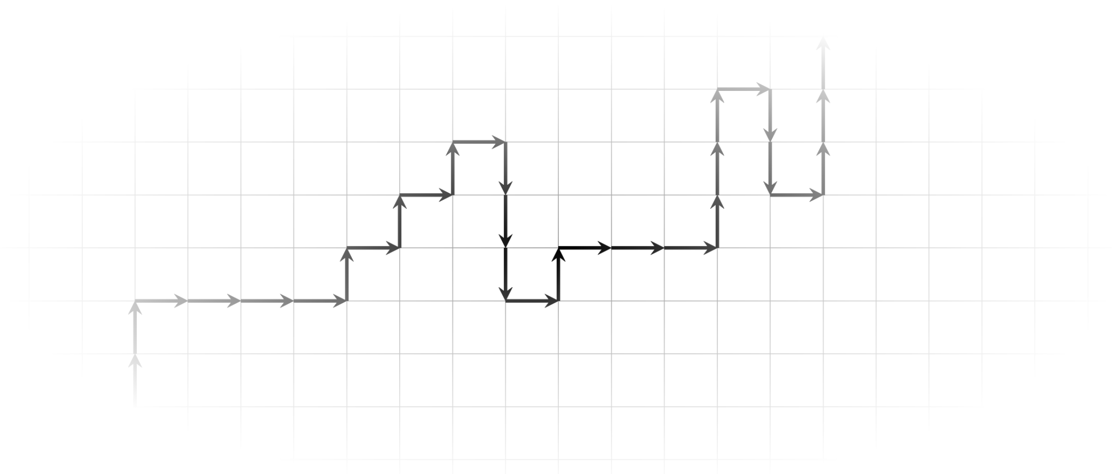
Definability of Combinatorial Functions



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Definability of Combinatorial Functions

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List of Publications

During my Ph.D. studies I have authored or co-authored the following publications, on which this thesis is based.

(i) I. Averbouch, T. Kotek, J.A. Makowsky, and E. Ravve. The universal edge elimination polynomial and the dichromatic polynomial. In *Electronic Notes in Discrete Mathematics*, volume 38, pages 77–82, 2011.

(ii) E. Fischer, T. Kotek, and J.A. Makowsky. Application of logic to combinatorial sequences and their recurrence relations. In *Model Theoretic Methods in Finite Combinatorics*, volume 558 of *Contemporary Mathematics*, Providence, RI, 2011. American Mathematical Society.

(iii) B. Godlin, T. Kotek, and J.A. Makowsky. Evaluations of graph polynomials. In *Workshop on Graph Theoretic Concepts in Computer Science (WG)*, pages 183–194, 2008.

(iv) T. Kotek. Complexity of Ising polynomials. Accepted to *Combinatorics, Probability and Computing.*

(v) T. Kotek. On the reconstruction of graph invariants. In *Electronic Notes in Discrete Mathematics*, volume 34, pages 375–379, 2009.

(vi) T. Kotek and J.A. Makowsky. A representation theorem for holonomic sequences based on counting lattice paths. *Fundamenta Informaticae*, 2012. To appear in a special issue for the 7th international conference on lattice path combinatorics and applications, Sienna, 2010.

(vii) T. Kotek and J.A. Makowsky. A representation theorem for (q-)holonomic sequences, Accepted to *Journal of Computer System Sciences*.

(viii) T. Kotek and J.A. Makowsky. Definability of combinatorial functions and their linear recurrence relations. In *Fields of Logic and Computation*, pages 444–462, 2010.

(ix) T. Kotek and J.A. Makowsky. Connection matrices and the definability of graph parameters. In *Computer Science Logic (CSL)*, 2012.

(x) T. Kotek, J.A. Makowsky, and B. Zilber. On counting generalized colorings. In *Computer Science Logic (CSL)*, pages 339–353, 2008.

(xi) T. Kotek, J.A. Makowsky, and B. Zilber. On counting generalized colorings. In *Model Theoretic Methods in Finite Combinatorics*, volume 558 of *Contemporary Mathematics*, Providence, RI, 2011. American Mathematical Society.

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Abstract

In recent years there has been growing interest in graph polynomials, functions from graphs to polynomial rings which are invariant to isomorphism. Graph polynomials encode information about the input graphs e.g. in their evaluations, coefficients, degree and roots. Many researchers studied specific graph polynomials such as the chromatic polynomial, the Tutte polynomial, the interlace polynomial or the matching polynomial. In this thesis we present a unified logical framework for graph polynomials. Using this framework we compare the definability of graph polynomials which count generalized colorings with that of graph polynomials defined as subset expansions. We show that many graph parameters cannot be evaluations or coefficients of the prominent graph polynomials studied in the literature. We consider the complexity of a specific graph polynomial, the Ising polynomial, as a case study of a conjecture by J.A. Makowsky from 2008 on the complexity of definable graph polynomials.

We also consider the definability of sequences of integers and of polynomials that have recurrence relations. We give a representation theorem for sequences satisfying recurrence relations in terms of positional weights. This theorem extends a theorem by N. Chomsky and M.P. Schützenberger for sequences of integers satisfying linear recurrence relations with constant coefficients. For P-recursive integer sequences, sequences which satisfy linear recurrences with polynomial coefficients, we give a representation in terms of counting lattice paths in appropriate lattices.

Abbreviations and Notations

$\lvert A\rvert$	—	The cardinality of the set A
$[n]$	—	The set $\{1,\dots,n\}$
$[n_1,n_2]$	—	The set $\{n_1,\dots,n_2\}$
k, t, x, y, ...	—	Indeterminates in polynomials or power series
$\bar{\mathsf{x}}$, $\bar{\mathsf{y}}$	—	Tuples of indeterminates $\bar{\mathsf{x}} = (\mathsf{x}_1,\dots,\mathsf{x}_\ell)$, $\ell \in \mathbb{N}$. ℓ is either clear from the context or is left unspecified.
$k_{(j)}$	—	The falling factorial $k_{(j)} = k\cdot(k-1)\cdots(k-(j-1))$
$\mathcal{R}[\mathsf{x}_1,\dots,\mathsf{x}_\ell]$	—	The ring of polynomials in indeterminates $\mathsf{x}_1,\dots,\mathsf{x}_\ell$ with coefficients in the ring $\mathcal{R}$. [1]
$\mathcal{R}(\mathsf{x}_1,\dots,\mathsf{x}_\ell)$	—	The field of rational functions in indeterminates $\mathsf{x}_1,\dots,\mathsf{x}_\ell$ with coefficients in the ring $\mathcal{R}$
$\dfrac{\mathcal{R}[\mathsf{x}_1,\dots,\mathsf{x}_\ell]}{\mathcal{R}[\mathsf{y}_1,\dots,\mathsf{y}_m]}$	—	The ring of rational functions of the form $\dfrac{a(\bar{\mathsf{x}})}{b(\bar{\mathsf{y}})}$ with $a(\bar{\mathsf{x}}) \in \mathcal{R}[\bar{\mathsf{x}}]$ and $b(\bar{\mathsf{y}}) \in \mathcal{R}[\bar{\mathsf{y}}]$

[1] Sometimes $\mathcal{R}$ is taken to be a semi-ring. In this case $\mathcal{R}[\bar{\mathsf{x}}]$ is a semi-ring as well. The case is similar for the next two notations.

Graph-theoretic notations

$V(G)$	—	For a graph G, $V(G)$ is the set of vertices of G
$E(G)$	—	For a graph G, $E(G)$ is the set of edges of G
n_G	—	$n_G = \|V(G)\|$ is the number of vertices in the graph G
m_G	—	$m_G = \|E(G)\|$ is the number of edges in the graph G
$E_G(S)$	—	For a graph G and $S \subseteq V(G)$, $E_G(S)$ is the set of edges of G with both end-points *in* S
$E_G(\bar{S})$	—	For a graph G and $S \subseteq V(G)$, $E_G(\bar{S})$ is the set of edges of G with both end-points *outside of* S
$[S, \bar{S}]_G$	—	For a graph G and $S \subseteq V(G)$, $[S, \bar{S}]_G$ is the set of edges of G with one end-point in S and one end-point outside of S
$\mathcal{G}$	—	the set of undirected (labeled) graphs $G = (V, E)$ where $V = [n]$ for some $n \in \mathbb{N}$
$\sqcup$	—	The disjoint union of two graphs
$\sqcup_k$	—	The k-sum of two graphs
$\bowtie$	—	The join of two graphs.

Special graphs

K_n	—	A clique on n vertices
P_n	—	A path on n vertices
C_n	—	A cycle on n vertices
E_n	—	A graph consisting of n isolated vertices
S_n	—	A star with n leaves
$K_{n,m}$	—	A complete bipartite graph with parts of sizes n and m
iK_j	—	The disjoint union of i cliques, each of size j

Logic

R, **E**, ...	—	Relation symbols
SOL(τ)	—	Second Order Logic with vocabulary τ
MSOL(τ)	—	Monadic Second Order Logic with vocabulary τ
MSOL_1	—	Monadic Second Order Logic on graphs $\langle V, E\rangle$, where $E \subseteq V$ is a binary edge relation
MSOL_2	—	Monadic Second Order Logic on graphs $\langle \overset{V \cup E}{\cancel{V}, \cancel{E}}, R_{inc}\rangle$, where $R_{inc} \subseteq V \times E \times V$ is a trinary incidence relation
Struct(τ)	—	The class of finite structures with vocabulary τ

Complexity classes

P	—	polynomial-time computable decision problems
NP	—	non-deterministically polynomial-time computable decision problems
FP	—	polynomial-time computable functions $f : \{0,1\}^\star \to \mathbb{N}$
#P	—	functions $f : \{0,1\}^\star \to \mathbb{N}$ which count accepting paths in non-deterministic polynomial-time Turing machines

Chapter 1

Introduction

The relationship between mathematical objects and the way they are represented is very often studied in classical logic. An early example of such a relationship is given by real functions which can be represented as polynomial functions; such functions satisfy various properties, e.g. they are always continuous. Another early example is A. Tarski's theorem which states that classes of algebras which are definable in First Order Logic (FOL) are closed under sub-algebras iff the defining formula is equivalent to a universal formula. In automata theory, Büchi's theorem states that a language is recognizable by a finite-state automaton iff it is definable in Monadic Second Order Logic (MSOL). B. Courcelle undertook a systematic study of graph properties definable in MSOL, which culminated in the monograph [43]. A celebrated theorem due to Courcelle states that verifying graph properties which are definable in MSOL on graphs of fixed tree-width can be done in linear time. The notion of MSOL definability was extended in [44] to graph parameters and graph polynomials, and Courcelle's theorem was generalized to graph polynomials. Graph polynomials definable in Second Order Logic (SOL) and its monadic fragment MSOL were further studied in [116, 118].

One of the main themes of this thesis is the study of definability properties of graph polynomials (Chapters 3, 4 and 5) and combinatorial functions (Chapters 9 and 10). The rest of this introduction explains this in further detail.

1.1 Graph polynomials

Graph parameters f are functions which take finite graphs to some numeric domain such as the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$ or the reals $\mathbb{R}$. Graph properties such as planarity or connectivity can be thought of as special cases where the value is 0 or 1.

Graph polynomials are functions p which take finite graphs to a polynomial ring, usually $\mathbb{Q}[\bar{\mathsf{x}}]$, where $\bar{\mathsf{x}} = (\mathsf{x}_1, \ldots, \mathsf{x}_\ell)$ is a tuple of fixed indeterminates[1]. Every evaluation $p(G; \bar{x}_0)$, $\bar{x}_0 \in \mathbb{Q}^\ell$, of the polynomial $p(G; \bar{\mathsf{x}})$ is a graph parameter. So are the coefficients of $p(G; \bar{\mathsf{x}})$, the total degree, and the zeros of $p(G; \bar{\mathsf{x}})$.

1.1.1 Counting colorings and subset expansions

The *chromatic polynomial* is the first graph polynomial which received attention in the literature. It was studied as early as 1912 by G. D. Birkhoff in [19]. Let $G = (V, E)$ be a graph with $|V| = n$. For any $k \in \mathbb{N}$, a *vertex-k-coloring of G* is a function $f : V \to [k]$, where $[k] = \{1, \ldots, k\}$. The coloring f is a *proper vertex-k-coloring of G* if additionally it satisfies that, whenever $(u, v) \in E$, then $f(u) \neq f(v)$. We denote by $\chi(G; k)$ the number of proper vertex-k-colorings of G. $\chi(G; k)$ is called the *counting function of proper vertex k-colorings*. For a fixed graph G this defines a function $\chi_G : \mathbb{N} \to \mathbb{N}$ which can be proven to be a polynomial in k with integer coefficients. We denote this polynomial by $\chi(G; \mathsf{k})$.

In the literature one finds various graph colorings which satisfy other constraints. For example *convex colorings* are colorings $f : [n] \to [k]$ for which every color $\ell \in [k]$ induces a connected graph. Convex colorings are usually not proper. Convex colorings were studied in [82, 122]. Another example is *proper edge colorings*, which are functions $f : E \to [k]$ constrained such that no two incident edges have the same color. One may also allow the colorings to have several color-sets, and to have coloring predicates which are not necessarily functions. The notion of *generalized graph colorings* covers all of these cases. Usually the constraints are expressible in SOL. In that case, we say that the generalized coloring is SOL-definable.

[1] We use *indeterminates* in polynomials and power series in order to distinguish between them from *variables*, which we use to refer to the variables of first and second order logic.

The first proof that $\chi(G;\mathsf{k})$ is a polynomial used the observation that $\chi(G;\mathsf{k})$ has a recursive definition, cf. [18, 25]. We give a general theorem via a different method which shows that many counting functions of generalized graph colorings are polynomials. The generalized graph colorings in question satisfy a simple property called *the extension property*.

Proposition A. *The counting function of any generalized graph coloring which satisfies the extension property is a polynomial.*

We actually prove this for τ-structures, so it covers colorings of digraphs and hypergraphs as well. We give a list of previously unnoticed graph polynomials which we obtain using Proposition A.

The chromatic polynomial can also be presented in more explicit ways. In [49, Theorem 1.4.1] an explicit description of $\chi(G;\mathsf{k})$ is given. Assume our graphs are equipped with a fixed linear ordering of the vertex set. Let b_m be the number of partitions of V into m independent sets of vertices. Then

$$\chi(G;\mathsf{k}) = \sum_{m=0}^{n} b_m \mathsf{k}_{(m)} \tag{1.1}$$

where $\mathsf{k}_{(m)}$ is the falling factorial $\mathsf{k}_{(m)} = \mathsf{k}\cdot(\mathsf{k}-1)\cdots(\mathsf{k}-(m-1))$. This can be rewritten as the following sum:

$$\chi(G;\mathsf{k}) = \sum_{M,A_M:\varphi_{indpart}(M,A_M)} \mathsf{k}_{(|A_M|)} \tag{1.2}$$

where the summation is over pairs (M, A_M) such that $M \subseteq V^2$ and $A_M \subseteq V$ for which $\varphi_{indpart}(M, A_M)$ is satisfied. $\varphi_{indpart}(M, A_M)$ says that "M is an equivalence relation on V", "each equivalence class induces an independent set" and "A_M consists of the first elements (with respect to the fixed ordering of V) of each equivalence class". $\varphi_{indpart}$ can be expressed in SOL. Such subset expansions definable in SOL are called SOL-*polynomials*.

Another explicit description for $\chi(G;\mathsf{k})$ is given in [49, Theorem 2.2.1]. It can be obtained from a bivariate graph polynomial $Z_{dichrom}(G;\mathsf{x},\mathsf{y})$ called *the dichromatic polynomial*. $Z_{dichrom}(G;\mathsf{x},\mathsf{y})$ is related to the Potts model in statistical mechanics and to the Tutte polynomial $T(G;\mathsf{x},\mathsf{y})$ by prefactors and substitutions, cf. [25]. $Z_{dichrom}(G; \mathsf{x}, \mathsf{y})$ is given by

$$Z_{dichrom}(G; \mathsf{x}, \mathsf{y}) = \sum_{S:S\subseteq E} \left(\prod_{v:\varphi_{fcomp}(v,S)} \mathsf{x} \cdot \prod_{e:e\in S} \mathsf{y} \right) = \sum_{S:S\subseteq E} \mathsf{x}^{k(S)} \cdot \mathsf{y}^{|S|}$$

where $\varphi_{fcomp}(v, S)$ is an SOL formula expressing that "v is the smallest vertex in a connected component of (V, S) with respect to some fixed linear order of the vertices", and $k(S)$ is the number of connected components in (V, S). $Z_{dichrom}$ is also said to be an SOL-polynomial.

It is well known, see e.g. [18], that

$$\chi(G; \mathsf{k}) = Z_{dichrom}(G; \mathsf{k}, -1) \tag{1.3}$$

Hence, $\chi(G; \mathsf{k})$ is a *substitution instance of an* SOL-*polynomial.* Both of the above explicit definitions of $\chi(G; \mathsf{k})$ are invariant to the ordering of V used.

We show that the classes of counting functions of generalized colorings and SOL-polynomials are essentially the same:

Theorem B. *A graph polynomial is a substitution instance of the counting function of a generalized coloring iff it is a substitution instance of an* SOL-*polynomial.*

1.1.2 MSOL-polynomials

The restriction of SOL to its monadic fragment MSOL gives rise to MSOL-polynomials, and to a more restricted class of graph polynomials, MSOL_1-polynomials. The class of MSOL-polynomials covers most of the known graph polynomials from the literature, including the Tutte polynomial, the matching polynomial, the independent set polynomial, the cover polynomial and the interlace polynomial. Yet MSOL-polynomials have decomposition properties which allow to prove strong *meta-theorems* for them:

Theorem 1.1.1 (Courcelle, Makowsky and Rotics, [115, 44]). *Every* MSOL-*polynomial p is computable in polynomial time on graphs of bounded tree-width. In fact, p is fixed parameter tractable with respect to the tree-width of the graph.*

For MSOL_1-polynomials this theorem can be extended to clique-width, a generalization of tree-width:

Theorem 1.1.2 (Courcelle, Makowsky and Rotics, [115, 44]).
Every MSOL_1*-polynomial* p *is computable in polynomial time on graphs of bounded clique-width. In fact,* p *is fixed parameter tractable with respect to the clique-width of the graph.*

Theorem 1.1.3 (Fischer and Makowsky, [60]).
Every MSOL*-polynomial* p *satisfies simple linear recurrence relations on recursive sequences of graphs, such as cycles, paths and ladders.*

We prove another combinatorial meta-theorem. Our theorem uses *connection matrices* to show that many graph invariants are not encoded as evaluations or fixed coefficients of MSOL-polynomials. Connection matrices are infinite matrices associated with a graph invariant f and a binary graph operation $\odot$ (e.g. disjoint union of graphs $\sqcup$). In [67] the connection matrix $M(f, \sqcup)$ of a graph parameter f and $\sqcup$ was introduced. It follows from [67] that for any b and c, if $(b-1)(c-1)$ is an integer, then the rank of the $\sqcup$-connection matrix of the evaluation $T(G; b, c)$ of the Tutte polynomial is finite. Therefore, no graph parameter g with infinite $\sqcup$-connection matrix rank is an integer-valued evaluation of $T(G; \mathsf{x}, \mathsf{y})$. The connection matrices of k-sums are further studied in [67]. They are used to characterize the graph parameters which count weighted homomorphisms.

We show the following:

Theorem C. MSOL*-polynomials have connection matrices of finite rank.*

We show that many graph parameters have connection matrices of infinite rank, implying that they cannot occur as evaluations or fixed coefficients of MSOL-polynomials. These graph parameters include for example the chromatic number, the clique number, the average degree and the girth.

We use a refinement of the same technique to give the first natural examples of graph polynomials which are provably not MSOL-polynomials. Sometimes we can use Theorems 1.1.1 and 1.1.2 to prove the non-definability of graph polynomials, subject to a complexity-theoretic assumption such as $\mathbf{P} \neq \mathbf{NP}$ or $\mathbf{FPT} \neq \mathbf{W}[\mathbf{1}]$. This is the case for the chromatic polynomial. It was shown in [64, 66] that the chromatic polynomial is not fixed paremeter tractable with respect to clique-width, implying using Theorem 1.1.2 that $\chi(G, \mathsf{k})$ is not an MSOL_1-polynomial unless $\mathbf{FPT} = \mathbf{W}[\mathbf{1}]$. The *harmonious polynomial*, which counts harmonious colorings, is not an MSOL-polynomial using Theorem 1.1.1 and [52], in which it is shown that computing the harmonious polynomial on trees is **NP**-hard. Our use of Theorem C eliminates the need to rely on complexity assumptions in these cases.

The method of showing non-definability using connection matrices can be applied to graph properties as well. The notion of cnnection matrices of graph properties can be thought of as a generalization of the notion of *Hankel matrices* from automata theory. J.W. Carlyle and A. Paz [37] and M. Fliess [63] showed that the rank of Hankel matrices is related to the size of a minimal representation of a regular language, see also [17]. The rank of Hankel matrices, the number of distinct rows in it and the maximum permutation matrix they contain serve as complexity measures for picture languages, cf. e.g. [72]. Hankel matrices were also used in communication complexity, see e.g. [93], and in machine learning, see e.g. [15, 98]

The classic tools for proving non-definability in the case of FOL and MSOL are the various *Ehrenfeucht-Fraïssé games*, also called *pebble games*. Two additional tools can be used to make the construction of such winning strategies easier and more transparent: the *composition of winning strategies* and the use of *locality properties* such as Hanf locality and Gaifman locality. However, proving non-definability using these methods can still be a challenging task, especially when allowing an additional order relation to be used. In contrast, proofs of non-definability using connection matrices are surprisingly simple and transparent, and allow the existence of an order relation. In Appendix A we consider an analog of Theorem C to graph properties, and use it to get non-definability results.

Makowsky conjectured in [116] that there is a meta-theorem for the complexity of evaluations $p(G; \bar{x}_0)$, $\bar{x}_0 \in \mathbb{Q}^\ell$, of MSOL-polynomials $p(G; \bar{\mathsf{x}})$. We say that a graph polynomial $p(G; \bar{\mathsf{x}})$ has the *Difficult Point Property (DPP)* if the evaluations $p(G; \bar{x}_0)$ are equivalent in terms of hardness of computation, except possibly for those in a "small and nice" exception set. Several of the prominent graph polynomials in the literature have the Difficult Point Property. This often remains true when the input graphs are restricted to be e.g. planar, bipartite or regular. Makowsky conjectured that MSOL-polynomials have the Difficult Point Property. This conjecture is called the *Difficult Point Conjecture (DPC)*. In many cases one can also prove that the evaluations belonging to the exception set are tractable, in which case we say the graph polynomial has the *Strong Difficult Point Property (SDPP)*, and the corresponding conjecture is called the *Strong Difficult Point Conjecture (SDPC)*. For some instances of the conjecture see [95], [23], [24], and [90]. The conjecture also covers the complexity of counting weighted homomorphisms as studied e.g. in [84].

As a case study of the conjecture, we consider two variations of a graph polynomial, called the *Ising polynomial*, which arises from the study of the Ising model in statistical mechanics. The trivariate Ising polynomial was studied with respect to its approximability by L. A. Goldberg, M. Jerrum and M. Paterson in [79]. It generalizes a bivariate Ising polynomial which was studied in by D. Andrén and K. Markström in [8].

We show:

Theorem D.

(i) Both Ising polynomials have the DPP, even when restricted to simple planar bipartite graphs.

(ii) The bivariate Ising polynomial has the SDPP on simple graphs.

H. Dell, T. Husfeldt and M. Wahlén introduced in [47] a new complexity assumption. It is a counting version of the Exponential Time Hypothesis (**ETH**) of R. Impagliazzo and R. Paturi. The new hypothesis, which we denote #**ETH**, says roughly that counting the number of satisfying assignments of a 3CNF formula requires exponential running time. Dell, Husfeldt and Wahlén use this hypothesis to study the complexity of the Tutte polynomial. In analogy to their results, we show a dichotomy theorem for the bivariate Ising polynomial:

Theorem E. *Unless the* #**ETH** *fails, the evaluations of the bivariate Ising polynomial are either polynomial time computable, or require exponential running time.*

1.2 Combinatorial counting sequences

Sequences that satisfy linear recurrence relations with constant coefficients have attracted considerable attention in the literature. Such sequences are called *C-finite* or *rational*. A classic example of such a recurrence relation is the Fibonacci numbers, given by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

and $F_1 = F_2 = 1$. The definability of languages whose counting sequences are C-finite was studied in 1963 by N. Chomsky and M.P. Schützenberger [40]. They proved that given a regular languages L, the sequence

$$d_L(n) = |\{w \in L : |w| = n\}|$$

which counts words of length n in L is C-finite. Together with the Büchi-Elgot-Trakhtenbrot theorem, which states that a language L is regular iff L is definable in MSOL, we get the following:

Theorem 1.2.1. *For every* MSOL*-definable language, $d_L(n)$ is C-finite.*

For details on the Büchi-Elgot-Trakhtenbrot theorem see for example [111] for a modern presentation.

In 2010 we proved a converse, which implies that the following theorem holds: [2]

Theorem 1.2.2. *For every sequence of integers $a(n)$, $a(n)$ is C-finite iff there exist an* MSOL*-definable language L and a constant $c \in \mathbb{N}$ such that $a(n) = d_L(n) - c^n$.*

The set of C-finite sequences of integers is understood quite well. It is well known that every C-finite sequence $a(n)$ of integers is a sum of exponential polynomials. There is a simple closed-form expression for $a(n)$ depending on the poles of the generating function of $a(n)$, see e.g. [62, Theorem IV.9].

The case is quite different for sequences of integers satisfying more liberal recurrences. We are particularly interested in *P-recursive sequences*, also called *holonomic sequences*. These are sequences $a(n)$ satisfying linear recurrence relations where the coefficients need not be constants; instead, the coefficients are allowed to be rational functions in n. A simple example is the factorial $n!$. The factorial is not C-finite because of a growth argument. It satisfies the P-recurrence

$$n! = n \cdot (n-1)! \tag{1.4}$$

[2] It turned out that this is implicitly already contained in [133].

is every non-negative C-finite of the form $d_L(n)$?

and $1! = 1$. In fact, this sequence is an example of a *Simple-P-recursive sequence* (or *SP-recursive* for short), since the coefficient of $(n-1)!$ in Equation (1.4) is a polynomial in n.

$n!$ has various combinatorial interpretations as the counting function of combinatorial objects e.g., $n!$ is the number of permutations of the set $[n]$. $n!$ can also be interpreted as counting a certain kind of lattice paths. However, $n!$ cannot be interpreted as counting the words of length n of some language, since any language over a finite alphabet Σ can contain at most $|\Sigma|^n$ words of any given length n.

Other P-recursive sequences have interpretations as counting lattice paths or walks, e.g. the Catalan numbers, the Motzkin numbers and the Schröder numbers, see e.g. [139], Gessel walks [28] and Kreweras' walks [107]. Criteria which imply the number of certain lattice walks is P-recursive were given in [29] and [121].

We show a combinatorial interpretation for P-recursive sequences using lattice paths:

Theorem F. *A sequence of integers is P-recursive iff it has an interpretation as counting a certain type of lattice paths.*

Sequences of polynomials may also be C-finite or P-recursive. For example, the Fibonacci polynomials $F_n(\mathsf{x})$ are a generalization of the Fibonacci numbers. They are given by the recurrence

$$F_n(\mathsf{x}) = \mathsf{x} \cdot F_{n-1}(\mathsf{x}) + F_{n-2}(\mathsf{x})$$

and $F_1(\mathsf{x}) = 1$ and $F_2(\mathsf{x}) = \mathsf{x}$, and we have $F_n(1) = F_n$. Another example is the falling factorials, whose evaluations $k_{(n)}$ for $k \in \mathbb{N}$ are the number of injective functions from $[n]$ to $[k]$. They satisfy the recurrence

$$\mathsf{k}_{(n)} = (\mathsf{k} - (n-1)) \cdot \mathsf{k}_{(n-1)}$$

which is a SP-recurrence.

We give combinatorial representation to the sets of C-finite and SP-recursive sequences of polynomials. These interpretations resemble MSOL-polynomials in that they are both weighted sums over definable objects. In this case, the sum is over words of MSOL-definable languages. The weights differ from those of MSOL-polynomials in that they may be *po-*

sitional weights, weights which depend on the positions in which certain letters occur in the word. We have:

Theorem G. *A sequence of polynomials is C-finite respectively SP-recursive iff it has an appropriate definable (positional) weights representation.*

Part I

Graph Polynomials

Chapter 2

Examples

This chapter presents examples from graph theory of graph polynomials which motivate our study of classes of graph polynomials. Section 2.1 considers with the most studied graph polynomials in the literature. Section 2.2 presents the Ising polynomials, whose complexity is studied in detail in Chapter 6. Section 2.3 shows examples of graph polynomials which arise from coloring graphs.

2.1 Classic graphs polynomials

2.1.1 Matching polynomial

Let $M \subseteq E$ be a set of edges. M is a *matching* if it consists of isolated edges. We denote by $cov(M)$ the set of vertices $v \in V$ such that there is an edge $e \in M$ which is incident to v. We note that $|cov(M)| = 2|M|$. Consider the graph parameter $m_i(G)$ which counts the number of matchings of G which consist of i (isolated) edges. The *bivariate matching polynomial* is defined as

$$M(G;\mathsf{x},\mathsf{y}) = \sum_{i=0}^{\lfloor \frac{n_G}{2} \rfloor} m_i(G)\mathsf{x}^i\mathsf{y}^{n_G-2i} = \sum_{M,C:M\subseteq E,C=V-cov(M)} \mathsf{x}^{|M|}\mathsf{y}^{|C|}$$

where the summation on the right-hand side is over all pairs (M, C) where $M \subseteq E$ is a matching and $C = V - cov(M)$. In particular, the coefficient of $\mathsf{x}^{\lceil \frac{n_G}{2} \rceil}$ in $M(G;\mathsf{x},\mathsf{y})$ is the number of perfect matchings in G. The sum of the coefficients of $\mathsf{x}^{\lfloor \frac{n_G}{2} \rfloor}$ and $\mathsf{x}^{\lfloor \frac{n_G}{2} \rfloor} y$ is the number of maximum matchings.

The properties "M is a matching" and "$C = V - cov(M)$" can be expressed by formulas in MSOL. Hence, $M(G;\mathsf{x},\mathsf{y})$ is an MSOL-polynomial.

Here M is a polynomial by definition. Two matching polynomials are obtained from $M(G;\mathsf{x},\mathsf{y})$ as substitution instances: the *generating matching polynomial*

$$g(G;\mathsf{x}) = M(G;\mathsf{x},1)$$

and the *defect matching polynomial* or *acyclic polynomial*

$$\mu(G;\mathsf{x}) = M(G;-1,\mathsf{x})\,.$$

Excellent surveys are [76, Chapter 1] and [113, Chapter 8.5]. The acyclic polynomial has attracted interest in chemistry, see for example [13, 144]. The matching polynomial is revisited in Example 3.1.1, Section 3.2.7 and Section 8.2.4.

2.1.2 Independent set polynomial

The independent set polynomial $I(G;\mathsf{x})$ is an example of a simple MSOL_1-polynomial. It is given by

$$I(G;\mathsf{x}) = \sum_{A \subseteq V : \varphi_{ind}(A)} \mathsf{x}^{|A|}$$

where $\varphi_{ind}(A)$ says that A is an independent set. For a survey see [109]. The independent set polynomial is closely related to the matching polynomial as follows:

$$g(G;\mathsf{x}) = I(L(G);\mathsf{x})$$

where $L(G)$ is the line graph of G. The degree of $I(G;\mathsf{x})$ is the independence number $i(G)$.

Another closely related graph polynomial is the clique polynomial $\Omega(G;\mathsf{x})$ given by

$$\Omega(G;\mathsf{x}) = \sum_{A \subseteq V : \varphi_{clique}(A)} \mathsf{x}^{|A|}$$

where $\varphi_{clique}(G)(A)$ says that A induces a clique in G. For every graph G and its complement $\overline{G}$, $I(\overline{G}; \mathsf{x}) = \Omega(G; \mathsf{x})$. The degree of the clique polynomial is $\omega(G)$, the clique number of G. $\Omega(G; \mathsf{x})$ is an MSOL_1-polynomial.

2.1.3 Characteristic polynomial

The characteristic polynomial $\mathrm{char}(G; \mathsf{x})$ of a graph is defined as the characteristic polynomial (in the sense of linear algebra) of the adjacency matrix A_G of G, cf. [18] or [46]. It is customary to denoted the coefficients of the characteristic polynomial as follows:

$$\mathrm{char}(G; \mathsf{x}) = \sum_{i=0}^{n_G} c_i \mathsf{x}^{n_G - i}.$$

An *elementary subgraph* of a graph G is a subgraph (not necessarily induced) which consists only of isolated edges and cycles. If H is an elementary subgraph of G, we denote by $k(H)$ its number of connected components, and $c(H)$ the number of its cycles. With this notation we have, [18, Proposition 7.4], that the coefficients of $\mathrm{char}(G; \mathsf{x})$ can be expressed as

$$c_i = (-1)^i \cdot \sum_{H : |V(H)| = i} (-1)^{k(H)} \cdot 2^{c(H)}$$

where the summation is over all elementary subgraphs $H = (V(H), E(H))$ of G of size i. Therefore we have

$$\mathrm{char}(G; \mathsf{x}) = \sum_{V_H \subseteq V(G), E_H \subseteq V(G) : \varphi_{elementary}} (-1)^{|V_H| + k(H)} \cdot 2^{c(H)} \cdot \mathsf{x}^{|V(G) - V_H|}$$

where the summation is over elementary subgraphs $H = (V_H, E_H)$. From this we get that $\mathrm{char}(G; \mathsf{x})$ is an evaluation of an MSOL-polynomial.

It is well known that the degree of $\mathrm{char}(G; \mathsf{x})$ is n_G, that $-c_2(G) = |E(G)|$, and that $-c_3(G)$ equals twice the number of triangles in G. Other coefficients have also been studied, e.g. [48]. The second largest zero $\lambda_2(G)$ of $\mathrm{char}(G; X)$ gives a lower bound to the conductivity of G, cf. [77]. It is known that $\mathrm{char}(G; \mathsf{x}) = \mu(G; \mathsf{x})$ iff G is a forest.

The characteristic polynomial will be revisited in Example 4.1.6 and Example 5.2.5.

2.1.4 The Tutte polynomial

The dichromatic polynomial was defined in Section 1.1 as follows:

$$Z_{dichrom}(\mathsf{x},\mathsf{y}) = \sum_{S:S\subseteq E}\left(\prod_{v:fcomp(v,S)} \mathsf{x} \cdot \prod_{e:e\in S} \mathsf{y}\right) = \sum_{S:S\subseteq E} \mathsf{x}^{k(S)} \cdot \mathsf{y}^{|S|}$$

where $fcomp(v, S)$ is the property "v is the smallest vertex in a connected component of $(V(G), S)$ with respect to some fixed linear order of the vertices" and $k(S)$ is the number of connected components of $(V(G), S)$. This is an MSOL-polynomial. The dichromatic polynomial is a generalization of the chromatic polynomial, see e.g. [136]:

$$\chi(G;\mathsf{k}) = Z_{dichrom}(\mathsf{k};-1) \tag{2.1}$$

However, unlike $\chi(G;\mathsf{k})$, $\chi_{dichrom}(G;\mathsf{x},\mathsf{y})$ is a polynomial by definition.

The dichromatic polynomial is closely related to the Tutte polynomial given by

$$T(G;\mathsf{x},\mathsf{y}) = \sum_{F\subseteq E(G)} (\mathsf{x}-1)^{r(E)-r(F)}(\mathsf{y}-1)^{n(F)} \tag{2.2}$$

where $k(F)$ is the number of connected components of the spanning subgraph $(V(G), F)$, $r(F) = |V| - k(F)$ is its rank and $n(F) = |F| - |V| + k(F)$ is its nullity. The Tutte polynomial and the dichromatic polynomial are co-reducible to each other via

$$T(G;\mathsf{x},\mathsf{y}) = \frac{Z_{dichrom}(G;(\mathsf{x}-1)(\mathsf{y}-1),\mathsf{y}-1)}{(\mathsf{x}-1)^{k(E)}(\mathsf{y}-1)^{n_G}}\,.$$

For a modern introduction to the Tutte polynomial see [25, Chapter X], [77] or [150].

The dichromatic polynomial and the Tutte polynomial are closely related to the partition function of the Potts model from statistical mechanics. The Tutte polynomial can also be considered a generalization of the Jones polynomial from knot theory.

The Tutte polynomial $T(G; \mathsf{x}, \mathsf{y})$ has remarkable evaluations which count certain configurations of the graph G, cf. [150], e.g:

(i) $T(G; 1, 1)$ is the number of spanning trees of G,

(ii) $T(G; 1, 2)$ is the number of connected spanning subgraphs of G,

(iii) $T(G; 2, 1)$ is the number of spanning forest of G,

(iv) $T(G; 1 - x, 0)$ is related to the evaluation $\chi(G; x)$ of the chromatic polynomial,

(v) $T(G; 2, 0)$ is related to the number of acyclic orientations of G,

(vi) $T(G; 0, -2)$ is the number of Eulerian orientations of G.

More sophisticated evaluations of the Tutte polynomial can be found in [80, 81].

One can prove that $T(G; \mathsf{x}, \mathsf{y})$ is an MSOL-polynomial, using the definition of $T(G; \mathsf{x}, \mathsf{y})$ via its spanning tree expansion, cf. [25, Chapter 10] and [44].

It is well-known that for every $\gamma, \delta \in \mathbb{Q}$, except for points (γ, δ) in a finite union of algebraic exception sets of dimension at most 1, computing the graph parameter $T(G; \gamma, \delta)$ is $\#\mathbf{P}$-hard on multi-graphs, see [95]. This holds even when restricted to bipartite planar graphs, see [148] and [147].

H. Dell, T. Husfeldt and M. Wahlén introduced in [47] a counting version of the Exponential Time Hypothesis ($\#\mathbf{ETH}$), which roughly states that counting the number of satisfying assignments to a 3CNF formula requires exponential time. This hypothesis is implied by the Exponential Time Hypothesis (**ETH**) for decision problems introduced by R. Impagliazzo and R. Paturi in [94]. Under $\#\mathbf{ETH}$, the authors of [47] show that the computation of the Tutte polynomial on simple graphs requires exponential time in $\frac{m_G}{\log^3 m_G}$ in general, where m_G in the number of edges of the graph.

We revisit the dichromatic polynomial in Section 3.2.8.

It is interesting to note a generalization of the dichromatic polynomial called the *edge elimination polynomial*, which was studied by I. Averbouch in his Ph.D. thesis [10] for its behavior regarding recurrence relations. The edge elimination polynomial generalizes simultaneously the dichromatic polynomial, the matching polynomial, the independent set polynomial and the bivariate chromatic polynomial, see [11, 12]. It was also studied in [146, 100].

2.2 The Ising polynomials

An Ising system is a simple graph $G = (V, E)$ together with vertex and edge weights. Every edge $(u, v) \in E$ has an *interaction energy* and every vertex $u \in V$ has an *external magnetic field strength* associated with them. A function $\sigma : V \to \{\pm 1\}$ is a *configuration* of the system or a *spin assignment*. The partition function of an Ising system is a generating function related to the probability that the system is in a certain configuration.

L. A. Goldberg, M. Jerrum and M. Paterson [79] studied the partition function of the Ising model in the case where both the interaction energies of an edge (u, v) and the external magnetic field strength of a vertex v are constant. This gives rise to a trivariate graph polynomial $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$. To define $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ we need a few definitions.

Given $S \subseteq V(G)$, we denote by $E_G(S)$ the set of edges in the graph induced by S in G and by $E_G(\bar{S})$ the set of edges in the graph obtained from G by deleting the vertices of S and their incident edges. We may omit the subscript and write e.g. $E(S)$ when the graph G is clear from the context. The *cut* $[S, \bar{S}]_G$ is the set of edges with one end-point in S and the other in $\bar{S} = V(G) \backslash S$.

The trivariate Ising polynomial is given by

$$Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z}) = \sum_{S \subseteq V(G)} \mathsf{x}^{|E_G(S)|} \mathsf{y}^{|S|} \mathsf{z}^{|E_G(\bar{S})|} .$$

For every G, $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ is a polynomial in $\mathbb{Z}[\mathsf{x}, \mathsf{y}, \mathsf{z}]$ with positive coefficients.

The bivariate Ising polynomial obtained from $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ by setting $\mathsf{x} = \mathsf{z} = \mathsf{t}$ was studied in [8]:

$$Z_{Ising}(G; \mathsf{t}, \mathsf{y}) = \sum_{S \subseteq V(G)} \mathsf{t}^{|E_G(S)| + |E_G(\bar{S})|} \mathsf{y}^{|S|} .$$

$Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ can be rewritten as follows, using that $E_G(S)$, $E_G(\bar{S})$ and $[S, \bar{S}]_G$ form a partition of $E(G)$:

$$Z_{Ising}(G; \mathsf{t}, \mathsf{y}) = t^{m_G} \sum_{S \subseteq V(G)} \mathsf{t}^{-|[S, \bar{S}]_G|} \mathsf{y}^{|S|} . \tag{2.3}$$

This presentation of $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ is closer to the way it was defined in [8].

The Ising polynomials encode information about the graph, including the size of the maximal cut, the number of components of the graph, the girth, the number of perfect matching and the matching polynomial.

For regular graphs they are equivalent and contain the independent set polynomial and the clique polynomial. A different bivariate Ising polynomial given by $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, 1)$ was studied by L. Borzacchini in [27]. This polynomial is again equivalent to the trivariate Ising polynomial $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ on regular graphs.

Both Ising polynomials $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ and $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ are MSOL-polynomials. However, the degrees of $Z_{Ising}(K_n; \mathsf{x}, 1, \mathsf{x})$ and $Z_{Ising}(K_n; \mathsf{t}, 1)$ are bounded from below by $\Omega(n_G^2)$, whereas MSOL_1-polynomials have degrees at most $O(n_G)$ by definition.

In Chapter 6 we will be interested in the complexity of the Ising polynomials, as a case study of Makowsky's Difficult Point Conjecture. We will also discuss the computation of $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ on graphs of bounded tree-width and clique-width in Appendix B.

L. A. Goldberg, M. Jerrum and M. Paterson [79] studied the evaluations of $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$. They considered the existence of fully polynomial randomized approximation schemes (FPRAS) for the graph parameters $Z_{Ising}(G; \gamma, \delta, \varepsilon)$, depending on the values of $(\gamma, \delta, \varepsilon) \in \mathbb{Q}^3$. They provided approximation schemes for some regions of $\mathbb{Q}^3$ while showing that other regions do not admit such approximation schemes. Approximation schemes for $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ were further studied in [154, 134]. M. Jerrum and A. Sinclair studied in [96] the approximability and $\#\mathbf{P}$-hardness of another case of the Ising model, where weights are provided as part of the input and no external field is present.

We show that the bivariate Ising polynomial $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ satisfies the SDPP and the trivariate Ising polynomial $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ satisfies the DPP. We show that outside of their respective exception sets, the evaluations of $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ and $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ are $\#\mathbf{P}$-hard. Since every evaluation of $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ is in $\mathbf{FP}^{\#\mathbf{P}}$, the evaluations are Turing equivalent. We also consider the complexity of $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ under $\#\mathbf{ETH}$ and prove an analogous theorem to that of Dell, Husfeldt and Wahlén for the Tutte polynomial.

The Potts model

The q-state Potts model deals with a similar scenario to the Ising model, except that the spins are not restricted to ± 1 but instead receive one of q possible values. The partition function of the Potts model in the case where no magnetic field is present is closely related to the dichromatic polynomial and the Tutte polynomial. The restriction of the Tutte polynomial to the so-called *Ising hyperbola* corresponds to the case of Ising model with no external field. The Tutte polynomial on the Ising hyperbola is tractable on planar graphs, see [61, 97, 95].

2.3 New graph polynomials: counting colorings

The literature contains many papers on various kinds colorings, and their authors are usually interested either in questions of extremal graph theory or in the complexity of deciding the existence of these colorings. Counting the number of generalized colorings is rarely studied. However, it turns out that counting the number of colorings with k colors very often gives rise to previously unnoticed graph polynomials. We list here a few examples, which we think deserve further investigations. All of the polynomials we show are SOL-polynomials.

2.3.1 Harmonious colorings

A proper vertex coloring $f : V(G) \to [k]$ is *harmonious*, if each pair of colors appears at most once along an edge. We denote by $\chi_{harm}(G)$ the least k such that G has a harmonious proper k-coloring and by $\chi_{harm}(G; k)$ the number of harmonious colorings of G using a color-set of size k. In contrast, a proper vertex coloring is *complete*, if each pair of colors appears *at least* once along an edge. We denote by $\chi_{comp}(G)$ the largest k such that G has a complete proper k-coloring. Let $\chi_{comp}(G; k)$ denote the number of complete k-colorings of G.

Proposition 2.3.1.

(i) $\chi_{harm}(G; k)$ *is a polynomial in* k.

(ii) $\chi_{comp}(G; k)$ *is not a polynomial in* k.

Proof.

(i) follows from Proposition A.

(ii) $\chi_{comp}(G; k) = 0$ for all large enough k but can be non-zero for small values of k.

$\square$

It is shown in [52], that computing $\chi_{harm}(G)$ is **NP**-complete already for trees. This, together with Theorem 1.1.1, shows that $\chi_{harm}(G; X)$ is not MSOL-polynomial, unless $\mathbf{P} = \mathbf{NP}$. We show in Section 5.3 that $\chi_{harm}(G; \mathsf{k})$ is not an MSOL-polynomial without the use of any complexity-theoretic assumption.

The chromatic polynomial is reducible to $\chi_{harm}(G; k)$:

Proposition 2.3.2. *Let G be connected. Let $S(G)$ be the stretch of G, obtained from G by replacing every edge $w = (u_1, u_2)$ with a path consisting of one new vertex v_e and two edges (u_1, v_e) and (u_2, v_e). Let $S'(G)$ be the graph obtained from $S(G)$ by forming a clique on the vertices v_e. Then*

$$\chi_{harm}(S'(G); k + |E|) = \chi(G; k) \cdot \binom{k + |E|}{|E|} |E|!$$

Proof. The clique on the v_e require that for any $e_1 \neq e_2$, v_{e_1} and v_{e_2} are colored differently, since the coloring must be proper. Let $w \in V(G)$ and $e \in E(G)$. The coloring must give w and v_e different colors using the requirement that no two edges are colored with the same pair of colors, since w is adjacent to some $v_{e'}$ and $v_{e'}$ is adjacent to v_e.

For every two vertices $w_1, w_2 \in V(G)$, if there is an edge e between them in G, then they are of distance 2 in $S'(G)$. Therefore, they must be of different colors, otherwise the edges (w_1, v_e) and (w_2, v_e) are colored the same. $\square$

Using Proposition 2.3.2 and the multiplicativity of the chromatic polynomial to disjoint union, one can compute the chromatic polynomial of any graph. Hence, computing the harmonious polynomial is #**P**-hard.

Similarly to the chromatic polynomial,

$$\chi_{harm}(G; \mathsf{k}) = \chi_{harm}(G \bowtie K_1; \mathsf{k} + 1) \cdot \frac{1}{\mathsf{k} + 1}\,.$$

Every evaluation, except possibly those in $\mathbb{N}$, is #**P**-hard. For every $k \in \mathbb{N}$, notice that if the G has more than $\binom{k}{2}$ many edges, then $\chi_{harm}(G;k) = 0$. There are finitely many graphs (up to isomorphism) with at most $\binom{k}{2}$ many edges up to the addition of isolated vertices. Hence, for every $k \in \mathbb{N}$, the graph parameter $\chi_{harm}(G;k)$ can be computed in polynomial-time.

2.3.2 $mcc(t)$-colorings

We denote by $\chi_{mcc(t)}(G;k)$ the number of vertex colorings with at most k colors for which no color induces a graph with a connected component of size larger than t (which are not necessarily proper). By Proposition A, $\chi_{mcc(t)}(G;k)$ is a polynomial in k. $\chi_{mcc(t)}(G;\mathsf{k})$ is a generalization of the chromatic polynomial since $\chi_{mcc(1)}(G;\mathsf{k}) = \chi(G;\mathsf{k})$.

Let $\chi_{mcc}(G;k,t) = \chi_{mcc(t)}(G;k)$ be the counting function of multicolorings satisfying the above condition, where t is considered a color-set. Proposition A cannot be applied to $\chi_{mcc}(G;k,t)$. Indeed, we will see in Example 3.2.12 that $\chi_{mcc}(G;k,t)$ is not a polynomial in t. *mcc*-colorings were studied in [3] and [112]. It follows from [56] that $mcc_t(G,2)$ is **NP**-hard.

We will see in Section 5.3 that $\chi_{mcc(t)}(G;\mathsf{k})$ is not an MSOL_1-polynomial. Whether $\chi_{mcc(t)}(G;\mathsf{k})$ is an MSOL-polynomial is still open.

2.3.3 More new graph polynomials

Proposition A allows us to easily show that the counting functions of many colorings from the literature are in fact polynomials in the number of colors. In this subsection we give a list of such colorings.

Path-rainbow colorings A function $c : E(G) \to [k]$ is a *path-rainbow* connected coloring if between any two vertices $u, v \in V(G)$ there exists a path where all the edges have different colors. These colorings were introduced in [39]. Finding the minimal k such that $\chi_{rainbow}(G;k) > 0$ is **NP**-hard, cf. [38]. We denote by $\chi_{rainbow}(G,k)$ the number of path-rainbow colorings of G with at most k colors.

By Proposition A, $\chi_{rainbow}(G;k)$ is a polynomial in k, and in Section 5.3 we show that $\chi_{rainbow}(G;\mathsf{k})$ is no an MSOL-polynomial.

Acyclic colorings A function $f : V(G) \to [k]$ is an *acyclic vertex coloring* if it is proper and there is no two colored cycle in G. Acyclic vertex colorings were introduced in [85] and A. V. Kostochka proved in 1978 in his thesis that it is **NP**-hard to decide for a given G and k if the there exists an acyclic vertex coloring with at most k colors, see [5].

Acyclic edge colorings, defined analogously to acyclic vertex colorings, were also studied, cf. e.g. [6, 135, 14].

Non-repetitive colorings A function $f : E(G) \to [k]$ is a *non-repetitive coloring* if the sequence of colors on any path in G is non-repetitive. A sequence $a_1, \dots, a_r$ is non-repetitive if there is no $i, j \geq 1$ such that $(a_i, \dots, a_{i+j-1}) = (a_{i+j}, \dots, a_{i+2j-1})$. Non-repetitive colorings of graphs were introduced in [4]. Their complexity was studied in [119]. The minimal number of colors needed to color G in a non-repetitive way is called the *Thue number of* G.

t-improper colorings Let $t \in \mathbb{N}$. A function $f : V(G) \to [k]$ is a *t-improper coloring* if every color induces a graph in which no vertex has valency[1] more than t. It is **NP**-hard to determine whether G is t-improperly 2-colorable for any fixed positive t (even if G is planar), cf. [45].

Convex colorings A coloring $f : V(G) \to [k]$ is a *convex coloring* if for every $c \in [k]$, $f^{-1}(c)$ induces a connected subgraph of G. The complexity of coloring graphs by convex colorings has been studied by several authors, e.g. [122]. The complexity of counting convex 2-colorings was studied by A. J. Goodall and S. D. Noble [82].

Co-colorings A function $f : V(G) \to [k]$ is a co-coloring if every color induces a graph which is either a clique or an independent set. It was introduced in [108].

Sub-colorings A coloring $f : V(G) \to [k]$ is a *sub-coloring* if every color induces a graph which is a disjoint union of cliques. These colorings were introduced in [123] and studied in [2].

Partitions into co-graphs A function $f : V(G) \to [k]$ is a *co-graph coloring* if each of the colors induces a co-graph. The partitions of graphs

[1]We use the term 'valency of v' rather than 'degree of v' to refer to the number of neighbors of v. We use the term 'degree' to refer to the degree of a polynomial.

into co-graphs was studied in [73]. The family of co-graphs is the smallest class of graphs that includes K_1 and is closed under complementation and disjoint union. They can be characterized as the P_4-free graphs.

G-free colorings A function $f : V(G) \to [k]$ is a *G-free coloring* if no color induces a graph which contains G as an induced subgraph. G-free colorings were studied e.g. in [31, 1].

$\mathcal{P}$-colorings A function $f : V(G) \to [k]$ is a *$\mathcal{P}$-k-coloring* if for all $i \in [k]$ which is in the range of f, the set $f^{-1}(i)$ induces a graph in $\mathcal{P}$. Here $\mathcal{P}$ is any graph property. F. Harary introduced the notion of *P-colorings*, cf. [87, 88, 32, 33]. Several of the colorings presented so far are $\mathcal{P}$-colorings for appropriate choices of $\mathcal{P}$.

We denote by $\chi_{\mathcal{P}}(G; k)$ the number of $\mathcal{P}$-k-colorings of G.

Chapter 3

Graph Polynomials that Count Colorings

In this chapter we define exactly the notion of generalized colorings and prove Proposition A, which gives a sufficient condition for showing that the counting functions of generalized colorings are polynomials. We then define SOL-polynomials and prove Theorem B, which says that the class of counting functions of generalized colorings is essentially the same as the class of SOL-polynomials.

3.1 Graphs, structures and the logic SOL

3.1.1 Graphs and structures

Let τ be a vocabulary and let $\mathfrak{M}$ be a finite τ-structure with universe M. We assume without loss of generality that the universes of our structures are $M = [n]$ for $n > 1$. We will further assume that all our structures are ordered, i.e. there exists a binary relation symbol $\mathbf{R}_{\leq}$ in τ which is always interpreted as the natural linear ordering of the universe. To simplify notation, we omit the order relation from structures.

The foremost example of such τ-structures is graphs. We think of (ordered) graphs as finite τ_1-structures, where $\tau_1 = \langle \mathbf{E}, \mathbf{R}_{\leq} \rangle$, $\mathbf{E}$ and $\mathbf{R}_{\leq}$ are binary relation symbols. The universe of such a τ_1-structure is the set of vertices and the interpretation of $\mathbf{E}$ is the edge relation. Further restrictions on the τ_1-structures can be imposed, such as symmetry and the lack selfloops for undirected simple graphs Hypergraphs are also τ-structures for the appropriate τ.

In Chapter 4 we will see another representation of graphs with vocabulary τ_2 that will be useful with the sub-logic MSOL of SOL. We defer the exact definition of τ_2 there, since $\mathrm{SOL}(\tau_1) = \mathrm{SOL}(\tau_2)$.

3.1.2 Second Order Logic SOL

We can now define Second Order Logic SOL for any vocabulary τ.

We define the logic $\mathrm{SOL}(\tau)$ for τ-structures inductively. Let $\mathfrak{M}$ be a τ-structure. We denote first order variables by $x_i : i \in \mathbb{N}$. They range over elements of universe. We denote second order variables by $U_{i,s} : i, s \in \mathbb{N}^+$. They range over s-ary relations over the universe.

Later we will be more liberal with the notation for first order and second order variables. Unless stated explicitly otherwise, we will keep to the rule that first order variables use lower-case letters, whereas second order variables use upper-case letters.

Terms t are either first order variables or constant symbols $a \in \tau$ of the vocabulary, or are obtained by applying the function symbols from τ on other terms inductively.

Definition 3.1.1 (SOL(τ)-formula).
Atomic SOL-formulas *are of one of the following forms:*

(i) $t_1 \approx t_2$

(ii) $R(t_1, \dots, t_s)$ *for every relation symbol of arity s in τ*

(iii) $U_{i,s}(t_1, \dots, t_s)$

where all the t_i are terms. The atomic formula have the natural interpretation.
Formulas are built inductively using the connectives $\vee, \wedge, \rightarrow, \leftrightarrow, \neg$, and the quantifiers $\forall x_i$, $\exists x_i$, $\forall U_{i,s}$, $\exists U_{i,s}$ with their natural interpretation.

Given a τ_1-structure G, the statement "G is a simple undirected graph" is SOL-definable by

$$\Phi_{s.u.} = \forall x \forall y \left(\mathbf{E}(x,y) \rightarrow (\mathbf{E}(y,x) \wedge \neg(x \approx y))\right) .$$

The following can be written as SOL-formulas for graphs $G = (V, E)$:

Example 3.1.1 (SOL on simple undirected graphs).

(i) The formula $\Phi_{match}(U)$ which says that $U \subseteq E$ is a matching:

$$\begin{aligned} \forall x \, \forall y \, \forall z \, \forall w \quad [\quad & (U(x,y) \wedge U(z,w)) \rightarrow \\ & (\mathbf{E}(x,y) \wedge \mathbf{E}(z,w) \wedge \\ & ((\neg(x \approx w) \wedge \neg(x \approx z) \wedge \neg(y \approx w) \wedge \neg(y \approx z)) \vee \\ & (x \approx z \wedge y \approx w) \vee (x \approx w \wedge y \approx z))) \;] \end{aligned}$$

(ii) The formula $\Phi_{pm}(U)$ which says that $U \subseteq E$ is a perfect matching:

$$\Phi pm(U) = \Phi_{match}(U) \wedge \forall x \, \exists y \, U(x,y)$$

(iii) The formula $\Phi_{fcomp}(x, U)$ which says that x is the first (minimal) element of some connected component of the spanning subgraph (V, U) with edge set $U \subseteq E$, or equivalently, that x has no neighbor y in (V, U) which such that y is smaller than x in the linear order of the vertices:

$$\Phi_{fcomp}(x, U) = \forall y \; ((\neg(y \approx x) \wedge \mathbf{R}_{\leq}(y, x)) \rightarrow \neg U(y, x))$$

(iv) The formula $\Phi_{ham}(U)$ which says that U is a Hamiltonian cycle, i.e. that (V, U) is a 2-regular connected graph.

$$\begin{aligned} \Phi_{2-reg}(U) = \forall x \, \exists y \, \exists z \quad [\quad & \neg(x \approx y) \wedge \neg(y \approx z) \wedge \neg(x \approx z) \wedge \\ & U(x,y) \wedge U(x,z) \wedge \\ & \forall w \, (U(x,w) \rightarrow (w \approx y \vee w \approx z)) \;] \end{aligned}$$

$$\begin{aligned} \Phi_{connected}(U) = \neg \exists B \quad [\quad & \forall x \forall y \, ((B(x) \wedge \neg B(y)) \rightarrow \neg U(x,y)) \wedge \\ & \wedge (\exists x B(x)) \wedge (\exists y \neg B(y)) \;] \end{aligned}$$

and $\Phi_{ham}(U) = \Phi_{2-reg}(U) \wedge \Phi_{connected}(U)$.

Example 3.1.2 (Modular sizes of sets).
For every a, b, we want to show that there is an SOL*-formula $\varphi_{a,b}(U)$ which*

says that the cardinality of the second order variable $U \subseteq V$ equals a modulo b. One way to do so would be to show that the universe of the structure can be partititioned into $a+b$ parts, a of them of size 1 and the rest of equal size. Saying that sets are of equal size can be done by existentially quatifying a bijection between the sets. However, we use another approach, so that the formula is actually in the fragment MSOL *of* SOL *which is discussed in Chapter 4.*

To show that $\varphi_{a,b}(U)$ is SOL*-definable we will use the order relation. The order relation induces an order on the elements of U. We will illustrate our approach using the case of $a = 1$ and $b = 2$. U is odd iff there is a subset A of U such that A contains the elements of U which are odd (with respect to the induced order on U) and the minimal and maximal elements of U belong to A. So, $\varphi_{Odd}(U) = \varphi_{1,2}(U)$ is given as*

$$\exists A \Big[\quad \forall x((U(x) \wedge (\forall y(U(y) \rightarrow \mathbf{R}_{\leq}(x,y))) \rightarrow A(x)) \wedge$$
$$\forall x((U(x) \wedge (\forall y(U(y) \rightarrow \mathbf{R}_{\leq}(y,x))) \rightarrow A(x)) \wedge (A \subseteq U) \wedge$$
$$\forall x \forall y \big([\exists z(\varphi_{succ}(x,z) \wedge \varphi_{succ}(z,y))] \rightarrow [A(x) \rightarrow (\neg A(z) \wedge A(y))] \big) \Big]$$

where $\varphi_{succ}(x_1, x_2)$ says that x_2 is the successor of x_1 in the order induced on U.

Similarly, one can show an SOL*-formula Φ_{Euler} which says that the graph is Eulerian, i.e. that every vertex has even valency and the graph is connected.*

3.2 Counting generalized colorings

3.2.1 φ-colorings

Let τ be a vocabulary and let $\mathfrak{M}$ be a τ-structure. Let k be a natural number. We denote by $\mathfrak{M}_{F,k}$ the two-sorted structure

$$\mathfrak{M}_{F,k} = \langle \mathfrak{M}, [k], F \rangle$$

where $F : M \rightarrow [k]$ is a function. We think of $\mathfrak{M}_{F,k}$ as the colored structure induced by the function F on $\mathfrak{M}$. The set $[k]$ will be referred to as the color set. Note that the order relation does not extend to the second sort $[k]$.

Let $\mathbf{F}$ be a unary function symbol and let $\tau_{\mathbf{F}} = \tau \cup \{\mathbf{F}\}$. $\mathfrak{M}_{F,k}$ is a two-sorted $\tau_{\mathbf{F}}$-structure.

Let $\mathcal{L}$ be a fragment of SOL.

Definition 3.2.1 (φ-colorings). *Let φ be an $\mathcal{L}(\tau_{\mathbf{F}})$-sentence. Let $\mathfrak{M}$ be a τ-structure with universe M. $\mathfrak{M}_{F,k}$ is called a* φ-colored τ-structure *and $F : M \to [k]$ is called a* φ-coloring of $\mathfrak{M}$ *if $\mathfrak{M}_{F,k}$ is a two-sorted $\tau_{\mathbf{F}}$-structure such that*

$$\mathfrak{M}_{F,k} \models \varphi\,.$$

Let $\mathcal{P}_{\varphi(\mathbf{F})}$ be the class

$$\mathcal{P}_{\varphi(\mathbf{F})} = \{\mathfrak{M}_{F,k} \mid \mathfrak{M}_{F,k} \models \varphi\}$$

of φ-colored τ-structures.

We usually write $\mathcal{P}_{\varphi}$ instead of $\mathcal{P}_{\varphi(\mathbf{F})}$.

Definition 3.2.2 (Coloring properties). *Let φ be an $\mathcal{L}(\tau_{\mathbf{F}})$-sentence. We say $\mathcal{P}_{\varphi(\mathbf{F})}$ is a* coloring property *if it satisfies the following properties:*

Permutation property *Let $\pi : [k] \to [k]$ be a permutation and let F_π be the function obtained from F by applying π, i.e. $F_\pi(v) = \pi(F(v))$. Then*

$$\mathfrak{M}_{F,k} \in \mathcal{P}_{\varphi(\mathbf{F})} \text{ iff } \mathfrak{M}_{F_\pi,k} \in \mathcal{P}_{\varphi(\mathbf{F})}.$$

In other words, $\mathcal{P}_{\varphi(\mathbf{F})}$ is closed under permutation of the color-set $[k]$.

Remark 3.2.1. It follows from the Permutation Property, that we can assume that $Range(F)$ is of the form $[k_0]$ for some $k_0 \leq k$.

Extension property *For every $\mathfrak{M}$, F with $Range(F) = [k_0]$, $k' \geq k_0$*

$$\mathfrak{M}_{F,k_0} \in \mathcal{P}_{\varphi(\mathbf{F})}$$

iff

$$\mathfrak{M}_{F,k'} \in \mathcal{P}_{\varphi(\mathbf{F})}$$

Namely, the extension property requires that increasing or decreasing the number of colors not in $Range(F)$ does not affect whether it belongs to the property.

Example 3.2.2 (Proper coloring and variations). *Let $G = (V, E, \leq)$ be a τ_1-structure:*

(i) A function $F : V \to [k]$ is a proper coloring*, if it satisfies the following* SOL*-sentence:*

$$\varphi_{proper} = \forall u \, \forall v \, (\mathbf{E}(u, v) \to \neg(\mathbf{F}(u) \approx \mathbf{F}(v)))$$

The class $\mathcal{P}_{\varphi_{proper}}$ is the class of tuples $\langle G, [k], F \rangle$ of graphs together with a color-set $[k]$ and a proper coloring F. As we know, the counting function of proper colorings is the chromatic polynomial.

(ii) Recall a proper coloring $F : V \to [k]$ is complete if every pair of colors appears on at least one edge. It is SOL*-definable by the following formula $\varphi_{complete}$:*

$$\varphi_{proper} \wedge \forall y_1 \, \forall y_2 \, \exists u \, \exists v \, (\mathbf{E}(u, v) \wedge \mathbf{F}(u) \approx y_1 \wedge \mathbf{F}(v) \approx y_2).$$

where y_1, y_2 range over the sort $[k]$ and u, v range over the set of vertices V.

$\mathcal{P}_{\varphi_{complete}}$ is not a coloring property, since it does not satisfy the extension property. The counting function of complete colorings is not a polynomial in k.

Definition 3.2.3 (Counting functions of φ-colorings).
Let φ be an $\mathcal{L}(\tau_{\mathbf{F}})$-sentence such that $\mathcal{P}_\varphi$ is a coloring property. Let $\chi_{\varphi(\mathbf{F})}$ be a function from pairs $\langle \mathfrak{M}, k \rangle$, which consists of a τ-structure $\mathfrak{M}$ and $k \in \mathbb{N}^+$, defined as follows:

$$\chi_{\varphi(\mathbf{F})}(\mathfrak{M}; k) = \mid \{F : \mathfrak{M}_{F,k} \in P_\varphi\} \mid$$

I.e., $\chi_{\varphi(\mathbf{F})}(\mathfrak{M}; k)$ counts the number of φ-colorings of $\mathfrak{M}$ with k colors.
We usually write χ_φ instead of $\chi_{\varphi(\mathbf{F})}$.

Denote by $c_\varphi(\mathfrak{M}, j)$ the number of φ-colorings of $\mathfrak{M}$ with color-set $[j]$ which use all the colors of $[j]$.

Proposition 3.2.3 (Special case of Proposition A).
Let φ be an $\mathcal{L}(\tau_{\mathbf{F}})$-sentence such that $\mathcal{P}_\varphi$ is a coloring property. For every $\mathfrak{M}$ the number $\chi_{\varphi(\mathbf{F})}(\mathfrak{M}; k)$ is a polynomial in k of the form

$$\sum_{j=1}^{|M|} c_\varphi(\mathfrak{M}, j)\binom{k}{j}$$

Proof. We first observe that any φ-coloring F uses at most $|M|$ of the k colors. By the permutation property, if F is a φ-coloring which uses j colors then any function obtained by permuting the colors is also a φ-coloring. By the extension property, given k colors, the number of φ-colorings that use exactly j of the k colors is the product of $c_\varphi(\mathfrak{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_\varphi(\mathfrak{M}; k) = \sum_{j=0}^{|M|} c_\varphi(\mathfrak{M}, j)\binom{k}{j}$$

The right side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k < j$ we have $\binom{k}{j} = 0$. $\square$

Remarks 3.2.4.

(i) Polynomials of the form

$$\sum_j a_j \binom{k}{j}$$

with indeterminate k where $a_j \in \mathbb{N}$ are called *Newton polynomials.*[1]

The polynomial guaranteed in Proposition 3.2.3 is a Newton polynomial.

(ii) The restriction to coloring properties in Proposition 3.2.3 is essential. Denote by $\chi_{onto}(G, k)$ the number of functions $f : V \to [k]$ which are onto. Clearly, this is not a polynomial in k since, for $k >| V |$, it always vanishes, so it should vanish identically, if it were a polynomial.

(iii) The proof of Proposition 3.2.3 does not guarantee that the coefficients of the power of k are integers. However, the proof of Proposition 3.2.5 below does guarantee it.

[1] The term 'Newton polynomials' is used in the context numerical analysis to refer to interpolation polynomials whose form resembles the form we use here.

In fact, it holds that $\chi_\varphi(\mathfrak{M};k)$ is in $\mathbb{Z}[k]$. For two functions $f_1, f_2 : [n] \to [k]$ we write $\sim_{perm}$ if there exists a permutation $\pi : [k] \to [k]$ such that for all $i \in [n]$ we have $\pi(f_1(i)) = f_2(i)$. In other words, f_1 and f_2 are equivalent if they can be obtained from one another by applying some permutation of the color set $[k]$. Let $d_\varphi(\mathfrak{M}, j)$ be the number of $\sim_{perm}$-equivalence classes of φ-colorings with color-set $[j]$ which use all of the colors in $[j]$.

In other words, the difference between $d_\varphi(\mathfrak{M}, j)$ and $c_\varphi(\mathfrak{M}, j)$ is that $c_\varphi(\mathfrak{M}, j)$ distinguishes between φ-colorings which are equivalent up to a permutation of the color-set. If P_φ satisfies the permutation property we have

$$c_\varphi(\mathfrak{M}, j) = j! \cdot d_\varphi(\mathfrak{M}, j)\,.$$

Proposition 3.2.5 (Special case of Proposition A). *Let φ be an $\mathcal{L}(\tau_{\mathbf{F}})$-sentence such that $\mathcal{P}_\varphi$ is a coloring property. For every $\mathfrak{M}$ the number $\chi_{\varphi(\mathbf{F})}(\mathfrak{M}, k)$ is a polynomial in k, namely*

$$\sum_{j=1}^{|M|} d_\varphi(\mathfrak{M}, j) \cdot k_{(j)}$$

where $k_{(j)}$ is the falling factorial.

Proof. By proposition 3.2.3,

$$\chi_\varphi(\mathfrak{M};k) = \sum_{j=1}^{|M|} c_\varphi(\mathfrak{M}, j)\binom{k}{j}\,.$$

By the Permutation Property, F is a φ-coloring iff all the functions which are $\sim_{perm}$ equivalent to F are φ-colorings. Therefore,

$$\chi_\varphi(\mathfrak{M};k) = \sum_{j=1}^{|M|} d_\varphi(\mathfrak{M}, j) \cdot j!\binom{k}{j} = \sum_{j=1}^{|M|} d_\varphi(\mathfrak{M}, j) \cdot k_{(j)}\,.$$

$\square$

Remarks 3.2.6.

(i) Polynomials of the form
$$\sum_j a_j k_{(j)}$$
where $a_j \in \mathbb{N}$ are called *Falling Factorial (FF) polynomials.* Every FF polynomial belongs to $\mathbb{Z}[k]$. The polynomial guaranteed in Proposition 3.2.5 is an FF polynomial.

(ii) There exist coloring properties which do not satisfy the extension property and yet their counting functions are polynomials. Let $P_{no\ ext}$ consist of all structures $\mathfrak{M}_{F,k}$ which satisfy the following condition:

— Let γ_1, γ_2 and γ_3 be the least, second least and third least elements in the linear ordering of $M = [n]$.
The function $F : M \to [k]$ satisfies either $F(\gamma_1) = F(\gamma_2) \neq F(\gamma_3)$ or $F(\gamma_1) = F(\gamma_3) \neq F(\gamma_2)$. If $F(\gamma_1) = F(\gamma_2)$ then F is onto and if $F(\gamma_1) = F(\gamma_3)$ then F is not onto.

Let $F : M \to [k]$ be a function such that $F(\gamma_1) = F(\gamma_2) \neq F(\gamma_3)$ and F is onto. The addition of an unused color to $[k]$ (i.e., looking at F as a function from M to $[k+1]$) implies that F is no longer onto and yet $F(\gamma_1) \neq F(\gamma_3)$. Hence, $\mathfrak{M}_{F,k+1} \notin P_{no\ ext}$, so $P_{no\ ext}$ does not satisfy the extension property. On the other hand, the number of such structures equals the number of functions $F : [n] \to [k]$ for which $F(1) = F(2) \neq F(3)$, so $\chi_{no\ ext}(\mathfrak{M}, k) = k^{n-2}(k-1)$ is a polynomial.

3.2.2 A syntactic definition of coloring properties

We call $\mathcal{P}_\varphi$ a coloring property if it satisfies the permutation and extension properties. Here we give an equivalent definition by placing a syntactic restriction on φ.

Definition 3.2.4 ($\mathcal{L}(\tau_{\mathbf{F}})$-Coloring sentence). *We say $\psi \in \mathcal{L}(\tau_{\mathbf{F}})$ is an* $\mathcal{L}(\tau_{\mathbf{F}})$-coloring sentence *if ψ is a $\mathcal{L}(\tau_{\mathbf{F}})$-sentence that does not quantify over the second sort, but instead all first and second order quantifiers are on the first sort only.*

Proposition 3.2.7. *Let τ be a vocabulary and let φ be an $\mathcal{L}(\tau_{\mathbf{F}})$-coloring sentence. The class $\mathcal{P}_\varphi$ is a coloring property.*

Proof. The class $\mathcal{P}_\varphi$ satisfies the permutation property because it is definable in SOL. For the extension property we prove a slightly stronger statement by induction on the number of connectives and quantifiers in φ:

(*) Let τ be a vocabulary. Let θ be an $\mathcal{L}(\tau_{\mathbf{F}})$-coloring sentence. Then for every τ-structure $\mathfrak{A}_1$ with universe A_1 and for every two-sorted $\tau_{\mathbf{F}}$-structure $\mathfrak{A} = \langle \mathfrak{A}_1, A_2, F \rangle$ we have

$$\mathfrak{A} \models \theta \text{ iff } \langle \mathfrak{A}_1, \mathit{Range}(F), F \rangle \models \theta$$

Note that the $\tau_{\mathbf{F}}$-structure $\langle \mathfrak{A}_1, \mathit{Range}(F), F \rangle$ is well-defined in the sense that $F : A_1 \to \mathit{Range}(F)$.

Basis Let τ be a vocabulary and let θ be an atomic formula. Assume first that θ does not contain $\mathbf{F}$. Any relation symbol, function symbol or constant symbol in θ is interpreted in $\mathfrak{A}$ over A_1 only. Since all the variables in θ range over the first sort only, the truth-value of θ does not depend on A_2.

On the other hand, if θ contains $\mathbf{F}$, then θ must be of the form $\mathbf{F}(t_1) \approx \mathbf{F}(t_2)$, where t_1 and t_2 are terms of τ which are interpreted as elements of the sort A_1. In this case, the truth-value of θ depends only on the elements of A_2 which can be obtained as $\mathbf{F}(a)$ for some $a \in A_1$. I.e., θ depends only on the range of the interpretation of $\mathbf{F}$.

Closure Assume θ_1 and θ_2 satisfy (*). It is enough to prove that (*) holds for $\theta = \theta_1 \vee \theta_2$, $\theta = \neg\theta_1$, $\theta = \forall x\, \theta_1$ and $\forall U\, \theta_1$.

$\theta = \theta_1 \vee \theta_2$:
$\mathfrak{A} \models \theta_1 \vee \theta_2$ if and only if $\mathfrak{A} \models \theta_1$ or $\mathfrak{A} \models \theta_2$. By the induction hypothesis, the latter holds iff $\langle \mathfrak{A}_1, \mathit{Range}(F), F \rangle \models \theta_1$ or $\langle \mathfrak{A}_1, \mathit{Range}(F), F \rangle \models \theta_2$, and this holds iff we have that $\langle \mathfrak{A}_1, \mathit{Range}(F), F \rangle \models \theta_1 \vee \theta_2$.

$\theta = \neg\theta_1$:
Similar to the previous case.

$\theta = \forall Z\, \theta_1$, where $Z = x$ or $Z = U$:
We extend the vocabulary τ with symbol $\mathbf{Z}$ as follows. The symbol $\mathbf{Z}$ is a constant symbol if $Z = x$ is a first order variable. The

symbol $\mathbf{Z}$ is a relation symbol of arity ρ if $Z = U$ is a second order variable of arity ρ. We note $\mathfrak{A} \models \theta$ iff for every interpretation Z of $\mathbf{Z}$ it holds that $\langle \mathfrak{A}, Z \rangle \models \theta_1$. By the induction hypothesis, this happens iff for every interpretation Z of $\mathbf{Z}$ it holds that $\langle \mathfrak{A}_1, Z, Range(F), F \rangle \models \theta_1$. The latter occurs if and only if $\langle \mathfrak{A}_1, Range(F), F \rangle \models \forall Z \, \theta_1$.

□

We want to prove a converse to Proposition 3.2.7. To do so, we need the following definition, which allows us to convert sentences which quantify over the second sort to related sentences which quantify over the first sort only.

Given a formula φ, we want to replace variables y which range over the second sort with $\mathbf{F}(x)$ for variables that range over the first sort. We require that x is interpreted as the minimal element with respect to the order $\mathbf{R}_\leq$ among those x' with the same value of $\mathbf{F}(x')$ as x. Similarly, for second order variables which range over $M^{m_1} \times [k]^{m_2}$, we want to replace $[k]^{m_2}$ with C^{m_2}, where C is the set of elements which are minimal among those elements with the same value of $\mathbf{F}$. We define this precisely here:

Definition 3.2.5. *Let τ be a vocabulary. We define a function* col *which assigns a formula* $\mathrm{col}(\varphi)$ *to every $\mathcal{L}(\tau)$-formula φ inductively below. Let ψ_φ be the following formula:*

$$\psi_\varphi = \exists C \left(\mathrm{col}(\varphi) \land (\forall x_1 \left(C(x_1) \leftrightarrow \forall x_2 ((\mathbf{F}(x_1) \approx \mathbf{F}(x_2)) \to (\mathbf{R}_\leq(x_1, x_2)))\right)\right)$$

ψ_φ requires that $\mathrm{col}(\varphi)$ *holds and, in addition, that C is the set of minimal elements among those elements with the same value of* $\mathbf{F}$.

We assume w.l.o.g that φ has no universal quantifiers. $\mathrm{col}(\varphi)$ *is defined inductively as follows:*

Basis *For every atomic formula with first order variables $y_1, \ldots, y_k$ which range over the second sort,* $\mathrm{col}(\varphi)$ *is obtained from φ by subtituting each y_i with $\mathbf{F}(y_i)$, and considering $y_1, \ldots, y_k$ now as variables which range over the first sort.*

Closure

— *If* $\varphi = (\varphi_1 \circ \varphi_2)$ *with* $\circ$ *a binary connective, then* $\mathrm{col}(\varphi) = (\mathrm{col}(\varphi_1) \circ \mathrm{col}(\varphi_2))$.

— *If* $\varphi = \neg\varphi_1$, *then* $\mathrm{col}(\varphi) = \neg\mathrm{col}(\varphi_1)$.

— *If* $\varphi = \exists x \varphi_1$ *where* x *ranges over the first sort, then* $\mathrm{col}(\varphi) = \exists x \mathrm{col}(\varphi_1)$.

— *If* $\varphi = \exists x \varphi_1$ *where* x *ranges over the second sort, then* $\mathrm{col}(\varphi) = \exists x (C(x) \wedge \mathrm{col}(\varphi_1))$, *where the quantifier* $\exists x$ *is now on the first sort.*

— *If* $\varphi = \exists U \varphi_1$ *where* U *ranges over* $M^{m_1} \times [k]^{m_2}$, *then* $\mathrm{col}(\varphi) = \exists U ((\forall \bar{x}, \bar{y}(U(\bar{x}, \bar{y}) \to \bigwedge_{i=1}^{m_2} C(y_i))) \wedge \mathrm{col}(\varphi_1))$, *where* $\forall \bar{x}, \bar{y}$ *stands for* $\forall x_1 \cdots \forall x_{m_1} \forall y_1 \cdots \forall y_{m_2}$.

Note that if φ *is a sentence, then* ψ_φ *is a coloring sentence.*

Example 3.2.8 (Complete colorings revisited). *Recall from Example 3.2.2(ii) that a complete coloring* $F : V \to [k]$ *is a proper coloring for which every pair of colors appears at least once along an edge. Let* $\varphi'_{complete}$ *be given by*

$$\varphi_{proper} \wedge \neg\exists y_1 \exists y_2 \neg \exists x_1 \exists x_2 (\mathbf{E}(x_1, x_2) \wedge \mathbf{F}(x_1) \approx y_1 \wedge \mathbf{F}(x_2) \approx y_2).$$

where y_1, y_2 *range over the sort* $[k]$ *and* u, v *range over the set of vertices* V. $\varphi'_{complete}$ *is equivalent to* $\varphi_{complete}$, *and has no universal quantifiers. Then we get that* $\mathrm{col}(\varphi_{complete})$ *is the conjungtion of* φ_{proper} *and*

$$\neg\exists y_1 (C(y_1) \wedge (\exists y_2 (C(y_2) \wedge (\neg \exists x_1 \exists x_2 (\mathbf{E}(x_1, x_2) \wedge \mathbf{F}(x_1) \approx y_1 \wedge \mathbf{F}(x_2) \approx y_2))))) .$$

Observation 3.2.9. *Let* $\mathfrak{M}_{F,k}$ *be a* $\tau_{\mathbf{F}}$*-structure. If* F *is onto* $[k]$, *then for every* $\varphi \in \mathcal{L}(\tau_{\mathbf{F}})$ *we have* $\mathfrak{M}_{F,k} \models \varphi$ *iff* $\mathfrak{M}_{F,k} \models \psi_\varphi$.

Proposition 3.2.10. *Let* τ *be a vocabulary and let* φ *be an* $\mathcal{L}(\tau_{\mathbf{F}})$*-sentence. The following are equivalent:*

(i) $\mathcal{P}_\varphi$ *is a coloring property.*

(ii) $\mathcal{P}_\varphi = \mathcal{P}_{\psi_\varphi}$, *where* ψ_φ *is from Definition 3.2.5.*

Proof. Proposition 3.2.7 give the direction (ii) to (i). Now assume (i) holds.

Let $\mathfrak{M}_{F,k}$ be a $\tau_{\mathbf{F}}$-structure. By the permutation property there exists $\mathfrak{M}_{F_0,k}$ such that $\mathfrak{M}_{F_0,k}$ is obtained from $\mathfrak{M}_{F,k}$ by applying a permutation on $[k]$ such that $Range(F_0) = [k_0]$, $k_0 \leq k$, and $\mathfrak{M}_{F,k} \models \varphi$ iff $\mathfrak{M}_{F_0,k} \models \varphi$. By the extension property, $\mathfrak{M}_{F_0,k} \models \varphi$ iff $\mathfrak{M}_{F_0,k_0} \models \varphi$.

Now consider ψ_φ. By Observation 3.2.9, $\mathfrak{M}_{F_0,k_0} \models \varphi$ iff $\mathfrak{M}_{F_0,k_0} \models \psi_\varphi$. ψ_φ is a coloring-sentence, so by Proposition 3.2.7, $\mathcal{P}_{\psi_\varphi}$ has the extension property, which implies $\mathfrak{M}_{F_0,k_0} \models \psi_\varphi$ iff $\mathfrak{M}_{F_0,k} \models \psi_\varphi$. Finally, by the permutation property of $\mathcal{P}_{\psi_\varphi}$, $\mathfrak{M}_{F_0,k} \models \psi_\varphi$ iff $\mathfrak{M}_{F,k} \models \psi_\varphi$, and the direction (i) to (ii) follows. □

3.2.3 Multi-colorings

To construct graph polynomials in several indeterminates, we extend in this and the next subsections the definitions in order to deal with several color-sets.

Let $\mathfrak{M}$ be a τ-structure with universe M. Let $\mathfrak{M}_{F,\bar{k}}$ be the $(\alpha+1)$-sorted structure $\langle \mathfrak{M}, [k_1] \ldots, [k_\alpha], F \rangle$ with

$$F : M^m \to [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}.$$

We denote by $\tau_{\alpha,\mathbf{F}}$ the corresponding vocabulary. Note $\mathbf{F}$ is now a relation symbol. We will revisit this issue in Remark 3.2.13.

We extend the definitions ofφ-colorings and coloring properties naturally to *φ-multi-colorings and multi-coloring properties*. Multi-coloring properties $\mathcal{P}_\varphi$ satisfy a version of the permutation and extension properties, and a new property called *the non-occurrence property*:

Permutation property Let $\bar{\pi} = (\pi_1, \ldots, \pi_\alpha)$ be permutations of the sets $[k_1], \ldots, [k_\alpha]$ respectively. Let $F : M^m \to [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$ and let $F_{\bar{\pi}}$ be the function obtained by applying the permutations $\pi_1, \ldots, \pi_\alpha$ on F. Then

$$\mathfrak{M}_{F,\bar{k}} \in \mathcal{P}_{\varphi(\mathbf{F})} \text{ iff } \mathfrak{M}_{F_{\bar{\pi}},\bar{k}} \in \mathcal{P}_{\varphi(\mathbf{F})}$$

Namely, $\mathcal{P}_{\varphi(\mathbf{F})}$ is closed under permutations of the color-sets.

Extension property For every $\mathfrak{M}$, $\bar{k} = k_1, \ldots, k_\alpha$, $\bar{k'} = k'_1, \ldots, k'_\alpha$, $k_1 \leq k'_1, \ldots, k_\alpha \leq k'_\alpha$, and $F : M^m \to [k'_1]^{m_1} \times \cdots \times [k'_\alpha]^{m_\alpha}$, we have

$$\mathfrak{M}_{F,\bar{k}} \in \mathcal{P}_{\varphi(\mathbf{F})}$$

iff

$$\mathfrak{M}_{F,\bar{k'}} \in \mathcal{P}_{\varphi(\mathbf{F})}.$$

Non-occurrence property Assume $F : M^m \to [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$ with $m_i = 0$. [2] Then for every $b \in \mathbb{N}$,

$$\langle \mathfrak{M}, [k_1], \ldots, [k_\alpha], F \rangle \in \mathcal{P}_{\varphi(\mathbf{F})}$$

iff

$$\langle \mathfrak{M}, [k_1], \ldots, [b], \ldots, [k_\alpha], F \rangle = \mathfrak{M}_{(k_1,\ldots,k_{i-1},b,k_{i+1},\ldots,k_\alpha),F} \in \mathcal{P}_{\varphi(\mathbf{F})}.$$

The extension property and the non-occurrence property require that increasing and decreasing the number of colors not used by F, respectively adding or removing unused color-sets does not affect whether $\mathfrak{M}_{F,\bar{k}}$ belongs to the multi-coloring property $\mathcal{P}_\varphi$.

$\chi_{\varphi(\mathbf{F})}(\mathfrak{M}; k_1, \ldots, k_\alpha)$ denotes the number of φ-multi-colorings with color-sets $[k_1], \ldots, [k_\alpha]$.

Proposition 3.2.11 (Special case of Proposition A).
Let φ be an $\mathcal{L}(\tau_{\alpha,\mathbf{F}})$-sentence such that $\mathcal{P}_\varphi$ is a multi-coloring property. For every $\mathfrak{M}$, $\chi_{\varphi(\mathbf{F})}(\mathfrak{M}; k_1, \ldots, k_\alpha)$ is an FF-polynomial in $k_1, \ldots, k_\alpha$ of the form

$$\sum_{j_1 \leq N_M} \sum_{j_2 \leq N_M} \cdots \sum_{j_\alpha \leq N_M} d_{\varphi(F)}(\mathfrak{M}, \bar{j}) \prod_{1 \leq \beta \leq \alpha} k_{\beta(j_\beta)}$$

where $\bar{j} = (j_1, \ldots, j_\alpha)$, $d_{\phi(R)}(\mathfrak{M}, \bar{j})$ is the number of φ-multi-colorings F with color-sets $[j_1], \ldots, [j_\alpha]$ which use all the colors in each color-set, up to permutations of the color sets, and N_M is a polynomial in $|M|$.

Proof. Similar to the one variable case. □

[2] For a set S the set S^0 is the singleton set which has as its unique element the empty tuple.

Example 3.2.12. *Recall from Section 2.3.2 that $\chi_{mcc(t)}(G;\mathsf{k})$ is the graph polynomial which denotes for every $k \in \mathbb{N}$, the number of vertex k-colorings for which no color induces a graph with a connected component of size larger than t.*

Let $F : V(G) \to [k]$ be any function. If $t \geq n_G$ then for every color $c \in [k]$ it holds that $|F^{-1}(c)| < t$ and, in particular, no color induces a graph which has a connected component larger than t. Therefore, for such t, $\chi_{mcc}(G;k,t) = k^{n_G}$. Assume $\chi_{mcc}(G;k,t)$ is a polynomial in t. Since k^{n_G} is a polynomial in t (of degree 0), and since k^{n_G} and $\chi_{mcc}(G;k,\mathsf{t})$ agree on infinitely many values of t, it must be true that $\chi_{mcc}(G;k,\mathsf{t}) = k^{n_G}$. However, $\chi_{mcc}(G;k,t)$ does not always agree with k^{n_G} on small values of $t \in \mathbb{N}$, and hence $\chi_{mcc}(G;k,t)$ cannot be a polynomial in t.

This example shows the motivation for requiring the non-occurrence property of coloring properties.

Remark 3.2.13 (A syntactic definition of multi-coloring properties). Instead of taking $\mathbf{F}$ to be a relation symbol, it is possible to consider ρ function symbols $\mathbf{F}_1, \ldots, \mathbf{F}_\rho$, where ρ is the arity of $\mathbf{F}$. Instead of a function $F : M^m \to [k_1]^{m_1} \times \cdots \times [k_\alpha]^{m_\alpha}$ we have $\rho = m_1 + \cdots + m_\alpha$ functions, $F_1, \ldots, F_\rho$ such that $F(\bar{a}) = (F_1(\bar{a}), \ldots, F_\rho(\bar{a}))$. It is possible to extend the notion of a coloring sentence to this case and to get a syntactic definition of multi-coloring properties. However, we will soon make a further extension of the notion of multi-colorings, which no longer seems to allow a syntactic definition.

3.2.4 Examples of multi-coloring properties

In Section 2.3 we gave several applications of Proposition A to counting colorings of graphs, but Proposition A also extends to other vocabularies as well. The counting functions of the following colorings are also polynomials:

(i) Given a graph $G = (V, E)$, a subset $A \subseteq V$, and a proper coloring $f|_A : A \to [k]$ of the subgraph $G[A]$ of G induced by A, $f : V \to [k]$ is a *proper extension of* $f|_A$ if $f|_A$ is the restriction of f to A, and in addition, f is a proper coloring of G, see e.g. [22]. [3]

[3] One could, of course, define Pre-coloring extensions not only for proper colorings, but also for proper edge colorings, for harmonious colorings, for acyclic colorings, etc.

(ii) Given hypergraph $H = (V, E)$ with $E \subseteq \wp(V)$, a function $f : V \to [k]$ is a *weak hypergraph proper coloring* if for every hyperegde e which is no a singleton, there exist $u, v \in e$ with $f(u) \neq f(v)$.

(iii) Given hypergraph $G = (H, E)$ with $E \subseteq \wp(V)$, a function $f : V \to [k]$ is a *strong hypergraph proper coloring* of H if for every $e \in E$ and for every $u, v \in e$ such that $u \neq v$ we have $f(u) \neq f(v)$.

(iv) Given hypergraph $H = (V, E, D)$ with $D, E \subseteq \wp(V)$, a function $f : V \to [k]$ is *mixed proper coloring* of H the following holds:

— f is a weak hypergraph proper coloring of E

— if $u, v \in d$ for some $d \in D$, then $f(u) = f(v)$.

The counting function of these colorings was shown to be a polynomial in k in [149].

3.2.5 Multi-colorings – the general cases

3.2.5.1 Multi-colorings with partial functions

We extend our definition in two ways. First, we now allow $F \subseteq M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$ to be a partial function. $\mathbf{F}$ will be a relation symbol. Second, we also allow several simultaneous coloring predicates $F_1, \ldots, F_s$ and the corresponding number of relation symbols. A coloring property P_φ will therefore consist of structures

$$\mathfrak{M}_{F_1,\ldots,F_s,k_1,\ldots,k_\alpha} = \langle \mathfrak{M}, [k_1], \ldots, [k_\alpha], F_1, \ldots, F_s \rangle$$

which satisfy φ and for which each F_i is a (possibly partial) function.

We may call multi-coloring properties and multi-coloring simply also coloring properties and colorings, if the situation is clear from the context. The permutation, extension and non-occurrence properties extend naturally to this case and Proposition A holds as well:

Proposition 3.2.14 (Proposition A for several partial functions).
Let φ be an $\mathcal{L}(\tau_{\mathbf{F_1},\ldots,\mathbf{F_s}})$ sentence such that $\mathcal{P}_{\varphi(\mathbf{F_1},\ldots,\mathbf{F_s})}$ is a multi-coloring property. For every $\mathfrak{M}$ the number $\chi_{\varphi(\bar{\mathbf{F}})}(\mathfrak{M};k)$ of φ-multi-colorings with several partial functions is an FF*-polynomial in k of the form*

$$\sum_{j_1 \le N_M} \sum_{j_2 \le N_M} \cdots \sum_{j_\alpha \le N_M} d_\varphi(\mathfrak{M}, \bar{j}) \prod_{1 \le \beta \le \alpha} k_{\beta(j_\beta)} \tag{3.1}$$

where $N_M \in \mathbb{N}$. Moreover, N_M is bounded by a polynomial in $|M|$.

3.2.5.2 Multi-colorings with bounded relations

Here we extend the multi-colorings with several partial functions by allowing several relations which are bounded in a certain way.

Definition 3.2.6 (Bounded relations and multi-coloring properties).

(i) *We say a relation $R \subseteq M^m \times [k_1]^{m_1} \times \cdots \times [k_\alpha]^{m_\alpha}$ is d-*bounded if the set of tuples of colors used by R,

$$\{\bar{c} \mid \text{ there exists } \bar{x} \in M^m \text{ such that } (\bar{x}, \bar{c}) \in R\},$$

is of size at most $|M|^d$.

(ii) A $\varphi(\bar{\mathbf{R}})$-multi-coloring property $P_{\varphi(\bar{\mathbf{R}})}$ is the class of τ-structures

$$\mathfrak{M}_{\bar{R},\bar{k}} = \left\langle \mathfrak{M}, [k_1], \ldots, [k_\alpha], \bar{R} \right\rangle$$

which satisfy φ. We say P_φ is *bounded* if there exists d such that each R in every structure $\mathfrak{M}_{\bar{R},\bar{k}}$ in P_φ is d-bounded.

Again, the permutation, extension and non-occurrence properties extend naturally and Proposition A holds. However, notice that we get only Newton polynomials. in this case.

Proposition 3.2.15 (Proposition A for several d-bounded relations).

Let $d \in \mathbb{N}$ and let φ be an $\mathcal{L}(\tau_{\mathbf{R_1},\ldots,\mathbf{R_s}})$-sentence such that $\mathcal{P}_{\varphi(\mathbf{R_1},\ldots,\mathbf{R_s})}$ is a multi-coloring property. For every $\mathfrak{M}$ the number $\chi_{\varphi(\bar{\mathbf{R}})}(\mathfrak{M},k)$ of

φ-multi-colorings with several d-bounded relations is a Newton polynomial in k of the form

$$\sum_{j_1 \leq N_M} \sum_{j_2 \leq N_M} \cdots \sum_{j_\alpha \leq N_M} c_\varphi(\mathfrak{M}, \bar{j}) \prod_{1 \leq \beta \leq \alpha} \binom{k_\beta}{j_\beta}$$

where $N_M \in \mathbb{N}$.

Remark 3.2.16. The syntactic definition of coloring properties with the extension property extends rather naturally to the case of Subsection 3.2.5.1, which allows several partial functions as coloring predicates. However, it seems that it does not extend to the case of bounded relations.

3.2.6 Closure properties for the general cases

The following proposition holds for both of the general cases in Subsection 3.2.5.

Proposition 3.2.17 (Sums and products). *Let ϕ, ψ be coloring properties. Then there are $\theta_1, \theta_2 \in$ SOL such that*

(i) $\chi_{\theta_1(\bar{\mathbf{F}}_3)}(\mathfrak{M}, \bar{k}, 1) = \chi_{\phi(\bar{\mathbf{F}}_1)}(\mathfrak{M}, \bar{k}) + \chi_{\psi(\bar{\mathbf{F}}_2)}(\mathfrak{M}, \bar{k})$

(ii) $\chi_{\theta_2(\bar{\mathbf{F}}_3)}(\mathfrak{M}, \bar{k}) = \chi_{\phi(\bar{\mathbf{F}}_1)}(\mathfrak{M}, \bar{k}) \cdot \chi_{\psi(\bar{\mathbf{F}}_2)}(\mathfrak{M}, \bar{k})$

Proof. Since we are dealing with ordered structures, we can define $\varphi_{total}(\mathbf{F}')$ which requires that F' is a total function $F' : M \to [k']$, where $[k']$ is a new color set. Let

$$\theta_1(\bar{\mathbf{F}}_1, \bar{\mathbf{F}}_2, \mathbf{F}') = \big(\varphi_{total}(\mathbf{F}') \wedge \phi(\bar{\mathbf{F}}_1) \wedge \mathbf{F}_2 = \emptyset\big) \vee \tag{3.2}$$

$$\big(\mathbf{F}' = \emptyset \wedge \mathbf{F_1} = \emptyset \wedge \psi(\bar{\mathbf{F}}_2)\big) \tag{3.3}$$

It holds that $\chi_{\theta_1}(\mathfrak{M}, \bar{k}, k') = (k')^{|M|} \cdot \chi_\phi(\mathfrak{M}, \bar{k}) + \chi_\psi(\mathfrak{M}, \bar{k})$. Taking k' to be 1, we get that $\chi_{\theta_1}(\mathfrak{M}, \bar{k}, 1)$ is the sum.

For the product we take $\chi_{\theta_2}(G, \lambda)$ with

$$\theta_2(\bar{\mathbf{F}}_3) = \big(\phi(\bar{\mathbf{F}}_1) \wedge \psi(\bar{\mathbf{F}}_2)\big)$$

where $\bar{\mathbf{F}}_3 = (\bar{\mathbf{F}}_1, \bar{\mathbf{F}}_2)$. ∎

3.2.7 The bivariate matching polynomial counts colorings

We want to define $M(G;\mathsf{x},\mathsf{y})$ as the counting function of colorings. We first do it for

$$g(G;\mathsf{x}) = M(G;\mathsf{x},1) = \sum_{M \subseteq E(G) \text{ is a matching}} \mathsf{x}^{|M|}. \quad (3.4)$$

There is an SOL formula $\varphi_1(R)$ which says that "$R \subseteq E(G) \times [k]$ is a partial function the domain of which is a matching of G". In other words R is a partial edge-coloring such that for each $i \in [k]$ the set $\{e \in E : (e,i) \in r\}$ is an independent set of edges. For each matching $M \subseteq E(G)$ there are $k^{|M|}$ many functions with domain M. Hence

$$g(G;k) = \mid \{R \subseteq E \times [k] : \mathfrak{M}_k \models \varphi_1(r)\} \mid \quad (3.5)$$

This shows that $g(G;\mathsf{x})$ counts the number of φ_1-colorings of G.

We can obtain a similar presentation for $g^*(G;k) = \sum_M k_{(|M|)}$ by writing

$$g^*(G;k) = \mid \{R \subseteq E \times [k] : \mathfrak{M}_k \models \varphi_2(r)\} \mid \quad (3.6)$$

where $\varphi_2(r)$ is the formula "$\varphi_1(R)$ and R is injective". This shows that $g^*(G;\mathsf{x})$ counts the number of φ_2-colorings of G.

To interpret the bivariate polynomial $M(G;\mathsf{x},\mathsf{y})$ as counting colorings we use two sorts of colors $[k_1]$ and $[k_2]$, the three-sorted structure $\mathfrak{M}_k = \langle V, [k_1], [k_2]; E, R_1, R_2 \rangle$, with two coloring relations $R_1 \subseteq E \times [k_1]$ and $R_2 \subseteq V \times [k_2]$ and a formula $\varphi_3(R_1, R_2)$ which says that "$R_1 \subseteq E \times [k_1]$ is a partial function the domain M of which is a matching of G" and "$R_2 \subseteq V \times [k_2]$ is a partial function with domain $V - cov(M)$".

M. Trinks gives another interpretation of the matching polynomial as the evaluations of the counting function of a type of generalized proper colorings [145].

3.2.8 The dichromatic polynomial $Z(G;\mathsf{x},\mathsf{y})$

Recall the dichromatic polynomial is defined as follows:

$$Z(G;\mathsf{x},\mathsf{y}) = \sum_{A \subseteq E} \mathsf{x}^{k(A)} \mathsf{y}^{|A|}$$

where $k(A)$ is the number of connected components of the spanning subgraph (V, A).

$Z(G; k_1, k_2)$ can be interpreted as counting three-sorted structure

$$G_{(A,F_1,F_2),(k_1,k_2)} = \langle G, [k_1], [k_2], A, F_1, F_2 \rangle$$

with $A \subseteq E$, $F_1 : V \to [k_1]$ and $F_2 : A \to [k_2]$ such that $(u, v) \in A$ implies $F_1(u) = F_1(v)$. This is expressed in the formula $\mathit{dichromatic}(A, F_1, F_2)$.

3.3 SOL-polynomials

We are now ready to introduce the SOL-polynomials, which generalize subset expansions and spanning tree expansions of graph polynomials as encountered in the literature.

3.3.1 SOL(τ)-polynomials

Let $\mathcal{R}$ be a commutative semi-ring, which contains the semi-ring of natural numbers $\mathbb{N}$. For our discussion $\mathcal{R} = \mathbb{Z}$ suffices, but the definitions generalize. Our polynomials have a fixed finite set of indeterminates, $\mathbf{X}$. We denote by $\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi(\bar{v}))$ the number of tuples $\bar{a}$ of elements the universe that satisfy φ.

Let $\mathfrak{M}$ be a τ-structure. We first define the *standard* SOL(τ)*-monomials* inductively.

Definition 3.3.1 (standard SOL-monomials).

(i) Let $\phi(\bar{v})$ be a formula in SOL(τ)*, where $\bar{v} = (v_1, \ldots, v_m)$ is a finite sequence of first order variables. Let $r \in \mathbf{X} \cup (\mathbb{Z} - \{0\})$ be either an indeterminate or an integer. Then*

$$r^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$$

is a standard SOL(τ)*-monomial (whose value depends on* $\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))$*).*

(ii) Finite products of standard SOL(τ)*-monomials are standard* SOL(τ)*-monomials.*

Remark 3.3.1.

(i) It is sometime convenient to write

$$\prod_{\bar{v}:\phi(\bar{v})} r \quad \text{or} \quad \prod_{\bar{v}:\phi} r$$

instead of $r^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$ when $\mathfrak{M}$ is clear from the context.

(ii) If $\phi = R(\bar{v})$ then we may write $r^{|R|}$ instead $r^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$.

(iii) If the tuple $\bar{v}$ in Definition 3.3.1(i) is actually of length 1, we write $r^{\mathrm{card}_{\mathfrak{M},v}(\phi(v))}$ instead of $r^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$.

The *falling factorial (*FF*)* SOL(τ)*-monomials* and the *Newton* SOL(τ)-*monomials* are defined similarly as follows:

Definition 3.3.2 (FF **SOL-monomials).**
The FF SOL(τ)*-monomials are defined as in Definition 3.3.1, except we replace the power*

$$r^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$$

with the falling factorial

$$r_{(\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v})))} \,.$$

Definition 3.3.3 (Newton SOL-monomials).
The Newton SOL(τ)*-monomials are defined as in Definition 3.3.1, except we replace the power*

$$r^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$$

with the binomial coefficient

$$\binom{r}{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))} \,.$$

Note the degree of a monomial is polynomially bounded by the cardinality of $\mathfrak{M}$.

Definition 3.3.4 (SOL-polynomials).
The polynomials definable in $\mathrm{SOL}(\tau)$ *are defined inductively:*

(i) *standard (respectively* FF *respectively Newton)* $\mathrm{SOL}(\tau)$*-monomials are standard (respectively* FF *respectively Newton)* $\mathrm{SOL}(\tau)$*-polynomials.*

(ii) *Let* ϕ *be a* $\tau \cup \{\bar{\mathbf{R}}\}$*-formula in* SOL *where* $\bar{\mathbf{R}} = (\mathbf{R}_1, \ldots, \mathbf{R}_m)$ *is a finite sequence of relation symbols not in* τ*. Let* t *be a standard (respectively* FF *respectively Newton)* $\mathrm{SOL}(\tau \cup \{\bar{\mathbf{R}}\})$*-polynomial. Then*

$$\sum_{\bar{R}:\langle\mathfrak{M},\bar{R}\rangle\models\phi(\bar{R})} t(\langle\mathfrak{M},\bar{R}\rangle\,;\bar{\mathsf{x}})$$

is a standard (respectively FF *respectively Newton)* $\mathrm{SOL}(\tau)$*-polynomial.*

For simplicity we refer to $\mathrm{SOL}(\tau)$-polynomials as SOL-polynomials when τ is clear from the context. Among the SOL-polynomials we find most of the known graph polynomials from the literature, cf. [116]. We will discuss the choice of basis of the SOL-polynomials in Section 3.4.

3.3.2 Properties of SOL-polynomials

Lemma 3.3.2.

(i) *Every indeterminate* $\mathsf{x} \in \mathbf{X}$ *can be written as a standard,* FF *and Newton* SOL*-monomial.*

(ii) *Every integer* c *can be written as a standard,* FF *and Newton* SOL*-monomial.*

Proof. The minimal element f_1 in the linear ordering of the universe is definable in SOL. For every $r \in (\mathbb{Z} - \{0\}) \cup \mathbf{X}$, the term r is a standard, FF and Newton SOL-monomial since

$$r = r^{\mathrm{card}_{\mathfrak{M},v}(v=f_1)} = r_{\mathrm{card}_{\mathfrak{M},v}(v=f_1)} = \binom{r}{\mathrm{card}_{\mathfrak{M},v}(v=f_1)}$$

For $r = 0$ note that $\sum_{R:\exists v\,(\neg(v\approx v))} 1$ is a standard, FF and Newton SOL-polynomial, and that $\sum_{R:\exists v\,(\neg(v\approx v))} 1 = 0$. $\square$

Lemma 3.3.3 (Normal form of monomials (1)).

(i) Let $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ be a standard SOL*-monomial. Then $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ can be written in the form*

$$r_1^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_1(\bar{v}))} \cdots r_t^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_t(\bar{v}))} \tag{3.7}$$

where $\varphi_1, \ldots, \varphi_t \in \mathrm{SOL}$ and $r_1, \ldots, r_t \in (\mathbb{Z} - \{0\}) \cup \mathbf{X}$ are distinct. The formulas $\varphi_1, \ldots, \varphi_t$ have the same tuple of free variables $\bar{v}$.

(ii) Let $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ be an FF *or Newton* SOL*-monomial. Then $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ can be written in the form of Equation (3.7), except we replace $r_i^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_i(\bar{v}))}$ with $r_{i(\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_i(\bar{v})))}$ or $\binom{r_i}{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_i(\bar{v}))}$ respectively, and the r_i's might not be distinct.*

Proof.

(i) From the definitions, it is easy to see that Φ is of the desired form, except $r_1, \ldots, r_t$ may not be distinct, i.e. $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ can be written as

$$r_1^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_{1,1}(\bar{v}))} \cdots r_1^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_{1,h_1}(\bar{v}))} \cdots . r_t^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_{t,1}(\bar{v}))} \cdots r_t^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_{t,h_t}(\bar{v}))}$$

We will prove that

$$r_i^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_{i,1}(\bar{v}))} \cdots r_i^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_{i,h_i}(\bar{v}))}$$

can be written as $r_i^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\theta(\bar{v}))}$ for some formula θ. Let m be the maximum number of free variables in any of the $\varphi_{i,j}$. Without loss of generality, we may assume that all $\varphi_{i,j}$ have m free variables. We may do so because otherwise we can add free variables $v_q, \ldots, v_m$ to those $\varphi_{i,j}$ which have less m free variables. We then change $\varphi_{i,j}$ to require also that each new variable is equal to v_1, thus keeping the same number of tuples satisfying $\varphi_{i,j}$. The first element f_1 and the second element f_2 of the linear ordering of $\mathfrak{M}$ are definable in SOL. Let $\theta(v_1, \ldots, v_m, u_1, \ldots, u_t)$ be the formula which holds for $(a_1, \ldots, a_m, b_1, \ldots, b_{h_i})$ iff:

(i.a) exactly one of the b_j is f_1 and all other b_j's are f_2, and

(i.b) if $b_j = f_1$ then $a_1, \ldots, a_m$ satisfy $\varphi_{i,j}$.

The formula θ uses the u_j variables to choose the formula $\varphi_{i,j}$. The tuples corresponding to the different $\varphi_{i,j}$ may assume to be disjoint which implies that

$$\mathsf{x}_i^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\theta(\bar{v}))} = \mathsf{x}_i^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_{i,1}(\bar{v}))} \cdots \mathsf{x}_i^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_{i,h_i}(\bar{v}))} .$$

(ii) It is easy to see that Φ is of the desired form.

$\square$

Lemma 3.3.4 (Normal form of monomials (2)).

(i) Let $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ be a standard SOL*-monomial. Then $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ can be written in the form*

$$\sum_{A_1} \cdots \sum_{A_{t-1}} \sum_{A_t : \varphi(A_1, \ldots, A_t)} r_1^{|A_1|} \cdots r_t^{|A_t|} \tag{3.8}$$

where $A_1, \ldots, A_t$ are second order variables of the same arity, $\varphi \in \mathrm{SOL}$ and $r_1, \ldots, r_t \in (\mathbb{Z} - \{0\}) \cup \mathbf{X}$ are distinct.

(ii) Let $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ be an FF *or Newton* SOL*-monomial. Then $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ can be written in the form of Equation (3.8), except we replace $r_i^{|A_i|}$ with $r_{i|A_i|}$ or $\binom{r_i}{|A_i|}$ respectively, and the r_i's might not be distinct.*

Proof.

(i) By Lemma 3.3.3, $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ can be written in the form

$$r_1^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_1(\bar{v}))} \cdots r_t^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_t(\bar{v}))}$$

where $\varphi_1, \ldots, \varphi_t \in \mathrm{SOL}$ and $r_1, \ldots, r_t \in (\mathbb{Z} - \{0\}) \cup \mathbf{X}$ are distinct. The formulas $\varphi_1, \ldots, \varphi_t$ have the same number m of free variables.

We define $\varphi(A_1, \ldots, A_t)$ as follows:

$$\varphi(A_1, \ldots, A_t) = \bigwedge_{i=1}^{t} \forall v_1 \cdots \forall v_m \left(A_i(v_1, \ldots, v_m) \leftrightarrow \varphi_i(v_1, \ldots, v_m) \right) .$$

$\varphi(A_1, \ldots, A_t)$ defines the tuple $(A_1, \ldots, A_t)$ uniquely, and the proof follows since we have that $|A_i| = \mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_i(\bar{v}))$.

(ii) Again, the proof is similar to the previous case.

□

Lemma 3.3.5 (Normal form of polynomials).

(i) Let $P(\mathfrak{M};\bar{\mathrm{x}})$ be a standard, FF *or Newton* SOL*−polynomial. Then $P(\mathfrak{M};\bar{\mathrm{x}})$ can be written in the form*

$$\sum_{R_1:\phi_1(R_1)} \cdots \sum_{R_s:\phi_s(R_1,\ldots,R_s)} \Phi(\left\langle\mathfrak{M}), \bar{R}\right\rangle;\bar{\mathrm{x}}), \tag{3.9}$$

where $\phi_1,\ldots,\phi_s \in$ SOL *and $\Phi(\left\langle\mathfrak{M}), \bar{R}\right\rangle;\bar{\mathrm{x}})$ is a standard,* FF *or Newton* SOL*-monomial respectively.*

(ii) In fact, Equation (3.9) can be rewritten as follows:

$$\sum_{R_1} \cdots \sum_{R_{s-1}} \sum_{R_s:\phi(R_1,\ldots,R_s)} \Phi(\left\langle\mathfrak{M}), \bar{R}\right\rangle;\bar{\mathrm{x}}), \tag{3.10}$$

where $\phi \in$ SOL*.*

The proof of Lemma 3.3.5 follows directly from the definitions.

Proposition 3.3.6. *The point-wise product of two standard,* FF *or Newton* SOL*-polynomials is again a standard,* FF *or Newton* SOL*-polynomial respectively.*

Proof. We prove it for standard SOL-polynomials. The proof for FF-polynomials and Newton polynomials is identical. Let $P_1(\mathfrak{M};\bar{\mathrm{x}})$ and $P_2(\mathfrak{M};\bar{\mathrm{x}})$ be standard SOL-polynomials. Every SOL-polynomial can be written in the form of Equation (3.9). Without loss of generality, we may assume P_1 and P_2 have the same number of sums (otherwise we add dummy sums of the form $\sum_{U:U=\emptyset}$). We proceed by induction on the number of summations in P_1 and P_2:
Base: By definition $P_1(\mathfrak{M};\bar{\mathrm{x}}) \cdot P_2(\mathfrak{M};\bar{\mathrm{x}})$ is an SOL-monomial.
Step: For $i = 1,2$ let

$$P_i(\mathfrak{M};\bar{\mathrm{x}}) = \sum_{R_i:\phi_i(R_i)} \Phi_i(\langle\mathfrak{M}, R_i\rangle;\bar{\mathrm{x}}).$$

Then

$$P_1(\mathfrak{M};\bar{\mathsf{x}}) \cdot P_2(\mathfrak{M};\bar{\mathsf{x}}) = \sum_{R_1} \sum_{R_2:\phi_1(R_1)\wedge\phi_2(R_2)} \Phi_1(\langle\mathfrak{M}, R_1\rangle;\bar{\mathsf{x}}) \cdot \Phi_2(\langle\mathfrak{M}, R_2\rangle;\bar{\mathsf{x}}).$$

By the induction hypothesis, this is a standard SOL-polynomial. □

Lemma 3.3.7. *Let τ be a vocabulary and let $\mathbf{S}$ be a relation symbol not in τ. Let $P(\mathfrak{M};\bar{\mathsf{x}})$ be a standard,* FF- *or Newton* SOL(τ)*-polynomial and let $A \in \{\emptyset, M\}$. Let $P^A(\langle\mathfrak{M}, S\rangle;\bar{\mathsf{x}})$ be graph polynomial which satisfies*

$$P^A(\langle\mathfrak{M}, S\rangle;\bar{\mathsf{x}}) = \begin{cases} P(\mathfrak{M};\bar{\mathsf{x}}) & S = A \\ 1 & \textit{otherwise} \end{cases}$$

Then P^A is a standard, FF- *respectively Newton* SOL($\tau \cup \{\mathbf{S}\}$)*-polynomial.*

Proof. We prove the lemma for standard SOL-polynomials by induction on the structure of $P(\mathfrak{M};\bar{\mathsf{x}})$. The proof for FF- and Newton SOL-polynomials is similar.
Base:

(i) Let $P(\mathfrak{M};\bar{\mathsf{x}}) = r^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi(\bar{v}))}$. Then $P^A(\mathfrak{M}, S;\bar{\mathsf{x}}) = r^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi(\bar{v})\wedge(S=A))}$ satisfies the conditions.

(ii) $P(\mathfrak{M};\bar{\mathsf{x}}) = P_1(\mathfrak{M};\bar{\mathsf{x}}) \cdot P_2(\mathfrak{M};\bar{\mathsf{x}})$. Then $P^A(\mathfrak{M};\bar{\mathsf{x}}) = P_1^A(\mathfrak{M};\bar{\mathsf{x}}) \cdot P_2^A(\mathfrak{M};\bar{\mathsf{x}})$ satisfies the conditions.

Step: Let

$$P(\mathfrak{M};\bar{\mathsf{x}}) = \sum_{\bar{R}:\langle\mathfrak{M},\bar{R}\rangle\models\phi(\bar{R})} t(\langle\mathfrak{M}, \bar{R}\rangle;\bar{\mathsf{x}}),$$

where t is a $(\tau\cup\{\bar{\mathbf{R}}\})$-polynomial and assume t^A is a standard, FF or Newton SOL($\tau \cup \{\bar{\mathbf{R}}, \mathbf{S}\}$)-polynomial. Let

$$P^A(\mathfrak{M}, S;\bar{\mathsf{x}}) = \sum_{\bar{R}:\langle\mathfrak{M},\bar{R}\rangle\models\phi(\bar{R})\wedge\mu(\bar{R},S)} t^A(\langle\mathfrak{M}, \bar{R}, S\rangle;\bar{\mathsf{x}}),$$

where $\mu(\bar{R}, S)$ says that either each relation in $\bar{R}$ is equal to $\emptyset$ or $S = A$ (or both). Then P^A satisfies the conditions. □

Proposition 3.3.8. *The point-wise sum of two standard,* FF *or Newton* SOL*-polynomials is again a standard,* FF *or Newton* SOL*-polynomial respectively.*

Proof. We prove it for standard SOL-polynomials. The proof for FF-polynomials and Newton polynomials is identical.

Let $P_1(\mathfrak{M};\bar{\mathsf{x}})$ and $P_1(\mathfrak{M};\bar{\mathsf{x}})$ be standard SOL-polynomials.

Let $P_1^{\emptyset}(\mathfrak{M},S;\bar{\mathsf{x}})$ and $P_2^{M}(\mathfrak{M},S;\bar{\mathsf{x}})$ be the SOL-polynomials given in Lemma 3.3.7. Then the sum of $P_1(\mathfrak{M};\bar{\mathsf{x}})$ and $P_2(\mathfrak{M};\bar{\mathsf{x}})$ is given by

$$\sum_{S\in\{\emptyset,M\}} P_1^{\emptyset}(\mathfrak{M},S;\bar{\mathsf{x}})\cdot P_2^{M}(\mathfrak{M},S;\bar{\mathsf{x}})$$

which is an SOL-polynomial. □

Proposition 3.3.9. *Let*

$$P_1(\mathfrak{M};\bar{\mathsf{x}}) = \sum_{\bar{R}:\theta}\prod_{\bar{b}:\psi}\sum_{\bar{a}:\phi} P_2(\langle\mathfrak{M},\bar{R},\bar{a},\bar{b}\rangle;\bar{\mathsf{x}}),$$

where $P_2(\mathfrak{A};\bar{\mathsf{x}})$ is a standard, FF *or Newton* SOL*-monomial and the product and inner summation are on tuples of first order variables. It holds that $P_1(\mathfrak{M};\bar{\mathsf{x}})$ is a standard,* FF *or Newton* SOL*-polynomial respectively.*

Proof. We can expand the product

$$\prod_{\bar{b}:\psi}\sum_{\bar{a}:\phi} P_2(\mathfrak{A};\bar{\mathsf{x}}) = \sum_{f:\vartheta}\prod_{\bar{a},\bar{b}:\varphi} P_2(\mathfrak{A};\bar{\mathsf{x}})\,,$$

where ϑ says the relation f is a function

$$f:\{\bar{b}\mid \langle\mathfrak{A},\bar{b}\rangle\models\psi\}\to\{\bar{a}\mid \langle\mathfrak{A},\bar{a},\bar{b}\rangle\models\phi\}\,,$$

and $\varphi = \big(f(\bar{b},\bar{a})\big)\wedge\psi\wedge\phi$. So the proposition holds. □

By induction the last proposition holds for functions defined by alternating $\prod$ and $\sum$, as long as all $\sum$ within the scope of a $\prod$ iterate over elements (and not over relations).

3.3.3 Combinatorial polynomials

It is noteworthy that the following combinatorial invariants can be written as standard, FF and Newton SOL-polynomials.

Cardinality, I The cardinality of a definable set

$$\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi(\bar{v})) = \sum_{\bar{v}:\varphi(\bar{v})} 1$$

is an an SOL-polynomial.

Cardinality, II Exponentiation of cardinalities

$$\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi(\bar{v}))^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\psi(\bar{v}))} = \prod_{\bar{v}:\psi(\bar{v})} \sum_{\bar{u}:\varphi(\bar{u})} 1$$

is an SOL-polynomial by proposition 3.3.9.

Factorials The factorial of the cardinality of a definable set

$$\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi(\bar{v}))! = \sum_{\pi:\varphi(\bar{v})\xrightarrow{1-1}\varphi(\bar{v})} 1$$

is an SOL-polynomial.

3.4 Standard vs FF vs Newton SOL-polynomials

We have introduced three notions of SOL-polynomials: standard, FF, and Newton SOL-polynomials. The sets of monomials $\mathsf{x}^i : i \in \mathbb{N}$ (powers of x) and $\mathsf{x}_{(i)} : i \in \mathbb{N}$ (falling factorials of x) each form a basis of the polynomial ring $\mathbb{Z}[\mathsf{x}]$. The set of monomials $\binom{\mathsf{x}}{i} : i \in \mathbb{N}$ (binomials of x) forms a basis in the polynomial ring $\mathbb{Q}[\mathsf{x}]$. Over $\mathbb{Q}$ each of these bases can be transformed into the other using linear transformations. In this section we discuss transformations of one basis into another using substitution by SOL-definable polynomials.

3.4.1 Standard vs FF polynomials

In the statement and proof of Proposition 3.2.5, the polynomial obtained is of the form

$$\sum_{j=1}^{|M|} d_{\varphi}(\mathfrak{M}, j) \cdot k_{(j)}.$$

In the literature on graph polynomials one can actually find graph polynomials in presentations that mix monomials of the form x^i with monomials of the form $\mathsf{x}_{(i)}$, e.g. the cover polynomial for directed graphs [41].

We extend the definition of SOL-polynomials by allowing both monomials of the form

$$r^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$$

and

$$r_{(\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v})))}$$

We call the polynomials obtained like this *extended* SOL-*polynomials.*

In the following we show that every extended SOL-polynomial on ordered structures can be written both as a standard SOL-polynomial and as an FF SOL-polynomial.

Proposition 3.4.1. *Let* $\Phi(\mathfrak{M}; \bar{\mathsf{x}}) = r^{\mathrm{card}_{\mathfrak{M},(v_1,\ldots,v_y)}(\phi(\bar{v}))}$ *be a standard* SOL-*monomial with* $r \in \mathbf{X} \cup (\mathbb{Z} - \{0\})$. *There is an* FF SOL-*polynomial* Φ' *such that for all structures* $\mathfrak{M}$ *we have*

$$\Phi(\mathfrak{M}; \bar{\mathsf{x}}) = \Phi'(\mathfrak{M}; \bar{\mathsf{x}}).$$

Proof. First assume r is a positive integer. The monomial $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ counts functions from the set

$$D_{\phi} = \{\bar{a} \mid \mathfrak{M} \models \phi(\bar{a})\}$$

to $[r]$. On the other hand, the monomial $(r)_{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$ counts *injective* functions from D_{ϕ} to $[r]$. Let $\Phi'(\mathfrak{M}; \bar{\mathsf{x}})$ be given by

$$\Phi'(\mathfrak{M}; \bar{\mathsf{x}}) = \sum_{A \subseteq V^y : \langle \mathfrak{M}, A \rangle \models \psi_1} \ \sum_{R \subseteq V^y : \langle \mathfrak{M}, A, R \rangle \models \psi_2} (r)_{\mathrm{card}_{\mathfrak{M},\bar{v}}(A(\bar{v}))}$$

where

(i) ψ_1 requires that A is a subset of D_ϕ, and

(ii) ψ_2 requires that R is an equivalence relation over D_ϕ such that for every two distinct tuples $\bar{a} \in A$ and $\bar{b} \in D_\phi$, if $\bar{a}$ and $\bar{b}$ belong to the same equivalence class in R, then $\bar{a} < \bar{b}$ with respect to the order on the structure $\mathfrak{M}$. Moreover, for every equivalence class in R there exists some $\bar{a} \in A$ which belongs to it.

Taking any injective function f from A to $[r]$, we may extend it to a function from D_ϕ to $[r]$ by assigning every $\bar{b} \in D_\phi - A$ with the same value as the $\bar{a} \in A$ for which $(\bar{a}, \bar{b}) \in R$. This extension is determined uniquely by f and R, and forms a bijection between the set of functions $g : D_\phi \to [r]$ and the set of triples (A, R, f) such that A and R satisfy ψ_1 and ψ_2 and $f : A \to [r]$ is injective. Hence, $\Phi'(\mathfrak{M}; \bar{\mathsf{x}}) = \Phi(\mathfrak{M}; \bar{\mathsf{x}})$. If $r \in \mathbf{X}$, we get that $\Phi'(\mathfrak{M}; \bar{\mathsf{x}})$ and $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ agree on every evaluation of r to a positive integer, and thus, by interpolation, $\Phi'(\mathfrak{M}; \bar{\mathsf{x}}) = \Phi(\mathfrak{M}; \bar{\mathsf{x}})$. Therefore, in particular $\Phi'(\mathfrak{M}; \bar{\mathsf{x}})$ and $\Phi(\mathfrak{M}; \bar{\mathsf{x}})$ also agree on all negative evaluations of r. □

Proposition 3.4.2. *Let* $\Phi(\mathfrak{M}; \bar{\mathsf{x}}) = r_{(\mathrm{card}_{\mathfrak{M},(v_1,\dots,v_y)}(\phi(\bar{v})))}$ *be an* FF SOL-*monomial for* $r \in \mathbf{X} \cup (\mathbb{Z} - \{0\})$. *There is a standard* SOL-*polynomial* Φ' *such that for all structures* $\mathfrak{M}$ *we have*

$$\Phi(\mathfrak{M}; \bar{\mathsf{x}}) = \Phi'(\mathfrak{M}; \bar{\mathsf{x}}).$$

Proof. By definition,

$$(r)_{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))} = \prod_{i=0}^{|D_\phi|-1} (r - i) \ .$$

where D_ϕ is as in Proposition 3.4.1 Therefore,

$$(r)_{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))} = \prod_{\bar{a}:\phi(\bar{a})} \left(r - \sum_{\bar{b}:\phi(\bar{b}) \wedge \bar{b} < \bar{a}} 1 \right) ,$$

where $\bar{b} < \bar{a}$ means $\bar{b}$ is smaller than $\bar{a}$ in the lexicographic order induced by the order on the elements of the structure $\mathfrak{M}$. By Proposition 3.3.9 we need only show that

$$r - \sum_{\bar{b}:\phi(\bar{b})\wedge\bar{b}<\bar{a}} 1 \tag{3.11}$$

is a standard SOL-polynomial with summation on elements only. The expression in (3.11) is given by

$$\sum_{\bar{b}:\phi(\bar{b})\wedge(\bar{b}\leq\bar{a})} r^{\mathrm{card}_{\mathfrak{M},\bar{w}}(\bar{w}=\bar{a}\wedge\bar{b}=\bar{a})} \cdot (-1)^{\mathrm{card}_{\mathfrak{M},\bar{w}}(\bar{w}=\bar{a}\wedge\bar{b}\neq\bar{a})}.$$

Hence, $(r)_{\mathrm{card}_{\mathfrak{M},\bar{v}}(\phi(\bar{v}))}$ is a standard SOL-polynomial. $\square$

3.4.2 Newton polynomials

In Proposition 3.2.3 we used monomials of the form $\binom{\mathsf{x}}{i}$. However, writing these as standard polynomials, they have rational but not integer coefficients; hence they are polynomials in $\mathbb{Q}[\mathsf{x}]$, but not in $\mathbb{Z}[\mathsf{x}]$. On the other hand, the coefficients of standard and FF SOL-polynomials are always integers by definition. Therefore, $\binom{\mathsf{x}}{i}$ cannot be written as a standard or FF SOL-polynomial.

On the other hand, FF SOL-monomials can be written as Newton SOL-polynomials. To see this we note

$$\mathsf{x}_{(|A|)} = |A|! \cdot \binom{\mathsf{x}}{|A|} = \sum_{R\subseteq A^2} \binom{\mathsf{x}}{|A|}$$

where R ranges over all permutations of A (as binary relations over A).

3.5 Equivalence of counting φ-colorings and SOL-polynomials

In this section we prove Theorem B, in two versions: Theorems 3.5.1 and 3.5.2 give one version and Theorem 3.5.3 gives another version.

Theorems 3.5.1 and 3.5.2 show that the class of the counting functions of SOL-definable $\varphi(\bar{F})$-multi-colorings with several partial functions $\bar{F}$ and the classes of standard and FF SOL-polynomials coincide.

Theorem 3.5.1. *Let $\mathcal{P}_\varphi(\bar{\mathbf{F}})$ be an* SOL*-definable multi-coloring property. The graph polynomial $\chi_{\varphi(\bar{\mathbf{F}})}(\mathfrak{M}; \bar{\mathsf{k}})$ is both a standard and an* FF SOL*-polynomial.*

The proof of Theorem 3.5.1 is given in Subsection 3.5.1

Theorem 3.5.2 states that every standard or FF SOL-polynomial is an evaluation of some $\chi_\varphi(\mathfrak{M}, \bar{k})$.

Theorem 3.5.2. *Let $P(\mathfrak{M}; \mathsf{k}_1, \ldots, \mathsf{k}_m)$ be either a standard or an* FF SOL*-polynomial. There exists an* SOL*-definable multi-coloring property $\mathcal{P}_\varphi$ with $m + l$ color-sets, $[k_1], \ldots, [k_{m+l}]$, and $a_1, \ldots, a_l \in \mathbb{Z}$ such that*

$$\chi_\varphi(\mathfrak{M}; \mathsf{k}_1, \ldots, \mathsf{k}_m, a_1, \ldots, a_l) = P(\mathfrak{M}; \mathsf{k}_1, \ldots, \mathsf{k}_m)$$

where $\chi_\varphi(\mathfrak{M}; \mathsf{k}_1, \ldots, \mathsf{k}_m, a_1, \ldots, a_l)$ is obtained by evaluating the indeterminates $\mathsf{k}_{m+1}, \ldots, \mathsf{k}_{m+l}$ to $a_1, \ldots, a_l$ respectively in $\chi_\varphi(\mathfrak{M}; \mathsf{k}_1, \ldots, \mathsf{k}_{m+l})$.

The proof of Theorem 3.5.2 is given in Subsection 3.5.2

Theorem 3.5.3 shows that the class of counting functions of SOL-definable $\varphi(\bar{R})$-multi-colorings with bounded relations $\bar{R}$ and the class of Newton SOL-polynomials coincide.

Theorem 3.5.3. *Let P be a function from the class of finite τ-structures to the ring $\mathbb{Q}[\bar{\mathsf{x}}]$. The following statements are equivalent:*

(i) P is a Newton SOL*-polynomial.*

(ii) P is an evaluation of the counting function $\chi_{\varphi(\bar{R})}(\mathfrak{M}; \bar{k})$ of an SOL*-definable multi-coloring where the relations in $\bar{R}$ are bounded.*

In Subsection 3.5.3 we motivate the need to extend partial functions to bounded relations in order to capture the Newton SOL-polynomials and sketch this direction of the proof. For the other direction of Theorem 3.5.3, the proof of Theorem 3.5.1 given in Subsection 3.5.1 can be augmented by using Proposition 3.2.3 instead of Proposition 3.2.5.

3.5.1 Proof of Theorem 3.5.1

We prove the theorem in the case of φ-colorings. The case of multiple indeterminates and several simultaneous functions is similar. Let $\mathcal{P}_\varphi$ be an

SOL-definable coloring property. From Proposition 3.2.5 we know that for every $\mathfrak{M}$ the number of elements given by $\chi_{\varphi(\mathbf{F})}(\mathfrak{M};k)$ is a polynomial in k of the form

$$\sum_{j=0}^{d\cdot|M|^m} d_\varphi(\mathfrak{M},j)\cdot k_{(j)}$$

where $d_\varphi(\mathfrak{M},j)$ is the number of φ-colorings F using all the colors in $[j]$, up to permutation of the color-set. In other words, if $[k]$ was ordered, $d_\varphi(\mathfrak{M},j)$ would count the number of φ-colorings F with a fixed set of j colors which are minimal lexicographically among φ-colorings F' obtained from F by permuting the color-set. The total number of colors used is bounded by $N = d\cdot \mid M \mid^m$. Hence we can interpret the set of colors used inside $\mathfrak{M}$ by the set $[M]^{d\cdot m}$. Since $\mathfrak{M}$ has a linear order $\leq_M$, a lexicographic order on $[M]^{d\cdot m}$ is definable in SOL.

We replace F by a relation where each occurrence of a color is substituted by a $(d\cdot m)$-tuple, and call this new relation S. We also modify the formula φ to a formula ψ by adding the requirement that all the colors used by S form an initial segment and that S is the smallest in the lexicographic order induced on the colors among its permutations. Let us denote by I_S the initial segment of this lexicographic ordering of the colors used by S. Clearly I_S is definable in SOL by a formula ρ.

We have that $\chi_\varphi(\mathfrak{M},k)$ is an FF SOL-polynomial given by

$$\sum_{S:\psi(S)}\sum_{I_S:\rho} x_{(\mathrm{card}_{\mathfrak{M},\bar{v}}(I_S(\bar{v})))} \tag{3.12}$$

By Proposition 3.4.2, $\chi_\varphi(\mathfrak{M},k)$ is also a standard SOL-polynomial. $\square$

3.5.2 Proof of Theorem 3.5.2

We prove Theorem 3.5.2 by induction on the structure of the set of standard SOL-polynomials. Using Proposition 3.4.2, the case of FF SOL-polynomials follows.

Base: Let $\Psi(\mathfrak{M};\bar{\mathrm{x}})$ be a standard SOL-monomial, then by Lemma 3.3.3, $\Psi(\mathfrak{M};\bar{\mathrm{x}})$ is of the form

$$r_1^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_1(\bar{v}))}\cdots r_t^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi_t(\bar{v}))}$$

$\varphi_1, \ldots, \varphi_t \in$ SOL and $r_1, \ldots, r_t \in (\mathbb{Z} - \{0\}) \cup \mathbf{X}$. Without loss of generality, assume $r_1 = \mathsf{k}_1, \ldots, r_{t'} = \mathsf{k}_{t'} \in \mathbf{X}$ are indeterminates and $r_{t'+1} = c_{t'+1}, \ldots, r_t = c_t \in \mathbb{Z} - \{0\}$ are constants. Then

$$\mathsf{k}_i^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi(\bar{v}))} = \chi_{\psi(\mathbf{F})}(\mathfrak{M}; \mathsf{k}_i)\,,$$

where $\psi(\mathbf{F})$ counts partial functions $F \subseteq M^m \times [k_i]$ whose domain is the set of m-tuples $\bar{a}$ which satisfy φ. Similarly, $c_i^{\mathrm{card}_{\mathfrak{M},\bar{v}}(\varphi(\bar{v}))}$ is obtained by evaluating k to c_i in $\chi_{\psi(\mathbf{F})}(\mathfrak{M}; \mathsf{k}_i)$. By Proposition 3.2.17, the set of counting functions for φ-multi-colorings is closed under finite product.

Closure: Let $P(\mathfrak{M}; \bar{\mathsf{k}})$ be a standard SOL-polynomial,

$$P(\mathfrak{M}; \bar{\mathsf{k}}) = \sum_{R : \langle \mathfrak{M}, R \rangle \models \phi(R)} t(\langle \mathfrak{M}, R \rangle\,; \bar{\mathsf{k}})$$

where R is a second order variable of arity ρ_R and $t(\langle \mathfrak{M}, R \rangle\,; \bar{\mathsf{k}})$ is an SOL-polynomial. The induction hypothesis says that $t(\langle \mathfrak{M}, R \rangle\,; \bar{\mathsf{k}})$ is the evaluation of some counting function of φ-multi-colorings:

$$t(\langle \mathfrak{M}, R \rangle\,; \bar{\mathsf{k}}) = \chi_{\theta(\bar{\mathbf{F}}')}(\langle \mathfrak{M}, R \rangle, \mathsf{k}_1, \ldots, \mathsf{k}_m, \bar{a})$$

Let

$$\chi_{\theta(\mathbf{F}, \bar{\mathbf{F}}') \wedge \exists R\, \varphi_{domain}(R, \mathbf{F}) \wedge \phi(R)}(\mathfrak{M}; \bar{\mathsf{k}}, \mathsf{k}')$$

be the counting function of tuples $(F, \bar{F}')$ of partial functions such that $\theta(\bar{F}')$ holds, $F : R \to [\mathsf{k}']$, $R \subseteq M^{\rho_R}$, $\varphi_{domain}(R, F)$ says "the domain of F is equal to R" and $\phi(R)$ holds. $[k']$ is a new color-set. Then we have

$$P(\mathfrak{M}, \bar{k}) = \chi_{\theta(\mathbf{F}, \bar{\mathbf{F}}') \wedge \exists R\, \varphi_{domain}(R, \mathbf{F}) \wedge \phi(R)}(\mathfrak{M}; \mathsf{k}_1, \ldots, \mathsf{k}_m, \bar{a}, 1)\,.$$

$\square$

3.5.3 From Newton SOL-polynomials to counting bounded relations

Now we prove direction (i) $\rightarrow$ (ii) of Theorem 3.5.3. Note that in this example the coloring relations cannot be replaced by partial functions.

Let $N(\mathfrak{M}; \bar{\mathsf{x}})$ be a Newton SOL-polynomial. By Lemmas 3.3.4 and 3.3.5, $N(\mathfrak{M}; \bar{\mathsf{x}})$ is a substitution of the following polynomial:

$$N'(\mathfrak{M}; \bar{\mathsf{x}}) = \sum_{R_1} \cdots \sum_{R_s} \sum_{A_1 \subseteq M} \cdots \sum_{A_{t-1} \subseteq M} \sum_{A_t \subseteq M : \phi(\bar{R}, \bar{A})} \binom{\mathsf{x}_1}{|A_1|} \cdots \binom{\mathsf{x}_t}{|A_t|} \tag{3.13}$$

where $R_1, \ldots, R_s, A_1, \ldots, A_t$ are second order variables and ϕ is a SOL-formula.

We will show how to transform $N'(\mathfrak{M}; \bar{\mathsf{x}})$ into a counting function of multi-colorings.

The term $\binom{k}{|A_i|}$ counts the number of ways to choose a set of colors of size $|A_i|$ from $[k_i]$. Equivalently, $\binom{k_i}{|A_i|}$ counts relations $S_i \subseteq A_i \times [k_i]$ such that there exists $I \subseteq [k_i]$ for which $S_i = A_i \times I$ and $|I| = |A_i|$. This can be expressed in SOL by a formula $\varphi_{choose,i}(A_i, S_i)$.

Hence, we have

$$N'(\mathfrak{M}; \bar{\mathsf{x}}) = \chi_{\theta(\bar{\mathbf{R}}, \bar{\mathbf{A}}, \bar{\mathbf{S}})}(\mathfrak{M}, k)$$

where $\theta = \phi(\bar{R}, \bar{A}) \wedge \bigwedge_{i=1}^{t} \varphi_{choose,i}(A_i, S_i)$.

Each of $S_1, \ldots, S_t$ is 1-bounded, and each of $R_1, \ldots, R_s, A_1, \ldots, A_t$ is 0-bounded, so the multi-coloring property P_θ is bounded. However, the S_i are not partial functions. $\square$

Chapter 4

MSOL-Polynomials

In this chapter we introduce the class of MSOL-polynomials, which contains most of the graph polynomials which were studied in the literature. In the next chapter the logic MSOL will be used for an extension of graphs, namely k-graphs, which we define here. We also discuss here a notion of MSOL-definable generalized colorings.

4.1 Graphs as structures for MSOL

Monadic Second Order Logic MSOL(τ) is the restriction of SOL(τ) which only allows second order variables to be unary. For graphs which are represented as τ_1-structures, this means that set variables can only range over sets of vertices (and not e.g. over sets of edges or other relations of arity more than one). We write MSOL_1 for MSOL(τ_1) on structures of τ_1.

However, in this chapter we are frequently interested in another representation of graphs which allows quantification over sets of edges. We write MSOL_2 to refer to monadic second order logic on structures of the following form:

Definition 4.1.1. *An* ordered graph *is a finite structure of the form:*

$$\langle D, V, E, R_{inc}, \leq \rangle$$

where V and E are unary relations which form a partition of D, $R_{inc} \subseteq V \times E \times V$ is the incidence relation of the graph, and $\leq$ is a linear *ordering of V. This allows us to have directed graphs with multiple edges as well.*

Let τ_2 be the vocabulary of such structures, $\tau_2 = \langle \mathbf{V}, \mathbf{E}, \mathbf{R_{inc}}, \mathbf{R_{\leq}} \rangle$.

In this chapter MSOL_2 *will refer to* $\mathrm{MSOL}(\tau_2)$.

One can also think of of τ_2-structures as two-sorted structures, with sorts V and E.

It is not hard to see that

Lemma 4.1.1. *If φ is expressible in* MSOL_1*, then φ is also expressible in* MSOL_2.

However, MSOL_2 is strictly more expressive than MSOL_1; e.g., existence of a perfect matching is MSOL_2-definable, but not MSOL_1-definable. In contrast to that, the expressive power of $\mathrm{SOL}(\tau_1)$ is the same as that of $\mathrm{SOL}(\tau_2)$. SOL is strictly more expressive on graphs than MSOL; e.g., the class of graphs of prime size is not MSOL-definable, but is SOL-definable. For details see [43].

All of the formulas in Examples 3.1.1 and 3.1.2 are actually expressible in MSOL_2.

Remark 4.1.2. There is a generalization $\mathrm{CMSOL}(\tau)$ of $\mathrm{MSOL}(\tau)$ which allows the usage of modular counting quantifiers $C_{a,b}$. The formula $C_{a,b}\bar{x}\,\varphi(\bar{x})$ requires that the number of tuples of elements which satisfy φ equals a modulo b. However, these quantifiers are definable in MSOL when an order is given, as is the case for us. In Example 4.1.6(i) we saw the MSOL-formula $\varphi_{1,2}(U)$, which says that $|U|$ equals 1 modulo 2. Hence, if $\bar{x}$ consists only of one variable x, we have that $C_{a,b}x\,\psi(x)$ is definable by

$$\exists U \left[\forall x \left(U(x) \leftrightarrow \psi(x)\right) \wedge \varphi_{a,b}(U)\right] .$$

$C_{a,b}\,\bar{x}\psi(\bar{\mathrm{x}})$ is MSOL-definable even if $\bar{x} = (x_1, \ldots, x_s)$ is a tuple consisting of more than one variable. For example, $C_{1,2}x_1, x_2\,\psi(x_1, x_2)$ is definable by

$$\exists U_1 \forall x_1 \left[x_1 \in U_1 \leftrightarrow \left(\exists U_2 \left(\forall x_2 \left(U_2(x_2) \leftrightarrow \psi(x_1, x_2)\right) \wedge \varphi_{1,2}(U_2)\right)\right) \wedge \varphi_{1,2}(U_1)\right] .$$

This formula says that the number of elements a_1 such that

$$|\{a_2 \mid \psi(a_1, a_2) \text{ holds}\}| \equiv 1 \pmod 2$$

is odd. For more detail see [42].

4.1.1 MSOL-Polynomials

We get MSOL-polynomials from standard SOL-polynomials by restricting Definitions 3.3.1 and 3.3.4.

Definition 4.1.2 (MSOL-monomials and MSOL-polynomials).

(i) *For any* τ, $\mathrm{MSOL}(\tau)$*-monomials are finite products of terms of the form*

$$r^{\mathrm{card}_{\mathfrak{M},v}(\phi(v))}$$

where $\phi \in \mathrm{MSOL}(\tau)$ *and* $r \in \mathbf{X} \cup (\mathbb{Z} - \{0\})$. *Note* v *is a single variable, as opposed to* $\bar{v}$ *in Definition 3.3.1.*

(ii) *For any* τ, $\mathrm{MSOL}(\tau)$*-polynomials are given as in Definition 3.3.4, with the following restrictions:*

(ii.a) *the monomial used in Definition 3.3.4(i) is an* $\mathrm{MSOL}(\tau)$*-monomial,*

(ii.b) *all of the iteration formulas in Definition 3.3.4(ii) are* $\mathrm{MSOL}(\tau)$*-formulas, and*

(ii.c) *all of the relation symbols in Definition 3.3.4(ii) are of arity* 1.

(iii) MSOL_i*-polynomials are* $\mathrm{MSOL}(\tau_i)$*-polynomials for* $i = 1, 2$.

We omit the word *standard* when discussing MSOL-polynomials since we are not interested in FF or Newton MSOL-polynomials.

The following normal form follows from the proofs of Lemma 3.3.3 and Lemma 3.3.5.

Lemma 4.1.3.

Let $P(\mathfrak{M})$ *be an* $\mathrm{MSOL}(\tau)$*-polynomial. Then there exist* $s, t \in \mathbb{N}$, MSOL*-formulas* $\Phi(U_1, \ldots, U_s)$ *and* $\phi_1(x), \ldots, \phi_t(x)$, *and* $r_1, \ldots, r_t \in (\mathbb{Z} - 0) \cup \mathbf{X}$ *such that* $P(\mathfrak{M})$ *equals in the form*

$$\sum_{U_1} \cdots \sum_{U_{s-1}} \sum_{U_s : \Phi(U_1, \ldots, U_s)} r_1^{\mathrm{card}_{\mathfrak{N},x}(\phi_1(x))} \cdots r_t^{\mathrm{card}_{\mathfrak{N},x}(\phi_t(x))} \tag{4.1}$$

where $\mathfrak{N} = \langle \mathfrak{M}, U_1, \ldots, U_s \rangle$.

From Lemma 4.1.1 we get

Lemma 4.1.4. *If $P(G;\bar{\mathsf{x}})$ is an* MSOL_1*-polynomial, then it is also an* MSOL_2*-polynomial.*

Example 4.1.5 (Simple MSOL-polynomials).

(i) The polynomials $I(G;\mathsf{x})$ and $\Omega(G;\mathsf{x})$ defined in Chapter 2 are MSOL_1*-polynomials.*

(ii) The polynomial $\mathsf{x}^{k(G)}$, where $k(G)$ is the number of connected components of G, can be written as

$$\sum_{S\subseteq V: S(x)\leftrightarrow\varphi_{fcomp}(v,S)} \mathsf{x}^{|S|} \tag{4.2}$$

where $\varphi_{fcomp}(v,S)$ is an MSOL *formula expressing that "v is the first vertex in the fixed ordering of V of some connected component of the spanning subgraph (V,S) induced by S". We get that $\mathsf{x}^{k(G)}$ is an* MSOL_1*-polynomial. Note that there us exactly one relation which satisfies the summation formula of Equation (4.2).*

Example 4.1.6 (The characteristic polynomial). *To see that the characteristic polynomial* $\mathrm{char}(G;\mathsf{x})$ *is of the required form, recall from Section 2.1.3 that*

$$\mathrm{char}(G;\mathsf{x}) = \sum_{\substack{V_H\subseteq V(G), E_H\subseteq V(G):\\ \varphi_{elem}(V_H,E_H)}} (-1)^{|V_H|+k(H)}\cdot 2^{\mathsf{c}(H)}\cdot \mathsf{x}^{|V(G)-V_H|}$$

where $H=(V_H,E_H)$ and $\varphi_{elem}(V_H,E_H)$ says the graph $H=(V_H,E_H)$ is an elementary subgraph. $\mathrm{char}(G;\mathsf{x})$ *is an evaluation of*

$$\widehat{\mathrm{char}}(G;\mathsf{w},\mathsf{x},\mathsf{y},\mathsf{z}) = \sum_{\substack{V_H\subseteq V(G), E_H\subseteq V(G):\\ \varphi_{elem}(V_H,E_H)}} \left(\mathsf{w}^{|V_H|}\cdot \mathsf{y}^{k(H)}\cdot \mathsf{z}^{\mathsf{c}(H)}\cdot \mathsf{x}^{|V(G)\setminus V_H|}\right)$$

at $\mathsf{w}=-1, \mathsf{y}=-1, \mathsf{z}=2$. $\mathrm{char}(G;\mathsf{x})$ *is an* MSOL_2*-polynomial since "being an elementary subgraph" is expressible in* MSOL_2*:*

$$\varphi_{elem}(V_H,E_H) = \forall x\big(V_H(x)\rightarrow(\varphi_{val0}(V_H,E_H,x)\vee\varphi_{val2}(V_H,E_H,x))\big)$$

where $\varphi_{val\,i}(V_H, E_H, x)$ says that x is of degree i in the subgraph (V_H, E_H). We also use that $V(G)\backslash V_H$ is MSOL_2-definable.

4.2 k-graphs and binary (k-)graph operations

In this chapter we will be interested in a slight generalization of graphs, namely *ordered k-graphs.*

Definition 4.2.1 (Ordered k-graphs).
An ordered k-graph *is a tuple $(G, a_1, \ldots, a_k)$ where G is an ordered graph and $a_1, \ldots, a_k$ are vertices of G.*
If G is a τ_i-structure for $i = 1, 2$, then $(G, a_1, \ldots, a_k)$ is a $\tau_{i,k}$-structure. $\tau_{i,k}$ is obtained from τ_i by adding k constant symbols.

We define the following useful graph operations which are based on the disjoint union $G \sqcup H$ of G and H.

Definition 4.2.2 (k-sum and join).

(i) Let G and H be k-graphs. The k-sum $G \sqcup_k H$ of G and H *is obtained from $G \sqcup H$ by pairwise identifying the vertices labeled a_i in G and H (and, in the case of τ_1, ignoring multiple edges). The resulting graph has $|V(G)| + |V(H)| - k$ vertices.*

(ii) Let G and H be graphs. The join $G \bowtie H$ of G and H *is obtained from $G \sqcup H$ by adding an edge between every $u \in V(G)$ and $v \in V(H)$.*

Notice $\sqcup_0$ is in fact the disjoint union.

4.3 A Feferman-Vaught type theorem

MSOL-polynomials are useful because they have strong decomposition properties. We give such a theorem due to Makowsky [115], which continues a line of research stemming from a theorem of S. Feferman and R. Vaught [58], For details see [115].

Let $\odot$ be a binary operation on τ-structures. For every τ-structure $\mathfrak{A}$ we denote by $Th^q(\mathfrak{A})$ the set of MSOL-formulas of quantifier rank at most q which hold on $\mathfrak{A}$.

Definition 4.3.1 ($\odot$-smooth). *We say $\odot$ is* MSOL(τ)-smooth *if for every $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1, \mathfrak{B}_2$ such that $Th^q(\mathfrak{A}_1) = Th^q(\mathfrak{A}_2)$ and $Th^q(\mathfrak{B}_1) = Th^q(\mathfrak{B}_2)$, we have $Th^q(\mathfrak{A}_1 \odot \mathfrak{B}_1) = Th^q(\mathfrak{A}_2 \odot \mathfrak{B}_2)$.*

Theorem 4.3.1.

(i) The disjoint union $\sqcup$ is smooth for τ_1-graphs and τ_2-graphs and for their respective k-graphs.

(ii) k-sum is smooth for k-graphs (again both for $\tau_{1,k}$ and $\tau_{2,k}$).

(iii) The join $\bowtie$ is smooth for τ_1. This remains true when extended to ordered $\tau_{1,k}$-graphs. However, $\bowtie$ is not smooth for τ_2.

Theorem 4.3.2 ([115]). *Let p be an* MSOL(τ)*-polynomial with indeterminates $\bar{\mathsf{x}}$. Let $\odot$ be a* MSOL(τ)*-smooth binary operation. Then there exist* MSOL(τ)*-polynomials $p_1, \ldots, p_\gamma$ and a polynomial $g(\mathsf{y}_1, \ldots, \mathsf{y}_\gamma, \mathsf{z}_1, \ldots, \mathsf{z}_\gamma)$ such that*

$$p(\mathfrak{M}_1 \odot \mathfrak{M}_2; \bar{\mathsf{x}}) = g(p_1(\mathfrak{M}_1; \bar{\mathsf{x}}), \ldots, p_\gamma(\mathfrak{M}_1; \bar{\mathsf{x}}), p_1(\mathfrak{M}_2; \bar{\mathsf{x}}), \ldots, p_\gamma(\mathfrak{M}_2; \bar{\mathsf{x}}))\,.$$

$g(\bar{\mathsf{y}}, \bar{\mathsf{z}})$ is a polynomial in the indeterminates $\bar{\mathsf{y}}$ and $\bar{\mathsf{z}}$ with coefficients in $\mathbb{Z}[\bar{\mathsf{x}}]$.

One can furthermore assume that g is linear in the indeterminates in $\bar{\mathsf{y}}$ and in $\bar{\mathsf{z}}$ by choosing the polynomials $p_1, \ldots, p_\gamma$ appropriately.

Remark 4.3.3. There are infinitely many MSOL-smooth graph operations. Any operation which can be expressed as a *quantifier-free non-vectorized transduction* of the rich disjoint union[1] of the input graphs is MSOL-smooth.

4.4 Generalized colorings and MSOL-polynomials

It is natural to restrict multi-colorings to MSOL and ask whether Theorems equivalent to 3.5.1 and 3.5.2 hold:

Definition 4.4.1.
A φ-multi-coloring is an MSOL_i*-multi-coloring if $\varphi \in \mathrm{MSOL}_i(\tau_{\bar{\mathbf{F}}})$.*

[1]The rich disjoint union of two graphs G and H is the disjoint union $G \sqcup H$ together with two unary relations V_G and V_H which distinguish the vertices of G and H.

Inspecting the proof of Theorem 3.5.2 one can verify the following:

Proposition 4.4.1. *Every* MSOL_i*-polynomial* $P(\mathfrak{M})$ *is an evaluation of some* MSOL_i*-multi-coloring* $\chi_\varphi(\mathfrak{M}, \bar{k})$.

The converse is, unfortunately, not true.

Example 4.4.2 (Harmonious colorings revisited). *It is not hard to see that being an harmonious coloring is definable by an* MSOL_1*-formula* φ_{harm}*,*

$$\varphi_{harm} = \forall x \forall y \big((\mathbf{E}(x, y) \wedge \neg(x \approx y)) \rightarrow \neg(\mathbf{F}(x) \approx \mathbf{F}(y)) \big) ,$$

and yet we will see that $\chi_{harm}(G; \mathsf{k})$ *is not an* MSOL*-polynomial.*

Chapter 5

Connection Matrices

In this chapter we define exactly the notion of connection matrices and prove Theorem C, which says that MSOL-polynomials and their evaluations and coefficients have connection matrices of finite rank. We give many examples of graph parameters which are not evaluations or coefficients of MSOL-polynomials. We give examples of natural graph polynomials which are not MSOL-polynomials.

5.1 Connection matrices

Let $\mathbb{F}$ be a field, usually $\mathbb{Q}$ or a field of rational functions $\mathbb{Q}(\bar{\mathrm{x}})$. Let τ be a vocabulary. Let $\mathrm{Struct}(\tau)$ be the class of finite structures with vocabulary τ. We are mostly interested in graphs and k-graphs. A function $f : \mathrm{Struct}(\tau) \to \mathbb{F}$ is said to be τ-invariant if isomorphic τ-structures receive the same value under f.

5.1.1 Graph invariants

Definition 5.1.1. *Let $f : \mathrm{Struct}(\tau) \to \mathbb{F}$ be τ-invariant and let $\odot$ be a binary τ-operation (such as the disjoint union $\sqcup$, the join $\bowtie$ for τ_1-graphs or k-sum $\sqcup_k$ for k-graphs).*

(i) f is $\odot$-additive *if for every two* $\mathrm{Struct}(\tau)$*-structures $\mathfrak{M}$ and $\mathfrak{N}$,*

$$f(\mathfrak{M} \odot \mathfrak{N}) = f(\mathfrak{M}) + f(\mathfrak{N})\,.$$

(ii) f *is* $\odot$-multiplicative *if for every two* Struct(τ)*-structures* $\mathfrak{M}$ *and* $\mathfrak{N}$*,*

$$f(\mathfrak{M} \odot \mathfrak{N}) = f(\mathfrak{M}) \cdot f(\mathfrak{N}) \,.$$

Let $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \ldots$ *be an infinite sequence of non-isomorphic* Struct(τ)*-structures.*

(iii) f *is* $\odot$-maximizing with respect to $\{\mathfrak{N}_i\}_{i=1}^{\infty}$ *if for every two* Struct(τ)*-structures* $\mathfrak{N}_i$ *and* $\mathfrak{N}_j$*,*

$$f(\mathfrak{N}_i \odot \mathfrak{N}_j) = \max\{f(\mathfrak{N}_i), f(\mathfrak{N}_j)\} \,.$$

(iv) f *is* $\odot$-minimizing with respect to $\{\mathfrak{N}_i\}_{i=1}^{\infty}$ *if for every two* Struct(τ)*-structures* $\mathfrak{N}_i$ *and* $\mathfrak{N}_j$*,*

$$f(\mathfrak{N}_i \odot \mathfrak{N}_j) = \min\{f(\mathfrak{N}_i), f(\mathfrak{N}_j)\} \,.$$

If $\{\mathfrak{N}_i\}_{i=1}^{\infty}$ *is an enumeration of all* Struct(τ)*-structures up to isomorphism, then we simply write* $\odot$*-maximizing or* $\odot$*-minimizing.*

Example 5.1.1 (Disjoint union).
Some graph parameters and their behavior regarding $\sqcup$*:*

$\sqcup$*-additive*	$\lvert V(G)\rvert$, $\lvert E(G)\rvert$, *the number of connected components, the number of blocks, the independence number.*
$\sqcup$*-multiplicative*	*The number of spanning forests, the number of perfect matchings, the Tutte polynomial.*
$\sqcup$*-maximizing*	*the chromatic number* $\chi(G)$*, the maximal degree the tree-width, the clique number.*

$\sqcup$-*minimizing* *The girth, the minimal valency.*

5.1.2 Connection matrices of graph invariants

For every τ-invariant f and binary τ-operation $\odot$, one can define an infinite matrix as follows.

Definition 5.1.2 ($\odot$-connection matrix). *Let $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \ldots$ be an enumeration of all* $\mathrm{Struct}(\tau)$*-structures up to isomorphism. Let $\mathbb{F}$ be a field and let $f : \mathrm{Struct}(\tau) \to \mathbb{F}$ be a τ-invariant. The infinite matrix $M(f,\odot)$ with entries in $\mathbb{F}$ is defined as follows:*
For every $i, j \in \mathbb{N}$,

$$(M(f,\odot))_{i,j} = f(\mathfrak{M}_i \odot \mathfrak{M}_j)\,.$$

We denote the rank of $M(f,\odot)$ by $r(f,\odot)$.

We are mostly interested in the rank of $M(f,\odot)$ rather than in $M(f,\odot)$ itself.

Proposition 5.1.2. *Let f be a τ-invariant. Let $\odot$ be a binary τ-operation.*

(i) *If f is $\odot$-multiplicative, then $r(f,\odot) \le 1$.*

(ii) *If f is $\odot$-additive, then $r(f,\odot) \le 2$.*

(iii) *Let $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \ldots$ be a sequence of non-isomorphic τ-structures. If f is $\odot$-maximizing ($\odot$-minimizing) with respect to $\{\mathfrak{N}_i\}_{i=1}^{\infty}$ and is unbounded on $\{\mathfrak{N}_i\}_{i=1}^{\infty}$, then $r(f,\odot)$ is infinite.*

Proof.

(i) The i-th row of $M(f,\odot)$ is given by the infinite vector

$$(f(\mathfrak{M}_i \odot \mathfrak{M}_1), f(\mathfrak{M}_i \odot \mathfrak{M}_2), f(\mathfrak{M}_i \odot \mathfrak{M}_3), \ldots) =$$

$$f(\mathfrak{M}_i)(f(\mathfrak{M}_1), f(\mathfrak{M}_2), f(\mathfrak{M}_3), \ldots)$$

so $M(f,\odot)$ is spanned by the vector $(f(\mathfrak{M}_1), f(\mathfrak{M}_2), f(\mathfrak{M}_3), \ldots)$.

(ii) The i-th row of $M(f, \odot)$ is given by the infinite vector

$$(f(\mathfrak{M}_i) + f(\mathfrak{M}_1), f(\mathfrak{M}_i) + f(\mathfrak{M}_2), f(\mathfrak{M}_i) + f(\mathfrak{M}_3), \dots)$$

so $M(f, \odot)$ is spanned by two vectors, the vector $\mathbf{1}$ and the vector $(f(\mathfrak{M}_1), f(\mathfrak{M}_2), f(\mathfrak{M}_3), \dots)$.

(iii) Let $\mathfrak{M}'_1, \mathfrak{M}'_2, \mathfrak{M}_3, \dots$ be an infinite sub-sequence of $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \dots$ such that the sequence $f(\mathfrak{M}'_i)$ is non-zero and strictly monotone increasing. Then for every i, j, $\max\{f(\mathfrak{M}'_i), f(\mathfrak{M}'_j)\} = f(\mathfrak{M}'_{\max\{i,j\}})$. So, for every i, $M(f, \odot)$ contains the following matrix as a sub-matrix:

$$M_i = \begin{pmatrix} f(\mathfrak{M}'_1) & f(\mathfrak{M}'_2) & \dots & f(\mathfrak{M}'_i) \\ f(\mathfrak{M}'_2) & f(\mathfrak{M}'_2) & \dots & f(\mathfrak{M}'_i) \\ \vdots & & & \vdots \\ f(\mathfrak{M}'_i) & f(\mathfrak{M}'_i) & \dots & f(\mathfrak{M}'_i) \end{pmatrix}$$

and by subtracting the first line from each of the rest we get a matrix which is non-zero exactly at the indices marked with $*$:

$$\begin{pmatrix} * & * & * & \dots & * & * \\ * & & & & & \\ * & * & & & & \\ \vdots & & \ddots & & & \\ * & * & * & \dots & * & \end{pmatrix}$$

M_i has rank i.

Hence, $M(f, \odot)$ cannot have finite rank.

$\square$

We need to make precise what we mean when considering graph polynomials in the context of connection matrices:

Definition 5.1.3 ($\odot$-connection matrix of graph polynomials).
Let $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \ldots$ *be an enumeration of all* $\mathrm{Struct}(\tau)$*-structures up to isomorphism. Let* $\mathbb{F}_1$ *be a field and let* $f : \mathrm{Struct}(\tau) \to \mathbb{F}_1[\mathsf{x}_1, \ldots, \mathsf{x}_r]$ *be a graph polynomial. The connection matrix* $M(f, \odot)$ *of* f *is defined as in Definition 5.1.2 and where* $\mathbb{F} = \mathbb{F}_1(\mathsf{x}_1, \ldots, \mathsf{x}_r)$ *is the field of rational functions over* $\mathbb{F}_1$ *with indeterminates* $\mathsf{x}_1, \ldots, \mathsf{x}_r$*.*

The following lemma is useful:

Lemma 5.1.3.

(i) The $\odot$-connection matrix of a linear combination of τ-invariants with $\odot$-connection matrices of finite rank is of finite rank.

(ii) The $\odot$-connection matrix of a finite product of τ-invariants with $\odot$-connection matrices of finite rank is of finite rank.

(iii) If $P(G; \mathsf{x})$ is a graph polynomial with $\odot$-connection matrix of finite rank, then the $\odot$-connection matrix of the Laurent polynomial $P\left(G; \frac{1}{\mathsf{z}}\right)$ has finite rank (in the appropriate field of rational functions).

Proof.

(i) Follows from the sub-additivity of the rank of matrices.

(ii) It is enough to prove the claim for the product of two graph invariants, f and g. Denote the $\odot$-connection matrices of f and g by M and N respectively. Since M and N are of finite rank, there exists $t \in \mathbb{N}$ such that for every $i \in \mathbb{N}$, the row M_i is a linear combination of the rows $M_1, \ldots, M_t$ and N_i is a linear combination of the rows $N_1, \ldots, N_t$. Hence, for every $i \in \mathbb{N}$ there exist $c_1, \ldots, c_t, d_1, \ldots, d_t$ such that for every $j \in \mathbb{N}$,

$$M_{i,j} = \sum_{r \le t} c_r M_{r,j} \quad \text{and} \quad N_{i,j} = \sum_{s \le t} d_s N_{s,j} \,.$$

Hence,

$$M_{i,j} N_{i,j} = \sum_{r,s \le t} c_r d_s M_{r,j} N_{s,j} \,.$$

So, $M(f \cdot g, \odot)$ is spanned by t^2 rows.

(iii) Denote the $\odot$-connection matrices of $P(G; \mathsf{x})$ and $P\left(G; \frac{1}{\mathsf{z}}\right)$ by M and M' respectively. Since M has finite rank, for every $i \in \mathbb{N}$ there exist polynomials $f_1, \ldots, f_t$ such that for every $j \in \mathbb{N}$,

$$M_{i,j} = \sum_{r \leq t} f_1(\mathsf{x}) M_{r,j} \,.$$

Hence,

$$M'_{i,j} = \sum_{r \leq t} f_1 \left(\frac{1}{\mathsf{z}}\right) M'_{r,j}$$

and the claim follows since $f_1\left(\frac{1}{\mathsf{z}}\right)$ are rational functions as required.

□

5.1.3 Connection matrices of MSOL-polynomials

Here we give precise statement of Theorem C:

Theorem 5.1.4. *Let $P(G; \bar{\mathsf{x}})$ be an* MSOL(τ)*-polynomial with m indeterminates and let $\odot$ be τ-smooth. There exists $\gamma_\odot \in \mathbb{N}$ depending on the polynomial P and on $\odot$ only such that $r(P(G; \bar{\mathsf{x}}); \odot) \leq \gamma_\odot$.*

Proof. Since $P(G; \bar{\mathsf{x}})$ is an MSOL(τ)-polynomial, it follows from Theorem 4.3.2 that there exists a finite set $\{p_1, p_2, \ldots, p_\gamma\}$ of graph polynomials in $\bar{\mathsf{x}}$ and a matrix $N^\odot$ of size $\gamma \times \gamma$ over $\mathbb{Z}[\bar{\mathsf{x}}]$, such that for every two (k-)graphs G and H,

$$P(G \odot H; \bar{\mathsf{x}}) = \sum_{1 \leq i,j \leq \gamma} p_i(G; \bar{\mathsf{x}}) N^\odot_{ij} p_j(H; \bar{\mathsf{x}}) \,.$$

Hence, the rows of the connection matrix $M(f, \odot)$ can be spanned by a base of size γ. In other words, $P(G_1 \odot G_2; \bar{\mathsf{x}})$ is given by a quadratic form defined by $N^\odot_{i,j}$ of rank at most γ. The number $\gamma = \gamma_\odot$ depends $\odot$. □

Using Theorem 5.1.4, we prove that many graph invariants cannot be obtained as evaluations or fixed coefficients of any MSOL-polynomial.

Theorem 5.1.5. *Let $P(G; \mathsf{x}_1, \ldots, \mathsf{x}_\ell)$ be an* MSOL(τ)*-polynomial with m indeterminates and let $\odot$ be τ-smooth. Let $f : \mathcal{G} \to \mathbb{R}$ be a graph parameter such that $r(f, \odot) = \infty$.*

(i) There does not exist an evaluation $\bar{x}_0 \in \mathbb{R}^\ell$ of $\bar{\mathsf{x}}$ such that we have $P(G;\bar{x}_0) = f(G)$ for every G.

(ii) For every tuple of natural numbers $\bar{i} \in \mathbb{N}^\ell$ let $c_{\bar{i}}(G)$ be the coefficient in $P(G;\bar{\mathsf{x}})$ of the monomial $\mathsf{x}_1^{i_1} \cdots \mathsf{x}_\ell^{i_\ell}$. There does not exist $\bar{i}$ such that $c_i(G) = f(G)$ for every G.

Proof.

(i) This follows directly from Theorem 5.1.4.

(ii) This holds since every fixed coefficient is an evaluation of an MSOL(τ)-polynomial. We prove it for an MSOL-polynomial $P(G;\mathsf{x},\mathsf{y})$ with two indeterminates x,y, and two iteration formulas for summation. The general case is similar. Let

$$P(G;\mathsf{x},\mathsf{y}) = \sum_{U_1:\Phi_1(U_1)} \sum_{U_2:\Phi_2(U_2)} \mathsf{x}^{\mathrm{card}_{\langle G,U_1,U_2\rangle,\bar{v}}(\varphi_1(\bar{v}))}\, \mathsf{y}^{\mathrm{card}_{\langle G,U_1,U_2\rangle,\bar{v}}(\varphi_2(\bar{v}))}$$

where $\Phi_1, \Phi_2, \varphi_1, \varphi_2$ are MSOL formulas. Let $\mathsf{x}^a \cdot \mathsf{y}^b$ be a fixed monomial with coefficient $c_{a,b}$. Then we have

$$c_{a,b} = \sum_{U_1:\Phi_1(U_1)} \sum_{U_2:\Phi_2(U_2)\wedge\varphi_{cards:a,b}(U_1,U_2)} 1$$

where $\varphi_{cards:a,b}(U_1,U_2)$ says

$$\mathrm{card}_{\langle G,U_1,U_2\rangle,\bar{v}}\,(\varphi_1(\bar{v})) = a \quad\text{and}\quad \mathrm{card}_{\langle G,U_1,U_2\rangle,\bar{v}}\,(\varphi_2(\bar{v})) = b\,.$$

Note that $\varphi_{cards:a,b}$ is MSOL-definable since a and b are fixed. Hence, this is an evaluation of an MSOL-polynomial.

□

5.2 Applications to graph parameters

5.2.1 Disallowed graph parameters: $\sqcup$-maximization and $\sqcup$-minimization

Using Theorem 5.1.5, we can show many well-known graph parameters are neither an evaluation nor a fixed coefficient of any MSOL-polynomial. For example, many graph parameters are $\sqcup$-maximizing with respect to cliques $G_i = K_i$, including:

— *Spectral radius*: The largest absolute value of an eigenvalue of G.

— *Chromatic number*: The minimum k such that V can be partitioned into k independent sets.

— *Minimum Partition into forests*: The minimum k such that V can be partitioned into $V_1, \ldots, V_k$ such that each V_i induces a forest.

— *Acyclic chromatic number*: The minimum k such that V can be partitioned into $V_1, \ldots, V_k$ such that the union of any two parts $V_i \cup V_j$ induce a forest.

— *Arboricity*: The minimum k such that E can be partitioned into $E_1, \ldots, E_k$ such that each subgraph (V, E_i) is a forest.

— *Star chromatic number*: The minimum k such that V can be partitioned into $V_1, \ldots, V_k$ such that the union of any two parts $V_i \cup V_j$ induce a star forest.

— *Clique number*: The size of the maximal clique in G. [1]

— *Hadwiger number*: The maximal k such that G contains the clique K_k as a minor.

— *Hajós number*: The maximal k such that G contains the clique K_k as a topological minor.

— *Tree-width (path-width)*: The minimum width of a tree- (path-) decomposition of G.

— *Clique-width*: The minimum k such that G has a k-expression.

— *Degeneracy* : The minimum k such that every induced subgraph of G has a vertex of valency at most k.

For other graph parameters we use other sequences of $G_1, G_2, \ldots, G_i, \ldots$:

[1] Note that the independence number of G, i.e. the size of the largest independent set in G, is the clique number of the complement graph $\overline{G}$.

— *Edge chromatic number*: The minimum k such that E can be partitioned into k matchings: $G_i = S_i$ (star of size i).

— *Thue number*: The minimum number of colors for which the edges of G can be colored by a non-repetitive coloring [4]: $G_i = S_i$.

— *Maximum valency*: $G_i = S_i$.

— *Circumference*: The size of the largest cycle in G: $G_i = C_i$ (cycle of size i)

— *Longest path*: Length of the longest path in G: $G_i = P_i$ (path of length i).

— *Maximal connected planar induced subgraph*: The size of the maximal induced graph of G which is connected and planar: $G_i = P_i$.

— *Maximal connected bipartite induced subgraph*: The size of the maximal induced graph of G which is connected and bipartite: $G_i = P_i$.

— *Boxicity*: The boxicity of G is minimum dimension d of an intersection graph H_{int} of boxes which are parallel to the axis such that H_{int} is isomorphic to G. Take G_i to be an enumeration of all graphs.

Many other graph parameters are $\sqcup$-minimizing:

— *Minimal eigenvalue*: The minimal eigenvalue of A_G: $G_i = K_{i,i}$ (complete bipartite graphs).

— *Spectral gap*: The smallest non-zero eigenvalue of the Laplacian matrix of G: $G_i = K_i$.

— *Girth*: The size of the smallest cycle in G: $G_i = C_i$

— *Minimum valency*: $G_i = S_i$.

— *Size of smallest connected component (block)*: $G_i = K_i$.

Some graph invariants are not $\sqcup$-maximizing or $\sqcup$-minimizing but behave in a similar way.

Example 5.2.1 (The number of cliques of maximal size).
Let $a_{\text{Cliques}}(G)$ be the number of cliques of maximum size in G. First notice that $a_{\text{Cliques}}(G)$ is neither $\sqcup$-maximizing nor $\sqcup$-minimizing with respect to cliques since $a_{\text{Cliques}}(K_i \sqcup K_i) = 2$ while $a_{\text{Cliques}}(K_i) = 1$.

Let $(G_i)_{i \in \mathbb{N}}$ be the sequence of graphs $2iK_{2i}$ which consist of $2i$ cliques of size $2i$. Similarly, let $(H_i)_{i \in \mathbb{N}}$ be the sequence $(2i+1)K_{2i+1}$. Notice that the maximum clique in $G_i \sqcup H_j$ is of size $\max\{2i, 2j+1\}$. The number of maximum cliques in $G_i \sqcup H_j$ is then either $2i$ or $2j+1$ and

$$a_{\text{Cliques}}(G_i \sqcup H_j) = \max\{a_{\text{Cliques}}(G_i), a_{\text{Cliques}}(H_j)\} .$$

Hence, while $a_{\text{Cliques}}(G)$ is not $\sqcup$-maximizing, $M(a_{\text{Cliques}}, \sqcup)$ contains a sub-matrix with infinite rank.

One can find other similar examples, such as the number of vertices of maximal valency and minimal valency, respectively, and the number of connected components or blocks of maximal or minimal size.

Remark 5.2.2. (Degrees and non-fixed coefficients) Note some of the graph parameters above may still be encoded in MSOL-polynomials. E.g., The independence number is the degree of the independent set polynomial and the clique number is the degree of the clique polynomial.

5.2.2 Disallowed graph parameters: Averages

Maximization and minimization can sometimes be thought of as a type of mean. For example, the maximal (minimal) valency of a graph is obtained from the generalized mean

$$\left(\frac{\sum_{v \in V(G)} \text{valency}(v)^p}{|V(G)|} \right)^{\frac{1}{p}}$$

when p tends to ∞ ($-\infty$).

Other instances of the generalized mean also lead to infinite connection matrices. In particular, below we show examples for $p = 1$, $p = 2$ and $p = -1$.

Arithmetic mean
Let avg valency(G) denote the average valency:

$$\operatorname{avg\,valency}(G) = \frac{\sum_{v \in V(G)} \operatorname{valency}(v)}{|V(G)|}.$$

Then $r(\operatorname{avg\,valency}, \sqcup)$ is infinite.

To see this, look at the connection matrix

$$M(\operatorname{avg\,valency}, \sqcup)_{i,j} = 2\frac{|E_i| + |E_j|}{|V_i| + |V_j|}.$$

The sub-matrix of $M(\operatorname{avg\,valency}, \sqcup)$ which consists only of rows and columns corresponding to graphs with exactly 1 edge is the Cauchy matrix $(\frac{2}{i+j})_{i,j}$, hence $r(\operatorname{avg\,valency}, \sqcup)$ is infinite.

Other similar average quantities have infinite connection matrices, including:

— For every $i \in \mathbb{N}$, the average size of the i-th open neighborhood of vertices $v \in V(G)$, $|\{u \mid 0 < \operatorname{dist}(u, v) \leq i\}|$.

— Average number of simple cycles in which vertices $v \in V(G)$ occur.

— Average number of triangles in which vertices $v \in V(G)$ occur.

— Average size of connected component in which vertices $v \in V(G)$ occur.

We can extend this approach to averages over $e \in E(G)$:

— Average number of edges incident to an edge $e \in E(G)$.

— Average number of cycles which include $e \in E(G)$.

— For every $i \geq 2$, the average number of i-paths which include $e \in E(G)$.

Quadratic mean
The quadratic mean of the valencies of vertices $v \in V(G)$,

$$\operatorname{quad\,avg\,valency}(G) = \left(\frac{\sum_{v \in V(G)} \operatorname{valency}(v)^2}{|V(G)|}\right)^{\frac{1}{2}},$$

has infinite connection matrix rank with respect to $\sqcup$. To see this, notice first that $(\text{quad avg valency}(G))^2$ has infinite connection matrix rank with respect to $\sqcup$ by looking again at graphs with exactly one edge. Again, $M(\text{quad avg valency}, \sqcup)$ has entries $M_{i,j} = \frac{2}{i+j}$. Hence, quad avg valency(G) has infinite connection matrix rank.

Harmonic mean

The harmonic mean of the valencies of vertices $v \in V(G)$,

$$\text{harm avg valency}(G) = \frac{|V(G)|}{\sum_{v \in V(G)} \frac{1}{\text{valency}(v)}}$$

has infinite connection matrix rank with respect to $\bowtie$. To see this, note that $K_{n,m} = E_n \bowtie E_m$ and consider

$$\begin{aligned}\text{harm avg valency}(K_{n,m}) &= \frac{n+m}{\frac{n}{m} + \frac{m}{n}} \\ &= \frac{1}{n^2+m^2} \cdot nm(n+m)\end{aligned}$$

For every function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$, let $N(f)$ be the matrix such that the entry in row i and column j is $f(i,j)$. Assume $M(\text{harm avg valency}, \bowtie)$ is of finite rank. Then $N_1 = N\left(\frac{ij(i+j)}{i^2+j^2}\right)$ is of finite rank. Clearly, $N_2 = N\left(\frac{j(i+j)}{i^2+j^2}\right)$ is also of finite rank, since N_2 is obtained from N_1 by multiplying each row i of N_1 by a scalar $\frac{1}{i}$. The matrix $N\left(\frac{1}{j}\right)$ is of row rank 1 because all of its rows are equal. So, $N_3 = N\left(\frac{i+j}{i^2+j^2}\right)$ is of finite rank by Lemma 5.1.3(ii). The matrix $N(i+j)$ is of row rank 2, since it is spanned by the vectors $(1\ 2\ 3\ 4\ \cdots)$ and $\mathbf{1}$. So, $N_4 = N\left(\frac{(i+j)^2}{i^2+j^2}\right)$ is of finite rank again by Lemma 5.1.3(ii). Now notice

$$\frac{(i+j)^2}{i^2+j^2} = 1 + \frac{2ij}{i^2+j^2}\,.$$

Hence, we have that $N_5 = N\left(\frac{1}{i^2+j^2}\right)$ is of finite rank, but N_5 is a Cauchy matrix and is therefore of infinite rank, in contradiction.

Hence, $M(\text{harm avg valency}, \bowtie)$ is of infinite rank.

5.2.3 Extending Theorem 5.1.5: derivatives and coefficients

Here we show that derivatives of MSOL-polynomials and various sums of their coefficients have connection matrices of finite rank, implying that the results from Subsections 5.2.1 and 5.2.2 hold for them as well.

Derivatives

The derivatives of various graph polynomials, including those of the independent set polynomial, the clique polynomial, the characteristic polynomial and the matching polynomial, were studied by several authors, cf. e.g. [46, 86, 110].

Theorem 5.2.3. *If $P(G;\bar{\mathsf{x}})$ is an* MSOL_i*-polynomial, $i = 1, 2$, then any fixed derivative* $\dfrac{\partial^{i_1+\cdots+i_\ell} P(G;\bar{\mathsf{x}})}{\partial \mathsf{x}_1^{i_1}\cdots\partial \mathsf{x}_\ell^{i_\ell}}$ *is also an* MSOL_i*-polynomial.*

Proof. We will show the proof for one case. The general proof is similar. Let

$$P(G;\bar{\mathsf{x}}) = \sum_{\bar{U}:\Phi(\bar{U})} \left(\prod_{\bar{v}_1:\phi_1(\bar{v}_1,\bar{U})} \mathsf{x}_1 \ldots \prod_{\bar{v}_\ell:\phi_\ell(\bar{v}_\ell,\bar{U})} \mathsf{x}_\ell \right).$$

Then

$$\frac{\partial^\ell P(G;\bar{\mathsf{x}})}{\partial \mathsf{x}_1\cdots\partial \mathsf{x}_\ell} = \sum_{\substack{\bar{U},\bar{a}_1,\ldots,\bar{a}_\ell: \\ \phi_1(\bar{a}_1,\bar{U})\wedge\cdots\wedge\phi_\ell(\bar{a}_\ell)\wedge\Phi(\bar{U})}} \left(\prod_{\bar{v}_1:\substack{\phi_1(\bar{v}_1,\bar{U})\wedge \\ (\bar{v}_1\neq\bar{a}_1)}} \mathsf{x}_1 \ldots \prod_{\bar{v}_\ell:\substack{\phi'_\ell(\bar{v}_\ell,\bar{U})\wedge \\ (\bar{v}_\ell\neq\bar{a}_\ell)}} \mathsf{x}_\ell \right) \quad (5.1)$$

Equation 5.1 holds since the degree of each indeterminate x_j in each monomial $\mathsf{x}_1^{i_1}\cdots\mathsf{x}_j^{i_j}\cdots\mathsf{x}_\ell^{i_\ell}$ is lowered by 1 but is counted i_j times. All of the formulas in Equation 5.1 can be expressed in MSOL_2. It remains to notice that, while the sum includes first order variables, one can instead sum over monadic second order variables of size 1. Since this is definable, we get that $\frac{\partial^\ell P(G;\bar{\mathsf{x}})}{\partial \mathsf{x}_1\cdots\partial \mathsf{x}_\ell}$ is an MSOL_2-polynomial. □

Sums of coefficients

Theorem 5.2.4. *Let $P(G;\mathsf{x}_1,\ldots,\mathsf{x}_\ell)$ be an* MSOL_i*-polynomial, $i = 1, 2$. The following graph invariants have $\odot$-connection matrices of finite rank for every τ_i-smooth operation:*

(i) The sum of the coefficients of $P(G; \bar{\mathsf{x}})$.

(ii) The sum of coefficients of monomials $\mathsf{x}_1^{i_1} \cdots \mathsf{x}_\ell^{i_\ell}$ such that i_1 equals $a \pmod b$.

(iii) The sum of coefficients of monomials $\mathsf{x}_1^{i_1} \cdots \mathsf{x}_\ell^{i_\ell}$ such that $i_1 + \cdots + i_\ell$ equals $a \pmod b$.

Proof. Let

$$P(G; \bar{\mathsf{x}}) = \sum_{\bar{U}:\Phi(\bar{U})} \left(\prod_{\bar{v}_1:\phi_1(\bar{v}_1,\bar{U})} \mathsf{x}_1 \cdots \prod_{\bar{v}_\ell:\phi_\ell(\bar{v}_\ell,\bar{U})} \mathsf{x}_\ell \right)$$

We prove (i)-(iii) by showing that all of the listed graph invariants are (evaluations of) MSOL-polynomials.

(i) The sum of coefficients is given by the evaluation of $P(G; \bar{\mathsf{x}})$ which assigns all of the indeterminates to 1.

(ii) The sum of coefficients $\mathsf{x}_1^{i_1} \cdots \mathsf{x}_\ell^{i_\ell}$ such that i_1 is odd is the evaluation which assigns 1 to all indeterminates of the MSOL_2-polynomial obtained from P by replacing $\phi_1(\bar{v}_1)$ with $\phi_1(\bar{v}_1) \wedge C_{1,2}\bar{x}\, \phi_1(\bar{x})$ where $C_{1,2}$ is the modular quantifier from Remark 4.1.2.

(iii) The proof is similar to the previous item.

$\square$

Example 5.2.5 (Coefficients of the characteristic polynomials). *Recall from Section 2.1.3 that the coefficients $c_i(G)$ of $\mathsf{x}^{|V(G)|-i}$ for small i of the characteristic polynomial $char(G; \mathsf{x})$ are well studied. For example, it is well known that the coefficient $c_2(G)$ of $x^{|V(G)|-2}$ is minus the number of edges of the graph and the coefficient c_3 of $x^{|V(G)|-3}$ is minus twice the number of triangles in the graph. Since $char(G; \mathsf{x})$ is an MSOL_2-polynomial, it has finite rank in $\odot \in \{\sqcup\} \cup \{\sqcup_k\}_{k=1}^{\infty}$. Hence, by Lemma 5.1.3 it holds that*

$$\widetilde{char}(G; \mathsf{x}) = \mathsf{x}^{|V(G)|} \cdot char\left(G; \frac{1}{\mathsf{x}}\right)$$

is of finite rank in such $\odot$. It is well-known that the highest coefficients of the characteristic polynomial satisfy $c_1(G) = 0$ and $c_0(G) = 1$. Hence

$\mathsf{x}^{-2} \cdot (\widetilde{char}(G;\mathsf{x}) - 1)$ *is a polynomial with constant term* $c_2(G)$*. Evaluating* x *to zero we get that* $c_2(G)$ *has finite rank in* $\odot$*.*

Similarly, $\mathsf{x}^{-3} \cdot (\widetilde{char}(G;\mathsf{x}) - c_2(G) \cdot \mathsf{x}^2 - 1)$ *is of finite rank in* $\odot$ *and, evaluating* x *to zero, we get that* $c_3(G)$ *has finite rank in* $\odot$*. Continuing in this fashion, we can show that any fixed coefficient* $c_i(G)$ *of* $char(G;\mathsf{x})$ *is of finite rank in* $\odot$*.*

5.3 Applications to graph polynomials

While most of the graph polynomials with a fixed finite number of indeterminates that we know from the literature are MSOL-polynomials, using Theorem 5.1.4 we can show some examples of natural graph polynomials which are not MSOL-polynomials. Other graph polynomials are not MSOL$_1$-polynomial, but may be MSOL-polynomials.

To show this, we prove the following lemma:

Lemma 5.3.1. *Given a* τ*-invariant* $p : \mathrm{Struct}(\tau) \to \mathbb{Q}[\mathsf{x}]$*, a binary* τ*-operation* $\odot$ *and an infinite sequence of non-isomorphic* τ*-structures* $\mathfrak{A}_i, i \in \mathbb{N}$*, let* $f : \mathbb{N} \to \mathbb{N}$ *be an unbounded function such that one of the following occurs:*

(A) *for every* $\lambda, i, j \in \mathbb{N}$*,* $p(\mathfrak{A}_i \odot \mathfrak{A}_j, \lambda) = 0$ *iff* $i + j > f(\lambda)$*, or*

(B) *for every* $\lambda, i, j \in \mathbb{N}$*,* $p(\mathfrak{A}_{2i} \odot \mathfrak{A}_{2j+1}, \lambda) = 0$ *iff* $i + j > f(\lambda)$*.*

Then the connection matrix $M(\odot, p)$ *has infinite rank.*

Proof. Let $\lambda \in \mathbb{N}$ and let p_λ be the graph parameter given by $p_\lambda(G) = p(G, \lambda)$. If (A) holds, consider the restriction N of the connection matrix $M(p_\lambda, \odot)$ to the rows and columns corresponding to $\mathfrak{A}_i$, $0 \leq i \leq f(\lambda) - 1$. If (B) holds, consider the restriction N of the connection matrix $M(p_\lambda, \odot)$ to the rows corresponding to $\mathfrak{A}_{2i}$ and columns corresponding to $\mathfrak{A}_{2i+1}$, $0 \leq i \leq f(\lambda) - 1$. In both cases N is a finite triangular matrix with non-zero diagonal. Hence the rank of $M(p_\lambda, \odot)$ is at least $f(\lambda) - 1$.

Using that f is unbounded, we get that $M(p, \odot)$ contains infinitely many finite sub-matrices with ranks which tend to infinity. Hence, the rank of $M(p, \odot)$ is infinite, □

We use Lemma 5.3.1 to prove non-definability of graph colorings counting colorings defined in Section 2.3:

Proposition 5.3.2.
The following graph polynomials are not MSOL_1*-polynomials:*

(i) *the chromatic polynomial* $\chi(G; \mathsf{k})$,

(ii) $\chi_{mcc(t)}(G; \mathsf{k})$ *for any fixed* $t > 0$,

(iii) $\chi_{v-acyclic}(G; \mathsf{k})$,

(iv) $\chi_{\mathcal{P}_{\text{Bipartite}}}(G; k)$, $\chi_{\mathcal{P}_{\text{Forest}}}(G; k)$, $\chi_{\mathcal{P}_{\text{Tree}}}(G; k)$ *and* $\chi_{\mathcal{P}_{\text{Planar}}}(G; k)$ *and* $\chi_{\mathcal{P}_{3-\text{regular}}}(G; k)$ *where* $\chi_{\mathcal{P}}(G; k)$ *is the number of* $\mathcal{P}$*-*k*-colorings.*

The following graph polynomials are not MSOL*-polynomials:*

(v) $\chi_{rainbow}(G; \mathsf{k})$,

(vi) $\chi_{convex}(G; \mathsf{k})$,

(vii) $\chi_{harm}(G; \mathsf{k})$,

(viii) $\chi_{t-improper}(G; \mathsf{k})$, *for any fixed* $t > 0$,

(ix) $\chi_{repetitive}(G; \mathsf{k})$,

Proof. From Theorem 5.1.4, it is enough to show that each of the graph polynomials has a connection matrix of infinite rank with respect to an $\mathrm{MSOL}(\tau_i)$-smooth for the appropriate $i = 1, 2$. To use Lemma 5.3.1 we need to specify the operation $\odot$ and structures $\mathfrak{A}_i$, and show the existence of f as required. We use case (A) of Lemma 5.3.1 unless otherwise specified.

(i) For the chromatic polynomial we use the join and cliques. We have that $K_i \bowtie K_j = K_{i+j}$ and $\chi(K_r; k) = 0$ iff $r > k$ (i.e., $f(k) = k$).

(ii) For $\chi_{mcc(t)}(G; k)$ we use the join and cliques. We have that $\chi_{mcc(t)}(K_r; k) = 0$ iff $r > kt$ (i.e., $f(k) = kt$).

(iii) For $\chi_{v-acyclic}(G; \mathsf{k})$, we use join and cliques. We have $\chi_{v-acyclic}(K_n; k) = 0$ iff $n > k$.

(iv) Here we use join and cliques. For the first three we have $\chi_{\mathcal{P}}(K_i; k) = 0$ iff $i > 2k$, and for the latter two we have $\chi_{\mathcal{P}_{\text{Planar}}}(K_i; k) = 0$ iff $i > 4k$ and$\chi_{\mathcal{P}_{3-\text{regular}}}(K_i; k) = 0$ iff $i > 3k$.

(v) For $\chi_{rainbow}(G;k)$, we use that the 1-sum of paths P_n with one end labeled is again a path with $P_i \sqcup_1 P_j = P_{i+j-1}$ and that $\chi_{rainbow}(P_r;k) = 0$ iff $r > k+3$.

(vi) For $\chi_{convex}(G;k)$, we use edgeless graphs and disjoint union $E_i \sqcup E_j = E_{i+j}$ and that $\chi_{convex}(E_r;k) = 0$ iff $r > k$.

(vii) Let iK_2 denote the graph which consists of i vertex disjoint edges. We have $\chi_{harm}(nK_2;k) = 0$ iff $n > \binom{k}{2}$.

(viii) For $\chi_{t-improper}(G;\mathsf{k})$, we use cliques and 1-sum, and that $\chi_{t-improper}(K_i \sqcup_1 K_j;k) = 0$ iff $\left\lceil \frac{i+j-2}{k} \right\rceil > t$.

(ix) For $\chi_{repetitive}(G;\mathsf{k})$, we use 1-sum and stars S_n. We have $S_i \sqcup_1 S_j = S_{i+j}$ and $\chi_{repetitive}(S_n;k) = 0$ iff $n > k$.

□

Chapter 6

Ising Polynomials: A Case Study of the Difficult Point Conjecture

In this chapter we prove Theorems D and E. In Section 6.1 we introduce some background on complexity including the exact statements of the Difficult Point Conjectures and the # Exponential Time Hypothesis, and consider the complexity of computing the Ising polynomials (as members of $\mathbb{Z}[\mathsf{t}, \mathsf{y}]$ and $\mathbb{Z}[\mathsf{x}, \mathsf{y}, \mathsf{z}]$ respectively). In Section 6.2 we prove that the bivariate Ising polynomial has the Strong DPP, even when we restrict the reductions so that do not increase the size of the graphs considerably. In Section 6.3 we show that the trivariate Ising polynomial has the DPP, even when restricted to input graphs which are simple, bipartite and planar.

This chapter is based on [101].

Overview of proofs

We prove the hardness results in Theorems D and E by showing that it is possible to interpolate an Ising polynomial in polynomial time with an oracle to any of the evaluations of the appropriate Ising polynomial except those in the exception set.

To prove the DPP of the trivariate Ising polynomial we essentially reduce any evaluation of it to the bivariate Ising polynomial. This will be done only for input graphs which are regular, and that will suffice to give Theorem D(i).

The reduction here is a combination of two reductions: attaching new leaves to each vertex, and replacing any edge by parallel paths of length 3.

For Theorem E we show that every evaluation of $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$, except those in the exception set, can be used to interpolate $Z_{Ising}(G; \mathsf{t}, 1)$, with the added restriction that the reductions used must not increase the size of the graph by more than a poly-log factor. Our reductions attach a certain type of tree to each vertex, and replace each edge by a so-called Phi graph. The trees and Phi graphs used will each be of poly-logarithmic size in n_G, and thus the graphs on which the oracle is queried will be of size quasi-linear in n_G. To show that the reduction here can be used to interpolate $Z_{Ising}(G; \mathsf{t}, 1)$ we need to show how to use it to compute the polynomial enough distinct points. Theorem D(ii) will follow as well.

The tractability of the evaluations in the exception set in Theorems D(ii) and E will be found in the literature.

6.1 Complexity background

6.1.1 The Difficult Point Conjecture

An exact formulation of Makowsky's conjecture from [116] is as follows:

Conjecture 6.1.1 (Difficult Point Conjecture, DPC).
Let $P(G; \mathsf{x}_1, \ldots, \mathsf{x}_\ell)$ be an MSOL-polynomial. There exists $B \subseteq \mathbb{Q}^\ell$ such that for every $\bar{x}_0, \bar{x}_1 \in \mathbb{Q}^\ell \backslash B$, $P(G; \bar{x}_0)$ and $P(G; \bar{x}_1)$ are Turing equivalent.[1] The set B is a finite union of semi-algebraic sets of dimension at most $\ell - 1$.

The strong version of the conjecture is as follows:

Conjecture 6.1.2 (Strong Difficult Point Conjecture, SDPC).
Let $P(G; \mathsf{x}_1, \ldots, \mathsf{x}_\ell)$ be an MSOL-polynomial. Then the Difficult Point Conjecture holds for $P(G; \bar{\mathsf{x}})$, and for every $\bar{x}_0 \in B$, $P(G; \bar{x}_0)$ is polynomial time computable.

Usually it is also the case that if $\bar{x}_1 \in \mathbb{Q} \backslash B$ then $P(G; \bar{x}_1)$ is hard to compute. Usually #**P**-hardness is considered.

[1] When we say "$P(G; \bar{x}_0)$ and $P(G; \bar{x}_1)$ are Turing equivalent" we mean that we can use $P(G; \bar{x}_i)$ to efficiently compute a $P(G; \bar{x}_{1-i})$, for $i = 0, 1$. More precisely, given an oracle $A_{\bar{x}_i}$ which on input H returns two pairwise independent natural numbers a and b such that $P(H; \bar{x}_i) = \frac{a}{b}$, one can compute $P(G; \bar{x}_{1-i})$ in polynomial-time.

6.1.2 Background on the complexity of the Ising polynomials

Computing the Ising polynomials[2] is #**P**-hard. In [8] it is shown that several graph invariants are encoded in $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$.

Proposition 6.1.3. *The following are polynomial time Turing reducible to* $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$*:*

— *the matching polynomial and the number of perfect matchings,*

— *the number of maximum and minimum cuts,*

and, for regular graphs,

— *the independent set polynomial and the vertex cover polynomial.*

The following propositions apply two hardness results from the literature to $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ using Proposition 6.1.3.

Proposition 6.1.4. $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ *is* #**P***-hard to compute, even when restricted to simple* 3*-regular bipartite planar graphs.*

Proof. The proposition follows from a result of [152] which states that it is #**P**-hard to compute $\#3\mathrm{RBP}-\mathrm{VC}$, the number of vertex covers on input graphs restricted to be 3-regular, bipartite and planar. □

For the next proposition we need the following definition which is introduced in [47] following [94]:

Definition 6.1.1 (# Exponential Time Hypothesis (#ETH)).
Let s *be the infimum of the set*

$$\{c : \textit{there exists an algorithm for } \#3\mathrm{SAT} \textit{ which runs in time } O(c^{n_G})\}$$

The # Exponential Time Hypothesis *is the conjecture that* $s > 1$*.*

Proposition 6.1.5. *There exists a constant* $c > 1$ *such that the computation of* $Z_{Ising}(G; \mathsf{t}, 1)$ *requires* $\Omega\left(c^{n_G}\right)$ *time on simple graphs, unless* #**ETH** *fails.*

[2]Here we mean computing, for each graph G the list of coefficient of the polynomial $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ or $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$.

Proof. This proposition follows from a result of [47] which states that there exists $c > 1$ for which computing the number of maximum cuts in simple graphs G takes at least $\Omega\left(c^{m_G}\right)$ time, unless the #**ETH** fails. It easy to see that the problem computing the number of maximum cuts of disconnected graphs can be reduced to that of connected graphs and get that no sub-exponential algorithm exists for connected graphs, and the proposition follows since for connected graphs $n_G = O(m_G)$.

$\square$

On the other hand, $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ and $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ can be computed naively in time which is exponential in n_G.

The trivariate and bivariate Ising polynomials fall under the general framework of partition functions, the complexity of which has been studied extensively starting with [51] and followed by [34, 78, 143, 35]. From [143, Theorem 6.1] and implicitly from [78] we get that the complexity of evaluations of the Ising polynomials satisfies a dichotomy theorem, saying that the graph parameter $Z(G; \gamma, \delta)$ is either polynomial-time computable or #**P**-hard. However, δ must be positive there.

6.2 Exponential Time Lower Bound

In this section we prove that in general the evaluations $(\gamma, \delta) \in \mathbb{Q}^2$ of $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$ require exponential time to compute under #**ETH**. In analogy with the use of Theta graphs to deal with the complexity of the Tutte polynomial, we define Phi graphs and use them to interpolate the indeterminate t in $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$. We interpolate y by a simple construction.

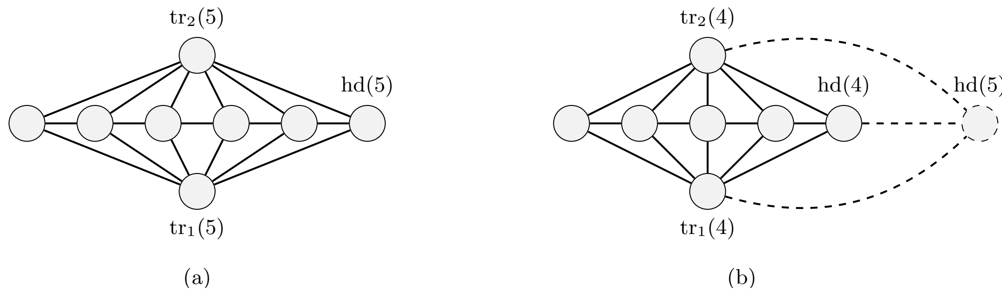
It will be convenient to use the following generalization of the bivariate Ising polynomial:

Definition 6.2.1. *For every $B, C \subseteq V(G)$ such that $B \cap C = \emptyset$ we define*

$$Z_{Ising}(G; B, C; \mathsf{t}, \mathsf{y}) = \sum_{S: B \subseteq S \subseteq V(G) \setminus C} \mathsf{t}^{|E_G(S)| + |E_G(\bar{S})|} \mathsf{y}^{|S|}.$$

Clearly, $Z_{Ising}(G; \emptyset, \emptyset; \mathsf{t}, \mathsf{y}) = Z_{Ising}(G; \mathsf{t}, \mathsf{y})$.

Figure 6.1: The graph L_5 and the construction of L_5 from L_4. L_5 is obtained from L_4 by adding the vertex $\mathrm{hd}(5)$ and its incident edges, and renaming $\mathrm{tr}_1(4)$ and $\mathrm{tr}_2(4)$ to $\mathrm{tr}_1(5)$ and $\mathrm{tr}_2(5)$ respectively.



6.2.1 Phi graphs

Our goal in this subsection is to define Phi graphs $\Phi_{\mathcal{H}}$ and compute the bivariate Ising polynomial at $\mathsf{y} = 1$ on graphs $G \otimes \Phi_{\mathcal{H}}$ to be defined below. In order to define Phi graphs we must first define L_h-graphs. For every $h \in \mathbb{N}$, the graph L_h is obtained from the path P_{h+1} with h edges as follows. Let $\mathrm{hd}(h)$ denote one of the end-points of P_{h+1}. Let $\mathrm{tr}_1(h)$ and $\mathrm{tr}_2(h)$ be two new vertices. L_h is obtained from P_{h+1} by adding edges to make both $\mathrm{tr}_1(h)$ and $\mathrm{tr}_2(h)$ adjacent to all the vertices of P_{h+1}.

We can also construct L_h recursively from L_{h-1} by

- adding a new vertex $\mathrm{hd}(h)$ to L_{h-1},
- renaming $\mathrm{tr}_i(h-1)$ to $\mathrm{tr}_i(h)$ for $i = 1, 2$, and
- adding three edges to make $\mathrm{hd}(h)$ adjacent to $\mathrm{hd}(h-1)$, $\mathrm{tr}_1(h)$ and $\mathrm{tr}_2(h)$.

Figure 6.1 shows L_5.

Let $B \sqcup C$ be a partition of the set $\{\mathrm{tr}_1, \mathrm{tr}_2, \mathrm{hd}\}$. Let $B(h)$ be the subset of $\{\mathrm{tr}_1(h), \mathrm{tr}_2(h), \mathrm{hd}(h)\}$ which corresponds to B and let $C(h)$ be defined similarly. We have that $B(h)$ and $C(h)$ form a partition of $\{\mathrm{tr}_1(h), \mathrm{tr}_2(h), \mathrm{hd}(h)\}$.

Definition 6.2.2. *We denote* $b_{B,C}(h) = Z_{Ising}(L_h; B(h), C(h); \mathsf{t}, 1)$.

The next two lemmas are devoted to computing $b_{B,C}(h)$.

Lemma 6.2.1.

$$\begin{aligned} b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h) &= b_{\{\mathrm{tr}_2,\mathrm{hd}\},\{\mathrm{tr}_1\}}(h) = \\ b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h) &= b_{\{\mathrm{tr}_2\},\{\mathrm{tr}_1,\mathrm{hd}\}}(h) = (\mathsf{t}^2+\mathsf{t})^h \cdot \mathsf{t}\,. \end{aligned}$$

Proof. We have

$$\begin{aligned} b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h) &= b_{\{\mathrm{tr}_2,\mathrm{hd}\},\{\mathrm{tr}_1\}}(h) = \\ b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h) &= b_{\{\mathrm{tr}_2\},\{\mathrm{tr}_1,\mathrm{hd}\}}(h) \end{aligned}$$

by symmetry. We compute $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h)$ by finding a simple linear recurrence relation which it satisfies and solving it. We divide the sum $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h)$ into two sums,

$$\begin{aligned} b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h) &= Z_{Ising}(L_h; \{\mathrm{tr}_1(h), \mathrm{hd}(h), \mathrm{hd}(h-1)\}, \{\mathrm{tr}_2(h)\}; \mathsf{t}, 1) + \\ &\quad Z_{Ising}(L_h; \{\mathrm{tr}_1(h), \mathrm{hd}(h)\}, \{\mathrm{tr}_2(h), \mathrm{hd}(h-1)\}; \mathsf{t}, 1) \end{aligned}$$

depending on whether $\mathrm{hd}(h-1)$ is in the iteration variable S of the sum $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h)$ (as in Definition 6.2.1). These two sums can be obtained from $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h-1)$ and $b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h-1)$ by adjusting for the addition of $\mathrm{hd}(h)$ and its incident edges:

- $\mathrm{hd}(h-1) \in S$: adding $\mathrm{hd}(h)$ (to the graph and to S) puts two new edges in $E(S) \sqcup E(\bar{S})$, namely $(\mathrm{tr}_1, \mathrm{hd}(h))$ and $(\mathrm{hd}(h-1), \mathrm{hd}(h))$. Hence,

$$Z_{Ising}(L_h; \{\mathrm{tr}_1(h), \mathrm{hd}(h), \mathrm{hd}(h-1)\}, \{\mathrm{tr}_2(h)\}; \mathsf{t}, 1)$$

$$= b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h-1) \cdot \mathsf{t}^2\,.$$

- $\mathrm{hd}(h-1) \notin S$: adding $\mathrm{hd}(h)$ puts just one new edge in $E(S) \sqcup E(\bar{S})$, namely $(\mathrm{tr}_1, \mathrm{hd}(h))$. Hence,

$$Z_{Ising}(L_h; \{\mathrm{tr}_1(h), \mathrm{hd}(h)\}, \{\mathrm{tr}_2(h), \mathrm{hd}(h-1)\}; \mathsf{t}, 1)$$

$$= b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h-1) \cdot \mathsf{t}\,.$$

Using that $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h-1) = b_{\{\mathrm{tr}_1\},\{\mathrm{tr}_2,\mathrm{hd}\}}(h-1)$, we get:

$$b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h) = b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(h-1) \cdot (\mathsf{t}^2 + \mathsf{t}) \tag{6.1}$$

and the lemma follows since $b_{\{\mathrm{tr}_1,\mathrm{hd}\},\{\mathrm{tr}_2\}}(0) = \mathsf{t}$ (note L_0 is simply a path of length 3). □

We are left with two distinct cases of $b_{B,C}(h)$ to compute, since by symmetry,

$$b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h) = b_{\emptyset,\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\}}(h) \text{ and } b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h) = b_{\{\mathrm{hd}\},\{\mathrm{tr}_1,\mathrm{tr}_2\}}(h)\,.$$

Lemma 6.2.2. *Let*

$$\begin{aligned}
\lambda_{1,2} &= \frac{\mathsf{t}}{2}\left(1 + \mathsf{t}^2 \pm \sqrt{5 - 2\mathsf{t}^2 + \mathsf{t}^4}\right),\\
c_1 &= \mathsf{t}^2 - c_2\,,\\
c_2 &= \frac{\mathsf{t}\left(-\mathsf{t}^3 - 2 + \mathsf{t} + \mathsf{t}\sqrt{5 - 2\mathsf{t}^2 + \mathsf{t}^4}\right)}{2\sqrt{5 - 2\mathsf{t}^2 + \mathsf{t}^4}}\,,\\
d_1 &= 1 - d_2\,,\\
d_2 &= \frac{-1 - 2\mathsf{t} + \mathsf{t}^2 + \sqrt{5 - 2\mathsf{t}^2 + \mathsf{t}^4}}{2\sqrt{5 - 2\mathsf{t}^2 + \mathsf{t}^4}}\,.
\end{aligned}$$

λ_1 corresponds to the $+$ case. If $\mathsf{t} \in \mathbb{R}$ then $c_1, c_2, d_1, d_2, \lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$, and

$$\begin{aligned}
b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h) &= c_1\lambda_1^h + c_2\lambda_2^h\\
b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h) &= d_1\lambda_1^h + d_2\lambda_2^h\,.
\end{aligned}$$

Proof. The content of the square root is always strictly positive for $\mathsf{t} \in \mathbb{R}$. Hence, $\lambda_1 \neq \lambda_2$ and $c_1, c_2, d_1, d_2, \lambda_1, \lambda_2 \in \mathbb{R}$.

The sequences $b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h)$ and $b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h)$ satisfy a mutual linear recurrence as follows:

$$\begin{aligned}
b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h) &= b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h-1) \cdot \mathsf{t}^3 + b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h-1) \cdot \mathsf{t}^2\\
b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h) &= b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h-1) + b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h-1) \cdot \mathsf{t}
\end{aligned}$$

This implies that both $b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(h)$ and $b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(h)$ satisfy linear recurrence relations with the following initial conditions:

$$b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(0) = t^2 \text{ and } b_{\{\mathrm{tr}_1,\mathrm{tr}_2,\mathrm{hd}\},\emptyset}(1) = t^5 + t^2$$
$$b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(0) = 1 \text{ and } b_{\{\mathrm{tr}_1,\mathrm{tr}_2\},\{\mathrm{hd}\}}(1) = t^2 + t\,.$$

These recurrences can be calculated and solved using standard methods, see e.g. [62] or [83]. □

Using the previous two lemmas, we get:

Lemma 6.2.3.

$$\begin{aligned} Z_{Ising}(L_h; \{\mathrm{tr}_1\}, \{\mathrm{tr}_2\}; \mathsf{t}, 1) &= Z_{Ising}(L_h; \{\mathrm{tr}_2\}, \{\mathrm{tr}_1\}; \mathsf{t}, 1) = 2\mathsf{t}(\mathsf{t}^2 + \mathsf{t})^h \\ Z_{Ising}(L_h; \{\mathrm{tr}_1, \mathrm{tr}_2\}, \emptyset; \mathsf{t}, 1) &= Z_{Ising}(L_h; \emptyset, \{\mathrm{tr}_1, \mathrm{tr}_2\}; \mathsf{t}, 1) \\ &= (c_1 + d_1)\lambda_1^h + (c_2 + d_2)\lambda_2^h\,, \end{aligned}$$

where $c_1, c_2, d_1, d_2, \lambda_1, \lambda_2$ are as in Lemma 6.2.2.

Proof. The lemma follows from Lemmas 6.2.1 and 6.2.2. □

Definition 6.2.3 (Phi graphs). *Let $\mathcal{H}$ be a finite set of positive integers. We denote by $\Phi_{\mathcal{H}}$ the graph obtained from the disjoint union of the graphs $L_h : h \in \mathcal{H}$ as follows. For each $i = 1, 2$, the vertices $\mathrm{tr}_i(h)$, $h \in \mathcal{H}$, are identified into one vertex denoted $\mathrm{tr}_i(\mathcal{H})$.*

The number of vertices in $\Phi_{\mathcal{H}}$ is $2 + \sum_{h\in\mathcal{H}}(h+1)$. Figure 6.2 shows $\Phi_{\{1,3,4\}}$.

Lemma 6.2.4. *Let $\mathcal{H}$ be a finite set of positive integers. Then*

$$Z_{Ising}(\Phi_{\mathcal{H}}; \{\mathrm{tr}_1(\mathcal{H})\}, \{\mathrm{tr}_2(\mathcal{H})\}; \mathsf{t}, 1) = (2\mathsf{t})^{|\mathcal{H}|} \prod_{h\in\mathcal{H}} (\mathsf{t}^2 + \mathsf{t})^h\,,$$

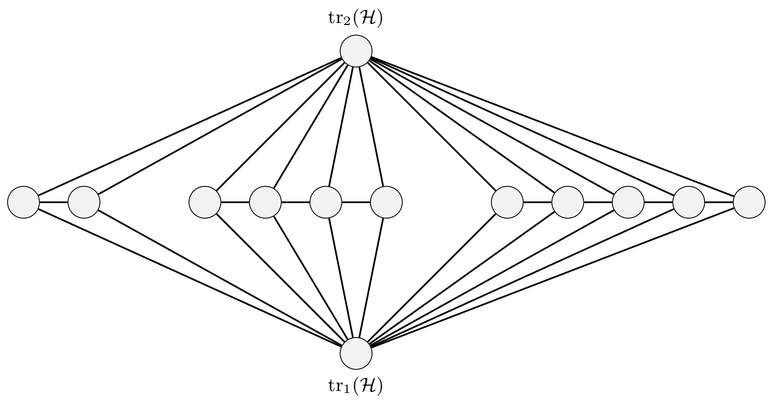
and

$$\begin{aligned} Z_{Ising}(\Phi_{\mathcal{H}}; \{\mathrm{tr}_1(\mathcal{H}), \mathrm{tr}_2(\mathcal{H})\}, \emptyset; \mathsf{t}, 1) &= \\ Z_{Ising}(\Phi_{\mathcal{H}}; \emptyset, \{\mathrm{tr}_1(\mathcal{H}), \mathrm{tr}_2(\mathcal{H})\}; \mathsf{t}, 1) &= \prod_{h\in\mathcal{H}} \Big((c_1 + d_1)\lambda_1^h + (c_2 + d_2)\lambda_2^h\Big)\,. \end{aligned}$$

Proof. It follows from Lemma 6.2.3 using that all edges are contained in some L_h. □

We can now define the graphs $G \otimes \mathcal{H}$:

Figure 6.2: An example of a Phi graph: the graph $\Phi_{\mathcal{H}}$ for $\mathcal{H} = \{1, 3, 4\}$.



Definition 6.2.4 **($G \otimes \mathcal{H}$).** *Let $\mathcal{H}$ be a finite set of positive integers. Let G be a graph. For every edge $e = (u_1, u_2) \in E(G)$, let $\Phi_{\mathcal{H},e}$ be a new copy of $\Phi_{\mathcal{H}}$. Each copy $\Phi_{\mathcal{H},e}$ has two vertices labeled $\mathrm{tr}_1(\mathcal{H}, e)$ and $\mathrm{tr}_2(\mathcal{H}, e)$ which correspond to $\mathrm{tr}_1(\mathcal{H})$ and $\mathrm{tr}_2(\mathcal{H})$ in Definition 6.2.3. Let $G \otimes \Phi_{\mathcal{H}} = G \otimes \mathcal{H}$ be the graph obtained from the disjoint union of the graphs*

$$\Phi_{\mathcal{H},e} : e \in E(G)$$

by identifying $\mathrm{tr}_i(\mathcal{H}, e)$ with u_i, $i = 1, 2$, for every edge $e = (u_1, u_2) \in E(G)$. [3]

Lemma 6.2.5. *Let $\mathcal{H}$ be a finite set of positive integers. Let $f_{\mathsf{t},\mathcal{H}}$ and $g_{p,\mathcal{H}}$ be the following functions:*

$$\begin{aligned} f_{\mathsf{t},\mathcal{H}}(e_1, e_2, r_1, r_2) &= \prod_{h \in \mathcal{H}} \left(e_1 r_1^h + e_2 r_2^h \right) \\ f_{p,\mathcal{H}}(\mathsf{t}) &= \left((2\mathsf{t})^{|\mathcal{H}|} \prod_{h \in \mathcal{H}} (\mathsf{t}^2 + \mathsf{t})^h \right)^{m_G} . \end{aligned}$$

[3]It does not matter how we identify u_1 and u_2 with $\mathrm{tr}_1(\mathcal{H}, e)$ and $\mathrm{tr}_2(\mathcal{H}, e)$ since the two possible alignments will give raise to isomorphic graphs.

Then

$$Z_{Ising}(G \otimes \mathcal{H}; \mathsf{t}, 1) \quad = \quad f_{p,\mathcal{H}}(\mathsf{t}) \cdot Z\left(G; f_{\mathsf{t},\mathcal{H}}\left(\frac{c_1+d_1}{2t}, \frac{c_2+d_2}{2t}, \frac{\lambda_1}{\mathsf{t}^2+\mathsf{t}}, \frac{\lambda_2}{\mathsf{t}^2+\mathsf{t}}\right), 1\right).$$

Proof. Let $\tilde{G} = G \otimes \mathcal{H}$. By definition,

$$Z_{Ising}(\tilde{G}; \mathsf{t}, 1) = \sum_{S \subseteq V(\tilde{G})} \mathsf{t}^{|E_{\tilde{G}}(S) \sqcup E_{\tilde{G}}(\bar{S})|}.$$

We can rewrite this sum as

$$\begin{aligned} Z_{Ising}(\tilde{G}; \mathsf{t}, 1) \quad = \quad & \sum_{S \subseteq V(G)} \left(\prod_{e \in [S,\bar{S}]_G} Z_{Ising}(\Phi_{\mathcal{H},e}; \{\mathrm{tr}_1(\mathcal{H},e)\}, \{\mathrm{tr}_2(\mathcal{H},e)\}; \mathsf{t}, 1) \right) \\ & \cdot \left(\prod_{e \in E_G(S) \sqcup E_G(\bar{S})} Z_{Ising}(\Phi_{\mathcal{H},e}; \{\mathrm{tr}_1(\mathcal{H},e), \mathrm{tr}_2(\mathcal{H},e)\}, \emptyset; \mathsf{t}, 1) \right), \end{aligned}$$

since edges only occur within some $\Phi_{\mathcal{H},e}$. Using lemma 6.2.4, the sum in the last equation can be written as

$$\begin{aligned} & \sum_{S \subseteq V(G)} \left((2\mathsf{t})^{|\mathcal{H}|} \prod_{h \in \mathcal{H}} (\mathsf{t}^2+\mathsf{t})^h \right)^{|[S,\bar{S}]_G|} \cdot \\ & \left(\prod_{h \in \mathcal{H}} \left((c_1+d_1)\lambda_1^h + (c_2+d_2)\lambda_2^h \right) \right)^{|E_G(S) \sqcup E_G(\bar{S})|}. \end{aligned}$$

Since $|[S,\bar{S}]_G = m_G - |E_G(S) \sqcup E_G(\bar{S})|$, we can rewrite the last equation as

$$\begin{aligned} & \left((2\mathsf{t})^{|\mathcal{H}|} \prod_{h \in \mathcal{H}} (\mathsf{t}^2+\mathsf{t})^h \right)^{m_G} \cdot \\ & \sum_{S \subseteq V(G)} \left(\frac{\prod_{h \in \mathcal{H}} \left((c_1+d_1)\lambda_1^h + (c_2+d_2)\lambda_2^h \right)}{(2\mathsf{t})^{|\mathcal{H}|} \prod_{h \in \mathcal{H}} (\mathsf{t}^2+\mathsf{t})^h} \right)^{|E_G(S) \sqcup E_G(\bar{S})|}. \end{aligned}$$

The last sum can be rewritten as

$$\sum_{S\subseteq V(G)}\left[\prod_{h\in\mathcal{H}}\left(\frac{c_1+d_1}{2t}\left(\frac{\lambda_1}{\mathsf{t}^2+\mathsf{t}}\right)^h+\frac{c_2+d_2}{2t}\left(\frac{\lambda_2}{\mathsf{t}^2+\mathsf{t}}\right)^h\right)\right]^{|E_G(S)\sqcup E_G(\bar{S})|}$$

and the lemma follows. □

The construction described above will be useful to deal with the evaluation of $Z_{Ising}(G;\mathsf{t},\mathsf{y})$ with $\mathsf{y}=-1$ due to the following lemma. For a graph G, let $G_{(1)}$ be the graph obtained from G by adding, for each $v\in V(G)$, a new vertex v' and an edge (v,v'). So v' is adjacent to v only. $G_{(1)}$ is a graph with $2n_G$ vertices.

Lemma 6.2.6. $Z_{Ising}(G;\mathsf{t},1)=(\mathsf{t}-1)^{-n_G}Z_{Ising}(G_{(1)};\mathsf{t},-1)$

Proof. By definition we have

$$\begin{aligned}Z_{Ising}(G_{(1)};\mathsf{t},-1) &= \sum_{S\subseteq V(G_{(1)})}\mathsf{t}^{|E_{G_{(1)}}(S)\sqcup E_{G_{(1)}}(\bar{S})|}(-1)^{|S|}\\ &= \sum_{S\subseteq V(G)}\mathsf{t}^{|E_G(S)\sqcup E_G(\bar{S})|}(\mathsf{t}-1)^{|S|}(\mathsf{t}-1)^{n_G-|S|}\end{aligned}$$

where the last equality is by considering the contribution of v' for each $v\in V(G)$: if $v\in S$ then v' contributes either $-\mathsf{t}$ or 1; if $v\notin S$ then v' contributes either t or -1. The last expression in the equation above equals

$$(\mathsf{t}-1)^{n_G}\sum_{S\subseteq V(G)}\mathsf{t}^{|E_G(S)\sqcup E_G(\bar{S})|} = (\mathsf{t}-1)^{n_G}\cdot Z_{Ising}(G;\mathsf{t},1)\,.$$

□

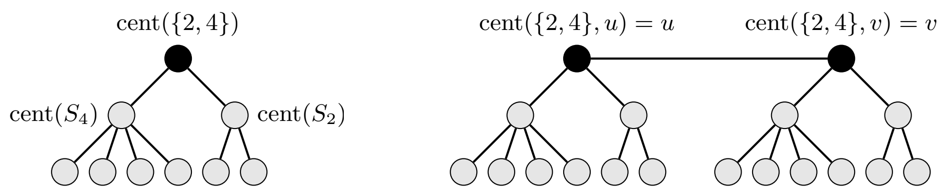
6.2.2 The Ising polynomials of certain trees

We denote be S_n the star with n leaves. Let $\mathrm{cent}(S_n)$ be the central vertex of the star S_n. A construction based on stars will be used to interpolate the y indeterminate from $Z_{Ising}(G;\gamma,\delta)$. First, notice the following:

Proposition 6.2.7. *For every $n\in\mathbb{N}^+$,*

$$\begin{aligned}Z_{Ising}(S_n;\{\mathrm{cent}(S_n)\},\emptyset;\mathsf{t},\mathsf{y}) &= \mathsf{y}\cdot(\mathsf{yt}+1)^n\\ Z_{Ising}(S_n;\emptyset,\{\mathrm{cent}(S_n)\};\mathsf{t},\mathsf{y}) &= (\mathsf{y}+\mathsf{t})^n\end{aligned}$$

Figure 6.3: Examples of $S_{\mathcal{H}}$ and $S_{\mathcal{H}}(G)$. In (b), the black vertices u, v are the vertices of K_2. They are also denoted $\mathrm{cent}(\mathcal{H}, u)$ and $\mathrm{cent}(\mathcal{H}, v)$ respectively.



(a) The graph $S_{\{2,4\}}$.

(b) The graph $S_{\{2,4\}}(K_2)$.

Proof. By definition,

$$Z_{Ising}(S_n; \{\mathrm{cent} S_n\}, \emptyset; \mathsf{t}, \mathsf{y}) = \sum_{S : \{\mathrm{cent}(S_n)\} \subseteq S \subseteq V(S_n)} \mathsf{t}^{|E_{S_n}(S) \sqcup E_{S_n}(\bar{S})|} \mathsf{y}^{|S|}$$

$$Z_{Ising}(S_n; \emptyset, \{\mathrm{cent} S_n\}; \mathsf{t}, \mathsf{y}) = \sum_{S \subseteq V(S_n) \setminus \{\mathrm{cent}(S_n)\}} \mathsf{t}^{|E_{S_n}(S) \sqcup E_{S_n}(\bar{S})|} \mathsf{y}^{|S|}$$

Consider a leaf v of S_n. For $Z_{Ising}(S_n; \{\mathrm{cent}(S_n)\}, \emptyset; \mathsf{t}, \mathsf{y})$, a leaf v has two options: either $v \in S$, in which case it contributes the weight of its incident edge, so its contribution is yt; or $v \notin S$, in which case it contributes 1. For $Z_{Ising}(S_n; \emptyset, \{\mathrm{cent}(S_n)\}; \mathsf{t}, \mathsf{y})$, v has two options: either $v \in S$, in which case it does not contribute the weight of its edge, so its contribution is y; or $v \notin S$, in which case its edge contributes t. □

Definition 6.2.5 (The graph $S_{\mathcal{H}}$). *Let $\mathcal{H}$ be a set of positive integers. The graph $S_{\mathcal{H}}$ is obtained from the disjoint union of $S_n : n \in \mathcal{H}$ and a new vertex* $\mathrm{cent}(\mathcal{H})$ *by adding edges between* $\mathrm{cent}(\mathcal{H})$ *and the centers* $\mathrm{cent}(S_n)$ *of all the stars $S_n : n \in \mathcal{H}$.*

See Figure 6.3(a) for an example.

Proposition 6.2.8. *Let $\mathcal{H}$ be a set of positive integers. Then,*

$$\begin{aligned} Z_{Ising}(S_{\mathcal{H}}; \{\mathrm{cent}(\mathcal{H})\}, \emptyset; \mathsf{t}, \mathsf{y}) &= \mathsf{y} \cdot \prod_{h \in \mathcal{H}} \left(\mathsf{yt} \cdot (\mathsf{yt}+1)^h + (\mathsf{y}+\mathsf{t})^h \right) \\ Z_{Ising}(S_{\mathcal{H}}; \emptyset, \{\mathrm{cent}(\mathcal{H})\}; \mathsf{t}, \mathsf{y}) &= \prod_{h \in \mathcal{H}} \left(\mathsf{y} \cdot (\mathsf{yt}+1)^h + \mathsf{t} \cdot (\mathsf{y}+\mathsf{t})^h \right) \end{aligned}$$

Proof. We have

$$\begin{aligned} Z_{Ising}(S_{\mathcal{H}}; \{\mathrm{cent}(\mathcal{H})\}, \emptyset; \mathsf{t}, \mathsf{y}) &= \mathsf{y} \cdot \prod_{h \in \mathcal{H}} \big(\mathsf{t} \cdot Z_{Ising}(S_h; \{\mathrm{cent}(S_h)\}, \emptyset; \mathsf{t}, \mathsf{y}) \\ &\qquad + Z_{Ising}(S_h; \emptyset, \{\mathrm{cent}(S_h)\}; \mathsf{t}, \mathsf{y}) \big) \\ Z_{Ising}(S_{\mathcal{H}}; \emptyset, \{\mathrm{cent}(\mathcal{H})\}; \mathsf{t}, \mathsf{y}) &= \prod_{h \in \mathcal{H}} \big(Z_{Ising}(S_h; \{\mathrm{cent}(S_h)\}, \emptyset; \mathsf{t}, \mathsf{y}) \\ &\qquad + \mathsf{t} \cdot Z_{Ising}(S_h; \emptyset, \{\mathrm{cent}(S_h)\}; \mathsf{t}, \mathsf{y}) \big) \end{aligned}$$

and by Proposition 6.2.7, the claim follows. □

Definition 6.2.6 (The graph $S_{\mathcal{H}}(G)$). *Let $\mathcal{H}$ be a set of positive integers and let G be a graph. For every vertex v of G, let $S_{\mathcal{H},v}(G)$ be a new copy of $S_{\mathcal{H}}$. We denote the center of each such copy of $S_{\mathcal{H}}$ by $\mathrm{cent}(\mathcal{H}, v)$. Let $S_{\mathcal{H}}(G)$ be the graph obtained from the disjoint union of the graphs in the set*

$$\{G\} \cup \{S_{\mathcal{H},v} : v \in V(G)\}$$

by identifying the pairs of vertices v and $\mathrm{cent}(\mathcal{H}, v)$.

In other words, $S_{\mathcal{H}}(G)$ is the *rooted product* of G and $(S_{\mathcal{H}}, \mathrm{cent}(\mathcal{H}))$. See Figure 6.3(b) for an example.

Proposition 6.2.9. *Let $\mathcal{H}$ be a set of positive integers. Let*

$$\begin{aligned} g_{p,\mathcal{H}}(\mathsf{t}, \mathsf{y}) &= \left(\prod_{h \in \mathcal{H}} \left(\mathsf{y} \cdot (\mathsf{yt}+1)^h + \mathsf{t} \cdot (\mathsf{y}+\mathsf{t})^h \right) \right)^{|V(G)} \\ g_{\mathsf{y},\mathcal{H}}(\mathsf{t}, \mathsf{y}) &= \mathsf{y} \prod_{h \in \mathcal{H}} \frac{\mathsf{yt} \cdot (\mathsf{yt}+1)^h + (\mathsf{y}+\mathsf{t})^h}{\mathsf{y} \cdot (\mathsf{yt}+1)^h + \mathsf{t} \cdot (\mathsf{y}+\mathsf{t})^h} \end{aligned}$$

Then

$$Z_{Ising}(S_{\mathcal{H}}(G);\mathsf{t},\mathsf{y}) = g_{p,\mathcal{H}}(\mathsf{t},\mathsf{y}) \cdot Z_{Ising}(G;\mathsf{t},g_{\mathsf{y},\mathcal{H}}(\mathsf{t},\mathsf{y}))\,.$$

Proof. By definition

$$Z_{Ising}(S_{\mathcal{H}}(G);\mathsf{t},\mathsf{y}) = \sum_{S\subseteq V(S_{\mathcal{H}}(G))} \mathsf{t}^{|E_{S_{\mathcal{H}}(G)}(S)\sqcup E_{S_{\mathcal{H}}(G)}(\bar{S})|}\mathsf{y}^{|S|}\,.$$

We would like to rewrite this sum as a sum over $S \subseteq V(G)$. By the structure of $S_{\mathcal{H}}(G)$,

$$\begin{aligned} Z_{Ising}(S_{\mathcal{H}}(G);\mathsf{t},\mathsf{y}) \;=\; & \sum_{S\subseteq V(G)} \mathsf{t}^{|E_G(S)\sqcup E_G(\bar{S})|} \\ & \left(\prod_{v\in S} Z_{Ising}(S_{\mathcal{H},v};\{\mathrm{cent}(\mathcal{H},v)\},\emptyset;\mathsf{t},\mathsf{y})+\right. \\ & \left.\prod_{v\in \bar{S}} Z_{Ising}(S_{\mathcal{H},v};\emptyset,\{\mathrm{cent}(\mathcal{H},v)\};\mathsf{t},\mathsf{y})\right) \end{aligned}$$

By Proposition 6.2.8,

$$\begin{aligned} Z_{Ising}(S_{\mathcal{H}}(G);\mathsf{t},\mathsf{y}) \;=\; & \sum_{S\subseteq V(G)} \mathsf{t}^{|E_G(S)\sqcup E_G(\bar{S})|} \\ & \left(\left(\mathsf{y}\cdot\prod_{h\in\mathcal{H}}\left(\mathsf{yt}\cdot(\mathsf{yt}+1)^h+(\mathsf{y}+\mathsf{t})^h\right)\right)^{|S|}\right. \\ & \left.\left(\prod_{h\in\mathcal{H}}\left(\mathsf{y}\cdot(\mathsf{yt}+1)^h+\mathsf{t}\cdot(\mathsf{y}+\mathsf{t})^h\right)\right)^{|V(G)\setminus|S|}\right) \end{aligned}$$

and the claim follows.

□

The following propositions will be useful:

Proposition 6.2.10. *Let $g_{\mathsf{y},\mathcal{H}}(\mathsf{t},\mathsf{y})$ be as in Proposition 6.2.9. Let $h_{\mathsf{y},\mathcal{H}}$ be the function given by*

$$h_{\mathsf{y},\mathcal{H}}(e_1,e_2,r) = \prod_{h\in\mathcal{H}} \left(1+\frac{1}{e_1+e_2\cdot r^h}\right).$$

Let $\gamma,\delta \notin \{-1,0,1\}$ such that $\gamma \neq -\delta$. There exist constants h_1,u_1,u_2,w (which depend on γ and δ) such that for every two finite sets of positive even numbers $\mathcal{H}_1$ and $\mathcal{H}_2$ which satisfy

- *$|\mathcal{H}_1| = |\mathcal{H}_2|$, and $\mathcal{H}_1,\mathcal{H}_2 \subseteq \mathbb{N}^+ \setminus \{1,\ldots,h_1\}$,*

we have

(i) $g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta), g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta), h_{\mathsf{y},\mathcal{H}_1}(u_1,u_2,w), h_{\mathsf{y},\mathcal{H}_2}(u_1,u_2,w) \in \mathbb{R}\setminus\{0\}$, *and*

(ii) $g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta) = g_{\mathsf{y},\mathcal{H}_2}(\gamma,\delta)$ *iff* $h_{\mathsf{y},\mathcal{H}_1}(u_1,u_2,w) = h_{\mathsf{y},\mathcal{H}_2}(u_1,u_2,w)$

Furthermore, u_1 and u_2 are non-zero and $w \notin \{-1,0,1\}$.

Proof. It cannot hold that $|\delta\gamma+1| = |\delta+\gamma|$. Furthermore we know that $\gamma,\delta \neq 0$. Hence, there is h_1 such that for every even $h > h_1$,the sequences $\delta(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h$ and $\delta\gamma(\delta\gamma+1)^h + (\delta+\gamma)^h$ are strictly ascending or descending, and in particular, are non-zero. Therefore we have $g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta), g_{\mathsf{y},\mathcal{H}_1}(\gamma,\delta) \in \mathbb{R}\setminus\{0\}$.

We have for $i=1,2$

$$\begin{aligned}
g_{\mathsf{y},\mathcal{H}_i}(\gamma,\delta) &= \delta \prod_{h\in\mathcal{H}_i} \frac{\delta\gamma\cdot(\delta\gamma+1)^h + (\delta+\gamma)^h}{\delta(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h} \\
&= \frac{\delta}{\gamma^{|\mathcal{H}_i|}} \prod_{h\in\mathcal{H}_i} \frac{\delta\gamma^2\cdot(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h}{\delta(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h} \\
&= \frac{\delta}{\gamma^{|\mathcal{H}_i|}} \prod_{h\in\mathcal{H}_i} \left(1+\frac{\delta(\gamma^2-1)\cdot(\delta\gamma+1)^h}{\delta(\delta\gamma+1)^h + \gamma\cdot(\delta+\gamma)^h}\right) \\
&= \frac{\delta}{\gamma^{|\mathcal{H}_i|}} \prod_{h\in\mathcal{H}_i} \left(1+\frac{1}{\frac{1}{\gamma^2-1} + \frac{\gamma}{\delta(\gamma^2-1)}\cdot\left(\frac{\delta+\gamma}{\delta\gamma+1}\right)^h}\right)
\end{aligned}$$

Let $u_1 = \frac{1}{\gamma^2-1}$, $u_2 = \frac{\gamma}{\delta(\gamma^2-1)}$ and $w = \frac{\delta+\gamma}{\delta\gamma+1}$. We have $u_1,u_2,w \in \mathbb{R}\setminus\{0\}$ and $w \notin \{-1,1\}$. Hence, we can take h_1 to be large enough so that $u_1+u_2+w^h$

non-zero. Since $u_1 + u_2 + w^h$ is strictly ascending or descending for even h, we have $h_{\mathsf{y},\mathcal{H}_1}(u_1, u_2, w), h_{\mathsf{y},\mathcal{H}_2}(u_1, u_2, w) \in \mathbb{R} \setminus \{0\}$ for large enough values of h. □

Proposition 6.2.11. *Let $\gamma, \delta \notin \{-1, 0, 1\}$ and $\gamma \neq -\delta$. Let $\mathcal{H}$ be a set of positive even integers. Let $g_{p,\mathcal{H}}(\mathsf{t}, \mathsf{y}))$ be from Proposition 6.2.9. Then there exists h_2 such that if $\mathcal{H} \subseteq \mathbb{N}^+ \setminus \{1, \dots, h_2\}$ then $g_{p,\mathcal{H}}(\gamma, \delta)) \neq 0$.*

Proof. Recall

$$g_{p,\mathcal{H}}(\gamma, \delta)) = \left(\prod_{h \in \mathcal{H}} \left(\delta(\delta\gamma + 1)^h + \gamma \cdot (\delta + \gamma)^h \right) \right)^{|V(G)|} .$$

We have that $\delta + \gamma$ is non-zero. If $\delta\gamma + 1 = 0$ then the claim holds even for $h_2 = 0$. Otherwise, using that $|\delta\gamma + 1| \neq |\delta + \gamma|$, at least one of $(\delta\gamma + 1)^h, (\delta + \gamma)^h$ becomes strictly larger in absolute value than the other for large enough h. □

6.2.3 Theorems D(ii) and E

The following lemma is a variation of Lemma 4 in [47].

For any $\mathcal{H}$, let $\sigma(\mathcal{H}) = \sum_{h \in \mathcal{H}} h$.

Lemma 6.2.12.
Let $\gamma \notin \{-1, 0, 1\}$, $\delta \neq 0$, $e_1, e_2 \neq 0$ and $r_1, r_2 \notin \{-1, 0, 1\}$ such that $|r_1| \neq |r_2|$. For every positive integer q' there exist $\hat{q} = \Omega(q')$ sets of positive even integers $\mathcal{H}_0, \dots, \mathcal{H}_{\hat{q}}$ such that

(i) $\sigma(\mathcal{H}_i) = O(\log^3 q')$ *for all* i,

(ii) $\sigma(\mathcal{H}_i) = \sigma(\mathcal{H}_j)$ *for all* $i \neq j$,

(iii) $f_{\mathsf{t},\mathcal{H}_i}(e_1, e_2, r_1, r_2) \neq f_{\mathsf{t},\mathcal{H}_j}(e_1, e_2, r_1, r_2)$ *for* $i \neq j$,

where $f_{\mathsf{t},\mathcal{H}}(e_1, e_2, r_1, r_2)$ is from Proposition 6.2.5. If additionally $\delta \notin \{-1, 1\}$ and $\gamma \neq -\delta$, we have

(iv) $g_{\mathsf{y},\mathcal{H}_i}(\gamma, \delta) \neq g_{\mathsf{y},\mathcal{H}_j}(\gamma, \delta)$ *for* $i \neq j$.

(v) $g_{p,\mathcal{H}_i}(\gamma, \delta) \neq 0$,

where $g_{\mathsf{y},\mathcal{H}}(e_1, e_2, r_1)$ *is from Proposition 6.2.9.*

The sets $\mathcal{H}_i$ *can be computed in polynomial time in* q'*.*

Proof.

Let $q = q' \log^3 q'$. First we define sets $\mathcal{H}'_0, \ldots, \mathcal{H}'_q$. We will use these sets to define the desired sets $\mathcal{H}_0, \ldots, \mathcal{H}_{\hat{q}}$.

For $i = 0, \ldots, q$, let $i[0], \ldots, i[\ell] \in \{0, 1\}$ be the binary expansion of i where[4] $\ell = \lfloor \log q \rfloor$. Let Δ be a positive even integer to be chosen later. Let $\tau \in \{1, 2\}$ be such that $|r_\tau| = \max\{|r_1|, |r_2|\}$. Then $|r_{3-\tau}| = \min\{|r_1|, |r_2|\}$. Let m_0 be an even integer such that $m_0 > h_1$ from Proposition 6.2.10 and $m_0 > h_2$ from Proposition 6.2.11. We choose $\mathcal{H}'_i$ as follows:

$$\mathcal{H}'_i = \{m_0 + \Delta\lceil \log q \rceil \cdot (2j + i[j]) : 0 \le j \le \ell\}.$$

The sets $\mathcal{H}'_i$ satisfy:

a. they are distinct,

b. they have equal cardinality $\ell+1$,

c. they contain only positive even integers between m_0 and $m_0 + \Delta(\log q + 1)(2\log q + 1)$, and

d. for i, j and any $a \in \mathcal{H}'_i$ and $b \in \mathcal{H}'_j$, either $a = b$ or $|a - b| \ge \Delta \log q$.

It is easy to see that $\sigma(\mathcal{H}'_i) = \Omega(\log q)$, $i = 0, \ldots, q$. On the other hand, since all the numbers in each of the $\mathcal{H}'_i$ are bounded by $O(\log^2 q)$ and the size of each $\mathcal{H}'_i$ is $O(\log q)$, we get that $\sigma(\mathcal{H}'_i) = O(\log^3 q)$ for each i. From this we get that at least $\hat{q} = \Omega\left(\frac{q' \log^3 q' + 1}{\log^3 q'}\right) = \Omega(q')$ of the sets $\mathcal{H}'_0, \ldots, \mathcal{H}'_q$ have the same sum value $\sigma(\mathcal{H}'_i)$. Let $\{\mathcal{H}_0, \ldots, \mathcal{H}_{\hat{q}}\}$ be a subset of $\{\mathcal{H}'_0, \ldots, \mathcal{H}'_q\}$ such that all the sets in $\{\mathcal{H}_0, \ldots, \mathcal{H}_{\hat{q}}\}$ have the same sum value $\sigma(\mathcal{H}_i)$. We have (i), (ii) and (v) for $\mathcal{H}_0, \ldots, \mathcal{H}_{\hat{q}}$.

We now turn to (iii) and (iv). The proofs of (iii) and (iv) are similar but not identical.

Let $0 \le i \ne j \le \hat{q}$, $\mathcal{H}_{i\setminus j} = \mathcal{H}_i \setminus \mathcal{H}_j$ and $\mathcal{H}_{j\setminus i} = \mathcal{H}_j \setminus \mathcal{H}_i$. Notice $\mathcal{H}_{i\setminus j} \cap \mathcal{H}_{j\setminus i} = \emptyset$.
Let $\sigma = \sigma(\mathcal{H}_{i\setminus j}) = \sigma(\mathcal{H}_{j\setminus i})$ and let $d = |\mathcal{H}_{i\setminus j}| = |\mathcal{H}_{j\setminus i}|$.

[4]Actually, we will also need that ℓ is larger than a constant depending on e_1, but this is true for large enough values of q.

(iii) We write $f_{\mathsf{t},\mathcal{H}_i}$ for short instead of $f_{\mathsf{t},\mathcal{H}_i}(e_1, e_2, r_1, r_2)$ in this proof. When other parameters are used instead of e_1, e_2, r_1, r_2 we write them explicitly. Since $f_{\mathsf{t},\mathcal{H}_i} = f_{\mathsf{t},\mathcal{H}_{i\setminus j}} \cdot f_{\mathsf{t},\mathcal{H}_i \cap \mathcal{H}_j}$, $f_{\mathsf{t},\mathcal{H}_j} = f_{\mathsf{t},\mathcal{H}_{j\setminus i}} \cdot f_{\mathsf{t},\mathcal{H}_i \cap \mathcal{H}_j}$ and $f_{\mathsf{t},\mathcal{H}_i \cap \mathcal{H}_j} \neq 0$, it is enough to show that $f_{\mathsf{t},\mathcal{H}_{i\setminus j}} - f_{\mathsf{t},\mathcal{H}_{j\setminus i}} \neq 0$.

Since $\sigma(\mathcal{H}_{i\setminus j}) = \sigma(\mathcal{H}_{j\setminus i})$ we have

$$f_{\mathsf{t},\mathcal{H}_{i\setminus j}} = f_{\mathsf{t},\mathcal{H}_{j\setminus i}} \text{ iff } f_{\mathsf{t},\mathcal{H}_{i\setminus j}}(e_1, e_2, r_2^{-1}, r_1^{-1}) = f_{\mathsf{t},\mathcal{H}_{j\setminus i}}(e_1, e_2, r_2^{-1}, r_1^{-1}) \,.$$

Hence we can assume from now on that $|r_\tau| > 1$ (otherwise we look at r_1^{-1} and r_2^{-1} instead).

For every $\mathcal{H}$, $f_{\mathsf{t},\mathcal{H}}$ can be rewritten as follows:

$$f_{\mathsf{t},\mathcal{H}} = \prod_{h \in \mathcal{H}} (e_\tau r_\tau^h + e_{3-\tau} r_{3-\tau}^h) = e_{3-\tau}^{\ell+1} \sum_{X \subseteq \mathcal{H}} s_{\mathcal{H}}(X) \,,$$

where

$$s_{\mathcal{H}}(X) = \left(\frac{e_\tau}{e_{3-\tau}} \right)^{|X|} r_\tau^{\sigma(X)} r_{3-\tau}^{\sigma(\mathcal{H}\setminus X)} \,.$$

We think of $h \in X$ (respectively $h \in \mathcal{H} \setminus X$) as corresponding to $e_\tau r_\tau^h$ (respectively $e_{3-\tau} r_{3-\tau}^h$).

It suffices to show that

$$\sum_{X_1 \subseteq \mathcal{H}_{i\setminus j}} s_{\mathcal{H}_{i\setminus j}}(X) - \sum_{X_2 \subseteq \mathcal{H}_{j\setminus i}} s_{\mathcal{H}_{j\setminus i}}(X) \neq 0 \,. \tag{6.2}$$

It holds that $s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j}) = s_{\mathcal{H}_{j\setminus i}}(\mathcal{H}_{j\setminus i}) = \left(\frac{e_\tau}{e_{3-\tau}} \right)^{\ell+1} r_\tau^\sigma$. Hence, $s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j})$ and $s_{\mathcal{H}_{j\setminus i}}(\mathcal{H}_{j\setminus i})$ cancel out in Inequality (6.2). Similarly, $s_{\mathcal{H}_{i\setminus j}}(\emptyset) = s_{\mathcal{H}_{j\setminus i}}(\emptyset) = r_{3-\tau}^\sigma$ cancel out. Let m_1 be the minimal element in $\mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i}$. Without loss of generality, assume $m_1 \in \mathcal{H}_{i\setminus j}$. We have

$$s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j} \setminus \{m_1\}) = \left(\frac{e_\tau}{e_{3-\tau}} \right)^{\ell} r_\tau^{\sigma - m_1} r_{3-\tau}^{m_1} \,.$$

$s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j} \setminus \{m_1\})$ has the largest exponent of r_τ out of all the monomials in both of the sums in Inequality (6.2), and any other exponent of r_τ is smaller by at least $\Delta \log q$. We will show that Inequality (6.2) holds by showing the following:

$$|s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j} \setminus \{m_1\})| > \sum_{X \subsetneq \mathcal{H}_{i\setminus j}\setminus\{m_1\}} |s_{\mathcal{H}_{i\setminus j}}(X)| + \sum_{X \subsetneq \mathcal{H}_{j\setminus i}} |s_{\mathcal{H}_{j\setminus i}}(X)| \,. \tag{6.3}$$

Each of the sums in Inequality (6.3) has at most $2^{\log q+1} = 2q$ monomials corresponding to the subsets of $\mathcal{H}_{i\setminus j}$ and $\mathcal{H}_{j\setminus i}$ respectively. The absolute value of each of these monomials can be bounded from above by $s \cdot |r_\tau|^{\sigma - m_1 - \Delta \log q} |r_{3-\tau}|^{m_1 + \Delta \log q}$, where s is the maximum of $\left|\frac{e_\tau}{e_{3-\tau}}\right|^\ell$ and 1. Hence, the right-hand side of Inequality (6.3) is at most

$$\begin{gathered} 4q \cdot s|r_\tau|^{\sigma - m_1 - \Delta \log q} |r_{3-\tau}|^{m_1 + \Delta \log q} = \\ 4qs \left(\frac{e_\tau}{e_{3-\tau}}\right)^{-\ell} \cdot \left|\frac{r_{3-\tau}}{r_\tau}\right|^{\Delta \log q} |s_{\mathcal{H}_{i\setminus j}}(\mathcal{H}_{i\setminus j} - \{m_1\})| \,. \end{gathered}$$

There exists a number Δ' which does not depend on q such that $4qs\left(\frac{e_\tau}{e_{3-\tau}}\right)^{-\ell} < (\Delta')^{\log q}$ and (iii) follows by setting Δ large enough so that $\Delta' \cdot \left|\frac{r_{3-\tau}}{r_\tau}\right|^{\Delta} < 1$.

(iv) By Proposition 6.2.10, there exist $u_1, u_2 \neq 0$ and $w \notin \{-1, 0, 1\}$ depending on γ, δ for which it is enough to show that $h_{\mathsf{y},\mathcal{H}_i}(u_1, u_2, w) \neq h_{\mathsf{y},\mathcal{H}_j}(u_1, u_2, w)$ to get (iv). We write $h_{\mathsf{y},\mathcal{H}_i}$ for short instead of $h_{\mathsf{y},\mathcal{H}_i}(u_1, u_2, w)$ in this proof.

Since we have $h_{\mathsf{y},\mathcal{H}_i} = h_{\mathsf{y},\mathcal{H}_{i\setminus j}} \cdot h_{\mathsf{y},\mathcal{H}_i \cap \mathcal{H}_j}$, $h_{\mathsf{y},\mathcal{H}_j} = h_{\mathsf{y},\mathcal{H}_{j\setminus i}} \cdot h_{\mathsf{y},\mathcal{H}_i \cap \mathcal{H}_j}$ and $h_{\mathsf{y},\mathcal{H}_i \cap \mathcal{H}_j} \neq 0$, it is enough to show that

$$h_{\mathsf{y},\mathcal{H}_{i\setminus j}} - h_{\mathsf{y},\mathcal{H}_{j\setminus i}} \neq 0 \,,$$

i.e.

$$\prod_{h \in \mathcal{H}_{i\setminus j}} \left(1 + \frac{1}{u_1 + u_2 \cdot w^h}\right) - \prod_{h \in \mathcal{H}_{j\setminus i}} \left(1 + \frac{1}{u_1 + u_2 \cdot w^h}\right) \neq 0 \,.$$

or equivalently,

$$\begin{aligned}\prod_{h\in\mathcal{H}_{i\setminus j}}\left(u_1+u_2\cdot w^h+1\right)\prod_{h\in\mathcal{H}_{j\setminus i}}\left(u_1+u_2\cdot w^h\right) &\quad - \\ \prod_{h\in\mathcal{H}_{j\setminus i}}\left(u_1+u_2\cdot w^h+1\right)\prod_{h\in\mathcal{H}_{i\setminus j}}\left(u_1+u_2\cdot w^h\right) &\neq \quad 0\end{aligned} \tag{6.4}$$

Consider a product of the form found in Inequality (6.4).

$$\begin{aligned}\prod_{h\in\mathcal{H}_a}\left(u_1+u_2\cdot w^h+1\right)\prod_{h\in\mathcal{H}_b}\left(u_1+u_2\cdot w^h\right) &= \\ \sum_{X\subseteq\mathcal{H}_a\cup\mathcal{H}_b}(u_1+1)^{|\mathcal{H}_a\setminus X|}u_1^{|\mathcal{H}_b\setminus X|}w^{\sigma(X)}u_2^{|X|}\end{aligned}$$

Let

$$\begin{aligned}p(X) \quad = \quad &\left((u_1+1)^{|\mathcal{H}_{i\setminus j}\setminus X|}u_1^{|\mathcal{H}_{j\setminus i}\setminus X|}-(u_1+1)^{|\mathcal{H}_{j\setminus i}\setminus X|}u_1^{|\mathcal{H}_{i\setminus j}\setminus X|}\right)\\ &\cdot u_2^{|X|}w^{\sigma(X)}\end{aligned}$$

It suffices to show that

$$\sum_{X\subseteq\mathcal{H}_{i\setminus j}\cup\mathcal{H}_{j\setminus i}}p(X)\neq 0\,. \tag{6.5}$$

We have $p(\emptyset)=p(\mathcal{H}_{i\setminus j}\sqcup\mathcal{H}_{j\setminus i})=0$, using that $|\mathcal{H}_{i\setminus j}|=|\mathcal{H}_{j\setminus i}|$. Let m_1 be the minimal element in $\mathcal{H}_{i\setminus j}\sqcup\mathcal{H}_{j\setminus i}$. Without loss of generality, assume $m_1\in\mathcal{H}_{i\setminus j}$. We have

$$\begin{aligned}\left|p(\mathcal{H}_{i\setminus j}\sqcup\mathcal{H}_{j\setminus i}-\{m_1\})\right| &= \left|u_2^{2d-1}w^{2\sigma-m_1}\right|\\ |p(\{m_1\})| &= \left|((u_1+1)u_1)^{d-1}u_2w^{m_1}\right|\end{aligned}$$

The largest exponent of w in Inequality (6.5) is $w^{2\sigma-m_1}$. For all other monomials in (6.5), the power of w is smaller by at least $\Delta\log q$. Similarly, the smallest exponent of w in Inequality (6.5) is w^{m_1}. For all other monomials in (6.5), the power of w is larger by at least $\Delta\log q$.

Let $X_0\subseteq\mathcal{H}_{i\setminus j}\sqcup\mathcal{H}_{j\setminus i})$ be maximal with respect to $|p(X_0)|$. Since

$d \leq \log q + 1$, we can choose Δ large enough so that we have $X_0 = \mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i} - \{m_1\}$ if $|w| > 1$ and $X_0 = \{m_1\}$ if $|w| < 1$.

We have the following:

$$\left| \sum_{X \subseteq \mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i} : X \neq X_0} p(X) \right| < |p(X_0)| \tag{6.6}$$

implying that Inequality (6.5) holds. To see that Inequality (6.6) holds, note that

$$\left| \sum_{\substack{X \subseteq \mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i} : \\ X \neq X_0}} p(X) \right| \leq 2^{2\log q + 2} \max_{X \subseteq \mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i} : X \neq X_0} |p(X)|$$

Let

$$k(d) = \max\left(\left| (u_1 + 1)^d \right|, 1 \right) \cdot \max\left(\left| u_1^d \right|, 1 \right) \cdot \max\left(\left| u_2^d \right|, 1 \right).$$

Then

$$\left| \sum_{\substack{X \subseteq \mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i} : \\ X \neq X_0}} p(X) \right| \leq \begin{cases} 4 \cdot 2^{\log q + 1} \cdot k(d) \cdot |w|^{2\sigma - m_1 - \Delta \log q}, & |w| > 1 \\ 4 \cdot 2^{\log q + 1} \cdot k(d) \cdot |w|^{m_1 + \Delta \log q}, & |w| < 1 \end{cases}$$

So, there exists a constant $c > 0$ depending on u_1, u_2, w such that (for large enough values of q),

$$\left| \sum_{\substack{X \subseteq \mathcal{H}_{i\setminus j} \sqcup \mathcal{H}_{j\setminus i} : \\ X \neq X_0}} p(X) \right| \leq \begin{cases} c^{\log q} |w|^{2\sigma - m_1 - \Delta \log q}, & |w| > 1 \\ c^{\log q} |w|^{m_1 + \Delta \log q}, & |w| < 1 \end{cases}$$

It remains to choose Δ large enough so that

$$\begin{cases} \frac{c^{\log q}}{u_2^{2d-1}} < |w|^{\Delta \log q}, & |w| > 1 \\ |w|^{\Delta \log q} < \frac{((u_1+1)u_1)^{d-1}}{c^{\log q}}, & |w| < 1 \end{cases}$$

□

We are now ready to prove Theorems D(ii) and E.

Theorem 6.2.13 (D(ii) and E).
Let $(\gamma, \delta) \in \mathbb{Q}^2$. If $\gamma \notin \{-1, 0, 1\}$ and $\delta \neq 0$, then

*(i) computing $Z_{Ising}(G; \gamma, \delta)$ is #**P**-hard, and*

*(ii) unless #**ETH** fails, computing $Z_{Ising}(G; \gamma, \delta)$ requires exponential time in $\frac{n_G}{\log^6 n_G}$.*

Otherwise, $Z_{Ising}(G; \gamma, \delta)$ is polynomial-time computable.

Proof. We set $\mathsf{t} = \gamma$ and $\mathsf{y} = \delta$ with $\gamma \notin \{-1, 0, 1\}$ and $\delta \neq 0$. By abuse of notation we refer to $c_1, c_2, d_1, d_2, \lambda_1, \lambda_2$ from Lemma 6.2.2 as the values they obtain when $\mathsf{t} = \gamma$. Since $\gamma \neq \{-1, 0, 1\}$, it is easy to verify that the following hold:

a. $c_1 + d_1, c_2 + d_2 \neq 0$,

b. $\lambda_1, \lambda_2 \neq 0$,

c. $\lambda_1, \lambda_2 \neq \pm(\gamma^2 + \gamma)$, and

d. $\lambda_1 \neq \pm\lambda_2$.

Let $e_i = \frac{c_i + d_i}{2\gamma}$ and $r_i = \frac{\lambda_i}{\gamma^2 + \gamma}$ for $i = 1, 2$. Let $q' = n_G^2$. Let $\mathcal{H}_0, \dots, \mathcal{H}_{\hat{q}}$ be the sets guaranteed in Lemma 6.2.12 with respect to $q', \gamma, \delta, e_1, e_2, r_1, r_2$.

First we deal with that case that $\gamma \neq -\delta$. We return to $\gamma = -\delta$ later.

We want to compute the $\hat{q} + 1$ values $Z_{Ising}(G \otimes \mathcal{H}_k; \gamma, 1)$. If $\delta = 1$ we simply do it using the oracle to Z at $(\gamma, 1)$. If $\delta = -1$ we use Lemma 6.2.6. Otherwise we proceed as follows.

By Proposition 6.2.9, for each $0 \leq i, k \leq \hat{q}$,

$$Z_{Ising}(S_{\mathcal{H}_i}(G \otimes \mathcal{H}_k); \gamma, \delta) = g_{p,\mathcal{H}_i}(\gamma, \delta) \cdot Z_{Ising}(G \otimes \mathcal{H}_k; \gamma, g_{\mathsf{y},\mathcal{H}_i}(\gamma, \delta)) \,. \tag{6.7}$$

It is guaranteed in Lemma 6.2.12 that for $i \neq j$, $g_{\mathsf{y},\mathcal{H}_i}(\gamma,\delta) \neq g_{\mathsf{y},\mathcal{H}_j}(\gamma,\delta)$.

We want to use Equation (6.7) to interpolate, for each $0 \leq k \leq m_G$, the univariate polynomials $Z_{Ising}(G \otimes \mathcal{H}_k; \gamma, \mathsf{y})$. We use the fact that the sizes of $G \otimes \mathcal{H}_k$, and therefore the y-degrees of $Z_{Ising}(G \otimes \mathcal{H}_k; \gamma, \mathsf{y})$, are at most $O(n_G \log^3 n_G)$, Since $g_{p,\mathcal{H}_i}(\gamma,\delta)$ is non-zero, we can interpolate in polynomial-time, for each $0 \leq k \leq m_G$, the $m_G + 1$ polynomials $Z_{Ising}(G \otimes \mathcal{H}_k; \gamma, \mathsf{y})$.

So, we computed $Z_{Ising}(G \otimes \mathcal{H}_k; \gamma, 1)$ for $0 \leq k \leq \hat{q}$. Now we use these values to interpolate t and get the univariate polynomial $Z_{Ising}(G \otimes \mathcal{H}_k; \mathsf{t}, 1)$. By Lemma 6.2.5,

$$Z_{Ising}(G; f_{\mathsf{t},\mathcal{H}_k}(e_1,e_2,r_1,r_2), 1) = Z_{Ising}(G \otimes \mathcal{H}_k; \gamma, 1) \cdot (f_{p,\mathcal{H}_k}(\gamma))^{-1} \ .$$

Since $\gamma \notin \{-1,0,1\}$, $f_{p,\mathcal{H}_k}(\gamma) \neq 0$. By Lemma 6.2.12, $f_{\mathsf{t},\mathcal{H}_k}(e_1,e_2,r_1,r_2)$ are distinct and polynomial time computable. Hence, the univariate polynomial $Z_{Ising}(G; \mathsf{t}, 1)$ can be interpolated. We get (i) by Proposition 6.1.4. Since $Z_{Ising}(-; \gamma, \delta)$ is only queried on graphs $S_{\mathcal{H}_i}(G \otimes \mathcal{H}_k)$ of sizes at most $O(n_G \log^6 n_G)$, (ii) holds by Proposition 6.1.5.

Consider the case $\gamma = -\delta$. By Proposition 6.2.9, for every G we have

$$Z_{Ising}(S_{\{1\}}(G); \gamma, \delta) = (\delta \cdot (1-\delta^2))^{n_G} \cdot Z_{Ising}(G; \gamma, -\delta^2) \,.$$

and the desired hardness results follow by the corresponding for $Z_{Ising}(G; \gamma, -\delta^2)$ (using that $\gamma \neq -(-\delta^2)$, $-\delta^2 \notin \{-1,0,1\}$ and that $(\delta \cdot (1-\delta^2))^{n_G}$ is non-zero).

Now we consider the cases where $\gamma \in \{-1,0,1\}$ or $\delta = 0$. Two cases are easily computed, namely $Z_{Ising}(G; 1, \delta) = (1+\delta)^{n_G}$ and $Z_{Ising}(G; \gamma, 0) = 1$.

The other two cases follows e.g. from Lemma 6.3 in [78]. In that lemma it is shown in particular that partition functions $Z_{A,D}(G)$ with a matrix A of edge-weights and a diagonal matrix D of vertex weights can be computed in polynomial time if A has rank 1 or is bipartite with rank 2. For $\gamma = 0$ we have

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}$$

so A is bipartite with rank 2. For $\gamma = -1$ we have

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}$$

so A has rank 1. Note Lemma 6.3 extends to negative values of δ. We refer the reader to [78] for details. $\square$

6.3 Simple Bipartite Planar Graphs

In this section we show that the evaluations of $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ are generally #**P**-hard to compute, even when restricted to simple graphs which are both bipartite and planar. To do so, we use that for 3-regular graphs, $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ is essentially equivalent to $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$. We use a two-dimensional graph transformation $R^{\ell,q}(G)$ which in applied to simple 3-regular bipartite planar graphs and emits simple bipartite planar graphs in order to interpolate $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$.

6.3.1 Definitions

The following is a variation of *k-thickening* for simple graphs:

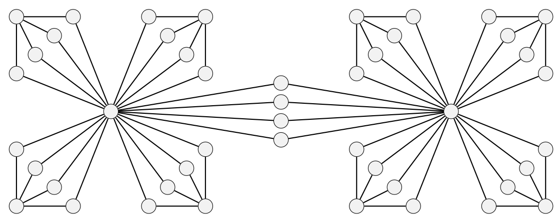
Definition 6.3.1 (k-Simple Thickening). *Given $\ell \in \mathbb{N}^+$ and a graph H, we define a graph $STh^{\ell}(H)$ as follows. For every edge $e = (u, w)$ in $E(H)$, we add 4ℓ new vertices $v_{e,1}, \ldots, v_{e,4\ell}$ to H. For each $v_{e,i}$, we add two new edges $(u, v_{e,i})$ and $(w, v_{e,i})$. Finally, we remove the edge e from the graph. Let $N_\ell(e)^+$ denote the subgraph of $STh^{\ell}(H)$ induced by the set of vertices $\{v_{e,1}, \ldots, v_{e,4\ell}, u, v\}$.*

The graph transformation used in the hardness proof is the following:

Definition 6.3.2 ($R^{\ell,q}(G)$). *Let G be a graph. For each $w \in V(G)$, let $G^q_w = (V^q_w, E^q_w)$ be a new copy of the star with $2q$ leaves. Denote by c_w the center of the star G^q_w. Let $R^{\ell,q}(G) = (V^{\ell,q}_R, E^{\ell,q}_R)$ be the graph obtained from the disjoint union of $STh^{\ell}(G)$ and $STh^{\ell}(G^q_w)$ for all $w \in V(G)$ by identifying w and c_w for all $w \in V(G)$.*

Remarks 6.3.1.

Figure 6.4: The construction of the graph $R^{\ell,q}(P_2)$ for $\ell = 1$ and $q = 2$, where P_2 is the path with two vertices and one edge.



(i) The construction of $R^{\ell,q}(G)$ can also be described as follows. Given G, we attach $2q$ new vertices to each vertex v of $V(G)$ to obtain a new simple graph G'. Then, $R^{\ell,q}(G) = STh^{\ell}(G')$.

(ii) For every simple planar graph G and $\ell, q \in \mathbb{N}^+$, $R^{\ell,q}(G)$ is a simple bipartite planar bipartite graph with n_R vertices and m_R edges, where $n_R = n_G(1 + 2q(1 + 4\ell)) + 4\ell m_G$ and and $m_R = 8\ell m_G + 16\ell q n_G$.

Figure 6.4 shows the graph $R^{1,2}(P_2)$.

In the following it is convenient to consider a multivariate version of $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ denoted $Z_{Ising}(G; \bar{\mathsf{x}}, \bar{\mathsf{y}}.\bar{\mathsf{z}})$. This approach was introduced for the Tutte polynomial by A. Sokal, see [136]. $Z_{Ising}(G; \bar{\mathsf{x}}, \bar{\mathsf{y}}.\bar{\mathsf{z}})$ has indeterminates which correspond to every $v \in V(G)$ and every $e \in E(G)$.

Definition 6.3.3. *Let* $\bar{\mathsf{x}} = (\mathsf{x}_e : e \in E(G))$, $\bar{\mathsf{y}} = (\mathsf{y}_u : u \in V(G))$ *and* $\bar{\mathsf{z}} = (\mathsf{z}_e : e \in E(G))$ *be tuples of distinct indeterminates. Let*

$$Z_{Ising}(G; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = \sum_{S \subseteq V(G)} \left(\prod_{e \in E_G(S)} \mathsf{x}_e \right) \left(\prod_{u \in S} \mathsf{y}_u \right) \left(\prod_{e \in E_G(\bar{S})} \mathsf{z}_e \right).$$

We may write $\mathsf{x}_{w,v}$ and $\mathsf{z}_{w,v}$ instead of x_e and z_e for an edge $e = (w, v)$. Clearly, by setting $\mathsf{x}_e = \mathsf{x}$ and $\mathsf{z}_e = \mathsf{z}$ for every $e \in E(G)$, and $\mathsf{y}_u = \mathsf{y}$ for every $u \in V(G)$ we get $Z_{Ising}(G; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$.

We furthermore define a variation of $Z_{Ising}(G; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$ obtained by restricting the range of the summation variable as follows:

Definition 6.3.4. *Given a graph H and $B, C \subseteq V(H)$ with B and C disjoint, let*

$$Z_{Ising}(H, B, C; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = \sum_{A\,:\,B\subseteq A\subseteq V(H)\,,\,A\cap C=\emptyset} \left(\prod_{e\in E_G(A)} \mathsf{x}_e\right)\left(\prod_{u\in A\setminus B} \mathsf{y}_u\right)\left(\prod_{e\in E_G(\bar{A})} \mathsf{z}_e\right) \tag{6.8}$$

where the summation is over all $A \subseteq V(H)$, such that A contains B and is disjoint from C.

We have $Z_{Ising}(H, \emptyset, \emptyset; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = Z_{Ising}(H; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$.

6.3.2 Lemmas, statement of Theorem D(i) and its proof

For every edge $e \in E(G)$ between u and v, let

$$\omega_1(e, S) = Z_{Ising}(N_\ell(e)^+, S \cap \{u, v\}, \{u, v\} \setminus S; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})\,,$$

and for every vertex $w \in V$, let

$$\omega_2(w, S) = Z_{Ising}(STh^\ell(G^q_w), S \cap \{w\}, \{w\} \setminus S; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})\,.$$

Let

$$\omega_1(S) = \prod_{e\in E(G)} \omega_1(e, S) \quad \text{and} \quad \omega_2(S) = \prod_{w\in V(G)} \omega_2(w, S)\,.$$

Let $\omega_{i,\mathrm{triv}}(S)$ for $i = 1, 2$ be the polynomials in x, y and z obtained from $\omega_i(S)$ by setting $\mathsf{x}_e = \mathsf{x}$ and $\mathsf{z}_e = \mathsf{z}$ for every $e \in E^{\ell,q}$ and $\mathsf{y}_v = \mathsf{y}$ for every $v \in V_R^{\ell,q}$.

Lemma 6.3.2.

$$Z_{Ising}(R^{\ell,q}(G); \mathsf{x}.\mathsf{y}.\mathsf{z}) = \sum_{S\subseteq V(G)} \omega_{1,\mathrm{triv}}(S) \cdot \omega_{2,\mathrm{triv}}(S) \cdot \mathsf{y}^{|S|}\,.$$

Proof. Each edge of $R^{\ell,q}(G)$ is either contained in some $N_\ell(e)^+$ for $e \in E(G)$, or in some $STh^\ell(G^q_w)$ for $w \in V(G)$. Hence, by the definitions of

$Z_{Ising}(R^{\ell,q}(G);\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}})$, $\omega_1(S)$ and $\omega_2(S)$,

$$Z_{Ising}(R^{\ell,q}(G);\bar{\mathsf{x}},\bar{\mathsf{y}},\bar{\mathsf{z}}) = \sum_{S\subseteq V(G)} \omega_1(S)\cdot\omega_2(S)\cdot\prod_{w\in S}\mathsf{y}_w$$

holds and the lemma follows. □

Lemma 6.3.3. *Let $e=(u,w)$ be an edge of G. Then*

$$\omega_{1,\mathrm{triv}}(e,S) = \begin{cases} (\mathsf{y}+\mathsf{z}^2)^{4\ell} & |\{u,v\}\cap S| = 0 \\ (\mathsf{xy}+\mathsf{z})^{4\ell} & |\{u,v\}\cap S| = 1 \\ (\mathsf{yx}^2+1)^{4\ell} & |\{u,v\}\cap S| = 2 \end{cases}$$

Proof. The value of $\omega_1(e,S)$ depends only on whether $u,w\in S$. Consider $A\subseteq V(N_\ell(e)^+)$ which satisfies the summation conditions in Equation (6.8) for $Z_{Ising}(N_\ell(e)^+, S\cap\{u,w\},\{u,w\}\setminus S;\mathsf{x},\mathsf{y},\mathsf{z})$.

(i) If $w\in S$ and $u\notin S$: Exactly one edge e' incident to $v_{e,i}$ crosses the cut $[A,\bar{A}]_{N_\ell(e)^+}$. The other edge e'' incident to $v_{e,i}$ belongs to $E(A)$ or $E(\bar{A})$, depending on whether $v_{e,i}\in A$. We get:

$$\omega_1(e,S) = \prod_{i=1}^{4\ell}(\mathsf{x}_{v_{e,i},w}\mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},u})\,.$$

(ii) If $w\notin S$ and $u\in S$: this case is similar to the previous case, and we get

$$\omega_1(e,S) = \prod_{i=1}^{4\ell}(\mathsf{x}_{v_{e,i},u}\mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},w})\,.$$

(iii) If $w,u\in S$: For each $v_{e,i}$, either $v_{e,i}\in A$, in which case both edges $(v_{e,i},w)$ and $(v_{e,i},u)$ are in $E(A)$, or $v_{e,i}\notin S$, and both edges $(v_{e,i},w)$ and $(v_{e,i},u)$ cross the cut. We get:

$$\omega_1(e,S) = \prod_{i=1}^{4\ell}(\mathsf{y}_{v_{e,i}}\mathsf{x}_{v_{e,i},u}\mathsf{x}_{v_{e,i},w} + 1)\,.$$

(iv) If $w,u\notin S$: For each $v_{e,i}$, either $v_{e,i}\in S$ and then both edges incident to $v_{e,i}$ cross the cut, or $v_{e,i} \notin S$ and none of the two edges cross the cut. We get:

$$\omega_1(e,S) = \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},w}\mathsf{z}_{v_{e,i},u})\,.$$

The lemma follows by setting $\mathsf{x}_e = \mathsf{x}$ and $\mathsf{z}_e = \mathsf{z}$ for every edge e and $\mathsf{y}_u = \mathsf{y}$ for every vertex u. $\square$

Lemma 6.3.4. *Let*

$$\begin{aligned} g_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z}) &= \mathsf{y}\cdot\left(\mathsf{y}\mathsf{x}^2+1\right)^{4\ell} + (\mathsf{y}\mathsf{x}+\mathsf{z})^{4\ell} \\ h_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z}) &= \left(\mathsf{y}+\mathsf{z}^2\right)^{4\ell} + \mathsf{y}\cdot(\mathsf{y}\mathsf{x}+\mathsf{z})^{4\ell}\,. \end{aligned}$$

Let w be a vertex of G. Then

$$\omega_{2,\mathrm{triv}}(w,S) = \begin{cases} (g_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z}))^{2q} & w \in S \\ (h_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z}))^{2q} & w \notin S \end{cases}$$

Proof. Consider A which satisfies the summation conditions in Equation (6.8) for $Z_{Ising}(STh^{\ell}(G^q_w), S\cap\{w\}, \{w\}\setminus S; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$.

(i) If $w \in S$ (or, equivalently, $c_w \in A$): Let $u \in V^q_w \setminus \{c_w\}$ and $e = \{u, c_w\}$. If $u \in A$, then the vertices u and $v_{e,1},\ldots,v_{e,4\ell}$ contribute

$$\mathsf{y}_u \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}}\mathsf{x}_{v_{e,i},w}\mathsf{x}_{v_{e,i},u} + 1)\,.$$

Otherwise, if $u \notin A$, then the vertices u and $v_{e,1},\ldots,v_{e,4\ell}$ contribute

$$\prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}}\mathsf{x}_{v_{e,i},w} + \mathsf{z}_{v_{e,i},u})\,.$$

Hence, $\omega_2(w,S)$ equals in this case

$$\prod_{u\in V^q_w}\left(\mathsf{y}_u \prod_{i=1}^{4\ell}(\mathsf{y}_{v_{e,i}}\mathsf{x}_{v_{e,i},w}\mathsf{x}_{v_{e,i},u}+1) + \prod_{i=1}^{4\ell}(\mathsf{y}_{v_{e,i}}\mathsf{x}_{v_{e,i},w}+\mathsf{z}_{v_{e,i},u})\right)\,.$$

(ii) If $w \notin S$ (or, equivalently, $c_w \notin A$): Let $u \in V^q_w \setminus \{c_w\}$ and $e = \{u, c_w\}$.

If $u \in A$, then the vertices u and $v_{e,1}, \ldots, v_{e,4\ell}$ contribute

$$\mathsf{y}_u \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} \mathsf{x}_{v_{e,i},u} + \mathsf{z}_{v_{e,i},w}) \,.$$

Otherwise, if $u \notin A$, then the vertices u and $v_{e,1}, \ldots, v_{e,4\ell}$ contribute

$$\prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},w} \mathsf{z}_{v_{e,i},u}) \,.$$

Hence, $\omega_2(w, S)$ equals in this case

$$\prod_{u \in V_w^q} \left(\prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} \mathsf{x}_{v_{e,i},u} + \mathsf{z}_{v_{e,i},w}) + \prod_{i=1}^{4\ell} (\mathsf{y}_{v_{e,i}} + \mathsf{z}_{v_{e,i},w} \mathsf{z}_{v_{e,i},u}) \right) .$$

The lemma follows by setting $\mathsf{x}_e = \mathsf{x}$ and $\mathsf{z}_e = \mathsf{z}$ for every edge e and $\mathsf{y}_u = \mathsf{y}$ for every vertex u. □

Lemma 6.3.5. *If G is d-regular, then*

$$\begin{aligned} Z_{Ising}(R^{\ell,q}(G); \mathsf{x}, \mathsf{y}, \mathsf{z}) &= f_{p,R}(\mathsf{x}, \mathsf{y}, \mathsf{z}, \ell, q) \cdot \\ &\quad Z_{Ising}(G; f_{\mathsf{t},R}(\mathsf{x}, \mathsf{y}, \mathsf{z}, \ell), f_{\mathsf{y},R}(\mathsf{x}, \mathsf{y}, \mathsf{z}, \ell, q)) \end{aligned}$$

where

$$\begin{aligned} f_{p,R}(\mathsf{x}, \mathsf{y}, \mathsf{z}, \ell, q) &= (h_{\ell,q}(\mathsf{x}, \mathsf{y}, \mathsf{z}))^{2qn_G} (\mathsf{y} + \mathsf{z}^2)^{2\ell d n_G} \,, \\ f_{\mathsf{t},R}(\mathsf{x}, \mathsf{y}, \mathsf{z}, \ell) &= \left(\frac{(\mathsf{y}\mathsf{x} + \mathsf{z})^2}{(\mathsf{y}\mathsf{x}^2 + 1)(\mathsf{y} + \mathsf{z}^2)} \right)^{2\ell} \,, \\ f_{\mathsf{y},R}(\mathsf{x}, \mathsf{y}, \mathsf{z}, \ell, q) &= \mathsf{y} \cdot \left(\frac{\mathsf{y}\mathsf{x}^2 + 1}{\mathsf{y} + \mathsf{z}^2} \right)^{2\ell d} \left(\frac{g_{\ell,q}(\mathsf{x}, \mathsf{y}, \mathsf{z})}{h_{\ell,q}(\mathsf{x}, \mathsf{y}, \mathsf{z})} \right)^{2q} . \end{aligned}$$

Proof. We want to rewrite $Z_{Ising}(R^{\ell,q}(G); \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$ as a sum over subsets S of vertices of G. Using Lemma 6.3.2, in order to compute $Z_{Ising}(R^{\ell,q}(G); \mathsf{x}, \mathsf{y}, \mathsf{z})$ we first need to find $\omega_{1,\mathrm{triv}}(S)$ and $\omega_{2,\mathrm{triv}}(S)$. Using Lemma 6.3.4, $\omega_{2,\mathrm{triv}}(S)$ is given by

$$\omega_{2,\mathrm{triv}}(S) = (g_{\ell,q}(\mathsf{x}, \mathsf{y}, \mathsf{z}))^{2q|S|} \cdot (h_{\ell,q}(\mathsf{x}, \mathsf{y}, \mathsf{z}))^{2qn_G - 2q|S|} \,.$$

In order to compute $\omega_{1,\mathrm{triv}}(S)$, consider $S \subseteq V(G)$. Since G is d-regular, the number of edges contained in S is $\frac{1}{2}(d \cdot |S| - |[S,\bar{S}]_G|)$, and the number of edges contained in $\bar{S}$ is $\frac{1}{2}(dn_G - d \cdot |S| - |[S,\bar{S}]_G|)$. Hence, by Lemma 6.3.3, $\omega_{1,\mathrm{triv}}(S)$ is given by

$$\omega_{1,\mathrm{triv}}(S) = (\mathsf{xy}+\mathsf{z})^{4\ell|[S,\bar{S}]_G|}(\mathsf{yx}^2+1)^{4\ell\cdot\frac{d\cdot|S|-|[S,\bar{S}]_G|}{2}}(\mathsf{y}+\mathsf{z}^2)^{4\ell\cdot\frac{dn_G-d\cdot|S|-|[S,\bar{S}]_G|}{2}}\,.$$

Using Lemma 6.3.2,

$$Z_{Ising}(R^{\ell,q}(G);\mathsf{x},\mathsf{y},\mathsf{z}) = \sum_{S\subseteq V(G)} \omega_{1,\mathrm{triv}}(S)\cdot\omega_{2,\mathrm{triv}}(S)\cdot \mathsf{y}^{|S|}$$

which is equal to $(\mathsf{y}+\mathsf{z}^2)^{4\ell\cdot\frac{dn_G}{2}}$ times

$$\sum_{S\subseteq V(G)} \left(\frac{(\mathsf{yx}+\mathsf{z})^2}{(\mathsf{yx}^2+1)(\mathsf{y}+\mathsf{z}^2)}\right)^{2\ell|[S,\bar{S}]_G|}\left(\mathsf{y}\cdot\left(\frac{\mathsf{yx}^2+1}{\mathsf{y}+\mathsf{z}^2}\right)^{2\ell d}\right)^{|S|}\cdot\omega_{2,\mathrm{triv}}(S)\,. \tag{6.9}$$

Plugging the expression for $\omega_{2,\mathrm{triv}}(S)$ in Equation (6.9), we get that $Z_{Ising}(R^{\ell,q}(G);\mathsf{x},\mathsf{y},\mathsf{z})$ equals $f_{p,R}(\mathsf{x},\mathsf{y},\mathsf{z},\ell,q)$ times

$$\sum_{S\subseteq V(G)} \left(\frac{(\mathsf{yx}+\mathsf{z})^2}{(\mathsf{yx}^2+1)(\mathsf{y}+\mathsf{z}^2)}\right)^{2\ell|[S,\bar{S}]_G|}\left(\mathsf{y}\cdot\left(\frac{\mathsf{yx}^2+1}{\mathsf{y}+\mathsf{z}^2}\right)^{2\ell d}\left(\frac{g_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})}{h_{\ell,q}(\mathsf{x},\mathsf{y},\mathsf{z})}\right)^{2q}\right)^{|S|}$$

and the lemma follows. $\square$

Lemma 6.3.6. *Let $e_1, e_2 \in \mathbb{Q}\setminus\{-1,0,1\}$ and let $a,b,c>0$ such that $b\neq c$, and in addition, $b>a$ or $c>a$. Then there is $c_1\in\mathbb{N}$ for which the sequence*

$$h(\ell) \quad = \quad \frac{e_1\cdot b^\ell + a^\ell}{c^\ell + e_2\cdot a^\ell}$$

is strictly monotone increasing or decreasing for $\ell \geq c_1$.

Proof. $h(\ell)$ can be rewritten as

$$h(\ell) \quad = \quad \frac{e_1\cdot\tilde{b}^\ell+1}{\tilde{c}^\ell+e_2}$$

by dividing both the numerator and the denominator of by a^ℓ and setting

$\tilde{b} = \frac{b}{a}$ and $\tilde{c} = \frac{c}{a}$. We have $\tilde{b} \neq \tilde{c}$ and $\tilde{b}, \tilde{c} > 0$.

If $\tilde{b}, \tilde{c} > 1$ then $h(\ell) = \Theta\left(\left(\frac{\tilde{b}}{\tilde{c}}\right)^\ell\right)$. If $\tilde{b} > 1$ and $\tilde{c} \leq 1$ then $h(\ell) = \Theta(\tilde{b}^\ell)$. If $\tilde{c} > 1$ and $\tilde{b} \leq 1$ then $h(\ell)^{-1} = \Theta(\tilde{c}^\ell)$. The lemma follows for each of the three cases. □

Lemma 6.3.7. *Let $e \in \mathbb{Q} \setminus \{-1, 0, 1\}$ and let $a, b, c > 0$ such that $b \neq c$. Then there is $c_1 \in \mathbb{N}$ for which the sequence*

$$d(\ell) \quad = \quad \frac{e \cdot b^\ell + a^\ell}{c^\ell + e \cdot a^\ell}$$

is strictly monotone increasing or decreasing for $\ell \geq c_1$.

Proof. If $b > a$ or $c > a$ then the lemma follows directly from Lemma 6.3.6. Assume $b, c \leq a$. We can write $d(\ell)$ in the following two ways:

$$d(\ell) \quad = \quad e^{-1}\left(1 - e^{-1} \cdot \frac{-e^2 \cdot b^\ell + c^\ell}{a^\ell + e^{-1} \cdot c^\ell}\right) = e^{-1}\left(1 - e \cdot \frac{-e^{-2} \cdot c^\ell + b^\ell}{a^\ell + e \cdot b^\ell}\right)^{-1}$$

We can now apply Lemma 6.3.6 to

$$\frac{-e^2 \cdot b^\ell + c^\ell}{a^\ell + e^{-1} \cdot c^\ell} \quad \text{or} \quad \frac{-e^{-2} \cdot c^\ell + b^\ell}{a^\ell + e \cdot b^\ell}$$

depending on whether $b > c$ or $c > b$, and the lemma follows.

□

Theorem D(i) is now given precisely and proved:

Theorem 6.3.8 (Theorem D(i)).
For all $(\gamma, \delta, \varepsilon) \in \mathbb{Q}^3$ such that

(i) $\delta \neq \{-1, 0, 1\}$,

(ii) $\delta + \varepsilon^2 \notin \{-1, 0, 1\}$,

(iii) $\delta + \varepsilon^2 \neq \pm(\delta\gamma^2 + 1)$,

(iv) $\delta\gamma^2 + 1 \neq 0$,

(v) $\gamma\delta + \varepsilon \neq 0$, *and*

(vi) $(\gamma\delta+\varepsilon)^4 \neq \left(\delta\gamma^2+1\right)^2\left(\delta+\varepsilon^2\right)^2$.

$Z_{Ising}(-;\gamma,\delta,\varepsilon)$ *is* $\#\mathbf{P}$*-hard on simple bipartite planar graphs.*

Proof. We will show that, on $3-$regular bipartite planar graphs G, the polynomial $Z_{Ising}(G;\mathsf{t},\mathsf{y})$ is polynomial-time computable using oracle calls to $Z_{Ising}(-;\gamma,\delta,\varepsilon)$. The oracle is only queried with input of simple bipartite planar graphs. Using Proposition 6.1.4, computing $Z_{Ising}(G;\mathsf{t},\mathsf{y})$ is $\#\mathbf{P}$-hard on 3-regular bipartite planar graphs.

Using (i) and (ii) it can be verified that there exists $c_0 \in \mathbb{N}^+$ such that for all $\ell \geq c_0$ and $q \in \mathbb{N}^+$, $f_{p,R}(\gamma,\delta,\varepsilon,2\ell,q) \neq 0$. We can use Lemma 6.3.5 to manufacture, in polynomial-time, evaluations of $Z_{Ising}(G;\mathsf{t},\mathsf{y})$ that will be used to interpolate $Z_{Ising}(G;\mathsf{t},\mathsf{y})$.

Let $\ell \geq c_0$ and let

$$E_{\mathsf{y},1} = \frac{\delta\gamma^2+1}{\delta+\varepsilon^2} \quad \text{and} \quad E_{\mathsf{y},2,\ell} = \frac{\delta(\delta\gamma^2+1)^{4\ell}+(\gamma\delta+\varepsilon)^{4\ell}}{(\delta+\varepsilon^2)^{4\ell}+\delta(\gamma\delta+\varepsilon)^{4\ell}}\,.$$

We have that $f_{\mathsf{y},R}(\gamma,\delta,\varepsilon,\ell,q) = \delta\left(E_{\mathsf{y},1}\right)^{2d\ell}\left(E_{\mathsf{y},2,\ell}\right)^{2q}$. Using (iv) we have $E_{\mathsf{y},1} \neq 0$.

Look at $E_{\mathsf{y},2,\ell}$ as a function of ℓ. Using (i), (ii), (iii) and (iv) and Lemma 6.3.7 with $a=(\gamma\delta+\varepsilon)^4$, $b=(\delta\gamma^2+1)^4$, $c=(\delta+\varepsilon^2)^4$ and $e=\delta$, there exists c_1 such that $E_{\mathsf{y},2,\ell}$ is strictly monotone increasing or decreasing. Hence, there exists $c_2 \geq c_1$ such that, for every $\ell \geq c_2$, $E_{\mathsf{y},2,\ell} \notin \{-1,0,1\}$. Moreover, $c_2 = c_2(\gamma,\delta,\varepsilon)$ is a function of γ, δ and ε.

We get that for $q_1 \neq q_2 \in [n_G+1]$ and $\ell > c_2$, $\left(E_{\mathsf{y},2,\ell}\right)^{2q_1} \neq \left(E_{\mathsf{y},2,\ell}\right)^{2q_2}$. Since $\delta\left(E_{y,1}\right)^{2d\ell}$ is not equal to 0 and does not depend on q, we get that for $q_1 \neq q_2 \in [n_G+1]$, $f_{\mathsf{y},R}(\gamma,\delta,\varepsilon,\ell,q_1) \neq f_{\mathsf{y},R}(\gamma,\delta,\varepsilon,\ell,q_2)$.

For every $\ell \in [m_G+c_2+1]\setminus[c_2]$, we can interpolate in polynomial-time the univariate polynomial $Z_{Ising}(G;f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell),\mathsf{y})$. Then, we can use the polynomial $Z_{Ising}(G;f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell),\mathsf{y})$ to compute $Z_{Ising}(G;f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell),j)$ for every $\ell \in [m_G+c_2+1]\setminus[c_2]$ and every $j \in [n_G+1]$. Let

$$E_{\mathsf{t}} = \left(\frac{(\gamma\delta+\varepsilon)^2}{(\delta\gamma^2+1)(\delta+\varepsilon^2)}\right)^2$$

and it holds that $f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell,q) = \left(E_{\mathsf{t}}\right)^{\ell}$. Clearly, $E_{\mathsf{t}} \neq -1$ and, by (v) and (vi), $E_{\mathsf{t}} \notin \{0,1\}$. Hence, for every $\ell_1 \neq \ell_2 \in \mathbb{N}^+$ we have

$f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell_1) \neq f_{\mathsf{t},R}(\gamma,\delta,\varepsilon,\ell_2)$. Therefore, we can compute the value of the bivariate polynomial $Z_{Ising}(G;\mathsf{t},\mathsf{y})$ on a grid of points of size $(m_G+1)\times(n_G+1)$ in polynomial-time using the oracle, and use them to interpolate $Z_{Ising}(G;\mathsf{t},\mathsf{y})$. $\square$

Part II

Recurrence Sequences

Chapter 7

Background on Recurrence Sequences

In this chapter we define exactly the type of recurrent sequences we are interested in, and give some useful properties.

7.1 Definitions

We now define exactly the types of recurrence relations we are interested in. Let $\mathcal{R}$ be a ring such as $\mathbb{Z}$ or $\mathbb{Q}[\mathsf{x}_1, \ldots, \mathsf{x}_\ell]$.

Definition 7.1.1. *Given a sequence $a(n)$ over $\mathcal{R}$ we say $a(n)$ is*

(i) C-finite *or* rational *if there is a fixed $q \in \mathbb{N}^+$ for which $a(n)$ satisfies for all $n > q$*

$$a(n) = \sum_{i=1}^{q} c_i a(n-i)$$

where each $c_i \in \mathcal{R}$.

(ii) P-recursive *or* holonomic *if there is a fixed $q \in \mathbb{N}^+$ for which $a(n)$ satisfies for all $n > q$*

$$p_0(n) \cdot a(n) = \sum_{i=1}^{q} p_i(n) a(n-i) \tag{7.1}$$

where each p_i is a polynomial in $\mathcal{R}[\mathsf{y}]$ and $p_0(n) \neq 0$ for any n. We call $p_0(\mathsf{y})$ the leading polynomial *of the recurrence.*

(iii) $a(n)$ *is* Simple-P-recursive or SP-recursive *if $a(n)$ is P-recursive with leading polynomial $p_0(\mathsf{y})$ such that $p_0(\mathsf{y}) = 1$.*

The terminology C-finite and holonomic are due to [153]. P-recursive is due to [138]. The term D-finite referring to generating functions whose sequence of coefficients is P-recursive also appears in the literature. P-recursive sequences were already studied in [20, 21].

7.2 Properties of recursive sequences

Here we present some useful lemmas on C-finite and P-recursive sequences.

The following two lemmas are well known, see e.g. [62, 55].

Lemma 7.2.1. *Let $a(n)$ be a sequence of integers.*

(i) If $a(n)$ is C-finite, then there is a constant $c \in \mathbb{Z}$ such that $a(n) \leq 2^{cn}$.

(ii) If $a(n)$ is P-recursive, then there is a constant $c \in \mathbb{N}$ such that $|a(n)| \leq n!^c$ for all $n \geq 2$.

(iii) The sets of C-finite, SP-recursive and P-recursive sequences over any ring $\mathcal{R}$ are closed under addition and point-wise multiplication.

In general, the bound on the growth rate of P-recursive integer sequences is best possible, since $a(n) = n!^m$ is easily seen to be P-recursive for any integer m, [68].

Lemma 7.2.2. *Let $a(n)$ be a sequence over a ring $\mathcal{R}$.*

(i) If $a(n)$ is C-finite then $a(n)$ is SP-recursive.

(ii) If $a(n)$ is SP-recursive then $a(n)$ is P-recursive.

Proof. The implications follow from the definitions.
□

The converses of Lemma 7.2.2 do not hold. For (i), $n!$ is SP-recursive but not C-finite due to Lemma 7.2.1(i). The Catalan numbers $C(n)$ in Example 8.1.5 are P-recursive but not SP-recursive due to their modular properties;

SP-recursive sequences are ultimately periodic modulo every m, yet $C(n)$ are not ultimately periodic modulo 2, cf. e.g. [7].

Lemma 7.2.3. *Let $a(n)$ be a sequence of integers and let $m > 1$ be a natural number. If $a(n)$ is SP-finite, then $a(n)$ is ultimately periodic modulo m. In other words, $a(n)$ satisfies a linear recurrence relation over $\mathbb{Z}_m$.*[1]

Proof. First note that for every polynomial $p(\mathsf{y})$ with integer coefficients, the sequence $p(1), p(2), \ldots, p(n), \ldots$ is ultimately periodic modulo every m (this follows from the closure of the set of sequences which are ultimately periodic modulo m to multiplication and addition). Therefore, we can assume $a(n)$ from Equation (7.1) is in fact given by a modular linear recurrence,

$$a(n + q(m)) = \sum_{i=0}^{q(m)-1} c_i(m) a(n + i) \mod m\,,$$

where $q(m)$ can be larger than q in Equation (7.1). The lemma follows since $a(n)$ depends on the $q(m)$ previous elements and the elements of the sequence $a(n) \mod m$ can only receive values from $0, \ldots, m - 1$. □

Lemma 7.2.4. *Let $a(n, \bar{\mathsf{x}})$ be a sequence over $\mathbb{Z}[\bar{\mathsf{x}}]$ and let $p_0(\mathsf{y}, \bar{\mathsf{x}})$ be a polynomial in $\mathbb{Z}[\mathsf{y}, \bar{\mathsf{x}}]$. Then:*

(i) If $a(n, \bar{\mathsf{x}})$ has a P-recurrence with leading polynomial $p_0(\mathsf{y}, \bar{\mathsf{x}})$ then there exists an SP-recursive sequence $b(n, \bar{\mathsf{x}})$ such that

$$a(n, \bar{\mathsf{x}}) = b(n, \bar{\mathsf{x}}) \prod_{j=1}^{n} \frac{1}{p_0(j, \bar{\mathsf{x}})}\,.$$

(ii) If there exists an SP-recursive sequence $b(n, \bar{\mathsf{x}})$ such that

$$a(n, \bar{\mathsf{x}}) = b(n, \bar{\mathsf{x}}) \prod_{j=1}^{n} \frac{1}{p_0(j, \bar{\mathsf{x}})}\,.$$

[1] By *ultimately periodic modulo m* we mean that there exists c such that the sequence $a(n+c)$, $n \geq 0$, is periodic modulo m. In [59] we referred to sequences which are ultimately periodic modulo every $m > 1$ as *MC-finite*.

then $a(n,\bar{\mathsf{x}})$ *has a P-recurrence with leading polynomial* $\prod_{j=0}^{q-1} p_0(\mathsf{y}-j,\bar{\mathsf{x}})$

Proof.

(i) Let

$$p_0(n,\bar{\mathsf{x}})a(n,\bar{\mathsf{x}}) = \sum_{i=1}^{q} p_i(n,\bar{\mathsf{x}})a(n-i,\bar{\mathsf{x}})\,.$$

We prove (i) by induction on n. Let

$$b(n,\bar{\mathsf{x}}) = \sum_{i=1}^{q} p_i(n,\bar{\mathsf{x}})b(n-i,\bar{\mathsf{x}}) \prod_{j=n-i+1}^{n-1} p_0(j,\bar{\mathsf{x}})$$

and

$$b(i,\bar{\mathsf{x}}) = a(i,\bar{\mathsf{x}}) \prod_{j=1}^{i} p_0(j,\bar{\mathsf{x}})\,,\ i \in [q]$$

Base For $n \in [q]$ the claim holds by definition of $b(n)$.

Step Assume the claim holds for all $n' < n$. $a(n,\bar{\mathsf{x}})$ satisfies the recurrence

$$a(n,\bar{\mathsf{x}}) = \frac{1}{p_0(n,\bar{\mathsf{x}})} \sum_{i=1}^{q} p_i(n,\bar{\mathsf{x}})a(n-i,\bar{\mathsf{x}})\,.$$

Using the induction hypothesis,

$$\begin{aligned} a(n,\bar{\mathsf{x}}) = & \quad \frac{1}{p_0(n,\bar{\mathsf{x}})} \sum_{i=1}^{q} \frac{p_i(n,\bar{\mathsf{x}})b(n-i,\bar{\mathsf{x}})}{\prod_{j=1}^{n-i} p_0(j,\bar{\mathsf{x}})} \\ = & \quad \left(\prod_{j=1}^{n} \frac{1}{p_0(j,\bar{\mathsf{x}})}\right) \sum_{i=1}^{q} p_i(n,\bar{\mathsf{x}})b(n-i,\bar{\mathsf{x}}) \prod_{j=n-i+1}^{n-1} p_0(j,\bar{\mathsf{x}}) \\ = & \quad \left(\prod_{j=1}^{n} \frac{1}{p_0(j,\bar{\mathsf{x}})}\right) b(n,\bar{\mathsf{x}}) \end{aligned}$$

(ii) Let an SP-recurrence for $b(n, \bar{\mathsf{x}})$ be:

$$b(n, \bar{\mathsf{x}}) = \sum_{i=1}^{q} p_i(n, \bar{\mathsf{x}}) b(n-i, \bar{\mathsf{x}}) .$$

Then

$$a(n, \bar{\mathsf{x}}) \prod_{j=1}^{n} p_0(j, \bar{\mathsf{x}}) = \sum_{i=1}^{q} p_i(n, \bar{\mathsf{x}}) a(n-i, \bar{\mathsf{x}}) \prod_{j=1}^{n-i} p_0(j, \bar{\mathsf{x}}) .$$

Dividing both sides of the last equation by $\prod_{j=1}^{n-q} p_0(j)$ we get

$$a(n, \bar{\mathsf{x}}) \prod_{j=n-q+1}^{n} p_0(j, \bar{\mathsf{x}}) = \sum_{i=1}^{q} p_i(n, \bar{\mathsf{x}}) a(n-i, \bar{\mathsf{x}}) \prod_{j=n-q+1}^{n-i} p_0(j, \bar{\mathsf{x}}) .$$

which we can rewrite as

$$a(n, \bar{\mathsf{x}}) \prod_{j=0}^{q-1} p_0(n-j, \bar{\mathsf{x}}) = \sum_{i=1}^{q} p_i(n, \bar{\mathsf{x}}) a(n-i, \bar{\mathsf{x}}) \prod_{j=0}^{q-1} p_0(n-j, \bar{\mathsf{x}}) .$$

and the claim follows.

□

Lemma 7.2.5. *Let $a(n, \bar{\mathsf{x}})$ be a P-recursive sequence over $\mathbb{Z}[\bar{\mathsf{x}}]$ with leading coefficient $p_0(\mathsf{y}) \in \mathbb{Z}[\mathsf{y}]$*

$$p_0(n) a(n, \bar{\mathsf{x}}) = \sum_{i=1}^{q} p_i(n, \bar{\mathsf{x}}) a(n-i, \bar{\mathsf{x}}) .$$

Then there is $s \in \mathbb{N}$ such that $b(n, \bar{\mathsf{x}}) = a(n+s, \bar{\mathsf{x}})$ is P-recursive with leading polynomial in $\mathbb{N}[\mathsf{y}]$.

Proof. Substituting $n+s$ for n in the recurrence relation for $a(n, \bar{\mathsf{x}})$, we get

$$p_0'(n) b(n, \bar{\mathsf{x}}) = \sum_{i=1}^{q} p_i'(n, \bar{\mathsf{x}}) b(n-i, \bar{\mathsf{x}}) ,$$

where $p_0'(n) = p_0(n+s)$ and $p_i'(n, \bar{\mathsf{x}}) = p_i(n+s, \bar{\mathsf{x}}) \in \mathbb{Z}[y, \bar{\mathsf{x}}]$ for $i \in [q]$. We need to show that for large enough $s \in \mathbb{N}$, $p_0(n+s)$ is of the desired form.

Let

$$p_0(\mathsf{y}) = \sum_{j=0}^{k} c_j \mathsf{y}^j$$

and let

$$p'_0(\mathsf{y}) = \sum_{j=0}^{k} c_j (\mathsf{y}+s)^j = \sum_{i=0}^{k} d_i \mathsf{y}^i \,.$$

Consider the coefficient d_i of each y^i in $p'_0(\mathsf{y})$:

$$d_i = \sum_{j=i}^{k} \binom{j}{i} c_j s^{j-i}$$

We will now see that for large enough s the sign of all of the d_i is the sign of c_k. It is enough to show that

$$\binom{k}{i} |c_k| s^{k-i} > \sum_{j=i}^{k-1} \binom{j}{i} |c_j| s^{j-i} \,.$$

Consider the right-hand side of the last equation.

$$\sum_{j=i}^{k-1} \binom{j}{i} |c_j| s^{j-i} < (k-1-i) \binom{k-1}{i} s^{k-1-i} \max_{i \le j \le k-1} |c_j| \tag{7.2}$$

and the right-hand side of Equation (7.2) is strictly smaller than $\binom{k}{i}|c_k|s^{k-i}$ provided s is chosen to be at least

$$s > \frac{(k-1-i)\binom{k-1}{i} \max_{i \le j \le k-1} |c_j|}{\binom{k}{i}|c_k|} \,.$$

Hence, either all of the coefficients of y_i in $p'_0(\mathsf{y})$ are positive or all are negative. Note that if all are negative we can simply multiply both sides of the recurrence relation by -1 to get the desired result. $\square$

Lemma 7.2.6. *Let $a_1(n, \bar{\mathsf{x}}), \ldots, a_s(n, \bar{\mathsf{x}})$ be P-recursive sequences such that each $a_i(n, \bar{\mathsf{x}})$ has a P-recurrence with leading polynomial $p_{0,i}(\mathsf{y})$ which belongs* *to* $\mathbb{Z}[\mathsf{y}]$*. Let* $c_1, \ldots, c_s \in \mathbb{Z}$*. Then the linear combination*

$$e(n, \bar{\mathsf{x}}) = \sum_{i=1}^{s} c_i a_i(n, \bar{\mathsf{x}})$$

has a P-recurrence with leading polynomial $p_{0,e}(\mathsf{y})$ *which belongs to* $\mathbb{Z}[\mathsf{y}]$*.*

Proof. By Lemma 7.2.4 there exist SP-recursive sequences $b_i(n, \bar{\mathsf{x}})$, $i = 1, \ldots, s$, such that for each i,

$$b_i(n, \bar{\mathsf{x}}) = a_i(n, \bar{\mathsf{x}}) \prod_{j=1}^{n} p_{0,i}(j)\,.$$

We can rewrite $e(n, \bar{\mathsf{x}})$ as follows:

$$\begin{aligned} e(n, \bar{\mathsf{x}}) &= \sum_{i=1}^{s} c_i b_i(n, \bar{\mathsf{x}}) \prod_{j=1}^{n} \frac{1}{p_{0,i}(j)} \\ &= \frac{\displaystyle\sum_{i=1}^{s} c_i b_i(n, \bar{\mathsf{x}}) \prod_{\substack{i'=1 \\ i' \neq i}}^{s} \prod_{j=1}^{n} p_{0,i'}(j)}{\displaystyle\prod_{j=1}^{n} p_{0,1}(j) \cdots p_{0,s}(j)} \end{aligned} \tag{7.3}$$

By the closure of SP-recursive sequences to addition and multiplication, the numerator of Equation (7.3) is SP-recursive. Since $p_{0,1}(\mathsf{y}) \cdots p_{0,s}(\mathsf{y})$ belongs to $\mathbb{Z}[\mathsf{y}]$, it follows from Lemma 7.2.4(ii) that $e(n, \bar{\mathsf{x}})$ has a P-recurrence with leading polynomial $p_{0,e}(\mathsf{y})$ in $\mathbb{Z}[\mathsf{y}]$. □

Chapter 8

Examples

This chapter presents examples of sequences of integers and polynomials from the literature of combinatorics and graph theory. These examples motivate the study of the classes of C-finite, SP-recurive and P-recursive sequences. In Section 8.1 we give examples of sequences of integers and in Section 8.2 of sequences of polynomials.

8.1 Integer sequences

8.1.1 C-finite with positive coefficients

The Fibonacci sequence $f(n)$ is defined by $f(n+2) = f(n+1) + f(n)$ with $f(1) = 1$ and $f(2) = 2$. It is therefore C-finite.

The Fibonacci sequence is a simple example of a logical representation of C-finite sequences of integers. Let L_{Fib} be given by the regular expression $(a \vee ab)^{\star}$ with counting function $d_{Fib}(n)$; it is easy to see that $d_{Fib}(n) = f(n)$.

Similarly, if $g(n+2) = 2g(n+1) + 3g(n)$ and L_g is given by

$$(a_1 \vee a_2 \vee b_1^2 \vee b_2^2 \vee b_3^2)^{\star}$$

with counting $d_g(n)$, then $g(n) = d_g(n)$.

8.1.2 Factorials and generalized factorials

The factorial function $n!$ is SP-recursive by $(n+1)! = (n+1) \cdot n!$, but $n!$ is not C-finite because it grows too fast.

$n!$ has several combinatorial interpretations: It counts the number of functions $f : [n] \to [n]$ which are bijective, and it also counts the number of functions $f : [n] \to [n]$ such that $f(j) \leq j + 1$.

Generalized factorials are functions of the form $\prod_{j=1}^{n} p(j)$ where $p(\mathsf{y})$ is a polynomial in some (semi-)ring $\mathcal{R}$. If $p \in \mathbb{N}[\mathsf{y}]$, then $\prod_{j=1}^{n} p(j)$ can be interpreted as counting functions $f : [n] \to [p(n)]$ such that $f(j) \leq p(j)$.

The factorial is revisited in Section 10.1 in the context of counting lattice paths.

8.1.3 Derangement numbers

The derangement numbers $D(n)$ are usually defined by their combinatorial interpretation f is bijective and for all $i \in [n]$ we have $f(i) \neq i$. Their explicit definition is given by

$$D(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor$$

They are SP-recursive by $D(n) = (n-1)(D(n-1)+D(n-2))$ with $D(0) = 1$ and $D(1) = 0$. But they are not C-finite, by the growth argument from Lemma 7.2.1 In [130, Example 8.6.1.] they are shown not to be hypergeometric. [1]

The derangement numbers are revisited in Section 10.4.

8.1.4 Central binomial coefficient

The function $\binom{2n}{n}$, the central binomial coefficient, is P-recursive and hypergeometric since

$$(n+1)^2 \cdot \binom{2(n+1)}{n+1} = 2 \cdot \binom{2(n+1)}{2} \cdot \binom{2n}{n}$$

$\binom{2n}{n}$ has many combinatorial interpretations. It counts the number of ordered partitions of $[2n]$ into two equal sized sets. If the partitions are not ordered, the counting function is $\frac{1}{2}\binom{2n}{n}$. $\binom{2n}{n}$ also counts the number of functions $f : [n+1] \to [n+1]$ such that $f(i+1) \geq f(i)$ and $f(n+1) = n+1$.

[1] A P-recursive sequence is *hypergeometric* if it has a P-recurrence with $q = 1$ in Equation (7.1).

$\frac{1}{2}\binom{2n}{n}$ also counts the number of graphs with vertex set $[2n]$ which consists of the disjoint union of two equal sized cliques. We denote this class of graphs by $\mathrm{EQ}_2\mathrm{CLIQUE}$.

Similarly, the class $\mathrm{EQ}_p\mathrm{CLIQUE}$ denotes the class of graphs with vertex set $[pn]$ which consist of p disjoint cliques of equal size. We denote by $b_p(n)$ the number of graphs with $[n]$ as a set of vertices which are in $\mathrm{EQ}_p\mathrm{CLIQUE}$. Clearly,

$$b_p(n) = \begin{cases} \frac{1}{p!}\binom{pn}{n} \cdot \binom{(p-1)n}{n} \cdot \ldots \cdot \binom{n}{n} & \text{if } p \text{ divides } n \\ 0 & \text{otherwise} \end{cases}$$

Congruence relations of binomial coefficients and related functions have received a lot of attention in the literature, starting with Lucas' famous result for $b_2(n)$, [114]. For $b_p(n)$ modulo p, a prime, we have:

Lemma 8.1.1. *For every $k > 1$, $b_p(pk) \equiv b_p(k) \pmod{p}$.*

The proof uses the method of combinatorial proof of Fermat's congruence theorem by J. Petersen from 1872, given in [70, page 157].

Proposition 8.1.2. *For every n which is not a power of the prime p, we have $b_p(n) \equiv 0 \pmod{p}$, and for every n which is a power of p we have $b_p(n) \equiv 1 \pmod{p}$. In particular, $b_p(n)$ is not ultimately periodic modulo p.*

Proof. By induction on n, where the basis is $n = p$ (for which $b_p(n) = 1$) and every n which is not divisible by p (for which $b_p(n) = 0$); the induction step follows from Lemma 8.1.1. □

By the previous Proposition, $b_2(n) = \frac{1}{2}\binom{2n}{n}$ is not SP-recursive using Lemma 7.2.3.

However, $b_2(n)$, and more generally $b_p(n)$, are P-recursive.

The central binomial coefficient is revisited in Section 10.1.

8.1.5 Catalan numbers

The Catalan numbers $C(n)$ are defined by $C(n) = \frac{1}{n+1} \cdot \binom{2n}{n}$. They satisfy the P-recurrence relation

$$C(n+1) = \frac{2(2n+1)}{n+2} \cdot C(n) \tag{8.1}$$

In [139] there is an abundance of combinatorial interpretations which are not pure. Many of these are based on functions $f : [n] \to [n]$ which represent lattice paths subject to various conditions. One of these is the set of weakly monotonic functions $f : [n] \to [n]$ such that $f(1) = 1$, $f(n) = n$ and $f(i) \leq i$.

It is known that $C(n)$ is not SP-recursive using Lemma 7.2.3, since $C(n)$ does not satisfy a modular linear recurrence relation modulo 2, cf. [99].

The Catalan numbers are revisited in Section 10.1.

8.1.6 Stirling numbers of the first kind

The Stirling numbers of the first kind $\left[{n \atop k}\right]$ count arrangements of $[n]$ into k non-empty cycles (where a single element and a pair of elements are considered cycles). In other words, $\left[{n \atop k}\right]$ counts permutations with k cycles. They satisfy the following recurrence relation

$$\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right] .$$

Using the growth argument from Lemma 7.2.1 we can see that the Stirling numbers of the first kind are not C-finite, e.g. $\left[{n \atop 1}\right]$ grows like the factorial $(n-1)!$. One can deduce from the above recurrence that for fixed k, the sequence $\left[{n \atop k}\right]$ is SP-recursive.

The Stirling numbers of the first kind are revisited in Section 10.4.

8.1.7 Stirling numbers of the second kind

The Stirling numbers of the second kind $\left\{{n \atop k}\right\}$ count the number of partitions of $[n]$ into k non-empty parts. $\left\{{n \atop k}\right\}$ are given explicitly by

$$\left\{{n \atop k}\right\} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n .$$

They satisfy the following recurrence relation

$$\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\} .$$

Similarly to the Stirling numbers of the first kind, one can deduce from the above equation that for fixed k, the sequence $\left\{{n \atop k}\right\}$ is C-finite.

One can show that $\left\{ n \atop k \right\}$ is C-finite without using the above recurrence relation, by exhibiting a regular language whose counting function equals the Stirling numbers of the second kind.

8.1.8 Bell numbers

The Bell numbers B_n count the number of partitions of an n-element set. Equivalently, they also count the number of equivalence relations on an n-element set. They satisfy the recurrence relation

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k,$$

but in [68] it is shown that they are not P-recursive. For more properties of Bell numbers, cf. [132], and for congruences, cf. [69].

Note that the Bell numbers can be expressed in terms of the Stirling numbers of the second kind:

$$B_n = \sum_{i=0}^{n} \left\{ n \atop i \right\} .$$

8.1.9 Trees and forests

Trees are (undirected) connected acyclic graphs. Denote by T_n the number of labeled trees on n vertices. Labeled trees were among the first objects to be counted explicitly, cf.[89, Theorem 1.7.2].

Theorem 8.1.3 (A. Cayley 1889). $T_n = n^{n-2}$.

S. Gerhold proved in [68] that T_n is not P-recursive. In [89, Chapter 3] there is a wealth of results on counting various labeled trees and tree-like structures.

It is well known, cf. [151], that the number of rooted forests on n vertices is $RF_n = (n+1)^{n-1}$. Again this is not P-recursive.

The number of forests F_n (of non-rooted trees) is more complicated. L. Takács, [141] showed that

$$F_n = \frac{n!}{n+1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{(2j+1)(n+1)^{n-2j}}{2^j \cdot j! \cdot (n-2j)!} \tag{8.2}$$

A simpler proof is given in [36]. Whether F_n is P-recursive or not seems to be open.

8.2 Polynomial sequences

8.2.1 Fibonacci and Lucas polynomials

The C-recurrence $F_n(\mathsf{x}) = \mathsf{x} \cdot F_{n-1}(\mathsf{x}) + F_{n-2}(\mathsf{x})$, $F_1(\mathsf{x}) = 1$ and $F_2(\mathsf{x}) = \mathsf{x}$ defines the Fibonacci polynomials. The Fibonacci numbers F_n can be obtained as an evaluation of the Fibonacci polynomial $F_n(1) = F_n$. The coefficients $F_{n,i}$ of x^i in $F_n(\mathsf{x})$ equal $\binom{(n+i-1)/2}{i}$ and can be interpreted as counting ordered sums of 1's and 2's in which 1 occurs exactly i times.

The Lucas polynomials $L_n(\mathsf{x})$ satisfy a similar recurrence. The Fibonacci polynomials are also matching polynomials, $F_n(\mathsf{x}) = M(P_n; 1, \mathsf{x})$ and $L_n(\mathsf{x}) = M(C_n; 1, \mathsf{x})$.

8.2.2 Falling and rising factorials

In Chapter 1 we discussed the the falling factorial $k \cdot (k-1) \cdots (k-(n-1))$, denoted $k_{(n)}$, $(k)_n$ or $k^{\underline{n}}$, which count injective functions from $[n]$ to $[k]$. For any fixed n, $k_{(n)}$ is a polynomial in k, $\mathsf{k}_{(n)}$ in $\mathbb{Z}[\mathsf{k}]$. The sequence $\mathsf{k}_{(n)} : n \in \mathbb{N}$ satisfies an SP-recurrence:

$$\mathsf{k}_{(n)} = (\mathsf{k} - n + 1) \cdot \mathsf{k}_{(n-1)} \, .$$

A related sequence of polynomials is the rising factorial given by $\mathsf{k} \cdot (\mathsf{k}+1) \cdots (\mathsf{k}+(n-1))$ and denoted $\mathsf{k}^{(n)}$ or $\mathsf{k}^{\overline{n}}$. It satisfies the SP-recurrence:

$$\mathsf{k}^{(n)} = (\mathsf{k} + n - 1) \cdot \mathsf{k}^{(n-1)} \, .$$

$k^{(n)}$ counts injective functions from $[n]$ to $[k + n - 1]$.

8.2.3 Touchard polynomials

The Touchard polynomials are defined

$$T_n(\mathsf{x}) \quad = \quad \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \mathsf{x}^k$$

where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind.

The Bell numbers B_n are given as an evaluation of $T_n(\mathsf{x})$, $B_n = T_n(1)$, which implies $T_n(\mathsf{x})$ is not P-recursive.

8.2.4 Matching polynomials

In this subsection we discuss sequences of polynomials which are the matching polynomials of recursive sequences of graphs. Our examples are also orthogonal polynomials (with respect to various inner products). For details see [92].

8.2.4.1 Rook polynomials and Laguerre polynomials

The rook polynomial $R_n(\mathsf{x})$ is the generating function

$$R_n(\mathsf{x}) = \sum_{i=0}^{n} r_{i,n} x^i$$

where $r_{i,n}$ is the number of ways of putting i non-attacking rooks on a square chess board of size $n \times n$, cf. [131]. The rook polynomial is the generating matching polynomial of the complete $n \times n$ bipartite graph, $R_n(\mathsf{x}) = g(K_{n,n}; \mathsf{x})$.

The rook polynomial is closely related to the Laguerre polynomials $L_n(\mathsf{x})$, which are a classical orthogonal polynomial sequence, by

$$R_n(\mathsf{x}) = n!\mathsf{x}^n L_n\left(-\frac{1}{\mathsf{x}}\right).$$

$L_n(\mathsf{x})$ is P-recursive, satisfying

$$(n+1) \cdot L_{n+1}(\mathsf{x}) = (2n+1-\mathsf{x}) \cdot L_n(\mathsf{x}) - n \cdot L_{n-1}(\mathsf{x})$$

with initial conditions $L_0(x) = 1$ and $L_1(\mathsf{x}) = 1 - \mathsf{x}$. As a consequence we get that $R_n(\mathsf{x})$ is P-recursive.

8.2.4.2 Hermite polynomials

The Hermite polynomials $He_n(\mathsf{x})$ and $H_n(\mathsf{x})$ are two sequences of orthogonal polynomials satisfying the following SP-recurrences:

$$\begin{aligned} He_{n+1}(\mathsf{x}) &= \mathsf{x}He_n(\mathsf{x}) - nHe_{n-1}(\mathsf{x}) \\ H_{n+1}(\mathsf{x}) &= 2\mathsf{x}H_n(\mathsf{x}) - 2nH_{n-1}(\mathsf{x}) \end{aligned}$$

with $He_0(\mathsf{x}) = 1$, $He_1(\mathsf{x}) = \mathsf{x}$, $H_0 = 1$ and $H_1 = 2\mathsf{x}$. They are related via

$$He_n(\mathsf{x}) = 2^{-\frac{n}{2}} H_n\left(\frac{\mathsf{x}}{\sqrt{2}}\right).$$

The Hermite polynomials arise in various areas of mathematics, including probability, combinatorics, and numerical analysis, as well as in physics in the context of the quantum harmonic oscillator, see e.g. [57, 142]. The Hermite polynomials $He_n(\mathsf{x})$ are in fact the acyclic matching polynomials of cliques $He_n(\mathsf{x}) = \mu(K_n; \mathsf{x})$.

The Hermite numbers H_n are defined as the evaluation at $\mathsf{x} = 0$ of the Hermite polynomials $H_n(\mathsf{x})$. The Hermite numbers have a combinatorial interpretation as the generating function of higher order matchings, [50]. H_n is SP-recursive with $H_n = -2(n-1)H_{n-2}$.

8.2.4.3 Chebyshev polynomials

The Chebyshev polynomials of the first kind can be defined by the C-recurrence $T_{n+1}(\mathsf{x}) = 2\mathsf{x}T_n(\mathsf{x}) - T_{n-1}(\mathsf{x})$, $T_0(\mathsf{x}) = 1$, and $T_1(\mathsf{x}) = \mathsf{x}$. The Chebyshev polynomials of the second kind $U_n(\mathsf{x})$ satisfy the same C-recurrence with different initial conditions. The Chebyshev polynomials are closely related to the acyclic matching polynomial $\mu(G; \mathsf{x})$ of cycles C_n and paths P_n:

$$T_n(\mathsf{x}) = \frac{1}{2}\mu(C_n; 2\mathsf{x}) \quad \text{and} \quad U_n(\mathsf{x}) = \mu(P_n; 2\mathsf{x}).$$

For details on the combinatorial properties of the Chebyshev polynomials see e.g. [76].

Chapter 9

A Representation Theorem for P-Recursive Sequences

9.1 Positional representations

In this chapter we prove Theorem G, which gives a representation of C-finite and P-recursive sequences of integers and polynomials in terms of *positional weights*. Positional weights are weights which are applied to words and which depend on the position of certain letters in the words.

Definition 9.1.1 (Positional weights). *Let $\Sigma = \{\sigma_1, \ldots, \sigma_\eta\}$, $\eta \in \mathbb{N}$, be a finite alphabet. Let $e_{\sigma_1}, \ldots, e_{\sigma_\eta} \in \mathbb{Z}(\mathsf{y}, \bar{\mathsf{x}})$ be rational functions over $\mathbb{Z}$ with indeterminates $\mathsf{x}_1, \ldots, \mathsf{x}_\ell, \mathsf{y}$. For every word w over Σ, the* weight of w *is*

$$\mathrm{weight}_{\bar{e}}(w) = \prod_{j=1}^{n} e_{w[j]}(j, \bar{\mathsf{x}}) .$$

Using the positional weights, we define:

Definition 9.1.2 (Positional Representation (PR)).

(i) Let L be a regular language over $\Sigma = \{\sigma_1, \ldots, \sigma_\eta\}$ and let $e_{\sigma_1}, \ldots, e_{\sigma_\eta} \in \mathbb{Z}(\mathsf{y}, \bar{\mathsf{x}})$. Then

$$W_{L,\bar{e}}(n, \bar{\mathsf{x}}) = \sum_{w \in L \colon |w|=n} \mathrm{weight}_{\bar{e}}(w)$$

(ii) *Let* $a(n,\bar{\mathsf{x}})$ *be a sequence over* $\mathbb{Z}[\mathsf{y},\bar{\mathsf{x}}]$*, let* $\Sigma = \{\sigma_1,\ldots,\sigma_\eta\}$ *and let* $e_{\sigma_1},\ldots,e_{\sigma_\eta} \in \mathbb{Z}(\mathsf{y},\bar{\mathsf{x}})$*. We say that* $a(n,\bar{\mathsf{x}})$ *has a* $\bar{e}$-Positional Representation ($\bar{e}$-PR) *if there exist* $s \in \mathbb{N}$ *and a regular language* L *over* Σ *such that*

$$a(n+s,\bar{\mathsf{x}}) = W_{L,\bar{e}}(n,\bar{\mathsf{x}})\,.$$

(iii) *We say* $a(n,\bar{\mathsf{x}})$ *has a* Positional Representation (PR) *if there exists a tuple of rational functions* $\bar{e}$ *such that* $a(n,\bar{\mathsf{x}})$ *has a* $\bar{e}$*-PR.*

Remark 9.1.1. The definitions in Chapter 7 were made over any ring $\mathcal{R}$, and we denoted recursive sequences e.g. $a(n)$. In this chapter we are only interested in recursive sequences over polynomial rings $\mathbb{Z}[\bar{\mathsf{x}}]$. By a slight abuse of notation, $\mathbb{Z}$ is considered a polynomial ring with 0 indeterminates. Therefore, we write e.g. $a(n,\bar{\mathsf{x}})$ to make the indeterminates explicit.

9.2 Theorem G: exact statement and proof

We can now state the main theorem of the chapter:

Theorem 9.2.1 (Theorem G). *Let* $a(n,\bar{\mathsf{x}})$ *be a sequence of polynomials with integer coefficients.*

(i) $a(n,\bar{\mathsf{x}})$ *is P-recursive over* $\mathbb{Z}[\bar{\mathsf{x}}]$ *iff* $a(n,\bar{\mathsf{x}})$ *is the difference of two sequences*

$$a(n,\bar{\mathsf{x}}) = v(n,\bar{\mathsf{x}}) - \prod_{j=1}^{n} u(j,\bar{\mathsf{x}}) \tag{9.1}$$

such that $v(n,\bar{\mathsf{x}})$ *has a* $\bar{e}$*-PR with* $e_{\sigma_1},\ldots,e_{\sigma_\eta},u \in \mathbb{N}(\mathsf{y},\bar{\mathsf{x}})$*.*

(ii) $a(n,\bar{\mathsf{x}})$ *is P-recursive over* $\mathbb{Z}[\bar{\mathsf{x}}]$ *with leading polynomial* $p_0(\mathsf{y}) \in \mathbb{Z}[\mathsf{y}]$ *iff Equation (9.1) holds and* $e_{\sigma_1},\ldots,e_{\sigma_\eta},u \in \frac{\mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]}{\mathbb{N}[\mathsf{y}]}$*.*

(iii) $a(n,\bar{\mathsf{x}})$ *is SP-recursive over* $\mathbb{Z}[\bar{\mathsf{x}}]$ *iff Equation (9.1) holds and* $e_{\sigma_1},\ldots,e_{\sigma_\eta},u \in \mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]$*.*

(iv) $a(n,\bar{\mathsf{x}})$ *is C-finite over* $\mathbb{Z}[\bar{\mathsf{x}}]$ *iff Equation (9.1) holds and* $e_{\sigma_1},\ldots,e_{\sigma_\eta},u \in \mathbb{N}[\bar{\mathsf{x}}]$*.*

The proof follows from the next two lemmas, which we will prove in Subsections 9.2.1 and 9.2.2 respectively. The proofs of the Lemmas use elementary concepts from the theory of regular languages, see for example [91].

Lemma 9.2.2. *Let $a(n,\bar{\mathsf{x}}) \in \mathbb{Z}[\bar{\mathsf{x}}]$ be a P-recursive sequence of polynomials with leading polynomial $p_0(\mathsf{y},\bar{\mathsf{x}})$.*

(i) There exist $s \in \mathbb{N}$, an alphabet Σ, regular languages L_1 and L_2 over Σ and a tuple $(e_\sigma : \sigma \in \Sigma)$ of rational functions $e_\sigma \in \mathbb{N}(\mathsf{y},\bar{\mathsf{x}})$ such that

$$a(n+s,\bar{\mathsf{x}}) = W_{L_1,\bar{e}}(n,\bar{\mathsf{x}}) - W_{L_2,\bar{e}}(n,\bar{\mathsf{x}})\,.$$

Furthermore,

(ii) If $p_0 \in \mathbb{Z}[\mathsf{y}]$, then the e_σ are rational functions in $\dfrac{\mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]}{\mathbb{N}[\bar{\mathsf{y}}]}$.

(iii) If $a(n,\bar{\mathsf{x}})$ is SP-recursive, then the e_σ are polynomials in $\mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]$.

(iv) If $a(n,\bar{\mathsf{x}})$ is C-finite, then the e_σ are polynomials in $\mathbb{N}[\bar{\mathsf{x}}]$.

Moreover, for each $\sigma \in \Sigma$, there exist $i_\sigma, j_{\sigma,1}, \ldots, j_{\sigma,\ell} \in \mathbb{N}$ such that

$$p_0(\mathsf{y},\bar{\mathsf{x}}) = \frac{\mathsf{y}^{i_\sigma}\mathsf{x}_1^{j_{\sigma,1}}\cdots\mathsf{x}_1^{j_{\sigma,\ell}}}{e_\sigma(\mathsf{y},\bar{\mathsf{x}})}\,.$$

Lemma 9.2.3. *Let L be a regular language. Let $\mathcal{S}$ be a $\mathbb{Z}(\mathsf{y},\bar{\mathsf{x}})$ or one of its sub-semi-rings. Let $(e_\sigma : \sigma \in \Sigma)$ be a tuple of rational functions from $\mathcal{S}$.*

(i) If $\mathcal{S} = \mathbb{Z}(\mathsf{y},\bar{\mathsf{x}})$, then $W_{L,\bar{e}}(n,\bar{\mathsf{x}})$ is P-recursive.

(ii) If $\mathcal{S} = \dfrac{\mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]}{\mathbb{N}[\bar{\mathsf{y}}]}$, then $W_{L,\bar{e}}(n,\bar{\mathsf{x}})$ has a P-recurrence with leading polynomial $p_0(\mathsf{y}) \in \mathbb{Z}[\mathsf{y}]$.

(iii) If $\mathcal{S} = \mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]$, then $W_{L,\bar{e}}(n,\bar{\mathsf{x}})$ is SP-recursive.

(iv) If $\mathcal{S} = \mathbb{N}[\bar{\mathsf{x}}]$, then $W_{L,\bar{e}}(n,\bar{\mathsf{x}})$ is C-finite.

Proof of Theorem 9.2.1. For one direction, (i), (iii) and (iv) follow from Lemma 9.2.3 using the fact that each of the sets of C-finite, SP-recursive and P-recursive sequences is closed under subtraction. For case (ii) of Theorem 9.2.1 we need Lemma 7.2.6 in addition to Lemma 9.2.3.

For the other direction, we have from Lemma 9.2.2 that

$$a(n+s,\bar{\mathsf{x}}) = W_{L_1,\bar{e}}(n,\bar{\mathsf{x}}) - W_{L_2,\bar{e}}(n,\bar{\mathsf{x}})\,.$$

where L_1 and L_2 are regular languages and $(e_\sigma : \sigma \in \Sigma)$ is a tuple of rational functions $e_\sigma \in \mathcal{S}$, and where $\mathcal{S} = \mathbb{Z}(\mathsf{y},\bar{\mathsf{x}})$ or $\mathcal{S} = \frac{\mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]}{\mathbb{N}[\bar{\mathsf{y}}]}$ or $\mathcal{S} = \mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]$ or $\mathcal{S} = \mathbb{N}[\bar{\mathsf{x}}]$, depending on the type of recurrence $a(n,\bar{\mathsf{x}})$ satisfies. We may assume w.l.o.g that there are disjoint sets $\Sigma_1,\Sigma_2 \subseteq \Sigma$ such that $L_1 \subseteq \Sigma_1^\star$ and $L_2 \subseteq \Sigma_2^\star$. The languages $L_1 \cup L_2^c$, where L_2^c is the complement of L_2, and $\Sigma_2^\star$ are regular languages and

$$W_{L_1,\bar{e}}(n,\bar{\mathsf{x}}) - W_{L_2,\bar{e}}(n,\bar{\mathsf{x}}) = W_{L_1\cup L_2^c,\bar{e}}(n,\bar{\mathsf{x}}) - W_{\Sigma_2^\star,\bar{e}}(n,\bar{\mathsf{x}})\,.$$

We have

$$W_{\Sigma_2^\star,\bar{e}}(n,\bar{\mathsf{x}}) = \sum_{w\in\Sigma^n}\prod_{j=1}^{n} e_{w[j]}(j,\bar{\mathsf{x}}) = \prod_{j=1}^{n}\left(\sum_{\sigma\in\Sigma_2} e_\sigma(j,\bar{\mathsf{x}})\right)$$

so the Theorem follows by taking $u(\mathsf{y},\bar{\mathsf{x}}) = \sum_{\sigma\in\Sigma_2} e_\sigma(\mathsf{y},\bar{\mathsf{x}})$.

□

9.2.1 Proof of Lemma 9.2.2

In this proof we use the notation $[\mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell}]\,p(\bar{\mathsf{x}})$, where $p(\bar{\mathsf{x}}) \in \mathbb{Z}(\bar{\mathsf{x}})$ and $\varepsilon_1,\ldots,\varepsilon_\ell \in \mathbb{N}$, to denote the coefficient of $\mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell}$ in $p(\bar{\mathsf{x}})$.

Let $a(n,\bar{\mathsf{x}})$ be a P-recursive sequence over $\mathbb{Z}[\bar{\mathsf{x}}]$. If the leading polynomial of the P-recurrence is in $\mathbb{Z}[\mathsf{y}]$, then by Lemma 7.2.5 there exists $s \in \mathbb{N}$ such that $b(n,\bar{\mathsf{x}}) = a(n+s,\bar{\mathsf{x}})$ and the leading polynomial p_0 in some P-recurrence of $b(n,\bar{\mathsf{x}})$ is in $\mathbb{N}[\mathsf{y}]$. If the leading polynomial of the P-recurrence is not in $\mathbb{Z}[\mathsf{y}]$, then we take $b(n,\bar{\mathsf{x}}) = a(n,\bar{\mathsf{x}})$.

We have:

$$p_0(n,\bar{\mathsf{x}})\cdot b(n,\bar{\mathsf{x}}) = \sum_{i=1}^{r} p_i(n,\bar{\mathsf{x}})b(n-i,\bar{\mathsf{x}}) \tag{9.2}$$

where $p_0(\mathsf{y},\bar{\mathsf{x}}),\ldots,p_r(\mathsf{y},\bar{\mathsf{x}}) \in \mathbb{Z}[\mathsf{y},\bar{\mathsf{x}}]$ are polynomials over $\mathbb{Z}$, $p_0(n) \neq 0$ for every $n \in \mathbb{N}$. The initial conditions of $b(n,\bar{\mathsf{x}})$ are then $b(1,\bar{\mathsf{x}}),\ldots,b(r,\bar{\mathsf{x}})$. Let $\deg_b$ be the maximal degree of any indeterminate in an initial condition.

Since the p_i are polynomials over $\mathbb{Z}$, we may write the recurrence as follows:

$$\begin{aligned} b(n,\bar{\mathsf{x}}) &= \tfrac{1}{p_0(n,\bar{\mathsf{x}})}\Big(\sum_{i=1}^{r_1} n^{\alpha_i}\mathsf{x}_1^{\gamma_{i,1}}\cdots\mathsf{x}_\ell^{\gamma_{i,\ell}} b(n-l_i,\bar{\mathsf{x}}) - \\ &\qquad \sum_{i=1}^{r_2} n^{\beta_i}\mathsf{x}_1^{\delta_{i,1}}\cdots\mathsf{x}_\ell^{\delta_{i,\ell}} b(n-g_i,\bar{\mathsf{x}}) \Big), \end{aligned} \tag{9.3}$$

where we have $\alpha_i,\beta_i,\gamma_{i,j},\delta_{i,j} \in \mathbb{N}$, and $l_i, g_i \in [r]$.

For every i, let $q_i(\bar{\mathsf{x}}) = \mathsf{x}_1^{\gamma_{i,1}}\cdots\mathsf{x}_\ell^{\gamma_{i,\ell}}$ and $t_i(\bar{\mathsf{x}}) = \mathsf{x}_1^{\delta_{i,1}}\cdots\mathsf{x}_\ell^{\delta_{i,\ell}}$.

Let $\varepsilon_1,\ldots,\varepsilon_\ell \in [0,\deg_b]$ and

$$sg(k,\bar{\varepsilon}) = \begin{cases} +, & [\mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell}]b(k,\bar{\mathsf{x}}) \geq 0 \\ -, & [\mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell}]b(k,\bar{\mathsf{x}}) < 0 \end{cases}.$$

Let

$$\Gamma_1 = \left\{\rho_1^-,\ldots,\rho_{r_1}^-,\rho_1^+,\ldots,\rho_{r_2}^+,c\right\}$$

and

$$\Gamma_2 = \left\{\pi_{k,t,\bar{\varepsilon}}^{sg(k,\bar{\varepsilon})} \mid k \in [r],\ \varepsilon_1,\ldots,\varepsilon_\ell \in [0,\deg_b] \text{ and } 1 \leq t \leq \left|[\mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell}]b(k,\bar{\mathsf{x}})\right|\right\}.$$

The alphabets Γ_1 and Γ_2 consist of previously unused letters.

A word $w = w_1w_2\cdots w_n$ over alphabet $\Gamma = \Gamma_1 \cup \Gamma_2$ is called a *recurrence path* with respect to Equation (9.3) if the following conditions holds:

Cond 1 $w_1\cdots w_r = c^{k-1}\pi_{k,t,\bar{\varepsilon}}^{sg(k,\bar{\varepsilon})}c^{r-k}$, where $\pi_{k,t,\bar{\varepsilon}}^{sg(k,\bar{\varepsilon})} \in \Gamma_2$.

Cond 2 $w_{r+1}\cdots w_n$ belongs to $\{\rho_1^-,\ldots,\rho_{r_1}^-,\rho_1^+,\ldots,\rho_{r_2}^+,c\}^\star$,

Cond 3 $w_n \in \Gamma_1 \backslash \{c\}$,

Cond 4 for every m, if $w_m = \rho_i^+$ for some i, then $w_{m-(l_i-1)}\cdots w_{m-1} = c^{l_i-1}$ and $w_{m-l_i} \neq c$, and

Cond 5 for every m, if $w_m = \rho_i^-$ for some i, then $w_{n-(g_i-1)}\cdots w_{m-1} = c^{g_i-1}$ and $w_{m-g_i} \neq c$.

A word w which is a recurrence path with respect to Equation (9.3) describes the recursive choices in one branch of the recurrence tree, from the root $b(n,\bar{\mathsf{x}})$ to a leaf, which represents a monomial in an initial condition. The letters ρ_i^+ respectively ρ_i^- are used to indicate the choice of the i-th element of the first respectively second sum as the next recursive step in Equation (9.3). The letter c is a placeholder for the indices that are skipped over by the recurrence. The initial conditions are dealt with similarly using the $\pi_{k,t,\bar{\varepsilon}}^{sg(k,\bar{\varepsilon})}$ letters. We consider every initial condition $b(k,\bar{\mathsf{x}})$ as a sum of monomials $\pm\mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell}$, depending on whether the coefficient $[\mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell}]\,b(k,\bar{\mathsf{x}})$ is positive or negative. To indicate that the monomial $\pm\mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell}$ in the initial condition $b(k,\bar{\mathsf{x}})$ is chosen we set $w_k = \pi_{k,t,\bar{\varepsilon}}^{sg(k,\bar{\varepsilon})}$ for some $1 \le t \le \left|[\mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell}]\,b(k,\bar{\mathsf{x}})\right|$.

We may now rewrite Equation (9.3) as the sum over all words w of length n which are recurrence paths with respect to Equation (9.3). Let L_{rec} be the language of recurrence paths. Let $\mathrm{sgn}_w \in \{-1,1\}$ be 1 if the parity of the number of letters in w of the form ρ_i^- or $\pi_{k,t,\bar{\varepsilon}}^-$ is even, and -1 otherwise. We get that

$$b(n,\bar{\mathsf{x}}) = \frac{d(n,\bar{\mathsf{x}})}{\prod_{j=1}^{n} p_0(j,\bar{\mathsf{x}})} \tag{9.4}$$

where

$$\begin{aligned} d(n,\bar{\mathsf{x}}) \quad = \quad \sum_{\substack{w\in L_{rec}:\\ |w|=n}} \mathrm{sgn}_w \Bigg[& \prod_{j:w[j]=\rho_1^+} j^{\alpha_1} q_1(\bar{\mathsf{x}}) \cdots \prod_{j:w[j]=\rho_{r_1}^+} j^{\alpha_{r_1}} q_{r_1}(\bar{\mathsf{x}}) \cdot \\ & \prod_{j:w[j]=\rho_1^-} j^{\beta_1} t_1(\bar{\mathsf{x}}) \cdots \prod_{j:w[j]=\rho_{r_2}^-} j^{\beta_{r_2}} t_{r_2}(\bar{\mathsf{x}}) \cdot \\ & \prod_{\pi_{t,k,\bar{\varepsilon}}^+\in\Gamma_2} \left(\prod_{j:w[j]=\pi_{t,k,\bar{\varepsilon}}^+} \mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell} \right) \cdot \\ & \prod_{\pi_{t,k,\bar{\varepsilon}}^-\in\Gamma_2} \left(\prod_{j:w[j]=\pi_{t,k,\bar{\varepsilon}}^-} \mathsf{x}_1^{\varepsilon_1}\cdots\mathsf{x}_\ell^{\varepsilon_\ell} \right) \Bigg] \end{aligned}$$

It is not hard to see that L_{rec} is a regular language: We denote by $\mathit{exp}_{initial}$ and $\mathit{exp}_{recurrence}$ the following regular expressions:

$$\mathit{exp}_{initial} = \sum_{\sigma\in\Gamma_2 \text{ of the form } \pi^+_{k,t,\bar{\varepsilon}} \text{or } \pi^-_{k,t,\bar{\varepsilon}}} c^{k-1}\sigma c^{r-k}$$

$$\mathit{exp}_{recurrence} = \sum_{i=1}^{r_1} c^{l_i-1}\rho_i^+ + \sum_{i=1}^{r_2} c^{g_i-1}\rho_i^- .$$

L_{rec} is given by the regular expression $\mathit{exp}_{initial} \cdot \mathit{exp}_{recurrence} \cdot \mathit{exp}_{recurrence}{}^{\star}$. Let L_{even} and L_{odd} be the languages which consist of all words over Γ such that the number of letters of the form ρ_i^- or $\pi^-_{k,t,\bar{\varepsilon}}$ in the word is even respectively odd. L_{even} and L_{odd} are regular languages. By the closure of regular languages with respect to intersection, $L_{rec} \cap L_{even}$ and $L_{rec} \cap L_{odd}$ are regular as well.

Let e_σ be defined as follows:

— For every $\rho_i^+ \in \Gamma_1$, let $e_{\rho_i^+}(\mathsf{y}, \bar{\mathsf{x}}) = \dfrac{\mathsf{y}^{\alpha_i} q_i(\bar{\mathsf{x}})}{p_0(\mathsf{y}, \bar{\mathsf{x}})}$.

— For every $\rho_i^- \in \Gamma_1$, let $e_{\rho_i^-}(\mathsf{y}, \bar{\mathsf{x}}) = \dfrac{\mathsf{y}^{\beta_i} t_i(\bar{\mathsf{x}})}{p_0(\mathsf{y}, \bar{\mathsf{x}})}$.

— For every $\sigma \in \Gamma_2$, σ is of the form $\pi^+_{t,k,\bar{\varepsilon}}$ or $\pi^-_{t,k,\bar{\varepsilon}}$, let $e_\sigma(\bar{\mathsf{x}}) = \dfrac{\mathsf{x}_1^{\varepsilon_1} \cdots \mathsf{x}_\ell^{\varepsilon_\ell}}{p_0(\mathsf{y}, \bar{\mathsf{x}})}$.

— Let $e_c(\bar{\mathsf{x}}) = \dfrac{1}{p_0(\mathsf{y}, \bar{\mathsf{x}})}$.

By Equation (9.4), $b(n, \bar{\mathsf{x}})$ is the difference of two sequences with Positional Representation,

$$b(n, \bar{\mathsf{x}}) = W_{Lrec\cap L_{even},\bar{e}}(n, \bar{\mathsf{x}}) - W_{Lrec\cap L_{odd},\bar{e}}(n, \bar{\mathsf{x}})$$

with $e_\sigma \in \mathbb{N}(\mathsf{y}, \bar{\mathsf{x}})$, $\sigma \in \Gamma_1 \cup \Gamma_2$.

If $a(n, \bar{\mathsf{x}})$ is C-finite, then so is $b(n, \bar{\mathsf{x}})$. So, $p_0(\bar{\mathsf{x}}) = 1$ and $b(n, \bar{\mathsf{x}}) = d(n, \bar{\mathsf{x}})$. Moreover, $\alpha_1 = \cdots \alpha_{r_1} = \beta_1 = \cdots = \beta_{r_2} = 0$. Hence, for every $\rho_i^+ \in \Gamma_1$, $e_{\rho_i^+}(\bar{\mathsf{x}}) = q_i(\bar{\mathsf{x}})$; for every $\rho_i^- \in \Gamma_1$, $e_{\rho_i^-}(\bar{\mathsf{x}}) = t_i(\bar{\mathsf{x}})$; and for every $\sigma \in \Gamma_2$, $e_\sigma(\bar{\mathsf{x}}) = 1$. Hence, for each $\sigma \in \Gamma_1 \cup \Gamma_2$, $e_\sigma \in \mathbb{N}[\bar{\mathsf{x}}]$.

If $a(n, \bar{\mathsf{x}})$ is SP-recursive, then so is $b(n, \bar{\mathsf{x}})$. So, $p_0(\bar{\mathsf{x}}) = 1$ and $b(n, \bar{\mathsf{x}}) = d(n, \bar{\mathsf{x}})$. Hence, the e_σ are polynomials in $\mathbb{N}[\mathsf{y}, \bar{\mathsf{x}}]$.

If $a(n, \bar{\mathsf{x}})$ has a P-recurrence with leading polynomial in $\mathbb{Z}[\mathsf{y}]$, then $b(n, \bar{\mathsf{x}})$ has leading polynomial $p_0(\mathsf{y})$ in $\mathbb{N}[\mathsf{y}]$. Hence, the e_σ are rational functions in $\frac{\mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]}{\mathbb{N}[\mathsf{y}]}$. □

9.2.2 Proof of Lemma 9.2.3

Let L be a regular language over a finite alphabet Σ and let $\bar{e} = (e_\sigma : \sigma \in \Sigma)$ be a tuple of rational functions $e_\sigma \in \mathbb{N}(\mathsf{y}, \bar{\mathsf{x}})$. We will consider the recurrences of $W_{L,\bar{e}}(n, \bar{\mathsf{x}})$.

The language L is accepted by a deterministic finite state automaton A with state set Q, an initial state q_0, a set of final states F and a state-transition function $\delta : Q \times \Sigma \to Q$. By abuse of notation we write $\delta(q, w)$ for the state reached by A on input w. For each $q \in Q$, let L_q be the language of the automaton A_q which is obtained from A by replacing the set F of final states with $\{q\}$. We will show that each $W_{L_q,\bar{e}}(n)$ satisfies the desired recurrence type depending on $\bar{e}$. The lemma follows from this, since $W_{L,\bar{e}}(n) = \sum_{q\in F} W_{L_q,\bar{e}}(n)$, and each of the four sets of sequences which appear in the statement of the lemma is closed under finite addition by Lemmas 7.2.1 and 7.2.6.

Let $w = u\xi$, where $\xi \in \Sigma$ and u is a word of length $n-1$. The state reached by A_q on w, $\delta(q_0, w)$, depends only on ξ and on the state reached by A_q on u, $\delta(q_0, u)$. More precisely, $\delta(q_0, w) = \delta(\delta(q_0, u), \sigma)$. Hence, by the definition of $W_{L_q,\bar{e}}(n, \bar{\mathsf{x}})$, we can write

$$\begin{aligned}
W_{L_q,\bar{e}}(n, \bar{\mathsf{x}}) &= \sum_{\xi\in\Sigma} \sum_{u\xi\in L_q : |u|=n-1} \prod_{j=1}^{n} e_{u\xi[j]}(j, \bar{\mathsf{x}}) \\
&= \sum_{\xi\in\Sigma} e_\xi(n, \bar{\mathsf{x}}) \sum_{u\xi\in L_q : |u|=n-1} \prod_{j=1}^{n-1} e_{u[j]}(j, \bar{\mathsf{x}}) \\
&= \sum_{\xi\in\Sigma} e_\xi(n, \bar{\mathsf{x}}) \sum_{q'\in Q:\, \delta(q',\xi)=q} \sum_{u\in L_{q'} : |u|=n-1} \prod_{j=1}^{n-1} e_{u[j]}(j, \bar{\mathsf{x}}) \\
&= \sum_{\substack{(\xi,q')\in\Sigma\times Q:\\ \delta(q',\xi)=q}} e_\xi(n, \bar{\mathsf{x}}) \sum_{u\in L_{q'} : |u|=n-1} \prod_{j=1}^{n-1} e_{u[j]}(j, \bar{\mathsf{x}}) \qquad (9.5)
\end{aligned}$$

The inner sum of Equation (9.5) equals to $W_{L_{q'},\bar{e}}(n-1,\bar{\mathsf{x}})$. Hence,

$$\begin{aligned} W_{L_q,\bar{e}}(n,\bar{\mathsf{x}}) &= \sum_{\substack{(\xi,q')\in\Sigma\times Q: \\ \delta(q',\xi)=q}} e_\xi(n,\bar{\mathsf{x}}) \cdot W_{L_{q'},\bar{e}}(n-1,\bar{\mathsf{x}}) \\ &= \sum_{q'\in Q} W_{L_{q'},\bar{e}}(n-1,\bar{\mathsf{x}}) \cdot \sum_{\substack{\xi\in\Sigma: \\ \delta(q',\xi)=q}} e_\xi(n,\bar{\mathsf{x}}) \qquad (9.6) \end{aligned}$$

We get that $W_{L_q,\bar{e}}(n)$ satisfies a recurrence, with coefficients which are rational functions, in $W_{L_{q'},\bar{e}}(n-1)$, $q' \in Q$. To prove that $W_{L_q,\bar{e}}(n)$ satisfies the desired recurrences for every $q \in Q$, we first write the recurrences in matrix form. Let $\mathbf{v_n}$ be the column vector of length n given by $\mathbf{v_n} = \left(W_{L_q,\bar{e}}(n) \,:\, q \in Q\right)^{tr}$. Then there is a matrix M of size $|Q| \times |Q|$ of rational functions in n over $\mathbb{Z}$ such that $\mathbf{v_n} = M\mathbf{v_{n-1}}$. The matrix M is given by $M = (m_{q_1,q_2})$ where

$$m_{q_1,q_2} = \sum_{\substack{\xi\in\Sigma: \\ \delta(q_2,\xi)=q_1}} e_\xi(n,\bar{\mathsf{x}})\,.$$

Let $\mathrm{char}_M(\lambda)$ be the characteristic polynomial of M,

$$\mathrm{char}_M(\lambda) = \det\left(\lambda \cdot \mathbf{1} - M\right) = \sum_{i=0}^{|Q|} c_i(n,\bar{\mathsf{x}})\lambda^i.$$

Let $\mathcal{R}_\mathcal{S}$ be the ring defined as follows:

$$\mathcal{R}_\mathcal{S} = \begin{cases} \mathbb{Z}(\mathsf{y},\bar{\mathsf{x}}) & \mathcal{S} = \mathbb{N}(\mathsf{y},\bar{\mathsf{x}}) \\ \frac{\mathbb{Z}[\mathsf{y},\bar{\mathsf{x}}]}{\mathbb{Z}[\mathsf{y}]} & \mathcal{S} = \frac{\mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]}{\mathbb{N}[\mathsf{y}]} \\ \mathbb{Z}[\mathsf{y},\bar{\mathsf{x}}] & \mathcal{S} = \mathbb{N}[\mathsf{y},\bar{\mathsf{x}}] \\ \mathbb{Z}[\mathsf{y}] & \mathcal{S} = \mathbb{N}[\bar{\mathsf{x}}] \end{cases}$$

Notice $c_i \in \mathcal{R}_\mathcal{S}$ for every i, since c_i is a linear combination of finite products of elements m_{q_1,q_2}. Moreover, $c_{|Q|} = 1$.

The Cayley-Hamilton theorem for commutative rings states that

$$\sum_{i=0}^{|Q|} c_i(n,\bar{\mathsf{x}})M^i = 0\,,$$

or equivalently,

$$M^{|Q|} = \sum_{i=0}^{|Q|-1} -c_i(n,\bar{\mathsf{x}})M^i\,.$$

Using that for every m, $\mathbf{v_m} = M^i\mathbf{v_{m-i}}$, we have

$$\begin{aligned}\mathbf{v_n} = M^{|Q|}\mathbf{v_{n-|Q|}} &= \sum_{i=0}^{|Q|-1} \left(-c_i(n,\bar{\mathsf{x}})\right) M^i\mathbf{v_{n-|Q|}} \\ &= \sum_{i=0}^{|Q|-1} \left(-c_i(n,\bar{\mathsf{x}})\right) \mathbf{v_{n-|Q|+i}}\,. \end{aligned} \tag{9.7}$$

Equation (9.7) gives $|Q|$ many recurrences, one for each row of the vector $\mathbf{v_n}$:

$$W_{L_q,\bar{e}}(n,\bar{\mathsf{x}}) = \sum_{i=1}^{|Q|} \left(-c_{|Q|-i}(n,\bar{\mathsf{x}})\right) \cdot W_{L_q,\bar{e}}(n-i,\bar{\mathsf{x}}) \tag{9.8}$$

We get that $W_{L_q,\bar{e}}(n,\bar{\mathsf{x}})$ is P-recursive.

If the e_σ are polynomials in $\mathbb{N}[\bar{\mathsf{x}}]$, so are the c_i, and by Equation (9.8), $W_{L_q,\bar{e}}(n,\bar{\mathsf{x}})$ is C-finite. If the e_σ are polynomials in $\mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]$, so are the c_i, and by Equation (9.8), $W_{L_q,\bar{e}}(n,\bar{\mathsf{x}})$ is SP-recursive.

Otherwise, multiplying both sides of Equation (9.8) by the common denominator of $c_0(n,\bar{\mathsf{x}}),\ldots,c_{|Q|-1}(n,\bar{\mathsf{x}})$, we get a P-recurrence for $W_{L_q,\bar{e}}(n,\bar{\mathsf{x}})$. If, furthermore, the e_σ are rational functions which belong to $\frac{\mathbb{N}[\mathsf{y},\bar{\mathsf{x}}]}{\mathbb{N}[\mathsf{y}]}$, then the leading polynomial in this P-recurrence, namely the common denominator of $c_0(n,\bar{\mathsf{x}}),\ldots,c_{|Q|-1}(n,\bar{\mathsf{x}})$, belongs to $\mathbb{Z}[\mathsf{y}]$. □

Chapter 10

Counting Lattice Paths and P-recursive Sequences

In this chapter we define a class of lattice paths and prove Theorem F, which gives a representation of P-recursive sequences of integers in terms of counting a certain type of lattice paths. As applications we consider sequences which count certain permutations.

10.1 Lattice paths interpretations of sequences

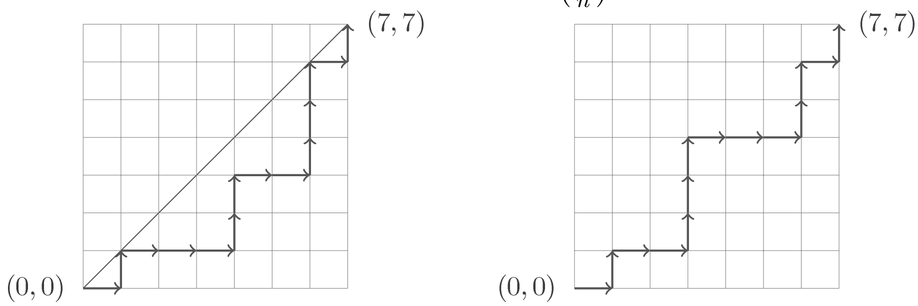
The Catalan numbers C_{n+1}, discussed in Section 8.1.5, count the number of Dyck paths in the grid $[0,n]^2 = \{0,\ldots,n\} \times \{0,\ldots,n\}$, paths from $(0,0)$ to (n,n) which consist of steps $\{\uparrow, \rightarrow\}$ and do not go above the diagonal line $y = x$. Such a path is depicted in Figure 10.1(a).

Recall the Catalan numbers are P-recursive.

The central binomial coefficients $\binom{2n}{n}$, discussed in Section 8.1.4, count paths similar to Dyck paths, except that they are no longer restricted to stay below the diagonal. Figure 10.1(b) shows such a path. They are P-recursive as well. However, both C_{n+1} and $\binom{2n}{n}$ are not SP-recursive due to their modular properties.

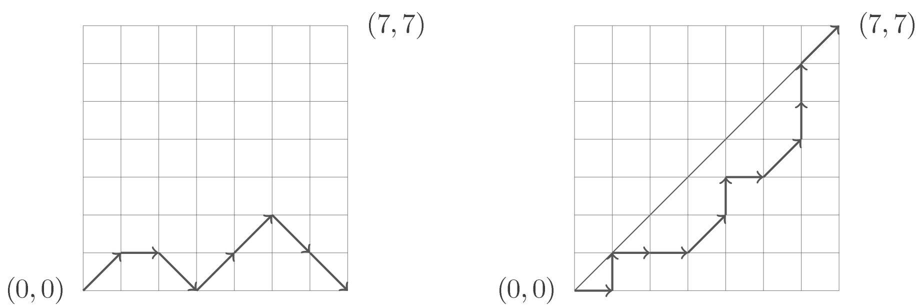
Some other P-recursive sequences have natural interpretations as counting various lattice paths, such as Motzkin numbers and Schröder numbers. Motzkin numbers count paths in $[0,n]^2$ from $(0,0)$ to $(n,0)$ with steps $\nearrow, \rightarrow, \searrow$. Schröder numbers count paths in $[0,n]^2$ from $(0,0)$ to (n,n) with

Figure 10.1: Paths counted by the Catalan numbers and the central binomial coefficient $\binom{2n}{n}$.



(a) Catalan numbers (b) The central binomial coefficient

Figure 10.2: Paths counted by Motzkin and Schröder numbers.

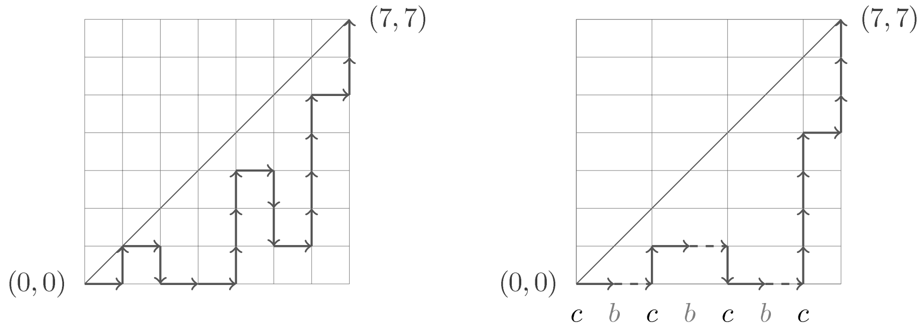


(a) Motzkin numbers (b) Schröder numbers

steps $\rightarrow, \uparrow, \nearrow$ which do not go above the diagonal. See examples in Figure 10.2.

The factorial $n!$ can be interpreted as the counting sequence of a certain type of lattice paths on the grid $[0,n]^2$. Let $d_1(n)$ be the number of paths from $(0,0)$ to (n,n) with step set $\{\uparrow, \downarrow, \rightarrow\}$ which do not cross the diagonal line and are self-avoiding. An example of such a path is given in Figure 10.3(a). Consider a path in $[0,n]^2$ which is counted by $d_1(n)$. At any point (i,j), the only step which the path can take and which changes the column j is $\rightarrow$. Since the path starts at $(0,0)$ and ends at (n,n), it must take exactly one $\rightarrow$ step at each of the columns $0,\ldots,n-1$. Since the paths enumerated

Figure 10.3: Paths count by the factorial and double factorial.



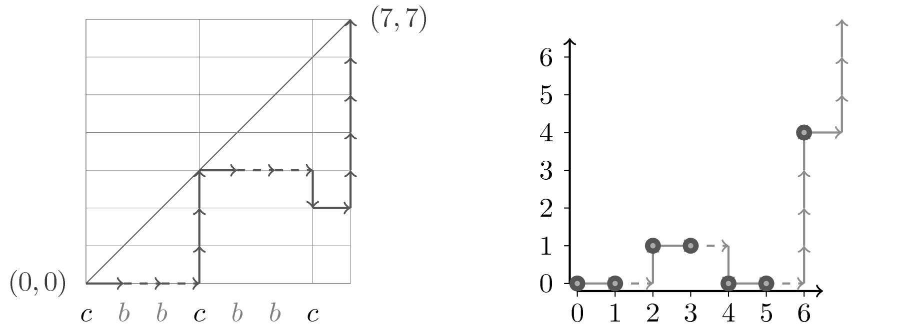
(a) Factorial (b) Double factorial

by $d_1(n)$ are self-avoiding, one can see that the paths can be determined uniquely by the points on the grid where $\rightarrow$ occurred.

Let $(0,0),(i_1,1),\ldots,(i_{n-1},n-1)$ be the points in which $\rightarrow$ occurs. We may think of these points as a function $f:[0,n-1]\rightarrow[0,n-1]$ which is defined as $f(j)=i_j$. The function f assigns each column j with the row i_j in which $\rightarrow$ occurs. It holds that $f(j)\leq j$ for every j. On the other hand, every function $f:[0,n-1]\rightarrow[0,n-1]$ for which $f(j)\leq j$ for every j is obtained by some legal lattice path of the paths counted by $d_1(n)$. Therefore, $d_1(n)$ is the number of functions $f:[0,n-1]\rightarrow[0,n-1]$ for which $f(j)\leq j$ for every j. Consequently, $d_1(n)=n!$.

Simple variants of $n!$ can be interpreted similarly. The double factorial $n!!$ is defined to be $n!!=n(n-2)(n-4)\cdots$. An interpretation in terms of lattice paths for $n!!$ can be obtained by augmenting the interpretation for $n!$ as follows. We label each column $j\in[0,n-1]$ of the grid $[0,n]^2$ by a letter $w[j+1]\in\{b,c\}$. The labeling of the columns induces a word $w=w[1]\cdots w[n]$ of length n over the alphabet $\{b,c\}$. The last column is left unlabeled. Let $d_2(n,w)$ be the number of paths from $(0,0)$ to (n,n) with step set $\{\uparrow,\downarrow,\rightarrow\}$ which do not cross the diagonal, are self-avoiding, and in addition, the steps $\uparrow$ and $\downarrow$ may occur only in columns labeled with c. Let w be a word of length n which belongs to the language $L_{!!}=L\left((bc)^{\star}+c(bc)^{\star}\right)$. The word w is of form either $bcbc\cdots bc$ or $cbcbc\cdots bc$. The paths enumerated by $d_2(n,w)$ can be distinguished uniquely by the points on the grid where

Figure 10.4: Paths counted by the double and triple factorials.



(c) Triple factorial

(d) The path in (b) as a function

$\rightarrow$ occurs in columns $j \in [0, n-1]$ which have the same parity as $n-1$, or equivalently, those columns $j \in [0, n-1]$ labeled c. Notice that on columns labeled b the only possible step is $\rightarrow$. The column n must consist of $\uparrow$ steps only in order to reach the upper right point of the grid. Let

$$d_2(n) = \sum_{w \in L_{!!}: |w|=n} d_2(n, w)\,.$$

We have $d_2(n) = n!!$. Figure 10.3(b) illustrates a path counted by $d_2(n)$. The dashed right arrows $\dashrightarrow$ are used to indicate that no other step starting at the same column is possible, since the column is labeled b. As in the case of the factorial, the double factorial can be considered as counting certain functions, see Figure 10.4(d). The triple factorial $n!!!$ can be treated similarly, see Figure 10.4(c).

Another simple variant of $n!$ which is SP-recursive is $n!^2$, which satisfies the recurrence $n!^2 = n^2 \cdot (n-1)!^2$. The sequence $n!^2$ can be interpreted as counting pairs of paths counted by $d_1(n)$.

10.2 Theorem F: exact statement

In this subsection we present Theorem F exactly. We begin by defining the type of lattice paths which we will use to characterize P-recursive sequences.

Let Σ be an alphabet.

Definition 10.2.1 ((w, σ)-path). *Let w be a word of length n over Σ and let $\sigma \in \Sigma$. A* (w, σ)-path *p is a lattice path in the grid $[0, n]^2$ satisfying the following conditions:*

(i) p starts at $(0, 0)$ and ends at (n, n),

(ii) every point (i, j) in p satisfies $i \leq j$, i.e. p does not cross the diagonal,

(iii) p consists of steps from $\{\rightarrow, \uparrow, \downarrow\}$,

(iv) p is self-avoiding, and

(v) $w[j+1] \neq \sigma$ implies the only step that occurs in column j is $\rightarrow$.

Definition 10.2.2 ($m_{L,\bar{s}}$). *Let L be a regular language over Σ, k be a natural number, and $\bar{s} = (s_1, \ldots, s_k)$ be a tuple of Σ letters. The sequence $m_{L,\bar{s}}(n)$ counts the number of tuples $(w, p_1, \ldots, p_k)$ where $w \in L$, $|w| = n$ and each p_i is a (w, s_i)-path.*

We can apply Definition 10.2.2 to $n!$, $n!!$ and $(n!)^2$. Let $L_c = L(c^\star)$. Then

$$n! = m_{L_c,c}(n),\ n!! = m_{L_{!!},c}(n),\ (n!)^2 = m_{L_c,(c,c)}(n)\,.$$

We are now ready to state Theorem F exactly:

Theorem 10.2.1 (Theorem F). *Let $a(n)$ be a sequence of integers.*

(i) $a(n)$ is SP-recursive iff there exist regular languages L_1 and L_2 and a tuple $\bar{s}$ of letters of Σ such that $a(n) = m_{L_1,\bar{s}}(n) - m_{L_2,\bar{s}}(n)$.

(ii) $a(n)$ is P-recursive iff there exist regular languages L_1 and L_2, a polynomial $q(\mathsf{y}) \in \mathbb{Z}[\mathsf{y}]$ and a tuple $\bar{s}$ of letters of Σ such that

$$a(n) = \frac{m_{L_1,\bar{s}}(n) - m_{L_2,\bar{s}}(n)}{\prod_{j=1}^{n} q(j)}\,.$$

(ii) follows from (i) in Theorem 10.2.1 by Lemma 7.2.4 and the closure of P-recursive sequences to finite product.

10.3 Proof of Theorem F

Here we prove 10.2.1. First note that (ii) follows from (i) in Theorem 10.2.1 by Lemma 7.2.4 and the closure of P-recursive sequences to finite product.

The proof of Theorem 10.2.1(i) follows from the following two lemmas, which we will prove in Subsections 10.3.1 and 10.3.2 respectively.

Lemma 10.3.1. *Let $a(n)$ be SP-recursive. Then there exist regular languages L_1 and L_2 over an alphabet Σ and a tuple $\bar{s}$ of Σ letters such that*

$$a(n) = m_{L_1,\bar{s}}(n) - m_{L_2,\bar{s}}(n)\,.$$

Lemma 10.3.2. *Let $a(n) = m_{L,\bar{s}}(n)$ with L a regular language and $\bar{s}$ a tuple of letters of Σ. The sequence $a(n)$ is SP-recursive.*

Using Lemma 10.3.1, Lemma 10.3.2 and the closure properties of SP-recursive sequences given in Lemma 7.2.1, the proof of Theorem 10.2.1 follows.

For the proof of the Lemmas we look at the (w, σ)-paths as functions.

Definition 10.3.1. *Let w be a word of length n. Let $B_{w,\sigma}$ be the set of all functions $f : [0, n-1] \to [0, n-1]$ which satisfy the following conditions:*

— *$f(j) \leq j$ for every $j \in [0, n-1]$, and*

— *$f(j-1) \neq f(j)$ implies $w[j+1] = \sigma$.*

Proposition 10.3.3. *For every w of length n over Σ and $\sigma \in \Sigma$,*

$$|B_{w,\sigma}| = \prod_{j:w[j]=\sigma} j\,.$$

Proposition 10.3.3 follows from the definition of $B_{w,\sigma}$.

Proposition 10.3.4. *Let L be a regular language and let $\bar{s}$ be a tuple of letters, $\bar{s} = (s_1, \ldots, s_k) \in \Sigma^k$. It holds that*

$$\sum_{w \in L:|w|=n} |B_{w,s_1}| \cdots |B_{w,s_k}| = m_{L,\bar{s}}(n)\,.$$

Proof. For every $\sigma \in \Sigma$ and $w \in L$, let $B'_{w,\sigma}$ be the set of (w,σ)-paths. We will construct a bijection between $B_{w,\sigma}$ and $B'_{w,\sigma}$. The proposition then follows since by definition of $m_{L,\bar{s}}(n)$,

$$m_{L,\bar{s}}(n) = \sum_{w \in L : |w| = n} |B'_{w,s_1}| \cdots |B'_{w,s_k}| \,.$$

Let $\sigma \in \Sigma$ and $w \in L$. Let p be a (w,σ)-path. The path p contains exactly $n \to$ steps. There is exactly one $\to$ step in each column $j \in [0, n-1]$ of the grid $[0,n]^2$. Let $(i_0, 0), \ldots, (i_{n-1}, n-1)$ be the positions of occurrences of $\to$ in p. For each $j \in [0, n-2]$, the steps between (i_j, j) and $(i_{j+1}, j+1)$ must all be $\uparrow$ steps or all $\downarrow$ steps, since the path is self-avoiding. All the steps in the last column n must be $\uparrow$ steps in order to reach (n,n). Hence, the path p is determined uniquely by $(i_0, 0), \ldots, (i_{n-1}, n-1)$. It holds that $i_j \leq j$, since the path does not cross the diagonal. Let $f_p : [0, n-1] \to [0, n-1]$ be given as $f_p(j) = i_j$. The function f_p belongs to $B_{w,\sigma}$. Let $h : B'_{w,\sigma} \to B_{w,\sigma}$ be given by $h(p) = f_p$. The function h is injective, since p is determined uniquely by $(i_0, 0), \ldots, (i_{n-1}, n-1)$. It is not hard to see that h is also surjective, since it holds, for every $f \in B_{w,\sigma}$, that for every $j \in [0, n-1]$, $f(j) \leq j$. □

Proposition 10.3.5. *Let w be a word over Σ and let σ be a letter of Σ which does not occur in w. Then the number of (w,σ)-paths is 1.*

Proof. Since σ does not occur in w, the only possible step in any of the columns $0, \ldots, |w|-1$ is $\to$. It is always the case that in the last column we have only $\uparrow$ steps up to the upper right corner of the grid. □

10.3.1 Proof of Lemma 10.3.1

Let $a(n)$ be a P-recursive integer sequence with leading polynomial $p_0(\mathsf{y}) \in \mathbb{Z}[\mathsf{y}]$. By Lemma 9.2.2, there exist regular languages $L_1, L_2 \subseteq \Sigma^\star$

and, for every $\sigma \in \Sigma$, $\alpha_\sigma, \beta_\sigma \in \mathbb{N}$ such that

$$a(n) = \frac{\displaystyle\sum_{\substack{w \in L_1: \\ |w|=n}} \prod_{\sigma \in \Sigma} \left(\prod_{j:w[j]=\sigma} j^{\alpha_\sigma} \right) - \sum_{\substack{w \in L_2: \\ |w|=n}} \prod_{\sigma \in \Sigma} \left(\prod_{j:w[j]=\sigma} j^{\beta_\sigma} \right)}{\displaystyle\prod_{j=1}^{n} p_0(j)} .$$

By Proposition 10.3.3, $\prod_{j:w[j]=\sigma} j$ is the number of functions in $B_{w,\sigma}$. Thus,

$$a(n) = \frac{\displaystyle\sum_{\substack{w \in L_1: \\ |w|=n}} \prod_{\sigma \in \Sigma} |B_{w,\sigma}|^{\alpha_\sigma} - \sum_{\substack{w \in L_2: \\ |w|=n}} \prod_{\sigma \in \Sigma} |B_{w,\sigma}|^{\beta_\sigma}}{\displaystyle\prod_{j=1}^{n} p_0(j)} ,$$

Without loss of generality, we may assume L_1 and L_2 are in fact languages over disjoint copies Σ_1 and Σ_2 of Σ given as $\Sigma_i = \{\sigma^{[i]} : \sigma \in \Sigma\}$, $i = 1, 2$. By Proposition 10.3.4,

$$a(n) = \frac{m_{L_1,\bar{s}_1}(n) - m_{L_2,\bar{s}_2}(n)}{\prod_{j=1}^{n} p_0(j)}$$

where

$$\begin{aligned} \bar{s}_1 &= \left(\overbrace{\sigma^{[1]}, \sigma^{[1]}, \ldots, \sigma^{[1]}}^{\alpha_\sigma \text{ times}} : \sigma \in \Sigma \right) \\ \bar{s}_2 &= \left(\overbrace{\sigma^{[2]}, \sigma^{[2]}, \ldots, \sigma^{[2]}}^{\beta_\sigma \text{ times}} : \sigma \in \Sigma \right) \end{aligned}$$

i.e. where $\bar{s}_1$ is a tuple of Σ letters in which every letter σ occurs exactly α_σ times and similarly for $\bar{s}_2$. Since Σ_1 and Σ_2 are disjoint, we can in fact let $\bar{s}$ be the tuple obtained by the concatenation of $\bar{s}_1$ and $\bar{s}_2$ by a similar argument to Proposition 10.3.5, and we have

$$a(n) = \frac{m_{L_1,\bar{s}}(n) - m_{L_2,\bar{s}}(n)}{\prod_{j=1}^{n} p_0(j)}$$

□

10.3.2 Proof of Lemma 10.3.2

Let L be a regular language and let $\bar{s} = (s_1, \ldots, s_k) \in \Sigma^k$. By Propositions 10.3.4 and 10.3.3

$$m_{L,\bar{s}}(n) = \sum_{w \in L : |w| = n} \prod_{j : w[j] = s_1} j \cdots \prod_{j : w[j] = s_k} j \,.$$

So, there exist polynomials $e_\sigma(\mathsf{y}) \in \mathbb{N}[\mathsf{y}]$, all of the form $e_\sigma(\mathsf{y}) = \mathsf{y}^{i_\sigma}$ for some $i_\sigma \in \mathbb{N}$, such that $W_{L,\bar{p}}(n) = m_{L,\bar{s}}(n)$. By Lemma 9.2.3, $m_{L,\bar{s}}(n)$ is SP-recursive. □

10.4 Applications to counting permutations

Interesting sets of P-recursive sequences arise from counting permutations. Gessel [71] and Noonan-Zeilberger [126] initiated the study of the P-recursiveness of counting permutations restricted by certain patterns. M. Bona [26], and independently [120], proved the following theorem.

Theorem 10.4.1. *The number $S_r(n)$ of permutations of length n containing exactly r subsequences of type 132 is P-recursive.*

In [126] it is conjectured that for any given subsequence q, rather than just 132, and for any given r, the number of n-permutations containing exactly r subsequences of type q is P-recursive. However, later evidence has caused Zeilberger to change his mind [53] and conjecture that it is not P-recursive for $q = 1324$.

In this section we give two examples of interpretations of SP-recursive sequences which arise from counting permutations as counting lattice paths, and where the lattice paths interpretation is intuitive. The first example counts permutations with a fixed number of cycles. The purpose of this example is to show how counting permutations can be naturally interpreted in terms of counting (w, σ)-paths. The second example counts derangements. The purpose of this example is to illustrate how to derive from the explicitly given recurrence relation a lattice paths interpretation. It gives a simplified proof for this case of Lemmas 9.2.2 and 10.3.1.

Various other classes of permutations have been studied for their recurrences, including permutations with longest increasing subsequence at most some fixed k [71, 140] and 321-hexagon-avoiding permutations [137].

10.4.1 Permutations with a fixed number of cycles

Recall from Subsection 8.1.6 that the Stirling numbers of the first kind $\left[{n \atop k}\right]$ count the number of permutations of $[n]$ with exactly k cycles and satisfy

$$\left[{n+1 \atop k}\right] = n \cdot \left[{n \atop k}\right] + \left[{n \atop k-1}\right] . \tag{10.1}$$

Using the Cayley-Hamilton theorem one can show that $\left[{n+1 \atop k}\right]$ is SP-recursive.

$\left[{n+1 \atop k}\right]$ can be interpreted naturally as counting (w, σ)-paths. Every arrangement of $[0, n]$ into k cycles can be obtained uniquely by the following process, which is used in [83] to prove Equation (10.1). The process augments permutations of $[0, t]$ to create permutations of $[0, t+1]$ by adding the element $t+1$. We start with the only permutation of $\{0\}$. Every subsequent element $1, 2, \ldots, n$ either forms a new cycle or is added to an existing cycle. We require that exactly $k-1$ elements form new cycles during the process, which ends with the element n. This implies the permutation obtained at the end of the process has k cycles, since we begin the process with a permutation which consists of a single cycle.

Let L_k be the language over alphabet $\Sigma = \{b, c\}$ which consists of all words with exactly $k-1$ occurrences of the letter b. The letter b is used to indicate which elements form a new cycle during the process. Equivalently, the b's in a word w of L_k indicate which elements of $[n]$ are minimal in their cycles in the resulting permutation. In addition, 0 is also minimal in its cycle. The language L_k is a regular language given by the regular expression

$$exp_k = (c^\star b)^{k-1}\, c^\star .$$

We would like to show that $m_{L_k,c}(n) = \left[{n+1 \atop k}\right]$. There are j ways to add the element j to an existing cycle, by putting j to the right of any of the elements already in the permutation, $[0, j-1]$. The addition of j into one of the existing cycles corresponds to the letter c. The sequence $m_{L_k,c}(n)$ counts paths which allow any step of $\{\rightarrow, \uparrow, \downarrow\}$ in columns labeled c. In a column $j \in [0, n-1]$ labeled c there are $j+1$ possibilities of movement for the lattice path before it leaves column j and does not return. On the other hand, there is only one way of forming a new cycle. The creation of a new cycle corresponds to the letter b, which occurs $k-1$ times in words of L_k. In a column j labeled b there is only one possibility of movement, since the only allowed step in this case is $\rightarrow$, which implies that upon arriving at the j-th column, the path leaves the column immediately (and does not return).

10.4.2 Derangements

Recall from Subsection 8.1.3 that the derangement numbers $D(n)$ count permutations of $[n]$ without fixed points. They satisfy the SP-recurrence

$$D(n) = (n-1) \cdot D(n-1) + (n-1) \cdot D(n-2) \tag{10.2}$$

with initial conditions $D(0) = 1$ and $D(1) = 0$. Let $D'(n)$ be the sequence such that $D(n+1) = D'(n)$, i.e. $D'(n)$ is the number of permutations of $[n+1]$ without fixed-points. The sequence $D'(n)$ satisfies the SP-recurrence

$$D'(n) = n \cdot D'(n-1) + n \cdot D'(n-2) \tag{10.3}$$

with initial conditions $D'(0) = D(1) = 0$ and $D'(1) = D(2) = 1$. Let the language L_{der} consist of all words w of length n over $\{a, b, c, d\}$ such that

(i) $w[1] = d$ and $w[2], \ldots, w[n] \in \{a, b, c\}$, and

(ii) $w[j] = c$ iff $w[j+1] = b$, for $2 \leq j \leq n-1$.

One can show that

$$D'(n) = \sum_{w \in L_{der} : |w| = n} \ \prod_{j : w[j] \in \{a,b\}} j \tag{10.4}$$

where the summation is over words of length n in L_{der}.

We can think of Equation (10.4) as the sum over all paths in the recurrence tree of equation (10.3) from the root to a leaf $D'(1)$ (notice a path in the recurrence tree that ends in $D'(0) = 0$ has value 0). Such a path can be described by $1 = t_1 \leq \ldots \leq t_r = n$ such that for each i, $1 < i \leq r$, the difference of subsequent elements $t_i - t_{i-1}$ is either 1 or 2. The elements t_i in $[n]$ for which a recurrence step of the form $i \cdot D'(i-1)$ was chosen (i.e., those for which $t_i - t_{i-1} = 1$) are assigned the letter a, whereas b is assigned to those elements t_i of $[n]$ which correspond to a choice of the form $i \cdot D'(i-2)$ (i.e., those t_i for which $t_i - t_{i-1} = 2$). We assign c to all the elements $i \in [n] - \{t_1, \ldots, t_r\}$, which are skipped by a recursive choice $j \cdot D'(j-2)$, where $j = i+1$. The letter d is assigned to the leafs $D'(1)$. By condition (i) the letter d is a place-holder for the leaf of the path in the recurrence tree, since it always occurs as $w[1]$ and never as any other $w[i]$. Condition (ii) requires that $i-1$ is skipped iff $i \cdot D'(i-2)$ is chosen for i. Notice this is a regular language, given by the regular expression

$$\mathit{exp}_{der} = d(cb + a)^\star .$$

Equation (10.4) can be interpreted as counting the number of tuples (w, p_0, p_1) where $w \in L_{der}$ is of length $|w| = n$, p_0 is a (w, a)-path and p_1 is a (w, b)-path. That is, $D(n+1) = m_{L_{der},(a,b)}(n)$.

Chapter 11

Conclusion and Open Problems

We studied the definability of two types of combinatorial functions: graph polynomials, and recursive sequences. Inspired by the chromatic polynomial, we studied the graph polynomials which arise from counting various kinds of graph colorings[1]. We gave a simple sufficient condition which implies that the counting function of a generalized coloring is indeed a graph polynomial. Using this condition we listed generalized colorings studied in the literature which were not noticed to be graph polynomials. We showed that the class of such graph polynomials is essentially equivalent to the class of graph polynomials given by subset expansions.

Next we considered the class of MSOL-polynomials. This class contains most of the prominent graph polynomials studied in the literature and has some nice combinatorial and complexity properties. We proved that the $\odot$-connection matrices of MSOL-polynomials for binary graph operations which are smooth, such as the disjoint union, the k-sum and the join, have finite ranks. We showed that many graph parameters are not evaluations or fixed coefficients of any MSOL-polynomial. We gave for the first time natural examples of graph polynomials with a fixed number of indeterminates

[1] A version of Proposition A and Theorem B appeared in the unpublished manuscript [118]. The results here consider standard and FF SOL-polynomials instead of just Newton polynomials, as was the case in [118]. They also replace some proofs, discuss a syntactic definition of the extension property and show the equivalence of FF and standard SOL-polynomials.

(indeed, one indeterminate) which are not MSOL-polynomials.

We studied the complexity of the bivariate and trivariate Ising polynomials as a case study of the Difficult Point Conjecture. We showed that the Ising polynomials have the Difficult Point Property (DPP), even on a very restricted class of graphs. Under a counting version of the exponential time hypothesis, we showed that the bivariate Ising polynomial satisfies an exponential running-time version of the DPP, an analog to a recent result of [47] for the Tutte polynomial. In Appendix B we show that the Ising polynomials require polynomial time to compute on graphs of bounded clique-width.

Next we turned our attention to sequences of integers or polynomials which satisfy recurrence relations. We showed that the sets of C-finite and P-recursive sequences have representation theorems in term of appropriate positionally weighted sums. This is a generalization of a classical theorem from 1963 of Chomsky and Schützenberger for C-finite sequences of integers.

We considered the set of P-recursive sequences of integers. Inspired by famous examples of lattice paths counting, we showed a representation theorem for P-recursive sequences of integers in terms of counting appropriate lattice paths and applied this representation to counting permutations. This gives an answer to an open question of M. Bousquet-Mélou stated in [30]. There Bousquet-Mélou asked whether there exists a class of objects whose counting functions capture the P-recursive sequences.

11.1 Further research and open problems

Graph polynomials

We have shown a list of new natural graph polynomials which count generalized colorings. While we considered their definability to some extent, some open questions remain:

Open Problem 1. *Are the graph polynomials which count the following generalized colorings* MSOL_2*-polynomials?*

— mcc_t *colorings, for* $t \in \mathbb{N}$*,*

— *Acyclic colorings,*

— *Non-repetitive colorings.*

Open Problem 2. *For which graphs G are the counting functions of G-free colorings* MSOL-*polynomials?*

Or, more generally,

Open Problem 3. *For which graph properties $\mathcal{P}$ are the counting functions of $\mathcal{P}$-k-colorings* MSOL-*polynomials?*

We have shown many graph parameters which cannot occur as evaluations or fixed coefficients of any MSOL-polynomials. Other ways of encoding graph parameters in graph polynomials are not covered by this. This leaves open, e.g.:

Open Problem 4. *What properties do the degrees and zeros of* MSOL-*polynomials have? Which graph parameters cannot occur as degrees and zeros of* MSOL-*polynomials?*

The study of the complexity of the Ising polynomials is not complete.

Open Problem 5. *Does the bivariate Ising polynomial require exponential running time, assuming* #**ETH** *holds, even when the input is restricted to graph classes $\mathcal{C}$ such as regular, planar or bipartite graphs?*

If the answer to the previous problem is affirmative for regular graphs, then we also get a partial answer to the following problem:

Open Problem 6. *Do the evaluations of the trivariate Ising polynomial require exponential running time assuming* #**ETH** *holds?*

Related questions remain open for other graph polynomials:

Open Problem 7. *Do other of the well-studied graph polynomials in the literature, such as the independent set polynomial, the interlace polynomial and the cover polynomial, require exponential running-time assuming* #**ETH** *holds? Does this hold when the input graphs are required to belong to various graph classes $\mathcal{C}$?*

Recurrence sequences

The representation theorem in Chapter 10 characterizes SP-recursive sequences of integers in terms of lattice paths with allowed steps $\{\rightarrow, \uparrow, \downarrow\}$ which depend on words of regular languages.

Open Problem 8. *Is it possible to represent SP-recursive sequences by counting lattice paths which do not depend on words?*

Open Problem 9. *Is it possible to represent P-recursive sequences by counting lattice paths without the need to divide by a generalized factorial?*

Open Problem 10. *Can other meaningful sets of sequences be represented by counting lattice paths with other step sets or with different definability criteria?*

A famous theorem of M. Soittola (see e.g. Theorem 3.1 in [16]) characterizes the sequences which count words of regular languages among all C-finite sequences of natural numbers in terms of the poles of their generating functions.

Open Problem 11. *Is there an analytic characterization of the P-recursive sequences which have $\bar{e}$-Positional Representations, where $\bar{e}$ is a tuple of rational functions in $\mathbb{N}(\mathsf{y})$ or polynomials in $\mathbb{N}[\mathsf{y}]$?*

The generating functions of P-recursive sequences are known to be exactly the differentiably finite (D-finite) generating functions. It is conceivable that an analytic characterization in Open Problem 11 could depend on the differential equation satisfied by the generating function of the sequence.

Appendix A

Non-definability of graph properties

In this appendix we give a general analog of Theorem C to graph properties, and demonstrate its vast applicability in simplifying known and give new non-definability results for graph properties. We prove a Feferman-Vaught-type Theorem for the logic CFOL, First Order Logic with the modular counting quantifiers with an auxiliary linear order relation[1], allowing us to prove non-definability results for CFOL.

In Section A.1 we extend our logical framework from Section 4.3 and give a version of Theorem C for graph properties. In Section A.2 we use this theorem to prove known and new non-definability results. In Section A.3 we consider the logic CFOL.

A.1 Connection Matrices for Properties: The Framework

Let τ be a purely relational finite vocabulary which may include constant symbols. A τ-*property* is a class of finite τ-structures closed under τ-isomorphisms. If the context is clear we just speak of properties and isomorphisms.

[1] Modular counting qunatifiers are not definable in FOL even in the presence of an auxiliary linear order relation. In contrast, MSOL and its extenion with modular counting qunatifiers, CMSOL, have the same expression power on ordered structures.

Let $\mathcal{L}$ be a subset of SOL. $\mathcal{L}$ is a *fragment of* SOL if the following conditions hold:

(i) For every finite relational vocabulary τ the set of $\mathcal{L}(\tau)$ formulas contains all the atomic τ-formulas and is closed under boolean operations and renaming of relation and constant symbols.

(ii) $\mathcal{L}$ is equipped with a notion of *quantifier rank* and we denote by $\mathcal{L}_q(\tau)$ the set of formulas of quantifier rank at most q. The quantifier rank is sub-additive under substitution of sub-formulas,

(iii) The set of formulas of $\mathcal{L}_q(\tau)$ with a fixed set of free variables is, up to logical equivalence, finite.

(iv) Furthermore, if $\phi(x)$ is a formula of $\mathcal{L}_q(\tau)$ with x a free variable of $\mathcal{L}$, then there is a formula ψ logically equivalent to $\exists x \phi(x)$ in $\mathcal{L}_{q'}(\tau)$ with $q' \geq q+1$.

Typical fragments are FOL, MSOL, CMSOL, the fixed point logics IFPL and FPL and their corresponding finite variable subsets.

For two τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ we define the *equivalence relation of* $\mathcal{L}_q(\tau)$-*non-distinguishability*, and we write $\mathfrak{A} \equiv_q^{\mathcal{L}} \mathfrak{B}$, if they satisfy the same sentences from $\mathcal{L}_q(\tau)$.

Definition A.1.1. *Let $s : \mathbb{N} \to \mathbb{N}$ be a function. A binary operation $\odot$ between τ-structures is called* $(s,\mathcal{L})$-smooth*, if for all $q \in \mathbb{N}$ whenever $\mathfrak{A}_1 \equiv_{q+s(q)}^{\mathcal{L}} \mathfrak{B}_1$ and $\mathfrak{A}_2 \equiv_{q+s(q)}^{\mathcal{L}} \mathfrak{B}_2$ then*

$$\mathfrak{A}_1 \odot \mathfrak{A}_2 \equiv_q^{\mathcal{L}} \mathfrak{B}_1 \odot \mathfrak{B}_2.$$

If $s(q)$ is identically 0 *we omit it.*

An $\mathcal{L}$-*transduction* of τ-structures into σ-structures is given by defining a σ-structure inside a given τ-structure. The universe of the new structure may be a definable subset of an m-fold Cartesian product of the old structure. If $m = 1$ we speak of *scalar* and otherwise of *vectorized* transductions. For every k-ary relation symbol $\mathbf{R} \in \sigma$ we need a τ-formula in $k \cdot m$ free individual variables to define it. We denote by Φ a sequence of τ-formulas which defines a transduction. We denote by $\Phi^{\star}$ the map sending τ-structures into

σ-structures induced by Φ. We denote by $\Phi^\sharp$ the map sending σ-formulas into τ-formulas induced by Φ. For a σ-formula $\Phi^\sharp(\theta)$ is the *backward translation* of θ into a τ-formula. Φ is *quantifier-free* if all its formulas are from $\mathrm{FOL}_0(\tau)$. We skip the details, and refer the reader to [111, 115].

A fragment $\mathcal{L}$ is *closed under scalar transductions*, if for Φ such that all the formulas of Φ are in $\mathcal{L}(\tau)$, Φ scalar, and $\theta \in \mathcal{L}(\sigma)$, the backward substitution $\Phi^\sharp(\theta)$ is also in $\mathcal{L}(\tau)$. A fragment of SOL is called *tame* if it is closed under scalar transductions. FOL, MSOL and CMSOL are all tame fragments. So are their finite variable versions.

FOL and SOL are also closed under vectorized transductions, but the monadic fragments MSOL and CMSOL are not.

We shall frequently use the following:

Proposition A.1.1. *Let Φ define a $\mathcal{L}$-transduction from τ-structures to σ-structures where each formula is of quantifier rank at most q. Let θ be a $\mathcal{L}(\sigma)_r$-formula. Then*

$$\Phi^\star(\mathfrak{A}) \models \theta \textit{ iff } \mathfrak{A} \models \Phi^\sharp(\theta)$$

and $\Phi^\sharp(\theta)$ is in $\mathcal{L}(\tau)_{q+r}$.

For digraphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ we denote by $G \times H = (V, E)$ the *Cartesian product*, given by

$$\begin{aligned} V &= V_G \times V_H \\ E &= \{((u_G, u_H), (v_G, v_H)) \in V \times V \mid (u_G, v_G) \in E_G, (u_H, v_H) \in E_H\} \end{aligned}$$

For ordered digraphs the Cartesian product is defined similarly and the resulting digraph has a lexicographic order on its vertices. Cartesian product extends naturally when allowing constants. This graph operation occurs in the literature of graph theory under other names including tensor product and direct product.

Proposition A.1.2 (Smooth operations).

(i) The rich disjoint union $\sqcup_{rich}$ of τ-structures and therefore also the disjoint union are FOL*-smooth,* MSOL_1*-smooth and* MSOL_2*-smooth. They are not* SOL*-smooth.*

(ii) The join $\bowtie$ on graphs is MSOL_1*-smooth but not* MSOL_2*-smooth.*

(iii) The Cartesian product $\times$ of graphs is FOL*-smooth, but not* MSOL_1*-smooth or* MSOL_2*-smooth.*

(iv) The concatenation of words is MSOL*-smooth.*

(v) Let Φ be a quantifier-free scalar transduction of τ-structures into τ-structures and let $\odot$ be an $\mathcal{L}$-smooth operation. Then the operation $\odot_\Phi(\mathfrak{A}, \mathfrak{B}) = \Phi^\star(\mathfrak{A} \odot \mathfrak{B})$ $\mathcal{L}$-smooth. If Φ has quantifier rank at most k, it is $(k, \mathcal{L})$-smooth.

Sketch of proof. (i) is shown for FOL and MSOL using the usual pebble games. For CMSOL one can use Courcelle's version of the Feferman-Vaught Theorem for CMSOL, cf. [43, 115]. (ii) is again shown using the pebble game for FOL. (iii) is shown using pebble games for MSOL. (iv) follows from Proposition A.1.1. □

Theorem A.1.3 (Feferman-Vaught Theorem for CFOL).

(i) The rich disjoint union $\sqcup_{rich}$ of τ-structures, and therefore the disjoint union, too, is CFOL*-smooth.*

(ii) The Cartesian product $\times$ of τ-structures is CFOL*-smooth.*

The proof does not use pebble games, but Feferman-Vaught-type reduction sequences. (i) can be proven using the same reduction sequences which are used in [43]. (ii) is proven using modifications of the reduction sequences as given in detail in [115, Theorem 1.6]. The exact modifications needed are given in Section A.3. To the best of our knowledge, (ii) of Theorem A.1.3 has not been stated in the literature before.

Remark A.1.4. We call this a Feferman-Vaught Theorem, because our proof actually computes the reduction sequences explicitly. However, this is not needed here, so we refer the reader to [115] for the definition of reduction sequences. One might also try to prove the theorem using the pebble games defined in [127], but at least for the case of the Cartesian product, the proof would be rather complicated and less transparent.

Theorem A.1.5 (Finite Rank Theorem for tame $\mathcal{L}$).
Let $\mathcal{L}$ be a tame fragment of SOL. *Let $\odot$ be a binary operation between τ-structures which is $\mathcal{L}$-smooth. Let $\mathcal{P}$ be a τ-property which is definable by a $\mathcal{L}$-formula ψ and $M(\odot, \psi)$ be the connection matrix defined by*

$$M(\odot, \mathcal{P})_{\mathfrak{A},\mathfrak{B}} = 1 \textit{ iff } \mathfrak{A} \odot \mathfrak{B} \models \psi \textit{ and } 0 \textit{ otherwise} .$$

Then

(i) There is a finite partition $\{U_1, \ldots, U_k\}$ of the (finite) τ-structures such that the sub-matrices obtained from restricting $M(\odot, \psi)$ to the rows corresponding to U_i and the columns corresponding to U_j, $M(\odot, \psi)^{[U_i,U_j]}$, have constant entries.

(ii) In particular, $M(\odot, \psi)$ has finite rank over any field $\mathbb{F}$.

Proof. (i) follows from the definition of smoothness and the fact that there are only finitely many formulas (up to logical equivalence) in $\mathcal{L}(\tau)_q$. (ii) follows from (i). □

A.2 Proving Non-definability of Properties

Non-definability on CFOL

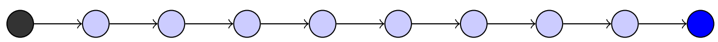
We will prove non-definability in CFOL using Theorem A.1.3 for Cartesian products combined with FOL transductions. It is useful to consider a slight generalization of the Cartesian product as follows. We add two constant symbols *start* and *end* to our graphs. In $G^1 \times G^2$ the symbol *start* is interpreted as the pair of vertices $(v^1_{start}, v^2_{start})$ from G^1 and G^2 respectively such that v^i_{start} is the interpretation of $start_i$ (i.e. *start* in G^i) for $i = 1, 2$.

The transduction $\Phi_{sym}(x, y) = E_D(x, y) \vee E_D(y, x)$ transforms a digraph $D = (V_D, E_D)$ into an undirected graph whose edge relation is the symmetric closure of the edge relation of the digraph.

The following transduction Φ_F transforms the Cartesian product of two directed graphs $G^i = (V_1, E_1, v^i_{start}, v^i_{end})$ with the two constants $start_i$ and end_i, $i = 1, 2$ into a certain digraph. It is convenient to describe Φ_F as a tranduction of the two input graphs G^1 and G^2:

$$\begin{aligned}\Phi_F\left(\left(v_1,v_2\right),\left(u_1,u_2\right)\right) &= \left(E_1(v_1,u_1)\wedge E_2(v_2,u_2)\right)\vee \\ &\quad \left(\left(v_1,v_2\right),\left(u_1,u_2\right)\right)=\left(\left(start_1,start_2\right),\left(end_1,end_2\right)\right)\end{aligned}$$

Consider the transduction obtained from Φ_F by applying Φ_{sym} when the input graphs are directed paths $P^i_{n_i}$ of length n_i. The input graphs look like this:



The result of the application of the transduction is given in Figure A.1. The result of the transduction has a cycle iff $n_1 = n_2$. The length of this

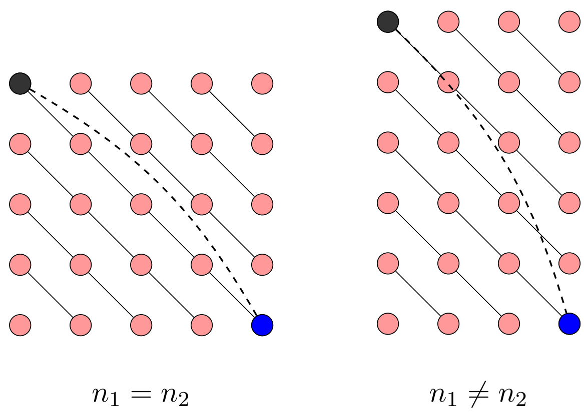


Figure A.1: The result of applying Φ_F and then Φ_{sym} on the two directed paths. There is a cycle iff the two directed paths are of the same length.

cycle is n_1. Hence, the connection sub-matrix with rows and columns labeled by directed paths of odd (even) length has ones on the main diagonal and zeros everywhere else, so it has infinite rank. Thus we have shown:

Theorem A.2.1. *The graphs without cycles of odd (even) length are not* CFOL-*definable even in the presence of a linear order.*

Corollary A.2.2. *Not definable in* CFOL *with order are:*

(i) Forests, bipartite graphs, chordal graphs, perfect graphs

(ii) *interval graphs (cycles are not interval graphs)*

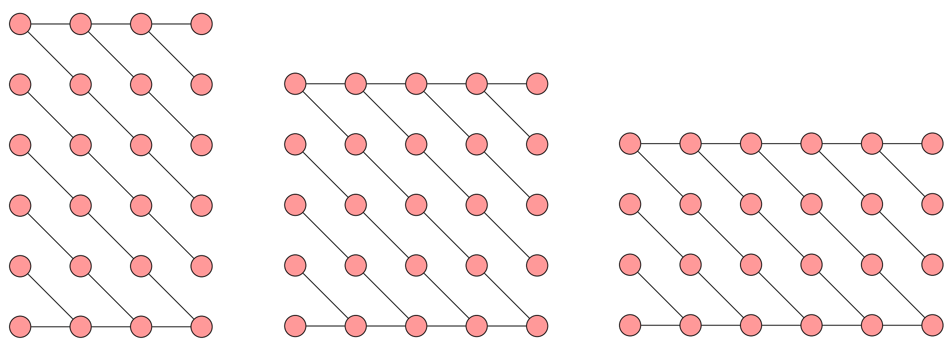
(iii) *Block graphs (every biconnected component is a clique)*

(iv) *Parity graphs (any two induced paths joining the same pair of vertices have the same parity)*

The transduction

$$\begin{aligned}\Phi_T\left((v_1, v_2), (u_1, u_2)\right) \quad &= \quad (E_1(v_1, u_1) \wedge E_2(v_2, u_2)) \vee \\ &\quad\quad (v_1 = u_1 = start_1 \wedge E(v_2, u_2)) \vee \\ &\quad\quad (v_1 = u_1 = end_1 \wedge E(v_2, u_2)),\end{aligned}$$

combined with Φ_{sym} transforms the two directed paths into the structures in Figure A.2.



$n_1 > n_2$ $\qquad$ $n_1 = n_2$ $\qquad$ $n_1 < n_2$

Figure A.2: The result of applying Φ_T and then Φ_{sym} on two directed paths. We get a tree iff the two directed paths are of equal length.

So, the result of the transduction is a tree iff $n_1 = n_2$. It is connected iff $n_1 \leq n_2$. Hence, both the connection matrices with directed paths as row and column labels of the property of being a tree and of connectivity have infinite rank.

Theorem A.2.3. *The properties of being a tree or a connected graph are not* CFOL*-definable even in the presence of linear order.*

For our next connection matrix we use the 2-sum of the following two 2-graphs:

(i) the 2-graph (G, a, b) obtained from K_5 by choosing two vertices a and b and removing the edge between them

(ii) the symmetric closure of the Cartesian product of the two digraphs $P^1_{n_1}$ and $P^2_{n_2}$:

We denote this transduction by Φ_P, see Figure A.3.

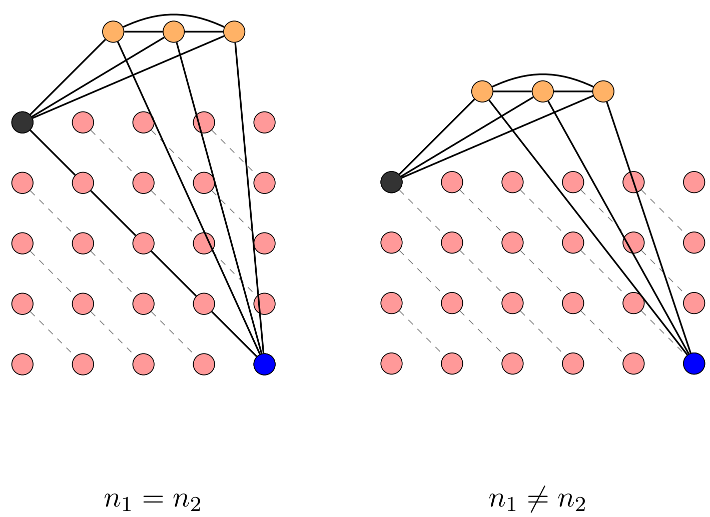


Figure A.3: The result of Φ_P on two directed paths. The graph obtained here is planar iff the two directed paths are of equal length.

So, the result of this construction has a clique of size 5 as a minor iff $n_1 = n_2$. It can never have a $K_{3,3}$ as a minor.

Theorem A.2.4. *The class of planar graphs is not* CFOL*-definable even on ordered graphs.*

If we modify the above construction by taking K_3 instead of K_5 and making $(start_1, start_2)$ and (end_1, end_2) adjacent, we get

Corollary A.2.5. *The following classes of graphs are not* CFOL*-definable even on ordered graphs.*

(i) Cactus graphs, i.e. graphs in which any two cycles have at most one vertex in common.

(ii) Pseudo-forests, i.e. graphs in which every connected component has at most one cycle.

The case of connected graphs was also shown non-definable in CFOL by. J. Nurmonen in [127] using his version of the pebble games for CFOL.

Non-definability in CMSOL

Considering the connection matrix where the rows and columns are labeled by the graphs on n vertices but without edges E_n, the graph $E_i \bowtie E_j = K_{i,j}$ is

(i) Hamiltonian iff $i = j$;

(ii) has a perfect matching iff $i = j$;

(iii) is a cage graph (a regular graph with as few vertices as possible for its girth) iff $i = j$;

(iv) is a well-covered graph (every minimal vertex cover has the same size as any other minimal vertex cover) iff $i = j$.

All of these connection matrices have infinite rank, so we get

Corollary A.2.6. *None of the properties above are* CMSOL*-definable as graphs even in the presence of an order.*

Using a modification $\widetilde{\bowtie}$ of the join operation used in [43, Remark 5.21] one can show the same for the class of graphs which have a spanning tree of degree at most 3. For any fixed natural number $d > 3$, by performing a transduction on $G\widetilde{\bowtie}H$ which attaches $d-3$ new vertices as pendants to each vertex of $G\widetilde{\bowtie}H$, one can extend the non-definability result to the class of graphs which have a spanning tree of degree at most d.

For the language of hypergraphs we cannot use the join operation, since it is not smooth. Note also that Hamiltonian and having a perfect matching are both definable in CMSOL in the language of hypergraphs. But using the connection sub-matrices of the disjoint union we still get:

(i) Regular: $K_i \sqcup K_j$ is regular iff $i = j$;

(ii) A generalization of regular graphs are *bi-degree* graphs, i.e., graphs where every vertex has one of two possible degrees. $K_i \sqcup (K_j \sqcup K_1)$ is a bi-degree graph iff $i = j$.

(iii) The average degree of $K_i \sqcup E_j$ is at most $\frac{|V|}{2}$ iff $i = j$;

(iv) A digraph is *aperiodic* if the common denominator of the lengths of all cycles in the graph is 1. We denote by C_i^d the directed cycle with i vertices. For prime numbers p, q the digraphs $C_p \sqcup C_q$ is aperiodic iff $p \neq q$.

(v) A graph is *asymmetric (or rigid)* if it has no non-trivial automorphisms. It was shown by P. Erdös and A. Rényi [54] that almost all finite graphs are asymmetric. So there is an infinite set $I \subseteq \mathbb{N}$ such that for $i \in I$ there is an asymmetric graph R_i of cardinality i. $R_i \sqcup R_j$ is asymmetric iff $i \neq j$.

Corollary A.2.7. *None of the properties above are* CMSOL-*definable as hypergraphs even in the presence of an order.*

Remark A.2.8. The case of asymmetric graphs illustrates that it is not always necessary to find explicit infinite families of graphs whose connection matrices are of infinite rank in order to show that such a family exists.

A.3 Proof of Theorem A.1.3

Recall that we denote by $\mathrm{D}_{m,i}$ the modular counting quantifiers $\mathrm{D}_{m,i}x\phi(x)$ which says that the number of elements satisfying ϕ equals i modulo m.

The proof of the Theorem follows exactly the proof of Theorem 1.6 in [115], in which an analogous statement was proven for the ordered sum of structures. We spell out the changes needed in the proof from [115] for the ordered product $\times$.

Proof of Theorem A.1.3. Given $G_1 = \langle V_1, E_2, <_1 \rangle$ and $G_2 = \langle V_2, E_2, <_2 \rangle$, their ordered product is given by $\langle V_1, E_2 \rangle \times \langle V_2, E_2 \rangle$ together with the lexicographic order $<$ on $V_1 \times V_2$ induced by $<_1$ and $<_2$.

The reduction sequences and Boolean functions of the atomic relations are given as follows:

For $E(u, v)$

Reduction sequence: $\langle E_1(u, v), E_2(u, v) \rangle$

Boolean function: $b_1^1 \wedge b_1^2$.

For $u \approx v$

Reduction sequence: $\langle u \approx_1 v, u \approx_2 v \rangle$

Boolean function: $b_1^1 \wedge b_1^2$.

For $u < v$

Reduction sequence: $\langle u <_1 v, u \approx_1 v, u <_2 v, u \approx_2 v \rangle$

Boolean function: $b_1^1 \vee \left(b_2^1 \wedge b_1^2 \right)$.

For $(\phi \wedge \psi)$**,** $\neg \phi$

We use the same reduction sequence and Boolean functions as in [115].

For $\exists x \phi$

Reduction sequence: $\left\langle \theta_1^A, \ldots, \theta_{m(J)}^A, \theta_1^B, \ldots, \theta_{m(J)}^B \right\rangle$

Boolean function: $B_\exists(\bar{c}) = \bigvee_{j \in J} (c_j^A \wedge c_j^B)$.

Now we turn to the counting quantifier $\mathrm{D}_{2,0}$.
Let $\Phi = \left\langle \phi_1^A, \ldots, \phi_m^A, \phi_1^B, \ldots, \phi_m^B \right\rangle$ be a reduction sequence for ϕ. We look at $B_\phi(\bar{b})$ in disjunctive normal form:

$$B_1 = \bigvee_{j \in J} C_j$$

with

$$C_j = C_j^A \wedge C_j^B$$

and

$$C_j^A = \left(\bigwedge_{i \in J(j,A,pos)} b_i^A \bigwedge_{i \in J(j,A,neg)} \neg b_i^A \right)$$

and

$$C_j^B = \left(\bigwedge_{i \in J(j,B,pos)} b_i^B \bigwedge_{i \in J(j,B,neg)} \neg b_i^B \right) .$$

B_1 has $2m$ Boolean variables, $b_1^A, \ldots, b_m^A, b_1^B, \ldots, b_m^B$. We assume without loss of generality that every C_j contains all of the variables. In other words, we have $\{1, \ldots, m\} \backslash J(j, A, pos) = J(j, A, neg)$ and $\{1, \ldots, m\} \backslash J(j, B, pos) = J(j, B, neg)$ for every $j \in J$.

Now let:

$$\alpha_j^A = \left(\bigwedge_{i \in J(j,A,pos)} \phi_i^A(x) \bigwedge_{i \in J(j,A,neg)} \neg \phi_i^A(x) \right)$$

and similarly

$$\alpha_j^B = \left(\bigwedge_{i \in J(j,B,pos)} \phi_i^B(x) \bigwedge_{i \in J(j,B,neg)} \neg \phi_i^B(x) \right)$$

Consider the formula $\mathrm{D}_{2,0}\, x\phi$. $\mathfrak{A} \times \mathfrak{B} \models \mathrm{D}_{2,0}\, x\phi$ iff the number of pairs $(a, b) \in A \times B$ such that $\langle \mathfrak{A} \times \mathfrak{B}, (a, b) \rangle \models \phi$ is even. $\langle \mathfrak{A} \times \mathfrak{B}, (a, b) \rangle \models \phi$ iff B_1 holds for $\langle \mathfrak{A}, a \rangle$ and $\langle \mathfrak{B}, b \rangle$. Note that $\langle \mathfrak{A}, a \rangle$ satisfies at most one of the α_j^A and similarly for $\langle \mathfrak{B}, b \rangle$.

The number of pairs (a, b) for which B_1 holds is even iff the number of C_j such that C_j holds for an odd number of pairs (a, b), is even. This holds iff the number of C_j such that C_j^A holds for an odd number of $a \in A$ and C_j^B holds for an odd number of $b \in B$, is even.

Let

$$\beta_j^A = \mathrm{D}_{2,0}\, x\alpha_j^A \quad \text{and} \quad \beta_j^B = \mathrm{D}_{2,0}\, x\alpha_j^B$$

and let

$$P = \{T \subseteq J \mid |T| \text{ is even } \} .$$

Finally we put

$$B_{\mathrm{D}_{2,0}}(\bar{c}) = \bigvee_{T \in P} \left(\bigwedge_{j \in T} (\neg c_j^A \wedge \neg c_j^B) \bigwedge_{j \notin T} (c_j^A \vee c_j^B) \right)$$

where $c_j^A = 1$ iff $\mathfrak{A} \models \beta_j^A$ and $c_j^B = 1$ iff $\mathfrak{B} \models \beta_j^B$.

So, we have:

$\mathrm{D}_{2,0}x\phi$

Reduction sequence: $\left\langle \beta_1^A, \ldots, \beta_{m(J)}^A, \beta_1^B, \ldots, \beta_{m(J)}^B \right\rangle$

Boolean function: $B_{\mathrm{D}_{2,0}}(\bar{c})$.

$\square$

A similar proof covers the quantifiers $\mathrm{D}_{3,0}, \mathrm{D}_{3,1}, \mathrm{D}_{3,2}$. Here we set, for each $i \in \{0,1,2\}$:

$$B_{\mathrm{D}_{3,i}}(\bar{c}) = \bigvee_{T_{1,1},T_{1,2},T_{2,1},T_{2,2}} \Bigg(\bigwedge_{j \in T_{1,1}} (c_{j,1}^A \wedge c_{j,1}^B) \wedge \bigwedge_{j \in T_{1,2}} (c_{j,1}^A \wedge c_{j,2}^B) \wedge \bigwedge_{j \in T_{2,1}} (c_{j,2}^A \wedge c_{j,1}^B) \wedge \bigwedge_{j \in T_{2,2}} (c_{j,2}^A \wedge c_{j,2}^B) \wedge \bigwedge_{j \in J \setminus (T_{1,1} \cup T_{1,2} \cup T_{2,1} \cup T_{2,2})} (c_{j,0}^A \vee c_{j,0}^B) \Bigg)$$

where the outer $\bigvee$ is over tuples $(T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2})$ of disjoint subsets of J, which additionally satisfy that

$$|T_{1,1}| + |T_{2,2}| + 2|T_{1,2}| + 2|T_{2,1}| \equiv i \mod 3 \,.$$

For every $i \in \{0,1,2\}$, $c_{j,i}^A = 1$ iff $\mathfrak{A} \models \mathrm{D}_{3,i}x\alpha_j^A$ and $c_{j,i}^B = 1$ iff $\mathfrak{B} \models \mathrm{D}_{3,i}x\alpha_j^A$. The reduction sequence of $\mathrm{D}_{3,i}x\phi$ is [2]

$$\Big\langle \mathrm{D}_{3,0}x\alpha_1^A, \mathrm{D}_{3,1}x\alpha_1^A, \mathrm{D}_{3,2}x\alpha_1^A, \ldots, \mathrm{D}_{3,0}x\alpha_{m(J)}^A, \mathrm{D}_{3,1}x\alpha_{m(J)}^A, \mathrm{D}_{3,2}x\alpha_{m(J)}^A, \mathrm{D}_{3,0}x\alpha_1^B, \mathrm{D}_{3,1}x\alpha_1^B, \mathrm{D}_{3,2}x\alpha_1^B, \ldots, \mathrm{D}_{3,0}x\alpha_{m(J)}^B, \mathrm{D}_{3,1}x\alpha_{m(J)}^B, \mathrm{D}_{3,2}x\alpha_{m(J)}^B \Big\rangle \,.$$

[2] If we were interested in making the reduction sequence shorter, we could have omitted $\mathrm{D}_{3,2}x\alpha_j^A$ and $\mathrm{D}_{3,2}x\alpha_j^B$ and express them using $\mathrm{D}_{3,0}x\alpha_j^A$, $\mathrm{D}_{3,1}x\alpha_j^A$, $\mathrm{D}_{3,0}x\alpha_j^B$ and $\mathrm{D}_{3,1}x\alpha_j^B$, in a similar way to our treatment of $\mathrm{D}_{2,0}$ and $\mathrm{D}_{2,1}$ in the proof above.

A.4 Merits and Limitations of Connection Matrices

Merits

The advantages of the Finite Rank Theorem for tame $\mathcal{L}$ in proving that a property is not definable in $\mathcal{L}$ are the following:

(i) It suffices to prove that certain binary operations on graphs (τ-structures) are $\mathcal{L}$-smooth operations.

(ii) Once the $\mathcal{L}$-smoothness of a binary operation has been established, proofs of non-definability become surprisingly simple and transparent. One of the most striking examples is the fact that asymmetric (rigid) graphs are not definable in CMSOL, cf. Corollary A.2.7.

(iii) Many properties can be proven to be non-definable using the same or similar sub-matrices, i.e., matrices with the same row and column indices. This is well illustrated in the examples of Section A.2.

Limitations

The classical method of proving non-definability in FOL using pebble games is complete in the sense that a property is $\mathrm{FOL}(\tau)_q$-definable iff the class of its models is closed under game equivalence of length q. Using pebble games one proves easily that the class of structures without any relations of even cardinality, EVEN, is not FOL-definable. This cannot be proven using connection matrices in the following sense:

Proposition A.4.1. *Let Φ a quantifier-free transduction between τ-structures and let $\odot_\Phi$ be the binary operation on τ-structures:*

$$\odot_\Phi(\mathfrak{A}, \mathfrak{B}) = \Phi^\star(\mathfrak{A} \sqcup_{rich} \mathfrak{B})$$

Then the connection matrix $M(\odot_\Phi, \mathrm{EVEN})$ satisfies the properties (i)-(iii) of Theorem A.1.5.

Appendix B

Ising polynomials and bounded clique-width

Although $Z_{Ising}(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is hard to compute in general, its computation on restricted classes of graphs can be tractable. Computing $Z_{Ising}(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is fixed parameter tractable with respect to tree-width using the general logical framework of [115]. This implies in particular that $Z_{Ising}(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is polynomial time computable on graphs of tree-width at most k, for any fixed k, which also follows from [125]. Likewise, the Tutte polynomial is known to be polynomial time computable on graphs of bounded tree-width, cf. [9, 124]. In contrast, for graphs of bounded clique-width, a width notion which generalizes tree-width, the best algorithm known for the Tutte polynomial is sub-exponential, cf. [74]. We show $Z_{Ising}(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is tractable on graphs of bounded clique-width:

Theorem B.0.2 (Tractability on graphs of bounded clique-width).

There exists a function $f(k)$ such that $Z(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is computable on graphs of clique-width at most k in running time $O\left({n_G}^{f(k)}\right)$.

$Z_{Ising}(G;\mathsf{x},\mathsf{y},\mathsf{z})$ is not fixed parameter tractable with respect to clique-width by [65].

Clique-width

We are interested in the clique-width of graphs. However, to discuss the clique-width of graphs we first need to discuss labeled graphs[1]. A *k-labeled graph* $(G, \bar{c})$ is a graph $G = (V, E)$ together with a tuple labels, $\bar{c} = (c_v : v \in V(G)) \in [k]^{n_G}$. The class $CW(k)$ of k-labeled graphs of clique-width at most k is defined inductively. Singletons (of any color) belong to $CW(k)$, and $CW(k)$ is closed under disjoint union $\sqcup$ and two other operations, $\rho_{i \to j}$ and $\mu_{i,j}$, to be defined next. For any $i, j \in [k]$, $\rho_{i \to j}(G, \bar{c})$ is obtained by relabeling any vertex with label i to label j. For any $i, j \in [k]$, $\mu_{i,j}(G, \bar{c})$ is obtained by adding all possible edges (u, v) such that $c_u = i$ and $c_v = j$.

The clique-width of a graph G is the minimal k such that there exists a labeling $\bar{c}$ for which $(G, \bar{c})$ belongs to $CW(k)$. We denote the clique-width of G by $cw(G)$.

A k-expression is a term t which consists of singletons, disjoint unions $\sqcup$, relabeling $\rho_{i \to j}$ and edge creations $\mu_{i,j}$, which witnesses that the graph $val(t)$ obtained by performing the operations on the singletons is of clique-width at most k.

Computation on graphs of bounded clique-width

In this section we prove Theorem B.0.2. Let G be a graph and let $cw(G)$ be its clique-width. While computing the clique-width of a graph is **NP**-hard, S. Oum and P. Seymour showed that given a graph of clique-width k, finding a $(2^{3k+2} - 1)$-expression is fixed parameter tractable with clique-width as parameter, cf. [128, 129]. Our algorithm uses this $(2^{3k+2} - 1)$-expression to compute $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ by *dynamic programming*. In fact, we need to compute a slight generalization $Z_{\text{labeled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$ of $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ described below. We compute for each graph obtained in the process of building G from its $(2^{3k+2}-1)$-expression the table of coefficients for Z_{labeled}, ending with the table for G, which gives us $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$.

Let $\bar{c} = (c_v : v \in V(G))$ be the labels from $[k]$ associated with the vertices of G by $t(G)$. We will show how to compute a multivariate polynomial

[1] Sometimes in the context of counting, the notion of "unlabeled" vs "labeled" graphs has a different meaning than here, referring to whether isomorphic graphs a counted up to isomorphism or not. This meaning was used in Section 8.1.9.

$Z_{\text{labeled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$ with indeterminate set

$$\left\{\mathsf{x}_{\{i,j\}}, \mathsf{y}_i, \mathsf{z}_{\{i,j\}} \mid i, j \in [k]\right\}$$

to be defined below. Note it is not the same multivariate polynomial as in Section 6.3. For simplicity of notation we write e.g. $\mathsf{x}_{i,j}$ or $\mathsf{x}_{j,i}$ for $\mathsf{x}_{\{i,j\}}$. The multivariate polynomial $Z_{\text{labeled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$ is defined as

$$Z_{\text{labeled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = \sum_{S \subseteq V(G)} \left(\prod_{v \in S} \mathsf{y}_{c_v}\right) \left(\prod_{(u,v) \in E_G(S)} \mathsf{x}_{c_u, c_v}\right) \left(\prod_{(u,v) \in E_G(\bar{S})} \mathsf{z}_{c_u, c_v}\right).$$

The left-most product in Equation (B.1) is over all vertices v in S. The two other products are over all edges in $E_G(S)$ and $E_G(\bar{S})$ respectively. It is not hard to see that $Z_{Ising}(G; \mathsf{x}, \mathsf{y}, \mathsf{z})$ is obtained from $Z_{\text{labeled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$ by substituting all the indeterminates $\mathsf{x}_{i,j}$, y_i and $\mathsf{z}_{i,j}$ by three indeterminate, x, y and z, respectively.

Given tuples of natural numbers $\bar{a} = (a_i : i \in [k])$, $\bar{b} = (b_{i,j} : i, j \in [k])$ and $\bar{c} = (c_{i,j} : i, j \in [k])$, we denote by $t_{\bar{a},\bar{b},\bar{c}}(G)$ the coefficient of the monomial

$$\prod_{i \in [k]} \mathsf{y}_i^{a_i} \prod_{i,j \in k} \mathsf{x}_{i,j}^{b_{i,j}} \mathsf{z}_{i,j}^{c_{i,j}}$$

in $Z_{\text{labeled}}(G; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$. We call a triple $(\bar{a}, \bar{b}, \bar{c})$ *valid* if $a_1 + \ldots + a_k \leq n_G$ and, for all $i, j \in [k]$, $b_{i,j}, c_{i,j} \leq m_G$. If $(\bar{a}, \bar{b}, \bar{c})$ is not valid, then $t_{\bar{a},\bar{b},\bar{c}}(G) = 0$. Therefore, to determine the polynomial $Z_{\text{labeled}}(G; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$ we need only to find $t_{\bar{a},\bar{b},\bar{c}}(G)$ for all valid triples $(\bar{a}, \bar{b}, \bar{c})$.

The $t_{\bar{a},\bar{b},\bar{c}}(G)$ form an $\left(k + 2k^2\right)$-dimensional array with no more than $(\max\{n_G, m_G\})^{k+2k^2}$ integer entries. Each entry in this table can be bounded from above by 2^{n_G} and thus can be written in polynomial space, so the size of the table is of the form ${n_G}^{p_1(cw(G))}$, where p_1 is a function of $cw(G)$ which does not depend on n_G.

We start by computing a $(2^{3cw(G)+2} - 1)$-expression for G, which determines a labeling $(G, \bar{c})$ of the vertices of G. We compute $Z_{\text{labeled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}})$ by dynamic programming on the structure of the k-expression of G.

Algorithm B.0.3.

(i) *If* (G, i) *is a singleton of any color* i, $Z_{\mathrm{labeled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = 1 + \mathsf{y}_i$.

(ii) *If* $(G, \bar{c})$ *is the disjoint union of* $(H, \bar{c}_{H_1})$ *and* $(H_2, \bar{c}_{H_2})$, *then*

$$Z_{\mathrm{labeled}}(G, \bar{c}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{\mathsf{z}}) = Z_{\mathrm{labeled}}(H_1, \bar{c}_{H_1}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{z}) \cdot Z_{\mathrm{labeled}}(H_2, \bar{c}_{H_2}; \bar{\mathsf{x}}, \bar{\mathsf{y}}, \bar{z}) \,.$$

(iii) *If* $(G, \bar{c}) = \mu_{p,r}(H, \bar{c}_H)$: *let* d_r *and* d_p *be the number of vertices of colors* r *and* p *in* H, *respectively.*

(iii.a) *For every valid* $(\bar{a}, \bar{b}, \bar{c})$, *if*

$$\begin{aligned} b_{p,r} &= \begin{cases} a_p \cdot a_r & p \neq r \\ \binom{a_p}{2} & p = r \end{cases} \\ c_{p,r} &= \begin{cases} (d_p - a_p) \cdot (d_r - a_r) & p \neq r \\ \binom{d_p - a_p}{2} & p = r \end{cases} \end{aligned} \tag{B.1}$$

set

$$t_{\bar{a},\bar{b},\bar{c}}(G) = \sum_{\bar{b}',\bar{c}'} t_{\bar{a},\bar{b}',\bar{c}'}(H)$$

where the summation is over all valid tuples $\bar{b}' = (b'_{i,j} \,:\, i,j \in [k])$ *and* $\bar{c}' = (c'_{i,j} \,:\, i,j \in [k])$ *such that* $b'_{i,j} = b_{i,j}$ *and* $c'_{i,j} = c_{i,j}$ *for all* $\{i,j\} \neq \{p,r\}$.

(iii.b) *For every valid* $(\bar{a}, \bar{b}, \bar{c})$, *if Equation (B.1) does not hold, set* $t_{\bar{a},\bar{b},\bar{c}}(G) = 0$.

(iv) *If* $(G, \bar{c}) = \rho_{p \to r}(H, \bar{c}_H)$:

(iv.a) *For every valid* $(\bar{a}, \bar{b}, \bar{c})$ *if* $a_p = 0$, *set*

$$t_{\bar{a},\bar{b},\bar{c}}(G) = \sum_{\bar{a}',\bar{b}',\bar{c}'} t_{\bar{a}',\bar{b}',\bar{c}'}(H)$$

where the summation is over all valid tuples $\bar{a}' = (a'_i \,:\, i \in [k])$, $\bar{b}' = (b'_{i,j} \,:\, i,j \in [k])$ *and* $\bar{c}' = (c'_{i,j} \,:\, i,j \in [k])$ *such that*

— $a_r = a'_p + a'_r$,

— $a_i = a'_i$ *for all* $i \notin \{p, r\}$,

— *for all* $j \in [k] \backslash \{p\}$,

$$b_{j,r} = \begin{cases} b'_{j,p} + b'_{j,r} & \text{if } j \neq r \\ b'_{r,r} + b'_{p,r} + b'_{p,p} & \text{if } j = r \end{cases}$$

and

$$c_{j,r} = \begin{cases} c'_{j,p} + c'_{j,r} & \text{if } j \neq r \\ c'_{r,r} + c'_{p,r} + c'_{p,p} & \text{if } j = r \end{cases}$$

and

— *for all* $i, j \in [k] \backslash \{p, r\}$, $b_{i,j} = b'_{i,j}$ *and* $c_{i,j} = c'_{i,j}$.

(iv.b) For every valid $(\bar{a}, \bar{b}, \bar{c})$ *if* $a_p \neq 0$, *set* $t_{\bar{a},\bar{b},\bar{c}}(G) = 0$.

Correctness

(i) By direct computation.

(ii) Proved in [8] for $Z_{Ising}(G; \mathsf{t}, \mathsf{y})$. The multivariate case is similar.

(iii) $G = \mu_{p,r}(H)$: Let S be a subset of vertices of $V(G) = V(H)$ with a_p and a_r vertices of colors p and r respectively. After adding all possible edges between vertices of color p and of color r in S, the number of edges between such vertices in $E_G(S)$ is $a_p \cdot a_r$ if $r \neq p$ and $\binom{a_p}{2}$ if $p = r$. Similarly, the number of edges between vertices colored p and r in $E_G(\bar{S})$ is $(d_p - a_p) \cdot (d_r - a_r)$ if $r \neq p$ and $\binom{d_p - a_p}{2}$ if $p = r$.

(iv) $G = \rho_{p \to r}(H)$: Let S be a subset of vertices of $V(G) = V(H)$. After re-coloring every vertex of color p in S to color r, we have $a_p = 0$. Every edge between a vertex colored p to any other vertex lies after the re-coloring between a vertex colored r and another vertex. There is one special case, which is the edges that lie between vertices colored r after the re-coloring. Before the re-coloring these edges were incident to vertices colored any combination of p and r.

Running Time

The size of the $(2^{3cw(G)+2}-1)$-expression is bounded by $n^c \cdot f_1(k)$ for some constant c, which does not depend on $cw(G)$, and for some function f_1 of $cw(G)$. Now we look at the possible operations performed by Algorithm B.0.3:

(i) The time does not depend on n since G is of size $O(1)$.

(ii) The time can be trivially bounded by the size of the table $t_{\bar{a},\bar{b},\bar{c}}$ to the power of 3, i.e. $n^{3p_1(cw(G))}$.

(iii) For $\mu_{p,r}$, the algorithm loops over all the values in the table $t_{\bar{a},\bar{b},\bar{c}}$, and for each entry (possibly) computes a sum over at most m_G elements. Then, the algorithm loops over all the values again and performs $O(1)$ operations.

(iv) For $\rho_{p\to r}$, the algorithm loops over all the values in the table $t_{\bar{a},\bar{b},\bar{c}}$, and for each entry possibly compute a sum over elements of the table $t_{\bar{a},\bar{b},\bar{c}}$. Then, the algorithm loops over all the values again and performs $O(1)$ operations.

Hence, Algorithm B.0.3 runs in time $O\left(n_G^{f(cw(G))}\right)$ for some f. [2]

[2]Running times of this kind are referred to as *Fixed parameter polynomial time* (**FPPT**) in [117], where the computation of various graph polynomials of graphs of bounded clique-width was treated.

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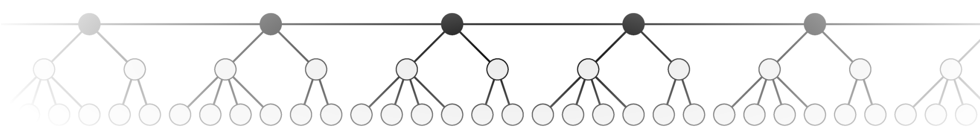
גדירות של פונקציות קומבינטוריות



תומר קוטק

גדירות של פונקציות קומבינטוריות

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר
דוקטור לפילוסופיה

תומר קוטק

הוגש לסנט הטכניון – מכון טכנולוגי לישראל
תמוז ה'תשע"ב חיפה יולי 2012

המחקר נעשה בהנחיית פרופ' יוהן מקובסקי בפקולטה למדעי המחשב.

בראש ובראשונה אני אסיר תודה למנחה הדוקטורט שלי, פרופ. ינוש מקובסקי, ממנו למדתי איך לעשות מחקר. אני מודה לו על שאיפשר לי את החופש לחקור כרצוני ובה בעת תמך בי בעת הצורך. היכולת היחודית שלו לראות את התמונה הגדולה ולהט המחקר שלו היוו השראה עבורי בעת לימודיי. אני מרגיש שאני בר מזל שהתברכתי בו כמנחה. כמו כן אני מודה לו על המימון הנדיב שהוא סיפק לי.

אני מודה לשותפיי לכתיבה, דר. איליה אברבוך, פרופ. אלדר פישר, בני גודלין, דר. אלנה רווה ופרופ. בוריס זילבר. אני אסיר תודה לבוחני הדוקטורט שלי, פרופ. אמיר שפילה, פרופ. רפאל יוסטר ופרופ. שמואל און על הערות שסייעו לשפר גרסה מוקדמת של החיבור הזה. אני מודה לפרופ. איירה גסל מאוניברסיטת ברנדייס שאירח אותי בקיץ של שנת 2010. כמו כן אני אסיר תודה לחבריי ללימודים, עמם התייעצתי לגבי מחקר וחלקתי רעיונות והפסקות קפה.

לבסוף, אני מודה למשפחתי על התמיכה הבלתי פוסקת ועל ייעוץ בעת הצורך. החיבור הזה מוקדש להם.

אני מודה לטכניון ולקרן פיין על התמיכה הכספית הנדיבה בהשתלמותי.

תקציר

הקשר בין אובייקטים מתמטיים לבין הדרך שבה הם מיוצגים, כלומר הגדירות שלהם, הוא נושא מחקר מרכזי בלוגיקה הקלאסית. הפונקציות הממשיות שניתנות לייצוג כפולינומים מהוות דוגמה מוקדמת לכך; כל פונקציה ממשית הניתנת ליצוג כפולינום מקיימת מגוון תכונות, לדוגמה פונקציה כזו היא תמיד רציפה. בתורת השפות הפורמליות, משפט בוכי אומר כי שפה היא רגולרית אם ורק אם היא גדירה בלוגיקה מונדית מסדר שני (Monadic Second Order Logic, MSOL). ב. קורסל בחן בצורה שיטתית את תכונות הגרפים הגדירות ב־ MSOL, כפי שמתואר בספרו [34]. משפט מפורסם של קורסל קובע כי ניתן להכריע בזמן לינארי האם גרפים בעלי רוחב עץ חסום מקיימים תכונות שגדירות ב־ MSOL. המושג 'גדירות' הורחב מתכונות של גרפים לפונקציות מגרפים לחוגי מספרים ופולינומים ונחקר בהקשר הזה ב־ [44], [611] ו־ [811].

נושא מרכזי בחיבור זה הוא גדירות של פונקציות קומבינטוריות משני סוגים: פולינומי־גרפים (פרקים 3, 4 ו־ 5) וסדרות שיש להן נוסחאות נסיגה לינאריות (פרקים 9 ו־ 10).

פולינומי־גרפים

פולינומי־גרפים הם פונקציות מאוסף הגרפים הסופיים לחוג פולינומים, למשל הפולינומים מעל המספרים השלמים, הרציונליים או הממשיים, שהן שמורות־גרפים, כלומר פונקציות המקבלות אותם ערכים לגרפים איזומורפיים.

פולינומי־גרפים מעוררים בשנים באחרונות עניין רב, אבל למחקר בנושא זה יש היסטוריה ארוכה. הפולינום הכרומטי נחקר כבר ב־ 1912 על ידי ג.ד. בירקהוף. הפולינום הכרומטי $\chi(G;k)$ הוא הפונקציה הסופרת את מספר הצביעות (proper colorings) של גרפים $G=(V,E)$, כלומר פונקציות $h:V\to[k]$ המקיימות כי צמתים שכנים נצבעים

בצבעים שונים, כאשר $[k] = \{1, \ldots, k\}$. בירקהוף הוכיח כי הפונקציה $\chi(G;k)$ היא פולינום ב־ k. ליתר דיוק, הוא הוכיח כי לכל גרף G קיים פולינום $\chi_G(\mathsf{k})$ השייך לחוג הפולינומים $\mathbb{Z}[\mathsf{k}]$ כך ש־ $\chi(G;k) = \chi_G(k)$ לכל $k \in \mathbb{N}$. אנו כותבים $\chi(G;\mathsf{k})$ כשאנחנו מתייחסים לפולינום הכרומטי בתור הפולינום הפורמלי במשתנה k, $\chi(G;\mathsf{k}) = \chi_G(\mathsf{k})$ (כלומר, $\chi(G;\mathsf{k})$ היא ההרחבה הטבעית של הפונקציה $\chi(G;k)$, $k \in \mathbb{N}$, המאפשרת קלטים שאינם בהכרח מספרים טבעיים).

ניתן למצוא בספרות גם צביעות מסוגים אחרים שפונקציות הספירה שלהן זכו לעניין, לעתים מבלי שהחוקרים ישימו לב שפונקציות הספירה הן פולינומים במספר הצבעים. לדוגמה, צביעות קמורות (convex colorings) הן שמורות־גרפים $h : V \to [k]$ כך שכל צבע $\ell \in [k]$ משרה תת גרף קשיר של G. דוגמה אחרת היא צביעות של קשתות (proper edge colorings).

צביעות של צמתים, צביעות קמורות וצביעות של קשתות הן שלוש דוגמאות לצביעות מוכללות. ניתן למצוא בספרות מגוון רחב של דוגמאות. חלקן מופיעות גם בחיבור זה. בדרך כלל התנאי (לדוגמה, אין צמתים שכנים בצבועים באותו צבע או כל צבע משרה תת גרף קשיר) גדיר בלוגיקה מסדר שני, (Second Order Logic, SOL).

ההוכחה הראשונה שהפולינום הכרומטי הוא פולינום השתמשה בהגדרה רקורסיבית שלו. אנחנו נותנים הוכחה כללית המשתמשת בשיטה אחרת. ההוכחה היא כללית במובן שהיא מתאימה לכל פונקצית ספירה של צביעה מוכללת, בתנאי שהצביעה המוכללת מקיימת תנאים פשוטים, שהחשוב ביניהם נקרא תנאי ההגדלה. זהו תנאי שנוגע לצבעים שאינם בשימוש על ידי הצביעה.

טענה א'. כל פונקצית ספירה של צביעה מוכללת המקיימת את תנאי ההגדלה היא פולינום־גרף.

למעשה אנחנו מוכיחים את הטענה הזו למבנים כלליים (למשל גם להיפר־גרפים) ולא רק לגרפים.

הפולינום הכרומטי ניתן לייצוג בדרכים אחרות, מפורשות יותר, למשל על ידי ביטוי מפורש של המקדמים. אם נתייחס לפולינום $\chi_G(\mathsf{x})$ כאשר הוא מיוצג מעל הבסיס $\mathsf{x}_{(m)} = \mathsf{x} \cdot (\mathsf{x}-1) \cdots (\mathsf{x}-(m-1))$, אז המקדם של $\mathsf{x}_{(m)}$ הוא b_m, מספר החלוקות של קבוצת הצמתים של הגרף ל־ m קבוצות בלתי תלויות. יצוג אחר הוא כהצבה בפולינום הדי־כרומטי הקשור הדוקות לפולינום של טאט (Tutte polynomial). בשני יצוגיים אלה אפשר לחשוב על הפולינום כסכום ממושקל של תתי קבוצות של קשתות, כאשר המשקלות הם פולינומים פשוטים כמו $\mathsf{x}_{(m)}$ או x^m והסכום הוא מעל קבוצה גדירה של תתי קבוצות

של קשתות. לדוגמה, הפולינום הדי־כרומטי מוגדר כך:

$$Z_{dichrom}(G;\mathsf{x},\mathsf{y}) = \sum_{S:S\subseteq E} \mathsf{x}^{k(S)} \cdot \mathsf{y}^{|S|}$$

כאשר $k(S)$ הוא מספר רכיבי הקשירות בגרף (V,S), ומתקיים

$$\chi(G;\mathsf{k}) = Z_{dichrom}(G;\mathsf{k},-1) \,.$$

ה־ $Z_{dichrom}$ הוא דוגמה לפולינום גדיר בלוגיקה מסדר שני.

אנו מראים כי אין זה מקרה שהפולינום הכרומטי ניתן לייצוג על ידי פולינום גדיר בלוגיקה מסדר שני.

משפט ב'. בהינתן פולינום גרפים p, (i) ו־ (ii) שקולים:

- (i) אפשר לקבל את p על ידי הצבת ערכים במשתנים של פולינום־גרפים שהוא פונקצית הספירה של צביעה מוכללת הגדירה בלוגיקה מסדר שני
- (ii) אפשר לקבל את p על ידי הצבת ערכים בפולינום גדיר מסדר שני.

ל־ MSOL יש תכונות מבניות מועילות והיא מסוגלת לבטא תכונות חשובות של גרפים כמו מישוריות, קשירות, המילטוניות ועוד. לכן, נעשה בלוגיקה הזו שימוש נרחב בתורת הגרפים. רוב פולינומי־הגרפים החשובים בספרות גדירים ב־ MSOL. עבור פולינומי־גרפים הגדירים בלוגיקה מונדית מסדר שני ידועים המטה־משפטים הבאים:

משפט (י.א. מקובסקי, ב. קורסל וא. רוטיץ'). כל פולינום־גרפים הגדיר בלוגיקה מונדית מסדר שני ניתן לחישוב בזמן פולינומי על גרפים בעלי רוחב עץ חסום.

משפט (י.א. מקובסקי וא. פישר). כל פולינום־גרפים שגדיר בלוגיקה מונדית מסדר שני מקיים נוסחאות נסיגה לינאריות פשוטות על סדרות רקורסיביות של גרפים כמו מעגלים, מסלולים וסולמות.

בחיבור הזה נראה מטה־משפט נוסף המשתמש במטריצות קשר (connection matrices) כדי להראות ששמורות גרפים רבות אינן מקודדות כהצבות או מקדמים של פולינומי־גרפים גדירים בלוגיקה מונדית מסדר שני. מטריצות קשר הוגדרו לראשונה על ידי פרידמן, לובש ושרייבר ב־ 2007 ב־ [76]. מדובר במטריצות אינסופיות שהשורות והעמודות שלהן מתאימות לגרפים (או מבנים דמויי גרפים). המטריצות מוגדרות ביחס לפעולה בינארית על גרפים $\odot$ ושמורת גרפים f. מטריצת הקשר $M(f,\odot)$

היא המטריצה שבתא המתאים לשורה המתויגת בגרף G ולעמודה המתויגת בגרף H נמצא הערך $f(G \odot H)$. הפעולה $\odot$ יכולה להיות למשל האיחוד הזר $\sqcup$ של G ו־H והשמורה היא בדרך כלל פונקציה עם טווח שהוא קבוצת מספרים או פולינומים.

אנחנו מראים את המשפט הבא:

משפט ג'. מטריצות הקשר של פולינומי־גרפים גדירים בלוגיקה מונדית מסדר שני הן בעלות דרגה סופית.

בחיבור מופיעות דוגמאות רבות לשמורות גרפים שדרגת מטריצת הקשר שלהן היא אינסופית, ולכן הן אינן יכולות להיות הצבות או מקדמים של אף פולינום־גרפים הגדיר בללוגיקה מונדית מסדר שני, לדוגמה מספר הצביעה, הדרגה הממוצעת ומספר הקליק של הגרף. כמו כן נראה לראשונה בעזרת עידון של השיטה הזו פולינומי־גרפים שאינם גדירים בלוגיקה מונדית מסדר שני.

מקובסקי העלה ב־ 2008 השערה כי יש מטה־משפט לגבי סיבוכיות החישוב של הצבות בפולינומי־גרפים הגדירים בלוגיקה מונדית מסדר שני. נאמר שלפולינום־גרפים $p(G;\bar{\mathrm{x}})$ יש את תכונת ההצבות הקשות לחישוב אם כמעט כל ההצבות $\bar{x}_0 \in \mathbb{Q}^\ell$ ב־p הן בעלות אותו קושי חישוב, כלומר ניתנות לרדוקציה פולינומית אחת לשניה. זאת, מלבד מספר קטן ומסודר של הצבות. השערת ההצבות הקשות לחישוב של מקובסקי המופיעה ב־ [611] גורסת כי לכל פולינום־גרפים הגדיר בלוגיקה מונדית מסדר שני יש את תכונת ההצבות הקשות לחישוב.

ניתן למצוא בספרות הוכחות שפולינומי־הגרפים החשובים ביותר אכן מקיימים את תכונת ההצבות הקשות לחישוב. בחיבור זה נתבונן בעוד פולינום־גרפים כמקרה מבחן להשערה. פולינום־הגרפים נקרא פולינוס אייזין (Ising polynomial) ומקורו במודל אייזין ממכניקה סטטיסטית. למעשה מדובר בשני פולינומי־גרפים שאחד הוא הרחבה של השני.

אנו מראים ששני הפולינומי אייזין הללו מקיימים את ההשערה בגרסה מורחבת:

משפט ד'. שני הפולינומי אייזין מקיימים את תכונת ההצבות הקשות לחישוב, אפילו כשמגבילים את הקלט לגרפים מישוריים דו־צדדים פשוטים.

ה. דל, ת. הוספלט ומ. וולן הציגו ב־ 2010 ([74]) הנחת סיבוכיות חדשה המסתמכת על הנחת הסיבוכיות של אימפגליאצו ופטורי מ־1999. ההנחה החדשה גורסת כי לוקח זמן אקספוננציאלי לחשב את מספר ההצבות המספקות בנוסחת 3CNF. ההנחה הזו למעשה חלשה יותר מזו של אימפגלאצו ופטורי. תחת ההנחה הזו, הנקראת הנחת הזמן האקספוננציאלי לבעיות ספירה (#Exponential Time Hypothesis, #**ETH**), חקרו

דל, הוספלט וולן את הסיבוכיות של הצבות של הפולינום הדי־כרומטי. אנחנו מראים משפט דיכוטומיה המקביל לתוצאות שלהם עבור הפולינום אייזין הפשוט:

משפט ה'. תחת הנחת הסיבוכיות האקספוננציאלית לבעיות ספירה, כל ההצבות של הפולינום אייזין הפשוט דורשות זמן ריצה אקספוננציאלי, מלבד קבוצה קטנה וסדורה היטב של הצבות הניתנות לחשוב בזמן פולינומי.

סדרות עם נוסחאות נסיגה

סדרות המקיימות נוסחאות נסיגה העסיקו מתמטיקאים כבר לפני מאות שנים. דוגמה מפורסמת לסדרה כזו היא סדרת מספרי פיבונאצ'י המקיימים את נוסחת הנסיגה:

$$F_n = F_{n-1} + F_{n-2}$$

כאשר $F_1 = F_2 = 1$. זו דוגמה לנוסחת נסיגה לינארית עם מקדמים שלמים. הגדירות של שפות שהסדרה של מספר המלים בכל אורך n שלהן מקיימת נוסחת נסיגה דומה נחקרה כבר ב־ 1963 על ידי נ. חומסקי ומ.פ. שוצנברז'ה. הם הוכיחו כי עבור שפות רגולריות סדרות אלה מקיימות נוסחאות נסיגה לינאריות עם מקדמים שלמים. ביחד עם משפט בוכי־אלגוט־טרכטנברוט, שאומר ששפה היא רגולרית אם ורק אם היא גדירה בלוגיקה מונדית בסדר שני, נקבל את המשפט הבא:

משפט. אם שפה היא גדירה בלוגיקה מונדית מסדר שני, אז הסדרה $d_L(n)$ של מספר המלים באורך n ב־ L מקיימת נוסחת נסיגה לינארית עם מקדמים שלמים.

למעשה ניתן להראות גם משפט הפוך, שכל סדרה המקיימת נוסחת נסיגה לינארית עם מקדמים שלמים היא הפרש $d_{L_1}(n) - d_{L_2}(n)$ עבור שפות רגולריות L_1 ו־ L_2.

סוג נוסף ומעניין של נוסחאות נסיגה נקרא נוסחאות נסיגה פולינומיות ($\mathrm{P-recurrence}$). מדובר בנוסחאות נסיגה לינארית שהמקדמים שלהן אינם קבועים, אלא הם פולינומים. לדוגמה העצרת מקיימת את נוסחת הנסיגה הבאה:

$$n! = n \cdot (n-1)!$$

כאשר $1! = 1$. זוהי נוסחת נסיגה פולינומית, ולעצרת אין נוסחת נסיגה עם מקדמים שלמים קבועים כי היא גדלה מהר מדי. מאותו טיעון נקבל שהעצרת לא סופרת מלים באף סוג של שפה. מצד שני, ידוע שהעצרת סופרת תמורות של קבוצות בגודל n. כמו כן, היא סופרת סוג מסויים של הילוכים בסריג. העצרת היא למעשה דוגמה לנוסחת

נסיגה פולינומית פשוטה כי המקדם של $n!$ בנוסחת הנסיגה הוא 1.

יש עוד דוגמאות של סדרות בעלות נוסחאות נסיגה פולינומיות הסופרות הילוכים בסריג, למשל סדרת מספרי קטלן, סדרת מספרי מוצקין, סדרת מספרי שרדר, הסדרה הסופרת הילוכי גסל והסדרה הסופרת הילוכי קרוורס. בחיבור זה נמצא משפט כללי המתאר את התופעה:

משפט ו'. סדרת מספרים שלמים מקיימת נוסחת נסיגה פולינומית אם ורק היא סופרת של סוג מסויים של הילוכים בסריג.

עד כה עסקנו בסדרות של מספרים המקיימות נוסחאות נסיגה, אך גם סדרות של פולינומים עשויות לקיים נוסחאות נסיגה. לדוגמה, פולינומי פיבונאצ'י הם סדרת הפולינומים המקיימים את הנוסחה הבאה:

$$F_n(\mathsf{x}) = \mathsf{x} \cdot F_{n-1}(\mathsf{x}) + F_{n-2}(\mathsf{x})$$

כאשר $F_1(\mathsf{x}) = 1$ ו־ $F_2(\mathsf{x}) = \mathsf{x}$. מתקיים כי $F_n(1) = F_n$. דוגמה נוספת היא הסדרה $\mathsf{x}_{(n)} = \mathsf{x} \cdot (\mathsf{x}-1) \cdots (\mathsf{x}-(n-1))$, שמקיימת את נוסחת הנסיגה

$$\mathsf{x}_{(n)} = (\mathsf{x}-(n-1)) \cdot \mathsf{x}_{(n-1)}$$

שהיא נוסחת נסיגה פולינומית פשוטה.

בחיבור זה נראה משפט ייצוג לסדרות של פולינומים המקיימות נוסחאות נסיגה עם מקדמים שלמים קבועים או נוסחאות נסיגה פולינומיות. יש דימיון בין הייצוגים האלה לבין פולינומי־הגרפים הגדירים בלוגיקה מונדית מסדר שני; אלא שבמקרה זה הסכום הוא מעל מלים בשפה הגדירה בלוגיקה מונדית מסדר שני והמשקלות הן תלויות־מקום, כלומר הן תלויות במקומות (באינדקסים) שבהם מופיעות אותיות מסויימות במלים עליהן סוכמים.

משפט ז'. סדרת פולינומים מקיימת נוסחת נסיגה לינארית עם מקדמים שלמים או נוסחת נסיגה פולינומית, בהתאמה, אם ורק אם הסדרה ניתנת לייצוג כסכום גדיר מתאים של משקולות תלויות־מקום.