

Detectability of labeled weighted (max-plus) automata

Kuize Zhang

Received: date / Accepted: date

Abstract Discrete-event systems (DESs) are generally composed of transitions between discrete states caused by spontaneous occurrences of partially-observed events. Detectability is a fundamental property in partially-observed dynamical systems, which describes whether one can use several observed output sequences to determine the internal states of a system. In this paper, we extend results on four fundamental notions of detectability, i.e., strong (periodic) detectability (SD and SPD) and weak (periodic) detectability (WD and WPD), from finite-state automata (FSAs) to weighted automata (WAs) and max-plus automata (MPAs), and show essentially different features on the notions in different classes of automata. FSAs are a widely studied untimed model of DESs, while WAs and MPAs can be regarded as timed models of DESs. It is known that SD and SPD of FSAs can be verified in P, while the problems of verifying WD and WPD of FSAs are PSPACE-complete. The contributions of the current paper are as follows. Firstly, we extend the notions of concurrent composition, observer, and detector from FSAs to WAs, and use them to give equivalent conditions for the four notions of detectability of WAs. Secondly, we prove that for a max-plus automaton \mathcal{A}^{mp} over semiring $\mathbb{N} \cup \{-\infty\}$, the self-composition and detector of \mathcal{A}^{mp} can be computed in NP, but the observer of \mathcal{A}^{mp} can be computed in 2-EXPTIME. As a result, we prove that SD and SPD of \mathcal{A}^{mp} can be verified in coNP, while WD and WPD can be verified in 2-EXPTIME. In addition, we also prove that the problems of verifying SD and SPD of \mathcal{A}^{mp} is coNP-hard.

Keywords labeled weighted automaton · labeled max-plus automaton · detectability · observer · detector · concurrent composition · complexity

This paper was partially supported by the Alexander von Humboldt Foundation.

K. Zhang
Control Systems Group, Technical University of Berlin, 10587 Berlin, Germany
E-mail: zkz0017@163.com, kuzhan@kth.se

1 Introduction

1.1 Literature review

The state detection problem of dynamical systems has been a fundamental problem in both computer science [12] and control science [5] since the 1950s and the 1960s, respectively. *Detectability* is a basic property of dynamic systems: when it holds one can use the *current* and *past* values of an observed output sequence generated by a system to reconstruct its *current* state [4, 17, 16, 14, 23]. This property plays a fundamental role in many related control problems such as observer design and controller synthesis. Hence for different applications, it is meaningful to characterize different notions of detectability.

For *discrete-event systems* (DESS) modeled by *finite-state automata* and *labeled Petri nets*, the detectability problem has been widely studied, see related results on finite-state automata [17, 16, 19, 21], and also see related results on labeled Petri nets [20, 11, 22], and on labeled bounded Petri nets [9]. Detectability has also been studied for probabilistic finite-state automata [6, 18].

In the above models, either logic models (finite-state automata and labeled Petri nets), or probabilistic finite-state automata, are untimed models. In such models, the time consumption of the occurrence of an event is not taken into account. Hence, when doing state estimation, which is a crucial step in determining a state, all subsequent states reachable through unobservable transitions are also considered. So, such a state estimate is not so accurate sometimes. Hence in order to record the time consumptions of occurrences of events, timed models are adopted, e.g., max-plus automata [3, 1, 7]. Recently, the notions of detectability have also been characterized for an interesting class of labeled unambiguous weighted automata [8].

1.2 Results in the literature

Two basic definitions are *strong detectability* and *weak detectability* [17]. The former implies that there exists a positive integer k such that for *every* infinite-length trajectory, each prefix of its output sequence of length no less than k allows reconstructing the current state. The latter weakens the former by changing “*every*” to “*some*”. Strong detectability and *strong periodic detectability* (a variant of strong detectability, requiring to determine states periodically along all output sequences) can be verified in polynomial time based on a *detector* method [16] (a variant of the classical powerset construction) under two widely-used assumptions of deadlock-freeness (which implies a system can always run) and divergence-freeness, i.e., having no unobservable reachable cycle (which implies the running of a system will always be observed). Strong detectability can also be verified in polynomial time without any assumption by a *concurrent-composition* method [21]. Unlike strong detectability, an exponential-time verification algorithm for weak detectability and *weak periodic detectability* (a variant of weak detectability, requiring to determine states periodically along some output sequence) based on an *observer* method (a type of powerset con-

struction) is designed in [17]. More precisely, it is PSPACE-complete to verify weak (periodic) detectability [19, 10].

For labeled Petri nets with inhibitor arcs, weak detectability is undecidable [20]. For labeled Petri nets, strong detectability is decidable under the above two assumptions, and it is EXPSpace-hard to verify strong detectability, but weak detectability is undecidable [11], which strengthens the related undecidability result proved in [20]; later, the decidability result for strong detectability is strengthened to hold under only the second of the above two assumptions [24].

Recently, the notion of observer has been extended to labeled unambiguous weighted automata [8] which is also of exponential complexity in the size of an unambiguous weighted automaton. By using the extended notion of observer, strong (periodic) detectability and weak (periodic) detectability are verified in exponential time, under the previously mentioned two assumptions.

1.3 Contribution of the paper

In this paper, we mainly characterize the four notions of detectability for labeled weighted automata and labeled max-plus automata, without any assumption. The first contribution is to extend the mathematical tools used to characterize the notions of detectability.

1. We extend the notions of concurrent composition, observer, and detector to labeled weighted automata. We use the notion of concurrent composition to give an equivalent condition for strong detectability, use the notion of observer to give equivalent conditions for weak detectability and weak periodic detectability, and use the notion of detector to give an equivalent condition for strong periodic detectability, all for labeled weighted automata.

Unlike unambiguous weighted automata in which for any state q and event sequence s there is at most one path labeled by s leading to q from an initial state, the notions of concurrent composition, observer, and detector may not be computed for general labeled weighted automata.

The subsequent contributions are on labeled max-plus automata over semiring $\mathbb{N} := \mathbb{N} \cup \{-\infty\}$.

2. For a labeled max-plus automaton \mathcal{A}^{mp} over \mathbb{N} , we prove that its observer can be computed in 2-EXPTIME, both its detector and its self-composition can be computed in NP, all in the size of \mathcal{A}^{mp} .
3. We prove that weak detectability and weak periodic detectability of \mathcal{A}^{mp} can be verified in 2-EXPTIME in the size of \mathcal{A}^{mp} . We also prove that the problems of verifying strong detectability and strong periodic detectability of \mathcal{A}^{mp} are both coNP-complete (see Tab. 1), where the coNP-hardness result even holds for labeled max-plus automata over $\mathbb{N} := \mathbb{N} \cup \{-\infty\}$.

	strong (periodic) detectability	weak (periodic) detectability
finite-state automata	P [16, 21] NL-hard [10]	PSPACE-complete [19]
labeled max-plus automata over semiring $\mathbb{N} \cup \{-\infty\}$	coNP-complete (Thms. 7, 11, 12)	2-EXPTIME (Thms. 9, 10)
labeled unambiguous weighed automata	EXPTIME [8]	EXPTIME [8]

Table 1 Results on complexity of verifying four notions of detectability of automata, where the results in [16, 8] are based on two widely-used assumptions of deadlock-freeness and divergence-freeness.

2 Preliminaries

2.1 Notations

Symbols \mathbb{N} , \mathbb{Z} , \mathbb{Z}_+ , and \mathbb{Q} denote the sets of natural numbers, integers, positive integers, and rational numbers, respectively. For a finite alphabet Σ , Σ^* and Σ^ω are used to denote the set of finite-length sequences (called *words*) of elements of Σ including the empty word ϵ and the set of infinite-length sequences (called *configurations*) of elements of Σ , respectively. As usual, $\Sigma^+ := \Sigma^* \setminus \{\epsilon\}$. For a word $s \in \Sigma^*$, $|s|$ stands for its length, and we set $|s'| = +\infty$ for all $s' \in \Sigma^\omega$. For $s \in \Sigma^+$ and natural number k , s^k and s^ω denote the concatenations of k copies of s and infinitely many copies of s , respectively. For a word (configuration) $s \in \Sigma^*$ (Σ^ω), a word $s' \in \Sigma^*$ is called a *prefix* of s , denoted as $s' \sqsubset s$, if there exists another word (configuration) $s'' \in \Sigma^*$ (Σ^ω) such that $s = s's''$. For two natural numbers $i \leq j$, $[i, j]$ denotes the set of all integers no less than i and no greater than j ; and for a set S , $|S|$ its cardinality and 2^S its power set.

We will use the known NP-complete *exact path length* (EPL) problem and *subset sum* (SS) problem in the literature to prove some of the main results.

2.2 The exact path length problem

Consider a k -dimensional weighted directed graph $G = (\mathbb{Z}^k, V, E)$, where $k \in \mathbb{Z}_+$, V is a finite set of vertices, $E \subset V \times \mathbb{Z}^k \times V$ a finite set of weighted edges with weights in \mathbb{Z}^k . For a path $v_1 \xrightarrow{z_1} \dots \xrightarrow{z_{n-1}} v_n$, its weight is defined by $\sum_{i=1}^{n-1} z_i$. The EPL problem [13] is stated as follows.

Problem 1 (EPL) Given a positive integer k , a k -dimensional weighted directed graph $G = (\mathbb{Z}^k, V, E)$, two vertices $v_1, v_2 \in V$, and a vector $z \in \mathbb{Z}^k$, determine whether there is a path from v_1 to v_2 with weight z .

We set as usual that for a natural number n , the size $\text{size}(n)$ of n to be the length of its binary representation, i.e., $\text{size}(n) = \lceil \log_2(n+1) \rceil$ if $n > 0$ ($\lceil \cdot \rceil$ is the ceiling function as usual), and $\text{size}(0) = 1$; for a negative integer $-n$, $\text{size}(-n) = \text{size}(n) + 1$ (here 1 is used to denote “-”); then for a vector $z \in \mathbb{Z}^k$, its size is the sum of the sizes of its entries. The size of an instance (k, G, v_1, v_2, z) of the EPL problem is defined by $\text{size}(k) + \text{size}(G) + 2 + \text{size}(z)$, where $\text{size}(G) = |V| + \text{size}(E)$, $\text{size}(E) = \sum_{(v_1, z', v_2) \in E} (2 + \text{size}(z'))$.

Proposition 1 ([13]) *The EPL problem is NP-complete. Particularly for fixed dimension $k = 1$, there is a pseudo-polynomial-time solution to the problem.*

2.3 The subset sum problem

The SS problem [2] is as follows.

Problem 2 (SS) Given positive integers n_1, \dots, n_m , and N , determine whether $N = \sum_{i \in I} n_i$ for some $I \subset [1, m]$.

Proposition 2 ([2]) *The SS problem is NP-complete.*

2.4 Weighted automata over semirings

A *semiring* is a tuple $\mathfrak{R} = (T, \oplus, \otimes, \mathbf{0}, \mathbf{1})$, where (T, \oplus) is a commutative monoid with identity element $\mathbf{0} \in T$, (T, \otimes) is a monoid with identity element $\mathbf{1} \in T$, $\mathbf{0} \otimes a = a \otimes \mathbf{0} = \mathbf{0}$ (i.e., $\mathbf{0}$ is the zero element of \mathfrak{R}), $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$, $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$ for all $a, b, c \in T$.

A *weighted automaton* is a tuple $\mathfrak{G} = (Q, E, \alpha, \mu)$ over semiring \mathfrak{R} , denoted by $(\mathfrak{R}, \mathfrak{G})$ for short, where Q is a finite set of *states*, E a finite *alphabet*, α is a map from Q to T , $q \in Q$ is called *initial* if $\alpha(q) \neq \mathbf{0}$, and denote the set of initial states by $Q_0 := \{q \in Q \mid \alpha(q) \neq \mathbf{0}\}$, map $\mu : E \rightarrow T^{Q \times Q}$ assigns to each *letter/event* $e \in E$ a *transition map/matrix* $\mu(e) : Q \times Q \rightarrow T$, where $(\mu(e))(q, q') \neq \mathbf{0}$ (also written as $\mu(e)_{qq'} \neq \mathbf{0}$) if and only if there is a *transition* from q to q' caused by occurrence of e , where such a transition is denoted by $q \xrightarrow{e/\mu(e)_{qq'}} q'$. For all $q \in Q$, we also regard $q \xrightarrow{e/\mathbf{1}} q$ as a transition, and $\mu(e) \equiv \mathbf{1}$. A transition $q \xrightarrow{e/\mu(e)_{qq'}} q'$ is called *instantaneous* if $\mu(e)_{qq'} = \mathbf{1}$, and called *noninstantaneous* otherwise. From now on, without loss of generality, we assume for each initial state $q_0 \in Q_0$, $\alpha(q_0) = \mathbf{1}$, because otherwise we can add a new transition $q_0 \xrightarrow{e/\alpha(q_0)} q_0$ such that $e \notin E$ and then reset $\alpha(q_0)$ to be equal to $\mathbf{1}$.

Particularly for \mathfrak{G} over *max-plus semiring* $\mathbb{N} := (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, which is called *max-plus automaton* $(\mathbb{N}, \mathfrak{G})$, for initial state $q \in Q_0$, $\alpha(q)$ denotes its initial time delay, and in a transition $q \xrightarrow{e/\mu(e)_{qq'}} q'$, $\mu(e)_{qq'}$ denotes its time delay. Hence the occurrence of an instantaneous transition has time delay 0, i.e., does not cost time. While the occurrence of a noninstantaneous transition has time delay a positive integer $\mu(e)_{qq'}$, i.e., costs time $\mu(e)_{qq'}$. As pointed out before, without loss of generality, we assume $\alpha(q_0) = 0$ for all $q_0 \in Q_0$.

2.5 Languages

For $q_0, \dots, q_n \in Q$ and $e_1, \dots, e_n \in E$, $n \in \mathbb{Z}_+$, we call

$$\pi := q_0 \xrightarrow{e_1} q_1 \xrightarrow{e_2} \dots \xrightarrow{e_n} q_n \quad (1)$$

a *path* if for all $i \in [0, n-1]$, $\mu(e_{i+1})_{q_i q_{i+1}} \neq \mathbf{0}$. A path π is called *simple* if q_0, \dots, q_{n-1} are pairwise different. A path π is called a *cycle* if $q_0 = q_n$. The set of paths starting at $q_0 \in Q$ and ending at $q \in Q$ (under event sequence $s \in E^+$) is denoted by $q_0 \rightsquigarrow q$ ($q_0 \xrightarrow{s} q$).

The *timed word* of path π is defined by

$$\tau(\pi) := (e_1, t_1)(e_2, t_2) \dots (e_n, t_n), \quad (2)$$

where for all $i \in [1, n]$, $t_i = \bigotimes_{j=1}^i \mu(e_j)_{q_{j-1} q_j}$. The *weight* of path π is defined by t_n . A path π is called *instantaneous* if $t_1 = \dots = t_n = \mathbf{1}$, and called *noninstantaneous* otherwise.

For max-plus automaton $(\mathbb{N}, \mathfrak{G})$, one has $t_i = \sum_{j=1}^i \mu(e_j)_{q_{j-1} q_j}$, hence weight t_i of path π can be used to denote the total time consumption of the occurrences of events e_1, \dots, e_i successively at state q_0 , $i \in [1, n]$.

The *timed language* $L(\mathfrak{R}, \mathfrak{G})$ generated by weighted automaton $(\mathfrak{R}, \mathfrak{G})$ is denoted by the set of the timed words of all paths of $(\mathfrak{R}, \mathfrak{G})$.

Analogously, for $q_0, q_1, \dots \in Q$ and $e_1, e_2, \dots \in E$, where $q_0 \in Q_0$, we call

$$\pi := q_0 \xrightarrow{e_1} q_1 \xrightarrow{e_2} \dots \quad (3)$$

an *infinite path* if for all $i \in \mathbb{N}$, $\mu(e_{i+1})_{q_i q_{i+1}} \neq \mathbf{0}$. The ω -*timed word* of infinite path π is defined by

$$\tau(\pi) := (e_1, t_1)(e_2, t_2) \dots, \quad (4)$$

where for all $i \in \mathbb{Z}_+$, $t_i = \bigotimes_{j=1}^i \mu(e_j)_{q_{j-1} q_j}$.

The ω -*timed language* $L^\omega(\mathfrak{R}, \mathfrak{G})$ generated by weighted automaton $(\mathfrak{R}, \mathfrak{G})$ is denoted by the set of the ω -timed words of all infinite paths of $(\mathfrak{R}, \mathfrak{G})$.

We define a labeling function $\ell : E \rightarrow \Sigma \cup \{\epsilon\}$, which represents when event $e \in E$ occurs, $\ell(e)$ will be observed if $\ell(e) \neq \epsilon$ (in this case we call e *observable*); while nothing will be observed if $\ell(e) = \epsilon$ (in this case we call e *unobservable*). A transition $q \xrightarrow{e/\mu(e)_{qq'}} q'$ is called *observable* (resp. *unobservable*) if e is observable (resp. unobservable). We denote by E_o and E_{uo} the sets of observable events and unobservable events, respectively. A path π (1) is called *unobservable* if $\ell(e_1 \dots e_n) = \epsilon$, and called *observable* otherwise. A labeled weighted automaton

$$\mathcal{A}^w := (\mathfrak{R}, \mathfrak{G}, \ell) \quad (5)$$

can be regarded as a partially-observed timed discrete-event system. Particularly, we denote a labeled max-plus automaton by

$$\mathcal{A}^{mp} := (\mathbb{N}, \mathfrak{G}, \ell). \quad (6)$$

Labeling function ℓ is recursively extended to $E^* \cup E^\omega \rightarrow \Sigma^* \cup \Sigma^\omega$ as $\ell(e_1 e_2 \dots) = \ell(e_1) \ell(e_2) \dots$. ℓ is also extended as follows: for all $(e, t) \in E \times T$, $\ell((e, t)) = (\ell(e), t)$ if $\ell(e) \neq \epsilon$, and $\ell((e, t)) = \epsilon$ otherwise. Hence ℓ is also recursively extended to $(E \times T)^* \cup (E \times T)^\omega \rightarrow (\Sigma \times T)^* \cup (\Sigma \times T)^\omega$.

The size of a given \mathcal{A}^{mp} is defined by $|Q| + |E| + \text{size}(\alpha) + \text{size}(\mu) + \text{size}(\ell)$, where $\text{size}(-\infty) = 1$, the size of a natural number has already been defined before,

$\text{size}(\alpha) = \sum_{q \in Q} \text{size}(\alpha(q))$, $\text{size}(\mu) = \sum_{e \in E} \sum_{q, q' \in Q} \text{size}(\mu(e)_{qq'})$, $\text{size}(\ell) = 2|\{(e, \ell(e)) | e \in E\}|$.

The *timed language* $\mathcal{L}(\mathcal{A}^w)$ and ω -*timed language* $\mathcal{L}^\omega(\mathcal{A}^w)$ generated by \mathcal{A}^w are defined by

$$\mathcal{L}(\mathcal{A}^w) := \{\gamma \in (\Sigma \times T)^* | (\exists w \in L(\mathfrak{R}, \mathfrak{G}))[\ell(w) = \gamma]\} \text{ and} \quad (7)$$

$$\mathcal{L}^\omega(\mathcal{A}^w) := \{\gamma \in (\Sigma \times T)^\omega | (\exists w \in L^\omega(\mathfrak{R}, \mathfrak{G}))[\ell(w) = \gamma]\}, \quad (8)$$

respectively.

Previously we assume without loss of generality that for each initial state $q_0 \in Q_0$, $\alpha(q_0) = \mathbf{1}$, because otherwise we can add a new transition $q_0 \xrightarrow{e/\alpha(q_0)} q_0$ such that $e \notin E$ and reset $\alpha(q_0)$ to be equal to $\mathbf{1}$. From now on we additionally assume that for all such e , $\ell(e) = \epsilon$ without loss of generality, because if an automaton starts at an initial state $q_0 \in Q_0$, then before the first occurrence of an event, one can observe nothing.

For labeled max-plus automaton $\mathcal{A}^{mp} = (\mathbb{N}, \mathfrak{G}, \ell)$, if it generates a path π as in (1), consider its timed word $\tau(\pi)$ as in (2), then at time t_i , one will observe $\ell(e_i)$ if $\ell(e_i) \neq \epsilon$; and observe nothing otherwise, where $i \in [1, n]$. We simply say one observes $\ell(\tau(\pi))$. With this intuitive observation, we next define the set of states consistent with observations.

3 Main results

3.1 Current-state estimate

For labeled weighed automaton \mathcal{A}^w , for output sequence $\gamma \in (\Sigma \times T)^+$, we define the *current-state estimate* as

$$\begin{aligned} \mathcal{M}(\mathcal{A}^w, \gamma) := \{q \in Q | (\exists q_0 \in Q_0)(\exists s \in E^+)(\exists \pi \in q_0 \xrightarrow{s} q) \\ [(\tau(\pi) = w_1(e_o, t)w_2) \wedge (e_o \in E_o) \wedge \\ (w_2 \in (E_{uo} \times \{t\})^*) \wedge (\ell(\tau(\pi)) = \gamma)]\}. \end{aligned} \quad (9)$$

Particularly for ϵ , we define the *instantaneous initial-state estimate* by

$$\begin{aligned} \mathcal{M}(\mathcal{A}^w, \epsilon) := Q_0 \cup \{q \in Q | (\exists q_0 \in Q_0)(\exists s \in (E_{uo})^+)(\exists \pi \in q_0 \xrightarrow{s} q) \\ [\tau(\pi) \in (E \times \{\mathbf{1}\})^+]\}. \end{aligned} \quad (10)$$

Analogously, for a subset $x \subset Q$, we define its *instantaneous state estimate* by

$$\begin{aligned} \mathcal{M}(\mathcal{A}^w, \epsilon | x) := x \cup \{q \in Q | (\exists q' \in x)(\exists s \in (E_{uo})^+)(\exists \pi \in q' \xrightarrow{s} q) \\ [\tau(\pi) \in (E \times \{\mathbf{1}\})^+]\}. \end{aligned} \quad (11)$$

Intuitively, for $\gamma = (\sigma_1, t_1) \dots (\sigma_n, t_n) \in (\Sigma \times T)^+$, $\mathcal{M}(\mathcal{A}^w, \gamma)$ denotes the set of states \mathcal{A}^w can be in when γ has just been generated by \mathcal{A}^w . While $\mathcal{M}(\mathcal{A}^w, \epsilon)$ denotes the set of states \mathcal{A}^w can be in when no output has been generated, but \mathcal{A}^w

might have started to run. Note that if for some $i \in [1, n]$, $t_i = \mathbf{0}$, then we must have $\mathcal{M}(\mathcal{A}^w, \gamma) = \emptyset$, because in this case $\gamma \notin \mathcal{L}(\mathcal{A}^w)$.

For labeled max-plus automaton $\mathcal{A}^{mp} = (\mathbb{N}, \mathfrak{G}, \ell)$, for $\gamma = (\sigma_1, t_1) \dots (\sigma_n, t_n) \in (\Sigma \times \mathbb{N})^+$, the current-state estimate $\mathcal{M}(\mathcal{A}^{mp}, \gamma)$ describes the set of states \mathcal{A}^{mp} can be in at time t_n if we observe σ_i at time t_i , $i \in [1, n]$. Since the current-state estimate is done at time t_n , after the occurrence of the event (corresponding to e_o in (9)) that generates σ_n , only instantaneous unobservable transitions are considered. For ϵ , $\mathcal{M}(\mathcal{A}^{mp}, \epsilon)$ denotes the set of states \mathcal{A}^{mp} can be in at the initial time when no output has been generated. Note that at the initial time, \mathcal{A}^{mp} might have started to run, so we only consider instantaneous unobservable transitions, which is similar to the consideration of only instantaneous unobservable transitions in (9) after the occurrence of the last observable event e_o .

Apparently, the state estimate defined here is more accurate than the versions for finite-state automata [17] and for labeled unambiguous weighted automata [8], because in the latter two versions, at the observation time, the subsequent states reachable through possible unobservable noninstantaneous transitions are also considered, but they actually have not been reached because the occurrences of these transitions cost time.

3.2 Notions of detectability

In this subsection, we formulate the four fundamental notions of detectability.

Definition 1 (SD) A labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ is called *strongly detectable* if there is $k \in \mathbb{N}$, for every ω -timed word $w \in L^\omega(\mathfrak{R}, \mathfrak{G})$, for each prefix γ of $\ell(w)$, if $|\gamma| \geq k$, then $|\mathcal{M}(\mathcal{A}^w, \gamma)| = 1$.

Definition 2 (SPD) A labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ is called *strongly periodically detectable* if there is $k \in \mathbb{N}$, for every ω -timed word $w \in L^\omega(\mathfrak{R}, \mathfrak{G})$, for every prefix $w' \sqsubset w$, there is $w'' \in (E \times T)^*$ such that $|\ell(w'')| < k$, $w'w'' \sqsubset w$, and $|\mathcal{M}(\mathcal{A}^w, \ell(w'w''))| = 1$.

Definition 3 (WD) A labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ is called *weakly detectable* if $L^\omega(\mathfrak{R}, \mathfrak{G}) \neq \emptyset$ implies that there is $k \in \mathbb{N}$, for some ω -timed word $w \in L^\omega(\mathfrak{R}, \mathfrak{G})$, for each prefix γ of $\ell(w)$, if $|\gamma| \geq k$, then $|\mathcal{M}(\mathcal{A}^w, \gamma)| = 1$.

Definition 4 (WPD) A labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ is called *weakly periodically detectable* if $L^\omega(\mathfrak{R}, \mathfrak{G}) \neq \emptyset$ implies that there is $k \in \mathbb{N}$, for some ω -timed word $w \in L^\omega(\mathfrak{R}, \mathfrak{G})$, for each prefix $w' \sqsubset w$, there is $w'' \in (E \times T)^*$ such that $|\ell(w'')| < k$, $w'w'' \sqsubset w$, and $|\mathcal{M}(\mathcal{A}^w, \ell(w'w''))| = 1$.

One observes that strong detectability and strong periodic detectability are incomparable. Consider a finite-state automaton \mathcal{A}_1 that contains only two states and they are both initial, and on each state, there is a self-loop with an unobservable event. \mathcal{A}_1 is strongly detectable, but not strongly periodically detectable. Consider another finite-state automaton \mathcal{A}_2 that contains three states q_0, q_1, q_2 such that only q_0 is initial, the transitions of \mathcal{A}_2 are $q_0 \xrightarrow{a} q_1$, $q_0 \xrightarrow{a} q_2$, $q_1 \xrightarrow{b} q_0$, $q_2 \xrightarrow{b} q_0$, where a and

b are observable. \mathcal{A}_2 is not strongly detectable but strongly periodically detectable. Particularly, if an automaton \mathcal{A}^w is deadlock-free and has no unobservable cycle, then strong detectability is stronger than strong periodic detectability.

Weak detectability and weak periodic detectability also have similar relations.

For labeled max-plus automaton $\mathcal{A}^{mp} = (\mathbb{N}, \mathcal{G}, \ell)$, if we assume that for every observable transition $q \xrightarrow{e/\mu(e)_{qq'}} q'$, $\mu(e)_{qq'} > 0$, then there will be no two consecutive observable observations occurring at the same time. Hence in Definition 1 and Definition 3, $|\gamma| \geq k$ implies that the time consumption is no less than k . Although this assumption meets practical requirements, we do not need it (actually we do not need any assumption) in order to characterize these notions of detectability.

3.3 Concurrent composition

In order to give an equivalent condition for strong detectability, we define a notion of *concurrent composition* of a labeled weighted automaton \mathcal{A}^w and itself (i.e., the self-composition of \mathcal{A}^w). This notion can be regarded as an extension of the notion of concurrent composition of finite-state automata proposed in [22]. In [21], the notion of concurrent composition is used to give a polynomial-time algorithm for verifying strong detectability of finite-state automata, without any assumption. Although the self-composition of a finite-state automaton \mathcal{A} can be computed in time polynomial of the size of \mathcal{A} , we will show that the self-composition of a max-plus automaton \mathcal{A}^{mp} (6) can be computed in time nondeterministically polynomial of the size of \mathcal{A}^{mp} .

Definition 5 Consider a labeled weighted automaton $\mathcal{A}^w = (\mathfrak{X}, \mathcal{G}, \ell)$. We define its *self-composition* by a nondeterministic finite-state automaton

$$\text{CC}_A(\mathcal{A}^w) = (Q', E', Q'_0, \delta', \Sigma, \ell'), \quad (12)$$

where $Q' = Q \times Q$; $E' = \{(e_1, e_2) \in E_o \times E_o \mid \ell(e_1) = \ell(e_2)\}$; $Q'_0 = Q_0 \times Q_0$; $\delta' \subset Q' \times E' \times Q'$ is the transition relation, for all states $(q_1, q_2), (q_3, q_4) \in Q'$ and events $(e_1, e_2) \in E'$, $((q_1, q_2), (e_1, e_2), (q'_1, q'_2)) \in \delta'$ if and only if in \mathcal{A}^w , there exist states $q_5, q_6, q_7, q_8 \in Q$, event sequences $s_1, s_2, s_3, s_4 \in (E_{uo})^*$, and paths

$$\begin{aligned} \pi_1 &:= q_1 \xrightarrow{s_1} q_5 \xrightarrow{e_1} q_7 \xrightarrow{s_3} q_3, \\ \pi_2 &:= q_2 \xrightarrow{s_2} q_6 \xrightarrow{e_2} q_8 \xrightarrow{s_4} q_4, \end{aligned} \quad (13)$$

such that $\tau(\pi_1) = w_1(e_1, t_1)w_3$, $\tau(\pi_2) = w_2(e_2, t_2)w_4$, $t_1 = t_2$, $w_3, w_4 \in (E_{uo} \times \{t_1\})^*$; for all $(e_1, e_2) \in E'$, $\ell'((e_1, e_2)) = \ell(e_1)$, and ℓ' is recursively extended to $(E')^* \cup (E')^\omega \rightarrow \Sigma^* \cup \Sigma^\omega$. For a state q' of $\text{CC}_A(\mathcal{A}^w)$, we write $q' = (q'(L), q'(R))$.

Intuitively, there is a transition $(q_1, q_2) \xrightarrow{(e_1, e_2)} (q'_1, q'_2)$ in $\text{CC}_A(\mathcal{A}^w)$ if and only if in \mathcal{A}^w , starting from q_1 and q_2 at the same time, after some common time delay, e_1 and e_2 occur as the unique observable events, state q_1 and q_2 can transition to q_3 and q_4 , respectively. Since we consider an observation at exactly the time observable events e_1, e_2 occur, we only consider unobservable instantaneous transitions after the occurrences of e_1, e_2 (see (13)).

3.4 Observer

We next define a notion of *observer* to concatenate current-state estimates. Later, we will use the notion of observer to give equivalent conditions for weak detectability and weak periodic detectability. This notion is an extension of the notion of observer of finite-state automata proposed in [17], in which the latter is used to give exponential-time verification algorithms for the four notions of detectability of finite-state automata, under the two previously mentioned assumptions. For a finite-state automaton \mathcal{A} , its observer can be computed in time exponential of the size of \mathcal{A} (actually powerset construction). While for a max-plus automaton \mathcal{A}^{mp} (6), we will show that its observer can be computed in time doubly exponential of the size of \mathcal{A}^{mp} .

Definition 6 For labeled weighted automaton $\mathcal{A}^w = (\mathfrak{A}, \mathfrak{G}, \ell)$, we define its *observer* as a deterministic finite-state automaton

$$\mathcal{A}_{obs}^w = (X, \Sigma \times T, x_0, \bar{\delta}_{obs}), \quad (14)$$

where $X \subset 2^Q$ is the state set, $\Sigma \times T$ the alphabet, $x_0 = \mathcal{M}(\mathcal{A}^w, \epsilon) \in X$ the unique initial state, $\bar{\delta}_{obs} \subset X \times (\Sigma \times T) \times X$ the transition relation. Note that $\Sigma \times T$ may be infinite. For all $x \subset Q$ different from x_0 , $x \in X$ if and only if there is $\gamma \in (\Sigma \times T)^+$ such that $x = \mathcal{M}(\mathcal{A}^w, \gamma)$. For all $x, x' \in X$ and $(\sigma, t) \in \Sigma \times T$, $(x, (\sigma, t), x') \in \bar{\delta}_{obs}$ if and only if

$$\begin{aligned} x' &= \{q \in Q \mid (\exists q' \in x)(\exists s_1 e_o s_2 \in E^+)(\exists \pi \in q' \xrightarrow{s_1 e_o s_2} q) \\ &\quad [(\ell(s_1 s_2) = \epsilon) \wedge (e_o \in E_o) \wedge (\ell(e_o) = \sigma) \wedge \\ &\quad (\tau(\pi) = w_1(e_o, t)w_2) \wedge (w_2 \in (E \times \{t\})^*)]\} \\ &=: \mathcal{M}(\mathcal{A}^w, (\sigma, t) \mid x). \end{aligned} \quad (15)$$

In Definition 6, after $\bar{\delta}_{obs}$ is recursively extended to $\bar{\delta}_{obs} \subset X \times (\Sigma \times T)^* \times X$ as usual, one has for all $x \in X$ and $(\sigma_1, t_1) \dots (\sigma_n, t_n) =: \gamma \in (\Sigma \times T)^+$, $(x_0, \gamma, x) \in \bar{\delta}_{obs}$ if and only if $\mathcal{M}(\mathcal{A}^w, \gamma') = x$, where $\gamma' = (\sigma_1, t'_1) \dots (\sigma_n, t'_n)$, $t'_i = \bigotimes_{j=1}^i t_j$, $i \in [1, n]$.

On the other hand, the alphabet $\Sigma \times T$ may not be finite, so generally we cannot compute the whole \mathcal{A}_{obs}^w . However, in order to study weak detectability, it is enough to compute a sub-automaton

$$^{sub}\mathcal{A}_{obs}^w = (X, \Sigma_{obs}^T, x_0, \delta_{obs}) \quad (16)$$

of \mathcal{A}_{obs}^w in which we consider a finite subset Σ_{obs}^T of $\Sigma \times T$ such that if there is a transition from $x \in X$ to $x' \in X$ in $\bar{\delta}_{obs}$ then there is also a transition from x to x' in δ_{obs} . Later we call $^{sub}\mathcal{A}_{obs}^w$ observer instead of \mathcal{A}_{obs}^w . Note that δ_{obs} may not be unique but can be finite.

3.5 Detector

In order to give an equivalent condition for strong periodic detectability, we define a notion of *detector*, which can be regarded as a simplified version of an observer. The notion of detector can also be regarded as an extension of the notion of detector of finite-state automata proposed in [16], in which the latter is used to give polynomial-time verification algorithms for strong detectability and strong periodic detectability of finite-state automata, also under two previously mentioned assumptions. The detector of a finite-state automaton \mathcal{A} can be computed in time polynomial of the size of \mathcal{A} . While we will show that the detector of a max-plus automaton \mathcal{A}^{mp} (6) can be computed in time nondeterministically polynomial of the size of \mathcal{A}^{mp} .

Definition 7 For labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$, we define its *detector* as a nondeterministic finite-state automaton

$$\mathcal{A}_{det}^w = (X, \Sigma \times T, x_0, \bar{\delta}_{det}), \quad (17)$$

where $X = \{x_0\} \cup \{x \subset Q \mid 1 \leq |x| \leq 2\}$ is the state set, $\Sigma \times T$ the alphabet, $x_0 = \mathcal{M}(\mathcal{A}^w, \epsilon)$ the unique initial state, $\bar{\delta}_{det} \subset X \times (\Sigma \times T) \times X$ the transition relation. For all $x \in X$, $x_0 \neq x' = \{q'_1, q'_2\} \in X$, and $(\sigma, t) \in \Sigma \times T$, where q'_1 may be equal to q'_2 , $(x, (\sigma, t), x') \in \bar{\delta}_{det}$ if and only if there exist $q_1, q_2 \in x$ (q_1 may be equal to q_2) such that there exist paths

$$q_1 \xrightarrow{s_1 e_1} q''_1 \xrightarrow{s'_1} q'_1, \quad (18)$$

$$q_2 \xrightarrow{s_2 e_2} q''_2 \xrightarrow{s'_2} q'_2 \quad (19)$$

such that $s_1, s'_1, s_2, s'_2 \in (E_{uo})^*$, $e_1, e_2 \in E_o$, $\ell(e_1) = \ell(e_2) = \sigma$, the weights of paths $q_1 \xrightarrow{s_1 e_1} q''_1$ and $q_2 \xrightarrow{s_2 e_2} q''_2$ are both equal to t , and paths $q''_1 \xrightarrow{s'_1} q'_1$ and $q''_2 \xrightarrow{s'_2} q'_2$ are unobservable and instantaneous.

Note that the alphabet $\Sigma \times T$ may not be finite either, so similarly to the notion of observer, we consider sub-automaton

$$^{sub}\mathcal{A}_{det}^w = (X, \Sigma_{det}^T, x_0, \delta_{det}) \quad (20)$$

of \mathcal{A}_{det}^w in which we consider a finite subset Σ_{det}^T of $\Sigma \times T$ such that if there is a transition from $x \in X$ to $x' \in X$ in $\bar{\delta}_{det}$ then there is also a transition from x to x' in δ_{det} . Later we call $^{sub}\mathcal{A}_{det}^w$ detector instead of \mathcal{A}_{det}^w .

For the relationship between observer $^{sub}\mathcal{A}_{obs}^w$ and detector $^{sub}\mathcal{A}_{det}^w$, we have the following proposition.

Proposition 3 Consider a labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ (5), its observer $^{sub}\mathcal{A}_{obs}^w$ (16) and detector $^{sub}\mathcal{A}_{det}^w$ (20). For every transition $(x, (\sigma, t), x') \in \delta_{obs}$, for every $\bar{x}' \subset x'$ satisfying $|\bar{x}'| = 2$ if $|x'| \geq 2$ and $|\bar{x}'| = 1$ otherwise, there is $\bar{x} \subset x$ such that (1) $|\bar{x}| = 2$ and $(\bar{x}, (\sigma, t), \bar{x}') \in \delta_{det}$ if $|x| > 1$ and (2) $|\bar{x}| = 1$ and $(\bar{x}, (\sigma, t), \bar{x}') \in \delta_{det}$ if $|x| = 1$.

Proof We only need to prove the case $|x| \geq 2$ and $|x'| \geq 2$, the other cases hold similarly. Arbitrarily choose $\{q_1, q_2\} = \bar{x}' \subset x'$ such that $q_1 \neq q_2$. By definition, there exist $q_3, q'_3, q_4, q_5 \in Q$, $e_1, e_2 \in E_o$, $s_1, s_2, s_3, s_4 \in (E_{uo})^*$, and paths

$$\begin{aligned} q_3 &\xrightarrow{s_1 e_1} q_4 \xrightarrow{s_3} q_1, \\ q'_3 &\xrightarrow{s_2 e_2} q_5 \xrightarrow{s_4} q_2 \end{aligned}$$

such that $\ell(e_1) = \ell(e_2) = \sigma$, the weights of paths $q_3 \xrightarrow{s_1 e_1} q_4$ and $q'_3 \xrightarrow{s_2 e_2} q_5$ are both equal to t , and paths $q_4 \xrightarrow{s_3} q_1$ and $q_5 \xrightarrow{s_4} q_2$ are unobservable and instantaneous. If $q_3 = q'_3$, we choose $\bar{x} = \{q_3, q_6\}$, where $q_6 \in x \setminus \{q_3\}$; otherwise, we choose $\bar{x} = \{q_3, q'_3\}$. Then by definition, one has $(\bar{x}, \sigma, \bar{x}') \in \delta_{det}$. \square

3.6 Equivalent conditions for detectability of weighted automata

In this subsection, we give equivalent conditions for the four notions of detectability of weighted automata by using the notions of observer, detector, and self-composition.

3.6.1 For strong detectability:

We next use the notion of self-composition to give an equivalent condition for strong detectability of weighted automata.

Theorem 1 A labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ (5) is not strongly detectable if and only if in its self-composition $\text{CC}_A(\mathcal{A}^w)$ (12),

there exists a transition sequence

$$q'_0 \xrightarrow{s'_1} q'_1 \xrightarrow{s'_2} q'_1 \xrightarrow{s'_3} q'_2 \text{ satisfying} \quad (21a)$$

$$q'_0 \in Q'_0; q'_1, q'_2 \in Q'; s'_1, s'_2, s'_3 \in (E')^+; q'_2(L) \neq q'_2(R); \quad (21b)$$

$$\text{and in } \mathcal{A}^w, \text{ there exists a cycle reachable from } q'_2(L). \quad (21c)$$

Proof By Definition 1, \mathcal{A}^w is not strongly detectable if and only if for all $k \in \mathbb{N}$, there exist $w_k \in L^\omega(\mathfrak{R}, \mathfrak{G})$ and $\gamma \sqsubset \ell(w_k)$, such that $|\gamma| \geq k$ and $|\mathcal{M}(\mathcal{A}^w, \gamma)| > 1$.

“if”: Arbitrarily given $k \in \mathbb{Z}_+$, consider $q'_0 \xrightarrow{s'_1} q'_1 \xrightarrow{(s'_2)^k} q'_1 \xrightarrow{s'_3} q'_2$, then by (21b), in \mathcal{A}^w there exists a path $q'_0(L) \xrightarrow{\bar{s}_1} q'_1(L) \xrightarrow{\bar{s}_2} q'_1(L) \xrightarrow{\bar{s}_3} q'_2(L) =: \pi_L$ such that $\ell(\bar{s}_1) = \ell'(s'_1)$, $\ell(\bar{s}_2) = \ell'((s'_2)^k)$, $\ell(\bar{s}_3) = \ell'(s'_3)$, and $\mathcal{M}(\mathcal{A}^w, \gamma) \supset \{q'_2(L), q'_2(R)\}$, where $\gamma = \ell(\tau(\pi_L))$; by (21c), there also exists a path $q'_2(L) \xrightarrow{\bar{s}_4} q_3 \xrightarrow{\bar{s}_5} q_3$, where $\bar{s}_5 \in E^+$. Note that $q_3 \xrightarrow{\bar{s}_5} q_3$ can be repeated for infinitely many times. Choose

$$w_k = \tau \left(q'_0(L) \xrightarrow{\bar{s}_1} q'_1(L) \xrightarrow{\bar{s}_2} q'_1(L) \xrightarrow{\bar{s}_3} q'_2(L) \xrightarrow{\bar{s}_4} q_3 \xrightarrow{(\bar{s}_5)^\omega} \right),$$

one has $w_k \in L^\omega(\mathfrak{R}, \mathfrak{G})$, $\gamma \sqsubset \ell(w_k)$ satisfies $|\gamma| \geq k + 2$, and $|\mathcal{M}(\mathcal{A}^w, \gamma)| > 1$. That is, \mathcal{A}^w is not strongly detectable.

“only if”: This implication holds because of the finiteness of the state set Q of \mathcal{A}^w and the Pigeonhole Principle. \square

3.6.2 For strong periodic detectability:

We first use the notion of observer to give an equivalent condition for strong periodic detectability of weighted automata, and furthermore represent the equivalent condition in terms of the notion of detector.

Theorem 2 *A labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ (5) is not strongly periodically detectable if and only if in its observer ${}^{sub}\mathcal{A}_{obs}^w$ (16), at least one of the two following conditions holds.*

- (i) *There is a reachable state $x \in X$ such that $|x| > 1$ and there exists a path $q \xrightarrow{s_1} q' \xrightarrow{s_2} q'$ in \mathcal{A}^w , where $q \in x$, $s_1 \in (T_{uo})^*$, $s_2 \in (T_{uo})^+$, $q' \in Q$.*
- (ii) *There is a reachable cycle in ${}^{sub}\mathcal{A}_{obs}^w$ such that no state in the cycle is a singleton.*

Proof By Definition 2, \mathcal{A}^w is not strongly periodically detectable if and only for all $k \in \mathbb{N}$, there is an ω -timed word $w_k \in L^\omega(\mathfrak{R}, \mathfrak{G})$ and a prefix $w' \sqsubset w_k$ such that for all $w'' \in (E \times T)^*$ satisfying $|\ell(w'')| < k$ and $w'w'' \sqsubset w_k$, one has $|\mathcal{M}(\mathcal{A}^w, \ell(w'w''))| > 1$.

“if”: Assume (i) holds. Then there exists a path $q_0 \xrightarrow{s_\gamma} q \xrightarrow{s_1} q' \xrightarrow{s_2} q'$ in \mathcal{A}^w such that $q_0 \in Q_0$ and $\mathcal{M}(\mathcal{A}^w, \ell(\tau(q_0 \xrightarrow{s_\gamma} q))) = x$. Denote $\tau(q_0 \xrightarrow{s_\gamma} q) =: w_1 \in L(\mathfrak{R}, \mathfrak{G})$ and $\tau(q_0 \xrightarrow{s_\gamma} q \xrightarrow{s_1} q' \xrightarrow{(s_2)^\omega}) =: w_1w_2 \in L^\omega(\mathfrak{R}, \mathfrak{G})$, then for every $w \sqsubset w_2$, one has $\ell(w) = \epsilon$ and $|\mathcal{M}(\mathcal{A}^w, \ell(w_1w))| = |\mathcal{M}(\mathcal{A}^w, \ell(w_1))| > 1$, which violates strong periodic detectability by definition.

Assume (ii) holds. Then there exists $\alpha\beta^\omega \in \mathcal{L}^\omega(\mathcal{A}^w)$ such that $\alpha \in (\Sigma \times T)^*$, $\beta \in (\Sigma \times T)^+$, $\mathcal{M}(\mathcal{A}^w, \alpha) = \mathcal{M}(\mathcal{A}^w, \alpha\beta)$, and for every prefix $\beta' \sqsubset \beta$, one has $|\mathcal{M}(\mathcal{A}^w, \alpha\beta')| > 1$. Choose $w_\alpha w_\beta \in L^\omega(\mathfrak{R}, \mathfrak{G})$ such that $\ell(w_\alpha) = \alpha$ and $\ell(w_\beta) = \beta^\omega$. Then for every $w'_\beta \sqsubset w_\beta$, one has $|\mathcal{M}(\mathcal{A}^w, \ell(w_\alpha w'_\beta))| > 1$, which also violates strong periodic detectability by definition.

“only if”: Assume \mathcal{A}^w is not strongly periodically detectable and (ii) does not hold, next we prove (i) holds.

Since \mathcal{A}^w is not strongly periodically detectable, by definition, choose integer $k > |2^Q|$, $w_k \in L^\omega(\mathfrak{R}, \mathfrak{G})$, and prefix $w' \sqsubset w_k$ such that for all $w'' \in (\Sigma \times T)^*$, $w'w'' \sqsubset w_k$ and $|\ell(w'')| < k$ imply $|\mathcal{M}(\mathcal{A}^w, \ell(w'w''))| > 1$. Since (ii) does not hold, one has $\ell(w_k) \in (\Sigma \times T)^*$ and $|\ell(w_k)| < k + |\ell(w')|$. Otherwise if $|\ell(w_k)| \geq k + |\ell(w')|$ or $\ell(w_k) \in (\Sigma \times T)^\omega$, we can choose \bar{w}'' such that $w'\bar{w}'' \sqsubset w_k$ and $|\ell(\bar{w}'')| = k$, then by the Pigeonhole Principle, there exist $\bar{w}_1'', \bar{w}_2'' \sqsubset \bar{w}''$ such that $|\ell(\bar{w}_1'')| < |\ell(\bar{w}_2'')|$ and $\mathcal{M}(\mathcal{A}^w, \ell(w'\bar{w}_1'')) = \mathcal{M}(\mathcal{A}^w, \ell(w'\bar{w}_2''))$, that is, (ii) holds. Then $w_k = w'\hat{w}_1''\hat{w}_2''$, where $\hat{w}_1'' \in (E \times T)^*$, $\hat{w}_2'' \in (E \times T_{uo})^\omega$. Moreover, one has $|\mathcal{M}(\mathcal{A}^w, \ell(w'\hat{w}_1''))| > 1$, and also by the Pigeonhole Principle there exists a path $q_0 \xrightarrow{w'\hat{w}_1''} q \xrightarrow{\hat{w}_1''} q' \xrightarrow{\hat{w}_2''} q'$ for some $q_0 \in Q_0$, $q, q' \in Q$, $\hat{w}_1'' \in (T_{uo})^*$, and $\hat{w}_2'' \in (T_{uo})^+$, i.e., (i) holds. \square

Theorem 3 *A labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ (5) is not strongly periodically detectable if and only if in its detector ${}^{sub}\mathcal{A}_{det}^w$ (20), at least one of the two following conditions holds.*

- (1) There is a reachable state $x' \in X$ such that $|x'| > 1$ and there exists a path $q \xrightarrow{s_1} q' \xrightarrow{s_2} q'$ in \mathcal{A}^w , where $q \in x'$, $s_1 \in (T_{uo})^*$, $s_2 \in (T_{uo})^+$, $q' \in Q$.
- (2) There is a reachable cycle in ${}^{sub}\mathcal{A}_{det}^w$ such that all states in the cycle have cardinality 2.

Proof We use Theorem 2 to prove this result.

We firstly prove (1) of this theorem is equivalent to (i) of Theorem 2.

“ \Rightarrow ”: Assume (1) holds. In ${}^{sub}\mathcal{A}_{det}^w$, choose a transition sequence $x_0 \xrightarrow{\alpha} x'$. Then one has $x' \subset \mathcal{M}(\mathcal{A}^w, \alpha) = \delta_{obs}(x_0, \alpha)$, hence (i) of Theorem 2 holds.

“ \Leftarrow ”: Assume (i) holds. In ${}^{sub}\mathcal{A}_{obs}^w$, choose a transition sequence $x_0 \xrightarrow{\alpha} x$. By Proposition 3, moving backward on $x_0 \xrightarrow{\alpha} x$ from x to x_0 , we can obtain a transition sequence $x_0 \xrightarrow{\alpha} x'$ of ${}^{sub}\mathcal{A}_{det}^w$ such that $q \in x' \subset x$, hence (1) of this theorem holds.

We secondly prove (2) of this theorem is equivalent to (ii) of Theorem 2.

“ \Rightarrow ”: Assume (2) holds. In ${}^{sub}\mathcal{A}_{det}^w$, choose a transition sequence $x_0 \xrightarrow{\alpha} x \xrightarrow{\beta} x$ such that in $x \xrightarrow{\beta} x$ all states are of cardinality 2 and $|\beta| > 0$. Without loss of generality, we assume $|\beta| > |2^Q|$, because otherwise we can repeat $x \xrightarrow{\beta} x$ for $|2^Q| + 1$ times. By definition, one has for all $\beta' \sqsubset \beta$, $|\mathcal{M}(\mathcal{A}^w, \alpha\beta')| > 1$. Then by the Pigeonhole Principle, there exist $\beta_1, \beta_2 \sqsubset \beta$ such that $|\beta_1| < |\beta_2|$ and $\mathcal{M}(\mathcal{A}^w, \alpha\beta_1) = \mathcal{M}(\mathcal{A}^w, \alpha\beta_2)$. Then in observer ${}^{sub}\mathcal{A}_{obs}^w$, one has $\delta_{obs}(x_0, \alpha\beta_1) = \mathcal{M}(\mathcal{A}^w, \alpha\beta_1) = \mathcal{M}(\mathcal{A}^w, \alpha\beta_2) = \delta_{obs}(x_0, \alpha\beta_2)$, and for every $\beta' \sqsubset \beta$, $\delta_{obs}(x_0, \alpha\beta') = \mathcal{M}(\mathcal{A}^w, \alpha\beta')$ has cardinality > 1 . Thus, (ii) of Theorem 2 holds.

“ \Leftarrow ”: Assume (ii) holds. In ${}^{sub}\mathcal{A}_{obs}^w$, choose a transition sequence $x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\beta_1} \dots \xrightarrow{\beta_n} x_{n+1}$ such that $n \geq |Q|^2$, $x_1 = x_{n+1}$, $|x_1|, \dots, |x_{n+1}| > 1$, and $\beta_1, \dots, \beta_n \in \Sigma \times T$. By using Proposition 3 from $n+1$ to 2, we obtain $x'_i \subset x_i$ for all $i \in [1, n+1]$ such that $|x'_1| = \dots = |x'_{n+1}| = 2$ and a transition sequence $x'_1 \xrightarrow{\beta_1} \dots \xrightarrow{\beta_n} x'_{n+1}$ of ${}^{sub}\mathcal{A}_{det}^w$. Moreover, also by Proposition 3, we obtain a transition sequence $x_0 \xrightarrow{\alpha} x'_1$ of ${}^{sub}\mathcal{A}_{det}^w$. By the Pigeonhole Principle, (2) of this theorem holds. \square

3.6.3 For weak detectability and weak periodic detectability:

Next we use the notion of observer to give equivalent conditions for weak detectability and weak periodic detectability of weighted automata.

Theorem 4 A labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ (5) is weakly detectable if and only if either one of the following three conditions holds.

- (i) $L^\omega(\mathfrak{R}, \mathfrak{G}) = \emptyset$.
- (ii) $L^\omega(\mathfrak{R}, \mathfrak{G}) \neq \emptyset$ and there exists $w \in L^\omega(\mathfrak{R}, \mathfrak{G})$ such that $\ell(w) \in (\Sigma \times T)^+$.
- (iii) $L^\omega(\mathfrak{R}, \mathfrak{G}) \neq \emptyset$ and in its observer ${}^{sub}\mathcal{A}_{obs}^w$, there is a reachable cycle in which all states are singletons.

Proof “if”: (i) naturally implies that \mathcal{A}^w is weakly detectable vacuously.

Assume (ii) holds. Then choose integer $k > |\ell(w)|$, one has \mathcal{A}^w is weakly detectable vacuously.

Assume (iii) holds. Then in $^{sub}\mathcal{A}_{obs}^w$, there is a transition sequence $x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} x_1$ such that $\gamma_1 \in (\Sigma_{obs}^T)^*$, $\gamma_2 \in (\Sigma_{obs}^T)^+$, and in $x_1 \xrightarrow{\gamma_2} x_1$, all states are singletons. Hence in \mathcal{A}^w , there exists an infinite path $q_0 \xrightarrow{s_1} q_1 \xrightarrow{s_2} q_1 \xrightarrow{(s_2)^\omega} \dots =: \pi$ such that $\tau(\pi) \in L^\omega(\mathfrak{R}, \mathfrak{G})$, $q_0 \in x_0$, $\{q_1\} = x_1$, $\ell(\tau(q_0 \xrightarrow{s_1} q_1)) = \gamma_1$, $\ell(\tau(q_1 \xrightarrow{s_2} q_1)) = \gamma_2$, and $\ell(\tau(\pi)) = \gamma_1(\gamma_2)^\omega$. For all prefixes $\gamma \sqsubset \gamma_1(\gamma_2)^\omega$ such that $|\gamma| \geq |\gamma_1|$, one has $|\mathcal{M}(\mathcal{A}^w, \gamma)| = 1$. Then \mathcal{A}^w is weakly detectable.

“only if”: This implication holds by definition and the Pigeonhole Principle. \square

Theorem 5 *A labeled weighted automaton $\mathcal{A}^w = (\mathfrak{R}, \mathfrak{G}, \ell)$ (5) is weakly periodically detectable if and only if either one of the following three conditions holds.*

- (i) $L^\omega(\mathfrak{R}, \mathfrak{G}) = \emptyset$.
- (ii) $L^\omega(\mathfrak{R}, \mathfrak{G}) \neq \emptyset$, there exists $w \in L^\omega(\mathfrak{R}, \mathfrak{G})$ such that $\ell(w) \in (\Sigma \times T)^+$ and $|\mathcal{M}(\mathcal{A}^w, \ell(w))| = 1$.
- (iii) $L^\omega(\mathfrak{R}, \mathfrak{G}) \neq \emptyset$ and in its observer $^{sub}\mathcal{A}_{obs}^w$, there is a reachable cycle in which at least one state is a singleton.

We omit a proof of Theorem 5 that is similar to that of Theorem 4.

3.7 Verification of notions of detectability for labeled max-plus automata

In this subsection, we show that for max-plus automaton $\mathcal{A}^{mp} = (\mathbb{N}, \mathfrak{G}, \ell)$ (6), its self-composition $\text{CC}_A(\mathcal{A}^{mp})$ (12), observer $^{sub}\mathcal{A}_{obs}^{mp}$ (16), and detector $^{sub}\mathcal{A}_{det}^{mp}$ (20) are computable with complexity upper bounds NP, 2-EXPTIME, and NP, by using the NP-complete multidimensional EPL problem [13] (Problem 1). As a result, the problems of verifying strong detectability and strong periodic detectability of \mathcal{A}^{mp} are proved to belong to coNP, and the problems of verifying weak detectability and weak periodic detectability of \mathcal{A}^{mp} are proved to belong to 2-EXPTIME. In addition, we also prove that the former two problems are both coNP-hard by using the NP-complete SS problem [2] (Problem 2).

3.7.1 Computation of self-composition $\text{CC}_A(\mathcal{A}^{mp})$ and verification of strong detectability

We first show how to compute $\text{CC}_A(\mathcal{A}^{mp})$. Given states $(q_1, q_2), (q_3, q_4) \in Q'$ and event $(e_1, e_2) \in E'$, we verify whether there is a transition

$$((q_1, q_2), (e_1, e_2), (q_3, q_4)) \in \delta'$$

as follows.

- (i) Guess states $q_5, q_6, q_7, q_8 \in Q$ such that there exist paths $q_5 \xrightarrow{e_1} q_7, q_6 \xrightarrow{e_2} q_8$ and unobservable instantaneous paths $q_7 \xrightarrow{s_3} q_3, q_8 \xrightarrow{s_4} q_4$, where $s_3, s_4 \in (E_{uo})^*$.
- (ii) Check whether there exist unobservable paths $q_1 \xrightarrow{s_1} q_5, q_2 \xrightarrow{s_2} q_6$, where $s_1, s_2 \in (E_{uo})^*$, such that the weights of paths $q_1 \xrightarrow{s_1} q_5 \xrightarrow{e_1} q_7, q_2 \xrightarrow{s_2} q_6 \xrightarrow{e_2} q_8$ are the same. If such paths $q_1 \xrightarrow{s_1} q_5, q_2 \xrightarrow{s_2} q_6$ exist, then one has $((q_1, q_2), (e_1, e_2), (q_3, q_4)) \in \delta'$.

Next we check the above (ii). Firstly, compute subgraphs $\mathcal{A}_{q_1}^{mp}$ (resp. $\mathcal{A}_{q_2}^{mp}$) of \mathcal{A}^{mp} starting at q_1 (resp. q_2) and passing through exactly all possible unobservable transitions. Secondly, compute asynchronous product $\mathcal{A}_{q_1}^{mp} \otimes \mathcal{A}_{q_2}^{mp}$ of $\mathcal{A}_{q_1}^{mp}$ and $\mathcal{A}_{q_2}^{mp}$, where the states of the product are exactly pairs (p_1, p_2) with p_1 and p_2 being states of $\mathcal{A}_{q_1}^{mp}$ and $\mathcal{A}_{q_2}^{mp}$, respectively; transitions are of the form

$$(p_1, p_2) \xrightarrow{(\epsilon, \epsilon) / -\mu(e)_{p_2 p_3}} (p_1, p_3),$$

where $p_2 \xrightarrow{e / \mu(e)_{p_2 p_3}} p_3$ is a transition of $\mathcal{A}_{q_2}^{mp}$, or of the form

$$(p_1, p_2) \xrightarrow{(\epsilon, \epsilon) / \mu(e)_{p_1 p_3}} (p_3, p_2),$$

where $p_1 \xrightarrow{e / \mu(e)_{p_1 p_3}} p_3$ is a transition of $\mathcal{A}_{q_1}^{mp}$. Finally, check in $\mathcal{A}_{q_1}^{mp} \otimes \mathcal{A}_{q_2}^{mp}$, whether there is a path from (q_1, q_2) to (q_5, q_6) whose weight is equal to $\mu(e_2)_{q_6 q_8} - \mu(e_1)_{q_5 q_7}$, which is actually a 1-dimensional EPL problem (Problem 1). Then since the EPL problem belongs to NP (Proposition 1), the following result holds.

Theorem 6 *The self-composition $\text{CC}_A(\mathcal{A}^{mp})$ of a max-plus automaton \mathcal{A}^{mp} (6) can be computed in NP in the size of \mathcal{A}^{mp} .*

One can see that the condition in Theorem 1 can be verified in time linear of the size of $\text{CC}_A(\mathcal{A}^{mp})$ by computing its strongly connected components (a similar check is referred to [22, Theorem 3]), then the following result holds.

Theorem 7 *The problem of verifying strong detectability of a max-plus automaton \mathcal{A}^{mp} (6) belongs to coNP.*

3.7.2 Computation of observer $^{sub}\mathcal{A}_{obs}^{mp}$ and verification of weak detectability and weak periodic detectability

We next show the complexity of computing $^{sub}\mathcal{A}_{obs}^{mp} = (X, \Sigma_{obs}^T, x_0, \delta_{obs})$ (shown in (16)) for max-plus automaton \mathcal{A}^{mp} (6). The initial state $x_0 = \mathcal{M}(\mathcal{A}^{mp}, \epsilon)$ can be directly computed by starting at an initial state of \mathcal{A}^{mp} and passing through all possible unobservable instantaneous transitions. We then start from x_0 , find all transitions step by step enough for verifying weak detectability and weak periodic detectability, which is equivalent to checking for all $x_1, x_2 \in X$ and $\sigma \in \Sigma$, whether there is a transition $x_1 \xrightarrow{(\sigma, t)} x_2$ for some $t \in \mathbb{N}$. If it does exist, then $x_1 \xrightarrow{(\sigma, t)} x_2$ is a transition of $^{sub}\mathcal{A}_{obs}^{mp}$, otherwise there is no transition $x_1 \xrightarrow{(\sigma, t')} x_2$ for any $t' \in \mathbb{N}$.

Choose a state $x_1 = \{q_1, \dots, q_n\} \in X$ that we have just computed, where $n \in \mathbb{Z}_+$, and $|x| = n$. Also choose $\sigma \in \Sigma$. For each $i \in [1, n]$, compute subgraph $\mathcal{A}_{q_i}^{mp}$ that consists of all paths of the form

$$q_i \xrightarrow{s_i^1} q_i^1 \xrightarrow{e_i} q_i^2 \quad (22)$$

of \mathcal{A}^{mp} with the corresponding weights such that $s_i^1 \in (E_{uo})^*$, $e_i \in E_o$, and $\ell(e_i) = \sigma$. Denote the set of all such q_i^2 by \bar{x}_2 . Note that one may have $|\bar{x}_2| > |x_1|$, $|\bar{x}_2| = |x_1|$, or $|\bar{x}_2| < |x_1|$.

We next check whether $(x_1, (\sigma, t), \mathcal{M}(\mathcal{A}^{mp}, \epsilon|\bar{x}_2)) \in \delta_{obs}$ for some $t \in \mathbb{N}$, where $\mathcal{M}(\mathcal{A}^{mp}, \epsilon|\bar{x}_2)$ is the instantaneous state estimate of \bar{x}_2 (defined in (11)).

- (1) For each $i \in [1, n]$, denote the number of states q_i^2 shown in (22) by $i_2 \in \mathbb{N}$, and denote these states by $q_{i,1}^2, \dots, q_{i,i_2}^2$. Here one may have $i_2 = 0$, which implies that there is no path of the form (22) starting from q_i .
- (2) Nondeterministically compute asynchronous product

$$\bigotimes_{i=1}^{i'_2} \mathcal{A}_{q_i}^{mp} \otimes \dots \otimes \bigotimes_{i=1}^{n'_2} \mathcal{A}_{q_n}^{mp}, \quad (23)$$

where $i'_2 \leq i_2$, $i \in [1, n]$, satisfy that states $q_{1,1}^2, \dots, q_{1,i'_2}^2, \dots, q_{n,1}^2, \dots, q_{n,n'_2}^2$ are pairwise different and

$$\{q_{1,1}^2, \dots, q_{1,i'_2}^2, \dots, q_{n,1}^2, \dots, q_{n,n'_2}^2\} = \{q_{1,1}^2, \dots, q_{1,i_2}^2, \dots, q_{n,1}^2, \dots, q_{n,i_2}^2\},$$

this also guarantees that $\sum_{i=1}^n i'_2 \leq |Q|$; the states of the product are

$$(q_{1,1}, \dots, q_{1,i'_2}, \dots, q_{n,1}, \dots, q_{n,n'_2}),$$

where $q_{i,1}, \dots, q_{i,i'_2}$ are states of $\mathcal{A}_{q_i}^{mp}$, $i \in [1, n]$; there is a transition

$$\begin{array}{c} (q_{1,1}, \dots, q_{1,i'_2}, \dots, q_{n,1}, \dots, q_{n,n'_2}) \\ (q'_{1,1}, \dots, q'_{1,i'_2}, \dots, q'_{n,1}, \dots, q'_{n,n'_2}) \end{array} \xrightarrow{(e_{1,1}, \dots, e_{1,i'_2}, \dots, e_{n,1}, \dots, e_{n,n'_2})}$$

if and only if either one of the two conditions holds.

- (a) For some $i \in [1, n]$ and $j \in [1, i'_2]$, $q_{i,j} \xrightarrow{e_{i,j}/\mu(e_{i,j})_{q_{i,j}} q'_{i,j}} q'_{i,j}$ is an unobservable transition of $\mathcal{A}_{q_i}^{mp}$, for all other pairs (k, l) , $e_{k,l}$ are equal to ϵ , and $q_{k,l} = q'_{k,l}$.
- (b) For all $i \in [1, n]$ and $j \in [1, i'_2]$, $q_{i,j} \xrightarrow{e_{i,j}/\mu(e_{i,j})_{q_{i,j}} q'_{i,j}} q'_{i,j}$ is an observable transition of $\mathcal{A}_{q_i}^{mp}$.
- (3) In product (23), guess transition

$$\begin{array}{c} (q_{1,1}^1, \dots, q_{1,i'_2}^1, \dots, q_{n,1}^1, \dots, q_{n,n'_2}^1) \\ (q_{1,1}^2, \dots, q_{1,i'_2}^2, \dots, q_{n,1}^2, \dots, q_{n,n'_2}^2) \end{array} \xrightarrow{(\bar{e}_{1,1}, \dots, \bar{e}_{1,i'_2}, \dots, \bar{e}_{n,1}, \dots, \bar{e}_{n,n'_2})}$$

where $\bar{e}_{1,1}, \dots, \bar{e}_{1,i'_2}, \dots, \bar{e}_{n,1}, \dots, \bar{e}_{n,n'_2}$ are observable (i.e., item (2b) is satisfied). Then check in product (23), whether there is an unobservable path

$$\underbrace{(q_1, \dots, q_1)}_{i'_2}, \dots, \underbrace{(q_n, \dots, q_n)}_{n'_2} \xrightarrow{\mathbf{s}} (q_{1,1}^1, \dots, q_{1,i'_2}^1, \dots, q_{n,1}^1, \dots, q_{n,n'_2}^1) \quad (24)$$

such that the weights of all components (that are actually paths of some $\mathcal{A}_{q_i}^{mp}$ of the form (22)) of

$$\begin{aligned} & \underbrace{(q_1, \dots, q_1, \dots, q_n, \dots, q_n)}_{1'_2} \xrightarrow{\mathbf{s}} (q_{1,1}^1, \dots, q_{1,1'_2}^1, \dots, q_{n,1}^1, \dots, q_{n,n'_2}^1) \\ & \xrightarrow{(\bar{e}_{1,1}, \dots, \bar{e}_{1,1'_2}, \dots, \bar{e}_{n,1}, \dots, \bar{e}_{n,n'_2})} (q_{1,1}^2, \dots, q_{1,1'_2}^2, \dots, q_{n,1}^2, \dots, q_{n,n'_2}^2) \end{aligned} \quad (25)$$

are the same. If Yes, then the weight is denoted by $t \in \mathbb{N}$, and we find a transition

$$(x_1, (\sigma, t), \mathcal{M}(\mathcal{A}^{mp}, \epsilon|\bar{x}_2)) \quad (26)$$

of δ_{obs} .

We need to do the above check (3) for at most $2^{|Q|}$ times (corresponding to non-deterministic computations of product (23)). Each check can be done by reducing it to the (multidimensional) EPL problem (Problem 1 and Proposition 1) (similar to the check of (ii) in computation of $\text{CC}_A(\mathcal{A}^{mp})$), hence can be done in NP in the size $O((|Q||Q|)^2(|E_o|^{E_o} + |Q||E_{uo}|)) = O(2^{2|Q|\log|Q|}(2^{|E_o|\log|E_o|} + |Q||E_{uo}|))$ of the product (23). Hence the checks of (1), (2), (3) can be done in 2-EXPTIME in the size of \mathcal{A}^{mp} .

If after checking the above (1), (2), (3), we obtain a transition (26) of δ_{obs} , then we continue to find transitions starting at $\mathcal{M}(\mathcal{A}^{mp}, \epsilon|\bar{x}_2)$ also by checking the above (1), (2), (3); otherwise, we need to choose a subset \bar{x}'_2 of \bar{x}_2 to check whether there is a transition from x_1 to $\mathcal{M}(\mathcal{A}^{mp}, \epsilon|\bar{x}'_2)$ in the order the cardinality of \bar{x}'_2 decreases from $|\bar{x}_2| - 1$. Note that if for some subset \bar{x}'_2 , we find a transition from x_1 to $\mathcal{M}(\mathcal{A}^{mp}, \epsilon|\bar{x}'_2)$, we do not need to check any proper subset \hat{x}'_2 of \bar{x}'_2 , because there will be no transition from x_1 to $\mathcal{M}(\mathcal{A}^{mp}, \epsilon|\hat{x}'_2)$ by definition.

When finishing the construction of $^{sub}\mathcal{A}_{obs}^{mp}$, in the worst case, x_1 may range over all subsets of Q . For each given x_1 , the corresponding \bar{x}_2 is unique. In the worse case, we may also execute the above steps (1), (2), (3) on all subsets of \bar{x}_2 when there is no transition from x_1 to $\mathcal{M}(\mathcal{A}^{mp}, \epsilon|\bar{x}'_2)$ for any subset \bar{x}'_2 of \bar{x}_2 . Hence, the total time consumption of computing the observer of \mathcal{A}^{mp} is 2-EXPTIME in the size of \mathcal{A}^{mp} . Then the following result holds.

Theorem 8 *The observer $^{sub}\mathcal{A}_{obs}^{mp}$ (16) of a max-plus automaton \mathcal{A}^{mp} (6) can be computed in 2-EXPTIME in the size of \mathcal{A}^{mp} .*

In Theorem 4, conditions (i) and (ii) can be verified in time linear of the size of \mathcal{A}^{mp} by computing its strongly connected components, and condition (iii) can be verified in time linear of the size of observer $^{sub}\mathcal{A}_{obs}^{mp}$. Then the following result holds.

Theorem 9 *The weak detectability of a max-plus automaton \mathcal{A}^{mp} (6) can be verified in 2-EXPTIME in the size of \mathcal{A}^{mp} .*

Similarly, by Theorem 5 and Theorem 8, the following result holds.

Theorem 10 *The weak periodic detectability of a max-plus automaton \mathcal{A}^{mp} (6) can be verified in 2-EXPTIME in the size of \mathcal{A}^{mp} .*

3.7.3 Computation of detector ${}^{sub}\mathcal{A}_{det}^{mp}$ and verification of strong periodic detectability

One directly sees that detector ${}^{sub}\mathcal{A}_{det}^{mp}$ is a simplified version of observer ${}^{sub}\mathcal{A}_{obs}^{mp}$, hence ${}^{sub}\mathcal{A}_{det}^{mp}$ can be computed similarly by starting from the initial state x_0 , and find all reachable states and transitions, where the states are of cardinality ≤ 2 . Hence in the process of looking for all reachable states, one can reduce the corresponding problems to a 1-dimensional EPL problem as in computation of self-composition $CC_A(\mathcal{A}^{mp})$ (see Section 3.7.1). We then conclude that the same as computation of self-composition $CC_A(\mathcal{A}^{mp})$, detector ${}^{sub}\mathcal{A}_{det}^{mp}$ can also be computed in NP in the size of \mathcal{A}^{mp} . Then by Theorem 3, the following result holds.

Theorem 11 *The problem of verifying strong periodic detectability of a max-plus automaton \mathcal{A}^{mp} (6) belongs to coNP.*

3.7.4 The complexity lower bounds on verifying strong (periodic) detectability of max-plus automata

Theorem 12 *The problems of verifying strong detectability and strong periodic detectability of a max-plus automaton \mathcal{A}^{mp} (6) are both coNP-hard.*

Proof We reduce the NP-complete SS problem (Problem 2) to negation of strong detectability and strong periodic detectability of max-plus automata.

Given positive integers n_1, \dots, n_m , and N , next we construct in polynomial time a max-plus automaton $\mathcal{A}_3^{mp} = (\mathbb{N}, \mathcal{E}, \ell)$ as illustrated in Fig. 1. q_0 is the unique initial state and has initial time delay 0. Event u is unobservable. Event e is observable, and $\ell(e) = e$. For all $i \in [0, m-1]$, there exists two unobservable transitions $q_i \xrightarrow{u/n_{i+1}} q_{i+1}$ and $q_i \xrightarrow{u/0} q_{i+1}$. The observable transitions are $q_m \xrightarrow{e/0} q_{m+1}^1$, $q_0 \xrightarrow{e/N} q_{m+1}^2$, and two self-loops $q_{m+1}^1 \xrightarrow{e/1} q_{m+1}^1$ and $q_{m+1}^2 \xrightarrow{e/1} q_{m+1}^2$.

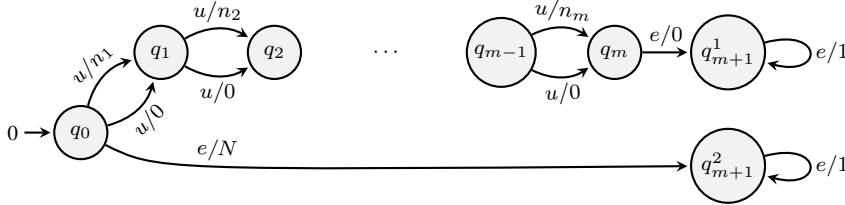


Fig. 1 Sketch of the reduction in the proof of Theorem 12.

Suppose there exists $I \subset [1, m]$ such that $N = \sum_{i \in I} n_i$. Then there is an unobservable path $\pi \in q_0 \xrightarrow{u^m} q_m$ whose weight is equal to N . Then we have

$$\ell(\tau(\pi \xrightarrow{e} q_{m+1}^1)) = (e, N), \quad (27)$$

$$\mathcal{M}(\mathcal{A}_3^{mp}, (e, N) \dots (e, N + i)) = \{q_{m+1}^1, q_{m+1}^2\} \quad (28)$$

for all $i \in \mathbb{N}$.

Choose

$$w = \tau(\pi \xrightarrow{e} q_{m+1}^1 \xrightarrow{e} q_{m+1}^1 \xrightarrow{e^\omega}) \in L^\omega(\underline{\mathbb{N}}, \mathfrak{G}).$$

Then

$$\ell(w) = (e, N)(e, N+1)(e, N+2) \dots$$

Choose prefix $\gamma_k = (e, N) \dots (e, N+k) \sqsubset \ell(w)$. Then we have $|\gamma_k| \geq k$ and $|\mathcal{M}(\mathcal{A}_3^{mp}, \gamma_k)| > 1$ by (28). Hence \mathcal{A}_3^{mp} is not strongly detectable.

For all $k \in \mathbb{N}$, choose the above w , choose $w' = \tau(\pi \xrightarrow{e} q_{m+1}^1) \sqsubset w$, for all w'' such that $w'w'' \sqsubset w$ and $|\ell(w'')| < k$, we have $|\mathcal{M}(\mathcal{A}_3^{mp}, \ell(w'w''))| > 1$ by (28). Hence \mathcal{A}_3^{mp} is not strongly periodically detectable.

Suppose for all $I \subset [1, m]$, $N \neq \sum_{i \in I} n_i$. Then for all $\pi \in q_0 \xrightarrow{u^m e} q_{m+1}^1$, one has

$$\begin{aligned} \ell(\tau(\pi)) &= (e, N') \text{ for some } N' \neq N, \\ \mathcal{M}(\mathcal{A}_3^{mp}, \ell(\tau(\pi))) &= \{q_{m+1}^1\}. \end{aligned}$$

One then has

$$\mathcal{L}(\mathcal{A}_3^{mp}) = \{\epsilon, (e, N') \dots (e, N' + k), (e, N) \dots (e, N + k) | k \in \mathbb{N}\},$$

and

$$\begin{aligned} \mathcal{M}(\mathcal{A}_3^{mp}, (e, N') \dots (e, N' + k)) &= \{q_{m+1}^1\}, \\ \mathcal{M}(\mathcal{A}_3^{mp}, (e, N) \dots (e, N + k)) &= \{q_{m+1}^2\} \end{aligned}$$

for all $k \in \mathbb{N}$. Hence \mathcal{A}_3^{mp} is strongly detectable and strongly periodically detectable. \square

4 conclusion

In this paper, we extended the notions of concurrent composition, observer, and detector from finite-state automata to weighted automata. By using these extended notions, we gave equivalent conditions for four fundamental notions of detectability, i.e., strong (periodic) detectability and weak (periodic) detectability, for weighted automata. Particularly, for a max-plus automaton \mathcal{A}^{mp} over semiring $\mathbb{N} \cup \{-\infty\}$, we proved that the extended notions can be computed with complexity upper bounds NP, 2-EXPTIME, and NP. Moreover, for \mathcal{A}^{mp} , we gave an 2-EXPTIME upper bound for verifying weak (periodic) detectability, and a coNP upper bound and a coNP lower bound on verifying strong (periodic) detectability.

The complexity upper bounds obtained in the paper are based on the NP-complete EPL problem proved in [13]. The technique used in [13] to give an NP upper bound on the EPL problem (actually over \mathbb{Z}^k) is to reduce the EPL problem to the existence of a nonnegative integer solution of a linear inequality with integer coefficients. Note that the existence of a nonnegative integer solution of a linear inequality with rational coefficients also belongs to NP [15, Cor. 18.1a], and the reduction

also works in the case extended to \mathbb{Q}^k , hence the EPL problem extended to \mathbb{Q}^k also belongs to NP. As a result, the upper bounds shown in Theorem 7, Theorem 10, and Theorem 11 are also valid for more general max-plus automata over semiring $\underline{\mathbb{Q}} := (\mathbb{Q} \cup \{-\infty\}, \max, +, -\infty, 0)$.

The lower bound for verifying weak (periodic) detectability of max-plus automata $\underline{\mathbb{Q}}$ is unknown and may also be 2-EXPTIME.

References

1. T. Colcombet. Unambiguity in automata theory. In Jeffrey Shallit and Alexander Okhotin, editors, *Descriptive Complexity of Formal Systems*, pages 3–18, Cham, 2015. Springer International Publishing.
2. M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., USA, 1990.
3. S. Gaubert. Performance evaluation of (max,+) automata. *IEEE Transactions on Automatic Control*, 40(12):2014–2025, 1995.
4. A. Giua and C. Seatzu. Observability of place/transition nets. *IEEE Transactions on Automatic Control*, 47(9):1424–1437, Sep 2002.
5. R.E. Kalman. Mathematical description of linear dynamical systems. *Journal of the Society for Industrial and Applied Mathematics Series A Control*, 1(12):152–192, 1963.
6. C. Keroglou and C. N. Hadjicostis. Verification of detectability in probabilistic finite automata. *Automatica*, 86:192–198, 2017.
7. J. Komenda, S. Lahaye, and J.-L. Boimond. Supervisory control of (max,+) automata: A behavioral approach. *Discrete Event Dynamic Systems*, 19(4):525–549, 2009.
8. A. Lai, S. Lahaye, and A. Giua. Verification of detectability for unambiguous weighted automata. *IEEE Transactions on Automatic Control*, page in press, 2020.
9. H. Lan, Y. Tong, J. Guo, and C. Seatzu. Verification of C-detectability using Petri nets. *Information Sciences*, 528:294–310, 2020.
10. T. Masopust. Complexity of deciding detectability in discrete event systems. *Automatica*, 93:257–261, 2018.
11. T. Masopust and X. Yin. Deciding detectability for labeled Petri nets. *Automatica*, 104:238–241, 2019.
12. E. F. Moore. Gedanken-experiments on sequential machines. *Automata Studies, Annals of Math. Studies*, 34:129–153, 1956.
13. M. Nykänen and E. Ukkonen. The exact path length problem. *Journal of Algorithms*, 42(1):41–53, 2002.
14. S. Sandberg. *1 Homing and Synchronizing Sequences*, pages 5–33. Springer Berlin Heidelberg, Berlin, Heidelberg, 2005.
15. A. Schrijver. *Theory of Linear and Integer Programming*. John Wiley & Sons, Inc., USA, 1986.
16. S. Shu and F. Lin. Generalized detectability for discrete event systems. *Systems & Control Letters*, 60(5):310–317, 2011.
17. S. Shu, F. Lin, and H. Ying. Detectability of discrete event systems. *IEEE Transactions on Automatic Control*, 52(12):2356–2359, Dec 2007.
18. X. Yin. Initial-state detectability of stochastic discrete-event systems with probabilistic sensor failures. *Automatica*, 80:127–134, 2017.
19. K. Zhang. The problem of determining the weak (periodic) detectability of discrete event systems is PSPACE-complete. *Automatica*, 81:217–220, 2017.
20. K. Zhang and A. Giua. Weak (approximate) detectability of labeled Petri net systems with inhibitor arcs. *IFAC-PapersOnLine*, 51(7):167–171, 2018. 14th IFAC Workshop on Discrete Event Systems WODES 2018.
21. K. Zhang and A. Giua. K -delayed strong detectability of discrete-event systems. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 7647–7652, Dec 2019.
22. K. Zhang and A. Giua. On detectability of labeled Petri nets and finite automata. *Discrete Event Dynamic Systems*, in press:33 pages, 2020.

-
23. K. Zhang, L. Zhang, and R. Su. A weighted pair graph representation for reconstructibility of Boolean control networks. *SIAM Journal on Control and Optimization*, 54(6):3040–3060, 2016.
 24. K. Zhang, L. Zhang, and L. Xie. *Discrete-Time and Discrete-Space Dynamical Systems*. Communications and Control Engineering. Springer International Publishing, 2020.