(3) To obtain the well known results about the "area" function of Littlewood Paley and its generalizations, one considers the operator valued function $k(x)$ which transforms the complex number a into the function $t^{-n-1/2}\varphi(x/t - z)$ of t and z, $0 < t < \infty$, $|z| \leqslant R$, where φ is the function described in (2). Then as in (2), we find that if

$$g(x) = \left[\int_0^\infty dt \int_{|z|<R} dz \left| \int t^{-n-1/2} \varphi\left(\frac{x-y}{t} - z\right) f(y)dy \right|^2\right]^{1/2}$$

then $\|g\|_P \leqslant c_P \|f\|_P$, $P = (p_1, \ldots . p_n)$, $1 < p_i < \infty$.

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† See the papers quoted below.

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*ENTSCHEIDUNGSPROBLEM REDUCED TO THE ∀∃∀ CASE**

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1. *The Main Theorem and Its Immediate Consequences.*—By an ∀∃∀ formula, we mean a formula of the predicate calculus of the form $\forall x \exists u \forall y Mxuy$, where $Mxuy$ is quantifier-free and contains neither the equality sign nor function symbols. By a restricted ∀∃∀ formula is meant an ∀∃∀ formula which contains only dyadic predicate letters and, for each dyadic G_i, only basic components of the forms G_ixy, G_iyx, G_iuy, G_iyu. By the decision problem of the (restricted) ∀∃∀ case, we mean the problem of finding a general algorithm to decide, for each given (restricted) ∀∃∀ formula, whether it is satisfiable (as always, in some nonempty domain).

MAIN THEOREM (this contains two parts). *MTa: The decision problem of the restricted* ∀∃∀ *case is unsolvable. MTb: The class of restricted* ∀∃∀ *formulas is a reduction type; in other words, there is a general algorithm by which each formula F of the predicate calculus, including the equality sign and function symbols, can be transformed into a restricted* ∀∃∀ *formula F^* such that F is satisfiable if and only if F^* is.*

As it happens, the proof of MTa and MTb can be modified to give slightly stronger results which deal with even narrower classes. In the specification of the class of restricted $\forall\exists\forall$ formulas, if we delete exactly one of the four forms of the basic components we get four subclasses of the class which will be called the distinguished subcases. The following is proved:

EXTENDED MAIN THEOREM. *If we use, instead of the restricted $\forall\exists\forall$ case, any of its four distinguished subcases, MTa and MTb remain true.*

It may be noted that we are reducing the predicate calculus in the broad sense to the special subcases of the $\forall\exists\forall$ case, thereby bringing out the full strength of the reduction.

The result yields a complete solution of the decision and the reduction problems of the predicate calculus so far as the classes are determined solely by the prefix forms of all formulas in the prenex normal form. The additional restrictions on the $\forall\exists\forall$ formulas touch on the further question of classifying formulas according to finer structures of the quantifiers and the quantifier-free matrices.

The problem of solving the $\forall\exists\forall$ case has, since the 30's, been mentioned implicitly or explicitly in various connections.[1] It is interesting for at least two reasons: on the one hand, the prefix appears to be simpler than many of the solvable cases; on the other hand, in common with the other unsolvable cases, it contains satisfiable formulas which are not satisfiable in any finite domains. Formally speaking, it is surprising that such a simple case turns out to be unsolvable.

In order to deduce certain immediate consequences from the Main Theorem, we bring together a few familiar facts.

By the method[2] of reducing every formula to one in the Skolem normal form, given any formula F of the form $\forall x\exists u\forall yMxuy$, we get a corresponding formula $\forall x[\forall u\forall y(Gxu \supset Mxuy)\ \&\ \exists zGxz]$, G not occurring in F, which is satisfiable if and only if F is. Since the new formula has a quantifier structure $\forall\exists\ \&\ \forall\forall\forall$ which can, in particular, be changed to $\forall\forall\forall\exists$ or $\forall\exists\forall\forall$ or $\forall\forall\exists\forall$, we have, by *MTb:*

COROLLARY 1. *The class of formulas of the form $\forall x(\forall u\forall yKxuy\ \&\ \exists zLxz)$ (or of the form $\forall x\forall u\forall z\exists yMxuzy$, etc.), each of which contains only dyadic predicates and, for each G_i, only basic components G_ixy, G_iyx, G_iuy, G_iyu, G_ixu, G_ixz, forms a reduction type; in fact, the same is true if we exclude basic components of exactly one of the first four forms.*

Given any formula F in the prenex normal form, we can insert any quantifiers into the prefix and add some trivial condition to get a formula F^* such that F is satisfiable if and only if F^* is. For example, if F is $\forall x\exists u\forall yMxuy$ and we wish to insert $\exists z$ after $\exists u$, we can take $\forall x\exists u\exists z\forall y[(Gz \vee \sim Gz)\ \&\ Mxuy]$ as F^*. In general, we have:

1.1. If a given prefix form class is decidable, then every prefix form class obtained by deletion of certain quantifiers from the given string is decidable; if a class is undecidable, any class obtained by inserting quantifiers is undecidable.

The following solvable cases are well-known:[3]

1.2. The class with prefix $\exists\ldots\exists\forall\ldots\forall$ and that with $\exists\ldots\exists\forall\forall\exists\ldots\exists$ are decidable. Moreover, each formula in these classes has the property $\emptyset$: if it is satisfiable, it is satisfiable in some finite domains.

From these we get, by *MTa* and *MTb:*

COROLLARY 2. *A prefix form class is a reduction type (and unsolvable) if and only if the prefix contains $\forall\exists\forall$ or $\forall\forall\forall\exists$ as an (order-preserving but not necessarily consecutive) substring.*

On the other hand, it is known that[4] there is an axiom of infinity, i.e., a formula not having the property ϕ, of the $\forall\exists\forall$ form. Hence, we have:

COROLLARY 3. *A prefix class is decidable if and only if every formula in it has the property* ϕ.

Schütte's formula is $\forall x \exists u \forall y[\sim Gxx \;\&\; Gxu \;\&\; (Guy \supset Gxy)]$. We can also get other axioms of infinity in which Gxx and Gxu do not occur, e.g., $\forall x\exists u\forall y[(Gyx \supset \sim Gxy) \;\&\; (\sim Gxy \supset Gyu)]$ and $\forall x\exists u\forall y[(Guy \supset \sim Gyu) \;\&\; (\sim Gxy \supset Gyu)]$. In fact, both formulas are satisfied by $<$ for G, and have no finite models because they imply Schütte's formula or a slight variant of it.

There is a standard argument to show that each unsolvable class (and therefore each reduction type) must contain some axiom of infinity: if every formula in it had the property ϕ, we would be able to solve its decision problem by testing each formula F successively for each number n, whether F has a refutation of length n or a finite model of size n. The interesting question of classifying axioms of infinity is wide open.

2. *Proof of the Main Theorem and the Domino Problems.*—The argument begins from the known fact that the halting problem for Turing machines is unsolvable and derives from this the unsolvability of a certain "domino problem" from which the unsolvability of the restricted $\forall\exists\forall$ case is deduced. The two steps are such that we have algorithms for getting a domino set from any Turing machine and a restricted $\forall\exists\forall$ formula from any domino set. Hence, assuming a decision procedure for the restricted $\forall\exists\forall$ case, we can retrace the constructions and settle the halting problem for all Turing machines. In this way, MTa can be proved. Since it is known that the decision problem for the whole predicate calculus can, by recursive function theory, similarly be reduced to the halting problem, MTb follows directly from MTa.[5]

A more informative derivation of MTb from MTa can be obtained from the "fundamental theorem of logic" due to Skolem, Herbrand, Gödel:[6]

2.1. There is an algorithm by which, given any formula F of the predicate calculus, we can find an infinite sequence of truth-functional formulas $F_1, F_2, \ldots,$ such that F is satisfiable if and only if every finite conjunction $F_1 \;\&\; \ldots \;\&\; F_n$ $(n = 1,2,\ldots)$ is satisfiable.

Since each finite conjunction can be tested for satisfiability (consistency) by, say, the truth table method, we can find a Turing machine M_F for each formula F such that F is satisfiable if and only if M_F never halts. All we need is to make M_F generate and test each finite conjunction and halt when one conjunction is found to be contradictory. Since we have a general method of finding M_F for each F, we can adjoin the method to the methods for reducing the halting problem of each Turing machine to the satisfiability of a restricted $\forall\exists\forall$ formula and derive MTb from MTa.[7]

In the usual formulation of 2.1, the equality sign and function symbols are not included in the predicate calculus. But 2.1 can be extended to include these. Alternatively, one can first eliminate the equality sign and function symbols by standard methods.[8]

Hence, we have to prove only MTa.

We observe first that in speaking of unsolvability we are as usual following Turing in assuming that any solvable problem is solvable by some Turing machine. For those who do not wish to accept this assumption, "unsolvability" should be replaced by "unsolvability by Turing machines" throughout.

Turing's original proof of the unsolvability of the general decision problem of the predicate calculus is by using an unsolvable printing problem on Turing machines.[9] It is observed that indeed the $\exists\forall\exists\forall\forall\forall\forall$ case is unsolvable because the printing problem for each machine can be expressed by a formula of such a form. Much of the complexity of the formula is caused by conditions to ensure properties of natural numbers.

Very recently, Büchi[10] achieved great simplifications by observing that we can delete these additional conditions by using the Skolem or Herbrand functions involved in the fundamental theorem of logic. Our proof of MTa depends heavily on this remark of Büchi's. For our purpose, it is sufficient to use the following special consequence of the fundamental theorem of logic:[11]

LEMMA 1. *A formula $\forall x \exists u \forall y Mxuy$, in which M is quantifier-free and contains neither the equality sign nor function symbols, is satisfiable if and only if $\forall x \forall y Mxx'y$, x' being short for $x + 1$, is satisfiable in the domain of natural numbers.*

While this lemma is well-known, Büchi's application of it is a new step that leads to a better understanding of the undecidability phenomenon in the predicate calculus. In this sense, Büchi may be said to have invented a new approach to the Entscheidungsproblem.

In obtaining our results from Lemma 1, the principal new tool we use is certain concepts on the domino problems.[12] Since these problems are central to our particular proof, we explain the notions in detail.

Assume we are given a finite set of square plates (the dominoes) of the same size (say, all of the unit area) with edges colored, each domino in a different manner. Suppose further there are infinitely many copies of each domino (domino type). We are not permitted to rotate or reflect a domino. Suppose now we wish to cover up the whole first quadrant of the infinite plane with such dominoes so that all corners fall on the lattice points.

2.2. A (finite, as always in this paper) set of dominoes is said to be solvable if and only if there is some way of covering the whole first quadrant by dominoes from the set so that any two adjoining edges always have the same color.

Using the ordinary Cartesian coordinates, we shall represent each unit square in the first quadrant by the coordinates of its upper right corner, and call the square (1,1) the origin. We have considered four domino problems.

2.3. The (unrestricted) domino problem. To find an algorithm to decide, for any given set of dominoes, whether it is solvable.

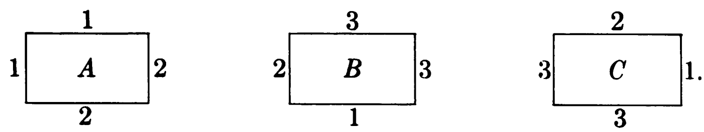
2.4 The origin-constrained (diagonal-constrained, row-constrained) domino problem. To decide, for any given set P of dominoes and a subset Q thereof, whether P has a solution with the origin (the main diagonal, the first row) occupied by a domino (dominoes) belonging to Q.

In the spring of 1960, we attacked the unrestricted problem without success but found that the origin-constrained problem is unsolvable. Recently, we noticed that, by Büchi's remark (Lemma 1 above), we can then infer from this that the $\exists$ & $\forall\exists\forall$ class is a reduction type, thereby providing an alternative proof of a result of Büchi's. Furthermore, it turned out that the construction can be improved to establish the unsolvability of the row-constrained problem from which by a rotation we got the unsolvability of the diagonal-constrained problem for which we finally obtained a direct proof. From this last result, we can derive MTa by the following lemma.

LEMMA 2. *If the diagonal-constrained domino problem is unsolvable, then the decision problem of the restricted* $\forall\exists\forall$ *case is unsolvable.*

In fact, we have a general procedure by which, given any set P of dominoes and a subset Q thereof, we can find a formula $\forall x \forall y Mxx'y$, where $\forall x \exists u \forall y Mxuy$ is a restricted $\forall\exists\forall$ formula, such that P has a diagonal-constrained solution relative to Q if and only if $\forall x \forall y Mxx'y$ is satisfiable in the domain of natural numbers. Hence, by Lemma 1 we get Lemma 2.

We explain the general procedure by a simple example:



Let $P = \{A,B,C\}$, $Q = \{A\}$. Then we say that P has a diagonal-constrained solution relative to Q if and only if the following formula F is satisfiable in the domain of natural numbers, i.e., there exist dyadic relations A,B,C of natural numbers such that F is true:

$$(F)\quad \forall x \forall y \{Axx \;\&\; [\sim(Axy \;\&\; Bxy) \;\&\; \sim(Bxy \;\&\; Cxy) \;\&\; \sim(Axy \;\&\; Cxy)] \;\&\; [(Axy \;\&\; Bx'y) \vee (Bxy \;\&\; Cx'y) \vee (Cxy \;\&\; Ax'y)] \;\&\; [(Ayx \;\&\; Byx') \vee (Byx \;\&\; Cyx') \vee (Cyx \;\&\; Ayx')]\}.$$

The clause Axx is for the constraint on the diagonal (i.e., the positions with coordinates of the form (x,x)). The next clause ensures that no position is occupied by two different dominoes. The third clause ensures that horizontally the adjacent edges have the same color. The last clause does the same for the vertical situation.

This is almost the desired formula except that we have not eliminated the diagonal expression Axx. To achieve the removal, we prove the following general statement:

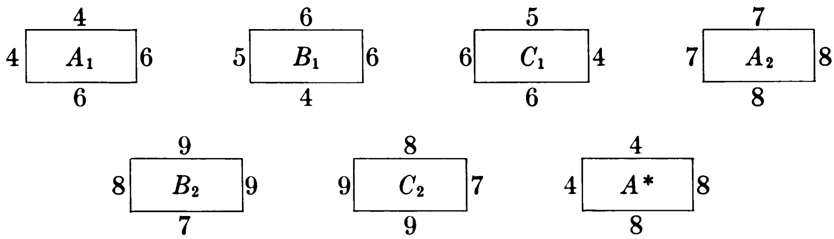
2.5. Given any set P and a subset Q of it, we can find two sets $P^* = \{A_1,\ldots, A_m, B_1,\ldots,B_n\}$ and $Q^* = \{A_1,\ldots,A_m\}$ such that P has a diagonal-constrained solution relative to Q if and only if P* has a diagonal-constrained solution relative to Q^* such that $\forall x \forall y[(B_1xy \supset \sim B_1yx) \;\&\; \ldots \;\&\; (B_nxy \supset \sim B_nyx)]$, i.e., the same domino never occurs at both (x, y) and (y, x).

We split every color i appearing in some domino of P into two colors i_a and i_b and obtain P_1 from P by substituting i_a for color i everywhere, P_2 from P by substituting i_b for i. Then, P^* is the union of P_1, P_2, and Q^*, which is obtained from Q by substituting in each domino every i_a for i on the top and left edges, i_b for i on the bottom and right edges. It is then clear that given any solution of P with diagonal-constraint in Q, we get for one P^* with constraint in Q^* with the desired property; and conversely that every constrained solution of P^* gives one of P.

For this transformed pair of sets P^* and Q^*, we can express the diagonal-constraint by $[(B_1xy \supset \sim B_1yx) \;\&\; \ldots \;\&\; (B_nxy \supset \sim B_nyx)]$ instead of $[A_1xx \vee \ldots$

$\vee A_mxx]$. Clearly the new condition excludes the possibility of any of the B's occurring on the diagonal. At the same time, by 2.5, no diagonal-constrained solutions are lost on account of the additional strength of the new condition.

Thus, in the example given, we can replace the pieces A, B, C by (taking $i_a = 3 + i$, $i_b = 6 + i$):



$P^* = \{A^*, A_1, B_1, C_1, A_2, B_2, C_2\}$ and $Q^* = \{A^*\}$.

We can then write a formula like F for this set except that instead of writing A^*xx, we write the longer condition: $[(A_1xy \supset \sim A_1yx) \& \ldots \& (C_2xy \supset \sim C_2yx)]$. This new formula is of the form $\forall x\forall yMxx'y$ whose corresponding formula (by Lemma 1) $\forall x\exists u\forall yMxuy$ is a restricted $\forall\exists\forall$ formula. Hence, the proof of Lemma 2 is completed.

Therefore, in order to complete the proof of MTa and therewith that of our Main Theorem, we have to prove only the unsolvability of the diagonal-constrained domino problem.

For the purpose of obtaining also the Extended Main Theorem, it is necessary to prove the unsolvability in a special form. We introduce here a few needed concepts and establish some obvious properties.

2.6. For any domino $A = (a,b,c,d)$ we shall call $\breve{A} = (d,c,b,a)$ its mirror image:

2.7. Given any set $P = \{A_1,\ldots,A_m, B_1,\ldots,B_n\}$ and a subset $Q = \{A_1,\ldots,A_m\}$ the pair (P,Q) is said to have the mirror property if:

(i) $\forall i(A_i = \breve{A}_i)$,

(ii) No A_i can occur below the main diagonal; e.g., no color on the right (bottom) of any B_j or A_k occurs on the left (top) of A_i,

(iii) For all k and j, no color which occurs on the top or bottom of some B_k occurs on the left or right edge of any B_j; hence, in particular $\forall k\forall j(B_k \neq \breve{B}_j)$.

$P^* = P \cup \{\breve{B}_1, \ldots, \breve{B}_n\}$ is called the mirror extension of P.

2.8. If (P,Q) has the mirror property and P^* is the mirror extension of P, then P has a diagonal-constrained solution over the first octant relative to Q if and only if P^* has a diagonal-constrained solution over the whole first quadrant relative to Q with the property that, for all j, $B_jxy \equiv \breve{B}_jyx$.

This is obvious from 2.7. On the one hand, given a solution of P in the first octant, we can fill up the second octant with mirror images (reflections about the main diagonal). On the other hand, if P^* has any solution over the whole quadrant, the first octant can contain only members of P.

The symmetric relation relative to mirror images enables us to eliminate any one of the component forms G_ixy, G_iyx, $G_ix'y$, G_iyx' in stating the condition that P^* has a solution with diagonal-constraint on Q:

2.9. If (P,Q) has the mirror property and P^* is the mirror extension of P, then there is a formula F in each of the four distinguished subclasses of the class of restricted $\forall\exists\forall$ formulas such that P^* has a diagonal-constrained solution relative to Q if and only if F is satisfiable.

Consider first the subclass obtained by deleting G_iyx'. We can take as F the conjunction of the following conditions:

(i) $\forall x \forall y[(A_1xy \equiv \breve{A}_1yx) \& \ldots \& (A_mxy \equiv \breve{A}_myx) \& (B_1xy \equiv \breve{B}_1yx) \& \ldots \& (B_nxy \equiv \breve{B}_nyx)]$,

(ii) $\forall x \forall y[(B_1xy \supset \sim B_1yx) \& \ldots \& (B_nxy \supset \sim B_nyx) \& (\breve{B}_1xy \supset \sim \breve{B}_1yx) \& \ldots \& (\breve{B}_nxy \supset \sim \breve{B}_nyx)]$,

(iii) The condition that no two dominoes occupy the same square,

(iv) The horizontal requirement using G_ixy and $G_ix'y$.

The vertical requirement becomes redundant because it can be derived from (i) and (iv). In other words, by (i), the horizontal requirement on one half of the quadrant ensures the vertical requirement on the other half.

Similarly, we can dispose of the subclass obtained by deleting $G_ix'y$.

To deal with the subclass obtained by deleting G_iyx, we observe that in conditions (i), (ii), a nd (iii), we have to use $G_ix'y$ and G_iyx' in place of G_ixy and G_iyx. As a result, the formula F imposes no restriction on the column with $x = 1$. If, however, F is satisfiable and therefore P^* has such a pseudo solution, we can cut off the first column and the first row and get a solution. Conversely, if we have a solution, then we certainly have such a pseudo solution since less is required, and therefore F is satisfiable.

Similarly, we can deal with the one remaining subclass obtained by deleting G_ixy.

In view of these considerations and the fact that the second half of the Extended Main Theorem follows from the first half as MTb from MTa, all we have to do to prove both theorems is to establish the following lemma:

LEMMA 3. *The unsolvability of the diagonal-constrained domino problem: we give a general procedure by which, given any Turing machine Z and a fixed-initial state q_1, we can find a pair (P,Q) of domino sets having the mirror property such that, beginning with a blank tape in the state q_1, Z halts eventually if and only if P has no diagonal-constrained solution in half of the first quadrant relative to Q.*

3. *Unsolvability of the Diagonal-Constrained Domino Problem: Proof of Lemma 3.*—Without loss of generality, we may modify the definition of Davis[13] for a Turing machine, so that we start the machine at the beginning of a one-way infinite blank tape, and we assume that the machine halts if it ever goes left from the beginning of the tape. Otherwise, our notation and definition follow Davis. In particular, each machine has a finite number of states (internal configurations) $q_1, \ldots, q_m$ and is capable of printing a finite number of symbols $S_0, S_1, \ldots, S_k$ of which S_0 is taken

as the symbol for a blank; the symbols R and L represent a move of the read-write head of one square to the right and to the left, respectively.

By the halting problem is meant specifically the problem of deciding, for each Turing machine, when beginning in a given state with a blank tape and the head scanning the leftmost square, whether it will eventually halt. The unsolvability of a stronger form of this problem is familiar from the literature.[14] For our purpose, it is sufficient to prove by a direct diagonal argument that the halting problem with any number input (i.e., a finite representation of some natural number on the initial tape) is unsolvable and then observe that the behavior of each machine with a fixed number input can be simulated separately by a machine beginning with a blank tape.

It is relatively easy to prove the unsolvability of the origin-constrained domino problem. Essentially, for each Turing machine Z, we specify a set of dominoes with a distinguished piece required to occur at the origin, so that for every t, the tth row of any proposed solution of P contains the whole situation of Z at t (tape, scanned square, state). As a result, P has no solution if and only if Z eventually halts. To extend the method to the row or diagonal-constrained case, we are faced with the difficulty that we do not have a fixed origin. Since, however, at each time t we need only be concerned with, say, no more than $t + 1$ squares of the tape, we can use a sequence of relative origins (or barriers) spaced with three squares between any two and double the distance between two barriers as we come to the next row or diagonal. In this way, we have, at each time t, an infinite number of copies of representations of the behavior of Z up to t, and a proposed solution cannot be completed if and only if Z halts. This is a rough indication of the basic idea behind our proof. It goes without saying that quite a number of details have to be handled to carry out the idea.

We will specify an effective procedure Π, such that the application of Π to any Turing machine Z results in the specification of a set of dominoes $\Pi(Z)$. To prove the lemma, we also exhibit a set Q such that for the set $P = \Pi(Z)$, the diagonal-constrained domino problem which constrains Q to the diagonal is unsolvable if and only if Z halts. Moreover, P and Q are so chosen that Q cannot occur in the region below the main diagonal. This will prove Lemma 3 and, as noted before, will establish the (Extended) Main Theorem.

Each domino of $\Pi(Z)$ is named uniquely by an ordered triple of symbols, each of which is from the alphabet D consisting of the numerals and the capital Greek letters Δ and Λ. Each of the symbols of these ordered triples is permitted to have a numeral for a subscript where necessary. Each of the edge colors of these dominoes is named uniquely by an ordered triple or an ordered pair (without subscripts) from the same alphabet D. In the names for the dominoes and the names for the colors, the symbol Λ is used to avoid the necessity of using blanks. This permits greater uniformity of notation, with each domino being named by a triple, even in the instances where fewer symbols would suffice. Also the colors are named more uniformly by triples and pairs. Whenever a symbol appears without subscript in the name of a domino, it can be understood to have the subscript Λ.

Making these conventions about Λ and blank, each domino is given a name of the form (σ_k, w_v, i_j). When the first symbol σ is a capital Greek letter, the domino serves the special purpose of marking off positions in the quadrant, around which the Turing machine construction will take place. When the first symbol σ is a numeral, it is understood to represent the tape symbol S_σ of the Turing machine Z. In particular, $\sigma = 0$ corresponds to the symbol S_0 of blank tape.

The subscript k indicates which kind of domino has been constructed. For each integer k between 1 and 26, there is a domino of kind k defined in Table 1. The dominoes of a given kind k all serve similar purposes in the construction. The exact procedure Π, as given below, indicates which dominoes of each kind are to be defined for the class $\Pi(Z)$ associated with any given Turing machine Z. In Table 1, each letter except Δ and Λ is a variable. The particular dominoes defined for $\Pi(Z)$ are obtained by replacing all of these variables by certain symbols from the alphabet D.

The variables w and v are called parity variables, and each of them has the range of values $\{1,2\}$. For each kind of domino shown in Table 1 with the variable w, dominoes having $w = 1$ and $w = 2$ are both assumed to be defined when we specify Π.

The variables i and j usually represent the present state and next state of the Turing machine Z, with i representing the state q_i, and j representing the state q_j.

We now specify the procedure Π in detail. For any Turing machine Z, we will define sets $\Pi_i(Z)$ of dominoes. The union of all the $\Pi_i(Z)$ defined will be $\Pi(Z)$. We let $\Pi_1(Z)$ be the set of all dominoes of kinds 19 through 26, together with all those dominoes of kinds 1 and 2 which replace η by each tape symbol of Turing machine Z.

We let $\Pi_2(Z)$ be the set of all dominoes of kinds 3, 4, 5, and 6, such that there exist ϵ, η, i, and j satisfying $q_iS_\eta S_\epsilon q_j$ is a quadruple of Z.

We let $\Pi_3(Z)$ be the set of all dominoes of kinds 7, 8, 9, and 10, such that there exist η, i, and j satisfying $q_iS_\eta Lq_j$ is a quadruple of Z.

We let $\Pi_4(Z)$ be the set of all dominoes of kinds 11 through 18, such that there exist ϵ, η, i, and j satisfying $q_iS_\eta Rq_j$ which is a quadruple of Z and S_ϵ is a tape symbol of Z.

Then $\Pi(Z) = \Pi_1(Z) \cup \Pi_2(Z) \cup \Pi_3(Z) \cup \Pi_4(Z)$ is the set of dominoes associated with the Turing machine Z. We also let Q be the set of dominoes of kinds 22, 23, and 24, which are to be permitted on the main diagonal.

The ordered set of lattice points (a,b) of the quadrant such that $a - b = k$ is called the kth diagonal. Note that the diagonal constraint applies to the zeroth diagonal (main diagonal). Note that if there is a solution to the diagonal-constrained domino problem for $\Pi(Z)$, the main diagonal and the first diagonal must each repeat with period 3 (ignoring the parity of the dominoes), and that the dominoes on the first diagonal must be of kinds 19, 20, and 21.

Since we have already considered how the main diagonal and the first diagonal must be filled in, let us call the remaining region (namely, the union of all kth diagonals for $k \geq 2$) the lower region. It can be proved that only dominoes of kinds 25, 26, and 1 through 18 can occur in the lower region.

Also, dominoes of kinds 25 or 26 must occur in every third position of the second diagonal. In each diagonal of the lower region, the constraints of the parity are transmitted along unchanged until a domino of kind 25 or 26 is reached, which then changes the parity. Thus, if a domino of kind 25 occurs on any diagonal, then a domino of kind 26 will occur along this diagonal before the next occurrence of a

TABLE 1

THE 26 KINDS OF DOMINOES

$(1,\Lambda)$	$(\eta,1,\Lambda)$	(η_1,w_1,Λ)	$(0,w,\Lambda)$	(w,Λ)	$(2,\Lambda)$	$(\eta,2,\Lambda)$	(η_2,w_2,Λ)	(η,w,Λ)	(w,Λ)
$(1,\Lambda)$	$(\eta,1,i)$	(η_3,w_1,i_j)	$(0,w,\Lambda)$	(w,Λ)	$(2,\Lambda)$	$(\eta,2,i)$	(η_4,w_2,i_j)	(ϵ,w,j)	(w,Λ)
$(1,i)$	$(\eta,1,\Lambda)$	(η_5,w_1,i_j)	$(0,w,\Lambda)$	(w,Λ)	$(2,i)$	$(\eta,2,\Lambda)$	(η_6,w_2,i_j)	(ϵ,w,j)	(w,Λ)
$(1,\Lambda)$	$(\eta,1,i)$	(η_7,w_1,i_j)	$(0,w,\Lambda)$	(w,Λ)	$(2,\Lambda)$	$(\eta,2,i)$	(η_8,w_2,i_j)	(η,w,Λ)	(w,j)
$(1,i)$	$(\eta,1,\Lambda)$	(η_9,w_1,i_j)	$(0,w,\Lambda)$	(w,j)	$(2,i)$	$(\eta,2,\Lambda)$	(η_{10},w_2,i_j)	(η,w,Λ)	(w,j)
$(1,\Lambda)$	$(\eta,1,i)$	(η_{11},w_1,i_j)	$(0,w,\Lambda)$	(w,Λ)	$(2,\Lambda)$	$(\eta,2,i)$	(η_{12},w_2,i_j)	$(\eta,w + 2,j)$	(w,Λ)
$(1,i)$	$(\eta,1,\Lambda)$	(η_{13},w_1,i_j)	$(0,w,\Lambda)$	(w,Λ)	$(2,i)$	$(\eta,2,\Lambda)$	(η_{14},w_2,i_j)	$(\eta,w + 2,j)$	(w,Λ)
$(3,j)$	$(\eta,3,j)$	(η_{15},w_1,j)	$(0,w,\Lambda)$	(w,Λ)	$(4,j)$	$(\eta,4,j)$	(η_{16},w_2,j)	(η,w,Λ)	(w,Λ)
$(1,\Lambda)$	$(\epsilon,1,\Lambda)$	(ϵ_{17},w_1,j)	$(0,w,\Lambda)$	$(w + 2,j)$	$(2,\Lambda)$	$(\epsilon,2,\Lambda)$	(ϵ_{18},w_2,j)	(ϵ,w,j)	$(w + 2,j)$
$(6,\Lambda)$	$(5,\Lambda,\Lambda)$	(Δ_{19},w,Λ)	$(0,w,1)$	(w,Λ)	$(7,\Lambda)$	$(6,\Lambda,\Lambda)$	(Δ_{20},w,Λ)	$(0,w,\Lambda)$	(w,Λ)
$(5,\Lambda)$	$(7,\Lambda,\Lambda)$	(Δ_{21},w,Λ)	(Λ,w,Λ)	(w,Λ)	$(5,\Lambda,\Lambda)$	$(5,\Lambda)$	$(\Delta_{22},\Lambda,\Lambda)$	$(5,\Lambda,\Lambda)$	$(5,\Lambda)$
$(6,\Lambda,\Lambda)$	$(6,\Lambda)$	$(\Delta_{23},\Lambda,\Lambda)$	$(6,\Lambda,\Lambda)$	$(6,\Lambda)$	$(7,\Lambda,\Lambda)$	$(7,\Lambda)$	$(\Delta_{24},\Lambda,\Lambda)$	$(7,\Lambda,\Lambda)$	$(7,\Lambda)$
$(2,\Lambda)$	$(\Lambda,1,\Lambda)$	$(\Delta_{25},w_2,\Lambda)$	(Λ,w,Λ)	(w,Λ)	$(1,\Lambda)$	$(\Lambda,2,\Lambda)$	$(\Lambda_{26},w_1,\Lambda)$	$(0,w,\Lambda)$	(w,Λ)

domino of kind 25. Since a domino of kind 26 *cannot* have a domino of kind 25 or 26 to the right of it, but a domino of kind 25 *must* have a domino of kind 25 or 26 to the right of it, this forces the length of the period between successive dominoes of kind 25 to be twice as great on each diagonal as it was on the preceding diagonal. In fact, we can prove that in any solution or partial solution in which the first $(k + 2)$ diagonals are filled in, the kth diagonal will have a periodic structure of period $3(2^k)$ and will have two occurrences of a domino of kind 25 and two occurrences of a domino of kind 26 in each period. Also, in any such case, any two solutions will have their kth diagonal agree beyond the first $3(2^k)$ positions, except for a translation. Thus, there is a sense in which the solution is almost unique.

Consider any solution to the diagonal-constrained domino problem for $\Pi(Z)$, or any partial solution in which the kth diagonal and all lower-numbered diagonals have been filled in. Call a domino of kinds 3 through 14 a key domino. Then we consider the sequence of those dominoes which occur on the kth diagonal between the occurrence of a domino of kind 25 and the next occurrence of a domino of kind 26, indicating only the first symbol of each domino in the sequence, except that we precede it by the third symbol in case of a key domino. The sequence obtained in this way will be the instantaneous description of the Turing machine Z at time $k - 1$.

At each point along the diagonal except near the key domino, the dominoes will be of kind 1 or 2, and will merely preserve the tape symbol which occurred to the left of them. But the key domino, which corresponds to the square of tape scanned by the Turing machine Z, will force a nearby domino on the next diagonal to be a key domino. The exact choice of which kind of key domino occurs will depend on whether the previous action was a move to the left or not and on whether the next action is a move to the left, a move to the right, or the writing of a new symbol.

For each k, all diagonals up to and including the kth diagonal can be filled in only if there is an instantaneous description of the machine Z at time $k - 1$. Thus the diagonal-constrained solution to the domino problem for $\Pi(Z)$ has a solution if and only if Z does not halt.

Most of the above statements can be proved by induction on k, and the others follow directly. In order that the reader may convince himself that this construction has all of the above properties, an explicit example has been worked out for the Turing machine:

$$Z = \{q_1S_0Rq_5,\ q_2S_0S_4q_3,\ q_3S_4Lq_6,\ q_5S_0Rq_2,\ q_6S_0Lq_8,\ q_8S_0Rq_3\}.$$

Since Z halts, a partial solution is shown.

The part near the origin of this solution has been printed as Table 2. However,

TABLE 2

THE SAMPLE PARTIAL SOLUTION

This position has coordinates (1,3)		$(\Delta_{22}, \Lambda, \Lambda)$
	$(\Delta_{24}, \Lambda, \Lambda)$	$(\Delta_{21}, 1, \Lambda)$
$(\Delta_{23}, \Lambda, \Lambda)$	$(\Delta_{20}, 1, \Lambda)$	$(0_1, 1_1, \Lambda)$

in order to indicate a large enough part of this solution to be of any value, it has been printed in an abbreviated form as Table 3. Thus, Table 2 serves only as a guide to the conventions by which Table 3 has been abbreviated.

In order to check the above solution, first the main diagonal and the first diagonal

TABLE 3

ABBREVIATED FORM OF A LARGER PART OF THE SAMPLE PARTIAL SOLUTION, WITH DOUBLE LINES NEAR THE LOWER LEFT CORNER TO SEPARATE THE PART SHOWN IN TABLE 2

$\Delta_{24},\Lambda,\Lambda$	$\Delta_{21},1,\Lambda$	$\Lambda_{25},2_2,\Lambda$						
$\Delta_{23},\Lambda,\Lambda$	$\Delta_{20},1,\Lambda$	$0_1,2_1,\Lambda$						
$\Delta_{22},\Lambda,\Lambda$	$\Delta_{19},1,\Lambda$	$0_{11},2_1,1_5$	$0_2,1_2,\Lambda$					
$\Delta_{24},\Lambda,\Lambda$	$\Delta_{21},2,\Lambda$	$\Lambda_{26},2_1,\Lambda$	$0_{18},1_2,2$					
$\Delta_{23},\Lambda,\Lambda$	$\Delta_{20},2,\Lambda$	$0_{18},2_2,5$	$0_{12},1_2,5_2$					
$\Delta_{22},\Lambda,\Lambda$	$\Delta_{19},2,\Lambda$	$0_{12},2_2,1_5$	$0_{16},1_2,5$	$0_1,2_1,\Lambda$				
$\Delta_{24},\Lambda,\Lambda$	$\Delta_{21},1,\Lambda$	$\Lambda_{25},1_2,\Lambda$	$\Lambda_{25},2_2,\Lambda$	$\Lambda_{26},2_1,\Lambda$				
$\Delta_{23},\Lambda,\Lambda$	$\Delta_{20},1,\Lambda$	$0_1,1_1,\Lambda$	$0_1,2_1,\Lambda$	$0_2,2_2,\Lambda$				
$\Delta_{22},\Lambda,\Lambda$	$\Delta_{19},1,\Lambda$	$0_{11},1_1,1_5$	$0_1,2_1,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$			
$\Delta_{24},\Lambda,\Lambda$	$\Delta_{21},2,\Lambda$	$\Lambda_{26},1_1,\Lambda$	$0_1,2_1,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$			
$\Delta_{23},\Lambda,\Lambda$	$\Delta_{20},2,\Lambda$	$0_{18},1_2,5$	$0_{11},2_1,5_2$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$			
$\Delta_{22},\Lambda,\Lambda$	$\Delta_{19},2,\Lambda$	$0_{12},1_2,1_5$	$0_{15},2_1,5$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$		
$\Delta_{24},\Lambda,\Lambda$	$\Delta_{21},1,\Lambda$	$\Lambda_{25},2_2,\Lambda$	$\Lambda_{26},2_1,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$		
$\Delta_{23},\Lambda,\Lambda$	$\Delta_{20},1,\Lambda$	$0_1,2_1,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	$0_2 2_2,\Lambda$		
$\Delta_{22},\Lambda,\Lambda$	$\Delta_{19},1,\Lambda$	$0_{11},2_1,1_5$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	
$\Delta_{24},\Lambda,\Lambda$	$\Delta_{21},2,\Lambda$	$\Lambda_{26},2_1,\Lambda$	$0_{18},2_2,2$	$0_4,2_2,2_3$	$4_8,2_2,3_6$	$4_2,2_2,\Lambda$	$4_2,2_2,\Lambda$	
$\Delta_{23},\Lambda,\Lambda$	$\Delta_{20},2,\Lambda$	$0_{18},2_2,5$	$0_{12},2_2,5_2$	$0_{16},2_2,2$	$0_2,2_2,\Lambda$	$0_{10},2_2,6_8$	$0_{18},2_2,3$	Stops here
$\Delta_{22},\Lambda,\Lambda$	$\Delta_{19},2,\Lambda$	$0_{12},2_2,1_5$	$0_{16},2_2,5$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	$0_2,2_2,\Lambda$	$0_{14},2_2,8_3$	$0_{16},2_2,3$
$\Delta_{24},\Lambda,\Lambda$	$\Delta_{21},1,\Lambda$	$\Lambda_{25},1_2,\Lambda$	$\Lambda_{25},1_2,\Lambda$	$\Lambda_{25},1_2,\Lambda$	$\Lambda_{25},1_2,\Lambda$	$\Lambda_{25},1_2,\Lambda$	$\Lambda_{25},1_2,\Lambda$	$\Lambda_{25},1_2,\Lambda$
$\Delta_{23},\Lambda,\Lambda$	$\Delta_{20},1,\Lambda$	$0_1,1_1,\Lambda$	$0_1,1_1,\Lambda$	$0_1,1_1,\Lambda$	$0_{17},1_1,5$	$0_1,1_1,\Lambda$	$0_{17},1_1,5$	$0_1,1_1,\Lambda$

are filled in. Then dominoes are filled in successively at the points whose coordinates are $(a,2)$ for each a, and then at coordinates (5,3), (6,3), (6,4), (7,4), (8,4), (8,5), (9,5), (10,5), (10,4), (11,3), (11,4), with other locations filled in as needed.

It should be remarked that in the construction of $\Pi(Z)$, the parity conditions on the dominoes permit more and more room for the Turing machine in each successive diagonal. By doubling the period at each diagonal, the amount of space available for the instantaneous description is $3(2^{k-1}) - 1$, which grows more rapidly than k, the amount of space required.

Many of the methods and techniques used in this paper are derived from developments made by Richard Büchi in connection with his work on closely related problems which is reported in the abstract[10] and will be published in a paper. We are heavily indebted to Büchi for an extended discussion on the content of his abstract.

Note added in proof: The paper by Büchi, mentioned above, is to be published in *Math. Annalen.* A preliminary draft of this paper has appeared as J. Richard Büchi, *Turing-Machines and the Entscheidungsproblem,* University of Michigan Technical Report 03105-22-T (under contract with the Office of Naval Research)(December 1961), 25 pp.

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Patrick Fischer and Dana Scott have suggested different ways of simplifying and improving the general treatment which are not included here because of our reluctance to make drastic revisions.

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[1] For example, Schütte, K., *Math. Annalen*, **109**, 575 (1934); Ackermann, W., *ibid.*, **112** 420 (1936); Ackermann, W., *Solvable Cases of the Decision Problem* (Amsterdam: North-Holland Publishing Company, 1954), p. 85; Church, A., *Revue philoscphique de Louvain*, **50**, 271 (1952); Suranyi, J., *Reduktionstheorie des Entscheidungsproblem* (Budapest: 1959), p. 177.

[2] See, e.g., Ackermann's book, *op. cit.*, p. 49.

[3] E.g., *ibid.*, pp. 71 and 82.

[4] Schütte, *op. cit.*

[5] More exactly, every recursively enumerable set is many-one (in fact, one-one) reducible to the set of satisfiable restricted $\forall\exists\forall$ formulas.

[6] See, e.g., Wang, H., Bell System Technical Journal, **40**, 4 (1961).

[7] These considerations were used by Büchi, J. R., *Notices Amer. Math. Soc.*, **8**, 354 (1961), for a similar purpose, and also previously used by L. Kalmár for a different purpose, see Surányi, *op. cit.*, p. 153.

[8] See, e.g., Surányi, *op. cit.*, pp. 45 and 50.

[9] Turing, A. M., *Proc. London Math. Soc.*, **42**, 259 (1936) and **43**, 544 (1937).

[18] Büchi, *op. cit.*

[11] For an informative discussion of matters relating to this, see Skolem, Th., *Les Entretiens de Zurich*, published by F. Gonseth, Zurich (1941), p. 25. In particular, more direct proofs of Lemma 1 are given there.

[12] Wang, H., *op. cit.*, p. 22.

[13] Davis, M., *Computability and Unsolvability* (New York: McGraw-Hill, 1958), p. 5.

[14] See, e.g., Davis, *op. cit.*, p. 70.

TRANSFORMATION OF STEROIDS WITH CHLORELLA PYRENOIDOSA

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The conversion of steroids by cultures of molds, yeasts, bacteria (including actinomycetes), and protozoa has produced a wide variety of products of oxidation and reduction.[1] To our knowledge, similar transformations performed by unicellular algae have not as yet been reported in the literature.

This is a report on (1) the reduction of 4-androstene-3,17-dione to testosterone by the metabolic activity of *Chlorella pyrenoidosa* C-37-2[2] propagated in a heterotrophic medium, (2) the modification of this strain of *Chlorella pyrenoidosa* C-37-2 so that it will grow in the presence of progesterone, and (3) some preliminary conversions of dehydroisoandrosterone by this same organism which indicate the possibility of oxidation (hydroxylation) of steroids.

Experimental.—The following is typical of the experiments conducted in this study of the transformations of steroids by algae. Two Fernbach flasks, each containing 750 ml of medium[3] composed of 3.75 gm of tryptone (Difco), 1.85 gm of yeast extract (Difco), and 0.75 gm of glucose, adjusted to pH 5.0 with 1.6 ml of 1 N sulfuric acid, were inoculated with 37.5 ml of an actively growing culture of *Chlorella pyrenoidosa* C-37-2. The cultures were incubated for 48 hr at 25°C on a reciprocating shaker (eighty-three $2\frac{1}{2}$-inch strokes per min). This strain of alga will grow in the dark on heterotrophic medium, but the growth rate is higher when