

A Variety Theorem for Relational Universal Algebra

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Abstract—We develop an analogue of universal algebra in which generating symbols are interpreted as relations. We prove a variety theorem for these relational algebraic theories, in which we find that their categories of models are precisely the ‘definable categories’. The syntax of our relational algebraic theories is string-diagrammatic, and can be seen as an extension of the usual term syntax for algebraic theories.

I. INTRODUCTION

Universal algebra is the study of what is common to algebraic structures, such as groups and rings, by algebraic means. The central idea of universal algebra is that of a *theory*, which is a syntactic description of some class of structures in terms of generating symbols and equations involving them. A *model* of a theory is then a set equipped with a function for each generating symbol in a way that satisfies the equations. There is a further notion of *model morphism*, and together the models and model morphisms of a given theory form a category. These categories of models are called *varieties*. Much of classical algebra can be understood as the study of a specific variety. For example, group theory is the study of the variety of groups, which arises from the theory of groups in the manner outlined above.

A given variety will in general arise as the models of more than one theory. A natural question to ask, then, is when two theories present the same category of varieties. To obtain a satisfying answer to this question it is helpful to adopt a more abstract perspective. Theories become categories with finite products, models become functors, and model morphisms become natural transformations. Our reward for this shift in perspective is the following answer to our question: Two theories (categories with finite products) present equivalent categories of varieties in case they have equivalent idempotent splitting completions. Thus, from a certain point of view universal algebra is the study of categories with finite products.

This point of view has developed into *categorical* universal algebra. For any sort of categorical structure we can treat categories with that structure as theories, functors that preserve it as models, and natural transformations thereof as model morphisms. The aim is then to figure out what sort of categories arise as models and model morphisms of this kind – that is, to determine the appropriate notion of variety. For example, if we take categories with finite limits to be our theories, then varieties correspond to locally finitely presentable categories [1].

The familiar syntax of classical algebra – consisting of *terms* built out of variables by application of the generating symbols – is inextricably bound to finite product structure.

In leaving finite products behind for more richly-structured settings, categorical universal algebra also leaves behind much of the syntactic elegance of its classical counterpart. While methods of specifying various sorts of theory (categories with structure) exist, these are often cumbersome, lacking the intuitive flavour of classical universal algebra.

The present paper concerns an analogue of classical universal algebra in which the generating symbols are understood as *relations* instead of functions. The role of classical terms is instead played by string diagrams, and categories with finite products become ‘cartesian bicategories of relations’ in the sense of [11] – an idea that first appears in [7]. This allows us to present relational algebraic theories in terms of generators and equations, in the style of classical universal algebra. Indeed, this approach to syntax for relational theories extends the classical syntax for algebraic theories, which admits a similar diagrammatic presentation.

Our development is best understood in the context of recent work on partial algebraic theories [14], in which the string-diagrammatic syntax for algebraic theories is modified to capture partial functions. This modification of the basic syntax coincides with an increase in the expressive power of the framework, corresponding roughly to the equalizer completion of a category with finite products [9]. The move to relational algebraic theories involves a further modification of the string-diagrammatic syntax, corresponding roughly to the regular completion of a category with finite limits [10]. Put another way, in [14] the (string-diagrammatic) syntax for algebraic theories is extended to express a certain kind of equality, and the resulting terms are best understood as partial functions. In this paper, we further extend the string-diagrammatic syntax to express existential quantification, and the resulting terms are best understood as relations.

The central contribution of this paper is a variety theorem characterizing the categories that arise as the models and model morphisms of some relational algebraic theory. Specifically, we will see that these are precisely the *definable* categories of [24]. We illustrate the use of our framework with a number of examples, including the extensively studied theory of regular semigroups [19], the theory of effectoids [30], and the theory of generalized separation algebras of [4].

While relations are a fundamental tool in computer science, algebraic structures described in relational terms are uncommon in comparison to structures described in terms of functions or partial functions. We speculate that part of the reason for this is that the algebra of relations is more difficult to work with in traditional syntax, and we hope that our approach to relational algebra will help alleviate this difficulty.

One area where relational structures play a fundamental role is the theory of databases [13]. A cartesian bicategory

of relations can be seen as a specific sort of database schema, with the arrows corresponding to possible queries [8]. This suggests a connection of the present work to the theory of relational databases: a database schema is a relational algebraic theory, queries are arrows therein, and possible instantiations of a schema are precisely models of the corresponding theory.

A. Related Work

The study of universal algebra in the modern sense began with the work of Birkhoff [6]. A few decades later, Lawvere introduced the categorical perspective in his doctoral thesis [28]. A modern account of universal algebra from the categorical perspective is [3]. A highlight of this account is the variety theorem for algebraic theories [2], which our variety theorem for relational algebraic theories is explicitly modelled on.

An important result in categorical algebra is Gabriel-Ulmer duality [18], which from our present perspective states that if we consider categories with finite limits as our notion of algebraic theory, then the corresponding notion of variety is that of a locally finitely presentable category [1]. Our development relies on the related notion of a definable category [24], [26], which recently arose in the development of an analogue of Gabriel-Ulmer duality for regular categories.

We use cartesian bicategories of relations [11] as our notion of relational algebraic theory. Our development relies on several results from the theory of allegories [17], in which cartesian bicategories of relations coincide with the notion of a unitary pre-tabular allegory. We also make use of the theory of regular and exact completions [10]. Of course, all of this relies on the theory of regular and exact categories [5].

The idea of using string diagrams to represent terms in more general notions of algebraic theories is relatively recent, and is heavily indebted to the work of Fox [16]. The present paper can be considered a generalisation of recent work on partial theories [14] to include relations. The idea to treat cartesian bicategories of relations as theories with models in the category of sets and relations has appeared previously in [7], although no variety theorem is provided therein.

B. Contributions

The central contribution of this paper is the variety theorem for relational algebraic theories (Theorems 6,7). Minor contributions include the examples of Section IV, which show how to formulate a number of structures from the wider literature as relational algebraic theories, and Lemma 2 – which offers a much simpler definition of cartesian bicategories of relations than the usual one.

Viewed from the perspective of categorical algebra, our variety theorem characterizes definable categories [24], which are a relatively new notion without much surrounding literature. It could also be seen as contributing to the project of a unified string-diagrammatic framework for partial, relational, and classical algebraic theories aspired to in [7], and partially fulfilled by [14].

$$\begin{array}{c} \overline{\square : 0 \rightarrow 0} \qquad \overline{\sigma : 2 \rightarrow 2} \qquad \overline{1 : 1 \rightarrow 1} \\[10pt] \frac{\gamma \in |\Gamma| \quad \delta_0(\gamma) = n \quad \delta_1(\gamma) = m}{\gamma : n \rightarrow m} \\[10pt] \frac{c : n \rightarrow m \quad c' : n' \rightarrow m'}{(c \otimes c') : n + n' \rightarrow m + m'} \qquad \frac{c : n \rightarrow k \quad c' : k \rightarrow m}{(c; c') : n \rightarrow m} \end{array}$$

Fig. 1. Sorting rules for terms over a monoidal signature Γ .

C. Organization

Section II introduces the basic machinery surrounding string diagrams and algebraic theories that we will require. In Section III we introduce cartesian bicategories of relations and the related idea of relations in a regular category. In Section IV we give the definition of a relational algebraic theory, and provide a number of examples. Section V contains the proof of the variety theorem for relational algebraic theories, and we briefly conclude in Section VI.

II. STRING DIAGRAMS AND MONOIDAL THEORIES

We assume some familiarity with category theory, in particular the theory of regular categories [5], string diagrams for monoidal categories [22], and some basic notions from the theory of 2-categories [23]. It will help to have a little experience with universal algebra, and we recommend [3] as a reference. In our development we will behave as though all monoidal categories are *strict* monoidal categories, justifying this behaviour in the usual way by appealing to the coherence theorem for monoidal categories [27].

In this section we introduce monoidal equational theories, and briefly explore how classical algebraic theories can be captured in the monoidal framework. In particular, this is done by forcing a monoidal equational theory to have coherent and natural commutative comonoid structure, with the resulting string diagrams corresponding to classical terms. This sets the tone for our development of relational theories, in which *relational terms* will correspond to a different sort of structure on a symmetric monoidal category. We restrict our presentation of monoidal syntax (string diagrams) to the single-sorted case, which we feel better captures the spirit of classical universal algebra. The multi-sorted syntax is a straightforward extension of the single-sorted case.

Definition 1. A *monoidal signature* $\Gamma = (|\Gamma|, \delta_0, \delta_1)$ consists of a set of operation symbols $|\Gamma|$ together with an *arity* $\delta_0 : |\Gamma| \rightarrow \mathbb{N}$ and a *coarity* $\delta_1 : |\Gamma| \rightarrow \mathbb{N}$.

Definition 2. Let Γ be a monoidal signature. We define a *term over Γ* to be a well-sorted word of the grammar:

$$c ::= \square \mid \sigma \mid 1 \mid c \otimes c \mid c; c \mid \gamma \in |\Gamma|$$

with the sorting discipline shown in Figure 1.

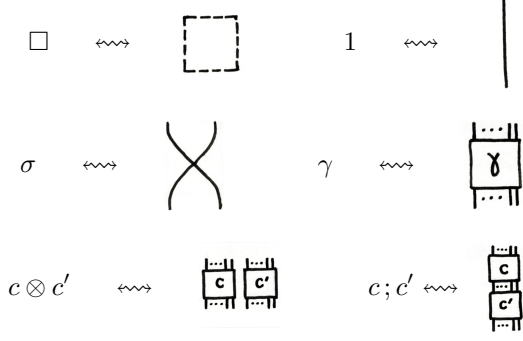


Fig. 2. Graphical notation for terms over a monoidal signature

$$\begin{aligned}
 (c_1 ; c_2) ; c_3 &= c_1 ; (c_2 ; c_3) & 1^n ; c &= c = c ; 1^m \\
 (c_1 \otimes c_2) \otimes c_3 &= c_1 \otimes (c_2 \otimes c_3) & \square \otimes c &= c = c \otimes \square \\
 (c_1 ; c_3) \otimes (c_2 ; c_4) &= (c_1 \otimes c_2) ; (c_3 \otimes c_4) \\
 \sigma ; \sigma &= 1 \otimes 1 & (c \otimes 1^k) ; \sigma_{m,k} &= \sigma_{n,k} ; (1^k \otimes c)
 \end{aligned}$$

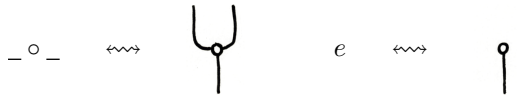
Fig. 3. Defining equations for string diagrams over a monoidal signature. $1^k = 1 \otimes \dots \otimes 1 : k \rightarrow k$ and $\sigma_{n,m} : n + m \rightarrow n + m$ are the multi-wire identity and symmetry morphisms, with $\sigma_{n,m}$ constructed inductively.

Terms over a monoidal signature are well-suited to the graphical notation of Figure 2, in which $c : n \rightarrow m$ is depicted as a wiring diagram with n wires coming in from the top of the diagram, and m output wires exiting the bottom. The sorting rules ensure that wires do not mysteriously appear or disappear in large compound terms.

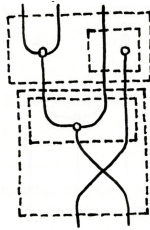
For example, consider the monoidal signature $M = (\Gamma_M, \delta_0, \delta_1)$ where $\Gamma_M = \{ _ \circ _, e \}$ with the (co)arities as in:

$$\delta_0(_ \circ _) = 2 \quad \delta_1(_ \circ _) = 1 \quad \delta_0(e) = 0 \quad \delta_1(e) = 1$$

Let us depict $_ \circ _$ and e in the graphical notation as follows:



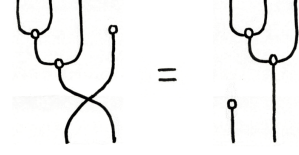
Then the term $(_ \circ _ \otimes (1 \otimes e)) ; ((_ \circ _ \otimes 1) ; \sigma) : 3 \rightarrow 2$ over M is



where the dashed line boxes correspond to the parentheses.

Definition 3. Let Γ be a monoidal signature. A *string diagram* over Γ is an equivalence class of terms over Γ in the equivalence relation generated by Figure 3. Write \mathbb{X}_Γ for the collection of string diagrams over Γ .

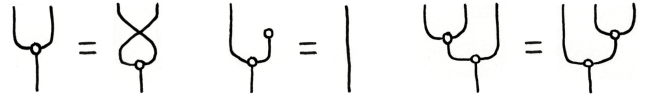
Two terms over Γ represent the same string diagram if and only if it is possible to continuously deform their corresponding diagrams into each other [22]. For example in \mathbb{X}_M we have:



If Γ is a monoidal signature then \mathbb{X}_Γ is a symmetric strict monoidal category. In this way, monoidal signatures present monoidal categories. We will need to impose further equations on these categories of string diagrams, which works as follows:

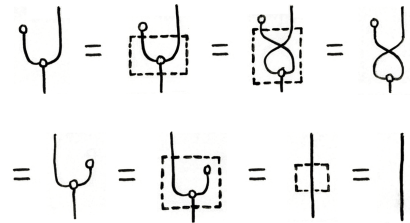
Definition 4 (Monoidal Equational Theory). Let Γ be a monoidal signature. An *equation* over Γ is a pair $([c_1], [c_2])$ of string diagrams over Γ with the same arity and coarity. We will often write ' $c_1 = c_2$ ' to denote such equations. A *monoidal equational theory* (Γ, E) consists of a monoidal signature Γ together with a set E of equations over Γ . Write $\mathbb{X}_{(\Gamma, E)}$ for the quotient of \mathbb{X}_Γ by the smallest congruence containing E .

Example 1. Recall our monoidal signature M , and let CM consist of the following equations over M :



Then (M, CM) is a monoidal equational theory, and $\mathbb{X}_{(M, CM)}$ is the PROP of commutative monoids [25]. The equations, of course, express commutativity, unitality, and associativity.

It is important to note that string diagrams over a monoidal equational theory (Γ, E) are amenable to equational reasoning, sometimes called *diagrammatic reasoning* in this context: If $([c_1], [c_2]) \in E$ then substituting c_1 for c_2 is sound in any context. For example in our equations CM above we ask for only the right unit law. Using diagrammatic reasoning, we can show that the left unit law holds in $\mathbb{X}_{(M, CM)}$ as follows:

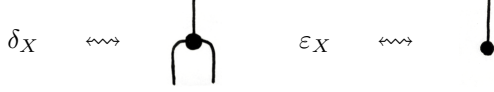


Notice that it is possible for two different monoidal equational theories to present equivalent categories of string diagrams. For example if we define CM' to be CM , but with the left unit law instead of the right unit law, then $\mathbb{X}_{(\Gamma, CM)} \cong \mathbb{X}_{(\Gamma, CM')}$. Our interest is therefore in the category of string diagrams corresponding to some monoidal equational theory, not the specific presentation in terms of generators and equations. For this reason, we move from talking about generators and equations to talking about symmetric strict monoidal categories more generally, which we call *monoidal theories* in this context.

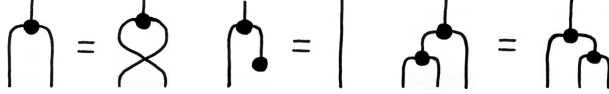
Algebraic theories can be understood in terms of monoidal theories. In particular, categories with finite products correspond to symmetric monoidal categories with certain additional structure, which we introduce now.

Definition 5. Let \mathbb{X} be a symmetric strict monoidal category.

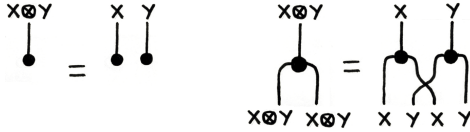
- (i) A *commutative comonoid* on an object X of \mathbb{X} is a triple $(X, \delta_X, \varepsilon_X)$ where $\delta_X : X \rightarrow X \times X$ and $\varepsilon_X : X \rightarrow I$, as in:



such that the following equations are satisfied:



- (ii) Suppose that each object X of \mathbb{X} is equipped with the structure of a commutative comonoid. Call this structure *coherent* in case for all objects X, Y of \mathbb{X} we have:



- (iii) Call such a coherent commutative comonoid structure on \mathbb{X} *natural* in case for every arrow $f : X \rightarrow Y$ of \mathbb{X} we have:



It turns out that a category with finite products is precisely a symmetric monoidal category with the above structure:

Theorem 1 ([16]). *A symmetric strict monoidal category is a cartesian monoidal category (a category with finite products) if and only if it is equipped with a coherent and natural commutative comonoid structure.*

Example 2. The category **Set** of sets and functions has finite products, with $\delta_X : X \rightarrow X \times X$ and $\varepsilon_X : X \rightarrow I$ defined by $\delta_X(x) = (x, x)$ and $\varepsilon_X(x) = *$, where $\{*\}$ is the unique element of the singleton set $I = \{*\}$.

We can use this characterisation of finite product structure to define algebraic theories and their associated notions in the language of monoidal categories as follows:

Definition 6 (Algebraic Theory). An *algebraic theory* is a symmetric strict monoidal category satisfying the conditions of Theorem 1. A *model* of an algebraic theory \mathbb{C} is a strict monoidal functor $F : \mathbb{C} \rightarrow \mathbf{Set}$ that preserves the commutative comonoid structure in the sense that $F(\delta_X) = \delta_{FX}$ and $F(\varepsilon_X) = \varepsilon_{FX}$. A *model morphism* is a natural transformation.

In particular, this means that any monoidal equational theory (Γ, E) satisfying the conditions of Theorem 1 has $\mathbb{X}_{(\Gamma, E)}$ an algebraic theory.

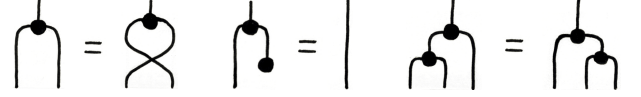
Example 3 (Commutative Monoids). We may extend our presentation of the monoidal theory of commutative monoids,

(M, CM) , to a presentation of the *algebraic theory* of commutative monoids. To do this, we add:

- (i) generators $\delta : 1 \rightarrow 2$ and $\varepsilon : 1 \rightarrow 0$, as in:



- (ii) equations making $(1, \delta, \varepsilon)$ a commutative comonoid:



- (iii) equations for each generator of M to ensure that our commutative comonoid structure will be a natural:



Write $\text{Alg}(M, CM)$ for the resulting monoidal equational theory. Then $\mathbb{X}_{\text{Alg}(M, CM)}$ is the algebraic theory of commutative monoids.

Compare this to the classical presentation of the algebraic theory of commutative monoids as the category of terms over signature

$$\Sigma = \{ _ \circ _ / 2, e / 0 \}$$

subject to equations

$$x_1 \circ x_2 = x_2 \circ x_1 \quad x_1 \circ e = x_1$$

$$x_1 \circ (x_2 \circ x_3) = (x_1 \circ x_2) \circ x_3$$

One way of understanding the commutative comonoid structure in $\mathbb{X}_{\text{Alg}(M, CM)}$ is that it allows us to work with variables in our string diagrams, with δ and ε allowing us to duplicate and discard variables, respectively. For example, the term $(x_1 \circ x_2) \circ x_1$ corresponds to the following string diagram:



We end this section with a brief discussion of the variety theorem for algebraic theories. To begin, we consider a category of categories of models and model morphisms:

Definition 7 (Varieties). If \mathbb{X} is an algebraic theory, Write $\text{Mod}(\mathbb{X})$ for the category of models of \mathbb{X} and model morphisms. Categories that arise as $\text{Mod}(\mathbb{X})$ for some \mathbb{X} are called *varieties*. If \mathcal{V}, \mathcal{W} are varieties, a *morphism of varieties* is a functor $R : \mathcal{V} \rightarrow \mathcal{W}$ such that

- (i) R admits a left adjoint $L : \mathcal{W} \rightarrow \mathcal{V}$, and
(ii) R commutes with sifted colimits.

This yields a 2-category Var of varieties, morphisms of varieties, and natural transformations. The variety theorem is:

Theorem 2 ([2]). *There is an adjunction of 2-categories*

$$\begin{array}{ccc} & \text{Mod} & \\ \text{CAT} & \xrightarrow{\quad} & \text{Var}^{\text{op}} \\ & \text{Th} & \end{array}$$

where CAT is the 2-category of categories with finite products, functors that preserve finite products, and natural transformations.

This neatly packages the relationship between theories and their categories of models. For example, since $\text{Th}(\text{Mod}(\mathbb{X}))$ is the idempotent splitting completion $\text{Split}(\mathbb{X})$, we obtain:

Theorem 3 ([2]). *Two algebraic theories \mathbb{X} and \mathbb{Y} have equivalent categories of models $\text{Mod}(\mathbb{X}) \simeq \text{Mod}(\mathbb{Y})$ if and only if they are Morita-equivalent as in $\text{Split}(\mathbb{X}) \simeq \text{Split}(\mathbb{Y})$.*

The rest of this paper is concerned with identifying an appropriate notion of relational theory and in proving an analogous variety theorem for it.

III. THE ALGEBRA OF RELATIONS

In the context of algebraic theories, finite product structure serves as an algebra of functions. In this section, we consider an analogous algebra of relations. There are two perspectives from which to consider this algebra of relations: As internal relations in a regular category, or through cartesian bicategories of relations. The two perspectives are very closely related, and we require both: it is through regular categories that our development connects to the wider literature on categorical algebra, but our syntax for relational theories will be the string-diagrammatic syntax for cartesian bicategories for relations.

To begin, we recall the category Rel of sets and relations, which will serve as the universe of models for relational theories in the same way that the category Set of sets and functions is the universe of models for classical algebraic theories. An elementary presentation of Rel is:

Definition 8. The category Rel has sets as objects, with arrows $f : X \rightarrow Y$ given by binary relations $f \subseteq X \times Y$. The composite of arrows $f : X \rightarrow Y$, $g : Y \rightarrow Z$ is defined by $fg = \{(x, z) \mid \exists y \in Y. (x, y) \in f \wedge (y, z) \in g\}$, and the identity relation on X is $\{(x, x) \mid x \in X\}$.

Rel will be the universe of models for our relational algebraic theories, and is therefore the motivating example of the more general notions in this section.

A. Categories of Internal Relations

In any regular category we can construct an abstract analogue of Definition 8. Instead of subsets $R \subseteq A \times B$, we represent relations as subobjects $R \rightrightarrows A \times B$. This approach to categorifying the theory of relations has a relatively long history [17], and integrates well with standard categorical logic due to the ubiquity of regular categories there.

Definition 9. Let \mathbb{C} be a regular category. The associated category of *internal relations*, $\text{Rel}(\mathbb{C})$, is defined as follows:

objects are objects of \mathbb{C}

arrows $r : A \rightarrow B$ are jointly monic spans

$$r = \langle f, g \rangle : R \rightrightarrows A \times B$$

modulo equivalence as subobjects of $A \times B$. That is, $r : R \rightrightarrows A \times B$ and $r' : R' \rightrightarrows A \times B$ are equivalent (and thus define the same arrow of $\text{Rel}(\mathbb{C})$) in case there exists an isomorphism $\alpha : R \rightarrow R'$ such that $\alpha r' = r$.

composition of two arrows $r : A \rightarrow B$ and $s : B \rightarrow C$ given respectively by $\langle f, g \rangle : R \rightrightarrows A \times B$ and $\langle h, k \rangle : S \rightrightarrows B \times C$ is defined by first constructing the pullback of h along g :

$$\begin{array}{ccc} R \times_B S & \xrightarrow{g'} & S \\ h' \downarrow & \lrcorner & \downarrow h \\ R & \xrightarrow{g} & B \end{array}$$

This defines an arrow $\langle h'f, g'k \rangle : R \times_B S \rightarrow A \times C$. The composite $rs : A \rightarrow C$ is defined to be the monic part of the image factorization of this arrow:

$$\begin{array}{ccc} R \times_B S & \xrightarrow{\langle h'f, g'k \rangle} & A \times C \\ & \searrow & \nearrow \\ & RS & \end{array}$$

identities $1_A : A \rightarrow A$ are given by the corresponding diagonal map $\langle 1_A, 1_A \rangle : A \rightrightarrows A \times A$.

Example 4. Set is a regular category, and the category of internal relations in $\text{Rel}(\text{Set})$ is precisely the usual category of sets and relations Rel.

B. Cartesian Bicategories of Relations

It is difficult to work with relations internal to a regular category directly. Routine calculations often involve complex interaction between pullbacks and image factorisations, and this quickly becomes intractable.

A much more tractable setting for working with relations is provided by cartesian bicategories of relations, which admit a convenient graphical syntax. A good way to think of this is by analogy with Theorem 1, with cartesian bicategories of relations providing a string-diagrammatic characterisation of categories of internal relations.

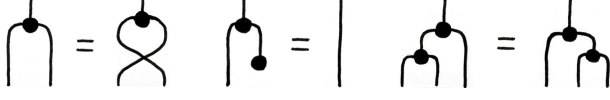
Cartesian bicategories of relations are defined in terms of commutative special frobenius algebras, which provide the basic syntactic scaffolding of our approach:

Definition 10. Let \mathbb{X} be a symmetric strict monoidal category. A *commutative special frobenius algebra* in \mathbb{X} is a 5-tuple $(X, \delta_X, \mu_X, \varepsilon_X, \eta_X)$, as in

$$\begin{array}{ccc} \delta_X & \leftarrow & \text{diagram of } \delta_X \\ \mu_X & \leftarrow & \text{diagram of } \mu_X \\ \varepsilon_X & \leftarrow & \text{diagram of } \varepsilon_X \\ \eta_X & \leftarrow & \text{diagram of } \eta_X \end{array}$$

such that

(i) $(X, \delta_X, \varepsilon_X)$ is a commutative comonoid:



(ii) (X, μ_X, η_X) is a commutative monoid:



(iii) μ_X and δ_X satisfy the special and frobienus equations:



A useful intermediate notion is that of a hypergraph category, in which objects are coherently equipped with commutative special frobienus algebra structure:

Definition 11. A symmetric strict monoidal category \mathbb{X} is called a *hypergraph category* [15] in case:

- (i) Each object X of \mathbb{X} is equipped with a commutative special frobienus algebra $(X, \delta_X, \mu_X, \varepsilon_X, \eta_X)$.
- (ii) The commutative comonoid structure is coherent (but not necessarily natural) in the sense of Definition 5.

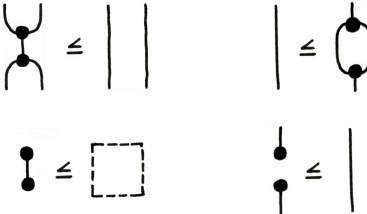
Now, a cartesian bicategory of relations is a hypergraph category with some additional properties:

Definition 12. A *cartesian bicategory of relations* [11] is a poset-enriched hypergraph category \mathbb{X} such that:

- (i) The commutative comonoid structure is *lax natural*. That is, for all arrows f of \mathbb{X} we have:



- (ii) Each of the frobienus algebras satisfy:



Example 5. The category Rel is a cartesian bicategory of relations with

$$\delta_X = \{(x, (x, x)) \mid x \in X\} \quad \mu_X = \{((x, x), x) \mid x \in X\}$$

$$\varepsilon_X = \{(x, *) \mid x \in X\} \quad \eta_X = \{(*, x) \mid x \in X\}$$

where $*$ is the unique element of the singleton set $I = \{*\}$.

Example 6. If \mathbb{C} is a regular category then $\text{Rel}(\mathbb{C})$ is a cartesian bicategory of relations with $X \otimes Y = X \times Y$, $I = 1$, and

$$\delta_X : X \rightarrow X \otimes X = \langle 1_X, \Delta_X \rangle : X \rightarrow X \times (X \times X)$$

$$\mu_X : X \otimes X \rightarrow X = \langle \Delta_X, 1_X \rangle : X \rightarrow (X \times X) \times X$$

$$\varepsilon_X : X \rightarrow I = \langle 1_X, !_X \rangle : X \rightarrow X \times I$$

$$\eta_X : I \rightarrow X = \langle !_X, 1_X \rangle : I \rightarrow I \times X$$

Where Δ_X is the diagonal morphism and $!_X$ is the unique morphism into the terminal object $I = 1$.

An important feature of cartesian bicategories of relations is that every hom-set admits a meet operation:

Lemma 1 ([7]). *Every cartesian bicategory of relations has meets of parallel arrows, with $f \cap g$ for $f, g : X \rightarrow Y$ defined by*

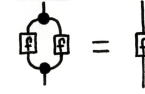


Further, the meet determines the poset-enrichment:

$$f \leq g \Leftrightarrow f \cap g = f$$

This allows us to give a much simpler presentation of the structure involved in a cartesian bicategory of relations:

Lemma 2. A hypergraph category \mathbb{X} is a cartesian bicategory of relations if and only if for each arrow f :



Proof. See Appendix A. □

A morphism of cartesian bicategories of relations is simply a structure-preserving functor:

Definition 13. A *morphism* of cartesian bicategories of relations $F : \mathbb{X} \rightarrow \mathbb{Y}$ is a strict monoidal functor that preserves the frobienus algebra structure:

$$F(\delta_X) = \delta_{FX} \quad F(\mu_X) = \mu_{FX}$$

$$F(\varepsilon_X) = \varepsilon_{FX} \quad F(\eta_X) = \eta_{FX}$$

We will require a 2-category of cartesian bicategories of relations in our development. While the 0- and 1- cells are cartesian bicategories of relations and their morphisms, respectively, the correct sort of 2-cell turns out to be a *lax* natural transformation, as in:

Definition 14. Let \mathbb{X}, \mathbb{Y} be cartesian bicategories of relations, and let $F, G : \mathbb{X} \rightarrow \mathbb{Y}$ be morphisms thereof. Then a *lax transformation* $\alpha : F \rightarrow G$ consists of an \mathbb{X}_0 -indexed family of arrows $\alpha_X : F(X) \rightarrow G(X)$ such that for each arrow $f : X \rightarrow Y$ of \mathbb{X} the following inequality holds in \mathbb{Y} :

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ Ff \downarrow & \leq & \downarrow Gf \\ FY & \xrightarrow{\alpha_Y} & GY \end{array}$$

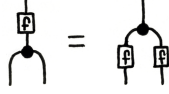
Definition 15. Let RAT be the 2-category with cartesian bicategories of relations as 0-cells, their morphisms as 1-cells, and lax transformations as 2-cells.

We call this 2-category RAT because in our variety theorem for relational algebraic theories it plays the role that CAT plays in Theorem 2 (the variety theorem for algebraic theories).

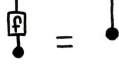
An important class of arrows in a cartesian bicategory of relations are the *maps*:

Definition 16 (Maps). An arrow $f : X \rightarrow Y$ in a cartesian bicategory of relations is called:

(i) *simple* in case



(ii) *total* in case



(iii) A *map* in case it is both simple and total.

Think of the maps in a cartesian bicategory of relations as relations that happen to be functions. The maps of a cartesian bicategory of relations always form a subcategory $\text{Map}(\mathbb{X})$. For example, $\text{Map}(\text{Rel}) \cong \text{Set}$. More generally:

Theorem 4 ([17]). For \mathbb{C} a regular category, there is an equivalence of categories $\mathbb{C} \simeq \text{Map}(\text{Rel}(\mathbb{C}))$.

We end this section with a remarkable fact about lax transformations: their components are always maps.

Lemma 3 ([7]). If \mathbb{X}, \mathbb{Y} are cartesian bicategories of relations, $F, G : \mathbb{X} \rightarrow \mathbb{Y}$ are morphisms thereof and $\alpha : F \rightarrow G$ is a lax transformation, then each component $\alpha_X : FX \rightarrow GX$ of α is necessarily a map.

IV. RELATIONAL ALGEBRAIC THEORIES

In this section we give the definition of relational algebraic theory and provide a number of examples.

Definition 17 ([7]). A *relational algebraic theory* is a cartesian bicategory of relations. A *model* of a relational algebraic theory \mathbb{X} is a morphism of cartesian bicategories of relations $F : \mathbb{X} \rightarrow \text{Rel}$. A *model morphism* $\alpha : F \rightarrow G$ is a lax transformation.

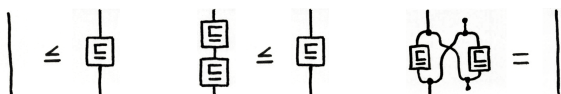
Now, from any monoidal equational theory we can obtain a relational algebraic theory by adding the required generators and equations (Definition 12) – much like the way we obtained the algebraic theory of monoids from a monoidal equational theory of monoids in Example 3.

Example 7 (Sets). The relational algebraic theory corresponding to empty monoidal equational theory $\mathbb{X}_{(\emptyset, \emptyset)}$ has sets as models and functions as model morphisms (see Lemma 3), and is therefore Set .

Example 8 (Posets). Consider the monoidal equational theory with a single generating symbol:

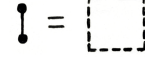


which is required to be reflexive, transitive, and antisymmetric:



Then models of the associated relational algebraic theory are precisely posets, with model morphisms precisely the monotone maps.

Example 9 (Nonempty Sets). Consider the monoidal equational theory with no generating symbols and a single equation:



Models of the associated relational algebraic theory are sets X such that the generating equation is satisfied in Rel :

$$\eta_X \varepsilon_X = \{(*, *)\} = \square_I$$

Where η_X and ε_X are defined as in Definition 8. If we calculate the relational composite, we find that:

$$\begin{aligned} \eta_X \varepsilon_X &= \{(*, *) \mid \exists x \in X. (*, x) \in \eta_X \wedge (x, *) \in \varepsilon_X\} \\ &= \{(*, *) \mid \exists x \in X\} \end{aligned}$$

and so models are nonempty sets, and model morphisms are functions thereof. Contrast this to the category of *pointed* sets, in which model morphisms must preserve the distinguished element.

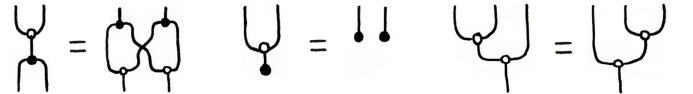
Example 10 (Regular Semigroups). A *semigroup* is a set equipped with an associative binary operation, denoted by juxtaposition. A semigroup S is *regular* [19] in case

$$\forall a \in S. \exists x \in S. axa = a$$

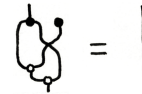
This is easily captured as a relational theory. Consider the monoidal equational theory with a single generating symbol:



Which is required to be simple, entire, and associative:



and finally must satisfy the axiom of regularity:



The associated relational algebraic theory is the category of regular semigroups and semigroup homomorphisms.

Example 11 (Von Neumann Regular Rings). A ring R is said to be *Von Neumann regular* [31] in case the multiplication monoid is a regular semigroup. The theory of rings is algebraic, and so adding the regularity axiom (from the semigroup example above) to its realization as a relational theory captures the Von Neumann regular rings.

Example 12 (Effectoids). An *effectoid* [30] is a set A equipped with a unary relation $\not\vdash _ \subseteq A$, a binary relation $_ \preceq _ \subseteq A \times A$, and a ternary relation $_ ; _ \vdash _ \subseteq A \times A \times A$ satisfying:

(Identity) For all $a, a' \in A$,

$$\begin{aligned} \exists x \in A. (\not x \mapsto x) \wedge (x; a \mapsto a') &\Leftrightarrow a \preceq a' \\ &\Leftrightarrow \exists y \in A. (\not x \mapsto y) \wedge (a; y \mapsto a') \end{aligned}$$

(Associativity) For all $a, b, c, d \in A$,

$$\exists x. (a; b \mapsto x) \wedge (x; c \mapsto d) \Leftrightarrow \exists y. (b; c \mapsto y) \wedge (a; y \mapsto d)$$

(Reflexive Congruence 1) For all $a \in A$, $a \preceq a$.

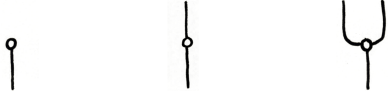
(Reflexive Congruence 2) For all $a, a' \in A$,

$$(\not x \mapsto a) \wedge (a \preceq a') \Rightarrow (\not x \mapsto a')$$

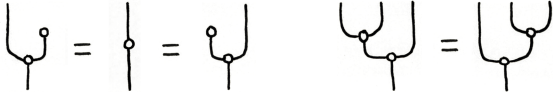
(Reflexive Congruence 3) For all $a, b, c \in A$,

$$\exists x. (a; b \mapsto x) \wedge (x \preceq c) \Rightarrow (a; b \preceq c)$$

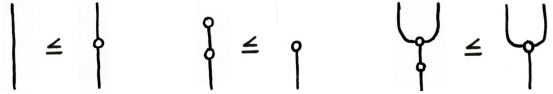
To obtain a relational algebraic theory of effectoids, we ask for three generating symbols in the generating monoidal equational theory, corresponding respectively to the unary, binary, and ternary relation:



Then the identity and associativity axioms become:



And the reflexive congruence axioms become:



Now the models of the corresponding relational algebraic theory are precisely the effectoids.

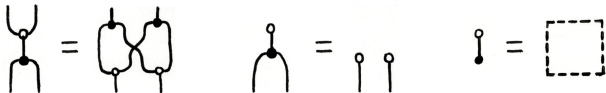
Example 13 (Generalized Separation Algebras). A *generalized separation algebra* [4] is a partial monoid satisfying the left and right cancellativity axioms, which further satisfies the conjugation axiom:

$$\forall x, y. (\exists z. x \circ z = y) \Leftrightarrow (\exists w. w \circ x = y)$$

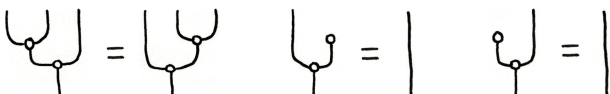
To capture generalized separation algebras as a relational algebraic theory, we require two generating symbols in the generating monoidal equational theory, corresponding to the monoid operation and the unit:



Both are required to be simple, and the unit is required to be total:



The associativity and unitality axioms become:



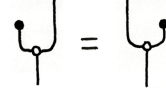
The left and right cancellativity axioms are:



where the upside-down generators are defined by:

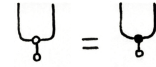


Finally, the axiom of conjugation is expressible as:



Then the corresponding relational algebraic theory has the category of generalized separation algebras and partial monoid homomorphisms as its category of models and model morphisms.

Example 14 (Generalized Pseudo-Effect Algebras). A *generalized pseudo-effect algebra* [4] is a generalized separation algebra in which $x \circ y = 0 \Rightarrow x = 0 = y$. That is,



Example 15 (Algebraic Theories). Let \mathbb{X} be an algebraic theory, and let $(\mathbb{X}_{\text{eq}})_{\text{reg/lex}}$ be the regular completion of \mathbb{X} [9], [10]. $\text{Rel}((\mathbb{X}_{\text{eq}})_{\text{reg/lex}})$ is a relational algebraic theory. Further, its models and model morphisms (as a relational algebraic theory) coincide with the models and model morphisms of \mathbb{X} (as an algebraic theory). Conversely, if \mathbb{X} is a relational algebraic theory, then the maps of \mathbb{X} form a subcategory $\text{Map}(\mathbb{X})$. By Theorem 1, $\text{Map}(\mathbb{X})$ has finite products. Thus the maps of a relational algebraic theory define an algebraic theory in the usual sense. Further, the notions of model and model morphism for relational algebraic theories restrict to the usual notions for algebraic theories on the category of maps.

Example 16 (Essentially Algebraic Theories). An *essentially algebraic theory* [29] is (among many equivalent presentations) a category \mathbb{X} with finite limits. Models are the finite-limit preserving functors $\mathbb{X} \rightarrow \text{Set}$, and model morphisms are natural transformations. For \mathbb{X} an essentially algebraic theory let $\mathbb{X}_{\text{reg/lex}}$ be the regular completion of \mathbb{X} [10]. Then $\text{Rel}(\mathbb{X}_{\text{reg/lex}})$ is a relational algebraic theory. Further, its models and model morphisms (as a relational algebraic theory) coincide with the models and model morphisms of \mathbb{X} (as an essentially algebraic theory). Conversely, if \mathbb{X} is a relational algebraic theory then the simple maps of \mathbb{X} are a partial algebraic theory in the sense of [14] – which turn out to be equivalent to essentially algebraic theories. The notions of model and model morphism for relational theories restrict to the notions of model and model morphism for partial theories.

V. THE VARIETY THEOREM

In this final section we prove the promised variety theorem for relational algebraic theories. We do this in phases: First we introduce some terminology concerning classes of idempotents

that will be important in what follows, and recall some of the details of the idempotent splitting completion. This done, we make the relationship between bicategories of relations and regular categories precise by constructing an adjunction of 2-categories. We then show how the situation extends to include exact categories, this being necessary because exactness is the difference between regular categories and definable categories. Finally, we introduce definable categories, which end up being the varieties of our relational theories, playing the role of Var in Theorem 2. This is structured so that once we have introduced the relevant parts of the theory of definable categories, the variety theorem follows immediately.

A. Flavours of Idempotent Splitting

We begin by introducing some classes of arrows that will play an important role in what follows:

Definition 18. An arrow $f : A \rightarrow B$ of a relational algebraic theory is called

(i) *reflexive* in case

$$\begin{array}{c} | \\ \hline \end{array} \leq \begin{array}{c} \boxed{f} \\ | \\ \hline \end{array}$$

(ii) *coreflexive* in case

$$\begin{array}{c} \boxed{f} \\ | \\ \hline \end{array} \leq \begin{array}{c} | \\ \hline \end{array}$$

(iii) *symmetric* in case

$$\begin{array}{c} \boxed{f} \\ | \\ \hline \end{array} \leq \begin{array}{c} \begin{array}{c} \curvearrowright \\ \boxed{f} \end{array} \\ | \\ \hline \end{array}$$

(iv) *transitive* in case

$$\begin{array}{c} \boxed{f} \\ \boxed{f} \\ | \\ \hline \end{array} \leq \begin{array}{c} \boxed{f} \\ | \\ \hline \end{array}$$

(v) A *partial equivalence relation* if it is symmetric and transitive.

(vi) An *equivalence relation* if it is reflexive, symmetric, and transitive.

Notice in particular that every partial equivalence relation is idempotent, and that every coreflexive is a partial equivalence relation.

Lemma 4 ([17]). *In a relational algebraic theory*

- (i) f is a partial equivalence relation if and only if f is symmetric and idempotent.
- (ii) f is coreflexive if and only if f is simple and f is a partial equivalence relation.

More specifically, we will need to split the above classes of idempotents in our relational algebraic theories. We recall the idempotent splitting completion:

Definition 19. Let \mathbb{X} be a category, and let \mathcal{E} be a collection of idempotents in \mathbb{X} . Define a category $\text{Split}_{\mathcal{E}}(\mathbb{X})$ by:

objects are pairs (X, a) where X is a object of \mathbb{X} and $a : X \rightarrow X$ is in \mathcal{E} .

arrows $(X, a) \xrightarrow{f} (Y, b)$ are arrows $f : X \rightarrow Y$ of \mathbb{X} such that $a f b = f$.

composition is composition in \mathbb{X} .

identities are $a = 1_{(X, a)} : (X, a) \rightarrow (X, a)$.

Of course, the point of the idempotent splitting completion is to split idempotents:

Observation 1. Every member of \mathcal{E} splits in $\text{Split}(\mathbb{X})$.

For our purposes in this section, it is important that the idempotent splitting completion behaves well with respect to relational algebraic theories. In particular, we have:

Proposition 1 ([17]). *If \mathbb{X} is a relational algebraic theory and \mathcal{E} is a class of partial equivalence relations in \mathbb{X} , then $\text{Split}(\mathcal{E})$ is also a relational algebraic theory.*

B. Tabulation and Regular Categories

We begin our exposition of the correspondence between regular categories and relational algebraic theories by recalling the notion of tabulation [11]. Intuitively, a tabulation of an arrow is a representation of it as a subobject in the category of maps.

Definition 20. A *tabulation* of an arrow $f : X \rightarrow Y$ in a relational algebraic theory \mathbb{X} consists of a pair of maps (h, k) such that the equation below on the left is holds in \mathbb{X} , and the map below on the right is monic in $\text{Map}(\mathbb{X})$:

$$\begin{array}{c} \begin{array}{c} \curvearrowright \\ \boxed{h} \end{array} \begin{array}{c} \boxed{k} \\ \curvearrowright \end{array} = \begin{array}{c} \boxed{f} \\ | \\ \hline \end{array} \end{array} \quad \begin{array}{c} \begin{array}{c} \boxed{h} \end{array} \begin{array}{c} \boxed{k} \end{array} \end{array}$$

If every arrow of \mathbb{X} admits a tabulation, say that \mathbb{X} is *tabular*. Further, define RAT_{tab} to be the full subcategory of RAT on the tabular 0-cells.

It turns out that if every arrow is representable in this way, then the category of maps is regular.

Proposition 2. *If \mathbb{X} is a tabular relational algebraic theory then $\text{Map}(\mathbb{X})$ is regular. This extends to a 2-functor $\text{Map} : \text{RAT}_{\text{tab}} \rightarrow \text{REG}$.*

Proof. See Appendix B. □

Conversely, in any regular category the category of internal relations is a tabular relational algebraic theory.

Proposition 3. *If \mathbb{C} is a regular category, then $\text{Rel}(\mathbb{C})$ is tabular. This extends to a 2-functor $\text{Rel} : \text{REG} \rightarrow \text{RAT}_{\text{tab}}$.*

Proof. See Appendix C. □

We may therefore treat relational algebraic theories and regular categories interchangeably at this level of abstraction.

Theorem 5. *There is an equivalence of 2-categories:*

$$\begin{array}{ccc} & \text{Map} & \\ \text{RAT}_{\text{tab}} & \xrightarrow{\quad} & \text{REG} \\ & \text{Rel} & \end{array} \quad \simeq$$

Proof. If we restrict the situation to include only the 0- and 1-cells, then the equivalence of the resulting categories is proven

in [11]. Extending the situation to include 2-cells involves no surprises, and the result is immediate. \square

Remarkably, we may complete any relational algebraic theory to a tabular one by formally splitting coreflexives.

Proposition 4. *Let \mathbb{X} be a relational algebraic theory, and let cor be the collection of coreflexives in \mathbb{X} . Then \mathbb{X} is tabular if and only if every member of cor splits. In particular, $\text{Split}_{\text{cor}}(\mathbb{X})$ is always tabular. This extends to a 2-adjunction:*

$$\begin{array}{ccc} & \text{Split}_{\text{cor}} & \\ \text{RAT} & \xrightarrow{\quad} & \text{RAT}_{\text{tab}} \\ & \perp & \\ & \xleftarrow{\quad} & \end{array}$$

where the right adjoint is the evident forgetful functor.

Proof. See Appendix D \square

Together, Theorem 5 and Proposition 4 tell us everything we need to know about the relationship between regular categories and relational algebraic theories. Next, we proceed to show how this situation extends to those regular categories that happen to be exact.

C. Effectivity and Exact Categories

It is possible to formulate exactness of a regular category succinctly as a property of the corresponding category of internal relations. Specifically, if we make the following definition:

Definition 21. Say that a relational algebraic theory \mathbb{X} is *effective* in case it is tabular and every equivalence relation in \mathbb{X} splits. Define RAT_{eff} to be the full 2-subcategory of RAT_{tab} on the effective 0-cells.

Then we may define exact categories as follows:

Definition 22. A regular category \mathbb{C} is said to be *exact* in case $\text{Rel}(\mathbb{C})$ is effective. Let EX be the full 2-subcategory of REG on the exact 0-cells.

At this point it should not be too surprising that we obtain the following analogue of Theorem 5 for exact categories and effective relational algebraic theories:

Proposition 5. *If \mathbb{X} is an effective relational algebraic theory, then $\text{Map}(\mathbb{X})$ is exact. Conversely, if \mathbb{C} is an exact category, then $\text{Rel}(\mathbb{C})$ is effective. This extends to an equivalence of 2-categories:*

$$\begin{array}{ccc} & \text{Map} & \\ \text{RAT}_{\text{eff}} & \xrightarrow{\quad} & \text{EX} \\ & \text{Rel} & \\ & \xleftarrow{\quad} & \end{array}$$

Proof. Immediate. \square

Further, we can obtain an effective relational algebraic theory from a tabular one by splitting the equivalence relations:

Proposition 6. *Let \mathbb{X} be a tabular relational algebraic theory, and let eq be the collection of equivalence relations in \mathbb{X} . Then $\text{Split}_{\text{eq}}(\mathbb{X})$ is effective. This extends to a 2-adjunction:*

$$\begin{array}{ccc} & \text{Split}_{\text{eq}} & \\ \text{RAT}_{\text{tab}} & \xrightarrow{\quad} & \text{RAT}_{\text{eff}} \\ & \perp & \\ & \xleftarrow{\quad} & \end{array}$$

where the right adjoint is the evident forgetful 2-functor.

Proof. Straightforward, similar to Proposition 4. \square

This approach to exactness also allows us to formulate the exact completion of a regular category succinctly:

Proposition 7 ([10], [26]). *If \mathbb{C} is regular, define the exact completion of \mathbb{C} by*

$$\mathbb{C}_{\text{ex/reg}} = \text{Map}(\text{Split}_{\text{eq}}(\text{Rel}(\mathbb{X})))$$

Then $\mathbb{C}_{\text{ex/reg}}$ is exact. In fact, this extends to a 2-adjunction:

$$\begin{array}{ccc} & \text{ex/reg} & \\ \text{REG} & \xrightarrow{\quad} & \text{EX} \\ & \perp & \\ & \xleftarrow{\quad} & \end{array}$$

where the right adjoint is the evident forgetful 2-functor.

We summarize the relationship of regular and exact categories to relational algebraic theories as follows:

Corollary 1. *The following diagram of left 2-adjoints commutes.*

$$\begin{array}{ccc} \text{REG} & \xrightarrow{\text{ex/reg}} & \text{EX} \\ \text{Map} \uparrow \sim & & \sim \uparrow \text{Map} \\ \text{RAT}_{\text{tab}} & \xrightarrow{\text{Split}_{\text{eq}}} & \text{RAT}_{\text{eff}} \end{array}$$

where the arrows marked with \sim are part of a 2-equivalence.

Similarly, splitting partial equivalence relations allows us to summarize the role of the idempotent splitting completion:

Proposition 8. *Write per to denote the collection of partial equivalence relations in a relational algebraic theory. There is a 2-adjunction:*

$$\begin{array}{ccc} & \text{Split}_{\text{per}} & \\ \text{RAT} & \xrightarrow{\quad} & \text{RAT}_{\text{eff}} \\ & \perp & \\ & \xleftarrow{\quad} & \end{array}$$

where the right adjoint is the evident forgetful functor. Further, for any relational algebraic theory \mathbb{X} , we have

$$\text{Split}_{\text{per}}(\mathbb{X}) \simeq \text{Split}_{\text{eq}}(\text{Split}_{\text{cor}}(\mathbb{X}))$$

and so the following diagram of left 2-adjoints commutes:

$$\begin{array}{ccc} \text{RAT}_{\text{tab}} & \xrightarrow{\text{Split}_{\text{eq}}} & \text{RAT}_{\text{eff}} \\ \text{Split}_{\text{cor}} \uparrow & \nearrow \text{Split}_{\text{per}} & \\ \text{RAT} & & \end{array}$$

Proof. The proof that $\text{Split}_{\text{per}}$ defines a 2-functor which is left adjoint to the forgetful 2-functor is straightforward, and similar to Proposition 4. A proof that $\text{Split}_{\text{per}}(\mathbb{X}) \simeq \text{Split}_{\text{eq}}(\text{Split}_{\text{cor}}(\mathbb{X}))$ can be found in [17], it follows immediately that our diagram of left 2-adjoints commutes. \square

D. Definable Categories

The final idea involved in our variety theorem is that of a definable category [24]. Definable categories come from categorical universal algebra. If we take regular categories as our notion of theory, regular functors into \mathbf{Set} as our notion of model, and natural transformations as our model morphisms, then definable categories are the corresponding varieties.

This is relevant in our development of relational algebraic theories because a consequence of Theorem 5 is that the categories that arise as the models of a relational algebraic theory are precisely the categories that arise as the models of the associated regular category, where category of models does not change when we split coreflexives because our universe of models \mathbf{Rel} is already tabular.

We follow the exposition of [26], and in particular we formulate definable categories via finite injectivity classes:

Definition 23 (Finite Injectivity Class). Let $h : A \rightarrow B$ be an arrow of \mathbb{X} . Then an object C of \mathbb{X} is said to be *h-injective* in case the function of hom-sets $\mathbb{X}(h, C) : \mathbb{X}(B, C) \rightarrow \mathbb{X}(A, C)$ defined by $X(h, C)(f) = hf$ is injective. If M is a finite set of arrows in \mathbb{X} , write $\text{inj}(M)$ for the full subcategory on the objects C of \mathbb{X} that are *h-injective* for each $h \in M$. We say that each $\text{inj}(M)$ is a *finite injectivity class* in \mathbb{X} .

Definable categories are now defined in relation to an ambient locally finitely presentable category. It is an open problem to give a free-standing characterization [24].

Definition 24. A category is said to be *definable* if it arises as a finite injectivity class in some locally finitely presentable category. If \mathbb{X} and \mathbb{Y} are definable categories, a functor $F : \mathbb{X} \rightarrow \mathbb{Y}$ is called an *interpretation* in case it preserves products and directed colimits. Let \mathbf{DEF} be the 2-category with definable categories as 0-cells, interpretations as 1-cells, and natural transformations as 2-cells.

From any definable category we can obtain an exact category by considering its interpretations into \mathbf{Set} .

Proposition 9 ([26]). *If \mathbb{X} is a definable category then the functor category $\mathbf{DEF}(\mathbb{X}, \mathbf{Set})$ is an exact category. This extends to a 2-functor $\mathbf{DEF}(_, \mathbf{Set}) : \mathbf{DEF}^{\text{op}} \rightarrow \mathbf{EX}$.*

Similarly, for any regular category the associated category of regular functors into \mathbf{Set} is definable.

Proposition 10 ([26]). *If \mathbb{C} is a regular category then the functor category $\mathbf{REG}(\mathbb{C}, \mathbf{Set})$ is definable. This extends to a 2-functor $\mathbf{REG}(_, \mathbf{Set}) : \mathbf{REG} \rightarrow \mathbf{DEF}^{\text{op}}$.*

If the category in question is exact, then considering interpretations of the resulting definable category into \mathbf{Set} yields the original exact category. This lifts to the 2-categorical setting.

Proposition 11 ([26]). *There is an adjunction of 2-categories:*

$$\begin{array}{ccc} & \mathbf{REG}(_, \mathbf{Set}) & \\ \mathbf{REG} & \xrightarrow{\quad} & \mathbf{DEF}^{\text{op}} \\ & \mathbf{DEF}(_, \mathbf{Set}) & \end{array} \quad \perp$$

Which specializes to a 2-equivalence:

$$\begin{array}{ccc} & \mathbf{REG}(_, \mathbf{Set}) & \\ \mathbf{EX} & \xrightarrow{\quad} & \mathbf{DEF}^{\text{op}} \\ & \mathbf{DEF}(_, \mathbf{Set}) & \end{array} \quad \simeq$$

This gives us yet another way to describe the regular completion of an exact category.

Proposition 12 ([26]). *Let \mathbb{C} be a regular category. Then:*

$$\mathbb{C}_{\text{ex/reg}} \simeq \mathbf{DEF}(\mathbf{REG}(\mathbb{C}, \mathbf{Set}), \mathbf{Set})$$

Thus, we may summarize the relationship between definable, regular, and exact categories as follows:

Corollary 2 ([26, Section 9,10]). *The following diagram of left 2-adjoints commutes.*

$$\begin{array}{ccc} & \mathbf{DEF}^{\text{op}} & \\ \mathbf{REG}(_, \mathbf{Set}) \nearrow & \sim & \nwarrow \mathbf{REG}(_, \mathbf{Set}) \\ \mathbf{REG} & \xrightarrow{\text{ex/reg}} & \mathbf{EX} \end{array}$$

where the arrow marked with \sim is part of a 2-equivalence.

At this point the ingredients of our variety theorem for relational algebraic theories are assembled. We present it as a straightforward consequence of the foregoing machinery:

Theorem 6. *There is an adjunction of 2-categories*

$$\begin{array}{ccc} & \mathbf{Mod} & \\ \mathbf{RAT} & \xrightarrow{\quad} & \mathbf{DEF}^{\text{op}} \\ & \mathbf{Th} & \end{array} \quad \perp$$

Proof. We obtain the desired 2-adjunction by pasting the commutative diagrams of left 2-adjoints from Corollary 1, Proposition 8, and Corollary 2 to obtain the following commutative diagram of left 2-adjoints:

$$\begin{array}{ccc} & \mathbf{DEF}^{\text{op}} & \\ \mathbf{REG}(_, \mathbf{Set}) \nearrow & \sim & \nwarrow \mathbf{REG}(_, \mathbf{Set}) \\ \mathbf{REG} & \xrightarrow{\text{ex/reg}} & \mathbf{EX} \\ \text{Map} \uparrow \sim & & \sim \uparrow \text{Map} \\ \mathbf{RAT}_{\text{tab}} & \xrightarrow{\text{Split}_{\text{eq}}} & \mathbf{RAT}_{\text{eff}} \\ \text{Split}_{\text{cor}} \uparrow & & \nwarrow \text{Split}_{\text{per}} \\ \mathbf{RAT} & & \end{array}$$

where the arrows marked with \sim are part of a 2-equivalence. \square

It may not be immediately clear what this tells us about the category of models and model morphisms of a regular algebraic theory, so a brief discussion is warranted. Consider an arbitrary relational algebraic theory \mathbb{X} . Our universe of models \mathbf{Rel} is tabular, so models of \mathbb{X} and models of $\text{Split}_{\text{cor}}(\mathbb{X})$ are the same thing since the image of any coreflexive in \mathbb{X} already splits in \mathbf{Rel} . Then the category of models of \mathbb{X} and model morphisms thereof is $\mathbf{RAT}_{\text{tab}}(\text{Split}_{\text{cor}}(\mathbb{X}), \mathbf{Rel})$. When we transport

this across the 2-equivalence $\text{Map} : \text{RAT}_{\text{tab}} \xrightarrow{\sim} \text{REG}$ it becomes $\text{REG}(\text{Map}(\text{Split}_{\text{cor}}(\mathbb{X}), \text{Set}))$, which is a definable category. Thus, categories of models and model morphisms of regular algebraic theories are definable categories.

Now, Set is exact, so Rel is effective, which means that much like the models of \mathbb{X} and $\text{Split}_{\text{cor}}(\mathbb{X})$, the models of \mathbb{X} and $\text{Split}_{\text{per}}(\mathbb{X})$ are the same. We have shown that $\text{RAT}_{\text{eff}} \simeq \text{EX} \simeq \text{DEF}^{\text{op}}$, and from this we obtain the following analogue of Theorem 3 – answering the question of when two relational algebraic theories generate the same category of models and model morphisms:

Theorem 7. *Two relational algebraic theories \mathbb{X} and \mathbb{Y} present equivalent definable categories if and only if $\text{Split}_{\text{per}}(\mathbb{X})$ and $\text{Split}_{\text{per}}(\mathbb{Y})$ are equivalent.*

Proof. By inspection of the proof of Theorem 6. \square

VI. CONCLUDING REMARKS

We have developed a theory of relational universal algebra and proved a variety theorem for it. In particular, the notion of variety corresponding to our relational algebraic theories was shown to be that of a definable category. We provided several examples to illustrate the use of our framework.

We outline two directions for future work. First, as discussed in the introduction the present paper should be thought of in the context of recent work on partial algebraic theories [14]. Some details of the connection between partial and relational theories remain to be worked out, and we plan to work them out, with the ultimate goal being a unified account of relational, partial, and classical algebraic theories in terms of string-diagrammatic syntax.

Second, there are a number of interesting constructions and correspondences involving classical algebraic theories that might be generalized to include relational (and partial) algebraic theories. In particular, work on composing algebraic theories via distributive laws [12], the correspondence between monads and algebraic theories [21], and the tensor product of algebraic theories [21]. Finally it would be interesting to know if recent work on the isotropy group of an (essentially) algebraic theory [20] extends to the relational setting.

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APPENDIX

We require the following lemmas from [17]:

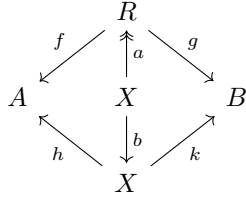
Lemma 5 ([17]). *Let $f, g : X \rightarrow Y$ be maps in a cartesian bicategory of relations. Then if $f \leq g$ or $g \leq f$ then $f = g$.*

Lemma 6 ([17]). *If (h, k) and (h', k') are both tabulations of an arrow f in a cartesian bicategory of relations, then there exists a unique isomorphism α such that $\alpha h = h'$ and $\alpha k = k'$. That is, tabulations are unique up to unique isomorphism.*

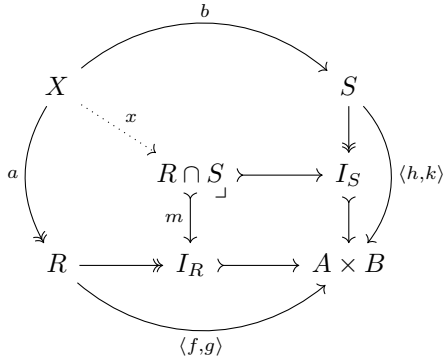
Along with the following (original) lemma concerning regular categories:

Lemma 7. *Two spans $\langle f, g \rangle : R \rightarrow A \times B$ and $\langle h, k \rangle : S \rightarrow A \times B$ are such that the image of $\langle f, g \rangle$ is a subobject of the*

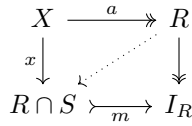
image of $\langle h, k \rangle$ if and only if there exists a cover $a : X \rightarrow R$ along with any morphism $b : X \rightarrow S$ such that



Proof. Suppose we have a commutative diagram of the appropriate form. We must show that the image I_R of $\langle f, g \rangle : R \rightarrow A \times B$ is isomorphic to $I_R \cap I_S$. Begin by constructing the meet, then we have:

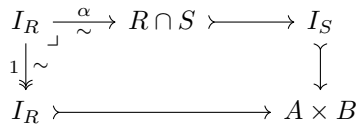


And since covers are left-orthogonal to monics we have:

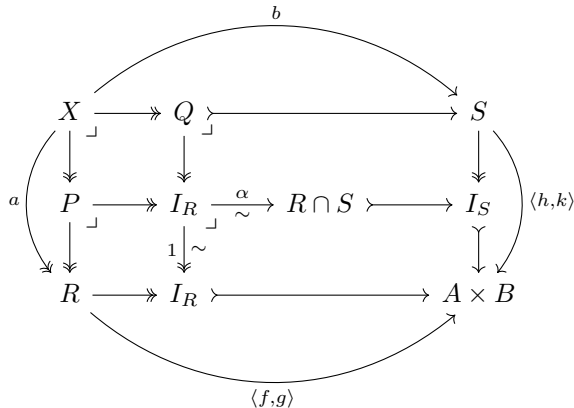


But then since $R \twoheadrightarrow I_R$ is an extremal epimorphism (being an equivalent characterisation of covers in a regular category) we have that $m : R \cap S \xrightarrow{\sim} R$ is an isomorphism, as required.

Conversely, suppose that there is an isomorphism $\alpha : I_R \xrightarrow{\sim} R \cap S$. Then the following diagram is a pullback:



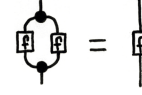
Now, define a, b by iterated pullback as in:



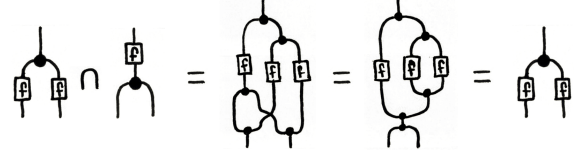
and the outer commutative square gives the required diamond shape. \square

A. Proof of Lemma 2

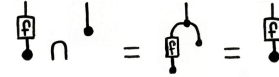
Let \mathbb{X} be a hypergraph category, and suppose that for all arrows f of \mathbb{X} we have



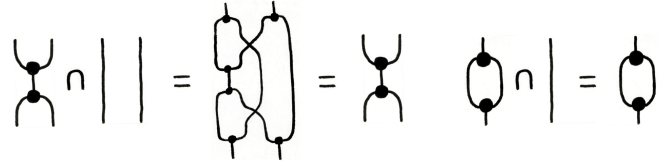
Define $f \cap g$ and $f \leq g$ as in Lemma 1. It is straightforward to verify that this is a preorder-enrichment of \mathbb{X} . We show that the conditions of Definition 12 are satisfied, beginning with lax naturality: For any arrow f of \mathbb{X} we have



and also



and so f is lax natural. Next, we show that the Frobenius algebra structure satisfies the required inequations:



Thus, \mathbb{X} is a cartesian bicategory of relations. For the converse, we know immediately that $f \cap f = f$, as required. \square

B. Proof of Proposition 2

A proof that $\text{Map}(\mathbb{X})$ is regular for any tabular relational algebraic theory \mathbb{X} can be found in [11]. The 2-functor Map is defined on 1-cells $F : \mathbb{X} \rightarrow \mathbb{Y}$ by $\text{Map}(F)(A) = FA$ and $\text{Map}(F)(f) = Ff$. For 2-cells $\alpha : F \rightarrow G$, $\text{Map}(\alpha)$ is defined by $\text{Map}(\alpha_A) = \alpha_A : FA \rightarrow GA$. Now if $f : A \rightarrow B$ in $\text{Map}(\mathbb{X})$, we use the fact that α is a lax transformation to obtain

$$\begin{aligned} \text{Map}(F)(f)\text{Map}(\alpha)_B &= Ff\alpha_B \\ &\leq \alpha_A Gf = \text{Map}(\alpha)_A\text{Map}(G)(f) \end{aligned}$$

By Lemma 3 we have that both sides of this inequation are maps, and then by Lemma 5 we have that $\text{Map}(\alpha)$ is a natural transformation. \square

C. Proof of Proposition 3

A proof that $\text{Rel}(\mathbb{C})$ is a tabular relational algebraic theory for any regular category \mathbb{C} can be found in [11]. The 2-functor Rel maps 1-cells $F : \mathbb{C} \rightarrow \mathbb{D}$ to the functor $\text{Rel}(F) : \text{Rel}(\mathbb{C}) \rightarrow \text{Rel}(\mathbb{D})$ defined to be the identity on objects, and defined on arrows by applying F to each component of the tabulation of the arrow in question. For 2-cells $\alpha : F \rightarrow G$ of REG , define $\text{Rel}(\alpha) : \text{Rel}(F) \rightarrow \text{Rel}(G)$ by setting $\text{Rel}(\alpha)_A$ to be the arrow tabulated by $\langle 1_{FA}, \alpha_A \rangle : FA \rightarrow GA$. ($\langle 1, x \rangle$ is always monic).

We must show that $\text{Rel}(\alpha)$ is a lax transformation. To that end, consider the two relevant composites

$$\text{Rel}(\alpha)_A; \text{Rel}(G)(\langle f, g \rangle) = \langle 1, \alpha_A \rangle; \langle Gf, Gg \rangle$$

and

$$\text{Rel}(F)(\langle f, g \rangle); \text{Rel}(\alpha)_B = \langle Ff, Fg \rangle; \langle 1, \alpha_B \rangle$$

which are defined by first construcing pullbacks

$$\begin{array}{ccc} P & \xrightarrow{p_1} & GR \\ p_0 \downarrow \lrcorner & & \downarrow Gf \\ FA & \xrightarrow{\alpha_A} & GA \end{array} \quad \text{and} \quad \begin{array}{ccc} Q & \xrightarrow{q_1} & FB \\ q_0 \downarrow \lrcorner & & \downarrow 1 \\ FR & \xrightarrow{Fg} & FB \end{array}$$

and then considering images as in

$$\begin{array}{ccc} P & \xrightarrow{\langle p_0, p_1 Gg \rangle} & FA \times GB \\ & \searrow & \nearrow \\ & I_P & \end{array}$$

and

$$\begin{array}{ccc} Q & \xrightarrow{\langle q_0 Ff, q_1 \alpha_B \rangle} & FA \times GB \\ & \searrow & \nearrow \\ & I_Q & \end{array}$$

Thus, it suffices to show that I_Q is a subobject of I_P . Since α is natural we have $q_0 \alpha_R Gf = q_0 Ff \alpha_A$, which induces an arrow $h : Q \rightarrow P$ as in:

$$\begin{array}{ccccc} Q & \xrightarrow{q_0} & FR & \xrightarrow{\alpha_R} & GR \\ q_0 \downarrow & \searrow h & & & \downarrow Gg \\ FR & & P & \xrightarrow{p_1} & GR \\ & \searrow Ff & \downarrow p_0 & & \downarrow Gf \\ & & FA & \xrightarrow{\alpha_A} & GA \end{array}$$

and then we have $h \langle p_0, p_1 Gg \rangle = \langle q_0 Ff, q_1 \alpha_B \rangle$ immediately in the first component, and for the second component:

$$\begin{array}{ccccc} & & P & \xrightarrow{p_1} & GR \\ & \nearrow h & & & \searrow Gg \\ Q & \xrightarrow{q_0} & FR & \xrightarrow{\alpha_R} & GB \\ & \searrow q_1 & & & \nearrow \alpha_B \\ & & FB & \xrightarrow{1} & FB \end{array}$$

and then we have

$$\begin{array}{ccccc} & & I_Q & & \\ & \swarrow & \uparrow & \searrow & \\ FA & & Q & & QB \\ & \swarrow & \downarrow \circ h & \nearrow & \\ & & I_P & & \end{array}$$

so by Lemma 7 I_Q is a subobject of I_P , as required, and we may conclude that $\text{Rel}(\alpha) : \text{Rel}(F) \rightarrow \text{Rel}(G)$ is a lax transformation. \square

D. Proof of Proposition 4

A proof that \mathbb{X} is tabular if and only if all coreflexives therein split can be found in [17]. $\text{Split}_{\text{cor}} : \text{RAT} \rightarrow \text{RAT}_{\text{tab}}$ is defined on 1-cells $F : \mathbb{X} \rightarrow \mathbb{Y}$ by $\text{Split}_{\text{cor}}(F)(X, a) = (FX, Fa)$ and $\text{Split}_{\text{cor}}(F)(f) = Ff$. For 2-cells $\alpha : F \rightarrow G$, $\text{Split}_{\text{cor}}(\alpha)$ is defined by $\text{Split}_{\text{cor}}(\alpha)_{(X, a)} = Fa \alpha_X Ga : (Fx, Fa) \rightarrow (Gx, Ga)$. This gives a lax transformation since for any $f : (X, a) \rightarrow (Y, b)$ in $\text{Split}_{\text{cor}}(\mathbb{X})$ we have

$$\begin{aligned} \text{Split}_{\text{cor}}(F)(f) \text{Split}_{\text{cor}}(\alpha)_{(Y, b)} &= Fa Ff \alpha_Y Gb \\ &\leq Fa \alpha_X Ga Gf = \text{Split}_{\text{cor}}(\alpha)_{(X, a)} \text{Split}_{\text{cor}}(G)(f) \end{aligned}$$

The proof that $\text{Split}_{\text{cor}}$ is a left 2-adjoint to the forgetful 2-functor is straightforward. \square