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Fine Analysis of the Quasi-Orderings on the Power Set

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Abstract. We pursue the fine analysis of the quasi-orderings $\preceq^{\exists}_{\forall}$ and $\preceq^{\forall}_{\exists}$ on the power set of a quasi-ordering $(Q, \preceq)$. We set $X \preceq^{\exists}_{\forall} Y$ if every $x \in X$ is majorized in $\preceq$ by some $y \in Y$, and $X \preceq^{\forall}_{\exists} Y$ if every $y \in Y$ is minorized in $\preceq$ by some $x \in X$. We show that both these quasi-orderings are α-wqo if and only if the original quasi-ordering is $(\alpha \cdot \omega)$-wqo. For $\preceq^{\forall}_{\exists}$ this holds also restricted to finite subsets, thus providing an example of a finitary operation on quasi-orderings which does not preserve wqo but preserves bqo.

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Let Q be a set quasi-ordered by the binary relation $\preceq$ (i.e. $\preceq$ is transitive and reflexive) and let $\mathcal{P}(Q)$ be the power set of Q. We denote by $\mathcal{P}_{\mathrm{f}}(Q)$ the set of all finite subsets of Q, and by $\mathcal{P}_{\mathrm{c}}(Q)$ the set of all countable subsets of Q. We study the following quasi-orderings induced by $\preceq$ on $\mathcal{P}(Q)$ (and hence on $\mathcal{P}_{\mathrm{f}}(Q)$ and $\mathcal{P}_{\mathrm{c}}(Q)$):

DEFINITION 1. If $X, Y \in \mathcal{P}(Q)$ let

$$X \preceq^{\exists}_{\forall} Y \iff \forall x \in X\ \exists y \in Y\ x \preceq y \quad \text{and}$$
$$X \preceq^{\forall}_{\exists} Y \iff \forall y \in Y\ \exists x \in X\ x \preceq y.$$

$\equiv^{\exists}_{\forall}$ and $\equiv^{\forall}_{\exists}$ will denote the equivalence relations induced by the two quasi-orderings.

Notice that $\preceq^{\exists}_{\forall}$ and $\preceq^{\forall}_{\exists}$ are mutually definable. Indeed if we denote by $X\uparrow$ and $X\downarrow$ respectively the upward closure and the downward closure of X (i.e. $X\uparrow = \{y \in Q \mid \exists x \in X\ x \preceq y\}$ and $X\downarrow = \{y \in Q\} \mid \exists x \in X\ y \preceq x)$ then

$$X \preceq^{\exists}_{\forall} Y \iff Q \setminus (\downarrow X) \preceq^{\forall}_{\exists} Q \setminus (\downarrow Y) \tag{1}$$

and

$$X \preceq^{\forall}_{\exists} Y \iff Q \setminus (X\uparrow) \preceq^{\exists}_{\forall} Q \setminus (Y\uparrow). \tag{2}$$

Notice also that for downward closed sets $\preceq^{\exists}_{\forall}$ coincides with $\subseteq$, while for upward closed sets $\preceq^{\forall}_{\exists}$ coincides with $\supseteq$. Moreover $X{\downarrow}\equiv^{\exists}_{\forall} X$ and $X{\uparrow}\equiv^{\forall}_{\exists} X$ for every $X \in \mathcal{P}(Q)$.

A quasi-ordering is a wqo (well quasi-ordering) if there are no infinite strictly descending chains and no infinite sets of mutually incomparable elements. The usual working definition of wqo is obtained with an application of the infinite Ramsey theorem:

DEFINITION 2. Let $\preceq$ be a quasi-ordering on Q. $\preceq$ is wqo if for every map $f : \mathbb{N} \to Q$ there exist $m < n$ such that $f(m) \preceq f(n)$.

The notion of bqo (better quasi-ordering) is a strengthening of wqo which was introduced by Nash-Williams in the 1960's and has proved to be very useful in showing that certain quasi-orderings are indeed wqo. We give the precise (and rather technical) definition of bqo below. The property of being bqo is preserved by a much wider class of operations than those that preserve wqo (see [7] for a survey; [12] and [10] are also introductions to bqo theory), the general pattern being that if a finitary operation preserves wqo then its infinitary version preserves bqo. The notion of bqo can be analyzed using a hierarchy of intermediate notions (called α-wqo, where α is a countable ordinal): this "fine analysis" has been carried out in [9] and [6]. We study the relationships between Q being wqo, bqo or α-wqo and the same properties for the quasi-orderings on $\mathcal{P}(Q)$ (and its subsets) defined above.

$\preceq^{\exists}_{\forall}$ (which is sometimes known as the Egli–Milner ordering) is arguably more natural and has widely been studied from the viewpoint of wqo and bqo theory: it turns out that if Q is wqo then $\mathcal{P}_{\mathrm{f}}(Q)$ is wqo with respect to $\preceq^{\exists}_{\forall}$ and that if Q is bqo then $\mathcal{P}(Q)$ is bqo with respect to $\preceq^{\exists}_{\forall}$ (the former result was proved probably for the first time by Erdős and Rado in [3], while the latter follows easily from the theorem on transfinite sequences proved by Nash-Williams in [8]). $\preceq^{\forall}_{\exists}$ was brought to the author's attention in December 1998 by P. Abdulla, who works in the area of reachability analysis (see [1, 2] and their references for the connection between this subject and wqo theory). Abdulla asked whether Q wqo with respect to $\preceq$ implies $\mathcal{P}_{\mathrm{f}}(Q)$ wqo with respect to $\preceq^{\forall}_{\exists}$. "$Q$ bqo implies $\supseteq$ bqo on upward closed subsets of Q" is folklore, and hence we can consider "Q bqo implies $\mathcal{P}(Q)$ bqo with respect to $\preceq^{\forall}_{\exists}$" as being known as well.

In this paper we provide a detailed analysis of the relationship between the properties of $\preceq$ and those of $\preceq^{\exists}_{\forall}$ and $\preceq^{\forall}_{\exists}$, in terms of the "fine analysis" described above. Our results include a sharper version of the result that can be deduced from Nash-Williams' theorem and a negative answer to Abdulla's question (the latter was independently obtained by P. Jančar [4]). In particular it turns out that $(Q, \preceq) \mapsto (\mathcal{P}_{\mathrm{f}}(Q), \preceq^{\forall}_{\exists})$ is an example of a finitary operation on quasi-orderings which does not preserve wqo but preserves bqo.

We follow notation and terminology of [6]; we refer to that paper for proofs or references for the results mentioned below.

If s is a finite sequence (i.e. a function with domain a natural number) we denote by $\mathrm{lh}(s)$ its length (which coincides with its domain) and, for every $i < \mathrm{lh}(s)$, by $s(i)$ its $(i+1)$-th element. Then we write s as $\langle s(0), \ldots, s(\mathrm{lh}(s)-1)\rangle$, so that in particular $\langle i \rangle$ is the sequence of length 1 whose only element is i. If s and t are finite sequences we write $s \sqsubseteq t$ if s is an initial segment of t, i.e. if $\mathrm{lh}(s) \leq \mathrm{lh}(t)$ and $\forall i < \mathrm{lh}(s)\ s(i) = t(i)$. We write $s \subseteq t$ if the range of s is a subset of the range of t, i.e. if $\forall i < \mathrm{lh}(s)\ \exists j < \mathrm{lh}(t)\ s(i) = t(j)$. $s \sqsubset t$ and $s \subset t$ are obtained from the above excluding equality, and we extend these notations also to the case where t is an infinite sequence (i.e. a function with domain $\mathbb{N}$). We write $s^\frown t$ for the concatenation of s and t, i.e. the sequence u such that $\mathrm{lh}(u) = \mathrm{lh}(s) + \mathrm{lh}(t)$, $u(i) = s(i)$ for every $i < \mathrm{lh}(s)$, and $u(\mathrm{lh}(s) + i) = t(i)$ for every $i < \mathrm{lh}(t)$. If $X \subseteq \mathbb{N}$ is infinite we denote by $[X]^{<\omega}$ (resp. $[X]^{\omega}$) the set of all finite (resp. infinite) subsets of X. We identify a subset of $\mathbb{N}$ with the unique sequence enumerating it in increasing order. For $s \in [\mathbb{N}]^{<\omega}$ and $n \leq \mathrm{lh}(s)$ let $s[n]$ be the finite set enumerated by $\langle s(0), \ldots, s(n-1)\rangle$. The following notation will also be useful: for $B \subseteq [\mathbb{N}]^{<\omega}$ let $S(B) = \{i \mid \langle i \rangle \in B\}$.

If B is a subset of $[\mathbb{N}]^{<\omega}$ we have

$$\bigcup B = \{n \mid \exists s \in B\ \exists i < \mathrm{lh}(s)\ s(i) = n\}.$$

A set $B \subseteq [\mathbb{N}]^{<\omega}$ is a block if:

(i) B is infinite (and hence $\bigcup B$ is also infinite);
(ii) $\forall X \in \left[\bigcup B\right]^{\omega}\ \exists s \in B\ s \sqsubset X$;
(iii) $\forall s, t \in B\ s \not\sqsubset t$.

B is a barrier if it satisfies (i), (ii) and

(iii)′ $\forall s, t \in B\ s \not\subset t$.

B is a smooth barrier if it is a barrier such that for all $s, t \in B$ with $\mathrm{lh}(s) < \mathrm{lh}(t)$ there exists $i < \mathrm{lh}(s)$ such that $s(i) < t(i)$.

It is immediate that every barrier is a block and from the clopen Ramsey theorem it follows that every block contains a barrier. Notice that if B is a block and $Y \in \left[\bigcup B\right]^{\omega}$ there exists a unique block $B' \subseteq B$ such that $\bigcup B' = Y$, namely $B' = \{s \in B \mid s \subset Y\}$. Notice also that if B is a barrier then B' is also a barrier and we say that B' is a subbarrier of B. It is obvious that every subbarrier of a smooth barrier is also smooth.

We define the lexicographic ordering on finite sequences of natural numbers by declaring $s <_{\mathrm{lex}} t$ if and only if either $s \sqsubset t$ or for some $i < \min\{\mathrm{lh}(s), \mathrm{lh}(t)\}$ we have $s[i] = t[i]$ and $s(i) < t(i)$. $<_{\mathrm{lex}}$ is a linear ordering and well-orders every block. If B is a block let $\mathrm{o.t.}(B)$ be the order type of the lexicographic ordering of B; $\mathrm{o.t.}(B)$ is thus an infinite countable ordinal. The order type of a block can be any countable limit ordinal, while if B is a barrier then $\mathrm{o.t.}(B)$ has the form $\omega^\beta \cdot n$ where $0 < \beta < \omega_1$, $0 < n < \omega$ and if $\beta < \omega$ then $n = 1$. Moreover, every barrier of order type $\omega^\beta \cdot n$ with $n > 1$ contains a subbarrier of order type ω^β, and hence the barriers of infinite indecomposable order type are most important. Furthermore, the order type of a smooth barrier is an indecomposable ordinal.

If B is a block and $i \in \bigcup B$ let $B(i) = \{t \mid \langle i \rangle ^\frown t \in B\}$. Notice that $\text{o.t.}(B) = \sum_{i \in \bigcup B} \text{o.t.}(B(i))$ and this implies that $\text{o.t.}(B(i)) < \text{o.t.}(B)$. If $i \notin S(B)$ then $B(i)$ is a block with $\bigcup(B(i)) = \{j \in \bigcup B \mid j > i\}$. Moreover if B is a barrier (resp. smooth barrier) and $i \notin S(B)$ then $B(i)$ is a barrier (resp. smooth barrier).

Let $s, t \in [\mathbb{N}]^{<\omega}$: we write $s \lhd t$ if for $u = s \cup t$ we have $s \sqsubseteq u$ and $t \sqsubseteq u \setminus \{u(0)\}$.

Let $\preceq$ be a quasi-ordering on a set Q, B be a block and $f : B \to Q$. We say that the map f is good with respect to $\preceq$ if there exist $s, t \in B$ such that $s \lhd t$ and $f(s) \preceq f(t)$. If f is not good then we say that it is bad.

DEFINITION 3. Let α be a countable ordinal and $\preceq$ a quasi-ordering on Q. $\preceq$ is α-wqo if for every barrier B with $\text{o.t.}(B) \leq \alpha$, every map $f : B \to Q$ is good with respect to $\preceq$. If $\preceq$ is α-wqo for every countable α, $\preceq$ is bqo.

Notice that ω-wqo is equivalent to wqo, because the barriers of order type ω consist only of singletons.

For the reasons mentioned above, the notion of α-wqo is most interesting when α is infinite indecomposable (notice in particular that $(\omega^\beta \cdot n)$-wqo coincides with ω^β-wqo for any $n > 0$). For any such α there exists a quasi-ordering which is β-wqo for every $\beta < \alpha$ and is not α-wqo ([6, Theorem 3.6]).

The following Theorem ([6, Theorem 4.3]) shows that smooth barriers suffice to give the definition of α-wqo:

THEOREM 4. *Let α be a countable ordinal and $\preceq$ a quasi-ordering on Q. $\preceq$ is α-wqo if and only if for every smooth barrier B with* $\text{o.t.}(B) \leq \alpha$*, every map $f : B \to Q$ is good with respect to $\preceq$.*

Given a block $B \subseteq [\mathbb{N}]^{<\omega}$ we construct two blocks associated to B: the first has already been widely used.

DEFINITION 5. If B is a block let $B^2 = \{s \cup t \mid s, t \in B \wedge s \lhd t\}$.

The main properties of B^2 are summarized by the following lemma:

LEMMA 6. *Let B be a block.*

(a) *B^2 is a block;*
(b) *for every $t \in B^2$ there exist unique $\pi_0(t), \pi_1(t) \in B$ such that $\pi_0(t) \lhd \pi_1(t)$ and $t = \pi_0(t) \cup \pi_1(t)$;*
(c) *if $t, t' \in B^2$ and $t \lhd t'$ then $\pi_1(t) = \pi_0(t')$;*
(d) *if B is a barrier then B^2 is a barrier;*
(e) *if B is a barrier and* $\text{o.t.}(B)$ *is indecomposable then* $\text{o.t.}(B^2) = \text{o.t.}(B) \cdot \omega$.

Proof. (a)–(d) are straightforward and well-known. To prove (e) we claim that for every $i \in \bigcup B$, we have $B^2(i) = \{s \in B \mid s(0) > i\}$. To prove the claim notice that if $s \in B^2(i)$ then obviously $s(0) > i$ and $t = \langle i \rangle ^\frown s \in B^2$. We have $\pi_0(t) \sqsubseteq t$: if $\pi_0(t) = t$ then $\pi_1(t) \subset \pi_0(t)$ against the definition of barrier; thus $\pi_0(t) \sqsubset t$ and $s = \pi_1(t) \in B$. On the other hand if $s \in B$ is such that $s(0) > i$ then (since B is a barrier) there exists $s' \in B$ such that $s' \sqsubset \langle i \rangle ^\frown s$. Then $s' \lhd s$ and $s' \cup s \in B^2$. Since $s' \cup s = \langle i \rangle ^\frown s$ we have $s \in B^2(i)$.

By the claim $B^2(i)$ is a final segment (with respect to $<_{\mathrm{lex}}$) of B: since o.t.(B) is indecomposable we have o.t.$(B^2(i)) =$ o.t.(B) and therefore o.t.$(B^2) = \sum_{i \in \cup B}$ o.t.$(B^2(i)) =$ o.t.$(B) \cdot \omega$. □

The second block we associate to B, as far as we know, has not been used before.

DEFINITION 7. If B is a block let $\overline{B}$ be the set of all $t \neq \langle\,\rangle$ such that

$$\exists i \ \ \langle i \rangle ^\frown t \in B \wedge \forall j < t(0) \Big(j \in \bigcup B \ \longrightarrow\ \exists n \leq \mathrm{lh}(t) \ \ \langle j \rangle ^\frown t[n] \in B \Big).$$

The main properties of $\overline{B}$ are summarized by the following lemma:

LEMMA 8. *Let B be a block.*

(a) *if* o.t.$(B) > \omega$ *then* $\overline{B}$ *is a block;*
(b) *if B is a barrier then* o.t.$(\overline{B}) \leq$ o.t.(B)*;*
(c) *if B is a smooth barrier and* o.t.$(B) = \alpha \cdot \omega$ *for some indecomposable* $\alpha \geq \omega$ *then* o.t.$(\overline{B}) \leq \alpha$.

Proof. (a) o.t.$(B) > \omega$ implies that $\bigcup B \neq S(B)$. If n is the least element of $\bigcup B \setminus S(B)$ let $Y = \{i \in \bigcup B \mid i > n\}$. Clearly Y is infinite and it is easy to check that $\bigcup \overline{B} \subseteq Y$. If $X \in [Y]^\omega$, for every $j \in \bigcup B$ with $j < X(0)$ by condition (ii) in the definition of block (applied to $\langle j \rangle ^\frown X \in \left[\bigcup B\right]^\omega$) there exists a unique (possibly empty) t_j such that $\langle j \rangle ^\frown t_j \in B$ and $t_j \sqsubset X$. Amongst these (finitely many) j's let i be such that $\mathrm{lh}(t_i)$ is maximal: then, since $n < X(0)$ and $\langle n \rangle \notin B$, we have $\mathrm{lh}(t_i) > 0$ and $t_i \in \overline{B}$. This implies both that $\bigcup \overline{B} \supseteq Y$ and that $\overline{B}$ contains an initial segment of every subset of Y. It follows from the definition of $\overline{B}$ that $t \not\sqsubset t'$ for every $t, t' \in \overline{B}$ and hence $\overline{B}$ is a block.

(b) Suppose o.t.$(\overline{B}) >$ o.t.(B): since $\bigcup \overline{B} \subseteq \bigcup B$ we can apply Lemma 3.2 of [6] which asserts (under the above hypothesis on the order types of the two blocks) the existence of $s \in B$ and $t \in \overline{B}$ such that $s \sqsubset t$. Then $s \subset \langle i \rangle ^\frown t \in B$ for some i, against the definition of barrier.

(c) If B is a smooth barrier then $B(i)$ is a smooth barrier for every $i \in \bigcup B \setminus S(B)$ and hence o.t.$(B(i)) <$ o.t.(B) is indecomposable. Therefore o.t.$(B(i)) \leq \alpha$ for every i and for $j > i$ we have o.t.$(B(i)(j)) < \alpha$. It is obvious that $\overline{B}(j) \subseteq \bigcup_{i<j} B(i)(j)$ and the indecomposability of α implies o.t.$(\overline{B}(j)) \leq \sum_{i<j}$ o.t.$(B(i)(j)) < \alpha$ for every $j \in \bigcup \overline{B}$. Thus, using the again the indecomposability of α, o.t.$(\overline{B}) = \sum_{j \in \bigcup \overline{B}}$ o.t.$(\overline{B}(j)) \leq \alpha$. □

We can now prove the main result of the paper.

THEOREM 9. *For any quasi-ordering $(Q, \preceq)$ and any infinite countable indecomposable ordinal α the following are equivalent:*

(i) *Q is $(\alpha \cdot \omega)$-wqo with respect to $\preceq$;*
(ii) *$\mathcal{P}(Q)$ is α-wqo with respect to $\preceq_\forall^\exists$;*
(iii) *$\mathcal{P}_c(Q)$ is α-wqo with respect to $\preceq_\forall^\exists$;*
(iv) *$\mathcal{P}(Q)$ is α-wqo with respect to $\preceq_\exists^\forall$;*
(v) *$\mathcal{P}_f(Q)$ is α-wqo with respect to $\preceq_\exists^\forall$.*

Proof. To prove that (i) implies (ii) assume that $\mathcal{P}(Q)$ is not α-wqo with respect to $\preceq_\forall^\exists$: by Theorem 4 there exists a smooth barrier B with $\text{o.t.}(B) \leq \alpha$ and a map $f : B \to \mathcal{P}(Q)$ which is bad with respect to $\preceq_\forall^\exists$. Let $g : B^2 \to Q$ be defined so that for every $t \in B^2$, $g(t) \in f(\pi_0(t))$ and $g(t) \npreceq y$ for every $y \in f(\pi_1(t))$. The existence of such a $g(t)$ follows from $f(\pi_0(t)) \npreceq_\forall^\exists f(\pi_1(t))$, which is a consequence of the badness of f. g is bad with respect to $\preceq$: indeed if $t, t' \in B^2$ are such that $t \lhd t'$ we have $\pi_1(t) = \pi_0(t')$ and hence $g(t) \npreceq g(t')$. By Lemma 6 and the fact that the order type of a smooth barrier is indecomposable it follows that $\text{o.t.}(B^2) = \text{o.t.}(B) \cdot \omega \leq \alpha \cdot \omega$. Hence Q is not $(\alpha \cdot \omega)$-wqo with respect to $\preceq$.

The proof that (i) implies (iv) is similar: given $f : B \to \mathcal{P}(Q)$ bad with respect to $\preceq_\exists^\forall$, let $g : B^2 \to Q$ be defined so that $g(t) \in f(\pi_1(t))$ and $x \npreceq g(t)$ for every $x \in f(\pi_0(t))$. Arguing as above we can prove that g is bad with respect to $\preceq$.

(ii) implies (iii) and (iv) implies (v) are trivial.

To prove that (iii) implies (i) suppose that Q is not $(\alpha \cdot \omega)$-wqo. By Theorem 4 there exist a smooth barrier B with $\text{o.t.}(B) \leq \alpha \cdot \omega$ and a bad map $f : B \to Q$. If $\text{o.t.}(B) < \alpha \cdot \omega$ then (since $\text{o.t.}(B)$ is indecomposable) $\text{o.t.}(B) \leq \alpha$ and $g : B \to \mathcal{P}_f(Q)$ defined by $g(s) = \{f(s)\}$ shows that $\mathcal{P}_f(Q)$, and a fortiori $\mathcal{P}_c(Q)$, is not α-wqo with respect to $\preceq_\forall^\exists$.

If $\text{o.t.}(B) = \alpha \cdot \omega > \omega$, define $g : \overline{B} \to \mathcal{P}_c(Q)$ by letting

$$g(t) = \{f(s) \mid s \in B \wedge t \sqsubset s\}.$$

We claim that g is bad with respect to $\preceq_\forall^\exists$: indeed if $t, u \in \overline{B}$ and $t \lhd u$ there exists $n \leq \text{lh}(u)$ such that $\langle t(0) \rangle ^\frown u[n] \in B$. Let $s = \langle t(0) \rangle ^\frown u[n]$, so that $s \lhd u$. Moreover $t \sqsubset s$ (otherwise $s \sqsubseteq t$ and so, by the definition of $\overline{B}$, $s \subset \langle i \rangle ^\frown t \in B$ for some i, against the fact that B is a barrier) and therefore $f(s) \in g(t)$. For every $f(s') \in g(u)$ we have $u \sqsubset s'$ and hence $s \lhd s'$, so that $f(s) \npreceq f(s')$. Therefore $g(t) \npreceq_\forall^\exists g(u)$.

By Lemma 8 $\overline{B}$ is a block and $\text{o.t.}(\overline{B}) \leq \alpha$. $\overline{B}$ contains a barrier B': obviously $\text{o.t.}(B') \leq \text{o.t.}(\overline{B})$ and g restricted to B' is also bad. Therefore $\mathcal{P}_c(Q)$ is not α-wqo with respect to $\preceq_\forall^\exists$.

The proof that (v) implies (i) follows the same pattern: the most interesting case is when there exists $f : B \to Q$ which is bad, where B is a smooth barrier with $\text{o.t.}(B) = \alpha \cdot \omega$. In this case define $g : \overline{B} \to \mathcal{P}_{\mathrm{f}}(Q)$ by letting

$$g(t) = \{f(\langle i\rangle ^\frown t[n]) \mid n \leq \mathrm{lh}(t) \wedge \langle i\rangle ^\frown t[n] \in B\}.$$

We claim that g is bad with respect to $\preceq^\forall_\exists$: indeed if $t, u \in \overline{B}$ and $t \vartriangleleft u$ we have $\langle t(0)\rangle ^\frown u[m] \in B$ for some $m \leq \mathrm{lh}(u)$. Hence $f(\langle t(0)\rangle ^\frown u[m]) \in g(u)$ and moreover $t \sqsubset \langle t(0)\rangle ^\frown u[m]$ (otherwise $\langle t(0)\rangle ^\frown u[m] \subseteq t \subset \langle i\rangle ^\frown t$ for some $\langle i\rangle ^\frown t \in B$). For every $f(\langle i\rangle ^\frown t[n]) \in g(t)$ we have $\langle i\rangle ^\frown t[n] \vartriangleleft \langle t(0)\rangle ^\frown u[m]$ and hence $f(\langle i\rangle ^\frown t[n]) \not\preceq f(\langle t(0)\rangle ^\frown u[m])$. Thus $g(t) \not\preceq^\forall_\exists g(u)$ and g is bad with respect to $\preceq^\forall_\exists$.

Arguing as above we conclude that $\mathcal{P}_{\mathrm{f}}(Q)$ is not α-wqo with respect to $\preceq^\forall_\exists$. $\square$

Notice that the equivalence of (ii) and (iv) follows from (1) and (2), but those formulae do not help in establishing any relation between (iii) and (v).

Since Q wqo implies $\mathcal{P}_{\mathrm{f}}(Q)$ wqo with respect to $\preceq^\exists_\forall$, we cannot replace $\mathcal{P}_{\mathrm{c}}(Q)$ with $\mathcal{P}_{\mathrm{f}}(Q)$ in (iii) of Theorem 9, and hence there is a certain asymmetry between $\preceq^\exists_\forall$ and $\preceq^\forall_\exists$. Another asymmetry arises from the fact that if Q is wqo then for every $X \in \mathcal{P}(Q)$ there exists $Y \in \mathcal{P}_{\mathrm{f}}(Q)$ such that $Y \equiv^\forall_\exists X$ (the easy proof means showing that Q has the finite basis property, which is actually equivalent to being wqo): hence the equivalence between (iv) and (v) is to be expected. On the other hand, even for Q a well-ordering (and hence bqo) it is not true that for every $X \in \mathcal{P}(Q)$ there exists $Y \in \mathcal{P}_{\mathrm{c}}(Q)$ such that $Y \equiv^\exists_\forall X$ (e.g., let Q be an ordinal of uncountable cofinality and take $X = Q$).

Theorem 9 has the following corollary, which includes some of the known results mentioned above:

COROLLARY 10. *For any quasi-ordering* $(Q, \preceq)$ *the following conditions are equivalent:*

(i) *Q is bqo with respect to $\preceq$;*
(ii) *$\mathcal{P}(Q)$ is bqo with respect to $\preceq^\exists_\forall$;*
(iii) *$\mathcal{P}_{\mathrm{c}}(Q)$ is bqo with respect to $\preceq^\exists_\forall$;*
(iv) *$\mathcal{P}(Q)$ is bqo with respect to $\preceq^\forall_\exists$;*
(v) *$\mathcal{P}_{\mathrm{f}}(Q)$ is bqo with respect to $\preceq^\forall_\exists$.*

Theorem 9 should be contrasted with the following fine version of Nash-Williams' theorem on transfinite sequences, where $Q^{<\alpha}$ denotes the set of sequences of elements of Q of length an ordinal less than α, quasi-ordered by order-preserving embeddability:

THEOREM 11 [6]. *For any quasi-ordering* $(Q, \preceq)$ *and any infinite countable indecomposable ordinal* α, *Q is α-wqo if and only if $Q^{<\alpha}$ is wqo.*

Another corollary of Theorem 9 answers Abdulla's question.

COROLLARY 12. *For any quasi-ordering* $(Q, \preceq)$ *the following are equivalent:*

(i) *Q is ω^2-wqo with respect to $\preceq$;*
(ii) *$\mathcal{P}(Q)$ is wqo with respect to $\preceq^{\exists}_{\forall}$;*
(iii) *$\mathcal{P}_c(Q)$ is wqo with respect to $\preceq^{\exists}_{\forall}$;*
(iv) *$\mathcal{P}(Q)$ is wqo with respect to $\preceq^{\forall}_{\exists}$;*
(v) *$\mathcal{P}_f(Q)$ is wqo with respect to $\preceq^{\forall}_{\exists}$;*
(vi) *Q is wqo with respect to $\preceq$ and does not contain any isomorphic copy of Rado's example* (*defined, e.g., in* [7, p. 492]).

Proof. Letting $\alpha = \omega$ in Theorem 9 we obtain the equivalence of conditions (i)–(v). By Theorem 11 (i) is equivalent to $Q^{<\omega^2}$ wqo. Higman's theorem (see, e.g., [7, Theorem 1.6]) states that if $(Q, \preceq)$ is wqo then $Q^{<\omega}$ is wqo: applying this to Q^{ω} we obtain that $Q^{<\omega^2}$ wqo is equivalent to Q^{ω} wqo. If Q is wqo the fact that Q^{ω} is wqo if and only if Q does not contain any isomorphic copy of Rado's example is due to Rado himself ([11]; see [5] for a simpler proof). □

The equivalence between (v) and (vi) of Corollary 12 (which answers negatively Abdulla's question) has been proved independently by P. Jančar ([4]) in a more direct way.

A consequence of Theorem 11 is that $Q^{<\omega_1}$ wqo is equivalent to Q bqo: this fact was first proved by Pouzet in [9] and contrasts with the situation described by Corollary 12, where a condition much weaker than bqo is shown to be equivalent to $\preceq^{\exists}_{\forall}$ and $\preceq^{\forall}_{\exists}$ being wqo.

Notice that the operation which associates to Q the quasi-ordering $(\mathcal{P}_f(Q), \preceq^{\forall}_{\exists})$ is finitary, yet does not preserve wqo. This is in disagreement with the widespread phenomenon that finitary operations fall within the realm of wqo theory, while bqo theory is necessary only for adequately dealing with infinitary operations.

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