PROOF THEORY OF RIESZ SPACES
AND MODAL RIESZ SPACES

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ABSTRACT. We design hypersequent calculus proof systems for the theories of Riesz spaces and modal Riesz spaces and prove the key theorems: soundness, completeness and cut–elimination. These are then used to obtain completely syntactic proofs of some interesting results concerning the two theories. Most notably, we prove a novel result: the theory of modal Riesz spaces is decidable. This work has applications in the field of logics of probabilistic programs since modal Riesz spaces provide the algebraic semantics of the Riesz modal logic underlying the probabilistic μ–calculus.

1. INTRODUCTION

Riesz spaces, also known as vector lattices, are real vector spaces equipped with a lattice order ($\leq$) such that the vector space operations of addition and scalar multiplication are compatible with the order in the following sense:

$$x \leq y \implies x + z \leq y + z \qquad x \leq y \implies rx \leq ry, \text{ for every positive scalar } r \in \mathbb{R}_{\geq 0}.$$

The simplest example of Riesz space is the linearly ordered set of real numbers $(\mathbb{R}, \leq)$ itself. More generally, for a given set X, the space of all functions $\mathbb{R}^X$ with operations and order defined pointwise is a Riesz space. If X carries some additional structure, such as a topology or a σ–algebra, then the spaces of continuous and measurable functions both constitute Riesz subspaces of $\mathbb{R}^X$. For this reason, the study of Riesz spaces originated at the intersection of functional analysis, algebra and measure theory and was pioneered in the 1930's by F. Riesz, G. Birkhoff, L. Kantorovich and H. Freudenthal among others. Today, the study of Riesz spaces constitutes a well–established field of research. We refer to [LZ71, JR77] as standard references.

The definition of Riesz spaces merges the notions of lattice order and that of real–vector space. The former is pervasive in logic and the latter is at the heart of probability theory (e.g., convex combinations, linearity of the expected value operator, *etc.*) Dexter Kozen was the first to observe in a series of seminal works (see, e.g., [Koz81, Koz85]) that, for the above reasons, the theory of Riesz spaces provides a convenient mathematical setting for the study and design of *probabilistic logics* which are formal languages conceived to express correctness properties of probabilistic transition systems (e.g., Markov chains, Markov decision processes, *etc.*) representing the formal semantics of computer programs using probabilistic operations such as random bit generation. In a series of recent works [Mio12, MS17, MFM17, Mio18, Mio14, FMM20], following Kozen's program, the second author has introduced a simple probabilistic modal logic called *Riesz Modal Logic.* Importantly, once extended with fixed–point operators in the style of the modal μ–calculus [Koz83], this logic is sufficiently expressive to interpret other popular probabilistic logics for verification such as *probabilistic CTL* (see, e.g., chapter 8 in [BK08] for an introduction to this logic). One key contribution from [MFM17, FMM20] is a duality theory which provides a bridge between the probabilisitic transition system semantics of the Riesz modal logic and its algebraic semantics given in terms of so–called *modal Riesz spaces.*

A *modal Riesz space* is a a structure $(V, \leq, \Diamond)$ such that $(V, \leq)$ is a Riesz space and $\Diamond$ is a unary operation $\Diamond : V \to V$ satisfying certain axioms (see Definition 2.16 for details). Terms without variables in the signature of modal Riesz spaces are exactly formulas of the Riesz modal logic of [MFM17, FMM20]. As a consequence of the duality theory, two formulas are equivalent in the transition semantics if and only if they are provably equal in the equational theory of modal Riesz spaces. This is a complete axiomatisation result (see [MFM17, FMM20] for details.)

One drawback of equational axiomatisations, such as that of [MFM17, FMM20], is that the underlying proof system of *equational logic* is not well–suited for proof–search. It is indeed often difficult to find proofs for even simple equalities. The source of this difficulty lies in the transitivity rules of equational logic:

$$\frac{A = B \qquad B = C}{A = C} \; Trans$$

For proving the equality $A = C$ it is sometimes necessary to come up with an additional term B and prove the two equalities $A = B$ and $B = C$. Since B ranges over all possible terms, the proof search endeavour faces an infinite branching in possibilities. It is therefore desirable to design alternative proof systems that are better behaved from the point of view of proof search, in the sense that the choices available during the proof–construction process are reduced to the bare minimum.

The mathematical field of *structural proof theory* (see [Bus98] for an overview), originated with the seminal work of Gentzen on his *sequent calculus* proof system LK for classical propositional logic [Gen34], investigates such proof systems. The key technical result regarding the sequent calculus, called the *cut–elimination theorem*, implies that when searching for a proof of a statement, only certain formulas need to be considered: the so–called *sub–formula property*. This simplifies significantly, in practice, the *proof search* endeavour.

The original system LK of Gentzen has been extensively investigated and generalised. For example, sequent–calculi for several substructural logics, linear logic, many modal logics and fixed–point temporal logics have been designed. One variant of sequent calculus, called *hyper–sequent calculus*, originally introduced by Avron in [Avr87] and independently

by Pottinger in [Pot83], allows for the manipulation of non–empty lists of sequents (hence the *hyper* adjective) rather than just sequents.

1.1. **First Contribution: Proof Theory of Riesz Spaces.**

The first contribution of this work is the design of a *hypersequent calculus* proof system **HR** for the theory of Riesz spaces, together with the proof of the cut–elimination theorem. From this we obtain new proofs, based on purely syntactic methods, of well–known results such as the fact that the equational theory of Riesz spaces is decidable and that the equational theory of Riesz spaces with real ($\mathbb{R}$) scalars is a conservative extension of the theory of Riesz spaces with rational ($\mathbb{Q}$) scalars. These results are presented in Section 3.

Our hypersequent calculus **HR** is based on, and extends, the hypersequent calculus **GA** for the theory of *lattice ordered Abelian groups* of [MOG05, MOG09]. From a technical point of view, the difficulty in extending their work to Riesz spaces lies mostly in the design of appropriate proof rules for dealing with real ($\mathbb{R}$) scalars. Our design choices have been driven by the two main goals: prove the cut–elimination theorem and preserve as much as possible the sub–formula property of the system. It is our belief that this type of rules might be of general interest in the field of proof theory and can be potentially re–used for designing proof systems for other quantitative logics.

Beside being the first structural proof system for Riesz spaces, a mathematically natural and well studied class of structures, the hypersequent calculus proof system **HR** is the basis for our second and most technically challenging contribution.

1.2. **Second Contribution: Proof Theory of Modal Riesz Spaces.**

The second and more technically challenging contribution of this work is the design of the *hypersequent calculus* proof system **HMR**, together with the proof of the cut–elimination theorem, for the theory of modal Riesz spaces or, equivalently (via the duality theory of [MFM17, FMM20]) for the Riesz modal logic. To the best of our knowledge, this is the first sound, complete and structural proof system for a probabilistic logic designed to express properties of probabilistic transition systems. From the cut–elimination theorem for **HMR** we derive a new result: the equational theory of modal Riesz spaces is decidable. Our proof is based on purely syntactical methods and does not rely, as it is often the case in decidability results of modal logics, on model theoretic properties (e.g., the finite model property) or on techniques from automata theory. These results are presented in Section 4.

The hypersequent calculus **HMR** is based and extends the hypersequent calculus **HR** with new rules dealing with the additional connectives ($\Diamond$ and 1) available in the signature of modal Riesz spaces. While this extension might superficially seem simple, it introduces significant complications in the proof of the cut–elimination theorem. The proof technique adopted in [MOG05, MOG09] for proving the cut–elimination theorem of **GA** does not seem to be applicable (as discussed in Subsection 4.1). We therefore follow a different approach based on a technical result (the *M–elimination* theorem) which states that one of the rules (M) of the system **HMR** can be safely removed from the system without affecting completeness. In order to simplify as much as possible the exposition of our cut–elimination proof for **HMR**, we prove the cut–elimination theorem for the system **HR** also using the technique based on the M–elimination theorem even though the cut–elimination theorem for **HR** could also be obtained following the approach of [MOG05, MOG09]. This will serve as a preparatory work for the more involved proof of cut–elimination for **HMR**.

1.3. **Organisation of this work.** This paper is structured in three main sections as follows:

Section 2 - Technical Background: in this section we give the basic definitions and results regarding Riesz spaces and modal Riesz spaces and fix some notational conventions.

Section 3 - Hypersequent Calculus for Riesz Spaces: this section is devoted to our hypersequent calculus **HR** proof system for the theory of Riesz spaces. This section is structured in several subsection, each presenting in details a result regarding **HR**.

Section 4 - Hypersequent Calculus for Modal Riesz Spaces: this section is devoted to our hypersequent calculus **HMR** proof system for the theory of modal Riesz spaces. The structure of this section matches exactly that of Section 3. This should allow for an easier comparison of the two systems and their technical differences.

2. TECHNICAL BACKGROUND

This section provides the necessary definitions and basic results regarding Riesz spaces (the articles [LZ71, JR77] are standard references) and modal Riesz spaces from [MFM17, FMM20], which play a key role in the the duality theory of the Riesz modal logic.

2.1. **Riesz Spaces.**

This section contains the basic definitions and results related to Riesz spaces. We refer to [LZ71] for a comprehensive reference to the subject.

A Riesz space is an algebraic structure $(R, 0, +, (r)_{r \in \mathbb{R}}, \sqcup, \sqcap)$ such that $(R, 0, +, (r)_{r \in \mathbb{R}})$ is a vector space over the reals, $(R, \sqcup, \sqcap)$ is a lattice and the induced order $(a \leq b \Leftrightarrow a \sqcap b = a)$ is compatible with addition and with the scalar multiplication, in the sense that: (i) for all $a, b, c \in R$, if $a \leq b$ then $a + c \leq b + c$, and (ii) if $a \geq b$ and $r \in \mathbb{R}_{\geq 0}$ is a non–negative real, then $ra \geq rb$. Formally we have:

Definition 2.1 (Riesz Space)**.** The *language* $\mathcal{L}_R$ of Riesz spaces is given by the (uncountable) signature $\{0, +, (r)_{r \in \mathbb{R}}, \sqcup, \sqcap\}$ where 0 is a constant, $+$, $\sqcup$ and $\sqcap$ are binary functions and r is a unary function, for all $r \in \mathbb{R}$. A *Riesz space* is a $\mathcal{L}_R$-algebra satisfying the set $\mathcal{A}_{\mathrm{Riesz}}$ of equational axioms of Figure 1. We use the standard abbreviations of $-x$ for $(-1)x$ and $x \leq y$ for $x \sqcap y = x$.

Remark 2.2. Note how the compatibility axioms have been equivalently formalised in Figure 1 as inequalities and not as implications by using $(x \sqcap y)$ and y as two general terms automatically satisfying the hypothesis $(x \sqcap y) \leq y$. Moreover the inequalities can be rewritten as equations using the lattice operations $(x \leq y \Leftrightarrow x \sqcap y = x)$ as follows:
- $(x \sqcap y) + z \leq y + z$ can be rewritten as $((x \sqcap y) + z) \sqcap (y + z) = (x \sqcap y) + z$ and
- $r(x \sqcap y) \leq ry$ can be rewritten as $r(x \sqcap y) \sqcap ry = r(x \sqcap y)$.

Since there is an equational axiomatization of Riesz spaces, the family of Riesz spaces is a variety in the sense of universal algebra.

(1) Axioms of real vector spaces:
- Additive group: $x + (y + z) = (x + y) + z$, $x + y = y + x$, $x + 0 = x$, $x - x = 0$,
- Axioms of scalar multiplication: $r_1(r_2 x) = (r_1 \cdot r_2)x$, $1x = x$, $r(x+y) = (rx)+(ry)$, $(r_1 + r_2)x = (r_1 x) + (r_2 x)$,

(2) Lattice axioms: (associativity) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$, $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$, (commutativity) $z \sqcup y = y \sqcup z$, $z \sqcap y = y \sqcap z$, (absorption) $z \sqcup (z \sqcap y) = z$, $z \sqcap (z \sqcup y) = z$.

(3) Compatibility axioms:
- $(x \sqcap y) + z \leq y + z$,
- $r(x \sqcap y) \leq ry$, for all scalars $r \geq 0$.

Figure 1: Set $\mathcal{A}_{\mathrm{Riesz}}$ of equational axioms of Riesz spaces.

Example 2.3. The real numbers $\mathbb{R}$ together with their standard linear order ($\leq$), expressed by taking $r_1 \sqcap r_2 = \min(r_1, r_2)$ and $r_1 \sqcup r_2 = \max(r_1, r_2)$, is a Riesz space. This is a fundamental example also due the following fact (see, e.g., [LvA07] for a proof): for any two terms A, B, we have that the equality $A = B$ holds in all Riesz spaces if and only if $A = B$ holds in the Riesz space $(\mathbb{R}, \leq)$. This provides a practical method for establishing if an equality is derivable from the axioms of Riesz spaces. For example, $-(\max(r_1, r_2)) = \min(-r_1, -r_2)$ holds universally in $\mathbb{R}$ and therefore $-(x \sqcup y) = (-x) \sqcap (-y)$ holds in all Riesz spaces.

Example 2.4. For a given set X, the set $\mathbb{R}^X$ of functions $f : X \to \mathbb{R}$ is a Riesz space when all operations are defined pointwise: $(rf)(x) = r(f(x))$, $(f + g)(x) = f(x) + g(x)$, $(f \sqcup g)(x) = f(x) \sqcup g(x)$, $(f \sqcap g)(x) = f(x) \sqcap g(x)$. Thus, for instance, the space of n-dimensional vectors $\mathbb{R}^n$ is a Riesz space whose lattice order is not linear.

Convention 2.5. We use the capital letters A, B, C to range over terms build from a set of variables ranged over by x, y, z. We write $A[B/x]$ for the term, defined as expected, obtained by substituting all occurrences of the variable x in the term A with the term B.

As observed in Remark 2.2, the family of Riesz spaces is a variety of algebras. This means, by Birkhoff completeness theorem, that two terms A and B are equivalent in all Riesz spaces if and only if the identity $A = B$ can be derived using the familiar deductive rules of equational logic, written as $\mathcal{A}_{\mathrm{Riesz}} \vdash A = B$.

Definition 2.6 (Deductive Rules of Equational Logic)**.** Rules for deriving identities between terms from a set $\mathcal{A}$ of equational axioms:

$$\frac{(A = B) \in \mathcal{A}}{\mathcal{A} \vdash A = B} \text{ Ax} \qquad \frac{}{\mathcal{A} \vdash A = A} \text{ refl} \qquad \frac{\mathcal{A} \vdash B = A}{\mathcal{A} \vdash A = B} \text{ sym} \qquad \frac{\mathcal{A} \vdash A = B}{\mathcal{A} \vdash C[A] = C[B]} \text{ ctxt}$$

$$\frac{\mathcal{A} \vdash A = B \quad \mathcal{A} \vdash B = C}{\mathcal{A} \vdash A = C} \text{ trans} \qquad \frac{\mathcal{A} \vdash f(\vec{A}, x, \vec{C}) = g(\vec{D}, x, \vec{E})}{\mathcal{A} \vdash f(\vec{A}, B, \vec{C}) = g(\vec{D}, B, \vec{E})} \text{ subst}$$

where A, B, C, D, E are terms of the algebraic signature under consideration built from a countable collection of variables, $C[\cdot]$ is a context and f, g are function symbols.

In what follows we denote with $\mathcal{A}_{\mathrm{Riesz}} \vdash A \leq B$ the judgment $\mathcal{A}_{\mathrm{Riesz}} \vdash A = A \sqcap B$. The following elementary facts (see, e.g., [LZ71, §2.12] for proofs) imply that, in the theory of Riesz spaces, a proof system for deriving equalities can be equivalently seen as a proof system for deriving equalities with 0 or inequalities.

Proposition 2.7. *The following assertions hold:*

- $\mathcal{A}_{\mathrm{Riesz}} \vdash A = B \iff \mathcal{A}_{\mathrm{Riesz}} \vdash A - B = 0$,
- $\mathcal{A}_{\mathrm{Riesz}} \vdash A = B \iff \big(\mathcal{A}_{\mathrm{Riesz}} \vdash A \leq B \text{ and } \mathcal{A}_{\mathrm{Riesz}} \vdash B \leq A\big).$

Convention 2.8. From now on, in the rest of this paper, it will be convenient to take the derived negation operation $(-A) = (-1)A$ as part of the signature and restrict all scalars r to be strictly positive $(r > 0)$. The scalar $0 \in \mathbb{R}$ can be of course be removed by rewriting $(0)A$ as 0.

Definition 2.9. A term A is in *negation normal form* (NNF) if the operator $(-)$ is only applied to variables.

For example, the term $(-x) \sqcap (-y)$ is in NNF, while the term $-(x \sqcup y)$ is not.

Lemma 2.10. *Every term A can be rewritten to an equivalent term in NNF.*

Proof. Negation can be pushed towards the variables by the following rewritings: $-(-A) = A$, $-(rA) = r(-A)$, $-(A + B) = (-A) + (-B)$, $-(A \sqcup B) = (-A) \sqcap (-B)$ and $-(A \sqcap B) = (-A) \sqcup (-B)$. $\qquad\square$

Negation can be defined on terms in NNF as follows.

Definition 2.11. Given a term A in NNF, the term $\overline{A}$ is defined as follows: $\overline{x} = -x$, $\overline{-x} = x$, $\overline{rA} = r\overline{A}$, $\overline{A + B} = \overline{A} + \overline{B}$, $\overline{A \sqcup B} = \overline{A} \sqcap \overline{B}$, $\overline{A \sqcap B} = \overline{A} \sqcup \overline{B}$.

The following are basic facts regarding negation of NNF terms.

Proposition 2.12. *For any term A in NNF, the term $\overline{A}$ is also in NNF and it holds that* $\mathcal{A}_{\mathrm{Riesz}} \vdash \overline{A} = -A.$

Proposition 2.13. *For any terms A, B in NNF, it holds that* $\overline{A}[B/x] = \overline{A[B/x]}.$

2.1.1. *Technical lemmas regarding Riesz spaces.* We now list some useful facts that will be used throughout the paper.

The following are useful derived operators frequently used in the theory of Riesz spaces:

Symbol	Terminology	Definition
A^+	The positive part	$A \sqcup 0$
A^-	The negative part	$(-A) \sqcup 0$
$|A|$	The absolute value	$A^+ + A^-$

Lemma 2.14. *The following equations hold:*

- *(1) For all A and $r > 0$, $r(A^-) = (rA)^-$.*
- *(2) For all A, B, $A + B \leq 2(A \sqcup B)$*
- *(3) For all A, B, if $A \leq B$ then $B^- \leq A^-$.*
- *(4) For all A, B, $(A + B)^- \leq A^- + B^-$.*
- *(5) For all $r > 0$, $0 \leq A$ if and only if $0 \leq rA$.*
- *(6) For all A, $A = 0$ if and only if $-A = 0$.*
- *(7) For all A, B, $-(A \sqcup B) = (-A) \sqcap (-B)$ and $-(A \sqcap B) = (-A) \sqcup (-B)$.*

Proof. As mentioned in Example 2.3, $\mathcal{A}_{\mathrm{Riesz}} \vdash A = B$ if and only if the equality $A = B$ holds universally in the Riesz space $(\mathbb{R}, \leq)$. It is then straightforward to check the validity of all equations in $\mathbb{R}$. $\qquad\square$

Lemma 2.15. *For all A, B, $A \sqcup B \geq 0$ if and only if $A^- \sqcap B^- = 0$.*

Proof. For all A, B we have:

$$
\begin{aligned}
0 \sqcap (A \sqcup B) &= (A \sqcap 0) \sqcup (B \sqcap 0) \\
&= -(((-A) \sqcup (-0)) \sqcap ((-B) \sqcup (-0))) \\
&= -(A^- \sqcap B^-)
\end{aligned}
$$

Hence $0 \sqcap (A \sqcup B) = 0$ if and only if $-(A^- \sqcap B^-) = 0$ if and only (by Lemma 2.14[6]) $(A^- \sqcap B^-) = 0$. The proof is complete recalling that $0 \leq A \sqcup B$ means, by definition, that $0 = 0 \sqcap (A \sqcup B)$. $\qquad\square$

2.2. **Modal Riesz Spaces.** This section contains the basic definitions and results related to modal Riesz spaces, as introduced in [MFM17, FMM20].

The language of modal Riesz spaces extends that of Riesz spaces with two symbols: a constant 1 and a unary operator $\Diamond$.

Definition 2.16 (Modal Riesz Space). The *language* $\mathcal{L}_R^\Diamond$ of modal Riesz spaces is $\mathcal{L}_R \cup \{1, \Diamond\}$ where $\mathcal{L}_R$ is the language of Riesz spaces as specified in Definition 2.1. A modal Riesz space is a $\mathcal{L}_R^\Diamond$-algebra satisfying the set $\mathcal{A}_{\text{Riesz}}^\Diamond$ of axioms of Figure 2.

Axioms of Riesz spaces	see Figure 1
$+$	
Positivity of 1:	$0 \leq 1$
Linearity of $\Diamond$:	$\Diamond(r_1 A + r_2 B) = r_1 \Diamond(A) + r_2 \Diamond(B)$
Positivity of $\Diamond$:	$\Diamond(0 \sqcup A) \geq 0$
1-decreasing property of $\Diamond$:	$\Diamond(1) \leq 1$

Figure 2: Set $\mathcal{A}_{\text{Riesz}}^\Diamond$ of equational axioms of modal Riesz spaces.

Example 2.17. Every Riesz space R can be made into a modal Riesz space by interpreting 1 with any positive element and by interpreting $\Diamond$ as the identity function ($\Diamond(x) = x$) or the constant 0 function $\Diamond(x) = 0$.

Example 2.18. The Riesz space $(\mathbb{R}, \leq)$ of linearly ordered real numbers becomes a modal Riesz space by interpreting 1 with the number 1, and $\Diamond$ by any linear (due to the linearity axiom) function $x \mapsto rx$ for a scalar $r \in \mathbb{R}$ such that $r \geq 0$ (due to the positivity axiom) and $r \leq 1$ (due to the 1-decreasing axiom).

Example 2.19. Generalising the previous example, the Riesz space $\mathbb{R}^n$ (with operations defined pointwise, see Example 2.4) becomes a modal Riesz space by interpreting 1 with the constant 1 vector and $\Diamond$ by a linear (due to the linearity axiom) map $M : \mathbb{R}^n \to \mathbb{R}^n$, thus representable as a square matrix,

$$
1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \Diamond = \begin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n,1} & r_{n,2} & \cdots & r_{n,n} \end{pmatrix}
$$

such that all entries $r_{i,j}$ are non–strictly positive (due to the positivity axiom) and where all the rows sum up to a value ≤ 1, i.e., for all $1 \leq i \leq n$ it holds that $\sum_{j=1}^{k} r_{i,j} \leq 1$ (due to the 1–decreasing axiom). Such matrices are known as sub-stochastic matrices. They can be regarded as a Markov chain whose set of states is $\{1, \ldots, n\}$ and from the state $i \in \{1, \ldots, n\}$ the probability of reaching at the next step the state j is $r_{i,j}$. This example, in fact, provides the motivation for the study of modal Riesz spaces in [MFM17, FMM20] in the context of logics for probabilistic programs. We refer to [FMM20] for a detailed exposition.

Example 2.20. Consider the equality $\Diamond(x \sqcup y) = \Diamond(x) \sqcup \Diamond(y)$. Does it hold in all modal Riesz spaces? In other words, does $\mathcal{A}_{\mathrm{Riesz}}^{\Diamond} \vdash \Diamond(x \sqcup y) = \Diamond(x) \sqcup \Diamond(y)$? The answer is negative. Take as example the modal Riesz space $\mathbb{R}^2$ with:

$$1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \Diamond = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 0 \end{pmatrix}$$

and let $a = (1, 0)$ and $b = (0, 1)$. One verifies that $\Diamond(a \sqcup b) = (1, 0)$ while $\Diamond(a) \sqcup \Diamond(b) = (\frac{2}{3}, 0)$. This example shows that unlike the theory of Riesz spaces (cf. Example 2.3), the theory of modal Riesz spaces cannot be generated by a linear model because, in any linear model, the equality $\Diamond(x \sqcup y) = \Diamond(x) \sqcup \Diamond(y)$ clearly holds.

We now expand the definitions and properties related to terms in negation normal form to modal Riesz spaces.

Definition 2.21. A term A is in *negation normal form* (NNF) if the operator $(-)$ is only applied to variables and the constant 1.

Lemma 2.22. *Every term A can be rewritten to an equivalent term in NNF.*

Proof. Negation can be pushed towards the variables by the following rewritings: $-\Diamond(A) = \Diamond(-A)$ (see Lemma 2.10 for the other operators). $\qquad\qquad\square$

Negation can be defined on terms in NNF as follows.

Definition 2.23. Given a term A in NNF, we expand the operator $\overline{A}$ as follows: $\overline{\Diamond A} = \Diamond \overline{A}$, $\overline{1} = -1$, $\overline{-1} = 1$.

The following are basic facts regarding negation of NNF terms.

Proposition 2.24. *For any term A in NNF, the term $\overline{A}$ is also in NNF and it holds that $\mathcal{A}_{\mathrm{Riesz}} \vdash \overline{A} = -A$.*

Proposition 2.25. *For any terms A, B in NNF, it holds that $\overline{A}[B/x] = \overline{A[B/x]}$.*

3. Hypersequent Calculus for Riesz Spaces

In this section we introduce the hypersequent calculus **HR** for the equational theory of Riesz spaces.

In what follows we proceed with a sequence of syntactical definitions and notational conventions necessary to present the rules of the system. We use the letters A, B, C to range over Riesz terms in negation normal form (NNF, see Definition 2.9) built from a countable set of variables x, y, z and negated variables $\overline{x}$, $\overline{y}$, $\overline{z}$. The scalars appearing in these terms are all strictly positive and are ranged over by the letters $r, s, t \in \mathbb{R}_{>0}$. From now on, the term scalar should always be understood as strictly positive scalar.

Definition 3.1. A *weighted term* is a formal expression $r.A$ where $r \in \mathbb{R}_{>0}$ and A is a term.

Given a weighted term $r.A$ and a scalar s we denote with $s.(r.A)$ the weighted term $(sr).A$. Thus we have defined (strictly positive) scalar multiplication on weighted terms.

We use the greek letters $\Gamma, \Delta, \Theta, \Sigma$ to range over possibly empty finite multisets of weighted terms. We often write these multisets as lists but they should always be understood as being taken modulo reordering of their elements. As usual, we write Γ, Δ for the concatenation of Γ and Δ.

We adopt the following notation:

- Given a sequence $\vec{r} = (r_1, \ldots r_n)$ of scalars and a term A, we denote with $\vec{r}.A$ the multiset $[r_1.A, \ldots, r_n.A]$. When $\vec{r}$ is empty, the multiset $\vec{r}.A$ is also empty.
- Given a multiset $\Gamma = [r_1.A_1, \ldots, r_n.A_n]$ and a scalar $s > 0$, we denote with $s.\Gamma$ the multiset $[s.r_1.A_1, \ldots, s.r_n.A_n]$.
- Given a sequence $\vec{s} = (s_1, \ldots s_n)$ of scalars and a multiset Γ, we denote with $\vec{s}.\Gamma$ the multiset $s_1.\Gamma, \ldots, s_n.\Gamma$.
- Given two sequences $\vec{r} = (r_1, \ldots r_n)$ and $\vec{s} = (s_1, \ldots s_m)$ of scalars, we denote $\vec{r}; \vec{s}$ the concatenation of the two sequences, i.e. the sequence $(r_1, \ldots r_n, s_1, \ldots s_m)$.
- Given a sequence $\vec{s} = (s_1, \ldots s_n)$ of scalars and a scalar r, we denote $(r\vec{s})$ the sequence $(rs_1, \ldots rs_n)$.
- Given two sequences $\vec{r} = (r_1, \ldots r_n)$ and $\vec{s} = (s_1, \ldots s_m)$ of scalars, we denote $\vec{r}\vec{s}$ the sequence $r_1\vec{s}; \ldots; r_n\vec{s}$.
- Given a sequence $\vec{s} = (s_1, \ldots s_n)$ of scalars, we denote $\sum \vec{s}$ the sum of all scalars in $\vec{s}$, i.e. the scalar $\sum_{i=1}^{n} s_i$.

Definition 3.2. A *sequent* is a formal expression of the form $\vdash \Gamma$.

If $\Gamma = \emptyset$, the corresponding empty sequent is simply written as $\vdash$.

Definition 3.3. A *hypersequent* is a non–empty finite multiset of sequents, written as $\vdash \Gamma_1 | \ldots | \vdash \Gamma_n$.

We use the letter G, H to range over hypersequents. Note that, under these notational conventions, the expression $\vdash \Gamma$ could either denote the sequent $\vdash \Gamma$ itself or the hypersequent $[\vdash \Gamma]$ containing only one sequent. The context will always determine which of these two interpretations is intended.

The hypersequent calculus **HR** is a deductive system for deriving hypersequents. The rules of **HR** are presented in Figure 3.

We write $\models_{\mathbf{HR}} G$ if the hypersequent G is derivable in the system **HR**.

Note that the axiom rule (INIT) allows for the derivation of $(\vdash)$, the hypersequent containing only the empty sequent. The contraction rule (C) allows to threat hypersequents as (non–empty) sets of sequents. Note that the logical rules are all presented using the syntactic sugaring $\vec{r}.A$ described above. For example, one valid instance of the rule $(+)$ is the following:

$$\frac{\vdash \Gamma, 2.3y, 2.x, \frac{1}{2}.3y, \frac{1}{2}.x}{\vdash \Gamma, 2.(3y + x), \frac{1}{2}(3y + x)} +$$

This effectively allows to apply the rule to several formulas in the sequent at the same time. This feature adds some flexibility in the process of proof construction and simplifies some proofs, but it is not strictly required. All our results hold even in a variant of the **HR** system where rules are allowed to act on only one formula at the time.

Axiom:
$$\overline{}\vdash \text{ INIT}$$

Structural rules:

$$\frac{G}{G \mid \vdash \Gamma} \text{ W} \qquad\qquad \frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma} \text{ C}$$

$$\frac{G \mid \vdash \Gamma_1, \Gamma_2}{G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2} \text{ S} \qquad\qquad \frac{G \mid \vdash \Gamma_1 \quad G \mid \vdash \Gamma_2}{G \mid \vdash \Gamma_1, \Gamma_2} \text{ M}$$

$$\frac{G \mid \vdash r.\Gamma}{G \mid \vdash \Gamma} \text{ T} \qquad\qquad \frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}} \text{ ID}, \sum r_i = \sum s_i$$

Logical rules:

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.0} \text{ 0} \quad \frac{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A+B)} \text{ +} \quad \frac{G \mid \Gamma \vdash \Gamma, (s\vec{r}).A}{G \mid \Gamma \vdash \Gamma, \vec{r}.(sA)} \text{ ×}$$

$$\frac{G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)} \text{ ⊔} \quad \frac{G \mid \vdash \Gamma, \vec{r}.A \quad G \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcap B)} \text{ ⊓}$$

CAN rule:

$$\frac{G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma} \text{ CAN}, \sum r_i = \sum s_i$$

Figure 3: Inference rules of **HR**.

Remark 3.4. On the one hand, we could have introduced appropriate exchange (i.e., re-ordering) rules and defined sequents and hypersequents as lists, rather than multisets. In the opposite direction, we could have defined hypersequents as (non–empty) sets and dispose of the rules (C). Our choice is motivated by a balance between readability and fine control over the derivation steps in the proofs.

Remark 3.5. Note that the following CUT rule is equivalent to the CAN rule of the **HR** hypersequent calculus:

$$\frac{G \mid \vdash \Gamma_1, \vec{r}.A \quad G \mid \vdash \Gamma_2, \vec{s}.\overline{A}}{G \mid \vdash \Gamma_1, \Gamma_2} \text{ CUT}, \sum \vec{r} = \sum \vec{s}$$

$$\frac{\dfrac{G \mid \vdash \Gamma_1, \vec{r}.A \quad G \mid \vdash \Gamma_2, \vec{s}.\overline{A}}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r}.A, \vec{s}.\overline{A}} \text{ M}}{G \mid \vdash \Gamma_1, \Gamma_2} \text{ CAN}, \sum \vec{r} = \sum \vec{s}$$

Figure 4: Derivability of the CUT rule.

$$\dfrac{\dfrac{\dfrac{\overline{}\ \text{INIT}}{\vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{Lemma 3.20}}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A} \quad G \mid \vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{W}^*}{\dfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma, \vec{r}.(A + \overline{A})}\ +}\ \text{CUT} \qquad \dfrac{\dfrac{\dfrac{\overline{}\ \text{INIT}}{\vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{Lemma 3.20}}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A} \quad G \mid \vdash \vec{s}.A, \vec{r}.\overline{A}}\ \text{W}^*}{\dfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma, \vec{r}.(\overline{A} + A)}\ +}\ \text{CUT}$$

$$\dfrac{\dfrac{G \mid \vdash \Gamma, \Gamma}{\dfrac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma}\ \text{C}}\ \text{S}}{}\ \text{CUT}$$

Figure 5: Derivability of the CAN rule.

Our choice (following [MOG09, MOG05]) of presenting the system **HR** using the CAN rule, rather than the equivalent CUT rule, is just motivated by elegance and technical convenience.

Example 3.6. Example of derivation of the hypersequent $\vdash 1.\big((2x + 2\overline{y}) \sqcup (y + \overline{x})\big)$ which consists of only one sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{}\ \text{INIT}}{\vdash 2.y, 2.\overline{y}}\ \text{ID}}{\vdash 2.x, 2.y, 2.\overline{x}, 2.\overline{y}}\ \text{ID}}{\vdash 2.x, 2.\overline{y} \mid \vdash 2.y, 2.\overline{x}}\ \text{S}}{\vdash 2.x, 2.\overline{y} \mid \vdash 1.y, 1.\overline{x}}\ \text{T(multiplication by 2)}}{\vdash 2.x, 2.\overline{y} \mid \vdash 1.(y + \overline{x})}\ +}{\vdash 2.x, 1.2\overline{y} \mid \vdash 1.(y + \overline{x})}\ \times}{\vdash 1.2x, 1.2\overline{y} \mid \vdash 1.(y + \overline{x})}\ \times}{\vdash 1.(2x + 2\overline{y}) \mid \vdash 1.(y + \overline{x})}\ +}{\vdash 1.\big((2x + 2\overline{y}) \sqcup (y + \overline{x})\big)}\ \sqcup$$

We now describe how sequents and hypersequents can be interpreted by Riesz terms. This means that **HR** is a *structural proof system*, i.e., by manipulating sequents and hypersequents it in fact deals with terms of a certain specific form.

Definition 3.7 (Interpretation). We interpret weighted terms $(r.A)$, sequents $\vdash \Gamma$ and hypersequents G as the Riesz terms $(\!|r.A|\!)$, $(\!|\vdash \Gamma|\!)$ and $(\!|G|\!)$, respectively, as follows:

	Syntax	Term interpretation $(\!|\_|\!)$
Weighted terms	$r.A$	rA
Sequents	$\vdash r_1.A_1, \ldots, r_n.A_n$	$(\!|r_1.A_1|\!) + \cdots + (\!|r_n.A_n|\!)$
Hypersequents	$\vdash \Gamma_1 \mid \ldots \mid \vdash \Gamma_n$	$(\!|\vdash \Gamma_1|\!) \sqcup \cdots \sqcup (\!|\vdash \Gamma_n|\!)$

Hence a weighted term is simply interpreted as the term scalar–multiplied by the weight. A sequent is interpreted as sum $(\sum)$ and a hypersequent is interpreted as a join of sums $(\bigsqcup \sum)$.

Example 3.8. The interpretation of the hypersequent:

$$\vdash 1.x, 2.(y \sqcap z) \mid \ \vdash 2.(3\overline{x} \sqcap y)$$

is the Riesz term:
$$\big(1x + 2(y \sqcap z)\big) \sqcup \big(2(3\overline{x} \sqcap y)\big).$$

In what follows we say that an hypersequent G has a CAN–free proof (resp., M–free, T–free, *etc.*) if it has a derivation that never uses the rule CAN (resp., rule M, rule T, *etc.*)

3.1. **Main results regarding the system HR.** We are now ready to state the main results regarding the hypersequent calculus **HR**. Each theorem will be proven in a separate subsection of this section.

Recall that we write $\mathcal{A}_{\mathrm{Riesz}} \vdash A \geq B$ if the inequality $A \geq B$ is derivable in equational logic from the axioms of Riesz spaces and that we write $\models_{\boldsymbol{HR}} G$ if the hypersequent G is derivable in the **HR** proof system.

Our first technical result states that the system **HR** can derive all and only those hypersequents G such that $\mathcal{A}_{\mathrm{Riesz}} \vdash (\!|G|\!) \geq 0$.

Theorem 3.9 (Soundness). *For every hypersequent G,*
$$\models_{\boldsymbol{HR}} G \quad \Longrightarrow \quad \mathcal{A}_{\mathrm{Riesz}} \vdash (\!|G|\!) \geq 0.$$

Theorem 3.10 (Completeness). *For every hypersequent G,*
$$\mathcal{A}_{\mathrm{Riesz}} \vdash (\!|G|\!) \geq 0 \quad \Longrightarrow \quad \models_{\boldsymbol{HR}} G.$$

Our next theorem states that all the logical rules of the hypersequent calculus **HR** are *CAN–free invertible.* This means that if an hypersequent G having the shape of the conclusion of a logical rule is derivable with a CAN–free proof, then also the premises of that logical rule are derivable by CAN–free proofs. So, for example, in the case of the $(\sqcap)$ rule, if the hypersequent
$$G \mid \vdash \Gamma, \vec{r}.(A \sqcap B)$$
has a CAN–free proof, then also
$$G \mid \vdash \Gamma, \vec{r}.A \quad \text{and} \quad G \mid \vdash \Gamma, \vec{r}.B$$
have CAN–free proofs.

Theorem 3.11 (CAN–free Invertibility). *All the logical rules are CAN–free invertible.*

The invertibility theorem is very important for proof search. When trying to derive a hypersequent G (without CAN applications) it is always possible to systematically apply the logical rules and reduce the problem of deriving G (without CAN applications) to the problem of deriving a number of hypersequents $G_1, \ldots G_n$ where no logical symbols appear. We call such reduced hypersequents without logical symbols *atomic hypersequents.*

$$\frac{\;\overline{G_1} \qquad \overline{G_n}\;}{G} \text{ Logical rules}$$

Figure 6: Systematic application of the logical rules to reduce the logical complexity.

As we will discuss later (Theorem 3.17), this procedure of simplification will lead to an algorithm for deciding if an arbitrary hypersequent G is derivable in **HR** or not.

The three theorems above are adaptations of similar results for the hypersequent calculus **GA** of [MOG09, MOG05] for the theory of lattice ordered abelian groups.

The following theorem, instead, appears to be novel. It is stated in the context of our system **HR** but a similar result can be proved for **GA** too.

Theorem 3.12 (M–elimination)**.** *If a hypersequent has a CAN–free proof, then it has a CAN–free and M–free proof.*

Our motivation for proving the above result is mostly technical. Indeed it allows to prove our main theorem (Theorem 3.13 below) in a rather simple way (different from that of [MOG09, MOG05]). However note how the M–elimination theorem is also useful from the point of view of proof search since it reduces the space of derivation trees to be explored.

We are now ready to state our main result regarding the system **HR**.

Theorem 3.13 (CAN elimination)**.** *If a hypersequent G has a proof, then it has a CAN–free proof.*

Proof sketch. The CAN rule has the following form:

$$\frac{G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma} \ \mathrm{CAN}, \sum \vec{r} = \sum \vec{s}$$

We show how to eliminate one application of the CAN rule. Namely, we prove that if the premise $G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}$ has a CAN–free proof then the conclusion $G \mid \vdash \Gamma$ also has a CAN–free proof. This of course implies the statement of the CAN–elimination theorem by using a simple inductive argument on the number of CAN's applications in a proof.

The proof proceeds by induction on the structure of A.

The base case in when $A = x$, i.e., when A is atomic. Proving this case is not at all straightforward in presence of the M rule, but it becomes much easier in the **HR** system without the M rule. This is why the M–elimination theorem, which asserts the equivalence between **HR** and **HR** $\setminus \{M\}$, is useful.

For the inductive case, when A is a complex term we invoke the invertibility theorem. For example, if $A = B + C$, the invertibility theorem states that $G \mid \vdash \Gamma, \vec{s}.B, \vec{s}.C, \vec{r}.\overline{B}, \vec{r}.\overline{B}$ must also have a CAN–free proof. We then note that, since B and C both have lower complexity than A, it follows from two applications of the inductive hypothesis that $G \mid \vdash \Gamma$ has a CAN-free proof, as desired. $\qquad\qquad\square$

Remark 3.14. Note, with reference to Remark 3.5, that theorems 3.12 and 3.13 together imply also a CUT–elimination theorem.

The CAN rule is not analytical, meaning that in its premise there is a formula not appearing (even as a subformula) in the conclusion. This is why the above CAN–elimination is of key importance, especially in the context of proof search.

However there is another rule of **HR** which is not analytical: the T rule. The following theorem shows that also the T rule is admissible if the scalars appearing in the end hypersequent G are all rational numbers.

Theorem 3.15 (Rational T–elimination)**.** *If a hypersequent G with only rational numbers has a CAN–free proof, then it has also a CAN–free and T–free proof.*

It can be shown, however, than in the general case where G contains irrational numbers, it is generally not possible to eliminate both rules CAN and T at the same time.

Proposition 3.16. *The system* **HR** *without the CAN and T rules is incomplete.*

As mentioned earlier, using the invertibility theorem, it is possible to reduce the problem of deriving an hypersequent G to the problem of deriving a number $G_1, \ldots G_n$ of atomic (i.e., without logical symbols) hypersequents. This leads to the following result.

Theorem 3.17 (Decidability). *There is an algorithm to decide whether or not a hypersequent has a proof.*

Interestingly, our approach to prove the above theorem (see Subsection 3.10), is based on considering a generalization of the concept of proof where the scalars appearing in the hypersequents can be variables, rather than numerical constants. For instance, the proof

$$\frac{\dfrac{}{\vDash} \text{ INIT}}{\vdash \Gamma, r.x, r.\overline{x}} \text{ ID}$$

is valid for any $r \in \mathbb{R}_{>0}$ and, similarly, the proof

$$\frac{\dfrac{}{\vDash} \text{ INIT}}{\vdash \Gamma, r.x, s.\overline{x}, t.\overline{x}} \text{ ID}, r = s + t$$

is valid for any values of reals $(r, s, t) \in \mathbb{R}_{>0}^3$ such that $r = s + t$. Lastly, the hypersequent containing two scalar–variables α, β and two concrete scalars s and t

$$\vdash (\alpha^2 - \beta).x, s.\overline{x}, t.\overline{x}$$

is derivable for any assignment of concrete assignments $r_1, r_2 \in \mathbb{R}_{>0}$ to α and β such that $(r_1)^2 - r_2 = s + t$. Hence a proof can be interpreted as describing the set of possible assignments to these real–valued variables that result in a valid concrete (i.e., where all scalars are numbers and not variables) proof.

The main idea behind the proof of Theorem 3.17 is that it is possible, given an arbitrary hypersequent G, to construct (automatically) a formula in the first order theory of the real closed field $(\text{FO}(\mathbb{R}, +, \times, \leq))$ describing the set of valid assignments. Since this theory is decidable and has quantifier elimination [Tar51], it is possible to verify if this set is nonempty and extract a valid assignment to variables.

3.2. **Relations with the hypersequent calculus GA of** [MOG09, MOG05] **and Lattice ordered Abelian groups.** As mentioned earlier, our hypersequent calculus system **HR** for the theory of Riesz spaces is an extension of the system **GA** of [MOG09, MOG05] for the theory of lattice–ordered Abelian groups (laG). The equational theory $(\mathcal{A}_{\text{laG}})$ of lattice–ordered Abelian groups can be defined by removing, from the signature of Riesz spaces, the scalar multiplication operations and, accordingly, the equational axioms regarding scalar multiplication. Integer scalars (e.g., $-3x$) can still be used as a short hand for repeated sums (e.g., $-(x + x + x)$). The system **HR** stripped out of scalars is essentially identical to the system **GA**.

From our Rational T–elimination theorem 3.15 we obtain as a corollary the fact that the theory of Riesz spaces is a proof–theoretic conservative extension of the theory of lattice–ordered Abelian groups.

Proposition 3.18. *Let A be a term in the signature of lattice–ordered Abelian groups (i.e., a Riesz term where all scalars are natural numbers). Then*

$$\mathcal{A}_{\mathrm{Riesz}} \vdash A \geq 0 \Leftrightarrow \mathcal{A}_{\mathrm{laG}} \vdash A \geq 0.$$

Proof. The ($\Leftarrow$) direction is trivial, since $\mathcal{A}_{\mathrm{Riesz}}$ is an extension of $\mathcal{A}_{\mathrm{laG}}$.

For the other direction, assume $\mathcal{A}_{\mathrm{Riesz}} \vdash A \geq 0$. Then, by the completeness theorem, the hypersequent $\vdash A$ has a **HR** proof. Then, by the CAN–elimination and the rational T–elimination theorems, $\vdash A$ has a CAN–free and T–free proof. This is essentially (the trivial translation details are omitted) translatable to a **GA** proof of $\vdash A$. Since the system **GA** is sound and complete with respect to $\mathcal{A}_{\mathrm{laG}}$ we deduce that $\mathcal{A}_{\mathrm{laG}} \vdash A \geq 0$ as desired. $\square$

Similarly, we could define the theory of Riesz spaces over rationals ($\mathcal{A}_{\mathbb{Q}-\mathrm{Riesz}}$), defined just as Riesz spaces but over the field $\mathbb{Q}$ of rational numbers instead of the field $\mathbb{R}$ of reals. Again, from 3.15, we get the following conservativity result.

Proposition 3.19. *Let A be a term in the signature of Riesz spaces over rationals. Then*

$$\mathcal{A}_{\mathrm{Riesz}} \vdash A \geq 0 \Leftrightarrow \mathcal{A}_{\mathbb{Q}-\mathrm{Riesz}} \vdash A \geq 0.$$

Both conservativity results are known as folklore in the theory of Riesz spaces. It is perhaps interesting, however, that here we obtain them in a completely syntactical (proof theoretic) way.

Compared to the proof technique used in [MOG09, MOG05] to prove the CAN–elimination theorem, our approach is novel in that our proof is based on the M–elimination theorem. We remark here that a proof of all the theorems stated in this section could have been obtained without using the M–elimination theorem, and instead following the proof structure adopted in [MOG09, MOG05]. The proof technique based on the M–elimination theorem will be however of great value in proving the CAN elimination of the system **HMR** in Section 4.

3.3. **Some technical lemmas.** Before embarking in the proofs of the theorems stated in Section 3, we prove in this subsection a few useful routine lemmas that will be used often.

Our first lemma states that the following variant of the ID rule (see Figure 3) where general terms A are considered rather than just variables, is admissible in the proof system **HR**.

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}} \ \mathrm{ID}, \sum \vec{r} = \sum \vec{s}$$

Formally, we prove the admissibility of a slightly more general rule which can act on several sequents of the hypersequent at the same time.

Lemma 3.20. *For all terms A, numbers $n > 0$, and vectors $\vec{r_i}$ and $\vec{s_i}$, for $1 \leq i \leq n$, such that $\sum \vec{r_i} = \sum \vec{s_i}$,*

$$\textit{if} \vDash_{\boldsymbol{HR}} \left[\vdash \Gamma_i\right]_{i=1}^n \ \textit{then} \vDash_{\boldsymbol{HR}} \left[\vdash \Gamma_i, \vec{r_i}.A, \vec{s_i}.\overline{A}\right]_{i=1}^n$$

Proof. We prove the result by induction on A.

- If A is a variable, we simply use the ID rule n times.
- If $A = 0$, we use the 0 rule n times.
- If $A = sB$, we use the $\times$ rule $2n$ times and conclude with the induction hypothesis.
- If $A = B + C$, we use the $+$ rule $2n$ times and conclude with the induction hypothesis.

- For the case $A = B \sqcap C$ or $A = B \sqcup C$, we first use the $\sqcap$ rule $2^n - 1$ times – one time on the conclusion, then again on the two premises, then on the four premises and so forth until we used the $\sqcap$ for all sequents – and then the $\sqcup$ rule n times on each premises and the W rule n times on each premises to remove the sequents with both B and C in them. We can then conclude with the induction hypothesis.

$$
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\begin{array}{c}
\ddots \quad
\cfrac{
\cfrac{
\cfrac{[\vdash \Gamma_i]_{i=0}^n}
{[\vdash \Gamma_i, \vec{r_i}.B, \vec{s_i}.\overline{B}]_{i=0}^k \mid [\vdash \Gamma_i, \vec{r_i}.C, \vec{s_i}.\overline{C}]_{i=k+1}^n} \; \text{IH}^2}
{\cfrac{[\vdash \Gamma_i, \vec{r_i}.B, \vec{s_i}.\overline{B}]_{i=0}^k \mid [\vdash \Gamma_i, \vec{r_i}.B, \vec{s_i}.\overline{C}]_{i=0}^k \mid [\vdash \Gamma_i, \vec{r_i}.C, \vec{s_i}.\overline{B}]_{i=k+1}^n \mid [\vdash \Gamma_i, \vec{r_i}.C, \vec{s_i}.\overline{C}]_{i=k+1}^n}
{[\vdash \Gamma_i, \vec{r_i}.B, \vec{s_i}.(\overline{B} \sqcup \overline{C})]_{i=0}^k \mid [\vdash \Gamma_i, \vec{r_i}.C, \vec{s_i}.(\overline{B} \sqcup \overline{C})]_{i=k+1}^n} \; \sqcup^n} \; \text{W}^n}
\quad \iddots
\end{array}
}
{[\vdash \Gamma_i, \vec{r_i}.(B \sqcap C), \vec{s_i}.(\overline{B} \sqcup \overline{C})]_{i=1}^n} \; \sqcap
$$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad \square$

The next result states that derivability in the **HR** system is preserved by substitution of terms for variables.

Lemma 3.21. *For all hypersequents G and formulas A, if $\vDash_{\textbf{HR}} G$ then $\vDash_{\textbf{HR}} G[A/x]$.*

Proof. We prove the result by induction on the proof of G. Most cases are quite straightforward, we simply use the induction hypothesis on the premises and then use the same rule. For instance, if the proof finishes with

$$\cfrac{G \mid \vdash \Gamma, \vec{r}.B, \vec{r}.C}{G \mid \vdash \Gamma, \vec{r}.(B + C)} \; +$$

by induction hypothesis $\vDash_{\textbf{HR}} G[A/x] \mid \vdash \Gamma[A/x], \vec{r}.B[A/x], \vec{r}.C[A/x]$ so

$$\cfrac{G[A/x] \mid \vdash \Gamma[A/x], \vec{r}.B[A/x], \vec{r}.C[A/x]}{G[A/x] \mid \vdash \Gamma[A/x], \vec{r}.(B + C)[A/x]} \; +$$

The only tricky case is when the ID rule is used on the variable x, where we conclude using Lemma 3.20. $\qquad \square$

The next lemma states that the logical rules are invertible using the CAN rule, meaning that if the conclusion is derivable, then the premises are also derivable. The difference with Theorem 3.11 is that the derivations of the premises introduce a CAN rule.

Lemma 3.22. *All logical rules are invertible.*

Proof. We simply use the CAN rule to introduce the operators. We will show the two most interesting cases, the other cases are trivial.

- The $\sqcap$ rule: if $G \mid \vdash \Gamma, \vec{r}.(A \sqcap B)$ is derivable. The proof of $G \mid \vdash \Gamma, \vec{r}.A$ is then:

$$
\cfrac{
G \mid \vdash \Gamma, \vec{r}.(A \sqcap B) \qquad
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{\vdash} \; \text{INIT}}{\vdash \vec{r}.A, \vec{r}.\overline{A}} \; \text{Lemma 3.20}}
{\vdash \vec{r}.A, \vec{r}.\overline{A} \mid \vdash \vec{r}.A, \vec{r}.\overline{B}} \; \text{W}}
{\vdash \vec{r}.A, \vec{r}.(\overline{A} \sqcup \overline{B})} \; \sqcup}
{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.(A \sqcap B), \vec{r}.(\overline{A} \sqcup \overline{B})} \; \text{M}}
{G \mid \vdash \Gamma, \vec{r}.A} \; \text{CAN}
$$

The proof of $G \mid \vdash \Gamma, \vec{r}.B$ is similar.

- The $\sqcup$ rule: if $G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)$ is derivable. The proof of $G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B$ is then:

$$
\dfrac{\dfrac{G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)}{G \mid \vdash \Gamma, \vec{r}.(A \sqcup B) \mid \vdash \Gamma, \vec{r}.B} \text{ W} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash} \text{ INIT}}{\vdash \vec{r}.A, \vec{r}.\overline{A}} \text{ Lemma 3.20}}{G \mid \vdash \vec{r}.A, \vec{r}.\overline{A} \mid \vdash \Gamma, \vec{r}.B} \text{ W}^* \quad \dfrac{\Pi}{G \mid \vdash \vec{r}.A, \vec{r}.\overline{B} \mid \vdash \Gamma, \vec{r}.B}}{G \mid \vdash \vec{r}.A, \vec{r}.(\overline{A} \sqcap \overline{B}) \mid \vdash \Gamma, \vec{r}.B} \sqcap}{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.(A \sqcup B), \vec{r}.(\overline{A} \sqcap \overline{B}) \mid \vdash \Gamma, \vec{r}.B} \text{ M}}{G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B} \text{ CAN}
$$

where Π is the following derivation:

$$
\dfrac{\dfrac{G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)}{G \mid \vdash \vec{r}.A, \vec{r}.\overline{B} \mid \vdash \Gamma, \vec{r}.(A \sqcup B)} \text{ W} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash} \text{ INIT}}{\vdash \vec{r}.B, \vec{r}.\overline{B}} \text{ Lemma 3.20}}{\vdash \vec{r}.A, \vec{r}.\overline{B}, \vec{r}.B, \vec{r}.\overline{A}} \text{ Lemma 3.20}}{\vdash \vec{r}.A, \vec{r}.\overline{B} \mid \vdash \vec{r}.B, \vec{r}.\overline{A}} \text{ S} \quad \dfrac{\dfrac{\dfrac{\overline{\vdash} \text{ INIT}}{\vdash \vec{r}.B, \vec{r}.\overline{B}} \text{ Lemma 3.20}}{\vdash \vec{r}.A, \vec{r}.\overline{B} \mid \vdash \vec{r}.B, \vec{r}.\overline{B}} \text{ W}}{\vdash \vec{r}.A, \vec{r}.\overline{B} \mid \vdash \vec{r}.B, \vec{r}.(\overline{A} \sqcap \overline{B})} \sqcap}{G \mid \vdash \vec{r}.A, \vec{r}.\overline{B} \mid \vdash \vec{r}.B, \vec{r}.(\overline{A} \sqcap \overline{B})} \text{ W}^*}{G \mid \vdash \vec{r}.A, \vec{r}.\overline{B} \mid \vdash \Gamma, \vec{r}.B, \vec{r}.(A \sqcup B), \vec{r}.(\overline{A} \sqcap \overline{B})} \text{ M}}{G \mid \vdash \vec{r}.A, \vec{r}.\overline{B} \mid \vdash \Gamma, \vec{r}.B} \text{ CAN}
$$

$\square$

Remark 3.23. The proof of invertibility does not introduce any new T rule, so if the conclusion of a logical rule has a T–free proof then the premises also have T–free proofs.

The next lemmas state that CAN–free derivability in the **HR** system is preserved by scalar multiplication.

Lemma 3.24. *Let $\vec{r} \in \mathbb{R}_{>0}$ be a non-empty vector and G a hypersequent. If $\vDash_{\mathbf{HR} \backslash \{CAN\}} G \mid \vdash \vec{r}.\Gamma$ then $\vDash_{\mathbf{HR} \backslash \{CAN\}} G \mid \vdash \Gamma$.*

Proof. We simply use the C,T and S rules :

$$
\dfrac{\dfrac{\dfrac{G \mid \vdash \vec{r}.\Gamma}{G \mid \vdash r_1.\Gamma \mid \ ... \ \mid \ r_n.\Gamma} \text{ S}^*}{G \mid \vdash \Gamma \mid \ ... \ \mid \vdash \Gamma} \text{ T}^*}{G \mid \vdash \Gamma} \text{ C}^*
$$

$\square$

Lemma 3.25. *Let $\vec{r} \in \mathbb{R}_{>0}$ be a vector and G a hypersequent. If $\vDash_{\mathbf{HR} \backslash \{CAN\}} G \mid \vdash \Gamma$ then $\vDash_{\mathbf{HR} \backslash \{CAN\}} G \mid \vdash \vec{r}.\Gamma$.*

Proof. We reason by induction on the size of $\vec{r}$.

If the size of $\vec{r}$ is 0: Since $\vdash \vec{r}.\Gamma \ = \vdash$, we simply use the W rule until we can use the INIT rule:

$$
\dfrac{\dfrac{}{\overline{\vdash}} \text{ INIT}}{G \mid \vdash} \text{ W}^*
$$

If the size of $\vec{r}$ is 1: we can use the T rule:

$$
\dfrac{G \mid \vdash \Gamma}{G \mid \vdash r_1.\Gamma} \text{ T}
$$

Otherwise: Let $(r_1, ..., r_{n+1}) = \vec{r}$. We can invoke the inductive hypothesis and conclude as follows:

$$\frac{\dfrac{G \mid \vdash \Gamma}{G \mid \vdash r_1.\Gamma, ..., r_n.\Gamma} \quad \dfrac{G \mid \vdash \Gamma}{G \mid \vdash r_{n+1}.\Gamma} \text{ T}}{G \mid \vdash r_1.\Gamma, ..., r_n.\Gamma, r_{n+1}.\Gamma} \text{ M}$$

$\square$

The above lemmas have two useful corollaries.

Corollary 3.26. *If* $\vDash_{\mathbf{HR}\backslash\{CAN\}} G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A$ *and* $\vDash_{\mathbf{HR}\backslash\{CAN\}} G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B$ *then* $\vDash_{\mathbf{HR}\backslash\{CAN\}} G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.B$.

Proof. If $\vec{r} = \emptyset$ or $\vec{s} = \emptyset$, the result is trivial. Otherwise

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A}{G \mid \vdash \vec{r}.\Gamma, (\vec{r}\vec{r}).A, (\vec{r}\vec{s}).A} \text{ Lemma 3.25} \quad \dfrac{G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B}{G \mid \vdash \vec{s}.\Gamma, (\vec{s}\vec{r}).B, (\vec{s}\vec{s}).B} \text{ Lemma 3.25}}{G \mid \vdash \vec{r}.\Gamma, \vec{s}.\Gamma, (\vec{r}\vec{r}).A, (\vec{s}\vec{r}).A, (\vec{r}\vec{s}).B, (\vec{s}\vec{s}).B} \text{ M}}{G \mid \vdash \vec{r}.\Gamma, (\vec{r}\vec{r}).A, (\vec{r}\vec{s}).B \mid \vec{s}.\Gamma, (\vec{s}\vec{r}).A, (\vec{s}\vec{s}).B} \text{ S}}{G \mid \vdash \vec{r}.\Gamma, (\vec{r}\vec{r}).A, (\vec{r}\vec{s}).B \mid \vdash \Gamma, \vec{r}.A, \vec{s}.B} \text{ Lemma 3.24}}{\dfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.B \mid \vdash \Gamma, \vec{r}.A, \vec{s}.B}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.B} \text{ C}} \text{ Lemma 3.24}$$

$\square$

Corollary 3.27. *If* $\vDash_{\mathbf{HR}\backslash\{CAN\}} G \mid \vdash \vec{r}.A, \vec{s}.A, \Gamma \mid \vdash \vec{r}.B, \vec{s}.B, \Gamma \mid \vdash \vec{r}.A, \vec{s}.B, \Gamma$, *then* $\vDash_{\mathbf{HR}\backslash\{CAN\}} G \mid \vdash \vec{r}.A, \vec{s}.A, \Gamma \mid \vdash \vec{r}.B, \vec{s}.B, \Gamma$.

Proof. If $\vec{r} = \emptyset$ or $\vec{s} = \emptyset$, the result is trivial. Otherwise

$$\frac{\dfrac{\dfrac{\dfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B \mid \vdash \Gamma, \vec{r}.A, \vec{s}.B}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B \mid \vdash \vec{r}.\Gamma, \vec{s}.\Gamma, (\vec{r}\vec{r}.)A, (\vec{r}\vec{s}).A, (\vec{s}\vec{r}).B, (\vec{s}\vec{s}).B} \text{ Lemma 3.25}}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B \mid \vdash \vec{r}.\Gamma, (\vec{r}\vec{r}).A, (\vec{r}\vec{s}).A \mid \vdash \vec{s}.\Gamma, (\vec{s}\vec{r}).B, (\vec{s}\vec{s}).B} \text{ S}}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B \mid \vdash \vec{r}.\Gamma, (\vec{r}\vec{r}).A, (\vec{r}\vec{s}).A \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B} \text{ Lemma 3.24}}{\dfrac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B}{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B} \text{ C}^2} \text{ Lemma 3.24}}$$

$\square$

3.4. **Soundness – Proof of Theorem 3.9.** We need to prove that if there exists a **HR** derivation d of a hypersequent G (written $d \vDash_{\mathbf{HR}} G$) then $(\!|G|\!) \geq 0$ is derivable in equational logic (written $\mathcal{A}_{\text{Riesz}} \vdash (\!|G|\!) \geq 0$). This is done in a straightforward way by showing that each deduction rule of the system **HR** is sound. The desired result then follows immediately by induction on d.

- For the rule

$$\frac{}{\vdash} \text{ INIT}$$

The semantics of the hypersequent consisting only of the empty sequent is $(\!|\vdash|\!) = 0$ and therefore $(\!|\vdash|\!) \geq 0$, as desired.

- For the rule

$$\frac{G}{G \mid \vdash \Gamma} \text{ W}$$

the hypothesis is $(\!(G)\!) \geq 0$ so

$$
\begin{aligned}
(\!(G \mid \vdash \Gamma)\!) &= (\!(G)\!) \sqcup (\!(\vdash \Gamma)\!) \\
&\geq (\!(G)\!) \\
&\geq 0
\end{aligned}
$$

- For the $C, ID, +, 0, \times$ and CAN rules, it is immediate to observe that the interpretation of the only premise and the interpretation of its conclusion are equal, therefore the result is trivial.
- For the rule

$$\frac{G \mid \vdash \Gamma_1, \Gamma_2}{G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2} \text{ S}$$

the hypothesis is $(\!(G \mid \vdash \Gamma_1, \Gamma_2)\!) \geq 0$ so according to Lemma 2.15, $(\!(G)\!)^- \sqcap (\!(\vdash \Gamma_1, \Gamma_2)\!)^- = 0$. Our goal is to prove that $(\!(G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2)\!) \geq 0$. Again, using Lemma 2.15, we equivalently need to prove that

$$(\!(G)\!)^- \sqcap (\!(\vdash \Gamma_1 \mid \vdash \Gamma_2)\!)^- = 0.$$

The above expression is of the form $A^- \sqcup B^-$, and since $A^- \geq 0$ always holds for every A (see Section 2.1.1), it is clear that $(\!(G)\!)^- \sqcap (\!(\vdash \Gamma_1 \mid \vdash \Gamma_2)\!)^- \geq 0$. It remains therefore to show that $(\!(G)\!)^- \sqcap (\!(\vdash \Gamma_1 \mid \vdash \Gamma_2)\!)^- \leq 0$. This is done as follows:

$$
\begin{aligned}
(\!(G)\!)^- \sqcap (\!(\vdash \Gamma_1 \mid \vdash \Gamma_2)\!)^- &\leq (\!(G)\!)^- \sqcap 2.(\!(\vdash \Gamma_1 \mid \vdash \Gamma_2)\!)^- &&\text{since } (\!(\vdash \Gamma_1 \mid \vdash \Gamma_2)\!)^- \geq 0 \\
&= (\!(G)\!)^- \sqcap 2.((\!(\vdash \Gamma_1)\!) \sqcup (\!(\vdash \Gamma_2)\!))^- &&\text{Lemma 2.14[1]} \\
&\leq (\!(G)\!)^- \sqcap ((\!(\vdash \Gamma_1)\!) + (\!(\vdash \Gamma_2)\!))^- &&\text{Lemma 2.14[2-3]} \\
&= (\!(G)\!)^- \sqcap ((\!(\vdash \Gamma_1, \Gamma_2)\!))^- \\
&= 0
\end{aligned}
$$

- For the rule

$$\frac{G \mid \vdash \Gamma_1 \quad G \mid \vdash \Gamma_2}{G \mid \vdash \Gamma_1, \Gamma_2} \text{ M}$$

the hypothesis is

$$
\begin{aligned}
(\!(G \mid \vdash \Gamma_1)\!) &\geq 0 \\
(\!(G \mid \vdash \Gamma_2)\!) &\geq 0
\end{aligned}
$$

so according to Lemma 2.15,

$$
\begin{aligned}
(\!(G)\!)^- \sqcap (\!(\vdash \Gamma_1)\!)^- &= 0 \\
(\!(G)\!)^- \sqcap (\!(\vdash \Gamma_2)\!)^- &= 0
\end{aligned}
$$

Following the same reasoning of the previous case (S rule) our goal is to show that $(\!(G)\!)^- \sqcap (\!(\vdash \Gamma_1, \Gamma_2)\!)^- \leq 0$. This is done as follows:

$$
\begin{aligned}
(\!(G)\!)^- \sqcap (\!(\vdash \Gamma_1, \Gamma_2)\!)^- &= (\!(G)\!)^- \sqcap ((\!(\vdash \Gamma_1)\!) + (\!(\vdash \Gamma_2)\!))^- \\
&\leq (\!(G)\!)^- \sqcap ((\!(\vdash \Gamma_1)\!)^- + (\!(\vdash \Gamma_2)\!)^-) &&\text{Lemma 2.14[4]} \\
&\leq (\!(G)\!)^- \sqcap (\!(\vdash \Gamma_1)\!)^- + (\!(G)\!)^- \sqcap (\!(\vdash \Gamma_2)\!)^- &&\text{distributivity of } \sqcap \text{ over } +
\end{aligned}
$$

- For the rule

$$\frac{G \mid \vdash r.\Gamma}{G \mid \vdash \Gamma} \; \mathrm{T}$$

  the hypothesis is $(\!|G \mid \vdash r.\Gamma|\!) \geq 0$ so using Lemma 2.15, we have

$$(\!|G|\!)^{-} \sqcap r.((\!|\vdash \Gamma|\!))^{-} = (\!|G|\!)^{-} \sqcap ((\!|\vdash r.\Gamma|\!))^{-} = 0$$

  Following the same reasoning of the S rule's case, our goal is to show that $(\!|G|\!)^{-} \sqcap (\!|\vdash \Gamma|\!)^{-} \leq 0$. To do so, we need to distinguish between two cases: whether or not $r \geq 1$.
  If $r \geq 1$, then

$$
\begin{aligned}
(\!|G|\!)^{-} \sqcap (\!|\vdash \Gamma|\!)^{-} &\leq& (\!|G|\!)^{-} \sqcap r.(\!|\vdash \Gamma|\!)^{-}\\
&=& 0
\end{aligned}
$$

  Otherwise, Lemma 2.14[5] states that $(\!|G|\!)^{-} \sqcap (\!|\vdash \Gamma|\!)^{-} \leq 0$ if and only if $r.((\!|G|\!)^{-} \sqcap (\!|\vdash \Gamma|\!)^{-}) \leq 0$, which is proven as follows:

$$
\begin{aligned}
r.((\!|G|\!)^{-} \sqcap (\!|\vdash \Gamma|\!)^{-}) &=& (r.(\!|G|\!)^{-}) \sqcap (r.(\!|\vdash \Gamma|\!)^{-})\\
&\leq& (\!|G|\!)^{-} \sqcap (r.(\!|\vdash \Gamma|\!)^{-})\\
&=& 0
\end{aligned}
$$

  In both cases $(\!|G|\!)^{-} \sqcap (\!|\vdash \Gamma|\!)^{-} \leq 0$.
- For the rule

$$\frac{G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)} \; \sqcup$$

  the hypothesis is $(\!|G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B|\!) \geq 0$. So :

$$
\begin{aligned}
(\!|G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)|\!) &=& (\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.(A \sqcup B)|\!)\\
&=& (\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.A|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.B|\!) \quad \text{distributivity of } \sqcup \text{ over } +\\
&\geq& 0
\end{aligned}
$$

- For the rule

$$\frac{G \mid \vdash \Gamma, \vec{r}.A \quad G \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcap B)} \; \sqcap$$

  the hypothesis is

$$
\begin{aligned}
(\!|G \mid \; \Gamma, \vec{r}.A \vdash \Gamma|\!) &\geq 0\\
(\!|G \mid \; \Gamma, \vec{r}.B \vdash \Gamma|\!) &\geq 0
\end{aligned}
$$

  So

$$
\begin{aligned}
(\!|G \mid \vdash \Gamma, \vec{r}.(A \sqcap B)|\!) &=& (\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.(A \sqcap B)|\!)\\
&=& (\!|G|\!) \sqcup ((\!|\vdash \Gamma, \vec{r}.A|\!) \sqcap (\!|\Gamma, \vec{r}.B|\!)) \quad \text{distributivity of } \sqcap \text{ over } +\\
&=& ((\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.A|\!)) \sqcap ((\!|G|\!) \sqcup (\!|\vdash \Gamma, \vec{r}.B|\!)) \quad \text{distributivity of } \sqcup \text{ over } \sqcap\\
&\geq& 0
\end{aligned}
$$

3.5. **Completeness – Proof of Theorem 3.10.** In order to prove Theorem 3.10 we first prove an equivalent result (Lemma 3.28 below) stating that if $\mathcal{A}_{\mathrm{Riesz}} \vdash A = B$ then the hypersequents $\vdash 1.A, 1.\overline{B}$ and $\vdash 1.B, 1.\overline{A}$ are both derivable. The advantage of this formulation is that it allows for a simpler proof by induction.

From Lemma 3.28 one indeed obtain Theorem 3.10 as a corollary.

Proof of Theorem 3.10. Recall that $\mathcal{A}_{\mathrm{Riesz}} \vdash (\!|G|\!) \geq 0$ is a shorthand for $\mathcal{A}_{\mathrm{Riesz}} \vdash 0 = (\!|G|\!) \sqcap 0$. Hence, from the hypothesis $\mathcal{A}_{\mathrm{Riesz}} \vdash (\!|G|\!) \geq 0$ we can deduce, by using Lemma 3.28, that $\vDash_{\mathbf{HR}} \vdash 1.(0 \sqcap (\!|G|\!)), 1.0$ is provable.

From this we can show that $\vDash_{\mathbf{HR}} G$ by invoking Lemma 3.22. Indeed, if G is $\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$ then $(\!|G|\!) = (\!|\Vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\Vdash \Gamma_n|\!)$ and

(1) by using the invertibility of the 0 rule, $\vdash 1.(0 \sqcap ((\!|\Vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\Vdash \Gamma_n|\!)))$ is derivable,
(2) by using the invertibility of the $\sqcap$ rule, $\vdash 1.((\!|\Vdash \Gamma_1|\!) \sqcup ... \sqcup (\!|\Vdash \Gamma_n|\!))$ is derivable,
(3) by using the invertibility of the $\sqcup$ rule $n - 1$ times, $\vdash 1.(\!|\Vdash \Gamma_1|\!) \mid \ ... \mid \vdash 1.(\!|\Vdash \Gamma_n|\!)$ is derivable,
(4) and finaly, by using the invertibility of the $+$ rule and $\times$ rule, $\vdash \Gamma_1 \mid \ ... \mid \vdash \Gamma_n$ is derivable.

$\square$

Lemma 3.28. *If $\mathcal{A}_{\mathrm{Riesz}} \vdash A = B$ then $\vdash 1.A, 1.\overline{B}$ and $\vdash 1.B, 1.\overline{A}$ are provable.*

Proof. We prove this result by induction on the proof, in equational logic (see Definition 2.6) of $\mathcal{A}_{\mathrm{Riesz}} \vdash A = B$.

- If the proof finishes with

$$\frac{}{\mathcal{A}_{\mathrm{Riesz}} \vdash A = A} \ \mathrm{refl}$$

  we can conclude with Lemma 3.20.
- If the proof finishes with

$$\frac{\mathcal{A}_{\mathrm{Riesz}} \vdash B = A}{\mathcal{A}_{\mathrm{Riesz}} \vdash A = B} \ \mathrm{sym}$$

  then the induction hypothesis allows us to conclude.
- If the proof finishes with

$$\frac{\mathcal{A}_{\mathrm{Riesz}} \vdash A = C \quad \mathcal{A}_{\mathrm{Riesz}} \vdash C = B}{\mathcal{A}_{\mathrm{Riesz}} \vdash A = B} \ \mathrm{trans}$$

  then the induction hypothesis is

$$\vDash_{\mathbf{HR}} 1.A, 1.\overline{C}$$
$$\vDash_{\mathbf{HR}} 1.C, 1.\overline{A}$$
$$\vDash_{\mathbf{HR}} 1.C, 1.\overline{B}$$
$$\vDash_{\mathbf{HR}} 1.B, 1.\overline{C}$$

  We will show that $\vDash_{\mathbf{HR}} 1.A, 1.\overline{B}$, the other one is similar.

$$\frac{\dfrac{\vdash 1.A, 1.\overline{C} \quad \vdash 1.C, 1.\overline{B}}{\vdash 1.A, 1.\overline{B}, 1.C, 1.\overline{C}} \ \mathrm{M}}{\vdash 1.A, 1.\overline{B}} \ \mathrm{CAN}$$

- If the proof finishes with

$$\frac{\mathcal{A}_{\text{Riesz}} \vdash f(x) = g(x)}{\mathcal{A}_{\text{Riesz}} \vdash f(A) = g(A)} \text{ subst}$$

  we conclude using the induction hypothesis and Lemma 3.21.

- If the proof finishes with

$$\frac{\mathcal{A}_{\text{Riesz}} \vdash A = B}{\mathcal{A}_{\text{Riesz}} \vdash C[A] = C[B]} \text{ ctxt}$$

  we prove the result by induction on C. For instance, if $C = rC'$ with $r > 0$, then the induction hypothesis is $\vDash_{\mathbf{HR}} 1.C'[A], 1.\overline{C'[B]}$ and $\vDash_{\mathbf{HR}} 1.C'[B], 1.\overline{C'[A]}$ so

$$\frac{\dfrac{\vdash (\frac{1}{r}r).C'[A], (\frac{1}{r}r).\overline{C'[B]}}{\vdash r.C'[A], r.\overline{C'[B]}} T}{\vdash 1.C[A], 1.\overline{C[B]}} \times^* \qquad\qquad \frac{\dfrac{\vdash (\frac{1}{r}r).C'[B], (\frac{1}{r}r).\overline{C'[A]}}{\vdash r.C'[B], r.\overline{C'[A]}} T}{\vdash 1.C[B], 1.\overline{C[A]}} \times^*$$

- It now remains to consider the cases when the the proof finishes with one of the axioms of Figure 1. We only show the nontrivial cases.

  – If the proof finishes with

$$\frac{}{\mathcal{A}_{\text{Riesz}} \vdash (r_1 + r_2)x = r_1 x + r_2 x} \text{ ax}$$

   then

$$\frac{\dfrac{\dfrac{\overline{\vdash} \text{ INIT}}{\vdash (r_1 + r_2).x, r_1.\overline{x}, r_2.\overline{x}} \text{ ID}}{\vdash 1.((r_1 + r_2)x), 1.r_1\overline{x}, 1.r_2\overline{x}} \times^*}{\vdash 1.((r_1 + r_2)x), 1.(r_1\overline{x} + r_2\overline{x})} +$$

   and

$$\frac{\dfrac{\dfrac{\overline{\vdash} \text{ INIT}}{\vdash r_1.x, r_2.x, (r_1 + r_2).\overline{x}} \text{ ID}}{\vdash 1.r_1 x, 1.r_2 x, 1.((r_1 + r_2)\overline{x})} \times^*}{\vdash 1.(r_1 x + r_2 x), 1.((r_1 + r_2)\overline{x})} +$$

  – If the proof finishes with

$$\frac{}{\mathcal{A}_{\text{Riesz}} \vdash (r(x \sqcap y)) \sqcap ry = r(x \sqcap y)} \text{ ax}$$

   then

$$\frac{\dfrac{\overline{\vdash} \text{ INIT}}{\vdash 1.(r(x \sqcap y)), 1.r(\overline{x} \sqcup \overline{y})} \text{ Lemma 3.20} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash} \Delta}{\vdash r.y, r.\overline{y}} \text{ ID}}{\vdash r.y, r.\overline{x} \mid \vdash r.y, r.\overline{y}} \text{ W}}{\vdash r.y, r.(\overline{x} \sqcup \overline{y})} \sqcup}{\vdash 1.ry, 1.r(\overline{x} \sqcup \overline{y})} \times^*}{\vdash 1.((r(x \sqcap y)) \sqcap ry), 1.r(\overline{x} \sqcup \overline{y})} \sqcap$$

   and

$$\frac{\dfrac{\overline{\vdash} \Delta}{\vdash 1.(r(x \sqcap y)), 1.(r(\overline{x} \sqcup \overline{y}))} \text{ Lemma 3.20}}{\vdash 1.(r(x \sqcap y)), 1.((r(\overline{x} \sqcup \overline{y})) \sqcup (r\overline{y}))} \sqcup - \text{W}$$

$\square$

Remark 3.29. By inspecting the proof of Lemma 3.28 it is possible to verify that the T rule is never used in the construction of $\vDash_{\mathbf{HR}} G$. This, together with the similar Remark 3.23 regarding Lemma 3.22, implies that the T rule is never used in the proof of the completeness Theorem 3.10. From this we get the following corollary.

Corollary 3.30. *The T rule is admissible in the system* **HR**.

It turns out, however, that there is no hope of eliminating both the T rule and the CAN rule from the **HR** system.

Lemma 3.31. *Let r_1 and r_2 be two irrational numbers that are algebraically independent over $\mathbb{Q}$ (so there is no $q \in \mathbb{Q}$ such that $qr_1 = r_2$). Then the atomic hypersequent G*

$$\vdash r_1.x \mid \vdash r_2.\overline{x}$$

does not have a CAN–free and T–free proof.

Proof. This is a corollary of the next Lemma 3.32. The idea is that in the **HR** system without the T rule and the CAN rule, the only way to derive G is by applying the structural rules S, C, W, M and the ID rule. Each of these rules can be seen as adding up the sequents in G or multiplying them up by a positive natural number scalar. The algebraic independence of r_1 and r_2 implies that it is not possible to construct a proof. $\square$

Lemma 3.32. *For all atomic hypersequent G formed using the variables and negated variables $x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}$ of the form*

$$\vdash \Gamma_1 \mid \ \ldots \mid \vdash \Gamma_m$$

where $\Gamma_i = \vec{r}_{i,1}.x_1, ..., \vec{r}_{i,k}.x_k, \vec{s}_{i,1}.\overline{x_1}, ..., \vec{s}_{i,k}.\overline{x_{i,k}}$, the following are equivalent:

(1) *G has a CAN–free and T–free proof.*
(2) *there exist natural numbers $n_1, ..., n_m \in \mathbb{N}$, one for each sequent in G, such that:*
   - *there exists $i \in [1..m]$ such that $n_i \neq 0$, i.e., the numbers are not all 0's, and*
   - *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*

   $$\sum_{i=1}^{m} n_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} n_i(\sum \vec{s}_{i,j})$$

   i.e., the scaled (by the numbers $n_1 \ldots n_m$) sum of the coefficients in front of the variable x_j is equal to the scaled sum of the coefficients in from of the covariable $\overline{x_j}$.

Proof. We prove (1) $\Rightarrow$ (2) by induction on the proof of G. We show only the M case, the other cases being trivial:

- If the proof finishes with

$$\frac{\vdash \Gamma_1 \mid ... \mid\vdash \Gamma_m \quad \vdash \Gamma_1 \mid ... \mid\vdash \Gamma'_m}{\vdash \Gamma_1 \mid ... \mid\vdash \Gamma_m, \Gamma'_m} \ \text{M}$$

  by induction hypothesis, there are $n_1, ..., n_m \in \mathbb{N}$ and $n'_1, ..., n'_m \in \mathbb{N}$ such that :
  – there exists $i \in [1..m]$ such that $n_i \neq 0$.
  – for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that $\sum_i n_i. \sum \vec{r}_{i,j} = \sum_i n_i. \sum \vec{s}_{i,j}$.
  – there exists $i \in [1..m]$ such that $n'_i \neq 0$.
  – for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that $\sum_{i=0}^{m-1} n'_i. \sum \vec{r}_{i,j} + n'_m. \sum \vec{r'}_{m,j} = \sum_{i=0}^{m-1} n'_i. \sum \vec{s}_{i,j} + n'_m. \sum \vec{s'}_{m,j}$.

If $n_m = 0$ then $n_1, ..., n_{m-1}, 0$ satisfies the property.

Otherwise if $n'_m = 0$ then $n'_1, ..., n'_{m-1}, 0$ satisfies the property.

Otherwise, $n_m.n'_1 + n'_m.n_1, n_m.n'_2 + n'_m.n_2, ..., n_m.n'_{m-1} + n'_m.n_{m-1}, n_m.n'_m$.

The other way $((2) \Rightarrow (1))$ is more straightforward. If there exist natural numbers $n_1, ..., n_m \in \mathbb{N}$, one for each sequent in G, such that:

- there exists $i \in [1..m]$ such that $n_i \neq 0$ and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} n_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} n_i(\sum \vec{s}_{i,j})$$

then we can use the W rule to remove the sequents corresponding to the numbers $n_i = 0$, and use the C rule $n_i - 1$ times then the S rule $n_i - 1$ times on the ith sequent to multiply it by n_i. If we assume that there is a natural number l such that $n_i = 0$ for all $i > l$ and $n_i \neq 0$ for all $i \leq l$, then the CAN–free T–free proof is:

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\vDash}{\vdash \Gamma_1{}^{n_1}, \ldots, \Gamma_l{}^{n_l}} \text{ID}^*
}{\vdash \Gamma_1{}^{n_1} \mid \ldots \mid \vdash \Gamma_l{}^{n_l}} \text{S}^*
}{\vdash \Gamma_1 \mid \ldots \mid \vdash \Gamma_l} \text{C-S}^*
}{\vdash \Gamma_1 \mid \ldots \mid \Gamma_m} \text{W}^* \quad \text{INIT}
$$

where Γ^n stands for $\underbrace{\Gamma, \ldots, \Gamma}_{n}$. $\qquad\qquad\qquad\square$

We can state a similar result regarding proofs that use the T rule. This will be useful in the proofs of the rational T–elimination and the decidability theorems. The only difference is that since the T rule can multiply a sequent by any strictly positive real number, the coefficients in the statement are arbitrary positive real numbers instead of natural numbers.

Lemma 3.33. *For all atomic hypersequent G formed using the variables and negated variables $x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}$ of the form*

$$\vdash \Gamma_1 \mid \ldots \mid \vdash \Gamma_m$$

where $\Gamma_i = \vec{r}_{i,1}.x_1, ..., \vec{r}_{i,k}.x_k, \vec{s}_{i,1}.\overline{x_1}, ..., \vec{s}_{i,k}.\overline{x_{i,k}}$, the following are equivalent:

(1) *G has a CAN–free proof.*

(2) *there exist numbers $t_1, ..., t_m \in \mathbb{R}_{\geq 0}$, one for each sequent in G, such that:*
   - *there exists $i \in [1..m]$ such that $t_i \neq 0$, i.e., the numbers are not all 0's, and*
   - *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*

$$\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} t_i(\sum \vec{s}_{i,j})$$

   i.e., the scaled (by the numbers $t_1 \ldots t_m$) sum of the coefficients in front of the variable x_j is equal to the scaled sum of the coefficients in from of the covariable $\overline{x_j}$.

Proof. We prove $(1) \Rightarrow (2)$ by induction on the proof of G. We will only deal with the case of T rule since every other cases are exactly the same as in Lemma 3.33. If the proof finishes

with

$$\frac{\vdash \Gamma_1 \mid \ \ldots \mid \vdash r.\Gamma_m}{\vdash \Gamma_1 \mid \ \ldots \mid \vdash \Gamma_m} \ \text{T}$$

then by induction hypothesis there are $t_1, ..., t_m \in \mathbb{R}$ such that :

- there exists $i \in [1..m]$ such that $t_i \neq 0$.
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that $\sum_{i=0}^{m-1} t_i. \sum \vec{r}_{i,j} + t_m. \sum r\vec{r}_{m,j} = \sum_{i=0}^{m-1} t_i. \sum \vec{s}_{i,j} + t_m. \sum r\vec{s}_{m,j}$.

so $t_1, \ldots, t_{m-1}, rt_m$ satisfies the property.

The other way $((2) \Rightarrow (1))$ is very similar to the previous lemma, only using the T rule instead of the C and S rules. If there exist numbers $t_1, ..., t_m \in \mathbb{R}$, one for each sequent in G, such that:

- there exists $i \in [1..m]$ such that $t_i \neq 0$ and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} t_i(\sum \vec{s}_{i,j})$$

then we can use the W rule to remove the sequents corresponding to the numbers $t_i = 0$, and use the T rule on the ith sequent to multiply it by t_i. If we assume that there is a natural number l such that $t_i = 0$ for all $i > l$ and $t_i \neq 0$ for all $i \leq l$, then the CAN–free proof is:

$$\frac{\dfrac{\dfrac{\overline{\vdash} \ \text{INIT}}{\vdash t_1.\Gamma_1, \ldots, t_l.\Gamma_l} \ \text{ID}^*}{\vdash t_1.\Gamma_1 \mid \ \ldots \mid \vdash t_l.\Gamma_l} \ \text{S}^*}{\dfrac{\vdash \Gamma_1 \mid \ \ldots \mid \vdash \Gamma_l}{\vdash \Gamma_1 \mid \ \ldots \mid \ \Gamma_m} \ \text{W}^*} \ \text{T}^*$$

$\square$

3.6. **CAN–free Invertibility – Proof of Theorem 3.11.** In this section, we go through the details of the proof of Theorem 3.11.

It is technically convenient, in order to carry out the inductive argument, to prove a slightly stronger result, expressed as the invertibility of more general logical rules that can act on the same formula on different sequents of the hypersequent, at the same time. The generalised rules are the following:

$$\boxed{\begin{array}{l} \textbf{Logical rules:} \\ \dfrac{[\vdash \Gamma_i]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.0]_{i=1}^n} \ 0 \quad \dfrac{[\vdash \Gamma_i, \vec{r}_i.A, \vec{r}_i.B]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.(A+B)]_{i=1}^n} \ + \quad \dfrac{[\vdash \Gamma_i, (s\vec{r}_i).A]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.(sA)]_{i=1}^n} \ \times \\[2ex] \dfrac{[\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=1}^n} \ \sqcup \quad \dfrac{[\vdash \Gamma_i, \vec{r}_i.A]_{i=1}^n \quad [\vdash \Gamma_i, \vec{r}_i.B]_{i=1}^n}{[\vdash \Gamma_i, \vec{r}_i.(A \sqcap B)]_{i=1}^n} \ \sqcap \end{array}}$$

Figure 7: Generalised logical rules

We conceptually divide the logical rules in three categories:

- The rules with only one premise and that do not change the number of sequents – the $0, +, \times$ rules.
- The rule with two premises – the $\sqcup$ rule.
- The rule with only one premise but that adds one sequent to the hypersequent – the $\sqcup$ rule.

Because of the similarities of the rules in each of these categories, we just prove the CAN–free invertibility of one rule in each category by means of a sequence of lemmas.

Lemma 3.34. *If d is a CAN–free proof of $[\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=1}^{n}$ then $[\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=1}^{n}$ has a CAN–free proof.*

Proof. By induction on d. Most cases are easy except the cases for when the proof ends with a M rule or a $\sqcap$ rule so we will only show those cases.

- If d finishes with

$$\frac{G \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcup B) \quad G \mid \vdash \Gamma_2, \vec{r}_2.(A \sqcup B)}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.(A \sqcup B), \vec{r}_2.(A \sqcup B)} \text{ M}$$

with $G = [\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=3}^{n}$ and $G' = [\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=3}^{n}$ then by induction hypothesis on the CAN–free proofs of the premises we have that

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_1, \vec{r}_1.B$$

and

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_2, \vec{r}_2.A \mid \vdash \Gamma_2, \vec{r}_2.B$$

are derivable by CAN–free proofs. We want to prove that both

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_2, \vec{r}_2.B$$

and

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_2, \vec{r}_2.A \mid \vdash \Gamma_1, \vec{r}_1.B$$

are CAN–free derivable, as this will allow us to conclude by application of the M rule as follows:

$$\frac{\dfrac{G \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_2, \vec{r_2}.B \quad G \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_2, \vec{r_2}.B}{G \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.B, \vec{r_2}.B} \text{ M} \quad \dfrac{G \mid \vdash \Gamma_2, \vec{r_2}.A \mid \vdash \Gamma_1, \vec{r_1}.B \quad G \mid \vdash \Gamma_2, \vec{r_2}.A \mid \vdash \Gamma_2, \vec{r_2}.B}{G \mid \vdash \Gamma_2, \vec{r_2}.A \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.B, \vec{r_2}.B} \text{ M}}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.A, \vec{r_2}.A \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.B, \vec{r_2}.B} \text{ M}$$

If $\vec{r}_1 = \emptyset$ or $\vec{r}_2 = \emptyset$, those two hypersequents are derivable using the W rule then the C rule.

Otherwise, by using the M rule, Lemma 3.25 and the W rule, we have

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_2, \vec{r}_2.B \mid \vdash \vec{r}_2.\Gamma_1, \vec{r}_1.\Gamma_2, (\vec{r}_1\vec{r}_2)A, (\vec{r}_1\vec{r}_2)B$$

and

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_2, \vec{r}_2.A \mid \vdash \Gamma_1, \vec{r}_1.B \mid \vdash \vec{r}_2.\Gamma_1, \vec{r}_1.\Gamma_2, (\vec{r}_1\vec{r}_2)A, (\vec{r}_1\vec{r}_2)B$$

We can then conclude using the C rule, Lemma 3.24 and the S rule.

- If d finishes with

$$\frac{G \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcup B), \vec{s}.C \quad G \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcup B), \vec{s}.D}{G \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcup B), \vec{s}.(C \sqcap D)} \sqcap$$

with $G = [\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=2}^n$ and $G' = [\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=2}^n$, then by induction hypothesis on the CAN–free proofs of the premises we have that

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{s}.C \mid \vdash \Gamma_1, \vec{r}_1.B, \vec{s}.C$$

and

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{s}.D \mid \vdash \Gamma_1, \vec{r}_1.B, \vec{s}.D$$

so by using the M rule and the W rule, we can derive

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{s}.C \mid \vdash \Gamma_1, \vec{r}_1.B, \vec{s}.D \mid \vdash \Gamma_1, \Gamma_1, \vec{r}_1.A, \vec{r}_1.B, \vec{s}.C, \vec{s}.D$$

and

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{s}.D \mid \vdash \Gamma_1, \vec{r}_1.B, \vec{s}.C \mid \vdash \Gamma_1, \Gamma_1, \vec{r}_1.A, \vec{r}_1.B, \vec{s}.C, \vec{s}.D$$

and then with the C rule and the S rule

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{s}.C \mid \vdash \Gamma_1, \vec{r}_1.B, \vec{s}.D$$

and

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{s}.D \mid \vdash \Gamma_1, \vec{r}_1.B, \vec{s}.C$$

We can then conclude with the $\sqcap$ rule.

$\square$

Lemma 3.35. *If d is a CAN–free-proof of $[\vdash \Gamma_i, \vec{r}_i.(A+B)]_{i=1}^n$ then $[\vdash \Gamma_i, \vec{r}_i.A, \vec{r}_i.B]_{i=1}^n$ has a CAN–free proof.*

Proof. Straightforward induction on d. For instance if d finishes with

$$\frac{G \mid \vdash \Gamma_1, \vec{r}_1.(A+B) \quad G \mid \vdash \Gamma_2, \vec{r}_2.(A+B)}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.(A+B), \vec{r}_2.(A+B)} \ \text{M}$$

with $G = [\vdash \Gamma_i, \vec{r}_i.(A+B)]_{i=3}^n$ and $G' = [\vdash \Gamma_i, \vec{r}_i.A, \vec{r}_i.B]_{i=3}^n$, then by induction hypothesis on the CAN–free proofs of the premises we have that

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}_1.B$$

and

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} G' \mid \vdash \Gamma_2, \vec{r}_2.A, \vec{r}_2.B$$

so

$$\frac{G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}_1.B \quad G' \mid \vdash \Gamma_2, \vec{r}_2.A, \vec{r}_2.B}{G' \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.A, \vec{r}_2.A, \vec{r}_1.B, \vec{r}_2.B} \ \text{M}$$

$\square$

Lemma 3.36. *If d is a CAN–free proof of $[\vdash \Gamma_i, \vec{r}_i.(A \sqcap B)]_{i=1}^n$ then $[\vdash \Gamma_i, \vec{r}_i.A]_{i=1}^n$ and $[\vdash \Gamma_i, \vec{r}_i.B]_{i=1}^n$ have a CAN–free proof.*

Proof. A straightforward induction on d. For instance if d finishes with

$$\frac{G \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcap B) \quad G \mid \vdash \Gamma_2, \vec{r}_2.(A \sqcap B)}{G \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.(A \sqcap B), \vec{r}_2.(A \sqcap B)} \ \text{M}$$

with $G = [\vdash \Gamma_i, \vec{r}_i.(A \sqcap B)]_{i=3}^n$, then by induction hypothesis on the CAN–free proofs of the premises we have that

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} [\vdash \Gamma_i, \vec{r}_i.A]_{i=3}^n \mid \vdash \Gamma_1, \vec{r}_1.A$$

and

$$\vDash_{\mathbf{HR}\backslash\{\text{CAN}\}} [\vdash \Gamma_i, \vec{r}_i.A]_{i=3}^n \mid \vdash \Gamma_2, \vec{r}_2.A$$

so

$$\frac{[\vdash \Gamma_i, \vec{r}_i.A]_{i=3}^n \mid \vdash \Gamma_1, \vec{r}_1.A \quad [\vdash \Gamma_i, \vec{r}_i.A]_{i=3}^n \mid \vdash \Gamma_2, \vec{r}_2.A}{[\vdash \Gamma_i, \vec{r}_i.A]_{i=3}^n \mid \vdash \Gamma_1, \Gamma_2, \vec{r}_1.A, \vec{r}_2.A} \; \text{M}$$

$\square$

3.7. M-elimination – Proof of Theorem 3.12. We need to show that for each hypersequent G and sequents Γ and Δ, if there exist CAN–free and M–free proofs d_1 of $G \mid \vdash \Gamma$ and d_2 of $G \mid \vdash \Delta$, then there exists also a CAN–free and M–free proof of $G \mid \vdash \Gamma, \Delta$.

The idea behind the proof is to combine d_1 and d_2 in a sequential way. First we take the proof d_1 and we modify it into a CAN–free and M–free preproof (i.e., an unfinished proof) of

$$G \mid \; G \mid \vdash \Gamma, \Delta$$

where all the leaves in the preproof are either terminated (by the INIT axiom) or non–terminated and of the form:

$$G \mid \vdash \vec{r}.\Delta$$

for some vector $\vec{r}$ of scalars. Then we use the proof d_2 to construct a CAN–free and M–free proof of each

$$G \mid \vdash \vec{r}.\Delta$$

hence completing the preproof of

$$G \mid \; G \mid \vdash \Gamma, \Delta$$

into a full proof. From this it is possible to obtain the desired CAN–free and M–free proof of $G \mid \vdash \Gamma, \Delta$ using several times the C rule:

$$\frac{G \mid \; G \mid \vdash \Gamma, \Delta}{G \mid \vdash \Gamma, \Delta} \; \text{C}^*$$

In what follows, the first step is formalized as Lemma 3.37 and the second step as Lemma 3.38.

Lemma 3.37. *Let d_1 be a CAN–free and M–free derivation of $G \mid \vdash \Gamma$ and let Δ be a sequent. Then there exists a preproof of*

$$G \mid \; G \mid \vdash \Gamma, \Delta.$$

where all non–terminated leaves are all of the form $G \mid \vdash \vec{r}.\Delta$ for some vector $\vec{r}$.

Proof. This is an instance of the slightly more general statement of Lemma 3.39 below. $\square$

Lemma 3.38. *Let d_2 be CAN–free and M–free derivation of $G \mid \vdash \Delta$. Then, for every vector $\vec{r}$, there exists a CAN–free and M–free proof of*

$$G \mid \vdash \vec{r}.\Delta$$

Proof. This is an instance of the slightly more general statement of Lemma 3.40 below. $\square$

Lemma 3.39. *Let d_1 be a CAN–free and M–free deriviation of $[\vdash \Gamma_i]_{i=1}^n$ and let G be a hypersequent and Δ be a sequent. Then for every sequence of vectors $\vec{r}_i$, there exists a preproof of*

$$G \mid \; [\vdash \Gamma_i, \vec{r}_i.\Delta]_{i=1}^n$$

where all non-terminated leaves are of the form $G \mid \; \vec{r}.\Delta$ for some vector $\vec{r}$.

Proof. By straightforward induction on d_1. $\qquad\square$

Lemma 3.40. *If d is a CAN–free M–free proof of $[\vdash \Delta_i]_{i=1}^n$ then for all $\vec{r_i}$, there is a CAN–free M–free proof of $[\vdash \vec{r_i}.\Delta_i]_{i=1}^n$.*

Proof. By induction on d. We show the only nontrivial case:

- If d finishes with

$$\frac{[\vdash \Delta_i]_{i=3}^n \mid \vdash \Delta_1, \Delta_2}{[\vdash \Delta_i]_{i=3}^n \mid \vdash \Delta_1 \mid \vdash \Delta_2} \text{ S}$$

By induction hypothesis there is CAN–free proof d' of

$$[\vdash \vec{r_i}.\Delta_i]_{i=3}^n \mid \vdash (\vec{r_1}\vec{r_2}).\Delta_1, (\vec{r_1}\vec{r_2}).\Delta_2$$

If $\vec{r_1} = \emptyset$ or $\vec{r_2} = \emptyset$, we have the empty sequent which is derivable. Otherwise,

$$\frac{\dfrac{[\vdash \vec{r_i}.\Delta_i]_{i=3}^n \mid \vdash (\vec{r_1}\vec{r_2}).\Delta_1, (\vec{r_1}\vec{r_2}).\Delta_2}{[\vdash \vec{r_i}.\Delta_i]_{i=3}^n \mid \vdash (\vec{r_1}\vec{r_2}).\Delta_1 \mid \vdash (\vec{r_1}\vec{r_2}).\Delta_2} \text{ S}}{[\vdash \vec{r_i}.\Delta_i]_{i=3}^n \mid \vdash \vec{r_1}.\Delta_1 \mid \vdash \vec{r_2}.\Delta_2} \text{ Lemma 3.24}$$

$\qquad\square$

3.8. CAN–elimination – Proof of Theorem 3.13.

The CAN rule has the following form:

$$\frac{G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}}{G \mid \vdash \Gamma} \text{ CAN}, \sum \vec{r} = \sum \vec{s}$$

We prove Theorem 3.13 by showing that if the hypersequent $G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}$ has a CAN–free derivation then also the hypersequent $G \mid \vdash \Gamma$ has a CAN–free derivation.

Our proof proceeds by induction on the complexity of the formula A. The base case is given by $A = x$ (or equivalently $A = \overline{x}$) for some variable x. The following lemma proves this base case.

Lemma 3.41. *If there is a CAN–free proof d of $G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}$, where $\sum \vec{r} = \sum \vec{s}$ then there exists a CAN–free proof of $G \mid \vdash \Gamma$.*

Proof. By the M–elimination Theorem 3.12, we can assume that d is CAN–free and also M–free. The statement then follows as a special case of Lemma 3.43 below. The formulation of Lemma 3.43 is more technical but allows for a simpler proof by induction on the structure of d. $\qquad\square$

For complex formulas A, we proceed by using the CAN–free invertibility Theorem 3.11 as follows:

- If $A = x$, we are in the base case of Lemma 3.41.
- If $A = 0$, we can conclude with the CAN–free invertibility of the 0 rule.
- If $A = B + C$, since the + rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.B, \vec{r}.C, \vec{s}.\overline{B}, \vec{s}.\overline{C}$ has a CAN–free proof. Therefore we can have a CAN–free proof of the hypersequent $G \mid \vdash \Gamma$ by invoking the induction hypothesis twice, since the complexity of B and C is lower than that of $B + C$.
- If $A = r'B$, since the $\times$ rule is CAN–free invertible, $G \mid \vdash \Gamma, (r'\vec{r}).B, (r'\vec{s}).\overline{B}$ has a CAN–free proof. Therefore we can have a CAN–free proof of the hypersequent $G \mid \vdash \Gamma$ by invoking the inductive hypothesis on the simpler formula B.

- If $A = B \sqcup C$, since the $\sqcup$ rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.(\overline{B} \sqcap \overline{C}) \mid \vdash \Gamma, \vec{r}.C, \vec{s}.(\overline{B} \sqcap \overline{C})$ has a CAN–free proof. Then, since the $\sqcap$ rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.\overline{B} \mid \vdash \Gamma, \vec{r}.C, \vec{s}.\overline{C}$ has a CAN–free proof. Therefore we can have a CAN–free proof of the hypersequent $G \mid \vdash \Gamma \mid \vdash \Gamma$ by invoking the induction hypothesis twice on the simpler formulas B and C.

  We can then derive the hypersequent $G \mid \vdash \Gamma$ as:
  $$\frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma} \; C$$

- If $A = B \sqcap C$, since the $\sqcup$ rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.(B \sqcap C), \vec{s}.\overline{B} \mid \vdash \Gamma, \vec{r}.(B \sqcap C), \vec{s}.\overline{C}$ has a CAN–free proof. Then, since the $\sqcap$ rule is CAN–free invertible, $G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.\overline{B} \mid \vdash \Gamma, \vec{r}.C, \vec{s}.\overline{C}$ has a CAN–free proof. Therefore we can have a CAN–free proof of the hypersequent $G \mid \vdash \Gamma \mid \vdash \Gamma$ by invoking the induction hypothesis twice on the simpler formulas B and C.

  We can then derive the hypersequent $G \mid \vdash \Gamma$ as:
  $$\frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma} \; C$$

  This concludes the proof of Theorem 3.13. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

One direct consequence of the CAN-elimination Theorem 3.13 is that we can strengthen the statement of Lemma 3.33 replacing "CAN–free proof" with just "proof", as follows.

Corollary 3.42. *For all atomic hypersequent G formed using the variables and negated variables $x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}$ of the form*

$$\vdash \Gamma_1 \mid \ldots \mid \vdash \Gamma_m$$

where $\Gamma_i = \vec{r}_{i,1}.x_1, ..., \vec{r}_{i,k}.x_k, \vec{s}_{i,1}.\overline{x_1}, ..., \vec{s}_{i,k}.\overline{x_k}$, the following are equivalent:

(1) *G has a proof.*
(2) *there exist real numbers $t_1, ..., t_m \in \mathbb{R}_{\geq 0}$, one for each sequent in G, such that:*
  - *there exists $i \in [1..m]$ such that $t_i \neq 0$, i.e., the numbers are not all 0's, and*
  - *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*

  $$\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} t_i(\sum \vec{s}_{i,j})$$

  i.e., the scaled (by the numbers $t_1 \ldots t_m$) sum of the coefficients in front of the variable x_j is equal to the sum of the coefficients in from of the covariable $\overline{x_j}$.

Lemma 3.43. *If there is a CAN–free and M–free proof of the hypersequent*

$$[\vdash \Gamma_i, \vec{r_i}.x, \vec{s_i}.\overline{x}]_{i=1}^{n}$$

then for all $\vec{r_i'}$ and $\vec{s_i'}$, with $\leq i \leq n$, such that $\sum \vec{r_i} - \sum \vec{s_i} = \sum \vec{r_i'} - \sum \vec{s_i'}$, there is a CAN–free, M–free proof of

$$\left[\vdash \Gamma_i, \vec{r_i'}.x, \vec{s_i'}.\overline{x}\right]_{i=1}^{n}$$

Proof. By induction on the derivation d of $[\vdash \Gamma_i, \vec{r_i}.x, \vec{s_i}.\overline{x}]_{i=1}^{n}$. Most cases are trivial, we just describe the more interesting one.

- If d finishes with:

$$\frac{[\vdash \Gamma_i, \vec{r_i}.x, \vec{r_i}.\overline{x}]_{i\geq 2} \mid \vdash \Gamma_1, \vec{c}.x, \vec{c'}.\overline{x}}{[\vdash \Gamma_i, \vec{r_i}.x, \vec{r_i}.\overline{x}]_{i\geq 2} \mid \vdash \Gamma_1, (\vec{a}; \vec{b}; \vec{c}).x, (\vec{a'}; \vec{b'}; \vec{c'}).\overline{x}} \text{ ID}$$

with $\vec{r_1} = \vec{b}; \vec{c}$ and $\vec{s_1} = \vec{b'}; \vec{c'}$, then $\sum \vec{c} - \sum \vec{c'} = \sum \vec{r'_1} + \sum \vec{a} - (\sum \vec{s'_1} + \sum \vec{a'})$, so by induction hypothesis, we have

$$\vDash_{\mathbf{HR}\setminus\{\mathrm{CAN}\}} \left[\vdash \Gamma_i, \vec{r'_i}.x, \vec{s'_i}.\overline{x}\right]_{i\geq 2} \mid \Gamma_1, (\vec{a}; \vec{r'_1}).x, (\vec{a'}; \vec{s'_1}).\overline{x}$$

which is the result we want.

$\square$

3.9. **Rational T-elimination – Proof of Theorem 3.15.** We need to prove that if a hypersequest sequent G, with all scalars in $\mathbb{Q}$, has a CAN–free derivation then it also has a CAN–free and T–free derivation.

Firstly, we observe that we can restrict to the case of G being an atomic hypersequent. Indeed, if G is not atomic, we can iteratively apply the logical rules (see Figure 6 on page 12) and reduce G to a number of atomic hypersequents $G_1, \ldots, G_n$. By the CAN–free invertibility Theorem 3.11, G is CAN–free derivable if and only if all G_i are CAN–free derivable.

Secondly, assume G is atomic and has a CAN–free derivation. Then, by application of Lemma 3.33 and using the same notation, there are $t_1, ..., t_m$ in $\mathbb{R}_{\geq 0}$ such that
- there exists $i \in [1..m]$ such that $t_i \neq 0$ and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} t_i(\sum \vec{s}_{i,j})$$

Since all coefficients are rational and the theory of linear arithmetic over $\mathbb{R}$ is a elementary extension of that of linear arithmetic over $\mathbb{Q}$ [FR75], there are $q_1, ..., q_m \in \mathbb{Q}_{\geq 0}$ satisfying the same property of $t_1, ..., t_m$. By multiplying all q_i by the least common multiple of their denominators, we get a solution $k_1, ..., k_m$ in $\mathbb{N}$. So according to Lemma 3.32, G has also a CAN–free and T–free derivation. This concludes the proof.

3.10. **Decidability – Proof of Theorem 4.11.** The previous results give us a simple algorithm for deciding if a hypersequent G is derivable in the system **HR**. The algorithm works in two steps:

(1) the problem of deciding if G is derivable is reduced to the problem of deciding if a finite number of atomic hypersequents $G_1, \ldots, G_n$ are derivable.
(2) A decision procedure for atomic hypersequents is executed and it verifies if all hypersequents computed at the first step are derivable.

The first step consists in applying recursively all possible logical rules to G until atomic premises $G_1, ..., G_n$ are obtained (see Figure 6 on page 12).
Indeed, the CAN–free invertibility Theorem 3.11 guarantees that G is derivable if and only if all the atomic hypersequents obtained in this way are derivable.

The second step can be performed using Corollary 3.42 which states that the hypersequent G_i is derivable if and only if there exist a sequence of real numbers $\vec{s} \in \mathbb{R}_{\geq 0}$ satisfying a system of (in)equations. This can be expressed directly by a (existentially quantified) formula in the first order theory of the real–closed field $FO(\mathbb{R}, +, \times, \leq)$. It is well known that this theory is decidable and admits quantifier elimination [Tar51]. Thus it is possible to decide if this formula is satifiable or not, that is, if the atomic hypersequent G_i is derivable or not.

The idea behind the above algorithm, reducing the problem of derivability to the problem of verifying the satisfiability of formulas in the first order theory of the real–closed field, can in fact be pushed forward. Not only we can decide if G is derivable or not, but we can return a formula $\phi \in FO(\mathbb{R}, +, \times, \leq)$ which describes the set of real–values assigned to the scalars in G that admits a derivation. For example, as explained in Subsection 3.1, consider the following simple hypersequent

$$\vdash r.x, r.\overline{x}$$

Not only this hypersequent is derivable for a fixed scalar $r \in \mathbb{R}_{>0}$, but the hypersequent

$$\vdash \alpha.x, \alpha.\overline{x}$$

is derivable for any assignment of concrete scalars in $\mathbb{R}_{>0}$ to the scalar–variable α.

Similarly, the hypersequent containing the scalar–variable α and two concrete scalars s and t

$$\vdash \alpha.x, s.\overline{x}, t.\overline{x}$$

is derivable for all concrete $r \in \mathbb{R}_{>0}$ assignments to α such that $r = s + t$.

Lastly, the hypersequent containing two scalar–variables α, β and two concrete scalars s and t

$$\vdash (\alpha^2 - \beta).x, s.\overline{x}, t.\overline{x}$$

is derivable for any assignment of concrete assignments $r_1, r_2 \in \mathbb{R}_{>0}$ to α and β such that $(r_1)^2 - r_2 = s + t$.

Hence we can generally consider hypersequents having polynomials (over a set $\alpha_1, \ldots, \alpha_l$ of scalar–variables) in place of concrete scalars.

We now describe an algorithm that takes a hypersequent G as input, having polynomials $R_1, \ldots, R_k \in \mathbb{R}_{>0}[\alpha_1, ..., \alpha_l]$ over scalar–variables $\alpha_1, \ldots, \alpha_l$ as coefficients in weighted formulas and returns a formula $\phi_G(\alpha_1, \ldots, \alpha_l) \in FO(\mathbb{R}, +, \times, \leq)$ with l variables $\alpha_1, \ldots, \alpha_l$, such that:

$$(s_1, \ldots, s_l) \in \mathbb{R}_{>0} \text{ is such that } \phi_G(s_1, \ldots, s_l) \text{ holds in } \mathbb{R}$$
$$\Leftrightarrow$$
$$G[s_j/\alpha_j] \text{ is derivable.}$$

where $G[s_j/\alpha_j]$ denotes the concrete hypersequent obtained by instantiating the scalar–variable α_j with the real number s_j.

The algorithm takes as input G and proceeds, again, in two steps:

(1) The algorithm returns

$$\phi_G = \bigwedge_{i=1}^{n} \phi_{G_i}$$

where $G_1, \ldots, G_n$ are the atomic hypersequents obtained by iteratively applying the logical rules, and ϕ_{G_i} is the formula recursively computed by the algorithm on input G_i.

(2) if G is atomic then G has the shape

$$\vdash \Gamma_1 \mid \; ... \mid \vdash \Gamma_m$$

where $\Gamma_i = \vec{R}_{i,1}.x_1, ..., \vec{R}_{i,k}.x_k, \vec{S}_{i,1}.\overline{x_1}, ..., \vec{S}_{i,k}.\overline{x_k}$. For all $I \subsetneq [1...m]$, we define

$$\phi_{G,I} = \exists \beta_1, ..., \beta_m, \bigwedge_{i \in I} (\beta_i = 0) \wedge \bigwedge_{i \notin I} (0 \le \beta_i \wedge \beta_i \ne 0) \wedge \bigwedge_{j=0}^{k} (\sum_{i=1}^{m} \beta_i \sum \vec{R}_{i,j} = \sum_{i=1}^{m} \beta_i \sum \vec{S}_{i,j})$$

and the formula ϕ_G is then constructed as follow:

$$\phi_G = \bigvee_{I \subsetneq [1...m]} \phi_{G,I}$$

The following theorem states the soundness of the above described algorithm.

Theorem 3.44. *Let G be a hypersequent having polynomials $R_1, \ldots, R_k \in \mathbb{R}_{>0}[\alpha_1, ..., \alpha_l]$ over scalar–variables $\alpha_1, \ldots, \alpha_l$. Let $\phi_G(\alpha_1, \ldots, \alpha_l)$ be the formula returned by the algorithm described above on input G. Then, for all $s_1, \ldots, s_l \in \mathbb{R}_{>0}$, the following are equivalent:*

(1) $\phi_G(s_1, \ldots, s_l)$ holds in $\mathbb{R}$,
*(2) $G[s_j/\alpha_j]$ is derivable in **HR**.*

Proof. If G is atomic, the theorem is a direct corollary of Lemma 3.33. So assume G is not atomic, i.e., the formulas in G contain some logical connective. Given any vector of scalars $s_1, \ldots, s_l \in \mathbb{R}_{>0}$, by using the CAN–free invertibility Theorem 3.11, $G[s_j/\alpha_j]$ is derivable if and only if all $G_i[s_j/\alpha_j]$ are derivable, where the hypersequents G_i are the atomic hypersequents obtainable from G by repeated applications of the logical rules, as show in Figure 6. Hence, the set of scalars $s_1, \ldots, s_l \in \mathbb{R}_{>0}$ that allows for a derivation of G is exactly the intersection of the scalars that allow derivations of each G_i. This is precisely the semantics of:

$$\phi_G = \bigwedge_{i=1}^{n} \phi_{G_i}$$

$\square$

4. Hypersequent Calculus for Modal Riesz Space

In this section we extend the system **HR** into the hypersequent calculus **HMR** for the equational theory of modal Riesz spaces. For that purpose we introduce to the system the two new rules of Figure 8 each dealing with the additional operators (the 1 constant and the unary $\Diamond$ modality) available in the syntax of modal Riesz spaces.

In the ($\Diamond$) rule, and in the rest of this section, the notation $\Diamond\Gamma$ stands for the sequence $r'_1.\Diamond A_1, ..., r'_n.\Diamond A_n$ when $\Gamma = r'_1.A_1, ..., r'_n.A_n$.

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}} \; 1, \; \sum \vec{r} \ge \sum \vec{s} \qquad\qquad \frac{\vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}}{\vdash \Diamond\Gamma, \vec{r}.1, \vec{s}.\overline{1}} \; \Diamond, \; \sum \vec{r} \ge \sum \vec{s}$$

Figure 8: Additional rules of **HMR**.

Definition 4.1 (System **HMR**). The hypersequent calculus proof system **HMR** consists of the rules of Figure 3 (i.e., those of the hypersequent calculus **HR**) plus the rules of Figure 8.

The (1) rule is quite similar to the ID rule but it reflects the axiom $0 \leq 1$ (see Figure 2) of modal Riesz spaces, and thus the side condition expresses an inequality, rather than an equality. The $\Diamond$ rule, as we will show in the soundness and completeness theorems below (Theorem 4.3 and Theorem 4.4), is remarkably capturing in one single rule all three axioms regarding the ($\Diamond$) modality (see Figure 2).

Remark 4.2. Note how the $\Diamond$ rule imposes strong constraints on the shape of its (single) premise and conclusion. First, both the conclusion and the premise are required to be hypersequents consisting of exactly one sequent. Furthermore, in the conclusion, all formulas, except those of the form 1 and $\overline{1}$ need to be of the form $r.\Diamond A$ for some weighted term $r.A$. These constraints determine main difficulties when trying adapt the proofs of Section 3 for the system **HMR**, but they are necessary. Indeed, for example, the following two alternative relaxed rules, while more natural looking, are in fact not sound:

$$\frac{G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}}{G \mid \vdash \Diamond\Gamma, \vec{r}.1, \vec{s}.\overline{1}} \qquad\qquad \frac{\vdash \Gamma, \vec{r'}.A, \vec{r}.1, \vec{s}.\overline{1}}{\vdash \Gamma, \vec{r'}.\Diamond A, \vec{r}.1, \vec{s}.\overline{1}}$$

as Remark 4.5 below shows.

The interpretation of **HMR** weighted terms, sequents and hypersequents is defined exactly as in Definition 3.7 for the system **HR**. That is, a weighted term is the term scalar-multiplied by the weight, a sequent is the sum of its weighted terms and a hypersequent is the join of its sequents. Throughout this section we adopt similar notation to that introduced in Section 3 for the system **HR** and we write $\vDash_{\textbf{HMR}} G$ if G is derivable using the rules of the system **HMR**.

4.1. **Main results regarding the system HMR.** This section presents our main results regarding the hypersequent calculus **HMR** and has the same pattern of Section 3.1 as we have tried to follow the same lines of presentation and reasoning and, whenever possible, to adapt the same proof techniques. We have been able to obtain variants of all the results proved for the system **HR** with the notable exception of the Rational T-elimination Theorem (Theorem 3.15) which remains an open problem in the context of the system **HMR**(see Section 4.9).

Our first two technical results about **HMR**, the soundness and completeness theorems, state that the system **HMR** can derive all and only those hypersequents G such that $\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash (\!|G|\!) \geq 0$.

Theorem 4.3 (Soundness). *For every hypersequent G,*

$$\vDash_{\textbf{HMR}} G \quad \Longrightarrow \quad \mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash (\!|G|\!) \geq 0.$$

Theorem 4.4 (Completeness). *For every hypersequent G,*

$$\mathcal{A}_{\text{Riesz}}^{\Diamond} \vdash (\!|G|\!) \geq 0 \quad \Longrightarrow \quad \vDash_{\textbf{HMR}} G.$$

Remark 4.5. The alternative rules considered in Remark 4.2 are unsound since the hypersequent $\vdash 1.\Diamond(\overline{x} \sqcap \overline{y}), 1.\Diamond(x) \sqcup \Diamond(y)$ would be derivable (see below) while $\mathcal{A}_{\mathrm{Riesz}}^{\Diamond} \not\vdash \Diamond(-x \sqcap -y) + \left(\Diamond(x) \sqcup \Diamond(y)\right) \geq 0$ or, equivalently, $\mathcal{A}_{\mathrm{Riesz}}^{\Diamond} \not\vdash \Diamond(x \sqcup y) \leq \Diamond(x) \sqcup \Diamond(y)$ is not derivable (see Example 2.20).

$$
\cfrac{
  \cfrac{
    \cfrac{\cfrac{\overline{\vphantom{1}} \;\text{INIT}}{\vdash 1.\overline{x}, 1.x}\;\text{ID}}{\vdash 1.\overline{x}, 1.x \mid \vdash 1.\overline{x} \sqcap \overline{y}, 1.y}\;\text{W}
  \quad
  \cfrac{
    \cfrac{\cfrac{\cfrac{\overline{\vphantom{1}} \;\text{INIT}}{\vdash 1.\overline{y}, 1.y}\;\text{ID}}{\vdash 1.\overline{x}, 1.\overline{y}, 1.x, 1.y}\;\text{ID}}{\vdash 1.\overline{y}, 1.x \mid \vdash 1.\overline{x}, 1.y}\;\text{S}
    \quad
    \cfrac{\cfrac{\cfrac{\overline{\vphantom{1}} \;\text{INIT}}{\vdash 1.\overline{y}, 1.y}\;\text{ID}}{\vdash 1.\overline{y}, 1.y}}{\vdash 1.\overline{y}, 1.x \mid \vdash 1.\overline{y}, 1.y}\;\text{W}
  }{\vdash 1.\overline{y}, 1.x \mid \vdash 1.\overline{x} \sqcap \overline{y}, 1.y}\;\sqcap
  }{\vdash 1.\overline{x} \sqcap \overline{y}, 1.x \mid \vdash 1.\overline{x} \sqcap \overline{y}, 1.y}\;\Diamond
}{
  \cfrac{\vdash 1.\overline{x} \sqcap \overline{y}, 1.x \mid \vdash 1.\Diamond(\overline{x} \sqcap \overline{y}), 1.\Diamond(y)}{
    \cfrac{\vdash 1.\Diamond(\overline{x} \sqcap \overline{y}), 1.\Diamond(x) \mid \vdash 1.\Diamond(\overline{x} \sqcap \overline{y}), 1.\Diamond(y)}{\vdash 1.\Diamond(\overline{x} \sqcap \overline{y}), 1.\Diamond(x) \sqcup \Diamond(y)}\;\sqcup
  }\;\Diamond
}
$$

Our next theorem states that all logical rules already present in the system **HR** and the $\Diamond$ rule are CAN–free invertible.

Theorem 4.6 (CAN–free Invertibility). *All the logical rules $\{0, +, \times, \sqcup, \sqcap, \Diamond\}$ are CAN–free invertible in **HMR**.*

The proof of this result is obtained by induction of the structure of derivations and is essentially identical to the one provided in Section 3.6. The new case represented by the $\Diamond$ rule presents no specific difficulties.

Remark 4.7. Note that, while the proof of CAN–free invertibility of the $\Diamond$ rule (Theorem 4.6 above) is not particularly difficult, the general invertibility property, i.e., that if the conclusion of a $\Diamond$ rule is derivable (possibly using CAN rules) then also its premise is derivable, appears to be quite nontrivial. We are able to prove this general form of invertibility only as a corollary of the CAN elimination theorem (Theorem 4.10 below) for **HMR**. Note how this contrasts with the case of the other logical rules already present in the system **HR** $(0, +, \times, \sqcup, \sqcap\})$ whose general invertibility is straightforward to prove (see Lemma 3.22 and 4.14 below) without invoking the CAN elimination theorem.

Note, instead, that the rule for the constant (1) is not CAN–free invertible in **HMR**. For example, the conclusion of the following valid instance of the rule:

$$
\cfrac{\vdash \frac{1}{3}.\overline{1}}{\vdash \frac{2}{3}.1, \frac{1}{3}.\overline{1}}\;1, \quad \tfrac{2}{3} \geq 0
$$

is derivable but its premise (whose semantics is $(\!\vdash \frac{1}{3}.\overline{1}\!) = \frac{1}{3}(-1))$ is, by the Soundness theorem, not derivable.

The importance of the invertibility theorem, in the context of the **HR** system (and also similar systems which inspired our work, such as the system GA of [MOG09, MOG05]), stems from the fact that it allows to reduce the logical complexity of formulas in a given hypersequent. This allowed to structure the proof of the CAN elimination theorem (see Section 3.8) as follows:

- first prove the atomic CAN elimination result (Lemma 3.41),
- then use the CAN–free invertibility of the logical rules to reduce the complexity of arbitrary hypersequents and CAN formulas to atomic hypersequents and atomic formulas.

This general proof technique is, however, not applicable in the context of the system **HMR**. This is because it is not possible to just invoke the CAN–free invertibility of the $\Diamond$–rule to reduce the complexity of a term of the form $\Diamond A$ in an arbitrary hypersequent due to the very constrained shape of the $\Diamond$–rule (see Figure 8) which requires the hypersequent to consist of only one sequent, and forces that only sequent to have only atoms, 1 terms, $\overline{1}$ terms and $\Diamond$ terms (i.e., terms whose outermost connective is a $\Diamond$)

It is still possible, however, to reduce the logical complexity of formulas in arbitrary hypersequents when the outermost connective of these formulas is in $\{0, +, \times, \sqcup, \sqcap\}$. By systematically applying these simplification steps to a complex hypersequent it is possible to obtain hypersequents having only atoms , 1 terms, $\overline{1}$ terms or $\Diamond$ terms. These simplified hypersequents are called *basic*.

Definition 4.8 (Basic Hypersequent)**.** A hypersequent G is *basic* if it contains only atoms, 1 terms, $\overline{1}$ terms or $\Diamond$ terms.

The following technical result is of key importance.

Theorem 4.9 (M elimination)**.** *If a hypersequent has a CAN–free proof, then it has a CAN–free and M–free proof.*

In the context of the system **HR**, the M elimination theorem allows for a very simple proof of CAN elimination for atomic CAN formulas (Lemma 3.41). Similarly, in the context of **HMR**, it will allow for a simple proof of a similar result regarding atomic CAN formulas (see Lemma 4.36).

However, compared to the situation in **HR**, where after being useful in proving Lemma 3.41, the M elimination theorem is not really needed to complete the proof of CAN elimination, in the context of **HMR** it appears to be of crucial importance. As already discussed above, in the context of **HMR**, it is not possible to simplify the complexity of CAN formulas of the form $\Diamond A$ simply by invoking the CAN–free invertibility of the $\Diamond$ rule. To address this limitation, it is possible to deal with the case of CAN formulas being $\Diamond$–formulas in a different way, by induction on the structure of the derivation d (in the style of the classic inductive proof techniques for eliminating CUT applications in sequent calculi, see, e.g., [Bus98]). In this inductive proof, however, there is a critically difficult case when the derivation ends with a M rule, as this rule breaks the proviso $\sum \vec{r} = \sum \vec{s}$ of the CAN rule. For instance, we do not know how to deal with the following instance of the M rule:

$$\dfrac{\dfrac{G \mid \vdash \Gamma_1, 2.\Diamond A, 3.\Diamond(\overline{A}) \quad G \mid \vdash \Gamma_2, 3.\Diamond A, 2.\Diamond(\overline{A})}{G \mid \vdash \Gamma_1, \Gamma_2, 2.\Diamond A, 3.\Diamond A, 2.\Diamond(\overline{A}), 3.\Diamond(\overline{A})} \text{ M}}{G \mid \vdash \Gamma_1, \Gamma_2} \text{ CAN}, 2 + 3 = 2 + 3$$

since we can not use the induction hypothesis on the two premises (because $2 \neq 3$).

The M elimination Theorem 4.9 is crucially important in eliminating this difficult case. The rest of the CAN elimination proof can then be carried out without serious technical difficulties. This is our main motivation for proving the M elimination theorem.

We can now state our main theorem regarding the system **HMR**.

Theorem 4.10 (CAN elimination)**.** *If a hypersequent G has a proof, then it has a CAN–free proof.*

Proof sketch. The full proof appears in Subsection 4.7. The CAN rule has the following form:

$$\frac{G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma} \text{ CAN}, \sum \vec{r} = \sum \vec{s}$$

Following the same proof structure of Theorem 3.13, we show how to eliminate one application of the CAN rule. Namely, we prove that if the premise $G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}$, with $\sum \vec{r} = \sum \vec{s}$, has a CAN–free proof d then the conclusion $G \mid \vdash \Gamma$ also has a CAN–free proof. This of course implies the statement of the CAN–elimination theorem by using a simple inductive argument on the number of CAN's applications in a proof.

The proof proceeds by double induction on the structure of A and the proof d.

The base cases are when $A = x$, i.e., when A is atomic, and when $A = 1$. Proving those cases is not at all straightforward in presence of the M rule, but it becomes much easier in the **HMR** system without the M rule.

For the inductive case, when A is a complex term which is not a $\Diamond$ formula we invoke the CAN–free invertibility theorem. For example, if $A = B + C$, the invertibility theorem states that $G \mid \vdash \Gamma, \vec{s}.B, \vec{s}.C, \vec{r}.\overline{B}, \vec{r}.\overline{B}$ must also have a CAN–free proof. We then note that, since B and C both have lower complexity than A, it follows from two applications of the inductive hypothesis that $G \mid \vdash \Gamma$ has a CAN-free proof, as desired.

Finally, when $A = \Diamond B$ for some B we prove the result by decreasing the complexity of the proof while keeping $\Diamond B$ as the CAN formula: we use the induction hypothesis on the premises of the last rule used in the proof - the only difficult case is when the last rule is the M rule. Fortunately this case can be removed by invoking the M elimination theorem. This simplification process is repeated until we reach an application of the $\Diamond$ rule, necessarily (due to the constraints of the $\Diamond$ rule) of the form:

$$\frac{\vdash \Gamma, \vec{r}.B, \vec{s}.\overline{B}, \vec{r'}.1, \vec{s'}.\overline{1}}{\vdash \Diamond \Gamma, \vec{r}.\Diamond B, \vec{s}.\Diamond \overline{B}, \vec{r'}.1, \vec{s'}.\overline{1}} \Diamond$$

We can then use the induction hypothesis on the simpler formula B. $\qquad\square$

Finally the algorithm introduced in the proof of Theorem 3.17 can be adapted to the **HMR** system to prove the following theorem.

Theorem 4.11 (Decidability). *There is an algorithm to decide whether or not a hypersequent has a proof.*

4.2. **Some technical lemmas.** All the lemmas presented in Subsection 3.3 in the context of the system **HR** need to be adapted to the new system **HMR**. In most cases, as in for instance Lemma 3.24, the proof is essentially identical and, for this reason, we omit it. In some cases, however, like Lemma 3.20, the $\Diamond$ rule makes the proof different and more complicated and, for this reason, we discuss how to prove the new difficult aspects of the proof.

We first adapt the extensionality property of the ID rule to the system **HMR**.

Lemma 4.12. *For all $A, \vec{r_i}, \vec{s_i}$ such that $\sum \vec{r_i} = \sum \vec{s_i}$, if $d \vDash_{\textbf{HMR}} [\vdash \Gamma_i]_{i=1}^n$ then $\vDash_{\textbf{HMR}} [\vdash \Gamma_i, \vec{r_i}.A, \vec{s_i}.\overline{A}]_{i=1}^n$*

Proof. We prove the result by double induction on (A, d). If A is not a $\Diamond$ formula, we prove the result as in Lemma 3.20 – which decreases the complexity of the formula each time. Otherwise we prove the result by induction on the proof d. $\qquad\square$

The next lemma states that if G is provable then the hypersequent obtained by substituting an atom for a formula in G is also provable.

Lemma 4.13. *If $\vDash_{HMR} G$ then for all formula A, $\vDash_{HMR} G[A/x]$.*

Proof. Similar to the proof of Lemma 3.21. $\qquad\square$

The following lemma, which will be useful in the proof of the completeness theorem, states that the rules $\{0, +, \times, \sqcup, \sqcap\}$, are invertible in **HMR**, in the sense that if the conclusion of one of these rules is derivable (possibly using CAN rules) then its premises are also derivable (possibly using CAN rules).

Lemma 4.14. *All the logical rules $\{0, +, \times, \sqcup, \sqcap\}$ rule are invertible.*

Proof. Similar to Lemma 3.22. $\qquad\square$

Remark 4.15. The proof of Lemma 4.14 does not introduce any new T rule, so if the conclusion of one of the logical rules $\{0, +, \times, \sqcup, \sqcap\}$ has a T–free proof, then the premises also have T–free proofs.

The next lemmas state that CAN–free derivability in the **HMR** system is preserved by scalar multiplication.

Lemma 4.16. *Let $\vec{r} \in \mathbb{R}_{>0}$ be a non-empty vector and G a hypersequent. If $\vDash_{HMR \backslash \{CAN\}} G \mid \vdash \vec{r}.\Gamma$ then $\vDash_{HMR \backslash \{CAN\}} G \mid \vdash \Gamma$.*

Proof. Similar to Lemma 3.24. $\qquad\square$

Lemma 4.17. *Let $\vec{r} \in \mathbb{R}_{>0}$ be a vector and G a hypersequent. If $\vDash_{HMR \backslash \{CAN\}} G \mid \vdash \Gamma$ then $\vDash_{HMR \backslash \{CAN\}} G \mid \vdash \vec{r}.\Gamma$.*

Proof. Simialr to Lemma 3.25. $\qquad\square$

The above lemmas have two useful corollaries.

Corollary 4.18. *If $\vDash_{HMR \backslash \{CAN\}} G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.A$ and $\vDash_{HMR \backslash \{CAN\}} G \mid \vdash \Gamma, \vec{r}.B, \vec{s}.B$ then $\vDash_{HMR \backslash \{CAN\}} G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.B$.*

Proof. Similar to Corollary 3.26 $\qquad\square$

Corollary 4.19. *If $\vDash_{HMR \backslash \{CAN\}} G \mid \vdash \vec{r}.A, \vec{s}.A, \Gamma \mid \vdash \vec{r}.B, \vec{s}.B, \Gamma \mid \vdash \vec{r}.A, \vec{s}.B, \Gamma$, then $\vDash_{HMR \backslash \{CAN\}} G \mid \vdash \vec{r}.A, \vec{s}.A, \Gamma \mid \vdash \vec{r}.B, \vec{s}.B, \Gamma$.*

Proof. Similar to Corollary 3.27. $\qquad\square$

4.3. **Soundness – Proof of Theorem 4.3.** The proof is similar to the one for Theorem 3.9: we prove that every rules is sound and then conclude by induction on the complexity derivations. Since the proofs of soundness for the rules already in **HR** are exactly the same as for the system **HR** done in Subsection 3.4, we only prove the soundness of the new rules.

• For the rule

$$\frac{\vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}}{\vdash \Diamond\Gamma, \vec{r}.1, \vec{s}.\overline{1}} \; \Diamond, \sum \vec{r} \geq \sum \vec{s}$$

the hypothesis is $(\!\vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}\!) \geq 0$ so

$$
\begin{aligned}
(\!\vdash \Diamond\Gamma, \vec{r}.1, \vec{s}.\overline{1}\!) \;&=\; (\!\vdash \Diamond\Gamma, (\sum \vec{r} - \sum \vec{s}).1\!) \text{ by distributivity} \\
&\geq\; (\!\vdash \Diamond\Gamma, (\sum \vec{r} - \sum \vec{s}).\Diamond 1\!) \text{ since } \Diamond 1 \leq 1 \\
&=\; \Diamond((\!\vdash \Gamma, (\sum \vec{r} - \sum \vec{s}).1\!)) \text{ by linearity of } \Diamond \\
&=\; \Diamond((\!\vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}\!)) \text{ by distributivity} \\
&\geq\; 0 \text{ by the hypothesis and the monotonocity of } \Diamond.
\end{aligned}
$$

• For the rule

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}} \; 1, \sum \vec{r} \geq \sum \vec{s}$$

the hypothesis is $(\!G \mid \vdash \Gamma\!) \geq 0$ so

$$
\begin{aligned}
(\!G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}\!) \;&\geq\; (\!G \mid \vdash \Gamma\!) \text{ since } \sum \vec{r} \geq \sum \vec{s} \text{ and } 0 \leq 1 \\
&\geq\; 0
\end{aligned}
$$

4.4. **Completeness – Proof of Theorem 4.4.** The proof follows the same pattern as in Subsection 3.5: we first prove a similar result that admits a simple proof by induction on the derivation of $\mathcal{A}^{\Diamond}_{\text{Riesz}} \vdash A = B$ and then we use it and the invertibility of the logical rules (Lemma 4.14) to prove Theorem 4.4, as shown in Subsection 3.5.

Lemma 4.20. *If $\mathcal{A}^{\Diamond}_{Riesz} \vdash A = B$ then $\vdash 1.A, 1.\overline{B}$ and $\vdash 1.B, 1.\overline{A}$ are provable in **HMR**.*

Proof. Since the other cases are proven in the exact same way as in Theorem 3.10, we will only derive the new axioms.

• For the axiom $0 \leq 1$.

$$\frac{\dfrac{\overline{\vdash} \; \text{INIT}}{\vdash 1.0} \; 0 \quad \dfrac{\overline{\vdash} \; \text{INIT}}{\vdash 1.1} \; 1, 1 \geq 0}{\dfrac{1.(0 \sqcap 1)}{\vdash 1.(0 \sqcap 1), 1.0} \; 0} \; \sqcap$$

and

$$\frac{\dfrac{\dfrac{\overline{\vdash} \; \text{INIT}}{\vdash 1.0} \; 0}{\dfrac{\vdash 1.0 \mid \vdash 1.\overline{1}}{\vdash 1.(0 \sqcup \overline{1})} \; \sqcup} \; W}{\vdash 1.0, 1.(0 \sqcup \overline{1})} \; 0$$

- For the axiom $\Diamond(1) \leq 1$.

$$
\cfrac{
\cfrac{
\cfrac{\overline{\vDash} \; \text{INIT}}{\vdash 1.1, 1.\overline{1}} \; 1
}{\vdash 1.\Diamond(1), 1.\Diamond(\overline{1})} \; \Diamond
\qquad
\cfrac{
\cfrac{\overline{\vDash} \; \text{INIT}}{\vdash 1.1, 1.\overline{1}} \; 1
}{\vdash 1.1, 1.\Diamond(\overline{1})} \; \Diamond
}{\vdash 1.(\Diamond(1) \sqcap 1), 1.\Diamond(\overline{1})} \; \sqcap
$$

and

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{\vDash} \; \text{INIT}}{1.1, 1.\overline{1}} \; 1
}{1.\Diamond(1), 1.\Diamond(\overline{1})} \; \Diamond
}{\vdash 1.\Diamond(1), 1.\Diamond(\overline{1}) \mid 1.\Diamond(1), 1.\overline{1}} \; W
}{\vdash 1.\Diamond(1), 1.(\Diamond(\overline{1}) \sqcup \overline{1})} \; \sqcup
$$

- For the axiom $\Diamond(r_1 x + r_2 y) = r_1 \Diamond(x) + r_2 \Diamond(y)$.

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{\vDash} \; \text{INIT}}{\vdash r_1.x, r_2.y, r_1.\overline{x}, r_2.\overline{y}} \; \text{ID}^2
}{\vdash 1.r_1 x, 1.r_2 y, r_1.\overline{x}, r_2.\overline{y}} \; \times^2
}{1.(r_1 x + r_2 y), r_1.\overline{x}, r_2.\overline{y}} \; +
}{\vdash 1.\Diamond(r_1 x + r_2 y), r_1.\Diamond(x), r_2.\Diamond(y)} \; \times^2
}{\vdash 1.\Diamond(r_1 x + r_2 y), 1.r_1 \Diamond(x), 1.r_2 \Diamond(y)} \; \Diamond
}{\vdash 1.\Diamond(r_1 x + r_2 y), 1.(r_1 \Diamond(\overline{x}) + r_2 \Diamond(\overline{y}))} \; +
$$

and

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{\vDash} \; \text{INIT}}{\vdash r_1.x, r_2.y, r_1.\overline{x}, r_2.\overline{y}} \; \text{ID}^2
}{\vdash r_1.x, r_2.y, 1.r_1\overline{x}, 1.r_2\overline{y}} \; \times^2
}{\vdash r_1.x, r_2.y, 1.(r_1\overline{x} + r_2\overline{y})} \; +
}{\vdash r_1.\Diamond(x), r_2.\Diamond(y), 1.\Diamond(r_1\overline{x} + r_2\overline{y})} \; \Diamond
}{\vdash 1.r_1\Diamond(x), 1.r_2\Diamond(y), 1.\Diamond(r_1\overline{x} + r_2\overline{y})} \; \times^2
}{\vdash 1.(r_1\Diamond(x) + r_2\Diamond(y)), 1.\Diamond(r_1\overline{x} + r_2\overline{y})} \; +
$$

- For the axiom $0 \leq \Diamond(0 \sqcup x)$.

$$
\cfrac{
\cfrac{\overline{\vDash} \; \text{INIT}}{\vdash 1.0} \; 0
\qquad
\cfrac{
\cfrac{
\cfrac{
\cfrac{\overline{\vDash} \; \text{INIT}}{\vdash 1.0} \; 0
}{\vdash 1.0 \mid \vdash 1.x} \; W
}{\vdash 1.(0 \sqcup x)} \; \sqcup
}{\vdash 1.\Diamond(0 \sqcup x)} \; \Diamond
}{
\cfrac{\vdash 1.(0 \sqcap \Diamond(0 \sqcup x))}{\vdash 1.(0 \sqcap \Diamond(0 \sqcup x)), 1.0} \; 0
} \; \sqcap
$$

and

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\vDash \text{INIT}}{\vdash 1.0}\ 0
}{\vdash 1.0 \mid\ \vdash 1.\Diamond(0 \sqcap \overline{x})}\ \text{W}
}{
\cfrac{1.(0 \sqcup \Diamond(0 \sqcap \overline{x}))}{}
}\ \sqcup
}{\vdash 1.0, 1.(0 \sqcup \Diamond(0 \sqcap \overline{x}))}\ 0
$$

$\square$

Remark 4.21. By inspecting the proof of Lemma 4.20 it is possible to verify that the T rule is never used in the construction of $\vDash_{\mathbf{HMR}} G$. This, together with the similar Remark 4.15 regarding the Lemma 4.14, implies that the T rule is never used in the proof of the completeness Theorem 4.4. From this we get the following corollary.

Corollary 4.22. *The T rule is admissible in the system* **HMR**.

As in the case of the system **HR** (see Lemma 3.31) there is no hope of eliminating both the T rule and the CAN rule from the **HMR** system.

Lemma 4.23. *Let r_1 and r_2 be two irrational numbers that are algebraically independent over $\mathbb{Q}$ (so there is no $q \in \mathbb{Q}$ such that $qr_1 = r_2$). Then the atomic hypersequent G*

$$\vdash r_1.x \mid\ \vdash r_2.\overline{x}$$

does not have a CAN–free and T–free proof.

Proof. The proof is similar to that of Lemma 3.31 but Lemma 4.24 below takes the place of Lemma 3.32. $\square$

Lemma 4.24. *For all basic hypersequent G formed using the variables and negated variables $x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}$ of the form*

$$\vdash \Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1} \mid\ \ldots \mid\ \vdash \Gamma_m, \Diamond\Delta_m, \vec{r'}_m.1, \vec{s'}_m.\overline{1}$$

where $\Gamma_i = \vec{r}_{i,1}.x_1, ..., \vec{r}_{i,k}.x_k, \vec{s}_{i,1}.\overline{x_1}, ..., \vec{s}_{i,k}.\overline{x_{i,k}}$, the following are equivalent:
(1) *G has a CAN–free and T–free proof.*
(2) *there exist natural numbers $n_1, ..., n_m \in \mathbb{N}$, one for each sequent in G, such that:*
   - *there exists $i \in [1..m]$ such that $n_i \neq 0$, i.e., the numbers are not all 0's, and*
   - *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*

$$\sum_{i=1}^m n_i\left(\sum \vec{r}_{i,j}\right) = \sum_{i=1}^m n_i\left(\sum \vec{s}_{i,j}\right)$$

   i.e., the scaled (by the numbers $n_1 \ldots n_m$) sum of the coefficients in front of the variable x_j is equal to the scaled sum of the coefficients in from of the covariable $\overline{x_j}$, and
   - *$\sum_{i=1}^m n_i \sum \vec{s'}_i \leq \sum_{i=1}^m n_i \sum \vec{r'}_i$, i.e., there are more 1 than $\overline{1}$, and*
   - *the hypersequent consisting of only one sequent*

$$\vdash \Delta_1^{n_1}, ..., \Delta_m^{n_m}, (\vec{r'}_1.1)^{n_1}, ..., (\vec{r'}_m.1)^{n_m}, (\vec{s'}_1.\overline{1})^{n_1}, ..., (\vec{s'}_m.\overline{1})^{n_m}$$

   has a CAN–free and T–free proof, where the notation Γ^n means $\underbrace{\Gamma, ..., \Gamma}_{n\ times}$.

Proof. We prove (1) $\Rightarrow$ (2) by induction on the proof of G. We show only the M case, the other cases being simple. We note Γ'_i for $\Gamma_i, \Diamond\Delta_i, \vec{r'}_i.1, \vec{s'}_i.\overline{1}$.

- If the proof finishes with

$$\frac{\vdash \Gamma'_1 \mid \ \dots \ \mid \vdash \Gamma'_m \quad \vdash \Gamma'_1 \mid \ \dots \ \mid \vdash \Gamma'_{m+1}}{\vdash \Gamma'_1 \mid \ \dots \ \mid \vdash \Gamma'_m, \Gamma'_{m+1}} \ \text{M}$$

by induction hypothesis, there are $n_1, ..., n_m \in \mathbb{N}$ such that :
– there exists $i \in [1..m]$ such that $n_i \neq 0$.
– for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that $\sum_i n_i. \sum \vec{r}_{i,j} = \sum_i n_i. \sum \vec{s}_{i,j}$.
– $\sum_{i=1}^m n_i \sum \vec{s'}_i \leq \sum_{i=1}^m n_i \sum \vec{r'}_i$.
– $\vdash \Delta_1^{n_1}, ..., \Delta_m^{n_m}, (\vec{r'}_1.1)^{n_1}, ..., (\vec{r'}_m.1)^{n_m}, (\vec{s'}_1.\overline{1})^{n_1}, ..., (\vec{s'}_m.\overline{1})^{n_m}$ has a CAN–free and T–free proof.

and $n'_1, ..., n'_m \in \mathbb{N}$ such that :
– there exists $i \in [1..m]$ such that $n'_i \neq 0$.
– for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that $\sum_{i=0}^{m-1} n'_i. \sum \vec{r}_{i,j} + n'_m. \sum \vec{r}_{m+1,j} = \sum_{i=0}^{m-1} n'_i. \sum \vec{s}_{i,j} + n'_m. \sum \vec{s}_{m+1,j}$.
– $\sum_{i=1}^{m-1} n'_i \sum \vec{s'}_i + n'_m \sum \vec{s'}_{m+1} \leq \sum_{i=1}^{m-1} n'_i \sum \vec{r'}_i + n'_m \sum \vec{r'}_{m+1}$.
– $\vdash \Delta_1^{n'_1}, ..., \Delta_{m+1}^{n'_m}, (\vec{r'}_1.1)^{n'_1}, ..., (\vec{r'}_{m+1}.1)^{n'_m}, (\vec{s'}_1.\overline{1})^{n'_1}, ..., (\vec{s'}_{m+1}.\overline{1})^{n_m}$ has a CAN–free and T–free proof.

If $n_m = 0$ then $n_1, ..., n_{m-1}, 0$ satisfies the property.
Otherwise if $n'_m = 0$ then $n'_1, ..., n'_{m-1}, 0$ satisfies the property.
Otherwise, $n_m.n'_1 + n'_m.n_1, n_m.n'_2 + n'_m.n_2, ..., n_m.n'_{m-1} + n'_m.n_{m-1}, n_m.n'_m$ satisfies the property.

The other way $((2) \Rightarrow (1))$ is more straightforward. If there exist natural numbers $n_1, ..., n_m \in \mathbb{N}$, one for each sequent in G, such that:

- there exists $i \in [1..m]$ such that $n_i \neq 0$ and
- for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that

$$\sum_{i=1}^m n_i \left(\sum \vec{r}_{i,j}\right) = \sum_{i=1}^m n_i \left(\sum \vec{s}_{i,j}\right)$$

  and
- $\sum_{i=1}^m n_i \sum \vec{s'}_i \leq \sum_{i=1}^m n_i \sum \vec{r'}_i$.and
- $\vdash \Delta_1^{n_1}, ..., \Delta_m^{n_m}, (\vec{r'}_1.1)^{n_1}, ..., (\vec{r'}_m.1)^{n_m}, (\vec{s'}_1.\overline{1})^{n_1}, ..., (\vec{s'}_m.\overline{1})^{n_m}$ has a CAN–free and T–free proof.

then we can use the W rule to remove the sequents corresponding to the numbers $n_i = 0$, and use the C rule $n_i - 1$ times then the S rule $n_i - 1$ times on the ith sequent to multiply it by n_i. If we assume that there is a natural number l such that $n_i = 0$ for all $i > l$ and $n_i \neq 0$ for all $i \leq l$, then the CAN–free T–free proof is:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash (\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1})^{n_1}, \dots, (\Delta_l, \vec{r'}_l.1, \vec{s'}_l.\overline{1})^{n_l}}{\vdash (\Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1})^{n_1}, \dots, (\Diamond\Delta_l, \vec{r'}_l.1, \vec{s'}_l.\overline{1})^{n_l}} \ \Diamond}{\vdash (\Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1})^{n_1}, \dots, (\Gamma_l, \Diamond\Delta_l, \vec{r'}_l.1, \vec{s'}_l.\overline{1})^{n_l}} \ \text{ID}^*}{\vdash (\Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1})^{n_1} \mid \ \dots \ \mid \vdash (\Gamma_l, \Diamond\Delta_l, \vec{r'}_l.1, \vec{s'}_l.\overline{1})^{n_l}} \ \text{S}^*}{\vdash \Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1} \mid \ \dots \ \mid \vdash \Gamma_l, \Diamond\Delta_l, \vec{r'}_l.1, \vec{s'}_l.\overline{1}} \ \text{C-S}^*}{\vdash \Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1} \mid \ \dots \ \mid \vdash \Gamma_m, \Diamond\Delta_m, \vec{r'}_m.1, \vec{s'}_m.\overline{1}} \ \text{W}^*$$

$\square$

The following similar result, involving the T rule, will be quite useful in proving the decidability of **HMR**. The only difference is that since the T rule can multiply a sequent by any strictly positive real number, the coefficients in the property are real numbers instead of natural numbers.

Lemma 4.25. *For all basic hypersequent G formed using the variables and negated variables $x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}$ of the form*

$$\vdash \Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1} \mid \ \ldots \ \mid \vdash \Gamma_m, \Diamond\Delta_m, \vec{r'}_m.1, \vec{s'}_m.\overline{1}$$

where $\Gamma_i = \vec{r}_{i,1}.x_1, ..., \vec{r}_{i,k}.x_k, \vec{s}_{i,1}.\overline{x_1}, ..., \vec{s}_{i,k}.\overline{x_{i,k}}$, the following are equivalent:

(1) *G has a CAN–free proof.*
(2) *there exist numbers $t_1, ..., t_m \in \mathbb{R}_{\geq 0}$, one for each sequent in G, such that:*
   - *there exists $i \in [1..m]$ such that $t_i \neq 0$, i.e., the numbers are not all 0's, and*
   - *for every variable and covariable $(x_j, \overline{x_j})$ pair, it holds that*

$$\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j}) = \sum_{i=1}^{m} t_i(\sum \vec{s}_{i,j})$$

   i.e., the scaled (by the numbers $t_1 \ldots t_m$) sum of the coefficients in front of the variable x_j is equal to the scaled sum of the coefficients in from of the covariable $\overline{x_j}$.
   - *$\sum_{i=1}^{n} t_i \sum \vec{s'}_i \leq \sum_{i=1}^{n} t_i \sum \vec{r'}_i$, i.e, there are more 1 than $\overline{1}$ and,*
   - *the hypersequent consisting of only one sequent*

$$\vdash t_1.\Delta_1, ..., t_m.\Delta_m, (t_1\vec{r'}_1).1, ..., (t_m\vec{r'}_m).1, (t_1\vec{s'}_1).\overline{1}, ..., (t_m\vec{s'}_m).\overline{1}$$

   has a CAN–free proof, where the notation $0.\Gamma$ means $\emptyset$.

Proof. Similar to Lemma 3.33. $\square$

4.5. **CAN–free Invertibility – Proof of Theorem 4.6.** The proofs presented in this subsection follow the same pattern of those in Subsection 3.6: we will prove the CAN–free invertibility of more general rules. The generalised non–modal rules are the same as those in Figure 7 from Subsection 3.6 and the generalised $\Diamond$ rule has the following shape:

$$\frac{[\vdash \Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}]_{i=1}^n}{[\vdash \Diamond\Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}]_{i=1}^n}$$

Remark 4.26. The generalized $\Diamond$ rule is unsound, the hypersequent $\vdash 1.\Diamond(\overline{x} \sqcap \overline{y}), 1.\Diamond(x) \sqcup \Diamond(y)$ is derivable using this rule (see Remark 4.5, a similar derivation can be used to derive the hypersequent). Yet, even if the generalized $\Diamond$ rule is not sound, it still enjoys CAN–free invertibility.

We will prove that those rules are CAN–free invertible by induction on the derivation of the conclusion. The proof steps dealing with the rules already present in **HR** are the same as in Subsection 3.6. In what follows we just show the details of the proof steps associated with the new cases associated with the $\Diamond$–rule and 1–rule of **HMR**.

Lemma 4.27. *If d is a CAN–free proof of $[\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=1}^n$ then $[\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=1}^n$ has a CAN–free proof.*

Proof. By induction on d.

• If d finishes with

$$\frac{[\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcup B)}{[\vdash \Gamma_i, \vec{r}_i.(A \sqcup B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcup B), \vec{r}.1, \vec{s}.\overline{1}} \ 1$$

then by induction hypothesis on the CAN–free proof of the premise we have that

$$\vDash_{\mathbf{HMR} \backslash \{\text{CAN}\}} [\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_1, \vec{r}_1.B$$

so

$$\frac{G' \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}.1, \vec{s}.\overline{1} \mid \vdash \Gamma_1, \vec{r}_1.B, \vec{r}.1, \vec{s}.\overline{1}}{G' \mid \vdash \Gamma_1, \vec{r}_1.A \mid \vdash \Gamma_1, \vec{r}_1.B} \ 1^*$$

with $G' = [\vdash \Gamma_i, \vec{r}_i.A \mid \vdash \Gamma_i, \vec{r}_i.B]_{i=2}^n$

• If d finishes with an application of the $\Diamond$ rule, the shape of the conclusion is

$$\vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$$

with $\vec{r_1} = \emptyset$ so the hypersequent

$$\vdash \Diamond\Gamma_1, \vec{r_1}.A, \vec{r}.1, \vec{s}.\overline{1} \mid \vdash \Diamond\Gamma_1, \vec{r_1}.B, \vec{r}.1, \vec{s}.\overline{1} = \vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1} \mid \vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$$

is derivable using the C rule.

$\square$

Lemma 4.28. *If d is a CAN–free proof of $[\vdash \Gamma_i, \vec{r}_i(A + B)]_{i=1}^n$ then $[\vdash \Gamma_i, \vec{r}_iA, \vec{r}_iB]_{i=1}^n$ has a CAN–free proof.*

Proof. By induction on d.

• If d finishes with

$$\frac{[\vdash \Gamma_i, \vec{r}_i.(A + B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.(A + B)}{[\vdash \Gamma_i, \vec{r}_i.(A + B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.(A + B), \vec{r}.1, \vec{s}.\overline{1}} \ 1$$

then by induction hypothesis on the CAN–free proof of the premise we have that

$$\vDash_{\mathbf{HMR} \backslash \{\text{CAN}\}} [\vdash \Gamma_i, \vec{r}_i.A, \vec{r}_i.B]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}_1.B$$

so

$$\frac{[\vdash \Gamma_i, \vec{r}_i.A, \vec{r}_i.B]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}_1.B}{[\vdash \Gamma_i, \vec{r}_i.A, \vec{r}_i.B]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}_1.B, \vec{r}.1, \vec{s}.\overline{1}} \ 1$$

• If d finishes with an application of the $\Diamond$ rule, the shape of the conclusion is

$$\vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$$

with $\vec{r_1} = \emptyset$ so the hypersequent $\vdash \Diamond\Gamma_1, \vec{r_1}.A, \vec{r_1}.B, \vec{r}.1, \vec{s}.\overline{1} = \vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$ is derivable.

$\square$

Lemma 4.29. *If d is a CAN–free proof of $\vdash \Gamma_i, \vec{r}_i.(A \sqcap B)]_{i=1}^n$ then $[\vdash \Gamma_i, \vec{r}_i.A]_{i=1}^n$ and $[\vdash \Gamma_i, \vec{r}_i.B]_{i=1}^n$ have a CAN–free proof.*

Proof. By induction on d. We will only show that $\vDash_{\mathbf{HMR} \backslash \{\text{CAN}\}} [\vdash \Gamma_i, \vec{r}_i.A]_{i=1}^n$, the other case is similar.

- If d finishes with

$$\frac{[\vdash \Gamma_i, \vec{r}_i.(A \sqcap B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcap B)}{[\vdash \Gamma_i, \vec{r}_i.(A \sqcap B)]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.(A \sqcap B), \vec{r}.1, \vec{s}.\overline{1}} \ 1$$

then by induction hypothesis on the CAN–free proof of the premise we have that

$$\vDash_{\mathbf{HMR}\backslash\{\text{CAN}\}} [\vdash \Gamma_i, \vec{r}_i.A]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.A$$

so

$$\frac{[\vdash \Gamma_i, \vec{r}_i.A]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.A}{[\vdash \Gamma_i, \vec{r}_i.A]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.A, \vec{r}.1, \vec{s}.\overline{1}} \ 1$$

- If d finishes with an application of the $\Diamond$ rule, the shape of the conclusion is

$$\vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$$

with $\vec{r_1} = \emptyset$ so the hypersequent $\vdash \Diamond\Gamma_1, \vec{r_1}.A, \vec{r}.1, \vec{s}.\overline{1} = \vdash \Diamond\Gamma_1, \vec{r}.1, \vec{s}.\overline{1}$ is derivable.

$\square$

Lemma 4.30. *If d is CAN–free proof of $[\vdash \Diamond\Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}]_{i=1}^n$ then $[\vdash \Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}]_{i=1}^n$ has a CAN–free proof.*

Proof. By induction on the proof. Since the hypersequent under consideration is a basic, we do not need to deal with any logical rule beside the $\Diamond$–rule, which leads immediately to the desired result, and the cases regarding the structural rules are very simple. For instance, if the proof finishes with the W rule:

$$\frac{[\vdash \Diamond\Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}]_{i=2}^n}{[\vdash \Diamond\Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}]_{i=2}^n \mid \vdash \Diamond\Gamma_1, \vec{r}_1.1, \vec{s}_1.\overline{1}} \ W$$

then by induction hypothesis

$$\vDash_{\mathbf{HMR}\backslash\{\text{CAN}\}} [\vdash \Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}]_{i=2}^n$$

so

$$\frac{[\vdash \Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}]_{i=2}^n}{[\vdash \Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.1, \vec{s}_1.\overline{1}} \ W$$

$\square$

4.6. **M-elimination – Proof of Theorem 4.9.** Following the same pattern of Subsection 3.7, we need to show that for each hypersequent G and sequents Γ and Δ, if there exist CAN–free and M–free proofs d_1 of $G \mid \vdash \Gamma$ and d_2 of $G \mid \vdash \Delta$, then there exists also a CAN–free and M–free proof of $G \mid \vdash \Gamma, \Delta$.

The general idea presented in Subsection 3.7 is to combine the derivations d_1 and d_2 in a sequential way, first constructing a preproof d_1' of $G \mid G \mid \vdash \Gamma, \Delta$ (using d_1) whose leaves are either axioms or hypersequents of the form $G \mid \vdash \vec{r}.\Delta$, and then by completing this preproof into a proof (using d_2). Finally, $G \mid G \mid \vdash \Gamma, \Delta$ can be easily turned into a proof of $G \mid \vdash \Gamma, \Delta$ as desired.

However, this technique can not be directly applied in the context of the system **HMR** due to the constraints imposed on the shape of the hypersequent by the $\Diamond$ rule. Indeed an application of the $\Diamond$ rule in d_1 acting on some hypersequent of the form

$$\vdash \Diamond\Gamma_1, \vec{r}_1.1, \vec{s}_1.\overline{1}$$

can not turned into an application of the $\Diamond$ rule on

$$G \mid \vdash \Delta, \Diamond\Gamma_1, \vec{r}_1.1, \vec{s}_1.\overline{1}$$

because this hypersequent can not be the conclusion of a $\Diamond$ rule as it does not satisfies the constraints.

For this reason, when constructing the preproof d_1' inductively from d_1, we stop at the applications of the $\Diamond$–rule. Hence, the inductive procedure takes the proof d_1 and produces a CAN–free and M–free preproof d_1' of

$$G \mid G \mid \vdash \Gamma, \Delta$$

where all the leaves in the preproof are either:

(1) terminated, or
(2) non–terminated and having the shape

$$G \mid \vdash \vec{r}.\Delta$$

which can then be completed using the derivation d_2 in the exact same way explained in Subsection 3.7, or

(3) non–terminated and having the shape:

$$G \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$$

for some sequent Γ' and vectors $\vec{r}, \vec{s}, \vec{t}$. For each of these leaves there is a corresponding proof of

$$\vdash \Gamma_1, \vec{r}_1.1, \vec{s}_1.\overline{1} \tag{4.1}$$

To obtain derivations, and thus complete, the leaves of d_1' of the third type, we proceed as follows. First, we use the proof d_2 to construct a CAN–free and M–free proof $d_{2,\vec{r}}$

$$G \mid \vec{r}.\Delta$$

for each vector of scalars $\vec{r}$ in the leaf. We then modify each proof $d_{2,\vec{r}}$ into a preproof d_2' of

$$G \mid G \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$$

using the exact same inductive procedure (which stops when reaching applications of $\Diamond$ formulas) introduced above for producing d_1' from d_1. Note that in this case, the leaves of the third kind in d_2' are of are of the form:

$$\vdash \Diamond\Gamma, \Diamond\Delta, \vec{s'}.1, \vec{t'}.\overline{1}$$

and have associated proofs of

$$\vdash \Gamma_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1} \tag{4.2}$$

Therefore, we can legitimately apply the $\Diamond$ rule (Lemma 4.31 below ensures that the proviso of the rule is respected) and reduce these leaves to leaves of the form

$$\vdash \Gamma, \Delta, \vec{s'}.1, \vec{t'}.\overline{1}$$

which, importantly, have a lower modal depth compared to the conclusion $G \mid \vdash \Gamma$ of the derivation d_1 we started with above.

In order the produce a derivation for the leaves $\vdash \Gamma, \Delta, \vec{s'}.1, \vec{t'}.\overline{1}$, and thus conclude the completion of d_1' into a full proof, it is sufficient to re–apply the whole the whole process using the derivations of Equation 4.1 and Equation 4.2 above. This process is well founded and eventually terminates because the modal depth is decreasing.

We now proceeds with the technical statements.

Lemma 4.31. *Let d_1 be a CAN–free and M–free derivation of $G \mid \vdash \Gamma$ using the $\Diamond$ rule and let Δ be a sequent. Then there exists a preproof of*

$$G \mid \ G \mid \vdash \Gamma, \Delta.$$

where all non–terminated leaves are all either of the form $G \mid \vdash \vec{r}.\Delta$ or of the form $G \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ for some sequent Γ' and vectors $\vec{r}, \vec{s}, \vec{t}$ such that

- $\sum \vec{s} \geq \sum \vec{t}$ *and*
- $\vdash \Gamma', \vec{s}.1, \vec{t}.\overline{1}$ *has a proof d_1' with a strictly lesser modal depth than d_1.*

Proof. This is an instance of the slightly more general statement of Lemma 4.34 below. $\square$

Lemma 4.32. *Let d_2 be CAN–free and M–free derivation of $G \mid \vdash \Delta$. Then, for every vector $\vec{r}$, there exists a CAN–free and M–free proof of*

$$G \mid \vdash \vec{r}.\Delta$$

with a lesser modal depth than d_2.

Proof. This is an instance of the slightly more general statement of Lemma 4.35 below. $\square$

Lemma 4.33. *Let d_1 be a CAN–free and M–free derivation of $G \mid \vdash \Gamma$ without any $\Diamond$ rule and let Δ be a sequent. Then there exists a preproof of*

$$G \mid \ G \mid \vdash \Gamma, \Delta.$$

where all non–terminated leaves are all of the form $G \mid \vdash \vec{r}.\Delta$ for some vector $\vec{r}$.

Proof. This is an other intance of Lemma 4.34. Since the leaves of the form $G \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ are generated only by the $\Diamond$ rule, and there is no $\Diamond$ rule in d_1, then all non–terminated leaves are all of the form $G \mid \vdash \vec{r}.\Delta$ for some vector $\vec{r}$. $\square$

Lemma 4.34. *Let d_1 be a CAN–free and M–free derivation of $[\vdash \Gamma_i]_{i=1}^n$ and let G be a hypersequent and Δ be a sequent. Then for every sequence of vectors $\vec{r_i}$, there exists a preproof of*

$$G \mid \ [\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=1}^n$$

where all non–terminated leaves are all of the form $G \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ for some sequent Γ' and vectors $\vec{r}, \vec{s}, \vec{t}$ such that

- $\sum \vec{s} \geq \sum \vec{t}$ *and*
- $\vdash \Gamma', \vec{s}.1, \vec{t}.\overline{1}$ *has a proof d_1' with a strictly lesser modal depth than d_1.*

Proof. We will only show the $\Diamond$ and the 1 rules, since all other cases are done in the same way as in Lemma 3.39.

- if d_1 finishes with:

$$\frac{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1}{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{s}.1, \vec{t}.\overline{1}} \ 1, \sum \vec{s} \geq \sum \vec{t}$$

  then by induction hypothesis, there is a preproof of $[\vdash \Gamma_i, \vec{r_i}.\Delta]_{i=2}^n \mid \vdash \Gamma_1, \vec{r_1}.\Delta$ where all non–terminated leaves are all of the form $G \mid \vdash \Diamond\Gamma', \vec{r}.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ for some sequent Γ' and vectors $\vec{r}, \vec{s}, \vec{t}$ such that
  - $\sum \vec{s} \geq \sum \vec{t}$ and
  - $\vdash \Gamma', \vec{s}.1, \vec{t}.\overline{1}$ has a proof d_1' with a strictly lesser modal depth than d_1.

We continue the preproof with

$$\frac{[\vdash \Gamma_i, \vec{r}_i.\Delta]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.\Delta}{[\vdash \Gamma_i, \vec{r}_i.\Delta]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.\Delta, \vec{s}.1, \vec{t}.\overline{1}} \ 1, \sum \vec{s} \geq \sum \vec{t}$$

- If d_1 finishes with:

$$\frac{\vdash \Gamma_1, \vec{s}.1, \vec{t}.\overline{1}}{\vdash \Diamond \Gamma_1, \vec{s}.1, \vec{t}.\overline{1}} \ \Diamond, \sum \vec{s} \geq \sum \vec{t}$$

then the preproof is simply the leaf $\vdash \Diamond \Gamma_1, \vec{r}_1.\Delta, \vec{s}.1, \vec{t}.\overline{1}$ which satisfies both
  - $\sum \vec{s} \geq \sum \vec{t}$ and
  - $\vdash \Gamma_1, \vec{s}.1, \vec{t}.\overline{1}$ is derivable using stricly less $\Diamond$ rule than in d_1.

$\square$

Lemma 4.35. *If d_2 is a CAN–free M–free proof of $[\vdash \Delta_i]_{i=1}^n$ then for all $\vec{r}_i$, there is a CAN–free M–free proof of $[\vdash \vec{r}_i.\Delta_i]_{i=1}^n$ with a lesser modal depth than d_2.*

Proof. We will only show the $\Diamond$ and 1 rules, the other cases being similar to Lemma 3.40 – and so does not introduce any new $\Diamond$ rule.

- if d_2 finishes with:

$$\frac{[\vdash \Delta_i]_{i=2}^n \mid \vdash \Delta_1}{[\vdash \Delta_i]_{i=2}^n \mid \vdash \Delta_1, \vec{s}.1, \vec{t}.\overline{1}} \ 1, \sum \vec{s} \geq \sum \vec{t}$$

then by induction hypothesis, there is a CAN–free M–free proof of $[\vdash \vec{r}_i.\Delta_i]_{i=2}^n \mid \vdash \vec{r}_1.\Delta_1$ with a lesser modal depth than d_2. We continue the proof with

$$\frac{[\vdash \vec{r}_i.\Delta_i]_{i=2}^n \mid \vdash \vec{r}_1.\Delta_1}{[\vdash \vec{r}_i.\Delta_i]_{i=2}^n \mid \vdash \vec{r}_1.\Delta_1, (\vec{r}_1\vec{s}).1, (\vec{r}_1\vec{t}).\overline{1}} \ 1, \sum \vec{r}_1\vec{s} \geq \sum \vec{r}_1\vec{t}$$

which does not increase the modal depth of the proof.
- If d_2 finishes with:

$$\frac{\vdash \Delta_1, \vec{s}.1, \vec{t}.\overline{1}}{\vdash \Diamond \Delta_1, \vec{s}.1, \vec{t}.\overline{1}} \ \Diamond, \sum \vec{s} \geq \sum \vec{t}$$

by induction hypothesis, there is a proof of $\vdash \vec{r}_1.\Delta_1, (\vec{r}_1\vec{s}).1, (\vec{r}_1\vec{t}).\overline{1}$ with modal depth strictly lesser than d_2. We continue the proof with

$$\frac{\vdash \vec{r}_1.\Delta_1, (\vec{r}_1\vec{s}).1, (\vec{r}_1\vec{t}).\overline{1}}{\vdash \vec{r}_1.\Diamond \Delta_1, (\vec{r}_1\vec{s}).1, (\vec{r}_1\vec{t}).\overline{1}} \ \Diamond, \sum \vec{r}_1\vec{s} \geq \sum \vec{r}_1\vec{t}$$

which gives a proof with lesser modal depth than d_2.

$\square$

4.7. CAN elimination – Proof of Theorem 4.10.

We remind that the CAN rule has the following form:

$$\frac{G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma} \ \text{CAN}, \sum \vec{r} = \sum \vec{s}$$

As in Subsection 3.8, we prove Theorem 4.10 by showing that if the hypersequent $G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}$ has a CAN–free derivation, then so does the hypersequent $G \mid \vdash \Gamma$.

As explained in the discussion before Theorem 4.10 and in its proof sketch, the proof can not just invoke the CAN–free invertibility Theorem 4.6 to simplify the logical complexity of the CAN formula, due to the constraints imposed by the $\Diamond$–rule.

To circumvent this issue, we prove the slightly more general Lemma 4.38 by double induction on both the formula A and the proof of $G \mid \vdash \Gamma, \vec{r}.A, \vec{s}.\overline{A}$.

We first prove the two basic cases where $A = x$ (or equivalently $A = \overline{x}$) and $A = 1$ (or again $A = \overline{1}$).

Lemma 4.36. *If there is a CAN–free d of $G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}$, where $\sum \vec{r} = \sum \vec{s}$ then there exists a CAN–free proof of $G \mid \vdash \Gamma$.*

Proof. By the M–elimination Theorem 4.9, we can assume that d is CAN–free and also M–free. The statement then follows as a special case of Lemma 4.39 below. The formulation of Lemma 4.39 is more technical but allows for a simpler proof by induction on the structure of d. $\qquad\square$

Lemma 4.37. *If there is a CAN–free d of $G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}$, where $\sum \vec{r} = \sum \vec{s}$ then there exists a CAN–free proof of $G \mid \vdash \Gamma$.*

Proof. As in the previous Lemma 4.36, we can assume that the proof is also M–free and we prove the more technical Lemma 4.40 that allows a simpler proof by induction on d. $\qquad\square$

In order to better handle the cases associated with the $\Diamond$ rule in the proof of Theorem 4.10, we prove the more general Lemma 4.38 of which the CAN elimination theorem rule is just an instance.

Lemma 4.38. *For all formulas A and number $n > 0$ and for all sequents Γ_i and vectors $\vec{r}_i, \vec{s}_i$ such that $\sum \vec{r}_i = \sum \vec{s}_i$, for $1 \leq i \leq n$,*

if $[\vdash \Gamma_i, \vec{r}_i.A, \vec{s}_i.\overline{A}]_{i=1}^n$ has a CAN–free M–free proof, then so does $[\vdash \Gamma_i]_{i=1}^n$.

Proof. For the basic cases $A = x$, $A = \overline{x}$, $A = 1$ and $A = \overline{1}$, we use Lemmas 4.36 and 4.37. For complex formulas A which are not $\Diamond$ formula, we proceed by invoking the CAN–free invertibility Theorem 4.6 as follows:

- If $A = 0$, we can conclude with the CAN–free invertibility of the rule 0.
- If $A = B + C$, since the $+$ rule is CAN–free invertible, $[\vdash \Gamma_i, \vec{r}_i.B, \vec{r}_i.C, \vec{s}_i.\overline{B}, \vec{s}_i.\overline{C}]$ has a CAN–free, M–free proof. Therefore we can have a CAN–free proof of the hypersequent $[\vdash \Gamma_i]_{i=1}^n$ by invoking the induction hypothesis twice, since the complexity of B and C is lower than that of $B + C$.
- If $A = r'B$, since the $\times$ rule is CAN–free invertible, $[\vdash \Gamma_i, (r'\vec{r}_i.).B, (r'\vec{s}_i.).\overline{B}]$ has a CAN–free, M–free proof. Therefore we can have a CAN–free proof of the hypersequent $[\vdash \Gamma_i]_{i=1}^n$ by invoking the induction hypothesis on the simpler formula B.
- If $A = B \sqcup C$, since the $\sqcup$ rule is CAN–free invertible,

$$\left[\vdash \Gamma_i, \vec{r}_i.B, \vec{s}_i.(\overline{B} \sqcap \overline{C})\right] \mid \left[\vdash \Gamma_i, \vec{r}_i.C, \vec{s}_i.(\overline{B} \sqcap \overline{C})\right]$$

  has a CAN–free, M–free proof. Then since the $\sqcap$ is CAN–free invertible, $[\vdash \Gamma_i, \vec{r}_i.B, \vec{s}_i.\overline{B}] \mid [\vdash \Gamma_i, \vec{r}_i.C, \vec{s}_i.\overline{C}]$ has a CAN–free, M–free proof. Therefore we can obtain a CAN–free proof of the hypersequent $[\vdash \Gamma_i]_{i=1}^n$ by invoking the induction hypothesis twice on the simpler formulas B and C.
- If $A = B \sqcap C$, we proceed in a similar way as for the case $A = B \sqcup C$.

  Now for the case $A = \Diamond B$, we distinguish two cases:

(1) the proof d ends with an application of the $\Diamond$ rule which simplifies $A = \Diamond B$ to B. In this case we can simply conclude by invoking the induction hypothesis on B.

(2) The proof d ends with some other rule (recall that no CAN rules and no M rules appear in d). In this case we decrease the complexity of d, keeping $\Diamond B$ as the CAN formula, and then invoke the induction hypothesis on the derivation having reduced complexity. This proof step is rather long to prove, as it requires analysing of all possible rules concluding the derivation d. We just illustrate the two case when d ends with a logical rule $(+)$ and a structural rule (C) to illustrate the general method.

- if the proof finishes with

$$\frac{[\vdash \Gamma_i, \vec{r}_i.\Diamond B, \vec{s}_i.\Diamond\overline{B}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.\Diamond B, \vec{s}_1.\Diamond\overline{B}, \vec{r'}.C, \vec{r'}.D}{[\vdash \Gamma_i, \vec{r}_i.\Diamond B, \vec{s}_i.\Diamond\overline{B}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.\Diamond B, \vec{s}_1.\Diamond\overline{B}, \vec{r'}.(C+D)} \; +$$

by induction hypothesis, there is a CAN–free M–free proof of

$$[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{r'}.C, \vec{r'}.D$$

We continue the proof with

$$\frac{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{r'}.C, \vec{r'}.D}{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1, \vec{r'}.(C+D)} \; +$$

- if the proof finishes with

$$\frac{[\vdash \Gamma_i, \vec{r}_i.\Diamond B, \vec{s}_i.\Diamond\overline{B}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.\Diamond B, \vec{s}_1.\Diamond\overline{B} \mid \vdash \Gamma_1, \vec{r}_1.\Diamond B, \vec{s}_1.\Diamond\overline{B}}{[\vdash \Gamma_i, \vec{r}_i.\Diamond B, \vec{s}_i.\Diamond\overline{B}]_{i=2}^n \mid \vdash \Gamma_1, \vec{r}_1.\Diamond B, \vec{s}_1.\Diamond\overline{B}} \; C$$

by induction hypothesis, there is a CAN–free M–free proof of

$$[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1 \mid \vdash \Gamma_1$$

We continue the proof with

$$\frac{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1 \mid \vdash \Gamma_1}{[\vdash \Gamma_i]_{i=2}^n \mid \vdash \Gamma_1} \; C$$

$\square$

Lemma 4.39. *If there is a CAN–free, M–free proof d of $[\vdash \Gamma_i, \vec{r}_i.x, \vec{s}_i.\overline{x}]_{i=1}^n$ then for all $\vec{r'_i}$. and $\vec{s'_i}$. such that for all $i, \sum \vec{r}_i - \sum \vec{s}_i = \sum \vec{r'}_i - \sum \vec{s'}_i$, there is a CAN–free, M–free proof of $\left[\vdash \Gamma_i, \vec{r'_i}.x, \vec{s'_i}.\overline{x}\right]_{i=1}^n$.*

Proof. By induction on d. $\square$

Lemma 4.40. *If there is a CAN–free, M–free proof d of $\left[\vdash \Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}\right]_{i=1}^n$ then for all $\vec{r'_i}$. and $\vec{s'_i}$. such that for all $i, \sum \vec{r}_i - \sum \vec{s}_i \leq \sum \vec{r'}_i - \sum \vec{s'}_i$, there is a CAN–free, M–free proof of $\left[\vdash \Gamma_i, \vec{r'_i}.1, \vec{s'_i}.\overline{1}\right]_{i=1}^n$.*

Proof. By induction on d. We show only the non–trivial case.

- If d finishes with:

$$\frac{\left[\vdash \Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}\right]_{i\geq 2} \mid \vdash \Gamma_1, \vec{c}1, \vec{c'}\overline{1}}{\left[\vdash \Gamma_i, \vec{r}_i.1, \vec{s}_i.\overline{1}\right]_{i\geq 2} \mid \vdash \Gamma_1, (\vec{a}; \vec{b}; \vec{c}).1, (\vec{a'}; \vec{b'}; \vec{c'}).\overline{1}} \; 1, \sum \vec{a} + \sum \vec{b} \geq \sum \vec{a'} + \vec{b'}$$

with $\vec{r_1} = \vec{b}; \vec{c}$ and $\vec{s_1} = \vec{b'}; \vec{c'}$, then $\sum \vec{c} - \sum \vec{c'} \leq \sum \vec{r'}_1 + \sum \vec{a} - (\sum \vec{s'}_1 + \sum \vec{a'})$ so by induction hypothesis

$$\vDash_{\mathbf{HMR}} \left[\vdash \Gamma_i, \vec{r'_i}.1, \vec{s'_i}.\overline{1} \right]_{i \geq 2} \ | \vdash \Gamma_1, (\vec{r'_1}; \vec{a}).1, (\vec{s'_1}; \vec{a'}).\overline{1}$$

which is the result we want.

$\square$

4.8. **Decidability – Proof of Theorem 4.11.** In this section we adapt the algorithm presented in Subsection 3.10 and prove the decidability of the **HMR** system.

The procedure takes a hypersequent G, where coefficients in weighted formulas are polynomials over scalar-variables $\vec{\alpha}$, and constructs a formula $\phi_G(\vec{\alpha}) \in FO(\mathbb{R}, +, \times, \leq)$ in the language of the first order theory of the real closed field. The procedure is recursive and terminates because each recursive call decreases the logical complexity and the modal complexity (i.e., the maximal modal depth of any formulas) of its input G. The key property is that a sequence of scalars $\vec{s} \in \mathbb{R}_{\geq 0}$ satisfies ϕ_G if and only if the hypersequent $G[s_j/\alpha_j]$ is derivable in the system **HMR**. The decidability then follows from the well–known fact that the theory $FO(\mathbb{R}, +, \times, \leq)$ admits quantifier elimination and is decidable [Tar51].

The algorithm to construct ϕ_G takes as input G and proceeds as follows:

(1) if G is not a basic hypersequent (i.e., if it contains any complex formula whose outermost connective is not $\Diamond$ or 1 or $\overline{1}$), then the algorithm returns

$$\phi_G = \bigwedge_{i=1}^{n} \phi_{G_i}$$

where $G_1, \ldots, G_n$ are the basic hypersequents obtained by iteratively applying the logical rules, and ϕ_{G_i} is the formula recursively computed by the algorithm on input G_i.

(2) if G has the shape $\vdash$ then $\phi_G = \top$.

(3) if G is a basic hypersequent which is not $\vdash$ then G has the shape

$$\vdash \Gamma_1, \Diamond \Delta_1, \vec{R'}_1.1, \vec{S'}_1.\overline{1} \ | \ \ldots \ | \vdash \Gamma_m, \Diamond \Delta_m, \vec{R'}_m.1, \vec{S'}_m.\overline{1}$$

where $\Gamma_i = \vec{R}_{i,1}.x_1, ..., \vec{R}_{i,k}.x_k, \vec{S}_{i,1}.\overline{x_1}, ..., \vec{S}_{i,k}.\overline{x_k}$. For all $I \subsetneq [1...m]$, we define:

- A formula $Z_I(\beta_1, ..., \beta_m)$ that states that for all $i \in I$, $\beta_i = 0$.

$$Z_I(\beta_1, ..., \beta_m) = \bigwedge_{i \in I}(\beta_i = 0)$$

- A formula $NZ_I(\beta_1, ..., \beta_m)$ that states that for all $i \notin I$, $0 < \beta_i$.

$$NZ_I(\beta_1, ..., \beta_m) = \bigwedge_{i \notin I}(0 \leq \beta_i) \wedge \neg(\beta_i = 0)$$

- A formula $A_I(\beta_1, ..., \beta_m)$ that states that all the atoms cancel each other.

$$A_I(\beta_1, ..., \beta_m) = \bigwedge_{j=0}^{k}(\sum_{i=1}^{m} \beta_i \sum \vec{R}_{i,j} = \sum_{i=1}^{m} \beta_i \sum \vec{S}_{i,j})$$

- A formula $O_I(\beta_1, ..., \beta_m)$ that states that there are more 1 than $\overline{1}$,

$$O_I(\beta_1, ..., \beta_m) = \sum_{i=1}^{m} \beta_i \sum \vec{S'}_i \leq \sum_{i=1}^{m} \beta_i \sum \vec{R'}_i$$

- A hypersequent $H_I(\beta_1, ..., \beta_m)$ which is the result of cancelling the atoms using $\beta_1, ..., \beta_m$ and then using the $\Diamond$ rule, i.e. is the leaf of the following preproof:

$$\dfrac{\dfrac{\dfrac{\vdash \beta_{k_1}.\Delta_{k_1}, (\beta_{k_1}\vec{R}'_{k_1}).1, (\beta_{k_1}\vec{S}'_{k_1}).\overline{1}, ..., \beta_{k_l}.\Delta_{k_l}, (\beta_{k_l}\vec{R}'_{k_l}).1, (\beta_{k_l}\vec{S}'_{k_l}).\overline{1}}{\vdash \beta_{k_1}.\Diamond\Delta_{k_1}, (\beta_{k_1}\vec{R}'_{k_1}).1, (\beta_{k_1}\vec{S}'_{k_1}).\overline{1}, ..., \beta_{k_l}.\Diamond\Delta_{k_l}, (\beta_{k_l}\vec{R}'_{k_l}).1, (\beta_{k_l}\vec{S}'_{k_l}).\overline{1}}\Diamond}{\vdash \beta_{k_1}.\Gamma_{k_1}, \beta_{k_1}.\Diamond\Delta_{k_1}, (\beta_{k_1}\vec{R}'_{k_1}).1, (\beta_{k_1}\vec{S}'_{k_1}).\overline{1}, ..., \beta_{k_l}.\Gamma_{k_l}, \beta_{k_l}.\Diamond\Delta_{k_l}, (\beta_{k_l}\vec{R}'_{k_l}).1, (\beta_{k_l}\vec{S}'_{k_l}).\overline{1}}\text{ID}^*}{\dfrac{\vdash \beta_{k_1}.\Gamma_{k_1}, \beta_{k_1}.\Diamond\Delta_{k_1}, (\beta_{k_1}\vec{R}'_{k_1}).1, (\beta_{k_1}\vec{S}'_{k_1}).\overline{1} \mid ... \mid \vdash \beta_{k_l}.\Gamma_{k_l}, \beta_{k_l}.\Diamond\Delta_{k_l}, (\beta_{k_l}\vec{R}'_{k_l}).1, (\beta_{k_l}\vec{S}'_{k_l}).\overline{1}}{\vdash \Gamma_{k_1}, \Diamond\Delta_{k_1}, \vec{R}'_{k_1}.1, \vec{S}'_{k_1}.\overline{1} \mid ... \mid \vdash \Gamma_{k_l}, \Diamond\Delta_{k_l}, \vec{R}'_{k_l}.1, \vec{S}'_{k_l}.\overline{1}}\text{T}^*}\text{S}^*}$$

where $\{k_1, ..., k_l\} = [1..m]\backslash I$.
- The formula $\phi_{H_I(\beta_1,...,\beta_m)}$ computed recursively from $H_I(\beta_1, ..., \beta_m)$ above.
- A formula $\phi_{G,I}$ that corresponds to $\phi_{G'}$ where G' is the hypersequent obtained on using the W rule on all i-th sequents for $i \in I$, i.e. the leaf of the following preproof:

$$\dfrac{\vdash \Gamma_{k_1}, \Diamond\Delta_{k_1}, \vec{R}'_{k_1}.1, \vec{S}'_{k_1}.\overline{1} \mid ... \mid \vdash \Gamma_{k_l}, \Diamond\Delta_{k_l}, \vec{R}'_{k_l}.1, \vec{S}'_{k_l}.\overline{1}}{\vdash \Gamma_1, \Diamond\Delta_1, \vec{R}'_1.1, \vec{S}'_1.\overline{1} \mid ... \mid \vdash \Gamma_m, \Diamond\Delta_m, \vec{R}'_m.1, \vec{S}'_m.\overline{1}}\text{W}^*$$

with $\{k_1, ..., k_l\} = [1..m]\backslash I$. Then $\phi_{G,I} =$

$$\exists\beta_1, ..., \beta_m, Z_I(\beta_1, ..., \beta_m) \wedge NZ_I(\beta_1, ..., \beta_m) \wedge A_I(\beta_1, ..., \beta_m) \wedge O_I(\beta_1, ..., \beta_m) \wedge \phi_{H_I(\beta_1,...,\beta_m)}$$

Finally, we return ϕ_G defined as follow:

$$\phi_G = \bigvee_{I \subsetneq [1...m]} \phi_{G,I}$$

The following theorem states the soundness of the above described algorithm.

Theorem 4.41. *Let G be a hypersequent having polynomials $R_1, ..., R_k \in \mathbb{R}_{>0}[\vec{\alpha}]$ over scalar–variables $\vec{\alpha}$. Let $\phi_G(\vec{\alpha})$ be the formula returned by the algorithm described above on input G. Then, for all $\vec{s} \in \mathbb{R}_{\geq 0}$, the following are equivalent:*

(1) $\phi_G(\vec{s})$ holds in $\mathbb{R}$,
*(2) $G[s_j/\alpha_j]$ is derivable in **HMR**.*

Proof. As in Theorem 3.44, by using the CAN–free invertibility Theorem 4.6, we can assume that G is a basic hypersequent. If G has the shape $\vdash$, the result is trivial. Otherwise, the result is a direct corollary of Lemma 4.25 since the formula NZ_I corresponds to the first property, the formula A_I corresponds to the second property, the formula O_I corresponds to the third one and the formula ϕ_{H_I} corresponds to the last one. $\qquad\square$

4.9. **One open problem regarding HMR.** We have not been able to prove or disprove the equivalent of Proposition 3.16 in the context of the system **HMR**. We leave this as an open problem.

Question. Let G be a hypersequent whose scalars are all rational numbers. Is it true that, if G has a CAN–free proof in **HMR** then G also has a CAN–free and T–free proof in **HMR**?

This is a question of practical importance. Indeed the T rule is the only non–analytical rule (beside the CAN rule which, however, can be eliminated) of the system **HMR** and, as a consequence, it makes the proof search endeavour more difficult.

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