Rosenberg-Type Completeness Criteria for Subclones of Slupecki's Clone

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***Abstract*—We describe all clones on a finite set with at least three elements, which are maximal for the property of not containing all nonsurjective operations. We deduce Rosenberg-type completeness criteria for every subclone of Slupecki's clone that contains all nonsurjective operations. As another application, we find all subclones of Słupecki's clone for which the associated R-relation has only finitely many classes.**

***Keywords*-completeness, Slupecki's clone, maximal clone**

I. Introduction

Słupecki's clone $\mathcal{S}$ consists of all operations on a finite set A ($|A| \geq 3$) which are either essentially unary or nonsurjective. The fact that $\mathcal{S}$ is the only maximal clone containing all unary operations on A underlies one of the oldest completeness criteria, due to Słupecki [11]. Despite the significance of $\mathcal{S}$, not much is known about its subclones, except for those that are fairly large, like the clones containing all permutations or all nonsurjective unary operations on A. These clones have been described by Haddad and Rosenberg [2], and by Szabó (unpublished, see Theorem 3), respectively. The results in [2] also yield a completeness criterion for $\mathcal{S}$.

In this paper we focus on clones that are not very large in the sense that they do not contain the clone $\mathcal{S}^-$ of all nonsurjective operations (a subclone of $\mathcal{S}$). Our main result (Theorem 7), which is stated in Section 3, is a description of the collection $\mathfrak{M}_A$ of all clones on A that are maximal for the property of not containing $\mathcal{S}^-$. Clearly, the maximal clones classified by Rosenberg [10], with the exception of $\mathcal{S}$, all belong to $\mathfrak{M}_A$; the novelty in Theorem 7 is that we also find all subclones of $\mathcal{S}$ that belong to $\mathfrak{M}_A$. In Section 4 we use this result to derive completeness criteria for all clones $\mathcal{U}$ such that $\mathcal{S}^- \subseteq \mathcal{U} \subseteq \mathcal{S}$.

Theorem 7 also contributes to our understanding of the family $\mathfrak{F}_A$ of all clones for which the associated $\mathcal{R}$-relation has only finitely many classes; here, by the $\mathcal{R}$-relation associated to a clone $\mathcal{C}$ we mean the equivalence relation that relates two operations on A if and only if they can be obtained from one another by substituting operations from $\mathcal{C}$ for their variables. It was proved in [7] that every clone $\mathcal{U}$ satisfying $\mathcal{S}^- \subseteq \mathcal{U} \subseteq \mathcal{S}$ belongs to $\mathfrak{F}_A$, but was left open whether there are any other subclones of $\mathcal{S}$ in $\mathfrak{F}_A$. In Section 5 we use Theorem 7 to prove that there are no other subclones of $\mathcal{S}$ in $\mathfrak{F}_A$.

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II. Preliminaries

A. Clones

Let A be a fixed set, and let m, n be positive integers.

An *n-ary operation* on A is a function $A^n \to A$. We will use the notation $\mathcal{O}^{(n)}$ for the set of all n-ary operations on A, and $\mathcal{O}$ for the set $\bigcup_{n=1}^{\infty} \mathcal{O}^{(n)}$ of all finitary operations on A.

For $1 \leq i \leq n$, the *i-th n-ary projection* on A is the operation $\pi_i^{(n)}\colon A^n \to A$, $(a_1, \ldots, a_n) \mapsto a_i$. The *composition* of an n-ary operation $f \in \mathcal{O}^{(n)}$ with an n-tuple $(g_1, \ldots, g_n)$ of m-ary operations $g_i \in \mathcal{O}^{(m)}$ is the m-ary operation $f(g_1, \ldots, g_n)\colon A^m \to A$, $\overline{a} \mapsto f\big(g_1(\overline{a}), \ldots, g_n(\overline{a})\big)$.

A *clone* on A is a set $\mathcal{C} \subseteq \mathcal{O}$ such that $\mathcal{C}$ contains all projections and is closed under composition. Thus, $\mathcal{O}$ is a clone on A, and so is the set $\mathcal{P}$ of all projections. If $\mathcal{C}$ and $\mathcal{D}$ are clones on A such that $\mathcal{C} \subseteq \mathcal{D}$, we say that $\mathcal{C}$ is a *subclone* of $\mathcal{D}$. The collection of all clones on A, ordered by $\subseteq$, is an algebraic lattice with largest element $\mathcal{O}$ and least element $\mathcal{P}$. Therefore, for every set $F \subseteq \mathcal{O}$ of operations, there is a least clone containing F, which is denoted by $\langle F \rangle$, and is called *the clone generated by* F.

For the *the full transformation monoid* $T := \mathcal{O}^{(1)}$ *on* A, the members of the clone $\mathcal{S}_0 := \langle T \rangle$ are exactly the operations of the form $f(\pi_i^{(n)})$ where $f \in T$ and $n \geq 1$, $1 \leq i \leq n$. The operations in $\mathcal{S}_0$ will be called *essentially unary operations*, and $\mathcal{S}_0$ will be referred to as *the clone of essentialy unary operations*.

An *m-ary relation* on A is a subset of A^m. For an n-ary operation f and an m-ary relation ρ on A, we say that f *preserves* ρ if whenever $\overline{a}_1, \ldots \overline{a}_n$ are m-tuples in ρ, then the m-tuple $f(\overline{a}_1, \ldots, \overline{a}_n)$ obtained by applying f coordinatewise also belongs to ρ. For arbitrary relation ρ on A, $\{\rho\}^\perp$ will denote the set of all operations on A that preserve ρ. It is well known and easy to check that $\{\rho\}^\perp$ is a clone on A.

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B. Completeness

Given a set A and a clone $\mathcal{C}$ on A, a subset F of $\mathcal{C}$ is said to be *complete* in $\mathcal{C}$ if $\mathcal{C} = \langle F \rangle$. For two clones $\mathcal{M}$ and $\mathcal{C}$ on A, $\mathcal{M}$ is said to be a *maximal subclone* of $\mathcal{C}$ if $\mathcal{M} \subsetneq \mathcal{C}$ and there is no clone $\mathcal{D}$ such that $\mathcal{M} \subsetneq \mathcal{D} \subsetneq \mathcal{C}$.

The following theorem serves as a background for finding efficient completeness criteria for finitely generated clones on finite sets.

Theorem 1 ([5],[8],[12])**.** *If $\mathcal{C}$ is a finitely generated clone on a finite set A, then*

1. *every proper subclone of $\mathcal{C}$ is contained in a maximal subclone of $\mathcal{C}$,*
2. *$\mathcal{C}$ has finitely many maximal subclones, and*
3. *every maximal subclone of $\mathcal{C}$ is of the form $\mathcal{C} \cap \{\rho\}^{\perp}$ for some relation ρ on A.*

It follows that, under the same assumptions on $\mathcal{C}$ as in Theorem 1, if we find a manageable finite set R of relations on A such that all maximal subclones of $\mathcal{C}$ are among the clones $\mathcal{C} \cap \{\rho\}^{\perp}$ ($\rho \in R$), then we have an efficient completeness criterion for $\mathcal{C}$, namely: $F \subseteq \mathcal{C}$ is complete in $\mathcal{C}$ if and only if $F \not\subseteq \{\rho\}^{\perp}$ holds for all $\rho \in R$. Optimally, R is such that the clones $\mathcal{C} \cap \{\rho\}^{\perp}$ ($\rho \in R$) are exactly the maximal subclones of $\mathcal{C}$. Therefore, a completeness criterion for $\mathcal{C}$ is nothing else than a description of the maximal subclones of $\mathcal{C}$.

C. Słupecki's clone and some of its subclones

In this subsection A will be a fixed finite set with k elements ($k \geq 3$). *Słupecki's clone* on A is the clone $\mathcal{S}$ consisting of all operations on A which are either nonsurjective or essentially unary.

More generally, for every integer r with $2 \leq r \leq k$, let $\mathcal{S}_r$ denote the clone consisting of all operations f on A such that either f has range of size $\leq r$, or f is essentially unary. In particular, $\mathcal{S}_k = \mathcal{O}$ and $\mathcal{S}_{k-1} = \mathcal{S}$. It is known from results of Słupecki [11] and Burle [1] that $\mathcal{S}_2 \subsetneq \cdots \subsetneq \mathcal{S}_{k-2} \subsetneq \mathcal{S}_{k-1} \subsetneq \mathcal{S}_k$ is an unrefinable chain (i.e., each $\mathcal{S}_{r-1}$ with $3 \leq r \leq k$ is a maximal subclone of $\mathcal{S}_r$), and there is a unique clone $\mathcal{S}_1$ such that $\langle T \rangle = \mathcal{S}_0 \subsetneq \mathcal{S}_1 \subsetneq \mathcal{S}_2$; $\mathcal{S}_1$ is called *Burle's clone*. Moreover, the clones $\mathcal{S}_r$ ($0 \leq r \leq k$) are the only clones on A that contain $T = \mathcal{O}^{(1)}$.

We will need the description of these clones via relations. Let

$$\beta = \big\{(a_1, a_2, a_3, a_4) \in A^4 : a_1 = a_i \text{ and } a_j = a_k \\ \text{for some } i, j, k \text{ with } \{1, i, j, k\} = \{1, 2, 3, 4\}\big\},$$

and for $3 \leq m \leq k$ let

$$\iota_m = \{(a_1, \ldots, a_m) \in A^m : a_i = a_j \text{ for some } i \neq j\}.$$

Proposition 2 ([5],[8])**.** *If A is a k-element set ($k \geq 3$), then $\mathcal{S}_1 = \{\beta\}^{\perp}$ and $\mathcal{S}_{r-1} = \{\iota_r\}^{\perp}$ for all $3 \leq r \leq k$.*

The results of Słupecki and Burle mentioned above describe all clones on A which contain the full transformation monoid T on A. Haddad and Rosenberg [2] extended this to a complete description of all clones on A which contain the symmetric group (the group of all permutations) on A.

An unpublished result of Szabó gives an analogous description for all clones on A which contain the monoid T^- consisting of the identity transformation $\pi_1^{(1)}$ and all nonsurjective transformations on A. Clearly, every submonoid of T containing T^- is of the form $T^- \cup G$ for some permutation group G on A. For any permutation group G on A, and for any $0 \leq r \leq k-1$, let $\mathcal{S}_r[G]$ denote the subclone of $\mathcal{S}_r$ obtained by omitting all essentially unary operations $f \in \mathcal{S}_r$ for which $f(x, \ldots, x)$ is a permutation not in G. It is easy to check that $\mathcal{S}_r[G]$ is indeed a clone, and it contains T^-.

Theorem 3 (Szabó)**.** *If A is a finite set with $k \geq 3$ elements, then the proper subclones of $\mathcal{O}$ containing T^- are exactly the clones $\mathcal{S}_r[G]$ where $0 \leq r \leq k-1$ and G is a permutation group on A.*

If $G = \{\pi_1^{(1)}\}$ is the one-element permutation group, then $T^- \cup G = T^-$, and the clone $\mathcal{S}_{k-1}[G]$ consists of the projections and all nonsurjective operations on A. We will denote this clone by $\mathcal{S}^-$, and will refer to it as *the clone of nonsurjective operations* on A (although the projections in $\mathcal{S}^-$ are surjective).

D. Rosenberg's Completeness Theorem

In this subsection A will be a finite set with $k \geq 2$ elements. It is well known ([5],[8],[12]) that the clone $\mathcal{O}$ of all operations is finitely generated. Rosenberg's theorem [10] is a completeness theorem for $\mathcal{O}$, that is, a description of the maximal subclones of $\mathcal{O}$. For the special cases $k = 2, 3, 4$ the maximal subclones of $\mathcal{O}$ were determined earlier by Post [9], Jablonskiĭ [3], and Mal'tsev (unpublished, see [5, p. 163]).

To state Rosenberg's theorem we need some terminology and notation. An m-ary relation ρ on A is said to be *totally reflexive*, if $\iota_m \subseteq \rho$, and *totally symmetric*, if it is invariant under any permutation of its coordinates. We will call an equivalence relation with exactly m blocks an *m-equivalence relation*. For an m-equivalence relation θ on A with $3 \leq m \leq k$ let

$$\lambda_\theta = \{(a_1, \ldots, a_m) \in A^m : a_i \,\theta\, a_j \text{ for some } i \neq j\}.$$

Definition 4. Let A be a k-element set ($k \geq 2$).

BPO is the set of all bounded partial orders on A.

Perm is the set of all fixed point free permutations of prime order on A.

Affn is the set of all quaternary relations $\{(a, b, c, a-b+c) : a, b, c \in A\}$ where $(A; +)$ is an elementary abelian p-group (p prime).

Eq is the set of all equivalence relations θ on A, such that θ is neither the equality relation nor the full relation A^2.

Centr is the set of all *central relations* on A; that is, all relations $\rho \subsetneq A^m$ ($1 \le m \le k-1$) such that ρ is totally reflexive and totally symmetric, and there exists at least one element $c \in A$ for which $\{c\} \times A^{m-1} \subseteq \rho$.

Reg is the set of all *regular relations* on A; that is, all relations $\rho \subseteq A^m$ ($3 \le m \le k$) of the form $\bigcap_{\theta \in E} \lambda_\theta$ where E is a nonempty set of m-equivalence relations on A such that $\bigcap_{\theta \in E} B_\theta \neq \emptyset$ whenever B_θ is a block of θ for each $\theta \in E$. (The last condition requires the equivalence relation $\bigcap_{\theta \in E} \theta$ to have $m^{|E|}$ equivalence classes, therefore $1 \le |E| \le \log_m k$.)

Theorem 5 (Rosenberg [10])**.** *If A is a finite set with $k \ge 2$ elements, then $\mathcal{M}$ is a maximal subclone of $\mathcal{O}$ if and only if $\mathcal{M} = \{\rho\}^\perp$ for some*

$$\rho \in \mathsf{BPO} \cup \mathsf{Perm} \cup \mathsf{Affn} \cup \mathsf{Eq} \cup \mathsf{Centr} \cup \mathsf{Reg}.$$

As we saw in subsection II-C, Słupecki's clone $\mathcal{S} = \{\iota_k\}^\perp$ is a maximal subclone of $\mathcal{O}$. The relation ι_k appears on Rosenberg's list as $\iota_k = \lambda_=$. Therefore $\iota_k \in \mathsf{Reg}$.

III. Separation theorem for the clone $\mathcal{S}^-$ of nonsurjective operations

From now on A will be a fixed finite set with $k \ge 3$ elements. Our main theorem is a criterion for a set F of operation on A to have the property that $\mathcal{S}^- \subseteq \langle F \rangle$. This is stronger than a completeness crierion for $\mathcal{S}^-$, because we are not restricting F to be a subset of $\mathcal{S}^-$. In fact, we will see in the next section that this result yields completeness criteria not only for $\mathcal{S}^-$, but also for all clones containing $\mathcal{S}^-$.

Let $\mathfrak{P}$ be the collection of all clones $\mathcal{C}$ on A such that $\mathcal{S}^- \not\subseteq \mathcal{C}$. Clearly, $\mathfrak{P}$ is partially ordered by $\subseteq$, and is not empty (e.g., $\mathcal{P} \in \mathfrak{P}$). Since $\mathcal{S}^-$ is finitely generated, a standard Zorn Lemma argument shows that every clone in $\mathfrak{P}$ is contained in a maximal member of $\mathfrak{P}$. Our goal is to explicitly describe a set R of relations on A such that the maximal members of $\mathfrak{P}$ are exactly the clones $\{\rho\}^\perp$ with $\rho \in R$. As a consequence, we get that for $F \subseteq \mathcal{O}$ we have $\mathcal{S}^- \subseteq \langle F \rangle$ if and only if $F \not\subseteq \{\rho\}^\perp$ holds for all $\rho \in R$.

It is easy to see that every maximal subclone of $\mathcal{O}$ not containing $\mathcal{S}^-$ must be a maximal member of $\mathfrak{P}$. Since $T^- \subseteq \mathcal{S}^-$, Słupecki's clone is the only maximal subclone of $\mathcal{O}$ that contains $\mathcal{S}^-$. Therefore, our set R of relations will contain every relation from Rosenberg's list, except ι_k.

To state our result we need some terminology and notation. For $0 \le m \le k$, $\binom{A}{m}$ will denote the set of all subsets of A of size m.

Definition 6. Let A be a k-element set ($k \ge 3$).

Reg^* is the set $\mathsf{Reg} \setminus \{\iota_k\}$ of all regular relations different from ι_k.

aCentr is the set of all *almost central relations* on A; that is, all relations $\rho \subsetneq A^m$ ($2 \le m \le k-2$) such that ρ is not a central relation on A, but for all sets $D \in \binom{A}{k-1}$, either $\rho|_D = D^m$ or $\rho|_D$ is a central relation on D.

$\mathsf{aReg}_{\le k-2}$ is the set of all relations $\rho \subseteq A^m$ ($4 \le m \le k-2$) of the form $\bigcap_{\theta \in E} \lambda_\theta$ where E is a set of m-equivalence relations on A such that $|E| \ge 2$ and $B \cap B' = \emptyset$ holds for arbitrary nonsingleton blocks B and B' of distinct members of E. (These conditions force the unions of the nonsingleton blocks of the equivalence relations $\theta \in E$ to be pairwise disjoint and to have sizes ≥ 3, therefore $m \ge \lceil k/2 \rceil + 1$ and $2 \le |E| \le k/3$.)

aReg_{k-1} is the set of all relations $\rho \subseteq A^{k-1}$ of arity $k-1 \ge 3$ which have the form

$$\rho = \iota_{k-1} \cup \big\{(a_1, \ldots, a_{k-1}) : \{a_1, \ldots, a_{k-1}\} \in \mathfrak{H}\big\}.$$

for some set $\mathfrak{H} \subseteq \binom{A}{k-1}$ such that $|\mathfrak{H}| < k-2$.

aReg is the union $\mathsf{aReg}_{\le k-2} \cup \mathsf{aReg}_{k-1}$.

Burle_3 is the one-element set $\{\beta\}$ if $k = 3$, and $\emptyset$ if $k > 3$.

Notice that for $k = 3$, all sets aCentr, $\mathsf{aReg}_{\le k-2}$, and aReg_{k-1} above are empty. The set $\mathsf{aReg}_{\le k-2}$ is empty even for $k = 4, 5$.

The notation aReg is justified by the fact that for every relation $\rho \in \mathsf{aReg}$, say ρ is m-ary, ρ is *almost regular* in the sense that for all sets $D \in \binom{A}{k-1}$, either $\rho|_D = D^m$ or $\rho|_D$ is a regular relation on D. In more detail, if $\rho \in \mathsf{aReg}_{\le k-2}$ and $D \in \binom{A}{k-1}$ is such that the unique element of $A \setminus D$ lies in a nonsingleton block of some $\theta \in E$, then there is a unique such θ, $\theta|_D$ is an m-equivalence relation on D, and $\rho|_D = \lambda_\theta|_D$ ($= \lambda_{\theta|_D}$ on D); otherwise, we have $\rho|_D = D^m$. If, in turn, $\rho \in \mathsf{aReg}_{k-1}$, then for $D \in \mathfrak{H}$ we have $\rho|_D = D^m$, while for $D \notin \mathfrak{H}$ we have $\rho|_D = \iota_{k-1}|_D$ ($= \iota_{|D|}$ on D).

Our main result can now be stated as follows.

Theorem 7 ([13])**.** *If A is a finite set with $k \ge 3$ elements, then a clone $\mathcal{M}$ on A is maximal for the property of not containing $\mathcal{S}^-$ if and only if $\mathcal{M} = \{\rho\}^\perp$ for some*

$$\rho \in \mathsf{BPO} \cup \mathsf{Perm} \cup \mathsf{Affn} \cup \mathsf{Eq} \cup \mathsf{Centr} \cup \mathsf{Reg}^* \cup \mathsf{aCentr} \cup \mathsf{aReg} \cup \mathsf{Burle}_3. \quad (1)$$

The proof of Theorem 7, which can be found in [13], is an expansion of Rosenberg's proof [10] for Theorem 5.

IV. Completeness criteria for clones containing $\mathcal{S}^-$

We can combine Theorem 7 with Theorem 3 to obtain completeness criteria for every clone $\mathcal{U}$ containing $\mathcal{S}^-$ on a finite set A with $k \ge 3$ elements. For the case when $\mathcal{U} = \mathcal{O}$, these considerations yield Rosenberg's Theorem 5, therefore from now on we will assume that $\mathcal{U} \neq \mathcal{O}$.

If $\mathcal{U}$ is a clone on A with $\mathcal{S}^- \subseteq \mathcal{U} \subsetneq \mathcal{O}$, then by Theorem 3, $\mathcal{U} = \mathcal{S}[G]$ for some permutation group G on A. Now let $\mathcal{M}$ be a maximal subclone of $\mathcal{U}$. If $\mathcal{S}^- \subseteq \mathcal{M}$, then Theorem 3 yields that (i) $\mathcal{M} = \mathcal{S}[H]$ for a maximal subgroup H of G. Otherwise, if $\mathcal{S}^- \not\subseteq \mathcal{M}$, then Theorem 7 implies that (ii) $\mathcal{M} = \mathcal{U} \cap \{\rho\}^\perp$ for some relation ρ satisfying (1). In addition, in case (ii), we must have $G \subseteq \{\rho\}^\perp$, because otherwise $\mathcal{U} \cap \{\rho\}^\perp \subsetneq \mathcal{S}[H] \subsetneq \mathcal{U}$ holds for a proper subgroup H of G, so $\mathcal{U} \cap \{\rho\}^\perp$ is not a maximal subclone of $\mathcal{U}$. One can also show that if $\rho \in \mathsf{Perm} \cup \mathsf{Affn}$, then $\mathcal{U} \cap \{\rho\}^\perp \subsetneq \mathcal{S}_{k-1}[G] \subsetneq \mathcal{U}$, so $\mathcal{U} \cap \{\rho\}^\perp$ is not a maximal subclone of $\mathcal{U}$. Thus, we get the following.

Corollary 8. *If A is a k-element set ($k \geq 3$), and $\mathcal{U} = \mathcal{S}[G]$ for some permutation group G on A, then every maximal subclone of $\mathcal{U}$ has the form*

(i) *$\mathcal{S}[H]$ for a maximal subgroup H of G, or*

(ii) *$\mathcal{U} \cap \{\rho\}^\perp$ for a relation*

$$\rho \in \mathsf{BPO} \cup \mathsf{Eq} \cup \mathsf{Centr} \cup \mathsf{Reg}^* \cup \mathsf{aCentr} \cup \mathsf{aReg} \cup \mathsf{Burle}_3 \quad (2)$$

such that $G \subseteq \{\rho\}^\perp$.

We note that not all clones $\mathcal{U} \cap \{\rho\}^\perp$ satisfying the restrictions in (ii) are maximal subclones of $\mathcal{U}$. A detailed analysis of which of them are maximal is given in [13].

In the special case when $\mathcal{U} = \mathcal{S}$ is Słupecki's clone, that is, when G is the symmetric group on A, then the only relation ρ in (2) satisfying the condition $G \subseteq \{\rho\}^\perp$ is $\rho = \iota_{k-1}$ if $k \geq 4$ and $\rho = \beta$ if $k = 3$. Therefore the maximal subclones of $\mathcal{S}$ are the clones of the form $\mathcal{S}[H]$ where H is a maximal subgroup of G, and $\{\iota_{k-1}\}^\perp$ if $k \geq 4$, resp., $\{\beta\}^\perp$ if $k = 3$. This special case of Corollary 8 can also be deduced from the results of Haddad and Rosenberg [2].

At the other extreme, when $\mathcal{U} = \mathcal{S}^-$ is the clone of nonsurjective operations, that is, when G is the one-element group, then G has no maximal proper subgroups, therefore every maximal proper subclone of $\mathcal{S}^-$ is of the form $\mathcal{U} \cap \{\rho\}^\perp$ for some ρ in (2).

V. Subclones of Słupecki's clone with finitely many relative $\mathcal{R}$-classes

As we mentioned in the introduction, a relativized version of Green's $\mathcal{R}$-relation on the set $\mathcal{O}$ of all operations on a finite set A can be defined as follows: given a clone $\mathcal{C}$ on A, we say that two operations $f, g \in \mathcal{O}$, where f is m-ary and g is n-ary, are *$\mathcal{C}$-equivalent,* and write $f \equiv_{\mathcal{C}} g$, if there exist n-ary operations $h_1, \ldots, h_m \in \mathcal{C}$ and m-ary operations $h'_1, \ldots, h'_n \in \mathcal{C}$ such that $f(h_1, \ldots, h_m) = g$ and $g(h'_1, \ldots, h'_n) = f$.

It is easy to show (see [6]) that the clones $\mathcal{C}$ for which $\equiv_{\mathcal{C}}$ has only finitely many equivalence classes form an order filter (up-closed set) $\mathfrak{F}_A$ in the lattice of all clones on A. In [7] we determined which maximal clones belong to $\mathfrak{F}_A$, and described the rough structure of $\mathfrak{F}_A$. In particular, we found that every clone containing the clone $\mathcal{S}^-$ of nonsurjective operations is in $\mathfrak{F}_A$.

Using Theorem 7 we can now prove that these are the only subclones of Słupecki's clone which belong to $\mathfrak{F}_A$.

Theorem 9. *Let A be a k-element set ($k \geq 3$). A subclone $\mathcal{C}$ of Słupecki's clone belongs to the order filter $\mathfrak{F}_A$ if and only if $\mathcal{S}^- \subseteq \mathcal{C}$.*

Before the proof we establish some sufficient conditions for a clone not to belong to $\mathfrak{F}_A$. In Lemma 10 below we restate a general condition from [7], and in Lemmas 11–13 we consider the clones $\{\rho\}^\perp$ where ρ is a relation of maximum arity in aReg or aCentr.

Lemma 10 **([7, Corollary 3.2]).** *Let ρ be a relation on A. If A has a nonempty subset B such that $\{\rho|_B\}^\perp \notin \mathfrak{F}_B$, then $\{\rho\}^\perp \notin \mathfrak{F}_A$.*

Lemma 11. *If $\rho \in \mathsf{aReg}_{k-1}$, then $\mathcal{S} \cap \{\rho\}^\perp \notin \mathfrak{F}_A$.*

Proof: Let $\rho \in \mathsf{aReg}_{k-1}$. Hence $k \geq 4$, and for each $D \in \binom{A}{k-1}$ we have $\rho|_D = \iota_{k-1}|_D$ or $\rho|_D = D^{k-1}$. The latter condition holds for less than $k-2$ distinct sets D, therefore we can choose and fix $B \in \binom{A}{k-1}$ such that $\rho|_B = \iota_{k-1}|_B$.

Claim. Let $f \in \mathcal{S} \cap \{\rho\}^\perp$. If the range of f contains B, then f is essentially unary.

Proof of Claim. Assume, for a contradiction, that f is an n-ary operation in $\mathcal{S} \cap \{\rho\}^\perp$ such that $\operatorname{Im} f$ contains B and f depends on at least two of its variables. Then $f \in \mathcal{S}$ and $|B| = k-1$ imply that B is the range of f. Let $B = \{b_1, \ldots, b_{k-1}\}$. By Jablonskiĭ's Lemma [4], there exist sets $C_1, \ldots, C_n \in \binom{A}{k-2}$ and n-tuples $\mathbf{a}_1, \ldots, \mathbf{a}_{k-1} \in C_1 \times \cdots \times C_n$ such that $f(\mathbf{a}_i) = b_i$ for all i ($1 \leq i \leq k-1$). Since $|C_i| = k-2$ for all $1 \leq i \leq n$ and $\iota_{k-1} \subseteq \rho$, we get that the n-tuples $\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}$ are coordinatewise ι_{k-1}-related, i.e., $(\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}) \in (\iota_{k-1})^n \subseteq \rho^n$. However, $\big(f(\mathbf{a}_1), \ldots, f(\mathbf{a}_{k-1})\big) = (b_1, \ldots, b_{k-1}) \in B^{k-1} \setminus \iota_{k-1}|_B = B^{k-1} \setminus \rho|_B = B^{k-1} \setminus \rho$. This contradicts the assumption that $f \in \{\rho\}^\perp$, and completes the proof of the claim. $\diamond$

Now, using the notation $A = \{0, 1, 2, \ldots, k-1\}$ and $B = \{1, 2, \ldots, k-1\}$ we can repeat the proof given in [7, Theorem 6.1] for $\mathcal{S}_{k-2} \notin \mathfrak{F}_A$ to show that $\mathcal{S} \cap \{\rho\}^\perp \notin \mathfrak{F}_A$. $\blacksquare$

Lemma 12. *If ρ is a $(k-2)$-ary relation in aCentr, then ρ satisfies the following condition for $r = k-2$:*

($*$) *There exist distinct elements $b, c \in A$ and further elements $u_2, \ldots, u_r \in A \setminus \{b, c\}$ and $v_2, \ldots, v_r \in A \setminus \{c\}$ such that*

$$(b, u_2, \ldots u_r) \in \rho, \qquad (c, u_2, \ldots u_r) \notin \rho,$$
$$(c, v_2, \ldots v_r) \in \rho, \qquad (b, v_2, \ldots v_r) \notin \rho.$$

Proof: Let $\rho \in \mathsf{aCentr}$ where ρ is $(k-2)$-ary. By definition, $\rho \subsetneq A^{k-2}$, ρ is not a central relation on A, but for each $D \in \binom{A}{k-1}$, $\rho|_D$ is either a central relation on D or is equal to D^{k-2}. It follows that ρ is totally reflexive and totally symmetric. Since $\rho \neq A^{k-2}$, there exists $B \in \binom{A}{k-1}$ such that $\rho|_B \neq B^{k-2}$. Hence $\rho|_B$ is a central relation on B, so there exists $b \in B$ such that $\{b\} \times B^{k-3} \subseteq \rho|_B$. Let b' denote the unique element of A such that $B = A \setminus \{b'\}$, and let $B' := A \setminus \{b\}$, $\overline{B} := A \setminus \{b, b'\} = B \cap B'$. Since $\rho|_B \left(\neq B^{k-2}\right)$ is a totally reflexive, totally symmetric relation on B which contains all $(k-2)$-tuples in which b occurs, $\rho|_B$ cannot contain any $(k-2)$-tuple whose coordinates are the $k-2$ elements of $\overline{B} = B \setminus \{b\}$ in some order. Thus, $\rho|_{\overline{B}} = \iota_{k-2}|_{\overline{B}}$. This implies that $\{x\} \times (B')^{k-3} \not\subseteq \rho|_{B'}$ if $x \in \overline{B} = B' \setminus \{b'\}$. Hence $\rho|_{B'} \neq (B')^{k-2}$, so $\rho|_{B'}$ is a central relation on B', and it must be that $\{b'\} \times (B')^{k-3} \subseteq \rho|_{B'}$.

Suppose now, for a contradiction, that $\rho|_D = D^{k-2}$ for each $D \in \binom{A}{k-1}$ such that $b, b' \in D$. Then every $(k-2)$-tuple containing both b and b' belongs to ρ. Since ρ is totally reflexive, totally symmetric, and satisfies $\{b\} \times B^{k-3} \subseteq \rho|_B \subseteq \rho$, it follows that $\{b\} \times A^{k-3} \subseteq \rho$. Hence ρ is a central relation on A, which contradicts our assumption on ρ.

This shows that there exists $C \in \binom{A}{k-1}$ such that $b, b' \in C$ and $\rho|_C \neq C^{k-2}$. Hence, we can repeat the argument for B from the previous paragraph to conclude that for the unique element c' in A with $C = A \setminus \{c'\}$ and for some $c \in C$, $\rho|_C$ is a central relation on C with $\{c\} \times C^{k-3} \subseteq \rho|_C$, $\rho|_{C'}$ is a central relation on $C' = A \setminus \{c\}$ with $\{c'\} \times (C')^{k-3} \subseteq \rho|_{C'}$, and for $\overline{C} = A \setminus \{c, c'\} = C \cap C'$ we have $\rho|_{\overline{C}} = \iota_{k-2}|_{\overline{C}}$. Clearly, $c' \neq b, b'$, because $b, b' \in C$ and $c' \notin C$. It follows also that $c \neq b, b'$ as we now show. Assuming $c = b$ we get that $B' = A \setminus \{b\} = A \setminus \{c\} = C'$, so $\rho|_{B'} = \rho|_{C'}$. As we saw above, $x = b'$ is the unique element of B' such that $\{x\} \times (B')^{k-3} \subseteq \rho|_{B'}$, and similarly, $y = c'$ is the unique element of $C' \,(= B')$ such that $\{y\} \times (C')^{k-3} \subseteq \rho|_{C'}$. Hence $b' = c'$, contradicting $c' \neq b'$. We get a contradiction in a similar way if we assume that $c = b'$. Thus, b, b', c, c' are four distinct elements of A.

Now we prove $(*)$ for $r = k-2$ and for the elements b, c chosen above. Let $w_1, \ldots, w_{k-4}$ be an enumeration of the $k-4$ elements of $A \setminus \{b, b', c, c'\}$. Then $c', w_1, \ldots, w_{k-4} \in A \setminus \{b'\} = B$ implies that $(b, c', w_1, \ldots, w_{k-4}) \in \{b\} \times B^{k-3} \subseteq \rho|_B$, hence $(b, c', w_1, \ldots, w_{k-4}) \in \rho$. On the other hand, $\{c, c', w_1, \ldots, w_{k-4}\} = A \setminus \{b, b'\} = \overline{B}$ implies that $(c, c', w_1, \ldots, w_{k-4}) \in \overline{B}^{k-2} \setminus \iota_{k-2}|_{\overline{B}} = \overline{B}^{k-2} \setminus \rho|_{\overline{B}}$, hence $(c, c', w_1, \ldots, w_{k-4}) \notin \rho$. Switching the roles of the b's and c's we obtain similarly that $(c, b', w_1, \ldots, w_{k-4}) \in \rho$ and $(b, b', w_1, \ldots, w_{k-4}) \notin \rho$. ■

Lemma 13. *Let ρ be an r-ary relation on a k-element set A $(k \geq 3,\ r \geq 2)$. If ρ satisfies condition $(*)$ from Lemma 12, then $\mathcal{S} \cap \{\rho\}^{\perp} \notin \mathfrak{F}_A$.*

Proof: Let $\mathcal{C} := \mathcal{S} \cap \{\rho\}^{\perp}$, and let us fix elements b, c, u_j, v_j in A such that $(*)$ holds. For any element $a \in A$ we will denote the constant tuples $(a, \ldots, a)$ by $\overline{a}$ (the length of the tuple will be clear from the context). We will prove $\mathcal{C} \notin \mathfrak{F}_A$ by exhibiting an infinite sequence f_n $(n \geq 2)$ of operations on A such that $f_m \not\equiv_{\mathcal{C}} f_n$ whenever $m \neq n$.

For $n \geq 2$, let f_n be the n-ary operation on A defined as follows: for arbitrary n-tuple $\mathbf{x} \in A^n$,

$$f_n(\mathbf{x}) := \begin{cases} a & \text{if } \mathbf{x} = \overline{a} \text{ for some } a \in A \setminus \{b, c\}, \\ b & \text{if } \mathbf{x} \in \{b, c\}^n, \\ c & \text{otherwise.} \end{cases}$$

Notice that f_n is invariant under any permutation of its variables. Since f_n is not constant, this implies that f_n depends on all of its variables.

To show that $f_m \not\equiv_{\mathcal{C}} f_n$ whenever $m \neq n$, let us assume, for a contradition, that there exist $n < m$ such that $f_m \equiv_{\mathcal{C}} f_n$. Then $f_m = f_n(\mathbf{h})$ for some tuple $\mathbf{h} = (h_1, \ldots, h_n)$ of m-ary operations in $\mathcal{C}$. This equality means that the function $\mathbf{h}\colon A^m \to A^n$, $\mathbf{a} \mapsto \mathbf{h}(\mathbf{a}) = \big(h_1(\mathbf{a}), \ldots, h_n(\mathbf{a})\big)$ maps the set $f_m^{-1}(a)$ into the set $f_n^{-1}(a)$ for each $a \in A$; indeed, if $\mathbf{x} \in f_m^{-1}(a)$, i.e., $f_m(\mathbf{x}) = a$, then $f_m = f_n(\mathbf{h})$ implies that $f_n(\mathbf{h}(\mathbf{x})) = a$, i.e., $\mathbf{h}(\mathbf{x}) \in f_n^{-1}(a)$. Applying this observation first to $a \in A \setminus \{b, c\}$ we see that $f_m^{-1}(a) = \{\overline{a}\}$ and $f_n^{-1}(a) = \{\overline{a}\}$, so

$$\mathbf{h}(\overline{a}) = \overline{a} \quad \text{for all } a \in A \setminus \{b, c\}. \tag{3}$$

Applying now the observation to $a = b$ we see that $f_m^{-1}(b) = \{b, c\}^m$ and $f_n^{-1}(b) = \{b, c\}^n$, therefore $\mathbf{x} \in \{b, c\}^m$ implies $\mathbf{h}(\mathbf{x}) \in \{b, c\}^n$ for all $\mathbf{x} \in A^m$. In particular,

$$\mathbf{h}(\overline{b}) \in \{b, c\}^n \quad \text{and} \quad \mathbf{h}(\overline{c}) \in \{b, c\}^n. \tag{4}$$

We have $(b, u_2, \ldots, u_r) \in \rho$ by assumption, where $u_2, \ldots, u_r \in A \setminus \{b, c\}$, so $(\overline{b}, \overline{u}_2, \ldots, \overline{u}_r) \in \rho^m$. Applying $\mathbf{h}$ and using (3) we get that

$$\big(\mathbf{h}(\overline{b}), \overline{u}_2, \ldots, \overline{u}_r\big) = \big(\mathbf{h}(\overline{b}), \mathbf{h}(\overline{u}_2), \ldots, \mathbf{h}(\overline{u}_r)\big) \in \rho^n,$$

since $\mathbf{h} \in \mathcal{C}$, and hence $\mathbf{h}$ preserves ρ. In view of (4) we have $\mathbf{h}(\overline{b}) \in \{b, c\}^n$, so in each coordinate the r-tuple $(\mathbf{h}(\overline{b}), \overline{u}_2, \ldots, \overline{u}_r) \in \rho^n$ is either $(b, u_2, \ldots, u_r)$ or $(c, u_2, \ldots, u_r)$. However, by assumption, $(c, u_2, \ldots, u_r) \notin \rho$. Therefore no coordinate of $\mathbf{h}(\overline{b})$ can be equal to c, proving that

$$\mathbf{h}(\overline{b}) = \overline{b}. \tag{5}$$

Similarly, we have $(c, v_2, \ldots, v_r) \in \rho$ by assumption, where $v_2 \ldots, v_r \in A \setminus \{c\}$, so $(\overline{c}, \overline{v}_2, \ldots, \overline{v}_r) \in \rho^m$. Applying $\mathbf{h}$ and using (3) and (5) we get that

$$\big(\mathbf{h}(\overline{c}), \overline{v}_2, \ldots, \overline{v}_r\big) = \big((\mathbf{h}(\overline{c}), \mathbf{h}(\overline{v}_2), \ldots, \mathbf{h}(\overline{v}_r)\big) \in \rho^n,$$

since $\mathbf{h}$ preserves ρ. In view of (4) we have $\mathbf{h}(\overline{c}) \in \{b, c\}^n$, so in each coordinate the r-tuple $\big(\mathbf{h}(\overline{c}), \overline{v}_2, \ldots, \overline{v}_r\big) \in \rho^n$ is either $(b, v_2, \ldots, v_r)$ or $(c, v_2, \ldots, v_r)$. However, by assumption, $(b, v_2, \ldots, v_r) \notin \rho$. Therefore no coordinate of $\mathbf{h}(\overline{c})$ can be equal to b, proving that

$$\mathbf{h}(\overline{c}) = \overline{c}. \tag{6}$$

Properties (3), (5), and (6) show that each component h_i of $\mathbf{h} = (h_1, \ldots, h_n)$ satisfies $h_i(\overline{a}) = a$ for all $a \in A$. Since $h_1, \ldots, h_n \in \mathcal{C} \subseteq \mathcal{S}$, each h_i $(1 \leq i \leq n)$ is a projection. Since $n < m$ and $f_m = f_n(\mathbf{h})$, we get that f_m depends on at most n variables. This contradicts the fact established earlier that f_m depends on all of its variables, and completes the proof of Lemma 13. ∎

Now we are ready to prove Theorem 9.

Proof of Theorem 9: The sufficiency was proved in [7, Theorem 6.1.]. For the necessity we will assume that $\mathcal{C}$ is a subclone of $\mathcal{S}$ such that $\mathcal{S}^- \not\subseteq \mathcal{C}$, and want to show that $\mathcal{C} \notin \mathfrak{F}_A$. By Theorem 7, the assumption $\mathcal{S}^- \not\subseteq \mathcal{C}$ is equivalent to the condition that $\mathcal{C} \subseteq \{\rho\}^\perp$ for one of the relations ρ satisfying (1). Thus $\mathcal{C} \subseteq \mathcal{S} \cap \{\rho\}^\perp$ for one of these relations ρ. Since $\mathfrak{F}_A$ is an order filter, it suffices to establish that for each ρ satisfying (1) we have that $\mathcal{S} \cap \{\rho\}^\perp \notin \mathfrak{F}_A$.

If $\rho \in \mathsf{BPO} \cup \mathsf{Perm} \cup \mathsf{Affn} \cup \mathsf{Eq} \cup \mathsf{Centr} \cup \mathsf{Reg}^*$, i.e., if $\{\rho\}^\perp$ is a maximal clone other than $\mathcal{S}$, then $\mathcal{S} \cap \{\rho\}^\perp \notin \mathfrak{F}_A$ was proved in [7, Theorems 7.1–7.2]. If $\rho \in \mathsf{Burle}_3$, then the desired conclusion $\mathcal{S} \cap \{\rho\}^\perp \notin \mathfrak{F}_A$ follows from [7, Corollary 3.8]. So, it remains to consider the cases when $\rho \in \mathsf{aCentr} \cup \mathsf{aReg}$.

If $\rho \in \mathsf{aCentr}$ and ρ has arity $k-2$, then Lemmas 12–13 show that $\mathcal{S} \cap \{\rho\}^\perp \notin \mathfrak{F}_A$ If $\rho \in \mathsf{aReg}_{k-1}$, then the same conclusion is proved in Lemma 11. Now let ρ be an m-ary relation in $\mathsf{aCentr} \cup \mathsf{aReg}$ such that $2 \leq m \leq k-3$ if $\rho \in \mathsf{aCentr}$ and $4 \leq m \leq k-2$ if $\rho \in \mathsf{aReg}$. Since $\rho \neq A^m$, there exists $B \in \binom{A}{k-1}$ such that $\rho|_B \neq B^m$. By Definition 6 and the subsequent remarks, if $\rho \in \mathsf{aCentr}$, then (i) $\rho|_B$ is a central relation on B of arity $2 \leq m \leq k-3 = |B|-2$, while if $\rho \in \mathsf{aReg}$, then (ii) $\rho|_B$ is a regular relation on B of arity $4 \leq m \leq k-2 = |B|-1$. It follows from [7, Theorem 7.1] that if (i) or (ii) holds for $\rho|_B$, then the maximal clone $\{\rho|_B\}^\perp$ on B does not belong to $\mathfrak{F}_B$. Therefore Lemma 10 implies that $\{\rho\}^\perp \notin \mathfrak{F}_A$, and hence also $\mathcal{S} \cap \{\rho\}^\perp \notin \mathfrak{F}_A$. ∎

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