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Formal Computations of Non Deterministic Recursive Program Schemes*

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Abstract. We extend to non deterministic recursive program schemes the methods and results which permit definition of the semantics of such schemes in the deterministic case. Under natural hypothesis the set of finite and infinite trees generated by a scheme is proved to be the greatest fixed point of the functional mapping usually attached to this scheme.

Introduction

Several authors have been recently interested in non deterministic recursive programs [1, 5, 11, 13, 14, 15, 16, 17].

These programs are obtained by introducing a connective *or* which has to be interpreted operationally as giving alternatives of a choice. As in the theory of deterministic recursive programs developed in [19], we shall consider non deterministic recursive program as interpretations of what we call non deterministic recursive programs schemes having the form $\{\phi_i = \tau_i / 1 \leqslant i \leqslant k\}$ where each τ_i is a term built up from constant function symbols, unknown function symbols ϕ_i and the binary symbol *or*. It is clear that every non deterministic recursive program can be obtained from such schemes by interpreting the constant function symbols.

Our aim in this paper is to define the one-many function computed by a non deterministic recursive program scheme under the Herbrand interpretation H, including all infinite computations of the program scheme (H is the interpretation which has as domain the set of finite and infinite trees and where the

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function f_H interpreting the symbol f of arity n maps the n trees $t_1, \ldots, t_n$ on the tree $f(t_1, \ldots, t_n)$). As shown in [21], the Herbrand interpretation has the same "initiality" property for non-deterministic as for deterministic recursive program schemes: the function computed under any interpretation I is the image of the function computed under H in the morphism induced by I.

It has been already noticed that non deterministic recursive program schemes look like context-free tree-grammars [1, 9, 11, 12], hence the results in this paper can be seen as generalizing those in [20] about context-free word-grammars.

Under the Herbrand interpretation a deterministic recursive program scheme naturally computes an infinite tree which is an element of the completion of the free magma considered as a partially ordered set. It is traditional to define the computed infinite tree as the least upper bound of some sequence or some directed set of finite approximations [8, 25].

In the present paper, to deal with nondeterminism we have to change the point of view by defining the computed infinite tree as the result of an infinite computation rather than the lub of the results of a collection of finite computations. This is equivalent to define the result of a successful infinite computation as an infinite intersection: it corresponds to the idea that the undefined element, usually denoted $\perp$ or Ω, represents indeed the whole set of possible values. To be precise, when we replace an unknown function symbol in a non terminal expression by "undefined", this really means that the sub-expression headed by this function symbol can take all the values in the computation domain. Thus an expression is given a set of values: the larger this set the more undefined is the expression. An expression is defined (and the computation leading to it is successful) if and only if its set of values is a singleton.

This new point of view matches the intuitive idea that one has of a non deterministic program which computes a one-many function. At the beginning of the computation the value of this one-many function at a point can be any subset of the computation domain and in the course of the computation the set of possible values decreases.

Naturally enough, when one knows the relationship between infinite computations and greatest fixed points [6, 18, 24], the set of trees computed by a nondeterministic recursive program scheme is proved to be the greatest fixed point of some functional associated with the scheme, when the powerset of trees is simply ordered by inclusion. This is at least true for Greibach schemes, the condition of being Greibach playing a central role as soon as one deals with infinite computations [6, 20].

This paper contains four parts. The first one contains the main definitions and notations used in this paper. In the second part are defined the non deterministic recursive program schemes, the successful computations of them and their results. The third part establishes some relations between the set of results and the greatest fixed point of the functional $\hat{S}$ associated with the scheme S. The fourth one is devoted to Greibach schemes. For these Greibach schemes we prove not only equality between the set of results of S and the greatest fixed point of $\hat{S}$ but also that this set is the least upper bound of a sequence of finite sets of finite trees in the Smyth preorder [26] which mixes inclusion and the usual order on the set of finite and infinite trees containing the "undefined" constant symbol Ω.

1. Definitions and Notations

We recall here some definitions and notations from [8].

We denote by $[n]$ the set of integers $\{1,\ldots,n\}$.

F is a set of function symbols, each $f\in F$ is given an arity $\rho(f)\in\mathbb{N}$, $V=\{x_i/i\geq 1\}$ is a set of variables disjoint from F; we denote $V_0=\emptyset$ and $V_k=\{x_1,\ldots,x_k\}$ for $k>0$; so $V=\bigcup_{k>0} V_K$.

For any set E, $M(F,E)$ is defined inductively by:
—$E\subset M(F,E)$;
—if $f\in F$ and $\rho(f)=0$, then $f\in M(F,E)$;
—if $f\in F$, $\rho(f)=n>0$ and $t_1,\ldots,t_n\in M(F,E)$ then $f(t_1,\ldots,t_n)\in M(F,E)$.

It is clear that $M(F,E)$ is the free F-magma generated by E, and that the elements of $M(F,E)$ can be seen as finite trees indexed by E. The elements of $M(F,V)$ will be called simply finite trees ("magma" is another word for "algebra").

We write $\tilde{t}$ a tree such that the list of its variables from left to right is exactly $x_1x_2\ldots x_m$ for some m, and we denote $\tilde{t}(u_1,\ldots,u_m)$ the tree obtained by substituting in $\tilde{t}$ a tree u_i to x_i for every $i\in[m]$. It is clear that every tree t can be written $\tilde{t}(x_{i_1},\ldots,x_{i_m})$ and, as a special case, $\tilde{t}=\tilde{t}(x_1,\ldots,x_m)$.

Let Ω be a symbol not in $F\cup V$; $M_\Omega(F,V)$ is the free magma $M(F\cup\{\Omega\},V)$ where $\rho(\Omega)=0$. We define on $M_\Omega(F,V)$ the syntactic order by:

$\prec$ is the least partial order such that
$\cdot\forall t\in M_\Omega(F,V)$, $\Omega\prec t$, and
$\cdot\forall v\in M_\Omega(F,V)$, $f(t_1,\ldots,t_n)\prec v$ iff $v=f(t'_1,\ldots,t'_n)$ and $t_i\prec t'_i$ for $i=1,\ldots,n$.

Now, if E is any set partially ordered by $\prec$, a subset A of E is said to be directed if for any $e_1,e_2\in A$, there exists $e_3\in A$ such that $e_1\prec e_3$ and $e_2\prec e_3$. The set E is said to be complete if (i) it has a least element and (ii) every directed subset of E has a least upper bound.

It is well known [4] that by using "completion by ideals" $M_\Omega(F,V)$ can be embedded in the free complete ordered F-magma $M_\Omega^\infty(F,V)$ which can be viewed as $M_\Omega(F,V)$ with added infinite trees which are the least upper bounds of the directed subsets of $M_\Omega(F,V)$. From now on we shall denote by a small letter $t,t',t_i,u,v\ldots$ a finite tree and by a capital letter $T,T',T_i,U,W\ldots$ a finite or infinite tree.

We define now a sub-F-magma of $M_\Omega^\infty(F,V)$ which contains $M(F,V)$ as a sub-F-magma. We say that a (finite or infinite) tree T of $M_\Omega^\infty(F,V)$ is maximal if $\forall T'\in M_\Omega^\infty(F,V)$, $T\prec T'\Rightarrow T=T'$. Seen as ideals of $M_\Omega(F,V)$ the maximal trees of $M_\Omega^\infty(F,V)$ are exactly the maximal ideals. From another, more intuitive, point of view, maximal trees are those which do not contain the symbol Ω. So it is clear that the set of maximal trees of $M_\Omega^\infty(F,V)$, denoted by $M^\infty(F,V)$ is an F-magma containing $M(F,V)$.

If $t\in M_\Omega(F,V_k)$ and $t_1,\ldots,t_k\in M_\Omega(F,V_{k'})$ we denote by $t.\langle t_1,\ldots,t_k\rangle$ the tree $t[t_1/x_1,\ldots,t_k/x_k]$ of $M_\Omega(F,V_{k'})$ obtained by substituting t_i to x_i in t. This operation can be extended to infinite trees in the following way.

If $T_0 \in M_\Omega^\infty(F, V_k)$ and $T_1, \ldots, T_k \in M_\Omega^\infty(F, V_{k'})$, the set $\{t_0.\langle t_1, \ldots, t_k\rangle / t_0 \in M_\Omega(F, V_k)$ and $t_0 \prec T_0; \forall_i \in [k], t_i \in M_\Omega(F, V_k)$ and $t_i \prec T_i\}$ is directed; the least upper bound of this set is, by definition, $T_0.\langle T_1, \ldots, T_k\rangle$ which belongs to $M_\Omega^\infty(F, V_{k'})$. Note that in the case where $T_0, T_1, \ldots, T_k$ are all in $M^\infty(F, V_k)$, i.e. are maximal, $T_0.\langle T_1, \ldots, T_k\rangle$ is maximal.

II. Non Deterministic Recursive Program Schemes

Let $\Phi = \{\phi_1, \ldots, \phi_N\}$ be a set of unknown function symbols disjoint from $F \cup V$. The arity of ϕ_i is denoted by n_i. A non deterministic recursive program scheme (abbreviated ndrps) is a system of equations of the form

$$S \begin{cases} \phi_i(x_1, \ldots, x_{n_i}) = \tau_i \\ i = 1, \ldots, N \end{cases}$$

where $t_i \in M(F \cup \Phi \cup \{or\}, \{x_1, \ldots, x_{n_i}\})$ for all $i \in [N]$.

The special function symbol *or* has arity 2 and is not in $F \cup \Phi \cup V$. The scheme S is deterministic if no τ_i contains *or*.

We first define a computation sequence in S. This notion is similar to the notion of computation sequence for an ordinary (deterministic) recursive program scheme: the role of *or* is to express a possibility of choice.

To be precise, a computation sequence is a possibly infinite sequence $u_1, u_2, \ldots, u_n, \ldots$ of elements of $M(F \cup \Phi \cup \{or\}, V)$ where for all i, $u_i \underset{S}{\rightarrow} u_{i+1}$ which means that either $u_i = \alpha\phi_j(v_1, \ldots, v_{n_j})\beta$ and $u_{i+1} = \alpha w \beta$ where $w = \tau_j.\langle v_1, \ldots, v_{n_j}\rangle$, or $u_i = \alpha\ or\ (v_1, v_2)\beta$ and either $u_{i+1} = \alpha v_1 \beta$ or $u_{i+1} = \alpha v_2 \beta$.

In fact what we define here is the most general notion of computation sequence which corresponds to "call by name". A more restricted notion corresponds to "call by value" which in the non deterministic case leads to completely different results, see [1] and will not be treated here. A treatment of call by value analogous to our present treatment of call by name is better carried out in the framework of the theory of magmoids, see [2], and will be given elsewhere by the authors.

What is clear from the definition is that there is a whole set of finite and infinite computation sequences of S from $t = u_1$. We now define what is the result of a computation sequence. A computation sequence $u_1, \ldots, u_n$ terminates at n with result u_n if and only if there does not exist v such that $u_n \underset{S}{\rightarrow} v$ that is if and only if there is no occurrence of *or* nor of any ϕ_i in u_n. We denote by $\mathrm{Res}_S(t)$ the set of results of all finite terminating computation sequences from t.

In order to define the result of non terminating (or infinite) computation sequences we need a definition and lemma.

Define the F-morphism γ of $M(F \cup \Phi \cup \{or\}, V)$ into $M_\Omega(F, V)$ by the induction:

—$\gamma(x) = x$ for $x \in V$
—$\rho(f) = 0 \Rightarrow \gamma(f) = f$
—$\gamma(f(u_1, \ldots, u_{\rho(f)})) = f(\gamma(u_1), \ldots, \gamma(u_{\rho(f)}))$
—$\gamma(\phi_j(u_1, \ldots, u_{n_j})) = \Omega$
—$\gamma(or(u_1, u_2)) = \Omega$

Lemma 1. *For all $u, v \in M(F \cup \Phi \cup \{or\}, V)$, $u \underset{S}{\to} v$ implies $\gamma(u) \prec \gamma(v)$.*

Proof. Let u and v be such that $u \underset{S}{\to} v$. We prove by induction on the size of u that $\gamma(u) \prec \gamma(v)$.

· If $u \in V \cup \{f \in F / \rho(f) = 0\}$ then $u \underset{S}{\to} v$ is impossible.

· If $u = \phi_i(u_1, \ldots, u_{n_i})$ or if $u = or(u_1, u_2)$, $\gamma(u) = \Omega$ and thus $\gamma(u) \prec \gamma(v)$.

· If $u = f(u_1, \ldots, u_n)$, where $n = \rho(f)$, there exist $i \leqslant n$ and u_i' such that $u_i \underset{S}{\to} u_i'$ and $v = f(u_1, \ldots, u_1', \ldots, u_n)$. By induction hypothesis $\gamma(u_i) \prec \gamma(u_i')$, hence $\gamma(u) = f(\gamma(u_1), \ldots, \gamma(u_i), \ldots, \gamma(u_n)) \prec f(\gamma(u_1), \ldots, \gamma(u_i'), \ldots, \gamma(u_n)) = \gamma(v)$.

From this lemma we get that for every infinite computation sequence $u_1, u_2, \ldots, u_n, \ldots$ the sequence $\gamma(u_1), \gamma(u_2), \ldots, \gamma(u_n), \ldots$ is increasing in $M_\Omega(F, V)$ ordered by $\prec$ and thus has a least upper bound $\operatorname{Sup}_i \gamma(u_i) \in M_\Omega^\infty(F, V)$. But in fact we are only interested in results belonging to $M^\infty(F, V)$ whence the definition.

We say that the infinite computation sequence $u(0), u(1), \ldots, u(n), \ldots$ is *successful* if and only if one of the three equivalent conditions holds:

1— $\bigcap_i \{v \in M_\Omega^\infty(F, V) / \gamma(u(i)) \prec v\}$ is a singleton.

2— $\bigcap_i \{v \in M_\Omega^\infty(F, V) / \gamma(u(i)) \prec v\} = \{\operatorname{Sup} \gamma(u(i))\}$.

3— $\operatorname{Sup}_i \gamma(u(i))$ is maximal.

The result of the infinite successful computation sequence $u(1), \ldots, u(n), \ldots$ is then defined as $\operatorname{Sup}_i \gamma(u(i))$ and we denote by $\operatorname{Res}_S^\omega(t)$ the set of results of all infinite successful computation sequences from t. We denote by $\operatorname{Res}_S^\infty(t)$ the set $\operatorname{Res}_S(t) \cup \operatorname{Res}_S^\omega(t)$. These definitions of computation sequences and their results correspond exactly to our intuitive idea of what can be computed by S, as a program operating on the domain $M^\infty(F, V)$. Especially one should remark that Ω acts as an auxiliary symbol needed for the definition of the result and must not appear in the result itself. This definition also covers the deterministic case and is equivalent to the definition of the computed tree given in that case [3, 8].

Examples.

$$1 - S_1 : \phi(x) = x \text{ } or \text{ } s(\phi(x))$$

(we also use *or* as an infix operator, in the most usual way). $\operatorname{Res}_{S_1}(\phi(x))$ is clearly $\{s^n(x) | n \in \mathbb{N}\}$ since all the terminating finite computation sequences are of the form $\phi(x) \to s(\phi(x)) \to s^2(\phi(x)) \to \cdots \to s^n(\phi(x)) \to s^n(x)$. Clearly also the only possible infinite computation sequence is $\phi(x) \to s(\phi(x)) \to \cdots \to s^n(\phi(x)) \to s^{n+1}(\phi(x)) \to \cdots$ and is successful since $\operatorname{Sup}_n s^n(\Omega) = s^\omega \in M^\infty(\{s\}, \{x\})$. Whence

$\mathrm{Res}^{\omega}_{S_1}(\phi(x))=\{s^{\omega}\}$.

$$2-S_2:\phi(x) = x \ \mathit{or}\ \phi(s(x))$$

We have $\mathrm{Res}_{S_2}(\phi(x))=\mathrm{Res}_{S_1}(\phi(x))$. But the only infinite computation sequence of S_2 from $\phi(x)$ is $\phi(x)\to\phi(s^1(x))\to\phi(s^2(x))\to\cdots\to\phi(s^n(x))\to\cdots$ and is not successful since $\gamma(\phi(s^n(x)))=\Omega$ for all n, whence $R^{\omega}_{S_2}(\phi(x))=\varnothing$.

The operational semantics of ndrps now defines the function computed by S as the mapping $\mathrm{val}\,S$ of $M^{\infty}(F,V)^{n_1}$ into $\mathcal{P}(M^{\infty}(F,V))$ given by $\mathrm{val}\,S(U_1,\ldots,U_{n_1})=\{T.\langle U_1,\ldots,U_{n_1}\rangle/T\in\mathrm{Res}^{\infty}_S(\phi_1(x_1,\ldots,x_{n_1}))\}$.

III. Denotational Semantics

In this section we try to give an alternative equivalent definition of the computed function $\mathrm{val}\,S$, for a ndrps S. Since this new definition amounts to define $\mathrm{val}\,S$ as the component of the fixed point of some functional $\hat{S}$ attached to S we shall call this semantics denotational as in [25]. We first notice that $\mathrm{val}\,S$, as a one-many function of $M^{\infty}(F,V)^{n_1}$ into $M^{\infty}(F,V)$, has the very special property of being represented by a subset of $M^{\infty}(F,V_{n_1})$, viz. by $\mathrm{Res}^{\infty}_S(\phi_1(x_1,\ldots,x_{n_1}))$. Reciprocally if A is a subset of $M^{\infty}(F,V_k)$ it defines the one many function $g_A(U_1,\ldots,U_k)=\{T.\langle U_1,\ldots,U_k\rangle/T\in A\}$. Since our aim is to retrieve $\mathrm{val}\,S$ as some fixed point of some functional $\hat{S}$ we can look for $\hat{S}$ being a mapping of $(\mathcal{P}(M^{\infty}(F,V)))^N$ into itself.

More precisely, let S be the ndrps

$$S\begin{cases}\phi_i(x_1,\ldots,x_{n_i})=\tau_i\\ i=1,\ldots,N\end{cases}$$

let $\vec{\Phi}$ be the sequence $\langle\phi_1,\ldots,\phi_N\rangle$, and let $\mathfrak{S}_S$ be $\mathcal{P}(M^{\infty}(F,V_{n_1}))\times\mathcal{P}(M^{\infty}(F,V_{n_2}))\times\cdots\times\mathcal{P}(M^{\infty}(F,V_{n_N}))$ included in $(\mathcal{P}(M^{\infty}(F,V)))^N$. Then $\hat{S}$ is the mapping from $\mathfrak{S}_S$ into itself defined by: let $\vec{A}=\langle A_1,\ldots,A_N\rangle\in\mathfrak{S}_S$, then $\hat{S}(\vec{A})=\langle\tau_1[\vec{A}/\vec{\Phi}],\ldots,\tau_N[\vec{A}/\vec{\Phi}]\rangle$ where the substitution $[\vec{A}/\vec{\Phi}]$ from $M(F\cup\Phi\cup\{\underline{or}\},V)$ into $\mathcal{P}(M^{\infty}(F,V))$ is defined inductively by:

- if $t\in V\cup\{f\in F/\rho(f)=0\}$, $t[\vec{A}/\vec{\Phi}]=\{t\}$
- if $t=f(u_1,\ldots,u_n)$, $t[\vec{A}/\vec{\Phi}]=\{f(v_1,\ldots,v_n)/v_i\in u_i[\vec{A}/\vec{\Phi}]\}$
- if $t=\mathit{or}\ (u_1,u_2)$, $t[\vec{A}/\vec{\Phi}]=u_1[\vec{A}/\vec{\Phi}]\cup u_2[\vec{A}/\vec{\Phi}]$
- if $t=\phi_i(u_1,\ldots,u_{n_i})$, $t[\vec{A}/\vec{\Phi}]=\bigcup_{T\in A_i} T.\ \langle u_1[\vec{A}/\vec{\Phi}],\ldots,u_{n_i}[\vec{A}/\vec{\Phi}]\rangle$.

The product $T.\langle T_1,\ldots,T_k\rangle$ has already been defined when $T_1,\ldots,T_k$ were finite or infinite trees. It remains to define $T.\langle\mathcal{U}_1,\ldots,\mathcal{U}_k\rangle$ when $\mathcal{U}_1,\ldots,\mathcal{U}_k$ are sets of maximal trees.

In the case where T is finite, $T.\langle \mathfrak{U}_1,\dots,\mathfrak{U}_k\rangle$ is the OI-substitution [11] (also called "greffe complète" [7]; see also [1]) which can be defined inductively by:

—if $t=x_i\in V_k$, $t.\langle \mathfrak{U}_1,\dots,\mathfrak{U}_k\rangle=\mathfrak{U}_i$
—if $t=f\in F$, with $\rho(f)=0$, $t.\langle \mathfrak{U}_1,\dots,\mathfrak{U}_k\rangle=\{f\}$
—if $t=f(u_1,\dots,u_n)$, $t.\langle \mathfrak{U}_1,\dots,\mathfrak{U}_k\rangle=\{f(v_1,\dots,v_n)/v_i\in u_i.\langle \mathfrak{U}_1,\dots,\mathfrak{U}_k\rangle\}$.

In the case where T is infinite we define $T.\langle \mathfrak{U}_1,\dots,\mathfrak{U}_k\rangle$ by $W\in T.\langle \mathfrak{U}_1,\dots,\mathfrak{U}_k\rangle$ if and only if there exist an increasing sequence $W_1,\dots,W_n,\dots$ and an increasing sequence $t_1,\dots,t_m,\dots$ such that $\operatorname{Sup}_m t_m=T$, $\operatorname{Sup}_n W_n=W$ and for all $i\in\mathbb{N}$, $W_i\in t_i.\langle \mathfrak{U}_1,\dots,\mathfrak{U}_k\rangle$.

It is clear from the definitions that if T is a (finite or infinite) maximal tree, and $\mathfrak{U}_1,\dots,\mathfrak{U}_k$ are sets of maximal trees, then $T.\langle \mathfrak{U}_1,\dots,\mathfrak{U}_k\rangle$ is a set of maximal trees.

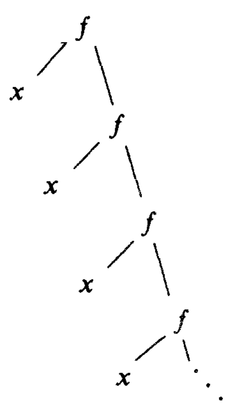
Hence it is easy to check that $\hat{S}$ is a mapping from $\mathfrak{E}_S$ into itself.

One should remark that if $\vec{A}=\langle A_1,\dots,A_N\rangle$ contains only finite trees, $\hat{S}(\vec{A})$ contains only finite trees. Thus $\hat{S}$ is also a mapping from $\mathfrak{E}'_S$ into itself, where $\mathfrak{E}'_S=\mathcal{P}(M(F,V_{n_1}))\times\cdots\times\mathcal{P}(M(F,V_{n_N}))\subset\mathfrak{E}_S$. Restricted to $\mathfrak{E}'_S$, $\hat{S}$ is nothing else than the mapping which allows to define algebraic tree languages as least fixed points and we can rephrase the theorem of algebraicity of [9].

Theorem 1. *The N-vector of subsets of $M(F,V)$ $\operatorname{Res}_S=\langle \operatorname{Res}_S(\phi_1(x_1,\dots,x_{n_1})),\dots,\operatorname{Res}_S(\phi_N(x_1,\dots,x_{n_N}))\rangle$ is the least fixed point of $\hat{S}$ in $\mathfrak{E}'_S$. Res_S is also the least fixed point of $\hat{S}$ in $\mathfrak{E}_S$.*

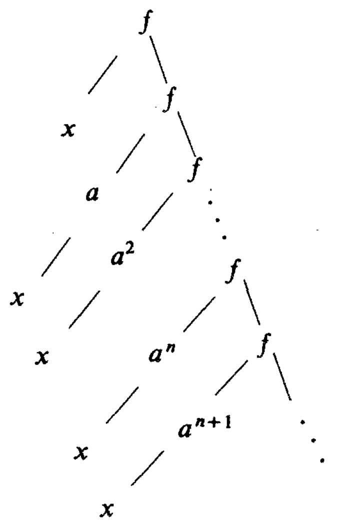
The mapping $\hat{S}$ can be easily proved to be an increasing mapping of $\mathfrak{E}_S$ into itself. But unlike its restriction to $\mathfrak{E}'_S$, it is not always weakly $\cup$-continuous, as shown by the following example. (We say that $\hat{S}$ is weakly $\cup$-continuous if $\hat{S}\left(\bigcup_n A_n\right)=\bigcup_n \hat{S}(A_n)$ for every sequence $(A_n)_n$ such that $A_i\subseteq A_{i+1}$ for all i).

Example. Let T be the infinite tree



where f is a binary symbol; let a be a unary symbol and let A_i be $\{a^n(x)/n<i\}$. So the sequence $\vec{B}_i=\langle\{T\},A_i\rangle$ is increasing and its limit is $\vec{B}=\langle\{T\},A\rangle$ where

$A=\{a^n(x)/n\in\mathbb{N}\}$. If we take $S=\{\phi_1(x)=\phi_1(x),\phi_2(x)=\phi_1(\phi_2(x))\}$, we have $\hat{S}(\vec{B}_i)=\langle\{T\},T.\langle A_i\rangle\rangle$ and $\bigcup_i \hat{S}(\vec{B}_i)\subset\hat{S}(\vec{B})=\hat{S}\left(\bigcup_i \vec{B}_i\right)=\langle\{T\},T.\langle A\rangle\rangle$; but we do not have equality because the infinite tree



is in $T.\langle A\rangle$ but is in none of the $T.\langle A_i\rangle$.

We want to characterize $\mathrm{Res}_S^\infty=\langle\mathrm{Res}_S^\infty(\phi_1(x_1,\ldots,x_{n_1})),\ldots,\mathrm{Res}_S^\infty(\phi_N(x_1,\ldots,x_{n_N}))\rangle\in\mathfrak{S}_S$ as a fixed point of $\hat{S}$. As Res_S^∞ may contain infinite trees, it follows from theorem 1 that Res_S^∞ cannot be the least fixed point of $\hat{S}$. So we turn our attention to the greatest fixed point of $\hat{S}$. Since $\mathfrak{S}_S$ is a complete lattice, this greatest fixed point does exist [22], namely

$$\nu(\hat{S})=\bigcup\{\vec{A}\mid\vec{A}\subset\hat{S}(\vec{A})\}.$$

One could be tempted to believe that $\nu(\hat{S})$ is equal to:

$$\lambda(\hat{S})=\bigcap_{i\in\mathbb{N}}\hat{S}^i(\vec{E}_S)$$

where $\vec{E}_S$ is the greatest element of $\mathfrak{S}_S$, i.e. $\langle M^\infty(F,V_{n_1}),\ldots,M^\infty(F,V_{n_N})\rangle$, but this cannot be proved in the usual fashion, since $\hat{S}$ is not weakly $\cap$-continuous as shown by the following example.

Example. Let A_n be the set $\{a^i(x)/i\geqslant n\}$ and S be the ndrps $\{\phi_1(x)=\phi_1(\phi_2(x)),\phi_2(x)=\phi_2(x)\}$. Then $\hat{S}(\langle A_n,\{a^\omega\}\rangle)=\langle A_n.\ \langle\{a^\omega\}\rangle,\{a^\omega\}\rangle=\langle\{a^\omega\},\{a^\omega\}\rangle$ thus $\bigcap_n \hat{S}(\langle A_n,\{a^\omega\}\rangle)=\langle\{a^\omega\},\{a^\omega\}\rangle$, while

$$\hat{S}\left(\bigcap_n\langle A_n,\{a^\omega\}\rangle\right)=\hat{S}\left(\langle\bigcap_n A_n,\{a^\omega\}\rangle\right)=\hat{S}(\langle\varnothing,\{a^\omega\}\rangle)=\langle\varnothing,\{a^\omega\}\rangle.$$

However the fact that $\lambda(\hat{S})=\nu(\hat{S})$ can be proved by means of topological considerations [4, 27] after having equipped $M^\infty(F,V)$ with a distance. The theorem which ends this section gives a sufficient condition in order to prove that $\nu(\hat{S})=\lambda(\hat{S})$ without any topology and in this case we get also that $\nu(\hat{S})=\lambda(\hat{S})=\mathrm{Res}_S^\infty$.

Lemma 2. *For every* $u\in M(F\cup\Phi\cup\{or\},V)$, $\mathrm{Res}_S^\infty(u)\subset u[\mathrm{Res}_S^\infty/\vec{\Phi}]$.

Proof. Let $u(0)=u,u(1),\ldots,u(k),\ldots$ be any finite terminating or infinite successful computation sequence with result u.

The general idea of this proof is the following. Let us select in u an occurrence of a symbol ϕ of Φ; this symbol will be replaced at some further step of the computation by a term; the non terminal symbols occurring in this term will be also derived at some further steps introducing new occurrences of non terminal symbols, and so on. In this way we can extract from the computation sequence $u(0),u(1),\ldots,u(k),\ldots$ a computation sequence $v(0)=\phi(x_1,\ldots,x_n),v(1),\ldots,v(l),\ldots$ with result W. Thus we can associate with each occurrence of a non terminal symbol its associated result. But in fact an occurrence of a non terminal symbol can be duplicated by a step of derivation and then from an occurrence of a non terminal symbol we can extract many computation sequences (e.g., with the equation $\phi(x)=f(x,x)$ we can have $\phi(\Psi(x))\underset{S}{\rightarrow}f(\Psi(x),\Psi(x))$ and the unique occurrence of Ψ in the first term can start two different computation sequences). Moreover, since we deal with infinite computation sequences, an occurrence can be duplicated an unbounded number of times, hence starting infinitely many extracted computation sequences.

More precisely the proof is done by induction on the size of u; thus we have to deal only with the occurrence of a non terminal symbol at the root of u: if $u(0)=u=\phi(u_1(0),\ldots,u_n(0))$ then $u(k)=\tilde{t}(u_1(k),\ldots,u_m(k))$ where $\phi(x_1,\ldots,x_n)\overset{*}{\underset{S}{\rightarrow}}\tilde{t}(x_{i_1},\ldots,x_{i_m})$ and for each $j\in[m]$, $u_{i_j}(0)\overset{*}{\underset{S}{\rightarrow}}u_j(k)$ and the technical part of the job is just keeping the "stories" of these derivations from the terms $\phi(x_1,\ldots,x_n),u_1(0),\ldots,u_n(0)$.

Then let us prove the result by induction on the size of u.

1—If $u\in V\cup\{f\in F/\rho(f)\}=0\}$, then the computation sequence is reduced to u and $U=u$. But $u[\mathrm{Res}_S^\infty/\vec{\Phi}]=\{u\}$, hence $U\in u[\mathrm{Res}_S^\infty/\vec{\Phi}]$.

2—If $u=f(u_1(0),\ldots,u_n(0))$, then for any i, $u(i)=f(u_1(i),\ldots,u_n(i))$, such that for any $k\leqslant n$, $u_k(i)\overset{*}{\underset{S}{\rightarrow}}u_k(i+1)$. Then for any $k\leqslant n$ we set $U_k=\sup_i\gamma(u_k(i))$. Since $\gamma(u_k(i))\prec U_k$, $\gamma(u(i))=f(\gamma(u_1(i)),\ldots,\gamma(u_n(i)))\prec f(U_1,\ldots,U_k)$, and thus $U=\mathrm{Sup}_i\,\gamma(u(i))\prec f(U_1,\ldots,U_n)$. But U being a maximal tree, so are $U_1,\ldots,U_n$ and $U=f(U_1,\ldots,U_n)$. The (finite or infinite) computation sequences $u_k(0),u_k(1),\ldots,u_k(l)\ldots$ having maximal trees as results, we can apply the induction hypothesis, whence $U_k\in u_k(0)[\mathrm{Res}_S^\infty/\vec{\Phi}]$. It follows that $U=f(U_1,\ldots,U_k)\in f(u_1(0),\ldots,u_k(0))[\mathrm{Res}_S^\infty/\vec{\Phi}]=u[\mathrm{Res}_S^\infty/\vec{\Phi}]$.

3—If $u=or\,(u_1(0),u_2(0))$, there exist an integer i_0 and $\varepsilon\in\{1,2\}$ such that:

—for $i\leqslant i_0$, $u_i=or\,(u_1(i),u_2(i))$ with for $j=1,2$ $u_j(i-1)\overset{*}{\underset{S}{\rightarrow}}u_j(i)$;
—$u(i_0+1)=u_\varepsilon(i_0)$.

Thus the computation sequence $u_\varepsilon(0), u_\varepsilon(1), \ldots, u_\varepsilon(i_0), \vec{u}(i_0+2), \ldots$ has result U and we apply the induction hypothesis: $U \in u_\varepsilon(0)[\mathrm{Res}_S^\infty/\vec{\Phi}]$ which, by definition, is included in $u[\mathrm{Res}_S^\infty/\vec{\Phi}]$.

4—The last case to examine is the case $u=\phi(u_1(0), \ldots, u_k(0))$ where $\phi \in \Phi$. We want to prove that it is possible to extract from the computation sequence $u, u(1), \ldots, u(n), \ldots$ a computation sequence from $\phi(x_1, \ldots, x_k)$ whose result is a maximal tree T such that U is obtained from T by substituting to every occurrence of x_i in T a maximal tree which is the result of a computation sequence from $u_i(0)$.

A—We put $u(0)=u$, $m_0=k$, θ_0 the identity on $[k]$, $\tilde{t}_0(x_1, \ldots, x_k)= \phi(x_1, \ldots, x_k)$ and we prove that for each $i \geqslant 1$ there exist an integer m_i, a mapping $\theta_i:[m_i]\to[k]$, a mapping $\eta_i=[m_i]\to[m_{i-1}]$, a finite tree $\tilde{t}_i(x_1, \ldots, x_{m_i})$ in $M(F \cup \Phi \cup \{or\}, V_{m_i})$, and m_i finite trees $u_1(i), \ldots, u_{m_i}(i)$, in $M(F \cup \Phi \cup \{or\}, V)$ such that

i) $u(i)=\tilde{t}_i(u_1(i), \ldots, u_{m_i}(i))$

ii) $\theta_i=\theta_{i-1}\circ\eta_i$

iii) $\tilde{t}_{i-1}(x_1, \ldots, x_{m_{i-1}}) \underset{S}{\overset{*}{\to}} \tilde{t}_i(x_{\eta_i(1)}, \ldots, x_{\eta_i(m_i)})$

iv) $\forall j \leqslant m_i$, $u_{\eta_i(j)}(i-1) \underset{S}{\overset{*}{\to}} u_j(i)$

This is proved by induction on i. For $i \geqslant 1$ we have $u(i-1)=\tilde{t}_{i-1}(u_1(i-1), \ldots, u_{m_{i-1}}(i-1))$, from the definition of $u(0)$ and m_0 if $i=1$, from the induction hypothesis otherwise. It is clear from the definition of $\underset{S}{\to}$ that, if $u(i-1)\underset{S}{\to}u(i)$, only two cases can occur.

1st case. In the derivation $u(i-1)\underset{S}{\to}u(i)$ the derived symbol occurs in one of the $u_j(i-1)$, i.e. there exists $j \leqslant m_{i-1}$ such that $u_j(i-1)\underset{S}{\to}v_j$ and $u(i)= \tilde{t}_{i-1}(u_1(i-1), \ldots, v_j, \ldots, u_{m_{i-1}}(i-1))$; then we put:

$$m_i = m_{i-1};$$

$$\tilde{t}_i(x_1, \ldots, x_{m_i}) = \tilde{t}_{i-1}(x_1, \ldots, x_{m_i});$$

$$\eta_i = \text{identity on } [m_i];\ \theta_i = \theta_{i-1};$$

$$u_l(i) = \begin{cases} u_l(i-1) & \text{if } l \neq j \\ v_j & \text{if } l=j \end{cases}$$

and it is easily checked that (i–iv) hold.

2nd case. In the derivation $u(i-1)\underset{S}{\to}u(i)$ the derived symbol occurs in $\tilde{t}_{i-1}(x_1, \ldots, x_{m_{i-1}})$. Thus there exists $\tilde{t}(x_{j_1}, \ldots, x_{j_n})$ such that

$$\tilde{t}_{i-1}(x_1, \ldots, x_{m_{i-1}}) \underset{S}{\to} \tilde{t}(x_{j_1}, \ldots, x_{j_n}) \quad \text{and} \quad u_i = \tilde{t}(u_{j_1}(i-1), \ldots, u_{j_n}(i-1)).$$

Then we put:

$m_i = n$

$\tilde{t}_i(x_1,\ldots,x_{m_i}) = \tilde{t}(x_1,\ldots,x_n)$

$\eta_i : [m_i] \to [m_{i-1}]$ with $\eta_i(l) = j_l$ for every $l \in [m_i]$.

$\theta_i = \theta_{i-1} \circ \eta_i$

$u_l(i) = u_{j_l}(i-1) = u_{\eta_i(l)}(i-1)$

and then (i–iv) also hold. This ends the induction on i.

Since $\tilde{t}_{i-1}(x_1,\ldots,x_{m_{i-1}}) \underset{S}{\overset{*}{\to}} \tilde{t}_i(x_{\eta_i(1)},\ldots,x_{\eta_i(m_i)})$,

$$\tilde{t}_{i-1}(x_{\theta_{i-1}(1)},\ldots,x_{\theta_{i-1}(m_{i-1})}) \underset{S}{\overset{*}{\to}} \tilde{t}_i(x_{\theta_{i-1}(\eta_i(1))},\ldots,x_{\theta_{i-1}(\eta_i(m_i))}) = \tilde{t}_i(x_{\theta_i(1)},\ldots,x_{\theta_i(m_i)}).$$

Hence there exists a finite or infinite computation sequence from $t_0(x_1,\ldots,x_{m_0}) = \phi(x_1,\ldots,x_k)$ with result $T = \operatorname{Sup}_i \gamma(\tilde{t}_i(x_{\theta_i(1)},\ldots,x_{\theta_i(m_i)}))$. If T is a maximal tree, then $T \in \mathrm{Res}_S^\infty(\phi(x_1,\ldots,x_k))$.

B—Let $Y_i \subset [m_i]$ be the set of indices of variables which occur in $\gamma(\tilde{t}_i(x_1,\ldots,x_{m_i}))$. If $j \in Y_i$ there exists one and only one $j' \in [m_{i+1}]$ such that $j = \eta_{i+1}(j')$ and moreover this integer j' is in Y_{i+1}. This comes from the fact that if $j \in Y_i$, then x_j occurs in no subtree of $\tilde{t}_i(x_i,\ldots,x_{m_i})$ with root labelled by *or* or by a symbol ϕ' in Φ and thus x_j occurs once in $\tilde{t}_{i+1}(x_{\eta_{i+1}(1)},\ldots,x_{\eta_{i+1}(m_{i+1})})$; whence there exists one and only one $j' \leqslant m_{i+1}$ such that $j = \eta_{i+1}(j')$. Moreover x_j occurs in $\tilde{t}_{i+1}(x_{\eta_{i+1}(1)},\ldots,x_{\eta_{i+1}(m_{i+1})})$ not below *or* or ϕ', thus $x_j = X_{\eta_{i+1}(j')}$ occurs in $\gamma(\tilde{t}_{i+1}(x_{\eta_{i+1}(1)},\ldots,x_{\eta_{i+1}(m_{i+1})})$ and $j' \in Y_{i+1}$.

It follows that for any i and for any j in Y_i, there exists a unique sequence of integers $\sigma_{i,j} = k_0, k_1, \ldots, k_l, \ldots$ such that

—$k_i = j$
—for any $l \geqslant 0$, $k_l \in [m_l]$ and $k_l = \eta_{l+1}(k_{l+1})$.

For $p \in [k]$ we denote by Σ_p the set of such sequences $\sigma_{i,j}$ such that $k_0 = p$. We have then: if $k_0, k_1, \ldots, k_q, \ldots, \in \Sigma_p, \theta_q(k_q) = p$; this is proved by induction on $m : \theta_0(k_0) = k_0 = p$ and if $\theta_q(k_q) = p$, $\theta_{q+1}(k_{q+1}) = \theta_q(\eta_{q+1}(k_{q+1})) = \theta_q(k_q)$.

To each sequence $\sigma = k_0, k_1, \ldots, k_q, \ldots$ in Σ_p we associate the sequence $u_{k_0}(0), u_{k_1}(1), \ldots, u_{k_q}(q), \ldots$. Since $k_q = \eta_{q+1}(k_{q+1})$ we have $u_{k_q}(q) = u_{\eta_{q+1}(k_{q+1})}(q) \underset{S}{\to} u_{k_{q+1}}(q+1)$.

So to each σ we associate a finite or infinite computation sequence from $u_{k_0}(0)$ with result $U_\sigma = \operatorname{Sup}_q \gamma(u_{k_q}(q))$. From the induction hypothesis, if U_σ is a maximal tree, $U_\sigma \in u_{k_0}(0)[\mathrm{Res}_S^\infty/\vec{\Phi}]$. Let $\mathfrak{U}_p$ be $\{U_\sigma / \sigma \in \Sigma_p\}$. Let us consider $u_q = \tilde{t}_q(u_1(q),\ldots,u_{m_q}(q))$. We define U'_q as the tree obtained by substituting $U_{\sigma_{q,j}}$ to x_j in $\gamma(\tilde{t}_q(x_1,\ldots,x_{m_q}))$ for all $j \in Y_q$. Since for j in Y_q, $\gamma(u_j(q)) \prec U_{\sigma_{q,j}}$ we have

$\gamma(u(q)) \prec U'_q$. Moreover since $\sigma_{q,j} \in \Sigma\theta_q(j)$, $U_{\sigma_{q,j}} \in \mathfrak{U}\theta_q(j)$, so that $U'_q \in \gamma(\tilde{t}_q(x_{\theta_q(1)}, \ldots, x_{\theta_q(m_q)})$. $\langle \mathfrak{U}_1, \ldots, \mathfrak{U}_n \rangle$. We prove now that $U'_q \prec U'_{q+1}$. Since $\tilde{t}_q(x_1, \ldots, x_{m_q}) \underset{S}{\to} \tilde{t}_{q+1}(x_{\eta_{q+1}(1)}, \ldots, x_{\eta_{q+1}(m_{q+1})})$, we have $\gamma(\tilde{t}_q(x_1, \ldots, x_{m_q})) \prec \gamma(\tilde{t}_{q+1}(x_{\eta_{q+1}(1)}, \ldots, x_{\eta_{q+1}(m_{q+1})})$. Hence $U'_q \prec U''_{q+1}$ where U''_{q+1} is obtained from $\gamma(\tilde{t}_{q+1}(x_1, \ldots, x_{m_{q+1}}))$ by substituting to every x_j such that $j \in Y_{q+1}$

$$\begin{cases} U_{\sigma_{q+1,j}} & \text{if } \eta_{q+1}(j) \notin Y_q \\ U_{\sigma_{q,\eta_{q+1}(j)}} & \text{if } \eta_{q+1}(j) \in Y_q. \end{cases}$$

But if $j \in Y_{q+1}$ and $\eta_{q+1}(j) \in Y_q$, we have no other j' such that $\eta_{q+1}(j') = \eta_{q+1}(j)$, hence $U_{\sigma_{q,\eta_{q+1}(j)}} = U_{\sigma_{q+1,j}}$ and $U''_{q+1} = U'_{q+1}$.

The increasing sequence $U'_0, U'_1, \ldots, U'_q, \ldots$ has a least upper bound W. Since $U'_q \in \gamma(\tilde{t}_q(x_{\theta_q(1)}, \ldots, x_{\theta_q(m_q)})$. $\langle \mathfrak{U}_1, \ldots, \mathfrak{U}_k \rangle$ and since $\gamma(\tilde{t}_q(x_{\theta_q(1)}, \ldots, x_{\theta_q(m_q)})$ is an increasing sequence with least upper bound T, $W \in T.\langle \mathfrak{U}_1, \ldots, \mathfrak{U}_k \rangle$. Moreover $\gamma(u(q)) \prec U'_q \prec W$, hence $U = \operatorname{Sup}_q(u(q)) \prec W$. By hypothesis U is maximal, so $W = U$ and T is also a maximal tree, which implies $T \in \mathrm{Res}_S^\infty(\phi(x_1, \ldots, x_k))$. We prove now that each $\mathfrak{U}_i$ is a set of maximal trees. Consider $U_{\sigma_{q,j}} \in \mathfrak{U}_i$. Since $U_{\sigma_{q,j}}$ is a subtree of U'_p for any $p \geq q$, it is a subtree of $W = U$. Since U is maximal tree, so is $U_{\sigma_{q,j}}$.

Each $\mathfrak{U}_i$ being a set a maximal trees, $\mathfrak{U}_i$ is included in $u_i(0)[\mathrm{Res}_S^\infty/\vec{\Phi}]$. Hence $U \in \mathrm{Res}_S^\infty(\phi(x_1, \ldots, x_k)).\langle u_1(0)[\mathrm{Res}_S^\infty/\vec{\Phi}], \ldots, u_k(0)[\mathrm{Res}_S^\infty/\vec{\Phi}]\rangle = \phi(u_1(0), \ldots, u_k(0))[\mathrm{Res}_S^\infty/\vec{\Phi}] = u[\mathrm{Res}_S^\infty/\vec{\Phi}]$.

Proposition 1.

$$\mathrm{Res}_S^\infty \subset \hat{S}(\mathrm{Res}_S^\infty)$$

Proof. Let $U \in \mathrm{Res}_S^\infty(\phi_i(x_1, \ldots, x_{n_i}))$. There exists a finite or infinite computation $u(0) = \phi_i(x_1, \ldots, x_{n_i}), u(1) = \tau_i, u(2), \ldots, u(n), \ldots$ with result U. From lemma 2, $U \in \tau_i[\mathrm{Res}_S^\infty/\vec{\Phi}]$ and thus $U \in [\hat{S}(\mathrm{Res}_S^\infty)]_i$ by definition of $\hat{S}$.

Proposition 2. *Let $\vec{A} \in \mathfrak{S}_S$ be such that $\vec{A} \subset \hat{S}(\vec{A})$, then $\vec{A} \subset \lambda(\hat{S})$.*

Proof. It is clear that $\vec{A} \subset \vec{E}_S$. If $\vec{A} \subset S^n(\vec{E}_S)$, because S is increasing, $\hat{S}(\vec{A}) \subset \hat{S}^{n+1}(\vec{E}_S)$ and thus $\vec{A} \subset \hat{S}^{n+1}(\vec{E}_S)$. It follows that $\vec{A} \subset \bigcap_n \hat{S}^n(\vec{E}_S)$.

Theorem 2. *If $\lambda(\hat{S})$ is included in Res_S^∞, then Res_S^∞ is the largest fixed point of $\hat{S}$, and is equal to $\lambda(\hat{S})$.*

Proof. Recall that $\nu(\hat{S}) = \cup\{\vec{A}/\vec{A} \subset \hat{S}(A)\}$ is the largest fixed point of $\hat{S}$. Hence, by proposition 1, $\mathrm{Res}_S^\infty \subset \nu(\hat{S})$ and by proposition 2, $\nu(\hat{S}) \subset \lambda(\hat{S})$. Consequently, if $\lambda(\hat{S}) \subset \mathrm{Res}_S^\infty$, then $\mathrm{Res}_S^\infty = \nu(\hat{S}) = \lambda(\hat{S})$.

IV. Greibach Schemes

This section contains our main theorem which concerns a class of schemes satisfying a condition which is very natural in the theory of languages: the Greibach condition on S amounts to saying that each occurrence of an unknown function symbol is included in a sub-expression headed by a basic function symbol. It ensures that every balanced infinite computation sequence is successful, where balanced means that all the branches of the tree are computed (this is the case of the Kleene's sequence or the sequence obtained by parallel outermost computation).

Definition. The set of contracting elements in $M(F \cup \Phi \cup \{or\}, V)$ is defined as the smallest subset C such that

$$V \subset C,$$
$$\rho(f)=0 \Rightarrow f \in C,$$
$$\forall t_1,\ldots,t_{\rho(f)} \in M(F \cup \Phi \cup \{or\}, V), f(t_1,\ldots,t_{\rho(f)}) \in C$$
$$\forall t_1, t_2 \in C,\ or\ (t_1,t_2) \in C.$$

The ndrps $S: \phi_i(x_1,\ldots,x_{n_i}) = \tau_i, i=1,\ldots,N$, is a *Greibach scheme* if and only if τ_i is contracting for all $i \in [N]$.

Example. The above example $S_1: \phi(x) = x\ or\ s(\phi(x))$ is a Greibach scheme and $S_2: \phi(x) = x\ or\ \phi(s(x))$ is not.

In proof of our following theorem, an essential tool is König's lemma which we take in the following form.

Lemma 3. *Let $\{A_i\}_{i \in \mathbb{N}}$ be a sequence of finite non empty subsets of the infinite set E. Let R be a relation on $E, R \subset E \times E$, such that*

$$\forall i \in \mathbb{N}, \forall a \in A_{i+1}, \exists b \in A_i \text{ such that } (b,a) \in R.$$

Then there exists an infinite sequence $\{a_i\}_{i \in \mathbb{N}}$ such that for all $i \in \mathbb{N}$, $a_i \in A_i$ and $(a_i, a_{i+1}) \in R$.

Theorem 3. *If S is a Greibach scheme,* Res_S^∞ *is the largest fixed point of $\hat{S}$ and is equal to* $\lambda(\hat{S}) = \bigcap_n \hat{S}^n(\vec{E}_S)$.

Proof. From theorem 2, it suffices to show that $\lambda(\hat{S}) \subset \mathrm{Res}_S^\infty$, that is $\lambda(\hat{S})_k$, the kth component of $\lambda(\hat{S})$, is included in $\mathrm{Res}_S^\infty(\phi_k(x_1,\ldots,x_{n_k}))$. We need first to define the parallel outermost derivation of depth i noted $\underset{i}{\Rightarrow}$ by

—$\forall v \in M(F \cup \Phi \cup \{or\}, V)\, v \underset{0}{\Rightarrow} v$
—if $v = x_j$ or $v = f$ with $\rho(f)=0$, $v \underset{i}{\Rightarrow} v$, $\forall i \geqslant 0$
—if $v = or(v_1, v_2), v \underset{i}{\Rightarrow} w$ iff there exists $\varepsilon \in \{1,2\}$ such that $v_\varepsilon \underset{i}{\Rightarrow} w$
—if $v = f(v_1,\ldots,v_n), v \underset{i}{\Rightarrow} w$ iff $w = f(w_1,\ldots,w_n)$ and $\forall j \in [n], v_j \underset{i}{\Rightarrow} w_j$
—if $v = \phi_l(v_1,\ldots,v_n), v \underset{i}{\Rightarrow} w$ iff $\tau_l.\langle v_1,\ldots,v_n\rangle \underset{i-1}{\Rightarrow} w$.

Let now U be in $\lambda(\hat{S})_k$. We prove that for any $i \geqslant 0$ there exists w such that $\phi_k(x_1,\ldots,x_{n_k}) \underset{i}{\Rightarrow} w$ and $\gamma(w) \prec U$. Since $U \in \phi_k(x_1,\ldots,x_{n_k})[\hat{S}^i(\vec{E}_S)/\vec{\Phi}]$ for all i, it is sufficient to prove by induction on i that if $U \in v[\hat{S}^i(\vec{E}_S)/\vec{\Phi}]$ for some v, then there exists w such that $v \underset{i}{\Rightarrow} w$ and $\gamma(w) \prec U$.

1st case $i=0$. Let U be in $v[\vec{E}_S/\vec{\Phi}]$. We prove by induction on v that $\gamma(v) \prec U$.

—If $v \in V \cup \{f \in F/\rho(f)=0\}$ then $v[\vec{E}_S/\vec{\Phi}] = \{v\}$ so $U=v$ and $\gamma(v)=U$.
—If $v = or(v_1,v_2)$, then $\gamma(v) = \Omega \prec U$;
—If $v = \phi(v_1,\ldots,v_n)$, then $\gamma(v) = \Omega \prec U$;
—If $v = f(v_1,\ldots,v_n)$, then $U = f(U_1,\ldots,U_n)$ with $U_j \in v_j[\vec{E}_S/\vec{\Phi}]$.

By the induction hypothesis $\gamma(v_j) \prec U_j$ and thus $\gamma(f(v_1,\ldots,v_n)) = f(\gamma(v_1),\ldots,\gamma(v_n)) \prec f(U_1,\ldots,U_n)$.

2nd case. Let U be in $v[\hat{S}^{i+1}(\vec{E}_S)/\vec{\Phi}]$. We yet prove the result by induction on v.

—If $v \in V \cup \{f \in F/\rho(f)=0\}$, $v[\hat{S}^{i+1}(\vec{E}_S)/\vec{\Phi}] = \{v\}$, so $U=v$ and $v \underset{i+1}{\Rightarrow} v$ and $\gamma(v)=v$.

—If $v = or(v_1,v_2)$, there exists $\varepsilon \in \{1,2\}$ such that $U \in v_\varepsilon[\hat{S}^{i+1}(\vec{E}_S)/\vec{\Phi}]$. By the induction hypothesis there exists w such that $v_\varepsilon \underset{i+1}{\Rightarrow} w$ and $\gamma(w) \prec U$, thus $v \underset{i+1}{\Rightarrow} w$.

—If $v = f(v_1,\ldots,v_n)$, $U = f(U_1,\ldots,U_n)$ with $U_j \in v_j[\hat{S}^{i+1}(\vec{E}_S)/\vec{\Phi}]$. By the induction hypothesis there exist $w_1,\ldots,w_n$ such that $v_j \underset{i+1}{\Rightarrow} w_j$ and $\gamma(w_j) \prec U_j$. Hence $f(v_1,\ldots,v_n) \underset{i+1}{\Rightarrow} f(w_1,\ldots,w_n)$ and $\gamma(f(w_1,\ldots,w_n)) \prec f(U_1,\ldots,U_n)$.

—If $v = \phi_l(v_1,\ldots,v_n)$ there exists $T \in [\hat{S}^{i+1}(\vec{E}_S)]_l = \tau_l[\hat{S}^i(\vec{E}_S)/\vec{\Phi}]$ such that $U \in T.\langle v_1[\hat{S}^{i+1}(\vec{E}_S)/\vec{\Phi}],\ldots,v_n[\hat{S}^{i+1}(\vec{E}_S)/\vec{\Phi}]\rangle$. By the induction hypothesis (on i) there exists $\tilde{t}(x_{j_1},\ldots,x_{j_m})$ such that $\tau_l \underset{i}{\Rightarrow} \tilde{t}(x_{j_1},\ldots,x_{j_m})$ and $\gamma(\tilde{t}(x_{j_1},\ldots,x_{j_m})) \prec T$. But then there exists $W \in \gamma(\tilde{t}(x_{j_1},\ldots,x_{j_m})).\langle\ldots,v_p[\hat{S}^{i+1}(\vec{E}_S)/\vec{\Phi}],\ldots\rangle$ with $W \prec U$. We have $W = \gamma(\tilde{t}(x_1,\ldots,x_m)).\langle U_1,\ldots,U_m\rangle$ with $U_p \in v_{j_p}[\hat{S}^{i+1}(\vec{E}_S)/\vec{\Phi}]$; so there exists w_p such that $v_{j_p} \underset{i+1}{\Rightarrow} w_p$ and $\gamma(w_p) \prec U_p$. We have then $\phi_l(x_1,\ldots,x_n) \to \tau_l \underset{i}{\Rightarrow} \tilde{t}(x_{j_1},\ldots,x_{j_m})$ hence $\phi_l(x_1,\ldots,x_n) \underset{i+1}{\Rightarrow} \tilde{t}(x_{j_1},\ldots,x_{j_m})$ and also $v_{j_p} \underset{i+1}{\Rightarrow} w_p$.

Thus the derivation $\phi_l(v_1,\ldots,v_n) \xrightarrow[S]{*} \tilde{t}(v_{j_1},\ldots,v_{j_m}) \xrightarrow[S]{*} \tilde{t}(w_1,\ldots,w_m)$ can be factorized through a parallel outermost derivation of depth $i+1$: $\phi_l(v_1,\ldots,v_n) \underset{i+1}{\Rightarrow} \tilde{t}(w'_1,\ldots,w'_m) \xrightarrow[S]{*} \tilde{t}(w_1,\ldots,w_m)$. Hence $\gamma(\tilde{t}(w'_1,\ldots,w'_m)) \prec \gamma(\tilde{t}(w_1,\ldots,w_m)) = \gamma(\tilde{t}(x_1,\ldots,x_m)).\langle\gamma(w_1),\ldots,\gamma(w_m)\rangle \prec \gamma(\tilde{t}(x_1,\ldots,w_m)).\langle U_1,\ldots,U_m\rangle = W \prec U$.

Let now G_i be the set $\{w/\phi_l(x_1,\ldots,x_n) \underset{i}{\Rightarrow} w$ and $\gamma(w) \prec U\}$. This set is obviously finite. We have just proved that it is not empty.

Moreover it is obvious that for any w in G_{i+1} there exists w' in G_i such that $w' \underset{1}{\Rightarrow} w$, and so $w' \overset{*}{\underset{S}{\rightarrow}} w$. We can apply König's lemma and obtain a parallel outermost derivation $w_0, w_1, \ldots, w_n, \ldots$ such that, for any $i, \phi_i(x_1, \ldots, x_n) \underset{i}{\Rightarrow} w_i$ and $\gamma(w_i) \prec U$. The result of the corresponding computation sequence is then $U' \prec U$. But, because S is a Greibach scheme, for any integer k there exists an integer i such that all the leaves labeled by Ω in $\gamma(w_i)$ are at a distance at least k from the root; this is an immediate consequence of the following property. Let $v(0) = v, v(1), \ldots, v(n), \ldots$ be a parallel outermost derivation; then there exists an integer k such that either $v(k) \in V$ or the root of $v(k)$ is in F; property which is proved by induction on the size of v:

—If $v(0) \in V$ or if the root of $v(0) \in F$ then there is nothing to prove.

—If $v(0) = or(u_1, u_2)$ then either $v(1) = u_1$ or $v(1) = u_2$ and we can apply the induction hypothesis.

—If $v(0) = \phi(u_1, \ldots, u_n)$ then since the scheme is a Greibach scheme, $v(1)$ has the form $t.\langle u_1, \ldots, u_n \rangle$ where t is a contracting term; hence the first steps of any parallel outmost derivation from $v(1)$ consist in deriving t in a parallel outermost way until t is rewritten either as $f(u'_1, \ldots, u'_{\rho(f)})$ (included the case $\rho(f) = 0$) and then there exists k such that the root of $v(k)$ is $f \in F$, or as x_i and then, for some $k, v(k) = u_i$ and we can apply the induction hypothesis.

It follows that U' is a maximal tree and that $U' = U$. We have thus proved that $U \in \mathrm{Res}_S^\infty(\phi_k(x_1, \ldots, x_{n_k}))$.

We now give an other characterization of $\lambda(\hat{S})$, when S is a non deterministic Greibach scheme, which looks like the usual definition of the function computed by a deterministic program scheme.

Remember that the least fixed point of the deterministic scheme S is the least upper bound of the increasing sequence $\{\hat{S}^n(\vec{\Omega}) / n \in \mathbb{N}\}$ of elements of $\vec{D}_S = M_\Omega^\infty(F, V_{n_1}) \times \ldots \times M_\Omega^\infty(F, V_{n_N})$ ordered by the syntactic order $\prec$, as proved by Nivat [19].

In the case of non deterministic schemes, $\hat{S}^n(\vec{\Omega})$ is an element of $\mathcal{D}_S = \mathcal{P}(M_\Omega^\infty(F, V_{n_1})) \times \cdots \times \mathcal{P}(M_\Omega^\infty(F, V_{n_N}))$. We consider on $\mathcal{P}(M_\Omega^\infty(F, V_i))$ the following preorder $\sqsubseteq$, defined by M. Smyth [26]: $A \sqsubseteq B$ if and only if $\forall u \in B, \exists v \in A$ such that $v \prec u$, which is extended componentwise to $\mathcal{D}_S$.

Proposition 3. *If S is a* ndrps, *$\{\hat{S}^n(\vec{\Omega}) / n \in \mathbb{N}\}$ is an increasing sequence for the Smyth preorder.*

Proof. It follows, from the definition of $\sqsubseteq$, that $\vec{\Omega} \sqsubseteq \vec{A}$, for any $\vec{A}$ in $\mathcal{D}_S$. Then, to obtain the result, since $\hat{S}^n(\vec{\Omega})$ contains only finite trees, it suffices to prove the following lemma.

Lemma 4. *If S is a* ndrps, *if $\vec{A}$ and $\vec{B}$ are in $\mathcal{D}_S$ and $\vec{A}$ contains only finite trees, $\vec{A} \sqsubseteq \vec{B}$ implies $S(\vec{A}) \sqsubseteq S(\vec{B})$.*

Proof. Let us prove by induction on t in $M(F \cup \Phi \cup \{or\}, V)$ that $\vec{A} \sqsubseteq \vec{B}$ implies $t[\vec{A}/\vec{\Phi}] \sqsubseteq t[\vec{B}/\vec{\Phi}]$ when $\vec{A}$ contains only finite trees.

—If $t=x_i$ or $t=a\in F$ with $\rho(a)=0$ then $t[\vec{A}/\vec{\Phi}]=t[\vec{B}/\vec{\Phi}]=\{t\}$

—If $t=or(t_1,t_2)$, by the induction hypothesis $t_\varepsilon[\vec{A}/\vec{\Phi}]\sqsubseteq t[\vec{B}/\vec{\Phi}]$ for $\varepsilon=1,2$ and thus,

$$t[\vec{A}/\vec{\Phi}]=t_1[\vec{A}/\vec{\Phi}]\cup t_2[\vec{A}/\vec{\Phi}]\sqsubseteq t_1[\vec{B}/\vec{\Phi}]\cup t_2[\vec{B}/\vec{\Phi}]=t[\vec{B}/\vec{\Phi}].$$

—If $t=f(t_1,\ldots,t_n)$, let U be in $t[\vec{B}/\vec{\Phi}]$. Then $U=f(U_1,\ldots,U_n)$ with $U_i\in t_i[\vec{B}/\vec{\Phi}]$ for every $i\in[n]$. By the induction hypothesis there exists $v_i\in t_i[\vec{A}/\vec{\Phi}]$ such that $v_i\prec U_i$, hence $v=f(v_1,\ldots,v_n)\prec U$ and $v\in t[\vec{A}/\vec{\Phi}]$; therefore $t[\vec{A}/\vec{\Phi}]\sqsubseteq t[\vec{B}/\vec{\Phi}]$.

—If $t=\phi_i(t_1,\ldots,t_{n_i})$ and if $U\in t[\vec{B}/\vec{\Phi}]$ there exists $T\in B_i$ such that $U\in T.\langle t_1[\vec{B}/\vec{\Phi}],\ldots,t_{n_i}[\vec{B}/\vec{\Phi}]\rangle$. But there exists $t'\prec T$ such that $t'\in A_i$ and we have also $t_j[\vec{A}/\vec{\Phi}]\sqsubseteq t_j[\vec{B}/\vec{\Phi}]$ by induction hypothesis, hence to prove the existence of v in $t[\vec{A}/\vec{\Phi}]$ such that $v\prec U$ we can choose this v in $t'.\langle t_1[\vec{A}/\vec{\Phi}],\ldots,t_{n_i}[\vec{A}/\vec{\Phi}]\rangle$ and then it is sufficient to prove that if $t'\prec T$ and $\vec{C}\sqsubseteq\vec{D}$ then $t'.\vec{C}\sqsubseteq T.\vec{D}$. This proof is done by induction on t':

—if $t'=\Omega$ then $t'.\vec{C}=\{\Omega\}\sqsubseteq T.\vec{D}$;

—if $t'=x_i$ then $T=x_i$ and $t'.\vec{C}=C_i\sqsubseteq D_i=T.\vec{D}$;

—if $t'=f$ with $\rho(f)=0$ then $T=f$ and $t'.\vec{C}=T.\vec{D}=\{f\}$;

—if $t'=f(t'_1,\ldots,t'_n)$ then $T=f(T_1,\ldots,T_n)$ with $t'_i\prec T_i$ and by the induction hypothesis $t'_i.\vec{C}\sqsubseteq T_i.\vec{D}$, hence $t'.\vec{C}=f.\langle t'_1.\vec{C},\ldots,t'_n.\vec{C}\rangle\sqsubseteq f.\langle T_1.\vec{D},\ldots,T_n.\vec{D}\rangle=T.\vec{D}$.

To the increasing sequence $\{\hat{S}^n(\Omega)/n\in\mathbb{N}\}$ for $\sqsubseteq$ we associate the element $\vec{B}_S$ of $\mathfrak{D}_S$ defined by $\vec{U}\in\vec{B}_S$ if and only if there exists an increasing sequence (for $\prec$) $u_1,\ldots,u_n,\ldots$ such that $u_n\in\hat{S}^n(\vec{\Omega})$ and $U=\operatorname{Sup}_i\vec{u}_i$. The aim of this kind of construction also used by Plotkin [23] and Arnold [1] is to give the least upper bound of the sequence $\{\hat{S}^n(\vec{\Omega})/n\in\mathbb{N}\}$ as show by the following proposition.

Proposition 4. *For any n, $\hat{S}^n(\vec{\Omega})\sqsubseteq\vec{B}_S$. For any $\vec{A}$ in $\mathfrak{D}_S$, $\hat{S}^n(\vec{\Omega})\sqsubseteq\vec{A}$ for every n implies $\vec{B}_S\sqsubseteq\vec{A}$.*

Proof. The first point of this proposition is immediate from the definition of $\vec{B}_S$. The second one is a direct consequence of the following lemma.

Lemma 5. *If $\hat{S}^n(\vec{\Omega})\sqsubseteq\vec{A}_n$ for every n, then $\vec{B}_S\sqsubseteq\bigcap_n\vec{A}_n$.*

Proof. Let $\vec{U}\in\bigcap_n\vec{A}_n$. For every n, $\vec{U}\in\vec{A}_n$, and since $\vec{S}^n(\vec{\Omega})\sqsubseteq\vec{A}_n$, there exists $\vec{u}_n\in\hat{S}^n(\vec{\Omega})$ such that $\vec{u}_n\prec\vec{U}$. So for any n the set $G_n=\{\vec{u}\in\hat{S}^n(\vec{\Omega})/\vec{u}\prec\vec{U}\}$ is non empty and finite since $\hat{S}^n(\vec{\Omega})$ is finite. Moreover, by proposition 3, for any $\vec{u}$ in $\hat{S}^{n+1}(\vec{\Omega})$ there exists $\vec{v}$ in $\hat{S}^n(\vec{\Omega})$ such that $\vec{v}\prec\vec{u}$; if $\vec{u}\prec\vec{U}$, that is $\vec{u}\in G_{n+1}$, we have $\vec{v}\prec\vec{u}\prec\vec{U}$ and thus $\vec{v}\in G_n$. So we can apply König's lemma and we obtain a sequence $\{\vec{u}_n/n\in\mathbb{N}\}$ with $\vec{u}_n \prec \vec{u}_{n+1}$, $\vec{u}_n\in\hat{S}^n(\vec{\Omega})$ and $\vec{u}_n\prec\vec{U}$. Hence we have $\operatorname{Sup}_i \vec{u}_i\in\vec{B}_S$ and $\operatorname{Sup}_i \vec{u}_i\prec\vec{U}$.

Theorem 4. *If S is a Greibach scheme, then $\vec{B}_S=\lambda(\hat{S})$.*

Proof. i) $\lambda(\hat{S})\subset\vec{B}_S$

From lemma 4, and since $\vec{\Omega}\sqsubseteq\vec{E}_S$, we have $\hat{S}^n(\vec{\Omega})\sqsubseteq\hat{S}^n(\vec{E}_S)$ for any n. From lemma 5 we get $\vec{B}_S\sqsubseteq\lambda(\vec{S})$; that is for any $\vec{U}$ in $\lambda(\vec{S})$ there exists a sequence $(\vec{u}_i)_i$ with $\vec{u}_i\in\hat{S}^i(\vec{\Omega})$ and $\operatorname{Sup}_i\vec{u}_i\prec\vec{U}$. But since S is a Greibach scheme $\operatorname{Sup}_i\vec{u}_i$ is maximal, hence $\vec{U}=\operatorname{Sup}_i\vec{u}_i\in\vec{B}_S$.

ii) $\vec{B}_S\subset\lambda(\hat{S})$

Let $\vec{U}$ be in $\vec{B}_S$. By definition there exists a sequence $(\vec{u}_i)_i$ with $\vec{u}_i\in\hat{S}^i(\vec{\Omega})$ such that $\vec{U}=\operatorname{Sup}_i\vec{u}_i$.

But exactly as in the proof of theorem 3 it can be shown that if $u\in[\hat{S}^n(\vec{\Omega})]_k$ there exists w such that $\phi_k(x_1,\ldots,x_{n_k})\underset{n}{\Rightarrow}w$ and $\gamma(w)\prec u$ (in fact this result still holds if $\vec{\Omega}$ is replaced by any vector of sets, like $\vec{E}_S$ in the case of theorem 3). Thus for each i there exists w_i such that $\phi_k(x_1,\ldots,x_{n_k})\underset{i}{\Rightarrow}w_i$ and $\gamma(w_i)\prec U$. By applying König's lemma and using the fact that S is a Greibach scheme, we get as in the proof of theorem 3 that U is the result of a (finite or infinite) successful computation from $\phi_k(x_1,\ldots,x_{n_k})$. Thus $\vec{B}_S\subset\operatorname{Res}_S^\infty=\lambda(\hat{S})$.

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