SOME TRANSCENDENTAL FUNCTIONS THAT YIELD TRANSCENDENTAL VALUES FOR EVERY ALGEBRAIC ENTRY

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Abstract. A transcendental function usually yields a transcendental value for an algebraic entry belonging to its domain, the algebraic exceptions forming the so-called *exceptional set*. For instance, the exceptional set of the function $\exp(z)$ is the unitary set $\{0\}$, which follows from the Hermite-Lindemann theorem. In this note, we give some explicit examples of transcendental entire functions having empty exceptional sets, i.e. functions that yield transcendental values for all algebraic entries, without exceptions.

1. Introduction

An algebraic function is a function $f(x)$ which satisfies $P(x, f(x)) = 0$, where $P(x, y)$ is a polynomial in x and y with integer coefficients. Functions that can be constructed using only a finite number of elementary operations are examples of algebraic functions. A function which

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is not algebraic is, by definition, a *transcendental function*, as e.g. the trigonometric functions, the exponential function, and their inverses. By evaluating a transcendental function at an algebraic point of its domain, one commonly finds a transcendental number, but exceptions can take place. All these exceptions (i.e., algebraic numbers at which the function assumes algebraic values) form the so-called *exceptional set*. This set plays an important role in transcendental number theory (see, e.g., Ref. [1] and references therein). It can be applied, e.g., for proofing that e and π are transcendental numbers [2].

The Hermite-Lindemann theorem (1884) was the first relevant, general result in this direction [3]. It can be written in the following simple form.

Lemma 1 (Hermite-Lindemann)**.** *The number e^z is transcendental for all non-null algebraic values of z.*

Since $e^0 = 1$, the exceptional set of this function consists only of $z = 0$. This 'almost empty' result led people to search for a transcendental function that could yield transcendental values for all algebraic points of its domain. The existence of such transcendental functions was first proved by Stäckel (1895), as a corollary of his main theorem [4].

Lemma 2 (Stäckel)**.** *For any countable set $A \subseteq \mathbb{C}$ and each dense set $T \subseteq \mathbb{C}$, there is a transcendental entire function f such that $f(A) \subseteq T$.*

By putting $A = \overline{\mathbb{Q}}$ and $T = \mathbb{C} \setminus \overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ into $\mathbb{C}$, as usual, it follows that there exist transcendental functions that assume transcendental values for all algebraic points in their domain. However, it is not so easy to present explicit examples of such transcendental functions.

In a recent paper, Surroca remarked that under the hypothesis of the Schanuel's conjecture the function $e^{e^{\alpha}}$ takes transcendental values for all $\alpha \in \overline{\mathbb{Q}}$ [5]. This conditional result has bothered the authors because, currently, a proof for Schanuel's conjecture seems to be far away, if one exists. Here in this work, we make use of some classical results of analytic number theory for exhibiting some examples of elementary transcendental functions whose exceptional set is empty. We also indicate how interpolation techniques can be applied for building such functions.

2. Functions with an empty exceptional set

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental function and let us denote by S_f the *exceptional set* of f, i.e. the set of all $\alpha \in \overline{\mathbb{Q}}$ for which $f(\alpha) \in \overline{\mathbb{Q}}$. We are interested in transcendental entire functions f for which $S_f = \emptyset$.

Let us start our analysis by showing the truth of the Surroca's remark, above. For that, we recall the Schanuel's conjecture, one of the main open problems in transcendental number theory.

Conjecture 1 (Schanuel). *If $z_1, \ldots, z_n$ are complex numbers linearly independent over $\mathbb{Q}$, then among the numbers $\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}$, at least n are algebraically independent.*

This conjecture was introduced in the 1960's by Schanuel in a course given by Lang [6]. It has several important consequences, as for instance: at least two distinct numbers among $\alpha, e^{\alpha}, e^{\alpha}, e^{e^{\alpha}}$ are transcendental. Therefore $e^{e^{\alpha}}$ is transcendental. The Surroca's result follows by noting that $e^{e^0} = e^1 = e$, which is a transcendental number, as first proved by Hermite (1873) [7].

In searching for unconditional examples of transcendental functions with empty exceptional sets, some known results from transcendental number theory are essential ingredients. Let us state them for making this text self-contained.

Lemma 3 (Lindemann-Weierstrass). *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers linearly independent over $\mathbb{Q}$. Then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

For a proof of this theorem, see Chap. 9 of Ref. [9].[1] Another important tool comes from the work of Baker on linear forms of logarithms of algebraic numbers. It states that [3, Chap. 1]:

Lemma 4 (Baker)**.** *Let* $\alpha_1, \ldots, \alpha_n$ *be non-zero algebraic numbers and let* $\beta_0, \ldots, \beta_n$ *be algebraic numbers, with* $\beta_0 \neq 0$. *Then the number* $e^{\beta_0} \cdot \alpha_1^{\beta_1} \cdot \ldots \cdot \alpha_n^{\beta_n}$ *is transcendental.*

As our last tool, let us present the Mahler's classification scheme. This classification demands the knowledge of the following definitions involving polynomials of a complex variable with integer coefficients [8, Chap. 3].

Definition 1 (Height and length of $P(z)$)**.** *For a polynomial* $P(z) = \sum_{k=0}^{n} a_k\, z^k$, *the* height *of* $P(z)$ *is*

$$\mathcal{H}(P) := \max\{|a_0|, \ldots, |a_n|\}$$

and the length of $P(z)$ *is*

$$\mathcal{L}(P) := \sum_{k=0}^{n} |a_k| \, .$$

[1]Note that the Lindemann-Weierstrass theorem is implied by Schanuel's conjecture.

Given positive integers H and n, we set

$$\mathcal{P}_{n,H} := \{P(z) \in \mathbb{Z}[z] : \deg(P) \leq n \text{ and } \mathcal{H}(P) \leq H\}.$$

Now, for any $\xi \in \mathbb{C}$ we define

$$\Omega_n(\xi, H) := \min\{|P(\xi)| : P(z) \in \mathcal{P}_{n,H} \text{ and } P(\xi) \neq 0\}$$

and take $\omega_n(\xi, H)$ as the exponent satisfying

$$\Omega_n(\xi, H) = H^{-\omega_n(\xi,H)\, n}\,. \tag{1}$$

Finally, define

$$\omega_n(\xi) := \lim_{H\to\infty} \sup\, \omega_n(\xi, H)$$

and

$$\omega(\xi) := \lim_{n\to\infty} \sup\, \omega_n(\xi).$$

Of course, $\omega(\xi)$ is either infinity or not. By defining $\nu(\xi)$ as the least positive integer n for which $\omega_n(\xi) = \infty$, we have four classes corresponding to the four possibilities for the values of $\omega(\xi)$ and $\nu(\xi)$:

- If $\omega(\xi) = 0$, then ξ is called an A-**number**.
- If $0 < \omega(\xi) < \infty$, then ξ is called an S-**number**.
- If $\omega(\xi) = \infty$ and $\nu(\xi) < \infty$, then ξ is called a U-**number**.
- If $\omega(\xi) = \infty$ and $\nu(\xi) = \infty$, then ξ is called a T-**number**.

This is the so-called *Mahler's classification* for complex numbers. The following useful properties of this classification were proved by Mahler himself [8, Chap. 3].

Lemma 5. *A given number $\xi \in \mathbb{C}$ is algebraic if and only if $\omega(\xi) = 0$.*

In other words, the set of A-numbers can be identified to $\overline{\mathbb{Q}}$. Therefore, a transcendental number can be defined as a complex number such that $\omega(\xi) \neq 0$. This conclusion is relevant since it yields an affirmative way for checking the transcendence of a given complex number, thus overcoming one of the greatest obstacles in transcendental number theory, namely the fact that transcendental numbers are not known by what they *are* but rather by what they *are not*.

Lemma 6. *For any algebraic number $\alpha \neq 0$, e^{α} is an S-number.*

We recall that a real number x is called a *Liouville number* if, for all positive integer n, there exist integer numbers $p > 0$ and $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}\,. \tag{2}$$

The Mahler classification of these numbers is well-known.

Lemma 7. *All Liouville numbers are U-numbers.*

A classical example of a Liouville number is the Liouville's constant ℓ, defined as a decimal with a 1 in each decimal place corresponding to $n!$ and 0 otherwise. It can be represented by the fast convergent series $\ell = \sum_{n=1}^{\infty} 1/10^{n!} = 0.1100010\ldots$

One of the most important properties of the Mahler's classification scheme is that two algebraically dependent numbers should belong to the same class. We note that the contrapositive of this fact enable us to produce the following attractive transcendental result.

Lemma 8. *If z_1 and z_2 are two complex numbers having different Mahler classifications, then for any polynomial $P(x,y) \in \mathbb{Z}[x,y]$, with degree greater than zero in both its variables, the number $P(z_1,z_2)$ is transcendental.*

The proofs of all these facts involving the Mahler's classification can be found in [8, Chap. 3].

We got now sufficient machinery for building some elementary functions whose exceptional set is empty.

Theorem 1 (Three explicit examples)**.** *Each elementary, transcendental function $f(z) = e^z + e^{1+z}$, $g(z) = e^{1+\pi\, z}$, and $h(z) = \ell + e^z$, ℓ being the Liouville constant, has an empty exceptional set.*

In other words, the functions $f(z)$, $g(z)$, and $h(z)$ all yield transcendental values for every algebraic z, i.e. $S_f = S_g = S_h = \emptyset$.

Proof. For the function $f(z) = e^z + e^{1+z}$, let us take $z \in \overline{\mathbb{Q}} \setminus \{0, -1\}$, so that 0, z, and $z+1$ are distinct algebraic numbers. By the Lindemann-Weierstrass theorem, the numbers e^0, e^z, and e^{z+1} are algebraically independent over $\mathbb{Q}$. Hence $e^z + e^{z+1}$ has to be transcendental. For the omitted cases, i.e. $z \in \{0, -1\}$, $f(z)$ is transcendental because $1 + e$ and $e^{-1} + 1$ are transcendental, since e is a transcendental number [7]. Therefore, $S_f = \emptyset$.

For the function $g(z) = e^{1+\pi z}$, suppose that $g(z)$ is algebraic for some $z \in \overline{\mathbb{Q}}$. Then by substituting $n = 2$, $\alpha_1 = g(z) = e^{1+\pi z}$, $\alpha_2 = -1$, $\beta_0 = -1$, $\beta_1 = 1$, and $\beta_2 = i\, z$ in the Baker's theorem, with $i = \sqrt{-1}$, one finds that the number $e^{\beta_0}\, \alpha_1^{\beta_1}\, \alpha_2^{\beta_2}$ has to be transcendental. However, from the fact that $e^{i\,\pi} = -1$, one has

$$e^{\beta_0}\, \alpha_1^{\beta_1}\, \alpha_2^{\beta_2} = e^{-1} \left(e^{1+\pi\, z}\right)^1 \times (-1)^{i\, z} = e^{\pi\, z} \times \left(e^{i\,\pi}\right)^{i\, z} = 1\,,$$

which is an algebraic number. This is a contradiction and then $g(z)$ can not be algebraic for any algebraic z, hence $S_g = \emptyset$.

For the function $h(z) = \ell + e^z$, since ℓ is a Liouville number then, by Lemma 6, it is a U-number. Hence $h(0)$ is a U-number, thus transcendental. For $z \neq 0$, by putting any algebraic z in Lemma 7 one finds that $h(z)$ is a sum of ℓ, which is an U-number, and e^z, which is a S-number. Therefore, by Lemma 8, this sum has to be a transcendental number. Therefore, $S_h = \emptyset$. $\square$

3. Further examples of transcendental functions with an empty exceptional set

We can use polynomial interpolation in searching for more examples of transcendental functions with an empty exceptional set. Let us introduce this creative method by establishing some fundamental rules.

Lemma 9. *Given $n \geq 1$ complex numbers $a_0, \ldots, a_n$, not all zero, the polynomial*

(3)

$$P_n(z) = a_0\,(z+1)\cdots(z^n+1) + \sum_{k=1}^{n} a_k\, z^k\,(z+1)\cdots(\widehat{z^k+1})\cdots(z^n+1)\,,$$

where the symbol $x_1 \cdots \widehat{x_k} \cdots x_n$ means that x_k is omitted in the product $x_1 \cdots x_n$, is not the null polynomial.

Proof. Suppose the contrary, i.e. that $P_n(0) = 0$. Then $a_0 = 0$ and

$$P_n(z) = \sum_{k=1}^{n} a_k\, z^k\,(z+1)\cdots(\widehat{z^k+1})\cdots(z^n+1)\,. \tag{4}$$

Of course, its derivative $P_n'(z)$ is also null at $z = 0$. Thus $P_n'(0) = a_1$. Thus

$$P_n(z) = \sum_{k=2}^{n} a_k\, z^k\, (z+1)\cdots\widehat{(z^k+1)}\cdots(z^n+1)\,. \tag{5}$$

Similarly, we get $a_2 = a_3 = \cdots = a_n = 0$ from the higher derivatives $P_n''(0)$, $P_n^{(3)}(0), \ldots, P_n^{(n)}(0)$, respectively. But this contradicts our hypothesis on the coefficients a_k of $P_n(z)$. $\square$

Lemma 10. *Let $P_n(z) \in \mathbb{C}[z]$ be a polynomial with degree n. Then*

$$|P_n(z)| \le \mathcal{L}(P)\, \max\{1, |z|\}^n\,, \tag{6}$$

for all $z \in \mathbb{C}$.

Proof. If $P_n(z) = a_0 + a_1 z + \ldots + a_n z^n$, with $a_n \neq 0$, then

$$\begin{aligned} |P_n(z)| &= |a_0 + a_1 z + \ldots + a_n z^n| \\ &\le |a_0| + |a_1|\,|z| + \ldots + |a_n|\,|z|^n \\ &\le \mathcal{L}(P)\max\{1, |z|\}^n\,. \end{aligned}$$

$\square$

In the next theorem, we shall make use of the following basic symmetric polynomials in n variables:

$$\begin{aligned}
\sigma_1(x_1,\ldots,x_n) &:= x_1+\ldots+x_n \\
&\vdots \\
\sigma_k(x_1,\ldots,x_n) &:= \sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k} \\
&\vdots \\
\sigma_n(x_1,\ldots,x_n) &:= x_1\cdots x_n\,,
\end{aligned}$$

which are known as the *elementary symmetric polynomials* in $x_1,\ldots,x_n$. Note that σ_k is the sum of all distinct products of k of the n variables. An important characteristic of these polynomials is that they satisfy the following identity:

(7)

$$(z-x_1)\cdots(z-x_n) = z^n-\sigma_1(x_1,\ldots,x_n)\, z^{n-1}+\ldots+(-1)^n\,\sigma_n(x_1,\ldots,x_n)\,.$$

Now, let $\{\alpha_1,\alpha_2,\ldots\}$ be an enumeration of $\overline{\mathbb{Q}}$. We define the function $\mathcal{U}:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ as

(8)

$$\mathcal{U}(w,z) := \sum_{n=1}^{\infty} \frac{w^n}{\left[1+\sum_{k=1}^{n}|\sigma_k(\alpha_1,\ldots,\alpha_n)|\right](|w|^n+1)\;n!}\,(z-\alpha_1)\cdots(z-\alpha_n)\,.$$

Now we can state our main result.

Theorem 2 (More examples from polynomial interpolation)**.** *For any complex number T and any algebraic number α, the number $\mathcal{U}(T,\alpha)$ is transcendental if and only if T is transcendental.*

Note that this implies that

$$S_{\mathcal{U}(T,\cdot)} = \begin{cases} \emptyset\,, & \text{if} \quad T \text{ is transcendental} \\ \overline{\mathbb{Q}}\,, & \text{if} \quad T \text{ is algebraic}\,. \end{cases}$$

Proof. First, we shall show that the function $\mathcal{U}$ is analytic in $\mathbb{C}^2$. Set $Q_n(z) = \prod_{k=0}^{n}(z-\alpha_k)$. Then, by Lemma 10, $|Q_n(z)| \le \mathcal{L}(Q_n)\, \max\{1,|z|\}^n$, for all $z \in \mathbb{C}$. On the other hand, due to the identity in Eq. (7), $\mathcal{L}(Q_n) = 1 + \sum_{k=0}^{n} |\sigma_k(\alpha_1,\ldots,\alpha_n)|$. Thus

$$\begin{aligned} |\mathcal{U}(w,z)| &\le \sum_{n=1}^{\infty} \frac{|w|^n}{(1+\sum_{k=1}^{n}|\sigma_k(\alpha_1,\ldots,\alpha_n)|)(|w|^n+1)n!}\mathcal{L}(Q_n)\max\{1,|z|\}^n \\ &\le \sum_{n=1}^{\infty} \frac{\max\{1,|z|\}^n}{n!} = e^{\max\{1,|z|\}} \end{aligned}$$

Therefore, this function is well defined and, in fact, analytic in $\mathbb{C}^2$ (Weierstrass M-test). Given a pair $(T,\alpha) \in \mathbb{C}\times\overline{\mathbb{Q}}$, we have that $\alpha = \alpha_{t+1}$ for some $t \ge 0$. Thus substituting $z=\alpha$ and $w=T$ in Eq. (8) and multiplying both sides by $(T+1)(T^2+1)\cdots(T^t+1)$, we get

$$\sum_{k=1}^{t} a_k T^k (T+1)\cdots\widehat{(T^k+1)}\cdots(T^t+1) = \mathcal{U}(T,\alpha)\prod_{k=1}^{t}\left(T^k+1\right) \tag{9}$$

where the coefficients

$$a_k = \frac{(\alpha-\alpha_1)\cdots(\alpha-\alpha_k)}{\left[1+\sum_{k=1}^{n}|\sigma_k(\alpha_1,\ldots,\alpha_n)|\right]\, n!}$$

are algebraic numbers. According to Lemma 9, the equality in Eq. (9) means that the number T is a root of the non-null polynomial

(10)

$$P_n(z) = -\mathcal{U}(T,\alpha)(z+1)\cdots(z^t+1)+\sum_{k=1}^{t} a_k z^k (z+1)\cdots(\widehat{z^k+1})\cdots(z^t+1)\,.$$

However the polynomial $P_n(z) \in \overline{\mathbb{Q}}[z]$ if and only if $\mathcal{U}(T,\alpha) \in \overline{\mathbb{Q}}$. As the set $\overline{\mathbb{Q}}$ is algebraically closed and $P_n(T) = 0$, we conclude that T is transcendental if and only if $\mathcal{U}(T,\alpha)$ is transcendental and this completes our proof. □

Clearly, this theorem allows one to construct an uncountable family of transcendental entire functions with an empty exceptional set, according to the following corollary.

Corollary 1 (Uncountable number of explicit examples)**.** *Every transcendental entire function of the family*

$$\mathcal{F} = \left\{\mathcal{U}(T,\cdot) : \mathbb{C} \to \mathbb{C},\ \textit{with}\, T \notin \overline{\mathbb{Q}}\,\right\}. \tag{11}$$

has an empty exceptional set.

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