

# Measuring robustness of dynamical systems. Relating time and space to length and precision

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**Abstract**—Verification of discrete time or continuous time dynamical systems over the reals is known to be undecidable. It is however known that undecidability does not hold for various classes of systems when considering *robust* systems: if robustness is defined as the fact that reachability relation is stable under infinitesimal perturbation, then their reachability relation is decidable. In other words, undecidability implies sensitivity under infinitesimal perturbation, a property usually not expected in systems considered “in practice”, and hence can be seen (somehow informally) as an artifact of the theory, that always assumes exactness. In a similar vein, it is known that, while undecidability holds for logical formulas over the reals, it does not hold when considering  $\delta$ -undecidability: one must determine whether a property is true, or  $\delta$ -far from being true.

We first extend the previous statements to a theory for general (discrete time, continuous-time, and even hybrid) dynamical systems, and we relate the two approaches. We also relate robustness to some geometric properties of reachability relation.

But mainly, when a system is robust, it then makes sense to quantify at which level of perturbation. We prove that assuming robustness to polynomial perturbations on precision leads to reachability verifiable in complexity class PSPACE, and even to a characterization of this complexity class. We prove that assuming robustness to polynomial perturbations on time or length of trajectories leads to similar statements, but with PTIME.

Actually, these results can also be read in another way, in terms of computational power of analog models. It has been recently unexpectedly shown that the length of a solution of a polynomial ordinary differential equation corresponds to a time of computation: PTIME corresponds to solutions of polynomial differential equations of polynomial length. Can we do something similar for PSPACE? How should we measure space robustly for dynamical systems? Our results argue that the answer is given by precision: space corresponds to the involved precision.

## I. INTRODUCTION

Several research communities have studied the relations between dynamical systems and computation. This includes studies of the unpredictability and undecidability in dynamical systems [27], [26], questions related to the hardness of verification for discrete, continuous and the so-called hybrid systems [2], questions related to the computational power of models of recurrent neural networks [31], or control theory questions [3], [10].

*Motivation related to verification:* while several undecidability results were stated for hybrid systems (such as Linear Hybrid Automata [22] or Piecewise Constant Derivative systems [2]), and while it was observed that some practical tools were “working well” (terminating) in practice, a folklore (sometimes informal) conjecture appeared in the literature of verification or in several talks, stating that this undecidability is due to non-stability, non-robustness, sensitivity to the initial values of the systems, and that it never occurs in “real” systems. There were several attempts to formalize and to prove (or to disprove) this conjecture, including [15], [23]. Refer to [1], [10] for some discussions.

In this article, we are inspired by the approach of [1], where several models for discrete, continuous time, and hybrid dynamical systems are considered: Turing machines, Piecewise affine maps, Linear hybrid automata and Piecewise Constant Derivative systems. To each of these models is associated a notion of perturbed dynamics by a small  $\epsilon$  (with respect to a suitable metrics), and a perturbed reachability relation is defined as the intersection of all reachability relations obtained by  $\epsilon$ -perturbations, for all possible values of  $\epsilon$ . The authors show that for all these models, the perturbed reachability relation is co-computably enumerable (co-c.e.,  $\Pi_1^0$ ), and that any co-c.e. relation can be defined as the perturbed reachability relation of such models. A corollary is, that if we define robustness as stability of the reachability relation under infinitesimal perturbations, then robust systems have a decidable reachability relation, and hence a decidable verification.

In a similar vein, [16] observed that some logics such as real arithmetic are decidable, but even the set of  $\Sigma_1$  sentences in a language extending real arithmetic with the sine function is already undecidable, and so deciding satisfaction of such “simple” formulas is impossible in the general case. However, such theoretical negative result only refers to the problem of deciding logic formula symbolically and precisely. If a relaxed notion of correctness is considered, then verification becomes algorithmically solvable: one asks to answer true when a given formula  $\phi$  is true, and to return false when it is  $\delta$ -robustly wrong, namely  $\phi$  is wrong, but actually some  $\delta$ -strengthening of  $\phi$  is false. In other words, undecidability happens only for

exact questions, whose decision is not stable by infinitesimal perturbations, while robust-satisfaction is decidable.

These approaches are relating robustness to the decidability of their verification, or robust satisfaction to a decidability of satisfaction, i.e. a *computability* question. Our purpose in the current article is to extend all this, and in particular [1], to more general classes of dynamical systems, but also, mainly to discuss also *complexity* issues: when is the verification of a system in PTIME or in PSPACE? How does it relate to the robustness of the system with respect to perturbations?

Basically, when a system is robust, it makes sense to quantify at which level it is: which level of perturbation  $\epsilon$  is allowed? We basically prove that assuming this perturbation is polynomial in the data leads to a characterization of PSPACE. This provides ways to guarantee complexity of the verification of a system. Actually, we also show that this idea can be used to define various complexity classes, by playing with various concepts of robustness: we introduce the concept of time and length perturbation (basically trajectories cannot be too long), and we prove that this leads to PTIME, and hence leads to systems where verification becomes polynomial.

*Motivation related to models of computation:* another field of research is related to understanding how analog (continuous-time) models of computation compare to more classical discrete models such as Turing machines. In particular, a famous historical celebrated mathematical model is the General Purpose Analog Computer (GPAC) model, introduced by Shanon in 1941 in [30], as a model for the Differential Analyzer [13], a mechanical programmable machine, on which he worked as an operator. It is known that functions computed by a GPAC correspond to a (component of the) solution of a system of Ordinary Differential Equations (ODE) with polynomial right-hand side (also called *pODE*) [30], [19].

Relating the computational power of this model to classical models such as the Turing machines, at the complexity level, is not a trivial task. In short, contrarily to discrete models of computation, continuous time models of computation (not only the GPAC, but many others) may exhibit the so-called “Zeno phenomena”, where time can be arbitrarily contracted in a continuous system, thus allowing an arbitrary speed-up of the computation. This forbids to take the naive approach of using the time variable of the ODE as a well-defined measure of time complexity: see [4], [10] for discussions.

A celebrated recent breakthrough has been obtained in [28], [7], where it has been proved that a key idea to solve this issue is, when using pODEs (in principle this idea can also be used for others continuous models of computation) to compute a given function  $f$ , the cost should be measured as a function of the length of the solution curve of the polynomial initial value computing the function  $f$ . As ODEs is a kind of universal language in experimental sciences, this breakthrough led to solve several open problems in various other contexts. This includes the proof of the existence of a universal (in the sense of Rubel) ODE [9]; proof of the strong Turing-completeness of biochemical reactions [14], or statements about the completeness of reachability problems (e.g. PTIME-

completeness of bounded reachability) for ODEs [7].

While it is now known that time should be measured by length, the question on how space should be measured is unclear. How should be measure the “memory” used by some ordinary differential equation? Can we provide a characterization of PSPACE using ODEs similar to the characterization of PTIME obtained in [7]? If time corresponds to length, what space corresponds to? We give some arguments to state that basically, over a compact domain space corresponds to precision, while it corresponds to the log of the size of some graph for systems over more general domains.

*Related work and main results:* the current article lies its foundations on some extensions of [1]. We reformulate some of their statements, and extend some of their results. In particular, we allow more general discrete time and continuous time dynamical systems, while [1] was restricting to Piecewise Affine Maps (PAM), and hybrid systems, both of them assumed to work on a bounded domain. Our statements are established in a wide generality by considering only computability of the domain or the dynamics. We know about generalizations that have also been obtained in [8], but focusing on dynamical systems as language recognizers, and mainly focusing somehow on generalizations of Theorem 14 ([1, Theorem 4]), while we discuss here reachability in a more natural and general way.

We also connect our approach to  $\delta$ -decidability introduced in [16]. Observe that the logics considered in the latter are not sufficiently expressive to cover our results (or [1]).

Another clear difference is also that we talk about complexity issues (space, time), while all these references are only talking about computability issues. Furthermore, we consider not only space perturbations, but also time and length perturbations, and we prove that many of the results, even for computability, can also be extended to this framework.

In Section II, we recall some known facts about the hardness of reachability (accessibility) problems in graphs, discussing in particular their encoding.

Before going to general discrete time dynamical systems, and later continuous and hybrid time dynamical systems, we consider a specific case of discrete time dynamical systems, namely Turing machines (TMs). We think this helps to understand the coming statements. We basically introduce in Section III the concept of (space) perturbed Turing machine, following [1], and recall some of the results established in this paper. The original contributions in this section are the extension of this framework to complexity. We prove that considering a polynomial robustness corresponds precisely to polynomial space, i.e. PSPACE (Theorem 17). We consider here space perturbations, but we prove later (in Section VIII) that a natural concept of time perturbation could also be considered in order to lead to a characterisation of polynomial time, i.e. PTIME (Theorem 58), using similar ideas.

Section IV, recalls how several undecidability results for dynamical systems are established using the embedding of a class of dynamical systems (e.g. TMs) into another. We believe this helps to understand the related sources of non-robustness.

Section V is considering general discrete time dynamical systems (over  $\mathbb{R}^d$ ). We use the concept of perturbation and robustness from [1]. But unlike the latter article, mainly restricting to Piecewise Affine Maps (PAMs), we consider general systems, discussing the importance of the domain, and of the preservation of rational numbers as in PAMs. A main technical result is then Theorem 24: this is an extension of [1, Theorem 5] to the general settings, established using similar ideas, but not exactly<sup>1</sup>. We also extend the framework to the case of systems that would not preserve rationals. This leads to an extension of main statement of [1]: robustness implies decidability of reachability (Corollary 25 and Corollary 39). Then we prove that this theory can actually be related to the approach of  $\delta$ -decidability proposed in [16]: ( $\epsilon$ -)robust means true or ( $\epsilon$ -)far from being true (Theorem 27). This is an also important original contribution relating the two approaches, giving arguments in the spirit of [16]. Our key point is then to be able to extend all at the complexity level. One key result is that polynomially robust implies reachability in PSPACE (Theorem 35). This is even a characterization of PSPACE (Theorem 36). Concretely, Section V is split into two subsections. In the first part, we only need to talk about rational numbers, and classical computability. In the second part, as functions may take non-rational values, we use the framework of Computable Analysis (CA). The first part helps for the intuition of the second. But actually, as a computable function in CA is continuous, the first is not a consequence of the second, even if several methods and reasoning are shared.

Section VI provides a nice and original view. Using various results established in the community of CA, we prove that robustness can also be seen as having a reachability relation that can be drawn. We use there the fact that the latter relates to computability for subsets of  $\mathbb{R}^p$ .

In Section VII, we state that similar statements hold also for continuous time, and even hybrid dynamical systems.

Section VIII is proving that other perturbations could be considered and would lead to talk about time complexity PTIME instead of PSPACE. Section IX discusses how the current work relates to the known characterizations of complexity classes with continuous time dynamical systems. Notice that some characterizations of PSPACE have been obtained recently in [17], [5] (discussed in this section), but at the price of technical conditions, that we believe not to provide a fully satisfying answer to previous questions.

*Preliminaries:* a  $*$  means that a (possibly more extended) proof can be found in appendix. We write  $d(\cdot, \cdot)$  for the norm-sup distance. A (open) (respectively close) *rational ball* is a subset of the form  $B(\mathbf{x}, \delta) = \{\mathbf{y} : d(\mathbf{x}, \mathbf{y}) < \delta\}$  (resp.  $\bar{B}(\mathbf{x}, \delta) = \{\mathbf{y} : d(\mathbf{x}, \mathbf{y}) \leq \delta\}$ ) for some rational  $\mathbf{x}$  and  $\delta$ . We could in principle use the Euclidean distance, but this distance has the advantage that its balls correspond directly to rounding at some precision. A set of the form  $\prod_{i=1}^d [a_i, b_i]$ , for rational  $(a_i), (b_i)$ , will be called a *rational closed box*. An

<sup>1</sup>As the proof there turns out to be ambiguous about whether “can make a transition” means in one step vs in one or many steps, and we are not sure to fully follow it in its exact form.

open rational box is obtained by considering open intervals in previous definition. The least closed set containing  $X$  is denoted by  $cls(X)$ . We write  $\ell(\cdot)$  for the function that measures the binary size of its argument.

## II. ABOUT REACHABILITY IN GRAPHS

Consider a directed graph  $G = (V, \rightarrow)$ , with  $\rightarrow \subseteq V^2$ . It is said *deterministic* if any node has at most one outgoing edge.

**Lemma 1** (Reachability for graphs). *Consider the following decision problem  $\text{PATH}(G, u, v)$ : given a directed graph  $G = (V, \rightarrow)$ , and some vertices  $u, v \in V$ , determine whether there is some path between  $u$  and  $v$  in  $G$ , denoted by  $u \xrightarrow{*} v$ .*

*Then  $\text{PATH}(G, u, v) \in \text{NLOGSPACE}$ .*

*Proof.* Basically, a non-deterministic TM can guess the intermediate nodes of the path: see e.g. [32, Example 8.19].  $\square$

**Lemma 2** (Immerman–Szelepcsényi’s theorem [24], [33]).  $\text{NLOGSPACE} = \text{coNLOGSPACE}$ .

*Proof.* See e.g. [32, Theorem 8.22].  $\square$

Actually, as we will see, we mainly focus its complement:

**Corollary 3.** *Consider the following decision problem  $\text{NOPATH}(G, u, v)$ : given a directed graph  $G = (V, \rightarrow)$ , and some vertices  $u, v \in V$ , determine whether there is no path between  $u$  and  $v$  in  $G$ .*

*Then  $\text{NOPATH}(G, u, v) \in \text{NLOGSPACE}$ .*

**Theorem 4** (Savitch’s theorem). *For any function  $f : \mathbb{N} \rightarrow \mathbb{N}$  with  $f(n) \geq \log n$ , we have  $\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n))$ .*

*Proof.* See e.g. [32, Theorem 8.5].  $\square$

**Corollary 5.**  $\text{PATH}(G, u, v) \in \text{SPACE}(\log^2(n))$  and  $\text{NOPATH}(G, u, v) \in \text{SPACE}(\log^2(n))$ .

**Remark 6.** *Notice that detecting whether there is no path between  $u$  and  $v$  is basically equivalent to determine whether all paths starting from  $u$  “loop”, i.e. remain disjoint from  $v$ . The above statement is established using a more subtle method that a simple depth of width search of the graph, using the trick of the proof of Savitch’s theorem, i.e. a recursive procedure (expressing reachability in less than  $2^t$  steps, called  $\text{CANYIELD}(C_1, C_2, t)$  in [32]) guaranteeing the above space complexity.*

Our purpose is to talk about various discrete time dynamical systems (and latter even continuous time dynamical systems).

**Definition 7** (Discrete Time Dynamical System). *A (general) discrete-time dynamical system  $\mathcal{P}$  is given by a set  $X$ , called domain, and some (possibly partial) function  $f$  from  $X$  to  $X$ .*

A trajectory of  $\mathcal{P}$  is a sequence  $(x_t)$  evolving according to  $f$ , i.e. such that  $x_{t+1} = f(x_t)$  for all  $t$ . We say that  $x^*$  (or a set  $X^*$ ) is reachable from  $x$  if there is a trajectory with  $x_0 = x$  and  $x_t = x^*$  (respectively  $x_t \in X^*$ ) for some  $t$ .

In other words, any discrete time dynamical system  $\mathcal{P}$  can be seen as a particular (deterministic) directed graph but where

$V$  is not necessarily finite. This graph corresponds to  $V = X$ , and  $\rightarrow$  corresponds to the graph of function  $f$ . If it remains finite, we can generalize some of the previous statements, but working over representations in order to make things feasible.

**Corollary 8** (Reachability for finite graphs). *Let  $s(n) \geq \log(n)$ . Assume the vertices of  $G$  can be encoded in binary using words of length  $s(n)$ . Assume the relation  $\rightarrow$  is decidable using a space polynomial in  $s(n)$  with this encoding.*

*Then, given the encoding of  $u \in V$  and of  $v \in V$ , we can decide whether there is a (respectively: no) path from  $u$  to  $v$ , in a space polynomial in  $s(n)$ .*

*Proof.* Use similar arguments and algorithms (in particular the trick of Savitch's theorem) as in previous corollary, but working over representations of vertices.  $\square$

We will still write (abusively) PATH and NOPATH for these problems. Notice that assuming that the vertices of  $G$  can be encoded in binary using words of length  $s(n)$  requires the graph  $G$  to be finite, with less than  $2^{s(n)}$  vertices.

**Remark 9.** *If write  $\log\text{-size}(G)$  for the log of the number of vertices of a finite graph, we see that this provides some relevant complexity measure of the hardness of the space complexity of these problems (with above assumptions on  $G$ ).*

### III. TURING MACHINES

Let us recall the definition of a (bi-infinite tape) Turing machine (TM): let  $\Sigma$  be a finite alphabet, and let  $B \notin \Sigma$  be the blank symbol. A TM over  $\Sigma$  is a tuple  $(Q, q_{\text{init}}, F, R, \Gamma)$  where  $Q$  is a finite set of control states,  $q_0 \in Q$  is the initial control state,  $F \subseteq Q$  (respectively  $R \subseteq Q$ ) is a set of accepting (respectively rejecting) states, with  $F \cap R = \emptyset$ , and  $\Gamma$  is a set of transitions of the form  $(q, a) \rightarrow (q', b, \delta)$  where  $q, q' \in Q$ ,  $a, b \in \Sigma \cup \{B\}$ , and  $\delta \in \{-1, 0, 1\}$ . When the machine has accepted or rejected, decision is unchanged: when  $q \in F$ , then  $q' \in F$ , and when  $q \in R$  then  $q' \in R$ .

A configuration  $C$  of the machine is given by the current control state  $q$ , and the current content of the bi-infinite tape:  $\cdots a_{-2}a_{-1}a_0a_1a_2 \cdots$ , where the  $a_i$ 's are symbols in  $\Sigma \cup \{B\}$ : this means that the head of the machine is in front of symbol  $a_0$ . We write  $\mathcal{C}_M$  for the set of the configurations of a TM, and write such a configuration as the triple  $(q, \cdots a_{-2}a_{-1}, a_0a_1a_2 \cdots)$ . Given a transition  $(q, a) \rightarrow (q', b, \delta)$  in  $\Gamma$ , if the control state is  $q$  and the symbol pointed by the head of the machine is equal to  $a$ , then the machine can change its configuration  $C$  to the configuration  $C'$  in the following manner: the control state is now  $q'$ , the symbol pointed by the head is replaced by  $b$  and then the head is moved to the left or to the right, or it stays at the same position according to whether  $\delta$  is  $-1, 1$ , or  $0$ , respectively. We write  $C \vdash C'$  when this holds, i.e.  $C'$  is the one-step next configuration of the configuration  $C$ . Then  $(\mathcal{C}_M, \vdash)$  corresponds to a particular dynamical system.

Word  $w = a_1 \cdots a_n \in \Sigma^*$  is accepted by  $M$  if, starting from the initial configuration  $C_0 = C_0[w] = (q_0, \cdots BBB, a_1a_2 \cdots a_n BBB \cdots)$  the machine eventually

stops in an accepting control state: that is, if we write  $\mathcal{F}$  for the configurations where  $q \in F$ , iff  $C_0 \vdash^* C^*$  for some  $C^* \in \mathcal{F}$ . Let  $L(M)$  denote the set of such words, i.e., the computably enumerable (c.e) language semi-recognized by  $M$ . We say that  $w$  is rejected by  $M$  if, starting from the configuration  $C_0$  the machine  $M$  eventually stops in a rejecting state.  $M$  is said to always halt if for all  $w$ , either  $w$  is accepted or  $w$  is rejected.

The article [1] introduces the concept of space perturbed Turing machine: the idea is, given  $n > 0$ , that the  $n$ -perturbed version of the machine  $M$  is unable to remain correct at distance more than  $n$  from the head of the machine. Namely, the  $n$ -perturbed version  $M_n$  of the machine is defined exactly as  $M$  except that before any transition all the symbols at the distance  $n$  or more from the head of the machine can be altered: given a configuration  $(q, \cdots a_{-n-1}a_{-n}a_{-n+1} \cdots a_{-1}, a_0a_1 \cdots a_{n-1}a_na_{n+1} \cdots)$ ,  $M_n$  may replace any symbol to the left of  $a_{-n}$  (starting from  $a_{-n-1}$ ) and to the right of  $a_n$  (starting from  $a_{n+1}$ ) by any other symbols in  $\Sigma \cup \{B\}$  before executing a transition of  $M$  at head position  $a_0$ . Hence  $M_n$  is nondeterministic.

A word  $w$  is accepted by the  $n$ -perturbed version of  $M$  iff there exists a run of this machine which stops in an accepting state. Let  $L_n(M)$  be the  $n$ -perturbed language of  $M$ , i.e., the set of words in  $\Sigma^*$  that are accepted by the  $n$ -perturbed version of  $M$ . From definitions, if a word is accepted by  $M$ , then it can also be recognized by all the  $n$ -perturbed versions of  $M$ , for every  $n > 0$ : perturbed machines have more behaviours. Moreover, if the  $(n+1)$ -perturbed version accepts a word  $w$ , the  $n$ -perturbed version will also accept it since  $n$ -perturbed machines have more behaviours than  $(n+1)$ -perturbed machines.

Let  $L_\omega(M) = \bigcap_n L_n(M)$ : this consists of all the words that can be accepted by  $M$  when subject to arbitrarily "small" perturbations. From definitions:

**Lemma 10** ([1, Lemma 2]).

$$L(M) \subseteq L_\omega(M) \subseteq \cdots \subseteq L_2(M) \subseteq L_1(M).$$

The  $\omega$ -perturbed language of a TM is the complement of a computably enumerable language:

**Theorem 11** (Perturbed reachability is co-c.e. [1, Theorem 3]).  *$L_\omega(M)$  is in the class  $\Pi_1^0$ .*

*Proof.* Given a bi-infinite configuration  $C$  of  $M$  of the form  $(q, \cdots a_{-n-1}a_{-n} \cdots a_{-1}, a_0a_1 \cdots a_na_{n+1} \cdots)$ , we define  $\varphi_n(C) = (q, a_{-n} \cdots a_{-1}, a_0a_1 \cdots a_n)$  made of words of length  $n$  and  $n+1$ .

For every  $n \in \mathbb{N}$ , we associate to the  $n$ -perturbed version  $M_n$  of TM  $M$  some graph  $G_n = (V_n, \rightarrow_n)$ : the vertices, denoted  $(\mathcal{V}_i)_i$ , of this graph correspond to the  $|Q| \times |\Sigma+1|^{2n+1}$  possible values of  $\varphi_n(C)$  for a configuration  $C$  of  $M$ . There is an edge between  $\mathcal{V}_i$  and  $\mathcal{V}_j$  in  $G_n$  iff  $M_n$  can go from configuration  $C$  to configuration  $C'$  in one step, with  $\varphi_n(C) = \mathcal{V}_i$  and  $\varphi_n(C') = \mathcal{V}_j$ .

Determining whether  $\mathcal{V}_i \rightarrow \mathcal{V}_j$  holds is easy (and in particular polynomial space computable) by considering that, when the head is moved to the left (resp. to the right) of

$\mathcal{V}_i$  a symbol in  $\Sigma \cup \{B\}$  is nondeterministically chosen and appended to the left (resp. right) of the configuration and the right-most (resp. left-most) one is lost (it belongs now to the perturbed area of the configuration and hence it can be replaced by any other symbol).

Let  $\mathcal{F}_n = \varphi_n(\mathcal{F})$  correspond to the accepting control states. By construction, the  $n$ -perturbed version  $\mathcal{M}_n$  of  $\mathcal{M}$  has an accepting run starting from a configuration  $C$ , iff  $\mathcal{F}_n$  is reachable from  $\varphi(C)$ , that is to say  $\text{PATH}(G_n, \varphi(C), \mathcal{F}_n)$ . By Corollary 8, this is decidable in a space polynomial in  $n$ .

Let  $\text{Basis}_n$  be the finite set of sequences  $s_n \in \Sigma^{n+1}$ , such that  $\mathcal{F}_n$  is reachable from  $C_0[s_n]$ . Let  $\text{Short}_n$  be the finite set of sequences  $s_k \in \Sigma^k$  with  $k < n$ , such that  $\mathcal{F}_n$  is reachable from  $C_0[s_k B^{n-k}]$ . Then  $L_n(\mathcal{M}) = \text{Short} \cup \text{Basis } \Sigma^*$ . Consequently,  $L_n(\mathcal{M})$  is decidable in space polynomial in  $n$ , and hence its complement also is. Thus,  $L_\omega(\mathcal{M})$  is c.e., as it is a (uniform in  $n$ ) union of decidable sets.  $\square$

Since a set that is c.e. and co-c.e. is decidable, robust languages (i.e.  $L_\omega(\mathcal{M}) = L(\mathcal{M})$ ) are necessarily decidable.

**Corollary 12** (Robust  $\Rightarrow$  decidable [1, Corollary 3]). *If  $L_\omega(\mathcal{M}) = L(\mathcal{M})$  then  $L(\mathcal{M})$  is decidable.*

The converse holds if another requirement on  $\mathcal{M}$  is added. Indeed, since  $L(\mathcal{M}) \subseteq L_\omega(\mathcal{M})$ ,  $L_\omega(\mathcal{M}) \neq L(\mathcal{M})$  means that there exists some word  $w$ , rejected by  $\mathcal{M}$ , but accepted by any  $n$ -perturbed version  $\mathcal{M}_n$ . This  $w$  is not rejected by  $\mathcal{M}$  in finite time, otherwise it would use finitely many cells of the tape, and with  $n$  sufficiently big,  $\mathcal{M}_n$  would still reject it. In other words, this  $w$  must not be accepted nor rejected by  $\mathcal{M}$ .

**Proposition 13** (Decidable  $\Rightarrow$  robust [1, Proposition 1]). *Assume  $\mathcal{M}$  always halts. Then  $L(\mathcal{M})$  is decidable and  $L_\omega(\mathcal{M}) = L(\mathcal{M})$ .*

In general,  $\omega$ -perturbed languages are not computable enumerable. Some of them are complete among co-r.e. languages: perturbed reachability is complete in  $\Pi_1^0$  [1, Theorem 4].

**Theorem 14** (Perturbed reachability is complete in  $\Pi_1^0$  [1, Theorem 4]). *For every TM  $\mathcal{M}$ , we can effectively construct another TM  $\mathcal{M}'$  such that  $L_\omega(\mathcal{M}') = \overline{L(\mathcal{M})}$ .*

Actually is possible to go to some complexity issues, and not only restrict to computability.

Indeed, when a language is robust, it makes sense to measure what level of perturbation  $f$  can be tolerated:

**Definition 15.** *Given some function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $L_{\{f\}}(\mathcal{M})$  for the set of words accepted by  $\mathcal{M}$  with space perturbation  $f$ :  $L_{\{f\}}(\mathcal{M}) = \{w \mid w \in L_{f(\ell(w))}(\mathcal{M})\}$ .*

The proof above of Theorem 11 establishes explicitly:

**Lemma 16.**  $L_n(\mathcal{M}) \in \text{SPACE}(\text{poly}(n))$ .

We get a characterization of PSPACE:

**Theorem 17** (Polynomial precision robust  $\Leftrightarrow$  PSPACE).  *$L \in \text{PSPACE}$  iff for some  $\mathcal{M}$  and some polynomial  $p$ ,  $L = L(\mathcal{M}) = L_{\{p\}}(\mathcal{M})$ .*

*Proof.* ( $\Rightarrow$ ) If  $M$  always terminates and works in polynomial space, then there exists a polynomial  $q(\cdot)$  that bounds the size of the used part of the tape of  $M$ . Considering a polynomial  $p \geq q + 2$ , we have for  $n \in \mathbb{N}$   $L_{p(\ell(w))}(\mathcal{M}) \subseteq L(M)$ . We always have the other inclusion.

( $\Leftarrow$ ) We always have  $L_{p(n)}(\mathcal{M}) \in \text{PSPACE}$  by previous lemma, and since  $L_{p(n)}(\mathcal{M}) = L$ , then  $L \in \text{PSPACE}$ .  $\square$

This considers space perturbations. Other types of perturbations are considered in Section VIII, leading to PTIME instead of PSPACE. For now, we keep to space perturbations.

#### IV. EMBEDDING DYNAMICAL SYSTEMS

Discussing issues for TMs has the advantage that related computability and complexity issues are well-known or easier to discuss. Many authors have then embedded TMs in various classes of dynamical systems in order to get hardness results, i.e. state that the difficulty of the reachability problem for the latter is at least as hard as for Turing machines.

Generally speaking, the trick is the following: if we fix the alphabet to  $\Sigma = \{0, 1\}$ , and  $Q = \{1, \dots, q\}$  for some integer  $q$ , and if we forget about blanks, we can always consider that  $\mathcal{C}_\mathcal{M} \subseteq \mathcal{C} = \mathbb{N} \times \Sigma^* \times \Sigma^*$ , i.e. that a configuration is given by some control state, and two finite words.

Fix some encoding function of configurations into a vector of real (or integer) numbers:  $\Upsilon : \mathcal{C} \rightarrow \mathbb{R}^d$ , with  $d \in \mathbb{N}$ . For example, one can consider,  $\Upsilon(q, w_1, w_2) = (q, \gamma(w_1), \gamma(w_2))$  with  $\gamma : \Sigma^* \rightarrow \mathbb{R}$  taken as:

- the encoding  $\gamma_\mathbb{N}$  that maps the word  $w = a_1 \dots a_n$  to the integer whose binary expression is  $w$ ,
- or the encoding  $\gamma_{[0,1]}$  that maps  $w$  to the real number of  $[0, 1]$  whose binary expansion is  $w$ ,
- or more generally, the encoding  $\gamma_{[0,1]}^k$  or  $\gamma_\mathbb{N}^k$ , using base  $k$  instead of base 2 for some  $k \geq 2$ ,
- or  $\gamma_{[0,1]}^k$  that maps  $w$  to  $(\gamma_{[0,1]}^k(w), \ell(w))$ .

Assume you have a function  $f : X \subseteq \mathbb{R}^d \rightarrow X$  such that for any configuration  $C$ , if we denote by  $C'$  the one step next configuration, we have  $f(\Upsilon(C)) = \Upsilon(C')$ : i.e. one step of the Turing machine corresponds to one step of the dynamical system  $(X, f)$  with respect to the encoding  $\gamma$ . That is, the following diagram commutes for one step:

$$\begin{array}{ccc} C & \xrightarrow{\quad \vdash \quad} & C' \\ \Upsilon \downarrow & & \downarrow \Upsilon \\ \Upsilon(C) & \xrightarrow{\quad f \quad} & \Upsilon(C') \end{array}$$

Then it will commutes for any number of steps:

$$\begin{array}{ccccccc} C & \xrightarrow{\quad \vdash \quad} & C' & \xrightarrow{\quad \vdash \quad} & C'' & \xrightarrow{\quad \vdash \quad} & C''' \cdots \cdots \\ \Upsilon \downarrow & & \downarrow \Upsilon & & \downarrow \Upsilon & & \downarrow \Upsilon \\ \Upsilon(C) & \xrightarrow{\quad f \quad} & \Upsilon(C') & \xrightarrow{\quad f \quad} & \Upsilon(C'') & \xrightarrow{\quad f \quad} & \Upsilon(C''') \cdots \cdots \end{array}$$

And hence, questions related to the existence of trajectories in the (dynamical system associated to the) Turing machine will be mapped to corresponding questions about the existence of trajectories over the dynamical system  $(X, f)$ .

In particular, as reachability is undecidable (c.e. complete, and hence c.e. hard) for Turing machines, this provides undecidability (c.e. hardness) of reachability for various classes of dynamical systems. As in most of the natural classes of dynamical systems, reachability is c.e. (just simulate the system to get a semi-algorithm), this leads to c.e. completeness.

Call such a situation a *step-by-step* emulation.

Such embedding strategies do provide undecidability results. But, encodings such as  $\gamma_{[0,1]}$  or  $\gamma_{[0,1]}^k$ , whose image is compact, map some intrinsically different configurations to points arbitrarily closed to each other (as a sequence over a compact must have some accumulation point). An encodings like  $\gamma_{\mathbb{N}}$  do not have a compact image, but involves emulations with arbitrarily big integers, which is another issue.

## V. DISCRETE TIME DYNAMICAL SYSTEMS

This leads to discuss now robustness issues for general dynamical systems over in  $\mathbb{R}^d$  for some  $d \in \mathbb{N}$ .

**Definition 18** (Discrete Time Dynamical System). A discrete-time dynamical system  $\mathcal{P}$  is given by a set  $X \subseteq \mathbb{R}^d$ , and some (possibly partial) function  $f$  from  $X$  to  $X$ .

The dynamical system will be called *rational* when  $f$  preserves rational numbers, i.e. whenever  $f(\mathbb{Q}^d) \subseteq \mathbb{Q}^d$ . We will say that a system is  $\mathbb{Q}$ -computable, if it is rational and  $f : \mathbb{Q}^d \rightarrow \mathbb{Q}^d$  is computable. We say that the system is (respectively: locally) *Lipschitz* when the function is.

To each rational discrete time dynamical system  $\mathcal{P}$  is associated its reachability relation  $R^{\mathcal{P}}(\cdot, \cdot)$  on  $\mathbb{Q}^d \times \mathbb{Q}^d$ . Namely, for two rational points  $\mathbf{x}$  and  $\mathbf{y}$ , the relation  $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  holds iff there exists a trajectory of  $\mathcal{P}$  from  $\mathbf{x}$  to  $\mathbf{y}$ .

### A. The case of rational systems

We first focus on the case of rational systems. Clearly the reachability relation of a  $\mathbb{Q}$ -computable system is computably enumerable: just simulate the dynamics with a TM. Actually, [1] considers only the special case of Piecewise affine maps, as representative of discrete time systems, which are particular  $\mathbb{Q}$ -computable Lipschitz systems.

**Definition 19 (PAM System).** A Piecewise affine map system (PAM) is a discrete-time dynamical system  $\mathcal{P}$  where  $f$  is a (possibly partial) function from  $X$  to  $X$  represented by a formula:  $f(\mathbf{x}) = A_i \mathbf{x} + \mathbf{b}_i$  for  $\mathbf{x} \in P_i$ ,  $i = 1 \dots N$  where  $A_i$  are rational  $d \times d$ -matrices,  $\mathbf{b}_i \in \mathbb{Q}^d$  and  $P_i$  are convex rational polyhedral sets in  $X$ .

In other words, a PAM system consists of partitioning the space into convex polyhedral sets (called *regions*), and assigning an affine update rule  $\mathbf{x} := A_i \mathbf{x} + \mathbf{b}_i$  to all the points sharing the same region.

**Remark 20.** All constants in the PAM definitions are assumed to be rational so that this remains a  $\mathbb{Q}$ -computable system. No form of continuity is assumed on function  $f$ .

The following result on the computational power of PAMs is known, and has been established using the technique of step-by-step emulation described in previous section (using  $\gamma_{[0,1]}$  and taking  $f$  as piecewise affine).

**Theorem 21** (Computational power of PAMs [27] [26]). Any c.e. language is reducible to the reachability relation of a PAM.

**Remark 22.** PAMs are introduced in [1] only for the case where  $X$  is necessarily some bounded polyhedral sets. Actually, from considered  $\gamma$ , the above result would also still hold when the regions  $P_i$  are assumed to be rational boxes.

This proves c.e.-completeness ( $\Sigma_1^0$ -completeness), and hence undecidability of reachability for  $\mathbb{Q}$ -computable systems. Let discuss whether this still holds for “robust systems”.

We can apply the paradigm of small perturbations: consider a discrete time dynamical system  $\mathcal{P}$  with function  $f$ . For any  $\varepsilon > 0$  we consider the  $\varepsilon$ -perturbed system  $\mathcal{P}_\varepsilon$ . Its trajectories are defined as sequences  $\mathbf{x}_t$  satisfying the inequality  $d(\mathbf{x}_{t+1}, f(\mathbf{x}_t)) < \varepsilon$  for all  $t$ . This non-deterministic system can be considered as  $\mathcal{P}$  submitted to a small noise with magnitude  $\varepsilon$ . For convenience, we write  $\mathbf{y} \in f_\varepsilon(\mathbf{x})$  as a synonym for  $d(f(\mathbf{x}), \mathbf{y}) < \varepsilon$ . We denote reachability in the system  $\mathcal{P}_\varepsilon$  by  $R_\varepsilon^{\mathcal{P}}(\cdot, \cdot)$ .

All trajectories of a non-perturbed system  $\mathcal{P}$  are also trajectories of the  $\varepsilon$ -perturbed system  $\mathcal{P}_\varepsilon$ . If  $\varepsilon_1 < \varepsilon_2$  then any trajectory of the  $\varepsilon_1$ -perturbed system is also a trajectory of the  $\varepsilon_2$ -perturbed PAM. Like for TMs we can pass to a limit for  $\varepsilon \rightarrow 0$ . Namely  $R_\omega^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  iff  $\forall \varepsilon > 0 R_\varepsilon^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ : this relation encodes reachability with arbitrarily small perturbing noise.

**Lemma 23** ([1, Lemma 3]). For any  $0 < \varepsilon_2 < \varepsilon_1$  and any  $\mathbf{x}$  and  $\mathbf{y}$  the following implications hold:  $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y}) \Rightarrow R_\omega^{\mathcal{P}}(\mathbf{x}, \mathbf{y}) \Rightarrow R_{\varepsilon_2}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}) \Rightarrow R_{\varepsilon_1}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ .

We prove the perturbed reachability relation of Lipschitz  $\mathbb{Q}$ -computable system is co-c.e., extending [1, Theorem 5].

**Theorem 24** (Perturbed reachability is co-c.e.). Consider a locally Lipschitz  $\mathbb{Q}$ -computable system whose domain  $X$  is a closed rational box.

Then the relation  $R_\omega^{\mathcal{P}}(\mathbf{x}, \mathbf{y}) \subseteq \mathbb{Q}^d \times \mathbb{Q}^d$  is in the class  $\Pi_1^0$ .

*Proof.* As  $f$  is locally Lipschitz, and  $X$  is compact, we know that  $f$  is Lipschitz: there exists some  $L > 0$  so that  $d(f(\mathbf{x}), f(\mathbf{y})) \leq L \cdot d(\mathbf{x}, \mathbf{y})$ .

For every  $\delta = 2^{-m}$ ,  $m \in \mathbb{N}$ , we associate some graph  $G_m = (V_\delta, \rightarrow_\delta)$ : its vertices, denoted by  $(\mathcal{V}_i)_i$ , correspond to some finite covering of compact  $X$  by rational open balls  $\mathcal{V}_i = B(\mathbf{x}_i, \delta_i)$  of radius  $\delta_i < \delta$ . There is an edge from  $\mathcal{V}_i$  to  $\mathcal{V}_j$  in this graph, that is to say  $\mathcal{V}_i \rightarrow_\delta \mathcal{V}_j$ , iff  $B(f(\mathbf{x}_i), (L+1)\delta) \cap \mathcal{V}_j \neq \emptyset$ . With our hypothesis on the domain, such a graph can be effectively obtained from  $m$ , considering a suitable discretization of the rational box  $X$ .

Claim 1: assume  $R_\epsilon^P(\mathbf{x}, \mathbf{y})$  with  $\mathbf{x} \in \mathcal{V}_i$  for  $\epsilon = 2^{-n}$ . Then  $\mathcal{V}_i \rightarrow_\epsilon \mathcal{V}_j$  for all  $\mathcal{V}_j$  with  $\mathbf{y} \in \mathcal{V}_j$ .

This basically holds as the graph for  $\delta = \epsilon$  is made to always have more trajectories/behaviours than  $R_\epsilon^P$ .

*Proof.* If  $\mathbf{y} \in \mathbf{f}_\epsilon(\mathbf{x})$ , then  $d(\mathbf{f}(\mathbf{x}_i), \mathbf{y}) \leq d(\mathbf{f}(\mathbf{x}_i), \mathbf{f}(\mathbf{x})) + d(\mathbf{f}(\mathbf{x}), \mathbf{y}) < Ld(\mathbf{x}_i, \mathbf{x}) + \epsilon \leq L\epsilon + \epsilon = (L+1)\epsilon$ , and hence there is an edge from  $\mathcal{V}_i \rightarrow_\epsilon \mathcal{V}_j$  to any  $\mathcal{V}_j$  containing  $\mathbf{y}$  by definition of the graph.  $\square$

Claim 2: for any  $\epsilon = 2^{-n}$ , there is some  $\delta = 2^{-m}$  so that if we have  $\mathcal{V}_i \rightarrow_\delta^* \mathcal{V}_j$  then  $R_\epsilon^P(\mathbf{x}, \mathbf{y})$  whenever  $\mathbf{x} \in \mathcal{V}_i, \mathbf{y} \in \mathcal{V}_j$ .

That is, Claim 2 says that  $\neg R_\epsilon^P(\mathbf{x}, \mathbf{y})$  implies  $\neg(\mathcal{V}_i \rightarrow_\delta^* \mathcal{V}_j)$  whenever  $\mathbf{x} \in \mathcal{V}_i, \mathbf{y} \in \mathcal{V}_j$ , for the corresponding  $\delta$ .

*Proof.* Consider  $\delta = 2^{-m}$  with  $\delta < \epsilon/(2L+2)$ : assume  $\mathcal{V}_{i_0} \rightarrow_\delta \mathcal{V}_{i_1} \dots \rightarrow_\delta \mathcal{V}_{i_\ell} = \mathcal{V}_j$  with  $\mathbf{x} \in \mathcal{V}_i, \mathbf{y} \in \mathcal{V}_j$ .

Assume by contradiction that  $\neg R_\epsilon^P(\mathbf{x}, \mathbf{y})$ , and let  $\ell$  be the least index such that  $\neg R_\epsilon^P(\mathbf{x}, \bar{\mathbf{z}})$  for some  $\bar{\mathbf{z}} \in \mathcal{V}_{i_{\ell+1}}$ .

As  $\mathcal{V}_{i_\ell} \rightarrow_\delta \mathcal{V}_{i_{\ell+1}}$  there is some  $\bar{\mathbf{y}} \in \mathcal{V}_{i_{\ell+1}}$  with  $d(\mathbf{f}(\mathbf{x}_{i_\ell}), \bar{\mathbf{y}}) < (L+1)\delta$ . Take  $\bar{\mathbf{z}} \in \mathcal{V}_{i_{\ell+1}}$ .

If  $\ell = 0$ , then  $d(\mathbf{f}(\mathbf{x}), \bar{\mathbf{z}}) \leq d(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}_{i_\ell})) + d(\mathbf{f}(\mathbf{x}_{i_\ell}), \bar{\mathbf{y}}) + d(\bar{\mathbf{y}}, \bar{\mathbf{z}}) < L\delta + (L+1)\delta + \delta = (2L+2)\delta < \epsilon$ , and hence  $R_\epsilon^P(\mathbf{x}, \bar{\mathbf{z}})$ : contradiction.

If  $\ell > 0$ , as  $\ell$  is the least index with the above property,  $R_\epsilon^P(\mathbf{x}, \mathbf{x}_{i_\ell})$ . But then  $d(\mathbf{f}(\mathbf{x}_{i_\ell}), \bar{\mathbf{z}}) \leq d(\mathbf{f}(\mathbf{x}_{i_\ell}), \bar{\mathbf{y}}) + d(\bar{\mathbf{y}}, \bar{\mathbf{z}}) < (L+1)\delta + \delta < (2L+2)\delta < \epsilon$ . And hence,  $R_\epsilon^P(\mathbf{x}_{i_\ell}, \bar{\mathbf{z}})$ , and since we have  $R_\epsilon^P(\mathbf{x}, \mathbf{x}_{i_\ell})$ , we get  $R_\epsilon^P(\mathbf{x}, \bar{\mathbf{z}})$  and a contradiction.  $\square$

From the two above items,  $\neg R_\omega^P(\mathbf{x}, \mathbf{y})$  holds iff for some  $\delta = 2^{-m}$ ,  $\neg(\mathcal{V}_i \rightarrow_\delta^* \mathcal{V}_j)$  for some  $\mathcal{V}_i, \mathcal{V}_j$  with  $\mathbf{x} \in \mathcal{V}_i, \mathbf{y} \in \mathcal{V}_j$ . This holds iff for some integer  $m$ ,  $\text{NOPATH}(G_m, \mathcal{V}_i, \mathcal{V}_j)$  for some  $\mathcal{V}_i, \mathcal{V}_j$  with  $\mathbf{x} \in \mathcal{V}_i, \mathbf{y} \in \mathcal{V}_j$ .

The latter property is c.e., as it corresponds to a union of decidable sets (uniform in  $m$ ), as  $\text{NOPATH}(G_m, \mathcal{V}_i, \mathcal{V}_j)$  is a decidable property over finite graph  $G_m$ .  $\square$

Notice this would work even only assuming the domain to be a computable compact: we will recall later what it is (using computable analysis), but let's say for now that the above proof only requires that given  $\delta = 2^{-m}$ , there is an effective way to determine an effective cover of it using finitely many rational balls of radius less than  $\delta$ .

**Corollary 25** (Robust  $\Rightarrow$  decidable [1, Corollary 5]). *Assume the hypotheses of Theorem 24.*

*If  $R_\omega^P = R^P$  then  $R^P$  is decidable.*

*Proof.*  $R^P$  is c.e. and we know from Theorem 24 that  $R_\omega^P$  is co-c.e.. If they are equal, then they are decidable, as a c.e. and co-c.e. set is decidable.  $\square$

**Remark 26.** *Notice that a similar statement holds even if  $X$  is not compact: from the proof, it is sufficient that there exists some family of graphs  $\mathcal{G} = (G_m)$  with  $G_m = (V_m, \rightarrow_m)$  to get a similar reasoning with the following properties:*

- 1)  $R_\epsilon^P(\mathbf{x}, \mathbf{y})$  with  $\mathbf{x} \in \mathcal{V}_i, \epsilon = 2^{-n}$ , implies  $\mathcal{V}_i \rightarrow_n \mathcal{V}_j$  for all  $\mathcal{V}_j$  containing  $\mathbf{y}$ .
- 2) For any  $\epsilon = 2^{-n}$ , there is some  $m$  such that if we have  $\mathcal{V}_i \rightarrow_m^* \mathcal{V}_j$  then  $R_\epsilon^P(\mathbf{x}, \mathbf{y})$  whenever  $\mathbf{x} \in \mathcal{V}_i, \mathbf{y} \in \mathcal{V}_j$ .

- 3) For all  $m$ ,  $G_m$  is a finite computable graph: determining whether  $\mathcal{V}_i \rightarrow_m \mathcal{V}_j$  in  $G_m$  can be effectively determined given integers  $m, i$ , and  $j$ .

When these three properties hold, we say that  $\mathcal{G}$  is a computable abstraction of the discrete time dynamical system.

There is a kind of converse property if some condition is added. Before stated this as Corollary 31, we relate robustness to the concept of  $\delta$ -decidability in [16], and also the existence of some witness of non-reachability.

Given  $\mathbf{x}$ , we write  $R^P(\mathbf{x})$  for the set of the points  $\mathbf{y}$  reachable from  $\mathbf{x}$ :  $R^P(\mathbf{x}) = \{\mathbf{y} | R^P(\mathbf{x}, \mathbf{y})\}$ . This is easily seen to also corresponds the smallest set such that  $\mathbf{x} \in R^P(\mathbf{x})$  and  $\mathbf{f}(R^P(\mathbf{x})) \subseteq R^P(\mathbf{x})$ .

We say that  $R^P(\mathbf{x}, \mathbf{y})$  is  $\epsilon$ -far from being true, if there is  $\mathcal{R}^* \subseteq X$  so that

- 1)  $\mathbf{x} \in \mathcal{R}^*$ ,
- 2)  $\mathbf{f}_\epsilon(\mathcal{R}^*) \subseteq \mathcal{R}^*$ ,
- 3)  $\mathbf{y} \notin \mathcal{R}^*$ .

When this holds, necessarily  $\neg R^P(\mathbf{x}, \mathbf{y})$ : indeed, for all  $\epsilon > 0$ , the set  $R_\epsilon^P(\mathbf{x}) = \{\mathbf{y} | R_\epsilon^P(\mathbf{x}, \mathbf{y})\}$  is the smallest set that satisfies  $\mathbf{x} \in R_\epsilon^P(\mathbf{x})$  and  $\mathbf{f}_\epsilon(R_\epsilon^P(\mathbf{x})) \subseteq R_\epsilon^P(\mathbf{x})$ . Consequently, as  $\mathcal{R}^*$  also satisfies these properties by the first two items,  $R_\epsilon^P(\mathbf{x}) \subseteq \mathcal{R}^*$ , and hence  $\mathbf{y} \notin R^P(\mathbf{x})$  as  $R^P(\mathbf{x}) \subseteq R_\epsilon^P(\mathbf{x}) \subseteq \mathcal{R}^*$  and  $\mathbf{y} \notin \mathcal{R}^*$  from the third item.

In other words,  $\mathcal{R}^*$  can be seen as a *witness* of the non-reachability of  $\mathbf{y}$  from  $\mathbf{x}$ . We will say that it is *at level  $\epsilon$* .

This provides a link to  $\delta$ -decidability [16]:

**Proposition 27** (Robust  $\Leftrightarrow$  reachability is true or  $\epsilon$ -far from being true). *We have  $R_\omega^P = R^P$  if and only if for all  $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$ , either*

- 1)  $R^P(\mathbf{x}, \mathbf{y})$  is true
- 2) or  $R^P(\mathbf{x}, \mathbf{y})$  is false, but furthermore, there exists  $\epsilon > 0$  such that it is  $\epsilon$ -far from being true.  
(i.e. there is a witness of it for some  $\epsilon > 0$  level).

*Proof.  $\Rightarrow$ :* As we said, for all  $\epsilon > 0$ , the set  $R_\epsilon^P(\mathbf{x})$  satisfies  $\mathbf{x} \in R_\epsilon^P(\mathbf{x})$  and  $\mathbf{f}_\epsilon(R_\epsilon^P(\mathbf{x})) \subseteq R_\epsilon^P(\mathbf{x})$  (this is even the smallest set such that this holds).

Let  $\mathbf{y} \in \mathbb{Q}^d$ , let us assume that  $R^P(\mathbf{x}, \mathbf{y}) = \bigcap_\epsilon R_\omega^P(\mathbf{x}, \mathbf{y})$  is not true. Then there exists  $\epsilon$  such that  $R_\epsilon^P(\mathbf{x}, \mathbf{y})$  is false, i.e.  $\mathbf{y} \notin R_\epsilon^P(\mathbf{x})$ . Consider  $\mathcal{R}^* = R_\epsilon^P(\mathbf{x})$ . Then,  $\mathbf{x} \in \mathcal{R}^*$  and from the first paragraph  $\mathbf{f}_\epsilon(\mathcal{R}^*) \subseteq \mathcal{R}^*$  and  $\mathbf{y} \notin \mathcal{R}^*$ .

$\Leftarrow$ : When  $R^P(\mathbf{x}, \mathbf{y})$  is true, then for all  $\epsilon > 0$ ,  $R_\epsilon^P(\mathbf{x}, \mathbf{y})$  is true, so  $R_\omega^P(\mathbf{x}, \mathbf{y})$  is. Now, when  $R^P(\mathbf{x}, \mathbf{y})$  is false, we know by hypothesis that  $R^P(\mathbf{x}, \mathbf{y})$  is  $\epsilon$ -far from being true for some  $\epsilon > 0$ : there exists a set  $\mathcal{R}^*$  satisfying  $\mathbf{x} \in \mathcal{R}^*$  and  $\mathbf{f}_\epsilon(\mathcal{R}^*) \subseteq \mathcal{R}^*$ . As  $R_\epsilon^P(\mathbf{x})$  is the smallest such set,  $R_\epsilon^P(\mathbf{x}) \subseteq \mathcal{R}^*$ . Now, as  $\mathbf{y} \notin \mathcal{R}^*$ , necessarily  $\mathbf{y} \notin R_\epsilon^P(\mathbf{x})$ . Hence  $R_\omega^P(\mathbf{x}, \mathbf{y})$  is false.  $\square$

We say that a subset  $\mathcal{R}^*$  of  $X$  is  $\epsilon$ -rejecting if it satisfies 2) and 3): that is to say,  $\mathbf{f}_\epsilon(\mathcal{R}^*) \subseteq \mathcal{R}^*$ , and  $\mathbf{y} \notin \mathcal{R}^*$ . A trajectory that reaches such a  $\mathcal{R}^*$  will never leave it. A system is *eventually decisional* if for all  $\mathbf{x}, \mathbf{y}$ , there is some  $\mathcal{R}^*$   $\epsilon$ -rejecting so that either the trajectory starting from  $\mathbf{x}$  reaches  $\mathbf{y}$ , or when not, it reaches  $\mathcal{R}^*$ .



We come back to the converse of Corollary 25: from Proposition 27, a robust dynamical system (i.e.  $R_\omega^P = R^P$ ) is eventually decisional, by considering  $R^* = \mathcal{R}^*$  for the  $\mathcal{R}^*$  given by item 2) there. Conversely:

**Lemma 28.** *Consider a rational system not robust, with  $\mathbf{f}$  continuous or Lipschitz: as  $R^P \subseteq R_\omega^P$ , this means that there exist some  $\mathbf{x}$  and  $\mathbf{y}$  with  $R_\omega^P(\mathbf{x}, \mathbf{y})$  but not  $R^P(\mathbf{x}, \mathbf{y})$ . The trajectory starting from  $\mathbf{x}$  can not reach any  $\epsilon$ -rejecting subset.*

*Proof.* By contradiction, assume the trajectory starting from  $\mathbf{x}$  reaches some  $\epsilon$ -rejecting  $R^*$ . Possibly by considering one more step, we can assume it reaches the interior of  $R^*$  for the first time at time  $t$ : indeed, if it reaches the frontier at  $\mathbf{x}^*$ , then we know that  $B(\mathbf{f}(\mathbf{x}^*), \epsilon) \subseteq R^*$ , and  $\mathbf{f}(\mathbf{x}^*)$  is in the interior of that ball. Now, from initial  $\mathbf{x}$  until the position at time  $t$ , it remains at some positive distance of  $\mathbf{y}$ . As  $\mathbf{f}$  is continuous or Lipschitz, the  $t$ -th iteration of  $\mathbf{f}$  also is. So there is some  $0 < \epsilon' < \epsilon$  taken sufficiently small so that  $R_{\epsilon'}^P$  intersects the interior of  $R^*$ , and remains at a positive distance of  $\mathbf{y}$ . Once in  $R^*$ ,  $\epsilon'$ -perturbed trajectories will remain in it, since we have  $\epsilon' < \epsilon$ . We get  $\mathbf{y} \notin R_{\epsilon'}^P$ , and consequently  $\neg R_\omega^P(\mathbf{x}, \mathbf{y})$ : contradiction.  $\square$

**Corollary 29.** *Consider a continuous or Lipschitz rational dynamical system. It is robust iff it is eventually decisional.*

We can even compute the witnesses under the hypotheses of Theorem 24. We say that some dynamical system is *effectively eventually decisional* when there is an algorithm such that, given  $\mathbf{x}$  and  $\mathbf{y}$ , it (terminates and) outputs such a  $R^*$  in the form of the union of rational balls.

**Proposition 30** (Reinforcement of Corollary 25). *Assume the hypotheses of Theorem 24. If  $R_\omega^P = R^P$  then  $R^P$  is computable, and the system is effectively eventually decisional.*

*Proof.* The proof of Theorem 24 shows that when  $R_\omega^P(\mathbf{x}, \mathbf{y})$  is false, then  $R_\epsilon^P(\mathbf{x}, \mathbf{y})$  is false for some  $\epsilon = 2^{-n}$ , and there is some  $\delta = 2^{-m}$ , with some graph  $G_m$  with some vertices  $\mathcal{V}_i$  and  $\mathcal{V}_j$  with  $\mathbf{x} \in \mathcal{V}_i$ ,  $\mathbf{y} \in \mathcal{V}_j$  and  $\neg(\mathcal{V}_i \xrightarrow{*}_\delta \mathcal{V}_j)$ . Denote by  $R^{G_m}$  the union of the vertices  $\mathcal{V}_k$  such that  $\mathcal{V}_i \xrightarrow{*}_\delta \mathcal{V}_k$ , for  $\mathbf{x} \in \mathcal{V}_i$  in that graph. Consider  $\mathcal{R}^* = R^{G_m}$ . This constitutes a witness at level  $\delta = 2^{-m}$  from the properties of the construction in that proof.

Then  $m$  can be found by testing increasing  $m$  until a graph with the above properties is found. The corresponding  $\mathcal{R}^* = R^{G_m}$  for the first graph found will be a witness at level  $\delta = 2^{-m}$  from above arguments.  $\square$

An effectively eventually decisional system has its reachability relation necessarily decidable (given  $\mathbf{x}$  and  $\mathbf{y}$  compute the path until it reaches  $\mathbf{y}$  (then accept), or  $R^*$  (then reject)):

**Corollary 31** (Decidable  $\Leftrightarrow$  robust, for eventually decisional systems). *Under the hypotheses of Theorem 24,  $R_\omega^P = R^P$  iff  $R^P$  is decidable and it is effectively eventually decisional iff it is effectively eventually decisional.*

We now go to complexity issues.

Assume the dynamical system is robust, i.e.  $R^P = R_\omega^P$ . That means that for all rationals  $\mathbf{x}, \mathbf{y}$ , we have  $R^P(\mathbf{x}, \mathbf{y}) \Leftrightarrow R_\omega^P(\mathbf{x}, \mathbf{y})$ . Consequently, for all rationals  $\mathbf{x}, \mathbf{y}$ , there exists some  $\epsilon$  (depending of  $\mathbf{x}, \mathbf{y}$ ) such that  $R^P(\mathbf{x}, \mathbf{y})$  and  $R_\epsilon^P(\mathbf{x}, \mathbf{y})$  have the same truth value (and unchanged by smaller  $\epsilon$ ).

It is then natural to quantify on the level of required robustness according to  $\mathbf{x}$  and  $\mathbf{y}$ , i.e. on the value  $\epsilon$ .

As we may always assume  $\epsilon = 2^{-n}$  for some integer  $n$ , we write  $R_n^P$  for  $R_{2^{-n}}^P$ , and we then introduce:

**Definition 32.** *Given some function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $R_{\{f\}}^P$  for the relation defined as follows: for any rational points  $\mathbf{x}$  and  $\mathbf{y}$  the relation holds iff  $R_{f(\ell(\mathbf{x})+\ell(\mathbf{y}))}^P(\mathbf{x}, \mathbf{y})$ .*

**Lemma 33.** *Consider a locally Lipschitz  $\mathbb{Q}$ -computable system, with  $\mathbf{f} : \mathbb{Q} \rightarrow \mathbb{Q}$  computable in polynomial time, whose domain  $X$  is a closed rational box. For  $\delta = 2^{-m}$ , consider the associated graph  $G_m$  considered in the proof of Theorem 24. Then  $\text{NOPATH}(G_m, \mathcal{V}_i, \mathcal{V}_j)$  is decidable using a space polynomial in  $m$ .*

*Proof.* This graph has less than  $\mathcal{O}(2^{d*m})$  vertices. The graph has a successor relation  $\rightarrow_\delta$  computable in space polynomial in  $m$ . Hence, the analysis of Corollary 8 applies, and we can determine whether  $\text{NOPATH}(G_m, \mathcal{V}_i, \mathcal{V}_j)$  using a space polynomial in  $m$ .  $\square$

**Theorem 34.** *Assume the hypotheses of Lemma 33. Assume  $p$  is some polynomial. Then  $R_{\{p\}}^P \in \text{PSPACE}$ .*

*Proof.* From the proof of Theorem 24, we know that for all  $n$  there exists some  $m$  (depending on  $n$ ), such that  $R_n^P(\mathbf{x}, \mathbf{y})$  and  $R^{G_m}(\mathbf{x}, \mathbf{y})$  have the same truth value, where  $R^{G_m}$  denotes reachability in the graph  $G_m$ .

With the hypotheses, given  $\mathbf{x}$  and  $\mathbf{y}$ , we can determine whether  $R_{\{p\}}^P(\mathbf{x}, \mathbf{y})$ , by determining the truth value of  $R_n^P(\mathbf{x}, \mathbf{y})$ , taking  $n$  polynomial in  $\ell(\mathbf{x}) + \ell(\mathbf{y})$ . From the proof of Theorem 24, the corresponding  $m$  is polynomially related to  $n$  (it is even affine in  $n$ ). Now the analysis of Lemma 33, shows that the truth value of  $R^{G_m}(\mathbf{x}, \mathbf{y})$  can be determined in space polynomial in  $m$ .  $\square$

**Theorem 35** (Polynomially robust to precision  $\Rightarrow$  PSPACE). *With same hypotheses, if  $R^P = R_{\{p\}}^P$  for some polynomial  $p$ , then  $R^P \in \text{PSPACE}$ .*

*Proof.* We have  $R_{\{p\}}^P \in \text{PSPACE}$  by Theorem (34) and since  $R^P = R_{\{p\}}^P$ , then  $R^P \in \text{PSPACE}$ .  $\square$

Actually, this is even a characterization of PSPACE (if one prefers: Reachability is PSPACE-complete for  $\mathbb{Q}$ -computable poly-time computational, polynomial robust to precision).

**Theorem 36** (Polynomially robust to precision  $\Leftrightarrow$  PSPACE). *Any PSPACE language is reducible to the reachability relation of PAM with  $R^P = R_{\{p\}}^P$  for some polynomial  $p$ .*

*Proof.* Let  $L \in \text{PSPACE}$ . There is a TM  $\mathcal{M}$  with  $L(\mathcal{M}) = L$  that works in polynomial space  $q(\cdot)$ . Its step-by-step emulation considered in Theorem 21, using  $\gamma_{[0,1]}$  is done using a precision  $\mathcal{O}(2^{-q(n)})$  on words of length  $n$ . The obtained



system satisfies  $R^{\mathcal{P}}_{\{q+\mathcal{O}(1)\}} = R^{\mathcal{P}}$  from the properties of the emulation. In other words, this comes from the fact that the involved emulation preserves robustness of TMs.  $\square$

Assuming the same hypotheses as in Theorem 35, when  $R^{\mathcal{P}} = R^{\mathcal{P}}_{\{p\}}$  for some polynomial  $p$ , we also see that we can determine a witness of the fact that  $\neg R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  in polynomial space (using a suitable representation of it).

### B. The case of computable systems

We consider now the case of general (possibly non-rational) discrete time dynamical systems. In that case,  $\mathbf{f}$  may take some non-rational values, and we need to talk about computability for functions over the reals. A system is said computable if the function  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is in computable analysis (CA).

A crash course on CA can be found in the appendix (see e.g. [34] or [11]), but in very short: a name for a point  $\mathbf{x} \in \mathbb{R}^d$  is a sequence  $(I_n)$  of nested open rational balls with  $I_{n+1} \subseteq I_n$  for all  $n \in \mathbb{N}$  and  $\{\mathbf{x}\} = \bigcap_{n \in \mathbb{N}} I_n$ . A name for a function  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$  is a list of all pairs of open rational balls  $(I, J)$  such that  $\mathbf{f}(\text{cls}(I)) \subseteq J$ . A name for a closed set  $F$  is a sequence  $(I_n)$  of open rational balls such that  $\text{cls}(I_n) \cap F = \emptyset$  and a sequence  $(J_n)$  of open rational balls such that  $J_n \cap F \neq \emptyset$ . A name for a compact  $K$  is a name of  $F$  as a closed set, and an integer  $L$  such that  $K \subseteq B(0, L)$ . All these names can be encoded as infinite sequences of symbols. The notion of computability involved is the one of Type 2 Turing machines, that is to say machines possibly working over infinite tapes, and outputting their results in possibly write-only output tapes. Then: a point  $\mathbf{x} \in \mathbb{R}^d$  is computable if it has a computable name. And similarly for defining the concept of computable function, computable closed set, or computable compact: we mean for example, that a closed set is computable if it has a computable name (and a compact is computable consequently also as a closed set). In particular this concept of computability for compacts implies the property discussed after Theorem 24 (see [34, Lemma 5.2.5] for a proof). If  $Y$  and  $Z$  are spaces with an associated naming system, then an operator  $f : Y \rightarrow Z$  is said computable if there is a computable function which associates each name of  $y \in Y$  to a name of  $f(y) \in Z$ .

From the model of CA, given the name<sup>2</sup> of  $\mathbf{f}$ , and (even for) some rational  $\mathbf{x}$  and  $\mathbf{y}$ , this is impossible to tell effectively if  $\mathbf{f}(\mathbf{x}) = \mathbf{y}$  in the general case. Consequently, given some rational ball  $B(\mathbf{y}, \delta)$ , we have to forbid “frontier reachability”, that is to say the case where  $B(\mathbf{y}, \delta)$  would not be reachable, but its frontier is, i.e.  $\overline{B}(\mathbf{y}, \delta) - B(\mathbf{y}, \delta)$  is reachable. A natural question arises then: given some ball such that either  $B(\mathbf{y}, \delta)$  is reachable (that case implies that  $\overline{B}(\mathbf{y}, \delta)$  is), or such that  $\overline{B}(\mathbf{y}, \delta)$  is not, decide which possibility holds. We call this the *ball (decision) problem*. Of course, from the above definitions, when  $R^{\mathcal{P}}(\mathbf{x})$  is a closed set,  $R^{\mathcal{P}}(\mathbf{x})$  is a computable closed set iff the associated ball problem is algorithmically solvable.

For a computable system, the ball decision problem is c.e.: simulate the evolution of the system starting from  $\mathbf{x}$  until step  $T$ , with increasing precision and  $T$ , until one finds the

guarantee that the position  $\mathbf{x}_T$  at time  $T$  remains in  $B(\mathbf{y}, \delta')$  for some  $\delta' < \delta$ . This works: if the ball is indeed reachable, this will terminate by eventually computing a sufficient approximation of the corresponding  $\mathbf{x}_T$ , and conversely it can't terminate without guaranteeing reachability. The ball problem is of course not co-c.e. in general.

To a discrete time system, we can also associate its reachability relation  $R^{\mathcal{P}}(\cdot, \cdot, \cdot)$  over  $\mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N}$ . Namely, for two rational points  $\mathbf{x}$  and  $\mathbf{y}$  and rational  $0 < \eta = 2^{-p}$ , encoded by the integer  $p$ , the relation  $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, p)$  holds iff there exists a trajectory of  $\mathcal{P}$  from  $\mathbf{x}$  to  $\overline{B}(\mathbf{y}, \eta)$ . We define  $R_{\epsilon}^{\mathcal{P}}$  similarly, and  $R_{\omega}^{\mathcal{P}} = \bigcap_{\epsilon} R_{\epsilon}^{\mathcal{P}}$ . This relation encodes reachability with arbitrarily small perturbing noise to some closed ball.

**Lemma 37.** *for any  $0 < \epsilon_2 < \epsilon_1$  and any  $\mathbf{x}$  and  $\mathbf{y}$ ,  $\eta$ , the following implications hold:  $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, p) \Rightarrow R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, p) \Rightarrow R_{\epsilon_2}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, p) \Rightarrow R_{\epsilon_1}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, p)$ .*

Given  $\mathbf{x}$ , and  $0 < \epsilon_2 < \epsilon_1$ , we have  $R^{\mathcal{P}}(\mathbf{x}) \subseteq R_{\epsilon_2}^{\mathcal{P}}(\mathbf{x}) \subseteq \text{cls}(R_{\epsilon_2}^{\mathcal{P}}(\mathbf{x})) \subseteq R_{\epsilon_1}^{\mathcal{P}}(\mathbf{x}) \subseteq \text{cls}(R_{\epsilon_1}^{\mathcal{P}}(\mathbf{x}))$ . Consequently,  $R_{\omega}^{\mathcal{P}}(\mathbf{x}) = \bigcap_{\epsilon > 0} R_{\epsilon}^{\mathcal{P}}(\mathbf{x}) = \bigcap_{\epsilon > 0} \text{cls}(R_{\epsilon}^{\mathcal{P}}(\mathbf{x}))$  is a closed set.

**Theorem 38** (Perturbed reachability is co-r.e.). *Consider a locally Lipschitz computable system whose domain  $X$  is a computable compact.  $R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, p) \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N}$  is in  $\Pi_1^0$ .*

This statement have similarities with [8, Theorem 13]: the result established there is about the language accepted by a system, but with very strong hypotheses on termination compared to ours which make their analysis really simpler.

*Proof.* As  $\mathbf{f}$  is locally Lipschitz, and  $X$  is compact, we know that  $\mathbf{f}$  is Lipschitz: there exists some  $L > 0$  so that  $d(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})) \leq L \cdot d(\mathbf{x}, \mathbf{y})$ .

For every  $\delta = 2^{-m}$ ,  $m \in \mathbb{N}$ , we associate some graph  $G_m = (V_{\delta}, \rightarrow_{\delta})$ : the vertices, denoted  $(\mathcal{V}_i)_i$ , of this graph correspond to some finite covering of compact  $X$  by rational open balls  $\mathcal{V}_i = B(\mathbf{x}_i, \delta_i)$  of radius  $\delta_i < \delta$ .

There is an edge from  $\mathcal{V}_i$  to  $\mathcal{V}_j$  in this graph, that is to say  $\mathcal{V}_i \rightarrow_{\delta} \mathcal{V}_j$ , iff  $B(\mathbf{f}_i, (L+2)\delta) \cap \mathcal{V}_j \neq \emptyset$ , given some rational  $\mathbf{f}_i$  given by some (computed)  $\delta$ -approximation of  $\mathbf{f}(\mathbf{x}_i)$ , i.e.  $\mathbf{f}_i$  such that  $\mathbf{f}(\mathbf{x}_i) \in B(\mathbf{f}_i, \delta)$ .

This is done to guarantee to cover  $B(\mathbf{f}(\mathbf{x}_i), (L+1)\delta)$ .

As we assumed compact  $X$  to be computable, such a graph can be effectively obtained from  $m$ , by computing suitable approximation  $\mathbf{f}_i$  of the  $\mathbf{f}(\mathbf{x}_i)$ 's at precision  $\delta$ .

We write, as expected,  $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  if there is a trajectory from  $\mathbf{x}$  to  $\mathbf{y}$ , allowing  $\mathbf{y}$  to be some real point (and similarly for  $R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ ).

- Claim 1: assume  $R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  with  $\mathbf{x} \in \mathcal{V}_i$  for  $\epsilon = 2^{-n}$ . Then  $\mathcal{V}_i \rightarrow_{\epsilon} \mathcal{V}_j$  for all  $\mathcal{V}_j$  with  $\mathbf{y} \in \mathcal{V}_j$ . This basically holds as the graph for  $\delta = \epsilon$  is made to always have more trajectories/behaviours than  $R_{\epsilon}^{\mathcal{P}}$ .

*Proof.* If  $\mathbf{y} \in \mathbf{f}_{\epsilon}(\mathbf{x})$ , then  $d(\mathbf{f}_i, \mathbf{y}) \leq d(\mathbf{f}_i, \mathbf{f}(\mathbf{x}_i)) + d(\mathbf{f}(\mathbf{x}_i), \mathbf{f}(\mathbf{x})) + d(\mathbf{f}(\mathbf{x}), \mathbf{y}) < \epsilon + Ld(\mathbf{x}_i, \mathbf{x}) + \epsilon = (L+2)\epsilon$ , and hence there is an edge from  $\mathcal{V}_i \rightarrow_{\epsilon} \mathcal{V}_j$  to any  $\mathcal{V}_j$  containing  $\mathbf{y}$  by definition of the graph.  $\square$

<sup>2</sup>Even assuming it is computable.

- Claim 2: for any  $\epsilon = 2^{-n}$ , there is some  $\delta = 2^{-m}$  so that if we have  $\mathcal{V}_i \xrightarrow{*}_{\delta} \mathcal{V}_j$  then  $R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  whenever  $\mathbf{x} \in \mathcal{V}_i$ ,  $\mathbf{y} \in \mathcal{V}_j$ .

*Proof.* Consider  $\delta = 2^{-m}$  with  $\delta < \epsilon/(2L+4)$ : assume  $\mathcal{V}_{i=i_0} \rightarrow_{\delta} \mathcal{V}_{i_1} \dots \rightarrow_{\delta} \mathcal{V}_{i_\ell=j}$  with  $\mathbf{x} \in \mathcal{V}_i$ ,  $\mathbf{y} \in \mathcal{V}_j$ .

Assume by contradiction that  $\neg R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ , and let  $\ell$  be the least index such that  $\neg R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \bar{\mathbf{z}})$  for some  $\bar{\mathbf{z}} \in \mathcal{V}_{i_{\ell+1}}$ . As  $\mathcal{V}_{i_\ell} \rightarrow_{\delta} \mathcal{V}_{i_{\ell+1}}$  there is some  $\bar{\mathbf{y}} \in \mathcal{V}_{i_{\ell+1}}$  with  $d(\mathbf{f}_{i_\ell}, \bar{\mathbf{y}}) < (L+2)\delta$  with  $d(\mathbf{f}_{i_\ell}, \mathbf{f}(x_{i_\ell})) < \delta$ . Take  $\bar{\mathbf{z}} \in \mathcal{V}_{i_{\ell+1}}$ .

If  $\ell = 0$ , then  $d(\mathbf{f}(\mathbf{x}), \bar{\mathbf{z}}) \leq d(\mathbf{f}(\mathbf{x}), \mathbf{f}(x_{i_\ell})) + d(\mathbf{f}(x_{i_\ell}), \mathbf{f}_{i_\ell}) + d(\mathbf{f}_{i_\ell}, \bar{\mathbf{y}}) + d(\bar{\mathbf{y}}, \bar{\mathbf{z}}) < L\delta + \delta + (L+2)\delta + \delta = (2L+4)\delta < \epsilon$ , and hence  $R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \bar{\mathbf{z}})$ : contradiction.

If  $\ell > 0$ , as  $\ell$  is the least index with the above property,  $R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, x_{i_\ell})$ . But then  $d(\mathbf{f}(x_{i_\ell}), \bar{\mathbf{z}}) \leq d(\mathbf{f}(x_{i_\ell}), \mathbf{f}_{i_\ell}) + d(\mathbf{f}_{i_\ell}, \bar{\mathbf{y}}) + d(\bar{\mathbf{y}}, \bar{\mathbf{z}}) < \delta + (L+2)\delta + \delta < (2L+4)\delta < \epsilon$ . And hence,  $R_{\epsilon}^{\mathcal{P}}(x_{i_\ell}, \bar{\mathbf{z}})$ , and since we have  $R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, x_{i_\ell})$ , we get  $R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \bar{\mathbf{z}})$  and a contradiction.  $\square$

That is, Claim 2 says that  $\neg R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  implies  $\neg(\mathcal{V}_i \xrightarrow{*}_{\delta} \mathcal{V}_j)$  whenever  $\mathbf{x} \in \mathcal{V}_i$ ,  $\mathbf{y} \in \mathcal{V}_j$ , for the corresponding  $\delta$ .

From the two above items,  $R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  holds iff for all  $\delta = 2^{-m}$ , we have  $\mathcal{V}_i \xrightarrow{*}_{\delta} \mathcal{V}_j$ , for all  $\mathcal{V}_i, \mathcal{V}_j$  with  $\mathbf{x} \in \mathcal{V}_i$ ,  $\mathbf{y} \in \mathcal{V}_j$ .

If one prefers,  $\neg R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  holds iff for some  $\delta = 2^{-m}$ ,  $\neg(\mathcal{V}_i \xrightarrow{*}_{\delta} \mathcal{V}_j)$  for some  $\mathcal{V}_i, \mathcal{V}_j$  with  $\mathbf{x} \in \mathcal{V}_i$ ,  $\mathbf{y} \in \mathcal{V}_j$ .

Then:

- Claim\* (compactness argument): Given a ball  $B(\mathbf{y}, \eta)$ , we have that  $\bar{B}(\mathbf{y}, \eta) \cap R_{\omega}^{\mathcal{P}}(\mathbf{x}) = \emptyset$  iff  $\bar{B}(\mathbf{y}, \eta) \cap \text{cls}(R_{\epsilon}^{\mathcal{P}}(\mathbf{x})) = \emptyset$  for some  $\epsilon > 0$ .

*Proof.*  $\Leftarrow$  (easy direction): If  $\bar{B}(\mathbf{y}, \eta) \cap R_{\epsilon}^{\mathcal{P}}(\mathbf{x}) = \emptyset$  for some  $\epsilon > 0$ , we cannot have  $\bar{B}(\mathbf{y}, \eta) \cap R_{\omega}^{\mathcal{P}}(\mathbf{x}) \neq \emptyset$ , as it would contain a point that would necessarily be in  $\bar{B}(\mathbf{y}, \eta) \cap R_{\epsilon}^{\mathcal{P}}(\mathbf{x})$ .

$\Rightarrow$  (compactness argument): Assume  $\bar{B}(\mathbf{y}, \eta) \cap R_{\omega}^{\mathcal{P}}(\mathbf{x}) = \emptyset$ . Since  $R_{\omega}^{\mathcal{P}}(\mathbf{x}) = \bigcap_{\epsilon>0} \text{cls}(R_{\epsilon}^{\mathcal{P}}(\mathbf{x}))$ , this means that  $\bigcup_{n \in \mathbb{N}} (\text{cls}(R_{2^{-n}}^{\mathcal{P}}(\mathbf{x})))^c$  is some covering of  $\bar{B}(\mathbf{y}, \eta)$ . As  $\bar{B}(\mathbf{y}, \eta)$  is closed and bounded, it is compact. Consequently, from the covering can be extracted some finite covering. Consequently,  $\bigcup_{n \neq n_0} (\text{cls}(R_{2^{-n}}^{\mathcal{P}}(\mathbf{x})))^c = (\text{cls}(R_{2^{-n_0}}^{\mathcal{P}}(\mathbf{x})))^c$  for some  $n_0$  is a covering of  $\bar{B}(\mathbf{y}, \eta)$ . In other words,  $\bar{B}(\mathbf{y}, \eta) \cap \text{cls}(R_{\epsilon}^{\mathcal{P}}(\mathbf{x})) = \emptyset$  for  $\epsilon = 2^{-n_0}$ . This proves the direction from left to right.  $\square$

Consequently, we basically can use arguments really similar to those of the proof of Theorem 24, where the role played by  $\mathbf{y}$  is now played by  $\bar{B}(\mathbf{y}, \eta)$ . With more details:

$R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, \eta)$  holds iff for all  $\delta = 2^{-m}$ , we have  $\mathcal{V}_i \xrightarrow{*}_{\delta} \mathcal{V}_j$ , for all  $\mathcal{V}_i, \mathcal{V}_j$  with  $\mathbf{x} \in \mathcal{V}_i$ ,  $\mathcal{V}_j \cap \bar{B}(\mathbf{y}, 2^{-p}) \neq \emptyset$ .

The direction from left to right is clear from Claim 1. Conversely, assume that for all  $\delta = 2^{-m}$ , we have  $\mathcal{V}_i \xrightarrow{*}_{\delta} \mathcal{V}_j$ , for all  $\mathcal{V}_i, \mathcal{V}_j$  with  $\mathbf{x} \in \mathcal{V}_i$ ,  $\mathcal{V}_j \cap \bar{B}(\mathbf{y}, 2^{-p}) \neq \emptyset$ . Assume by contradiction that  $\neg R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, \eta)$ . From Claim\*, we know that  $\bar{B}(\mathbf{y}, \eta) \cap \text{cls}(R_{\epsilon}^{\mathcal{P}}(\mathbf{x})) = \emptyset$  for some  $\epsilon > 0$ . In particular  $\bar{B}(\mathbf{y}, \eta) \cap R_{\epsilon}^{\mathcal{P}}(\mathbf{x}) = \emptyset$ . Then for the corresponding  $\delta = 2^{-m}$  from Claim 2, we cannot have  $\mathcal{V}_i \xrightarrow{*}_{\delta} \mathcal{V}_j$ , for any  $\mathcal{V}_i, \mathcal{V}_j$  with

$\mathbf{x} \in \mathcal{V}_i$ ,  $\mathcal{V}_j \cap \bar{B}(\mathbf{y}, 2^{-p}) \neq \emptyset$ . This proves the direction from right to left.

If one prefers,  $\neg R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, \eta)$  holds iff for some integer  $m$ , the following property  $P_m$  holds:  $\text{NOPATH}(G_m, \mathcal{V}_i, \mathcal{V}_j)$  for any  $\mathcal{V}_i$  and  $\mathcal{V}_j$  with  $\mathbf{x} \in \mathcal{V}_i$  and  $\mathcal{V}_j \cap \bar{B}(\mathbf{y}, 2^{-p}) \neq \emptyset$ .

The latter property is computably enumerable, as it corresponds to a union of decidable sets (uniform in  $m$ ), as the property  $P_m$  is a decidable property over finite graph  $G_m$ .  $\square$

**Corollary 39** (Robust  $\Rightarrow$  decidable). *Assume the hypotheses of Theorem 38. Assume that for all rational  $\mathbf{x}$ ,  $R^{\mathcal{P}}(\mathbf{x})$  is closed, and  $R^{\mathcal{P}}(\mathbf{x}) = R_{\omega}^{\mathcal{P}}(\mathbf{x})$ . Then the ball decision problem is decidable.*

*Proof.* Given some instance  $B(\mathbf{y}, \delta)$  of the ball problem, run in parallel the c.e. algorithm for it (and when its termination is detected, accepts) and the c.e. algorithm for  $(R^{\mathcal{P}}(\mathbf{x}))^c = (R_{\omega}^{\mathcal{P}}(\mathbf{x}))^c$  (and when its termination is detected, rejects).  $\square$

**Definition 40.** *Given some function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $R_{f\{f\}}^{\mathcal{P}}$  as: for two rational points  $\mathbf{x}$  and  $\mathbf{y}$ , and  $p$ , the relation holds iff  $R_{f(\ell(\mathbf{x})+\ell(\mathbf{y})+p)}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, p)$ .*

We need also the concept of polynomial (poly.) time computable function: see [25]. In short, a quickly converging name of  $\mathbf{x} \in \mathbb{R}^d$  is a name of  $\mathbf{x}$ , with  $I_n$  of radius  $< 2^{-n}$ . Then  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$  is computable in poly. time, if there is some oracle TM  $M$ , such that, for all  $\mathbf{x}$ , given any fast converging name of  $\mathbf{x}$  as oracle, given  $n$ ,  $M$  produces some open rational ball of radius  $< 2^{-n}$  containing  $\mathbf{f}(\mathbf{x})$ , in a time poly. in  $n$ .

**Theorem 41.** *Consider a locally Lipschitz system, with  $\mathbf{f}$  polynomial time computable, whose domain  $X$  is a closed rational box. Then  $R_{f\{f\}}^{\mathcal{P}} \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N} \in \text{PSPACE}$ .*

*Proof.* The proof of Theorem 38 (similar to the one of Theorem 24) shows that when  $R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, p)$  is false, then  $R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}, p)$  is false for some  $\epsilon = 2^{-n}$ . With the hypotheses, given  $\mathbf{x}, \mathbf{y}$  and  $p$ , we can take  $n$  polynomial in  $\ell(\mathbf{x}) + \ell(\mathbf{y}) + p$ . From the proof, the corresponding  $m$  is polynomially related to  $n$  (it is even affine in  $n$ ). Now an analysis similar to the one of Lemma 33, shows that the truth value of  $R^{G_m}(\mathbf{x}, \mathbf{y}, p)$  can be determined in space polynomial in  $m$ .  $\square$

**Theorem 42** (Polynomially robust to precision  $\Rightarrow$  PSPACE). *Consider the hypotheses of Theorem 41. Assume that for all rational  $\mathbf{x}$ ,  $R^{\mathcal{P}}(\mathbf{x})$  is closed, and  $R^{\mathcal{P}}(\mathbf{x}) = R_{f\{f\}}^{\mathcal{P}}$  for some polynomial  $p$ . Then the ball decision problem is in PSPACE.*

*Proof.* We have  $R_{f\{f\}}^{\mathcal{P}} \in \text{PSPACE}$  by Theorem 41 and since  $R^{\mathcal{P}} = R_{f\{f\}}^{\mathcal{P}}$ , then  $R^{\mathcal{P}} \in \text{PSPACE}$ .  $\square$

## VI. RELATING ROBUSTNESS TO DRAWABILITY

We can go even further, and go to geometric properties: in the previous sections, we associated to every discrete time dynamical system a reachability relation over the rationals. But we could also see it as a relation over the reals, and use the framework of CA, talking about subsets of  $\mathbb{R}^d \times \mathbb{R}^d$ . A closed subset of  $\mathbb{R}^d$  is called c.e. closed if one can effectively

enumerate the rational open balls intersecting it. From the statements of [34], the following holds:

**Theorem 43.** *Consider a computable discrete time system  $\mathcal{P}$  whose domain is a computable compact.*

*For all  $\mathbf{x}$ ,  $\text{cls}(R^{\mathcal{P}}(\mathbf{x})) \subseteq \mathbb{R}^d$  is a c.e. closed subset.*

*Proof.* We do the proof for the case of discrete time dynamical system. Write  $R^{\mathcal{P},T}(\mathbf{x}, \mathbf{y})$  iff there exists a trajectory of  $\mathcal{P}$  from  $\mathbf{x}$  to  $\mathbf{y}$  in less than  $T$  steps. We can write  $R^{\mathcal{P},0}(\mathbf{x}, \mathbf{y})$  as  $\{(\mathbf{x}, \mathbf{y}) | \mathbf{x} = \mathbf{y}\}$ , which is a computable closed subset [34, Example 5.1.3]). We can then also write  $R^{\mathcal{P},T+1}(\mathbf{x}, \cdot) = \mathbf{F}(R^{\mathcal{P},T}(\mathbf{x}, \cdot))$  where  $\mathbf{F}(K) := K \cup \mathbf{f}(K)$ . As  $\mathbf{f}$  is computable, we know it is continuous, and by induction on  $T$ ,  $R^{\mathcal{P},T+1}(\mathbf{x}, \mathbf{y})$  is a compact: indeed, as  $\mathbf{f}$  is computable, it is continuous, and as  $K$  is a closed subset living in a compact by induction, it is compact, and the image of a compact by some continuous function is compact. As  $\mathbf{f}$  is computable, and  $K$  is compact, we know that  $\mathbf{f}(K)$  is computable ([34, Theorem 6.2.4]), and hence also  $\mathbf{F}(K)$  ([34, Theorem 5.1.13]). And then by induction on  $T$ , that  $R^{\mathcal{P},T}(\mathbf{x}, \mathbf{y})$  is a closed computable subset. A computable closed set is computably enumerable-closed: we can enumerate the rational balls intersecting it ([11, Proposition 5.16]).

Furthermore, as it can be checked in all the above references of theorems above from [34], (see also [34, Theorem 6.2.1] for the required iteration) the above reasoning is even effective: we can even produce effectively in  $T$  a name of it (even effectively from a name of  $\mathbf{f}$ ). This means consequently by doing things in parallel (i.e. dovetailing) that we can effectively enumerate the rational balls intersecting  $\text{cls}(\bigcup_T R^{\mathcal{P},T}(\cdot, \cdot))$ , by considering increasing  $T$  and the balls in these enumerations.  $\square$

A closed set is called co-c.e. closed if one can effectively enumerate the rational closed balls in its complement. Using arguments similar to the proof of Theorems 38 and 24:

**Theorem 44.** *Consider a computable locally Lipschitz discrete time system whose domain  $X$  is a computable compact.*

*For all  $\mathbf{x}$ ,  $\text{cls}(R_{\omega}^{\mathcal{P}}(\mathbf{x})) \subseteq \mathbb{R}^d$  is a co-c.e. closed subset.*

*Proof.* From the proof of Theorem 38, for all the  $\mathbf{x}, \mathbf{y}$  such that  $R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  is false, in a easily controllable computable neighborhood of  $\mathbf{x}$ , with  $\epsilon = 2^{-n}$ , there exists some  $\delta = 2^{-m}$  and some witness  $\mathcal{R}^* = \mathcal{R}_{\delta}^*$  at level  $\delta$  of that fact: this witness guarantees  $R^{\mathcal{P}}(\mathbf{x}) \subseteq R_{\epsilon}^{\mathcal{P}}(\mathbf{x}) \subseteq \mathcal{R}_{\delta}^*$ , and  $\overline{B}(\mathbf{y}, \eta) \cap R_{\epsilon}^{\mathcal{P}}(\mathbf{x}) = \emptyset$  implies  $\overline{B}(\mathbf{y}, \eta) \cap \mathcal{R}_{\delta}^* = \emptyset$ .

Then a strategy to produce all the rational balls whose closure is not intersecting  $\text{cls}(R^{\mathcal{P}}(\mathbf{x}))$ , for increasing  $n$ , generate in parallel all such balls in the corresponding witness. This will exhaust all such balls.  $\square$

**Corollary 45** (Robust  $\Rightarrow$  computable)). *Assume the hypotheses of Theorem 44.*

*If  $R_{\omega}^{\mathcal{P}} = R^{\mathcal{P}}$  then  $\text{cls}(R^{\mathcal{P}}) \subseteq \mathbb{R}^d \times \mathbb{R}^d$  is computable.*

*Proof.* This follows from the fact that a closed set is computable iff it is c.e. closed and co-c.e. closed ([11, Proposition

5.16]), observing that above statements are effective given a name of  $\mathbf{x}$ .  $\square$

For closed sets, the notion of computability can be also interpreted as the possibility of being plotted with arbitrarily chosen precision: here the intuition is that  $\mathbf{z}/2^n$  corresponds to some pixel at precision  $2^n$ , and that 1 is black (i.e. the pixel is plotted black), 0 is white (i.e. the pixel is plotted white).

**Theorem 46** ([11, Proposition 5.7],[34, pages 127–128]). *For a closed set  $A \subseteq \mathbb{R}^k$ ,  $A$  is computable iff it can be plotted: there exists a computable function  $f : \mathbb{N} \times \mathbb{Z}^k \rightarrow \mathbb{N}$  with  $\text{range}(f) \subseteq \{0, 1\}$  and such that for all  $n \in \mathbb{N}$  and  $\mathbf{z} \in \mathbb{Z}^k$*

$$f(n, \mathbf{z}) = \begin{cases} 1 & \text{if } B(\frac{\mathbf{z}}{2^n}, 2^{-n}) \cap A \neq \emptyset, \\ 0 & \text{if } B(\frac{\mathbf{z}}{2^n}, 2 \cdot 2^{-n}) \cap A = \emptyset, \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases}$$

This is called *local computability* in [12].

**Corollary 47** (Robust  $\Rightarrow$  drawable)). *Assume the hypotheses of Theorem 44.*

*If  $R_{\omega}^{\mathcal{P}} = R^{\mathcal{P}}$  then  $\text{cls}(R^{\mathcal{P}}) \subseteq \mathbb{R}^d \times \mathbb{R}^d$  can be plotted.*

*Proof.* This follows from Corollary 45 and Theorem 46 (that is to say [11, Proposition 5.7],[34, page 127–128]).  $\square$

This is even effective in a name of  $\mathbf{f}$ . Actually, the converse is true, if some topological properties are assumed.

**Theorem 48.** *Assume  $R^{\mathcal{P}}$  is closed, and can be plotted effectively in a name of  $\mathbf{f}$ . Then the system is robust, i.e.  $R_{\omega}^{\mathcal{P}} = R^{\mathcal{P}}$ .*

Actually, we prove the stronger statement that, if  $\text{cls}(R^{\mathcal{P}})$  can be plotted effectively in a name of  $\mathbf{f}$ , then  $R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y}) = R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  except maybe for some  $(\mathbf{x}, \mathbf{y})$  in  $\text{cls}(R^{\mathcal{P}}) - R^{\mathcal{P}}$ .

*Proof.* By Theorem 46, we know that  $\text{cls}(R^{\mathcal{P}})$  is computable, and it is known to be equivalent to the fact that the distance function  $d(\cdot, \text{cls}(R^{\mathcal{P}}))$  is computable [34, Corollary 5.1.8]. That means that given some rational ball, a name for  $\mathbf{x}$ , and for  $\mathbf{y}$ , with  $\neg R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ , the following procedure is guaranteed to terminate, when  $(\mathbf{x}, \mathbf{y})$  is not in  $\text{cls}(R^{\mathcal{P}}) - R^{\mathcal{P}}$ : compute a name of  $d((\mathbf{x}, \mathbf{y}), \text{cls}(R^{\mathcal{P}}(\mathbf{x})))$  until a proof that it is strictly positive is found:  $d((\mathbf{x}, \mathbf{y}), \text{cls}(R^{\mathcal{P}}(\mathbf{x}))) = 0$  would mean that  $(\mathbf{x}, \mathbf{y}) \in \text{cls}(R^{\mathcal{P}})$ , but not in  $R^{\mathcal{P}}$ .

It answers by reading a finite part, say  $m$  cells, of the names of  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{f}$ , and hence give the same answer if the names are altered after symbol number  $m$ . That means there exists some precision  $\epsilon$  (related to  $m$ , basically  $2^{-m}$  if we consider names converging exponentially fast) so that  $\neg R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  remains true for some  $\epsilon$ -neighborhood of  $\mathbf{x}$  and  $\mathbf{y}$ , and unchanged by a small variation of  $\mathbf{f}$ . In other words, for all  $\mathbf{x}, \mathbf{y}$ , when  $\neg R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ , there exists some  $\epsilon$  such that  $\neg R_{\epsilon}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ , i.e.  $\neg R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ . When  $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  holds, we always have that  $R_{\omega}^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  holds.  $\square$

This is also possible to adapt this at the complexity level with hypotheses in the spirit of previous results. This would lead to a concept of *local poly-space computability* in the

spirit of the local poly-time complexity introduced in [12]. The latter is devoted to discussing equivalence at the poly-time complexity of various representations of compact sets.

## VII. CONTINUOUS TIME AND HYBRID SYSTEMS

The previous approaches have a very high level of applicability, and are able to talk about systems that could be even continuous time, or hybrid.

**Definition 49.** A continuous-time dynamical system  $\mathcal{P}$  is given by a set  $X \subseteq \mathbb{R}^d$ , and some Ordinary Differential Equation (ODE) of the form  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  on  $X$ .

It is known that the maximal interval of existence of solutions can be non-computable, even for computable ODEs [20]. To simplify the discussion, we assume in this article that the ODEs have solutions<sup>3</sup> defined over all  $\mathbb{R}$ . A trajectory of  $\mathcal{P}$  starting at some  $\mathbf{x}_0 \in X$  is a solution of the differential equation with initial condition  $\dot{\mathbf{x}} = \mathbf{x}_0$ , defined as a continuous right derivable function  $\xi : \mathbb{R}^+ \rightarrow X$  such that  $\xi(0) = \mathbf{f}(\mathbf{x}_0)$  and for every  $t$ ,  $\mathbf{f}(\xi(t))$  is equal to the right derivative of  $\xi(t)$ . To each continuous time dynamical system  $\mathcal{P}$  we associate its reachability relation  $R^{\mathcal{P}}$  as before.

For any  $\varepsilon > 0$  the  $\varepsilon$ -perturbed system  $\mathcal{P}_\varepsilon$  is described by the differential inclusion  $d(\dot{\mathbf{x}}, \mathbf{f}(\mathbf{x})) < \varepsilon$ . This non-deterministic system can be considered as  $\mathcal{P}$  submitted to a small noise of magnitude  $\varepsilon$ . We denote reachability in the system  $\mathcal{P}_\varepsilon$  by  $R_\varepsilon^{\mathcal{P}}$ . The limit reachability relation  $R_\omega^{\mathcal{P}}$  is introduced as before.

**Theorem 50** (Perturbed reachability is co-r.e.). *Consider a continuous time dynamical system, with  $\mathbf{f}$  locally Lipschitz and computable, and whose domain is a computable compact.*

*Then, for all  $\mathbf{x}$ ,  $\text{cls}(R_\omega^{\mathcal{P}}(\mathbf{x})) \subseteq \mathbb{R}^d$  is a co-c.e. closed subset.*

Its proof can be considered as the main technical result established in [29]. Independently:

*Proof.* The proof is similar to the proof of Theorems 38 and 24: adapt the construction of the involved graph  $G_m$  to cover the flow of the trajectory. With our hypotheses the solutions are defined over all  $\mathbb{R}$ . It is proved in [20, Theorem 3.1] that Lipschitz (and even effectively locally Lipschitz) homogeneous computable ODEs have computable solutions over their maximal domain, so this is feasible.  $\square$

**Corollary 51** (Robust  $\Rightarrow$  decidable). *Assume the hypotheses of Theorem 50. If  $R_\omega^{\mathcal{P}} = R^{\mathcal{P}}$  then  $\text{cls}(R^{\mathcal{P}}) \subseteq \mathbb{R}^d \times \mathbb{R}^d$  is computable.*

We can also state the equivalent of previous statements at the complexity level, assuming basic hypotheses that make computability of the solutions to remain in polynomial space.

Actually, we can even deal with the so-called hybrid systems. Various models have been considered in literature, but one common point is that they all correspond to continuous time dynamical systems, where the dynamics might be discontinuous (hence it is not computable). To be very general,

a dynamical system can be described by its flow  $\phi(\mathbf{x}, t)$  (the idea is that, given  $\mathbf{x}$ , and time  $t$ ,  $\phi$  maps the trajectory starting from  $\mathbf{x}$  to the position at time  $t$ ). By considering  $T$  to be discrete, this even covers discrete time dynamical systems, and  $T = [0, +\infty)$ , continuous and hybrid time systems.

**Definition 52.** A hybrid system  $\mathcal{P}$  is given by a set  $X \subseteq \mathbb{R}^d$ , and a semi-group  $T$ , and some flow function  $\phi : X \times T \rightarrow X$  satisfying  $\phi(\mathbf{x}, 0) = \mathbf{x}$  and  $\phi(\phi(\mathbf{x}, t), t') = \phi(\mathbf{x}, t + t')$ .

Previous proofs are basically using the fact that

- 1) reachability  $R^{\mathcal{P}}$  is c.e.;
- 2) perturbed reachability is co-c.e.

The first point is usually very clear in any of the considered models, as it is, roughly speaking, always expected that one can at least simulate the model. The second point is, on a given class of models, usually less clear, but if we look at our proof methods, typically the proof of Theorems 38 and 24, we see that we only need to be able to construct some computable abstraction satisfying Claim 1 and Claim 2.

The key remark is that these properties are not talking about function  $\mathbf{f}$ , but its graph. The major point is that assuming a function has the closure of its graph computable is a way more general concept than assuming computability. For example, the characteristic function  $\chi_{[0, \infty)}$  is not computable, as it is not continuous. But its graph, as well as its closure, is easy to draw (made of two segments): see discussions for e.g. in [12]. One usually expects to be able to draw the closure of the graph of the flow, and this is sufficient to get results similar to the previous ones, relating robustness to computability. In particular, this allowed us to talk about discontinuous functions, in particular not computable in CA, as we did.

## VIII. OTHER PERTURBATIONS

We can also consider time perturbed TM: the idea, is that given  $n > 0$ , the  $n$ -perturbed version of the machine  $\mathcal{M}$  is unable to remain correct after a time  $n$ : given an integer  $n > 0$ , the  $n$ -perturbed version of the machine  $\mathcal{M}$  is defined exactly as  $\mathcal{M}$  except that after a time greater than  $n$  then its internal state  $q$  can change in a non-deterministic manner: given a configuration  $(q, \dots a_{-n-1}a_{-n}a_{-n+1} \dots a_{-1}, a_0a_1 \dots a_{n-1}a_na_{n+1} \dots)$  (with  $\neg(q \in F \cup R)$ ) the  $n$ -perturbed version of  $\mathcal{M}$  may go to  $(q', \dots a_{-n-1}a_{-n}a_{-n+1} \dots a_{-1}, a_0a_1 \dots a_{n-1}a_na_{n+1} \dots)$  for any  $q' \in Q$ .

Let  $L^n(\mathcal{M})$  be the time  $n$ -perturbed language of  $\mathcal{M}$ , i.e., the set of words in  $\Sigma^*$  that are accepted by the time  $n$ -perturbed version of  $\mathcal{M}$ . From definitions, and using similar ideas:

**Lemma 53.**  $L(\mathcal{M}) \subseteq L^\omega(\mathcal{M}) \subseteq \dots \subseteq L^2(\mathcal{M}) \subseteq L^1(\mathcal{M})$ .

**Theorem 54.**  $L^\omega(\mathcal{M})$  is in the class  $\Pi_1^0$ .

*Proof.* For a word  $w$ ,  $w \notin L^\omega(\mathcal{M})$ , iff there exists  $n \in \mathbb{N}$  such that  $w \notin L^n(\mathcal{M})$ . As  $L^n(\mathcal{M})$  is decidable uniformly in

<sup>3</sup>Notice that a non-total solution must necessarily leave any compact, see e.g. [21], so when  $X$  is compact this is not a restriction.

<sup>4</sup>In particular, may accept. More subtle perturbations can be considered, keeping the results valid.

$n$ , the complement of  $L^\omega(\mathcal{M})$  is computably enumerable, as it is the uniform in  $n$  union of decidable sets. We get that  $L^\omega(\mathcal{M}) \in \Pi_1^0$  (co-computably enumerable).  $\square$

**Corollary 55** (Length robust  $\Rightarrow$  decidable). *If  $L^\omega(\mathcal{M}) = L(\mathcal{M})$  then  $L(\mathcal{M})$  is decidable.*

**Theorem 56.** *When  $M$  always stops,  $L^\omega(\mathcal{M}) = L(\mathcal{M})$ .*

*Proof.* We directly have  $L(\mathcal{M}) \subseteq L^\omega(\mathcal{M})$ .

Let  $w \in L^\omega(\mathcal{M}) = \bigcap_{n \in \mathbb{N}} L^n(\mathcal{M})$ , so  $\forall n \in \mathbb{N}, w \in L^n(\mathcal{M})$ . By contradiction, we assume that  $L(\mathcal{M}) \subsetneq L^\omega(\mathcal{M})$ . So there exists  $w \in L^\omega(\mathcal{M})$  such that  $w \notin L(\mathcal{M})$ . Since  $\mathcal{M}$  always terminates, it rejects  $w$  after using a time  $q(\ell(w))$ . But, then  $w \notin L^n(\mathcal{M})$  for any  $n \geq q(\ell(w)) + 2$ , and hence  $w \notin L^\omega(\mathcal{M})$ , a contradiction.  $\square$

**Definition 57.** *Given some function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $L^{\{f\}}(\mathcal{M})$  for the set of words accepted by  $\mathcal{M}$  with time perturbation  $f$ :  $L^{\{f\}}(\mathcal{M}) = \{w \mid w \in L^{f(\ell(w))}(\mathcal{M})\}$ .*

**Theorem 58** (Polynomially robust to time  $\Leftrightarrow$  PTIME). *A language  $L$  is in PTIME iff for some  $\mathcal{M}$  and some polynomial  $p$ ,  $L = L(\mathcal{M}) = L^{\{p\}}(\mathcal{M})$ .*

*Proof.* This can be established as for space perturbation. In an independent view, the intuition of the proof is that the polynomial in  $n$  can be seen as a time-out.  $M$  works in polynomial time  $p(n)$ , so in at most  $p(n)$  steps, so the machine for  $L^n(\mathcal{M})$  can reject if it has not accepted or rejected in  $p(n)$  steps.

( $\Rightarrow$ ) If  $M$  always terminates and works in polynomial time, then there exists a polynomial  $q$  that bounds the execution time of  $M$ , so we have a polynomial  $p$  ( $p \geq q$ ) such that, for  $n \in \mathbb{N}$ ,  $L^p(\mathcal{M}) \subseteq L(\mathcal{M})$ . We have the other inclusion by definition.

( $\Leftarrow$ ) We always have  $L^{p(n)}(\mathcal{M}) \in \text{PTIME}$  and since  $L^{p(n)}(\mathcal{M}) = L$ , then  $L \in \text{PTIME}$ .  $\square$

**Theorem 59** (Polynomially robust to length  $\Leftrightarrow$  PTIME). *Any PTIME language is reducible to the reachability relation of PAM with  $R^P = L^{\{p\}}(\mathcal{P})$  for some polynomial  $p$ .*

Fix some distance  $\delta(\cdot, \cdot)$  over the domain  $X$ . A finite trajectory of discrete time dynamical system  $\mathcal{P}$  is a finite sequence  $(x_t)_{t \in 0 \dots T}$  such that  $x_{t+1} = f(x_t)$  for all  $0 \leq t < T$ . Its associated length is defined as  $\mathcal{L} = \sum_{i=0}^{T-1} \delta(x_i, x_{i+1})$ .

We could also consider length perturbed discrete time dynamical system: the idea, is that given  $L > 0$ , the  $L$ -perturbed version of the system is unable to remain correct after a length  $L$ . We then define  $R^{P,L}(\mathbf{x}, \mathbf{y})$  as there exists a finite trajectory of  $\mathcal{P}$  from  $\mathbf{x}$  to  $\mathbf{y}$  of length  $\mathcal{L} \leq L$ .

When considering TMs as such dynamical systems,  $\delta(\cdot, \cdot)$  is basically some distance over configurations of TMs. Word  $w$  is said to be accepted in length  $d$  if the trajectory starting from  $C_0[w]$  to the accepting configuration has length  $\leq d$ .

**Definition 60.** *Distance  $\delta(C, C')$  is called time metric iff for  $C \vdash C'$ , we have  $\delta(C, C') \leq p(\ell(C))$ , and  $\delta(C, C') \geq \frac{1}{p(\ell(C))}$  for some polynomial  $p$ .*

Write  $\mathcal{L}(M, t)$  for the set of words accepted by  $M$  in a length less than  $t$ . Given some function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $L^{(f)}(\mathcal{M})$  for  $L^{(f)}(\mathcal{M}) = \{w \mid w \in \mathcal{L}(M, f(\ell(w)))\}$ .

**Theorem 61** (Length robust for some time-metric distance  $\Leftrightarrow$  PTIME). *Assume  $\delta(\cdot, \cdot)$  is time metric. Then, a language  $L$  is in PTIME iff for some Turing machine  $\mathcal{M}$ , and some polynomial  $p(n)$ ,  $L = L(\mathcal{M}) = L^{(f)}(\mathcal{M})$ .*

*Proof.* Let  $w$  be the input of size  $n$ . The execution of a Turing machine is a sequence  $(C_i) = (q_i, l_i, r_i)$ .

( $\Rightarrow$ ) If  $L$  is in PTIME, so there is a Turing machine  $\mathcal{M}$  that computes  $L$  in polynomial time  $p(n)$ . Since the distance between two successive configurations is bounded by a polynomial  $q(n)$ , we have that the total length  $\mathcal{L}$  is  $\mathcal{L} = \sum_{i=0}^{p(n)-1} d(C_i, C_{i+1}) \leq \sum_{i=0}^{p(n)-1} q(n)$ , which is a polynomial in  $n$ .

Thus  $L$  is computable in polynomial length.

( $\Leftarrow$ ) If  $L$  is computable in polynomial length  $p(n)$ .

Let  $T$  to be fixed. Then we have, for all  $i \in \{1 \dots T\}$ :  $d(C_i, C_{i+1}) \geq \frac{1}{\text{poly}(\ell(C_i))}$ , thus  $\sum_{i=0}^{T-1} d(C_i, C_{i+1}) \geq \frac{T}{\text{poly}(\ell(C_{\min}))}$ , where  $C_{\min}$  is chosen to minimize the previous lower bound.

Take  $T = p(n) \times \text{poly}(\ell(C_{\min}))$ , we have that a Turing machine simulating the trajectory will accepts or rejects after a polynomial number of steps, thus  $L \in \text{PTIME}$ .  $\square$

One way to obtain such a distance  $\delta(C, C')$  is to take the Euclidean between  $\Upsilon(C)$  and  $\Upsilon(C')$  for  $\gamma = \gamma_{[0,1]}$ .

**Proposition 62.** *The obtained distance is time metric.*

*Proof.* We consider  $C_i = (q_i, l_i, r_i)$  and  $C_{i+1} = (q_{i+1}, l_{i+1}, r_{i+1})$ , with  $C_i \vdash C_{i+1}$ . We write  $\bar{l}_i = \Upsilon(l_i)$  and  $\bar{r}_i = \Upsilon(r_i)$ .

Then, from the definition of  $\Upsilon$  and  $\gamma_{[0,1]}$ :

- $|\bar{r}_{i+1} - \bar{r}_i| \leq 1$ .
- $|\bar{l}_{i+1} - \bar{l}_i| \leq 1$ .
- And the gaps between  $\bar{r}_{i+1}$  and  $\bar{r}_i$ , and  $\bar{l}_{i+1}$  and  $\bar{l}_i$  remain polynomial in the size of a configuration.

This provides property 1.

By the encoding of the real numbers over the tapes of the Turing machines, the gap between two consecutive configurations is at least  $\frac{1}{2}$  (we assume that the Turing machine is not allowed not to do anything: that would clearly corresponds to a looping situation). This provides property 2.  $\square$

Given some function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $R^{P,(f)}$  for the set of words accepted by  $\mathcal{M}$  with length perturbation  $f$ :  $R^{P,(f)} = \{w \mid w \in R^{P,f(\ell(w))}\}$ .

**Theorem 63** (Polynomially length robust  $\Leftrightarrow$  PTIME). *Assume distance  $d$  is time metric. Assume  $R^P = R^{P,(p)}$  for some polynomial  $p$ . Then  $R^P$  is in PTIME.*

*Proof.* Since  $d$  is time metric, a polynomial time and polynomial length are essentially the same, so the proof is very analogous to the one of Theorem 61.  $\square$

## IX. ANALOG COMPLEXITY UNDER ROBUSTNESS PRISM

The following was established in [7] ( $\text{len}_y(0, t)$  stands for the length of the curve between time 0 and  $t$ ):

**Theorem 64** (Analog characterization of PTIME [7, Theorem 2.2]). *A decision problem (language)  $\mathcal{L}$  belongs to the class PTIME if and only if it is poly-length-analog-recognizable. That is to say: there exist vectors  $\mathbf{p}$ , and  $\mathbf{q}$  of polynomials with rational coefficients and a polynomial  $\Pi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , such that for all  $w \in \Sigma^*$ , there is a (unique)  $\mathbf{y} : \mathbb{R}_+ \rightarrow \mathbb{R}^d$  such that for all  $t \in \mathbb{R}_+$ :*

- 1)  $\mathbf{y}(0) = \mathbf{q}(\gamma_{[0,1]}^k(w))$  and  $\mathbf{y}'(t) = \mathbf{p}(\mathbf{y}(t))$
- 2) if  $|\mathbf{y}_1(t)| \geq 1$  then  $|\mathbf{y}_1(u)| \geq 1$  for all  $u \geq t$
- 3) if  $w \in \mathcal{L}$  (resp.  $\notin \mathcal{L}$ ) and  $\text{len}_y(0, t) \geq \Pi(|w|)$  then  $\mathbf{y}_1(t) \geq 1$  (resp.  $\leq -1$ )
- 4)  $\text{len}_y(0, t) \geq t$

Conditions 2) is basically stating that this is a length-robust system. Condition 4) is the equivalent of second condition of Definition 60, that guarantees the equivalence with length for Turing machines. And the result is obtained using a step-by-step emulation using encoding  $\gamma_{[0,1]}^k$ .

Very recently, a characterization of PSPACE has also been obtained: see [17], for full details. Basically, of the form.

**Theorem 65** ([6, Theorem 3.11],[17]). *A language  $L \subseteq \Gamma^*$  belongs to PSPACE iff there are ODEs  $\mathbf{y}' = \mathbf{p}(\mathbf{y}, \mathbf{z})$  and  $\mathbf{z}' = \mathbf{q}(\mathbf{y}, \mathbf{z})$ , where  $\mathbf{p}, \mathbf{q}$  are vectors of polynomials, there is a polynomial  $u$ , (vector-valued) functions  $\mathbf{r}, \mathbf{s} \in \text{GPVAL}$ , a  $\gamma_{\mathbb{N}}^k$ -bound  $\phi$ , and  $\varepsilon > 0$ ,  $\tau \geq \alpha > 0$ ,  $\alpha, \tau \in \mathbb{Q}$ , such that, for all  $w \in \Sigma^*$ , one has that the solution  $(\mathbf{y}, \mathbf{z})$  of the ODEs with the initial condition  $\mathbf{y}(0) = \mathbf{r}(x)$  and  $\mathbf{z}(0) = \mathbf{s}(\mathbf{y}(0)) = \mathbf{s} \circ \mathbf{r}(0)$ , with  $d(\gamma_{\mathbb{N}}^k(w), \mathbf{x}) < 1/4$ , satisfies:*

1. If  $\bar{t}_1 > 0$  is such that  $|\mathbf{y}_1(\bar{t}_1)| \geq 1$ , then  $|\mathbf{y}_1(t)| \geq 1$  for all  $t \geq \bar{t}_1$  and  $|\mathbf{y}_1(t)| \geq 3/2$  for all  $t \geq \bar{t}_1 + 1$ ;
2. If  $w \in L$  (respectively  $\notin L$ ) then there is some  $\bar{t}_1 > 0$  such that  $\mathbf{y}_1(\bar{t}_1) \geq 1$  (respectively  $\leq -1$ );
3.  $\|(\mathbf{y}(t), \mathbf{z}(t))\| \leq \phi \circ u(|w|)$ , for all  $t \geq 0$ ;
4. Suppose that  $(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$  satisfies the ODE with  $\mathbf{y}(0) = \mathbf{r}(x)$ ,  $\mathbf{z}(0) = \mathbf{s}(\mathbf{y}(0)) = \mathbf{s} \circ \mathbf{r}(x)$ , except possibly at time instants  $t_i$ , for  $i = 1, 2, \dots$ , satisfying: a)  $t_i - t_{i-1} \geq \tau$ , where  $t_0 = 0$ ; b)  $\|\tilde{\mathbf{y}}(t_i) - \lim_{t \rightarrow t_i^-} \tilde{\mathbf{y}}(t)\| \leq \varepsilon$  and  $\tilde{\mathbf{z}}(t_i) = \mathbf{s}(\tilde{\mathbf{y}}(t_i))$ ; c)  $\lim_{t \rightarrow t_i^-} \tilde{\mathbf{z}}_1(t) > 1$ . Then conditions 1,2,3,5 hold for  $(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})$ ;
- 5) For any  $b > a \geq 0$  such that  $|b - a| \geq \tau$ , there is an interval  $I = [c, d] \subseteq [a, b]$ , with  $|d - c| \geq \alpha$ , such that  $\mathbf{z}_1(t) \geq 3/2$  for all  $t \in I$ .

Condition  $d(\gamma_{\mathbb{N}}^k(w), x) < 1/4$ , 4) and 5) basically impose that there exists some abstraction graph. Conditions 3) makes it keep a polynomial log-size, and conditions 1) and 2) impose to the system to be eventually decisional. This guarantees PSPACE, and allows a robust emulation of a TM. However, the space used by the ODEs cannot be “read” easily (as by a concept such as length in previous theorem).

This is using a rather natural encoding, but the system is not living in a compact. If one wants to remain bounded, an alternative is to use the trick of [8], based on a change

of variable, at the price of a rather adhoc encoding. This is basically based on the following:

**Theorem 66** (Robust simulation of a TM over  $\mathbb{R}^6$  [18]). *For any TM  $M$ , there is an analytic and computable ODE  $y' = g_M(y)$  defined over  $\mathbb{R}^6$  which simulates  $M$  using the encoding  $\gamma_{\mathbb{N}}$ , and remains valid for perturbations less than  $\varepsilon \leq 1/4$ .*

The idea is that if  $\phi$  is a solution of  $y' = g_M(y)$  simulating  $M$  on  $\mathbb{R}^6$ , we can consider  $\phi_1 = \frac{2}{\pi} \arctan \phi$  as a corresponding emulation of  $M$  on  $(-1, 1)^6$ . Then  $\phi_1$  will be solution of the ODE  $\phi_1' = f_M(\phi_1)$  with  $f_M(x) = \frac{2}{\pi} \frac{1}{1 + \tan^2(\frac{x\pi}{2})} g_M(\tan(\frac{x\pi}{2}))$ . Consequently, the continuous time dynamical system given by  $y' = f_M(y)$  simulates TM  $M$  on  $X$ , if the input word  $w = a_1 \dots a_n$  is encoded in  $X$  by  $\gamma_{\arctan}(w) = \frac{2}{\pi} \arctan(\gamma_{\mathbb{N}}(w)) = \frac{2}{\pi} \arctan(w_0 + w_1 2 + \dots + w_n 2^n)$ . Then a computation will remain correct if the states are not perturbed more than  $\arctan(s(w) + \varepsilon) - \arctan(s(w))$ , where  $s(w)$  is an upper bound of the size of tape on word  $w$ .

However, the used encoding  $\gamma_{\arctan}$  is rather artificial, and mainly the system is defined over some open bounded domain, not compact. Furthermore, robustness holds for points close to the image by  $\Upsilon$  of configurations, but not for arbitrary points.

The question of a simpler characterization of PSPACE, over a compact, using a simple encoding remains open. However, we believe that our results help to understand that space-complexity is related to robustness to precision over a compact, or more generally to the log-size of some abstraction graph over general domains.

Notice that we can prove that reachability is PSPACE-complete for polynomially robust pODE systems, But the point of the previous results (and open question) is to get a *uniform* embedding: given some machine  $M$ , we would like a same ODE that works for any input size.

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We recall here the very basics of computable analysis: see [34] or [11].

Basically, the idea behind classical computability and complexity is to fix some representations of objects (such as graphs, integers, etc., ...) using finite words over some finite alphabet, say  $\Sigma = \{0, 1\}$ , and to say that such an object is computable when such a representation can be produced using a Turing machine. The aim of computable analysis is to be able to talk also about objects such as real numbers, functions over the reals, closed subsets, compact subsets, ..., which cannot be represented by finite words over  $\Sigma$  (a clear reason for it is that such words are countable while the set  $\mathbb{R}$  for example is not). However, they can be represented by some infinite words over  $\Sigma$ , and the idea is to fix such representations for these various objects, called *names*, with suitable computable properties. In particular, in all the following proposed representations, it can be proved that an object is computable iff it has some computable representation.

**Remark 67.** *Here the notion of computability involved is the one of Type 2 Turing machines, that is to say computability over possibly infinite words: the idea is that such a machine has some read-only input tape(s), that contain the input(s), which can correspond to either a finite or infinite word(s), a read-write working tape, and one (or several) write only output tape(s). It evolves as a classical Turing machine, the only difference being that we consider it outputs an infinite words when it writes forever the symbols of that words on its (or one of its) write-only infinite output tape(s): see [34] for details.*

A name for a point  $\mathbf{x} \in \mathbb{R}^d$  is a sequence  $(I_n)$  of nested open rational balls with  $I_{n+1} \subseteq I_n$  for all  $n \in \mathbb{N}$  and  $\{\mathbf{x}\} = \bigcap_{n \in \mathbb{N}} I_n$ . Such a name can be encoded as infinite sequence of symbols.

We call a real function  $f : \subseteq \mathbb{R} \rightarrow \mathbb{R}$  computable, iff some (Type 2 Turing) machine maps any name of any  $x \in \text{dom}(f)$  to a name of  $f(x)$ . For real functions  $\mathbf{f} : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  we consider machines reading  $n$  names in parallel.

It can be proved that a computable function is necessarily continuous. A name for a function  $\mathbf{f}$  is a list of all pairs of open rational balls  $(I, J)$  such that  $\mathbf{f}(\bar{I}) \subseteq J$ . Following above remark, one can prove that a real function is computable iff it has some computable name.

A name for a closed set  $F$  is a sequence  $(I_n)$  of open rational balls such that  $\text{cls}(I_n) \cap F = \emptyset$  and a sequence  $(J_n)$  of open rational balls such that  $J_n \cap F \neq \emptyset$ .

Given some closed set  $F$ , the distance function  $d_F : \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by  $d_F(x) := \inf_{y \in F} d(x, y)$ . Closed subset  $F \subseteq \mathbb{R}^n$  is computable iff its distance function  $d_A : \mathbb{R}^n \rightarrow \mathbb{R}$  is ([34, Corollary 5.1.8]). A name for a compact  $K$  is a name of  $F$  as a closed set, and an integer  $L$  such that  $K \subseteq B(0, L)$ .

A closed set is called computably-enumerable closed if one can effectively enumerate the rational open balls intersect-

ing it:  $\{(q, \varepsilon) \in \mathbb{Q}^n \times \mathbb{Q}_+ \mid B(q, \varepsilon) \cap A \neq \emptyset\}$  is computably enumerable ([11, Definition 5.13], [34, Definition 5.1.1]). A closed set is called co-computably-enumerable closed if one can effectively enumerate the rational closed balls in its complement: the set  $\{(q, \varepsilon) \in \mathbb{Q}^n \times \mathbb{Q}_+ \mid \bar{B}(q, \varepsilon) \subseteq U\}$  is computably enumerable ([11, Definition 5.10], [34, Definition 5.1.1]).

We need also the concept of polynomial time computable function in computable analysis: see [25]. In short, a quickly converging name of  $\mathbf{x} \in \mathbb{R}^d$  is a name of  $\mathbf{x}$ , with  $I_n$  of radius  $< 2^{-n}$ . A function  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$  is said to be computable in polynomial time, if there is some oracle TM  $M$ , such that, for all  $\mathbf{x}$ , given any fast converging name of  $\mathbf{x}$  as oracle, given  $n$ ,  $M$  produces some open rational ball of radius  $< 2^{-n}$  containing  $\mathbf{f}(\mathbf{x})$ , in a time polynomial in  $n$ .