Arch. Math. Logic 43, 215–264 (2004)
Digital Object Identifier (DOI):
10.1007/s00153-003-0207-x

Mathematical Logic

Wim Veldman

An intuitionistic proof of Kruskal's theorem

Received: 31 October 2001 /
Published online: 15 January 2004 –

0. Introduction

In 1960, J.B. Kruskal published a proof of a conjecture due to A. Vazsonyi. Vazsonyi's conjecture, to be explained in detail in Section 8, says that the collection of all finite trees is *well-quasi-ordered* by the relation of embeddability, that is, for every infinite sequence $\alpha(0), \alpha(1), \alpha(2), \ldots$ of finite trees there exist i, j such that $i < j$ and $\alpha(i)$ embeds into $\alpha(j)$. Kruskal established an even stronger statement that he called the Tree Theorem. His proof extends an argument developed by G. Higman in 1952.

In 1963, a much shorter proof of Kruskal's Theorem was given by C.St.J.A. Nash-Williams, who introduced the elegant and powerful *minimal-bad-sequence* argument.

The purpose of this paper is to show that the proofs given by Higman and Kruskal are essentially constructive and acceptable from an intuitionistic point of view and that the later argument given by Nash-Williams has to be called into question.

The paper consists of the following 11 Sections.

1. Dickson's Lemma
2. Almost full relations
3. Some induction principles and Brouwer's Thesis
4. Ramsey's Theorem
5. The Finite Sequence Theorem
6. Vazsonyi's Conjecture for binary trees
7. Higman's Theorem
8. Vazsonyi's Conjecture and the Tree Theorem
9. Minimal-Bad-Sequence Arguments
10. The Principle of Open Induction
11. Concluding Remarks

Except for Section 9, we will argue intuitionistically.

W. Veldman: Subfaculteit Wiskunde, Katholieke Universiteit Nijmegen, Postbus 9010, 6500 GL Nijmegen, The Netherlands. e-mail: `veldman@math.kun.nl`

1. Dickson's Lemma

Kruskal's Theorem belongs to a large and growing family of results. Our involvement with this family started when, many years ago, John Burgess asked for a constructive proof of the following statement:

For all infinite sequences α, β of natural numbers
there exist i, j such that $i < j$ and both $\alpha(i) \leq \alpha(j)$ and $\beta(i) \leq \beta(j)$.

(We are using $i, j, m, n, \ldots$ as variables over the set $\mathbb{N}$ of natural numbers, and $\alpha, \beta, \ldots$ as variables over the set $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$, that is, all infinite sequences of natural numbers.)

We now prove an immediate generalization of the above statement.

Theorem 1.1. *For every $p > 0$, for all infinite sequences $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$ of natural numbers, there exist i, j such that $i < j$ and for every $k < p : \alpha_k(i) \leq \alpha_k(j)$.*

Proof. We use induction on p. First observe that for every α there exists $i \leq \alpha(0)$ such that $\alpha(i) \leq \alpha(i+1)$. This observation proves the case $p = 1$ of the statement of the Theorem. Now assume that $p > 1$ and that we proved the case $p - 1$ of the statement of the Theorem. We handle the case p as follows. Let $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$ be infinite sequences of natural numbers. Define the proposition QED (with a meaning slightly different from the usual one, "quod *est* demonstrandum", "what *is* to be proved", rather than "quod *erat* demonstrandum", "what *was* to be proved") as follows:

QED := *there exist i, j such that $i < j$ and for every $k < p$:$\alpha_k(i) \leq \alpha_k(j)$*

We *claim*:

For every n, for every m, either QED or there exists $i > m$ such that $\alpha_0(i) \geq n$.

Observe that the claim holds if $n = 0$. Let $n > 0$ be a natural number and assume that we proved already:

For every m, either QED or there exists $i > m$ such that $\alpha_0(i) \geq n - 1$.

Let m be a natural number. Applying the assumption repeatedly, we find a strictly increasing sequence $\gamma(0), \gamma(1), \gamma(2), \ldots$ of natural numbers such that $m < \gamma(0)$ and for every ℓ: either QED or $\alpha_0\big(\gamma(\ell)\big) \geq n - 1$. Using the induction hypothesis, we calculate i, j such that $i < j$ and for every k, if $0 < k < p$, then $\alpha_k\big(\gamma(i)\big) \leq \alpha_k\big(\gamma(j)\big)$. Now observe: *either* QED, *or* $\alpha_0\big(\gamma(i)\big) = n-1 \leq \alpha_0\big(\gamma(j)\big)$ and therefore QED, *or* $\alpha_0\big(\gamma(i)\big) \geq n$, therefore: either QED or there exists $i > m$ such that $\alpha_0(i) \geq n$.

We conclude that our claim is valid.

There are two ways to complete the proof.

First way :
Using the claim repeatedly, we build a strictly increasing sequence γ of natural numbers such that for every i: either QED or $\alpha_0\big(\gamma(i+1)\big) \geq \alpha_0\big(\gamma(i)\big)$. We again apply the induction hypothesis and calculate i, j such that $i < j$ and for every k, if $0 < k < p$, then $\alpha_k\big(\gamma(i)\big) \leq \alpha_k\big(\gamma(j)\big)$. Now observe: *either* QED, *or* for all $k < p$, $\alpha_k\big(\gamma(i)\big) \leq \alpha_k\big(\gamma(j)\big)$, and also QED; therefore in any case QED.

Second way :
Slightly generalizing the result of our claim, we conclude: for every $k < p$, for every m, there exists $i > m$ such that either QED or $\alpha_k(i) \geq m$. We now first build a strictly increasing sequence γ_0 of natural numbers such that, for every i, either QED or $\alpha_0(0) \leq \alpha_0\big(\gamma_0(i)\big)$. We then build a strictly increasing subsequence γ_1 of γ_0 such that, for every i, either QED or $\alpha_1(0) \leq \alpha_1\big(\gamma_1(i)\big)$. Continuing in this way, we build, for every positive $k < p$, a strictly increasing subsequence γ_k of γ_{k-1} such that, for every i, either QED or $\alpha_k(0) \leq \alpha_k\big(\gamma_k(i)\big)$. Now observe: *either* QED, *or* for every $k < p$, $\alpha_k(0) \leq \alpha_k\big(\gamma_{p-1}(1)\big)$, and also QED; therefore in any case QED. □

1.1. Theorem 1.1 is known as Dickson's Lemma. (See [3].) It is often cited in the form: every subset of $\mathbb{N}^p$ contains a finite number of $\leq^p$-minimal elements, where, for all $(a_0, a_1, \ldots, a_{p-1})$ and $(b_0, b_1, \ldots, b_{p-1})$ in $\mathbb{N}^p$, $(a_0, a_1, \ldots, a_{p-1}) \leq^p (b_0, b_1, \ldots, b_{p-1})$ if and only if, for every $k < p$, $a_k \leq b_k$.

Formulated in this way, however, it is not true constructively, as we are to see now from some examples in Brouwer's style. Let $d : \mathbb{N} \to \{0, 1, \ldots, 9\}$ be the decimal expansion of π, so $\pi = 3 + \sum_{n=1}^{\infty} d(n) \cdot 10^{-n}$.

Let A be the set of all natural numbers n such that there exists i such that for all $k < 99$, $d(i+k) = 9$. The set A may be empty and the set A may coincide with the whole set $\mathbb{N}$, but we do not know in which case we are. The set of the minimal elements of A may be empty and it may also coincide with the set $\{0\}$, but we do not know in which case we are. We are unable to determine the number of elements of this set and have no proof that it is (effectively) finite.

It does not help to add the condition that the given set is a decidable subset of $\mathbb{N}$. Let B be the set of all natural numbers n such that there exists $i < n$ with the property that, for all $k < 99$, $d(i + k) = 9$. We are unable to decide if the set B is empty or not, and also if it has a first element or not.

If however a subset of $\mathbb{N}$ is both a decidable subset of $\mathbb{N}$ and inhabited, that is, at least one natural number is known to belong to it, then this set will have a first element.

Unfortunately, these two conditions do not help in higher dimensions.

Let p be a natural number and let A be a decidable subset of $\mathbb{N}^p$. For every $(a_0, a_1, \ldots, a_{p-1})$ in $\mathbb{N}^p$ we may decide if there exists $(b_0, b_1, \ldots, b_{p-1})$ in A different from $(a_0, a_1, \ldots, a_{p-1})$ such that $(b_0, b_1, \ldots, b_{p-1}) \leq^p (a_0, a_1, \ldots, a_{p-1})$, or not. Therefore the set $M(A)$ consisting of the $\leq^p$-minimal elements of A is also a decidable subset of $\mathbb{N}^p$. $M(A)$ need not be a finite subset of $\mathbb{N}^p$. In order to see this, we let A be the set $\{\langle 1, 1\rangle\} \cup \{\langle 0, n\rangle |$ There exists $i < n$ such that for all $k < 99,\ d(i + k) = 9\}$.

Observe that $\langle 1, 1\rangle$ belongs to $M(A)$. If there exists i such that for all $k < 99$, $d(i + k) = 9$, and i_0 is the least such i, then $M(A)$ will also contain the ordered pair $\langle 0, i_0 + 1\rangle$. Therefore, $M(A)$ has at least one and at most two elements. The statement: "$M(A)$ has exactly one element" implies that there is no i such that for all $k < 99$, $d(i + k) = 9$. The statement: "$M(A)$ has two elements" implies that there exists i such that for all $k < 99$, $d(i + k) = 9$. We have no proof of either statement, and are unable to show that $M(A)$ is finite.

Theorem 1.1 implies however that, for every decidable subset A of $\mathbb{N}^p$, the set $M(A)$ is *almost-finite* in the following sense:

For every $\gamma : \mathbb{N} \to M(A)$ *there exist* i, j *such that* $i < j$ *and* $\gamma(i) = \gamma(j)$.

The notion of an almost-finite subset of $\mathbb{N}$ is studied in [26] and [30].

1.2. A not unusual classical proof of Theorem 1.1 uses the following fact:

For every $\alpha : \mathbb{N} \to \mathbb{N}$ *there exists a strictly increasing function* $\gamma : \mathbb{N} \to \mathbb{N}$ *such that, for every* i, $\alpha(\gamma(i)) \le \alpha(\gamma(i+1))$.

(That is, every sequence of natural numbers contains an infinite monotone subsequence.)

The following example shows that this statement fails constructively.

Let $\alpha : \mathbb{N} \to \mathbb{N}$ *be such that for each* n, $\alpha(n) \le 1$, *and* $\alpha(n) = 1$ *if and only if there is no* $i < n$ *such that for all* $k < 99$, $d(i+k) = 9$.

Suppose $\gamma : \mathbb{N} \to \mathbb{N}$ is strictly increasing and for every i, $\alpha(\gamma(i)) \le \alpha(\gamma(i+1))$. If $\alpha(\gamma(0)) = 0$, then we have found i such that for every $k < 99$, $d(i+k) = 9$; if $\alpha(\gamma(0)) = 1$ we are sure that no such i exists. We are unable to find such i and we cannot find an infinite monotone subsequence of α.

2. Almost full relations

2.1. Let A be an inhabited set and let R be a binary relation on A. Let $\alpha : \mathbb{N} \to A$. We say that α *meets* R if and only if there exist i, j such that $i < j$ and $\alpha(i) R \alpha(j)$. We say that R is *almost full* on A if and only if every $\alpha : \mathbb{N} \to A$ meets R.

Observe that every relation R that is almost full on A must be reflexive. (For any given a in A one may consider the sequence $\alpha : \mathbb{N} \to A$ such that for every i, $\alpha(i) = a$. As α meets R we have aRa.)

We might have defined: R is *almost full* on A if and only if every $\alpha : \mathbb{N} \to A$ either meets R or the equality relation $=$. We then also would allow some non-reflexive relations to be almost full. In fact, we did so in [25].

A (reflexive) almost full relation on A that is also transitive is called a *well-quasi-ordering* of A. The importance of this notion has been stressed repeatedly, see [10].

2.2. Let A, B be sets and let $R \subseteq A \times A$ and $T \subseteq B \times B$ be binary relations on A, B respectively.

We define a binary relation on the set $A \times B$, called the *product* $R \times T$ of the relations R and T, as follows:

for all $\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle$ in $A \times B$,
$\langle a_0, b_0 \rangle R \times T \langle a_1, b_1 \rangle$ if and only if both $a_0 R a_1$ and $b_0 T b_1$.

Theorem 2.1. *Let* R *be an almost full (reflexive) relation on* $\mathbb{N}$. *Then* $\le \times R$ *is almost full on* $\mathbb{N} \times \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 1.1. Let α, β be infinite sequences of natural numbers. Define the proposition QED as follows:

QED := *there exist* i, j *such that* $i < j$ *and both* $\alpha(i) \leq \alpha(j)$ *and* $\beta(i) R \beta(j)$.

We prove first, by induction, as in the proof of Theorem 1.1:

For every n, for every m, either QED or there exists $i > m$ such that $\alpha(i) \geq n$.

We then build a strictly increasing sequence γ of natural numbers such that for every i, either QED or $\alpha\big(\gamma(i)\big) \leq \alpha\big(\gamma(i+1)\big)$. Now determine i, j such that $i < j$ and $\beta\big(\gamma(i)\big) R \beta\big(\gamma(j)\big)$. Observe: either QED or both $\alpha\big(\gamma(i)\big) \leq \alpha\big(\gamma(j)\big)$ and $\beta\big(\gamma(i)\big) R \beta\big(\gamma(j)\big)$, and therefore in any case QED. □

Observe that Theorem 1.1 easily follows from Theorem 2.1.

3. Some induction principles and Brouwer's Thesis

In Section 4 we want to show that the product of any two almost full relations on $\mathbb{N}$ is almost full. This statement, that we want to call *Ramsey's Theorem*, will be proved by transfinite induction. This paragraph is devoted to some principles of transfinite induction and to Brouwer's Thesis which allows us to use these principles in proving our results.

3.1. We first introduce the notion of a *stump*.

We have taken the word "stump" from Brouwer's paper [1] but are using it in a sense which is different from Brouwer's. Stumps are certain decidable subsets of the set $\mathbb{N}^*$ of finite sequences of natural numbers.

$*$ denotes the binary operation on $\mathbb{N}^*$ which consists in the concatenation of finite sequences.

If s belongs to $\mathbb{N}^*$ and A is a subset of $\mathbb{N}^*$ we let $s * A$ be the set of all finite sequences of the form $s * t$, where t belongs to A.

The set **Stp** of stumps is given by the following inductive definition:

(i) The empty set, $\emptyset$, is a stump. We sometimes call this set the *basic stump*.

(ii) If $\sigma_0, \sigma_1, \sigma_2, \ldots$ is an infinite sequence of stumps, then the set $\{\langle\ \rangle\} \cup \bigcup_{n \in \mathbb{N}} \langle n \rangle * \sigma_n$ is also a stump.

(iii) Every stump is obtained from the empty stump by repeated applications of the construction step mentioned in (ii).

The stumps $\sigma_0, \sigma_1, \sigma_2, \ldots$ are called the *immediate substumps* of the stump

$$\sigma := \{\langle\ \rangle\} \cup \bigcup_{n \in \mathbb{N}} \langle n \rangle * \sigma_n.$$

We may view a non-empty stump σ as an ω-sequence of stumps, that is, as a function from the set $\mathbb{N}$ of natural numbers to the set **Stp** of stumps associating to every natural number n the n-th immediate substump of σ. We therefore sometimes write $\sigma(n)$ for the set $\{s | s \in \mathbb{N}^* | \langle n \rangle * s \in \sigma\}$.

3.2. Once we accept the inductive definition of the set of stumps we will recognize the validity of the following *principle of induction on the set of stumps:*

Let A be a subset of the set **Stp** *of stumps.*
If every stump σ belongs to A as soon as every immediate substump of σ belongs to A, then every stump belongs to A.

We now mention some other principles of induction on stumps that follow from this first and basic one.

3.3. We first consider a *principle of induction on the set of ordered pairs of stumps*.

Let us call an ordered pair $\langle \sigma_0, \sigma_1 \rangle$ of stumps *more simple* than an ordered pair $\langle \tau_0, \tau_1 \rangle$ of stumps if *either* $\sigma_0 = \tau_0$ and σ_1 is an immediate substump of τ_1 *or* $\sigma_1 = \tau_1$ and σ_0 is an immediate substump of τ_0.

Let B be a subset of the set $\mathbf{Stp} \times \mathbf{Stp}$ *of ordered pairs of stumps.*
If every ordered pair $\langle \sigma_0, \sigma_1 \rangle$ of stumps belongs to B as soon as every ordered pair of stumps more simple than $\langle \sigma_0, \sigma_1 \rangle$ belongs to B, then every ordered pair of stumps belongs to B.

In order to prove this principle define, assuming that B is a subset of $\mathbf{Stp} \times \mathbf{Stp}$ satisfying the hypothesis, the set A of all stumps σ such that for every stump τ the ordered pair $\langle \sigma, \tau \rangle$ belongs to B, and show, using the principle of induction on the set of stumps, that A coincides with **Stp**.

3.4. Let m be a natural number. A finite sequence of length m will be called an m-sequence. Let us call an m-sequence $\langle \sigma_0, \sigma_1, \ldots, \sigma_{m-1} \rangle$ of stumps *easier than* an m-sequence $\langle \tau_0, \tau_1, \ldots, \tau_{m-1} \rangle$ of stumps, if there exists $i < m$ such that σ_i is an immediate substump of τ_i and, for all j such that $i < j < m$, $\sigma_j = \tau_j$. We now formulate, for each positive integer m, a *principle of induction on the set of m-sequences of stumps:*

Let m be a natural number and C be a subset of the set $\mathbf{Stp}^m$ *of m-sequences of stumps.*
If every m-sequence $\langle \sigma_0, \sigma_1, \ldots, \sigma_{m-1} \rangle$ of stumps belongs to C as soon as every m-sequence of stumps easier than $\langle \sigma_0, \sigma_1, \ldots, \sigma_{m-1} \rangle$ belongs to C, then every m-sequence of stumps belongs to C.

One proves the correctness of this sequence of principles by complete induction. In case $m = 1$, the principle coincides with the basic principle of induction on the set of stumps. Dealing with the case m, where $m > 1$, define, assuming that C is a subset of $\mathbf{Stp}^m$ satisfying the hypothesis, the set A of all stumps σ such that for every $(m-1)$-sequence $\langle \tau_0, \tau_1, \ldots, \tau_{m-2} \rangle$ of stumps the sequence $\langle \tau_0, \tau_1, \ldots, \tau_{m-2}, \sigma \rangle$ belongs to C, and show, using the principle of induction on the set of stumps and the principle of induction on the set of $(m-1)$-sequences of stumps, that A coincides with **Stp**.

3.5. We also introduce a *principle of induction on the set of finite sequences of stumps*.

Let us call a finite sequence $\langle \sigma_0, \sigma_1, \ldots, \sigma_{m-1} \rangle$ of stumps *more facile* than a finite sequence $\langle \tau_0, \tau_1, \ldots, \tau_{n-1} \rangle$ of stumps if either $m = n$ and $\langle \sigma_0, \sigma_1, \ldots, \sigma_{m-1} \rangle$ is easier than $\langle \tau_0, \tau_1, \ldots, \tau_{n-1} \rangle$, or $m > n$ and σ_{m-1} is an immediate substump of τ_{n-1}.

Let C be a subset of the set $\mathbf{Stp}^*$ *of finite sequences of stumps.*
If every finite sequence $\langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \rangle$ of stumps belongs to C as soon as every finite sequence of stumps more facile than $\langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \rangle$ belongs to C, then every finite sequence of stumps belongs to C.

In order to prove this principle, define, assuming that C is a subset of $\mathbf{Stp}^*$ satisfying the hypothesis, the set B of all stumps σ with the property that every non-empty finite sequence of stumps $\langle\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\rangle$ such that $\sigma_{n-1} = \sigma$ belongs to C, and show, using the series of principles of induction treated in section 3.4 and the principle of induction on the set of stumps, that B coincides with $\mathbf{Stp}$.

3.6. We find it useful to slightly reformulate the induction principles from Sections 3.4 and 3.5.

Let m be a nonzero natural number.

We call a stump σ *m-ary* if and only if $\sigma \neq \emptyset$ and for every $j > m$, $\sigma(j) = \emptyset$.

Let σ, τ be m-ary stumps. We say that σ is *easier* than τ if and only if there exists i such that $\sigma(i)$ is an immediate substump of $\tau(i)$ and for every $j > i$, $\sigma(j) = \tau(j)$.

3.6.1. Principle of induction on the set of m-ary stumps:

> *Let m be a positive natural number and let C be a collection of m-ary stumps.*
> *If every m-ary stump σ belongs to C as soon as every m-ary stump easier than σ belongs to C, then every m-ary stump belongs to C.*

One may prove this principle in the same way as the principle mentioned in Section 3.4.

For every stump σ one may decide if $\sigma = \emptyset$ or not. We call a stump σ *finitary* if σ is non-empty and has only finitely many non-empty immediate substumps.

For every finitary stump σ one may decide if $\sigma = \{\langle\,\rangle\}$ or not, that is, assuming that $\sigma \neq \emptyset$, if for every i, $\sigma(i) = \emptyset$, or not. A finitary stump σ will be called *nontrivial* if $\sigma \neq \{\langle\,\rangle\}$.

Let σ be a nontrivial finitary stump. There exists exactly one natural number i such that $\sigma(i) \neq \emptyset$ and for every $j > i$, $\sigma(j) = \emptyset$. We will call this number the *characteristic number* of σ, notation: $\mathrm{char}(\sigma)$.

Let σ, τ be nontrivial finitary stumps. We say that σ is *more facile* than τ if either $\mathrm{char}(\sigma) = \mathrm{char}(\tau)$ and σ is easier than τ or $\mathrm{char}(\sigma) > \mathrm{char}(\tau)$ and $\sigma\big(\mathrm{char}(\sigma)\big)$ is an immediate substump of $\tau\big(\mathrm{char}(\tau)\big)$.

3.6.2. Principle of induction on the set of finitary stumps:

> *Let C be a collection of the set of finitary stumps.*
> *If every nontrivial finitary stump σ belongs to C as soon as every nontrivial finitary stump more facile than σ belongs to C, then every nontrivial finitary stump belongs to C.*

One may prove this principle in the same way as the principle mentioned in Section 3.5.

Let σ be a finitary stump. The set of all natural numbers i such that $\sigma(i) \neq \emptyset$ will be called the *domain* of σ, notation: $\mathrm{Dom}(\sigma)$.

3.7. We call the following statement *Brouwer's Thesis*.

> *Let P be a subset of the set $\mathbb{N}^*$ of finite sequences of natural numbers. If for every infinite sequence α of natural numbers there exists n such that $\langle\alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$ belongs to P, then there exists a stump σ such that for every infinite sequence α of natural numbers there exists n such that $\langle\alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$ belongs to both σ and P.*

Brouwer came to this Thesis by reflecting on the possible structure of a proof of the statement "Every infinite sequence α has a finite initial part in the set P". The argument for his Thesis has been the subject of much debate in the foundations of intuitionistic mathematics. Our formulation of the Thesis is not literally to be found in Brouwer's writings, but seems to come close to his intentions.

4. Ramsey's Theorem

4.1. We want to consider at-most-binary relations rather than just binary relations.

Let A be a set. For each natural number n, A^n is the set of all n-sequences of elements of A. An *at-most-unary* relation on A is a subset of $\{\langle\ \rangle\} \cup A^1$. An *at-most-binary* relation on A is a subset of $\{\langle\rangle\} \cup A^1 \cup A^2$. Every at-most-unary relation A is also an at-most-binary relation on A.

4.2. We want to extend the notion "almost full" from binary to at-most-binary relations.

Let R be an at-most-binary relation on the set $\mathbb{N}$ of natural numbers.

Let $s = \langle s(0), s(1), \ldots, s(n-1)\rangle$ be a finite sequence of natural numbers. We say that s *meets* R if and only if some subsequence of s belongs to R, that is, either the empty sequence $\langle\rangle$ belongs to R or there exists $i < n$ such that $\langle s(i)\rangle$ belongs to R, or there exist $i, j < n$ such that $i < j$ and $\langle s(i), s(j)\rangle$ belongs to R.

Let α be an infinite sequence of natural numbers. We say that α *meets* R if and only if, for some n, the finite sequence $\langle\alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$ meets R. We say that R is *almost full* if and only if every infinite sequence α of natural numbers meets R.

4.3. Let R be an at-most-binary relation on $\mathbb{N}$, and let σ be a stump. We say that σ *secures* that R is almost full if and only if for every α in $\mathcal{N}$ there exists n such that $\langle\alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$ belongs to σ and meets R.

Brouwer's Thesis implies: for every at-most-binary relation R on the set $\mathbb{N}$ of natural numbers, if R is almost full, then there exists a stump σ that secures that R is almost full.

4.4. Let R, T be at-most-binary relations on $\mathbb{N}$.

We let $R \sqcap T$ be the set of all finite sequences s such that either s belongs to R and some initial part of s belongs to T or s belongs to T and some initial part of s belongs to R.

We call $R \sqcap T$ the *open-intersection* of R and T, for the following reason. With any at-most-binary relation R on $\mathbb{N}$ we may associate the open subset $R^\#$ of Baire space consisting of all infinite sequences α in $\mathcal{N}$ having an initial part in R. Observe that for all at-most-binary relations R, T on $\mathbb{N}$, $(R \sqcap T)^{\#} = R^{\#} \cap T^{\#}$.

The operation $\sqcap$ of open-intersection is easily seen to be idempotent, commutative and associative. For every finite sequence $R_0, R_1, \dots, R_n$ of at-most-binary relations on $\mathbb{N}$, we denote the open-intersection of $R_0, R_1, \dots, R_n$ by $\underset{k \le n}{\sqcap} R_k$.

4.5. Let R be an at-most-binary relation on $\mathbb{N}$.

For every n, we let R^n be the set of all finite sequences s such that $\langle n \rangle * s$ belongs to R.

Observe that R^n is an at-most-unary relation on $\mathbb{N}$.

Remark finally that for every stump σ, for every at-most-binary relation R on $\mathbb{N}$, if σ secures that R is almost full then, for every n, *either* $\sigma(n) = \emptyset$ and $\langle\,\rangle$ belongs to R *or* $\sigma(n)$ secures that the at-most-binary relation $R^n \cup R$ is almost full.

Theorem 4.1. *For every stump σ, for all at-most-binary relations R, T on $\mathbb{N}$, if σ secures that R is almost full, and if also T is almost full, then $R \sqcap T$ is almost full.*

Proof. We use induction on the set of stumps. We may assume that σ is a stump different from $\emptyset$ and that the statement of the Theorem has been verified for every immediate substump $\sigma(n)$ of σ. We also assume that R, T are almost full at-most-binary relations on $\mathbb{N}$ and that σ secures that R is almost full. Observe that, for each n, *either* $\sigma(n) = \emptyset$ and $\langle\,\rangle$ belongs to R and $R \sqcap T$ is almost full, *or* $\sigma(n)$ secures that the at-most-binary relation $R \cup R^n$ is almost full.

Using this fact repeatedly we conclude first that $(R \cup R^0) \sqcap T$ is almost full, then that $(R \cup R^1) \sqcap (R \cup R^0) \sqcap T$ is almost full, then that $(R \cup R^2) \sqcap (R \cup R^1) \sqcap (R \cup R^0) \sqcap T$ is almost full, and so on. Now let α be an infinite sequence of natural numbers.

We define the proposition QED by:

$$\text{QED} \quad := \quad \alpha \text{ meets } R \sqcap T.$$

We construct a sequence γ of natural numbers, by induction. We define $\gamma(0) := 0$.

Let n be a natural number and suppose we defined already the first $n+1$ values of γ, say $\gamma(0), \gamma(1), \dots, \gamma(n)$.

As $\underset{k \le n}{\sqcap} (R \cup R^{\alpha(\gamma(k))}) \sqcap T$ is almost full, we determine i, j such that $\gamma(n) < i < j$ and some initial part of $\langle \alpha(i), \alpha(j) \rangle$ belongs to $\underset{k \le n}{\sqcap} (R \cup R^{\alpha(\gamma(k))}) \sqcap T$.

Observe that some initial part of $\langle \alpha(i), \alpha(j) \rangle$ belongs to T. We now distinguish two cases. *Case (i):* Some initial part of $\langle \alpha(i), \alpha(j) \rangle$ belongs to R. Then QED. *Case (ii):* for every $k \le n$, some initial part of $\langle \alpha(i) \rangle$ belongs to $R^{\alpha(\gamma(k))}$, that is some initial part of, $\langle \alpha(\gamma(k)), \alpha(i) \rangle$ belongs to R.

We now define: $\gamma(n+1) := i$.

Observe that γ is a strictly increasing sequence of natural numbers and that for each i, j, if $i < j$, then *either* QED *or* some initial part of $\langle \alpha(\gamma(i)), \alpha(\gamma(j)) \rangle$ belongs to R. We now again use the fact that also T is almost full and determine i, j such that $i < j$ and some initial part of $\langle \alpha(\gamma(i)), \alpha(\gamma(j)) \rangle$ belongs to T.

Observe: either QED, or some initial part of $\langle \alpha(\gamma(i)), \alpha(\gamma(j)) \rangle$ belongs to $R \sqcap T$, and therefore, in any case, QED.

It follows that every infinite sequence α of natural numbers meets $R \sqcap T$, that is, $R \sqcap T$ is almost full. □

Corollary 4.2. (Intuitionistic Ramsey Theorem)
For all at-most-binary relations R, T on $\mathbb{N}$, if both R and T are almost full, then $R \sqcap T$ is almost full.

Proof. Use Brouwer's Thesis and apply Theorem 4.1. □

4.6. We now explain why Corollary 4.2 is called the Intuitionistic Ramsey Theorem.

The *classical* (*infinite*) *Ramsey Theorem*, see [15], reads as follows:

> *For every binary relation R on $\mathbb{N}$ there exists a strictly increasing sequence γ of natural numbers such that* either *for all i, j, if $i < j$, then $\gamma(i) R \gamma(j)$,* (*the sequence γ then is called R*-homogeneous) or *for all i, j, if $i < j$, then not $\gamma(i) R \gamma(j)$* (*the sequence γ then is called* $\mathbb{N} \times \mathbb{N} \backslash R$-homogeneous).

The following example shows that this statement is false constructively.

> *Let R be a binary relation on $\mathbb{N}$ such that for all m, n, mRn if and only if there exists $i \leq \max(m, n)$ such that for all $k < 99$, $d(i + k) = 9$.*

If there exists an R-homogeneous strictly increasing sequence γ, then there exists i such that for every $k < 99$, $d(i+k) = 9$, and if there exists a $\mathbb{N} \times \mathbb{N} \backslash R$-homogeneous strictly increasing sequence γ, then there exists no such i.

We have no proof of either alternative, and can not prove that there exists a strictly increasing sequence of natural numbers that is either R-homogeneous or $\mathbb{N} \times \mathbb{N} \backslash R$-homogeneous.

Using classical logic, we may reformulate the classical Ramsey Theorem as follows:

> *For every binary relation R on $\mathbb{N}$, it is impossible that both R and $\mathbb{N} \times \mathbb{N} \backslash R$ are almost full.*

Observe that we may draw this conclusion, also constructively, from the Intuitionistic Ramsey Theorem.

Conversely, one may prove the Intuitionistic Ramsey Theorem from the classical one, using classical logic, as follows.

Suppose R, T are almost full at-most-binary relations on $\mathbb{N}$. Let α be an infinite sequence of natural numbers. Determine a strictly increasing sequence γ of natural numbers such that for all i, j if $i < j$, then some initial part of $\langle \alpha(\gamma(i)), \alpha(\gamma(j)) \rangle$ belongs to R. Now determine i, j such that $i < j$ and some initial part of $\langle \alpha(\gamma(i)), \alpha(\gamma(j)) \rangle$ belongs to T. It is clear that α meets $R \sqcap T$.

We may conclude that $R \sqcap T$ is almost full.

4.7. We want to give a second proof of the Intuitionistic Ramsey Theorem. This second proof is in two steps. We first prove a result on at-most-unary relations on $\mathbb{N}$.

Theorem 4.3. *For all stumps σ, τ, for all at-most-unary relations A, B on $\mathbb{N}$, if σ secures that A is almost full and τ secures that B is almost full, then $A \sqcap B$ is almost full.*

Proof. We use the principle of induction on the set of ordered pairs of stumps introduced in Section 3.3.

Assume that $\langle\sigma_0, \sigma_1\rangle$ is an ordered pair of stumps and that the statement of the Theorem has been verified for every pair $\langle\tau_0, \tau_1\rangle$ of stumps that is more simple than $\langle\sigma_0, \sigma_1\rangle$, that is, either $\tau_0 = \sigma_0$ and τ_1 is an immediate substump of σ_1, or $\tau_1 = \sigma_1$ and τ_0 is an immediate substump of σ_0. Let A, B be at-most-unary relations on $\mathbb{N}$ such that σ_0 secures that A is almost full and σ_1 secures that B is almost full. We now prove that $A \sqcap B$ is almost full. Let α be an infinite sequence of natural numbers. Define the proposition QED as follows:

$$\text{QED} := \alpha \text{ meets } A \sqcap B.$$

Consider $\alpha(0)$. Observe that *either* $\sigma_0\big(\alpha(0)\big) = \emptyset$ and $\langle\,\rangle$ belongs to A and $A \sqcap B$ is almost full, *or* $\sigma_0\big(\alpha(0)\big)$ secures that $A^{\alpha(0)} \cup A$ is almost full. Recall that σ_1 secures that B is almost full. Consider $\alpha \circ S$, the composition of the sequence α and the successor function S. Observe that $\alpha \circ S$ meets $(A^{\alpha(0)} \cup A) \sqcap B$. Therefore, either QED or $\langle\,\rangle$ belongs to $A^{\alpha(0)}$, that is, $\langle\alpha(0)\rangle$ belongs to A. Observe that σ_0 secures that A is almost full and that *either* $\sigma_1\big(\alpha(0)\big) = \emptyset$ and $\langle\,\rangle$ belongs to B and $A \sqcap B$ is almost full, *or* $\sigma_1\big(\alpha(0)\big)$ secures that $B^{\alpha(0)} \cup B$ is almost full. Therefore $\alpha \circ S$ meets $A \sqcap (B^{\alpha(0)} \cup B)$. Therefore either QED or $\langle\,\rangle$ belongs to $B^{\alpha(0)}$, that is, $\langle\alpha(0)\rangle$ belongs to B. Combining our conclusions, we find either QED or $\langle\alpha(0)\rangle$ belongs to $A \sqcap B$.

So in any case QED.

We conclude that every infinite sequence α meets $A \sqcap B$, that is, $A \sqcap B$ is almost full. □

Theorem 4.4. *For all stumps σ_0, σ_1, for all at-most-binary relations R, T on $\mathbb{N}$, if σ_0 secures that R is almost full and σ_1 secures that T is almost full, then $R \sqcap T$ is almost full.*

Proof. We again use the principle of induction on the set ordered pairs of of stumps introduced in Section 3.3.

Assume that σ_0, σ_1 are stumps and that the statement of the Theorem has been verified for every pair of stumps more simple than $\langle\sigma_0, \sigma_1\rangle$. Let R, T be at-most-binary relations on $\mathbb{N}$ such that σ_0 secures that R is almost full and σ_1 secures that T is almost full. We now prove that $R \sqcap T$ is almost full. Let α be an infinite sequence of natural numbers. Define the proposition QED as follows:

$$\text{QED} := \alpha \text{ meets } R \sqcap T.$$

Observe that *either* $\sigma_0\big(\alpha(0)\big) = \emptyset$ and $\langle\,\rangle$ belongs to R and $R \sqcap T$ is almost full *or* $\sigma\big(\alpha(0)\big)$ secures that $R^{\alpha(0)} \cup R$ is almost full. Recall that σ_1 secures that T is almost full.

So for every infinite sequence β of natural numbers there exist i, j such $i < j$ and some initial part of $\langle\alpha\big(\beta(i)\big), \alpha\big(\beta(j)\big)\rangle$ belongs to $R^{\alpha(0)} \cup R$ and some initial part of $\langle\alpha\big(\beta(i)\big), \alpha\big(\beta(j)\big)\rangle$ belongs to T. Spelling out the various possibilities we find: either QED or $\langle\alpha(0)\rangle$ belongs to R or $\langle\alpha(0), \alpha\big(\beta(i)\big)\rangle$ belongs to R. Similarly, using the fact that σ_0 secures that R is almost full and that *either* $\sigma_1\big(\alpha(0)\big) = \emptyset$ *or* $\sigma_1\big(\alpha(0)\big)$ secures that $T^{\alpha(0)} \cup T$ is almost full, we find that for every infinite sequence β of natural numbers there exists i such that either QED or $\langle\alpha(0)\rangle$ belongs to T or $\langle\alpha(0), \alpha\big(\beta(i)\big)\rangle$ belongs to T.

Using the previous Theorem we find i such that either QED or both an initial part of $\langle\alpha(0), \alpha(i)\rangle$ belongs to R and an initial part of $\langle\alpha(0), \alpha(i)\rangle$ belongs to T, therefore again QED.

We conclude that every infinite sequence α meets $R \sqcap T$, that is, $R \sqcap T$ is almost full. □

4.8. The method of proof of Theorem 4.4 is more powerful than the method of proof of Theorem 4.1. We may use a similar double induction to obtain the corresponding result for at-most-ternary relations. It seems impossible to prove this by an argument in the style of the proof of Theorem 4.1. One may go on and prove the result for at-most-n-ary relations,where n is a natural number. The strongest result in this direction is the so-called Clopen Ramsey Theorem. We do not go into details, as we will not make use of this Theorem when dealing with the main subject of this paper.

Corollary 4.5. (*i*) *Let A, B be subsets of $\mathbb{N}$ such that every strictly increasing sequence of natural numbers meets both A and B.*
Then every strictly increasing sequence of natural numbers meets $A \cap B$.
(*ii*) *Let R, T be binary relations on $\mathbb{N}$ such that every strictly increasing sequence of natural numbers meets both R and T. Then every strictly increasing sequence of natural numbers meets $R \cap T$.*

Proof. (i) Define at-most-binary relations A', B' on $\mathbb{N}$ by: $A' := \{\langle m, n\rangle | m \in A \text{ or } m = n\}$ and $B' := \{\langle m, n\rangle | m \in B \text{ or } m = n\}$.
Observe that both A' and B' are almost full.
(We show that A' is almost full. Let α be an infinite sequence of natural numbers. Define an infinite sequence α' of natural numbers, as follows: $\alpha'(0) := \alpha(0)$ and for each $n > 0$, $\alpha'(n) := \alpha(n)$ if there does not exist $i < n$ such that $\alpha'(i) = \alpha(n)$, and $\alpha'(n) := \operatorname{Max}_{i<n}(\alpha'(i) + 1)$ otherwise. Observe that α' is one-to-one, so it has a strictly increasing subsequence and will meet A, say $\alpha'(i)$ belongs to A. If $\alpha'(i) = \alpha(i)$, then α meets A and therefore also A', if $\alpha'(i) \neq \alpha(i)$, then α meets $=$ and therefore also A'.)
From Corollary 4.2, the Intuitionistic Ramsey Theorem, we conclude that $A' \sqcap B'$ is almost full.
So every strictly increasing sequence meets $A' \sqcap B'$, and, as it does not meet $=$, it will meet $A \cap B$.
(ii) We leave the proof of part (ii) to the reader. □

4.9. The second statement of Corollary 4.5 is called the Intuitionistic Ramsey Theorem in [25].

Corollary 4.6. *Let R, T be binary relations on $\mathbb{N}$. If both R, T are almost full on $\mathbb{N}$, then $R \times T$ is almost full on $\mathbb{N} \times \mathbb{N}$.*

Proof. Let $J : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be some pairing function, so let $K, L : \mathbb{N} \to \mathbb{N}$ be such that, for each n, $J\big(K(n), L(n)\big) = n$. Now define binary relations R' and T'' on $\mathbb{N}$ by $R' := \{\langle m, n\rangle | \langle K(m), K(n)\rangle$ belongs to $R\}$ and $T'' := \{\langle m, n\rangle | \langle L(m), L(n)\rangle$ belongs to $T\}$.

Observe that R' and T'' are almost full on $\mathbb{N}$, therefore $R' \cap T''$ is almost full on $\mathbb{N}$, and also $R \times T$ is almost full on $\mathbb{N} \times \mathbb{N}$. □

4.10. Let A, B be sets. The *sum* or *disjoint union* of the sets A, B is the set $A \times \{0\} \cup B \times \{1\}$, notation $A \biguplus B$ or $A + B$. Let $R \subseteq A \times A$ and $T \subseteq B \times B$ be binary relations on A, B, respectively. We define a binary relation $R + T$, called the *sum* of the relations R and T as follows: for all $\langle c_0, i_0\rangle, \langle c_1, i_1\rangle$ in $A + B$, $\langle c_0, i_0\rangle(R + T)\langle c_1, i_1\rangle$ if and only if either $i_0 = i_1 = 0$ and c_0Rc_1 or $i_0 = i_1 = 1$ and c_0Tc_1.

Corollary 4.7. *Let R, T be binary relations on $\mathbb{N}$. If both R, T are almost full on $\mathbb{N}$, then $R + T$ is almost full on $\mathbb{N} \biguplus \mathbb{N}$.*

Proof. Observe that $=$ is almost full on $\{0, 1\}$, therefore, by Corollary 4.6, $R \times T \times =$ is almost full on $\mathbb{N} \times \mathbb{N} \times \{0, 1\}$. Therefore, for every sequence $\alpha : \mathbb{N} \to \mathbb{N}$ and every sequence $\beta : \mathbb{N} \to \{0, 1\}$ there exist i, j such that $\alpha(i)R\alpha(j)$ and $\alpha(i)T\alpha(j)$ and $\beta(i) = \beta(j)$, that is: $R + T$ is almost full on $\mathbb{N} + \mathbb{N}$. □

5. The Finite Sequence Theorem

5.1. We consider the set $\mathbb{N}^*$ of all finite sequences of natural numbers. $*$ denotes the binary operation of concatenation of finite sequences. For every nonempty element s of $\mathbb{N}^*$ there exist a natural number $s(0)$ and a finite sequence of natural numbers $\mathrm{Rem}(s)$ ("the remainder of s") such that $s = \langle s(0)\rangle * \mathrm{Rem}(s)$. We now define a binary relation $\leq^*$ on $\mathbb{N}^*$ as follows:

For all s, t in $\mathbb{N}^*$:
$s \leq^* t$ if and only if either $s = \langle\,\rangle$ or both s and t are non-empty and
either $s \leq^*$ *Rem*(t) or $s(0) \leq t(0)$ *and Rem*$(s) \leq^*$ *Rem*(t).

This definition may be thought of as a definition by recursion on length(t).

It is useful to think of a finite sequence $s = \langle s(0), s(1), \ldots, s(n-1)\rangle$ as a function with domain $n = \{0, 1, \ldots, n-1\}$. We will write: $\mathrm{Dom}(s) = n$. So the domain of a finite sequence is the same as its length.

One may verify without difficulty, that for all s, t in $\mathbb{N}^*$, $s \leq^* t$ if and only if there exists a strictly increasing function h from $\mathrm{Dom}(s)$ to $\mathrm{Dom}(t)$ such that for all i in $\mathrm{Dom}(s) : s(i) \leq t\big(h(i)\big)$.

For example: $\langle 1, 3, 5\rangle \leq^* \langle 0, 0, 2, 1, 8, 3, 3, 6, 1\rangle$.

We want to prove that $\leq^*$ is almost full on $\mathbb{N}^*$. To this end, we define, for each n, k, an element $\langle n\rangle^k$ of $\mathbb{N}^*$ as follows: $\langle n\rangle^0 := \langle\,\rangle$ and for each k, $\langle n\rangle^{k+1} := \langle n\rangle * \langle n\rangle^k$. So $\langle n\rangle^k$ is the finite sequence of length k with the constant value n. Remark that for all s_0, t_0, s_1, t_1 in $\mathbb{N}^*$: If $s_0 \leq^* s_1$ and $t_0 \leq^* t_1$, then $s_0 * t_0 \leq^* s_1 * t_1$.

Theorem 5.1. *For every infinite sequence α of finite sequences of natural numbers there exist i, j such that $i < j$ and $\alpha(i) \leq^* \alpha(j)$.*

Proof. For all natural numbers n, k we define the proposition $P(n, k)$ as follows:

$P(n, k) :=$ For every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist i, j such that $i < j$ and either $\alpha(i) \leq^* \alpha(j)$ or $\langle n \rangle^{k+1} \leq^* \alpha(i)$.

We want to prove: for all n, k, $P(n, k)$, and do so by double induction.

Observe that $P(0, 0)$ is true.

We now show:

$$\boxed{\text{For all } n, k, \text{ if } P(n, k), \text{ then } P(n, k+1).}$$

So let n, k be natural numbers and assume $P(n, k)$ and let $\alpha : \mathbb{N} \to \mathbb{N}^*$. We construct two functions β_0, β_1 from $\mathbb{N}$ to $\mathbb{N}^*$ and a function γ from $\mathbb{N}$ to $\mathbb{N}$ such that for every i, if $\alpha(i) \neq \langle\rangle$, then $\alpha(i) = \beta_0(i) * \langle\gamma(i)\rangle * \beta_1(i)$ and, if not $\langle n \rangle^{k+1} \leq^* \alpha(i)$, then $\beta_1(i) = \langle\ \rangle$, and, if $\langle n \rangle^{k+1} \leq^* \alpha(i)$, then not $\langle n \rangle^{k+1} \leq^* \beta_0(i)$ but $\langle n \rangle^k \leq^* \beta_0(i)$ and $n \leq \gamma(i)$, and, if $\alpha(i) = \langle\rangle$, then $\beta_0(i) = \beta_1(i) = \langle\rangle$.

Now observe: for every i, not $\langle n \rangle^{k+1} \leq^* \beta_0(i)$, and therefore, by the assumption $P(n, k)$, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist i, j such that $i < j$ and $\beta_0\big(\delta(i)\big) \leq^* \beta_0\big(\delta(j)\big)$.

Also, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist i, j such that $i < j$ and $\gamma\big(\delta(i)\big) \leq \gamma\big(\delta(j)\big)$.

Finally, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist i, j such that $i < j$ and either $\beta_1\big(\delta(i)\big) \leq^* \beta_1\big(\delta(j)\big)$ or $\langle n \rangle \leq^* \beta_1\big(\delta(i)\big)$. (This follows from the assumption $P(n, k)$.)

Applying Ramsey's Theorem, we calculate i, j such that $i < j$ and simultaneously $\beta_0(i) \leq^* \beta_0(j)$ and $\gamma(i) \leq \gamma(j)$ and either $\beta_1(i) \leq^* \beta_1(j)$ or $\langle n \rangle \leq^* \beta_1(i)$, and therefore *either* $\alpha(i) = \langle\rangle$ or $\alpha(j) = \langle\rangle$, and α meets $\leq^*$, *or* $\alpha(i) \leq^* \alpha(j)$ or $\langle n \rangle \leq^* \beta_1(i)$. Suppose $\langle n \rangle \leq^* \beta_1(i)$, then reconsidering our construction, we find $\langle n \rangle^k \leq^* \beta_0(i)$ and $n \leq \gamma(i)$, therefore $\langle n \rangle^{k+2} \leq^* \alpha(i)$.

We conclude: for every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist i, j such that $i < j$ and either $\alpha(i) \leq^* \alpha(j)$ or $\langle n \rangle^{k+2} \leq^* \alpha(i)$, that is $P(n, k+1)$.

We now want to prove:

$$\boxed{\text{For all } n, \text{ if for all } k,\ P(n, k), \text{ then } P(n+1, 0).}$$

So let n be a natural number and assume: for every k, $P(n, k)$. We want to prove: $P(n+1, 0)$.

For every finite sequence of natural numbers s we define a finite sequence s' of natural numbers of the same length as s such that for every $j < \text{length}(s)$, $s'(j) = \min(s(j), n)$.

Let $\alpha : \mathbb{N} \to \mathbb{N}^*$. Let $\alpha' : \mathbb{N} \to \mathbb{N}^*$ be such that for all j, $\alpha'(j) = (\alpha(j))'$. Calculate $k = \text{length}\big(\alpha(0)\big)$. We may assume $k > 0$. Applying $P(n, k-1)$ we find i, j such that $0 < i < j$ and either $\alpha'(i) \leq^* \alpha'(j)$ or $\langle n \rangle^k \leq^* \alpha'(i)$.

Now observe: if $\alpha'(i) \neq \alpha(i)$ or $\alpha'(j) \neq \alpha(j)$, then $\langle n+1 \rangle \leq^* \alpha(i)$ or $\langle n+1 \rangle \leq^* \alpha(j)$.

On the other hand, if both $\alpha'(i) = \alpha(i)$ and $\alpha'(j) = \alpha(j)$ then either $\alpha(i) \leq^* \alpha(j)$ or $\langle n \rangle^k \leq^* \alpha(i)$. If $\langle n \rangle^k \leq^* \alpha(i)$, then also $\alpha'(0) \leq^* \alpha(i)$, as $\alpha'(0) \leq^* \langle n \rangle^k$, and if $\alpha(0)) \neq \alpha'(0)$, then $\langle n+1 \rangle \leq^* \alpha(0)$.

We conclude: for every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist i, j such that $i < j$ and either $\alpha(i) \leq^* \alpha(j)$ or $\langle n+1 \rangle \leq^* \alpha(i)$, that is, $P(n+1, 0)$.

Clearly then, for all n, k, $P(n, k)$. Now let $\alpha : \mathbb{N} \to \mathbb{N}^*$. Calculate $k :=$ length$\big(\alpha(0)\big)$ and $n := \max\{\big(\alpha(0)\big)(j) \mid j < k\}$. Applying $P(n, k)$ we find i, j such that $0 < i < j$ and either $\langle n \rangle^k \leq^* \alpha(i)$ or $\alpha(i) \leq^* \alpha(j)$, and therefore either $\alpha(0) \leq^* \alpha(i)$ or $\alpha(i) \leq^* \alpha(j)$.

Therefore, every infinite sequence of finite sequences of natural numbers meets $\leq^*$, that is, $\leq^*$ is almost full on $\mathbb{N}^*$. □

5.2. We intend to generalize Theorem 5.1.

Let R be an at-most-binary relation on the set $\mathbb{N}$ of natural numbers. We now define a binary relation R^* on the set $\mathbb{N}^*$ of finite sequences of natural numbers, as follows.

> For all s, t in $\mathbb{N}^*$:
> sR^*t if and only if either $s = \langle\,\rangle$ or both s and t are non-empty and *either* $sR^*\mathrm{Rem}(t)$ *or* one of the three sequences $\langle\,\rangle$, $\langle s(0)\rangle$ and $\langle s(0), t(0)\rangle$ belongs to R and $\mathrm{Rem}(s)R^*\mathrm{Rem}(t)$.

This definition may be thought of as a definition by recursion on length(t). One may verify without difficulty that, for all s, t in $\mathbb{N}^*$, sR^*t if and only if there exists a strictly increasing function h from Dom(s) to Dom(t) such that for all i in Dom(s), one of the three sequences $\langle\,\rangle$, $\langle s(i)\rangle$ and $\langle s(i), t\big(h(i)\big)\rangle$ belongs to R.

We want to prove: for every at-most-binary relation R on $\mathbb{N}$, if R is almost full on $\mathbb{N}$, then R^* is almost full on $\mathbb{N}^*$.

We first consider the case that R is decidable, that is, we may decide, for every finite sequence of length at most 2, if s belongs to R or not.

Theorem 5.2. *For every stump σ, for every at-most-binary decidable relation R on $\mathbb{N}$, if σ secures that R is almost full on $\mathbb{N}$, then R^* is almost full on $\mathbb{N}^*$.*

Proof. We use induction on the set of stumps. The statement of the Theorem is obviously true if $\sigma = \emptyset$ as there is no relation R such that the empty stump secures that R is almost full.

Let us assume that σ is a non-basic stump and that the statement of the Theorem has been proved for every one of its immediate substumps $\sigma(n)$.

Let R be a decidable at-most-binary relation on $\mathbb{N}$ such that σ secures that R is almost full.

Observe that for every n, *either* $\sigma(n) = \emptyset$, and therefore $\langle\,\rangle$ belongs to R, and R^* is almost full, *or* $\sigma(n)$ secures that $R \cup R^n$ is almost full, and we may assume, by the induction hypothesis, that $(R \cup R^n)^*$ is almost full on $\mathbb{N}^*$.

For every finite sequence s of natural numbers we define the proposition $P(s)$ as follows:

$$P(s) := \text{For every } \alpha : \mathbb{N} \to \mathbb{N}^* \text{ there exist } i, j \text{ such that } i < j \text{ and either } \alpha(i)R^*\alpha(j) \text{ or } sR^*\alpha(i).$$

We want to prove: for every finite sequence s of natural numbers, $P(s)$, and do so by induction on length(s).

Observe that $P(\langle\,\rangle)$ is trivially true.

Observe also that, for every n, the proposition $P(\langle n\rangle)$ is equivalent to the statement that $(R \cup R^n)^*$ is almost full on $\mathbb{N}^*$, and therefore true by the induction hypothesis.

Now assume that s is a finite sequence of length at least 2 and that we proved $P\big(\mathrm{Rem}(s)\big)$.

We want to prove $P(s)$. So assume $\alpha : \mathbb{N} \to \mathbb{N}^*$.

We construct two functions β_0, β_1 from $\mathbb{N}$ to $\mathbb{N}^*$ and a function γ from $\mathbb{N}$ to $\mathbb{N}$ such that for every i, if $\alpha(i) \neq \langle\ \rangle$, then $\alpha(i) = \beta_0(i) * \langle\gamma(i)\rangle * \beta_1(i)$ and not $\langle s(0)\rangle R^*\beta_0(i)$, but if $\langle s(0)\rangle R^*\alpha(i)$, then some initial part of the sequence $\langle s(0), \gamma(i)\rangle$ belongs to R and, if $\alpha(i) = \langle\ \rangle$, then $\beta_0(i) = \beta_1(i) = \langle\ \rangle$. Also, for every i, if not $\langle s(0)\rangle R^*\alpha(i)$, then $\beta_1(i) = \langle\,\rangle$.

Observe that for every i, not $\langle s(0)\rangle R^*\beta_0(i)$, and by the induction hypothesis, $(R \cup R^{s(0)})^*$ is almost full on $\mathbb{N}^*$, therefore, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist i, j such $i < j$ and $\beta_0\delta(i)R^*\beta_0\delta(j)$. Also, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist i, j such that $i < j$ and some initial part of $\langle\gamma\big(\delta(i)\big), \gamma\big(\delta(j)\big)\rangle$ belongs to R. Finally, as we are assuming $P\big(\mathrm{Rem}(s)\big)$, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist i, j such that $i < j$ and either $\beta_1\big(\delta(i)\big)R^*\beta_1\big(\delta(j)\big)$ or $\mathrm{Rem}(s)R^*\beta_1\big(\delta(i)\big)$.

Applying Ramsey's Theorem we calculate i, j such that $i < j$ and simultaneously $\beta_0(i)R^*\beta_0(j)$ and some initial part of $\langle\gamma(i), \gamma(j)\rangle$ belongs to R and either $\beta_1(i)R^*\beta_1(j)$ or $\mathrm{Rem}(s)R^*\beta_1(i)$, therefore *either* $\alpha(i) = \langle\ \rangle$ or $\alpha(j) = \langle\ \rangle$ and α meets R^* *or* $\alpha(i)R^*\alpha(j)$ or $\mathrm{Rem}(s)R^*\beta_1(i)$.

Suppose $\mathrm{Rem}(s)R^*\beta_1(i)$. Reconsidering our construction we see that $\beta_1(i) \neq \langle\,\rangle$, therefore some initial part of $\langle s(0), \gamma(i)\rangle$ belongs to R, so $sR^*\alpha(i)$.

We conclude: for every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist i, j such that $i < j$ and either $\alpha(i)R^*\alpha(j)$ or $sR^*\alpha(i)$, that is: $P(s)$. Clearly then, for every finite sequence s of natural numbers, $P(s)$.

Let $\alpha : \mathbb{N} \to \mathbb{N}^*$. $P\big(\alpha(0)\big)$ implies that α meets R^*.

Therefore, every infinite sequence of finite sequences of natural numbers meets R^*, that is, R^* is almost full on $\mathbb{N}^*$. □

5.3. Theorem 5.1 may easily be derived from Theorem 5.2.

Also the idea at work in the proof of Theorem 5.1 is related to the leading idea in the proof of Theorem 5.2. In the case of Theorem 5.1 think of the stump consisting of all finite sequences of natural numbers that just meet the relation $\leq$. (A finite sequence of natural numbers *just meets* a relation R if it meets the relation R while none of its proper initial parts meets the relation R.) Observe that the finite sequences in the n-th immediate substump of this stump have length at most $n+2$. Proving for every k, $P(n, k)$ in the proof of Theorem 5.1 corresponds to showing that this n-th immediate substump has the property mentioned in Theorem 5.2.

5.4. We want to get rid of the assumption that R is a *decidable* at-most-binary relation on $\mathbb{N}$ in Theorem 5.2. Observe that we managed to prove Theorem 4.3 and its Corollary, Theorem 4.4, the Intuitionistic Ramsey Theorem, without making such an assumption. Our main tool in achieving our goal will be the Fan Theorem. The Fan Theorem probably is the best-known consequence of Brouwer's Thesis.

5.4.1. Let δ be an infinite sequence of natural numbers.

We let F_δ, the *fan determined by* δ, be the collection of all infinite sequences γ of natural numbers such that, for every i, $\gamma(i) \leq \delta(i)$. We let K_δ be the set of all initial parts of members of F_δ, that is, K_δ is the set of all finite sequences c of natural numbers such that for every $i < \text{length}(c)$, $c(i) \leq \delta(i)$. We also define a mapping G_δ of the set $\mathcal{N}$ of all infinite sequences of natural numbers into the set F_δ, as follows: for every α, for every n, $\big(G_\delta(\alpha)\big)(i) = \min\big(\alpha(i), \delta(i)\big)$. Observe that for every γ in F_δ, $G_\delta(\gamma) = \gamma$. G_δ is called a *retraction* of $\mathcal{N}$ onto F_δ. For every α in $\mathcal{N}$, n in $\mathbb{N}$, we define: $\overline{\alpha}n := \langle \alpha(0), \alpha(1), \ldots \alpha(n-1)\rangle$.

Lemma 5.41. *For every stump σ, for every infinite sequence δ of natural numbers, the set $\sigma \cap K_\delta$ is a finite set of finite sequences of natural numbers.*

Proof. We use induction on the set of stumps. Observe that the statement of the Lemma is obviously true in case $\sigma = \emptyset$. Assume that σ is not the basic stump and that the statement of the Theorem holds for every immediate substump of σ.

Let δ be an infinite sequence of natural numbers.

Observe that $\sigma \cap K_\delta = \{\langle\,\rangle\} \cup \bigcup_{i \leq \delta(0)} \langle i \rangle * (\sigma(i) \cap K_{\delta \circ S})$, therefore $\sigma \cap K_\delta$ is a finite subset of $\mathbb{N}^*$. □

Theorem 5.4.2. *Let δ be an infinite sequence of natural numbers.*

Let P be a subset of $\mathbb{N}^*$ such that every γ in F_δ has an initial segment in P.

There exists a finite subset Q of P such that every γ in F_δ has an initial segment in Q.

Proof. Assume that every γ in F_δ has an initial segment in P. Let G_δ be the retraction of $\mathcal{N}$ onto F_δ as defined in Section 5.4.1. Observe that for every γ there exists n such that $\overline{G_\delta(\gamma)}n$ belongs to P, therefore either $\overline{G_\delta(\gamma)}n \neq \overline{\gamma}n$ (and therefore: $\overline{\gamma}n \notin K_\delta$), or $\overline{\gamma}n$ belongs to P. Using Brouwer's Thesis, determine a stump σ such that for every γ there exists n such that $\overline{\gamma}n$ belongs to P or to $\mathbb{N}^* \backslash K_\delta$, and $\overline{\gamma}n$ belongs to σ. Consider the finite set $\sigma \cap K_\delta$ and determine for every finite sequence in this set of maximal length an initial segment in P. Let Q consist of all initial segments obtained in this way. □

Theorem 5.4.2 is a restricted version of the Fan Theorem. This version suffices for the results of this Section. We need a more general version in the later Sections of the paper.

Let δ be a function from $\mathbb{N}^*$ to $\mathbb{N}^*$ such that, for each s in $\mathbb{N}^*$, $\delta(s)$ is a non-empty finite sequence of natural numbers. We let L_δ, *the fan dictated by* δ, be the set of all α in $\mathcal{N}$ such that for each n there exists $i < \text{Dom}(\delta(\overline{\alpha}n))$ such that $\alpha(n) = (\delta(\overline{\alpha}n))(i)$.

Theorem 5.4.3 (Fan Theorem). *Let δ be a function from $\mathbb{N}^*$ to $\mathbb{N}^* \backslash \{\langle\,\rangle\}$.*

Let P be a subset of $\mathbb{N}^*$ such that every γ in L_δ has an initial segment in P.

There exists a finite subset Q of P such that every γ in L_δ has an initial segment in Q.

Proof. The proof is similar to the proof of Theorem 5.4.2 and left to the reader. □

5.5. For every non-empty t in $\mathbb{N}^*$ and every c in $\mathbb{N}$, $c < \text{length}(t)$, we let $A_0(t,c)$ and $A_1(t,c)$ be the elements of $\mathbb{N}^*$ such that $t = A_0(t,c) * \langle t(c)\rangle * A_1(t,c)$ and $A_0(t,c)$ has length c.

Theorem 5.5.1. *For every stump σ, for every at-most-binary relation R on $\mathbb{N}$, if σ secures that R is almost full on $\mathbb{N}$, then R^* is almost full on $\mathbb{N}^*$.*

Proof. We use induction on the set of stumps. If $\sigma = \emptyset$, then the statement of the Theorem is obviously true. Let us assume that σ is a non-basic stump and that the statement of the Theorem has been proved for every one of its immediate substumps.

Let R be an at-most-binary relation on $\mathbb{N}$ such that σ secures that R is almost full.

For every finite sequence s of natural numbers we define the proposition $P(s)$ as follows:

$$P(s) := \text{For every } \alpha : \mathbb{N} \to \mathbb{N}^* \text{ there exist } i, j \text{ such that } i < j \text{ and either } \alpha(i) R^* \alpha(j) \text{ or } s R^* \alpha(i).$$

We want to prove: for every finite sequence s of natural numbers, $P(s)$, and do so by induction on length(s).

Observe that $P(\langle\,\rangle)$ is trivially true.

Observe also that, for every n, the proposition $P(\langle n\rangle)$ is equivalent to the statement that $(R \cup R^n)^*$ is almost full on $\mathbb{N}^*$, and therefore true by the induction hypothesis.

Now assume that s is a finite sequence of natural numbers of length at least 2 and that we proved $P\big(\text{Rem}(s)\big)$.

We want to prove $P(s)$. So assume $\alpha : \mathbb{N} \to \mathbb{N}^*$. We define $\delta : \mathbb{N} \to \mathbb{N}$ as follows.

For each i, $\delta(i) := \text{length}\big(\alpha(i)\big) - 1$ if $\alpha(i) \neq \langle\,\rangle$ and $\delta(i) := 0$ if $\alpha(i) = \langle\,\rangle$. For every γ in the fan F_δ we define functions $B_0(\alpha,\gamma)$ and $B_1(\alpha,\gamma)$ from $\mathbb{N}$ to $\mathbb{N}^*$ and a function $m(\alpha,\gamma)$ from $\mathbb{N}$ to $\mathbb{N}$ such that for every i, if $\alpha(i) \neq \langle\,\rangle$, then $(B_0\big(\alpha,\gamma)\big)(i) := A_0\big(\alpha(i),\gamma(i)\big)$ and $\big(m(\alpha,\gamma)\big)(i) := \big(\alpha(i)\big)\big(\gamma(i)\big)$ and $(B_1\big(\alpha,\gamma)\big)(i) := A_1\big(\alpha(i),\gamma(i)\big)$, and therefore $\alpha(i) = \big(B_0(\alpha,\gamma)\big)(i) * \langle\big(m(\alpha,\gamma)\big)(i)\rangle * \big(B_1(\alpha,\gamma)\big)(i)$, and if $\alpha(i) = \langle\,\rangle$, then $\big(B_0(\alpha,\gamma)\big)(i) = \big(B_1(\alpha,\gamma)\big)(i) = \langle\,\rangle$ and $\big(m(\alpha,\gamma)\big)(i) = 0$.

Observe that for every γ in the fan F_δ, every subsequence of $B_0(\alpha,\gamma)$ meets $(R \cup R^{s(0)})^*$, every subsequence of $m(\alpha,\gamma)$ meets R, and every subsequence of $B_1(\alpha,\gamma)$ meets R^* or contains a member b such that $\text{Rem}(s) R^* b$.

Applying Ramsey's Theorem we find i, j such that $i < j$ and simultaneously: $\big(B_0(\alpha,\gamma)\big)(i) R^* \big(B_0(\alpha,\gamma)\big)(j)$ or $\langle s(0)\rangle R^* \big(B_0(\alpha,\gamma)\big)(i)$, and $\big(m(\alpha,\gamma)\big)(i) R \big(m(\alpha,\gamma)\big)(j)$, and $\big(B_1(\alpha,\gamma)\big)(i) R^* \big(B_1(\alpha,\gamma)\big)(j)$ or $\text{Rem}(s) R^* B_1(\alpha,\gamma)(j)$.

Observe that of the infinite sequence γ only the values $\gamma(i), \gamma(j)$ are involved in this long statement; let us denote it by $C\big(\gamma(i),\gamma(j)\big)$. Applying the Fan Theorem we find a natural number N such that for every γ in the fan F_δ there exist i, j such that $i < j < N$ and $C\big(\gamma(i),\gamma(j)\big)$.

We now consider the finite set of finite sequences $\{\overline{\gamma}N | \gamma \in F_\delta\}$. Let c, d belong to this set. We say that c is *earlier* than d if there exists $i < N$ such that $c(i) < d(i)$ and for every $j < N$, $j \neq i$, $c(j) = d(j)$.

We say that c is *safe* if for every $i < N$, if $c(i) < \delta(i)$, then some initial part of $\langle s(0), \big(\alpha(i)\big)\big(c(i)\big)\rangle$ belongs to R.

We define a proposition QED as follows:

QED:= There exist i, j such that $i < j$ and either $\alpha(i) R^* \alpha(j)$ or $s R^* \alpha(i)$.

Now observe the following two facts: (i) $\overline{\delta}N$ is safe, and (ii) for every safe element c of $\{\overline{\gamma}N | \gamma \in F_\delta\}$, either QED or there exists a safe element d of $\{\overline{\gamma}N | \gamma \in F_\delta\}$ such that d is earlier than c.

We offer a proof of (ii): suppose c is safe. We may assume that for every $i < N$, $\alpha(i)$ is a *non-empty* finite sequence. Determine i, j such that $i < j < N$ and $C\big(c(i), c(j)\big)$. Now observe: either QED or $\langle s(0)\rangle R^* A_0\big(\alpha(i), c(i)\big)$ or $\mathrm{Rem}(s) R^* A_1\big(\alpha(i), c(i)\big)$. Assume the latter, that is: $\mathrm{Rem}(s) R^* A_1\big(\alpha(i), c(i)\big)$; then, as $\mathrm{Rem}(s)$ is non-empty and c is safe, also some initial part of $\langle s(0), \big(\alpha(i)\big)\big(c(i)\big)\rangle$ belongs to R, therefore $s R^* \alpha(i)$ and QED.

So either QED or $\langle s(0)\rangle R^* A_0\big(\alpha(i), c(i)\big)$. Choose $k < c(i)$ such that some initial part of $\langle s(0), \big(\alpha(i)\big)(k)\rangle$ belongs to R and define the finite sequence d of length N by: $d(i) := k$ and for every $j < N$, if $j \neq i$, then $d(j) = c(j)$. d belongs to $\{\overline{\gamma}N | \gamma \in F_\delta\}$ and d is safe and d is earlier than c.

It follows from (i) and (ii) that we reach the conclusion QED within finitely many steps.

We conclude: for every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist i, j such that $i < j$ and either $\alpha(i) R^* \alpha(j)$ or $s R^*(i)$, that is: $P(s)$.

Clearly then, for every finite sequence s of natural numbers, $P(s)$, and R^* is almost full on $\mathbb{N}^*$. □

Corollary 5.3 (Finite Sequence Theorem, sometimes called Higman's Lemma). *Let R be an at-most-binary relation on $\mathbb{N}$.*

If R is almost full on $\mathbb{N}$, then R^* is almost full on $\mathbb{N}^*$.

Proof. Use Brouwer's Thesis and apply Theorem 5.5.1. □

6. Vazsonyi's Conjecture for binary trees

6.1. We define the set $\mathcal{T}_{\{2\}}$ of binary trees by means of the following inductive definition:

(i) The empty set $\emptyset$ is a binary tree. (We do not distinguish between the empty set $\emptyset$ and the empty sequence $\langle\rangle$.)
(ii) For all binary trees T, U, the ordered pair $\langle T, U\rangle$ is also a binary tree.
(iii) Every binary tree is obtained from the empty set by finitely many applications of step (ii).

(The above definition may be applied within any domain V where we have a non-surjective one-to-one mapping $\langle\,\rangle$ from $V \times V$ into V. If we think of the set-theoretical Wiener-Kuratowski definition of ordered pair, we take for V the collection of the hereditarily finite sets. But we might also start from the set $\mathbb{N}$ of natural numbers with a suitable pairing function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}\backslash\{0\}$, for instance: $\langle m, n\rangle := 2^m(2n+1)$. Binary trees then are natural numbers.)

It is convenient to think of an ordered pair $\langle T, U\rangle$ as a function on the set $\{0, 1\}$.

Every non-empty binary tree T is of the form $T = \langle T(0), T(1)\rangle$. The trees $T(0), T(1)$ are called the *immediate subtrees* of X the tree T.

We define a binary relation $\preccurlyeq$ on the set $\mathcal{T}_{\{2\}}$ of binary trees as follows, by induction:

> For all binary trees T, U, $T \preccurlyeq U$ ("T *neatly embeds into* U") if and only if either $T = \emptyset$ or both T and U are non-empty and *either* $T \preccurlyeq U(0)$ *or* $T \preccurlyeq U(1)$ *or* both $T(0) \preccurlyeq U(0)$ and $T(1) \preccurlyeq U(1)$.

We want to show, in this Section, that $\preccurlyeq$ is almost full on $\mathcal{T}_{\{2\}}$. This is a special case of Vazsonyi's Conjecture, mentioned in the Introduction.

Observe that we may decide, for all binary trees T, U, if T embeds into U or not.

6.2. We define a mapping B on the set $\mathcal{T}_{\{2\}}$ that associates to every binary tree T a finite subset $B(T)$ of $\{0, 1\}^*$:

(i) $B(\emptyset) := \{\langle\,\rangle\}$.
(ii) For every non-empty binary tree T, $B(T) := \{\langle\,\rangle\} \cup \langle 0\rangle * B\big(T(0)\big) \cup \langle 1\rangle * B\big(T(1)\big)$.

(We may be said to apply the definition from 6.1 in the domain V consisting of the finite sets of finite sequences of natural numbers, where the pairing operation $\langle\,\rangle$ is defined by: $\langle T, U\rangle := \{\langle\,\rangle\} \cup \langle 0\rangle * T \cup \langle 1\rangle * U$.)

One may prove that for all binary trees T, U,

> T neatly embeds into U if and only if there exists a function f from $B(T)$ to $B(U)$ such that, for all a in $B(T)$, $f(a) * \langle 0\rangle$ is an initial part of $f(a * \langle 0\rangle)$ and $f(a) * \langle 1\rangle$ is an initial part of $f(a * \langle 1\rangle)$.

6.3. We want to prove that the relation $\preccurlyeq$ is almost full on the set $\mathcal{T}_{\{2\}}$, that is, for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$ there exist i, j such that $i < j$ and $\alpha(i) \preccurlyeq \alpha(j)$.

For every binary tree T we define the proposition $P(T)$ as follows:

> $P(T) :=$ For every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$ there exist i, j such that $i < j$ and either $\alpha(i) \preccurlyeq \alpha(j)$ or $T \preccurlyeq \alpha(i)$.

We intend to show: for every binary tree T, $P(T)$.

This obviously implies that $\preccurlyeq$ is almost full on $\mathcal{T}_{\{2\}}$.

We want to reach our goal by induction on $\mathcal{T}_{\{2\}}$.

It suffices to show: $P(\emptyset)$ and for every non-empty binary tree T, if both $P\big(T(0)\big)$ and $P\big(T(1)\big)$, then $P(T)$.

Observe that $P(\emptyset)$ is true.

6.4. Assume that T is a non-empty binary tree and that we proved both $P\big(T(0)\big)$ and $P\big(T(1)\big)$. We wish to prove $P(T)$, that is:

for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$ there exist i, j such that $i < j$ and either $\alpha(i) \preccurlyeq \alpha(j)$ or $T \preccurlyeq \alpha(i)$.

Our strategy for proving this is based upon the following observation: for every binary tree U, if T does not neatly embed into U, then T is non-empty and either U is empty or U is also non-empty and *either* $T(0)$ does not neatly embed into $U(0)$ and T does not neatly embed into $U(1)$, *or* T does not neatly embed into $U(0)$ and $T(1)$ does not neatly embed into $U(1)$. We are using the fact that we may decide, for all binary trees T, U, if T neatly embeds into U or not.

We now let A_0, A_1, respectively be the set of all binary trees U such that $T(0), T(1)$, respectively does not neatly embed into U.

We now consider the set $(A_0 \dot{\bigcup} A_1)^*$ consisting of all finite sequences of elements of the set $A_0 \dot{\bigcup} A_1 = A_0 \times \{0\} \cup A_1 \times \{1\}$.

We define a so-called *evaluation mapping* Ev from the set $(A_0 \dot{\bigcup} A_1)^*$ to the set $\mathcal{T}_{\{2\}}$ of binary trees, as follows:

(i) $Ev(\emptyset) := \emptyset$.

(ii) For every non-empty finite sequence $s = \langle s(0)\rangle * \mathrm{Rem}(s)$ from $(A_0 \dot{\bigcup} A_1)^*$:
if $s(0)$ has the form $\langle U, 0\rangle$, then $Ev(s) := \langle U, Ev\big(\mathrm{Rem}(s)\big)\rangle$, and
if $s(0)$ has the form $\langle U, 1\rangle$, then $Ev(s) := \langle Ev\big(\mathrm{Rem}(s)\big), U\rangle$.

Remark that for every U in $\mathcal{T}_{\{2\}}$, if not $T \preccurlyeq U$, then there exists s in $(A_0 \dot{\bigcup} A_1)^*$ such that $Ev(s) = U$.

Now observe that by assumption $\preccurlyeq$ is almost full on both A_0 and A_1, therefore by Corollary 4.7, $\preccurlyeq + \preccurlyeq$ is almost full on $A_0 \dot{\bigcup} A_1$, and therefore, by the Finite Sequence Theorem, Corollary 5.3, $(\preccurlyeq + \preccurlyeq)^*$ is almost full on $(A_0 \dot{\bigcup} A_1)^*$.

We now *claim* the following:

For all s, t in $(A_0 \dot{\bigcup} A_1)^*$, if $s(\preccurlyeq + \preccurlyeq)^* t$, then $Ev(s) \preccurlyeq Ev(t)$.

We prove this claim by induction on length(s) + length(t).

Observe that for all t in $(A_0 \dot{\bigcup} A_1)^*$, $\emptyset = Ev(\emptyset) \preccurlyeq Ev(t)$.

Assume now that s, t are non-empty elements of $(A_0 \dot{\bigcup} A_1)^*$ and that we proved already: for all u, v in $(A_0 \dot{\bigcup} A_1)^*$, such that length$(u)$ + length(v) < length(s) + length(t), if $u(\preccurlyeq + \preccurlyeq)^* v$, then $Ev(u) \preccurlyeq Ev(v)$. Assume $s(\preccurlyeq + \preccurlyeq)^* t$. There are two cases to distinguish.

Case (i). $s(\preccurlyeq + \preccurlyeq)^*\mathrm{Rem}(t)$, and therefore $Ev(s) \preccurlyeq Ev\big(\mathrm{Rem}(t)\big)$, and, as $Ev\big(\mathrm{Rem}(t)\big) \preccurlyeq Ev(t)$, also $Ev(s) \preccurlyeq Ev(t)$.

Case (ii). $s(0)(\preccurlyeq + \preccurlyeq)t(0)$ and $\mathrm{Rem}(s)(\preccurlyeq + \preccurlyeq)^*\mathrm{Rem}(t)$. We may assume: $s(0) = \langle U, 0\rangle$ and $t(0) = \langle V, 0\rangle$. Then $U \preccurlyeq V$ and $Ev\big(\mathrm{Rem}(s)\big) \preccurlyeq Ev\big(\mathrm{Rem}(t)\big)$.

Now observe $Ev(s) = \langle U, Ev\big(\mathrm{Rem}(s)\big)\rangle$ and $Ev(t) = \langle V, Ev\big(\mathrm{Rem}(t)\big)\rangle$, and therefore $Ev(s) \preccurlyeq Ev(t)$.

This ends the proof of our claim.

We now establish the proposition $P(T)$ as follows:

Let $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$. Determine $\beta : \mathbb{N} \to (A_0 \dot{\bigcup} A_1)^*$ such that for every n, if not $T \preccurlyeq \alpha(n)$, then $Ev\big(\beta(n)\big) = \alpha(n)$. Determine i, j such that $i < j$ and $\beta(i)(\preccurlyeq + \preccurlyeq)^*\beta(j)$. Then *either* $T \preccurlyeq \alpha(i)$ *or* $T \preccurlyeq \alpha(j)$ *or* $Ev\big(\beta(i)\big) = \alpha(i)$ and $Ev\big(\beta(j)\big) = \alpha(j)$, and therefore $\alpha(i) \preccurlyeq \alpha(j)$.

Theorem 6.1 (Vazsonyi's Conjecture for binary trees). *$\preccurlyeq$ is almost full on $\mathcal{T}_{\{2\}}$, that is, for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$ there exist i, j such that $i < j$ and $\alpha(i) \preccurlyeq \alpha(j)$.*

Proof. See Sections 6.3 and 6.4. Observe that for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$, $P\big(\alpha(0)\big)$, so there exist i, j such that $0 < i < j$ and either $\alpha(0) \preccurlyeq \alpha(i)$ or $\alpha(i) \preccurlyeq \alpha(j)$. □

7. Higman's Theorem

7.1. Vazsonyi's Conjecture is also true for ternary trees.

How should one prove it?

The set $\mathcal{T}_{\{3\}}$ of ternary trees is defined as follows:

(i) The empty set $\emptyset$ belongs to $\mathcal{T}_{\{3\}}$.

(ii) For all T_0, T_1, T_2 in $\mathcal{T}_{\{3\}}$, the 3-sequence $\langle T_0, T_1, T_2\rangle$ belongs to $\mathcal{T}_{\{3\}}$.

(iii) Every element of $\mathcal{T}_{\{3\}}$ is obtained from the empty set by finitely many applications of step (ii).

We consider every non-empty element T of $\mathcal{T}_{\{3\}}$ as a function with domain $3 = \{0, 1, 2\}$ and write: $T = \langle T(0), T(1), T(2)\rangle$.

We define a binary relation $\preccurlyeq$ on $\mathcal{T}_{\{3\}}$ as follows:

For all T, U in $\mathcal{T}_{\{3\}}$, $T \preccurlyeq U$ ("T *neatly embeds into* U") if and only if either $T = \emptyset$ or U is non-empty and *either* there exists $i < 3$ such that $T \preccurlyeq U(i)$ *or* both T, U are non-empty and for every $i < 3$, $T(i) \preccurlyeq U(i)$.

Suppose that we want to prove that $\preccurlyeq$ is almost full on $\mathcal{T}_{\{3\}}$ and try the approach of the proof of Theorem 5.3. We then are led to consider binary trees. For assume that T is a non-empty ternary tree and that we got so far as to prove: for every $i < 3$, for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{3\}}$ there exist j, k such that $j < k$ and either $T(i) \preccurlyeq \alpha(j)$ or $\alpha(j) \preccurlyeq \alpha(k)$. We then want to prove: for every $\alpha : \mathbb{N} \to T_{\{3\}}$ there exist i, j such that $i < j$ and either $T \preccurlyeq \alpha(i)$ or $\alpha(i) \preccurlyeq \alpha(j)$, and study the set of all ternary trees U such that T does not neatly embed into U. Let us call this set $\mathcal{T}_{\{3\}} \upharpoonright T$. Observe that T does not neatly embed into U if and only if either U is empty or U is non-empty and there exists $i < 3$ such that $T(i)$ does not neatly embed into $U(i)$, and for all $j < 3$, if $j \neq i$, then T does not neatly embed into $U(j)$. Therefore every member X of $\mathcal{T}_{\{3\}} \upharpoonright T$ is obtained from two earlier constructed members U, V of $\mathcal{T}_{\{3\}} \upharpoonright T$ in one of the following three ways: we choose W such that $T(0)$ does not neatly embed into W and form $X := \langle W, U, V\rangle$, or we choose W such that $T(1)$ does not neatly embed into W and form $X := \langle U, W, V\rangle$, or we choose W such that $T(2)$ does not neatly embed into W and form $X := \langle U, V, W\rangle$. We find it useful to consider the extra tree W as a label. In this way the study of ternary trees leads to the study of labeled binary trees. Further reflection brings one to consider at-most-binary trees rather than just binary trees.

7.2. Let A be a non-empty finite set of natural numbers. We introduce the set $\mathcal{T}_A$ of A-ary trees as follows:

(i) The empty set $\emptyset$ belongs to $\mathcal{T}_A$.
(ii) For every k in A, for all $T_0, T_1, \dots, T_{k-1}$ in $\mathcal{T}_A$, the k-sequence $\langle T_0, T_1, \dots, T_{k-1}\rangle$ belongs to $\mathcal{T}_A$.
(iii) Every element of $\mathcal{T}_A$ is obtained from the empty set by finitely many applications of step (ii).

Every non-empty element T of $\mathcal{T}_A$ is a k-sequence $T = \langle T(0), T(1), \dots, T(k-1)\rangle$ of elements of $\mathcal{T}_A$ where $k = \mathrm{Dom}(T)$ is an element of A. The trees $T(0), T(1), \dots, T(k-1)$ are called the *immediate subtrees* of the tree T.

We define the set $\mathcal{T}$ of *trees* by: $\mathcal{T} := \bigcup_{n\in\mathbb{N}} \mathcal{T}_{\{0,1,\dots,n\}}$.

We define a binary relation $\preccurlyeq$ on $\mathcal{T}$ as follows:

For all trees T, U, $T \preccurlyeq U$ ("T *neatly embeds into* U") if and only if either $T = \emptyset$ or U is non-empty and *either* there exists $i \in \mathrm{Dom}(U)$ such that $T \preccurlyeq U(i)$, *or* both T and U are non-empty and $\mathrm{Dom}(T) = \mathrm{Dom}(U)$ and, for each i in $\mathrm{Dom}(T)$, $T(i) \preccurlyeq U(i)$.

We consider the mapping that we defined in Section 6.2 and extend it to a mapping B that associates to every tree T a finite subset $B(T)$ of the set $\mathbb{N}^*$ of finite sequences of natural numbers:

(i) $B(\emptyset) := \{\langle\,\rangle\}$.
(ii) For every non-empty tree T:

$$B(T) := \{\langle\,\rangle\} \cup \bigcup_{j<\mathrm{Dom}(T)} \langle j\rangle * B\big(T(j)\big).$$

One may prove the following:

For all trees T, U, T neatly embeds into U if and only if there exists a function f from $B(T)$ to $B(U)$ such that for all s in $\mathbb{N}^*$, j in $\mathbb{N}$, the following two conditions are fulfilled:
(i) If $s * \langle j\rangle$ belongs to $B(T)$, then $f(s) * \langle j\rangle$ is an initial part of $f(s * \langle j\rangle)$, and
(ii) $s * \langle j\rangle$ belongs to $B(T)$ if and only if $f(s) * \langle j\rangle$ belongs to $B(U)$.

Our aim, in this Section, is to prove that for each non-empty finite set A of natural numbers, $\preccurlyeq$ is almost full on $\mathcal{T}_A$.

Observe that $\preccurlyeq$ is not almost full on $\mathcal{T}$.

7.3. Let A be a non-empty set of natural numbers.

We introduce the set $\mathcal{LT}_A$ of *A-ary labeled trees* as follows:

(i) For every natural number m, the ordered pair $\langle m, \emptyset\rangle$ belongs to $\mathcal{LT}_A$. We sometimes call a pair $\langle m, \emptyset\rangle$ a *basic* labeled tree.
(ii) For every natural number m, for every k in A, for every k-sequence $T = \langle T(0), T(1), \dots, T(k-1)\rangle$ of elements of $\mathcal{LT}_A$, the ordered pair $\langle m, T\rangle$ belongs to $\mathcal{LT}_A$.

(iii) Every member of $\mathcal{LT}_A$ is obtained by finitely many applications of step (ii) from trees of the form $\langle m, \emptyset\rangle$.

For every natural number m, for every k, for every k-sequence $T = \langle T(0), \dots, T(k-1)\rangle$ of elements of $\mathcal{LT}_A$, the trees $T(0), T(1), \dots, T(k-1)$ are called the *immediate subtrees* of the tree $\langle m, T\rangle$. If T, U both belong to $\mathcal{LT}_A$, then T is called a *subtree* of U if and only if there exists a natural number l, and an l-sequence $V = \langle V(0), \dots, V(l-1)\rangle$ of elements of $\mathcal{LT}_A$ such that $V(0) = T$ and $V(l-1) = U$ and for each $i < l-1$, $V(i)$ is an immediate subtree of $V(i+1)$.

In this Section, the set A will always be a *finite* set of natural numbers. Every element T of $\mathcal{LT}_A$ is an ordered pair $\langle m, T\rangle$ where m is a natural number and $T = \langle T(0), T(1), \dots, T(k-1)\rangle$ is a k-sequence of elements of $\mathcal{LT}_A$ and $k = \mathrm{Dom}(T)$ belongs to $A \cup \{0\}$. We define the set $\mathcal{LT}$ of labeled trees by $\mathcal{LT} := \bigcup_{n\in\mathbb{N}} \mathcal{LT}_{\{0,1,\dots,n\}}$.

Let R be an at-most-ternary relation on $\mathbb{N}$, that is, $R \subseteq \bigcup_{i<4} \mathbb{N}^i$. For each k in $\mathbb{N}$ we let R^k be the set of all elements a of $\mathbb{N}^*$ such that $\langle k\rangle * a$ belongs to R. Observe that R^k is an at-most-binary relation on $\mathbb{N}$.

We define a binary relation $\preccurlyeq_R$ on $\mathcal{LT}$ as follows:

For all labeled trees $\langle m, T\rangle$ and $\langle n, U\rangle$, $\langle m, T\rangle \preccurlyeq_R \langle n, U\rangle$ ("$\langle m, T\rangle$ *neatly embeds into* $\langle n, U\rangle$ *with respect* to R") if and only if *either* there exists k in $\mathbb{N}$ such that $\mathrm{Dom}(T) = \mathrm{Dom}(U) = \{0, 1, \dots, k-1\}$ and some initial part of $\langle m, n\rangle$ belongs to R^k and for all $i < k$, $T(i) \preccurlyeq_R U(i)$, *or* there exists $i \in \mathrm{Dom}(U)$ such that $\langle m, T\rangle \preccurlyeq_R U(i)$.

We define a mapping B that associates to every labeled tree $\langle m, T\rangle$ in $\mathcal{LT}$ a finite subset $B(T)$ of the set $\mathbb{N}^*$ of finite sequences of natural numbers:

(i) $B(\langle m, \emptyset\rangle) := \{\langle\,\rangle\}$.
(ii) For every labeled tree $\langle m, T\rangle$ such that T is non-empty
$B(\langle m, T\rangle) := \{\langle\,\rangle\} \cup \bigcup_{j<\mathrm{Dom}(T)} \langle j\rangle * B\big(T(j)\big)$.

We also define a mapping L that associates to every labeled tree $\langle m, T\rangle$ in $\mathcal{LT}$ a function $L(\langle m, T\rangle)$ from $B(T)$ to $\mathbb{N}$, a so-called *labeling* of $B(T)$:

(i) $\big(L(\langle m, T\rangle)\big)(\langle\,\rangle) := m$.
(ii) For every finite sequence $\langle j\rangle * s$ in $B(T)$, $\big(L(\langle m, T\rangle)\big)(\langle j\rangle * s) := \Big(L\big(T(j)\big)\Big)(s)$.

Observe that, for all labeled trees $\langle m, T\rangle$, $\langle n, U\rangle$, for every at-most-ternary relation R on $\mathbb{N}$,

$\langle m, T\rangle$ neatly embeds into $\langle n, U\rangle$ with respect to R if and only if there exists a function f from $B(\langle m, T\rangle)$ into $B(\langle n, U\rangle)$ such that

(i) for all s in $\mathbb{N}^*$, j in $\mathbb{N}$,
if $s * \langle j\rangle$ belongs to $B(\langle m, T\rangle)$, then $f(s) * \langle j\rangle$ is an initial part of $f(s * \langle j\rangle)$.
(ii) for all s in $\mathbb{N}^*$, k in $\mathbb{N}$,
if s has k immediate extensions in $B(\langle m, T\rangle)$ then $f(s)$ has k immediate extensions in $B(\langle n, U\rangle)$ and some initial part of $\langle \big(L(\langle m, T\rangle)\big)(s), \big(L(\langle n, U\rangle)\big)\big(f(s)\big)\rangle$ belongs to R^k.

Actually, what we shall prove in this Section is the following extension of the statement presented as our aim at the end of Subsection 7.2:

For each non-empty finite set A of natural numbers, for every at-most-ternary relation R on $\mathbb{N}$, if for each k in $A \cup \{0\}$ the at-most-binary relation R^k is almost full on $\mathbb{N}$, then $\preccurlyeq_R$ is almost full on $\mathcal{LT}_A$.

7.4. Let A be a non-empty finite set of natural numbers.

For every nonzero element k of A we want to define a so-called evaluation map $Ev_{A,k}$ from $\mathcal{LT}_{A\cup\{k-1\}}$ to $\mathcal{LT}_A$.

We first consider the case that $k-1$ belongs to A, and then the case that $k-1$ does not belong to A.

7.4.1. Suppose that $k-1$ belongs to A.

We let $f_{A,k}$ be a fixed one-to-one enumeration of the set $\mathbb{N} \times (\{0\} \cup \bigcup_{j<k} \mathcal{LT}_A \times \{j\})$. So, for every n, $f_{A,k}(n)$ either has the form $\langle m, 0\rangle$ where m is a natural number, or the form $\langle m, V, j\rangle$ where m is a natural number and V belongs to $\mathcal{LT}_A$ and j is some natural number smaller than k.

We now define the evaluation map $Ev_{A,k}$ from $\mathcal{LT}_{A\cup\{k-1\}} = \mathcal{LT}_A$ to $\mathcal{LT}_A$ as follows:

(i) Let ℓ be an element of A that differs both from 0 and from $k-1$.
Then, for every m, for every ℓ-sequence U of elements of $\mathcal{LT}_A$, $Ev_{A,k}(\langle m, U\rangle) := \langle m, W\rangle$ where W is an ℓ-sequence of elements of $\mathcal{T}_A$ and, for each $j < \ell$, $W(j) := Ev_{A,k}\big(U(j)\big)$.

(ii) For every n, m, for every $(k-1)$-sequence U of elements of $\mathcal{LT}_A$, if $f_{A,k}(n) = \langle m, 0\rangle$, then $Ev_{A,k}(\langle n, U\rangle) := \langle m, W\rangle$ where W is a $(k-1)$-sequence of elements of $\mathcal{LT}_A$ and for each $j < k-1$, $W(j) := Ev_{A,k}\big(U(j)\big)$.
For every n, for every V in $\mathcal{LT}_A$, for every $j < k$, for every $(k-1)$-sequence U of elements of $\mathcal{LT}_A$, if $f_{A,k}(n) = \langle m, V, j\rangle$, then $Ev_{A,k}(\langle n, U\rangle) := \langle m, W\rangle$ where W is a k-sequence of elements of $\mathcal{LT}_A$, and for each i, if $i < j$, then $W(i) := Ev_{A,k}\big(U(i)\big)$, and, if $i > j$, then $W(i) := Ev_{A,k}\big(U(i-1)\big)$, and $W(j) := V$.

7.4.2. Suppose that $k-1$ does not belong to A. We now let $f_{A,k}$ be a fixed one-to-one enumeration of the set $\mathbb{N} \times \bigcup_{j<k} \mathcal{LT}_A \times \{j\}$. So, for every n, $f_{A,k}(n)$ has the form $\langle m, V, j\rangle$, where m is a natural number and V belongs to $\mathcal{LT}_A$ and j is some natural number smaller than k.

We define the evaluation map $Ev_{A,k}$ from $\mathcal{LT}_{A\cup\{k-1\}}$ to $\mathcal{LT}_A$ as in Section 7.4.1.

7.4.3. The following observation will be important in the sequel:

for every finite set A of natural numbers and every nonzero element k of A:

For every V in $\mathcal{LT}_A$ there exists at least one but only finitely many U in $\mathcal{LT}_{A\cup\{k-1\}}$ such that $Ev_{A,k}(U) = V$.

Theorem 7.1 (G. Higman, 1952, see [6]). *For every positive natural number p, for every p-ary stump σ such that $\sigma(0)$ is non-empty, for every at-most-ternary relation R on $\mathbb{N}$, if for each k in $A := Dom(\sigma) = \{i | i \in \mathbb{N} | \sigma(i) \neq \emptyset\}$ the stump $\sigma(k)$ secures that R^k is almost full on $\mathbb{N}$, then $\preccurlyeq_R$ is almost full on $\mathcal{LT}_A$.*

Proof. We use for every positive natural number p, the principle of induction on the set of p-ary stumps explained in Section 3.6.1.

Let p be a positive natural number and let σ be a p-ary stump and assume that the statement of the Theorem has been verified for every p-ary stump τ that is easier than σ in the sense of Section 3.6, that is, such that there exists k such that $\tau(k)$ is an immediate substump of $\sigma(k)$ and for each $n > k$, $\tau(n) = \sigma(n)$.

Define $A := \mathrm{Dom}(\sigma)$.

Let R be an at-most-ternary relation on $\mathbb{N}$ such that for each k in A, $\sigma(k)$ secures that R^k is almost full on $\mathbb{N}$. For every A-ary labeled tree $\langle m, T\rangle$ we define the proposition $P(\langle m, T\rangle)$ as follows:

> $P(\langle m, T\rangle)$ ("$\langle m, T\rangle$ *has the property* P") := For every $\alpha : \mathbb{N} \to \mathcal{LT}_A$ there exist i, j such that $i < j$ and either $\alpha(i) \preccurlyeq_R \alpha(j)$ or $\langle m, T\rangle \preccurlyeq_R \alpha(i)$.

We now show that every A-ary labeled tree $\langle m, T\rangle$ has the property P.

We assume that k is either zero or a nonzero element of A, and that we are given a tree of the form $\langle m, \emptyset\rangle$, or, if k is positive, of the form $\langle m, T\rangle$ where T is a k-sequence of elements of $\mathcal{LT}_A$ such that for every $i < k$, $T(i)$ has the property P. We want to prove that $\langle m, T\rangle$ itself has the property P.

Observe that both R^k and $\sigma(k)$ are non-empty.

We define an at-most-ternary relation R' on $\mathbb{N}$ as follows.

If k is positive, then for each n_0, n_1 in $\mathbb{N}$, $\langle n_0, n_1\rangle$ belongs to $(R')^{k-1}$ if and only if *either* there exist p_0, p_1 such that $f_{A,k}(n_0) = \langle p_0, 0\rangle$ and $f_{A,k}(n_1) = \langle p_1, 0\rangle$ and some initial part of $\langle p_0, p_1\rangle$ belongs to R^{k-1} *or* there exist p_0, p_1 in $\mathbb{N}$, U_0, U_1 in $\mathcal{LT}_A$ and j_0, j_1 in $\mathbb{N}$ such that $f_{A,k}(n_0) = \langle p_0, U_0, j_0\rangle$ and $f_{A,k}(n_1) = \langle p_1, U_1, j_1\rangle$, and some initial part of $\langle p_0, p_1\rangle$ belongs to R^k, and $j_0 = j_1$ and either $U_0 \preccurlyeq_R U_1$ or $T(j_0) \preccurlyeq_R U_0$. Further, if k is either positive or zero, then, for each n_0, n_1 in $\mathbb{N}$, $\langle n_0, n_1\rangle$ belongs to $(R')^k$ if and only if either $\langle n_0, n_1\rangle$ belongs to R^k or $\langle m, n_0\rangle$ belongs to R^k. Finally, for each i in A such that i differs from both k and $k-1$ we define: $(R')^i = R^i$.

Observe that, if k is positive, then $(R')^{k-1}$ is an almost full at-most-binary relation on $\mathbb{N}$. (This follows from the fact that, for each $i < k$, $T(i)$ has the property P and the fact that both R^k and R^{k-1} are almost full at-most-binary relations on $\mathbb{N}$, and Ramsey's Theorem.)

We form a p-ary stump τ such that $\mathrm{Dom}(\tau) = \mathrm{Dom}(\sigma) \cup \{k-1\}$ and $\tau(k) = \big(\sigma(k)\big)(m)$ and for each $i > k$, $\tau(i) = \sigma(i)$ and for each i in $\mathrm{Dom}(\tau)$, if $\tau(i) \neq \emptyset$, then $\tau(i)$ secures that $(R')^i$ is almost full on $\mathbb{N}$. Observe that τ is easier than σ.

Applying the induction hypothesis, we conclude that $\preccurlyeq_{R'}$ is almost full on $\mathcal{LT}_{A\cup\{k-1\}}$.

Now let α be a function from $\mathbb{N}$ to $\mathcal{LT}_A$.

We want to show: there exist i, j such that $i < j$ and either $\langle m, T\rangle \preccurlyeq_R \alpha(i)$ or $\alpha(i) \preccurlyeq_R \alpha(j)$.

To this end we consider the set F of all functions $\beta : \mathbb{N} \to \mathcal{LT}_{A\cup\{k-1\}}$ such that for each i, $Ev_{A,k}\big(\beta(i)\big) = \alpha(i)$.

It follows from Remark 7.4.3 that F is a fan.

We use the induction hypothesis and the fan theorem and we determine a natural number N such that for every β in F there exist i, j such that $i < j < N$ and $\beta(i) \preccurlyeq_{R'} \beta(j)$.

Let $\langle n, U\rangle$ be an element of $\mathcal{LT}_{A\cup\{k-1\}}$. We call $\langle n, U\rangle$ an *analysis* of its own evaluation $Ev_{A,k}(\langle n, U\rangle)$. We call $\langle n, U\rangle$ *disappointing* if and only if *either* $\mathrm{Dom}(U) = k$ and some initial part of $\langle m, n\rangle$ belongs to R^k *or* $\mathrm{Dom}(U) = k - 1$ and there exist p in $\mathbb{N}$, V in $\mathcal{LT}_A$ and j in $\mathbb{N}$ such that $f_{A,k}(n) = \langle p, V, j\rangle$ and $T(j) \preccurlyeq V$. In general, we are unable to decide if $\langle n, U\rangle$ is disappointing or not.

One may prove, by induction on the construction of the elements of $\mathcal{LT}_A$, that for every $\langle q, W\rangle$ in $\mathcal{LT}_A$, if every analysis $\langle n, U\rangle$ of $\langle q, W\rangle$ contains a disappointing subtree, then $\langle m, T\rangle \preccurlyeq_R \langle q, W\rangle$. We need the following *combinatorial principle*:

> Let N be a natural number and let $A_0, A_1, \dots, A_{N-1}$ be an N-sequence of finite sets.
> Let P be a subset of $A_0 \times A_1 \times \dots \times A_{N-1}$ and let, for each $i < N$, B_i be a subset of A_i.
> Assume that for every element $\langle a_0, a_1, \dots, a_{N-1}\rangle$ of $A_0 \times A_1 \times \dots \times A_{N-1}$ either $\langle a_0, a_1, \dots, a_{N-1}\rangle$ belongs to P or there exists $i < N$ such that a_i belongs to B_i.
> Then either there exists an element $\langle a_0, a_1, \dots, a_{N-1}\rangle$ of P or there exists $i < N$ such that B_i coincides with A_i.

Leaving the proof of this combinatorial principle to the reader, we apply it and complete our argument.

We let N be the natural number that we found by applying the Fan Theorem. For each $i < N$ we let A_i be the set of all elements $\langle n, U\rangle$ of $\mathcal{LT}_{A\cup\{k-1\}}$ such that $Ev_{A,k}(\langle n, U\rangle) = \alpha(i)$.

We let B_i be the set of all elements of A_i that contain a disappointing subtree.

We let P be the set of all elements $\langle\langle n_0, U_0\rangle, \dots, \langle n_{N-1}, U_{N-1}\rangle\rangle$ of $A_0 \times A_1 \times \dots \times A_{N-1}$ such that there exist i, j such that $i < j < N$ and $Ev_{A,k}(\langle n_i, U_i\rangle) \preccurlyeq_R Ev_{A,k}(\langle n_j, U_j\rangle)$. Observe that for all $\langle n_0, U_0\rangle, \langle n_1, U_1\rangle$ in $\mathcal{LT}_{A\cup\{k-1\}}$, if $\langle n_0, U_0\rangle \preccurlyeq_{R'} \langle n_1, U_1\rangle$ then either $Ev_{A,k}(\langle n_0, U_0\rangle) \preccurlyeq_R Ev_{A,k}(\langle n_1, U_1\rangle)$ or the tree $\langle n_0, U_0\rangle$ contains a disappointing subtree. One may prove this by induction on the construction of the elements of $\mathcal{LT}_{A\cup\{k-1\}}$.

We conclude:

Either there exists β in F and $i, j < N$ such that $Ev_{A,k}\big(\beta(i)\big) \preccurlyeq_R Ev_{A,k}\big(\beta(j)\big)$, that is, $\alpha(i) \preccurlyeq_R \alpha(j)$, or there exists $i < N$ such that for every β in F, $\beta(i)$ contains a disappointing subtree, and therefore $\langle m, T\rangle \preccurlyeq_R \alpha(i)$. □

Corollary 7.2. *For every n, $\preccurlyeq$ is almost full on $\mathcal{T}_{\{0,1,\dots,n-1\}}$.*

Proof. Apply Higman's Theorem to $\mathcal{LT}_{\{0,1,\dots,n-1\}}$, where the relation R coincides with the set $\{\langle i\rangle | i \in \mathbb{N}\}$, in particular, for every $i < n$, the empty sequence $\langle\,\rangle$ belongs to R^i. □

8. Vazsonyi's Conjecture and the Tree Theorem

We consider the set $\mathcal{T}$ of all finite trees, as we defined it in Section 7.2. Every finite tree T is a finite sequence $T = \langle T(0), T(1), \dots, T(k-1)\rangle$ of earlier-constructed finite trees.

The empty sequence $\langle\,\rangle = \emptyset$ is also a finite tree.

We define a binary relation $\sqsubseteq$ on the set $\mathcal{T}$, as follows:

For all T, U in $\mathcal{T}$, $T \sqsubseteq U$ ("T *embeds into* U") if and only if either $T = \emptyset$ or U is non-empty and *either* there exists $i \in \mathrm{Dom}(U)$ such that $T \sqsubseteq U(i)$, *or* both T and U are non-empty and $T(\sqsubseteq)^*U$, that is, there exists a strictly increasing function h from $\mathrm{Dom}(T)$ to $\mathrm{Dom}(U)$ such that for all i in $\mathrm{Dom}(T)$, $T(i) \sqsubseteq U\big(h(i)\big)$.

Observe that, for all finite trees T, U, T embeds into U if and only if there exists a mapping f from $B(T)$ into $B(U)$ such that

(i) for all s in $\mathbb{N}^*$, j in $\mathbb{N}$,
if $s * \langle j\rangle$ belongs to $B(T)$, then $f(s * \langle j\rangle)$ is a proper extension of $f(s)$.

(ii) for all s in $\mathbb{N}^*$, j in $\mathbb{N}$,
if both $s * \langle j\rangle$ and $s * \langle j+1\rangle$ belong to $B(T)$, then there exist k_0, k_1 such that $k_0 < k_1$ and $f(s) * \langle k_0\rangle$ is an initial part of $f(s * \langle j\rangle)$ and $f(s) * \langle k_1\rangle$ is an initial part of $f(s * \langle j+1\rangle)$.

Theorem 8.1 (Vazsonyi's Conjecture). $\sqsubseteq$ *is almost full on* $\mathcal{T}$, *that is, for every* $\alpha : \mathbb{N} \to \mathcal{T}$ *there exist* i, j *such that* $i < j$ *and* $\alpha(i) \sqsubseteq \alpha(j)$.

Proof. For every T in $\mathcal{T}$ we define a proposition $P(T)$, as follows:

$P(T)$ ("*T has the property P*") :=
For every $\alpha : \mathbb{N} \to \mathcal{T}$ there exist i, j such that $i < j$ and either $T \sqsubseteq \alpha(i)$ or $\alpha(i) \sqsubseteq \alpha(j)$.

We want to prove that every finite tree has the property P and use induction on the set $\mathcal{T}$ of finite trees.

It is obvious that the empty sequence $\emptyset$ has the property P, as $\emptyset$ embeds into every finite tree.

Now assume that $T = \langle T(0), T(1), \dots, T(k-1)\rangle$ is a finite tree and that for every $j < k$, $T(j)$ has the property P.

Observe that, for every finite tree U, T does not embed into U if and only if *either* $\mathrm{Dom}(U) < k$ and for each j in $\mathrm{Dom}(U)$, T does not embed into $U(j)$, *or* $\mathrm{Dom}(U) \geq k$ and there exists a strictly increasing function h from $k-1$ into $\mathrm{Dom}(U)$ such that for each $j < k-1$, T does not embed into $U\big(h(j)\big)$, and for each $i \in \mathrm{Dom}(U)$, if $i < h(0)$, then $T(0)$ does not embed into $U(i)$, and for each $j < k-2$, if $h(j) < i < h(j+1)$, then $T(j+1)$ does not embed into $U(i)$, and if $i > h(k-2)$, then $T(k-1)$ does not embed into $U(i)$.

It follows that we have to consider labeled at-most-$(k-1)$-ary trees. For each $j < k$, we let $\mathcal{T} \upharpoonright T(j)$ denote the set of all finite trees U such that $T(j)$ does not embed into U. Observe that $T(j)$ has the property P, therefore $\sqsubseteq$ is almost full on $\mathcal{T} \upharpoonright T(j)$, and $\sqsubseteq^*$ is almost full on $\big(\mathcal{T} \upharpoonright T(j)\big)^*$.

We let f be an enumeration of the set $\prod_{j<k} \big(\mathcal{T} \restriction T(j)\big)^*$. So, for every m, $f(m)$ is a k-sequence of finite sequences of trees such that for each $j < \text{length}(k)$, for each $i < \text{length}\big(\big(f(m)\big)(j)\big)$, the tree $T(j)$ does not embed into $\big(\big(f(m)\big)(j)\big)(i)$.

We now consider the set $\mathcal{LT}_A$ where $A := \{0, 1, \ldots, k-1\}$. We define an evaluation map Ev from the set $\mathcal{LT}_A$ to the set $\mathcal{T}$, as follows:

(i) For every natural number m, $Ev(\langle m, \emptyset\rangle) := \emptyset$.

(ii) Let U be a finite sequence of elements of $\mathcal{LT}_A$ of length $< k-1$, and let m be a natural number.
Then $Ev(\langle m, U\rangle) := W$ where W is a finite tree such that $\text{Dom}(W) = \text{Dom}(U)$ and for each j in $\text{Dom}(W)$, $W(j) := Ev\big(U(j)\big)$.

(iii) Let U be a finite sequence of elements of $\mathcal{LT}_A$ of length $k-1$, and let m be a natural number. Consider $V := f(m)$. V is a finite sequence of length k and for each $j < k$, $V(j)$ belongs to $\big(\mathcal{T} \restriction T(j)\big)^*$. We define:
$Ev(\langle m, U\rangle) := V(0) * \langle Ev\big(U(0)\big)\rangle * V(1) * \langle Ev\big(U(1)\big)\rangle * \cdots * \langle Ev\big(U(k-2)\big)\rangle * V(k-1)$.

We define a ternary relation R on the set $\mathbb{N}$ of natural numbers as follows: for each $i \neq k-1$, $R^i := \mathbb{N} \times \mathbb{N}$, and for all m_0, m_1 in $\mathbb{N}$, $\langle m_0, m_1\rangle$ belongs to R^{k-1} if and only if for each $j < k$, $\big(f(m_0)\big)(j) \sqsubseteq^* f(m_1)(j)$.

We now make some remarks:

(i) Ev is a surjective map from the set $\mathcal{LT}_A$ onto the set $\mathcal{T} \restriction T$ of all finite trees U such that T does not embed into U.

(ii) For all finite sequences U, V of elements of $\mathcal{LT}_A$, for all natural numbers m, n, if $\langle m, U\rangle \preccurlyeq_R \langle n, V\rangle$, then $Ev(\langle m, U\rangle) \sqsubseteq Ev(\langle n, V\rangle)$.

(iii) $\preccurlyeq_R$ is almost full on $\mathcal{LT}_A$.

((ii) may be proved by spelling out the definitions, and (iii) follows by Higman's Theorem from the fact that for each i, R^i is almost full on $\mathbb{N}$. R^{k-1} is almost full on $\mathbb{N}$ by Ramsey's Theorem, as, for each $j < k$, $\sqsubseteq^*$ is almost full on $\big(\mathcal{T} \restriction T(j)\big)^*$.)

It is now easy to conclude: $\sqsubseteq$ is almost full on $\mathcal{T} \restriction T$, that is, T has the property P.

It follows that every finite tree has the property P and that $\sqsubseteq$ is almost full on $\mathcal{T}$.

$\square$

8.1. Observe that the effort needed to prove Vazsonyi's Conjecture from Higman's Theorem is relatively small. J.B. Kruskal, at the time of writing [9], would perhaps have been surprised by this short argument.

We now want to follow him and extend Theorem 8.1 to labeled trees.

Let R be an at-most-ternary relation on the set $\mathbb{N}$ of natural numbers, and let k be a natural number.

We define a binary relation $\sqsubseteq_{R,k}$ on the set $\mathcal{LT}$ of labeled finite trees as follows:

For all labeled trees $\langle m, T\rangle$, $\langle n, U\rangle$,
$\langle m, T\rangle \sqsubseteq_{R,k} \langle n, U\rangle$ ("*The tree $\langle m, T\rangle$ embeds into the tree $\langle n, U\rangle$ with respect to R up to k*") if and only if *either* there exists i in $\text{Dom}(U)$ such that

$\langle m, T\rangle \sqsubseteq_{R,k} U(i)$ *or* there exists $j < k$ such that $\mathrm{Dom}(T) = \mathrm{Dom}(U) = j$ and some initial part of $\langle m, n\rangle$ belongs to R^j and for each i in $\mathrm{Dom}(T)$, $T(i) \sqsubseteq_{R,k} U(i)$ *or* both $\mathrm{Dom}(T) \geq k$ and $\mathrm{Dom}(U) \geq k$ and some initial part of $\langle m, n\rangle$ belongs to R^k and $T(\sqsubseteq_{R,k})^* U$, that is, there is a strictly increasing function from $\mathrm{Dom}(T)$ to $\mathrm{Dom}(U)$ such that for each i in $\mathrm{Dom}(T)$, $T(i) \sqsubseteq_{R,k} U\big(h(i)\big)$.

One may prove the following:

For all labeled trees $\langle m, T\rangle$, $\langle n, U\rangle$, for every at-most-ternary relation R on $\mathbb{N}$, for every k in $\mathbb{N}$, $\langle m, T\rangle$ embeds into $\langle n, U\rangle$ with respect to R up to k if and only if there exists a mapping f from $B(\langle m, T\rangle)$ into $B(\langle n, U\rangle)$ such that

(i) for all s in $\mathbb{N}^*$, j in $\mathbb{N}$, if $s * \langle j\rangle$ belongs to $B(\langle m, T\rangle)$, then $f(s * \langle j\rangle)$ is a proper extension of $f(s)$.
(ii) For all s in $\mathbb{N}^*$, j in $\mathbb{N}$, if both $s * \langle j\rangle$ and $s * \langle j+1\rangle$ belong to $B(\langle m, T\rangle)$, then there exist k_0, k_1 such that $k_0 < k_1$ and $f(s) * \langle k_0\rangle$ is an initial part of $f(s * \langle j\rangle)$ and $f(s) * \langle k_1\rangle$ is an initial part of $f(s * \langle j+1\rangle)$.
(iii) for all s in $\mathbb{N}^*$, if s has fewer than k immediate extensions in $B(\langle m, T\rangle)$ then the number of immediate extensions of s in $B(\langle m, T\rangle)$ is equal to the number of immediate extensions of $f(s)$ in $B(\langle n, U\rangle)$ (and therefore, in view of (ii), for each j such that $s * \langle j\rangle$ belongs to $B(\langle m, T\rangle)$, $f(s) * \langle j\rangle$ is an initial part of $f(s * \langle j\rangle)$.)
(iv) For all s in $\mathbb{N}^*$, j in $\mathbb{N}$, if $j < k$ and s belongs to $B(\langle m, T\rangle)$ and has j immediate extensions in $B(\langle m, T\rangle)$ then some initial part of $\langle \big(L(\langle m, T\rangle)\big)(s), \big(L(\langle n, U\rangle)\big)\big(f(s)\big)\rangle$ belongs to R^j; for all s in $\mathbb{N}^*$, if s has at least k immediate extensions in $B(\langle m, T\rangle)$ then some initial part of $\langle \big(L(\langle m, T\rangle)\big)(s), \big(L(\langle n, U\rangle)\big)\big(f(s)\big)\rangle$ belongs to R^k.

We intend to prove the following statement (*Kruskal's Tree Theorem*):

For every co-finite set A of natural numbers, for every at-most-ternary relation R on $\mathbb{N}$, for every natural number k, if for each $i \leq k$ such that i belongs to $A \cup \{0\}$ the at-most-binary relation R^i is almost full on $\mathbb{N}$, and every $i \geq k$ belongs to A, then the relation $\sqsubseteq_{R,k}$ is almost full on $\mathcal{LT}_A$.

8.2. Let A be a co-finite set of natural numbers, and let k be a nonzero natural number such that every natural number $n \geq k$ belongs to A.

For each nonzero natural number i we want to define a so-called evaluation map $Ev^*_{A,k,i}$ from $\mathcal{LT}_{A\cup\{i-1\}}$ to $\mathcal{LT}_A$.

We distinguish several cases:

(i) $i \geq k$ and $i-1$ belongs to A.
(ii) $i = k$ and $k-1$ does not belong to A.
(iii) $i < k$ and $i-1$ belongs to A.
(iv) $i < k$ and $i-1$ does not belong to A.

8.2.1. We first consider the case that $i \geq k$ and $i-1$ belongs to A. We let $f^*_{A,k,i}$ be an enumeration of the set $\mathbb{N} \times \left(\{0\} \cup \left((\mathcal{LT}_A)^*\right)^i \times \{1\}\right)$ so, for every n, $f^*_{A,k,i}(n)$ has either the form $\langle m, 0\rangle$ where m is a natural number, or the form $\langle m, V, 1\rangle$ where m is a natural number and V is an i-sequence of elements of $(\mathcal{LT}_A)^*$. We now define the map $Ev^*_{A,k,i}$ from $\mathcal{LT}_A$ to $\mathcal{LT}_A$ as follows.

(i) Let ℓ be a element of A, $\ell \neq i-1$. Then, for every n, for every ℓ-sequence U of elements of $\mathcal{LT}_A$, $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle n, W\rangle$ where W is an ℓ-sequence of elements of $\mathcal{LT}_A$ and for each $j < \ell$, $W(j) := Ev^*_{A,k,i}\big(U(j)\big)$.

(ii)$_1$ For every n, m, for every $(i-1)$-sequence U of elements of $\mathcal{LT}_A$, if $f^*_{A,k,i}(n) = \langle m, 0\rangle$, then $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle m, W\rangle$ where W is an $(i-1)$-sequence of elements of $\mathcal{LT}_A$ such that for every $j < i-1$, $W(j) := Ev^*_{A,k,i}\big(U(j)\big)$.

For every n, m, for every V from $\left((\mathcal{LT}_A)^*\right)^i$, for every $(i-1)$-sequence U of elements of $\mathcal{LT}_A$, if $f^*_{A,k,i}(n) = \langle m, V, 1\rangle$, then $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle m, W\rangle$ where
$W := V(0) * \langle Ev^*_{A,k,i}\big(U(0)\big)\rangle * \cdots * \langle Ev^*_{A,k,i}\big(U(i-2)\big)\rangle * V(i-1)$.

8.2.2. We now consider the case that $i = k$ and $k-1$ does not belong to A. We let $f^*_{A,k,k}$ be an enumeration of the set $\mathbb{N} \times \left((\mathcal{LT}_A)^*\right)^k$. The definition of the map $Ev^*_{A,k,k}$ is almost the same as in case 8.2.1. We only replace (ii)$_1$ by:

(ii)$_2$ For every n, m, for every V from $\left((\mathcal{LT}_A)^*\right)^k$, for every $(k-1)$-sequence U of elements of $\mathcal{LT}_{A\cup\{k-1\}}$, if $f^*_{A,k,k}(n) = \langle m, V\rangle$ then $Ev^*_{A,k,k}(\langle n, U\rangle) := \langle m, W\rangle$ where
$W := V(0) * \langle Ev^*_{A,k,k}\big(U(0)\big)\rangle * \cdots * \langle Ev^*_{A,k,k}\big(U(k-2)\big)\rangle * V(k-1)$.

8.2.3. We then consider the case that $i < k$ and $i-1$ belongs to A. We let $f^*_{A,k,i}$ be an enumeration of the set $\mathbb{N} \times (\{0\} \cup \bigcup_{j<i} \mathcal{LT}_A \times \{j\})$. So, for every n, $f^*_{A,k,i}(n)$ has either the form $\langle m, 0\rangle$, where m is a natural number, or the form $\langle m, V, j\rangle$ where V belongs to $\mathcal{LT}_A$ and m, j are natural numbers, $j < i$.

We define the map $Ev^*_{A,k,i}$ from $\mathcal{LT}_A$ to $\mathcal{LT}_A$ as follows:

(i) is as in Section 8.2.1, but we replace (ii)$_1$ by:

(ii)$_3$ For every n, m, for every $(i-1)$-sequence U of elements of $\mathcal{LT}_A$, if $f^*_{A,k,i}(n) = \langle m, 0\rangle$, then $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle m, W\rangle$ where W is an $(i-1)$-sequence of elements of $\mathcal{LT}_A$ such that for every $j < i-1$, $W(j) = Ev^*_{A,k,i}\big(U(j)\big)$.
For every n, m, j such that $j < i$, for every V in $\mathcal{LT}_A$, for every $(i-1)$-sequence U of elements of $\mathcal{LT}_A$, if $f^*_{A,k,i}(n) = \langle m, V, j\rangle$, then $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle m, W\rangle$ where W is an i-sequence of elements of $\mathcal{LT}_A$ such that $W(j) := V$ and for every $\ell < j$, $W(\ell) := Ev^*_{A,k,i}\big(U(\ell)\big)$ and for every ℓ such that $j < \ell < i$, $W(\ell) := Ev^*_{A,k,i}\big(U(\ell-1)\big)$.

8.2.4. We finally consider the case that $i < k$ and $i-1$ does not belong to A. We now let $f^*_{A,k,i}$ be an enumeration of the set $\mathbb{N} \times \bigcup_{j<i}(\mathcal{LT}_A \times \{j\})$. The definition of the mapping $Ev^*_{A,k,i}$ is the same as in the previous Section 8.2.3.

Theorem 8.2 (Tree Theorem, J.B. Kruskal, 1960, see [9]). *Let σ be a finitary stump such that $\sigma(0)$ is non-empty. Let $k := char(\sigma)$ be the greatest natural number i such that $\sigma(i)$ is non-empty. Let A be the set of all natural numbers i such that either $\sigma(i)$ is non-empty or $i \geq k$. If for every i in A such that $i \leq k$, $\sigma(i)$ secures that R^i is almost full on $\mathbb{N}$, then $\sqsubseteq_{R,k}$ is almost full on $\mathcal{LT}_A$.*

Proof. We use the principle of induction on the set of finitary stumps, introduced in Section 3.6.2. Let σ be a finitary stump and assume that the statement of the Theorem has been verified for every finitary stump τ that is more facile than σ in the sense of Section 3.6, that is, either $\text{char}(\tau) = \text{char}(\sigma)$ and there exists i such that $\tau(i)$ is an immediate substump of $\sigma(i)$ and for every $n > i$, $\tau(n) = \sigma(n)$, or $\text{char}(\tau) > \text{char}(\sigma)$ and $\tau\big(\text{char}(\tau)\big)$ is an immediate substump of $\sigma\big(\text{char}(\sigma)\big)$. Define $k := \text{char}(\sigma)$. Let A be the set of all natural numbers i such that either $\sigma(i)$ is non-empty or $i \geq k$, and let R be an at-most-ternary relation on $\mathbb{N}$ such that for every i in A, if $i \leq k$, then $\sigma(i)$ secures that R^i is almost full on $\mathbb{N}$.

For every A-ary labeled tree $\langle m, T\rangle$ we define the proposition $P(\langle m, T\rangle)$, to be pronounced as: "$\langle m, T\rangle$ *has the property* P", as follows:

$$P(\langle m, T\rangle) := \text{For every } \alpha : \mathbb{N} \to \mathcal{LT}_A \text{ there exist } i, j \text{ such that } i < j \\ \text{and either } \alpha(i) \sqsubseteq_{R,k} \alpha(j) \text{ or } \langle m, T\rangle \sqsubseteq_{R,k} \alpha(i).$$

We wish to prove that every A-ary labeled tree $\langle m, T\rangle$ has the property P and do so by induction.

So assume that $\langle m, T\rangle$ is an A-ary labeled tree, and that for every j in $\text{Dom}(T)$, $T(j)$ has the property P. (In particular, T might be the empty sequence).

We consider $i := \text{Dom}(T)$ and distinguish the cases $i < k$ and $i \geq k$.

We first consider the case $i < k$. Reminding ourselves of the proof of Higman's Theorem we easily see what should be our course of action.

We define an at-most-ternary relation R' on $\mathbb{N}$ as follows.

If $i > 0$, then, for all n_0, n_1 in $\mathbb{N}$, $\langle n_0, n_1\rangle$ belongs to $(R')^{i-1}$ if and only if *either* there exist p_0, p_1 such that $f^*_{A,k,i}(n_0) = \langle p_0, 0\rangle$ and $f^*_{A,k,i}(n_1) = \langle p_1, 0\rangle$ and some initial part of $\langle p_0, p_1\rangle$ belongs to R^{i-1}, *or* there exist p_0, p_1, j_0, j_1 in $\mathbb{N}$ and V_0, V_1 in $\mathcal{LT}_A$ such that $f^*_{A,k,i}(n_0) = \langle p_0, V_0, j_0\rangle$ and $f^*_{A,k,i}(n_1) = \langle p_1, V_1, j_1\rangle$ and $j_0 = j_1 < i$ and some initial part of $\langle p_0, p_1\rangle$ belongs to R^i, and either $V_0 \sqsubseteq_{R,k} V_1$ or $T(j_0) \sqsubseteq_{R,k} V_0$.

For every finite sequence s of natural numbers of length at most 2, s belongs to $(R')^i$ if and only if s belongs to R^i or $\langle m\rangle * s$ belongs to R^i.

Finally, for each ℓ in A such that ℓ differs from both i and $i - 1$, we define $(R')^\ell := R^\ell$.

Observe that $(R')^{i-1}$ is an almost full at-most-binary relation on $\mathbb{N}$. Observe that also $(R')^i$ is an almost full on $\mathbb{N}$, so, for every ℓ in $A \cup \{i - 1\}$, if $\ell \leq k$, then $(R')^\ell$ is almost full.

We form a finitary stump τ such that $\tau(i) = \big(\sigma(i)\big)(m)$, and for all ℓ in $A \cup \{i-1\}$, if $\ell \leq k$, then $\tau(\ell)$ secures that $(R')^\ell$ is almost full on $\mathbb{N}$, and if $i < \ell \leq k$, then $\tau(\ell) = \sigma(\ell)$, and if $\ell > k$, then $\tau(\ell) = \emptyset$.

Observe that τ is more facile than σ.

We define $A' := A \cup \{i-1\}$, in case $\sigma(i)(m) \neq \emptyset$, and $A' := A \backslash \{i\} \cup \{i-1\}$, in case $\sigma(i)(m) = \emptyset$.

Applying the induction hypothesis, we conclude that $\sqsubseteq_{R',k}$ is almost full on $\mathcal{LT}_{A'}$.

The argument is now completed as in the proof of Higman's Theorem.

Let $\langle n, V\rangle$ be an element of $\mathcal{LT}_{A'}$.

We call $\langle n, V\rangle$ *disappointing* if *either* $\mathrm{Dom}(V) = i$ and some initial segment of $\langle m, n\rangle$ belongs to R^i, *or* $\mathrm{Dom}(V) = i-1$ and there exist p in $\mathbb{N}$, U in $\mathcal{LT}_A$ and $j < i$ such that $f^*_{A,k,i}(n) = \langle p, U, j\rangle$ and $T(j) \sqsubseteq_{A'} U$. We shall call $\langle n, V\rangle$ an *analysis* of its own evaluation $Ev^*_{A,k,i}(\langle n, V\rangle)$.

We first observe that for all $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ in $\mathcal{LT}_{A'}$:

If $\langle n_0, U_0\rangle \sqsubseteq_{R',k} \langle n_1, U_1\rangle$, then either $Ev^*_{A,k,i}(\langle n_0, U_0\rangle) \sqsubseteq_{R,k} Ev^*_{A,k,i}(\langle n_1, U_1\rangle)$ or the tree $\langle n_0, U_0\rangle$ has a disappointing subtree.

We now show that $\langle m, T\rangle$ has the property P.

Let $\alpha : \mathbb{N} \to \mathcal{LT}_A$. We consider the fan F consisting of all functions $\beta : \mathbb{N} \to \mathcal{LT}_{A'}$ such that for every n, $Ev^*_{A,k,i}\big(\beta(n)\big) = \alpha(n)$.

Using the Fan Theorem we determine a natural number N such that for every β in F there exist p, q such that $p < q < N$ and $\beta(p) \sqsubseteq_{R',k} \beta(q)$. Consider the finite set $B := \{\overline{\beta}N | \beta \in F\}$. For each sequence b in B we determine p_b, q_b such that $p_b < q_b < N$ and $b(p_b) \sqsubseteq_{R',k} b(q_b)$ or the tree $b(p_b)$ has a disappointing subtree. Inspection of the set of pairs $\{\langle b(p_b), b(q_b)\rangle | b \in B\}$ will lead us to find either p, q such that $p < q < N$ and $\alpha(p) \sqsubseteq_{R,k} \alpha(q)$ or some p such that $p < N$ and for every $b \in B$, $b(p)$ contains a disappointing subtree. We conclude that every analysis of $\alpha(p)$ contains a disappointing subtree and therefore $\langle m, T\rangle \sqsubseteq_{R,k} \alpha(p)$.

We now study the case $i \geq k$.

We distinguish two subcases: $\big(\sigma(k)\big)(m) \neq \emptyset$ and $\big(\sigma(k)\big)(m) = \emptyset$. Let us first assume that $\big(\sigma(k)\big)(m) \neq \emptyset$.

We define an at-most-ternary relation R' on $\mathbb{N}$ as follows:

If $i > 0$, then, for all n_0, n_1 in $\mathbb{N}$, $\langle n_0, n_1\rangle$ belongs to $(R')^{i-1}$ if and only if *either* there exist p_0, p_1 such that $f^*_{A,k,i}(n_0) = \langle p_0, 0\rangle$ and $f^*_{A,k,i}(n_1) = \langle p_1, 0\rangle$ and some initial part of $\langle p_0, p_1\rangle$ belongs to R^{i-1}, *or* there exist p_0, p_1 in $\mathbb{N}$, and V_0, V_1 in $((\mathcal{LT}_A)^*)^i$ such that $f^*_{A,k,i}(n_0) = \langle p_0, V_0, 1\rangle$ and $f^*_{A,k,i}(n_1) = \langle p_1, V_1, 1\rangle$ and $\langle p_0, p_1\rangle$ belongs to R^k and for each $j < i$, there exists a strictly increasing function h from $\mathrm{Dom}\big(V_0(j)\big)$ to $\mathrm{Dom}\big(V_1(j)\big)$ such that for each ℓ in $\mathrm{Dom}\big(V_0(j)\big)$, $\big(V_0(j)\big)(\ell) \sqsubseteq_{R,k} \big(V_1(j)\big)\big(h(\ell)\big)$ or $T(j) \sqsubseteq_{R,k} \big(V_0(j)\big)(\ell)$.

For every finite sequence s of natural numbers of length at most 2, s belongs to $(R')^i$ if and only if $\langle m\rangle * s$ belongs to R^i.

Finally, for each ℓ such that $\ell < k$ and $\ell < i-1$, we define $(R')^\ell := R^\ell$, and for each ℓ such that $k \leq \ell < i-1$, we define $(R')^\ell := R^k$.

Observe that $(R')^{i-1}$ is almost full on $\mathbb{N}$. (This follows from the fact that for each $j < i$, $T(j)$ has the property P, and the Finite Sequence Theorem, and Ramsey's Theorem.)

Observe that for each ℓ in $A \cup \{i-1\}$, if $\ell \leq i$, then $(R')^\ell$ is almost full on $\mathbb{N}$.

We consider the finitary stump τ such that for all ℓ, if $\ell < k$ and $\ell < i-1$, then $\tau(\ell) = \sigma(\ell)$, and for all ℓ, if $k \leq \ell < i-1$, then $\tau(\ell) = \sigma(k)$, and $\tau(i) = \big(\sigma(k)\big)(m)$ and for all ℓ, if $\ell > i$, then $\tau(\ell) = \emptyset$.

Observe that, for each ℓ, if $\ell \leq i-1$, then $\tau(\ell)$ secures that $(R')^\ell$ is almost full on $\mathbb{N}$, and that τ is more facile than σ.

We may assume, therefore, that $\sqsubseteq_{R',i}$ is almost full on $\mathcal{LT}_{A\cup\{i-1\}}$.

Let $\langle n, U\rangle$ be an element of $\mathcal{LT}_{A\cup\{i-1\}}$.

We call $\langle n, U\rangle$ *disappointing* if *either* $\mathrm{Dom}(U) \geq i$ and some initial part of $\langle m, n\rangle$ belongs to R^k *or* $\mathrm{Dom}(U) = i-1$ and there exist p in $\mathbb{N}$ and V in $\big((\mathcal{LT}_A)^*\big)^k$ such that $f^*_{A,k,i}(n)$ equals $\langle p, V, 1\rangle$ and for some $j < i$, for some $q < \mathrm{Dom}\big(V(j)\big)$, $T(j) \sqsubseteq_{R,k} \big(V(j)\big)(q)$.

We make two observations:

(i) Assume that $\langle n, U\rangle$ belongs to $\mathcal{LT}_A$ and that $\mathrm{Dom}(U) \geq i$. If every $\langle p, W\rangle$ in $\mathcal{LT}_{A\cup\{i-1\}}$ such that $Ev^*_{A,k,i}(\langle p, W\rangle) = \langle n, U\rangle$, that is, every analysis of $\langle n, U\rangle$, is disappointing, then $\langle m, T\rangle \sqsubseteq_{R,k} \langle n, U\rangle$.

(ii) Assume that $\langle n, U\rangle$ belongs to $\mathcal{LT}_A$. If every analysis of $\langle n, U\rangle$ contains a disappointing subtree, then $\langle m, T\rangle \sqsubseteq_{R,k} \langle n, U\rangle$.

We also need the following remark:

> For all $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ in $\mathcal{LT}_{A\cup\{i-1\}}$, if $\langle n_0, U_0\rangle \sqsubseteq_{R',k} \langle n_1, U_1\rangle$ then *either* $Ev^*_{A,k,i}(\langle n_0, U_0\rangle) \sqsubseteq_{R,k} Ev^*_{A,k,i}(\langle n_1, U_1\rangle)$ *or* the tree $\langle n_0, U_0\rangle$ contains a disappointing subtree.

We now prove that $\langle m, T\rangle$ has the property P, as follows.

Let $\alpha : \mathbb{N} \to \mathcal{LT}_A$. Consider the fan F consisting of all functions $\beta : \mathbb{N} \to \mathcal{LT}_{A\cup\{i-1\}}$ such that for every n, $Ev^*_{A,k,i}\big(\beta(n)\big) = \alpha(n)$. Using the Fan Theorem we determine a natural number N such that for every β in F there exist p, q such that $p < q < N$ and $\beta(p) \sqsubseteq_{R',k} \beta(q)$. Reasoning as in the first part of this proof, we conclude: either there exist p, q such that $p < q < N$ and $\alpha(p) \sqsubseteq_{R,k} \alpha(q)$, or for some $p < N$, every analysis of $\alpha(p)$ contains a disappointing subtree, and therefore $\langle m, T\rangle \sqsubseteq_{R,k} \alpha(p)$.

Let us now consider the case $\big(\sigma(k)\big)(m) = \emptyset$. We conclude that the empty sequence $\langle\,\rangle$ belongs to R^k. We define $B := \{\ell | \ell \in A | \ell \leq i\}$ and consider $\mathcal{LT}_B$. Observe that every labeled tree $\langle n, U\rangle$ that belongs to $\mathcal{LT}_{A\cup\{i-1\}}$ but not to $\mathcal{LT}_A$ contains a disappointing subtree, and that every labeled tree in $\mathcal{LT}_A$ has at least one analysis in $\mathcal{LT}_B$.

We define an at-most-ternary relation R' on $\mathbb{N}$ exactly as in the previous case $\big(\sigma(k)\big)(m) \neq \emptyset$.

Observe that for each ℓ in B, $(R')^\ell$ is almost full on $\mathbb{N}$.

Applying Higman's Theorem, we conclude that $\preccurlyeq_{R'}$ is almost full on $\mathcal{LT}_B$.

We need the following observations:

(i) For all $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ in $\mathcal{LT}_B$, if $\langle n_0, U_0\rangle \preccurlyeq_{R'} \langle n_1, U_1\rangle$, then $\langle n_0, U_0\rangle \sqsubseteq_{R',k} \langle n_1, U_1\rangle$ and *either* $Ev^*_{A,k,i}(\langle n_0, U_0\rangle) \sqsubseteq_{R,k} Ev^*_{A,k,i}(\langle n_1, U_1\rangle)$ *or* the tree $\langle n_0, U_0\rangle$ contains a disappointing subtree.

(ii) For all $\langle p, W\rangle$ in $\mathcal{LT}_B$, if every analysis $\langle n, U\rangle$ of $\langle p, W\rangle$ that belongs to $\mathcal{LT}_B$ has a disappointing subtree, then $\langle m, T\rangle \preccurlyeq_{R,k} \langle p, W\rangle$.

The proof that $\langle m, T\rangle$ has the property P is from here on almost the same as in the case $\big(\sigma(k)\big)(m) \neq \emptyset$ and is left to the reader. □

9. Minimal-bad-sequence arguments

We show how some of the results proved in this paper are obtained more easily by the minimal-bad-sequence argument due to Nash-Williams. We freely use classical logic in this Section. In Section 10 we shall discuss the problem if we could do something similar constructively.

9.1. The Finite Sequence Theorem

9.1.1.

$$\boxed{\leq^* \text{ is almost full on } \mathbb{N}^*.} \qquad (\mathit{Cf.\ Theorem}\ 5.1)$$

Let $\alpha : \mathbb{N} \to \mathbb{N}^*$. We say α is *bad* if α does not meet $\leq^*$. Suppose there exists at least one bad $\alpha : \mathbb{N} \to \mathbb{N}^*$. We define $\alpha_0 : \mathbb{N} \to \mathbb{N}^*$ in such a way that α_0 is bad and for each i, for each $\alpha : \mathbb{N} \to \mathbb{N}^*$, if for each $j < i$, $\alpha(j) = \alpha_0(j)$, but $\alpha(i) = \mathrm{Rem}\big(\alpha_0(i)\big)$, then α is good, that is, α meets $\leq^*$.

α_0 is called a *minimal bad sequence*. Observe that for each i, $\alpha_0(i) \neq \langle\,\rangle$. We consider the sequences $\gamma : \mathbb{N} \to \mathbb{N}$ and $\beta : \mathbb{N} \to \mathbb{N}^*$ such that for each i, $\gamma(i) = \big(\alpha_0(i)\big)(0)$ and $\beta(i) = \mathrm{Rem}\big(\alpha_0(i)\big)$, so $\alpha_0(i) = \langle\gamma(i)\rangle * \beta(i)$. We claim that for every strictly increasing $\delta : \mathbb{N} \to \mathbb{N}$ the sequence $\beta \circ \delta$ meets $\leq^*$.

For suppose $\delta : \mathbb{N} \to \mathbb{N}$ is strictly increasing.

Consider the sequence: $\alpha_0(0), \alpha_0(1), \ldots, \alpha_0(\delta(0)-1), \beta\circ\delta(0), \beta\circ\delta(1), \ldots$.

This sequence meets $\leq^*$. There are several possibilities.

(i) There exists i, j such that $i < j < \delta(0)$ and $\alpha_0(i) \leq^* \big(\alpha_0(j)\big)$.
This will not happen, as α_0 is bad.

(ii) There exist i, j such that $i < \delta(0) \leq \delta(j)$ and $\alpha_0(i) \leq^* \beta\circ\delta(j)$. Then also $\alpha_0(i) \leq^* \alpha_0\big(\delta(j)\big)$.
This will not happen, as α_0 is bad.

(iii) There exist i, j such that $i < j$ and $\beta\circ\delta(i) \leq^* \beta\circ\delta(j)$.
Then $\beta\circ\delta$ meets $\leq^*$.

We now determine $\delta : \mathbb{N} \to \mathbb{N}$ such that δ is strictly increasing and for each i, $\gamma\big(\delta(i)\big) \leq \gamma\big(\delta(i+1)\big)$. We calculate i, j such that $i < j$ and $\beta\big(\delta(i)\big) \leq^* \beta\big(\delta(j)\big)$ and conclude: $\alpha_0(\delta(i)) \leq^* \alpha_0(\delta(j))$.

Contradiction, as α_0 is bad.

We conclude that there is no bad sequence. Therefore, every $\alpha : \mathbb{N} \to \mathbb{N}^*$ will meet $\leq^*$.

9.1.2.

For every binary relation R on $\mathbb{N}$: If R is almost full on $\mathbb{N}$, then R^* is almost full on $\mathbb{N}^*$.	(*Cf. Theorem* 5.2)

Suppose R is almost full on $\mathbb{N}$.

Let $\alpha : \mathbb{N} \to \mathbb{N}^*$. We say α is *bad* if α does not meet R^*. Suppose there exists at least one bad $\alpha : \mathbb{N} \to \mathbb{N}^*$. Determine $\alpha_0 : \mathbb{N} \to \mathbb{N}^*$ such that α_0 is bad and for every i in $\mathbb{N}$, for each $\alpha : \mathbb{N} \to \mathbb{N}^*$, if for each $j < i$, $\alpha(j) = \alpha_0(j)$ but $\alpha(i) = \mathrm{Rem}(\alpha_0(i))$, then α meets R^*. Observe that for every i, $\alpha_0(i) \neq \langle\,\rangle$.

Determine $\gamma : \mathbb{N} \to \mathbb{N}$ and $\beta : \mathbb{N} \to \mathbb{N}^*$ such that for every i, $\alpha_0(i) = \langle \gamma(i) \rangle * \beta(i)$.

Arguing as in Section 9.1.1, we prove that for every increasing $\delta : \mathbb{N} \to \mathbb{N}$ there exist i, j such that $i < j$ and $\beta\big(\delta(i)\big) R^* \beta\big(\delta(j)\big)$.

Also, for every strictly increasing $\delta : \mathbb{N} \to \mathbb{N}$ there exist i, j such that $i < j$ and $\gamma\big(\delta(i)\big)\ R\ \gamma\big(\delta(j)\big)$.

Using Ramsey's Theorem, we conclude that there exist i, j such that $i < j$ and both $\gamma(i) R \gamma(j)$ and $\beta(i) R^* \beta(j)$, and therefore $\alpha_0(i) R^* \alpha_0(j)$. So α_0 is not bad. Contradiction.

We conclude that there is no bad $\alpha : \mathbb{N} \to \mathbb{N}^*$. Therefore, every $\alpha : \mathbb{N} \to \mathbb{N}^*$ will meet R^*.

9.2. Higman's Theorem

9.2.1.

$\preccurlyeq$ is almost full on $\mathcal{T}_2$.	(*Cf. Theorem* 6.1)

Let α be a function from the set $\mathbb{N}$ of natural numbers to the set $\mathcal{T}_2$ of strictly binary trees. We say α is *bad* if α does not meet $\preccurlyeq$. Suppose there exists at least one bad $\alpha : \mathbb{N} \to \mathbb{N}^*$.

Determine $\alpha_0 : \mathbb{N} \to \mathcal{T}_2$ such that α_0 is bad and for every i, for every $\alpha : \mathbb{N} \to \mathcal{T}_2$, if for each $j < i$, $\alpha(j) = \alpha_0(j)$ and $\alpha(i)$ is an immediate subtree of $\alpha_0(i)$, then α is good, that is, α meets $\preccurlyeq$. Observe that for each i, $\alpha_0(i)$ is non-empty.

Consider the sequence

$$\big(\alpha_0(0)\big)(0),\ \big(\alpha_0(0)\big)(1),\ \big(\alpha_0(1)\big)(0),\ \big(\alpha_0(1)\big)(1), \ldots$$

consisting of the immediate subtrees of the elements of α_0, in their natural order.

Let us call this sequence β. So, for every i, $\beta(2i) = \big(\alpha_0(i)\big)(0)$ and $\beta(2i+1) = \big(\alpha_0(i)\big)(1)$.

We claim that for every strictly increasing sequence δ, the sequence $\beta \circ \delta$ meets $\preccurlyeq$.

For suppose $\delta : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Consider $\delta(0)$. Calculate i_0 such that $\delta(0) = 2i_0$ or $\delta(0) = 2i_0 + 1$.

Consider the sequence: $\alpha_0(0), \alpha_0(1), \ldots, \alpha_0(i_0 - 1), \beta \circ \delta(0), \beta \circ \delta(1), \ldots$.

This sequence meets $\preccurlyeq$. There are several possibilities.

(i) There exist i, j such that $i < j < i_0$ and $\alpha_0(i) \preccurlyeq \alpha_0(j)$.
This will not happen, as α_0 is bad.

(ii) There exist i, j such that $i < \delta(0)$ and $\alpha_0(i) \preccurlyeq \beta \circ \delta(j)$. Calculate i_1 such that $\delta(j) = 2i_1$ or $\delta(j) = 2i_1 + 1$.
Then $\alpha_0(i) \preccurlyeq \beta \circ \delta(j) \preccurlyeq \alpha_0(i_1)$, and, as $\beta \circ \delta(j)$ is a proper subtree of $\alpha_0(i_1)$, $i < i_1$ and $\alpha_0(i) \preccurlyeq \alpha_0(i_1)$.
This will not happen, as α_0 is bad.

(iii) There exist i, j such that $i < j$ and $\beta \circ \delta(i) \preccurlyeq \beta \circ \delta(j)$.
Then $\beta \circ \delta$ meets $\preccurlyeq$.

Observe that the sequences $\big(\alpha_0(0)\big)(0), \big(\alpha_0(1)\big)(0), \big(\alpha_0(2)\big)(0), \ldots$ and $\big(\alpha_0(0)\big)(1), \big(\alpha_0(1)\big)(1), \big(\alpha_0(2)\big)(1), \ldots$ are subsequences of β.

Using Ramsey's Theorem, we determine i, j such that $i < j$ and both $\big(\alpha_0(i)\big)(0) \preccurlyeq \big(\alpha_0(j)\big)(0)$ and $\big(\alpha_0(i)\big)(1) \preccurlyeq \big(\alpha_0(j)\big)(1)$, and therefore $\alpha_0(i) \preccurlyeq \alpha_0(j)$.

Contradiction, as α_0 is bad.

We conclude that there is no bad sequence. Therefore, every $\alpha : \mathbb{N} \to \mathcal{T}_2$ will meet $\preccurlyeq$.

9.2.2.

For every finite subset A of $\mathbb{N}$ containing 0, for every at-most-ternary relation R on $\mathbb{N}$, if, for each k in A, R^k is almost full on $\mathbb{N}$, then $\preccurlyeq_R$ is almost full on $\mathcal{LT}_A$.

(*Cf. Theorem 7.1.*) Let A be a finite subset of $\mathbb{N}$ containing 0, and R an at-most-ternary relation on $\mathbb{N}$ such that for each k in A, R^k is almost full on $\mathbb{N}$. Let $\alpha : \mathbb{N} \to \mathcal{LT}_A$. We say that α is *bad* if α does not meet $\preccurlyeq_R$. Suppose that there exists at least one bad sequence $\alpha : \mathbb{N} \to \mathcal{LT}_A$.

We determine a sequence $\alpha_0 : \mathbb{N} \to \mathcal{LT}_A$ such that α_0 is bad and for every i, for every $\alpha : \mathbb{N} \to \mathcal{LT}_A$, if for every $j < i, \alpha(j) = \alpha_0(j)$ and $\alpha(i)$ is an immediate subtree of $\alpha_0(i)$, then α meets $\preccurlyeq_R$.

We determine $\gamma : \mathbb{N} \to \mathbb{N}$ and $\tau : \mathbb{N} \to \bigcup\limits_{k \in A} (\mathcal{LT}_A)^k$ such that for every i, $\alpha_0(i) = \langle \gamma(i), \tau(i) \rangle$.

We let $\beta : \mathbb{N} \to \mathcal{LT}_A$ be an enumeration of the set $\{\big(\tau(i)\big)(j) | i \in \mathbb{N},\ j \in \mathrm{Dom}\big(\tau(i)\big)\}$. Arguing as in Section 9.2.1 we prove that for every strictly increasing $\delta : \mathbb{N} \to \mathbb{N}$ there exist i, j such that $i < j$ and $\beta(i) \preccurlyeq_R \beta(j)$.

We determine a strictly increasing function $\delta : \mathbb{N} \to \mathbb{N}$ and k in A such that for every i, $\mathrm{Dom}\Big(\tau\big(\delta(i)\big)\Big) = k$.

Using Ramsey's Theorem, we find i, j such that $i < j$ and for each $p < k$, $\Big(\tau\big(\delta(i)\big)\Big)(p) \preccurlyeq_R \Big(\tau\big(\delta(j)\big)\Big)(p)$ and $\langle \gamma\big(\delta(i)\big), \gamma\big(\delta(j)\big)\rangle$ belongs to R^k, and therefore: $\alpha_0\big(\delta(i)\big) \preccurlyeq_R \alpha_0\big(\delta(j)\big)$.

Contradiction, as α_0 is bad.

We conclude that there is no bad sequence. Therefore, every $\alpha : \mathbb{N} \to \mathcal{LT}_A$ will meet $\preccurlyeq_R$.

9.3. Kruskal's Theorem

9.3.1.

Vazsonyi's Conjecture: $\sqsubseteq$ is almost full on $\mathcal{T}$.	($Cf.\ Theorem$ 8.1)

Let α be a function from the set $\mathbb{N}$ of natural numbers to the set $\mathcal{T}$ of finite trees. We say α is *bad* if α does not meet $\sqsubseteq$. Suppose there exists at least one bad $\alpha : \mathbb{N} \to \mathcal{T}$.

Determine $\alpha_0 : \mathbb{N} \to \mathcal{T}$ such that α_0 is bad and for every i, for every $\alpha : \mathbb{N} \to \mathcal{T}$, if for each $j < i$, $\alpha(j) = \alpha_0(j)$ and $\alpha(i)$ is an immediate subtree of $\alpha_0(i)$, then α meets $\sqsubseteq$.

Let B be the set $\{\big(\alpha_0(i)\big)(j) | i \in \mathbb{N}, j \in \mathrm{Dom}\big(\alpha_0(i)\big)\}$ of all immediate subtrees of the trees $\alpha_0(0), \alpha_0(1), \ldots$.

Arguing as in the previous Sections, we prove that $\sqsubseteq$ is almost full on B.

Using the Finite Sequence Theorem we conclude that $\sqsubseteq^*$ is almost full on B^*.

Observe that B^* is a subset of $\mathcal{T}$ and that the trees $\alpha_0(0), \alpha_0(1), \ldots$ belong to B^*.

So there exist i, j such that $i < j$ and $\alpha_0(i) \sqsubseteq^* \alpha_0(j)$, that is $\alpha_0(i) \sqsubseteq \alpha_0(j)$.

Contradiction, as α_0 is bad.

We conclude that there is no bad sequence. Therefore, every $\alpha : \mathbb{N} \to \mathcal{T}$ will meet $\sqsubseteq$.

9.3.2.

For every at-most-ternary relation R on $\mathbb{N}$, for every k, if, for every $i \leq k$, R^i is almost full on $\mathbb{N}$, then $\sqsubseteq_{R,k}$ is almost full on $\mathcal{LT}_{\mathbb{N}}$.

($Cf.\ Theorem$ 8.2)

Suppose that R is an at-most-ternary relation on $\mathbb{N}$, and k is a natural number and for every $i \leq k$, R^i is almost full on $\mathbb{N}$. Let α be a function from the set $\mathbb{N}$ of natural numbers to the set $\mathcal{LT}_{\mathbb{N}}$ of labeled finite trees. We say α is *bad* if α does not meet $\sqsubseteq_{R,k}$. Assume that there exists at least one bad $\alpha : \mathbb{N} \to \mathcal{LT}_{\mathbb{N}}$. We determine $\alpha_0 : \mathbb{N} \to \mathcal{LT}_{\mathbb{N}}$ such that for every i, for every $\alpha : \mathbb{N} \to \mathcal{LT}_{\mathbb{N}}$, if, for each $j < i$, $\alpha(j) = \alpha_0(j)$ and $\alpha(i)$ is an immediate subtree of $\alpha_0(i)$, then α meets $\sqsubseteq_{R,k}$.

We determine $\gamma : \mathbb{N} \to \mathbb{N}$ and $\tau : \mathbb{N} \to (\mathcal{LT}_{\mathbb{N}})^*$ such that for every i, $\alpha_0(i) = \langle \gamma(i), \tau(i) \rangle$.

Let $B := \{\big(\tau(i)\big)(j) | i \in \mathbb{N}, | j \in \mathrm{Dom}\big(\tau(i)\big)\}$ be the set of all immediate subtrees of the elements of α_0. Arguing as before, we conclude that $\sqsubseteq_{R,k}$ is almost full on B.

We now distinguish two cases:

(i) There exists a strictly increasing function $\delta : \mathbb{N} \to \mathbb{N}$ and a natural number $n_0 < k$ such that for every i, $\mathrm{Dom}\Big(\tau\big(\delta(i)\big)\Big) = n_0$. Applying Ramsey's Theorem we find i, j such that $i < j$ and some initial part of $\langle \gamma\big(\delta(i)\big), \gamma\big(\delta(j)\big)\rangle$ belongs to R^{n_0}, and for each $q < n_0$, $\Big(\tau\big(\delta(i)\big)\Big)(q) \sqsubseteq_{R,k} \Big(\tau\big(\delta(j)\big)\Big)(q)$, therefore $\alpha_0\big(\delta(i)\big) \sqsubseteq_{R,k} \alpha_0\big(\delta(j)\big)$.
Contradiction, as α_0 is bad.

(ii) There exists a strictly increasing function $\delta : \mathbb{N} \to \mathbb{N}$ such that for every i, $\mathrm{Dom}\big(\tau(\delta(i)\big) \geq k$.
Applying the Finite Sequence Theorem we find i, j such that $i < j$ and some initial part of $\langle \gamma\big(\delta(i)\big), \gamma\big(\delta(j)\big)\rangle$ belongs to R^k and $\tau\big(\delta(i)\big)(\sqsubseteq_{R,k})^* \tau\big(\delta(j)\big)$ and therefore $\alpha_0\big(\delta(i)\big) \sqsubseteq_{R,k} \alpha_0\big(\delta(j)\big)$.
Contradiction, as α_0 is bad.

We conclude that there is no bad sequence. Therefore, every $\alpha : \mathbb{N} \to \mathcal{LT}_{\mathbb{N}}$ will meet $\sqsubseteq_{R,k}$.

9.4. Extending Kruskal's Theorem

We discuss a famous extension of Kruskal's Theorem found by H. Friedman. For each nonzero natural number k we introduce a subset $\mathcal{LT}_k$ of the set $\mathcal{LT}$ of labeled trees as follows:

(i) For each $j > k$, the ordered pair $\langle j, \emptyset\rangle$ belongs to $\mathcal{LT}_k$.
(ii) For each $j \leq k$, for each non-empty finite sequence T of elements $\mathcal{LT}_k$, the ordered pair $\langle j, T\rangle$ belongs to $\mathcal{LT}_k$.
(iii) Clauses (i), (ii) produce all elements of $\mathcal{LT}_k$.

Let R be a binary relation on $\mathbb{N}$.

For each nonzero natural number k we define a binary relation $\sqsubseteq^{\#}_{k,R}$ on the set $\mathcal{LT}_k$ as follows:

For all $\langle m, T\rangle$, $\langle n, U\rangle$ in $\mathcal{LT}_k$, $\langle m, T\rangle \sqsubseteq^{\#}_{k,R} \langle n, U\rangle$ if and only if *either* $T = U = \emptyset$ and $\langle m, n\rangle$ belongs to R, *or* $m = n \leq k$ and both T, U are non-empty and $T(\sqsubseteq^{\#}_{k,R})^* U$, that is, there exists a strictly increasing function h from $\mathrm{Dom}(T)$ to $\mathrm{Dom}(U)$ such that for each i in $\mathrm{Dom}(T)$, $T(i) \sqsubseteq^{\#}_{k,R} U\big(h(i)\big)$,
or $m \leq n < k$ and for some i in $\mathrm{Dom}(U)$, $\langle m, T\rangle \sqsubseteq^{\#}_{k,R} U\big(i\big)$.

Let $\langle m, T\rangle$ be some labeled tree.

An element s of $B(\langle m, T\rangle)$ is called an *interior* point of $B(\langle m, T\rangle)$ if $s * \langle 0\rangle$ belongs to $B(\langle m, T\rangle)$. An element s of $B(\langle m, T\rangle)$ is called an *endpoint* of $B(\langle m, T\rangle)$ if it is not an interior point of $B(\langle m, T\rangle)$.

One may prove the following:

For every binary relation R on $\mathbb{N}$, for every nonzero natural number k, for all $\langle m, T\rangle$, $\langle n, U\rangle$ in $\mathcal{LT}_k$, $\langle m, T\rangle \sqsubseteq^{\#}_{k,R} \langle n, U\rangle$ if and only if there exists a mapping from $B(\langle m, T\rangle)$ into $B(\langle n, U\rangle)$ such that

(i) for all s in $\mathbb{N}^*$, j in $\mathbb{N}$, if $s * \langle j\rangle$ belongs to $B(\langle m, T\rangle)$ then $f(s * \langle j\rangle)$ is a proper extension of $f(s)$.
(ii) for all s in $\mathbb{N}^*$, j in $\mathbb{N}$, if both $s * \langle j\rangle$ and $s * \langle j+1\rangle$ belong to $B(\langle m, T\rangle)$, then there exist k_0, k_1 such that $k_0 < k_1$ and $f(s) * \langle k_0\rangle$ is an initial part of $f(s * \langle j\rangle)$, and $f(s) * \langle k_1\rangle$ is an initial part of $f(s * \langle j+1\rangle)$.
(iii) for all s in $\mathbb{N}^*$, if s is an endpoint of $B(\langle m, T\rangle)$, then $f(s)$ is an endpoint of $B(\langle n, U\rangle)$ and the ordered pair $\langle \big(\ell(\langle m, T\rangle)\big)(s),\ \big(\ell(\langle n, U\rangle)\big)\big(f(s)\big)\rangle$ belongs to R.

(iv) for all s in $\mathbb{N}^*$, if s is an interior point of $B(\langle m, T\rangle)$, then $\big(\ell(\langle m, T\rangle)\big)(s) = \big(\ell(\langle n, U\rangle)\big)\big(f(s)\big)$.
(v) For all t in $\mathbb{N}^*$, if t is an initial part of $f(\langle\,\rangle)$, then $\big(\ell(\langle n, U\rangle)\big)(t) \geq m$. Also, for all s in $\mathbb{N}^*$, j in $\mathbb{N}$, if both s and $s * \langle j\rangle$ belong to $B(\langle m, T\rangle)$, then for every t in $\mathbb{N}^*$, if t properly extends $f(s)$ and is a proper initial part of $f(s * \langle j\rangle)$, then $\big(\ell(\langle n, U\rangle)\big)(t) \geq \big(\ell(\langle n, U\rangle)\big)\big(f(s * \langle j\rangle)\big)$.

Condition (v) is often called *Friedman's gap condition*.

Observe that for all trees $\langle 0, T\rangle$, $\langle 0, U\rangle$ in $\mathcal{LT}_1$, if there exists a subtree $\langle j, V\rangle$ of $\langle 0, U\rangle$ such that $\langle 0, T\rangle \sqsubseteq^{\#}_{1,R} \langle j, V\rangle$ then $\langle 0, T\rangle \sqsubseteq^{\#}_{1,R} \langle 0, U\rangle$. This statement is not true for trees $\langle 1, T\rangle$, $\langle 0, U\rangle$ in $\mathcal{LT}_1$.

Friedman's extension of Kruskal's Theorem says the following:

For every nonzero natural number k, for every binary relation R on $\mathbb{N}$, if R is almost full on $\mathbb{N}$, then $\sqsubseteq^{\#}_{k,R}$ is almost full on $\mathcal{LT}_k$.

This theorem is proved by induction on k, by a repeated minimal-bad-sequence-argument. I do not see how to replace this minimal-bad-sequence argument by a constructive argument of the same degree of credibility as the argument given for Kruskal's Theorem. The question if there is such an argument seems to be the most important question arising from this paper. In Section 10.7, the last Section of the paper, we will see that Friedman's result might be saved by a somewhat speculative extension of the axioms of intuitionistic analysis.

We now sketch Friedman's argument for the case $k = 1$. We have to make some preparations.

We let $\mathcal{LT}_1^0$ be the set of all trees of the form $\langle 0, T\rangle$ where T is a non-empty finite sequence of elements of $\mathcal{LT}_1$.

Let A be a subset of $\mathcal{LT}_1$. We let $\mathcal{LT}_1[A]$ be the subset of $\mathcal{LT}_1$ that is given by the following definition:

(i) Every element of A belongs to $\mathcal{LT}_1[A]$.
(ii) For every non-empty finite sequence T of elements of $\mathcal{LT}_1[A]$ the ordered pair $\langle 1, T\rangle$ belongs to $\mathcal{LT}_1[A]$.
(iii) Every element of $\mathcal{LT}_1[A]$ is produced from elements of A by repeated applications of step (ii).

The following statement is easily proved from Kruskal's Theorem:

For every subset A of $\mathcal{LT}$,
If $\sqsubseteq^{\#}_{1,R}$ is almost full on A, then $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}_1[A]$.

Let α be a function from $\mathbb{N}$ to $\mathcal{LT}_1$. We say α is *bad* if α does not meet $\sqsubseteq^{\#}_{1,R}$.

We now first prove that $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}_1^0$, as follows. Suppose that there exists at least one bad $\alpha : \mathbb{N} \to \mathcal{LT}_1^0$. Determine $\alpha_0 : \mathbb{N} \to \mathcal{LT}_1^0$ such that α_0 is bad and for each i, for each $\alpha : \mathbb{N} \to \mathcal{LT}_1^0$, if for each $j < i$, $\alpha(j) = \alpha_0(j)$, and $\alpha(i)$ is a proper subtree, but not necessarily an immediate subtree, of $\alpha_0(i)$, then α meets $\sqsubseteq^{\#}_{1,R}$. Let A be the set of all elements of $\mathcal{LT}_1^0$ that are a proper subtree of one of the trees $\alpha_0(0), \alpha_0(1), \ldots$. Reasoning as in earlier such cases,we conclude that $\sqsubseteq^{\#}_{1,R}$ is almost full on A.

Let B be the set of all basic trees in $\mathcal{LT}_1$, that is of all trees $\langle j, \emptyset\rangle$, where $j > 1$. As R is almost full on $\mathbb{N}$, $\sqsubseteq^{\#}_{1,R}$ is almost full on B.

Using Ramsey's Theorem we conclude that $\sqsubseteq^{\#}_{1,R}$ is almost full on $A \cup B$.

Using the remark we just made, we conclude that $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}_1[A \cup B]$.

We now reconsider α_0. Observe that there exists $\tau : \mathbb{N} \to (\mathcal{LT}_1[A \cup B])^*$ such that for each i, $\alpha_0(i) = \langle 0, \tau(i)\rangle$.

As, by the Finite Sequence Theorem, τ meets $(\sqsubseteq^{\#}_{1,R})^*$, α_0 will meet $\sqsubseteq^{\#}_{1,R}$.

Contradiction, as α_0 is bad.

We conclude that there is no bad $\alpha : \mathbb{N} \to \mathcal{LT}^0_1$.

Therefore $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}^0_1$.

It now follows easily that $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}_1$ as a whole.

It suffices to remark that $\mathcal{LT}_1$ coincides with $\mathcal{LT}_1[\mathcal{LT}^0_1]$.

10. The Principle of Open Induction

10.1. We consider the set $\mathcal{N}$ of all infinite sequences of natural numbers. For every α in $\mathcal{N}$, n in $\mathbb{N}$, we will write $\overline{\alpha}n := \langle \alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$.

Let A be a subset of the set $\mathbb{N}^*$ of all finite sequences of natural numbers. As in Section 4 we let $A^{\#}$ be the set of all α in $\mathcal{N}$ such that there exists n such that $\overline{\alpha}n$ belongs to A. α is A-*good* if α belongs to $A^{\#}$, α is A-*bad* otherwise.

Let α, β be elements of $\mathcal{N}$.

We define: α *comes before* β, notation $\alpha < \beta$, if and only if there exists i such that $\overline{\alpha}i = \overline{\beta}i$ and $\alpha(i) < \beta(i)$.

The argument underlying the results of the previous Section may be said to be the following

Minimal Bad Sequence Principle:
For every subset A of $\mathbb{N}^*$, if there exists α such that α does not belong to $A^{\#}$, then there exists α such that α does not belong to $A^{\#}$ while every β coming before α does belong to $A^{\#}$.

This principle is false when read as it stands and interpreted constructively.

It would imply that every inhabited subset A of the set $\mathbb{N}$ of natural numbers has a least element.

10.2. We consider the following contrapositive formulation of the Minimal Bad Sequence Principle:

Open Induction Principle:
For every subset A of $\mathbb{N}^*$, if every α belongs to $A^{\#}$ as soon as every β coming before α belongs to $A^{\#}$, then every α belongs to $A^{\#}$.

If we should accept this principle as an axiom of intuitionistic analysis, we could retain, after a slight revision, the arguments and results from Section 9.

Unfortunately, we fail to see why the principle in general is true, although we are to establish an important special case in Section 10.7.

10.3. It is useful to compare the Open Induction Principle with the well-known Principle of Induction on Monotone Bars.

Let P be a subset of $\mathbb{N}^*$. P is called a *bar* (or: a *bar in* $\mathcal{N}$) if and only if for every α there exists n such that $\langle \alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$ belongs to P, that is, $\mathcal{N}$ coincides with $P^\#$. P is called *monotone* if and only if for every s in $\mathbb{N}^*$, i in $\mathbb{N}$, if s belongs to P, then $s * \langle i \rangle$ belongs to P. P is called *hereditary* if and only if for every s in $\mathbb{N}^*$, if, for every i, $s * \langle i \rangle$ belongs to P, then s belongs to P.

Theorem 10.3.1 (Principle of Induction on Monotone Bars). *For every subset P of $\mathbb{N}^*$, if P is a monotone bar in $\mathcal{N}$ and a hereditary subset of $\mathbb{N}^*$, then the empty sequence $\langle\,\rangle$ belongs to P.*

Proof. Brouwer's Thesis guarantees that for every subset P of $\mathbb{N}^*$, if P is a bar, then there exists a stump σ such that $P \cap \sigma$ is a bar. So it suffices to prove, by induction on the set of stumps, that for every stump σ, for every subset P of $\mathbb{N}^*$, if $P \cap \sigma$ is a bar and P is a monotone and hereditary subset of $\mathbb{N}^*$, then the empty sequence $\langle\,\rangle$ belongs to P. We leave the straightforward proof to the reader. $\square$

10.4. We want to prove an extension of the Principle of Induction on Monotone Bars. Let T be a subset of $\mathbb{N}^*$. T is called a *frame* if and only if the empty sequence $\langle\,\rangle$ belongs to T and for every s in $\mathbb{N}^*$, s belongs to T if and only if there exists i such that $s * \langle i \rangle$ belongs to T.

Let $T \subseteq \mathbb{N}^*$ be a frame. We let $[T]$ be the set of all α in $\mathcal{N}$ such that, for every n, $\langle \alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$ belongs to T.

Let $T \subseteq \mathbb{N}^*$ be a frame and let P be a subset of T. P is called a *bar in* $[T]$ if and only if for every α in $[T]$ there exists n such that $\langle \alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$ belongs to P. P is called *monotone in T* if and only if for every s in P, $i \in \mathbb{N}$, if $s*\langle i \rangle$ belongs to T, then $s*\langle i \rangle$ belongs to P. P is called *hereditary in T* if and only if for every s in T, if, for every i such that $s*\langle i \rangle$ belongs to T, $s*\langle i \rangle$ belongs to P, then s belongs to P.

Theorem 10.4.1 (Principle of Induction on Monotone Bars in Decidable Frames). *Let T be a decidable subset of $\mathbb{N}^*$ and a frame. For every subset P of T, if P is a bar in $[T]$, and monotone in T and hereditary in T, then the empty sequence $\langle\,\rangle$ belongs to P.*

Proof. Let T be a decidable subset of $\mathbb{N}^*$ and a frame. We define a function R from $\mathbb{N}^*$ to T, as follows: $R(\langle\,\rangle) := \langle\,\rangle$ and for each s in $\mathbb{N}^*$, i in $\mathbb{N}$, if $s * \langle i \rangle$ belongs to T, then $R(s * \langle i \rangle) := s * \langle i \rangle$, and if $s * \langle i \rangle$ does not belong to T, then $R(s*\langle i \rangle) := R(s)*\langle i_0 \rangle$ where i_0 is the least j such that $R(s)*\langle j \rangle$ belongs to T. We let ρ be the function from $\mathcal{N}$ to $[T]$ such that for all α, for all n, $\overline{\rho(\alpha)}n = R(\overline{\alpha}n)$. For every α in $[T]$, $\rho(\alpha) = \alpha$. ρ is called a *retraction* from $\mathcal{N}$ onto $[T]$.

Now let P be a subset of T and a bar in $[T]$. Let Q be the set of all s in $\mathbb{N}^*$ such that $R(s)$ belongs to P. Observe that for every α we may determine n such that $\overline{\rho(\alpha)}n$ belongs to P. It follows that Q is a bar in $[\mathbb{N}^*] = \mathcal{N}$. If, in addition, P is monotone in T and hereditary in T, then Q is monotone and hereditary in $\mathbb{N}^*$, and the empty sequence will belong to Q and therefore to P. $\square$

A decidable frame is called a *spread direction* in [1].

10.5. It has been asked if the principle that we obtain by removing the condition of decidability from the Principle of Induction on Monotone Bars in Decidable Frames, is acceptable as an axiom of intuitionistic analysis.

> *Principle of Induction on Monotone Bars in Frames:*
> Let $T \subseteq \mathbb{N}^*$ be a frame. For every subset P of T, if P is a bar in $[T]$, and monotone in T and hereditary in T, then the empty sequence $\langle\,\rangle$ belongs to P.

Before going into this question we first show that this principle entails the principle of Open Induction.

Theorem 10.5.1. *The Principle of Induction on Monotone Bars in Frames implies the Open Induction Principle.*

Proof. Let A be a subset of $\mathbb{N}^*$ such that, for every α in $\mathcal{N}$, if every β coming before α has an initial part in A, then α has an initial part in A. Let α belong to $\mathcal{N}$ and s to $\mathbb{N}^*$. We define:

> α *comes before* s if and only if there exists $i < \mathrm{Dom}(s)$ such that $\alpha(i) < s(i)$ and for all $j < i$, $\alpha(j) = s(j)$.

Observe that for every α in $\mathcal{N}$, for every s in $\mathbb{N}^*$, α comes before $s * \langle 0\rangle$ if and only if α comes before s, and, for each n in $\mathbb{N}$, α comes before $s * \langle n+1\rangle$ if and only if α comes before $s * \langle n\rangle$ or $s * \langle n\rangle$ is an initial part of α. Let T be the set of all s in $\mathbb{N}^*$ such that every α coming before s has an initial part in A. Observe that T is a frame, and that every α in $[T]$ has an initial part in A. Let P be the set of all s in T such that every α in $\mathcal{N}$ that has s as an initial part has an initial part in A. Observe that P is a bar in $[T]$ and that P is monotone in T. P is also hereditary in T. For suppose s belongs to T, and for every i, if $s * \langle i\rangle$ belongs to T, then $s * \langle i\rangle$ belongs to P. Then $s * \langle 0\rangle$ belongs to T and therefore to P, so $s * \langle 1\rangle$ belongs to T and therefore to P, so $s * \langle 2\rangle$ belongs to T and therefore to $P, \ldots$. We conclude that, for every i, $s * \langle i\rangle$ belongs to P, so s belongs to P. Using the Principle of Induction on Monotone Bars in Frames we conclude that the empty sequence belongs to P, that is, every α has an initial part in A. □

If one is prepared to accept Brouwer's Thesis and its consequence, the Principle of Induction on Monotone Bars in Decidable Frames, what reason could one have not to accept the more general Principle of Induction on Monotone Bars in Frames?

We should reflect on the meaning of the statement:

$$P \text{ is a bar in } [T]$$

which says:

> For every α in $\mathcal{N}$, **if,** for every m, $\overline{\alpha}m$ belongs to T,
> **then** there exists n such that $\overline{\alpha}n$ belongs to P.

This is a weak statement. It says that, if we are given some sequence α, we will be able to produce a natural number n such that $\overline{\alpha}n$ belongs to P, *but only if we* *are first given a proof that for every m, $\overline{\alpha}m$ belongs to T.* As long as we have no such proof, we may very well be unable to calculate n such that, if for every m, $\overline{\alpha}m$ belongs to T, then $\overline{\alpha}n$ belongs to P.

(A logical scheme enabling one to conclude draw a conclusion $\exists x[A \to B]$ from a hypothesis of the form $A \to \exists x[B]$ is sometimes called an *independence-of-premiss* scheme. There is no constructive justification for such schemes).

For this reason, a monotone bar in a frame T does not always give rise to a monotone bar in $\mathbb{N}^*$.

10.6. Nevertheless, the Principle of Induction Monotone Bars admits of a further extension.

Theorem 10.6.1 (Principle of Induction on Monotone Bars in Enumerable Frames). *Let T be an enumerable subset of $\mathbb{N}^*$ and a frame. For every subset P of T, if P is a bar in $[T]$, and monotone in T and hereditary in T, then the empty sequence $\langle\ \rangle$ belongs to P.*

Proof. Let T be an enumerable subset of $\mathbb{N}^*$ and a frame. Let f be a function from $\mathbb{N}$ to $\mathbb{N}^*$ enumerating T. Observe that, for each i, j in $\mathbb{N}$, if $f(i) * \langle j \rangle$ belongs to T, then there exists k in $\mathbb{N}$ such that $f(k) = f(i) * \langle j \rangle$. We define a function R from $\mathbb{N}^*$ to T, as follows: $R(\langle\ \rangle) := \langle\ \rangle$ and for each s in $\mathbb{N}^*$, i in $\mathbb{N}$, if there exists j such that $f(i) = R(s) * \langle j \rangle$, then $R(s * \langle i \rangle) = f(i)$, and if not, then $R(s * \langle i \rangle) = f(i_0)$, where i_0 is the least k such that, for some j, $f(k) = R(s) * \langle j \rangle$. We let ρ be the function from $\mathcal{N}$ to $\mathcal{N}$ such that for all α, for all n, $\overline{\rho(\alpha)(n)} = R(\overline{\alpha}n)$. Observe that, for every α, $\rho(\alpha)$ belongs to $[T]$. Also, for every β in $[T]$ there exists α in $\mathcal{N}$ such that $\rho(\alpha) = \beta$. Let P be a bar in $[T]$, and monotone in T and hereditary in T. Let Q be the set of all s in $\mathbb{N}^*$ such that $R(s)$ belongs to P. Observe that Q is a bar in $[\mathbb{N}^*]$ and monotone in $\mathbb{N}^*$, and hereditary in $\mathbb{N}^*$. The empty sequence will belong to Q and therefore to P. □

The following surprising consequence of Theorem 10.6.1 was first proved by Thierry Coquand.

Theorem 10.6.2 (Open Induction for Cantor space, Th. Coquand, 1997). *For every decidable subset A of $\{0, 1\}^*$, if every α in $\mathcal{C}$ belongs to $A^\#$ as soon as every β in $\mathcal{C}$ coming before α belongs to $A^\#$, then every α in $\mathcal{C}$ belongs to $A^\#$.*

Proof. Let T be the set of all finite sequences s in $\{0, 1\}^*$ such that every α in $\mathcal{C}$ coming before s belongs to $A^\#$.

Observe that for every s in $\{0, 1\}^*$, the set of all α in $\mathcal{C}$ coming before s is a fan. Therefore s belongs to T if and only if there exists n in $\mathbb{N}$ such that for every α in $\mathcal{C}$ coming before s there exists $m \leq n$ such that $\overline{\alpha}m$ belongs to A. It follows that T is enumerable. T is also a frame as for every s in $\{0, 1\}^*$, s belongs to T if and only if $s * \langle 0 \rangle$ belongs to T.

We let P be the set of all s in T such that for all α in $\mathcal{C}$, if s is an initial part of α, then α belongs to $A^\#$. Observe that P is a bar in $[T]$, and a monotone subset of T.

P is also a hereditary subset of T. For suppose that s belongs to T, and that for every n, if $s * \langle n \rangle$ belongs to T, then $s * \langle n \rangle$ belongs to P. As $s * \langle 0 \rangle$ belongs to T, it follows that every α in $\mathcal{C}$ such that $s * \langle 0 \rangle$ is an initial part of α belongs to $A^{\#}$. Also, every α in $\mathcal{C}$ coming before s belongs to $A^{\#}$, as s belongs to T. We conclude that every α coming before $s * \langle 1 \rangle$ belongs to $A^{\#}$, therefore $s * \langle 1 \rangle$ belongs to T and also to P. We conclude that every α such that s is an initial part of α belongs to $A^{\#}$, that is, s belongs to P.

Using Theorem 10.6.1 we conclude that the empty sequence $\langle\,\rangle$ belongs to P. Therefore, every α in $\mathcal{C}$ belongs to $A^{\#}$. □

10.6.1. The Principle of Open Induction for Cantor space extends to a Principle of Open Induction for the real closed interval $[0, 1]$ and the set $[0, \infty)$ in the following way.

Let $\mathbb{Q}$ be the set of rational numbers and let $\rho : \mathbb{N} \to \mathbb{Q}$ be some enumeration of $\mathbb{Q}$.

Let $J : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijective map, a so-called pairing function, with inverse functions K, L, such that for every n, $n = J\big(K(n), L(n)\big)$. For every natural number m we define: $m' := \rho\big(K(m)\big)$ and $m'' := \rho\big(L(m)\big)$. For every α in $\mathcal{N}$ we define functions α', α'' from $\mathbb{N}$ to $\mathbb{Q}$ such that, for every n, $\alpha'(n) := (\alpha(n))'$ and $\alpha''(n) := (\alpha(n))''$. An element α of $\mathcal{N}$ is called a *real number* if and only if, for each n, $\alpha'(n) \leq \alpha'(n+1) \leq \alpha''(n+1) \leq \alpha''(n)$ and for each p there exists n such that $\alpha''(n) - \alpha'(n) \leq \frac{1}{p}$.

Let α, β be real numbers. α *really-coincides with* β if and only if for each n, $\alpha'(n) \leq \beta''(n)$ and $\beta'(n) \leq \alpha''(n)$. We denote the set of real numbers by $\mathbb{R}$. Let X, Y be subsets of $\mathbb{R}$. We say that X *really-coincides* with Y if and only if every member of X really-coincides with a member of Y and every member of Y really-coincides with a member of X.

$[0, 1]$ is the set of all real numbers α such that for each n, $0 \leq \alpha''(n)$ and $\alpha'(n) \leq 1$.

We now consider the set $\{0, 1, 2\}^*$ of all finite sequences assuming values in the set $\{0, 1, 2\}$. We construct a mapping H from $\{0, 1, 2\}^*$ to $\mathbb{N}$ as follows. We let $H(\langle\,\rangle)$ be the natural number m such that $m' = 0$ and $m'' = 1$. Assume that s belongs to $\{0, 1, 2\}^*$ and that we defined $H(s)$. For each $i < 3$ we now define $H(s * \langle i \rangle)$, in such a way that $(H(s * \langle 0 \rangle))' = (H(s))'$,

$(H(s*\langle 1 \rangle))' = H((s))' + \frac{1}{4}((H(s))'' - (H(s))'')$ and $(H(s*\langle 2 \rangle))' = (H(s))' + \frac{1}{2}((H(s))'' - (H(s)'))$ and, for each $i < 3$, $(H(s * \langle i \rangle))'' - (H(s * \langle i \rangle))' = \frac{1}{2}((H(s))'' - (H(s))')$.

We define a mapping h from the fan $\{0, 1, 2\}^{\mathbb{N}}$ to $[0, 1]$ as follows: for every α, n, $\big(h(\alpha)\big)(n) = H(\overline{\alpha}n)$. Observe that, for every α in $\{0, 1, 2\}^{\mathbb{N}}$, $h(\alpha)$ belongs to $[0, 1]$, and that for every β in $[0, 1]$ there exists α in $\{0, 1, 2\}^{\mathbb{N}}$ such that $h(\alpha)$ really coincides with β.

Let A be a subset of $\mathbb{N}$. We let $A^{\flat}$ be the set of all real numbers α such that there exists n, m such that m belongs to A and $m' < \alpha'(n) \leq \alpha''(n) < m''$. We might call $A^{\flat}$ the (real) open set determined by A.

Let α, β be real numbers. We say that α is *smaller* than β, notation $\alpha < \beta$, if and only if there exists n such that $\alpha''(n) < \beta'(n)$.

We let $[0, \infty)$ be the set of all real numbers α such that for each n, $0 \leq \alpha(n)''$. A subset B of $[0, 1]$, (or $[0, \infty)$, respectively), is called *progressive in* $[0, 1]$ (or: *in* $[0, \infty)$) if and only if every α in $[0, 1]$, (or $[0, \infty)$, respectively) belongs to B as soon as every β smaller than α belongs to B.

Theorem 10.6.3 (Open Induction for $[0, 1]$ **and** $[0, \infty)$**, Th. Coquand, 1997).**

(i) For every decidable subset A of $\mathbb{N}$, if $A^\flat$ is progressive in $[0, 1]$, $[0, 1]$ is a subset of $A^\flat$.

(ii) For every decidable subset A of $\mathbb{N}$,
if $A^\flat$ is progressive in $[0, \infty)$, then $[0, \infty)$ is a subset of $A^\flat$.

Proof. (i) Let A be a decidable subset of $\mathbb{N}$ such that $A^\flat$ is progressive in $[0, 1]$.
We let B be the set of all finite sequences s in $\{0, 1, 2\}^*$ such that there exists $m \leq \text{length}(s)$, m in A and $m' < H(s) < H(s)'' < m''$.
Observe that B is a decidable subset of $\{0, 1, 2\}^*$ and that $B^\#$ consists of all α in $\{0, 1, 2\}^\mathbb{N}$ such that $h(\alpha)$ belongs to $A^\flat$. A moment's reflection shows that every α in $\{0, 1, 2\}^\mathbb{N}$ belongs to $B^\#$ as soon as every β in $\{0, 1, 2\}^\mathbb{N}$ coming before α belongs to $B^\#$, therefore, by a straightforward extension of Theorem 10.6.2, $B^\#$ coincides with $\{0, 1, 2\}^\mathbb{N}$, and $[0, 1]$ is a subset of $A^\flat$.

(ii) Let A be a decidable subset of $\mathbb{N}$ such that $A^\flat$ is progressive in $[0, \infty)$.
One proves, by induction, using (i), that, for each n, $A^\flat$ is progressive in $[n, n + 1)$, and every member of $[n, n + 1)$ belongs to $A^\flat$. Therefore, $[0, \infty)$ is a subset of $A^\flat$. □

Another proof of the principle of Open Induction for $[0, 1]$ may be found in [29] and [30].

10.7. Does not an extension of the argument given for Theorem 10.6.2 prove the Open Induction Principle in general? We used the fact that, given any decidable subset A of $\{0, 1\}^*$, the set of all s in $\{0, 1\}^*$ such that every α in $\mathcal{C}$ coming before s belongs to $A^\#$ is enumerable. Now suppose A is a decidable subset of $\mathbb{N}^*$ rather than $\{0, 1\}^*$ and consider the set of all s in $\mathbb{N}^*$ such that every α in $\mathcal{N}$ coming before s belongs to $A^\#$. This set is a Π^1_1 subset of $\mathbb{N}^*$ and it seems difficult to show that it is enumerable, if one faces the following two facts. Firstly, in classical recursion theory, a Π^1_1 set is not always recursively enumerable. Secondly, in intuitionistic as well as in classical descriptive set theory a co-analytic set is not always strictly analytic, that is, for some inhabited co-analytic subsets X of Baire space $\mathcal{N}$ there does not exist a continuous function from $\mathcal{N}$ *onto* X, see [28]. On the other hand it is argued in [5] that every inhabited and "determinate" subset of $\mathbb{N}$ must be enumerable, irrespectively of the logical complexity of its definition. The notion of a "determinate" subset of $\mathbb{N}$ is not very precise, but it would entail that for every such subset X of $\mathbb{N}$, and every natural number n, the proposition "$n \in X$" is "determinate", that is, it does not involve "incomplete objects" like an infinite sequence of natural numbers that is constructed step by step and not dictated by an algorithm.

Following J.J. de Iongh, the authors of [5] suggest that to any determinate proposition P the so-called *Brouwer-Kripke-axiom* may be applied, that is, one may assume that there exists α in $\mathcal{C}$ such that

P if and only if, for some n in $\mathbb{N}$, $\alpha(n) = 1$.

They also have at their disposal the following

Axiom of Countable Choice:
For every subset R of $\mathbb{N} \times \mathcal{N}$, if for every n in $\mathbb{N}$ there exists α in $\mathcal{N}$ such that $A(n, \alpha)$, then there exists α such that, for every n, $A(n, \alpha^n)$. (For each α, for each n, for each m, $\alpha^n(m) := \alpha(J(n, m))$.)

It follows that for every determinate subset X of $\mathbb{N}$ there exists α in $\mathcal{C}$ such that, for every n, n belongs to X if and only if, for some m, $\alpha^n(m) = 1$. If X is also inhabited one easily constructs a function f from $\mathbb{N}$ to $\mathbb{N}$ enumerating X.

The authors of [5] use this result in both of their two intuitionistic versions of Souslin's extension of Cantor's Theorem. Souslin's theorem says (classically) that not only every closed, but also every analytic subset of $\mathbb{R}$ is either at most countable or at least as big as $\mathcal{C}$. Its two intuitionistic versions in [5] are due to Wim Gielen and John Burgess, respectively.

We formulate a final theorem that does not depend on the Brouwer-Kripke axiom.

Theorem 10.7.1 (Intuitionistic Principle of Open Induction). *For every decidable subset A of $\mathbb{N}^*$, if every α belongs to $A^\#$ as soon as every β coming before α belongs to $A^\#$, and the set of all s in $\mathbb{N}^*$ such that every α coming before s belongs to $A^\#$ is enumerable, then every α belongs to $A^\#$.*

Proof. Like the proof of Theorem 10.6.2, using Theorem 10.6.1. □

Friedman's extension of Kruskal's Theorem follows from Theorem 10.7.1 if we are prepared to use along with the principle of induction on monotone bars, a restricted form of the Brouwer-Kripke-axiom and the usual axiom of countable choice.

11. Concluding remarks

This paper was elicited by a purported intuitionistic proof of Kruskal's Theorem given by Thierry Coquand. He used the principle of Open Induction explained in Section 10. I felt dissatisfied with this proof as it exceeds the bounds of intuitionistic analysis as formalized in [7]. I had the impression that the original proofs given by Higman and Kruskal were more constructive notwithstanding the fact that these authors freely use classical logic. I wrote this paper in order to verify this impression in detail.

I discussed these matters with Thierry Coquand when visiting him in Göteborg in February 1997. He then discovered the two special cases of principle of Open Induction mentioned and proved in Section 10. Some years before, we had exchanged views on possible intuitionistic versions of Ramsey's Theorem, see [25], and [2]. Kruskal's Theorem is of course a Ramseyan Theorem, so it was natural that we should study its constructive content.

J.H. Gallier, in his survey paper [4] mentions the finding of a constructive proof of Kruskal's Theorem as a major problem. Various people were searching for constructive proofs of Ramseyan theorems, see for instance [13], and [17]. In [16] the Finite Sequence Theorem is proved for decidable relations on $\mathbb{N}$.

Kruskal's Theorem plays an important role in proof theory. There are deep connections with proof theoretic ordinals and the project of Reverse Mathematics initiated by H. Friedman, see [22]. It seems that ordinals made their entry in the discussions about Kruskal's Theorem in [19]. Monica Seisenberger succeeded in developing a constructive proof, avoiding ordinals, from the ordinal-theoretic proof in [15], see [21]. She restricts herself to the case of decidable relations on $\mathbb{N}$. Another difference with the present paper is that she avoids Brouwer's Thesis, thereby following a line recommended by P. Martin-Löf, see [12]. If one prefers not to use Brouwer's Thesis one may *define* a relation R to be *almost full* or *unavoidable* if and only if there exists a stump σ such that every finite sequence of natural numbers not belonging to σ meets R. This of course is a difference in style mainly, the problem of how to prove Kruskal's Theorem remains the same.

The question if the generalized principle of induction on monotone bars mentioned in Section 10 is intuitionistically acceptable was raised already by G. Kreisel, see the preface to Volume II of [8]. Such an extension would enable one to give an intuitionistic consistency proof for classical analysis. H. Luckhardt defended the extension as a natural one in [11]. The extension is also discussed by A.S. Troelstra in [24]. He carefully distinguishes between various possible formulations of the extension.

It seems that Friedman's extension of Kruskal's Theorem, mentioned in Section 9, came to be thought of in connection with the large project of proving the Graph Minor Theorem, see [18].

The special case of this extension discussed in [20] is provable intuitionistically as well as classically.

I want to express my thanks to Andreas Blass, Mark Bickford, Victor Krivtsov and an anonymous referee for calling my attention to some mistakes and many inaccuracies in an earlier version of this paper. Their observations sometimes led to substantial improvements.

References

1. Brouwer, L.E.J.: Points and spaces. Canad. J. Math. **6**, 1–17 (1954)
   also In: L.E.J. Brouwer, Collected Works, Vol.I. Philosophy and Foundations of Mathematics, A. Heyting, (ed.), North Holland Publ. Co., Amsterdam etc., 1975, pp. 522–538
2. Coquand, Th.: An analysis of Ramsey's Theorem. Inf. Comp. **110**, 297–304 (1994)
3. Dickson, L.E.: Finiteness of the odd perfect and primitive abundant numbers with n distinct factors. Am. J. Math. **35**, 413–426 (1913)
4. Gallier, J.H.: What's so special about Kruskal's theorem and the ordinal Γ_0? A survey of some results in proof theory. Ann. Pure Appl. Logic **53**, 199–260 (1991)
5. Gielen, W., de Swart, H., Veldman, W.: The continuum hypothesis in intuitionism. J. Symb. Logic **46**, 121–136 (1981)

6. Higman, G.: Ordering by divisibility in abstract algebras. Proc. London Math. Soc. **2**, 326–336 (1952)
7. Kleene, S.C., Vesley, R.E.: The Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions. North Holland Publ. Co., Amsterdam etc., 1965
8. Kreisel, G.: Reports of the Seminar on Foundations of Analysis. Stanford University, 1963
9. Kruskal, J.B.: Well-quasi-ordering, the tree theorem, and Vazsonyi's Conjecture. Trans. Am. Math. Soc. **95**, 210–225 (1960)
10. Kruskal, J.B.: The Theory of Well-Quasi-Ordering: A Frequently Discovered Concept. J. Combinatorial Theory, Series A **13**, 297–305 (1972)
11. Luckhardt, H.: Extensional Gödel Functional Interpretation, a Consistency Proof of Classical Analysis. Lecture Notes in Mathematics, Vol. 306, Springer-Verlag, Berlin etc., 1973
12. Martin-Löf, P.: Notes on Constructive Mathematics. Almquist and Wiksell, Stockholm, 1970
13. Murthy, C.R., Russell, J.R.: A constructive proof of Higman's Lemma. 5th Annual Symposium on Logic in Computer Science, Philadelphia PA. (IEEE Computer Soc. Press, Silver Spring, MD) pp. 257-267
14. Nash-Williams, C.St.J.A.: On well-quasi-ordering finite trees. Proc. Cambridge Phil. Soc. **59**, 833–835 (1963)
15. Ramsey, F.P.: On a problem of formal logic. Proc. London Math. Soc. **30**, 264–286 (1928)
16. Rathjen, M., Weiermann, A.: Proof-theoretic investigations on Kruskal's theorem. Ann. Pure Appl. Logic **60**, 49–88 (1993)
17. Richman, F., Stolzenberg, G.: Well Quasi-Ordered Sets. Adv. Math. **97**, 145–153 (1993)
18. Robertson, N., Seymour, P.D.: Graph Minors. IV. Tree-Width and Well-Quasi-Ordering. J. Combinatorial Theory, Series B **48**, 227–254 (1990)
19. Schmidt, D.: Well-Partial Orderings and Their Maximal Order Types. Habilitationsschrift, Heidelberg, 1979
20. Schütte, K., Simpson, S.G.: Ein in der reinen Zahlentheorie unbeweisbarer Satz über endliche Folgen von natürlichen Zahlen. Arch. Math. Logik 25, 75–89 (1985)
21. Seisenberger, M.: Kruskal's tree theorem in a constructive theory of inductive definitions. In: P. Schuster, U. Berger and H. Osswald, (eds.), Reuniting the Antipodes, Constructive and Nonstandard Views of the Continuum, Proceedings of a Symposium held in San Servolo/Venice, May 16-22, 1999, Kluwer Academic Publishers, Dordrecht etc., 2001, pp. 241–255
22. Simpson, S.G.: Nonprovability of certain combinatorial properties of finite trees. In: L.A. Harrington et al., (eds.), Harvey Friedman's research on the foundations of mathematics, Studies in Logic and the foundations of mathematics, Vol. 117, North Holland Publ.Co., Amsterdam etc., 1975, pp. 87–117
23. Simpson, S.: Ordinal numbers and the Hilbert basis theorem. J. Symb. Logic **53**, 961–974 (1988)
24. Troelstra, A.S.: Extended Bar Induction of type zero. In: J. Barwise, H.J. Keisler and K. Kunen, (eds.), The Kleene Symposium, Proc., Madison/Wis. USA, 1978, Studies in Logic and the Foundations of Mathematics, Vol. 101, North Holland Publ. Co., Amsterdam etc., 1980, pp. 277–316
25. Veldman, W., Bezem, M.: Ramsey's Theorem and the pigeonhole principle in intuitionistic mathematics. J. London Math. Soc. **47**, 193–211 (1993)
26. Veldman, W.: Some intuitionistic variations on the notion of a finite set of natural numbers. In: H.C.M. de Swart and L.J.M. Bergmans, (eds.), Perspectives on Negation, essays

in Honour of Johan J. de Iongh on the Occasion of His 80th Birthday, Tilburg University Press, Tilburg, 1995, pp. 177–202
27. Veldman, W.: On sets enclosed between a set and its double complement. In: A. Cantini et al., (eds.), Logic and Foundations of Mathematics, Proceedings Xth International Congress on Logic, Methodology and Philosophy of Science, Florence 1995, Volume III, Kluwer Academic Publishers, Dordrecht etc., 1999, pp. 143–154
28. Veldman, W.: The Borel Hierarchy and the Projective Hierarchy in Intuitionistic Mathematics. Report No. 0103, Department of Mathematics, University of Nijmegen, 2001
29. Veldman, W.: Almost the Fan Theorem. Report No. 0113, Department of Mathematics, University of Nijmegen, 2001
30. Veldman, W.: Bijna de waaierstelling. Nieuw Archief voor Wiskunde, vijfde serie, deel **2**, 330–339 (2001)