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Testing Equality in Differential Ring Extensions Defined by PDE's and Limit Conditions

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Abstract. We present here methods to test equality in differential extensions of effective rings. The extensions considered are obtained by adjunction of formal power series solutions of given (non linear) systems of PDE's with initial/limit conditions. The equality test is indeed the only operation that is not trivial in such extensions, and is hence a central problem for formally manipulating solutions of differential systems without computing or studying them completely. The problem has been solved in [9] for ordinary differential systems and in [17] for solutions of some partial differential systems with a finite set of initial conditions at the origin. We first recall and refine the results of [17] and then give a method to handle some partial differential systems with limit conditions on the axis $x_1 = 0$. In particular, we reduce the study of the regular cases of the latter differential algebraic problem to a purely (computable) algebraic one, and the singular cases to a topological one, the Ritt problem concerning the distribution of the singular zeros of a differential system among its irreducible components.

Keywords: Partial differential equations, Tests of identities, Differential algebra.

1 Introduction

In the purely algebraic case, we know that an algebraic extension of an effective ring is effective, i.e. one can actually compute in such a ring the usual arithmetical operation +,* and =. Of course, we are interested in knowing in which cases a differential extension of an effective differential ring R is effective (for

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example $R = \mathbb{Q}[x]$ is effective, and we would like to know if $\mathbb{Q}[x, \sin(x)]$ is effective). More precisely, let k be an effective field of constants (notice that this means in particular that we can test equalities in k, which can be a difficult problem in number theory), we will focus on the equality test in extensions of $R = k[x]$ obtained by adjunction of formal power series $f_1, \ldots, f_m$. It is in fact the only operation that is not theoretically trivial.

Of course, the methods we expose are not to be used in case the f_i's are well-known special functions (like sin, exp, Γ ...): one would then rather utilize the great amount of formulas involving them. We show hereafter how to deal with any set of formal power series $f_1, \ldots, f_m$, only known to be solutions of (partial) differential equations, without performing a complete study of their properties. However, for the sake of simplicity, we gave some examples involving special functions.

D. Zeilberger gave an algorithm for proving function identities when the series are holonomic in one variable (for example, see [26]). The case when the f_i's are in $k[[x]]$ and defined by a system of (non linear) algebraic differential equations and a finite set of initial conditions, was (at least theoretically) solved by J. Denef and L. Lipshitz [9] (algebraic differential equations are equations of the form $P\left(x, f_1, f_1', \ldots, f_1^{(m_1)}, \ldots, f_m, \ldots, f_m^{(n_m)}\right) = 0$ where P is a polynomial in all its variables with coefficients in k). In the special case of a triangular ≈ Pfaffian non singular system, J. Shackell [23] gave two algorithms.

However, if we wish to consider series in many variables and satisfying PDE's, we must be much more cautious. The first problem we encounter is due to undecidability and uncomputability results (see [9]). H. Wilf and D. Zeilberger gave an algorithm to test multisum and/or integral identities involving hypergeometric series (with several variables) (see [25]).

I gave in [17] an algorithm for series defined by a system of non linear PDE's and a finite set of initial conditions at the origin. In this paper, we recall and state this method more precisely, and we apply it to series in two variables, defined by systems of PDE's with limit conditions on the axis $x_1 = 0$. We do not give here a complete algorithm, because some singular cases remain unsolved.

In Section 2, we give basic notions in differential algebra (which is our main tool), and specify the class of systems to be considered: they roughly correspond to auto-reduced and coherent sets in rings of differential polynomials, and they are called formally integrable systems in other formalisms. The interest of using differential algebra lies in the fact that differential problems can be solved by means of algebraic methods. As we do not wish to consider all the solutions of given systems, but only a specified one, we also introduce initial/limit conditions.

In Section 3, we give the outlines of the methods exposed in Sections 4 and 5: $f = (f_1, \ldots, f_m)$ is the solution of a given system Σ, one adds to Σ the relations to be tested and compute another system Σ', which is auto-reduced and coherent. We then study this new system.

In Section 4, we recall the method given in [17] for the case of a system of PDE's with a finite set of initial conditions at the origin and we give a somewhat more precise result. Roughly speaking, we reduce the problem to testing whether the initial conditions belong to an algebraic set for which there is an effective membership test.

In Section 5, the f_i's are series in $k[[x_1, x_2]]$ satisfying a system of PDE's Σ and limit conditions on the axis $x_1 = 0$. We first precise in 5.1 the class of systems to be considered. Following the general method given in Section 3, we consider a new system Σ'. We treat in Section 5.2 the case when Σ' is regular on the axis $x_1 = 0$ (we precise later what is meant by "regular"). In this case, the problem can be solved using the algorithm exposed in Section 4. In other words, it reduces again to testing whether the initial conditions belong to an algebraic set with an effective membership test. We are not able to show that the problem is decidable when Σ' is singular on the axis $x_1 = 0$, but we show in Section 5.3 that the identity problem is then equivalent to a difficult topological problem. More precisely, we reduce it to Ritt's problem concerning the distribution of the singular zeros of a differential system among its irreducible components. (We give a very short exposition of Ritt's problem in Section 5.3.4)

2 Preliminaries

To study differential equations in a computable way, we shall use differential algebra, a generalization of commutative algebra to differential equations. In this section we recall its outlines. For a complete exposition, the reader is referred to [20, 16, 4], and for outlines to [15, 2, 18].

2.1 Some Usual Notations and Terminology

Throughout what follows, k will denote a fixed effective field of constants of characteristic zero, and we will consider the ring R of polynomials $R = k[x_1, \ldots, x_n]$, equipped with the derivations $\delta_i = \dfrac{\partial}{\partial x_i}$, that commute one with each other; thus R is an effective differential ring. Θ will denote the free monoid generated by the δ_i's; an element $\delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}$ of Θ will be denoted either by θ, or by ∂_α when more precision is needed. For example $\partial_{(1,2)} = \delta_1 \delta_2^2$. Consider the ring of polynomials $\mathcal{R} = k[x_1, \ldots, x_n][\theta y_i, i = 1, \ldots, m, \ \theta \in \Theta]$: we define an action of Θ on $\mathcal{R}$ by $\theta(\theta' y_i) = (\theta\theta') y_i$, hence $\mathcal{R}$ is a differential ring, which will be denoted by $\mathcal{R} = k[x_1, \ldots, x_n]\{y_1, \ldots, y_m\}$, and called the *ring of partial differential polynomials over* $k[X]$. If I is an ideal in $\mathcal{R}$, I is a *differential ideal* if it is stable under $\delta_i, i = 1, \ldots, n$. If $P_1, \ldots, P_t$ are in $\mathcal{R}$, the *differential ideal* generated by the P_i's will be denoted by $[P_1, \ldots, P_t]$ (it is in fact the algebraic ideal generated by all the θP_i's for $\theta \in \Theta$).

In the sequel, we will assume that we have defined an *admissible ordering* on the set of derivatives $\Gamma = \{\theta y_i, i = 1, \ldots, m, \theta \in \Theta\}$, i.e. an ordering such that ($u < v \Rightarrow \theta u < \theta v$), and $v \leq \theta v$ for all $v \in \Gamma$ and all $\theta \in \Theta$. Such an ordering is a well ordering, i.e.: there is no infinite decreasing sequence in Γ (see [16], chap. 0 § 17 lemma 15 c).

For the examples below, we choose the following order: if $\partial_\alpha, \partial_{\alpha'}$ are in Θ, we will say that $\mathrm{ord}(\partial_\alpha) = |\alpha| = \sum_{i=1,\ldots,n} \alpha_i$. Then $\partial_\alpha < \partial_{\alpha'}$ if $|\alpha| < |\alpha'|$ or $|\alpha| = |\alpha'|$ and $\alpha < \alpha'$ for the lexicographical order. This order on Θ induces an order on the $\partial_\alpha y_i$'s: $\partial_\alpha y_i < \partial_{\alpha'} y_{i'}$ if $\partial_\alpha < \partial_{\alpha'}$ or $\partial_\alpha = \partial_{\alpha'}$ and $i < i'$.

Let P be a partial differential polynomial in $\mathcal{R}$, (hereafter: "p.d.p."), its *leader* is the highest derivative $\partial_\alpha y_i$ involved in P and will be denoted by v_P, its *order* is $\mathrm{ord}(v_P) = |\alpha|$, its *degree* is $\deg(P, v_P)$ (i.e.: the degree of P with respect to v_P), its *initial*, denoted by I_P, is the coefficient of $(v_P)^{deg(P)}$ (it is a p.d.p. involving only lower derivatives than v_P) and its *separant* is $S_P = \dfrac{\partial P}{\partial v_P}$ which is also $I_{\theta P}$ for every operator $\theta \neq Id$.

Example 1. If $\mathcal{R} = \mathbb{Q}[x_1, x_2]\{y_1, y_2\}$ and $P = x_1 x_2 y_1 (\partial_{(1,1)} y_2)^2 + y_2$, then $v_P = \partial_{(1,1)} y_2$, $\mathrm{ord}(P) = 2$, $\deg(P) = 2$, $I_P = x_1 x_2 y_1$, $S_P = 2 x_1 x_2 y_1 \partial_{(1,1)} y_2$.

Let us now introduce the notion of *reduction.* If P, Q are two p.d.p., P is said to be *reduced with respect to* Q if

- P contains no derivative $\partial_\alpha(v_Q)$ with $|\alpha| > 0$ (such a derivative is also called a proper derivative of v_Q);
- $\deg(P, v_Q) < \deg(Q) = \deg(Q, v_Q)$.

For example, the p.d.p. P written above is not reduced with respect to $Q = y_1^2 \partial_{(0,1)} y_2 + y_1$, because $v_P = \partial_{(1,0)} v_Q$.

We can reduce one p.d.p. with respect to another: for all $P, Q \in \mathcal{R}$ there exists a p.d.p. T reduced with respect to Q such that

$$(I_Q)^\nu (S_Q)^{\nu'} P = \left(\sum_{i=1\cdots r} M_i \cdot \theta_i Q \right) + T, \tag{1}$$

where ν, ν' are positive integers, the p.d.p. $M_1, \ldots, M_r$ are in $\mathcal{R}$, and $\theta_1, \ldots, \theta_r$ are in Θ.

There exist algorithms to compute reduction, see e.g. [20], page 165, or [16], page 77. We will denote the reduction process by $P \overset{Q}{\rightarrow} T$, or $T =$ Remainder(P, Q), assuming that we have chosen an algorithm. We will show briefly on an example how such an algorithm works. Consider the p.d.p. P and Q given above. The highest derivative of v_Q involved in P is $u = \partial_{x_1} v_Q$, and the degree of P with respect to u is 2. Let C be the coefficient of u^2 in P: $C = x_1 x_2 y_1$. Then $S_Q P - Cu \cdot (\partial_{x_1} Q) = T_1$ is of degree at most 1 with respect to u, and no derivative of v_Q higher than u is involved in T_1. We shall say that we performed an elementary reduction of P with respect to Q. Proceeding repeatedly, we get a remainder reduced with respect to Q. Here we would obtain $T = x_1 x_2 y_1 (\partial_{(1,0)} y_1)^2 + y_1^4 y_2$.

Notice that if Q belongs to the ground ring $k[x_1, \ldots, x_n]$ then $\forall P \in \mathcal{R}$, $P \overset{Q}{\to} 0$.

The following lemma illustrates the use of performing such a reduction.

Lemma 1. *Let P and Q be two elements of $\mathcal{R}$, and let $f \in k[[x_1, \ldots, x_n]]$ such that $S_Q(f) \neq 0$ and $I_Q(f) \neq 0$. Let T be the remainder of P with respect to Q. Then*

$$\begin{cases} Q(f) = 0 \\ P(f) = 0 \end{cases} \Leftrightarrow \begin{cases} Q(f) = 0 \\ T(f) = 0. \end{cases}$$

Proof. It is an easy consequence of equation 1. Indeed, $Q(f) = 0 \Leftrightarrow (\theta Q(f) = 0,\ \forall \theta \in \Theta)$. $\square$

We can also use this process in order to reduce a p.d.p. P with respect to a set $\mathcal{A} = \{A_1, \ldots, A_r\}$ of p.d.p.: we reduce P with respect to the A_i's until we get a remainder reduced with respect to all of them. The result will of course depend on the way we proceeded; for example we could reduce P first with respect to A_1 and then with respect to A_2, or we could begin with A_2. We will denote this by $P \overset{\mathcal{A}}{\to} T$, or $\mathrm{red}(P, \mathcal{A}) = T$, assuming again that an algorithm has been chosen.

Remark 1. Let I be an ideal, and Q a polynomial, then we denote by $I : Q^\infty$ the ideal $\{Q | \exists a \in \mathbb{N}\ Q^a \in I\}$. With this usefull notation, $\mathrm{red}(P, \mathcal{A}) = 0$ implies that $P \in [\mathcal{A}] : H_{\mathcal{A}}^\infty$.

2.2 Systems of PDE's Associated with Initial Conditions

2.2.1 Auto-Reduced and Coherent Sets

Let us now define *auto-reduced and coherent sets* of p.d.p., which are in some sense analogous to Groebner bases in commutative algebra (although the algebraic properties of auto-reduced coherent sets are much weaker).

Definition 1. *A set $\mathcal{A} = \{A_1, \ldots, A_r\}$ of p.d.p. is* auto-reduced *(or a* chain *in [20], chap. 1) if each polynomial in $\mathcal{A}$ is reduced with respect to the others. We adopt the standard notations: $H_{\mathcal{A}} = \prod_{i=1\ldots r} I_{A_i} S_{A_i}$, and $S_{\mathcal{A}} = \prod_{i=1\ldots r} S_{A_i}$.*

Remark 2. If an auto-reduced set $\mathcal{A}$ contains a non-zero p.d.p A in the ground ring $k[[x_1, \ldots, x_n]]$, then $\mathcal{A} = \{A\}$, for every non zero p.d.p. is reducible by a non-zero p.d.p. in $k[[x_1, \ldots, x_n]]$.

Let P, Q be two p.d.p. such that there exist θ, θ' with $v_{\theta P} = v_{\theta' Q}$ and let us choose the smallest possible θ, θ'. Then the *S-polynomial* of P and Q is defined by

$$\mathrm{SPol}(P, Q) = \frac{S_Q}{\gcd(S_P, S_Q)} \cdot \theta P - \frac{S_P}{\gcd(S_P, S_Q)} \cdot \theta' Q.$$

Notice that $v_{(\mathrm{SPol}(P,Q))} < v_{\theta P} = v_{\theta' Q}$.

Definition 2. *If $\mathcal{A} = \{A_1, \ldots, A_r\}$ is such that SPol$(A_i, A_j) \xrightarrow{\mathcal{A}} 0$ then $\mathcal{A}$ is said to be* coherent.

Autoreduced and coherent sets of p.d.p. would be called formally integrable systems in other formalisms.

2.2.2 Solution of a System of Algebraic Partial Differential Equations

Let $\Sigma = \{P_1, \ldots, P_r\}$ be a set of p.d.p. in $\mathcal{R}$. Let $f = (f_1, \ldots, f_m)$ be an m-tuple of series in $k[[x_1, \ldots, x_n]]$. When substituting f_i to y_i for $i = 1, \ldots, m$ in one of the P_j's, we obtain a series, denoted by $P_j(f) \in k[[x_1, \ldots, x_n]]$. We say that f *is a solution of the system* $\Sigma(f) = 0$ if all the series $P_j(f)$, for $j = 1, \ldots r$, are the zero series. In this paper, we will focus on that kind of solution for systems of p.d.p.

Lemma 2. *Let $P \in \mathcal{R}$ be a p.d.p. and let $f = (f_1, \ldots, f_m)$ be a m-tuple of series in $k[[x_1, \ldots, x_n]]$. Then $P(f) = 0$ is equivalent to $(\theta P(f)(\underline{0}) = 0$, $\forall \theta \in \Theta)$, where $\underline{0}$ is the n-tuple $(0, \ldots, 0) \in \mathbb{R}^n$.*

Proof. $P(f) \in k[[x_1, \ldots, x_n]]$, so $P(f) = \sum_{\alpha \in \mathbb{N}^n} \frac{p_\alpha}{\alpha!} x^\alpha$, with the usual multi-index notation: $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We know that $p_\alpha = \partial_\alpha(P(f))\,(\underline{0}) \in k$. Of course, $P(f) = 0$ if and only if $(p_\alpha = 0, \forall \alpha \in \mathbb{N}^n)$, and this proves the lemma. □

In order to define a unique solution of a system of partial differential equations, we must specify initial/limit conditions.

Definition 3. *Let $\mathcal{A}$ be a subset of $\mathcal{R}$. A derivative θy_i is said to be* under the stair of *$\mathcal{A}$ if it is not the leader of any $\theta' A$, where $ord(\theta') > 0$, and $A \in \mathcal{A}$. We denote this by $\theta y_i \sqsubseteq \mathcal{A}$. The set $\mathcal{A}$ is said to be* closed *if the set of derivatives under the stair of $\mathcal{A}$ is finite.*
Consider the system $\mathcal{A}(f) = 0$, where the unknown f is an m-tuple $(f_1, \ldots, f_m)$ of series in $k[[x_1, \ldots, x_n]]$. A set of initial conditions *$I.C.$ for this system is a set of equations of the following form:*

$$I.C. = \{(\partial_\alpha f_i)(\underline{0}) = c_{(i,\alpha)} | \quad \partial_\alpha y_i \sqsubseteq \mathcal{A}\}, \; where \; c_{(i,\alpha)} \in k.$$

A set of initial conditions is proper for *$\mathcal{A}$ if it is computable and satisfies the following additional conditions:*

$$A(f)(\underline{0}) = 0, \quad S_A(f)(\underline{0}) \neq 0, \quad I_A(f)(\underline{0}) \neq 0, \quad for\ all\ A \in \mathcal{A}.$$

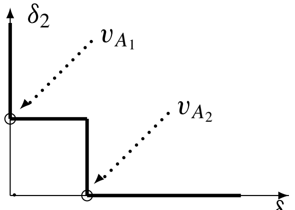
Remark 3. One can easily check that the values of $A(f)(\underline{0})$, $S_A(f)(\underline{0})$ and $I_A(f)(\underline{0})$ only depend on the coefficients (of the f_i's) determined by the initial conditions.

2.2.3 An Example

Consider the system:

$$\left\{A_1(f) = (\delta_2 f)^2 + f^2 - 1 = 0,\ A_2(f) = (\delta_1 f)^2 + f^2 - 1 = 0\right\},$$

with the initial conditions: $f(\underline{0}) = 0$, $\delta_1 f(\underline{0}) = 1$, $\delta_2 f(\underline{0}) = 1$. The following diagram shows the derivatives of y in the "plane" $\mathbb{N}^2$. The "stair" is represented by the thick lines. The derivatives of y situated "above" or "on" these lines are derivatives of v_{A_1} and v_{A_2}, i.e. they can be written θv_{A_i} with $\theta \in \Theta$ ($i = 1$ or $i = 2$).



Derivatives of y

Remark 4. According to definition 3, we consider as "under the stair" those derivatives that are not proper derivatives of v_{A_1} or v_{A_2}: this definition includes v_{A_1} and v_{A_2} themselves. The reason is that p.d.p. are in general of degree greater than 1, and so the values of $\delta_i f(\underline{0})$, $i = 1, 2$ are not uniquely determined by $\mathcal{A}$ and must be given by the set of initial conditions. In the example above, we have chosen $\delta_1 f(\underline{0}) = \delta_2 f(\underline{0}) = 1$, but $\delta_1 f(\underline{0}) = \delta_2 f(\underline{0}) = -1$ was another possible choice. Of course, if all the polynomials in $\mathcal{A}$ are linear, or quasi linear (meaning linear in their leading derivative), we may – and will sometimes – omit to give the values of the leading derivatives.

$\mathcal{A} = \{A_1(y), A_2(y)\}$ is an auto-reduced coherent set in $\mathcal{R} = \mathbb{Q}[x_1, x_2]\{y\}$ and the set of initial conditions is proper for $\mathcal{A}$. This system has a unique solution in $\mathbb{Q}[[x_1, x_2]]$ (which is simply $f(x_1, x_2) = \sin(x_1 + x_2)$). Indeed, differentiating the partial differential equations, we obtain $\delta_i(f)(\delta_i^2(f) + f) = 0$. As $\delta_i(f)(\underline{0}) \neq 0$, if f is a solution of the system, then of course $\delta_i(f) \neq 0$, and so we have $\delta_i^2(f) + f = 0$. This clearly gives us recurrence relations satisfied by the coefficients of an eventual solution f in $\mathbb{Q}[[x_1, x_2]]$. The coherence of $\mathcal{A}$ ensures us that there is no contradiction between the different recurrence relations of the form $\theta(A_i)(f)(\underline{0}) = 0$. E.g. to determine $\delta_1\delta_2 f(\underline{0})$ we can either use $\delta_1 A_2(f)$ or $\delta_2 A_1(f)$. $\mathrm{SPol}(A_1, A_2) \xrightarrow{\mathcal{A}} 0$ simply expresses that the result will be the same. Hence the system with the initial conditions has a unique solution in $\mathbb{Q}[[x_1, x_2]]$, and it is possible to compute the coefficients of the series f up to any order. Although this example is very simple, it illustrates a general result, stated in the next paragraph.

2.2.4 A Fundamental Theorem

The example exposed in the previous paragraph illustrates the following general result:

Theorem 1. *Let $\mathcal{A}$ be an auto-reduced coherent set in $\mathcal{R}$, and $I.C.$ a proper set of initial conditions for $\mathcal{A}$. Then there is a unique m-tuple of computable series $f = (f_1, \dots, f_m)$, $f_i \in k[[x_1, \dots, x_n]]$ satisfying the equations in $I.C.$ and $\mathcal{A}(f) = 0$.*

Proof. This is in fact a well known result, see [14, 21]. The reader can also find a simple sketch of the proof in [17, 18]. □

Notice that the set of initial conditions is not required to be finite. If it is infinite, we shall also speak of *limit conditions*.

In the sequel, we shall not only consider auto-reduced coherent sets, but also sets with weaker algebraic properties:

Definition 4. *A set $\mathcal{A} \subset \mathcal{R} = k[x_1, \dots, x_n]\{y_1, \dots, y_m\}$ is* block-wise auto-reduced and coherent *if $\mathcal{A} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_s$ where:*

$$\begin{cases} \bullet\ \mathcal{A}_1 \subset k[x_1, \dots, x_n]\{y_1, \dots, y_{r_1}\}, \\ \mathcal{A}_2 \subset k[x_1, \dots, x_n]\{y_1, \dots, y_{r_2}\}, \text{ and } v_A \in \{y_{r_1+1}, \dots, y_{r_2}\},\ \forall A \in \mathcal{A}_2, \\ \vdots \\ \mathcal{A}_i \subset k[x_1, \dots, x_n]\{y_1, \dots, y_{r_i}\}, \text{ and } v_A \in \{y_{r_{i-1}+1}, \dots, y_{r_i}\},\ \forall A \in \mathcal{A}_i, \\ \vdots \\ \mathcal{A}_s \subset k[x_1, \dots, x_n]\{y_1, \dots, y_m\}, \text{ and } v_A \in \{y_{r_{s-1}+1}, \dots, y_m\},\ \forall A \in \mathcal{A}_s. \\ \\ \bullet\ \text{Each } \mathcal{A}_i \text{ is auto-reduced and coherent.} \end{cases}$$

Theorem 2. *Let $\mathcal{A}$ be blockwise auto-reduced and coherent, and let $I.C.$ be a proper set of initial conditions for $\mathcal{A}$. Then there is a unique computable m-tuple of series $f = (f_1, \dots, f_m) \in (k[[x_1, \dots, x_n]])^m$, satisfying $\mathcal{A}(f) = 0$ and the equations in $I.C.$*

Proof. Using theorem 1, one can see that there is a unique r_1-tuple of series $g_1, \dots, g_{r_1}$ satisfying the equations in $\mathcal{A}_1$ and the initial conditions. Let

$g_i = \sum_{\alpha \in \mathbb{N}} \frac{d_{(i,\alpha)}}{|\alpha!|} x^\alpha$ (where $|\alpha| = \sum_{i=1}^n \alpha_i$), for $i = 1, \ldots r_1$. We can add to $I.C.$ the set of equations

$$\left\{\partial_\alpha f_i(\underline{0}) = d_{(i,\alpha)},\ \text{for } i = 1, \ldots r_1,\ \text{and for all } \alpha \in \mathbb{N}^n\right\}.$$

This gives us a new set of initial conditions $(I.C.)_1$, and the system $(\mathcal{A}, I.C.)$ is equivalent to the system $(\mathcal{A}_2 \cup \cdots \cup \mathcal{A}_s, (I.C.)_1)$. Proceeding repeatedly, we prove the theorem by induction. □

Remark 5. On should be carefull when using this theorem: a block-wise auto-reduced and coherent set, and even an autoreduced coherent set may very well admit no solution. This is not a contradiction, for in such a case there is no proper set of initial conditions.

Remark 6. In fact, as we use proper initial conditions, we only need to assume the set $\mathcal{A}$ to be coherent. But one should notice that testing in a naive way that the initial conditions are proper would lead to perform more or less the same computations that one would do to get an auto-reduced coherent set associated to $\mathcal{A}$ (see definition 5 in the next paragraph).

2.2.5 Computation of Auto-reduced and Coherent Sets

The previous theorems illustrate the interest of auto-reduced coherent sets of p.d.p. Consider a set $\mathcal{L}$ of p.d.p. In order to have more information about the solutions of the system defined by $\mathcal{L}$, the classical method is to compute a particular auto-reduced coherent set of p.d.p., having "more or less" the same solutions as $\mathcal{L}$. This method is sometimes called "completion of the system".

Lemma 3. *Let $\mathcal{L}$ be a finite subset of $\mathcal{R}$. One can compute an auto-reduced coherent set of p.d.p. $\mathcal{M}$ such that*

$$\begin{cases} [\mathcal{M}] \subseteq [\mathcal{L}], \\ L \xrightarrow{\mathcal{M}} 0, \ \textit{for all } L \in \mathcal{L}. \end{cases}$$

(Recall that $[\mathcal{M}]$ denotes the differential ideal generated by $\mathcal{M}$.)

Proof. A complete proof of this lemma (and the corresponding algorithm) can be found in [2, 17], or in [18], page 50. The principle of the method is analogous to the computation of Groebner bases: the syzygies (in Buchberger algorithm) are replaced by the S-polynomials, and one uses Ritt's reduction (exposed in Section 2.1). □

Definition 5. *Let $\mathcal{L}$ be a finite subset of $\mathcal{R}$. An auto-reduced coherent set $\mathcal{M}$ fulfilling the hypotheses of lemma 3 will be called* an auto-reduced coherent set associated to *$\mathcal{L}$ and denoted by* **Associated(**$P_1, \ldots, P_t$**)**.

Remark 7. Notice that such a set is not unique, but this abuse of notation may be justified if we assume that an algorithm computing such a set has been chosen.

Remark 8. Let f be a solution of $\mathcal{L}$. As $[\mathcal{M}] \subseteq [\mathcal{L}]$, f is necessarily a solution of $\mathcal{M}$. We have a partial converse: let f be a solution of $\mathcal{M}$, such that $H_{\mathcal{M}}(f) \neq 0$. Then f is also a solution of $\mathcal{L}$. (It is a straightforward generalisation of lemma 1, page 261.)

3 Outlines of the Method

Our aim is to be able to compute in differential extensions of $R = k[x_1, \ldots, x_n]$, obtained by adjunction of formal power series $f_1, \ldots, f_m \in k[[x_1, \ldots x_n]]$. We will asume that the m-tuple of series $f = (f_1, \ldots, f_m)$ is defined as the unique solution of a system $(\mathcal{A}, I.C.)$ satisfying the hypotheses of theorem 2. The set of initial conditions may be finite or infinite (see example 5.1.1), but it is supposed to be computable.

Let $P_1, \ldots, P_t$ be given p.d.p. in $\mathcal{R}$, we want to test whether $P_1(f) = \cdots = P_t(f) = 0$.

If one of the P_j's is in $k[x_1, \ldots, x_n]$, by hypothesis (k is an effective ring) we know how to test $P_j(f) = 0$. So, we assume that all the P_j's do indeed involve at least one of the indeterminates y_i. We first test whether $P_j(f)(\underline{0}) = 0$ for $j = 1, \ldots, t$. (That is, we test whether the constant coefficient of the series $P_j(f)$ is zero. We can do it because f is computable). If one of them is not zero, it is finished. Otherwise, we wish in fact to test whether f is a solution of the system $\{\mathcal{A}(f) = 0, P_1(f) = \cdots P_t(f) = 0\}$. We compute an auto-reduced coherent set $\mathcal{B}$ associated to $\mathcal{A} \cup \{P_1, \ldots, P_t\}$ (see definition 5). Recall that it means:

1. $[\mathcal{B}] \subset [A, P_1, \ldots, P_t]$,
2. for all $j = 1, \ldots, t$, $P_j \xrightarrow{\mathcal{B}} 0$,
3. for all $A \in \mathcal{A}$, $A \xrightarrow{\mathcal{B}} 0$.

Condition 1. implies that if $P_j(f) = 0$, $j = 1, \ldots, t$, then $\mathcal{B}(f) = 0$ ($A(f) = 0$ for each $A \in \mathcal{A}$, by hypothesis). In particular, if $\mathcal{B}$ contains a single p.d.p. $B \in k[x_1, \ldots, x_n]$, then $\mathcal{B}(f) \neq 0$, thus there is at least one i such that $P_i(f) \neq 0$. Suppose that $\mathcal{B}$ is not reduced to a single p.d.p. $B \in k[x_1, \ldots, x_n]$, or in other words, all elements in $\mathcal{B}$ do indeed involve at least one of the indeterminates y_i. Let us now investigate under which condition we have:

$$\mathcal{B}(f) = 0 \Longrightarrow P_j(f) = 0, \ \ j = 1, \ldots, t.$$

Conditions 2. and 3. imply that there exist positive integers ν_j, ν'_j such that

$$(H_{\mathcal{B}})^{\nu_j} P_j = \sum_{B \in \mathcal{B},\ \theta \in \Theta} M_{B,\theta} \cdot \theta B, \tag{2}$$

$$(H_{\mathcal{B}})^{\nu'_j} A_j = \sum_{B \in \mathcal{B},\ \theta \in \Theta} M'_{B,\theta} \cdot \theta' B_\ell, \tag{3}$$

where $M_{B,\theta} \in \mathcal{R}$, $M'_{B,\theta} \in \mathcal{R}$ and only a finite number of $M_{B,\theta}$ (resp. $M_{B,\theta}$) are non zero.

So if $H_{\mathcal{B}}(f) \neq 0$ and $\mathcal{B}(f) = 0$, then $P_j(f) = 0$, $j = 1 \ldots t$. Reciprocally, condition 1. implies that if $\mathcal{B}(f) \neq 0$, then there is at least one i such that $P_i(f) \neq 0$.

Remark 9. Notice that if one knows that at least one of the series $P_i(f)$ is non zero, it is easy to find such a P_i. Indeed, as the series $f_1, \ldots, f_m$ are computable, one can compute successively the sets

$$\text{Coeff}\ (\theta) = \left\{\theta P_1(f)(\underline{0}), \ldots, \theta P_t(f)(\underline{0})\right\}$$

for increasing θ until one finds $\theta P_i(f)(\underline{0}) \neq 0$, for some $i \in 1 \ldots t$ (see lemma 2). We shall call "**Which**" a procedure:

$$\begin{aligned} \text{Which}\ &: \text{List p.d.p.} \longrightarrow \text{p.d.p.} \\ &[P_1, \ldots, P_t] \longrightarrow P_i,\ 1 \leq i \leq t, \end{aligned}$$

where $P_i(f) \neq 0$. Of course, such a procedure stops if and only if there is at least one i such that $P_i(f) \neq 0$.

Hence our first aim will be to test whether $\mathcal{B}(f) = 0$, when $H_{\mathcal{B}}(f) \neq 0$.

Suppose $H_{\mathcal{B}}(f) \neq 0$. As mentioned in lemma 2, $\mathcal{B}(f) = 0$ is equivalent to

$$\theta B(f)(\underline{0}) = 0, \quad \forall \theta \in \Theta, \quad \forall B \in \mathcal{B}. \tag{4}$$

Indeed, $B(f) \in k[[x_1, \ldots, x_n]]$, so $B(f) = \sum_{\alpha \in \mathbb{N}^n} \frac{b_\alpha}{|\alpha|!} x^\alpha$, where $b_\alpha = \partial_\alpha B(f)$ $(\underline{0}) \in k$. As $B(f) = 0$ if and only if $(b_\alpha = 0, \forall \alpha \in \mathbb{N}^n)$, it is equivalent to equation (4). But we can't test this for an infinite set of $\theta \in \Theta$. The main tool that will be used to reduce this problem to a finite test is the following lemma, that we proved in [17]:

Lemma 4. *For all θ in Θ, and every B in $\mathcal{B}$, there exist positive integers ν, ν' such that:*

$$(H_{\mathcal{B}})^{\nu} (S_{\mathcal{A}})^{\nu'} \theta B = \sum_{\theta \in \Theta, A \in \mathcal{A}} M_{\theta,A} \cdot \theta A \ + \sum_{\theta B \in \overline{\mathcal{B}}} M'_{\theta B} \cdot \theta B, \tag{5}$$

where the M_ℓ's are in $\mathcal{R}$ and $\overline{\mathcal{B}} = \{\theta B$ such that $B \in \mathcal{B}$ and $v_{\theta B} \sqsubseteq \mathcal{A}\}$.

Proof. See [17]. The demonstration of the lemma only uses the facts that $\mathcal{B}$ is auto-reduced and coherent and that $\mathcal{A}$ is reduced to zero by $\mathcal{B}$. Although it was used in [17] only in the case when $\mathcal{A}$ is auto-reduced and coherent, it is also true if $\mathcal{A}$ is only block-wise auto-reduced and coherent. $\square$

What does this roughly mean? Equation (4) gives us the recurrence relations satisfied by a potential solution of $\mathcal{B}$. But as we know that f is a solution of $\mathcal{A}$, and as every solution of $\mathcal{B}$ which is not a solution of $H_{\mathcal{B}}$ is also a solution of $\mathcal{A}$, some of the recurrence relations given by (4) are implied by the system $\mathcal{A}$. Hence, we will only have to test that f satisfies the recurrence relations given by the system $\mathcal{B}$ and that are not already implied by the equations $\mathcal{A}(f) = 0$.

In the "regular cases"(we precise later what this means), we will only have to check that $\theta B(f)(\underline{0}) = 0,\ \forall B \in \mathcal{B},\ \forall \theta \in \Theta$ for $\theta B \in \overline{\mathcal{B}}$. If the set of initial conditions is finite (as in Section 4), the set $\overline{\mathcal{B}}$ is finite, and so we only have a finite set of algebraic equations to test. And we shall see in Section 5 how to use this approach if the set of initial conditions is infinite (we will then speak of limit conditions).

4 Review of a Known Method for a Particular Case

In this section, we consider an m-tuple of computable series $f = (f_1, \dots, f_m)$ which is the unique solution of a system given by a block-wise auto-reduced coherent set in $\mathcal{R} = k[x_1, \dots, x_n]\{y_1, \dots, y_m\}$ and a proper *finite* set of initial conditions. As in the previous section, we want to test whether $P_1(f) = \dots = P_t(f) = 0$, where the P_i's are *given* p.d.p. in $\mathcal{R}$. I gave in [17] a method and I recall it here briefly. A precise study of this algorithm will enable us to get more information.

Remark 10. In [17], I supposed that $f = (f_1, \dots, f_m)$ was defined as the unique solution of a system given by an auto-reduced coherent set in $\mathcal{R}$, and a proper finite set of initial conditions. These assumptions were just made to ensure that the system had a unique and computable m-tuple solution. However, the algorithm only requires the following hypotheses:

- existence, unicity and computability of the solution;
- finiteness of the set of initial conditions;
- the separants and the initials of the p.d.p. do not vanish at the origin (i.e. the initial conditions imply that $S_A(f)(\underline{0}) \neq 0,\ I_A(f)(\underline{0}) \neq 0,\ \forall A \in \mathcal{A}$).

One can easily check that the algorithm in [17] also "works" for systems given by a block wise auto-reduced coherent set and a proper finite set of initial conditions.

Suppose $P_i(f)(\underline{0}) = 0,\ i = 1, \dots t$. We compute the set $\mathcal{B}$ associated to $\mathcal{A} \cup \{P_1, \dots, P_t\}$, as exposed in Section 3. Notice that, in this Section, the set $\overline{\mathcal{B}} = \{\theta B$ such that $B \in \mathcal{B}$ and $v_{\theta B} \sqsubseteq \mathcal{A}\}$ is finite.

4.1 Degenerate Case

If $\mathcal{B}$ is reduced to a p.d.p. $B \in k[x_1, \ldots, x_n]$, of course $\mathcal{B}(f) \neq 0$. Thus, the system is not consistent and there is at least one i such that $P_i(f) \neq 0$.

4.2 Regular Case: $H_{\mathcal{B}}(f)(\underline{0}) \neq 0$

Using lemma 4, it is clear that if $H_{\mathcal{B}}(f)(\underline{0}) \neq 0$, then $\mathcal{B}(f) = 0$ if and only $\theta B(f)(\underline{0}) = 0$ for $\theta B \in \overline{\mathcal{B}}$, i.e. for a finite number of p.d.p. θB, which is very easy to test.

This can be formulated in a more algebraic language. Let N be the number of initial conditions (N is equal to the number of derivatives θy_i that are under the stair of $\mathcal{A}$). Now consider the vector c of k^N: $c = (c_{(i,\alpha)})$ where the $c_{(i,\alpha)}$'s are the initial conditions at the origin. We will call c: *vector of initial conditions*; f is the unique solution of the system $\mathcal{A}(f) = 0, \quad \partial_\alpha f_i = c_{i,\alpha}$.

Lemma 5. *Under the above hypotheses, let $Q \in \mathcal{R}$, and $Q_1 =$ Remainder $(Q, \mathcal{A})$, then*

$$Q(f)(\underline{0}) = 0 \Longleftrightarrow Q_1(f)(\underline{0}) = 0. \tag{6}$$

The second member of (6) is an algebraic equation depending only of the coefficients of c defined above.

Proof. $Q_1 = \text{Remainder}(Q, \mathcal{A})$ means that

$$(H_{\mathcal{A}})^\nu Q = \sum_{A \in \mathcal{A},\ \theta \in \Theta} M_{\theta,A} \cdot \theta A \; + \; Q_1,$$

where $H_{\mathcal{A}}$ is the product of the initials and separants of the elements of $\mathcal{A}$ and $\nu \in \mathbb{N}$. As $\mathcal{A}(f) = 0$, then $\theta A(f)(\underline{0}) = 0$, $\forall \theta \in \Theta$ and $\forall A \in \mathcal{A}$. We have assumed above that $H_{\mathcal{A}}(f)(\underline{0}) \neq 0$, hence $Q(f)(\underline{0}) = 0 \Leftrightarrow Q_1(f)(\underline{0}) = 0$. Moreover, Q_1 is reduced with respect to $\mathcal{A}$, it only involves derivatives that are under the stair of $\mathcal{A}$, so the p.d.p. $Q_1(f)(\underline{0}) = 0$ depends only of the coefficients of c. □

This lemma enables us to speak of the algebraic variety of k^N defined by a set of equations of the type $Q(f)(\underline{0}) = 0$ where $Q \in \mathcal{R}$. (Recall that $\mathcal{R} = k[x_1, \ldots, x_n]\{y_1, \ldots, y_m\}$.) Call W the variety of k^N defined by the equations $\{Q(f)(\underline{0}) = 0$ for $Q \in \overline{\mathcal{B}}\}$, and W' the variety defined by $H_{\mathcal{B}}(f)(\underline{0}) = 0$. Saying that we are in the regular case means that $c \notin W'$. And in this case $\mathcal{B}(f) = 0$ if and only if $c \in W$. Hence we showed that

Lemma 6. *If $c \notin W'$, then $\mathcal{B}(f) = 0 \Leftrightarrow c \in W$.*

We shall call "**TestRegular**" the following procedure:

$$\begin{array}{rcl} \text{TestRegular} : \text{List p.d.p.} & \longrightarrow & \text{Boolean} \\ \mathcal{B} & \longrightarrow & \begin{cases} \text{true if } c \in W, \\ \text{false if } c \notin W, \end{cases} \end{array}$$

where W is the variety defined by the equations $Q(f)(\underline{0}) = 0$ for $Q \in \overline{\mathcal{B}}$. This procedure can only be used when the set $\overline{\mathcal{B}}$ is finite.

4.3 Semi-Singular Case: $H_{\mathcal{B}}(f)(\underline{0}) = 0$, but $H_{\mathcal{B}}(f) \neq 0$.

In this case, we can test $\mathcal{B}(f) = 0$ by a continuity argument. I proved in [17] the following lemma:

Lemma 7. *Let f be the m-tuple of series defined by $\mathcal{A}$ and the vector of initial conditions c. If $H_{\mathcal{B}}(f)(\underline{0}) = 0$ but $H_{\mathcal{B}}(f) \neq 0$, then $\mathcal{B}(f) = 0$ if and only if $c \in \overline{W - W'}$, where $\overline{W - W'}$ denotes the Zariski closure of $W - W'$ in k^N.*

Proof. First notice that the Zariski and the metric closure of $(W - W')$ in k^N coincide. We already know that, if c is a vector of initial conditions

$$c \in (W - W') \Rightarrow \mathcal{B}(f) = 0.$$

(See Section 4.2, lemma 6). In other words:

$$c \in (W - W') \Longrightarrow \forall \theta \in \Theta,\ \forall B \in \mathcal{B},\ \theta B(f)(\underline{0}) = 0. \tag{7}$$

It was proved in [17] that for each $B \in \mathcal{B}$ and for each $\theta \in \Theta$, the value of $\theta B(f)(\underline{0})$ continuously depends on the coefficients of c. Hence we obtain

$$c \in \overline{W - W'} \Longrightarrow \forall \theta \in \Theta,\ \forall B \in \mathcal{B},\ \theta B(f)(\underline{0}) = 0.$$

Let us now prove that if c is a vector of initial conditions such that $H_{\mathcal{B}}(f) \neq 0$, then $(\mathcal{B}(f) = 0 \Rightarrow c \in \overline{W - W'})$. We shall prove it by abstract nonsense. Unfortunately, we must introduce some more notation. Indeed, we must "translate" in an algebraic language the fact that $H_{\mathcal{B}}(f) \neq 0$. We call $\mathcal{R}^\star$ the ring of polynomials in $n + N$ indeterminates

$$\mathcal{R}^\star = k[x_1, \ldots, x_n, Y_{(i,\alpha)},\ \text{where } \partial_\alpha y_i \text{ is under the stair of } \mathcal{A}].$$

If P is a p.d.p. of $\mathcal{R}$ involving only derivatives that are under the stair of $\mathcal{A}$, we shall note $P^\star$ the element of $\mathcal{R}^\star$ obtained by substituting $Y_{(i,\alpha)}$ to $\partial_\alpha y_i$. Let $I^\star$ be the ideal of $\mathcal{R}^\star$ generated by the following set of polynomials:

$$\left\{(\theta B)^\star,\ \text{ for } \theta B \in \overline{\mathcal{B}}\right\}.$$

Note that $c \in W$ if and only if $(\underline{0}, c) \in k^{N+n}$ is in the algebraic variety defined by $I^\star$. ($\underline{0}$ still denotes $(0, \dots, 0) \in k^n$.) In the same way, $c \notin W'$ if and only if $(\underline{0}, c) \in k^{N+n}$ is not in the algebraic variety defined by $H_{\mathcal{B}}^\star$.

Consider a vector of initial conditions c such that $\mathcal{B}(f) = 0$, $H_{\mathcal{B}}(f) \neq 0$ and assume $c \notin \overline{W - W'}$. This means that there is a polynomial $Q^\star$ in $I^\star : (H_{\mathcal{B}}^\star)^\infty$ such that $Q^\star((\underline{0}, c)) \neq 0$, and there exists $\nu \in \mathbb{N}$ such that

$$(H_{\mathcal{B}}^\star)^\nu Q^\star \in I^\star.$$

So

$$(H_{\mathcal{B}}^\star)^\nu Q^\star = \sum_{\theta B \in \overline{\mathcal{B}}} M_{\theta B}^\star \cdot (\theta B)^\star. \tag{8}$$

To each polynomial $P^\star$ of $\mathcal{R}^\star$, we can associate $P \in \mathcal{R}$ by substituting $\partial_\alpha y_i$ to $Y_{(i,\alpha)}$. The equation (8) then becomes:

$$(H_{\mathcal{B}})^{\nu_1} Q = \sum_{\theta B \in \overline{\mathcal{B}}} M_{\theta B} \cdot \theta B. \tag{9}$$

We assumed that $Q(f)(\underline{0}) = Q^\star(c) \neq 0$ (hence $Q(f) \neq 0$). Moreover $H_{\mathcal{B}}(f) \neq 0$ and $\mathcal{B}(f) = 0$ (hence $\theta B(f) = 0$ for all $\theta B \in \overline{\mathcal{B}}$). This is not compatible with equation (9), a contradiction which proves the lemma. □

Notice that this can be tested by means of Groebner bases for example (see [1, 17] or [8], chap. 4). We call "**TestSingular** " the following procedure:

$$\begin{array}{lcl} \text{TestSingular} : \text{List p.d.p.} & \longrightarrow & \text{Boolean} \\ \mathcal{B} & \longrightarrow & \begin{cases} \text{true if } c \in \overline{W - W'}, \\ \text{false if } c \notin \overline{W - W'}, \end{cases} \end{array}$$

where W is the variety defined by the equations $\{Q(f)(\underline{0}) = 0 \text{ for } Q \in \overline{\mathcal{B}}\}$, and W' the variety defined by $H_{\mathcal{B}}(f)(\underline{0}) = 0$. Notice that, (as TestRegular) the procedure TestSingular can only be used when $\overline{\mathcal{B}}$ is finite.

4.4 The Algorithm

The results of the three preceding paragraphs enable us to conclude in all cases. Suppose $f_1, \dots, f_m$ are defined by a closed blockwise auto-reduced coherent set $\mathcal{A}$ and a proper finite set of initial conditions. We give herafter an algorithm called **FiniteCase**. The input $\mathcal{P}_0$ is a set of p.d.p. and the ouput contains a boolean and two sets of p.d.p. The boolean is the answer to the test. If negative, the second set is $\mathcal{P}_0$. The third set is non void if the answer is positive. In such a case, it contains an auto-reduced coherent set $\mathcal{B}$ associated to $\mathcal{A}$ and the second set $\mathcal{P}$, which contains $\mathcal{P}_0$. This is due to the fact that the set of relations to be tested may be enlarged during the algorithm. Moreover, $\mathcal{P}(f) = 0$, $\mathcal{B}(f) = 0$ and $H_{\mathcal{B}}(f) \neq 0$.

FiniteCase: List p.d.p. $\longrightarrow$ (Boolean, List p.d.p., List p.d.p.)
$\mathcal{P}_0 \quad \longrightarrow$ R = (R.reply, R.ToTest, R.Equations)

Initialisation : $\mathcal{P} := \mathcal{P}_0$, $\mathcal{B} :=$ [ ], flag := true, $i := 1$

1. *We test whether the constant coefficients of the series $P_i(f)$ are zero or not.*
   For $P \in \mathcal{P}$, compute $P(f)(0)$.
   If one of them is non zero,
   Then **Return** R := (false, $\mathcal{P}$, $\mathcal{B}$)
   Else go to the next step.
2. *We compute an auto-reduced coherent set $\mathcal{B}$ associated to $\mathcal{A} \cup \{P_1, \dots, P_t\}$ and test whether the system is consistent.*
   $\mathcal{B}$:= Associated ($\mathcal{A}$, $\mathcal{P}$)
   If $\mathcal{B} = [B]$ where $B \in k[x_1, \dots, x_n]$
   Then **Return** R := (false, $\mathcal{P}$, [ ])
   Else go to the next step.
3. Compute $H := H_{\mathcal{B}}(f)(\underline{0})$
   If $H \neq 0$, (*this means that we are in the regular case*)
   Then flag := TestRegular($\mathcal{B}$) ; **Return** R := (flag, $\mathcal{P}$, $\mathcal{B}$)
   Else go to the next step
4. *We proceed recursively to test whether* $P_1(f) = \cdots = P_t(f) = Q(f) = 0$ *where Q is the initial or the separant of a p.d.p. $B \in \mathcal{B}$.*
   Let $\mathcal{H} = \{\text{initial}(B) \text{ for } B \in \mathcal{B}\} \cup \{\text{separant}(B) \text{ for } B \in \mathcal{B}\}$.
   For $Q \in \mathcal{H}$ repeat
   $\mathcal{P}$:= concat($\mathcal{P}_0$, Q)
   R' := FiniteCase($\mathcal{P}$)
   If (R'.reply = true) (*This means that $P(f) = 0$ for all $P \in \mathcal{P}$*)
   Then **Return** R'
   If (R'.reply = false) (*There is at least one $P \in \mathcal{P}$ such that $P(f) \neq 0$*)
   Then pol := Which($\mathcal{P}$) (*We find $P \in \mathcal{P}$ such that $P(f) \neq 0$*)
   If (pol $\in \mathcal{P}_0$ then **Return** R := (false, $\mathcal{P}_0$, [ ])
5. *We are in the semi-singular case, and we use the procedure* TestSingular
   flag := TestSingular($\mathcal{B}$)
   Return R := (flag, $\mathcal{P}$, $\mathcal{B}$)

Theorem 3. *Let $\mathcal{P}_0$ be a given set of p.d.p.*

1. *The algorithm FiniteCase terminates.*
2. *Call R=FiniteCase($\mathcal{P}_0$). Then we are in one of the following two cases:*

$$(i) \begin{cases} -R.reply = false \\ -\exists P \in \mathcal{P}_0 \text{ such that } P(f) \neq 0, \end{cases}$$
$$(ii) \begin{cases} -R.reply = true, \\ -\text{each } P \in \mathcal{P}_0 \text{ is an element of } R.ToTest, \\ -Q(f) = 0,\ \forall Q \in R.ToTest, \\ -\mathcal{B} = R.Equations \text{ is an auto-reduced coherent set associated} \\ \text{to } \mathcal{A} \cup R.ToTest. \text{ Moreover } H_{\mathcal{B}}(f) \neq 0, \text{ and } \mathcal{B}(f) = 0. \end{cases}$$

Proof. The first part of the theorem was already proved in [17] (Section 3.4). Indeed, the algorithm exposed in [17] performed exactly the same computations as this one, the only difference being that we store more information in this present version. We only provide a sketch of the arguments.

The only problem is that apparently infinitely many recursive calls of the algorithm may occur. But at each recursive call, we add an initial or a separant of the set of equations, so that we compute an auto-reduced coherent set of strictly lower rank than the preceding set computed. It is known that there is no infinite sequence of auto-reduced sets (see [20], page 164, or [16], chap. 1, page 81, or in [17], Section 2.2.).

To prove part 2 of the theorem, we must study at each step of the algorithm the way the value of R is assigned. The result of a recursive call is supposed correct by induction.

Step 1 only tests if the $P(f)$'s are trivially non zero: if there is one $P \in \mathcal{P}_0$ such that $P(f)(\underline{0}) \neq 0$, then the algorithm returns (false, $\mathcal{P}_0$,[ ]). Thus we are in case (i).

If not, then, as explained in Section 3 and at the beginning of Section 4, we compute at step 2 an auto-reduced coherent set $\mathcal{B}$ associated to $\mathcal{A} \cup \{P_1, \dots, P_t\}$ (see remark 5). Then, we check that we are not in a degenerate case: if $\mathcal{B}$ is reduced to a polynomial in $k[x_1, \dots, x_n]$, then of course, $\mathcal{B}(f) \neq 0$. In other words the system is not compatible and $P(f) \neq 0$ for some $P \in \mathcal{P}_0$. The algorithm then returns (false, $\mathcal{P}_0$,[ ]), thus we are again in case (i).

At step 3, we test whether we are in a regular case. If $H_{\mathcal{B}}(f)(\underline{0}) \neq 0$, we just use the procedure TestRegular. If TestRegular($\mathcal{B}$) is "false", the returned result corresponds to case (i) (indeed, R.ToTest=$\mathcal{P}$ and at this stage, $\mathcal{P} = \mathcal{P}_0$). If TestRegular($\mathcal{B}$) is "true", then the returned result corresponds to case (ii).

At step 4, we test recursively whether $H_{\mathcal{B}}(f) = 0$. In fact, we apply the procedure FiniteCase successively to the sets of p.d.p. $\{P_1, \dots, P_t, Q\}$, where Q is an initial or a separant of some $B \in \mathcal{B}$. Call R' the answer of such a recursive call.

- If R'.reply = true, then R'.ToTest contains at least $P_1, \dots, P_t, Q$. Indeed, at each recursive call of the algorithm, we add an element to the set of p.d.p. $\mathcal{P}$ but no element is suppressed. As there may have been many "nested" recursive calls of the algorithm, R'.ToTest may contain many other p.d.p. $Q_1, \dots, Q_s$, but contains at least the p.d.p. initially present in $\mathcal{P}$. By recursion hypothesis, R'.reply satisfies all the remaining conditions of (ii).
- If R'.reply = false, then it means that one $P \in \mathcal{P}$ satisfies $P(f) \neq 0$, where $\mathcal{P} = \{P_1, \dots, P_t, Q\} = \mathcal{P}_0 \cup \{Q\}$. So we must decide whether there is $P \in \mathcal{P}_0$ such that $P(f) \neq 0$ or whether $Q(f) \neq 0$. Hence we apply the procedure Which (see remark 9). If the answer is an element of $\mathcal{P}_0$, the algorithm FiniteCase returns a result corresponding to case (i). Otherwise, we just go on testing whether f annuls the initials and the separants of the B's.

So we only come to step 5 of the algorithm when we know that $H_{\mathcal{B}}(f) \neq 0$. We then just apply the procedure TestSingular. If TestSingular($\mathcal{B}$) is "false", we are in case (i) (indeed, R.ToTest only contains the elements of $\mathcal{P}_0$). And if TestSingular($\mathcal{B}$) is "true", the returned result corresponds to case (ii). Moreover notice that, as in step 3, the set R.Equations contains in fact an auto-reduced coherent set $\mathcal{B}$ which is associated to $\{\mathcal{A} \cup \text{R.ToTest}\}$, and such that $H_{\mathcal{B}}(f) \neq 0$. □

4.5 An Example

Let $\mathcal{R} = \mathbb{Q}[x]\{f, g\}$, which is an ordinary differential ring, with $\delta = \frac{\partial}{\partial x}$. Consider the system of ordinary differential equations:

$$A_1 = \delta^2 f - xf$$
$$A_2 = \delta^4 g + 2g \cdot (\delta^2 g) - (\delta g)^2 - 8x(\delta f)^2 - 12f \cdot (\delta f) - 4xg^2 - 8x^2 f^2$$

with the set of initial conditions:

$$f(0) = 0,\ (\delta f)(0) = 1,\ \ g(0) = 0, (\delta g)(0) = 0,\ (\delta^2 g)(0) = 2,\ (\delta^3 g)(0) = 0.$$

We choose the ranking respecting the order of derivation, and such that $f < g$. As the system is quasi linear, we do not need here to assign a value to $\delta^2 f$ and $\delta^4 g$ in order to define a unique solution (*cf* remark 4). We shall test that this unique solution (f, g) of the system satisfies the equation: $P(f, g) = (\delta g) - 2f \cdot (\delta f) = 0$. One easily tests that $P(f, g)(0) = 0$, and so we have to compute an auto-reduced coherent set $\mathcal{B}$ associated to $\{A_1, A_2, P\}$ (step 2 of the algorithm). We obtain $\mathcal{B} = \{B_1, B_2\}$, with

$$B_1 = x^3 f^{15} g^2 - 2x^3 f^{17} g + x^3 f^{19},$$
$$B_2 = (x^3 f^{15} g - x^3 f^{17})(\delta f)^2 - x^4 f^{17} g + x^4 f^{19}.$$

For a complete exposition of the algorithm used to compute $\mathcal{B}$, the reader is referred to [2, 17, 18].

$\mathcal{B}$ is not reduced to an element of $\mathbb{Q}[x]$, and so we go to the third step of the algorithm, and we test whether the product $H_{\mathcal{B}}$ of the initials and the separants of the elements of $\mathcal{B}$ vanishes at the origin. We have $I_{B_1} = x^3 f^{15}$, $I_{B_2} = (x^3 f^{15} g - x^3 f^{17})$, $S_{B_1} = 2x^3 f^{15} g - 2x^3 f^{17}$, $S_{B_2} = (2x^3 f^{15} g - 2x^3 f^{17})(\delta f)$. They all vanish at the origin, and so we have to go to the next step. Now we shall test successively whether $Q(f, g) = 0$, where Q is an initial or a separant of B_i. We call the algorithm to test whether $(P(f, g) = 0, I_{B_1}(f, g) = 0)$, and we obtain that $I_{B_1}(f, g) \neq 0$. So we call the algorithm to test whether $(P(f, g) = 0, I_{B_2}(f, g) = 0)$ and we obtain the answer (true, $[P, I_{B_2}]$, $[x^3 f^{15} g - x^3 f^{17}, \delta^2 f - xf]$) which means that $P(f, g) = 0$ and $I_{B_2}(f, g) = 0$ and that the second list of p.d.p.'s is an auto-reduced coherent set associated to $\{P, I_{B_2}\}$. Let us detail this. To test $P(f, g) = I_{B_2}(f, g) = 0$, the algorithm computes an auto-reduced coherent set $\mathcal{C}$ associated to $\{P, I_{B_2}\}$ (step 2): $\mathcal{C} = \{C_1 = x^3 f^{15} g - x^3 f^{17},\ C_2 = f^{(2)} - xf\}$. We now go to step 3: we have $I_{C_1} = S_{C_1} = x^3 f^{15}$, and $I_{C_2} = S_{C_2} = 1$. We see that $I_{C_1}(f, g)(0) = S_{C_1}(f, g)(0) = 0$ and that $I_{C_2}(f, g)(0) \neq 0,\ S_{C_2}(f, g)(0) \neq 0$. Hence we have to go to step 4 of the algorithm. We thus test recursively whether $x^3 f^{15} = 0$, and we obtain that this series is not the zero series. Hence we can go to step 5 of the algorithm: we know that $\mathcal{C}$ is associated to $\{P, I_{B_2}\}$, and that $H_{\mathcal{C}}(f, g) \neq 0$ but that $H_{\mathcal{C}}(f, g)(0) = 0$. So we are in a semi-singular case. We compute the derivatives of the elements of $\mathcal{C}$ whose leaders are under the stair of $\mathcal{A}$, i.e. $\overline{\mathcal{C}} = \{C_1,\ \delta C_1,\ \delta^2 C_1,\ \delta^3 C_1,\ \delta^4 C_1,\ C_2\}$. We now consider these polynomials as elements of $\mathbb{Q}[x,\ f, \delta f, \delta^2 f,\ g, \delta g, \delta^2 g, \delta^3 g, \delta^4 g, z]$ where z is a new indeterminate, higher than all the others. We now have to test whether the initial conditions belong to $\overline{W - W'}$ where W is the variety defined by the equations in $\overline{\mathcal{C}^\star}$ and W' is the variety defined by the equation $zH_{\mathcal{C}^\star} - 1 = 0$. So we compute a Groebner basis of the ideal generated by $\overline{\mathcal{C}^\star} \cup \{zH_{\mathcal{C}^\star} - 1\}$ for an ordering eliminating z, and we only keep in this basis the elements not containing the indeterminate z: $\{g - f^2, \delta g - 2f \cdot \delta f, \delta^2 f - xf, \delta^2 g - 2(\delta f)^2 - 2xf^2, \delta^3 g - 8xf \cdot \delta f - 2f^2\}$. One easily checks that all these polynomials vanish when the indeterminates are replaced by the corresponding initial conditions. The whole computation lasted 102 seconds on an IBM RS/6000 360.

Remark 11. One may notice on this example that the autoreduced and coherent sets computed are not in general char sets of some ideal, e.g. $\mathcal{B}$ is not. In fact the computations made in this algorithm to test that the non vanishing of initials and separants are quite similar to those done in the diffalg algorithm to test that they are invertible (see [13, 2, 3]).

5 More General Extensions of $k[x_1, x_2]$

In this section, we study extensions of $k[x_1, x_2]$ obtained by adjunction of series defined by systems of p.d.p., with limit conditions given on the axis $x_1 = 0$. The formalism we use enables us to proceed as above, basically following the outlines exposed in Section 3. But $\mathcal{R}$ will here denote the differential ring $\mathcal{R} = k\{y_1, \ldots y_m\}$. Our approach could be extended to the case of systems in more variables, with limit conditions given on one axis of coordinate. In order to "simplify" the notations, we restrict ourselves to systems involving only two derivations. Our purpose is not to give an exhaustive method, but to show that the formalism already exposed in this work makes it possible to deal with identity problems for quite a large class of problems, although some singular cases remain unsolved. The resolution of such cases is shown to be equivalent to the so called Ritt problem: to test whether a given singular solution of a system belongs to its general solution (see Section 5.3.4).

5.1 Introduction of a Particular Class of Extensions

Before giving some more definitions to specify our framework, let us work out an example.

5.1.1 An Introductory Example

Consider the 3-tuple:

$$f_1 = \exp(x_2), \quad f_2 = \sin(x_1), \quad f_3 = \exp(-x_2)\sin(x_1).$$

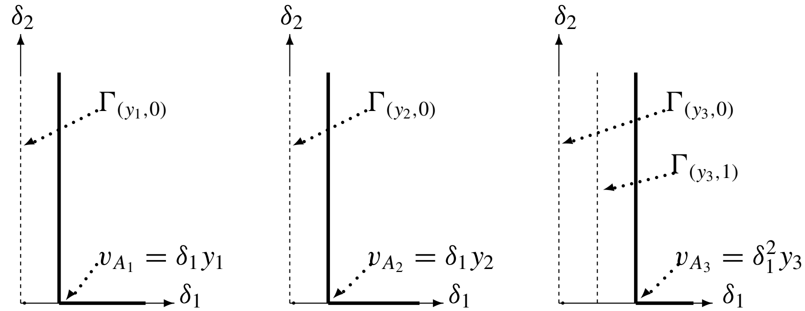
It can be defined as the solution of a system of partial differential equations with limit conditions given on the axis $x_1 = 0$. Consider

$\mathcal{A} \subset \mathbb{Q}\{y_1, y_2, y_3\} : \mathcal{A} = \{A_1 = \delta_1 y_1,\ A_2 = (\delta_1 y_2)^2 + y_2^2 - 1,\ A_3 = \delta_1^2 y_3 + y_3\}$. The set E of derivatives that are under the stair of $\mathcal{A}$ can be described in the following way:

$$\begin{aligned} E = \{v_{A_1}\, v_{A_2},\ v_{A_3}\} &\cup \{\delta_2^\ell y_1,\ \forall \ell \in \mathbb{N}\} \cup \{\delta_2^\ell y_2,\ \forall \ell \in \mathbb{N}\} \\ &\cup \{\delta_2^\ell y_3,\ \forall \ell \in \mathbb{N}\} \cup \{\delta_2^\ell \delta_1 y_3,\ \forall \ell \in \mathbb{N}\} \end{aligned}$$

$$\text{or} \begin{cases} E = \text{Leaders}(\mathcal{A}) \cup \Gamma_{(y_1,0)} \cup \Gamma_{(y_2,0)} \cup \Gamma_{(y_3,0)} \cup \Gamma_{(y_3,1)} \\ \qquad \text{where } \Gamma_{(y_i,s)} = \{\delta_2^\ell \delta_1^s y_i,\ \forall \ell \in \mathbb{N}\}. \end{cases}$$

The following three diagrams represent the derivatives of y_1, y_2 and y_3. Let us detail the first one (on the left). We represented in the plane $\mathbb{N}^2$ the derivatives of y_1. The thick lines represent the stair. We have an infinite number of θy_1 that are situated "under the stair": all the $\delta_2^\ell y_1$ (i.e. $\Gamma_{(y_1,0)}$) and v_{A_1}. Thus we will have to give initial conditions corresponding to these derivatives, or in other words, we will have to set numerical values for the corresponding coefficients of the series.



Derivatives of y_1 Derivatives of y_2 Derivatives of y_3

We need to define $\theta f_i(0,0)$ for each θy_i in E. The polynomials A_1 and A_3 are linear with respect to their leaders, so we only need to fix initial conditions for the derivatives of f_1 and f_3 that are strictly under the stair of $\mathcal{A}$.

A set of equations of the form: $\{\delta_2^\ell \delta_1^s f_i(0,0) = c_{i,(s,\ell)},\ \text{ for } \ell \in \mathbb{N}\}$ is equivalent to the single equation $(\delta_1^s f_i)(0, x_2) = \sum_{\ell \in \mathbb{N}} \frac{c_{i,(s,\ell)}}{\ell!}(x_2)^\ell = \tilde{f}_{(i,s)}(x_2)$. Hence we shall here define the following series in $\mathbb{Q}[x_2]$:

$$\tilde{f}_{(i,s)}(x_2) = \delta_1^s f_i(0, x_2) \text{ for } \Gamma_{(y_i,s)} \subset E. \tag{10}$$

The $\tilde{f}_{(i,s)}$'s can be defined by a system of differential equations and a finite set of initial conditions: $\tilde{\mathcal{A}} \subset \mathbb{Q}[x_2]\{y_{(1,0)},\ y_{(2,0)},\ y_{(3,0)},\ y_{(3,1)}\}$,

$$\tilde{\mathcal{A}} = \{\delta_2 y_{(1,0)} - y_{(1,0)},\ y_{(2,0)},\ y_{(3,0)},\ \delta_2 y_{(3,1)} + y_{(3,1)}\},$$

$$\text{and } I.C. = \{\tilde{f}_{(1,0)}(0) = 1,\ \tilde{f}_{(2,1)}(0) = 1,\ \tilde{f}_{(3,1)}(0) = 1\}.$$

Let $\mathcal{I.C.} = I.C. \cup \{\delta_1 f_{2,}(0,0) = 1\}$. It is easy to see that this system has a unique 5-tuple of solutions in $\mathbb{Q}[x_2]$. The limit conditions determined by equations (10) are proper. The system $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{I.C.})$ completely defines (f_1, f_2, f_3).

5.1.2 Our Framework

Definition 6. *Let $\mathcal{A}$ be an auto-reduced and coherent set in $\mathcal{R}$. $\mathcal{A}$ is* admissible *if $\mathcal{A} = \{A_1, \ldots, A_m\}$, and the set $Leaders(\mathcal{A}) = \{v_A,\ \text{for } A \in \mathcal{A}\}$ is of the following type: $Leaders(\mathcal{A}) = \{\delta_1^{\nu_1} y_1, \ldots, \delta_1^{\nu_m} y_m\}$, where the ν_i's are in $\mathbb{N}$.*

In Section 5.1.1, $\mathcal{A}$ was admissible. In the sequel, $\mathcal{A}$ will always be an admissible auto-reduced coherent set in $\mathcal{R}$. The set E of derivatives that are under the stair of $\mathcal{A}$ is very easy to describe:

$$E = \text{Leaders}(\mathcal{A}) \cup \left(\bigcup_{s<\nu_i} \Gamma_{(y_i,s)} \right),$$

where $\Gamma_{(y_i,s)} = \{\delta_1^s \delta_2^\ell y_i \text{ for } \ell \in \mathbb{N}\}$. (See example 5.1.1). Suppose for example that $\Gamma_{(y_1,2)} \subset E$. In other words, $\delta_1^2 \delta_2^\ell y_1$ is under the stair of $\mathcal{A}$ for all $\ell \in \mathbb{N}$. We have to specify a set of initial conditions containing, among others, the equations of the form:

$$(\delta_1^2 \delta_2^\ell f_1)(\underline{0}) = c_{1,(2,\ell)} \text{ for all } \ell \in \mathbb{N}. \tag{11}$$

And, as in example 5.1.1, this means that the series $\sum_{\ell \in \mathbb{N}} \frac{(\delta_1^2 \delta_2^\ell f_1)(\underline{0})}{\ell!} x_2^\ell$ must be entirely determined. This latter series is exactly $(\delta_1^2 f_1)(0, x_2)$. Thus giving the set of equations (11) is equivalent to giving a series $\tilde{f}_{1,2} \in k[[x_2]]$ and the equation $(\delta_1^2 f_1)(0, x_2) = \tilde{f}_{1,2}$. So, for each $\Gamma_{(y_i,s)} \subset E$, we need to define

$$\delta_1^s f_i(0, x_2) = \tilde{f}_{i,s} \in k[[x_2]]. \tag{12}$$

We shall consider in this paper the case when the series $\tilde{f}_{i,s}$ are defined as the unique solution of a differential system with a finite set of initial conditions. The series $\tilde{f}_{i,s}$ can be seen as limit conditions on the axis $x_1 = 0$.

Remark 12. This assumption is not a great limitation. In fact, we only need to assume that for all y_i, $\mathcal{A}$ contains a p.d.p. whose leader is $\delta_1^{\nu_i} y_i$. The other elements of $\mathcal{A}$ may be considered as elements of $\tilde{\mathcal{A}}$, defining initial condition. We may also remark that this assumption is generic. E.g. consider a change of derivation operators $d_i = \sum_{i=1}^{2} c_{i,j}\delta_j$, then for any p.d.p. $P(\delta, y)$ and for almost all values of the constants $c_{i,j}$, we get a new p.d.p $P'(d, y)$ whose leader is of the form $d_1^{\nu} y$.

Lemma 8. *Let $\mathcal{A}$ be an admissible auto-reduced and coherent set in $\mathcal{R}$. Let E be the set of derivatives that are under the stair of $\mathcal{A}$:*

$$E = Leaders(\mathcal{A}) \cup \left(\bigcup_{s<\nu_i} \Gamma_{(y_i,s)} \right) = Leaders(\mathcal{A}) \cup \left(\bigcup_{s<\nu_i} \{\delta_1^s \delta_2^\ell y_i \; for \; \ell \in \mathbb{N}\} \right)$$

Let $(\tilde{f}_{(i,s)}$, for $s < \nu_i)$ be a $(\nu_1 + \cdots + \nu_m)$-tuple of series in $k[[x_2]]$ defined by a block-wise auto-reduced coherent set $\tilde{\mathcal{A}} \subset \tilde{\mathcal{R}} = k[x_2]\{y_{(i,s)}$, for $s < \nu_i\}$ with a proper finite *set of initial conditions $I.C$. Let $c_1, \ldots, c_m$ be m elements in k. The set of equations*

$$\{\delta_1^{\nu} f_i(0, x_2) = \tilde{f}_{(i,\nu)}(x_2) \; for \; s < \nu_i, \; \delta_1^{\nu_i} f_i(\underline{0}) = c_i, \;\; for \; i = 1, \ldots, m\} \quad (13)$$

entirely determines the values of all the $\theta f_i(\underline{0})$ such that $\theta y_i \sqsubseteq \mathcal{A}$. If these values are such that $S_A(f)(\underline{0}) \neq 0$, $I_{\mathcal{A}}(f)(\underline{0}) \neq 0$, and $A(f)(\underline{0}) = 0$, then the set of equations (13) is a proper set of initial conditions for $\mathcal{A}$.

Proof. The set (13) contains one equation for each derivative under the stair of $\mathcal{A}$, if we consider an equation between series as a set of equations between their coefficients. As $I.C.$ is a proper set of initial conditions for $\tilde{\mathcal{A}}$, according to theorem 1, $\mathcal{I.C}.$ is computable. With the additional hypotheses $S_A(f)(\underline{0}) \neq 0$, $I_{\mathcal{A}}(f)(\underline{0}) \neq 0$, and $A(f)(\underline{0}) = 0$ this implies that the set of equations (13) is a proper set of initial conditions for $\mathcal{A}$.

Definition 7. *Let $\mathcal{A}$ be an admissible auto-reduced and coherent set in $\mathcal{R}$. Let $\tilde{\mathcal{A}}$, $I.C.$ be as in lemma 8, and $c_1, \ldots c_m$ be elements of k. Call $\mathcal{I.C}.$ the set $I.C. \cup \{\delta_1^{\nu_i} f_i(\underline{0}) = c_i, \;\; for \; i = 1, \ldots, m\}$. If the set of equations (13) is a proper set of initial conditions for $\mathcal{A}$, then the system $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{I.C}.)$ is then called a* complete system.

Of course, the system considered in example 5.1.1 was complete, according to this definition.

Theorem 4. *Let $\Sigma = (\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{I.C.})$ be a complete system. Σ defines a unique m-tuple $f = (f_1, \ldots, f_m)$ of computable series in $k[[x_1, x_2]]$. The coefficients of the f_i's continuously depend on the initial conditions in $\mathcal{I.C}$.*

Proof. The existence and unicity of the solutions are straightforward consequences of theorem 1. Let us detail the way we can compute any coefficient of f_i. Let $\theta \in \Theta$. Let us compute $\theta f_i(\underline{0})$. If $\theta f_i(\underline{0})$ is given by an equation in $\mathcal{I.C.}$, there is nothing to do. If $\theta y_i \in \Gamma_{(y_i,s)}$ where $s < \nu_i$, then $\theta f_i(\underline{0}) = \delta_1^{\alpha_1}\delta_2^{\alpha_2} f_i(0,0) = \delta_2^{\alpha_2}\tilde{f}_{(i,\alpha_1)}(0)$. There is $\tilde{A} \in \tilde{\mathcal{A}}$ such that $\delta_2^{\alpha_2} y_{(i,\alpha_1)} = \delta_2^{\ell} v_{\tilde{A}}$, hence $\delta_2^{\ell}\tilde{A}(\tilde{f})(0) = (S_{\tilde{A}}(\tilde{f})(0))\ \theta f_i(0) + (\tilde{T}(\tilde{f})(0)) = 0$ where $\tilde{T}$ contains only derivatives of lower order than $v_{\delta_2^{\ell}\tilde{A}}$. As $S_{\tilde{A}}$ only involves derivatives that are under the stair of $\tilde{\mathcal{A}}$, it is clear that $S_{\tilde{A}}(\tilde{f})(0)$ is a polynomial in the initial conditions, which is non zero by hypothesis. Hence $\theta f_i(\underline{0}) = -\left(\frac{1}{S_{\tilde{A}}(\tilde{f})(0)}\right)(\tilde{T}(\tilde{f})(0))$ and we can apply the same process to the derivatives involved in $\tilde{T}$. As there is no infinite sequence of derivatives of decreasing order, the procedure will stop. At the end, we obtain $\theta f_i(\underline{0}) = \frac{\mathrm{Num}(\tilde{f})(0)}{\mathrm{Denom}(\tilde{f})(0)}$ where $\mathrm{Denom}(\tilde{f})(0)$ is a product of the separants of the elements of $\tilde{\mathcal{A}}$, and $Num \in \tilde{\mathcal{R}}$ involves only derivatives that are under the stair of $\tilde{\mathcal{A}}$.

Proceeding in the same way, if $\theta y_i = v_{\theta' A}$ for some $A \in \mathcal{A}$, we see that $\theta f_i(\underline{0}) = \frac{Num(f)(\underline{0})}{Denom(f)(\underline{0})}$, where $Denom$ is a product of the separants of the elements of $\mathcal{A}$, and Num involves only derivatives that are under the stair of $\mathcal{A}$. Thus all the $\theta' f(\underline{0})$ involved in these expressions can be computed as exposed just above and can be expressed as a rational function of the initial conditions. □

Remark 13. Notice that this framework enables us to consider extensions of $k[x_1, x_2]$ defined by equations involving explicitely x_1 and x_2, and to test equations involving explicitely x_1 and x_2. These variables belong to the ground ring, so we may use them freely. We can also, when needed, consider the ground ring to be $k(x_1, x_2)$ if working in a field may help, e.g. for Groebner bases computations. However, not all computer algebra system allow to use such a field when computing standard bases. Except in strongly typed system such as Axiom or Magma, the choice of possible fields is in general very limited.

In such cases, we can bypass this trouble by considering the 2-tuple (x_1, x_2) as the unique solution of the system $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{I.C.})$ where

$$\mathcal{A} = \{\delta_1 y_1 - 1,\ \delta_1 y_2\} \subset k\{y_1,\ y_2\},$$

$$\tilde{\mathcal{A}} = \{\delta_2 y_{(1,0)},\ \delta_2 y_{(2,0)} - 1\} \subset k\{y_{(1,0)}, y_{(2,0)}\},$$

$$\mathcal{I.C.} = \{\tilde{f}_{(1,0)}(0) = 0,\ \tilde{f}_{(2,0)}(0) = 0\}.$$

Consider now $f_3 = x_2 \sin(x_1)$. The 3-tuple (x_1, x_2, f_3) is the unique solution of the system: $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{I.C.})$ where

$$\mathcal{A} = \{\delta_1 y_1 - 1,\ \delta_1 y_2,\ \delta_1^2 f_3 + f_3\}$$

$$\tilde{\mathcal{A}} = \{\delta_2 y_{(1,0)},\ \delta_2 y_{(2,0)} - 1,\ y_{(3,0)},\ \delta_2 y_{(3,1)} - 1\}$$

$$\mathcal{I.C.} = \{\tilde{f}_{(1,0)}(0) = 0,\ \tilde{f}_{(2,0)}(0) = 0,\ \tilde{f}_{(3,1)}(0) = 0\}.$$

If we want to test that $x_2\delta_2 f_3 - f_3 = 0$, using this formalism we would test that $f_2 \cdot \delta_2 f_3 - f_3 = 0$.

5.2 *Effective Test in a Regular Case*

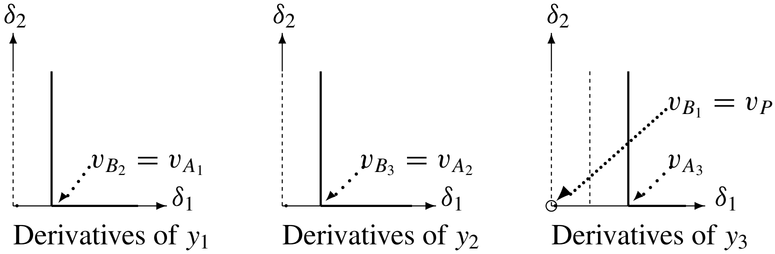
Let us first state a preliminary lemma (very similar to lemma 2, page 262).

Lemma 9. *Let Q be a p.d.p. in $\mathcal{R}$. Let g be a series in $k[[x_1, x_2]]$. The series $Q(g)$ is the zero series if and only if all the series $(\delta_1^\ell Q(g)(0, x_2),\ \text{for } \ell \in \mathbb{N})$ are the zero series in $k[[x_2]]$.*

Proof. The series $Q(g) \in k[[x_1, x_2]]$ can be considered as a series in $k[[x_2]][[x_1]]$. We write this $Q(g) = \sum_{i\in\mathbb{N}} \frac{b_i(x_2)}{i!} x_1^i$, where $b_i(x_2) \in k[[x_2]]$. Thus $Q(g) = 0$ if and only if $b_i(x_2) = 0,\ \forall i \in \mathbb{N}$. As $b_i(x_2) = \delta_1^i Q(g)(0, x_2)$, the lemma is proved. □

5.2.1 An Introductory Example

Come back to the example exposed in Section 5.1.1 and let us try to test whether $P(f) = f_1 f_3 - f_2 = 0$. We first compute $P(f)(\underline{0})$. (Using the equations in $\tilde{\mathcal{A}}$ and in $I.C.$, we know that $f_1(\underline{0}) = 1$, $f_2(\underline{0}) = 0$, $f_3(\underline{0}) = 0$.) We obtain that $P(f)(\underline{0}) = 0$. So we compute $\mathcal{B} = \text{Associated}(\mathcal{A} \cup \{P\})$ (see déf. 5). We obtain $\mathcal{B} = \{B_1 = P = f_1 f_3 - f_2,\ B_2 = A_1,\ B_3 = A_2\}$.



One can easily check that $H_{\mathcal{B}}(f)(\underline{0}) \neq 0$. Thus $H_{\mathcal{B}}(f)(0, x_2) \neq 0$. Moreover, $H_{\mathcal{B}}(f) \neq 0$, and so $P(f) = 0 \Leftrightarrow \mathcal{B}(f) = 0$.

Using lemma 9, we see that $P(f) = 0$ if and only if $\delta_1^i P(f)(0, x_2) = 0$ for all $i \in \mathbb{N}$.

Now, as P is an element of $\mathcal{B}$, using lemma 4, page 267, we see that for all $i \in \mathbb{N}$

$$(H_{\mathcal{B}})^{\nu}(H_{\mathcal{A}})^{\nu'}(\delta_1^i P) = \sum_{\theta\in\Theta, A\in\mathcal{A}} M_{\theta,A}\cdot\theta A + \sum_{\theta B\in\overline{\mathcal{B}}} M'_{\theta B}\cdot\theta B, \qquad (14)$$

where the M's and M''s are in $\mathcal{R}$ and $\overline{\mathcal{B}} = \{\theta B$ such that $B \in \mathcal{B}$ and $v_{\theta B} \sqsubseteq \mathcal{A}\}$.

The set $\overline{\mathcal{B}}$ is here

$$\overline{\mathcal{B}} = \{A_1,\ A_2\} \cup \{\delta_2^{\ell} P,\ \delta_2^{\ell}\delta_1 P\ \forall \ell \in \mathbb{N}\} \cup \{\delta_1^2 P\}.$$

Equation (14) then becomes:

$$(H_{\mathcal{B}})^{\nu}(H_{\mathcal{A}})^{\nu'}\delta_1^i P = \sum_{\theta\in\Theta, A\in\mathcal{A}} M_{\theta,A}\cdot\theta A + \sum_{\ell\in\mathbb{N}} N_{0,\ell}\cdot\delta_2^{\ell}P + \sum_{\ell\in\mathbb{N}} N_{1,\ell}\cdot\delta_2^{\ell}\delta_1 P + N_2\cdot\delta_1^2 P. \qquad (15)$$

By hypothesis, $\theta A_{\ell}(f)(0, x_2) = 0$ for all $\theta \in \Theta$ and for $1 \le \ell \le 3$.

It is easilly checked that $\delta_1^2 P(f)(0, 0) = 0$. Suppose now that

$$P(f)(0, x_2) = 0,\ \delta_1 P(f)(0, x_2) = 0.$$

The latter series are in $k[[x_2]]$ and so the above equations imply that

$$\delta_2^{\ell}(P)(f)(0, x_2) = 0,\ \delta_2^{\ell}\delta_1 P(f)(0, x_2) = 0.$$

As $H_{\mathcal{B}}(f)(0, x_2) \neq 0$, and $H_{\mathcal{A}}(f)(0, x_2) \neq 0$, equation (15) thus implies that $\delta_1^i P(f)(0, x_2) = 0$ for all $i \in \mathbb{N}$.

We thus proved that $P(f) = 0$ if and only if

$$P(f)(0, x_2) = 0,\ \delta_1(P)(f)(0, x_2) = 0,\ \delta_1^2(P)(f)(0, 0) = 0.$$

Consider now the equation $\delta_1 P(f)(0, x_2) = 0$. We have

$$(\delta_1 P(f)(0, x_2) = 0) \Leftrightarrow (\mathrm{Remainder}(\delta_1 P, \mathcal{A})(f)(0, x_2) = 0),$$

where $\mathrm{Remainder}(\delta_1 P, \mathcal{A})(f)(0, x_2) = (f_1(\delta_1 f_3) - \delta_1 f_2)(0, x_2)$. All the series involved in this equation are defined by the limit conditions, except $\tilde{f}_{(2,1)} = \delta_1 f_2(0, x_2)$. But using the equation A_2, we can find a differential equation for $\tilde{f}_{(2,1)}$. Indeed, $A_2(f) = 0 \Rightarrow \tilde{f}_{(2,1)}^2 + \tilde{f}_{(2,0)}^2 - 1 = 0$. Adding this equation to $\tilde{\mathcal{A}}$, we obtain a new blockwise auto-reduced set $\widehat{\mathcal{A}}$ in $\mathbb{Q}\{y_{(1,0)}, y_{(2,0)}, y_{(2,1)}, y_{(3,0)}, y_{(3,1)}\}$. One can easily check that $(\widehat{\mathcal{A}}, \mathcal{I.C.})$ uniquely defines all the series involved in the equations to test, and we can rewrite them:

$$P(f)(0, x_2) = \tilde{f}_{(1,0)}\tilde{f}_{(3,0)} - \tilde{f}_{(2,0)},$$

$$\mathrm{Remainder}(\delta_1 P, \mathcal{A})(f)(0, x_2) = \tilde{f}_{(1,0)}\tilde{f}_{(3,1)} - \tilde{f}_{(2,1)}.$$

where $\widehat{f} = (\tilde{f}_{(i,s)})_{(s\le\nu_i)}$ is the unique solution of a system $\widehat{\mathcal{A}}(\widehat{f}) = 0$ associated to a finite set of initial conditions. We can use the procedure FiniteCase described above to test them and conclude.

5.2.2 The Test

Now suppose that we are given an m-tuple of series $f = (f_1, \dots, f_m)$ in $k[[x_1, x_2]]$ as the solution of a complete system, and we want to test if $P_1(f) = \cdots = P_t(f) = 0$, where the P_i's are given p.d.p. in $k\{y_1, \dots, y_m\}$. We shall proceed exactly as exposed in Section 3.

Suppose $P_i(f)(\underline{0}) = 0$ for $i = 1, \dots, t$. We compute an auto-reduced coherent set $\mathcal{B}$ in $k\{y_1, \dots, y_m\}$, associated to $\mathcal{A} \cup \{P_1, \dots, P_t\}$. As mentioned in Section 3, if $\mathcal{B}$ contains only a non zero polynomial in k, then there is i such that $P_i(f) \neq 0$.

In this Section, we suppose that all p.d.p. in $\mathcal{B}$ involve at least one of the differential indeterminates y_i, and that $H_{\mathcal{B}}(f)(0, x_2) \neq 0$.

As $H_{\mathcal{B}}(f) \neq 0$, we know that $P_1(f) = \cdots = P_t(f) = 0$ if and only if $\mathcal{B}(f) = 0$.

Theorem 5. *If $H_{\mathcal{B}}(f)(0, x_2) \neq 0$, then $\mathcal{B}(f) = 0$ if and only if*

$$\delta_1^i B(f)(0, x_2) = 0, \text{ for all } B \in \mathcal{B} \text{ and for all } i \in \mathbb{N} \text{ such that } \delta_1^i B \in \overline{\mathcal{B}} \tag{16}$$

where $\overline{\mathcal{B}} = \{\theta B$ such that $B \in \mathcal{B}$ and $v_{\theta B} \sqsubseteq \mathcal{A}\}$.

Proof. As stated in lemma 9 (page 280), $\mathcal{B}(f) = 0$ is equivalent to

$$\delta_1^i B(f)(0, x_2) = 0, \text{ for all } B \in \mathcal{B} \text{ and for all } i \in \mathbb{N}. \tag{17}$$

The latter implies of course that, in particular

$$\delta_1^i B(f)(0, x_2) = 0, \text{ for all } B \in \mathcal{B} \text{ and for all } i \text{ such that } \delta_1^i B \in \overline{\mathcal{B}}. \tag{18}$$

Let us now prove the converse. Assume that

$$\delta_1^i B(f)(0, x_2) = 0, \ \forall B \in \mathcal{B}, \ \forall i \text{ such that } \delta_1^i B \in \overline{\mathcal{B}}.$$

Let $\delta_1^i B$ be an element of $\overline{\mathcal{B}}$. As $\delta_1^i B(f)(0, x_2)$ is an element of $k[[x_2]]$, the equation $\delta_1^i B(f)(0, x_2) = 0$ implies that $\delta_2^\ell\big(\delta_1^i B(f)(0, x_2)\big) = 0$, for all $\ell \in \mathbb{N}$. This in turn means that $\delta_2^\ell \delta_1^i B(f)(0, x_2) = 0$ for all $\ell \in \mathbb{N}$.

Let $B \in \mathcal{B}$ and $j \in \mathbb{N}$ such that $\delta_1^j B \notin \overline{\mathcal{B}}$. Using lemma 4, page 267, we know that there exist positive integers ν, ν' such that

$$H_{\mathcal{A}}^{\nu} H_{\mathcal{B}}^{\nu'} \cdot \delta_1^j B = \sum_{\theta \in \Theta, A \in \mathcal{A}} M_{\theta, A} \cdot \theta A + \sum_{\theta B \in \overline{\mathcal{B}}} M'_{\theta B} \cdot \theta B. \tag{19}$$

We know that $H_{\mathcal{A}}(f)(0, x_2)$ and $H_{\mathcal{B}}(f)(0, x_2)$ are nonzero series of $k[[x_2]]$. Moreover, we just proved that $\delta_2^\ell\big(\delta_1^i B(f)(0, x_2)\big) = 0$, for all $\ell \in \mathbb{N}$, and for all $i \leq \nu_i$; this means that $\theta B(f)(0, x_2) = 0$ for all $\theta B \in \overline{\mathcal{B}}$. Thus equation (19) implies that the series $\delta_1^j B(f)(0, x_2)$ is zero. □

Let us now explain how to test a finite set of equations of the type $Q(f)(0, x_2)$ where $Q \in \mathcal{R}$.

We will use the following ring morphism:

$$\begin{array}{rcl}\phi : k\,[\theta y_i \text{ for } \theta y_i \sqsubseteq \mathcal{A}] & \longrightarrow & k\,\{y_{(i,s)} \text{ for } i \le m \text{ and } s \le \nu_i\} \\ \delta_1^{\alpha_1}\delta_2^{\alpha_2} y_i & \longrightarrow & \delta_2^{\alpha_2} y_{(i,\alpha_1)}\end{array}$$

The ring $k\,\{y_{(i,s)}$ for $i \le m$ and $s \le \nu_i\}$ is here a differential ring with respect to the sole derivation δ_2.

Notice that $\phi(A)$ is defined for all $A \in \mathcal{A}$. The set

$$\widehat{\mathcal{A}} = \tilde{\mathcal{A}} \cup \{\phi(A_1), \ldots, \phi(A_m)\}$$

is blockwise auto-reduced and $(\widehat{\mathcal{A}}, \mathcal{I.C.})$ uniquely defines $\delta_1^s f_i(0, x_2)$ for all $s \le \nu_i$. We call $\widehat{f}$ the $(m + \nu_1 + \cdots + \nu_m)$-tuple of series in $k[[x_2]]$: $\widehat{f} = (\delta_1^s f_i(0, x_2)$ for all $s \le \nu_i)$. Let Q be a p.d.p. in $\mathcal{R}$. Reasoning as in lemma 5, page 269, the following equivalences are trivial.

$$\begin{aligned} Q(f)(0, x_2) = 0 &\Leftrightarrow Q_1(f)(0, x_2) = \text{Remainder}(Q, \mathcal{A})(f)(0, x_2) = 0 \\ &\Leftrightarrow Q_2(\widehat{f})(x_2) = \phi(Q_1)(\widehat{f})(x_2) = 0, \end{aligned}$$

and we use the procedure FiniteCase to test the last equation (as in Section 5.2.1).

Let us summarize the results if this section.

Let $\widehat{f}$ be the following $(m + \nu_1 + \cdots + \nu_m)$-tuple of series in $k[[x_2]]$:

$$\widehat{f} = (\delta_1^s f_i(0, x_2) \text{ for } s \le \nu_i).$$

Let Ω be the following set of equations in $\widehat{\mathcal{R}} = k\{y_{(i,s)}$ for $s \le \nu_i\}$ (this ring being an ordinary differential ring with respect to δ_2):

$$\Omega = \{\phi(\text{Remainder}(\delta_1^s B, \mathcal{A})) \text{ for } B \in \mathcal{B} \text{ and } s \le \nu_i\},$$

where ϕ is the morphism defined on page 283.

We have $\mathcal{B}(f) = 0$ if and only if $R(\widehat{f})(x_2)$ is the zero series, for all R in Ω. The tuple $\widehat{f}$ is the unique solution of a blockwise auto-reduced coherent system in the ordinary differential ring $\widehat{\mathcal{R}} = k\,\{y_{(i,s)}$ for $i \le m$ and $s \le \nu_i\}$, associated to a *finite* set of initial condition. The procedure FiniteCase (exposed in the previous chapter) enables to test the equations $R(\widehat{f})(x_2) = 0$.

5.3 Semi-Singular Case : $H_{\mathcal{B}}(f)(0, x_2) = 0$ but $H_{\mathcal{B}}(f) \neq 0$

5.3.1 Introduction

In the previous section, we proved that

$$\begin{cases} H_{\mathcal{B}}(f)(0, x_2) \neq 0, \\ R(\widehat{f}) = 0 \text{ for all } R \in \Omega \end{cases} \Longrightarrow \mathcal{B}(f) = 0.$$

Call $\mathcal{I}$ the differential ideal of $k\{y_{(i,s)}$ for $s \le \nu_i\}$ generated by Ω. Call $\widehat{H_{\mathcal{B}}}$ the p.d.p. $\phi(H_{\mathcal{B}})$ of $\widehat{\mathcal{R}}$. We thus proved that if $\widehat{f}$ is a zero of $\mathcal{I}$ which does not annul $\widehat{H_{\mathcal{B}}}$, then $\mathcal{B}(f) = 0$.

We now wish to study what happens if $\widehat{f}$ is a zero of $\mathcal{I}$ *and* of $\widehat{H_{\mathcal{B}}}$. We will say, following Ritt, that $\widehat{f}$ is a *singular zero* of $\mathcal{I}$.

We encountered a similar situation in Section 4 : we first treated in Section 4.2 the regular case. Using a topological argument, we were then able in Section 4.3 to treat the semi-singular case. We used the Zariski topology in k^N, i.e. closed sets of k^N are defined by algebraic varieties.

The situation encountered in the present section is more complicated, and requires the use of the *differential* Zariski topology.

We very briefly recall (without proof) in Section 5.3.2 a few classical results about the differential Zariski topology. For more details, the reader is referred to [16], chap. IV § 1 p. 146 or to [4], chap. 3, or to [18], pages 43–46.

In Section 5.3.3, we show that our test in the semi-singular case reduces to a version of Ritt's problem, briefly exposed in Section 5.3.4.

5.3.2 The Differential Zariski Topology

Introducing the differential Zariski topology is quite technical. The main difficulty is that one cannot usefully define the "differential algebraic closure" of a differential field. Kolchin introduced "universal extensions" of a differential field, which play an equivalent role in the theory, in spite of the fact that they are not unique. In particular, every differential extension $K \supset k$ of finite type can be embedded into a universal extension of k. Nevertheless, the results that we shall use for our purpose are very close to the usual theorems relative to the (algebraic) Zariski topology.

Theorem 6. *Let $\mathcal{U}$ be a universal extension of a differential field k. Let m be an integer. Let S be a subset of $\mathcal{U}^m$. Let $Id(S)$ be the set*

$$Id(S) = \{P \in k\{y_1, \dots, y_m\},\ P(f) = 0\ \forall f \in S\}.$$

Then $Id(S)$ is a radical differential ideal, describing the closure of S for the differential Zariski topology of $\mathcal{U}$ relative to k. In other words, the variety defined by the ideal $Id(S)$ is the smallest differential algebraic variety of $\mathcal{U}$ over k containing S. This can be denoted by :

$$\overline{S} = \{g \in \mathcal{U}^m \text{ such that } P(g) = 0,\quad \forall P \in Id(S)\}.$$

Proof. See [16] chap. IV § 3 th. 2 p. 147.

Remark 14. Let us notice the following easy consequence of this theorem. Let Λ be a set of p.d.p.'s in $k\{y_1, \dots, y_m\}$. Let S be a subset of $\mathcal{U}^n$. If :

$$\forall f \in S,\quad \forall P \in \Lambda,\quad P(f) = 0$$

then $\Lambda \subset Id(S)$ and so the differential Zarisky closure $\overline{S}$ of S is a subset of the variety defined by Λ. Thus

$$\forall f \in \overline{S},\quad \forall P \in \Lambda,\quad P(f) = 0.$$

Theorem 7. *Let $\mathcal{I}$ be a differential ideal and H a p.d.p. in $k\{y_1, \ldots, y_m\}$. Let W be the differential algebraic variety defined by $\mathcal{I}$ and W' the differential algebraic variety defined by H. Then the differential Zariski closure of $W - W'$ is defined by the differential ideal $\mathcal{I} : H^\infty$.*

Proof. The differential Zariski closure of $W - W'$ is the union of the irreducible components of W that are not contained in W'. The other components are contained in $W \cap W'$ and so are solutions of the ideal $[\mathcal{I} \cup H]$. Now, we have the classical decomposition $\{\mathcal{I}\} = \{\mathcal{I} : H^\infty\} \cap \{\mathcal{I} \cup H\}$, hence the result. □

5.3.3 Application to the Semi-Singular Case

We now come back to the hypotheses of Section 5.3.1 : suppose $H_\mathcal{B}(f) \neq 0$ but $H_\mathcal{B}(f)(0, x_2) = 0$.

Theorem 8. *If $H_\mathcal{B}(f) \neq 0$, then $\mathcal{B}(f) = 0$ if and only if $\widehat{f}$ is a zero of the ideal $\mathcal{I} : \widehat{H_\mathcal{B}}^\infty$.*

Proof. Let us first prove that if $\widehat{f}$ is a zero of the ideal $\mathcal{I} : \widehat{H_\mathcal{B}}^\infty$ then $\mathcal{B}(f) = 0$.

We shall use the results exposed in Section 5.3.2.

Let B be an element of $\mathcal{B}$. The following equivalences are very easy.

$$\begin{aligned} B(f) = 0 &\Leftrightarrow \forall s \in \mathbb{N},\ \delta_1^s B(f)(0, x_2) = 0 \\ &\Leftrightarrow \forall s \in \mathbb{N},\ Q_s(f)(0, x_2) = 0,\ \text{where } Q_s = \text{Remainder}(\delta_1^s B, \mathcal{A}) \\ &\Leftrightarrow \forall s \in \mathbb{N},\ R_s(\widehat{f}) = 0 \text{ where } R_s = \phi(Q_s) \end{aligned}$$

Thus $\mathcal{B}(f) = 0$ if and only if $\forall R \in \Lambda,\ R(\widehat{f}) = 0$ where

$$\Lambda = \{\phi(\text{Remainder}(\delta_1^s B, \mathcal{A})),\ \forall B \in \mathcal{B},\ \forall s \in \mathbb{N}\}.$$

We know that

$$\begin{cases} \widehat{H_\mathcal{B}}(\widehat{f}) \neq 0, \\ R(\widehat{f}) = 0 \text{ for all } R \in \Omega \end{cases} \Longrightarrow \mathcal{B}(f) = 0.$$

This is equivalent to

$$\left(\widehat{f} \in V(\mathcal{I}) - V(\widehat{H_\mathcal{B}})\right) \Rightarrow \forall R \in \Lambda,\ R(\widehat{f}) = 0$$

where $V(\mathcal{I})$ is the differential algebraic variety defined by the ideal $\mathcal{I}$ of $k\{y_1, \ldots, y_m\}$, and $V(\widehat{H_\mathcal{B}})$ is the differential algebraic variety defined by $\widehat{H_\mathcal{B}}$. Using remark 14, we claim that

$$\widehat{f} \in \overline{V(\mathcal{I}) - V(\widehat{H_\mathcal{B}})} \Rightarrow \forall R \in \Lambda,\ R(\widehat{f}) = 0.$$

As $\overline{V(\mathcal{I}) - V(\widehat{H_\mathcal{B}})} = V(\mathcal{I} : \widehat{H_\mathcal{B}}^\infty)$ (see theorem 7), the implication

$$\left(\widehat{f} \in V(\mathcal{I} : \widehat{H_\mathcal{B}}^\infty) \Rightarrow \mathcal{B}(f) = 0\right)$$

is proved.

Let us now prove the converse by abstract nonsense. Assume that $H_{\mathcal{B}}(f) \neq 0$, $\mathcal{B}(f) = 0$ and $\widehat{f}$ is not a zero of the ideal $\mathcal{I} : \widehat{H_{\mathcal{B}}}^{\infty}$.

This means that there exists a differential polynomial $\widehat{Q} \in \mathcal{I} : \widehat{H_{\mathcal{B}}}^{\infty}$ such that $\widehat{Q}(\widehat{f}) \neq 0$. There exists an integer $\nu \in \mathbb{N}$ such that

$$\widehat{H_{\mathcal{B}}}^{\nu}\,\widehat{Q} = \sum_{\widehat{R}\in\Omega,\ \ell\in\mathbb{N}} \widehat{M}_{\widehat{R},\ell} \cdot \delta_2^{\ell}\widehat{R}. \tag{20}$$

This equation involves differential polynomials in the ordinary differential ring $k\{y_{(i,s)},\ \text{for } s \leq \nu_i\}$ (the sole derivation is δ_2). As in the proof of lemma 7 (page 270), we substitute to each indeterminate $\delta_2^{\ell} y_{(i,s)}$ the indeterminate $\delta_2^{\ell}\delta_1^{s} y_i$. Equation (20) thus becomes

$$H_{\mathcal{B}}^{\nu} Q = \sum_{\theta B\in\overline{\mathcal{B}}} M_{\theta B} \cdot \mathrm{Remainder}(\theta B, \mathcal{A}). \tag{21}$$

We assumed that $Q(f)(0, x_2) = \widehat{Q}(\widehat{f})(x_2) \neq 0$. Moreover $H_{\mathcal{B}}(f) \neq 0$, and $\mathcal{B}(f) = 0$ (thus $\theta B(f) = 0$ for all $\theta B \in \overline{\mathcal{B}}$). This is not compatible with equation (21). This proves the theorem. □

5.3.4 Connection with Ritt's Problem

Let us first briefly recall Ritt's problem. For a complete exposition, the reader is referred to [16], chap. IV § 16. Let $\mathcal{I}$ and $\mathcal{J}$ be two prime differential ideals, each of them being defined by a characteristic set. In other words, we are given $\mathcal{I} = [\mathcal{A}] : H_{\mathcal{A}}^{\infty}$ and $\mathcal{J} = [\mathcal{B}] : H_{\mathcal{B}}^{\infty}$, where $\mathcal{A}$ and $\mathcal{B}$ are auto-reduced coherent sets of p.d.p.

Ritt's problem is the following: how can we decide whether $\mathcal{I} \subset \mathcal{J}$?

This problem can also be formulated in other terms: how can we test that every zero of $\mathcal{J}$ is also a zero of $\mathcal{I}$? Let us first study the zeros of $\mathcal{I}$. We know that $\mathcal{A} \subset \mathcal{I}$ and so every zero f of $\mathcal{I}$ satisfies the equation $\mathcal{A}(f) = 0$. Conversely, every zero f of $\mathcal{A}$ such that $H_{\mathcal{A}}(f) \neq 0$ is a zero of $\mathcal{I}$. The problem is to characterize the zeros of $\mathcal{I}$ that are also zeros of $H_{\mathcal{A}}$. These are called *singular zeros of* $\mathcal{A}$. In particular, if there is an initial or a separant of an element of $\mathcal{A}$ that belongs to $\mathcal{J}$, then every zero of $\mathcal{J}$ is a singular zero of $\mathcal{A}$. The difficulty is to know whether a singular component is included in the adherence of the regular zeros, or not. In the first case, it belongs to the singular locus of the main component and in the second it is an isolated singular component.

The original Ritt's problem can be extended to a larger class of ideals, introduced by Boulier (see [1, 2, 3]). We shall consider *regular ideals*, i.e. ideals of the form $[\mathcal{A}] : H^{\infty}$, where $\mathcal{A}$ is an auto-reduced coherent set, and H is the product of $H_{\mathcal{A}}$ and of a p.d.p. reduced with respect to $\mathcal{A}$. From an algorithmic point of view, these ideals are more useful than prime ideals given by characteristic sets. It is clear that Ritt's problem can be easily formulated with regular ideals instead of prime ideals (see [12]). The difficulty is that one can test ideal membership with characteristic sets of regular ideals, but they are in general not a basis of the ideal they define. Solving the Ritt problem is in fact equivalent to computing a basis from a characteristic set, i.e. to have an effective version of the Ritt–Raudenbush basis theorem (see [18] I.4.3 p. 61–62).

Let us come back to the problem exposed in Section 5.3.3. We are led to test whether $\widehat{f}$ is a zero of the ideal $\mathcal{I} : \widehat{H_\mathcal{B}}^\infty$, knowing that $\widehat{H_\mathcal{B}}(\widehat{f}) = 0$. Using the diffalg algorithm exposed in [3], and available in the Maple distribution (see also [13]) we can find a decomposition of $\mathcal{I} : \widehat{H_\mathcal{B}}^\infty$ of the form:

$$\mathcal{I} : \widehat{H_\mathcal{B}}^\infty = \bigcap_{i=1,\dots,R} \mathcal{J}_i : H_i^\infty$$

where each $\mathcal{J}_i : H_i^\infty$ is a regular differential ideal (a very simple algorithm may also be found in citesad). We need then to test whether :

$$\exists i \text{ such that } \widehat{f} \in V(\mathcal{J}_i : H_i^\infty),$$

which is the extension of Ritt's problem exposed just above.

Unfortunately, Ritt's problem is far from being solved in the general case, although some partial results exists. See [11], [20] chap. 3 § II, [16] chap. 4 §15 and 16, or [5, 6, 7].

6 Conclusion

The last Section showed that testing equality in differential ring extensions reduces to a version of Ritt's problem. Much work has been done on the subject and one can sometimes conclude by specific methods, but the general problem is still open. The automatic manipulation of solutions of non-linear differential systems with limit conditions is thus far from being easy. The difficulty is specific to the non-linear situation. The reader is referred to [19] for a similar approach with linear systems. This paper nevertheless shows a way to deal with initial/limit conditions in computer algebra which applies to the "generic" non-singular case.

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