Automatica 95 (2018) 207–212

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

# Minimal realizations of nonlinear systems ☆

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ARTICLE INFO

*Article history:*
Received 23 September 2016
Received in revised form 19 March 2018
Accepted 6 April 2018

*Keywords:*
Nonlinear systems
Time-varying systems
State space realization
Reduction
Accessibility
Polynomial methods

ABSTRACT

The nonlinear realization theory is recasted for time-varying single-input single-output nonlinear systems. The concept of realization has been extended to cover also the realizations with order greater than the order of input–output equation. The minimal realization problem is studied. The state realization is said to be minimal if it is either accessible and observable or its state dimension is minimal. In the linear case the two definitions are equivalent, but not for nonlinear time-invariant systems. It is shown that the two definitions remain equivalent for nonlinear systems under certain technical assumptions. Two alternative methods are presented for finding the minimal realization.



## 1. Introduction

There exist numerous papers where the realization of nonlinear time-invariant systems is studied, see for instance Belikov, Kotta, and Tõnso (2014), Conte, Moog, and Perdon (2007), Delaleau and Respondek (1995) and van der Schaft (1987), but to the best knowledge of the authors there is no contribution to the realization problem of nonlinear time-varying systems. We follow the algebraic approach of differential one-forms (Conte et al., 2007), combined with the theory of non-commutative polynomial rings (Belikov et al., 2014; Halás, 2008; Zhang, Moog, & Xia, 2010; Zheng, Willems, & Zhang, 2001), adapted from time-invariant to time-varying single-input single-output (SISO) case. In the present paper (i) a new definition of (transfer) equivalence and realization are given, (ii) realizability conditions in Proposition 3 have been generalized from time-invariant to time-varying systems and more importantly, extended also for the case when the dimension of realization is greater than the order of input–output (i/o) equation. Recall that in the literature two definitions of minimality of the state space realization are used. First, one may require minimality of the state dimension. Second, the realization is said to be minimal when it is both observable and accessible (controllable), see for instance Kailath (1980), 363. Though in the linear time-invariant case these two definitions are equivalent, this is no longer true in the class of nonlinear time-invariant systems, as shown via examples in Zhang et al. (2010). The latter points to the inconsistency of linear and nonlinear theories. We will show that these two definitions remain equivalent under certain technical assumptions.

In general, the direct application of the realization algorithm does not necessarily provide a realization with minimal state dimension. To find the minimal realization of time-varying nonlinear system, two alternative approaches are considered in the paper. The *first* approach is based on the fact that if one starts from the irreducible i/o equation, then the realization will be accessible.[1] Reduction theory of nonlinear systems is based on the notion of irreducible variable $\varphi$, i.e. a variable, satisfying certain autonomous differential equation $F(\varphi, \varphi^{(1)}, \dots, \varphi^{(\mu)}) = 0$, see for instance Conte et al. (2007) and Zhang et al. (2010); Zheng et al. (2001). The irreducible equation is formed from the assumption $F(0, \dots, 0) = 0$, taking $\varphi = 0$.[2] The *second* method starts from a non-minimal realization, followed by the decomposition of the latter into non-accessible and accessible subsystems. Then the equations of the accessible subsystem may be taken as the minimal observable realization of the given i/o equation after substituting into it the solution of the non-accessible part.

☆ The work of Ü. Kotta and M. Tõnso was supported by the Estonian Research Council grant PUT481. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Alessandro Astolfi under the direction of Editor Daniel Liberzon.

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1 The realization algorithms always yield an observable realization.

2 This means that zero is a solution of the autonomous differential equation. The assumption is reasonable for majority of nonlinear systems and for all linear systems.

https://doi.org/10.1016/j.automatica.2018.05.007

The paper is organized as follows. Section 2 recalls the basics of the algebraic approach. In Section 3 the notion of equivalent one-forms is introduced. Section 4 deals with the reduction problem. In Section 5 the general realization of arbitrary dimension is considered and in Section 6 the minimal realization is discussed. Finally, Section 7 draws conclusions.

## 2. Preliminaries

In this paper two types of SISO nonlinear time-varying equations are considered. First, the i/o equation in the form

$$y^{(n)}(t) = \phi(t,\ y(t), \dots, y^{(n-1)}(t), u(t), \dots, u^{(r)}(t)), \tag{1}$$

and second, the state equations

$$\dot{x}(t) = f(t, x(t), u(t)),\ \ y(t) = h(t, x(t)), \tag{2}$$

where $u(t) \in \mathbb{R}$ is input, $y(t) \in \mathbb{R}$ is output and $x(t) \in \mathbb{R}^n$ is state variable. For the sake of compactness the argument $t$ will be omitted from now on. The special case, where input $u$ is missing in systems (1) or (2), is not treated in this paper. Sometimes also the i/o equation in implicit form are considered

$$\psi(t, y, \dots, y^{(n)}, u, \dots, u^{(r)}) = 0, \tag{3}$$

where $\psi(\cdot) = y^{(n)} - \phi(\cdot)$. Additionally, we assume that $\psi(\cdot)$ and $f(t, x, u)$ (in the expanded form) do not include the terms, depending only on $t$. This requirement is consistent with linear theory which considers $\dot{x} = A(t)x + B(t)u$, and not $\dot{x} = A(t)x + B(t)u + C(t)$. Below we briefly recall the approach of differential 1-forms from Conte et al. (2007), extending it to the time-varying case, i.e. for the case when the system equations depend explicitly on time $t$. Formally, this means that the ground field $k = \mathbb{R}(t)$ is a field of meromorphic functions of $t$ and not just $\mathbb{R}$ as in the case of time-invariant systems. The approach of 1-forms is based on the idea of working with differentials of nonlinear system equations rather than with equations themselves. This allows to linearize the intermediate computations.

Let $\mathcal{A}_\infty$ be the ring of analytic functions in a finite number of variables from the set $\{t,\ y^{(\ell)}, \ell \geqslant 0,\ u^{(k)}, k \geqslant 0\}$. The ring $\mathcal{A}_\infty$ is an integral domain. Let $\mathcal{K}_\infty$ be the field of fractions of the ring $\mathcal{A}_\infty$. The elements of $\mathcal{K}_\infty$ are the meromorphic functions. Let $\mathrm{d}/\mathrm{d}t : \mathcal{K}_\infty \to \mathcal{K}_\infty$ be the time-derivation operator. For the sake of compactness we write $\mathrm{d}/\mathrm{d}t(a) := \dot{a}$, $(\mathrm{d}/\mathrm{d}t)^2(a) := \ddot{a}$ and $(\mathrm{d}/\mathrm{d}t)^n(a) := a^{(n)}$ for $n > 2$, $a \in \mathcal{K}_\infty$. Then the pair $(\mathcal{K}_\infty, \mathrm{d}/\mathrm{d}t)$ is differential field (Kolchin, 1973). Over the field $\mathcal{K}_\infty$ a differential vector space $\mathcal{E}_\infty := \mathrm{sp}_{\mathcal{K}_\infty}\{\mathrm{d}\zeta \mid \zeta \in \mathcal{K}_\infty\}$ is defined, where sp denotes linear span. Consider a 1-form $\omega \in \mathcal{E}_\infty$ such that $\omega = \sum_i \alpha_i \mathrm{d}\zeta_i$, $\alpha_i, \zeta_i \in \mathcal{K}_\infty$. Its derivative $\dot{\omega}$ is defined by $\dot{\omega} = \sum_i(\dot{\alpha}_i \mathrm{d}\zeta_i + \alpha_i \mathrm{d}\dot{\zeta}_i)$. The same notations are used for derivative operators in $\mathcal{K}_\infty$ and $\mathcal{E}_\infty$. The space $\mathcal{E}_\infty$ is closed under derivative operator. One says that $\omega \in \mathcal{E}_\infty$ is an *exact* 1-form if $\omega = \mathrm{d}\alpha$ for some $\alpha \in \mathcal{K}_\infty$. A 1-form $\nu$ for which $\mathrm{d}\nu = 0$ is said to be *closed* (locally exact). A subspace $\mathcal{V}$ is said to be closed or *completely integrable*, if it admits locally an exact basis $\mathcal{V} = \mathrm{sp}_{\mathcal{K}_\infty}\{\mathrm{d}\zeta_1, \dots, \mathrm{d}\zeta_r\}$ (Choquet-Bruhat, DeWitt-Morette, & Dillard-Bleichi, 1982). Integrability of $\mathcal{V} = \mathrm{sp}_{\mathcal{K}_\infty}\{\omega_1, \dots, \omega_r\}$ can be checked by *Frobenius theorem*: $\mathcal{V}$ is completely integrable if and only if $\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_r = 0$ for $i = 1, \dots, r$. Here d is exterior differential operator and $\wedge$ denotes the wedge product, see Choquet-Bruhat et al. (1982).

Next, the algebraic approach of 1-forms is supplemented by the theory of non-commutative polynomial ring. Polynomials allow to represent the 1-forms as well as the operations with them in a compact form; such tools have been used to address many problems for nonlinear time-invariant systems (Belikov, Kotta, & Tõnso, 2015; Halás, 2008; Zheng et al., 2001). The field $\mathcal{K}_\infty$ and the operator $\mathrm{d}/\mathrm{d}t$ induce a non-commutative ring of left differential polynomials $\mathcal{K}_\infty[s]$. A polynomial $p \in \mathcal{K}_\infty[s]$ can be uniquely written as $p = p_\kappa s^\kappa + p_{\kappa-1}s^{\kappa-1} + \cdots + p_1 s + p_0$, where $s$ is a formal variable and $p_i \in \mathcal{K}_\infty$ for $i = 0, \dots \kappa$. Polynomial $p \neq 0$ if and only if at least one of the functions $p_i$ is non-zero. If $p_\kappa \not\equiv 0$, then the integer $\kappa$ is called the *degree* of $p$ and denoted by $\deg(p)$. We set additionally $\deg(0) = -\infty$. The addition of the polynomials is defined in the standard way. However, for $a \in \mathcal{K}_\infty \subset \mathcal{K}_\infty[s]$ the multiplication is defined by the commutation rule $s \cdot a := a s + \dot{a}$. In $\mathcal{K}_\infty[s]$ the following *left Ore condition* holds: For all non-zero $a, b \in \mathcal{K}_\infty[s]$ there exist non-zero $\widetilde{a}, \widetilde{b} \in \mathcal{K}_\infty[s]$ such that $\widetilde{a}a = \widetilde{b}b$. From the ring $\mathcal{K}_\infty[s]$ one can construct a non-commutative *field of fractions*. Define a set $\mathcal{V} := \mathcal{K}_\infty[s] \setminus \{0\}$. Consider the set of left fractions denoted by $\mathcal{K}_\infty\,(s) = \mathcal{V}^{-1}\mathcal{K}_\infty[s]$. Elements of $\mathcal{K}_\infty\,(s)$ are left fractions in the form $b^{-1}\,a$, where $a \in \mathcal{K}_\infty[s]$, $b \in \mathcal{V}$. Since the ring $\mathcal{K}_\infty[s]$ is an integral domain, thus $\mathcal{K}_\infty\,(s)$ is a field.

A left differential polynomial $a \in \mathcal{K}_\infty[s]$ may be interpreted as an operator $a(s) : \mathcal{E}_\infty \to \mathcal{E}_\infty$. Define for $\mathrm{d}y, \mathrm{d}u, \mathrm{d}t \in \mathcal{E}_\infty$

$$\begin{aligned} &s\mathrm{d}y := (\mathrm{d}/\mathrm{d}t)\mathrm{d}y = \mathrm{d}\dot{y},\quad s\mathrm{d}u := (\mathrm{d}/\mathrm{d}t)\mathrm{d}u = \mathrm{d}\dot{u},\\ &s\mathrm{d}t := (\mathrm{d}/\mathrm{d}t)\mathrm{d}t = 0. \end{aligned} \tag{4}$$

It is natural to extend (4) for $a = \sum_{i=0}^k a_i s^i$ as $a(s)(\alpha \mathrm{d}\zeta) := \sum_{i=0}^k a_i(s^i \cdot \alpha)\mathrm{d}\zeta$ with $a_i, \alpha \in \mathcal{K}_\infty$ and $\mathrm{d}\zeta \in \{\mathrm{d}y, \mathrm{d}u, \mathrm{d}t\}$. Using (4) every 1-form $\omega = \sum_{\alpha=0}^k a_\alpha \mathrm{d}y^{(\alpha)} + \sum_{\beta=0}^\ell b_\beta \mathrm{d}u^{(\beta)} + c_0\mathrm{d}t \in \mathcal{E}_\infty$, where $a_\alpha, b_\beta, c_0 \in \mathcal{K}_\infty$, may be expressed in terms of the left differential polynomials as $\omega = \sum_{\alpha=0}^k a_\alpha s^\alpha \mathrm{d}y + \sum_{\beta=0}^\ell b_\beta s^\beta \mathrm{d}u + c_0\mathrm{d}t := a(s)\mathrm{d}y + b(s)\mathrm{d}u + c\mathrm{d}t$, where $a, b \in \mathcal{K}_\infty[s]$ and $c = c_0 \in \mathcal{K}_\infty$. It is easy to see that $s\,\omega = \dot{\omega}$, for $\omega \in \mathcal{E}_\infty$.

## 3. Equivalence of 1-forms

In Section 4 we associate with each nonlinear system its 1-form and on the set of 1-forms $\mathcal{E}_\infty$ the equivalence relation is defined. In linear case the equivalence relation reduces to the equality of transfer functions.

**Definition 1.** The 1-forms $\omega_1, \omega_2 \in \mathcal{E}_\infty$ are called *equivalent*, denoted $\omega_1 \cong \omega_2$, if there exist non-zero polynomials $\lambda, \mu \in \mathcal{K}_\infty[s]$ such that

$$\lambda(s)\omega_1 = \mu(s)\omega_2. \tag{5}$$

**Proposition 1.** *Relation (5) defines an equivalence relation on $\mathcal{E}_\infty$.*

**Proof.** Symmetry and reflexivity are obvious. To show transitivity we assume that $\omega \cong \widetilde{\omega}$ and $\widetilde{\omega} \cong \hat{\omega}$. Due to (5) we may write

$$\alpha(s)\omega = \beta(s)\widetilde{\omega},\ \ \gamma(s)\widetilde{\omega} = \delta(s)\hat{\omega}, \tag{6}$$

By the left Ore condition one can find, for arbitrary non-zero polynomials $\beta, \gamma \in \mathcal{K}_\infty[s]$, two non-zero $\widetilde{\beta}, \widetilde{\gamma} \in \mathcal{K}_\infty[s]$ such that $\widetilde{\beta}\beta = \widetilde{\gamma}\gamma$. Multiplying the relations (6), respectively, by $\widetilde{\beta}$ and $\widetilde{\gamma}$ from left, we obtain $\widetilde{\beta}(s)\alpha(s)\omega = \widetilde{\beta}(s)\beta(s)\widetilde{\omega}$, $\widetilde{\gamma}(s)\gamma(s)\widetilde{\omega} = \widetilde{\gamma}(s)\delta(s)\hat{\omega}$. Adding these equalities and regarding that $\widetilde{\beta}\beta = \widetilde{\gamma}\gamma$ yields $\widetilde{\beta}(s)\alpha(s)\omega = \widetilde{\gamma}(s)\delta(s)\hat{\omega}$, thus $\omega \cong \hat{\omega}$. □

The equivalence relation divides 1-forms in $\mathcal{E}_\infty$ into equivalence classes.

**Definition 2.** A differential form $\omega \in \mathcal{E}$ is called *irreducible*, if $\omega = \gamma(s)\pi$, where $\gamma \in \mathcal{K}_\infty[s]$, $\pi \in \mathcal{E}_\infty$, and $\pi \neq 0$ yields $\deg(\gamma) = 0$.

**Definition 3.** Given $\omega \in \mathcal{E}_\infty$, the form $\pi_\omega$ is an irreducible form of $\omega$ whenever $\pi_\omega \cong \omega$ and $\pi_\omega$ is irreducible.

Algorithm 1 finds $\pi_\omega = \tilde{a}(s)\mathrm{d}y + \tilde{b}(s)\mathrm{d}u + \tilde{c}\mathrm{d}t$ for the given 1-form $\omega = a(s)\mathrm{d}y + b(s)\mathrm{d}u + c\mathrm{d}t$.

**Algorithm 1.**

(1) Compute polynomial $\gamma$ as the greatest left common divisor of $a$ an $b$.
(2) Find $\tilde{a}, \tilde{b}$ such that $a = \gamma\,\tilde{a},\ b = \gamma\,\tilde{b}$.
(3) Determine $\tilde{c}$ by solving the linear differential equation $\gamma(s)\tilde{c} = c$.

**Proposition 2.** *Any $\omega \in \mathcal{E}_\infty$ satisfies $\omega \cong \pi_\omega$.*

The proof follows directly from Definition 2.

**Lemma 1.** *Let $\omega_1, \omega_2 \in \mathcal{E}_\infty$ and let $\pi_{\omega_1}$ and $\pi_{\omega_2}$ be, respectively, their irreducible forms, i.e., $\omega_i = \gamma_i(s)\pi_{\omega_i}$ for $i = 1, 2$. Also, let $\beta \in \mathcal{K}_\infty$ and $c_i(t) \in k = \mathbb{R}(t)$, $i = 1, 2$, be such that $\gamma_i(s)c_i(t) \equiv 0$. Then, $\omega_1 \cong \omega_2$ if and only if*

$$\pi_{\omega_1} = \beta\pi_{\omega_2} + [c_1(t) + c_2(t)]\mathrm{d}t. \tag{7}$$

**Proof.** *Sufficiency*. By Ore condition, one can find polynomials $\delta_1, \delta_2$ such that $\delta_1\gamma_1 = \delta_2\gamma_2$. Now $\gamma_1(s)\pi_{\omega_1} = \gamma_1(s)\beta\pi_{\omega_2} + \gamma_1(s)[c_1(t) + c_2(t)]\mathrm{d}t = \gamma_1(s)\beta\pi_{\omega_2} + \gamma_1(s)c_2(t)\mathrm{d}t$ and $\delta_1(s)\gamma_1(s)\pi_{\omega_1} = \delta_1(s)\gamma_1(s)\beta\pi_{\omega_2} + \delta_1(s)\gamma_1(s)c_2(t)\mathrm{d}t = \delta_1(s)\gamma_1(s)\beta\pi_{\omega_2} + \delta_2(s)\gamma_2(s)c_2(t)\mathrm{d}t = \delta_1(s)\gamma_1(s)\beta\pi_{\omega_2}$.Thus, $\pi_{\omega_1} \cong \pi_{\omega_2}$ and since $\pi_{\omega_i} \cong \omega_i$, $i = 1, 2$, by the transitivity of relation $\cong$, one has $\omega_1 \cong \omega_2$.

*Necessity*. Since $\gamma_i(s)c_i(t) \equiv 0$ for $i = 1, 2$, one can write

$$\begin{aligned}\omega_1 &= \gamma_1(s)\pi_{\omega_1} - \gamma_1(s)c_1(t)\mathrm{d}t\\ \omega_2 &= \gamma_2(s)\pi_{\omega_2} + \gamma_2(s)\frac{c_2(t)}{\beta}\mathrm{d}t,\end{aligned} \tag{8}$$

where $\beta \in \mathcal{K}_\infty$, $\beta \neq 0$ is specified later. Because $\omega_1 \cong \omega_2$, there exist polynomials $\alpha_1, \alpha_2 \in \mathcal{K}_\infty[s]$ such that

$$\alpha_1(s)\omega_1 = \alpha_2(s)\omega_2. \tag{9}$$

Denote by $\lambda := \alpha_1\gamma_1$ and $\mu := \alpha_2\gamma_2$. Then (8) and (9) yield $\lambda(s)\pi_{\omega_1} - \lambda(s)c_1(t)\mathrm{d}t = \mu(s)\pi_{\omega_2} + \mu(s)(c_2(t)/\beta)\mathrm{d}t$. Regarding that $1 = \lambda\lambda^{-1} \in \mathcal{K}_\infty(s)$ allows to rewrite previous equality as $\lambda(s)\left[\pi_{\omega_1} - c_1(t)\mathrm{d}t - \lambda^{-1}(s)\mu(s)\pi_{\omega_2} - \lambda^{-1}(s)\mu(s)(c_2(t)/\beta)\mathrm{d}t\right] = 0$. Since $\lambda \neq 0$, the expression in square brackets has to be zero and we obtain

$$\pi_{\omega_1} = \lambda^{-1}(s)\mu(s)\pi_{\omega_2} + [c_1(t) + \lambda^{-1}(s)\mu(s)\tfrac{c_2(t)}{\beta}]\mathrm{d}t. \tag{10}$$

Next we show that $\lambda^{-1}\mu \in \mathcal{K}_\infty$. Note that $c_i(t)$ can be anything satisfying $\gamma_i(s)c_i(t) \equiv 0$, thus we can take $c_i(t) = 0$ for $i = 1, 2$. Since $\pi_{\omega_1}, \pi_{\omega_2} \in \mathcal{E}_\infty$, then $\lambda^{-1}\mu$ has to belong to $\mathcal{K}_\infty[s]$. Let $\lambda^{-1}\mu = \beta$, then, because $\pi_{\omega_1}$ is irreducible $\deg(\beta) = 0$, i.e., $\beta \in \mathcal{K}_\infty$. Finally, (10) becomes (7). □

## 4. Reduction of i/o equations

In this section the notions from Section 3 will be associated with system (1). Let $\mathcal{Q}$ be the quotient ring of $\mathcal{A}_\infty$, where the variables $y^{(n+k)}$, $k \geqslant 0$ are considered to be dependent variables and as such are substituted by independent ones, using (perhaps repeatedly) the relation $y^{(n)} = \phi(\cdot)$. This is different from the definition of $\mathcal{A}_\infty$, where $y^{(n+k)}$, $k \geqslant 0$ are considered to be independent variables. To conclude, the elements of $\mathcal{Q}$ are functions of variables from the set $\{t, y^{(\ell)}, 0 \leqslant \ell \leqslant n-1, u^{(k)}, k \geqslant 0\}$. The derivative operator $\mathrm{d}/\mathrm{d}t : \mathcal{Q} \to \mathcal{Q}$ is defined by $\mathrm{d}/\mathrm{d}t(y^{(\ell)}) := y^{(\ell+1)}$ for $\ell = 0, \ldots, n-2$, $\mathrm{d}/\mathrm{d}t(y^{(n-1)}) := \phi(\cdot)$ and $\mathrm{d}/\mathrm{d}t(u^{(k)}) := u^{(k+1)}$ for $k \geqslant 0$. Denote by $\mathcal{K}$ the field of fractions of the ring $\mathcal{Q}$. The derivative operator can be extended from $\mathcal{Q}$ to $\mathcal{K}$. The pair $(\mathcal{K}, \mathrm{d}/\mathrm{d}t)$ is a differential field. In analogy to the construction in Section 3, we define the differential vector space $\mathcal{E} := \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}\zeta \mid \zeta \in \mathcal{K}\}$ and the non-commutative polynomial ring $\mathcal{K}[s]$.

By applying the operator d to Eq. (1) we obtain

$$\bar{\omega} := \mathrm{d}y^{(n)} - \sum_{i=0}^{n-1}\frac{\partial\phi}{\partial y^{(i)}}\mathrm{d}y^{(i)} - \sum_{j=0}^{r}\frac{\partial\phi}{\partial u^{(j)}}\mathrm{d}u^{(j)} - \frac{\partial\phi}{\partial t}\mathrm{d}t = 0, \tag{11}$$

called the *globally linearized* i/o equation. Moreover, $\bar{\omega}$ is called the *differential form of system* (1). The 1-form $\bar{\omega}$ can be represented in terms of three non-commutative polynomials from the ring $\mathcal{K}[s]$ by rewriting (11) as

$$\bar{\omega} \equiv p(s)\mathrm{d}y + q(s)\mathrm{d}u + \varrho\mathrm{d}t = 0, \tag{12}$$

where

$$p = s^n - \sum_{i=0}^{n-1}\frac{\partial\phi}{\partial y^{(i)}}s^i,\ q = -\sum_{j=0}^{r}\frac{\partial\phi}{\partial y^{(j)}}s^j,\ \varrho = -\frac{\partial\phi}{\partial t}. \tag{13}$$

Assume that $\bar{\omega}$ satisfies $\bar{\omega} = \gamma(s)\pi_{\bar{\omega}}, \gamma \in \mathcal{K}[s], \deg(\gamma) \geqslant 1$, where $\pi_{\bar{\omega}}$ is irreducible in $\mathcal{E}$. The 1-forms $\pi_{\bar{\omega}}$, found by Algorithm 1, involve integrating constant(s) $C_1, C_2 \ldots$, originating from the solution of $\gamma(s)\tilde{c} = c$. To obtain a unique irreducible equation that is consistent with linear theory, in what follows we take $C_1 = C_2 = \cdots = 0$ in $\pi_{\bar{\omega}}$, making it unique up to multiplication by function from $\mathcal{K}$. Taking $C_1 = C_2 = \cdots = 0$ in $\pi_{\bar{\omega}}$ corresponds to finding a particular solution of linear differential equation in step (3) of the Algorithm 1. It is possible to show, like in the time-invariant case in Bartosiewicz, Kotta, Tõnso, and Wyrwas (2016), that $\pi_{\bar{\omega}}$ is exact or can be made exact by multiplying it with an integrating factor $\alpha \in \mathcal{K}$. By abuse of notation one may choose new $\gamma \in \mathcal{K}[s]$ such that $\pi_{\bar{\omega}} = \mathrm{d}\varphi, \varphi \in \mathcal{K}$ is exact and

$$\bar{\omega} = \gamma(s)\mathrm{d}\varphi,\ \deg(\gamma) \geqslant 1. \tag{14}$$

The function $\varphi$, obtained in the reduction process, is not necessarily in a form that allows to construct a new irreducible i/o equation $\varphi = 0$, as demonstrated by the examples below. To exclude such systems we introduce Assumptions 1 and 2.

If (14) holds, then there exist a function $F$ and non-zero $k = k(t, u, \dot{u}, \ldots) \in \mathcal{K}$ such that

$$\psi = kF(\varphi, \dot{\varphi}, \ldots, \varphi^{(\mu)}), \quad \mu \geqslant 1, \tag{15}$$

see van der Schaft (1987), Lemma 6.2. When $k \neq 0$, then from $\psi(\cdot) = 0$ it follows $F(\cdot) = 0$.

**Assumption 1.** $F(0, \ldots, 0) = 0$.

This is the technical assumption made in most papers and means that zero is a solution of the autonomous differential equation $F(\varphi, \dot{\varphi}, \ldots, \varphi^{(\mu)}) = 0$. In general, the non-zero solution of autonomous differential equation is difficult to find; moreover, analytic solutions may not exist.

**Assumption 2.** The equation $\varphi = 0$ in the form (3) can be uniquely transformed into the explicit form (1), where the result depends on at least one of the input derivatives $u^{(\ell)}$, $\ell \geqslant 0$.

**Example 1.** Consider the i/o equation $\psi := \ddot{y} + y\dot{u} + u\dot{y} + 1/(\dot{y} + uy) = 0$. If $\varphi = \dot{y} + uy$, then system can be rewritten in the form $\psi = F(\varphi, \dot{\varphi}) = \dot{\varphi} + 1/\varphi = 0$, However, the assumption $F(0, 0) = 0$ is not satisfied, since 0 does not satisfy the equation $\dot{\varphi} + 1/\varphi = 0$.

**Example 2.** Consider Example 3.3 from Zhang et al. (2010)

$$\psi := \ddot{y}(2\dot{y} - 3u) - \dot{u}(3\dot{y} - 4u) = 0. \tag{16}$$

The function $\psi$ can be rewritten in the form $\psi = kF(\dot{\varphi}, \varphi) = \dot{\varphi}$, where the irreducible variable $\varphi = (\dot{y} - u)(\dot{y} - 2u)$ cannot be uniquely transformed into explicit form (1).

**Example 3.** Consider Example 5.4 from Zhang et al. (2010)

$$\psi := u\ddot{y} - \dot{u}\dot{y} = 0. \tag{17}$$

The function $\psi$ can be rewritten in the form $\psi = kF(\dot{\varphi}, \varphi) = u^2\dot{\varphi}$, where $\varphi = \dot{y}/u$. However, when we transform the new equation $\varphi = \dot{y}/u = 0$ into the explicit form (1), we obtain a degenerate system $\dot{y} = 0$, not involving input $u$.

**Definition 4.** Under Assumptions 1 and 2 the system (1) is said to be *irreducible* if its 1-form $\bar{\omega} \in \mathcal{E}$, defined by (11), is irreducible in $\mathcal{E}$. Otherwise system (1) is called *reducible*. We call $\pi \in \mathcal{E}$ *reduced 1-form of* (1), if $\bar{\omega} = \beta(s)\pi$, where $\beta \in \mathcal{K}$ and $1 \leqslant \deg(\beta) \leqslant \deg(\gamma)$.

**Definition 5.** A non-zero function $\varphi \in \mathcal{K}$, satisfying Assumptions 1 and 2, is said to be a *irreducible variable* for system (1), if there exist a polynomial $\gamma \in \mathcal{K}[s]$ such that (14) holds.

**Definition 6.** We call $\varphi = 0$ *irreducible i/o equation*, being equivalent to the original equation (1), if $\varphi$ is irreducible variable of (1) and $\varphi$ satisfies Assumptions 1 and 2.

## 5. Realization

Application of state elimination to observable system (2) results in the i/o equation $\psi_x(\cdot) = 0$. The irreducible i/o equation of $\psi_x(\cdot) = 0$ is denoted by $\varphi_x(\cdot) = 0$. For system (2) we also define its differential form $\omega_x := \mathrm{d}\psi_x$ and irreducible form $\pi_{\omega_x} := \mathrm{d}\varphi_x$.

**Definition 7.** The observable system (2) is said to be *accessible* if $\psi_x(\cdot) = 0$ is irreducible.

Note that Definition 7 is actually equivalent to the lack of autonomous element $\varphi(x)$ for system (2). If such $\varphi$ exists, then by observability of (2), $\varphi$ can be rewritten as $\varphi(x(y, \dot{y}, \ldots, u, \dot{u}, \ldots))$ where the latter is the solution of the autonomous differential equation (15) where on the left hand side is $\psi_x$. This contradicts the fact that $\psi_x$ is irreducible.

Define the spaces $\mathcal{Y} := \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}t, \mathrm{d}y^{(\ell)},\ \ell \geqslant 0\}, \mathcal{U} := \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}u^{(\ell)},\ \ell \geqslant 0\}, \mathcal{X} := \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x\}$. The subspace $\mathcal{O}_\infty := \mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$ is called the *observable space* of system (2).

**Definition 8.** The system (2) is said to be *observable* if $\mathcal{O}_\infty = \mathcal{X}$.

The *relative degree* $\rho$ of a 1-form $\omega \in \mathcal{E}$ is defined to be the least integer such that $\omega^{(\rho)} \notin \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}t, \mathrm{d}y, \ldots, \mathrm{d}y^{(n-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(r)}\}$. If such an integer does not exist, we set $\rho := \infty$. Next, we define for Eq. (1) the nonincreasing sequence of subspaces $\mathcal{H}_k$ of $\mathcal{E}$ by recursive formula

$$\begin{aligned} &\mathcal{H}_1 = \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}t,\ \mathrm{d}y, \ldots, \mathrm{d}y^{(n-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(r)}\}, \\ &\mathcal{H}_{k+1} = \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\},\ k \geqslant 1. \end{aligned} \tag{18}$$

Denote its limit by $\mathcal{H}_\infty$ so that $\mathcal{H}_1 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty$, where $k^* \in \mathbb{N}$. Each $\mathcal{H}_k$ contains the 1-forms with relative degrees equal to $k$ or higher than $k$. The subspace $\mathcal{H}_\infty$ contains the 1-forms with infinite relative degree and can be shown to be integrable. The proof is analogous to that given in Aranda-Bricaire, Moog, and Pomet (1995), Proposition 3.3 for time-invariant systems.

**Lemma 2.** *The 1-form $\omega \in \mathcal{H}_\infty$ if and only if $\omega$ is a linear combination of reduced 1-form(s) of* (2) *and* $\mathrm{d}t$.

The proof is an extension of Proposition 3.12 in Conte et al. (2007).

From (18) it follows that the subspaces $\mathcal{H}_k$, $k = 1, \ldots, r+1$ have the following structure:

$$\mathcal{H}_k = \mathcal{H}_{r+2} \oplus \mathrm{sp}\{\mathrm{d}u, \mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(r-k+1)}\}. \tag{19}$$

**Definition 9.** Eq. (2) is called a state space *realization* of Eq. (1), if their irreducible forms are equal up to multiplication by a function in $\mathcal{K}$, or alternatively, $\omega_x \cong \bar{\omega}$.

**Proposition 3.** *The i/o equation* (1) *has the state space realization in the form* (2) *if and only if the subspace $\mathcal{H}_{r+2}$, computed for* Eq. (1), *is completely integrable.*

**Proof.** *Sufficiency.* Assume the subspace $\mathcal{H}_{r+2}$ is integrable, i.e. $\mathcal{H}_{r+2} = \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}t, \mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$. Due to structure (19) the subspace $\mathcal{H}_{r+1} = \mathcal{H}_{r+2} \oplus \{\mathrm{d}u\}$, thus the derivative of any 1-form $\mathrm{d}x_i \in \mathcal{H}_{s+2}$ can be represented in the form $\mathrm{d}\dot{x}_i = \sum_{j=1}^{n} \alpha_j \mathrm{d}x_j + \beta \mathrm{d}u + \gamma \mathrm{d}t$. This yields locally $\dot{x} = f(t, x, u)$.

*Necessity.* We assume that state equation exist for (1) and show that $\mathcal{H}_{r+2}$ is integrable. In general, the realization can be unobservable. In this case we decompose the state equations into observable and unobservable subsystems and from now on in this proof we refer by the word 'realization' to its observable subsystem since unobservable part is not reflected in the i/o equation. Let $\nu$ be the dimension of (the observable part of) the realization and let $x = [x_1, \ldots, x_\nu]^{\mathrm{T}}$.

**Case 1.** If $\nu = n$, then $\mathcal{H}_{r+2} = \mathrm{sp}\{\mathrm{d}x_1, \ldots, \mathrm{d}x_\nu, \mathrm{d}t\}$.

**Case 2.** If $\nu < n$, then $\mathrm{sp}_{\mathcal{K}}\{\mathrm{d}t, \mathrm{d}x\} \subset \mathcal{H}_{r+2}$. In this case $\mathcal{H}_{r+2} = \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x\} + \mathcal{H}_\infty$. Note that $\mathcal{H}_\infty$ and $\mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}t\}$ are always integrable. Since the sum of two integrable spaces is always integrable, the space $\mathcal{H}_{r+2}$ has to be integrable as well.

**Case 3.** $\nu > n$. In general, the i/o equation $\psi = 0$ is reducible, therefore in the proof we consider three i/o equations together with the corresponding 1-forms: (a) Eq. (3) of order $n$, i.e. $\psi = 0$ and its differential form $\bar{\omega}$; (b) the irreducible i/o equation of (3) $\varphi = 0$ of order $\mu$ and its differential form $\pi_{\bar{\omega}} = \mathrm{d}\varphi$; (c) the i/o equation $\psi_x = 0$ of order $\nu$, obtained from (2) via state elimination, and its differential form $\omega_x = \mathrm{d}\psi_x$.

By Definition 9, the irreducible forms of the realization (2) and that of (3) are equal, i.e. $\pi_{\omega_x} = \pi_{\bar{\omega}}$. Next, we compute for each of three i/o equations the particular subspace $\mathcal{H}_k$, which determines the state coordinates for their realization, assuming that the dimension of the realization equals with the order of the i/o equation. *(i)* For $\varphi = 0$ consider a (possibly non-integrable) subspace $\mathrm{sp}_{\mathcal{K}}\{\mathrm{d}t, \theta_1, \ldots, \theta_\mu\}$. *(ii)* Using Lemma 2 and the notion of relative degree, one can show that for $\psi = 0$ the subspace $\mathcal{H}_{r+2} = \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}t, \theta_1, \ldots, \theta_\mu, \pi_{\bar{\omega}}, \ldots, \pi_{\bar{\omega}}^{(n-\mu-1)}\}$, where $r$ is the highest order of input derivative in $\psi$ and $\bar{\omega} = \gamma(s)\pi_{\bar{\omega}}$. The 1-forms $\pi_{\bar{\omega}}, \ldots, \pi_{\bar{\omega}}^{(n-\mu-1)}$ are exact, since they correspond to $\mathrm{d}\varphi, \ldots, \mathrm{d}\varphi^{(n-\mu-1)}$. *(iii)* For $\psi_x = 0$ we obtain (using $\pi_{\bar{\omega}} = \pi_{\omega_x}$) $\mathrm{sp}\{\mathrm{d}t, \mathrm{d}x\} = \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}t, \theta_1, \ldots, \theta_\mu, \pi_{\bar{\omega}}, \ldots, \pi_{\bar{\omega}}^{(n-\mu-1)}, \pi_{\bar{\omega}}^{(n-\mu)}, \ldots, \pi_{\bar{\omega}}^{(\nu-\mu-1)}\}$, which is integrable, since realization exists for $\psi_x = 0$.

Assume now, contrary to our claim that $\mathcal{H}_{r+2}$ is not integrable. Then by Frobenius theorem there exists $\theta_\kappa$, $1 \leqslant \kappa \leqslant \mu$ such that $\mathrm{d}\theta_\kappa \wedge \mathrm{d}t \wedge \Theta \wedge \pi_{\bar{\omega}} \wedge \cdots \wedge \pi_{\bar{\omega}}^{(n-\mu-1)} \neq 0$, where $\Theta := \theta_1 \wedge \cdots \wedge \theta_\mu$. Next, we check integrability of the subspace

$$\mathrm{sp}_{\mathcal{K}}\{\mathrm{d}t, \theta_1, \ldots, \theta_\mu, \pi_{\bar{\omega}}, \ldots, \pi_{\bar{\omega}}^{(\nu-\mu-1)}\}. \tag{20}$$

By Frobenius theorem we compute the wedge products

$$\mathrm{d}\theta_\kappa \wedge \mathrm{d}t \wedge \Theta \wedge \pi_{\bar{\omega}} \wedge \cdots \wedge \pi_{\bar{\omega}}^{(\nu-\mu-1)} \tag{21}$$

for $\kappa = 1, \ldots, \mu$. Consider the differentials of highest output derivative in each on 1-form in (21). For $\pi_{\bar{\omega}}$ it is $\mathrm{d}y^{(\mu)}$, since the order of $\varphi$ is $\mu$. Differentiating $\pi_{\bar{\omega}}$ repeatedly yields that for $\pi_{\bar{\omega}}^{(1)}$ it is $\mathrm{d}y^{(\mu+1)}$, and so on, until for $\pi_{\bar{\omega}}^{(\nu-\mu-1)}$ it is $\mathrm{d}y^{(\nu-1)}$. Thus $\mathrm{d}y^{(\nu-1)}$ appears only in the last 1-form $\pi_{\bar{\omega}}^{(\nu-\mu-1)}$ and is not represented in the other 1-forms of (21). Due to the properties of the wedge product the expression (21) is non-zero unless $d\theta_\kappa \wedge dt \wedge \Theta \wedge \pi_{\bar\omega} \wedge \cdots \wedge \pi_{\bar\omega}^{(n-\mu-1)} = 0$. Thus (20) is not integrable. This is in contradiction with the fact that realization exists for $\psi_x = 0$. Hence, our assumption was wrong and $\mathcal{H}_{r+2}$ is integrable. □

## 6. Minimal realization

In this section, it is shown that under Assumptions 1 and 2, a realization as defined in Definition 9 has a minimal dimension if and only if it is both accessible and observable. This is exactly expected from the well established linear systems theory. However, in the nonlinear context, such result was not available to date. For the proof of this claim we define for system (2) the field $\hat{\mathcal{K}}$ and vector space $\hat{\mathcal{E}}$ in the similar manner as $\mathcal{K}$ and $\mathcal{E}$ are defined for system (1). Now $\hat{\mathcal{K}}$ is the field of functions in the finite number of the variables $\{x, u^{(k)}, k \geqslant 0\}$. For system (2) we also define the sequence of subspaces $\hat{\mathcal{H}} \in \hat{\mathcal{E}}$ by $\hat{\mathcal{H}}_1 = \mathrm{sp}_{\hat{\mathcal{K}}}\{dx\}$, $\hat{\mathcal{H}}_{k+1} = \{\omega \in \hat{\mathcal{H}}_k \,|\, \dot\omega \in \hat{\mathcal{H}}_k\}$, $k \geqslant 1$.

**Definition 10.** The realization is called minimal if its dimension is minimal among all realizations.

**Theorem 1.** *Let (2) be the realization of (1). Under Assumptions 1 and 2, the realization (2) has the smallest dimension if and only if it is accessible and observable.*

**Proof.** *Sufficiency.* One has to show that whenever state equations are accessible and observable, then there is no equivalent realization of smaller dimension. Equivalently, one may prove that if the dimension of the realization can be reduced, then either the given realization is not accessible or it is not observable. Given state equations (2) of dimension $n$, assume that there exists an equivalent realization of smaller dimension. The state elimination in (2) yields the i/o equation $\psi_x(y, \ldots, y^{(k)}, u, \dot u, \ldots) = 0$, where $k$ is the observability index. If $k < n$, then the original system is not observable. If $k = n$, then the system is observable. However, from the assumption that the dimension can be reduced, due to Definition 7, $\psi_x = F(\varphi, \dot\varphi, \ldots)$ for some $F$ and $\varphi$. That is, the original system admits an autonomous element $\varphi$ and therefore is not accessible.
*Necessity.* The proof is by contradiction. *First*, assume the minimal dimension realization (2) is not accessible. Because of non-accessibility of the realization it may be decomposed into non-accessible and accessible subsystems as described in Conte et al. (2007) for time-invariant systems. Since $\hat{\mathcal{H}}_\infty$ is integrable there exist locally $\nu$ functions, say $\xi_1, \ldots, \xi_\nu$, with infinite relative degree, such that $\hat{\mathcal{H}}_\infty = \mathrm{sp}_{\hat{\mathcal{K}}}\{dt, d\xi_1, \ldots, d\xi_\nu\}$. The functions $\xi_1, \ldots, \xi_\nu$ are called *non-accessible state coordinates* and the subspace $\chi_{na} := \mathrm{sp}_{\hat{\mathcal{K}}}\{d\xi_1, \ldots, d\xi_\nu\}$ is called *non-accessible subspace* of (2). Since the space $\chi_{na}$ is invariant with respect to the time-derivative operator up to 1-form $dt$, one may write $\dot\xi_i = \tilde f_i(t, \xi_1, \ldots, \xi_\nu)$, $i = 1, \ldots, \nu$. Choosing the *accessible state coordinates* $\xi_{\nu+1}, \ldots, \xi_n$ from the *accessible subspace* $\chi_a := \mathrm{sp}_{\hat{\mathcal{K}}}\{dx\} \setminus \chi_{na} = \mathrm{sp}_{\hat{\mathcal{K}}}\{d\xi_{\nu+1}, \ldots, d\xi_n\}$ yields the system in decomposed form:

$$\dot\xi_i = \tilde f_i(t, \xi_1, \ldots, \xi_\nu), \qquad i = 1, \ldots, \nu \tag{22a}$$
$$\dot\xi_j = \tilde f_j(t, \xi, u), \qquad j = \nu + 1, \ldots, n \tag{22b}$$
$$y = h(t, \xi). \tag{22c}$$

For chosen coordinates $\xi_1, \ldots, \xi_n$ the functions $\tilde f_{\nu+1}, \ldots, \tilde f_n$ are single-valued due to Assumption 2. The dynamics of the accessible subsystem (22b) depends also on the components of non-accessible subsystem, i.e. $\xi_1, \ldots, \xi_\nu$. Our goal is to find the representation of accessible subsystem, not depending on these components. Due to Assumption 1, zero is the solution of $F(\varphi, \ldots, \varphi^{(\nu)}) = 0$, thus, it is also solution of non-accessible subsystem (22a). Substituting $\xi_1 = 0, \ldots, \xi_\nu = 0$ into (22b) gives us the accessible subsystem of (2), not depending any more on non-accessible components. The accessible subsystem (22b) is a realization of (1) of dimension $n - \nu$. This yields a contradiction.

*Second*, assume that the minimal dimension realization (2) is not observable. We can again decompose it into unobservable and observable subsystems

$$\dot\zeta_k = \hat f_k(t, \zeta, u), \qquad k = 1, \ldots, \mu \tag{23a}$$
$$\dot\zeta_\ell = \hat f_\ell(t, \zeta_{\mu+1}, \ldots, \zeta_n, u), \qquad \ell = \mu + 1, \ldots, n \tag{23b}$$
$$y = h(t, \zeta_{\mu+1}, \ldots, \zeta_n). \tag{23c}$$

The observable subsystem (23b) is a realization of dimension $n-\mu$. This yields a contradiction. □

**Corollary 1.** *Under Assumptions 1 and 2 the dimension of minimal realization is the order of the irreducible i/o equation.*

We will demonstrate below that the systems, not satisfying Assumption 2, do not possess the realization, satisfying both definitions of minimality.

**Example 4** (*Continuation of Example 2*). . State variables for i/o equation (16) can be defined as $x_1 = 2u^2 - 3u\dot y + \dot y^2$, $x_2 = y$, yielding a non-minimal realization

$$\dot x_1 = 0,\ \dot x_2 = \tfrac{1}{2}(3u \pm \sqrt{u^2 + 4x_1}),\ y = x_2, \tag{24}$$

where $u_2 + 4x_1 \geqslant 0$. Since $\varphi = (\dot y - u)(\dot y - 2u)$ does not satisfy Assumption 2, by Definition 6, $\varphi = 0$ cannot be called the irreducible equation of (16) and $\psi = 0$ is not reducible. Therefore realization (24) is minimal.

**Example 5** (*Continuation of Example 3*). . The non-accessible realization of (17) is $\dot x_1 = 0, \dot x_2 = ux_1, y = x_2$, where $x_1 = \dot y/u, x_2 = y$. Since $\varphi = \dot y/u$ does not satisfy Assumption 2, the system (17) is not reducible, and so the realization is minimal.

Minimal realization can be found either using the irreducible i/o equation, defined in Section 4, or from the decomposition of non-minimal realization, as described in the proof of Theorem 1.

**Example 6.** Consider the system

$$\psi := y^{(3)} - ty - uy + \dot y - 2\dot u\dot y - y\ddot u - t\ddot y - u\ddot y. \tag{25}$$

**Solution 1.** *By (11), the one-form $\bar\omega = a(s)dy + b(s)du + c\,dt$, where $a = s^3 - (u+t)s^2 - (2\dot u + 1)s - (\ddot u + u + t)$, $b = -ys^2 - 2\dot y s - (\ddot y + y)$, $c = -\ddot y - y$. From Algorithm 1, we obtain $\gamma = s^2 + 1$. On step (3) we have to solve differential equation $\ddot{\tilde c} + \tilde c = y + \ddot y$, yielding general solution $\tilde c = C_1 \cos t + C_2 \sin t - y$. Thus, the non-unique irreducible form $\pi_{\bar\omega} = d\dot y - (u + t)dy - y du + (C_1 \cos t + C_2 \sin t - y)dt$. After taking $C_1 = C_2 = 0$ we obtain $\pi_{\bar\omega} = d(\dot y - ty - uy)$, thus $\varphi = \dot y - (t + u)y$. Note that $\varphi$ satisfies $\psi = \ddot\varphi + \varphi$. The irreducible equation of (25) is $\dot y = ty + uy$, resulting in the minimal realization $\dot x = (t + u)x$, $y = x$.*

**Solution 2.** *A non-minimal realization of (25) is*

$$\begin{aligned}&\dot x_1 = ux_1 + x_2,\ \dot x_2 = tux_1 + x_3,\\ &\dot x_3 = (t + u + t^2u)x_1 + x_2 + tx_3,\ y = x_1,\end{aligned} \tag{26}$$

*where $x_1 = y$, $x_2 = \dot y - uy$, $x_3 = \ddot y - u\dot y - y\dot u - tuy$. To find the minimal realization, we follow the proof of Theorem 1 and compute for system (26) the subspaces $\hat{\mathcal{H}}_1 = \mathrm{sp}_{\hat{\mathcal{K}}}\{dt, dx_1, dx_2, dx_3\}$, $\hat{\mathcal{H}}_2 = \mathrm{sp}_{\hat{\mathcal{K}}}\{dt, -t dx_1 + dx_2, -(t^2 + 1)dx_1 + dx_3\} =: \hat{\mathcal{H}}_\infty$. Finding the exact basis in $\hat{\mathcal{H}}_\infty$ allows us to rewrite $\hat{\mathcal{H}}_\infty = \mathrm{sp}_{\hat{\mathcal{K}}}\{dt, d\xi_1, d\xi_2\}$,* *where $\xi_1 = x_2 - tx_1$ and $\xi_2 = x_3 - (t^2+1)x_1$. After choosing $\xi_3 = x_1$ we can find the decomposed system*

$$\dot{\xi}_1 = \xi_2 - t\xi_1,\ \dot{\xi}_2 = t(\xi_2 - t\xi_1), \tag{27a}$$

$$\dot{\xi}_3 = \xi_1 + (t+u)\xi_3,\ y = \xi_3. \tag{27b}$$

*Like in the Solution 1, we are not going to use the general solution of non-accessible subsystem (27a) but its particular solution $\xi_1 = \xi_2 = 0$. Substituting $\xi_1 = 0,\ \xi_2 = 0$ into (27b) gives the accessible subsystem of (25), not depending any more on non-accessible components $\xi_1$ and $\xi_2$: $\dot{\xi}_3 = (t+u)\xi_3,\ y = \xi_3$. The latter system is the minimal realization of (25), which coincides with Solution 1.*

## 7. Conclusion

The paper addressed the realization problem of nonlinear time-varying systems. In particular, we have introduced the new definitions of the systems' equivalence and the realization, based on differential 1-forms. Necessary and sufficient realizability condition has been generalized so that it involves the realizations with the dimension greater than the order of i/o equation—this case is new even for time-invariant systems. We have also specified the class of nonlinear systems, where the realization has the smallest dimension if and only if it is accessible and observable. In the given class of systems, the theory of nonlinear systems remains consistent with the theory of linear systems. Two alternative methods have been presented for finding the minimal realization of the time-varying systems. The obtained results can be extended to the multi-input multi-output systems.

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