Decidability of All Minimal Models

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Abstract. We consider a simply typed λ-calculus with constants of ground types, and assume that for one ground type o, there are finitely many constants of type o. We call *minimal model* the quotient by observational equivalence of the set of all closed terms whose type is of terminal subformula o. We show that this model is decidable: all classes of any given type are recursively representable, and observational equivalence on closed terms is a decidable relation. In particular, this result solves the question raised by R.Statman on the decidability of this model in the case of a unique ground type and two constants.

Observational equivalence on simply typed λ-terms is defined as the least binary relation $\|$ including β-equivalence and such that $t : A \to B \,\|\, t' : A \to B$ iff for all closed $u : A, u' : A$, if $u : A \,\|\, u' : A$ then $(t)u : B \,\|\, (t')u' : B$.

Suppose we are dealing with a simply typed λ-calculus with constants which are all of ground type, and assume for one ground type o there are finitely many constants of type o. Then, consider the set of all closed terms whose type is either o or of the form $A_1 \dots A_n \to o$. We call *minimal model* of simply typed λ-calculus the quotient of this set by observational equivalence.

We prove in this paper that this model is decidable *i.e* if we call *type* of a class the type of its elements then there are only finitely many classes of any given type, and there exists a computable function which, given an arbitrary type A, returns a set that contains a unique representative of each class of type A. We prove also that observational equivalence on closed terms is a decidable relation. In particular, this result solves the question raised by R.Statman on the decidability of this model in the case of a unique ground type and two constants. This question can be seen as a simplification of the *Lambda Definability* problem ([9], [10]) which is known to be undecidable ([5]).

The results presented here were found while studying the *Higher Order Matching* problem ([2], [6], [7], [8]) whose decidability is still open. As a corollary of the decidability of all minimal models, we prove also the decidability of a particular case of the higher order matching problem: the problem of solving finite sets of matching equations whose right-members are all constants of ground types.

1 Terms

We quote in this first part the definition of simply typed terms. The reader is assumed to be familiar with the notions of λ-term, α, β-conversions (see [4], [3] or [1]). We shall admit the well-known results of strong normalization and confluence of the β-reduction on simply typed terms.

1.1 Types

We let $\mathcal{F}$ be the set of all formulas of a language consisting of an arbitrary set of constants $\mathcal{O}$ and a binary connective $\to$:

$\mathcal{O} \subset \mathcal{F}$, and if $A, B \in \mathcal{F}$ then $(A \to B) \in \mathcal{F}$.

The formula $A = (A_1 \to (\ldots A_n \to o)\ldots)$ where $o \in \mathcal{O}$ will be denoted as $A_1 \ldots A_n \to o$. The constant o is called the *terminal subformula* (t.s.f.) of A. We call *order* of A the integer defined by:

if $n = 0$ then Ord $(A) = 1$, else $\text{Ord}(A) = \sup_{i=1}^{n}(\text{Ord}(A_i)) + 1$.

1.2 Simply Typed Terms

Assume that there is given:

- an infinite, countable set of variables, $x, y, z, \ldots$
- an infinite, countable set of constants, $a, b, c \ldots$
- an application from the set of all variables and constants to the set $\mathcal{F}$, mapping each symbol to a formula called its *type*, such that:
  - for each $A \in \mathcal{F}$, there exists an infinite number of variables of type A,
  - all constants are of ground type *i.e.* their types belong to $\mathcal{O}$.

We call *typed variables* all pairs of the form (x, A), written $x : A$, where A is the type of x, and *typed constants* all pairs of the form $a : o$ where o is the ground type of a. The set of *simply typed terms* is defined as the least set $\mathcal{S}$ satisfying:

0. all typed variables and all typed constants belong to $\mathcal{S}$,
1. if $t : B \in \mathcal{S}$ and if $x : A$ is a typed variable, then $\lambda x\, t : A \to B \in \mathcal{S}$,
2. if $t : A \to B, u : A \in \mathcal{S}$ then $(t)u : B \in \mathcal{S}$.

The *context* of a typed term $t : A$ is defined as the set of all free variables and constants of t. We call *order* of $t : A$ the order of A. A *closed* term contains no free variable (*e.g.* a typed constant $a : o$ is a closed term).

We denote as $\overline{\mathcal{S}}$ the quotient of $\mathcal{S}$ by α-equivalence *i.e.* renamings of bound variables by fresh variables of same type. By convention, elements of $\overline{\mathcal{S}}$ and $\mathcal{S}$ will be called *terms* and $\mathcal{S}$*-terms* respectively. Greek letters shall be used to denote arbitrary $\mathcal{S}$-terms. An $\mathcal{S}$-term τ of the α-class (the term) t will be called *a representative* of t.

2 Minimal Models

2.1 Observational Equivalence

Let t, t' be closed terms of type $A_1 \dots A_n \to \circ$. Let 0, 1 be distinct constants of type $\circ$. We say that t and t' are *observationally equivalent* if and only if for all closed $u_1 : A_1, \dots, u_n : A_n$ whose constants of type $\circ$ belong to $\{1, 0\}$, $(t)u_1 \dots u_n \equiv_\beta (t')u_1 \dots u_n$[1]. We write $\|$ the relation thus defined.

Lemma 1. *Let t, t' be closed terms of type $A_1 \dots A_n \to \circ$. Then $t \| t'$ if and only if for all closed $u_1 : A_1, \dots, u_n : A_n$, $(t)u_1 \dots u_n \equiv_\beta (t')u_1 \dots u_n$.*

Proof. Suppose $u_1, \dots, u_n$ closed and such that $(t)u_1 \dots u_n \,\beta\, a$, $(t')u_1 \dots u_n \,\beta\, a'$ with $a \neq a'$. Let 0, 1 be distinct constants of type $\circ$. Call v_i the term obtained by the substitution in u_i of 1 for a, 0 for all other constants. Then $(t)v_1 \dots v_n$ is of normal form a or 1 while $(t')v_1 \dots v_n$ is of normal form a' or 0. □

Lemma 2. *$t \| t'$ if and only if for all u of ground type such that $u[t/x]$, $u[t'/x]$ are closed and well-typed, $u[t/x] \equiv_\beta u[t'/x]$.*

Proof. Suppose $t \| t'$ and $u[t/x]$ closed of ground type. We prove $u[t/x] \equiv_\beta u[t'/x]$ by induction on the length of the left-normalization of $u[t/x]$. The only case where we apply the hypothesis on t, t' is $u[t/x] = (t)u_1[t/x] \dots u_n[t/x]$. If $t \,\beta\, t'$ or if $t = \lambda y\, t_0$ then by induction hypothesis $u[t/x] \equiv_\beta (t)u_1[t'/x] \dots u_n[t'/x] = u_0[t'/x]$. By lemma 22 and by hypothesis on t and t', $u_0[t'/x] \equiv_\beta u[t/x]$. □

We denote as $[t : A]$ the class of observational equivalence of $t : A$, letting $App([u : A \to B], [v : A]) = [(u)v : B]$. It follows from lemma 2 that this notion of application of a class to another is well-defined.

2.2 Minimal Models

Let $\mathcal{C}$ be a finite, non-empty set of constants of same ground type $\circ$. For all types A of t.s.f. $\circ$, we write $\mathcal{T}(A, \mathcal{C})$ the set of all closed terms of type A whose constants of type $\circ$ belong to $\mathcal{C}$. The quotient set $\mathcal{T}(A, \mathcal{C}) / \|$ is denoted as $\mathcal{M}(A, \mathcal{C})$.

The pair $\mathcal{M}_\mathcal{C} = (\{\mathcal{M}(A, \mathcal{C}) \mid t.s.f.(A) = \circ\}, App)$ will be called a *minimal model* of simply typed λ-calculus. The present paper intends to show that *all minimal models are decidable i.e.* we shall prove that:

1. for all pairs $(A, \mathcal{C})$, $\mathcal{M}(A, \mathcal{C})$ is a finite set,
2. there exists a computable function which, given a pair $(A, \mathcal{C})$, returns a complete set of $\|$-representatives for $\mathcal{M}(A, \mathcal{C})$ *i.e.* returns a set that contains a unique representative of each element of $\mathcal{M}(A, \mathcal{C})$
3. $\|$ is a decidable relation.

[1] $\equiv_\beta$ denotes β-equivalence *i.e.* the reflexive and transitive closure of β-reduction.

3 Pure Types

Our first aim is to prove that if all minimal models are decidable in the particular case of $\mathcal{O} = \{o\}$, then all minimal models are decidable in the case where $\mathcal{O}$ is an arbitrary set.

3.1 Pure Forms

Definition 3. We call *pure form* of a type $B = B_1 \dots B_p \to o$ the type A satisfying:

1. if all ground types appearing in B are equal to o then $A = B$,
2. if *t.s.f.*$(B_j) \neq o$ then A is equal to the pure form of $B_1 \dots B_{j-1} B_{j+1} \dots B_p \to o$.
3. if *t.s.f.*$(B_1) = \dots =$ *t.s.f.*$(B_p) = o$ then $A = A_1 \dots A_p \to o$ where A_j is the pure form of B_j.

Remark. If $B_1 \dots B_p \to o$ is of pure form $A_1 \dots A_n \to o$, then there exists a unique sequence $I = (i_1, \dots, i_n)$ such that $1 \leq i_1 < \dots < i_n \leq n$, B_{i_k} is of pure form A_k and $\forall j \notin I$, *t.s.f.*$(B_j) \neq o$. Consequently, every type has a unique pure form.

Definition 4. We define *Nil* as a set that contains, for each $o \in \mathcal{O}$, a unique constant *nil* of type o. For any type A, we denote as $\lambda.nil : A$ the unique term of type A of the form $\lambda x_1 \dots x_n\, nil$ where $nil \in Nil$.

Definition 5. Let $A = A_1 \dots A_n \to o$. Let $B = B_1 \dots B_p \to o$ be a type of pure form A. Let $I = (i_1, \dots, i_n)$ such that $1 \leq i_1 < \dots < i_n \leq n$ with B_{i_k} of pure form A_k and $\forall j \notin I$, *t.s.f.*$(B_j) \neq o$.

1. (a) for any typed variable $Y : B$, let $Y_B^A = \lambda x_1 \dots \lambda x_n (Y) v_1 \dots v_p : A$ with

   $$v_{i_k} = {x_k}_{A_k}^{B_{i_k}} \ (1 \leq k \leq n) \text{ and } v_j = \lambda.nil : B_j \ (j \notin I)$$

   (b) for any typed variable $X : A$, let $X_A^B = \lambda y_1 \dots \lambda y_p (X) w_1 \dots w_n : B$ with

   $$w_k = {y_{i_k}}_{B_{i_k}}^{A_k} \ (1 \leq k \leq n).$$

2. (a) for any closed $t : B$, let $t_B^A : A = Y_B^A[t/Y]$,

   (b) for any closed $u : A$, let $u_A^B : B = X_A^B[u/X]$.

Lemma 6. Let B be a type of pure form A.

1. *For all closed $t : B$, $t \parallel (t_B^A)_A^B$.*
2. *For all closed $u : B$, $u \parallel (u_A^B)_B^A$.*

Proof. Suppose A, B and I are defined as they are in definition 5. We prove (1) and (2) by induction on B. If $p = 0$ *i.e.* if $B = \circ$ then $A = \circ$ and for all closed $t : \circ$, $u : \circ$, $t = t^{\circ}_{\circ}$, and $u = u^{\circ}_{\circ}$. Suppose $p > 0$.

1. Let $t : B$, $v_1 : B_1, \ldots, v_p : B_p$ be closed terms. Let $\overline{v}_{i_k} = ({v_{i_k}}^{A_k}_{B_{i_k}})^{B_{i_k}}_{A_k}$ $(1 \leq k \leq n)$. Let $\overline{v}_j = \lambda.nil : B_j$ $(j \notin I)$.
   - by definition, $(t)\overline{v}_1 \ldots \overline{v}_p \equiv_\beta ((t^A_B)^B_A)v_1 \ldots v_p$,
   - by induction hypothesis, for each $k \in [1 \ldots n]$, $\overline{v}_{i_k} \parallel v_{i_k}$ therefore, for all M such that $(\lambda z M)v_{i_k} : \circ$ be closed and well typed, $(\lambda z\, M)v_{i_k} \equiv_\beta (\lambda z\, M)\overline{v}_{i_k}$,
   - if $j \notin I$ then $t.s.f.(B_j) \neq \circ$ therefore, for all M such that $(\lambda z\, M)v_j : \circ$ be closed and well-typed, z cannot be free in the normal form of M and $(\lambda z\, M)v_j \equiv_\beta (\lambda z\, M)\overline{v}_j$.

   Thus $(t)v_1 \ldots v_p \equiv_\beta (t)\overline{v}_1 \ldots \overline{v}_p \equiv_\beta ((t^A_B)^B_A)v_1 \ldots v_p$. Since $v_1, \ldots, v_p$ are arbitrary, $t \parallel (t^A_B)^B_A$.

2. Let $u : A$, $w_1 : A_1, \ldots, w_n : A_n$ be closed terms. Let $\overline{w}_k = ({w_k}^{B_{i_k}}_{A_k})^{A_k}_{B_{i_k}}$ $(1 \leq k \leq n)$.
   - by definition, $(u)\overline{w}_1 \ldots \overline{w}_n \equiv_\beta ((u^B_A)^A_B)w_1 \ldots w_n$,
   - by induction hypothesis, for each $k \in [1 \ldots n]$, $\overline{w}_k \parallel w_k$ therefore, for all M such that $(\lambda z M)v_{i_k} : \circ$ be closed and well typed, $(\lambda z\, M)w_k \equiv_\beta (\lambda z\, M)\overline{w}_k$.

   Thus $(u)w_1 \ldots w_n \equiv_\beta (u)\overline{w}_1 \ldots \overline{w}_n \equiv_\beta ((u^B_A)^A_B)w_1 \ldots w_n$. Since $w_1, \ldots, w_p$ are arbitrary, $u \parallel (u^B_A)^A_B$.

□

Lemma 7. *Call* $\parallel$-compatible *every function F satisfying $F(t) \parallel F(t') \Leftrightarrow t \parallel t'$. Let B be an arbitrary type. Let A be the pure form of B. Then, for every finite, non-empty set of constants $\mathcal{C}$ of type* t.s.f.(A):

1. $(u : A \mapsto u^B_A : B)$ *is a $\parallel$-compatible function from $\mathcal{T}(A, \mathcal{C})$ to $\mathcal{T}(B, \mathcal{C})$.*
2. $(t : B \mapsto t^A_B : A)$ *is a $\parallel$-compatible function from $\mathcal{T}(B, \mathcal{C})$ to $\mathcal{T}(A, \mathcal{C})$.*

Proof. For all typed variables $X : A$, $Y : B$, the constants of Y^A_B, X^B_A belong to $Nil - \{nil : \circ\}$ hence, if $t \in \mathcal{T}(B, \mathcal{C})$ then $t^A_B \in \mathcal{T}(A, \mathcal{C})$ and if $u \in \mathcal{T}(A, \mathcal{C})$ then $u^B_A \in \mathcal{T}(B, \mathcal{C})$. It follows from definition 5 that if $u : A \parallel u' : A$ then $u^B_A \parallel u'^B_A$. By lemma 6(2), if $u^B_A \parallel u'^B_A$ then $u \parallel u'$. Similarly, if $t : B \parallel t' : B$ then $t^A_B \parallel t'^A_B$. By lemma 6(1), if $t^A_B \parallel t'^A_B$ then $t \parallel t'$. □

3.2 Reduction to the Case of a Unique Ground Type

Lemma 8. If all minimal models are decidable in the particular case of $\mathcal{O} = \{o\}$, then all minimal models are decidable in the case where $\mathcal{O}$ is an arbitrary set.

Proof. Let $\mathcal{R}$ be any function such that for all pairs $(A, \mathcal{C})$ where A contains a unique ground type, $\mathcal{R}(A, \mathcal{C})$ is a complete set of $\|$-representatives for $\mathcal{M}(A, \mathcal{C})$. Call $\mathcal{R}^*$ the function defined as follows:

> Let B be any type of t.s.f. o, let $\mathcal{C}$ be any non-empty, finite set of constants of type o, let A be the pure form of B. Let $\mathcal{R}^*(B, \mathcal{C}) = \{u_A^B : B \mid u \in \mathcal{R}(A, \mathcal{C})\}$.

Note that if $\mathcal{R}$ is computable the $\mathcal{R}^*$ is also computable. For all pairs $(B, \mathcal{C})$ with B of pure form A and for all $t \in \mathcal{T}(B, \mathcal{C})$, $(t_B^A)_A^B \| t$ and there exists $u \in \mathcal{R}(A, \mathcal{C})$ such that $(t_B^A)_A^B \| u_A^B$, hence, $\mathcal{R}^*(B, \mathcal{C})$ is a complete set of $\|$-representatives for $\mathcal{M}(B, \mathcal{C})$. Furthermore, for all t, $t' \in \mathcal{T}(B, \mathcal{C})$, $t \| t'$ if and only if $t_B^A \| t'^A_B$. Hence, if the relation $\|$ is decidable on the set $\mathcal{T}(A, \mathcal{C})$, then this relation is also decidable on the set $\mathcal{T}(B, \mathcal{C})$. □

4 Atomic Interpolation

Through sections 4, 5, 6 and 7, the set $\mathcal{O}$ is assumed to be equal to $\{o\}$. We shall give now the definition of an *(atomic) interpolation problem* and prove that for every closed term t, there exists an interpolation problem Φ of which t is a solution, and such that every solution of Φ is observationally equivalent to t.

Definition 9. An *interpolation equation* E is defined as an equation of the form $[(x)u_1 \dots u_n = a]$ where $u_1, \dots, u_n$ are closed terms and a is a constant. A *solution* of E is a closed term t such that $(t)u_1 \dots u_n, \beta\, a$. Two interpolation equations are *equivalent* if and only if they have the same set of solutions.

Remark. $[(x)u_1 \dots u_n = a]$ and $[(x)u'_1 \dots u'_n = a]$ are equivalent $\Leftrightarrow$ for each i, $[(x_i)u_i = a]$ and $[(x_i)u'_i] = a$ are equivalent $\Leftrightarrow$ for each i, $u_i \| u'_i$.

Definition 10. An *interpolation problem* Φ is defined as a finite set of interpolation equations containing the same variable. A *solution* of Φ is a closed term which is a solution of each $E \in \Phi$.

Proposition 11. *Let $E = [(x)u_1 \dots u_n = a]$ be an interpolation equation. Let $0, 1$ be two distinct constants. Let $E^* = [(x)u_1^* \dots u_n^* = a^*] = E[1/0, 0/1]$. Then, for all closed t:*

1. *If t is a solution of $\{E, E^*\}$ then for any constant b, $t[b/0, b/1]$ is a solution of $\{E, E^*\}$.*

2. If 0, 1 are not free in t then t is a solution of $E \Leftrightarrow t$ is a solution of E^*.

Proof. 1. Suppose t is a solution of $\{E, E^*\}$. Let $t^* = t[2/0, 3/1]$ where $2, 3$ are two new constants. If, for instance, $(t^*)u_1 \dots u_n \,\beta\, 2$ then $(t^*)u^* \dots u^* \,\beta\, 2$ hence $(t)u_1 \dots u_n \,\beta\, 0$, $a = 0$, $(t)u^* \dots u^* \,\beta\, 0$ and $a^* = 0$, a contradiction. Similarly, the normal form of $(t^*)u_1 \dots u_n$ cannot be 3, and the normal form of $(t^*)u_1^* \dots u_n^*$ cannot be 2 nor 3. Consequently, for any constant b, $t[b/0, b/1] = t^*[b/2, b/3]$ is a solution of $\{E, E^*\}$.

2. Suppose 0, 1 are not free in t. Let $\sigma = [1/0, 0/1]$. Then $(t)u_1 \dots u_n \,\beta\, a$ iff $\sigma((t)u_1 \dots u_n) \,\beta\, \sigma(a)$ iff $(t)\sigma(u_1) \dots \sigma(u_n) \,\beta\, \sigma(a)$ iff $(t)u_1^* \dots u_n^* \,\beta\, a^*$. □

Lemma 12. *For any $(A, \mathcal{C})$, there exists a finite set P of interpolation problems satisfying:*

> *Let Φ be any element of P. If Φ is solvable, then there exists a solution of Φ that belongs to $\mathcal{T}(A, \mathcal{C})$.*
>
> *Let $\mathcal{R} \subset \mathcal{T}(A, \mathcal{C})$ be any set which contains a unique solution of each solvable problem in P. Then $\mathcal{R}$ is a complete set of $||$-representatives for $\mathcal{M}(A, \mathcal{C})$.*

Proof. By induction of the order of A. If $A = \circ$ and $\mathcal{C} = \{a_1, \dots, a_m\}$ then $P = \{\{[x = a_1]\}, \dots, \{[x = a_m]\}\}$. Suppose $A = A_1 \dots A_n \to \circ$. Let 0, 1 be two new constants. By induction hypothesis $\mathcal{M}_i(A_i, \{0, 1\})$ is finite. Let U_i be a complete set of $||$-representatives for this latter set. Let F be the set of all functions from $U_1 \times \dots \times U_n$ to $\mathcal{C} \cup \{0, 1\}$. Let P_0 be the set of all problems $\{[(x)u_1 \dots u_n = f(u_1, \dots, u_n)] \mid \forall i\; u_i \in U_i\}$ with $f \in F$. For all equations E, write E^* the equation $E[1/0, 0/1]$. We let P be the set of all problems $\Psi^* = \{E \mid E \in \Phi\} \cup \{E^* \mid E \in \Phi\}$ with $\Psi \in P_0$.

Let $\Psi^* \in P$. If Ψ^* is solved by a closed term t then: obviously, if $b \notin \mathcal{C} \cup \{0, 1\}$ then for any $a \in \mathcal{C}$, the term $t[a/b]$ is a solution of Ψ^* ; by proposition 11 (1) and by definition of Ψ^*, for any $a \in \mathcal{C}$, the term $t[a/1, a/0]$ is a solution of Ψ^*. Therefore, there exists a solution of Φ^* that belongs to $\mathcal{T}(A, \mathcal{C})$.

Furthermore, if $Q \in \mathcal{M}(A, \mathcal{C})$ then by definition of $||$, there exists a unique problem $\Psi \in P_0$ of which all elements of Q are solutions; by proposition 11 (2), Ψ^* is also the unique element of P of which all elements of Q are solutions. □

Remark. The preceding lemma proves also that for each pair $(A, \mathcal{C})$, the quotient set $\mathcal{M}(A, \mathcal{C}) = \mathcal{T}(A, \mathcal{C}) / \,||$ is a finite set.

5 Accessibility

The notion of *accessibility* of an address, in a simply typed term on η-long form, will be used extensively in section 6.

5.1 η-long Forms

Let $t = \lambda x_1 \dots x_m\,(u)v_1 \dots v_p : A_1 \dots A_n \to o$ where $m \leq n$ and u is either a variable, a constant or a term of the form $\lambda y\, w$. We call *η-long form* of t the unique term of same type of the form:

$\lambda x_1 \dots x_m x_{m+1} \dots x_n.(u^*)v_1^* \dots v_p^* x_{m+1}^* \dots x_n^*$, where:

- v_i^* is the η-long form of v_i,
- x_j^* is the η-long form of x_j,
- if u is a variable or a constant then $u^* = u$, else u^* is the η-long form of u.

In the remaining, all terms will be supposed to be on η-long form *i.e.* all terms will be assumed to belong to the least set $\overline{\mathcal{L}}$ satisfying:

0. all typed constants and all typed variables of ground type belong to $\overline{\mathcal{L}}$,
1. if $t : o \in \overline{\mathcal{L}}$ is of ground type and if $y_1 : A_1, \dots, y_n : A_n$ are typed variables then $\lambda y_1 \dots y_n.t : A_1 \dots A_n \to o \in \overline{\mathcal{L}}$,
2. if $t_1 : A_1, \dots, t_n : A_n \in \overline{\mathcal{L}}$ and if $x : A_1 \dots A_n \to o$ is a typed variable where o is a ground type, then $(x)\, t_1 \dots t_n : o \in \overline{\mathcal{L}}$,
3. if $t : A_1 \dots A_n \to o$, $u_1 : A_1, \dots, u_n : A_n \in \overline{\mathcal{L}}$ where o is a ground type then $(t)\, u_1 \dots u_n : o \in \overline{\mathcal{L}}$.

Remark. If $t : A \in \overline{S}$ then $t\,\beta\,u$ if and only if $t^*\,\beta\,u^*$, where t^*, u^* are the η-long forms of t, u respectively. Furthermore, if $v : A \in \overline{\mathcal{L}}$ and if $v\,\beta\,w$ then $w : A \in \overline{\mathcal{L}}$. Therefore, we may assume whitout loss of generality that the set of simply typed terms is restricted to $\overline{\mathcal{L}}$.

5.2 Addresses

Definition 13. Let L be the set of all lists of integers. We let $(\tau, \Delta) \mapsto \tau/\Delta$ be the least application from $\mathcal{L} \times L$ to the set of all $\mathcal{L}$-terms of type o, which satisfies:

- $\lambda.\mathcal{Y}.\varepsilon/\langle\rangle = \varepsilon$,
- if $\tau = \lambda\mathcal{Y}.(\varepsilon_0)\varepsilon_1 \dots \varepsilon_n$ where $n > 0$ and ε_0 is a variable, a constant or an element of $\mathcal{L}$ then $\forall\, \Delta \in L$,

  $$\tau/\langle i\rangle\Delta = \varepsilon_i/\Delta \quad \text{and} \quad \text{if } \varepsilon_0 \in \mathcal{L} \text{ then } \tau/\langle 0\rangle\Delta = \varepsilon_0/\Delta.$$

We call set of *addresses* in τ the set of all Δ such that τ/Δ is defined. We denote by $Sub(\tau, \Delta)$ the α-class of τ/Δ.

For any term t, we call set of *addresses* in t the set of addresses in any of its representatives and *depth* of t the maximal length of an address in t. We say that Δ is a *free occurrence* of the variable or constant z in t if and only if for every τ representative of t, τ/Δ is of the form $(z)\varepsilon_1 \ldots \varepsilon_n$. For any context Γ, we call *Γ-occurrences* in t all free occurrences of elements of Γ in t.

Definition 14. Let a be a constant. Let $\tau = \lambda\mathcal{Y}.(\varepsilon_0)\varepsilon_1 \ldots \varepsilon_n$. Let Δ be any address in τ. We call *pruning of τ by a at Δ* the $\mathcal{L}$-term $\tau(a/\Delta)$ defined by:

- $\tau(a/\langle\rangle) = \lambda\mathcal{Y}.a$,
- $\tau(a/\langle i\rangle\Delta)) = \lambda\mathcal{Y}.(\varepsilon'_0)\varepsilon'_1 \ldots \varepsilon'_n$ where
  $\varepsilon'_i = \varepsilon_i(a/\Delta)$ and $\varepsilon'_j = \varepsilon_j \quad (j \neq i)$.

Since $\tau \equiv_\alpha \tau'$ implies $\tau(a,\Delta) \equiv_\alpha \tau'(a,\Delta)$, we may define the *pruning of t by a at Δ* as $\tau(a/\Delta)$ where τ is an arbitrary representative of t.

For any set of constants $\mathcal{C}$, we call *$\mathcal{C}$-prunings* of t all terms obtained by successive prunings of t by elements of $\mathcal{C}$. A $\mathcal{C}$-pruning $\bar{t}$ is said to be *strict* if and only if at least one $\mathcal{C}$-occurrence in $\bar{t}$ is not a $\mathcal{C}$-occurrence in t.

5.3 Accessibility

Definition 15. An address Δ is said to be *β-accessible* in a term w iff:

> let $\overline{w}$ be the pruning of w at Δ by a constant a which does not appear in w. Then a appears in the normal form of $\overline{w}$.

Remark. If w is closed and if Δ is β-accessible in w, then for any a, the pruning of w by a at Δ is of normal form a.

6 Transferring Terms

We define now a class of closed terms of a simple structure, called *transferring terms.* The key-result presented in this section is the following: for every closed term t, there exists a transferring term observationally equivalent to t. As a corollary of this result, we will prove in section 7 the existence of an algorithm which takes as an input a pair $(A, \mathcal{C})$ and returns a set that contains a unique transferring representative of each element of $\mathcal{M}(A, \mathcal{C})$.

We give at first the definition of a transferring term. Next, we give the definition of an *approximation* of a solution of an interpolation problem. The links between these two definitions will be explained in section 6.2.

Definition 16. We say that a term $t : A$ is *transferring* if and only if t is closed, on normal form, and of the form:

1. $t = \lambda y_1 \dots y_n.a$ where a is a constant, or,
2. $t = \lambda y_1 \dots y_n.(y_i)v_1 \dots v_p[w'/0, w''/1]$ where:
   (a) 0, 1 are constants,
   (b) $v_1 \dots v_p$ are *closed* and their constants belong to $\{0, 1\}$
   (c) $\lambda y_1 \dots y_n.w'$ et $\lambda y_1 \dots y_n.w''$ are transferring.

Remark. If $t = \lambda \mathcal{Y}.w$ is transferring and if Δ is a $\mathcal{Y}$-occurrence in w then for every representative ε of w, all free variables of ε/Δ belong to $\mathcal{Y}$. Therefore, if t is a solution of $E = [(x)\, u_1 \dots u_n = a]$ and if $\langle 0 \rangle \Delta$ is β-accessible in $(t)u_1 \dots u_n$ then $(\lambda \mathcal{Y}.\varepsilon/\Delta))u_1 \dots u_n \, \beta \, a$.

6.1 Approximations

Definition 17. A *vector* is by definition a sequence $(t_1, \dots, t_m)$ of closed terms of same type, denoted by $< t_1, \dots, t_m >$. We call *type* of a vector the type of its elements. If $\overline{V} = (V_1, \dots, V_n) = (< u_i^1, \dots, u_i^m >)_{i=1}^n$, then:

- for $W =< a_1, \dots, a_m >$, $[(x)\overline{V} = W]$ denotes the interpolation problem $\{[(x)u_1^j \dots u_n^j = a_j] \mid j \in [1 \dots m]\}$,
- for any closed $t : A_1 \dots A_n \to o$ where A_i is the type of V_i, $[(x)\overline{V}][x \leftarrow t]$ denotes the normal form of $< (t)u_1^1 \dots u_n^1, \dots, (t)u_1^m \dots u_n^m >$.

Definition 18. Let $W =< a_1, \dots, a_m >$ be any vector of constants that contains at least two distincts elements. Let 0, 1 be two new constants. The set of *(0, 1)-approximations* of W is defined as the set of all elements of $\Pi_{j=1}^m \{a_j, 0, 1\}$ that contain at least two distinct constants.

We say that $< u_1, \dots, u_m >$ of type $A_1 \dots A_n \to o$ is *W-splitting* if and only if there exists in $\Pi_{j=1}^m \mathcal{T}(A_i, \{0, 1\})$ at least one $(v_1, \dots v_n)$ such that the normal form of $< (u_1)v_1 \dots v_p, \dots, (u_m)v_1 \dots v_p >$ is an approximation of W.

Definition 19. Let $\overline{V} = (< u_i^1, \dots, u_i^m >)_{i=1}^n$. Let t be any closed term such that $[(x)\overline{V}][x \leftarrow t]$ is defined. Let $w_j = (t)u_1^j \dots u_n^j$. An address Δ is said to be:

- *totally $\overline{V}$-accessible* in t iff for each j, $\langle 0 \rangle \Delta$ is β-accessible in w_j,
- *partially $\overline{V}$-accessible* in t iff Δ is not totally $\overline{V}$-accessible in t and there exists j such that $\langle 0 \rangle \Delta$ is β-accessible in w_j,
- *$\overline{V}$-inaccessible* otherwise.

Lemma 20. If $[(x)\overline{V}][x \leftarrow \lambda \mathcal{Y}.w]$ is a (0, 1)-approximation of W and if no strict $\{0, 1\}$-pruning of $\lambda \mathcal{Y}.w$ is a (0, 1)-approximation of W then: all partially $\overline{V}$-accessible addresses in $\lambda \mathcal{Y}.w$ are $\{0, 1\}$-occurrences; all constants of w belong to $\{0, 1\}$

Proof. Suppose $\overline{V} = (< u_i^1, ..., u_i^m >)_{i=1}^n$, $[(x)\overline{V}][x \leftarrow \lambda\mathcal{Y}.w] =< b_1, \ldots, b_m >$, $W =< a_1, \ldots, a_m >$. Let Δ be any partially $\overline{V}$-accessible address in $\lambda\mathcal{Y}.w$. Let j, k be such that $\langle 0\rangle\Delta$ be β-accessible in $(\lambda\mathcal{Y}.w)u_1^j \ldots u_n^j$ and be not β-accessible in $(\lambda\mathcal{Y}.w)u_1^k \ldots u_n^k$. Let $c = 1$ if $b_k \in \{0, a_k\}$, 0 otherwise. Let $\lambda\mathcal{Y}.\overline{w}$ be the c-pruning of $\lambda\mathcal{Y}.w$ at δ. Then $[(x)\overline{V}][x \leftarrow \lambda\mathcal{Y}.\overline{w}] =< c_1, \ldots c_m >$ where $\forall l\ c_l = \{b_l, 0, 1\}$, $c_k = b_k$ and $c_j = c$ with $c \neq c_k$. Hence, $< c_1, \ldots c_m >$ is still an approximation of W. By hypothesis. $\overline{w}$ is not a strict pruning of w, therefore Δ is a {0,1}-occurrence in w.

Since there exists at least one approximation of W, this vector contains at least two distinct constants. Therefore, all occurrences of a constant in $\lambda\mathcal{Y}.w$ are partially $\overline{V}$-accessible, and all constants of $\lambda\mathcal{Y}.w$ belong to $\{0, 1\}$. □

6.2 Existence of Transferring Representatives

We prove now that for every closed term t, there exists a transferring term observationally equivalent to t. If we assume that this property holds for all terms of depth at most $h - 1$, and consider a term t of depth h, then the next lemma proves that given an arbitrary interpolation problem Φ of which t is a solution, the problem Φ contains a splitting row; this latter property allows us to split Φ into two smaller interpolation problems Φ_0 and Φ_1, so that if we also assume the induction hypothesis that there exists transferring solutions of Φ_0 and Φ_1, then from these solutions and from this splitting row, one can build a transferring solution of Φ. The conclusion follows from the fact that for every t, there exists an interpolation problem such that every solution of this problem is observationally equivalent to t.

Lemma 21. *Let $t = \lambda y_1 \ldots y_n.(y_i)\lambda\mathcal{X}_1.v_1 \ldots \lambda\mathcal{X}_p.v_p = \lambda\mathcal{Y}.w$ where all $\lambda\mathcal{Y}\mathcal{X}_k.v_k$ are transferring. If t is a solution of $[(x)\overline{V} = W]$ where W contains at least two distinct constants, then at least one element of $\overline{V}$ is W-splitting.*

Proof. Let 0, 1 be two new constants. Let $\bar{t} = \lambda\mathcal{Y}.\overline{w}$ be the maximal $(0, 1)$-pruning of t such that $[(x)\overline{V}][x \leftarrow \bar{t}]$ is an approximation of W. By lemma 20, all constants of $\bar{t}$ belong to $\{0, 1\}$, and every $\mathcal{Y}$-occurrence in $\overline{w}$ is totally $\overline{V}$-accessible.

We shall prove by induction on the number P of $\mathcal{Y}$-occurrences in $\overline{w}$ that at least one element of $(V_1, \ldots, V_n) = \overline{V}$ is W-splitting. If $P = 1$ then V_i is splitting. Suppose $P > 1$. Let $\Delta = \langle k\rangle\Delta'$ be any $\mathcal{Y}$-occurrence in w of non-null length. As $\lambda\mathcal{Y}\mathcal{X}_k.v_k$ is transferring, $Sub(\tau, \Delta)$ is of the form $(y_j)w_1 \ldots w_q[w'/0, w''/1]$ where $w_1, \ldots, w_m$ are closed and $\lambda\mathcal{Y}\mathcal{X}_k.w'$, $\lambda\mathcal{Y}\mathcal{X}_k.w''$ are transferring.

1. Since Δ is totally $\overline{V}$-accessible, if $w', w'' \in \{0, 1\}$ then $[(x)\overline{V}][x \leftarrow \lambda\mathcal{Y}.(y_j)w_1...w_q[w'/0, w''/1] = [(x)\overline{V}][x \leftarrow \bar{t}]$ and thereby V_j is W-splitting.

2. Otherwise, for instance, $w' \notin \{0,1\}$. By hypothesis on $\bar{t}$, there exists at least one 0-occurrence Δ_0 in $(y_j)w_1 \dots w_q$ such that $\Delta\Delta_0$ is totally $\overline{V}$-accessible in $\bar{t}$. Thereby $\forall\, u \in V_j$, $(u)w_1 \dots w_q\, \beta\, 0$ hence $(u)w_1 \dots w_q[w'/0, w''/1]\, \beta\, w'$.

   Let $\bar{t}' = \lambda\mathcal{Y}.\overline{w}'$ be the normal term obtained by the substitution at δ in $\overline{w}$ of any element of V_j for the free occurrence of y_j. Then $[(x)\overline{V}][x \leftarrow \bar{t}] = [(x)V][x \leftarrow \bar{t}']$, $\overline{w}'$ contains $(P-1)$ $\mathcal{Y}$-occurrences and by induction hypothesis at least one element of $\overline{V}$ is W-splitting. □

Theorem 22. *For any closed t, there exists a transferring term $t' \parallel t$.*

Proof. We assume that t is on normal form and of type $A = A_1 \dots A_n \to o$. Let $\mathcal{C}$ be the set of all constants of t if this set is not empty, $\{nil\}$ where nil is an arbitrary constant otherwise. Let P be a problem satisfying for the pair $(A, \mathcal{C})$ the conditions of lemma 12. Let $car(t)$ be the element of P of which t is a solution.

We shall prove by induction on (H, m) where H is the depth of t, that for every $\Phi \subset car(t)$ of cardinal at most m, there exists a transferring solution of Φ. This result is clear if all right-members of Φ are equal to a unique constant (in particular, if $|\Phi| \leq 1$). Suppose $\Phi = [(x)V_1 \dots V_n = W] = \{E_1, \dots, E_m\}$ where W contains at least two distinct elements, with t of the form $\lambda\mathcal{Y}.(y_i)w_1 \dots w_p$.

By induction hypothesis on H, there exist transferring terms $\lambda\mathcal{Y}\, t_1, \dots, \lambda\mathcal{Y}\, t_p$ such that $\lambda\mathcal{Y}\, t_k \parallel \lambda\mathcal{Y}\, w_k$. By proposition 2, $\lambda\mathcal{Y}.(y_j)t_1 \dots t_p \parallel t$. By lemma 21, at least one element of $\overline{V} = \{V_1 \dots V_n\}$ is W-splitting.

Let $V_j =< u^1, \dots, u^m >$ be any W-splitting element of $\overline{V}$. Let 0, 1 be two new constants. Let $v_1 \dots v_p$ be closed terms whose constants belong to $\{0,1\}$ such that the normal form $< b_1, \dots, b_m >$ of $< (u^1)v_1 ... v_p, \dots, (u^m)v_1 ... v_p >$ is an approximation of W. Let Φ_0 be the problem which contains each E_j such that $\mathsf{D}_j = 0$, let Φ_1 be the problem which contains each E_j such that $\mathsf{D}_j = 1$. Since Φ_0 and Φ_1 contain at most $m-1$ equations, by induction hypothesis there exist $\lambda\mathcal{Y}.w'$ and $\lambda\mathcal{Y}.w''$ which are transferring solutions of Φ_0 and Φ_1 respectively. The term $\lambda\mathcal{Y}.(y_j)v_1 \dots v_p[w'/0, w''/1]$ is then a transferring solution of Φ. □

Remark. Theorem 22 is equivalent to the following property : every non-trivial, solvable interpolation problem contains at least one splitting row.

7 Computation of Transferring Representatives

It remains to prove, for every order N, the existence of a computable function which, given an arbitrary pair $(A, \mathcal{C})$ where A is of order at most N, returns a complete set of representatives for $\mathcal{M}(A, \mathcal{C})$. We prove the existence of such a function by giving constructive proofs of lemma 12 and theorem 22.

Lemma 23. *Let N be any integer. Let $\mathcal{C}$ be any finite, non-empty set of constants. Suppose that there exists a computable function $\mathcal{R}_{N-1}$ such that for all pairs $(B, \mathcal{C})$ where B is a type of order at most $N-1$, $\mathcal{R}_{N-1}(B, \mathcal{C})$ is a complete set of $\|$-representatives for $\mathcal{M}(B, \mathcal{C})$. Then,*

1. *There exists a computable function Car_N such that for all pairs $(A, \mathcal{C})$ where A is a type of order at most N, $Car_N(A, \mathcal{C})$ is a finite set of of interpolation problems satisfying:*

   each solvable problem in $Car_N(A, \mathcal{C})$ is solved by some element of $\mathcal{T}(A, \mathcal{C})$. Any set which contains, for each solvable problem $\Phi \in Car_N(A, \mathcal{C})$, a unique solution of Φ that belongs to $\mathcal{T}(A, \mathcal{C})$, is a complete set of $\|$-representatives for $\mathcal{M}(A, \mathcal{C})$.

2. *There exists a computable function which, given two terms t, $t' \in \mathcal{T}(A, \mathcal{C})$ where A is of order at most N, determines whether $t \parallel t'$ or not.*

Proof. For $A = A_1 \ldots A_n \to o$, the proof of (1) is similar to the proof of lemma 12 where $\mathcal{R}_{N-1}(B_i, \{0, 1\})$ replaces U_i ($i \in [1 \ldots n]$). The set $Car_N(A, \mathcal{C})$ is then defined as the resulting set P. The second part of the lemma follows from the fact that for all t, $t' \in \mathcal{T}(A, \mathcal{C})$, then $t \parallel t'$ if and only if there exists $Car_N(A, \mathcal{C})$ such that t and t' are solutions of Φ. □

Lemma 24. *Let N be any integer. Let $\mathcal{C}$ be any finite, non-empty set of constants. If:*

there exists a computable function $\mathcal{R}_{N-1}$ such that for all pairs $(B, \mathcal{C})$ where B is a type of order at most $N-1$, $\mathcal{R}_{N-1}(B, \mathcal{C})$ is a complete set of $\|$-representatives for $\mathcal{M}(B, \mathcal{C})$,

then

there exists a computable function $\mathcal{R}_N$ such that for all pairs $(A, \mathcal{C})$ where A is a type of order at most $N-1$, $\mathcal{R}_N(A, \mathcal{C})$ is a complete set of $\|$-representatives for $\mathcal{M}(A, \mathcal{C})$,

Proof. For $A = A_1 \ldots A_n \to o$, let M be the maximal cardinal of all problems in $Car_N(A, \mathcal{C})$. For each $m \in [1 \ldots M]$, let $\mathcal{T}_m$ be the set defined as follows:

1. $\mathcal{T}_1$ is the set of all terms of the form $\lambda y_1 \ldots y_n . a : A$ where $a \in \mathcal{C}$.
2. for $m > 1$, $\mathcal{T}_m$ is equal to the union of $\mathcal{T}_{m-1}$ and the set of all terms of the form $\lambda y_1 \ldots y_n . (y_j) v_1 \ldots v_p [w'/0, w''/1]$ where:

   (a) 0, 1 are two new constants,

   (b) for $B_j = D_1 \ldots D_p \to o$, $\quad v_k \in \mathcal{R}_{N-2}(D_k, \{0, 1\})$,

   (c) $\lambda y_1 \ldots y_n . w', \lambda y_1 \ldots y_n . w'' \in \mathcal{T}_{m-1}$

It follows from the proof of theorem 22 where we take $P = Car_N(A, \mathcal{C})$ and where for V_j of type $B_j = D_1 \ldots D_p \to \circ$, v_k is assumed to belong to $\mathcal{R}_{N-2}(D_k, \{0,1\})$, that every solvable element of $Car_N(B, \Gamma)$ has a solution in $\mathcal{T}_M$. Therefore, $\mathcal{R}_N(B, \Gamma)$ can be defined as a set that contains, for each $\Phi \in Car_N(B, \Gamma)$ solved by some element of $\mathcal{T}_M$, a unique element of $\mathcal{T}_M$ which is a solution of Φ. $\square$

8 Decidability of all Minimal Models

Theorem 25. All minimal models are decidable.

Proof. By lemma 8, it is sufficient to prove this result in the particular case of $\mathcal{O} = \{\circ\}$. If we let $\mathcal{R}_1(\circ, \mathcal{C}) = \mathcal{C}$ then from lemma 24 we infer the existence of a computable function which, given an arbitrary pair $(A, \mathcal{C})$ where A is a type and $\mathcal{C}$ is a finite, non-empty set of constants, returns a complete set of $||$-representatives for $\mathcal{M}(A, \mathcal{C})$. By lemma 23 (2), there exists also a computable function which determines, given two closed terms, whether these terms are observationally equivalent or not. $\square$

9 Decidability of Atomic Matching

Definition 26. An *atomic matching problem* Φ is defined as a finite set of equations of the form $[u = a]$ where a is a constant of ground type. Let $x_1 \ldots x_n$ be the set of all free variables of Φ. We call *solution* of Φ every sequence of closed terms $(t_1 \ldots t_n)$ such that t_i and x_i be of same type, and such that for every $[u = a] \in \Phi$, $u[t_1/x_1 \ldots t_n/x_n]$ be of normal form a.

Remark. If a is of type $\circ$ and x is of type A with *t.s.f.*$(A) = \diamond \neq \circ$ then for any constant $b : \diamond$, $[u = a]$ and $[u[\lambda x_1 \ldots x_n.b : A/x] = a]$ have same set of solutions. Thus, we may assume w.l.o.g that for every $[u = a] \in \Phi$, every free variable in u is of *t.s.f.* equal to the type of a. Consequently, we may also assume that all right-members of Φ are of same ground type, since Φ is solvable iff for each ground type $\circ$, $\Phi_\circ = \{[u = a] \in \Phi \mid a \text{ is of type } \circ\}$ is solvable.

Theorem 27. Atomic Matching is decidable

Proof. Let $\mathcal{C}$ be any finite, non-empty set of constants of type $\circ$. Let $x_1 : A_1, \ldots, x_n : A_n$ be such that *t.s.f.*$(A_i) = \circ$. By theorem 25, there exists a computable function that, given the pair $(A_i, \mathcal{C})$ returns a set $\mathcal{R}_i$ which is a complete set of $||$-representatives for $\mathcal{M}(A_i, \mathcal{C})$ $(1 \leq n \leq n)$.

Let Φ be any atomic matching problem of free variables $x_1 \ldots x_n$, such that all right-members of Φ belong to $\mathcal{C}$. Suppose $(w_1 \ldots w_n)$ is a solution of Φ. We may assume that for each i, every constant of type $\circ$ appearing in w_i belongs to

$\mathcal{C}$. Then for each i, there exists $t_i \in \mathcal{R}_i$ such that $w_i \parallel t_i$, and $(t_1 \ldots t_n)$ is still a solution of Φ. Thus, if Φ is solvable then $\mathcal{R}_1 \times \ldots \times \mathcal{R}_n$ contains at least one solution of Φ. □

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