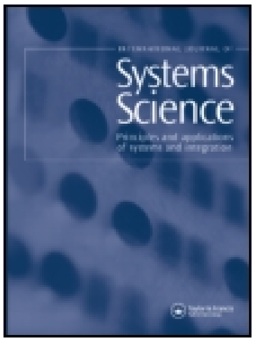

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Control synthesis for polynomial discrete-time systems under input constraints via delayed-state Lyapunov functions

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This paper presents a discrete-time control design methodology for input-saturating systems using a Lyapunov function with dependence on present and past states. The approach is used to bypass the usual difficulty with full polynomial Lyapunov functions of expressing the problem in a convex way. Also polynomial controllers are allowed to depend on both present and past states. Furthermore, by considering saturation limits on the control action, the information about the relationship between the present and past states is introduced via Positivstellensatz multipliers. Sum-of-squares techniques and available semi-definite programming (SDP) software are used in order to find the controller.

Keywords: delayed Lyapunov function; discrete time; polynomial systems; control design; SOS approach; stabilisation; convex optimisation

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1. Introduction

Many smooth nonlinear systems can be transformed to polynomial ones, either by a Taylor-series approximation (Sala & Ariño, 2009) or by a change of variable and state augmentation (Papachristodoulou & Prajna, 2005). As convex programming tools for polynomial systems have been recently developed (see references below), the polynomial approach may be used to obtain alternative nonlinear control solutions to others in literature (Khalil, 2002; Koshkouei & Burnham, 2011; Slotine, 1991), based on a systematic modelling and convex-programming approach. The approach, however, involves some conservative choices in order to get a reasonable computational cost: finite degree of Lyapunov functions and finite degree and number of Karush-Kuhn-Tucker (KKT) like multipliers associated to algebraic constraints (Jarvis-Wloszek, Feeley, Tan, Sun, & Packard 2005).

Stability analysis and control design for polynomial systems have received attention in recent literature, both in continuous-time (Chesi, 2011; Pozo & Rodellar, 2010) and discrete-time settings (Tanaka, Ohtake, & Wang, 2008; Xu, Xie, & Wang, 2007). The basic framework uses sum-of-squares (SOS) techniques (Balas, Packard, Seiler, & Topcu, 2012; Seiler, Zheng, & Balas, 2013) and Positivstellensatz theorems (Jarvis-Wloszek et al., 2005) to prove local stability. The reader is referred to Chesi (2010) for a survey and additional literature regarding the main ideas in the approach.

In this polynomial control framework, if the controller and a Lyapunov function have to be simultaneously found, the discrete-time design case usually leads to a non-convex problem which has to be solved by V-K iterations or any other similar algorithms (Xu et al., 2007). To avoid this problem, Prajna, Papachristodoulou, and Wu (2004) and Tanaka et al. (2008) proposed restricting the dependence of the Lyapunov function to the subset $\tilde{x}$ of the states which are not directly affected by the control action (i.e. $\tilde{x}_{k+1}$ does not depend on u_k). This outperforms the classical quadratic case control design but it is still quite restricted.

In this work, the stabilisation problem for polynomial systems with input bounds is addressed in a convex way, using the whole state information. The main idea is introducing delayed states in the Lyapunov function which breaks up some bilinear terms and also provides the state-feedback controller with extra degrees of freedom (rationally depending on present and past state values). The use of Lyapunov functions with dependence on delayed scheduling parameters has been successfully applied in the Takagi-Sugeno linear matrix inequality (LMI) framework (Guerra, Kerkeni, Lauber, & Vermeiren, 2012). In the discrete-time case here considered, due to the construction of the involved matrices, there is no need of Krasovskii-like terms in Lyapunov functions, as other developments need (Gassara, Hajjaji, Kchaou, & Chaabane, 2014).

In this paper, the delay idea is applied to include a full delayed state in polynomial systems. Information about the

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relationship between present and past state values is introduced by specifying bounds in the control action and Positivstellensatz multipliers. The approach improves over existent ones in literature, if restricted to convex optimisation set-ups. The recent work by Valmorbida, Tarbouriech, and Garcia (2013) proposes a similar development addressing the polynomial control synthesis for discrete-time systems under actuator saturation. However, their proposal leads to a non-convex bilinear matrix inequality problem, which needs to be solved iteratively without guarantees of global optimality, as widely known (Fukuda & Kojima, 2001). Intentionally, analysis and comparison with bilinear matrix inequality (BMI) approaches has been left out of the scope of this paper, because the BMI results depend on the initial conditions and iteration step sizes.

The objective of the paper is, hence, lifting conservativeness in polynomial control by using delayed-state Lyapunov functions and saturation bounds while keeping the resulting SOS conditions convex.

The structure of the paper is as follows: next section states the notation followed in the rest of the paper as well as summarises the existent preliminary results related to the current issue and presents the problem statement, Section 3 presents a convex control design methodology by delayed polynomial Lyapunov functions, Section 4 shows an academic example demonstrating the effectiveness of the proposed approach and, finally, a conclusions section closes the paper.

2. Preliminaries and notation

Let us first introduce some notation and basic SOS results to be used throughout the paper. The set of polynomials in a variable $x \in \mathbb{R}^n$ will be denoted by $\mathcal{R}_x$, and the n-dimensional vectors of polynomials in x as $\mathcal{R}_x^n$. The corresponding element of a polynomial symmetric expression will be denoted as $(*)$. The frontier of a semi-algebraic region Ω will be denoted by $\partial\Omega$.

Polynomials in some variables, say, x, which can be decomposed as an SOS of other polynomials will be denoted by Σ_x and the $N \times N$ SOS polynomial matrices in x (see Proposition 1) by Σ_x^N. SOS decompositions of (matrix) polynomials can be found by searching for a positive semi-definite scalar matrix using semi-definite programming (SDP) software:

Proposition 1 (Scherer & Hol, 2005)**:** *Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$. $F(x)$ is an SOS polynomial matrix if and only if there exist a constant matrix $Q \succeq 0$ satisfying*

$$F(x) = (I \otimes z(x))^T \, Q \, (I \otimes z(x)) \, \forall x \in \mathbb{R}^n \tag{1}$$

with $z(x)$ being a column vector whose entries are all monomials in x with degree no greater than d.

Evidently, SOS polynomials in independent variables x are non-negative, and SOS polynomial matrices are positive semi-definite matrices, for all values of x. Note also that the condition in Proposition 1 can be cast as a non-strict LMI in the elements of Q.

The notation used in Jarvis-Wloszek et al., (2005) will be also used in the rest of the paper: given polynomials $\{f_1, \ldots, f_t\}$, $\mathcal{M}(f_1, \ldots, f_t)$ will denote the multiplicative monoid, $\wp(f_1, \ldots, f_t)$ denotes the cone and $\mathcal{I}(f_1, \ldots, f_t)$ denotes the ideal generated by the set of $f\prime_k s$. Denote also by $\mathcal{I}^N(f_1, \ldots, f_t)$ the set of $N \times N$ matrices whose elements belong to the ideal $\mathcal{I}(f_1, \ldots, f_t)$. Denote also by $\wp^N(f_1, \ldots, f_t)$ the subset of $\mathcal{I}^N(f_1, \ldots, f_t)$ formed by the cone of matrices which are positive semi-definite for any values of the argument variables.

Using the above notation, the cited work recasts the so-called *Positivstellensatz* theorem (Stengle, 1974) to assert local non-negativeness of polynomials in a region with polynomial boundary. The lemma below generalises the concept to local positive semi-definiteness of polynomial matrices:

Lemma 1: *The polynomial matrix $P(x) \in \mathcal{R}_x^{N\times N}$ is positive semi-definite in a region $\Omega = \{x : g_i(x) > 0, h_j(x) = 0, i : 1, \ldots, r, j : 1, \ldots v\}$ if there exist polynomial matrices $G(x) \in \wp^N(g_1, \ldots, g_r)$ and $H(x) \in \mathcal{I}^N(h_1, \ldots, h_v)$ which verify:*

$$P(x) - G(x) + H(x) \in \Sigma_x^N \tag{2}$$

Proof: Multiplying Equation (2) by auxiliary variables $v \in \mathbb{R}^N$ on the left and right, it results in a polynomial SOS condition $v^T(P(x) - G(x) + H(x))v \in \Sigma_{x,v}$ so that, if it holds, $v^T P(x) v$ is non-negative in Ω as required. □

Note that computational checking of Equation (2) can be done with the LMIs deriving from Proposition 1. For instance, one choice of matrices above for computations may be

$$G(x) = \sum_{i=1}^{r} S_i(x) g_i(x) \quad H(x) = \sum_{j=1}^{v} Z_j(x) h_j(x)$$

where $S_i(x)$ are SOS matrices and $Z_j(x)$ are arbitrary ones, both with unknown coefficients. $S_i(x)$ and $Z_j(x)$ can be full polynomial matrices or, for instance, only diagonal ones depending on the available computing resources.

2.1. *Stability of polynomial systems*

Consider a discrete-time polynomial system on the form

$$x_{k+1} = A(x_k) z(x_k) + B(x_k) u_k \tag{3}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^c$ are the state and control-input vectors at time instant k respectively, $A(x_k) \in \mathcal{R}_{x_k}^{n\times m}$ and $B(x_k) \in \mathcal{R}_{x_k}^{n\times c}$ are polynomial matrices and

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$z(x_k) \in \mathcal{R}^m_{x_k}$ is a polynomial vector in the states. On the sequel, shorthand $z_k = z(x_k)$ will be used for brevity.

Define a candidate Lyapunov function $V : \mathcal{D} \to \mathbb{R}$ as

$$V(x_k) = z_k^T Q^{-1}(x_k) z_k \tag{4}$$

where $\mathcal{D} \in \mathbb{R}^n$ is an open set, $0 \in \mathcal{D}$ and $Q(x_k) \in \mathcal{R}^{m\times m}_{x_k}$ is a polynomial matrix in the states. Note that $Q^{-1}(x_k)$ appears in the Lyapunov function instead of $Q(x_k)$ to adapt the standard technique of change of variables for control in LMI framework (Bernussou, Peres, & Geromel, 1989) to polynomial cases, setting $\rho = Q^{-1}(x_k) z_k$, $V = \rho^T Q(x_k) \rho$. Consider now a state-feedback controller in the form

$$u_k = -K(x_k) z_k \tag{5}$$

where $K(x_k) = M(x_k) Q^{-1}(x_k)$ is the feedback gain and $M(x_k) \in \mathcal{R}^{c\times m}_{x_k}$. According to Lyapunov theory, the controller (5) stabilises the system (3) if conditions

$$V(0) = 0 \tag{6}$$

$$V(x_k) > 0, \forall x_k \in \mathcal{D}, x_k \neq 0 \tag{7}$$

$$\Delta V = V(x_{k+1}) - V(x_k) \leq 0, \{x_{k+1}, x_k\} \in \mathcal{D} \tag{8}$$

are satisfied (Khalil, 2002). Further, if the inequality (8) is strict for $x_k \in \mathcal{D}\setminus\{0\}$, then the system is asymptotically stable. Moreover, if $\mathcal{D} = \mathbb{R}^n$, stability is global.

In the controller synthesis problem (i.e. the controller has to be found simultaneously with the Lyapunov function), some conservative assumptions are addressed in literature (Tanaka et al., 2008; Xu et al., 2007) to cast the problem in a convex way.

- If Q is constant and $z_k = x_k$, the controller synthesis problem becomes convex by Schur complement, resulting in finding Q and coefficients of polynomials in $M(x_k)$ such that, for an arbitrary $\epsilon > 0$:

$$\begin{bmatrix} Q & (*)^T \\ A(x_k) Q - B(x_k) M(x_k) & Q \end{bmatrix} - \epsilon I \in \Sigma^{2m}_{x_k}$$

- Following the idea introduced in continuous-time in Prajna et al. (2004), consider a Lyapunov function defined by $Q(\tilde{x}_k)$, where $\tilde{x}_k = E x_k \in \mathbb{R}^L$, being E a constant matrix fulfilling[1] $EB(x_k) = 0$. If z_k can be expressed as

$$z_k = T(\tilde{x}_k) x_k \tag{9}$$

with $T(\tilde{x}_k) \in \mathcal{R}^{m\times n}_{\tilde{x}_k}$, the problem is still convex.

If the above problems render infeasible, local stability conditions can be posed based on modifying conditions (7) and (8) to make them hold locally in a so-called *region of interest* $\Omega \subset \mathbb{R}^n$. Lemma 1 enables checking such conditions with SOS programming (sufficient conditions). For instance, the local stability results in Xu et al. (2007) can be adapted to the notation here as follows:

Corollary 1: *If polynomial matrices $G(x_k)$, $H(x_k)$ as defined in Lemma 1 can be found fulfilling*

$$\begin{bmatrix} Q & (*)^T \\ A(x_k) Q - B(x_k) M(x_k) & Q \end{bmatrix} - \epsilon I - G(x_k) + H(x_k) \in \Sigma^{2m}_{x_k} \tag{10}$$

with $\epsilon > 0$, then $\Delta V(x_k)$ is locally negative in a region of the state space Ω except at the origin.

When conditions (6), (7) and (8) hold for all $x \in \Omega$, the system is said to be *locally stable* in Ω, implying that all level sets $\{x : V(x) \leq \gamma\} \subset \Omega$ are invariant (Khalil, 2002). SOS procedures also allow expanding the proven domain of attraction to sets larger than the referred level sets (Pitarch, Sala, & Ariño, 2014).

2.2. *Problem statement*

The nonlinear nature of Equations (4) and (8) is a fundamental difficulty with non-quadratic Lyapunov functions in discrete-time systems. Up to the authors' knowledge, the general problem of finding a Lyapunov function $Q(x_k)$ and a controller gain $K(x_k)$ together has not been posed in convex form. The Lyapunov functions with dependence on delayed scheduling parameters in Guerra et al. (2012), inspired using a full delayed state polynomial Lyapunov function to reduce the conservativeness of the above results, as discussed next.

3. Main result

Consider a *delayed*-rational candidate Lyapunov function $V(x_k, x_{k-1})$ in the form

$$V(x_k, x_{k-1}) = z_k^T Q^{-1}(\tilde{x}_k, x_{k-1}) z_k \tag{11}$$

and a state-feedback control law which can depend on present and past states

$$u_k = -K(x_k, x_{k-1}) z_k \tag{12}$$

where, $K(x_k, x_{k-1}) = M(x_k, x_{k-1}) Q^{-1}(\tilde{x}_k, x_{k-1})$, being $Q(\tilde{x}_k, x_{k-1}) \in \mathcal{R}^{m\times m}_{\tilde{x}_k, x_{k-1}}$ and $M(x_k, x_{k-1}) \in \mathcal{R}^{c\times m}_{x_k, x_{k-1}}$. It will be assumed that there exists a constant matrix $E \in \mathbb{R}^{L\times n}$ such that z_k can be expressed as Equation (9) and another

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constant matrix $E^{\perp}$ such that $E^T E^{\perp} = 0$ and the rows of E and $E^{\perp}$ form a basis of $\mathbb{R}^n$. Obviously, by definition, the columns of B belong to the row space of $E^{\perp}$.

Consider a region Ω of the augmented state space:

$$\Omega_0 = \left\{x : z(x)^T U z(x) \leq R^2\right\} \tag{13}$$

$$\Omega = \{x_k, x_{k-1} : x_k \in \Omega_0, x_{k-1} \in \Omega_0\} \tag{14}$$

and a second region Φ, $\Phi \subset \Omega$, where initial conditions are supposed to lie in, described as

$$\Phi = \left\{x_0, x_{-1} : \max\left(z_0^T Y z_0, z_{-1}^T Y z_{-1}\right) \leq \beta^2\right\} \tag{15}$$

here U and Y are constant user-defined matrices with suitable dimension. Consider also that each individual control input has known saturation bounds

$$|e_j u_k| \leq \mu_j, \quad \mu_j \in \mathbb{R}, j : 1, \ldots, c \tag{16}$$

where e_j is the standard canonical row vector in $\mathbb{R}^c$ whose j-th component is one and the rest are zero. Hence, a set of vectors $\overline{u_i}$, $i : 1, \ldots, 2^c$ can be constructed such that the control action u belongs to its convex hull.

Theorem 1: *Assume* $\{x_0, x_{-1}\} \in \Phi$. *Then, the system* (3) *with the control law* (12) *is locally stable in region* (14)*, satisfies the control input saturation* (16) *and* Φ *belongs to the domain of attraction of the origin if the following SOS problem is feasible for all* $i : 1, \ldots, 2^c$ *and* $j : 1, \ldots, c$*:*

$$\begin{bmatrix} Q(Ex_k, x_{k-1}) & (*)^T \\ \Psi(x_k, x_{k-1}) & Q(E \cdot A(x_k) z_k, x_k) \end{bmatrix} - \epsilon I - \Upsilon_{1i} \in \Sigma^{2m}_{x_k, x_{k-1}} \tag{17}$$

$$Q(Ex_k, x_{k-1}) - \epsilon I - W_1 - \mathcal{H}_1 \in \Sigma^m_{x_k, x_{k-1}} \tag{18}$$

$$\begin{bmatrix} Q(Ex_k, x_{k-1}) & (*)^T \\ e_j M(x_k, x_{k-1}) & \mu_j^2 \end{bmatrix} - \Upsilon_{2i} \in \Sigma^{m+1}_{x_k, x_{k-1}} \tag{19}$$

$$R^2 U^{-1} - Q(Ex_k, x_{k-1}) - W_2 - \mathcal{H}_2 \in \Sigma^m_{x_k, x_{k-1}} \tag{20}$$

$$\begin{bmatrix} \beta^{-2} z_k^T Y z_k & z_k^T \\ z_k & Q(Ex_k, x_{k-1}) \end{bmatrix} - \Upsilon_{3i} \in \Sigma^{m+1}_{x_k, x_{k-1}} \tag{21}$$

where

$$\Psi(x_k, x_{k-1}) = T(E \cdot A(x_k))(A(x_k) Q(Ex_k, x_{k-1}) - B(x_k) M(x_k, x_{k-1})),$$

$$\Upsilon_{di} = S_{di}\mathcal{H}_{di} + \sum_{b=1}^{n-L} H_{db}\phi_{bi}, \quad d : 1, \ldots, 3 \tag{22}$$

being $\epsilon > 0$ *and*:

$$\tilde{\phi}_b = e_b(Ex_k - E \cdot A(x_{k-1}) z_{k-1}),$$

$$\phi_{bi} = e_b\left(E^{\perp} x_k - E^{\perp} A(x_{k-1}) z_{k-1} + E^{\perp} B(x_{k-1}) \bar{u}_i\right),$$

$$W_1 \in \wp^m\left(R^2 - z_k^T U z_k, R^2 - z_{k-1}^T U z_{k-1}\right),$$
$$W_2 \in \wp^m\left(R^2 - z_{k-1}^T U z_{k-1}\right),$$

$$S_{1i} \in \wp^{2m}\left(R^2 - z_k^T U z_k, R^2 - z_{k-1}^T U z_{k-1}\right),$$

$$S_{2i} \in \wp^{m+1}\left(R^2 - z_k^T U z_k, R^2 - z_{k-1}^T U z_{k-1}\right),$$
$$S_{3i} \in \wp^{m+1}\left(\beta^2 - z_{k-1}^T Y z_{k-1}\right),$$

$$\mathcal{H}_1 \in \mathcal{I}^m\left(\tilde{\phi}_1, \ldots, \tilde{\phi}_L\right),$$
$$\mathcal{H}_2 \in \mathcal{I}^m\left(R^2 - z_k^T U z_k,\ \tilde{\phi}_1, \ldots, \tilde{\phi}_L\right),$$

$$\mathcal{H}_{1i} \in \mathcal{I}^{2m}\left(\tilde{\phi}_1, \ldots, \tilde{\phi}_L\right),$$
$$\{\mathcal{H}_{2i}, \mathcal{H}_{3i}\} \in \mathcal{I}^{m+1}\left(\tilde{\phi}_1, \ldots, \tilde{\phi}_L\right),$$

$$H_{1b} \in \mathcal{R}^{(2m)\times(2m)}_{x_k, x_{k-1}}, \{H_{2b}, H_{3b}\} \in \mathcal{R}^{(m+1)\times(m+1)}_{x_k, x_{k-1}}.$$

The decision variables in the above problem are the coefficients (note that degrees are chosen beforehand) of the polynomial matrices $W_1, W_2, S_{1i}, S_{2i}, S_{3i}$, $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_{1i}, \mathcal{H}_{2i}, \mathcal{H}_{3i}, H_{1b}, H_{2b}, H_{3b}$, $Q(Ex_k, x_{k-1})$ and $M(x_k, x_{k-1})$.

Proof: Using the candidate Lyapunov function (11), stability condition (8) now becomes

$$\Delta V = z_{k+1}^T Q^{-1}(\tilde{x}_{k+1}, x_k) z_{k+1} - z_k^T Q^{-1}(\tilde{x}_k, x_{k-1}) z_k < 0$$

Substituting z_{k+1} by its value

$$z_{k+1} = T(E \cdot A(x_k))(A(x_k) - B(x_k) K(x_k, x_{k-1})) z_k,$$

performing the well-known change of variable

$$\rho = Q^{-1}(\tilde{x}_k, x_{k-1}) z_k \tag{23}$$

and applying Schur complement, it leads to

$$\eta^{\mathrm{T}} \begin{bmatrix} Q(\tilde{x}_k, x_{k-1}) & (*)^T \\ \Psi(x_k, x_{k-1}) & Q(\tilde{x}_{k+1}, x_k) \end{bmatrix} \eta \geq 0$$

being η a vector of independent variables.

The relationship between present and past states is

$$\begin{aligned} &E^{\perp}(x_k - A(x_{k-1}) z_{k-1} - B(x_{k-1}) u_{k-1}) = 0, \\ &E(x_k - A(x_{k-1}) z_{k-1}) = 0 \end{aligned} \tag{24}$$

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This information can be introduced in the SOS constraints with terms $\mathcal{H}(x_k, x_{k-1})$ belonging to the ideals associated to the above equalities. However, to avoid introducing new variables u in the SOS program, equalities in Equation (24) depending on $E^\perp$ must be introduced with arbitrary multiplier matrices $H_{db}(x_k, x_{k-1})$, conforming the rightmost summation in the definition of Υ_{di} in Equation (22), but keeping linearity in ϕ_{bi}. In fact, to actually get Equation (22), a last step is needed: as the resulting expressions are affine in u_{k-1}, they will hold if they do in all the vertices given by vectors $\bar{u}_i$, from convexity arguments. Note that multipliers H_{db} must be shared between all vertices.

Now, positive semi-definite matrix multipliers $W_i(x_k, x_{k-1})$, $S_{di}(x_k, x_{k-1})$ are provided to add information about Ω in SOS conditions so that they need to hold only locally (note that multipliers S_{di} can actually be different for different $\bar{u}_i$). After these steps, Equations (17) and (18) are obtained, so Equation (11) is a valid Lyapunov function as Equations (6)–(8) hold locally in Ω.

Define now Θ as the Lyapunov level set

$$\Theta = \{x_k, x_{k-1} : V(x_k, x_{k-1}) \leq 1\}.$$

Conditions (19) ensure that u does not take values larger than the saturation bounds μ inside the region $\Pi = \Theta \cap \Omega$. They are obtained from the inequality

$$z_k^T Q^{-1}(\tilde{x}_k, x_{k-1}) z_k - z_k^T e_j^T K(x_k, x_{k-1})^T \mu_j^{-1} I(*) \geq 0$$

in a similar way to the quadratic case (Boyd, 1994) for Θ, but relaxed with local information on Ω and system dynamics analogous to the above discussed multipliers.[2]

As a last step in the proof, as locality conditions only hold in Ω, we need to ensure that there exists an invariant subset of Ω containing the initial set Φ.

Let us assume $V(x_k, x_{k-1}) \geq 1 \forall x_k \in \partial\Omega_0, x_{k-1} \in \Omega_0$ which is enforced by Equation (20) as later shown. Let us prove that $\Pi = \Theta \cap \Omega$ is invariant. Indeed, the points $x_k \in \partial\Omega_0$ and $x_{k-1} \in \Omega_0$ are outside Π, so the trajectories will never leave Π through that part of $\partial\Omega$.

If $x_k \in \Omega_0, x_{k-1} \in \partial\Omega_0, V(x_k, x_{k-1}) \leq 1$ then $x_{k+1} \in \Omega_0, x_k \in \Omega_0$ and $V(x_{k+1}, x_k) < 1$. Indeed, $V(x_{k+1}, x_k) < 1$ from Equation (17); then expression (20), from the above paragraph, discards the option of x_{k+1} leaving Ω_0. Hence, if $\{x_k, x_{k-1}\} \in \Pi$, we have $\{x_{k+1}, x_k\} \in \Pi$.

To enforce $V(x_k, x_{k-1}) \geq 1 \forall x_k \in \partial\Omega_0, x_{k-1} \in \Omega_0$, similar issues to those arising in Equation (19) discussed in footnote 2 apply. Thus, resorting to similar argumentations gives Equation (20).

The last set of SOS constraints must ensure the initial condition set $\Phi \subset \Pi$. As $\Phi \subset \Omega$ by assumption, $\Phi \subset \Theta$ has to be ensured, too. It can be proved by enforcing $V(x_k, x_{k-1}) \leq 1 \forall \{x_k, x_{k-1}\} \in \Phi$. A sufficient condition for this to hold is

$$\frac{1}{\beta^2} z_k^T Y z_k - z_k^T Q^{-1}(\tilde{x}_k, x_{k-1}) z_k \geq 0$$

enforced locally in Ω by Equation (21), after applying Schur complement and Positivstellensatz (details omitted for brevity).

So, $\{x_0, x_{-1}\} \in \Phi \subset \Pi \subset \Omega$, invariance of Π has been ensured by SOS constraints and $\Pi \subset \Theta$ ensures the control action bounds (16) are met, so multipliers arising from Equation (24) are valid.

Now, the proven invariant set in the augmented space is *not* a Lyapunov function level set: the level set Θ can actually extend outside the local-stability region Ω, removing conservativeness. So, the discrete-time analog to La-Salle invariance theorem needs to be invoked: the system will converge to the largest invariant set in $\Delta V = 0$, and only the origin verifies the zero-increment condition (details omitted for brevity). □

Remark 1: With $Q(\tilde{x}_k)$, $z_k = T(\tilde{x}_k) x_k$ and $u = -K(x_k) z_k$, Theorem 1 reduces to cases in Xu et al. (2007) and Tanaka et al. (2008). A more general version encompassing the 'natural' case $V = z_k^T Q(x_k) z_k$ may be crafted by letting $Q(x_k, x_{k-1})$. In that case, the SOS problems would involve variables (x_k, x_{k-1}, x_{k+1}). However, to keep convexity, new multipliers analogous to Equation (22) are needed, with additional ϕ_b and ϕ_{bi} now referring to the relationship between x_k and x_{k+1}. Details are omitted because the usefulness of the approach is limited, as the controller cannot depend on future x_{k+1}.

Remark 2: In discrete-time, Lyapunov-Krasovskii (LK) functionals are actually a particular case of generic Lyapunov functions of an augmented finite-dimensional realisation incorporating delayed states (well known, for instance in Gonzalez, Sala, Garcia, & Albertos 2013; Hetel, Daafouz, & Iung 2008 and). From the realisation $\psi_k = (x_k x_{k-1})^T$ considering a delayed controller $u_k = -(K_1(x_k, x_{k-1}) K_2(x_k, x_{k-1}))\psi_k$, then the closed loop is

$$\psi_{k+1} = \tilde{A}(x_k, x_{k-1})\psi_k,$$

$$\tilde{A}(x_k, x_{k-1}) = \begin{bmatrix} A(x_k) - B(x_k) K_1(x_k, x_{k-1}) & -B(x_k) K_2(x_k, x_{k-1}) \\ I & 0 \end{bmatrix}$$

If a candidate 'full' Lyapunov function (encompassing any quadratic LK choices for unit delay) $V(x_k, x_{k-1}) = \psi^T Q(\tilde{x}_k, x_{k-1})^{-1} \psi$ (being $Q(\tilde{x}_k, x_{k-1})$ a suitably partitioned 2×2 block-polynomial matrix) is chosen, and changes of variable leading to Equation (17) are enforced, then the Lyapunov discrete increment

$$V_{k+1} - V_k = \psi_k^T (\tilde{A}(x_k, x_{k-1})^T Q(\tilde{x}_{k+1}, x_k)^{-1} \tilde{A}(x_k, x_{k-1}) - Q(\tilde{x}_k, x_{k-1})^{-1})\psi_k$$

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leads to a 4×4 block-polynomial matrix condition $\Pi(x_k, x_{k-1}) > 0$ by Schur complement (details omitted for brevity). Then, a necessary condition to ensure $\Pi(x_k, x_{k-1}) > 0$ is, taking the minor obtained from its first and third rows and columns, the following inequality:

$$\begin{bmatrix} Q_{11}(\tilde{x}_k, x_{k-1}) & (*)^T \\ A(x_k)\, Q_{11}(\tilde{x}_k, x_{k-1}) - B(M_1 + M_2) & Q_{11}(\tilde{x}_{k+1}, x_k) \end{bmatrix} > 0, \tag{25}$$

$$M_1 = K_1(x_k, x_{k-1})\, Q_{11}(\tilde{x}_k, x_{k-1}),$$
$$M_2 = K_2(x_k, x_{k-1})\, Q_{12}(\tilde{x}_k, x_{k-1}),$$

which proves that the developments in this paper do not lose generality with respect to the full controllers and Lyapunov matrices above. Indeed, if $z_k = x_k$ (so $T(x_k) = I$), Equation (25) is condition (17) without Positivstellesatz terms Υ_{1i}. Therefore, if Equation (25) holds with particular M_1, M_2, so it will with a single M_1, $M_2 = 0$, i.e. there will exist a single controller gain K fulfilling Equation (17). The case $T \neq I$ can also be easily set up. So, this is the motivation on why Equation (11) is taken as an LK candidate (equivalent Lyapunov function of the augmented system) instead of other more complex constructions which would not be useful with the proposed developments.

Remark 3: Presence of x_{k-1} in Q instead of only $Q(\tilde{x}_k)$ (or $Q(x_k)$, remark above), allows controller $M(x_k, x_{k-1})$ to take into account present and past information, so it provides more degrees of freedom to find a solution which does not violate the saturation constraints. Note also that, even if, of course, an undelayed controller $u(x_k)$ achieving the same performance and constraints will likely exist, maybe it cannot be obtained with convex SOS conditions.

In this approach, the bilinearity has been resolved by conceiving a full-rank matrix $[E\ E^T]$ and an implicit change of coordinates, so that:

(1) In the nullspace of B, we can add an arbitrary multiplier because the control action and the matrix K do not appear. Also, the Lyapunov function can depend on Ex_k due to the nullification of B. So, no conservatism from the 'delay' trick is induced in this subspace.
(2) In the image space of B, to avoid decision variables in 'K', the actual control variable must be kept. Then, as $H(u - Kz)$ would be bilinear (due to the product of Positivstellensatz multiplier H and controller K decision variables), saturation constraints on u should be added either by Positivstellensatz conditions or, as we chose, by convex-hull argumentations. This may be conservative (we are only considering bounds on u, considered independent of decision variables, instead of $u = Kz$) but allows for more general Lyapunov functions and controllers which effectively achieved improved results. See the example in Section 4.

4. Example

Consider the following polynomial system:

$$x_{k+1} = \begin{bmatrix} -0.7 & 0.05 \\ 0.3x_{2k}\left(1 - 0.166x_{1k}^2\right) & 0.8 \end{bmatrix} x_k + \begin{bmatrix} -0.02 \\ 0.05x_{1k} \end{bmatrix} u_k$$

The goal will be to obtain the largest possible region of initial conditions Φ, with a predefined shape, for a fixed degree in the Lyapunov function and multipliers. Given the model, as $E = 0$, $E^{\perp} = I$, then $z_k = x_k$ is the only option.

Conditions to find a global controller with a quadratic $V(x_k)$, i.e. constant Q, are infeasible. Note that setting a polynomial $Q(\tilde{x}_k)$ is not a viable option, as $E = 0$, so $\tilde{x}_k$ is empty.

Now define, for instance, a spherical state-space region of interest Ω and a spherical region of initial conditions Φ as

$$\Omega = \max\left(x_{1_k}^2 + x_{2_k}^2,\, x_{1_{k-1}}^2 + x_{2_{k-1}}^2\right) \leq 4.5^2$$

$$\Phi = \max\left(x_{1_k}^2 + x_{2_k}^2,\, x_{1_{k-1}}^2 + x_{2_{k-1}}^2\right) \leq \beta^2$$

The objective will be maximising the size parameter β for fixed control action bounds μ, $|u_k| \leq \mu$, while proving that Φ belongs to the domain of attraction of the origin, enforcing the existence of a Lyapunov level set larger than Φ included in Ω. The maximum degree for $M(x_k, x_{k-1})$ and $Q(x_{k-1})$ is set to two. The parameterisations of Positivstellensatz terms are

$$W_1 = \psi_1\left(4.5^2 - x_{1_{k-1}}^2 - x_{2_{k-1}}^2\right),$$
$$W_2 = \psi_2\left(4.5^2 - x_{1_{k-1}}^2 - x_{2_{k-1}}^2\right),$$
$$S_{di} = \varrho_{di}\left(4.5^2 - x_{1_k}^2 - x_{2_k}^2\right) + \psi_{di}\left(4.5^2 - x_{1_{k-1}}^2 - x_{2_{k-1}}^2\right), \quad d : 1, 2$$

$$S_{3i} = \psi_{3i}\left(\beta^2 - x_{1_{k-1}}^2 - x_{2_{k-1}}^2\right) \text{ and } \mathcal{H}_1 = \mathcal{H}_2 = 0,$$

where ψ_1, ψ_2, ψ_{di} and ϱ_{di} are diagonal matrices of appropriate dimension whose entries belong to $\Sigma_{x_k, x_{k-1}}$, with a maximum degree of 4. Similarly, terms H_{di} in Theorem 1 are taken again as diagonal matrices whose entries belong to $\mathcal{R}_{x_k, x_{k-1}}$ of degree 4.

Table 1. Comparison of different approaches.

β	$\mu = \infty$	$\mu = 6.3$	$\mu = 1.05$
Q, M	*Inf*	*Inf*	*Inf*
$Q, M(x_k)$	1.273	1.272	0.937
$Q(x_{k-1}), M(x_k, x_{k-1})$	1.275	1.383	1.162

$*Inf \equiv infeasible$

Table 2. Approximate computational resources with the different approaches.

	Problem size	RAM	Parser time	Solver time
Q, M	774×201	10 Mb	1.19 s	0.96 s
$Q, M(x_k)$	784×201	10 Mb	1.08 s	0.76 s
$Q(x_{k-1}), M(x_k, x_{k-1})$	21041×3320	230 Mb	25.84 s	65.26 s

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Once the problem is set up, constant Lyapunov functions from literature are compared to the delayed approach here proposed. Feasible solutions were found by software SOSOPT using the image representation of the SOS problem (Balas et al., 2012; Seiler et al., 2013).

The largest β obtained until infeasibility with the different approaches is shown in Table 1. Row 1 presents results with constant decision variables Q, M; row 2 presents results obtained using Xu et al. (2007) approach; row 3 presents results with more flexible parameterisations allowed by Theorem 1.

Table 2 shows the amount of RAM memory, the time spent in the parsing phase and the time employed by the solver to obtain a solution for each of the considered approaches with $\mu = 6.3$ (i.e. to compute the figures in the centre column of Table 1). The code was executed in an Intel® Core™2 Duo CPU P8600 2.4GHz, 4 Gb DDR3 RAM machine running MATLAB R2011b with SOSOPT 2.01 and SeDuMi 1.3.

Looking at Table 1, it can be seen that a linear controller cannot be proven to stabilise the system in region Ω. Then, a polynomial controller $M(x_k)$, using Xu et al. (2007), keeps obtaining the same β for any $6.3 < \mu < \infty$.

The last row shows that improvement with respect to Xu et al. (2007) has been achieved with rational controllers arising from Theorem 1 (8.73% increase of β with $\mu = 6.3$ and 24% with a 6 times lower bound $\mu = 1.05$). Analysing the results, it is shown that, without saturation constraints ($\mu = \infty$), there is not enough information between past and present states, so there is practically no improvement over prior literature results. On the other hand, if saturation bound is low (rightmost column), the percent improvement over previous work is high. However, the proved region remains small because there is not enough input power to stabilise the system from initial conditions far away from the origin.

Note that the reported improvements come at the expense of a significantly increased computational cost: the second row considers SOS problems in x_k; the last row doubles the number of independent variables and increases the number of multipliers.

5. Conclusions

This paper develops a convex stabilisation design for polynomial systems, which reduces some sources of conservatism in previous literature results. An extension from the classical polynomial Lyapunov function is given, based on including delayed states and knowledge about limits on the control input. The percentage improvement in performance with respect to prior results increases as input bounds get smaller. The input bound can be actually considered as a design parameter, with a maximum value given by actual physical saturation limits.

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Notes

1. A particular case (Tanaka *et al.* 2008) is choosing E to be a row-selector matrix extracting the state variables whose corresponding row of $B(x)$ is zero (i.e. $\tilde{x}$ are states that do not directly depend on the control input).
2. Actually, we should prove $\mu_j - z e_j^T K^T K e_j z > 0$ via multipliers in the cone $(1 - z^T Q^{-1} z)$ and the rest of constraints defining (14) and system dynamics (24). However, the need of the change of variable (23) forces the use of some constant (S-procedure like) multipliers because relationship between ρ and χ is lost (details omitted for brevity).

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