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On Nash equilibria and improvement cycles in pure positional strategies for Chess-like and Backgammon-like n-person games

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ABSTRACT

We consider n-person positional games with perfect information modeled by finite directed graphs that may have directed cycles, assuming that all infinite plays form a single outcome c, in addition to the standard outcomes $a_1, \dots, a_m$ formed by the terminal positions. (For example, in the case of Chess or Backgammon $n = 2$ and c is a draw.) These $m+1$ outcomes are ranked arbitrarily by n players. We study existence of (subgame perfect) Nash equilibria and improvement cycles in pure positional strategies and provide a systematic case analysis assuming one of the following conditions:

(i) there are no random positions; (ii) there are no directed cycles; (iii) the ïnfinite outcome" c is ranked as the worst one by all n players; (iv) $n = 2$; (v) $n = 2$ and the payoff is zero-sum.

1. Summary

In this article we study Nash-solvability (NS) and improvement acyclicity (AC) of finite n-person positional games with perfect information in pure positional strategies. A game is modeled by a finite directed graph (digraph) $G = (V, E)$ whose vertex-set V is partitioned into $n+2$ subsets $V = V_1 \cup \cdots \cup V_n \cup V_R \cup V_T$, where V_i are the positions of a player $i \in I = \{1, \dots, n\}$, while V_R and $V_T = \{a_1, \dots, a_m\}$ are the random and terminal positions. A game is called Chess-like if $V_R = \emptyset$ and Backgammon-like in general.

Digraph G may have directed cycles (di-cycles), yet, we assume that all infinite plays form a single outcome c, in addition to the terminal outcomes V_T.

The standard Chess and Backgammon are two-person ($n = 2$) zero-sum games in which c is defined as a draw. Yet, in a Chess-like game, n players $I = \{1, \dots, n\}$ may rank $m+1$ outcomes $A = \{a_1, \dots, a_m, c\}$ arbitrarily. Furthermore, an arbitrary real-valued *utility* (called also payoff) function $u : I \times A \to \mathbb{R}$ is defined for a Backgammon-like game.

Remark 1. The Backgammon-like games can be viewed as *transition-free stochastic games with perfect information*, in which the local reward $r(i, e)$ is 0 for each player $i \in I$ and move e, unless e is a *terminal move*, which results in a terminal position

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Table 1
Main diagram. Notation: NS = Nash-solvability, AC = Acyclicity; I = Initialized: for a given initial position, U = Uniform: for all initial positions simultaneously; Y = the property holds for the class, N = it does not, ? = the problem is open; the letter is calligraphic if the corresponding result is new and it is bold if the table contains no stronger result, otherwise, the font is standard. Columns of the table are partitioned in accordance with the assumption (i), which holds for the left and fails for the right four columns; furthermore, row 6 is related to assumption (ii), rows 2 and 4 to (iii), rows 3 and 4 to (iv), and row 5 to (v).

INS	IAC	UNS	UAC	games	INS	IAC	UNS	UAC
?	N	N	N	n-person	N	N	N	N
?	**N**	**N**	N	n-person and c is the worst	N	N	N	N
Y	**Y**	N	N	2-person	N	N	N	N
Y	Y	$\mathcal{N}$	N	2-person and c is the worst	**N**	N	N	N
Y	Y	Y	$\mathcal{Y}$	2-person zero-sum	Y	**N**	**Y**	N
Y	Y	Y	Y	n person with no di-cycles	Y	Y	Y	$\mathcal{Y}$
Chess-like games; No random moves				In any case, all di-cycles Form one outcome c	Backgammon-like games; May have random moves			

$v \in V_T$. Obviously, in this case all infinite plays are equivalent, since the effective payoff is 0 for every such play. If $n = 2$ and $m = 1$ then the zero-sum Backgammon-like games turn into the so-called *simple stochastic games* introduced by Condon in [11,12].

All players are restricted to their pure positional strategies; see Section 2.8.

We will consider two concepts of solution: the classical Nash equilibrium (NE) and also a stronger one: the absence of the (best response) improvement cycles.

The concept of *solvability* is defined as the existence of a solution for *any* utility function u. Respectively, we arrive to the notions of *Nash-solvability* (*NS*) and *improvement acyclicity* (*AC*). Both concepts will be considered in two cases: *initialized* (I), that is, with respect to a fixed initial position $v_0 \in V \setminus V_T$ and *uniform* (U), that is, for all possible initial positions simultaneously. In the literature, the latter case is frequently referred to as *ergodic* or *subgame perfect*; see Remark 6.

For the obtained four concepts INS, UNS, IAC, UAC the following implications hold:

$$\text{UAC} \Rightarrow \text{IAC} \Rightarrow \text{INS}, \qquad \text{UAC} \Rightarrow \text{UNS} \Rightarrow \text{INS}.$$

Remark 2. All four concepts will be formally defined and discussed in details in the next section. Yet, AC (unlike NS) requires several immediate comments, as an anonymous referee suggested. In each situation (strategy profile) $x = (x_i \mid i \in I)$ we restrict the players $i \in I$ by their "uniformly best" responses $x_i^* = x_i^*(x)$ satisfying an additional "laziness" condition. Here, "uniformly best" means the best with respect to all possible initial positions simultaneously. The existence of a uniformly best response is guaranteed by Theorem 2. Furthermore, a response $x_i^* = x_i^*(x)$ is called *lazy* if the set of positions in which the decision x_i is changed in order to improve the result of the player i cannot be reduced without changing the result itself.

The state of art is summarized in Table 1 that contains criteria for AC and NS based on the following assumptions: (i) there are no random positions; (ii) there are no directed cycles; (iii) the "infinite outcome" c is ranked as the worst one by all n players; (iv) $n = 2$; (v) $n = 2$ and the payoff is zero-sum.

Let us briefly discuss the most important results of Table 1.

The classical backward induction algorithm [16,30,31], searching for some special uniform Nash equilibria in the n-person Chess-like games on trees, can be just slightly modified to prove the UAC for the n-person Backgammon-like games on acyclic digraphs; see the last row of the table and Section 6.

Another modification of the backward induction was recently applied to the two-person zero-sum Chess-like games on arbitrary finite digraphs to show their UNS (and INS) [2,8]. In Section 4 we slightly strengthen this result by proving UAC as well; see row 5 in the left-hand (Chess-like) side of the table.

As for the right-hand side of this row, that is, for the two-person zero-sum Backgammon-like games, the situation is more complicated: UNS (and INS) still hold. This result was derived in [8] from the existence of the uniformly optimal strategies for the classical two-person zero-sum stochastic games with perfect information and limiting mean payoff [18,34]; see Theorem 5 of Section 5. Yet, IAC (and UAC) fail for the two-person zero-sum Backgammon-like games, the corresponding example was constructed in [12].

Furthermore, IAC (and INS) hold for the two-person Chess-like games, that is, (iv), even without (iii) or (v), implies IAC [1] and, in particular, INS [7]; see row 3 of the table. However, these results cannot be extended to the uniform case or to Backgammon-like two-person, but non-zero-sum, games. In other words, INS (and IAC) (respectively, UNS (and UAC)) may fail, even under assumption (iii), for the two-person Backgammon-like (respectively, Chess-like) games; see row 4 of the table and Section 3.

Remark 3. Interestingly, the rows 1 and 2, as well as 3 and 4, of Table 1 are identical. In other words, it is not clear whether condition (iii) matters at all. Definitely, it helps a lot in proving NS, because forcing a di-cycle becomes a punishing strategy when (iii) holds.

Finally, the fundamental question, whether a game with a fixed initial position and without random moves is Nash-solvable (that is, INS for Chess-like games) remains open. For the two-person case, $n \leq 2$, the answer is positive [7]; see also [1] and the last section of [9]. Yet, in general, there is no counterexample and no proof, even under assumption (iii). Although, in the latter case some partial results are known:

(j) INS holds when $m = |V_T| \leq 3$, that is, for at most 4 outcomes [10];
(jj) INS also holds for the *play-once games*, in which each player is in control of at most one position, $|V_i| \leq 1$ for all $i \in I$ [7].

In [7], INS was conjectured for the n-person Chess-like games satisfying (iii) and, in fact, for a much larger class of games with the so-called *additive costs*; see [7] and Section 8 for the definitions and more details. It was shown in [7] that (iii) holds for these games whenever all local costs are non-negative and that INS may fail otherwise; in other words, the assumption (iii) seems essential in the considered case.

2. Introduction

2.1. Positional game structures

Given a finite digraph $G = (V, E)$ in which loops and multiple arcs are allowed; a vertex $v \in V$ is interpreted as a *position* and a directed edge (arc) $e = (v, v') \in E$ as a *move* from v to v'. A position of out-degree 0 (with no moves) is called *terminal*. Let $V_T = \{a_1, \ldots, a_m\}$ be the set of all terminal positions.

Let us also introduce a set of n players $I = \{1, \ldots, n\}$ and a partition $D : V = V_1 \cup \cdots \cup V_n \cup V_R \cup V_T$; each player $i \in I$ is in control of all positions of V_i, while V_R is the set of random positions, called also positions of chance. For each $v \in V_R$ a probability distribution over the set of outgoing arcs is fixed.

An *initial* position $v_0 \in V$ may be fixed. The triplet (G, D, v_0) or pair (G, D) is called a *positional game structure*, initialized or non-initialized, respectively.

A positional game structure is called Chess-like if there are no positions of chance, $V_R = \emptyset$, and it is called Backgammon-like, in general. Eight examples of the non-initialized Chess-like game structures $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6$ and initialized Backgammon-like game structures $\mathcal{G}'_2, \mathcal{G}'_3, \mathcal{G}'_4, \mathcal{G}'_6$ are given in Fig. 1.

2.2. Plays, outcomes, preferences, and payoffs

Given an initialized positional game structure (G, D, v_0), a *play* is defined as a directed path that begins in v_0 and either ends in a terminal position $a \in V_T$ or is infinite.

In this article we assume that All Infinite Plays Form One Outcome c, in addition to the standard Terminal outcomes of V_T. (In [8], this condition was referred to as AIPFOOT.)

Remark 4. In contrast, in [9,20,19,21] it was assumed that all di-cycles form pairwise distinct outcomes. A criterion of Nash-solvability was obtained for the two-person symmetric case, that is, when $n = 2$ and $e = (v, v')$ is an arc whenever $e' = (v', v)$ is.

A *utility (or payoff)* function is a mapping $u : I \times A \to \mathbb{R}$, whose value $u(i, a)$ is interpreted as a profit of the player $i \in I = \{1, \ldots, n\}$ in the case of the outcome $a \in A = \{a_1, \ldots, a_m, c\}$.

A payoff is called *zero-sum* if $\sum_{i \in I} u(i, a) = 0$ for every $a \in A$. Two-person zero-sum games will play an important role. For example, the standard Chess and Backgammon are two-person zero-sum games in which every infinite play is a draw, $u(1, c) = u(2, c) = 0$. It is not difficult to see that $u(i, c) = 0$ for all $i \in I$ can be always assumed without any loss of generality.

An important class of (non-zero-sum) payoffs is defined by the condition $u(i, c) < u(i, a)$ for all $i \in I$ and $a \in V_T$; in other words, the infinite outcome c is ranked as the worst one by all players. Several possible motivations for this assumption are discussed in [7,8].

A quadruple (G, D, v_0, u) and triplet (G, D, u) will be called a Backgammon-like (or Chess-like, in the case $V_R = \emptyset$) game, initialized and non-initialized, respectively.

For Chess-like games only n total pre-orders $o = \{o(u_i) \mid i \in I\}$ over A matter rather than the payoffs $u(i, a)$ themselves. Moreover, in this case, ties can be excluded; in other words, we can assume that $o(u_i)$ is a *total order* over A, called the *preference* of the player i; for more details, see Remarks 7 and 9. The set of all preferences $o = \{o(u_i) \mid i \in I\}$ is called the *preference profile*. In contrast, for Backgammon-like games the payoffs $u_i = u(i, a)$ do matter, since their probability combinations are compared.

2.3. Strategies

Given a positional game structure (G, D), a *strategy* x_i of a player $i \in I$ is defined as a mapping $x_i : V_i \to E_i$ that assigns to each position $v \in V_i$ a move (v, v') from this position.

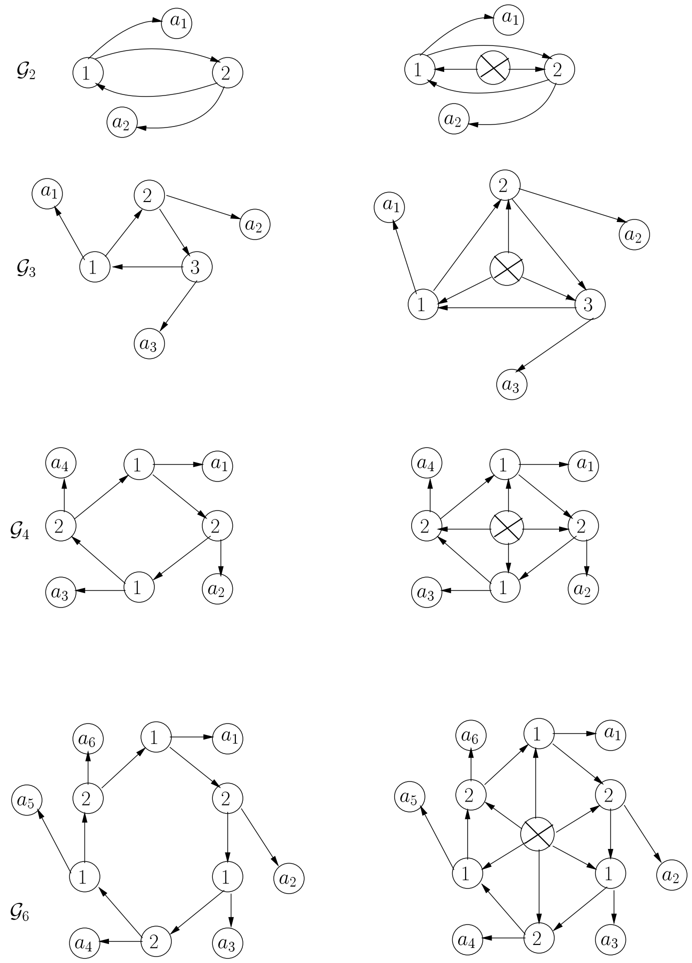


Fig. 1. The non-initialized Chess-like game structures $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6$ and the corresponding initialized Backgammon-like game structures $\mathcal{G}'_2, \mathcal{G}'_3, \mathcal{G}'_4, \mathcal{G}'_6$. For $j \in \{2, 3, 4, 6\}$, the positional game structure $\mathcal{G}_j$ contains a unique di-cycle of the length j. In $\mathcal{G}_3$, there are three players controlling one position each, while in $\mathcal{G}_2$, $\mathcal{G}_4$, and $\mathcal{G}_6$ there are only two players who take turns; hence, each of them controls $j/2$ positions. In each position v_ℓ, the corresponding player has only two options: (p) to proceed to $v_{\ell+1}$ and (t) to terminate at a_ℓ, where $\ell = 1, \ldots, j$ and $j + 1 = 1$, by a convention. Furthermore, for $j \in \{2, 3, 4, 6\}$, the positional game structure $\mathcal{G}'_j$ is obtained from $\mathcal{G}_j$ by adding one position v_0, which is random, initial, and marked by "X". There is a move from v_0 to each position v_ℓ of the di-cycle. The probabilities are not fixed yet assumed to be strictly positive. To save space we show only symbols a_ℓ but not v_ℓ.

Remark 5. In this article, we restrict the players to their *pure positional* strategies. In other words, the move (v, v') of a player $i \in I$ in a position $v \in V_i$ is deterministic (not random) and it depends only on the position v itself (not on the preceding positions or moves). Every considered strategy is assumed to be pure and positional unless it is explicitly said otherwise.

Let X_i be the set of all strategies of a player $i \in I$ and $X = \prod_{i \in I} X_i$ be the direct product of these sets. An element $x = (x_1, \ldots, x_n) \in X$ is called a *strategy profile or situation*.

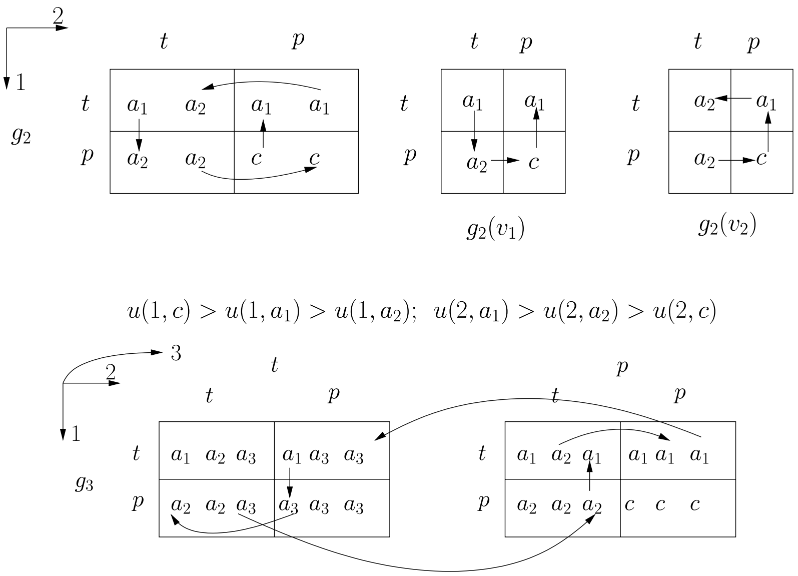
2.4. Normal forms

A positional game structure $\mathcal{G}$ can be represented in normal (or strategic) form.

We distinguish four cases, considering whether $\mathcal{G}$ is Chess- or Backgammon-like and initialized or not.

Given a *Chess-like initialized* positional game structure $\mathcal{G} = (G, D, v_0)$ and a strategy profile $x \in X$, a play $p(x)$ is naturally and uniquely defined by the following rules: it begins in v_0 and in each position $v \in V_i$ proceeds with the arc (v, v') according to the strategy x_i. Obviously, $p(x)$ either ends in a terminal position $a \in V_T$, or $p(x)$ is infinite. In the latter case $p(x)$ is a *lasso*, i.e., it consists of an initial path that leads to a di-cycle repeated infinitely. The latter holds, because all players are restricted to their positional strategies.

Anyway, an outcome $a =\in A = \{a_1, \ldots, a_m, c\}$ is assigned to every strategy profile $x \in X$. Thus, a game form $g_{v_0} : X \to A$ is defined. It is called the *normal form* of the positional game structure $\mathcal{G}$.



$$u(1,c) > u(1,a_1) > u(1,a_2);\quad u(2,a_1) > u(2,a_2) > u(2,c)$$

$$u(1,a_2) > u(1,a_1) > u(1,a_3) > u(1,c);$$
$$u(2,a_3) > u(2,a_2) > u(2,a_1) > u(2,c);$$
$$u(3,a_1) > u(3,a_3) > u(3,a_2) > u(3,c)$$

Fig. 2. The normal forms g_2 and g_3 of the positional game structures $\mathcal{G}_2$ and $\mathcal{G}_3$ from Fig. 1. Each player has only two strategies: to terminate (t) or proceed (p). Respectively, g_2 and g_3 are represented by the 2×2 and $2 \times 2 \times 2$ tables each entry of which contains 2 and 3 terminals corresponding to 2 and 3 (non-terminal) potential initial positions of $\mathcal{G}_2$ and $\mathcal{G}_3$, respectively. The rows and columns are the strategies of the players 1 and 2, while two strategies of the player 3 in g_3 are the left and right 2×2 subtables. For each game, a preference profile is fixed and the corresponding lazy uniformly BR im-cycle is shown. Thus, UAC fails in both cases. Two initialized game forms, corresponding to v_1 and v_2, are given for g_2. In each of them the im-cycle is broken, because of a tie. In fact, IAC hold for both game forms.

If the game structure $\mathcal{G} = (G, D)$ is not initialized then we apply the above construction for each initial position $v_0 \in V \setminus V_T$ to obtain a mapping $g : X \times (V \setminus V_T) \to A$, which is the *normal form* of $\mathcal{G}$ in this case.

We will use the notation $p = p(x, v_0)$ and $a = a(x, v_0) = g(x, v_0) = g_{v_0}(x)$ for the corresponding plays and outcomes, respectively.

For Chess-like game structures in Fig. 1 their normal forms are given in Figs. 2 and 3.

In a Backgammon-like non-initialized game form $\mathcal{G} = (G, D)$, each strategy profile $x \in X$ uniquely defines a Markov chain on V. Given also an initial position $v_0 \in V \setminus V_T$, this chain defines a unique limit probability distribution $q = q(x, v_0)$ on $A = \{a_1, \ldots, a_m, c\}$, where $q(a) = q(x, v_0, a)$ is the probability to stop at $a \in V_T$, while $q(c) = q(x, v_0, c)$ is the probability of an infinite play. Let Q denote the set of all probability distributions over A. The obtained mappings $g_{v_0} : X \to Q$ and $g : X \times (V \setminus V_T) \to Q$ are the *normal game forms* of the above Backgammon-like game forms, initialized and non-initialized, respectively.

Given also a payoff $u : I \times A \to \mathbb{R}$, the normal form games for the above two cases are defined by the pairs (g_{v_0}, u) and (g, u), respectively. These games can be also represented by real-valued mappings

$$f_{v_0} : I \times X \to \mathbb{R} \quad \text{and} \quad f : I \times X \times (V \setminus V_T) \to \mathbb{R}.$$

Indeed, let us fix $i \in I, x \in X$, and $v_0 \in V \setminus V_T$; having a probability distribution $q(x, v_0)$ over A and a real-valued payoff function $u_i : A \to \mathbb{R}$, where $u_i(a) = u(i, a)$, we compute the expected payoff by formula

$$f(i, x, v_0) = \sum_{a \in A} q(x, v_0, a) u(i, a).$$

In the case of Chess-like games, Markov chains and the corresponding limit probability distributions $q = q(x, v_0) \in Q$ over A are replaced by deterministic plays and the corresponding outcomes $a = g(x, v_0) \in A$; respectively, $f(i, x, v_0) = u(i, g(x, v_0))$.

2.5. Nash equilibria

Let us recall the standard concept of the Nash equilibrium (NE) for the normal form games.

First, let us consider the initialized case. Given $f_{v_0} : I \times X \to \mathbb{R}$, a strategy profile $x \in X$ is called a NE if $f_{v_0}(i, x) \ge f_{v_0}(i, x')$ for each player $i \in I$ and every strategy profile x' that can differ from x only in the ith component. In other words, no player

g_4

	tt	tp	pt	pp
tt	$a_1\ a_2\ a_3\ a_4$	$a_1\ a_2\ a_3\ a_1$	NE $a_1\ a_3\ a_3\ a_4$	$a_1\ a_3\ a_3\ a_1$
tp	$a_2\ a_2\ a_3\ a_2$	$a_2\ a_2\ a_3\ a_4$	$a_3\ a_3\ a_3\ a_4$	$a_3\ a_3\ a_3\ a_3$
pt	$a_1\ a_2\ a_4\ a_4$	$a_1\ a_2\ a_1\ a_1$	$a_1\ a_4\ a_4\ a_4$	$a_1\ a_1\ a_1\ a_1$
pp	$a_2\ a_2\ a_4\ a_4$	$a_2\ a_2\ a_2\ a_2$ NE	$a_4\ a_4\ a_4\ a_4$	$c\ c\ c\ c$

$o_1:\ a_2 > a_1 > a_3 > a_4 > c, \qquad o_2:\ a_3 > a_2 > a_4 > a_1 > c.$

g_6

	ttt	ttp	tpt	tpp	ptt	ptp	ppt	ppp
ttt	$a_1a_2a_3a_4a_5a_6$	$a_1a_2a_3a_4a_5a_1$	$a_1a_2a_3a_5a_5a_6$	$a_1a_2a_3a_5a_5a_1$	○ $a_1a_3a_3a_4a_5a_6$	■ $a_1a_3a_3a_4a_5a_1$	$a_1a_3a_3a_5a_5a_6$	$a_1a_3a_3a_5a_5a_1$
ttp	$a_1a_2a_3a_4a_6a_6$	$a_1a_2a_3a_4a_1a_1$	$a_1a_2a_3a_6a_6a_6$	$a_1a_2a_3a_1a_1a_1$	■ $a_1a_3a_3a_4a_6a_6$	$a_1a_3a_3a_4a_1a_1$	○ $a_1a_3a_3a_6a_6a_6$	$a_1a_3a_3a_1a_1a_1$
tpt	○ $a_1a_2a_4a_4a_5a_6$	$a_1a_2a_4a_4a_5a_1$	$a_1a_2a_5a_5a_5a_6$	$a_1a_2a_5a_5a_5a_1$	$a_1a_4a_4a_4a_5a_6$	$a_1a_4a_4a_4a_5a_1$	$a_1a_5a_5a_5a_5a_6$	$a_1a_5a_5a_5a_5a_1$
tpp	$a_1a_2a_4a_4a_6a_6$	$a_1a_2a_4a_4a_1a_1$	○ $a_1a_2a_6a_6a_6a_6$	$a_1a_2a_1a_1a_1a_1$	$a_1a_4a_4a_4a_6a_6$	$a_1a_4a_4a_4a_1a_1$	$a_1a_6a_6a_6a_6a_6$	$a_1a_1a_1a_1a_1a_1$
ptt	$a_2a_2a_3a_4a_5a_6$	■ $a_2a_2a_3a_4a_5a_2$	$a_2a_2a_3a_5a_5a_6$	$a_2a_2a_3a_5a_5a_2$	$a_3a_3a_3a_4a_5a_6$	○ $a_3a_3a_3a_4a_5a_3$	$a_3a_3a_3a_5a_5a_6$	$a_3a_3a_3a_5a_5a_3$
ptp	■ $a_2a_2a_3a_4a_6a_6$	$a_2a_2a_3a_4a_2a_2$	$a_2a_2a_3a_6a_6a_6$	$a_2a_2a_3a_2a_2a_2$	$a_3a_3a_3a_4a_6a_6$	$a_3a_3a_3a_4a_3a_3$	$a_3a_3a_3a_6a_6a_6$	○ $a_3a_3a_3a_3a_3a_3$
ppt	$a_2a_2a_4a_4a_5a_6$	○ $a_2a_2a_4a_4a_5a_2$	$a_2a_2a_5a_5a_5a_6$	■ $a_2a_2a_5a_5a_5a_2$	$a_4a_4a_4a_4a_5a_6$	$a_4a_4a_4a_4a_5a_4$	$a_5a_5a_5a_5a_5a_6$	■ $a_5a_5a_5a_5a_5a_5$
ppp	$a_2a_2a_4a_4a_6a_6$	$a_2a_2a_4a_4a_2a_2$	■ $a_2a_2a_6a_6a_6a_6$	○ $a_2a_2a_2a_2a_2a_2$	$a_4a_4a_4a_4a_6a_6$	$a_4a_4a_4a_4a_4a_4$	■ $a_6a_6a_6a_6a_6a_6$	$c\ c\ c\ c\ c\ c$

$o_1:\ a_6 > a_5 > a_2 > a_1 > a_3 > a_4 > c, \qquad o_2:\ a_3 > a_2 > a_6 > a_4 > a_5 > c;\ \ a_6 > a_1 > c.$

Fig. 3. The normal forms g_4 and g_6 of the positional game structures $\mathcal{G}_4$ and $\mathcal{G}_6$ from Fig. 1. In each case there are two players, who control 2 positions each in $\mathcal{G}_4$ and 3 positions each in $\mathcal{G}_6$. Again, in every position there are only two options: to terminate (t) or proceed (p). Hence, in $\mathcal{G}_4$ and $\mathcal{G}_6$, each player has, respectively, 4 and 8 strategies, which are naturally coded by the 2- and 3-letter words in the alphabet $\{t, p\}$. Respectively, g_4 and g_6 are represented by the 4×4 and 8×8 tables each entry of which contains 4 and 6 terminals corresponding to 4 and 6 (non-terminal) potential initial positions of $\mathcal{G}_4$ and $\mathcal{G}_6$. Again, players 1 and 2 control the rows and columns, respectively. In each game a preference profile is fixed. More precisely, 3 profiles are fixed in $\mathcal{G}_6$, yet, all three define the same cycle. Moreover, for each such profile u, the corresponding game (g_6, u) has *no uniform (subgame perfect) NE*. Indeed, the (unique) uniformly best response of the player 1 (respectively, 2) to each strategy of 2 (respectively, 1) is shown by a white disc (respectively, black square); the obtained two sets are disjoint. In contrast, UNS holds for $\mathcal{G}_4$, as a tedious verification shows. In particular, for the given preferences there are two uniform Nash equilibria, shown in the figure together with a lazy uniformly BR im-cycle.

$i \in I$ can profit by choosing a new strategy if all opponents keep their strategies unchanged. In the non-initialized case, the similar property is required for each $v_0 \in V \setminus V_T$. Given $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$, a strategy profile $x \in X$ is called a *uniform NE* if $f(i, x, v_0) \geq f(i, x', v_0)$ for each $i \in I$, for every x' defined as above, and for all $v_0 \in V \setminus V_T$, too.

Remark 6. In the literature, this concept is also called "a *subgame perfect* NE". This name is justified when the digraph $G = (V, E)$ is acyclic and each vertex $v \in V$ can be reached from v_0. Indeed, in this case (G, D, v, u) is a subgame of (G, D, v_0, u) for each v. However, if G has a di-cycle then any two of its vertices v' and v'' can be reached one from the other; in other words, (G, D, v', u) is a subgame of (G, D, v'', u) and vice versa. Thus, the name *a uniform (or ergodic)* NE is more accurate.

Remark 7. Let us also notice that for a Chess-like game, the inequality $f_{v_0}(i, x) \geq f_{v_0}(i, x')$ is reduced to $u(i, g(v_0, x)) \geq u(i, g(v_0, x'))$, which can be verified whenever a total pre-order o_i over A is known. It should be verified for all $i \in I$ and for a given (respectively, for each) $v_0 \in V \setminus T$ in the initialized (respectively, non-initialized) case. Moreover, without any loss of generality, total pre-orders can be replaced by (total) orders, as far as the NS of Chess-like games is concerned. Indeed, if a NE exists with respect to all preference profiles (total orders) then it exists for all payoffs (total pre-orders) as well. It is enough to notice that the concept of a NE is based on the absence of individual improvements, defined by strong inequalities. Hence, any "merging of payoffs" may result only in creating some new equilibria, while the old ones remain.

2.6. On UNS of Chess-like and INS of Backgammon-like games

From the above definitions, it follows immediately that UNS implies INS.

The non-initialized Chess-like positional game structures $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6$ given in Fig. 1 are not UNS. The corresponding payoffs u^2, u^3, u^4, u^6 will be constructed in Section 3. Moreover, u^3 and u^6 will satisfy condition (iii): any di-cycle is the worst outcome for all players. In contrast, games $(\mathcal{G}_2, u^2)$ and even $(\mathcal{G}_4, u^4)$ have uniform NE whenever u^2 and u^4 satisfy (iii).

For this reason, a larger example $(\mathcal{G}_6, u^6)$ is needed to show that UNS may fail for the two-person Chess-like games, even under assumption (iii).

Let us also note that all obtained initialized game structures are INS for any choice of the initial position. Moreover, it is an open problem, whether INS holds for all Chess-like games. This problem remains open even if (iii) is assumed. Yet, it is solved in the affirmative for the two-person games, $n = 2$, even without assumption (iii); see [7] and also [1,9]. Thus, we can fill the first four positions of the INS- and UNS-columns of the Chess-like (left-hand) side of Table 1.

Furthermore, the UNS of Chess-like games can be reduced to the INS of some special Backgammon-like games as follows. Given a non-initialized Chess-like game $\mathcal{G} = (G, D, u)$, let us add to its digraph $G = (V, E)$ a random initial position v_0 and a move (v_0, v) from v_0 to *each* position $v \in V \setminus V_T$. Furthermore, let us define probabilities $q(v_0, v)$ such that $\sum_{v \in V \setminus V_T} q(v_0, v) = 1$ and denote the obtained probability distribution and initialized Backgammon-like game by $q(v_0)$ and $\mathcal{G}'$, respectively. By construction, games $\mathcal{G}$ and $\mathcal{G}'$ have the same set X of the strategy profiles. The following relations between $\mathcal{G}$ and $\mathcal{G}'$ hold.

Theorem 1. (i) *If $x \in X$ is a uniform NE in $\mathcal{G}$ then it is a NE for every $q(v_0)$ in $\mathcal{G}'$.*
(ii) *If $x \in X$ is a NE in $\mathcal{G}'$ for some strictly positive $q(v_0)$ then x is a uniform NE in $\mathcal{G}$.*

Part (i) is obvious; part (ii) will be proved in Section 7. The next corollary is immediate.

Corollary 1. *The following three claims are equivalent:*

(j) *$\mathcal{G}$ is UNS;*
(jj) *$\mathcal{G}'$ is INS for some strictly positive $q(v_0)$;*
(jj′) *$\mathcal{G}'$ is INS for each strictly positive $q(v_0)$.* □

This corollary and the previous results allow us to fill with "*N*" the first four positions in the INS (and UNS) columns of the Backgammon (right-hand) side of Table 1.

The fifth row of this table shows that UNS (and INS) hold for the two-person zero-sum games. This statement will be derived from the classical Gillette theorem [18]; see also [8,24], and Section 6.

Finally, according to the last row of the table, UNS holds for acyclic digraphs. In this case uniform Nash equilibria can be obtained by a slight modification of the classical backward induction procedure [16,30,31]; see Section 4.

2.7. Uniform and lazy best responses

Again, let us start with the initialized case. Given the normal form $f_{v_0} : I \times X \to \mathbb{R}$ of an initialized Backgammon-like game, a player $i \in I$, and a pair of strategy profiles x, x' such that x' may differ from x only in the ith component, we say that *x' improves x* (for the player i) if $f_{v_0}(i, x) < f_{v_0}(i, x')$.

Let us underline that the inequality is strict and remark that a situation $x \in X$ is a NE if and only if it can be improved by no player $i \in I$. In other words, any sequence of improvements (an *im-path*) either cycles, or terminates in a NE.

Given a player $i \in I$ and situation $x = (x_i \mid i \in I)$, a strategy $x_i^* \in X_i$ is called a *best response (BR)* of i in x, if $f_{v_0}(i, x^*) \geq f_{v_0}(i, x')$ for any x', where x^* and x' are both obtained from x by replacement of its ith component x_i by x_i^* and x_i', respectively. A BR x_i^* is not necessarily unique but the corresponding best achievable value $f_{v_0}(i, x^*)$ is, of course, unique. Moreover, somewhat surprisingly, such best values can be achieved by a BR x_i^* simultaneously for all initial positions $v_0 \in V \setminus V_T$.

Theorem 2. *Let us fix a player $i \in I$ and situation $x \in X$ in the normal form $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$ of a non-initialized Backgammon-like game. Then, there is a (pure positional) strategy $x_i^* \in X_i$ which is the best response of i in x for all initial positions $v_0 \in V \setminus V_T$ simultaneously.*

We will call such a strategy x_i^* a *uniformly* BR of the player i in the situation x. Obviously, the non-strict inequality $f_v(i, x) \leq f_v(i, x^*)$ holds for each position $v \in V$. We will say that x_i^* improves x if this inequality is strict, $f_{v_0}(i, x) < f_{v_0}(i, x^*)$, for at least one $v_0 \in V$. This statement will serve as the definition of a uniform improvement for the non-initialized case. Let us remark that, by this definition, $x \in X$ is a uniform NE if and only if it can be uniformly improved by no player $i \in I$. In other words, any sequence of uniform improvements (a *uniform im-path*) either cycles or terminates in a uniform NE.

Remark 8. We will prove Theorem 2 in Section 5, by viewing a Backgammon-like game as a stochastic game with perfect information and transition-free payoff. Namely, the local reward $r(i, e)$ equals $u(i, a)$ for each terminal move $e = (v, a)$, $a \in V_T$, and $r(i, e') = 0$ for any non-terminal move $e' = (v, v')$, $v' \notin V_T$. Then, the search for a BR in a given situation $x \in X$ is reduced to a controlled Markov chain problem. It is well known that in this case a uniformly optimal (pure) strategy exists and can be found by linear programming; moreover, even if the player i is allowed to use *history dependent* strategies, still there is a (uniformly) best response in *positional* strategies; see, for example, [4,35], also [5,6,40], and Section 2.8.

For Chess-like games, Theorem 2 can be proved by the following, much simpler, arguments. Given a non-initialized Chess-like game $\mathcal{G} = (G, D, u)$, a player $i \in I$, and a strategy profile $x \in X$, in every position $v \in V \setminus (V_i \cup V_T)$ let us fix a move (v, v') in accordance with x and delete all other moves.

Let us order A according to the preference $u_i = u(i, *)$. Let $a^1 \in A$ be the best outcome. (Note that $a^1 = c$ might hold.) Let V^1 be the set of positions from which player i can reach a^1; in particular, we have $a^1 \in V^1$. Let us fix corresponding moves in $V^1 \cap V_i$. By definition, there are no moves to a^1 from $V \setminus V^1$. Moreover, if $a^1 = c$ then player i can reach no di-cycle beginning from $V \setminus V^1$; in particular, the induced digraph $G_1 = G[V \setminus V^1]$ contains no di-cycles. Then, let us consider an outcome a^2 that is the next best for i in A, after a^1, and repeat the same arguments for G_1 and a^2, etc. This procedure will result in a uniformly best response x_i^* of i in x, since all chosen moves of player i are optimal independently of v_0. □

Remark 9. Again, without any loss of generality, total pre-orders can be replaced with the (total) orders, as far as the AC (or NS) of Chess-like games are concerned. Indeed, an improvement requires a strict inequality (for at least one initial position). Hence, if AC holds for all preference profiles (total orders) then it holds for all payoffs (total pre-orders) as well. To show this, it is enough to notice that any "merging of payoffs" may result only in elimination of improvement cycles but no new one can appear.

Remark 10. Let us also mention in passing that a uniformly BR exists and it can be found by the same procedure when di-cycles form not one but several distinct outcomes, like in [9,20,19,21].

Let x_i^* be a BR of the player i in a situation $x = (x_1, \dots, x_i, \dots, x_n)$. It may happen that x_i and x_i^* recommend two distinct moves, (v, v') and (v, v''), in a position $v \in V_i$, although the original strategy x_i is already optimal with respect to v; in other words, $g(i, x, v)$ is a (unique) optimal outcome for the initial position v but the considered BR changes an optimal move in v, which is not necessary.

A BR x_i^* is called *lazy* if this happens for no $v \in V_i$. A similar definition was introduced in [1] for the initialized Chess-like games. In general, for Backgammon-like games, a BR will be called *lazy* if the subset $V_i' \subseteq V_i$ in which x_i and x_i^* require the same move cannot be strictly increased.

Remark 11. We will not discuss how to verify the laziness condition. In fact, we will make use of it only in Section 5, for Backgammon-like games on acyclic digraphs and in this case the problem becomes simple. It is also simple for Chess-like games on arbitrary digraphs.

In this article, we restrict the players by their lazy uniformly BRs unless the opposite is explicitly said.

2.8. Best response and Nash equilibria in history dependent strategies

In this paper, all players are restricted to their positional strategies. To justify this restriction, in this subsection we eliminate it and show that the result of the game will not change.

Let (G, D, v_0) be an initialized positional game structure. A sequence of positions $d = (v_0, v_1, \dots, v_k)$ such that $(v_{j-1}, v_j) \in E$ is a move for $j = 1, \dots, k$ is called a *debut*. Let us notice that the same position may occur in d several times.

A *general, history dependent, strategy* x_i of a player $i \in I$ is a mapping that assigns a move (v, v') to every debut $d = (v_0, v_1, \dots, v_k)$ in which $v_k = v \in V_i$.

Let us recall that strategy x_i is *positional* if in each position $v \in V_i$ the chosen move depends only on v but not on the preceding positions of the debut. Let X_i and X_i' denote the sets of all positional and general (history dependent) strategies of a player $i \in I$; by definition, $X_i \subseteq X_i'$. Furthermore, let $X = \prod_{i\in I} X_i$ and $X' = \prod_{i\in I} X_i'$ be the sets of positional and general strategy profiles; clearly, $X \subseteq X'$.

Now, we can strengthen Theorem 2 by simply noting that the positional strategy $x_i^* \in X_i$, which is a uniform BR of the player $i \in I$ in the situation $x \in X$, is still a uniform BR of i, even if we extend X_i to X_i'.

The new version of Theorem 2 immediately implies that each (uniform) NE in positional strategies is a general (uniform) NE; in other words, if $x^* \in X$ is a (uniform) NE with respect to X then it is a (uniform) NE with respect to X', as well. The claim holds for both Chess- and Backgammon-like games, initialized and non-initialized cases. In particular, history dependent strategies may be ignored in the case of the two-person zero-sum games, since a saddle point in (uniform) positional strategies always exists in this case. Yet in general, there may be no NE in X, while in X' its existence is guaranteed by the so-called "folk theorems".

Finally, let us mention that NS in history dependent strategies was recently proven in [15]. More precisely, the authors showed that an ε-NE in pure (but history dependent) strategies exists in n-person Backgammon-like games and, moreover, $\varepsilon = 0$ can be chosen for Chess-like games, provided all infinite plays are equivalent and ranked as the worst outcome for each player. Let us also recall that INS in *positional* strategies in the considered case remains an open problem.

2.9. Improvement cycles and acyclicity

Let $f_{v_0} : I \times X \to \mathbb{R}$ (respectively, $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$) be the normal form of an initialized (respectively, non-initialized) Backgammon-like game. A sequence $x^0, \dots, x^{k-1} \in X$ of strategy profiles is called a *uniformly best improvement cycle*, or just an im-cycle, for short, if x^{j+1} is a uniformly BR improvement of x^j by a player $i = i(j) \in I$, for $j = 0, \dots, k-1$, where the indices are taken modulo k.

Let us recall that in both initialized and non-initialized cases the players are restricted to their lazy uniformly best responses.

Remark 12. The concept of an improvement cycle (im-cycle) was introduced in 1838 by Cournot [13] (112 years before Nash's equilibria [36,37]). Yet, AC of *positional* games was considered just recently, by Kukushkin [32,33]. He restricted himself to IAC and to the games modeled by trees and demonstrated that even in this case im-cycles may appear. In fact, they exist already in a very simple two-person zero-sum game, with only four terminals; see Example 3 of [32]. Then, to reduce the set of im-cycles he restricted the improvements, allowing the individuals (players) to change their decisions (moves) only in the positions within the actual play. Indeed, other changes make no sense, except for preparing further improvements. Making use of this restriction, Kukushkin derived several interesting criteria of IAC.

In [1], these concepts and results were extended from trees to arbitrary finite digraphs. The individual improvements can be restricted in many ways. For positional games, several new restrictions were suggested in [1] and an umbrella concept of a *restricted* improvement (ri) and *ri-acyclicity* were introduced.

In particular, Kukushkin's "actual play restriction" was replaced with a stronger "*laziness rule*": an individual improvement is *lazy* if the set of positions in which the individual decision was changed in order to get a better result cannot be reduced without changing this result. In this article, we restrict the players to their lazy uniformly best responses, both concepts of IAC and UAC being subject to this restriction. Obviously, even in this case, IAC and UAC imply INS and UNS, respectively.

Let us also recall that an improvement is *uniform* if the new result is not worse than the original one with respect to each initial position and is strictly better for at least one of them. Since such a position may vary from one improvement to another, we conclude that UAC implies IAC but not vice versa.

The existence of a uniformly best response is guaranteed by Theorem 2. Although several such responses may exist, yet, the BR profile of outcomes is unique; see, for example, Figs. 2 and 3.

A game is called *im-acyclic* (AC) if it has no im-cycles. Thus, the concepts of IAC and UAC appear for the initialized and non-initialized games. Obviously, UAC implies IAC. Indeed, an initialized im-cycle is also a non-initialized one, but not vice versa. The inverse holds only if all k improvements correspond to one initial position v_0, while in a non-initialized im-cycle they may correspond to different positions $v(j) \in V$ for $j = 0, 1, \dots, k-1$; see, for example, g_2 in Fig. 2.

The implications (IAC $\Rightarrow$ INS) and (UAC $\Rightarrow$ UNS) also follow immediately from the above definitions.

Examples of im-cycles will be given in Section 3 for the non-initialized Chess-like games in Fig. 1. These examples allow us to fill with "N" the first four positions of the UAC-column in the left-hand side of Table 1. Furthermore, we refer to [1] to fill the IAC-column. Indeed, Theorems 2 and 3 of [1] imply that IAC holds for the acyclic and two-person cases; see also Sections 4 and 6, respectively.

Let us now consider the concepts of IAC and UAC for Backgammon-like games, that is, the right-hand side of Table 1. UAC (and IAC) hold in the case of the acyclic digraphs; see [1] and also Section 4. Yet, in presence of di-cycles, IAC (and UAC) fail in all five cases. The first four follow from Theorem 1 and examples of Section 3, while for two-person zero-sum games a corresponding example was constructed in 1993 by Condon in [12].

3. Games with a unique di-cycle and without uniform Nash equilibria

Let us consider the non-initialized Chess-like positional game structures $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_6$ given in Fig. 1. Each digraph $G_j = (V_j, E_j)$ consists of a unique di-cycle C_j of length j and a matching connecting each vertex v^j_ℓ of C_j to a terminal a^j_ℓ, where $\ell = 1, \dots, j$ and $j = 2, 3, 4, 6$. The digraphs G_2, G_4 and G_6 are bipartite; respectively, $\mathcal{G}_2$, $\mathcal{G}_4$ and $\mathcal{G}_6$ are two-person positional game structures in which two players move taking turns; $\mathcal{G}_3$ is a three-person game structure; $\mathcal{G}_2$ and $\mathcal{G}_3$ are play-once, that is, each player controls a unique position. In every non-terminal position v^j_ℓ there are only two moves; one of them (t) immediately terminates in a^j_ℓ, while the other one (p) proceeds along the di-cycle to $v^j_{\ell+1}$; by a convention, we set $j + 1 = 1$.

Remark 13. In Figs. 1–3 the symbols a^j_ℓ for the terminal positions are replaced by a_ℓ, to save space we omit the superscript j. For the same purpose, in Fig. 1 we show only symbols a_ℓ but not v^j_ℓ.

Thus, in $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$, and $\mathcal{G}_6$ each player has 2, 2, 4, and 8 strategies, respectively. For $\mathcal{G}_2$ and $\mathcal{G}_3$ each player has two strategies coded by the letters t and p, while for $\mathcal{G}_4$ and $\mathcal{G}_6$ the strategies are coded by, respectively, the 2- and 3-letter words in the alphabet $\{t, p\}$. For example, the strategy (tpt) of player 2 in $\mathcal{G}_6$ requires to proceed to v^2_5 from v^2_4 and to terminate in a^2_2 from v^2_2 and in a^2_6 from v^2_6.

The corresponding normal game forms g_2, g_3, g_4, g_6 of size 2×2, $2 \times 2 \times 2$, 4×4, and 8×8 are shown in Figs. 2 and 3. All four game forms are non-initialized, that is, each situation is a set of 2, 3, 4, and 6 terminals, respectively. These terminals correspond to the non-terminal positions of $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$, and $\mathcal{G}_6$, each of which can serve as an initial position.

Let us start with g_2 and consider the preference profile

$$o_1 : c > a_1 > a_2, \qquad o_2 : a_1 > a_2 > c;$$

in other words, player 1 likes and player 2 dislikes c the most, while both prefer a_1 to a_2. It is easily seen that all four situations of g_2 form an im-cycle; hence, none of them is a uniform NE; see Fig. 1. Thus, UNS fails for this example. In contrast, INS holds.

In other words, if an initial position, v_1 or v_2, is fixed then the obtained initialized positional game structure has a NE for every payoff. Both corresponding game forms, $g_2(v_1)$ and $g_2(v_2)$, are shown in Fig. 2. It is easy to verify that the considered im-cycle is broken in each one by a tie. This example appeared in [1], Fig. 10; see also Fig. 1 of [8].

In addition to the above counterclockwise im-cycle, the clockwise one is defined by the preference profile

$$o = (o_1, o_2) \quad \text{such that } o_1 : a_2 > a_1 > c,\ o_2 : c > a_2 > a_1.$$

Obviously, there can be no more im-cycles in a 2×2 game form. In both cases, the preferences o_1 and o_2 are not opposite and the outcome c is the best for one player and the worst for the other.

Based on these simple observations, one could boldly conjecture that UNS holds for the two-person non-initialized Chess-like games whenever:

(A) they are zero-sum or
(A) any di-cycle is the worst outcome c for both players.

As it will be shown in Section 6, Conjecture (A) holds, indeed, not only for Chess-like but even for Backgammon-like games. Yet, (B), unfortunately, fails. Although, $\mathcal{G}_2$ and even $\mathcal{G}_4$ are too small to provide a counterexample, however, $(\mathcal{G}_6, o)$ has no uniform NE whenever the preferences $o = (o_1, o_2)$ satisfy:

$$o_1 : a_6 > a_5 > a_2 > a_1 > a_3 > a_4 > c, \qquad o_2 : a_3 > a_2 > a_6 > a_4 > a_5 > c, \qquad a_6 > a_1 > c.$$

To verify this, let us consider the normal form g_6 in Fig. 3. By Theorem 2, there is a uniform BR of player 2 to each strategy of player 1 and vice versa. It is not difficult to check that the obtained two sets of situations (denoted by the white discs and black squares in Fig. 3) are disjoint. Hence, there is no uniform NE. Furthermore, it is not difficult to verify that the obtained 16 situations induce an im-cycle of length 10 and two im-paths of lengths 2 and 4 that end in this im-cycle.

Remark 14. Im-cycles exist already in g_4; one of them is shown in Fig. 3 for the preferences

$$o_1 : a_2 > a_1 > a_3 > a_4 > c, \qquad o_2 : a_3 > a_2 > a_4 > a_1 > c.$$

Yet, two uniform Nash equilibria also exist in this case; see Fig. 3. Moreover, a tedious case analysis shows that the game form $\mathcal{G}_4$ (and, of course, $\mathcal{G}_2$) are UNS, unlike $\mathcal{G}_6$.

For $n = 3$ a simpler counterexample to Conjecture (B) was obtained in [7]. It is given by the game structure $\mathcal{G}_3$ and the preference profile

$$o = (o_1, o_2, o_3) \quad \text{such that } o_1 : a_2 > a_1 > a_3 > c, \qquad o_2 : a_3 > a_2 > a_1 > c \qquad o_3 : a_1 > a_3 > a_2 > c.$$

In other words, for each player $i \in I = \{1, 2, 3\}$ to terminate is an average outcome; it is better (worse) when the next (previous) player terminates; finally, if nobody does then the di-cycle c appears, which is the worst outcome for all. The considered game has an im-cycle of length 6, which is shown in Fig. 2. Indeed, let player 1 terminate at a_1, while 2 and 3 proceed. The corresponding situation (a_1, a_1, a_1) can be improved by 2 to (a_1, a_2, a_1), which in its turn can be improved by 1 to (a_2, a_2, a_2).

Let us repeat the same procedure two times more to obtain the im-cycle of length six shown in Fig. 2.

The remaining two situations (a_1, a_2, a_3) and (c, c, c) appear when all three players simultaneously terminate or, respectively, proceed. These situations are not NE either. Moreover, each of them can be improved by every player. Thus, there is no uniform NE in the obtained game.

A Backgammon-like non-INS game corresponding to this example was shown in [7].

Let us recall that, in general, an initialized Backgammon-like game is assigned to each non-initialized Chess-like game, as in Fig. 1; furthermore, the former is INS if and only if the latter is UNS, by Theorem 1. Thus, the above examples allow us to fill with "*N*" the first four positions of columns 3 and 5 of Table 1.

Theorem 3. *IAC and UAC fail, respectively, for Backgammon- and Chess-like two-person games in which c is ranked as the worst outcome by both players.* □

4. Backward induction for Backgammon-like games on acyclic digraphs

The classical backward induction procedure [30,31,16] searching for Nash equilibria in the n-person Chess-like games, modeled by the trees, can be easily extended to Backgammon-like games on acyclic digraphs; see, for example, [1,8]. Moreover, this modification implies UAC, as well.

Theorem 4. *Backgammon-like games on acyclic digraphs are uniformly acyclic, that is, UAC (together with UNS, IAC, and INS) hold.*

Proof. In fact, all uniform NE can be obtained by the standard backward induction procedure.

Let game $\mathcal{G} = (G, D, u)$ be a minimal counterexample, that is, $\mathcal{G}$ has an im-cycle Y, the digraph $G = (V, E)$ has no di-cycle, and there is no similar example $\mathcal{G}'$ with a digraph G' smaller than G.

By minimality of G, for every move $e = (v, v') \in E$ of a player i (that is, $v \in V_i$) there are two successive strategy profiles $x, x' \in Y$ such that e is chosen by x_i and not chosen by x'_i. Indeed, otherwise we can eliminate e from G (if e is not chosen by any strategy of Y) or all other arcs from v (if e is chosen by all strategies of Y), and Y remains an im-cycle in the reduced game $\mathcal{G}'$, as well.

By acyclicity of G, there is a position $v \in V \setminus V_T$ such that each move (v, v') from it leads to a terminal position $v' \in V_T$. Let us choose such a position v, delete all arcs (v, v') from G, and denote the obtained reduced graph by G' and the corresponding game by $\mathcal{G}'$. By construction, v is a terminal position in G'. Let us consider two cases:

Case 1: $v \in V_R$ is a position of chance. Then, for each player $i \in I$, let us define the terminal payoff $u(i, v)$ as an average of $u(i, v')$ for all moves (v, v') from v in G.

Case 2: $v \in V_i$ is a private position of a player $i \in I$. Then, $u(i, v)$ will be defined as the maximum of $u(i, v')$ for all moves (v, v') from v. Let such a maximum be realized in v'_0. Then $u(i', v) = u(i', v'_0)$ for all $i' \in I$. Obviously, every strategy profile of an im-cycle will choose in v a move (v, v'_0). Such a move may be not unique, yet, by the laziness condition, it will remain unchanged in the considered im-cycle.

In both cases, games $\mathcal{G}$ and $\mathcal{G}'$ are equivalent. In particular, the im-cycle in $\mathcal{G}$ remains an im-cycle in $\mathcal{G}'$, in contradiction with the minimality of G. Thus, UAC holds for the game $\mathcal{G} = (G, D, u)$ whenever G is an acyclic digraph. $\square$

In fact, the above arguments imply stronger and more constructive statements. Let us choose successively the pre-terminal positions $v \in V \setminus V_T$ and make them terminal, as shown above. Furthermore, we will fix an optimal move, maximizing $u(i, *)$, whenever $v \in V_i$ is a private position. Obviously, in $k = |V \setminus V_T|$ steps, we obtain a strategy profile x^*. This recursive procedure is known as *backward induction*; it was suggested in the early fifties by Kuhn [30,31] and Gale [16]; the profile x^* is called the *Backward Induction (or sophisticated) Equilibrium* (BIE). Since a sequence of lazy BR improvements (a lazy im-path) cannot cycle, as shown above, it must terminates in a BIE.

Furthermore, it is also clear that every BIE is a uniform NE. Moreover, the inverse is true too.

Corollary 2. *The sets of all BIE and NE coincide in each Backgammon-like game on an acyclic digraph.*

Finally, the BIE is unique, in absence of positions of chance and ties.

Corollary 3. *A Chess-like game on an acyclic digraph has a unique BIE if payoffs $u(i, *)$ have no ties.*

Proof. Both above statements easily follow from an analysis of the backward induction procedure. $\square$

Remark 15. It is a curious combinatorial problem to bound the length of a shortest (or longest) im-path from a situation x to a BIE x^*. Yet, these problems have no practical value, since it is much easier to obtain a BIE x^* by backward induction (in $|V \setminus V_T|$ steps) rather than by constructing an im-path to x^* from x.

Remark 16. Let us note that the laziness assumption is essential. Indeed, otherwise, an im-cycle may appear, as the example in Fig. 4 shows. It is easy to verify that the obtained im-cycle contains only BR uniform improvements, yet, they are not lazy.

Let us also notice that the payoffs have ties. However, even if the terminals are totally ordered, in presence of random positions, ties still may appear in the backward induction procedure.

The example in Fig. 4 is slightly more complicated than Example 3 of [32] showing that an im-cycle can appear if the players are allowed to change their decisions beyond the actual play. However, some improvements of the obtained im-cycle are not uniform and not lazy; see Sections 2.7–2.9 for the definition. We had to modify Example 3, since we restrict the players to their uniform and lazy best responses.

Finally, let us note that the backward induction procedure (and even INS and UNS) may fail in presence of a di-cycle; see examples in Section 3. Still, a modified backward induction works for the two-person zero-sum Chess-like games; see Section 6.

5. Mean payoff *n*-person games

Let us recall the model of Section 2 and introduce the following modifications. Let (G, D) be a non-initialized positional game structure, in which $G = (V, E)$ is a digraph and $D : V = V_1 \cup \cdots \cup V_n \cup V_R \cup V_T$ is a partition of its positions. Without any loss of generality, we may assume that there are no terminals, $V_T = \emptyset$, since each terminal $v \in V_T$ can be replaced by a loop $e_v = (v, v)$.

Let us introduce a local reward function $r : I \times E \to \mathbb{R}$, whose value $r(i, e)$ is interpreted as a profit obtained by the player $i \in I$ whenever the play passes the arc $e \in E$.

As before, all players are restricted to their pure positional strategies. A strategy profile $x \in X$ uniquely defines a Markov chain on V. Whenever we fix also an initial position $v_0 \in V$, this chain defines a unique limit probability distribution $q = q(x, v_0)$ on V, that is, $q(v) = q(x, v_0, v) = \lim_{t\to\infty} t(v)/t$, where $t(v)$ is the expected fraction of the total time t

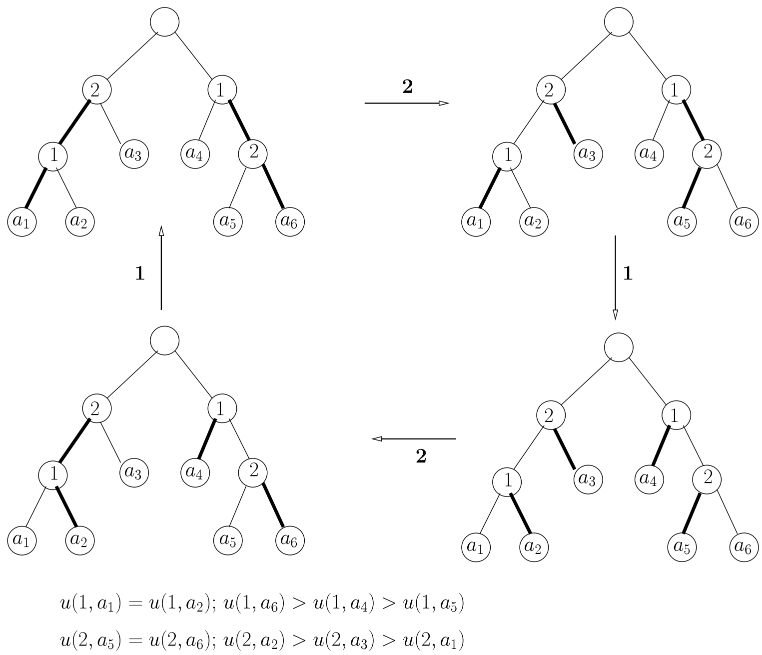


Fig. 4. A uniformly BR (but not lazy) im-cycle. This example shows that the laziness assumption is essential for UAC. Solid lines define the strategies of the players. The initial position (on the top) can belong to any player or be random. This example is similar to Example 3 of [32]. Yet, unlike Example 3, it represents a uniform and lazy BR im-cycle.

that the play spends in v. Respectively, for a move $e = (v, v')$ in a random position $v \in V_R$, we obtain the limit probability $q(e) = q(x, v_0, e) = q(v)q(v, v')$, where $q(v, v')$ is the probability of the considered move (v, v') in v.

Then, the effective limit payoff (or in other words, the normal form of the considered positional games) is defined as a real-valued function $f : I \times X \times (V \setminus V_T) \to \mathbb{R}$, where $f(i, x, v_0) = \sum_{e \in E} q(x, v_0, e) r(i, e)$.

The deterministic case, $V_R = \emptyset$, is also of interest. In this case, given $x \in X$ and $v_0 \in V$, we obtain a play $p = p(x, v_0)$, which can be viewed as a deterministic Markov chain. Since all players are restricted to their positional strategies, $p(x)$ is a lasso, which consists of an initial part and a di-cycle $C = C(x, v_0)$ in G repeated infinitely. Then, $q(e) = q(x, v_0, e) = |C|^{-1}$ for $e \in C$ and $q(e) = 0$ for $e \notin C$.

In the literature on mean payoff games, usually the two-person zero-sum case is considered, since not much is known for other cases. The most important statement is as follows.

Theorem 5. *Two-person zero-sum mean payoff games are UNS; in other words, every such game has a saddle point (NE) in uniformly optimal strategies.* $\square$

This is just a reformulation of the fundamental result of [18,34]. For the two-person zero-sum case, the above model (first introduced in [25]) and the classical Gillette model [18] are equivalent [5,8].

However, IAC (and UAC) may fail for two-person zero-sum mean payoff games even in the absence of random positions. Let us consider the complete bipartite 2 by 2 digraph G on four vertices $v_1, v_3 \in V_1$ and $v_2, v_4 \in V_2$; see Fig. 5. Let the local rewards of player 1 be

$$r(1, (v_1, v_2)) = r(1, (v_2, v_1)) = r(1, (v_3, v_4)) = r(1, (v_4, v_3)) = 1,$$
$$r(1, (v_1, v_4)) = r(1, (v_4, v_1)) = r(1, (v_2, v_3)) = r(1, (v_3, v_2)) = -1,$$

while $r(2, e) + r(1, e) \equiv 0$ for all arcs $e \in E$, since the game is zero-sum.

The corresponding normal form is of size 4×4.

It is not difficult to verify that the strategy profile $x = (x_1, x_2)$ defined by the strategies

$x_1 = \{(v_1, v_2), (v_3, v_4)\}$ and $x_2 = \{(v_2, v_3), (v_4, v_1)\}$ is a uniform saddle point, which results in the di-cycle (v_1, v_2, v_3, v_4); see Fig. 5. Yet, the following four strategy profiles define an im-cycle

$$x^1 = (x'_1, x'_2), \quad x^2 = (x'_1, x''_2), \quad x^3 = (x''_1, x''_2), \quad x^4 = (x''_1, x'_2), \quad \text{where}$$
$$x'_1 = \{(v_1, v_2), (v_3, v_2)\}, \quad x''_1 = \{(v_1, v_4), (v_3, v_4)\}, \quad x'_2 = \{(v_2, v_1), (v_4, v_1)\}, \quad x''_2 = \{(v_2, v_3), (v_4, v_3)\},$$

which is invariant with respect to all possible initial positions.

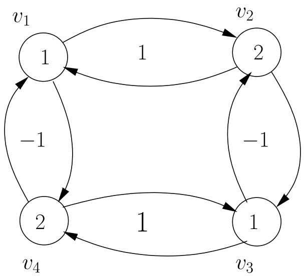


Fig. 5. An im-cycle in a mean payoff game. The corresponding four strategy profiles are shown in Section 5.

Remark 17. It is easily seen that the above im-cycle is uniformly BR but not lazy. A lazy uniformly BR im-cycle was constructed by Oudalov [38] for a two-person zero-sum mean payoff game without random moves. This (computer-generated and still unpublished) example required a 4×4 complete bipartite digraph.

It is also known that INS (and UNS) may fail for a two-person (but not zero-sum) mean payoff game, even in absence of random positions. In other words, the zero-sum assumption is essential in Theorem 5.

To see this, let us recall an example from [22]; see also [25,8]. Let G be the complete bipartite 3×3 digraph and the local rewards of players 1 and 2 on its 18 arcs be given by the following two 3×3 matrices:

$$\begin{matrix} 0 & 0 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-\varepsilon & 0 & 1 \end{matrix}$$

where $r(i, (v, v')) = r(i, (v', v))$ for each player $i \in I = \{1, 2\}$ and every pair of oppositely directed arcs (v, v'), (v', v), and where ε is a small positive number, say, $\varepsilon = 0.1$.

Let us fix an arbitrary initial position, construct the corresponding normal form of size $3^3 \times 3^3 = 27 \times 27$, and choose the BR of player 1 in every row and BR of player 2 in every column. It was shown in [22] that the obtained two sets of di-cycles are disjoint. Hence, there is no NE. In [23], it was shown that the above example is minimal, since INS holds for all games on 2 by k bipartite digraphs.

Let us now demonstrate that n-person Backgammon-like games form a very special subclass of n-person mean payoff games [8]. The reduction is simple. Let $\mathcal{G} = (G, D, u)$ be a non-initialized Backgammon-like n-person game. First, without any loss of generality, we can assume that $u(i, c) = 0$ for all $i \in I$. Indeed, to enforce this condition it is sufficient to subtract $u(i, c)$ from the payoff $u(i, *)$ of each player $i \in I$. More precisely, given a payoff $u : I \times A \to \mathbb{R}$ let us define a payoff $u' : I \times A \to \mathbb{R}$ by formula $u'(i, a) = u(i, a) - u(i, c)$ for all $i \in I,\ a \in V_T$, and denote the obtained game by $\mathcal{G}' = (G, D, u')$.

Then, let us get rid of V_T by adding a loop $e_v = (v, v)$ to each terminal position $v \in V_T$, setting the local reward $r(i, e_v) = u'(i, v)$ for each such loop, and $r(i, e) \equiv 0$ for all other arcs $e \in E$ and all players $i \in I$. Thus, we obtain a non-initialized mean payoff game $\mathcal{G}'' = (G', D', r)$. The following statements are obvious:

- all three games $\mathcal{G}$, $\mathcal{G}'$, and $\mathcal{G}''$ are equivalent;
- $\mathcal{G}$ is a Chess-like game if and only if $\mathcal{G}''$ (and $\mathcal{G}'$) are deterministic (that is, $V_R = \emptyset$);
- all rewards $r(i, e)$ are positive (non-negative) in $\mathcal{G}''$ if and only if c is the worst (maybe, not strictly) outcome for all players $i \in I$ of $\mathcal{G}$;
- $\mathcal{G}$, $\mathcal{G}'$, and $\mathcal{G}''$ can be two-person and/or zero-sum only simultaneously.

6. Two-person (zero-sum) Chess- and Backgammon-like games

Let us recall that UNS (and UAC) may fail for two-person Chess-like games, even when c is the worst outcome for all players; see examples g_2 and g_6 of Section 3. Respectively, by Theorem 1, INS may fail for the corresponding Backgammon-like games.

In contrast, IAC (and INS) hold for the Chess-like, not necessarily zero-sum, two-person games. This follows from Theorem 2 of [1]; see also [7,8] for the case of INS.

Remark 18. It is a fundamental open problem whether INS still holds for n-person Chess-like games; see Section 8. It is known that UNS may fail already when $n = 3$ and c is the worst outcome for all three players; see example g_3 in Section 3. For IAC, a similar example is known [1] only for $n = 4$, while for $n = 3$ the question remains open.

For the rest of this section, we assume that all considered games are two-person zero-sum unless the opposite is said explicitly.

Let us recall that IAC (and UAC) may fail for Backgammon-like games, as it was shown in 1993 by Condon [12]. Yet, UNS (and INS) hold and can be derived from Theorem 5 and the reduction of Section 5.

Remark 19. Such a proof looks like "a big gun for a small fly". Indeed, all known proofs of Theorem 5 are sophisticated and based on the Hardy–Littlewood Tauberian theorems [27] that summarize relations between the Abel and Cesaro averages. The first proof was given in 1957 by Gillette [18]. Yet, he did not verify the "Tauberian conditions" accurately and this flaw was repaired only in 1969 by Liggett and Lippman [34].

However, we have no simpler proof for the UNS of Backgammon-like games, although they represent just a very special case of the mean payoff games. In contrast, for Chess-like games, UAC (together with the remaining three properties) hold and can be proved much easier as follows.

First, let us show that in a Chess-like game $\mathcal{G} = (G, D, u)$ a uniform NE (that is, a saddle point) $x^* = (x_1^*, x_2^*)$ exists and it can be found by a modified backward induction algorithm, which was recently and independently suggested in [2,8].

Remark 20. In contrast to n-person Backgammon-like games on acyclic digraphs, in this case we obtain some, but not necessarily all, uniform Nash equilibria.

In every position $v \in V \setminus V_T = V_1 \cup V_2$ we select one or several arcs such that a strategy profile $x = (x_1, x_2)$ is a uniform saddle point whenever the strategy x_i requires a selected arc in each $v \in V_i,\ i \in I = \{1, 2\}$.

Without any loss of generality, we can make the following two assumptions:

(a) There are no parallel arcs in G, i.e., for all $v, v' \in V$ there is at most one arc $e = (v, v')$ from v to v'. Indeed, if there are several such arcs we can just identify them.
(b) Player 1 ranks the m terminal outcomes in the order $a_1 > \cdots > a_m$, while 2 ranks them in the reverse order. We assume that the order is strict, i.e., there is no tie; see Remarks 7 and 9. Indeed, if several successive outcomes form a tie then we can just identify the corresponding terminals in G.

Then, the infinite outcome c should be placed somewhere between $a_1, \ldots, a_m$ and we will assume, for simplicity, that there are no ties, again.

Remark 21. This time we say "for simplicity" rather than "without loss of generality", because some uniformly optimal strategies might be lost due to the last assumption. However, this may happen even if we allow a tie between c and some terminal a_j.

In particular, c might be the best outcome for player 1 or 2, but obviously not for both unless $m = 0$. (This is, due to the last assumption; otherwise $m = 1$ would be also possible). In the case $m = 0$ there are no terminals at all. Hence, each situation $x \in X$ results in c and x is a uniform NE. Thus, in each position every move is optimal. Otherwise, by assumption (b), terminal a_1 is the best outcome for player 1. If a_1 is an isolated vertex, we eliminate it from G; otherwise, let us fix a move (v, a_1) in G and consider the following two cases. If $v \in V_1$, let us select the move (v, a_1) in G and then reduce G by contracting (v, a_1) and eliminating all other moves from v.

Remark 22. At this moment, some optimal moves in v might be lost.

If $v \in V_2$ then let us eliminate the arc (v, a_1) from G unless there are no other moves from v in G. Then, v becomes a new terminal. In this case, let us identify it with a_1 and also select (v, a_1) in the original digraph G. If (v, a_1) is a unique move to a_1 then we just eliminate the obtained isolated terminal from G.

In all cases the original digraph G is reduced. Let us denote the obtained digraph by G' and repeat the above procedure again, etc. Each time we choose the best terminal of player 1 or 2 until they are all eliminated and only infinite plays remain. Then, let us select all remaining arcs in the original digraph G.

Let us notice that arcs are eliminated in the current (reduced) digraphs, while they are selected in the original digraph G, the copy of which we keep all time.

By construction, in every position $v \in V_1 \cup V_2$ at least one arc is selected and a strategy profile x is a uniform NE whenever x uses a selected arc in each position $v \in V_1 \cup V_2$.

Remark 23. The obtained procedure and the standard backward induction are somewhat similar. Yet, the latter gives *all* uniform NE in acyclic n-person games, while the former gives some uniform NE in (two-person zero-sum) Chess-like games; see Remarks 20 and 22 below.

Remark 24. In contrast, the complexity of getting a uniform NE (saddle point) in a Backgammon-like game is a fundamental open problem, although the existence is implied by Theorem 5 and by the reduction of Section 5. Unfortunately, the above algorithm cannot be extended to Backgammon-like games, which, already in the case $m = 1$ are equivalent with the so-called simple stochastic games [11,12]. The latter are known to be in the intersection of NP and co-NP and, yet, no polynomial, or even pseudo-polynomial, algorithm for their solution is known. Andersson and Miltersen recently proved in [3] that simple stochastic games and general Gillette's stochastic games with perfect information are polynomially equivalent; see also [26].

To prove UAC for Chess-like games, we just combine the above arguments, which prove UNS, and the arguments Section 4, which prove UAC for Backgammon-like n-person games on acyclic digraphs. Again, in a minimal counterexample $\mathcal{G} = (G, D, u)$ with an im-cycle Y, for every edge $e = (v, v') \in V_i,\ i \in I = \{1, 2\}$, there are two strategy profiles $x, x' \in Y$ such that e is chosen by x_i and not chosen by x'_i. Yet, the arc (v, a_1) (considered in the above proof of UNS) cannot satisfy this property. Indeed, if $v \in V_1$ then this move will be chosen by every strategy profile $x \in Y$, by the optimality of a_1 for player 1.

Here, it is important to recall that the players are restricted to their lazy uniformly best responses.

In contrast, if $v \in V_2$ then move (v, a_1) will be chosen by no $x \in Y$ unless (v, a_1) is a unique (forced) move in v. In both cases, we get a contradiction with the minimality of G.

Thus, a sequence of uniform improvements cannot cycle and, hence, it must terminate in a uniform saddle point. This implies UAC, which, in its turn, implies UNS, IAC, and INS.

The results of this section are summarized by the following two statements.

Theorem 6. • *Two-person zero-sum Chess-like games are uniformly acyclic (UAC).*
• *Some uniform Nash equilibria can be found by the above modified backward induction procedure.* □

7. Proofs of Theorems 1 and 2

Clearly, Theorem 2 is implied by Theorem 5. Indeed, given an n-person mean payoff game in which all players but one have their strategies fixed, we obtain a mean payoff game of the remaining single player i, or in other words, a controlled Markov chain. It is well known that in this case a uniformly optimal strategy x_i^* exists; see, for example, [4,35]. This and the above reduction imply the statement of Theorem 2 for Backgammon-like games.

Furthermore, part (i) of Theorem 1 is obvious. In its turn, part (ii) can be derived from the Chess-like case of Theorem 2 (see Section 2.7) as follows. Let us assume indirectly that a Chess-like game $\mathcal{G}$ has no uniform NE. Then, for each situation $x \in X$ of its normal form there is a BR improvement for a player $i \in I$, that is, a situation $y = y(i, x) \in X$ such that y may differ from x only in the ith coordinate, $u(i, g(x, v_0)) \le u(i, g(y, v_0))$ for all $v_0 \in V \setminus T$, and the inequality is strict for at least one v_0.

By assumption, $q(v_0)$ is strictly positive, that is, $q(v_0, v) > 0$ for all $v \in V \setminus (V_T \cup \{v_0\})$. Hence, in $\mathcal{G}'$, situation y is still an improvement of x (although not necessarily BR) for the same player i. Since this holds for all $x \in X$, we conclude that there is no NE in $\mathcal{G}'$. □

Remark 25. The strict positivity of $q(v_0)$ is essential in the above arguments. For example, if $q(v_0, v) = 0$ for all but one $v \in V \setminus (V_T \cup \{v_0\})$ then $\mathcal{G}'$ is a Chess-like game too and its INS (with respect to v_0) holds for all known examples, perhaps, even in general.

8. Chess-like *n*-person games and the INS-conjecture for additive payoffs

According to Table 1, INS of n-person Chess-like games remains an open problem. There is no counterexample and no proof, even under assumption (iii) of Section 1: c is the worst outcome for all n players. Yet, we conjecture that under this assumption INS holds.

In this section we shall recall a much stronger conjecture suggested in [7]. Given an initialized Chess-like positional game structure (G, D, v_0), let us introduce a local reward function $r : I \times E \to \mathbb{R}$. The real number $r(i, e)$ is interpreted as the profit obtained by player $i \in I$ when the play passes arc $e \in E$.

Let us recall that, in absence of random moves, each strategy profile $x \in X$ defines a unique play $p = p(x)$ that begins in the initial position v_0 and either (j) terminates at $a = g(x) \in V_T$ or (jj) results in a lasso that consists of an initial path and a simple di-cycle $c = c(x) \in C(G)$. The *additive effective payoff* $u : I \times X \to \mathbb{R}$ is defined in case (j) as the sum of all local rewards of the obtained play, $u(i, x) = \sum_{e \in p(x)} r(i, e)$, and in case (jj) $u(i, x) \equiv -\infty$, for all $i \in I$. In other words, in case (jj) all infinite plays are equivalent and ranked as the worst outcome by all players, in agreement with (iii). Yet, in case (j) payoffs may depend not only on the terminal position $a = g(x)$ but on the entire play $p(x)$.

The following two equivalent assumptions were considered in [7]:

(t) all local rewards are non-positive, $r(i, e) \le 0$ for all $e \in E,\ i \in I$, and
(tt) all di-cycles are non-positive, $\sum_{e \in c} r(i, e) \le 0$ for all $c \in C = C(G),\ i \in I$.

Obviously, (t) implies (tt). Moreover, it was shown in 1958 by Gallai [17] that in fact these two assumptions are equivalent, since (t) can be enforced by a potential transformation whenever (tt) holds; see [17] and also [7] for the definitions and more details.

Remark 26. In [7], players $i \in I$ minimize cost functions $-u(i, x)$ rather than maximize payoffs $u(i, x)$. Hence, (t) and (tt) turn into non-negativity conditions for local costs.

It was proven in [7] that INS holds under conditions (t) and (tt) for the so-called play-once games defined by the following extra condition (ttt) $|V_i \leq 1|\ \forall i \in I$; in other words, each player controls at most one position.

It was conjectured in [7] that assumptions (t) and (tt) imply INS in general, even without (ttt), and this INS-conjecture was verified for several non-trivial examples. It was also shown in [7] that conditions (t) and (tt) are essential, that is, without (t) and (tt), INS may fail, even when (ttt) holds.

It is easily seen that each terminal payoff is a special case of an additive one [7]. To show this, let us just set $r(i, e) = u(i, a)$ whenever $e = (v, a)$ is a terminal move in G and $r(i, e) \equiv 0$ otherwise. Let us also notice that no terminal move can belong to a di-cycle. Hence, conditions (t) and (tt) hold for a terminal payoff automatically.

Obviously, a proof of the INS-conjecture for additive payoffs would answer in positive the second question of Table 1, yet, the first one would still remain open.

Remark 27. The concept of an additive payoff can be naturally extended to n-person Back-gammon-like games. Yet, INS for these games may fail already for terminal payoffs; see Section 3.

9. Why Chess and Backgammon can be solved in pure positional uniformly optimal strategies

Chess and Backgammon can be viewed as two-person zero-sum stochastic games with perfect information and limiting mean effective payoff; see Section 6 or, for example, [8]. Solvability of such games in pure positional uniformly optimal strategies was proven in 1957 by Gillette [18]. The proof is quite involved; it is based on the classical Hardy–Littlewood (1931) Tauberian theorems [27] that summarize relations between the Abel and Cesaro averages. Conditions of [27] were not accurately verified in [18] and this flaw was repaired only in 1969 by Liggett and Lippman [34]. Let us recall that the above reduction results in a very special local rewards for Chess and Backgammon (as well as for n-person Chess- and Backgammon-like games): $r(i, e) = 0$ unless $e = (v, v)$ is the loop corresponding to a terminal $v \in V_T$.

Nevertheless, in the case of Backgammon, we are not aware of any simpler proof of solvability.

Furthermore, in presence of random positions, no efficient solution algorithm is known. Andersson and Miltersen [3] (see also [26]) recently demonstrated that the general Gillette model (and, in particular, the two-person zero-sum Backgammon-like games with m terminals) can be polynomially reduced to the case $m = 1$, that is, to simple stochastic games, introduced in 1992 by Condon [11,12]. Although the latter class is in the intersection of NP and co-NP [11], yet, even a pseudo-polynomial algorithm is not currently known.

In contrast, the situation with two-person zero-sum Chess-like games is much better. Proofs of solvability were given in the early works by Zermelo [42], König [29], and Kalmar [28]. Surveys of these results and, in particular, of their relations to the classical Chess defined by the FIDE rules can be found in [14,41,39]. Yet, somewhat surprisingly, an efficient algorithm was given only in 2009, independently in [2,8]. This algorithm (a modified backward induction) finds some, but not all, positional uniformly optimal strategies of both players in (almost) linear time; see [2] for more details. Let us note that not only UNS but also UAC can be derived for the considered games; see Section 6.

Two-person zero-sum Chess- and Backgammon-like games can naturally be extended to the n-person case. In this case, the concept of a saddle point is replaced with an equilibrium (NE) suggested in 1950 by Nash [36,37]. Soon after Nash's work, Kuhn [30,31] and Gale [16] suggested an algorithm (backward induction) which finds *all* uniform (subgame perfect) NE in positional strategies for the n-person Chess-like games modeled by trees. This algorithm can be easily extended to the Backgammon-like games on acyclic digraphs. Moreover, UNS can be strengthened to UAC; see Section 4.

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