satisfies the given condition, and consider σ-ideals A, C not contained in I. Choose $a\in A-I$ and $c\in C-I$; then there exist $b\in R$ and $t\in\mathbb{Z}$ such that $ab\sigma^t(c)\notin I$. If $t\geqslant 0$, then $\sigma^t(c)\in C$ and we have $AC\nsubseteq I$. If $t<0$, then $\sigma^{-t}(a)\sigma^{-t}(b)c\notin\sigma^{-t}(I)=I$. In this case, $\sigma^{-t}(a)\in A$ and again $AC\nsubseteq I$. Thus I is σ-prime.

Conversely, suppose that I is σ-prime, and consider $a, c\in R-I$. The sets

$$A=\sum_{i=0}^{\infty} R\sigma^i(a)R \qquad \text{and} \qquad C=\sum_{j=0}^{\infty} R\sigma^j(c)R$$

are then σ-ideals not contained in I, whence $AC\nsubseteq I$. Consequently, $\sigma^i(a)b\sigma^j(c)\notin I$ for some $i, j\geqslant 0$, and thus $a\sigma^{-i}(b)\sigma^{j-i}(c)\notin I$.

(b) See [17, Remarks 4*, 5*, p. 338]. ∎

As is well known, δ-prime noetherian rings need not be prime (e.g., $k[x]/x^2k[x]$, where k is a field of characteristic 2, and $\delta=d/dx$). However, such rings are primary, and they have artinian classical quotient rings (cf. [16, Theorem 1 and Lemma 2; 31, Theorem 2.2]). We aim at an analogous statement for (σ,δ)-prime noetherian rings (Theorem 2.3), using Jordan's method of proof. The following lemma will help keep the computations manageable.

LEMMA 2.2. *Let (σ,δ) be a skew derivation on a ring R, and let I be a σ-ideal of R. Set $I_0=R$ and $I_1=I$, and for $j=2, 3, \ldots$ set*

$$I_j=\{r\in I\mid \delta\sigma^{m(1)}\delta\sigma^{m(2)}\cdots\delta\sigma^{m(i)}(r)\in I \text{ for all } i=1, \ldots, j-1$$
$$\text{and } m(1), \ldots, m(i)=0, 1, \ldots\}.$$

Then each I_j is a σ-ideal of R. Moreover, $II_j+I_jI\subseteq I_{j+1}$ for all j and $\delta(I_j)\subseteq I_{j-1}$ for all $j>0$.

Proof. Obviously $\sigma(I_j)\subseteq I_j$ *for all j and* $\delta(I_j)\subseteq I_{j-1}$ for all $j>0$.

Certainly I_0 and I_1 are ideals of R. Now assume, for some $j\geqslant 1$, that I_j is an ideal. Given $a\in I_{j+1}$ and $r\in R$, we have $a\in I_j$ and so $ra\in I_j$, whence

$$\delta\sigma^{m(1)}\delta\sigma^{m(2)}\cdots\delta\sigma^{m(i)}(ra)\in I$$

for all $i<j$ and $m(k)\geqslant 0$. For any $m(j)\geqslant 0$, we have $\sigma^{m(j)}(a)\in I_{j+1}$ and so $\sigma^{m(j)}(a)$ and $\delta\sigma^{m(j)}(a)$ are both in I_j. Hence,

$$\delta\sigma^{m(j)}(ra)=\sigma^{m(j)+1}(r)\delta\sigma^{m(j)}(a)+\delta\sigma^{m(j)}(r)\sigma^{m(j)}(a)\in I_j,$$

from which we obtain

$$\delta\sigma^{m(1)}\delta\sigma^{m(2)}\cdots\delta\sigma^{m(j)}(ra)\in I$$

for all $m(1), ..., m(j-1)\geqslant 0$. Thus $ra\in I_{j+1}$, and a similar argument shows that $ar\in I_{j+1}$. Therefore I_{j+1} is an ideal.

We now have that all the I_j are σ-ideals. It remains to prove that $II_j+I_jI\subseteq I_{j+1}$ for all j.

Certainly $II_0+I_0I\subseteq I_1$. Now assume, for some $j\geqslant 1$, that $II_{j-1}+I_{j-1}I\subseteq I_j$. Given $a\in I$ and $b\in I_j$, we have $b\in I_{j-1}$ and so $ab\in I_j$, whence

$$\delta\sigma^{m(1)}\,\delta\sigma^{m(2)}\cdots\delta\sigma^{m(i)}(ab)\in I$$

for all $i<j$ and $m(k)\geqslant 0$. For any $m(j)\geqslant 0$, we have

$$\delta\sigma^{m(j)}(ab)=\sigma^{m(j)+1}(a)\delta\sigma^{m(j)}(b)+\delta\sigma^{m(j)}(a)\sigma^{m(j)}(b)\in II_{j-1}+RI_j=I_j,$$

from which we obtain

$$\delta\sigma^{m(1)}\delta\sigma^{m(2)}\cdots\delta\sigma^{m(j)}(ab)\in I$$

for all $m(1), ..., m(j-1)\geqslant 0$. Thus $ab\in I_j$, and a similar argument shows that $ba\in I_j$. Therefore $II_j+I_jI\subseteq I_{j+1}$, completing our second induction. ∎

THEOREM 2.3. *Let (σ, δ) be a skew derivation on a right noetherian ring R, with σ an automorphism. Let N be the prime radical of R. If R is (σ, δ)-prime, then N is σ-prime and $\mathscr{C}(0)=\mathscr{C}(N)$, whence R has a right artinian classical right quotient ring.*

Proof. Note that 0 is a σ-ideal of R with nonzero right annihilator. Choose a σ-ideal I of R maximal with respect to the property $\mathrm{r.ann}(I)\neq 0$. If A and B are any σ-ideals properly containing I, then $\mathrm{r.ann}(A)=\mathrm{r.ann}(B)=0$, whence $\mathrm{r.ann}(AB)=0$ and so $AB\not\subseteq I$. Thus I is σ-prime, and we shall show that $N=I$.

Define $I_0, I_1, ...$ as in Lemma 2.2. Then $I_0\supseteq I_1\supseteq ...$ is a descending chain of σ-ideals, and the corresponding ascending chain of right annihilators must stabilize. Hence, $\mathrm{r.ann}(I_m)=\mathrm{r.ann}(I_{m+1})$ for some $m>0$. Since I_m is a σ-ideal, we have $\sigma(I_m)=I_m$, from which we see that $\mathrm{r.ann}(I_m)$ is a σ-ideal. Consider any $b\in\mathrm{r.ann}(I_m)$. For $a\in I_{m+1}$, we have $a, \delta(a)\in I_m$ and so $ab=\delta(a)b=0$, whence $0=\delta(ab)=\sigma(a)\,\delta(b)$. Thus $\sigma(I_{m+1})\,\delta(b)=0$, from which we obtain $\delta(b)\in\mathrm{r.ann}(I_m)$. Therefore $\mathrm{r.ann}(I_m)$ is a (σ, δ)-ideal.

If $H=\mathrm{l.ann}(\mathrm{r.ann}(I_m))$, we similarly see that H is a (σ, δ)-ideal. As R is (σ, δ)-prime, either $H=0$ or $\mathrm{r.ann}(I_m)=0$. However, $\mathrm{r.ann}(I_m)\supseteq\mathrm{r.ann}(I)\neq 0$, whence $H=0$ and consequently $I_m=0$. Lemma 2.2 shows that $I^m\subseteq I_m$, and so $I^m=0$. Now $I\subseteq N$. Since I is σ-prime and hence semiprime, $I=N$. Therefore N is σ-prime.

Since the inclusion $\mathscr{C}(0)\subseteq\mathscr{C}(N)$ holds in general (e.g., [21, Lemma 10.8; 37, Proposition 4.1.3]), it remains to prove the reverse inclusion; that R

then has a right artinian classical right quotient ring is Small's Theorem (e.g., [21, Theorem 10.9; 37, Theorem 4.1.4]). Proving the inclusion $\mathscr{C}(N) \subseteq \mathscr{C}(0)$ amounts to showing that R is $\mathscr{C}(N)$-torsionfree on each side. Since $I_m = 0$, it is enough to show that each of the ideal factors I_j/I_{j+1} is $\mathscr{C}(N)$-torsionfree on each side. Note that because $\sigma(N) = N$ we have $\sigma(\mathscr{C}(N)) = \mathscr{C}(N)$.

As $I_0/I_1 = R/N$, it is certainly $\mathscr{C}(N)$-torsionfree. Now assume, for some $j > 0$, that I_{j-1}/I_j is $\mathscr{C}(N)$-torsionfree. Consider $a \in I_j$ and $c \in \mathscr{C}(N)$ such that $ca \in I_{j+1}$. For any $t \geqslant 0$, we have $\sigma^t(a) \in I_j$ and

$$\sigma^{t+1}(c)\, \delta\sigma^t(a) + \delta\sigma^t(c)\sigma^t(a) = \delta\sigma^t(ca) \in \delta\sigma^t(I_{j+1}) \subseteq I_j,$$

whence $\sigma^{t+1}(c)\, \delta\sigma^t(a) \in I_j$. Since $\sigma^{t+1}(c) \in \mathscr{C}(N)$ and $\delta\sigma^t(a) \in I_{j-1}$, it follows from our induction hypothesis that actually $\delta\sigma^t(a) \in I_j$. This being true for all $t \geqslant 0$, we find that $a \in I_{j+1}$. Thus I_j/I_{j+1} is $\mathscr{C}(N)$-torsionfree on the left, and similarly on the right. This completes the induction. ▌

Turning to commutative rings, we first note that commutativity places sharp restrictions on skew derivations, as the following trivial lemma shows. (See, e.g., [12, Lemma, p. 539; 48, Lemme 3] for previous uses.)

LEMMA 2.4. *Let R be a ring with a skew derivation (σ, δ).*

(a) *If R is commutative, then $(a - \sigma(a))\, \delta(b) = (b - \sigma(b))\, \delta(a)$ for all $a, b \in R$.*

(b) *If there exists a central element $c \in R$ such that $\sigma(c)$ is central and $c - \sigma(c)$ is invertible, then δ is inner.*

Proof. (a) Applying δ to the equation $ba = ab$ yields

$$\sigma(b)\, \delta(a) + \delta(b)a = \sigma(a)\, \delta(b) + \delta(a)b,$$

from which the desired equation follows.

(b) Note that the element $u = c - \sigma(c)$ is a central unit. For $r \in R$, applying δ to the equation $rc = cr$ yields

$$\sigma(r)\, \delta(c) + \delta(r)c = \sigma(c)\, \delta(r) + \delta(c)r,$$

whence $u\delta(r) = \delta(c)r - \sigma(r)\, \delta(c)$. Thus $\delta(r) = u^{-1}\delta(c)r - \sigma(r)u^{-1}\delta(c)$ for all $r \in R$. ▌

LEMMA 2.5. *Let (σ, δ) be a skew derivation on a commutative ring R, with σ an automorphism, and let J be a σ-prime ideal of R. If σ does not induce the identity on R/J, then $\delta(J) \subseteq J$.*

Proof. By assumption, $(1 - \sigma)(R) \not\subseteq J$, and so the image of $(1 - \sigma)(R) \cdot R$ in R/J is thus a nonzero ideal A such that $\sigma(A) = A$. The annihilator

$B=\operatorname{ann}_{R/J}(A)$ is a σ-ideal as well, and so by σ-primeness $B=0$. Thus the image of $(1-\sigma)(R)$ in R/J has zero annihilator. In view of Lemma 2.4,

$$(1-\sigma)(R)\cdot\delta(J)=(1-\sigma)(J)\cdot\delta(R)\subseteq J,$$

and therefore $\delta(J)\subseteq J$. ∎

THEOREM 2.6. *Let (σ,δ) be a skew derivation on a commutative noetherian ring R, with σ an automorphism. If R is (σ,δ)-prime, then R must be either σ-prime or δ-prime. Moreover, if R is (σ,δ)-prime but not σ-prime, then R has a unique associated prime N, and $(1-\sigma)(R)\subseteq N$ while $\delta(N)\nsubseteq N$.*

Proof. Let N be the prime radical of R. By Theorem 2.3, N is σ-prime and R has an artinian classical quotient ring S, which of course is also (σ,δ)-prime. Since it suffices to verify the desired conclusions for S, we may thus assume that R is artinian, with radical N. Moreover, we are done if $N=0$, and so we may also assume that $N\neq 0$.

Since N is nilpotent and R is (σ,δ)-prime, N cannot be a (σ,δ)-ideal, whence $\delta(N)\nsubseteq N$. Lemma 2.5 then shows that σ induces the identity on R/N, that is, $(1-\sigma)(R)\subseteq N$. Consequently, the σ-prime ideal N must actually be prime. Thus R is local, and N is the only prime of R.

It remains to show that R is δ-prime. Choose $x\in N$ such that $\delta(x)\notin N$; then $\delta(x)$ is a unit. For all $r\in R$, we have

$$(r-\sigma(r))\,\delta(x)=(x-\sigma(x))\,\delta(r)$$

and so $\sigma(r)=r+v\delta(r)$, where $v=\delta(x)^{-1}(\sigma(x)-x)$. Thus $\sigma=1+v\delta$, whence all δ-ideals of R are actually (σ,δ)-ideals. Therefore R is δ-prime. ∎

The results of Theorem 2.6 are special to the commutative case, as the following examples show. The first of these examples was independently constructed for another purpose by Bergman and Isaacs in answer to a question of Kamal: there exists a central idempotent in the skew polynomial ring $R[\theta;\sigma,\delta]$ which does not lie in R [32, Example 6.4].

EXAMPLE 2.7. There is an artinian ring R with a skew derivation (σ,δ) such that σ is an automorphism, R is δ-simple but not σ-prime, and R has two associated primes on each side.

Proof. Choose a field k and an indeterminate x. In $M_2(k(x))$, let σ be the inner automorphism induced by the matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, and let δ be the inner σ-derivation induced by the matrix $\left(\begin{smallmatrix}0&x^{-1}\\0&0\end{smallmatrix}\right)$. Then

$$\sigma\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}d&c\\b&a\end{pmatrix}\quad\text{and}\quad\delta\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}cx^{-1}&0\\0&-bx^{-1}\end{pmatrix}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k(x))$. Next set $S = \begin{pmatrix} k[x] & xk[x] \\ xk[x] & k[x] \end{pmatrix}$, and observe that S is closed under σ and δ. Since $\sigma^2 = 1$, the restriction of σ to S is an automorphism.

If x is identified with the diagonal matrix $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ in S, then $\sigma(x) = x$ and $\delta(x) = 0$. Hence, Sx is a (σ, δ)-ideal of S, and the ring $R = S/Sx$ inherits an induced skew derivation. Note that R is a 4-dimensional k-algebra, and that R is not semiprime, whence R is not σ-prime. Note also that R has two maximal ideals, which appear as right and left annihilators of the ideals

$$\begin{pmatrix} xk[x] & xk[x] \\ x^2k[x] & xk[x] \end{pmatrix} \Big/ Sx \quad \text{and} \quad \begin{pmatrix} xk[x] & x^2k[x] \\ xk[x] & xk[x] \end{pmatrix} \Big/ Sx.$$

Thus each maximal ideal of R is both a right and a left associated prime of R.

It remains to show that R is δ-simple, that is, that Sx is a maximal δ-ideal of S. Consider a δ-ideal I in S that properly contains Sx, and choose an element $u \in I - Sx$. After multiplying u by matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then subtracting off terms from Sx and multiplying by a scalar, we may asume that u equals either $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$. If $u = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, then the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -\delta(u)$ is in I, whence the matrix $\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$ is in I, and finally the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \delta\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$ is in I. Thus $I = S$ in this case, and similarly if $u = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$. Therefore Sx is a maximal δ-ideal. ∎

EXAMPLE 2.8. There is an artinian ring R with a skew derivation (σ, δ) such that σ is an automorphism and R is (σ, δ)-simple but R is neither σ-prime nor δ-prime.

Proof. Choose a field k, and let τ be the "left shift" automorphism of k^3 given by the rule $\tau(a, b, c) = (b, c, a)$. Let $S = k^3[x; \tau]$ and $T = k^3[x, x^{-1}; \tau]$, and extend τ to automorphisms of S and T where $\tau(x) = x$. Then set $\sigma = \tau^2 = \tau^{-1}$, and let δ be the inner σ-derivation on T given by the rule

$$\delta(t) = (0, 0, 1)x^{-1}t - \sigma(t)(0, 0, 1)x^{-1}.$$

For $(a, b, c) \in k^3$ and $n \in \mathbb{Z}$, we compute that

$$\delta((a, b, c)x^n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \\ (0, -a, b)x^{n-1} & \text{if } n \equiv 1 \pmod 3 \\ (-c, 0, b)x^{n-1} & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

In particular, $\delta(S) \subseteq S$.

Since $\sigma(x^3) = x^3$ and $\delta(x^3) = 0$, the set Sx^3 is a (σ, δ)-ideal of S, and so the ring $R = S/Sx^3$ inherits an induced skew derivation. Note that R is a

9-dimensional k-algebra, and that R is not semiprime, whence R is not σ-prime. Since $\delta((0,0,1)x^2)=0$, the set

$$[k(0,0,1)x^2+Sx^3]/Sx^3$$

is a nilpotent δ-ideal of R, and so R is not δ-prime.

It remains to show that R is (σ,δ)-simple, that is, that Sx^3 is a maximal (σ,δ)-ideal of S. Consider a (σ,δ)-ideal I in S that properly contains Sx^3, and choose an element $u\in I-Sx^3$. After multiplying u by x or x^2 if necessary, then subtracting off terms from Sx^3, we may assume that $u=(a,b,c)x^2$ for some nonzero $(a,b,c)\in k^3$. Next, after replacing u by $\sigma(u)$ or $\sigma^2(u)$ if necessary, we may assume that $a\neq 0$. Finally, after replacing u by $(a^{-1},0,0)u$, we may assume that $u=(1,0,0)x^2$. Then $\delta(u)=(0,-1,0)x$ and $\delta^2(u)=(0,0,-1)$, whence $(0,0,1)\in I$. Since $\sigma(0,0,1)$ and $\sigma^2(0,0,1)$ then also belong to I, we conclude that $I=S$. Therefore Sx^3 is a maximal (σ,δ)-ideal. ∎

For R, σ, δ as constructed in Example 2.8, it can be shown that the skew polynomial ring $R[\theta;\sigma,\delta]$ is not prime. A similar example (with k^3 replaced by k^4), for which the skew polynomial ring is not even semiprime, is constructed in [19].

Example 2.8 can also be presented in a matrix form analogous to Example 2.7. Namely, in $M_3(k(x))$ let σ be the inner automorphism given by the rule

$$\sigma\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}=\begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix},$$

let δ be the inner σ-derivation induced by the matrix

$$\begin{pmatrix} 0 & 0 & x^{-1}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

set

$$S=\begin{pmatrix} k[x] & xk[x] & x^2k[x]\\ x^2k[x] & k[x] & xk[x]\\ xk[x] & x^2k[x] & k[x]\end{pmatrix},$$

and let $R=S/Sx$.

3. Contractions to Commutative Coefficient Rings

In this section we concentrate on a skew polynomial ring $T = R[\theta; \sigma, \delta]$ in the case that R is a commutative noetherian ring and σ is an automorphism, and we characterize those ideals I of R that can occur as contractions of prime ideals of T; that is, we determine the ideals of the form $P \cap R$ where $P \in \operatorname{Spec}(T)$. For this case, our results extend and complete work of Irving [25], who considered the case that R is commutative but not necessarily noetherian and σ is not necessarily an automorphism. In a few places, we reproduce short portions of his arguments in order to make our presentation smoother. In several other places, modifications of his methods provide key steps to our approach.

We now state the main theorem of this section, which will be proved via a number of subsidiary results.

THEOREM 3.1. *Let $T = R[\theta; \sigma, \delta]$ where R is a commutative noetherian ring and σ is an automorphism.*

(I) *If P is a prime ideal of T and $I = P \cap R$, then one of the following cases must hold:*

(a) *I is a (σ, δ)-prime ideal of R. In this case, either*

(i) *I is a σ-prime (σ, δ)-ideal of R, or*

(ii) *I is a δ-prime (σ, δ)-ideal of R and R/I has a unique associated prime ideal, which contain $(1 - \sigma)(R)$.*

(b) *I is a prime ideal of R and $\sigma(I) \neq I$.*

(II) *Conversely, if I is any ideal of R satisfying* (a) *or* (b), *then $I = P \cap R$ for some prime ideal P of T. More specifically, in case* (a), *$IT \in \operatorname{Spec}(T)$, while in case* (b), *there exists a unique $P \in \operatorname{Spec}(T)$ such that $P \cap R = I$, and T/P is a commutative domain.*

Convention. For all skew derivations (σ, δ) considered in this section, we assume that σ is an *automorphism*. (This assumption will be used in all results except Lemma 3.8.)

LEMMA 3.2. *Let $T = R[\theta; \sigma, \delta]$. Suppose that for all nonzero $a, b \in R$ there exists a nonnegative integer n such that either $aR\sigma^n(b) \neq 0$ or $aR\delta^n(b) \neq 0$. Then T is a prime ring.*

Proof. If not, T contains nonzero ideals A and B such that $AB = 0$. Without loss of generality, $A = \operatorname{l.ann}(B)$ and $B = \operatorname{r.ann}(A)$.

Choose a nonzero element $u \in A$ with minimal degree m and leading coefficient u_m. We claim that $\sigma^{-m}(u_m) \in A$, that is, $\sigma^{-m}(u_m)B = 0$. To see this, it is enough to show that $\sigma^{-m}(u_m)C = 0$ where $C = \operatorname{r.ann}(u)$. If

$\sigma^{-m}(u_m)C \neq 0$, choose $v \in C$ with $\sigma^{-m}(u_m)v \neq 0$, with minimal degree n for this possibility, and with leading coefficient v_n. Since $v \in C$, we have $uv = 0$ and so $u_m\sigma^m(v_n) = 0$, whence $\deg(uv_n) < m$. Then by minimality of m we obtain $uv_n = 0$, and hence $u(v - v_n\theta^n) = 0$. Now $v - v_n\theta^n$ is an element of C with degree less than n, whence $\sigma^{-m}(u_m)(v - v_n\theta^n) = 0$. However, since $u_m\sigma^m(v_n) = 0$ we also have $\sigma^{-m}(u_m)v_n = 0$ and so $\sigma^{-m}(u_m)v = 0$, contradicting our choice of v. Thus $\sigma^{-m}(u_m) \in A$, as claimed.

In particular, $A \cap R \neq 0$, and a symmetric argument shows that $B \cap R \neq 0$. Choose nonzero elements $a \in A \cap R$ and $b \in B \cap R$. By hypothesis, there exist a nonnegative integer n and an element $r \in R$ such that either $ar\sigma^n(b) \neq 0$ or $ar\delta^n(b) \neq 0$. If $n = 0$ then $arb \neq 0$, while if $n > 0$ then

$$ar\theta^n b = ar\sigma^n(b)\theta^n + [\text{intermediate terms}] + ar\delta^n(b) \neq 0.$$

In either case this contradicts the assumption that $AB = 0$. Therefore T is prime. ▌

The σ-prime case of the following proposition was proved by Irving [25, Theorem 2.2]. This case also holds with a noncommutative coefficient ring [5, Proposition 2.1]. However, the general case does not hold for arbitrary noncommutative noetherian coefficient rings, as shown by examples in [19].

PROPOSITION 3.3. *Let $T = R[\theta; \sigma, \delta]$ where R is commutative noetherian. If I is a (σ, δ)-prime ideal of R, then IT is a prime ideal of T.*

Proof. Since $T/IT \cong (R/I)[\theta; \sigma, \delta]$, there is no loss of generality in assuming that $I = 0$. By Lemma 3.2, it is enough to show that given any nonzero $a, b \in R$, there exists a nonnegative integer n such that either $a\sigma^n(b) \neq 0$ or $a\delta^n(b) \neq 0$. According to Theorem 2.6, either R is σ-prime or R is δ-prime with a unique associated prime ideal.

Assume first that R is σ-prime. Set $B = \sum_{n=0}^{\infty} \sigma^n(b)R$, and observe that B is a nonzero σ-ideal of R. Then $\sigma(B) = B$ and so $\operatorname{ann}(B)$ is a σ-ideal, whence $\operatorname{ann}(B) = 0$ by σ-primeness. Consequently, $aB \neq 0$, and thus $a\sigma^n(b) \neq 0$ for some $n \geqslant 0$.

Now assume that R is δ-prime, and that R has a unique associated prime N. Then N equals the prime radical of R and R is $\mathscr{C}(N)$-torsionfree. Set $B = \sum_{n=0}^{\infty} R\delta^n(b)$, and observe that B is a nonzero δ-ideal of R. By δ-primeness, B cannot be nilpotent, whence $B \nsubseteq N$, and so $B \cap \mathscr{C}(N)$ is nonempty. As R is $\mathscr{C}(N)$-torsionfree, we conclude that $aB \neq 0$, and therefore $a\delta^n(b) \neq 0$ for some $n \geqslant 0$. ▌

LEMMA 3.4. *Let $T = R[\theta; \sigma]$, let $P \in \operatorname{Spec}(T)$, and let $I = P \cap R$. If $\theta \in P$ then I is prime, while if $\theta \notin P$ then $\sigma(I) = I$ and I is σ-prime.*

Proof. See [17, Lemmas 1.2, 1.3]. ▌

DEFINITION. Given an automorphism σ on a ring R and an ideal I in R, set

$$\mathscr{C}^{\sigma}(I)=\mathscr{C}^{\sigma}_{R}(I)=\{a\in R \mid \sigma^{i}(a)\in\mathscr{C}(I) \text{ for all } i\in\mathbb{Z}\}.$$

Then $\mathscr{C}^{\sigma}(I)$ is the largest multiplicative set $X\subseteq\mathscr{C}(I)$ satisfying $\sigma(X)=X$.

PROPOSITION 3.5. *Let $T=R[\theta;\sigma,\delta]$ where R is commutative noetherian, and let I be a prime of R such that $\sigma(I)\neq I$.*

(a) *Let x be an indeterminate, and extend (σ,δ) to a skew derivation on $R[x]$ such that $\sigma(x)=x$ and $\delta(x)=0$. Let $X=\mathscr{C}^{\sigma}_{R[x]}(I[x])$, set $R^{\circ}=R[x]X^{-1}$ and $T^{\circ}=T[x]X^{-1}$, and identify T° with $R^{\circ}[\theta;\sigma,\delta]$. Then there exists $b\in R^{\circ}$ such that $\delta=\delta_b$ on R°.*

(b) *There exists a unique prime P in T such that $P\cap R=I$. Moreover, T/P is a commutative domain, and $PT^{\circ}=IR^{\circ}+(\theta-b)T^{\circ}$.*

Proof. We are relying on Lemma 1.2 to extend (σ,δ) to $R[x]$, and on Lemma 1.4 to see that X is a denominator set in $T[x]$ and that T° is naturally isomorphic to $R^{\circ}[\theta;\sigma,\delta]$.

(a) Since $\sigma(I)\neq I$, we have $\sigma(I)\nsubseteq I$ and hence $(1-\sigma)(R)\nsubseteq I$. Then $(1-\sigma)(R)\nsubseteq\sigma^{i}(I)$ for all $i\in\mathbb{Z}$. Choose $c_0,...,c_n\in R$ such that the elements $c_0-\sigma(c_0),...,c_n-\sigma(c_n)$ generate the ideal $(1-\sigma)(R)\cdot R$. Then

$$\{c_0-\sigma(c_0),...,c_n-\sigma(c_n)\}\nsubseteq\sigma^{i}(I)$$

for all $i\in\mathbb{Z}$. Setting $c=c_0+c_1x+\cdots+c_nx^n$, we see that $c-\sigma(c)\in X$, whence $c-\sigma(c)$ becomes invertible in R°. Thus, by Lemma 2.4, δ is inner on R°.

(b) By Lemma 1.5, we may identify T° with $R^{\circ}[\theta^{\circ};\sigma]$, where $\theta^{\circ}=\theta-b$.

Observe that $P^{\circ}=IR^{\circ}+\theta^{\circ}T^{\circ}$ is an ideal of T° such that $P^{\circ}\cap R^{\circ}=IR^{\circ}$ and

$$T^{\circ}/P^{\circ}\cong R^{\circ}/IR^{\circ}\cong(R[x]/I[x])\,X^{-1}.$$

Thus P° is prime and T°/P° is a commutative domain. If P is the contraction of P° to T, then P is a prime of T such that $P\cap R=I$ and T/P is a commutative domain.

Now consider any prime Q in T such that $Q\cap R=I$. Then $Q[x]$ is a prime of $T[x]$ such that $Q[x]\cap R[x]=I[x]$, whence $Q[x]$ is disjoint from X. Consequently, the ideal $QT^{\circ}=Q[x]X^{-1}$ is a prime of T° such that $QT^{\circ}\cap R^{\circ}=IR^{\circ}$. Since $\sigma(I)\neq I$, we have $\sigma(IR^{\circ})\neq IR^{\circ}$, and so Lemma 3.4 shows that $\theta^{\circ}\in QT^{\circ}$. As $QT^{\circ}\cap R^{\circ}=IR^{\circ}$, it follows that $QT^{\circ}=IR^{\circ}+\theta^{\circ}T^{\circ}=P^{\circ}$. Therefore $Q=P$, proving that P is unique, and that $PT^{\circ}=P^{\circ}=IR^{\circ}+\theta^{\circ}T^{\circ}$. ∎

Propositions 3.3 and 3.5 prove part (II) of Theorem 3.1. In proving part (I), we distinguish between cases where δ becomes inner or outer after suitable localization; in the former case we can reduce to the situation that $\delta=0$ and apply Lemma 3.4.

The following lemma was proved by Irving in case char(R) is either zero or at least as large as the number of minimal primes of R [25, Proposition 2.4].

LEMMA 3.6. *Let R be a commutative noetherian ring with an automorphism σ such that $\sigma \neq 1$ and R is σ-prime. If R is not isomorphic to a direct product of an odd number of copies of $\mathbb{Z}/2\mathbb{Z}$, there exists $c \in R$ such that $c-\sigma(c)$ is regular.*

Proof. If R is a domain, just choose any $c \in R$ such that $\sigma(c) \neq c$.

Now assume that R is not a domain. Since R is σ-prime, there exist a minimal prime P in R and a positive integer n such that $\sigma^n(P)=P$ and $P \cap \sigma(P) \cap \cdots \cap \sigma^{n-1}(P)=0$; in particular, R is semiprime and the minimal primes of R are P, $\sigma(P), \ldots, \sigma^{n-1}(P)$. We may assume that $\sigma^i(P) \neq P$ for $i=1, \ldots, n-1$. As R is not a domain, $n \geqslant 2$.

Set $B=\sigma(P) \cap \cdots \cap \sigma^{n-1}(P)$, and note that $B \neq 0$. Observe that given any nonzero elements $b_i \in \sigma^i(B)$ for $i=0, \ldots, n-1$, the sum $b_0+\cdots+b_{n-1}$ is not in any minimal prime and so is regular.

If n is even, choose a nonzero element $b \in B$ and set

$$c=b+\sigma^2(b)+\sigma^4(b)+\cdots+\sigma^{n-2}(b).$$

Then $c-\sigma(c)=b-\sigma(b)+\sigma^2(b)-\sigma^3(b)+\cdots+\sigma^{n-2}(b)-\sigma^{n-1}(b)$, and the observation of the previous paragraph shows that $c-\sigma(c)$ is regular.

If n is odd, and if there exist distinct nonzero elements $b, b' \in B$, set

$$c=b+\sigma^2(b)+\sigma^4(b)+\cdots+\sigma^{n-3}(b)+\sigma^{-1}(b').$$

Then $c-\sigma(c)=(b-b')-\sigma(b)+\sigma^2(b)-\sigma^3(b)+\cdots+\sigma^{n-3}(b)-\sigma^{n-2}(b)+\sigma^{-1}(b')$. Since $b-b'$ is a nonzero element of B and $\sigma^{-1}(b')$ is a nonzero element of $\sigma^{n-1}(B)$, we again see that $c-\sigma(c)$ is regular.

Finally, suppose that n is odd and that B contains just one nonzero element e. Since R is semiprime, $B^2 \neq 0$, whence $e^2=e$. Then e, $\sigma(e), \ldots, \sigma^{n-1}(e)$ are pairwise orthogonal idempotents, and so the element

$$f=e+\sigma(e)+\cdots+\sigma^{n-1}(e)$$

is an idempotent. On the other hand, f is regular by our observation above, and so $f=1$. For $i=0, \ldots, n-1$, the set $\{0, \sigma^i(e)\}$ equals the ideal $\sigma^i(B)$, whence $\sigma^i(e)R=\{0, \sigma^i(e)\}$. But then $\sigma^i(e)R \cong \mathbb{Z}/2\mathbb{Z}$ as rings, and thus $R \cong (\mathbb{Z}/2\mathbb{Z})^n$, contradicting our hypotheses. Therefore the lemma is proved. ∎

LEMMA 3.7. *Let R be a commutative artinian ring with an automorphism σ such that $\sigma \neq 1$ and R is σ-prime. Then any σ-derivation δ on R is inner.*

Proof. If R is not isomorphic to a direct product of copies of the field $k = \mathbb{Z}/2\mathbb{Z}$, then by Lemma 3.6 there exists $c \in R$ such $c - \sigma(c)$ is regular. Since R is artinian, $c - \sigma(c)$ must be invertible, and then Lemma 2.4 shows that δ is inner.

Now assume that $R = k^n$ for some positive integer n, and observe that σ and δ must be k-linear. Since $\sigma \neq 1$, we have $n > 1$. Let $e_1, ..., e_n$ be the primitive idempotents in R. The e_i must be permuted by σ, and due to our σ-primeness assumption σ must act transitively on them. Thus after renumbering we may assume that $\sigma(e_i) = e_{i+1}$ for all i, where we interpret indices modulo n.

For all i, observe that $\delta(e_i) = \delta(e_i^2) = e_{i+1}\delta(e_i) + \delta(e_i)e_i$, whence $\delta(e_i) = a_i e_i + b_i e_{i+1}$ for some $a_i, b_i \in k$. Also,

$$0 = \delta(e_i e_{i+1}) = e_{i+1}\delta(e_{i+1}) + \delta(e_i)e_{i+1} = (a_{i+1} + b_i)e_{i+1},$$

whence $a_{i+1} + b_i = 0$. Thus $\delta(e_i) = a_i e_i - a_{i+1}e_{i+1}$ for all i. Setting $d = a_1 e_1 + \cdots + a_n e_n$, we conclude that $de_i - \sigma(e_i)d = \delta(e_i)$ for all i. Therefore δ is inner in this case also. ∎

LEMMA 3.8. *Let $T = R[\theta; \sigma, \delta]$, and let Q be an ideal of R.*

(a) $\delta(Q\sigma(Q) \cdots \sigma^m(Q)) \subseteq \sigma(Q)\sigma^2(Q) \cdots \sigma^m(Q)$ *for all* $m = 1, 2, ...$

(b) $Q\sigma(Q) \cdots \sigma^m(Q)\theta^m \subseteq TQ$ *for all* $m = 0, 1, ...$

Proof. (a) First, $\delta(Q\sigma(Q)) \subseteq \sigma(Q)\delta\sigma(Q) + \delta(Q)\sigma(Q) \subseteq \sigma(Q)$. If the inclusion holds for some m, then

$$\begin{aligned}\delta(Q\sigma(Q) \cdots \sigma^{m+1}(Q)) &\subseteq \sigma(Q\sigma(Q) \cdots \sigma^m(Q))\delta\sigma^{m+1}(Q) \\ &\quad + \delta(Q\sigma(Q) \cdots \sigma^m(Q))\sigma^{m+1}(Q) \\ &\subseteq \sigma(Q)\sigma^2(Q) \cdots \sigma^{m+1}(Q).\end{aligned}$$

(b) Obviously $Q\theta^0 \subseteq TQ$ and $Q\sigma(Q)\theta \subseteq Q[\theta Q + \delta(Q)] \subseteq TQ$. If the inclusion holds for some $m > 0$, then using (a) we see that

$$\begin{aligned}&Q\sigma(Q) \cdots \sigma^{m+1}(Q)\theta^{m+1} \\ &\quad = Q[\sigma(Q\sigma(Q) \cdots \sigma^m(Q))\theta]\theta^m \\ &\quad \subseteq Q[\theta Q\sigma(Q) \cdots \sigma^m(Q) + \delta(Q\sigma(Q) \cdots \sigma^m(Q))]\theta^m \\ &\quad \subseteq [TQ\sigma(Q) \cdots \sigma^m(Q) + Q\sigma(Q) \cdots \sigma^m(Q)]\theta^m \subseteq TQ. \quad ∎\end{aligned}$$

LEMMA 3.9 (cf. [25, Proposition 4.3 and Corollary 4.4]). *Let $T=R[\theta;\sigma,\delta]$ where R is noetherian. Let $P\in\operatorname{Spec}(T)$, set $I=P\cap R$, and let Q be an annihilator prime of ${}_R(R/I)$.*

(a) *$Q\sigma(Q)\cdots\sigma^m(Q)\subseteq I$ for some nonnegative integer m.*

(b) *If Q_1 is any annihilator prime of ${}_R(R/I)$, then $Q_1=\sigma^j(Q)$ for some nonnegative integer j.*

(c) *If R is commutative, then all annihilator primes of R/I are minimal over I.*

Proof. (a) There is a left ideal $A>I$ such that $Q=\operatorname{ann}_R(A/I)$. Set

$$B=\{b\in T\mid Q\sigma(Q)\cdots\sigma^i(Q)b\subseteq P \text{ for some } i\geqslant 0\},$$

and observe that B is a right ideal of T. Lemma 3.8 shows that

$$Q\sigma(Q)\cdots\sigma^i(Q)R\theta^iA\subseteq TQA\subseteq TI\subseteq P$$

for all $i\geqslant 0$, and hence $TA\subseteq B$. Since T is noetherian, B is finitely generated, and so there exists $m\geqslant 0$ such that $Q\sigma(Q)\cdots\sigma^m(Q)B\subseteq P$. Thus $Q\sigma(Q)\cdots\sigma^m(Q)TA\subseteq P$, whence $Q\sigma(Q)\cdots\sigma^m(Q)\subseteq P$ (because $A\nsubseteq P$), and therefore $Q\sigma(Q)\cdots\sigma^m(Q)\subseteq I$.

(b) Since $I\subseteq Q_1$, it follows from (a) that $\sigma^j(Q)\subseteq Q_1$ for some $j\in\{0,...,m\}$. By symmetry, $\sigma^k(Q_1)\subseteq Q$ for some $k\geqslant 0$, and so $\sigma^{j+k}(Q_1)\subseteq\sigma^j(Q)\subseteq Q_1$. Since R is noetherian and σ^{j+k} is an automorphism, we must have $\sigma^{j+k}(Q_1)=Q_1$, and thus $\sigma^j(Q)=Q_1$.

(c) Choose a prime $Q_1\subseteq Q$ which is minimal over I. Since R is commutative, Q_1 is an associated prime of R/I, and hence an annihilator prime. By (b), $Q_1=\sigma^j(Q)$ for some $j\geqslant 0$, whence $\sigma^j(Q)\subseteq Q$, and consequently $Q=\sigma^j(Q)=Q_1$. Thus Q is minimal over I, and similarly all annihilator primes of R/I are minimal over I. ∎

PROPOSITION 3.10. *Let $T=R[\theta;\sigma,\delta]$ where R is commutative noetherian. Let $P\in\operatorname{Spec}(T)$, set $I=P\cap R$, and assume that I is neither prime nor σ-prime. Then R/I has a unique associated prime Q, and $(1-\sigma)(R)\subseteq Q$.*

Proof. Let H be the largest (σ,δ)-ideal contained in I. After factoring H and HT out of R and T, we may assume that $H=0$. Since $\mathscr{C}^\sigma(I)$ is a denominator set in R such that $\sigma(\mathscr{C}^\sigma(I))=\mathscr{C}^\sigma(I)$ and $\mathscr{C}^\sigma(I)$ is disjoint from P, we may localize with respect to $\mathscr{C}^\sigma(I)$. Thus there is no loss of generality in assuming that all elements of $\mathscr{C}^\sigma(I)$ are invertible in R.

If δ is inner, then using Lemma 1.5 we may assume that $\delta=0$. But then Lemma 3.4 shows that I is either prime or σ-prime, contradicting our hypotheses. Therefore δ must be outer.

Let N be the prime radical of I. According to Lemma 3.9, all annihilator primes of R/I are minimal over I, whence $\mathscr{C}(I)=\mathscr{C}(N)$. Choose a prime Q minimal over I. Since all primes minimal over I are annihilator primes of R/I, Lemma 3.9 shows that the primes minimal over I are among Q, $\sigma(Q)$, $\sigma^2(Q)$, ... In particular,

$$\mathscr{C}^\sigma(Q)=\bigcap_{i\in\mathbb{Z}}\mathscr{C}(\sigma^i(Q))\subseteq\mathscr{C}(N)=\mathscr{C}(I),$$

and so $\mathscr{C}^\sigma(Q)\subseteq\mathscr{C}^\sigma(I)$. Thus all elements of $\mathscr{C}^\sigma(Q)$ are invertible in R.

We next claim that if $\sigma^n(Q)=Q$ for some $n>0$, then $\sigma(Q)=Q$. Suppose not.

Set $J=Q\cap\sigma(Q)\cap\cdots\cap\sigma^{n-1}(Q)$, and observe that J is a σ-prime ideal of R. Since $\sigma(Q)\neq Q$ we also have $(1-\sigma)(R)\nsubseteq J$, and so Lemma 2.5 shows that $\delta(J)\subseteq J$. Hence, $JT=TJ$. Since $\sigma^n(Q)=Q$, we see that $J\subseteq\sigma^i(Q)$ for all i, whence J is contained in all the primes minimal over I, and so $J^t\subseteq I$ for some $t\geqslant 0$. Now $(JT)^t=J^tT\subseteq IT\subseteq P$, yielding $JT\subseteq P$ and $J\subseteq I$. Since J is a (σ,δ)-ideal, we must have $J=0$. As a result, R is semiprime and all the $\sigma^i(Q)$ are minimal primes of R, whence $\mathscr{C}_R(0)\subseteq\mathscr{C}^\sigma(Q)$. Thus all elements of $\mathscr{C}_R(0)$ are invertible in R, and so R is artinian. However, since R is σ-prime and $\sigma\neq 1$, Lemma 3.7 now says that δ is inner, a contradiction. This verifies the claim.

If Q_1 is a prime minimal over I but distinct from Q, Lemma 3.9 shows that $Q_1=\sigma^j(Q)$ and $Q=\sigma^k(Q_1)$ for some $j,k>0$. Then $\sigma^{j+k}(Q)=Q$ but $\sigma(Q)\neq Q$, contradicting the claim just established. Therefore Q is the only prime minimal over I. Hence, $Q=N$, and by Lemma 3.9 Q is the unique associated prime of R/I.

A final use of Lemma 3.9 yields $Q\sigma(Q)\cdots\sigma^m(Q)\subseteq I$ for some $m\geqslant 0$. Since I is not prime, $Q\nsubseteq I$, and so there is some $j\in\{1,...,m\}$ such that $\sigma^j(Q)$ is disjoint from $\mathscr{C}(I)$. As $\mathscr{C}(I)=\mathscr{C}(N)=\mathscr{C}(Q)$, we obtain $\sigma^j(Q)\subseteq Q$ and so $\sigma^j(Q)=Q$. Thus $\sigma(Q)=Q$ by the claim established above.

Now $\mathscr{C}(Q)=\mathscr{C}^\sigma(Q)$, and so all elements of $\mathscr{C}(Q)$ are invertible in R. Since δ is not inner, we conclude from Lemma 2.4 that $(1-\sigma)(R)$ is disjoint from $\mathscr{C}(Q)$, and therefore $(1-\sigma)(R)\subseteq Q$. ∎

The case of Proposition 3.10 in which $R\supseteq\mathbb{Q}$ and the associated primes of I are all assumed to have the same height is contained in [25, Theorems 4.1, 4.2]; however, the proof of the second of these theorems appears to be incomplete.

PROPOSITION 3.11. *Let $T=R[\theta;\sigma,\delta]$ where R is commutative noetherian, let $P\in\operatorname{Spec}(T)$, and set $I=P\cap R$. Assume that R/I has a unique associated prime Q, and that $(1-\sigma)(R)\subseteq Q$. Then I is a (σ,δ)-ideal, and I is the largest δ-ideal contained in Q.*

Proof. Assume first that $\delta(Q) \subseteq Q$. Then Q is a (σ, δ)-ideal, and so $QT = TQ$. Since $Q^m \subseteq I$ for some $m \geqslant 0$, it follows that $(QT)^m \subseteq IT \subseteq P$, whence $QT \subseteq P$ and so $Q \subseteq I$. Thus in this case $I = Q$ and we are done.

Now assume there exists $c \in Q$ such that $\delta(c) \notin Q$. Since Q is the unique associated prime of R/I, we have $\mathscr{C}(Q) = \mathscr{C}(I)$, and so $\delta(c) \in \mathscr{C}(I)$.

Suppose there exists $a \in I$ such that $\delta(a) \notin I$. Since Q/I is nilpotent, the annihilator of Q in R/I is essential in R/I, and so there exists $r \in R$ such that $r\delta(a) \notin I$ while $Qr\delta(a) \subseteq I$. Note that $\sigma^{-1}(r)a \in I$ and that

$$\delta(\sigma^{-1}(r)a) = r\delta(a) + \delta\sigma^{-1}(r)a \equiv r\delta(a) \qquad (\mathrm{mod}\ I),$$

whence $\delta(\sigma^{-1}(r)a) \notin I$ and $Q\delta(\sigma^{-1}(r)a) \subseteq I$. Thus, after replacing a by $\sigma^{-1}(r)a$, we may assume that $Q\delta(a) \subseteq I$. By Lemma 2.4,

$$(a - \sigma(a))\, \delta(c) = (c - \sigma(c))\, \delta(a) \in Q\delta(a) \subseteq I.$$

Since $\delta(c) \in \mathscr{C}(I)$, it follows that $a - \sigma(a) \in I$, and so $\sigma(a) \in I$. But now $a, \sigma(a) \in P$ and so $\delta(a) = \theta a - \sigma(a)\theta \in P$, contradicting the assumption that $\delta(a) \notin I$.

Thus $\delta(I) \subseteq I$. For any $b \in I$, we now have

$$(b - \sigma(b))\, \delta(c) = (c - \sigma(c))\, \delta(b) \in I,$$

whence $b - \sigma(b) \in I$, and so $\sigma(b) \in I$. Therefore I is a (σ, δ)-ideal.

Finally, consider any δ-ideal $J \subseteq Q$. Let K/J be the $\mathscr{C}(Q)$-torsion ideal of R/J, and note that $K \subseteq Q$. For any $a \in K$, there exists $d \in \mathscr{C}(Q)$ such that $da \in J$. Then $\sigma(d) \in \mathscr{C}(Q)$ and

$$\sigma(d)\, \delta(a) + \delta(d)a = \delta(da) \in J,$$

whence $d\sigma(d)\, \delta(a) \in J$ and so $\delta(a) \in K$. Thus K is a δ-ideal. Since R/K is $\mathscr{C}(Q)$-torsionfree, it follows as in the previous paragraph that K is actually a (σ, δ)-ideal, whence $KT = TK$. Moreover, $K^t \subseteq I$ for some $t \geqslant 0$, and so $(KT)^t \subseteq IT \subseteq P$. Then $KT \subseteq P$ and $K \subseteq I$. Therefore I is the largest δ-ideal contained in Q. ∎

Proof of Theorem 3.1. As we have already noted, Propositions 3.3 and 3.5 prove part (II).

Now consider any $P \in \mathrm{Spec}(T)$, and set $I = P \cap R$.

Assume first that I is a σ-prime ideal. For any $a \in I$, we have $a, \sigma(a) \in P$ and so $\delta(a) = \theta a - \sigma(a)\theta \in P$, whence $\delta(a) \in I$. Thus I is a (σ, δ)-ideal, whence I is (σ, δ)-prime and (a) holds. Assume next that I is a prime ideal. If $\sigma(I) = I$, then I is σ-prime, and by the previous case, (a) holds. Otherwise, $\sigma(I) \neq I$ and (b) holds.

If I is neither prime nor σ-prime, Propositions 3.10 and 3.11 show that I is a (σ, δ)-ideal, that R/I has a unique associated prime Q, and that I is the largest δ-ideal contained in Q. It follows that I is (σ, δ)-prime, and again (a) holds.

Finally, if I is (σ, δ)-prime, then Theorem 2.6 shows that either (i) or (ii) holds. ∎

We close this section with an example illustrating the three types of ideals occurring in Theorem 3.1. (This example is a "quantized Weyl algebra" $A_1(k, -1)$; see Section 8 for a more detailed discussion of primes in such algebras.)

Let R be a polynomial ring $k[x]$ where k is an algebraically closed field of characteristic different from 2, and let σ be the k-algebra automorphism of R sending x to $-x$. In view of Lemma 1.2, there is a k-linear σ-derivation δ on R such that $\delta(x)=1$. Note that $\delta(x^2)=\sigma(x)+x=0$.

(a)(i) The σ-prime (σ, δ)-ideals of R consist of 0 together with the ideals $(x^2-\alpha^2)R$ for nonzero $\alpha \in k$.

(a)(ii) There is just one δ-prime (σ, δ)-ideal I in R for which R/I has a unique associated prime and that associated prime contains $(1-\sigma)(R)$, namely, $I=x^2R$.

(b) The primes of R not invariant under σ are the maximal ideals $(x-\alpha)R$ for nonzero $\alpha \in k$.

4. Factors Satisfying Polynomial Identities

We continue to investigate a skew polynomial ring $T=R[\theta; \sigma, \delta]$ where R is commutative noetherian and σ is an automorphism, aiming toward conditions guaranteeing that certain prime factors of T satisfy polynomial identities. The main theorem of the section says that whenever P and Q are distinct comparable primes of T with the same contraction in R, then T/P and T/Q are "usually" P.I. More precisely:

THEOREM 4.1. *Let $T=R[\theta; \sigma, \delta]$ where R is commutative noetherian and σ is an automorphism. Let $P>Q$ be comparable prime ideals of T, and suppose that $P \cap R = Q \cap R = I$. Then I is a (σ, δ)-ideal, $Q=IT$, and one of the following holds:*

(a) *T/Q is a P.I. ring.*

(b) *T/P is commutative, and the prime ideal $(P/Q)\mathscr{C}_R(I)^{-1}$ in $(T/Q)\mathscr{C}_R(I)^{-1}$ is generated by a normal element of the form $\theta - a$, where $a \in (R/I)\mathscr{C}_R(I)^{-1}$.*

Convention. For all skew derivations (σ, δ) considered in this section, we assume that σ is an *automorphism.* (This assumption will be used in all results except Lemma 4.4.)

The assumption in Theorem 4.1 that two primes of T contract to I rules out case (b) of Theorem 3.1. Of the two cases of Theorem 3.1(a), case (i) is easier to handle, and so we treat it first.

THEOREM 4.2. *Let $T = R[\theta; \sigma, \delta]$ where R is commutative noetherian. Let I be a σ-prime (σ, δ)-ideal of R such that $(1-\sigma)(R) \nsubseteq I$, and suppose there exist prime ideals $P > Q$ in T such that $P \cap R = Q \cap R = I$.*

(a) *$Q = IT$.*

(b) *If T/P is not commutative, then the automorphism induced by σ on R/I has finite order, and T/Q is a P.I. ring.*

(c) *If T/P is commutative, then $(P/Q)\mathscr{C}_R(I)^{-1}$ is generated in $(T/Q)\mathscr{C}_R(I)^{-1}$ by a normal element of the form $\theta - a$, for some $a \in (R/I)\mathscr{C}_R(I)^{-1}$.*

Proof. We may assume that $I = 0$, and that all elements of $\mathscr{C}_R(0)$ are invertible in R. Since R is σ-prime and hence semiprime, it follows that R is now artinian. By Lemma 3.7, $\delta = \delta_a$ for some $a \in R$. Using Lemma 1.5, we see that after replacing θ by $\theta - a$ we may assume that $\delta = 0$.

(a) Since R is artinian, $\mathrm{K.dim}(T) = 1$ (e.g., [37, Proposition 6.5.4]), and so T cannot contain a chain of three distinct primes. On the other hand, 0 is a prime of T (Proposition 3.3), and hence $Q = 0$.

(b) Since T/P is not commutative, $\theta \notin P$, and so [24, Theorem 4.3] shows that σ has finite order, say n. Then $R[\theta^n]$ is a commutative subring of T, and T is finitely generated as a right or left $R[\theta^n]$-module, whence T is P.I.

(c) Choose $r \in R$ such that $\sigma(r) \neq r$, and so $\sigma(r) - r \notin P$. Since T/P is commutative, $(\sigma(r) - r)\theta \in P$, and hence $\theta \notin \mathscr{C}(P)$. As θ is a normal element, it follows that $\theta \in P$, and therefore $P = \theta T$ (because $P \cap R = 0$). ∎

The analog of Theorem 4.2 for the case that $(1-\sigma)(R) \subseteq I$ is a consequence of the following standard result.

THEOREM 4.3. *Let $T = R[\theta; \delta]$ where R is a field. If T is not simple, then T is a finitely generated module over a central subring of the form $k[u]$, where k is the subfield of δ-constants of R, and hence T is a P.I. ring.*

Proof. If $\mathrm{char}(R) = 0$, then $\delta = 0$ (e.g., [37, Theorem 1.8.4]), and so T

itself has the form $k[u]$. Now assume that $\mathrm{char}(R)=p>0$. In this case, there exist elements $a_0, ..., a_{n-1}$ in k such that

$$\delta^{p^n}+a_{n-1}\delta^{p^{n-1}}+\cdots+a_1\delta^p+a_0\delta=0$$

(e.g., [30, Theorem 4.1.6; 36, Theorem 4]), and consequently the element

$$u=\theta^{p^n}+a_{n-1}\theta^{p^{n-1}}+\cdots+a_1\theta^p+a_0\theta$$

is central in T. Moreover, since R satisfies a nontrivial homogeneous linear differential equation over k, it follows from [1, Theorem 1] (or [20, Proposition 2.1]) that $\dim_k(R)<\infty$. Therefore T is module-finite over the central subring $k[u]$. ∎

In treating the case corresponding to Theorem 3.1(a)(ii), we shall of course again factor out I and localize with respect to $\mathscr{C}(I)$. Then we will be in the case that R is artinian local, say with maximal ideal M, and $(1-\sigma)(R)\subseteq M$. By δ-primeness, any nonzero δ-ideal of R is nonnilpotent and so contains a unit. Thus R is actually δ-simple in this case. In these circumstances, we shall prove that T is module-finite over a subring which is isomorphic to an ordinary differential operator ring over R/M, after which we can proceed as in Theorem 4.3.

LEMMA 4.4. *Let (σ, δ) be a skew derivation on a ring R, and let M be a left ideal of R. Set $M_0=R$, and set*

$$M_j=\{r\in R \mid \delta^i(r)\in M \text{ for all } i=0, 1, ..., j-1\}$$

for $j=1, 2, ...$ Then each M_j is a left ideal of R. If also M is a σ-ideal of R, then $MM_j\subseteq M_{j+1}$ for all j.

Proof. Since $M_0=R$ and $M_1=M$, they are certainly left ideals of R. Now assume, for some $j\geqslant 1$, that M_j is a left ideal. Given $r\in R$ and $a\in M_{j+1}$, observe that $a\in M_j$ and so $ra\in M_j$, whence $\delta^i(ra)\in M$ for all $i<j$. Since $a, \delta(a)\in M_j$, we also have

$$\delta(ra)=\sigma(r)\,\delta(a)+\delta(r)a\in M_j$$

and so $\delta^j(ra)\in M$, whence $ra\in M_{j+1}$. Thus M_{j+1} is a left ideal.

Now assume that M is a σ-ideal. Obviously $MM_0\subseteq M_1$. Suppose, for some $j\geqslant 0$, that $MM_j\subseteq M_{j+1}$. Given $a\in M$ and $b\in M_{j+1}$, observe that since $b\in M_j$ we have $ab\in M_{j+1}$ and hence $\delta^i(ab)\in M$ for all $i\leqslant j$. Also,

$$\delta(ab)=\sigma(a)\,\delta(b)+\delta(a)b\in MM_j+RM_{j+1}=M_{j+1},$$

and so $\delta^{j+1}(ab)\in M$, whence $ab\in M_{j+2}$. Therefore $MM_{j+1}\subseteq M_{j+2}$. ∎

LEMMA 4.5. *Let R be a commutative ring with a skew derivation (σ, δ), and let M be a maximal ideal of R such that $\sigma(M) \subseteq M$. Define $M_0, M_1, \ldots$ as in Lemma 4.4. For all $j = 1, 2, \ldots$, either $M_j = M_{j+1}$ or else $M_j/M_{j+1} \cong R/M$ and δ induces an additive isomorphism of M_j/M_{j+1} onto M_{j-1}/M_j.*

Proof. Note that $M_i = M_{i-1} \cap \delta^{-1}(M_{i-1})$ for all $i > 0$. Hence, for each $j > 0$ we see that δ induces an additive embedding

$$f_j \colon M_j/M_{j+1} \to M_{j-1}/M_j.$$

For all $r \in R$ and $a \in M_j$, we have

$$\delta(ra) = \sigma(r)\,\delta(a) + \delta(r)a \equiv \sigma(r)\,\delta(a) \qquad (\mathrm{mod}\ M_j),$$

whence $f_j(rb) = \sigma(r) f_j(b)$ for all $r \in R$ and $b \in M_j/M_{j+1}$. Since σ is an automorphism, it follows that f_j of any submodule of M_j/M_{j+1} is a submodule of M_{j-1}/M_j. In particular, whenever M_{j-1}/M_j is simple and f_j is nonzero, f_j will be an additive isomorphism and M_j/M_{j+1} will be simple.

Now consider an index j for which $M_j \neq M_{j+1}$. If $j = 1$, then $M_{j-1}/M_j = R/M$, while if $j > 1$ we may assume by induction that $M_{j-1}/M_j \cong R/M$. In either case, M_{j-1}/M_j is simple, whence f_j is an isomorphism and M_j/M_{j+1} is simple. By Lemma 4.5, $M(M_j/M_{j+1}) = 0$, and therefore $M_j/M_{j+1} \cong R/M$. ∎

PROPOSITION 4.6. *Let $T = R[\theta; \sigma, \delta]$ where R is commutative, artinian, local, and δ-simple. Let M be the maximal ideal of R, and assume that $(1 - \sigma)(R) \subseteq M$.*

(a) *$T \cong M_t(S)$ where $t = \mathrm{length}(R)$ and $S = \mathrm{End}_T(T/MT)$.*

(b) *There exist a derivation δ' on R/M and a subring $S' \subseteq S$ such that $S' \cong (R/M)[\theta'; \delta']$ and S is a finitely generated left S'-module.*

Proof. Since there is nothing to prove if $M = 0$, assume that $M \neq 0$, whence $t \geqslant 2$. By δ-simplicity, there exists $a \in M$ such that $\delta(a) \notin M$. Then $\delta(a)$ is a unit, and the element $v = \delta(a)^{-1}(\sigma(a) - a)$ lies in M because $(1 - \sigma)(R) \subseteq M$. For any $r \in R$, we have

$$(r - \sigma(r))\,\delta(a) = (a - \sigma(a))\,\delta(r)$$

by Lemma 2.4, whence $\sigma(r) - r = v\delta(r)$. Thus $\sigma = 1 + v\delta$.

(a) Define $M_0, M_1, \ldots$ as in Lemma 4.4, and observe that $v\delta(M_j) \subseteq MM_j \subseteq M_j$ for all $j > 0$. Since $\sigma = 1 + v\delta$, it follows that all the M_j are σ-ideals. The descending chain $M_0 \supseteq M_1 \supseteq \ldots$ must stop at some stage; say s is the first index such that $M_s = M_{s+1}$. Then M_s is a δ-ideal, and consequently $M_s = 0$. By Lemma 4.5, $M_j/M_{j+1} \cong R/M$ for all $j = 0, \ldots, s-1$, whence $s = \mathrm{length}(R) = t$.

Now Lemma 4.5 also shows that δ^{t-1} induces an additive isomorphism of M_{t-1} onto R/M, and so there exists $z \in M_{t-1}$ such that $\delta^{t-1}(z) \equiv 1 \pmod{M}$. Note that $Mz = 0$ and $M_{t-1} = Rz$. We claim that

$$\theta^i z \in \sum_{j=0}^{i} M_{t-1-j}\theta^{i-j}$$

for all $i = 0, ..., t-1$. This is clear in case $i = 0$. If the claim holds for some $i < t-1$, then

$$\begin{aligned}\theta^{i+1}z \in \sum_{j=0}^{i} \theta M_{t-1-j}\theta^{i-j} &\subseteq \sum_{j=0}^{i} (M_{t-1-j}\theta + M_{t-2-j})\theta^{i-j} \\ &\subseteq \sum_{j=0}^{i+1} M_{t-1-j}\theta^{i+1-j},\end{aligned}$$

because $\sigma(M_{t-1-j}) = M_{t-1-j}$ and $\delta(M_{t-1-j}) \subseteq M_{t-2-j}$. This verifies the claim.

The case $i = t-1$ of this claim yields

$$\theta^{t-1}z = z_{t-1}\theta^{t-1} + z_{t-2}\theta^{t-2} + \cdots + z_0$$

with $z_j \in M_j$ for each j. Thus $z_j \in M$ for all $j > 0$. Also, $z_0 = \delta^{t-1}(z)$, whence $z_0 - 1 \in M$, and so $\theta^{t-1}z - 1 \in MT$. Since $zM = 0$, we obtain $z\theta^{t-1}z = z$. Therefore the element $e = z\theta^{t-1}$ is an idempotent such that $eT = zT$.

For $j = 0, ..., t-1$, we have

$$\begin{aligned}M_jT/M_{j+1}T &\cong (M_j/M_{j+1}) \otimes_R T \cong (R/M) \otimes_R T \cong M_{t-1} \otimes_R T \\ &\cong M_{t-1}T = zT = eT.\end{aligned}$$

Since these modules are all projective, $T_T \cong (eT)^t$, and hence $T \cong M_t(eTe)$. As $eT \cong T/MT$, we also have $eTe \cong S$, and therfore $T \cong M_t(S)$.

(b) Let U be the idealizer of MT in T, so that $MT \subseteq U \subseteq T$ and $U/MT \cong S$. Since $R \subseteq U$ and $R \cap MT = M$, the set $K = (R + MT)/MT$ is a subring of U/MT and $K \cong R/M$. As T is not artinian, neither is S nor U/MT, and so $U/MT \neq K$, that is, $U \neq R + MT$.

Choose an element $u \in U - (R + MT)$ of minimal degree n. Then $n > 0$ and the leading coefficient u_n of u is not in M. Since $u_n^{-1} \in R \subseteq U$, we have $u_n^{-1}u \in U$. Hence, there is no loss of generality in assuming that $u_n = 1$. Given any $r \in R$, we have $ur - ru \in U$ and

$$ur - ru = (\sigma^n(r) - r)\theta^n + [\text{lower terms}].$$

Since $(1-\sigma)(R) \subseteq M$, we also have $\sigma^n(r) - r \in M$, and hence it follows from the minimality of n that $ur - ru \in R + MT$. Thus $[u, R] \subseteq R + MT$.

If $w=u+MT$, then $[w, K]\subseteq K$. Hence, $[w, -]$ induces a derivation δ' on K, and the set $W=K+Kw+Kw^2+\ldots$ is a subring of U/MT. Observing that $1, w, w^2, \ldots$ are left linearly independent over K, we see that $W\cong K[\theta'; \delta']$. Thus it suffices to show that U/MT is a finitely generated left W-module. Now T/MT is a left W-module, and it is clear that it is generated by $1, \theta+MT, \ldots, \theta^{n-1}+MT$. Since W is noetherian, we conclude that U/MT is a finitely generated left W-module, as desired. ∎

We shall see later that in the situation of Proposition 4.6, T is often (perhaps always) isomorphic to $M_t((R/M)[\theta'; \delta'])$ for some derivation δ' on R/M (Theorem 7.7 and Corollary 7.8).

THEOREM 4.7. *Let $T=R[\theta; \sigma, \delta]$ where R is commutative, artinian, local, and δ-simple. Let M be the maximal ideal of R, and assume that $(1-\sigma)(R)\subseteq M$. If T is not a simple ring, then it is a finitely generated left module over a commutative noetherian subring, and hence is a P.I. ring.*

Proof. Note from Proposition 3.3 that T is a prime ring. Let X be the multiplicative set of monic elements of T. By [42, Proposition 2.2 and Theorem 2.4] (or [37, Proposition 7.9.3 and Theorem 7.9.4]), X is an Ore set in T and $\mathrm{K.dim}(TX^{-1})=\mathrm{K.dim}(R)=0$. Thus TX^{-1} is a simple artinian ring.

By assumption, T contains a proper nonzero ideal I. Since TX^{-1} is simple and X consists of regular elements, I must contain an element of X, whence T/I is finitely generated as a right or left R-module. Thus T/I is a P.I. ring.

In view of Proposition 4.6, T is finitely generated as a left module over a subring S' which is isomorphic to $K[\theta'; \delta']$ where $K=R/M$ and δ' is a derivation on K. If S' is simple, it embeds in T/I and so is P.I. But then S' is artinian by Kaplansky's Theorem, which gives a contradiction. Thus, S' is not simple. By Theorem 4.3, S' is module-finite over a central subring $k[u]$, where k is the subfield of δ'-constants of K. Therefore T is a finitely generated left module over $k[u]$. ∎

Proof of Theorem 4.1. Since two distinct primes of T contract to I, we cannot be in case (b) of Theorem 3.1. Consequently, I is a (σ, δ)-ideal of R and IT is a prime ideal of T. Hence, we may assume that $I=0$ and that T is a prime ring. We may also assume that all elements of $\mathscr{C}_R(0)$ are invertible in R.

Suppose first that R is σ-prime. As in the proof of Theorem 4.2(a), it follows that R is artinian and that $Q=0$. If $\sigma\neq 1$, Theorem 4.2 gives the desired conclusions. If $\sigma=1$, then R must be a field, and in this case Theorem 4.3 shows that T is a P.I. ring.

Now suppose that R is not σ-prime. By Theorem 3.1, R is δ-prime, R has a unique associated prime M, and $(1-\sigma)(R)\subseteq M$. Since M is the unique associated prime of R, we have $\mathscr{C}(0)=\mathscr{C}(M)$, and hence R is artinian local with M as its maximal ideal. It now follows from δ-primeness that R is actually δ-simple. Therefore Theorem 4.7 says that T is a P.I. ring. Moreover, since R is artinian we may use the argument of Theorem 4.2(a) once again to conclude that $Q=0$. ▌

5. Cliques and Localizability

Building on the P.I. results of the previous section, we study cliques of primes in skew polynomial ring of the type $T=R[\theta;\sigma,\delta]$ we have been considering. We show that cliques in Spec(T) have bounded uniform rank, and that T satisfies the strong second layer condition. (See, e.g., [21, 28, 37] for definitions of links, cliques, the second layer conditions, and localizability.) These conditions have strong effects on the ideal theory of T (see e.g., [11; 21, Chapter 12; 35]). Moreover, it follows that if R contains an uncountable subfield k over which σ and δ are linear, then all cliques of primes in T are classically localizable.

THEOREM 5.1. *Let $T=R[\theta;\sigma,\delta]$ where R is commutative noetherian and σ is an automorphism.*

(a) *T satisfies the right and left strong second layer conditions.*

(b) *If X is a clique of prime ideals in T, then there exists a positive integer b such that* $\operatorname{rank}(T/P)\leqslant b$ *for all $P\in X$.*

(c) *Assume that R contains an uncountable subfield k such that σ and δ are k-linear. Then all cliques of prime ideals in T are classically localizable.*

Convention. For all skew derivations (σ,δ) considered in this section, we assume that σ is an *automorphism.*

To obtain the second layer condition, we follow the method used by Bell [7, Theorem 7.3] and Sigurdsson [44, Proposition 2.4] to prove this for various differential operator rings. Namely, given comparable primes $P>Q$ in T, we show that after possibly adjoining a central indeterminate and then localizing with respect to a suitable Ore set, we can separate P from Q by an AR-ideal.

LEMMA 5.2. *Let $T=R[\theta;\sigma,\delta]$ where R is commutative noetherian. If I is a (σ,δ)-ideal of R, then IT is an AR-ideal of T.*

Proof. (This is exactly analogous to [7, Lemma 7.1].) Let x be an indeterminate and identify $T[x]$ with $R[x][\theta;\sigma,\delta]$ where (σ,δ) has been extended to $R[x]$ with $\sigma(x)=x$ and $\delta(x)=0$. Observe that $I^2, I^3, \ldots$ are all (σ, δ)-ideals, whence the Rees ring

$$\mathscr{R}(I)=\sum_{j=0}^{\infty} I^j x^j \subseteq R[x]$$

is closed under σ and δ. In particular, σ restricts to an automorphism of $\mathscr{R}(I)$. Moreover, $IT=TI$ and so $(IT)^j=I^jT$ for all $j \geqslant 0$. It follows that $\mathscr{R}(I)[\theta;\sigma,\delta]=\mathscr{R}(IT)$. Since R is commutative noetherian, $\mathscr{R}(I)$ is noetherian, and then $\mathscr{R}(IT)$ is noetherian. Therefore IT is an AR-ideal (see, e.g., [6, Lemma 6.1] or [21, Lemma 11.12]). ∎

PROPOSITION 5.3. *Let $T=R[\theta;\sigma,\delta]$ where R is commutative noetherian. Let $P>Q$ be primes of T, set $I=P\cap R$, and assume that $\sigma(I)=I$. Then one of the following holds*:

(a) *In T/Q the ideal P/Q contains a nonzero AR-ideal.*

(b) *T/P is commutative, and in $(T/Q)\mathscr{C}_R(I)^{-1}$ the ideal $(P/Q)\mathscr{C}_R(I)^{-1}$ is generated by a normal element of the form $\theta-b$, where b is the image of some element from $R\mathscr{C}_R(I)^{-1}$.*

Proof. If $Q\cap R=I$, then Theorem 4.1 shows that either T/Q is a P.I. ring or else (b) holds. In case T/Q is P.I., the ideal P/Q must contain a nonzero central element c (e.g., [37, Theorem 13.6.4]), and c generates a nonzero AR-ideal.

Now suppose that $Q\cap R\neq I$, whence $IT \nsubseteq Q$. Since $\sigma(I)=I$, case (b) of Theorem 3.1 is ruled out, and so I is a (σ,δ)-ideal. By Lemma 5.2, IT is an AR-ideal of T, and therefore $(IT+Q)/Q$ is a nonzero AR-ideal of T/Q. ∎

PROPOSITION 5.4. *Let $T=R[\theta;\sigma,\delta]$ where R is commutative noetherian. Let $P>Q$ be primes of T, set $I=P\cap R$, and assume that $\sigma(I)\neq I$. Let x be an indeterminate, and set $X=\mathscr{C}^{\sigma}_{R[x]}(I[x])$. Then in $(T[x]/Q[x])X^{-1}$ the ideal $(P[x]/Q[x])X^{-1}$ contains a nonzero normal element of the form $\theta-b$, where b is the image of some element from $R[x]X^{-1}$.*

Proof. By Theorem 3.1, I is a prime of R. Let R° and T° be as in Proposition 3.5, and use Lemma 1.5 to identify T° with $R^\circ[\theta-b;\sigma]$. In particular, $\theta-b$ is a normal element of T°, and $\theta-b\in PT^\circ$ by Proposition 3.5.

If T/Q is commutative, any nonzero element of $(P[x]/Q[x])X^{-1}$ of the form $\theta-b'$ will do. Thus we may assume that T/Q is noncommutative. Now $T^\circ/(\theta-b)T^\circ$ is commutative while T°/QT° is not, and so $\theta-b\notin QT^\circ$. Therefore the image of $\theta-b$ in T°/QT° is a nonzero normal element contained in PT°/QT°. ∎

Proof of Theorem 5.1(a). By symmetry (Lemma 1.5(a)), it is enough to prove the right strong second layer condition. If that fails, then by [21, Proposition 11.3] there exist a finitely generated uniform right T-module M with an affiliated series $0 < U < M$ and corresponding affiliated primes P, Q such that $P > Q$ and $MQ = 0$. Note that since M/U is a faithful (T/Q)-module, $\operatorname{ann}_T(M) = Q$. Also, U is essential in M because M is uniform.

Set $I = P \cap R$. Suppose first that T/P is noncommutative. Then Theorem 3.1 says that $\sigma(I) = I$, whence by Proposition 5.3 there exists an ideal J in T such that $P \supseteq J > Q$ and J/Q has the AR-property. Since M is a finitely generated (T/Q)-module with an essential submodule U annihilated by J/Q, it follows that $M(J/Q)^n = 0$ for some $n > 0$ (see, e.g., [21, Lemma 11.11] or [37, Theorem 4.2.2]). But then $J^n \subseteq \operatorname{ann}_T(M) = Q$, which is impossible.

Thus T/P must be commutative. Since U is a fully faithful (T/P)-module, it must be torsionfree.

Let x be an indeterminate, let $X = \mathscr{C}^{\sigma}_{R[x]}(I[x])$, and set $T° = T[x]X^{-1}$. Since X is disjoint from $P[x]$ and $Q[x]$, the extensions $PT°$ and $QT°$ are primes of $T°$. Moreover, $X \subseteq \mathscr{C}(P[x])$ (e.g., [8, Lemma 2.1] or [21, Lemma 9.21]), from which we infer that $U[x]$ is X-torsionfree. Observe that $U[x]$ is essential in $M[x]$ as T-modules, and hence also as $T[x]$-modules. Consequently, $M[x]$ is X-torsionfree. Now if $U° = U[x]X^{-1}$ and $M° = M[x]X^{-1}$, then $U°$ is an essential $T°$-submodule of $M°$, and $M°$ is a faithful $(T°/QT°)$-module.

By either Proposition 5.3 or 5.4, the ideal $PT°/QT°$ in $T°/QT°$contains a nonzero AR-ideal. However, since $U°(PT°) = 0$, this leads to a contradiction as in the case that T/P is noncommutative.

Therefore the strong second layer condition holds in T. ▮

LEMMA 5.5. *Let $T = R[\theta; \sigma, \delta]$ where R is commutative noetherian, and let P, Q be distinct primes of T such that either $P \rightsquigarrow Q$ or $Q \rightsquigarrow P$.*

(a) *If $\sigma(P \cap R) \neq P \cap R$, then either $\sigma(P \cap R) = Q \cap R$ or $\sigma(Q \cap R) = P \cap R$; in particular, $\sigma(Q \cap R) \neq Q \cap R$.*

(b) *If $P \cap R$ is a (σ, δ)-ideal, then $P \cap R = Q \cap R$.*

Proof. By symmetry, we need only prove the case that $P \rightsquigarrow Q$. Then T contains an ideal A such that $P \cap Q > A \supseteq PQ$ and the bimodule $B = (P \cap Q)/A$ is torsionfree as a left (T/P)-module and as a right (T/Q)-module.

(a) Set $I = P \cap R$, and recall from Theorem 3.1 that I is prime. Let X, $R°$, $T°$, b be as in Proposition 3.5, and use Lemma 1.5 to identify $T°$ with $R°[\theta°; \sigma]$ where $\theta° = \theta - b$. Then $PT° = IR° + \theta° T°$ and $PT° \cap R° = IR°$. Note that the natural map $R/I \to R°/IR°$ is injective.

Observe that $(P[x] \cap Q[x])/A[x] \cong T[x] \otimes_T B$ as left $T[x]$-modules, whence $(P[x] \cap Q[x])/A[x]$ is torsionfree on the left over $T[x]/P[x]$. Similarly, it is torsionfree on the right over $T[x]/Q[x]$, and so $P[x] \rightsquigarrow Q[x]$ in $T[x]$.

Since X is a denominator set in $T[x]$ disjoint from $P[x]$, we must have $X \subseteq \mathscr{C}(P[x])$ (e.g., [8, Lemma 2.1; 21, Lemma 9.21]), and then it follows from the link $P[x] \rightsquigarrow Q[x]$ that $X \subseteq \mathscr{C}(Q[x])$ (e.g., [28, Theorem 5.4.5; 21, Lemma 12.17]). Now PT° and QT° are primes of T°, and the link $P[x] \rightsquigarrow Q[x]$ localizes to a link $PT^\circ \rightsquigarrow QT^\circ$ (e.g., [28, Theorem 5.4.4; 21, Exercise 11S]).

In T°, the element θ° is normal, and so $\theta^\circ T^\circ$ is an AR-ideal. Moreover, $\theta^\circ T^\circ \subseteq PT^\circ$ and so it follows from the link $PT^\circ \rightsquigarrow QT^\circ$ that $\theta^\circ T^\circ \subseteq QT^\circ$ (e.g., [28, Proposition 5.3.9; 21, Proposition 11.16]). Thus $QT^\circ = J^\circ + \theta^\circ T^\circ$, where $J^\circ = QT^\circ \cap R^\circ$. Set $J = Q \cap R$, and observe that the natural map

$$R/J \to R^\circ/J^\circ \to T^\circ/QT^\circ$$

can be factored as a composition of embeddings

$$R/J \to T/Q \to T[x]/Q[x] \to T^\circ/QT^\circ.$$

Consequently, the natural map $R/J \to R^\circ/J^\circ$ is injective.

Because of the link $PT^\circ \rightsquigarrow QT^\circ$, there is an ideal A° in T° such that

$$PT^\circ \cap QT^\circ > A^\circ \supseteq PT^\circ QT^\circ$$

and the bimodule $(PT^\circ \cap QT^\circ)/A^\circ$ is torsionfree on the left over T°/PT° and on the right over T°/QT°. (Actually we could take $A^\circ = AT^\circ$.) Since $PT^\circ/\theta^\circ T^\circ$ and $QT^\circ/\theta^\circ T^\circ$ are distinct primes in the commutative ring $T^\circ/\theta^\circ T^\circ$, they cannot be linked, and so $\theta^\circ T^\circ \nsubseteq A^\circ$. Thus if $D = \theta^\circ T^\circ \cap A^\circ$, the bimodule $\theta^\circ T^\circ/D$ is nonzero. As $\theta^\circ T^\circ/D$ embeds in $(PT^\circ \cap QT^\circ)/A^\circ$, it must be torsionfree over T°/PT° on the left and over T°/QT° on the right, whence

$$\operatorname{l.ann}_{T^\circ}(\theta^\circ T^\circ/D) = PT^\circ \qquad \text{and} \qquad \operatorname{r.ann}_{T^\circ}(\theta^\circ T^\circ/D) = QT^\circ.$$

For $r \in R^\circ$, we have $(\theta^\circ T^\circ/D)r = 0$ if and only if $\theta^\circ r \in D$, if and only if $\sigma(r)\theta^\circ \in D$, if and only if $\sigma(r)(\theta^\circ T^\circ/D) = 0$. Thus

$$IR^\circ = PT^\circ \cap R^\circ = \sigma(QT^\circ \cap R^\circ) = \sigma(J^\circ).$$

Since the natural maps $R/I \to R^\circ/IR^\circ$ and $R/J \to R^\circ/J^\circ$ are injective, we conclude that $I = \sigma(J)$, that is, $P \cap R = \sigma(Q \cap R)$.

(b) Since $\sigma(P \cap R) = P \cap R$, it follows from (a) that $\sigma(Q \cap R) = Q \cap R$, and then Theorem 3.1 shows that $Q \cap R$ is a (σ, δ)-ideal. By

Lemma 5.2, $(P \cap R)T$ and $(Q \cap R)T$ are AR-ideals of T. It follows from the link $P \rightsquigarrow Q$ that $(P \cap R)T \subseteq Q$ and $(Q \cap R)T \subseteq P$ (e.g., [28, Proposition 5.3.9; 21, Proposition 11.16]), and therefore $P \cap R = Q \cap R$. ∎

Proof of Theorem 5.1(b), (c). (b) Since this is obvious if X is finite, assume that X is infinite. Choose a prime $P \in X$, and set $I = P \cap R$. If $\sigma(I) \neq I$, then by Lemma 5.5, $\sigma(Q \cap R) \neq Q \cap R$ for all $Q \in X$. In this case, Theorem 3.1 shows that T/Q is commutative for all $Q \in X$, and so we have the bound $b = 1$.

Now assume that $\sigma(I) = I$. By Theorem 3.1, I is a (σ, δ)-ideal, and then Lemma 5.5 shows that $Q \cap R = I$ for all $Q \in X$. In this case, it will suffice to show that T/IT is a P.I. ring, since the ranks of the prime factors in a P.I. ring are bounded by the P.I. degree (see, e.g., [41, Proof of Theorem, p. 180; 43, Theorem 6.1.30]).

Suppose that I is σ-prime. If S is the classical quotient ring of R/I, then $S[\theta; \sigma, \delta]$ is a prime ring containing infinitely many nonzero prime ideals that contract to 0 in S. In case $\sigma \neq 1$ on S, Lemma 3.7 shows that δ is inner, and so $S[\theta; \sigma, \delta] \cong S[\theta; \sigma]$. Then by [24, Theorem 4.3], σ must have finite order on S, whence $S[\theta; \sigma]$ is P.I. and consequently T/IT is P.I. On the other hand, if $\sigma = 1$ on S, then S is a field and $S[\theta; \sigma, \delta] = S[\theta; \delta]$. Then $S[\theta; \delta]$ is P.I. by Theorem 4.3, and again T/IT is P.I.

If I is not σ-prime, then by Theorem 3.1, I is δ-prime, R/I has a unique associated prime M, and $(1-\sigma)(R) \subseteq M$. Then R/I has an artinian local classical quotient ring S, with maximal ideal MS, the ring S is δ-simple, and $(1-\sigma)(S) \subseteq MS$. Since $S[\theta; \sigma, \delta]$ has infinitely many prime ideals, Theorem 4.7 shows that $S[\theta; \sigma, \delta]$ is P.I. Therefore T/IT is P.I. in this case also.

(c) Note that T is a k-algebra. Since we have the second layer condition and bounded ranks for the prime factors corresponding to any clique, the classical localizability follows from [45, Proposition 4.5] or [47, Lemma 1(iii) and Theorem 8] (see also [28, Theorem 7.2.15]). ∎

Following Bell [7, Proposition 7.10], we can in particular use Theorem 5.1 to obtain a dimension inequality for skew polynomial rings as follows.

COROLLARY 5.6. *Let $T = R[\theta; \sigma, \delta]$ where R is commutative noetherian (and σ is an automorphism). Assume that R contains an uncountable subfield k such that σ and δ are k-linear. If* gl.dim(T) *is finite, then* cl.K.dim$(T) \leqslant$ gl.dim(T).

Proof. See Theorem 5.1 and [10, Theorem 8]. ∎

6. Quantized Derivations

Recent work on "quantized analogs" of many standard algebras has revived interest in general skew polynomial extensions, since many of these quantized algebras can be represented as (iterated) skew polynomial rings. One basic example is the *quantized Weyl algebra* $A_1(k, q)$ over a field k: this is the k-algebra generated by elements x, y subject to the sole relation $xy - qyx = 1$, where q is some nonzero element of k (see, e.g., [15, 34, 39]). (See also the end of Section 8 for a competing class of quantized Weyl algebras.) Just as the ordinary Weyl algebra is a differential operator ring over a polynomial ring, $A_1(k, q)$ is a skew polynomial ring $k[y][x; \sigma, \delta]$, where σ is the k-algebra automorphism of $k[y]$ such that $\sigma(y) = qy$, and δ is the (unique) k-linear σ-derivation on $k[y]$ such that $\delta(y) = 1$. A special feature of this skew derivation (σ, δ) is that $\delta\sigma = q\sigma\delta$, as can be quickly checked using Lemma 1.1 (cf. Lemma 8.1 and [15, Proposition 1.1]).

An example with similar features occurs in the representation theory of the "q-enveloping algebra" $U_q(sl_2(\mathbb{C}))$ (see [14, 29]). This is a $\mathbb{C}$-algebra with generators E, F, K, K^{-1} such that $KE = q^2EK$ and $KF = q^{-2}FK$, while $EF - FE = (K^2 - K^{-2})/(q^2 - q^{-2})$. (Here q is any nonzero complex number which is not a root of unity.) Let R be the skew polynomial ring $\mathbb{C}[y][x; \tau]$ where τ is the $\mathbb{C}$-algebra automorphism of $\mathbb{C}[y]$ satisfying $\tau(y) = q^2y$, and let σ be the $\mathbb{C}$-algebra automorphism of R satisfying $\sigma(x) = qx$ and $\sigma(y) = q^{-1}y$. (The algebra R is known as the "coordinate ring of the quantum plane.") Then there exist unique $\mathbb{C}$-linear σ^2-derivations δ_1 and δ_2 on R such that $\delta_1(x) = 0$ and $\delta_1(y) = x$ while $\delta_2(x) = y$ and $\delta_2(y) = 0$, and R can be made into a module over $U_q(sl_2(\mathbb{C}))$ on which E acts as $\delta_1\sigma^{-1}$ and F as $\delta_2\sigma^{-1}$, while K acts as σ (see [38, Theorem 4.3]). In this example, $\delta_1\sigma^2 = q^{-4}\sigma^2\delta_1$ and $\delta_2\sigma^2 = q^4\sigma^2\delta_2$ (cf. [38, Example 2.2]).

We anticipate that just as ordinary derivations play a key role in the representation theory of enveloping algebras, skew derivations (σ, δ) for which $\delta\sigma$ is a constant multiple of $\sigma\delta$ will play a corresponding role in the representation theory of q-enveloping algebras. Hence, we propose to use the term *q-skew derivation* for such skew derivations, and we will investigate the corresponding *q-skew differential operator rings.*

Our purpose in this section is only to initiate the analysis of q-skew differential operator rings, partly to illustrate the computational advantages of q-skew derivations over arbitrary skew derivations, and partly to prepare for some computations in q-skew differential operator rings that arise in studying skew polynomial rings with commutative artinian local δ-simple coefficients, in the following section. We shall also give a detailed analysis of prime ideals in quantized Weyl algebras in Section 8. For a study of prime ideals in general noetherian q-skew differential operator rings, see [19].

DEFINITION. Let (σ, δ) be a skew derivation on a ring R. A (σ, δ)-*constant* is any element $q \in R$ such that $\sigma(q) = q$ and $\delta(q) = 0$. (The set of all (σ, δ)-constants forms a unital subring of R, but we shall not need this observation.) By a *(left) q-skew derivation* on R we shall mean a triple (σ, δ, q) such that (σ, δ) is a (left) skew derivation, q is a central (σ, δ)-constant in R, and $\delta\sigma = q\sigma\delta$. Of course if q is already specified, we refer to the pair (σ, δ) itself as a q-skew derivation.

The main computational advantage of a q-skew derivation (σ, δ, q) over an arbitrary skew derivation is the existence of formulas such as a q-analog of Leibniz's Rule (Lemma 6.2), in which ordinary binomial coefficients are replaced by some q-analog binomial coefficients. These latter coefficients are best described as evaluations of Gaussian polynomials, as follows.

DEFINITION. Let t be indeterminate. For integers $n \geqslant m \geqslant 0$, the *t-binomial coefficient* or *Gaussian polynomial* $\binom{n}{m}_t$ is the rational function defined as

$$\binom{n}{m}_t = \frac{(t^n - 1)(t^{n-1} - 1) \cdots (t - 1)}{(t^m - 1)(t^{m-1} - 1) \cdots (t - 1)(t^{n-m} - 1)(t^{n-m-1} - 1) \cdots (t - 1)}.$$

The numerator and denominator each contain n factors of $t - 1$. Eliminating these, we obtain an alternate formula

$$\binom{n}{m}_t = \frac{(n!)_t}{(m!)_t((n-m)!)_t},$$

where $(j!)_t = (t^{j-1} + t^{j-2} + \cdots + 1)(t^{j-2} + t^{j-3} + \cdots + 1) \cdots (t + 1)(1)$ for $j > 0$ and $(0!)_t = 1$.

LEMMA 6.1. (a) *For all integers $n \geqslant m \geqslant 0$, the t-binomial coefficient $\binom{n}{m}_t$ is a polynomial in t with nonnegative integer coefficients.*

(b) $\binom{n}{0}_t = \binom{n}{n}_t = 1$ *for all* $n \geqslant 0$.

(c) $\binom{n}{m}_t = \binom{n-1}{m}_t + t^{n-m}\binom{n-1}{m-1}_t = \binom{n-1}{m-1}_t + t^m\binom{n-1}{m}_t$ *for all* $n > m > 0$.

Proof. See [2, p. 35; 46, p. 26]. ∎

DEFINITION. Let q be an element in a ring R. For all integers $n \geqslant m \geqslant 0$, the *q-binomial coefficient* $\binom{n}{m}_q$ in R is just the evaluation of the polynomial

$\binom{n}{m}_t$ at $t=q$. In particular, it is immediate from the second of our defining formulas for $\binom{n}{m}_t$ that $\binom{n}{m}_1$ in $\mathbb{Z}$ equals the ordinary binomial coefficient $\binom{n}{m}$.

We can now give the following *q-Leibniz Rules* for *q*-skew derivations.

LEMMA 6.2. *Let (σ, δ, q) be a q-skew derivation on a ring R, and let $T=R[\theta; \sigma, \delta]$.*

(a) *$\delta^n(ab)=\sum_{i=0}^{n} \binom{n}{i}_q \sigma^{n-i}\delta^i(a)\delta^{n-i}(b)$ for all $a, b\in R$ and $n=0, 1, \ldots$.*

(b) *$\theta^n a=\sum_{i=0}^{n} \binom{n}{i}_q \sigma^{n-i}\delta^i(a)\theta^{n-i}$ for all $a\in R$ and $n=0, 1, \ldots$.*

Proof. (a) For $n=0$ this is clear, while for $n=1$, this is just the definition of a σ-derivation. If the rule holds for some n, then using Lemma 6.1(c) we compute that

$$\begin{aligned}
\delta^{n+1}(ab) &= \delta\delta^n(ab)\\
&= \sum_{i=0}^{n} \binom{n}{i}_q [\sigma^{n+1-i}\delta^i(a)\,\delta^{n+1-i}(b)+\delta\sigma^{n-i}\delta^i(a)\,\delta^{n-i}(b)]\\
&= \sum_{i=0}^{n} \binom{n}{i}_q [\sigma^{n+1-i}\delta^i(a)\,\delta^{n+1-i}(b)+q^{n-i}\sigma^{n-i}\delta^{i+1}(a)\,\delta^{n-i}(b)]\\
&= \sigma^{n+1}(a)\,\delta^{n+1}(b)+\delta^{n+1}(a)b\\
&\quad+\sum_{i=1}^{n}\left[\binom{n}{i}_q+q^{n+1-i}\binom{n}{i-1}_q\right]\sigma^{n+1-i}\delta^i(a)\,\delta^{n+1-i}(b)\\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}_q \sigma^{n+1-i}\delta^i(a)\,\delta^{n+1-i}(b).
\end{aligned}$$

(b) This is proved in the same fashion. ∎

The q-Leibniz Rules allow us to obtain analogs for a number of standard results about ordinary derivations. We are particularly concerned with the result that δ-prime noetherian rings in characteristic zero are necessarily prime (e.g., [16, Corollary to Lemma 2; 31, Lemma 2.1 and Theorem 2.2; 20, Corollary 1.4]). An analog for a q-skew derivation (σ, δ, q) is that if q is not a root of unity, a (σ, δ)-prime noetherian ring is necessarily σ-prime (Proposition 6.5). We start with a corollary of Lemma 2.2.

LEMMA 6.3. *Let (σ, δ, q) be a q-skew derivation on a ring R, and let I be a σ-ideal of R. Set $I_0=R$, set*

$$I_j=\{r\in R \mid \delta^i(r)\in I \text{ for all } i=0, \ldots, j-1\}$$

for $j=1, 2, \ldots$, and set $J=\{r\in R \mid \delta^i(r)\in I \text{ for all } i=0, 1, \ldots\}$. Then each I_j is *a σ-ideal of R, while J is a (σ, δ)-ideal and is the largest δ-ideal contained in I. Moreover, if $\sigma^{-1}(I)=I$ and $q \in \mathscr{C}(I)$, then $\sigma^{-1}(I_j)=I_j$ for all j and $\sigma^{-1}(J)=J$.*

Proof. Observe that if $r \in R$ and $\delta^i(r) \in I$ for some $i \geqslant 0$, then

$$\delta\sigma^{m(1)}\delta\sigma^{m(2)} \cdots \delta\sigma^{m(i)}(r) = q^{m(1)+2m(2)+\cdots+im(i)}\sigma^{m(1)+m(2)+\cdots+m(i)}\delta^i(r) \in I$$

for all $m(1), ..., m(i) \geqslant 0$. Hence, the sets I_j are the same as those in Lemma 2.2, from which we see that each I_j is a σ-ideal and that J is a (σ, δ)-ideal. It is clear that J is the largest δ-ideal contained in I.

Now assume that $\sigma^{-1}(I)=I$ and that $q \in \mathscr{C}(I)$. Obviously $\sigma^{-1}(I_0)=I_0$, and we have just proved that $I_j \subseteq \sigma^{-1}(I_j)$ for all j. If $a \in \sigma^{-1}(I_j)$ for some $j>0$, then $q^i\sigma\delta^i(a)=\delta^i\sigma(a) \in I$ for all $i<j$. It follows from our assumptions on I and q that $\delta^i(a) \in I$ for all $i<j$, and so $a \in I_j$. Therefore $\sigma^{-1}(I_j)=I_j$ for all j, and likewise $\sigma^{-1}(J)=J$. ∎

In the proof of the next lemma, we proceed in parallel with [20, Proposition 1.1].

LEMMA 6.4. *Let (σ, δ, q) be a q-skew derivation on ring R, with σ an automorphism. Let I be an ideal of R such that $\sigma(I)=I$ and I is σ-prime, and assume that $q^i+q^{i-1}+\cdots+1 \notin I$ for all $i=0, 1, ...$ If*

$$J=\{r \in R \mid \delta^i(r) \in I \text{ for all } i=0, 1, ...\},$$

then $\sigma(J)=J$ and J is a σ-prime (σ, δ)-ideal of R. Thus if I is minimal among σ-prime ideals of R, it must be a (σ, δ)-ideal.

Proof. If $q \in I$, then $\delta(R)=\delta\sigma(R)=q\sigma\delta(R) \subseteq I$, in which case I is a (σ, δ)-ideal and there is nothing to prove. Therefore we may assume that $q \notin I$. Then qR is a σ-ideal not contained in I, and so since I is σ-prime it follows that $q \in \mathscr{C}(I)$. Likewise, $q^i+q^{i-1}+\cdots+1 \in \mathscr{C}(I)$ for all $i \geqslant 0$, and hence $\binom{n}{m}_q \in \mathscr{C}(I)$ for all $n \geqslant m \geqslant 0$.

By Lemma 6.3, J is a (σ, δ)-ideal and $\sigma^{-1}(J)=J$, whence $\sigma(J)=J$.

To prove that J is σ-prime, we verify the criterion given in Lemma 2.1(a). Given elements $a', c' \in R-J$, choose nonnegative integers m, n minimal such that $\delta^m(a')$ and $\delta^n(c')$ are not in I. Since I is σ-prime, there exist $b' \in R$ and $t \in \mathbb{Z}$ such that $\delta^m(a')b'\sigma^t\delta^n(c') \notin I$. Set $a=\sigma^{-n}(a')$ and $b=\sigma^{-n}(b')$ while $c=\sigma^t(c')$. Then

$$\begin{aligned} q^{mn}\sigma^n\delta^m(a)\sigma^n(b)\,\delta^n(c) &= \delta^m\sigma^n(a)\sigma^n(b)\,\delta^n(c) \\ &= q^{nt}\delta^m(a')b'\sigma^t\delta^n(c') \notin I, \end{aligned}$$

whence $\sigma^n\delta^m(a)\sigma^n(b)\,\delta^n(c) \notin I$. Since $\sigma^{-1}(I)=I$ and $\delta^i(a')=q^{in}\sigma^n\delta^i(a)$ for all $i \geqslant 0$, we see that $\delta^i(a) \in I$ for all $i < m$, and similarly $\delta^j(c) \in I$ for all $j < n$.

Now expand

$$\delta^{m+n}(abc) = \sum_{i=0}^{m+n} \binom{m+n}{i}_q \sigma^{m+n-i}\delta^i(ab)\, \delta^{m+n-i}(c)$$
$$= \sum_{i=0}^{m+n} \sum_{j=0}^{i} \binom{m+n}{i}_q \binom{i}{j}_q \sigma^{m+n-i}(\sigma^{i-j}\delta^j(a)\, \delta^{i-j}(b))\, \delta^{m+n-i}(c)$$
$$= \sum_{i=0}^{m+n} \sum_{j=0}^{i} \binom{m+n}{i}_q \binom{i}{j}_q$$
$$\times \sigma^{m+n-j}\delta^j(a)\sigma^{m+n-i}\delta^{i-j}(b)\, \delta^{m+n-i}(c).$$

For $i > m$ we have $m+n-i < n$ and so $\delta^{m+n-i}(c) \in I$. For $i \leqslant m$ and $j < m$ we have $\delta^j(a) \in I$ and so $\sigma^{m+n-j}\delta^j(a) \in I$. Hence,

$$\delta^{m+n}(abc) \equiv \binom{m+n}{m}_q \binom{m}{m}_q \sigma^n\delta^m(a)\sigma^n(b)\, \delta^n(c) \qquad (\text{mod } I).$$

Since $\sigma^n\delta^m(a)\sigma^n(b)\, \delta^n(c) \notin I$ and the q-binomial coefficients are all in $\mathscr{C}(I)$, we conclude that $\delta^{m+n}(abc) \notin I$, and so $abc \notin J$.

Thus $a'b'\sigma^{n+t}(c') = \sigma^n(abc) \notin J$, and therefore J is σ-prime. The final statement of the lemma is clear. ▌

PROPOSITION 6.5. *Let (σ, δ, q) be a q-skew derivation on a right noetherian ring R, with σ an automorphism. Assume that R is (σ, δ)-prime, and that $q^i + q^{i-1} + \cdots + 1 \neq 0$ for all $i = 0, 1, \ldots$. Then R is σ-prime.*

Proof. Set $C = \{q^i + q^{i-1} + \cdots + 1 \mid i = 0, 1, \ldots\}$. Each $c \in C$ is a nonzero central (σ, δ)-constant, whence cR is a nonzero (σ, δ)-ideal. Since R is (σ, δ)-prime, such ideals must have zero annihilator, and hence $C \subseteq \mathscr{C}(0)$. Thus if N is the prime radical of R, we obtain $C \subseteq \mathscr{C}(N)$ (e.g., [21, Lemma 10.8] or [37, Proposition 4.1.3]), and so $C \subseteq \mathscr{C}(P)$ for all minimal primes P (e.g., [21, Proposition 6.5] or [37, Proposition 3.2.4]).

Partition the minimal primes of R into disjoint σ-orbits $\Pi_1, \ldots, \Pi_n$, and for $j = 1, \ldots, n$ set $I_j = \cap \Pi_j$. Then $I_1, \ldots, I_n$ are σ-prime ideals of R and $I_1 \cap \cdots \cap I_n = N$. In particular, C is disjoint from each I_j, and $I_1 \cap \cdots \cap I_n$ is nilpotent. Now set

$$J_j = \{r \in R \mid \delta^i(r) \in I_j \text{ for all } i = 0, 1, \ldots\}$$

for all j. By Lemma 6.4, each J_j is a σ-prime (σ, δ)-ideal of R. Since

$J_1 \cap \cdots \cap J_n$ is nilpotent and R is (σ, δ)-prime, we conclude that some $J_j = 0$. Therefore R is σ-prime. ∎

A special case of Proposition 6.5 (with $q = 1$ and R assumed δ-prime) was obtained by Kamal [32, Corollary 5.11].

We now obtain the following general statement for contractions of prime ideals in noetherian skew polynomial rings. The non-q-skew part of the statement was proved by Bell in the case that $P \cap R$ is a (σ, δ)-ideal [4, Corollary 6.2]; his proof works in the more general situation, and we have merely rearranged his argument slightly.

THEOREM 6.6. *Let $T = R[\theta; \sigma, \delta]$ where R is right noetherian and σ is an automorphism. Let P be a prime ideal of T, let H be the largest (σ, δ)-ideal of R contained in $P \cap R$, and let N be the prime radical of H. Then N is σ-prime and $\mathscr{C}_R(H) = \mathscr{C}_R(N)$. Moreover, if (σ, δ, q) is a q-skew derivation and $q^i + q^{i-1} + \cdots + 1 \notin H$ for all $i = 0, 1, \ldots$, then H itself is σ-prime.*

Proof. Without loss of generality, $H = 0$. If A and B are any nonzero (σ, δ)-ideals of R, then TA and BT are nonzero ideals of T, and TA, $BT \nsubseteq P$ because $A, B \nsubseteq P \cap R$. Hence, $TABT \nsubseteq P$ and so $AB \neq 0$. Therefore R is (σ, δ)-prime, and the desired conclusions follow from Theorem 2.3 and Proposition 6.5. ∎

We can in particular use Theorem 6.6 to see that in the q-skew case of Theorem 3.1, if q is not a root of unity modulo I then only cases (a)(i) and (b) occur.

COROLLARY 6.7. *Let $T = R[\theta; \sigma, \delta]$ where R is commutative noetherian, σ is an automorphism, and (σ, δ, q) is a q-skew derivation. Let P be a prime ideal of T, and set $I = P \cap R$. If $q^i + q^{i-1} + \cdots + 1 \notin I$ for all $i = 0, 1, \ldots$, then either I is a σ-prime (σ, δ)-ideal or I is a prime ideal and $\sigma(I) \neq I$.*

Proof. If I is a (σ, δ)-ideal, Theorem 6.6 shows that I is σ-prime. Otherwise, Theorem 3.1 says that I is prime and $\sigma(I) \neq I$. ∎

7. COMMUTATIVE ARTINIAN COEFFICIENTS

Returning to our consideration of prime ideals in $T = R[\theta; \sigma, \delta]$ when R is commutative noetherian and σ is an automorphism, we look further at case (a) of Theorem 3.1, since that is the case in which it is possible for more than one prime of T to contract to the same ideal I in R. Thus, we start with a (σ, δ)-prime ideal I of R, and we ask for a description of the primes of T that contract to I. The first step is of course to factor I out and then localize. Namely, let R_1 be the classical quotient ring of R/I, let $T_1 = R_1[\theta; \sigma, \delta]$, and let $\phi: T \to T_1$ be the natural map. Then

$$\{P \in \operatorname{Spec}(T) \mid P \cap R = I\}$$
$$= \{\phi^{-1}(P_1) \mid P_1 \in \operatorname{Spec}(T_1) \text{ and } P_1 \cap R_1 = 0\}.$$

Moreover, R_1 is artinian by Theorem 2.3.

Thus we may now assume that we are dealing with an artinian coefficient ring R, and we look for the primes of T that contract to zero in R. We approach this by trying to represent T as a ring of matrices. In view of Theorem 2.6, there are two cases to consider. In the first case, R is σ-prime, and here we can represent T in terms of matrices over a skew polynomial ring over a field. (While a description of the primes of T can in this case be obtained directly from [24, Theorems 4.3, 4.4], the matrix description of T helps to clarify the picture.) In the second case, R is δ-prime with a unique associated prime, which contains $(1-\sigma)(R)$. As R is artinian, it must actually be local and δ-simple. Here we extend the analysis given in Proposition 4.6 and show that in fairly wide circumstances T must be isomorphic to a full matrix ring over an ordinary differential operator ring over a field; in fact we have no examples where this fails to happen.

For the σ-prime case we extract the following general result from a theorem of Jategaonkar [27, Theorem 2.1(a)].

PROPOSITION 7.1. *Let σ be an automorphism of a ring R, and assume that R contains orthogonal central idempotents $f_1, \ldots, f_n$ such that $f_1 + \cdots + f_n = 1$ and $\sigma(f_i) = f_{i+1}$ for $i = 1, \ldots, n-1$ while $\sigma(f_n) = f_1$. Let S denote the ring Rf_1, and let τ be the automorphism $\sigma^n|_S$ on S. Set $U = S[\theta'; \tau]$ and*

$$V = \{v \in M_n(U) \mid v_{ij} \in \theta' U \text{ for all } i < j\}.$$

Then there exists an isomorphism of $R[\theta; \sigma]$ onto V which maps R onto the subring of diagonal matrices with entries from S.

Proof. Set $T = R[\theta; \sigma]$. Since there is nothing to prove if $n = 1$, assume that $n > 1$.

Note that the restrictions of σ to the rings Rf_i give ring isomorphisms

$$Rf_1 \to Rf_2 \to \cdots \to Rf_n \to Rf_1,$$

the composition of which is τ. Hence, there is no loss of generality in assuming that $R = S^n$ and that

$$\sigma(s_1, \ldots, s_n) = (\tau(s_n), s_1, s_2, \ldots, s_{n-1})$$

for all $(s_1, \ldots, s_n) \in R$.

Let $\phi: R \to M_n(U)$ be the diagonal embedding, that is,

$$\phi(s_1, ..., s_n) = \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & s_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & s_n \end{pmatrix}$$

for all $(s_1, ..., s_n) \in R$. Set

$$z = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \theta' \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in M_n(U),$$

and check that $z\phi(r) = \phi\sigma(r)z$ for all $r \in R$. Thus ϕ extends to a ring homomorphism $T \to M_n(U)$ such that $\phi(\theta) = z$.

In terms of the standard matrix units $e_{ij} \in M_n(U)$, we have

$$z = \theta' e_{1n} + e_{21} + e_{32} + \cdots + e_{n,n-1}.$$

For $m = 2, ..., n-1$, it follows that

$$\begin{aligned} z^m = \theta'(&e_{1,n-m+1} + e_{2,n-m+2} + \cdots + e_{mn}) \\ &+ (e_{m+1,1} + e_{m+2,2} + \cdots + e_{n,n-m}), \end{aligned}$$

while $z^n = \theta'(e_{11} + e_{22} + \cdots + e_{nn})$. From this we conclude that ϕ is injective, and that $\phi(T) = V$. ∎

COROLLARY 7.2. *Let $T = R[\theta; \sigma, \delta]$ where R is commutative artinian σ-prime, σ is an automorphism, and $\sigma \neq 1$. Set $S = Rf$ where f is a primitive idempotent in R, and let n be the least positive integer such that $\sigma^n(f) = f$ (this exists because R is σ-prime). Let τ be the automorphism $\sigma^n|_S$ on S, and set $U = S[\theta'; \tau]$ and*

$$V = \{v \in M_n(U) \mid v_{ij} \in \theta' U \text{ for all } i < j\}.$$

Then there exists an isomorphism of T onto V which maps R onto the subring of diagonal matrices with entries from S.

Proof. Since δ is inner by Lemma 3.7, we may assume that $\delta = 0$. Note that $f, \sigma(f), ..., \sigma^{n-1}(f)$ are distinct primitive idempotents, and hence they are orthogonal. By σ-primeness, $f+\sigma(f)+\cdots+\sigma^{n-1}(f)=1$. Therefore the corollary follows from Proposition 7.1. ∎

In the situation of Corollary 7.2, the primes of V that contract to zero in the image of R are the ideals PV where P is a prime of U not containing θ'. Thus finding the primes of T reduces to finding the primes of U, which was done by Irving in [24]. In case τ has infinite order, the only primes of U are 0 and $\theta' U$ [24, Theorem 4.3]. In case τ has finite order m, the primes of U correspond to the primes of the polynomial ring $S^\tau[(\theta')^m]$, where S^τ is the subfield of elements of S fixed by τ [24, Theorem 4.4].

We now turn to the case of a commutative artinian local δ-simple coefficient ring. Here q-skew derivations enter in, and we require some computations with q-binomial coefficients.

LEMMA 7.3. *Let q be a central element in a ring R, and let $s>1$ be an integer. Assume that $q^{s-1}+q^{s-2}+\cdots+1=0$ while $q^{i-1}+q^{i-2}+\cdots+1$ is a non-zero-divisor for $i=1,\ldots,s-1$. Then $\binom{s}{i}_q=0$ for all $i=1,\ldots,s-1$ and $\binom{m}{s-1}_q=0$ for all integers $m\geqslant s-1$ such that $m\not\equiv -1 \pmod{s}$.*

Proof. Set $f(i)=q^{i-1}+q^{i-2}+\cdots+1$ for all $i>0$. For $i=1,\ldots,s-1$, we observe that

$$f(i)f(i-1)\cdots f(1)\binom{s}{i}_q=f(s)f(s-1)\cdots f(s-i+1)=0,$$

whence $\binom{s}{i}_q=0$ (because $f(i), f(i-1), \ldots, f(1)$ are non-zero-divisors).

We next claim that $f(ns)=0$ for all $n>0$. To see this, choose an indeterminate x, and note that in the polynomial ring $\mathbb{Z}[x]$, we have

$$\begin{aligned}
&(x-1)(x^{ns-1}+x^{ns-2}+\cdots+1)\\
&\quad=x^{ns}-1=(x^s-1)(x^{(n-1)s}+{}^{(n-2)s}+\cdots+1)\\
&\quad=(x-1)(x^{s-1}+x^{s-2}+\cdots+1)(x^{(n-1)s}+x^{(n-2)s}+\cdots+1),
\end{aligned}$$

whence $x^{s-1}+x^{s-2}+\cdots+1$ divides $x^{ns-1}+x^{ns-2}+\cdots+1$. Hence, $f(s)$ divides $f(ns)$ in R, and so $f(ns)=0$, as claimed.

Now consider any $m\geqslant s-1$ such that $m\not\equiv -1 \pmod{s}$. Then $m=ns+j$ for some $n>0$ and some $j\in\{0,1,\ldots,s-2\}$. Consequently, $ns\in\{m,m-1,\ldots,m-s+2\}$. Thus

$$f(s-1)f(s-2)\cdots f(1)\binom{m}{s-1}_q=f(m)f(m-1)\cdots f(m-s+2)=0$$

(because $f(ns)=0$), and therefore $\binom{m}{s-1}_q=0$. ∎

LEMMA 7.4. *Let R be a ring with a q-skew derivation (σ, δ, q), and suppose that there exists a central element $x \in R$ such that $\sigma(x) = qx$ while $\delta(x) = 1$. Then*

$$\sigma^n = \sum_{i=0}^{n} q^{i(i-1)/2}(q-1)^i \binom{n}{i}_q x^i\delta^i$$

for all $n = 1, 2, \ldots$.

Proof. For all $r \in R$, observe that

$$\sigma(r) + \delta(r)x = \delta(rx) = \delta(xr) = qx\delta(r) + r,$$

whence $\sigma(r) + (q-1)x\delta(r)$. Thus $\sigma = 1 + (q-1)x\delta$, which gives the desired formula in case $n = 1$.

Now assume that the formula holds for some n. For $i > 0$, it follows from Lemma 1.1 that

$$\delta(x^i) = (q^{i-1} + q^{i-2} + \cdots + 1)x^{i-1},$$

and so $(q-1)x\delta(x^i r) = (q-1)q^i x^{i+1}\delta(r) + (q^i - 1)x^i r$ for all $r \in R$. This equation holds when $i = 0$ as well. Using Lemma 6.1(c), we conclude that

$$\begin{aligned}
\sigma^{n+1} &= (1 + (q-1)x\delta)\sigma^n \\
&= \sum_{i=0}^{n} q^{i(i-1)/2}(q-1)^i \binom{n}{i}_q x^i\delta^i \\
&\quad + \sum_{i=0}^{n} q^{i(i-1)/2}(q-1)^i \binom{n}{i}_q [(q-1)q^i x^{i+1}\delta^{i+1} + (q^i - 1)x^i\delta^i] \\
&= \sum_{i=0}^{n} q^{i(i-1)/2}(q-1)^i \binom{n}{i}_q q^i x^i\delta^i \\
&\quad + \sum_{i=0}^{n} q^{(i+1)i/2}(q-1)^{i+1} \binom{n}{i}_q x^{i+1}\delta^{i+1} \\
&= 1 + \sum_{i=1}^{n} q^{i(i-1)/2}(q-1)^i \left[q^i \binom{n}{i}_q + \binom{n}{i-1}_q\right] x^i\delta^i \\
&\quad + q^{(n+1)n/2}(q-1)^{n+1}x^{n+1}\delta^{n+1} \\
&= \sum_{i=0}^{n+1} q^{i(i-1)/2}(q-1)^i \binom{n+1}{i}_q x^i\delta^i,
\end{aligned}$$

which completes the induction step. ∎

PROPOSITION 7.5. *Let $T=R[\theta;\sigma,\delta]$ and suppose there exist central elements $q, x\in R$ such that $\delta(q)=0$ and $\delta(x)=1$ while $\sigma(x)=qx$ and $x^s=0$ for some integer $s>1$. Assume that $q^{s-1}+q^{s-2}+\cdots+1=0$ while $q^{i-1}+q^{i-2}+\cdots+1$ is a unit for $i=1, ..., s-1$. Then σ^s is the identity, δ^s is an ordinary derivation leaving xR invariant, and*

$$T\cong M_s((R/xR)[\theta';\delta^s]).$$

Proof. As in Lemma 7.4, $\sigma=1+(q-1)x\delta$. In particular, from $\delta(q)=0$ we get $\sigma(q)=q$. Also,

$$\delta\sigma=\delta+(q-1)[qx\delta^2+\delta]=q\delta+q(q-1)x\delta^2=q\sigma\delta,$$

and thus (σ,δ,q) is a q-skew derivation.

By Lemma 7.3, $\binom{s}{i}_q=0$ for all $i=1, ..., s-1$, and so Lemma 7.4 shows that $\sigma^s=1$. Also, using Lemma 6.2 we see that δ^s is a derivation and that $\theta^s r=r\theta^s+\delta^s(r)$ for all $r\in R$. Since $s\geqslant 2$, we have $\delta^s(x)=0$, and hence $\delta^s(xR)\subseteq xR$.

Since $\theta^s r=r\theta^s+\delta^s(r)$ for all $r\in R$, the set

$$U=\sum_{n=0}^{\infty} R\theta^{ns}$$

is a subring of T and $U\cong R[\theta';\delta^s]$. We also have $\delta^s(x)=0$, whence x is central in U and $U/xU\cong(R/xR)[\theta';\delta^s]$. Hence, it suffices to show that $T\cong M_s(U/xU)$.

Set $c=\delta^{s-1}(x^{s-1})$. Using the formula for $\delta(x^i)$ in Lemma 1.1, we find that

$$c=(q^{s-2}+q^{s-3}+\cdots+1)(q^{s-3}+\cdots+1)\cdots(q+1)(1),$$

which is a central unit as well as a (σ,δ)-constant. Hence, if $y=c^{-1}x^{s-1}$ then $\delta^{s-1}(y)=1$. Note that $\delta^i(y)\in xR$ for all $i\leqslant s-2$.

Obviously $yx=0$. On the other hand, given $r\in R$ such that $yr=0$, we see by expanding the equation $\delta^{s-1}(yr)=0$ and invoking the previous paragraph that $r\in xR$. Thus $\operatorname{ann}_R(y)=xR$. Now if $r\in R$ and $x^i r\in x^{i+1}R$ for some $i<s$, then after multiplying by $c^{-1}x^{s-1-i}$ we obtain $yr=0$ and so $r\in xR$. Therefore

$$x^iR/x^{i+1}R\cong R/xR\cong yR$$

for all $i=0, ..., s-1$.

We next expand

$$\theta^{s-1}y=\sum_{i=0}^{s-1}\binom{s-1}{i}_q\sigma^{s-1-i}\delta^i(y)\theta^{s-1-i}.$$

Since $\sigma^{s-1-i}\delta^i(y)\in xR$ for $i<s-1$, and since $\delta^{s-1}(y)=1$, we obtain $y\theta^{s-1}y=y$. Thus the element $e=y\theta^{s-1}$ is an idempotent such that $yT=eT$. For $i=0,...,s-1$, we have

$$x^iT/x^{i+1}T\cong(x^iR/x^{i+1}R)\otimes_R T\cong yR\otimes_R T\cong yT=eT,$$

whence $T_T\cong(eT)^s$ and so $T\cong M_s(eTe)$. Consequently, it will be enough to show that $eTe\cong U/xU$.

Since x is central in U, so is y. Hence, $Ue=yU\theta^{s-1}$ and so $Ue=eUe$. Thus Ue is a subring of eTe and the rule $u\mapsto ue$ provides a ring homomorphism of U onto Ue. For $u\in U$, observe that $ue=0$ if and only if $yu\theta^{s-1}=0$, if and only if $yu=0$, if and only if $u\in xU$ (because $\operatorname{ann}_R(y)=xR$). Therefore $U/xU\cong Ue$ (as rings), and we will be done provided $Ue=eTe$.

We approach this by showing that $y\theta^m e\in Ue$ for all $m\geqslant 0$. If $m=ns-1$ for some $n>0$, then since y commutes with θ^s we obtain

$$y\theta^m e=y\theta^{(n-1)s}\theta^{s-1}e=\theta^{(n-1)s}y\theta^{s-1}e=\theta^{(n-1)s}e^2=\theta^{(n-1)s}e\in Ue.$$

Now suppose that $m\not\equiv -1 \pmod s$, and consider the equation

$$\theta^m y=\sum_{i=0}^{m}\binom{m}{i}_q\sigma^{m-i}\delta^i(y)\theta^{m-i}.$$

We have $\delta^i(y)=0$ for $i>s-1$ and $\sigma^{m-i}\delta^i(y)\in xR$ for $i<s-1$, while if $m\geqslant s-1$, Lemma 7.3 shows that $\binom{m}{s-1}_q=0$. Thus $\theta^m y\in xT$ and so $y\theta^m y=0$, whence $y\theta^m e=0$. This establishes the claim. As a result,

$$eTe\subseteq yTe=\sum_{m=0}^{\infty}Ry\theta^m e\subseteq Ue,$$

and therefore $Ue=eTe$ as desired. ∎

In the case $s=2$ of Proposition 7.5, slightly weaker hypotheses suffice, as follows. For an analogous result in the context of $\mathbb{Z}_2$-graded algebras, with the element x being supercentral rather than central, see [9, Lemma 1.5].

COROLLARY 7.6. *Let $T=R[\theta;\sigma,\delta]$ and suppose there exists a central element $x\in R$ such that $x^2=0$ and $\delta(x)=1$. Then σ^2 is the identity, δ^2 is an ordinary derivation leaving xR invariant, and $T\cong M_2((R/xR)[\theta';\delta^2])$.*

Proof. Observe that $0=\delta(x^2)=\sigma(x)+x$, whence $\sigma(x)=-x$. Thus if $q=-1$ and $s=2$, the hypotheses of Proposition 7.5 are satisfied. ∎

Proposition 7.5 and Corollary 7.6 can be used to give examples in characteristic zero of an interesting homological phenomenon which has previously been observed only in positive characteristic, namely the existence of ring extensions $R \subseteq T$ such that T is a free R-module on each side, and R is a direct summand of T as a right or left R-module, yet $\mathrm{gl.dim}(T) < \mathrm{gl.dim}(R)$. The original example (see [18, p. 317] or [37, Example 7.2.7]) is a differential operator ring $T = R[\theta; \delta]$ where $R = k[x]/x^2k[x]$ for a field k of characteristic 2 and $\delta = d/dx$.

The construction carries over to the corresponding q-skew differential operator ring in characteristic zero, or in fact in any characteristic. Choose a field k, and choose an indeterminate z. Let σ be the k-algebra automorphism of $k[z]$ such that $\sigma(z) = -z$, and let δ be the k-linear σ-derivation on $k[z]$ such that $\delta(z) = 1$. Then $(\sigma, \delta, -1)$ is a q-skew derivation on $k[z]$. Moreover, $\delta(z^2) = \sigma(z) + z = 0$, and so (σ, δ) induces a q-skew derivation on the ring $R = k[z]/z^2k[z]$. If x denotes the coset of z in R, then $\sigma(x) = -x$ and $\delta(x) = 1$. Observe that $R/xR \cong k$, and that $\delta^2 = 0$ on R. Now if $T = R[\theta; \sigma, \delta]$, Corollary 7.6 shows that $T \cong M_2(k[\theta'])$. Therefore $\mathrm{gl.dim}(T) = 1$ even though $\mathrm{gl.dim}(R) = \infty$.

After seeing this example, Bell pointed out a simpler one, namely the embedding of $k[z]/z^2k[z]$ into $M_2(k)$ where $z \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. His example is actually a quotient of ours: in the notation above, θ^2 is central in T and $T/\theta^2T \cong M_2(k)$.

Using Proposition 7.5, similar examples can be constructed resulting in larger matrix rings. Namely, if k contains a primitive nth root of unity q, use the automorphism σ of $k[z]$ such that $\sigma(z) = qz$, and use $R = k[z]/z^nk[z]$. In this case, $T \cong M_n(k[\theta'])$. (See Theorem 8.6.)

THEOREM 7.7. *Let $T = R[\theta; \sigma, \delta]$ where σ is an automorphism and R is commutative, artinian, local, and δ-simple, with maximal ideal M. Suppose there exist $q \in R$ and $x \in M$ such that $\delta(q) = 0$ and $\delta(x) \notin M$ while $\sigma(x) = qx$. Then there exists a derivation δ' on R/M such that $T \cong M_t((R/M)[\theta'; \delta'])$, where $t = \mathrm{length}(R)$. Moreover, if $\mathrm{char}(R/M) = 0$ then $M = xR$ and t equals the index of nilpotence of x.*

Proof. Since we may replace δ by $\delta(x)^{-1}\delta$, there is no loss of generality in assuming that $\delta(x) = 1$.

We proceed by induction on t, the first case being $t = 2$. Then $M = xR$ and $x^2 = 0$, and the desired conclusions are given by Corollary 7.6.

Now let $t > 2$ and assume the theorem holds for coefficient rings of length less than t. Note that the ideal generated by any δ-constant in R is a δ-ideal, whence by δ-simplicity all nonzero δ-constants in R are units.

Let s be the index of nilpotence of x. By Lemma 1.1,

$$0 = \delta(x^s) = (q^{s-1} + q^{s-2} + \cdots + 1)x^{s-1},$$

whence $q^{s-1} + q^{s-2} + \cdots + 1$ is not a unit and so $q^{s-1} + q^{s-2} + \cdots + 1 = 0$.

On the other hand, for $i=1, ..., s-1$ the element x^i is a nonzero non-unit and so

$$0 \neq \delta(x^i) = (q^{i-1} + q^{i-2} + \cdots + 1)x^{i-1}.$$

Then $q^{i-1}+q^{i-2}+\cdots+1 \neq 0$, whence $q^{i-1}+q^{i-2}+\cdots+1$ is a unit.

Proposition 7.5 now shows that $T \cong M_s((R/xR)[\theta''; \delta''])$ for some derivation δ'' on R/xR. Since if $M = xR$ we are done, suppose that $M > xR$. In the proof of Proposition 7.5 we saw that $x^iR/x^{i+1}R \cong R/xR$ for all $i=0, ..., s-1$, and so if $r = \text{length}(R/xR)$ then $rs = t$.

By Proposition 3.3, T is a prime ring, and hence so is $(R/xR)[\theta''; \delta'']$, from which it follows that R/xR is δ''-prime [31, Lemma 1.3]. Then R/xR has no nonzero nilpotent δ''-ideals, whence R/xR is actually δ''-simple. Since $M/xR \neq 0$, there exists $z \in M/xR$ such that $\delta''(z) \notin M/xR$. By our induction hypothesis,

$$(R/xR)[\theta''; \delta''] \cong M_r((R/M)[\theta'; \delta'])$$

for some derivation δ' on R/M, and therefore $T \cong M_t((R/M)[\theta'; \delta'])$.

Finally, if m is the index of nilpotence of z, then $0 = \delta''(z^m) = mz^{m-1}\delta''(z)$ and so $mz^{m-1} = 0$, whence $m \cdot 1 \in M/xR$. Thus M can differ from xR only in case $\text{char}(R/M) > 0$. This completes the induction step. ∎

In the situation of Theorem 7.7, all primes of T contract to zero in R. Given the isomorphism of the theorem, the primes of T correspond to the primes of the differential operator ring $U = (R/M)[\theta'; \delta']$. If U is not simple, then by Theorem 4.3, U is a finitely generated module over its center. Since U is a principal ideal domain, the nonzero primes of U correspond to the irreducible elements of its center.

In both Proposition 7.5 and Theorem 7.7, our hypotheses imply that the given skew derivation must be a q-skew derivation. We now reverse direction and show that q-skew derivations lead to the setup of Theorem 7.7.

COROLLARY 7.8. *Let $T = R[\theta; \sigma, \delta]$ where σ is an automorphism and R is commutative, artinian, local, and δ-simple, with maximal ideal M. Assume that (σ, δ, q) is a q-skew derivation, and that $(1-\sigma)(R) \subseteq M$. Then there exists a derivation δ' on R/M such that $T \cong M_t((R/M)[\theta'; \delta'])$, where $t = \text{length}(R)$.*

Proof. If $M = 0$, then $\sigma = 1$ and the conclusion is clear. Thus we may assume that $M \neq 0$, and then $t \geqslant 2$. As R is δ-simple, $\delta \neq 0$, whence $\delta\sigma \neq 0$ and so $q \neq 0$. Another application of δ-simplicity now shows that q is a unit.

Define $M_0, M_1, ...$ as in Lemma 4.4, and note from Lemma 6.3 that each M_j is a σ-ideal. As in the proof of Proposition 4.6, $M_t = 0$ and $M_j/M_{j+1} \cong R/M$ for all $j = 0, ..., t-1$; moreover, there exists $z \in M_{t-1}$ such that $\delta^{t-1}(z) \equiv 1 \pmod{M}$. Since $M_{t-1} = Rz$ and M_{t-1} is a σ-ideal, $\sigma(z) = az$ for some $a \in R$.

Now observe that

$$q^{t-1}\sigma\delta^{t-1}(z) = \delta^{t-1}\sigma(z) = \delta^{t-1}(az) = \sum_{i=0}^{t-1} \binom{t-1}{i}_q \sigma^{t-1-i}\delta^i(a)\,\delta^{t-1-i}(z).$$

Since $\delta^{t-1-i}(z) \in M$ for all $i > 0$, and since $\delta^{t-1}(z) \equiv 1 \pmod{M}$, we obtain $q^{t-1} \equiv \sigma^{t-1}(a) \pmod{M}$. On the other hand, $\sigma^{t-1}(a) \equiv a \pmod{M}$, because $(1-\sigma)(R) \subseteq M$. Hence, $a \equiv q^{t-1} \pmod{M}$. Since $Mz = 0$, we obtain $\sigma(z) = q^{t-1}z$.

Finally, set $x = \delta^{t-2}(z)$. Then $x \in M$ and $\delta(x) \notin M$. Since

$$q^{t-2}\sigma(x) = q^{t-2}\sigma\delta^{t-2}(z) = \delta^{t-2}\sigma(z) = \delta^{t-2}(q^{t-1}z) = q^{t-1}x,$$

we have $\sigma(x) = qx$. The desired conclusion is now given by Theorem 7.7. ∎

We conjecture that Corollary 7.8 remains valid even without the assumption that (σ, δ) is a q-skew derivation. However, our method of proof—via Theorem 7.7—cannot be used in general, since there need not exist $q \in R$ and $x \in M$ satisfying the hypotheses of Theorem 7.7, as the following example shows.

EXAMPLE 7.9. There is a commutative artinian local ring R with maximal ideal M and a skew derivation (σ, δ) such that

(a) σ is an automorphism and $(1-\sigma)(R) \subseteq M$.

(b) R is δ-simple.

(c) There do not exist $q \in R$ and $x \in M$ with $\delta(q) = 0$ and $\delta(x) \notin M$ while $\sigma(x) = qx$.

Proof. Choose a field k of characteristic $p \geqslant 3$. Let z be an indeterminate, let σ be the k-algebra *endomorphism* of $k[z]$ such that $\sigma(z) = z + z^2$, and let δ be the k-linear σ-derivation on $k[z]$ such that $\delta(z) = 1$. By Lemma 1.1,

$$\delta(z^p) = \sum_{i=0}^{p-1} \sigma(z)^i z^{p-1-i} = z^{p-1} \sum_{i=0}^{p-1} (1+z)^i,$$

which lies in $z^p k[z]$ because of characteristic p. Thus $z^p k[z]$ is a (σ, δ)-ideal of $k[z]$, and so (σ, δ) induces a skew derivation on the ring $R = k[z]/z^p k[z]$. It is clear that R is commutative artinian local with maximal ideal yR, where y denotes the coset of z in R.

(a) Since $R=k\oplus yR$ and $\sigma(y)\in yR$, we have $(1-\sigma)(R)\subseteq M$. Also, $\sigma(y^i)=y^i(1+y)^i$ for all $i=1, ..., p-1$, from which it follows that σ is an automorphism on R.

(b) The ideals of R are R, yR, $y^2R, ..., y^pR=0$. For $m=1, ..., p-1$, Lemma 1.1 shows that

$$\delta(y^m)=y^{m-1}\sum_{i=0}^{m-1}(1+y)^i,$$

and since $\sum_{i=0}^{m-1}(1+y)^i$ is a unit, y^mR is not a δ-ideal. Thus R is δ-simple.

(c) From the formulas for $\delta(y^m)$ in the previous paragraph, we see that $\ker(\delta)=k$. Hence if there exist $q\in R$ and $x\in M$ with the given properties, we must have $q\in k$ and

$$x=a_1y+a_2y^2+\cdots+a_{p-1}y^{p-1}$$

for some $a_i\in k$, with $a_1\neq 0$. Then

$$\sum_{i=1}^{p-1}qa_iy^i=qx=\sigma(x)=\sum_{i=1}^{p-1}a_iy^i(1+y)^i,$$

whence $qa_1=a_1$ and $qa_2=a_1+a_2$. However, since $a_1\neq 0$ these equations are incompatible. Therefore there do not exist q, x with the given properties. ∎

The skew polynomial rings $T=R[\theta;\sigma,\delta]$ constructed from R,σ,δ as in Example 7.9 do not seem to contradict the conjecture above. In case $p=3$, it can be shown that $\theta^3-\theta^2$ is central in T and that the idealizer of yT is $yT+k[\theta^3-\theta^2]$, whence Proposition 4.6(a) shows that $T\cong M_3(k[\theta^3-\theta^2])$. Similarly, if $p=5$ then $T\cong M_5(k[\theta^5-2\theta^3-4\theta^2])$.

8. Quantized Weyl Algebras

In this section we analyze prime ideals and prime factors in quantized Weyl algebras, concentrating on the case in which the coefficient ring is a field and the quantization is nontrivial.

Definition. Let S be a ring and q a central unit in S. The *quantized Weyl algebra* $A_1(S,q)$ is the ring generated by S together with two additional elements x and y such that

(a) x and y commute with all elements of S;

(b) $xy-qyx=1$.

Clearly $A_1(S, q)$ is isomorphic to a skew polynomial ring $S[y][x; \sigma, \delta]$ where σ is the automorphism of $S[y]$ such that $\sigma = 1$ on S and $\sigma(y) = qy$, while δ is the (unique) S-linear σ-derivation on $S[y]$ such that $\delta(y) = 1$. (We restrict q to be a central unit of S in order that σ will be an automorphism.)

When viewing $A_1(S, q)$ as a skew polynomial ring, we prefer to use x for the initial polynomial variable and θ for the skew polynomial variable. Thus throughout this section we shall write our quantized Weyl algebras in the form

$$A_1(S, q) = S[x][\theta; \sigma, \delta],$$

where $\sigma = 1$ and $\delta = 0$ on S, while $\sigma(x) = qx$ and $\delta(x) = 1$. This skew derivation δ can be described explicitly in the following manner:

$$\delta(f) = \frac{\sigma(f) - f}{qx - x} = \frac{f(qx) - f(x)}{qx - x}$$

for all $f \in S[x]$. (To see this, either check that the given formula defines a σ-derivation sending x to 1, or take the existence of δ from Lemma 1.2 and then apply Lemma 2.4(a).) The operator

$$f \mapsto \frac{f(qx) - f(x)}{qx - x}$$

is sometimes called the *q-difference operator*. It was introduced in the case $S = \mathbb{R}$ in 1908 by Rev. F. H. Jackson [26].

LEMMA 8.1. *Let $A_1(S, q) = S[x][\theta; \sigma, \delta]$. Then $\delta(x^m) = (q^{m-1} + q^{m-2} + \cdots + 1)x^{m-1}$ for all $m = 1, 2, \ldots$, and consequently $\delta\sigma = q\sigma\delta$.*

Proof. The formula for $\delta(x^m)$ is immediate from Lemma 1.1, and then it is clear that $\delta\sigma(x^m) = q\sigma\delta(x^m)$ for all m. Since σ and δ are S-linear, we conclude that $\delta\sigma = q\sigma\delta$. ∎

PROPOSITION 8.2. *Let $T = A_1(S, q) = S[x][\theta; \sigma, \delta]$, and set $u = \theta x - x\theta = (q - 1)x\theta + 1$. Then u is a normal element in T. If $q - 1$ is a unit in S, then $T/uT \cong S[x, x^{-1}]$.*

Proof (cf. [3, Lemma 2.2]). As is easily verified, $ux = qxu$ and $\theta u = qu\theta$, from which it follows that $uT = Tu$. Now assume that $q - 1$ is a unit. In the Laurent polynomial ring $S[x, x^{-1}]$, observe that

$$((1 - q)^{-1}x^{-1})x = (1 - q)^{-1} = q(1 - q)^{-1} + 1 = qx((1 - q)^{-1}x^{-1}) + 1.$$

Hence, the inclusion map $S[x] \to S[x, x^{-1}]$ extends (uniquely) to a ring homomorphism $\phi: T \to S[x, x^{-1}]$ such that $\phi(\theta) = (1-q)^{-1}x^{-1}$. Clearly ϕ is surjective and $\phi(u) = 0$. An obvious induction with respect to degree in θ establishes that

$$T = T \oplus (S[x] + S\theta + S\theta^2 + \dots)$$

as abelian groups. Since $(S[x] + S\theta + S\theta^2 + \dots) \cap \ker(\phi) = 0$, we conclude that $\ker(\phi) = uT$. Therefore $T/uT \cong S[x, x^{-1}]$. ∎

Proposition 8.2 shows that quantized Weyl algebras generally have a far richer ideal structure than ordinary Weyl algebras. In particular, they have higher Krull and global dimensions, as follows.

PROPOSITION 8.3. *Let $T = A_1(k, q)$ where k is a field. If $q \neq 1$, then the left and right Krull dimensions of T, the classical Krull dimension of T, and the global dimension of T are all equal to* 2.

Proof. Note that T is a domain. By Proposition 8.2, there is a nonzero normal element $u \in T$ such that $T/uT \cong k[x, x^{-1}]$, which is a domain of Krull dimension 1. Thus $\mathrm{cl.K.dim}(T) \geqslant 2$. On the other hand,

$$\mathrm{r.K.dim}(T) \leqslant \mathrm{r.K.dim}(k[x]) + 1 = 2$$

by [37, Proposition 6.5.4]. Therefore

$$\mathrm{r.K.dim}(T) = \mathrm{cl.K.dim}(T) = 2,$$

and similarly $\mathrm{l.K.dim}(T) = 2$. Since hereditary noetherian prime rings have Krull dimension at most 1 (e.g., [37, Corollary 6.2.8]), T cannot be hereditary. On the other hand,

$$\mathrm{gl.dim}(T) \leqslant \mathrm{gl.dim}(k[x]) + 1 = 2$$

by [37, Theorem 7.5.3]. Therefore $\mathrm{gl.dim}(T) = 2$. ∎

THEOREM 8.4. *Let $T = A_1(k, q) = k[x][\theta; \sigma, \delta]$ where k is a field. Assume that $q \neq 1$ and that q is not a root of unity.*

(a) *Every nonzero prime ideal of T contains the normal element $u = \theta x - x\theta$.*

(b) $\mathrm{Spec}(T) = \{0\} \cup \{uT + QT \mid Q \in \mathrm{Spec}(k[x])$ *and* $x \notin Q\}$.

Proof. (a) See [3, Lemma 2.2].

(b) Set $R = k[x]$. As shown in the proof of Proposition 8.2, the inclusion map $R \to k[x, x^{-1}]$ extends to a surjective ring homomorphism

$\phi: T \to k[x, x^{-1}]$ such that $\ker(\phi) = uT$. Since the primes of $k[x, x^{-1}]$ are exactly the ideals $Qk[x, x^{-1}]$ where $Q \in \operatorname{Spec}(R)$ and $x \notin Q$, the desired description of $\operatorname{Spec}(T)$ follows. ∎

COROLLARY 8.5. *Let $T = A_1(k, q) = k[x][\theta; \sigma, \delta]$ where k is an algebraically closed field, and set $u = (q-1)x\theta + 1$. Assume that $q \neq 1$ and that q is not a root of unity. Then*

$$\operatorname{Spec}(T) = \{0, uT\} \cup \{(x-\alpha)T + (\theta + (q-1)^{-1}\alpha^{-1})T \mid \alpha \in k - \{0\}\}.$$

There are many more prime ideals in $A_1(k, q)$ when q is a root of unity, since then the center is quite large. Note first that if q is a primitive nth root of unity for some $n > 1$, then $\delta(x^n) = 0$ by Lemma 8.1 and so x^n is central in $A_1(k, q)$. Moreover, it follows from Lemmas 7.3 and 6.2 that θ^n is central as well. Thus $A_1(k, q)$ is a finitely generated module over the central subring $k[x^n, \theta^n]$. In fact, the center of $A_1(k, q)$ is precisely $k[x^n, \theta^n]$ [3, Lemma 2.2]. (Cf. also [39, Proposition 1; 15, Theorem 2.10].)

THEOREM 8.6. *Let $T = A_1(k, q) = k[x][\theta; \sigma, \delta]$ where k is a field, and assume that q is a primitive nth root of unity for some $n > 1$.*

(I) *If $P \in \operatorname{Spec}(T)$ and $I = P \cap k[x]$, then one of the following cases must hold:*

(a) $I = 0$.

(b) $I = Q \cap \sigma(Q) \cap \cdots \cap \sigma^{n-1}(Q)$ *for some maximal ideal Q of $k[x]$ such that $x \notin Q$.*

(c) $I = x^n k[x]$.

(d) *I is a maximal ideal of $k[x]$ and $x \notin I$.*

(II) *If I is an ideal of $k[x]$ satisfying* (b) *or* (c) *above, then $IT \in \operatorname{Spec}(T)$. In case* (b) *holds and k is algebraically closed,*

$$T/IT \cong \{v \in M_n(k[z]) \mid v_{ij} \in zk[z] \text{ for all } i < j\}$$

for some indeterminate z. In case (c), *$T/IT \cong M_n(k[z])$ for some indeterminate z.*

Proof. We apply Theorem 3.1, with $R = k[x]$.

(I) Note that since xR is a σ-ideal but not a δ-ideal, $I \neq xR$. Thus $x \notin I$.

If I is a σ-prime (σ, δ)-ideal, then either (a) or (b) holds, while if I is a prime of R and $\sigma(I) \neq I$ then (d) holds.

Now assume that I is a δ-prime (σ, δ)-ideal and that R/I has a unique associated prime. Since the case $I = 0$ is covered, we may assume that $I \neq 0$. Then I is contained in a unique maximal ideal M of R, and since I is a

σ-ideal so is M. There is only one possibility: $M = xR$. Thus $I \subseteq xR$ and I contains a power of x. Since $x^n R$ is a δ-ideal (recall that $\delta(x^n) = 0$) and I is δ-prime, we obtain $x^n R \subseteq I$. In view of Lemma 8.1, $x^{m-1} \in \delta(x^m)R$ for all $m = 1, ..., n-1$, whence $x^{n-1} \notin I$. Therefore $I = x^n R$ in this case, and (c) holds.

(II) Suppose first that $I = Q \cap \sigma(Q) \cap \cdots \cap \sigma^{n-1}(Q)$ for some maximal ideal Q of $k[x]$ such that $x \notin Q$. Then I is a σ-prime ideal, and σ does not induce the identity on T/I. By Lemma 2.5, $\delta(I) \subseteq I$. Thus I is a σ-prime (σ, δ)-ideal, whence $IT \in \operatorname{Spec}(T)$.

If also k is algebraically closed, then

$$I = (x-\alpha)R \cap (qx-\alpha)R \cap \cdots \cap (q^{n-1}x-\alpha)R$$

for some nonzero $\alpha \in k$. In this case, $R/I \cong k^n$. Let f be a primitive idempotent of R/I. Since I is σ-prime and σ is k-linear, we see that $\sigma^i(f) \neq f$ for all $i = 1, ..., n-1$. The desired description of T/IT now follows from Corollary 7.2 (because $\sigma^n = 1$).

Finally, suppose that $I = x^n R$. Since $\sigma(x^n) = x^n$ and $\delta(x^n) = 0$, we see that I is a (σ, δ)-ideal. Moreover, $x^{m-1} \in \delta(x^m)R$ for all $m = 1, ..., n-1$ (as above), whence R/I is δ-simple and so I is δ-prime. Thus $IT \in \operatorname{Spec}(T)$ in this case also. Moreover,

$$T/IT \cong (R/I)[\theta; \sigma, \delta] \cong M_n(k[z])$$

by Proposition 7.5, because δ^n vanishes on R/xR. ∎

Observe that if $T = A_1(k[t, t^{-1}], t)$ for a field k and an indeterminate t, then for all nonzero $q \in k$, the element $t - q$ is central in T and $T/(t-q)T \cong A_1(k, q)$. Provided k contains primitive nth roots of unity for all n, it follows from Theorem 8.6 that T has factor rings isomorphic to $M_n(k[z])$ for each $n = 1, 2, ...$

In conclusion, we mention that $A_1(k, q)$ is not the only candidate for the title of *q-analog Weyl algebra*. Several authors—e.g., [22, 23, 33]—have proposed algebras which may be described as finite extensions of localizations of $A_1(k, q)$'s, in the following manner. Start with $A_1(k, q^{-r})$ for some field k (usually $\mathbb{C}$), some nonzero $q \in k$, and some integer $r \geqslant 2$. Then invert the normal element $\theta x - x\theta$, and adjoin an rth root z for $(\theta x - x\theta)^{-1}$ satisfying $z\theta = q^{-1}\theta z$ and $zx = qxz$. (The construction may be started with $A_1(k, q^r)$ instead; an isomorphic algebra results. Namely, there is an isomorphism of $A_1(k, q^r)[z]$ onto $A_1(k, q^{-r})[z]$ fixing x and z and sending θ to $z^{-r}\theta$.) The case $r = 4$ of this construction yields Hayashi's *q-Weyl algebra* $\mathcal{A}_q^-(1)$ [22, p. 131], while the case $r = 2$ yields the algebra A_q studied by Kirkman and Small [33, Lemma 1.3]. The latter case, but with q replaced by q^2, also yields the algebra that Hodges labels A_q [23, Sect. 3].

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