

A species approach to Rota's twelvefold way

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Abstract

An introduction to Joyal's theory of combinatorial species is given and through it an alternative view of Rota's twelvefold way emerges.

1. Introduction



In how many ways can n balls be distributed into k urns? If there are no restrictions given, then each of the n balls can be freely placed into any of the k urns and so the answer is clearly k^n . But what if we think of the balls, or the urns, as being identical rather than distinct? What if we have to place at least one ball, or can place at most one ball, into each urn?

The twelve cases resulting from exhaustively considering these options are collectively referred to as the *twelvefold way*, an account of which can be found in Section 1.4 of Richard Stanley's [8] excellent *Enumerative Combinatorics*, Volume 1. He attributes the idea of the twelvefold way to Gian-Carlo Rota and its name to Joel Spencer.

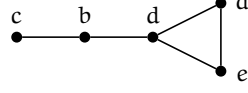
Stanley presents the twelvefold way in terms of counting functions, $f : U \rightarrow V$, between two finite sets. If we think of U as a set of balls and V as a set of urns, then requiring that each urn contains at least one ball is the same as requiring that f is surjective, and requiring that each urn contain at most one ball is the same as requiring that f is injective. To say that the balls are identical, or that the urns are identical, is to consider the equality of such functions up to a permutation of the elements of U , or up to a permutation of the elements of V .

We provide a uniform treatment of the twelvefold way, as well as some extensions of it, using Joyal's theory of combinatorial species, and in doing so we will endeavor to give an accessible introduction to this beautiful theory. Section 2 contains this introduction to species and it is based on two articles of Joyal [5, 6] and the book *Combinatorial species and tree-like structures* by Bergeron, Labelle, and Leroux [1]. In Section 3 we give the species interpretation of the twelvefold way.

2. Combinatorial species

Before defining what a combinatorial species is we shall consider simple graphs as a motivating example.

A simple undirected (labeled) graph is a pair (U, E) , where U is a set of vertices and E is a set of edges, which are 2-element subsets of U . For instance, the graph with vertex set $U = \{a, b, c, d, e\}$ and edge set $E = \{\{c, b\}, \{b, d\}, \{d, a\}, \{a, e\}, \{d, e\}\}$ is depicted here:



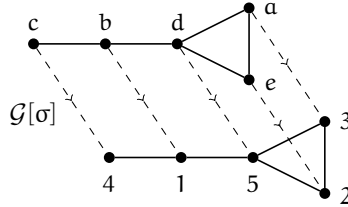
Informally, a graph in which vertices are indistinguishable is said to be unlabeled:



Formally, an *isomorphism* of graphs (U, E) and (V, E') is a bijection $\sigma : U \rightarrow V$ that preserves the adjacency relation: $\{x, y\} \in E$ if and only if $\{\sigma(x), \sigma(y)\} \in E'$. Then an *unlabeled graph* is an isomorphism class.

A species defines not only a class of (labeled) combinatorial objects but also how those objects are affected by relabeling. This mechanism is called *transport of structure*. In particular, a species carries the necessary information for defining the unlabeled counterpart of itself.

For graphs one defines $\mathcal{G}[U]$ as the set of graphs on the finite set U , and for a bijection $\sigma : U \rightarrow V$ one defines the transport of structure, $\mathcal{G}[\sigma] : \mathcal{G}[U] \rightarrow \mathcal{G}[V]$, by $\mathcal{G}[\sigma](U, E) = (V, \sigma(E))$, where $\sigma(E) = \{\{\sigma(x), \sigma(y)\} : \{x, y\} \in E\}$. For instance, if $\sigma = \begin{pmatrix} a & b & c & d \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}$ is the bijection from $U = \{a, b, c, d\}$ to $V = \{1, 2, 3, 4\}$ mapping a to 3, b to 1, etc., then



In terms of transport of structure, two graphs $s \in \mathcal{G}[U]$ and $t \in \mathcal{G}[V]$ are isomorphic if there is a bijection $\sigma : U \rightarrow V$ such that $\mathcal{G}[\sigma](s) = t$.

It is easy to check that for all bijections $\sigma : U \rightarrow V$ and $\tau : V \rightarrow W$ we have $\mathcal{G}[\sigma \circ \tau] = \mathcal{G}[\sigma] \circ \mathcal{G}[\tau]$. And, for the identity $\text{Id}_U : U \rightarrow U$, it holds that $\mathcal{G}[\text{Id}_U] = \text{Id}_{\mathcal{G}[U]}$.

A *species* is a rule which

- produces, for each finite set U , a finite set $F[U]$;
- produces, for each bijection $\sigma : U \rightarrow V$, a function $F[\sigma] : F[U] \rightarrow F[V]$.

These data satisfy the following conditions: $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$ for all bijections $\sigma : U \rightarrow V$ and $\tau : V \rightarrow W$, and $F[\text{Id}_U] = \text{Id}_{F[U]}$ for the identity map $\text{Id}_U : U \rightarrow U$. For instance, \mathcal{G} as defined above is a species.

While this definition of species is adequate, the natural setting for species is category theory. A *category* consists of the following data:

- A collection of *objects*: U, V, W, \dots
- A collection of *arrows*: $\sigma, \tau, \rho, \dots$, each having a *domain* and a *codomain*; writing $\sigma : U \rightarrow V$ we express that U is the domain and V is the codomain of σ .
- For each pair of arrows $\sigma : U \rightarrow V$ and $\tau : V \rightarrow W$, a composite arrow $\tau \circ \sigma : U \rightarrow W$.
- For each object U , an identity arrow $\text{Id}_U : U \rightarrow U$.

These data satisfy the following axioms:

- Associativity: for σ and τ as above and ρ an arrow with domain W we have $\rho \circ (\tau \circ \sigma) = (\rho \circ \tau) \circ \sigma$.
- Identity laws: for σ as above, $\sigma \circ \text{Id}_U = \sigma = \text{Id}_V \circ \sigma$.

Let us now give three examples of categories, the latter two of which are essential to the definition of a species. The category Set has sets as objects and functions as arrows; the identity arrow Id_U is the identity function on U . The category FinSet is defined similarly but has only *finite* sets as objects. The category \mathbb{B} has finite sets as objects and bijections as arrows.

Morphisms of categories are called a functors. Eilenberg and Mac Lane stress their importance in *General theory of natural equivalences* [4]:

It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of a natural transformation [...]

The definition goes as follows: Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a mapping of objects to objects and arrows to arrows such that

- If $\sigma : U \rightarrow V$ then $F[\sigma] : F[U] \rightarrow F[V]$;
- $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$;
- $F[\text{Id}_U] = \text{Id}_{F[U]}$.

Armed with these notions from category theory a species is simply a functor $F : \mathbb{B} \rightarrow \text{FinSet}$. This definition was first given by Joyal in 1981 [5]. An element $s \in F[U]$ is called an *F-structure* on U . Note that if F is a species and $\sigma : U \rightarrow V$ is a bijection, then $\text{Id}_{F[U]} = F[\text{Id}_U] = F[\sigma^{-1} \circ \sigma] = F[\sigma^{-1}] \circ F[\sigma]$ and, similarly, $\text{Id}_{F[V]} = F[\sigma] \circ F[\sigma^{-1}]$. Thus, the transport of structure is always a bijection and a species can also be regarded as a functor $\mathbb{B} \rightarrow \mathbb{B}$. In particular, if U and V are two finite sets of the same cardinality $|U| = |V|$, and $\sigma : U \rightarrow V$ is a bijection witnessing this, then $F[\sigma] : F[U] \rightarrow F[V]$ is a bijection, proving that $|F[U]| = |F[V]|$. In other words, the number of F-structures on U only depends on the number of elements of U .

For ease of notation, we write $F[n]$ for $F[[n]]$, the set of F-structures on $[n] = \{1, 2, \dots, n\}$. The *generating series* of F is the formal power series

$$F(x) = \sum_{n \geq 0} |F[n]| \frac{x^n}{n!}.$$

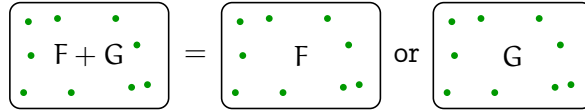
Two F -structures $s \in F[U]$ and $t \in F[V]$ are *isomorphic*, and we write $s \sim t$, if there is a bijection $\sigma : U \rightarrow V$ such that $F[\sigma](s) = t$. Let $F[U]/\sim$ denote the set of equivalence classes under this relation. An unlabeled F -structure is then simply a member of $F[U]/\sim$. Two isomorphic F -structures are also said to have the same *isomorphism type* and the *type generating series* of F is

$$\tilde{F}(x) = \sum_{n \geq 0} |F[n]/\sim| x^n.$$

We have only seen one concrete species so far, that of simple graphs. Another example is the species L of linear orders. For a finite set U with n elements, let $L[U]$ consist of all bijections $f : [n] \rightarrow U$. For instance, if we represent f by its images $f(1)f(2)\dots f(n)$ then $L[3] = \{123, 132, 213, 231, 312, 321\}$. Transport of structure is defined by $L[\sigma](f) = \sigma \circ f$, or in terms of images $L[\sigma](f(1)\dots f(n)) = \sigma(f(1))\dots \sigma(f(n))$. Clearly, there are $n!$ linear orders on $[n]$. Thus $|L[n]| = n!$ and $L(x) = 1/(1-x)$. Any two linear orders $f, g \in L[U]$ are isomorphic. Indeed, $\sigma = g \circ f$ defines a bijection from U to U and $g = L[\sigma](f)$. Thus $|L[n]/\sim| = 1$ and $\tilde{L}(x) = 1/(1-x)$.

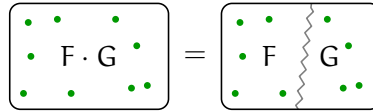
Let E be the species of sets. It is defined by $E[U] = \{U\}$ and, for bijections $\sigma : U \rightarrow V$, $E[\sigma](U) = V$. That is, for any finite set U there is exactly one E -structure, the set U itself. Thus $E(x) = e^x$ and $\tilde{E}(x) = 1/(1-x)$.

We can build new species from existing species using certain operations. The simplest operation is addition, which is just disjoint union. An $(F + G)$ -structure is thus either an F -structure or a G -structure:



That is, for any finite set U we define $(F + G)[U] = F[U] \sqcup G[U]$. For any bijection $\sigma : U \rightarrow V$, transport of structure is defined by $(F + G)[\sigma](s) = F[\sigma](s)$ if $s \in F[U]$ and $(F + G)[\sigma](s) = G[\sigma](s)$ if $s \in G[U]$.

We can also form the product of two species. An $(F \cdot G)$ -structure on a set U is a pair (s, t) , where s is an F -structure on a subset U_1 of U and t is a G -structure on the remaining elements $U_2 = U \setminus U_1$:



Formally, $(F \cdot G)[U] = \bigsqcup (F[U_1] \times G[U_2])$ in which the union is over all pairs (U_1, U_2) such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. The transport of structure is defined by $(F \cdot G)[\sigma](s, t) = (F[\sigma_1](s), G[\sigma_2](t))$, where $\sigma_1 = \sigma|_{U_1}$ is the restriction of σ to U_1 and $\sigma_2 = \sigma|_{U_2}$ is the restriction of σ to U_2 .

It is clear that if F and G are species, then $(F + G)(x) = F(x) + G(x)$. It also holds that $(FG)(x) = F(x)G(x)$. To see this, recall that, for any species F and any finite set U , the number $|F[U]|$ of F -structures on U only depends on $|U|$. Thus, the coefficient in front of

$x^n/n!$ in $(FG)(x)$ is

$$|(FG)[n]| = \left| \bigsqcup_{S \subseteq [n]} F[S] \times G[[n] \setminus S] \right| = \sum_{k=0}^n \binom{n}{k} |F[k]| |G[n-k]|,$$

which is also the coefficient in front of $x^n/n!$ in $F(x)G(x)$. For the type generating series one can similarly show that $\widetilde{(F+G)}(x) = \widetilde{F}(x) + \widetilde{G}(x)$ and $\widetilde{(FG)}(x) = \widetilde{F}(x)\widetilde{G}(x)$.

As a simple example, multiplying the set species with itself we get $E^2[U] = \{(A, U \setminus A) : A \subseteq U\}$. This is called the subset (or power set) species, and is denoted \mathcal{P} . Note that $\mathcal{P}(x) = E(x)^2 = e^{2x} = \sum_{n \geq 0} 2^n x^n/n!$, reflecting the elementary fact that there are 2^n subsets of an n -element set. Also, $\widetilde{\mathcal{P}}(x) = 1/(1-x)^2 = \sum_{n \geq 0} (n+1)x^n$, reflecting that the isomorphism type of a set is determined by its cardinality and that an n -element set has subsets of $n+1$ different cardinalities.

Another, perhaps more interesting, example is that of derangements: Define the species \mathcal{S} of permutations as follows. For any finite set U , let $\mathcal{S}[U]$ be the set of bijections $f : U \rightarrow U$; and, for any bijection $\sigma : U \rightarrow V$, define the transport of structure by $\mathcal{S}[\sigma](f) = \sigma \circ f \circ \sigma^{-1}$. A *derangement* is a permutation without fixed points. Let \mathcal{D} denote the species of derangements, so that $\mathcal{D}[U] = \{f \in \mathcal{S}[U] : f(x) \neq x \text{ for all } x \in U\}$ with transport of structure defined the same way as for the species \mathcal{S} . Note that any permutation on U can be seen as a set of fixed points on a subset U_1 of U together with a derangement of the remaining elements $U_2 = U \setminus U_1$. This is most apparent when writing a permutation in disjoint cycle form. For instance, the permutation $(1)(2\ 7\ 3)(4)(5)(6\ 9)(8)$ can be identified with a the pair consisting of the fixed point set $\{1, 4, 5, 8\}$ together with the derangement $(2\ 7\ 3)(6\ 9)$. Thus $\mathcal{S} = E \cdot \mathcal{D}$ and $\mathcal{S}(x) = E(x)\mathcal{D}(x)$. It immediately follows that

$$\begin{aligned} \mathcal{D}(x) &= \mathcal{S}(x) \cdot E(x)^{-1} = \frac{1}{1-x} \cdot e^{-x} \\ &= \sum_{n \geq 0} n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!} \right) \frac{x^n}{n!}. \end{aligned}$$

In particular, $|\mathcal{D}[n]|/n! \rightarrow e^{-1}$ as $n \rightarrow \infty$, which the reader may recognize as the answer to Montmort's (1713) hat-check problem [3].

Define the singleton species X by $X[U] = \{U\}$ if $|U| = 1$ and $X[U] = \emptyset$ otherwise. Similarly, define the species 1, characteristic of the empty set, by $1[U] = \{U\}$ if $U = \emptyset$ and $1[U] = \emptyset$ otherwise. The combinatorial equality $L = 1 + XL$ informally states that a linear order is either empty or it consists of a first element together with the remaining elements. A computer scientist would recognize this as the same recursive pattern as in the definition of a singly linked list of nodes: it is either the null reference or it is a pair consisting of a data element and a reference to the next node in the list. To make formal sense of this combinatorial equality we first need to define what it means for two species to be equal.

Two species F and G are (*combinatorially*) *equal* if there is a family of bijections

$\alpha_U : F[U] \rightarrow G[U]$ such that for any bijection $\sigma : U \rightarrow V$ the diagram

$$\begin{array}{ccc} F[U] & \xrightarrow{\alpha_U} & G[U] \\ F[\sigma] \downarrow & & \downarrow G[\sigma] \\ F[V] & \xrightarrow{\alpha_V} & G[V] \end{array}$$

commutes. That is, for any F -structure s on U we have $G[\sigma](\alpha_U(s)) = \alpha_V(F[\sigma](s))$. In category theory this is called natural isomorphism of functors.

To prove that $L = 1 + XL$ we define the family of bijections $\alpha_U : L[U] \rightarrow (1 + XL)[U]$ by $\alpha_\emptyset(\epsilon) = \emptyset$ and $\alpha_U(a_1 a_2 \cdots a_n) = (a_1, a_2 \cdots a_n)$ for $U \neq \emptyset$. In the case $U = \emptyset$ the diagram above trivially commutes and for any nonempty finite set U we have

$$\begin{array}{ccc} a_1 a_2 \cdots a_n & \xrightarrow{\alpha_U} & (a_1, a_2 \cdots a_n) \\ L[\sigma] \downarrow & & \downarrow (1 + XL)[\sigma] \\ \sigma(a_1)\sigma(a_2) \cdots \sigma(a_n) & \xrightarrow{\alpha_V} & (\sigma(a_1), \sigma(a_2) \cdots \sigma(a_n)) \end{array}$$

which concludes the proof of $L = 1 + XL$. From this it immediately follows that $L(x) = 1 + xL(x)$ and we rediscover the identity $L(x) = 1/(1 - x)$. Similarly, $\tilde{L}(x) = 1 + x\tilde{L}(x)$ and we also rediscover that $\tilde{L}(x) = 1/(1 - x)$.

Next we shall define substitution of species. As a motivating example, consider the species \mathcal{S} of permutations. For instance, $(1\ 7)(2\ 8\ 3\ 4)(5\ 9)(6)$ is an element of $\mathcal{S}[9]$ written in disjoint cycle form. Here we have listed the cycles in increasing order with respect to the smallest element in each cycle, but this is just a convention; since the cycles are disjoint they commute and we may list them in any order. In other words, any permutation can be represented by a set of cycles. Using species this can be expressed as $\mathcal{S} = E \circ \mathcal{C} = E(\mathcal{C})$, in which a \mathcal{C} -structure is a permutation with a single cycle.

Another example of substitution of species is that of set partitions; this species is denoted Par . Define E_+ as the species of nonempty sets, so that $E = 1 + E_+$. Then $\text{Par} = E(E_+)$, which we can read as stating that a partition is a set of nonempty sets. Similarly, the species of *ballots*, also called ordered set partitions, is defined by $\text{Bal} = L(E_+)$.

In general, we can think of an $F(G)$ -structure as a generalized partition in which each block of the partition carries a G -structure, and the blocks are structured by F :

Formally, we define the *substitution* of G in F , also called *partitional composition*, as follows. For species F and G , with $G[\emptyset] = \emptyset$,

$$(F \circ G)[U] = \bigsqcup_{\substack{\beta = \{B_1, B_2, \dots, B_k\} \\ \text{partition of } U}} F[\beta] \times G[B_1] \times G[B_2] \times \cdots \times G[B_k].$$

Transport of structure is defined in a natural way, but there are quite a few details to keep track of. To a first approximation, an $(F \circ G)$ -structure on U is a tuple (s, t_1, \dots, t_k) , where $s \in F[\beta]$ and $t_i \in G[B_i]$. To guarantee that the union is disjoint we will, however, also “tag” these tuples with their associated set partition. Hence a typical $(F \circ G)$ -structure is of the form $(\beta, s, t_1, \dots, t_k)$. To describe the image of this tuple under the transport of structure we first need some auxiliary definitions. Assume that $\sigma : U \rightarrow V$ is a bijection. Let $\bar{\beta} = \{\bar{B}_1, \dots, \bar{B}_k\}$ be the set partition of V obtained from β by transport of structure. That is, with $\bar{\sigma} = \text{Par}[\sigma]$ we have $\bar{\beta} = \bar{\sigma}(\beta)$. Further, let $\bar{s} = F[\bar{\sigma}](s) \in F[\bar{\beta}]$ and $\bar{t}_i = G[\sigma_i](t_i) \in G[\bar{B}_i]$, where $\sigma_i = \sigma|_{B_i}$ is the restriction of σ to the block B_i . Then

$$(\bar{\beta}, \bar{s}, \bar{t}_1, \dots, \bar{t}_k) = (F \circ G)[\sigma](\beta, s, t_1, \dots, t_k).$$

Concerning generating series, one can show that $F(G)(x) = F(G(x))$. As an example, recall that $\mathcal{S} = E \circ \mathcal{C}$ and hence $\mathcal{S}(x) = \exp(\mathcal{C}(x))$. Thus

$$\mathcal{C}(x) = \log(\mathcal{S}(x)) = \log(1 - x)^{-1} = \sum_{n \geq 1} \frac{x^n}{n} = \sum_{n \geq 1} (n-1)! \frac{x^n}{n!},$$

which is unsurprising since there clearly are $(n-1)!$ ways to make a cycle using n distinct elements.

For type generating series the situation is more complicated. The series $\widetilde{F(\widetilde{G})}(x)$ is, as a rule, different from $\widetilde{F}(\widetilde{G}(x))$. To state the true relation between these entities a more general generating series is needed. The *cycle index series* for a species F is defined by

$$Z_F(x_1, x_2, x_3, \dots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}[n]} |\text{Fix } F[\sigma]| x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} x_3^{c_3(\sigma)} \dots$$

where $\text{Fix } F[\sigma] = \{s \in F[n] : F[\sigma](s) = s\}$ is the set of F -structures on $[n]$ fixed by $F[\sigma]$, and $c_i(\sigma)$ denotes the number of cycles of length i in σ . To illustrate this definition we give three examples:

Example 1. Consider the species X of singletons. The fixed point set $\text{Fix } X[\sigma]$ is empty unless $n = 1$ and there is only one permutation on $[1] = \{1\}$, the identity, so the cycle index series consists of a single term:

$$Z_X(x_1, x_2, \dots) = |\text{Fix } X[\text{Id}_1]| x_1^{c_1(\text{Id}_1)} x_2^{c_2(\text{Id}_1)} \dots = x_1.$$

Example 2. For any species F we denote by F_n the restriction of F to sets of cardinality n . In other words, $F_n[U] = F[U]$ if $|U| = n$ and $F_n[U] = \emptyset$ otherwise. With this notation the singleton species X is identical to the species E_1 . Let us now calculate the cycle index series for E_2 , the species of sets of cardinality 2:

$$\begin{aligned} Z_{E_2}(x_1, x_2, \dots) &= \frac{1}{2} \left(|\text{Fix } E_2[\text{Id}_2]| x_1^{c_1(\text{Id}_2)} x_2^{c_2(\text{Id}_2)} + |\text{Fix } E_2[(1\ 2)]| x_1^{c_1(1\ 2)} x_2^{c_2(1\ 2)} \right) \\ &= \frac{1}{2} (x_1^2 + x_2). \end{aligned}$$

Example 3. For the species L of linear orders, $\text{Fix } L[\sigma]$ is empty unless σ is the identity map, in which case $\text{Fix } L[\sigma] = L[n]$. Thus

$$\begin{aligned} Z_L(x_1, x_2, \dots) &= \sum_{n \geq 0} \frac{1}{n!} |\text{Fix } L[\text{Id}_n]| x_1^{c_1(\text{Id}_n)} x_2^{c_2(\text{Id}_n)} \dots \\ &= \sum_{n \geq 0} \frac{1}{n!} |L[n]| x_1^n = \frac{1}{1 - x_1}. \end{aligned}$$

It should be noted that the cycle index series encompasses both the generating series and the type generating series of a species. To be more precise, for any species F ,

$$F(x) = Z_F(x, 0, 0, \dots) \quad \text{and} \quad \tilde{F}(x) = Z_F(x, x^2, x^3, \dots).$$

Let us prove this. For the generating series we have

$$\begin{aligned} Z_F(x, 0, 0, \dots) &= \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in S[n]} |\text{Fix } F[\sigma]| x^{c_1(\sigma)} 0^{c_2(\sigma)} 0^{c_3(\sigma)} \dots \right) \\ &= \sum_{n \geq 0} \frac{1}{n!} |\text{Fix } F[\text{Id}_n]| x^n = \sum_{n \geq 0} \frac{|F[n]|}{n!} x^n = F(x). \end{aligned}$$

For the type generating series we have

$$\begin{aligned} Z_F(x, x^2, x^3, \dots) &= \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in S_n} |\text{Fix } F[\sigma]| x^{c_1(\sigma) + 2c_2(\sigma) + 3c_3(\sigma) + \dots} \\ &= \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in S_n} |\text{Fix } F[\sigma]| x^n = \sum_{n \geq 0} \frac{|F[n]|}{n!} x^n = \tilde{F}(x) \end{aligned}$$

by Burnside's lemma.

Further, for species F and G we have $Z_{F+G} = Z_F + Z_G$, $Z_{F \cdot G} = Z_F \cdot Z_G$, and

$$Z_{F \circ G}(x_1, x_2, \dots) = Z_F(Z_G(x_1, x_2, \dots), Z_G(x_2, x_4, \dots), \dots).$$

In particular, $(F \circ G)(x) = F(G(x))$ while $\widetilde{F \circ G}(x) = Z_F(\tilde{G}(x), \tilde{G}(x^2), \tilde{G}(x^3), \dots)$. As an example, consider the species $F = E_2 \circ (X + X^2)$ and isomorphism types of F -structures. For each of the sizes 2, 3, and 4 there is exactly one type: using pictures, those are $\{\bullet, \bullet\}$, $\{\bullet, \bullet\bullet\}$, and $\{\bullet\bullet, \bullet\bullet\}$. There is no type of any other size and thus the type generating series is $\tilde{F}(x) = x^2 + x^3 + x^4$. Note that the type generating series for E_2 and $X + X^2$ are x^2 and $x + x^2$, and naively composing them we get $x^2 + 2x^3 + x^4$, which is different from $\tilde{F}(x)$. The correct procedure to calculate $\tilde{F}(x)$ is

$$\begin{aligned} \tilde{F}(x) &= Z_{E_2}(\widetilde{(X + X^2)}(x), \widetilde{(X + X^2)}(x^2), \dots) \\ &= Z_{E_2}(x + x^2, x^2 + x^4) = \frac{1}{2} \left((x + x^2)^2 + x^2 + x^4 \right) = x^2 + x^3 + x^4. \end{aligned}$$

Changing the topic, but continuing with the species $F = E_2 \circ (X + X^2)$, say that we want to refine the count of F -structures by keeping track of how many ordered pairs, stemming from the X^2 term, such a structure contains. We can accomplish this by putting a

“weight”, say y , on each X^2 -structure. The type generating series for F would in this case be $x^2 + yx^3 + y^2x^4$. Two more interesting examples of this kind of refinement would be to count permutations while keeping track of the number of cycles, or to count trees while keeping track of the number of leaves. There is a variant of combinatorial species, called weighted species, that allows us to keep track of various parameters. Before giving its definition we’ll need to introduce weighted sets and their morphisms.

Let R be a ring. An R -weighted set is a pair (A, w) , where A is a finite set and $w : A \rightarrow R$ is a function. We think of w as assigning a weight to each element of A . A morphism of R -weighted sets (A, w) and (B, v) is a weight-preserving function $\varphi : A \rightarrow B$; that is, $w = v \circ \varphi$, or, equivalently, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ & \searrow w & \swarrow v \\ & R & \end{array}$$

commutes. If, in addition, ψ is a morphism of R -weighted sets from (B, v) to (C, u) then it’s easy to check that the composition $\psi \circ \varphi$ is a morphism from (A, w) to (C, u) . The identity map is of course also weight-preserving and so we have all the ingredients of a category. More generally, given categories \mathcal{A} and \mathcal{D} , an object $D \in \mathcal{D}$ and a functor $F : \mathcal{A} \rightarrow \mathcal{D}$, the slice category (F/D) is defined as follows: Objects are pairs (A, w) with $A \in \mathcal{A}$ and $w : F[A] \rightarrow D$ in \mathcal{D} . Arrows $(A, w) \rightarrow (A', w')$ are arrows $\varphi : A \rightarrow A'$ making the following diagram commute:

$$\begin{array}{ccc} F[A] & \xrightarrow{F[\varphi]} & F[A'] \\ & \searrow w & \swarrow w' \\ & D & \end{array}$$

Recall that Set is the category of sets and functions and that FinSet is the category of finite sets and functions. By slight abuse of notation, let the inclusion functor from FinSet to Set taking objects and arrows to themselves also be denoted by FinSet . In terms of the definition of a slice category, let $F = \text{FinSet}$, $\mathcal{A} = \text{FinSet}$, and $\mathcal{D} = \text{Set}$. Assuming that we for the moment identify the ring R with its underlying set, forgetting about the two operations on R , we also let $D = R$. Then (FinSet/R) is the (slice) category of R -weighted sets with morphisms of R -weighted sets as arrows, and we define an R -weighted species as a functor $F : \mathbb{B} \rightarrow (\text{FinSet}/R)$.

As an example, let us define a $\mathbb{Z}[y]$ -weighted version of the species \mathcal{S} of permutations. Define the weight of $f \in \mathcal{S}[U]$ as $w_U(f) = y^{\text{cyc}(f)}$, where $\text{cyc}(f)$ denotes the number of cycles in f , and let \mathcal{S}_w denote the resulting weighted species. To be more specific,

- for any finite set U , let $\mathcal{S}_w[U]$ be the $\mathbb{Z}[y]$ -weighted set $(\mathcal{S}[U], w_U)$;
- for any bijection $\sigma : U \rightarrow V$, let $\mathcal{S}_w[\sigma] : \mathcal{S}_w[U] \rightarrow \mathcal{S}_w[V]$ be the morphism of $\mathbb{Z}[y]$ -weighted sets defined by $\mathcal{S}[\sigma] : \mathcal{S}[U] \rightarrow \mathcal{S}[V]$.

Recall that the transport of structure $\mathcal{S}[\sigma]$ is conjugation by σ ; that is, $\mathcal{S}[\sigma](f) = \sigma \circ f \circ \sigma^{-1}$. It is well known that two permutations are conjugate if and only if they have the same cycle structure. In particular, conjugate permutations have the same number of cycles. Thus, $\mathcal{S}[\sigma]$ is weight-preserving, or, equivalently, the diagram

$$\begin{array}{ccc} \mathcal{S}[U] & \xrightarrow{\mathcal{S}[\sigma]} & \mathcal{S}[V] \\ & \searrow w_U \quad \swarrow w_V & \\ & \mathbb{Z}[y] & \end{array}$$

commutes.

$\in \mathcal{R}$

For any weighted set (A, w) let $|A|_w = \sum_{a \in A} w(a)$; it's called the *total weight of A* . Note that if w is the trivial weight function, assigning 1 to each element, then $|A|_w = |A|$. With this definition in hand, the power series associated with a weighted species F_w are defined as expected:

$$\begin{aligned} Z_{F_w}(x_1, x_2, x_3, \dots) &= \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}[n]} |\text{Fix } F[\sigma]|_w x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} x_3^{c_3(\sigma)} \dots, \\ F_w(x) &= Z_{F_w}(x, 0, 0, \dots) = \sum_{n \geq 0} |F[n]|_w \frac{x^n}{n!}, \\ \tilde{F}_w(x) &= Z_{F_w}(x, x^2, x^3, \dots) = \sum_{n \geq 0} |F[n]/\sim|_w x^n, \end{aligned}$$

where $|F_w[U]/\sim|_w$ is the total weight of any member of $F_w[U]/\sim$. Since $F_w[\sigma]$ is weight-preserving the choice of representative is immaterial.

It is straightforward to extend addition, multiplication, and composition to weighted species. All we need to do is define disjoint union and (Cartesian) product of weighted sets: If (A, w) and (B, v) are weighted sets, then $(A, w) \sqcup (B, v) = (A \sqcup B, w \sqcup v)$ and $(A, w) \times (B, v) = (A \times B, w \times v)$, where $(w \sqcup v)(x)$ is $w(x)$ or $v(x)$ depending on whether $x \in A$ or $x \in B$, and $(w \times v)(x, y) = w(x)v(y)$. With these definitions we have $|A \sqcup B|_{w \sqcup v} = |A|_w + |B|_v$ and $|A \times B|_{w \times v} = |A|_w |B|_v$.

Returning to the \mathcal{S}_w example, we find that $\mathcal{S}_w = E(y\mathcal{C})$, where $y\mathcal{C}$ is suggestive notation for the weighted species of cycles in which each cycle is given the weight y . Thus $\mathcal{S}_w(x) = \exp(y\mathcal{C}(x)) = \exp(y \log(1-x)^{-1}) = (1-x)^{-y}$ and the coefficient in front of $y^k x^n / n!$ in $\mathcal{S}_w(x)$ is an unsigned Stirling number of the first kind. That is,

$$\mathcal{S}_w(x) = \sum_{n \geq 0} \left(\sum_{k=1}^n c(n, k) y^k \right) \frac{x^n}{n!},$$

where $c(n, k)$ is the number of permutations of $[n]$ with k cycles.

As a more complex example we shall now calculate the cycle index series of the set species. We have $|\text{Fix } E[\sigma]| = 1$ for every $\sigma \in \mathcal{S}[n]$ and so

$$Z_E(x_1, x_2, x_3, \dots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}[n]} x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} x_3^{c_3(\sigma)} \dots$$

Or, rewriting this slightly,

$$Z_E(x_1, x_2, x_3, \dots) = \left[\sum_{n \geq 0} \left(\sum_{\sigma \in \mathcal{S}[n]} x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} x_3^{c_3(\sigma)} \dots \right) \frac{x^n}{n!} \right]_{x=1}.$$

Inside the outermost bracket is an exponential generating series for permutations in which each cycle of length i carries the weight x_i . Next, we recast this example in terms of weighted species. Let \mathcal{S}_w be the $\mathbb{Z}[x_1, x_2, \dots]$ -weighted species of permutations in which $w(\sigma) = x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} \dots$ for each permutation σ . Then $Z_E(x_1, x_2, \dots) = \mathcal{S}_w(1)$. On the other hand, we have $\mathcal{S}_w = E(x_1 \mathcal{C}_1 + x_2 \mathcal{C}_2 + \dots)$ in which $x_i \mathcal{C}_i$ is the species of cycles of length i , each with weight x_i . Thus $\mathcal{S}_w(x) = \exp(\frac{x_1 x}{1} + \frac{x_2 x^2}{2} + \frac{x_3 x^3}{3} + \dots)$ and setting $x = 1$ we arrive at

$$Z_E(x_1, x_2, x_3, \dots) = \exp\left(\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \dots\right).$$

Let us use this expression to investigate the type generating series of the set partition species Par . The isomorphism type of a set partition is determined by the sizes of its blocks; those sizes form an integer partition and hence $\widetilde{\text{Par}}(x) = \prod_{k \geq 1} (1 - x^k)^{-1}$. On the other hand, $\text{Par} = E(E_+)$ which in terms of type generating series means

$$\widetilde{\text{Par}}(x) = Z_E(\widetilde{E}_+(x), \widetilde{E}_+(x^2), \widetilde{E}_+(x^3), \dots).$$

Clearly, $\widetilde{E}_+(x) = x/(1 - x)$ and we arrive at the intriguing identity

$$\prod_{k \geq 1} \frac{1}{1 - x^k} = \exp \sum_{k \geq 1} \frac{1}{k} \frac{x^k}{1 - x^k}.$$

Let Par_w be the species of set partitions weighted by the number of blocks: the weight function $w_U : \text{Par}[U] \rightarrow \mathbb{Z}[y]$ is defined by $w_U(\beta) = y^k$, where $\beta = \{B_1, \dots, B_k\}$ is a partition of U with k blocks. Then $\text{Par}_w = E(yE_+)$, in which yE_+ denotes the species of nonempty sets each with weight y . Thus $\text{Par}_w(x) = \exp(y(e^x - 1))$, and the coefficient in front of $y^k x^n / n!$ in $\text{Par}_w(x)$ is a Stirling number of the second kind:

$$\text{Par}_w(x) = \sum_{n \geq 0} \left(\sum_{k=1}^n S(n, k) y^k \right) \frac{x^n}{n!},$$

where $S(n, k)$ is the number of set partitions of $[n]$ with k blocks. Transitioning to type generating series we also find that

$$\prod_{k \geq 1} \frac{1}{1 - yx^k} = \exp \sum_{k \geq 1} \frac{1}{k} \frac{y^k x^k}{1 - x^k}, \quad (1)$$

which is a refined generating series for integer partitions in which y keeps track of the number of parts.

A two-sort species is a functor $F : \mathbb{B} \times \mathbb{B} \rightarrow \text{FinSet}$ in which $\mathbb{B} \times \mathbb{B}$ denotes a product category. In general, if \mathcal{A} and \mathcal{B} are two categories, then $\mathcal{A} \times \mathcal{B}$ has pairs (A, B) as objects, where A is an object of \mathcal{A} and B is an object of \mathcal{B} . Similarly, the arrows are

pairs (f, g) , where f and g are arrows of \mathcal{A} and \mathcal{B} , respectively. Composition of arrows is componentwise and $\text{Id}_{(\mathcal{A}, \mathcal{B})} = (\text{Id}_{\mathcal{A}}, \text{Id}_{\mathcal{B}})$. We shall write $F(X, Y)$ to indicate that F is a two-sort species and that the two sorts are called X and Y . For natural numbers n and k , let $F[n, k] = F[[n], [k]]$. We define the *generating series* of F by

$$F(x, y) = \sum_{n, k \geq 0} |F[n, k]| \frac{x^n}{n!} \frac{y^k}{k!}.$$

Note that any unisort species F can be considered a two-sort species by specifying of what sort it is: saying that F is a species of sort X means that, as a two-sort species, $F[U, V] = F[U]$ if V is empty, and $F[U, V] = \emptyset$ otherwise. To be a species of sort Y is interpreted similarly.

Addition and multiplication of species is straightforwardly extended to the two-sort context. As an example, consider the species $F(X, Y) = (1 + X)Y$, where X is the species of singletons of sort X and Y is the species of singletons of sort Y . If $|U| \leq 1$ and $|V| = 1$ then there is exactly one F -structure, namely the pair (U, V) ; otherwise, there are no F -structures. For instance, $F[\emptyset, \{1\}] = \{(\emptyset, \{1\})\}$ and $F[\{1, 2\}, V] = \emptyset$. Similarly,

$$(E(X)Y)[U, V] = \begin{cases} \{(U, V)\} & \text{if } |V| = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is less clear how to extend the substitution operation to the two-sort context and so we detail it here. Let $\text{Par}[U, V]$ denote the set of partitions of the disjoint union $U \sqcup V$. For instance, if $U = \{1, 2\}$ and $V = \{a, b, c\}$, then $\pi = \{\{1\}, \{2, a, c\}, \{b\}\}$ is a member of $\text{Par}[U, V]$. In the definition of $F(G, H)$ below, we view a block B of such a partition as a pair of sets (S, T) , where $S = B \cap U \subseteq U$ and $T = B \cap V \subseteq V$. Thus, the blocks of the example partition, π , will be viewed as the pairs $(\{1\}, \emptyset)$, $(\{2, a, c\})$, and $(\emptyset, \{b\})$. We are now in a position to define the substitution of G and H in F , also called *partitional composition*. Let $F(X, Y)$, $G(X, Y)$, and $H(X, Y)$ be two-sort species. Then

$$F(G, H) = \bigsqcup_{\substack{\pi \in \text{Par}[U, V] \\ \kappa: \pi \rightarrow \{X, Y\}}} F[\beta, \gamma] \times G[B_1] \times \cdots \times G[B_k] \times H[C_1] \times \cdots \times H[C_\ell],$$

in which $\beta = \{B_1, \dots, B_k\} = \kappa^{-1}(X)$ and $\gamma = \{C_1, \dots, C_\ell\} = \kappa^{-1}(Y)$. Here we can think of κ as assigning one of two colors to each block of π ; the partition β consists of the X -colored blocks, and γ consists of the Y -colored blocks. Intuitively, an $F(G, H)$ -structure is obtained from an F -structure by inflating each element of sort X into a G -structure and each element of sort Y into an H -structure.

Let us now consider the species $\Phi(X, Y)$ whose structures on (U, V) are all functions $f: U \rightarrow V$, and whose transport of structure is defined by $\Phi[\sigma, \tau](f) = \tau \circ f \circ \sigma^{-1}$. We shall prove the combinatorial equality $\Phi(X, Y) = E(E(X)Y)$. Let $\Psi(X, Y) = E(E(X)Y)$ denote the right-hand side and define $\alpha_{UV}: \Phi[U, V] \rightarrow \Psi[U, V]$ by

$$\alpha_{UV}(f) = \{(f^{-1}(v), v) : v \in V\}.$$

To show that $\Phi(X, Y) = \Psi(X, Y)$ then amounts to proving that, for any pair of bijections $\sigma : U \rightarrow U'$ and $\tau : V \rightarrow V'$, the diagram

$$\begin{array}{ccc} \Phi[U, V] & \xrightarrow{\alpha_{UV}} & \Psi[U, V] \\ \Phi[\sigma, \tau] \downarrow & & \downarrow \Psi[\sigma, \tau] \\ \Phi[U', V'] & \xrightarrow{\alpha_{U'V'}} & \Psi[U', V'] \end{array}$$

commutes. By direct calculation we have

$$\begin{aligned} \Psi[\sigma, \tau](\alpha_{UV}(f)) &= \{((\sigma \circ f^{-1})(v), \tau(v)) : v \in V\}; \\ \alpha_{U'V'}(\Phi[\sigma, \tau](f)) &= \{((\sigma \circ f^{-1} \circ \tau^{-1})(v), v) : v \in V'\}. \end{aligned}$$

Since τ is a bijection these two sets are clearly equal.

Let $F(X, Y)$ be a two-sort species. Two F -structures $s \in F[U, V]$ and $t \in F[U', V']$ are said to have the same *isomorphism type*, and one writes $s \sim t$, if there are bijections $\sigma : U \rightarrow U'$ and $\tau : V \rightarrow V'$ such that $F[\sigma, \tau](s) = t$. The *type generating series* of $F(X, Y)$ is

$$\tilde{F}(x, y) = \sum_{n, k \geq 0} |F[n, k]/\sim| x^n y^k,$$

where $F[n, k]/\sim$ denotes the set of equivalence classes of $F[n, k]$ with respect to \sim .

Two F -structures $s \in F[U, V]$ and $t \in F[U', V']$ are said to have the same *isomorphism type according to the sort X* , and one writes $s \sim_X t$, if $V = V'$ and there is a bijection $\sigma : U \rightarrow U'$ such that $F[\sigma, \text{Id}_V](s) = t$. Similarly, s and t are said to have the same *isomorphism type according to the sort Y* , and one writes $s \sim_Y t$, if $U = U'$ and there is a bijection $\tau : V \rightarrow V'$ such that $F[\text{Id}_U, \tau](s) = t$. It's easy to see that $\sim_X \circ \sim_Y = \sim = \sim_Y \circ \sim_X$, in which the circle denotes composition of relations, and thus \sim_X and \sim_Y refine (factor) \sim in a natural way. Define $[s]_X = \{t : t \sim_X s\}$ and $[s]_Y = \{t : t \sim_Y s\}$ as the equivalence classes of s with respect to \sim_X and \sim_Y .

As an example, let us revisit the species $\Phi(X, Y)$ of functions $f : U \rightarrow V$. For $f \in \Phi[U, V]$ and $g \in \Phi[U', V]$ we have

$$\begin{aligned} g \sim_X f &\iff \Phi[\sigma, \text{Id}_V](g) = f \quad \text{for some bijection } \sigma : U \rightarrow U' \\ &\iff g = f \circ \sigma \quad \text{for some bijection } \sigma : U \rightarrow U'. \end{aligned}$$

To be even more concrete, let $U = V = \{1, 2, 3\}$ and represent $f \in \Phi[U, V]$ by $f(1)f(2)f(3)$. As we have just seen, $g \sim_X f$ in $\Phi[U, V]$ precisely when f and g are equal up to a permutation of U . Thus the set of equivalence classes of $\Phi[U, V]$ with respect to \sim_X is

$$\{[111]_X, [112]_X, [113]_X, [122]_X, [123]_X, [133]_X, [222]_X, [223]_X, [233]_X, [333]_X\},$$

where we have chosen the lexicographically smallest representative for each equivalence class. Why are there 10 equivalence classes in this case? Well, up to a permutation of U , a function from U to $V = \{1, 2, 3\}$ is determined by the number of elements, x_i , that f

maps to $i \in V$. Every element of U has to be mapped to exactly one element of V and thus $x_1 + x_2 + x_3 = |U| = 3$. From elementary combinatorics we know that the number of nonnegative solutions to this equation is $\binom{|U|+|V|-1}{|U|} = \binom{5}{3} = 10$.

For $f \in \Phi[U, V]$ and $g \in \Phi[U, V']$, we have

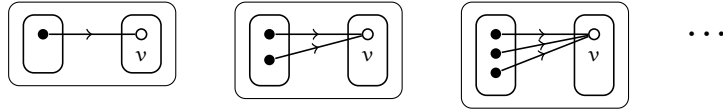
$$\begin{aligned} g \sim_Y f &\iff \Phi[\text{Id}_U, \tau](g) = f \quad \text{for some bijection } \tau : V \rightarrow V' \\ &\iff \tau \circ g = f \quad \text{for some bijection } \tau : V \rightarrow V'. \end{aligned}$$

In particular, if $V = V'$ then $f \sim_Y g$ precisely when f and g are equal up to a permutation of V . Now, up to a permutation of V , a function $f \in \Phi[U, V]$ is determined by the set of its nonempty fibres $\{f^{-1}(v) : v \in f(U)\}$, sometimes called the coimage of f . This set forms a partition of U into at most $|V|$ blocks and hence the number of equivalence classes of $\Phi[U, V]$ with respect to \sim_Y is $S(n, 0) + S(n, 1) + \dots + S(n, k)$, where $n = |U|$ and $k = |V|$. If $U = V = \{1, 2, 3\}$, then the $S(3, 1) + S(3, 2) + S(3, 3) = 1 + 3 + 1 = 5$ equivalence classes are $[111]_Y$, $[112]_Y$, $[121]_Y$, $[122]_Y$, and $[123]_Y$.

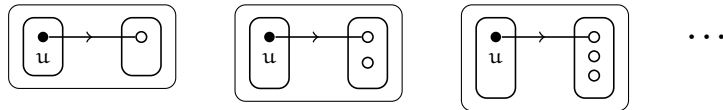
We now continue our development of the general framework. Following Joyal [6] we shall use the notation

$$\begin{aligned} \widetilde{\sum}_U F[U, V] &= \{ [s]_X : s \in F[U, V] \text{ for some finite set } U \}; \\ \widetilde{\sum}_V F[U, V] &= \{ [s]_Y : s \in F[U, V] \text{ for some finite set } V \}. \end{aligned}$$

We would like to define a unisort “species of types” whose structures are precisely the elements of $\widetilde{\sum}_U F[U, V]$. This doesn’t, however, quite fit our current species framework, the reason being that the set $\widetilde{\sum}_U F[U, V]$ could be infinite. For instance, consider (again) the species $\Phi(X, Y)$ of functions $f : U \rightarrow V$. For any nonempty finite sets U and V , both $\widetilde{\sum}_U \Phi[U, V]$ and $\widetilde{\sum}_V \Phi[U, V]$ are infinite. Indeed, if for simplicity we assume that $V = \{v\}$ is a singleton, then $\widetilde{\sum}_U \Phi[U, V]$ contains one element per positive size of the domain:



Similarly, if $U = \{u\}$ is a singleton, then $\widetilde{\sum}_V \Phi[U, V]$ contains one element per positive size of the codomain:



We’ll handle this using weighted species, but first we need to generalize the notion of an R -weighted species. Recall that we defined an R -weighted set to be a pair (A, w) , where A is a finite set and $w : A \rightarrow R$. Let us say that (A, w) is a *summable R -weighted set* if A is a (possibly infinite) set, $w : A \rightarrow R$, and the total weight $|A|_w = \sum_{a \in A} w(a)$ exists. Then an R -weighted species is a functor $F : \mathbb{B} \rightarrow (\text{Set}/R)$, where (Set/R) is the category of summable R -weighted sets with morphisms of R -weighted sets as arrows.

We are now ready to define the desired species of types. For any two-sort species $F(X, Y)$ we define the $\mathbb{Z}[[x]]$ -weighted species $\widetilde{\sum}_U F[U, Y]$ as follows: For any finite set V , let the

collection of structures on V be the summable $\mathbb{Z}[[x]]$ -weighted set $(\widetilde{\sum}_U F[U, V], w)$, where for any s in $F[U, V]$ the weight of $[s]_X$ in $\widetilde{\sum}_U F[U, V]$ is $w(s) = x^{|U|}$. For any bijection $\tau : V \rightarrow V'$, define the transport of structure $\widetilde{\sum}_U F[U, \tau]$ as the map

$$[s]_X \longmapsto [F[\text{Id}_U, \tau](s)]_X,$$

which is clearly weight-preserving. The resulting species, $\widetilde{\sum}_U F[U, Y]$, is called the *species of types according to X* ; the species $\widetilde{\sum}_V F(X, V]$ of types according to Y is defined analogously. We'll use the following notation for their generating series:

$$\begin{aligned}\widetilde{\sum}_U F[U, y] &= \sum_{n, k \geq 0} |F[n, k]/\sim_X| x^n \frac{y^k}{k!}, \\ \widetilde{\sum}_V F(x, V] &= \sum_{n, k \geq 0} |F[n, k]/\sim_Y| \frac{x^n}{n!} y^k,\end{aligned}$$

where $F[U, V]/\sim_X = \{[s]_X : s \in F[U, V]\}$ and $F[U, V]/\sim_Y = \{[s]_Y : s \in F[U, V]\}$ are the sets of equivalence classes with respect to the relations \sim_X and \sim_Y . Let us briefly return to the $\Phi(X, Y)$ species for an example. We discussed the equivalence classes of $\Phi[U, V]$ with respect to \sim_X and \sim_Y in some detail above, and from that discussion it is clear that

$$\begin{aligned}\widetilde{\sum}_U \Phi[U, y] &= \sum_{n, k \geq 0} \binom{n+k-1}{n} x^n \frac{y^k}{k!}; \\ \widetilde{\sum}_V \Phi(x, V] &= \sum_{n, k \geq 0} (S(n, 0) + \dots + S(n, k)) \frac{x^n}{n!} y^k.\end{aligned}$$

The cycle index series for a two-sort species $F(X, Y)$ is defined by

$$Z_F(x_1, x_2, \dots; y_1, y_2, \dots) = \sum_{\substack{n \geq 0 \\ k \geq 0}} \frac{1}{n!k!} \sum_{\substack{\sigma \in S[n] \\ \tau \in S[k]}} |\text{Fix } F[\sigma, \tau]| x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} \dots y_1^{c_1(\tau)} y_2^{c_2(\tau)} \dots$$

One can show that if $F(X, Y)$, $G(X, Y)$, and $H(X, Y)$ are two-sort species, then $Z_{F+G} = Z_F + Z_G$, $Z_{FG} = Z_F Z_G$, and

$$\begin{aligned}Z_{F(G, H)}(x_1, x_2, x_3, \dots; y_1, y_2, y_3, \dots) &= \\ Z_F(Z_G(x_1, x_2, x_3, \dots), Z_G(x_2, x_4, x_6, \dots), Z_G(x_3, x_6, x_9, \dots), \dots; \\ Z_H(y_1, y_2, y_3, \dots), Z_H(y_2, y_4, y_6, \dots), Z_H(y_3, y_6, y_9, \dots), \dots)\end{aligned}$$

The major generating series associated with a two-sort species relate to the cycle index series as follows:

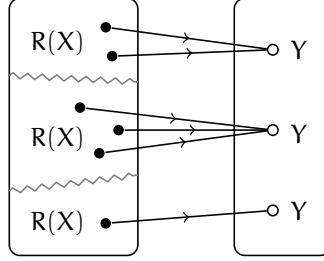
$$\begin{aligned}F(x, y) &= Z_F(x, 0, 0, 0, \dots; y, 0, 0, 0, \dots); \\ \widetilde{\sum}_U F[U, y] &= Z_F(x, x^2, x^3, \dots; y, 0, 0, 0, \dots); \\ \widetilde{\sum}_V F(x, V] &= Z_F(x, 0, 0, 0, \dots; y, y^2, y^3, \dots); \\ \widetilde{F}(x, y) &= Z_F(x, x^2, x^3, \dots; y, y^2, y^3, \dots).\end{aligned}$$

3. The twelvefold way

Let $R(X)$ be a unisort species and consider the two-sort species

$$\Phi_R(X, Y) = E(R(X)Y)$$

of functions with R -enriched fibres [1, p. 113]:



Note that if $R(X) = E(X)$ is the set species, then $\Phi_R(X, Y) = \Phi(X, Y)$ is the species of all functions $f : U \rightarrow V$ from above. If $R(X) = E_+(X)$ is the species of nonempty sets, then we are requiring that each fibre $f^{-1}(v) = \{u \in U : f(u) = v\}$ is nonempty, and so f is surjective. If $R(X) = 1 + X$, then each fibre is empty or a singleton, and we get injective functions.

Lemma 1. *Let $\Phi_R(X, Y) = E(R(X)Y)$ be as above. Then*

$$\Phi_R(x, y) = \exp(yR(x));$$

$$\widetilde{\sum}_U \Phi_R[U, y] = \exp(y\widetilde{R}(x));$$

$$\widetilde{\sum}_V \Phi_R(x, V] = \exp(y(R(x) - R(0)))(1 - y)^{-R(0)};$$

$$\widetilde{\Phi}_R(x, y) = \exp \sum_{k \geq 1} \frac{y^k}{k} \widetilde{R}(x^k).$$

Proof. The first equality is immediate. Calculating the cycle index series we find that

$$Z_{\Phi_R}(x_1, x_2, x_3, \dots; y_1, y_2, y_3, \dots) = \exp \sum_{k \geq 1} \frac{y_k}{k} Z_R(x_k, x_{2k}, x_{3k}, \dots) \quad (2)$$

and the remaining three identities follow from calculating $Z_{\Phi_R}(x, x^2, x^3, \dots; y, 0, 0, 0, \dots)$, $Z_{\Phi_R}(x, 0, 0, 0, \dots; y, y^2, y^3, \dots)$, and $Z_{\Phi_R}(x, x^2, x^3, \dots; y, y^2, y^3, \dots)$. Let us do the second of these three calculations. By isolating the first term in the sum on the right-hand side of (2) it follows that

$$\begin{aligned} \widetilde{\sum}_V \Phi_R(x, V] &= Z_{\Phi_R}(x, 0, 0, 0, \dots; y, y^2, y^3, \dots) \\ &= \exp \left(y Z_R(x, 0, 0, \dots) + \sum_{k \geq 2} \frac{y^k}{k} Z_R(0, 0, 0, \dots) \right). \end{aligned}$$

Since $R(x) = Z_R(x, 0, 0, \dots)$ this can be further simplified as follows:

$$\begin{aligned}\widetilde{\sum}_V \Phi_R(x, V) &= \exp\left(yR(x) + R(0) \sum_{k \geq 2} \frac{y^k}{k}\right) \\ &= \exp\left(yR(x) - yR(0) + R(0) \log(1 - y)^{-1}\right) \\ &= \exp\left(y(R(x) - R(0))\right) (1 - y)^{-R(0)}.\end{aligned}$$

We leave verification of the remaining two identities to the reader. \square

As an example, let us apply the above lemma when $R(X) = X$. Then each fibre is a singleton and Φ_X is the species of bijections $f : U \rightarrow V$. The four generating series are

$$\Phi_X(x, y) = e^{xy} = \sum_{n \geq 0} n! \cdot \frac{x^n y^n}{(n!)^2}; \quad (3)$$

$$\widetilde{\sum}_U \Phi_X[U, y] = e^{xy} = \sum_{n \geq 0} 1 \cdot x^n \frac{y^n}{n!}; \quad (4)$$

$$\widetilde{\sum}_V \Phi_X(x, V] = e^{xy} = \sum_{n \geq 0} 1 \cdot \frac{x^n}{n!} y^n; \quad (5)$$

$$\widetilde{\Phi}_X(x, y) = \exp \sum_{k \geq 1} \frac{x^k y^k}{k} = \frac{1}{1 - xy} = \sum_{n \geq 0} 1 \cdot x^n y^n. \quad (6)$$

In other words, to distribute n balls into k urns so that each urn gets exactly one ball is impossible unless there are as many urns as there are balls, that is, unless $n = k$. And, if $n = k$, then there are $n!$ ways to distribute the balls if balls and urns are distinct, and there is exactly one way if the balls or the urns are identical.

Lemma 2. *Let $A(X)$ and $B(X)$ be two unisort species, and let the two-sort species $\Phi_R(X, Y) = E(R(X)Y)$ be as above. Then*

$$\Phi_{A+B}(X, Y) = \Phi_A(X, Y) \Phi_B(X, Y).$$

In particular, if $R(X) = 1 + R_+(X)$, where $R_+(X)$ is the species of nonempty R -structures, then

$$\Phi_R(X, Y) = E(Y) \Phi_{R_+}(X, Y).$$

Proof. For the first statement we have

$$\Phi_{A+B} = E((A + B)(X)Y) = E(A(X)Y + B(X)Y) = E(A(X)Y)E(B(X)Y) = \Phi_A \Phi_B.$$

The second statement is simply $\Phi_1(X, Y) = E(Y)$. \square

Corollary 3. *Let $A(X)$ and $B(X)$ be two unisort species, and let $\Phi_R(X, Y)$ be as above. Then $Z_{\Phi_{A+B}} = Z_{\Phi_A} Z_{\Phi_B}$. In particular, if $R(X) = 1 + R_+(X)$ then $Z_{\Phi_R} = Z_{E(Y)} Z_{\Phi_{R_+}}$, and consequently*

$$\begin{aligned}
\Phi_R(x, y) &= E(y) \cdot \Phi_{R_+}(x, y), \\
\widetilde{\sum}_U \Phi_R[U, y] &= E(y) \cdot \widetilde{\sum}_U \Phi_{R_+}[U, y], \\
\widetilde{\sum}_V \Phi_R(x, V] &= \widetilde{E}(y) \cdot \widetilde{\sum}_V \Phi_{R_+}(x, V], \\
\widetilde{\Phi}_R(x, y) &= \widetilde{E}(y) \cdot \widetilde{\Phi}_{R_+}(x, y),
\end{aligned}$$

in which $E(y) = e^y$ and $\widetilde{E}(y) = 1/(1 - y)$.

The twelfefold way can be seen as the twelve generating series obtained from calculating, for each $R(X) \in \{E(X), E_+(X), 1 + X\}$, the four generating series in Lemma 1. Note that it suffices to know $R(x)$ and $\widetilde{R}(x)$ to calculate those four series. For reference we record $R(x)$ and $\widetilde{R}(x)$ in Table 1. We shall run through the cases of this table going from top

$R(X)$	$R(x)$	$\widetilde{R}(x)$
X	x	x
$1 + X$	$1 + x$	$1 + x$
$E_+(X)$	$e^x - 1$	$x(1 - x)^{-1}$
$E(X)$	e^x	$(1 - x)^{-1}$
$L_+(X)$	$x(1 - x)^{-1}$	$x(1 - x)^{-1}$
$L(X)$	$(1 - x)^{-1}$	$(1 - x)^{-1}$
$\mathcal{C}(X)$	$\log(1 - x)^{-1}$	$x(1 - x)^{-1}$

Table 1: Some (type) generating series

to bottom. Having already done $R(X) = X$, we proceed with $R(X) = 1 + X$. By direct calculation, or from Corollary 3 and equations (3) through (6), we immediately get

$$\Phi_{1+X}(x, y) = e^{y(1+x)} = \sum_{n, k \geq 0} (k)_n \frac{x^n y^k}{n! k!}, \quad (7)$$

$$\widetilde{\sum}_U \Phi_{1+X}[U, y] = e^{y(1+x)} = \sum_{n, k \geq 0} \binom{k}{n} x^n \frac{y^k}{k!}, \quad (8)$$

$$\widetilde{\sum}_V \Phi_{1+X}(x, V] = \frac{1}{1-y} e^{xy} = \sum_{n, k \geq 0} [n \leq k] \frac{x^n}{n!} y^k, \quad (9)$$

$$\widetilde{\Phi}_{1+X}(x, y) = \frac{1}{1-y} \cdot \frac{1}{1-xy} = \sum_{n, k \geq 0} [n \leq k] x^n y^k, \quad (10)$$

where $(k)_n = k(k-1) \cdots (k-n+1)$ denotes the falling factorial and $[P]$ is the Iverson bracket, which is 1 if the proposition P is true and 0 if it is false.

Assuming that each urn contains at most one ball, there are thus $(k)_n$ ways to distribute n distinct balls into k distinct urns; there are $\binom{k}{n}$ ways to distribute n identical balls into k distinct urns; if $n \leq k$, then there's exactly one way to distribute n balls (distinct or identical) into k identical urns, and if $n > k$ there's no way to distribute the balls.

As a side note, the generating series in (8) is the same as for the $\mathbb{Z}[x]$ -weighted species $\mathcal{P}_w(Y) = E_w(Y)E(Y)$ in which the weight of an E_w -structure is $w(A) = x^{|A|}$. Indeed, those species are combinatorially equal: $\widetilde{\sum}_U \Phi_{1+X}[U, Y] = \mathcal{P}_w(Y)$.

Let us now consider $R(X) = E_+(X)$. Using Lemma 1 we find that

$$\Phi_{E_+}(x, y) = \exp(y(e^x - 1)) = \sum_{n, k \geq 0} k! S(n, k) \frac{y^k x^n}{k! n!}, \quad (11)$$

$$\widetilde{\sum}_U \Phi_{E_+}[U, y] = \exp\left(\frac{xy}{1-x}\right) = \sum_{n, k \geq 0} \binom{n-1}{n-k} x^n \frac{y^k}{k!}, \quad (12)$$

$$\widetilde{\sum}_V \Phi_{E_+}(x, V) = \exp(y(e^x - 1)) = \text{Par}_w(x) = \sum_{n, k \geq 0} S(n, k) \frac{x^n}{n!} y^k, \quad (13)$$

$$\widetilde{\Phi}_{E_+}(x, y) = \exp \sum_{k \geq 1} \frac{y^k}{k} \frac{x^k}{1-x^k} = \prod_{k \geq 1} \frac{1}{1-yx^k} = \sum_{n, k \geq 0} p_k(n) x^n y^k, \quad (14)$$

where $S(n, k)$ is a Stirling number of the second kind and $p_k(n)$ is the number of integer partitions of n with k parts. Note that the second equality in (14) is the same as (1) on page 11, above. The interpretation of the formulas above in terms of balls and urns should at this stage be clear.

The last case of the twelvefold way is $R(X) = E(X)$. Since $E(X) = 1 + E_+(X)$ we may use Corollary 3 when convenient, and we have

$$\Phi_E(x, y) = \exp(ye^x) = \sum_{n, k \geq 0} k^n \frac{x^n y^k}{n! k!}; \quad (15)$$

$$\widetilde{\sum}_U \Phi_E[U, y] = \exp\left(\frac{y}{1-x}\right) = \sum_{n, k \geq 0} \binom{n+k-1}{n} x^n \frac{y^k}{k!}; \quad (16)$$

$$\widetilde{\sum}_V \Phi_E(x, V) = \frac{1}{1-y} \widetilde{\sum}_V \Phi_{E_+}(x, V) = \sum_{n, k \geq 0} (S(n, 0) + \dots + S(n, k)) \frac{x^n}{n!} y^k; \quad (17)$$

$$\widetilde{\Phi}_E(x, y) = \frac{1}{1-y} \widetilde{\Phi}_{E_+}(x, y) = \sum_{n, k \geq 0} (p_0(n) + \dots + p_k(n)) x^n y^k. \quad (18)$$

Bogart's [2] twentyfold way is a natural extension of Rota's twelvefold way, which, in our setting, amounts to having more choices for $R(X)$. For the twelvefold way $R(X)$ is one of $E(X)$, $E_+(X)$, or $1 + X$. For the twentyfold way there are three additional options: X , $L(X)$, and $L_+(X)$. At first glance this appears to give us $6 \cdot 4 = 24$ cases, but since $\widetilde{E}(x) = \widetilde{L}(x)$ and $\widetilde{E}_+(x) = \widetilde{L}_+(x)$ there are only 20 distinct cases.

Let us explore these additional cases, and let us start with $R(X) = L_+(X)$. This gives us two new series (the other two series being the same as for $R(X) = E_+(X)$):

$$\Phi_{L_+}(x, y) = \exp\left(\frac{xy}{1-x}\right) = \sum_{n, k \geq 0} k! L(n, k) \frac{x^n y^k}{n! k!},$$

$$\widetilde{\sum}_V \Phi_{L_+}(x, V) = \exp\left(\frac{xy}{1-x}\right) = \sum_{n, k \geq 0} L(n, k) \frac{x^n}{n!} y^k,$$

$R(X)$	$\Phi_R(x, y)$	$\widetilde{\sum}_U \Phi_R[U, y]$	$\widetilde{\sum}_V \Phi_R(x, V)$	$\widetilde{\Phi}_R(x, y)$
X	$\exp(xy)$	$\exp(xy)$	$\exp(xy)$	$\frac{1}{1-xy}$
$1 + X$	$\exp(y(1+x))$	$\exp(y(1+x))$	$\frac{1}{1-y} \exp(xy)$	$\frac{1}{1-y} \cdot \frac{1}{1-xy}$
$E_+(X)$	$\exp(y(e^x - 1))$	$\exp\left(\frac{xy}{1-x}\right)$	$\exp(y(e^x - 1))$	$\prod_{k \geq 1} \frac{1}{1-yx^k}$
$E(X)$	$\exp(ye^x)$	$\exp\left(\frac{y}{1-x}\right)$	$\frac{1}{1-y} \exp(y(e^x - 1))$	$\frac{1}{1-y} \prod_{k \geq 1} \frac{1}{1-yx^k}$
$L_+(X)$	$\exp\left(\frac{xy}{1-x}\right)$	See $R(X) = E_+(X)$	$\exp\left(\frac{xy}{1-x}\right)$	See $R(X) = E_+(X)$
$L(X)$	$\exp\left(\frac{y}{1-x}\right)$	See $R(X) = E(X)$	$\frac{1}{1-y} \exp\left(\frac{xy}{1-x}\right)$	See $R(X) = E(X)$
$\mathcal{C}(X)$	$(1-x)^{-y}$	See $R(X) = E_+(X)$	$(1-x)^{-y}$	See $R(X) = E_+(X)$

Table 2: The twenty-two-fold way

where $L(n, k) = \binom{n}{k} (n-1)_{n-k}$ denotes a Lah number [7]. Using the language of species, the standard combinatorial interpretation of these numbers is as follows: Define the species $\text{Lah} = E(L_+)$, so that a Lah-structure is a set of nonempty disjoint linear orders, such as $\{63, 9, 4815, 27\} \in \text{Lah}[9]$. The number of Lah-structures on $[n]$ with k blocks / linear orders is $L(n, k)$.

With $R(X) = L(X)$ the two new series are

$$\Phi_L(x, y) = \exp\left(\frac{y}{1-x}\right) = \sum_{n, k \geq 0} (n+k-1)_n \frac{x^n y^k}{n! k!},$$

$$\widetilde{\sum}_V \Phi_L(x, V) = \frac{1}{1-y} \widetilde{\sum}_V \Phi_{L_+}(x, V) = \sum_{n, k \geq 0} (L(n, 0) + \cdots + L(n, k)) \frac{x^n y^k}{n!},$$

and this concludes the twentyfold way.

We have seen Stirling numbers of the second kind ($R(X) = E_+(X)$). We have also seen Lah numbers ($R(X) = L_+(X)$), which are sometimes referred to as Stirling numbers of the third kind [9]. But what about the Stirling numbers of the first kind? Is there a choice of $R(X)$ which gives those numbers? Note that the species constructions for the Stirling numbers of the second and third kind follow a similar pattern: $\text{Par}(X) = E(E_+(X))$ and $\text{Lah}(X) = E(L_+(X))$. Assume, more generally, that the species $R(X)$ and $R^c(X)$ are related by $R(X) = E(R^c(X))$, so that an R -structure is a set of disjoint R^c -structures. We may then call R^c the species of *connected* R -structures [1, p. 46].

Proposition 4. *Assume that $R(X) = E(R^c(X))$ and define the $\mathbb{Z}[y]$ -weighted species $R_w(X) = E(yR^c(X))$, where $yR^c(X)$ is like $R^c(X)$ but with a weight y given to each structure. Then*

$$\widetilde{\sum}_V \Phi_{R^c}(X, V) = R_w(X).$$

Proof. A typical $\widetilde{\sum}_V \Phi_{R^c}(X, V)$ -structure on a finite set U is an equivalence class $[s]_Y$, where $s \in \Phi_{R^c}[U, V]$ for some finite set $V = \{v_1, \dots, v_k\}$. This representative, s , can in turn be written $s = \{(s_v, v) : s_v \in R^c[B_v], v \in V\}$ for some partition $\{B_{v_1}, \dots, B_{v_k}\}$ of U . Define $\alpha_U : \widetilde{\sum}_V \Phi_{R^c}[U, V] \rightarrow R_w[U]$ by $\alpha_U(s) = \{s_v : v \in V\}$. It's easy to see that this definition is independent of the representative for $[s]_Y$. It preserves weights too: the weight of $[s]_Y$ is $y^{|V|} = y^k$, which is also the weight of $\{s_v : v \in V\}$. Further, for any bijection $\sigma : U \rightarrow U'$, the relevant diagram commutes and thus α_U is the desired natural isomorphism. \square

This proposition applies to every other row of Table 2: the connected components of a set, $E(X)$, are the points, X ; the connected components of a partition, $\text{Par}(X)$, are the blocks, $E_+(X)$; the connected components of a Lah-structure are the linear orders, $L_+(X)$; and the connected components of a permutation are the cycles, $\mathcal{C}(X)$. See also Figure 1 for an illustration of these cases. Letting $E_w(X) = E(yX)$, $\text{Par}_w(X) = E(yE_+(X))$, $\text{Lah}_w(X) = E(yL_+(X))$, and $\mathcal{S}_w(X) = E(y\mathcal{C}(X))$, and applying Proposition 4 we have

$$\begin{aligned} \widetilde{\sum}_V \Phi_X(X, V) &= E_w(X); & \widetilde{\sum}_V \Phi_{E_+}(X, V) &= \text{Par}_w(X); \\ \widetilde{\sum}_V \Phi_{L_+}(X, V) &= \text{Lah}_w(X); & \widetilde{\sum}_V \Phi_{\mathcal{C}}(X, V) &= \mathcal{S}_w(X). \end{aligned}$$

We could add to this list by considering other natural pairs $R(X)$ and $R^c(X)$ such as graphs and connected graphs, or forests and trees.

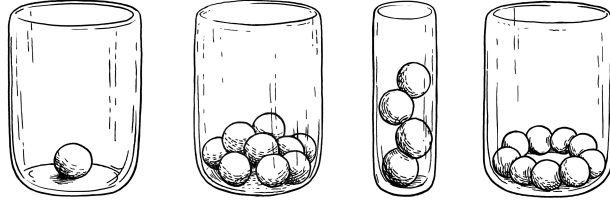


Figure 1: Urns illustrating $R(X) = X$, $E_+(X)$, $L_+(X)$, and $\mathcal{C}(X)$

Returning to the Stirling numbers, we see from above that

$$\widetilde{\sum}_V \Phi_{\mathcal{C}}(x, V) = \mathcal{S}_w(x) = (1-x)^{-y} = \sum_{n,k \geq 0} c(n, k) \frac{x^n}{n!} y^k,$$

where $c(n, k)$ denotes an unsigned Stirling number of the first kind. The other three series for $R(X) = \mathcal{C}(X)$ are $\widetilde{\sum}_U \Phi_{\mathcal{C}}[U, y] = \widetilde{\sum}_U \Phi_{E_+}[U, y]$, $\widetilde{\Phi}_{\mathcal{C}}(x, y) = \widetilde{\Phi}_{E_+}(x, y)$, and $\Phi_{\mathcal{C}}(x, y) = (1-x)^{-y} = \sum_{n,k \geq 0} k! c(n, k) \frac{x^n}{n!} \frac{y^k}{k!}$. Each choice of $R(X)$ that we have discussed up to this point can be found in Table 1, and the four series associated with that choice are listed in Table 2.

We will stop here, but note that there are, of course, other choices for $R(X)$ that may be natural, but we have omitted. For instance, setting

$$R(X) = E_{\text{odd}}(X) = E_1(X) + E_3(X) + \dots$$

would require that each urn contains an odd number of balls.

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