European Journal of Control ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

European Journal of Control

journal homepage: www.elsevier.com/locate/ejcon

Realization of nonlinear MIMO system on homogeneous time scales

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ARTICLE INFO

Article history:
Received 5 March 2014
Received in revised form
6 January 2015
Accepted 25 January 2015
Recommended by MingQing Xiao

Keywords:
Nonlinear control systems
Time scales
Polynomial methods
Input–output equations
State-space realization

ABSTRACT

The paper addresses a state space realization problem of a set of higher order delta-differential input–output equations, defined on a homogeneous time scale. The algebraic framework of differential one-forms is applied to formulate necessary and sufficient solvability conditions. This approach applies the total differential operator to analytic system equations to obtain the infinitesimal system description in terms of one-forms. This representation can be converted into polynomial system description by interpreting the polynomial indeterminate as the delta derivative acting on one-forms. The system description in terms of two matrices over skew polynomial ring is then used to derive explicit formulas for the differentials of state coordinates that significantly simplify the calculations. The formulas are found from the left quotients computed by the left Euclidean division algorithm.

1. Introduction

The state-space realization problem of a set of nonlinear input–output (i/o) equations has been addressed in numerous papers, see [6,22,26,28] for continuous- and [7,21,25] for discrete-time systems. There are many various approaches developed to solve this problem – based either on the sequence of subspaces of differential one-forms [13], on the sequence of distributions of vector fields as in [26], or on the iterative Lie brackets of the vector fields as in [14]. The comparison of these methods and the explicit relations between them have been reported in [17,18] for single-input single-output (SISO) and multi-input multi-output (MIMO) cases, respectively.

Majority of results suggest different step-by-step algorithms to find the realization. However, the papers [6,21] are an exception since they give explicit formulas to compute the differentials of state coordinates. Both papers rely on the polynomial tools built upon the algebraic framework of differential one-forms [13] to solve the realization problem. The name 'polynomial framework' does not point to the fact that systems are described by polynomial i/o equations, but to the fact that one first associates to the set of analytic i/o equations the polynomial system description. For that one has to first linearize the nonlinear equations globally applying the total differential operator to analytic system equations and represent the infinitesimal system description in terms of non-commutative polynomials having coefficients that depend on meromorphic functions in independent system variables. This can be done by interpreting the polynomial indeterminate as the delta derivative acting on one-forms. The most powerful argument in favor of polynomial approach is the computational simplicity. Indeed, using polynomial system description, it is possible to derive explicit polynomial formulas that represent differentials of the state coordinates in a compact and simple form (whenever the realization problem is solvable). There are two ways to derive formulas – either using the concept of adjoint polynomials as in [6] or employing the method based on the left quotients computed by the left Euclidean division algorithm [4]. The method of adjoint polynomials seems to be more efficient in case of continuous-time systems. However, in the shift-operator based discrete-time case the technique based on the left quotients is more suitable, since it reduces to the application of the very efficient cut-and-shift operator [21]. Though computations in case of both approaches are completely different, they will yield the same results. Note that the polynomial approach is especially well-suited for checking realizability conditions. However, in order to find the state coordinates, one has to integrate differential one-forms. The integration of (integrable in principle) differential one-forms is known to be a difficult task, in general.

The goal of this paper is to use time scale calculus [8] to unify the polynomial realization theories of continuous- and discrete-time (described in terms of difference operator) nonlinear systems, presenting both simultaneously in the same language. The results for both special cases will follow then from a general result. In fact, time scale approach incorporates more possibilities, but the mentioned two cases are just the most important for control theory. The results of this paper can be understood as extension of those presented in

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http://dx.doi.org/10.1016/j.ejcon.2015.01.006
0947-3580/

[4,10], where the problem was studied for the SISO systems. In [10] the differentials of the state coordinates were computed using a recursive algorithm, that requires to solve at each step a set of nonlinear equations. The latter is known to be a difficult task even for the systems of medium-complexity. In [4] the explicit polynomial formulas have been given that allow one to compute the differentials of the state coordinates using left quotients based on the left Euclidean division algorithm. In the conference paper [5] the MIMO case was partly covered with a strong focus on software implementation in computer algebra systems *Mathematica* rather than precise theoretical justification of applicability of the proposed tools to realization problem. Finally, let us note that the results of this paper are new for the difference operator based discrete-time case, and they complement those in [6] for continuous-time case based on the concept of adjoint polynomials.

The paper is organized as follows. Section 2 recalls the essential notions from time scale calculus and the algebraic framework of differential one-forms. In the next section the polynomial formalism is presented. In Section 4 the explicit polynomial formulas that allow one to calculate the differentials of the state coordinates are presented and illustrated by several examples. Concluding remarks are drawn in the last section.

2. Preliminaries

2.1. Time scale calculus

The following definitions and a general introduction to time scale calculus can be found in [8]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The cases, important for control systems, include continuous-time case $\mathbb{T}=\mathbb{R}$, discrete-time case $\mathbb{T}=T\mathbb{Z}$ for $T>0$, and a more complicated case of non-uniformly sampled time scale $\mathbb{T}=\{a=t_1<t_2<\cdots<t_n=b\}$ with $t_\ell\in\mathbb{R}$.

Definition 1. The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ are defined by $\sigma(t):=\inf\{s\in\mathbb{T}: s>t\}$ and $\rho(t):=\sup\{s\in\mathbb{T}: s<t\}$, respectively.

Definition 2. The graininess function $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t):=\sigma(t)-t$.

A time scale $\mathbb{T}$ is called *homogeneous* if $\mu\equiv\text{const}$. In this paper we assume that the time scale $\mathbb{T}$ is homogeneous. *Delta derivative* of $\xi(t)$, denoted by $\xi^{\Delta}(t)=:\xi^{\langle 1\rangle}$ is the extension of ordinary time derivative in the continuous-time case. For the formal definition of delta derivative see [8]. Table 1presents the operators/functions for two typical cases of $\mathbb{T}$, where id means identity operator.

For a function $\xi:\mathbb{T}\to\mathbb{R}$ we define the lth-order ($l\geq 2$) delta-derivative by $\xi^{\langle l\rangle}:=\left(\xi^{\langle l-1\rangle}\right)^{\Delta}$. In addition, for notational convenience, denote $\xi^{\langle l\ldots n\rangle}:=\left(\xi^{\langle l\rangle},\ldots,\xi^{\langle n\rangle}\right)$ for $l=0,\ldots,n$, where $\xi^{\langle 0\rangle}$ stands for ξ. Furthermore, by $\zeta(\{\xi_l\})$ is understood $\zeta(\xi_1,\ldots,\xi_n)$. Finally, the ℓ-fold application of the operators σ and ρ is denoted by $\xi^{\sigma^\ell}:=\left(\xi^{\sigma^{\ell-1}}\right)^{\sigma}$ and $\xi^{\rho^\ell}:=\left(\xi^{\rho^{\ell-1}}\right)^{\rho}$, respectively.

Table 1
Basic types of operators/functions.

$\mathbb{T}$	σ	$\xi^{\sigma}(t)$	ρ	$\xi^{\rho}(t)$	Δ	$\xi^{\Delta}(t)$	μ
$\mathbb{R}$	id	$\xi(t)$	id	$\xi(t)$	$\frac{d}{dt}$	$\frac{d\xi(t)}{dt}$	0
$T\mathbb{Z}$	σ	$\xi(t+T)$	σ^{-1}	$\xi(t-T)$	$\frac{\sigma-\text{id}}{\mu}$	$\frac{\xi(t+T)-\xi(t)}{T}$	T

2.2. Algebraic framework

Throughout the paper we assume that $i,j=1,\ldots,p$ and $\kappa=1,\ldots,m$. Consider a nonlinear system, described by a set of higher order i/o delta-differential equations on the homogeneous time scale $\mathbb{T}$, relating the inputs u_κ, the outputs y_j, and a finite number of their delta derivatives

$$y_i^{\langle n_i\rangle}=\phi_i\left(y_j,y_j^{\langle 1\rangle},\ldots,y_j^{\langle n_{ij}\rangle},u_\kappa,u_\kappa^{\langle 1\rangle},\ldots,u_\kappa^{\langle r_{i\kappa}\rangle}\right),\tag{1}$$

where $u:\mathbb{T}\to\mathcal{U}\subset\mathbb{R}^m$ is the vector of input signals, $y:\mathbb{T}\to\mathcal{Y}\subset\mathbb{R}^p$ is the vector of output signals, and ϕ_i are meromorphic functions. Notations $n:=n_1+\cdots+n_p$ and $r:=\max\{r_{i\kappa}\}$ are used below for system (1). Moreover, we assume that the indices in (1) satisfy relations

$$\begin{aligned}&n_1\leq n_2\leq\cdots\leq n_p,\quad n_{ij}<n_j,\\&n_{ij}<n_i,\quad j\leq i,\quad n_{ij}\leq n_i,\quad j>i,\\&r_{i\kappa}<n_i.\end{aligned}\tag{2}$$

The conditions (2) mean that Eqs. (1) are in the so-called doubly reduced form, suggested as a suitable starting point for realization in [27]. According to Proposition 2.6 in [27], an arbitrary set of i/o equations can be (at least locally) transformed into this form under mild rank conditions. Therefore, Eq. (2) are not restrictive and are made to simplify the presentation and, in particular, computations. In principle, they can be removed, but in such a case one has to rely on complicated technical algebraic constructions as in [3] suitable for implicit system representation. Though in [27] only the continuous-time case was addressed, the extension to the homogeneous time scale case can be done in the similar manner.

Definition 3. The state-space description

$$\begin{aligned}&x^{\Delta}=f(x,u),\\&y=h(x),\end{aligned}\tag{3}$$

with $x(t)\in\mathbb{R}^n$ is said to be realization of the set of the i/o equations (1) if Eqs. (1) and (3) have the same solution sets $\{(u(t),y(t)),\ t\geq 0\}$.

The system (1) is said to be *realizable* if it admits a realization in the sense of Definition 3.

Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of independent system variables from the set $\mathcal{C}=\left\{y_i,y_i^{\langle 1\rangle},\ldots,y_i^{\langle n_i-1\rangle},\ u_\kappa^{\langle\beta\rangle},\ \beta\geq 0\right\}$. Define $\phi:=(\phi_1,\ldots,\phi_p)^{\mathrm{T}}$. We use the index ϕ to indicate the dependence of delta-derivative and shift operators Δ_ϕ and σ_ϕ (acting on elements of $\mathcal{K}$) on system equations (1). That is, $y_i^{\langle n_i\rangle}$ should be replaced, according to (1), by ϕ_i, whenever it appears in some expression, since it is not an independent variable of $\mathcal{K}$. For $\zeta(\{y_i^{\langle 0\ldots n_i-1\rangle},u_\kappa^{\langle 0\ldots\beta\rangle}\})\in\mathcal{K}$ the forward-shift operator $\sigma_\phi:\mathcal{K}\to\mathcal{K}$ is defined by

$$\zeta^{\sigma_\phi}\left(\left\{y_i^{\langle 0\ldots n_i-1\rangle},u_\kappa^{\langle 0\ldots\beta+1\rangle}\right\}\right):=\zeta\left(\left\{\left(y_i^{\langle 0\ldots n_i-1\rangle}\right)^{\sigma_\phi},\left(u_\kappa^{\langle 0\ldots\beta\rangle}\right)^{\sigma_\phi}\right\}\right),$$

where $\left(y_i^{\langle 0\ldots n_i-1\rangle}\right)^{\sigma_\phi}=\left(y_i+\mu y_i^{\langle 1\rangle},\ldots,y_i^{\langle n_i-1\rangle}+\mu y_i^{\langle n_i\rangle}\right)$, $\left(u_\kappa^{\langle 0\ldots\beta\rangle}\right)^{\sigma_\phi}=u_\kappa^{\langle 0\ldots\beta\rangle}+\mu u_\kappa^{\langle 1\ldots\beta+1\rangle}$. The delta-derivative operator $\Delta_\phi:\mathcal{K}\to\mathcal{K}$ is defined as

$$\zeta^{\Delta_\phi}\left(\left\{y_i^{\langle 0\ldots n_i-1\rangle},u_\kappa^{\langle 0\ldots\beta+1\rangle}\right\}\right):=\begin{cases}\frac{1}{\mu}\left(\zeta^{\sigma_\phi}-\zeta\right) & \text{if } \mu\neq 0,\\ \sum_{i=1}^{p}\sum_{\alpha=0}^{n_i-1}\frac{\partial\zeta}{\partial y_i^{\langle\alpha\rangle}}y_i^{\langle\alpha+1\rangle}+\sum_{\kappa=1}^{m}\sum_{\beta\geq 0}\frac{\partial\zeta}{\partial u_\kappa^{\langle\beta\rangle}}u_\kappa^{\langle\beta+1\rangle} & \text{if } \mu=0.\end{cases}$$

Herein after, the ℓ-fold application of the forward-shift and delta derivative operators is denoted by $\zeta^{\sigma_\phi^\ell}:=\left(\zeta^{\sigma_\phi^{\ell-1}}\right)^{\sigma_\phi}$ and $\zeta^{\Delta_\phi^\ell}:=\left(\zeta^{\Delta_\phi^{\ell-1}}\right)^{\Delta_\phi}$, respectively.

Proposition 1 (Bartosiewicz et al. [2]). *For $F,G\in\mathcal{K}$ the delta derivative and forward-shift operators satisfy the following properties:*

(i) $F^{\sigma_\phi} = F + \mu F^{\Delta_\phi}$,
(ii) $(aF + bG)^{\Delta_\phi} = aF^{\Delta_\phi} + bG^{\Delta_\phi}$, *for any* a, b *from* $\mathbb{R}$,
(iii) $(FG)^{\Delta_\phi} = F^{\sigma_\phi} G^{\Delta_\phi} + F^{\Delta_\phi} G$,
(iv) *on homogeneous time scale the operators* Δ_ϕ *and* σ_ϕ *commute, i.e.* $(F^{\sigma_\phi})^{\Delta_\phi} = (F^{\Delta_\phi})^{\sigma_\phi}$.

Example 1. Consider the coupled nonlinear Van der Pol oscillators

$$\begin{aligned} y_1^{\langle 3\rangle} &= 2ku_1 + k^2 y_2 + u_1^{\Delta} - y_1^{\Delta} - ky_1^{\Delta} \\ &\quad + 2k\epsilon y_1^{\Delta} - y_1\left(k(2+k) + 2\epsilon\left(y_1^{\Delta}\right)^2\right) - 2ky_1^{\langle 2\rangle} \\ &\quad + \epsilon y_1^{\langle 2\rangle} - \epsilon y_1^2\left(2ky_1^{\Delta} + y_1^{\langle 2\rangle}\right) \\ y_2^{\langle 2\rangle} &= y_1 - u_1 + u_2 + ky_1 - (1+k)y_2 - \epsilon y_1^{\Delta} + \epsilon y_1^2 y_1^{\Delta} \\ &\quad + \epsilon y_2^{\Delta} - \epsilon y_2^2 y_2^{\Delta} + y_1^{\langle 2\rangle}, \end{aligned} \tag{4}$$

where ϵ and k are positive constants. Note that we intentionally describe the system in terms of i/o delta-differential equations (4), since the model of coupled Van der Pol oscillators was studied in different time domains [23], for $\mathbb{T} = \mathbb{R}$ when $\xi^{\Delta}(t) = \dot{\xi}(t)$, and for $\mathbb{T} = T\mathbb{Z}$ (based on the Euler sampling method) with $\xi^{\Delta}(t) = (\xi(\sigma(t)) - \xi(t))/\mu(t) = (\xi(t+T) - \xi(t))/T$.

Theorem 1 (Kotta et al. [20]). *The nonlinear control system of the form (1) is submersive if and only if the condition*

$$\operatorname{rank}_{\mathcal{K}} \begin{pmatrix} 1+\alpha_{11} & \cdots & \alpha_{1p} & \beta_{11} & \cdots & \beta_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{p1} & \cdots & 1+\alpha_{pp} & \beta_{p1} & \cdots & \beta_{pm} \end{pmatrix} = p \tag{5}$$

holds, where

$$\alpha_{ij} = \sum_{k=0}^{n_j-1} (-1)^{n_j-k-1} \mu^{n_j-k} \frac{\partial \phi_i}{\partial y_j^{\langle k\rangle}}, \quad \beta_{i\kappa} = \sum_{l=0}^{r_\kappa} (-1)^{r_\kappa - l+1} \mu^{r_\kappa - l+2} \frac{\partial \phi_i}{\partial u_\kappa^{\langle l\rangle}}.$$

In general, the field $\mathcal{K}$ is not inversive, i.e., not every element of $\mathcal{K}$ has a pre-image with respect to σ_ϕ. Under (5) the operator σ_ϕ is injective, and therefore there exists a σ_ϕ-differential overfield $\mathcal{K}^*$, called the *inversive closure* of $\mathcal{K}$, such that σ_ϕ can be extended to $\mathcal{K}^*$ and this extension is an automorphism of $\mathcal{K}^*$, see [2,12]. The backward-shift operator $\rho_\phi : \mathcal{K}^* \to \mathcal{K}^*$ is the inverse of σ_ϕ, i.e., $\rho_\phi := \sigma_\phi^{-1}$. Thus, denoting by ρ_ϕ^{ℓ} the ℓ-fold application of the backward-shift operator, we have $\zeta = (\zeta^{\rho_\phi})^{\sigma_\phi}$ and $\zeta^{\rho_\phi^{\ell}} = (\zeta^{\rho_\phi^{\ell+1}})^{\sigma_\phi}$ for $\zeta \in \mathcal{K}^*$. Note that in the continuous-time case when $\sigma_\phi = \mathrm{id}_{\mathcal{K}}$, $\mathcal{K}^* = \mathcal{K}$. An explicit construction of inversive closure is given in [2,3]. We assume that the inversive closure of the σ_ϕ-differential field $\mathcal{K}$ is given.

Consider next the infinite set of differentials

$$\mathrm{d}\mathcal{C} = \left\{\mathrm{d}y_i, \mathrm{d}y_i^{\langle 1\rangle}, \ldots, \mathrm{d}y_i^{\langle n_i-1\rangle},\ \mathrm{d}u_\kappa^{\langle\beta\rangle},\ \beta \geq 0\right\}$$

and denote by $\mathcal{E}$ the vector space spanned over the field $\mathcal{K}^*$ by the elements of $\mathrm{d}\mathcal{C}$, namely $\mathcal{E} := \operatorname{span}_{\mathcal{K}^*} \mathrm{d}\mathcal{C}$. An arbitrary element of $\mathcal{E}$ has a form

$$\omega = \sum_{i=1}^{p} \sum_{\alpha=0}^{n_i-1} a_{i,\alpha} \mathrm{d}y_i^{\langle\alpha\rangle} + \sum_{\kappa=1}^{m} \sum_{\beta \geq 0} b_{\kappa,\beta}\, \mathrm{d}u_\kappa^{\langle\beta\rangle},$$

where $a_{i,\alpha}, b_{\kappa,\beta} \in \mathcal{K}^*$ and only a finite number of coefficients $b_{\kappa,\beta}$ are nonzero. The elements of $\mathcal{E}$ are called the differential *one-forms*.

For $\zeta \in \mathcal{K}^*$ the operator $\mathrm{d} : \mathcal{K}^* \to \mathcal{E}$ is defined as

$$\mathrm{d}\zeta = \sum_{i=1}^{p} \sum_{\alpha=0}^{n_i-1} \frac{\partial \zeta}{\partial y_i^{\langle\alpha\rangle}} \mathrm{d}y_i^{\langle\alpha\rangle} + \sum_{\kappa=1}^{m} \sum_{\beta \geq 0} \frac{\partial \zeta}{\partial u_\kappa^{\langle\beta\rangle}} \mathrm{d}u_\kappa^{\langle\beta\rangle}.$$

Starting from the space $\mathcal{E}$ it is possible to build up the structures used in the exterior differential calculus. We refer to [1] for details, whereas here we just recall some basic notions. Define the set $\wedge \mathrm{d}\mathcal{C} = \{\mathrm{d}\zeta \wedge \mathrm{d}\eta | \zeta, \eta \in \mathcal{C}\}$, where $\wedge$ denotes the wedge product with the standard properties $\mathrm{d}\zeta \wedge \mathrm{d}\eta = -\mathrm{d}\eta \wedge \mathrm{d}\zeta$ and $\mathrm{d}\zeta \wedge \mathrm{d}\zeta = 0$ for $\zeta, \eta \in \mathcal{C}$. Introduce the space $\mathcal{E}^2 = \operatorname{span}_{\mathcal{K}^*} \wedge \mathrm{d}\mathcal{C}$ of two-forms. The operator $\mathrm{d} : \mathcal{E} \to \mathcal{E}^2$, called the exterior derivative operator, is defined for $\omega = \sum_{\ell=1}^{k} a_\ell(\zeta_1, \ldots, \zeta_k) \mathrm{d}\zeta_\ell \in \mathcal{E}$, where $\zeta_1, \ldots, \zeta_k \in \mathcal{C}$, by the rule $\mathrm{d}\omega := \sum_{\ell,\ell'} (\partial a_\ell / \partial \zeta_{\ell'})\, \mathrm{d}\zeta_\ell \wedge \mathrm{d}\zeta_{\ell'}$. The notion of two-form is generalized to the p-form and wedge product is defined for arbitrary p-forms.

One says that $\omega \in \mathcal{E}$ is an *exact* one-form, if $\omega = \mathrm{d}\zeta$ for some $\zeta \in \mathcal{K}^*$. A one-form ω for which $\mathrm{d}\omega = 0$ is said to be *closed*. Note that exact one-forms are closed, whereas closed one-forms are only locally exact. A subspace is said to be closed or integrable, if it has a basis which consists only of closed one-forms. Integrability of the subspace of one-forms can be checked by the Frobenius theorem below.

Theorem 2 (Choquet-Bruhat et al. [11]). *Let* $\Omega = \operatorname{span}_{\mathcal{K}^*}\{\omega_1, \ldots, \omega_\nu\} \subset \mathcal{E}$, *where* $\omega_1, \ldots, \omega_\nu$ *are linearly independent over* $\mathcal{K}^*$. *The subspace* Ω *is integrable if and only if for all* $\ell = 1, \ldots, \nu$

$$\mathrm{d}\omega_\ell \wedge \omega_1 \wedge \cdots \wedge \omega_\nu = 0.$$

For the one-form $\omega = \sum_k \lambda_k\, \mathrm{d}\xi_k$, where $\lambda_k \in \mathcal{K}^*$ and $\xi_k \in \mathcal{C}$, we define the operators $\Delta_\phi : \mathcal{E} \to \mathcal{E}$ and $\sigma_\phi : \mathcal{E} \to \mathcal{E}$ as

$$\omega^{\Delta_\phi} := \sum_k \left(\lambda_k^{\Delta_\phi}\, \mathrm{d}\xi_k + \lambda_k^{\sigma_\phi}\, \mathrm{d}(\xi_k^{\Delta_\phi})\right) = \sum_k \left(\lambda_k^{\Delta_\phi}\, \mathrm{d}\xi_k + (\lambda_k + \mu\lambda_k^{\Delta_\phi})\, \mathrm{d}(\xi_k^{\Delta_\phi})\right)$$

and

$$\omega^{\sigma_\phi} := \sum_k \lambda_k^{\sigma_\phi}\, \mathrm{d}(\xi_k^{\sigma_\phi}).$$

Proposition 2 (Bartosiewicz et al. [2]). *The following statements hold on homogeneous time scale* $\mathbb{T}$:

(i) $\mathrm{d}(\xi^{\Delta_\phi}) = (\mathrm{d}\xi)^{\Delta_\phi}$,
(ii) $\mathrm{d}(\xi^{\sigma_\phi}) = (\mathrm{d}\xi)^{\sigma_\phi}$.

3. Polynomial system description

A *left* polynomial can be uniquely written in the form $\pi(\mathfrak{z}) = \sum_{\ell=0}^{k} \pi_\ell \mathfrak{z}^\ell$, $\pi_\ell \in \mathcal{K}^*$. If $\pi_0 \neq 0$, then k is called the degree of π, denoted by $\deg(\pi(\mathfrak{z}))$.

Definition 4. The skew polynomial ring, induced by $(\mathcal{K}^*, \sigma_\phi, \Delta_\phi)$, is the non-commutative ring $\mathcal{K}^*[\mathfrak{z}; \sigma_\phi, \Delta_\phi]$ of polynomials in $\mathfrak{z}$ with usual addition and multiplication satisfying, for any $\zeta \in \mathcal{K}^* \subset \mathcal{K}^*[\mathfrak{z}; \sigma_\phi, \Delta_\phi]$, the commutation rule

$$\mathfrak{z}\zeta := \zeta^{\sigma_\phi}\mathfrak{z} + \zeta^{\Delta_\phi}. \tag{6}$$

Example 2. Consider two common special cases:

1. The choice of $\mathbb{T} = \mathbb{R}$ results[1] in $\mathcal{K}[\mathfrak{z}; \mathrm{id}_{\mathcal{K}}, \mathrm{d}/\mathrm{d}t]$ with commutation rule (6) being defined as $\mathfrak{z}\zeta = \zeta\mathfrak{z} + \mathrm{d}\zeta/\mathrm{d}t$ for $\zeta \in \mathcal{K}$.
2. The choice of $\mathbb{T} = T\mathbb{Z}$ yields the ring $\mathcal{K}^*[\mathfrak{z}; \sigma, (\sigma - \mathrm{id}_{\mathcal{K}^*})/T]$ with commutation rule (6) being $z\zeta = \zeta(t+T)\mathfrak{z} + (\zeta(t+T) - \zeta(t))/T$, $T \in \mathbb{R}$.

Example 3. In order to illustrate the commutation rule (6), consider multiplication of two polynomials $p(\mathfrak{z}) = \mathfrak{z}^2 + 1$ and $q(\mathfrak{z}) = \zeta\mathfrak{z} - 1$ with

[1] Recall that in the continuous-time case $\mathcal{K}^* = \mathcal{K}$.

$\zeta \in \mathcal{K}^*$:

$$(\mathfrak{z}^2+1)(\zeta\mathfrak{z}-1) = \mathfrak{z}^2\zeta\mathfrak{z}-\mathfrak{z}^2+\zeta\mathfrak{z}-1$$
$$= \mathfrak{z}\left(\zeta^{\sigma_\phi}\mathfrak{z}^2+\zeta^{\Delta_\phi}\mathfrak{z}\right)-\mathfrak{z}^2+\zeta\mathfrak{z}-1$$
$$= \zeta^{\sigma_\phi^2}\mathfrak{z}^3+\left(2\zeta^{\sigma_\phi\Delta_\phi}-1\right)\mathfrak{z}^2+\left(\zeta^{\langle 2\rangle}+\zeta\right)\mathfrak{z}-1.$$

Since the operators σ_ϕ and Δ_ϕ are not independent (see property (i) from Proposition 1), we can rewrite the obtained result in terms of Δ_ϕ only as

$$\left(\mu^2\zeta^{\Delta_\phi^2}+2\mu\zeta^{\Delta_\phi}+\zeta\right)\mathfrak{z}^3+\left(2\mu\zeta^{\Delta_\phi^2}+2\zeta^{\Delta_\phi}-1\right)\mathfrak{z}^2+\left(\zeta^{\Delta_\phi^2}+\zeta\right)\mathfrak{z}-1.$$

Define

$$\mathfrak{z}^k \mathrm{d}y_j := \mathrm{d}y_j^{\langle k\rangle}, \quad \mathfrak{z}^l\,\mathrm{d}u_\kappa := \mathrm{d}u_\kappa^{\langle l\rangle} \tag{7}$$

for $k,l \geq 0$ to represent the nonlinear system (1) in terms of two polynomial matrices. Differentiate (1) to obtain the infinitesimal system description

$$\mathrm{d}y_i^{\langle n_i\rangle} - \sum_{j=1}^{p}\sum_{\alpha=0}^{n_{ij}} \frac{\partial\phi_i}{\partial y_j^{\langle\alpha\rangle}}\mathrm{d}y_j^{\langle\alpha\rangle} - \sum_{\kappa=1}^{m}\sum_{\beta=0}^{r_{i\kappa}} \frac{\partial\phi_i}{\partial u_\kappa^{\langle\beta\rangle}}\mathrm{d}u_\kappa^{\langle\beta\rangle} = 0 \tag{8}$$

and use relations (7) to rewrite (8) as

$$P(\mathfrak{z})\,\mathrm{d}y + Q(\mathfrak{z})\,\mathrm{d}u = 0, \tag{9}$$

where $P(\mathfrak{z})$ and $Q(\mathfrak{z})$ are $p\times p$- and $p\times m$-dimensional matrices, respectively, whose elements $p_{ij}(\mathfrak{z}), q_{i\kappa}(\mathfrak{z})$ are from $\mathcal{K}^*[\mathfrak{z};\sigma_\phi,\Delta_\phi]$ and

$$p_{ij}(\mathfrak{z}) = \delta_{ij}\mathfrak{z}^{n_i} - \sum_{\alpha=0}^{n_{ij}} p_{ij,\alpha}\mathfrak{z}^\alpha, \quad p_{ij,\alpha} = \frac{\partial\phi_i}{\partial y_j^{\langle\alpha\rangle}} \in \mathcal{K}^*,$$
$$q_{i\kappa}(\mathfrak{z}) = -\sum_{\beta=0}^{r_{i\kappa}} q_{i\kappa,\beta}\mathfrak{z}^\beta, \quad q_{i\kappa,\beta} = \frac{\partial\phi_i}{\partial u_\kappa^{\langle\beta\rangle}} \in \mathcal{K}^* \tag{10}$$

with δ_{ij} being Kronecker delta. Eq. (9) describes the globally linearized system, corresponding to Eqs. (1). Further, the notations $p_{i\cdot}(\mathfrak{z}) := [p_{i1}(\mathfrak{z}),\ldots,p_{ip}(\mathfrak{z})]$ and $q_{i\cdot}(\mathfrak{z}) := [q_{i1}(\mathfrak{z}),\ldots,q_{im}(\partial)]$ are used for row vectors of $P(\mathfrak{z})$ and $Q(\mathfrak{z})$, respectively.

Example 4 (*Continuation of Example 1*). Applying operator d to (4) and using relations (7) yield the polynomial system description (9) with

$$p_{11}(\mathfrak{z}) = \mathfrak{z}^3+(\epsilon y_1^2+2k-\epsilon)\mathfrak{z}^2+\left(4\epsilon y_1 y_1^\Delta+2k\epsilon y_1^2-2k\epsilon+k+1\right)\mathfrak{z}$$
$$+\left(2ky_1^\Delta+y_1^{\langle 2\rangle}\right)2\epsilon y_1+2\epsilon\left(y_1^\Delta\right)^2+k(k+2),$$
$$p_{12}(\mathfrak{z}) = -k^2,$$
$$p_{21}(\mathfrak{z}) = -\mathfrak{z}^2+(\epsilon-\epsilon y_1^2)\mathfrak{z}-2\epsilon y_1 y_1^\Delta-k-1,$$
$$p_{22}(\mathfrak{z}) = \mathfrak{z}^2+\epsilon(y_2^2-1)\mathfrak{z}+2\epsilon y_2 y_2^\Delta+k+1,$$

and $q_{11}(\mathfrak{z}) = -\mathfrak{z}-2k, q_{12}(\mathfrak{z}) = 0, q_{21}(\mathfrak{z}) = 1, q_{22}(\mathfrak{z}) = -1$.

4. Realization

Definition 5. The relative degree υ of a one-form $\omega \in \mathcal{E}$ is defined to be the least integer such that

$$\omega^{\langle\upsilon\rangle} \notin \mathrm{span}_{\mathcal{K}^*}\left\{\mathrm{d}y_i, \mathrm{d}y_i^{\langle 1\rangle},\ldots,\mathrm{d}y_i^{\langle n_i-1\rangle}, \mathrm{d}u_\kappa, \mathrm{d}u_\kappa^{\langle 1\rangle},\ldots,\mathrm{d}u_\kappa^{\langle r\rangle}\right\}.$$

If such an integer does not exist, we set $\upsilon = \infty$.

Define for system (1) the nonincreasing sequence of subspaces $\{\mathcal{H}_k\}_{k=1}^{r+2}$ of $\mathcal{E}$ as follows:

$$\mathcal{H}_1 = \mathrm{span}_{\mathcal{K}^*}\left\{\mathrm{d}y_i^{\langle 0\ldots n_i-1\rangle}, \mathrm{d}u_\kappa^{\langle 0\ldots r\rangle}\right\},$$

$$\mathcal{H}_{k+1} = \left\{\omega \in \mathcal{H}_k | \omega^{\Delta_\phi} \in \mathcal{H}_k\right\}, \quad k \geq 1, \tag{11}$$

which plays the key role in the study of realizability problem. Note that $\mathcal{H}_k$ contains the one-forms whose relative degree is greater than or equal to k.

Definition 6. The system (3) is said to be generically observable if the rank of observability matrix is equal to n, i.e.,

$$\mathrm{rank}_{\mathcal{K}^*}\frac{\partial\left(h, h^{\langle 1\rangle},\ldots,h^{\langle n-1\rangle}\right)}{\partial x} = n.$$

Theorem below is the extension of that in [10], given for the SISO case. Its proof is given in the Appendix.

Theorem 3. *The nonlinear system, described by the set of i/o equations (1) under constraints (2), has an observable state-space realization if and only if the subspaces $\mathcal{H}_k$, defined by (11), are completely integrable for $k = 1,\ldots,r+2$. Moreover, the state coordinates can be found by integrating the exact basis one-forms of $\mathcal{H}_{r+2}$.*

Note that to calculate the subspaces $\mathcal{H}_k$ directly from (11) is not an easy task. Our goal is to find the exact formulas to compute the basis vectors for subspaces $\mathcal{H}_k$, and in particular those of $\mathcal{H}_{r+2}$ that define the differentials of the state coordinates. Doing so we rely on the so-called *left division* operation of non-commutative polynomials. Since σ_ϕ is an automorphism on $\mathcal{K}^*$, the left division operation is well-defined in $\mathcal{K}^*[\mathfrak{z};\sigma_\phi,\Delta_\phi]$. Given two polynomials $p(\mathfrak{z}), q(\mathfrak{z}) \in \mathcal{K}^*[\mathfrak{z};\sigma_\phi,\Delta_\phi]$, $q(\mathfrak{z}) \neq 0$ with $\deg p(\mathfrak{z}) > \deg q(\mathfrak{z})$, then there exist a unique *left quotient* polynomial $\gamma(\mathfrak{z})$ and a unique *left remainder* polynomial $\eta(\mathfrak{z})$ such that $p(\mathfrak{z}) = q(\mathfrak{z})\gamma(\mathfrak{z})+\eta(\mathfrak{z})$ and $\deg \eta(\mathfrak{z}) < \deg q(\mathfrak{z})$. The left quotient and the left remainder polynomials can be computed by *left Euclidean division algorithm*, see [9].

Introduce the certain one-forms in terms of which one of our main results will be formulated. Let

$$\omega_{i,l} = \begin{bmatrix} p_{i\cdot,l}(\mathfrak{z}) & q_{i\cdot,l}(\mathfrak{z}) \end{bmatrix}\begin{bmatrix}\mathrm{d}y \\ \mathrm{d}u\end{bmatrix} \tag{12}$$

for $i = 1,\ldots,p$, $l = 1,\ldots,n_i$, where $p_{i\cdot,l}(\mathfrak{z})$ and $q_{i\cdot,l}(\mathfrak{z})$ are vectors of polynomials from $\mathcal{K}^*[\mathfrak{z};\sigma_\phi,\Delta_\phi]$, which can be recursively calculated from the equalities

$$p_{i\cdot,l-1}(\mathfrak{z}) = \mathfrak{z}p_{i\cdot,l}(\mathfrak{z})+\xi_{i\cdot,l}, \quad \deg \xi_{i\cdot,l} = 0,$$
$$q_{i\cdot,l-1}(\mathfrak{z}) = \mathfrak{z}q_{i\cdot,l}(\mathfrak{z})+\gamma_{i\cdot,l}, \quad \deg \gamma_{i\cdot,l} = 0 \tag{13}$$

with initial polynomials $p_{i\cdot,0}(\mathfrak{z}) := p_{i\cdot}(\mathfrak{z})$ and $q_{i\cdot,0}(\mathfrak{z}) := q_{i\cdot}(\mathfrak{z})$.

Example 5 (*Continuation of Example 4*). From (4), $n = n_1+n_2 = 3+2 = 5$ and $r = \max\{r_{11}, r_{12}, r_{21}, r_{22}\} = \{1,0,0,0\} = 1$. Let $p_{ij,0} := p_{ij}$ and $q_{ij,0} := q_{ij}$ for $i = 1,2$ and $j = 1,2$. At the first step, from the equality $p_{11,0}(\mathfrak{z}) = \mathfrak{z}p_{11,1}(\mathfrak{z})+\eta_1$ one can compute the left quotient

$$p_{11,1}(\mathfrak{z}) = \mathfrak{z}^2+\left(\epsilon(y_1^\rho)^2+2k-\epsilon\right)\mathfrak{z}-2k\epsilon$$
$$+k+1+\frac{4\epsilon y_1 y_1^\rho+2(\mu k-2k)\epsilon(y_1^\rho)^2}{\mu}$$

and the left remainder[2]

$$\eta_1 = k(k+2)-\frac{4\epsilon y_1^\rho y_1^\Delta}{\mu}+2\epsilon\left(y_1^\Delta\right)^2+2\epsilon y_1\left(2ky_1^\Delta+y_1^{\langle 2\rangle}\right).$$

At the next step, from the equality $p_{11,1}(\mathfrak{z}) = \mathfrak{z}p_{11,2}(\mathfrak{z})+\eta_2$ one obtains that $p_{11,2}(\mathfrak{z}) = \mathfrak{z}+\epsilon(y_1^{\rho^2})^2-\epsilon+2k$. Finally, one gets that $p_{11,3} = 1$. Proceeding in the similar manner with the other entries of $P(\mathfrak{z})$ and

[2] In principle, the left remainder is not required for the computation of one-forms $\omega_{i,l}$ in (12). However, we calculated it for illustrative purposes.

$Q(\mathfrak{z})$, one gets the following sequences of left quotients as

$$\{p_{12,l}\}_{l=1}^{3}=\{0,0,0\},$$
$$\{p_{21,l}\}_{l=1}^{2}=\left\{-\mathfrak{z}-\epsilon\left(y_1^{\rho}\right)^2+\epsilon,-1\right\},$$
$$\{p_{22,l}\}_{l=1}^{2}=\left\{\mathfrak{z}+\epsilon\left(y_1^{\rho}\right)^2-\epsilon,1\right\},$$

and

$$\{q_{11,l}\}_{l=1}^{3}=\{-1,0,0\},\quad \{q_{12,l}\}_{l=1}^{3}=\{0,0,0\},$$
$$\{q_{21,l}\}_{l=1}^{2}=\{0,0\},\quad \{q_{22,l}\}_{l=1}^{2}=\{0,0\}.$$

Next, using (12), we calculate the one-forms of the subspace $\mathcal{H}_{r+2}=\mathcal{H}_3=\mathrm{span}_{\mathcal{K}^*}\{\omega_{i,j}\}$ for $i=1,2$, $j=1,\dots,n_i$ as follows:

$$\omega_{1,1}=\mathrm{d}y_1^{\langle 2\rangle}+\left(\epsilon\left(y_1^{\rho}\right)^2+2k-\epsilon\right)\mathrm{d}y_1^{\Delta}+\Big(k-2k\epsilon+1+\left[4\epsilon y_1y_1^{\rho}+2(\mu k-2k)\epsilon\left(y_1^{\rho}\right)^2\right]/\mu\Big)\mathrm{d}y_1-\mathrm{d}u_1,$$
$$\omega_{1,2}=\mathrm{d}y_1^{\Delta}+\left(\epsilon\left(y_1^{\rho^2}\right)^2-\epsilon+2k\right)\mathrm{d}y_1,$$
$$\omega_{1,3}=\mathrm{d}y_1,$$
$$\omega_{2,1}=-\mathrm{d}y_1^{\Delta}-\left(\epsilon\left(y_1^{\rho}\right)^2-\epsilon\right)\mathrm{d}y_1+\mathrm{d}y_2^{\Delta}+\left(\epsilon\left(y_2^{\rho}\right)^2-\epsilon\right)\mathrm{d}y_2,$$
$$\omega_{2,2}=-\mathrm{d}y_1+\mathrm{d}y_2.$$

4.1. Main result

Now we are ready to state and prove our main result.

Theorem 4. *For the i/o model (1), the subspaces $\mathcal{H}_k$, for $k=1,\dots,r+2$, can be calculated as*

$$\mathcal{H}_k=\mathrm{span}_{\mathcal{K}^*}\{\omega_{i,l},\ i=1,\dots,p,l=1,\dots,n_i,\ \mathrm{d}u_\kappa,\mathrm{d}u_\kappa^{\langle 1\rangle},\dots,\mathrm{d}u_\kappa^{\langle r-k+1\rangle},\ \kappa=1,\dots,m\}. \tag{14}$$

Proof. The proof is by mathematical induction. First, we show that formula (14) holds for $k=1$. Taking $k=1$ in (14) yields

$$\mathcal{H}_1=\mathrm{span}_{\mathcal{K}^*}\{\omega_{i,l},\ l=1,\dots,n_i,\ \mathrm{d}u_\kappa,\mathrm{d}u_\kappa^{\langle 1\rangle},\dots,\mathrm{d}u_\kappa^{\langle r\rangle}\}. \tag{15}$$

In order to simplify the following discussion note that (13), regarding that $p_{i\cdot,0}(\mathfrak{z})=p_{i\cdot}(\mathfrak{z})$ and $q_{i\cdot,0}(\mathfrak{z})=q_{i\cdot}(\mathfrak{z})$, can be rewritten for $l=1,\dots,n_i$ explicitly as

$$p_{i\cdot,0}(\mathfrak{z})=\mathfrak{z}^l p_{i\cdot,l}(\mathfrak{z})+\Xi_{i\cdot,l}(\mathfrak{z}),\quad \deg \Xi_{i\cdot,l}(\mathfrak{z})<l,$$
$$q_{i\cdot,0}(\mathfrak{z})=\mathfrak{z}^l q_{i\cdot,l}(\mathfrak{z})+\Gamma_{i\cdot,l}(\mathfrak{z}),\quad \deg \Gamma_{i\cdot,l}(\mathfrak{z})<l. \tag{16}$$

Due to the structure of the i/o equations (1), $\deg(p_{ii}(\mathfrak{z}))=n_i$, $\deg(p_{ij}(\mathfrak{z}))\le n_i-1$, for $i\ne j$ and $\deg(q_{i\kappa}(\mathfrak{z}))\le r$. Taking into account the relations (16) yields that $\deg(p_{ii,l}(\mathfrak{z}))=\deg(p_{ii}(\mathfrak{z}))-l=n_i-l$, $\deg(p_{ij,l}(\mathfrak{z}))\le n_i-l-1$ for $i\ne j$ and $\deg(q_{i\cdot,l}(\mathfrak{z}))\le r-l$. Moreover, due to (10), polynomials $p_{ii}(\mathfrak{z})$ are monic, thus $p_{ii,l}(\mathfrak{z})$ are monic as well. Hence the polynomials $p_{ij,l}(\mathfrak{z})$ and $q_{i\kappa,l}(\mathfrak{z})$ have the form

$$p_{ij,l}(\mathfrak{z})=\delta_{ij}\mathfrak{z}^{n_i-l}+\sum_{\alpha=0}^{n_i-l-1}p_{ij,l,\alpha}\mathfrak{z}^{\alpha},\quad q_{i\kappa,l}(\mathfrak{z})=\sum_{\beta=0}^{r-l}q_{i\kappa,l,\beta}\mathfrak{z}^{\beta}.$$

By (12) we may write

$$\omega_{i,l}=\mathrm{d}y_i^{\langle n_i-l\rangle}+\sum_{j=1}^{p}\sum_{\alpha=0}^{n_i-l-1}p_{ij,l,\alpha}\,\mathrm{d}y_j^{\langle\alpha\rangle}+\sum_{\kappa=1}^{m}\sum_{\beta=0}^{r-l}q_{i\kappa,l,\beta}\,\mathrm{d}u_\kappa^{\langle\beta\rangle}$$

for $l=1,\dots,n_i$. Taking $l=n_i$ in the latter expression we obtain $\omega_{i,n_i}=\mathrm{d}y_i$. In addition, letting $l=n_i-1$ yields

$$\omega_{i,n_i-1}=\mathrm{d}y_i^{\Delta}+\sum_{j=1}^{p}p_{ij,n_i-1,0}\mathrm{d}y_j+\sum_{\kappa=1}^{m}q_{i\kappa,n_i-1,0}\,\mathrm{d}u_\kappa,$$

etc. It means we have represented $\omega_{i,l}$ as a linear combination of $\mathrm{d}y_i,\mathrm{d}y_i^{\langle 1\rangle},\dots,\mathrm{d}y_i^{\langle n_i-1\rangle}$, $\mathrm{d}u_\kappa,\mathrm{d}u_\kappa^{\langle 1\rangle},\dots,\mathrm{d}u_\kappa^{\langle r\rangle}$ being the basis vectors of $\mathcal{H}_1$ defined by (11). Thus, Eq. (15) agrees with $\mathcal{H}_1$ in (11).

Assume next that formula (14) holds for k and we prove it to be valid for $k+1$. The proof is based on the definition of the subspaces $\mathcal{H}_k$. We have to prove that

$$\mathcal{H}_{k+1}=\mathrm{span}_{\mathcal{K}^*}\left\{\omega_{i,l},\mathrm{d}u_\kappa,\mathrm{d}u_\kappa^{\langle 1\rangle},\dots,\mathrm{d}u_\kappa^{\langle r-k\rangle}\right\}, \tag{17}$$

calculated according to formula (14), satisfies condition (11). Observe first that the one-forms $\omega_{i,l},\mathrm{d}u_\kappa,\mathrm{d}u_\kappa^{\langle 1\rangle},\dots,\mathrm{d}u_\kappa^{\langle r-k\rangle}\in\mathcal{H}_k$, since we have assumed formula (14) to hold for k. Second, we have to prove that the delta-derivatives of the basis one-forms of (17) belong to $\mathcal{H}_k$. By (12), we have

$$\omega_{i,l}^{\Delta_\phi}=\left[\mathfrak{z}p_{i\cdot,l}(\mathfrak{z})\ \ \mathfrak{z}q_{i\cdot,l}(\mathfrak{z})\right]\begin{bmatrix}\mathrm{d}y\\ \mathrm{d}u\end{bmatrix},$$

that yields, using the relations (13),

$$\omega_{i,l}^{\Delta_\phi}=\left[p_{i\cdot,l-1}(\mathfrak{z})-\xi_{i\cdot,l}\ \ q_{i\cdot,l-1}(\mathfrak{z})-\gamma_{i\cdot,l}\right]\begin{bmatrix}\mathrm{d}y\\ \mathrm{d}u\end{bmatrix}. \tag{18}$$

After reordering terms in (18) we get

$$\omega_{i,l}^{\Delta_\phi}=\left[p_{i\cdot,l-1}(\mathfrak{z})\ \ q_{i\cdot,l-1}(\mathfrak{z})\right]\begin{bmatrix}\mathrm{d}y\\ \mathrm{d}u\end{bmatrix}-\left[\xi_{i\cdot,l}\ \ \gamma_{i\cdot,l}\right]\begin{bmatrix}\mathrm{d}y\\ \mathrm{d}u\end{bmatrix}. \tag{19}$$

The one-form $\omega_{i,l}^{\Delta_\phi}$ is now represented as a sum of two terms. To deal with the first term, we consider two separate cases. In case $l=1$ the first term yields $p_{i\cdot,0}(\mathfrak{z})\,\mathrm{d}y+q_{i\cdot,0}(\mathfrak{z})\,\mathrm{d}u=p_{i\cdot}(\mathfrak{z})\,\mathrm{d}y+q_{i\cdot}(\mathfrak{z})\,\mathrm{d}u=0$ due to polynomial system description (9). In case $l=2,\dots,n_i$, the first term of (19) equals $\omega_{i,l-1}$ by (12), thus it is in $\mathcal{H}_k$. As for the second term of (19), it is a linear combination of $\mathrm{d}y_i,\mathrm{d}u_\kappa\in\mathcal{H}_k$, since the elements of $\xi_{i\cdot,l}$ and $\gamma_{i\cdot,l}$ are functions from $\mathcal{K}^*$. Therefore, $\omega_{i,l}^{\Delta_\phi}\in\mathcal{H}_k$ for $l=1,\dots,n_i$. Finally, observe that the delta-derivatives of the remaining basis one-forms in (17) are $\mathrm{d}u_\kappa^{\langle 1\rangle},\dots,\mathrm{d}u_\kappa^{\langle r-k+1\rangle}$, which are also in $\mathcal{H}_k$. Thus, we have proved that $\mathcal{H}_{k+1}$, computed according to (14), agrees with definition (11). □

Corollary 1. *The subspaces $\mathcal{H}_1,\dots,\mathcal{H}_{r+1}$ are integrable if $\mathcal{H}_{r+2}$ is integrable.*

Proof. Assume that $\mathcal{H}_{r+2}$ is integrable. According to Theorem 4 the subspace $\mathcal{H}_k$, $k=1,\dots,r+1$ can be represented in form (14). In addition, we note that the one-forms $\mathrm{d}u_\kappa,\mathrm{d}u_\kappa^{\langle 1\rangle},\dots,\mathrm{d}u_\kappa^{\langle r-k+1\rangle}$, $k=1,\dots,r+1$ are integrable. Therefore, we can conclude that $\mathcal{H}_1,\dots,\mathcal{H}_{r+1}$ are also integrable. □

Note that Corollary 1 is important from the computational point of view. It states that to verify whether the system (1) is realizable in the state-space form (3) or not it is enough to calculate only the subspace $\mathcal{H}_{r+2}$. The latter can be easily done using explicit formulas (14) from Theorem 4.

Next, three examples are used to illustrate the realization method. For system in the first example $\mathcal{H}_{r+2}$ appears to be completely integrable in continuous- and discrete-time cases. For the second example the subspace $\mathcal{H}_{r+2}$ is integrable only for $\mathbb{T}=\mathbb{R}$. The last example demonstrates non-integrability for both cases.

Example 6 (*Continuation of Example 5*). The subspace $\mathcal{H}_3$ can be simplified, resulting in $\mathcal{H}_3=\mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y_1,\mathrm{d}y_1^{\Delta},\mathrm{d}y_1^{\langle 2\rangle}-\mathrm{d}u_1,\mathrm{d}y_2,\mathrm{d}y_2^{\Delta}\}$, which is, according to Theorem 2, integrable. The choice $x_1=y$, $x_2=y_1^{\Delta}$, $x_3=y_2$, $x_4=y_2^{\Delta}$, $x_5=y_1^{\langle 2\rangle}-u_1$ yields the state equations

$$x_1^{\Delta}=x_22$$
$$x_2^{\Delta}=x_5+u_1$$
$$x_3^{\Delta}=x_4$$
$$x_4^{\Delta}=kx_1-(k+1)x_3+u_2+\epsilon x_2x_1^2-\epsilon x_2-\epsilon x_3^2x_4+\epsilon x_4+x_1+x_5$$
$$x_5^{\Delta}=k^2x_3+2k\epsilon x_2-x_1\left(k(k+2)+2\epsilon x_2^2\right)$$

$$-\epsilon(2kx_2+x_5)x_1^2-kx_2-2kx_5+(\epsilon-\epsilon x_1^2)u_1+\epsilon x_5-x_2$$
$$y_1=x_1$$
$$y_2=x_3.$$

Example 7. Consider the i/o equation

$$y^{\langle 2\rangle}=y^{\Delta}u^{\Delta}+uy \tag{20}$$

that may be described as in (9) by two polynomials $p=\mathfrak{z}^2-u^{\langle 1\rangle}\mathfrak{z}-u$ and $q=-y^{\langle 1\rangle}\mathfrak{z}-y$. From (20), $n=2$ and $r=1$. Note that due to the fact that (20) is a SISO system with $p=m=1$, one can simplify notation as $p_{11,0}=p_0, q_{11,0}=q_0$, etc. Next, compute, according to (13), two sequences of the left quotients as

$$p_1=\mathfrak{z}-\left(u^{\Delta}\right)^{\rho_\phi},\quad q_1=-\left(y^{\Delta}\right)^{\rho_\phi},$$
$$p_2=1,\quad q_2=0.$$

By (12), the one-forms of the subspace $\mathcal{H}_{s+2}=\mathcal{H}_3=\text{span}_{\mathcal{K}^*}\{\omega_1,\omega_2\}$ are

$$\omega_1=p_1\,\mathrm{d}y+q_1\,\mathrm{d}u=\left(\mathfrak{z}-\left(u^{\Delta}\right)^{\rho_\phi}\right)\mathrm{d}y-\left(y^{\Delta}\right)^{\rho_\phi}\mathrm{d}u,$$
$$\omega_2=p_2\,\mathrm{d}y+q_2\,\mathrm{d}u=\mathrm{d}y.$$

Since $\mathrm{d}y$ is the basis vector of the subspace $\mathcal{H}_3$, ω_1 may be simplified, resulting in

$$\mathcal{H}_3=\text{span}_{\mathcal{K}^*}\left\{\mathrm{d}y,\mathrm{d}y^{\Delta}-\left(y^{\Delta}\right)^{\rho_\phi}\mathrm{d}u\right\}.$$

Note that integrability of $\mathcal{H}_3$ depends on the coefficient $(y^{\Delta})^{\rho_\phi}$. Next, we separately consider two typical cases. In the continuous-time case ($\mathbb{T}=\mathbb{R}$ and $\sigma=\rho_\phi=\text{id}_{\mathcal{K}}$), the subspace $\mathcal{H}_3=\text{span}_{\mathcal{K}^*}\{\mathrm{d}y,\mathrm{d}\dot{y}-\dot{y}\,\mathrm{d}u\}$ is, by Theorem 2, integrable. The choice $x_1=y$, $x_2=e^{-u}\dot{y}$ yields the classical state equations

$$\dot{x}_1=e^{u}x_2$$
$$\dot{x}_2=e^{-u}ux_1$$
$$y=x_1.$$

In the discrete-time case ($\mathbb{T}=T\mathbb{Z}$ and $\Delta=(\sigma-\text{id}_{\mathcal{K}^*})/T$), the subspace

$$\mathcal{H}_3=\text{span}_{\mathcal{K}^*}\left\{\mathrm{d}y,\frac{1}{T}\mathrm{d}\sigma(y)+\frac{1}{T}(\sigma^{-1}(y)-y)\mathrm{d}u\right\}$$

is, according to Theorem 2, not integrable, since $\mathrm{d}\omega_2\wedge\omega_1=(1/T)\,\mathrm{d}\sigma^{-1}(y)\wedge\mathrm{d}u\wedge\mathrm{d}y\neq 0$. Recall that either $\sigma^{-1}(y)$ or $\sigma^{-1}(u)$ may be chosen as the independent variable of $\mathcal{K}^*$. In the latter case $\sigma^{-1}(y)=(\sigma(y)-2y-yu+\sigma^{-1}(u)y)/\ ((T^2+1)\sigma^{-1}(u)-u-1)$, yielding again that $\mathrm{d}\omega_2\wedge\omega_1\neq 0$.

Example 8. Consider the "ball and beam" system with input being angle of the beam and output being position of the ball. The input–output equation of the system is

$$y^{\langle 2\rangle}=\vartheta\left(y\left(u^{\Delta}\right)^2-g\sin(u)\right), \tag{21}$$

where $\vartheta=mR^2/(J+mR^2)$, the constant parameters J,R and m represent, respectively, the inertia, radius and mass of the ball, and g is the gravitational constant. Usually, system (21) is considered separately for continuous- and discrete-time cases, see, for example, [15,24], respectively. Here, however, we consider the time scale based system description which accommodates both: continuous- and discrete-time models.

System (21) can be described as in (9) by two polynomials $p(\mathfrak{z})=\mathfrak{z}^2-\vartheta\left(u^{\Delta}\right)^2$ and $q(\mathfrak{z})=-\vartheta yu^{\Delta}\mathfrak{z}+\vartheta\cos(u)$. Note that $n=2$ and $r=1$. Next, we compute two sequences of the left quotients as

$$p_1=\mathfrak{z},\quad q_1=-\vartheta\left(yu^{\Delta}\right)^{\rho_\phi},$$
$$p_2=1,\quad q_2=0.$$

Then, the one-forms of the subspace $\mathcal{H}_{s+2}=\mathcal{H}_3=\text{span}_{\mathcal{K}^*}\{\omega_1,\omega_2\}$ are

$$\omega_1=\mathrm{d}y^{\Delta}-\vartheta\left(yu^{\Delta}\right)^{\rho_\phi}\mathrm{d}u,$$
$$\omega_2=\mathrm{d}y.$$

Next, we consider separately two typical cases. In the continuous-time case the subspace

$$\mathcal{H}_3=\text{span}_{\mathcal{K}^*}\{\mathrm{d}y,\mathrm{d}\dot{y}-\vartheta y\dot{u}\,\mathrm{d}u\}$$

is, by Theorem 2, not integrable. In the discrete-time case the subspace

$$\mathcal{H}_3=\text{span}_{\mathcal{K}^*}\left\{\mathrm{d}y,\frac{1}{T}\mathrm{d}\sigma(y)-\frac{\vartheta}{T}(\sigma^{-1}(y)(u-\sigma^{-1}(u)))\mathrm{d}u\right\}$$

is not integrable.

Thus, we may conclude that it is not possible to find the classical state-space realization of system (21) for the cases listed above.

5. Conclusions

The linear algebraic framework of differential forms [1,2] is used to study the realization problem of nonlinear control system, described by the set of higher order delta differential equations, defined on homogeneous time scales. The latter allows us to unify the study of nonlinear control systems, defined in different time domains. The necessary and sufficient solvability conditions are formulated in terms of integrability of certain subspaces $\mathcal{H}_k$ of differential one-forms. The polynomial formulas are used to explicitly calculate the basis one-forms of subspace $\mathcal{H}_{r+2}$ (with r being the maximal delta derivative in i/o equations) directly from the polynomial system description. The latter is obtained from the globally linearized nonlinear system equations when we interpret the polynomial indeterminate as the delta derivative operator acting on input and output differentials. The basis elements of $\mathcal{H}_{r+2}$ define the differentials of the state coordinates. The explicit formulas allow us to replace the previously developed recursive algorithm [10] and significantly simplify calculations.

The solution proposed in this paper can be combined with the result from [19] (see also [16] for the SISO case) that constructs the irreducible representation for system (1), since both rely on system description in terms of two polynomial matrices. If one assumes the set of i/o equations (1) to be in the irreducible form, then the resulting nth-order realization (3) will be minimal (accessible). Note that the results obtained in this paper are valid only for systems defined on homogeneous time scales. The extension to more general cases like regular, but non-homogeneous time scale [3] (that corresponds to non-uniformly sampled system) will make the subject for the future research.

Acknowledgments

The work was supported by the European Union through the European Regional Development Fund (fund No. 3.2.01.01.11-0037) and by the ESF Grant N8787. Ü. Kotta and M. Tõnso were additionally supported by the Estonian Research Council, personal research funding Grant PUT481.

Appendix

Proof of Theorem 3. *Sufficiency*. According to (11), construct the subspaces $\mathcal{H}_1,\dots,\mathcal{H}_{r+2}$ and assume that all of them are completely integrable. Let $\{\mathrm{d}\zeta_1,\dots,\mathrm{d}\zeta_n\}$ be a basis for $\mathcal{H}_{r+2}$. Note that $\mathrm{d}\zeta_1,\dots,\mathrm{d}\zeta_n$ are one-forms with relative degree equal to or greater than $r+2$. From the structure of (1), the relative degree of $\mathrm{d}u_\kappa$ is equal to $r+1$, and therefore, $\mathrm{d}u_\kappa \in \mathcal{H}_{r+1}$. Note that the control variables are independent system variables; otherwise, $\mathrm{d}u_\kappa = \sum_{\ell=1}^{n} \alpha_{\ell\kappa}\,\mathrm{d}\zeta_\ell$ would be in $\mathcal{H}_{r+2}$ yielding a contradiction. Hence, $\mathcal{H}_{r+1} = \mathcal{H}_{r+2} \oplus \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}u_1, \ldots, \mathrm{d}u_m\}$. In a similar manner it can be shown that

$$\mathcal{H}_l = \mathcal{H}_{r+2} \oplus \mathrm{span}_{\mathcal{K}^*}\left\{\mathrm{d}u_1^{\langle r-l+1\rangle}, \ldots, \mathrm{d}u_m^{\langle r-l+1\rangle}\right\}$$

for $l=1,\ldots,r$. Next, choose the state coordinates as

$$x_\ell = \zeta_\ell\left(y_1^{\langle n-1\rangle}, \ldots, y_p^{\langle n-1\rangle}, u_1^{\langle r\rangle}, \ldots, u_m^{\langle r\rangle}\right)$$

for $\ell = 1, \ldots, n$. According to (11), one has

$$\mathrm{d}x_\ell^\Delta = \mathrm{d}\zeta_\ell^\Delta \in \mathcal{H}_{r+2} \oplus \mathrm{span}_{\mathcal{K}^*}\{u_1, \ldots, u_m\}.$$

Hence, $\mathrm{d}x_\ell^\Delta = \sum_{k=1}^{n} \beta_{\ell k}\,\mathrm{d}x_k + \sum_{l=1}^{m} \gamma_{\ell l}\,\mathrm{d}u_l$ for $\ell = 1, \ldots, n$. Thus, we get the following state-space description of the i/o equation (1):

$$\begin{aligned}
x_1^\Delta &= f_1(x_1, \ldots, x_n, u_1, \ldots, u_m)\\
&\vdots\\
x_n^\Delta &= f_n(x_1, \ldots, x_n, u_1, \ldots, u_m)\\
y_j &= x_l, \quad j = 1, \ldots, p, \ l \in \{1, \ldots, n\}
\end{aligned} \tag{22}$$

for some meromorphic functions f_i. The i/o map in the new coordinates remains the same. Therefore, the sets of solutions of (1) and (22) coincide. The proof can be concluded observing that, by construction, the state-space representation is observable. □

Necessity. Assume that (1) has an observable state-space realization of the form (3). Compute

$$\begin{aligned}
y &= h(x),\\
y^\Delta &= h^{\Delta_\phi}(x, u_1, \ldots, u_m),\\
&\vdots\\
y^{\langle n-1\rangle} &= h^{\Delta_\phi^{n-1}}\left(x, u_1^{\langle 0\ldots n-2\rangle}, \ldots, u_m^{\langle 0\ldots n-2\rangle}\right).
\end{aligned} \tag{23}$$

The set of equations (23) can be solved (generically), according to Definition 6, with respect to the state variables, yielding

$$x = \zeta\left(\left\{y_i^{\langle 0\ldots n-1\rangle}, u_\kappa^{\langle 0\ldots n-2\rangle}\right\}\right). \tag{24}$$

Next, compute $y^{\langle n\rangle}$ and substitute x from (24) to get

$$y^{\langle n\rangle} = h^{\langle n\rangle}\left(\zeta\left(\left\{y_i^{\langle 0\ldots n-1\rangle}, u_\kappa^{\langle 0\ldots n-2\rangle}\right\}\right), u_1^{\langle 0\ldots n-1\rangle}, \ldots, u_m^{\langle 0\ldots n-1\rangle}\right) \tag{25}$$

having observable state-space realization of the form (3). Note that $s<n$ by definition of (1). Thus, for the (maximal possible) case $r=n-1$ the subspaces $\mathcal{H}_1, \ldots, \mathcal{H}_{r+2}$, defined by $\mathcal{H}_i = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_n, \mathrm{d}u_\kappa, \ldots, \mathrm{d}u_\kappa^{\langle n-i\rangle}\}$ and $\mathcal{H}_{n+1} = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_n\}$, are completely integrable. □

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