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Causal Semantics for BPP Nets with Silent Moves

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Abstract. BPP nets, a subclass of finite Place/Transition Petri nets, are equipped with some causal behavioral semantics, which are variations of fully-concurrent bisimilarity [3], inspired by weak [28] or branching bisimulation [12] on labeled transition systems. Then, we introduce novel, efficiently decidable, distributed semantics, inspired by team bisimulation [17] and h-team bisimulation [19], and show how they relate to these variants of fully-concurrent bisimulation.

Keywords: Petri nets, BPP, Causality, Team bisimulation, Weak and branching bisimulation.

1. Introduction

A BPP net is a simple type of finite Place/Transition Petri net [32, 6, 36, 16] whose transitions have singleton pre-set. Nonetheless, as a transition can produce more tokens than the only one consumed, the reachable markings of a BPP net can be infinitely many. BPP is the acronym of *Basic Parallel Processes* [4], a simple CCS [28, 15] subcalculus (without the restriction operator) whose processes cannot communicate. In [16] a variant of BPP, which requires guarded summation (as in Simple BPP [7], SBPP [9] or BPP$_g$ [4]) and also that the body of each process constant is guarded (i.e., guarded recursion), is actually shown to represent *all and only* the BPP nets, up to net isomorphism, and this explains the name of this class of nets.

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In recent papers [17, 19, 20], we have proposed novel behavioral equivalences for BPP nets, based on a suitable generalization of the concept of bisimulation [31, 28], originally defined over labeled transition systems (LTSs, for short; see, e.g., [15]). A *team bisimulation* [17] R over the places of an *unmarked* BPP net is a relation such that if two places s_1 and s_2 are related by R, then if (one token in place) s_1 performs a and reaches the marking m_1, then (one token in place) s_2 may perform a reaching a marking m_2 such that m_1 and m_2 are element-wise, bijectively related by R (and vice versa if s_2 moves first). *Team bisimilarity* is the largest team bisimulation over the places of the *unmarked* BPP net, and then such a relation is lifted to markings by *additive closure*: if place s_1 is team bisimilar to place s_2 and the marking m_1 is team bisimilar to m_2 (the base case relates the empty marking to itself), then also $s_1 \oplus m_1$ is team bisimilar to $s_2 \oplus m_2$, where $\_ \oplus \_$ is the operator of multiset union. Hence, team bisimilar markings have the same size. A slight weakening of this equivalence is *h-team bisimilarity* [19, 20], which may equate markings with different size, that differ only for the number of deadlock places.

These equivalences are sensible, indeed, as we proved in [19, 20] that h-team bisimilarity coincides with (strong) fully-concurrent bisimilarity [3] (fc-bisimilarity for short), while team bisimilarity coincides with a slight strengthening of fc-bisimilarity, called *state-sensitive* fc-bisimilarity, requiring that the related markings have the same size. Moreover, these equivalences are decidable in polynomial time. First, by using a (slight generalization of the) algorithm in [25], we can check, given an equivalence relation R on the set of places, whether two markings m_1 and m_2 are element-wise, bijectively related by R in $O(n)$ time, where n is the number of places. Second, by adapting the classic Kanellakis-Smolka algorithm for standard bisimilarity over LTSs [23, 24], the equivalence classes of (h-)team bisimulation equivalence over places can be computed in $O(m \cdot n^2)$ time, where m is the number of net transitions. Finally, once (h-)team bisimilarity on places has been computed, checking whether two markings are (h-)team bisimilar can be done in $O(n)$ time.

In this paper, we study the causal semantics of BPP nets in the presence of silent moves, taking inspiration from *weak* bisimulation [28] and *branching* bisimulation [12] (originally defined on LTSs). Therefore, starting from the definition of strong fully-concurrent bisimulation [3], we first define *weak fully-concurrent bisimulation* and *branching fully-concurrent bisimulation*. These behavioral causal semantics have been provided for BPP nets, but they can be easily adapted for general Place/Transition nets [32, 6, 36, 16].

Then, we elaborate on the definitions of team [17] and h-team bisimulations [19, 20], in order to define their weak/branching variants. It is interesting to observe that, for weak/branching (h-)team bisimilarity, silent transitions are really unobservable only if they do not change the *observable size* of the current marking, where by observable size of a marking m we mean either the total number of tokens in m for weak/branching team bisimilarity, or the total number of tokens on the places in m which are able to perform some observable action for weak/branching h-team bisimilarity. Indeed, we consider as really interesting only the so-called τ*-(h-)sequential* BPP nets, i.e., BPP nets whose silent transitions produce exactly the same number of *observable* tokens they consume, i.e., exactly one token for weak team bisimilarity (τ-sequential) or only one token in the only place of the reached marking which is able to perform some observable action for weak h-team bisimilarity (τ-h-sequential).

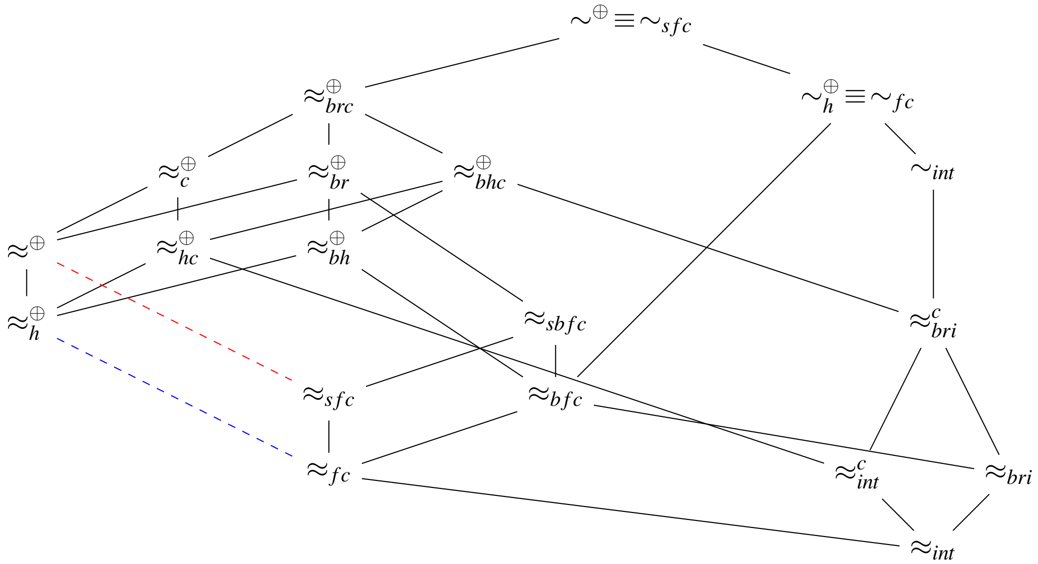
These equivalences can be checked in polynomial time. Weak (h-)team bisimilarity on places can be computed by first saturating the net transitions and then checking strong (h-)team bisimilarity on the saturated net. Moreover, by adapting the Groote-Vaandrager algorithm in [13] for computing branching bisimilarity on finite-state LTSs, branching (h-)team bisimilarity on the places of a BPP net can be computed in $O(l + m \cdot n^2)$, where n is the number of places, m the number of transitions and l the number of labels. Finally, with the algorithm in [25], we can check whether two markings are weak/branching (h-)team bisimilar in $O(n)$ time.

Of course, we prove that weak team bisimilar markings respect the global behavior; in particular, the sequential behavior (weak team bisimilarity implies weak interleaving bisimilarity) and the causal behavior (weak team bisimilarity implies state-sensitive weak fc-bisimilarity for τ-sequential BPP nets; and we also provide some argument supporting our conjecture that they are actually the same on τ-sequential BPP nets). Moreover, we introduce *rooted* weak team bisimulation equivalence, in the same line of rooted weak bisimilarity on LTSs [28, 15]. This equivalence is useful because it is possible to prove that it is a congruence over the operators of the BPP calculus [4, 16].

Finally, we extend the approach to *branching (h-)team bisimulation*, following the intuition of branching bisimulation on LTSs, proposed by van Glabbeek and Weijland in [12], proving that branching (h-)team bisimilarity implies branching interleaving bisimilarity and that branching team bisimilarity implies state-sensitive branching fc-bisimilarity; and we also provide an example showing that the reverse implication does not hold. We argue, by means of examples, that branching (h-)team bisimilarity is more appropriate than weak (h-)team bisimilarity for BPP nets with silent moves. Moreover, we introduce *rooted* branching (h-)team bisimulation equivalence, because it is a congruence over all the operators of the BPP calculus [4, 16].

The diagram in Figure 1 shows all the 21 behavioral equivalences studied in this paper, where the top element is the most discriminating one (namely, team bisimilarity $\sim^{\oplus}$, which coincides with state-sensitive fully-concurrent bisimilarity $\sim_{sfc}$) and the bottom element is the coarsest one (namely, weak interleaving bisimilarity $\approx_{int}$). Each edge from high to low is an implication, where the two dotted edges are for expressing a *conditional* implication; e.g., weak team bisimilarity $\approx^{\oplus}$ implies state-sensitive weak fully-concurrent bisimilarity $\approx_{sfc}$ *only for* τ-sequential BPP nets. These 21 equivalences are roughly divided into three groups: five related to interleaving semantics are outlined in the right part of the diagram; six related to the causal semantics of BPP nets in the middle; and ten related to the various variants of strong/weak/branching (h-)team bisimilarity (eight of which are original of this paper) are outlined in the left part of the diagram.

The paper is organized as follows. Section 2 introduces the basic definitions about BPP nets with silent moves. Section 3 recalls the usual interleaving behavioral equivalences, i.e., strong, weak and branching interleaving bisimilarities. Section 4 discusses the causal semantics of BPP nets. First, our own variant definition of *strong fully-concurrent bisimulation* (sfc-bisimulation, for short) is recalled from [20]. Then, *weak fc-bisimulation* and *branching fc-bisimulation* are introduced, together with their novel, stronger variants, called *state-sensitive*, requiring additionally that all the related markings have the same size. Section 5 recalls the main definitions and results about team/h-team bisimilarity from [17, 19, 20]. We describe (a slight generalization of) the algorithm in [25] for checking whether two markings are element-wise bijectively related by an equivalence place relation R and, moreover,



Legenda

$\sim_{int}$	interleaving bisimilarity	$\sim_{sfc}$	state-sensitive fully-concurrent bisimilarity	$\sim^{\oplus}$	team bisimilarity
$\approx_{int}$	weak interleaving bisimilarity	$\sim_{fc}$	fully-concurrent bisimilarity	$\sim^{\oplus}_{h}$	h-team bisimilarity
$\approx^{c}_{int}$	rooted weak interleaving bisimilarity	$\approx_{sfc}$	state-sensitive weak fully-concurrent bisimilarity	$\approx^{\oplus}$	weak team bisimilarity
$\approx_{bri}$	branching interleaving bisimilarity	$\approx_{fc}$	weak fully-concurrent bisimilarity	$\approx^{\oplus}_{c}$	rooted weak team bisimilarity
$\approx^{c}_{bri}$	rooted branching interleaving bisimilarity	$\approx_{sbfc}$	state-sensitive branching fully-concurrent bisimilarity	$\approx^{\oplus}_{h}$	weak h-team bisimilarity
$\approx^{\oplus}_{br}$	branching team bisimilarity	$\approx_{bfc}$	branching fully-concurrent bisimilarity	$\approx^{\oplus}_{hc}$	rooted weak h-team bisimilarity
$\approx^{\oplus}_{brc}$	rooted branching team bisimilarity	$\approx^{\oplus}_{bh}$	branching h-team bisimilarity	$\approx^{\oplus}_{bhc}$	rooted branching h-team bisimilarity

Figure 1. The diagram with the 21 behavioral equivalences studied in this paper

we state that team bisimilarity coincides with state-sensitive sfc-bisimilarity and that h-team bisimilarity coincides with sfc-bisimilarity for BPP nets.

Section 6 copes with the distributed weak equivalence checking problem; first, we discuss weak team bisimulation over places of an unmarked BPP net, showing that the classic results of weak bisimulation over LTSs also hold in this case; moreover, a few examples discussing its pros and cons are presented. Section 6.3 discusses the lifting of weak team bisimilarity to markings and proves that weak team bisimilarity is finer than state-sensitive weak fc-bisimilarity for τ-sequential BPP nets. Section 6.4 defines the minimization of BPP nets w.r.t. weak team bisimilarity on places. In Section 6.5 rooted weak team bisimilarity is introduced. Section 6.6 defines weak h-team bisimilarity on places and show that its lifting to markings is finer than weak fully-concurrent bisimilarity for τ-h-sequential BPP nets. Section 7 copes with the distributed branching equivalence checking problem; first, we discuss branching team bisimulation over the places of an unmarked BPP net, showing that the classic results of branching bisimulation over LTSs also hold in this case; moreover, a few examples discussing its pros and cons are presented. In Section 7.2 we discuss the lifting of branching team bisimilarity to markings, and we prove that branching team bisimilarity is finer than state-sensitive branching fully-concurrent bisimilarity for BPP nets. Then, the minimization of BPP nets w.r.t. branching team bisimilarity is defined in Section 7.3, while in Section 7.4, rooted branching team bisimilarity is also introduced. Then, in Section 7.5 we present branching h-team equivalence on places and show that its lifting to markings is finer than branching fully-concurrent bisimilarity for BPP nets. Finally, Section 8 discusses related literature, some future research and open problems.

2. Basic definitions

Definition 2.1. (Multiset) Let $\mathbb{N}$ be the set of natural numbers. Given a finite set S, a *multiset* over S is a function $m : S \to \mathbb{N}$. The *support* set $dom(m)$ of m is $\{s \in S \mid m(s) \neq 0\}$. The set of all multisets over S, denoted by $\mathscr{M}(S)$, is ranged over by m. We write $s \in m$ if $m(s) > 0$. The *multiplicity* of s in m is given by the number $m(s)$. The *size* of m, denoted by $|m|$, is the number $\sum_{s \in S} m(s)$, i.e., the total number of its elements. A multiset m such that $dom(m) = \emptyset$ is called *empty* and is denoted by θ. We write $m \subseteq m'$ if $m(s) \leq m'(s)$ for all $s \in S$.

Multiset union $\_ \oplus \_$ is defined as follows: $(m \oplus m')(s) = m(s) + m'(s)$; it is commutative, associative and has θ as neutral element. *Multiset difference* $\_ \ominus \_$ is defined as follows: $(m_1 \ominus m_2)(s) = max\{m_1(s) - m_2(s), 0\}$. The *scalar product* of a number j with m is the multiset $j \cdot m$ defined as $(j \cdot m)(s) = j \cdot (m(s))$. By s_i we also denote the multiset with s_i as its only element. Hence, a multiset m over $S = \{s_1, \dots, s_n\}$ can be represented as $k_1 \cdot s_1 \oplus k_2 \cdot s_2 \oplus \dots \oplus k_n \cdot s_n$, where $k_j = m(s_j) \geq 0$ for $j = 1, \dots, n$. □

Definition 2.2. (BPP net) A labeled *BPP net* is a tuple $N = (S, A, T)$, where

- S is the finite set of *places*, ranged over by s (possibly indexed),
- A is the finite set of *labels*, ranged over by ℓ (possibly indexed), and
- $T \subseteq S \times A \times \mathscr{M}(S)$ is the finite set of *transitions*, ranged over by t (possibly indexed).

Given a transition $t = (s, \ell, m)$, we use the notation:

- $^\bullet t$ to denote its *pre-set* s (which is a single place) of tokens to be consumed;
- $l(t)$ for its *label* ℓ, and
- $t^\bullet$ to denote its *post-set* m (a multiset, possibly empty) of tokens to be produced.

Hence, a transition t can be also represented as $^\bullet t \xrightarrow{l(t)} t^\bullet$. We also define pre-sets and post-sets for places as follows: $^\bullet s = \{t \in T \mid s \in t^\bullet\}$ and $s^\bullet = \{t \in T \mid s \in {^\bullet t}\}$. Note that while the pre-set (post-set) of a transition is, in general, a multiset, the pre-set (post-set) of a place is a set. □

Graphically, a place is represented by a little circle, a transition by a little box, which is connected by a directed arc from the place in its pre-set and to the places in its post-set (if any); the arcs may be labeled by a positive integer, called the *weight*, to denote the number of tokens consumed/produced by the transition (if the number is omitted, then the weight default value of the arc is 1); for BPP nets, only the arcs from transitions to places may have a weight larger than 1.

Definition 2.3. (Marking, BPP net system) A multiset over S is called a *marking*. Given a marking m and a place s, we say that the place s contains $m(s)$ *tokens*, graphically represented by $m(s)$ bullets inside place s. A *BPP net system* $N(m_0)$ is a tuple (S, A, T, m_0), where (S, A, T) is a BPP net and m_0 is a marking over S, called the *initial marking*. We also say that $N(m_0)$ is a *marked* net. $\square$

Definition 2.4. (Enabling, firing sequence, transition sequence, reachable marking) Given a BPP net $N = (S, A, T)$, a transition t is *enabled* at marking m, denoted by $m[t\rangle$, if ${}^\bullet t \subseteq m$. The execution (or *firing*) of t enabled at m produces the marking $m' = (m \ominus {}^\bullet t) \oplus t^\bullet$. This is written $m[t\rangle m'$. A *firing sequence* starting at m is defined inductively as follows:

- $m[\varepsilon\rangle m$ is a firing sequence (where ε denotes an empty sequence of transitions) and
- if $m[\sigma\rangle m'$ is a firing sequence and $m'[t\rangle m''$, then $m[\sigma t\rangle m''$ is a firing sequence.

If $\sigma = t_1 \ldots t_n$ (for $n \geq 0$) and $m[\sigma\rangle m'$ is a firing sequence, then there exist $m_1, \ldots, m_{n+1}$ such that $m = m_1[t_1\rangle m_2[t_2\rangle \ldots m_n[t_n\rangle m_{n+1} = m'$, and $\sigma = t_1 \ldots t_n$ is called a *transition sequence* starting at m and ending at m'. The definition of pre-set and post-set can be extended to transition sequences as follows: ${}^\bullet\varepsilon = \theta$, ${}^\bullet(t\sigma) = {}^\bullet t \oplus ({}^\bullet\sigma \ominus t^\bullet)$, $\varepsilon^\bullet = \theta$, $(t\sigma)^\bullet = \sigma^\bullet \oplus (t^\bullet \ominus {}^\bullet\sigma)$. We say that a transition sequence σ is *sequential* if $|{}^\bullet\sigma| \leq 1$. The set of *reachable markings* from m is $[m\rangle = \{m' \mid \exists \sigma. m[\sigma\rangle m'\}$. Note that the reachable markings can be countably infinitely many. A BPP net system $N(m_0) = (S, A, T, m_0)$ is *safe* if each marking m reachable from the initial marking m_0 is a set, i.e., $\forall m \in [m_0\rangle, m(s) \leq 1$ for all $s \in S$. The set of *reachable places* from s is $reach(s) = \bigcup_{m \in [s\rangle} dom(m)$. Note that $reach(s)$ is always a finite set, even if $[s\rangle$ is infinite. $\square$

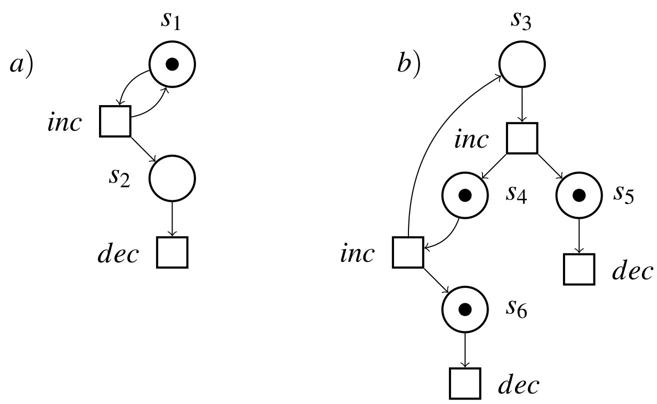


Figure 2. The net representing a semi-counter in (a), and a variant in (b)

Example 2.5. By using the drawing convention mentioned above, Figure 2 shows in (a) the simplest BPP net representing a semi-counter, i.e., a counter which cannot test for zero. Note that the number represented by this semi-counter is given by the number of tokens which are present in place s_2, i.e., in the place ready to perform *dec*; hence, Figure 2(a) represents a semi-counter holding number 0; note also that the number of tokens which can be accumulated in s_2 is unbounded. Indeed, the set of reachable markings for a BPP net can be countably infinite. In (b), a variant semi-counter is outlined, which holds number 2 (i.e., two tokens are ready to perform *dec*). $\square$

Proposition 2.6. Given a BPP net $N = (S,A,T)$ and a firing sequence $m[\sigma\rangle m'$, where $m = s_1 \oplus \ldots \oplus s_n$, we can find sequential transition sequences $\sigma_1,\ldots,\sigma_n$ and markings $m_1,\ldots,m_n$ such that $s_i[\sigma_i\rangle m_i$, for $i = 1,\ldots,n$, σ is a permutation of $\sigma_1\sigma_2\ldots\sigma_n$ and $m' = m_1 \oplus m_2 \oplus \ldots \oplus m_n$.

Proof:
By induction on the length of σ. If $\sigma = \varepsilon$, then $\sigma_i = \varepsilon$ and $m_i = s_i$ for $i = 1,\ldots,n$. If $\sigma = t\sigma'$, then take $s_1 = {}^\bullet t$ and $t^\bullet = p_1 \oplus \ldots \oplus p_k$. In such a case, $p_1 \oplus \ldots \oplus p_k \oplus s_2 \oplus \ldots \oplus s_n[\sigma'\rangle m'$. As σ' is shorter, by induction, there exist $\sigma_2,\ldots,\sigma_n$, with $m_2,\ldots,m_n$, such that $s_i[\sigma_i\rangle m_i$, for $i = 2,\ldots,n$; and there also exist $\beta_1,\ldots,\beta_k$, with $m'_1,\ldots,m'_k$, such that $p_i[\beta_i\rangle m'_i$, for $i = 1,\ldots,k$; and additionally, $m' = m'_1 \oplus \ldots \oplus m'_k \oplus m_2 \oplus \ldots \oplus m_n$ and σ' is a permutation of $\beta_1\ldots\beta_k\sigma_2\ldots\sigma_n$. Therefore, the thesis follows by choosing $\sigma_1 = t\beta_1\ldots\beta_k$. □

Proposition 2.7. Given a BPP net $N = (S,A,T)$, a marking $m = s_1 \oplus \ldots \oplus s_k$, with $k \geq 1$, and the sequential transition sequences σ_i such that $s_i[\sigma_i\rangle m_i$, for $i = 1,\ldots,k$, it follows that $m[\sigma\rangle m'$, where $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ and $m' = m_1 \oplus m_2 \oplus \ldots \oplus m_k$.

Proof:
By induction on the size of m. The base case is when $m = s_1$, and thesis follows trivially. The inductive case, by assuming that the thesis holds for $m = s_1 \oplus \ldots \oplus s_k$, demonstrates the thesis for $\overline{m} = m \oplus s_{k+1}$. We know that there are sequential transition sequences σ_i such that $s_i[\sigma_i\rangle m_i$, for $i = 1,\ldots,k+1$. We also know, by induction, that $m[\sigma\rangle m'$, where $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ and $m' = m_1 \oplus m_2 \oplus \ldots \oplus m_k$. Therefore, it follows that $\overline{m} = m \oplus s_{k+1}[\sigma\rangle m' \oplus s_{k+1}[\sigma_{k+1}\rangle m' \oplus m_{k+1} = \overline{m}'$, i.e., $\overline{m}[\sigma\sigma_{k+1}\rangle \overline{m}'$, as required. □

Definition 2.8. (BPP net with silent moves, observable label, h-observable submarking) A BPP net $N = (S,A,T)$ such that $A_\tau = A \setminus \{\tau\}$, where τ is the only invisible action that can be used to label transitions, is called a BPP net *with silent moves*.

Given a transition sequence σ, its *observable label* $o(\sigma)$ is computed inductively as follows.

$$o(\varepsilon) = \varepsilon \qquad o(t\sigma) = \begin{cases} l(t)o(\sigma) & \text{if } l(t) \neq \tau \\ o(\sigma) & \text{otherwise.} \end{cases}$$

We also define the auxiliary function $o_\tau(\sigma)$ as follows: In case $o(\sigma) \neq \varepsilon$ or σ is empty, then $o_\tau(\sigma) = o(\sigma)$; in case $o(\sigma) = \varepsilon$ and σ is not empty, then $o_\tau(\sigma) = \tau$.

Let $o(S) = \{s \in S \mid \exists \sigma. s[\sigma\rangle m', o(\sigma) \in A_\tau\}$ be the set of *observable* places, i.e., places that can perform some observable action. Then, given a marking m, we denote by $o(m)$ the marking

$$o(m)(s) = \begin{cases} m(s) & \text{if } s \in o(S) \\ 0 & \text{otherwise.} \end{cases}$$

For instance, let us consider the nets in Figure 3. Then, $o(2 \cdot s_4 \oplus s_6 \oplus s_7) = 2 \cdot s_4$, or $o(s_{12} \oplus s_{13}) = \theta$. Of course, $o(m)$ is a multiset on $o(S)$ and it is called the *h-observable submarking* of m. □

Definition 2.9. (τ-simple, τ-sequential, τ-h-sequential) A net $N = (S,A,T)$ with silent moves is

- τ-*simple* if $\forall s \in S$ and $\forall \sigma$ such that $o(\sigma) = \varepsilon$, in case $s[\sigma\rangle m$ and $s \in m$, then $m = s$;

- τ-*sequential* if $\forall t \in T$, $l(t) = \tau$ implies $|t^\bullet| = |{}^\bullet t|$;
- τ-*h-sequential* if $\forall t \in T$, $l(t) = \tau$ implies $|o(t^\bullet)| = |o({}^\bullet t)|$. □

A τ-simple BPP net is a net where τ labeled transitions cannot form a cycle such that the effect of traversing that cycle is the production of additional tokens: any existing silent cycle does not generate new tokens. This property will be important in Section 6.2, when we will provide a characterization of weak team bisimilarity in terms of strong team bisimilarity.

A τ-sequential BPP net is a net whose τ-labeled transitions produce exactly one token, so that their execution does not change the size of the current marking. Of course, if a BPP net is τ-sequential, then it is also τ-simple. The definition of τ-sequential can be weakened by requiring that the number of the produced tokens may be more than one, but the number of observable places is preserved by the transition (τ-h-sequential). We will argue that only silent transitions with these properties can be really unobservable: if a τ-labeled transition changes the number of currently available tokens, then it has a visible effect on the structure of the system.

3. Interleaving semantics

In this section we survey some standard behavioural interleaving equivalences for BPP nets, inspired by the original definitions [28, 12, 15] on LTSs.

Definition 3.1. (Interleaving Bisimulation) Let $N = (S, A, T)$ be a BPP net. An *interleaving bisimulation* is a relation $R \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ such that if $(m_1, m_2) \in R$ then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that $m_2[t_2\rangle m_2'$ with $l(t_1) = l(t_2)$ and $(m_1', m_2') \in R$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that $m_1[t_1\rangle m_1'$ with $l(t_1) = l(t_2)$ and $(m_1', m_2') \in R$.

Two markings m_1 and m_2 are *interleaving bisimilar* (or *interleaving bisimulation equivalent*), denoted by $m_1 \sim_{int} m_2$, if there exists an interleaving bisimulation R such that $(m_1, m_2) \in R$. □

Interleaving bisimilarity $\sim_{int}$, which is defined as the union of all the interleaving bisimulations, is the largest interleaving bisimulation and also an equivalence relation.

Example 3.2. Continuing Example 2.5 about Figure 2, it is easy to realize that relation $R = \{(s_1 \oplus k \cdot s_2, s_3 \oplus k_1 \cdot s_5 \oplus k_2 \cdot s_6) \mid k = k_1 + k_2 \text{ and } k, k_1, k_2 \geq 0\} \cup \{(s_1 \oplus k \cdot s_2, s_4 \oplus k_1 \cdot s_5 \oplus k_2 \cdot s_6) \mid k = k_1 + k_2$ and $k, k_1, k_2 \geq 0\}$ is an interleaving bisimulation. □

Definition 3.3. (Weak Interleaving Bisimulation) Let $N = (S, A, T)$ be a BPP net with silent moves. A *weak* interleaving bisimulation is a relation $R \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ such that if $(m_1, m_2) \in R$ then

- $\forall t_1$ such that $l(t_1) \neq \tau$, $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that $m_2[\sigma_2\rangle m_2'$ with $l(t_1) = o(\sigma_2)$ and $(m_1', m_2') \in R$,
- $\forall t_1$ such that $l(t_1) = \tau$ and $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that $m_2[\sigma_2\rangle m_2'$ with $o(\sigma_2) = \varepsilon$ and $(m_1', m_2') \in R$,

- $\forall t_2$ such that $l(t_2) \neq \tau$, $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ such that $m_1[\sigma_1\rangle m_1'$ with $l(t_2) = o(\sigma_1)$ and $(m_1', m_2') \in R$,
- $\forall t_2$ such that $l(t_2) = \tau$ and $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ such that $m_1[\sigma_1\rangle m_1'$ with $o(\sigma_1) = \varepsilon$ and $(m_1', m_2') \in R$.

Two markings m_1 and m_2 are *weak* interleaving bisimilar, denoted by $m_1 \approx_{int} m_2$, if there exists a weak interleaving bisimulation R such that $(m_1, m_2) \in R$. □

Note that an invisible transition performed by one of the two markings may be matched by the other one also by idling, i.e., by performing an empty sequence of transitions. Weak interleaving bisimilarity $\approx_{int}$, which is defined as the union of all the weak interleaving bisimulations, is the largest weak interleaving bisimulation and also an equivalence relation. Of course, (strong) interleaving bisimilarity $\sim_{int}$ is finer than weak interleaving bisimilarity $\approx_{int}$; the two equivalences coincide if the BPP net has no silent moves.

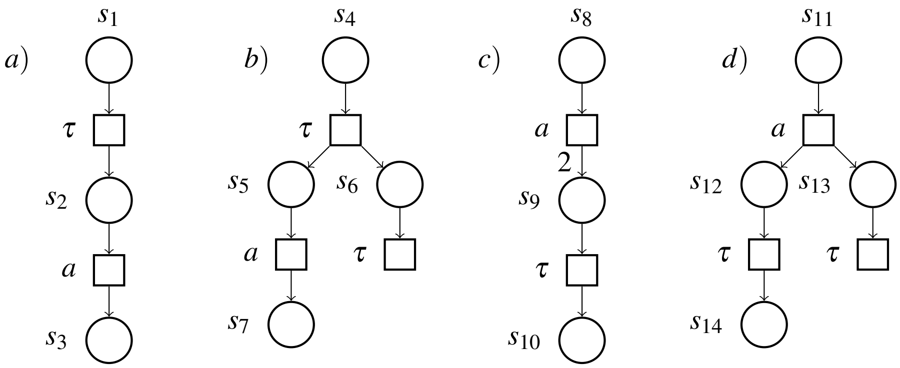


Figure 3. Four weakly interleaving bisimilar BPP nets

Example 3.4. Consider the four nets in Figure 3. It is not difficult to realize that $s_1 \approx_{int} s_4$, because $R = \{(s_1, s_4), (s_2, s_5 \oplus s_6), (s_2, s_5), (s_3, s_6 \oplus s_7), (s_3, s_7)\}$ is a weak interleaving bisimulation. Similarly, one can check that $s_1 \approx_{int} s_8$ and $s_1 \approx_{int} s_{11}$, by building suitable weak interleaving bisimulations. □

Weak interleaving bisimilarity is not a congruence for the choice operator of the process algebra BPP [4, 15]. The coarsest congruence relation included into $\approx_{int}$ is as follows.

Definition 3.5. (Rooted Weak Interleaving Bisimilarity) Let $N = (S, A, T)$ be a BPP net with silent moves. Two markings m_1 and m_2 are *rooted weak* interleaving bisimilar, denoted $m_1 \approx_{int}^c m_2$, if

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ s.t. $m_2[\sigma_2\rangle m_2'$ with $l(t_1) = o_\tau(\sigma_2)$ and $m_1' \approx_{int} m_2'$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ s.t. $m_1[\sigma_1\rangle m_1'$ with $l(t_2) = o_\tau(\sigma_1)$ and $m_1' \approx_{int} m_2'$. □

Therefore, if two markings are rooted weak interleaving bisimilar, in case one of the two initially performs an invisible transition (e.g., $l(t_1) = \tau$), then the other is able to respond with a nonempty sequence of invisible transitions (e.g., $o_\tau(\sigma_2) = \tau$); since the reached markings are simply weakly interleaving bisimilar (e.g., $m_1' \approx_{int} m_2'$), future invisible transitions performed by one of the two can be matched by the other one also by idling. Hence, rooted weak interleaving bisimilarity $\approx^c_{int}$ is slightly finer than weak interleaving bisimilarity $\approx_{int}$. Nonetheless, if two weak interleaving bisimilar markings cannot perform any silent transition *initially*, then these two markings are also rooted weak interleaving bisimilar.

Example 3.6. Continuing Example 3.4 about the four BPP nets in Figure 3, it is not difficult to realize that $s_1 \approx^c_{int} s_4$, while $s_1 \not\approx^c_{int} s_8$, even if $s_1 \approx_{int} s_8$. $\square$

Definition 3.7. (Branching interleaving bisimulation) Let $N = (S, A, T)$ be a BPP net with τ-moves. A *branching* interleaving bisimulation is a relation $R \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ such that if $(m_1, m_2) \in R$ then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$,
  - either $l(t_1) = \tau$ and $\exists \sigma_2$ such that $o(\sigma_2) = \varepsilon$, $m_2[\sigma_2\rangle m_2'$ with $(m_1, m_2') \in R$ and $(m_1', m_2') \in R$,
  - or $\exists \sigma, t_2$. $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $m_2[\sigma\rangle m[t_2\rangle m_2'$ with $(m_1, m) \in R$ and $(m_1', m_2') \in R$,
- and, symmetrically, $\forall t_2$ such that $m_2[t_2\rangle m_2'$
  - either $l(t_2) = \tau$ and $\exists \sigma_1$ such that $o(\sigma_1) = \varepsilon$, $m_1[\sigma\rangle m_1'$ with $(m_1', m_2) \in R$ and $(m_1', m_2') \in R$,
  - or $\exists \sigma, t_1$. $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $m_1[\sigma\rangle m[t_1\rangle m_1'$ with $(m, m_2) \in R$ and $(m_1', m_2') \in R$.

Two markings m_1 and m_2 are *branching interleaving bisimilar*, denoted $m_1 \approx_{bri} m_2$, if there exists a branching interleaving bisimulation R that relates them. $\square$

Note that a silent transition performed by one of the two markings may be matched by the other one also by idling: this is due to the *either* case when $\sigma_2 = \varepsilon$ (or $\sigma_1 = \varepsilon$). Branching interleaving bisimilarity $\approx_{bri}$, which is defined as the union of all the branching interleaving bisimulations, is the largest branching interleaving bisimulation and also an equivalence relation. Of course, (strong) interleaving bisimilarity $\sim_{int}$ is finer than branching interleaving bisimilarity $\approx_{bri}$; the two equivalences coincide if the BPP net has no silent moves: this is due to the *or* case. Branching interleaving bisimilarity is finer than weak interleaving bisimilarity $\approx_{int}$ because a branching interleaving bisimulation is also a weak interleaving bisimulation.

Example 3.8. Consider the nets in Figure 4. It is not difficult to see that $s_1 \approx_{int} s_4$. However, $s_1 \not\approx_{bri} s_4$, because to transition $s_4 \xrightarrow{a} s_5$, place s_1 can only try to respond with $s_1 \xrightarrow{\tau} s_2 \xrightarrow{a} s_3$, but not all the conditions required are satisfied; in particular, $s_2 \not\approx_{bri} s_4$, because only s_4 can do b. $\square$

Remark 3.9. (Stuttering property) It is not difficult to prove that, given a silent firing sequence $m_1[t_1\rangle m_2[t_2\rangle m_3 \ldots m_n[t_n\rangle m_{n+1}$, with $l(t_i) = \tau$ for $i = 1, \ldots, n$, if $m_1 \approx_{bri} m_{n+1}$, then $m_i \approx_{bri} m_j$ for $i, j = 1, \ldots, n+1$. This is sometimes called the *stuttering property*.

An important property holds for $\approx_{bri}$. Consider the *either* case: since $(m_1, m_2) \in \approx_{bri}$ by hypothesis, and $m_2[\sigma_2\rangle m_2'$ with $(m_1, m_2') \in \approx_{bri}$ and $(m_1', m_2') \in \approx_{bri}$, it follows that $(m_2, m_2') \in \approx_{bri}$ because $\approx_{bri}$ is an equivalence relation. This implies that all the markings in the silent path from m_2 to m_2' are branching interleaving bisimilar (by the *stuttering property*). Similarly for the *or* case: if $m_1[t_1\rangle m_1'$

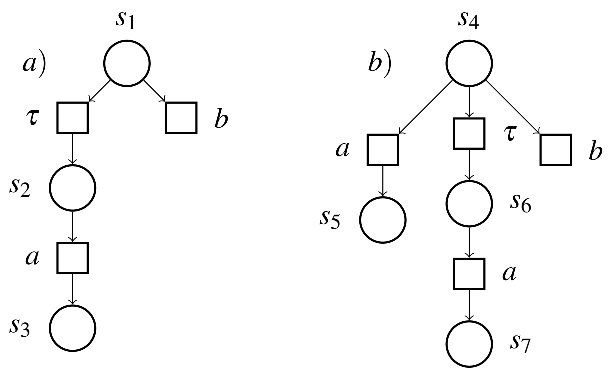


Figure 4. Two non-branching bisimilar BPP nets

(with $l(t_1)$ that can be τ) and m_2 responds by performing $m_2[\sigma\rangle m[t_2\rangle m_2'$ with $m_1 \approx_{bri} m$, then, by transitivity, $m_2 \approx_{bri} m$; hence, by the stuttering property, m_1 is branching interleaving bisimilar to each marking in the path from m_2 to m. This stuttering property holds for all the branching-style bisimulations we are going to define in the following.

These constraints are not required by weak interleaving bisimilarity $\approx_{int}$ (cf. Example 3.8): given $m_1 \approx_{int} m_2$, when matching $m_1[t_1\rangle m_1'$ with $m_2[\sigma\rangle m'[t_2\rangle m''[\sigma'\rangle m_2'$, where $o(\sigma) = o(\sigma') = \varepsilon$ and $l(t_1) = l(t_2)$, weak bisimilarity only requires that $m_1' \approx_{int} m_2'$, but does not impose any condition on the intermediate states; in particular, it is not required that $m_1 \approx_{int} m'$, or that $m_1' \approx_{int} m''$. $\square$

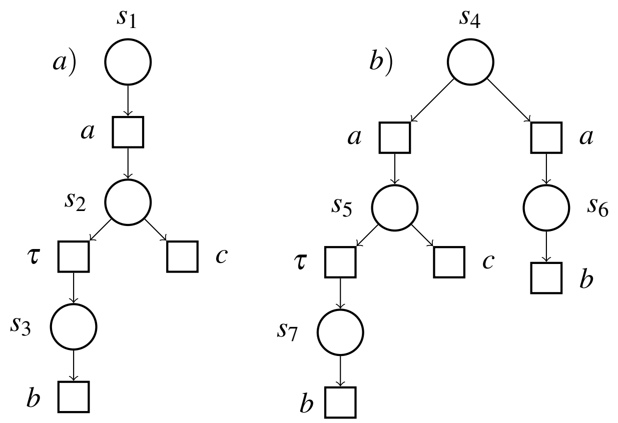


Figure 5. Two weakly interleaving bisimilar nets, which are not branching interleaving bisimilar

Example 3.10. To show that $\approx_{bri}$ does better respect the timing of choices, consider the nets in Figure 5. A weak interleaving bisimulation is $R = \{(s_1, s_4), (s_2, s_5), (s_3, s_6), (s_3, s_7)\}$, hence s_1 is weak interleaving bisimilar to s_4. However, in the net (a), in each computation the choice between b and c is made after the a-labeled transition, while in the net (b) there is a computation where c is already discarded after a. In fact, $s_1 \not\approx_{bri} s_4$: to transition $s_4 \xrightarrow{a} s_6$, place s_1 can only try to respond with $s_1 \xrightarrow{a} s_2$, but s_2 and s_6 are not equivalent, because only s_2 can do c. $\square$

Branching interleaving bisimilarity is not a congruence for the choice operator of the process algebra BPP [4, 15]. The coarsest congruence relation included into $\approx_{bri}$ is as follows.

Definition 3.11. (Rooted branching interleaving bisimilarity) Two markings m_1 and m_2 are rooted branching interleaving bisimilar, denoted $m_1 \approx^c_{bri} m_2$, if

- $\forall t_1$ such that $m_1[t_1\rangle m'_1$, $\exists t_2$ such that $l(t_1) = l(t_2)$, $m_2[t_2\rangle m'_2$ and $m'_1 \approx_{bri} m'_2$,
- $\forall t_2$ such that $m_2[t_2\rangle m'_2$, $\exists t_1$ such that $l(t_1) = l(t_2)$, $m_1[t_1\rangle m'_1$ and $m'_1 \approx_{bri} m'_2$. □

Note that rooted branching interleaving bisimilarity is finer than branching interleaving bisimilarity: in the first step, the two markings should be able to match their transitions as in strong interleaving bisimilarity; after this first step, the reached markings have to be simply related by branching bisimilarity. Nonetheless, if two branching interleaving bisimilar markings cannot perform any silent transition *initially*, then these two markings are also rooted branching interleaving bisimilar. Note also that interleaving bisimilarity $\sim_{int}$ is finer than rooted branching interleaving bisimilarity $\approx^c_{bri}$.

Example 3.12. Consider again the nets in Figure 3. it is not difficult to realize that $s_1 \approx^c_{bri} s_4$ and $s_8 \approx^c_{bri} s_{11}$; on the contrary, $s_1 \not\approx^c_{bri} s_8$ and $s_4 \not\approx^c_{bri} s_{11}$. □

4. Causality-based behavioral semantics

In order to define causality-based semantics for BPP nets, we need some auxiliary definitions, adapting those in, e.g., [2, 3, 29, 11].

Definition 4.1. (Isomorphism) Given two BPP nets $N_1 = (S_1, A, T_1)$ and $N_2 = (S_2, A, T_2)$, we say that N_1 and N_2 are *isomorphic via* f if there exists a bijection $f : S_1 \cup T_1 \to S_2 \cup T_2$ such that $f(S_1) = S_2$ and $f(T_1) = T_2$, satisfying the following condition: $\forall t \in T_1$, if $t = (^\bullet t, \ell, t^\bullet)$, then $f(t) = (f(^\bullet t), \ell, f(t^\bullet))$, where f is applied element-wise to each component of the marking. Two BPP net systems $N_1(m_1)$ and $N_2(m_2)$ are *rooted isomorphic* if the isomorphism f ensures, additionally, that $f(m_1) = m_2$. □

Definition 4.2. (Acyclic net) A BPP net $N = (S, A, T)$ is *acyclic* if there exists no sequence $x_1 x_2 \ldots x_n$ such that $n \geq 3$, $x_i \in S \cup T$ for $i = 1, \ldots, n$, $x_1 = x_n$, $x_1 \in S$ and $x_i \in {^\bullet x_{i+1}}$ for $i = 1, \ldots, n-1$, i.e., the arcs of the net do not form any cycle. □

The concurrent semantics of a marked net is defined by a class of particular acyclic safe nets, where places are not branched (hence they represent a single run) and all arcs have weight 1. This kind of net is called *causal net*. We use the name C to denote a causal net, the set B to denote its places (called *conditions*), the set E to denote its transitions (called *events*), and L to denote its labels.

Definition 4.3. (Causal net) A causal net is a marked BPP net $C(\mathsf{m}_0) = (B, L, E, \mathsf{m}_0)$ satisfying the following conditions:

1. C is acyclic;

2. $\forall b \in B \ |{}^\bullet b| \leq 1 \wedge |b^\bullet| \leq 1$ (i.e., the places are not branched);

3. $\forall b \in B \ \mathsf{m}_0(b) = \begin{cases} 1 & \text{if } {}^\bullet b = \emptyset \\ 0 & \text{otherwise;} \end{cases}$

4. $\forall e \in E \ \ e^\bullet(b) \leq 1$ for all $b \in B$ (i.e., all the arcs have weight 1).

We denote by $Min(C)$ the set m_0, and by $Max(C)$ the set $\{b \in B \mid b^\bullet = \emptyset\}$. □

Note that a causal net, being a BPP net, is finite; as it is acyclic, it represents a finite computation. Note also that any reachable marking of a BPP causal net is a set, i.e., this net is *safe*; in fact, the initial marking is a set and, assuming by induction that a reachable marking m is a set and enables t, i.e., $\mathsf{m}[t\rangle\mathsf{m}'$, then also $\mathsf{m}' = (\mathsf{m} \ominus {}^\bullet t) \oplus t^\bullet$ is a set, because the net is acyclic and because of the condition on the shape of the post-set of t (weights can only be 1). As the initial marking of a causal net is fixed by its shape (according to item 3 of Definition 4.3), in the following, in order to make the notation lighter, we often omit the indication of the initial marking, so that the causal net $C(\mathsf{m}_0)$ is denoted by C.

Definition 4.4. (Moves of a causal net) Given two BPP causal nets $C = (B, L, E, \mathsf{m}_0)$ and $C' = (B', L, E', \mathsf{m}_0)$, we say that C moves in one step to C' through e, denoted by $C[e\rangle C'$, if ${}^\bullet e \subseteq Max(C)$, $E' = E \cup \{e\}$ and $B' = B \cup e^\bullet$; in other words, C' extends C by one event e. □

Definition 4.5. (Folding and Process) A *folding* from a BPP causal net $C = (B, L, E, \mathsf{m}_0)$ into a BPP net system $N(m_0) = (S, A, T, m_0)$ is a function $\rho : B \cup E \to S \cup T$, which is type-preserving, i.e., such that $\rho(B) \subseteq S$ and $\rho(E) \subseteq T$, satisfying the following:

- $L = A$ and $\mathsf{l}(e) = l(\rho(e))$ for all $e \in E$;
- $\rho(\mathsf{m}_0) = m_0$, i.e., $m_0(s) = |\rho^{-1}(s) \cap \mathsf{m}_0|$;
- $\forall e \in E, \rho({}^\bullet e) = {}^\bullet \rho(e)$, i.e., $\rho({}^\bullet e)(s) = |\rho^{-1}(s) \cap {}^\bullet e|$ for all $s \in S$;
- $\forall e \in E, \rho(e^\bullet) = \rho(e)^\bullet$, i.e., $\rho(e^\bullet)(s) = |\rho^{-1}(s) \cap e^\bullet|$ for all $s \in S$.

A pair (C, ρ), where C is a BPP causal net and ρ a folding from C to a BPP net system $N(m_0)$, is a *process* of $N(m_0)$, written also as π. □

Definition 4.6. (Isomorphic processes) Given a BPP net system $N(m_0)$, two of its processes (C_1, ρ_1) and (C_2, ρ_2) are *isomorphic via* f if C_1 and C_2 are rooted isomorphic via f and $\rho_1 = \rho_2 \circ f$. □

Definition 4.7. (Moves of a process) Let $N(m_0) = (S, A, T, m_0)$ be a net system and let (C_i, ρ_i), for $i = 1, 2$, be two processes of $N(m_0)$. We say that (C_1, ρ_1) moves in one step to (C_2, ρ_2) through e, denoted by $(C_1, \rho_1) \xrightarrow{e} (C_2, \rho_2)$, if $C_1[e\rangle C_2$ and $\rho_1 \subseteq \rho_2$. This is also written as $\pi_1 \xrightarrow{e} \pi_2$, where $\pi_i = (C_i, \rho_i)$ for $i = 1, 2$. We can extend the definition of move to transition sequences as follows:

- $\pi \stackrel{\varepsilon}{\Longrightarrow} \pi$, where ε is the empty transition sequence, is a move sequence and
- if $\pi \xrightarrow{e} \pi'$ and $\pi' \stackrel{\sigma}{\Longrightarrow} \pi''$, then $\pi \stackrel{e\sigma}{\Longrightarrow} \pi''$ is a move sequence.

□

Definition 4.8. (Partial orders of events from a process) From a causal net $C = (B,L,E)$, we can extract the *partial order of its events* $\mathsf{E_C} = (E, \preceq)$, where $e_1 \preceq e_2$ iff there exists a sequence $x_1x_2x_3 \ldots x_n$ such that $n \geq 3$, $x_i \in B \cup E$ for $i = 1, \ldots, n$, $e_1 = x_1, e_2 = x_n$, and $x_i \in {}^\bullet x_{i+1}$ for $i = 1, \ldots, n-1$; in other words, $e_1 \preceq e_2$ if there is a path from e_1 to e_2. We can also extract the *abstract* partial order of its *observable* events $\mathsf{O}_C = (E', \preceq')$, where $E' = \{e \in E \mid \mathsf{l}(e) \neq \tau\}$ and $\preceq' = \preceq \restriction E'$.

Two partial orders $(E_1, \preceq_1)$ and $(E_2, \preceq_2)$ are isomorphic if there is a label-preserving, order-preserving bijection $g : E_1 \to E_2$, i.e., a bijection such that $\mathsf{l}_1(e) = \mathsf{l}_2(g(e))$ and $e \preceq_1 e'$ if and only if $g(e) \preceq_2 g(e')$. We also say that g is an *abstract* (or *concrete*) *event isomorphism* between the causal nets C_1 and C_2 if it is an isomorphism between their associated abstract (or concrete) partial orders of events O_{C_1} and O_{C_2} (or E_{C_1} and E_{C_2}). □

4.1. Strong (state-sensitive) fully-concurrent bisimulation

Fully-concurrent bisimulation (fc-bisimulation, for short) was originally proposed in [3], and its definition was inspired by previous notions of equivalence on other models of concurrency: *history-preserving bisimulation*, originally defined in [34] under the name of *behavior-structure bisimulation*, and then elaborated on in [10] (who called it by this name) and [5] (who called it by *mixed ordering bisimulation*). Besides (strong) fc-bisimulation equivalence, we define also a novel, slightly stronger version, called *state-sensitive* fc-bisimulation equivalence, that is equivalent to a form of bisimulation equivalence on the BPP causal nets, called *causal-net bisimilarity*, as proved in [20].

Definition 4.9. (Fully-concurrent bisimulation) Let $N = (S,A,T)$ be a BPP net. A (strong) *fc-bisimulation* is a relation R, composed of triples of the form (π_1, g, π_2), where, for $i = 1,2$, $\pi_i = (C_i, \rho_i)$ is a process of $N(m_{0i})$ for some m_{0i} and g is an event isomorphism between E_{C_1} and E_{C_2}, such that if $(\pi_1, g, \pi_2) \in R$ then

i) $\forall t_1, \pi_1'$ such that $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$, $\exists t_2, \pi_2', g'$ such that

1. $\pi_2 \xrightarrow{e_2} \pi_2'$ with $\rho_2'(e_2) = t_2$;
2. $g' = g \cup \{(e_1, e_2)\}$, and finally,
3. $(\pi_1', g', \pi_2') \in R$;

ii) and symmetrically, if π_2 moves first.

Two markings m_1 and m_2 of N are sfc-bisimilar, denoted by $m_1 \sim_{fc} m_2$, if there exists a fully-concurrent bisimulation R containing a triple $((C_1^0, \rho_1^0), g_0, (C_2^0, \rho_2^0))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_i$ for $i = 1,2$. □

Let us denote by $\sim_R^{fc} = \{(m_1, m_2) \mid m_1 \text{ is sfc-bisimilar to } m_2 \text{ thanks to } R\}$. Of course, $\sim_{fc} = \bigcup\{\sim_R^{fc} \mid R \text{ is a strong fully-concurrent bisimulation}\} = \sim_{\mathscr{R}}^{fc}$, where relation $\mathscr{R} = \bigcup\{R \mid R \text{ is a strong fully-concurrent bisimulation}\}$ is the largest strong fully-concurrent bisimulation.

Proposition 4.10. [19, 20] For each BPP net $N = (S,A,T)$, relation $\sim_{fc} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.

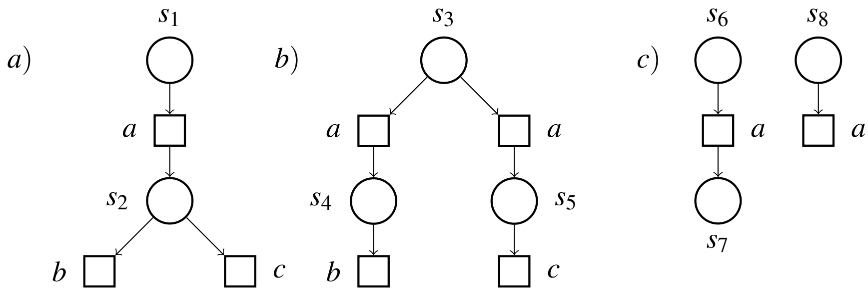


Figure 6. Some BPP nets

Example 4.11. Consider Figure 6. Of course, $s_1 \not\sim_{fc} s_3$, even if they generate the same causal nets. In fact, transition $s_1 \xrightarrow{a} s_2$ might be matched by s_3 either with $s_3 \xrightarrow{a} s_4$ or with $s_3 \xrightarrow{a} s_5$, so that $s_2 \sim_{fc} s_4$ or $s_2 \sim_{fc} s_5$ must hold; but this is impossible, because only s_2 can perform both b and c. □

Proposition 4.12. (Fully-concurrent bisimilarity is finer than interleaving bisimilarity) Let $N = (S, A, T)$ be a BPP net. If $m_1 \sim_{fc} m_2$, then $m_1 \sim_{int} m_2$.

Proof:
If $m_1 \sim_{fc} m_2$, then there exists a fully-concurrent bisimulation R with a triple $((C_1^0, \rho_1^0), g_0, (C_2^0, \rho_2^0))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i^0(Max(C_i^0)) = m_i$ for $i = 1, 2$. Relation $R' = \{(\rho_1(Max(C_1)), \rho_2(Max(C_2))) \mid ((C_1, \rho_1), g, (C_2, \rho_2)) \in R\}$ is an interleaving bisimulation. As the pair (m_1, m_2) is in R', it follows that $m_1 \sim_{int} m_2$. □

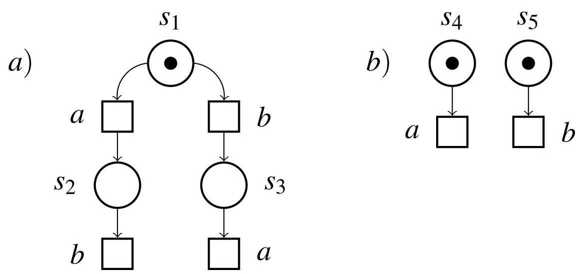


Figure 7. Two non-fully-concurrent bisimilar BPP nets

The implication above is strict. Consider the nets in Figure 7. Of course, $s_1 \sim_{int} s_4 \oplus s_5$ because both markings can generate the two traces ab and ba, but $s_1 \not\sim_{fc} s_4 \oplus s_5$ because the two markings generate different partial orders.

Definition 4.13. (State-sensitive strong fully-concurrent bisimulation) A strong fc-bisimulation R is *state-sensitive* if for each triple $((C_1, \rho_1), g, (C_2, \rho_2)) \in R$, the maximal markings have equal size, i.e., $|\rho_1(Max(C_1))| = |\rho_2(Max(C_2))|$.

Two markings m_1 and m_2 of N are ssfc-bisimilar, denoted by $m_1 \sim_{sfc} m_2$, if there exists a state-sensitive strong fc-bisimulation R containing a triple $((C_1^0, \rho_1^0), g_0, (C_2^0, \rho_2^0))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_i$ for $i = 1, 2$. □

Of course, also the above definition can be defined coinductively; moreover, $\sim_{sfc}$ is an equivalence relation, too. State-sensitive fc-bisimilarity is equivalent to a form of bisimulation equivalence on the BPP causal nets, called *causal-net bisimilarity* [19, 20]. Note that if $m_1 \sim_{sfc} m_2$, then $|m_1| = |m_2|$.

Example 4.14. Consider Figure 6(c). Of course, $s_6 \not\sim_{sfc} s_8$, because they do not generate the same causal net. However, $s_6 \sim_{fc} s_8$ because still they generate an isomorphic concrete partial order of events. Consider also Figure 3. Note that $s_8 \not\sim_{sfc} s_{11}$, but $s_8 \sim_{fc} s_{11}$, because even if the two markings generate different causal nets, the underlying concrete partial orders of events are isomorphic. □

4.2. Weak (state-sensitive) fully-concurrent bisimulation

Definition 4.15. Let $N = (S, A, T)$ be a BPP net. A *weak fully-concurrent bisimulation* is a relation R, composed of triples of the form (π_1, g, π_2), where, for $i = 1, 2$, $\pi_i = (C_i, \rho_i)$ is a process of $N(m_{0i})$ for some m_{0i}, and g is an abstract isomorphism between C_1 and C_2, such that if $(\pi_1, g, \pi_2) \in R$ then

i) $\forall t_1, \pi_1'$ such that $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$, $\exists \sigma_2, \pi_2', g'$ such that

1. $\pi_2 \stackrel{\sigma_2}{\Longrightarrow} \pi_2'$,
2. if $l(e_1) = \tau$, then $o(\sigma_2) = \varepsilon$ and $g' = g$; otherwise, $l(e_1) = o(\sigma_2)$ and there is a transition e_2 in σ_2 such that $l(e_1) = l(e_2)$ and $g' = g \cup \{(e_1, e_2)\}$; and finally,
3. $(\pi_1', g', \pi_2') \in R$;

ii) and symmetrically, if π_2 moves first.

Two markings m_1 and m_2 of N are wfc-bisimilar, denoted by $m_1 \approx_{fc} m_2$, if there exists a weak fully-concurrent bisimulation R containing a triple $((C_1^0, \rho_1), g_0, (C_2^0, \rho_2))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i(Min(C_i^0)) = \rho_i(Max(C_i^0)) = m_i$ for $i = 1, 2$. □

Let us denote by $\approx_R^{fc} = \{(m_1, m_2) \mid m_1$ is wfc-bisimilar to m_2 due to $R\}$. Of course, $\approx_{fc} = \bigcup\{\approx_R^{fc} \mid R$ is a weak fully-concurrent bisimulation$\} = \approx_{\mathscr{R}}^{fc}$, where relation $\mathscr{R} = \bigcup\{R \mid R$ is a weak fully-concurrent bisimulation$\}$ is the largest weak fully-concurrent bisimulation by item 4 of Proposition 4.17, based on the following lemma.

Lemma 4.16. Given a BPP net $N = (S, A, T)$, let R be a weak fc-bisimulation such that $(\pi_1, g, \pi_2) \in R$. Then, the following hold:

(*i*) For all σ_1 such that $\pi_1 \stackrel{\sigma_1}{\Longrightarrow} \pi_1'$ and $o(\sigma_1) = \varepsilon$, there exist σ_2, π_2' such that $\pi_2 \stackrel{\sigma_2}{\Longrightarrow} \pi_2'$, $o(\sigma_2) = \varepsilon$, and, finally, $(\pi_1', g, \pi_2') \in R$.

(*ii*) Symmetrically, if π_2 moves first.

Proof:
The proof is by induction on the length of σ_1. We prove only case (*i*), as the other one is symmetric. The base case is $\pi_1 \stackrel{\varepsilon}{\Longrightarrow} \pi_1$; in such a case, π_2 replies by idling, $\pi_2 \stackrel{\varepsilon}{\Longrightarrow} \pi_2$, and $(\pi_1, g, \pi_2) \in R$, as

required. In general, we can assume that $\pi_1 \xrightarrow{e_1} \pi_1' \overset{\sigma_1}{\Longrightarrow} \pi_1''$, where $l(e_1) = \tau$ and $o(\sigma_1) = \varepsilon$. Since $(\pi_1, g, \pi_2) \in R$, for the move $\pi_1 \xrightarrow{e_1} \pi_1'$, by Definition 4.15, we have that $\exists \sigma_2, \pi_2'$ such that $\pi_2 \overset{\sigma_2}{\Longrightarrow} \pi_2'$, $o(\sigma_2) = \varepsilon$ and $(\pi_1', g, \pi_2') \in R$. Now, induction can be applied to $\pi_1' \overset{\sigma_1}{\Longrightarrow} \pi_1''$ as σ_1 is a shorter path. Hence, we can conclude that $\exists \sigma_2', \pi_2''$ such that $\pi_2' \overset{\sigma_2'}{\Longrightarrow} \pi_2''$, $o(\sigma_2') = \varepsilon$ and also $(\pi_1'', g, \pi_2'') \in R$. Summing up, if $\pi_1 \xrightarrow{e_1} \pi_1' \overset{\sigma_1}{\Longrightarrow} \pi_1''$, with $l(t_1) = \tau$ and $o(\sigma_1) = \varepsilon$, then $\pi_2 \overset{\sigma_2}{\Longrightarrow} \pi_2' \overset{\sigma_2'}{\Longrightarrow} \pi_2''$, $o(\sigma_2 \sigma_2') = \varepsilon$ and, finally, $(\pi_1'', g, \pi_2'') \in R$, as required. □

Proposition 4.17. For each BPP net $N = (S, A, T)$, the following hold:

1. the identity relation $\mathscr{I} = \{((C,\rho), id, (C,\rho)) \mid \exists m.(C,\rho)$ is a process of $N(m)$ and id is the identity abstract event isomorphism on $C\}$ is a weak fully-concurrent bisimulation;
2. the inverse relation $R^{-1} = \{((C_2,\rho_2), g^{-1}, (C_1,\rho_1)) \mid ((C_1,\rho_1), g, (C_2,\rho_2)) \in R\}$ of a weak fully-concurrent bisimulation R is a weak fully-concurrent bisimulation;
3. the relational composition, up to net isomorphism, $R_1 \circ R_2 = \{((C_1,\rho_1), g, (\overline{C}_3, \overline{\rho}_3)) \mid ((C_1,\rho_1), g_1, (C_2,\rho_2)) \in R_1 \wedge ((\overline{C}_2, \overline{\rho}_2), g_2, (\overline{C}_3, \overline{\rho}_3)) \in R_2 \wedge (C_2,\rho_2)$ and $(\overline{C}_2, \overline{\rho}_2)$ are rooted isomorphic via $f_2 \wedge g = g_2 \circ (f_2 \circ g_1)\}$ of two weak fc-bisimulations R_1 and R_2 is a weak fc-bisimulation;
4. the union $\bigcup_{i \in I} R_i$ of a family of weak fully-concurrent bisimulations R_i is a weak fc-bisimulation.

Proof:
The proof of cases 1, 2 and 4 is trivial. The proof of 3, which exploits Lemma 4.16, is omitted because it follows the same steps of the following Proposition 4.28. □

Proposition 4.18. For each BPP net $N = (S, A, T)$, $\approx_{fc} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.

Proof:
Similar to the proof of the following Proposition 4.29 and so omitted.

Proposition 4.19. (Strong fc-bisimilarity is finer than weak fc-bisimilarity) For each BPP net $N = (S, A, T)$, if $m_1 \sim_{fc} m_2$, then $m_1 \approx_{fc} m_2$.

Proof:
If $m_1 \sim_{fc} m_2$, then there exists an sfc-bisimulation R containing a triple $((C_1^0, \rho_1^0), g_0, (C_2^0, \rho_2^0))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i^0(Min(C_i^0)) = m_i$ for $i = 1, 2$.

Relation $R' = \{((C_1,\rho_1), g', (C_2,\rho_2)) \mid ((C_1,\rho_1), g, (C_2,\rho_2)) \in R\}$, where g' is the restriction of g on the observable events, is a weak fully-concurrent bisimulation. As the triple $((C_1^0, \rho_1^0), g_0, (C_2^0, \rho_2^0))$ is in R', it follows that $m_1 \approx_{fc} m_2$. □

Example 4.20. Consider the nets in Figure 3. Of course, $s_1 \not\sim_{fc} s_4$ because the generated concrete partial orders are not isomorphic. However, $s_1 \approx_{fc} s_4$ because the generated abstract partial orders are isomorphic. For the same reason, $s_1 \not\sim_{fc} s_8$ but $s_1 \approx_{fc} s_8$, as well as $s_1 \not\sim_{fc} s_{11}$ but $s_1 \approx_{fc} s_{11}$. Finally, note that $s_8 \sim_{fc} s_{11}$ and so $s_8 \approx_{fc} s_{11}$, too. □

Proposition 4.21. (Weak fc-bisimilarity is finer than weak interleaving bisimilarity) For each BPP net $N = (S, A, T)$, if $m_1 \approx_{fc} m_2$, then $m_1 \approx_{int} m_2$.

Proof:
Similar to the proof of Proposition 4.12. The implication is strict. Consider the nets in Figure 7: of course, $s_1 \approx_{int} s_4 \oplus s_5$, but $s_1 \not\approx_{fc} s_4 \oplus s_5$. □

Definition 4.22. (State-sensitive weak fully-concurrent bisimulation) A weak fully-concurrent bisimulation R is *state-sensitive* if for each triple $((C_1, \rho_1), g, (C_2, \rho_2)) \in R$, the maximal markings have equal size, i.e., $|\rho_1(Max(C_1))| = |\rho_2(Max(C_2))|$.

Two markings m_1 and m_2 of N are swfc-bisimilar, denoted by $m_1 \approx_{sfc} m_2$, if there exists a state-sensitive weak fully-concurrent bisimulation R containing a triple $((C_1^0, \rho_1^0), g_0, (C_2^0, \rho_2^0))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_i$ for $i = 1, 2$. □

Proposition 4.23. For each BPP net $N = (S, A, T)$, if $m_1 \sim_{sfc} m_2$, then $m_1 \approx_{sfc} m_2$. □

Example 4.24. Consider the nets in Figure 3. We argued in Example 4.14 that $s_8 \sim_{fc} s_{11}$, because the generated concrete partial orders are isomorphic. However, $s_8 \not\approx_{sfc} s_{11}$ because the markings related by $\sim_{fc}$ do not always have the same size; in particular, the final markings $2 \cdot s_{10}$ and s_{14}, which are reached after performing all the transitions, have different size. □

4.3. Branching (state-sensitive) fully-concurrent bisimulation

Definition 4.25. Given a BPP net $N = (S, A, T)$, a *branching fully-concurrent bisimulation* is a relation R, composed of triples of the form (π_1, g, π_2), where, for $i = 1, 2$, $\pi_i = (C_i, \rho_i)$ is a process of $N(m_{0i})$ for some m_{0i}, and g is an abstract isomorphism between C_1 and C_2, such that if $(\pi_1, g, \pi_2) \in R$ then

i) $\forall t_1, \pi_1'$ such that $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$,

- *either* $l(e_1) = \tau$ and $\exists \sigma_2$ (with $o(\sigma_2) = \varepsilon$), π_2' such that $\pi_2 \xRightarrow{\sigma_2} \pi_2'$, $(\pi_1, g, \pi_2') \in R$ and $(\pi_1', g, \pi_2') \in R$;
- *or* $\exists \sigma$ (with $o(\sigma) = \varepsilon$), e_2, π_2', π_2'', g' such that
  1. $\pi_2 \xRightarrow{\sigma} \pi_2' \xrightarrow{e_2} \pi_2''$;
  2. if $l(e_1) = \tau$, then $l(e_2) = \tau$ and $g' = g$; otherwise, $l(e_1) = l(e_2)$ and $g' = g \cup \{(e_1, e_2)\}$;
  3. and finally, $(\pi_1, g, \pi_2') \in R$ and $(\pi_1', g', \pi_2'') \in R$;

ii) symmetrically, if π_2 moves first.

Two markings m_1 and m_2 of N are bfc-bisimilar, denoted by $m_1 \approx_{bfc} m_2$, if there exists a branching fully-concurrent bisimulation R with a triple $((C_1^0, \rho_1), g_0, (C_2^0, \rho_2))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i(Min(C_i^0)) = \rho_i(Max(C_i^0)) = m_i$ for $i = 1, 2$. □

Example 4.26. Consider the nets in Figure 5. We argued in Example 3.10 that s_1 is weak interleaving bisimilar to s_4. It is not too difficult to prove that $s_1 \approx_{fc} s_4$, too. However, $s_1 \not\approx_{bfc} s_4$, by the same argument described in Example 3.10: indeed, branching fc-bisimilarity is more appropriate than weak fc-bisimilarity as it does fully respect the timing of choices. $\square$

Let us denote by $\approx_R^{bfc} = \{(m_1, m_2) \mid m_1$ is bfc-bisimilar to m_2 due to $R\}$. Of course, $\approx_{bfc} = \bigcup\{\approx_R^{bfc} \mid R$ is a branching fully-concurrent bisimulation$\} = \approx_{\mathscr{R}}^{bfc}$, where relation $\mathscr{R} = \bigcup\{R \mid R$ is a branching fully-concurrent bisimulation$\}$ is the largest branching fully-concurrent bisimulation by item 4 of Proposition 4.28, based on the following lemma.

Lemma 4.27. Let $N = (S, A, T)$ be a BPP net and let R be a branching fc-bisimulation such that $(\pi_1, g, \pi_2) \in R$. Then, the following hold:

(i) For all σ_1 such that $\pi_1 \stackrel{\sigma_1}{\Longrightarrow} \pi_1'$ and $o(\sigma_1) = \varepsilon$, there exist σ_2, π_2' such that $\pi_2 \stackrel{\sigma_2}{\Longrightarrow} \pi_2'$, $o(\sigma_2) = \varepsilon$, and, finally, $(\pi_1', g, \pi_2') \in R$.

(ii) Symmetrically, if π_2 moves first.

Proof:
The proof, by induction on the length of σ_1, is very similar to that of Lemma 4.16, and so omitted. $\square$

Proposition 4.28. For each BPP net $N = (S, A, T)$, the following hold:

1. the identity relation $\mathscr{I} = \{((C, \rho), id, (C, \rho)) \mid \exists m.(C, \rho)$ is a process of $N(m)$ and id is the identity abstract event isomorphism on $C\}$ is a branching fully-concurrent bisimulation;

2. the inverse relation $R^{-1} = \{((C_2, \rho_2), g^{-1}, (C_1, \rho_1)) \mid ((C_1, \rho_1), g, (C_2, \rho_2)) \in R\}$ of a branching fully-concurrent bisimulation R is a branching fully-concurrent bisimulation;

3. the composition, up to isomorphism, $R_1 \circ R_2 = \{((C_1, \rho_1), g, (\overline{C}_3, \overline{\rho}_3)) \mid ((C_1, \rho_1), g_1, (C_2, \rho_2)) \in R_1 \land ((\overline{C}_2, \overline{\rho}_2), g_2, (\overline{C}_3, \overline{\rho}_3)) \in R_2 \land (C_2, \rho_2)$ and $(\overline{C}_2, \overline{\rho}_2)$ are rooted isomorphic via $f_2 \land g = g_2 \circ (f_2 \circ g_1)\}$ of two branching fc-bisimulations R_1 and R_2 is a branching fc-bisimulation;

4. the union $\bigcup_{i \in I} R_i$ of a family of branching fc-bisimulations R_i is a branching fc-bisimulation.

Proof:
The proof of 1, 2 and 4 is trivial and so omitted. For case 3, assume that $((C_1, \rho_1), g, (\overline{C}_3, \overline{\rho}_3)) \in R_1 \circ R_2$ and that $(C_1, \rho_1) \xrightarrow{e_1} (C_1', \rho_1')$. Since $(\pi_1, g_1, \pi_2) \in R_1$, where $\pi_i = (C_i, \rho_i)$ for $i = 1, 2$, and R_1 is a branching fully-concurrent bisimulation, we have that for the move $\pi_1 \xrightarrow{e_1} \pi_1'$

- *either* $l(e_1) = \tau$ and $\exists \sigma_2$ (with $o(\sigma_2) = \varepsilon$), π_2' such that $\pi_2 \stackrel{\sigma_2}{\Longrightarrow} \pi_2'$, $(\pi_1, g_1, \pi_2') \in R_1$ and, moreover, $(\pi_1', g_1, \pi_2')) \in R_1$, where $\pi_i' = (C_i', \rho_i')$ for $i = 1, 2$;

- *or* $\exists \sigma$ (with $o(\sigma) = \varepsilon$), $e_2, \pi_2', \pi_2'', g_1'$ such that

   1. $\pi_2 \stackrel{\sigma}{\Longrightarrow} \pi_2' \xrightarrow{e_2} \pi_2''$;

2. if $l(e_1) = \tau$, then $l(e_2) = \tau$ and $g_1' = g_1$; otherwise, $l(e_1) = l(e_2)$ and $g_1' = g_1 \cup \{(e_1, e_2)\}$;
3. and finally, $(\pi_1, g_1, \pi_2') \in R_1$ and $(\pi_1', g_1', \pi_2'') \in R_1$.

Let us consider the *either* case, first. Since (C_2, ρ_2) and $(\overline{C}_2, \overline{\rho}_2)$ are isomorphic via f_2, it follows that $\overline{\pi}_2 \stackrel{\overline{\sigma}_2}{\Longrightarrow} \overline{\pi}_2'$, where $\overline{\pi}_2 = (\overline{C}_2, \overline{\rho}_2)$ and $\overline{\pi}_2' = (\overline{C}_2', \overline{\rho}_2')$, such that π_2' and $\overline{\pi}_2'$ are isomorphic via f_2', where f_2' extends f_2 in the obvious way (notably, by mapping the transitions of σ_2 to the corresponding transitions of $\overline{\sigma}_2$).

As $(\overline{\pi}_2, g_2, \overline{\pi}_3) \in R_2$, where $\overline{\pi}_3 = (\overline{C}_3, \overline{\rho}_3)$, and R_2 is a branching fully-concurrent bisimulation, by Lemma 4.27, for $\overline{\pi}_2 \stackrel{\overline{\sigma}_2}{\Longrightarrow} \overline{\pi}_2'$, we have that there exist $\overline{\sigma}_3, \overline{\pi}_3'$ such that $\overline{\pi}_3 \stackrel{\overline{\sigma}_3}{\Longrightarrow} \overline{\pi}_3'$, with $(\overline{\pi}_2', g_2, \overline{\pi}_3') \in R_2$. Note that g is not modified, as $g = g_2 \circ (f_2' \circ g_1)$ (no observable event has been added). Hence, by $(\pi_1, g_1, \pi_2')) \in R_1$ and $(\overline{\pi}_2',\ g_2,\ \overline{\pi}_3') \in R_2$, we have $(\pi_1, g, \overline{\pi}_3') \in R_1 \circ R_2$. Similarly, one can conclude that $(\pi_1', g, \overline{\pi}_3') \in R_1 \circ R_2$ because of $(\pi_1', g_1, \pi_2')) \in R_1$ and $(\overline{\pi}_2',\ g_2,\ \overline{\pi}_3') \in R_2$. Summing up, if $((C_1, \rho_1), g, (\overline{C}_3, \overline{\rho}_3)) \in R_1 \circ R_2$ and $(C_1, \rho_1) \xrightarrow{e_1} (C_1', \rho_1')$, then $\exists \overline{\sigma}_3, \overline{C}_3', \overline{\rho}_3'$ such that $(\overline{C}_3, \overline{\rho}_3) \stackrel{\overline{\sigma}_3}{\Longrightarrow} (\overline{C}_3', \overline{\rho}_3')$, $((C_1, \rho_1), g, (\overline{C}_3', \overline{\rho}_3')) \in R_1 \circ R_2$ and $((C_1', \rho_1'), g, (\overline{C}_3', \overline{\rho}_3')) \in R_1 \circ R_2$, as required.

The *or* case is as follows. Since (C_2, ρ_2) and $(\overline{C}_2, \overline{\rho}_2)$ are isomorphic via f_2, it follows that $\overline{\pi}_2 \stackrel{\overline{\sigma}}{\Longrightarrow} \overline{\pi}_2' \xrightarrow{\overline{e}_2} \overline{\pi}_2''$ such that π_2' and $\overline{\pi}_2'$ are isomorphic via f_2', where f_2' extends f_2 in the obvious way (notably, by mapping the transitions of σ to the corresponding transitions of $\overline{\sigma}$), as well as π_2'' and $\overline{\pi}_2''$ are isomorphic via f_2'', where f_2'' extends f_2' in the obvious way (notably, by mapping e_2 to the corresponding event $\overline{e}_2$). As $(\overline{\pi}_2, g_2, \overline{\pi}_3) \in R_2$ and R_2 is a branching fully-concurrent bisimulation, by Lemma 4.27, for the move $\overline{\pi}_2 \stackrel{\overline{\sigma}}{\Longrightarrow} \overline{\pi}_2'$, there exist $\overline{\sigma}', \overline{\pi}_3'$ such that $\overline{\pi}_3 \stackrel{\overline{\sigma}'}{\Longrightarrow} \overline{\pi}_3'$ with $(\overline{\pi}_2', g_2, \overline{\pi}_3') \in R_2$. Now, since $(\overline{\pi}_2', g_2, \overline{\pi}_3') \in R_2$ and $\overline{\pi}_2' \xrightarrow{\overline{e}_2} \overline{\pi}_2''$, we have that

- *either* $l(\overline{e}_2) = \tau$ and $\exists \overline{\sigma}_3'$ (with $o(\overline{\sigma}_3') = \varepsilon$), $\overline{\pi}_3''$ such that $\overline{\pi}_3' \stackrel{\overline{\sigma}_3'}{\Longrightarrow} \overline{\pi}_3''$, $(\overline{\pi}_2', g_2, \overline{\pi}_3'') \in R_2$ and, moreover, $(\overline{\pi}_2'', g_2, \overline{\pi}_3'')) \in R_2$;
- *or* $\exists \underline{\sigma}$ (with $o(\underline{\sigma}) = \varepsilon$), $\overline{e}_3, \overline{\pi}_3'', \overline{\pi}_3''', g_2'$ such that

  1. $\overline{\pi}_3' \stackrel{\underline{\sigma}}{\Longrightarrow} \overline{\pi}_3'' \xrightarrow{\overline{e}_3} \overline{\pi}_3'''$;
  2. if $l(\overline{e}_2) = \tau$, then $l(\overline{e}_3) = \tau$ and $g_2' = g_2$; otherwise, $l(\overline{e}_2) = l(\overline{e}_3)$ and $g_2' = g_2 \cup \{(\overline{e}_2, \overline{e}_3)\}$;
  3. and finally, $(\overline{\pi}_2', g_2, \overline{\pi}_3'') \in R_2$ and $(\overline{\pi}_2'', g_2', \overline{\pi}_3''') \in R_2$.

Summing up, if the *either* case applies, if $((C_1, \rho_1), g, (\overline{C}_3, \overline{\rho}_3)) \in R_1 \circ R_2$ and $(C_1, \rho_1) \xrightarrow{e_1} (C_1', \rho_1')$, then there exist $\overline{\sigma}', \overline{\sigma}_3', \overline{C}_3'', \overline{\rho}_3''$ such that $(\overline{C}_3, \overline{\rho}_3) \stackrel{\overline{\sigma}'\overline{\sigma}_3'}{\Longrightarrow} (\overline{C}_3'', \overline{\rho}_3'')$, $((C_1, \rho_1), g, (\overline{C}_3'', \overline{\rho}_3'')) \in R_1 \circ R_2$ and $((C_1', \rho_1'), g, (\overline{C}_3'', \overline{\rho}_3'')) \in R_1 \circ R_2$, as required, where $((C_1, \rho_1), g, (\overline{C}_3'', \overline{\rho}_3'')) \in R_1 \circ R_2$ because $((C_1, \rho_1), g_1, (C_2', \rho_2')) \in R_1$ and $((\overline{C}_2', \overline{\rho}_2'), g_2, (\overline{C}_3'', \overline{\rho}_3'')) \in R_2$, and, similarly, $((C_1', \rho_1'), g, (\overline{C}_3'', \overline{\rho}_3'')) \in R_1 \circ R_2$ because $((C_1', \rho_1'), g_1', (C_2'', \rho_2'')) \in R_1$ and $((\overline{C}_2'', \overline{\rho}_2''), g_2, (\overline{C}_3'', \overline{\rho}_3'')) \in R_2$. The other sub-case (i.e., when the *or* case applies) is similar and so omitted. The symmetric case when $(\overline{C}_3, \overline{\rho}_3)$ moves first is analogous, hence omitted. Therefore, $R_1 \circ R_2$ is a branching fully-concurrent bisimulation, indeed. □

Proposition 4.29. For each BPP net $N = (S, A, T)$, $\approx_{bfc} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.

Proof:
Reflexivity: The identity relation $\mathscr{I} = \{((C,\rho),id,(C,\rho)) \mid \exists m.(C,\rho)$ is a process of $N(m)$ and id is the identity abstract event isomorphism on $C\}$ is a branching fully-concurrent bisimulation by Proposition 4.28(1). Therefore, $m \approx_{bfc} m$ for all m.

Symmetry: For any $(m_1,m_2) \in \approx_{bfc}$, there exists a branching fully-concurrent bisimulation R with a triple $((C_1^0,\rho_1^0),g_0,(C_2^0,\rho_2^0))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_i$ for $i = 1,2$. By Proposition 4.28(2), relation R^{-1} is a branching fully-concurrent bisimulation containing the triple $((C_2^0,\rho_2^0),g_0^{-1},(C_1^0,\rho_1^0))$, and so $(m_2,m_1) \in \approx_{bfc}$.

Transitivity also holds for $\approx_{bfc}$. Assume $(m_1,m_2) \in \approx_{bfc}$ and $(m_2,m_3) \in \approx_{bfc}$; hence, there exist two branching fully-concurrent bisimulations R_1 and R_2 such that R_1 has a triple $((C_1^0,\rho_1^0),g_0,(C_2^0,\rho_2^0))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_i$ for $i = 1,2$; and R_2 has a triple $((\overline{C}_2^0,\overline{\rho}_2^0),g_0,\ (\overline{C}_3^0,\overline{\rho}_3^0))$, where $\overline{C}_i^0$ contains no transitions and $\overline{\rho}_i^0(Min(\overline{C}_i^0)) = \overline{\rho}_i^0(Max(\overline{C}_i^0)) = m_i$ for $i = 2,3$. Note tht (C_2^0,ρ_2^0) and $(\overline{C}_2^0,\overline{\rho}_2^0)$ are isomorphic via some bijection f. Hence, by Proposition 4.28(3), relation $R_1 \circ R_2$ is a branching fully-concurrent bisimulation containing the triple $((C_1^0,\rho_1^0),g_0,(\overline{C}_3^0,\overline{\rho}_3^0))$ so that $(m_1,m_3) \in \approx_{bfc}$. □

Proposition 4.30. For each BPP net $N = (S,A,T)$, if $m_1 \sim_{fc} m_2$, then $m_1 \approx_{bfc} m_2$.

Proof:
If $m_1 \sim_{fc} m_2$, then there exists an fc-bisimulation R with a triple $((C_1^0,\rho_1^0),g_0,(C_2^0,\rho_2^0))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i^0(Min(C_i^0)) = m_i$ for $i = 1,2$. $R' = \{((C_1,\rho_1),g',(C_2,\rho_2)) \mid ((C_1,\rho_1),g,\ (C_2,\rho_2)) \in R\}$, where g' is the restriction of g on the observable events, is a branching fc-bisimulation. As R' has the triple $((C_1^0,\rho_1^0),g_0,(C_2^0,\rho_2^0))$, it follows that $m_1 \approx_{bfc} m_2$. □

Proposition 4.31. (Branching fc-bisimilarity is finer than branching interleaving bisimilarity) For each BPP net $N = (S,A,T)$, if $m_1 \approx_{bfc} m_2$, then $m_1 \approx_{bri} m_2$.

Proof:
Similar to the proof of Proposition 4.12. The implication is strict. Consider the nets in Figure 7: of course, $s_1 \approx_{bri} s_4 \oplus s_5$, but $s_1 \not\approx_{bfc} s_4 \oplus s_5$. □

Of course, we also have that $\approx_{bfc}$ is finer than $\approx_{fc}$ because a branching fc-bisimulation is also a weak fc-bisimulation.

Definition 4.32. (State-sensitive branching fully-concurrent bisimulation) A branching fully concurrent bisimulation R is *state-sensitive* if for each triple $((C_1,\rho_1),g,(C_2,\rho_2)) \in R$, the maximal markings have equal size, i.e., $|\rho_1(Max(C_1))| = |\rho_2(Max(C_2))|$.

Two marking m_1 and m_2 of N are sbfc-bisimilar, denoted by $m_1 \approx_{sbfc} m_2$, if there exists a state-sensitive branching fully-concurrent bisimulation R containing a triple $((C_1^0,\rho_1^0),g_0,(C_2^0,\rho_2^0))$, where C_i^0 contains no transitions, g_0 is empty and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_i$ for $i = 1,2$. □

Example 4.33. Consider the nets in Figure 3. We argued in Example 4.14 that $s_8 \sim_{fc} s_{11}$, because the generated concrete partial orders are isomorphic. However, $s_8 \not\approx_{sbfc} s_{11}$ because the markings related by $\sim_{fc}$ do not always have the same size; in particular, the final markings $2 \cdot s_{10}$ and s_{14}, which are reached after performing all the transitions, have different size. □

Finally, to complete the picture, we have that $\approx_{sbfc}$ is obviously finer than $\approx_{sfc}$ because a sbfc-bisimulation is also a swfc-bisimulation.

5. A distributed approach to strong equivalence checking

In this section, we recall the main definitions and results about strong (h-)team bisimulation equivalence, outlined in [17, 19, 20] and we also describe (a slight generalization of) the algorithm in [25].

5.1. Additive closure and its properties

Definition 5.1. (Additive closure) Given a BPP net $N = (S, A, T)$ and a *place relation* $R \subseteq S \times S$, we define a *marking relation* $R^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$, called the *additive closure* of R, as the least relation induced by the following axiom and rule.

$$\frac{}{(\theta,\theta) \in R^{\oplus}} \qquad\qquad \frac{(s_1,s_2) \in R \quad (m_1,m_2) \in R^{\oplus}}{(s_1 \oplus m_1, s_2 \oplus m_2) \in R^{\oplus}}$$

□

It follows, by induction, that two markings are related by $R^{\oplus}$ only if they have the same size.

Proposition 5.2. For any BPP net $N = (S, A, T)$ and any place relation $R \subseteq S \times S$, if $(m_1, m_2) \in R^{\oplus}$, then $|m_1| = |m_2|$. □

An alternative way to define that m_1 and m_2 are related by $R^{\oplus}$ is to state that m_1 can be represented as $s_1 \oplus s_2 \oplus \ldots \oplus s_k$, m_2 can be represented as $s'_1 \oplus s'_2 \oplus \ldots \oplus s'_k$ and $(s_i, s'_i) \in R$ for $i = 1, \ldots, k$.

Proposition 5.3. [17] For each BPP net $N = (S, A, T)$ and each place relation $R, R_1, R_2 \subseteq S \times S$:

1. If R is an equivalence relation, then $R^{\oplus}$ is an equivalence relation.
2. If $R_1 \subseteq R_2$, then $R_1^{\oplus} \subseteq R_2^{\oplus}$, i.e., the additive closure is monotone.
3. If $(m_1, m_2) \in R^{\oplus}$ and $(m'_1, m'_2) \in R^{\oplus}$, then $(m_1 \oplus m'_1, m_2 \oplus m'_2) \in R^{\oplus}$.
4. If R is an equivalence, $(m_1 \oplus m'_1, m_2 \oplus m'_2) \in R^{\oplus}$ and $(m_1, m_2) \in R^{\oplus}$, then $(m'_1, m'_2) \in R^{\oplus}$. □

Now we list some useful, and less obvious, properties of additively closed place relations.

Proposition 5.4. [17] For any BPP net $N = (S, A, T)$ and any family of place relations $R_i \subseteq S \times S$, the following hold:

1. $\emptyset^{\oplus} = \{(\theta, \theta)\}$, i.e., the additive closure of the empty place relation is a singleton marking relation, relating the empty marking to itself.
2. $(\mathscr{I}_S)^{\oplus} = \mathscr{I}_M$, i.e., the additive closure of the identity relation on places $\mathscr{I}_S = \{(s,s) \mid s \in S\}$ is the identity relation on markings $\mathscr{I}_M = \{(m,m) \mid m \in \mathscr{M}(S)\}$.

3. $(R^{\oplus})^{-1} = (R^{-1})^{\oplus}$, i.e., the inverse of an additively closed relation R is the additive closure of its inverse R^{-1}.

4. $(R_1 \circ R_2)^{\oplus} = (R_1^{\oplus}) \circ (R_2^{\oplus})$, i.e., the additive closure of the composition of two place relations is the compositions of their additive closures.

5. $\bigcup_{i \in I}(R_i^{\oplus}) \subseteq (\bigcup_{i \in I} R_i)^{\oplus}$, i.e., the union of additively closed relations is included into the additive closure of their union. □

When R is an equivalence relation, it is rather easy to check whether two markings are related by $R^{\oplus}$. An algorithm, described in [17], establishes whether an R-preserving bijection between the two markings exists, by first implementing the equivalence relation R as an adjacency matrix A of size n (the entry $A[s,s']$ is marked 1 if $(s,s') \in R$, 0 otherwise), and then by checking whether for each place/token s in m_1 there exists a place/token s' in m_2 such that the entry $A[s,s']$ is marked 1. The complexity of this algorithm is not very high: first, the generation of the adjacency matrix takes $O(n^2)$ time, and then checking whether $m_1 \sim^{\oplus} m_2$ takes $O(k^2)$ time, if k is the size of m_1 and m_2. Note that if we want to perform additional team equivalence checks on the same net, we can reuse the already computed matrix A, so that the new checks will take only $O(k^2)$ time from the second check on.

Algorithm 1 Algorithm for checking whether $(m_1, m_2) \in R^{\oplus}$

Let $N = (S, A, T)$ be an *BPP*, with $S = \{s_1, \ldots, s_n\}$.
Let m_1 and m_2 be two markings on S.
Let $R \subseteq S \times S$ be an *equivalence* place relation.
Let $P = \{B_1, \ldots, B_l\}$, $1 \leq l \leq n$, be the partition of S in the equivalence classes (called *blocks*) of R: $B_i \cap B_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{l} B_i = S$, $\forall s, s' \in B_i.(s,s') \in R$ for $i = 1, \ldots, l$ and, finally, $\forall s \in B_i, \forall s' \in B_j$ if $i \neq j$, then $(s,s') \notin R$.

Let $count_1, count_2$ be two integer variables
for all blocks in P **do**
  $count_1, count_2 = 0$
  for all places s in the current block **do**
    $count_1 = count_1 + m_1(s)$
    $count_2 = count_2 + m_2(s)$
  end for
  if not$(count_1 == count_2)$ **then**
    return $false$
  end if
end for
return $true$

However, this algorithm can be improved. Algorithm 1 (which is a slight generalization of the algorithm proposed originally in [25]) checks whether $(m_1, m_2) \in R^{\oplus}$ simply by checking if, for each equivalence class of R, the number of places/tokens of m_1 in that class equals the number of places/tokens of m_2 in the same class. In this way, we are sure that there is an R-preserving, bijective mapping between the two markings. The complexity of this new algorithm is $O(n)$, because we have essentially to scan all the (equivalence classes and then the) places (in these classes), and this complexity holds already for the first check. Therefore, this new algorithm is better than the old one for the first check, while it may be less performant than the original one, from the second check onwards, only if the markings are small compared to the size of the net: more precisely, if $k < \sqrt{n}$. The reason why Algorithm 1 usually outperforms the old one in [17] is that, by exploiting the partition of S induced by R, there is no need to build any auxiliary data structure for representing R. As we will check whether $(m_1, m_2) \in R^\oplus$ for equivalence relations that are computed by means of variations of the Kanellakis-Smolka algorithm [23, 24] (whose output is directly a partition of $S \cup \{\theta\}$), we have that Algorithm 1 can be really exploited successfully. Nonetheless, the old algorithm, if successful, computes a set of pairs of matched places, while the new one simply gives a boolean result.

5.2. Team bisimulation on places

Definition 5.5. (Team bisimulation) Let $N = (S, A, T)$ be a BPP net. A *team bisimulation* is a place relation $R \subseteq S \times S$ such that if $(s_1, s_2) \in R$ then for all $\ell \in A$

- $\forall m_1$ such that $s_1 \xrightarrow{\ell} m_1$, $\exists m_2$ such that $s_2 \xrightarrow{\ell} m_2$ and $(m_1, m_2) \in R^\oplus$,
- $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$, $\exists m_1$ such that $s_1 \xrightarrow{\ell} m_1$ and $(m_1, m_2) \in R^\oplus$.

Two places s and s' are *team bisimilar* (or *team bisimulation equivalent*), denoted $s \sim s'$, if there exists a team bisimulation R such that $(s, s') \in R$. □

Example 5.6. Continuing Example 2.5 about the semi-counters in Figure 2, it is easy to see that relation $R = \{(s_1, s_3), (s_1, s_4), (s_2, s_5), (s_2, s_6)\}$ is a team bisimulation. In fact, the pair (s_1, s_3) is a team bisimulation pair because, to transition $s_1 \xrightarrow{inc} s_1 \oplus s_2$, s_3 can respond with $s_3 \xrightarrow{inc} s_4 \oplus s_5$, and $(s_1 \oplus s_2, s_4 \oplus s_5) \in R^\oplus$; symmetrically, if s_3 moves first. Also the pair (s_1, s_4) is a team bisimulation pair because, to transition $s_1 \xrightarrow{inc} s_1 \oplus s_2$, s_4 can respond with $s_4 \xrightarrow{inc} s_3 \oplus s_6$, and $(s_1 \oplus s_2, s_3 \oplus s_6) \in R^\oplus$; symmetrically, if s_4 moves first. Also the pair (s_2, s_5) is a team bisimulation pair: to transition $s_2 \xrightarrow{dec} \theta$, s_5 responds with $s_5 \xrightarrow{dec} \theta$, and $(\theta, \theta) \in R^\oplus$. Similarly for the pair (s_2, s_6). Hence, relation R is a team bisimulation, indeed. The team bisimulation above is a very simple, finite relation, proving that $s_1 \sim s_3$. In Example 3.2, in order to show that s_1 and s_3 are interleaving bisimilar, we had to introduce a rather complex relation, with infinitely many pairs. □

Example 5.7. Consider Figure 8. It is easy to realize that relation $R = \{(s_1, s_4), (s_2, s_5), (s_2, s_6), (s_2, s_7), (s_3, s_8), (s_3, s_9)\}$ is a team bisimulation. □

Proposition 5.8. [17] For any BPP net $N = (S, A, T)$, the following hold:

1. The identity relation $\mathscr{I}_S = \{(s, s) \mid s \in S\}$ is a team bisimulation;
2. the inverse relation $R^{-1} = \{(s', s) \mid (s, s') \in R\}$ of a team bisimulation R is a team bisimulation;

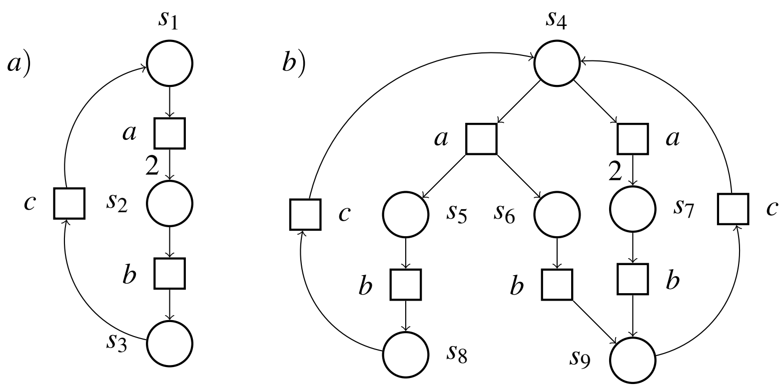


Figure 8. Two team bisimilar BPP nets

3. the relational composition $R_1 \circ R_2 = \{(s, s'') \mid \exists s'.(s, s') \in R_1 \wedge (s', s'') \in R_2\}$ of two team bisimulations R_1 and R_2 is a team bisimulation;
4. the union $\bigcup_{i \in I} R_i$ of team bisimulations R_i is a team bisimulation. □

Remember that $s \sim s'$ if there exists a team bisimulation containing the pair (s, s'). This means that $\sim$ is the union of all team bisimulations, i.e.,

$$\sim = \bigcup \{R \subseteq S \times S \mid R \text{ is a team bisimulation}\}.$$

By Proposition 5.8(4), $\sim$ is also a team bisimulation, hence the largest such relation.

Proposition 5.9. For each BPP net $N = (S, A, T)$, relation $\sim \subseteq S \times S$ is the largest team bisimulation relation. □

Proposition 5.10. For each BPP net $N = (S, A, T)$, relation $\sim \subseteq S \times S$ is an equivalence relation. □

Remark 5.11. (Complexity of $\sim$) It is well-known that the classic Kanellakis-Smolka algorithm for computing bisimulation equivalence over a finite-state LTS with n states and m transitions has $O(m \cdot n)$ time complexity [23, 24]. This very same partition refinement algorithm can be easily adapted also for computing team bisimilarity $\sim$ over BPP nets: it is enough to consider the empty marking θ as an additional, special place which is team bisimilar to itself only, and to consider the little additional cost due to the fact that the reached markings are to be related by the additive closure of the equivalence relation induced by the current partition over places; this extra check can be done with Algorithm 1 in $O(n)$ time, so that the overall time complexity is $O(m \cdot n^2)$, where m is the number of the net transitions and n is the number of the net places. □

5.3. Team bisimilarity over markings

Starting from team bisimulation equivalence $\sim$, which has been computed *over the places of an unmarked* BPP net N, we can lift it *over the markings* of N in a distributed way: m_1 is team bisimulation equivalent to m_2 if these two markings are related by the additive closure of $\sim$, i.e., if $(m_1, m_2) \in \sim^{\oplus}$, usually denoted by $m_1 \sim^{\oplus} m_2$.

Proposition 5.12. For any BPP net $N = (S, A, T)$, if $m_1 \sim^{\oplus} m_2$, then $|m_1| = |m_2|$.

Proof:
By Proposition 5.2. □

Proposition 5.13. For any BPP net $N = (S, A, T)$, $\sim^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.

Proof:
By Proposition 5.3: since $\sim$ is an equivalence relation (Proposition 5.10), its additive closure $\sim^{\oplus}$ is also an equivalence relation. □

Remark 5.14. (Complexity of $\sim^{\oplus}$) Once the place relation $\sim$ has been computed once and for all for the given net (in $O(m \cdot n^2)$ time; cf. Remark 5.11)), Algorithm 1 checks whether two markings m_1 and m_2 are team bisimulation equivalent in $O(n)$ time. □

Example 5.15. Continuing Example 5.6 about the semi-counters, the marking $s_1 \oplus 2 \cdot s_2$ is team bisimilar to the following markings of the net in (b): $s_3 \oplus 2 \cdot s_5$, or $s_3 \oplus s_5 \oplus s_6$, or $s_3 \oplus 2 \cdot s_6$, or $s_4 \oplus 2 \cdot s_5$, or $s_4 \oplus s_5 \oplus s_6$, or $s_4 \oplus 2 \cdot s_6$. □

The following theorem provides a characterization of team bisimulation equivalence $\sim^{\oplus}$ as a suitable bisimulation-like relation over markings. It is interesting to observe that this characterization gives a dynamic interpretation of team bisimulation equivalence, while Definition 5.1 gives a structural definition of team bisimulation equivalence $\sim^{\oplus}$ as the additive closure of $\sim$.

Theorem 5.16. [17] Let $N = (S, A, T)$ be a BPP net. Two markings m_1 and m_2 are team bisimulation equivalent, $m_1 \sim^{\oplus} m_2$, if and only if $|m_1| = |m_2|$ and

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that ${}^\bullet t_1 \sim {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \sim^{\oplus} t_2^\bullet$, $m_2[t_2\rangle m_2'$ and $m_1' \sim^{\oplus} m_2'$,
- and symmetrically, $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that ${}^\bullet t_1 \sim {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \sim^{\oplus} t_2^\bullet$, $m_1[t_1\rangle m_1'$ and $m_1' \sim^{\oplus} m_2'$. □

By the theorem above, it is clear that $\sim^{\oplus}$ is an interleaving bisimulation; hence, team bisimilarity does respect the sequential behavior of BPP nets.

Corollary 5.17. (Team bisimilarity is finer than interleaving bisimilarity) Let $N = (S, A, T)$ be a BPP net. If $m_1 \sim^{\oplus} m_2$, then $m_1 \sim_{int} m_2$. □

Moreover, team bisimilarity does respect the causal semantics of BPP nets, as it coincides with state-sensitive strong fully concurrent bisimilarity, as proved in [20].

Theorem 5.18. (Team bisimilarity and ssfc-bisimilarity coincide) Let $N = (S, A, T)$ be a BPP net. Then, $m_1 \sim_{sfc} m_2$ if and only if $m_1 \sim^{\oplus} m_2$. □

Therefore, our characterization of ssfc-bisimilarity, which is, in our opinion, the intuitively correct strong causal semantics for BPP nets (as it preserves the causal nets [2, 29], as proved in [19, 20]), is quite appealing because it is based on the very simple technical definition of team bisimulation on the places of the unmarked net, and, moreover, offers a very efficient algorithm to check whether two markings are ssfc-bisimilar (see Remark 5.14).

5.4. H-team bisimilarity

In order to provide the definition of *h-team bisimulation on places* for unmarked BPP nets, adapting the definition of team bisimulation on places (cf. Definition 5.5), we need first to extend the domain of a place relation: the empty marking θ is considered as an additional place, so that a place relation is defined not on S, rather on $S \cup \{\theta\}$. Therefore, the symbols p_1 and p_2 that occur in the following definitions can only denote either the empty marking θ or a single place s.

First of all, we extend the idea of additive closure to these more general place relations, yielding *h-additive closure*, still denoted by $-^{\oplus}$ with abuse of notation.

Definition 5.19. (H-additive closure) Given a BPP net $N = (S, A, T)$ and a *place relation* $R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\})$, we define a *marking relation* $R^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$, called the *h-additive closure* of R, as the least relation induced by the following axiom and rule.

$$\frac{}{(\theta, \theta) \in R^{\oplus}} \qquad \frac{(p_1, p_2) \in R \quad (m_1, m_2) \in R^{\oplus}}{(p_1 \oplus m_1, p_2 \oplus m_2) \in R^{\oplus}}$$

□

Note that if two markings are related by $R^{\oplus}$ (i.e., by the h-additive closure of R), then they may have different size; in fact, even if the axiom relates the empty marking to itself (so two markings with the same size), as $R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\})$, it may be the case that $(\theta, s) \in R$, so that, assuming $(m_1', m_2') \in R^{\oplus}$ with $|m_1'| = |m_2'|$, we get $(m_1', s \oplus m_2') \in R^{\oplus}$, as θ is the identity for the operator of multiset union. Hence, Proposition 5.2, which is valid for place relations defined over S, is not valid for place relations defined over $S \cup \{\theta\}$. However, the properties in Propositions 5.3 and 5.4 hold also for these more general place relations. In particular, if $R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\})$ is an equivalence relation, then $R^{\oplus}$ is also an equivalence relation.

Remark 5.20. (Complexity of h-additive closure) Given an equivalence relation $R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\})$, the complexity of checking whether two markings m_1 and m_2 are related by $R^{\oplus}$ is still $O(n)$, where n is the size of S, because Algorithm 1 can be easily adapted to this case. First, P is a partition of $S \cup \{\theta\}$, so that one of its blocks contains θ. Then, in line 2 of the algorithm, it is enough to state

2: **forall** *blocks in P (except that of* θ*)* **do**

so that, for each class (except for the class of θ), it checks whether the number of tokens in the places of m_1 belonging to this class equals the number of tokens in the places of m_2 in the same class; if this holds for all the considered equivalence classes, then $(m_1, m_2) \in R^{\oplus}$. □

Definition 5.21. (H-team bisimulation) Let $N = (S, A, T)$ be a BPP net. An *h-team bisimulation* is a place relation $R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\})$ such that if $(p_1, p_2) \in R$ then for all $\ell \in A$

- $\forall m_1$ such that $p_1 \xrightarrow{\ell} m_1$, $\exists m_2$ such that $p_2 \xrightarrow{\ell} m_2$ and $(m_1, m_2) \in R^{\oplus}$,
- $\forall m_2$ such that $p_2 \xrightarrow{\ell} m_2$, $\exists m_1$ such that $p_1 \xrightarrow{\ell} m_1$ and $(m_1, m_2) \in R^{\oplus}$.

p_1 and p_2 are *h-team bisimilar* (or *h-team bisimulation equivalent*), denoted $p_1 \sim_h p_2$, if there exists an h-team bisimulation R such that $(p_1, p_2) \in R$. □

Since a team bisimulation is also an h-team bisimulation, we have that team bisimilarity $\sim$ implies h-team bisimilarity $\sim_h$. This implication is strict as illustrated in the following examples.

Example 5.22. Consider the nets in Figure 6. It is not difficult to realize that s_6 and s_8 are h-team bisimilar because $R = \{(s_6, s_8), (s_7, \theta)\}$ is a h-team bisimulation. In fact, s_6 can reach s_7 by performing a, and s_8 can reply by reaching the empty marking θ, and $(s_7, \theta) \in R$. In Example 4.11 we argued that $s_6 \sim_{fc} s_8$ and in fact we will state that h-team bisimilarity coincide with fc-bisimilarity. This example shows that h-team bisimulation equivalence is not sensitive to the kind of termination of a process: even if s_7 is a stuck place, denoting a deadlock situation, it is equivalent to the empty marking θ, i.e., the marking denoting a properly terminated process. This is in contrast with the definition of team bisimulation on place (cf. Definition 5.5), which is sensitive to the kind of termination. In fact, $s_6 \nsim s_8$, and indeed $s_6 \nsim_{sfc} s_8$. □

We now list some obvious properties.

Proposition 5.23. For any BPP net $N = (S, A, T)$, the following hold:

1. The identity relation $\mathscr{I}_S = \{(p, p) \mid p \in S \cup \{\theta\}\}$ is an h-team bisimulation;
2. the inverse relation $R^{-1} = \{(p', p) \mid (p, p') \in R\}$ of an h-team bisimulation R is an h-team bisimulation;
3. the relational composition $R_1 \circ R_2 = \{(p, p'') \mid \exists p'.(p, p') \in R_1 \wedge (p', p'') \in R_2\}$ of two h-team bisimulations R_1 and R_2 is an h-team bisimulation;
4. the union $\bigcup_{i \in I} R_i$ of h-team bisimulations R_i is an h-team bisimulation. □

Relation $\sim_h$ is the union of all h-team bisimulations, i.e.,

$$\sim_h = \bigcup \{R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\}) \mid R \text{ is an h-team bisimulation}\}.$$

By Proposition 5.23(4), $\sim_h$ is also an h-team bisimulation, hence the largest such relation. By direct application of Proposition 5.23, the following follows.

Proposition 5.24. For each BPP net $N = (S, A, T)$, $\sim_h \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\})$ is an equivalence. □

Starting from h-team bisimulation equivalence $\sim_h$, which has been computed over the places (and the empty marking) of an *unmarked* BPP net N, we can lift it over *the markings* of N in a distributed way: m_1 is h-team bisimulation equivalent to m_2 if these two markings are related by the additive closure of $\sim_h$, i.e., if $(m_1, m_2) \in \sim_h^{\oplus}$, usually denoted by $m_1 \sim_h^{\oplus} m_2$.

Proposition 5.25. For each BPP net $N = (S, A, T)$, relation $\sim_h^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.

Proof:
By Proposition 5.3: since $\sim_h$ is an equivalence relation (Proposition 5.24), its additive closure $\sim_h^{\oplus}$ is also an equivalence relation. □

Remark 5.26. **(Complexity of $\sim_h^{\oplus}$)** About complexity, we note that computing $\sim_h$ is not more difficult than computing $\sim$ (cf. Remark 5.11). The Kanellakis-Smolka partition refinement algorithm [23, 24] can be adapted also in this case. It is enough to consider the empty marking θ as an additional, special place which is h-team bisimilar to each deadlock place. Hence, the initial partition considers two sets: one composed of all the deadlock places and θ, the other one with all the non-deadlock places. Therefore, the time complexity is also in this case $O(m \cdot n^2)$, where m is the number of the net transitions and n is the number of the net places. Once $\sim_h$ has been computed once and for all for the given net, the complexity of checking whether two markings m_1 and m_2 are h-team bisimulation equivalent is $O(n)$, where n is the number of places (cf. Remark 5.20). □

The main motivation for introducing h-team bisimilarity is the following theorem stating that h-team bisimilarity coincides with strong fully-concurrent bisimilarity for BPP nets.

Theorem 5.27. [20]**(Fully concurrent bisimilarity and h-team bisimilarity coincide)** Given a BPP net $N = (S, A, T)$, $m_1 \sim_{fc} m_2$ if and only if $m_1 \sim_h^{\oplus} m_2$. □

Therefore, our characterization of sfc-bisimilarity, which is considered by many [3, 10, 5, 34] the intuitively correct strong causal semantics, is quite appealing because it is based on the very simple technical definition of h-team bisimulation, and, moreover, offers a rather efficient algorithm to check whether two markings of a BPP net are sfc-bisimilar (see Remark 5.26).

Remark 5.28. **(Resource-aware behavioral equivalence)** We think that state-sensitive strong fc-bisimilarity (hence, also strong team bisimilarity) is more accurate than strong fc-bisimilarity (hence, strong h-team bisimilarity) because it is *resource-aware*, in the following sense. In the implementation of a system, a token is an instance of a sequential process, so that a processor is needed to execute it. If two markings differ for the number of tokens, i.e., they have different size, then a different number of processors are necessary for their execution. Hence, a semantics such as ssfc-bisimilarity which relates only markings of the same size is more accurate as it equates distributed systems only if they require the same amount of execution resources. □

6. A distributed approach to weak equivalence checking

In this section, we extend the approach to BPP nets with silent moves, by introducing *weak team bisimilarity* $\approx$ (as well as *weak h-team bisimilarity* $\approx_h$) on places of an unmarked BPP nets and then its additive closure $\approx^{\oplus}$ (and $\approx_h^{\oplus}$) on markings.

6.1. Weak team bisimulation on places

In order to adapt the definition of weak bisimulation on LTSs [28, 15] for unmarked BPP nets, we need some auxiliary notation.

We can define relation $\stackrel{\varepsilon}{\Longrightarrow} \subseteq S \times \mathscr{M}(S)$ as (a generalization of) the reflexive and transitive closure of the silent transition relation; formally, for every place $s \in S$, we have that $s \stackrel{\varepsilon}{\Longrightarrow} s$, denoting that each place can silently reach itself with zero steps; moreover, if $s \stackrel{\varepsilon}{\Longrightarrow} m$, $s' \in m$ and $s' \stackrel{\tau}{\longrightarrow} m'$, then

$s \stackrel{\varepsilon}{\Longrightarrow} (m \ominus s') \oplus m'$. Note that $s \stackrel{\varepsilon}{\Longrightarrow} m$ if and only if there exists a *sequential* transition sequence σ such that $s[\sigma\rangle m$ and $o(\sigma) = \varepsilon$. If $s \stackrel{\varepsilon}{\Longrightarrow} m$ and the sequence of silent moves is not empty, we may also denote this by $s \stackrel{\tau}{\Longrightarrow} m$. Hence, this is the same as saying that there exists a *nonempty*, sequential transition sequence σ such that $s[\sigma\rangle m$ and $o_\tau(\sigma) = \tau$. We also write $m \stackrel{\varepsilon}{\Longrightarrow} m'$ if there exists a transition sequence σ such that $m[\sigma\rangle m'$ and $o(\sigma) = \varepsilon$. In particular, we have $\theta \stackrel{\varepsilon}{\Longrightarrow} \theta$. Finally, for any $\ell \in A_\tau = A \setminus \{\tau\}$, we write $m_1 \stackrel{\ell}{\longrightarrow} m_2$ if there exists a transition t such that $m_1[t\rangle m_2$ and $l(t) = \ell$. We also write $s \stackrel{\ell}{\Longrightarrow} m$ if there exist two markings m_1 and m_2 such that $s \stackrel{\varepsilon}{\Longrightarrow} m_1 \stackrel{\ell}{\longrightarrow} m_2 \stackrel{\varepsilon}{\Longrightarrow} m$. Note that $s \stackrel{\ell}{\Longrightarrow} m$ if and only if there exists a sequential transition sequence σ such that $s[\sigma\rangle m$ and $o(\sigma) = \ell$.

Definition 6.1. (Weak team bisimulation on places) Let $N = (S, A, T)$ be a BPP net with silent moves. A *weak team bisimulation* is a relation $R \subseteq S \times S$ such that if $(s_1, s_2) \in R$ then for all $\ell \in A_\tau$

- $\forall m_1$ such that $s_1 \stackrel{\ell}{\longrightarrow} m_1$, $\exists m_2$ such that $s_2 \stackrel{\ell}{\Longrightarrow} m_2$ and $(m_1, m_2) \in R^\oplus$,
- $\forall m_1$ such that $s_1 \stackrel{\tau}{\longrightarrow} m_1$, $\exists m_2$ such that $s_2 \stackrel{\varepsilon}{\Longrightarrow} m_2$ and $(m_1, m_2) \in R^\oplus$,
- $\forall m_2$ such that $s_2 \stackrel{\ell}{\longrightarrow} m_2$, $\exists m_1$ such that $s_1 \stackrel{\ell}{\Longrightarrow} m_1$ and $(m_1, m_2) \in R^\oplus$, and, finally,
- $\forall m_2$ such that $s_2 \stackrel{\tau}{\longrightarrow} m_2$, $\exists m_1$ such that $s_1 \stackrel{\varepsilon}{\Longrightarrow} m_1$ and $(m_1, m_2) \in R^\oplus$.

Two places s and s' are *weakly team bisimilar*, denoted by $s \approx s'$, if there exists a weak team bisimulation R such that $(s, s') \in R$. □

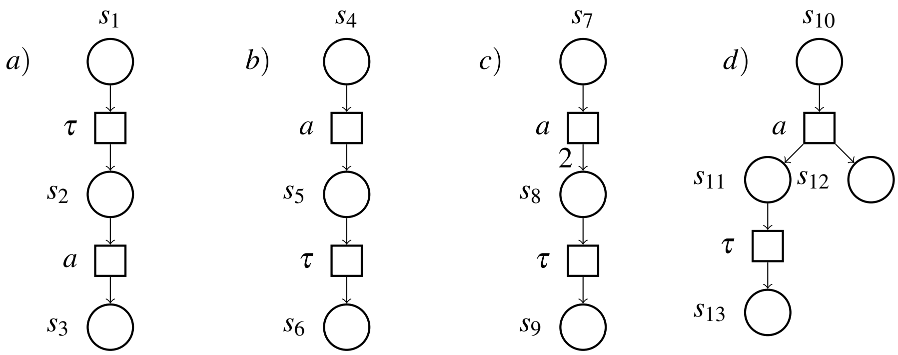


Figure 9. Two pairs of weakly team bisimilar BPP nets

Example 6.2. Consider Figure 9. $R_1 = \{(s_1, s_4), (s_2, s_4), (s_3, s_5), (s_3, s_6)\}$ is a weak team bisimulation. Also $R_2 = \{(s_7, s_{10}), (s_8, s_{11}), (s_8, s_{12}), (s_9, s_{12}), (s_9, s_{13})\}$ is a weak team bisimulation. □

Example 6.3. Consider Figure 3. Note that $s_1 \not\approx s_4$ because s_1 cannot match transition $s_4 \stackrel{\tau}{\longrightarrow} s_5 \oplus s_6$, as it cannot reach silently any marking of size 2. For the same reason, it is not difficult to realize that also $s_4 \not\approx s_8$: the apparently silent transition $s_4 \stackrel{\tau}{\longrightarrow} s_5 \oplus s_6$ cannot be matched silently by s_8, as the only silent transition it can perform is $s_8 \stackrel{\varepsilon}{\Longrightarrow} s_8$, but the size of the reached markings is different. Moreover, $s_8 \not\approx s_{11}$ because $s_9 \not\approx s_{13}$; in fact, s_9 cannot match transition $s_{13} \stackrel{\tau}{\longrightarrow} \theta$, because it cannot reach silently the empty marking. These examples show that τ-labeled transitions changing the number of currently available tokens, either by adding tokens (as in the first example) or by removing the token (as in the last example), are not really unobservable. In order to support our claim, note that it is not possible to find a τ-free net weak team bisimilar to s_4, or to s_{13}, while this is possible w.r.t. weak interleaving bisimilarity. As discussed in Examples 3.4 and 4.20, s_1, s_4, s_8 and s_{11} are all pairwise interleaving bisimilar and also weak fully-concurrent bisimilar. However, we think that such silent transitions cannot be considered as unobservable, as they do change the structure of the system. □

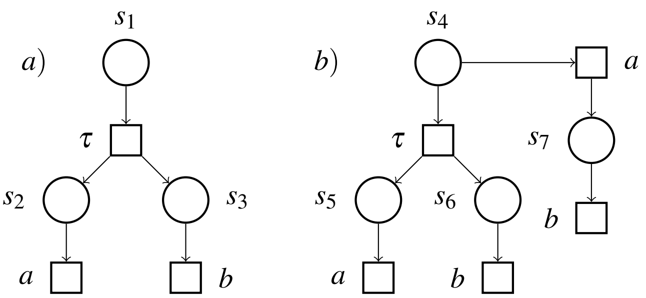


Figure 10. Two weakly team bisimilar BPP nets with different causal behavior

Example 6.4. Consider Figure 10. $R = \{(s_1,s_4),(s_2,s_5),(s_3,s_6),\ (s_3,s_7)\}$ is a weak team bisimulation; in particular, if $s_4 \xrightarrow{a} s_7$, then $s_1 \stackrel{a}{\Longrightarrow} s_3$ and $(s_3,s_7) \in R^{\oplus}$. However, s_1 and s_4 do not offer the same causal behavior, as s_4 may perform b caused by a, while for s_1 actions a and b are always causally independent. So, weak team bisimulation may not respect causality if the BPP net is not τ-sequential (see Definition 2.9). However, as argued also in the previous example, we claim that a silent transition that is not preserving the number of tokens is not really unobservable and so the step $s_1 \stackrel{a}{\Longrightarrow} s_3$, which passes through a marking of different size (i.e., $s_2 \oplus s_3$) is not really acceptable. Indeed, this example is one of the main motivation for introducing branching team bisimilarity in Section 7. □

We now list some useful properties of weak team bisimulation relations, one of which is based on the following lemma.

Lemma 6.5. Let $N = (S,A,T)$ be a BPP net with silent moves and let R be a weak team bisimulation such that $(s,s') \in R$. Then, the following hold:

(i) For all m such that $s \stackrel{\varepsilon}{\Longrightarrow} m$, there exists m' such that $s' \stackrel{\varepsilon}{\Longrightarrow} m'$ and $(m,m') \in R^{\oplus}$;

(ii) for all m, for all $\ell \in A_\tau$ such that $s \stackrel{\ell}{\Longrightarrow} m$, there exists m' such that $s' \stackrel{\ell}{\Longrightarrow} m'$ and $(m,m') \in R^{\oplus}$;

and symmetrically, if s' moves first.

Proof:
The proof is by induction on the length of the computation. (i) The base case is $s \stackrel{\varepsilon}{\Longrightarrow} s$; in such a case, $s' \stackrel{\varepsilon}{\Longrightarrow} s'$ and $(s,s') \in R^{\oplus}$, as required, because by assumption $(s,s') \in R$. If the sequence is of length one, then we can assume $s \xrightarrow{\tau} m$; since $(s,s') \in R$, we have that $s' \stackrel{\varepsilon}{\Longrightarrow} m'$ such that $(m,m') \in R^{\oplus}$, as required.

In general, we can assume that $s \xrightarrow{\tau} m_1 \stackrel{\varepsilon}{\Longrightarrow} m$; since $(s,s') \in R$, we have that $s' \stackrel{\varepsilon}{\Longrightarrow} m_2$ with $(m_1,m_2) \in R^{\oplus}$. The move $m_1 \stackrel{\varepsilon}{\Longrightarrow} m$ is derivable if there exists a transition sequence σ such that $m_1[\sigma\rangle m$ and $o(\sigma) = \varepsilon$. By Proposition 2.6, if $m_1 = s_1 \oplus \ldots \oplus s_k$, then there exist silent sequential transition sequences σ_i and markings m_i^1 such that $s_i[\sigma_i\rangle m_i^1$, for $i = 1,\ldots,k$, and $m = m_1^1 \oplus \ldots \oplus m_k^1$. Therefore, $s_i \stackrel{\varepsilon}{\Longrightarrow} m_i^1$ for $i = 1,\ldots,k$, and these sequences are all shorter, so that induction can be applied. In fact, as $(m_1,m_2) \in R^{\oplus}$, marking m_2 can be represented as $s_1' \oplus \ldots \oplus s_k'$ such that $(s_i,s_i') \in R$, for $i = 1,\ldots,k$. Hence, by induction, as $(s_i,s_i') \in R$ and $s_i \stackrel{\varepsilon}{\Longrightarrow} m_i^1$, we can conclude that $s_i' \stackrel{\varepsilon}{\Longrightarrow} m_i^2$, with $(m_i^1,m_i^2) \in R^{\oplus}$ for $i = 1,\ldots,k$. By Proposition 2.7, $m_2 \stackrel{\varepsilon}{\Longrightarrow} m' = m_1^2 \oplus \ldots \oplus m_k^2$ and $(m,m') \in R^{\oplus}$ by Proposition 5.3, because $(m_i^1,m_i^2) \in R^{\oplus}$ for $i = 1,\ldots,k$.

(ii) The base case is $s \xrightarrow{\ell} m$. In such a case, since $(s,s') \in R$, we have $s' \stackrel{\ell}{\Longrightarrow} m'$, with $(m,m') \in R^{\oplus}$, as required. If the sequence is longer, then we can distinguish two subcases: either $s \xrightarrow{\ell} m_1 \stackrel{\varepsilon}{\Longrightarrow} m$ or $s \xrightarrow{\tau} m_1 \stackrel{\ell}{\Longrightarrow} m$. In the former subcase, as $(s,s') \in R$, we have that $s' \stackrel{\ell}{\Longrightarrow} m_2$ with $(m_1,m_2) \in R^{\oplus}$. The move $m_1 \stackrel{\varepsilon}{\Longrightarrow} m$ is derivable if there exists a transition sequence σ such that $m_1[\sigma\rangle m$ and $o(\sigma) = \varepsilon$. By Proposition 2.6, if $m_1 = s_1 \oplus \ldots \oplus s_k$, then there exist sequential transition sequences σ_i and markings m_i^1 such that $s_i[\sigma_i\rangle m_i^1$, for $i = 1,\ldots,k$, and $m = m_1^1 \oplus \ldots \oplus m_k^1$. Therefore, $s_i \stackrel{\varepsilon}{\Longrightarrow} m_i^1$ for $i = 1,\ldots,k$. As $(m_1,m_2) \in R^{\oplus}$, marking m_2 can be represented as $s_1' \oplus \ldots \oplus s_k'$ such that $(s_i,s_i') \in R$, for $i = 1,\ldots,k$. Hence, by item (i), as $(s_i,s_i') \in R$ and $s_i \stackrel{\varepsilon}{\Longrightarrow} m_i^1$, we can conclude that $s_i' \stackrel{\varepsilon}{\Longrightarrow} m_i^2$, with $(m_i^1,m_i^2) \in R^{\oplus}$ for $i = 1,\ldots,k$. By Proposition 2.7, $m_2 \stackrel{\varepsilon}{\Longrightarrow} m' = m_1^2 \oplus \ldots \oplus m_k^2$ and $(m,m') \in R^{\oplus}$ by Proposition 5.3, because $(m_i^1,m_i^2) \in R^{\oplus}$ for $i = 1,\ldots,k$.

In the latter case, since $(s,s') \in R$, we have that $s' \stackrel{\varepsilon}{\Longrightarrow} m_2$ with $(m_1,m_2) \in R^{\oplus}$. The move $m_1 \stackrel{\ell}{\Longrightarrow} m$ is derivable if there exists a transition sequence σ such that $m_1[\sigma\rangle m$ and $o(\sigma) = \ell$. By Proposition 2.6, if $m_1 = s_1 \oplus \ldots \oplus s_k$, then there exist sequential transition sequences σ_i and markings m_i^1 such that $s_i[\sigma_i\rangle m_i^1$, for $i = 1,\ldots,k$, and $m = m_1^1 \oplus \ldots \oplus m_k^1$. W.l.o.g., we can assume that $o(\sigma_1) = \ell$, while $o(\sigma_i) = \varepsilon$ for $i = 2,\ldots,k$. Therefore, $s_1 \stackrel{\ell}{\Longrightarrow} m_1^1$ and $s_i \stackrel{\varepsilon}{\Longrightarrow} m_i^1$ for $i = 2,\ldots,k$. As $(m_1,m_2) \in R^{\oplus}$, marking m_2 can be represented as $s_1' \oplus \ldots \oplus s_k'$ such that $(s_i,s_i') \in R$, for $i = 1,\ldots,k$. Therefore, as $(s_1,s_1') \in R$ and $s_1 \stackrel{\ell}{\Longrightarrow} m_1^1$, by induction (as the computation is shorter), we have $s_1' \stackrel{\ell}{\Longrightarrow} m_1^2$ and $(m_1^1,m_1^2) \in R^{\oplus}$. Moreover, by item (i), as $(s_i,s_i') \in R$ and $s_i \stackrel{\varepsilon}{\Longrightarrow} m_i^1$, we can conclude that $s_i' \stackrel{\varepsilon}{\Longrightarrow} m_i^2$, with $(m_i^1,m_i^2) \in R^{\oplus}$ for $i = 2,\ldots,k$. By Proposition 2.7, $m_2 \stackrel{\ell}{\Longrightarrow} m' = m_1^2 \oplus \ldots \oplus m_k^2$ and $(m,m') \in R^{\oplus}$ by Proposition 5.3, because $(m_i^1,m_i^2) \in R^{\oplus}$ for $i = 1,\ldots,k$. □

Proposition 6.6. For each BPP net $N = (S,A,T)$ with silent moves, the following hold:

1. the identity relation $\mathscr{I} = \{(s,s) \mid s \in S\}$ is a weak team bisimulation;
2. the inverse relation $R^{-1} = \{(s',s) \mid (s,s') \in R\}$ of a weak team bisimulation R is a weak team bisimulation;
3. the relational composition $R_1 \circ R_2 = \{(s,s'') \mid \exists s'.(s,s') \in R_1 \land (s',s'') \in R_2\}$ of two weak team bisimulations R_1 and R_2 is a weak team bisimulation;
4. the union $\bigcup_{i \in I} R_i$ of weak team bisimulations R_i is a weak team bisimulation.

Proof:
The proof of (1) is immediate: $(s,s) \in \mathscr{I}$ is a weak bisimulation pair as whatever transition s performs (say, $s \xrightarrow{\ell} m$), the other s in the pair does exactly the same transition $s \xrightarrow{\ell} m$ and $(m,m) \in \mathscr{I}^{\oplus}$.

The proof of (2) is also immediate: if $(s_2,s_1) \in R^{-1}$, then $(s_1,s_2) \in R$; since R is a weak team bisimulation, the third condition ensures that $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$, $\exists m_1$ such that $s_1 \stackrel{\ell}{\Longrightarrow} m_1$ and $(m_1,m_2) \in R^{\oplus}$, i.e., $(m_2,m_1) \in (R^{\oplus})^{-1} = (R^{-1})^{\oplus}$ by Proposition 5.4(3); similarly, the fourth condition ensures that $\forall m_2$ such that $s_2 \xrightarrow{\tau} m_2$, $\exists m_1$ such that $s_1 \stackrel{\varepsilon}{\Longrightarrow} m_1$ and $(m_1,m_2) \in R^{\oplus}$, i.e., $(m_2,m_1) \in (R^{\oplus})^{-1} = (R^{-1})^{\oplus}$ by Proposition 5.4(3); symmetrically, if s_1 moves first. Hence, R^{-1} is a weak bisimulation, too.

The proof of (3) is also easy, thanks to Lemma 6.5: given a pair $(s,s'') \in R_1 \circ R_2$, there exists a place s' such that $(s,s') \in R_1$ and $(s',s'') \in R_2$; as $(s,s') \in R_1$, if $s \xrightarrow{\ell} m_1$ (or $s \xrightarrow{\tau} m_1$), there exists m_2 such that $s' \stackrel{\ell}{\Longrightarrow} m_2$ (or $s' \stackrel{\varepsilon}{\Longrightarrow} m_2$) with $(m_1,m_2) \in R_1^{\oplus}$. Since $(s',s'') \in R_2$, we have also that there exists m_3 such that $s'' \stackrel{\ell}{\Longrightarrow} m_3$ (or $s'' \stackrel{\varepsilon}{\Longrightarrow} m_3$) with $(m_2,m_3) \in R_2^{\oplus}$ (by Lemma 6.5). Summing up, for any pair $(s,s'') \in R_1 \circ R_2$, if $s \xrightarrow{\ell} m_1$ (or $s \xrightarrow{\tau} m_1$), then there exists a marking m_3 such that $s'' \stackrel{\ell}{\Longrightarrow} m_3$ (or $s'' \stackrel{\varepsilon}{\Longrightarrow} m_3$) with $(m_1,m_3) \in R_1^{\oplus} \circ R_2^{\oplus} = (R_1 \circ R_2)^{\oplus}$ by Proposition 5.4(4), as required.

The proof of (4) is trivial, too: assume $(s,s') \in \bigcup_{i \in I} R_i$; then, there exists $j \in I$ such that (s,s') belongs to R_j. If $s \xrightarrow{\ell} m_1$ (or $s \xrightarrow{\tau} m_1$), then there must exist m_2 such that $s' \stackrel{\ell}{\Longrightarrow} m_2$ (or $s' \stackrel{\varepsilon}{\Longrightarrow} m_2$) with $(m_1,m_2) \in R_j^{\oplus}$. By Proposition 5.4(5), $R_j^{\oplus} \subseteq (\bigcup_{i \in I} R_i)^{\oplus}$ and so $(m_1,m_2) \in (\bigcup_{i \in I} R_i)^{\oplus}$, as required. So $\bigcup_{i \in I} R_i$ is a weak team bisimulation, too. $\square$

Remember that $s \approx s'$ if there exists a weak team bisimulation containing the pair (s,s'). This means that $\approx$ is the union of all weak team bisimulations, i.e.,

$$\approx = \bigcup \{R \subseteq S \times S \;\big|\; R \text{ is a weak team bisimulation}\}.$$

By Proposition 6.6(4), $\approx$ is also a weak team bisimulation, hence the largest such relation.

Proposition 6.7. For each BPP net $N = (S,A,T)$, relation $\approx \subseteq S \times S$ is the largest weak team bisimulation relation. $\square$

Observe that a weak team bisimulation relation need not be reflexive, symmetric, or transitive. Nonetheless, the largest weak team bisimulation relation $\approx$ is an equivalence relation. As a matter of fact, as the identity relation $\mathscr{I}$ is a weak team bisimulation by Proposition 6.6(1), we have that $\mathscr{I} \subseteq \approx$, and so $\approx$ is reflexive. Symmetry derives from the following argument. For any $(s,s') \in \approx$, there exists a weak team bisimulation R such that $(s,s') \in R$; by Proposition 6.6(2), relation R^{-1} is a weak team bisimulation containing the pair (s',s); hence, $(s',s) \in \approx$ because $R^{-1} \subseteq \approx$. Transitivity also holds for $\approx$. Assume $(s,s') \in \approx$ and $(s',s'') \in \approx$; hence, there exist two weak team bisimulations R_1 and R_2 such that $(s,s') \in R_1$ and $(s',s'') \in R_2$; by Proposition 6.6(3), relation $R_1 \circ R_2$ is a weak team bisimulation containing the pair (s,s''); hence, $(s,s'') \in \approx$, because $R_1 \circ R_2 \subseteq \approx$. Summing up, we have the following.

Proposition 6.8. For each BPP net $N = (S,A,T)$, relation $\approx \subseteq S \times S$ is an equivalence relation. $\square$

6.2. Saturated net

Now we give a characterization of weak team bisimulation on a BPP net N as a (strong) team bisimulation over a suitably enriched variant of N, called its saturated net. The saturated net of N may be not a BPP net, because it may be infinite (more precisely, with infinitely many transitions); however, if the net is τ-simple (see Definition 2.9), then the saturated net of a BPP net N is still a BPP net. A τ-simple BPP net is such that it is not possible to have a silent cycle that produces new tokens; so if a silent cycle is present, it may only be of the form $s \stackrel{\varepsilon}{\Longrightarrow} s$, for some $s \in S$.

Definition 6.9. (Saturated net) Let $N = (S, A, T)$ be a BPP net, with $A_\tau = A \setminus \{\tau\}$. Its *saturated* net is the net $N' = (S, A_\tau \cup \{\varepsilon\}, T')$, where $T' = \{(s, \delta, m) \mid \delta \in A_\tau \cup \{\varepsilon\} \text{ and } s \stackrel{\delta}{\Longrightarrow} m\}$. □

This saturated net has the same set of places as N, but the transitions are computed by means of a generalization of the (partial) reflexive/transitive closure $\stackrel{\delta}{\Longrightarrow}$ of the transition relation $\longrightarrow$. Note that the set T' can be countably infinite if the net is not τ-simple. For instance, if N contains a transition $t = (s_1, \tau, s_1 \oplus s_2)$, then T' contains the countably infinite set of transitions $\{(s_1, \tau, s_1 \oplus k \cdot s_2) \mid k \geq 1\}$. In general, if $s \stackrel{\varepsilon}{\Longrightarrow} s \oplus m$ is a cycle in N, then also $(s, \varepsilon, s \oplus k \cdot m)$ is a transition in T', for each $k \geq 1$.

Proposition 6.10. Let $N = (S, A, T)$ be a BPP net, with $A_\tau = A \setminus \{\tau\}$. Its saturated net $N' = (S, A_\tau \cup \{\varepsilon\}, T')$ is a BPP net if and only if N is τ-simple.

Proof:
The transitions in T' are all with a singleton pre-set by construction. So, N' is a BPP net iff T' is finite, so that the thesis is actually: T' is finite iff N is τ-simple. As observed above, if N is not τ-simple, then T' is infinite. The reverse implication (if N is τ-simple, then T' is finite) holds by construction. In fact, since the net is τ-simple, the transitions of the form $s \stackrel{\varepsilon}{\Longrightarrow} \overline{m}$ are finitely many, as no cycle producing new tokens is present and the net N is finite.

Similarly, if $m = s_1 \oplus \ldots \oplus s_k$, one can conclude that the number of transitions of the form $m \stackrel{\varepsilon}{\Longrightarrow} \overline{m}$ is finite, because, by Propositions 2.6 and 2.7, these are generated by transitions of the form $s_i \stackrel{\varepsilon}{\Longrightarrow} m_i$, for $i = 1, \ldots, k$, such that $\overline{m} = m_1 \oplus \ldots \oplus m_k$, and these latter transitions are finitely many by the argument above.

Finally, if $s \stackrel{\ell}{\Longrightarrow} m$, then there exist m_1 and m_2 such that $s \stackrel{\varepsilon}{\Longrightarrow} m_1 \stackrel{\ell}{\longrightarrow} m_2 \stackrel{\varepsilon}{\Longrightarrow} m$; the thesis follows also in this case by observing that (i) the number of possible m_1's is finite, (ii) the number of ℓ-labeled transitions is finite, (iii) the number of possible m's is finite as well, where (i) and (iii) hold by the arguments above. □

It is possible to offer an alternative, yet equivalent, definition of weak team bisimulation on places over the net N as a *strong* team bisimulation on places over its saturated net N'.

Proposition 6.11. Let $N = (S, A, T)$ be a BPP net, with $A_\tau = A \setminus \{\tau\}$ and let $R \subseteq S \times S$ be a weak team bisimulation. If $(s_1, s_2) \in R$ then for all $\delta \in A_\tau \cup \{\varepsilon\}$

- $\forall m_1$ such that $s_1 \stackrel{\delta}{\Longrightarrow} m_1$, there exists m_2 such that $s_2 \stackrel{\delta}{\Longrightarrow} m_2$ and $(m_1, m_2) \in R^\oplus$,

- $\forall m_2$ such that $s_2 \stackrel{\delta}{\Longrightarrow} m_2$, there exists m_1 such that $s_1 \stackrel{\delta}{\Longrightarrow} m_1$ and $(m_1, m_2) \in R^{\oplus}$.

Proof:
Since R is a weak team bisimulation, if $(s_1, s_2) \in R$ and $s_1 \stackrel{\delta}{\Longrightarrow} m_1$, then, by Lemma 6.5, $s_2 \stackrel{\delta}{\Longrightarrow} m_2$ and $(m_1, m_2) \in R^{\oplus}$; symmetrically, if s_2 moves first. □

A consequence of this alternative characterization is that it is possible to define $\approx$ as the greatest fixed point of a suitable functional over binary place relations, as done for (strong) team bisimulation over BPP nets in [17].

Remark 6.12. (Complexity of $\approx$) Another consequence of Proposition 6.11 is that it is also possible to check whether two places are weak team bisimilar on a τ-simple BPP net N by checking whether they are strong team bisimilar on its saturated net N'. Hence, the two-step algorithm is as follows:

1. First compute the saturated net N'. We conjecture that T' can be computed in polynomial time; our conjecture is based on the fact that the simplified problem of computing the (partial) reflexive/transitive closure over automata can be solved in $O(n^3)$ time, where n is the number of states of the automaton, if the classic Floyd-Warshall algorithm [8] is exploited.
2. Then check whether two places are strong team bisimilar on N' in time complexity $O(m' \cdot n^2)$, where m' is the size of T'.

Hence, we claim that $\approx$ can be computed in polynomial time. □

6.3. Lifting weak team bisimilarity to markings

Starting from weak team bisimilarity $\approx$, which has been computed *over the places of an unmarked* BPP net N, we can lift it *over the markings* of N in a structural, distributed way: m_1 is weak team bisimulation equivalent to m_2 if these two markings are related by the additive closure of $\approx$, i.e., if $(m_1, m_2) \in \approx^{\oplus}$, usually denoted by $m_1 \approx^{\oplus} m_2$. Hence, weak team bisimulation equivalent markings have the same size.

Proposition 6.13. For each BPP net $N = (S, A, T)$, if $m_1 \approx^{\oplus} m_2$, then $|m_1| = |m_2|$. □

Proposition 6.14. For each BPP net $N = (S, A, T)$, $\approx^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.

Proof:
Since $\approx$ is an equivalence relation, by Proposition 5.3, $\approx^{\oplus}$ is an equivalence relation, too. □

Note that, once $\approx$ has been computed in polynomial time (cf. Remark 6.12), checking whether two markings are weak team bisimulation equivalent takes only $O(n)$ time with Algorithm 1.

The following theorem provides a characterization of weak team bisimulation equivalence as a suitable bisimulation-like relation over markings, i.e., over a global model of the overall behavior. Indeed, this result gives evidence of the fact that weak team bisimulation equivalence does respect the global behavior of the net. It is interesting to observe that this characterization gives a dynamic interpretation of weak team bisimulation equivalence, while the definition of weak team bisimulation equivalence as the additive closure of $\approx$ gives a structural definition.

Theorem 6.15. Let $N = (S, A, T)$ be a BPP net. Two markings m_1 and m_2 are weak team bisimulation equivalent, $m_1 \approx^{\oplus} m_2$, if and only if $|m_1| = |m_2|$ and

1. $\forall t_1$ such that $l(t_1) \neq \tau$ and $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that σ_2 is sequential, ${}^\bullet t_1 \approx {}^\bullet \sigma_2$, $l(t_1) = o(\sigma_2)$, $t_1^\bullet \approx^{\oplus} \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^{\oplus} m_2'$,

2. $\forall t_1$ such that $l(t_1) = \tau$ and $m_1[t_1\rangle m_1'$, *either* $\exists \sigma_2$ such that σ_2 is nonempty and sequential, ${}^\bullet t_1 \approx {}^\bullet \sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx^{\oplus} \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^{\oplus} m_2'$, *or* $\exists s_2 \in m_2$ such that ${}^\bullet t_1 \approx s_2$, $t_1^\bullet \approx s_2$ and $m_1' \approx^{\oplus} m_2$,

3. $\forall t_2$ such that $l(t_2) \neq \tau$ and $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ such that σ_1 is sequential, ${}^\bullet \sigma_1 \approx {}^\bullet t_2$, $o(\sigma_1) = l(t_2)$, $\sigma_1^\bullet \approx^{\oplus} t_2^\bullet$, $m_1[\sigma_1\rangle m_1'$ and $m_1' \approx^{\oplus} m_2'$,

4. $\forall t_2$ such that $l(t_2) = \tau$ and $m_2[t_2\rangle m_2'$, *either* $\exists \sigma_1$ such that σ_1 is nonempty and sequential, ${}^\bullet \sigma_1 \approx {}^\bullet t_2$, $o(\sigma_1) = \varepsilon$, $\sigma_1^\bullet \approx^{\oplus} t_2^\bullet$, $m_1[\sigma_1\rangle m_1'$ and $m_1' \approx^{\oplus} m_2'$, *or* $\exists s_1 \in m_1$ such that $s_1 \approx {}^\bullet t_2$, $s_1 \approx t_2^\bullet$ and $m_1 \approx^{\oplus} m_2'$.

Proof:
($\Rightarrow$) If $m_1 \approx^{\oplus} m_2$, then $|m_1| = |m_2|$ by Proposition 6.13. Moreover, for each t_1 such that $m_1[t_1\rangle m_1'$, we have that $m_1 = s_1 \oplus \overline{m}_1$, where $s_1 = {}^\bullet t_1$. As $m_1 \approx^{\oplus} m_2$, by Definition 5.1, it follows that there exist s_2 and $\overline{m}_2$ such that $m_2 = s_2 \oplus \overline{m}_2$, $s_1 \approx s_2$ and $\overline{m}_1 \approx^{\oplus} \overline{m}_2$. Since $s_1 \approx s_2$, by Definition 6.1, we have to consider two cases for the shape of t_1:

(*i*) if $t_1 = s_1 \xrightarrow{\ell} \underline{m}_1$, then there exists $\underline{m}_2$ such that $s_2 \stackrel{\ell}{\Longrightarrow} \underline{m}_2$ and $\underline{m}_1 \approx^{\oplus} \underline{m}_2$. This means that for transition t_1, there exists a sequential transition sequence σ_2 such that $o(\sigma_2) = \ell = l(t_1)$, ${}^\bullet \sigma_2 = s_2$, $\sigma_2^\bullet = \underline{m}_2$, hence with ${}^\bullet t_1 \approx {}^\bullet \sigma_2$ and $t_1^\bullet \approx^{\oplus} \sigma_2^\bullet$. Hence, $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx^{\oplus} m_2'$ by Proposition 5.3. Hence, this corresponds to item 1 of the bisimulation conditions.

(*ii*) if $t_1 = s_1 \xrightarrow{\tau} \underline{m}_1$, then there exists $\underline{m}_2$ such that $s_2 \stackrel{\varepsilon}{\Longrightarrow} \underline{m}_2$ and $\underline{m}_1 \approx^{\oplus} \underline{m}_2$. This means that for transition t_1, *either* there exists a nonempty sequential transition sequence σ_2 such that $o(\sigma_2) = \varepsilon$, ${}^\bullet \sigma_2 = s_2$ and $\sigma_2^\bullet = \underline{m}_2$, hence with ${}^\bullet t_1 \approx {}^\bullet \sigma_2$ and $t_1^\bullet \approx^{\oplus} \sigma_2^\bullet$; *or* s_2 responds by idling, i.e., ${}^\bullet t_1 \approx s_2$ and $t_1^\bullet \approx s_2$. The *either* case is analogous to the previous one, and so omitted; this ensures the first part of item 2 of the bisimulation conditions. The *or* case, instead, accounts for the second part of item 2: since $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2 = s_2 \oplus \overline{m}_2$, $\overline{m}_1 \approx^{\oplus} \overline{m}_2$ and $t_1^\bullet \approx s_2$, it follows that $m_1' \approx^{\oplus} m_2$.

The case when m_2 moves first is symmetric, hence omitted. These cases accounts for items 3 and 4 of the bisimulation conditions.

($\Leftarrow$) Let us assume that $|m_1| = |m_2|$ and that the four bisimulation-like conditions hold; then, we prove that $m_1 \approx^{\oplus} m_2$. First of all, assume that no transition t_1 is enabled at m_1; in such a case, no observable transition is enabled at m_2; in fact, if $m_2[t_2\rangle m_2'$ with $l(t_2) \neq \tau$, then, by the third condition, a nonempty transition sequence σ_1 must be executable at m_1, contradicting the assumption that no transition is enabled at m_1. However, m_2 may enable silent τ-sequential transitions: by the fourth condition, m_1 can reply by idling. This means that each place in m_1 is a deadlock, and similarly each place in m_2 is weakly bisimilar to a deadlock; therefore, all the places in m_1 and m_2 are pairwise weakly bisimilar; hence, the condition $|m_1| = |m_2|$ is enough to ensure that $m_1 \approx^{\oplus} m_2$.

Now, assume that $m_1[t_1\rangle m_1'$ for some t_1. If $l(t_1) \neq \tau$, then the first condition ensures that there exists a sequential transition sequence σ_2 such that $^\bullet t_1 \approx {}^\bullet\sigma_2$, $l(t_1) = o(\sigma_2)$, $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^\oplus m_2'$. We have that $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, $m_1 = {}^\bullet t_1 \oplus \overline{m}_1$, $m_2 = {}^\bullet\sigma_2 \oplus \overline{m}_2$. Since $m_1' \approx^\oplus m_2'$ and $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, by Proposition 5.3, it follows that $\overline{m}_1 \approx^\oplus \overline{m}_2$, and so $m_1 \approx^\oplus m_2$, because $^\bullet t_1 \approx {}^\bullet\sigma_2$. The second condition, accounting for the case when $l(t_1) = \tau$, is analogous, and so omitted.

Symmetrically, if we start from a transition t_2 enabled at m_2. □

By the theorem above, it is clear that $\approx^\oplus$ is a weak interleaving bisimulation; hence, the following corollary follows trivially.

Corollary 6.16. (Weak team bisimilarity is finer than weak interleaving bisimilarity) Let $N = (S, A, T)$ be a BPP net with silent moves. If $m_1 \approx^\oplus m_2$, then $m_1 \approx_{int} m_2$. □

The above implication is strict. In fact, considering Figure 3, we have that $s_1 \approx_{int} s_4$, but $s_1 \not\approx s_4$, as discussed in Example 6.3.

Now we want to argue that weak team bisimilarity $\approx^\oplus$ implies state-sensitive weak fully-concurrent bisimilarity $\approx_{sfc}$ if the BPP net is τ-sequential. To this aim, we first propose a lemma stating that if the BPP net is τ-sequential, then –(i) and (ii)– the execution of silent moves does not activate any parallel activity and, by starting from a singleton marking, –(iii) and (iv)– in the execution of a visible, one-action step, the only visible transition causes all the future visible actions executable from the reached marking.

Lemma 6.17. Let $N = (S, A, T)$ be a τ-sequential BPP net. The following hold:

(i) If $s[\sigma\rangle m$ and $o(\sigma) = \varepsilon$, then $|m| = 1$.

(ii) If $m[\sigma\rangle m'$ and $o(\sigma) = \varepsilon$, then $|m| = |m'|$.

(iii) If $s[\sigma\rangle m$ and $o(\sigma) = \ell$, then there exist $\sigma_1, t, \sigma_2, s', m'$ such that $\sigma = \sigma_1 t \sigma_2$, $l(t) = \ell$, $o(\sigma_1) = o(\sigma_2) = \varepsilon$ and $s[\sigma_1\rangle s'[t\rangle m'[\sigma_2\rangle m$, where $|m'| = |m|$.

(iv) If $s[\sigma_1\rangle m[\sigma_2\rangle m'$, $o(\sigma_1) = \ell_1$ and $o(\sigma_2) = \ell_2$, then $t_1 \preceq t_2$, where, for $i = 1, 2$, $t_i \in \sigma_i$ and $l(t_i) = \ell_i$.

Proof:
(i) By induction on the length of σ. The base case is $s[\varepsilon\rangle s$ and the thesis follows trivially. For the inductive case, $\sigma = t\sigma'$; hence, $s[t\rangle m'[\sigma'\rangle m$. Since the net is τ-sequential, $|m'| = 1$, i.e., $m' = s'$ for some $s' \in S$. By induction, as the computation $s'[\sigma'\rangle m$ is shorter, we have that that $|m| = 1$, as required.

(ii) If $m[\sigma\rangle m'$ and $m = s_1 \oplus \ldots \oplus s_k$, by Proposition 2.6, there exist $\sigma_1 \ldots \sigma_k$ such that $s_i[\sigma_i\rangle m_i$ for $i = 1, \ldots, k$, where σ is a permutation of $\sigma_1 \ldots \sigma_k$ and $m' = m_1 \oplus \ldots \oplus m_k$. By item (i) above, each m_i is such that $|m_i| = 1$, so that $|m| = |m'|$.

(iii) By induction on the length of σ. The base case is $s[t\rangle m$, with $o(t) = \ell$, and the thesis follows trivially by choosing $\sigma_1 = \sigma_2 = \varepsilon$. For the inductive case, $\sigma = t\sigma'$ and $s[t\rangle m_1[\sigma'\rangle m$. We have two subcases: either $l(t) = \tau$ and $o(\sigma') = \ell$, or $l(t) = \ell$ and $o(\sigma') = \varepsilon$.

In the former subcase, since the net is τ-sequential, $m_1 = s_1$ for some $s_1 \in S$. By induction, as the computation $s_1[\sigma'\rangle m$ is shorter, we can conclude that there exist σ_1, σ_2, t' such that $\sigma' = \sigma_1 t' \sigma_2$, $l(t') = \ell$ and $s_1[\sigma_1\rangle s'[t'\rangle m'[\sigma_2\rangle m$, where $|m'| = |m|$; summing up, $s[t\sigma_1\rangle s'[t'\rangle m'[\sigma_2\rangle m$, where $|m'| = |m|$, as required. In the latter subcase, by item (ii), the step $m_1[\sigma'\rangle m$ is such that $|m_1| = |m|$; summing up, $s[\varepsilon\rangle s[t\rangle m_1[\sigma'\rangle m$, where $|m_1| = |m|$, as required.

(iv) If $s[\sigma_1\rangle m$, then by item (iii), there exist $\sigma_1^1, t_1, \sigma_2^1$ such that $\sigma_1 = \sigma_1^1 t_1 \sigma_2^1$, $l(t_1) = \ell_1$, $o(\sigma_1^1) = o(\sigma_2^1) = \varepsilon$ and $s[\sigma_1^1\rangle s'[t_1\rangle m'[\sigma_2^1\rangle m$, where $|m'| = |m|$. This means that the event t_1 is causing each transition that m can enable. Since $m[\sigma_2\rangle m'$, the observable event $t_2 \in \sigma_2$ is such that $t_1 \preceq t_2$. □

Remark 6.18. Note that tem (iv) above does not hold if the BPP net is not τ-sequential. Example 6.4 shows the computation $s_1[t_\tau t_a\rangle s_3[t_b\rangle \theta$, but the transitions t_a and t_b are causally independent. □

Theorem 6.19. (Weak team bisimilarity implies state-sensitive weak fc-bisimilarity) Let $N = (S, A, T)$ be a τ-sequential BPP. If $m_1 \approx^{\oplus} m_2$, then $m_1 \approx_{sfc} m_2$.

Proof:
Let $R = \{((C_1, \rho_1), g, (C_2, \rho_2)) \mid (C_1, \rho_1)$ is a process of $N(m_1)$, (C_2, ρ_2) is a process of $N(m_2)$ and g is an abstract event isomorphism between C_1 and C_2, such that $\rho_1(Max(C_1)) \approx^{\oplus} \rho_2(Max(C_2))\}$. We want to prove that R is a state-sensitive weak fc-bisimulation.

First, observe that the triple $((C_1^0, \rho_1^0), g_0, (C_2^0, \rho_2^0))$ where g_0 is empty and, for $i = 1, 2$, C_i^0 is a BPP causal net which contains no transitions and $\rho_i^0(Max(C_i^0)) = m_i$, belongs to relation R because $m_1 \approx^{\oplus} m_2$ by hypothesis. Note also that if the relation R is a state-sensitive weak fc-bisimulation, then this triple ensures that $m_1 \approx_{sfc} m_2$. It is enough to check that R is a weak fc-bisimulation, because, since for each triple $((C_1, \rho_1), g, (C_2, \rho_2)) \in R$ we have that $\rho_1(Max(C_1)) \approx^{\oplus} \rho_2(Max(C_2))$, Proposition 6.13 ensures that R is state-sensitive.

Now assume $(\pi_1, g, \pi_2) \in R$, where $\pi_i = (C_i, \rho_i)$ for $i = 1, 2$. In order to be a weak fully-concurrent bisimulation triple, it is necessary that

(i) $\forall t_1, \pi_1'$ such that $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$, $\exists \sigma_2', \pi_2', g'$ such that

1. $\pi_2 \stackrel{\sigma_2'}{\Longrightarrow} \pi_2'$,
2. if $l(e_1) = \tau$, then $o(\sigma_2') = \varepsilon$ and $g' = g$; otherwise, $l(e_1) = o(\sigma_2')$ and there is a transition e_2 in σ_2' such that $l(e_1) = l(e_2)$ and $g' = g \cup \{(e_1, e_2)\}$; and finally,
3. $(\pi_1', g', \pi_2') \in R$;

(ii) and symmetrically, if π_2 moves first.

Let us consider any transition t_1 such that $\rho_1(Max(C_1))[t_1\rangle m_1'$ and $l(t_1) \neq \tau$. Since $\rho_1(Max(C_1)) \approx^{\oplus} \rho_2(Max(C_2))$, by Theorem 6.15, there exists a sequential σ_2 such that ${}^\bullet t_1 \approx {}^\bullet \sigma_2$, $l(t_1) = o(\sigma_2)$, $t_1^\bullet \approx^{\oplus} \sigma_2^\bullet$, $\rho_2(Max(C_2))[\sigma_2\rangle m_2'$ and $m_1' \approx^{\oplus} m_2'$. Therefore, it is really possible to extend the causal net C_1 to the causal net C_1' through a suitable transition e_1, as well as to extend the causal net C_2 to the causal net C_2' through a suitable transition sequence σ_2', including the observable transition e_2, as required above: indeed, $g' = g \cup \{(e_1, e_2)\}$. Note that, by Lemma 6.17, since σ_2 is sequential, it can be represented as ${}^\bullet\sigma_2[\sigma'\rangle s[t_2\rangle\overline{m}[\sigma''\rangle\sigma_2^\bullet$, where $|\overline{m}| = |\sigma_2^\bullet|$; hence, all the transitions enabled by $\sigma_2^\bullet$ are caused by t_2; note also that $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, so that a future event, in the bisimulation game, which is caused by t_1 can only be matched by a future event caused by t_2. Summing up, for the move $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$, we add the triple $(\pi_1', g', \pi_2') \in R$, where $\pi_i' = (C_i', \rho_i')$ for $i = 1, 2$, so that $\rho_1'(Max(C_1')) = m_1' \approx^\oplus m_2' = \rho_2'(Max(C_2'))$ and $|m_1'| = |m_2'|$, as required.

A similar argument is necessary when $l(t_1) = \tau$. As $\rho_1(Max(C_1)) \approx^\oplus \rho_2(Max(C_2))$, by Theorem 6.15, *either* $\exists\sigma_2$ such that σ_2 is nonempty and sequential, ${}^\bullet t_1 \approx {}^\bullet\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^\oplus m_2'$, *or* $\exists s_2 \in m_2$ such that ${}^\bullet t_1 \approx s_2$, $t_1^\bullet \approx s_2$ and $m_1' \approx^\oplus m_2$. The *either* case is very similar to the above, and so omitted; we simply observe that $|t_1^\bullet| = 1 = |\sigma_2^\bullet|$, so that the causality relation is strictly respected. For the *or* case, it is really possible to extend the causal net C_1 to the causal net C_1' through a suitable transition e_1, while the causal net C_2 is not modified. Summing up, for the move $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$, we add the triple (π_1', g, π_2) to R, so that $\rho_1'(Max(C_1')) = m_1' \approx^\oplus m_2 = \rho_2(Max(C_2))$ and $|m_1'| = |m_2|$, as required.

Symmetrically, if $\rho_2(Max(C))$ moves first. □

Now we want to expose an informal argument in favor of the following conjecture: for each τ-sequential BPP net, if $m_1 \approx_{sfc} m_2$, then $m_1 \approx^\oplus m_2$.

Let R be a state-sensitive weak fc-bisimulation. We want to argue that for each triple $((C_1, \rho_1), g, (C_2, \rho_2))$ in R, we have that $\rho_1(Max(C_1)) \approx^\oplus \rho_2(Max(C_2))$. Let us assume, towards a contradiction, that a triple $((C_1, \rho_1), g, (C_2, \rho_2))$ in R is such that $\rho_1(Max(C_1)) \not\approx^\oplus \rho_2(Max(C_2))$. By Theorem 6.15, this implies that for some t_1 such that $\rho_1(Max(C))[t_1\rangle m_1'$,

1. either $l(t_1) \neq \tau$ and does not exist a sequential σ_2 such that ${}^\bullet t_1 \approx {}^\bullet\sigma_2$, $l(t_1) = o(\sigma_2)$, $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^\oplus m_2'$;
2. or $l(t_1) = \tau$ and does not exist σ_2 such that σ_2 is nonempty and sequential, ${}^\bullet t_1 \approx {}^\bullet\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^\oplus m_2'$, *nor* does exist $s_2 \in m_2$ such that ${}^\bullet t_1 \approx s_2$, $t_1^\bullet \approx s_2$ and $m_1' \approx^\oplus m_2$.

Let us consider item 1), i.e., the case $l(t_1) \neq \tau$. (The other case is similar, and so omitted.) Of course, if $\rho_1(Max(C_1))[t_1\rangle m_1'$, then for all π_1' such that $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$, by Definition 4.15, there exist $\overline{\sigma}_2, \pi_2', g'$ such that

1. $\pi_2 \stackrel{\overline{\sigma}_2}{\Longrightarrow} \pi_2'$, with $\pi_2' = (C_2', \rho_2')$;
2. if $l(e_1) = \tau$, then $o(\overline{\sigma}_2) = \varepsilon$ and $g' = g$; otherwise, $l(e_1) = o(\overline{\sigma}_2)$ and there is a transition e_2 in $\overline{\sigma}_2$ such that $l(e_1) = l(e_2)$ and $g' = g \cup \{(e_1, e_2)\}$;
3. and finally, $(\pi_1', g', \pi_2') \in R$;

However, among the possibly many $\overline{\sigma}_2$'s satisfying the condition above, since R is state sensitive and the net is τ-sequential, it seems necessary that one of them is a *sequential* transition sequence such that $|{}^\bullet t_1| = |{}^\bullet\overline{\sigma}_2|$ and $|t_1^\bullet| = |\overline{\sigma}_2^\bullet|$. Actually, it seems even necessary that such a sequential $\overline{\sigma}_2$ is such that ${}^\bullet t_1 \approx {}^\bullet\overline{\sigma}_2$, $l(t_1) = o(\overline{\sigma}_2)$, $t_1^\bullet \approx^\oplus \overline{\sigma}_2^\bullet$, $\rho_2(Max(C_2))[\overline{\sigma}_2\rangle m_2''$ and $m_1' \approx^\oplus m_2''$, thus contradicting the nonexistence of such a transition sequence postulated above. In fact, if this is not the case, it seems that otherwise $((C_1,\rho_1),g,(C_2,\rho_2))$ would not satisfy the state-sensitive weak fc-bisimulation requirements. Of course, this argument is not a formal proof as a characterization of all the possible transition sequences $\overline{\sigma}_2$ is lacking, in order to really prove that one of them has the required property.

If $m_1 \approx_{sfc} m_2$, then there exists a state-sensitive weak fully-concurrent bisimulation R containing a triple $((C_1^0,\rho_1^0),g_0,(C_2^0,\rho_2^0))$, where C_i^0 is a BPP causal net which has no transitions, g_0 is empty and $\rho_i^0(Max(C_i^0)) = m_i$ for $i = 1,2$. By the argument above, $m_1 \approx^\oplus m_2$. The conclusion of this informal argument is that state-sensitive weak fc-bisimilarity implies weak team bisimilarity, so that we conjecture that these two equivalences do coincide for τ-sequential BPP nets.

Corollary 6.20. Let $N = (S,A,T)$ be a τ-sequential net. If $m_1 \approx^\oplus m_2$, then $m_1 \approx_{fc} m_2$.

Proof:
By Theorem 6.19 and the fact that $\approx_{sfc} \subseteq \approx_{fc}$. □

The inclusion, proved in this corollary, is strict, i.e., the converse does not hold in general. E.g., the τ-free (hence also τ-sequential) BPP net in Figure 6(c) is such that $s_6 \approx_{fc} s_8$, but $s_6 \not\approx^\oplus s_8$.

If the BPP net is not τ-sequential, then weak team bisimilarity is incomparable to (state-sensitive) weak fully-concurrent bisimilarity. On the one hand, Example 6.4 discusses the two nets in Figure 10, such that $s_1 \approx s_4$ but $s_1 \not\approx_{fc} s_4$, because the partial order of observable events they generate are not isomorphic. On the other hand, consider the nets in Figure 3(c) and (d), which are not τ-sequential. Note that $s_8 \not\approx s_{11}$: transition $s_8 \xrightarrow{a} 2 \cdot s_9$ cannot be matched by s_{11}, as, by doing a weakly, it can reach only either s_{12}, or s_{14}, or $s_{12} \oplus s_{13}$, or $s_{14} \oplus s_{13}$, but none of these markings is weak team bisimilar to $2 \cdot s_9$, because either they have a different size or $s_9 \not\approx s_{13}$. Nonetheless, $s_8 \approx_{fc} s_{11}$, as the partial order of observable events they generate are isomorphic.

As discussed in Example 6.4, only τ-sequential nets are intuitively correct, as a silent transition whose execution does not preserve the number of tokens cannot be considered as unobservable. Therefore, for correct nets, weak team bisimilarity seems an alternative characterization of the resource-aware (as it is sensitive to the size of the involved markings) strengthening of weak fully-concurrent bisimilarity, which is more easily defined and more easily computable.

6.4. Minimizing nets w.r.t. $\approx$

In [17], we showed how to compute, for a given BPP net N, its reduced net $N_\sim$, i.e., the minimized net according to (strong) team bisimilarity $\sim$ on places (see Definition 5.5), where the places of the reduced net $N_\sim$ are equivalence classes of the places of N. We proved that this reduction is correct, i.e., it relates, via $\sim$, each place s in N to its corresponding place $[s]_\sim$ in $N_\sim$, and similarly for markings of the two nets, via team equivalence $\sim^\oplus$. Moreover, we argued that $N_\sim$ is really the net with the least number of places exhibiting the same behavior.

By Proposition 6.11 we noted that weak team bisimilarity on the places of a τ-simple net N can be equivalently characterized as strong team bisimulation on the places of the saturated net N'. Therefore, it is possible to minimize the net N w.r.t. the weak team bisimulation equivalence $\approx$ over places by minimizing the saturated net N' w.r.t. $\sim$. Since N and its saturated net N' have the same set of places, the equivalence classes computed over N' w.r.t. $\sim$ are the same equivalence classes over N w.r.t. $\approx$.

A direct construction of the reduced net w.r.t. $\approx$, which minimizes the number of places *and the number of transitions*, can be also defined as follows.

Definition 6.21. (Reduced net) Let $N = (S, A, T)$ be a BPP net and let $\approx$ be the weak team bisimulation equivalence relation over its places. The *reduced* net $N_{\approx} = (S_{\approx}, A, T_{\approx})$ is defined as follows:

- $S_{\approx} = \{[s] \mid s \in S\}$, where $[s] = \{s' \in S \mid s \approx s'\}$;
- $T_{\approx} = \{([s], \ell, [m]) \mid (s, \ell, m) \in T, \ell \neq \tau\} \cup \{([s], \tau, [m]) \mid (s, \tau, m) \in T, [s] \neq [m]\}$,

where $[m]$ is defined as follows: $[\theta] = \theta$ and $[m_1 \oplus m_2] = [m_1] \oplus [m_2]$. If the net N has initial marking $m_0 = k_1 \cdot s_1 \oplus \ldots \oplus k_n \cdot s_n$, then $N_{\approx}$ has initial marking $[m_0] = k_1 \cdot [s_1] \oplus \ldots \oplus k_n \cdot [s_n]$. □

Lemma 6.22. Let $N = (S, A, T)$ be a BPP net and let $N_{\approx} = (S_{\approx}, A, T_{\approx})$ be its reduced net w.r.t. $\approx$. Relation $R = \{(s, [s]) \mid s \in S\}$ is a weak team bisimulation.

Proof:
If $s \xrightarrow{\ell} m$ with $\ell \neq \tau$, then also $[s] \xrightarrow{\ell} [m]$ by definition of $T_{\approx}$ and, as required, $(m, [m]) \in R^{\oplus}$. If $s \xrightarrow{\tau} m$ and $[s] = [m]$, then $[s]$ replies by idling, and $(m, [s]) \in R^{\oplus}$, because $[s] = [m]$. Finally, if $s \xrightarrow{\tau} m$ and $[s] \neq [m]$, then $[s] \xrightarrow{\tau} [m]$ by definition of $T_{\approx}$ and $(m, [m]) \in R^{\oplus}$, as required.

The case when $[s]$ moves first is slightly more complex for the freedom in choosing the representative in an equivalence class. Transition $[s] \xrightarrow{\ell} [m]$ is possible, by Definition of $T_{\approx}$, if there exist $s' \in [s]$ and $m' \in [m]$ such that $s' \xrightarrow{\ell} m'$; as $s \approx s'$, there must exists a transition $s \xRightarrow{\ell} m''$ (in case $\ell \neq \tau$) or $s \xRightarrow{\varepsilon} m''$ (in case $\ell = \tau$) such that $m' \approx^{\oplus} m''$; summing up, if $[s] \xrightarrow{\ell} [m]$, then $s \xRightarrow{\ell} m''$ (or $s \xRightarrow{\varepsilon} m''$, in case $\ell = \tau$) with $(m'', [m]) \in R^{\oplus}$, as required, because $[m] = [m'] = [m'']$. □

Theorem 6.23. Let $N = (S, A, T)$ be a BPP net and let $N_{\approx} = (S_{\approx}, A, T_{\approx})$ be its reduced net w.r.t. $\approx$. For any $m \in \mathscr{M}(S)$, we have that $m \approx^{\oplus} [m]$.

Proof:
By induction on the size of m. If $m = \theta$, then $[m] = \theta$ and the thesis follows trivially. If $m = s \oplus m'$, then $[m] = [s] \oplus [m']$; by Lemma 6.22, $s \approx [s]$ and, by induction, $m' \approx^{\oplus} [m']$; therefore, by the rule in Definition 5.1, $m \approx^{\oplus} [m]$. □

As a consequence of this theorem, we would like to point out that the reduced net w.r.t. $\approx$ is indeed the *least* net offering the same weak team bisimilar behavior as the original net: no further fusion of places can be done, as there are not two places in the reduced net which are weak team bisimilar. Moreover, silent transitions relating weak team bisimilar places in the original net do not generate any silent transition in the reduced net, so that the number of transitions is minimized, too.

As an example, consider the net in Figure 11(a). The equivalence classes w.r.t. $\approx$ are $\{s_1, s_2\}$, $\{s_3\}$ and $\{s_4, s_5\}$. Hence, the reduced net has only three places and is actually isomorphic to the net

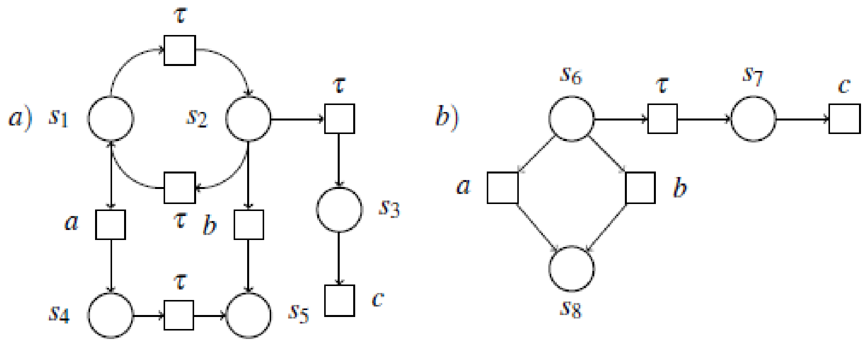


Figure 11. A BPP net in (a) and its reduced net w.r.t. $\approx$ in (b)

in (b). Note that the transitions $s_1 \xrightarrow{\tau} s_2$, $s_2 \xrightarrow{\tau} s_1$ and $s_4 \xrightarrow{\tau} s_5$, which connect weak team bisimilar places, do not originate any silent transition in the reduced net. The marking $s_1 \oplus s_2 \oplus s_3$ is weak team bisimilar to $2 \cdot s_6 \oplus s_7$.

6.5. Rooted weak team bisimilarity

Definition 6.24. (Rooted weak team bisimilarity on places) Let $N = (S, A, T)$ be a BPP net with silent moves. Two places s_1 and s_2 are *rooted weakly team bisimilar*, denoted $s_1 \approx_c s_2$, if for all $\ell \in A$

- $\forall m_1$ such that $s_1 \xrightarrow{\ell} m_1$, $\exists m_2$ such that $s_2 \stackrel{\ell}{\Longrightarrow} m_2$ and $m_1 \approx^{\oplus} m_2$,
- $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$, $\exists m_1$ such that $s_1 \stackrel{\ell}{\Longrightarrow} m_1$ and $m_1 \approx^{\oplus} m_2$. □

Note that if $s_1 \xrightarrow{\tau} m_1$, then s_2 must be able to respond with a nonempty sequence of silent moves: $s_2 \stackrel{\tau}{\Longrightarrow} m_2$. However, after this initial step, the reached markings are to be related by weak team bisimilarity, so that future silent moves of one of the two can be matched by the other one also by idling. Therefore, rooted weak team bisimilarity is a slightly finer variant of weak team bisimilarity.

Proposition 6.25. Let $N = (S, A, T)$ be a BPP net with silent moves. If $s_1 \approx_c s_2$, then $s_1 \approx s_2$. □

Nonetheless, if two weakly team bisimilar places cannot perform any silent transition *initially*, then these two places are also rooted weakly team bisimilar.

Example 6.26. Consider the nets in Figure 12. Of course, $s_1 \approx s_2$, but $s_1 \not\approx_c s_2$; however, $s_1 \approx_c s_6$ as well as $s_2 \approx_c s_4$. □

Proposition 6.27. Let $N = (S, A, T)$ be a BPP net with silent moves. Relation $\approx_c$ is an equivalence.

Proof:
Reflexivity is easy: for any move that s performs, the other s does the same transition and so the reached marking m is the same; hence, the condition is satisfied because $m \approx^{\oplus} m$, which holds because $\approx^{\oplus}$ is reflexive. Symmetry derives from the fact that $\approx^{\oplus}$ is symmetric and that the two conditions in Definition 6.24 are symmetric.

Transitivity: if $s_1 \approx_c s_2$ and $s_2 \approx_c s_3$, then $s_1 \approx_c s_3$, because $\approx^\oplus$ is transitive. In fact, since $s_1 \approx_c s_2$, for all m_1 such that $s_1 \xrightarrow{\ell} m_1$, there exists m_2 such that $s_2 \stackrel{\ell}{\Longrightarrow} m_2$ and $m_1 \approx^\oplus m_2$. Since $s_2 \approx_c s_3$, we have also that $s_2 \approx s_3$ by Proposition 6.25; hence, by Lemma 6.5, $s_3 \stackrel{\ell}{\Longrightarrow} m_3$ and $m_2 \approx^\oplus m_3$. Summing up, if $s_1 \xrightarrow{\ell} m_1$, there exists m_3 such that $s_3 \stackrel{\ell}{\Longrightarrow} m_3$ and $m_1 \approx^\oplus m_3$, as $\approx^\oplus$ is transitive. The case when s_2 moves first is symmetric, hence omitted. Summing up, $s_1 \approx_c s_3$. □

We can also define rooted weak team bisimulation equivalence *on markings* as the additive closure of rooted weak team bisimilarity on places, i.e., $\approx_c^\oplus$. Of course, by Proposition 5.2, rooted weak team bisimulation equivalence relates markings of the same size only; moreover, $\approx_c^\oplus$ is an equivalence relation, by Proposition 5.3, as $\approx_c$ is an equivalence relation (by Proposition 6.27). Also in this case, once $\approx_c$ has been computed, checking whether two markings of size k are related by $\approx_c^\oplus$ takes only $O(n)$ time, where n is the number of places.

Proposition 6.28. (Rooted weak team bisimilarity is finer than weak team bisimilarity) Let $N = (S, A, T)$ be a BPP net. If $m_1 \approx_c^\oplus m_2$, then $m_1 \approx^\oplus m_2$.

Proof:
By Proposition 6.25, we have that $\approx_c \subseteq \approx$. Since the additive closure is monotone (by Proposition 5.3(4)), the thesis follows trivially. □

The following theorem provides a characterization of rooted weak team bisimilarity as a suitable bisimulation-like relation over markings, i.e., over a global model of the overall behavior.

Theorem 6.29. Let $N = (S, A, T)$ be a BPP net. If two markings m_1 and m_2 are rooted weak team bisimulation equivalent, $m_1 \approx_c^\oplus m_2$, then $|m_1| = |m_2|$ and

1. $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that σ_2 is sequential, ${}^\bullet t_1 \approx_c {}^\bullet \sigma_2$, $l(t_1) = o_\tau(\sigma_2)$, $t_1^\bullet \approx^\oplus \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx^\oplus m_2'$,
2. $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ such that σ_1 is sequential, ${}^\bullet \sigma_1 \approx_c {}^\bullet t_2$, $o_\tau(\sigma_1) = l(t_2)$, $\sigma_1^\bullet \approx^\oplus t_2^\bullet$, $m_1[\sigma_1\rangle m_1'$ and $m_1' \approx^\oplus m_2'$.

Proof:
If $m_1 \approx_c^\oplus m_2$, then $|m_1| = |m_2|$ by Proposition 5.2. Moreover, for any t_1 such that $m_1[t_1\rangle m_1'$, we have that $m_1 = s_1 \oplus \overline{m}_1$, where $s_1 = {}^\bullet t_1$. As $m_1 \approx_c^\oplus m_2$, by Definition 5.1, it follows that there exist s_2 and $\overline{m}_2$ such that $m_2 = s_2 \oplus \overline{m}_2$, $s_1 \approx_c s_2$ and $\overline{m}_1 \approx_c^\oplus \overline{m}_2$.

Since $s_1 \approx_c s_2$, by Definition 6.24, we have that for transition $t_1 = s_1 \xrightarrow{\ell} \underline{m}_1$, there must exist $\underline{m}_2$ such that $s_2 \stackrel{\ell}{\Longrightarrow} \underline{m}_2$ and $\underline{m}_1 \approx^\oplus \underline{m}_2$. This means that for transition t_1, there exists a sequential transition sequence σ_2 such that $o_\tau(\sigma_2) = \ell = l(t_1)$, ${}^\bullet\sigma_2 = s_2$, $\sigma_2^\bullet = \underline{m}_2$, hence with ${}^\bullet t_1 \approx_c {}^\bullet \sigma_2$ and $t_1^\bullet \approx^\oplus \sigma_2^\bullet$. Therefore, $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx^\oplus m_2'$ by Proposition 5.3.

The case when m_2 moves first is symmetric, hence omitted. □

Note that, contrary to Theorem 6.15, we do not have an *if-and-only-if* condition. As a matter of fact, it is not true that if two markings m_1 and m_2 of the same size are such that they satisfy

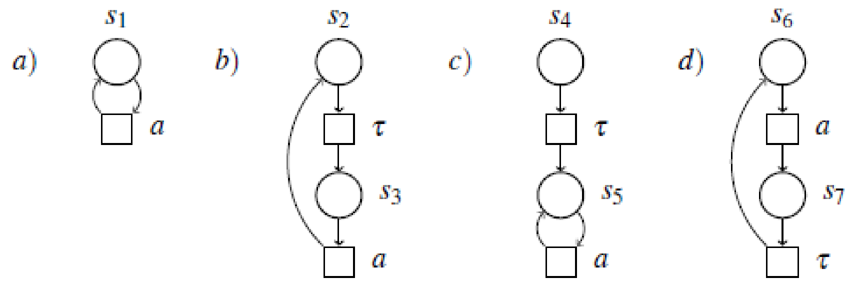


Figure 12. Some weak team bisimilar BPP nets

the two bisimulation conditions of Theorem 6.29, then they are rooted weak team bisimilar. As a counterexample, consider the net in Figure 12 and the two markings $2 \cdot s_1 \oplus s_2$ and $s_1 \oplus 2 \cdot s_2$. As s_1 cannot perform any silent transition, we have that $s_1 \not\approx_c s_2$, even if $s_1 \approx s_2$, and so $2 \cdot s_1 \oplus s_2 \not\approx_c^{\oplus} s_1 \oplus 2 \cdot s_2$, even if $2 \cdot s_1 \oplus s_2 \approx^{\oplus} s_1 \oplus 2 \cdot s_2$. However, the two bisimulation conditions are satisfied for these markings. In one direction, to transition $2 \cdot s_1 \oplus s_2 \xrightarrow{a} 2 \cdot s_1 \oplus s_2$, the other marking can reply with $s_1 \oplus 2 \cdot s_2 \xrightarrow{a} s_1 \oplus 2 \cdot s_2$, where $s_1 \approx_c s_1$ and $2 \cdot s_1 \oplus s_2 \approx^{\oplus} s_1 \oplus 2 \cdot s_2$. Similarly, to transition $2 \cdot s_1 \oplus s_2 \xrightarrow{\tau} 2 \cdot s_1 \oplus s_3$, the other marking can reply with $s_1 \oplus 2 \cdot s_2 \xrightarrow{\tau} s_1 \oplus s_2 \oplus s_3$, where $s_2 \approx_c s_2$, $s_3 \approx s_3$ and $2 \cdot s_1 \oplus s_3 \approx^{\oplus} s_1 \oplus s_2 \oplus s_3$. Symmetrically, if $s_1 \oplus 2 \cdot s_2$ moves first.

Corollary 6.30. (Rooted weak team bisimilarity is finer than rooted weak interleaving bisimilarity) Let $N = (S, A, T)$ be a BPP net. If $m_1 \approx_c^{\oplus} m_2$, then $m_1 \approx_{int}^c m_2$.

Proof:
We want to prove that if $m_1 \approx_c^{\oplus} m_2$, then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ s.t. $m_2[\sigma_2\rangle m_2'$ with $l(t_1) = o_\tau(\sigma_2)$ and $m_1' \approx_{int} m_2'$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ s.t. $m_1[\sigma_1\rangle m_1'$ with $o_\tau(\sigma_1) = l(t_2)$ and $m_1' \approx_{int} m_2'$,

so that $m_1 \approx_{int}^c m_2$ follows directly by Definition 3.5. However, this implication is obvious, due to Theorem 6.29 and Corollary 6.16. □

6.6. Weak H-team bisimilarity

We provide the definition of *weak h-team bisimulation on places*, adapting the definition of weak team bisimulation on places (cf. Definition 6.1). In this definition (and in the following ones), the empty marking θ is considered as an additional place, so that the relation is defined not on S, rather on $S \cup \{\theta\}$; therefore, the symbols p_1 and p_2 that occur in the definition below can only denote either the empty marking θ or a single place, because of the shape of BPP net transitions.

Definition 6.31. (Weak h-team bisimulation on places) Let $N = (S, A, T)$ be a BPP net with silent moves, such that $A_\tau = A \setminus \{\tau\}$. A *weak h-team bisimulation* (or wh-team bisimulation, for short) is a relation $R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\})$ such that if $(p_1, p_2) \in R$ then for all $\ell \in A_\tau$

- $\forall m_1$ such that $p_1 \xrightarrow{\ell} m_1$, $\exists m_2$ such that $p_2 \overset{\ell}{\Longrightarrow} m_2$ and $(m_1, m_2) \in R^{\oplus}$,
- $\forall m_1$ such that $p_1 \xrightarrow{\tau} m_1$, $\exists m_2$ such that $p_2 \overset{\varepsilon}{\Longrightarrow} m_2$ and $(m_1, m_2) \in R^{\oplus}$,
- $\forall m_2$ such that $p_2 \xrightarrow{\ell} m_2$, $\exists m_1$ such that $p_1 \overset{\ell}{\Longrightarrow} m_1$ and $(m_1, m_2) \in R^{\oplus}$, and, finally,
- $\forall m_2$ such that $p_2 \xrightarrow{\tau} m_2$, $\exists m_1$ such that $p_1 \overset{\varepsilon}{\Longrightarrow} m_1$ and $(m_1, m_2) \in R^{\oplus}$.

p and p' are *weakly h-team bisimilar* (or *wh-team bisimulation equivalent*), denoted by $p \approx_h p'$, if there exists a weak h-team bisimulation R such that $(p, p') \in R$. □

Of course, the definition of weak h-team bisimulation is coinductive and it is possible to prove, following the same steps done for $\approx$ in Section 6.1, that also $\approx_h$ is an equivalence relation. Moreover, since a weak team bisimulation is a weak h-team bisimulation, we have that $\approx$ implies $\approx_h$. This implication is strict, as illustrated in the following example.

Example 6.32. Consider the nets in Figure 3. Note that $s_6 \approx_h \theta$ because $R_1 = \{(s_6, \theta), (\theta, \theta)\}$ is a wh-team bisimulation. Therefore, we also have that $s_1 \approx_h s_4$ because $R_2 = \{(s_1, s_4), (s_2, s_5), (\theta, s_6), (\theta, \theta), (s_3, s_7)\}$ is a weak h-team bisimulation. In fact, if s_4 moves with $s_4 \xrightarrow{\tau} s_5 \oplus s_6$, s_1 can reply with $s_1 \xrightarrow{\tau} s_2$ and $(s_2, s_5 \oplus s_6) \in R_2^{\oplus}$. For the same reason, $s_8 \approx_h s_{11}$ because $s_9 \approx_h s_{13}$; in fact, s_9 can match transition $s_{13} \xrightarrow{\tau} \theta$, because $s_{10} \approx_h \theta$. More intriguing is the following case: $s_4 \approx_h s_8$. If $s_8 \xrightarrow{a} 2 \cdot s_9$, then s_4 can reply with $s_4 \overset{a}{\Longrightarrow} s_6 \oplus s_7$ and $2 \cdot s_9 \approx_h^{\oplus} s_6 \oplus s_7$. Moreover, the silent transition $s_4 \xrightarrow{\tau} s_5 \oplus s_6$ can be matched silently by s_8 (the only silent transition it can perform is $s_8 \overset{\varepsilon}{\Longrightarrow} s_8$), and $s_5 \oplus s_6 \approx_h^{\oplus} s_8$ because $s_5 \approx_h s_8$ and $s_6 \approx_h \theta$. These examples show that, contrary to what happens for weak team bisimilarity (cf. Example 6.3), τ-labeled transitions changing the number of currently available tokens may be really unobservable. □

Example 6.33. Consider the two nets in Figure 10. Relation $R = \{(s_1, s_4), (s_2, s_5), (s_3, s_6), (s_3, s_7)\}$ is a weak h-team bisimulation, as we already argued in Example 6.4 that R is a weak team bisimulation. However, s_1 and s_4 do not offer the same causal behavior, because s_4 may perform b caused by a, while for s_1 actions a and b are always causally independent. This example shows that weak h-team bisimulation may not respect causality if the BPP net is not τ-h-sequential (see Definition 2.9). As a matter of fact, the main difference w.r.t. the example above, is that here the silent transition is not preserving the number of tokens on places that can perform some observable action, i.e., it is not τ-h-sequential, while the silent transitions of the previous example are all τ-h-sequential. □

As done for weak team bisimilarity, also weak h-team bisimilarity can be characterized by means of strong h-team bisimilarity, as explained by the following proposition.

Proposition 6.34. Let $N = (S, A, T)$ be a BPP net, with $A_\tau = A \setminus \{\tau\}$ and let $R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\})$ be a weak h-team bisimulation. If $(p_1, p_2) \in R$ then for all $\delta \in A_\tau \cup \{\varepsilon\}$

- $\forall m_1$ such that $p_1 \overset{\delta}{\Longrightarrow} m_1$, there exists m_2 such that $p_2 \overset{\delta}{\Longrightarrow} m_2$ and $(m_1, m_2) \in R^{\oplus}$,
- $\forall m_2$ such that $p_2 \overset{\delta}{\Longrightarrow} m_2$, there exists m_1 such that $p_1 \overset{\delta}{\Longrightarrow} m_1$ and $(m_1, m_2) \in R^{\oplus}$.

Proof:
As R is a wh-team bisimulation, if $(p_1, p_2) \in R$ and $p_1 \stackrel{\delta}{\Longrightarrow} m_1$, then $p_2 \stackrel{\delta}{\Longrightarrow} m_2$ and $(m_1, m_2) \in R^{\oplus}$, by (the analogous of) Lemma 6.5; symmetrically, if p_2 moves first. $\square$

Remark 6.35. (Complexity of $\approx_h$) A consequence of Proposition 6.34 is that it is possible to check whether two places are weak h-team bisimilar on a τ-simple BPP net N by checking whether they are strong h-team bisimilar on its saturated net N' (cf. Remark 6.12). $\square$

Starting from weak h-team bisimilarity $\approx_h$, we can lift it over the markings of N in a structural, distributed way: m_1 is weak h-team bisimulation equivalent to m_2 if these two markings are related by the h-additive closure of $\approx_h$, i.e., if $m_1 \approx_h^{\oplus} m_2$. Note that weak h-team bisimulation equivalent markings may not have the same size. Once $\approx_h$ has been computed, checking whether two markings are weak h-team bisimulation equivalent takes only $O(n)$ time. Of course, since $\approx \subseteq \approx_h$, we have that $\approx^{\oplus} \subseteq \approx_h^{\oplus}$, by Proposition 5.3.

The following theorem provides a characterization of weak h-team bisimulation equivalence as a suitable bisimulation-like relation over markings.

Theorem 6.36. Let $N = (S, A, T)$ be a BPP net. Two markings m_1 and m_2 are weak h-team bisimulation equivalent, $m_1 \approx_h^{\oplus} m_2$, if and only if

1. $\forall t_1$ such that $l(t_1) \neq \tau$ and $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that σ_2 is sequential, ${}^{\bullet}t_1 \approx_h {}^{\bullet}\sigma_2$, $l(t_1) = o(\sigma_2)$, $t_1^{\bullet} \approx_h^{\oplus} \sigma_2^{\bullet}$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx_h^{\oplus} m_2'$,

2. $\forall t_1$ such that $l(t_1) = \tau$ and $m_1[t_1\rangle m_1'$, *either* $\exists \sigma_2$ such that σ_2 is nonempty and sequential, ${}^{\bullet}t_1 \approx_h {}^{\bullet}\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^{\bullet} \approx_h^{\oplus} \sigma_2^{\bullet}$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx_h^{\oplus} m_2'$, *or* $\exists s_2 \in m_2$ such that ${}^{\bullet}t_1 \approx_h s_2$, $t_1^{\bullet} \approx_h s_2$ and $m_1' \approx_h^{\oplus} m_2$,

3. $\forall t_2$ such that $l(t_2) \neq \tau$ and $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ such that σ_1 is sequential, ${}^{\bullet}\sigma_1 \approx_h {}^{\bullet}t_2$, $o(\sigma_1) = l(t_2)$, $\sigma_1^{\bullet} \approx_h^{\oplus} t_2^{\bullet}$, $m_1[\sigma_1\rangle m_1'$ and $m_1' \approx_h^{\oplus} m_2'$,

4. $\forall t_2$ such that $l(t_2) = \tau$ and $m_2[t_2\rangle m_2'$, *either* $\exists \sigma_1$ such that σ_1 is nonempty and sequential, ${}^{\bullet}\sigma_1 \approx_h {}^{\bullet}t_2$, $o(\sigma_1) = \varepsilon$, $\sigma_1^{\bullet} \approx_h^{\oplus} t_2^{\bullet}$, $m_1[\sigma_1\rangle m_1'$ and $m_1' \approx_h^{\oplus} m_2'$, *or* $\exists s_1 \in m_1$ such that $s_1 \approx_h {}^{\bullet}t_2$, $s_1 \approx_h t_2^{\bullet}$ and $m_1 \approx_h^{\oplus} m_2'$.

Proof:
The proof follows the same steps of Theorem 6.15. $\square$

By the theorem above, it is clear that $\approx_h^{\oplus}$ is a weak interleaving bisimulation; hence, the following corollary follows trivially.

Corollary 6.37. (Weak h-team bisimilarity is finer than weak interleaving bisimilarity) Let $N = (S, A, T)$ be a BPP net. If $m_1 \approx_h^{\oplus} m_2$, then $m_1 \approx_{int} m_2$. $\square$

This implication is strict. In Figure 7, we have that $s_1 \approx_{int} s_4 \oplus s_5$ (actually, they are even strong interleaving bisimilar), but $s_1 \not\approx_h^{\oplus} s_4 \oplus s_5$ because the two markings have different *observable* size.

Now we want to argue that weak h-team bisimilarity $\approx_h^{\oplus}$ implies weak fully-concurrent bisimilarity $\approx_{fc}$ if the BPP net is τ-h-sequential. To this aim, we first propose a lemma stating that if the BPP net is τ-h-sequential, then –(i) and (ii)– the execution of silent moves does not activate any parallel *observable* activity and, starting from a singleton marking, –(iii) and (iv)– in the execution of a visible, one-action step, the only visible transition causes all the future visible actions executable from the reached marking.

Lemma 6.38. Let $N = (S, A, T)$ be a τ-h-sequential BPP net. The following hold:

(i) If $s[\sigma\rangle m$ and $o(\sigma) = \varepsilon$, then $|o(m)| = |o(s)|$.

(ii) If $m[\sigma\rangle m'$ and $o(\sigma) = \varepsilon$, then $|o(m)| = |o(m')|$.

(iii) If $s[\sigma\rangle m$ and $o(\sigma) = \ell$, then there exist σ_1, t, σ_2 such that $\sigma = \sigma_1 t \sigma_2$, $l(t) = \ell$, $o(\sigma_1) = o(\sigma_2) = \varepsilon$ and $s[\sigma_1\rangle m''[t\rangle m'[\sigma_2\rangle m$, where $o(m'') = {}^\bullet t$, $|o(m')| = |o(m)|$.

(iv) If $s[\sigma_1\rangle m[\sigma_2\rangle m'$, $o(\sigma_1) = \ell_1$ and $o(\sigma_2) = \ell_2$, then $t_1 \preceq t_2$, where, for $i = 1, 2$, $t_i \in \sigma_i$ and $l(t_i) = \ell_i$.

Proof:
Similar to the proof of Lemma 6.17 and so omitted. □

Theorem 6.39. (Weak h-team bisimilarity implies weak fc-bisimilarity) Let $N = (S, A, T)$ be a τ-h-sequential BPP net with silent moves. If $m_1 \approx_h^{\oplus} m_2$, then $m_1 \approx_{fc} m_2$.

Proof:
Very similar to the proof of Theorem 6.19 and so omitted. □

We actually conjecture that for each τ-h-sequential BPP net, if $m_1 \approx_{fc} m_2$, then $m_1 \approx_h^{\oplus} m_2$. The argument exposed at the end of Section 6.3 can be adapted to support this claim. Therefore, we conjecture that these two equivalences do coincide for τ-h-sequential BPP nets.

However, if the net is not τ-h-sequential, then the two equivalences are different. On the one hand, if $m_1 \approx_{fc} m_2$, then it may happen that $m_1 \not\approx_h^{\oplus} m_2$: an example illustrating this fact is described in the conclusions (see Figure 16). On the other hand, if $m_1 \approx_h^{\oplus} m_2$, then it may happen that $m_1 \not\approx_{fc} m_2$: Example 6.4 illustrates this fact.

A BPP net can be minimized w.r.t. weak h-team bisimilarity, too. The h-reduced net w.r.t. $\approx_h$, which minimizes the number of places and the number of transitions, can be defined as follows, where by $o(S)$ we denote the set of places that can perform some observable action (cf. Definition 2.8).

Definition 6.40. (H-reduced net) Let $N = (S, A, T)$ be a BPP net and let $\approx_h$ be the weak h-team bisimulation equivalence relation over its places. The *h-reduced* net $N_h = (S_h, A, T_h)$ is defined as:

- $S_h = \{[s] \mid s \in o(S)\}$, where $[s] = \{s' \in o(S) \mid s \approx_h s'\}$;
- $T_h = \{([s], \ell, [o(m)]) \mid (s, \ell, m) \in T, \ell \neq \tau\} \cup \{([s], \tau, [o(m)]) \mid (s, \tau, m) \in T, [s] \neq [o(m)]\}$,

where $[m]$ is defined as follows: $[\theta] = \theta$ and $[m_1 \oplus m_2] = [m_1] \oplus [m_2]$. If the net N has initial marking m_0, then N_h has initial marking $[o(m_0)]$. □

Given a BPP net $N = (S, A, T)$ with silent moves, it is an easy exercise to prove that relation $R = \{(s, [s]) \mid s \in o(S)\}$ is a weak team bisimulation between the net $o(N) = (o(S), A, o(T))$, where $o(T) = \{(s, \ell, o(m)) \mid s \in o(S), (s, \ell, m) \in T\}$, and the net N_h. Indeed, the h-reduced net N_h of N w.r.t. $\approx_h$ is isomorphic to the reduced net $o(N)_\approx$ of $o(N)$ w.r.t. $\approx$ (cf. Definition 6.21).

Of course, also weak h-team bisimilarity $\approx_h$ is not a congruence for the choice operator of BPP, so that it is necessary to define a slight strengthening of this equivalence.

Definition 6.41. (Rooted weak h-team bisimilarity on places) Let $N = (S, A, T)$ be a BPP net. We have that p_1 and p_2 are *rooted weakly h-team bisimilar*, denoted $p_1 \approx_{hc} p_2$, if for all $\ell \in A$

- $\forall m_1$ such that $p_1 \xrightarrow{\ell} m_1$, $\exists m_2$ such that $p_2 \stackrel{\ell}{\Longrightarrow} m_2$ and $m_1 \approx_h^\oplus m_2$,
- $\forall m_2$ such that $p_2 \xrightarrow{\ell} m_2$, $\exists m_1$ such that $p_1 \stackrel{\ell}{\Longrightarrow} m_1$ and $m_1 \approx_h^\oplus m_2$. □

Of course, we lift rooted weak h-team bisimilarity to markings by additive closure, yielding $\approx_{hc}^\oplus$, which is slightly coarser than rooted weak team bisimilarity $\approx_c^\oplus$. E.g., consider Figure 3: $s_1 \approx_{hc}^\oplus s_4$ because $s_2 \approx_h^\oplus s_5 \oplus s_6$, but $s_1 \not\approx_c^\oplus s_4$ because $s_2 \not\approx^\oplus s_5 \oplus s_6$.

The following theorem provides a characterization of $\approx_{hc}^\oplus$ as a suitable bisimulation-like relation over markings, whose proof is omitted as it is very similar to that of Theorem 6.29.

Theorem 6.42. Let $N = (S, A, T)$ be a BPP net. If two markings m_1 and m_2 are rooted weak h-team bisimulation equivalent, $m_1 \approx_{hc}^\oplus m_2$, then

1. $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ such that σ_2 is sequential, ${}^\bullet t_1 \approx_{hc} {}^\bullet\sigma_2$, $l(t_1) = o_\tau(\sigma_2)$, $t_1^\bullet \approx_h^\oplus \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx_h^\oplus m_2'$,
2. $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ such that σ_1 is sequential, ${}^\bullet\sigma_1 \approx_{hc} {}^\bullet t_2$, $o_\tau(\sigma_1) = l(t_2)$, $\sigma_1^\bullet \approx_h^\oplus t_2^\bullet$, $m_1[\sigma_1\rangle m_1'$ and $m_1' \approx_h^\oplus m_2'$. □

Corollary 6.43. (Rooted weak h-team bisimilarity implies rooted weak interleaving bisimilarity) Let $N = (S, A, T)$ be a BPP net. If $m_1 \approx_{hc}^\oplus m_2$, then $m_1 \approx_{int}^c m_2$.

Proof:
We want to prove that if $m_1 \approx_{hc}^\oplus m_2$, then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists \sigma_2$ s.t. $m_2[\sigma_2\rangle m_2'$ with $l(t_1) = o_\tau(\sigma_2)$ and $m_1' \approx_{int} m_2'$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists \sigma_1$ s.t. $m_1[\sigma_1\rangle m_1'$ with $o_\tau(\sigma_1) = l(t_2)$ and $m_1' \approx_{int} m_2'$,

so that $m_1 \approx_{int}^c m_2$ follows directly by Definition 3.5. However, this implication is obvious, due to Theorem 6.42 and Corollary 6.37. □

7. A distributed approach to branching equivalence checking

In this section, we introduce a novel behavioral semantics for BPP nets with silent moves, by introducing *branching team bisimilarity* $\approx_{br}$ on places of an unmarked BPP nets (and then its additive closure $\approx_{br}^\oplus$ on markings), by adapting the definition of branching bisimulation on LTSs [12, 15].

7.1. Branching team bisimulation on places

In order to define this new relation on places, we need an auxiliary notation: by $s \Rrightarrow s'$ we mean that there exists a silent path $s = s_0 \xrightarrow{\tau} s_1 \xrightarrow{\tau} s_2 \dots s_{n-1} \xrightarrow{\tau} s_n = s'$ (with $n \geq 0$), i.e., a τ-sequential transition sequence from s to s'.

Definition 7.1. (Branching team bisimulation on places) Let $N = (S, A, T)$ be a BPP net. A *branching team bisimulation* is a relation $R \subseteq S \times S$ such that if $(s_1, s_2) \in R$ then for all $\ell \in A$

- $\forall m_1$ such that $s_1 \xrightarrow{\ell} m_1$,
  - either $\ell = \tau$ and $\exists s'_2$ such that $s_2 \Rrightarrow s'_2$ with $(s_1, s'_2) \in R$ and $(m_1, s'_2) \in R$,
  - or $\exists s, m_2$ such that $s_2 \Rrightarrow s \xrightarrow{\ell} m_2$ with $(s_1, s) \in R$ and $(m_1, m_2) \in R^{\oplus}$,
- and, symmetrically, $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$,
  - either $\ell = \tau$ and $\exists s'_1$ such that $s_1 \Rrightarrow s'_1$ with $(s'_1, s_2) \in R$ and $(s'_1, m_2) \in R$,
  - or $\exists s, m_1$ such that $s_1 \Rrightarrow s \xrightarrow{\ell} m_1$ with $(s, s_2) \in R$ and $(m_1, m_2) \in R^{\oplus}$.

Two places s and s' are *branching team bisimilar* (or *branching team bisimulation equivalent*), denoted by $s \approx_{br} s'$, if there exists a branching team bisimulation R such that $(s, s') \in R$. □

This definition is not a rephrasing of the original definition on LTS in [12], rather of a slight variant called *semi-branching bisimulation* [12, 1, 15], which gives rise to the same equivalence as the original definition but has better mathematical properties; in particular it ensures [1] that the composition of branching bisimulation on places is a branching bisimulation on places (see Proposition 7.11(3)).

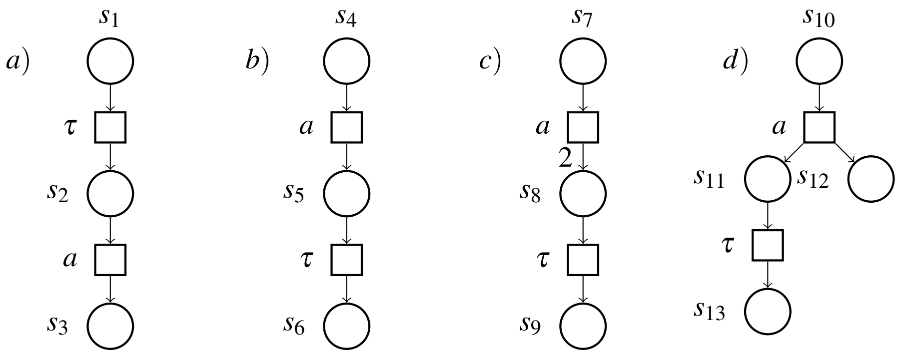


Figure 13. Two pairs of branching team bisimilar BPP nets

Example 7.2. Consider Figure 13. $R_1 = \{(s_1, s_4), (s_2, s_4), (s_3, s_5), (s_3, s_6)\}$ is a branching team bisimulation. $R_2 = \{(s_7, s_{10}), (s_8, s_{11}), (s_8, s_{12}), (s_9, s_{13})\}$ is also a branching team bisimulation. □

Example 7.3. Consider Figure 5. Note that $s_1 \not\approx_{br} s_4$. In fact, to transition $s_4 \xrightarrow{a} s_6$, place s_1 can only try to respond with $s_1 \xrightarrow{a} s_2$, but s_2 and s_6 are clearly not equivalent, because only s_2 can do c. □

Example 7.4. Consider the nets in Figure 3. Note that $s_1 \not\approx_{br} s_4$ because s_1 cannot match transition $s_4 \xrightarrow{\tau} s_5 \oplus s_6$, as it cannot reach silently any marking of size 2. For the same reason, it is not difficult to realize that also $s_4 \not\approx_{br} s_8$. Moreover, $s_8 \not\approx_{br} s_{11}$ because $s_9 \not\approx_{br} s_{13}$; in fact, s_9 cannot match transition $s_{13} \xrightarrow{\tau} \theta$, because it cannot reach silently the empty marking. These examples show that τ-labeled transitions that change the number of currently available tokens are not really unobservable. However, s_1, s_4, s_8 and s_{11} are all pairwise branching interleaving bisimilar and also branching fully-concurrent bisimilar (but not state-sensitive bfc-bisimilar). However, we think that such silent transitions do change the structure of the system and so they cannot be considered as unobservable. □

Example 7.5. (Branching team bisimilarity is better than weak team bisimilarity) Consider the two nets in Figure 10. We argued that $s_1 \approx s_4$, even if s_1 and s_4 do not offer the same causal behavior. However, note that $s_1 \not\approx_{br} s_4$ because if $s_4 \xrightarrow{a} s_7$, then s_1 can try to respond with $s_1 \xrightarrow{\tau} s_2 \oplus s_3 \xrightarrow{a} s_3$, but $s_4 \not\approx_{br} s_2 \oplus s_3$ as a place cannot be branching team bisimilar to a marking of size 2. □

Remark 7.6. (Stuttering Property, again) It is not difficult to prove that, given a τ-sequential transition sequence $s_1 \xrightarrow{\tau} s_2 \xrightarrow{\tau} s_3 \ldots s_n \xrightarrow{\tau} s_{n+1}$, if $s_1 \approx_{br} s_{n+1}$, then $s_i \approx_{br} s_j$ for $i, j = 1, \ldots, n+1$. This is the *stuttering property*, also discussed in Remark 3.9.

An important property holds for $\approx_{br}$. Consider $s_1 \approx_{br} s_2$. Then, suppose $s_1 \xrightarrow{\tau} m_1$ and that s_2 responds by performing the τ-sequential path $s_2 \Rrightarrow s_2'$ with $s_1 \approx_{br} s_2'$ and $m_1 \approx_{br} s_2'$. By transitivity (this will be proved in Proposition 7.13), we have that also $s_2 \approx_{br} s_2'$. Hence, by the stuttering property, s_1 is branching team bisimilar to each place in the path from s_2 to s_2', and so all the places traversed in the τ-sequential path from s_2 to s_2' are branching team bisimilar. Similarly, if $s_1 \xrightarrow{\ell} m_1$ and s_2 responds with $s_2 \Rrightarrow s \xrightarrow{\ell} m_2$ (with $s_1 \approx_{br} s$ and $m_1 \approx_{br}^{\oplus} m_2$), then by transitivity $s_2 \approx_{br} s$, so that all the places traversed in the τ-sequential path from s_2 to s are branching team bisimilar. □

A branching team bisimulation is also a weak team bisimulation, and so $\approx_{br}$ is finer than $\approx$.

Proposition 7.7. Let $N = (S, A, T)$ be a BPP net with silent moves. If $s_1 \approx_{br} s_2$, then $s_1 \approx s_2$. □

Example 7.8. Consider the nets in Figure 4. It is not difficult to see that $s_1 \approx s_4$. However, $s_1 \not\approx_{br} s_4$, because to transition $s_4 \xrightarrow{a} s_5$, place s_1 can only try to respond with $s_1 \xrightarrow{\tau} s_2 \xrightarrow{a} s_3$, but not all the conditions required are satisfied; in particular, $s_2 \not\approx_{br} s_4$, because only s_4 can do b. Hence, contrary to weak team bisimilarity, branching team bisimilarity does respect the timing of choices. □

Example 7.9. Consider the nets in Figure 14. It is easy to realize that $R = \{(s_1, s_4), (s_2, s_5), (s_3, s_5)\}$ is a branching team bisimulation. Note that to transition $s_2 \xrightarrow{\tau} s_3$, place s_5 responds by idling. Note also that to move $s_5 \xrightarrow{c} \theta$, place s_2 responds with $s_2 \xrightarrow{\tau} s_3 \xrightarrow{c} \theta$ and, indeed, by performing the τ move, the system passes through branching team bisimilar places only, i.e., $s_2 \approx_{br} s_3$. □

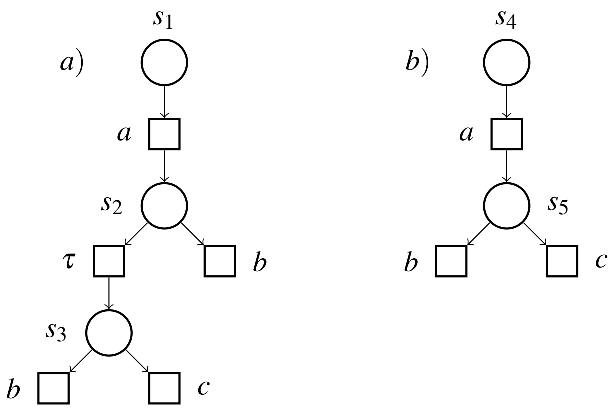


Figure 14. Two branching team bisimilar BPP nets

We now list some useful properties of branching team bisimulation relations, one of which is based on the following lemma.

Lemma 7.10. Let $N = (S, A, T)$ be a BPP net with silent moves and let R be a branching team bisimulation such that $(s_1, s_2) \in R$. Then, the following hold:

(i) For all s_1' such that $s_1 \Rightarrow s_1'$, there exists s_2' such that $s_2 \Rightarrow s_2'$ and $(s_1', s_2') \in R$; and symmetrically,

(ii) For all s_2' such that $s_2 \Rightarrow s_2'$, there exists s_1' such that $s_1 \Rightarrow s_1'$ and $(s_1', s_2') \in R$.

Proof:
The proof is by induction on the length of the computation. We prove only case (i), as the other one is symmetric. The base case is when the sequence is of length zero, i.e., $s_1 \Rightarrow s_1$; in such a case, $s_2 \Rightarrow s_2$ and $(s_1, s_2) \in R$, as required.

In general, we can assume that $s_1 \xrightarrow{\tau} s_1' \Rightarrow s_1''$; since $(s_1, s_2) \in R$, we have that *either* $\exists s_2'$ such that $s_2 \Rightarrow s_2'$ with $(s_1', s_2') \in R$, *or* $\exists s, m_2$ such that $s_2 \Rightarrow s \xrightarrow{\tau} m_2$, with $(s_1', m_2) \in R^{\oplus}$; note that, since $R^{\oplus}$ relates marking of the same size only, this means that m_2 must be a place and so $(s_1', m_2) \in R$. In any case, induction can be applied to either $(s_1', s_2') \in R$ or $(s_1', m_2) \in R$, as $s_1' \Rightarrow s_1''$ is a shorter path; hence, in the former case, we can conclude that $\exists s_2''$ such that $s_2' \Rightarrow s_2''$ with $(s_1'', s_2'') \in R$; and, similarly, in the latter case, we can conclude that $\exists s_2''$ such that $m_2 \Rightarrow s_2''$ with $(s_1'', s_2'') \in R$. Summing up, if $s_1 \xrightarrow{\tau} s_1' \Rightarrow s_1''$, then $s_2 \Rightarrow s_2''$ such that $(s_1'', s_2'') \in R$, as required. □

Proposition 7.11. For each BPP net $N = (S, A, T)$ with silent moves, the following hold:

1. the identity relation $\mathscr{I} = \{(s, s) \mid s \in S\}$ is a branching team bisimulation;
2. the inverse relation $R^{-1} = \{(s', s) \mid (s, s') \in R\}$ of a branching team bisimulation R is a branching team bisimulation;
3. the relational composition $R_1 \circ R_2 = \{(s, s'') \mid \exists s'.(s, s') \in R_1 \wedge (s', s'') \in R_2\}$ of two branching team bisimulations R_1 and R_2 is a branching team bisimulation;

4. the union $\bigcup_{i\in I} R_i$ of branching team bisimulations R_i is a branching team bisimulation.

Proof:
The proof of (1) is immediate: $(s,s) \in \mathscr{I}$ is a branching team bisimulation pair because whatever transition s performs (say, $s \xrightarrow{\ell} m$), the other s in the pair does exactly the same transition $s \xrightarrow{\ell} m$ and $(m,m) \in \mathscr{I}^{\oplus}$. This is ensured by the *or* condition in Definition 7.1.

The proof of (2) is also immediate: if $(s_2,s_1) \in R^{-1}$, then $(s_1,s_2) \in R$; since R is a branching team bisimulation, the second condition ensures that $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$,

– either $\ell = \tau$ and $\exists s_1'$ such that $s_1 \Rightarrow s_1'$ with $(s_1',s_2) \in R$ and $(s_1',m_2) \in R$, (i.e., with $(s_2,s_1') \in R^{-1}$ and $(m_2,s_1') \in R^{-1}$)

– or $\exists s, m_1$ such that $s_1 \Rightarrow s \xrightarrow{\ell} m_1$ with $(s,s_2) \in R$ and $(m_1,m_2) \in R^{\oplus}$, (i.e., with $(s_2,s) \in R^{-1}$ and $(m_2,m_1) \in (R^{\oplus})^{-1} = (R^{-1})^{\oplus}$ by Proposition 5.4(3)).

And symmetrically, if s_1 moves first. Hence, R^{-1} is a branching team bisimulation, too.

The proof of (3) is less immediate, but not too difficult, thanks to Lemma 7.10. Given a pair $(s_1,s_3) \in R_1 \circ R_2$, there exists a place s_2 such that $(s_1,s_2) \in R_1$ and $(s_2,s_3) \in R_2$. If $s_1 \xrightarrow{\ell} m_1$, since $(s_1,s_2) \in R_1$, it follows that *either* $\ell = \tau$ and $\exists s_2'$ such that $s_2 \Rightarrow s_2'$ with $(s_1,s_2') \in R_1$ and $(m_1,s_2') \in R_1$, *or* $\exists s, m_2$ such that $s_2 \Rightarrow s \xrightarrow{\ell} m_2$ with $(s_1,s) \in R_1$ and $(m_1,m_2) \in R_1^{\oplus}$.

In the former case, since $(s_2,s_3) \in R_2$ and $s_2 \Rightarrow s_2'$, by Lemma 7.10, there exists s_3' such that $s_3 \Rightarrow s_3'$ with $(s_2',s_3') \in R_2$; in such a case, we have that, to transition $s_1 \xrightarrow{\tau} m_1$, s_3 replies with $s_3 \Rightarrow s_3'$ such that $(s_1,s_3') \in R_1 \circ R_2$ and $(m_1,s_3') \in R_1 \circ R_2$, as required.

In the latter case, since $(s_2,s_3) \in R_2$ and $s_2 \Rightarrow s$, by Lemma 7.10, there exists s' such that $s_3 \Rightarrow s'$ with $(s,s') \in R_2$. Now, since $(s,s') \in R_2$ and $s \xrightarrow{\ell} m_2$, we have two subcases: *either* $\ell = \tau$ and $\exists s_3'$ such that $s' \Rightarrow s_3'$ with $(s,s_3') \in R_2$ and $(m_2,s_3') \in R_2$, *or* $\exists \bar{s}, m_3$ such that $s' \Rightarrow \bar{s} \xrightarrow{\ell} m_3$ with $(s,\bar{s}) \in R_2$ and $(m_2,m_3) \in R_2^{\oplus}$. In the former subcase, to transition $s_1 \xrightarrow{\tau} m_1$, s_3 replies with $s_3 \Rightarrow s_3'$ such that $(s_1,s_3') \in R_1 \circ R_2$ and $(m_1,s_3') \in R_1 \circ R_2$[1], as required. In the latter subcase, to transition $s_1 \xrightarrow{\ell} m_1$, s_3 replies with the sequence $s_3 \Rightarrow \bar{s} \xrightarrow{\ell} m_3$ such that $(s_1,\bar{s}) \in R_1 \circ R_2$ and $(m_1,m_3) \in (R_1)^{\oplus} \circ (R_2)^{\oplus} = (R_1 \circ R_2)^{\oplus}$ by Proposition 5.4(4), as required. The case when s_3 moves first is symmetric, and so omitted. Hence, $R_1 \circ R_2$ is a branching team bisimulation, too.

The proof of (4) is trivial, too: assume $(s_1,s_2) \in \bigcup_{i\in I} R_i$; then, $j \in I$ exists such that (s_1,s_2) belongs to R_j. If $s_1 \xrightarrow{\ell} m_1$, then *either* $\ell = \tau$ and $\exists s_2'$ such that $s_2 \Rightarrow s_2'$ with $(s_1,s_2') \in R_j$ and $(m_1,s_2') \in R_j$, *or* $\exists s, m_2$ such that $s_2 \Rightarrow s \xrightarrow{\ell} m_2$ with $(s_1,s) \in R_j$ and $(m_1,m_2) \in R_j^{\oplus}$. In the former case, $\{(s_1,s_2'),(m_1,s_2')\} \subseteq \bigcup_{i\in I} R_i$ as $R_j \subseteq \bigcup_{i\in I} R_i$. In the latter case, we have that $(s_1,s) \in \bigcup_{i\in I} R_i$ because $R_j \subseteq \bigcup_{i\in I} R_i$; moreover, also $(m_1,m_2) \in (\bigcup_{i\in I} R_i)^{\oplus}$ as $R_j^{\oplus} \subseteq (\bigcup_{i\in I} R_i)^{\oplus}$ by Proposition 5.4(5). So $\bigcup_{i\in I} R_i$ is a branching team bisimulation, too. □

Remember that $s \approx_{br} s'$ if there exists a branching team bisimulation containing the pair (s,s'). This means that $\approx_{br}$ is the union of all branching team bisimulations, i.e.,

$$\approx_{br} = \bigcup \{R \subseteq S \times S \mid R \text{ is a branching team bisimulation}\}.$$

[1] As $(m_2,s_3') \in R_2$, this means that m_2 is a place; hence, the condition $(m_1,m_2) \in R_1^{\oplus}$ is equivalent to $(m_1,m_2) \in R_1$.

By Proposition 7.11(4), $\approx_{br}$ is also a branching team bisimulation, hence the largest such relation.

Proposition 7.12. For each BPP net $N = (S, A, T)$, relation $\approx_{br} \subseteq S \times S$ is the largest branching team bisimulation relation. □

The largest branching team bisimulation relation $\approx_{br}$ is an equivalence relation. As a matter of fact, as the identity relation $\mathscr{I}$ is a branching team bisimulation by Proposition 7.11(1), we have that $\mathscr{I} \subseteq \approx_{br}$, and so $\approx_{br}$ is reflexive. Symmetry derives from Proposition 7.11(2). Transitivity also holds for $\approx_{br}$ by Proposition 7.11(3). Summing up, we have the following.

Proposition 7.13. For each BPP net $N = (S, A, T)$, relation $\approx_{br} \subseteq S \times S$ is an equivalence relation. □

Remark 7.14. (Complexity of $\approx_{br}$) From a complexity point of view, branching team bisimilarity is the easiest equivalence to decide over BPP nets with silent moves. According to [13, 12], it can be checked on finite-state LTSs with time complexity $O(l + n \cdot m)$ and space complexity $O(n+m)$, where l is the number of labels, n the number of states and m the number of transitions, by means of a coarsest partition refinement algorithm in the style of [23, 24]. Therefore, essentially the same complexity is necessary to compute branching team bisimilarity on places of an unmarked BPP net, with the usual adaptation of counting the empty marking as an additional, dummy place and with the extra cost due to the fact that the reached markings are to be related by the additive closure of the current partition over places (by means of Algorithm 1). Hence, the overall time complexity is $O(l + n^2 \cdot m)$, where n is the number of places, m the number of transitions and l the number of labels. □

7.2. Lifting branching team bisimilarity to markings

Starting from branching team bisimilarity over the places of an unmarked BPP net, we can define *branching team bisimulation equivalence* over its markings in a structural, distributed way, as the additive closure of $\approx_{br}$, i.e., $\approx_{br}^{\oplus}$. Hence, branching team bisimilar markings have the same size.

Proposition 7.15. For each BPP net $N = (S, A, T)$, if $m_1 \approx_{br}^{\oplus} m_2$, then $|m_1| = |m_2|$. □

Proposition 7.16. For each BPP net $N = (S, A, T)$, $\approx_{br}^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence.

Proof:
By Proposition 7.13, $\approx_{br}$ is an equivalence relation. Hence, by Proposition 5.3, $\approx_{br}^{\oplus}$ is an equivalence relation, too. □

Note that, once $\approx_{br}$ has been computed in $O(l + n^2 \cdot m)$, where n is the number of places, m the number of transitions and l the number of labels (cf. Remark 7.14), checking whether two markings are branching team bisimulation equivalent takes only $O(n)$ time with Algorithm 1. Of course, since by Proposition 7.41 we have that $\approx_{br} \subseteq \approx$, then by Proposition 5.3 we have that $\approx_{br}^{\oplus} \subseteq \approx^{\oplus}$. This implication is strict (cf., e.g., Example 7.8).

The following theorem provides a characterization of branching team equivalence as a suitable bisimulation-like relation over markings. Indeed, this result gives evidence of the fact that branching team bisimulation equivalence does respect the global behavior of the net.

Theorem 7.17. Let $N = (S, A, T)$ be a BPP net with silent moves. Two markings m_1 and m_2 are branching team bisimilar, $m_1 \approx_{br}^{\oplus} m_2$, if and only if $|m_1| = |m_2|$ and

1. $\forall t_1$ such that $m_1[t_1\rangle m_1'$,
   - either $l(t_1) = \tau$ and
     (i) either $\exists \sigma_2$ nonempty and τ-sequential, such that $^\bullet t_1 \approx_{br} {}^\bullet\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx_{br} \sigma_2^\bullet$, $^\bullet t_1 \approx_{br} \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ with $m_1 \approx_{br}^{\oplus} m_2'$ and $m_1' \approx_{br}^{\oplus} m_2'$,
     (ii) or $\exists s_2 \in m_2$ such that $^\bullet t_1 \approx_{br} s_2$, $t_1^\bullet \approx_{br} s_2$, with $m_1' \approx_{br}^{\oplus} m_2$,
   - or $\exists \sigma, t_2$ such that σt_2 is sequential, σ is τ-sequential, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $^\bullet t_1 \approx_{br} {}^\bullet\sigma t_2$, $^\bullet t_1 \approx_{br} {}^\bullet t_2$, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$, $m_2[\sigma\rangle m[t_2\rangle m_2'$ with $m_1 \approx_{br}^{\oplus} m$ and $m_1' \approx_{br}^{\oplus} m_2'$;
2. and, symmetrically, $\forall t_2$ such that $m_2[t_2\rangle m_2'$,
   - either $l(t_2) = \tau$ and
     (i) either $\exists \sigma_1$ nonempty and τ-sequential, such that $^\bullet\sigma_1 \approx_{br} {}^\bullet t_2$, $o(\sigma_1) = \varepsilon$, $\sigma_1^\bullet \approx_{br} t_2^\bullet$, $\sigma_1^\bullet \approx_{br} {}^\bullet t_2$, $m_1[\sigma_1\rangle m_1'$ with $m_1' \approx_{br}^{\oplus} m_2$ and $m_1' \approx_{br}^{\oplus} m_2'$,
     (ii) or $\exists s_1 \in m_1$ such that $s_1 \approx_{br} {}^\bullet t_2$, $s_1 \approx_{br} t_2^\bullet$, with $m_1 \approx_{br}^{\oplus} m_2'$,
   - or $\exists \sigma, t_1$ such that σt_1 is sequential, σ is τ-sequential, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $^\bullet\sigma t_1 \approx_{br} {}^\bullet t_2$, $^\bullet t_1 \approx_{br} {}^\bullet t_2$, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$, $m_1[\sigma\rangle m[t_1\rangle m_1'$ with $m \approx_{br}^{\oplus} m_2$ and $m_1' \approx_{br}^{\oplus} m_2'$.

Proof:
($\Rightarrow$) If $m_1 \approx_{br}^{\oplus} m_2$, then $|m_1| = |m_2|$ by Proposition 7.15. Moreover, for any t_1 such that $m_1[t_1\rangle m_1'$, we have that $m_1 = s_1 \oplus \overline{m}_1$, where $s_1 = {}^\bullet t_1$. As $m_1 \approx_{br}^{\oplus} m_2$, by Definition 5.1, it follows that there exist s_2 and $\overline{m}_2$ such that $m_2 = s_2 \oplus \overline{m}_2$, $s_1 \approx_{br} s_2$ and $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$. Since $s_1 \approx_{br} s_2$, by Definition 7.1, if $t_1 = s_1 \xrightarrow{\ell} p_1$, we have to consider two cases:

(i) either $\ell = \tau$ and $\exists p_2$ such that $s_2 \Rrightarrow p_2$ with $s_1 \approx_{br} p_2$ and $p_1 \approx_{br} p_2$,

(ii) or $\exists p, p_2$ such that $s_2 \Rrightarrow p \xrightarrow{\ell} p_2$, with $s_1 \approx_{br} p$ and $p_1 \approx_{br}^{\oplus} p_2$.

Case (i): We have to consider two subcases: (a) Either there exists a nonempty τ-sequential transition sequence σ_2 such that $o(\sigma_2) = \varepsilon$, $^\bullet\sigma_2 = s_2$, $\sigma_2^\bullet = p_2$, hence with $^\bullet t_1 \approx_{br} {}^\bullet\sigma_2$, $^\bullet t_1 \approx_{br} \sigma_2^\bullet$ and $t_1^\bullet \approx_{br} \sigma_2^\bullet$. ($b$) Or s_2 replies by idling, i.e., $p_2 = s_2$; in such a case, $^\bullet t_1 \approx_{br} s_2$ and $t_1^\bullet \approx_{br} s_2$.

In subcase (a), $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx_{br}^{\oplus} m_2'$ by Definition 5.1. For the same reason, $m_1 \approx_{br}^{\oplus} m_2'$, as $^\bullet t_1 \approx_{br} \sigma_2^\bullet$.

Similarly, in subcase (b), $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2 = s_2 \oplus \overline{m}_2$ and so $m_1' \approx_{br}^{\oplus} m_2$.

Case (ii): This means that for transition t_1, there exists a (possibly empty) τ-sequential transition sequence σ and a transition t_2 such that $^\bullet\sigma t_2 = s_2$, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $^\bullet t_2 = p = \sigma^\bullet$, $t_2^\bullet = p_2$, and so $^\bullet t_1 \approx_{br} {}^\bullet t_2 = \sigma^\bullet$ and $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$. Now, $m = p \oplus \overline{m}_2$ and so $m_1 \approx_{br}^{\oplus} m$ by Definition 5.1. Similarly, $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = t_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx_{br}^{\oplus} m_2'$ by Proposition 5.3.

The case when m_2 moves first is symmetric, hence omitted.

($\Leftarrow$) Let us assume that $|m_1| = |m_2|$ and that the bisimulation-like conditions hold; then, we prove that $m_1 \approx_{br}^{\oplus} m_2$. First of all, assume that no transition t_1 is enabled at m_1; in such a case, no observable transition is enabled at m_2; in fact, if $m_2[t_2\rangle m_2'$ with $l(t_2) \neq \tau$, then, by the (2-or) condition, a nonempty, sequential transition sequence σt_1 must be executable at m_1, contradicting the assumption that no transition is enabled at m_1. However, m_2 may enable τ-sequential transitions: by the (2-either-(ii)) condition, m_1 can reply by idling. This means that each place in m_1 is a deadlock, and similarly each place in m_2 is branching team bisimilar to a deadlock; therefore, all the places in m_1 and m_2 are pairwise branching team bisimilar; hence, the condition $|m_1| = |m_2|$ is enough to ensure that $m_1 \approx_{br}^{\oplus} m_2$.

Now, assume that $m_1[t_1\rangle m_1'$ for some t_1. Let us consider first the (1-either) condition, i.e., with $l(t_1) = \tau$. This case is actually composed of two subcases.

In subcase (i), we know that there exists a nonempty τ-sequential transition sequence σ_2 such that $^\bullet t_1 \approx_{br} {}^\bullet\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx_{br} \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ and $m_1' \approx_{br}^{\oplus} m_2'$. Therefore, we have that $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2' = \sigma_2^\bullet \oplus \overline{m}_2$, $m_1 = {}^\bullet t_1 \oplus \overline{m}_1$, $m_2 = {}^\bullet\sigma_2 \oplus \overline{m}_2$. Since $m_1' \approx_{br}^{\oplus} m_2'$ and $t_1^\bullet \approx_{br} \sigma_2^\bullet$, it follows that $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$ by Proposition 5.3, and so $m_1 \approx_{br}^{\oplus} m_2$, because $^\bullet t_1 \approx_{br} {}^\bullet\sigma_2$.

In subcase (ii), we have that $\exists s_2 \in m_2$ such that $^\bullet t_1 \approx_{br} s_2$, $t_1^\bullet \approx_{br} s_2$, with $m_1' \approx_{br}^{\oplus} m_2$. Note that $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_1 = {}^\bullet t_1 \oplus \overline{m}_1$ and $m_2 = s_2 \oplus \overline{m}_2$. Since $m_1' \approx_{br}^{\oplus} m_2$ and $t_1^\bullet \approx_{br} s_2$, it follows that $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$, and so $m_1 \approx_{br}^{\oplus} m_2$, because $^\bullet t_1 \approx_{br} s_2$.

Let us now consider the (1-or) condition. This means that $\exists \sigma, t_2$ such that σt_2 is sequential, σ is τ-sequential, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $^\bullet t_1 \approx_{br} {}^\bullet\sigma t_2$, $^\bullet t_1 \approx_{br} {}^\bullet t_2 = \sigma^\bullet$, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$, $m_2[\sigma\rangle m[t_2\rangle m_2'$ with $m_1 \approx_{br}^{\oplus} m$ and $m_1' \approx_{br}^{\oplus} m_2'$. Note that $m_1 = {}^\bullet t_1 \oplus \overline{m}_1$, $m_1' = t_1^\bullet \oplus \overline{m}_1$, $m_2 = {}^\bullet\sigma t_2 \oplus \overline{m}_2$ and $m_2' = t_2^\bullet \oplus \overline{m}_2$. Since $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$ and $m_1' \approx_{br}^{\oplus} m_2'$, by Proposition 5.3, we have that $\overline{m}_1 \approx_{br}^{\oplus} \overline{m}_2$. Hence, $m_1 \approx_{br}^{\oplus} m_2$, because $^\bullet t_1 \approx_{br} {}^\bullet\sigma t_2$.

Symmetrically, if we start from a transition t_2 enabled at m_2. □

By the theorem above, it is clear that $\approx_{br}^{\oplus}$ is a branching interleaving bisimulation; hence, the following corollary follows trivially.

Corollary 7.18. (Branching team equivalence is finer than branching interleaving bisimilarity) Let $N = (S, A, T)$ be a BPP net. If $m_1 \approx_{br}^{\oplus} m_2$, then $m_1 \approx_{bri} m_2$, i.e., we have that $\approx_{br}^{\oplus} \subseteq \approx_{bri}$. □

Example 7.19. The containment in the above corollary is strict. Consider the nets in Figure 3. Clearly, the markings s_1 and s_4 are branching interleaving bisimilar; however, they are not branching team equivalent: to the transition $s_4 \xrightarrow{\tau} s_5 \oplus s_6$, place s_1 can try to respond only with $s_1 \xrightarrow{\tau} s_2$, however $s_2 \not\approx_{br}^{\oplus} s_5 \oplus s_6$, because they have different size. □

Now we want to prove that branching team bisimilarity is finer than state-sensitive, branching fully-concurrent bisimilarity.

Theorem 7.20. (Branching team bisimilarity is finer than state-sensitive branching fully concurrent bisimilarity) Let $N = (S, A, T)$ be a BPP net with silent moves. If $m_1 \approx_{br}^{\oplus} m_2$, then $m_1 \approx_{sbfc} m_2$.

Proof:
Let $R = \{((C_1,\rho_1),g,(C_2,\rho_2)) \mid (C_1,\rho_1)$ is a process of $N(m_1)$, (C_2,ρ_2) is a process of $N(m_2)$ and g is an abstract event isomorphism between C_1 and C_2, such that $\rho_1(Max(C_1)) \approx_{br}^{\oplus} \rho_2(Max(C_2))\}$. We want to prove that R is a state-sensitive branching fc-bisimulation. First, observe that the triple $((C_1^0,\rho_1^0),g_0,(C_2^0,\rho_2^0))$ where g_0 is empty and, for $i=1,2$, C_i^0 is a BPP causal net which contains no transitions and $\rho_i^0(Max(C_i^0)) = m_i$, belongs to relation R because $m_1 \approx_{br}^{\oplus} m_2$ by hypothesis. Note also that if the relation R is a state-sensitive branching fc-bisimulation, then this triple ensures that $m_1 \approx_{sbfc} m_2$. It is enough to check that R is a branching fc-bisimilarity, because, since for each triple $((C_1,\rho_1),g,(C_2,\rho_2)) \in R$ we have that $\rho_1(Max(C_1)) \approx_{br}^{\oplus} \rho_2(Max(C_2))$, Proposition 7.15 ensures that R is state-sensitive. Now assume $(\pi_1,g,\pi_2) \in R$, where $\pi_i = (C_i,\rho_i)$ for $i=1,2$. In order to be a branching fully-concurrent bisimulation triple, it is necessary that

i) $\forall t_1,\pi_1'$ such that $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$,

- *either* $l(e_1) = \tau$ and $\exists\sigma_2'$ (with $o(\sigma_2') = \varepsilon$), π_2' such that $\pi_2 \stackrel{\sigma_2'}{\Longrightarrow} \pi_2'$, $(\pi_1,g,\pi_2') \in R$ and $(\pi_1',g,\pi_2') \in R$;
- *or* $\exists\sigma'$ (with $o(\sigma') = \varepsilon$), e_2,π_2',π_2'',g' such that
  1. $\pi_2 \stackrel{\sigma'}{\Longrightarrow} \pi_2' \xrightarrow{e_2} \pi_2''$;
  2. if $l(e_1) = \tau$, then $l(e_2) = \tau$ and $g' = g$; otherwise, $l(e_1) = l(e_2)$ and $g' = g \cup \{(e_1,e_2)\}$;
  3. and finally, $(\pi_1,g,\pi_2') \in R$ and $(\pi_1',g',\pi_2'') \in R$;

ii) symmetrically, if π_2 moves first.

Let us consider any transition t_1 such that $\rho_1(Max(C_1))[t_1\rangle m_1'$ and $l(t_1) \neq \tau$. Since $\rho_1(Max(C_1)) \approx_{br}^{\oplus} \rho_2(Max(C_2))$, by Theorem 7.17, $\exists\sigma,t_2$ such that σt_2 is sequential, σ is τ-sequential, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, ${}^\bullet t_1 \approx_{br} {}^\bullet\sigma t_2$, ${}^\bullet t_1 \approx_{br} {}^\bullet t_2$, $t_1^\bullet \approx_{br}^{\oplus} t_2^\bullet$, $m_2[\sigma\rangle m[t_2\rangle m_2'$ with $m_1 \approx_{br}^{\oplus} m$ and $m_1' \approx_{br}^{\oplus} m_2'$. Therefore, it is really possible to extend the causal net C_1 to the causal net C_1' through a suitable event e_1, as well as to extend the causal net C_2 to the causal net C_2' through a suitable transition sequence σ', followed by the observable event e_2, as required above: indeed, $g' = g \cup \{(e_1,e_2)\}$. Summing up, for the move $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$, we add the triples (π_1,g,π_2') and (π_1',g',π_2'') to R, so that $\rho_1(Max(C_1)) = m_1 \approx_{br}^{\oplus} m = \rho_2'(Max(C_2'))$ and $\rho_1'(Max(C_1')) = m_1' \approx_{br}^{\oplus} m_2' = \rho_2''(Max(C_2''))$, as required. A similar argument is necessary when $l(t_1) = \tau$. As $\rho_1(Max(C_1)) \approx_{br}^{\oplus} \rho_2(Max(C_2))$, by Theorem 7.17, besides the case similar to the above (omitted), it also possible that

(i) either $\exists\sigma_2$ nonempty and τ-sequential, such that ${}^\bullet t_1 \approx_{br} {}^\bullet\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx_{br} \sigma_2^\bullet$, ${}^\bullet t_1 \approx_{br} \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ with $m_1 \approx_{br}^{\oplus} m_2'$ and $m_1' \approx_{br}^{\oplus} m_2'$,

(ii) or $\exists s_2 \in m_2$ such that ${}^\bullet t_1 \approx_{br} s_2$, $t_1^\bullet \approx_{br} s_2$, with $m_1' \approx_{br}^{\oplus} m_2$.

The *either* case (i) ensures that it is possible to extend the causal net C_1 to the causal net C_1' through a suitable silent transition e_1, as well as to extend the causal net C_2 to the causal net C_2' through a suitable τ-sequential transition sequence σ_2', as required above. Summing up, for the move $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$, we add the two triples (π_1,g,π_2') and (π_1',g,π_2') to R, so that $\rho_1(Max(C_1)) = m_1 \approx_{br}^{\oplus} m_2' = \rho_2'(Max(C_2'))$ and $\rho_1'(Max(C_1')) = m_1' \approx_{br}^{\oplus} m_2' = \rho_2'(Max(C_2'))$, as required. For the *or* case (ii), it is really possible to extend the causal net C_1 to the causal net C_1' through a suitable transition e_1, while the causal net C_2 is not modified. Summing up, for the move $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho_1'(e_1) = t_1$, we add (π_1', g, π_2) to R, so that $\rho_1'(Max(C_1')) = m_1' \approx^{\oplus} m_2 = \rho_2(Max(C_2))$, as required.

Symmetrically, if $\rho_2(Max(C))$ moves first. □

However, the reverse implication of Theorem 7.20 does not hold in general: it may happen that if $m_1 \approx_{sbfc} m_2$, then $m_1 \not\approx^{\oplus}_{br} m_2$, as the following example shows.

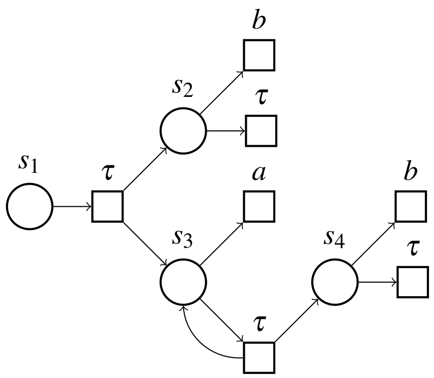


Figure 15. A BPP net with $s_1 \approx_{sbfc} s_3$

Example 7.21. Consider the net in Figure 15. It is not difficult to realize that $s_1 \approx_{sbfc} s_3$. Informally, if $s_1 \xrightarrow{\tau} s_2 \oplus s_3$, s_3 can reply with $s_3 \xrightarrow{\tau} s_3 \oplus s_4$ and $s_2 \oplus s_3 \approx_{sbfc} s_3 \oplus s_4$, as required. Symmetrically, besides the move above, s_3 can also do $s_3 \xrightarrow{a} \theta$, and s_1 can reply with $s_1 \stackrel{\tau}{\Longrightarrow} s_3 \xrightarrow{a} \theta$ with $s_3 \approx_{sbfc} s_3$ and $\theta \approx_{sbfc} \theta$. However, $s_1 \not\approx^{\oplus}_{br} s_3$: if $s_3 \xrightarrow{a} \theta$, then s_1 responds with $s_1 \xrightarrow{\tau} s_2 \oplus s_3 \xrightarrow{\tau} s_3 \xrightarrow{a} \theta$, but the silent path $s_1 \xrightarrow{\tau} s_2 \oplus s_3 \xrightarrow{\tau} s_3$ is not composed of τ-sequential transitions, so that $s_1 \not\Rightarrow s_3$. □

However, if the BPP net is τ-sequential, we conjecture that also the reverse implication holds, so that the two equivalences coincide for τ-sequential BPP nets.

Corollary 7.22. (Branching team bisimilarity is finer than branching fully-concurrent bisimilarity) Let $N = (S, A, T)$ be a BPP net with silent moves. If $m_1 \approx^{\oplus}_{br} m_2$, then $m_1 \approx_{bfc} m_2$.

Proof:
Since $\approx_{sbfc} \subseteq \approx_{bfc}$, the thesis follows by Theorem 7.20. □

The reverse implication of Corollary 7.22 does not hold in general. For instance, the BPP nets in Figure 3(c) and (d) are such that $s_8 \approx_{bfc} s_{11}$, but $s_8 \not\approx^{\oplus}_{br} s_{11}$. In fact, after the execution of action a, the two reached markings are $2 \cdot s_9$ and $s_{12} \oplus s_{13}$, but $s_9 \not\approx_{br} s_{13}$, because $s_{10} \not\approx_{br} \theta$.

7.3. Minimizing nets w.r.t. $\approx_{br}$

In this section, we propose the construction of a reduced net, i.e., a net obtained by merging together branching team bisimilar places. We will show that this technique is correct: a marking of the original net is branching team bisimilar to the corresponding marking of the reduced net.

Definition 7.23. (Reduced net) Let $N = (S, A, T)$ be a BPP net and let $\approx_{br}$ be the branching team bisimulation equivalence relation over its places. The *reduced* net $N_{br} = (S_{br}, A, T_{br})$ is defined as:

- $S_{br} = \{[s] \mid s \in S\}$, where $[s] = \{s' \in S \mid s \approx_{br} s'\}$;
- $T_{br} = \{([s], \ell, [m]) \mid (s, \ell, m) \in T, \ell \neq \tau\} \cup \{([s], \tau, [m]) \mid (s, \tau, m) \in T, [s] \neq [m]\}$,

where $[m]$ is defined as follows: $[\theta] = \theta$ and $[m_1 \oplus m_2] = [m_1] \oplus [m_2]$. If the net N has initial marking $m_0 = k_1 \cdot s_1 \oplus \ldots \oplus k_n \cdot s_n$, then N_{br} has initial marking $[m_0] = k_1 \cdot [s_1] \oplus \ldots \oplus k_n \cdot [s_n]$. □

Lemma 7.24. Let $N = (S, A, T)$ be a BPP net and let $N_{br} = (S_{br}, A, T_{br})$ be its reduced net w.r.t. $\approx_{br}$. Relation $R = \{(s, [s]) \mid s \in S\}$ is a branching team bisimulation.

Proof:
If $s \xrightarrow{\ell} m$ with $\ell \neq \tau$, then $[s] \xrightarrow{\ell} [m]$ by definition of T_{br} and $(m, [m]) \in R^{\oplus}$, as required. If $s \xrightarrow{\tau} m$ and $[s] = [m]$, then $[s]$ replies by idling, and $(m, [s]) \in R^{\oplus}$, because $[s] = [m]$. Finally, if $s \xrightarrow{\tau} m$ and $[s] \neq [m]$, then $[s] \xrightarrow{\tau} [m]$ by definition of T_{br} and $(m, [m]) \in R^{\oplus}$, as required.

The case when $[s]$ moves first is slightly more complex for the freedom in choosing the representative in an equivalence class. Transition $[s] \xrightarrow{\ell} [m]$ is possible, by Definition of T_{br}, if there exist $s' \in [s]$ and $m' \in [m]$ such that $s' \xrightarrow{\ell} m'$; as $s \approx_{br} s'$ and $s' \xrightarrow{\ell} m'$, then

- either $\ell = \tau$ and $\exists p_1$ such that $s \Rrightarrow p_1$ with $p_1 \approx_{br} s'$ and $p_1 \approx_{br} m'$,
- or $\exists \overline{s}, m_1$ such that $s \Rrightarrow \overline{s} \xrightarrow{\ell} m_1$ with $\overline{s} \approx_{br} s'$ and $m_1 \approx_{br}^{\oplus} m'$.

Summing up, if $[s] \xrightarrow{\ell} [m]$, then

- either $\ell = \tau$ and $\exists p_1$ such that $s \Rrightarrow p_1$ with $p_1 \in [s']$ and $p_1 \in [m']$ (i.e., with $(p_1, [s]) \in R$ and $(p_1, [m]) \in R$, because $[s] = [s'] = [p_1] = [m'] = [m]$);
- or $\exists \overline{s}, m_1$ such that $s \Rrightarrow \overline{s} \xrightarrow{\ell} m_1$ with $\overline{s} \in [s']$ (i.e., with $(\overline{s}, [s]) \in R$, as $[s] = [s'] = [\overline{s}]$) and $m_1 \in [m']$ (i.e., with $(m_1, [m]) \in R^{\oplus}$, as $[m_1] = [m'] = [m]$).

Hence, $R = \{(s, [s]) \mid s \in S\}$ is a branching team bisimulation. □

Theorem 7.25. Let $N = (S, A, T)$ be a BPP net and let $N_{br} = (S_{br}, A, T_{br})$ be its reduced net w.r.t. $\approx_{br}$. For any $m \in \mathscr{M}(S)$, we have that $m \approx_{br}^{\oplus} [m]$.

Proof:
By induction on the size of m. □

As a consequence of this theorem, we would like to point out that the reduced net w.r.t. $\approx_{br}$ is indeed the *least* net offering the same branching team bisimilar behavior as the original net: no further fusion of places can be done, as there are not two places in the reduced net which are branching team bisimilar. Moreover, silent transitions relating branching team bisimilar places in the original net do not generate any silent transition in the reduced net, so that the number of transitions is minimized, too.

7.4. Rooted branching team bisimilarity

Definition 7.26. (Rooted branching team bisimilarity on places) Let $N = (S, A, T)$ be a BPP net. Two places s_1 and s_2 are *rooted branching team bisimilar*, denoted $s_1 \approx_{brc} s_2$, if for all $\ell \in A$

- $\forall m_1$ such that $s_1 \xrightarrow{\ell} m_1$, there exists m_2 such that $s_2 \xrightarrow{\ell} m_2$ and $m_1 \approx_{br}^{\oplus} m_2$,
- $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$, there exists m_1 such that $s_1 \xrightarrow{\ell} m_1$ and $m_1 \approx_{br}^{\oplus} m_2$. □

The peculiar feature of rooted branching team bisimilarity is that initial moves are matched as in strong team bisimulation, while subsequent moves are matched as for branching team bisimilarity. Therefore, rooted branching team bisimilarity is a slightly finer variant of branching team bisimilarity.

Proposition 7.27. if $s_1 \approx_{brc} s_2$, then $s_1 \approx_{br} s_2$. □

Nonetheless, if two branching team bisimilar places cannot perform any silent transition *initially*, then these two places are also rooted branching team bisimilar.

Example 7.28. Considering again Figure 14, we have that $s_1 \approx_{brc} s_4$ because $s_2 \approx_{br} s_5$; however, note that $s_2 \not\approx_{brc} s_5$. □

Proposition 7.29. Let $N = (S, A, T)$ be a BPP net. Relation $\approx_{brc}$ is an equivalence relation.

Proof:
Standard: it follows from the fact that $\approx_{br}$ and $\approx_{br}^{\oplus}$ are equivalence relations. □

Proposition 7.30. If $s_1 \approx_{brc} s_2$, then $s_1 \approx_c s_2$.

Proof:
Trivial, as $\approx_{br} \subseteq \approx$. □

Example 7.31. Consider Figure 4. It is easy to see that $s_1 \approx_c s_4$, however $s_1 \not\approx_{brc} s_4$. □

Proposition 7.32. If $s_1 \sim s_2$, then $s_1 \approx_{brc} s_2$.

Proof:
Trivial, as $\sim \subseteq \approx_{br}$. □

We can also define rooted branching team bisimulation equivalence *on markings* as the additive closure of rooted branching team bisimilarity on places, i.e., $\approx_{brc}^{\oplus}$. Of course, by Proposition 5.2, rooted branching team bisimulation equivalence relates markings of the same size only; moreover, $\approx_{brc}^{\oplus}$ is an equivalence relation, by Proposition 5.3, as $\approx_{brc}$ is an equivalence relation (by Proposition 7.29). Also in this case, once $\approx_{brc}$ has been computed, checking whether two markings are related by $\approx_{brc}^{\oplus}$ takes $O(n)$ time.

Proposition 7.33. (Rooted branching team bisimilarity is finer than branching team bisimilarity) Let $N = (S, A, T)$ be a BPP net. If $m_1 \approx_{brc}^{\oplus} m_2$, then $m_1 \approx_{br}^{\oplus} m_2$.

Proof:
By Proposition 7.27, we have that $\approx_{brc} \subseteq \approx_{br}$. Since the additive closure is monotone (by Proposition 5.3(4)), the thesis follows trivially. □

Proposition 7.34. (Rooted branching team bisimilarity is finer than rooted weak team bisimilarity) Let $N = (S,A,T)$ be a BPP net. If $m_1 \approx^{\oplus}_{brc} m_2$, then $m_1 \approx^{\oplus}_{c} m_2$.

Proof:
Similar to the previous one, as by Proposition 7.30, we have that $\approx_{brc} \subseteq \approx_{c}$. □

Proposition 7.35. (Strong team bisimilarity is finer than rooted branching team bisimilarity) Let $N = (S,A,T)$ be a BPP net. If $m_1 \sim^{\oplus} m_2$, then $m_1 \approx^{\oplus}_{brc} m_2$.

Proof:
Similar to the previous one, as by Proposition 7.32, we have that $\sim \subseteq \approx_{brc}$. □

The following theorem provides a characterization of rooted branching team bisimilarity as a suitable bisimulation-like relation over markings, i.e., over a global model of the overall behavior.

Theorem 7.36. Let $N = (S,A,T)$ be a BPP net. If two markings m_1 and m_2 are rooted branching team bisimulation equivalent, $m_1 \approx^{\oplus}_{brc} m_2$, then $|m_1| = |m_2|$ and

1. $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that $^\bullet t_1 \approx_{brc} {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \approx^{\oplus}_{br} t_2^\bullet$, $m_2[t_2\rangle m_2'$ and $m_1' \approx^{\oplus}_{br} m_2'$,
2. $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that $^\bullet t_1 \approx_{brc} {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \approx^{\oplus}_{br} t_2^\bullet$, $m_1[t_1\rangle m_1'$ and $m_1' \approx^{\oplus}_{br} m_2'$.

Proof:
If $m_1 \approx^{\oplus}_{brc} m_2$, then $|m_1| = |m_2|$ by Proposition 5.2. Moreover, for any t_1 such that $m_1[t_1\rangle m_1'$, we have that $m_1 = s_1 \oplus \overline{m}_1$, where $s_1 = {}^\bullet t_1$. As $m_1 \approx^{\oplus}_{brc} m_2$, by Definition 5.1, it follows that there exist s_2 and $\overline{m}_2$ such that $m_2 = s_2 \oplus \overline{m}_2$, $s_1 \approx_{brc} s_2$ and $\overline{m}_1 \approx^{\oplus}_{brc} \overline{m}_2$. Since $s_1 \approx_{brc} s_2$, by Definition 7.26, we have that for transition $t_1 = s_1 \xrightarrow{\ell} p_1$, there must exist p_2 such that $s_2 \xrightarrow{\ell} p_2$ and $p_1 \approx^{\oplus}_{br} p_2$. This means that for transition t_1, there exists a transition t_2 such that $l(t_2) = \ell = l(t_1)$, $^\bullet t_2 = s_2$, $t_2^\bullet = p_2$, hence with $^\bullet t_1 \approx_{brc} {}^\bullet t_2$ and $t_1^\bullet \approx^{\oplus}_{br} t_2^\bullet$. Note that $m_1' = t_1^\bullet \oplus \overline{m}_1$ and $m_2' = t_2^\bullet \oplus \overline{m}_2$, and so $m_1' \approx^{\oplus}_{br} m_2'$ by Proposition 5.3. The case when m_2 moves first is symmetric, hence omitted. □

Note that, contrary to Theorem 7.17, we do not have an *if-and-only-if* condition. In fact, it is not true that if two markings of the same size are such that they satisfy the two bisimulation conditions of Theorem 7.36, then they are rooted branching team bisimilar. As counterexample, consider the net in Figure 12 and the two markings $2 \cdot s_1 \oplus s_2$ and $s_1 \oplus 2 \cdot s_2$ (cf. the discussion after Theorem 6.29).

Corollary 7.37. (Rooted branching team bisimilarity is finer than rooted branching interleaving bisimilarity)
Let $N = (S,A,T)$ be a BPP net. If $m_1 \approx^{\oplus}_{brc} m_2$, then $m_1 \approx^{c}_{bri} m_2$.

Proof:
We want to prove that if $m_1 \approx^{\oplus}_{brc} m_2$, then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ s.t. $m_2[t_2\rangle m_2'$ with $l(t_1) = l(t_2)$ and $m_1' \approx_{bri} m_2'$,

- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ s.t. $m_1[t_1\rangle m_1'$ with $l(t_1) = l(t_2)$ and $m_1' \approx_{bri} m_2'$,

so that $m_1 \approx_{bri}^{c} m_2$ follows directly by Definition 3.11. However, this implication is obvious, due to Theorem 7.36 and Corollary 7.18. □

7.5. Branching H-team bisimilarity

We provide the definition of *branching h-team bisimulation on places* for unmarked BPP nets, adapting the definition of branching team bisimulation on places (cf. Definition 7.1), as a relation not on S, rather on $S \cup \{\theta\}$ (cf. Definition 6.31).

The auxiliary notation $s \Rrightarrow s'$, meaning a τ-sequential transition sequence from place s to place s', is here relaxed to $s \rightrightarrows m$, meaning a τ-h-sequential transition sequence from s to marking m, i.e., a transition sequence $s = s_1[t_1\rangle m_2[t_2\rangle m_3 \dots m_n[t_n\rangle m_{n+1} = m$ (with $n \geq 0$) such that $l(t_i) = \tau$ and t_i is τ-h-sequential, for $i = 1, \dots, n$.

Definition 7.38. (Branching h-team bisimulation on places) Let $N = (S, A, T)$ be a BPP net. A *branching h-team bisimulation* is a relation $R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\})$ such that if $(p_1, p_2) \in R$ then for all $\ell \in A$

- $\forall m_1$ such that $p_1 \xrightarrow{\ell} m_1$,
  - either $\ell = \tau$ and there exists m_2 such that $p_2 \rightrightarrows m_2$ with $(p_1, m_2) \in R^{\oplus}$ and $(m_1, m_2) \in R^{\oplus}$,
  - or $\exists m, m_2$ such that $p_2 \rightrightarrows m \xrightarrow{\ell} m_2$ with $(p_1, m) \in R^{\oplus}$ and $(m_1, m_2) \in R^{\oplus}$,
- and, symmetrically, $\forall m_2$ such that $p_2 \xrightarrow{\ell} m_2$,
  - either $\ell = \tau$ and there exists m_1 such that $p_1 \rightrightarrows m_1$ with $(m_1, p_2) \in R^{\oplus}$ and $(m_1, m_2) \in R^{\oplus}$,
  - or $\exists m, m_1$ such that $p_1 \rightrightarrows m \xrightarrow{\ell} m_1$ with $(m, p_2) \in R^{\oplus}$ and $(m_1, m_2) \in R^{\oplus}$.

p and p' are *branching h-team bisimilar* (or *bh-team bisimulation equivalent*), denoted by $p \approx_{bh} p'$, if there exists a branching h-team bisimulation R such that $(p, p') \in R$. □

Since a branching team bisimulation is a branching h-team bisimulation, we have that $\approx_{br}$ implies $\approx_{bh}$. This implication is strict, as illustrated in the following example.

Example 7.39. Consider the nets in Figure 3. In Example 6.32 we observed that $s_6 \approx_h \theta$ because $R_1 = \{(s_6, \theta), (\theta, \theta)\}$ is a wh-team bisimulation. Actually, R_1 is also a bh-team bisimulation (but not a br-team bisimulation) so that $s_6 \approx_{bh} \theta$. Therefore, we also have that $s_1 \approx_{bh} s_4$ because $R_2 = \{(s_1, s_4), (s_2, s_5), (\theta, s_6), (\theta, \theta), (s_3, s_7)\}$ is a bh-team bisimulation. In fact, if s_4 moves with $s_4 \xrightarrow{\tau} s_5 \oplus s_6$, s_1 can reply with $s_1 \xrightarrow{\tau} s_2$ and $(s_2, s_5 \oplus s_6) \in R_2^{\oplus}$. For the same reason, $s_8 \approx_{bh} s_{11}$ because $s_9 \approx_{bh} s_{13}$; in fact, s_9 can match transition $s_{13} \xrightarrow{\tau} \theta$, because $s_{10} \approx_{bh} \theta$. Similarly, one can observe that $s_4 \approx_{bh} s_8$: the silent transition $s_4 \xrightarrow{\tau} s_5 \oplus s_6$ can be matched silently by s_8 with $s_8 \rightrightarrows s_8$, and $s_5 \oplus s_6 \approx_{bh}^{\oplus} s_8$ because $s_5 \approx_{bh} s_8$ and $s_6 \approx_{bh} \theta$. These examples show that, contrary to what happens for branching team bisimilarity, τ-labeled transitions changing the number of currently available tokens are really unobservable if they are τ-h sequential. □

Remark 7.40. (Stuttering property revisited) It is not difficult to prove that, given a τ-h-sequential transition sequence $s_1[t_1\rangle m_2[t_2\rangle m_3 \ldots m_n[t_n\rangle m_{n+1}$ (with $n \geq 0$) such that $l(t_i) = \tau$ and t_i is τ-h-sequential for $i = 1, \ldots, n$, so that $s_1 \rightrightarrows m_{n+1}$, if $s_1 \approx_{bh}^{\oplus} m_{n+1}$, then $s_1 \approx_{bh}^{\oplus} m_j$ for all $j = 2, \ldots n$.

To show that this holds, let us prove the thesis for $n = 2$, i.e., if $s_1[t_1\rangle m_2[t_2\rangle m_3$ and $s_1 \approx_{bh}^{\oplus} m_3$, then $s_1 \approx_{bh}^{\oplus} m_2$. First note that, since $s_1 \approx_{bh}^{\oplus} m_3$, we have two cases: *either* $s_1 \approx_{bh} \theta$, *or* there exists s_3 such that $m_3 = s_3 \oplus \overline{m}_3$, $s_1 \approx_{bh} s_3$ and $\theta \approx_{bh}^{\oplus} \overline{m}_3$. In the former case, the thesis follows trivially, because m_2 must be equivalent to θ. In the latter case, since t_1 is τ-h-sequential, we have that $|s_1| = |o(m_2)|$; hence, there exists s_2 such that $m_2 = s_2 \oplus \overline{m}_2$, and $\theta \approx_{bh}^{\oplus} \overline{m}_2$. Now, we have two subcases: either $s_2 = s_3$, or $s_2 \xrightarrow{\tau} s_3 \oplus m_2'$ so that $\overline{m}_3 = \overline{m}_2 \oplus m_2'$. In the former subcase, the thesis follows trivially, as we already know that $s_1 \approx_{bh} s_3$ and $\theta \approx_{bh}^{\oplus} \overline{m}_2$. In the latter subcase, if $s_1 \xrightarrow{\ell} m_1$, then s_2 can respond by first performing $s_2 \xrightarrow{\tau} s_3 \oplus m_2'$ and then the corresponding move of s_3, as we already know that $s_1 \approx_{bh} s_3$; on the other side of the bisimulation game, if $s_2 \xrightarrow{\ell} m$, then s_1 can respond with $s_1 \xrightarrow{\tau} s_2 \oplus \overline{m}_2 \xrightarrow{\ell} m \oplus \overline{m}_2$, so that $s_2 \approx_{bh}^{\oplus} s_2 \oplus \overline{m}_2$ and $m \approx_{bh}^{\oplus} m \oplus \overline{m}_2$, because $\theta \approx_{bh}^{\oplus} \overline{m}_2$.

Therefore, by requiring that the silent transitions be τ-h-sequential, we have that branching h-team bisimilarity $\approx_{bh}$ enjoys the stuttering property. In fact, assuming $p_1 \approx_{bh} p_2$ and that $p_1 \xrightarrow{\ell} m_1$, then, e.g., in the *either* case, p_2 responds with $p_2 \rightrightarrows m_2$ with $p_1 \approx_{bh}^{\oplus} m_2$, so that, by transitivity, $p_2 \approx_{bh}^{\oplus} m_2$; hence, by the stuttering property, we are sure that, in passing from place p_2 to marking m_2, we traverse only markings that are all bh-team equivalent to p_2 and m_2. □

It is easy to observe that a branching h-team bisimulation is also a weak h-team bisimulation, and so $\approx_{bh}$ is finer than $\approx_h$.

Proposition 7.41. Let $N = (S, A, T)$ be a BPP net with silent moves. If $s_1 \approx_{bh} s_2$, then $s_1 \approx_h s_2$. □

Example 7.42. (Branching h-team bisimilarity is better than weak h-team bisimilarity) Consider the two nets in Figure 10. In Example 6.4 we observed that s_1 and s_4 are weak (h-)team bisimilar, even if s_1 and s_4 do not offer the same causal behavior. In Example 7.5 we argued that $s_1 \not\approx_{br} s_4$ because if $s_4 \xrightarrow{a} s_7$, then s_1 can try to respond with $s_1 \xrightarrow{\tau} s_2 \oplus s_3 \xrightarrow{a} s_3$, but $s_4 \not\approx_{br} s_2 \oplus s_3$ as a place cannot be branching team bisimilar to a marking of size 2. Of course, also the coarser bh-team bisimilarity cannot equate s_1 and s_4 because the τ-labeled transition is not τ-h-sequential. □

We now list some useful properties of branching h-team bisimulation relations, one of which is based on the following lemma.

Lemma 7.43. Let $N = (S, A, T)$ be a BPP net with silent moves and let R be a branching h-team bisimulation such that $(s_1, s_2) \in R$. Then, the following hold:

(i) For all m_1 such that $s_1 \rightrightarrows m_1$, there exists m_2 such that $s_2 \rightrightarrows m_2$ and $(m_1, m_2) \in R^{\oplus}$.

(ii) For all m_2 such that $s_2 \rightrightarrows m_2$, there exists m_1 such that $s_1 \rightrightarrows m_1$ and $(m_1, m_2) \in R^{\oplus}$.

Proof:
The proof is by induction on the length of the path $s_1 \rightrightarrows m_1$. We prove only case (i), as the other one is symmetric. The base case is the empty path: if $s_1 \rightrightarrows s_1$, then $s_2 \rightrightarrows s_2$ and $(s_1, s_2) \in R^{\oplus}$, as required. In general, we can assume $s_1 \xrightarrow{\tau} m'_1 \xRightarrow{\varepsilon} m_1$, where all the transitions in the path are τ-h-sequential, so that $|o(s_1)| = |o(m'_1)| = |o(m_1)|$. Since $(s_1, s_2) \in R$, we have that *either* $\exists m'_2$ such that $s_2 \rightrightarrows m'_2$ with $(m'_1, m'_2) \in R^{\oplus}$, *or* $\exists m, m'_2$ such that $s_2 \rightrightarrows m \xrightarrow{\tau} m'_2$, with $(s_1, m) \in R^{\oplus}$ and $(m'_1, m'_2) \in R^{\oplus}$.

In the former case, by Proposition 2.6, assuming $k = max\{|m'_1|, |m'_2|\}$ and $m'_1 = p'_1 \oplus \ldots \oplus p'_k$ (where p'_i can also be θ), the path $m'_1 \xRightarrow{\varepsilon} m_1$, composed only of τ-h-sequential transitions, can be decomposed in $p'_i \rightrightarrows \overline{m}_i$, for $i = 1, \ldots, k$, so that $m_1 = \overline{m}_1 \oplus \ldots \oplus \overline{m}_k$. Since $(m'_1, m'_2) \in R^{\oplus}$, we have that there exist $p''_1, \ldots, p''_k$ (where p''_i can also be θ) such that $m'_2 = p''_1 \oplus \ldots \oplus p''_k$ and $(p'_i, p''_i) \in R$, for $i = 1, \ldots, k$. Therefore, induction can be applied to (p'_i, p''_i) and $p'_i \rightrightarrows \overline{m}_i$, to conclude that there exists $\underline{m}_i$ such that $p''_i \rightrightarrows \underline{m}_i$ with $(\overline{m}_i, \underline{m}_i) \in R^{\oplus}$. Summing up, if $s_1 \xrightarrow{\tau} m'_1 \xRightarrow{\varepsilon} m_1$, then $s_2 \rightrightarrows m_2$, where $m_2 = \underline{m}_1 \oplus \ldots \oplus \underline{m}_k$, so that $(m_1, m_2) \in R^{\oplus}$, as required.

In the latter case, as $|o(s_1)| = |o(m)|$ and $|o(m'_1)| = |o(m'_2)|$, we can conclude that also $m \xrightarrow{\tau} m'_2$ is due to a τ-h-sequential transition because $|o(m)| = |o(m'_2)|$ by transitivity, so that we can write $s_2 \rightrightarrows m'_2$. Moreover, by an argument similar to the above, we can assume that $m'_1 = p'_1 \oplus \ldots \oplus p'_k$ (where p'_i can also be θ) and that the path $m'_1 \xRightarrow{\varepsilon} m_1$ can be decomposed in $p'_i \rightrightarrows \overline{m}_i$, for $i = 1, \ldots, k$, so that $m_1 = \overline{m}_1 \oplus \ldots \oplus \overline{m}_k$. Since $(m'_1, m'_2) \in R^{\oplus}$, we have that there exist $p''_1, \ldots, p''_k$ (where p''_i can also be θ) such that $m'_2 = p''_1 \oplus \ldots \oplus p''_k$ and $(p'_i, p''_i) \in R$, for $i = 1, \ldots, k$. Therefore, induction can be applied to (p'_i, p''_i) and $p'_i \rightrightarrows \overline{m}_i$, to conclude that there exists $\underline{m}_i$ such that $p''_i \rightrightarrows \underline{m}_i$ with $(\overline{m}_i, \underline{m}_i) \in R^{\oplus}$. Summing up, if $s_1 \xrightarrow{\tau} m'_1 \xRightarrow{\varepsilon} m_1$, then $s_2 \rightrightarrows m_2$, where $m_2 = \underline{m}_1 \oplus \ldots \oplus \underline{m}_k$, so that $(m_1, m_2) \in R^{\oplus}$, as required. □

Proposition 7.44. For each BPP net $N = (S, A, T)$ with silent moves, the following hold:

1. the identity relation $\mathscr{I} = \{(p, p) \mid p \in S \cup \{\theta\}\}$ is a branching h-team bisimulation;
2. the inverse relation $R^{-1} = \{(p', p) \mid (p, p') \in R\}$ of a branching h-team bisimulation R is a branching h-team bisimulation;
3. the relational composition $R_1 \circ R_2 = \{(p, p'') \mid \exists p'.(p, p') \in R_1 \wedge (p', p'') \in R_2\}$ of two branching h-team bisimulations R_1 and R_2 is a branching h-team bisimulation;
4. the union $\bigcup_{i \in I} R_i$ of branching h-team bisimulations R_i is a branching h-team bisimulation.

Proof:
The proofs of (1), (2) and (4) are immediate. The proof of (3) is less immediate, but not too difficult, thanks to Lemma 7.43. Given a pair $(p_1, p_3) \in R_1 \circ R_2$, there exists a p_2 such that $(p_1, p_2) \in R_1$ and $(p_2, p_3) \in R_2$. If $p_1 \xrightarrow{\ell} m_1$, since $(p_1, p_2) \in R_1$, it follows that
either $\ell = \tau$ and $\exists m_2$ such that $p_2 \rightrightarrows m_2$ with $(p_1, m_2) \in R_1^{\oplus}$ and $(m_1, m_2) \in R_1^{\oplus}$,
or $\exists m, m_2$ such that $p_2 \rightrightarrows m \xrightarrow{\ell} m_2$ with $(p_1, m) \in R_1^{\oplus}$ and $(m_1, m_2) \in R_1^{\oplus}$.

In the former case, since $(p_2, p_3) \in R_2$ and $p_2 \rightrightarrows m_2$, by Lemma 7.43, there exists m_3 such that $p_3 \rightrightarrows m_3$ with $(m_2, m_3) \in R_2^{\oplus}$; in such a case, we have that, to transition $p_1 \xrightarrow{\tau} m_1$, p_3 replies with

$p_3 \rightrightarrows m_3$ such that $(p_1, m_3) \in R_1^{\oplus} \circ R_2^{\oplus} = (R_1 \circ R_2)^{\oplus}$ by Proposition 5.4(4), and $(m_1, m_3) \in R_1^{\oplus} \circ R_2^{\oplus} = (R_1 \circ R_2)^{\oplus}$, as required.

In the latter case, since $(p_2, p_3) \in R_2$ and $p_2 \rightrightarrows m$, by Lemma 7.43, there exists m' such that $p_3 \rightrightarrows m'$ with $(m, m') \in R_2^{\oplus}$. Since $(p_1, m) \in R_1^{\oplus}$, it follows that $m = s_2 \oplus \overline{m}$ with $(p_1, s_2) \in R_1$ and $(\theta, \overline{m}) \in R_1^{\oplus}$. Since $(m, m') \in R_2^{\oplus}$, it follows that $m' = s_3 \oplus \overline{m}'$ with with $(s_2, s_3) \in R_2$ and $(\overline{m}, \overline{m}') \in R_2^{\oplus}$. Now, since $m \xrightarrow{\ell} m_2$, this step must be due to transition $s_2 \xrightarrow{\ell} \overline{m}_2$, so that $m_2 = \overline{m} \oplus \overline{m}_2$. Since $(s_2, s_3) \in R_2$, to transition $s_2 \xrightarrow{\ell} \overline{m}_2$, s_3 may respond in two ways:
either $\ell = \tau$ and $\exists m''$ such that $s_3 \rightrightarrows m''$ with $(s_2, m'') \in R_2^{\oplus}$ and $(\overline{m}_2, m'') \in R_2^{\oplus}$,
or $\exists m'', \overline{m}_3$ such that $s_3 \rightrightarrows m'' \xrightarrow{\ell} \overline{m}_3$ with $(s_2, m'') \in R_2^{\oplus}$ and $(\overline{m}_2, \overline{m}_3) \in R_2^{\oplus}$.

In the former subcase, to transition $p_1 \xrightarrow{\tau} m_1$, p_3 replies with $p_3 \rightrightarrows m'' \oplus \overline{m}'$ such that $(p_1, m'' \oplus \overline{m}') \in R_1^{\oplus} \circ R_2^{\oplus} = (R_1 \circ R_2)^{\oplus}$ and $(m_1, m'' \oplus \overline{m}') \in R_1^{\oplus} \circ R_2^{\oplus} = (R_1 \circ R_2)^{\oplus}$ as required. In fact, $(p_1, m'' \oplus \overline{m}') \in R_1^{\oplus} \circ R_2^{\oplus}$ follows by the fact that $(p_1, s_2 \oplus \overline{m}) \in R_1^{\oplus}$ and, moreover, as $(s_2, m'') \in R_2^{\oplus}$ and $(\overline{m}, \overline{m}') \in R_2^{\oplus}$, by the fact that $(s_2 \oplus \overline{m}, m'' \oplus \overline{m}') \in R_2^{\oplus}$ by additivity. Similarly, $(m_1, m'' \oplus \overline{m}') \in R_1^{\oplus} \circ R_2^{\oplus}$ follows by the fact that $(m_1, m_2) \in R_1^{\oplus}$, $m_2 = \overline{m}_2 \oplus \overline{m}$, $(\overline{m}_2, m'') \in R_2^{\oplus}$ and $(\overline{m}, \overline{m}') \in R_2^{\oplus}$.

In the latter subcase, to $p_1 \xrightarrow{\ell} m_1$, p_3 replies with $p_3 \rightrightarrows m'' \oplus \overline{m}' \xrightarrow{\ell} \overline{m}_3 \oplus \overline{m}'$ such that $(p_1, m'' \oplus \overline{m}') \in (R_1)^{\oplus} \circ (R_2)^{\oplus} = (R_1 \circ R_2)^{\oplus}$ and $(m_1, \overline{m}_3 \oplus \overline{m}') \in (R_1)^{\oplus} \circ (R_2)^{\oplus} = (R_1 \circ R_2)^{\oplus}$ as required. In fact, $(m_1, \overline{m}_3 \oplus \overline{m}') \in (R_1)^{\oplus} \circ (R_2)^{\oplus}$ follows by the fact that $(m_1, m_2) \in R_1^{\oplus}$, $m_2 = \overline{m}_2 \oplus \overline{m}$, $(\overline{m}_2, \overline{m}_3) \in R_2^{\oplus}$ and $(\overline{m}, \overline{m}') \in R_2^{\oplus}$.

The case when p_3 moves first is symmetric, and so omitted. Hence, $R_1 \circ R_2$ is a branching h-team bisimulation, too. □

Remember that $p \approx_{bh} p'$ if there exists a branching h-team bisimulation containing the pair (p, p'). This means that $\approx_{bh}$ is the union of all branching h-team bisimulations, i.e.,

$$\approx_{bh} = \bigcup \{R \subseteq (S \cup \{\theta\}) \times (S \cup \{\theta\}) \mid R \text{ is a branching h-team bisimulation}\}.$$

By Proposition 7.44(4), $\approx_{bh}$ is also a branching h-team bisimulation, hence the largest such relation. Another direct consequence of Proposition 7.44 is that $\approx_{bh}$ is an equivalence relation.

Remark 7.45. (Complexity of $\approx_{bh}$) From a complexity point of view, branching h-team bisimilarity $\approx_{bh}$ is not harder than $\approx_{br}$ (cf. Remark 7.14). The partition refinement algorithm checking $\approx_{br}$ (by extending the LTS algorithm in [13] to BPP nets) can be adapted to consider an initial partition of $S \cup \{\theta\}$ in two blocks: one composed of $o(S)$ and the other with $\{\theta\} \cup (S \setminus o(S))$. Hence, also in this case the time complexity is essentially $O(l + n^2 \cdot m)$, where n is the number of places, m the number of transitions and l the number of labels. □

Once branching h-team bisimilarity $\approx_{bh}$ over the places of an unmarked BPP net has been computed, checking whether two markings are branching h-team bisimulation equivalent $\approx_{bh}^{\oplus}$ can be done in $O(n)$ time. Of course, $\approx_{bh}^{\oplus}$ is coarser than $\approx_{br}^{\oplus}$, while it is finer than $\approx_{h}^{\oplus}$.

The following theorem provides a characterization of branching h-team equivalence as a suitable bisimulation-like relation over markings.

Theorem 7.46. Let $N = (S, A, T)$ be a BPP net with silent moves. Two markings m_1 and m_2 are branching h-team bisimilar, $m_1 \approx_{bh}^{\oplus} m_2$, if and only if

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$,
  - either $l(t_1) = \tau$ and
    - (i) either $\exists \sigma_2$ nonempty and τ-h-sequential, such that $^\bullet t_1 \approx_{bh} {}^\bullet\sigma_2$, $o(\sigma_2) = \varepsilon$, $t_1^\bullet \approx_{bh}^{\oplus} \sigma_2^\bullet$, $^\bullet t_1 \approx_{bh}^{\oplus} \sigma_2^\bullet$, $m_2[\sigma_2\rangle m_2'$ with $m_1 \approx_{bh}^{\oplus} m_2'$ and $m_1' \approx_{bh}^{\oplus} m_2'$,
    - (ii) or $\exists p_2 \in m_2$ such that $^\bullet t_1 \approx_{bh} p_2$, $t_1^\bullet \approx_{bh}^{\oplus} p_2$, with $m_1' \approx_{bh}^{\oplus} m_2$,
  - or $\exists \sigma, t_2$ such that σt_2 is sequential, σ is τ-h-sequential, $o(\sigma) = \varepsilon$, $l(t_1) = l(t_2)$, $^\bullet t_1 \approx_{bh} {}^\bullet\sigma t_2$, $^\bullet t_1 \approx_{bh}^{\oplus} \sigma^\bullet$, $^\bullet t_1 \approx_{bh} {}^\bullet t_2$, $t_1^\bullet \approx_{bh}^{\oplus} t_2^\bullet$, $t_1^\bullet \approx_{bh}^{\oplus} \sigma t_2^\bullet$, $m_2[\sigma\rangle m[t_2\rangle m_2'$ with $m_1 \approx_{bh}^{\oplus} m$ and $m_1' \approx_{bh}^{\oplus} m_2'$;
- and, symmetrically, if m_2 moves first.

Proof:
Similar to the proof of Theorem 7.17 and so omitted. □

A consequence of this theorem is that branching h-team bisimilarity is finer than branching fully-concurrent bisimilarity.

Theorem 7.47. (Branching h-team bisimilarity is finer than branching fully-concurrent bisimilarity) Let $N = (S, A, T)$ be a BPP net with silent moves. If $m_1 \approx_{bh}^{\oplus} m_2$, then $m_1 \approx_{bfc} m_2$.

Proof:
Similar to the proof of Theorem 7.20 and so omitted. □

However, the reverse implication of Theorem 7.47 does not hold: if $m_1 \approx_{bfc} m_2$, then it may happen that $m_1 \not\approx_{bh}^{\oplus} m_2$. The same example discussed in Example 7.21 applies also to this case. Nonetheless, if the BPP net is τ-h-sequential, we conjecture that also the reverse implication holds, so that the two equivalences coincide for τ-h-sequential BPP nets.

Corollary 7.48. (Branching h-team bisimilarity is finer than branching interleaving bisimilarity) Let $N = (S, A, T)$ be a BPP net with silent moves. If $m_1 \approx_{bh}^{\oplus} m_2$, then $m_1 \approx_{bri} m_2$.

Proof:
It follows from Theorem 7.47 and Proposition 4.31. □

The implication above is strict. Consider again the nets in Figure 7: of course, $s_1 \approx_{bri} s_4 \oplus s_5$, however $s_1 \not\approx_{bh}^{\oplus} s_4 \oplus s_5$.

A BPP net can be minimized w.r.t. branching h-team bisimilarity, too. The bh-reduced net w.r.t. $\approx_{bh}$, which minimizes the number of places and the number of transitions, can be defined as follows.

Definition 7.49. (Bh-reduced net) Let $N = (S, A, T)$ be a BPP net and let $\approx_{bh}$ be the branching h-team bisimulation equivalence relation over its places. The *bh-reduced* net $N_{bh} = (S_{bh}, A, T_{bh})$ is defined as follows:

- $S_{bh} = \{[s] \mid s \in o(S)\}$, where $[s] = \{s' \in o(S) \mid s \approx_{bh} s'\}$;
- $T_{bh} = \{([s], \ell, [o(m)]) \mid (s, \ell, m) \in T, \ell \neq \tau\} \cup \{([s], \tau, [o(m)]) \mid (s, \tau, m) \in T, [s] \neq [o(m)]\}$,

where $[m]$ is defined as follows: $[\theta] = \theta$ and $[m_1 \oplus m_2] = [m_1] \oplus [m_2]$. If the net N has initial marking m_0, then N_{bh} has initial marking $[o(m_0)]$. □

Given a BPP net $N = (S, A, T)$ with silent moves, it is an easy exercise to prove that relation $R = \{(s, [s]) \mid s \in o(S)\}$ is a branching team bisimulation between the net $o(N) = (o(S), A, o(T))$, where $o(T) = \{(s, \ell, o(m)) \mid s \in o(S), (s, \ell, m) \in T\}$, and the net N_{bh}. Indeed, the bh-reduced net N_{bh} of N w.r.t. $\approx_{bh}$ is isomorphic to the reduced net $o(N)_{\approx_{br}}$ of $o(N)$ w.r.t. $\approx_{br}$ (cf. Definition 7.23).

Of course, also branching h-team bisimilarity $\approx_{bh}$ is not a congruence for the choice operator of BPP, so that it is necessary to define a slight strengthening of this equivalence.

Definition 7.50. (Rooted branching h-team bisimilarity on places) Let $N = (S, A, T)$ be a BPP net. p_1 and p_2 are *rooted branching h-team bisimilar*, denoted $p_1 \approx_{bhc} p_2$, if for all $\ell \in A$

- $\forall m_1$ such that $s_1 \xrightarrow{\ell} m_1$, there exists m_2 such that $s_2 \xrightarrow{\ell} m_2$ and $m_1 \approx_{bh}^{\oplus} m_2$,
- $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$, there exists m_1 such that $s_1 \xrightarrow{\ell} m_1$ and $m_1 \approx_{bh}^{\oplus} m_2$. □

Of course, not only $\approx_{bhc} \subseteq \approx_{bh}$, but also $\approx_{bhc} \subseteq \approx_{hc}$, and even $\approx_{brc} \subseteq \approx_{bhc}$, and these inclusions are preserved by additive closure (by Proposition 5.3), so that $\approx_{bhc}^{\oplus} \subseteq \approx_{bh}^{\oplus}$, $\approx_{bhc}^{\oplus} \subseteq \approx_{hc}^{\oplus}$ as well as $\approx_{brc}^{\oplus} \subseteq \approx_{bhc}^{\oplus}$.

The following theorem provides a characterization of rooted branching h-team bisimilarity as a suitable bisimulation-like relation over markings, i.e., over a global model of the overall behavior.

Theorem 7.51. Let $N = (S, A, T)$ be a BPP net. If two markings m_1 and m_2 are rooted branching h-team bisimulation equivalent, $m_1 \approx_{bhc}^{\oplus} m_2$, then

1. $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that ${}^\bullet t_1 \approx_{bhc} {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \approx_{bh}^{\oplus} t_2^\bullet$, $m_2[t_2\rangle m_2'$ and $m_1' \approx_{bh}^{\oplus} m_2'$,
2. $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that ${}^\bullet t_1 \approx_{bhc} {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \approx_{bh}^{\oplus} t_2^\bullet$, $m_1[t_1\rangle m_1'$ and $m_1' \approx_{bh}^{\oplus} m_2'$.

Proof:
Similar to the proof of Theorem 7.36 and so omitted. □

Corollary 7.52. (**Rooted branching h-team bisimilarity implies rooted branching interleaving bisimilarity**) Let $N = (S, A, T)$ be a BPP net. If $m_1 \approx_{bhc}^{\oplus} m_2$, then $m_1 \approx_{bri}^{c} m_2$.

Proof:
We want to prove that if $m_1 \approx_{bhc}^{\oplus} m_2$, then

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ s.t. $m_2[t_2\rangle m_2'$ with $l(t_1) = l(t_2)$ and $m_1' \approx_{bri} m_2'$,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ s.t. $m_1[t_1\rangle m_1'$ with $l(t_1) = l(t_2)$ and $m_1' \approx_{bri} m_2'$,

so that $m_1 \approx_{bri}^{c} m_2$ follows directly by Definition 3.11. However, this implication is obvious, due to Theorem 7.51 and Corollary 7.48. □

8. Conclusion, related literature and future research

The ten team-style bisimulation-based behavioral equivalences proposed in this paper are truly concurrent equivalences which seem the most natural, intuitive and simple extension to BPP nets with silent moves of the corresponding interleaving bisimulation-based behavioral equivalences on LTSs. Each of these equivalences has rather low complexity, actually much lower than the corresponding interleaving behavioral equivalence for BPP nets. For instance, (strong) interleaving bisimilarity $\sim_{int}$ is PSPACE-complete [22] on BPP nets, while (strong) (h-)team bisimilarity $\sim^{\oplus}$ can be checked in polynomial time. Moreover, they have a clear correspondence with causality-based semantics for BPP nets:

- Strong team bisimilarity $\sim^{\oplus}$ coincides with state-sensitive strong fully-concurrent bisimilarity $\sim_{sfc}$ (Theorem 5.18).
- Strong h-team bisimilarity $\sim_h^{\oplus}$ coincides with strong fully-concurrent bisimilarity $\sim_{fc}$ (Theorem 5.27);
- Weak team bisimilarity $\approx^{\oplus}$ implies state-sensitive weak fully-concurrent bisimilarity $\approx_{sfc}$ for τ-sequential BPP nets (Theorem 6.19). We also conjecture that the converse implication holds for τ-sequential BPP nets.
- Weak h-team bisimilarity $\approx_h^{\oplus}$ implies weak fully-concurrent bisimilarity $\approx_{fc}$ for τ-h-sequential BPP nets (Theorem 6.39). We also conjecture that the reverse implication holds for τ-h-sequential BPP nets.
- Branching team bisimilarity $\approx_{br}^{\oplus}$ implies state-sensitive branching fully-concurrent bisimilarity $\approx_{sbfc}$ for BPP nets (Theorem 7.20). We also conjecture that the reverse implication holds for τ-sequential BPP nets.
- Branching h-team bisimilarity $\approx_{bh}^{\oplus}$ implies branching fully-concurrent bisimilarity $\approx_{bfc}$ for BPP nets (Theorem 7.47). We also conjecture that the reverse implication holds for τ-h-sequential BPP nets.

From a technical point of view, these very simple (and decidable in polynomial time) team-style bisimulation-based behavioral equivalences seem a sort of *egg of Columbus*: a simple (actually, a bit surprising in its simplicity) solution for a presumedly very hard problem.

Our definitions of the causal semantics for BPP nets, described in Section 4, were inspired by previous work on *fully-concurrent bisimilarity* [3], *history-preserving bisimilarity* [10, 33] and *structure-preserving bisimilarity* [11].

The research outlined in Sections 6 and 7 is a generalization of the our previous work [18], where we approached the problem of defining *weak* team bisimilarity and *branching* team bisimilarity for the simpler Petri net class of finite-state machines (nets whose transitions have singleton pre-set and singleton or empty post-set). There we also showed that rooted weak team bisimilarity $\approx_c^{\oplus}$ and rooted branching team bisimilarity $\approx_{brc}^{\oplus}$ are congruences for the operators of CFM [16], a process algebra that can represent all the finite-state machines, up to net isomorphism, and, moreover, we provide a sound and complete, finite axiomatization of these congruences. Similarly, in [20] we provided a sound and complete, finite axiomatization of strong h-team bisimilarity $\sim_h^{\oplus}$ for BPP.

As a future work, it would be interesting to investigate a (possibly finite) axiomatization of rooted weak (h-)team bisimilarity and rooted branching (h-)team bisimilarity over the process algebra BPP. These axiomatizations might be obtained by adding, to the sound and complete set of axioms of rooted weak/branching bisimilarities for finite-state CCS [26, 27, 12, 15], the expected axioms for the parallel operator, stating that it is associative, commutative, with $\mathbf{0}$ as its identity. We conjecture that the same set of axioms in [18, 20] can be slightly adapted also for BPP.

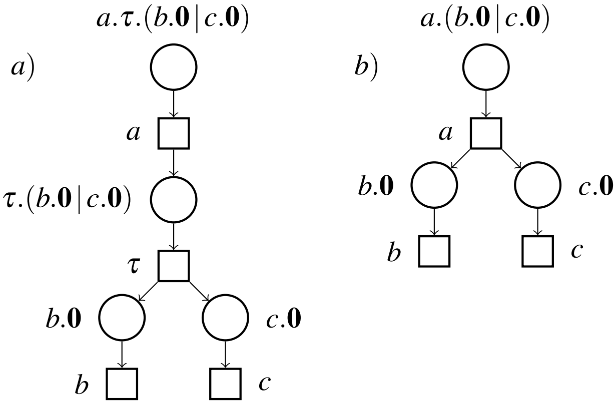


Figure 16. The first τ-law is invalid for non-τ-sequential BPP nets

However, note that these axiomatizations would be sound only for τ-*sequential* BPP, i.e., the fragment of BPP where a τ action can be used to prefix sequential terms only. For instance, the first τ-law $a.\tau.p \approx a.p$ is not sound if p is not a sequential term. In Figure 16 we show an instance of this law when $p = b.\mathbf{0}\,|\,c.\mathbf{0}$, where the nets are defined according to the semantics in [16]. If $a.\tau.(b.\mathbf{0}\,|\,c.\mathbf{0}) \xrightarrow{a} \tau.(b.\mathbf{0}\,|\,c.\mathbf{0})$, then $a.(b.\mathbf{0}\,|\,c.\mathbf{0})$ cannot reply because its only possibility is the move $a.(b.\mathbf{0}\,|\,c.\mathbf{0}) \stackrel{a}{\Longrightarrow} b.\mathbf{0} \oplus c.\mathbf{0}$, but then $\tau.(b.\mathbf{0}\,|\,c.\mathbf{0}) \not\approx^{\oplus} b.\mathbf{0} \oplus c.\mathbf{0}$, as the two markings have different size. Note that the net for $a.\tau.(b.\mathbf{0}\,|\,c.\mathbf{0})$ is weak fully-concurrent bisimilar to the net for $a.(b.\mathbf{0}\,|\,c.\mathbf{0})$; hence, this example shows that, for general (i.e., non-τ-h-sequential) BPP nets, weak fully-concurrent bisimilarity $\approx_{fc}$ does not imply weak h-team bisimilarity $\approx_h^{\oplus}$.

Another line of open problems is related to better investigate the algorithmics of the ten team-style behavioral equivalences we have discussed in this paper.

For instance, we have mentioned in Remark 5.11 that the classic Kanellakis-Smolka algorithm can be easily adapted to compute strong team bisimilarity on places $\sim$ in $O(m \cdot n^2)$, where n is the number of places and m the number of transitions. Further study is necessary to see whether the Paige-Tarjan-Valmari algorithm [30, 37] for computing bisimilarity on LTSs in $O(m \cdot log\, n)$, where n is the number of states and m of transitions, can be generalized to BPP nets in order to compute $\sim$.

Similarly, in Remark 7.14 we have mentioned that the Groote-Vaandrager algorithm [13] (based on a partition refinement algorithm in Kanellakis-Smolka style) can be easily adapted to compute branching team bisimilarity on places $\approx_{br}$ in $O(l+n^2 \cdot m)$, where n is the number of places, m the number of transitions and l the number of labels. Further study is necessary to see whether the more efficient algorithm in [14, 21] for branching bisimulation on LTSs, which runs in $O(m \cdot log\, n)$ (where n is the number of states and m of transitions), can be adapted to BPP nets in order to compute $\approx_{br}$.

Finally, further work is necessary to define a precise algorithm for computing, given a BPP net N, its saturated net N' (cf. Remark 6.12), in order to substantiate our claim that weak team bisimilarity on places $\approx$ can be computed in polynomial time. Moreover, further work is necessary to see whether the more efficient algorithm in [35] for computing weak bisimulation equivalence over a finite-state LTS (which does not require to explicitly compute the whole transitive closure beforehand and has $O(m \cdot n)$ time complexity, where n is the number of the states and m of the transitions) can be generalized to BPP nets in order to compute $\approx$.

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