ON THE PROFINITE TOPOLOGY ON A FREE GROUP

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ABSTRACT

If F is a free abstract group, its profinite topology is the coarsest topology making F into a topological group, such that every group homomorphism from F into a finite group is continuous. It was shown by M. Hall Jr that every finitely generated subgroup of F is closed in that topology. Let $H_1, H_2, \ldots, H_n$ be finitely generated subgroups of F. J.-E. Pin and C. Reutenauer have conjectured that the product $H_1 H_2 \ldots H_n$ is a closed set in the profinite topology of F; also, they have shown that this conjecture implies a conjecture of J. Rhodes on finite semigroups. In this paper we give a positive answer to the conjecture of Pin and Reutenauer. Our method is based on the theory of profinite groups acting on graphs.

Introduction

Let F be a free abstract group. Let $\mathcal{N}$ be the set of all normal subgroups N of F of finite index. Then $\mathcal{N}$ can be considered as a basis of neighbourhoods of the identity element of F that determines a topology for F making it into a topological group. This is the so-called profinite topology or Hall topology for the group F. Properties of F related to this topology were studied by Marshall Hall Jr (see [**5**]).

In [**5**, p. 429], Hall proves that if H is a finitely generated subgroup of F, then H is the intersection of those subgroups of F of finite index that contain H; or, equivalently (see [**4**, p. 131]), that H is closed in the profinite topology of F. In connection with this result of Hall, J.-E. Pin and C. Reutenauer (see [**7**]) raise the following conjecture: let $H_1, H_2, \ldots, H_n$ be a finite sequence of finitely generated subgroups of F; then the subset $H_1 H_2 \ldots H_n$ (the product of the groups $H_1, H_2, \ldots, H_n$ in F) is closed in the topology of F defined above. As is shown in [**7**] and [**6**], a positive answer to this conjecture implies in turn a positive answer to a conjecture of J. Rhodes on the existence of a certain algorithm for finite semigroups.

In this paper we prove the conjecture of Pin and Reutenauer (in fact, a slightly more general version of it; see Theorem 2.1) using the theory of profinite groups acting on boolean graphs developed in [**3**] and [**11**].

1. *Notation and preliminaries*

For the benefit of the reader, we collect in this section some of the basic concepts and facts about profinite groups and graphs in a profinite context. For more details on profinite groups, see [**9**], [**2**] or [**8**], for example; more information on the theory of profinite groups acting on (boolean) graphs can be found in [**3**] and [**11**].

Let G be a residually finite group and $\mathcal{N}$ the set of normal subgroups of G of finite index. Consider the topology on G such that for $g \in G$, a fundamental system of

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neighbourhoods of g consists of the cosets gN, for $N \in \mathcal{N}$. This makes G into a topological group, with a Hausdorff topology, the so-called *profinite* topology on G. The *completion* $\hat{G} = \varprojlim G/N$, where N runs through $\mathcal{N}$, is a *profinite group*, that is, a compact, Hausdorff, totally disconnected topological group. Then G is canonically embedded as a subgroup of $\hat{G}$ by the map $g \to (gN)$. If X is a subset of the topological group $\hat{G}$, we denote by $\bar{X}$ its topological closure in $\hat{G}$. The profinite topology on G is precisely the one induced by the topology of $\hat{G}$. We restate this in the following result.

LEMMA 1.1. *Let G be a residually finite group, K be a subset of G and $\bar{K}$ be the closure of K in $\hat{G}$. Then K is closed in the profinite topology of G if and only if $\bar{K} \cap G = K$.*

COROLLARY 1.2. *Let F be a free abstract group and K a finitely generated subgroup of F. Then $\bar{K} \cap F = K$.*

Proof. By the result of Hall mentioned in the Introduction (see [**4**, p. 131]), K is closed in the profinite topology of F.

Next we recall the basic terminology and establish some results on groups acting on trees, in a profinite context; see [**3**] and [**11**] for details. A *boolean space* (or profinite space) X is a compact, Hausdorff, totally disconnected topological space. Such a space is a projective limit of finite discrete spaces. We say that a profinite group G *acts* on a boolean space X if it acts on it continuously. For the needs in this paper, we follow the terminology in [**3**], and we define an (oriented) *boolean graph* Γ to consist of two boolean spaces, $V = V(\Gamma)$, the *space of vertices*, and $E = E(\Gamma)$, the *space of edges*, and two continuous maps $d_0, d_1 : E \to V$ (for $e \in E$, $d_1(e)$ and $d_2(e)$ are the *initial* and *end* points of the edge e, respectively). A profinite group G acts on the boolean graph Γ if it acts on the spaces V and E in such a way that the maps d_0 and d_1 become G-maps. A boolean graph Γ can be expressed, in a natural way, as a projective limit of finite graphs. If this can be done in such a way that these finite graphs are connected (in the usual sense of abstract graphs), one says that the boolean graph Γ is *connected.*

To explain the notion of a boolean tree, we need to introduce some additional notation. Let $\hat{\mathbb{Z}}$ be the profinite completion of the group of integers $\mathbb{Z}$. (Observe that $\hat{\mathbb{Z}}$ is a topological ring; in fact, $\hat{\mathbb{Z}} \approx \Pi \mathbb{Z}_p$, where p runs through the set of prime numbers, and $\mathbb{Z}_p$ denotes the ring of p-adic integers.) For a finite set T of cardinality t, define $\hat{\mathbb{Z}}[T]$ to be the direct sum of t copies of $\hat{\mathbb{Z}}$; then $\hat{\mathbb{Z}}[T]$ is an abelian profinite group, the so-called profinite abelian free group of rank t. If $S = \varprojlim S_i$ (where each S_i is a finite discrete space) is a boolean space, then the groups $\hat{\mathbb{Z}}[S_i]$ form, in a natural way, a projective system of abelian profinite groups, and we define

$$\hat{\mathbb{Z}}[[S]] = \varprojlim \hat{\mathbb{Z}}[S_i].$$

It is not hard to see that $\hat{\mathbb{Z}}[[S]]$ is well-defined, and that S is naturally embedded in $\hat{\mathbb{Z}}[[S]]$. In fact, $\hat{\mathbb{Z}}[[S]]$ is characterized by the following universal property: whenever A is an abelian profinite group and θ: $S \to A$ is a continuous mapping, there exists a unique continuous homomorphism $\bar{\theta}$: $\hat{\mathbb{Z}}[[S]] \to A$ extending θ. Next, let Γ be a non-empty boolean graph with incidence maps d_0, d_1, and with vertex space V and edge space E. Consider the following sequence of abelian profinite groups and homomorphisms:

$$0 \longrightarrow \hat{\mathbb{Z}}[[E]] \xrightarrow{d} \hat{\mathbb{Z}}[[V]] \xrightarrow{\varepsilon} \hat{\mathbb{Z}} \longrightarrow 0, \tag{$*$}$$

where d and ε are the continuous homomorphisms defined by $d(e) = d_1(e) - d_0(e)$ for each $e \in E$, and $\varepsilon(v) = 1$ for each $v \in V$. One says that Γ is a *boolean tree* if the above sequence is exact. Observe that this is a natural definition, for it extends the geometric notion of an abstract tree: if Γ is an abstract oriented graph with sets of edges and vertices E and V respectively, and we substitute for $\hat{\mathbb{Z}}$ by, for example, $\mathbb{Z}$, and one thinks of $\hat{\mathbb{Z}}[[E]]$ and $\hat{\mathbb{Z}}[[V]]$ as the free abstract abelian groups on E and V respectively, then Γ is an abstract tree if and only if the above short sequence is exact (see [**1**, Chapter 1, Lemmas 6.3 and 6.4]). Also, it is easily checked that a boolean graph is connected (that is, it is a projective limit of finite connected graphs) if and only if the sequence $(*)$ is exact at $\hat{\mathbb{Z}}[[V]]$.

For example, a finite tree, in the usual abstract sense, is also a boolean tree. Moreover, the projective limit of finite trees is a boolean tree, as it easily follows from the above definition and the fact that the functor $\varprojlim$ preserves exactness (see [**8**, Proposition 3.6, p. 35]). However, not every boolean tree is a projective limit of finite trees. (For example, consider the Cayley graph $\Gamma = \Gamma(\hat{\mathbb{Z}}, \{1\})$ of the free profinite group of rank 1 with respect to its basis $\{1\}$, as described in the next paragraph; then Γ is a boolean tree, but its finite quotient graphs contain cycles necessarily.)

Next we describe a more substantial example of a boolean tree that, in fact, will be needed for the understanding of the rest of this paper. Let F be an abstract free group on a finite basis B, and let $\hat{F}$ be its profinite completion; then $\hat{F}$ is a free profinite group on the basis B. Define the vertex space V of a boolean graph $\Gamma(\hat{F})$ (the *Cayley graph* of $\hat{F}$ with respect to B) to be the space $\hat{F}$, and the edge space E to be the cartesian product $E = \hat{F} \times B$; finally, define the incidence maps $d_0, d_1 : E \to V$ by $d_0(f, b) = f$ and $d_1(f, b) = fb$, where $f \in \hat{F}$ and $b \in B$. Then $\Gamma(\hat{F})$ is a boolean tree (see [**3**, Theorem 1.2]). It should be noted that there is a natural *left* action of $\hat{F}$ on $\Gamma(\hat{F})$: $\hat{F}$ acts on V by left multiplication, and on E by left multiplication on the first component: $f(f', b) = (ff', b)$, where $f, f' \in \hat{F}$ and $b \in B$.

Next we state a result that will be used several times in this paper. The proof is a straightforward consequence of the definition of a boolean tree.

LEMMA 1.3 (Proposition 1.18 in [**11**]).

(a) *Let $\{\Delta_i \mid i \in I\}$ be a collection of boolean subtrees of a boolean tree T with non-empty intersection. Then $\bigcap_{i \in I} \Delta_i$ is a boolean tree.*

(b) *If T_1 and T_2 are boolean subtrees of a boolean tree T with a common vertex, then $T_1 \cup T_2$ is also a boolean subtree of T.*

Let Γ be a boolean tree. If $x, y \in V(\Gamma)$, then, according to Lemma 1.3, the intersection of all boolean subtrees of Γ containing x and y is a subtree of Γ denoted by $[x, y]$, the *chain* connecting x and y. If $[x, y]$ has only finitely many vertices, we say that it has finite length; then $[x, y]$ can be described, as in the discrete situation, as a minimal (unique) sequence of vertices and edges $x = v_0, e_1, v_1, \ldots, e_n, v_n = y$, with $\{d_0(e_i), d_1(e_i)\} = \{v_{i-1}, v_i\}$.

LEMMA 1.4. *Let Γ be a boolean tree, T and T' subtrees of Γ, $T \cap T' \neq \varnothing$, $x \in V(T)$, $x' \in V(T')$. Then $[x,x'] \cap T \cap T' \neq \varnothing$.*

Proof. By Lemma 1.3, $P = [x,x'] \cap T$, $P' = [x,x'] \cap T'$ and $T \cup T'$ are boolean trees. Hence $[x,x'] \subseteq T \cup T'$ and $[x,x'] = P \cup P'$. Since $[x,x']$ is a connected graph, so is any of its finite quotient graphs. Suppose that $P \cap P' = \varnothing$, and consider the graph Δ consisting of two vertices, v and v', and two edges, e and e', with $d_i(e) = v$ and $d_i(e') = v'$, $i = 1,2$. Then there is an epimorphism of boolean graphs $[x,x'] \to \Delta$ that sends the vertices and edges of P to v and e, respectively, and the vertices and edges of P' to v' and e', respectively; however, Δ is disconnected. From this contradiction we deduce that $P \cap P' \neq \varnothing$, as desired.

Let G be a profinite group that acts on a boolean tree Γ. Suppose that the quotient graph $G\backslash\Gamma$ is finite, and let $\phi:\Gamma \to G\backslash\Gamma$ be the canonical epimorphism of graphs. Clearly $G\backslash\Gamma$ has a subtree T' whose vertices are those of $G\backslash\Gamma$ (a maximal tree). Choose a vertex v of Γ. Since T' is finite, there exists a connected 'lifting' T of T' (that is, T is a subtree of Γ such that ϕ sends T isomorphically to T'), with v as one of its vertices. Moreover, each edge e' of $G\backslash\Gamma$ not in T' can be lifted to an edge e of Γ such that one of the vertices of e is in T. The finite structure Σ, consisting of T and the set of those edges e, is called a *connected transversal containing* v, of the action of G on Γ. Note that Σ contains exactly one representative of each of the G-orbits of the vertices and of the edges of Γ; however, Σ is not a graph in general. (See Proposition 2.6 in Chapter 1 of [1] for the situation in the case of abstract graphs, which is similar in the profinite context when $G\backslash\Gamma$ is finite.)

2. *Products of subgroups*

THEOREM 2.1. *Let G be a finite extension of a free abstract group F, and let K and $H_1, H_2, \ldots, H_n$ be finitely generated subgroups of G. Then $S = H_1 H_2 \ldots H_n K$ is a closed subset in the profinite topology of G.*

Proof. We shall start with a series of reductions. Since F has finite index in G, the profinite topology of F is precisely the topology induced by the profinite topology of G; in addition, F is open and closed in the profinite topology of G. We shall show first that we may assume that $H_1, H_2, \ldots, H_n, K \leqslant F$. Indeed, since H_i is finitely generated and $F \cap H_i$ has finite index in H_i, we have that $F \cap H_i$ is also finitely generated; say $H_i = \bigcup_j h(ij)(F \cap H_i)$ (a disjoint union). Therefore

$$H_1 H_2 \ldots H_n K = \bigcup_j h(ij) H_1^{h(ij)} H_2^{h(ij)} \ldots H_{i-1}^{h(ij)} (F \cap H_i) H_{i+1} \ldots H_n K,$$

and obviously each of $H_1^{h(ij)}, H_2^{h(ij)}, \ldots, H_{i-1}^{h(ij)}, (F \cap H_i), H_{i+1}, \ldots, H_n, K$ is finitely generated. If $H_1^{h(ij)} H_2^{h(ij)} \ldots H_{i-1}^{h(ij)} (F \cap H_i) H_{i+1} \ldots H_n K$ is closed in G, so is the finite union

$$H_1 H_2 \ldots H_n K = \bigcup_j h(ij) H_1^{h(ij)} H_2^{h(ij)} \ldots H_{i-1}^{h(ij)} (F \cap H_i) H_{i+1} \ldots H_n K.$$

Hence we may substitute for H_i by $F \cap H_i$ ($i = 1, \ldots, n$) and for K by $F \cap K$, and so we may assume that the H_i and K are subgroups of F.

Next, since F is a closed and open subset of G, if $H_1 H_2 \ldots H_n K$ is closed in F, then $H_1 H_2 \ldots H_n K$ is closed in G. Therefore, from now on we may assume that $G = F$ is a free group.

Since K is finitely generated, by a theorem of M. Hall there is a subgroup of finite index U of F such that K is a free factor of U (see [4]). By the argument used above, we may assume that the H_i and K are subgroups of U, and then it suffices to prove that $H_1 H_2 \ldots H_n K$ is closed in U. So we may substitute for F by U.

Thus from now on we shall assume, in addition, that $G = F = U$ is a free group and that $F = K * L$ (free product of abstract groups) for some subgroup L of F.

One easily sees that $\hat{F} = \hat{K} \amalg \hat{L}$ (profinite free product, that is, the coproduct of $\hat{K}$ and $\hat{L}$ in the category of profinite groups), and that $\bar{K} = \hat{K}$ and $\bar{H}_i = \hat{H}_i$ $(i = 1, \ldots, n)$. Choose bases for K and L to form a basis for F, and hence for the free profinite groups $\hat{K}$, $\hat{L}$ and $\hat{F}$ respectively.

Consider the abstract Cayley graphs $\Gamma(F)$ and $\Gamma(K)$ (see, for example, [10, 1]), and the profinite Cayley graphs $\Gamma(\hat{F})$ and $\Gamma(\hat{K})$ (see [3, §1]) with respect to the chosen bases. Then $\Gamma(F)$ and $\Gamma(K)$ are trees as ordinary graphs (see [10, Proposition 15, p. 39]), $\Gamma(\hat{F})$ and $\Gamma(\hat{K})$ are boolean trees (see [3, Theorem 1.2]), and clearly $\Gamma(F) \subset \Gamma(\hat{F})$ and $\Gamma(K) \subset \Gamma(\hat{K}) \subset \Gamma(\hat{F})$.

The profinite topology on F is the one induced by the topology of $\hat{F}$. Consider two compact subsets, A and B, of $\hat{F}$; from Tichonov's theorem and the continuity of multiplication, one deduces that the product AB is compact and hence a closed subset of $\hat{F}$; it follows that the topological closure of $H_1 H_2 \ldots H_n K$ in $\hat{F}$ is $\bar{H}_1 \bar{H}_2 \ldots \bar{H}_n \bar{K}$. Therefore, according to Lemma 1.1, to prove that $H_1 H_2 \ldots H_n K$ is closed in the profinite topology of F, it suffices to show that $(\bar{H}_1 \bar{H}_2 \ldots \bar{H}_n \bar{K}) \cap F = H_1 H_2 \ldots H_n K$. Suppose that $h_i \in \bar{H}_i$ and $k \in \bar{K}$, and that $h_1 h_2 \ldots h_n k \in F$; one must show that $h_1 h_2 \ldots h_n k \in H_1 H_2 \ldots H_n K$. We shall do this by induction on n.

The case $n = 0$ is precisely the result of M. Hall (see Corollary 1.2 above). Assume then that $n \geqslant 1$, and that the result holds when the number of H_i is less than n. Since $h_1 h_2 \ldots h_n k \in F$, the chain $[1, h_1 h_2 \ldots h_n k]$ in $\Gamma(F)$ is finite, and hence so is

$$h_n^{-1} \ldots h_1^{-1} [1, h_1 \ldots h_n k] = [h_n^{-1} \ldots h_1^{-1}, k].$$

Let D_i be the minimal $\bar{H}_i$-invariant boolean subtree of $\Gamma(\hat{F})$ containing the vertex 1 $(i = 1, \ldots, n)$. Observe that $D_i = \bigcup_j \bar{H}_i [1, r_j]$, where the r_j constitute a finite set of generators of H_i. Since for each j the chain $[1, r_j]$ is finite, it follows that the quotient graph $\bar{H}_i \backslash D_i$ is finite. Consider the boolean tree

$$D = h_n^{-1} \ldots h_2^{-1} D_1 \cup h_n^{-1} \ldots h_3^{-1} D_2 \cup \ldots \cup h_n^{-1} D_{n-1} \cup D_n.$$

(D is indeed a boolean tree by Lemma 1.3(b), since

$$h_n^{-1} \ldots h_{i+1}^{-1} D_i \cap h_n^{-1} \ldots h_{i+2}^{-1} D_{i+1} \neq \varnothing, \quad \text{for } i = 1, \ldots, n-1,$$

where $h_{n+1} = 1$.) Then $[h_n^{-1} \ldots h_1^{-1}, k]$ is contained in

$$[h_n^{-1} \ldots h_1^{-1}, 1] \cup \Gamma(\bar{K}) \subset D \cup \Gamma(\bar{K}).$$

(These are, in fact, boolean trees by Lemma 1.3(b), since $[h_n^{-1} \ldots h_1^{-1}, 1] \cap \Gamma(\bar{K}) \neq \varnothing$ and $D \cap \Gamma(\bar{K}) \neq \varnothing$.) If $h_n^{-1} \ldots h_1^{-1} \in \bar{K}$, then $h_1 \ldots h_n k \in \bar{K} \cap F = K$ and we are done. Hence we may assume that $h_n^{-1} \ldots h_1^{-1} \notin \bar{K}$. Now, since $[h_n^{-1} \ldots h_1^{-1}, k]$ is finite, there must exist a first vertex, say v, of $[h_n^{-1} \ldots h_1^{-1}, k]$ which is in $\Gamma(\bar{K})$. Let e be the last edge of $[h_n^{-1} \ldots h_1^{-1}, k]$ not in $\Gamma(\bar{K})$. Then v is one of the vertices of e, and e is an edge of $[h_n^{-1} \ldots h_1^{-1}, 1]$. So $[h_n^{-1} \ldots h_1^{-1}, v]$ is a finite subchain of $[h_n^{-1} \ldots h_1^{-1}, 1]$, and $[v, k]$ is a finite chain in $\Gamma(\bar{K})$. Thus there exists $t \in K$ such that $vt = k$. It follows that $h_1 \ldots h_n k \in F$ if and only if $h_1 \ldots h_n v \in F$; and $h_1 \ldots h_n k \in H_1 \ldots H_n K$ if and only if

$h_1 \dots h_n v \in H_1 \dots H_n K$. Therefore we may consider v instead of k, and so we shall assume from now on that

$$k \in [h_n^{-1} \dots h_1^{-1}, 1] \subset D = h_n^{-1} \dots h_2^{-1} D_1 \cup h_n^{-1} \dots h_3^{-1} D_2 \cup \dots \cup h_n^{-1} D_{n-1} \cup D_n.$$

For the sake of clarity and to avoid some notational difficulties, we shall consider separately the case $n = 1$.

Case 1: $n = 1$. Then $D = D_1$. Denote by $\pi : D_1 \to \bar{H}_1 \backslash D_1$ the canonical epimorphism of graphs. Consider now the subtree $T = D_1 \cap \Gamma(\bar{K})$, and the natural action of $\bar{H}_1 \cap \bar{K}$ on T. We shall see that the quotient graph $(\bar{H}_1 \cap \bar{K}) \backslash T$ is isomorphic to $\pi(T) \subset \bar{H}_1 \backslash D_1$, in particular, that it is finite. Indeed, note first that π induces an epimorphism $\bar{\pi} : (\bar{H}_1 \cap \bar{K}) \backslash T \to \pi(T)$. Now, suppose $t, t' \in T$ and $xt = t'$ for some $x \in \bar{H}_1$. If t is a vertex, then $t, t' \in \bar{K}$, and so $x \in \bar{H}_1 \cap \bar{K}$. If t is an edge, then its vertices are in $\bar{K}$, and again we deduce that $x \in \bar{H}_1 \cap \bar{K}$. Thus $\bar{\pi}$ is an isomorphism. Let $\rho : T \to (\bar{H}_1 \cap \bar{K}) \backslash T$ be the canonical epimorphism of graphs. Since $(\bar{H}_1 \cap \bar{K}) \backslash T$ is finite, there exists a connected transversal Σ of ρ in T containing the vertex 1. Note that $\Sigma \subset \Gamma(F) \cap \Gamma(\hat{K}) = \Gamma(K)$. Now, k is a vertex of T. Therefore there exists $g \in \bar{H}_1 \cap \bar{K}$ such that $gk \in \Sigma$. So $gk \in K$. Then $h_1 g^{-1} gk = h_1 k \in F$, and therefore $h_1 g^{-1} \in F \cap \bar{H}_1 = H_1$. Thus $h_1 k \in H_1 K$ as desired.

We shall assume from now on that $n \geqslant 2$. Furthermore, as a convention, we shall agree that if $i = n$, then $h_n^{-1} \dots h_{i+1}^{-1} = 1$.

Case 2: $k \in h_n^{-1} \dots h_{i+1}^{-1} D_i$, *where* $1 < i \leqslant n$. Then there exist $h' \in \bar{H}_i$ and a vertex f of $\bigcup_j [1, r_j] \subset \Gamma(F)$ (the r_j are a finite set of generators of the abstract group H_i) such that $h' h_{i+1} \dots h_n k = f \in F$. From the induction hypothesis, it follows that

$$h' h_{i+1} \dots h_n k \in H_i H_{i+1} \dots H_n K;$$

and since $h_1 h_2 \dots h_i h'^{-1} h' h_{i+1} \dots h_n k = h_1 h_2 \dots h_n k \in F$, one obtains that $h_1 h_2 \dots h_i h'^{-1} \in F$. Then, again by the induction hypothesis (H_n plays now the rôle of K), $h_1 h_2 \dots h_i h'^{-1} \in H_1 H_2 \dots H_i$. Thus $h_1 h_2 \dots h_n k \in H_1 H_2 \dots H_n K$, as desired.

Case 3: $k \in h_n^{-1} \dots h_2^{-1} D_1 (n \geqslant 2)$. Since the chain $[k, 1]$ is in

$$D = h_n^{-1} \dots h_2^{-1} D_1 \cup h_n^{-1} \dots h_3^{-1} D_2 \cup \dots \cup h_n^{-1} D_{n-1} \cup D_n,$$

it follows from Lemma 1.4 that

$$[k, 1] \cap h_n^{-1} \dots h_2^{-1} D_1 \cap (h_n^{-1} \dots h_3^{-1} D_2 \cup \dots \cup h_n^{-1} D_{n-1} \cup D_n) \neq \varnothing.$$

Therefore there exist some i $(2 \leqslant i \leqslant n)$ such that

$$[k, 1] \cap h_n^{-1} \dots h_2^{-1} D_1 \cap h_n^{-1} \dots h_{i+1}^{-1} D_i \neq \varnothing.$$

Let k' be a vertex in this intersection. Then also $k' \in \bar{K}$, since $[k, 1] \subset \Gamma(\bar{K})$. Now, the group $\bar{H} = h_n^{-1} \dots h_2^{-1} \bar{H}_1 h_2 \dots h_n$ acts on the tree $h_n^{-1} \dots h_2^{-1} D_1$, and $\bar{H} \cap \bar{K}$ acts on the tree $T = h_n^{-1} \dots h_2^{-1} D_1 \cap \Gamma(\bar{K})$, in a natural way. Denote by

$$\pi : h_n^{-1} \dots h_2^{-1} D_1 \to \bar{H} \backslash h_n^{-1} \dots h_2^{-1} D_1$$

the canonical epimorphism of graphs. Note that the quotient $\bar{H} \backslash h_n^{-1} \dots h_2^{-1} D_1$ is a finite graph since $\bigcup_j [1, r_j]$ is a finite graph, and

$$h_n^{-1} \dots h_2^{-1} D_1 = h_n^{-1} \dots h_2^{-1} \bar{H}_1 \left(\bigcup_j [1, r_j] \right) = \bar{H} h_n^{-1} \dots h_2^{-1} \left(\bigcup_j [1, r_j] \right).$$

(Here the r_j are a finite set of generators of the abstract group H_1.) Next we show that the quotient graph $(\bar{K} \cap \bar{H})\backslash T$ is isomorphic to $\pi(T) \subset \bar{H}\backslash h_n^{-1} \dots h_2^{-1} D_1$. To see this, as in Case 1, note first that π induces an epimorphism $\bar{\pi}:(\bar{K} \cap \bar{H})\backslash T \to \pi(T)$. Now, suppose $t, t' \in T$ and $xt = t'$ for some $x \in \bar{H}$. If t and t' are vertices, then $t, t' \in \bar{K}$, and so $x \in \bar{K} \cap \bar{H}$. If t is an edge, then its vertices are in $\bar{K}$, and again we deduce that $x \in \bar{K} \cap \bar{H}$. Thus $\bar{\pi}$ is an isomorphism. Therefore $(\bar{K} \cap \bar{H})\backslash T$ is a finite graph. Let $\rho: T \to (\bar{H} \cap \bar{K})\backslash T$ be the canonical epimorphism of graphs. Since $(\bar{K} \cap \bar{H})\backslash T$ is finite, there exists a connected transversal Σ of ρ in T containing the vertex k' of T. Now, k is also a vertex of T. Therefore there exists $g \in \bar{K} \cap \bar{H}$ such that $gk = k'' \in \Sigma$, and $[k', k'']$ is a chain in Σ, and hence finite. Since $\Sigma \subset \Gamma(\bar{K})$, there exists an element $r \in K$ such that $k'r = k''$. Then

$$h_1 h_2 \dots h_n k = h_1 h_2 \dots h_n g^{-1} gk = h_1 h_2 \dots h_n g^{-1} k'' = h_1 h_2 \dots h_n g^{-1} k' r \in F.$$

It follows that $h_1 h_2 \dots h_n g^{-1} k' \in F$. Observe that since $g^{-1} \in \bar{H}$, one has

$$h_1 h_2 \dots h_n g^{-1} = \bar{h}_1 h_2 \dots h_n \in \bar{H}_1 \bar{H}_2 \dots \bar{H}_n,$$

where $\bar{h}_1 \in \bar{H}_1$; on the other hand, $k' \in h_n^{-1} \dots h_{i+1}^{-1} D_i$. Since $1 < i \leqslant n$, by Case 2 we deduce that $h_1 h_2 \dots h_n g^{-1} k' \in H_1 H_2 \dots H_n K$. It follows that

$$h_1 h_2 \dots h_n k = h_1 h_2 \dots h_n g^{-1} k' r \in H_1 H_2 \dots H_n K,$$

as needed.

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