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Asymptotic Solutions of $Y'' = F(x)Y$

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It is shown how to find asymptotic expansions of solutions of the given equation as $x \to +\infty$ under modest assumptions, namely that $F(x)$ is an element of a Hardy field and there exists at least one solution of the given equation that is nonzero for sufficiently large x. © 1995 Academic Press, Inc.

1. INTRODUCTION

Recall that a Hardy field [2] is a field of germs of real-valued functions on positive half-lines in $\mathbb{R}$ that is closed under differentiation. Examples of these are the field of rational functions $\mathbb{R}(x)$ and, for any Hardy field k, the field obtained by adjoining to k any set of real germs that are exponentials of germs of k, or antiderivatives of elements of k, in particular of logarithms of positive elements of k, or that are algebraic over k. Each element of a Hardy field has a definite limit in $\mathbb{R} \cup \{+\infty, -\infty\}$ as $x \to +\infty$. A Hardy field is an ordered field, its positive elements being those that are ultimately positive, that is, positive for x sufficiently large. A Hardy field also has a canonical valuation denoted ν, which is a homomorphism from the multiplicative group of the Hardy field onto an ordered abelian group, with certain properties enumerated in [2], the most important of which are that for nonzero elements a, b of the Hardy field such that $\nu(a) > \nu(b) \neq 0$, we have $\nu(a + b) = \nu(b)$ and $\nu(a') > \nu(b')$. For u a nonzero element of a Hardy field, $\nu(u) > 0$ means that $\lim_{x\to+\infty} u(x) = 0$.

The results of this section are either more or less well known, or due independently to Boshernitzan and Rosenlicht, or due to Boshernitzan; most of Theorem 2 and the last part of Theorem 3 are due to Boshernitzan

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[1, Sect. 17]. We give here a simplified exposition adapted to our purposes.

Associated with the differential equation $Y'' = F(x)Y$ is its Riccati equation $V' + V^2 = F(x)$, which is satisfied by $v = y'/y$ whenever y is a nonvanishing solution of the linear equation. If $F(x)$ is in a Hardy field and the Riccati equation has a solution on a positive half-line then the linear equation has two linearly independent solutions in some extension Hardy field [2, Theorem 2, Corollary 2]. Note that the more general differential equation $Y'' + p(x)Y' + q(x)Y = 0$ with coefficients $p(x)$, $q(x)$ in a Hardy field is equivalent to the differential equation $U'' = F(x)U$ with coefficient $F(x)$ in a (possibly larger) Hardy field under the change of variable $y = u \exp(-\frac{1}{2}\int^x p(t)\,dt)$.

THEOREM 1. *Suppose that the linear differential equation $Y'' = F(x)Y$, where $F(x)$ is in a Hardy field, has two linearly independent solutions in a Hardy field. Then there are linearly independent solutions y_1, y_2 such that $\nu(y_1) > \nu(y_2)$. y_1 is uniquely determined up to a nonzero real factor, as is $(y_2/y_1)'$. If y_1, y_2 are chosen positive, then $y_1y_2' - y_2y_1'$ is a positive constant C, $(y_1/y_2)' = -C/y_2^2$ and $(y_2/y_1)' = C/y_1^2$. If $v_i = y_i'/y_i$ for $i = 1, 2$, then v_1 is unique and $v_2 - v_1 = C/y_1y_2 > 0$.*

Any two linearly independent solutions of the differential equation with the same ν-value have a quotient that approaches a nonzero real limit as $x \to +\infty$, therefore they have a nonzero real linear combination with higher ν-value. This shows the existence of y_1, y_2 as desired. We have $(y_1y_2' - y_2y_1')' = y_1y_2'' - y_2y_1'' = 0$, so that $y_1y_2' - y_2y_1'$ is a constant $C \in \mathbb{R}$. This $C \neq 0$, for otherwise y_1/y_2 would be constant. If we take y_1, $y_2 > 0$, as we may, then y_1/y_2 is a positive germ approaching zero as $x \to +\infty$, hence decreasing, so that $(y_1/y_2)' = (y_2y_1' - y_1y_2')/y_2^2 = -C/y_2^2 < 0$, so that $C > 0$. Then $(y_2/y_1)' = (y_1y_2' - y_2y_1')/y_1^2 = C/y_1^2$ and $v_2 - v_1 = y_2'/y_2 - y_1'/y_1 = C/y_1y_2 > 0$. Since all solutions of the original differential equation are linear combinations of y_1 and y_2, the uniqueness statements for y_1, $(y_2/y_1)'$, and v_1 are clear.

THEOREM 2. *Let F, Φ be elements of a Hardy field in which each of the differential equations $Y'' = F(x)Y$ and $Y'' = \Phi(x)Y$ has two linearly independent solutions. Let y_1, y_2 and η_1, η_2, respectively, be linearly independent solutions of the given differential equations with $\nu(y_1) > \nu(y_2)$ and $\nu(\eta_1) > \nu(\eta_2)$, and suppose that $\nu(y_1) > \nu(\eta_1)$. Then $\nu(y_2) < \nu(\eta_2)$, $y_1'/y_1 < \eta_1'/\eta_1$, $y_2'/y_2 > \eta_2'/\eta_2$, and $F > \Phi$.*

Since $\nu(y_1/\eta_1) > 0 > \nu(\eta_2/\eta_1)$, we can differentiate to get $\nu((\eta_1y_1' - y_1\eta_1')/\eta_1^2) > \nu((\eta_1\eta_2' - \eta_2\eta_1')/\eta_1^2)$ or $\nu(\eta_1y_1' - y_1\eta_2') > \nu(\eta_1\eta_2' - \eta_2\eta_1') = 0$. Assuming, as we may, that y_1, $\eta_1 > 0$, we have y_1/η_1 positive and approaching zero on a positive half-line, hence with negative derivative, so that $\nu(\eta_1y_1' - y_1\eta_1') > 0$ and $\eta_1y_1' - y_1\eta_1' < 0$, or $y_1'/y_1 < \eta_1'/\eta_1$. Since

$\eta_1 y_1' - y_1\eta_1'$ is negative and approaches zero on a positive half-line, it has a positive derivative, so that $0 < (\eta_1 y_1' - y_1\eta_1')' = \eta_1 y_1'' - y_1\eta_1'' = y_1\eta_1(F - \Phi)$, so $F > \Phi$. To prove that $\nu(y_2) < \nu(\eta_2)$ it suffices to prove that $\nu(1/y_2^2) > \nu(1/\eta_2^2)$, hence that $\nu((y_1/y_2)') > \nu((\eta_1/\eta_2)')$, hence that $\nu(y_1/y_2) > \nu(\eta_1/\eta_2)$, hence that $\nu(y_2/y_1) < \nu(\eta_2/\eta_1)$, or $\nu((y_2/y_1)') < \nu((\eta_2/\eta_1)')$, or $\nu(1/y_1^2) < \nu(1/\eta_1^2)$, or $\nu(y_1) > \nu(\eta_1)$, which was assumed. Finally, since $\nu(y_2) < \nu(\eta_2)$, we have $\nu(\eta_2/y_2) > 0$. Taking η_2, y_2 positive, as we may, the positive function η_2/y_2 approaches zero as $x \to +\infty$, hence has negative derivative, so that $y_2\eta_2' - \eta_2 y_2' < 0$, or $\eta_2'/\eta_2 < y_2'/y_2$.

A consequence of the Sturm comparison theorem is that if f, ϕ are continuous real-valued functions on an interval I of $\mathbb{R}$ and $f(x) > \phi(x)$ for all $x \in I$ then between any two zeros of a nonzero solution of $Y'' = f(x)Y$ on I lies at least one zero of any solution on the same interval I of $Y'' = \phi(x)Y$. Hence if F, ϕ are in a Hardy field and $F > \Phi$ the differential equation $Y'' = F(x)Y$ has nonzero solutions in a Hardy field if the differential equation $Y'' = \Phi(x)Y$ does. For $F = 0$, the differential equation $Y'' = F(x)Y$ has nonzero solutions in a Hardy field, for $F = -1$ it does not. Hence the problem of finding a lower bound for germs F in a Hardy field such that $Y'' = F(x)Y$ has nonzero solutions in a Hardy field. For this we pass to the Riccati equation $V' + V^2 = F(x)$, noting that the linear differential equation has nonzero solutions in a Hardy field if and only if there is a germ v such that $v' + v^2 = F(x)$. In the latter case, v is in an extension Hardy field of any Hardy field containing F. For a "small" F we must have $F < 0$, so for a solution v of the Riccati equation we have $v' + v^2 < 0$, so $v' < -v^2$, so $|v'| > v^2$, so $\nu(v') \leq \nu(v^2)$, so $\nu((1/v)') = \nu(-v'/v^2) \leq 0 = \nu(x')$, giving $\nu(1/v) \leq \nu(x)$, or $\nu(v) \geq \nu(1/x)$. Therefore we can write $v = (\alpha + z)/2x$, for some $\alpha \in \mathbb{R}$ and z a Hardy field element with $\nu(z) > 0$. Then $v' + v^2 = ((\alpha - 1)^2 - 1 + 2(\alpha - 1)z + 2z'x + z^2)/4x^2$. Since $\nu(z) > 0 > \nu(\log x)$, we have $\nu(z') > \nu(1/x)$, so that $v' + v^2$ is near $((\alpha - 1)^2 - 1)/4x^2$ and is as small as possible only if $\alpha = 1$. In this case $v = (1 + z)/2x$ and $v' + v^2 = -1/4x^2 + z'/2x + z^2/4x^2$. Thus $v' + v^2$ is small for v in a Hardy field only if $v = (1 + z)/2x$ and $z'/2x + z^2/4x^2$ is small for elements z in a Hardy field such that $\nu(z) > 0$. As before, for minimality we have $z'/2x + z^2/4x^2 < 0$, so $|z'/2x| > z^2/4x^2$, so $\nu(z'/2x) \leq \nu(z^2/4x^2)$, and we can continue as before, and then repeat this process. This leads in a natural way to the expressions in the following lemma. The common notation $l_0(x) = x$, $l_1(x) = \log x$, $l_2(x) = \log\log x$, etc., is employed. The lemma itself is easily proved by induction on n. Note that for $n = 0$ the sum for v has only one term.

LEMMA. *For any nonnegative integer n and*

$$v = \frac{1}{2x} + \frac{1}{2xl_1(x)} + \cdots + \frac{1}{2xl_1(x)\ldots l_{n-1}(x)} + \frac{1+z}{2xl_1(x)\ldots l_n(x)}$$

we have

$$v' + v^2 = -\frac{1}{(2x)^2} - \frac{1}{(2xl_1(x))^2} - \cdots - \frac{1}{(2xl_1(x)\ldots l_n(x))^2} + \frac{z'}{2xl_1(x)\ldots l_n(x)} + \frac{z^2}{(2xl_1(x)\ldots l_n(x))^2}.$$

THEOREM 3. *Let n be a nonnegative integer, z an element of a Hardy field, and*

$$v = \frac{1}{2x} + \frac{1}{2xl_1(x)} + \cdots + \frac{1}{2xl_1(x)\ldots l_{n-1}(x)} + \frac{1+z}{2xl_1(x)\ldots l_n(x)}$$

Then

(1) $v' + v^2 \le -1/(2x)^2 - \cdots - 1/(2xl_1(x)\ldots l_n(x))^2$ *only if* $\nu(z) \ge \nu(1/l_{n+1}(x))$;

(2) *if* $\nu(z) \ge \nu(1/l_{n+1}(x))$, *then*

$$\nu\left(v' + v^2 + \frac{1}{(2x)^2} + \cdots + \frac{1}{(2xl_1(x)\ldots l_n(x))^2}\right) \ge \nu(1/(xl_1(x)\ldots l_{n+1}(x))^2);$$

(3) *if F is an element of a Hardy field each infinitely increasing element of which exceeds some repeated logarithm of x and the differential equation $Y'' = F(x)Y$ has a nonzero solution in some Hardy field, then for some n we have*

$$F > -\frac{1}{(2x)^2} - \frac{1}{(2xl_1(x))^2} - \cdots - \frac{1}{(2xl_1(x)\ldots l_n(x))^2}.$$

The statement in (1) is equivalent to

$$\frac{z'}{2xl_1(x)\ldots l_n(x)} \le \frac{-z^2}{(2xl_1(x)\ldots l_n(x))^2},$$

so that if $z \ne 0$ then $|z'/z^2| \ge 1/2xl_1(x)\ldots l_n(x)$, so $\nu((1/z)') = \nu(z'/z^2) \le \nu(1/2xl_1(x)\ldots l_n(x)) = \nu((l_{n+1}(x))')$. If $\nu(z) = 0$, then for some $\alpha \in \mathbb{R}$ we have $\nu((1/z) - \alpha) > 0 > \nu(l_{n+1}(x))$, so $\nu((1/z)') > \nu((l_{n+1}(x))')$. Therefore $\nu(z) \ne 0$, and by [3, Lemma to Proposition 6] we have $\nu(1/z) \le \nu(l_{n+1}(x))$, proving the first part. If now $\nu(z) \ge \nu(1/l_{n+1}(x))$, then $\nu(z') \ge \nu(1/xl_1(x)\ldots l_n(x)(l_{n+1}(x))^2)$, so that

$$\nu\left(v' + v^2 + \frac{1}{(2x)^2} + \cdots + \frac{1}{(2xl_1(x) \ldots l_n(x))^2}\right)$$
$$= \nu\left(\frac{z'}{2xl_1(x) \ldots l_n(x)} + \frac{z^2}{(2xl_1(x) \ldots l_n(x))^2}\right)$$

which is $\geq \nu(1/(xl_1(x) \ldots l_{n+1}(x))^2)$, as claimed for the second part. For the third part, the assumptions made show that there are solutions y_1, y_2 of $Y'' = F(x)Y$ having the properties given in Theorem 1. In particular, $\nu(1/y_1^2) = \nu((y_2/y_1)')$. Since $\nu(y_2/y_1) < 0$, y_2/y_1 is infinitely increasing, so that $y_2/y_1 > l_n(x)$ for some n, so that $\nu(y_2/y_1) \leq \nu(l_n(x))$, so $\nu(1/y_1^2) = \nu((y_2/y_1)') \leq \nu((l_n(x))') = \nu(1/xl_1(x) \ldots l_{n-1}(x))$ and therefore $\nu(y_1) \geq \nu((xl_1(x) \ldots l_{n-1}(x))^{1/2}) > \nu((xl_1(x) \ldots l_n(x))^{1/2})$. Now the logarithmic derivative of $(xl_1(x) \ldots l_n(x))^{1/2}$ is $1/2x + \cdots + 1/2xl_1(x) \ldots l_n(x)$ which is a solution of the Riccati equation

$$V' + V^2 = -\frac{1}{(2x)^2} - \cdots - \frac{1}{(2xl_1(x) \ldots l_n(x))^2},$$

so that $(xl_1(x) \ldots l_n(x))^{1/2}$ is a solution of the linear differential equation

$$Y'' = \left(-\frac{1}{(2x)^2} - \cdots - \frac{1}{(2xl_1(x) \ldots l_n(x))^2}\right) Y.$$

This solution is also the first of a pair of solutions of the latter differential equation in the sense of the pair y_1, y_2 of Theorem 1, since $\int (1/(xl_1(x) \ldots l_n(x))) = l_{n+1}(x)$, which is infinitely increasing. Comparing the solution y_1 of $Y'' = F(x)Y$ with the solution $(xl_1(x) \ldots l_n(x))^{1/2}$ of the other linear differential equation and using the last part of Theorem 2 gives (3), which completes the proof.

Note that the special condition on F in the first part of (3) above is automatically satisfied if F lies in a Hardy field each element of which is contatined in a Hardy field of finite rank, in particular if F satisfies an algebraic differential equation [3, Theorem 2].

It is useful to work out all the data of Theorem 1 for the differential equation

$$Y'' = \left(-\frac{1}{(2x)^2} - \frac{1}{(2xl_1(x))^2} - \cdots - \frac{1}{(2xl_1(x) \ldots l_n(x))^2}\right) Y$$

which occurs at the end of the last proof. We know that we can take $y_1 = (xl_1(x) \ldots l_n(x))^{1/2}$. We can then take $y_2 = y_1 l_{n+1}(x)$, getting $C = 1$, $v_1 = 1/2x + 1/2xl_1(x) + \cdots + 1/2xl_1(x) \ldots l_n(x)$, $v_2 = v_1 + 1/xl_1(x) \ldots l_{n+1}(x)$.

2. Asymptotic Expansions

Suppose that the differential equation $Y'' = F(x)Y$ has two linearly independent solutions in a Hardy field and let y_1, y_2, v_1, v_2 be as in Theorem 1. Then $v_1 = y_1'/y_1$ is unique and may be characterized as the smallest solution (in a suitably large Hardy field) of the Riccati equation $V' + V^2 = F(x)$. If we are given an asymptotic approximation of v_1, that is, an element of a Hardy field containing v_1 that approximates v_1 in some sense depending on the valuation of the Hardy field, then from the equation $y_1'/y_1 = v_1$ we can obtain some sort of approximation of y_1, up to a constant factor, and from the equation $(y_2/y_1)' = C/y_1^2$ we can obtain some sort of approximation for y_2, up to a constant factor and the addition of an arbitrary multiple of y_1. Therefore a good approximation for v_1, in some suitable sense, can give us good approximations for all solutions of the equations $Y'' = F(x)Y$ and $V' + V^2 = F(x)$. An arbitrary solution of the Riccati equation can be written $v = y'/y$, where y is a nonzero solution of $Y'' = F(x)Y$, and if $v \neq v_1$ then y will not be a multiple of y_1, hence will be a multiple of $y_2 + cy_1$, for some $c \in \mathbb{R}$, so $v = (y_2' + cy_1')/(y_2 + cy_1)$. Thus for any solution $v \neq v_1$ of the Riccati equation we have $v - v_2 = (y_2' + cy_1')/(y_2 + cy_1) - y_2'/y_2 = -Cc/y_2(y_2 + cy_1)$, which can be approximated by the partial sums of the series

$$-\frac{C}{y_2^2}\left(c - c^2\frac{y_1}{y_2} + c^3\left(\frac{y_1}{y_2}\right)^2 - c^4\left(\frac{y_1}{y_2}\right)^3 + \cdots\right).$$

This is how the nonminimal solutions of the Riccati equation form a family parametrized by a real number c. Another way to look at the variation of $v \neq v_1$ is to note that $\nu(v - v_2) = \nu(-cC/y_2(y_2 + cy_1)) = \nu(1/y_2^2)$, so that if y_2 is large then the variation in v is small.

Let v and F lie in a Hardy field and let $v' + v^2 = F$. If $\nu(v) < 0$, then $\nu(1/v) > 0 > \nu(x)$, so that $\nu(v'/v^2) = \nu((1/v)') > \nu(x') = 0$, so that $\nu(v') > \nu(v^2)$ and $v^2 \sim F$, so that $\lim_{x\to+\infty} F(x) = +\infty$. For $\nu(v) \geq 0$, we have $\nu(v) > \nu(x)$, so that $\nu(v') > v(x') = 0$. Therefore if $\nu(v) = 0$ then $v^2 \sim F$ and $\lim_{x\to+\infty} F(x)$ is some positive real number. If $\nu(v) > 0$ then $\lim_{x\to+\infty} F(x) = 0$.

Asymptotic solutions of $V' + V^2 = F(x)$ in the case $\lim_{x\to+\infty} F(x) = +\infty$ are given by [2, Theorem 5], which we reproduce here as Theorem 4 for the convenience of the reader.

Theorem 4. *Let f be an element of a Hardy field such that $\nu(f) < 0$. Then any solution of the differential equation $V' + V^2 = f^2$ in an extension Hardy field is $\sim \pm f$ and there exists a solution v in a suitable extension Hardy field such that $v \sim f$. Furthermore, there exist differential polynomials $A_1(G)$, $A_2(G)$, ..., independent of f, in the ring $\mathbb{Q}[G, G', G''$,*

...], *where* G *is a differential indeterminate, with each* $A_i(G)$ *homogeneous of weight* i *if the ring is graded so that* $G, G', G'', \dots$ *are homogeneous of weights* $1, 2, 3, \dots$, *respectively, such that if* $f'/f = g$ *and* $a_i = A_i(g)$ *for* $i \geq 1$, *then for any real* $\varepsilon > 0$ *we have* $\nu(a_i) > \nu(|f|^{\varepsilon})$ *and*

$$\nu(v - f(1 + a_1 f^{-1} + \cdots + a_i f^{-i})) > \nu(|f|^{\varepsilon - i})$$

for all $i \geq 0$. *If* $\nu(g) \geq \nu(x^{-1})$, *which case necessarily arises if* $\nu(f) > \nu(x^{\varepsilon})$ *for all real* $\varepsilon > 0$, *we have*

$$\nu(v - f(1 + a_1 f^{-1} + \cdots + a_i f^{-i})) > \nu(x^{-i-1})$$

for all $i > 0$.

In applying this result to the equation $V' + V^2 = F(x)$, we of course have $f^2 = F$. We get the expansion of the unique v_1 by taking $f = -\sqrt{F}$ and we get v_2 by taking $f = \sqrt{F}$. For any solution $v \neq v_1$ of $V' + V^2 = F(x)$, the theorem gives the same asymptotic expansions for v and v_2. This seeming contradiction is explained as follows. $\nu(y_2'/y_2) = \nu(v_2) = \nu(f) < \nu(f'/f)$ (and $<\nu(1/x)$ in the last contingency), so that by [3, Propositions 3 and 4] the comparability class of y_2 exceeds that of f (and of x in the last case). Also, $v_2 \sim \sqrt{F}$ is infinitely increasing, hence also $\int v_2$, hence also $e^{\int v_2} = y_2$. Thus $1/y_2^2$ is infinitely smaller than either $1/f$ or $1/x$, that is, $\nu(1/y_2^2) = \nu(v - v_2) \geq \nu(f^{-N}), \nu(x^{-N})$ for any integer N. The above unique asymptotic expansion for v_2 is not sufficiently fine to distinguish between v and v_2.

Consider now the equation $V' + V^2 = F(x)$, where $F(x)$ belongs to a Hardy field, $F(x) > 0$, and $\nu(F(x)) = 0$. Here $\lim_{x \to +\infty} F(x)$ is a positive real number. In this case the argument of [2, pp. 306–308], which led to Theorem 4, can be mimicked, with certain appropriate modifications, as follows. First, for large x_0 we have a solution y of $Y'' = F(x)Y$ on $[x_0, +\infty)$ with any prescribed positive values for $y(x_0)$ and $y'(x_0)$ and this y will have no zeros, hence lie in a Hardy field. Therefore the equation $Y'' = F(x)Y$ will have two linearly independent solutions in a Hardy field. The equation $V' + V^2 = F(x)$ will have a solution v in this Hardy field. For any such v, $\nu(v) = 0$ and $v^2 \sim F(x)$, so $v \sim f = \pm\sqrt{F(x)}$, if f is taken with the appropriate sign, giving

$$v = f(1 - v'/f^2)^{1/2}.$$

If we let $g = f'/f = F'/2F$, then $\nu(f - \lim_{x \to +\infty} f(x)) > 0 > \nu(\log x)$, so that $\nu(g) = \nu(f') > \nu(1/x)$. Thus $\nu(g') > \nu(1/x^2)$ and for all $i \geq 0$ we have $\nu(g^{(i)}) > \nu(1/x^{i+1})$.

If w is in our Hardy field and $w \sim f$, then $\nu(w'/f) > 0$ and an easy computation, as in [2], shows that

$$\nu(v - f(1 - w'/f^2)^{1/2}) = \nu((w - v)').$$

As in [3], let $P_r(t)$ be the sum of the first $r + 1$ terms of the binomial expansion of $(1 - t)^{1/2}$. Then if $w \sim f$ we get

$$\nu((1 - w'/f^2)^{1/2} - P_r(w'/f^2)) \geq (r + 1)\nu(w'/f^2) = (r + 1)\nu(w').$$

Therefore

$$\nu(v - fP_r(w'/f^2)) \geq \min\{\nu((w - v)'), (r + 1)\nu(w')\}.$$

As in [2], we set $w_1 = f$ and $w_{i+1} = fP_i(w'/f^2)$ if $i \geq 1$. By induction

$$\nu(v - w_i) > \nu(x^{l-i})$$

for all $i \geq 0$. As a consequence, if y_1, y_2, v_1, v_2 are as in Theorem 1 and $v_1 \sim v_2$, then $\nu(y_1'/y_1 - y_2'/y_2) = \nu(v_1 - v_2) > \nu(1/x^2)$, so that for some $A \in \mathbb{R}$ we have $\log(y_1/y_2)$ bounded for $x > A$, so that y_1/y_2 lies between two positive constants, implying the falsehood $\nu(y_1/y_2) = 0$. Thus $v_1 \nsim v_2$. Therefore $v_1 \sim -\sqrt{F}$, $v_2 \sim \sqrt{F}$. Now note that each w_i/f is an element of the differential ring $\mathbb{Q}[1/f, g, g', g'', \ldots]$. The reasoning of [2, p. 308] can be followed through, to obtain the results summarized in the next theorem.

THEOREM 5. *Let f be an element of a Hardy field such that $\nu(f) = 0$. Then any solution of the differential equation $V' + V^2 = f^2$ in an extension Hardy field is $\sim \pm f$ and there exists a solution v in a suitable extension Hardy field such that $v \sim f$. Furthermore, there exist differential polynomials $A_1(G)$, $A_2(G)$, $\ldots$, exactly the same ones as in Theorem 4, independent of f, in the ring $\mathbb{Q}[G, G', G'', \ldots]$, where G is a differential indeterminate, with each $A_i(G)$ homogeneous of weight i if the ring is graded so that G, G', G'', $\ldots$ are homogeneous of weights 1, 2, 3, $\ldots$, respectively, such that if $f'/f = g$ and $a_i = A_i(g)$ for $i \geq 1$, then $\nu(a_i) > \nu(1/x^{i+1})$ for all $i \geq 1$ and for all $i \geq 1$ we have*

$$\nu(v - f(1 + a_1f^{-1} + a_2f^{-2} + \cdots + a_if^{-i})) > \nu(1/x^{i+1}).$$

In the context of this result, let the positive real number α be such that $|f| \sim \alpha$. If we use the notation of Theorem 1, then for any real $\varepsilon > 0$ we have $-\alpha - \varepsilon < v_1 < -\alpha + \varepsilon$, $\alpha - \varepsilon < v_2 < \alpha + \varepsilon$, and $y_1 = e^{(-\alpha+\varepsilon_1)x}$, $y_2 = e^{(\alpha+\varepsilon_2)x}$ where $\varepsilon_1(x)$, $\varepsilon_2(x)$ are germs approaching zero as $x \to +\infty$. If $v \neq v_1$ is another solution of the Riccati equation, the variation $v - v_2$ is infinitesimal compared to x^{-i}, for each i, since $\nu(v - v_2) = \nu(1/y_2^2) = \nu(e^{-2(\alpha+\varepsilon_2)x}) > \nu(x^{-i})$, so it is not unreasonable for v and v_2 to have the same asymptotic expansions. However, the asymptotic expansions of the theorem may be of little value in specific cases, as for example for the equation $V' + V^2 = 1 + e^{-x^n}$, for any positive number n. The next result enables us to find very sharp estimates of v_1, together with error estimates, by using successive approximations.

THEOREM 6. *Let α be a positive real number and let F, v_1, w be elements of a Hardy field such that $F \sim \alpha^2$, $v_1 \sim -\alpha$ is a solution of the equation $V' + V^2 = F(x)$ and $w \sim -\alpha$, $w \neq v_1$. Then $v_1 = w + h$, where $h \sim b^2/(b' + 2bw)$, with $b = F - w' - w^2$.*

h is the unique field element with the properties that $\nu(h) > 0$ and $h' + 2wh + h^2 = b$. Let $I = e^{\int 2w}$, where $\int$ denotes any antiderivative. Since $-\alpha - \varepsilon < w < -\alpha + \varepsilon$ for any real $\varepsilon > 0$, $(-\alpha - \varepsilon)x < \int w < (-\alpha + \varepsilon)x$, so that $e^{2(-\alpha-\varepsilon)x} < I < e^{2(-\alpha+\varepsilon)x}$, so $\nu(I) > 0$. We have the equation

$$(Ih)' + h^2 I = bI.$$

If we had $\nu((Ih)') \geq \nu(h^2 I)$, it would follow that $\nu((1/Ih)') = \nu((Ih)'/(Ih)^2) \geq \nu(1/I) = \nu((1/I)')$, and since $\nu(Ih)$, $\nu(I) \neq 0$, we would have $\nu(1/Ih) = \nu(1/I)$, contrary to $\nu(h) > 0$. Therefore $\nu((Ih)') < \nu(h^2 I)$, so that $(Ih)' \sim bI$. Now $\nu(b) > 0$, so the positive germ $|b|$, which approaches zero as $x \to +\infty$, is decreasing, giving $|b|' < 0$. Since $(bI)' = bI(b'/b + 2w) = bI(|b|'/|b| + 2w)$, we get $|(bI)'| > |bI\alpha|$, or $\nu((bI)') \leq \nu(bI)$. Let u be such that $u' = bI$ and $\nu(u) \neq 0$. If $\nu(u) < \nu(bI)$, then $\nu(u') < \nu((bI)')$, which is false, so $\nu(u) \geq \nu(bI)$. Therefore $\nu(u/bI) \geq 0$, so that $\nu((u/bI)') > 0$, that is, $\nu((bIu' - u(bI)') > \nu(b^2I^2)$, or $\nu(b^2I^2 - u(bI)') > \nu(b^2I^2)$, so that $b^2I^2 \sim u(bI)'$ and $u \sim (bI)^2/(bI)'$. Going back to $(Ih)' \sim bI$ we get $(Ih)' \sim u'$, so that $Ih \sim u \sim (bI)^2/(bI)'$, whence $h \sim b^2/(b' + 2bw)$.

The proof of the last theorem fails at several points if we try to use it to find $v_2 \sim \alpha$ such that $v_2' + v_2^2 = F$, as might be guessed from the fact that v_2 is not unique. If $w \sim \alpha$ and we set $v_2 = w + h$ we have an element h such that $\nu(h) > 0$ which satisfies the same equation $h' + 2wh + h^2 = b$, with b as before. But now $\nu(I) < 0$, so possibly $\nu(Ih) = 0$, scuttling the proof that $(Ih)' \sim bI$. Even if the latter is true, the next step $\nu((bI)') \leq \nu(bI)$ can also fail.

In certain cases we can use Theorem 6 to get explicit asymptotic expansions for v_1. For example, if $F = 1 + e^{-x}$ we can start with $w = -1$ and use successive approximations to obtain approximants to v_1 that are polynomials in e^{-x} with rational coefficients. These approximate v_1 to within arbitrarily high powers of e^{-x}. To calculate we might set $e^{-x} = t$ with $t' = -t$, set $v_1 = -1 + a_1t + a_2t^2 + \ldots$ with undetermined constant coefficients, and obtain the coefficients from the equation $v_1' + v_1^2 = 1 + t$. We get $v_1 \sim -1 - t/3 + t^2/36 - t^3/270 + \cdots = -1 - e^{-x}/3 + e^{-2x}/36 - e^{-3x}/270 + \ldots$. But if we try to find $v_2 \sim 1 + b_1 t + b_2 t^2 + \ldots$ with constant coefficients such that $v_2' + v_2^2 = 1 + t$ the computation breaks down at t^2.

Consider finally the case of equations $V' + V^2 = F(x)$, assumed to have solutions in a Hardy field, where $\nu(F) > 0$. To find asymptotic approximations for the minimal solution v_1 of this equation we effect a well-known change of variable, possibly repeated, to reduce the present problem to one of the cases previously considered. If $y'' = F(x)y$ and we set $x = e^\xi$, $y = \eta e^{\xi/2}$ (so that $\xi = \log x$, $\eta = yx^{-1/2}$), then

$$\frac{d^2\eta}{d\xi^2} = \left(\frac{1}{4} + x^2F(x)\right)\eta = \left(\frac{1}{4} + e^{2\xi}F(e^\xi)\right)\eta.$$

That is, if to x, y, y_1, y_2, v, v_1, v_2, $F(x)$ for the original equation we make correspond ξ, η, η_1, η_2, ω, ω_1, ω_2, $\Phi(\xi)$ for the second equation, we have $\Phi(\xi) = 1/4 + e^{2\xi}F(e^\xi)$. Also, $\nu(y)$ is maximal if $y = y_1$ and the corresponding $\nu(\eta)$ is maximal if $\eta = \eta_1$. Corresponding to any v, we have $\omega = (d\eta/d\xi)/\eta = -\frac{1}{2} + vx$, and, in particular, $\omega_1 = -\frac{1}{2} + v_1 x$. Therefore if we have an approximation for ω_1 in terms of ξ, we can get from it an approximation for v_1 in terms of x. If $Y'' = F(x)Y$ has solutions in a Hardy field and the condition of Theorem 3(3) holds, then for some n we have

$$F(x) > -\frac{1}{(2x)^2} - \frac{1}{(2xl_1(x))^2} - \cdots - \frac{1}{(2xl_1(x)\ldots l_n(x))^2},$$

and then

$$\begin{aligned}\Phi(\xi) = \frac{1}{4} + x^2F(x) &> -\frac{1}{(2l_1(x))^2} - \cdots - \frac{1}{(2l_1(x)\ldots l_n(x))^2}\\ &= -\frac{1}{(2\xi)^2} - \cdots - \frac{1}{(2\xi l_1(\xi)\ldots l_{n-1}(\xi))^2},\end{aligned}$$

so that $\Phi(\xi)$ involves a smaller n than $F(x)$. After repeating this process, if necessary, we reduce to the case $F(x) > 0$ and then $\Phi(\xi) > 1/4$. As an example, corresponding to the differential equation $V' + V^2 = -x^{-3}$ we have $\Omega' + \Omega^2 = 1/4 - e^{-\xi}$. By the argument in the last paragraph, ω_1 will have an asymptotic expansion $\omega_1 \sim -\frac{1}{2} + a_1e^{-\xi} + a_2e^{-2\xi} + \ldots$, for certain numerical coefficients a_1, a_2, $\ldots$, so that $v_1 = 1/2x + \omega_1/x \sim a_1x^{-2} + a_3x^{-3} + \ldots$, and this works out to be $v_1 \sim 1/2x^2 + 1/12x^3 + 1/48x^4 + \ldots$.

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