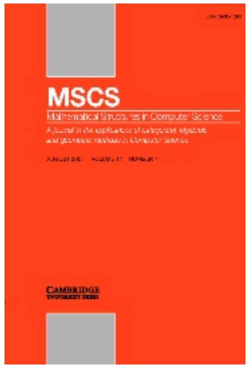

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Mathematical Structures in Computer Science / Volume 17 / Issue 04 / August 2007, pp 775 - 817
DOI: 10.1017/S0960129507006056, Published online: 06 July 2007

Math. Struct. in Comp. Science (2007), *vol.* 17, *pp.* 775–817.
doi:10.1017/S0960129507006056 Printed in the United Kingdom

Restriction categories III: colimits, partial limits and extensivity

ROBIN COCKETT[†] and STEPHEN LACK[‡]

[†]*Department of Computer Science, University of Calgary, Calgary, Alberta, T2N 1N4, Canada*
[‡]*School of Computing and Mathematics, University of Western Sydney, Sydney, Australia*

Received 7 April 2005; revised 29 September 2006

A restriction category is an abstract formulation for a category of partial maps, defined in terms of certain specified idempotents called the restriction idempotents. All categories of partial maps are restriction categories; conversely, a restriction category is a category of partial maps if and only if the restriction idempotents split. Restriction categories facilitate reasoning about partial maps as they have a purely algebraic formulation.
In this paper we consider colimits and limits in restriction categories. As the notion of restriction category is not self-dual, we should not expect colimits and limits in restriction categories to behave in the same manner. The notion of colimit in the restriction context is quite straightforward, but limits are more delicate. The suitable notion of limit turns out to be a kind of lax limit, satisfying certain extra properties.
Of particular interest is the behaviour of the coproduct, both by itself and with respect to partial products. We explore various conditions under which the coproducts are 'extensive' in the sense that the total category (of the related partial map category) becomes an extensive category. When partial limits are present, they become ordinary limits in the total category. Thus, when the coproducts are extensive we obtain as the total category a lextensive category. This provides, in particular, a description of the extensive completion of a distributive category.

1. Introduction

In a category $\mathscr{C}$ with a suitable class $\mathscr{M}$ of monomorphisms, one can define a category $\mathsf{Par}_{\mathscr{M}}(\mathscr{C})$ of partial maps in $\mathscr{C}$ whose domain of definition lies in $\mathscr{M}$. The resulting category has further structure, which determines, among other things, the extent of the partiality involved. This is necessary, as an abstract category can arise as a category of partial maps in more than one way. For example, any category $\mathbf{X}$ can be regarded as the category of partial maps in $\mathbf{X}$ where the class $\mathscr{M}$ consists only of the isomorphisms (the 'total subobjects'); thus, if $\mathbf{X} = \mathsf{Par}_{\mathscr{M}}(\mathscr{C})$, we have $\mathsf{Par}_{\mathscr{M}}(\mathscr{C}) = \mathsf{Par}_{\mathsf{Iso}}(\mathbf{X})$.

A variety of techniques have been employed to describe this extra structure. We recall below four possible approaches to capturing this further structure, indicating how the 'trivial case' where $\mathscr{M}$ is just the isomorphisms can be identified.

[†] Partially supported by NSERC, Canada.
[‡] Supported by the Australian Research Council.

A. Given partial maps $f, g : A \to B$, we define $f \leqslant g$ if g is defined whenever f is, and they then agree. This makes $\mathsf{Par}_{\mathscr{M}}(\mathscr{C})$ into a bicategory, and is the approach taken in Carboni (1987); it is also closely related to Freyd's notion of allegory (Freyd and Scedrov 1990). The trivial case is characterised by the fact that the partial order is discrete, in the sense that $f \leqslant g$ only if $f = g$.
B. If the category $\mathscr{C}$ of total maps has finite products, this induces a symmetric monoidal structure on $\mathsf{Par}_{\mathscr{M}}(\mathscr{C})$, which is given on objects by the product in $\mathscr{C}$. The trivial case can be characterised by the fact that this symmetric monoidal structure on the category of partial maps is in fact cartesian (that is, given by the categorical product). This approach was taken in Robinson and Rosolini (1988) and Curien and Obtulowicz (1989).
C. If $\mathscr{C}$ has a strict initial object, and the unique map out of the initial object is in $\mathscr{M}$, then $\mathsf{Par}_{\mathscr{M}}(\mathscr{C})$ has zero maps, given by the 'nowhere defined' partial maps. These were fundamental in the approach of Di Paola and Heller (1987). The presence of these zero maps means that only when the category itself is trivial can the partiality be trivial.
D. To every partial map $f : A \to B$ we can associate the partial map $\overline{f} : A \to A$ (which is defined whenever f is) in which it acts as the identity. This operation is taken as fundamental in the notion of restriction category studied in the earlier instalments (Cockett and Lack 2002; Cockett and Lack 2003) of this sequence of papers, and again here. The maps of the form $\overline{f}$ are always idempotents, and are called restriction idempotents. This time the trivial case is characterised by the fact that the restriction idempotents are just the identity maps.

The assignment of $\overline{f} : A \to A$ to $f : A \to B$ mentioned above satisfies four axioms:

[R.1] $f\overline{f} = f$ for all $f : A \to B$;
[R.2] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ for all $f : A \to B$ and $g : A \to C$;
[R.3] $\overline{g\overline{f}} = \overline{g}\,\overline{f}$ for all $f : A \to B$ and $g : A \to C$;
[R.4] $\overline{g}f = f\overline{gf}$ for all $f : A \to B$ and $g : B \to C$.

A key property of restriction categories, which is not shared by the axiomatics of Di Paola and Heller (1987), Robinson and Rosolini (1988), Curien and Obtulowicz (1989) and Carboni (1987), is that any full subcategory of a restriction category has an induced restriction structure; in fact, the restriction categories are precisely the full subcategories of categories of partial maps. Conversely, a restriction category is a category of partial maps if and only if the restriction idempotents split. Restriction categories facilitate reasoning about partial maps as they have a purely algebraic formulation, which does not involve having any structure on the types.

In this paper we consider the structure on a restriction category arising from limits and colimits on the category of total maps. As the notion of restriction category is not self-dual, we should not expect colimits and limits in restriction categories to behave in the same manner. The notion of coproduct in the restriction context is quite straightforward: a restriction category with *restriction coproducts* is just a cocartesian object in the 2-category rCat of restriction categories. This means that the diagonal $\mathbf{X} \to \mathbf{X} \times \mathbf{X}$ and the unique map $\mathbf{X} \to \mathbf{1}$ to the terminal restriction category both have left adjoints in the 2-category rCat. This is described in more concrete terms in Section 2; it means that the category **X** has coproducts that satisfy certain conditions involving the restriction structure.

On the other hand, a *cartesian* object in rCat necessarily has a trivial restriction structure. The suitable notion of a restriction category with restriction products turns out to be a cartesian object in a 2-category rCatl with the same objects and 1-cells as rCat, but with a certain type of 'lax natural transformation' as 2-cells. This time the underlying category of a restriction category with restriction products does not in general have products, although the category of total maps does; a concrete description is given in Section 4. The resulting structure turns out to be equivalent to the p-categories of Robinson and Rosolini (1988), and indeed to other different formulations by a variety of authors. A restriction category can also have products, which are entirely independent of the restriction structure. The presence of such products does have the slightly surprising effect of ensuring that the lattices of restriction idempotents have finite joins over which the meets distribute.

More generally, the suitable notion of limit turns out to be a certain type of lax limit, and we briefly explore these in Section 4.4. Once again, restriction limits in a restriction category become ordinary limits in the category of total maps.

Of particular interest is the behaviour of the coproduct, both by itself and with respect to partial products. We explore in Section 3 various conditions under which the coproducts are 'extensive' in the sense that the total category (of the related partial map category) becomes an extensive category. When partial limits are present, they become ordinary limits in the total category. Thus, when the coproducts are extensive we obtain as the total category a lextensive category. This provides, in particular, an alternative description of the extensive completion of a distributive category to that given in Cockett and Lack (2001). This is described in Section 5.4. But what is the importance of being extensive? Section 2 answers this question for partial map categories very concretely: extensivity means that there is a 'calculus of matrices'. This is critical to understanding and manipulating the maps in these settings.

Notation. The identity morphism on an object A is denoted by A or 1_A. We write $\langle f|g\rangle$: $A+B\to C$ for the morphism induced by $f : A\to C$ and $g : B\to C$, and $\langle f,g\rangle : A\to B\times C$ for the morphism induced by $f : A\to B$ and $g : A\to C$. We also write $\langle f_\lambda\rangle : \sum_{\lambda\in\Lambda} A_\lambda\to B$ for the morphism induced by a Λ-indexed family of morphisms $f_\lambda : A_\lambda\to B$. Our notation for coproduct injections is more flexible: sometimes we write i and j for the two injections of a binary coproduct, and sometimes we use i with a suitable subscript. We write $\tau : A+B\to B+A$ for the canonical isomorphism. The projections of a product are usually denoted by π with a suitable subscript.

2. Coproducts and matrices

It is well known that in the category of sets and binary relations the disjoint union (of a finite family of sets) serves both as coproduct and product, so there is a 'calculus of matrices': see Carboni (1991), for example. In this section we consider the extent to which this can be adapted to deal not with relations but with partial functions. We then consider when such a calculus is available in an abstract category of partial maps, or a restriction category.

Given finite families $(A_\lambda)_{\lambda\in\Lambda}$ and $(B_\kappa)_{\kappa\in K}$ of sets, and a partial function $f : \sum_\lambda A_\lambda \to \sum_\kappa B_\kappa$, we may define a partial function $f_{\lambda\kappa} : A_\lambda \to B_\kappa$ for each $\lambda \in \Lambda$ and $\kappa \in K$ by declaring $f_{\lambda\kappa}(x)$ to be defined if and only if $f(x)$ is defined and lies in B_κ, in which case $f_{\lambda\kappa}(x) = f(x)$. Conversely, a matrix $(f_{\lambda\kappa})_{\lambda\in\Lambda,\kappa\in K}$, with $f_{\lambda\kappa}$ a partial function from A_λ to B_κ for each λ and κ, determines a relation f from $\sum_\lambda A_\lambda$ to $\sum_\kappa B_\kappa$, where, if $x \in A_\lambda$ and $y \in B_\kappa$, we have $f(x) = y$ if and only if $f_{\lambda\kappa}(x) = y$. The relation f is in fact a partial function precisely when, for each $\lambda \in \Lambda$, if $f_{\lambda\kappa}(x)$ and $f_{\lambda\kappa'}(x)$ are both defined, then $\kappa = \kappa'$: in other words, if for each x and λ there is at most one κ for which $f_{\lambda\kappa}(x)$ is defined.

Not only can we represent partial functions by matrices, we can represent the composition of partial functions by matrix multiplication, in the following sense. If $f : \sum_\lambda A_\lambda \to \sum_\kappa B_\kappa$ and $g : \sum_\kappa B_\kappa \to \sum_k C_k$ are partial functions with matrices $(f_{\lambda\kappa})_{\lambda\in\Lambda,\kappa\in K}$ and $(g_{\kappa\mu})_{\kappa\in K,\mu\in M}$, then the matrix of gf is $(\vee_\kappa g_{\kappa\mu} f_{\lambda\kappa})_{\lambda\in\Lambda,\mu\in M}$, where $\vee_\kappa g_{\kappa\mu} f_{\lambda\kappa}$ is the partial function $h : A_\lambda \to C_k$ with $h(x) = g_{\kappa\mu} f_{\lambda\kappa}(x)$ if the right-hand side is defined for some (necessarily unique) κ, and undefined otherwise.

If f is defined by $t : E \to \sum_\kappa B_\kappa$ with domain $m : E \to \sum_\lambda A_\lambda$, then $f_{\lambda\kappa}$ can be computed as a pullback, as in

$$\begin{tikzcd}
 & & & E_{\lambda\kappa} \arrow[dl] \arrow[dr] & & & \\
 & & E_{\lambda-} \arrow[dl] \arrow[dr] & & E_{-\kappa} \arrow[dl] \arrow[dr] & & \\
 & A_\lambda \arrow[dl, "1"'] \arrow[dr, "i_\lambda"] & & E \arrow[dl, "m"'] \arrow[dr, "t"] & & B_\kappa \arrow[dl, "i_\kappa"'] \arrow[dr, "1"] & \\
A_\lambda & & \sum_\lambda A_\lambda & & \sum_\kappa B_\kappa & & B_\kappa .
\end{tikzcd}$$

In effect, we are composing f with the injection $i_\lambda : A_\lambda \to \sum_\lambda A_\lambda$, seen as a total partial map, and the partial map $i_\kappa^* : \sum_\kappa B_\kappa \to B_\kappa$, which is defined as the identity on B_κ and is undefined elsewhere. More abstractly, i_κ^* is the (unique) map satisfying $i_\kappa^* i_\kappa = 1$ and $i_\kappa i_\kappa^* = \overline{i_\kappa^*}$. (We shall say that i_κ^* is the restriction retraction of i_κ.)

We can recover f from the $f_{\lambda\kappa}$ as the composite

$$\sum_\lambda A_\lambda \xrightarrow{\sum_\lambda h_\lambda} \sum_{\lambda\kappa} A_\lambda \xrightarrow{\sum_{\lambda\kappa} f_{\lambda\kappa}} \sum_{\lambda\kappa} B_\kappa \xrightarrow{\sum_\kappa \nabla} \sum_\kappa B_\kappa$$

where $h_\lambda : A_\lambda \to \sum_\kappa A_\lambda$ is defined by $h_\lambda(x) = (x, \kappa)$ if $f_{\lambda\kappa}(x)$ is defined for some (necessarily unique) κ, and undefined otherwise. Once again, there is also a more abstract characterisation of h_λ: it is the unique map satisfying $h'_\lambda h_\lambda = \overline{h_\lambda}$ and $h_\lambda h'_\lambda = \overline{h'_\lambda}$, where $h'_\lambda : \sum_\kappa A_\lambda \to A_\lambda$ is $\langle \overline{f_{\lambda\kappa}} \rangle_{\kappa\in K}$. (We shall say that h_λ is the restriction inverse of h'_λ.)

What structure does a restriction category **X** need in order to support such a calculus of matrices? Obviously, **X** must have finite coproducts, and the coproduct injections must have restriction retractions. Also, given a morphism $f : A \to \sum_\kappa B_\kappa$, the map $\langle \overline{i_\kappa^* f} \rangle_{\kappa\in K} : \sum_\kappa A \to A$ must have a restriction inverse. This sets up a pair of functions between:

— the set of morphisms from $\sum_\lambda A_\lambda$ to $\sum_\kappa B_\kappa$; and
— the set of matrices $(f_{\lambda\kappa} : A_\lambda \to B_\kappa)_{\lambda\in\Lambda,\kappa\in K}$ with the property that for each λ, the map $\langle i_\kappa^* \overline{f i_\lambda} \rangle_{\kappa\in K} : \sum_\kappa A_\lambda \to A_\lambda$ has a restriction inverse.

Finally, we need these functions to be mutually inverse and to respect composition. We shall investigate when this occurs in the remainder of Section 2.

2.1. *Restriction coproducts*

In the previous section we saw that for a restriction category $\mathbf{X}$ with coproducts to admit a calculus of matrices, it is necessary that the coproduct injections be restriction monics, and so, in particular, be total. In this section we examine the situation in which the coproduct injections are total.

Lemma 2.1. Let $\mathbf{X}$ be a restriction category with coproducts, and suppose that the injections of every binary coproduct $A+B$ are total. Then:

(i) the unique arrow $z_A : 0 \to A$ is total for every object A;
(ii) the codiagonal $\nabla : A+A \to A$ is total for every object A;
(iii) $\overline{f+g} = \overline{f} + \overline{g}$ for all arrows f and g.

Proof. To prove (iii), let $f : A \to A'$ and $g : B \to B'$ and write $i : A \to A+B$, $j : B \to A+B$, $i' : A' + B'$, and $j' : B' \to A' + B'$ for the injections. Then $\overline{(f+g)i} = i\overline{(f+g)i} = i\overline{i'f} = i\overline{f}$ since i' is total, and, similarly, $\overline{(f+g)j} = j\overline{g}$; thus $\overline{f+g} = \overline{f} + \overline{g}$. The proof of (ii) is similar, and (i) follows immediately from the fact that z_A is an injection of the coproduct $A+0$. $\square$

We say that such a restriction category has *restriction coproducts*. A more abstract point of view is that such a restriction category is just a *cocartesian object* in the 2-category rCat. Recall that an ordinary category $\mathscr{C}$ has binary coproducts if and only if the diagonal functor $\mathscr{C} \to \mathscr{C} \times \mathscr{C}$ has a left adjoint. More generally, an object $\mathscr{C}$ of a 2-category with finite products is said to be cocartesian if the diagonal $\mathscr{C} \to \mathscr{C} \times \mathscr{C}$ has a left adjoint in the 2-category. Thus, a cocartesian restriction category is a restriction category $\mathbf{X}$ for which the diagonal restriction functor $\mathbf{X} \to \mathbf{X} \times \mathbf{X}$ and the unique restriction functor $\mathbf{X} \to \mathbf{1}$ to the terminal restriction category both have left adjoints in the 2-category rCat. Any 2-functor takes adjunctions to adjunctions, and a finite-product-preserving 2-functor takes cocartesian objects to cocartesian objects. For instance, there is a 2-functor $\mathsf{Total} : \mathsf{rCat} \to \mathsf{Cat}$ that sends a restriction category $\mathbf{X}$ to its category of total maps, and, clearly, Total preserves finite products. Then again, there is a 2-functor $K_r : \mathsf{rCat} \to \mathsf{rCat}$ that sends a restriction category $\mathbf{X}$ to the restriction category $K_r(\mathbf{X})$ obtained by splitting the restriction idempotents of $\mathbf{X}$.

Proposition 2.2. If $\mathbf{X}$ is a restriction category with restriction coproducts, $\mathsf{Total}(\mathbf{X})$ and $\mathsf{Total}(K_r(\mathbf{X}))$ have coproducts. If $F : \mathbf{X} \to \mathbf{Y}$ is a coproduct-preserving restriction functor between restriction categories with restriction coproducts, then $\mathsf{Total}(F) : \mathbf{X} \to \mathbf{Y}$ and $\mathsf{Total}(K_r(F)) : \mathsf{Total}(K_r(\mathbf{X})) \to \mathsf{Total}(K_r(\mathbf{Y}))$ preserve coproducts.

Proof. The 2-functors Total and $\mathsf{Total}(K_r)$ send cocartesian objects to cocartesian objects, and so send restriction categories with restriction coproducts to categories with coproducts.

Similarly, if F preserves coproducts, it commutes with the left adjoints $\mathbf{1} \to \mathbf{X}$ and $\mathbf{X} \times \mathbf{X} \to \mathbf{X}$, so $\mathsf{Total}(F)$ commutes with the induced left adjoints $1 \to \mathsf{Total}(\mathbf{X})$ and $\mathsf{Total}(\mathbf{X}) \times \mathsf{Total}(\mathbf{X}) \to \mathsf{Total}(\mathbf{X})$; that is, $\mathsf{Total}(F)$ preserves coproducts. The case of $\mathsf{Total}(K_r(F))$ is entirely analogous. □

This proposition has a converse when the restriction category is classified. Recall (Cockett and Lack 2002) that an arrow $r : A \to B$ in a restriction category is said to be a *restriction retraction* if there is an arrow $i : B \to A$ with $ri = 1$ and $\bar{r} = ir$; such an i is unique. Recall further (Cockett and Lack 2003) that a restriction category $\mathbf{X}$ is *classified* if the inclusion $\mathsf{Total}(\mathbf{X}) \to \mathbf{X}$ has a right adjoint R, and for each object A the counit $\epsilon_A : RA \to A$ is a restriction retraction. The promised converse is now given by the following proposition.

Proposition 2.3. If $\mathbf{X}$ is a classified restriction category and $\mathsf{Total}(\mathbf{X})$ has coproducts, then $\mathbf{X}$ has restriction coproducts. An arbitrary functor $F : \mathbf{X} \to \mathscr{C}$ preserves coproducts if and only if its restriction $\mathsf{Total}(\mathbf{X}) \to \mathscr{C}$ to the total maps preserves coproducts. In particular, for a restriction category $\mathbf{Y}$ with restriction coproducts, a restriction functor $F : \mathbf{X} \to \mathbf{Y}$ preserves coproducts if and only if $\mathsf{Total}(F) : \mathsf{Total}(\mathbf{X}) \to \mathsf{Total}(\mathbf{Y})$ does so also.

Proof. Since $\mathbf{X}$ is classified, the inclusion $\mathsf{Total}(\mathbf{X}) \to \mathbf{X}$ is a left adjoint, and so preserves all existing colimits. Since it is also bijective on objects, $\mathbf{X}$ has coproducts if $\mathsf{Total}(\mathbf{X})$ has coproducts; and the injections are clearly total.

Since the inclusion $I : \mathsf{Total}(\mathbf{X}) \to \mathbf{X}$ is bijective on objects, a functor $G : \mathsf{Total}(\mathbf{X}) \to \mathscr{C}$ preserves coproducts if and only if GI does so also. Since FI is just the composite of $\mathsf{Total}(F)$ and the inclusion $\mathsf{Total}(\mathbf{Y}) \to \mathbf{Y}$, it follows that F preserves coproducts. □

We also have the following proposition.

Proposition 2.4. If $\mathbf{X}$ is a restriction category with coproducts and a zero object, then $\mathbf{X}$ has restriction coproducts.

Proof. If $\mathbf{X}$ has a zero object, the injection $i : A \to A + B$ has a retraction $\langle 1|0 \rangle : A + B \to A$, and hence is monic; but monomorphisms are always total. □

Example 2.5. If $\mathscr{D}$ is a distributive category, then the endofunctor $+1$ of $\mathscr{D}$ has a well-known monad structure, and the Kleisli category $\mathscr{D}_{+1}$ of this monad has a restriction structure, which is described in Example 7 of Cockett and Lack (2002, Section 2.1.3). Since $\mathscr{D}$ has coproducts and the left adjoint $I : \mathscr{D} \to \mathscr{D}_{+1}$ is bijective on objects, $\mathscr{D}_{+1}$ has coproducts; the injections are in the image of I and so are total. Thus $\mathscr{D}_{+1}$ has restriction coproducts.

It is well known (see Carboni *et al.* (1993) and Cockett (1993), for example) that the free completion under (finite) coproducts of a category $\mathscr{C}$ can be formed as the category $\mathsf{Fam}(\mathscr{C})$ of finite families of objects of $\mathscr{C}$. Explicitly, an object of $\mathsf{Fam}(\mathscr{C})$ is a finite family $(A_\lambda)_{\lambda \in \Lambda}$ of objects of $\mathscr{C}$, and a morphism from $(A_\lambda)_{\lambda \in \Lambda}$ to $(B_\kappa)_{\kappa \in K}$ consists of a function $\varphi : \Lambda \to K$ and a family $(f_\lambda : A_\lambda \to B_{\varphi\lambda})_{\lambda \in I}$ of morphisms in $\mathscr{C}$. The universal property of $\mathsf{Fam}(\mathscr{C})$ is expressed in terms of the fully faithful functor $J : \mathscr{C} \to \mathsf{Fam}(\mathscr{C})$ sending an object of $\mathscr{C}$ to the corresponding singleton family.

Remark 2.6. If $\mathbf{X}$ is a restriction category, $\mathsf{Fam}(\mathbf{X})$ has a canonical restriction structure, with $\overline{(\varphi, f)} = \big(1_\Lambda, (\overline{f_\lambda})_{\lambda\in\Lambda}\big)$. Then $J : \mathbf{X} \to \mathsf{Fam}(\mathbf{X})$ is clearly a restriction functor. Furthermore, $\mathbf{X}$ has restriction coproducts if and only if $J : \mathbf{X} \to \mathsf{Fam}(\mathbf{X})$ has a left adjoint in rCat. A purely formal consequence is that $\mathsf{Fam}(\mathbf{X})$ is the free restriction category with restriction coproducts on $\mathbf{X}$; we leave the precise formulation of the universal property to the reader. Another straightforward observation is that the restriction category $\mathsf{Fam}(\mathbf{X})$ is classified whenever $\mathbf{X}$ is classified.

2.2. *Restriction zero objects*

To begin with, we allow $\mathbf{X}$ to be an arbitrary restriction category. Given arrows $f : A \to B$ and $g : B \to A$ in $\mathbf{X}$, recall (Cockett and Lack 2002) that g is *restriction inverse* to f (and f to g) if $gf = \overline{f}$ and $fg = \overline{g}$. A restriction inverse is unique if it exists. In the special case where f is total, we have $gf = \overline{f} = 1$; then f is said to be a *restriction monic* and g its *restriction retraction*, and we often write f^* for f.

We say that a zero object 0 in a restriction category is a *restriction zero* if for every object A the zero map $0_{AA} : A \to A$ is a restriction idempotent; that is, $\overline{0_{AA}} = 0_{AA}$.

Lemma 2.7. For a restriction category $\mathbf{X}$, the following are equivalent:

(i) $\mathbf{X}$ has a restriction zero;
(ii) $\mathbf{X}$ has an initial object 0 and a terminal object 1, and each $z_A : 0 \to A$ is a restriction monic;
(iii) $\mathbf{X}$ has a terminal object 1 and each $t_A : A \to 1$ is a restriction retraction.

Proof.

(i) $\Rightarrow$ (ii). If 0 is a restriction zero, it is both initial and terminal, and for any object A there is a unique $z_A : 0 \to A$ and a unique $t_A : A \to 0$. Clearly, $t_A z_A = 1$, since 0 is initial, while $z_A t_A = 0_{AA} = \overline{0_{AA}} = \overline{z_A t_A}$.

(ii) $\Rightarrow$ (iii). Let $z_1^* : 1 \to 0$ be the restriction retraction of $z_1 : 0 \to 1$, and $t_A : A \to 1$ be the unique map. Then $t_A z_A z_1^* = 1$, since 1 is terminal; and we must show that $z_A z_1^* t_A$ is a restriction idempotent. Now $t_A = z_1 z_A^*$, since 1 is terminal, so $z_A z_1^* t_A = z_A z_1^* z_1 z_A^* = z_A z_A^*$, which is indeed a restriction idempotent.

(iii) $\Rightarrow$ (i). For each object A, choose $s_A : 1 \to A$ satisfying ($t_A s_A = 1$ and) $\overline{s_A t_A} = s_A t_A$. Then $s_0 : 1 \to 0$ is inverse to $z_1 : 0 \to 1$, so 0 is a zero object. Finally, $\overline{0_{AA}} = \overline{s_A t_A} = s_A t_A = 0_{AA}$. □

We now suppose once again that $\mathbf{X}$ has coproducts.

Proposition 2.8. Let $\mathbf{X}$ be a restriction category with coproducts in which the coproduct injections are restriction monics. Then the initial object 0 is a restriction zero if and only if the maps $i^* : A + B \to A$ are natural in B; they are always natural in A.

Proof. The i^* can be seen as $1_A + z_B^* : A + B \to A + 0$, which are clearly natural in A, and will be natural in B if and only if the $z_B^* : B \to 0$ are also. But this will be the case if and only if 0 is not just initial but also terminal, and the result then follows by Lemma 2.7. □

We now observe that in order to have a calculus of matrices, the category **X** must have a restriction zero object. We have already seen that the coproduct injections must be restriction monics, and thus, in particular, that $z_A : 0 \to A$ must be one. To deal with empty coproducts, every map $f : A \to 0$ should be representable as an 'empty matrix', which clearly means that there can be at most one such map. Thus, in this case 0 is not just an initial object but a zero object (that is, an initial and a terminal object). By Lemma 2.7, it follows that the initial object is a restriction zero.

Example 2.9. If $\mathscr{D}$ is a distributive category, the initial object of $\mathscr{D}$ is a restriction zero in $\mathscr{D}_{+1}$. To see this, observe that the left adjoint $I : \mathscr{D} \to \mathscr{D}_{+1}$ preserves colimits, so 0 is initial in $\mathscr{D}_{+1}$. For every object A, there is a unique arrow $A \to 0 + 1 = 1$ in $\mathscr{D}$, so 0 is also terminal in $\mathscr{D}_{+1}$. The zero map $0_{AA} : A \to A$ in $\mathscr{D}_{+1}$ is

$$A \xrightarrow{!} 1 \xrightarrow{i_2} A + 1\ ,$$

and its restriction is

$$A \xrightarrow{\langle 1, i_2 ! \rangle} A \times (A+1) \xrightarrow{\delta^{-1}} A \times A + A \xrightarrow{\pi_1 + !} A + 1.$$

The fact that these two maps agree is an easy exercise in distributive categories.

Lemma 2.10. If **X** is a restriction category with restriction coproducts and a restriction zero, then:

(i) each coproduct injection $i : A \to A+B$ is a restriction monic, with restriction retraction $i^* : A + B \to A$ equal to $\langle 1|0 \rangle : A + B \to A$, so the restriction idempotent ii^* is $1 + 0 : A + B \to A + B$;
(ii) if $f : C \to A + B$ is total, and the restriction idempotent $\overline{i^* f}$ splits, then the section $k : C_A \to C$ of the splitting is the pullback in $\mathsf{Total}(\mathbf{X})$ of the injection $i : A \to A + B$ along f;
(iii) the natural transformations in $\mathsf{Total}(\mathbf{X})$ whose components are the coproduct injections are cartesian.

Proof.

(i) We can regard i as $1_A + z_B$. Then $(1_A + z_B^*)(1_A + z_B) = 1$, while

$$(1_A + z_B)(1_A + z_B^*) = 1_A + 0_{BB} = \overline{1_A} + \overline{0_{BB}} = \overline{1_A + 0_{BB}}.$$

(ii) Suppose that $k : C_A \to C$ and $k^* : C \to C_A$ provide the splitting, so that $k^* k = 1$ and $kk^* = \overline{ii^* f}$. Then

$$ii^* f kk^* = (1+0) f \overline{(1+0)f} = (1+0)f = \overline{1+0} f = f \overline{(1+0)f} = fkk^*,$$

so $ii^* fk = fk$. We claim that the commutative square

$$\begin{tikzcd} C_A \arrow[r, "k"] \arrow[d, "i^* fk"'] & C \arrow[d, "f"] \\ A \arrow[r, "i"'] & A + B \end{tikzcd}$$

is in fact a pullback in $\mathsf{Total}(\mathbf{X})$. Since i and k are monic, it will suffice to show that a total map $u : D \to C$ factorises through k if fu factorises through i. But if fu factorises through i, we have $ii^*fu = fu$, and now $kk^*u = \overline{ii^*f}u = u\overline{ii^*fu} = u\overline{fu} = u$.

(iii) We need to show that the square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & A+B \arrow[d, "f+g"] \\ A' \arrow[r, "i'"'] & A'+B' \end{tikzcd}$$

is a pullback in $\mathsf{Total}(\mathbf{X})$. Since $\overline{i'^*(f+g)} = \overline{fi^*} = \overline{i^*} = ii^*$, the result follows by part (ii). $\square$

As we saw above, in the category of sets and relations a coproduct $A + B$ is also a product, but in the case of sets and partial functions this is no longer the case. We now describe the trace that remains of this product structure. In a restriction category $\mathbf{X}$ with restriction coproducts and a restriction zero, we have a functor $+ : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$, and natural transformations $i^* : A + B \to A$ and $j^* : A + B \to B$. If there were a natural diagonal $\Delta : A \to A + A$ satisfying the triangle equations, this would exhibit $A + B$ as the product of A and B. Although there is no such Δ, we shall see that there are various maps that 'try' to be the diagonal; we shall call them decisions.

2.3. *The calculus of matrices*

In this section we consider a restriction category $\mathbf{X}$ with restriction coproducts and a restriction zero. The main aim of this section is to establish, under further conditions still to be determined, a bijection between arrows $f : \sum_\lambda A_\lambda \to \sum_\kappa B_\kappa$ and matrices $(f_{\lambda\kappa})$ with the property that for each λ the map $(\overline{f_{\lambda\kappa}})_\kappa : \sum_\kappa A_\lambda \to A_\lambda$ has a restriction inverse h_λ. This bijection should send f to $(i^*_\kappa f i_\lambda)_{\lambda,\kappa}$ and $(f_{\lambda\kappa})_{\lambda,\kappa}$ to the composite

$$\textstyle\sum_\lambda A_\lambda \xrightarrow{\sum_\lambda h_\lambda} \sum_{\lambda\kappa} A_\lambda \xrightarrow{\sum_{\lambda\kappa} f_{\lambda\kappa}} \sum_{\lambda\kappa} B_\kappa \xrightarrow{\nabla} \sum_\kappa B_\kappa.$$

The universal property of the coproduct $\sum_\lambda A_\lambda$ reduces this to the case where Λ is a singleton. Thus we are to establish a bijection between the set of morphisms $f : A \to \sum_\kappa B_\kappa$ and the set of those K-tuples $(f_\kappa : A \to B_\kappa)$ for which $(\overline{f_\kappa})_\kappa : \sum_\kappa A \to A$ has a restriction inverse h. For any $f : A \to \sum_\kappa B_\kappa$, the induced map $\langle \overline{i^*_\kappa f}\rangle_\kappa : \sum_\kappa A \to A$ will clearly need to have a restriction inverse h. Moreover, $\overline{h}$ will have to be $\overline{f}$. This is because if h is restriction inverse to $\langle \overline{i^*_\kappa}\rangle_\kappa$,

$$\overline{h} = \langle \overline{i^*_\kappa f}\rangle_\kappa h = \nabla \sum_\kappa \overline{i^*_\kappa f} h = \nabla \overline{\sum_\kappa i^*_\kappa f} h = \nabla h \overline{(\sum_\kappa i^*_\kappa f) h},$$

but for our bijection we need $(\sum_\kappa i^*_\kappa f)h = f$, so $\overline{h} = \nabla h \overline{f}$. But then

$$\overline{h} = \overline{\nabla h \overline{f}} = \overline{h\overline{f}} = \overline{h}\,\overline{f} = \overline{h}\,\overline{(\sum_\kappa i^*_\kappa f) h} = \overline{(\sum_\kappa i^*_\kappa f) h} = \overline{f},$$

as claimed.

If $\langle\overline{i_\kappa^* f}\rangle_\kappa$ does have a restriction inverse h and $\overline{h}$ is $\overline{f}$, we write $[f]$ for h, and call it a *decision* for f or *f-decision*, for reasons that will become clearer below.

Proposition 2.11. An arrow $h : A \to \sum_\kappa A$ is the decision of $f : A \to \sum_\kappa B_\kappa$ if and only if $\nabla h = \overline{f}$ and the square

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & \sum_\kappa A \arrow[d, "\sum_\kappa f"] \\
\sum_\kappa B_\kappa \arrow[r, "\sum_\kappa i_\kappa"'] & \sum_{\kappa,\kappa' \in K} B_\kappa
\end{tikzcd}$$

commutes.

We will leave the proof of the proposition to the next section. Observe, however, that it helps to explain the name 'decision'. Since $\overline{[f]} = \overline{\nabla[f]} = \overline{f}$, the decision $[f]$ is defined whenever h is defined, and the effect of $[f]$ is 'to send an element $a \in A$ to the element in the component of $\sum_\kappa A$ corresponding to the component of $f(a) \in \sum_\kappa B_\kappa$'.

Theorem 2.12. Let **X** be a restriction category with restriction coproducts and a restriction zero, and in which every map $f : A \to \sum_\kappa B_\kappa$ has a decision. Then there is a bijection between the set of all maps $f : \sum_\lambda A_\lambda \to \sum_\kappa B_\kappa$ and the set of those matrices $(f_{\lambda\kappa} : A_\lambda \to B_\kappa)_{\lambda,\kappa}$ for which $\langle\overline{f_{\lambda\kappa}}\rangle_\kappa : \sum_\kappa A_\lambda \to A_\lambda$ has a restriction inverse for every λ. The bijection sends f to the matrix $(i_\kappa^* f i_\lambda)_{\lambda,\kappa}$.

Proof. Write Φ for the function computing the matrix of a map $f : \sum_\lambda A_\lambda \to \sum_\kappa B_\kappa$, and Ψ for the purported inverse, which sends $(f_{\lambda\kappa})_{\lambda,\kappa}$ to the composite

$$\sum_\lambda A_\lambda \xrightarrow{\sum_\lambda h_\lambda} \sum_{\lambda\kappa} A_\lambda \xrightarrow{\sum_{\lambda\kappa} f_{\lambda\kappa}} \sum_{\lambda\kappa} B_\kappa \xrightarrow{\nabla} \sum_\kappa B_\kappa$$

where h_λ is restriction inverse to $\langle\overline{f_{\lambda\kappa}}\rangle_\kappa : \sum_\kappa A_\lambda \to A_\lambda$.

Starting with $f : \sum_\lambda A_\lambda \to \sum_\kappa B_\kappa$, we get the matrix $\Phi(f) = (i_\kappa^* f i_\lambda : A_\lambda \to B_\kappa)$; and then $\Psi(\Phi(f))$ is the composite

$$\sum_\lambda A_\lambda \xrightarrow{\sum_\lambda [f i_\lambda]} \sum_{\lambda\kappa} A_\lambda \xrightarrow{\sum_{\lambda\kappa} i_\kappa^* f i_\lambda} \sum_{\lambda\kappa} B_\kappa \xrightarrow{\nabla} \sum_\kappa B_\kappa.$$

To see that this is just f, observe that in the diagram

$$\begin{tikzcd}
& \sum_{\lambda\kappa} A_\lambda \arrow[r, "\sum_{\lambda\kappa} i_\lambda"] & \sum_{\lambda\kappa\lambda'} A_{\lambda'} \arrow[r, "\sum_{\lambda\kappa} f"] & \sum_{\lambda\kappa\kappa'} B_{\kappa'} \arrow[dr, "\sum_{\lambda\kappa} i_\kappa^*"] & \\
\sum_\lambda A_\lambda \arrow[ur, "\sum_\lambda [f i_\lambda]"] \arrow[r, "\sum_\lambda i_\lambda"] \arrow[dr, "1"'] & \sum_{\lambda\lambda'} A_\lambda \arrow[r, "\sum_\lambda f"] \arrow[d, "\nabla"] & \sum_{\lambda\kappa} B_\kappa \arrow[ur, "\sum_{\lambda\kappa} i_\kappa"] \arrow[rr, "1"'] & & \sum_{\lambda\kappa} B_\kappa \arrow[d, "\nabla"] \\
& \sum_\lambda A_\lambda \arrow[rrr, "f"'] & & & \sum_\kappa B_\kappa
\end{tikzcd}$$

the large upper parallelogram commutes by Proposition 2.11, the upper triangle commutes since $i_\kappa^* i_\kappa = 1$, the lower triangle commutes by one of the triangle equations, and the large lower rectangle commutes by naturality of ∇. Thus the entire diagram commutes and $\Psi(\Phi(f)) = f$.

Suppose on the other hand that we are given $f_{\lambda\kappa} : A_\lambda \to B_\kappa$ for each $\lambda \in \Lambda$ and $\kappa \in K$, and that $\langle \overline{f_{\lambda\kappa}} \rangle_\kappa : \sum_\kappa A_\lambda \to A_\lambda$ has a restriction inverse h_λ for each $\lambda \in \Lambda$. Then $\Phi\Psi$ sends the matrix $\langle f_{\lambda\kappa} \rangle_{\lambda,\kappa}$ to the composite

$$A_\lambda \xrightarrow{i_\lambda} \textstyle\sum_\lambda A_\lambda \xrightarrow{\sum_\lambda h_\lambda} \sum_{\lambda\kappa} A_\lambda \xrightarrow{\quad \sum_{\lambda\kappa} f_{\lambda\kappa} \quad} \sum_{\lambda\kappa} B_\kappa \xrightarrow{\nabla} \sum_\kappa B_\kappa \xrightarrow{i^*_\kappa} B_\kappa,$$

which, by the naturality of i^*_λ and the definition of $\nabla : \sum_\lambda \kappa B_\kappa \to B_\kappa$, is just

$$A_\lambda \xrightarrow{h_\lambda} \textstyle\sum_\kappa A_\lambda \xrightarrow{\sum_\kappa f_{\lambda\kappa}} \sum_\kappa B_\kappa \xrightarrow{i^*_\kappa} B_\kappa.$$

Naturality of i^*_κ gives $i^*_\kappa(\sum_\kappa f_{\lambda\kappa}) = f_{\lambda\kappa} i^*_\kappa$, thus we must show that $f_{\lambda\kappa} i^*_\kappa h_\lambda = f_{\lambda\kappa}$.

Now i^*_κ is restriction inverse to i_κ, and h_λ is restriction inverse to $\langle \overline{f_{\lambda\kappa}} \rangle_\kappa$, so $i^*_\kappa h_\lambda$ is restriction inverse to $\langle \overline{f_{\lambda\kappa}} \rangle_\kappa i_\kappa$, which is just $\overline{f_{\lambda\kappa}}$. But restriction idempotents are their own restriction inverses, so $i^*_\kappa h_\lambda = \overline{f_{\lambda\kappa}}$. Thus $f_{\lambda\kappa} i^*_\kappa h_\lambda = f_{\lambda\kappa}\overline{f_{\lambda\kappa}} = f_{\lambda\kappa}$, so $\Phi\Psi$ is indeed the identity, and the bijection is established. □

We end this section by showing how to 'multiply' matrices.

Proposition 2.13. Under the hypotheses of Theorem 2.12, if $f : \sum_\lambda A_\lambda \to \sum_\kappa B_\kappa$ has matrix $(f_{\lambda\kappa})_{\lambda,\kappa}$, and $g : \sum_\kappa B_\kappa \to \sum_\mu C_\mu$ has matrix $(g_{\kappa\mu})_{\kappa,\mu}$, then the composite gf has matrix $(\vee_\kappa g_{\kappa\mu} f_{\lambda\kappa})_{\lambda,\mu}$, where $\vee_\kappa g_{\kappa\mu} f_{\lambda\kappa} : A_\lambda \to C_\mu$ is given by

$$A_\lambda \xrightarrow{h_\lambda} \textstyle\sum_\kappa A_\lambda \xrightarrow{\quad \langle g_{\kappa\mu} f_{\lambda\kappa} \rangle_\kappa \quad} C_\mu$$

and h_λ is the restriction inverse of $\langle \overline{f_{\lambda\kappa}} \rangle_\kappa : \sum_\kappa A_\lambda \to A_\lambda$.

Proof. We must show that

$$i^*_\mu g f i_\lambda = \langle g_{\kappa\mu} f_{\lambda\kappa} \rangle_\kappa h_\lambda.$$

By the theorem, $f i_\lambda = \sum_\kappa (i^*_\kappa f i_\lambda) h_\lambda$, so

$$i^*_\mu g f i_\lambda = i^*_\mu g \sum_\kappa (i^*_\kappa f i_\lambda) h_\lambda = \langle i^*_\mu g i_\kappa i^*_\kappa f i_\lambda \rangle_\kappa h_\lambda = \langle g_{\kappa\mu} f_{\lambda\kappa} \rangle_\kappa h_\lambda,$$

as required. □

2.4. *Decisions*

In this section we further explore decisions in a restriction category $\mathbf{X}$ with restriction coproducts and restriction zero; the main goal is to prove Proposition 2.11. Recall that $h : A \to \sum_\kappa A$ is the decision of $f : A \to \sum_\kappa B_\kappa$ if it is restriction inverse to $\langle \overline{i^*_\kappa f} \rangle_\kappa : \sum_\kappa A \to A$ and $\overline{h} = \overline{f}$. We say that $h : A \to \sum_\kappa A$ is a decision if it is the decision of some map $f : A \to \sum_\kappa B_\kappa$.

Example 2.14.

(i) If K is a singleton, so that we have a single map $f : A \to B$, a decision for f is a map $h : A \to A$ that is restriction inverse to $\overline{f}$: this is just $\overline{f}$ itself.

(ii) If K is empty, so that f is the unique map $A \to 0$, a decision for f is a map $h : A \to 0$ that is restriction inverse to the unique map $z_A : 0 \to A$: then $f = h = z_A^*$.
(iii) Let f be a coproduct injection $i_\lambda : A_\lambda \to \sum_\lambda A_\lambda$. Then $\langle \overline{i_\kappa^* i_\lambda} \rangle_\kappa : \sum_\kappa A_\lambda \to A_\lambda$ is i_λ^*, which has restriction inverse i_λ. Thus i_λ is its own decision.

Proposition 2.15. For a map $h : A \to \sum_{\kappa \in K} A$, the following are equivalent:

(i) h is its own decision;
(ii) h is a decision;
(iii) h has a restriction inverse $g : \sum_\kappa A \to A$ and $gi_\kappa : A \to A$ is a restriction idempotent for each κ.

Proof. The downward implications are trivial; we must show that given restriction idempotents $e_\kappa : A \to A$ for each $\kappa \in K$, if $\langle e_\kappa \rangle_\kappa : \sum_\kappa A \to A$ has a restriction inverse h, then h is its own decision.

Since h is restriction inverse to $\langle e_\kappa \rangle_\kappa$ and i_κ^* is restriction inverse to i_κ, we see that $i_\kappa^* h$ is restriction inverse to $\langle e_\kappa \rangle_\kappa i_\kappa$; but the latter is just e_κ which is its own restriction inverse. So $i_\kappa^* h = e_\kappa$, and hence $\overline{i_\kappa^* h} = e_\kappa$. But then h is restriction inverse to $\langle \overline{i_\kappa^* h} \rangle_\kappa$, which is just to say that h is its own decision. $\square$

The next result says that we can 'conjugate' decisions by restriction inverses.

Corollary 2.16. If $h : A \to \sum_\kappa A$ is a decision and $f : A \to B$ a map with restriction inverse $g : B \to A$, then

$$B \xrightarrow{g} A \xrightarrow{h} \textstyle\sum_\kappa A \xrightarrow{\sum_\kappa f} \sum_\kappa B$$

is a decision and $\overline{(\sum_\kappa g)hf} = \overline{hf}$.

Proof. Since h is a decision, it is restriction inverse to $\langle \overline{i_\kappa^* h} \rangle_\kappa : \sum_\kappa A \to A$. Since g is restriction inverse to f and $\sum_\kappa f$ is restriction inverse to $\sum_\kappa g$, we also have $(\sum_\kappa f)hg$ is restriction inverse to $f \langle \overline{i_\kappa^* h} \rangle_\kappa (\sum_\kappa g)$. Now

$$f \langle \overline{i_\kappa^* h} \rangle_\kappa (\sum_\kappa g) i_\kappa = f \langle \overline{i_\kappa^* h} \rangle_\kappa i_\kappa g = f \overline{i_\kappa^* h} g = f g \overline{i_\kappa^* h g} = \overline{g i_\kappa^* h g},$$

which is a restriction idempotent, thus $(\sum_\kappa f)hg$ is a decision by the Proposition.

Finally,

$$\overline{(\sum_\kappa g)hf} = \overline{\nabla(\sum_\kappa g)hf} = \overline{g \nabla hf} = \overline{g\overline{hf}} = \overline{gf\overline{hf}} = \overline{\overline{f}\,\overline{hf}} = \overline{f}\,\overline{hf} = \overline{hf}.$$

$\square$

Corollary 2.17. If $h : \sum_\lambda A_\lambda \to \sum_\kappa \sum_\lambda A_\lambda$ is a decision, then so is $k_\lambda = (\sum_\kappa i_\lambda^*) h i_\lambda$ for each λ, and h is the composite

$$\textstyle\sum_\lambda A_\lambda \xrightarrow{\sum_\lambda k_\lambda} \sum_\lambda \sum_\kappa A_\lambda \xrightarrow{\sigma} \sum_\kappa \sum_\lambda A_\lambda$$

where σ is the canonical isomorphism.

Proof. The fact that k_λ is a decision is immediate from the previous corollary. On the other hand,

$$\begin{aligned}
\sigma(\sum_\lambda k_\lambda)i_\lambda &= \sigma i_\lambda k_\lambda \\
&= (\sum_\kappa i_\lambda)k_\lambda \\
&= (\sum_\kappa i_\lambda)(\sum_\kappa i^*_\lambda)hi_\lambda \\
&= (\sum_\kappa \overline{i^*_\kappa})hi_\lambda \\
&= hi_\lambda\overline{(\sum_\kappa i^*_\kappa)hi_\lambda} \\
&= hi_\lambda\overline{hi_\lambda} \\
&= hi_\lambda
\end{aligned}$$

for each λ, where the penultimate step uses the previous corollary. Thus $\sigma(\sum_\lambda k_\lambda) = h$, as claimed. □

We are now ready to prove Proposition 2.11. We shall make frequent use of the naturality of i^*_κ.

Proof of Proposition 2.11. First we simplify the condition for $h : A \to \sum_\kappa A$ to be the decision of $f : A \to \sum_\kappa B_\kappa$. This will be the case if $h\langle\overline{i^*_\kappa f}\rangle_\kappa = \overline{\langle\overline{i^*_\kappa f}\rangle_\kappa}$ and $\langle\overline{i^*_\kappa f}\rangle_\kappa h = \overline{h}$. Now $\langle\overline{i^*_\kappa f}\rangle_\kappa = \nabla(\sum_\kappa \overline{i^*_\kappa f}) = \nabla\overline{\sum_\kappa(i^*_\kappa f)}$, so the first condition becomes

$$h\overline{i^*_\kappa f} = i_\kappa\overline{i^*_\kappa f}.$$

Suppose that $(\sum_\kappa f)h = (\sum_\kappa i_\kappa)f$ and $\nabla h = \overline{f}$. Then $\overline{h} = \overline{\nabla h} = \overline{f}$. Now

$$(\sum_\kappa \overline{f})h = \overline{\sum_\kappa f h} = h\overline{(\sum_\kappa f)h} = h\overline{(\sum_\kappa i_\kappa)h} = h\overline{h} = h,$$

so

$$\overline{i^*_\kappa h} = \overline{i^*_\kappa(\sum_\kappa \overline{f})h} = \overline{\overline{f}i^*_\kappa h} = \overline{f i^*_\kappa h} = \overline{i^*_\kappa(\sum_\kappa f)h} = \overline{i^*_\kappa(\sum_\kappa i_\kappa)f} = \overline{i_\kappa i^*_\kappa f} = \overline{i^*_\kappa f},$$

but now

$$h\overline{i^*_\kappa f} = h\overline{i^*_\kappa h} = \overline{i^*_\kappa}h = i_\kappa i^*_\kappa h = i_\kappa\nabla i_\kappa i^*_\kappa h = i_\kappa\nabla\overline{i^*_\kappa}h = i_\kappa\nabla h\overline{i^*_\kappa h} = i_\kappa\overline{h}\,\overline{i^*_\kappa h} = i_\kappa\overline{i^*_\kappa h},$$

giving the first condition. As for the second,

$$\begin{aligned}
\langle\overline{i^*_\kappa f}\rangle_\kappa h &= \nabla(\sum_\kappa \overline{i^*_\kappa f})h \\
&= \nabla\overline{\sum_\kappa (i^*_\kappa f)h} \\
&= \nabla\overline{h(\sum_\kappa i^*_\kappa f)h} \\
&= \overline{h}\overline{(\sum_\kappa i^*_\kappa)(\sum_\kappa f)h} \\
&= \overline{h}\overline{(\sum_\kappa i^*_\kappa)(\sum_\kappa i_\kappa)f} \\
&= \overline{h}\,\overline{f} \\
&= \overline{h},
\end{aligned}$$

so h is the decision of f.

Suppose conversely that h is the decision of f. Then

$$\overline{\sum_\kappa i^*_\kappa f h} = h\overline{(\sum_\kappa i^*_\kappa f)h} = h\overline{\overline{\sum_\kappa i^*_\kappa f}h} = h\overline{\langle\overline{i^*_\kappa f}\rangle_\kappa h} = h\overline{h} = h,$$

so

$$\nabla h = \nabla\overline{\sum_\kappa i^*_\kappa f h} = \nabla(\sum_\kappa \overline{i^*_\kappa f})h = \langle\overline{i^*_\kappa f}\rangle_\kappa h = \overline{h} = \overline{f}.$$

On the other hand,

$$(\sum_\kappa f)h = (\sum_\kappa f)\overline{\sum_\kappa i^*_\kappa f h} = \sum_\kappa (f\overline{i^*_\kappa f})h = \sum_\kappa (\overline{i^*_\kappa} f)h = (\sum_\kappa \overline{i^*_\kappa})(\sum_\kappa f)h,$$

so

$$\begin{aligned}
(\sum_\kappa i_\kappa)f &= (\sum_\kappa i_\kappa)f\overline{h} \\
&= (\sum_\kappa i_\kappa)f\nabla\overline{\sum_\kappa i^*_\kappa f h} \\
&= (\sum_\kappa i_\kappa)\nabla(\sum_\kappa f)\overline{(\sum_\kappa i^*_\kappa)(\sum_\kappa f)h} \\
&= (\sum_\kappa i_\kappa)\nabla\overline{\sum_\kappa i^*_\kappa}(\sum_\kappa f)h \\
&= (\sum_\kappa i_\kappa)\nabla(\sum_\kappa i_\kappa i^*_\kappa f)h \\
&= (\sum_\kappa i_\kappa)(\sum_\kappa i^*_\kappa f)h \\
&= (\sum_\kappa \overline{i^*_\kappa})(\sum_\kappa f)h \\
&= (\sum_\kappa f)h.
\end{aligned}$$

$\square$

We saw in Example 2.14 that a decision for $f : A \to \sum_{\kappa \in K} B_\kappa$ always exists if K is empty or a singleton. We end this section by proving that all decisions exist provided that binary ones do.

Proposition 2.18. A restriction category $\mathbf{X}$ with restriction coproducts and a restriction zero has all decisions provided it has a decision for each $f : A \to B + C$.

Proof. Let $f : A \to \sum_{\kappa \in K} B_\kappa$ be given, where K is a finite set of cardinality greater than 2. Choose $\lambda \in K$, and regard $\sum_{\kappa \in K} B_\kappa$ as the coproduct of B_λ and $\sum_{\kappa \neq \lambda} B_\kappa$ with injections i and j. By assumption, $f : A \to B_\lambda + (\sum_{\kappa \neq \lambda} B_\kappa)$ has a decision $h_\lambda : A \to A + A$. Suppose as induction hypothesis that $j^* f : A \to \sum_{\kappa \neq \lambda} B_\kappa$ has a decision $h' : A \to \sum_{\kappa \neq \lambda} A$. We shall show that

$$A \xrightarrow{h_\lambda} A + A \xrightarrow{1+h'} A + \textstyle\sum_{\kappa \neq \lambda} A = \sum_{\kappa \in K} A$$

is a decision for $f : A \to \sum_{\kappa \in K} B_\kappa$.

Commutativity of

$$\begin{tikzcd}
A \arrow[r, "h_\lambda"] \arrow[d, "f"'] & A + A \arrow[r, "1+h'"] \arrow[d, "f+f"'] & A + \sum_{\kappa \neq \lambda} A \arrow[d, "f+\sum_{\kappa \neq \lambda} f"'] \arrow[dr, equal] & \\
B_\lambda + \sum_{\kappa \neq \lambda} B_\kappa \arrow[r, "i+j"'] \arrow[dr, equal] & \sum_{\kappa'} B_{\kappa'} + \sum_\kappa B_\kappa \arrow[r, "1+\sum_{\kappa \neq \lambda} i_\kappa"'] & \sum_{\kappa'} B_{\kappa'} + \sum_{\kappa \neq \lambda, \kappa'} B_{\kappa'} \arrow[dr, equal] & \sum_\kappa A \arrow[d, "\sum_\kappa f"] \\
& \sum_\kappa B_\kappa \arrow[rr, "\sum_\kappa i_\kappa"'] & & \sum_{\kappa, \kappa'} B_{\kappa'}
\end{tikzcd}$$

gives one of the conditions in Proposition 2.11; it remains to show that $\nabla(1 + h')h_\lambda = \overline{f}$.

Commutativity of

$$\begin{tikzcd}
A \arrow[r, "h_\lambda"] \arrow[dr, "\overline{(1+j^*f)h_\lambda}"'] & A + A \arrow[r, "1+h'"] \arrow[dr, "1+\overline{j^*f}"'] & A + \sum_{\kappa \neq \lambda} A \arrow[d, "1+\nabla"] \\
& A \arrow[r, "h_\lambda"'] \arrow[dr, "f"'] & A + A \arrow[d, "\nabla"] \\
& & A
\end{tikzcd}$$

reduces this to proving that $\overline{f(1 + j^* f)h_\lambda} = \overline{f}$.

To do so, first observe that

$$\begin{aligned}
\overline{(i^* f + 1)(1 + j^* f)h_\lambda} &= \overline{(i^* f + j^* f)h_\lambda} \\
&= \overline{\langle \overline{i^* f} | \overline{j^* f} \rangle h_\lambda} \\
&= \overline{h_\lambda} \\
&= \overline{f}
\end{aligned}$$

so

$$\begin{aligned} f\overline{(1+j^*f)h_\lambda} &= f\overline{f}\,\overline{(1+j^*f)h_\lambda} \\ &= f\overline{(i^*f+1)(1+j^*f)h_\lambda}\,\overline{(1+j^*f)h_\lambda} \\ &= f\overline{(i^*f+1)(1+j^*f)h_\lambda} \\ &= f\overline{(i^*f+j^*f)h_\lambda} \\ &= \overline{f}, \end{aligned}$$

as required. □

Finally, we record the following result, which will be needed later.

Proposition 2.19. If $f : A \to B + C$ and $f' : A' \to B' + C'$ have decisions h and h', then

$$(1+\tau+1)(f+f') : A + A' \to (B+B') + (C+C')$$

has decision $(1+\tau+1)(h+h')$.

Proof. Let $i : B \to B+C$, $j : C \to B+C$, $i' : B' \to B'+C'$ and $j' : C' \to B'+C'$ be the various injections. Then the injection $k : B+B' \to B+B'+C+C'$ is given by $(1+\tau+1)(i+i')$. Similarly, write l for the injection

$$(1+\tau+1)(j+j') : C+C' \to B+B'+C+C'.$$

First observe that

$$\overline{(1+\tau+1)(h+h')} = \overline{h+h'} = \overline{h}+\overline{h'} = \overline{f}+\overline{f'} = \overline{f+f'} = \overline{(1+\tau+1)(f+f')}.$$

Now h is restriction inverse to $\langle\overline{i^*f}|\overline{j^*f}\rangle$ and h' is restriction inverse to $\langle\overline{i'^*f'}|\overline{j'^*f'}\rangle$, thus $h+h'$ is restriction inverse to $\langle\overline{i^*f}|\overline{j^*f}\rangle + \langle\overline{i'^*f'}|\overline{j'^*f'}\rangle$, and $(1+\tau+1)(h+h')$ is restriction inverse to $\left(\langle\overline{i^*f}|\overline{j^*f}\rangle + \langle\overline{i'^*f'}|\overline{j'^*f'}\rangle\right)(1+\tau+1)$. But

$$\begin{aligned} \left(\langle\overline{i^*f}|\overline{j^*f}\rangle + \langle\overline{i'^*f'}|\overline{j'^*f'}\rangle\right)(1+\tau+1) &= (\nabla+\nabla)(\overline{i^*f}+\overline{j^*f}+\overline{i'^*f'}+\overline{j'^*f'})(1+\tau+1) \\ &= (\nabla+\nabla)(1+\tau+1)(\overline{i^*f}+\overline{i'^*f'}+\overline{j^*f}+\overline{j'^*f'}) \\ &= \nabla(\overline{i^*f+i'^*f'}+\overline{j^*f+j'^*f'}) \\ &= \nabla(\overline{(i+i')^*(f+f')}+\overline{(j+j')^*(f+f')}) \\ &= \langle\overline{(i+i')^*(f+f')}|\overline{(j+j')^*(f+f')}\rangle, \end{aligned}$$

so $(1+\tau+1)(h+h')$ is the decision of $(1+\tau+1)(f+f')$, as claimed. □

3. Extensive restriction categories

3.1. *Extensivity*

In the previous section we saw that a restriction category $\mathbf{X}$ admits a calculus of matrices if it has restriction coproducts, a restriction zero and decisions. In this section we relate this structure to the question of when $\mathsf{Total}(\mathbf{X})$ and $\mathsf{Total}(K_r(\mathbf{X}))$ are extensive.

Proposition 3.1. If $\mathbf{X}$ is a restriction category with restriction coproducts and a restriction zero, then $\mathsf{Total}(\mathbf{X})$ is extensive if and only if for every total arrow $f : C \to A + B$ the restriction idempotent $\overline{(1+0)f}$ splits and an f-decision exists. If $\mathbf{X}$ has an object 1 that is terminal in $\mathsf{Total}(\mathbf{X})$, then it suffices to consider the case $A = B = 1$.

Proof. We know that $\mathsf{Total}(\mathbf{X})$ has coproducts since $\mathbf{X}$ has restriction coproducts, and we know that the coproduct injections in $\mathsf{Total}(\mathbf{X})$ are cartesian, since $\mathbf{X}$ has a restriction zero. Thus $\mathsf{Total}(\mathbf{X})$ will be extensive if and only if it has pullbacks along coproduct injections, and coproducts are stable.

Suppose that $\overline{(1+0)f}$ splits for every $f : C \to A + B$, and that an f-decision exists. Let $k : C_A \to C$ and $k^*C \to C_A$ provide the splitting for $\overline{(1+0)f}$. Let $l : C_B \to C$ and $l^* : C \to C_B$ provide the splitting for $\overline{(0+1)f}$, which exists since $\overline{(0+1)f} = \overline{\tau(0+1)f} = \overline{(1+0)\tau f}$. We are to show that $\langle k|l\rangle : C_A + C_B$ is invertible.

The f-decision $h : C \to C + C$ is restriction inverse to $\langle kk^*|ll^*\rangle$, so $h\langle kk^*|ll^*\rangle = kk^* + ll^*$ and $\langle kk^*|ll^*\rangle h = \overline{h} = \overline{f} = 1$. Thus $h\langle k|l\rangle = k + l$, so $(k^* + l^*)h\langle k|l\rangle = (k^* + l^*)(k + l) = 1$, while $\langle k|l\rangle(k^* + l^*)h = \langle kk^*|ll^*\rangle h = 1$, as required.

Suppose conversely that $\mathsf{Total}(\mathbf{X})$ is extensive. Then any map $C \to A + B$ has the form $f + g : A' + B' \to A + B$, and now $if = (f + g)i'$, so f is total, and, similarly, g is total. Also $\overline{(1+0)(f+g)} = \overline{i^*(f+g)} = \overline{fi'^*} = \overline{i'^*}$ so that i' and i'^* provide a splitting for $\overline{(1+0)(f+g)}$. Finally, the identity $A' + B' \to A' + B'$ is easily seen to be a decision for $f + g$.

If 1 is terminal in $\mathsf{Total}(K(\mathbf{X}))$, it suffices to show stability of the coproduct $1 + 1$; see Carboni *et al.* (1993) or Cockett (1993). $\square$

Corollary 3.2. If $\mathbf{X}$ is a restriction category with restriction coproducts and a restriction zero, then $\mathsf{Total}(K_r(\mathbf{X}))$ is extensive if and only if every arrow $f : C \to A + B$ in $\mathbf{X}$ has a decision map. If $\mathbf{X}$ has an object 1 that is terminal in $\mathsf{Total}(K_r(\mathbf{X}))$, then it suffices to consider the case $A = B = 1$.

Proof. Since $\mathbf{X}$ has restriction coproducts, so does $K_r(\mathbf{X})$, and since $\mathbf{X}$ has a restriction zero, so does $K_r(\mathbf{X})$. All restriction idempotents split in $K_r(\mathbf{X})$, so by Proposition 3.1, $\mathsf{Total}(K_r(\mathbf{X}))$ will be extensive if and only if every total arrow $f : (C, e) \to (A + B, e_1 + e_2)$ has a decision. To say that f is total is to say that $\overline{f} = e$.

A decision h for $f : (C, e) \to (A + B, e_1 + e_2)$ is an arrow $h : C \to C + C$ in $\mathbf{X}$ satisfying $(\overline{f} + \overline{f})h = h = h\overline{f}$, $\nabla h = \overline{f}$ and $(f + f)h = (i + j)f$; that is, a decision map for f in $\mathbf{X}$. $\square$

In light of the proposition, we say that a restriction category $\mathbf{X}$ is *extensive* if it has restriction coproducts and a restriction zero, and every map $f : C \to A+B$ has a decision. By the uniqueness of decisions and the characterisation of Proposition 2.11, the existence of these decisions can be viewed as a combinator assigning to each $f : C \to A + B$ a map $\langle f\rangle : C \to C + C$ satisfying the decision axioms:

[D.1] $\nabla\langle f\rangle = \overline{f}$;

[D.2] $(f + f)\langle f\rangle = (i + j)f$.

This decision structure is equational, and so can be added freely. It would be interesting to have a description of the free extensive restriction category on a restriction category, or the free extensive restriction category on a mere category.

Of course, to say that $\mathbf{X}$ is extensive as a restriction category is quite different from saying that it is extensive as a mere category. In fact, as an extensive restriction category has a zero object, it cannot be an extensive category unless it is the trivial category with a single object and a single arrow. The connection between extensive restriction categories and extensive categories is, rather (see Corollary 3.2), that if $\mathbf{X}$ is an extensive restriction category, then $\mathsf{Total}(K_r(\mathbf{X}))$ is an extensive category.

Example 3.3. If $\mathscr{D}$ is a distributive category, then $\mathscr{D}_{+1}$ is an extensive restriction category, so $\mathsf{Total}(K_r(\mathscr{D}_{+1}))$ is an extensive category. We have already seen that $\mathscr{D}_{+1}$ has restriction coproducts and a restriction zero, thus we may apply Corollary 3.2. If $f : C \to A + B + 1$ is an arrow in $\mathscr{D}_{+1}$ from C to $A + B$, let h be the composite

$$C \xrightarrow{\langle C,f\rangle} C \times (A + B + 1) \xrightarrow{\delta^{-1}} C \times A + C \times B + C \xrightarrow{\pi_1+\pi_1+!} C + C + 1.$$

Verification of the commutativity of the diagrams

$$\begin{tikzcd}
C \arrow[r, "\langle C,f\rangle"] \arrow[drrr, bend right, "\overline{f}"'] & C \times (A+B+1) \arrow[r, "\delta^{-1}"] \arrow[dr, "\delta^{-1}"'] & C \times A + C \times B + C \arrow[r, "\pi_1+\pi_1+!"] & C + C + 1 \arrow[d, "\nabla+1"] \\
 & & C \times (A+B) + C \arrow[u, "\delta^{-1}+C"] \arrow[r, "\pi_1+!"] & C + 1
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "\langle C,f\rangle"] \arrow[d, "f"'] & C \times (A+B+1) \arrow[r, "\delta^{-1}"] \arrow[d, "f\times(A+B+1)"] & C \times A + C \times B + C \arrow[d, "\pi_1+\pi_1+!"] \\
A+B+1 \arrow[r, "\Delta"] \arrow[ddrr, bend right, "i+j+1"'] & (A+B+1)\times(A+B+1) \arrow[d, "\delta^{-1}"] & C + C + 1 \arrow[d, "\nabla+1"] \\
 & \begin{array}{c}(A+B+1)\times A + (A+B+1)\times B \\ +(A+B+1)\end{array} \arrow[r, "\pi_1+\pi_1+!"'] & A+B+1+A+B+1+1 \arrow[d, "t"] \\
 & & A+B+A+B+1
\end{tikzcd}$$

is a straightforward exercise in distributive categories; here

$$t : A + B + 1 + A + B + 1 + 1 \to A + B + A + B + 1$$

is the composite of the twist map

$$A + B + 1 + A + B + 1 + 1 \to A + B + A + B + 1 + 1 + 1$$

and

$$A + B + A + B + !.$$

Thus h is the required decision for f.

3.2. *Extensive maps*

As well as considering when $\mathsf{Total}(\mathbf{X})$ or $\mathsf{Total}(K_r(\mathbf{X}))$ is extensive, we can look at subcategories that are extensive. To this end, we say that *the map* $f : A \to B$ *in* $\mathbf{X}$ *is extensive* if for any decision $h : B \to B + B$ there is an hf-decision $k : A \to A + A$.

Lemma 3.4.

(i) Restriction isomorphisms are extensive.
(ii) Restriction idempotents are extensive.
(iii) Decision maps are extensive.
(iv) Coproduct injections and codiagonals are extensive.

Proof. First we show that restriction isomorphisms are extensive. If $f : A \to B$ has restriction inverse $g : B \to A$, and $h : B \to B + B$ is a decision, then $(g+g)hf$ is a decision and $\nabla(g+g)hf = \overline{(g+g)hf} = \overline{hf}$ by Corollary 2.16. Thus

$$\begin{aligned}
(hf + hf)(g+g)hf &= (hfg + hfg)hf \\
&= (h\overline{g} + h\overline{g})hf \\
&= (h+h)\overline{(g+g)}hf \\
&= (h+h)hf\overline{(g+g)hf} \\
&= (h+h)hf\overline{hf} \\
&= (h+h)hf \\
&= (i+j)hf,
\end{aligned}$$

so $(g+g)hf$ is an hf-decision, and f is extensive.

Every restriction idempotent is restriction inverse to itself, and is therefore extensive. Similarly, decisions and coproduct injections are restriction isomorphisms, and thus extensive. As for the codiagonal, if $h : A \to A + A$ is a decision, consider the composite

$$A + A \xrightarrow{h+h} A + A + A + A \xrightarrow{1+\tau+1} A + A + A + A.$$

On the one hand we have

$$\begin{aligned}
\nabla(1+\tau+1)(h+h) &= (\nabla + \nabla)(h+h) \\
&= \overline{h} + \overline{h} \\
&= \overline{h+h} \\
&= \overline{\nabla(h+h)} \\
&= \overline{h\nabla},
\end{aligned}$$

and on the other,

$$\begin{aligned}
(h\nabla + h\nabla)(1+\tau+1)(h+h) &= (h+h)\nabla(h+h) \\
&= (h+h)h\nabla \\
&= (i+j)h\nabla;
\end{aligned}$$

thus $(1+\tau+1)(h+h)$ is an $h\nabla$-decision. $\square$

Proposition 3.5. Let $\mathbf{X}$ be a restriction category with restriction coproducts and a restriction zero. Then the extensive maps in $\mathbf{X}$ form a restriction subcategory $\mathsf{Ex}(\mathbf{X})$ of $\mathbf{X}$, which is closed under finite coproducts and contains the decisions; and $\mathsf{Total}(K_r(\mathsf{Ex}(\mathbf{X})))$ is extensive. Furthermore, $\mathsf{Ex}(\mathbf{X})$ is maximal among restriction subcategories of $\mathbf{X}$ with these properties.

Proof. By Lemma 3.4, we know that $\mathsf{Ex}(\mathbf{X})$ contains the identities, the restriction idempotents, the coproduct injections and the codiagonals. Thus, it will be a restriction subcategory provided it is closed under composition, and it will be closed under finite coproducts provided the extensive maps are. By Lemma 3.4 once again, we know that $\mathsf{Ex}(\mathbf{X})$ contains the decisions; while the fact that $\mathsf{Total}(K_r(\mathsf{Ex}(\mathbf{X})))$ is extensive and the maximality of $\mathsf{Ex}(\mathbf{X})$ will follow from Corollary 3.2. Thus we need only show that the extensive maps are closed under composition and coproducts.

If $f : A \to B$ and $g : B \to C$ are extensive and $h : C \to C + C$ is a decision, let $k : B \to B + B$ be an hg-decision, and let $l : A \to A + A$ be a kf-decision. Then $\nabla l = \overline{kf} = \overline{\nabla kf} = \overline{\overline{hg}f} = \overline{hgf}$ and

$$\begin{aligned}
(hgf + hgf)l &= (hg + hg)(f + f)l \\
&= (hg + hg)(\overline{hg} + \overline{hg})(f + f)l \\
&= (hg + hg)(\nabla k + \nabla k))(f + f)l \\
&= (hg\nabla + hg\nabla)(kf + kf)l \\
&= (hg\nabla + hg\nabla)(i + j)kf \\
&= (hg + hg)kf \\
&= (i + j)hgf,
\end{aligned}$$

so l is also an hgf-decision.

Finally, let $f : A \to B$ and $f' : A' \to B'$ be extensive, and let $h : B+B' \to B+B'+B+B'$ be a decision. By Corollary 2.17, h can be written as $(1+\tau+1)(k+k')$, where $k : B \to B+B$ and $k' : B' \to B'+B'$ are decisions. Since f and f' are extensive, kf and $k'f'$ have decisions, so, by Proposition 2.19, $(1 + \tau + 1)(kf + k'f')$ has a decision; but $(1 + \tau + 1)(kf + k'f') = h(f + f')$, so we have proved that $f + f'$ is extensive. $\square$

Clearly, $\mathsf{Total}(K_r(\mathbf{X}))$ is extensive if and only if $\mathsf{Ex}(\mathbf{X}) = \mathbf{X}$; that is, if every map is extensive. Note, however, that the construction $\mathsf{Ex}(\mathbf{X})$ is not functorial in $\mathbf{X}$.

4. Limits in restriction categories

We saw in Section 2 that cocartesian objects in rCat give a good notion of restriction category with coproducts. We now turn to products, and the first thing to observe is that cartesian objects in rCat are *not* a good notion.

If $\mathbf{X}$ is a restriction category, and the unique restriction functor $! : \mathbf{X} \to \mathbf{1}$ has a right adjoint in rCat, then $\mathbf{X}$ has a terminal object 1, and for each object A, the unique map $t_A : A \to 1$ is total. But if $f : A \to B$ is any map, then $\overline{f} = \overline{\overline{t_B}f} = \overline{t_B f} = \overline{t_A} = 1$, so f is total. Thus $! : \mathbf{X} \to \mathbf{1}$ can have a right adjoint in rCat only if the restriction structure on $\mathbf{X}$ is trivial.

The situation for binary products is much the same. Suppose that $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ has a right adjoint in rCat. Explicitly, this means that $\mathbf{X}$ has binary products as a mere category, the diagonal and projections are total, and $\overline{f \times g} = \overline{f} \times \overline{g}$ for any maps f and g. Let $f : A \to B$ be any map, and let $p, q : A \times A \to A$ be the projections. Then $\overline{1_A \times f} = \overline{\overline{p}(1_A \times f)} = \overline{p(1_A \times f)} = \overline{p} = 1$, so $1_A \times f : A \times A \to A \times B$ is total; and now $\overline{f} = \overline{f} q\Delta = q(1_A \times \overline{f})\Delta = q\overline{1_A \times f}\Delta = q\Delta = 1$, so f is total. Thus once again $\Delta : \mathbf{X} \to \mathbf{X}\times\mathbf{X}$ can have a right adjoint in rCat only if the restriction structure on $\mathbf{X}$ is trivial.

We shall now look at other possible notions of products in restriction categories, and, more generally, at limits.

4.1. *Cartesian objects in* rCatl

One possible approach to the unsatisfactory nature of cartesian objects in rCat is to change the 2-category rCat. In Cockett and Lack (2003) we defined a 2-category rCatl with the same objects and arrows as rCat, namely, the restriction categories and restriction functors, but with a larger class of 2-cells. For restriction functors $F, G : \mathbf{X} \to \mathbf{Y}$, a 2-cell in rCatl from F to G consists of a total map $\alpha_X : FX \to GX$ in $\mathbf{Y}$ for each object X of $\mathbf{X}$ such that for each $f : X \to Y$ in $\mathbf{X}$, the diagram

$$\begin{tikzcd} FX \arrow[rr, "Ff"] \arrow[d, "F\overline{f}"'] & & FY \arrow[d, "\alpha_Y"] \\ FX \arrow[r, "\alpha_X"'] & GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$

commutes. The reason for the name rCatl is that if one thinks of a restriction category $\mathbf{X}$ as a 2-category (where there is a 2-cell $f \leqslant g$ if and only if $f = g\overline{f}$) and restriction functors as 2-functors, then a 2-cell in rCatl is precisely a lax natural transformation from F to G whose components are total.

We now define a *restriction terminal object* in a restriction category $\mathbf{X}$ to be an object T for which the corresponding restriction functor $\mathbf{1} \to \mathbf{X}$ is right adjoint in rCatl to the unique functor $\mathbf{X} \to \mathbf{1}$. In more explicit terms, this amounts to giving, for each object A of $\mathbf{X}$, a total map $t_A : A \to T$ such that $t_T = 1_T$, and for each arrow $f : A \to B$, we have $t_B f = t_A \overline{f}$.

Proposition 4.1. A restriction terminal object in $\mathbf{X}$ is terminal in Total($\mathbf{X}$). Conversely, if $\mathbf{X}$ is a classified restriction category, a terminal object in Total($\mathbf{X}$) is restriction terminal in $\mathbf{X}$.

Proof. The first statement follows immediately from the fact that Total : rCatl $\to$ Cat is a 2-functor, and so preserves adjunctions; alternatively, it is equally easy to verify directly.

The second statement is an instance of Cockett and Lack (2003, Proposition 3.7). □

This proposition means, in particular, that there is no ambiguity in saying 'T is a restriction terminal object' since the total maps $t_A : A \to T$ are unique. It also shows that restriction terminal objects are unique up to a unique isomorphism.

Another point of view on restriction terminal objects may be obtained by consideration of the functor $\mathrm{RId} : \mathbf{X}^{\mathrm{op}} \to \mathsf{Set}$, which is defined in Cockett and Lack (2002). This sends an object A to the set of all restriction idempotents on A, and a morphism $f : A \to B$ to the function sending a restriction idempotent $e : B \to B$ to $\overline{ef} : A \to A$.

Proposition 4.2. A restriction terminal object is precisely a representation of the functor $\mathrm{RId} : \mathbf{X}^{\mathrm{op}} \to \mathsf{Set}$.

Proof. If T is a restriction terminal object, then any $f : A \to T$ determines a restriction idempotent $\overline{f}$ on A, while any restriction idempotent $e : A \to A$ determines a map $t_A e : A \to T$. These processes are inverse, since $\overline{t_A e} = \overline{e} = e$ and $t_A \overline{f} = t_T f = 1_T f = f$.

On the other hand, if T is an object equipped with an isomorphism $\alpha : \mathbf{X}(-, T) \cong \mathrm{RId}$, then for each A there is a unique $t_A : A \to T$ with $\alpha_A(t_A) = 1_A$. For any $f : A \to B$ we have

$$\alpha_A(t_B f) = \mathrm{RId}(f)\alpha_B(t_B) = \mathrm{RId}(f)(1_B) = \overline{f}$$

and

$$\alpha_A(t_A \overline{f}) = \mathrm{RId}(\overline{f})\alpha_A(t_A) = \mathrm{RId}(\overline{f})(1_A) = \overline{f}.$$

Thus $t_B f = t_A \overline{f}$, since α_A is invertible. It remains to show that $t_T = 1_T$. Let e be the restriction idempotent $\alpha_T(1_T)$. Then for any $g : A \to T$ we have

$$\alpha_A(g) = \alpha_A(1_T g) = \mathrm{RId}(g)(\alpha_T(1_T)) = \mathrm{RId}(g)(e) = \overline{eg}.$$

In particular, $\alpha_T(e) = \overline{ee} = e = \alpha_T(1_T)$, so $e = 1_T$; and now $\alpha_T(t_T) = 1_T = e = \alpha_T(1_T)$, so $t_T = 1_T$. □

Next we turn to the case of a restriction category $\mathbf{X}$ for which the diagonal restriction functor $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ has a right adjoint in rCatl. We then say that $\mathbf{X}$ has *binary restriction products*. Explicitly, this means that there is a restriction functor $\mathbf{X} \times \mathbf{X} \to \mathbf{X}$, whose value at an object (A, B) we denote by $A \times B$ and whose value at an arrow (f, g) we denote by $f \times g$; and total maps $\Delta : A \to A \times A$, $p : A \times B \to A$, and $q : A \times B \to B$ satisfying

$$\begin{tikzcd}
 & A \arrow[dl, "1"'] \arrow[d, "\Delta"] \arrow[dr, "1"] & \\
A & A \times A \arrow[l, "p"] \arrow[r, "q"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[d, "\Delta"'] \arrow[dr, "1"] & \\
A \times B \times A \times B \arrow[r, "p \times q"'] & A \times B
\end{tikzcd}$$

$$\begin{tikzcd}
A \times B \arrow[d, "p"'] & A \times B \arrow[l, "\overline{f} \times \overline{g}"'] \arrow[r, "\overline{f} \times \overline{g}"] \arrow[dd, "f \times g"] & A \times B \arrow[d, "q"] \\
A \arrow[d, "f"'] & & B \arrow[d, "g"] \\
A' & A' \times B' \arrow[l, "p"] \arrow[r, "q"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\overline{f}"] \arrow[dd, "f"'] & A \arrow[d, "\Delta"] \\
 & A \times A \arrow[d, "f \times f"] \\
A' \arrow[r, "\Delta"'] & A' \times A'
\end{tikzcd}$$

Once again, $\mathsf{Total} : \mathsf{rCatl} \to \mathsf{Cat}$ preserves products and adjunctions, so $\mathsf{Total}(\mathbf{X})$ will have binary products whenever $\mathbf{X} \to \mathbf{X} \times \mathbf{X}$ has a right adjoint in rCatl.

It turns out that if $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ does have a right adjoint in rCatl, it automatically satisfies certain further conditions, as the following proposition shows. In particular, the diagonal maps $\Delta : A \to A \times A$ are not just lax natural, but natural.

Proposition 4.3. If $\mathbf{X}$ is a restriction category, and $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ has a right adjoint in rCatl, then:

(i) $\overline{(f \times g)\Delta} = \overline{f}\,\overline{g}$ for all f and g with the same domain; and
(ii) the maps $\Delta : A \to A \times A$ are natural in A.

Proof.

(i) Since $\overline{(f \times g)\Delta} = \overline{\overline{(f \times g)}\Delta} = \overline{(\overline{f} \times \overline{g})\Delta}$, it will suffice to show that $\overline{(e \times e')\Delta} = ee'$ for all restriction idempotents e and e'.
First observe that $\overline{(e \times e')\Delta} = p\Delta\overline{(e \times e')\Delta} = p\overline{e \times e'\Delta} = p(e \times e')\Delta$, and, similarly, $\overline{(e \times e')\Delta} = q(e \times e')\Delta$. Using lax naturality of p, we have $e\overline{(e \times e')\Delta} = ep(e \times e')\Delta = p(e \times e')\Delta = \overline{(e \times e')\Delta}$, and using lax naturality of q, we have $e'\overline{(e \times e')\Delta} = \overline{(e \times e')\Delta}$.
Thus

$$\begin{aligned}
\overline{(e \times e')\Delta} &= ee'\overline{(e \times e')\Delta} \\
&= \overline{(e \times e')\Delta}ee' \\
&= \overline{(e \times e')\Delta ee'} \\
&= \overline{(eee' \times e'ee')\Delta ee'} \\
&= \overline{(ee' \times ee')\Delta ee'} \\
&= \overline{\Delta ee'} \\
&= \overline{ee'} \\
&= ee'.
\end{aligned}$$

(ii) This follows from (i) and lax naturality of Δ, since

$$(f \times f)\Delta = (f \times f)\Delta\overline{(f \times f)\Delta} = (f \times f)\Delta\overline{f}\,\overline{f} = (f \times f)\Delta\overline{f} = \Delta f. \qquad \square$$

We say that $\mathbf{X}$ has restriction products if it is a cartesian object in rCatl; that is, if it has binary restriction products and a restriction terminal. If $\mathbf{X}$ and $\mathbf{Y}$ are restriction categories with restriction products, then a restriction functor $F : \mathbf{X} \to \mathbf{Y}$ is said to preserve restriction products if it commutes with the right adjoints $\mathbf{1} \to \mathbf{X}$ and $\mathbf{X} \times \mathbf{X} \to \mathbf{X}$ in rCatl. This definition can be made more explicit. If T and S denote the restriction terminal objects of $\mathbf{X}$ and $\mathbf{Y}$, then there is a unique total map $\varphi : FT \to S$, and F preserves the restriction terminal object if and only if $\varphi : FT \to S$ is invertible. Similarly, for any objects X and Y of $\mathbf{X}$ there is a unique total map $\psi_{X,Y} : F(X \times Y) \to FX \times FY$ commuting with the projections, and F preserves binary restriction products if and only if each $\psi_{X,Y}$ is invertible. We now have the following proposition.

Proposition 4.4. If $\mathbf{X}$ is a restriction category with restriction products, then $\mathsf{Total}(\mathbf{X})$ and $\mathsf{Total}(K_r(\mathbf{X}))$ have products; if $\mathbf{Y}$ is another such restriction category and $F : \mathbf{X} \to \mathbf{Y}$ is a restriction functor that preserves restriction products, then $\mathsf{Total}(F) : \mathsf{Total}(\mathbf{X}) \to \mathsf{Total}(\mathbf{Y})$ and $\mathsf{Total}(K_r(F)) : \mathsf{Total}(K_r(\mathbf{X})) \to \mathsf{Total}(K_r(\mathbf{Y}))$ preserve products.

4.2. *p-categories*

In this section we recall Robinson and Rosolini's notion of a *p-category* (Robinson and Rosolini 1988), in order to compare it to the various structures considered above.

A *p-category* is a category $\mathbf{X}$ equipped with a functor $\times : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$, a natural family of maps $\Delta : A \to A \times A$, and families $p_{A,B} : A \times B \to A$ natural in A, and $q_{A,B} : A \times B \to B$ natural in B, required to make the following diagrams

$$
\begin{tikzcd}
 & X \arrow[dl, "1"'] \arrow[d, "\Delta"] \arrow[dr, "1"] & \\
X & X \times X \arrow[l, "p"] \arrow[r, "q"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times Y \arrow[d, "\Delta"'] \arrow[dr, "1"] & \\
X \times Y \times X \times Y \arrow[r, "p\times q"'] & X \times Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & X \times (Y \times Z) \arrow[dl, "1\times p"'] \arrow[d, "p"] \arrow[dr, "1\times q"] & \\
X \times Y \arrow[r, "p"'] & X & X \times Z \arrow[l, "p"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (X \times Y) \times Z \arrow[dl, "p\times 1"'] \arrow[d, "q"] \arrow[dr, "q\times 1"] & \\
X \times Z \arrow[r, "q"'] & Z & Y \times Z \arrow[l, "q"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \times (Y \times Z) \arrow[r, "f\times(g\times h)"] \arrow[d, "\Delta"'] & X' \times (Y' \times Z') \arrow[d, "\Delta"] \\
(X \times (Y \times Z)) \times (X \times (Y \times Z)) \arrow[d, "(1\times p)\times q"'] & (X' \times (Y' \times Z')) \times (X' \times (Y' \times Z')) \arrow[d, "(1\times p)\times q"] \\
(X \times Y) \times (Y \times Z) \arrow[d, "1\times q"'] & (X' \times Y') \times (Y' \times Z') \arrow[d, "1\times q"] \\
(X \times Y) \times Z \arrow[r, "(f\times g)\times h"] & (X' \times Y') \times Z'
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \times Y \arrow[r, "\Delta"] \arrow[d, "f\times g"'] & (X \times Y) \times (X \times Y) \arrow[r, "q\times p"] & Y \times X \arrow[d, "f'\times g'"] \\
X' \times Y' \arrow[r, "\Delta"] & (X' \times Y') \times (X' \times Y') \arrow[r, "q'\times p'"] & Y' \times X'
\end{tikzcd}
$$

commutative for all arrows f, g and h. The last two diagrams provide a natural associativity isomorphism $\alpha_{X,Y,Z} : X \times (Y \times Z) \to (X \times Y) \times Z$ and a natural symmetry isomorphism $\tau_{X,Y} : X \times Y \to Y \times X$.

Given a map $f : X \to X'$, Robinson and Rosolini define $\mathrm{dom} f : X \to X$ to be

$$X \xrightarrow{\Delta} X \times X \xrightarrow{1\times f} X \times X' \xrightarrow{p} X,$$

and their Proposition 1.4 verifies that this makes **X** into a restriction category. As Robinson and Rosolini observe (using slightly different terminology), although a p-category structure on a category may not be unique, a p-category structure on a restriction category is. Thus it makes sense to ask which restriction categories are p-categories.

Proposition 4.5. A restriction category is a p-category if and only if it has binary restriction products.

Proof. First suppose that **X** is a p-category. Robinson and Rosolini (1988, Proposition 1.4) proved that $\times : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ is a restriction functor, and that each instance of p, q and Δ is total. The diagonal is natural by assumption, and the 'triangle equations' linking Δ with p and q hold by assumption. Thus it remains only to check that the projections p and q are lax natural. In the case of p, lax naturality amounts to the equation $p(f \times g) = fp(\overline{f} \times \overline{g})$ for all arrows f and g. Consider first the special case where f is the identity. In the diagram

$$\begin{tikzcd}
X \times Y \arrow[r, "1\times\Delta"] \arrow[dr, "1\times g"'] & X \times (Y \times Y) \arrow[r, "1\times(1\times g)"] \arrow[dr, "1\times g\times g"'] & X \times (Y \times Y') \arrow[r, "1\times p"] \arrow[d, "1\times(g\times 1)"] & X \times Y \arrow[d, "1\times g"] & \\
& X \times Y' \arrow[r, "1\times\Delta"] \arrow[rr, bend right, "1"'] & X \times (Y' \times Y') \arrow[r, "1\times p"] & X \times Y' \arrow[r, "p"] & X
\end{tikzcd}$$

the left square commutes by naturality of Δ, the triangle by functoriality of $\times$, the right square by (one-sided) naturality of p, and the curved region by one of the triangle equations. Thus the exterior commutes, which is to say that $p(1 \times g) = p(1 \times \overline{g})$. As for the general case, in the diagram

$$\begin{tikzcd}
X \times Y \arrow[r, "f\times g"] \arrow[d, "\overline{f}\times 1"] \arrow[dd, bend right=60, "\overline{f}\times\overline{g}"'] & X' \times Y' \arrow[dr, "p"] & \\
X \times Y \arrow[r, "1\times g"] \arrow[d, "1\times\overline{g}"] & X \times Y \arrow[u, "f\times 1"] \arrow[d, "p"] & X' \\
X \times Y \arrow[r, "p"'] & X \arrow[ur, "f"'] &
\end{tikzcd}$$

the left and top regions commute by functoriality of $\times$, the bottom region by the special case just considered, and the right region by the one-sided naturality of p. Commutativity of the exterior is then the desired equation $p(f \times g) = fp(\overline{f} \times \overline{g})$.

Lax naturality of q states that $q(f \times g) = gq(\overline{f} \times \overline{g})$; we leave the verification to the reader.

Now suppose conversely that **X** has binary restriction products. We must show that $p : X \times Y \to X$ is natural in X, that $q : X \times Y \to Y$ is natural in Y, and that $\alpha_{X,Y,Z}$ and

$\tau_{X,Y}$ are natural in all variables. The equations involving only instances of p, q and Δ all hold because the binary restriction products are actual products in $\mathsf{Total}(\mathbf{X})$.

For naturality of p, we use lax naturality of p and naturality of Δ to see that $p(f \times 1) = fp\overline{f \times 1} = fp\overline{f}\,\overline{pq} = fp$; the case of q is similar.

As for τ, first observe that

$$\overline{(g \times f)\tau} = \overline{(g \times f)(q \times p)\Delta} = \overline{(gq \times fp)\Delta} = \overline{gq}\,\overline{fp} = \overline{g \times f}.$$

Now

$$(g \times f)\tau = (g \times f)\tau\overline{(g \times f)\tau} = (g \times f)\tau\overline{g \times f} = \tau(f \times g).$$

The case of α is similar but more complicated. Since

$$\begin{aligned} \overline{((f \times g) \times h)\alpha} &= \overline{((f \times g) \times h)((1 \times p) \times qq)\Delta} \\ &= \overline{((f \times gp) \times hqq)\Delta} \\ &= \overline{f \times gp}\,\overline{hqq} \\ &= \overline{fp}\,\overline{gpq}\,\overline{hqq} \end{aligned}$$

and

$$\begin{aligned} \overline{f \times (g \times h)} &= \overline{fp}\,\overline{(g \times h)q} \\ &= \overline{fp}\,\overline{\overline{g \times h}q} \\ &= \overline{fp}\,\overline{\overline{gp}\,\overline{hq}q} \\ &= \overline{fp}\,\overline{q\overline{gpq}\,\overline{hqq}} \\ &= \overline{fp}\,\overline{gpq}\,\overline{hqq}, \end{aligned}$$

we have $\overline{((f \times g) \times h)\alpha} = \overline{f \times (g \times h)}$, and now we deduce

$$\begin{aligned} ((f \times g) \times h)\alpha &= ((f \times g) \times h)\alpha\overline{((f \times g) \times h)\alpha} \\ &= ((f \times g) \times h)\alpha\overline{f \times (g \times h)} \\ &= \alpha(f \times (g \times h)). \end{aligned}$$

□

If $\mathbf{X}$ is a p-category, Robinson and Rosolini define a *one-element object* to be an object T with a family $t_X : X \to T$ of maps in $\mathbf{X}$ for which $p : X \times T \to X$ is invertible, with inverse

$$X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times t_X} X \times T.$$

Proposition 4.6. If $\mathbf{X}$ is a p-category, an object T of $\mathbf{X}$ is a one-element object if and only if it is a restriction terminal object; the map $t_X : X \to T$ in the definition of one-element object is the unique total map from X to T.

Proof. To say that

$$X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times t_X} X \times T \xrightarrow{p} X$$

is the identity is precisely to say that t_X is total. The fact that the t_X are lax natural and $t_T = 1_T$ for a one-element object T is part of Robinson and Rosolini (1988, Theorem 3.3).

Conversely, if T is a restriction terminal object, we have a family $t_X : X \to T$ of total maps; it then remains to show that

$$X \times T \xrightarrow{p} X \xrightarrow{\Delta} X \times X \xrightarrow{1 \times t_X} X \times T$$

is the identity. But this follows from the fact that restriction products in $\mathbf{X}$ are genuine products in $\mathsf{Total}(\mathbf{X})$. □

The relationship between p-categories with one-element object and various other structures is analysed in some detail in Robinson and Rosolini (1988). Translating this into our nomenclature, the restriction categories with restriction products are exactly the *partial cartesian categories* in the sense of Curien and Obtulowicz (1989), or, alternatively, the *pre-dht-symmetric categories* of Hoehnke (1977), and they are a special case of the *bicategories of partial maps* of Carboni (1987). For more details on these correspondences, see Robinson and Rosolini (1988).

4.3. *Categories with products and a restriction*

Before leaving our discussion of products in restriction categories, it is worth discussing a quite different type of product that sometimes exists. While restriction products are tensor products that are actual products in the total map category, it is also possible that a restriction category could have products in the ordinary sense. A well-known example of this is provided by the category of sets and partial maps. The restriction product of A and B is just their product $A \times B$ as sets; this is not the categorical product in the partial map category, which is given by $A + A \times B + B$. (This is more generally true for the partial map category of any lextensive category (Carboni *et al.* 1993; Cockett 1993), where the $\mathscr{M}$-maps are taken to be the coproduct injections.)

Recall that the restriction idempotents associated with a particular object A in a restriction category $\mathbf{X}$ form a meet semi-lattice with $e_1 \wedge e_2 = e_1 e_2$ and greatest element $\top = 1_A$. These lattices sit over each object to give the *restriction fibration* (Cockett and Lack 2002, Section 4) $\partial : \mathscr{R}(\mathbf{X}) \to \mathbf{X}$ over all the maps: the substitutions preserve the meet but not the greatest element. When this fibration is restricted to the total maps, one obtains a meet-semilattice fibration.

If a category has both a restriction structure and a terminal object 1 (and here we emphasise that we do not assume any relation between the two structures) then we may consider the restriction $\overline{!_A} : A \to A$ of the unique map $!_A : A \to 1$. Then for any $f : A \to B$ we have $\overline{!_A}\,\overline{f} = \overline{!_A \overline{f}} = \overline{!_A}$, and thus $\overline{!_A}$ must be the least restriction idempotent in the above ordering: in terms of partial maps, this determines the smallest possible domain. Thus the presence of a terminal object forces each object A to have a least element in its lattice of restriction idempotents. Furthermore, it is clear that the substitution functors of the fibration mentioned above preserve these least elements.

When a category has both a restriction structure and finite products, $\overline{\langle e, e' \rangle}$ is the *join* of e and e' in the lattice of restriction idempotents of an object. To see this, first observe that

$$e\overline{\langle e, e' \rangle e} = p\langle e, e' \rangle \overline{\langle e, e' \rangle} = p\langle e, e' \rangle = e,$$

so $e \leqslant \overline{\langle e, e' \rangle}$, and, similarly, $e' \leqslant \overline{\langle e, e' \rangle}$. Now if d is a restriction idempotent and $e, e' \leqslant d$, then

$$\overline{\langle e, e' \rangle} d = \overline{\overline{\langle e, e' \rangle} d} = \overline{\langle ed, e'd \rangle} = \overline{\langle e, e' \rangle},$$

so $\overline{\langle e, e' \rangle} \leqslant d$.

This proves that the semilattices of restriction idempotents are lattices. In fact they are distributive lattices since

$$e \wedge (e_1 \vee e_2) = \overline{\langle e_1, e_2 \rangle} e = \overline{\langle e_1, e_2 \rangle e} = \overline{\langle e_1 e, e_2 e \rangle} = e_1 e \vee e_2 e = (e \wedge e_1) \vee (e \wedge e_2)$$

and

$$e \wedge \bot = \overline{!_A} e = \overline{!_A e} = \overline{!_A} = \bot.$$

Proposition 4.7. If $\mathbf{X}$ is a category with a restriction structure and (finite) products, the fibration of restriction idempotents

$$\partial : \mathscr{R}(\mathbf{X}) \to \mathbf{X}$$

is a fibred join-semilattice and the fibration

$$\partial_t : \mathscr{R}_t(\mathbf{X}) \to \mathsf{Total}(\mathbf{X})$$

is a fibred distributive lattice.

Proof. It remains only to check that the inverse image functors preserve the relevant structure. It was proved in Cockett and Lack (2002, Section 4.1) that binary meets are always preserved, while the top element is preserved by the total maps. Thus it will suffice to show that for an arbitrary map $f : X \to Y$, the induced functor $\mathrm{RId}(f) : \mathrm{RId}(Y) \to \mathrm{RId}(X)$ preserves finite joins. For the bottom element we have $\mathrm{RId}(f)(\overline{!_Y}) = \overline{\overline{!_Y} f} = \overline{!_Y f} = \overline{!_X}$; for binary joins we have:

$$\begin{aligned}
\mathrm{RId}(f)(e \vee e') &= \mathrm{RId}(f)\overline{\langle e, e' \rangle} \\
&= \overline{\langle e, e' \rangle f} \\
&= \overline{\langle ef, e'f \rangle} \\
&= ef \vee e'f \\
&= \mathrm{RId}(f)(e) \vee \mathrm{RId}(f)(e').
\end{aligned}$$

□

In a split restriction category with products this means that the M-subobjects in the total category must already have finite joins that are preserved by pulling back. Thus *products* in the restriction category lead to *colimits* in the lattices of $\mathscr{M}$-subobjects.

4.4. *Restriction limits*

We have already discussed products and coproducts in restriction categories. Now we turn briefly to more general notions of limit. Once again, these will be analysed in terms of adjunctions in rCatl.

Let $\mathbf{X}$ be a restriction category and $\mathscr{C}$ be a finite category. We shall define the restriction limit of a functor $S : \mathscr{C} \to \mathbf{X}$ to be a cone $p_C : L \to SC$ over S with total components and satisfying the following universal property. If $q_C : M \to SC$ is a lax cone over S (that is, $Sc.q_C = q_D\overline{Sc.q_C}$ for any $c : C \to D$), then there is a unique arrow $f : M \to L$ satisfying $p_C f = q_C e$, where e is the composite of the restriction idempotents $\overline{q_C}$.

It follows immediately from the definition that restriction limits are unique up to unique isomorphism. Equally immediate is the fact that if S takes its values in $\mathsf{Total}(\mathbf{X})$, then a restriction limit of $\mathbf{X}$ is a genuine limit in $\mathsf{Total}(\mathbf{X})$.

Example 4.8. The restriction limit of the empty diagram is precisely a restriction terminal object. The restriction limit of a diagram on the discrete category with two objects is the restriction product of the corresponding objects.

The following proposition provides a new example.

Proposition 4.9. The restriction limit of an arrow f is precisely a splitting for the idempotent $\overline{f}$.

Proof. A restriction limit of f amounts to a monomorphism $p : P \to X$ for which fp is total, having the property that for any arrows $q : Q \to X$ and $q' : Q \to Y$ satisfying $q' = fq\overline{q'}$, there is a unique $r : Q \to P$ satisfying $pr = qe$ and $fpr = q'e$, where $e = \overline{q}\overline{q'}$. In fact, $\overline{q}\overline{q'} = \overline{q}\overline{fq\overline{q'}} = \overline{q}\,\overline{fqq'} = \overline{fqq'} = \overline{q'}$, and $pr = qe$ implies $fpr = fqe = fq\overline{q'} = q' = q'\overline{q'} = q'e$, so the only condition on r is that $pr = q\overline{q'}$.

Taking $q = 1_X$ and $q' = f$, we obtain a unique $s : X \to P$ satisfying $ps = \overline{f}$. Taking $q = p$ and $q' = fp$, we obtain a unique $t : P \to P$ satisfying $pt = p\overline{fp} = p$. Since $psp = \overline{f}p = p\overline{fp} = p$, we deduce by uniqueness of t that $sp = 1$. Thus p and s provide a splitting for $\overline{f}$.

On the other hand, if $p : P \to X$ and $s : X \to P$ split $\overline{f}$, while q and q' satisfy $q' = fq\overline{q'}$, then $psq\overline{q'} = \overline{f}q\overline{q'} = q\overline{fq\overline{q'}} = q\overline{fq\overline{q'}} = q\overline{q'}$, and $sq\overline{q'}$ is unique with this property, since p is monic. Thus $p : P \to X$ exhibits P as the restriction limit of f. $\square$

We now, as promised, analyse these restriction limits in terms of adjunctions in rCatl. We continue to suppose that $\mathbf{X}$ is a restriction category, and now allow $\mathscr{C}$ to be an arbitrary category that is not necessarily finite. We shall define a restriction category $\mathbf{X}^{\mathscr{C}}$ and a restriction functor $\Delta : \mathbf{X} \to \mathbf{X}^{\mathscr{C}}$, and show that if $\mathscr{C}$ is finite, this Δ has a right adjoint if and only if $\mathbf{X}$ has restriction limits of functors with domain $\mathscr{C}$.

As a category, $\mathbf{X}^{\mathscr{C}}$ consists of functors from $\mathscr{C}$ to $\mathbf{X}$ and lax natural transformations between them. More explicitly, given functors $F, G : \mathscr{C} \to \mathbf{X}$, an arrow $\alpha : F \to G$ in $\mathbf{X}^{\mathscr{C}}$ consists of an arrow $\alpha_A : FA \to GA$ in $\mathbf{X}$ for each object A of $\mathscr{C}$, such that $\alpha_B.Ff = Gf.\alpha_A\overline{\alpha_B.Ff}$ for every arrow $f : A \to B$ in $\mathscr{C}$. Composition is defined pointwise: $(\beta\alpha)_A = \beta_A\alpha_A$. The restriction structure is also defined pointwise: $\overline{\alpha} : F \to F$ has $\overline{\alpha}_A = \overline{\alpha_A}$. The only thing to check is that $\overline{\alpha}$ is in fact an arrow of the category. To do this, note that

$$
\begin{aligned}
\overline{\alpha_B.Ff} &= \overline{Gf.\alpha_A.\overline{\alpha_B.Ff}} \\
&= \overline{Gf.\alpha_A}\,\overline{\alpha_B.Ff} \\
&= \overline{\alpha_A}.\overline{Gf.\alpha_A}.\overline{\alpha_B.Ff} \\
&= \overline{\alpha_A}.\overline{\alpha_B.Ff}
\end{aligned}
$$

and now

$$\begin{aligned}
\overline{\alpha_B}.Ff &= Ff.\overline{\alpha_B.Ff} \\
&= Ff.\overline{\overline{\alpha_A}.\alpha_B.Ff} \\
&= Ff.\overline{\alpha_A}.\overline{\overline{\alpha_B}.Ff}.
\end{aligned}$$

The restriction functor $\Delta : \mathbf{X} \to \mathbf{X}^{\mathscr{C}}$ sends an object X of $\mathbf{X}$ to the functor $\mathscr{C} \to \mathbf{X}$ constant at X, and sends an arrow $f : X \to Y$ to the family of arrows $X \to Y$, each of which is just f.

Proposition 4.10. If $\Delta : \mathbf{X} \to \mathbf{X}^{\mathscr{C}}$ has a right adjoint in rCatl, then $\mathsf{Total}(\mathbf{X})$ has ordinary $\mathscr{C}$-limits.

Proof. Applying the 2-functor $\mathsf{Total} : \mathsf{rCatl} \to \mathsf{Cat}$ to the adjunction gives an adjunction

$$\begin{tikzcd}
\mathsf{Total}(\mathbf{X}^{\mathscr{C}}) \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathsf{Total}(\mathbf{X}) \arrow[l, bend left, "\mathsf{Total}(\Delta)"]
\end{tikzcd}$$

of categories. The functor $\mathsf{Total}(\Delta)$ lands in the full subcategory $\mathsf{Total}(\mathbf{X})^{\mathscr{C}}$ of $\mathsf{Total}(\mathbf{X}^{\mathscr{C}})$ consisting of the functors $\mathscr{C} \to \mathbf{X}$ landing in $\mathsf{Total}(\mathbf{X})$. It follows that $\Delta : \mathsf{Total}(\mathbf{X}) \to \mathsf{Total}(\mathbf{X})^{\mathscr{C}}$ has a right adjoint, and thus $\mathsf{Total}(\mathbf{X})$ has $\mathscr{C}$-limits. □

In order to compare the two approaches to restriction limits, we assume that the category $\mathscr{C}$ is finite.

Proposition 4.11. If $\mathscr{C}$ is a finite category and $\mathbf{X}$ a restriction category, then to give a right adjoint in rCatl to $\Delta : \mathbf{X} \to \mathbf{X}^{\mathscr{C}}$ is precisely to give a restriction limit in $\mathbf{X}$ of each functor $S : \mathscr{C} \to \mathbf{X}$.

Proof. Let $R : \mathbf{X}^{\mathscr{C}} \to \mathbf{X}$ be right adjoint in rCatl to Δ. If $S : \mathscr{C} \to \mathbf{X}$ is an object of $\mathbf{X}^{\mathscr{C}}$, let $L = R(S)$, and let $p : L \to S$ be the component at S of the counit. Then p is a lax cone, and its components $p_C : L \to SC$ are total. But to say that p is a lax cone is to say, for each $f : C \to D$, that $p_D = Sf.p_C\overline{p_D}$, and since p_D is total, this means that p is in fact a cone.

We now show that the cone $p : L \to S$ is a restriction limit cone. If $q : M \to S$ is a lax cone (that is, an arrow $\Delta M \to S$ in $\mathbf{X}^{\mathscr{C}}$), let $f : M \to R(S) = L$ be given by $R(q) : R\Delta M \to R(S)$ composed with the unit $n : M \to R\Delta M$. In the diagram

$$\begin{tikzcd}
\Delta M \arrow[r, "\Delta n"] \arrow[d, "\Delta(\overline{Rq.n})"'] & \Delta R\Delta M \arrow[rr, "\Delta Rq"] \arrow[d, "\Delta\overline{Rq}"] & & \Delta RS \arrow[d, "p"] \\
\Delta M \arrow[r, "\Delta n"] \arrow[rr, bend right, "1"'] & \Delta R\Delta M \arrow[r, "p"] & \Delta M \arrow[r, "q"'] & S
\end{tikzcd}$$

the left square commutes by a restriction category axiom, the right rectangle by lax naturality of p and the curved region by one of the triangle equations. Commutativity of the exterior amounts to the equation $p_C f = q_C.\overline{Rq.n} = q_C\overline{f}$.

We shall now show that $\overline{f}$ is the composite of the $\overline{q_C}$, which we henceforth denote e. For each C we have $\overline{f} = \overline{p_C f} = \overline{q_C \overline{f}} = \overline{q_C}\,\overline{f}$, so $\overline{f} = e\overline{f}$. On the other hand, $\overline{q_C} e = e$, so $\overline{q}.\Delta e = \Delta e$ and $\overline{Rq.R\Delta e} = \overline{R(\overline{q}.\Delta e)} = \overline{R\Delta e}$. Thus

$$e\overline{f} = e\overline{Rq.n} = e\overline{Rq.n.e} = e\overline{Rq.R\Delta e.n.e} = e.\overline{R\Delta e.n.e} = e.\overline{n.e} = e.$$

This proves that $\overline{f}$ is the composite of the $\overline{q_C}$, and hence that f provides the desired factorisation.

Finally, we must prove that the factorisation f is unique. To do this, we shall show that an arrow $f : M \to RS$ is determined by the $p_C f$ and by $\overline{f}$; then if $p_C f = p_C f'$, we also have $\overline{f} = \overline{p_C f} = \overline{p_C f'} = \overline{f'}$. Now in

$$\begin{tikzcd}
M \arrow[r, "n"] & R\Delta M \arrow[r, "R\Delta f"] & R\Delta RS \arrow[r, "Rp"] & RS \\
Mf \arrow[u, "\overline{f}"] \arrow[rr, "x"'] & & RS \arrow[u, "n"'] \arrow[ur, "1"'] &
\end{tikzcd}$$

the rectangle commutes by lax naturality of n, and the triangle by one of the triangle equations; commutativity of the exterior confirms that f is determined by $p.\Delta f$ and $\overline{f}$, that is, by the $p_C f$ and by $\overline{f}$. This completes the construction of restriction $\mathscr{C}$-limits in $\mathbf{X}$.

Suppose conversely that $\mathbf{X}$ has restriction $\mathscr{C}$-limits. For each $S : \mathscr{C} \to \mathbf{X}$, define $R(S)$ to be the restriction limit of S, and define the component at S of the counit $\Delta R \to 1$ to be the restriction limit cone $p : \Delta R(S) \to S$. If $\sigma : S \to T$ is an arrow in $\mathbf{X}^{\mathscr{C}}$, then for each object C of $\mathscr{C}$, let $q_C = \sigma_C p_C : R(S) \to TC$. If $f : C \to D$ is an arrow of $\mathscr{C}$, in the following diagram

$$\begin{tikzcd}
R(S) \arrow[r, "p_C"] & SC \arrow[r, "\sigma_C"] & TC \arrow[dr, "Tf"] & \\
R(S) \arrow[u, "\overline{\sigma_D.Sf.p_C}"] \arrow[r, "p_C"] \arrow[rr, bend right, "p_D"'] & SC \arrow[u, "\overline{\sigma_D.Sf}"] \arrow[r, "Sf"] & SD \arrow[r, "\sigma_D"] & TD
\end{tikzcd}$$

the left square commutes by one of the restriction category axioms, the right rectangle by lax naturality of σ, and the curved region by naturality of p. Finally, $\overline{\sigma_C.Sf.p_C} = \overline{\sigma_D.p_D}$ by naturality of p once again, so $Tf.\sigma_C.p_C.\overline{\sigma_D.p_D} = \sigma_D.p_D$, that is, $Tf.q_C.\overline{q_D} = q_D$, and hence the q form a lax cone. We now define $R(\sigma) : R(S) \to R(T)$ to be the unique arrow for which $\overline{R(\sigma)}$ is the composite of the restriction idempotents $\overline{\sigma_C.p_C}$ and the diagram

$$\begin{tikzcd}
R(S) \arrow[rr, "R(\sigma)"] \arrow[d, "\overline{R(\sigma)}"'] & & R(T) \arrow[d, "p_C"] \\
R(S) \arrow[r, "p_C"'] & SC \arrow[r, "\sigma_C"'] & TC
\end{tikzcd}$$

commutes.

The unit $n : M \to R\Delta M$ is defined to be the unique arrow satisfying $p_C n = 1$ for each leg $p_C : R\Delta M \to M$ of the restriction limit cone of ΔM.

We leave to the reader the various straightforward verifications: that R is a restriction functor, that the unit and counit are lax natural, and that the triangle equations hold. $\square$

Proposition 4.12. A restriction category $\mathbf{X}$ has all (finite) restriction limits if and only if $\mathbf{X}$ is split as a restriction category and $\mathsf{Total}(\mathbf{X})$ has finite limits.

Proof. We have already seen that $\mathsf{Total}(\mathbf{X})$ has $\mathscr{C}$-limits if $\mathbf{X}$ has restriction $\mathscr{C}$-limits; and that restriction idempotents split in $\mathbf{X}$ if $\mathbf{X}$ has restriction $\mathbb{2}$-limits. Thus it remains to show that if $\mathbf{X}$ is a split restriction category and $\mathsf{Total}(\mathbf{X})$ has finite limits, then $\mathbf{X}$ has restriction limits.

Let $S : \mathscr{C} \to \mathbf{X}$ be given. Define a new functor $S' : \mathscr{C} \to \mathbf{X}$ as follows. For an object C of $\mathscr{C}$ let e_C be the composite of all the restriction idempotents $\overline{Sf}$ where $f : C \to D$ is an arrow in $\mathscr{C}$ with domain C. Let $i_C : S'C \to SC$ and $r_C : SC \to S'C$ be the splitting of e_C. Given an arrow $f : C \to D$, we have $e_D.Sf.e_C = Sf.e_C$, so Sf restricts to an arrow $S'f : S'C \to S'D$ satisfying $i_D.S'f = Sf.i_C$; and this defines a functor $S' : \mathscr{C} \to \mathbf{X}$ with a natural transformation $i : S' \to S$. Since S' lands in $\mathsf{Total}(\mathbf{X})$, we may form its limit $p_C : L \to S'C$ in $\mathsf{Total}(\mathbf{X})$, and now $i_C p_C : L \to SC$ give a cone over S with total components; we shall show that it is a restriction limit cone.

Let $q_C : M \to SC$ be the components of a lax cone over S, and write $d : M \to M$ for the composite of the restriction idempotents $\overline{q_C}$. We must show that there is a unique arrow $f : M \to L$ satisfying $i_C.p_C.f = q_C.d$. Let $i : M' \to M$ and $r : M \to M'$ be a splitting of d. Each composite $q_C i$ is total, and for an arrow $f : C \to D$ in $\mathscr{C}$ we have $q_D.i = Sf.q_C.\overline{q_D}i = Sf.q_C.i$; thus the $q_C i$ form the components of a cone. For each $f : C \to D$ we have $\overline{Sf}.q_C.i = q_C.i.\overline{Sf.q_C.i} = q_C.i.\overline{q_D.i} = q_C.i$, so $i_C.r_C.q_C.i = q_C.i$; but this means that $r_C.q_C.i$ is total, and forms a cone over S'. Thus, by the universal property of the limit $p_C : L \to S'C$, there is a unique total map $g : M' \to L$ satisfying $p_C.g = r_C.q_C.i$. Now $i_C.p_C.gr = i_C.r_C.q_C.ir = q_C.ir = q_C.d$, so gr shows the existence of an f.

For the uniqueness, let f be any map satisfying $i_C.p_C.f = q_C.d$; then $\overline{f} = \overline{i_C.p_C.f} = \overline{q_C.d} = \overline{\overline{q_C}.d} = \overline{d} = d$. Now fi satisfies $i_C.p_C.fi = q_C.di = q_C.i$, and $\overline{fi} = \overline{\overline{f}i} = \overline{di} = \overline{i} = 1$. Thus, by the universal property of the limit L in $\mathsf{Total}(\mathbf{X})$, we have $fi = g$, and now $f = f\overline{f} = fd = fir = gr$. $\square$

Finally, we observe that under a further assumption, restriction products and splitting of restriction idempotents suffice to obtain all (finite) limits in the restriction category.

Let $\mathbf{X}$ be a cartesian restriction category. We say that an object X is *separable* if the diagonal $\Delta : X \to X \times X$ is a restriction monic; that is, if there is a map $r : X \times X \to X$ with $r\Delta = 1$ and $\Delta r = \overline{r}$. (Recall that such an r is unique if it exists, and is called the restriction retraction of Δ.)

Proposition 4.13. If $\mathbf{X}$ is a split cartesian restriction category in which every object is separable, then $\mathsf{Total}(\mathbf{X})$ has all finite limits.

Proof. We already know that $\mathsf{Total}(\mathbf{X})$ has finite products; it remains to show that it has equalisers. Suppose then that $f, g : X \to Y$ are given in $\mathsf{Total}(\mathbf{X})$, and let $h : X \to Y \times Y$ be the induced map and $r : Y \times Y \to Y$ be the restriction retraction of $\Delta : Y \to Y \times Y$. Now consider the restriction idempotent $\overline{rh}$, and let $i : E \to X$ and $s : X \to E$ be its splitting. We shall show that i is the desired equaliser. First, i has a retraction, so is a monomorphism, and so in turn is total. We must show that $fi = gi$, and that if j is any total map with $fj = gj$, then j factorises through i.

Write $p, q : Y \times Y \to Y$ for the projections. Observe first that

$$p\bar{r} = p\Delta r = r = q\Delta r = q\bar{r}$$

and now

$$fi = fisi = f\overline{rhi} = ph\overline{rhi} = p\bar{r}hi = q\bar{r}hi = qh\overline{rhi} = gisi = gi.$$

On the other hand, if j is total and $fj = gj$, then $hj = \Delta k$ for a (unique total) map k, so

$$isj = \overline{rh}j = j\overline{rhj} = j\overline{r\Delta k} = j\bar{k} = j,$$

and sj gives the required factorisation of j through i. □

5. Counital copy categories

We have seen that there are many different ways of describing the structure we call a restriction category with restriction products, but we shall actually add one more to this list: the *counital copy categories*, which we introduce in this section. (The slightly weaker structure of *copy category* will not be considered in this paper.)

5.1. *Restriction products revisited*

The starting point is the observation that if $\mathbf{X}$ is a restriction category with restriction products, then, as a category, $\mathbf{X}$ has a symmetric monoidal structure, with tensor product given by restriction product. The associativity isomorphism is the α appearing in the definition of a p-category, while the symmetry is the τ. The unit is the restriction terminal object, and the unit constraint $X \times T \cong X$ is the projection. In light of the coherence results for monoidal categories (Mac Lane 1971), we shall omit explicit mention of the associativity isomorphisms, and write as if the tensor product were strictly associative.

As observed in Carboni (1987), for each object X, the diagonal map $\Delta : X \to X \times X$ is coassociative and cocommutative, and has a counit given by $t_X : X \to T$. Thus, every object has a canonical cocommutative comonoid structure in the symmetric monoidal category. Furthermore, since the Δ are natural, every morphism $f : X \to Y$ is a morphism of cosemigroups, although it may not preserve the counit. It will preserve the counit if it is a total map; conversely, if f preserves the counit, that is, if $t_Y f = t_X$, then $\bar{f} = \overline{t_Y f} = \overline{t_X} = 1$, and hence f is total. Thus the total maps are precisely the counit-preserving ones.

There are two further further conditions that necessarily hold in a restriction category with restriction products: the diagonal $\Delta : T \to T \times T$ must be inverse to the unit isomorphism $r = l : T \times T \to T$ of the monoidal structure, and the composite

$$X \times Y \xrightarrow{\Delta \times \Delta} X \times X \times Y \times Y \xrightarrow{1 \times \tau \times 1} X \times Y \times X \times Y$$

must be $\Delta : X \times Y \to X \times Y \times X \times Y$. Together, these conditions say that $\Delta : X \to X \times X$ is not just a natural transformation, but a *monoidal natural transformation*; it can also be viewed as an instance of the 'middle four interchange' law for bicategories. To see that these conditions must hold in a restriction category with restriction products, it suffices to observe that they are all equations in $\mathsf{Total}(\mathbf{X})$, where the tensor product is a genuine product, and that such equations always hold in a symmetric monoidal category for which the tensor product is the categorical (cartesian) product. We call a symmetric monoidal category equipped with maps $\Delta : X \to X \times X$ that are monoidally natural, coassociative, cocommutative and have counits, a *counital copy category*. (We shall have cause to look in Cockett and Lack (in preparation) at a slightly weaker structure, called a *copy category*, in which the assumption that the cosemigroups have counits is dropped.) It will turn out that a symmetric monoidal category can have at most one counital copy structure, and has such a structure if and only if it arises from a restriction category with restriction products.

If $\mathscr{V}$ is an arbitrary symmetric monoidal category, let $\mathsf{Copy}(\mathscr{V})$ be the category whose objects are the cocommutative comonoids in $\mathscr{V}$ and whose morphisms are the homomorphisms of cosemigroups. Then $\mathsf{Copy}(\mathscr{V})$ has a canonical symmetric monoidal structure: the tensor product of cocommutative comonoids $(C, \delta : C \to C \otimes C, \epsilon : C \to I)$ and $(D, \delta : D \to D \otimes D, \epsilon : D \to I)$ has underlying $\mathscr{V}$-object $C \otimes D$, with comultiplication and counit given by

$$C \otimes D \xrightarrow{\delta \otimes \delta} C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes \tau \otimes 1} C \otimes D \otimes C \otimes D$$

$$C \otimes D \xrightarrow{\epsilon \otimes \epsilon} I \otimes I \xrightarrow{r} I.$$

But now the map $\delta : C \to C \otimes C$ in $\mathscr{V}$ is a map $\Delta : (C, \delta, \epsilon) \to (C, \delta, \epsilon) \otimes (C, \delta, \epsilon)$ in $\mathsf{Copy}(\mathscr{V})$, which is coassociative, cocommutative and counital by definition of the objects of $\mathsf{Copy}(\mathscr{V})$, natural by definition of morphisms in $\mathsf{Copy}(\mathscr{V})$, and monoidally natural by definition of the monoidal structure on $\mathsf{Copy}(\mathscr{V})$. Thus $\mathsf{Copy}(\mathscr{V})$ is a counital copy category. On the other hand, there is an evident forgetful functor $U : \mathsf{Copy}(\mathscr{V}) \to \mathscr{V}$ that strictly preserves the symmetric monoidal structure, and if $\mathscr{V}$ *is* a counital copy category, this U is clearly an equivalence of categories. This proves the following proposition.

Proposition 5.1. The counital copy categories are precisely the symmetric monoidal categories of the form $\mathsf{Copy}(\mathscr{V})$ for some symmetric monoidal $\mathscr{V}$.

We then have the following theorem.

Theorem 5.2. The following structures on a category $\mathbf{X}$ are equivalent:

(i) restriction category with restriction products;
(ii) p-category with a one-element object;
(iii) partial cartesian category in the sense of Curien and Obtulowicz;
(iv) counital copy category;
(v) symmetric monoidal structure with $U : \mathsf{Copy}(\mathbf{X}) \to \mathbf{X}$ an equivalence;
(vi) symmetric monoidal structure for which there exists some equivalence $\mathbf{X} \simeq \mathsf{Copy}(\mathscr{V})$.

All the structure is determined by either the restriction category structure or the symmetric monoidal structure.

5.2. *Classified restriction categories and equational lifting categories*

In this brief section we revisit the analysis of classified restriction categories given in Cockett and Lack (2003): in particular, their connection with the *equational lifting monads* of Bucalo *et al.* (1999).

Let $\mathscr{C}$ be a symmetric monoidal category with tensor product $\otimes$, unit I and symmetry τ. The associativity and unit isomorphisms will be suppressed where possible. A *symmetric monoidal monad* (Kock 1972) on $\mathscr{C}$ is a monad $T = (T, \eta, \mu)$ equipped with a natural transformation $\varphi_{A,B} : TA \otimes TB \to T(A \otimes B)$ satisfying the equations

$$\begin{tikzcd} TA \otimes TB \arrow[r, "\varphi_{A,B}"] \arrow[d, "\tau"'] & T(A \otimes B) \arrow[d, "\tau"] \\ TB \otimes TA \arrow[r, "\varphi_{B,A}"'] & T(B \otimes A) \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[r, "\eta_A \otimes \eta_B"] \arrow[dr, "\eta_{A \otimes B}"'] & TA \otimes TB \arrow[d, "\varphi_{A,B}"] \\ & T(A \otimes B) \end{tikzcd}$$

$$\begin{tikzcd} T^2A \otimes T^2B \arrow[r, "\varphi_{TA,TB}"] \arrow[d, "\mu_A \otimes \mu_B"'] & T(TA \otimes TB) \arrow[r, "T\varphi_{A,B}"] & T^2(A \otimes B) \arrow[d, "\mu_{A \otimes B}"] \\ TA \otimes TB \arrow[rr, "\varphi_{A,B}"'] & & T(A \otimes B). \end{tikzcd}$$

In fact, the structure on T of symmetric monoidal monad can be given either by φ or by a natural family of maps $\psi_{A,B} : A \times TB \to T(A \times B)$ satisfying equations given in Kock (1972). One obtains $\psi_{A,B}$ from $\varphi_{A,B}$ by composing with $\eta_A \times 1_{TB}$, and one obtains $\varphi_{A,B}$ from $\psi_{A,B}$ as the composite

$$\begin{tikzcd} TA \times TB \arrow[r, "\psi_{TA,B}"] & T(TA \times B) \arrow[r, "T\tau"] & T(B \times TA) \arrow[r, "T\psi_{B,A}"] & T^2(B \times A) \arrow[r, "T^2\tau"] & T^2(A \times B) \arrow[d, "\mu_{A \times B}"] \\ & & & & T(A \times B). \end{tikzcd}$$

When the maps ψ rather than φ are used, one sometimes speaks of a *commutative strong monad* rather than a *symmetric monoidal monad.*

A symmetric monoidal monad T on $\mathscr{C}$ induces a symmetric monoidal structure on the Kleisli category $\mathscr{C}_T$. If we regard the objects of $\mathscr{C}_T$ as being the objects of $\mathscr{C}$, and arrows in $\mathscr{C}_T$ from A to B as being arrows in $\mathscr{C}$ from A to TB, then the product of objects A and A' is $A \otimes A'$, while the product of arrows $f : A \to TB$ and $f' : A' \to TB'$ is the composite

$$A \otimes A' \xrightarrow{f \otimes f'} TB \otimes TB' \xrightarrow{\varphi_{B,B'}} T(B \otimes B').$$

The left adjoint $I : \mathscr{C} \to \mathscr{C}_T$ strictly preserves the symmetric monoidal structure.

If $\mathscr{C}$ is not just symmetric monoidal but a counital copy category, then, applying $I : \mathscr{C} \to \mathscr{C}_T$ to the cocommutative comonoid structures on objects of $\mathscr{C}$, one obtains a canonical cocommutative structure on each object of $\mathscr{C}_T$. If the resulting copy maps $A \to A \otimes A$ in $\mathscr{C}_T$ are natural, they will certainly be monoidally natural, so $\mathscr{C}_T$ will be a counital copy category. As for the naturality, this amounts to commutativity of the exterior of

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "\Delta"'] & & TB \arrow[dl, "\Delta"'] \arrow[d, "T\Delta"] \\
A \otimes A \arrow[r, "f\otimes f"'] & TB \otimes TB \arrow[r, "\varphi_{B,B}"'] & T(B \otimes B)
\end{tikzcd}
$$

for every $f : A \to TB$ in $\mathscr{C}$. Now the quadrilateral commutes by naturality of the copy maps in $\mathscr{C}$, so the exterior will commute provided the triangular region does so. We therefore define a symmetric monoidal monad T on a counital copy category $\mathscr{C}$ to be a *copy monad* if $\varphi_{B,B}\Delta = T\Delta$ for all objects B.

Proposition 5.3. If T is a copy monad on a counital copy category $\mathscr{C}$, then $\mathscr{C}_T$ is a counital copy category.

Example 5.4. If $\mathscr{D}$ is a distributive category, then the monad $+1$ on $\mathscr{D}$ is a symmetric monoidal monad, via the maps

$$(A+1)\times(B+1) \xrightarrow{\delta^{-1}} A\times B + A + B + 1 \xrightarrow{A\times B+!} A\times B + 1.$$

The fact that $+1$ is a copy monad amounts to commutativity of the exterior of

$$
\begin{tikzcd}
(A+1)\times(A+1) \arrow[r, "\delta^{-1}"] & A\times A + A + A + 1 \arrow[dr, "A\times A+!"] & \\
A+1 \arrow[u, "\Delta"] \arrow[r, "\Delta+1"'] & A\times A+1 \arrow[u, "i_{14}"] \arrow[r, "1"'] & A\times A+1
\end{tikzcd}
$$

It follows that $\mathscr{D}_{+1}$ is a counital copy category, and so that $\mathsf{Total}(K_r(\mathscr{D}_{+1}))$ has finite products.

In Bucalo *et al.* (1999), a symmetric monoidal monad on a category with finite products $\mathscr{C}$ is called an *equational lifting monad* if $\psi_{A,B} : A\times TB \to T(A\times B)$ satisfies

$$
\begin{tikzcd}
TA \arrow[r, "\Delta"] \arrow[d, "T\Delta"'] & TA\times TA \arrow[d, "\psi_{TA,A}"] \\
T(A\times A) \arrow[r, "T(\eta_A\times 1)"'] & T(TA\times A)
\end{tikzcd}
$$

We observe that this implies commutativity of

$$
\begin{tikzcd}
TA \arrow[r, "T\Delta"] \arrow[rr, bend left, "T\Delta"] \arrow[d, "\Delta"'] & T(A\times A) \arrow[r, "T\tau"] \arrow[d, "T(\eta_A\times 1)"] & T(A\times A) \arrow[d, "T(1\times\eta_A)"'] \arrow[dr, "T\eta_{A\times A}"'] \arrow[drr, "1"] & & \\
TA\times TA \arrow[r, "\psi_{TA,A}"] \arrow[rrrr, bend right, "\varphi_{A,A}"'] & T(TA\times A) \arrow[r, "T\tau"] & T(A\times TA) \arrow[r, "T\psi_{A,A}"'] & T^2(A\times A) \arrow[r, "\mu_{A\times A}"] & T(A\times A)
\end{tikzcd}
$$

which is to say that T is a copy monad. Thus every equational lifting monad is a copy monad.

Question 5.5. Is there a copy monad on a category with finite products that is not an equational lifting monad?

In Cockett and Lack (2003), we defined the notion of a *classifying monad* on a category $\mathscr{C}$ and gave various characterisations. To give a monad T the structure of a classifying monad is to give its Kleisli category $\mathscr{C}_T$ the structure of a restriction category for which $F_T : \mathscr{C} \to \mathscr{C}_T$ takes its values among the total maps and the components of the counit $\epsilon_T : F_T U_T \to 1$ are restriction retractions. We now prove the following proposition.

Proposition 5.6. An equational lifting monad is a classifying monad.

Proof. We have already seen that $\mathscr{C}_T$ is a counital copy category, and thus, in particular, a restriction category. The restriction of $f : A \to TB$ is given by

$$A \xrightarrow{\langle 1,f\rangle} A \times TB \xrightarrow{\eta_A \times 1} TA \times TB \xrightarrow{\varphi_{A,B}} T(A \times B) \xrightarrow{Tp} TA.$$

By Cockett and Lack (2003, Proposition 3.15), T will be a classifying monad if and only if the restriction of $\eta_B f : A \to TB$ is η_A, for each $f : A \to B$ in $\mathscr{C}$; and the restriction of $1 : TA \to TA$ is $T\eta_A :: TA \to T^2A$. The restriction of $\eta_B f : A \to TB$ is given by

$$A \xrightarrow{\langle 1,\eta_B f\rangle} A \times TB \xrightarrow{\eta_A \times 1_T B} TA \times TB \xrightarrow{\varphi_{A,B}} T(A \times B) \xrightarrow{Tp} TA$$

and

$$\begin{aligned} T(p)\varphi_{A,B}(\eta_A \times 1_{TB})\langle 1, \eta_B f\rangle &= T(p)\varphi_{A,B}(\eta_A \times \eta_B)\langle 1, f\rangle \\ &= T(p)\eta_{A\times B}\langle 1, f\rangle \\ &= \eta_A p\langle 1, f\rangle \\ &= \eta_A, \end{aligned}$$

as required. For the latter, the restriction of $1 : TA \to TA$ is

$$TA \xrightarrow{\Delta} TA \times TA \xrightarrow{\eta_{TA} \times 1} T^2A \times TA \xrightarrow{\varphi_{TA,A}} T(TA \times A) \xrightarrow{Tp} T^2A$$

and

$$T(p)\varphi_{TA,A}(\eta_{TA} \times 1)\Delta = T(p)\psi_{A,A}\Delta = T(p)T(\eta_A \times 1)T(\Delta) = T\eta_A,$$

as required. □

5.3. *Distributive copy categories*

A counital copy category $\mathbf{X}$ is a restriction category with restriction products; if $\mathbf{X}$ also has restriction coproducts and the canonical maps $\delta : A \times B + A \times C \to A \times (B + C)$ are invertible for all objects A, B and C, we call $\mathbf{X}$ a *distributive copy category*.

Proposition 5.7. For a counital copy category $\mathbf{X}$ with restriction coproducts, the following are equivalent:

(i) $\mathbf{X}$ is a distributive copy category;
(ii) $\mathsf{Total}(\mathbf{X})$ is a distributive category;
(iii) $K_r(\mathbf{X})$ is a distributive copy category;
(iv) $\mathsf{Total}(K_r(\mathbf{X}))$ is a distributive category.

Proof. The equivalence of (i) and (ii) is immediate from the fact that restriction products and restriction coproducts in $\mathbf{X}$ are products and coproducts in $\mathsf{Total}(\mathbf{X})$; the equivalence of (iii) and (iv) is a special case of this. The fact that (iii) implies (i) is trivial; it remains only to show that if the canonical map $A\times B+A\times C\to A\times(B+C)$ is invertible for every object in $\mathbf{X}$, then it is invertible for every object in $K_r(\mathbf{X})$. This follows easily from the fact that the objects of $K_r(\mathbf{X})$ are retracts of the objects of $\mathbf{X}$. $\square$

Since in a distributive category the unique map $0\to A\times 0$ is invertible for any object A, the proposition implies that the same is true for a distributive copy category.

Our main result about distributive copy categories is the following theorem.

Theorem 5.8. If $\mathbf{X}$ is a counital copy category with restriction coproducts, then $\mathbf{X}$ is an extensive restriction category if and only if it is a distributive copy category and has a restriction zero.

Proof. If $\mathbf{X}$ is an extensive restriction category with restriction products then it has a restriction zero by definition of extensivity for restriction categories; and $\mathsf{Total}(K_r(\mathbf{X}))$ is extensive with finite products, and hence distributive, so $\mathbf{X}$ is a distributive copy category by the proposition.

Suppose conversely that $\mathbf{X}$ is a distributive copy category with a restriction zero. We must show that every map $f:C\to 1+1$ has a decision. Let h be the composite

$$C\xrightarrow{\Delta}C\times C\xrightarrow{C\times f}C\times(1+1)\xrightarrow{\delta^{-1}}C+C.$$

Then $\nabla h=p(C\times f)\Delta=p(C\times\overline{f})\Delta=p\Delta\overline{(C\times\overline{f})\Delta}=p\Delta\overline{f}=\overline{f}$, giving one condition for h to be an f-decision. The second follows from commutativity of:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\times C \arrow[r, "C\times f"] \arrow[d, "\Delta\times C"'] & C\times(1+1) \arrow[r, "\delta^{-1}"] \arrow[d, "\Delta\times(1+1)"'] & C+C \arrow[d, "\Delta+\Delta"] \\
C\times C \arrow[r, "C\times\Delta"'] \arrow[dr, "C\times f"'] & C\times C\times C \arrow[r, "C\times C\times f"'] & C\times C\times(1+1) \arrow[r, "\delta^{-1}"] \arrow[d, "C\times f\times(1+1)"'] & C\times C+C\times C \arrow[d, "C\times f+C\times f"] \\
 & C\times(1+1) \arrow[r, "C\times\Delta"'] \arrow[d, "\delta^{-1}"'] & C\times(1+1)\times(1+1) \arrow[r, "\delta^{-1}"'] & C\times(1+1)+C\times(1+1) \arrow[d, "\delta^{-1}+\delta^{-1}"] \\
 & C+C \arrow[rr, "i+j"'] & & C+C+C+C.
\end{tikzcd}$$

$\square$

Thus we have the following examples of distributive copy categories.

Example 5.9.

(i) For a distributive category $\mathscr{D}$ we saw in Example 5.4 that $\mathscr{D}_{+1}$ is a counital copy category and in Example 3.3 that it is an extensive restriction category. By the theorem, then, $\mathscr{D}_{+1}$ is a distributive copy category.

(ii) If $\mathscr{V}$ is a symmetric monoidal category, $\mathsf{Copy}(\mathscr{V})$ is a counital copy category. If $\mathscr{V}$ also has coproducts, and the tensor product distributes over the coproducts, then $\mathsf{Copy}(\mathscr{V})$ is a distributive copy category.

(iii) The category CRng of commutative rings can of course be regarded as the category of commutative monoids in the monoidal category Ab of abelian groups. We can therefore regard $\mathsf{CRng}^{\mathrm{op}}$as the category of cocommutative comonoids in the monoidal category $\mathsf{Ab}^{\mathrm{op}}$. Now the tensor product in $\mathsf{Ab}^{\mathrm{op}}$distributes over coproducts, so $\mathsf{Copy}(\mathsf{Ab}^{\mathrm{op}})$ is a distributive copy category. An object of $\mathsf{Copy}(\mathsf{Ab}^{\mathrm{op}})$ is a cocommutative comonoid in $\mathsf{Ab}^{\mathrm{op}}$; that is, a commutative ring. In fact $\mathsf{Copy}(\mathsf{Ab}^{\mathrm{op}})$ is just $\mathsf{CRng}_{\times}^{\mathrm{op}}$, where $\mathsf{CRng}_{\times}$ is the category whose objects are the commutative rings and whose morphisms are the functions preserving $+$, $\times$ and 0, but not necessarily preserving 1. Thus $\mathsf{CRng}_{\times}^{\mathrm{op}}$ is a distributive copy category. It is not hard to see that idempotents split in $\mathsf{CRng}_{\times}^{\mathrm{op}}$, and that the category of total maps is just $\mathsf{CRng}^{\mathrm{op}}$, and thus we recover the well-known fact that $\mathsf{CRng}^{\mathrm{op}}$ is extensive.

5.4. *The extensive completion of a distributive category*

In this section we apply the results obtained above to give a description of the extensive completion of a distributive category. There is a 2-category Dist of distributive categories, functors preserving finite products and coproducts, and natural transformations; and there is a full sub-2-category $\mathsf{Ext}_{\mathrm{pr}}$ of Dist consisting of those distributive categories that are also extensive. The inclusion has a left biadjoint, and the value at a distributive category $\mathscr{D}$ of this left biadjoint is what we mean by the extensive completion of the distributive category $\mathscr{D}$. An explicit construction of the extensive completion was given in Cockett and Lack (2001); here we shall give an alternative, more conceptual, description.

Given a distributive category $\mathscr{D}$, we have seen that there is a monad $+1$ on $\mathscr{D}$ whose Kleisli category $\mathscr{D}_{+1}$ has a restriction structure. We may now split the restriction idempotents in $\mathscr{D}_{+1}$, and then take the total maps in this new restriction category, to give a category $\mathsf{Total}(K_r(\mathscr{D}_{+1}))$. The image of the left adjoint $\mathscr{D} \to \mathscr{D}_{+1}$ lands in $\mathsf{Total}(\mathscr{D}_{+1})$, and if we compose the resulting functor $I : \mathscr{D} \to \mathsf{Total}(\mathscr{D}_{+1})$ with the map $\mathsf{Total}(J) : \mathsf{Total}(\mathscr{D}_{+1}) \to \mathsf{Total}(K_r(\mathscr{D}_{+1}))$ induced by the inclusion $J : \mathscr{D}_{+1} \to K_r(\mathscr{D}_{+1})$, we obtain a functor $N : \mathscr{D} \to \mathsf{Total}(K_r(\mathscr{D}_{+1}))$. It turns out that $N : \mathscr{D} \to \mathsf{Total}(K_r(\mathscr{D}_{+1}))$ exhibits $\mathsf{Total}(K_r(\mathscr{D}_{+1}))$ as the extensive completion of $\mathscr{D}$, as we shall see below.

We saw in Example 3.3 that $\mathsf{Total}(K_r(\mathscr{D}_{+1}))$ is extensive, and we saw in Example 5.4 that it has finite products, and so lies in $\mathsf{Ext}_{\mathrm{pr}}$. The inclusion $J : \mathscr{D}_{+1} \to K_r(\mathscr{D}_{+1})$ preserves restriction products and restriction coproducts, so the induced map $\mathsf{Total}(J) :$ $\mathsf{Total}(\mathscr{D}_{+1}) \to \mathsf{Total}(K_r(\mathscr{D}_{+1}))$ preserves products and coproducts. The left adjoint $\mathscr{D} \to$ $\mathscr{D}_{+1}$ preserves coproducts, and the inclusion $\mathsf{Total}(\mathscr{D}_{+1}) \to \mathscr{D}_{+1}$ preserves and reflects them, so $I : \mathscr{D} \to \mathsf{Total}(\mathscr{D}_{+1})$ preserves coproducts. On the other hand, the left adjoint $\mathscr{D} \to \mathscr{D}_{+1}$ sends products to restriction products, so $I : \mathscr{D} \to \mathsf{Total}(\mathscr{D}_{+1})$ also preserves products. Thus $N : \mathscr{D} \to \mathsf{Total}(K_r(\mathscr{D}_{+1}))$ preserves products and coproducts, and hence is a morphism in Dist.

It remains to check the universal property. To do this, we use the theory of *effective completions of classifying monads* developed in Cockett and Lack (2003, Section 5). Recall that a restriction category $\mathbf{X}$ is *classified* if the inclusion $\mathsf{Total}(\mathbf{X}) \to \mathbf{X}$ has a right adjoint R for which the components $\epsilon_A : RA \to A$ are restriction retractions. The induced comonad on $\mathbf{X}$ is called the *classifying comonad*. A monad T on a category $\mathscr{C}$ was defined in Cockett and Lack (2003) to be a *classifying monad* if it is equipped with the requisite structure to make the Kleisli category $\mathscr{C}_T$ into a classified restriction category whose classifying comonad is the comonad induced by the Kleisli adjunction. The classifying monad T is said to be *effective* if the restriction category $\mathscr{C}_T$ is split and the left adjoint $F_T : \mathscr{C} \to \mathscr{C}_T$ exhibits $\mathscr{C}$ as the category of total maps in $\mathscr{C}_T$. In other words, a classifying monad is effective if it is the partial map classifier for a category of partial maps. Various characterisations of effective classifying monads were given in Cockett and Lack (2003, Theorem 5.8).

Given a classifying monad T on a category $\mathscr{C}$, the restriction category $\mathscr{C}_T$ is classified; the split restriction category $K_r(\mathscr{C}_T)$ need not be classified in general, although it will be if T is an *interpreted classifying monad* in the sense of Cockett and Lack (2003, Section 4). The precise details of this definition are unimportant in the present context, but it *is* important to know that the monad $+1$ on a distributive category $\mathscr{D}$ is an interpreted classifying monad, as observed in Cockett and Lack (2003, Example 4.15). For a general classifying monad T there is, nonetheless, a universal way to obtain a split classified restriction category from $\mathscr{C}_T$: it is obtained by splitting more idempotents than just the restriction ones, and is denoted by $K_{\mathrm{cr}}(\mathscr{C}_T)$ – see Cockett and Lack (2003, Section 3.3).

Since $K_{\mathrm{cr}}(\mathscr{C}_T)$ is a split classified restriction category, the induced monad we get on $\mathsf{Total}(K_{\mathrm{cr}}(\mathscr{C}_T))$ is an effective classifying monad. It is, in fact, the universal way of associating an effective classifying monad to the classifying monad T, in a sense made precise in Cockett and Lack (2003, Section 5), and so is called the *effective completion* of the classifying monad T. In the case where the monad T is interpreted (such as the monad $+1$ on a distributive category), the effective completion may be described more simply as $\mathsf{Total}(K_r(\mathscr{C}_T))$. We shall use the universal property of the effective completion to show that $\mathsf{Total}(K_r(\mathscr{D}_{+1}))$ is the extensive completion of the distributive category $\mathscr{D}$.

Consider distributive categories $\mathscr{D}$ and $\mathscr{E}$ equipped with the corresponding interpreted classifying monads $+1$. A morphism of classifying monads (in the sense of Cockett and Lack (2003)) from $(\mathscr{D}, +1)$ to $(\mathscr{E}, +1)$ consists of a functor $H : \mathscr{D} \to \mathscr{E}$ equipped with a family of maps $\varphi : H(A+1) \to HA + 1$ natural in A and rendering the diagrams

$$
\begin{tikzcd}
HA \arrow[r, "Hi_A"] \arrow[dr, "i_{HA}"'] & H(A+1) \arrow[d, "\varphi_A"] \\
& HA+1
\end{tikzcd}
\qquad
\begin{tikzcd}
H(A+1+1) \arrow[r, "\varphi_{A+1}"] \arrow[d, "H(A+\nabla)"'] & H(A+1)+1 \arrow[r, "\varphi_A+1"] & HA+1+1 \arrow[d, "HA+\nabla"] \\
H(A+1) \arrow[rr, "\varphi_A"'] & & HA+1
\end{tikzcd}
$$

$$\begin{tikzcd} HA \arrow[r, "H\langle A,f\rangle"] \arrow[d, "\langle HA,Hf\rangle"'] & H(A\times(B+1)) \arrow[r, "H\delta^{-1}"] & H(A\times B+A) \arrow[r, "H(\pi_1+!)"] & H(A+1) \arrow[d, "\varphi_A"] \\ HA\times H(B+1) \arrow[r, "HA\times\varphi_B"'] & HA\times(HB+1) \arrow[r, "\delta^{-1}"'] & HA\times HB+HA \arrow[r, "\pi_1+!"'] & HA+1 \end{tikzcd}$$

commutative for all objects A and all morphisms $f : A \to B+1$. A straightforward argument shows that the condition involving a morphism $f : A \to B+1$ holds for all such f if and only if it holds for all f with $B = 1$; that is, for all $a : A \to 1+1$. The resulting diagram is

$$\begin{tikzcd} HA \arrow[r, "H\langle A,a\rangle"] \arrow[d, "\langle HA,Ha\rangle"'] & H(A\times(1+1)) \arrow[r, "H\delta^{-1}"] & H(A+A) \arrow[r, "H(A+!)"] & H(A+1) \arrow[d, "\varphi_A"] \\ HA\times H(1+1) \arrow[r, "HA\times\varphi_1"'] & HA\times(H1+1) \arrow[r, "\delta^{-1}"'] & HA\times H1+HA \arrow[r, "\pi_1+!"'] & HA+1 \end{tikzcd}$$

Given morphisms (H,φ) and (K,ψ) of classifying monads, a transformation from (H,φ) to (K,ψ) consists of a natural transformation $\alpha : H \to K$ rendering

$$\begin{tikzcd} H(A+1) \arrow[r, "\varphi_A"] \arrow[d, "\alpha_{A+1}"'] & HA+1 \arrow[d, "\alpha_A+1"] \\ K(A+1) \arrow[r, "\psi_A"'] & KA+1 \end{tikzcd}$$

commutative for all A. There is now a 2-category $\mathsf{Dist}_{\mathrm{cl}}$ consisting of the distributive categories, the morphisms of classifying monads and the transformations of these. It is a full sub-2-category of the 2-categories icMnd and cMnd defined in Cockett and Lack (2003). The biadjunctions constructed in Cockett and Lack (2003, Section 5.2) now show that $N : \mathscr{D} \to \mathsf{Total}(K_r(\mathscr{D}_{+1}))$ has a canonical structure of morphism of classifying monads, and exhibits $\mathsf{Total}(K_r(\mathscr{D}_{+1}))$ as the bireflection of $\mathscr{D}$ into the full sub-2-category of $\mathsf{Dist}_{\mathrm{cl}}$ consisting of the extensive categories with finite products.

We now turn to an analysis of the notion of a morphism of classifying monads. Suppose, as above, that $\mathscr{D}$ and $\mathscr{E}$ are distributive.

Lemma 5.10. If $H : \mathscr{D} \to \mathscr{E}$ preserves finite coproducts, there is a unique φ making H into a morphism of classifying monads, namely $HA+! : HA + H1 \to HA + 1$.

Proof. If $\varphi_A : HA + H1 \to HA + 1$ makes H into such a morphism, it must have the form $\langle \alpha_A | \beta_A \rangle$ where $\alpha_A : HA \to HA + 1$ and $\beta_A : H1 \to HA + 1$ are both natural in A. Compatibility of α_A with the first injection gives $\alpha_A = i_{HA}$, while naturality of β_A gives commutativity of

$$\begin{tikzcd} H1 \arrow[r, "\beta_0"] \arrow[dr, "\beta_A"'] & H0+1 \arrow[d, "H!+1"] \\ & HA+1 \end{tikzcd}$$

But $H0$ is initial, so $\beta_0 : H1 \to H0 + 1$ is the unique map, and φ_A must be $HA{+}!$, as claimed.

Conversely, we must show that $\varphi_A = HA{+}!$ satisfies the various conditions. It is clearly natural and satisfies the compatibility conditions with $i_A : A \to A + 1$ and $A + \nabla : A + 1 + 1 \to A + 1$, so we need only show compatibility with maps $a : A \to 1 + 1$. Commutativity of

$$
\begin{tikzcd}
HA \times H1 + HA \times H1 \arrow[r, "\delta"] \arrow[d, "\pi_1+\pi_1"'] & HA \times (H1 + H1) \arrow[r, "HA\times\theta"] & HA \times H(1+1) \\
HA + HA \arrow[r, "\theta"'] & H(A+A) \arrow[r, "H\delta"'] & H(A \times (1+1)) \arrow[u, "\langle H\pi_1, H\pi_2\rangle"']
\end{tikzcd}
$$

where θ denotes the canonical isomorphisms expressing the fact that H preserves coproducts, gives commutativity of

$$
\begin{tikzcd}
 & & HA \times (H1+1) \arrow[r, "\delta^{-1}"] & HA \times H1 + HA \arrow[ddr, "\pi_1+!"] & \\
 & HA \times H(1+1) \arrow[ur, "HA\times\varphi_1"] \arrow[r, "HA\times\theta^{-1}"] & HA \times (H1+H1) \arrow[u, "HA\times(H1+!)"'] \arrow[r, "\delta^{-1}"'] & HA \times H1 + HA \times H1 \arrow[u, "1+\pi_1"] \arrow[d, "\pi_1+\pi_1"'] \arrow[dr, "\pi_1+!"] & \\
HA \arrow[ur, "\langle HA, Ha\rangle"] \arrow[r, "H\langle A,a\rangle"'] & H(A \times (1+1)) \arrow[u, "\langle H\pi_1, H\pi_2\rangle"'] \arrow[r, "H\delta^{-1}"'] & H(A+A) \arrow[r, "\theta^{-1}"] \arrow[dr, "H(A+!)"'] & HA + HA \arrow[r, "HA+!"] \arrow[dr, "HA+H!"] & HA + 1 \\
 & & & H(A+1) \arrow[r, "\theta^{-1}"'] & HA + H1 \arrow[u, "HA+!"']
\end{tikzcd}
$$

$\square$

Corollary 5.11. If $H, K : \mathscr{D} \to \mathscr{E}$ preserve finite coproducts, any natural transformation $\alpha : H \to K$ is a transformation of classifying monads.

On the other hand, the coproduct-preserving functors are not the only morphisms of classifying monads, as shown by the following example.

Example 5.12. If X is any object of $\mathscr{E}$, the constant functor $\Delta X : \mathscr{D} \to \mathscr{E}$ at X becomes a morphism of classifying monads if we define $\varphi_A : H(A+1) \to HA + 1$ to be the injection $X \to X + 1$ for any A.

We may now prove the following theorem.

Theorem 5.13. The functor $N : \mathscr{D} \to \mathsf{Total}(K_r(\mathscr{D}_{+1}))$ exhibits $\mathsf{Total}(K_r(\mathscr{D}_{+1}))$ as the extensive completion of the distributive category $\mathscr{D}$.

Proof. We know from Cockett and Lack (2003) that composition with N induces, for any extensive category $\mathscr{E}$ with products, an equivalence between the category of morphisms of classifying monads from $\mathscr{D}$ to $\mathscr{E}$ and the category of morphisms of classifying monads from $\mathsf{Total}(K_r(\mathscr{D}_{+1}))$ to $\mathscr{E}$. It remains to prove that if $G : \mathscr{D} \to \mathscr{E}$ preserves finite products and coproducts, and $(H, \varphi) : \mathsf{Total}(K_r(\mathscr{D}_{+1})) \to \mathscr{E}$ is the induced morphism of classifying monads, then H preserves finite products and coproducts. But since H may be constructed as the composite of $\mathsf{Total}(K_r(G_{+1})) : \mathsf{Total}(K_r(\mathscr{D}_{+1})) \to \mathsf{Total}(K_r(\mathscr{E}_{+1}))$ and the canonical equivalence $\mathsf{Total}(K_r(\mathscr{E}_{+1})) \to \mathscr{E}$, it will suffice to prove that $\mathsf{Total}(K_r(G_{+1}))$ preserves finite products and coproducts. Since G preserves coproducts, G_{+1} preserves restriction coproducts, and thus $\mathsf{Total}(K_r(G_{+1}))$ preserves coproducts by Proposition 2.2. Similarly, G_{+1} preserves restriction products, and hence $\mathsf{Total}(K_r(G_{+1}))$ preserves products by Proposition 4.4. □

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