An $O(n \log n)$-Space Decision Procedure for Intuitionistic Propositional Logic

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Abstract

We present a certain calculus LG which is equivalent to intuitionistic propositional logic and in which lengths of deductions are linearly bounded in terms of the size of the endsequent. Backwards application of the rules of LG thus gives rise to an $O(n \log n)$-SPACE decision algorithm for intuitionistic propositional logic. The system LG is easily implemented as a tableau system for intuitionistic logic. It is therefore of interest in its own right.

Keywords: Decision procedure, intuitionistic logic, sequent calculus.

1 Introduction

The first decision procedure for intuitionistic propositional logic has been given by Gentzen along with his sequent calculus formulation of intuitionistic and classical logic [4]. Gentzen's procedure is based on backwards construction of deduction trees in his calculus LJ, and it requires looking for repetitions of sequents on the actual branch under construction. Computationally it thus behaves very badly. (For a possible improvement see [1].)

Ladner [6] has shown that intuitionistic propositional logic is in PSPACE by giving an $O(n^4)$-SPACE decision procedure for the modal logic S4, which, by a well known linear translation is able to interpret intuitionistic propositional logic. Ladner's procedure is based on the construction of Kripke models. Shortly afterwards Statman [7] proved the PSPACE completeness of LJ by translating classical second-order propositional logic into intuitionistic propositional logic.

In this paper we present an $O(n \log n)$-SPACE decision procedure which is based on backwards application of the rules of a certain sequent calculus. The essential feature of this calculus is the fact that for every one of its rules all premisses are in some sense less complex than the conclusion. This ensures that every deduction in this calculus of a given sequent is bounded in length by some number depending only on the form of this sequent. Thus every attempt to construct a deduction tree of a given sequent is automatically guaranteed to terminate. One does not have to look for repetitions while constructing branches of the deduction tree.

We now give more details about our construction. We will use several different calculi, all of them deriving sequents of the form $M \Rightarrow r$, where M is a sequence of formulas and r is a single formula. We start with the following calculus LJ, consisting of axioms $M, p, N \Rightarrow p$ (with p a propositional variable) and $M, \bot, N, \Rightarrow r$ and rules:

$$\frac{M \Rightarrow v_0 \quad M \Rightarrow v_1}{M \Rightarrow v_0 \wedge v_1}\,(JI\wedge) \qquad \frac{M, v_0, v_1, N \Rightarrow r}{M, v_0 \wedge v_1, N \Rightarrow r}\,(JE\wedge)$$

$$\frac{M \Rightarrow v_i}{M \Rightarrow v_0 \vee v_1}\,(JI\vee) \qquad \frac{M, v_0, N \Rightarrow r \quad M, v_1, N \Rightarrow r}{M, v_0 \vee v_1, N \Rightarrow r}\,(JE\vee)$$

$$\frac{M, v_0 \Rightarrow v_1}{M \Rightarrow v_0 \to v_1}\,(JI \to) \qquad \frac{M, v_0 \to v_1, N \Rightarrow v_0 \quad M, v_1, N \Rightarrow r}{M, v_0 \to v_1, N \Rightarrow r}\,(JE \to).$$

It is well known that LJ is equivalent to the propositional part of Gentzen's homonymous calculus for intuitionistic logic. It thus suffices to deduce all intuitionistically deducible sequents. But attempts to obtain a decision procedure from this calculus suffer from the fact that the left premiss of a $(JE \to)$-inference may be more complex than its conclusion, because the principal formula is retained when going from the conclusion to the premiss. Thus backwards construction of a deduction tree for a given sequent in LJ will usually not terminate, unless the sequent is deducible.

To overcome this difficulty one appropriately divides the $(JE \to)$-rule into four rules depending on the form of the premiss of the principal formula: for this purpose one uses the well known intuitionistic equivalences $p \wedge p \to v \equiv p \wedge v$, $(u \wedge v) \to w \equiv u \to (v \to w)$, $(u \vee v) \to w \equiv (u \to w) \wedge (v \to w)$ and $u \wedge (u \to v) \to w \equiv u \wedge (v \to w)$ (cf. [8].) In this way the $(JE \to)$-rule is becoming finer by one degree. This results in the calculus LI consisting of the axioms of LJ, the rules $II\wedge := JI\wedge, II\vee := JI\vee, II \to := JI \to, IE\wedge := JE\wedge, IE\vee := JE\vee$ and the four rules:

$$\frac{L, p, M, v, N \Rightarrow r}{L, p, M, p \to v, N \Rightarrow r} \quad \frac{L, v, M, p, N \Rightarrow r}{L, p \to v, M, p, N \Rightarrow r}\,(IE \to P)$$

$$\frac{M, v_0 \to (v_1 \to w), N \Rightarrow r}{M, (v_0 \wedge v_1) \to w, N \Rightarrow r}\,(IE \to \wedge)$$

$$\frac{M, v_0 \to w, v_1 \to w, N \Rightarrow r}{M, (v_0 \vee v_1) \to w, N \Rightarrow r}\,(IE \to \vee)$$

$$\frac{M, v_0, v_1 \to w, N \Rightarrow v_1 \quad M, w, N \Rightarrow r}{M, (v_0 \to v_1) \to w, N \Rightarrow r}\,(IE \to\to).$$

(Equivalence of LJ and LI is proved in [2] and [5].)

In LI the principal formulas are replaced by simpler formulas when going backwards. Thus all premisses of all rules have smaller degree than the corresponding conclusions, where the degree is defined as follows: $deg(p) := 2$ if p is a propositional variable, $deg(u \wedge v) := deg(u)(1 + deg(v))$, $deg(u \vee v) := 1 + deg(u) + deg(v)$, $deg(u \to v) := 1 + deg(u)deg(v)$ and the degree of a sequent is the sum of the degrees of its formulas. So every deduction of a sequent s is bounded in length by $deg(s)$. LI thus solves the termination problem of LJ. Hence LI provides a decision procedure for intuitionistic propositional logic without looking for repetitions. But unfortunately $deg(s)$ grows exponentially in the size of s. This means that deductions in LI may be of exponential length in the size of their endsequent. This in turn means that every decision procedure based on LI requires exponential space. Moreover it is clear that we cannot find a slower growing function solving the task of deg, because in the rules $(IE \to \vee)$ and $(IE \to\to)$ some subformula of the conclusion in one premiss appears twice.

The strategy used to avoid this doubling of subformulas is different for both rules: For $(IE \to \vee)$ we use a strategy that has proved convenient in some other areas, e.g. conversion of formulas to disjunctive normal form—replace a complex formula by a new propositional variable.

We thus replace our principal formula $(u \vee v) \to w$ by the three formulas $u \to p, v \to p, p \to w$. Now we have only added three propositional variables instead of doubling a complex formula. Unfortunately this strategy does not work for the case of $(IE \to\to)$. But looking at the structure of deductions in LI quickly reveals that problems only arise when parts of the doubled subformula to the right of the sequent arrow return to the left-hand side via some backwards application of $(II \to)$. There is, however, one case in which this does not cause harm, because we can simultaneously throw out the corresponding formula on the left-hand side: suppose that we have a sequent $M, (u \to v) \to w, N \Rightarrow u \to v$, then if the formula u is shifted to the left, we can instantly replace the two formulas $u, (u \to v) \to w$ by the equivalent formulas $u, v \to w$ resulting in the sequent $M, u, v \to w, N \Rightarrow v$, where now we have a single u on the left-hand side. In order to be able to always proceed in this way we therefore have to keep our sequents of the form $M, u \to v, N \Rightarrow u$. The left premiss of every application of $(IE \to\to)$ *is* of this form, but it usually gets destroyed by backwards applications of the II-rules, unless we simultaneously alter the corresponding formula on the left-hand side. For instance going backwards from a sequent of the form $M, (u \wedge v) \to w, N \Rightarrow u \wedge v$ now results in the sequents $M, u \to (v \to w), N \Rightarrow u$ and $M, v \to (u \to w), N \to v$, and going backwards from $M, (u \vee v) \to w, N \Rightarrow u \vee v$ results in e.g. $M, u \to p, v \to p, p \to w, N \Rightarrow u$. In this manner sequents are always kept in a convenient form so that if we come to a sequent $M, (u \to v) \to w, N \Rightarrow u \to v$ we can apply the above reduction, thus cancelling one occurrence of u. This gives us the new calculus LG whose I-rules now come in pairs, vic. for sequents of the form $M, u \to v, N \Rightarrow u$ resp. for sequents not of this form. (Actually the $(GI2\wedge)$-rule is slightly different from what has been announced here—the difference is explained below.) In LG now every deduction of a sequent s is bounded in length by $deg(s)$, where this time deg is a linearly growing function defined below. Thus deductions are linearly bounded in terms of their endsequent. Implementing backwards application of LG in a straightforward way thus provides an $O(n \log n)$-SPACE decision procedure for intuitionistic propositional logic.

Notice that the usual tableau procedure for classical propositional logic requires a similar amount of space. The difference, which makes intuitionistic logic PSPACE complete is, of course, that we do need backtracking whereas in classical logic we don't.

2 The calculus

To prove equivalence of LJ and LG we introduce another calculus LH consisting of axioms $M, p, N \Rightarrow p$ and $M, \bot, N \Rightarrow r$ and rules

$$\frac{M \Rightarrow v_0 \quad M \Rightarrow v_1}{M \Rightarrow v_0 \wedge v_1}\,(HI\wedge) \qquad \frac{M, v_0, v_1, N \Rightarrow r}{M, v_0 \wedge v_1, N \Rightarrow r}\,(HE\wedge)$$

$$\frac{M \Rightarrow v_i}{M \Rightarrow v_0 \vee v_1}\,(HI\vee) \qquad \frac{M, v_0, N \Rightarrow r \quad M, v_1, N \Rightarrow r}{M, v_0 \vee v_1, N \Rightarrow r}\,(HE\vee)$$

$$\frac{M, v_0, N \Rightarrow v_1}{M, N \Rightarrow v_0 \to v_1}\,(HI \to)$$

$$\frac{L, p, M, v, N \Rightarrow r}{L, p, M, p \to v, N \Rightarrow r} \quad \frac{L, v, M, p, N \Rightarrow r}{L, p \to v, M, p, N \Rightarrow r}\,(HE \to P)$$

$$\frac{M, v_0 \to w, N \Rightarrow v_0 \quad M, v_1 \to w, N \Rightarrow v_1 \quad M, w, N \Rightarrow r}{M, (v_0 \wedge v_1) \to w, N \Rightarrow r} \; (HE \to \wedge)$$

$$\frac{M, v_0 \to p, v_1 \to p, p \to w, N \Rightarrow v_i \quad M, w, N \Rightarrow r}{M, (v_0 \vee v_1) \to w, N \Rightarrow r} \; (HE \to \vee) \; [p \text{ not in the conclusion}]$$

$$\frac{M, v_0, v_1 \to w, N \Rightarrow v_1 \quad M, w, N \Rightarrow r}{M, (v_0 \to v_1) \to w, N \Rightarrow r} \; (HE \to\to).$$

General restrictions

1. p is a propositional variable.
2. Conclusions of HI rules must not be of the form $M, v \to w, N \Rightarrow v$.

We need:

FOLK LORE: Every LJ-deducible sequent has an LJ-deduction in which every left premiss of a $(JE \to)$-inference is either an axiom or the conclusion of a JI-rule.

PROOF. (For a proof see [5].) ∎

Furthermore it holds that

LEMMA 2.1
If a sequent $M, v \to p, p \to w, N \Rightarrow r$ is deducible by LJ, where p is a propositional variable which does not occur in M, v, w, N or r, then the sequent $M, v \to w, N \Rightarrow r$ is also deducible.

PROOF. If $M, v \to p, p \to w, N \Rightarrow r$ is deducible, then so is the result of replacing every occurrence of p in this sequent by the formula w. But since p is not in M, N, r, v or w the result of this replacement is $M, v \to w, w \to w, N \Rightarrow r$. But by cutting out $w \to w$ this sequent is deducible iff the sequent $M, v \to w, N \Rightarrow r$ is deducible. ∎

From this follows

THEOREM 2.2
LJ and LH are equivalent.

PROOF. First we show that all rules of LH are admisible for LJ: This is obvious for the HI-rules and for $(HE\wedge)$, $(HE\vee)$ and $(HE \to P)$. For $(HE \to \wedge)$ suppose we have LJ-deductions of $M, v_0 \to w, N \Rightarrow v_0$ and $M, v_1 \to w, N \Rightarrow v_1$. Then we also have an LJ-deduction of $M, N \Rightarrow ((v_0 \to w) \to v_0) \wedge ((v_1 \to w) \to v_1)$. But since LJ deduces $M, (v_0 \wedge v_1) \to w, ((v_0 \to w) \to v_0) \wedge ((v_1 \to w) \to v_1), N \Rightarrow v_0 \wedge v_1$ cut gives us an LJ-deduction of $M, (v_0 \wedge v_1) \to w, N \Rightarrow v_0 \wedge v_1$. Together with the LJ-deduction of $M, w, N \Rightarrow r$ this gives an LJ-deduction of $M, (v_0 \wedge v_1) \to w, N \Rightarrow r$.

For $(HE \to \vee)$ we apply cut to the left premiss and to $(v_0 \vee v_1) \to p \Rightarrow (v_0 \to p) \wedge (v_1 \to p)$, obtaining a deduction of $M, (v_0 \vee v_1) \to p, p \to w, N \Rightarrow v_i$, and to this sequent we apply the lemma resulting in a deduction of $M, (v_0 \vee v_1) \to w, N \Rightarrow v_i$. From this we obtain the required deduction by applications of $(JI\vee)$ and $(JE \to)$.

For $(HE \to\to)$ we apply cut to the left premiss and to $(v_0 \to v_1) \to w \Rightarrow v_1 \to w$ resulting in a deduction of $M, v_0, (v_0 \to v_1) \to w, N \Rightarrow v_1$. From this we obtain the required deduction by applications of $(JI \to)$ and $(JE \to)$.

To prove that every LJ-deducible sequent is deducible by LH we define a degree function for sequents by setting $deg(p) := 0$ for p a propositional variable $deg(v \to w) := deg(v) + deg(w) + 1, deg(v \wedge w) := deg(v) + deg(w) + 2, deg(v \vee w) := deg(v) + deg(w) + 3, deg(M, v \to w \Rightarrow v) :=$ sum of the degrees of the formulas on the left-hand side and $deg(s) :=$ sum of the degrees of all formulas in s for sequents s not of this form. We now prove completeness by induction on the degree of a given LJ-deducible sequent and by distinguishing cases according to the last inference rule in a given LJ-deduction which obeys the restriction expressed in the folk lore result above:

If this last inference is an application of a JI-rule and s is not of the form $M, v \to w, N \Rightarrow v$, or it is an application of $(JE\wedge)$ or $(JE\vee)$, then all premisses have smaller degrees, hence they are deducible by LH, and by using the corresponding HI-rules or $(HE\wedge)$ and $(HE\vee)$, respectively, we again obtain the given sequent.

If the last rule applied is a $(JE \to)$ with principal formula $v \Rightarrow w$, then we distinguish cases according to the form of v:

If v is a propositional variable, then the left-hand side of the given sequent contains v, and the right premiss of this inference is the premiss of an application of $(HE \to P)$ leading to the given sequent. In the other cases the right premiss of the $(JE \to)$-inference is also the right premiss of the corresponding $(HE \to)$-inference leading to the given sequent.

If v is $v_0 \wedge v_1$, then the left premiss has itself two premisses of the form $M, (v_0 \wedge v_1) \to w, N \Rightarrow v_0$ and $M, (v_0 \wedge v_1) \to w, N \Rightarrow v_1$. But LJ deduces $v_0 \to w \Rightarrow (v_0 \wedge v_1) \to w$ and $v_1 \to w \Rightarrow (v_0 \wedge v_1) \to w$. Hence LJ deduces all premisses of an application of $(HE \to \wedge)$ leading to the given sequent s. Moreover all these premisses are of smaller degree.

If v is of the form $v_0 \vee v_1$, then the left premiss has itself a premiss of the form $M, (v_0 \vee v_1) \to w, N \Rightarrow v_i$ and since LJ deduces $v_0 \to p, v_1 \to p, p \to w \Rightarrow (v_0 \vee v_1) \to w$, it also deduces $M, v_0 \to p, v_1 \to p, p \to w, N \Rightarrow v_i$, which is the left premiss of an application of $(HE \to \vee)$ leading to s and which is moreover of smaller degree, hence deducible by LH.

If v is of the form $v_0 \to v_1$, then the left premiss of the last inference itself has one premiss $M, v_0, (v_0 \to v_1) \to w, N \Rightarrow v_1$. But LJ deduces $v_0, v_1 \to w \Rightarrow (v_0 \to v_1) \to w$, hence it also deduces $M, v_0, v_1 \to w, N \Rightarrow v_1$, which is the left premiss of an application of $(HE \to\to)$ leading to s and which is of smaller degree than s.

If s is of the form $M, v \to w, N \Rightarrow v$ and the final inference is by a JI-rule, then v cannot be a propositional variable and we again distinguish cases according to the form v:

If v is of the form $v_0 \wedge v_1$, then the premisses of this inference are of the form $M, (v_0 \wedge v_1) \to w, N \Rightarrow v_0$ and $M, (v_0 \wedge v_1) \to w, N \to v_1$. Now, again, LJ deduces $v_0 \to w \Rightarrow (v_0 \wedge v_1) \to w$ and $v_1 \to w \Rightarrow (v_0 \wedge v_1) \to w$, hence by cut LJ also deduces the first two premisses of an application of $(HE \to \wedge)$ leading to the given sequent s. But since LJ deduces $w \Rightarrow (v_0 \wedge v_1) \to w$, too, it also deduces $M, w, N \Rightarrow v_0 \wedge v_1$ which is the third premiss of an application of $(HE \to \wedge)$ leading to s. Again by the induction hypothesis all these premisses are deducible by LH, hence so is s.

If v is of the form $v_0 \vee v_1$ then the premiss is of the form $M, (v_0 \vee v_1) \to w, N \Rightarrow v_i$ and since LJ deduces $v_0 \to p, v_1 \to p, p \to w \Rightarrow (v_0 \vee v_1) \to w$, it also deduces $M, v_0 \to p, v_1 \to p, p \to w, N \Rightarrow v_i$. Moreover as before LJ deduces $M, w, N \Rightarrow v_0 \vee v_1$, and both these sequents are of smaller degree, hence deducible by LH. Thus by an application of $(HE \to \vee)$ s is also deducible by LH.

If v is of the form $v_0 \to v_1$, then the premiss is $M, v_0, (v_0 \to v_1) \to w, N \Rightarrow v_1$. Now as LJ deduces $v_0, v_1 \to w \Rightarrow (v_0 \to v_1) \to w$, it also deduces $M, v_0, v_1 \to w, N \Rightarrow v_1$. Then again LJ deduces $M, w, N \Rightarrow v_0 \to v_1$ and both sequents are provable by LH. Hence by an application of $(HE \to\to)$ so is s. ∎

REMARK 2.3

1. It is easily seen that the rules $(HI\wedge)$, $(HI \to)$, $(HE\wedge)$, $(HE\vee)$, and $(HE \to P)$ are invertible.
2. It is also easily seen that in the case w is atomic the three formulas $v_0 \to p, v_1 \to p$ and $p \to w$ may be replaced by the two formulas $v_0 \to w$ and $v_1 \to w$ without destroying completeness or lengthening deductions.

Inspection of the rules of LH shows:

PROPERTY 2.4
All the conclusions of LH have greater degree than their corresponding premisses.

This means that every LH-deduction of a sequent s has length at most $deg(s)$, which is atmost three times the number of connectives of s.

Deductions in LH may be shortened by subdividing the $(HE \to)$-rules according to whether the left immediate subformula of the principal formula appears on the right-hand side of the conclusion or not:

LEMMA 2.5

1. If LH deduces the two sequents $M, v_0 \to w, N \Rightarrow v_0$ and $M, v_1 \to w, \Rightarrow v_1$, then it also deduces $M, w, N \Rightarrow v_1 \wedge v_1$.
2. If LH deduces the sequent $M, v_0 \to p, v_1 \to p, p \to w, N \Rightarrow v_i$, where p is a propositional variable occurring only in the indicated positions, then it also deduces $M, w, N \Rightarrow v_0 \vee v_1$.
3. If LH deduces the sequent $M, v_0, v_1 \to w, N \Rightarrow v_1$, then it also deduces $M, w, N \Rightarrow v_0 \to v_1$.

PROOF.

1. Since LH is equivalent to LJ, cut is an admissible rule of LH. Now LH deduces $w \Rightarrow v_i \to w$. Hence under the conditions of (1) it also deduces $M, w, N \Rightarrow v_0$ and $M, w, N \Rightarrow v_1$. Thus by $(HI\wedge)$ LH also deduces $M, w, N \Rightarrow v_0 \wedge v_1$.
2. Since LH is equivalent to LJ, substitution of formulas for propositional variables is also an admissible rule for LH. Thus if LH deduces a sequent $M, v_0 \to p, v_1 \to p, p \to w, N \Rightarrow v_i$, then it also deduces the sequent $M, v_0 \to w, v_1 \to w, w \to w, N \Rightarrow v_i$, and by cut it also deduces $M, w, N \Rightarrow v_i$. Therefore by $(HI\vee)$ LH deduces the sequent $M, w, N \Rightarrow v_0 \vee v_1$.
3. If LH deduces $M, v_0, v_1 \to w, N \Rightarrow v_1$, then by $(HI \to)$ it also deduces $M, v_1 \to w, N \Rightarrow v_0 \to v_1$ and finally by cut LH deduces $M, w, N \Rightarrow v_0 \to v_1$.

Therefore if the left immediate subformula of the principal formula of an $(HE \to \wedge)$-inference appears on the right-hand side of the conclusion, then the rightmost premiss of the $(HE \to \wedge)$-rule is deducible iff the other two premisses are, and if the left immediate subformula of the principal formula of an $(HE \to \vee)$ or $(HE \to\to)$-inference appears on the right-hand side of the conclusion, then the right premiss is deducible if the left one is. Thus in any case we need not mention these rightmost premisses. If they do not appear there, then we may use one of the reductions $(v_0 \wedge v_1) \to w \longrightarrow v_0 \to (v_1 \to w)$ or

$(v_0 \vee v_1) \to w \longrightarrow v_0 \to p, v_1 \to p, p \to w$. (The first one is well known to be correct, and the second one has been shown correct above.) This finally gives us the calculus LG, consisting of the usual axioms and rules

$$\frac{M \Rightarrow v_0 \quad M \Rightarrow v_1}{M \Rightarrow v_0 \wedge v_1}\,(GI1\wedge) \qquad \frac{M, v_0 \to w, N \Rightarrow v_0 \quad M, v_1 \to w, N \Rightarrow v_1}{M, (v_0 \wedge v_1) \to w, N \Rightarrow v_0 \wedge v_1}\,(GI2\wedge)$$

$$\frac{M \Rightarrow v_i}{M \Rightarrow v_0 \vee v_1}\,(GI1\vee) \qquad \frac{M, v_0 \to p, v_1 \to p, p \to w, N \Rightarrow v_i}{M, (v_0 \vee v_1) \to w, N \Rightarrow v_0 \vee v_1}\,(GI2\vee) \quad [p \text{ not in the conclusion}]$$

$$\frac{M, v_0, N \Rightarrow v_1}{M, N \Rightarrow v_0 \to v_1}\,(GI1 \to) \qquad \frac{M, v_0, v_1 \to w, N \Rightarrow v_1}{M, (v_0 \to v_1) \to w, N \Rightarrow v_0 \to v_1}\,(GI2 \to)$$

$$\frac{M, v_0, v_1, N \Rightarrow r}{M, v_0 \wedge v_1, N \Rightarrow r}\,(GE\wedge) \qquad \frac{M, v_0, N, \Rightarrow r \quad M, v_1, N \Rightarrow r}{M, v_0 \vee v_1, N \Rightarrow r}\,(GE\vee)$$

$$\frac{L, p, M, v, N \Rightarrow r}{L, p, M, p \to v, N \Rightarrow r} \qquad \frac{L, v, M, p, N \Rightarrow r}{L, p \to v, M, p, N \Rightarrow r}\,(GE \to P)$$

$$\frac{M, v_0 \to (v_1 \to w), N \Rightarrow r}{M, (v_0 \wedge v_1) \to w, N \Rightarrow r}\,(GE \to \wedge) \quad [r \neq v_0 \wedge v_1]$$

$$\frac{M, v_0 \to p, v_1 \to p, p \to w, N \Rightarrow r}{M, (v_0 \vee v_1) \to w, N \Rightarrow r}\,(GE \to \vee) \quad [p \text{ not in the conclusion}, r \neq v_0 \vee v_1]$$

$$\frac{M, v_0, v_1 \to w, N \Rightarrow v_1 \quad M, w, N \Rightarrow r}{M, (v_0 \to v_1) \to w, N \Rightarrow r}\,(GE \to\to) \quad [r \neq v_0 \to v_1].$$

General restrictions

1. p is a propositional variable.
2. Conclusions of $(GI1)$-rules must not be of the form $M, v \to w \Rightarrow v$.

In LG all rules except $(GI1\vee)$, $(GI2\vee)$ and $(GE \to\to)$ are invertible. ∎

Figure 1 shows an example of an LG-deduction. (This is not the shortest existing LG-deduction of its endsequent—but we know that the length of every possible deduction is bounded by three times the number of connectives of its endsequent.) Deductions in both LG and LH are of course found by backwards application of the rules. This procedure gives rise to an $O(n \log n)$-SPACE tableau-like decision algorithm for intuitionistic propositional logic which—unlike the usual intuitionistic tableaux (cf. [3])—does not require loop checking.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a,p,d\to p,d\Rightarrow d}{a,p,d\to p,p\to d\Rightarrow d}\,(GE\to P)}{a,a\to p,d\to p,p\to d\Rightarrow d}\,(GE\to P) \qquad \dfrac{\dfrac{\dfrac{\dfrac{a,p,e,c\to q,d\to q,q\to p,d\Rightarrow d}{a,p,e,c\to q,d\to q,q\to p,p\to d\Rightarrow d}\,(GE\to P)}{a,a\to p,e,c\to q,d\to q,q\to p,p\to d\Rightarrow d}\,(GE\to P)}{a,a\to p,e,(c\vee d)\to p,p\to d\Rightarrow c\vee d}\,(GI2\vee)}{a,a\to p,(e\to(c\vee d))\to p,p\to d\Rightarrow e\to(c\vee d)}\,(GI2\to)}{a,a\to p,(d\wedge(e\to(c\vee d)))\to p,p\to d\Rightarrow d\wedge(e\to(c\vee d))}\,(GI2\wedge)}{a,(a\vee(d\wedge(e\to(c\vee d))))\to d\Rightarrow(d\wedge(e\to(c\vee d)))}\,(GE\to\vee)}{(a\vee(d\wedge(e\to(c\vee d))))\to d\Rightarrow a\to(d\wedge(e\to(c\vee d)))}\,(GI1\to)}{(a\vee(d\wedge(e\to(c\vee d))))\to d\Rightarrow(a\to(d\wedge(e\to(c\vee d))))\vee f}\,(GI1\vee)}{}$$

FIG. 1. An LG-deduction

3 Implementation

To describe such a decision algorithm we consider two formulations LF and LE of LG, and LE will be seen to be directly translatable into such an algorithm. The formulas of both LF and LE are those of LG plus a new propositional variable Λ plus all the formulas $v \to^\circ w$, where v and w are LG formulas (the connective $\to^\circ$ is called the marked arrow). The sequents contain at most one marked arrow and at most one Λ; the marked arrows only occur on the left-hand side and the Λ only on the right hand side. (An LF/LE sequent of the form $M, v \to^\circ w, N \Rightarrow \Lambda$ is meant to stand for the LG sequent $M, v \to w, N \Rightarrow v$.) Now the axioms of LF are all sequents of the form $M, p, N \Rightarrow p$ or $L, p, M, p \to^\circ v, N \Rightarrow \Lambda$, or $L, p \to^\circ v, M, p, N \Rightarrow \Lambda$, where p is a propositional variable and $M, \bot, N \Rightarrow r$ and its rules are

$$\frac{M \Rightarrow v_0 \quad M \Rightarrow v_1}{M \Rightarrow v_0 \wedge v_1}\,(FI1\wedge) \qquad \frac{M, v_0 \to^\circ w, N \Rightarrow \Lambda \quad M, v_1 \to^\circ w, N \Rightarrow \Lambda}{M, (v_0 \wedge v_1) \to^\circ w, N \Rightarrow \Lambda}\,(FI2\wedge)$$

$$\frac{M \Rightarrow v_i}{M \Rightarrow v_0 \vee v_1}\,(FI1\vee)$$

$$\frac{M, v_0 \to p, v_1 \to^\circ p, p \to w, N \Rightarrow \Lambda}{M, (v_0 \vee v_1) \to^\circ w, N \Rightarrow \Lambda} \quad \frac{M, v_0 \to^\circ p, v_1 \to p, p \to w, N \Rightarrow \Lambda}{M, (v_0 \vee v_1) \to^\circ w, N \Rightarrow \Lambda}\,(FI2\vee)^*$$

*: [p not occurring in the conclusion]

$$\frac{M, v_0, N \Rightarrow v_1}{M, N \Rightarrow v_0 \to v_1}\,(FI1 \to) \qquad \frac{M, v_0, v_1 \to^\circ w, N \Rightarrow \Lambda}{M, (v_0 \to v_1) \to^\circ w, N \Rightarrow \Lambda}\,(FI2 \to)$$

$$\frac{M, v_0, v_1, N \Rightarrow r}{M, v_0 \wedge v_1, N \Rightarrow r}\,(FE\wedge)$$

$$\frac{L, p, M, v, N \Rightarrow r}{L, p, M, p \to v, N \Rightarrow r} \qquad \frac{L, v, M, p, N \Rightarrow r}{L, p \to v, M, p, N \Rightarrow r}\,(FE \to P)$$

$$\frac{M, v_0, N \Rightarrow r \quad M, v_1, N \Rightarrow r}{M, v_0 \vee v_1, N \Rightarrow r} (FE\vee)$$

$$\frac{M, v_0 \to (v_1 \to w), N \Rightarrow r}{M, (v_0 \wedge v_1) \to w, N \Rightarrow r} (FE \to \wedge)$$

$$\frac{M, v_0 \to p, v_1 \to p, p \to w, N \Rightarrow r}{M, (v_0 \vee v_1) \to w, N \Rightarrow r} (FE \to \vee) \quad [p \text{ not in the conclusion}]$$

$$\frac{M', v_0, v_1 \to^\circ w, N' \Rightarrow \Lambda \quad M, w, N \Rightarrow r}{M, (v_0 \to v_1) \to w, N \Rightarrow r} (FE \to\to).$$

[M' and N' result from M and N by replacing marked arrows by arrows]

One easily proves:

LEMMA 3.1
LF deduces all sequents which LG deduces.

PROOF. First, a straightforward induction on the length of LG deductions shows:

(*) if LG deduces a sequent of the form $M, v \to w, N \Rightarrow v$, then LF deduces the sequent $M, v \to^\circ w, N \Rightarrow \Lambda$.

Now the general case is proved by induction on the degree of the given sequent s: If s is an LG axiom, then it is also an LF axiom. If it is the conclusion of one of the $(GI1)$-rules or of a GE-rule different from $(GE \to\to)$, then the premisses are of smaller degree and thus are deducible by LF. But these are also the premisses of the corresponding LF rules leading to the conclusion s. Therefore s is deducible by LF. If s is the conclusion of an application of $(GI2\wedge)$, then s is of the form $M, (v_0 \wedge v_1) \to w, N \Rightarrow v_0 \wedge v_1$. Hence if s is deducible by LG, so is the sequent $M, v_0 \to (v_1 \to w), N \Rightarrow v_0 \wedge v_1$, which is of smaller degree, i.e. deducible by LF. But this latter sequent is the premiss of an $(FE \to \wedge)$-inference leading to s: (Note that the restriction $r \neq v_0 \wedge v_1$ has been dropped.) Thus s is deducible by LF. Similarly, if s is the conclusion of a $(GI2\vee)$-inference, then s is of the form $M, (v_0 \vee v_1) \to w, N \Rightarrow v_0 \vee v_1$ and, since LG is equivalent to LJ the sequent $M, v_0 \to p, v_1 \to p, p \to w, N \Rightarrow v_0 \vee v_1$ is also deducible. This sequent is of smaller degree than s, hence deducible by LF and is the premiss of an application of $(FE \to \vee)$ leading to s. Thus s is also deducible by LF. If s is the conclusion of a $(GI2 \to)$-inference, then it is of the form $M, (v_0 \to v_1) \to w, N \Rightarrow v_0 \to v_1$. Then the sequent $M, w, N \Rightarrow v_0 \to v_1$ is deducible by LG and by induction hypothesis also by LF. Also the sequent $M, v_0, v_1 \to w, N \Rightarrow v_1$ is deducible by LG, hence, according to $(*)$, the sequent $M, v_0, v_1 \to^\circ w, N \Rightarrow \Lambda$ is deducible by LF. But these two sequents are the premisses of an $(FE \to\to)$-inference leading to s. Hence s is deducible by LF. Finally, if s is the conclusion of an application of $(GE \to\to)$, then its left premiss is of a form to which $(*)$ may be applied. Furthermore application of $(*)$ to this sequent results in a sequent which is the left premiss of an application of $(FE \to\to)$ leading to s. But the right premiss of such an $(FE \to\to)$-inference coincides with the right premiss of the given $(GE \to\to)$-inference, hence it is deducible by the induction hypothesis. Therefore s is deducible by LF. ∎

Furthermore it holds that

LEMMA 3.2
If LF deduces a sequent s without marked arrows, then LJ deduces s, too.

PROOF. Let us call a sequent 'ordinary', if it either contains *no* marked arrow or it contains a marked arrow and its right-hand side is Λ. Let us furthermore consider a transformation τ on ordinary sequents, which sends every sequent without a marked arrow into itself and every sequent of the form $M, v \rightarrow^{\circ} w, N \Rightarrow \Lambda$ into $M, v \rightarrow w, N \Rightarrow v$. Then induction on the length of LF deductions shows that if LF deduces an ordinary sequent s, then LJ deduces the sequent $\tau(s)$:

This is obvious for LF-axioms. Now let s be the conclusion of some LF-inference R. Then inspection of the rules of LF shows that the premisses of s are also ordinary sequents. Thus by the induction hypothesis the transformed premisses are deducible by LJ. But further inspection of the rules shows that for all rules

$$\frac{S}{T} \text{ and } \frac{S \quad S'}{T}$$

of LF the rules

$$\frac{\tau(S)}{\tau(T)} \text{ and } \frac{\tau(S) \quad \tau(S')}{\tau(T)}$$

either are LG rules or result from the rules $(GE \rightarrow \wedge)$, $(GE \rightarrow \vee)$ and $(GE \rightarrow\rightarrow)$ by dropping the restriction on r. All these rules are, of course, admissible for LJ. Therefore $\tau(s)$ is deducible by LJ. Since for sequents without marked arrows τ is the identity, this implies the result. ∎

Now LJ and LG are equivalent, hence

THEOREM 3.3
LF deduces exactly the same sequents without marked arrows as does LG.

Now we call a 'prededuction' in a given calculus any tree of sequents of this calculus where successive nodes obey the rules of the calculus and we consider the other new calculus LE: its objects are classical, i.e. multi-succedent sequents, its axioms are the sequents of the form $M, p, N \Rightarrow p, R$ or $L, p, M, p \rightarrow^{\circ} v, N \Rightarrow \Lambda R$, or $L, p \rightarrow^{\circ} v, M, p, N \Rightarrow \Lambda, R$ where p is a propositional variable and $M, \bot, N \Rightarrow R$ and its rules are

$$\frac{M \Rightarrow v_0, R, v_1 \quad M \Rightarrow v_1, R, v_0}{M \Rightarrow v_0 \wedge v_1, R} (EI1\wedge)$$

$$\frac{M, v_0 \rightarrow^{\circ} w, N \Rightarrow \Lambda, R, v_1 \quad M, v_1 \rightarrow^{\circ} w, N \Rightarrow \Lambda, R, v_0}{M, (v_0 \wedge v_1) \rightarrow^{\circ} w, N \Rightarrow \Lambda, R} (EI2\wedge)$$

$$\frac{M \Rightarrow v_i, R, v_{1-i}}{M \Rightarrow v_0 \vee v_1, R} (EI1\vee)$$

$$\frac{M, v_0 \rightarrow^{\circ} p, v_1 \rightarrow p, p \rightarrow w, N \Rightarrow \Lambda, R}{M, (v_0 \vee v_1) \rightarrow^{\circ} w, N \Rightarrow \Lambda, R} \quad \frac{M, v_0 \rightarrow p, v_1 \rightarrow^{\circ} p, p \rightarrow w, N \Rightarrow \Lambda, R}{M, (v_0 \vee v_1) \rightarrow^{\circ} w, N \Rightarrow R} (EI2\vee)^{*}$$

*: [p not occurring in the conclusion]

$$\frac{M, v_0 \Rightarrow v_1, R}{M \Rightarrow v_0 \rightarrow v_1, R} (EI1 \rightarrow)$$

$$\frac{M, v_0, v_1, N \Rightarrow R}{M, v_0 \wedge v_1, N \Rightarrow R}\,(EE\wedge) \qquad \frac{M, v_0, v_1 \rightarrow^\circ w, N \Rightarrow \Lambda, R}{M, (v_0 \rightarrow v_1) \rightarrow^\circ w, N \Rightarrow \Lambda, R}\,(EI2 \rightarrow)$$

$$\frac{L, p, M, v, N \Rightarrow R, p}{L, p, M, p \rightarrow v, N \Rightarrow R} \qquad \frac{L, v, M, p, N \Rightarrow R, p}{L, p \rightarrow v, M, p, N \Rightarrow R}\,(EE \rightarrow P)$$

$$\frac{M, v_0, N \Rightarrow R, v_1 \quad M, v_1, N \Rightarrow R, v_0}{M, v_0 \vee v_1, N \Rightarrow R}\,(EE\vee)$$

$$\frac{M, v_0 \rightarrow (v_1 \rightarrow w), N \Rightarrow R}{M, (v_0 \wedge v_1) \rightarrow w, N \Rightarrow R}\,(EE \rightarrow \wedge)$$

$$\frac{M, v_0 \rightarrow p, v_1 \rightarrow p, p \rightarrow w, N \Rightarrow R}{M, (v_0 \vee v_1) \rightarrow w, N \Rightarrow R}\,(EE \rightarrow \vee) \quad [p \text{ not in the conclusion}]$$

$$\frac{M', v_0, v_1 \rightarrow^\circ w, N' \rightarrow \Lambda, R, r \quad M, w, N \Rightarrow r, R, v_0, v_1}{M, (v_0 \rightarrow v_1) \rightarrow w, N \Rightarrow r, R}\,(EE \rightarrow\rightarrow).$$

[M' and N' result from M and N by replacing marked arrows by arrows]

We immediately observe:

1. If LF deduces a sequent $M \Rightarrow r$, then LE deduces every sequent $M \Rightarrow r, R$.
2. If LE deduces a sequent $M \Rightarrow r, R$, then LF deduces the sequent $M \Rightarrow r$.

Taken together these observations imply that on intuitionistic sequents LF and LE coincide, i.e. LF deduces a sequent $M \Rightarrow r$ iff LE deduces this sequent.

Moreover, given an LE inference

$$\frac{S}{T}\,(R)$$

together with the positions of its principal formula and the marked arrow in the conclusion, it is obvious how to get back the sequent T from its premiss S. Likewise, given an LE inference

$$\frac{S \quad S'}{T}\,(R)$$

and the positions of the principal formula and the marked arrow the sequent T may be obtained both from S and from S'. This shows that the names and premisses of all LE inferences together with the positions of principal formulas and marked arrows uniquely determine their conclusions. Similarly a whole branch of an LE prededuction may be represented by its topmost sequent together with a list of names of inferences and positions of principal formulas and marked arrows.

Therefore we need not store whole branches of predeductions during backward proof search in LE, but only a single sequent and three lists of natural numbers.

Now the familiar depth-first left-to-right implementation of backward proof search in a calculus with non invertible rules like LE uses—besides the sequents themselves–lists of both branching and backtracking points. These lists contain one entry for each line of the current branch of the current prededuction. But as we already have our list of positions of principal formulas we may at the same time use this list as our list of backtracking points: if a deduction could not be obtained using as principal formula the formula at place n, try formula $n+1$ as new principal formula. Neither do we need to store the list of branching points separately, but instead we may simply add one more bit to the names of inferences indicating whether we proceed with the left or right premiss of our current sequent.

In order to determine the size of the memory space needed to store our sequents and the three lists of positions of principal formulas, positions of marked arrows and (augmented) names of inferences we define a measure μ on formulas via $\mu(p) := 1$, for p a propositional variable, $\mu(u \wedge v) := \mu(u) + \mu(v) + 2$, $\mu(u \vee v) := \mu(u) + \mu(v) + 6$ and $\mu(u \to v) := \mu(u \to^{\circ} v) := \mu(u) + \mu(v) + 1$, and for t a sequent $\mu(t)$ is defined as the sum of all $\mu(v)$ for the formulas v of t. Then all the premisses of LE-rules have smaller measure than their conclusions. But obviously there is some linear function $l(n)$ such that $\mu(t) \leq l(\lambda(t))$, where $\lambda(t)$ is the number of symbols of a sequent t. This means that no branch of an LE-prededuction of a sequent s is longer than $l(\lambda(s))$. Hence the lists of positions of principal formulas and marked arrows and the list of names of inferences are also not longer than this number. But the names of inferences take constant memory space and the positions of principal formulas and marked arrows are numbers $\leq l(\lambda(s))$. These numbers may be represented in $O(\log(\lambda(s)))$ space. Thus the size of the memory space needed to store the three lists is of order $O(\lambda(s)\log(\lambda(s)))$. Now the memory space needed to store a sequent depends on the size of each symbol. For the symbols which occur in the endsequent of an LE-deduction the size does not depend on the length of the sequent; it is therefore to be considered as constant. But the number of new propositional variables introduced during backward applications of $(EI2\vee)$ and $(EE \to \vee)$ on a branch of a prededuction does depend on the length of this branch. But this number is not greater than $l(\lambda(s))$. The new propositional variables may therefore be represented in $O(\log(\lambda(s)))$ space and the whole sequents may be represented in $O(\lambda(s)\log(\lambda(s)))$ space.

Thus backward proof search in LE may be performed by an $O(n \log n)$-SPACE algorithm.

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