ORDINAL DISTANCES IN TRANSFINITE GRAPHS

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Abstract — An ordinal-valued metric, taking its values in the set $\aleph_1$ of all countable ordinals, can be assigned to a metrizable set $\mathcal{M}$ of nodes in any transfinite graph. $\mathcal{M}$ contains all the nonsingleton nodes, as well as certain singleton nodes. Moreover, this yields a graphical realization of Cantor's countable ordinals, as well as of the Aristotelian ideas of "potential" and "actual" infinities, the former being represented by the arrow ranks and the latter by the ordinal ranks of transfiniteness. This construct also extends transfinitely the ideas of nodal eccentricities, radii, diameters, centers, peripheries, and blocks for graphs, and the following generalizations are established. With ν denoting the rank of a ν-graph $\mathcal{G}^\nu$, the ν-nodes of $\mathcal{G}^\nu$ comprise the center of a larger ν-graph. Also, when there are only finitely many ν-nodes and when those ν-nodes are "pristine" in the sense that they do not embrace nodes of lower ranks, the infinitely many nodes of all ranks have eccentricities of the form $\omega^\nu \cdot p$, where ω is the first transfinite ordinal and p lies in a finite set of natural numbers. Furthermore, the center is contained in a single block of highest rank. Also, when each loop of the ν-graph is confined within a $(\nu - 1)$-section, the center either is a single node of highest rank, or is the set of internal nodes of a $(\nu - 1)$-section, or is the union of the latter two kinds of centers.

Key Words: Distances in graphs, transfinite graphs, ordinal-valued distances, eccentricities, blocks, centers.

1. Introduction

The idea of distances in connected finite graphs has been quite fruitful, with much research directed toward both theory and applications. See, for example, [2], [3], [5], and the references therein. Such distances are given by a metric that assigns to each pair of nodes

the minimum number of branches among all paths connecting those two nodes. Thus, the metric takes its values in the set $\aleph_0$ of natural numbers. That distance considerations can be so fruitful in the theory of finite graphs inspires the question of whether distance constructs can be devised for transfinite graphs. Transfinite graphs [7], [8] represent a generalization of graphs that is roughly analogous to Cantor's extension of the natural numbers to the transfinite ordinals.

An affirmative answer to that question was achieved in [4], wherein a real-valued metric was devised for the purpose of ascertaining limit points at infinite extremities of a conventionally infinite, electrical, resistive network, through which points electrical current could flow into other such networks. This construct was extended to higher ranks of transfiniteness [8] with an infinite hierarchy of metrics, one for each rank of transfiniteness. These metrics take their values in the nonnegative real line, are quite different from the standard branch-count metric mentioned above, require a variety of restriction such as local finiteness, and do not reduce to the branch-count metric for finite graphs.

Let us emphasize why, we feel, that for this paper it is inappropriate to use a real-valued metric that makes infinite extremities of a conventionally infinite graph look as though they are only finitely distant from any node of the graph. If branch counts are to determine distances between nodes, then no node is closer to any infinite extremity than any other node. Thus, all nodes in a conventionally infinite graph should be viewed as equally distant from any extremity, and that distance should be ω, the first transfinite ordinal. This property cannot be avoided if branch counts are to prevail.

The problem attacked in this work is the invention of a single metric that extends the standard branch-count metric to transfinite graphs, one that holds for all ranks of transfiniteness, and reduces to the standard branch-count metric for finite graphs. In closer analogy to Cantor's work, the metric proposed in this paper assigns countable ordinals to pairs of nodes in a connected transfinite graph; that is, it takes its values in the set $\aleph_1$ of all countable ordinals. Moreover, the metric is applicable even when the graph is not locally finite and may even have uncountably many branches.

As a consequence, the ideas of nodal eccentricities, radii, diameters, centers, peripheries,

and blocks are herein extended to transfinite graphs. However, to do so, the set $\aleph_1$ has to be enlarged by inserting an "arrow rank" [7, page 4], [8, page 4] immediately preceding each limit ordinal. These arrow ranks reflect the Aristotelian idea of a "potential infinity" as distinct from the other Aristotelian idea of an "actual infinity" typified by the ordinals.

Several theorems concerning these ideas are proven, some of which lift results concerning finite graphs to transfinite graphs. For example, with $\mathcal{G}^\nu$ henceforth denoting a transfinite graph of rank ν (i.e., a ν-graph) [7, Chap. 2], the transfinite radius and diameter of $\mathcal{G}^\nu$ are related, but now in a more complicated way (Theorem 7.2). Nodes of highest rank (i.e., ν-nodes) are shown to comprise the center of a larger ν-graph (Corollary 7.5). A finite range of possible transfinite ordinals for the eccentricities of the nodes of $\mathcal{G}^\nu$ is established (Corollary 8.10). The center of $\mathcal{G}^\nu$ lies in a block of highest rank (Theorem 9.5), and that center is identified either as a single ν-node or as a certain infinite set of nodes of all ranks (Theorem 10.3). To obtain these and other results, we employ some restrictions on $\mathcal{G}^\nu$, which are introduced when needed and are then assumed to hold throughout the rest of the paper.

Various properties of transfinite graphs are used in this work. These can be found in the book [7]. A simplified but more restrictive rendition of the subject is given in [8]. We will work in the generality of [7] and will refer to specific pages in that book when invoking various concepts and results. In this paper, we do not allow any branch to be a self-loop; thus, every branch is incident to two different nodes. However, parallel branches are allowed. We use the standard notations for ordinals and cardinals [1].

Furthermore, any transfinite node x^α may (but need not) contain exactly one node of lower rank x^β $(\beta < \alpha)$; x^β in turn may contain exactly one other node x^γ $(\gamma < \beta)$, and so forth through finitely many decreasing ranks. We say that x^α *embraces* itself and x^β, x^γ, ... , as well. On the other hand, if x^α is not embraced by a node of higher rank, we call x^α a *maximal* node. It is the maximal nodes we will be primarily concerned with because connectedness to x^α implies connectedness to x^β, x^γ, ... , as well. Rather than repeating the adjective "maximal," we let it be understood throughout that any node discussed is maximal unless the opposite is explicitly stated. This implies that different (maximal) nodes

must be "totally disjoint," that is, they embrace no common elements [7, Lemma 2.2-1].

Throughout this work we restrict the rank ν of $\mathcal{G}^\nu$ to $1 \leq \nu \leq \omega$, $\nu \neq \vec{\omega}$. (Here, $\vec{\omega}$ is the arrow rank immediately preceding ω [7, Sec. 2.3].) All our results can be extended to many ordinal ranks higher than ω by modifying our arguments in obvious ways. However, it is not apparent whether this extension can be made throughout all the ordinal ranks in $\aleph_1$ [7, Sec. 2.5].

Given any ordinal rank ρ no higher than ν, two nodes (resp. two branches) of $\mathcal{G}^\nu$ are said to be ρ-connected if there is a path of rank ρ or less that terminates at those two nodes (resp. terminates at 0-nodes of those branches).[1] If this holds for $\rho = \nu$ and for all branches in $\mathcal{G}^\nu$, $\mathcal{G}^\nu$ is said to be ν-connected. We always assume that $\mathcal{G}^\nu$ is ν-connected.

2. Lengths of Paths

Throughout this paper, we use the natural sum of transfinite ordinals to obtain the normal expansion of that sum [1, pages 354-355].

0-Paths:

A (nontrivial) 0-path P^0 is an alternating sequence

$$P^0 = \{\ldots, x_m^0, b_m, x_{m+1}^0, b_{m+1}, \ldots\} \tag{1}$$

of branches b_m and conventional nodes x_m^0 (also called "0-nodes") in which no term repeats and each branch is incident to the two 0-nodes adjacent to it in the sequence. If the sequence terminates on either side, it terminates at a 0-node. This is the conventional definition of a path. (The 0-nodes of (1) need not be maximal when P^0 occurs within a transfinite graph.) When P^0 is one-ended (i.e., one-way infinite), its length is defined to be $|P^0| = \omega$. When P^0 is endless (i.e., two-way infinite), its length is taken to be $|P^0| = \omega \cdot 2$. If P^0 is two-ended (i.e., has only finitely many 0-nodes), we set $|P^0| = \tau_0$, where τ_0 is the number of branches in P^0. We might motivate these definitions by noting that we are using ω to denote the infinity of branches in a one-ended 0-path and using $\omega \cdot 2$ to represent to fact that an endless 0-path is the union of two one-ended paths. Equivalently, we can identify ω with each 0-tip traversed; a one-ended 0-path has one 0-tip, and an endless 0-path has two 0-tips—hence,

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[1]Paths of various ranks are defined in the next section. See also [7, Chap. 2].

the length $\omega \cdot 2$. (See [7, page 20] for the definition of a 0-tip.)

1-Paths.

A (nontrivial) 1-path P^1 [7, page 28] is an alternating sequence

$$P^1 \; = \; \{\ldots, x_m^1, P_m^0, x_{m+1}^1, P_{m+1}^0, \ldots\} \tag{2}$$

of 1-nodes x_m^1 and 0-paths P_m^0 that represents a tracing through a transfinite graph of rank 1 or greater in which no node is met more than once in the tracing. If the sequence terminates on either side, it terminates at a 0-node or 1-node. See [7, page 28] for the full definition of a 1-path. The length $|P^1|$ of P^1 is defined as follows. When P^1 is one-ended, $|P^1| = \omega^2$, and, when P^1 is endless, $|P^1| = \omega^2 \cdot 2$. When P^1 is two-ended (i.e., when it has only finitely many 1-nodes), we set $|P^1| = \sum_m |P_m^0|$, where the sum is over the finitely many 0-paths P_m^0 in (2); thus, in this case, $|P^1| = \omega \cdot \tau_1 + \tau_0$, where τ_1 is the number of 0-tips P^1 traverses, and τ_0 is the number of branches in all the 0-paths in (2) that are two-ended. It is important here to write $|P^1|$ as indicated and not as $\tau_0 + \omega \cdot \tau_1$ because ordinal addition is not commutative [1, page 327]. Thus, $\omega \cdot \tau_1 + \tau_0$ takes into account the lengths of all the 0-paths in (2), but $\tau_0 + \omega \cdot \tau_1$ fails to do so. As specified above, $\omega \cdot \tau_1 + \tau_0$ is the "normal expansion" [1, pages 354-355] of $|P^1|$.

μ-Paths:

Now, let μ be any positive natural number. A μ-path [7, page 33] is an alternating sequence

$$P^\mu \; = \; \{\ldots, x_m^\mu, P_m^{\alpha_m}, x_{m+1}^\mu, P_{m+1}^{\alpha_{m+1}}, \ldots\} \tag{3}$$

of μ-nodes x_m^μ and α_m-paths $P_m^{\alpha_m}$, where $0 \leq \alpha_m < \mu$. (The natural numbers α_m may vary with m, and the μ-nodes need not be maximal.) As before, P^μ represents a tracing through a transfinite graph of rank μ or larger in which no node is met more than once in the tracing. Termination on either side of (3) occurs at a node of rank μ or less. When P^μ is one-ended, its length $|P^\mu|$ is defined to be $\omega^{\mu+1}$, and, when P^μ is endless , we set $|P^\mu| = \omega^{\mu+1} \cdot 2$. When, however, P^μ is two-ended (i.e., has only finitely many μ-nodes), we set $|P^\mu| = \sum_m |P_m^{\alpha_m}|$, where as always this sum denotes a normal expansion of an ordinal

obtained through a natural summation of ordinals [1, pages 354-355]. Recursively, this gives

$$|P^\mu| = \omega^\mu \cdot \tau_\mu + \omega^{\mu-1} \cdot \tau_{\mu-1} + \ldots + \omega \cdot \tau_1 + \tau_0, \tag{4}$$

where $\tau_\mu, \tau_{\mu-1}, \ldots, \tau_0$ are natural numbers. τ_μ is the number of $(\mu - 1)$-tips among all the one-ended and endless $(\mu - 1)$-paths (i.e., when $\alpha_m = \mu - 1$) appearing in (3); τ_μ is not 0. For $k = \mu - 1, \mu - 2, \ldots, 1$, we set τ_k equal to the number of $k - 1$-tips generated by these recursive definitions. Finally, τ_0 is one-half the number of elementary tips [7, page 9] generated recursively by these definitions. Thus, τ_0 is a number of branches because each branch has exactly two elementary tips. Any τ_k $(k < \mu)$ can be 0.

Example 2.1. Let P^3 be the two-ended 3-path:

$$P^3 = \{x_1^2, P_1^2, x_2^3, P_2^2, x_3^3, P_3^2, x_4^3\}$$

Here, P_1^2 is assumed to be a one-ended 2-path terminating on the left with x_1^2 and reaching x_2^3 through a 2-tip. Hence, $|P_1^2| = \omega^3$. We take P_2^2 to be the two-ended 2-path

$$P_2^2 = \{y_1^2, Q_1^1, y_2^2, Q_2^0, y_3^2\},$$

where y_1^2 and y_3^2 are members of x_2^3 and x_3^3 respectively. Q_1^1 is an endless 1-path reaching the 2-nodes y_1^2 and y_2^2 with 1-tips, and Q_2^0 is a finite 0-path with four branches, whose terminal 0-nodes are members of y_2^2 and y_3^2. Hence, $|Q_1^1| = \omega^2 \cdot 2$ and $|Q_2^0| = 4$. Finally, we take P_3^2 to be an endless 2-path reaching x_3^3 and x_4^3 through 2-tips. Hence, $|P_3^2| = \omega^3 \cdot 2$.

Altogether then, with a rearrangement of the following ordinal sum to get a normal-expansion, we may write

$$\begin{aligned} |P^3| &= |P_1^2| + |P_2^2| + |P_3^2| \\ &= \omega^3 + |Q_1^1| + |Q_2^0| + \omega^3 \cdot 2 \\ &= \omega^3 + \omega^2 \cdot 2 + 4 + \omega^3 \cdot 2 \\ &= \omega^3 \cdot 3 + \omega^2 \cdot 2 + 4 \end{aligned}$$

□

$\vec{\omega}$-*paths:*

$\vec{\omega}$-paths occur within paths of ranks ω and higher, but they are never two-ended [7, pages 40-41]. The length of an $\vec{\omega}$-path $P^{\vec{\omega}}$ is defined to be $|P^{\vec{\omega}}| = \omega^\omega$ when $P^{\vec{\omega}}$ is one-ended, and $|P^{\vec{\omega}}| = \omega^\omega \cdot 2$ when $P^{\vec{\omega}}$ is endless.

ω-paths:

A (nontrivial) ω-path P^ω [7, page 44]

$$P^\omega = \{\ldots, x_m^\omega, P_m^{\alpha_m}, x_{m+1}^\omega, P_{m+1}^{\alpha_{m+1}}, \ldots\} \tag{5}$$

is an alternating sequence of (not necessarily maximal) ω-nodes x_m^ω and α_m-paths $P_m^{\alpha_m}$ ($0 \le \alpha_m \le \vec{\omega}$) that represents a tracing through a graph of rank ω (or larger) in which no node is met more than once and a termination on either side is at a node of rank ω or less. By definition, when P^ω is one-ended, $|P^\omega| = \omega^{\omega+1}$; also, when P^ω is endless, $|P^\omega| = \omega^{\omega+1} \cdot 2$. When P^ω is two-ended (i.e., has only finitely many ω-nodes), we set

$$|P^\omega| = \sum_m |P_m^{\alpha_m}| = \omega^\omega \cdot \tau_\omega + \sum_{k=0}^{\infty} \omega^k \cdot \tau_k \tag{6}$$

with the natural summation being understood. Here, τ_ω is the number of $\vec{\omega}$-tips among all the one-ended and endless $\vec{\omega}$-paths appearing as elements of $P_m^{\alpha_m}$ in (5) (i.e., when $\alpha_m = \vec{\omega}$); τ_ω is not 0. On the other hand, the τ_k are determined recursively, as they are in (4). There are only finitely many nonzero terms in the summation within (6) because there are only finitely many paths $P_m^{\alpha_m}$ in a two-ended ω-path and each $|P_m^{\alpha_m}|$ is a finite sum as in (4).

An immediate result of all these definitions is the following.

Lemma 2.2. *If Q^β is a subpath of a γ-path P^γ ($0 \le \beta \le \gamma$), then $|Q^\beta| \le |P^\gamma|$.*

It is easy to add ordinals when they are in normal-expansion form—simply add their corresponding coefficients. Thus, the length of the union of two paths that are totally disjoint except for incidence at a terminal node (a "series connection") is obtained by adding their lengths in normal expansion form. Similarly, if Q is a proper subpath of P, the part $P \backslash Q$ of P that is not in Q has the total length $|P| - |Q|$, which is obtained by subtracting the coefficients of $|Q|$ from the corresponding coefficients of $|P|$.

3. Metrizable Sets of Nodes

In a connected finite graph, for every two nodes there is at least one path terminating at them. This is not in general true for transfinite graphs.

Example 3.1. The 1-graph of Fig. 1 provides an example. In that graph, x_a^1 (resp. x_b^1) is a nonsingleton 1-node containing the 0-tip t_a^0 (resp. t_b^0) for the one-ended path of a_k branches (resp. b_k branches) and also embracing an elementary tip of branch d (resp. e). There are, in addition, uncountably many 0-tips for paths that alternate infinitely often between the a_k and b_k branches by passing through c_k branches; those tips are contained in singleton 1-nodes, one for each. x_{abc}^1 denotes one such singleton 1-node; the others are not shown. Note that there is no path connecting x_{abc}^1 to x_a^1 (or to any other 1-node) because any tracing between x_{abc}^1 and x_a^1 must repeat 0-nodes. Thus, our definition (given in the next section) of the distance between two nodes as the minimum path length for all paths connecting those nodes cannot be applied to x_{abc}^1 and x_a^1. We seek some means of applying this distance concept to at least some pairs of nodes. □

To this end, we impose the following condition on the transfinite graph $\mathcal{G}^\nu$, which is understood to hold henceforth.

Condition 3.2. *If two tips (perforce of ranks less than ν and possibly differing) are nondisconnectable,[2] then either they are shorted together (i.e., are embraced by the same node) or at least one of them is open (i.e., is the sole member of a singleton node).*

The 1-graph of Fig. 1 satisfies this condition.

The following results ensue: As specified above, $\mathcal{G}^\nu$ is ν-*connected*, which means that for any two branches there is a two-ended path P^ρ of some rank ρ ($\rho \le \nu$) that meets those two branches. Nevertheless, there may be two nodes not having any path that meets them (i.e., the two nodes are not ν-connected). For instance, the 1-graph of Fig. 1 is 1-connected, but there is no path that meets x_a^1 and x_{abc}^1. Now, as will be established by Lemma 3.3 below, if $\mathcal{G}^\nu$ satisfies Condition 3.2, then, for any two nonsingleton nodes, there will be at least one two-ended path terminating at them. As a result, we will be able to define distances between nonsingleton nodes. Furthermore, some singleton nodes may be amenable to such distance measurements, as well. To test this, we need merely append a new branch b to a singleton node x^α by adding an elementary tip of b to x^α as an embraced elementary tip

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[2]Two tips are called *nondisconnectable* if their representative (one-ended) paths continue to meet no matter how far along the representative paths one proceeds [7, page 58]. Two tips are called *disconnectable* if they have representative paths that are totally disjoint.

to get a nonsingleton node $\hat{x}^\alpha$, with the other elementary tip of b left open (i.e., b is added as an end branch)—and then check to see if Condition 3.2 is maintained. More generally, with $\mathcal{G}^\nu$ being ν-connected and satisfying Condition 3.2, let $\mathcal{M}$ be a set consisting of all the nonsingleton (maximal) nodes in $\mathcal{G}^\nu$ and possibly other singleton (maximal) nodes having the property that, if end branches are appended to those singleton nodes simultaneously, Condition 3.2 is still satisfied by the resulting network. Any such set $\mathcal{M}$ will be call a *metrizable* set of nodes.

Lemma 3.3. *Let $\mathcal{M}$ be a metrizable set of nodes in $\mathcal{G}^\nu$. Then, for any two nodes of $\mathcal{M}$, there exists a two-ended path terminating to those nodes.*

Proof. Let x_a^α and x_b^β be two different nodes in $\mathcal{M}$. Since they are maximal, they must be totally disjoint. Then, by Condition 3.2, any tip in x_a^α is disconnectable from every tip in x_b^β; indeed, if they were nondisconnectable, they would have to be shorted together, making x_a^α and x_b^β the same node. Thus, we can choose a representative path P_a for that tip in x_a^α that is totally disjoint from a representative path P_b for a tip in x_b^β. By the definition of ν-connectedness, there will be a path P_{ab} connecting a branch of P_a and a branch of P_b. By [7, Corollary 3.5-4], there is in the subgraph $P_a \cup P_{ab} \cup P_b$ induced by the branches of those three paths a two-ended path terminating at x_a^α and x_b^β. $\square$

Example 3.4. For an illustration, remove in Fig. 1 branches d and e along with the 0-nodes y_a^0 and y_b^0, thereby making x_a^1 and x_b^1 singleton 1-nodes. Then, the remaining 0-nodes along with x_a^1 and x_b^1 comprise a metrizable set. Also, those 0-nodes along with x_{abc}^1 comprise another metrizable set. However, those 0-nodes along with x_a^1, x_b^1, and x_{abc}^1 do not comprise a metrizable set. $\square$

4. Distances Between Nodes

Our objective now is to define ordinal distances between nodes whereby the metric axioms are satisfied. Let $\mathcal{M}$ be a metrizable set of nodes in $\mathcal{G}^\nu$. We define the distance function $d\colon \mathcal{M} \times \mathcal{M} \rightsquigarrow \aleph_1$ as follows: If x_a^α and x_b^β are different nodes in $\mathcal{M}$, we set

$$d(x_a^\alpha, x_b^\beta) \;=\; \min\{|P_{ab}|\colon P_{ab} \text{ is a two ended path terminating at } x_a^\alpha \text{ and } x_b^\beta\}. \qquad (7)$$

If $x_a^\alpha = x_b^\beta$, we set $d(x_a^\alpha, x_b^\beta) = 0$. By our constructions in Sec. 2, $|P_{ab}|$ is a countable ordinal no larger than $\omega^\omega \cdot k$, where k is a natural number. Moreover, any set of ordinals is

well-ordered and thus has a least member. Therefore, the minimum indicated in (7) exists, and is a countable ordinal.

Obviously, $d(x_a^\alpha, x_b^\beta) > 0$ if $x_a^\alpha \neq x_b^\beta$. Moreover, $d(x_a^\alpha, x_b^\beta) = d(x_b^\beta, x_a^\alpha)$. It remains to prove the triangle inequality; namely, if x_a^α, x_b^β, and x_c^γ are any three (maximal) nodes in $\mathcal{M}$, then

$$d(x_a^\alpha, x_b^\beta) \leq d(x_a^\alpha, x_c^\gamma) + d(x_c^\gamma, x_b^\beta). \tag{8}$$

This is easily done by invoking Lemma 2.2 and using [7, Corollary 3.5-4], whose rather long proof needs Condition 3.2.

Proposition 4.1. *d satisfies the metric axioms.*

Clearly, d reduces to the standard (branch-count) distance function when $\mathcal{G}^\nu$ is replaced by a finite graph. We have achieved one of the objectives of this paper by showing that the branch-count distance function can be extended transfinitely to any metrizable set of nodes in $\mathcal{G}^\nu$.

Example. 4.2. For the 1-graph of Fig. 1 and with $\mathcal{M}$ consisting of all the 0-nodes along with x_a^1 and x_b^1, we have $d(x_1^0, x_2^0) = 1$, $d(x_1^0, x_a^1) = d(x_1^0, x_b^1) = \omega$, $d(x_1^0, y_a^0) = d(x_1^0, y_b^0) = \omega + 1$, and $d(y_a^0, y_b^0) = \omega \cdot 2 + 2$. □

Because the minimum in (7) is achieved, we can sharpen Lemma 3.3 as follows.

Lemma 4.3. *Given any two nodes x and y in $\mathcal{M}$, there exists a path Q_{xy} terminating at x and y for which $|Q_{xy}| = d(x, y)$.*

There may be more than one such path. We call each of them an *x-to-y geodesic*.

5. Ordinals and Ranks

As we have seen, the distance between any two nodes of $\mathcal{M}$ is a countable ordinal. However, given any $x \in \mathcal{M}$, the set $\{d(x, y) : y \in \mathcal{M}\}$ may have no maximum. For example, this is the case for a one-ended 0-path P^0 where x is any fixed node of P^0 and y ranges through all the 0-nodes of P^0. On the other hand, for finite graphs the said maximum exists and is the "eccentricity" of x. We will be able to define an "eccentricity" for every node of $\mathcal{M}$ if we expand the set $\aleph_1$ of countable ordinals into the set $\mathcal{R}$ of ranks [7, page 4], [8, page 4]. This is done by inserting an *arrow rank $\vec{\rho}$* immediately before each limit-ordinal

rank $\rho \in \aleph_1$. $\mathcal{R}$ looks like[3]

$$\mathcal{R} \; = \; \{\vec{0}, 0, 1, 2, \ldots, \vec{\omega}, \omega, \omega + 1, \ldots, \overrightarrow{\omega \cdot 2}, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \overrightarrow{\omega \cdot n}, \omega \cdot n, \omega \cdot n + 1, \ldots,$$

$$\overrightarrow{\omega^2}, \omega^2, \omega^2 + 1, \ldots, \overrightarrow{\omega^k}, \omega^k, \omega^k + 1, \ldots, \overrightarrow{\omega^\omega}, \omega^\omega, \omega^\omega + 1, \ldots\}.$$

Note that the set of all ranks is well-ordered. Indeed, there is an order-preserving bijection from $\mathcal{R}$ to $\aleph_1$ obtained by replacing each rank by its successor rank. Since $\aleph_1$ is well-ordered, so, too, is $\mathcal{R}$.

In accordance with two Aristotelian ideas [6, page 3], we can view each transfinite (successor or limit) ordinal as an "actual infinity" because distances between nodes can assume those values, whereas each arrow rank (other than $\vec{0}$) can be viewed as a "potential infinity" because distances can only increase toward and approach an arrow rank without achieving it.

The arrow ranks served as a notational convenience in the prior works [7] and [8], but, for the purposes of this paper, we wish to define arrow ranks in terms of sequences of countable ordinals.

Let $\mathcal{A}$ be any set of countable ordinals having a countable ordinal ζ as an upper bound (i.e., $\zeta \geq \alpha$ for all $\alpha \in \mathcal{A}$). Let $\mathcal{D}$ be the set of countable ordinals, each of which is greater than every member of $\mathcal{A}$ and is no greater than ζ. If $\mathcal{D}$ is empty, $\mathcal{A}$ has a greatest member, namely, ζ. So, assume $\mathcal{D}$ is not empty. By well-ordering, $\mathcal{D}$ has a least member λ. If λ is a successor ordinal, then $\mathcal{A}$ has a greatest member, namely, $\lambda - 1$; in this case, $\lambda - 1$ is either a successor ordinal or a limit ordinal. We also denote $\lambda - 1$ by $\sup \mathcal{A}$. If λ is a limit ordinal, then there exists an increasing sequence $\{\alpha_k\}_{k=0}^\infty$ contained in $\mathcal{A}$ such that, for each $\gamma \in \mathcal{A}$, $\alpha_k > \gamma$ for all k sufficiently large (i.e., there exists a k_0 such that $\alpha_k > \gamma$ for all $k \geq k_0$).

With λ being a nonzero limit ordinal, we define the arrow-rank $\vec{\lambda}$ as an equivalence class of such increasing sequences of ordinals, where two such sequences $\{\alpha_k\}_{k=0}^\infty$ and $\{\beta_k\}_{k=0}^\infty$ (not necessarily in $\mathcal{A}$ now) are taken to be equivalent if, for each γ less than λ, there exists a natural number k_0 such that $\gamma < \alpha_k, \beta_k < \lambda$ for all $k > k_0$. The axioms of an equivalence

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[3]As was done in the prior works, we treat 0 as the first limit ordinal and $\vec{0}$ as the first arrow rank, but in this paper $\vec{0}$ will never be used. All our arrow ranks will be understood to be other than $\vec{0}$.

relationship are clearly satisfied. Each such sequence $\{\alpha_k\}_{k=0}^{\infty}$ is a representative of $\vec{\lambda}$, and we say that $\{\alpha_k\}_{k=0}^{\infty}$ *reaches* $\vec{\lambda}$. In this case, we let $\sup\mathcal{A}$ denote $\vec{\lambda}$.

Note that this equivalence class of increasing sequences is different from the set of ordinals less than λ. The latter is λ itself by the definition of ordinals. Also, we are distinguishing this equivalence class from the limit λ of any such sequence in the equivalence class [1, pages 165-166].

We summarize these definitions as follows.

Lemma 5.1. *If $\mathcal{A}$ is any set of countable ordinals that are bounded above by a countable ordinal ζ, then $\sup\mathcal{A}$ exists either as a (successor or limit) ordinal or as an arrow rank.*

6. Eccentricities and Related Ideas

First of all, note that the lengths of all paths in a ν-graph $\mathcal{G}^{\nu}$ are bounded by $\omega^{\nu+1} \cdot 2$ because the longest possible paths in $\mathcal{G}^{\nu}$ are the endless paths of rank ν. Therefore, all distances in $\mathcal{G}^{\nu}$ are also bounded above by $\omega^{\nu+1} \cdot 2$.

The eccentricity $e(x)$ of any node $x \in \mathcal{M}$ is defined by

$$e(x) \;=\; \sup\{d(x,y) \colon y \in \mathcal{M}\}. \tag{9}$$

Two cases arise: First, the supremum is achieved at some node $\hat{y} \in \mathcal{M}$. In this case, $e(x)$ is an ordinal; so, we can replace "sup" by "max" in (9) and write $e(x) = d(x,\hat{y})$. Second, the supremum is not achieved at any node in $\mathcal{M}$. In this case, $e(x)$ is an arrow rank.

The ideas of radii and diameters for finite graphs [3, page 32], [5, page 21] can also be extended transfinitely. Given $\mathcal{G}^{\nu}$ and $\mathcal{M}$, the *radius* $\mathrm{rad}(\mathcal{G}^{\nu}, \mathcal{M})$ is the least eccentricity among the nodes of $\mathcal{M}$:

$$\mathrm{rad}(\mathcal{G}^{\nu}, \mathcal{M}) \;=\; \min\{e(x) \colon x \in \mathcal{M}\} \tag{10}$$

We also denote this simply by rad with the understanding that $\mathcal{G}^{\nu}$ and $\mathcal{M}$ are given. The minimum exists as a rank (either as on ordinal or as an arrow rank) because the set of ranks is well-ordered. Thus, there will be at least one $x \in \mathcal{M}$ with $e(x) = \mathrm{rad}$.

Furthermore, the *diameter* $\mathrm{diam}(\mathcal{G}^{\nu}, \mathcal{M})$ is defined by

$$\mathrm{diam}(\mathcal{G}^{\nu}, \mathcal{M}) \;=\; \sup_{x \in \mathcal{M}} \sup_{y \in \mathcal{M}} \{d(x,y)\} \;=\; \sup\{d(x,y) \colon x, y \in \mathcal{M}\}. \tag{11}$$

With $\mathcal{G}^\nu$ and $\mathcal{M}$ understood, we denote the diameter simply by diam. As we have noted before, each $d(x, y)$ is no greater than $\omega^{\nu+1} \cdot 2$. So, by Lemma 5.1, diam exists either as an ordinal or as an arrow rank. In effect, we are defining diam as the "largest" of the eccentricities.

The ideas of the center and periphery of finite graphs can also be extended. The *center* of $(\mathcal{G}^\nu, \mathcal{M})$ is the set of nodes in $\mathcal{M}$ having the least eccentricity, namely, rad. The center is never empty.

The *periphery* of $(\mathcal{G}^\nu, \mathcal{M})$ is the set of nodes in $\mathcal{M}$ having the greatest eccentricity, namely, diam. If diam is an ordinal, there will be at least two nodes of $\mathcal{M}$ in the periphery. Indeed, if there did not exist at least two nodes in the periphery, then the ordinal diam could only be approached from below by distances between pairs of nodes that are less than diam; thus, the supremum in (11) would have to be an arrow rank—a contradiction. On the other hand, if diam is an arrow rank, the periphery can have any positive number of nodes—even just one or an infinity of them, as the following examples will show. It seems that the periphery will never be empty, but presently this is only a conjecture.

Example 6.1. Let $\mathcal{G}^0$ be a one-ended 0-path with $\mathcal{M}$ being the set of all 0-nodes. (We do not assign a 1-node at the path's infinite extremity.) Then, every 0-node has an eccentricity of $\vec{\omega}$. Thus, rad $=$ diam $= \vec{\omega}$, and $\mathcal{M}$ is both the center and the periphery of $(\mathcal{G}^0, \mathcal{M})$.

We have here a graphical realization of Aristotle's potential infinity represented by the eccentricity $\vec{\omega}$. However, this potential infinity can be made into an actual infinity by appending a 1-node at the infinite extremity of this 0-path. In Examples 6.4 and 6.5 below, it will not be possible to convert the potential infinities therein into actual infinities. $\square$

Example 6.2. Consider the 1-graph of Fig. 1 with $\mathcal{M}$ being the set of all 0-nodes along with x_a^1 and x_b^1. (Ignore x_{abc}^1 and all other 1-nodes.) The 0-nodes to the left of the 1-nodes all have the eccentricity $\omega + 1$. Also, $e(x_a^1) = e(x_b^1) = \omega \cdot 2 + 1$, and $e(y_a^0) = e(y_b^0) = \omega \cdot 2 + 2$. Thus, rad $= \omega + 1$ and diam $= \omega \cdot 2 + 2$. The center consists of all the 0-nodes to the left of the 1-nodes, and the periphery is $\{y_a^0, y_b^0\}$. By appending more "end" branches incident to either x_a^1 or x_b^1, we can increase the number of nodes in the periphery. $\square$

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Example 6.3. Now, consider the 1-graph obtained from Fig. 1 by deleting the branches d and e and the 0-nodes y_a^0 and y_b^0 but appending a new branch incident to x_1^0 and x_a^1. Let $\mathcal{M}$ be all the nodes. Then, the eccentricity of every node is ω. Thus, rad $=$ diam $= \omega$, and the center and periphery are the same, namely, $\mathcal{M}$. $\square$

Example 6.4. This time, let $\mathcal{G}^\nu$ consist of a one-ended 0-path P_a^0 and an endless 0-path P_b^0 forming a 1-loop, as shown in Fig. 2. $\mathcal{M}$ is now the set of all (maximal) nodes. P_a^0 starts at the nonmaximal 0-node z^0 embraced by the 1-node x^1 and reaches the 1-node y^1. P_b^0 reaches both x^1 and y^1. Let v^0 be a 0-node of P_a^0 at a distance of k from x^1. (For z^0, $k = 0$.) The shortest distance between v^0 and any node w^0 of P_b^0 is provided by a path that passes through x^1; it has the length $\omega + k$. (The path passing through y^1 and terminating at v^0 and w^0 has length $\omega \cdot 2$.) Thus, $e(v^0) = \omega + k$. On the other hand, $e(w^0) = \omega \vec{\cdot} 2$; indeed, $d(w^0, v^0) = \omega + k$, which increases indefinitely but never achieves $\omega \cdot 2$ as v^0 approaches y^1. Furthermore, $e(x^1) = e(y^1) = \omega$. Thus, rad $= \omega$, diam $= \omega \vec{\cdot} 2$, the center is $\{x^1, y^1\}$, and the periphery is the set of all the 0-nodes of P_b^0.

Here is another graphical realization of Aristotle's potential infinity, this time one represented by the eccentricity $\omega \vec{\cdot} 2$. In contrast to Example 6.1, there is no way of appending another transfinite node in order to convert this potential infinity into an actual infinity $\omega \cdot 2$. Thus, we have here an incontrovertible representation of Aristotle's potential infinity. $\square$

Example 6.5. Consider now the 1-graph of Fig. 3. The eccentricities of the nodes are as follows: $e(x_k) = \omega \cdot 2 + k$ for $k = 1, 2, 3, \ldots$; $e(y^1) = \omega \cdot 2$; $e(z_k)^0 = \omega \vec{\cdot} 2$ for $k = \ldots, -1, 0 - 1, \ldots$; $e(w^1) = \omega \vec{\cdot} 3$. Thus, rad $= \omega \vec{\cdot} 2$, diam $= \omega \vec{\cdot} 3$, the center is $\{z_k : k = \ldots, -1, 0, 1, \ldots\}$, and the periphery is the singleton $\{w^1\}$.

Here, we have two different representations of Aristotle's potential infinity, given by the eccentricities $\omega \vec{\cdot} 2$ and $\omega \vec{\cdot} 3$; these, too, cannot be converted into ordinals by appending transfinite nodes. Note also that the periphery has only one node—in contrast to the peripheries of finite graphs, which must have two or more nodes. $\square$

These examples can immediately be converted into examples for graphs of higher ranks by replacing branches by endless paths, all of the same rank. For instance, if every branch

is replaced by an endless path of rank $\nu - 2$, then every 0-node becomes a $(\nu - 1)$-node, and every 1-node becomes a ν-node. Of course, there are far more complicated ν-graphs.

7. Some General Results

Henceforth, let it be understood that the metrizable set $\mathcal{M}$ of nodes has been chosen and fixed for the ν-graph $\mathcal{G}^\nu$ at hand and that any node we refer to is in $\mathcal{M}$.

For any rank ρ with $0 \leq \rho \leq \nu$, a ρ-section $\mathcal{S}^\rho$ of $\mathcal{G}^\nu$ is defined as the subgraph of $\mathcal{G}^\nu$ induced by a maximal set of branches that are ρ-connected.[4] By virtue of Condition 3.2, the ρ-sections partition $\mathcal{G}^\nu$ (that is, each branch is in one and only one ρ-section) [7, Corollary 3.5-6].

We now define a bordering node of $\mathcal{S}^\rho$ to be a node of rank larger than ρ that is incident to $\mathcal{S}^\rho$. Thus, the bordering node embraces as α-tip ($\alpha \leq \rho$) traversed by $\mathcal{S}^\rho$; in other words, there is a one-ended α-path in $\mathcal{S}^\rho$ whose α-tip is embraced by the bordering node. Also, we define an internal node of $\mathcal{S}^\rho$ to be a (maximal) node of rank ρ or less contained in $\mathcal{S}^\rho$.

The idea of a component is similar to but different from a $(\nu - 1)$-section. A component of a subgraph $\mathcal{H}$ of $\mathcal{G}^\nu$ is a subgraph of $\mathcal{H}$ induced by a maximal set of branches in $\mathcal{H}$ that are ν-connected [7, page 49]. Because $\mathcal{G}^\nu$ is ν-connected, it has just one component, namely, itself. However, a proper subgraph $\mathcal{H}$ of $\mathcal{G}^\nu$ may have many components. For example, if $\mathcal{H}$ consists of two $(\nu - 1)$-sections that do not share any bordering nodes, then each of them is a component of $\mathcal{H}$.

In the next theorem, $\mathcal{S}^\rho$ is any ρ-section whose bordering nodes are incident to $\mathcal{S}^\rho$ only through ρ-tips. In Fig. 1, x_a^1 and x_b^1 are bordering nodes of the 0-section to the left of those nodes, and the condition is satisfied, that is, those 1-nodes are incident to that 0-section only through 0-tips. However, branch d induces a 0-section by itself, and the condition is not satisfied because d reaches x_a^1 through a (-1)-tip (i.e., a tip of branch d); similarly for e and x_b^1. In Fig. 2, P_a^0 and P_b^0 are different 0-sections; P_b^0 satisfies the condition, but P_a^0 does not because of node x^1 and its embraced (-1)-tip.

Theorem 7.1. *Let $\mathcal{S}^\rho$ be a ρ-section in $\mathcal{G}^\nu$ ($0 \leq \rho < \nu$) all of whose bordering nodes are incident to $\mathcal{S}^\rho$ only through ρ-tips. Then, all the internal nodes of $\mathcal{S}^\rho$ have the same*

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[4]This is the same definition of a ρ-section as that given in [7, page 49] but is somewhat more general than that of [8, page 36].

eccentricity.

Proof. By virtue of our hypothesis and the ρ-connectedness of $\mathcal{S}^\rho$, for any internal node x^α ($\alpha \leq \rho$) and any bordering node z^γ ($\gamma > \rho$) of $\mathcal{S}^\rho$ in $\mathcal{M}$, there is a representative ρ-path P^ρ for a ρ-tip embraced by z^γ and lying in $\mathcal{S}^\rho$, and there also is a two-ended path Q lying in $\mathcal{S}^\rho$ and terminating at x^α and a node of P^ρ. So, by Condition 3.2 and [7, Corollary 3.5-4], there is in $P \cup Q$ a one-ended ρ-path R^ρ that terminates at x^α and reaches z^γ through a ρ-tip. Moreover, all paths that terminate at x^α, that lie in $\mathcal{S}^\rho$, and that reach z^γ must be one-ended ρ-paths. Therefore, $d(x^\alpha, z^\gamma) = \omega^{\rho+1}$. For any other node y^β ($\beta \leq \rho$) in $\mathcal{S}^\rho$, we have $d(x^\alpha, y^\beta) < \omega^{\rho+1}$ by the ρ-connectedness of $\mathcal{S}^\rho$. So, if $\mathcal{G}^\nu$ consists only of $\mathcal{S}^\rho$ and its bordering nodes (so that $\nu = \rho + 1$), we can conclude that $e(x^\alpha) = \omega^{\rho+1}$, whatever be the choice of the internal node x^α in $\mathcal{S}^\rho$ and in $\mathcal{M}$.

Next, assume that there is a node v^δ of $\mathcal{G}^\nu$ lying outside of $\mathcal{S}^\rho$ and different from all the bordering nodes of $\mathcal{S}^\rho$. By the ν-connectedness of $\mathcal{G}^\nu$, there is a path P_{xv} terminating at x^α and v^δ. Let z^γ now be the last bordering node of $\mathcal{S}^\rho$ that P_{xv} meets. Let P_{zv} be that part of P_{xv} lying outside of $\mathcal{S}^\rho$. Then, by what we have shown above, there is a one-ended ρ-path Q^ρ_{xz} that terminates at x^α, lies in $\mathcal{S}^\rho$, and reaches z^γ through a ρ-tip. Then, $R_{xv} = Q_{xz} \cup P_{zv}$ is a two-ended path that terminates at x^α and v^δ. Moreover, $|R_{xv}| \leq |P_{xv}|$.

Now, let y^β ($\beta \leq \rho$) be any other internal node of $\mathcal{S}^\rho$ in $\mathcal{M}$ (i.e., different from x^α). Again, there is a one-ended ρ-path Q^ρ_{yz} satisfying the same conditions as Q^ρ_{xz}. We have $d(x^\alpha, z^\gamma) = d(y^\beta, z^\gamma) = \omega^{\rho+1}$. Let $R_{yv} = Q_{yz} \cup P_{zv}$. Thus, $|R_{xv}| = |R_{yv}|$. We have shown that, for each one-ended path R_{xv} terminating at x^α and v^δ and passing through exactly one bordering node of z^γ of $\mathcal{S}^\rho$, there is another path R_{yv} of the same length terminating at y^β and v^δ and identical to R_{xv} outside $\mathcal{S}^\rho$. It follows that $d(x^\alpha, v^\delta) = d(y^\beta, v^\delta)$. We can conclude that $e(x^\alpha) = e(y^\beta)$ whatever be the choices of x^α and y^β in $\mathcal{S}^\rho$ and $\mathcal{M}$. $\square$

Figs. 1, 2, and 3 provide examples for Theorem 7.1. In Fig. 1, all the 0-nodes to the left of the 1-nodes have the same eccentricity $\omega + 1$ in accordance with the theorem. In Fig. 2, all the nodes of P^0_b have the same eccentricity $\omega \overset{\rightarrow}{\cdot} 2$, whereas the eccentricities of the nodes of P^0_a vary; this, too, conforms with Theorem 7.1. Similarly, in Fig. 3, the nodes z^0_k have the same eccentricities, but the nodes x^0_k have differing eccentricities.

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A standard result [5, page 21] can be extended to the transfinite case, albeit in a more complicated way. Given $\mathcal{G}^\nu$ and $\mathcal{M}$, rad may be either an ordinal or an arrow rank. If it is an arrow rank, we let rad$^+$ denote the limit ordinal immediately following rad.

Theorem 7.2.

(i) *If rad is an ordinal, then* rad $\leq$ *diam* $\leq$ *rad·2.*

(ii) *If rad is an arrow rank, then* rad $\leq$ *diam* $\leq$ *rad*$^+$ $\cdot$ 2.

Proof. The proofs of (i) and (ii) are much the same. So, let us consider (ii) alone. That rad $\leq$ diam follows directly from the definitions (9), (10), and (11). Next, by the definition of the diameter (11), we can choose two sequences $\{y_k\}_{k=0}^\infty$ and $\{z_k\}_{k=0}^\infty$ of nodes such that the sequence $\{d(y_k, z_k)\}_{k=0}^\infty$ approaches or achieves diam. Let x be any node in the center. By the triangle inequality,

$$d(y_k, z_k) \ \leq \ d(y_k, x) + d(x, z_k).$$

Now, $d(y_k, x) \leq$ rad $\leq$ rad$^+$, and similarly for $d(x, z_k)$. Therefore,

$$d(y_k, z_k) \leq \ \text{rad}^+ \ + \ \text{rad}^+ \ = \text{rad}^+ \ \cdot 2.$$

□

Another standard result is that the nodes of any finite graph comprise the center of some finite connected graph [5, page 22].[5] This, too, can be extended transfinitely—in fact, in several ways, but the proofs are more complicated than that for finite graphs. Nonetheless, the scheme of the proofs remains the same. First, we need the following lemma. In the following, $\nu - 1$ denotes $\vec{\omega}$ when $\nu = \omega$.

Lemma 7.3. *Let* $\mathcal{S}^{\nu-1}$ *be a* $(\nu - 1)$*-section of* $\mathcal{G}^\nu$*, where* $1 \leq \nu \leq \omega$ *and* $\nu \neq \vec{\omega}$*. Let* u^ν *be a* ν*-node incident to* $\mathcal{S}^{\nu-1}$ *(thus, a bordering node of* $\mathcal{S}^{\nu-1}$*), and let* x^α *($\alpha < \nu$) be an* α*-node in* $\mathcal{S}^{\nu-1}$ *(thus, an internal node of* $\mathcal{S}^{\nu-1}$*). Then, there exists in* $\mathcal{S}^{\nu-1}$ *a two-ended path of length no larger than* ω^ν *connecting* x^α *and* u^ν*.*

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[5]This result extends immediately to infinite 0-graphs with infinitely many 0-nodes. We are now considering transfinite graphs of ranks 1 or greater.

Proof. That u^ν is incident to $\mathcal{S}^{\nu-1}$ means that there is in $\mathcal{S}^{\nu-1}$ a one-ended β-path P^β with $\beta \leq \nu - 1$ whose β-tip is embraced by u^ν. Let $P^{\beta+1}$ be the two-ended path obtained by appending to P^β the $(\beta+1)$-node $y^{\beta+1}$ embraced by u^ν and reached by P^β. ($y^{\beta+1}$ will not be maximal if $\beta + 1 < \nu$; otherwise, $y^{\beta+1} = u^\nu$.) The length $|P^{\beta+1}|$ of $P^{\beta+1}$ is equal to $\omega^{\beta+1}$ because $P^{\beta+1}$ traverses only one β-tip; all other tips traversed by $P^{\beta+1}$ are of lesser rank. Let z^γ be any node (not necessarily maximal) of P^β; thus, $\gamma \leq \beta$. By the $(\nu-1)$-connectedness of $\mathcal{S}^{\nu-1}$, there is in $\mathcal{S}^{\nu-1}$ a two-ended λ-path Q^λ ($0 \leq \lambda \leq \nu - 1$) terminating at x^α and z^γ. The tips traversed by Q^λ have ranks no greater than $\lambda - 1$, hence, no greater than $\nu - 2$. By [7, Corollary 3.5-4], there is a two-ended path R^δ in $P^{\beta+1} \cup Q^\lambda$ terminating at x^α and $y^{\beta+1}$. All the tips traversed by R^δ are of ranks no greater than $\nu - 1$, and there is at most one traversed tip of rank $\nu - 1$. Hence, the length of R^δ satisfies $|R^\delta| \leq \omega^\nu$. $\square$

Given any ν-graph $\mathcal{G}^\nu$ with $1 \leq \nu \leq \omega$ and $\nu \neq \vec{\omega}$, let us construct a larger ν-graph $\mathcal{H}^\nu$ by appending six additional ν-nodes p_i^ν and q_i^ν ($i = 1, 2, 3$) and also appending isolated endless $(\nu-1)$-paths[6] that reach ν-nodes as shown in Fig. 4. Such paths connect p_1^ν to p_2^ν, p_2^ν to p_3^ν, p_3^ν to every ν-node in $\mathcal{G}^\nu$, and similarly for p_i^ν replaced by q_i^ν. Note that the singleton end-nodes p_1^ν and q_1^ν can be included in the chosen metrizable set $\mathcal{M}$ for $\mathcal{H}^\nu$. All the other ν-nodes of $\mathcal{H}^\nu$ are nonsingletons and therefore are in $\mathcal{M}$, too.

Theorem 7.4. *The ν-nodes of $\mathcal{G}^\nu$ ($1 \leq \nu \leq \omega$, $\nu \neq \vec{\omega}$) comprise the center of $\mathcal{H}^\nu$, and the periphery of $\mathcal{H}^\nu$ is $\{p_1^\nu, q_1^\nu\}$.*

Proof. We look for bounds on the eccentricities of all the nodes in $\mathcal{M}$. Let x^α and y^β be any two nodes whose ranks satisfy $0 \leq \alpha, \beta < \nu$. It follows that x^α (resp. y^β) is an internal node of a $(\nu-1)$-section in $\mathcal{G}^\nu$, and that section has at least one ν-node u^ν (resp. v^ν) as a bordering node because $\mathcal{G}^\nu$ is ν-connected. By the triangle inequality,

$$d(x^\alpha, y^\beta) \;\leq\; d(x^\alpha, u^\nu) + d(u^\nu, p_3^\nu) + d(p_3^\nu, v^\nu) + d(v^\nu, y^\beta).$$

By Lemma 7.3, $d(x^\alpha, u^\nu) \leq \omega^\nu$ and $d(v^\nu, y^\beta) \leq \omega^\nu$. Clearly, $d(u^\nu, p_3^\nu) = d(p_3^\nu, v^\nu) = \omega^\nu \cdot 2$. Thus, $d(x^\alpha, y^\beta) \leq \omega^\nu \cdot 6$. This also shows that, for any ν-node v^ν in $\mathcal{G}^\nu$, $d(x^\alpha, v^\nu) \leq \omega^\nu \cdot 5$. Since $d(u^\nu, p_1^\nu) = \omega^\nu \cdot 6$, we have $d(x^\alpha, p_1^\nu) = d(x^\alpha, u^\nu) + d(u^\nu, p_1^\nu) \leq \omega^\nu + \omega^\nu \cdot 6 = \omega^\nu \cdot 7$.

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[6]An isolated endless path embraces no tips other than the ones it traverses. Thus, to reach any other part of a graph in which the isolated path is a subgraph, one must proceed through a terminal tip of that path.

Now, $d(x^\alpha, u^\nu) \geq 1$ because there is at least one branch in any path connecting x^α and u^ν. Thus, we also have $d(x^\alpha, p_1^\nu) \geq \omega^\nu \cdot 6 + 1$. Note also that the distance from x^α to any node of the appended endless paths is strictly less than $\omega^\nu \cdot 7$. All these results hold for p_i^ν replaced by q_i^ν. Altogether then, we can conclude the following: For any node in $\mathcal{G}^\nu$ of rank less than ν, say, x^α, the eccentricity $e(x^\alpha)$ of x^α is bounded as follows:

$$\omega^\nu \cdot 6 + 1 \ \leq \ e(x^\alpha) \ \leq \ \omega^\nu \cdot 7$$

Next, consider any two ν-nodes of $\mathcal{G}^\nu$, say, u^ν and v^ν again. By what we have already shown, $d(u^\nu, v^\nu) \leq \omega^\nu \cdot 4$, and $d(u^\nu, p_1^\nu) = d(u^\nu, q_1^\nu) = \omega^\nu \cdot 6$. The distance from u^ν to any node of the appended endless $(\nu - 1)$-paths is less than $\omega^\nu \cdot 6$. Also, for any node y^β in $\mathcal{G}^\nu$ of rank less than ν, $d(u^\nu, y^\beta) \leq \omega^\nu \cdot 5$. So, the largest distance between u^ν and any other node in $\mathcal{H}^\nu$ is equal to $\omega^\nu \cdot 6$; that is, $e(u^\nu) = \omega^\nu \cdot 6$.

Finally, we have $e(p_3^\nu) = e(q_3^\nu) = \omega^\nu \cdot 8$, $e(p_2^\nu) = e(q_2^\nu) = \omega^\nu \cdot 10$, and $e(p_1^\nu) = e(q_1^\nu) = \omega^\nu \cdot 12$. The eccentricities of the nodes of the appended endless paths lie between these values.

We have considered all cases. Comparing these equalities and inequalities for all the eccentricities, we can draw the conclusion of the theorem. □

As an immediate corollary, we have the following generalization of a result for finite graphs.

Corollary 7.5. *The ν-nodes of $\mathcal{G}^\nu$ ($1 \leq \nu \leq \omega$, $\nu \neq \vec{\omega}$) comprise the center of some ν-graph $\mathcal{H}^\nu$.*

Variations of Corollary 7.5 can also be established through much the same proofs. For instance, all the nodes of $\mathcal{G}^\nu$ of one or more specified ranks can be made to comprise the center of some ν-graph. This is because the $(\nu - 1)$-sections of $\mathcal{G}^\nu$ partition $\mathcal{G}^\nu$. Still more generally, if $\mathcal{G}^\nu$ has only finitely many ν-nodes, any arbitrary set of nodes of $\mathcal{G}^\nu$ in $\mathcal{M}$ can be made the center simply by appending enough endless $(\nu - 1)$-paths in series.

8. When All the Nodes of Highest Rank Are Pristine

The nodes of highest rank in $\mathcal{G}^\nu$ are the ν-nodes, of course. A ν-node is said to be pristine if it does not embrace a node of lower rank. Thus, a pristine ν-node consists only of $(\nu - 1)$-tips. Henceforth, we assume the following.

Condition 8.1. *All the ν-nodes are pristine.*

Because of this, we can view the ν-nodes as lying only at infinite extremities of the $(\nu-1)$-sections to which they are incident because they can be reached only through $(\nu-1)$-tips of such sections; they are, in fact, the bordering nodes of such $(\nu-1)$-sections. All the other nodes are of ranks less than ν and are internal nodes of $(\nu-1)$-sections. Given any $(\nu-1)$-section $\mathcal{S}^{\nu-1}$, the set of all internal nodes of $\mathcal{S}^{\nu-1}$ will be denoted by $i(\mathcal{S}^{\nu-1})$ and will be called the *interior* of $\mathcal{S}^{\nu-1}$.

A boundary ν-node of $\mathcal{S}^{\nu-1}$ is defined to be a bordering ν-node that contains $(\nu-1)$-tips of $\mathcal{S}^{\nu-1}$ and also $(\nu-1)$-tips of one or more other $(\nu-1)$-sections of $\mathcal{G}^\nu$. Thus, a boundary node lies at the infinite extremities of two or more $(\nu-1)$-sections and thereby connects them. A ν-path can pass from the interior of one $(\nu-1)$-section into the interior of another $(\nu-1)$-section only by passing through a boundary node.

Another assumption we henceforth impose is the following.

Condition 8.2. *There are only finitely many boundary ν-nodes throughout $\mathcal{G}^\nu$.*

Nevertheless, each $(\nu-1)$-section may have infinitely many incident non-boundary bordering ν-nodes, and each boundary ν-node may be incident to infinitely many $(\nu-1)$-sections.

Lemma 8.3. *If P is a two-ended ν-path, then $|P| = \omega^\nu \cdot k$, where k is the number of $(\nu-1)$-tips traversed by P.*

Proof. This follows directly from the definition of the length $|P|$ and the fact that all ν-nodes are pristine. □

Because all ν-nodes are now pristine, we can strengthen Lemma 7.3 as follows.

Lemma 8.4.

(a) *Let x^ν be a bordering node of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$, and let z be an internal node of $\mathcal{S}^{\nu-1}$. Then, there exists a two-ended ν-path $P^\nu_{z,x}$ within $\mathcal{S}^{\nu-1}$ that terminates at z and reaches x^ν through its one and only $(\nu-1)$-tip. Moreover, the length $|P^\nu_{z,x}|$ of $P^\nu_{z,x}$ is ω^ν. $P^\nu_{z,x}$ is a z-to-x geodesic. Finally, all two-ended ν-paths within $\mathcal{S}^{\nu-1}$ terminating at an internal node of $\mathcal{S}^{\nu-1}$ and at a bordering node of $\mathcal{S}^{\nu-1}$ have the length ω^ν.*

(b) *Let x^ν and y^ν be two bordering nodes of $\mathcal{S}^{\nu-1}$. Then, there exists an two-ended ν-path*

$P_{x,y}^\nu$ within $\mathcal{S}^{\nu-1}$ that reaches x^ν and y^ν through its two $(\nu-1)$-tips. Moreover, the length $|P_{x,y}^\nu|$ of $P_{x,y}^\nu$ is $\omega^\nu \cdot 2$. $P_{x,y}^\nu$ is an x^ν-to-y^ν geodesic. Finally, all two-ended ν-paths in $\mathcal{S}^{\nu-1}$ reaching two bordering nodes of $\mathcal{S}^{\nu-1}$ have the length $\omega^\nu \cdot 2$.

Proof. The proof of part (a) is much the same as that of Lemma 7.3 except that now $P_{z,x}^\nu$ is incident to x^ν only through its one and only $(\nu-1)$-tip.

Part (b) is proven similarly, but now we use two representative one-ended paths, one for each of x^ν and y^ν. $\square$

Lemma 8.5. *Let x^ν be a bordering node of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$, let z be an internal node of $\mathcal{S}^{\nu-1}$, and let y be any node of $\mathcal{G}^\nu$. Then, $|d(x^\nu, y) - d(z, y)| \leq \omega^\nu$.*

Proof. By Lemma 8.4(a), $d(z, x^\nu) = \omega^\nu$. Since, d is a metric,

$$d(x^\nu, y) \;\leq\; d(x^\nu, z) + d(z, y) \;=\; \omega^\nu + d(z, y).$$

Also,

$$d(z, y) \;\leq\; d(z, x^\nu) + d(x^\nu, y) \;=\; \omega^\nu + d(x^\nu, y).$$

These inequalities yield the conclusion. $\square$

We will show below (Theorem 8.7) that, as a consequence of Conditions 8.1 and 8.2, no node of $\mathcal{G}^\nu$ can have an arrow-rank eccentricity and that all eccentricities comprise a finite set of ordinal values. But, first note that three examples of the occurrence of arrow-rank eccentricities are given in Examples 6.1, 6.4, and 6.5. Each of these examples violates either Condition 8.1 or Condition 8.2.

Theorem 8.6. *The eccentricities of all the nodes are contained within the following finite set of ordinals:*

$$\{\omega^\nu \cdot p \colon 1 \leq p \leq 2m + 2\} \tag{12}$$

Here, p and m are natural numbers, and m is the number of boundary ν-nodes.

Proof. The eccentricity of any node is at least as large as the distance between any internal node of a $(\nu-1)$-section and any bordering ν-node of that $(\nu-1)$-section. Therefore, Lemma 8.4(a) implies that the eccentricity of any node of $\mathcal{G}^\nu$ is at least ω^ν, whence the lower bound in (12). The proof of the upper bound requires more effort.

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First of all, we can settle two simple cases by inspection. If $\mathcal{G}^\nu$ consists of a single $(\nu - 1)$-section with exactly one bordering ν-node (in $\mathcal{M}$, of course), then all the nodes of $\mathcal{G}^\nu$ have the eccentricity ω^ν. If that one and only $(\nu - 1)$-section for $\mathcal{G}^\nu$ has two or more (possibly infinitely many) bordering nodes, the internal nodes have eccentricity ω^ν, and the bordering nodes have eccentricity $\omega^\nu \cdot 2$. In both cases, the conclusion of the theorem is fulfilled with $m = 0$.

We now turn to the general case where $\mathcal{G}^\nu$ has at least one boundary ν-node and therefore at least two $(\nu - 1)$-sections. $\mathcal{G}^\nu$ will have a two-ended ν-path of the following form:

$$P_{0,k}^\nu \;=\; \{x_0, P_0^{\nu-1}, x_1^\nu, P_1^{\nu-1}, \ldots, x_{k-1}^\nu, P_{k-1}^{\nu-1}, x_k\} \tag{13}$$

Because all ν-nodes are pristine, the x_i^ν ($i = 1, \ldots, k-1$) are nonsingleton bordering ν-nodes (possibly boundary ν-nodes), and the $P_i^{\nu-1}$ ($i = 2, \ldots, k-2$) are endless $(\nu - 1)$-paths. The same is true of x_0, x_k, $P_1^{\nu-1}$, and $P_{k-1}^{\nu-1}$ if x_0 and x_k are ν-nodes, too. If x_0 (resp. x_k) is of lower rank, then it is an internal node, and $P_1^{\nu-1}$ (resp. $P_{k-1}^{\nu-1}$) is a one-ended $(\nu - 1)$-path.

Let us first assume that x_0 and x_k are internal nodes in different $(\nu - 1)$-sections. Let $\mathcal{S}_0^{\nu-1}$ be the $(\nu - 1)$-section containing x_0. Let $x_{i_1}^\nu$ be the last ν-node in (13) that is incident to $\mathcal{S}_0^{\nu-1}$. $x_{i_1}^\nu$ will be a boundary ν-node because it is also incident to another $(\nu - 1)$-section, say, $\mathcal{S}_1^{\nu-1}$. If need be, we can replace the subpath of (13) between x_0 and x_{i_1} by a ν-path $\{x_0, Q_0^{\nu-1}, x_{i_1}^\nu\}$, where $Q_0^{\nu-1}$ is a one-ended $(\nu - 1)$-path and resides in $\mathcal{S}_0^{\nu-1}$, to get a shorter overall ν-path terminating at x_0 and x_k.

Now, let $\mathcal{S}_1^{\nu-1}$ be the next $(\nu - 1)$-section after $\mathcal{S}_0^{\nu-1}$ through which our (possibly) reduced path proceeds. Also, let $x_{i_2}^\nu$ be the last ν-node in that path that is incident to $\mathcal{S}_1^{\nu-1}$. $x_{i_2}^\nu$ will be a boundary ν-node incident to $\mathcal{S}_1^{\nu-1}$ and another $(\nu - 1)$-section $\mathcal{S}_2^{\nu-1}$. If need be, we can replace the subpath between x_{i_1} and x_{i_2} by a ν-path $\{x_{i_1}^\nu, Q_1^{\nu-1}, x_{i_2}^\nu\}$, where $Q_1^{\nu-1}$ is an endless $(\nu - 1)$-path residing in $\mathcal{S}_1^{\nu-1}$. This will yield a still shorter overall ν-path terminating at x_0 and x_k.

Continuing this way, we will find a boundary ν-node $x_{i_j}^\nu$ that is incident to the $(\nu - 1)$-section $\mathcal{S}_j^{\nu-1}$ containing x_k. Finally, we let $Q_j^{\nu-1}$ be a one-ended $(\nu - 1)$-path in $\mathcal{S}_j^{\nu-1}$ terminating at $x_{i_j}^\nu$ and x_k. Altogether, we will have the following two-ended ν-path, which

is not longer than $P_{0,k}^{\nu}$ (actually shorter if the aforementioned replacements were needed).

$$Q_{0,k}^{\nu} = \{x_0, Q_0^{\nu-1}, x_{i_1}^{\nu}, Q_1^{\nu-1}, x_{i_2}^{\nu}, \ldots, x_{i_j}^{\nu}, Q_j^{\nu-1}, x_k\} \qquad (14)$$

Because all the ν-nodes herein are boundary nodes and pristine, the length $|Q_{0,k}|$ is obtained simply by counting the $(\nu-1)$-tips traversed by $Q_{0,k}^{\nu}$ and multiplying by ω^{ν} (see Lemma 8.4). We get $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j)$, where $j \leq m$. Finally, we note that any geodesic path between x_0 and x_k has a length no larger than than $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j)$.

Next, consider the case where x_0 is a bordering ν-node of $\mathcal{S}_0^{\nu-1}$ and x_k remains an internal node of $\mathcal{S}_j^{\nu-1}$. $P_0^{\nu-1}$ in (13) will be an endless $(\nu-1)$-path residing in some $(\nu-1)$-section $\mathcal{S}_0^{\nu-1}$. We let $x_{i_1}^{\nu}$ be the last ν-node in (13) incident to $\mathcal{S}_0^{\nu-1}$. Otherwise our procedure is as before, and we can now conclude that $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j+1)$ because the passage from x_0 into $\mathcal{S}_0^{\nu-1}$ traverses one $(\nu-1)$-tip.

The same conclusion, namely, $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j+1)$ holds if x_k is a bordering ν-node and x_0 is an internal node. Finally, if both x_0 and x_k are bordering ν-nodes, we get $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j+2)$. For all cases, we can assert that the geodesic between x_0 and x_k has a length no larger than $|Q_{0,k}^{\nu}| = \omega^{\nu} \cdot (2j+2)$.

Now, the eccentricity $e(x_0)$ for x_0 is the supremum of the lengths of all geodesics starting at x_0 and terminating at all other nodes x_k. Since there are only finitely many boundary nodes and since every geodesic will have the form of (14), every eccentricity will be a multiple of ω^{ν} (there are no arrow-rank eccentricities). Also, since $j \leq m$ where m is the number of boundary ν-nodes in $\mathcal{G}^{\nu}$, we can conclude that $e(x_0) \leq \omega^{\nu} \cdot (2m+2)$, whatever be the node x_0. $\square$

The lengths of all geodesics will reside in the finite set of values (12). Consequently, for every node x_0 of $\mathcal{G}^{\nu}$ there will be at least one geodesic of maximum length starting at x_0 and terminating at some other node z of $\mathcal{G}^{\nu}$. Such a geodesic is called an *eccentric path* for x_0, and z is called an *eccentric node* for x_0. In general, there are many eccentric paths and eccentric nodes for a given x_0.

Corollary 8.7. *Let x^{ν} be any bordering ν-node of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ with the eccentricity $e(x^{\nu}) = \omega^{\nu} \cdot k$, and let z be an internal node of $\mathcal{S}^{\nu-1}$ with the eccentricity $e(z) = \omega^{\nu} \cdot p$. Then, $|k - p| \leq 1$.*

Proof. Let $P_{z,x}$ be the two-ended ν-path obtained by appending x^ν to a one-ended $(\nu - 1)$-path in $\mathcal{S}^{\nu-1}$ that reaches x^ν and terminates at the internal node z. By Lemma 8.4(a), $P_{z,x}^\nu$ is a z-to-x^ν geodesic, and $|P_{z,x}^\nu| = d(z, x^\nu) = \omega^\nu$. Now, let w be any node. By the triangle inequality for the metric d,

$$d(z, w) \leq d(z, x^\nu) + d(x^\nu, w) = \omega^\nu + d(x^\nu, w).$$

Next, let w be an eccentric node for z. We get $d(z, w) = e(z)$ and $e(z) \leq \omega^\nu + d(x^\nu, w)$. Moreover, $d(x^\nu, w) \leq e(x^\nu)$. Therefore,

$$e(z) \leq \omega^\nu + e(x^\nu). \tag{15}$$

By a similar argument with w now being an eccentric node for x^ν, we get

$$e(x^\nu) \leq \omega^\nu + e(z). \tag{16}$$

So, with (15) we have $\omega^\nu \cdot p \leq \omega^\nu + \omega^\nu \cdot k = \omega^\nu \cdot (k + 1)$, or $p \leq k + 1$. On the other hand, with (16) we have in the same way $k \leq p + 1$. Whence our conclusion. $\square$

That $k - p$ can equal 0 is verified by the next example.

Example 8.8. Consider the 1-graph of Fig. 5 consisting of a one-ended 0-path of 0-nodes w_k^0 and an endless 0-path of 0-nodes y_k^0 connected in series to two 1-nodes x^1 and z^1 as shown. The eccentricities are as follows: $e(w_k^0) = \omega \cdot 3$ for $k = 1, 2, 3, \dots$, $e(x^1) = \omega \cdot 2$, $e(y_k^0) = \omega \cdot 2$ for $k = \dots, -1, 0, 1, \dots$, and $e(z^1) = \omega \cdot 3$. Thus, $e(x^1) - e(y_k^0) = 0$, as asserted. $\square$

An immediate consequence of Theorem 8.6 and Corollary 8.7 is the following.

Corollary 8.9. *The eccentricities of all the nodes form a consecutive set of values in (12), with the minimum (resp. maximum) eccentricity being the radius (resp. diameter) of $\mathcal{G}^\nu$.*

By virtue of Theorem 7.1 and Condition 8.1, we have that, if an ordinal is the eccentricity of an internal node of a $(\nu - 1)$-section $\mathcal{S}^{\nu-1}$, then there will be infinitely many nodes with the same eccentricity, for example, all the internal nodes of $\mathcal{S}^{\nu-1}$. Moreover, a boundary node of $\mathcal{S}^{\nu-1}$ may also have that same eccentricity; the 1-node x^1 in Example 8.8 illustrates this. Furthermore, it is possible for the radius of $\mathcal{G}^\nu$ to be the eccentricity of only on node

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in $\mathcal{G}^\nu$; this occurs when the center consists of only one node. Can another eccentricity occur for only one node? No, by virtue of Conditions 8.1 and 8.2. There must be at least two nodes for each eccentricity larger than the radius. The proof of this is virtually the same as a proof of Lesniak for finite graphs [5, page 176].

We will need two more results. They hold except for the trivial case where $\mathcal{G}^\nu$ has only one $(\nu-1)$-section and only one ν-node.

Lemma 8.10. *Except for the trivial case just noted, a non-boundary bordering ν-node x^ν of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ has an eccentricity that is exactly ω^ν larger than the eccentricity of the internal nodes of $\mathcal{S}^{\nu-1}$.*

Proof. This follows from Theorem 7.1 and the fact that any eccentric path starting at x^ν and entering $\mathcal{S}^{\nu-1}$ must pass through exactly one $(\nu-1)$-tip. $\square$

An *end $(\nu-1)$-section* is a $(\nu-1)$-section having exactly one boundary ν-node.

Lemma 8.11. *Except for the trivial case, the eccentricity of the internal nodes of an end $(\nu-1)$-section is exactly ω^ν larger than the eccentricity of its boundary ν-node.*

Proof. Any eccentric path of any node of $i(\mathcal{S}^{\nu-1})$ must pass through that boundary ν-node. So, the argument of the preceding proof works again. $\square$

9. The Center Lies in a ν-Block

This is a known result for finite graphs [3, Theorem 2.2], [5, Theorem 2.9], which we now extend transfinitely. As before, the *center* of $\mathcal{G}^\nu$ is the set of nodes having the minimum eccentricity. To define a "ν-block," we first define the *removal* of a pristine nonsingleton ν-node x^ν to be the following procedure: x^ν is replaced by two or more singleton ν-nodes, each containing exactly one of the $(\nu-1)$-tips of x^ν and with every $(\nu-1)$-tip of x^ν being so assigned. We denote the resulting ν-graph by $\mathcal{G}^\nu - x^\nu$. Then, a subgraph $\mathcal{H}$ of $\mathcal{G}^\nu$ will be called a *ν-block* of $\mathcal{G}^\nu$ if $\mathcal{H}$ is a maximal ν-connected subgraph such that, for every x^ν, all the branches of $\mathcal{H}$ lie in the same component of $\mathcal{G}^\nu - x^\nu$. A more explicit way of defining a ν-block is as follows: For any ν-node x^ν, $\mathcal{G}^\nu - x^\nu$ consists of one or more components. Choose one of those components. Repeat this for every ν-node, choosing one component for each ν-node. Then, take the intersection[7] of all those chosen components. That intersection

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[7]This is the subgraph induced by those branches, each of which lie in all the chosen components.

may be empty, but, if it is not empty, it will be a ν-block of $\mathcal{G}^\nu$. Upon taking all possible intersections of components, one component from each $\mathcal{G}^\nu - x^\nu$, and then choosing the nonempty intersections, we will obtain all the ν-blocks of $\mathcal{G}^\nu$.

Furthermore, we define a *cut-node* as a nonsingleton ν-node x^ν such that $\mathcal{G}^\nu - x^\nu$ has two or more components. It follows that the cut ν-nodes *separate* the ν-blocks in the sense that any path that terminates at two branches in different ν-blocks must pass through at least one cut ν-node. (Otherwise, the two branches would be in the same component of $\mathcal{G}^\nu - x^\nu$ for every x^ν and therefore in the same ν-block.) In summary, we have the following:

Lemma 9.1. *The ν-blocks of $\mathcal{G}^\nu$ partition $\mathcal{G}^\nu$, and the cut ν-nodes separate the ν-blocks.*

Proof. For each x^ν, each branch will be in at least one of the components of $\mathcal{G}^\nu - x^\nu$, and therefore in at least one of the ν-blocks. On the other hand, no branch can be in two different ν-blocks because then there would be a cut ν-node that separates a branch from itself—an absurdity. $\square$

Lemma 9.2. *Each $(\nu - 1)$-section $\mathcal{S}^{\nu-1}$ is contained in a ν-block.*

Proof. Since every ν-node is pristine, any two branches of $\mathcal{S}^{\nu-1}$ are connected through a two-ended path of rank no greater than $\nu - 1$, and that path will not meet any ν-node. Thus, $\mathcal{S}^{\nu-1}$ will lie entirely within a single component of $\mathcal{G}^\nu - x^\nu$, whatever be the choice of x^ν. By the definition of a ν-block, we have the conclusion. $\square$

By definition, all the bordering nodes of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$ will be ν-nodes. Moreover, every $(\nu - 1)$-section $\mathcal{S}^{\nu-1}$ will have at least one bordering node, and all the bordering nodes of $\mathcal{S}^{\nu-1}$ will be nodes of $\mathcal{S}^{\nu-1}$. Thus, by Lemma 9.2, every ν-block $\mathcal{H}$ will contain the bordering ν-nodes of its $(\nu - 1)$-sections, and therefore the rank of $\mathcal{H}$ is ν. So, henceforth we denote $\mathcal{H}$ by $\mathcal{H}^\nu$. In general, a ν-node can belong to more than one $(\nu - 1)$-section and also to more than one ν-block.

Example 9.3. Fig. 6 shows a 1-graph in which the P_k^0 $(k = 1, 2, 3, 4, 5)$ are endless 0-paths and v^1, w^1, x^1, y^1, and z^1 are 1-nodes. There are two 1-blocks: One of them consists of P_1^0 along with v^1 and w^1, and the other consists of the P_k^0 $(k = 2, 3, 4, 5)$ along with w^1, x^1, y^1, and z^1. The only cut 1-node is w^1. Also, there are five 0-sections, each consisting of one endless 0-path along with its two bordering 1-nodes. $\square$

Example 9.4. The condition that the ν-nodes are pristine is needed for Lemma 9.2 to hold. For example, Fig. 7 shows a 1-graph with a nonpristine 1-node x^1, three 1-blocks,[8] and two 0-sections. Branch b_1 induces one 1-block, branch b_2 indices another 1-block, and all the branches of the one-ended 0-path P^0 induce the third 1-block. However, b_1 and b_2 together induce a single 0-section $\mathcal{S}^0$, and the branches of P^0 induce another 0-section. $\mathcal{S}^0$ lies in the union of two 1-blocks. □

We are finally ready to verify the title of this section concerning the *center* of $\mathcal{G}^\nu$, which by definition is the set of nodes having the minimum eccentricity. According to Theorem 8.6, such nodes exist. Having set up appropriate definitions and preliminary results for the transfinite case, we can now use a proof that is much the same as that for finite graphs [3, Theorem 2.2], [5, Theorem 2.9].

Theorem 9.5. *The center of $\mathcal{G}^\nu$ lies in a ν-block.*

Proof. Suppose the center of $\mathcal{G}^\nu$ lies in two or more ν-blocks. Let $\mathcal{H}_1^\nu$ and $\mathcal{H}_2^\nu$ be two of them. By Lemma 9.1, there is a cut ν-node x^ν separating them. Let u be an eccentric node for x^ν, and let $P_{x,u}$ be an x^ν-to-u geodesic. Thus, $|P_{x,u}| = e(x^\nu)$. $P_{x,u}$ cannot contain any node different from x^ν in at least one of $\mathcal{H}_1^\nu$ and $\mathcal{H}_2^\nu$, say, $\mathcal{H}_1^\nu$. Let w be a center node in $\mathcal{H}_1^\nu$ other than x^ν, and let $P_{w,x}$ be a w-to-x^ν geodesic. Then, $P_{w,x} \cup P_{x,u}$ is a path whose length satisfies $|P_{w,x} \cup P_{x,u}| = |P_{w,x}| + |P_{x,u}| \geq 1 + e(x)$. This shows that the eccentricity of w is greater than the minimum eccentricity, that is, w is not a center node—a contradiction that proves the theorem. □.

10. The Centers of Cycle-Free ν-Graphs

We now specialize our study to a certain kind of ν-graph that encompasses the class of transfinite trees as a special case. (A *transfinite ν-tree* is a ν-connected ν-graph having no loops.) The kind of ν-graph we now deal with is one having no ν-loop that passes through more than one $(\nu - 1)$-section. All other loops are allowed. Let us be more specific.

Because all the ν-nodes are pristine, every loop of rank less than ν must lie within a single $(\nu-1)$-section $\mathcal{S}^{\nu-1}$, that is, all its nodes are internal nodes of $\mathcal{S}^{\nu-1}$. Such loops are allowed. Moreover, a ν-loop might also lie in a single $(\nu - 1)$-section $\mathcal{S}^{\nu-1}$ in the sense that all its

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[8]Here, we are extending the definition of a 1-block by requiring that the elementary tips of x^1 also be placed in singleton nodes.

nodes of ranks less than ν are internal nodes of $\mathcal{S}^{\nu-1}$ and all its ν-nodes are bordering nodes of $\mathcal{S}^{\nu-1}$; thus, all its branches lie in $\mathcal{S}^{\nu-1}$. Such a loop will pass through a closed sequence $\{x_1^\nu, x_2^\nu, \ldots, x_{k-1}^\nu, x_1^\nu\}$ of bordering ν-nodes of $\mathcal{S}^{\nu-1}$ alternating with endless $(\nu-1)$-paths within $\mathcal{S}^{\nu-1}$. The possibility of such endless $(\nu-1)$-paths within $\mathcal{S}^{\nu-1}$ is implied by Lemma 8.4(b). Such ν-loops within $\mathcal{S}^{\nu-1}$ are also allowed. On the other hand, it is possible in general for a ν-loop to pass through two or more $(\nu-1)$-sections. For the sake of a succinct terminology, we shall call the latter kind of ν-loop a *cycle*. We will henceforth assume that $\mathcal{G}^\nu$ is so structured that it does not have any cycle and will say that $\mathcal{G}^\nu$ is *cycle-free*.

In conformity with our definition of an *end $(\nu-1)$-section* as a $(\nu-1)$-section having exactly one boundary ν-node, we now define a *non-end $(\nu-1)$-section* as a $(\nu-1)$-section having two or more boundary ν-nodes. Note that, when $\mathcal{G}^\nu$ is cycle-free, two $(\nu-1)$-sections cannot share more than one boundary ν-node, for otherwise $\mathcal{G}^\nu$ would contain a cycle. However, still more is implied by the cycle-free condition.

Lemma 10.1. *Assume that $\mathcal{G}^\nu$ is cycle-free. Then, $\mathcal{G}^\nu$ has only finitely many non-end $(\nu-1)$-sections.*

Proof. If there are no non-end $(\nu-1)$-sections, the conclusion is trivially satisfied. So, assume otherwise, and choose any non-end $(\nu-1)$-section. Label it and all its boundary ν-nodes by "1." Label by "2" all the non-end $(\nu-1)$-sections that share boundary ν-nodes with that 1-labeled $(\nu-1)$-section (if such exist), and label their unlabeled boundary ν-nodes by "2," as well. Each 2-labeled section shares exactly one boundary ν-node with the 1-labeled $(\nu-1)$-section and does not share any 2-labeled ν-node with any other 2-labeled $(\nu-1)$-section, for otherwise $\mathcal{G}^\nu$ would contain a cycle. It follows that the number of labeled ν-nodes is no less than the number of labeled $(\nu-1)$-sections. Next, label by "3" all the non-end $(\nu-1)$-sections that share boundary ν-nodes with the 2-labeled sections (if such exist), and label their unlabeled boundary ν-nodes by "3," as well. Again, each 3-labeled $(\nu-1)$-section shares exactly one boundary ν-node with exactly one 2-labeled $(\nu-1)$-section and does not share any 3-labeled ν-node with any other 3-labeled $(\nu-1)$-section, for otherwise $\mathcal{G}^\nu$ would contain a cycle. Here, too, it follows that the number of labeled ν-nodes is no less than the number of labeled $(\nu-1)$-sections. Continue this way. At each step, the number

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of labeled ν-nodes will be no less than the number of labeled $(\nu - 1)$-sections. Since there are only finitely many boundary ν-nodes (Condition 8.2) and since these are the ν-nodes that have been labeled, our conclusion follows. $\square$

Our next objective is to replace our cycle-free ν-graph $\mathcal{G}^\nu$ ($\nu \geq 1$) by a conventional finite tree (i.e., a 0-connected 0-graph having no loop and only finitely many branches), a correspondence that will be exploited in the proof of our final theorem. The nodal eccentricities for $\mathcal{T}^0$ will be related to the nodal eccentricities for $\mathcal{G}^\nu$ in a simple way (see Lemma 10.2 below).

In the following, the index k will number the boundary ν-nodes of $\mathcal{G}^\nu$ (one number for each) as well as sets of non-boundary bordering ν-nodes. The index m will number the interiors of non-end $(\nu - 1)$-sections (one number for each interior) as well as sets of the interiors of end $(\nu - 1)$-sections.

Consider, first of all, a non-end $(\nu - 1)$-section $\mathcal{S}^{\nu-1}$. Each of its boundary ν-nodes x_k^ν is replaced by a 0-node x_k^0 having the same index number k. Also, all the nonboundary bordering ν-nodes of $\mathcal{S}^{\nu-1}$ (if such exist) are replaced by a single 0-node $x_{k'}^0$. Finally, the interior $i(\mathcal{S}^{\nu-1})$ is replaced by a single 0-node y_m^0. A branch is inserted between y_m^0 and each of the x_k^0 and between y_m^0 and $x_{k'}^0$ as well. Thus, $\mathcal{S}^{\nu-1}$ is replaced by a star 0-graph. We view x_k^0 (resp. $x_{k'}^0$, resp. y_m^0) as representing x_k^ν (resp. every nonboundary bordering node $x_{k'}^\nu$ of $\mathcal{S}^{\nu-1}$, resp. every internal node of $\mathcal{S}^{\nu-1}$). For any $y^\gamma \in i(\mathcal{S}^{\nu-1})$, we have $d(y^\gamma, x_k^\nu) = d(y^\gamma, x_{k'}^\nu) = \omega^\nu$ (Lemma 8.4(a)) and $d(y_m^0, x_k^0) = d(y_m^0, x_{k'}^0) = 1$.

Next, consider all the end $(\nu-1)$-sections that are incident to a single boundary node $x_{k''}^\nu$. We represent all of their interiors by a single 0-node $y_{m'}^0$, and we insert a branch between $y_{m'}^0$ and $x_{k''}^0$. If at least one of those end $(\nu - 1)$-sections has a non-boundary bordering ν-node $x_{k'''}^\nu$, we represent all of them by another single 0-node $x_{k'''}^0$, and we insert another branch between $y_{m'}^0$ and $x_{k'''}^0$. So, all of these end $(\nu - 1)$-sections incident to the chosen $x_{k''}^0$ are represented either by a single branch incident at $x_{k''}^0$ and $y_{m'}^0$ or by two branches in series incident at $x_{k''}^0$, $y_{m'}^0$, and $x_{k'''}^0$. Here, too, we have replaced the said set of end $(\nu-1)$-sections by either a one-branch or two-branch (elementary) star 0-graph. Again, we view $x_{k'''}^0$ (resp. $y_{m'}^0$) as representing every nonboundary bordering node $x_{k'''}^\nu$ (resp. every internal

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node) of the said end $(\nu - 1)$-sections. For any internal node y^γ in any one of those end $(\nu - 1)$-sections, we have $d(y^\gamma, x_{k''}^\nu) = d(y^\gamma, x_{k'''}^\nu) = \omega^\nu$ and $d(y_{m'}^0, x_{k''}^0) = d(y_{m'}^0, x_{k'''}^0) = 1$.

We now connect all these star 0-graphs together at their end 0-nodes in the same way that the $(\nu - 1)$-sections are connected together at their boundary ν-nodes. The result is a finite 0-tree $\mathcal{T}^0$. Indeed, since $\mathcal{G}^\nu$ is cycle-free, $\mathcal{T}^0$ has no loops. Also, by Condition 8.2 and Lemma 10.1, $\mathcal{T}^0$ has only finitely many branches.

Lemma 10.2. *A node z of any rank in $\mathcal{G}^\nu$ has an eccentricity $e(z) = \omega^\nu \cdot p$ if and only if its representative 0-node z^0 in $\mathcal{T}^0$ has the eccentricity $e(z^0) = p$.*

(Here again, p is a natural number.)

Proof. An eccentric path P^ν of any node z of any rank in $\mathcal{G}^\nu$ passes alternately through $(\nu - 1)$-sections and bordering ν-nodes and terminates at z and an eccentric node for z. Because all ν-nodes are pristine, the length $|P^\nu|$ is obtained by counting the $(\nu - 1)$-tips traversed by P^ν and multiplying by ω^ν (Lemma 8.3). Furthermore, corresponding to P^ν there is a unique path Q^0 in $\mathcal{T}^0$ whose nodes x_k^0 and y_m^0 alternate in Q^0 and represent the bordering nodes x_k^ν and interiors of $(\nu - 1)$-sections $\mathcal{S}_m^{\nu-1}$ traversed by P^ν. Each branch of Q^0 corresponds to one traversal of a $(\nu - 1)$-tip in P^ν, and conversely. Thus, we have $|P^\nu| = \omega^\nu \cdot p$ and $|Q^0| = p$, where p is the number of branches in Q^0. Also, since P^ν is an eccentric path in $\mathcal{G}^\nu$, Q^0 is an eccentric path in Q^0. Whence our conclusion. $\square$

Here is our principal result concerning the centers of cycle-free ν-graphs.

Theorem 10.3. *The center of any cycle-free ν-graph $\mathcal{G}^\nu$ has one of the following forms:*

(a) *A single ν-node x^ν.*

(b) *The interior $i(\mathcal{S}^{\nu-1})$ of a single $(\nu - 1)$-section $\mathcal{S}^{\nu-1}$.*

(c) *The set $i(\mathcal{S}^{\nu-1}) \cup \{x^\nu\}$, where $i(\mathcal{S}^{\nu-1})$ is as in (b) and x^ν is one of the bordering ν-nodes of $\mathcal{S}^{\nu-1}$.*

Proof. In the trivial case where $\mathcal{G}^\nu$ has just one $(\nu - 1)$-section and just one ν-node, all the nodes have the same eccentricity and form (c) holds. So, consider the case where $\mathcal{G}^\nu$ has at least two $(\nu - 1)$-sections or at least two ν-nodes. Because of Lemmas 8.10 and 8.11, neither a non-boundary bordering ν-node nor the interior of an end $(\nu - 1)$-section can

be in the center. Thus, any center node of $\mathcal{G}^\nu$ is either a boundary ν-node or an internal node of a non-end $(\nu - 1)$-section. Also, the correspondence between boundary ν-nodes x_k^ν and their representative 0-nodes x_k^0 in $\mathcal{T}^0$ is a bijection, and so, too, is the correspondence between the interiors $i(\mathcal{S}_m^{\nu-1})$ of non-end $(\nu - 1)$-sections and their representative 0-nodes y_m^0 in $\mathcal{T}^0$. The eccentricities of these entities in $\mathcal{G}^\nu$ are related to the eccentricities of their representatives in $\mathcal{T}^0$ as stated in Lemma 10.2.

We now invoke an established theorem for finite 0-trees [3, Theorem 2.1]; namely, the center of such a tree is either a single 0-node of a pair of adjacent 0-nodes. When the center of $\mathcal{T}^0$ is a single 0-node x_k^0, form (a) holds. When the center of $\mathcal{T}^0$ is a single 0-node y_m^0, form (b) holds. Finally, when the center of $\mathcal{T}^0$ is a pair of adjacent 0-nodes, one of them will be a 0-node x_k^0 and the other will be a 0-node y_m^0, and thus form (c) holds. $\square$

Finally, let us note in passing that, except for the trivial case mentioned above, the periphery of a cycle-free ν-graph, defined as consisting of those nodes with the maximum eccentricity, is comprised either of nonboundary bordering ν-nodes of end $(\nu - 1)$-sections, or the interiors of end $(\nu - 1)$-sections, or both.

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Figure Captions

Fig. 1. A 1-graph consisting of a one-way infinite ladder along with two branches, d and e, connected to infinite extremities of the ladder. x_a^1 and x_b^1 are the only nonsingleton 1-nodes; all the other 1-nodes are singletons.

Fig. 2. A 1-loop having two 0-sections.

Fig. 3. The 1-graph of Example 6.5.

Fig. 4. The ν-graph $\mathcal{H}^\nu$. The lines (other than those of the rectangle) denote isolated endless $(\nu - 1)$-paths.

Fig. 5. The 1-graph of Example 8.9.

Fig. 6. The 1-graph of Example 9.3.

Fig. 7. The 1-graph of Example 9.4.

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