Introductory note to 1913 and D. König 1927b

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Zermelo's *1913, Über eine Anwendung der Mengenlehre auf die Theorie des Schachpiels,* is an account of an address given at the Fifth International Congress of Mathematicians in Cambridge in 1912. It is often cited as the first mathematical analysis of strategies in games. While the paper claims to be an application of set theory, and while it would have appeared that way to Zermelo's contemporaries, the set-theoretic notions in the paper have since become part of standard mathematical practice, and to modern eyes the arguments in the paper are more combinatorial than set-theoretic.[1] The notion of "Zermelo's Theorem" (usually described as a variant of "in chess, either White or Black has a winning strategy, or both can force a draw") derives from this paper. Although statements of this sort follow from the claims made in the paper, Zermelo's arguments for these claims are incomplete. As we shall see below, there are other gaps in the paper, one of which was fixed by Dénes König in his *1927a*. König's paper also contains two paragraphs, *1927b*, on arguments of Zermelo fixing this gap, using ideas similar to König's.

At the beginning of his *1913*, Zermelo notes that although he will discuss chess, his arguments apply to a wider class of games. Initially he describes this class as those two-player games "of reason" in which chance has no role. In the second paragraph of the paper, he makes the assumption that the game has only finitely many possible positions (or, rather, invokes the fact that this is true of chess, where a position of the game consists of the positions of all the pieces plus the identity of the player to move next and information such as which players have castled[2]), and in the third paragraph he says that the rules of the game allow infinite runs, which should be considered ties. In the first paragraph he mentions that there are many positions in the game of chess for which it is known that one player or the other can force a win in a certain number of moves, and proposes investigating whether such an analysis is possible in principle for arbitrary positions.

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[1] As an example of how set-theoretic language was perceived at the time (and even much later), note that von Neumann and Morgenstern spend Sections 8–10 of their *1943* on the importance of set-theoretic notions for studying games.

[2] If one includes the list of previous moves then the set of positions becomes infinite.

Zermelo's analysis begins by letting P denote the set of all possible positions of the game, and letting $P^{\mathfrak{a}}$ denote the set of countable sequences of positions, finite or infinite. Fixing a position q, he lets Q be the set of all sequences in $P^{\mathfrak{a}}$ starting with q such that for each successive pair of positions in the sequence the latter member is obtained by a legal move from the former, and such that the sequence either continues infinitely or ends with a stalemate or a win for one player or the other. Zermelo notes that given a position q and a natural number r (Zermelo is not explicit about whether the case $r = 0$ is to be included), White can force a win from q in at most r moves if and only if there is a nonempty set $U_r(q)$ of members of Q such that each $\mathbf{q}$ in $U_r(q)$ is a continuation of q in which White wins in at most r moves (starting from q), and such that for each $\mathbf{q} \in U_r(q)$ and each position in $\mathbf{q}$ where it is Black's turn to move, and for each possible move for Black at that point, there is a $\mathbf{q}' \in U_r(q)$ which agrees with $\mathbf{q}$ up to this point, and has the position resulting from Black making this move as its next member. In more modern terminology (appearing no later than *Kuhn 1953*), Zermelo has introduced here a *game-tree* with root q of *height* at most $r+1$, in which all terminal nodes are wins for White, and in which all nodes for which it is Black's turn to move have successors corresponding to every move available to Black at that position. The existence of such a tree is indeed equivalent to the existence of a *quasi-strategy*[3] for White guaranteeing a win in r moves or fewer: White simply plays to maintain the condition that the continuation of the game starting from q is an initial segment of a member of $U_r(q)$.

Still fixing q and r, Zermelo notes that the union of all such sets $U_r(q)$ would also satisfy the conditions on $U_r(q)$. He calls this union $\overline{U}_r(q)$, and notes that as r increases the sets $\overline{U}_r(q)$ also increase under $\subseteq$ (though of course they may eventually all be the same, and may all be the empty set). For each q such that $\overline{U}_r(q)$ is nonempty for some natural number r, Zermelo lets ρ_q be the least such r, and he lets $U^*(q)$ denote $\overline{U}_{\rho_q}(q)$. He also lets τ denote the maximum of the set of defined values ρ_q.

Zermelo lets t be the integer such that $t+1$ is the size of P, and presents an argument to the effect that $\tau \leq t$. The idea behind this argument is that if some position is repeated during a play by a winning quasi-strategy for White, then one could adjust the quasi-strategy to play from the first occurrence of this position in the way that one played from the second, thus winning in fewer moves. This argument was later shown by König in *1927a* to be incomplete, as it does not account for all possible sequences of moves for Black; that is, playing with the same strategy does not guarantee the same resulting sequence of moves.

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[3] A *strategy* for White specifies a move for White in each position obtainable by the strategy; a quasi-strategy merely specifies an acceptable set of moves (see *Kechris 1995*). The distinction is important when the Axiom of Choice fails, but is less important here, since P is finite. Nonetheless, we will use the term "quasi-strategy" for the sets of sequences described by Zermelo in this paper.

For each q, Zermelo lets $U(q)$ denote $\overline{U}_\tau(q)$, and claims that $U(q)$ being nonempty is equivalent to the assertion that q is a winning position for White. In fact, $U(q)$ being nonempty is equivalent to the existence of some natural number r such that White has a quasi-strategy guaranteeing a win in r moves or fewer. Zermelo does not address in this paper what it means for a player to have a quasi-strategy guaranteeing a win without specifying an upper bound on the number of moves needed to win. König's subsequent work *1927a* would show that in games in which each player has only finitely many moves available in each position, having a winning quasi-strategy in this more general sense implies having one with a fixed upper bound for the number of moves needed.

Zermelo then defines sets $V_s(q)$, analogous to $U_s(q)$ except that the corresponding quasi-strategies merely guarantee that White does not lose in fewer than s moves, though they allow that White loses on the s-th move. So each $V_s(q)$ is a set of members of Q such that each $\mathbf{q}$ in $V_s(q)$ is a continuation of q in which White does not lose in fewer than s moves starting from q, and such that for each $\mathbf{q} \in V_s(q)$ and each position in $\mathbf{q}$ where it is Black's turn to move, and for each possible move for Black at that point, there is a $\mathbf{q}' \in V_s(q)$ which agrees with $\mathbf{q}$ up to this point, and has the position resulting from Black making this move as its next member. Again, the union $\overline{V}_s(q)$ of all such $V_s(q)$ satisfies these conditions. Now, however, the sets $\overline{V}_s(q)$ are shrinking as s increases.

Zermelo now remarks that, given q, if $\overline{V}_s(q)$ is empty for any positive integer s, then, letting σ be the maximal s for which $\overline{V}(s)$ is nonempty, $\sigma \leq \tau$ (he also lets $V^*(q)$ denote $\overline{V}_\sigma(q)$ in this case). The argument for this is not given (Zermelo also reiterates here that $\tau \leq t$, which, as we noted above, is not satisfactorily demonstrated in this paper, but that issue does not affect this one.) The first missing claim is that if $\overline{V}_s(q)$ is empty, then Black has a quasi-strategy which guarantees a win in $s-1$ moves or fewer, starting from q. Modulo precise notions of *game* and *strategy*, this fact is sometimes called *determinacy for fixed finite length games of perfect information*; indeed, this assertion is often called Zermelo's Theorem, referring to the arguments in this paper (a generalization is called "the theorem of Zermelo-von Neumann" in *Kuhn 1953*). Granting this point, one needs to see that if Black has a quasi-strategy guaranteeing a win in $s-1$ moves or fewer, then he or she has a quasi-strategy guaranteeing a win in τ moves or fewer. Given the definition of τ this is clear for suitably symmetric games,[4] but it need not hold in general. Finally, it is clear that if Black has a quasi-strategy that guarantees a win in τ moves or fewer starting from q, then $\overline{V}_{\tau+1}(q)$ is empty.

[4] That is, games where for every position where it is White's term to move there is a position where is it Black's turn such that the game trees below the two conditions are isomorphic. Strictly speaking this is not true of chess, since it can only be White's turn when the pieces are in their initial position.

Zermelo lets $V(q)$ denote $\overline{V}_{\tau+1}(q)$, and claims that $V(q)$ being nonempty is equivalent to White being able to force a draw from the position q. This claim is missing the same steps as the corresponding claim for $U(q)$ above. Given that the game is suitably symmetric, the statement that $V(q)$ is nonempty is equivalent to the statement that White can delay a loss by any specified amount he or she chooses, which again by the subsequent work of König and the finiteness of chess means being able to delay a loss indefinitely.

The penultimate paragraph of the paper provides a partial summary, and asserts in a roundabout manner that in chess, either one player or the other has a winning strategy, or both players can force a draw. Zermelo notes that in each position q, $U(q) \subseteq V(q)$, and if $U(q)$ is nonempty, then White can force a win from q. If $U(q)$ is empty but $V(q)$ is not, then White can force a draw (as we mentioned in the previous paragraph, this is true but not supported by the arguments in the paper). If both sets are empty, then White can delay a loss until the σ-th move, for the value of σ corresponding to q. Furthermore, the two sets $U^*(q)$ (in the case where White can force a win) and $V^*(q)$ (otherwise) make up the set of "correct" moves for White from the position q. Zermelo notes than an analogous situation holds for Black, so that there exists a subset $W(q)$ of Q consisting of all continuations of the game (starting from q) in which both players can be said to have played correctly.

The final paragraph of the paper notes that the paper gives no means of determining in general which player has a winning strategy from which positions in chess, and that, given such a method, chess would in some sense cease to be a game.

As mentioned above, König's *1927a* points out that Zermelo's argument for the statement $\tau \leq t$ is incomplete. König's proof of this statement uses the following statement, which had appeared in his *1926*:[5] if E_i $(i \in \mathbb{N})$ are nonempty finite sets and R is a binary relation such that for each $i \in \mathbb{N}$ and each $x \in E_{i+1}$ there is a $y \in E_i$ such that $(y, x) \in R$, then there exists a sequence $\langle x_i : i \in \mathbb{N} \rangle$ such that each $x_i \in E_i$ and $(x_i, x_{i+1}) \in R$ for all $i \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of natural numbers). This principle is now known as *König's lemma*, often rephrased as "every infinite finitely-branching tree has an infinite branch." König uses this principle to prove that if G is a game in which each player has only finitely many available moves at each point, and one player has a winning strategy in this game, then this player has a strategy guaranteeing a win within a fixed number of moves. König credits this application of his lemma to a suggestion of von Neumann.

Before publishing his paper, König wrote to Zermelo, pointing out the gap in Zermelo's argument for $\tau \leq t$, and providing a correct proof. Zermelo then replied with a correct proof of his own. König's *1927b* consists of two paragraphs in the final section pertaining to Zermelo's corrected proof.

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[5] As translated in *Schwalbe and Walker 2001* from *D. König 1927a*; see *Franchella 1997* for much more on the history of this statement.

The first paragraph was apparently written by König, summarizing Zermelo's argument. It contains a proof that if White can force a win from a given position within some fixed number of moves, then White has a winning strategy that guarantees a win in fewer than t moves, where t is the number of positions in the game where it is White's turn to move (note that the definition of t has changed; this t is smaller than the t from *Zermelo 1913*, as we are counting only the number of moves that White makes). To show this, Zermelo lets m_r, for each positive integer r, be the number of such positions from which White can force a win in at most r moves, but cannot force a win in fewer moves (though Zermelo does not give a name for the set of such positions, let us call it M_r). Since the corresponding sets of positions are disjoint, and since there are only finitely many possible positions in the game, m_r is nonzero for only finitely many values of r. Furthermore, if p is a position from which White can win in at most r moves (for some $r > 1$) by first playing w_1, then there must be a response by Black such that the resulting position is in M_{r-1}, since from every such position White can win in at most $r - 1$ many moves, but if he could win in fewer moves from every such position, then White could win in fewer than r moves from the position p. Zermelo concludes that the set of values r such that m_r is positive is an initial sequence of the set of positive integers, so if λ is the largest integer r such that m_r is positive, then $m_r \geq 1$ for all positive integers $r \leq \lambda$. Then $m = \sum_{r=1}^{\lambda} m_r$ is smaller than the number of positions in which it is White's turn to play (since, for instance, Black can force a win from some such positions), so λ must be smaller than this version of t. This establishes that if p is a position from which White can force a win in at most r moves (for White), for some positive integer r, then r is less than the total number of positions in which it is White's turn to move.

In the second paragraph, König quotes Zermelo directly. Zermelo gives a proof that if White has a strategy guaranteeing a win, then he has one guaranteeing a win in a fixed number of moves. This is shown by König using his lemma, and Zermelo's argument uses the same idea (and implicitly includes a proof of the lemma). In brief, suppose that p is a position from which White cannot force a win in a fixed number of moves. Then no matter how White plays, there must be a move for Black such that White cannot force a win in a fixed number of moves from the resulting position (if each resulting position p' were in some $M_{r'}$, then p would be in M_{r+1} for r the supremum of these values r'—this uses the fact that each player has just finitely many possible available moves at each point). This observation gives a strategy for Black to postpone a loss forever, by always moving to ensure that the resulting position is not in any set M_r, contradicting the assumption that White has a winning strategy.

Kalmár (in *1928/1929*) extended König's analysis to games where there may be infinitely many possible moves at some points. In this paper he proved what is now known as Zermelo's Theorem for these games, the statement that in each position of such a game, either one player or the other has a strategy guaranteeing a win, or both players can force a draw. His proof uses a ranking of nodes in the game tree by transfinite ordinals, which was to become an important method in descriptive set theory (see *Kechris 1995*). Using this method, he was able to show that if a player has a winning strategy in such a game, then he has one in which no position is repeated, thus realizing Zermelo's idea from his *1913*.

Aside from the work of König and Kalmár, Zermelo's *1913* would seem to have been forgotten for several decades after it was written. In the interval between Zermelo's paper and König's, Émile Borel published several notes on game theory (for example, *1921, 1924, 1927*), none of which mentions Zermelo. Von Neumann's work in game theory began during this period, and though he was informed of Zermelo's work by König,[6] he does not cite it in his *1928c*. Zermelo's *1913* is not mentioned in von Neumann and Morgenstern's book *1943*, which is often cited as the birthplace of game theory. Many authors credit the birth of game theory to some combination of Borel, von Neumann and Morgenstern (for instance, *Luce and Raiffa 1957, Zagare 1984, Straffin 1993, Vorob'ev 1994, Jones 2000, Ritzberger 2002, Mendelson 2004*, and among these only *Luce and Raiffa 1957, Zagare 1984,* and *Jones 2000* credit Borel). Aside from the work of König and Kalmár, the earliest citation of Zermelo's *1913* that we have been able to find is Kuhn's *1953*. Kuhn credits Zermelo with proving that "a zero-sum two-person game with perfect information always has a saddle-point in pure strategies." As we have seen, the argument that Zermelo gives for this fact in his *1913* is incomplete. Although Zermelo's focus was on other issues, it seems fair to say that this fact is the most significant contribution of his paper.

[6] According to König in a 1927 letter to Zermelo; see *Ebbinghaus 2007b*.

Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels

1913

Die folgenden Betrachtungen sind unabhängig von den besonderen Regeln des Schachspiels und gelten prinzipiell ebensogut für alle ähnlichen Verstandesspiele, in denen zwei Gegner unter Ausschluss des Zufalls gegeneinander spielen; es soll aber der Bestimmtheit wegen hier jeweilig auf das Schach als das bekannteste aller derartigen Spiele exemplifiziert werden. Auch handelt es sich nicht um irgend eine Methode des praktischen Spiels, sondern lediglich um die Beantwortung der Frage: kann der Wert einer beliebigen während des Spiels möglichen Position für eine der spielenden Parteien sowie der bestmögliche Zug mathematisch-objektiv bestimmt oder wenigstens definiert werden, ohne dass auf solche mehr subjektiv-psychologischen wie die des „vollkommenen Spielers" und dergleichen Bezug genommen zu werden brauchte? Dass dies wenigstens in einzelnen besonderen Fällen möglich ist, beweisen die sogenannten „Schachprobleme", d. h. Beispiele von Positionen, in denen der Anziehende *nachweislich* in einer vorgeschriebenen Anzahl von Zügen das Matt erzwingen kann. Ob aber eine solche Beurteilung der Position auch in anderen Fällen, wo die genaue Durchführung der Analyse in der unübersehbaren Komplikation der möglichen Fortsetzungen ein praktisch unüberwindliches Hindernis findet, wenigstens theoretisch denkbar ist und überhaupt einen Sinn hat, scheint mir doch der Untersuchung wert zu sein, und erst diese Feststellung dürfte für die praktische Theorie der „Endspiele" und der „Eröffnungen", wie wir sie in den Lehrbüchern des Schachspiels finden, die sichere Grundlage bilden. Die im folgenden zur Lösung des Problems verwendete Methode ist der „Mengenlehre" und dem „logischen Kalkül" entnommen und erweist die Fruchtbarkeit dieser mathematischen Disziplinen in einem Falle, wo es sich fast ausschliesslich um *endliche* Gesamtheiten handelt.

Da die Anzahl der Felder, sowie die der ziehenden Steine endlich ist, so ist es auch die Menge P der möglichen Positionen $p_0, p_1, p_2 \ldots p_t$, wobei immer Positionen als verschieden aufzufassen sind, je nachdem Weiss oder Schwarz am Zuge ist, eine der Parteien schon rochiert hat, ein gegebener Bauer bereits verwandelt ist u. s. w. Es sei nun q eine dieser Positionen, dann sind von q aus „Endspiele" möglich $\mathfrak{q} = (q, q_1, q_2 \ldots)$, nämlich Folgen von Positionen, die mit q beginnen und im Einklang mit den Spielregeln auf einander folgen, sodass jede Position q_λ aus der vorhergehenden $q_{\lambda-1}$ abwechselnd durch einen zulässigen Zug von Weiss oder Schwarz hervorgeht. Solch ein mögliches End-
502 spiel $\mathfrak{q}$ kann entweder in einer „Matt" oder | „Patt" Stellung sein natürliches Ende finden, oder aber auch — theoretisch wenigstens — unbegrenzt verlaufen, in welchem Falle die Partie zweifellos als unentschieden oder „remis" zu gelten hätte. Die Gesamtheit Q aller dieser zu q gehörenden „Endspiele" $\mathfrak{q}$

On an application of set theory to the theory of the game of chess

1913

The following considerations are independent of the particular rules of the game of chess. They are in principle also valid for all similar games of reason in which two opponents play against each other and from which chance is excluded. But, for concreteness's sake, chess, the most well-known among games of this kind, will be used as an example here. Also, we shall not be concerned with any method for actually playing the game, but only with an answer to the following question: Is it possible to determine or, at least define, in a mathematically objective manner the value of an arbitrary position possible in a game of chess for one of the players as well as the best move possible without invoking more subjectively psychological notions such as that of the "perfect player" and similar ones? That this is at least possible in individual special cases is shown by the so-called "chess problems", i.e., examples of positions for which it is *possible to demonstrate* that the player whose turn it is to move can force checkmate in a specified number of moves. It certainly seems worthwhile to me to investigate whether it is at least theoretically possible, and whether it makes sense at all, to evaluate a position also in cases where it is practically impossible to carry out a precise analysis on account of the unsurveyable complication posed by the possible continuations of the game. Only such a determination would provide a firm foundation for the applied theory of "endgames" and "openings" as it can be found in chess textbooks. In what follows, we shall use a method taken from "set theory" and from the "calculus of logic" in order to solve the problem, thereby showing that these mathematical disciplines can be fruitfully applied in a case in which almost exclusively *finite* totalities are concerned.

Since the number of squares and that of the moving pieces is finite, so is the set P of the possible positions $p_0, p_1, p_2, \ldots p_t$, where positions are always to be considered distinct depending on whether it is White's or Black's turn to move, whether one of the players has already castled, whether a given pawn has already been promoted, etc. Now let q be one of these positions. Then, proceeding from q, "endgames" are possible $\mathfrak{q} = (q, q_1, q_2, \ldots)$, that is, sequences of positions beginning with q and succeeding one another according to the rules of the game so that each position q_λ results from the preceding one $q_{\lambda-1}$ by alternate legal moves of White and Black. A possible endgame $\mathfrak{q}$ of this kind can either come to a natural end in a "checkmate" or "stalemate" position, or—theoretically, at least—go on indefinitely, in which case the game would undoubtedly have to be considered a draw or "remis". The totality Q of all such "endgames" $\mathfrak{q}$ belonging to q is always ist stets eine wohldefinierte, endliche oder unendliche Untermenge der Menge $P^{\mathfrak{a}}$, welche alle möglichen abzählbaren Folgen gebildet aus Elementen p von P umfasst.

Unter diesen Endspielen $\mathfrak{q}$ können einige in r oder weniger „Zügen“ (d. h. einfachen Positionswechseln $p_{\lambda-1} \longrightarrow p_\lambda$, nicht etwa Doppelzügen) zum Gewinn von Weiss führen, doch wird dies in der Regel auch noch vom Spiel des Gegners abhängen. Wie muss aber eine Position q beschaffen sein, damit Weiss, wie Schwarz auch spielt, in höchstens r Zügen den Gewinn *erzwingen* kann? Ich behaupte, die notwendige und hinreichende Bedingung hierfür ist die Existenz einer nicht verschwindenden Untermenge $U_r(q)$ der Menge Q von folgender Beschaffenheit:

1. Alle Elemente $\mathfrak{q}$ von $U_r(q)$ enden in höchstens r Zügen mit dem Gewinn von Weiss, sodass keine dieser Folgen mehr als $r+1$ Glieder enthält und daher $U_r(q)$ jedenfalls endlich ist.

2. Ist $\mathfrak{q} = (q, q_1, q_2 \dots)$ ein beliebiges Element von $U_r(q)$, q_λ ein beliebiges Glied dieser Reihe, welches einem ausgeführten Zuge von Schwarz entspricht, also entweder immer ein solches gerader oder eines ungerader Ordnung, je nachdem bei q Weiss oder Schwarz am Zuge ist, sowie endlich q'_λ eine mögliche Variante, sodass Schwarz von $q_{\lambda-1}$ aus ebensogut nach q'_λ wie nach q_λ hätte ziehen können, so enthält $U_r(q)$ noch mindestens ein Element der Form $\mathfrak{q}'_\lambda = (q, q_1, \dots q_{\lambda-1}, q'_\lambda, \dots)$, welches mit $\mathfrak{q}$ die ersten λ Glieder gemein hat. In der Tat kann in diesem und nur in diesem Falle Weiss mit einem beliebigen Elemente $\mathfrak{q}$ von $U_r(q)$ beginnen und jedesmal, wo Schwarz q'_λ statt q_λ spielt, mit einem entsprechenden $\mathfrak{q}'_\lambda$ weiterspielen, also unter allen Umständen in höchstens r Zügen gewinnen.

Solcher Untermengen $U_r(q)$ kann es freilich mehrere geben, aber die Summe je zweier ist stets von derselben Beschaffenheit, und ebenso auch die Vereinigung $\overline{U}_r(q)$ aller solchen $U_r(q)$, welche durch q und r eindeutig bestimmt ist und jedenfalls von 0 verschieden sein, d. h. mindestens ein Element enthalten muss, sofern überhaupt solche $U_r(q)$ existieren. Somit ist $\overline{U}_r(q) \neq 0$ die notwendige und hinreichende Bedingung dafür, dass Weiss den Gewinn in höchstens r Zügen erzwingen kann. Ist $r < r'$ so ist stets $\overline{U}_r(q)$ Untermenge von $\overline{U}_{r'}(q)$, weil dann jede Menge $U_r(q)$ sicher auch die an $U_{r'}(q)$ gestellten Anforderungen erfüllt, also in $\overline{U}_{r'}(q)$ enthalten sein muss, und dem kleinsten $r = \rho$, für welches noch $\overline{U}_r(q) \neq 0$ ist, entspricht der gemeinsame Bestandteil $U^*(q) = \overline{U}_\rho(q)$ aller solchen $\overline{U}_r(q)$; dieser umfasst alle solche Fortsetzungen, mit denen Weiss in der kürzesten Zeit gewinnen muss. Nun besitzen aber diese Minimalwerte $\rho = \rho_q$ ihrerseits ein von q unabhängiges Maximum $\tau \leqq t$, wo $t+1$ die Anzahl der möglichen Positionen ist, sodass $U(q) = \overline{U}_\tau(q) \neq 0$ die notwendige und hinreichende Bedingung dafür darstellt, dass in der Position q irgend ein $\overline{U}_r(q)$ nicht verschwindet und Weiss überhaupt „auf Gewinn steht.“ Ist nämlich in einer Position q der Gewinn überhaupt zu erzwingen, so ist er es auch, wie wir zeigen wollen, in
503 höchsten t | Zügen. In der Tat müsste jedes Endspiel $\mathfrak{q} = (q, q_1, q_2 \dots q_n)$

a well-defined finite or infinite subset of the set $P^{\mathfrak{a}}$ comprising all possible countable sequences formed from elements p of P.

Some among these endgames $\mathfrak{q}$ can lead to a win for White in r or fewer "moves" (i.e., simple changes in position $p_{\lambda-1} \to p_\lambda$, never in double moves), whereas this will usually also depend on how the opponent plays. But how must a position q be constituted so that White can *force* a win in at most r moves regardless of how Black plays? I claim that the necessary and sufficient condition for this is the existence of a nonempty subset $U_r(q)$ of the set Q with the following properties:

1. All elements $\mathfrak{q}$ of $U_r(q)$ end with a win for White in at most r moves so that none of these sequences contains more than $r+1$ terms, and hence $U_r(q)$ is certainly finite.

2. Let $\mathfrak{q} = (q, q_1, q_2, \ldots)$ be an arbitrary element of $U_r(q)$, and let q_λ be an arbitrary term of this sequence which corresponds to a move made by Black, and hence is always one of either even or odd order depending on whether it is White's or Black's turn to move at q. Finally, let q'_λ be a possible variant so that Black might as well have moved from $q_{\lambda-1}$ to q'_λ instead of to q_λ. Then $U_r(q)$ still contains at least one element of the form $\mathfrak{q}'_\lambda = (q, q_1, \ldots q_{\lambda-1}, q'_\lambda, \ldots)$ which has the first λ terms in common with $\mathfrak{q}$. In fact, it is in this and only this case that White can begin with an arbitrary element $\mathfrak{q}$ of $U_r(q)$ and, whenever Black plays q'_λ instead of q_λ, continue playing with a corresponding $\mathfrak{q}'_\lambda$, and hence win in at most r moves under all circumstances.

To be sure, there can be several such subsets $U_r(q)$. But the sum of any two of them is always constituted in the same way, and the same holds of the union $\overline{U}_r(q)$ of all such $U_r(q)$, which is uniquely determined by q and r and which, in any case, must be different from 0, i.e., contain at least one element, provided that such $U_r(q)$ exist at all. Thus $\overline{U}_r(q) \neq 0$ is the necessary and sufficient condition for White being able to force a win in at most r moves. If $r < r'$, then $\overline{U}_r(q)$ is always a subset of $\overline{U}_{r'}(q)$ since, in this case, every set $U_r(q)$ certainly meets the demands placed on $U_{r'}(q)$ as well, and hence must be contained in $\overline{U}_{r'}(q)$. And to the least $r = \rho$ for which still $\overline{U}_r(q) \neq 0$ there corresponds the common component $U^*(q) = \overline{U}_\rho(q)$ of all such $\overline{U}_r(q)$; it comprises all such continuations by means of which White has to win in the shortest time possible. But now these minimal values $\rho = \rho_q$ themselves possess a maximum $\tau \leq t$ independent of q, where $t+1$ is the number of the possible positions, so that $U(q) = \overline{U}_\tau(q) \neq 0$ is the necessary and sufficient condition for some $\overline{U}_r(q)$ not vanishing in the position q and for White being in a "winning position" at all. For if it is at all possible in a position q to force a win, then it is also possible to do so, as we will show, in at most t moves. In fact, every endgame $\mathfrak{q} = (q, q_1, q_2, \ldots q_n)$ with $n > t$ would mit $n > t$ mindestens eine Position $q_\alpha = q_\beta$ doppelt enthalten, und Weiss hätte beim ersten Erscheinen derselben ebenso weiter spielen können wie beim zweiten Male und jedenfalls schon früher als beim n ten Zuge gewinnen, also $\rho \leqq t$.

Ist andererseits $U(q) = 0$, so kann Weiss, wenn der Gegner richtig spielt, höchstens remis machen, er kann aber auch „auf Verlust stehen“ und wird dann versuchen, dass „Matt“ möglichst hinauszuschieben. Soll er sich noch bis zum s^{ten} Zuge halten können, so muss eine Untermenge $V_s(q)$ von Q existieren von folgender Beschaffenheit:

1. In keinem der in $V_s(q)$ enthaltenen Endspiele verliert Weiss vor dem s^{ten} Zuge.

2. Ist $\mathfrak{q}$ ein beliebiges Element von $V_s(q)$ und in $\mathfrak{q}$ durch einen erlaubten Zug von Schwarz q_λ ersetzbar durch q'_λ, so enthält $V_s(q)$ noch mindestens ein Element der Form

$$\mathfrak{q}'_\lambda = (q, q_1, q_2, \dots q_{\lambda-1}, q'_\lambda \dots),$$

welches mit q bis zum λ^{ten} Gliede übereinstimmt und dann mit q'_λ weitergeht.

Auch diese Mengen $V_s(q)$ sind sämtlich Untermengen ihrer Vereinigung $\overline{V}_s(q)$, welche durch q und s eindeutig bestimmt ist und die gleiche Eigenschaft besitzt wie V_s selbst, und für $s > s'$ wird jetzt $\overline{V}_s(q)$ Untermenge von $\overline{V}_{s'}(q)$. Die Zahlen s, für welche $\overline{V}_s(q)$ von 0 verschieden ausfällt, sind entweder unbegrenzt oder $\leqq \sigma \leqq \tau \leqq t$, da der Gegner, wenn überhaupt, den Gewinn in höchstens τ Zügen müsste erzwingen können. Somit kann Weiss dann und nur dann mindestens remis machen, wenn $V(q) = \overline{V}_{r+1}(q) \neq 0$ ist, und im anderen Falle kann er vermöge $V^*(q) = \overline{V}_\sigma(q)$ den Verlust noch mindestens $\sigma \leqq \tau$ Züge hinausschieben. Da jedes $U_r(q)$ gewiss auch den an $V_s(q)$ gestellten Anforderungen genügt, so ist jedes $\overline{U}_r(q)$ Untermenge jeder Menge $\overline{V}_s(q)$, und $U(q)$ Untermenge von $V(q)$. Das Ergebnis unserer Betrachtung ist also das folgende:

Jeder während des Spiels möglichen Position q entsprechen zwei wohldefinierte Untermengen $U(q)$ und $V(q)$ aus der Gesamtheit Q der mit q beginnenden Endspiele, deren zweite die erste umschliesst. Ist $U(q)$ von 0 verschieden, so kann Weiss, wie Schwarz auch spielt, den Gewinn erzwingen und zwar in höchstens ρ Zügen vermöge einer gewissen Untermenge $U^*(q)$ von $U(q)$, aber nicht mit Sicherheit in weniger Zügen. Ist $U(q) = 0$ aber $V(q) \neq 0$, so kann Weiss wenigstens remis machen vermöge der in $V(q)$ enthaltenen Endspiele. Verschwindet aber auch $V(q)$, so kann Weiss, wenn der Gegner richtig spielt, den Verlust höchstens bis zum σ^{ten} Zuge hinausschieben vermöge einer wohldefinierten Menge $V^*(q)$ von Fortsetzungen. Auf alle Fälle sind nur die in U^*, bezw. V^* enthaltenen Partieen im Interesse von Weiss als „korrekt“ zu betrachten, mit jeder anderen Fortsetzung würde er, wenn in Gewinnstellung, bei richtigem Gegenspiel den gesicherten Gewinn verscherzen oder verzögern, sonst aber den Verlust der Partie ermöglichen oder beschleunigen. Ganz analoge Betrachtungen gelten natürlich auch für Schwarz, und als „korrekt“ zu have to contain at least one position $q_\alpha = q_\beta$ twice, and White could have continued to play at the first occurrence ⟦of the position⟧ just as well as at the second occurrence and certainly could have already won earlier than at the nth move, and hence $\rho \leq t$.

If, on the other hand, $U(q) = 0$, then White can at most achieve a draw, assuming the opponent plays correctly. But he also can be "in a losing position", in which case he will try to defer a "checkmate" for as long as possible. If he is to survive until the sth move, then there must exist a subset $V_s(q)$ of Q constituted as follows:

1. In none of the endgames contained in $V_s(q)$ does White lose before the sth move.

2. If $\mathfrak{q}$ is an arbitrary element of $V_s(q)$ and if it is possible to replace q_λ by q'_λ in $\mathfrak{q}$ by a legal move by Black, then $V_s(q)$ still contains at least one element of the form

$$\mathfrak{q}'_\lambda = (q, q_1, q_2, \ldots q_{\lambda-1}, q'_\lambda \ldots),$$

which is identical to $\mathfrak{q}$ up to the λth term and then continues with q'_λ.

These sets $V_s(q)$, too, are all subsets of their union $\overline{V}_s(q)$, which is uniquely determined by q and s and possesses the same property as V_s itself. For $s > s'$, $\overline{V}_s(q)$ now becomes a subset of $\overline{V}_{s'}(q)$. The numbers s for which $\overline{V}_s(q)$ differs from 0 are either unlimited or $\leq \sigma \leq \tau \leq t$, since the opponent would have to be able to force a win in at most τ moves, if at all. Thus White can at least make a draw if and only if $V(q) = \overline{V}_{\tau+1}(q) \neq 0$. In the other case, he can postpone defeat for at least $\sigma \leq \tau$ moves by virtue of $V^*(q) = \overline{V}_\sigma(q)$. Since every $U_r(q)$ certainly meets the demands placed on $V_s(q)$, every $\overline{U}_r(q)$ is a subset of every set $\overline{V}_s(q)$, and $U(q)$ a subset of $V(q)$. Our consideration thus has the following result:

To each position q possible during the game there correspond two well-defined subsets $U(q)$ and $V(q)$ from the totality Q of the endgames beginning with q, with the latter including the former. If $U(q)$ differs from 0, then White can force the win regardless of how Black plays, namely in at most ρ moves by virtue of a certain subset $U^*(q)$ of $U(q)$, but not with certainty in fewer moves. If $U(q) = 0$ but $V(q) \neq 0$, then White can at least make a draw by virtue of the endgames contained in $V(q)$. But if $V(q)$ vanishes as well, then, assuming that the opponent plays correctly, White can postpone defeat to no later than the σth move by virtue of a well-defined set $V^*(q)$ of continuations. In any event, only those games contained in U^* and V^* respectively are to be considered "correct" from White's point of view. With any other continuation, White, when in a winning position, would either throw away or delay a certain win, assuming that the opponent plays correctly, or else make possible the loss of the game or accelerate it. Very similar considerations are of course valid for Black, and we would have to consider those games as having been

Ende geführte Partieen hätten diejenigen zu gelten, welche *gleichzeitig* den beiderseitigen Bedingungen entsprechen, sie bilden also in jedem Falle wieder eine wohldefinierte Untermenge $W(q)$ von Q.

504 | Die Zahlen t und τ sind von der Position unabhängig und lediglich durch die Spielregeln bestimmt. Jeder möglichen Position entspricht eine τ nicht überschreitende Zahl $\rho = \rho_q$ oder $\sigma = \sigma_q$, je nachdem Weiss oder Schwarz in ρ bezw. σ Zügen, aber nicht in weniger, den Gewinn erzwingen kann. Die spezielle Theorie des Spiels hätte diese Zahlen, soweit dies möglich ist, zu bestimmen oder wenigstens in Grenzen einzuschliessen, was bisher allerdings nur in besonderen Fällen, wie bei den „Problemen“ oder den eigentlichen „Endspielen“ gelungen ist. Die Frage, ob die Anfangsposition p_0 bereits für eine der spielenden Parteien eine „Gewinnstellung“ ist, steht noch offen. Mit ihrer exacten Beantwortung würde freilich das Schach den Charakter eines Spieles überhaupt verlieren.

ended "correctly" which satisfy the conditions on both sides *simultaneously*. Hence, they certainly form a well-defined subset $W(q)$ of Q again.

The numbers t and τ are independent of the position and solely determined by the rules of the game. To every possible position there corresponds a number $\rho = \rho_q$ or $\sigma = \sigma_q$ not exceeding τ, depending on whether White or Black can force the win in ρ or σ, but not fewer, moves. The special theory of the game would have to determine these numbers to the extent to which this is possible, or at least determine the limits within which they must lie. So far this has been done only in special cases such as the "problems" and the "endgames" proper. The question as to whether the initial position p_0 already is a "winning position" for one of the players is still open. Its precise answer would of course deprive chess of its character as a game.