Higher-Order Probabilistic Programming

A Tuotorial at POPL 2019

Part III

Ugo Dal Lago

(Based on joint work with *Flavien Breuvart*, *Raphaëlle Crubillé*, *Charles Grellois*, *Davide Sangiorgi*,. . .)

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Equivalence and Distance Checking

- How could we check two higher-order probabilistic programs to be (context) equivalent?

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Equivalence and Distance Checking

- How could we check two higher-order probabilistic programs to be (context) equivalent?
- How about their distance?
- Contextual equivalence and contextual distance are good answers, *definitionally*.
    - They are the coarsest compatible and adequate relation and metric between programs.
    - There is however an explicit quantification over all contexts, which make argument inherently complicated.

Examples

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples

$\lambda x.x$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples

$\Delta\Delta = (\lambda x.xx)(\lambda x.xx)$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples

Not Context Equivalent: $C = [\cdot]$.

Context Distance? Consider $C_n = (\lambda x.\ \underbrace{x \ldots x}_{n \text{ times}})[\cdot]$.

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

$$I \oplus \Omega \qquad \text{vs.} \qquad \Omega$$

Examples

Not Context Equivalent: $C = [\cdot]$.
Context Distance? Cannot Easily Amplify.

$I \oplus \Omega$ vs. I

$I \oplus \Omega$ vs. Ω

Examples

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

$$I \oplus \Omega \qquad \text{vs.} \qquad \Omega$$

$$(\lambda x.I) \oplus (\lambda x.\Omega) \qquad \text{vs.} \qquad \lambda x.I \oplus \Omega$$

Examples

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Not Context Equivalent in CBV: $C = (\lambda x.x(xI))[\cdot]$
Apparently Context Equivalent in CBN.

$$I \oplus \Omega \qquad \text{vs.} \qquad \Omega$$

$$(\lambda x.I) \oplus (\lambda x.\Omega) \qquad \text{vs.} \qquad \lambda x.I \oplus \Omega$$

Examples

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$$Y_1 \qquad \text{vs.} \qquad Y_2$$

Examples

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

$$\begin{aligned} Y_1 M &\to^* M(Y_2 M) \oplus M(Y_3 M) \\ Y_2 M &\to^* M(Y_1 M) \oplus M(Y_3 M) \\ Y_3 M &\to^* M(Y_1 M) \oplus M(Y_2 M) \end{aligned}$$

$$Y_1 \qquad \text{vs.} \qquad Y_2$$

Probabilistic Bisimulation in the Abstract [LS1992]

- **Labelled Markov Chain** (LMC): a triple $\mathcal{M} = (\mathcal{S}, \mathcal{L}, \mathcal{P})$, where
  - $\mathcal{S}$ is a countable set of *states*;
  - $\mathcal{L}$ is a set of *labels*;
  - $\mathcal{P}$ is a *transition probability matrix*, i.e., a function $\mathcal{P} : \mathcal{S} \times \mathcal{L} \times \mathcal{S} \to \mathbb{R}$ such that for every state t and for every label ℓ, $\mathcal{P}(t, \ell, \mathcal{S}) = \sum_{s \in \mathcal{S}} \mathcal{P}(t, \ell, s) \leq 1$;
- **Bisimulation**: equivalence relation $\mathcal{R}$ on $\mathcal{S}$ such that whenever $t \mathrel{\mathcal{R}} s$, it holds that $\mathcal{P}(t, \ell, E) = \mathcal{P}(s, \ell, E)$ for every equivalence class E of $\mathcal{S}$ modulo $\mathcal{R}$.
- Variation: **Simulation**, which is required to be a preorder.
- Bisimilarity and Similarity can always be formed.

Proposition

$\sim \, = \, \precsim \cap \precsim^{op}$.

Terms

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms **Values**

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms	**Values**
M	

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms **Values**

$$\begin{tikzcd}
M \arrow[r, "{eval,\ [\![M]\!](V)}"] \arrow[dr, "{eval,\ [\![M]\!](W)}"] \arrow[ddr, "{eval,\ [\![M]\!](Z)}"'] & V \\
& W \\
& Z \\
& \vdots
\end{tikzcd}$$

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms	**Values**
	$\lambda x.N$

A Labelled Markov Chain for $\Lambda_{\oplus}$

$$
\begin{array}{ccc}
\textbf{Terms} & & \textbf{Values} \\
N\{W/x\} & \xleftarrow{\quad W,\ 1 \quad} & \lambda x.N
\end{array}
$$

Probabilistic Applicative Bisimulation

$$\lambda x.M \;\; \mathcal{R} \;\; \lambda x.N$$

Probabilistic Applicative Bisimulation

$$\begin{tikzcd} \lambda x.M \arrow[rr, bend left, "L"] & \mathcal{R} \;\; \lambda x.N & M\{L/x\} \end{tikzcd}$$

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$$M \quad \mathcal{R} \quad N$$

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\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & \llbracket M \rrbracket \\
M \arrow[urrr, bend left, "eval"] & \mathcal{R} & N &
\end{tikzcd}
$$

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\end{tikzcd}
$$

$$
\begin{tikzcd}
& & \llbracket M \rrbracket \arrow[dr, bend left] & \\
M \arrow[urr, bend left, "eval"] & \mathcal{R} & N \arrow[d, bend right, "eval"'] & \llbracket M \rrbracket (E) \\
& & \llbracket N \rrbracket &
\end{tikzcd}
$$

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$$
\begin{tikzcd}
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 & & & \llbracket N \rrbracket \arrow[urr, bend right] & &
\end{tikzcd}
$$

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$$\begin{tikzcd}
 & & & \llbracket M \rrbracket \arrow[dr, bend left] & \\
M \arrow[urrr, bend left, "eval"] & \mathcal{R} & N \arrow[dr, bend right, "eval"'] & & \llbracket M \rrbracket(E) = \llbracket N \rrbracket(E) \\
 & & & \llbracket N \rrbracket \arrow[ur, bend right] &
\end{tikzcd}$$

Applicative Bisimilarity vs. Context Equivalence

- **Bisimilarity**: the union $\sim$ of all bisimulation relations.
- Is it that $\sim$ is included in $\equiv$? How to prove it?
- Natural strategy: is $\sim$ a congruence?
  - If this is the case:

$$
\begin{aligned}
M \sim N \implies C[M] \sim C[N] &\implies \sum [\![C[M]]\!] = \sum [\![C[N]]\!] \\
&\implies M \equiv N.
\end{aligned}
$$

  - This is a necessary sanity check anyway.
- The naïve proof by induction **fails**, due to application: from $M \sim N$, one cannot directly conclude that $LM \sim LN$.

Howe's Technique

$\mathcal{R}$ $\qquad\qquad\qquad\qquad$ $\mathcal{R}^H$

Howe's Technique

$$\begin{tikzcd} \mathcal{R} \arrow[rr, bend left, "\subseteq"] & & \mathcal{R}^H \end{tikzcd}$$

Howe's Technique

$\mathcal{R}^H$ is a
Congruence
whenever $\mathcal{R}$ is
an equivalence

$$\mathcal{R} \xrightarrow{\subseteq} \mathcal{R}^H$$

Howe's Technique

$\sim^H$ is a **Congruence**

$$\sim \xrightarrow{\subseteq} \sim^H$$

Howe's Technique

$\sim^H$ is a **Congruence**

$$\begin{tikzcd} \sim \arrow[rr, bend left, "\subseteq"] & & \sim^H \arrow[ll, bend left, "\supseteq"] \end{tikzcd}$$

Key Lemma

Our Neighborhood

- Λ, where we observe **convergence**

	$\sim\ \subseteq\ \equiv$	$\equiv\ \subseteq\ \sim$
CBN	✓	✓
CBV	✓	✓

[Abramsky1990, Howe1993]

- $\Lambda_{\oplus}$ with nondeterministic semantics, where we observe **convergence**, in its **may** or **must** flavors.

	$\sim\ \subseteq\ \equiv$	$\equiv\ \subseteq\ \sim$
CBN	✓	✗
CBV	✓	✗

[Ong1993, Lassen1998]

The Probabilistic Case

- $\Lambda_{\oplus}$ with probabilistic semantics.

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- Counterexample for CBN: $(\lambda x.I) \oplus (\lambda x.\Omega) \not\sim \lambda x.I \oplus \Omega$
- **Where** these discrepancies come from?

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- From **testing**!

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- Counterexample for CBN: $(\lambda x.I) \oplus (\lambda x.\Omega) \not\sim \lambda x.I \oplus \Omega$
- **Where** these discrepancies come from?
- From **testing**!
- Bisimulation can be characterized by testing equivalence as follows:

Calculus	**Testing**
Λ	$T ::= \omega \mid a \cdot T$
probabilistic $\Lambda_\oplus$	$T ::= \omega \mid a \cdot T \mid \langle T, T \rangle$
nondeterministic $\Lambda_\oplus$	$T ::= \omega \mid a \cdot T \mid \wedge_{i \in I} T_i \mid \ldots$

The Probabilistic Case

- $\Lambda_{\oplus}$ with probabilistic semantics.

	$\precsim \subseteq \leq$	$\leq \subseteq \precsim$
CBN	✓	✗
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- Probabilistic simulation can be characterized by testing as follows:

$$T ::= \omega \;\Big|\; a \cdot T \;\Big|\; \langle T, T \rangle \;\Big|\; T \vee T$$

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CBV	✓	×

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$$T ::= \omega \;\big|\; a \cdot T \;\big|\; \langle T, T \rangle \;\big|\; T \vee T$$

- Full abstraction can be recovered if endowing $\Lambda_{\oplus}$ with parallel disjunction [CDLSV2015].

	$\precsim \subseteq \leq$	$\leq \subseteq \precsim$
CBN	✓	×
CBV	✓	✓

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$$\frac{}{\Gamma, x \vdash x} \qquad \frac{x, \Gamma \vdash M}{\Gamma \vdash \lambda x.M} \qquad \frac{\Gamma \vdash M \qquad \Delta \vdash N}{\Gamma, \Delta \vdash MN} \qquad \frac{\Gamma \vdash M \qquad \Gamma \vdash N}{\Gamma \vdash M \oplus N}$$

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  - The maximum distance induced by traces, i.e., sequences of actions: $\delta^t(M, N) = \sup_{\mathsf{T}} |Pr(M, \mathsf{T}) - Pr(N, \mathsf{T})|$.

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- **Soundness and Completeness Results:**

$\delta^b \leq \delta^c$	$\delta^c \leq \delta^b$	$\delta^t \leq \delta^c$	$\delta^c \leq \delta^t$
✓	×	✓	✓

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✓	✗	✓	✓

- **Example**: $\delta^t(I, I \oplus \Omega) = \delta^t(I \oplus \Omega, \Omega) = \frac{1}{2}$.

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- **A Tuple LMC**.
  - **Preterms**: $M ::= x \;\big|\; \lambda x.M \;\big|\; \lambda !x.M \;\big|\; MM \;\big|\; M \oplus M \;\big|\; !M$
  - **Terms**: any preterm M such that $\Gamma \vdash M$.
  - **States**: *sequences* of terms, rather than terms.
  - **Actions** not only model parameter passing, but also *copying* of terms.

Context

$$\frac{}{!\Gamma, x \vdash x} \qquad \frac{}{!\Gamma, !x \vdash x} \qquad \frac{x, \Gamma \vdash M}{\Gamma \vdash \lambda x.M} \qquad \frac{!x, \Gamma \vdash M}{\Gamma \vdash \lambda !x.M}$$

$$\frac{!\Gamma \vdash M}{!\Gamma \vdash !M} \qquad \frac{\Gamma, !\Theta \vdash M \qquad \Delta, !\Theta \vdash N}{\Gamma, \Delta, \Theta \vdash MN} \qquad \frac{\Gamma \vdash M \qquad \Gamma \vdash N}{\Gamma \vdash M \oplus N}$$

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- **Examples**: $\delta^t(!(I \oplus \Omega), !\Omega) = \frac{1}{2}$ $\qquad \delta^t(!(I \oplus \Omega), !I) = 1.$
- **Trivialisation** does not hold in general, but becomes true in *strongly normalising* fragments or in presence of *parellel disjuction*.

Denotational Models

- **Probabilistic Powerdomains** [JonesPlotkin1991, JungTix1998].
  - Probabilistic effects are interpreted in *monadic* style [Moggi1989].
  - Inherently difficult endeavour, because the category of measurable spaces is not cartesian closed.

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  - Can adequately model continuous distributions *and* conditioning.
- **Game and GoI Models** [DanosHarmer2002, DLFVY2017].
  - Higher-order programs are interpreted as strategies or automata.
  - Game models are fully abstract, in presence of states.

Thank You!

Questions?