Unforgettable Forgetful Determinacy

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Abstract

This paper presents a relatively compact yet complete proof of the Forgetful Determinacy Theorem (FDT) of Yuri Gurevich and Leo Harrington. The original motivation for this theorem was to provide a simpler proof of a powerful decidability result by Michael Rabin. Rabin's result and related techniques have found considerable application in computer science in dealing with the satisfiability problem for modal logics of programs. The FDT asserts the existence of a special kind of winning strategy in a particular class of infinite games. Although it first appeared as part of an alternative proof of Rabin's theorem, the FDT is a game-theoretic result that applies to more general situations than those in which it was originally used. Both the FDT and an earlier result by Richard Büchi and Lawrence Landweber address the issue of what kind of information is necessary to execute the winning strategy. Recently infinite games have been used to model non-terminating computations, and common methods of specification for such computations produce games to which the FDT applies. The original proof of the FDT was sketchy. Other proofs have been given, including one by Alexander and Vladimir Yakhnis that strengthened the result by providing more explicit strategies for the players. Nevertheless, there was still need for a more compact proof. To produce such a proof, we apply a modified version of the one by Yakhnis and Yakhnis to the slightly more general setting of graph-games. We use the notion of graph-games to emphasize the relationship between the FDT and the earlier result by Büchi and Landweber.

Keywords: Gurevich–Harrington, forgetful determinacy, infinite games, restricted-memory strategies, monadic second-order theories.

1 Introduction

The Forgetful Determinacy Theorem (FDT) of Yuri Gurevich and Leo Harrington asserts that in certain infinite games, it is always the case that one of the players has a special kind of winning strategy. The original version [2] appeared as part of a simpler proof of the most difficult part of Michael Rabin's proof that the monadic second-order theory of the infinite binary tree is decidable [8]. Independently, Andre Muchnik found another less complicated proof of Rabin's theorem [6]. Rabin's decidability result is extremely powerful and has application in computer science, since many decision problems, including the satisfiability problem for many modal logics of programs, reduce to it. Although the original motivation for the FDT was to simplify Rabin's proof, the FDT applies in more general settings than the one in which it was originally used. Infinite games between two players have recently found more direct use in computer science as models of nonterminating concurrent computations in which a system must interact with a potentially hostile environment, e.g. [7]. Natural specifications on such computations give rise to a class of games to which the Forgetful Determinacy Theorem applies.

The original proof of the FDT was sketchy and became a source of frustration for many. J. Donald Monk gave an expanded and complete version of the proof in lecture notes [5]. Recently, Alexander and Vladimir Yakhnis gave a proof that strengthened the FDT, in part, by providing more explicit strategies for the players [10]. Still, there seems to be interest in a more compact proof. This paper gives such a proof by using the approach of [10] in the manner described below. The goal of [10], to provide a wider class of winning strategies and a more explicit criterion for the winning player, was different from ours. In order to achieve its goal, [10] introduced the notion of priority automata and allowed the strategies for the player to be partially defined. In order to produce a more compact proof, we alter the approach of [10] in two stages.

In the first stage, we make only the following modifications. First, we make it unnecessary to use the concept of mutually perpetual strategies introduced in [10] by requiring our strategies to be totally defined. Secondly, we use the latest appearance record, which was introduced in [2] and generalized in [10] to 'intersecting colours', rather than the more general priority automaton. Finally, in adapting the proof to these changes, we make explicit the fact that the only information from past play required by our strategies is the latest appearance record for the set of colours used to define the game. (Compare the construction of the strategy for Mr δ in our proof of Lemma 3.3 to the section 'Winning strategies for the player' in [10].)

In the second stage, we introduce the slightly more general setting of graph-games and the idea of a forgetful colouring. Then we translate the modified proof described above into this framework. The proof thus obtained, unlike its predecessors, is free from the necessity to repeatedly show the invariance of various notions for positions related in certain ways. We do show a similar invariance once (Corollary 4.1).

The proof presented here is the final result of both stages of this alteration. It incorporates much from the one by Yakhnis and Yakhnis, but can be read independently of their paper.

When the graph (arena) of a graph-game is finite, it can also be seen as the transition diagram of a finite automaton. This correspondence is used to illustrate the relationship between the FDT and an earlier result by Richard Büchi and Lawrence Landweber [1]. The notion of a graph-game generalizes the notion of a game guided by a finite game automaton found in [9] and [11]. For a finite arena, the notion of forgetful strategies introduced here for graph-games coincides with that of finite automata discernible strategies in [9] and [11]. A different treatment of graph-games with finite graphs is given by Robert McNaughton in [4].

Section 2 introduces the notion of graph-games and a version of the FDT for such games. Section 3 contains the proof of this theorem. In Section 4, it is shown that the original version of the FDT and the result by Büchi and Landweber follow from the FDT for graph-games.

2 Graph-games

We first define the game and state the result. Let $MOVE$ be a finite alphabet. Intuitively, $MOVE$ is the set from which legal moves for the players of the game are drawn. Define $MOVE^*$ as usual and let $MOVE^\omega$ be the set of infinite sequences formed from elements of $MOVE$. An *arena* A is a coloured (finite or infinite) multidigraph (a directed graph in which parallel edges are allowed) that satisfies the following conditions:

1. The vertices are divided into two disjoint subsets, the *even* vertices and the *odd* vertices, and edges go only from even vertices to odd vertices or from odd vertices to even ones.
2. There is a distinguished even vertex called the *start vertex* of A, there is a path from the start vertex to every other vertex, and there is at least one outgoing edge from each vertex.
3. The edges of A are labelled by elements of $MOVE$ in such a way that no two outgoing edges from the same vertex have the same label.
4. Some of the vertices of A are coloured with one or more colours from a finite set of colours called S.

Associated with any sequence of edges $e_1, e_2, \dots, e_n$ (possibly with repeated vertices and edges) that forms a directed path in A is the element $p = \mu_1\mu_2 \dots \mu_n$ of $MOVE^*$, where for

$1 \leq i \leq n$, μ_i is the label on e_i. If this path begins at the start vertex, then we call p a *position* in the game. By condition (3), if p is a position, then the corresponding path is uniquely determined, and we say that p *ends* at the last vertex in this path. (The labels on the edges leading out of the last vertex are the possible moves at the position.) A *play* in A is an element of $MOVE^{\omega}$ such that all of its finite prefixes are positions. We denote the set of plays in A by $PLAY(A)$. A game is played by two players, Mr 0 and Mr 1, who alternately choose elements of $MOVE$ so that after each turn the element of $MOVE^*$ produced so far is a position. In other words, the players together construct an element of $PLAY(A)$. Winning a graph-game for a player means producing a play in a given subset of $PLAY(A)$ called his *winning set*. Let C^s denote the set of vertices of colour s. The games we are interested in are those in which the winning sets are Boolean combinations of the sets $[C^s]$ for $s \in S$, where $[C^s]$ is the set of plays in A that have infinitely many finite prefixes that end at a vertex in C^s.

Formally, a graph-game is a triple $\Gamma = (A, \epsilon, W_\epsilon)$, where A is an arena, $\epsilon \in \{0, 1\}$ (denoting the player who goes first), and W_ϵ is a Boolean combination of $[C^s]$ sets (the winning set for player ϵ). Set $W_{(1-\epsilon)} = PLAY(A) \setminus W_\epsilon$. Mr ϵ wins if the element of $PLAY(A)$ formed by the infinite sequence of moves the players choose to make is in W_ϵ; otherwise (in case the play is in $W_{(1-\epsilon)}$), Mr $(1-\epsilon)$ wins.

It is Mr ϵ's turn to move at the even vertices. For $\delta \in \{0, 1\}$, let $TURNS(\delta)$ denote the vertices at which Mr δ makes a move. A *forgetful strategy* f for Mr δ in Γ is a function f from $TURNS(\delta)$ to the powerset of $MOVE$ such that for all v in $TURNS(\delta)$, $f(v) \neq \emptyset$ and $f(v)$ contains only elements of $MOVE$ that label an outgoing edge from v. (The strategy is 'forgetful' in the sense that it depends only on v and not on how v was reached.) From now on, f denotes a forgetful strategy for Mr δ, and g denotes a forgetful strategy for Mr $(1-\delta)$. An element $\mu_0\mu_1\ldots$ in $PLAY(A)$ is *consistent* with f and g after position $p_1 = \mu_0\mu_1\ldots\mu_n$ if for all $m \geq n$ where $\mu_0\mu_1\ldots\mu_m$ ends at vertex v, $\mu_{m+1} \in f(v)$ if $v \in TURNS(\delta)$, and $\mu_{m+1} \in g(v)$ if $v \in TURNS(1-\delta)$. We say that f *wins* Γ *against* g at v if for all positions p_1 that end at v, all plays consistent with f and g after p_1 are in W_δ. We say that f *wins* Γ for Mr δ if f wins Γ against g at the start vertex where g is the strategy for Mr $(1-\delta)$ that allows any possible move at each vertex in $TURNS(1-\delta)$. Another strategy f_1 for Mr δ is a *refinement* of f if for all $v \in TURNS(\delta)$, $f_1(v) \subseteq f(v)$.

We adapt from [10] the notion of the set of winning positions for Mr δ with restraints (referred to here as *restrictions*) to the framework of graph-games as follows. We define the set of *winning vertices* for Mr δ with restrictions f and g, $WIN(\Gamma, \delta, f/g)$, to be the set of vertices v such that Mr δ has a refinement of f that wins Γ against g at v. Note that playing a winning refinement from a vertex in $WIN(\Gamma, \delta, f/g)$ keeps play within the set of winning vertices. Indeed, suppose that f_1 is a refinement of f that wins against g at v_1, and that v_2 is a vertex reached by play consistent with f_1 and g after position p_1 that ends at v_1. Then v_2 is in $WIN(\Gamma, \delta, f/g)$ because any play consistent with f_1 and g after a position ending at v_2 differs from play consistent with f_1 and g after p_1 only with respect to a finite prefix, and thus is also in W_δ since membership in this set depends only on the set of colours that are reached by a play infinitely often.

We wish to show that whenever the colouring of the vertices satisfies an extra condition, graph-games are determined in the strong sense that one of the players has a winning forgetful strategy. The extra condition involves the notion of the LAR or *latest appearance record* (see [2] and [10]). The LAR at a given position is a list of the elements of S in some order. We begin by considering the elements of S to be ordered in some fixed way so that the colours of the start vertex come after the remaining colours. Let l be the sequence of all elements of S in increasing order. If p is the empty string, then $LAR(p) = l$. In general, at position p that ends at v, let

$L = \{s \in S \mid v \in C^s\}$, and let l'' be the sequence of elements of L in increasing order. If p is not empty, then $p = p_1\mu$ for some $\mu \in MOVE$, and $LAR(p) = l'l''$ where l' is the result of removing all elements of L from $LAR(p_1)$. We say that the colouring of A is *forgetful* if for each vertex v of A, the LAR of every position that ends at v is the same. In this case we speak of this value as being the value of the LAR at vertex v. We now state the Forgetful Determinacy Theorem (FDT).

THEOREM 2.1
Let $\Gamma = (A, \epsilon, W_\epsilon)$ be a graph-game in which the colouring of A is forgetful. Then one of the players has a forgetful strategy that wins Γ.

In the original version of the FDT, the arena is defined as a subset of $MOVE^*$ that forms an infinite tree without 'dead ends'. This is a special case of the arena described above in which positions and vertices can be identified since there is a unique position that ends at each vertex. A forgetful strategy in this case is the same as a strategy in the ordinary sense, that is, one defined on positions in the game rather than on vertices. Because our strategies are defined on vertices rather than positions, that is, they are forgetful, determinacy of the games defined here does not follow from classical determinacy results (such as the determinacy of Borel games [3]) in which strategies are defined on positions. The original version of the FDT follows from the theorem above by considering an auxiliary graph-game associated with the original game.

3 Proof of the forgetful determinacy theorem

The proof in this section is adapted from [10], with corresponding notions defined here in terms of vertices rather than positions. Lemmas 3.1 and 3.3 and the construction of the fixed point below all have analogues in [10]. The notions of rank and related basic strategies below are as they appear in [10], where they are adapted from [2]. We now state the lemma from which the forgetful determinacy of the graph-game follows.

LEMMA 3.1 (cf. Theorem 4.2 of [10])
Let $\Gamma = (A, \epsilon, W_\epsilon)$ be a graph-game in which the colouring of A is forgetful, f a forgetful strategy for Mr δ, and g a forgetful strategy for Mr $(1-\delta)$ in Γ. Then Mr δ has a single refinement of f that wins against g at every vertex in $WIN(\Gamma, \delta, f/g)$, and Mr $(1-\delta)$ has a single refinement of g that wins against f at every vertex in $WIN(\Gamma, 1-\delta, g/f)$. Furthermore, every vertex of A is either in $WIN(\Gamma, \delta, f/g)$ or in $WIN(\Gamma, 1-\delta, g/f)$.

The Forgetful Determinacy Theorem stated in the last section follows from a special case of Lemma 3.1. Suppose that our original forgetful strategies, f and g, merely assign to each vertex at which they are defined the set of all μ in $MOVE$ that label an edge from that vertex. According to Lemma 3.1, the start vertex of A is either in $WIN(\Gamma, \delta, f/g)$ or in $WIN(\Gamma, 1-\delta, g/f)$. In the first case, there is a forgetful strategy that wins Γ for Mr δ, and in the second case there is a forgetful strategy that wins Γ for Mr $(1-\delta)$. We now turn to the proof of Lemma 3.1.

We begin by showing that it is sufficient to prove a slightly different statement. (The argument is from Section 5 of [10].) We can consider the winning sets W_δ and $W_{(1-\delta)}$ to be expressed in canonical form, that is, as the union of some number of terms each of which is the intersection of sets of the form $[C^s]$ and the complements of such sets, that is, sets of the form $[C^s]^c$. Let $\beta \in \{0, 1\}$ be such that W_β has the fewest of such terms.

CLAIM 3.2
Either W_β or its complement can be expressed in the form

$$W = (U^1 \cup [B^1]) \cap \cdots \cap (U^m \cup [B^m]) = ([B^1] \cap \cdots \cap [B^m]) \cup \left(\bigcup_{i=1}^{m} U^i\right)$$

where each $B^i = \bigcup_{j \in I_i} C^j$ and each I_i is a subset of S. We call the sets U^i the *derived winning sets.* Also each U^i is empty, or has fewer terms than W_β, or its complement has the same number of terms and contains a term in which no set of the form $[C^s]^c$ occurs (that is, all are uncomplemented).

Proof of the claim is an exercise in Boolean manipulation except for use of the fact that for any subset I of S, $\bigcup_{j \in I} [C^j] = [\bigcup_{j \in I} C^j]$. First note that if W_β contains a term in which no $[C^s]$ set is complemented, say $W_\beta = ([C^{s_1}] \cap [C^{s_2}] \cap \cdots \cap [C^{s_k}]) \cup U'$, then W_β can easily be expressed in the desired manner with each $U^i = U'$. Otherwise W_β contains some number of terms, say m, each of which contains some sets of the form $[C^s]^c$. Let B^i be the union of the C^s sets that appear in complemented form in the ith term. If there are uncomplemented $[C^s]$ sets in the same term, let V^i be their intersection; otherwise let V^i be all of $PLAY(A)$. Then $W_\beta = \bigcup_{i=1}^{m} (V^i \cap [B^i]^c) = \bigcup_{i=1}^{m} ((V^i \cup (\bigcup_{j \neq i} (V^j \cap [B^j]^c))) \cap [B^i]^c)$. Define U^i as the complement of the set $V^i \cup (\bigcup_{j \neq i} (V^j \cap [B^j]^c))$. Each U^i set is empty, or its complement has m terms and contains one, namely V^i, with no complemented sets. Also $W_{(1-\beta)} = (U^1 \cup [B^1]) \cap \cdots \cap (U^m \cup [B^m]) = ([B^1] \cap \cdots \cap [B^m]) \cup (\bigcup_{i=1}^{m} U^i)$. This establishes the claim.

Lemma 3.1 is obviously true if one of the winning sets is empty, that is, has no terms. Otherwise the winning set with the fewest terms or its complement can be expressed in the special form in which either the derived winning sets are empty, or they contain fewer terms, or their complements can easily be expressed in a form in which the new derived winning sets have fewer terms. An argument using induction on the minimum number of terms in either of the winning sets shows that in order to prove Lemma 3.1, it is sufficient to prove the following:

LEMMA 3.3 (cf. Lemma 5.1 of [10])
If $W_\delta = (U^1 \cup [B^1]) \cap \cdots \cap (U^m \cup [B^m]) = ([B^1] \cap \cdots \cap [B^m]) \cup (\bigcup_{i=1}^{m} U^i)$ where for $1 \leq i \leq m$, $B^i = \bigcup_{j \in I_i} C^j$ and I_i is a subset of S, and Lemma 3.1 holds for each Γ^i, which is the game in which U^i replaces W_δ as the winning set for Mr δ, then it also holds for the game Γ in which W_δ is the winning set for Mr δ.

In order to prove Lemma 3.3, we need to develop notions that express necessary information about the future, in particular, that measure the ability of a given player to reach a given C^s set. With this motivation, we define *ranks* with respect to certain subsets of vertices. For a subset of vertices X, player δ, and forgetful strategies f and g, define the sets D^i as follows:

1. $D^0 = X$.
2. If $v \in TURNS(\delta)$ and there is an edge labelled by $\mu \in f(v)$ to a vertex in D^n, then $v \in D^{n+1}$.
3. If $v \in TURNS(1-\delta)$, there is an edge labelled by $\mu \in g(v)$ to a vertex in D^n, and for all $\mu \in g(v)$, the edge labelled by μ from v is to a vertex in D^k for some $k \leq n$, then $v \in D^{n+1}$.

Once these sets are defined, we define the notion of $RANK$ with respect to X, δ, f, and g:

$$RANK(X, \delta, f/g)(v) = \begin{cases} 0 & \text{if } v \in D^0 \setminus \bigcup_{k \geq 1} D^k \\ \min\{k \mid k \geq 1,\, v \in D^k\} & \text{if } v \in \bigcup_{k \geq 1} D^k. \end{cases}$$

Also let $DOM(X, \delta, f/g) = \bigcup_{k\geq 0} D^k$, and let $DOM^+(X, \delta, f/g) = \bigcup_{k\geq 1} D^k$. The value of $RANK(X, \delta, f/g)$ at a given vertex indicates the ability of Mr δ to reach a vertex in X in *future* play according to f and g from that vertex. If $RANK(X, \delta, f/g)(v) = k$, then Mr δ has a refinement of f that allows him to reach a vertex in X from v within k moves as long as Mr $(1-\delta)$ plays g, and Mr $(1-\delta)$ has a strategy that is a refinement of g that keeps Mr δ from reaching X from v in fewer than k moves as long as Mr δ plays f. In particular, from any vertex in $DOM^+(X, \delta, f/g)$, Mr δ has a refinement of f that allows him to reach a vertex in X as long as Mr $(1-\delta)$ plays according to g. If $RANK(X, \delta, f/g)$ is undefined or is 0 at v, then Mr $(1-\delta)$ has a strategy that is a refinement of g that keeps Mr δ from ever reaching X from v as long as Mr δ plays f.

Our strategies are built from the following two basic ones. The first is described as one for Mr δ. Here and below, when defining a strategy for a given player at vertex v, we always implicitly assume it is that player's turn to move at v. Let $DECR(X, \delta, f/g)(v)$ be $f(v)$ if $v \notin DOM^+(X, \delta, f/g)$; otherwise $DECR(X, \delta, f/g)(v)$ is the set of $\mu \in f(v)$ that label an outgoing edge from v to a vertex v_1 with $RANK(X, \delta, f/g)(v_1) < RANK(X, \delta, f/g)(v)$ or with $v_1 \in X$. The second strategy, described here as one for Mr $(1-\delta)$, is $AVOID(X, 1-\delta, g/f)$, which assigns to vertex v the set of $\mu \in g(v)$ that label an outgoing edge from v to a vertex not in $DOM(X, \delta, f/g)$ if this set is not empty, and is the same as $g(v)$ otherwise. Note that any play consistent with f and $AVOID(X, 1-\delta, g/f)$ after a position p that ends at a vertex not in $DOM^+(X, \delta, f/g)$ remains out of $DOM(X, \delta, f/g)$ after p.

We now proceed with the proof of Lemma 3.3. The set of winning vertices for Mr δ is the fixed point of a monotonic operator on vertices of A. For $1 \leq i \leq m$ and X a subset of the vertices of A, define $F_i(X) = DOM^+(X \cap B^i, \delta, f/g) \cup WIN(\Gamma^i, \delta, f/g^{iX})$, where $g^{iX} = AVOID(X \cap B^i, 1-\delta, g/f)$.

CLAIM 3.4 (cf. Lemma 6.1 of [10])
$X \subseteq Y \Rightarrow F_i(X) \subseteq F_i(Y)$.

PROOF. Assume $v \in F_i(X)$ and $v \notin DOM^+(Y \cap B^i, \delta, f/g)$. Since $DOM^+(X \cap B^i, \delta, f/g) \subseteq DOM^+(Y \cap B^i, \delta, f/g)$, $v \notin DOM^+(X \cap B^i, \delta, f/g)$, which means it must be the case that v is in $WIN(\Gamma^i, \delta, f/g^{iX})$. Let f_1 be the refinement of f that wins against g^{iX} at v. Since $v \notin DOM^+(Y \cap B^i, \delta, f/g)$, any play consistent with f_1 and g^{iY} after position p_1 that ends at v is also consistent with f_1 and g^{iX} after p_1, so f_1 wins against g^{iY} at v. Thus, $v \in WIN(\Gamma^i, \delta, f/g^{iY})$, that is, $v \in F_i(Y)$. ∎

Let $H(X) = \bigcap_{1\leq i\leq m} F_i(X)$. Then $X \subseteq Y \Rightarrow H(X) \subseteq H(Y)$. Define the sequence X_λ as follows. Here $X_{<\lambda} = \bigcap_{t<\lambda} X_t$. We begin by letting X_0 be the entire vertex set of A. Then $X_\lambda = H(X_{<\lambda})$. It is easy to see that this sequence is non-increasing, and therefore there is a first ordinal η such that $X_\eta = X_{\eta+1}$. Let $P = X_\eta$. Then

$$P = \bigcap_{1\leq i\leq m} (DOM^+(P \cap B^i, \delta, f/g) \cup WIN(\Gamma^i, \delta, f/g^{iP})).$$

A Strategy for Mr $(1-\delta)$: Mr $(1-\delta)$ has a forgetful strategy refining g that wins against f at all vertices not in P.

PROOF. (cf. proof of Lemma 7.1 in [10]) Fix λ and i and consider vertices in $WIN(\Gamma^i, 1-\delta, g^{iX_{<\lambda}}/f)$. By the assumption that Lemma 3.1 holds for each Γ^i, there is a forgetful strategy $g^{\lambda i}$ refining $g^{iX_{<\lambda}}$ and winning Γ^i against f at all such vertices. We now define the strategy

G for Mr $(1-\delta)$. For vertices in P, we can allow G to be the same as g. Consider a vertex $v \notin P$. Then there is a smallest λ such that $v \notin X_\lambda$. Then $v \in X_{<\lambda}$, but $v \notin H(X_{<\lambda})$. Thus, there is a smallest i such that $v \notin F_i(X_{<\lambda})$. We associate the pair $\langle \lambda, i\rangle$ with v. Then $v \notin DOM^+(X_{<\lambda} \cap B^i, \delta, f/g)$ and $v \notin WIN(\Gamma^i, \delta, f/g^{iX_{<\lambda}})$. By our assumption for Γ^i, $v \in WIN(\Gamma^i, 1-\delta, g^{iX_{<\lambda}}/f)$. Let $G(v) = g^{\lambda i}(v)$. Consider any play consistent with f and G after a position that ends at vertex v not in P. Play consistent with f and $g^{\lambda i}$ after a position that ends in a vertex that is not in $F_i(X_{<\lambda})$ stays out of $F_i(X_{<\lambda})$, so that as play continues from v, each pair associated with a vertex in the play must be lexicographically greater than or equal to the pair associated with the next vertex. Consequently, there must be some point of the play after which the pairs associated with vertices do not change. Without loss of generality, we can assume that from some point on in the play, the pair associated with each vertex is $\langle \lambda, i\rangle$. From this point on all vertices reached are in $X_{<\lambda}$, and G at these vertices, as a refinement of $g^{iX_{<\lambda}}$, not only stays out of $DOM(X_{<\lambda} \cap B^i, \delta, f/g)$, but actually stays out of B^i. Also after this point the same winning strategy for Mr $(1-\delta)$ for Γ^i is played. Thus, play after this point is consistent with f and $g^{\lambda i}$ and wins Γ^i for Mr $(1-\delta)$. This means that the play is in the complement of U^i and in the complement of $[B^i]$, and therefore wins Γ for Mr $(1-\delta)$. ∎

A Strategy for Mr δ: Mr δ has a forgetful strategy refining f that wins against g at any vertex $v \in P$.

PROOF. (cf. proof of Lemma 8.2 of [10]) By assumption, for each i, there is a forgetful strategy f^i refining f that wins Γ^i against g^{iP} at all vertices in $WIN(\Gamma^i, \delta, f/g^{iP})$. For $v \in P$, define the desired strategy F for Mr δ as follows. (For vertices not in P, we can let F be the same as f.) We wish to choose a value of i which we call the *goal* at v using the value of the LAR at v. We construct a sequence of not necessarily distinct values between 1 and m as follows. Suppose the value of the LAR at v is $s_{i_1}s_{i_2}\ldots s_{i_n}$. Replace each s_{i_j} by the values of k in increasing order such that $s_{i_j} \in I_k$ where $B^k = \bigcup_{s \in I_k} C^s$. Now consider the sequence obtained from this list by discarding all but the rightmost occurrence of each value. Choose the goal i so that it is the leftmost value in this sequence. In the sequence of distinct values between 1 and m used to pick the goal at a vertex v in B^j for a given j, the value j appears to the right of any value k such that v is not in B^k. Also, if there is a point in the play after which a certain B^k is not hit at all, the number of values to the right of k in the sequences obtained for vertices from this point on cannot decrease. In fact, if there is such a point, the goal chosen along this play eventually does not change, and equals the index j (not necessarily equal to k), of a set B^j that is never reached after some point on this play.

We define F at vertex $v \in P$ as follows:

$$F(v) = \begin{cases} f^i(v) & \text{if } v \notin DOM^+(P \cap B^i, \delta, f/g) \\ DECR(P \cap B^i, \delta, f/g)(v) & \text{otherwise,} \end{cases}$$

where i is the goal picked at v for Mr δ. Any play consistent with F and g remains in P whenever play reaches a vertex in P. It is easy to see that any move made by Mr $(1-\delta)$ according to g from a vertex in $F_i(P)$ must be to a vertex in $F_i(P)$, and that any move made by Mr δ according to F from a vertex in P at which the goal is i must also be to a vertex in $F_i(P)$. For each i, it is impossible for a vertex to be in $F_i(P)$ and not in P, since Mr $(1-\delta)$ has a strategy that forces play to stay out of P and wins Γ at any vertex outside of P, and from inside $F_i(P)$ it is possible either for Mr δ to reach P, or to win Γ^i (and hence Γ since $U^i \subseteq W_\delta$) as long as Mr $(1-\delta)$ continues to stay out of $DOM(P \cap B^i, \delta, f/g)$. This means that play stays in P, since $F_i(P) \subseteq P$ for each i. We now show that F wins Γ for Mr δ against g at any vertex $v \in P$.

Consider an element of $PLAY(A)$ that is consistent with F and g after a position p that ends at vertex v in P as it continues after v.
Case 1: There is a point in the play after which the i chosen does not change, and $DOM(P \cap B^i, \delta, f/g)$ is not reached. Then play after this point is that of f^i against g^{iP} and wins Γ^i (and hence Γ) for Mr δ.
Case 2: There is no point after which i remains the same. This means that the play keeps hitting all of the sets B^i for $1 \leq i \leq m$, and this constitutes a win for Mr δ.
Case 3: There is a point after which the goal i does not change, but there is no point after which $DOM(P \cap B^i, \delta, f/g)$ is not reached. Then it must still be the case that the play keeps hitting all of the sets B^i for $1 \leq i \leq m$, and this constitutes a win for Mr δ. ∎

We have exhibited a forgetful strategy for each player that is a refinement of his initial strategy, and that wins against the opponent's initial strategy at any of his winning vertices. Also $P = WIN(\Gamma, \delta, f/g)$, and any vertex not in P is in $WIN(\Gamma, 1-\delta, g/f)$. QED

4 More or less forgetful strategies

In the original version of the Forgetful Determinacy Theorem, the arena is an infinite tree and the winning strategy is entirely determined by the LAR at a position (vertex) and certain kinds of information about the future at that position. Of course, any colouring in a tree is forgetful. We show that the existence of such a strategy in any graph-game with a forgetful colouring follows from the Forgetful Determinacy Theorem for graph-games. In order to make the notion of information about the future precise, we define an equivalence relation on vertices of A called $SAMEFUTURE$. Consider an arena A with a forgetful colouring. For a single vertex v and a subset of vertices X, let X_v be the elements of $MOVE^*$ that label a sequence of edges from v to a vertex in X. If $v \in X$, then X_v contains the empty string. Let V denote the set of vertices of A. Vertices v_1 and v_2 are related by $SAMEFUTURE$ exactly when both are odd vertices or both are even vertices, $V_{v_1} = V_{v_2}$, and for all $s \in S$, $C^s_{v_1} = C^s_{v_2}$. The strategies we have constructed are 'forgetful' in that they do not depend on knowledge of the entire sequence of moves made so far, only on the vertex at which that sequence ends. Because of our assumption that the colouring is forgetful, the identity of that vertex incorporates some information about past play, namely, the (unique) value of the LAR of all positions that end at that vertex. In order to extend this notion of forgetfulness, we define a second equivalence relation $SAMEPAST$ on vertices of A so that two vertices v_1 and v_2 are related by $SAMEPAST$ precisely when the value of the LAR at v_1 is the same as it is at v_2. Finally, we define an equivalence relation $SAMESTRAT$ so that two vertices are related by $SAMESTRAT$ if and only if they are related both by $SAMEPAST$ and by $SAMEFUTURE$. The original version of the Forgetful Determinacy Theorem says that when the arena is a tree, one of the players has a winning strategy that is the same for any two positions related by $SAMESTRAT$. We now state the corollary from which the original FDT follows.

COROLLARY 4.1
Let $\Gamma = (A, \epsilon, W_\epsilon)$ be a graph-game in which the colouring of A is forgetful. Then one of the players has a forgetful strategy winning Γ that assigns the same value to any two vertices related by $SAMESTRAT$.

PROOF. We first look at an auxiliary game $\widetilde{\Gamma} = (\widetilde{A}, \epsilon, \widetilde{W}_\epsilon)$. The arena $\widetilde{A}$ has as its vertex set the set of equivalence classes of the $SAMESTRAT$ relation on vertices of A; that is, the vertices of $\widetilde{A}$ are of the form $\widetilde{v}$ where $\widetilde{v}$ denotes the equivalence class of the vertex v in A. The new start vertex is the equivalence class of the original start vertex. There is an edge labelled by $\mu \in MOVE$ from $\widetilde{v}$ to $\widetilde{v'}$ in $\widetilde{A}$ if there is an outgoing edge labelled μ from a vertex in $\widetilde{v}$ to a vertex in $\widetilde{v'}$. Vertex $\widetilde{v}$ of $\widetilde{A}$ is coloured with colour $s \in S$ if there is a vertex in $\widetilde{v}$ that is in C^s. For $s \in S$, let $\widetilde{C}^s$ denote the set of vertices of $\widetilde{A}$ of colour s. Also, $\widetilde{v}$ is an odd vertex if there is an odd vertex in $\widetilde{v}$; otherwise $\widetilde{v}$ is even. The notions above (edges in $\widetilde{A}$, the colouring of vertices, and the definition of even and odd vertices of $\widetilde{A}$) that were defined in terms of the existence of a vertex in $\widetilde{v}$ with a certain property could equally well have been defined by requiring that all vertices in $\widetilde{v}$ have that property, since for vertices v_1 and v_2 related by $SAMESTRAT$,

1. for $s \in S, v_1 \in C^s \iff v_2 \in C^s$;
2. for $\mu \in MOVE$, if there is an edge labelled by μ from v_1 to a vertex v', then there is an edge labelled by μ from v_2 to a vertex related by $SAMESTRAT$ to v'; and
3. v_1 and v_2 are either both odd or both even vertices.

If p is a position that ends at v in A, then p is a position in $\widetilde{A}$ that ends at $\widetilde{v}$. Furthermore $PLAY(A) = PLAY(\widetilde{A})$. Let $[\widetilde{C}^s]$ denote the set of plays in $\widetilde{A}$ that have infinitely many finite prefixes that end at a vertex of $\widetilde{A}$ that is in $\widetilde{C}^s$, and let $\widetilde{W}_\epsilon$ be the Boolean combination of $[\widetilde{C}^s]$ sets that corresponds to W_ϵ, that is, in which each $[C^s]$ set in W_ϵ is replaced by the corresponding $[\widetilde{C}^s]$ set. Since for all $s \in S$ and vertices v of A, $v \in C^s$ if and only if $\widetilde{v} \in \widetilde{C}^s$, $[C^s]$ equals $[\widetilde{C}^s]$, and W_ϵ equals $\widetilde{W}_\epsilon$. This completes the definition of $\widetilde{\Gamma} = (\widetilde{A}, \epsilon, \widetilde{W}_\epsilon)$. Note that the colouring of $\widetilde{A}$ is forgetful. By Theorem 2.1, there is a winning forgetful strategy $\widetilde{F}$ for one of the players in $\widetilde{\Gamma}$. The forgetful strategy F for Γ defined so that $F(v) = \widetilde{F}(\widetilde{v})$ is a winning strategy that has the same value at any two vertices of A related by $SAMESTRAT$, that is, F is the desired strategy. ∎

In [1], Büchi and Landweber consider the following type of game. The first player at his turn is allowed to pick any letter from alphabet I, and the second player picks from alphabet J. The winning sets are defined by a deterministic finite-state automaton, called a Muller automaton, that operates on infinite strings. A Muller automaton on alphabet Σ is of the form $M = (Q, q_0, \Delta, F)$ where Q is a finite set of states, q_0 is the designated start state, $F \subseteq 2^Q$, and Δ is a (not necessarily total) function from $Q \times \Sigma$ to Q. A run of M on an element $\alpha_0\alpha_1 \ldots$ of Σ^ω is a sequence of states $r = q_0q_1 \ldots$ such that $\Delta(q_i, \alpha_i) = q_{i+1}$ for all $i \geq 0$. Let $In(r)$ denote the set of states that occur infinitely often in r. A run r is accepting if $In(r) \in F$. An element of Σ^ω is accepted by M if there is an accepting run of M on it. The transition diagram of M is the directed multigraph with vertex set equal to Q, and an edge labelled α from q_1 to q_2 whenever $\Delta(q_1, \alpha) = q_2$. (Usually, the transition function is assumed to be total, but here we allow it to be partial to make it easier to view the arenas of graph-games as transition diagrams of automata.) In the Büchi–Landweber setting, each pair of successive moves by the two players produces an element of the alphabet $I \times J$, and the winning set for the second player is the set of sequences from this alphabet accepted by some given (fixed) Muller automaton with total transition function. In [1], it is shown that in this case, there is a winning strategy for one of the players that can be executed by a finite-state automaton, that is, there is a deterministic winning strategy for one of the players such that the appropriate move for that player is a function of the state reached by some deterministic finite-state automaton on his opponent's alphabet after reading the sequence of his opponent's previous moves.

We now examine this result in the context of graph-games in which the arena is finite (see also [4, 9, 11]). Suppose $M = (Q, q_0, \Delta, F)$ is the Muller automaton on alphabet $I \times J$ that accepts the winning set for the second player. (Without loss of generality we assume that all states of

M are reachable from q_0.) We construct another Muller automaton $M_\cup$ on alphabet $I \cup J$ that accepts the sequences in $(I \cup J)^\omega$ that correspond to those accepted by M. States of $M_\cup$ are of the form (q, x) or (q) where $x \in I$ and $q \in Q$. The start state of $M_\cup$ is (q_0). There is a transition on x from (q) to (q, x) for every $q \in Q$ and $x \in I$. There is a transition on $y \in J$ from (q_1, x) to (q_2) whenever there is a transition from q_1 to q_2 on (x, y) in M. The automaton $M_\cup$ accepts whenever the set of states of the form (q) that occur infinitely often is in F. We assign each state of the form (q) a distinct colour s_q. One can view the arena in this case as an infinite tree and derive the Büchi–Landweber result from the fact that the $SAMESTRAT$ relation has only finitely many equivalence classes. Since we have graph-games at our disposal, we choose as our arena the transition diagram of $M_\cup$ with start vertex (q_0), in which each vertex of the form (q), that is, each even vertex, is coloured with s_q. The second player, who moves at the odd vertices, wins this game if the set of colours occurring infinitely often along the path produced by the play corresponds to a set of states in F. More precisely, the winning set for the second player is $\bigcup_{I \in F}\big((\bigcap_{q \in I}[s_q]) \cap (\bigcap_{q \in Q \setminus I}[s_q]^c)\big)$. Note that all elements of $(I \cup J)^\omega$ that correspond in the natural way to elements of $(I \times J)^\omega$ are plays in this game, and each play is in the winning set of a given player if and only if it represents a win for the same player in the original Büchi–Landweber game. We cannot immediately apply Theorem 2.1 because the colouring of the transition diagram of $M_\cup$ is not necessarily forgetful. If, however, we enlarge $M_\cup$ to a new Muller automaton M_{lar} (in which each state is a pair consisting of a state of $M_\cup$ and the LAR of some position ending at that state) that also keeps track of the LAR at each position, and consider the transition diagram of M_{lar} in which states corresponding to any value of the LAR and a state of the form (q) are coloured with s_q, then we do obtain a graph-game with a forgetful colouring. In the graph-game whose arena is the coloured transition diagram of M_{lar} (with the same winning set for each player), there is a winning forgetful strategy for one of the players, and this strategy, since it is defined on vertices which are states of M_{lar}, can be thought of as being executed by the finite state automaton M_{lar}. Here the strategy is not necessarily deterministic, and the set of moves assigned by the strategy is a function of the state reached when the entire sequence of moves, that is, the current position in the game, is read by M_{lar}.

In the traditional setting in which the arena is an infinite tree rather than a graph, strategies are defined on positions rather than on vertices. We can also consider positional strategies in graph-games. Let $POS(\delta)$ be the set of positions that end at a vertex in $TURNS(\delta)$. A *positional strategy* f for Mr δ in Γ is a function f from $POS(\delta)$ to the powerset of $MOVE$ such that for all p in $POS(\delta)$, $f(p) \neq \emptyset$, and $f(p)$ contains only letters that label an outgoing edge from the vertex at which p ends. The plays in A consistent with f are defined in the natural way, and f is winning if all plays consistent with it are in the winning set for Mr δ. Then the forgetful winning strategy in the new graph-game in which the arena is the transition diagram of M_{lar} induces a winning positional strategy, that is, one defined on positions rather than on vertices, in the game in which the arena is the transition diagram of $M_\cup$. Also in this case, the strategy can be thought of as being executed by M_{lar}. In [1], the constructed strategy is deterministic and the finite state automaton that executes the strategy consumes only the moves of the opponent. Clearly, we can force the strategy executed by M_{lar} to be deterministic and then construct such an automaton for the winning player from M_{lar}. Thus the setting of Büchi and Landweber corresponds to a special graph-game in which there are only finitely many vertices.

The technique used above of expanding the arena to include information about the LAR can be applied to produce winning positional strategies that require remembering only the LAR in any graph-game (with a finite or infinite arena) whose colouring is not forgetful. Thus we have

COROLLARY 4.2
Let $\Gamma = (A, \epsilon, W_\epsilon)$ be a graph-game. Then one of the players has a winning positional strategy for Γ that assigns the same value to any two positions that end at the same vertex and have the same LAR.

Even in a graph-game with a colouring that is not forgetful, it makes sense to define the $SAMEFUTURE$ relation on vertices in the arena. In the expanded arena there is a separate copy of each vertex of the original arena for each value of the LAR that is the LAR of a position that ends at that vertex. If two vertices in the original arena are related by the $SAMEFUTURE$ relation, any copies of these vertices in the expanded arena corresponding to the same value of the LAR are related by the $SAMESTRAT$ relation in the expanded arena. Finally, if we require that the forgetful strategy in the expanded arena is the same for any two vertices related by the $SAMESTRAT$ relation in the expanded arena, we obtain the following corollary that is the analogue of the original version of the FDT for arbitrary graph-games.

COROLLARY 4.3
Let $\Gamma = (A, \epsilon, W_\epsilon)$ be a graph-game. Then one of the players has a winning positional strategy for Γ that assigns the same value to any two positions that have the same LAR and end at vertices related by $SAMEFUTURE$.

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