On Computability of Equilibria in Markets with Production

Jugal Garg*   Vijay V. Vazirani†

Abstract

Even though production is an integral part of the Arrow-Debreu market model, most of the work in theoretical computer science has so far concentrated on markets without production, i.e., the *exchange economy*. This paper takes a significant step towards understanding computational aspects of markets with production.

For markets with separable, piecewise-linear concave (SPLC) utilities and SPLC production, we obtain a linear complementarity problem (LCP) formulation that captures exactly the set of equilibria, and we further give a complementary pivot algorithm for finding an equilibrium. This settles a question asked by Eaves in 1975 [14]. Since this is a path-following algorithm, we obtain a proof of membership of this problem in PPAD, using Todd, 1976. We also obtain an elementary proof of existence of equilibrium (i.e., without using a fixed point theorem), rationality, and oddness of the number of equilibria. We further give a proof of PPAD-hardness for this problem and also for its restriction to markets with linear utilities and SPLC production. Experiments show that our algorithm is practical. Also, it is strongly polynomial when the number of goods or the number of agents and firms is constant. This extends the result of Devanur and Kannan (2008) to markets with production.

Finally, we show that an LCP-based approach cannot be extended to PLC (non-separable) production, by constructing an example which has only irrational equilibria.

1 Introduction

Among the most novel and significant additions to the theory of algorithms and computational complexity over the last decade has been deep insights into the computability of equilibria, both Nash and market. However, within the study of market equilibria, most of this work was concentrated on the economies without production, [11, 7, 6, 8, 3, 5, 42, 10, 37, 32, 18, 38, 4, 13][1]; as described in Section 1.2, the results obtained so far for markets with production are quite rudimentary. Production is, of course, central to the Arrow-Debreu model [1] and to most economies, and this represents a crucial gap in the current theory. The purpose of this paper is to address this gap.

For the exchange economy (i.e., the Arrow-Debreu model without production), once the case of linear utilities was settled with polynomial time algorithms [12, 24, 20, 23, 43], the next case was understanding the computability of equilibria in markets with separable, piecewise-linear concave (SPLC) utilities. A series of remarkable works resolved this long-standing open problem and showed it is PPAD-complete [3, 5, 37], not only for Arrow-Debreu markets but also for Fisher markets[2] [2]. However, the proof of membership in PPAD, given in [37], was indirect – using the characterization of PPAD via the class FIXP [17].

[18] gave a direct proof of membership in PPAD by giving a path-following algorithm. This is a complementary pivot algorithm using Lemke's scheme [27], and it builds on a classic result of Eaves [15] giving a similar algorithm for the linear exchange market. Another importance of the result of [18] was that experimental results conducted on randomly generated instances showed that their algorithm is practical. In the face of PPAD-completeness of the problem, and the current status of the P = PPAD question, this is the best one can hope for.

For the case of 2-Nash, the problem of finding a Nash equilibrium in a 2-player bimatrix game, a path-following algorithm and membership in PPAD are provided by the classic Lemke-Howson algorithm [28]; this is also a complementary pivot algorithm. We note that whereas several complementary pivot algorithms have since been given for Nash equilibrium [27, 36, 22, 26, 41], for market equilibrium, [18] and ours are the only such algorithms since Eaves' work.

An example of Mas-Colell, having only irrational equilibria for Leontief utilities (stated in [15]), shows that a similar approach is not feasible for arbitrary piecewise-linear, concave utilities. The latter case lies in the class FIXP and proving it FIXP-hard remains an open problem.

Our work on markets with production is inspired by this development and brings it on par with the current status of exchange markets. On the one hand, this development made our task easier. On the other hand, unlike the case of exchange markets, where a

*College of Computing, Georgia Institute of Technology, Atlanta, GA 30332-0280. Email: jugal.garg@gmail.com. Supported by Algorithms and Randomness Center (ARC) Postdoctoral Fellowship.

†College of Computing, Georgia Institute of Technology, Atlanta, GA 30332–0280. Email: vazirani@cc.gatech.edu. Research supported by NSF Grants CCF-0914732 and CCF-1216019, and a Guggenheim Fellowship.

[1] This list is by no means complete.

[2] Fisher market is a special case of exchange economy.

demarcation between rational and irrational equilibria was already well known before [18] embarked on their work, for markets with production, no such results were known, making our task harder.

For markets with separable, piecewise-linear concave (SPLC) utilities and SPLC production, we obtain a linear complementarity problem (LCP) formulation that captures exactly the set of equilibria, and we further give a complementary pivot algorithm for finding an equilibrium. This settles a question asked by Eaves in 1975 of extending his work to markets with production.

Since this is a path-following algorithm, we obtain a proof of membership of this problem in PPAD, using Todd, 1976. We also obtain an elementary proof of existence of equilibrium (i.e., without using a fixed point theorem), rationality, and oddness of the number of equilibria. We further give a proof of PPAD-hardness for this problem and also for its restriction to Arrow-Debreu markets with linear utilities and SPLC production.

Experiments show that our algorithm is practical. We note that market equilibrium algorithms, especially those involving production, are important in practice [34, 16]. Because of the PPAD-completeness of the problem, ours is among the best available avenues to designing equilibrium computing algorithms that can have an impact in practice. We further show that our algorithm is strongly polynomial when the number of goods or the number of agents and firms is constant. This extends the result of [10] to markets with production.

Finally, we show that an LCP-based approach cannot be extended to PLC (non-separable) production, by constructing an example which has only irrational equilibria. We expect this case to be FIXP-complete, even if utilities are linear.

Besides being practical, complementary pivot algorithms have the additional advantage that they have provided deep insights into the problems studied in the past. A case in point is the classic Lemke-Howson algorithm [28] for computing a Nash equilibrium of a 2-person bimatrix game, where besides oddness of the number of equilibria, it yielded properties such as index, degree, and stability [39, 33, 21, 40]. As stated above, we have already established that our problem has an odd number of equilibria. We expect our algorithm to yield additional insights as well.

1.1 Salient features Our result involves two main steps. The first is deriving an LCP whose solutions are exactly the set of equilibria of our market with production. The second is ensuring that Lemke's scheme is guaranteed to converge to a solution.

Lemke's scheme involves following the unique path that starts with the primary ray on the one-skeleton of the associated polyhedron (see Section A for detailed explanation of these terms). Such a path can end in two ways, either a solution to the LCP or a secondary ray. In the latter case, the scheme provides no recourse and simply aborts without finding a solution. We show that the associated polyhedron of our LCP has no secondary rays and therefore Lemke's scheme is guaranteed to give a solution.

Several classes of LCPs have been identified for which Lemke's scheme converges to a solution [31, 9]. However, none of these classes captures our LCP, or the LCPs of Eaves [15] for linear exchange markets or [18] for SPLC exchange markets, even though they resort to a similar approach. In the progression of these three works, the LCPs have become more involved and proving the lack of secondary rays has become harder and harder. Clearly, this calls for further work to understand the underlying structure in these LCPs.

Some new ideas were needed for deriving the LCP. The LCP has to capture: (i) optimal production plans for each firm, (ii) optimal bundle for each agent, and (iii) market clearing conditions. Given prices, the optimal production plan of each firm can be obtained through a linear program (LP) using variables capturing amount of raw and produced goods. Then, using complementary slackness and feasibility conditions of these LPs, we obtain an LCP to capture the production. Next, an LCP for consumption and market clearing is sought using variables capturing amount of goods consumed in order to merge it with the production LCP, however it turned out to be unlikely and the correct way is to use variables capturing value of goods. The only way out was to somehow convert amount variables in production LCP to value variables, which fortunately was doable.

The resulting LCP captures market equilibria, however it has non-equilibrium solutions as well and more importantly, it is homogeneous – the corresponding polyhedron forms a cone, with origin being the only vertex. Similar issues arise in [15, 18] too, and they deal with these by simply imposing a lower bound of 1 on every price variable[3]. It turns out that such a trivial lower bound does not work for our LCP; prices where no firm can make positive profit are needed (see Section 5 for details). We recourse to the sufficiency condition *no production out of nothing* [1, 29] for the existence of equilibria and show that such prices exist (using Farkas' lemma) and can be obtained. After im-

[3]This is without loss of generality as market equilibrium prices are scale invariant [1].

posing such a lower bound, the resulting LCP exactly captures market equilibria (up to scaling).

Next we show that our algorithm is strongly polynomial when either the number of goods or the number of agents and firms are constant, extending the result of Devanur and Kannan [10] to markets with production. For this, we decompose a constant dimensional space into polynomially many regions and show that every region can contain at most one vertex traversed by our algorithm.

Since our algorithm follows a complementary path, it together with Todd's result [35] on locally orienting such paths, proves that the problem is in PPAD. Since every LCP has a vertex solution (if a solution exists) in the polyhedron associated with it, there is an equilibrium with rational prices. In the absence of secondary rays, all but one of the equilibria get paired up through complementary paths; the remaining one with the primary ray. Therefore there are odd number of equilibria.

Market equilibrium computation in exchange markets with SPLC utilities is known to be PPAD-hard [3, 37]. We reduce such a market to a market with linear utilities and SPLC production, thereby proving its hardness too. This reduction is general enough, in a sense that an exchange market with concave utilities can be reduced to an equivalent market with linear utilities and concave production. Since linear is a special case of SPLC functions, we obtain PPAD-completeness for markets with SPLC utilities and SPLC production.

1.2 Related work Jain and Varadarajan [25] studied the Arrow-Debreu markets with production, and gave a polynomial time algorithm for production and utility functions coming from a subclass of CES (constant elasticity of substitution) functions; i.e., constant returns to scale (CRS) production. They also gave a reduction from the exchange market with CES utilities to a linear utilities market in which firms have CES production. Our reduction from the exchange market to a linear utilities market with arbitrary production is inspired by their reduction but is more general.

We note that CRS production is relatively easy to deal with, since there is no positive profit to any firm at an equilibrium. To the best of our knowledge, no computational work has been done for the original Arrow-Debreu market with decreasing returns to scale production.

In next few pages, we present main ideas, techniques and results of the paper. For details, we refer the reader to full version [19].

2 The Arrow-Debreu Market Model

The market model defined by Arrow and Debreu [1] consists of the following: A set $\mathcal{G}$ of divisible goods, a set $\mathcal{A}$ of agents and a set $\mathcal{F}$ of firms. Let n denote the number of goods in the market.

The production capabilities of firm f is defined by a set of production possibility vectors (PVVs) $\mathcal{Y}^f$; in a vector negative coordinates represent inputs and positive coordinates represents output. The set of input and output goods for each firm is disjoint. Standard assumptions on set $\mathcal{Y}^f$ are (see [1]):

1. Set $\mathcal{Y}^f$ is closed and convex; convexity captures law of diminishing returns.
2. *Downward close* - additional raw goods do not decrease the production.
3. *No production out of nothing* - firms together can not produce something out of nothing, i.e., $\oplus_{f\in\mathcal{F}}\mathcal{Y}^f \cap \mathbb{R}^n_+ = \mathbf{0}$.

The goal of a firm is to produce as per a profit maximizing (optimal) schedule. Firms are owned by agents: θ^i_f is the profit share of agent i in firm f such that $\forall f \in \mathcal{F},\ \sum_{i\in\mathcal{A}} \theta^i_f = 1$.

Each agent i comes with an initial endowment of goods; w^i_j is amount of good j with agent i. The preference of an agent i over bundles of goods is captured by a non-negative, non-decreasing and concave utility function $U^i : \mathbb{R}^n_+ \to \mathbb{R}_+$. Non-decreasingness is due to free disposal property, and concavity captures the law of diminishing marginal returns. Each agent wants to buy a (optimal) bundle of goods that maximizes her utility to the extent allowed by her earned money – from initial endowment and profit shares in the firms. Without loss of generality, we assume that total initial endowment of every good is 1, i.e, $\sum_{i\in\mathcal{A}} w^i_j = 1, \forall j \in \mathcal{G}$[4].

Given prices of goods, if there is an assignment of optimal production schedule to each firm and optimal affordable bundle to each agent so that there is neither deficiency and nor surplus of any good, then such prices are called *market clearing* or *market equilibrium* prices. The market equilibrium problem is to find such prices when they exist. In a celebrated result, Arrow and Debreu [1] proved that market equilibrium always exists under some mild conditions, however the proof is non-constructive and uses heavy machinery of Kakutani fixed point theorem.

Note that operating point of a firm, at any given prices, is on the boundary of $\mathcal{Y}^f$, which can be defined

[4]This is like redefining the unit of goods by appropriately scaling utility and production parameters.

by a concave function/correspondence. To work under finite precision model concave is generally approximated with piecewise-linear concave.

A well studied restriction of Arrow-Debreu model is *exchange economy*, i.e., markets without production firms.

2.1 PLC production and irrationality In this section we demonstrate an example of a market with (non-separable) PLC production and the simplest utility functions, namely *linear*, having only irrational equilibrium prices and allocations.

Consider a market with three goods, three agents and one firm. The initial endowments of agents are $w^1 = w^2 = w^3 = (1, 1, 0)$. Each utility function has one linear segment; $U_1 = x_1^1$, $U_2 = x_2^2$ and $U_3 = x_3^3$. The firm is owned by agent 3, i.e., $\theta_1^3 = 1$. It has exactly one production segment without any upper limit on the raw material used, and needs two units of good 1 and a unit of good 2 to produce a unit of good 3. This is a Leontief (PLC) production function where the quantities of raw goods are needed in a fixed proportion. Let r_j's and s_j's respectively be the amount of goods used and produced by the firm on its only segment, then they should satisfy: $2 \cdot s_3 \leq r_1$ and $s_3 \leq r_2$.

Let $\boldsymbol{p} = (p_1, p_2, p_3)$ denote the price of goods. Note that at an equilibrium of this market all prices must be positive, otherwise the demand of zero priced goods will be infinite. Since equilibrium prices are scale invariant [1], we set $p_1 = 1$. The firm will produce at equilibrium due to positive demand of good 3 from agent 3, however its profit will be zero otherwise it will want to produce infinite amount. Hence we have $p_3 = 2 + p_2$. From the market clearing conditions, we get $p_2^2 + 2p_2 - 2 = 0$. Thus the only equilibrium prices of this market are $p_1 = 1, p_2 = \sqrt{3} - 1$ and $p_3 = {}^{(1+\sqrt{3})}\!/_2$. At equilibrium the allocation and production variables are: $x_1^1 = \sqrt{3}$, $x_2^2 = {}^{\sqrt{3}}\!/_{(\sqrt{3}-1)}$ and $x_3^3 = s_3 = {}^{r_1}\!/_2 = r_2 = {}^{\sqrt{3}}\!/_{(\sqrt{3}+1)}$.

This rules out the possibility of linear complementarity problem (LCP) formulation for PLC production. The next logical step is to consider separable PLC (SPLC) production instead, together with SPLC utilities, as PLC utility functions are already known to have irrationality [15].

3 Markets with SPLC Utility & SPLC Production

In this section, we define parameters representing separable piecewise-linear concave (SPLC) utility and SPLC production functions. All the parameters are assumed to be rational numbers.

For each pair of agent i and good j we are specified a non-decreasing, piecewise-linear and concave (PLC) function $U_j^i : \mathbb{R}_+ \to \mathbb{R}_+$, which gives the utility that i derives as a function of the amount of good j that she receives. Her overall utility, $U^i(\boldsymbol{x})$, for a bundle $\boldsymbol{x} = (x_1, \ldots, x_n)$ of goods is additively separable over the goods, i.e., $U^i(\boldsymbol{x}) = \sum_{j\in\mathcal{G}} U_j^i(x_j)$. The number of segments in function U_j^i is denoted by $|U_j^i|$, and the k^{th} segment of U_j^i by (i, j, k). The slope of a segment specifies the rate at which the agent derives utility per unit of additional good received. Suppose segment (i, j, k) has domain $[a, b] \subseteq \mathbb{R}_+$, and slope c. Then, we define $u_{jk}^i = c$ and $l_{jk}^i = b - a$. The length of the last segment is infinity. Since U_j^i is concave, $u_{j(k-1)}^i > u_{jk}^i, \ \forall k \geq 2$.

In this paper we consider the case where every firm produces a good using a set of goods as raw material and the production function is additively separable over goods. For simplicity, we assume that each firm produces exactly one good[5]. Let firm f produces good j_f.

For each pair of firm f and good j we are specified a production function $P_j^f : \mathbb{R}_+ \to \mathbb{R}_+$ which is non-negative, non-decreasing, PLC, and defines f's ability to produce good j_f as a function of the amount of good j. The overall production of the firm f from a bundle $\boldsymbol{x} = (x_1, \ldots, x_n)$ of goods is: $P^f(\boldsymbol{x}) = \sum_{j\in\mathcal{G}} P_j^f(x_j)$. The number of segments in P_j^f is denoted by $|P_j^f|$, and the k^{th} segment by (f, j, k). The slope of a segment specifies the rate at which good j_f can be produced from a unit of additional good j. Suppose segment (f, j, k) has domain $[a, b] \subseteq \mathbb{R}_+$, and slope c, then we define $\alpha_{jk}^f = c$ and $o_{jk}^f = b - a$. This implies that on segment (f, j, k), firm f can produce α_{jk}^f amount of good j_f from a unit amount of good j and at this rate it can use up to o_{jk}^f units of good j. The length of the last segment is infinity. Since P_j^f is concave, $\alpha_{j(k-1)}^f > \alpha_{jk}^f \ \forall k \geq 2$.

Given prices $\boldsymbol{p} = (p_1, \ldots, p_n)$ for the goods, each firm operates on a production schedule that maximizes its profit $-$ money earned from the production minus the money spent on the raw material. Let $\mathcal{E}^f$ denote the profit of firm j. Agent i earns $\sum_{j\in\mathcal{G}} w_j^i p_j$ from the initial endowment and $\sum_{j\in\mathcal{G}} \theta_f^i \mathcal{E}^f$ from the profit shares in firms, and buys a bundle of goods that maximizes her utility. Prices $\boldsymbol{p}$ gives an equilibrium if market clears when each firm produces at an optimal plan and each agent buys an optimal bundle. We will denote this market, with SPLC production and SPLC utilities, by $\mathcal{M}$.

[5] This is without loss of generality since production is separable. For a firm producing multiple goods we can create as many firms as number of produced goods with agent's shares being duplicated.

4 Equilibrium Characterization and LCP Formulation

For equilibrium characterization, we need to capture (i) optimal production plans for each firm, (ii) optimal bundles for each agent, and (iii) market clearing conditions.

Optimal production. Recall that on segment (f, j, k), α^f_{jk} units of good j_f can be produced using a unit of good j. Given prices $\boldsymbol{p}$, the optimal production plan of firm f is given by the following linear program (LP), where x^f_{jk} denote the amount of raw good j used by firm f on (f, j, k):

$$(4.1) \quad \text{maximize} \quad \sum_{j,k} x^f_{jk}(\alpha^f_{jk} p_{j_f} - p_j)$$
$$\text{subject to} \quad 0 \le x^f_{jk} \le o^f_{jk}, \quad \forall (j, k)$$

Note that since $\alpha^f_{jk} > \alpha^f_{j(k+1)}, \forall k$, an optimal solution of this LP will have $x^f_{jk} = o^f_{jk}$ whenever $x^f_{j(k+1)} > 0$, as required. Let β^f_{jk} be the dual variable corresponding to inequality $x^f_{jk} \le o^f_{jk}$. From the optimality conditions, we get the following linear constraints and complementarity conditions (We will refer to these as follows: the equation number will refer to the constraint and the equation number with a prime will refer to the complementarity condition, e.g., (4.2) refers to the first constraint below and (4.2') refers to the corresponding complementarity condition.). All variables introduced will have a non-negativity constraint; for the sake of brevity, we will not write them explicitly.

$$(4.2) \ \forall (j, k): \quad \alpha^f_{jk} p_{j_f} - p_j \le \beta^f_{jk} \quad \text{and}$$
$$x^f_{jk}(\alpha^f_{jk} p_{j_f} - p_j - \beta^f_{jk}) = 0$$

$$(4.3) \ \forall (j, k): \quad x^f_{jk} \le o^f_{jk} \quad \text{and} \quad \beta^f_{jk}(x^f_{jk} - o^f_{j'k}) = 0$$

Note that (4.2) and (4.3) are equivalent to the above LP due to strong duality. Combining these for all the firms gives us a linear complementarity problem (LCP) formulation that capture optimal production. The amount produced on segment (f, j, k) is

$$y^f_{jk} = \alpha^f_{jk} x^f_{jk}.$$

Next we need to characterize and derive an LCP to capture optimal bundles of each agent, and market clearing conditions. Suppose x^i_{jk} denote the amount of good j obtained by agent i on segment (i, j, k), then i spends $\sum_{j,k} x^i_{jk} p_j$, which is a quadratic term. It turned out to be unlikely to capture this through an LCP in amount variables. Therefore we need to use q^i_{jk} in place of $x^i_{jk} p_j$, representing money spent by agent i on segment (i, j, k). Further, we need same type of variables in production LCP as well to tie everything in goods side market clearing condition.

The only recourse is to convert amount variables in production LCP to money variables. It turns out to be doable, though not immediately clear, using complementarity and change of variables. We multiply both the equations in (4.3) by p_j and replace the expression $x^f_{jk} p_j$ by r^f_{jk} denoting the money spent on raw material j on segment (f, j, k). Assuming that $p_j > 0, \forall j \in \mathcal{G}$ at equilibrium, (4.2) and (4.3) for all the firms are equivalent to:

$$(4.4) \quad \forall (f, j, k): \quad \alpha^f_{jk} p_{j_f} - p_j \le \beta^f_{jk} \quad \text{and}$$
$$r^f_{jk}(\alpha^f_{jk} p_{j_f} - p_j - \beta^f_{jk}) = 0$$

$$(4.5) \quad \forall (f, j, k): \quad r^f_{jk} \le o^f_{jk} p_j \quad \text{and}$$
$$\beta^f_{jk}(r^f_{jk} - o^f_{jk} p_j) = 0$$

To capture produced amount let s^f_{jk} denote the revenue of firm f on segment (f, j, k), namely $y^f_{jk} p_{j_f}$. Directly replacing y^f_{jk} with ${s^f_{jk}}/{p_{j_f}}$ and x^f_{jk} with ${r^f_{jk}}/{p_j}$ in $y^f_{jk} = \alpha^f_{jk} x^f_{jk}$ will give quadratic equality. Instead, we observe that β^f_{jk} captures the profit per unit of raw material on segment (f, j, k) when $r^f_{jk} > 0$. Further if $\beta^f_{jk} > 0$, then the firm utilizes segment (f, j, k) completely. Putting these together, we include the following equalities, where $\mathcal{E}^f$ captures the profit of firm f.

$$(4.6) \ \forall (f, j, k): \quad s^f_{jk} = r^f_{jk} + o^f_{jk} \beta^f_{jk} \quad \text{and}$$
$$\forall f: \quad \mathcal{E}^f = \sum_{j,k} o^f_{jk} \beta^f_{jk}$$

REMARK 4.1. *For (non-separable) PLC production functions, even though an LP can be written to capture optimal production, we show impossibility of an LCP formulation for markets with PLC production by constructing an example of such a market in Section 2.1, which has only irrational equilibria.*

Market clearing. The market clearing constraints are easier now. Let q^i_{jk} denotes the money spent by agent

i on segment (i, j, k). The following constraints capture the market clearing, where $\mathcal{F}(j)$ denote the set of firms producing good j, and λ_i is related to optimal bundle and defined below; we have included the corresponding complementarity conditions in order to obtain an LCP in the standard form.

(4.7)
$$\forall j \in \mathcal{G}: \quad \sum_{i,k} q^i_{jk} + \sum_{f,k} r^f_{jk} \le p_j + \sum_{f \in \mathcal{F}(j), j', k} s^f_{j'k} \quad \text{and}$$
$$p_j\Big(\sum_{i,k} q^i_{jk} + \sum_{f,k} r^f_{jk} - p_j - \sum_{f \in \mathcal{F}(j), j', k} s^f_{j'k}\Big) = 0$$

(4.8)
$$\forall i \in \mathcal{A}: \quad \sum_j w^i_j p_j + \sum_f \theta^i_f \mathcal{E}^f \le \sum_{j,k} q^i_{jk} \quad \text{and}$$
$$\lambda_i\Big(\sum_j w^i_j p_j + \sum_f \theta^i_f \mathcal{E}^f - \sum_{j,k} q^i_{jk}\Big) = 0$$

Optimal bundle. Given prices $\boldsymbol{p}$, since agent i's earning, $\sum_j w^i_j p_j + \sum_f \theta^i_f \mathcal{E}^f$ from initial endowment and profit shares in the firms, is fixed, she will spend on those goods, where her utility per unit of money – bang-per-buck – is maximum; on segment (i, j, k) it is u^i_{jk}/p_j. For this, she will sort her segments in decreasing order of bang-per-buck and start buying from the first until she runs out of money. Partition her segments by equality in classes $Q^i_1, Q^i_2, \ldots$ in order. Let the partition where she runs out of money be called *flexible*, all before that are called *forced*, and all after that are called *undesirable*.

Introduce a variable λ_i that captures inverse of the bang-per-buck of the flexible partition. In order to capture segments of forced partition, variable γ^i_{jk} is introduced so that if (i, j, k) is forced, then $1/\lambda_i = u^i_{jk}/p_j + \gamma^i_{jk}$; supplement prices. The following constraints from [18] ensure optimal bundle to each agents.

$$\text{(4.9)} \quad \forall (i, j, k): \quad u^i_{jk}\lambda_i \le p_j + \gamma^i_{jk} \quad \text{and}$$
$$q^i_{jk}(u^i_{jk}\lambda_i - p_j - \gamma^i_{jk}) = 0$$

$$\text{(4.10)} \quad \forall (i, j, k): \quad q^i_{jk} \le l^i_{jk} p_j \quad \text{and}$$
$$\gamma^i_{jk}(q^i_{jk} - l^i_{jk} p_j) = 0$$

Let us denote the LCP defined by the sets of constraints and complementarity conditions given in (4.4) through (4.10), together with non-negativity on all variables, as **AD-LCP**.

LEMMA 4.1. *Any equilibrium of market $\mathcal{M}$ yields a solution to AD-LCP.*

Since LCPs always have rational solutions (if one exists), next corollary follows from Lemma 4.1.

COROLLARY 4.1. *In a market with SPLC utilities and SPLC production functions, equilibrium prices and allocations are rational (up to scaling) if all input parameters are rational.*

AD-LCP suffers from two shortcomings. First, since the rhs vector of the constraints, denoted by $\boldsymbol{q}$ in Section A, is zero (homogeneous system), the polyhedron is highly degenerate – in fact, it is a cone with its vertex at the origin. Second, it may admit solutions that do not correspond to equilibria.

5 Non-homogeneous LCP and the Algorithm

In this section, first we dehomogenize AD-LCP by imposing appropriate lower bounds on price variables and show that it exactly captures the market equilibria. Next we derive an algorithm for computing market equilibrium using the new LCP, and its consequences.

Recall the condition that firms together can not produce something out of nothing (see Section 2). We characterize this condition for SPLC production, where firm f needs at least $1/\alpha^f_{j1}$ units of good j to produce one unit of good j_f. Let $G_{\mathcal{F}}(\mathcal{M})$ be a weighted directed graph, where goods are nodes and the weight of an edge from j to j' is $\max_{f, j_f = j'} 1/\alpha^f_{j1}$.

DEFINITION 5.1. (NO PRODUCTION OUT OF NOTHING) *We say that market $\mathcal{M}$ satisfies no production out of nothing if weights of edges in every cycle of $G_{\mathcal{F}}(\mathcal{M})$ multiply to strictly less than one.*

For a good j, let desire(j) be the total amount represented by its non-zero utility segments, *i.e.*, desire$(j) = \sum_{(i,k): u^i_{jk} > 0} l^i_{jk}$. If desire$(j) > 1$, $\forall j \in \mathcal{G}$ then $\boldsymbol{p} > 0$ in every equilibrium of the market[6]. This is because, under the *no production out of nothing* condition at least one good has to diminish on any production cycle or path. Further, since firms never operate at losses if the price of the produced good is zero then so is the price of used goods.

From now on we assume that desire$(j) > 1$, $\forall j \in \mathcal{G}$, and call this condition *enough demand*. In that case since the equilibrium prices are positive, and they are known to be scale invariant, we can lower bound them with positive numbers. This will lead to a non-zero rhs in the LCP. Further, we want negative rhs only in the agent side market clearing condition (4.8). This is needed to ensure that all the equilibrium conditions

[6] Stronger condition can be derived to ensure non-zero prices at equilibrium, however we stick to this one for simplicity.

except market clearing are satisfied on the path followed by the algorithm, which is crucial to prove no secondary rays and in turn convergence of the algorithm.

Suppose, we lower bound p_j by a positive number c_j. We do this by replacing p_j with $p'_j + c_j$ in AD-LCP. Then to keep the rhs of (4.4) non-negative, we need $c_j - \alpha^f_{jk} c_{j_f} \geq 0,\ \forall (f, j, k)$, i.e., no positive profit on segment (f, j, k) at prices $\boldsymbol{c}$. We solve the following to compute such a vector $\boldsymbol{c}$,

$$\forall (f, j, k) : \alpha^f_{jk} c_{j_f} \leq c_j; \qquad \forall j : c_j \geq 1 \tag{5.11}$$

The second condition in (5.11) is to get a non-zero c_j's as the first condition is homogeneous. We show the next lemma using *no production out of nothing* and Farkas' lemma.

LEMMA 5.1. *Polyhedron of (5.11) is non-empty and has a non-empty interior.*

Take a vector $\boldsymbol{c}$ from the interior of (5.11) (Lemma 5.1). Replace p_j with $p'_j + c_j$ in AD-LCP, and the resulting LCP, call it **NHAD-LCP**, is as in Table 1. There are non-negativity constraints on all the variables, however for brevity we omit them.

The following theorem establishes strong connection between NHAD-LCP and market equilibria of $\mathcal{M}$ (see Section 6.1 for a proof outline).

THEOREM 5.1. (ME AND LCP) *The solutions of NHAD-LCP capture exactly the equilibria of market $\mathcal{M}$ with SPLC utilities and SPLC production (up to scaling).*

5.1 Algorithm From Theorem 5.1 computing an equilibrium of market $\mathcal{M}$ reduces to solving NHAD-LCP, which has the same form as the formulation given in (A.1) in Section A; equalities (5.14) can be removed by replacing s^f_{jk}s and $\mathcal{E}^f$ with the corresponding expressions in the LCP. Let M and $\boldsymbol{q}$ be the matrix and rhs vector formed by the inequalities of NHAD-LCP, and let $\boldsymbol{y}$ be the variable vector such that NHAD-LCP can be written as $M\boldsymbol{y} \leq \boldsymbol{q},\ \boldsymbol{y} \geq 0,\ \boldsymbol{y}^T(M\boldsymbol{y} - \boldsymbol{q}) = 0$.

Since the rhs vector $\boldsymbol{q}$ does have negative entries, namely in (5.16), Lemke's algorithm is applicable (Refer to Section A for detailed description of Lemke's algorithm). We will add the z variable only in the constraints and complementarity conditions that have a negative rhs. Thus we need to make two changes to NHAD-LCP to obtain the augmented LCP, which we call **NHAD-LCP'**. First, we change (5.16) as follows:

$$\begin{gathered}\forall i \in \mathcal{A} : \sum_j w^i_j p'_j + \sum_f \theta^i_f \mathcal{E}^f - \sum_{j,k} q^i_{jk} - z \leq - \sum_j w^i_j c_j \\ \text{and } \lambda_i(\textstyle\sum_j w^i_j (p'_j + c_j) + \sum_f \theta^i_f \mathcal{E}^f - \sum_{j,k} q^i_{jk} - z) = 0\end{gathered} \tag{5.19}$$

Second, we impose non-negativity on z. Let the polyhedron of NHAD-LCP' be denoted by $\mathcal{P}'$. Let M' be the augmented matrix of NHAD-LCP' and $\boldsymbol{y'}$ be the corresponding variable vector $(\boldsymbol{y}, z)$. Recall from Section A that the set of solutions of NHAD-LCP', called S, consists of paths and cycles. Our algorithm traverses one such path starting from the *primary ray* – unbounded edge of S where $\boldsymbol{y} = 0$. Except for the *primary ray* all other unbounded edges in S with $z > 0$ are called *secondary rays.* Clearly, $z = \max_i \sum_j w^i_j c_j$ and all other variables zero is a solution vertex of NHAD-LCP', call it $\boldsymbol{y}'_0$; it is also the vertex of the *primary ray.*

1. Initialization: Let $\boldsymbol{y'} \leftarrow \boldsymbol{y}'_0$
2. **While** $z > 0$ in the current solution $\boldsymbol{y'}$, **do**
   - Suppose at $\boldsymbol{y'}$ we have $y'_i = 0$ and $(M'\boldsymbol{y'} - \boldsymbol{q})_i = 0$, i.e., i is the double label.
   - **If** $(M'\boldsymbol{y'} - \boldsymbol{q})_i$ just became 0 at the current vertex, **then** pivot by relaxing $y'_i = 0$.
   - **Else**, pivot by relaxing $(M'\boldsymbol{y'} - \boldsymbol{q})_i = 0$.
   - **If** a new vertex is reached, **then** reinitialize $\boldsymbol{y'}$ with it. **Else** output 'Secondary ray'. **Exit**.
3. Output solution $\boldsymbol{y'}$.

The algorithm can never cycle or get stuck (no double label found) as discussed in Section A. It terminates when either z becomes zero or a *secondary ray* is reached. In the former case we obtain a solution of the original NHAD-LCP and hence a market equilibrium (Theorem 5.1). We need to show that the latter case never happens as it leads to failure. For this, market $\mathcal{M}$ has to satisfy the weakest known sufficiency conditions for the existence of an equilibrium [29]. By construction $\mathcal{M}$ satisfies all except *strong connectivity.* It is defined in Section 6.2 with the proof of the following theorem.

THEOREM 5.2. (NO SECONDARY RAY) *The polyhedron of NHAD-LCP', corresponding to a market $\mathcal{M}$ with SPLC utilities SPLC production, satisfying strong connectivity, no production out of nothing and enough demand, has no secondary rays.*

REMARK 5.1. *If we run our algorithm on an arbitrary instance, without sufficiency conditions, then we may*

$$(5.12)\quad \forall(f,j,k):\quad \alpha^f_{jk}p'_{j_f} - p'_j - \beta^f_{jk} \le c_j - \alpha^f_{jk}c_{j_f} \quad \text{and} \quad r^f_{jk}\left(\alpha^f_{jk}(p'_{j_f}+c_{j_f}) - (p'_j+c_j) - \beta^f_{jk}\right) = 0$$

$$(5.13)\quad \forall(f,j,k):\quad r^f_{jk} - o^f_{jk}p'_j \le o^f_{jk}c_j \quad \text{and} \quad \beta^f_{jk}(r^f_{jk} - o^f_{jk}(p'_j+c_j)) = 0$$

$$(5.14)\quad \forall(f,j,k):\quad s^f_{jk} = r^f_{jk} + o^f_{jk}\beta^f_{jk} \quad \text{and} \quad \forall f \in \mathcal{F}:\quad \mathcal{E}^f = \sum_{j,k} o^f_{jk}\beta^f_{jk}$$

$$\forall j \in \mathcal{G}:\quad \sum_{i,k} q^i_{jk} + \sum_{f,k} r^f_{jk} - p'_j - \sum_{f\in\mathcal{F}(j),j',k} s^f_{j'k} \le c_j \quad \text{and}$$

$$(5.15)\quad p'_j(\sum_{i,k} q^i_{jk} + \sum_{f,k} r^f_{jk} - (p'_j+c_j) - \sum_{f\in\mathcal{F}(j),j',k} s^f_{j'k}) = 0$$

$$\forall i \in \mathcal{A}:\quad \sum_j w^i_j p'_j + \sum_f \theta^i_f \mathcal{E}^f - \sum_{j,k} q^i_{jk} \le -\sum_j w^i_j c_j \quad \text{and}$$

$$(5.16)\quad \lambda_i(\sum_j w^i_j(p'_j+c_j) + \sum_f \theta^i_f\mathcal{E}^f - \sum_{j,k} q^i_{jk}) = 0$$

$$(5.17)\quad \forall(i,j,k):\quad u^i_{jk}\lambda_i - p'_j - \gamma^i_{jk} \le c_j \quad \text{and} \quad q^i_{jk}(u^i_{jk}\lambda_i - (p'_j+c_j) - \gamma^i_{jk}) = 0$$

$$(5.18)\quad \forall(i,j,k):\quad q^i_{jk} - l^i_{jk}p'_j \le l^i_{jk}c_j \quad \text{and} \quad \gamma^i_{jk}(q^i_{jk} - l^i_{jk}(p'_j+c_j)) = 0$$

Table 1: **NHAD-LCP**

end up on a secondary ray, however that does not imply anything whether equilibrium exists or not. This is expected since checking existence even in its restriction to exchange markets with SPLC utilities is NP-complete [37], and any such implication leads to showing NP=co-NP [30].

As consequences of Theorem 5.2 we obtain a number of important results on computation, complexity and structure of market equilibria.

COROLLARY 5.1. *For market $\mathcal{M}$ with SPLC utilities and SPLC production satisfying strong connectivity, no production out of nothing and enough demand,*

- *there exists a complementary pivot algorithm to compute an equilibrium.*
- *market equilibrium computation is in PPAD.*
- *number of market equilibria is odd.*

Our algorithm gives an elementary proof (without using the fixed-point theorem) of the existence of equilibrium in such markets. Further, we get the first algorithm, without going through fixed-point, to find an equilibrium in such a general setting. This settles the question asked by Eaves in 1975 for LCP and pivoting algorithm to compute equilibrium in markets with production. In the full version, we show that this simple algorithm is strongly polynomial when number of goods, or number of agents and firms is a constant. On randomly generated instances, our algorithm takes near linear number of pivots, hence is practical (see Section 7).

Finally, we obtain a reduction from an exchange market to a market with linear utilities and derive the following theorem.

THEOREM 5.3. *The problem of computing an equilibrium of an Arrow-Debreu market with linear utilities and SPLC production is PPAD-complete, assuming the weakest known sufficiency conditions by Maxfield [29]. In general checking existence of an equilibrium in these markets is NP-complete.*

6 Proof Overviews

6.1 The Market Equilibrium and LCP Theorem Here we present an overview of the proof of Theorem 5.1. The proof of every market equilibrium is a solution of NHAD-LCP (up to scaling) follows easily from Lemma 4.1 and enough demand condition. For the other direction, next we sketch a proof to show that at any solution of NHAD-LCP, each firm produces as per an optimal plan and each agent gets an optimal bundle.

At a solution of NHAD-LCP, the price of good j is $p_j = p'_j + c_j$. Let $x^f_{jk} = {r^f_{jk}}/{p_j}$ and $y^f_{jk} = {s^f_{jk}}/{p_{j_f}}$ be the amount of used and produced goods on segment (f,j,k). These are well defined since p_j's are positive. For a firm f, observe that x^f_{jk}'s and β^f_{jk}'s are non-negative and satisfy conditions (4.2) and (4.3) since (5.12) and (5.13) are satisfied at the given solution. Therefore they form a solution of LP (4.1) at prices p_j, and hence x^f_{jk}'s give the amounts to be used at optimal production plan.

Next we show that $y^f_{jk} = \alpha^f_{jk} x^f_{jk}$ indeed holds. If $x^f_{jk} > 0$ then from (5.12) we have $\beta^f_{jk} = \alpha^f_{jk} p_{j_f} - p_j$. If $\beta^f_{jk} = 0$ then from (5.14) we have $s^f_{jk} = r^f_{jk}$. In this case,

$$y^f_{jk} = \frac{s^f_{jk}}{p_{j_f}} = \alpha^f_{jk} \frac{r^f_{jk}}{p_j} = \alpha^f_{jk} x^f_{jk}.$$

If $\beta^f_{jk} > 0$ then using (5.13') we have $r^f_{jk} = o^f_{jk} p_j$ and $x^f_{jk} = o^f_{jk}$, and using (5.14) $s^f_{jk} = r^f_{jk} + o^f_{jk}\beta^f_{jk}$.

$$y^f_{jk} = \frac{s^f_{jk}}{p_{j_f}} = \frac{r^f_{jk} + o^f_{jk}\beta^f_{jk}}{p_{j_f}} = \frac{o^f_{jk} p_j + o^f_{jk}(\alpha^f_{jk} p_{j_f} - p_j)}{p_{j_f}}$$
$$= \alpha^f_{jk} x^f_{jk}.$$

For optimal bundle, let $x^i_{jk} = {}^{q^i_{jk}}/_{p_j}$. We show that agents buy segments in decreasing order of their bang-per-buck at prices $\boldsymbol{p}$. Recall that fully-bought segments are called forced, partial ones are flexible and the rest are undesired. Using both (5.17) and (5.18), we conclude that for segment (i, j, k)

$$0 < x^i_{jk} < l^i_{jk} \quad \Rightarrow \quad \frac{u^i_{jk}}{p_j} = \frac{1}{\lambda_i} \quad \text{ and}$$
$$x^i_{jk} = l^i_{jk} \quad \Rightarrow \quad \frac{u^i_{jk}}{p_j} \geq \frac{1}{\lambda_i}, \quad \text{ and}$$
$$x^i_{jk} = 0 \quad \Rightarrow \quad \frac{u^i_{jk}}{p_j} \leq \frac{1}{\lambda_i}.$$

As needed we have bang-per-buck of fully-bought segments is more than that of partially bought, which is more than that of unutilized. Market clearing follows from (5.15) and (5.16).

6.2 The No Secondary Ray Theorem Here we briefly discuss an approach to show that there are no *secondary rays* in the polyhedron $\mathcal{P}'$ of NHAD-LCP'. First we define the *strong connectivity* condition. We say that *agent i is non-satiated by good j* if the last segment of U^i_j has positive slope, i.e., $u^i_{j|U^i_j|} > 0$. Similarly, firm f is non-satiated by good j if $\alpha^f_{j|P^f_j|} > 0$. We note that length of the last segment is infinity.

Definition 6.1. **(Strong connectivity)** *Construct a directed graph $G(\mathcal{M})$ whose nodes correspond to agents and firms of market $\mathcal{M}$ and there is an edge from node a to b if there is a good possessed/produced by node a for which node b is non-satiated. Market $\mathcal{M}$ satisfies strong connectivity if this graph has a strongly connected component containing all the agent nodes.*

The proof is by contradiction. Suppose there is a *secondary ray*, say R, in $\mathcal{P}'$. Recall that a secondary ray is an unbounded edge in the solution set of NHAD-LCP' with $z > 0$. Let R be incident on the vertex $(\boldsymbol{y_*}, z_*)$, with $z_* > 0$, and has the direction vector $(\boldsymbol{y_\circ}, z_\circ)$. Then $R = \{(\boldsymbol{y_*}, z_*) + \delta(\boldsymbol{y_\circ}, z_\circ) \mid \forall \delta \geq 0\}$. Since each of these points is a solution of NHAD-LCP', we have $\boldsymbol{y_\circ} \geq 0$ and $z_\circ \geq 0$ to start with, due to non-negativity constraint on all the variables; no variable decreases on R.

Next we derive contradiction for each of the three cases: (i) $\boldsymbol{p}'_\circ > 0$, (ii) $\boldsymbol{p}'_\circ = 0$, and (iii) $\boldsymbol{p}'_\circ \neq 0$, $\boldsymbol{p}'_\circ \not> 0$. Here we discuss the most involved case (iii).

Assume that $\boldsymbol{p}'_\circ \not> 0$ and $\boldsymbol{p}'_\circ \neq 0$. Let $S \subset \mathcal{G}$ be the set of goods for which the vector $\boldsymbol{p}'_\circ$ is zero and $\overline{S}$ be the remaining goods; by assumption, both these sets are non-empty. Let $A_1 \subseteq A$ be the set of agents who are non-satiated by at least one good in S. Clearly, the prices of goods in S remain constant throughout R and those of goods in $\overline{S}$ go to infinity. Hence eventually, the bang-per-buck of all segments corresponding to goods from S will dominate that of goods from $\overline{S}$.

Let F be the set of firms producing a good from S and $\overline{F}$ be the remaining firms. Similarly, for any firm in $\overline{F}$ all the segments corresponding to S will be profitable and will dominate that of goods from $\overline{S}$. Further, firms in F cannot produce anything using goods from $\overline{S}$ and their production does not change on R. Therefore, their revenue remains constant on R.

Note that if $\lambda_i = 0$ then $q^i_{jk} = 0$, $\forall (j,k)$ due to (5.17'). Using this fact and (5.19'), we conclude that at any solution of NHAD-LCP' surplus of every agent is at most z, and it is exactly z if $\lambda_i > 0$. By (5.15), each good in $\overline{S}$ is fully sold. Now, since only goods in S can remain unsold and their total amount in the market is constant, the total surplus of all agents (money earned − money spent) is bounded. Since $z_\circ \geq 0$, each agent has a non-negative surplus, and hence the surplus of each agent is bounded.

Now, consider an agent i who has a good from $\overline{S}$ in her initial endowment. Since her earnings go to infinity and her surplus is bounded, she must eventually buy up all segments corresponding to goods in S for which she has positive utility. Similarly, consider a firm in $\overline{F}$. Since the price of the good it produces go to infinity, all its non-zero segments corresponding to goods in S will be profitable, and hence will be produced fully. Next we derive contradictions based on what A_1 consist of.

Suppose $A_1 = A$, then by the observation made above, any agent having a non-zero amount of a good from $\overline{S}$ must eventually demand more than the available amount of some good in S. Since there can be no production cycle at any prices due to *no production out of nothing* condition, the available amount of every good is bounded, contradicting (5.15).

If $A_1 = \emptyset$, then consider an arbitrary agent i. For strong connectivity to hold, there must be some agent i_1 or a firm $f \in \overline{F}$ such that i has a good for which i_1 or f is non-satiated. Since $A_1 = \emptyset$ and all the firms in $\overline{F}$ are satiated for all the goods in S, this good is from $\overline{S}$. Hence each agent has a good from $\overline{S}$ in her initial endowment. Let $j \in S$. Now, by the observation made above, all agents will eventually buy all segments of j for which they have positive utility. Further, since there can be no production cycles there is at least one good $j \in S$ not getting produced, contradicting (5.15), since desire$(j) > 1$ (due to *enough demand* condition).

Finally, suppose $\emptyset \subset A_1 \subset A$. Agents of A_1 do not own any good from $\overline{S}$, otherwise by observation made above, demand of some good in S eventually goes to infinity, contradicting (5.15). Further, for the same reason firms of $\overline{F}$ are satiated for goods in S. Therefore, in graph $G(\mathcal{M})$ of *strong connectivity* definition, there is no edge from A_1 to $A \setminus A_1$, A_1 to $\overline{F}$, and F to $\overline{F}$, contradicting *strong connectivity.*

7 Experimental Results

We coded our algorithm in Matlab and ran it on randomly generated instances of markets with SPLC production and SPLC utility functions. Number of segments are kept the same in all the utility and production functions, let it be denoted by $\#seg$. An instance is created by picking values uniformly at random – w_j^is from [0, 1], θ_f^i's from [0, 1], u_{jk}^i's from [0, 1], l_{jk}^i's from $[0,\ 10/\#seg]$, α_{jk}^f from [0, 1] (in order to avoid positive cycles in production) and o_{jk}^f's from $[0,\ 10/\#seg]$. For simplicity we assume that firm a produces good a. For every firm f, θ_f^i's are scaled so that they sum up to one. Similarly, for every good j, w_j^i's are scaled so that they sum up to one. For each pair of agent i and good j, u_{jk}^i's are sorted in decreasing order to get PLC U_j^i, and similarly for each pair of firm f and good j, α_{jk}^f's are sorted in decreasing order to get PLC P_j^f. The experimental results are given in Table 2. Note that, even in the worst case the number of iterations is always linear in the total number of segments in all the input functions. Total number of segments in a market with n goods, m agents and l firms is $(mn + l(n-1))\#seg$.

References

[1] K. Arrow and G. Debreu. Existence of an equilibrium for a competitive economy. *Econometrica*, 22:265–290, 1954.

[2] W. C. Brainard and H. E. Scarf. How to compute equilibrium prices in 1891. *Cowles Foundation Discussion Paper*, (1270), 2000.

[3] X. Chen, D. Dai, Y. Du, and S.-H. Teng. Settling the complexity of Arrow-Debreu equilibria in markets with additively separable utilities. In *FOCS*, 2009.

[4] X. Chen, D. Paparas, and M. Yannakakis. The complexity of non-monotone markets. In *STOC*, pages 181–190, 2013.

[5] X. Chen and S.-H. Teng. Spending is not easier than trading: on the computational equivalence of Fisher and Arrow-Debreu equilibria. In *ISAAC*, pages 647–656, 2009.

[6] B. Codenotti, B. McCune, and K. Varadarajan. Market equilibrium via the excess demand function. In *STOC*, pages 74–83, 2005.

[7] B. Codenotti, S. Pemmaraju, and K. Varadarajan. On the polynomial time computation of equilibria for certain exchange economies. In *SODA*, pages 72–81, 2005.

[8] B. Codenotti, A. Saberi, K. Varadarajan, and Y. Ye. Leontief economies encode two-player zero-sum games. In *SODA*, 2006.

[9] R. Cottle, J. Pang, and R. Stone. *The Linear Complementarity Problem.* Academic Press, Boston, 1992.

[10] N. Devanur and R. Kannan. Market equilibria in polynomial time for fixed number of goods or agents. In *FOCS*, pages 45–53, 2008.

[11] N. Devanur, C. H. Papadimitriou, A. Saberi, and V. V. Vazirani. Market equilibrium via a primal-dual algorithm for a convex program. *JACM*, 55(5), 2008.

[12] N. Devanur and V. Vazirani. An improved approximation scheme for computing Arrow-Debreu prices for the linear case. In *Proceedings of 23rd FSTTCS*, 2003.

[13] R. Duan and K. Mehlhorn. A combinatorial polynomial algorithm for the linear Arrow-Debreu market. In *ICALP*, 2013.

[14] B. C. Eaves. A finite algorithm for the linear exchange model. *Cowles Foundation Discussion Paper No. 389*, 1975.

[15] B. C. Eaves. A finite algorithm for the linear exchange model. *Journal of Mathematical Economics*, 3:197–203, 1976.

[16] B. Ellickson. *Competitive equilibrium: Theory and applications.* Cambridge University Press, 1994.

[17] K. Etessami and M. Yannakakis. On the complexity of Nash equilibria and other fixed points. *SIAM Journal on Computing*, 39(6):2531–2597, 2010.

[18] J. Garg, R. Mehta, M. Sohoni, and V. V. Vazirani. A complementary pivot algorithm for market equilibrium under separable piecewise-linear concave utilities. In *STOC*, 2012.

[19] J. Garg and V. V. Vazirani. On computability of equilibria in markets with production. In *arXiv:1308.5272*, 2013.

[20] R. Garg and S. Kapoor. Auction algorithms for market equilibrium. In *STOC*, 2004.

[21] S. Govindan and R. Wilson. Equivalence and invariance of the index and degree of Nash equilibria. *Games*

#agents, #goods, #firms, #seg	#Instances	Min	Avg	Max
2, 2, 2, 2	100	8	13.61	18
5, 5, 5, 2	100	34	68.85	99
5, 5, 5, 5	100	163	227.58	291
10, 5, 5, 2	100	70	119.95	148
10, 5, 5, 5	100	104	396.98	472
10, 10, 10, 2	100	118	224.67	275
10, 10, 10, 5	10	260	714.5	905
10, 10, 10, 10	10	326	1486.3	2210
15, 5, 5, 2	100	141	173.24	214
15, 5, 5, 5	100	500	581.42	684
15, 10, 10, 2	100	219	295.84	374
15, 10, 10, 5	10	1186	1934.3	2833
15, 10, 10, 10	10	2678	2853.2	3190

Table 2: Experimental Results

and Economic Behavior, 21(1):56–61, 1997.

[22] P. J.-J. Herings and R. Peeters. Homotopy methods to compute equilibria in game theory. *Econ Theory*, 42:119–156, 2010.

[23] K. Jain. A polynomial time algorithm for computing the Arrow-Debreu market equilibrium for linear utilities. *SIAM Journal on Computing*, 37(1):306–318, 2007.

[24] K. Jain, M. Mahdian, and A. Saberi. Approximating market equilibrium. In *Proceedings of 6th APPROX*, 2003.

[25] K. Jain and K. Varadarajan. Equilibria for economies with production: Constant-returns technologies and production planning constraints. In *SODA*, pages 688–697, 2006.

[26] D. Koller, N. Megiddo, and B. V. Stengel. Efficient computation of equilibria for extensive two-person games. *Games and Economic Behavior*, 14(2):247–259, 1996.

[27] C. E. Lemke. Bimatrix equilibrium points and mathematical programming. *Management Science*, 11(7):681–689, 1965.

[28] C. E. Lemke and J. J. T. Howson. Equilibrium points of bimatrix games. *SIAM J. on Applied Mathematics*, 12(2):413–423, 1964.

[29] R. R. Maxfield. General equilibrium and the theory of directed graphs. *Journal of Mathematical Economics*, 27(1):23–51, 1997.

[30] N. Megiddo. A note on the complexity of P-matrix LCP and computing an equilibrium. IBM Research Report 6439, 1988.

[31] K. Murty. *Linear Complementarity, Linear and Nonlinear Programming.* Heldermann Verlag, 1988.

[32] J. B. Orlin. Improved algorithms for computing Fisher's market clearing prices. In *STOC*, pages 291–300, 2010.

[33] L. S. Shapley. A note on the Lemke-Howson algorithm. In *Mathematical Programming Studies*, pages 175–189, 1974.

[34] J. B. Shoven and J. Whalley. *Applying general equilibrium.* Cambridge University Press, 1992.

[35] M. J. Todd. Orientation in complementary pivot algorithms. *Mathematics of Operation Research*, 1(1):54–66, 1976.

[36] A. H. van den Elzen and A. J. J. Talman. A procedure for finding Nash equilibria in bimatrix games. *ZOR-Methods and Models of Operations Research*, 35:27–43, 1991.

[37] V. V. Vazirani and M. Yannakakis. Market equilibrium under separable, piecewise-linear, concave utilities. *Journal of ACM*, 58(3):10:1–10:25, 2011.

[38] L. A. Vegh. Concave generalized flows with applications to market equilibria. In *FOCS*, 2012.

[39] A. von Schemde. *Index and Stability in Bimatrix Games: A Geometric-Combinatorial Approach.* Springer, 2005.

[40] A. von Schemde and B. von Stengel. Strategic characterization of the index of an equilibrium. In *SAGT*, pages 242–254, 2008.

[41] B. von Stengel, A. H. van den Elzen, and A. J. J. Talman. Computing normal form perfect equilibria for extensive two-person games. *Econometrica*, 70(2):693–715, 2002.

[42] Y. Ye. Exchange market equilibria with Leontief's utility: Freedom of pricing leads to rationality. *Theor. Comput. Sci.*, 378(2):134–142, 2007.

[43] Y. Ye. A path to the Arrow-Debreu competitive market equilibrium. *Math. Program.*, 111(1-2):315–348, 2008.

A The Linear Complementarity Problem and Lemke's Algorithm

Given an $n \times n$ matrix $\boldsymbol{M}$, and a vector $\boldsymbol{q}$, the linear complementarity problem asks for a vector $\boldsymbol{y}$ satisfying the following conditions:

$$\text{(A.1)} \quad \boldsymbol{My} \le \boldsymbol{q}, \quad \boldsymbol{y} \ge 0, \quad \boldsymbol{q} - \boldsymbol{My} \ge 0 \quad \text{and} \quad \boldsymbol{y} \cdot (\boldsymbol{q} - \boldsymbol{My}) = 0.$$

The problem is interesting only when $\boldsymbol{q} \not\geq 0$, since otherwise $\boldsymbol{y} = 0$ is a trivial solution. Let us introduce slack variables $\boldsymbol{v}$ to obtain the equivalent formulation

$$\text{(A.2)} \quad \boldsymbol{My}+\boldsymbol{v} = \boldsymbol{q}, \quad \boldsymbol{y} \ge 0, \quad \boldsymbol{v} \ge 0 \quad \text{and} \quad \boldsymbol{y}\cdot\boldsymbol{v} = 0.$$

Let $\mathcal{P}$ be the polyhedron in $2n$ dimensional space defined by the first three conditions; we will assume that $\mathcal{P}$ is non-degenerate. Under this condition, any solution to (A.2) will be a vertex of $\mathcal{P}$, since it must satisfy $2n$ equalities. Note that the set of solutions may be disconnected.

An ingenious idea of Lemke was to introduce a new variable and consider the system:

$$\text{(A.3)} \quad \boldsymbol{My} + \boldsymbol{v} - z\boldsymbol{1} = \boldsymbol{q}, \quad \boldsymbol{y} \ge 0, \quad \boldsymbol{v} \ge 0, \quad z \ge 0 \quad \text{and} \quad \boldsymbol{y} \cdot \boldsymbol{v} = 0.$$

Let $\mathcal{P}'$ be the polyhedron in $2n + 1$ dimensional space defined by the first four conditions; again we will assume that $\mathcal{P}'$ is non-degenerate. Since any solution to (A.3) must still satisfy $2n$ equalities, the set of solutions, say S, will be a subset of the one-skeleton of $\mathcal{P}'$, i.e., it will consist of edges and vertices of $\mathcal{P}'$. Any solution to the original system must satisfy the additional condition $z = 0$ and hence will be a vertex of $\mathcal{P}'$.

Now S turns out to have some nice properties. Any point of S is *fully labeled* in the sense that for each i, $y_i = 0$ or $v_i = 0$. We will say that a point of S *has double label* i if $y_i = 0$ and $v_i = 0$ are both satisfied at this point. Clearly, such a point will be a vertex of $\mathcal{P}'$ and it will have only one double label. Since there are exactly two ways of relaxing this double label, this vertex must have exactly two edges of S incident at it. Clearly, a solution to the original system (i.e., satisfying $z = 0$) will be a vertex of $\mathcal{P}'$ that does not have a double label. On relaxing $z = 0$, we get the unique edge of S incident at this vertex.

As a result of these observations, we can conclude that S consists of paths and cycles. Of these paths, Lemke's algorithm explores a special one. An unbounded edge of S such that the vertex of $\mathcal{P}'$ it is incident on has $z > 0$ is called a *ray*. Among the rays, one is special – the one on which $\boldsymbol{y} = 0$. This is called the *primary ray* and the rest are called *secondary rays*. Now Lemke's algorithm explores, via pivoting, the path starting with the primary ray. This path must end either in a vertex satisfying $z = 0$, i.e., a solution to the original system, or a secondary ray. In the latter case, the algorithm is unsuccessful in finding a solution to the original system; in particular, the original system may not have a solution.

Remark: Observe that $z\boldsymbol{1}$ can be replaced by $z\boldsymbol{a}$, where vector $\boldsymbol{a}$ has a 1 in each row in which $\boldsymbol{q}$ is negative and has either a 0 or a 1 in the remaining rows, without changing its role; in our algorithm, we will set a row of $\boldsymbol{a}$ to 1 if and only if the corresponding row of $\boldsymbol{q}$ is negative. As mentioned above, if $\boldsymbol{q}$ has no negative components, (A.1) has the trivial solution $\boldsymbol{y} = 0$. Additionally, in this case Lemke's algorithm cannot be used for finding a non-trivial solution, since it is simply not applicable.