POLYNOMIALS WITH MULTIPLE ZEROS

R. GÜTING

1. *Introduction.* Estimates involving polynomials can often naturally be given in terms of the discriminants of these polynomials or of the resultants of pairs of polynomials. Since the discriminant of a polynomial with multiple zeros vanishes as does the resultant of two polynomials with common zeros these results become trivial when applied to such polynomials or pairs of polynomials. Therefore it is often necessary to exclude polynomials with multiple zeros from a given investigation. In theoretical studies of a measure-theoretical nature this often does not affect the results; however for the purpose of constructing polynomials with specified properties it can be an advantage if it is not necessary to restrict the attention to polynomials without multiple zeros.

The following lemmas and theorems have many applications in the theory of transcendental numbers (see *e.g.* [1] and [5]). Many of them are sharper forms of earlier results.

We consider polynomials P, Q, R. Let

$$P(x) = a_p x^p + a_{p-1} x^{p-1} + \dots + a_0 = a_p \prod_{i=1}^{p} (x - \alpha_i), \quad a_p \neq 0$$

be a polynomial with complex coefficients $a_p, a_{p-1}, \dots, a_0$. Then $p = \deg(P)$ denotes the degree of P, $s(P) = |a_p| + |a_{p-1}| + \dots + |a_0|$ is called the size of P and $h(P) = \max(|a_p|, |a_{p-1}|, \dots, |a_0|)$ is called the height of P. Also

$$D(P) = a_p^{2p-2} \prod_{i<j} (\alpha_i - \alpha_j)^2$$

is the discriminant of P and

$$\operatorname{Res}(P, Q) = a_p^{q} \prod_{i=1}^{p} Q(\alpha_i)$$

denotes the resultant of P and Q. Note that p, q, r are used as abbreviations for $\deg(P)$, $\deg(Q)$, $\deg(R)$ throughout the paper.

The following estimates are given in terms of the degrees, the sizes, the discriminants and resultants. We do not use the height $h(P)$. The size characterises a polynomial better than the height does and, if necessary, one can, of course, always use the relation $h(P) \leqslant s(P) \leqslant (p+1)h(P)$.

In the following estimates it was the objective of the author to use methods which are designed basically to keep the absolute values of the exponents of the sizes as small as possible. But this given, an effort was also made to obtain estimates with reasonably good "coefficients", that is factors in these estimates depending only on the degrees of the polynomials.

In a first section several lemmas are derived which mainly serve to yield good "coefficients" in the later estimates. These lemmas are preceded by two results of K. Mahler [4] of which extensive use is made. In a further section we prove seven theorems on polynomials without multiple zeros. These form the foundation for the last section which contains the proofs of seven theorems on polynomials with multiple zeros which are, except for the "coefficients", direct generalisations of the preceding theorems. A further result is proved independently.

In the general case of polynomials with arbitrary complex coefficients and in the special case of real coefficients, the results can frequently be stated in a single formula by means of the symbol ω which equals 1 in the former case and $\frac{1}{2}$ in the latter. Since the assumptions for the real case are not the same throughout, they are stated explicitly in each result.

Of special interest for applications in the theory of algebraic and transcendental numbers is the case of polynomials with (rational) integral coefficients. In all significant results this case is dealt with separately since our theorems lead to refinements of earlier results.

2. *Preparatory investigations.* We start with two very useful lemmas of K. Mahler [**4**]:

LEMMA A. *Let $Q(x)$ be a polynomial with leading coefficient b and the zeros $\alpha_1, \alpha_2, \ldots, \alpha_q$ which may or may not be all distinct. Then, if the subscripts $i_1, i_2, \ldots, i_m$ are distinct,*

$$|\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_m}| \leqslant \frac{s(Q)}{|b|}$$

and hence

$$\max(1, |\alpha_{i_1}|)\max(1, |\alpha_{i_2}|)\ldots\max(1, |\alpha_{i_m}|) \leqslant \frac{s(Q)}{|b|}.$$

Proof. See Mahler [**4**].

LEMMA B. *Let P_i, $i = 1, 2, \ldots, m$, be arbitrary polynomials and let $P = P_1 P_2 \ldots P_m$. Then*

$$\prod_{i=1}^{m} s(P_i) \leqslant 2^p s(P).$$

Proof. See Mahler [**4**].

Next we derive an upper estimate for the resultant of two polynomials which will be needed in the proof of Theorem 1′.

LEMMA C. *Let R and T be arbitrary polynomials. Then*

$$|\mathrm{Res}(R, T)| \leqslant s(R)^t . s(T)^r.$$

Proof. Let $R = d\prod_{i=1}^{r}(x-\delta_i)$. Then

$$\begin{aligned}|\mathrm{Res}(R, T)| &= |d|^t \prod_{i=1}^{r} |T(\delta_i)| \\ &\leqslant |d|^t \prod_{i=1}^{r} s(T)\max(1, |\delta_i|)^t \\ &= |d|^t s(T)^r \prod_{i=1}^{r} \max(1, |\delta_i|)^t.\end{aligned}$$

Applying Lemma A to the last product we obtain the result.

The following Lemmas D, E, F and G deal with elementary symmetric functions. Their only purpose is to prove Lemma I which enters directly or indirectly into nearly all later results. We use $\sigma_m(\gamma_1, \gamma_2, \ldots, \gamma_l)$ for $m = 0, 1, \ldots, l$ to denote the m-th elementary symmetric function of the variables $\gamma_1, \gamma_2, \ldots, \gamma_l$.

LEMMA D. *Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be any variables. Then*

$$\sigma_m(\alpha_{n+2}, \ldots, \alpha_p) = \sum_{i=0}^{m} (-1)^{m-i} \alpha_{n+1}^{m-i} \sigma_i(\alpha_{n+1}, \ldots, \alpha_p) \tag{1}$$

and also alternatively

$$\sigma_m(\alpha_{n+2}, \ldots, \alpha_p) = \sum_{i=m+1}^{p-n} (-1)^{m+1+i} \alpha_{n+1}^{m-i} \sigma_i(\alpha_{n+1}, \ldots, \alpha_p). \tag{2}$$

Proof. Let A_n denote the vector $(\alpha_n, \ldots, \alpha_p)$. Then we can write $\sigma_m(A_{n+1})$ as an abbreviation for $\sigma_m(\alpha_{n+1}, \ldots, \alpha_p)$. The proof is by induction on m. Using the relation

$$\sigma_{m+1}(A_{n+2}) = \sigma_{m+1}(A_{n+1}) - \alpha_{n+1} \sigma_m(A_{n+2}) \tag{3}$$

one finds no difficulty in proving formula (1). To prove formula (2) by induction we start with the identity

$$0 = (\alpha_{n+1} - \alpha_{n+1})(\alpha_{n+1} - \alpha_{n+2}) \ldots (\alpha_{n+1} - \alpha_p).$$

Thus

$$0 = \sum_{i=0}^{p-n} (-1)^i \alpha_{n+1}^{p-n-i} \sigma_i(A_{n+1}),$$

$$0 = \sum_{i=0}^{p-n} (-1)^{i+1} \alpha_{n+1}^{-i} \sigma_i(A_{n+1}),$$

and

$$\sigma_0(A_{n+2}) = 1 = \sum_{i=1}^{p-n} (-1)^{i+1} \alpha_{n+1}^{-i} \sigma_i(A_{n+1}).$$

This proves (2) in the case $m = 0$. The induction step is again a consequence of (3): replacing $\sigma_m(A_{n+2})$ in (3) by the right-hand side of (2) gives the result.

LEMMA E. *Suppose that $\alpha_1, \ldots, \alpha_p$ are the zeros of a polynomial $P(x)$ with leading coefficient a. Then for $m = 0, 1, \ldots, p-1$*

$$|\sigma_m(\alpha_2, \ldots, \alpha_p)| \leqslant \frac{s(P)}{|a| \max(1, |\alpha_1|)}.$$

Proof. Since $\sigma_i(A_1) = (-1)^i a_i / a$ it follows from formula (1) with $n = 0$ that, in the case $|\alpha_1| \leqslant 1$,

$$|\sigma_m(A_2)| \leqslant \sum_{i=0}^{m} |\sigma_i(A_1)| \leqslant \frac{1}{|a|} \sum_{i=0}^{m} |a_i| \leqslant \frac{s(P)}{|a|}.$$

In the case $|a_1| > 1$ we obtain from formula (2) the estimate

$$|\sigma_m(A_2)| \leqslant \frac{1}{|\alpha_1|} \sum_{i=m+1}^{p} |\sigma_i(A_1)| \leqslant \frac{1}{|a\alpha_1|} \sum_{i=m+1}^{p} |a_i| \leqslant \frac{s(P)}{|a\alpha_1|}.$$

LEMMA F. *Under the assumptions of Lemma E we have for $m = 0, 1, \ldots, p-2$*

$$|\sigma_m(\alpha_3, \ldots, \alpha_p)| \leqslant \begin{cases} \dfrac{(m+1)\, s(P)}{|a| \max(1, |\alpha_1|)} & \text{if } |\alpha_2| \leqslant 1 \\[2ex] \dfrac{(p-m-1)\, s(P)}{|a| \max(1, |\alpha_1|)\, |\alpha_2|} & \text{if } |\alpha_2| > 1. \end{cases}$$

Proof. Use Lemmas D and E.

LEMMA G. *Under the assumptions of Lemma E*

$$|\sigma_m(\alpha_4, \ldots, \alpha_p)| \leqslant \frac{(m+2)(m+1)s(P)}{2|a| \prod_{i=1}^{3} \max(1, |\alpha_i|)}$$

if at most one of the numbers $|\alpha_1|, |\alpha_2|, |\alpha_3|$ *is greater than* 1 *and*

$$|\sigma_m(\alpha_4, \ldots, \alpha_p)| \leqslant \frac{(p-m-1)(p-m-2)s(P)}{2|a| \prod_{i=1}^{3} \max(1, |\alpha_i|)}$$

otherwise.

Proof. To prove the first estimate we may suppose, without loss of generality, that the absolute values of both α_2 and α_3 are at most equal to 1. Thus we can apply formula (1) with $n = 2$ together with the first estimate of Lemma F.

To prove the second estimate we may suppose that the absolute values of both α_2 and α_3 are greater than 1. Then we can apply formula (2) with $n = 2$ in conjunction with the second estimate of Lemma F.

LEMMA H. *Let* $P = P_1 P_2$ *and suppose that*

$$P_1(x) = a_1 \prod_{j=1}^{n} (x - \alpha_j).$$

If $n \leqslant 3$ *then*

$$s(P_2) \leqslant \binom{p}{n} \frac{s(P)}{|a_1| \prod_{j=1}^{n} \max(1, |\alpha_j|)}.$$

Proof. Let

$$P_2(x) = a_2 \prod_{j=n+1}^{p} (x - \alpha_j).$$

Now use the relation

$$s(P_2) = |a_2| \sum_{m=0}^{p-n} |\sigma_m(\alpha_{n+1}, \ldots, \alpha_p)|$$

and apply for the cases $n = 1, 2$ and 3 the Lemmas E, F and G respectively.

COROLLARY. *If* $P = P_1 P_2$ *and* $n = \deg(P_1) \leqslant 3$ *then*

$$s(P_1)s(P_2) \leqslant 2^n \binom{p}{n} s(P).$$

Proof. If P_1 is represented as in Lemma H then

$$s(P_1) \leqslant |a_1| \prod_{j=1}^{n} (1 + |\alpha_j|) \leqslant 2^n |a_1| \prod_{j=1}^{n} \max(1, |\alpha_j|).$$

Note that if p is large this corollary is a sharper form of Lemma B in the special case that $m = 2$. We are now ready to prove

LEMMA I. *If $\alpha_1, \ldots, \alpha_p$ are the zeros of the polynomial P with leading coefficient a and if β is any complex number then for $n = 0, 1, 2, 3$ $(n \leqslant p)$ we have*

$$|a(\beta-\alpha_{n+1})\ldots(\beta-\alpha_p)| \leqslant \binom{p}{n} \frac{s(P)\max(1,|\beta|)^{p-n}}{\prod_{j=1}^{n} \max(1,|\alpha_j|)}.$$

Proof. In the notation of Lemma H, the left-hand side equals $|a_1|\,|P_2(\beta)|$ and is estimated by $|a_1|\,s(P_2)\max(1,|\beta|)^{p-n}$. Lemma H now yields the result.

COROLLARY. *Let α_i be any zero of the polynomial $P(x)$. Then the derivative $P'(x)$ satisfies*

$$|P'(\alpha_i)| \leqslant ps(P)\max(1,|\alpha_i|)^{p-2} \quad (i = 1, 2, \ldots, p).$$

To prove the result of Lemma H for $n > 3$ appears to be more difficult. However there is the following more general, but for large p weaker

LEMMA J. *Under the conditions of Lemma I the estimate*

$$|a(\beta-\alpha_{n+1})(\beta-\alpha_{n+2})\ldots(\beta-\alpha_p)| \leqslant \frac{2^{p-n}s(P)\max(1,|\beta|)^{p-n}}{\prod_{j=1}^{n}\max(1,|\alpha_j|)}$$

holds for arbitrary n $(0 \leqslant n \leqslant p)$. (Empty products have the value 1).

Proof. The proof of this weaker estimate is very simple:

$$|a(\beta-\alpha_{n+1})\ldots(\beta-\alpha_p)| \leqslant 2^{p-n}|a|\max(1,|\beta|)^{p-n}\prod_{j=n+1}^{p}\max(1,|\alpha_j|)$$

$$= 2^{p-n}|a|\max(1,|\beta|)^{p-n}\frac{\prod_{j=1}^{p}\max(1,|\alpha_j|)}{\prod_{j=1}^{n}\max(1,|\alpha_j|)}.$$

Now apply Lemma A to the product in the numerator.

3. *Simple zeros of polynomials.* We are now ready to derive a number of results for polynomials with simple zeros. Most of them are refinements of earlier results (*cf.* Theorems 1, 3, 6 and 7 with **[2]**, Theorem 1, Lemma 3, Theorems 3 and 4 respectively, and Theorems 2 and 4 with **[3]**, Hilfssatz 3 and Hilfssatz 4 respectively). They actually remain valid if the zeros are multiple zeros, but they become trivial in this case or else the results of §4 are generalisations of them and improvements in the sense that the exponents of the sizes are sharpened.

The modulus of the value of a polynomial with rational integral coefficients at algebraic points cannot be arbitrarily small. This fact is illustrated and stated in more general terms in the following

THEOREM 1. *Let $Q(x)$ and $R(x)$ be polynomials and let α denote a zero of $Q(x)$. Then*

$$|R(\alpha)| \geqslant |\mathrm{Res}(Q,R)|^{\omega}s(Q)^{-\omega r}s(R)^{-\omega q+1}\max(1,|\alpha|)^{r} \tag{1}$$

with $\omega = 1$. Moreover, ω may be replaced by $\frac{1}{2}$ if both Q and R have real coefficients and α is not real.

Proof. Suppose that Q and R have real coefficients and that α is not real. The assumptions imply that the complex conjugate $\bar{\alpha}$ of α is a second zero of $Q(x)$ and that $|R(\alpha)| = |R(\bar{\alpha})|$. Thus, denoting the leading coefficient of Q by b and putting $\alpha_1 = \alpha$ and $\alpha_2 = \bar{\alpha}$ we obtain

$$|\mathrm{Res}(Q, R)| = |b|^r \prod_{i=1}^{q} |R(\alpha_i)| = |b|^r |R(\alpha)|^2 \prod_{i=3}^{q} |R(\alpha_i)|. \tag{2}$$

Since† $|R(\alpha_i)| \leqslant s(R) m(\alpha_i)^r$ we find, utilising Lemma A, that

$$\prod_{i=3}^{q} |R(\alpha_i)| \leqslant s(R)^{q-2} \left(\prod_{i=3}^{q} m(\alpha_i) \right)^r \leqslant \frac{s(R)^{q-2} s(Q)^r}{|b|^r m(\alpha_1)^r m(\alpha_2)^r}.$$

Hence, from (2),

$$|\mathrm{Res}(Q, R)| \leqslant |R(\alpha)|^2 s(R)^{q-2} s(Q)^r m(\alpha)^{-2r}.$$

This proves (1) with $\omega = \frac{1}{2}$. The general estimate with $\omega = 1$ is obtained by writing

$$|\mathrm{Res}(Q, R)| = |b|^r |R(\alpha)| \prod_{i=2}^{q} |R(\alpha_i)|$$

instead of (2) and proceeding similarly.

COROLLARY. *Let $Q(x)$ and $R(x)$ be polynomials with integral coefficients and without common zeros. If α denotes a zero of $Q(x)$ then*

$$|R(\alpha)| \geqslant s(Q)^{-\omega r} s(R)^{-\omega q+1} \max(1, |\alpha|)^r$$

with $\omega = 1$. Moreover, if α is not real, ω may be replaced by $\frac{1}{2}$.

Proof. Since Q and R have integral coefficients the resultant $\mathrm{Res}(Q, R)$ is an integer and it is different from zero, since Q and R have no common zeros. Thus its absolute value is at least 1.

We now turn to the derivative Q' of a polynomial Q. From Theorem 1 it may be expected that its value at a zero of Q is not arbitrarily small. In this special case the result of Theorem 1 can be sharpened:

THEOREM 2. *Let $Q(x)$ be a polynomial without multiple zeros and let α denote a zero of $Q(x)$. Then*

$$|Q'(\alpha)| \geqslant c_1(q) |D(Q)|^{1/4} s(Q)^{-(q-3)/2} |\mathrm{Im}\,\alpha|^{1/2} \max(1, |\alpha|)^{q-3},$$

if Q has real coefficients and α is not real, and

$$|Q'(\alpha)| \geqslant c_2(q) |D(Q)|^{1/2} s(Q)^{-(q-2)} \max(1, |\alpha|)^{q-2}$$

generally. Here $D(Q)$ denotes the discriminant of Q, $c_1(q) = \sqrt{2} \binom{q}{3}^{-(q-2)/4}$ and $c_2(q) = \binom{q}{2}^{-(q-1)/2}$.

Proof. Suppose first that Q has real coefficients and that α is not real. Let $Q(x) = b \prod_{i=1}^{q} (x - \alpha_i)$ with $\alpha_1 = \alpha$ and $\alpha_2 = \bar{\alpha}$. Then the discriminant $D(Q)$ is given by

$$D(Q) = b^{2q-2} \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

† Note that we use the abbreviation $m(\alpha)$ for $\max(1, |\alpha|)$.

Hence
$$|D(Q)| = \frac{|Q'(\alpha)|^2|Q'(\bar{\alpha})|^2}{|\alpha-\bar{\alpha}|^2}|b|^{2q-6}\cdot \prod_{\substack{i<j\\ i,j\neq 1,2}} |\alpha_i-\alpha_j|^2,$$

so that
$$|D(Q)| = \frac{|Q'(\alpha)|^4}{4|\mathrm{Im}\,\alpha|^2}|b|^{2q-6}\cdot \prod_{\substack{i<j\\ i,j\neq 1,2}} |\alpha_i-\alpha_j|^2. \tag{3}$$

Now
$$\prod_{\substack{i<j\\ i,j\neq 1,2}} |\alpha_i-\alpha_j|^2 = \prod_{i=3}^{q}\prod_{\substack{j=3\\ j\neq i}}^{q} |\alpha_i-\alpha_j|.$$

By Lemma I with $n = 3$, and for $i \geqslant 3$,
$$\prod_{\substack{j=3\\ j\neq i}}^{q} |\alpha_i-\alpha_j| \leqslant \binom{q}{3}\frac{s(Q)\,m(\alpha_i)^{q-3}}{|b|\,m(\alpha_1)\,m(\alpha_2)\,m(\alpha_i)} = \binom{q}{3}\frac{s(Q)\,m(\alpha_i)^{q-4}}{|b|\,m(\alpha)^2}.$$

Therefore,
$$\prod_{\substack{i<j\\ i,j\neq 1,2}} |\alpha_i-\alpha_j|^2 \leqslant \binom{q}{3}^{q-2}\frac{s(Q)^{q-2}}{|b|^{q-2}m(\alpha)^{2q-4}}\prod_{i=3}^{q} m(\alpha_i)^{q-4}.$$

Applying Lemma A we now find the estimate
$$\prod_{\substack{i<j\\ i,j\neq 1,2}} |\alpha_i-\alpha_j|^2 \leqslant \binom{q}{3}^{q-2}\frac{s(Q)^{2q-6}}{|b|^{2q-6}m(\alpha)^{4q-12}}.$$

Using this in the inequality (3) yields:
$$|D(Q)| \leqslant \binom{q}{3}^{q-2}\frac{|Q'(\alpha)|^4 s(Q)^{2q-6}}{4|\mathrm{Im}\,\alpha|^2 m(\alpha)^{4q-12}}.$$

This proves the first inequality of Theorem 2. Similarly, if Q has arbitrary complex coefficients,
$$|D(Q)| = |Q'(\alpha)|^2|b|^{2q-4}\prod_{\substack{i<j\\ i\neq 1}} |\alpha_i-\alpha_j|^2.$$

Now
$$\prod_{\substack{i<j\\ i\neq 1}} |\alpha_i-\alpha_j|^2 = \prod_{i=2}^{q}\prod_{\substack{j=2\\ j\neq i}}^{q} |\alpha_i-\alpha_j|.$$

By Lemma I, for $i \geqslant 2$,
$$\prod_{\substack{j=2\\ j\neq i}}^{q} |\alpha_i-\alpha_j| \leqslant \binom{q}{2}\frac{s(Q)\,m(\alpha_i)^{q-2}}{|b|\,m(\alpha_1)\,m(\alpha_i)} = \binom{q}{2}\frac{s(Q)\,m(\alpha_i)^{q-3}}{|b|\,m(\alpha)}.$$

Hence
$$|D(Q)| \leqslant |Q'(\alpha)|^2|b|^{2q-4}\binom{q}{2}^{q-1}\frac{s(Q)^{q-1}}{|b|^{q-1}m(\alpha)^{q-1}}\prod_{i=2}^{q} m(\alpha_i)^{q-3}$$
$$\leqslant |Q'(\alpha)|^2\binom{q}{2}^{q-1}s(Q)^{2q-4}m(\alpha)^{-(2q-4)}.$$

Taking positive square roots one obtains the required inequality.

Next we turn to the following question. Let a complex number α and a polynomial R be given. If $|R(\alpha)|$ is small it may be expected that some zero β of R is fairly close to α. What can be said about the distance $|\alpha - \beta|$? The following two results give a lower and an upper bound for $|\alpha - \beta|$ in terms of $|R(\alpha)|$.

THEOREM 3. *Let β be a zero of the polynomial R and let α be any complex number. Then*

$$|R(\alpha)| \leqslant |\alpha - \beta| \, rs(R) \max(1, |\alpha|)^{r-1} \max(1, |\beta|)^{-1}. \tag{4}$$

Moreover, if R has real coefficients, α is real and β is not real then

$$|R(\alpha)| \leqslant |\alpha - \beta|^2 \binom{r}{2} s(R) \max(1, |\alpha|)^{r-2} \max(1, |\beta|)^{-2}. \tag{5}$$

Proof. We may write $R(\alpha) = (\alpha - \beta)[c(\alpha - \beta_2) \dots (\alpha - \beta_r)]$ and apply Lemma I to the product in brackets to obtain the first estimate. To find the second estimate write $R(\alpha) = (\alpha - \beta)(\alpha - \bar{\beta})[c(\alpha - \beta_3) \dots (\alpha - \beta_r)]$ and apply Lemma I to the new product in brackets. Notice that $|\alpha - \beta| = |\alpha - \bar{\beta}|$ and $|\beta| = |\bar{\beta}|$.

THEOREM 4. *Let Q be a polynomial without multiple zeros and let γ be an arbitrary complex number. Let α denote a zero of Q at minimal distance from γ. Then*

$$|\gamma - \alpha| \leqslant \frac{c_3(q) \, |Q(\gamma)| \, s(Q)^{(q-3)/2}}{|D(Q)|^{1/4} |\operatorname{Im} \alpha|^{1/2} \max(1, |\alpha|)^{q-3}}$$

if Q has real coefficients and α is not real; and

$$|\gamma - \alpha| \leqslant \frac{c_4(q) \, |Q(\gamma)| \, s(Q)^{q-2}}{|D(Q)|^{1/2} \max(1, |\alpha|)^{q-2}}$$

in all cases. The constants c_3 and c_4 are given by

$$c_3(q) = 2^{q-(3/2)} \binom{q}{3}^{(q-2)/4} \quad \text{and} \quad c_4(q) = 2^{q-1} \binom{q}{2}^{(q-1)/2}.$$

Proof. By the choice of α, every zero α_i of Q satisfies the inequality

$$|\alpha - \alpha_i| \leqslant |\gamma - \alpha_i| + |\gamma - \alpha| \leqslant 2|\gamma - \alpha_i|.$$

Hence, putting $\alpha = \alpha_1$,

$$\begin{aligned} |Q(\gamma)| &= |\gamma - \alpha| \left| a \prod_{i=2}^{q} (\gamma - \alpha_i) \right| \geqslant \frac{|\gamma - \alpha|}{2^{q-1}} \left| a \prod_{i=2}^{q} (\alpha - \alpha_i) \right| \\ &= |\gamma - \alpha| \, 2^{-q+1} |Q'(\alpha)|. \end{aligned}$$

Now apply Theorem 2.

The result of Theorem 3 may be somewhat sharpened in the case where R is the derivative P' of a polynomial P which is evaluated at a zero of the polynomial P (compare the exponents of $\max(1, |\alpha|)$ and note that this result is applied in Theorem 7 to reduce the exponent of the size of the polynomial Q occurring there).

THEOREM 5. *Let α_1 and α_2 be zeros of the polynomial Q. Then*

$$|Q'(\alpha_1)| \leqslant |\alpha_1 - \alpha_2| \binom{q}{2} s(Q) \max(1, |\alpha_1|)^{q-3} \max(1, |\alpha_2|)^{-1}. \tag{6}$$

Moreover, if Q has real coefficients, α_1 is real and α_2 is not real, then

$$|Q'(\alpha_1)| \leqslant |\alpha_1 - \alpha_2|^2 \binom{q}{3} s(Q) \max(1, |\alpha_1|)^{q-4} \max(1, |\alpha_2|)^{-2}. \tag{7}$$

Proof. To prove the inequality (6) apply Lemma I to the second factor in the decomposition

$$Q'(\alpha_1) = (\alpha_1 - \alpha_2)[b(\alpha_1 - \alpha_3) \dots (\alpha_1 - \alpha_q)].$$

To prove the inequality (7) let $\alpha_3 = \bar{\alpha}_2$. Since α_1 is real we have

$$|\alpha_1 - \alpha_2| = |\alpha_1 - \bar{\alpha}_2| = |\alpha_1 - \alpha_3|,$$

hence

$$|Q'(\alpha_1)| = |\alpha_1 - \alpha_2|^2 \,|[b(\alpha_1 - \alpha_4) \dots (\alpha_1 - \alpha_q)]|.$$

Now we apply Lemma I to the expression inside the square brackets.

We conclude this section with two results which illustrate the fact that the distance between algebraic numbers of given sizes and given degrees cannot be arbitrarily small. Note that Theorem 7 gives a better exponent of the size even in the general inequality than an earlier result of the author [**2**; Theorem 4].

THEOREM 6. *Let α be a zero of the polynomial $Q(x)$ and β a zero of the polynomial $R(x)$. Then*

$$|\alpha - \beta| \geqslant \frac{|\mathrm{Res}(Q, R)|^{\omega} \max(1, |\alpha|) \max(1, |\beta|)}{\min(q, r) s(Q)^{\omega r} s(R)^{\omega q}} \tag{8}$$

with $\omega = 1$ generally and $\omega = \frac{1}{2}$ if both Q and R have real coefficients and at least one of the numbers α, β is not real.

Proof. To prove (8) with $\omega = \frac{1}{2}$ suppose that α is not real. Then we can combine formula (1) with $\omega = \frac{1}{2}$ and formula (4) to find the inequality

$$|\alpha - \beta| \geqslant \frac{c_5(r) |\,\mathrm{Res}(Q, R)|^{1/2} m(\alpha) m(\beta)}{s(Q)^{r/2} s(R)^{q/2}} \tag{9}$$

with $c_5(r) = 1/r$. If α is real and β is not real we can combine the inequality (1) with $\omega = 1$ and (5) to obtain (9) with $c_5(r) = \binom{r}{2}^{-1/2} > \dfrac{1}{r}$. Thus (9) holds always with $c_5(r) = 1/r$. Interchanging the roles of α and β we can derive the inequality (9) with $c_5(r)$ replaced by $1/q$. This proves (8) with $\omega = 1/2$. The case $\omega = 1$ is obtained from combining (1) with $\omega = 1$ and (4).

THEOREM 7. *Let α_1 and α_2 be two zeros of the polynomial $Q(x)$. Then*

$$|\alpha_1 - \alpha_2| \geqslant \frac{|D(Q)|^{\omega/2} \max(1, |\alpha_1|) \max(1, |\alpha_2|)}{\binom{q}{2} q^{(\omega q/2) - 1} s(Q)^{\omega(q-1)}} \tag{10}$$

where $\omega = 1$. But ω may be replaced by $\frac{1}{2}$ if all of the following conditions are satisfied: Q has real coefficients, at least one of α_1, α_2 is not real and $\alpha_2 \neq \bar{\alpha}_1$.

Proof. We first carry out the proof of formula (10) with $\omega = \frac{1}{2}$ under the given additional assumptions. Without loss of generality we may assume that α_1 is not real. We distinguish two cases:

(a). Suppose α_2 is not real. Let $\alpha_3 = \bar{\alpha}_1$ and $\alpha_4 = \bar{\alpha}_2$. By formula (6) of Theorem 5 we may estimate $|Q'(\alpha_1)|$, $|Q'(\alpha_2)|$, $|Q'(\alpha_3)|$ and $|Q'(\alpha_4)|$ in terms of $|\alpha_1 - \alpha_2|$, $|\alpha_2 - \alpha_1|$, $|\alpha_3 - \alpha_4|$ and $|\alpha_4 - \alpha_3|$ respectively. Multiplying the corresponding inequalities together and observing that $|\alpha_1 - \alpha_2| = |\bar{\alpha}_1 - \bar{\alpha}_2| = |\alpha_3 - \alpha_4|$ we obtain

$$\prod_{i=1}^{4} |Q'(\alpha_i)| \leqslant |\alpha_1 - \alpha_2|^4 \binom{q}{2}^4 s(Q)^4 \prod_{i=1}^{4} m(\alpha_i)^{q-4}.$$

Now using $D(Q) = \pm b^{q-2} \prod_{i=1}^{q} Q'(\alpha_i)$ gives

$$|D(Q)| \leqslant |\alpha_1 - \alpha_2|^4 \binom{q}{2}^4 s(Q)^4 \prod_{i=1}^{4} m(\alpha_i)^{q-4} |b|^{q-2} \prod_{i=5}^{q} |Q'(\alpha_i)|.$$

Applying the Corollary of Lemma I to the two products and utilizing Lemma A we find successively

$$\prod_{i=5}^{q} |Q'(\alpha_i)| \prod_{i=1}^{4} m(\alpha_i)^{q-4} \leqslant q^{q-4} s(Q)^{q-4} \prod_{i=1}^{q} m(\alpha_i)^{q-2} \prod_{i=1}^{4} m(\alpha_i)^{-2}$$
$$\leqslant q^{q-4} |b|^{-q+2} s(Q)^{2q-6} m(\alpha_1)^{-4} m(\alpha_2)^{-4}.$$

Hence

$$|D(Q)| \leqslant |\alpha_1 - \alpha_2|^4 \binom{q}{2}^4 q^{q-4} s(Q)^{2q-2} m(\alpha_1)^{-4} m(\alpha_2)^{-4}.$$

which is the required result.

(b). Suppose α_2 is real. Let $\alpha_3 = \bar{\alpha}_1$. By formula (6) of Theorem 5 we may estimate $|Q'(\alpha_1)|$ and $|Q'(\alpha_3)|$ in terms of $|\alpha_1 - \alpha_2|$ and $|\alpha_3 - \alpha_2|$ respectively and by formula (7) of the same theorem we may estimate $|Q'(\alpha_2)|$ in terms of $|\alpha_2 - \alpha_1|^2$. Proceeding as in part (a) we obtain the inequality

$$|D(Q)| \leqslant |\alpha_1 - \alpha_2|^4 \binom{q}{3} \binom{q}{2}^2 q^{q-3} s(Q)^{2q-2} m(\alpha_1)^{-4} m(\alpha_2)^{-4}$$

which is slightly better than the assertion.

This completes the proof of (10) with $\omega = \frac{1}{2}$. The general formula is obtained by estimating $|Q'(\alpha_1)|$ and $|Q'(\alpha_2)|$ by formula (6) of Theorem 5 in terms of $|\alpha_1 - \alpha_2|$ and $|\alpha_2 - \alpha_1|$ and proceeding similarly.

4. *Multiple zeros of polynomials.* We now proceed to generalise the results of the last section to polynomials with multiple zeros. In order to make the relation between the theorems apparent we denote the theorems of this section by the same numbers as the corresponding theorems of the previous section, but we add a dash to distinguish them. However the last result has no equivalent in the previous section.

THEOREM 1′. *Let $P(x)$ and $R(x)$ be polynomials and let α denote a zero of order k of $P(x)$. Then*

$$|R(\alpha)| \geqslant \frac{|\mathrm{Res}(P, R)|^{\omega/k} \max(1, |\alpha|)^r}{2^{\omega pr/k} s(P)^{\omega r/k} s(R)^{(\omega p/k)-1}} \tag{1}$$

with $\omega = 1$. Also, (1) holds with $\omega = \frac{1}{2}$ if P and R have real coefficients and α is not real.

Proof. We can write $P(x)$ as a product $P(x) = Q(x)^k T(x)$ with $Q(\alpha) = 0$, $T(\alpha) \neq 0$. If P has real coefficients we may suppose that the polynomials $Q(x)$ and $T(x)$ in this factorisation have real coefficients. By Theorem 1,

$$|R(\alpha)| \geqslant |\mathrm{Res}(Q, R)|^{\omega} s(Q)^{-\omega r} s(R)^{-\omega q+1} m(\alpha)^r. \tag{2}$$

Since $|\mathrm{Res}(P, R)| = |\mathrm{Res}(Q, R)|^k |\mathrm{Res}(T, R)|$ we find from Lemma C:

$$|\mathrm{Res}(P, R)| \leqslant |\mathrm{Res}(Q, R)|^k s(R)^t s(T)^r,$$

i.e.

$$|\mathrm{Res}(Q, R)| \geqslant |\mathrm{Res}(P, R)|^{1/k} s(R)^{-t/k} s(T)^{-r/k}.$$

Substituting this in (2) we obtain

$$|R(\alpha)| \geqslant \frac{|\mathrm{Res}(P, R)|^{\omega/k} m(\alpha)^r}{s(Q)^{\omega r} s(R)^{\omega q - 1 + (\omega t/k)} s(T)^{\omega r/k}}.$$

From Lemma B,

$$s(Q)^r s(T)^{r/k} = [s(Q)^k s(T)]^{r/k} \leqslant 2^{pr/k} s(P)^{r/k}.$$

Using also in the exponent of $s(R)$ the relation $kq + t = p$ we get the desired estimate.

THEOREM 2′. *Let P be a polynomial with arbitrary complex coefficients and let α be a zero of order k of P. Then for any decomposition $P = Q^k R$ with polynomials Q, R such that Q has no multiple zeros, $Q(\alpha) = 0$, $R(\alpha) \neq 0$, $s(R) \geqslant 1$ the following holds*:

$$|P^{(k)}(\alpha)| \geqslant c_6(p, k) |\mathrm{Im}\,\alpha|^{k/2} |D(Q)|^{k/4} |\mathrm{Res}(Q, R)|^{1/2} s(P)^{-(p-3k)/2k} \max(1, |\alpha|)^{p-3k},$$

if P, Q, R have real coefficients and α is not real;

$$|P^{(k)}(\alpha)| \geqslant c_7(p, k) |D(Q)|^{k/2} |\mathrm{Res}(Q, R)| s(P)^{-(p-2k)/k} \max(1, |\alpha|)^{p-2k}$$

generally. Here $P^{(k)}$ denotes the k-th derivative of P,

$$c_6(p, k) = k!\,(p/k)^{-3(p-2k)/4} 2^{-(p^2-3pk-k^2)/2k}$$

and

$$c_7(p, k) = k!\,(p/k)^{-(p-k)} 2^{-p(p-2k)/k}.$$

Proof. Suppose that P, Q, R have real coefficients and that α is not real. From the relation

$$P^{(k)}(\alpha) = k![Q'(\alpha)]^k R(\alpha)$$

we derive, applying Theorem 2 to $Q(x)$ and Theorem 1 to $R(x)$,

$$|P^{(k)}(\alpha)| \geqslant \frac{k!\,c_1(q)^k |\mathrm{Im}\,\alpha|^{k/2} |D(Q)|^{k/4} |\mathrm{Res}(Q, R)|^{1/2} m(\alpha)^{kq-3k+r}}{s(Q)^{[k(q-3)+r]/2} s(R)^{(q/2)-1}}.$$

Now, using the relation $kq + r = p$ and the assumption $s(R) \geqslant 1$,

$$|P^{(k)}(\alpha)| \geqslant k!\,c_1(q)^k |\mathrm{Im}\,\alpha|^{k/2} |D(Q)|^{k/4} |\mathrm{Res}(Q, R)|^{1/2} s(Q)^{-(p-3k)/2} m(\alpha)^{p-3k}.$$

According to Lemma B,

$$s(Q)^k \leqslant s(Q)^k s(R) \leqslant 2^p s(P).$$

Hence

$$|P^{(k)}(\alpha)| \geqslant c_6(p, k) |\mathrm{Im}\,\alpha|^{k/2} |D(Q)|^{k/4} |\mathrm{Res}(Q, R)|^{1/2} s(P)^{-(p-3k)/2k} m(\alpha)^{p-3k}.$$

Using the fact that $kq \leqslant p$ we obtain $c_6(p, k)$ in the form given in the theorem. The general formula is found similarly.

COROLLARY. *Suppose that P has integral coefficients and that α is a zero of order k of P. Then, if α is not real,*

$$|P^{(k)}(\alpha)| \geqslant c_6(p, k) |\mathrm{Im}\,\alpha|^{k/2} s(P)^{-(p-3k)/2k} \max(1, |\alpha|)^{p-3k};$$

and if α is real

$$|P^{(k)}(\alpha)| \geqslant c_7(p, k)s(P)^{-(p-2k)/k}\max(1, |\alpha|)^{p-2k}.$$

Proof. Let $Q(x)$ denote the minimal polynomial† of α. Then $R(x)$ defined by $P(x) = Q^k(x)R(x)$ is a polynomial with integral coefficients. Hence $D(Q)$ and $\text{Res}(Q, R)$ are integers. Since Q is irreducible and has no zeros in common with R, both $D(Q)$ and $\text{Res}(Q, R)$ are different from 0 and thus have absolute value at least equal to 1.

THEOREM 3′. *Let β be a zero of order l of a polynomial $R(x)$ and let α be any complex number. Then*

$$|R(\alpha)| \leqslant 2^{r-l}|\alpha-\beta|^l s(R)\max(1, |\alpha|)^{r-l}\max(1, |\beta|)^{-l}. \tag{3}$$

Moreover, if R has real coefficients, α is real and β is not real then

$$|R(\alpha)| \leqslant 2^{r-2l}|\alpha-\beta|^{2l} s(R)\max(1, |\alpha|)^{r-2l}\max(1, |\beta|)^{-2l}. \tag{4}$$

Proof. The proof is analogous to the proof of Theorem 3. However instead of Lemma I the more general Lemma J is used.

THEOREM 4′. *Let $P(x)$ be an arbitrary polynomial and γ any complex number. Let α be a zero of $P(x)$ whose distance from γ is minimal and denote the order of α by k. Let $P = Q^kR$ with polynomials Q, R chosen so that Q has no multiple zeros, $Q(\alpha) = 0$, $R(\alpha) \neq 0$ and $s(R) \geqslant 1$. Then, if P, Q, R have real coefficients and α is not real,*

$$|\gamma-\alpha|^k \leqslant \frac{c_8(p, k)|P(\gamma)|\, s(P)^{(p-3k)/2k}}{|\text{Res}(Q, R)|^{1/2}|D(Q)|^{k/4}\max(1, |\alpha|)^{p-3k}|\text{Im}\,\alpha|^{k/2}} \tag{5}$$

and in any case

$$|\gamma-\alpha|^k \leqslant \frac{c_9(p, k)|P(\gamma)|\, s(P)^{(p-2k)/k}}{|\text{Res}(Q, R)|\,|D(Q)|^{k/2}\max(1, |\alpha|)^{p-2k}}. \tag{6}$$

The factors c_8 and c_9 are given by

$$c_8(p, k) = \sqrt{2}^{(p^2/k)-p-3k}\left(\frac{p}{k}\right)^{3(p-2k)/4}$$

and

$$c_9(p, k) = 2^{(p^2/k)-p-k}\left(\frac{p}{k}\right)^{p-k}.$$

Proof. By our assumption about the choice of α, every zero β_i of R satisfies the inequality $|\gamma-\beta_i| \geqslant |\gamma-\alpha|$. Therefore $|\alpha-\beta_i| \leqslant |\alpha-\gamma| + |\beta_i-\gamma| \leqslant 2|\beta_i-\gamma|$. Hence, denoting the leading coefficient of R by d, we find that

$$|R(\gamma)| = |d|\prod_{i=1}^{r}|\gamma-\beta_i| \geqslant |d|\prod_{i=1}^{r}\frac{|\alpha-\beta_i|}{2} = \frac{|R(\alpha)|}{2^r}.$$

Theorem 1 implies that

$$|R(\gamma)| \geqslant 2^{-r}|\text{Res}(Q, R)|^{\omega}s(Q)^{-\omega r}s(R)^{-\omega q+1}m(\alpha)^r \tag{7}$$

holds with $\omega = 1$, and, provided that Q and R have real coefficients and α is not real,

† As the minimal polynomial of an algebraic number α we select from the polynomials, which have integral coefficients and vanish at α, that one of smallest degree whose leading coefficient is positive.

with $\omega = \frac{1}{2}$. Next, we apply Theorem 4 to Q. For convenience we write the two estimates of Theorem 4 in one formula:

$$|\gamma - \alpha| \leqslant c_{10}(q, \alpha; \omega)\,|Q(\gamma)|\, s(Q)^{\omega q - 1 - \omega}\,|D(Q)|^{-\omega/2} m(\alpha)^{-q+4-2\omega}, \tag{8}$$

where ω is defined as above and $c_{10}(q, \alpha; \omega)$ is given by

$$c_{10}(q, \alpha; \tfrac{1}{2}) = c_3(q)/|\mathrm{Im}\,\alpha|^{1/2},$$
$$c_{10}(q, \alpha; 1) = c_4(q).$$

With this notation we find, taking (8) to the k-th power and using the relations $|Q(\gamma)|^k = |P(\gamma)|/|R(\gamma)|$ and (7),

$$|\gamma - \alpha|^k \leqslant \frac{2^r c_{10}(q, \alpha; \omega)^k\,|P(\gamma)|\, s(Q)^{\omega r + k(\omega q - 1 - \omega)} s(R)^{\omega q - 1}}{|\mathrm{Res}(Q, R)|^{\omega}\,|D(Q)|^{k\omega/2} m(\alpha)^{r + kq - k(4 - 2\omega)}}. \tag{9}$$

By Lemma B, $s(Q)^k s(R) \leqslant 2^p s(P)$. Hence also $s(Q)^k \leqslant 2^p s(P)$ since the size of R is at least 1. Taking the first of these two inequalities to the power $\omega q - 1$ and the second to the power $\omega(r - k)/k$ and substituting the results in (9), we obtain, since $p = kq + r$,

$$|\gamma - \alpha|^k \leqslant \frac{2^{r + p[(\omega p/k) - 1 - \omega]} c_{10}(q, \alpha; \omega)^k\,|P(\gamma)|\, s(P)^{(\omega p/k) - 1 - \omega}}{|\mathrm{Res}(Q, R)|^{\omega}\,|D(Q)|^{k\omega/2} m(\alpha)^{p - k(4 - 2\omega)}}.$$

In the case $\omega = 1$,

$$2^{r + p[(p/k) - 2]}\left[2^{q-1}\binom{q}{2}^{[(q-1)/2]}\right]^k \leqslant 2^{(p^2/k) - p - k}\left(\frac{p}{k}\right)^{p-k}.$$

In the case $\omega = \frac{1}{2}$,

$$2^{r + p[(p/2k) - (3/2)]}\left[2^{q-(3/2)}\binom{q}{3}^{[(q-2)/4]}\right]^k |\mathrm{Im}\,\alpha|^{-k/2} \leqslant \sqrt{2}^{(p^2/k) - p - 3k}\left(\frac{p}{k}\right)^{3(p-2k)/4} |\mathrm{Im}\,\alpha|^{-k/2}.$$

Hence the result.

COROLLARY. *Suppose that $P(x)$ is a polynomial with integral coefficients. Let γ be any complex number. Denote by α a zero of $P(x)$ at minimal distance from γ and let k be the order of α. Then*

$$|\gamma - \alpha|^k \leqslant c_8(p, k)\,|\mathrm{Im}\,\alpha|^{-k/2} \max(1, |\alpha|)^{-p+3k} s(P)^{(p-3k)/2k}\,|P(\gamma)|$$

if α is not real; and

$$|\gamma - \alpha|^k \leqslant c_9(p, k) \max(1, |\alpha|)^{-p+2k} s(P)^{(p-2k)/k}\,|P(\gamma)|$$

for any α.

Proof. Proceed as in the proof of the Corollary to Theorem 2′.

THEOREM 5′. *Let α_1 and α_2 be zeros of the polynomial P of orders k_1 and k_2 respectively. Then*

$$|P^{(k_1)}(\alpha_1)| \leqslant k_1!\, 2^p\, |\alpha_1 - \alpha_2|^{k_2/\omega} s(P) \max(1, |\alpha_1|)^{p - 2k_1 - (k_2/\omega)} \max(1, |\alpha_2|)^{-k_2/\omega}$$

with $\omega = 1$. If P has real coefficients, α_1 is real and α_2 is not real ω may be replaced by $\frac{1}{2}$.

Proof. We can write $P(x) = (x - \alpha_1)^{k_1}(x - \alpha_2)^{k_2} R(x)$ for some polynomial $R(x)$. Since $P^{(k_1)}(\alpha_1) = k_1!\,(\alpha_1 - \alpha_2)^{k_2} R(\alpha_1)$, we find

$$|P^{(k_1)}(\alpha_1)| \leqslant k_1!\,|\alpha_1 - \alpha_2|^{k_2}\, s(R)\, m(\alpha_1)^r.$$

By Lemma B, $(1+|\alpha_1|)^{k_1}(1+|\alpha_2|)^{k_2}s(R) \leqslant 2^p s(P)$. Therefore

$$|P^{(k_1)}(\alpha_1)| \leqslant k_1!\,2^p|\alpha_1-\alpha_2|^{k_2}s(P)m(\alpha_1)^r(1+|\alpha_1|)^{-k_1}(1+|\alpha_2|)^{-k_2}$$
$$\leqslant k_1!\,2^p|\alpha_1-\alpha_2|^{k_2}s(P)m(\alpha_1)^{p-2k_1-k_2}m(\alpha_2)^{-k_2},$$

since $r = p-k_1-k_2$.

If P has real coefficients, α_1 is real and α_2 is not real we can factorise:

$$P(x) = (x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}(x-\bar{\alpha}_2)^{k_2}T(x).$$

Now $|P^{(k_1)}(\alpha_1)| = k_1!\,|\alpha_1-\alpha_2|^{2k_2}s(T)m(\alpha_1)^t$. Thus we may proceed as before.

THEOREM 6′. *Let α be a zero of order k of the polynomial $P(x)$ and let β be a zero of order l of the polynomial $R(x)$. Then*

$$|\alpha-\beta| \geqslant \frac{|\mathrm{Res}(P,R)|^{\omega/kl}\max(1,|\alpha|)\max(1,|\beta|)}{2^{\min(r/l,\,p/k)+(\omega pr/kl)-1}s(P)^{\omega r/kl}s(R)^{\omega p/kl}}$$

with $\omega = 1$, and, provided that P and R have real coefficients and at least one of the numbers α, β is not real, with $\omega = \frac{1}{2}$.

Proof. The proof is completely analogous to the proof of Theorem 6, but instead of formulae (1), (4) and (5) we use respectively the formulae (1), (3) and (4) of this section.

THEOREM 7′. *Let α_1 and α_2 be two zeros of the polynomial $P(x)$ of orders k_1 and k_2 respectively. Write $P(x) = Q_1(x)^{k_1}Q_2(x)^{k_2}R(x)$ with polynomials Q_1, Q_2, R satisfying $Q_1(\alpha_1) = Q_2(\alpha_2) = 0$ and $s(R) \geqslant 1$. Then*

$$|\alpha_1-\alpha_2| \geqslant \frac{|\mathrm{Res}(Q_1,Q_2)|^{\omega}\max(1,|\alpha_1|)\max(1,|\alpha_2|)}{p\,.\,2^{\omega p^2/k_1k_2}s(P)^{\omega\max(q_1/k_2,\,q_2/k_1)}} \tag{10}$$

with $\omega = 1$ in all cases and with $\omega = \frac{1}{2}$ if Q_1 and Q_2 have real coefficients and at least one of the numbers α_1, α_2 is not real. Further, if $k_1 = k_2 = k$ write $P(x) = Q(x)^kR(x)$ with $Q(\alpha_1) = Q(\alpha_2) = 0$ and $s(R) \geqslant 1$. Then

$$|\alpha_1-\alpha_2| \geqslant \frac{|D(Q)|^{\omega/2}\max(1,|\alpha_1|)\max(1,|\alpha_2|)}{(p/k)^{1+(\omega p/2k)}2^{\omega p(p-k)/k^2}s(P)^{\omega(p-k)/k^2}} \tag{11}$$

with $\omega = 1$, and ω may be replaced by $\frac{1}{2}$ if Q has real coefficients, at least one of α_1, α_2 is not real and $\alpha_1 \neq \bar{\alpha}_2$.

Proof. To prove inequality (10) we deduce from Theorem 6

$$|\alpha_1-\alpha_2| \geqslant \frac{|\mathrm{Res}(Q_1,Q_2)|^{\omega}m(\alpha_1)m(\alpha_2)}{\min(q_1,q_2)s(Q_1)^{\omega q_2}s(Q_2)^{\omega q_1}}\,. \tag{12}$$

From Lemma B we obtain further $s(Q_1)^{k_1}s(Q_2)^{k_2}s(R) \leqslant 2^ps(P)$. Hence,

$$s(Q_1)^{q_2}s(Q_2)^{q_1} \leqslant \left(\frac{2^ps(P)}{s(Q_2)^{k_2}s(R)}\right)^{q_2/k_1}\left(\frac{2^ps(P)}{s(Q_1)^{k_1}s(R)}\right)^{q_1/k_2}$$

$$\leqslant \frac{2^{p(p-r)/k_1k_2}s(P)^{(p-r)/k_1k_2}}{[s(Q_1)^{k_1}s(Q_2)^{k_2}s(R)]^{\min(q_1/k_2,\,q_2/k_1)}}$$

where we used the relation $p = k_1q_1+k_2q_2+r$. Utilizing the trivial inequality $s(P) \leqslant s(Q_1)^{k_1}s(Q_2)^{k_2}s(R)$ it follows that

$$s(Q_1)^{q_2}s(Q_2)^{q_1} \leqslant 2^{p(p-r)/k_1k_2}s(P)^{[p-r-\min(k_1q_1,\,k_2q_2)]/k_1k_2},$$

which, substituted in (12), yields the estimate (10). For the proof of (11) write now $P(x)$ as required and apply Theorem 7 to obtain

$$|\alpha_1 - \alpha_2| \geqslant \frac{|D(Q)|^{\omega/2} m(\alpha_1)\, m(\alpha_2)}{\binom{q}{2} q^{-1+(\omega q/2)} s(Q)^{\omega(q-1)}}. \tag{13}$$

Using the inequality $s(Q)^k \leqslant s(Q)^k s(R) \leqslant 2^p s(P)$ and the equation $kq + r = p$ we get

$$s(Q)^{q-1} \leqslant 2^{p(q-1)/k} s(P)^{(q-1)/k} \leqslant 2^{p(p-k)/k^2} s(P)^{(p-k)/k^2}$$

and $\binom{q}{2} q^{-1+(\omega q/2)} \leqslant (p/k)^{1+(\omega p/2k)}$. These two estimates and (13) yield the result (11).

Corollary. *Suppose that the polynomial P has integral coefficients and that α_1 and α_2 are zeros of P of orders k_1 and k_2 respectively. If α_1 and α_2 are not conjugate algebraic numbers,*

$$|\alpha_1 - \alpha_2| \geqslant \frac{\max(1, |\alpha_1|)\max(1, |\alpha_2|)}{p \,.\, 2^{\omega p^2/(k_1 k_2)} s(P)^{\omega[p-\min(k_1, k_2)]/(k_1 k_2}}; \tag{14}$$

and if α_1 and α_2 are conjugate algebraic numbers,

$$|\alpha_1 - \alpha_2| \geqslant \frac{\max(1, |\alpha_1|)\max(1, |\alpha_2|)}{(p/k)^{1+(\omega p/2k)} 2^{\omega p(p-k)/k^2} s(P)^{\omega(p-k)/k^2}}, \tag{15}$$

where $k = k_1 = k_2$. The value of ω can always be taken as 1 and ω may be replaced by $\frac{1}{2}$ if at least one of the numbers α_1 and α_2 is not real and $\alpha_1 \neq \bar{\alpha}_2$.

Proof. Note that, if α_1 and α_2 are not conjugate algebraic numbers, then $P = Q_1^{k_1} Q_2^{k_2} R$, where Q_1 and Q_2 are the minimal polynomials of α_1 and α_2 respectively and R is a polynomial with integral coefficients. Hence $\mathrm{Res}(Q_1, Q_2)$ is an integer and different from zero since Q_1 and Q_2 have no common zeros. Also, $s(R) \geqslant 1$. Obviously the exponent of $s(P)$ in (10) may be replaced by that of $s(P)$ in (14). If α_1 and α_2 are conjugate algebraic numbers, P has a factorisation $P = Q^k R$, where Q is the minimal polynomial of α_1 and α_2 and R is a polynomial with integral coefficients. Since Q has no multiple zeros, $D(Q) \neq 0$. As it is an integer, $|D(Q)| \geqslant 1$.

The following result, though correct even in the case $k = 1$, is only of interest in the case $k > 1$. It shows how, in the neighbourhood of a multiple zero of a polynomial P, the derivative $P'(x)$ may be estimated in terms of $P(x)$.

Theorem 8′. *Let P be an arbitrary polynomial. Let γ be any complex number and let α be a zero of P at minimal distance from γ. If the order of α is denoted by k, then*

$$|P'(\gamma)| \leqslant 2^{p/k} p(1+|\alpha|)^{-1} s(P)^{1/k} \max(1, |\gamma|)^{(p/k)-1} |P(\gamma)|^{1-(1/k)}.$$

Proof. If $P(\gamma) = 0$ then clearly $\gamma = \alpha$. If in addition $k > 1$ then there is nothing to prove. If however $P(\gamma) = 0$ and $k = 1$, we obtain the result readily from the corollary to Lemma I.

Let us suppose therefore, that $P(\gamma) \neq 0$. We write $P(x) = (x-\alpha)^k Q(x)$. Then

$$\begin{aligned} P'(x) &= (x-\alpha)^{k-1}\left(kQ(x) + (x-\alpha)\,Q'(x)\right) \\ &= \left(\frac{P(x)}{Q(x)}\right)^{(k-1)/k}\left(kQ(x) + (x-\alpha)\,Q'(x)\right). \end{aligned}$$

In particular,

$$P'(\gamma) = P(\gamma)^{1-(1/k)}Q(\gamma)^{1/k}\left(k + (\gamma - \alpha)\frac{Q'(\gamma)}{Q(\gamma)}\right). \tag{16}$$

Now $\dfrac{Q'(x)}{Q(x)}$ is the logarithmic derivative of $Q(x)$. Therefore if

$$Q(x) = b(x - \beta_1)^{k_1} \dots (x - \beta_m)^{k_m},$$

$k_1 + \dots + k_m = q$, then

$$\frac{Q'(x)}{Q(x)} = \frac{k_1}{x - \beta_1} + \dots + \frac{k_m}{x - \beta_m}.$$

Hence

$$\left|\frac{Q'(\gamma)}{Q(\gamma)}\right| \leqslant q \max_{j=1}^{m} \frac{1}{|\gamma - \beta_j|} = \frac{q}{\min\limits_{j=1}^{m} |\gamma - \beta_j|} \leqslant \frac{q}{|\gamma - \alpha|},$$

since by definition of α, $|\gamma - \alpha| \leqslant \min\limits_{j=1}^{m} |\gamma - \beta_j|$. Therefore, (16) implies that

$$|P'(\gamma)| \leqslant |P(\gamma)|^{1-(1/k)}|Q(\gamma)|^{1/k}(k + q) = |P(\gamma)|^{1-(1/k)}|Q(\gamma)|^{1/k}p,$$

whence

$$|P'(\gamma)| \leqslant p|P(\gamma)|^{1-(1/k)}m(\gamma)^{q/k}s(Q)^{1/k}. \tag{17}$$

To estimate $s(Q)$ we use Lemma B. We find that

$$[s(x - \alpha)^k s(Q)] \leqslant 2^p s(P).$$

Hence $s(Q) \leqslant (1 + |\alpha|)^{-k} 2^p s(P)$. Substituting this in (17) and writing $q = p - k$ yields the required result.

References

1. A. O. Gelfond, *Transcendental and algebraic numbers* (Dover: New York, 1960).
2. R. Güting, " Approximation of algebraic numbers by algebraic numbers ", *Michigan Math. J.* 8 (1961), 149–159.
3. F. Kasch and B. Volkmann, " Zur Mahlerschen Vermutung über *S*-Zahlen ", *Math. Ann.*, 136 (1958), 442–453.
4. K. Mahler, "An application of Jensen's formula to polynomials", *Mathematika*, 7 (1960), 98–100.
5. V. G. Sprindžuk, " Proof of Mahler's conjecture on the measure of the set of *S*-numbers ", *Izv. Akad. Nauk SSSR, Ser. Mat.*, 29 (1965), 379–436 (Russian).

Department of Mathematics,
Makerere University College,
Kampala, Uganda.

(*Received on the* 17*th of April*, 1967.)