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COMPUTING AND DOMINATING THE RYLL-NARDZEWSKI FUNCTION

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For a countably categorical theory $\mathtt{T}$*, we study the complexity of computing and the complexity of dominating the function specifying the number of* n*-types consistent with* $\mathtt{T}$*.*

1. PRELIMINARIES

Independently in 1959, E. Engeler [1], C. Ryll-Nardzewski [2], and L. Svenonius [3] provided a myriad of necessary and sufficient conditions on a first-order theory for it to be countably categorical.[1] Of these, perhaps the best remembered condition is the following: for each $n \in \mathbb{N}$, there exist only finitely many n-types consistent with the theory.

Definition. A theory $\mathtt{T}$ is *countably categorical* (alternately $\aleph_0$-*categorical*) if $\mathtt{T}$ has, up to isomorphism, a unique countable model.

Ryll-Nardzewski **THEOREM** [1-3].[2] A theory $\mathtt{T}$ is countably categorical if and only if for each $n \in \mathbb{N}$ there are only finitely many n-types consistent with $\mathtt{T}$.

For a countably categorical theory $\mathtt{T}$, the Ryll-Nardzewski theorem implies that the function mapping an integer n to the number of n-types consistent with $\mathtt{T}$ is a well-defined function from $\mathbb{N}$ to $\mathbb{N}$. In this paper, we study the complexity of this function.

Definition. For an arbitrary theory $\mathtt{T}$, the *Ryll-Nardzewski function* for $\mathtt{T}$ is the function $\mathrm{RN}_{\mathtt{T}} : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ such that $\mathrm{RN}_{\mathtt{T}}(n)$ gives the number of n-types consistent with $\mathtt{T}$.

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[1] In this paper, a theory is always first-order, complete, and consistent.

[2] Though usually called the Ryll-Nardzewski theorem, it should be noted that the result was independently and nearly simultaneously proved by three mathematicians [1-3]. It is only because of historical reasons that its name attributes it to Ryll-Nardzewski.

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By the Ryll-Nardzewski theorem, the function $\mathrm{RN}_{\mathtt{T}}$ has a range inside $\mathbb{N}$ if and only if $\mathtt{T}$ is countably categorical. The main result of the present paper provides sharp upper bounds on the complexity of computing and the complexity of dominating $\mathrm{RN}_{\mathtt{T}}$ for a countably categorical structure. First, we recall the analogous result for a countably categorical theory.

THEOREM 1 [4]. Let $\mathtt{T}$ be a countably categorical theory. Then $\mathrm{RN}_{\mathtt{T}} \leq_T \mathtt{T}'$. Moreover, this bound is sharp.

Proof. For any theory $\mathtt{T}$, we have $\mathrm{RN}_{\mathtt{T}}(n) \geq m$ if and only if

$$(\exists\psi_1(\bar{x}))\dots(\exists\psi_m(\bar{x})) \bigwedge_{1\leq i\leq m} \mathtt{T} \vdash (\exists\bar{a})\,[\psi_i(\bar{a})] \wedge (\forall\bar{x}) \left[\psi_i(\bar{x}) \implies \bigwedge_{j\neq i} \neg\psi_j(\bar{x})\right], \tag{1}$$

where $\psi_i(\bar{x})$ has exactly n free variables. For a countably categorical theory $\mathtt{T}$, the value of $\mathrm{RN}_{\mathtt{T}}(n)$ is finite for all n by the Ryll-Nardzewski theorem. To compute $\mathrm{RN}_{\mathtt{T}}(n)$, therefore, it suffices to find the greatest m for which $\mathrm{RN}_{\mathtt{T}}(n) \geq m$. Since the outer conjunction in (1) is finitary, it is immediate that $\mathtt{T}'$ suffices as an oracle to do so.

We refer the reader to [4] for sharpness. Alternately, it follows from Theorem 2. □

THEOREM 2. There is a computable structure with a countably categorical theory $\mathtt{T}$ such that any function f dominating $\mathrm{RN}_{\mathtt{T}}$ computes $\varnothing^{(\omega+1)}$. In particular, the Ryll-Nardzewski function $\mathrm{RN}_{\mathtt{T}}$ satisfies $\mathrm{RN}_{\mathtt{T}} \equiv_T \varnothing^{(\omega+1)}$.

By Theorem 1, the result obtained in Theorem 2 is sharp. Theorem 2 is proved in Sec. 2. Before delving into its proof, we mention some related literature. A countably categorical theory $\mathtt{T}$ such that $\mathtt{T} \equiv_T \varnothing^{(\omega)}$ is exemplified in [5]. In [6], for any $\mathbf{d} \leq_{tt} \varnothing^{(\omega)}$, a countably categorical theory $\mathtt{T}$ such that $\mathtt{T} \equiv_{tt} \mathbf{d}$ is constructed using a finite language. In both cases, however, there is a computable function f dominating $\mathrm{RN}_{\mathtt{T}}$. Therefore, those theories are inadequate to establish Theorem 2.

We refer the reader to [7] for background on model theory (especially Sec. 6.1 which covers Fraïssé constructions) and to [8] for background on computability theory and computable model theory.

2. PROOF OF THEOREM 2

Our construction of a theory $\mathtt{T}$ witnessing the theorem relies heavily on the existence of a $\mathbf{0}^{(\omega+1)}$-computable function possessing an approximation satisfying various properties. In Sec. 2.1, we demonstrate the existence of such a function and approximation. In Sec. 2.2, we exhibit the model $\mathcal{M}$ and verify that it has the requisite properties.

2.1. The function to dominate. We include a proof of Lemma 3 as the form of h is important for showing Lemma 4.

LEMMA 3 [9, Thm. 4.13].[3] There is a total $\varnothing^{(\omega+1)}$-computable function $h : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall g : \mathbb{N} \to \mathbb{N}) \left[(\forall x \in \mathbb{N}) \, [g(x) > h(x)] \implies g \geq_T \varnothing^{(\omega+1)} \right].$$

Proof. Let $h_1 : \mathbb{N} \to \mathbb{N}$ be the function given by

$$h_1(\langle i, j \rangle) := \begin{cases} s & \text{if } j \text{ enters } \left(\varnothing^{(i-1)}\right)' \text{ at stage } s, \\ 0 & \text{otherwise, i.e., if } j \notin \left(\varnothing^{(i-1)}\right)'. \end{cases}$$

Let $h_2 : \mathbb{N} \to \mathbb{N}$ be the function given by

$$h_2(x) := (\mu s) \left[\varnothing^{(\omega+1)} \restriction x = K^{\varnothing^{(\omega)}}[s] \restriction x \right].$$

Define $h : \mathbb{N} \to \mathbb{N}$ by $h(x) := h_1(x) + h_2(x)$.

It follows immediately from the definition that $h \leq_T \varnothing^{(\omega+1)}$. Therefore, we need only argue that any function g dominating h computes $\varnothing^{(\omega+1)}$. As a first step, we show that any function g dominating h_1 computes $\varnothing^{(\omega)}$. Indeed, given i and j, using g, we can determine whether $j \in \varnothing^{(i)}$ by seeing if the computation $\varphi_j^{\varnothing^{(i-1)}}(j)[g(\langle i, j \rangle)]$ converges. Of course, the computation $\varphi_j^{\varnothing^{(i-1)}}(j)[g(\langle i, j \rangle)]$ converges if and only if $\varphi_j^{\varnothing^{(i-1)}}(j)[h_1(\langle i, j \rangle)]$ converges as g dominates h. The computation $\varphi_j^{\varnothing^{(i-1)}}(j)[g(\langle i, j \rangle)]$ may query $\varnothing^{(i-1)}$ as an oracle on a finite set of numbers. Having reduced the question whether j is in $\varnothing^{(i)}$ to a finite set of questions about $\varnothing^{(i-1)}$, repeating as such, we eventually reduce to questions about $\varnothing$, which are computable.

Thus if g dominates h_1, then it computes $\varnothing^{(\omega)}$. As a second step, we show that g computes $\varnothing^{(\omega+1)}$. Indeed, given x, h_1, and h_2, we can determine whether $x \in \varnothing^{(\omega+1)}$ by computing $\varphi_x^{\varnothing^{(\omega)}}(x)[g(x)]$. Since g dominates h_2, this converges if and only if $x \in \varnothing^{(\omega+1)}$. Moreover, the computation is g-computable as g dominates h_1 and hence computes $\varnothing^{(\omega)}$. □

When building the theory T, it will be necessary to approximate the function h. Though perhaps not strictly necessary, it simplifies later arguments if we impose strong constraints on how the approximations behave. Essentially, it is helpful to assume that the approximations computed by $\varnothing^{(n)}$ for $n \in \mathbb{N}$ do not increase too rapidly nor require the full computational power of the oracle.

LEMMA 4. There is a sequence of functions $\{f_n : \mathbb{N} \to \mathbb{N}\}_{n \in \mathbb{N}}$ such that:

(F1) the function $f_n : \mathbb{N} \to \mathbb{N}$ is uniformly $\varnothing^{(n-4)}$-computable;

(F2) the functions $\{f_n\}_{n \in \mathbb{N}}$ satisfy $h(m) = \lim_{n \to \infty} f_n(m)$;

(F3) the function f_0 satisfies $f_0(m) = 0$ for all m;

(F4) the functions $\{f_n\}_{n \in \mathbb{N}}$ satisfy $f_n(n+3) = 0$;

(F5) for all $n, m \in \mathbb{N}$, $0 \leq f_{n+1}(m) - f_n(m) \leq 1$;

(F6) for all $n \in \mathbb{N}$, $|\{m : f_{n+1}(m) - f_n(m) = 1\}| \leq 1$.

[3] Though we reference Jockusch and McLaughlin [9] for the next result, it was known before then, at least implicitly. For example, it follows from the fact that for every $x \in \mathcal{O}$, there is a Π_1^0-singleton f in Baire space with $f \equiv_T H_x$ (Rogers [10]), and the fact that the Π_1^0-singletons coincide with the uniformly majorreducible functions (Kuznetsov and Trakhtenbrot [11]).

For notational convenience, we let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function given by $f(m,n) := f_n(m)$.

Proof. It is enough to satisfy (F1) and (F2) since (F3)-(F6) can be easily achieved by slowing down and distributing any increases in the approximation. We describe how to approximate h_1 and h_2 separately, denoting their nth respective approximation functions by $f_{1,n}$ and $f_{2,n}$. Then $f_n := f_{1,n} + f_{2,n}$ gives an approximation to h.

For approximating $h_1(\langle i,j\rangle)$, it suffices to take

$$f_{1,n}(\langle i,j\rangle) := \begin{cases} s & \text{if } n > i+3 \text{ and } j \text{ enters } \left(\varnothing^{(i-1)}\right)' \text{ at stage } s, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_{1,n}(\langle i,j\rangle)$ is uniformly $\varnothing^{(n-4)}$-computable: The value is zero unless $n > i{+}3$, i.e., $n{-}4 > i{-}1$, and so $\varnothing^{(n-4)}$ knows if and when j will enter $\left(\varnothing^{(i-1)}\right)'$. Since $f_{1,n}(\langle i,j\rangle) = h_1(\langle i,j\rangle)$ if $n > i+3$, we have $h_1(\langle i,j\rangle) = \lim_{n\to\infty} f_{1,n}(\langle i,j\rangle)$.

For approximating $h_2(x)$, it suffices to take

$$f_{2,n}(x) := (\mu s)(\forall j < x) \left[\text{if } j \in K^{\varnothing^{(\omega)}} \text{ with use contained in } \varnothing^{(n-5)}, \text{ then } j \in K_s^{\varnothing^{(\omega)}}\right].$$

Then $f_{2,n}(x)$ is uniformly $\varnothing^{(n-4)}$-computable: For each j less than x, the oracle $\varnothing^{(n-4)}$ can determine if and when j enters $K^{\varnothing^{(\omega)}}$ with use contained in $\varnothing^{(n-5)}$. The value of $f_{2,n}(x)$ is then the maximum of the stages for those j that enter. Also, for any j, if j enters $K^{\varnothing^{(\omega)}}$, the computation uses a bounded number M_j of jumps. Let M be the maximum of the number of such jumps for j less than x, i.e., $M := \max\{M_j\}_{j<x}$. Then $f_{2,n}(x) = h_2(x)$ for all $n > M+1$. Therefore, $h_2(x) = \lim_{n\to\infty} f_{2,n}(x)$. $\square$

2.2. The Fraïssé construction. In a manner similar to [6], we will employ a Fraïssé construction to create a countably categorical theory $\mathtt{T}$. The theory $\mathtt{T}$ will be such that $\mathrm{RN}_{\mathtt{T}}$ dominates the function h. The language of $\mathtt{T}$ will be a reduct of the language

$$L := \{U, V\} \cup \{R_i \mid i \in \omega, i \geq 3\} \cup \{Q_{j,k} \mid j,k \in \omega\},$$

where U and V are binary relations, R_i is an i-ary relation, and $Q_{j,k}$ is a j-ary relation.

The intuition is that the presence of the relation $Q_{j,k}$ (on some tuple) will code that $f(j,k) = f(j,k-1)+1$; the absence of the relation $Q_{j,k}$ (on every tuple) will code that $f(j,k) = f(j,k-1)$. The remaining relations serve to create a countably categorical theory (after taking a Fraïssé limit) such that the full theory is a definitional expansion of the theory restricted to the language $\{U, V, R_3\}$. In the next definition, unfortunately, this intuition may be masked to a reader unfamiliar with similar constructions.

Definition. Let $\mathcal{K}$ be the class of finite L-structures $\mathcal{C}$ for which the following hold:

(K1) each relation on $\mathcal{C}$ is symmetric and holds only on tuples of distinct elements;

(K2) the structure $\mathcal{C}$ satisfies

$$\neg(\exists\bar{x})(\exists y)(\exists z)\left[R_i(\bar{x}) \wedge U(y,z) \wedge \bigwedge_{\substack{\bar{w}\subset\bar{x} \\ |\bar{w}|=i-2}} (R_{i-1}(\bar{w},y) \wedge R_{i-1}(\bar{w},z))\right];$$

(K3) if $f(j,n) > f(j,n-1)$, then $\mathcal{C}$ satisfies

$$\neg(\exists x_1 \ldots \exists x_j)(\exists y_1 \ldots \exists y_{n-j})\left[Q_{j,n}(\bar{x}) \wedge R_n(\bar{x},\bar{y}) \wedge \bigwedge_{y_i,y_j} V(y_i,y_j)\right];$$

(K4) if $f(j,n) = f(j,n-1)$, then $\mathcal{C}$ satisfies

$$\neg(\exists\bar{x})\,[Q_{j,n}(\bar{x})]\,.$$

To use the Fraïssé construction, we need to verify that $\mathcal{K}$ has the hereditary property, the amalgamation property, and the joint embedding property.

LEMMA 5. The class $\mathcal{K}$ satisfies the hereditary property, the amalgamation property, and the joint embedding property.

Proof. The class $\mathcal{K}$ has the hereditary property since it is defined via universal formulae.

For the amalgamation property, we show that if $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are L-structures in $\mathcal{K}$ with $\mathcal{A} \subseteq \mathcal{B}, \mathcal{C}$, then there are an L-structure $\mathcal{D} \in \mathcal{K}$ and embeddings $g : \mathcal{B} \to \mathcal{D}$ and $h : \mathcal{C} \to \mathcal{D}$ with $g \restriction_A = h \restriction_A$. Fixing $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, we let $\mathcal{D}$ be the free join of $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$, i.e., the structure with universe $B \cup C$ and with relations $R^{\mathcal{B}\cup\mathcal{C}} := R^{\mathcal{B}} \cup R^{\mathcal{C}}$ for every $R \in L$. Then $\mathcal{D}$ satisfies (K1) as both $\mathcal{B}$ and $\mathcal{C}$ satisfy (K1). Also $\mathcal{D}$ satisfies (K2)-(K4) as both $\mathcal{B}$ and $\mathcal{C}$ do and no relations hold in D other than those in B and C. In particular, every two elements in the disallowed tuple are in some realization of some relation, and so the disallowed tuple, were it to exist in D, would have to be a subset of B or C. Thus we conclude that $\mathcal{D} \in \mathcal{K}$, showing that $\mathcal{K}$ has the amalgamation property.

Taking $A = \varnothing$, we see that $\mathcal{K}$ has the joint embedding property. □

Let $\mathcal{M}$ be the unique Fraïssé limit (see, e.g., [7, Thm. 6.1.2]) of the class $\mathcal{K}$. The theory $\mathtt{T}$ will be the theory for an appropriate reduct of $\mathcal{M}$. Since $\mathcal{M}$ will be a definitional expansion of the reduct, we verify various facts about $\mathcal{M}$ rather than the reduct.

LEMMA 6. The theory of $\mathcal{M}$ is countably categorical. Therefore, the theory of any reduct of $\mathcal{M}$ is countably categorical.

Proof. Being a Fraïssé limit, the structure $\mathcal{M}$ is ultrahomogeneous and, hence, admits quantifier elimination. Thus, the number of n-types is determined by the number of quantifier-free n-types. For each n, there are only finitely many relations among P, R_i, and $Q_{j,k}$ whose arity is at most n and which have occurrences in the types considered (since $h(n)$ is finite). Consequently, the theory of $\mathcal{M}$ is countably categorical.

Also, the reduct of any countably categorical theory is countably categorical. □

LEMMA 7. The function $\mathrm{RN}_{Th(\mathcal{M})}$ dominates h. Therefore, in any theory $\mathtt{T}$ for which $\mathrm{Th}(\mathcal{M})$ is a definitional expansion of $\mathtt{T}$, the function $\mathrm{RN}_{\mathtt{T}}$ dominates h.

Proof. Fix j. In view of (F2), (F3), and (F5), there are at least $h(j)$ many n such that $f(j,n) > f(j,n-1)$. For each of these n, the relation $Q_{j,n}$ will hold on some tuple on which no other relation $Q_{j,n'}$ holds for $n' \neq n$. Of course, this exploits the ultrahomogeneity of $\mathcal{M}$. Consequently, there are at least $h(j)$ many distinct n-types, and so $\mathrm{RN}_{Th(\mathcal{M})}(j) \geq h(j)$. □

We now show that we can restrict our attention to an appropriate reduct of $\mathcal{M}$.

LEMMA 8. If $i > 3$, then

$$\mathcal{M} \models (\forall \bar{x}) \left[R_i(\bar{x}) \iff \neg(\exists y)(\exists z) \left(U(y,z) \wedge \bigwedge_{\substack{\bar{w}\subset\bar{x} \\ |\bar{w}|=i-2}} (R_{i-1}(\bar{w},y) \wedge R_{i-1}(\bar{w},z)) \right) \right];$$

if $f(j,n) > f(j,n-1)$, then

$$\mathcal{M} \models (\forall \bar{x}) \left[Q_{j,n}(\bar{x}) \iff \neg(\exists \bar{y}) \left(R_n(\bar{x},\bar{y}) \wedge \bigwedge_{y_i,y_k} V(y_i,y_k) \right) \right];$$

if $f(j,n) = f(j,n-1)$, then

$$\mathcal{M} \models (\forall \bar{x}) \left[\neg Q_{j,n}(\bar{x}) \right].$$

Therefore, the structure $\mathcal{M}$ is a definitional expansion of its reduct to the language $\{U, V, R_3\}$. Thus the reduct has the same Ryll-Nardzewski function as $\mathcal{M}$.

Proof. The rightward directions follow immediately from (K2)-(K4). We show the leftward directions via the contrapositive.

Suppose $\mathcal{M} \models \neg R_i(\bar{x})$. By ultrahomogeneity, it suffices to see that there is some $\mathcal{C} \in \mathcal{K}$ which extends $\bar{x}$ so that

$$\mathcal{C} \models (\exists y)(\exists z) \left[U(y,z) \wedge \bigwedge_{\substack{\bar{w}\subset\bar{x} \\ |\bar{w}|=i-2}} (R_{i-1}(\bar{w},y) \wedge R_{i-1}(\bar{w},z)) \right].$$

Let $\mathcal{C}$ be the structure comprising $\bar{x}$ and two new elements y and z. Relations on the new structure are the relations on $\bar{x}$, the relation $U(y,z)$, and the relations $R_{i-1}(\bar{w},y)$ and $R_{i-1}(\bar{w},z)$ for $\bar{w}$ a subset of $\bar{x}$ of the appropriate size. As we added no occurrences of Q or V, we see that $\mathcal{C} \in \mathcal{K}$, and we are done.

Similarly, suppose $\mathcal{M} \models \neg Q_{j,n}(\bar{x})$. Let $\mathcal{C}$ be the structure consisting of $\bar{x}$ and a tuple $\bar{y}$ whose relations are the relations on $\bar{x}$, $R_n(\bar{x},\bar{y})$, and $V(y_i,y_k)$ for each $y_i, y_k \in \bar{y}$. It is easy to see that $\mathcal{C} \in \mathcal{K}$, and by the ultrahomogeneity of $\mathcal{M}$, we obtain $\mathcal{M} \models (\exists \bar{y}) \left[R_n(\bar{x},\bar{y}) \wedge \bigwedge_{y_i,y_k} V(y_i,y_k) \right]$. □

Let $\mathtt{T}$ be the theory of $\mathcal{M}$ in the language with signature $\{U, V, R_3\}$. The reason for taking the reduct of $\mathcal{M}$ is the fact that the countable model of $\mathtt{T}$ is computable, which we will verify using the following:

THEOREM 9. [12]. Let $\mathtt{T}$ be a countably categorical theory. If $\mathtt{T} \cap \exists_{n+1}$ is Σ^0_n uniformly in n, then $\mathtt{T}$ has a computable model.

LEMMA 10. The reduct of the structure $\mathcal{M}$ to the language $\{U, V, R_3\}$ is computable.

Proof. Uniformly in n, the fragment $\mathtt{T} \cap \exists_n$ is computable in $\varnothing^{(n-1)}$. The salient point is that n-quantifier formulae in $\mathtt{T}$ are equivalent to quantifier-free formulae in the language $\{U, V\} \cup \{R_i \mid i \leq n+3\} \cup \{Q_{j,k} \mid k \leq n+2\}$. The n-quantifier theory of $\mathtt{T}$ is therefore determined by whether or not $f(j,k) > f(j,k-1)$ for $k \leq n+2$. This in turn depends uniformly on information computable in $\varnothing^{(n+2-4)} = \varnothing^{(n-2)}$.

It remains to observe that the n-quantifier formulae in $\mathtt{T}$ are equivalent to quantifier-free formulae in the language $\{U, V\} \cup \{R_i \mid i \leq n+3\} \cup \{Q_{j,k} \mid k \leq n+2\}$. This follows by playing an Ehrenfeucht–Fraïssé game of length n. Given a pair of tuples $\bar{a}, \bar{b}$ which have the same quantifier-free $\{U, V\} \cup \{R_i \mid i \leq n+3\} \cup \{Q_{j,k} \mid k \leq n+2\}$-types, and given a tuple $\bar{c}$, it suffices to show the existence of a tuple $\bar{d}$ such that $\bar{a}\bar{c}$ and $\bar{b}\bar{d}$ have the same $\{U, V\} \cup \{R_i \mid i \leq n+2\} \cup \{Q_{j,k} \mid k \leq n+1\}$-types. It is easy to check that such $\bar{b}\bar{d}$ exists in $\mathcal{K}$, and the rest is done by the ultrahomogeneity of $\mathcal{M}$. □

Taken together, Lemmas 3, 6, 7, and 10 show that the theory $\mathtt{T}$ witnesses Theorem 2.

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