International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: http://www.tandfonline.com/loi/tcon20

Transformation of nonlinear discrete-time system into the extended observer form

V. Kaparin & Ü. Kotta

To cite this article: V. Kaparin & Ü. Kotta (2017): Transformation of nonlinear discrete-time system into the extended observer form, International Journal of Control, DOI: 10.1080/00207179.2017.1294264

To link to this article: http://dx.doi.org/10.1080/00207179.2017.1294264

To appear in the *International Journal of Control*
Vol. 00, No. 00, Month 20XX, 1–19

Publisher: Taylor & Francis
Journal: *International Journal of Control*
DOI: http://dx.doi.org/10.1080/00207179.2017.1294264

RESEARCH ARTICLE

Transformation of nonlinear discrete-time system into the extended observer form

V. Kaparin[a∗] and Ü. Kotta[a]

[a] *Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia*

(*v5.0 released February 2015*)

The paper addresses the problem of transforming discrete-time single-input single-output nonlinear state equations into the extended observer form, which, besides the input and output, also depends on a finite number of their past values. Necessary and sufficient conditions for the existence of both the extended coordinate and output transformations, solving the problem, are formulated in terms of differential one-forms, associated with the input-output equation, corresponding to the state equations. An algorithm for transformation of state equations into the extended observer form is proposed and illustrated by an example. Moreover, the considered approach is compared with the method of dynamic observer error linearisation, which likewise is intended to enlarge the class of systems transformable into an observer form.

Keywords: discrete-time system; nonlinear control system; extended coordinate transformation; output transformation; extended observer form

1. Introduction

The construction of nonlinear observer with linearisable error dynamics is relatively easy once the state equations are in the observer form. The methods of transforming discrete-time state equations into the observer form, relying on the state transformation only, impose restrictive conditions on nonlinear systems (see, for example, Lee and Nam (1991)). The intention to enlarge the class of systems, for which observers with linear error dynamics can be constructed, motivates various extensions of the observer forms and generalisations of the transformations (Besançon and Bornard (1995); Califano, Monaco, and Normand-Cyrot (2003, 2009); Huijberts (1999); Huijberts, Nijmeijer, and Pogromsky (1999)). In Besançon and Bornard (1995); Califano et al. (2003, 2009), for instance, the matrix A (the coefficient matrix of a state vector in the matrix representation of the observer form) is allowed to depend on the control variable u. In Besançon and Bornard (1995) only the state transformation is employed, whereas the authors of Califano et al. (2003) use also

∗Corresponding author. Email: vkaparin@cc.ioc.ee

the output transformation. The paper by Califano et al. (2009) extends the results of Califano et al. (2003) into the multi-input multi-output case. Both Huijberts (1999) and Huijberts, Nijmeijer, and Pogromsky (1999) address the problem of transforming a discrete-time system without inputs into the extended observer form, i.e., an observable linear system interconnected with a nonlinearity, which, besides the current output value, also depends on a finite number of its past values. A corollary of the results considered in Huijberts, Lilge, and Nijmeijer (1999) states that a system can be always transformed into the extended observer form with the number of past output values equal to $n-1$ (where n is the dimension of the state space of the system under consideration), provided the system under consideration is strongly observable. However, in general, the minimal number of past values, necessary for transformation, is preferred, since it requires less memory. The case of the input dependent extended observer form was considered in Kaparin, Kotta, and Mullari (2013), where necessary and sufficient conditions for the existence of both the state and output coordinate transformations were formulated in terms of certain partial derivatives related to the input-output equation of a system. Though the conditions presented in Kaparin et al. (2013) are simple, they are not intrinsic, requiring to check the existence of a certain unknown function.

Regarding continuous-time systems, the methods of transforming the state equations into various extended observer forms can be found in Boutat (2015); Califano and Moog (2014); López Morales, Plestan, and Glumineau (1999) and references therein. Note, however, that though the approaches to continuous- and discrete-time systems are often similar, the results might not coincide due to different properties of the derivative and shift operators.

The purpose of this paper is to present intrinsic necessary and sufficient conditions for the existence of the extended coordinate and output transformations, allowing to bring discrete-time single-input single-output nonlinear state equations into the extended observer form. The conditions are formulated in terms of differential one-forms, associated with the input-output equation, corresponding to the state equations. Though preliminary results were published in the conference article Kaparin and Kotta (2011), this paper contains additional contribution. First, we slightly modified the formulation of the conditions to make the proof of their sufficiency more rigorous and simple. Second, we added the algorithm for transformation of state equations into the extended observer form. Moreover, we provide a comparison of our approach with a method, called dynamic observer error linearisation and proposed in Zhang, Feng, and Xu (2010), where in order to transform a system into the generalised observer form it is suggested to augment the system by means of the so-called auxiliary dynamic of the specific linear form.

2. Preliminaries

Consider a single-input single-output nonlinear discrete-time system, described by the state equations

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) : \mathbb{Z} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector, $u(t) : \mathbb{Z} \to \mathbb{U} \subset \mathbb{R}$ is an input, $y(t) : \mathbb{Z} \to \mathbb{Y} \subset \mathbb{R}$ is an output, $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ and $h : \mathbb{X} \to \mathbb{Y}$ are assumed to be real meromorphic functions[1]. Hereinafter, to simplify the exposition of the paper we leave out the time argument t and use symbols $^{[i]}$ instead of the shifted time arguments, so $x := x(t)$ and $x^{[i]} := x(t+i)$. From now on, we consider system (1) to be observable in the sense of the rank condition, see Sontag (1984) (and also Kotta (2005)).

[1] The assumption that the functions are meromorphic is necessary for the construction of the algebraic framework in Subsection 2.1.

The approach of this paper rests on the analysis of the input-output (i/o) representation of the system. Thus, we assume that application of the state elimination algorithm to the state equations (1) leads to the following higher-order i/o difference equation

$$y^{[n]} = \phi\left(y^{[0]}, \ldots, y^{[n-1]}, u^{[0]}, \ldots, u^{[n-1]}\right). \tag{2}$$

Remark 1: Note that under observability assumption, one may always find the i/o representation (2), at least locally, using the state elimination algorithm. However, the global state elimination problem is a difficult task that results generally in an implicit i/o equation accompanied with a number of inequations, see Diop (1991).

2.1 *Algebraic framework*

Below we recall necessary facts from the linear algebraic approach based on differential forms, focusing on the equation (2) (see Aranda-Bricaire, Kotta, and Moog (1996) for details). Let $\mathcal{K}^*$ denote the field of meromorphic functions (i.e., the field of fractions of the ring of analytic functions) in a finite number of independent system variables from the infinite set $\mathcal{C}^* = \left\{y^{[0]}, \ldots, y^{[n-1]}; u^{[k]}, k \geq 0; v^{[-l]}, l > 0\right\}$, where $^{[-l]}$ denotes the lth backward shift and the variable v can be chosen either to be y or u. The choice can be briefly described as follows. If $\partial\phi/\partial y \not\equiv 0$, then one may choose $v = u$. In this case, using the i/o equation (2), the variables $y^{[-l]}$, $l > 0$ can be expressed through the independent variables from $\mathcal{C}^*$. If $\partial\phi/\partial u \not\equiv 0$, then one may set $v = y$ and consider the variables $u^{[-l]}$, $l > 0$ as dependent. If both $\partial\phi/\partial y \not\equiv 0$ and $\partial\phi/\partial u \not\equiv 0$ hold, then one has freedom of choice.

For the function $F \in \mathcal{K}^*$ the forward-shift operator $\sigma : \mathcal{K}^* \to \mathcal{K}^*$ is defined as

$$\sigma\left(F\left(y^{[0]}, \ldots, y^{[n-2]}, y^{[n-1]}, u^{[0]}, \ldots, u^{[k]}, v^{[-1]}, \ldots, v^{[-l]}\right)\right) = \\ = F\left(y^{[1]}, \ldots, y^{[n-1]}, \phi, u^{[1]}, \ldots, u^{[k+1]}, v^{[0]}, \ldots, v^{[-l+1]}\right),$$

meaning that σ shifts forward every argument of the function F and then replaces $y^{[n]}$ by ϕ, according to (2). Hereinafter, the i-fold application of the forward-shift operator we denote by $^{[i]}$. Furthermore, the backward-shift operator $\rho : \mathcal{K}^* \to \mathcal{K}^*$ is defined as the inverse of σ, i.e., $\rho := \sigma^{-1}$. Thus, denoting by $^{[-i]}$ the i-fold application of the backward-shift operator, we have $F = \left(F^{[-1]}\right)^{[1]}$ and $F^{[-i]} = \left(F^{[-i+1]}\right)^{[1]}$.

Consider the infinite set of differentials $\mathrm{d}\mathcal{C}^* = \left\{\mathrm{d}y, \mathrm{d}y^{[1]}, \ldots, \mathrm{d}y^{[n-1]}; \mathrm{d}u^{[k]}, k \geq 0; \mathrm{d}v^{[-l]}, l > 0\right\}$ and define $\mathcal{E} = \operatorname{span}_{\mathcal{K}^*} \mathrm{d}\mathcal{C}^*$. The elements of $\mathcal{E}$ are called the *differential one-forms*. Any element of $\mathcal{E}$ has the form $\omega = \sum_i \alpha_i \mathrm{d}\zeta_i$, where $\zeta_i \in \mathcal{C}^*$ and only a finite number of the coefficients $\alpha_i \in \mathcal{K}^*$ are non-zero.

For $F \in \mathcal{K}^*$ define the operator $\mathrm{d} : \mathcal{K}^* \to \mathcal{E}$ by

$$\mathrm{d}F\left(y^{[0]}, \ldots, y^{[n-1]}, u^{[0]}, \ldots, u^{[k]}, v^{[-1]}, \ldots, v^{[-l]}\right) = \\ = \sum_{i=0}^{n-1} \frac{\partial F}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \sum_{j=0}^{k} \frac{\partial F}{\partial u^{[j]}} \mathrm{d}u^{[j]} + \sum_{r=1}^{l} \frac{\partial F}{\partial v^{[-r]}} \mathrm{d}v^{[-r]}.$$

Starting from the space $\mathcal{E}$ it is possible to build up the structures used in the exterior differential calculus. We refer to the book by Choquet-Bruhat, DeWitt-Morette, and Dillard-Bleick (1982) for details, whereas here we just recall some basic notions. Define the set $\wedge\mathrm{d}\mathcal{C}^* = \{\mathrm{d}\zeta \wedge \mathrm{d}\eta \mid \mathrm{d}\zeta, \mathrm{d}\eta \in$

$\mathrm{d}\mathcal{C}^*\}$, where $\wedge$ denotes the wedge product with the standard properties $\mathrm{d}\zeta \wedge \mathrm{d}\eta = -\mathrm{d}\eta \wedge \mathrm{d}\zeta$ and $\mathrm{d}\zeta \wedge \mathrm{d}\zeta = 0$ for $\mathrm{d}\zeta, \mathrm{d}\eta \in \mathrm{d}\mathcal{C}^*$. Introduce the space $\mathcal{E}^2 = \mathrm{span}_{\mathcal{K}^*} \wedge \mathrm{d}\mathcal{C}^*$ of two-forms. The operator $\mathrm{d} : \mathcal{E} \to \mathcal{E}^2$, called the exterior derivative operator, is defined for $\omega = \sum_{\ell=1}^k a_\ell(\zeta_1, \ldots, \zeta_k)\mathrm{d}\zeta_\ell \in \mathcal{E}$, where $\zeta_1, \ldots, \zeta_k \in \mathcal{C}^*$, by the rule $\mathrm{d}\omega := \sum_{\ell,\ell'} \partial a_\ell / \partial \zeta_{\ell'}\ \mathrm{d}\zeta_\ell \wedge \mathrm{d}\zeta_{\ell'}$. The notion of two-form is generalized to the p-form and the wedge product is defined for arbitrary p-forms.

2.2 *Problem statement*

The purpose is to find conditions under which there exist an extended coordinate transformation[2] $\psi(\cdot, \xi_1, \ldots, \xi_{2N}) : \mathbb{X} \to \mathbb{X}$, parameterized by $(\xi_1, \ldots, \xi_{2N})$ and defined by

$$z = \psi\left(x, y^{[-1]}, \ldots, y^{[-N]}, u^{[-1]}, \ldots, u^{[-N]}\right) \tag{3}$$

as well as an output transformation $\Psi : \mathbb{Y} \to \mathbb{Y}$, defined by

$$Y = \Psi(y) \tag{4}$$

such that in the new coordinates the state equations (1) are in the following extended observer form[3] with a buffer $N \in \{1, \ldots, n-2\}$

$$\begin{aligned}
z_1^{[1]} &= z_2 + \varphi_1\left(Y^{[0]}, \ldots, Y^{[-N]}, u^{[0]}, \ldots, u^{[-N]}\right)\\
&\vdots\\
z_{n-N}^{[1]} &= z_{n-N+1} + \varphi_{n-N}\left(Y^{[0]}, \ldots, Y^{[-N]}, u^{[0]}, \ldots, u^{[-N]}\right)\\
z_{n-N+1}^{[1]} &= z_{n-N+2}\\
&\vdots\\
z_{n-1}^{[1]} &= z_n\\
z_n^{[1]} &= 0\\
Y &= z_1,
\end{aligned} \tag{5}$$

where the forward shift of the coordinates z depends besides the input u and the output y also on their past values $u^{[-1]}, \ldots, u^{[-N]}$, and $y^{[-1]}, \ldots, y^{[-N]}$. The canonical form (5) without inputs was considered earlier in Huijberts (1999), Huijberts, Lilge, and Nijmeijer (1999). We do not address the case when the buffer $N = n-1$ since, as shown in Huijberts, Lilge, and Nijmeijer (1999), a system can be always transformed into such form (even without the output transformation (4)), whenever the system under consideration is strongly observable. The proof carries over to input-dependent systems up to some singularities caused by the presence of the input in the observability matrix. Therefore, it is obvious that the results of this paper address only the case $n \geq 3$, which is a quite common assumption for different extensions of the observer form (see, for example, Boutat (2015)).

[2]The details about the properties of the extended coordinate transformation can be found in Huijberts (1999) for the case of autonomous systems.

[3]The extended observer form without inputs was considered earlier in Huijberts (1999).

Observe that application of the state elimination algorithm to (5) yields the following i/o equation

$$Y^{[n]} = \sum_{l=1}^{n-N} \varphi_l \left(Y^{[n-l]}, \ldots, Y^{[n-l-N]}, u^{[n-l]}, \ldots, u^{[n-l-N]} \right)$$

and that (4) yields $Y^{[n]} = \Psi\left(y^{[n]}\right)$. Therefore, one may claim that if the state equations (1) can be transformed into the extended observer form (5) by means of the extended coordinate change (3) and the output transformation (4), then the i/o equation (2), corresponding to (1), can be written in the form

$$\Psi(\phi) = \sum_{l=1}^{n-N} \varphi_l \left(Y^{[n-l]}, \ldots, Y^{[n-l-N]}, u^{[n-l]}, \ldots, u^{[n-l-N]} \right). \tag{6}$$

The converse holds too, since (6) is always realizable into the extended observer form (5). Indeed, if (6) holds, it is direct to check that one may choose the new state variables as

$$\begin{aligned} z_1 &= Y, \\ z_i &= Y^{[i-1]} - \sum_{l=1}^{\min(i-1,n-N)} \varphi_l \Big(Y^{[i-1-l]}, \ldots, Y^{[i-1-l-N]}, \\ &\qquad u^{[i-1-l]}, \ldots, u^{[i-1-l-N]} \Big), \quad i = 2, \ldots, n, \end{aligned} \tag{7}$$

leading to the state equations in the extended observer form (5).

3. Necessary and sufficient conditions

In order to present the conditions for transformation of a system into the extended observer form (5), we introduce the following notation. Define, for $i, j = 0, \ldots, n-1$, the differential one-forms

$$\omega_i := \frac{\partial \phi}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial \phi}{\partial u^{[i]}} \mathrm{d}u^{[i]} \tag{8}$$

and wedge multipliers

$$\Omega_i := \prod_{\substack{k=\max(0,i-N) \\ k \neq i}}^{\min(i+N,n-1)} \wedge \mathrm{d}y^{[k]} \wedge \mathrm{d}u^{[k]}, \tag{9}$$

$$\Omega_{i,j} := \left(\prod_{\substack{k=\max(0,i-N) \\ k \neq i,j}}^{\min(i+N,n-1)} \wedge \mathrm{d}y^{[k]} \wedge \mathrm{d}u^{[k]} \right) \wedge \left(\prod_{\substack{l=\max(0,j-N) \\ l \neq i,j; l \neq k, \forall k}}^{\min(j+N,n-1)} \wedge \mathrm{d}y^{[l]} \wedge \mathrm{d}u^{[l]} \right). \tag{10}$$

For example, if $n = 4$ and $N = 1$, then

$$\Omega_{0,0} = \Omega_0 = \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]}, \qquad \Omega_{0,2} = \Omega_{2,0} = \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]} \wedge \mathrm{d}y^{[3]} \wedge \mathrm{d}u^{[3]},$$
$$\Omega_{1,1} = \Omega_1 = \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]}, \qquad \Omega_{0,3} = \Omega_{3,0} = \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]} \wedge \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]},$$
$$\Omega_{2,2} = \Omega_2 = \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]} \wedge \mathrm{d}y^{[3]} \wedge \mathrm{d}u^{[3]}, \qquad \Omega_{1,2} = \Omega_{2,1} = \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}y^{[3]} \wedge \mathrm{d}u^{[3]},$$
$$\Omega_{3,3} = \Omega_3 = \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]}, \qquad \Omega_{1,3} = \Omega_{3,1} = \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]},$$
$$\Omega_{0,1} = \Omega_{1,0} = \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]}, \qquad \Omega_{2,3} = \Omega_{3,2} = \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]}.$$

Next, denote as $\bar{\varphi}_l$ the composite functions of φ_l and the output transformation (4), such that

$$\bar{\varphi}_l\left(y^{[0]}, \ldots, y^{[-N]}, u^{[0]}, \ldots, u^{[-N]}\right) := \varphi_l\left(Y^{[0]}, \ldots, Y^{[-N]}, u^{[0]}, \ldots, u^{[-N]}\right),$$

and define the vector argument

$$\nu_l := \left[y^{[n-l]}, \ldots, y^{[n-l-N]}, u^{[n-l]}, \ldots, u^{[n-l-N]}\right] \tag{11}$$

for $l = 1, \ldots, n-N$. Notation (11) is employed merely for compactness of the exposition. In order to prove the main result, that is Theorem 1 below, we need the following lemma, the proof of which is given in the Appendix.

Lemma 1: *For functions $\bar{\varphi}_1(\nu_1), \ldots, \bar{\varphi}_{n-N}(\nu_{n-N})$ the following holds*

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \sum_{i=0}^{n-1} \Upsilon_i, \tag{12}$$

where

$$\Upsilon_i := \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \left(\frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[i]}} \mathrm{d}u^{[i]} \right). \tag{13}$$

Now we are ready to prove our main result.

Theorem 1: *The system* (1) *can be transformed by the extended coordinate change* (3) *and the output transformation* (4) *into the extended observer form* (5) *with a buffer $N \in \{1, \ldots, n-2\}$ if and only if for all $i, j = 0, \ldots, n-1$ the following holds*

$$\mathrm{d}\left(\omega_i + \omega_j\right) \wedge \left(\omega_i + \omega_j\right) \wedge \Omega_{i,j} = 0. \tag{14}$$

Remark 2: Note that in (14) the buffer N is hidden inside the definition of the codistributions $\Omega_{i,j}$.

Proof. Necessity. Assume that the system (1) is transformable into the extended observer form (5). Consequently, the i/o equation (2), corresponding to (1), can be rewritten in the form (6), the total differential of which reads as

$$\Psi'(\phi)\mathrm{d}\phi = \Psi'(\phi) \sum_{i=0}^{n-1} \omega_i = \sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l\left(\nu_l\right),$$

where $\Psi'(\phi) := \partial\Psi(\phi)/\partial\phi$. According to Lemma 1,

$$\Psi'(\phi)\sum_{i=0}^{n-1}\omega_i = \sum_{i=0}^{n-1}\Upsilon_i,$$

which yields

$$\Psi'(\phi)\omega_i = \Upsilon_i \tag{15}$$

for $i = 0,\dots,n-1$. Consider the functions $\bar{\varphi}_{n-N-l}(\nu_{n-N-l})$ for $l = \max(0, i-N),\dots,\min(i, n-1-N)$. Taking into account (11) for new index $n-N-l$, one can write

$$\mathrm{d}\bar{\varphi}_{n-N-l}(\nu_{n-N-l}) = \sum_{s=0}^{N}\left(\frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[l+s]}}\mathrm{d}y^{[l+s]} + \frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[l+s]}}\mathrm{d}u^{[l+s]}\right). \tag{16}$$

Note that one may replace the multiplication index k in (9) by the sum $s+l$ to obtain

$$\Omega_i = \prod_{\substack{s+l=\max(0,i-N)\\ s+l\neq i}}^{\min(i+N,n-1)} \wedge\mathrm{d}y^{[s+l]}\wedge\mathrm{d}u^{[s+l]}.$$

As a consequence, taking into account that $s+l$ in (16) takes values from $\max(0, i-N)$ to $\min(i+N, n-1)$, one obtains

$$\mathrm{d}\bar{\varphi}_{n-N-l}(\nu_{n-N-l})\wedge\Omega_i = \left(\frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[i]}}\mathrm{d}y^{[i]} + \frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[i]}}\mathrm{d}u^{[i]}\right)\wedge\Omega_i,$$

which, by (15) and (13), leads to

$$\Psi'(\phi)\omega_i\wedge\Omega_i = \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)}\mathrm{d}\bar{\varphi}_{n-N-l}(\nu_{n-N-l})\wedge\Omega_i.$$

Application of the exterior derivative to the equality above yields

$$\left(\mathrm{d}\left(\Psi'(\phi)\right)\wedge\omega_i + \Psi'(\phi)\mathrm{d}\omega_i\right)\wedge\Omega_i = 0$$

or alternatively

$$\left(\mathrm{d}\ln\left|\Psi'(\phi)\right|\wedge\omega_i + \mathrm{d}\omega_i\right)\wedge\Omega_i = 0. \tag{17}$$

Next, comparing the definitions (9) and (10), one may observe that the multipliers $\mathrm{d}y^{[j]}$ and $\mathrm{d}u^{[j]}$ are not represented in Ω_i only for $j < i-N$ or $j > i+N$, whereas $\Omega_{i,j}$ does not contain them for every j. Consequently, the replacement of Ω_i in (17) by $\Omega_{i,j}$ changes the equality only for $i-N \le j \le i+N$. Thus, taking into account the definition (8), one may obtain

$$\left(\mathrm{d}\ln\left|\Psi'(\phi)\right|\wedge\omega_i + \mathrm{d}\omega_i\right)\wedge\Omega_{i,j} = A_{i,j}\wedge\Omega_{i,j}, \tag{18}$$

where

$$A_{i,j} := \begin{cases} 0, & \text{if } j < i-N \text{ or } j > i+N, \\ \left(\dfrac{\partial^2\phi}{\partial u^{[i]}\partial u^{[j]}} + \dfrac{\Psi''(\phi)}{\Psi'(\phi)}\dfrac{\partial\phi}{\partial u^{[j]}}\dfrac{\partial\phi}{\partial u^{[i]}}\right)\mathrm{d}u^{[j]}\wedge\mathrm{d}u^{[i]}+ & \\ +\left(\dfrac{\partial^2\phi}{\partial u^{[i]}\partial y^{[j]}} + \dfrac{\Psi''(\phi)}{\Psi'(\phi)}\dfrac{\partial\phi}{\partial y^{[j]}}\dfrac{\partial\phi}{\partial u^{[i]}}\right)\mathrm{d}y^{[j]}\wedge\mathrm{d}u^{[i]}+ & \\ +\left(\dfrac{\partial^2\phi}{\partial y^{[i]}\partial u^{[j]}} + \dfrac{\Psi''(\phi)}{\Psi'(\phi)}\dfrac{\partial\phi}{\partial u^{[j]}}\dfrac{\partial\phi}{\partial y^{[i]}}\right)\mathrm{d}u^{[j]}\wedge\mathrm{d}y^{[i]}+ & \\ +\left(\dfrac{\partial^2\phi}{\partial y^{[i]}\partial y^{[j]}} + \dfrac{\Psi''(\phi)}{\Psi'(\phi)}\dfrac{\partial\phi}{\partial y^{[j]}}\dfrac{\partial\phi}{\partial y^{[i]}}\right)\mathrm{d}y^{[j]}\wedge\mathrm{d}y^{[i]}, & \text{otherwise.} \end{cases}$$

Since the wedge product is anticommutative, one may conclude that $A_{i,j} = -A_{j,i}$ and therefore, using (18), one may obtain

$$\left(\mathrm{d}\ln\left|\Psi'(\phi)\right| \wedge (\omega_i+\omega_j) + \mathrm{d}\omega_i + \mathrm{d}\omega_j\right)\wedge\Omega_{i,j} = 0,$$

which confirms (14).

Sufficiency. Assume that the conditions (14) are satisfied. Then there exist integrating factors $\lambda_{i,j}\left(y^{[0]},\dots,y^{[n-1]},u^{[0]},\dots,u^{[n-1]}\right)$ such that

$$(\omega_i+\omega_j)\wedge\Omega_{i,j} = \lambda_{i,j}\mathrm{d}\chi_{i,j}\wedge\Omega_{i,j} \tag{19}$$

for some functions $\chi_{i,j}$. Take the exterior derivative of (19) and then replace in the obtained equality $\mathrm{d}\chi_{i,j}\wedge\Omega_{i,j}$ using (19) to get

$$\left(\mathrm{d}\left(\omega_i+\omega_j\right) - \mathrm{d}\ln|\lambda_{i,j}| \wedge (\omega_i+\omega_j)\right)\wedge\Omega_{i,j} = 0. \tag{20}$$

Observe that the coefficient of $\mathrm{d}u^{[i]}\wedge\mathrm{d}y^{[i]}$ on the left-hand side of (20) yields

$$\frac{\partial\lambda_{i,j}}{\partial u^{[i]}}\frac{\partial\phi}{\partial y^{[i]}} - \frac{\partial\lambda_{i,j}}{\partial y^{[i]}}\frac{\partial\phi}{\partial u^{[i]}} = 0. \tag{21}$$

Similarly, replacing in (20) the index j by $s = 0,\dots,n-1$, $s\neq j$ and considering the coefficients of $\mathrm{d}u^{[i]}\wedge\mathrm{d}y^{[i]}$ on the left-hand side of the obtained equality, one may obtain

$$\frac{\partial\lambda_{i,s}}{\partial u^{[i]}}\frac{\partial\phi}{\partial y^{[i]}} - \frac{\partial\lambda_{i,s}}{\partial y^{[i]}}\frac{\partial\phi}{\partial u^{[i]}} = 0 \tag{22}$$

for $s = 0,\dots,n-1$, $s\neq j$. The comparison of (21) and (22) leads to the following partial differential equation

$$\frac{\partial\lambda_{i,j}}{\partial y^{[i]}}\frac{\partial\lambda_{i,s}}{\partial u^{[i]}} = \frac{\partial\lambda_{i,j}}{\partial u^{[i]}}\frac{\partial\lambda_{i,s}}{\partial y^{[i]}},$$

one solution of which is $\lambda_{i,j} = \lambda_{i,s}$. In this manner it is possible to show that all $\lambda_{i,j}$, $i,j = 0,\dots,n-1$ can be mutually equal. Consequently there exists common integrating factor $\lambda := \lambda_{i,j}$, in terms of which (20) reads

$$\left(\mathrm{d}\left(\omega_i+\omega_j\right) - \mathrm{d}\ln|\lambda| \wedge (\omega_i+\omega_j)\right)\wedge\Omega_{i,j} = 0. \tag{23}$$

Now, considering the coefficients of $\mathrm{d}y^{[i]} \wedge \mathrm{d}y^{[j]}$, $\mathrm{d}u^{[i]} \wedge \mathrm{d}u^{[j]}$, $\mathrm{d}u^{[j]} \wedge \mathrm{d}y^{[i]}$, and $\mathrm{d}u^{[i]} \wedge \mathrm{d}y^{[j]}$ on the left-hand side of (23) we obtain the following equalities

$$
\begin{aligned}
\frac{\partial \ln|\lambda|}{\partial y^{[i]}}\frac{\partial \phi}{\partial y^{[j]}} - \frac{\partial \ln|\lambda|}{\partial y^{[j]}}\frac{\partial \phi}{\partial y^{[i]}} &= 0, \quad i \neq j,\\
\frac{\partial \ln|\lambda|}{\partial u^{[i]}}\frac{\partial \phi}{\partial u^{[j]}} - \frac{\partial \ln|\lambda|}{\partial u^{[j]}}\frac{\partial \phi}{\partial u^{[i]}} &= 0, \quad i \neq j,\\
\frac{\partial \ln|\lambda|}{\partial u^{[j]}}\frac{\partial \phi}{\partial y^{[i]}} - \frac{\partial \ln|\lambda|}{\partial y^{[i]}}\frac{\partial \phi}{\partial u^{[j]}} &= 0,\\
\frac{\partial \ln|\lambda|}{\partial u^{[i]}}\frac{\partial \phi}{\partial y^{[j]}} - \frac{\partial \ln|\lambda|}{\partial y^{[j]}}\frac{\partial \phi}{\partial u^{[i]}} &= 0
\end{aligned}
$$

for all $i, j = 0, \ldots, n-1$, which guarantees the fulfillment of

$$\mathrm{d}\lambda \wedge \mathrm{d}\phi = 0. \tag{24}$$

According to Cartan's Lemma, from (24) follows $\mathrm{d}\lambda \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi\}$, and therefore λ can be represented as a composite function of ϕ and some other function. We will show below that the choice

$$\lambda = \frac{1}{\Psi'(\phi)} \tag{25}$$

guarantees that the composite function $\Psi(\phi)$ has the form (6). Taking into account the definition (10) and using (25), one may rewrite (23) for $j = i$ as

$$\big(\Psi'(\phi)\mathrm{d}\omega_i + \mathrm{d}\Psi'(\phi) \wedge \omega_i\big) \wedge \Omega_i = 0,$$

whose coefficients yield

$$
\begin{aligned}
\Psi'(\phi)\frac{\partial^2 \phi}{\partial y^{[i]}\partial y^{[k]}} + \frac{\partial \Psi'(\phi)}{\partial y^{[k]}}\frac{\partial \phi}{\partial y^{[i]}} &= 0,\\
\Psi'(\phi)\frac{\partial^2 \phi}{\partial u^{[i]}\partial u^{[k]}} + \frac{\partial \Psi'(\phi)}{\partial u^{[k]}}\frac{\partial \phi}{\partial u^{[i]}} &= 0,\\
\Psi'(\phi)\frac{\partial^2 \phi}{\partial y^{[i]}\partial u^{[k]}} + \frac{\partial \Psi'(\phi)}{\partial u^{[k]}}\frac{\partial \phi}{\partial y^{[i]}} &= 0,\\
\Psi'(\phi)\frac{\partial^2 \phi}{\partial u^{[i]}\partial y^{[k]}} + \frac{\partial \Psi'(\phi)}{\partial y^{[k]}}\frac{\partial \phi}{\partial u^{[i]}} &= 0
\end{aligned}
\tag{26}
$$

for $i, k = 0, \ldots, n-1$, $k \neq i - N, \ldots, i + N$. It is easy to verify that (26) can be rewritten as

$$
\begin{aligned}
\frac{\partial^2 \Psi(\phi)}{\partial y^{[i]}\partial y^{[k]}} &= 0, & \frac{\partial^2 \Psi(\phi)}{\partial u^{[i]}\partial u^{[k]}} &= 0,\\
\frac{\partial^2 \Psi(\phi)}{\partial y^{[i]}\partial u^{[k]}} &= 0, & \frac{\partial^2 \Psi(\phi)}{\partial u^{[i]}\partial y^{[k]}} &= 0,
\end{aligned}
$$

implying that the function $\Psi(\phi)$ may be represented in the form (6). This completes the proof. □

4. Algorithm

In this section we represent the algorithm for transformation of the system (1) into the observer form (5), whenever possible. First, taking into account (8) and (13), compare the coefficients of $\mathrm{d}y^{[i]}$ and $\mathrm{d}u^{[i]}$ at both sides of equality (15) to obtain

$$\begin{aligned} \left(\Psi'(\phi)\right)\frac{\partial\phi}{\partial y^{[i]}} &= \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \frac{\partial\bar{\varphi}_{n-N-l}\left(\nu_{n-N-l}\right)}{\partial y^{[i]}}, \\ \left(\Psi'(\phi)\right)\frac{\partial\phi}{\partial u^{[i]}} &= \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \frac{\partial\bar{\varphi}_{n-N-l}\left(\nu_{n-N-l}\right)}{\partial u^{[i]}} \end{aligned} \tag{27}$$

for $i = 0, \ldots, n-1$.

The algorithm is applied to the i/o representation (2) of the system (1) (see Remark 1).

Algorithm 1:

Step 1. Check the validity of the conditions (14). If they are not satisfied, the problem is not solvable; stop.

Step 2. Under the conditions (14) the following holds for $i, j = 0, \ldots, n-1$, $j \neq i-N, \ldots, i+N$

$$\begin{aligned} \left(\ln\left|\Psi'(\phi)\right|\right)' &= -\left(\frac{\partial\phi}{\partial y^{[j]}}\right)^{-1}\frac{\partial}{\partial y^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial y^{[i]}}\right|\right) = -\left(\frac{\partial\phi}{\partial y^{[j]}}\right)^{-1}\frac{\partial}{\partial y^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial u^{[i]}}\right|\right) = \\ &= -\left(\frac{\partial\phi}{\partial u^{[j]}}\right)^{-1}\frac{\partial}{\partial u^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial u^{[i]}}\right|\right) = -\left(\frac{\partial\phi}{\partial u^{[j]}}\right)^{-1}\frac{\partial}{\partial u^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial y^{[i]}}\right|\right). \end{aligned} \tag{28}$$

From (28) it is easy to observe that in general there is a freedom of choice in computation of $(\ln|\Psi'(\phi)|)'$. However, the variables and the indices j and i should be chosen such that the function ϕ depends on both $y^{[j]}$, $y^{[i]}$ (alternatively $y^{[j]}$, $u^{[i]}$ or $u^{[j]}$, $u^{[i]}$ or $u^{[j]}$, $y^{[i]}$). Next, solving the i/o equation (2) with respect to an arbitrary variable $y^{[i]}$ or $u^{[i]}$, find the replacement rule $y^{[i]} = F(\cdot)$ or $u^{[i]} = G(\cdot)$, respectively. Application of the replacement rule to $(\ln|\Psi'(\phi)|)'$ yields $\left(\ln\left|\Psi'\left(y^{[n]}\right)\right|\right)'$, which can be shifted backward n times to obtain $(\ln|\Psi'(y)|)'$, where now prime means the derivative with respect to y. Thus, the output transformation can be computed as

$$Y = \Psi(y) = \int \mathrm{e}^{\int(\ln|\Psi'(y)|)'\mathrm{d}y}\mathrm{d}y.$$

Step 3. Solve, if possible, the system of partial differential equations (27) to find the functions $\bar{\varphi}_1, \ldots, \bar{\varphi}_{n-N}$, from which the functions $\varphi_1, \ldots, \varphi_{n-N}$ can be obtained applying the output transformation.

Step 4. Using the functions $\varphi_1, \ldots, \varphi_{n-N}$ and the output transformation (4), construct the system in the extended observer form (5).

5. Example

Examine the following state equations

$$
\begin{aligned}
x_1^{[1]} &= x_2 u \\
x_2^{[1]} &= x_3 \\
x_3^{[1]} &= (x_1 + x_2 u + u)(x_2 u + x_3)(x_1 u + x_4) \\
x_4^{[1]} &= x_1 + u \\
y &= x_1.
\end{aligned}
\tag{29}
$$

The i/o equation, corresponding to (29), is

$$
y^{[4]} = \left(y + u + y^{[1]}u^{[1]}\right)\left(y^{[1]} + u^{[1]} + y^{[2]}\right) u^{[3]} \left(y^{[2]} + \frac{y^{[3]}}{u^{[2]}}\right). \tag{30}
$$

Note that once the system is transformable into the extended observer form with some arbitrary buffer N it is also transformable into the extended observer forms with the buffers that are greater than N. Therefore, our goal is to find the least buffer N, for which the system (29) is transformable into the extended observer form (5). Consequently, it is reasonable to initiate Algorithm 1 with $N = 1$.

Step 1. Compute, according to (8),

$$
\begin{aligned}
\omega_0 &= \left(y^{[1]} + u^{[1]} + y^{[2]}\right) u^{[3]} \left(y^{[2]} + \frac{y^{[3]}}{u^{[2]}}\right)(\mathrm{d}y + \mathrm{d}u), \\
\omega_1 &= u^{[3]} \left(y^{[2]} + \frac{y^{[3]}}{u^{[2]}}\right) \left(\left(y + u + u^{[1]}\left(u^{[1]} + 2y^{[1]} + y^{[2]}\right)\right) \mathrm{d}y^{[1]} + \right. \\
&\quad \left. + \left(y + u + y^{[1]}\left(y^{[1]} + 2u^{[1]} + y^{[2]}\right)\right) \mathrm{d}u^{[1]}\right), \\
\omega_2 &= u^{[3]}\left(y + u + u^{[1]}y^{[1]}\right)\left(\left(\left(y^{[1]} + u^{[1]} + 2y^{[2]} + \frac{y^{[3]}}{u^{[2]}}\right)\mathrm{d}y^{[2]} - \right.\right. \\
&\quad \left.\left. - \left(\left(y^{[1]} + u^{[1]} + y^{[2]}\right)\frac{y^{[3]}}{\left(u^{[2]}\right)^2}\right)\mathrm{d}u^{[2]}\right), \\
\omega_3 &= \frac{\left(y + u + y^{[1]}u^{[1]}\right)\left(u^{[1]} + y^{[1]} + y^{[2]}\right)}{u^{[2]}}\left(u^{[3]}\mathrm{d}y^{[3]} + \left(y^{[2]}u^{[2]} + y^{[3]}\right)\mathrm{d}u^{[3]}\right).
\end{aligned}
$$

For the case $n = 4$ and $N = 1$ the conditions (14) are the following

$$\begin{aligned}
\mathrm{d}\omega_0 \wedge \omega_0 \wedge \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]} &= 0,\\
\mathrm{d}\omega_1 \wedge \omega_1 \wedge \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]} &= 0,\\
\mathrm{d}\omega_2 \wedge \omega_2 \wedge \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]} \wedge \mathrm{d}y^{[3]} \wedge \mathrm{d}u^{[3]} &= 0,\\
\mathrm{d}\omega_3 \wedge \omega_3 \wedge \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]} &= 0,\\
\mathrm{d}\left(\omega_0 + \omega_1\right) \wedge \left(\omega_0 + \omega_1\right) \wedge \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]} &= 0,\\
\mathrm{d}\left(\omega_0 + \omega_2\right) \wedge \left(\omega_0 + \omega_2\right) \wedge \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]} \wedge \mathrm{d}y^{[3]} \wedge \mathrm{d}u^{[3]} &= 0,\\
\mathrm{d}\left(\omega_0 + \omega_3\right) \wedge \left(\omega_0 + \omega_3\right) \wedge \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]} \wedge \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]} &= 0,\\
\mathrm{d}\left(\omega_1 + \omega_2\right) \wedge \left(\omega_1 + \omega_2\right) \wedge \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}y^{[3]} \wedge \mathrm{d}u^{[3]} &= 0,\\
\mathrm{d}\left(\omega_1 + \omega_3\right) \wedge \left(\omega_1 + \omega_3\right) \wedge \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}y^{[2]} \wedge \mathrm{d}u^{[2]} &= 0,\\
\mathrm{d}\left(\omega_2 + \omega_3\right) \wedge \left(\omega_2 + \omega_3\right) \wedge \mathrm{d}y^{[1]} \wedge \mathrm{d}u^{[1]} &= 0.
\end{aligned}$$

By direct computations one can confirm that the conditions above are satisfied, which means that the system (29) is transformable via the extended coordinate and output transformations into the extended observer form with the buffer $N = 1$.

Step 2. According to (28)

$$\left(\ln\left|\Psi'\left(\phi\right)\right|\right)' = \frac{-u^{[2]}}{u^{[3]}\left(u + y + u^{[1]}y^{[1]}\right)\left(u^{[1]} + y^{[1]} + y^{[2]}\right)\left(u^{[2]}y^{[2]} + y^{[3]}\right)}.$$

The easiest way is to solve the i/o equation (30) with respect to $u^{[3]}$. This yields the following replacement rule

$$u^{[3]} = \frac{u^{[2]}y^{[4]}}{\left(u + y + u^{[1]}y^{[1]}\right)\left(u^{[1]} + y^{[1]} + y^{[2]}\right)\left(u^{[2]}y^{[2]} + y^{[3]}\right)},$$

applying which to $\left(\ln\left|\Psi'\left(\phi\right)\right|\right)'$, one obtains

$$\left(\ln\left|\Psi'\left(y^{[4]}\right)\right|\right)' = -\frac{1}{y^{[4]}}. \tag{31}$$

The equality (31) shifted backward 4 times leads to $\left(\ln\left|\Psi'\left(y\right)\right|\right)' = -1/y$ yielding the output transformation

$$Y = \Psi\left(y\right) = \int \mathrm{e}^{\int -\frac{1}{y}\mathrm{d}y}\mathrm{d}y = \ln y. \tag{32}$$

Step 3. The system of partial differential equations (27) for $n = 4$, $N = 1$ reads as

$$
\begin{aligned}
\frac{1}{u+y+u^{[1]}y^{[1]}} &= \frac{\partial\bar{\varphi}_3}{\partial y},\\
\frac{u^{[1]}}{u+y+u^{[1]}y^{[1]}} + \frac{1}{u^{[1]}+y^{[1]}+y^{[2]}} &= \frac{\partial\bar{\varphi}_3}{\partial y^{[1]}} + \frac{\partial\bar{\varphi}_2}{\partial y^{[1]}},\\
\frac{1}{u^{[1]}+y^{[1]}+y^{[2]}} + \frac{u^{[2]}}{u^{[2]}y^{[2]}+y^{[3]}} &= \frac{\partial\bar{\varphi}_2}{\partial y^{[2]}} + \frac{\partial\bar{\varphi}_1}{\partial y^{[2]}},\\
\frac{1}{u^{[2]}y^{[2]}+y^{[3]}} &= \frac{\partial\bar{\varphi}_1}{\partial y^{[3]}},\\
\frac{1}{u+y+u^{[1]}y^{[1]}} &= \frac{\partial\bar{\varphi}_3}{\partial u},\\
\frac{y^{[1]}}{u+y+u^{[1]}y^{[1]}} + \frac{1}{u^{[1]}+y^{[1]}+y^{[2]}} &= \frac{\partial\bar{\varphi}_3}{\partial u^{[1]}} + \frac{\partial\bar{\varphi}_2}{\partial u^{[1]}},\\
\frac{y^{[3]}}{\left(u^{[2]}\right)^2 y^{[2]}+u^{[2]}y^{[3]}} &= \frac{\partial\bar{\varphi}_2}{\partial u^{[2]}} + \frac{\partial\bar{\varphi}_1}{\partial u^{[2]}},\\
\frac{1}{u^{[3]}} &= \frac{\partial\bar{\varphi}_1}{\partial u^{[3]}},
\end{aligned}
$$

leading to

$$
\begin{aligned}
\bar{\varphi}_1 &= \ln\left|u^{[3]}\right| + \ln\left|y^{[2]} + \frac{y^{[3]}}{u^{[2]}}\right|,\\
\bar{\varphi}_2 &= \ln\left|y^{[1]} + u^{[1]} + y^{[2]}\right|,\\
\bar{\varphi}_3 &= \ln\left|y + u + y^{[1]}u^{[1]}\right|,
\end{aligned}
$$

which, due to the output transformation (32), yields

$$
\begin{aligned}
\varphi_1 &= \ln\left|u^{[3]}\right| + \ln\left|e^{Y^{[2]}} + \frac{e^{Y^{[3]}}}{u^{[2]}}\right|,\\
\varphi_2 &= \ln\left|e^{Y^{[1]}} + u^{[1]} + e^{Y^{[2]}}\right|,\\
\varphi_3 &= \ln\left|e^{Y} + u + e^{Y^{[1]}}u^{[1]}\right|.
\end{aligned}
$$

Step 4. Using (7), one can define the new state variables

$$
\begin{aligned}
z_1 &= Y, \\
z_2 &= Y^{[1]} - \ln|u| - \ln\left|e^{Y^{[-1]}} + \frac{e^Y}{u^{[-1]}}\right|, \\
z_3 &= Y^{[2]} - \ln\left|u^{[1]}\right| - \ln\left|e^{Y} + \frac{e^{Y^{[1]}}}{u}\right| - \ln\left|e^{Y^{[-1]}} + u^{[-1]} + e^Y\right|, \\
z_4 &= Y^{[3]} - \ln\left|u^{[2]}\right| - \ln\left|e^{Y^{[1]}} + \frac{e^{Y^{[2]}}}{u^{[1]}}\right| - \ln\left|e^{Y} + u + e^{Y^{[1]}}\right| - \\
&\quad - \ln\left|e^{Y^{[-1]}} + u^{[-1]} + e^Y u\right|,
\end{aligned}
$$

which, due to the output transformation (32) and state equations (29), can be rewritten as

$$
\begin{aligned}
z_1 &= \ln|x_1|, \\
z_2 &= \ln|x_2| - \ln\left|x_1^{[-1]} + \frac{x_1}{u^{[-1]}}\right|, \\
z_3 &= \ln|x_3| - \ln|x_1 + x_2| - \ln\left|x_1^{[-1]} + u^{[-1]} + x_1\right|, \\
z_4 &= \ln|x_1 u + x_4| - \ln\left|x_1^{[-1]} + u^{[-1]} + x_1 u\right|,
\end{aligned}
$$

that leads to the state equations in the extended observer form

$$
\begin{aligned}
z_1^{[1]} &= z_2 + \ln|u| + \ln\left|e^{Y^{[-1]}} + \frac{e^Y}{u^{[-1]}}\right| \\
z_2^{[1]} &= z_3 + \ln\left|e^{Y^{[-1]}} + u^{[-1]} + e^Y\right| \\
z_3^{[1]} &= z_4 + \ln\left|e^{Y^{[-1]}} + u^{[-1]} + e^Y u\right| \\
z_4^{[1]} &= 0 \\
Y &= z_1.
\end{aligned}
$$

6. Comparison with dynamic observer error linearisation

In this section we recall form Zhang et al. (2010) the dynamic observer error linearisation technique and compare it with transformation of a system into the extended observer form.

Like the authors of Zhang et al. (2010), here we consider the system (1) without inputs, i.e.

$$
\begin{aligned}
x^{[1]} &= f(x) \\
y &= h(x).
\end{aligned} \tag{33}
$$

Definition 1: The system (1) is called *dynamically observer error linearisable* if there exist an auxiliary dynamics

$$
\begin{aligned}
\xi^{[1]} &= \sigma(\xi, y), \quad \xi \in \mathbb{R}^d, \\
\bar{y} &= \bar{h}(\xi, y), \quad \bar{y} \in \mathbb{R}
\end{aligned} \tag{34}
$$

and a local diffeomorphism $\zeta = \Phi(\xi, x)$ defined in a neighborhood of U of $(0,0)$, which transform the augmented system

$$\begin{aligned} x^{[1]} &= f(x) \\ \xi^{[1]} &= \sigma(\xi, y) \\ \bar{y} &= \bar{h}(\xi, y) \end{aligned} \tag{35}$$

into the observer form.

In Zhang et al. (2010) the auxiliary dynamics (34) were restricted to a linear system of the special form

$$\begin{aligned} \xi_i^{[1]} &= \begin{cases} \xi_{i+1}, & i = 1, \dots, d-1 \\ \sum_{j=1}^{d} \lambda_j \xi_j + \rho y, & i = d \end{cases} \\ \bar{y} &= \xi_1 \end{aligned} \tag{36}$$

where λ_j, $j = 1, \dots, d$ are coefficients satisfying the Schur-Cohn criterion and ρ is a design parameter. Then the dynamic observer error linearisation problem was considered as transformation of the augmented system (35) with auxiliary dynamic (36) into the following generalised observer form

$$\begin{aligned} \zeta_1^{[1]} &= \zeta_2 \\ &\vdots \\ \zeta_d^{[1]} &= \zeta_{d+1} \\ \zeta_{d+1}^{[1]} &= \zeta_{d+2} + \widehat{\varphi}_1\left(\bar{y}, \bar{y}^{[1]}, \dots, \bar{y}^{[d]}\right) \\ &\vdots \\ \zeta_{n+d-1}^{[1]} &= \zeta_{n+d} + \widehat{\varphi}_{n-1}\left(\bar{y}, \bar{y}^{[1]}, \dots, \bar{y}^{[d]}\right) \\ \zeta_{n+d}^{[1]} &= \widehat{\varphi}_n\left(\bar{y}, \bar{y}^{[1]}, \dots, \bar{y}^{[d]}\right) \\ \bar{y} &= \zeta_1. \end{aligned} \tag{37}$$

Theorem 2: *The system* (35)*, augmented by the auxiliary dynamics* (36) *with* $\lambda_1 = \cdots = \lambda_d = 0$, $d \in \{1, \dots, n-2\}$ *is transformable into the generalised observer form* (37) *with* $\widehat{\varphi}_{n-d+1} = \cdots = \widehat{\varphi}_n = 0$ *if and only if the initial system* (33) *is transformable into the extended observer form* (5) *(without input) with the buffer* $N = d$ *and the output transformation* $Y = \mathrm{id}$.

Proof. Necessity. Assume that the augmented system (35) is transformable into the generalised observer form (37) with $\widehat{\varphi}_{n-d+1} = \cdots = \widehat{\varphi}_n = 0$. The i/o equation, corresponding to (37), is

$$\bar{y}^{[n+d]} = \sum_{l=1}^{n-d} \widehat{\varphi}_l\left(\bar{y}^{[n-l+d]}, \dots, \bar{y}^{[n-l]}\right). \tag{38}$$

The auxiliary dynamic (36) with $\lambda_1 = \cdots = \lambda_d = 0$ yields

$$\bar{y}^{[d]} = \rho y,$$

which allows to rewrite (38) as

$$y^{[n]} = \frac{1}{\rho}\sum_{l=1}^{n-d} \widehat{\varphi}_l \left(\rho y^{[n-l]}, \ldots, \rho y^{[n-l-d]}\right) = \sum_{l=1}^{n-d} \varphi_l \left(y^{[n-l]}, \ldots, y^{[n-l-d]}\right), \tag{39}$$

being the i/o equation, corresponding to (33). The state variables (7) lead to the realization of the i/o equation (39) in the extended observer form (5) (without input) with $N = d$ and $Y = \mathrm{id}$.

Sufficiency. If the system (33) is transformable into the extended observer form with $N = d$ and $Y = \mathrm{id}$, then the i/o equation, corresponding to (33), can be rewritten in the form (39). Thus, all the steps from the necessity part can be performed in the reverse order. □

Note that, unlike the conditions from Theorem 1, those from Zhang et al. (2010) are not intrinsic, since the procedure in Zhang et al. (2010) requires first to find a candidate of the auxiliary dynamics (36) and then to check whether the system (35) augmented by this candidate can be transformed into the generalized observer form (37).

Example 1 (Zhang et al. (2010))**:** Consider a system, described by the state equations

$$\begin{aligned} x_1^{[1]} &= x_2 \\ x_2^{[1]} &= x_3 \\ x_3^{[1]} &= x_1 + x_2 x_3 \\ y &= x_1. \end{aligned} \tag{40}$$

The authors of Zhang et al. (2010) showed that, using the auxiliary dynamic $\xi_1^{[1]} = y$ and the new output $\bar{y} = \xi_1$, one may define the new state variables

$$\begin{aligned} \zeta_1 &= \xi_1, \\ \zeta_2 &= x_1, \\ \zeta_3 &= x_2 - \xi_1 x_1, \\ \zeta_4 &= x_3 - x_1 x_2, \end{aligned}$$

such that the system (40) augmented by $\xi_1^{[1]} = y$ and $\bar{y} = \xi_1$ can be represented in the following generalised observer form

$$\begin{aligned} \zeta_1^{[1]} &= \zeta_2 \\ \zeta_2^{[1]} &= \zeta_3 + \bar{y}\bar{y}^{[1]} \\ \zeta_3^{[1]} &= \zeta_4 \\ \zeta_4^{[1]} &= \bar{y}^{[1]} \\ \bar{y} &= \zeta_1. \end{aligned} \tag{41}$$

To simplify the comparison of (41) with the extended observer form, we rewrite the equations (41) in the alternative but equivalent form

$$\begin{aligned} \zeta_1^{[1]} &= \zeta_2 \\ \zeta_2^{[1]} &= \zeta_3 + \bar{y}\bar{y}^{[1]} \\ \zeta_3^{[1]} &= \zeta_4 + \bar{y} \\ \zeta_4^{[1]} &= 0 \\ \bar{y} &= \zeta_1. \end{aligned} \tag{42}$$

Now we show that the original system (40) is transformable into the extended observer form with $Y = \mathrm{id}$ and $N = 1$. The i/o equation, corresponding to (40), is

$$y^{[3]} = y + y^{[1]}y^{[2]},$$

from which one can find by simple inspection $\varphi_1 = y^{[1]}y^{[2]}$ and $\varphi_2 = y$, leading to the state equations in the extended observer form

$$\begin{aligned} z_1^{[1]} &= z_2 + yy^{[-1]} \\ z_2^{[1]} &= z_3 + y^{[-1]} \\ z_3^{[1]} &= 0 \\ y &= z_1. \end{aligned} \tag{43}$$

Keeping in mind that $\bar{y} = \xi_1 = y^{[-1]}$, the comparison of (42) and (43) demonstrates clearly that the auxiliary dynamic $\xi_1^{[1]} = y$ just increases the dimension of the system but does not provide any benefits compared to the extended observer form.

7. Conclusions

Necessary and sufficient conditions are presented for the existence of the extended coordinate and output transformations that allow one to transform discrete-time single-input single-output nonlinear state equations into the extended observer form. The results in a sense generalise and simplify those stated in Huijberts (1999). The advantages of the presented conditions are the following. First, unlike those in Huijberts (1999), our conditions do not require the calculation of the Lie derivatives of dual vector fields, corresponding to certain one-forms, as well as the interior products of one-forms and vector fields. Moreover, in order to simplify the result, we suggest to use the different set of one-forms, which contain less terms than those in Huijberts (1999). As a consequence, our conditions are easier to check and implement. Second, in Huijberts (1999) the case without inputs was considered, whereas the conditions given in the present paper are applicable also for input dependent systems. Finally, in comparison with the alternative conditions from Kaparin et al. (2013) and the dynamic observer error linearization method from Zhang et al. (2010), the presented conditions are intrinsic, not equation dependent. Furthermore, in this paper we improved the algorithm from Kaparin et al. (2013), making the step of finding the output transformation significantly simpler.

Funding

This work was supported by the Estonian Research Council, personal research funding grant PUT481.

References

Aranda-Bricaire, E., Kotta, Ü., & Moog, C. H. (1996). Linearization of discrete-time systems. *SIAM Journal on Control and Optimization*, *34*(6), 1999–2023. doi:

Besançon, G., & Bornard, G. (1995, September). Time-varying linearization up to output injection of discrete-time nonlinear systems. In *The 3rd european control conference* (pp. 3772–3776). Rome, Italy.

Boutat, D. (2015). Extended nonlinear observer normal forms for a class of nonlinear dynamical systems. *International Journal of Robust and Nonlinear Control*, *25*(3), 461–474.

Califano, C., Monaco, S., & Normand-Cyrot, D. (2003). On the observer design in discrete-time. *Systems & Control Letters*, *49*(4), 255–265.

Califano, C., Monaco, S., & Normand-Cyrot, D. (2009). Canonical observer forms for multi-output systems up to coordinate and output transformations in discrete time. *Automatica*, *45*(11), 2483–2490.

Califano, C., & Moog, C. H. (2014). The observer error linearization problem via dynamic compensation. *IEEE Transactions on Automatic Control*, *59*(9), 2502–2508.

Choquet-Bruhat, Y., DeWitt-Morette, C., & Dillard-Bleick, M. (1982). *Analysis, manifolds and physics, part 1: Basics*. Amsterdam, The Netherlands: Elsevier.

Diop, S. (1991). Elimination in control theory. *Mathematics of Control, Signals, and Systems*, *4*(1), 17–32.

Huijberts, H. J. C. (1999). On existence of extended observers for nonlinear discrete-time systems. In H. Nijmeijer & T. I. Fossen (Eds.), *New directions in nonlinear observer design* (Vol. 244, pp. 73–92). London, UK: Springer. doi:

Huijberts, H. J. C., Lilge, T., & Nijmeijer, H. (1999, August, September). Synchronization and observers for nonlinear discrete time systems. In *European control conference* (pp. 1–9). Karlsruhe, Germany.

Huijberts, H. J. C., Nijmeijer, H., & Pogromsky, A. Y. (1999). Discrete-time observers and synchronization. In G. Chen (Ed.), *Controlling chaos and bifurcations in engineering systems* (pp. 439–456). Boca Raton, FL, USA: CRC Press.

Kaparin, V., & Kotta, Ü. (2011, December). Extended observer form for discrete-time nonlinear control systems. In *The 9th ieee international conference on control and automation* (pp. 507–512). Santiago, Chile. doi:

Kaparin, V., Kotta, Ü., & Mullari, T. (2013). Extended observer form: Simple existence conditions. *International Journal of Control*, *86*(5), 794–803. doi:

Kotta, Ü. (2005). Decomposition of discrete-time nonlinear control systems. *Proceedings of the Estonian Academy of Sciences. Physics. Mathematics*, *54*(3), 154–161.

Lee, W., & Nam, K. (1991). Observer design for autonomous discrete-time nonlinear systems. *Systems & Control Letters*, *17*(1), 49–58.

López Morales, V., Plestan, F., & Glumineau, A. (1999). Linearization by completely generalized input-output injection. *Kybernetika*, *35*(6), 793–802.

Sontag, E. D. (1984). A concept of local observability. *Systems & Control Letters*, *5*(1), 41–47.

Zhang, J., Feng, G., & Xu, H. (2010). Observer design for nonlinear discrete-time systems: Immersion and dynamic observer error linearization techniques. *International Journal of Robust and Nonlinear Control*, *20*(5), 504–514.

Appendix A. Proof of Lemma 1

Proof. It is easy to observe that

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \sum_{l=1}^{n-N}\sum_{s=0}^{N}\left(\frac{\partial\bar{\varphi}_l(\nu_l)}{\partial y^{[n-l-s]}}\mathrm{d}y^{[n-l-s]} + \frac{\partial\bar{\varphi}_l(\nu_l)}{\partial u^{[n-l-s]}}\mathrm{d}u^{[n-l-s]}\right).$$

On the right-hand side of the relationship, given above, replace the summation index l by $l+1$. In this case $l = 0,\ldots,n-N-1$ and one can change the summation order

$$\sum_{l=1}^{n-N}\sum_{s=0}^{N} a_{l,s} = \sum_{l=0}^{n-N-1}\sum_{s=0}^{N} a_{n-N-l,N-s},$$

which yields

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \sum_{l=0}^{n-N-1}\sum_{s=0}^{N}\left(\frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[l+s]}}\mathrm{d}y^{[l+s]} + \frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[l+s]}}\mathrm{d}u^{[l+s]}\right).$$

Change the summation indices l and s for $i = l+s$ and l. It is easy to see, that in this case i changes from 0 to $n-1$ and $l = i-s$. Since $s = 0,\ldots,N$, the minimal and maximal values of $i-s$ are $i-N$ and i, respectively. On the other hand, l changes from 0 to $n-N-1$. Thus, we take $l = \max(0,i-N),\ldots,\min(i,n-1-N)$. As a result, one can use the following relation

$$\sum_{l=0}^{n-N-1}\sum_{s=0}^{N} a_{l,l+s} = \sum_{i=0}^{n-1}\sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} a_{l,i},$$

which leads to (12). □