Hadamard grade of power series

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À Vera Sós pour son 80e anniversaire, avec admiration et amitié

Abstract

The Hadamard product of two power series $\sum a_n z^n$ and $\sum b_n z^n$ is the power series $\sum a_n b_n z^n$. We define the (Hadamard) grade of a power series A to be the least number (finite or infinite) of algebraic power series, the Hadamard product of which equals A. We study and discuss this notion.

Keywords: Hadamard product, algebraic power series, Hadamard grade, automatic sequences.

AMS Subject Classification: 11R58, 11D88, 13F25, 11B85.

1   Introduction

The Hadamard product of two power series is simply the power series obtained by termwise multiplication of their coefficients. Since the seminal paper of Hadamard [19] this product occurred in various, sometimes unexpected, fields (look for example at the introduction of [5]). We revisit the question of what can be said about the Hadamard product of "simple" series. We introduce in particular the closure under finite Hadamard products of the set of algebraic power series.

2 The early work of Hadamard

Definition 1 Let $A(z) := \sum_{n \geq 0} a_n z^n$ and $B(z) := \sum_{n \geq 0} b_n z^n$ be two power series with coefficients in some commutative field K. Their *Hadamard product* (sometimes also called *child product*, see [26, Section 8, p. 251], *Schur product*, or *quasi-inner product*, see [5, p. 36]) is the power series $A * B(z)$ defined by

$$A * B(z) := \sum_{n \geq 0} a_n b_n z^n.$$

Note that this definition makes sense either by considering series A and B as formal power series, or, if K is a subfield of the field of complex numbers $\mathbb{C}$, as series possibly converging in a neighborhood of 0 (if A and B converge in a neighborhood of 0, then obviously $A * B$ also converges in some neighborhood of 0). If A and B are two functions, we let $A * B$ denote the Hadamard product of their respective power series, provided they exist.

2.1 Singularities of Hadamard products

Starting from the easy observation that for a and b nonzero

$$\frac{1}{a-z} * \frac{1}{b-z} = \frac{1}{ab-z}$$

one may ask whether the singularities of the Hadamard product of two series with coefficients in $\mathbb{C}$ are related to the product of singularities of these series. Hadamard realized that this is indeed the case.

Theorem 1 (Hadamard [19]) *Let $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$ be two power series holomorphic in a neighborhood of 0. Let σ_A and σ_B be their respective sets of singularities. Then the singularities of $A * B$ are among the values $\alpha\beta$, where $\alpha \in \sigma_A$ and $\beta \in \sigma_B$. In other words $\sigma_{A*B} \subset \sigma_A \cdot \sigma_B := \{\alpha\beta, \ \alpha \in \sigma_A, \ \beta \in \sigma_B\}$.*

The proof, which is by no means easy, involves the study of the integral

$$\oint A(u) B\left(\frac{z}{u}\right) \frac{\mathrm{d}u}{u} = \sum a_n b_n z^n$$

extended to a circle around the origin. The holomorphy of $A(z)$ and $B(z)$ in a neighborhood of 0 is therefore an essential hypothesis for the result to hold. And indeed in Section 2.2 below we show what may happen if say $\limsup_{n+\infty} |a_n|^{1/n} = \infty$.

Remark 1 Other papers on singularities of Hadamard product may be of interest for the reader: see in particular [4, 11, 12].

2.2   A counter-example: the Euler series

The hypothesis that both A and B converge in some neighborhood of the origin is necessary for the validity of Theorem 1. Indeed consider the Euler $E(z)$ formal power series defined by

$$E(z) := \sum_{n \geq 0} (-1)^n n! z^n.$$

This series, considered as a complex series, diverges everywhere except at 0. As will be explained below, $E(z)$ has the integral representation $I(z)$ where

$$(\#) \qquad I(z) = \int_0^\infty \frac{e^{-u}}{1 + zu} \, du = -\frac{1}{z} e^{1/z} \log \frac{1}{z} + S\left(\frac{1}{z}\right)$$

with

$$S(y) := -y e^y \left(\gamma + \sum_{n \geq 1} \frac{(-1)^n y^n}{n.n!} \right)$$

and γ is the Euler constant. The integral above is a function well-defined in the complex plane except on the real negative axis $\{z = s + it, \ t = 0, \ s < 0\}$.

Now let $B(z) := e^z$. We have $E * B(z) = \sum_{n \geq 0} (-1)^n z^n = \frac{1}{1+z}$ which has only one singularity at $z = -1$, while e^z does not have any singularity.

In order to demystify the integral representation $E(z) = I(z)$, compute the derivatives of $I(z)$:

$$I^{(n)}(z) = (-1)^n \int_0^\infty \frac{n! e^{-u}}{(1 + zu)^{n+1}} \, du$$

which reduces at $z = 0$ to $I^{(n)}(0) = (-1)^n (n!)^2$. Therefore the formal Taylor series for $I(z)$ is

$$\sum_{n \geq 0} (-1)^n n! z^n$$

which indeed coincides with $E(z)$.

The last part of identity $(\#)$ is an illuminating computation (see Hardy's book [20, p. 26–27]) that describes precisely the singularities of $E(z)$.

Theorem 2 ([20]) *The integral rerpresentation of the Euler series $\sum_{n \geq 0} (-1)^n n! z^n$ can be continued for all z to a many-valued function with an infinity of branches differing by integral multiples of $2i\pi z^{-1} e^{1/z}$. One branch tends to 1 as z tends to 0 through positive values. More precisely*

$$I(z) = -\frac{1}{z} e^{1/z} \log \frac{1}{z} + S\left(\frac{1}{z}\right)$$

where

$$S(y) := -y e^y \left(\gamma + \sum_{n \geq 1} \frac{(-1)^n y^n}{n.n!} \right)$$

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and γ is the Euler constant. $I(z)$ lives on the same Riemann surface as $\log z$. In particular, for $z = 1$, [Euler and Hardy write]

$$\sum_{n \geq 0}(-1)^n n! = -e\left(\gamma + \sum_{n \geq 1}\frac{(-1)^n}{nn!}\right) = .5963...$$

Proof. (This proof is attributed to Euler by Hardy in [20]).

Suppose $z = x$ is a positive real number. Performing the change of variable $t = \frac{x}{1+xu}$, we have

$$I(x) = \int_0^\infty \frac{e^{-u}}{1+xu}\,du = -\frac{1}{x}e^{1/x}\,\mathrm{li}(e^{-1/x})$$

where $\mathrm{li}(\xi) := \int_0^\xi \frac{ds}{\log s}$, for $\xi \in [0,1)$, is the *logarithmic integral* (*logarithm-integral* in Hardy's terminology). Now

$$
\begin{aligned}
-\mathrm{li}(e^{-y}) &= \int_y^\infty \frac{e^{-u}}{u}\,du = \int_1^\infty \frac{e^{-u}}{u}\,du - \int_0^1 \frac{1-e^{-u}}{u}\,du - \int_1^y \frac{du}{u} + \int_0^y \frac{1-e^{-u}}{u}\,du \\
&= -\gamma - \log y - \sum_{n \geq 1}\frac{(-1)^n y^n}{n.n!}
\end{aligned}
$$

which concludes the proof. $\square$

3 Digression: elementary optics

Consider N transparent plates one against the other with respective thicknesses $a_1, a_2, \ldots, a_N$ ($a_n > 0$) and indices of refraction $n_1, n_2, \ldots, n_N$ ($n_k > 1$).

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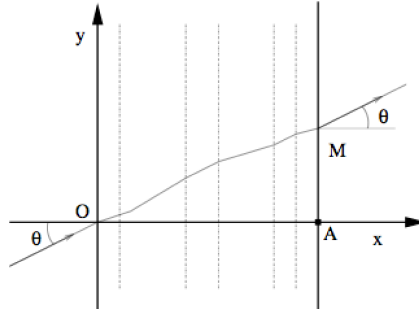
An incoming ray hits the system at the origin with an incident angle θ and reappears after N refractions at M on the opposite side. Let θ_k be the angle at which the luminous path meets the kth plate. The Snell-Descartes law states that $n_k \sin \theta_k$ is independent of k. An easy computation shows that the length $AM = \varphi(\theta)$ is given by

$$\varphi(\theta) = \sum_{1 \leq k \leq N} \frac{a_k \sin \theta}{\sqrt{n_k^2 - \sin^2 \theta}}.$$

Put $z := \sin \theta$ and $\psi(z) := \varphi(\theta)$, thus

$$\psi(z) = \sum_{1 \leq k \leq N} \frac{a_k z}{\sqrt{n_k^2 - z^2}}.$$

In [23] the second named author and A. Sebbar notice that

$$\psi(z) = \frac{z}{\sqrt{1 - z^2}} \; * \sum_{1 \leq k \leq N} \frac{a_k n_k z}{n_k^2 - z^2}.$$

If none of the a_k vanish and if the n_k are pairwise distinct, then ψ is an algebraic function of degree 2^n. The above Hadamard representation is somewhat simpler than the original ψ since one factor is rational whereas the other is quadratic.

The above identity is even valid for $N = \infty$, a_k and $n_k \in \mathbb{C}$, provided the series converges. The second factor can often be summed, especially if a_k and n_k are rational functions of k. Choose for example $n_k = (2k + 1)$ and $a_k = (2k + 1)^{-1}$. Then the identity reads:

$$\sum_{k \geq 0} \frac{z}{(2k + 1)\sqrt{(2k + 1)^2 - z^2}} = \frac{\pi}{4} \frac{z}{\sqrt{1 - z^2}} * \tan \frac{\pi z}{2}.$$

This could be thought as of a quasi-summation of the lefthand side series.

Let us test the formula. For $z \to 0$ it reduces to

$$\sum_{k \geq 0} \frac{1}{(2k + 1)^2} = \frac{\pi^2}{8}.$$

More generally, put $\tan X := \sum_{j \geq 0} t_{2j+1} X^{2j+1}$. Observe that

$$\frac{1}{(2k + 1)\sqrt{(2k + 1)^2 - z^2}} = \frac{1}{(2k + 1)^2} \sum_{j \geq 0} (-1)^j \binom{-\frac{1}{2}}{j} \frac{z^{2j}}{(2k + 1)^2 j}.$$

Then the above Hadamard product leads to the well known equality

$$\sum_{k \geq 0} \frac{1}{(2k + 1)^{2j+2}} = \frac{\pi^{2j+2}}{2^{2j+3}} t_{2j+1}.$$

The example $a_k = (2k + 1)^{-1}$ and $n_k = (2k + 1)$ corresponds to an optical system with infinitely many thinner and thinner plates but with no physical meaning since $n_k = (2k + 1)$ tends to infinity. In real materials the highest indices seem not to go beyond 4 or 5.

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One last remark before closing this section. The identity showing that ψ is a Hadamard product of two simple functions can easily be extended. Let $H(z) := \sum_{j \geq 0} h_j z^{2j+1}$ be any odd function holomorphic around the origin ($|z| < R$). Then, for all complex nonzero a_k and n_k, and for $N \leq +\infty$,

$$\sum_{1 \leq k \leq N} a_k H\left(\frac{z}{n_k}\right) = H(z) \; * \; \sum_{1 \leq k \leq N} \frac{a_k n_k z}{n_k^2 - z^2}$$

provided $|z| < R \inf_k |n_k|$. The proof consists in expanding both sides of the identity into Taylor series, just as in the case $H(z) = z(1-z^2)^{-1/2}$. Needless to say that similar identities hold if H is even, or neither odd nor even.

4 Recalling some definitions and results

In what follows we will need the definition of *D-finite* power series and the definition of *automatic* sequences.

Definition 2 ([24]) A power series is *D-finite* (or *differentially finite* or *holonomic*) if it satisfies a linear differential equation with polynomial coefficients.

Example 1 The power series $F := \sum_{n \geq 0} \frac{z^n}{n!}$ is D-finite, since it satisfies $F' - F = 0$. The series $\sum_{n \geq 0} \sqrt{n} z^n$ is not D-finite [16]. The series $\sum_{n \geq 1} \log n \; z^n$ and $\sum_{n \geq 1} p_n z^n$ where p is the n-th prime are not D-finite [13]. A nice example of a non-D-finite power series is $\sum_{n \geq 1} \zeta(2n+1) z^n$ where ζ is the Riemann function (see [3]).

Definition 3 Let q be an integer ≥ 2. A sequence $(a_n)_{n \geq 0}$ is *q-automatic* if the q-kernel of $(a_n)_{n \geq 0}$, i.e., the set of subsequences $\{(a_{q^k n + j})_{n \geq 0}, \; k \geq 0, \; j \in [0, q^k - 1]\}$, is finite.

5 The Hadamard grade of a power series

Definition 4 A power series $A(z) := \sum a_n z^n$ with coefficients in a commutative field K is said to have *finite Hadamard grade over* K if there exist finitely many power series $B_1(z)$, $B_2(z)$, ..., $B_k(z)$ that are algebraic (over $K(z)$) such that $A = B_1 * B_2 * \cdots * B_k$. The least k having this property is called the *Hadamard grade over* K of A.

We give some first properties of power series with finite Hadamard grade.

Theorem 3

- *The Hadamard product of two power series having finite Hadamard grade has finite Hadamard grade.*

- *If A is a power series that is algebraic over $K(z)$, then A has Hadamard grade 1.*

- *If A is a power series with finite Hadamard grade, then A is D-finite.*

Proof. The first two assertions are clear. For the third, recall that a power series is *D-finite* (or *differentially finite* or *holonomic*) if it satisfies a linear differential equation with polynomial coefficients (see [24]); also recall that any algebraic power series is *D*-finite and that the Hadamard product of two *D*-finite power series is also *D*-finite (see [24]). □

Remark 2 The *D*-finiteness of the Hadamard product of two *D*-finite power series can be found in the paper [24], but as noted by Stanley this is already in the paper of Jungen [22] who indicates that this result can be found in the unpublished manuscripts of Hurwitz (available at École Polytechnique de Zurich). Jungen also gives the same characterization as Stanley's for the coefficients of *D*-finite power series.

The next result shows that in positive characteristic the definition of the grade is not really pertinent (that is why in the next sections we will focus on the case where K is the field of complex numbers $\mathbb{C}$ or the field of rational numbers $\mathbb{Q}$).

Theorem 4 *A power series on a field of positive characteristic has finite Hadamard grade if and only if it is algebraic.*

Proof. Only one direction need to be proven. This was done by Furstenberg [15] for a finite field. Fliess [14] noted that the proof still works for a perfect field of positive characteristic. Deligne [8] gave a general proof. □

Remark 3 For more around algebraic power series the reader can look for example at [1, 17, 18, 26] and the references therein.

6 Hadamard grade of formal power series with complex or with rational coefficients

6.1 The base field is $\mathbb{C}$

From now on we will restrict ourselves to the case where the field K is either $\mathbb{C}$ or $\mathbb{Q}$. We begin with a theorem stating that a minimal Hadamard decomposition of an irrational power series into finitely many algebraic power series only involves algebraic irrational power series.

Theorem 5 *Let $A(z)$ be a power series with complex coefficients. If A is irrational and has finite Hadamard grade d, then in any Hadamard decomposition $A = B_1 * B_2 * \ldots * B_d$ with algebraic B_i's, all the B_i's are irrational.*

Proof. It suffices to use the minimality of d and a result of Jungen [22] stating that the Hadamard product of a rational power series and an algebraic power series is algebraic. □

The following result is classical.

Theorem 6 *Let $F = \sum_{n \geq 0} a_n z^n$ be a formal power series with coefficients in the complex field $\mathbb{C}$ that is algebraic over $\mathbb{C}(z)$. Then the radius of convergence of F is > 0 and F has finitely many singularities in its disk of convergence.*

Using this theorem and Hadamard's Theorem 1 yields the following corollary.

Corollary 1 *Let $A(z)$ be a formal power series with coefficients in $\mathbb{C}$ which has finite Hadamard grade. Then A has a positive radius of convergence. Furthermore A has only finitely many singularities in its disk of convergence.*

There is a simple necessary condition for a complex power series to be of finite Hadamard grade in terms of diagonals of rational functions.

Theorem 7 *If A is a complex power series with finite Hadamard grade d, then A is the (complete) diagonal of a complex rational power series with $2d$ variables.*

Proof. The proof is done by induction on d. The case $d = 1$ can be found in the paper of Furstenberg [15]. Now if $A_1, A_2, \ldots, A_d$ are complex algebraic power series, then there exist a two-variable rational power series $B(X,Y) := \sum b_{k,\ell} X^k Y^\ell$ and a rational power series $C(Z_1, Z_2, \ldots, Z_{2d-2}) := \sum c_{j_1,j_2,\ldots,j_{2d-2}} Z_1^{j_1} Z_2^{j_2} \ldots Z_{2d-2}^{j_{2d-2}}$ such that $A_d(z) = \sum b_{k,k} z^k$ and $A_1(z) * A_2(z) * \ldots A_{d-1}(z) = \sum c_{j,j,\ldots,j} z^j$. The (ordinary Cauchy) product of the series B and C, i.e., $B(X,Y)C(Z_1, Z_2, \ldots, Z_{2d-2}) = \sum b_{k,\ell} c_{j_1,j_2,\ldots,j_{2d-2}} X^k Y^\ell Z_1^{j_1} Z_2^{j_2} \ldots Z_{2d-2}^{j_{2d-2}}$ is a rational power series whose complete diagonal $\sum b_{j,j} c_{j,j,\ldots,j} z^j$ is equal to $A_1(z) * A_2(z) * \ldots A_d(z)$. $\square$

6.2 The base field is $\mathbb{Q}$

What can be said of a power series with rational coefficients and finite Hadamard grade over $\mathbb{Q}$? We begin with a simple remark.

Remark 4 The family of power series with rational coefficients and with finite Hadamard grade over $\mathbb{Q}$ is countable.

Our next result links finite Hadamard grade over $\mathbb{Q}$ and automaticity for the case of power series with rational coefficients (for more about *automatic sequences* the reader can look at [2]).

Theorem 8 *Let $A(z) := \sum a_n z^n$ be a power series with rational coefficients. If A has finite Hadamard grade over the field of rational numbers $\mathbb{Q}$, then there are only finitely many primes dividing at least one of the denominators of the a_n's. Furthermore for all primes p outside these finitely many primes, the series $\overline{A}(z) := \sum (a_n \bmod p) z^n$ is algebraic over $\mathbb{F}_p(z)$. In particular the sequence $(a_n \bmod p)_{n\geq 0}$ is p-automatic. Even more, for these primes p and for any $r \geq 1$, the sequence $(a_n \bmod p^r)_{n\geq 0}$ is p-automatic.*

Proof. Write $A = B_1 * B_2 * \cdots * B_d$ where the B_i's are algebraic over $\mathbb{C}(z)$. From Eisenstein's theorem [10] (see Remark 5 below) the denominators of the coefficients of each B_i have only finitely many distinct prime factors. Let $\mathcal{P}$ be the (finite) set of primes that divide the denominator of at least one B_i. For each prime $p \notin \mathcal{P}$ one can write $(A \bmod p) = (B_1 \bmod p) * (B_2 \bmod p) * \cdots * (B_d \bmod p)$. Each $(B_i \bmod p)$ is algebraic over $\mathbb{F}_p(z)$, so is their Hadamard product (Theorem 4). Actually for any $r \geq 1$ the sequence $(a_n \bmod p^r)_{n\geq 0}$ is p-automatic: this was proved by Christol (see [6, 7], see also [9]). $\square$

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Remark 5

- The Eisenstein theorem states: *Let $\sum a_n z^n$ be an algebraic power series with rational coefficients. Then there exist two integers A and C such that, for all $n \geq 0$, the number $CA^n a_n$ is an integer.* This theorem was actually proved by Heine [21].

- Christol (see, e.g., [7]) calls *globally bounded* any series $f = \sum a_n z^n$ with rational coefficients such that f has a positive radius of convergence (when seen as a power series on $\mathbb{C}$) and such that there exist $\alpha, \beta \in \mathbb{Q}$ such that $\alpha f(\beta z)$ belongs to $\mathbb{Z}[[z]]$. He shows that that any algebraic power series $\sum a_n z^n$ is *globally automatic*, i.e., for all but finitely many primes p and for all integers $r \geq 2$, the sequence $(a_n \mod p^r)_{n \geq 0}$ is p-automatic.

7 Examples, Counterexamples, and Questions

Using the results of the previous sections we can give examples and counterexamples of power series having finite Hadamard grade: we will restrict to power series **with rational coefficients**, and consider their **Hadamard grade over** $\mathbb{Q}$, although some of the statements and questions below could be formulated for the Hadamard grade over K of power series with coefficients in K, where K is any field of characteristic zero.

(a) *The power series e^z, $\log(1+z)$, $\sum \pm z^n$ where the sequence ± 1 is not ultimately periodic, all have infinite Hadamard grade.*

Proof. The denominators of the power series e^z and $\log(1+z)$ are divisible by infinitely many primes. If the sequence $\pm$ is not ultimately periodic, then the series $\sum \pm z^n$ admits the unit circle as natural boundary (this was proved by Szegő in [25]). □

(b) *There exist power series with Hadamard grade ≥ 2 (i.e., non-algebraic sequences with finite Hadamard grade, since all series with Hadamard grade ≥ 2 are transcendental).*

Proof. There is a classical example of a nonalgebraic Hadamard product of two algebraic power series (see [22]). Namely the Hadamard square of $\sum \binom{2n}{n} z^n = (1-4z)^{-1/2}$, is equal to $\sum \binom{2n}{n}^2 z^n = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{(1-16z \sin^2 \theta)^{1/2}}$ which is transcendental. □

(c) *There exist D-finite power series that have infinite Hadamard grade. There even exist D-finite power series with integer coefficients that have infinite Hadamard grade.*

Proof. First the exponential series $\sum_{n \geq 0} z^n/n!$ has infinite Hadamard grade since the denominators of its coefficients are divisible by infinitely many primes (actually *all* the primes); it is of course D-finite. Second the Euler series $E(z) := \sum_{n \geq 0} (-1)^n n! z^n$ is clearly D-finite since $z^2 E'(z) + (1+z)E(z) = 1$, while it has infinite Hadamard grade (its radius of convergence is 0). □

Question. Is there a D-finite power series with integer coefficients and positive radius of convergence which is algebraic on $\mathbb{F}_p(z)$ for all primes p when reduced modulo p, but has infinite Hadamard grade?

Question. Is it true that algebraic power series can be decomposed into finite Hadamard products of quadratic power series?

(d) *There exist two power series A and B having infinite Hadamard grade, such that $A * B$ has finite Hadamard grade.*

Proof. Of course if A is even and B odd, then $A * B = 0$. But, from Section 2.2, we have even more than the claim above: take $A(z) := E(z)$ the Euler series and $B(z) := e^z$. These power series both have infinite Hadamard grade, and their Hadamard product is rational, hence of Hadamard grade 1. Another example is given by the Hadamard square of $\sum \pm z^n$, where the sequence of $\pm$'s is not ultimately periodic. $\square$

Questions.

- Is it true that for all $k > 0$ there exists a power series whose grade is k?
- Is it true that if B is a nonzero rational power series, then for every power series A the Hadamard grade of $A + B$ and of AB are both equal to the Hadamard grade of A?
- Is it true that if the Hadamard grade of a power series A is infinite, then for any nonzero power series B of finite Hadamard grade, the Hadamard grades of $A + B$ and AB are both infinite?
- Let a be an irrational number (real or complex). Is it true that the grade of $(1 + z)^a$ is infinite?
- Let $\sum a_n z^n$ be algebraic irrational. Is it true that for all integers $k \geq 1$ the Hadamard grade of $\sum a_n^k z^n$ is equal to k? Is it the case if $a_n := \binom{2n}{n}$?
- Let $A(z) := \sum a_n z^n$. Let $\mathcal{H}_k$ denote the Hadamard grade of the Hadamard product of k factors equal to A. If there exists $C > 0$ such that $\mathcal{H}_k < C$ for all k, is it true that $A(z)$ must be a rational function? What if A satisfies the weaker condition $\mathcal{H}_k = o(k)$? or even the weaker condition that there exists an integer $k \geq 2$ such that $\mathcal{H}_k < k$? (Note that conversely if A is rational then $\mathcal{H}_k = 1$ for all $k \geq 1$.)

Remark 6 More examples (where $\mathbb{Q}$ is replaced by $\mathbb{C}$) can be obtained from results of [3] on non-D-finite power series.

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