Computational Complexity of Ehrenfeucht–Fraïssé Games on Finite Structures

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Abstract. We show that deciding the winner of the r–moves Ehrenfeucht–Fraïssé game on two finite structures A and B, over any fixed signature Σ that contains at least one binary and one ternary relation, is PSPACE complete. We consider two natural modifications of the EF game, the one–sided r–moves EF game, where the spoiler can choose from the first structure A only, and therefore the duplicator wins only if B satisfies all the existential formulas of rank at most r that A satisfies; and the k–alternations r–moves EF game (for each fixed k), where the spoiler can choose from either structure, but he can switch structure at most k times, and therefore the duplicator wins iff A and B satisfy the same first order formulas of rank at most r and quantifier alternation at most k (defined in the paper). We show that deciding the winner in both the one–sided EF game and the k–alternations EF game is also PSPACE complete.

1 Introduction

Two structures A and B are r–equivalent ($A \equiv_r B$) if they satisfy the same first order formulas of quantifier depth at most r; A and B are $L^s_{\infty\omega}$–equivalent ($A \equiv^s_\infty B$) if they satisfy the same $L_{\infty\omega}$ (or even just first order, for finite structures) formulas with at most s variables (free or bound). Equivalently A and B are r–equivalent or $L^s_{\infty\omega}$–equivalent if the duplicator wins the r–moves Ehrenfeucht–Fraïssé game (EF game) or the s–pebbles pebble game on A and B, respectively. r–equivalence and $L^s_{\infty\omega}$–equivalence play an important role in logic and computer science, and in particular in descriptive complexity theory, which aims to characterize the queries in a given complexity class by means of a logic in which they can be described. To show that a class K of structures is not axiomatizable in first order logic (or in $L_{\infty\omega}$), in fact, it is sufficient to show that for every r (s) there are structures $A \in K$ and $B \notin K$, with $A \equiv_r B$ ($A \equiv^s_\infty B$). See for example [1]. We are interested in determining the complexity of deciding whether two finite structures are r–equivalent, and we will call this the Ehrenfeucht–Fraïssé problem (EF problem), or $L^s_{\infty\omega}$–equivalent, and we will call this the pebble problem. If the number r of moves is fixed then it is easy to see that the EF problem is in LOGSPACE. Martin Grohe [4] has shown that, if the number s of pebbles is fixed, the pebble problem is complete for PTIME. If r and s are part of the input, the exact complexity of both the EF problem and

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the pebble problem was so far open. It is easy to see that the EF problem is in PSPACE, and that the pebble problem is in EXPTIME, but neither had been proved complete yet. In this paper we show that the EF problem is complete for PSPACE. We also consider two natural modifications of the EF problem on two finite structures A and B: the one-sided EF problem, where the spoiler can choose from the first structure A only, and therefore the duplicator wins only if B satisfies all the existential formulas of rank at most r that A satisfies; and the k–alternations r–moves EF game (for each fixed k), where the spoiler can choose from either structure, but he can switch structure at most k times, and therefore the duplicator wins iff A and B satisfy the same first order formulas of rank at most r and quantifier alternation (defined below) at most k, and show that they are also PSPACE complete. One can view these EF problems and the pebble problem as an approximation to the structure isomorphism problem [6], whose complexity is very problematic; instead of asking whether two finite structures are isomorphic, that is whether they agree on all first order formulas, we change the class of formulas they have to agree on, and consider the complexity of the corresponding problem.

In this paper we will provide only sketches of the proofs. Full proofs will appear elsewhere [2],[3].

2 Background

We give here the definition of Ehrenfeucht–Fraïssé game, one sided Ehrenfeucht–Fraïssé game, and k–alternations Ehrenfeucht–Fraïssé game.

Definition 1. *Let* (G, H) *be a pair of structures over a given vocabulary, and* $\boldsymbol{\gamma}$, $\boldsymbol{\theta}$ *strings of elements in* G *and* H *respectively ; the* r–moves EF game (EF game) *on* $(G, \boldsymbol{\gamma}, H, \boldsymbol{\theta})$ *is played by two players called the* spoiler *and the* duplicator*. Each player has to make* r *moves in the course of the play. The players take turns. In the* i^{th} *round the spoiler selects one of the structure* G *or* H *and an element from that structure; the duplicator answers by choosing an element from the structure not chosen by the spoiler. At the end of* r *rounds the duplicator wins iff* $(\boldsymbol{\gamma}, g_1, \ldots g_r) \to (\boldsymbol{\theta}, h_1, \ldots h_r)$ *is a partial isomorphism from* G *to* H*, where* $g_1, \ldots, g_r$ *are the elements chosen from* G *by either the spoiler or the duplicator in round* $1, \ldots, r$ *and* $h_1, \ldots h_r$ *are the elements chosen from* H*. Otherwise the spoiler wins.*

The connection with logic comes from the following:

Definition 2. *The* quantifier rank *(or simply* rank*) of a first order formula* ϕ *is the maximum number of nested quantifiers in it, defined by induction by:*

1. $qr(\phi) = 0$ *if* ϕ *is atomic.*
2. $qr(\neg\phi) = qr(\phi)$.
3. $qr(\phi \vee \psi) = qr(\phi \wedge \psi) = \max\{qr(\phi), qr(\psi)\}$.
4. $qr(\exists x\psi) = qr(\forall x\phi) = 1 + qr(\phi)$.

Theorem 1. *The duplicator wins the r–moves EF game on $(G, \boldsymbol{\gamma}, H, \boldsymbol{\theta})$ iff $\boldsymbol{\gamma}$ and $\boldsymbol{\theta}$ satisfy the same formulas of rank at most r, that is iff*

$$G \models \phi(\boldsymbol{\gamma}) iff H \models \phi(\boldsymbol{\theta})$$

for all ϕ of quantifier rank at most r.

For a proof see [1].
We also want to consider the following game:

Definition 3. *Let G, H, $\boldsymbol{\gamma}$, $\boldsymbol{\theta}$ be as in Definition 1. The* r–moves one–sided EF game *on $(G, \boldsymbol{\gamma}, H, \boldsymbol{\theta})$ is played like the EF game, except that the spoiler can choose from structure G only. The winning condition is as in the EF game.*

The only difference between the one–sided EF game and the ordinary EF game is that in the one–sided EF game the spoiler is restricted to choose from the first input structure only. It is possible to give a logical charcterization of the one–sided EF game in the spirit of Theorem 1.

Definition 4. Existential formulas *are defined by the clauses:*

1. *All quantifier free formulas are existential.*
2. *If ϕ and ψ are existential then $\phi \vee \psi$ and $\phi \wedge \psi$ are existential.*
3. *If ϕ is existential then $\exists x \phi$ is existential*

Theorem 2. *The duplicator wins the n–moves one–sided EF game on (G, γ, H, θ) iff θ satisfies all the existential formulas of rank $\leq n$ that γ satisfies.*

Proof: the proof is similar to that of Theorem 1.
A generalization of the one–sided EF game consists in allowing the spoiler to choose from either structure, but with a bound k on the number of times he alternates, that is chooses in one round from a different structure than the previous round.

Definition 5. *Let k be a natural number. Let G, H, $\boldsymbol{\gamma}$, $\boldsymbol{\theta}$ be as in Definition 1. The* k alternations r–moves one–sided EF game *on $(G, \boldsymbol{\gamma}, H, \boldsymbol{\theta})$ is played like the EF game, except that the spoiler is allowed at most k alternations. The winning condition is as in the EF game.*

Again it is possible to relate this game to logic.

Definition 6. *The* external quantifier set and alternation number *of a first order formula are defined by induction as follows:*

1. *The external quantifier set of an atomic formula is empty; the alternation number is 0.*
2. *The external quantifier set Q_ξ of a formula $\xi = \phi\, b\, \psi$, where $b = \wedge$ or $\vee$ is equal to $Q_\phi \cup Q_\psi$; the alternation number is the maximum of the alternation numbers of ϕ and ψ.*

3. *The external quantifier set of $\neg\phi$ is obtained by replacing $\exists$ with $\forall$ and vice versa in the external quantifier set of ϕ; the alternation number is the same as that of ϕ.*
4. *The external quantifier set of $\exists x\, \phi$ is $\{\exists\}$; the alternation number is the same as that of ϕ if Q_ϕ does not contain $\forall$, and it is one plus the alternation number of ϕ otherwise.*
5. *The external quantifier set of $\forall x\, \phi$ is $\{\forall\}$; the alternation number is the same as that of ϕ if Q_ϕ does not contain $\exists$, and it is one plus the alternation number of ϕ otherwise.*

Theorem 3. *The duplicator wins the r-moves, k alternation EF game on $(G, \boldsymbol{\gamma}, H, \boldsymbol{\theta})$ iff $\boldsymbol{\gamma}$ and $\boldsymbol{\theta}$ satisfy the same formulas of rank at most r and alternation number at most k.*

Proof: A proof is given in [2]

Definition 7. *Given a fixed signature Σ, we will call the EF problem (for Σ), the one sided EF problem (for Σ) and the k–alternations EF problem (for Σ) the following problems, respectively: given as input two finite structures A and B over Σ, and a number r, determine who wins the r–moves EF game, one–sided EF game, k–alternations EF game on A and B.*

3 Complexity of the EF problem

It is easy to show that the EF problem is in PSPACE (for any signature Σ, actually even if Σ is part of the input); our goal here is to show hardness for PSPACE. We will use a reduction from the following game theoretic version of Quantified Boolean Formula (QBF).

Definition 8. The Quantified Boolean Formula game *is played by two players, I and II; on input a formula of the form $\phi = \exists x_1 \forall x_2 \ldots \exists x_{2r-1} \forall x_{2r} (C_1 \wedge \ldots \wedge C_n)$ the game continues for r rounds. In round i player I chooses a truth assignment for variable x_{2i-1} and then player II chooses a truth value for x_{2i}. At the end of the r rounds, player I wins iff the assignment that has been produced makes $C_1 \wedge \ldots \wedge C_n$ true.*

We call *the QBF problem* the problem of deciding who wins the QBF game on a certain input formula.

Theorem 4 ([7]). *The QBF problem is PSPACE complete.*

Theorem 5. *The EF problem (for finite structures over any fixed signature Σ that contains at least one binary and one ternary relation) is PSPACE complete.*

The plan of the proof is to show that the QBF problem reduces to the EF problem. Given any quantified boolean formula of the form

$$\phi = \exists x_1 \forall x_2 \ldots \exists x_{2r-1} \forall x_{2r} (C_1 \wedge \ldots \wedge C_n)$$

we will show how to construct structures A and B over $\Sigma = \{E, H\}$, where E is a binary relation and H is a ternary relation, such that player I wins the QBF game on input ϕ iff the spoiler wins the $2r + 1$–moves EF game on (A, B). The main difficulty originates from the fact that in each round of the QBF game player I and II can assign a truth value to one given variable only, while the spoiler and the duplicator have much more freedom and the spoiler ,in particular, in an y round of the EF game can choose any vertex he wants. The proof will proceed by first imposing additional constraints on the spoiler and the duplicator, for example requiring them in each round to choose from a certain subset of the vertices of A and B only, as explained below, so that the EF game with constraints will closely reflect the QBF game. Then it will be easy to show that player I wins QBF game on input ϕ iff the spoiler wins the $2r+1$–moves EF game with constraints on (A, B). To complete the proof it will be necessary to show that the first player that does not respect the constraints is going to lose the EF game. Both structures A and B consist of r blocks; the idea is that choosing some of the vertices in block i of A or B in a move of the EF game corresponds to assigning truth value T or F to variable x_i or to variable x_{i+1} of ϕ, (we will label such vertices by $T(x_i)$,$F(x_i)$,$T(x_{i+1})$ or $F(x_{i+1})$ or sometimes simply by T and F), choosing other vertices in block i will correspond to recording that a certain truth value has been assigned to x_i in a previous move, and to choosing a truth value for x_{i+1}, (we will label such elements by $TF(x_i x_{i+1}), TT(x_i x_{i+1}), \ldots$); then there are also vertices in block i that do not correspond to any truth assignment to the variables of ϕ (we will denote any such vertex simply by v*). We then consider constraining the spoiler and the duplicator to play from block i in round i and $i + 1$, i odd, and in the following way:

round i	round $i + 1$	
$s : T(x_i)$	$d : F(x_i)$	A
$d : TF(x_i x_{i+1})$	$s : v*$	B

This means that in round i the spoiler must first assign a truth value to variable x_i (by choosing an element $T(x_i)$ or $F(x_i)$ in block i of structure A), duplicator must record the spoiler's assignment and assign a truth value to variable x_{i+1} (by choosing an element $TT(x_i x_{i+1})$ or $TF(x_i x_{i+1})$ or $FT(x_i x_{i+1})$ or $FF(x_i x_{i+1})$ in block i of structure B); then the spoiler must play some vertex $v*$, in block i of structure B, which does not correspond to a truth assignment, and the duplicator must record the truth assignment to variable x_{i+1} in structure A as well (by choosing some element $T(x_{i+1})$ or $F(x_{i+1})$ in block i of A). At the end of the first $2r$ rounds played in this fashion, a truth assignment of the variables of ϕ has been determined by the two players of the EF game. In the last move the spoiler will have a chance to win iff the assignment makes ϕ true. The main difficulty is so as to ensure that the spoiler loses if he does not follow the rules.

In the next section we will construct preliminary structures $\bar{A_k}$ and $\bar{B_k}$ over a signature containing a binary relation only, and prove some useful facts about them. Structures A and B will be obtained by introducing a ternary relation on the vertices of $\bar{A}_k$ and $\bar{B}_k$ (after some minor modification). $\bar{A}_k$ and $\bar{B}_k$ consist of r blocks; the blocks are gadgets I_j (see Figure 3), introduced in the next section, which are in turn built out of other gadgets J_m and L_n.

We will say that in the s–moves EF game on two structures, *the spoiler forces a pair* (y, y') if he can play y (or y') and the duplicator must answer with y' (or y) not to lose the game.

3.1 The structures $\bar{A}_k$ and $\bar{B}_k$

We first need to introduce gadgets J_k and L_k shown in Fig. 1. They are somehow similar to gadgets used in [5] and [4]. An edge between a vertex v and the symbol k or $k-1$ means that v has k or $k-1$ additional neighbours, not shown in the figure. They will be called *special neighbours*. Special neighbours of distinct vertices are all distinct. Vertices with special neighbours will be called *vertices in the middle*. Then gadget I_k is built using gadgets J_k and L_k as follows (see Fig. 2): I_k has vertices x, x', y, y'; x and x' have each 16 neighbours; again we will call them *vertices in the middle*; the vertices in the middle have k or $k-1$ additional neighbours (besides x or x'); again they will be called *special neighbours*; in addition each vertex v in the middle is glued to a separate copy of a gadget J_{k-1} or L_{k-1}, disjoint from all others, so that v coincides with z and y with t and y' with t' as follows:

1. Eight of the vertices in the middle connected with x have k special neighbours; four of them are glued to an L_{k-1} gadget, and four to an J_{k-1}.
2. Eight of the vertices in the middle connected with x have $k-1$ special neighbours; four are glued to a L_{k-1} gadget, and four to a J_{k-1} gadget.
3. Four of the vertices in the middle connected with x' have k special neighbours, and they are glued to a L_{k-1} gadget.
4. The remaining 12 vertices in the middle connected with x' have $k-1$ special neighbours; four of them are glued to a L_{k-1} gadget, and eight to a J_{k-1} gadget.

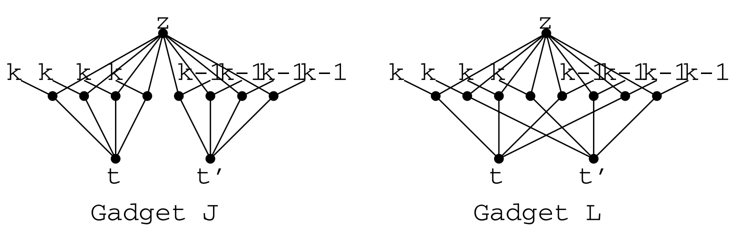

Fig. 1. The gadgets J_k and L_k

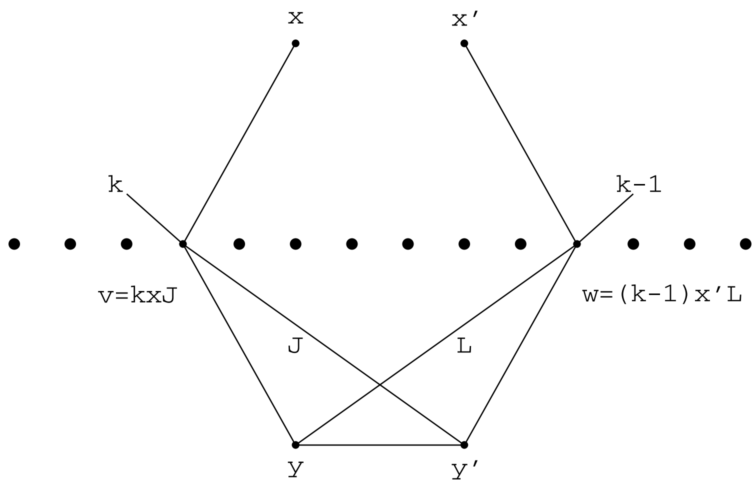

Fig. 2. The gadget I_k

We will denote a vertex in the middle of gadget I J and L by giving its neighbours, or the gadget it is glued to; so, for example, if v is a vertex in the middle of gadget I_k, $v = kxJ$ means that v is any of the 4 vertices with k special neighbours and glued to an J_{k-1} gadget; $(k-1)vy$ stands for a vertex in the middle of gadget J, having $k-1$ special neighbours and connected to v and y. We need the following lemma:

Lemma 1. *In the $k+1$–moves EF game on $(I_k,\ x, I_k,\ x')$, the spoiler can force the pair (y, y'), but the duplicator has a strategy to win the k– moves EF game that allows him to answer y with y and y' with y'.*

Proof:(sketch) In the $k+1$–moves game the spoiler can start by playing $v = kxJ$ and the duplicator must answer with $w = kx'L$, then the spoiler can select $w(k-1)y'$ in gadget L_{k-1} and the duplicator must answer $v(k-1)y$ in gadget J_{k-1}. With only k moves the duplicator can follow a partial isomorphism that maps x to x', y to y and y' to y'. The only problem maybe if the spoiler plays all k special neighbours of, say, some kxJ; but then the duplicator can play all $k-1$ neighbours of $(k-1)x'J$ and any additional vertex not connected to x'.

Definition 9. *Let k be even and let $\bar{A}_k$ consist of $k/2$ copies of I gadgets, I_k, I_{k-2}, $I_{k-4}, \ldots, I_2$ with the y (y') vertex of the i^{th} gadget coinciding with the x (x') vertex of the $(i+1)^{th}$ gadget, plus an additional vertex connected to the x vertex of the first gadget I_k, as shown in figure 3, and $\bar{B}_k$ be the same as $\bar{A}_k$ except that it has an additional vertex connected to x'_1 and not x_1.*

We will say that gadget $I_{k-2(i-1)}$ is the i^{th} block of $\bar{A}_k$ or $\bar{B}_k$. Now we want to consider the $k+1$–moves EF game on $\bar{A}_k$ and $\bar{B}_k$, and show that the spoiler can

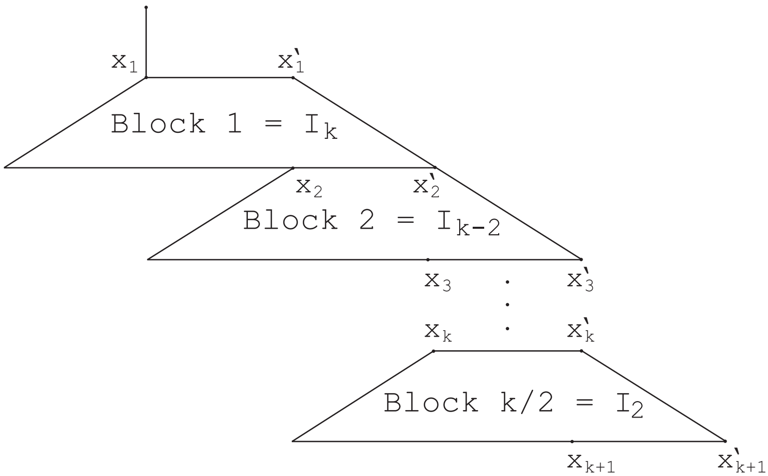

Fig. 3. The structure $\bar{A}_k$

force the pair (x_{k+1}, x'_{k+1}) only if he follows a precise strategy of first choosing certain elements in block 1, then certain elements in block 2, and so on. Once we have constructed structures A and B form $\bar{A}$ and $\bar{B}$ this strategy will correspond to assigning truth value to the variables of ϕ, and so on as explained at the end of Section 2.

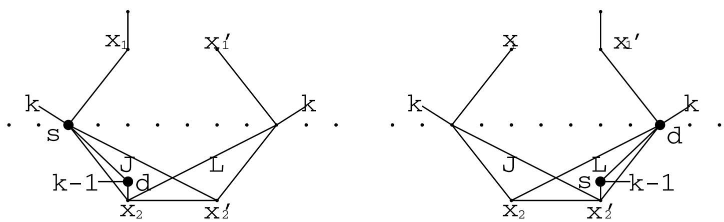

Fig. 4. The first two lawful moves

Definition 10. *Consider the $k+1$–moves EF game on $\bar{A}_k$ and $\bar{B}_k$; we will say that* the players play lawfully *if in round i, $i < k+1$ and i odd, the spoiler chooses a vertex v in the i^{th} block, $v = (k-2(i-1))x_iJ$ (or a special neighbour of v) from the middle of gadget $I_{k-2(i-1)}$ and the duplicator answers with some $w = (k-2(i-1))x'_iL$ (or a special neighbour of w). In the next round the spoiler plays lawfully if he chooses vertex $(k-2(i-1)-1)wx'_{i+1}$ (or a special neighbour) or vertex $(k-2(i-1)-2)wx_{i+1}$ in gadget L (or a special neighbour), and the* *duplicator plays lawfully if he plays vertex* $(k-2(i-1)-1)vx_{i+1}$ *(if the spoiler has played* $(k-2(i-1)-1)wx'_{i+1}$ *or a special neighbour if the spoiler has played a special neighbour of* $(k-2(i-1)-1)wx'_{i+1}$*) or vertex* $(k-2(i-1)-2)vx'_{i+1}$ *(if the spolier has played* $(k-2(i-1)-2)wx_{i+1}$ *or a special neighbour if the spoiler has played a special neighbour of* $(k-2(i-1)-2)wx_{i+1}$*) in gadget* J*; (See Fig 4, where the spoiler's moves are marked s, the duplicator 's d.) In the last round the spoiler plays lawfully if he chooses* x_{k+1} *or* x'_{x+1}*, from either* A *or* B*, and the duplicator if he plays* x'_{k+1} *(if the spoiler has played* x_{k+1}*) or* x_{k+1} *(if the spoiler has played* x'_{k+1}*) from* B *or* A*.*

Theorem 6. *In the the* $k+1$*–moves EF game on* $\bar{A}_k$ *and* $\bar{B}_k$ *we have that if the spoiler plays lawfully, he can force a pair* (x_{k+1}, x'_{k+1})*; but if he does not play lawfully the duplicator has a strategy to win the game and answer* x_{k+1} *with* x_{k+1} *and* x'_{k+1} *with* x'_{k+1}*.*

Proof(sketch): let move j be the first unlawful move of the spoiler. Then for the rest of the game the duplicator can play according to the partial isomorphism that maps x_j to x'_j and x_l to x_l, x'_l to x'_l for $l > j$.

3.2 The Main Theorem

We are now ready to describe the structures A and B. Recall that our goal is to show that player I wins the QBF game on input

$$\phi = \exists x_1 \forall x_2 \ldots \exists x_{2r-1} \forall x_{2r} (C_1 \wedge \ldots \wedge C_n).$$

iff the spoiler wins the $2r+1$–moves EF game on (A, B).

A and B are obtained by introducing a ternary relation H on the vertices of structures $\bar{A}_k$ and $\bar{B}_k$ (after minor modifications), with $k = 2r$. In order to motivate the definition of H, consider the $k+1$–moves EF game on $\bar{A}_k$ and $\bar{B}_k$. For simplicity , consider a *lawful strategy* defined as in Definition 10, except that the players must play vertices in the middle, and not special neighbours. If the players play according to such a lawful strategy a run of the game may look like:

$$\begin{array}{llllll} s : T(x_1) & d : F(x_2) & \ldots & s : F(x_{2r-1}) & d : F(x_{2r}) & d : x'_{k+1} \\ d : TF(x_1 x_2) & s : v* & \ldots & d : FF(x_{2r-1} x_{2r}) & s : v* & s : x_{k+1} \end{array}$$

That is, the spoiler has first chosen an element in the middle of gadget I_k labelled T, the duplicator has answered with an element labelled TF , then the spoiler has chosen an element with no label, and so on. The first k (lawful) rounds determine a truth assignment for the variables of ϕ. Of course the duplicator wins the game on $\bar{A}_k$ and $\bar{B}_k$, while we want the duplicator to win a run of the game iff the truth assignment determined by the first k rounds of the run does not make ϕ true. We will achieve this by taking advantage of the fact that in the last move the spoiler can force a pair (x_{k+1}, x'_{k+1}). We first replace x_{k+1} and x'_{k+1} with two sets $V_{x_{k+1}}$ and $V_{x'_{k+1}}$, each having $k+1$ new vertices labelled C_1, $k+1$ new vertices labelled C_2,...,$k+1$ new vertices labelled C_n, where $C_1, \ldots, C_n$

are the clauses of ϕ (again, no unary relation corresponding to these labels are present in the signature).$V_{x_{k+1}}$ has also an additional new vertex $w*$, with no label. There are no edges between vertices in the same set $V_{x_{k+1}}$ or $V'_{x_{k+1}}$, and there is an edge between any vertex $v \in V_{x_{k+1}}$ ($V_{x'_{k+1}}$) and any vertex w that was connected to x_{k+1} (x'_{k+1}) in $\bar{A}_k$ or $\bar{B}_k$. So we have:

- $V_{x_{k+1}} = \{w*, C_1, \ldots, C_1, \ldots, C_n, \ldots, C_n\}$.
- $V_{x'_{k+1}} = \{C_1, \ldots, C_1, \ldots, C_n, \ldots, C_n\}$.

Definition 11. *Let C_k* and *D_k be the structures obtained by replacing x_{k+1} and x'_{k+1} with the sets $V_{x_{k+1}}$ and $V_{x'_{k+1}}$, as described above.*

On C_k and D_k a lawful run of the $k+1$–moves EF game may look like:

$$\begin{array}{llllll} s : T(x_1) & d : F(x_2) & \ldots & s : F(x_{2r-1}) & d : F(x_{2r}) & d : C_j \\ d : TF(x_1x_2) & s : v* & \ldots & d : FF(x_{2r-1}x_{2r}) & s : v* & s : w* \end{array}$$

In the last round the spoiler has played an unlabelled element $w*$ from set $V_{x_{k+1}}$ and the duplicator has responded with an element of $V_{x'_{k+1}}$, that is has exhibited a clause C_j of ϕ. The duplicator must lose the run if C_j is not falsified by the assignment. To ensure this, we add a ternary relation H; no triple $(*, *, w*)$ is in H, but for example we will have $H(T(x_1), F(x_2), C_j)$ iff assigning T to x_1 and F to x_2 in ϕ makes clause C_j true. For the sake of exposition in introducing ternary relation H we will first label explicitly some of the vertices in A_k and B_k with the labels T,F,TT,TF,FT,FF; this is just to facilitate the exposition and introduce ternary relation H below; no unary relations are part of the signature Σ. First we label the vertices in the middle of gadget I_i and gadgets J_{i-1} and L_{i-1} for each $i = k$, $k-2, \ldots, 2$.

1. Of the four vertices ixJ, two are labelled T and the other two F; of the four vertices $(i-1)xJ$, ixL, $(i-1)xL$,or $ix'J$,$(i-1)x'J$, $(i-1)x'L$, one is labelled TT, one TF, one FF, one FT.
2. Of the four verices in the middle of any gadget J_{i-1} with $i-1$ special neighbours, or $i-2$ special neighbours, two are labelled T and two F;
3. In gadget L_{i-1} the two vertices $(i-1)zt'$ and the two vertices $(i-2)zt$ are not labelled; of the two remaining verices $(i-1)zt$ and the two $(i-2)zt'$, one is labelled T and the other F.

We will say that two vertices v and w are consecutive in block i iff v is a vertex in the middle of gadget $I_{k-2(i-1)}$ and w is a vertex in the middle of the L or J gadget glued to v.

Definition 12. *Relation H is defined as follows on the vertices of C_k and D_k:*

- *$H(u, v, C_j)$ iff u is labelled a (a =T or F) v is labelled b, u and v are consecutive in block i, and assigning x_i to a and x_{i+1} to b makes clause C_j true in ϕ.*
- *$H(w, v*, C_j)$ w is labelled ab (a, $b = T$ or F), w and $v*$ are consecutive in block i and assigning x_i to a and x_{i+1} to b makes clause C_j true in ϕ.*

– $H(z, v, C_j)$ *z is labelled ac, v is labelled b, z and v are consecutive in block i and assigning* x_i *to a and* x_{i+1} *to b makes clause* C_j *true in* ϕ.

Definition 13. *Structures* A and B *are obtained from* C_k *and* D_k *by introducing ternary relation* H.

Now the proof of Theorem 5 that is the proof that the spoiler wins the $k+1$–moves EF game on A and B iff player I wins the QBF game on ϕ proceeds as follows; we have to consider three cases:

– The spoiler plays lawfully: he wins the game iff ϕ is true.
– The spoiler does not play lawfully, but he can still force pair (x_{k+1}, x'_{k+1}) in the last move; the spoiler has not gained anything by not playing lawfully, he could have played lawfully.
– The spoiler does not play lawfully and cannot force pair (x_{k+1}, x'_{k+1}). In this case the duplicator can follow a winning strategy for the game on $\bar{A}_k$ and $\bar{B}_k$ as in Theorem 6. The only thing we need to check is that the relation H does not cause any problem.

Question: Can we eliminate relation H and show that the EF problem is complete if played on structures over a signature containing a binary relation only?

4 The one–sided and the k–alternations EF games

We show here that the one–sided EF game and the k–alternations EF game are also PSPACE complete.

Theorem 7. *The one–sided EF game is PSPACE complete.*

Proof (sketch): We just show hardness. We reduce the r–moves EF game on structures A and B over some signature Σ to the $r+n$-moves one–sided EF game on structures C and D, on a signature $\Sigma' = \Sigma \cup \{R, B\}$ where R and B are new unary relations, and $n = |V_A| + |V_B|$.(here V_A , V_B are the sets of vertices of structures A and B) Order the elements of $V_A \cup V_B$, so that $V_A = \{1, \ldots, m\}$ and $V_B = \{m+1, \ldots, m+k\}$ for some m and k,then:

– For any $x \in V_A \cup V_B$ there is an element $\bar{x}$ in V_C with x additional new neighbours colored R and n additional new neighbours colored B. V_C also contains many elements with x new neighbours colored R and $n-1$ neighbours colored B. All relations except for R and B are empty in C.
– For any element $(x, y) \in V_A \times V_B \cup V_B \times V_A$ there is a vertex $(\overline{x, y})$ in V_D with x new neighbours colored R and n new neighbours colored B. V_D also contains many elements with x new neighbours colored R and $n-1$ neighbours colored B. We have $E(\overline{a, b})(\overline{c, d})$ iff Exy in A but not Ezt in B, or vice versa; where x is the cohordinate a or b of the pair (a, b) that belongs to A, and y is the one that belongs to B. Similarly for all other relations in Σ.

Theorem 8. *The one–sided EF game reduces to the k alternations EF game , for any* $k \geq 0$.

A proof is given in [2].

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