Algorithms for analyzing and verifying infinite-state recursive probabilistic systems

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LATA'13
Bilbao, April 5th, 2013

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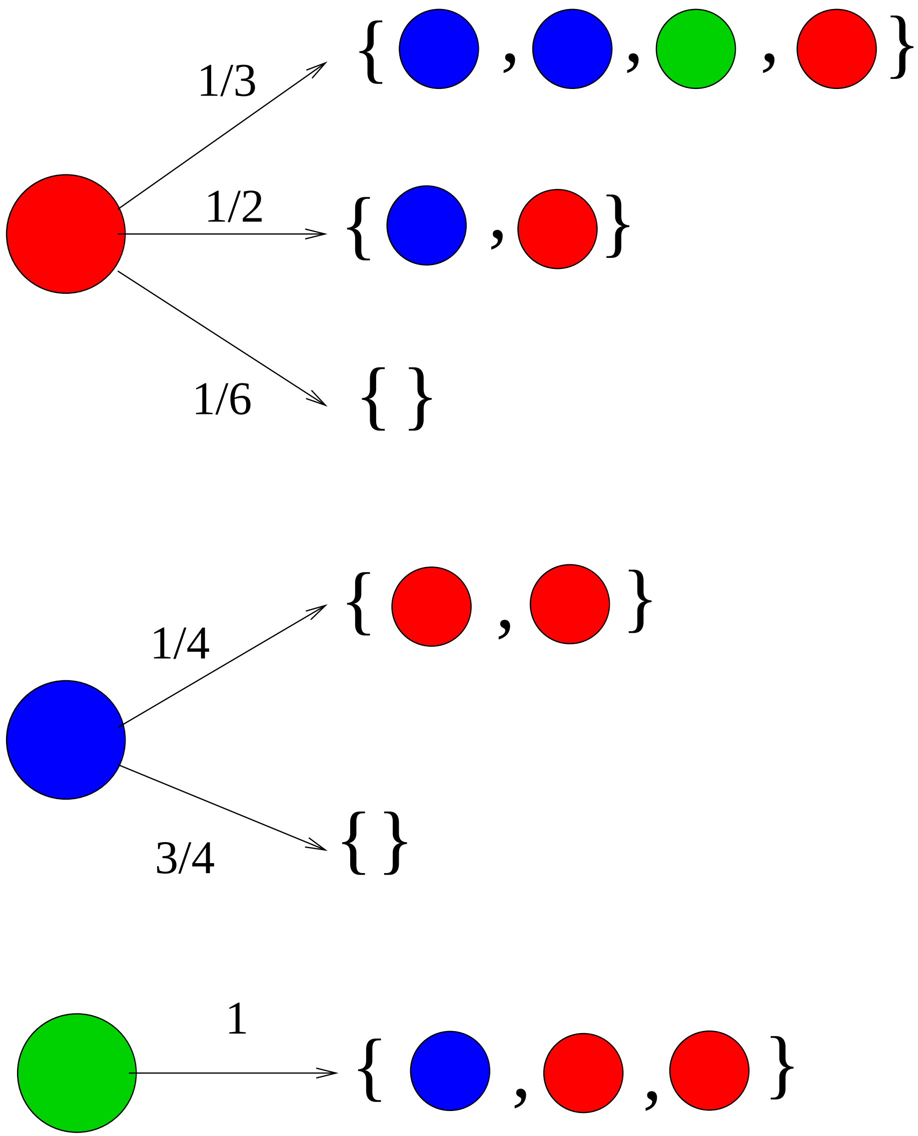
- Such models can also be captured by probabilistic extensions to classic infinite-state automata-theoretic models, like context-free grammars, pushdown automata, and one-counter automata.
- The algorithmic theory, and complexity, of analyzing such recursive MCs and their extension to Markov decision processes and stochastic games, has turned out to be an extremely rich subject.
- In this talk, I will survey only one fragment of this theory (focusing mainly on recent joint work with Alistair Stewart and Mihalis Yannakakis).

Multi-type Branching Processes (Kolmogorov,1940s)



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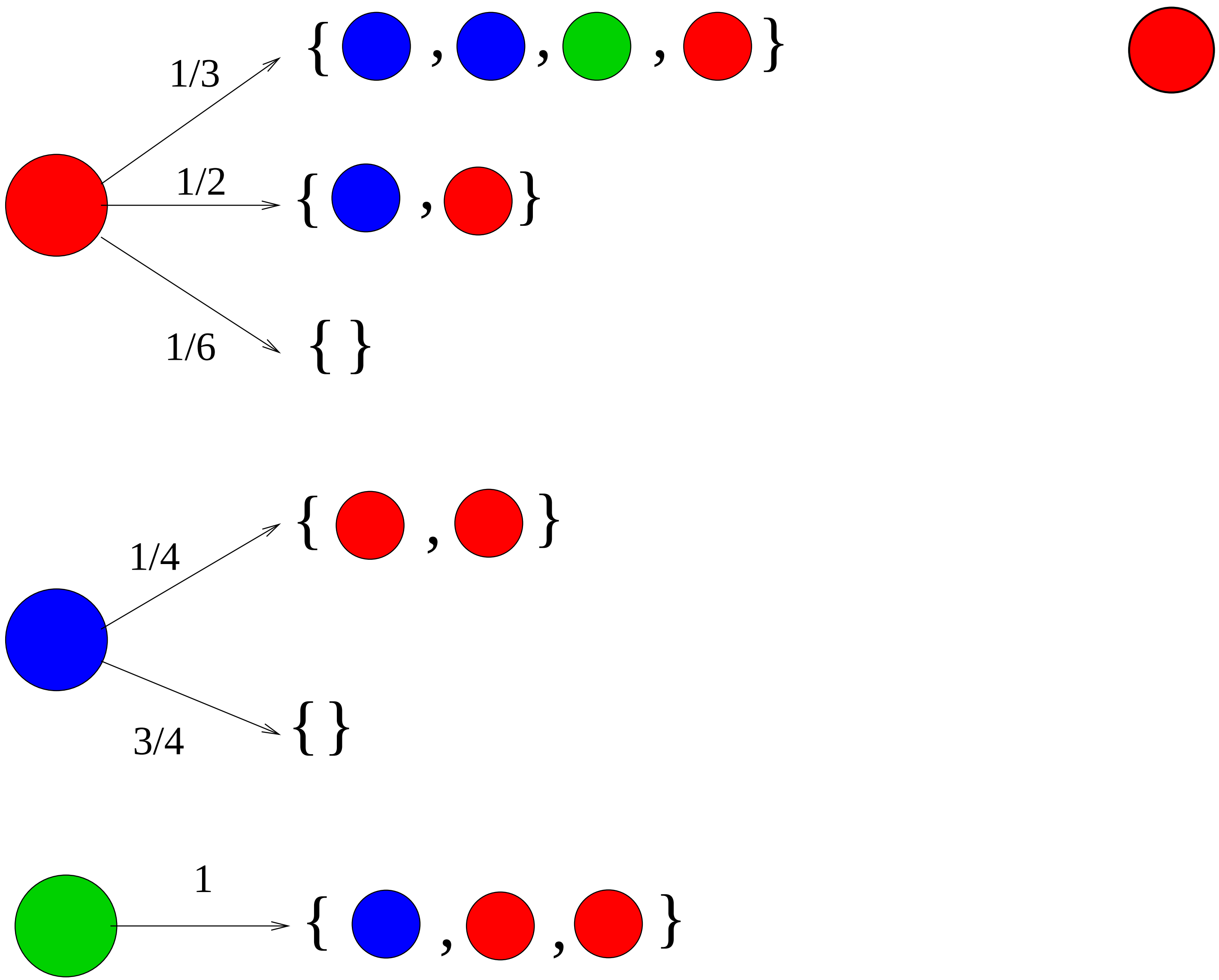

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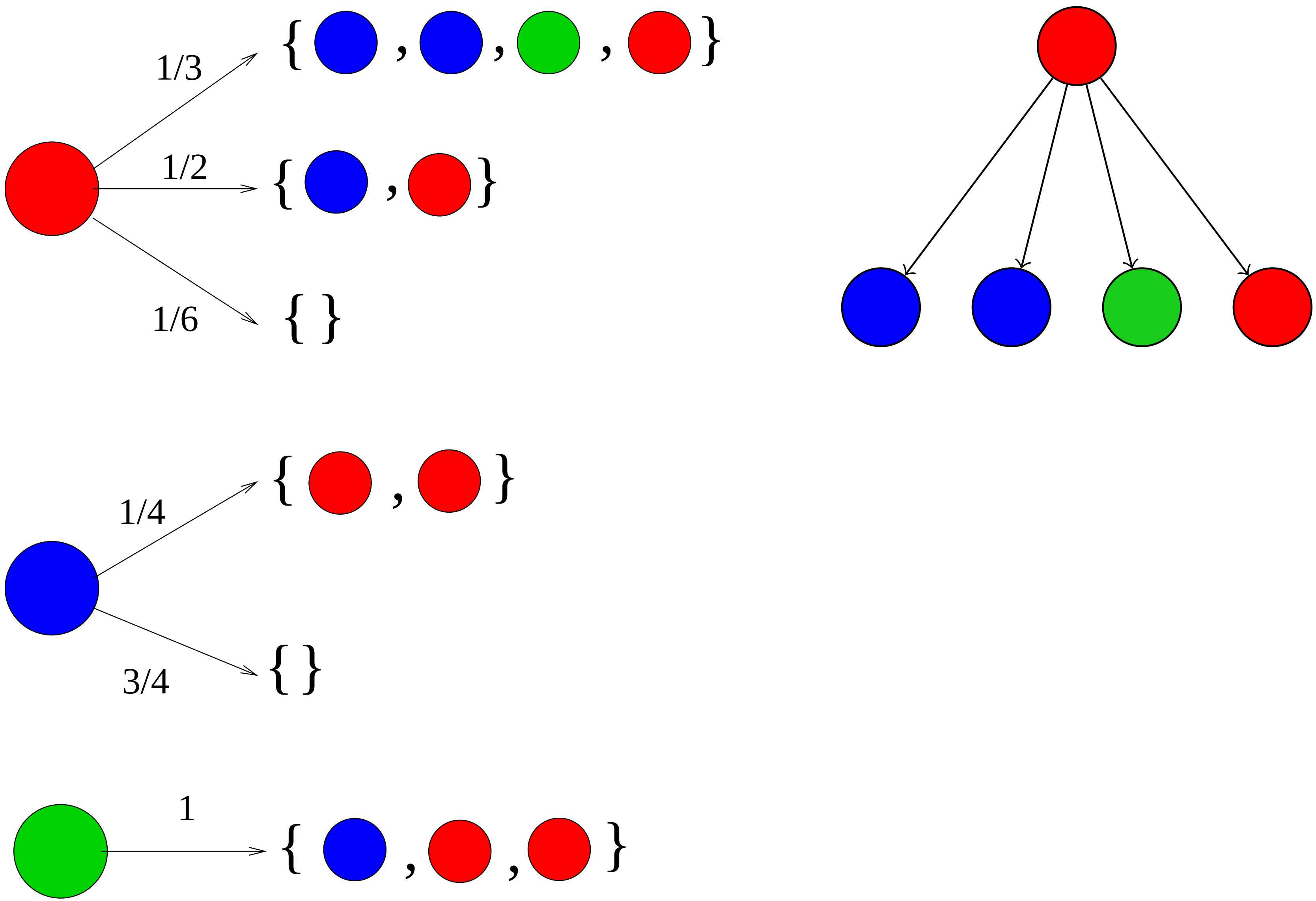

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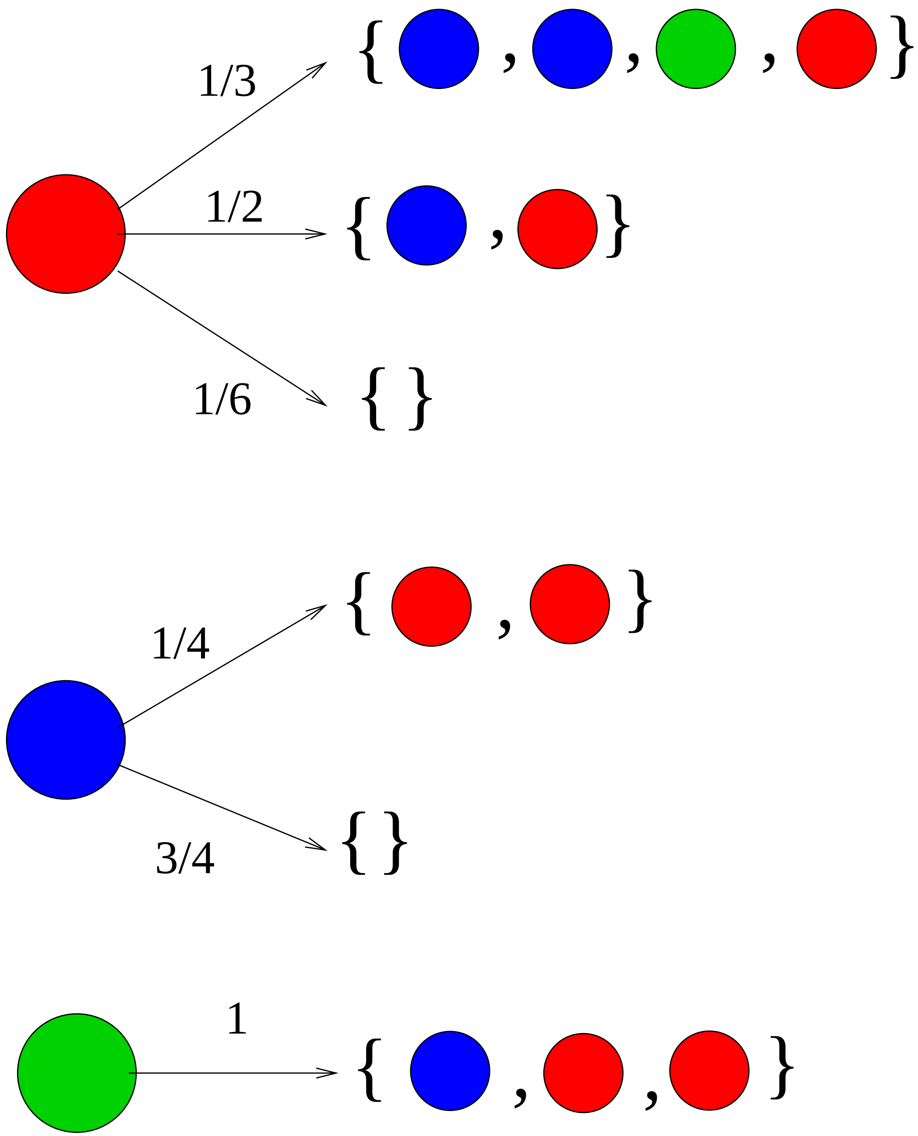

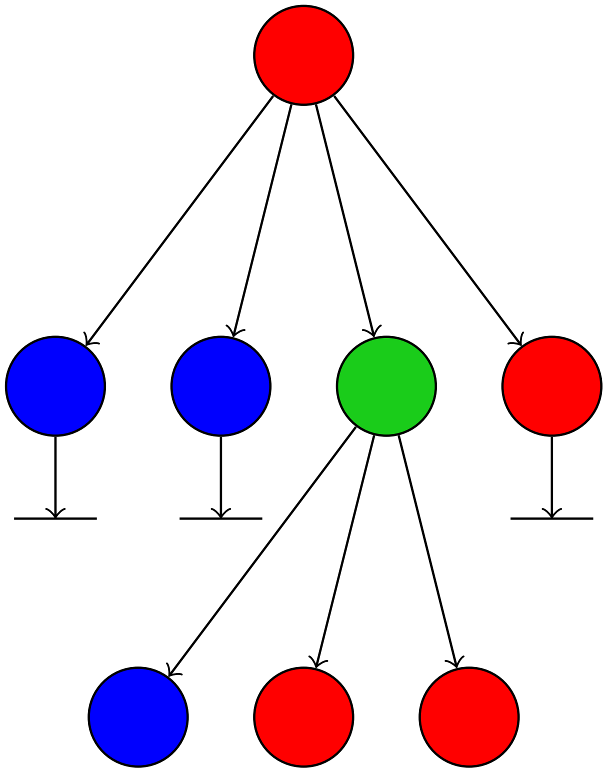

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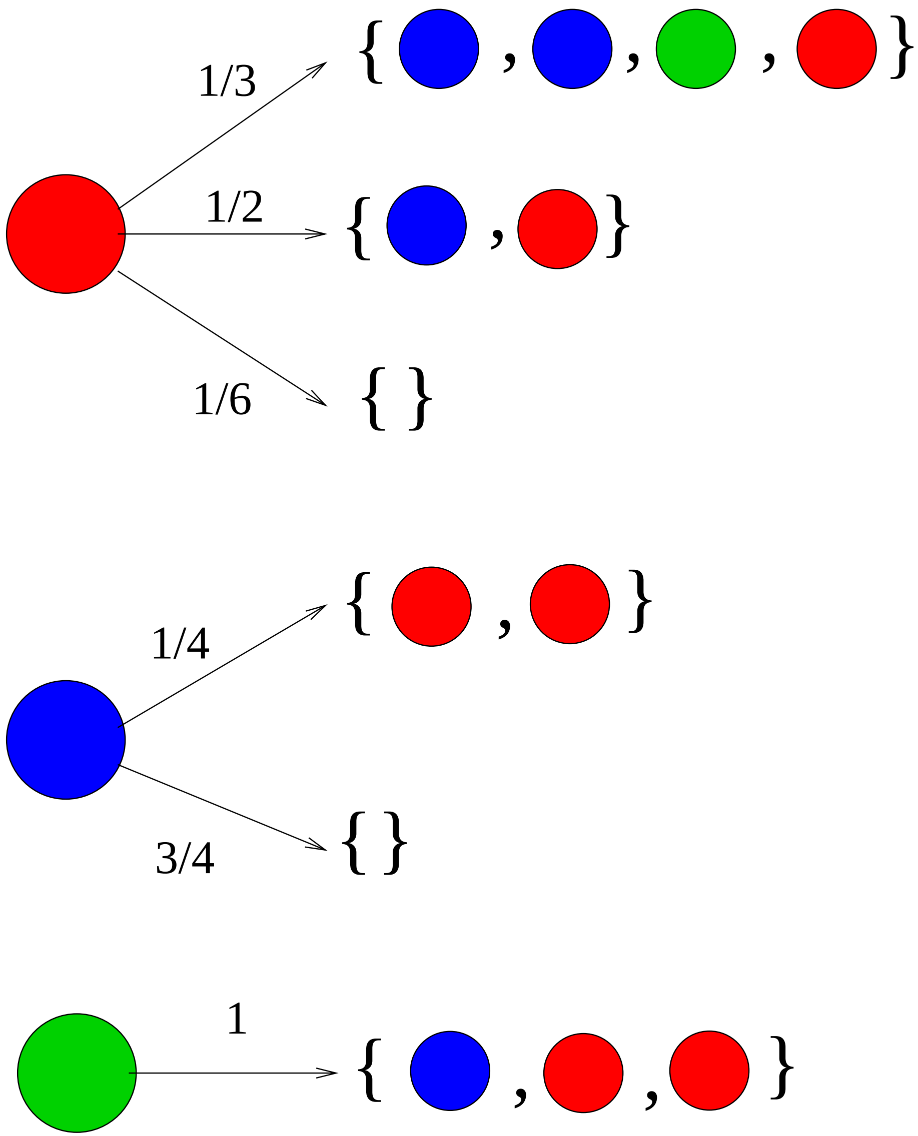

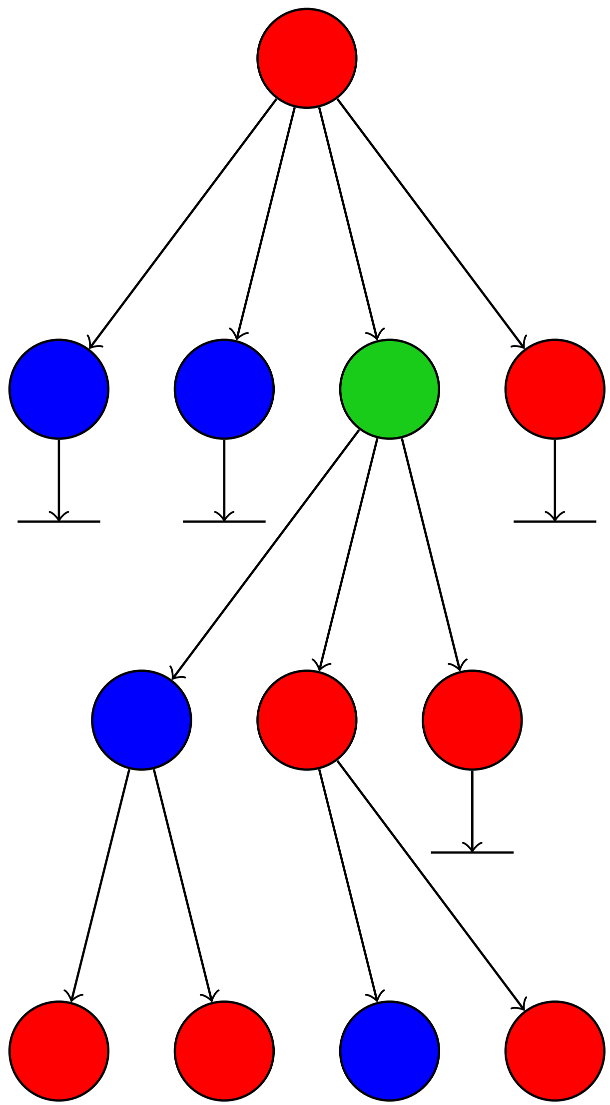

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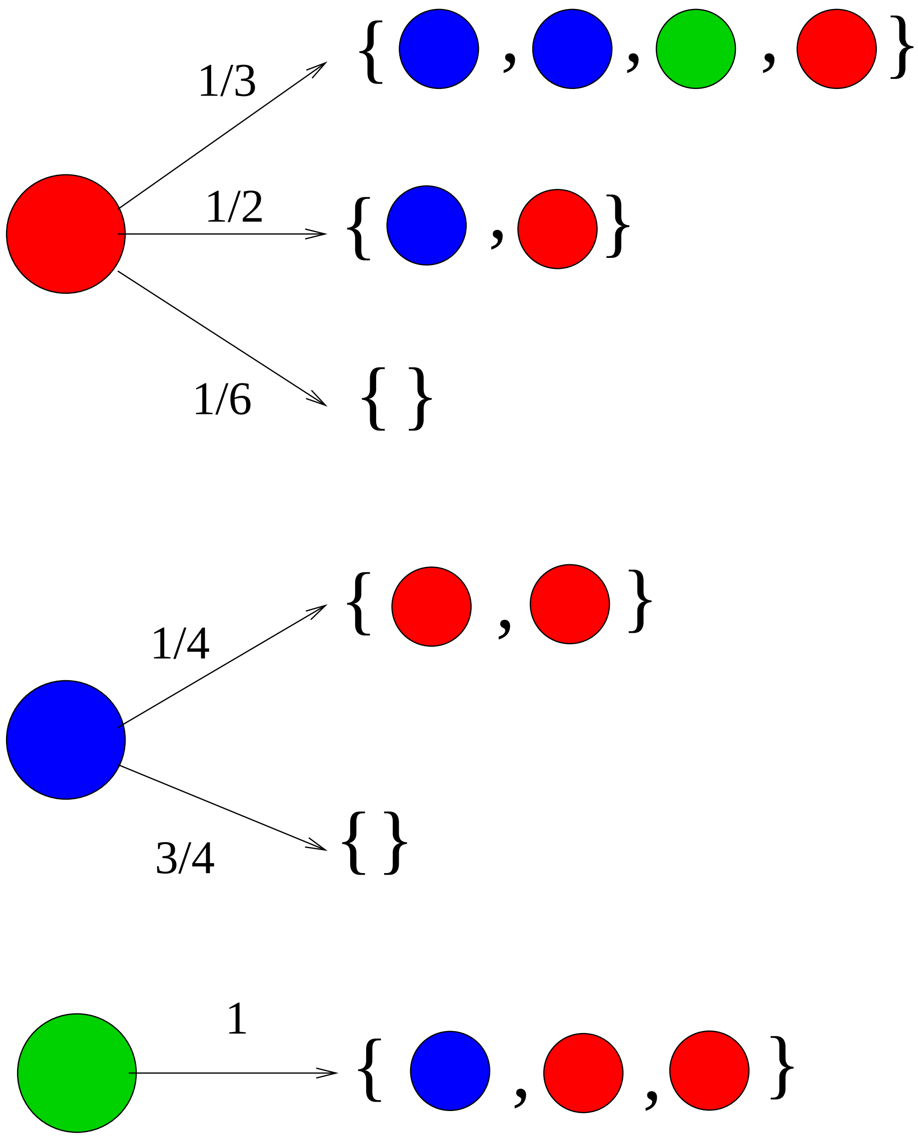
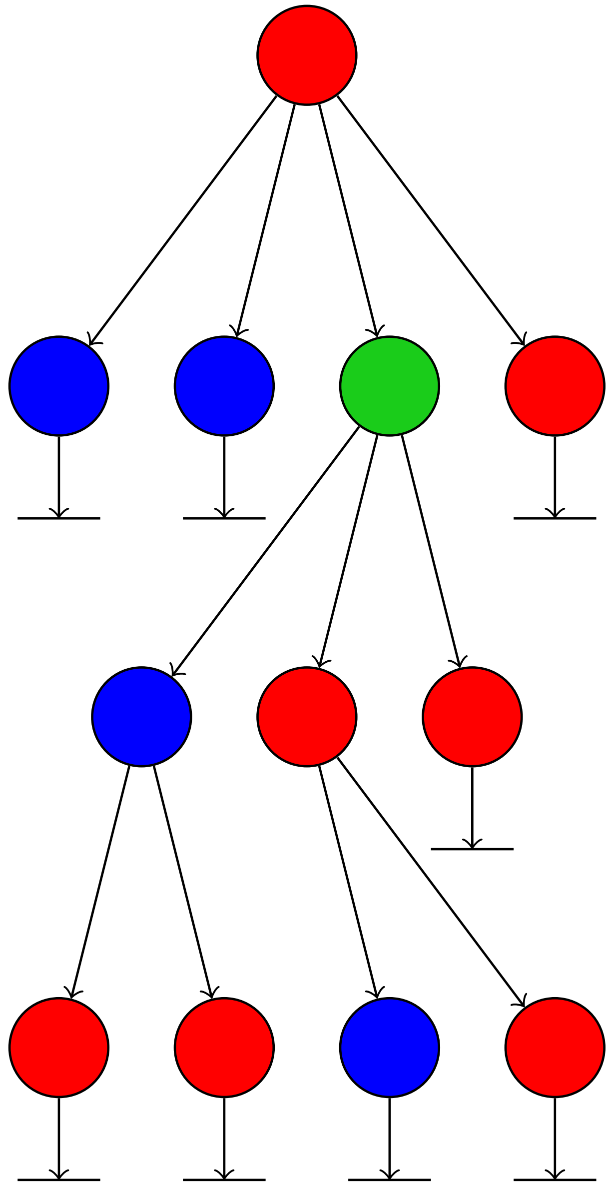
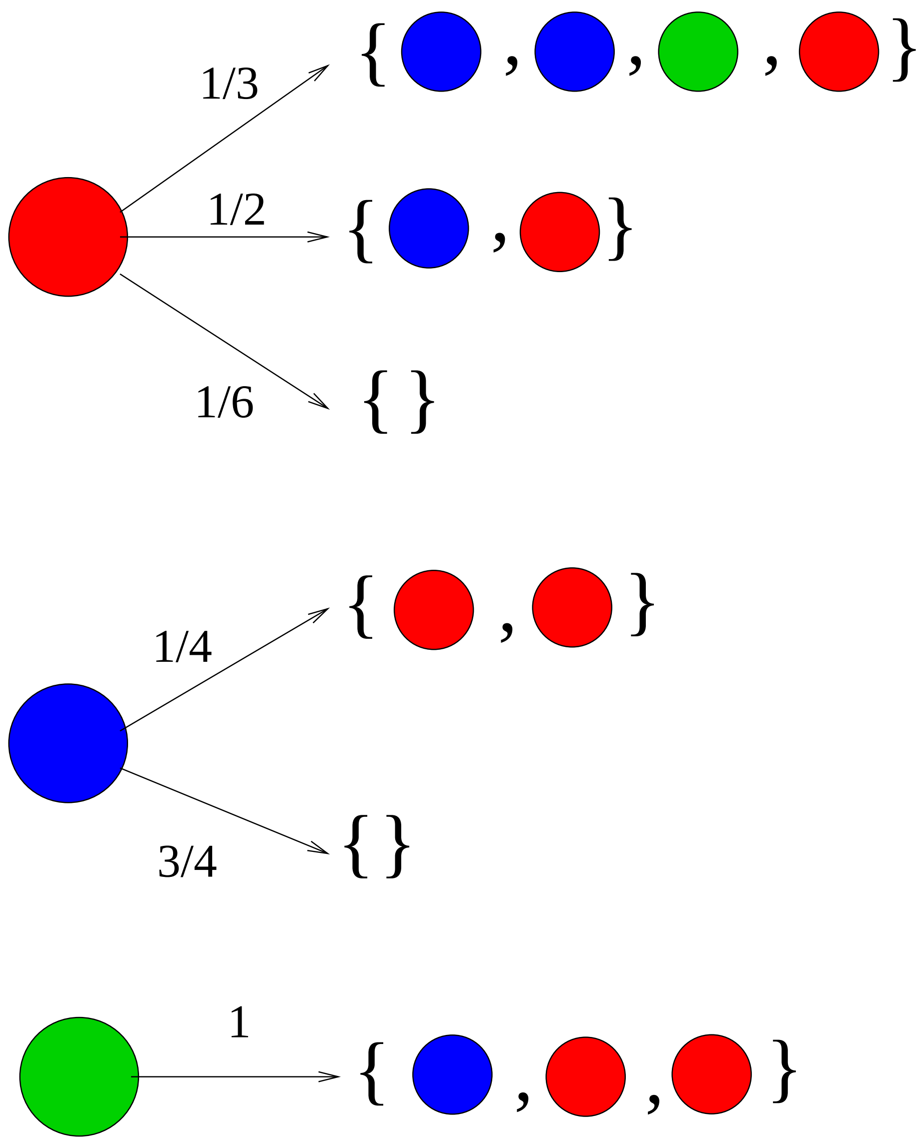
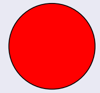
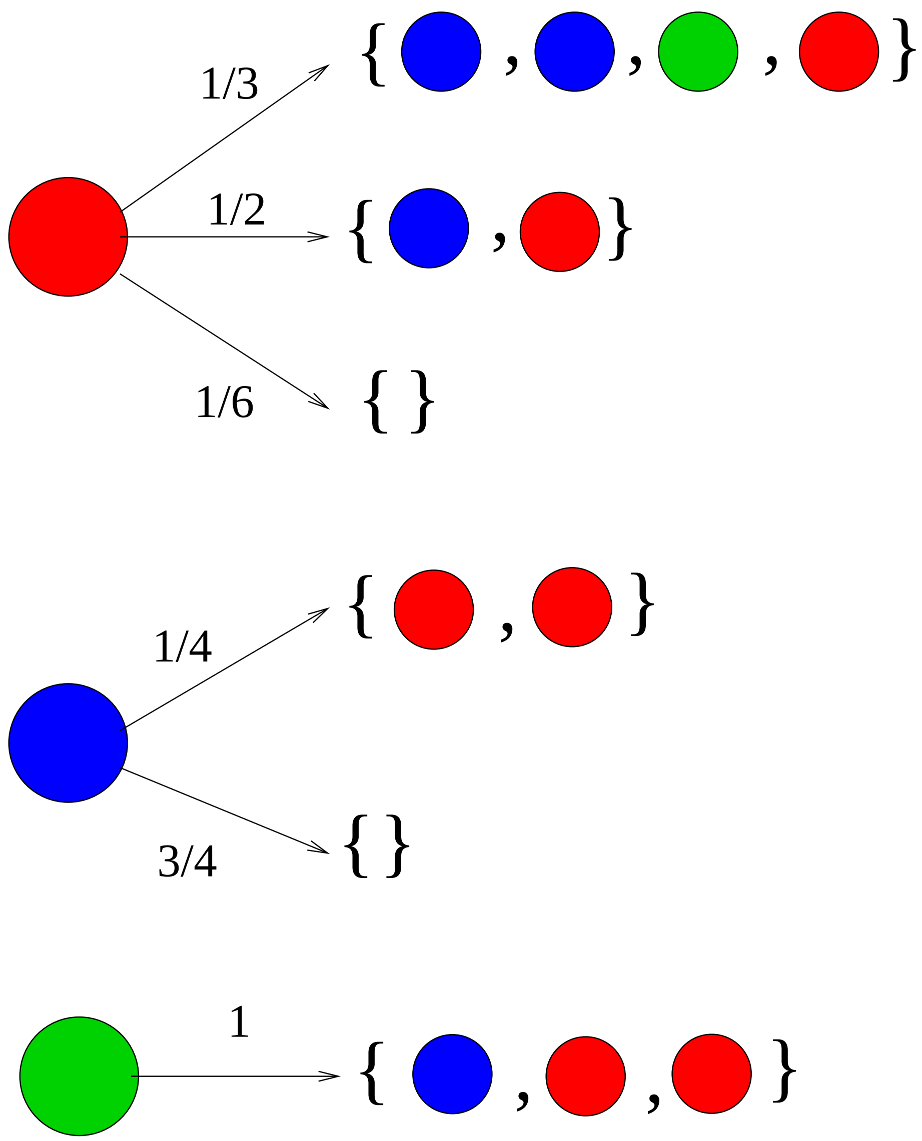
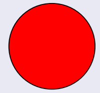




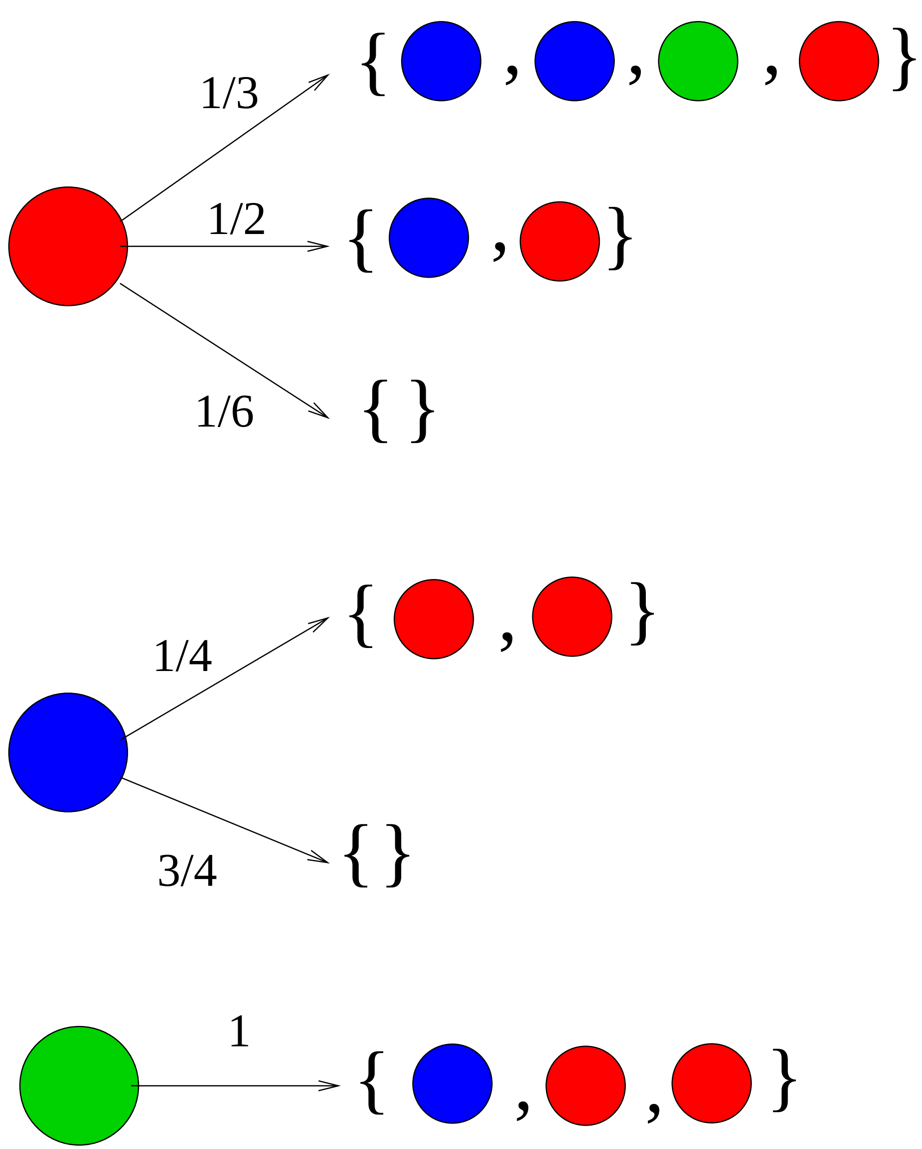


Question: What is the probability of eventual extinction, starting with one  ?



Question: What is the probability of eventual extinction, starting with one  ?

$$x_R \;=$$



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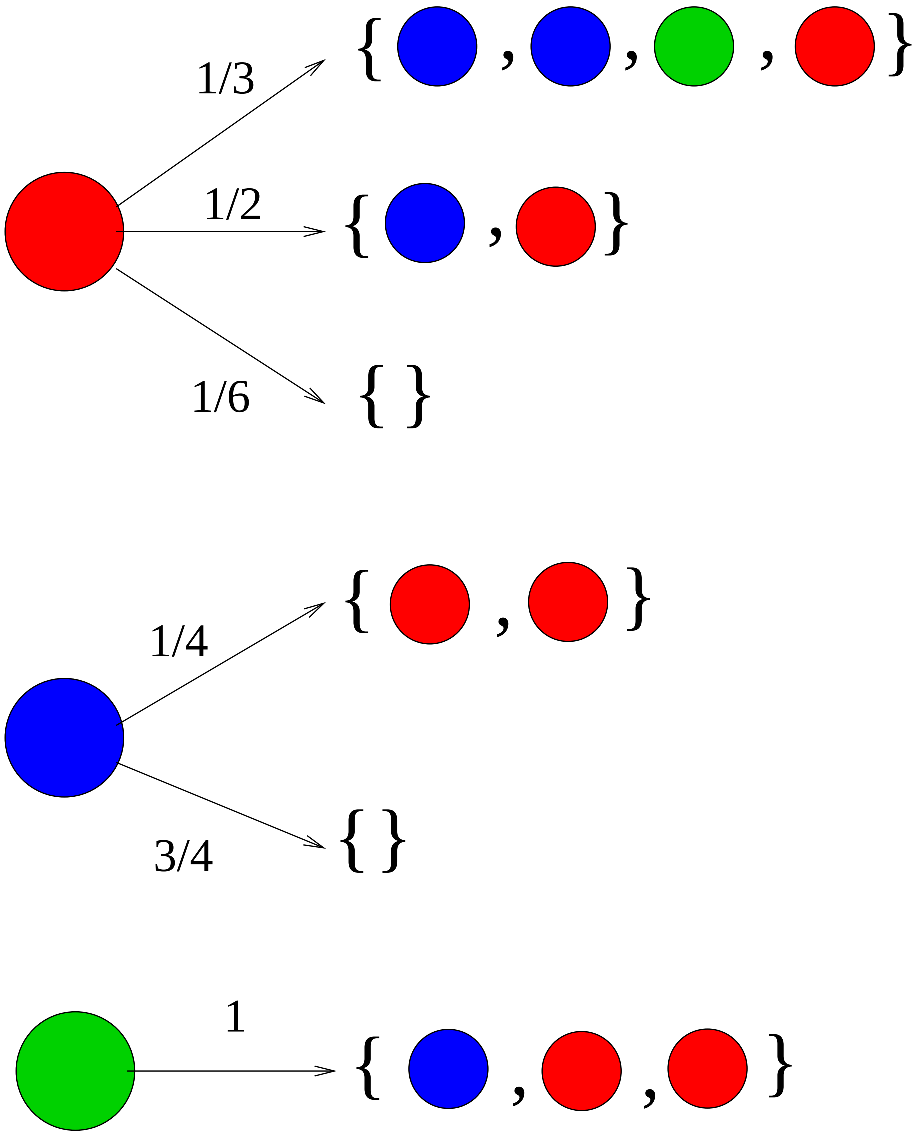
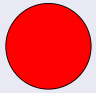


Question: What is the probability of eventual extinction, starting with one  ?

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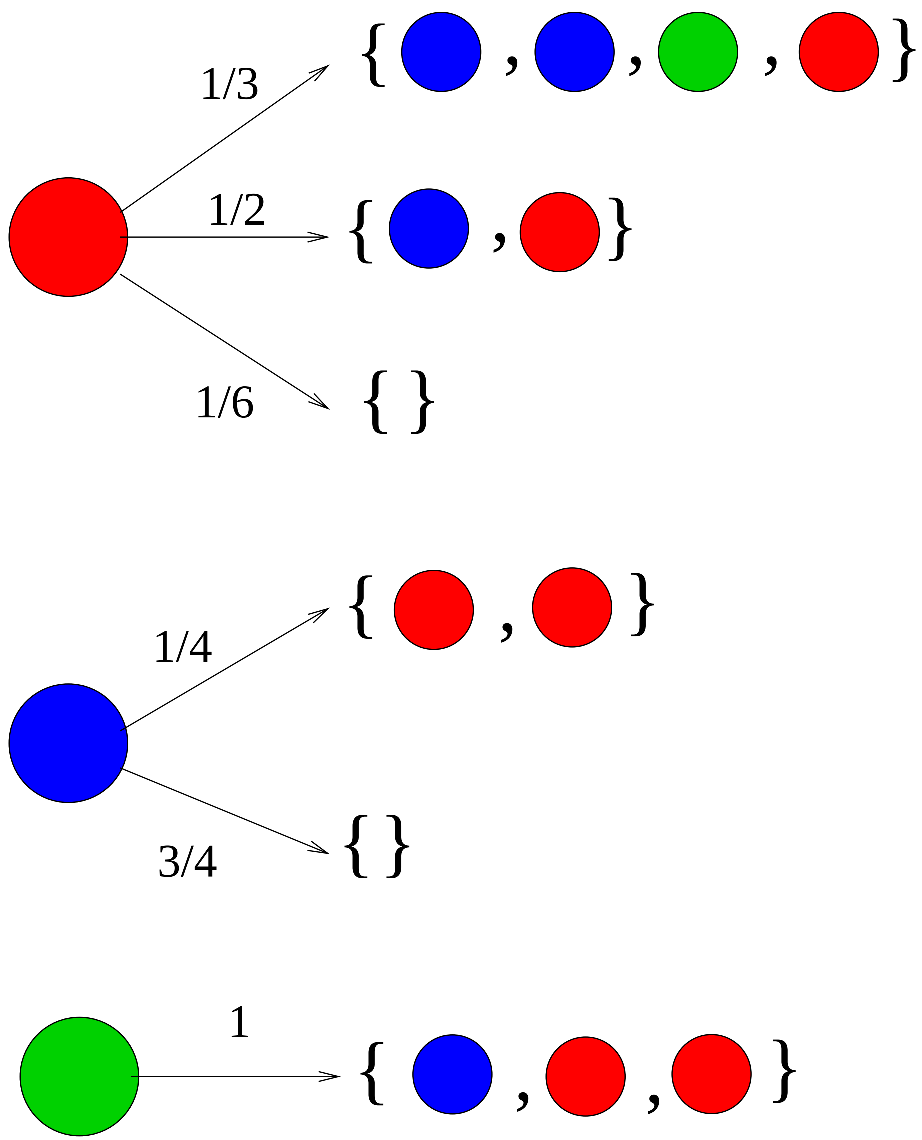
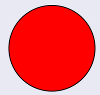


Question: What is the probability of eventual extinction, starting with one  ?

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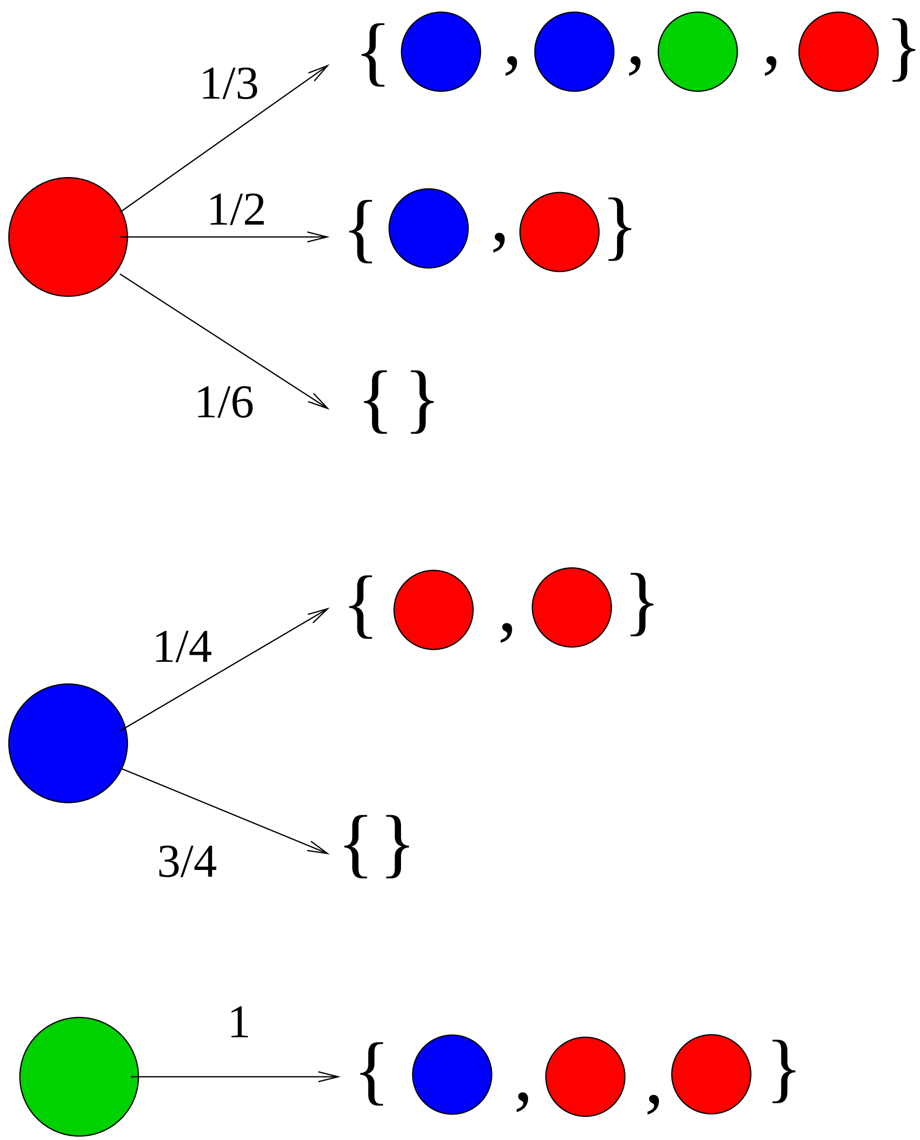

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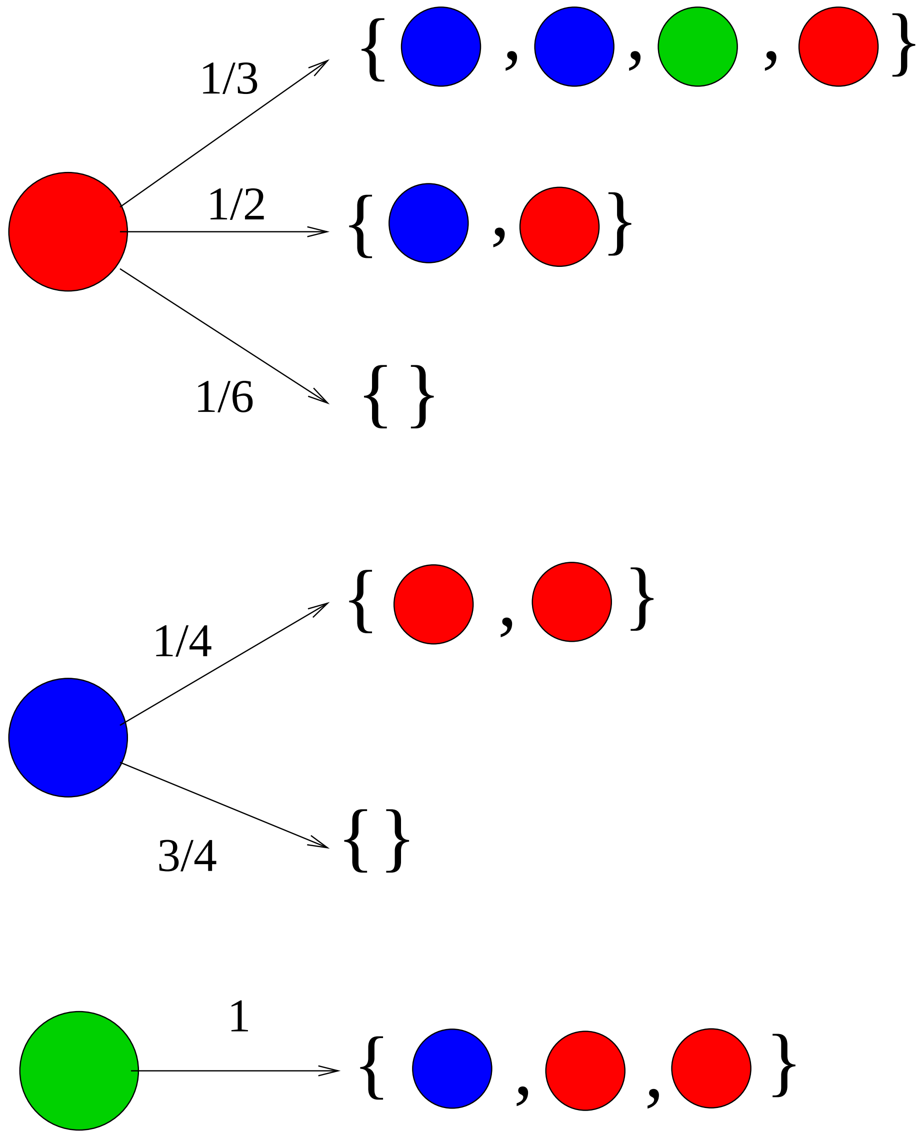
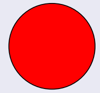


Question: What is the probability of eventual extinction, starting with one  ?

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The extinction probabilities are the least fixed point, $\mathbf{q}^* \in [0,1]^3$, of $\bar{\mathbf{x}} = P(\bar{\mathbf{x}})$.
$q_R^* = 0.276$; $q_B^* = 0.769$; $q_G^* = 0.059$.

Stochastic Context-Free Grammars

$$R \xrightarrow{1/3} aBBcGdR$$

$$R \xrightarrow{1/2} bcBbR$$

$$R \xrightarrow{1/6} d$$

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probability of this derivation: $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{6}^3$

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Fact

Termination probabilities (also called the partition function of the SCFG) are the least fixed point, $\mathbf{q}^* \in [0,1]^3$, of $\bar{\mathbf{x}} = P(\bar{\mathbf{x}})$.

Some other key computations for SCFGs

- string ("inside") probability: Given an SCFG, G, and a string w, what is the probability that G generates w?
- regular language probability: Given a SCFG, G, and given a DFA, D, what is the probability that G generates a string in $L(D)$?
- ω-regular model checking: Given a stochastic context-free process and a Büchi automaton, B, what is the probabilty that a run of G generates an ω-word in $L(B)$?

For general SCFGs, G, all these questions are at least as hard as computation of SCFG termination probabilities.

Probabilistic Polynomial Systems of Equations

$$\frac{1}{3}x_B^2 x_G x_R + \frac{1}{2}x_B x_R + \frac{1}{6}$$

is a Probabilistic Polynomial: the coefficients are positive and sum to 1.

A Probabilistic Polynomial System (PPS), is a system of n equations

$$\mathbf{x} = P(\mathbf{x})$$

in n variables where each $P_i(x)$ is a probabilistic polynomial.

Every multi-type Branching Process (BP) with n types, and every SCFG with n nonterminals, corresponds to a PPS, and vice-versa.

Basic properties of PPSs, $\mathbf{x} = P(\mathbf{x})$

For every PPS, $P : [0,1]^n \rightarrow [0,1]^n$ defines a monotone map on $[0,1]^n$.

Proposition

- *A PPS, $x = P(x)$ has a least fixed point, $q^* \in [0,1]^n$.*
  (q^* can be irrational.)
- $q^* = \lim_{k\rightarrow\infty} P^k(\mathbf{0})$.
- *q^* is vector of extinction/termination probabilities for the BP (SCFG).*

Question

Can we compute the probabilities q^* efficiently (in P-time)?

First considered by Kolmogorov & Sevastyanov (1940s).

Newton's method

Newton's method

Seeking a solution to $F(\mathbf{x}) = 0$, we start at a guess $\mathbf{x}^{(0)}$, and iterate:

$$\mathbf{x}^{(k+1)} := \mathbf{x}^{(k)} - (F'(\mathbf{x}^{(k)}))^{-1} F(\mathbf{x}^{(k)})$$

Here $F'(\mathbf{x})$, is the **Jacobian matrix**:

$$F'(\mathbf{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} \dots \frac{\partial F_1}{\partial x_n} \\ \vdots \vdots \vdots \\ \frac{\partial F_n}{\partial x_1} \dots \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

For PPSs, $F(x) \equiv (P(x) - x)$, and Newton iteration looks like this:

$$\mathbf{x}^{(k+1)} := \mathbf{x}^{(k)} + (I - P'(\mathbf{x}^{(k)}))^{-1}(P(\mathbf{x}^{(k)}) - \mathbf{x}^{(k)})$$

where $P'(\mathbf{x})$ is the Jacobian of $P(\mathbf{x})$.

Newton on PPSs

We can decompose $\mathbf{x} = P(\mathbf{x})$ into its strongly connected components (SCCs), based on variable dependencies, and eliminate "0" variables.

Theorem [E.-Yannakakis'05]

Decomposed Newton's method converges monotonically to the LFP $\mathbf{q}^*$ for PPSs, and for more general Monotone Polynomial Systems (MPSs).

But...

- In [E.-Yannakakis'05,'09], we gave no upper bounds on # of iterations needed for PPSs or MPSs.
- We proved hardness results (PosSLP-hardness) for obtaining any nontrivial approximation of the LFP of MPSs for recursive Markov chains.

What is Newton's worst case behavior for PPSs?

[Esparza,Kiefer,Luttenberger,'10] studied Newton's method on MPSs further:

- Gave bad examples of PPSs, $\mathbf{x} = P(\mathbf{x})$, where $q^* = 1$, requiring exponentially many iterations, as a function of the encoding size $|P|$ of the equations, to converge to within additive error $< 1/2$.
- For strongly-connected equation systems they gave an exponential upper bound in $|P|$.
- But they gave no upper bounds for arbitrary PPSs or MPSs in terms of $|P|$.

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- But they gave no upper bounds for arbitrary PPSs or MPSs in terms of $|P|$.
- (Recently [Stewart-E.-Yannakakis'13], we give a matching exponential upper bound in $|P|$ for arbitrary PPSs and MPSs.)

P-time approximation for PPSs

Theorem ([E.-Stewart-Yannakakis,STOC'12])

Given a PPS, $\mathbf{x} = P(\mathbf{x})$, with LFP $\mathbf{q}^* \in [0,1]^n$, we can compute a rational vector $\mathbf{v} \in [0,1]^n$ such that

$$\|\mathbf{v} - \mathbf{q}^*\|_\infty \leq 2^{-j}$$

in time polynomial in both the encoding size $|P|$ of the equations and in j (the number of "bits of precision").

We use Newton's method..... but how?

Qualitative decision problems for PPSs are in P-time

Theorem ([Kolmogorov-Sevastyanov'47,Harris'63])

For certain classes of strongly-connected PPSs, $q_i^* = 1$ *for all* i *iff the* *spectral radius* $\varrho(P'(\mathbf{1}))$ *for the* *moment matrix* $P'(\mathbf{1})$ *is* ≤ 1*, and otherwise* $q_i^* < 1$ *for all* i*.*

Theorem ([E.-Yannakakis'05])

Given a PPS, $\mathbf{x} = P(\mathbf{x})$*, deciding whether* $q_i^* = 1$ *is in P-time.*

(Deciding whether $q_i^* = 0$ *is also in P-time (and a lot easier).)*

Algorithm for approximating the LFP q^* for PPSs

1. Find and remove all variables x_i such that $q_i^* = 0$ or $q_i^* = 1$.
2. On the resulting system of equations, run Newton's method starting from $\mathbf{0}$.

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Theorem ([ESY'12])

Given a PPS $\mathbf{x} = P(\mathbf{x})$ *with LFP* $\mathbf{0} < \mathbf{q}^* < \mathbf{1}$, *if we apply Newton starting at* $\mathbf{x}^{(0)} = \mathbf{0}$, *then*

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Algorithm with rounding

1. Find and remove all variables x_i such that $q_i^* = 0$ or $q_i^* = 1$.
2. On the resulting system of equations, run Newton's method starting from $\mathbf{0}$.
3. After each iteration, round down to a multiple of 2^{-h}

Theorem ([ESY'12])

If, after each Newton iteration, we round down to a multiple of 2^{-h} where $h := 4|P| + j + 2$, then after h iterations $\|\mathbf{q}^* - \mathbf{x}^{(h)}\|_\infty \leq 2^{-j}$.

Thus, we obtain a P-time algorithm (in the standard Turing model) for approximating q^*.

High level picture of proof

- For a PPS, $x = P(x)$, with LFP $\mathbf{0} < \mathbf{q}^* < \mathbf{1}$, $P'(q^*)$ is a non-negative square matrix, and (we show)

$$\text{(spectral radius of } P'(q^*) \text{)} \equiv \varrho(P'(q^*)) < 1$$

- So, $(I - P'(q^*))$ is non-singular, and $(I - P'(q^*))^{-1} = \sum_{i=0}^{\infty}(P'(q^*))^i$.

- We can show the $\#$ of Newton iterations needed to get within $\epsilon > 0$ is

$$\approx\approx \log \|(I - P'(q^*))^{-1}\|_\infty + \log \frac{1}{\epsilon}$$

- $\|(I - P'(q^*))^{-1}\|_\infty$ is tied to the distance $|1 - \varrho(P'(q^*))|$, which in turn is related to $\min_i(1 - q_i^*)$, which we can lower bound.

- Uses lots of Perron-Frobenius theory.

Proof outline: some key lemmas

$(\mathbf{1} - \mathbf{q}^*)$ is the vector of survival probabilities.

Lemma

If $\mathbf{q}^* - \mathbf{x}^{(k)} \leq \lambda(\mathbf{1} - \mathbf{q}^*)$ *for some* $\lambda > 0$, *then* $\mathbf{q}^* - \mathbf{x}^{(k+1)} \leq \frac{\lambda}{2}(\mathbf{1} - \mathbf{q}^*)$.

Lemma

For any PPS with LFP $\mathbf{q}^*$, *such that* $\mathbf{0} < \mathbf{q}^* < \mathbf{1}$, *for any* i,

$$q_i^* \leq 1 - 2^{-4|P|}.$$

The complexity of quantitative decision problems for BPs

Proposition

Given a PPS, $x = P(x)$, and a probability p, deciding whether $q_i^* \leq p$ is in PSPACE.

Proof.

$$\exists \mathbf{x}(\mathbf{x} = P(\mathbf{x}) \wedge x_i \leq p)$$

is expressible in the existential theory of reals. There are PSPACE decision procedures for $\exists\mathbb{R}$ ([Canny'89,Renegar'92]). $\square$

Now some bad news:

Theorem ([E.-Yannakakis,'05,'07])

Given a PPS, $x = P(x)$, deciding whether $q_i^* \leq 1/2$ (or $q_i^* \leq p$ for any $p \in (0, 1)$), is both Sqrt-Sum-hard and PosSLP-hard.

two "hard" problems

Sqrt-Sum: the square-root sum problem is the following decision problem:

Given $(d_1, \ldots, d_n) \in \mathbb{N}^n$ and $k \in \mathbb{N}$, decide whether $\sum_{i=1}^n \sqrt{d_i} \leq k$.

Solvable in PSPACE.

Open problem ([GareyGrahamJohnson'76]) whether it is in NP (or even the polynomial time hierarchy).

PosSLP: Given an arithmetic circuit (Straight Line Program) with gates $\{+, *, -\}$ with integer inputs, decide whether the output is > 0.

PosSLP captures all of polynomial time in the unit-cost arithmetic RAM model of computation.

[Allender, Bürgisser, Kjeldal-Petersen, Miltersen,2006] Gave a (Turing) reduction from Sqrt-Sum to PosSLP and showed both can be decided in the Counting Hierarchy: $P^{PP^{PP^{PP}}}$. Nothing better is known.

The quantitative decision problem for PPSs is PosSLP-equivalent

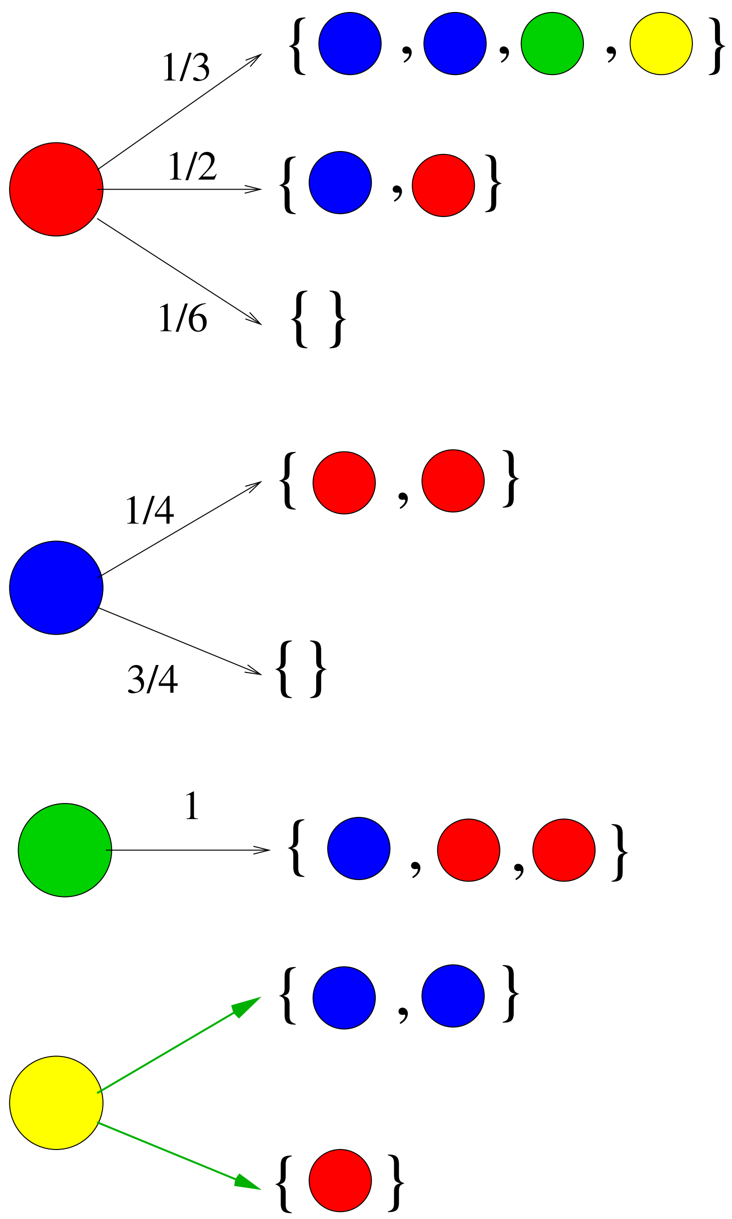
Theorem ([E.-Stewart-Yannakakis'12])

Given a PPS, $x = P(x)$, and a probability p, deciding whether $q_i^* < p$ is P-time (many-one) reducible to PosSLP. (And thus PosSLP-equivalent.)

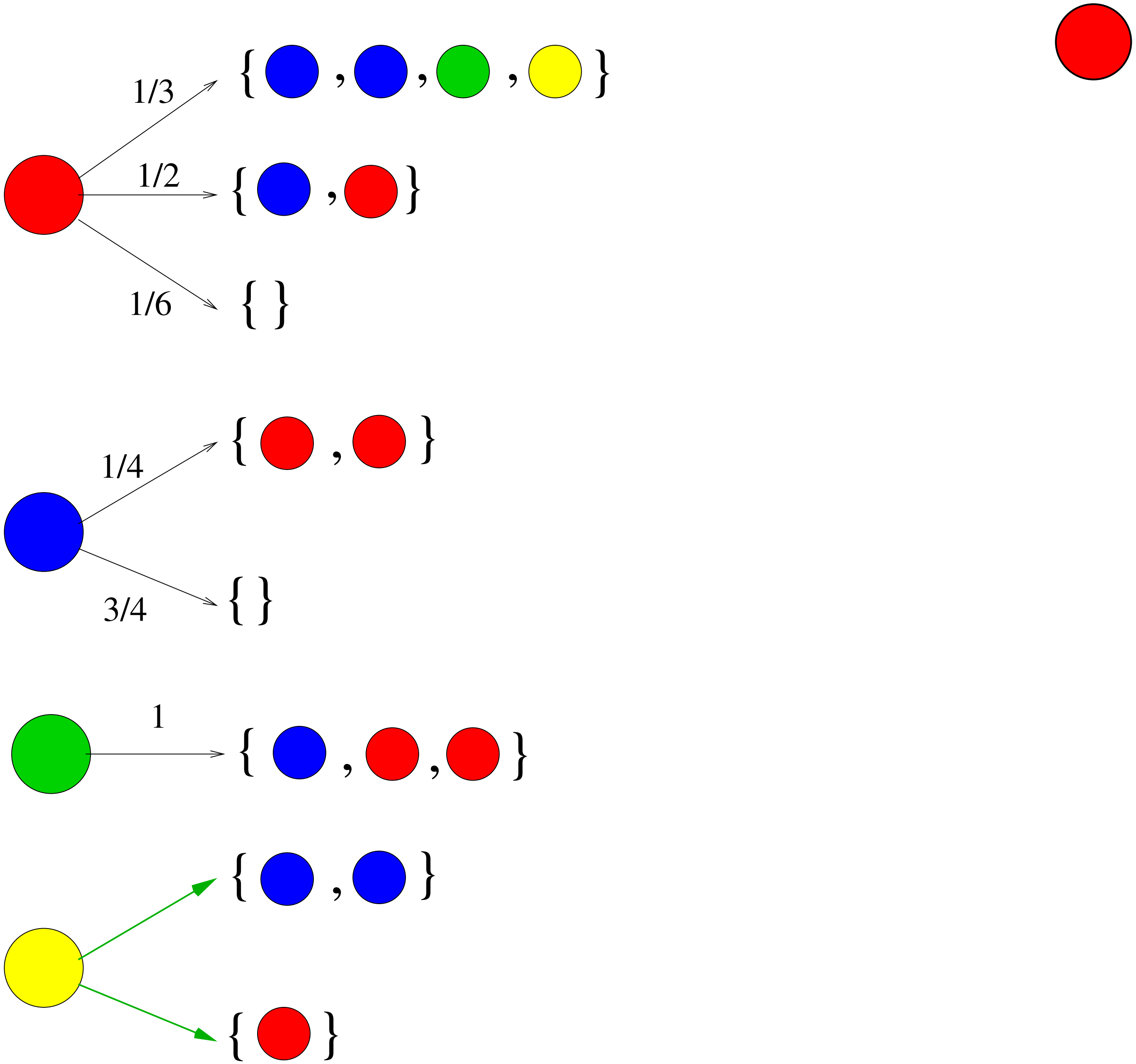
- Thus it captures the full power of polynomial time in the unit-cost arithmetic RAM model of computation.

  And by [Allender, et. al.'06], it is also in the Counting Hierarchy.

Branching Markov Decision Processes



Branching Markov Decision Processes



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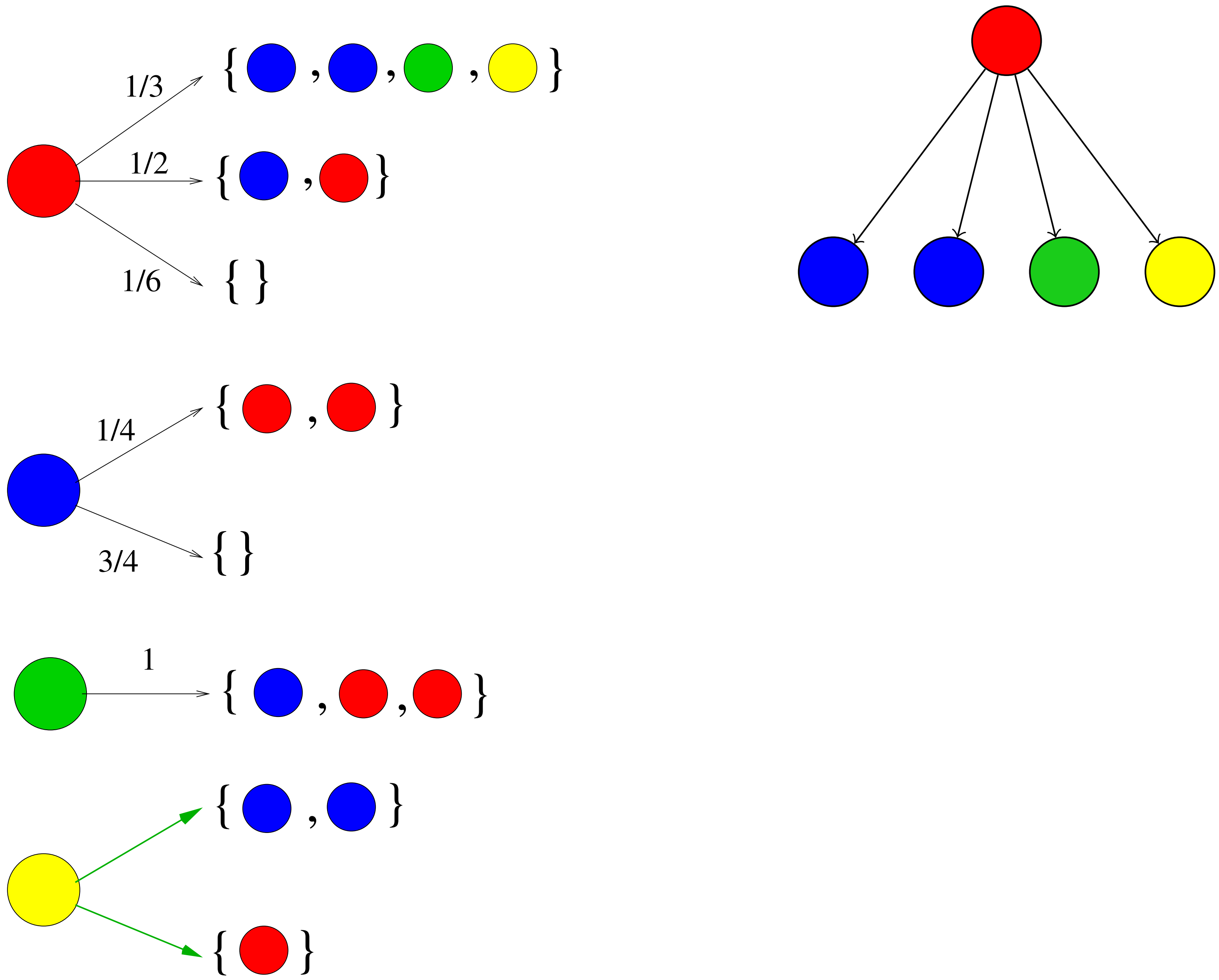

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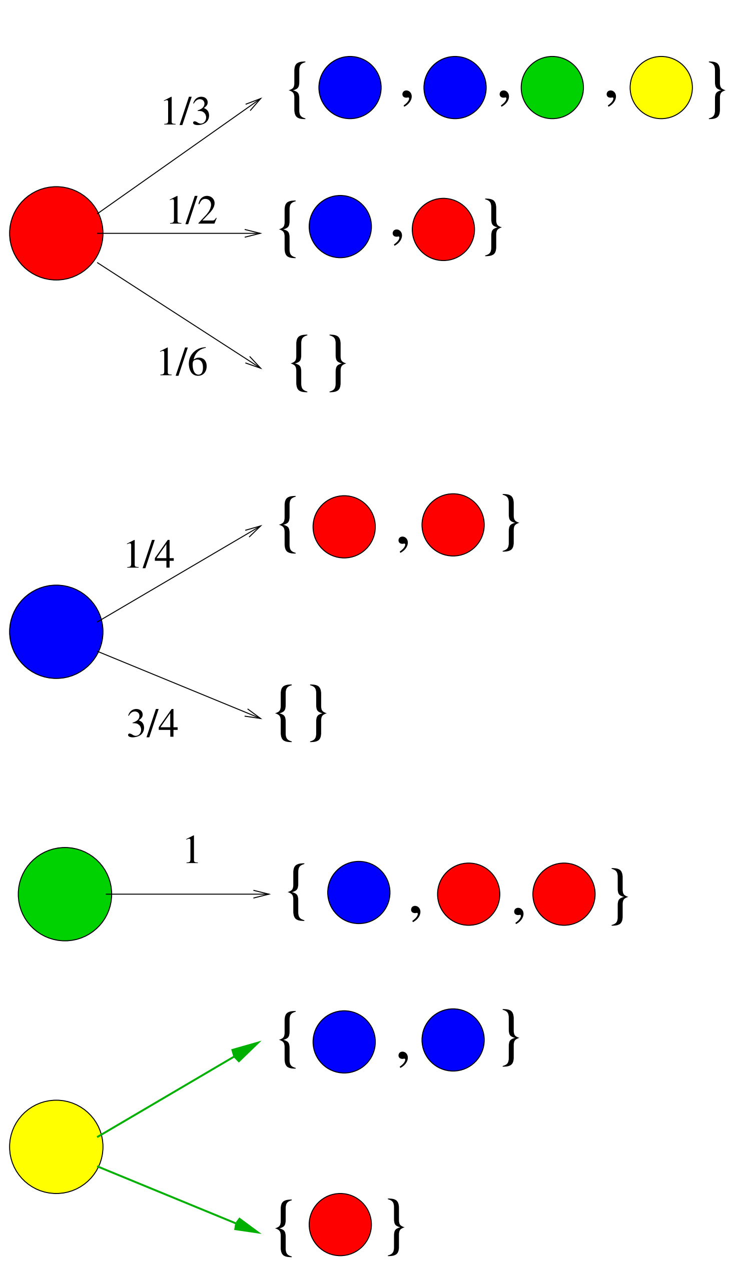

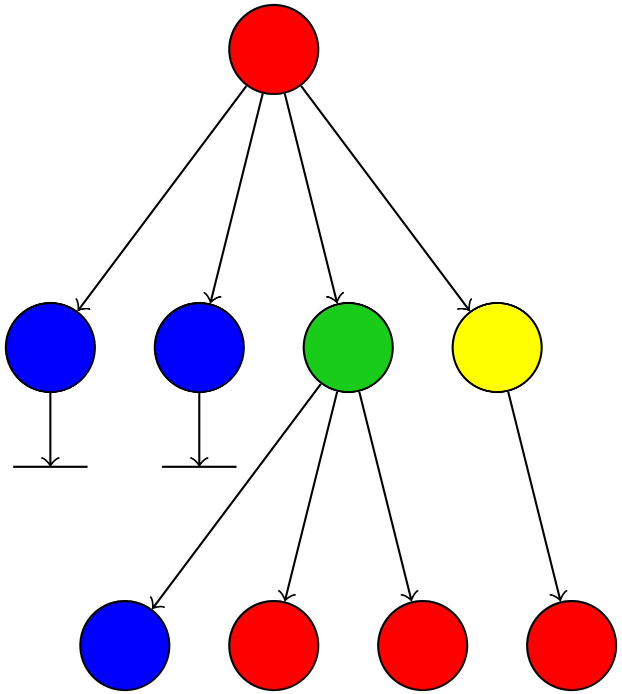

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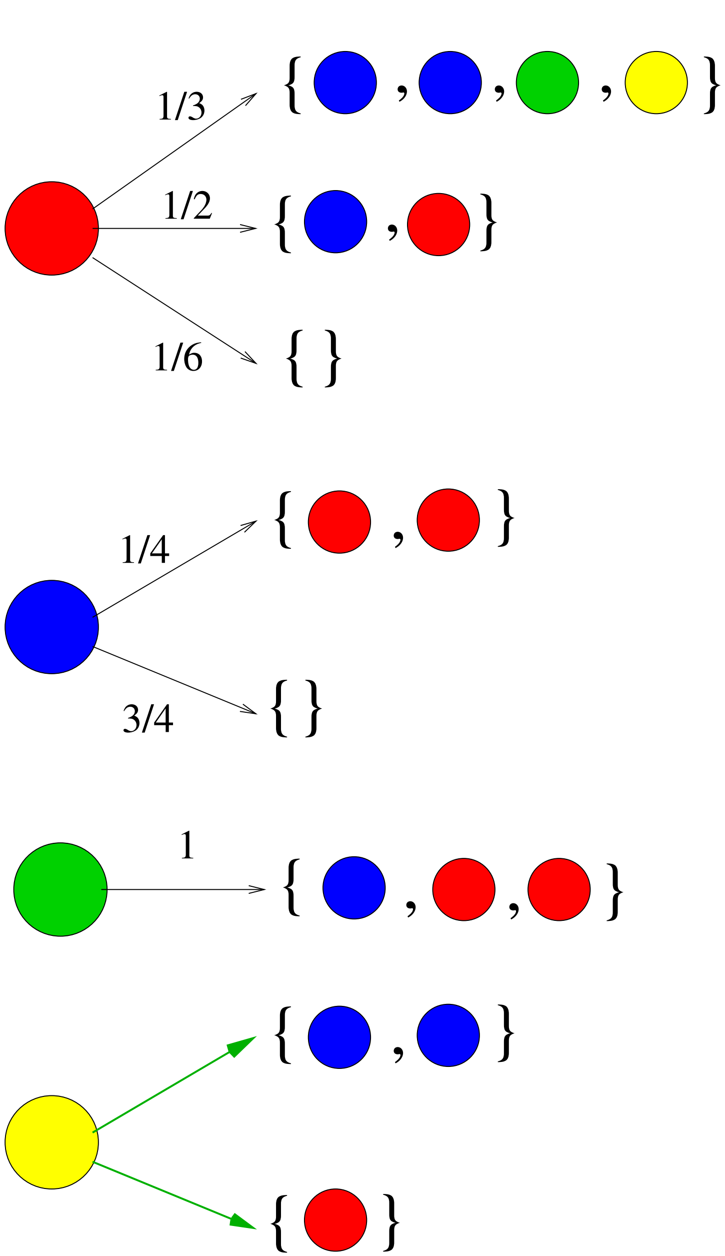

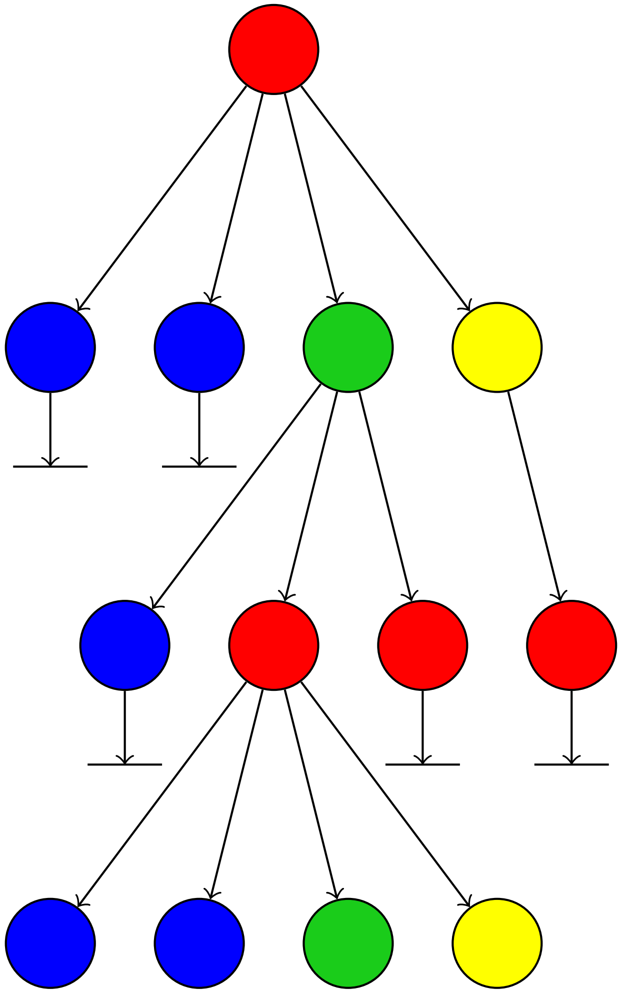

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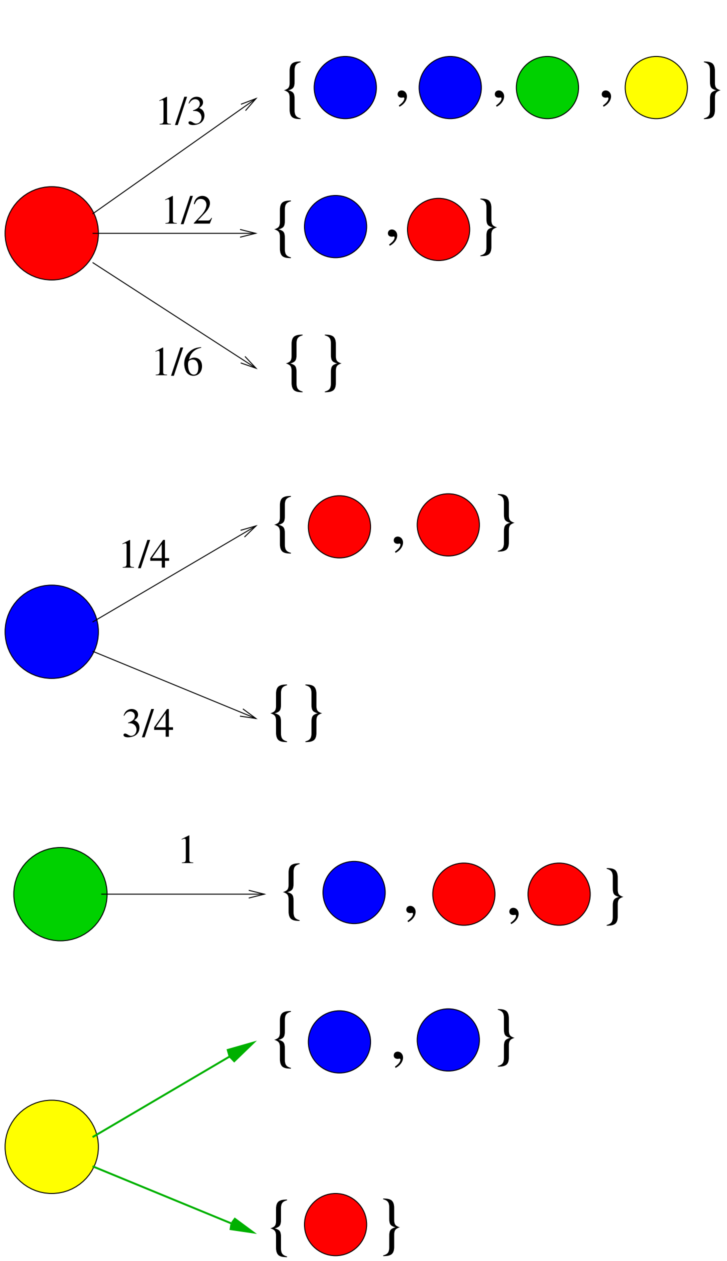

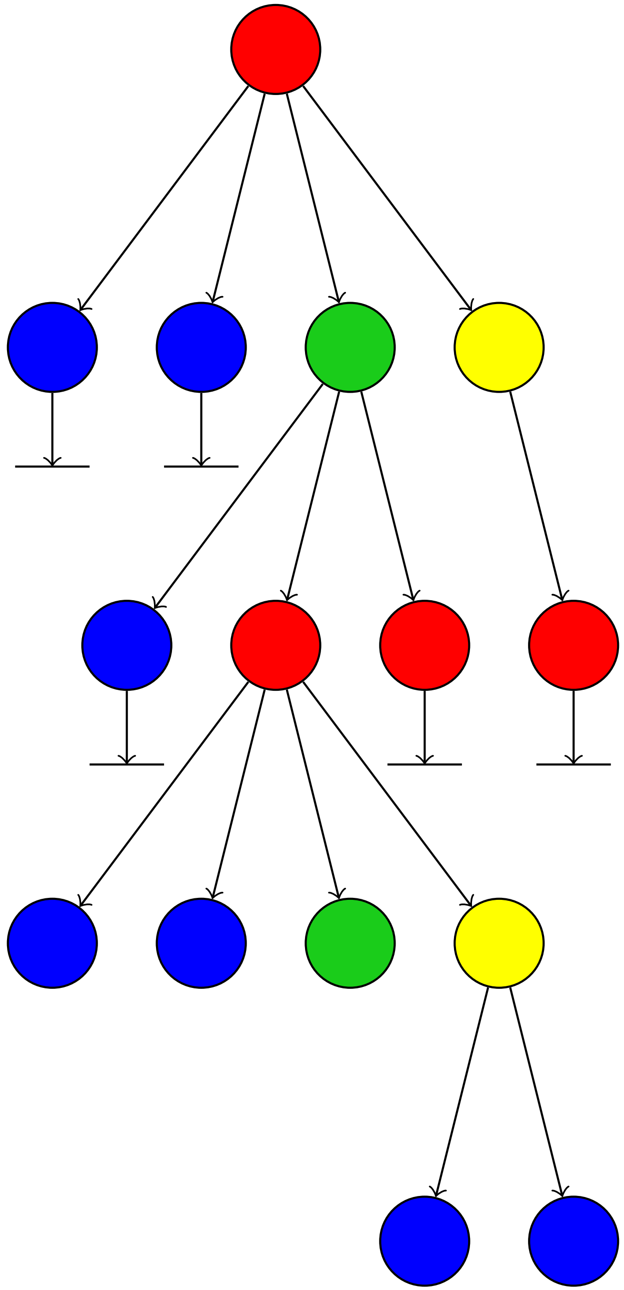

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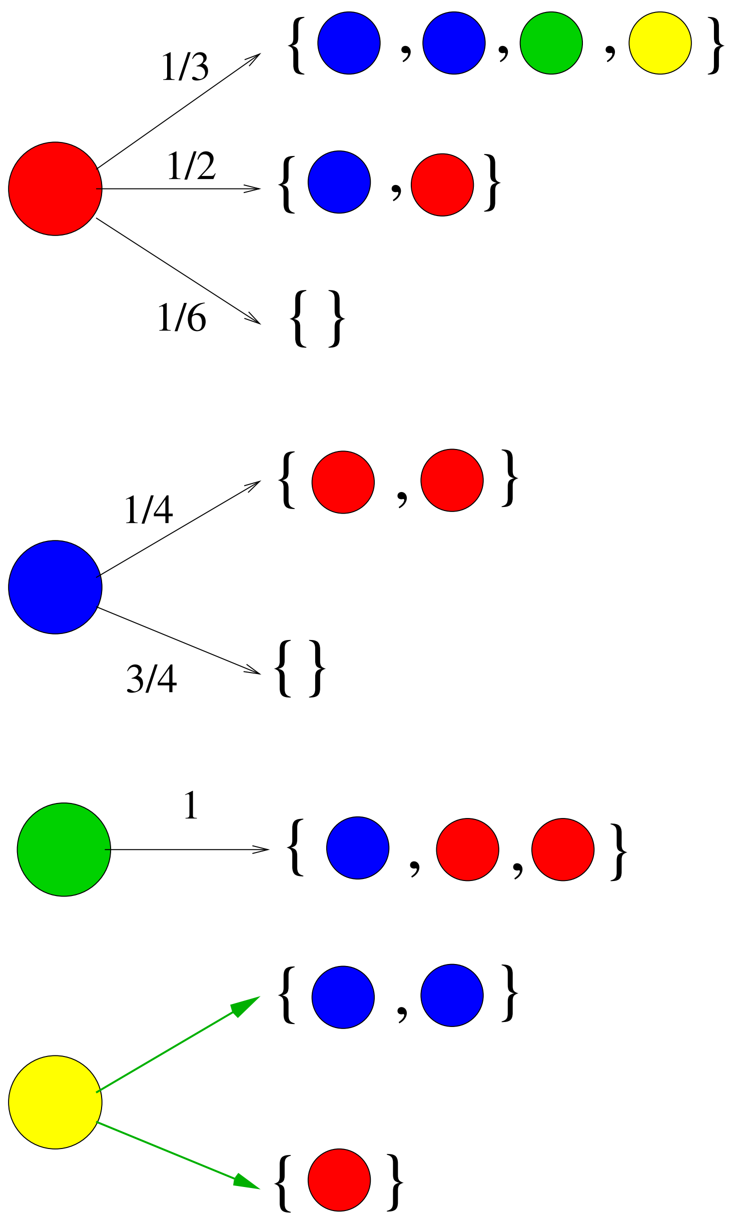
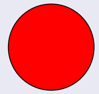
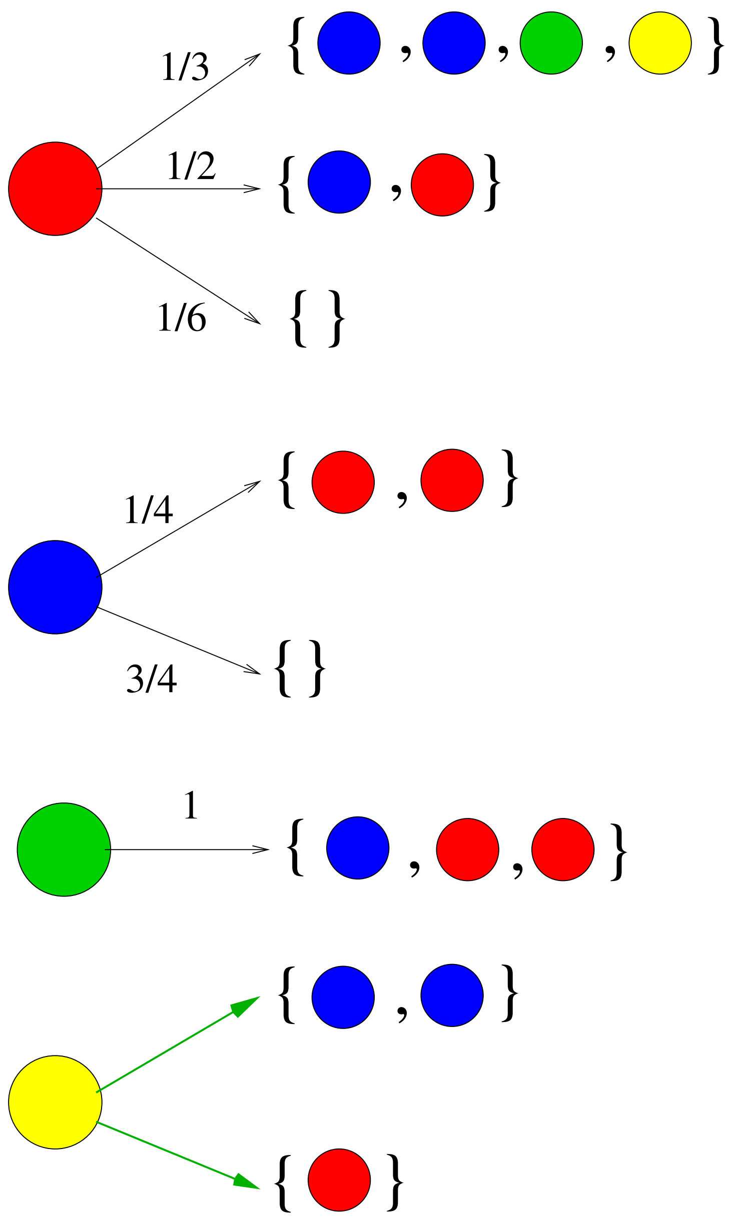


Question

What is the maximum probability of extinction, starting with one  ?

Branching Markov Decision Processes

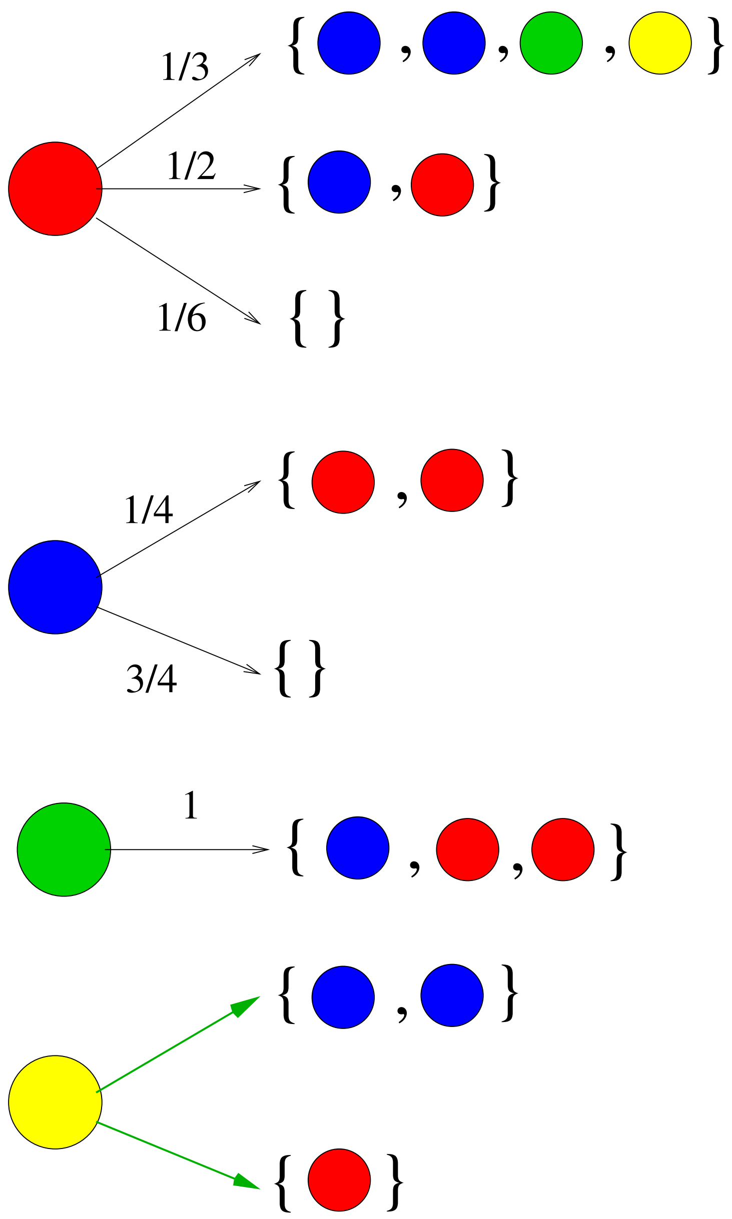


Question

What is the maximum probability of extinction, starting with one ?

$$
\begin{aligned}
x_R &= \frac{1}{3}x_B^2 x_G x_Y + \frac{1}{2}x_B x_R + \frac{1}{6}\\
x_B &= \frac{1}{4}x_R^2 + \frac{3}{4}\\
x_G &= x_B x_R^2\\
x_Y &=
\end{aligned}
$$

Branching Markov Decision Processes



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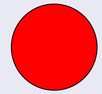
What is the maximum probability of extinction, starting with one  ?

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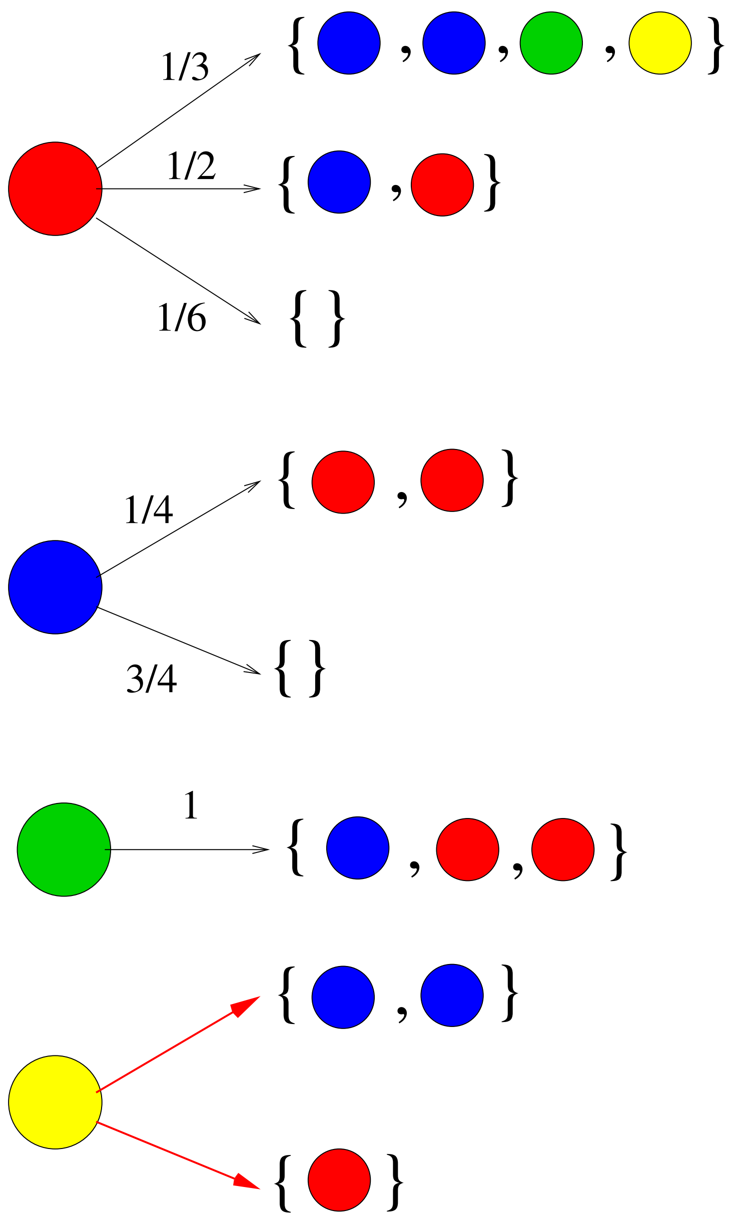
$$x_Y = \max\{x_B^2, x_R\}$$

We get fixed point equations, $\bar{\mathbf{x}} = P(\bar{\mathbf{x}})$.

Fact [E.-Yannakakis'05]

The maximum extinction probabilities are the least fixed point, $\mathbf{q}^* \in [0,1]^3$, of $\bar{\mathbf{x}} = P(\bar{\mathbf{x}})$.

Branching Markov Decision Processes



Question

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$$
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The minimum extinction probabilities are the least fixed point, $\mathbf{q}^* \in [0,1]^3$, of $\bar{\mathbf{x}} = P(\bar{\mathbf{x}})$.

A Maximum Probabilistic Polynomial System (maxPPS) is a system

$$\mathbf{x}_i = \max\{p_{i,j}(\mathbf{x}) : j = 1, \ldots, m_i\} \qquad i = 1, \ldots, n$$

of n equations in n variables, where each $p_{i,j}(x)$ is a probabilistic polynomial. We denote the entire system by:

$$\mathbf{x} = P(\mathbf{x})$$

Minimum Probabilistic Polynomial Systems (minPPSs) are defined similarly.

These are Bellman optimality equations for maximizing (minimizing) extinction probabilities in a BMDP.

We use max/minPPS to refer to either a maxPPS or an minPPS.

Basic properties of max/minPPSs, $\mathbf{x} = P(\mathbf{x})$

$P : [0,1]^n \rightarrow [0,1]^n$ defines a monotone map on $[0,1]^n$.

Proposition. [E.-Yannakakis'05]

- *Every max/minPPS, $x = P(x)$ has a least fixed point, $q^* \in [0,1]^n$.*
- $q^* = \lim_{k \rightarrow \infty} P^k(\mathbf{0})$.
- q^* *is vector of optimal extinction probabilities for the BMDP.*

Question

Can we compute the probabilities q^* efficiently (in P-time)?

P-time approximation for BMDPs and max/minPPSs

Theorem ([E.-Stewart-Yannakakis,ICALP'12])

Given a max/minPPS, $\mathbf{x} = P(\mathbf{x})$, with LFP $\mathbf{q}^* \in [0,1]^n$, we can compute a rational vector $\mathbf{v} \in [0,1]^n$ such that

$$\|\mathbf{v} - \mathbf{q}^*\|_\infty \leq 2^{-j}$$

in time polynomial in the encoding size $|P|$ of the equations, and in j.

We establish this via a Generalized Newton's Method that uses linear programming in each iteration.

Newton iteration as a first-order (Taylor) approximation

An iteration of Newton's method on a PPS, applied on current vector $y \in \mathbb{R}^n$, solves the equation

$$P^{\mathbf{y}}(\mathbf{x}) = \mathbf{x}$$

where $P^{\mathbf{y}}(\mathbf{x}) \equiv P(\mathbf{y}) + P'(\mathbf{y})(\mathbf{x} - \mathbf{y})$ is a linear (first-order Taylor) approximation of $P(x)$.

Generalised Newton's method

Linearisation

Given a maxPPS

$$(P(\mathbf{x}))_i = \max\{p_{i,j}(\mathbf{x}) : j = 1, \ldots, m_i\} \qquad i = 1, \ldots, n$$

We define the linearisation, $P^y(x)$, by:

$$(P^{\mathbf{y}}(\mathbf{x}))_i = \max\{p_{i,j}(\mathbf{y}) + \nabla p_{i,j}(\mathbf{y}).(\mathbf{x} - \mathbf{y}) : j = 1, \ldots, m_i\} \qquad i = 1, \ldots, n$$

Generalised Newton's method

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Generalised Newton's method, applied at vector y

For a maxPPS, minimize $\sum_i x_i$ subject to $P^{\mathbf{y}}(\mathbf{x}) \leq \mathbf{x}$;

For a minPPS, maximize $\sum_i x_i$ subject to $P^{\mathbf{y}}(\mathbf{x}) \geq \mathbf{x}$;

These can both be phrased as linear programming problems. Their optimal solution solves $P^{\mathbf{y}}(\mathbf{x}) = \mathbf{x}$, and yields the GNM iteration we need.

Algorithm for max/minPPSs

1. Find and remove all variables x_i such that $q_i^* = 0$ or $q_i^* = 1$.
   ($q_i^* = 1$ decidable in P-time using LP [E.-Yannakakis'06]: reduces to a spectral radius optimization problem for non-negative square matrices.)

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2. On the resulting system of equations, run Generalized Newton's Method, starting from $\mathbf{0}$. After each iteration, round down to a multiple of 2^{-h}.
   Each iteration of GNM can be computed in P-time by solving an LP.

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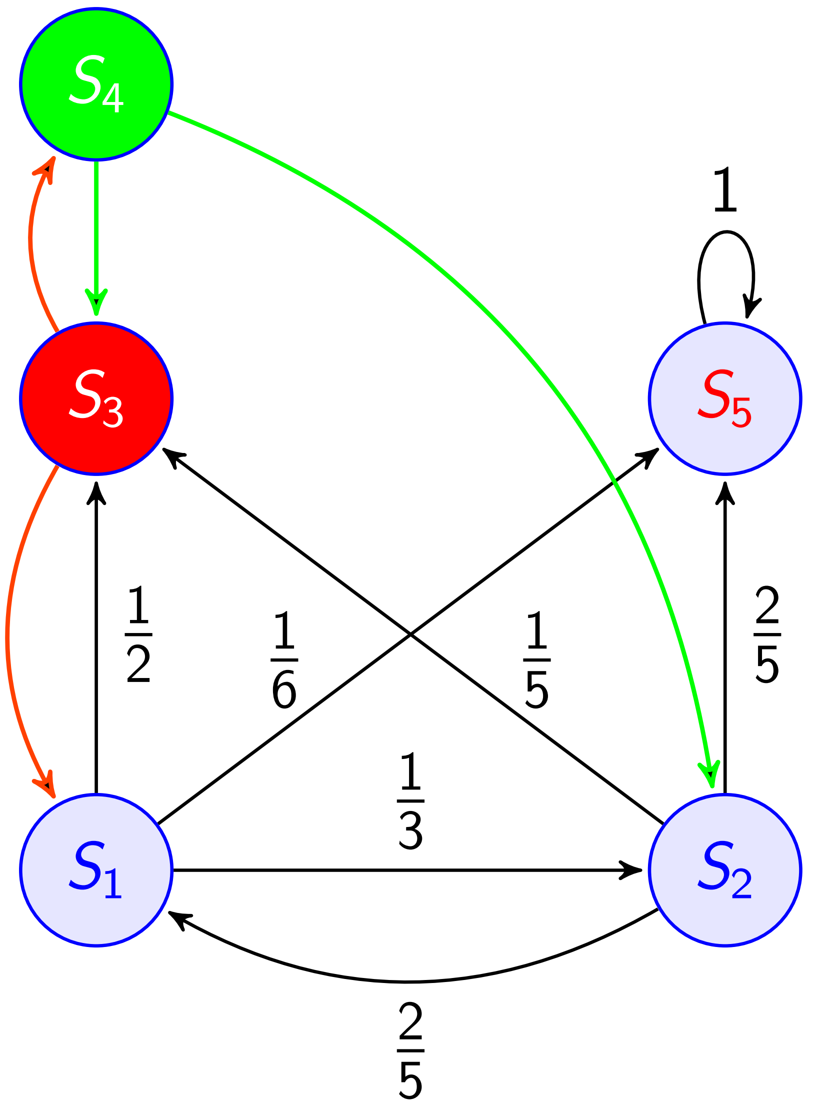
Theorem [ESY'12]

Given a max/minPPS $\mathbf{x} = P(\mathbf{x})$ with LFP $\mathbf{0} < \mathbf{q}^* < \mathbf{1}$, if we apply rounded GNM starting at $\mathbf{x}^{(0)} = \mathbf{0}$, using $h := 4|P| + j + 1$ bits of precision, then

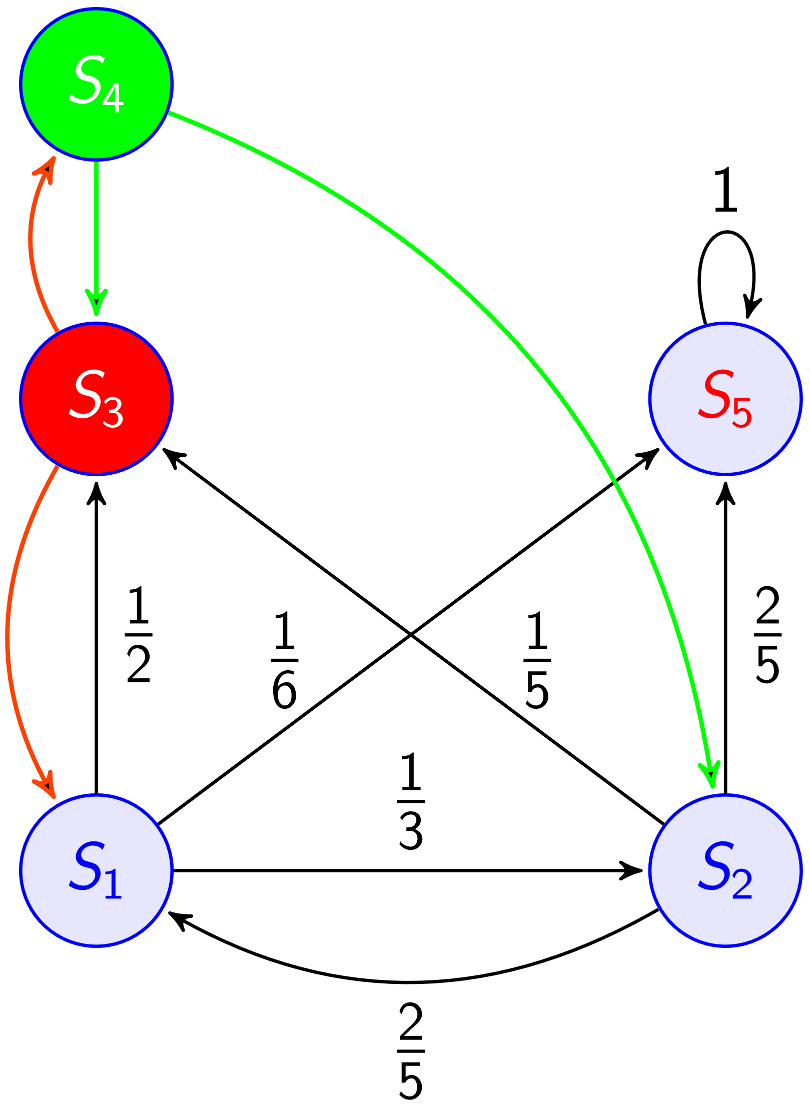
$$\|\mathbf{q}^* - \mathbf{x}^{(4|P|+j+1)}\|_\infty \leq 2^{-j}.$$

We can do all this in time polynomial in $|P|$ and j.

finite-state Simple Stochastic Games



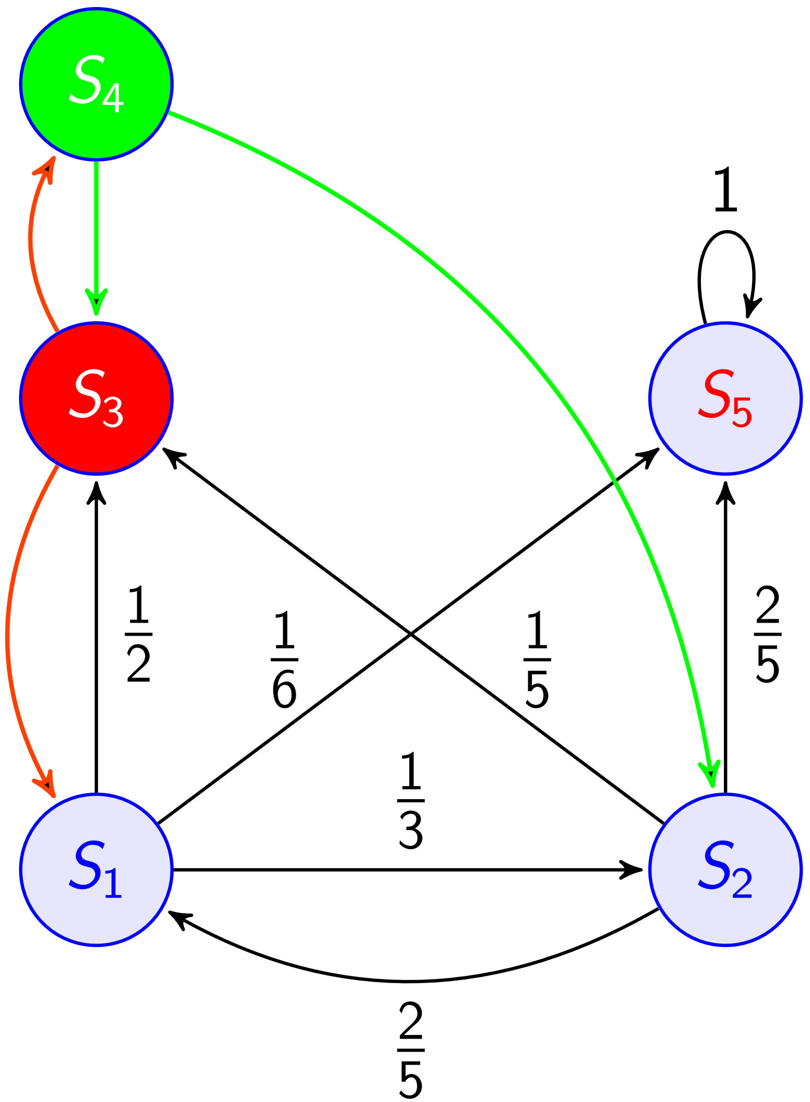
finite-state Simple Stochas



What is the value of the game for hitting S_5 starting at S_1?
(These games are determined.)

$$x_1 \quad =$$

finite-state Simple Stochas



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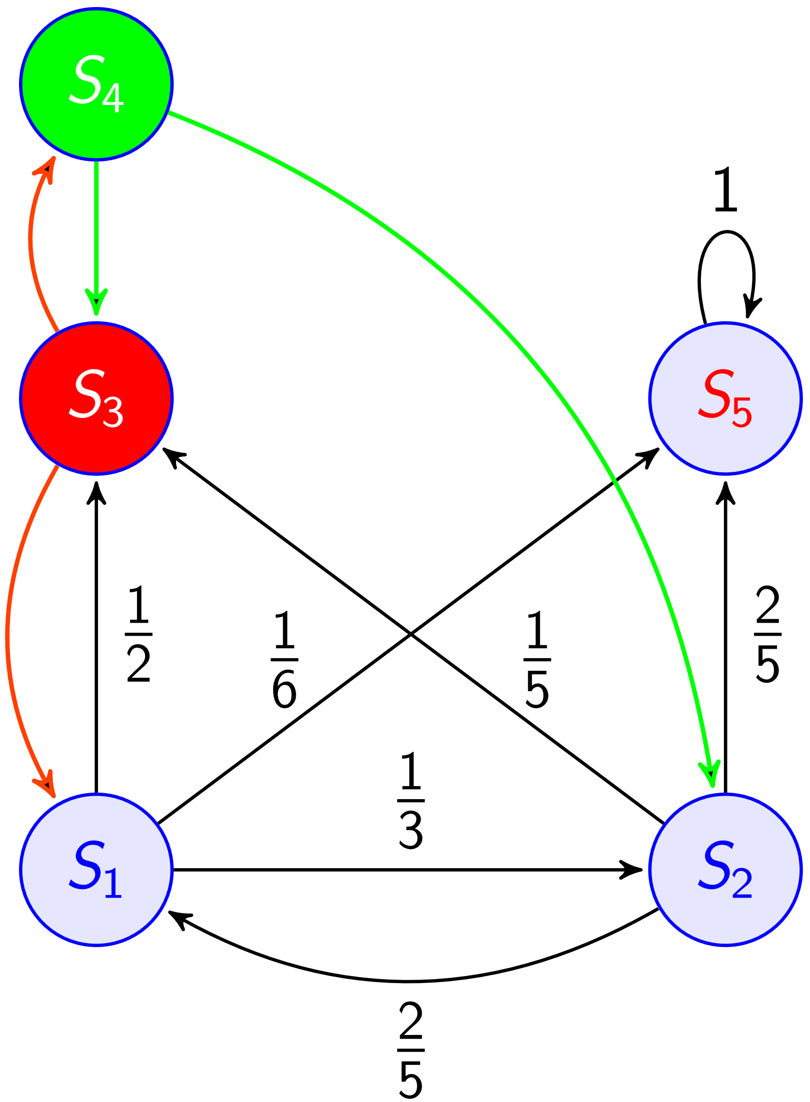

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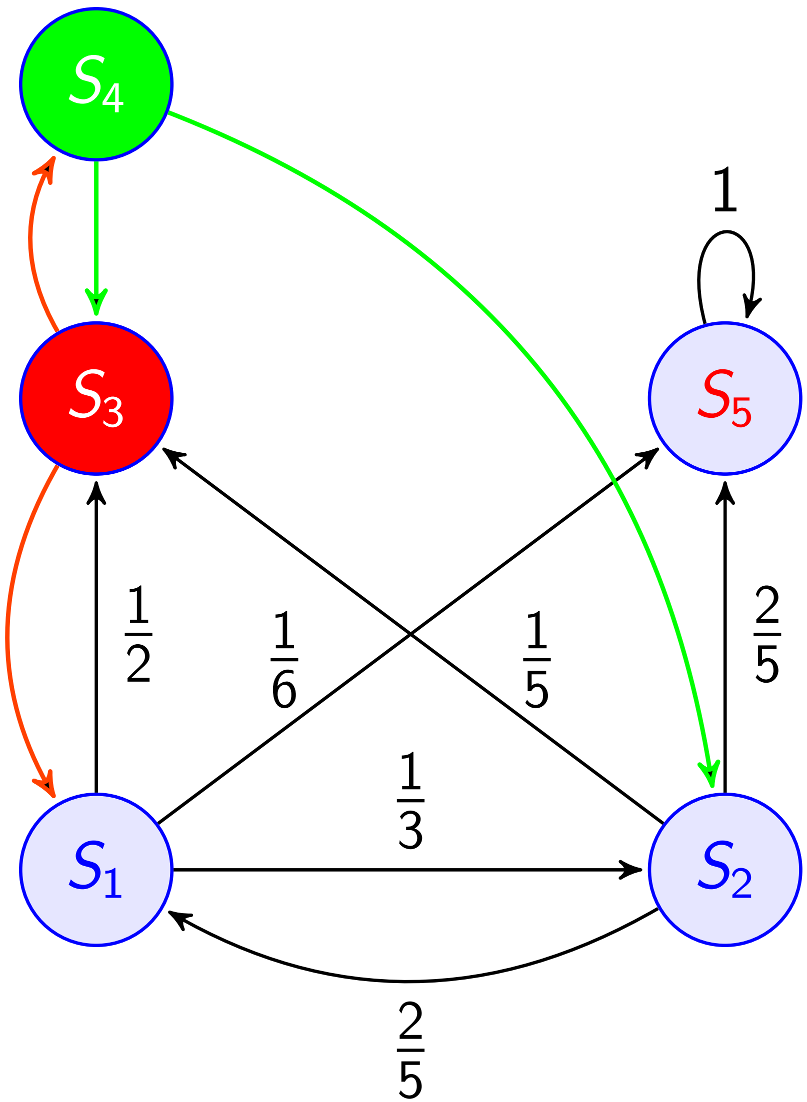
We get linear-min-max equations,

$$\bar{\mathbf{x}} = P(\bar{\mathbf{x}}).$$



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We get linear-min-max equations,

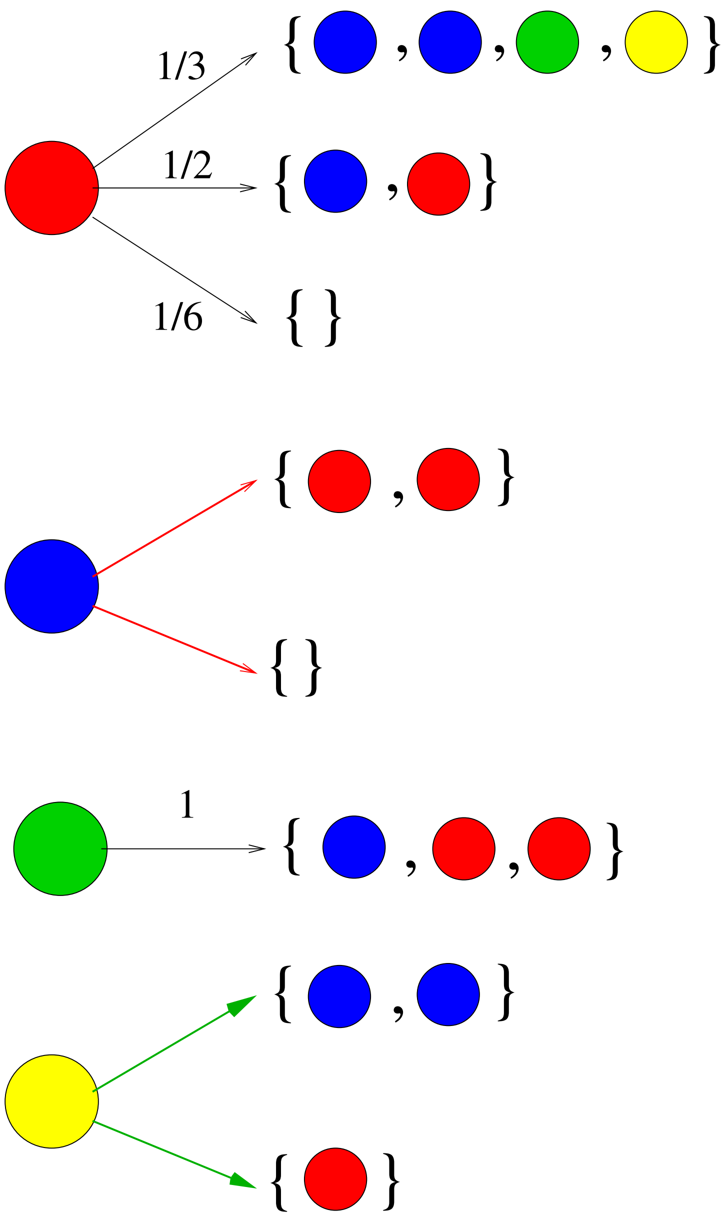
$$\bar{\mathbf{x}} = P(\bar{\mathbf{x}}).$$

Fact: [Shapley'53,Condon'92]
Hitting values are the least fixed point, $q^* \in [0,1]^4$, of $\mathbf{x} = P(\mathbf{x})$.

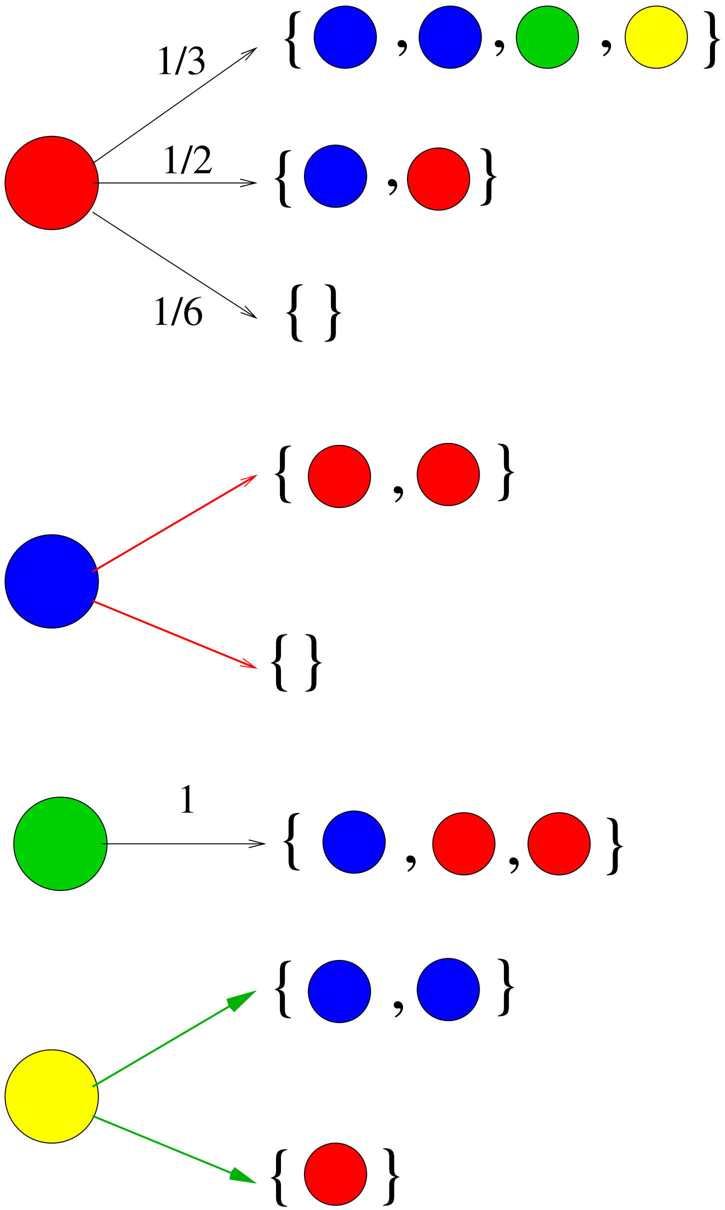
- In any finite-state SSG, both max and min, have optimal positional strategies (i.e., deterministic and memoryless optimal strategies).
- Thus [Condon'92]: deciding whether the game value $q_i^* \leq 1/2$, is in **NP** $\cap$ **coNP**.

  And computing the (exact, rational) values q^* is in **FNP**.
- Long standing open problem whether SSGs are solvable in **P**-time. (Subsumes parity games and mean payoff games.)

Branching Simple Stochastic Games



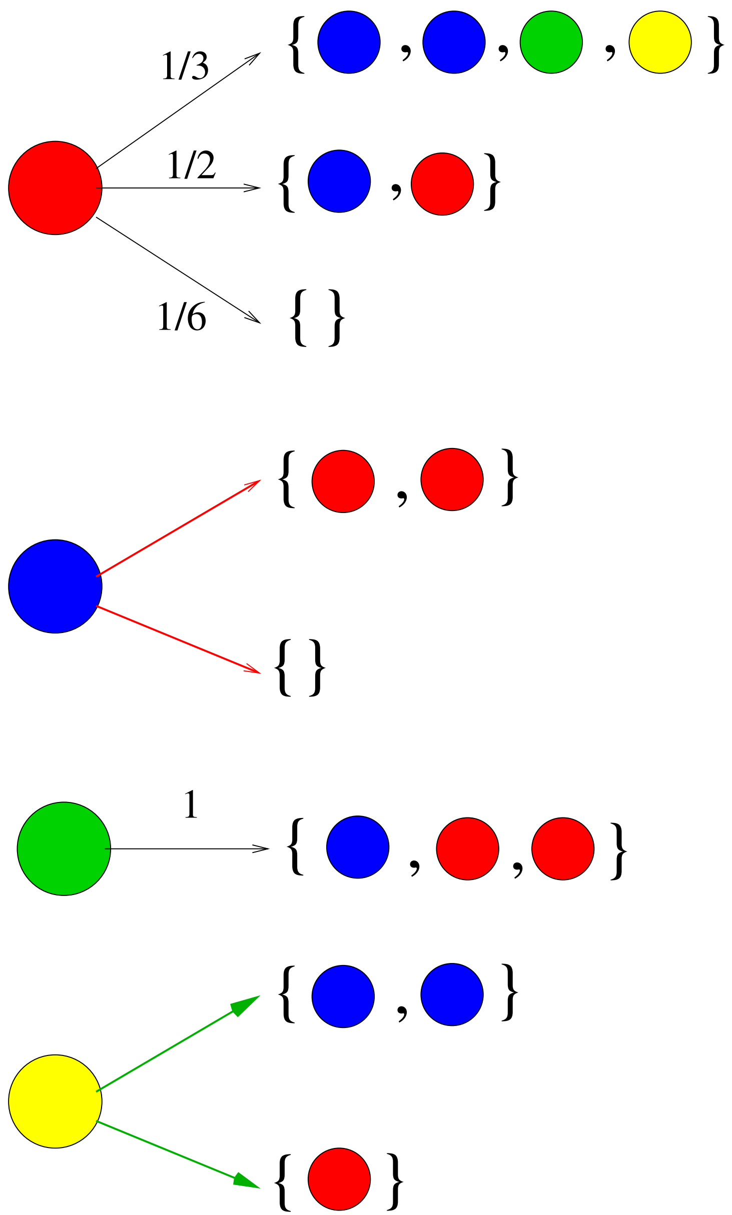
Branching Simple Stochastic Games



Types belonging to min:

Types belonging to max:

Branching Simple Stochastic Games



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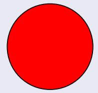
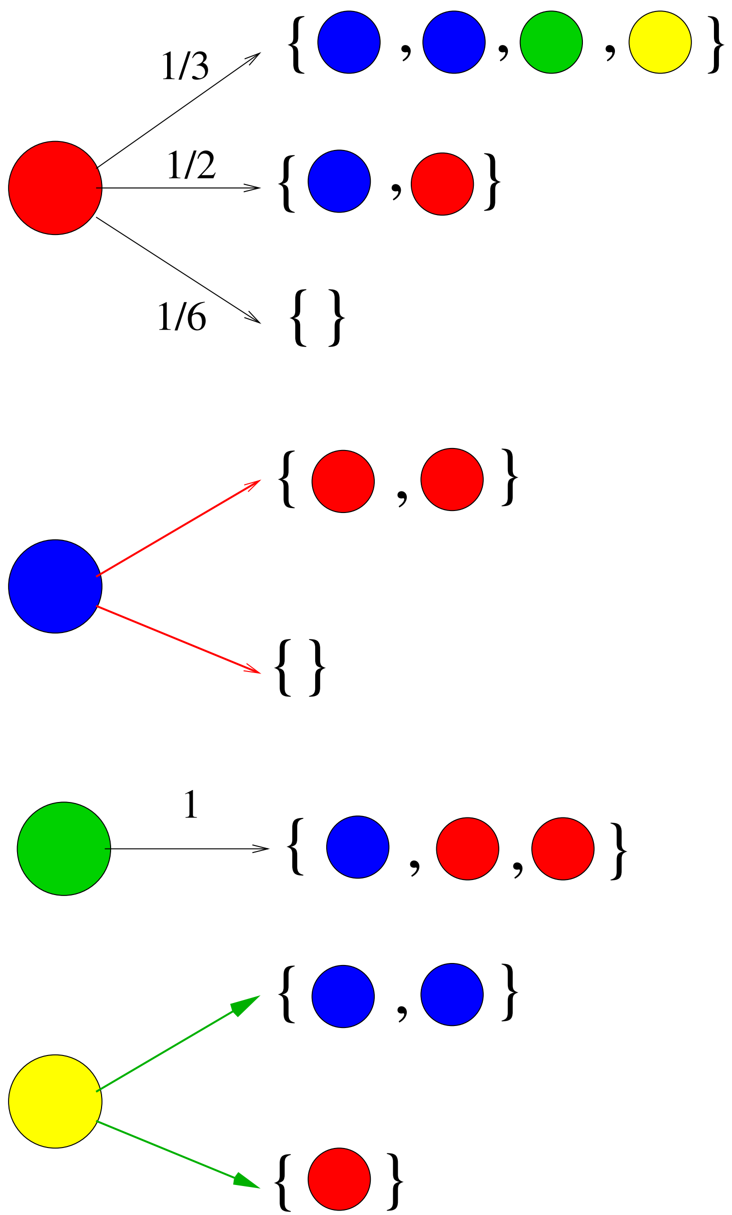
What is the value of extinction, starting with one  ?

Branching Simple Stochastic Games



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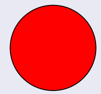
What is the value of extinction, starting with one  ?

$$x_R = \frac{1}{3}x_B^2 x_G x_Y + \frac{1}{2}x_B x_R + \frac{1}{6}$$
$$x_B = \min\{x_R^2, 1\}$$
$$x_G = x_B x_R^2$$
$$x_Y = \max\{x_B^2, x_R\}$$

We get fixed point equations, $\bar{\mathbf{x}} = P(\bar{\mathbf{x}})$.

Fact [E.-Yannakakis'05]

The extinction values are the LFP, $\mathbf{q}^* \in [0,1]^3$ of $\bar{\mathbf{x}} = P(\bar{\mathbf{x}})$.

Qualitative and Quantitative problems for BSSGs

Theorem ([E.-Yannakakis'05])

For any BSSG, both players have static positional optimal strategies for maximizing (minimizing) extinction probability.

A static positional strategy is one that, for every type belonging to the player, always deterministically chooses the same single rule.
(i.e., it is deterministic, memoryless, and "context-oblivious".)

Theorem ([E.-Yannakakis'06])

Given a BSSG, deciding if the extinction value is $q_i^* = 1$ is in **NP** $\cap$ **coNP**, *& is at least as hard as computing the exact value for a finite-state SSG.*

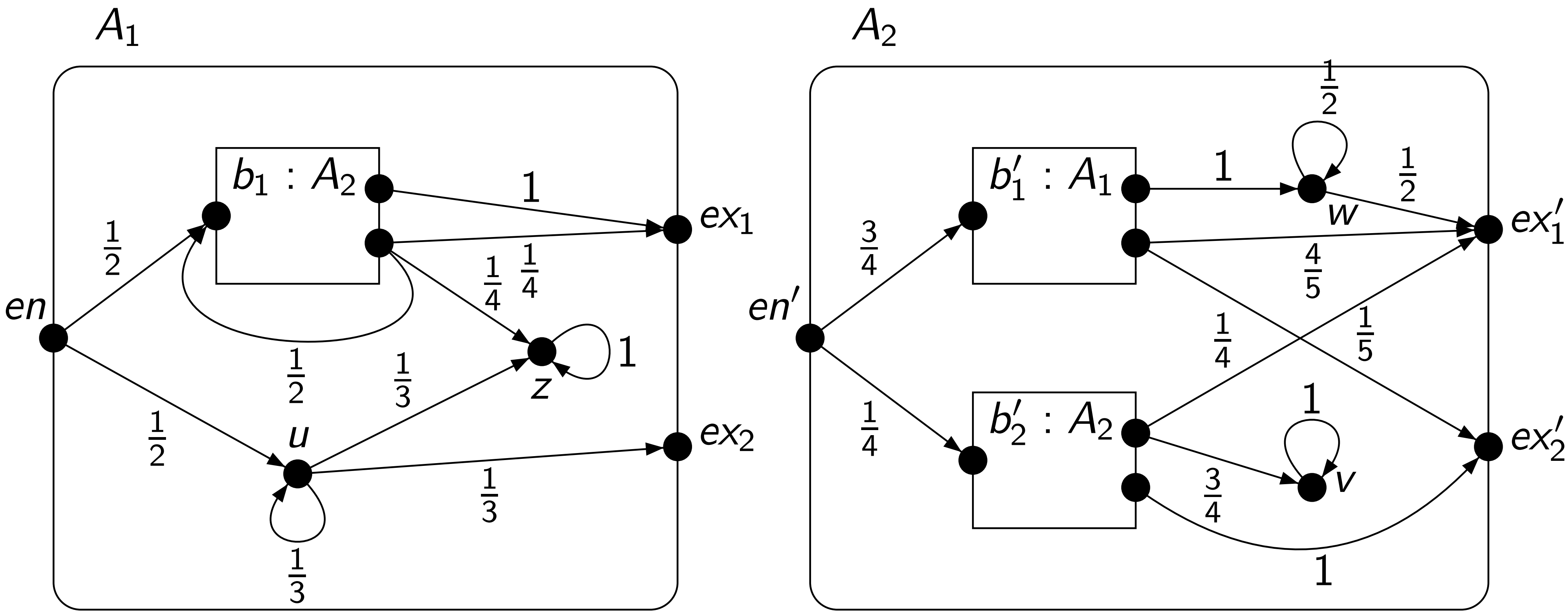
Theorem ([ESY'12])

Given a BSSG, and given $\epsilon > 0$, we can compute a vector $v \in [0,1]^n$, such that $\|v - q^*\|_\infty \leq \epsilon$, in **FNP**.

One piece of a larger story

- Many other analyses: expected total reward, discounted reward, expected limiting average reward, model checking.
- Many analyses require termination probabilities q^* as a prerequisite, but they also require non-trivial additional work.
- Recursive Markov Chains (RMCs) form a more general class of countable infinite-state discrete-time MCs. (BPs and SCFGs correspond to 1-exit RMCs.)

Recursive Markov Chain



- RMCs also have MPSs (not PPSs) whose LFP $q^* \in [0,1]^n$ gives their termination probabilities.
- However, any non-trivial approximation of q^* for RMCs is PosSLP-hard ([E.-Yannakakis'07]).
- For RMDPs and RSSGs any non-trivial approximation of their value vector is uncomputable! ([E.-Yannakakis'05]).

- But other subclasses of RMCs, corresponding to other important stochastic processes, are analyzable.
- 1-box RMCs correspond to (discrete-time) Quasi-Birth-Death processes (QBDs), and to probabilistic one-counter automata (OC-MCs).
- For QBDs we can approximate q^* in P-time ([E.-Wojtczak-Yannakakis'08], [Stewart-E.-Yannakakis'13]).
- Many problems for OC-MDPs and OC-SSGs are also decidable ([Brazdil-Brozek-E.-Kucera-Wojtczak'10,'10,'11]), but for many we don't know good complexity bounds.

Conclusion

- A very rich landscape, with still many open questions.
- Can we solve finite-state SSGs in **P**-time?
- Can we obtain any better upper bounds for PosSLP??
- Deciding $q^* \geq 1/2$ for Branching SSGs subsumes both of these problems.