Half-Positional Determinacy of Infinite Games

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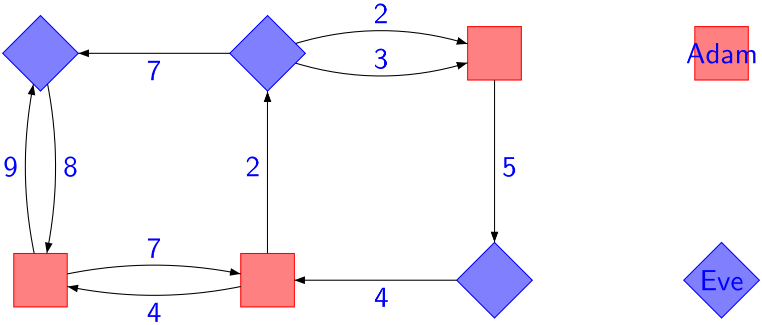
May 25, 2006

Games



Eve wins iff the greatest number appearing infinitely often during an infinite play is even.

Games

Game = arena + winning condition

Arena:
$G = (\mathrm{Pos}_A, \mathrm{Pos}_E, C, \mathrm{Mov})$ where $\mathrm{Pos} = \mathrm{Pos}_A \cup \mathrm{Pos}_E$,
$\mathrm{Mov} \subseteq \mathrm{Pos} \times \mathrm{Pos} \times C$

Winning condition:
Subset $W \subseteq C^\omega$; we assume that it is prefix independent, i.e.
$u \in W \iff cu \in W$

Plays and strategies

A play π is a sequence of moves such that $\mathrm{source}(\pi_{n+1}) = \mathrm{target}(\pi_n)$.

A strategy for Eve (Adam) is a partial function $s : \mathrm{Pos} \cup \mathrm{Mov}^* \to \mathrm{Mov}$ which tells Eve (Adam) what they should do in a given situation (the current position, moves so far).

A strategy s is winning for X if each play consistent with s is winning for X.

A strategy s is positional if $s(\pi)$ depends only on $\mathrm{target}(\pi)$.

Definition

A game (G, W) is determined if for each starting position one of players has a winning strategy. (Not all games are determined.) If the game is determined, we have $\mathrm{Pos} = \mathrm{Win}_E \cup \mathrm{Win}_A$ and strategies s_E and s_A such that each play π with $\mathrm{source}(\pi) \in \mathrm{Win}_X$ and consistent with s_X is winning for X.

Determinacy types

Definition

A determinacy type is given by three parameters:

- admissible strategies for Eve (positional, arbitrary)
- admissible strategies for Adam (positional, arbitrary)
- admissible arenas (finite, infinite)

Definition

A winning condition W is (α, β, γ)-determined, if for each γ-arena G the game (G, W) is (α, β)-determined, i.e. for each starting position either Eve has a winning α-strategy or Adam has a winning β-strategy.

Half-positional conditions

For short, we call (positional, arbitrary, infinite)-determined conditions half-positional, and (positional, arbitrary, finite)-determined conditions finitely half-positional.

We will focus on half-positional and finitely half-positional winning conditions, but first we will present some facts about all determinacy types.

Let D be a determinacy type.

Theorem

Let $W \subseteq C^\omega$ be a winning condition such that for each nonempty D-arena G over C exists a nonempty set $M \subseteq G$ such that in the game (G, W) one of the players has a D-strategy winning from M (i.e. from each starting position in M). Then W is D-determined.

Facts about D-determinacy

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Theorem

Let $W \subseteq C^\omega$ be a D-determined winning condition, and $S \subseteq C$. Then $W \cup (C^* S)^\omega$ also is a D-determined winning condition.

Union

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Let $\mathcal{W}$ be a (finite, countable) family of (finitely) half-positional conditions. Is $\bigcup \mathcal{W}$ also a half-positional winning condition?

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We know already that a union of any half-positional condition and $(C^*S)^\omega$ is half-positional.

There is a subclass of half-positional conditions called positional/suspendable conditions. A countable union of such winning conditions is also positional/suspendable.

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A union of an uncountable family of half-positional conditions need not be half-positional — even for Büchi conditions:

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Arena: One Eve's position E and ω Adam's positions (A_n)
In E Eve chooses n and moves to A_n.
In A_n Adam chooses r and returns to E. This move is colored with (n, r).

For each $f : \omega \to \omega$, W_f is the Büchi condition given by $S_f = \{(n, f(n)) : n \in \omega\}$: Eve wins W_f if Adam uses moves colored with S_f infinitely many times.

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Eve can win $\bigcup_{f:\omega\to\omega} W_f$, but only if she uses a non-positional strategy.

Examples of half-positional winning conditions

Concavity

Definition

A winning condition W is convex if for all sequences of words (u_n), $u_n \in C^*$, if

- $u_1 u_3 u_5 u_7 \ldots \in W$,
- $u_2 u_4 u_6 u_8 \ldots \in W$,

then $u_1 u_2 u_3 u_4 \ldots \in W$.

A winning condition is concave if its complement is convex.

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Not all half-positional conditions are concave.

Geometrical conditions

Let $C = [0,1]^n$.
For $u \in C^+$, let $P(u)$ be the average color of u, i.e.

$$P(u) = \frac{1}{|u|} \sum_{k=1}^{|u|} u_k. \tag{1}$$

For $w \in C^\omega$, let $P_n(w) = P(w_{|n})$.

Let A be a subset of C. Let $WF(A) \subset C^\omega$ be a set of w such that each cluster point of $(P_n(w))$ is an element of A, and $WF'(A)$ be a set of w such that at least one cluster point of $(P_n(w))$ is an element of A.

For which A's $WF(A)$ and $WF'(A)$ are half-positional?

Half-positional determinacy vs geometry

For which A's $WF(A)$ and $WF'(A)$ are half-positional?

No.	A	condition	finite	infinite	concavity
0	trivial	$WF'(A)$ or $WF(A)$	yes	yes	yes
1	not co-convex	$WF'(A)$ or $WF(A)$	no	no	no
2	co-convex	$WF'(A)$	yes	no	yes
3	co-convex, not open	$WF(A)$	yes?	no	weak only
4	co-convex, open	$WF(A)$	yes?	yes?	weak only
5	open half-space	$WF(A)$	yes	yes	weak only

Monotonic automata

Consider the languages: $C^*a^nC^*$, $C^*a^{n-1}bC^*$, $C^*ba^{n-1}C^*$ over $C = \{a, b, c\}$.

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They can be recognized by a deterministic finite automaton satisfying the following special conditions:

- The set of states $Q = \{0, \dots, n\}$;
- 0 is the initial state, n is the only accepting state;
- The transition function σ is monotonic, i.e. $q \geq q'$ implies $\sigma(q, c) \geq \sigma(q', c)$.

We call such an automaton a monotonic automaton $A = (n, \sigma)$ over C.

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For $L_A = C^* a^2 C^*$ the resulting WM_A is not concave: $(babab)^\omega$ is a combination of $(bbbaa)^\omega$ and $(aabbb)^\omega$. (However, monotonic conditions are weakly concave.)

the end

thank you