


# Towards an arboretum of monadically stable classes of graphs

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## Abstract

Logical transductions provide a very useful tool to encode classes of structures inside other classes of structures. In this paper we study first-order transductions and the quasi-order they induce on infinite hereditary classes of finite graphs. Surprisingly, this quasi-order is very complex, though shaped by the locality properties of first-order logic. This contrasts with the conjectured simplicity of the MSO transduction quasi-order. Main dividing lines inherited from classical model theory are the notions of monadic stability and monadic dependence.

It appears that the FO transduction quasi-order has a great expressive power and many class properties studied earlier (such as having bounded pathwidth, bounded shrubdepth, etc.) may be equivalently defined by it. In this paper we study properties of the FO transduction quasi-order in detail, particularly in the monadically stable domain. Among other things we prove that the classes with given pathwidth form a strict hierarchy in the FO transduction quasi-order. Our basic results are based on a new normal form of transductions. We formulate several old and new conjectures. For example it is a challenging problem whether the classes with given treewidth form a strict hierarchy. This is illustrating the rich spectrum of an emerging area.

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## 1 Introduction and statement of results

Interpretations are a model theoretical tool to construct relational structures from other relational structures by means of logical formulas. Simple interpretations are a special type of interpretations where the universe of the target structure is a definable subset of the universe of the source structure. A transduction provides a tool to encode a class of structures inside another class of structures by means of creating a bounded number of copies of the elements of a structure, coloring the vertices and applying a fixed simple interpretation (see Section 2 for more details). In this paper we restrict ourselves to transductions of graphs, unless otherwise specified. The existence of a transduction of a class  $\mathcal{D}$  in a class  $\mathcal{C}$  is denoted by  $\mathcal{C} \sqsubseteq \mathcal{D}$ . We write  $\mathcal{C} \sqsubset \mathcal{D}$  for  $\mathcal{C} \sqsubseteq \mathcal{D}$  and  $\mathcal{D} \not\sqsubseteq \mathcal{C}$ ,  $\mathcal{C} \equiv \mathcal{D}$  for  $\mathcal{C} \sqsubseteq \mathcal{D}$  and  $\mathcal{D} \sqsubseteq \mathcal{C}$ , and  $\mathcal{C} \triangleleft \mathcal{D}$  for the property that  $(\mathcal{C}, \mathcal{D})$  is a *cover*, that is that  $\mathcal{C} \sqsubset \mathcal{D}$  and there is no class  $\mathcal{F}$  with  $\mathcal{C} \sqsubset \mathcal{F} \sqsubset \mathcal{D}$ .

Whatever logic is considered for the interpretation part of the transductions, it is easily checked that  $\sqsubseteq$  is a quasi-order. We study here the first-order (FO) and monadic second-order (MSO) transduction quasi-orders  $\sqsubseteq_{\text{FO}}$  and  $\sqsubseteq_{\text{MSO}}$ . These quasi-orders are very different and have a sound combinatorial and model theoretic relevance. From the definition it easily follows that we can restrict our attention to infinite *hereditary* classes, that is, infinite classes that are closed under taking induced subgraphs.

MSO transductions are basically understood. Let us write  $\mathcal{E}$  for the class of edgeless graphs,  $\mathcal{T}_n$  for the class of forests of depth  $n$ ,  $\mathcal{P}$  for the class of all paths,  $\mathcal{T}$  for the class of all trees and  $\mathcal{G}$  for the class of all graphs. The MSO transduction quasi-order is conjectured to be simply the chain  $\mathcal{E} \triangleleft_{\text{MSO}} \mathcal{T}_1 \triangleleft_{\text{MSO}} \dots \triangleleft_{\text{MSO}} \mathcal{T}_n \triangleleft_{\text{MSO}} \dots \triangleleft_{\text{MSO}} \mathcal{P} \triangleleft_{\text{MSO}} \mathcal{T} \triangleleft_{\text{MSO}} \mathcal{G}$  [3].

In a combinatorial setting this hierarchy has a very concrete meaning and it was investigated using the following notions: a class  $\mathcal{C}$  has *bounded shrubdepth* if  $\mathcal{C} \sqsubseteq_{\text{MSO}} \mathcal{T}_n$  for some  $n$ ;  $\mathcal{C}$  has *bounded linear cliquewidth* if  $\mathcal{C} \sqsubseteq_{\text{MSO}} \mathcal{P}$ ;  $\mathcal{C}$  has *bounded cliquewidth* if  $\mathcal{C} \sqsubseteq_{\text{MSO}} \mathcal{T}$ . These definitions very nicely illustrate the treelike structure of graphs from the above mentioned classes from a logical point of view, which is combinatorially captured by the existence of treelike decompositions with certain properties. It is still open whether the MSO transduction quasi-order is as shown above [3, Open Problem 9.3], though the initial fragment  $\mathcal{E} \triangleleft_{\text{MSO}} \mathcal{T}_1 \triangleleft_{\text{MSO}} \mathcal{T}_2 \triangleleft_{\text{MSO}} \dots \triangleleft_{\text{MSO}} \mathcal{T}_n$  has been proved to be as stated in [13]. We are essentially left with the following questions: Does  $\mathcal{C} \sqsubset_{\text{FO}} \mathcal{P}$  imply  $(\exists n) \mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{T}_n$ ? This is equivalent to the question whether one can transduce with MSO arbitrary long paths from any class of unbounded shrubdepth (see [17] for a proof of the CMSO version). Is the pair  $(\mathcal{P}, \mathcal{T})$  a cover? Is the pair  $(\mathcal{T}, \mathcal{G})$  a cover? This last question is related to a famous conjecture of Seese [27] and the CMSO version has been proved in [6].

Note that MSO collapses to FO on classes of (colored) trees of bounded depth [13], hence the bounded shrubdepth parts of the MSO and FO transduction quasi-orders coincide.

The above hierarchy is related to the complexity of the model-checking problem for MSO. Every MSO property can be tested in cubic time on classes with bounded cliquewidth [7], while on the class  $\mathcal{G}$  of all graphs there are formulas for which model-checking is NP-hard. Moreover, the dependency on the quantifier rank of the formula is double exponential on classes with bounded shrubdepth [13], while on  $\mathcal{P}$  it is non-elementary [18] (under standard assumptions from complexity theory).

Also the FO model-checking problem has received a lot of attention in the literature. Unlike for MSO, each FO formula can be tested in polynomial time. However, the degree of the polynomial has to depend on the formula (under standard assumptions from complexity theory). Hence, the question in this context is to determine the limit of fixed-parameter

tractability of FO model-checking. In this setting there are two kinds of results. First, there are classes for which model-checking is known to be fixed-parameter tractable. These include for example nowhere dense graph classes [15] and classes with bounded cliquewidth [7]. Then, there are classes for which model-checking is fixed-parameter tractable modulo the existence of some polynomial time preprocessing algorithm. These include structurally bounded expansion classes (for which the preprocessing needs to compute a depth-2 low shrubdepth cover) [12] and classes with bounded twin-width (for which the preprocessing needs to compute an appropriate elimination sequence) [4]. Note that whenever we can efficiently solve the FO model-checking problem on classes with a certain property, then we can also solve it efficiently on classes that locally have this property. This observation follows from Gaifman's Locality Theorem for first-order logic. We elaborate on the consequences of the locality properties of first-order logic in Section 3.1 and Appendix B. For instance, we give there a normal form for FO transductions and prove that the local versions of monadic stability and monadic dependence collapse with the non-local versions.

It is likely that, as in the MSO case, the FO transduction quasi-order will allow to draw important algorithmic and structural dividing lines. First note that FO transductions give alternative characterizations of the graph class properties mentioned above. A class  $\mathcal{C}$  has

- bounded shrubdepth if and only if  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{T}_n$  for some  $n$  [14],
- bounded linear cliquewidth if and only if  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{H}$ , where  $\mathcal{H}$  denotes the class of *half-graphs* (bipartite graphs with vertex set  $\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\}$  and edge set  $\{a_i b_j : 1 \leq i \leq j \leq n\}$  for some  $n$ ) [5], and
- bounded cliquewidth if and only if  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{TP}$ , where  $\mathcal{TP}$  denotes the class of *trivially perfect graphs* (comparability graphs of rooted trees) [5].

Also, FO transductions allow to give an alternative characterizations of classical model theoretical properties [2]: a class  $\mathcal{C}$  is *monadically stable* if  $\mathcal{C} \not\sqsubseteq_{\text{FO}} \mathcal{H}$  and *monadically dependent* if  $\mathcal{C} \not\sqsubseteq_{\text{FO}} \mathcal{G}$ . We further call a class  $\mathcal{C}$  *monadically straight* if  $\mathcal{C} \not\sqsubseteq_{\text{FO}} \mathcal{TP}$ . To the best of our knowledge this property has not been studied in the literature but seems to play a key role in the study of FO transductions.

The FO transduction quasi-order is much more complicated than the MSO transduction quasi-order. This is outlined in Figure 1, and it is the goal of this paper to explore this quasi-order.

We are motivated by three aspects of the  $\sqsubseteq_{\text{FO}}$  quasi-order, which have been specifically considered in the past and appeared to be highly non-trivial. The first aspect concerns the antichains formed by the class of paths and classes with bounded shrubdepth.

► **Conjecture 1.** *A class  $\mathcal{C}$  has bounded shrubdepth if and only if  $\mathcal{C} \not\sqsubseteq_{\text{FO}} \mathcal{P}$ .*

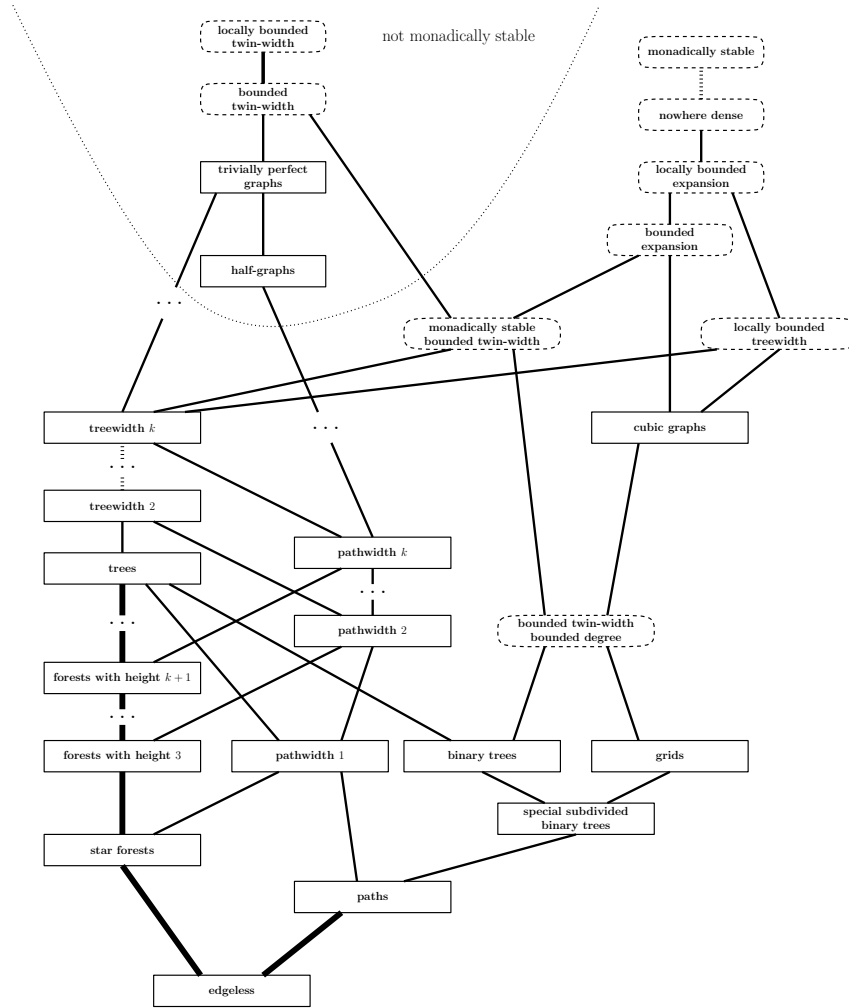
This conjecture may be seen to be equivalent to the following one.

► **Conjecture 2.** *The pairs  $\{\mathcal{T}_n, \mathcal{P}\}$  with  $n > 1$  are all the maximal antichains containing  $\mathcal{P}$ .*

Note that the maximal antichains play a key role in the theory of finite dualities (and equivalently, FO definable CSP classes, see e.g. [26]).

The second aspect that was studied in detail concerns the chain formed by classes with bounded pathwidth, which is eventually covered by the class of half-graphs. This is related to the following result, which we reformulate here in our setting.

► **Theorem 1** ([24, 25]). *In the FO transduction quasi-order there is no class  $\mathcal{C}$  between the classes with bounded pathwidth and the class of half-graphs. Formally, if  $\mathcal{H} \not\sqsubseteq_{\text{FO}} \mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{H}$  (that is,  $\mathcal{C}$  is monadically stable and has bounded linear cliquewidth), then  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{PW}_n$  for some  $n$ , where  $\mathcal{PW}_n$  denotes the class of graphs with pathwidth at most  $n$ .*



■ **Figure 1** Partial outline of the FO transduction quasi-order. The special subdivided binary trees are those subdivisions of binary trees that are subgraphs of the grid. Dashed boxes correspond to families of not necessarily transduction equivalent graph classes sharing a common property. Fat lines correspond to covers, normal lines correspond to strict containment, dotted lines correspond to containment (with a possible collapse).

The last aspect concerns the chain of classes with bounded treewidth, which is eventually covered by the class of trivially perfect graphs. This is related to the following result.

► **Theorem 2** ([20]). *If  $\mathcal{H} \not\sqsubseteq_{\text{FO}} \mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{TP}$  (that is,  $\mathcal{C}$  is monadically stable and  $\mathcal{C}$  has bounded cliquewidth), then  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{TW}_n$  for some  $n$ , where  $\mathcal{TW}_n$  denotes the class of graphs with treewidth at most  $n$ .*

In this paper, we establish the following results, greatly expanding known properties of the FO transduction quasi-order.

► **Theorem 3.** *The FO transduction quasi-order has the following properties.*

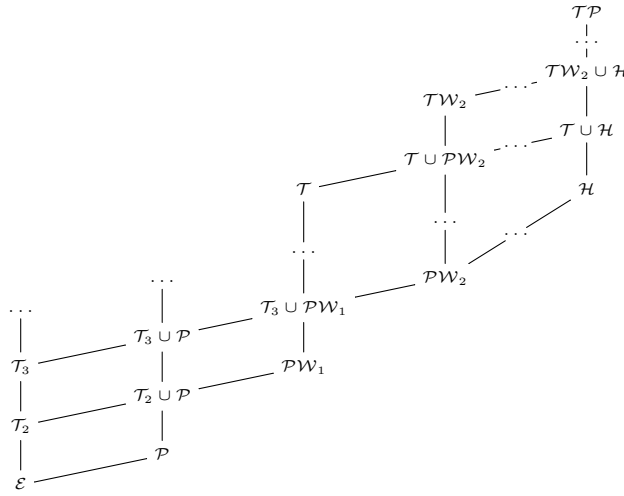
1.  $\mathcal{E} \triangleleft_{\text{FO}} \mathcal{P} \triangleleft_{\text{FO}} \mathcal{P} \cup \mathcal{T}_2 \triangleleft_{\text{FO}} \cdots \triangleleft_{\text{FO}} \mathcal{P} \cup \mathcal{T}_n$  and  $\mathcal{T}_n \triangleleft_{\text{FO}} \mathcal{P} \cup \mathcal{T}_n$ ;
2.  $\mathcal{T}_{n+1} \sqsubseteq_{\text{FO}} \mathcal{PW}_n$  but  $\mathcal{T}_{n+2} \not\sqsubseteq_{\text{FO}} \mathcal{PW}_n$ ;
3.  $\mathcal{PW}_n$  is incomparable with  $\mathcal{T}$  for  $n \geq 2$ ;

4.  $\mathcal{PW}_n \sqsubset_{\text{FO}} \mathcal{PW}_{n+1}$ ;
5.  $\mathcal{T} \sqsubset_{\text{FO}} \mathcal{TW}_2$ ;
6.  $\mathcal{PW}_n \sqsubset_{\text{FO}} \mathcal{TW}_n$ ;
7.  $\mathcal{BT} \sqsubset_{\text{FO}} \mathcal{T}$ , where  $\mathcal{BT}$  is the class of binary trees;
8.  $\mathcal{BT}$  and the class of grids are incomparable.

Item 4 (strictness of the pathwidth hierarchy) was a motivating example for this study, and we conjecture that a similar statement holds with treewidth. This would be a consequence of the following conjecture.

► **Conjecture 3.**  $\mathcal{PW}_{n+1}$  and  $\mathcal{TW}_n$  are non-comparable.

Hence, the structure of the FO transduction quasi-order is much more complex than what is sketched on Figure 1. We illustrate in Figure 2.



■ **Figure 2** A more detailed fragment of the FO transduction quasi-order

A main tool to prove Theorem 3 will be the following normal form for FO transductions, where the missing terminology will be introduced in Section 2.

► **Theorem 4** (Transduction normal form). *Every transduction  $\mathsf{T}$  is subsumed by the composition of a copying operation, an immersive transduction and a perturbation. If  $\mathsf{T}$  is non-copying, then no copying operation is needed.*

Using this result we also characterize the equivalence class of the class of paths in the FO transduction quasi-order (see Section 4 for the definition of bandwidth).

► **Theorem 5.** *A class  $\mathcal{C}$  is FO transduction equivalent to the class of paths if and only if it is a perturbation of a class with bounded bandwidth containing graphs with arbitrarily large connected components.*

Finally, in Section 6 we prove the next result, as an application of Theorem 4 with independent interest.

► **Theorem 6.** *For a class  $\mathcal{C}$  of graphs we have the following equivalences:*

1.  $\mathcal{C}$  is locally monadically dependent if and only if  $\mathcal{C}$  is monadically dependent;
2.  $\mathcal{C}$  is locally monadically straight if and only if  $\mathcal{C}$  is monadically straight;
3.  $\mathcal{C}$  is locally monadically stable if and only if  $\mathcal{C}$  is monadically stable.

This theorem is completed by a non-monadic version in the appendix.

## 2 Preliminaries

We assume familiarity with first-order logic and graph theory and refer e.g. to [8, 16] for background and for all undefined notation. The vertex set of a graph  $G$  is denoted as  $V(G)$  and its edge set  $E(G)$ . The *complement* of a graph  $G$  is the graph  $\overline{G}$  with the same vertex set, in which two vertices are adjacent if they are not adjacent in  $G$ . The *disjoint union* of two graphs  $G$  and  $H$  is denoted as  $G + H$ , and their *complete join*  $\overline{G} + \overline{H}$  as  $G \oplus H$ . The *lexicographic product*  $G \bullet H$  of two graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$ , in which  $(u, v)$  is adjacent to  $(u', v')$  if either  $u$  is adjacent to  $u'$  in  $G$  or  $u = u'$  and  $v$  is adjacent to  $v'$  in  $H$ . The *pathwidth* of a graph  $G$  is  $\text{pw}(G) = \min\{\omega(H) - 1 : \exists \text{ interval graph } H \text{ with } H \supseteq G\}$ . The *treewidth* of a graph  $G$  is  $\text{tw}(G) = \min\{\omega(H) - 1 : \exists H \in \mathcal{TP} \text{ with } H \supseteq G\}$ . We write  $G^k$  for the  $k$ -th *power* of  $G$  (which has the same vertex set as  $G$  and two vertices are connected if their distance is at most  $k$  in  $G$ ). The *bandwidth* of a graph  $G$  is  $\text{bw}(G) = \min\{\ell : \exists P \in \mathcal{P} \text{ with } P^\ell \supseteq G\}$ .

In this paper we consider either graphs or  $\Sigma$ -*expanded graphs*, that is, graphs with additional unary relations in  $\Sigma$  (for a set  $\Sigma$  of unary relation symbols). We usually denote graphs by  $G, H, \dots$  and  $\Sigma$ -expanded graphs by  $G^+, H^+, G^*, H^*, \dots$ , but sometimes we will use  $G, H, \dots$  for  $\Sigma$ -expanded graphs as well. In formulas, the adjacency relation will be denoted as  $E(x, y)$ . For each non-negative integer  $r$  we can write a formula  $\delta_{\leq r}(x, y)$  such that for every graph  $G$  and all  $u, v \in V(G)$  we have  $G \models \delta_{\leq r}(u, v)$  if and only if the distance between  $u$  and  $v$  in  $G$  is at most  $r$ . For improved readability we write  $\text{dist}(x, y) \leq r$  for  $\delta_{\leq r}(x, y)$ . For  $U \subseteq V(G)$  we write  $B_r^G(U)$  for the subgraph of  $G$  induced by the closed  $r$ -neighborhoods of the vertices in  $U$ . We write  $N^G(v)$  for the open neighborhood of  $v$  (as a set of vertices). For the sake of simplicity we use the notation  $B_r^G(v)$  instead of  $B_r^G(\{v\})$  and, if  $G$  is clear from the context, we drop the superscript  $G$ . For a class  $\mathcal{C}$  and an integer  $r$ , we denote by  $\mathcal{B}_r^{\mathcal{C}}$  the class of all the balls of radius  $r$  of graphs in  $\mathcal{C}$ :  $\mathcal{B}_r^{\mathcal{C}} = \{B_r^G(v) \mid G \in \mathcal{C} \text{ and } v \in V(G)\}$ . For a formula  $\varphi(x_1, \dots, x_k)$  and a graph (or a  $\Sigma$ -expanded graph)  $G$  we define

$$\varphi(G) := \{(v_1, \dots, v_k) \in V(G)^k : G \models \varphi(v_1, \dots, v_k)\}.$$

Let  $\Sigma$  be a set of unary relation symbols. A *simple interpretation*  $\mathsf{I}$  of graphs in  $\Sigma$ -expanded graphs is a pair  $(\nu(x), \eta(x, y))$  consisting of two formulas (in the language of  $\Sigma$ -expanded graphs), where  $\eta$  is symmetric and anti-reflexive (i.e.  $\models \eta(x, y) \leftrightarrow \eta(y, x)$  and  $\models \eta(x, y) \rightarrow \neg(x = y)$ ). If  $G^+$  is a  $\Sigma$ -expanded graph, then  $H = \mathsf{I}(G^+)$  is the graph with vertex set  $V(H) = \nu(G^+)$  and edge set  $E(H) = \rho(G) \cap \nu(G)^2$ . A *non-copying transduction*  $\mathsf{T}$  (from graphs to graphs) is a pair  $(\Sigma_{\mathsf{T}}, \mathsf{I}_{\mathsf{T}})$ , where  $\Sigma_{\mathsf{T}}$  is a finite set of unary relation symbols and  $\mathsf{I}_{\mathsf{T}}$  is a simple interpretation of graphs in  $\Sigma_{\mathsf{T}}$ -expanded graphs. A graph  $H$  is a  $\mathsf{T}$ -*transduction* of a graph  $G$  if there exists a  $\Sigma_{\mathsf{T}}$ -expansion  $G^+$  of  $G$  with  $\mathsf{I}_{\mathsf{T}}(G^+) = H$ . A class  $\mathcal{C}$  of graphs is a  $\mathsf{T}$ -*transduction* of a class  $\mathcal{D}$  of graphs if for every graph  $G \in \mathcal{C}$  there exists a graph  $H \in \mathcal{D}$  such that  $G$  is a  $\mathsf{T}$ -transduction of  $H$ . We also say that  $\mathsf{T}$  is a transduction from  $\mathcal{D}$  onto  $\mathcal{C}$ . A class  $\mathcal{C}$  of graphs is a *transduction* of a class  $\mathcal{D}$  of graphs if it is a  $\mathsf{T}$ -*transduction* of  $\mathcal{D}$  for some transduction  $\mathsf{T}$ . Note that non-copying transductions compose, and that the relation “ $\mathcal{C}$  is a non-copying transduction of  $\mathcal{D}$ ” defines a quasi-order on classes of graphs and we denote by  $\mathcal{C} \sqsubseteq_{\text{FO}}^{\circ} \mathcal{D}$  the property that  $\mathcal{C}$  is a non-copying transduction of  $\mathcal{D}$ . For a positive integer  $k$ , the  $k$ -*copy operation*  $\mathsf{C}_k$  maps a graph  $G$  to the graph  $\mathsf{C}_k(G)$  obtained from  $k$  copies of  $G$  by making corresponding vertices adjacent. A *transduction* is any composition of copy operations and non-copying transductions. Note that transductions compose, and that the relation “ $\mathcal{C}$  is a transduction of  $\mathcal{D}$ ” defines a quasi-order on classes of graphs and we denote by  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{D}$  the property that  $\mathcal{C}$  is a transduction of  $\mathcal{D}$ . Intuitively, if  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{D}$ , then  $\mathcal{C}$  is at most as complex as  $\mathcal{D}$ . For a class  $\mathcal{C}$  and a transduction  $\mathsf{T}$  we write  $\mathsf{T}(\mathcal{C})$  for all

graphs that can be obtained by applying the transduction  $T$  to graphs in  $\mathcal{C}$ . We say that a transduction  $T'$  *subsumes* a transduction  $T$  if, for every graph  $G$  we have  $T'(\{G\}) \supseteq T(\{G\})$ . We denote by  $T' \geq T$  the property that  $T'$  subsumes  $T$ . The following fact is easy to establish (see for instance [12]).

► **Observation 1.** *Every transduction  $T$  is subsumed by the composition of a copy operation and a non-copying transduction.*

We call a class  $\mathcal{C}$  *easy* if for every integer  $k$  the class  $C_k(\mathcal{C})$  is a non-copying transduction of  $\mathcal{C}$ . In particular, it is easily checked that if  $\mathcal{C}$  is closed under adding pendant vertices, then  $\mathcal{C}$  is easy. In Figure 1 only the classes  $\mathcal{E}$  and  $\mathcal{T}_n$  are not easy. As  $\mathcal{C}$  is obviously a non-copying transduction of  $C_k(\mathcal{C})$ , we deduce that if  $\mathcal{C}$  is easy and  $\mathcal{C} \equiv_{FO} \mathcal{D}$ , then  $\mathcal{D}$  is easy. The following observation follows from Observation 1.

► **Observation 2.** *Let  $\mathcal{C}, \mathcal{D}$  be classes. If  $\mathcal{D}$  is easy, then  $\mathcal{C} \sqsubseteq_{FO} \mathcal{D}$  if and only if  $\mathcal{C} \sqsubseteq_{FO}^\circ \mathcal{D}$ .*

This observation justifies that we shall mainly study non-copying transductions in this paper. Let  $r$  be a non-negative integer. A formula  $\varphi(x_1, \dots, x_k)$  is  *$r$ -local* if for every  $(\Sigma$ -expanded) graph  $G$  and all  $v_1, \dots, v_k \in V(G)$  we have

$$G \models \varphi(v_1, \dots, v_k) \iff B_r^G(\{v_1, \dots, v_k\}) \models \varphi(v_1, \dots, v_k).$$

An  $r$ -local formula  $\varphi(x_1, \dots, x_k)$  is *strongly  $r$ -local* if  $\models \varphi(x_1, \dots, x_k) \rightarrow \text{dist}(x_i, x_j) \leq r$  for all  $1 \leq i < j \leq k$  (see [23]).

► **Lemma 7** (Gaifman's Locality Theorem [10]). *Every formula  $\varphi(x_1, \dots, x_m)$  is equivalent to a Boolean combination of  $t$ -local formulas and so-called local sentences of the form*

$$\exists x_1 \dots \exists x_k \left( \bigwedge_{1 \leq i \leq k} \chi(x_i) \wedge \bigwedge_{1 \leq i < j \leq k} \text{dist}(x_i, x_j) > 2r \quad (\text{where } \chi \text{ is } r\text{-local}). \right)$$

Furthermore, if the quantifier-rank of  $\varphi$  is  $q$ , then  $r \leq 7^{q-1}$ ,  $t \leq 7^{q-1}/2$ , and  $k \leq q + m$ .

We call a transduction  $T$  *immersive* if it is non-copying and the interpretation associated to  $T$  is strongly local. A *subset complementation* transduction is defined by the quantifier-free interpretation on a  $\Sigma$ -expansion (with  $\Sigma = \{M\}$ ) by  $\rho(x, y) := \neg(E(x, y) \leftrightarrow (M(x) \wedge M(y)))$ . In other words, the subset complementation transduction complements the adjacency inside the subset of the vertex set defined by  $M$ . We denote by  $\oplus M$  the subset complementation defined by the unary relation  $M$ . A *perturbation* is a (bounded) composition of subset complementations.

### 3 Basic properties of FO transductions

#### 3.1 A normal form

We now establish a normal form for first-order transductions that captures the local character of first-order logic. The normal form is based on Gaifman's Locality Theorem and uses only *strongly local formulas*, while the basic-local formulas are handled by subset complementations. The normal form (Theorem 4) states that every transduction  $T$  is subsumed by the composition of a copying operation  $C$ , an immersive transduction  $T_{\text{imm}}$  and a perturbation  $P$ , that is  $T \leq P \circ T_{\text{imm}} \circ C$ , and that the copy operation can be omitted if  $T$  is non-copying.



**Proof of Theorem 4.** According to Observation 1 we can restrict to the case where the transduction is non-copying.

Let  $T = (\Sigma_T, I_T)$  be a non-copying transduction from graphs to graphs. Without loss of generality, we may assume that the interpretation  $I_T$  defines the domain directly from the  $\Sigma_T$ -expansion. Then the only non-trivial part of the interpretation is the adjacency relation, which is defined by a symmetric and anti-reflexive formula  $\eta(x, y)$ .

We shall prove that the transduction  $T$  is subsumed by the composition of an immersive transduction  $T_\psi$  and a perturbation  $T_q$ .

We define  $\Sigma_{T_\psi}$  as the disjoint union of  $\Sigma_T$  and a set  $\Sigma_0 = \{T_i \mid 1 \leq i \leq n_1\}$ , for some integer  $n_1$  we shall precise later and let  $\Sigma_{T_q} = \{Z_j \mid 1 \leq j \leq n_2\}$ , for some integer  $n_2$  we shall also precise later.

Let  $r$  be the quantifier rank of  $\eta(x, y)$ . According to Lemma 7,  $\eta$  is logically equivalent to a formula in Gaifman normal form, that is, to a Boolean combination of  $t$ -local formulas and basic-local sentences  $\theta_1, \dots, \theta_{n_1}$ . To each  $\theta_i$  we associate a unary predicate  $T_i \in \Sigma_{T_\psi}$ .

We consider the formula  $\tilde{\eta}(x, y)$  obtained from the Gaifman normal form of  $\eta(x, y)$  by replacing the sentence  $\theta_i$  by the formula  $T_i(x)$ . Note that  $\tilde{\eta}$  is  $t$ -local. It follows that under the assumption  $\text{dist}(x, y) > 2t$  the formula  $\tilde{\eta}$  is equivalent to a formula  $\tilde{q}(x, y)$  of the form  $\bigvee_{(i,j) \in \mathcal{F}} \zeta_i(x) \wedge \zeta_j(y)$ , where  $\mathcal{F} \subseteq [n_2] \times [n_2]$  for some integer  $n_2$  and the formulas  $\zeta_i$  ( $1 \leq i \leq n_2$ ) are  $t$ -local. Note that  $\mathcal{F}$  is symmetric as  $\eta$  (hence  $\tilde{\eta}$  and  $\tilde{q}$ ) are symmetric.

We define  $\psi(x, y) := \neg(\tilde{\eta}(x, y) \leftrightarrow \tilde{q}(x, y)) \wedge (\text{dist}(x, y) \leq 2t)$ , which is  $2t$ -strongly local, and we define  $I_{T_\psi}$  as the interpretation of graphs in  $\Sigma_{T_\psi}$ -structures by using the same definitions as in  $I_T$  for the domain, then defining the adjacency relation by  $\psi(x, y)$ .

To each formula  $\zeta_i$  we associate a unary predicate  $Z_i \in \Sigma_{T_q}$ . We define the formula  $q$  by  $q(x, y) := \bigvee_{(i,j) \in \mathcal{F}} Z_i(x) \wedge Z_j(y)$ , and we define  $I_q$  as the interpretation defining the adjacency by the formula  $q(x, y)$ . Observe that the transduction  $T_q$  is equivalent to the sequence of the subset complementations  $\oplus Z_i$  (for  $(i, i) \in \mathcal{F}$ ) and of  $\oplus(Z_i \vee Z_j) \oplus Z_i \oplus Z_j$  (for  $(i, j) \in \mathcal{F}$  and  $i < j$ ).

Now assume that a graph  $H$  is a  $T$ -transduction of a graph  $G$ , and let  $G^+$  be a  $\Sigma_T$ -expansion of  $G$  such that  $H = I_T(G^+)$ . We define the  $\Sigma_0$ -expansion  $G^*$  of  $G^+$  (which is thus a  $\Sigma_{T_\psi}$ -expansion of  $G$ ) by defining, for each  $i \in [n_1]$ ,  $T_i(G^*) = V(G)$  if  $G^+ \models \theta_i$  and  $T_i(G^*) = \emptyset$  otherwise. Let  $K = I_{T_\psi}(G^*)$ . We define the  $\Sigma_{T_q}$ -expansion of  $K$  by defining, for each  $j \in [n_2]$ ,  $Z_j(K^+) = \zeta_j(G^+)$ . According to these definitions, we have  $\eta(G^+) = \tilde{\eta}(G^*) = (\neg(\psi(x, y) \leftrightarrow \tilde{q}(x, y)))(G^*) = q(K^+)$  hence  $I_{T_q}(K^+) = H$ .

It follows that the transduction  $T$  is subsumed by the composition of the immersive transduction  $T_\psi$  and a sequence of subset complementations, the perturbation  $T_q$ . ◀

► **Corollary 8.** *For every immersive transduction  $T$  and every integer  $k$ , there exists an immersive transduction  $T'$  and an integer  $k'$  such that the transduction consisting in complementing  $k$  subsets and then applying  $T$  is subsumed by the transduction consisting in applying  $T'$  and then  $k'$  subset complementations.*

Immersive transductions have very strong properties. However, the existence of an immersive transduction from a class onto another can be characterized by a (seemingly) weaker property, which is the existence of a transduction that does not shrink the distances too much, as we prove now.

► **Lemma 9.** *Assume there is a non-copying transduction  $T$  encoding  $\mathcal{C}$  in  $\mathcal{D}$  with associated interpretation  $I$  and a positive  $\epsilon > 0$  with the property that for every graph  $G \in \mathcal{C}$  there is a coloring  $H^+$  of  $H \in \mathcal{D}$  with  $G = I(H^+)$  and  $\text{dist}_G \geq \epsilon \text{dist}_H$ . Then there exists an immersive transduction encoding  $\mathcal{C}$  in  $\mathcal{D}$  that subsumes  $T$ .*



**Proof.** Let  $l = (\nu(x), \eta(x, y))$ . By Gaifman's locality theorem, there is a formula  $\varphi(x, y)$  on a monadic lift, such that for every colored graph  $H^+$  there is a coloring  $H^*$  of  $H^+$  with  $H^* \models \varphi(x, y)$  if and only if  $H^+ \models \eta(x, y)$ , where  $\varphi$  is  $t$ -local for some  $t$  (as in the proof of Theorem 4). We further define a new mark  $M$  and let  $l' = (M(x), \varphi(x, y) \wedge \delta_{[1/\epsilon]}(x, y))$ . The transduction  $T'$  associated to the interpretation  $l'$  is immersive and subsumes the transduction  $T$ .  $\blacktriangleleft$

► **Lemma 10.** *Let  $\mathcal{C}, \mathcal{F}$  be graph classes, and let  $T$  be an immersive transduction encoding a class  $\mathcal{D}$  with  $\mathcal{D} \supseteq \{G \oplus K_1 \mid G \in \mathcal{C}\}$  in  $\mathcal{F}$ . Then there exists an integer  $r$  such that  $\mathcal{C} \sqsubseteq_{FO}^{\circ} \mathcal{B}_r^{\mathcal{F}}$ .*

**Proof.** Let  $T$  be an immersive transduction encoding  $\mathcal{D}$  in  $\mathcal{F}$ , with associated interpretation  $l$ . For every graph  $G \in \mathcal{C}$  there exists a graph  $F \in \mathcal{F}$  such that  $G \oplus K_1 = l(F^+)$ , where  $F^+$  is a coloring of  $F$ . Let  $v$  be the apex of  $G \oplus K_1$ . By strong locality of  $l$  we get  $l(F^+) = l(B_\ell^{F^+}(v))$  for some fixed  $\ell$  depending on  $T$ . Let  $U$  be a transduction allowing to take an induced subgraph, then  $G$  can be obtained by the non-copying transduction  $U \circ T$  applied on the class  $\mathcal{B}_\ell^{\mathcal{F}}$ .  $\blacktriangleleft$

### 3.2 Further basic properties

► **Observation 3.** *If  $\mathcal{C}$  and  $\mathcal{D}$  are incomparable in  $\sqsubseteq_{FO}$ , then we have  $\mathcal{C} \cap \mathcal{D} \sqsubseteq_{FO} \mathcal{C} \sqsubseteq_{FO} \mathcal{C} \cup \mathcal{D}$ . If  $\mathcal{C}$  and  $\mathcal{D}$  are incomparable in  $\sqsubseteq_{FO}^{\circ}$ , then we have  $\mathcal{C} \cap \mathcal{D} \sqsubseteq_{FO}^{\circ} \mathcal{C} \sqsubseteq_{FO}^{\circ} \mathcal{C} \cup \mathcal{D}$ .*

► **Lemma 11.** *The quasi-orders  $\sqsubseteq_{FO}^{\circ}$  and  $\sqsubseteq_{FO}$  are join semi-lattices. In both quasi-orders the join of two classes  $\mathcal{C}$  and  $\mathcal{D}$  is the class  $\mathcal{C} \cup \mathcal{D}$ .*

**Proof.** Of course we have  $\mathcal{C} \sqsubseteq_{FO}^{\circ} \mathcal{C} \cup \mathcal{D}$  and  $\mathcal{D} \sqsubseteq_{FO}^{\circ} \mathcal{C} \cup \mathcal{D}$  (and the same with  $\sqsubseteq_{FO}$ ).

Now assume  $\mathcal{F}$  is a class with  $\mathcal{C} \sqsubseteq_{FO}^{\circ} \mathcal{F}$  and  $\mathcal{D} \sqsubseteq_{FO}^{\circ} \mathcal{F}$ . Let  $l_1 = (\nu_1, \eta_1)$  and  $l_2 = (\nu_2, \eta_2)$  be the interpretation parts of the non-copying transductions encoding  $\mathcal{C}$  in  $\mathcal{F}$  and encoding  $\mathcal{D}$  in  $\mathcal{F}$ , respectively. By relabeling the colors, we can assume that no unary relation is used by both  $l_1$  and  $l_2$ . Let  $M$  be a new unary relation. We define the interpretation  $l = (\nu, \eta)$  by  $\nu := (\exists v M(v)) \wedge \nu_1 \vee \neg(\exists v M(v)) \wedge \nu_2$  and  $\eta := (\exists v M(v)) \wedge \eta_1 \vee \neg(\exists v M(v)) \wedge \eta_2$ . Let  $G \in \mathcal{C} \cup \mathcal{D}$ . If  $G \in \mathcal{C}$ , then there exists a coloring  $H^+$  of  $H \in \mathcal{F}$  with  $G = l_1(H^+)$ . We define  $H^*$  as the monadic lift of  $H^+$  where all vertices also belong to the unary relation  $M$ . Then  $G = l(H^*)$ . Otherwise, if  $G \in \mathcal{D}$  then there exists a coloring  $H^+$  of  $H \in \mathcal{F}$  with  $G = l_2(H^+)$  thus  $G = l(H^+)$ . We deduce  $\mathcal{C} \cup \mathcal{D} \sqsubseteq_{FO}^{\circ} \mathcal{F}$ .

Last, assume  $\mathcal{F}$  is a class with  $\mathcal{C} \sqsubseteq_{FO} \mathcal{F}$  and  $\mathcal{D} \sqsubseteq_{FO} \mathcal{F}$ . Then there is a copy operation  $C$  and two non-copying transductions  $T_1$  and  $T_2$  such that  $T_1$  is a transduction encoding  $C(\mathcal{C})$  in  $\mathcal{F}$  and  $T_2$  is a transduction encoding  $C(\mathcal{D})$  in  $\mathcal{F}$ . According to the previous case, there is a non-copying transduction  $T$  encoding  $C(\mathcal{C}) \cup C(\mathcal{D}) = C(\mathcal{C} \cup \mathcal{D})$  in  $\mathcal{F}$ .  $\blacktriangleleft$

On the opposite direction, we have the following easy fact.

► **Observation 4.** *Assume  $\mathcal{D} \sqsubseteq_{FO}^{\circ} \mathcal{C}_1 \cup \mathcal{C}_2$ . Then there is a partition  $\mathcal{D}_1 \cup \mathcal{D}_2$  of  $\mathcal{D}$  with  $\mathcal{D}_1 \sqsubseteq_{FO}^{\circ} \mathcal{C}_1$  and  $\mathcal{D}_2 \sqsubseteq_{FO}^{\circ} \mathcal{C}_2$ . Consequently, a similar statement with  $\sqsubseteq_{FO}$  instead of  $\sqsubseteq_{FO}^{\circ}$  also holds.*

From this observation, we deduce the following useful lemma.

► **Lemma 12.** *If  $\mathcal{C}_1 \triangleleft_{FO} \mathcal{C}_2$  and  $\mathcal{D}$  is incomparable with  $\mathcal{C}_2$  in  $\sqsubseteq_{FO}$ , then  $\mathcal{C}_1 \cup \mathcal{D} \triangleleft_{FO} \mathcal{C}_2 \cup \mathcal{D}$ ; if  $\mathcal{C}_1 \triangleleft_{FO}^{\circ} \mathcal{C}_2$  and  $\mathcal{D}$  is incomparable with  $\mathcal{C}_2$  in  $\sqsubseteq_{FO}^{\circ}$  then  $\mathcal{C}_1 \cup \mathcal{D} \triangleleft_{FO}^{\circ} \mathcal{C}_2 \cup \mathcal{D}$ .*

**Proof.** Let  $\mathcal{F}$  be a class with  $\mathcal{C}_1 \cup \mathcal{D} \sqsubseteq_{\text{FO}} \mathcal{F} \sqsubseteq_{\text{FO}} \mathcal{C}_2 \cup \mathcal{D}$ . According to Observation 4 there exists a partition  $\mathcal{F}_1 \cup \mathcal{F}_2$  of  $\mathcal{F}$  with  $\mathcal{F}_1 \sqsubseteq_{\text{FO}} \mathcal{C}_2$  and  $\mathcal{F}_2 \sqsubseteq_{\text{FO}} \mathcal{D}$ . As  $\mathcal{C}_1 \triangleleft_{\text{FO}} \mathcal{C}_2$  either  $\mathcal{F}_1 \sqsubseteq_{\text{FO}} \mathcal{C}_1$  (and then  $\mathcal{F} \equiv_{\text{FO}} \mathcal{C}_1 \cup \mathcal{D}$ ) or  $\mathcal{F}_1 \equiv_{\text{FO}} \mathcal{C}_2$  (and then  $\mathcal{F} \equiv_{\text{FO}} \mathcal{C}_2 \cup \mathcal{D}$ ). The proof with  $\sqsubseteq_{\text{FO}}^\circ$  follows the same lines.  $\blacktriangleleft$

► **Lemma 13.** *Let  $\mathcal{C}$  be a class closed by disjoint union. Assume  $\mathcal{C} \sqsubseteq_{\text{FO}}^\circ \mathcal{D}$ . Then there exists an immersive transduction encoding  $\mathcal{C}$  in  $\mathcal{D}$ .*

**Proof.** According to Theorem 4, the transduction of  $\mathcal{C}$  in  $\mathcal{D}$  is subsumed by the composition of an immersive transduction  $\mathsf{T}$  (with associated interpretation  $\mathsf{l}$ ) and a perturbation (with associated interpretation  $\mathsf{l}_P$ ). Let  $r$  be such that  $\text{dist}_{\mathsf{l}(H)} \geq \text{dist}_H/r$ . Let  $c$  be the number of unary relations used in the perturbation. Let  $G$  be a graph in  $\mathcal{C}$ , let  $n > 3 \cdot 2^{c|G|}$  and let  $K = nG$  ( $n$  disjoint copies of  $G$ ). By assumption there exists a coloring  $H^+$  of a graph  $H$  in  $\mathcal{D}$  with  $K = \mathsf{l}_P \circ \mathsf{l}(H^+)$ . By the choice of  $n$ , at least 3 copies  $G_1, G_2$ , and  $G_3$  of  $G$  in  $K$  have the same marks at the same vertices. For  $a \in \{2, 3\}$  and  $v \in V(G_1)$ , we denote by  $\tau_a(v)$  the vertex of  $G_a$  corresponding to the vertex  $v$  of  $G_1$ . Let  $u, v$  be vertices of  $G_1$ . By assumption there exists a path  $P = (w_0 = u, \dots, w_\ell = v)$  of length  $\ell \leq D$  linking  $u$  and  $v$ . Assume the adjacency of  $w_i$  and  $w_{i+1}$  has been complemented by  $\mathsf{l}_P$ . Then so are the adjacencies of  $w_i$  and  $\tau_2(w_{i+1})$ , of  $\tau_3(w_i)$  and  $\tau_2(w_{i+1})$ , and of  $\tau_3(w_i)$  and  $w_{i+1}$ . It follows that in  $\mathsf{l}(H^+)$  the distance of  $u$  and  $v$  is at most 3. It follows that  $\text{dist}_{G_1} \geq \text{dist}_H/3r$ . Hence the transduction obtained by composing  $\mathsf{T}$  with the extraction of the induced subgraph  $G_1$  implies the existence of a strongly local transduction from  $\mathcal{D}$  to  $\mathcal{C}$ , according to Lemma 9.  $\blacktriangleleft$

► **Corollary 14** (Elimination of the perturbation). *Let  $\mathcal{C}$  be a class closed by disjoint union. Assume  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{D}$ . Then there exists a copy operation  $\mathsf{C}$  and an immersive transduction  $\mathsf{T}_{\text{imm}}$  such that  $\mathsf{T}_{\text{imm}} \circ \mathsf{C}$  is a transduction encoding  $\mathcal{C}$  in  $\mathcal{D}$ .*

Let  $\mathcal{C}$  be a class of graphs, we denote by  $\mathcal{C} \oplus K_1$  the class obtained from  $\mathcal{C}$  by adding an apex to each graph.

Let  $f$  be a graph invariant. A hereditary class  $\mathcal{C}$  has *locally bounded*  $f$  if there exists a function  $g$  with  $f(G) \leq g(r)$  for every  $G \in \mathcal{C}$  with radius  $r$ . The invariant  $f$  is *apex-friendly* if  $f(G \oplus K_1)$  is bounded by a function of  $f(G)$ .

► **Lemma 15.** *Assume  $f_1$  and  $f_2$  are graph invariants, and that  $f_1$  is apex-friendly. Assume that every class with locally bounded  $f_1$  is a non-copying transduction of a class with locally bounded  $f_2$ . Then every class with bounded  $f_1$  is a non-copying transduction of a class with bounded  $f_2$ .*

**Proof.** Let  $\mathcal{C}$  be a class with bounded  $f_1$ . Let  $\mathcal{C}'$  be the closure of  $\{G \oplus K_1 \mid G \in \mathcal{C}\}$  by disjoint union. Then  $\mathcal{C}'$  has locally bounded  $f_1$  (as  $f_1$  is apex-friendly). By assumption there exists a class  $\mathcal{D}$  with locally bounded  $f_2$  such that  $\mathcal{C}' \sqsubseteq_{\text{FO}}^\circ \mathcal{D}$ . It follows from Lemma 13 that there is an immersive transduction of  $\mathcal{C}'$  in  $\mathcal{D}$  (with interpretation  $\mathsf{l}$ ). Thus for every  $G \in \mathcal{C}$  there exists a vertex coloring  $H^+$  of  $H \in \mathcal{D}$  such that  $G \oplus K_1 = \mathsf{l}(H^+)$ . By strong locality, the radius of  $H$  is at most some  $r$ , thus  $f_2(H) \leq C(r)$ . We deduce (by composing with the generic induced subgraph transduction) that  $\mathcal{C}$  is a non-copying transduction of the class of graphs  $H$  with  $f_2(H) \leq C(r)$ .  $\blacktriangleleft$

The assumption that  $f_1$  is apex-friendly is necessary. Consider for instance  $f_1$  to be the maximum degree, while  $f_2$  is the treewidth. A class with locally bounded maximum degree is simply a class with bounded maximal degree, thus has locally bounded treewidth. However, the class of grids (which has bounded maximum degree) cannot be transduced in any class of bounded tree-width as it cannot be transduced in the class of tree-orders.

## 4 Transduction of bounded degree classes

As a warm-up we now characterize transductions of the class  $\mathcal{E}$ .

► **Lemma 16.** *A class  $\mathcal{C}$  is a transduction of  $\mathcal{E}$  (or, equivalently,  $\mathcal{C} \equiv_{\text{FO}} \mathcal{E}$ ) if and only if  $\mathcal{C}$  is a perturbation of a class whose members have uniformly bounded size connected components. However, a class  $\mathcal{C}$  is a non-copying transduction of  $\mathcal{E}$  if and only if  $\mathcal{C}$  is a perturbation of  $\mathcal{E}$ .*

**Proof.** Assume that  $\mathcal{C}$  is a perturbation of the class  $\mathcal{D}$  and that all graphs in  $\mathcal{D}$  have connected components of size at most  $n$ . Then  $\mathcal{D}$  can obviously be obtained from  $\mathcal{E}$  by the composition of an  $n$ -copy transduction composed with some (immersive) transduction.

Conversely, if  $\mathcal{C}$  is a transduction of  $\mathcal{E}$  then, according to Theorem 4,  $\mathcal{C}$  is a perturbation of a class  $\mathcal{D} \subseteq \text{T}_{\text{imm}} \circ \mathcal{C}(\mathcal{E})$ , where  $\text{T}_{\text{imm}}$  is immersive and  $\mathcal{C}$  is a  $k$ -copy transduction. By the strong locality of the interpretation associated to  $\text{T}_{\text{imm}}$  no connected component of a graph in  $\mathcal{D}$  can have size greater than  $k$ .

The last statement is obvious, as every immersive transduction from  $\mathcal{E}$  has  $\mathcal{E}$  as its image (as we consider only infinite classes). ◀

We continue with the following lemma, which is the basis of the model checking algorithm presented in [11].

► **Lemma 17.** *A class is a transduction of a class with bounded degree if and only if it is a perturbation of a class with bounded degree.*

Consequently, if  $\mathcal{D}$  has bounded maximum degree,  $\mathcal{C}$  is closed by disjoint union and  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{D}$ , then  $\mathcal{C}$  has bounded maximum degree.

**Proof.** According to Theorem 4, the transduction of  $\mathcal{C}$  in  $\mathcal{D}$  is subsumed by the composition of a copy operation, an immersive transduction  $\text{T}$  (with associated interpretation  $\text{I}$ ) and  $k$  subset complementations (with associated interpretation  $\text{I}_P$ ). As copy operations and immersive transductions preserve the property to have bounded maximum degree, the first part of the statement holds. The second part of the statement follows from Lemma 13. ◀

► **Lemma 18.** *Assume  $\mathcal{C}$  is closed by disjoint union and  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{D}$ . Then there exist positive integers  $c, k$  such that for every integer  $r \geq 1$  we have*

$$\sup_{H \in \mathcal{C}} \max_{v \in V(H)} |B_r^H(v)| \leq k \cdot \sup_{G \in \mathcal{D}} \max_{v \in V(G)} |B_{cr}^G(v)|. \quad (1)$$

**Proof.** If  $\mathcal{D}$  does not have bounded maximum degree, then the statement is obviously true. Otherwise, according to Lemma 17, there exists a  $k$ -copy operation  $\mathcal{C}$  and an immersive transduction  $\text{T}_{\text{imm}}$  such that  $\text{T}_{\text{imm}} \circ \mathcal{C}$  is a transduction of  $\mathcal{C}$  in  $\mathcal{D}$ . By the strong locality of  $\text{T}_{\text{imm}}$ , if  $H \in \text{T}_{\text{imm}} \circ \mathcal{C}(\{G\})$ , for every vertex  $v$  of  $H$  (corresponding, before copy, to a vertex  $u$  in  $G$ ) we have  $|B_r^H(v)| \leq k |B_{cr}^G(u)|$ , for some constant  $c$  depending only on  $\text{T}_{\text{imm}}$ . ◀

We now consider the special case of transductions of paths and give a complete characterization of the classes that are transductions of the class of all paths and prove Theorem 5. Precisely, we prove that a class  $\mathcal{C}$  is a transduction of the class of paths if and only if there exist integers  $k, \ell$  such that every graph  $G \in \mathcal{C}$  can be obtained from a graph with bandwidth at most  $\ell$  by complementing at most  $k$  subsets.

**Proof of Theorem 5.** Let  $\mathcal{C}$  be a transduction of the class of all paths. According to Theorem 4, this transduction is subsumed by the composition of a copy operation, an immersive transduction, and a perturbation  $\mathcal{P}$ . By the strong locality property of immersive transductions, every class obtained from  $\mathcal{P}$  by the composition of a copy operation and an immersive transduction has its image included in the class of all the subgraphs of the  $\ell$ -power of paths, for some integer  $\ell$  depending on the transduction.

Conversely, assume that  $\mathcal{C}$  is a perturbation of a class  $\mathcal{D}$  containing graphs with bandwidth at most  $\ell$ . Then  $\mathcal{D}$  is a subclass of the monotone closure of the class  $\mathcal{P}^\ell$  of  $\ell$ -powers of paths, which has bounded star chromatic number. As taking the  $\ell$ -th power is obviously a transduction, and as the monotone closure of a class with bounded star chromatic number can be obtained by a transduction (cf Lemma 21 in Appendix A) we get that  $\mathcal{C}$  is a transduction of the class of paths.  $\blacktriangleleft$

## 5 The transduction quasi-order

This section is devoted to the proof of Theorem 3.

### Proof of Theorem 3.

*Item 1.* Obviously  $\mathcal{E} \sqsubseteq_{\text{FO}} \mathcal{P}$ . Assume for contradiction that there exists a class  $\mathcal{C}$  with  $\mathcal{E} \sqsubseteq_{\text{FO}} \mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{P}$ . According to Theorem 5, the class  $\mathcal{C}$  is a perturbation of a class  $\mathcal{D}$  with bounded bandwidth. The connected components of the graphs in  $\mathcal{D}$  have unbounded size as otherwise  $\mathcal{D}$  (and thus  $\mathcal{C}$ ) can be transduced in  $\mathcal{E}$  (according to Lemma 16). But then the diameters of the connected components of the graphs in  $\mathcal{D}$  are unbounded, thus  $\mathcal{P}$  can be transduced in  $\mathcal{D}$ , thus can be transduced in  $\mathcal{C}$ , contradicting our assumption. Hence  $\mathcal{E} \not\sqsubseteq_{\text{FO}} \mathcal{P}$ . Let  $n \geq 2$ . According to Lemma 12, as  $\mathcal{P}$  is incomparable with  $\mathcal{T}_n$  we have  $\mathcal{E} \cup \mathcal{T}_n \not\sqsubseteq_{\text{FO}} \mathcal{P} \cup \mathcal{T}_n$ , that is  $\mathcal{T}_n \not\sqsubseteq_{\text{FO}} \mathcal{P} \cup \mathcal{T}_n$ . Also, for  $n \geq 1$ . As  $\mathcal{T}_n \not\sqsubseteq_{\text{FO}} \mathcal{T}_{n+1}$  and  $\mathcal{T}_{n+1}$  is incomparable with  $\mathcal{P}$  we deduce from Lemma 12 that  $\mathcal{T}_n \cup \mathcal{P} \not\sqsubseteq_{\text{FO}} \mathcal{T}_{n+1} \cup \mathcal{P}$ .

*Item 2.*  $\mathcal{T}_{n+1} \sqsubseteq_{\text{FO}} \mathcal{PW}_n$  as  $\mathcal{T}_{n+1} \sqsubseteq_{\text{FO}} \mathcal{P} \cup \mathcal{T}_{n+1} \subseteq \mathcal{PW}_n$ . We now prove  $\mathcal{T}_{n+2} \not\sqsubseteq_{\text{FO}} \mathcal{PW}_n$  by induction on  $n$ . This is clear if  $n = 1$ . Now assume that we have proved the statement for  $n \geq 1$  and assume for contradiction that  $\mathcal{T}_{n+3} \sqsubseteq_{\text{FO}} \mathcal{PW}_{n+1}$ . The class  $\mathcal{PW}_{n+1}$  is the monotone closure of the class  $\mathcal{I}_{n+2}$  of interval graphs with clique number at most  $n + 2$ . As the class  $\mathcal{I}_{n+2}$  has bounded star chromatic number, it follows from Lemma 21 that  $\mathcal{PW}_{n+1} \sqsubseteq_{\text{FO}} \mathcal{I}_{n+2}$ . Let  $\mathsf{T}$  be the composition of the respective transductions, that is, a transduction of  $\mathcal{T}_{n+3}$  in  $\mathcal{I}_{n+2}$ . As  $\mathcal{T}_{n+3}$  is closed by disjoint unions, it follows from Lemma 13 that there is an immersive transduction  $\mathsf{T}_0$  of  $\mathcal{T}_{n+3}$  in  $\mathcal{I}_{n+2}$ . Now consider any graph  $H \in \mathcal{T}_{n+1}$  and let  $H' \in \mathcal{T}_{n+2}$  be the graph obtained from  $2d$  copies of  $H$  by adding a universal vertex  $v$  and let  $G$  be a graph in  $\mathcal{I}_{n+2}$  such that  $H'$  is a  $\mathsf{T}$ -transduction of  $G$ . As  $\mathsf{T}_0$  is strongly local, we can assume that all the vertices of  $G$  are at distance at most  $d$  from  $v$ , where  $d$  depends only on  $\mathsf{T}_0$ . It is easily checked that there exists a path  $P$  of length at most  $2d$  of  $G$  that dominates  $G$ , and such that  $G - P \in \mathcal{I}_{n+1}$ . By encoding the adjacencies in  $G$  to the vertices of  $P$  by a monadic expansion, we get that there exists a transduction  $\mathsf{T}_1$  (independent of our choice of  $G$  and  $H$ ) such that  $H$  is a  $\mathsf{T}_1$ -transduction of  $G - P$ . In particular,  $\mathsf{T}_1$  is a transduction of  $\mathcal{T}_{n+2}$  in  $\mathcal{PW}_n$ , what contradicts our induction hypothesis.

*Item 3.* The class  $\mathcal{T}$  has unbounded linear cliquewidth hence  $\mathcal{T} \not\sqsubseteq \mathcal{H}$ , thus  $\mathcal{T} \not\sqsubseteq \mathcal{PW}_n$  for every integer  $n$ . Assume for contradiction that  $\mathcal{PW}_2 \sqsubseteq_{\text{FO}} \mathcal{T}$ . Then there exists a strongly local transduction of  $\mathcal{PW}_2$  in  $\mathcal{T}$ . In particular, there exists an integer  $r$  such that the class  $\{K_1 \oplus P_n : n \in \mathbb{N}\}$  is the transduction of  $\mathcal{T}_r$ , what contradicts the property  $\mathcal{P} \not\sqsubseteq_{\text{FO}} \mathcal{T}_r$ . It follows that, for  $n \geq 2$ ,  $\mathcal{PW}_n \not\sqsubseteq_{\text{FO}} \mathcal{T}$  thus  $\mathcal{PW}_n$  and  $\mathcal{T}$  are incomparable.

*Item 4.* We have  $\mathcal{PW}_n \sqsubseteq_{\text{FO}} \mathcal{PW}_{n+1}$  as  $\mathcal{PW}_n \subset \mathcal{PW}_{n+1}$  and  $\mathcal{PW}_{n+1} \not\sqsubseteq_{\text{FO}} \mathcal{PW}_n$  as  $\mathcal{T}_{n+2} \subset \mathcal{PW}_{n+1}$  and  $\mathcal{T}_{n+2} \not\sqsubseteq_{\text{FO}} \mathcal{PW}_n$ , as proved in Item 2.

*Item 5.*  $\mathcal{TW}_2 \not\sqsubseteq_{\text{FO}} \mathcal{T}$  follows from  $\mathcal{PW}_2 \not\sqsubseteq_{\text{FO}} \mathcal{T}$  (Item 3).

*Item 6.* That  $\mathcal{TW}_n \not\sqsubseteq_{\text{FO}} \mathcal{PW}_n$  follows from the fact that  $\mathcal{PW}_n$  has bounded linear cliquewidth (thus  $\mathcal{PW}_n \sqsubseteq_{\text{FO}} \mathcal{H}$ ) but  $\mathcal{TW}_n$  does not (thus  $\mathcal{TW}_n \not\sqsubseteq_{\text{FO}} \mathcal{H}$ ).

*Item 7.* That  $\mathcal{T} \not\sqsubseteq \mathcal{BT}$  directly follows from Lemma 17.

*Item 8.* That the class of grids cannot be transduced in the class of binary trees follows from the fact that binary trees have bounded cliquewidth (i.e.  $\mathcal{BT} \sqsubseteq_{\text{FO}} \mathcal{TP}$ ) but grids do not. That the class of binary trees cannot be transduced in the class of grids directly follows from Lemma 18.  $\blacktriangleleft$

## 6 Local monadically stable, straight, and dependent classes

We now consider local variants of monadic stability, monadic straightness, and monadic dependence. A class  $\mathcal{C}$  is *locally monadically dependent* if, for every integer  $r$ , the class  $\mathcal{B}_r^{\mathcal{C}}$  is monadically dependent; a class  $\mathcal{C}$  is *locally monadically stable* if, for every integer  $r$ , the class  $\mathcal{B}_r^{\mathcal{C}}$  is monadically stable; a class  $\mathcal{C}$  is *locally monadically straight* if, for every integer  $r$ , the class  $\mathcal{B}_r^{\mathcal{C}}$  is monadically straight.

► **Lemma 19.** *Let  $\mathcal{C}$  be a class such that the class  $\mathcal{C}' = \{n(G \oplus K_1) \mid n \in \mathbb{N}, G \in \mathcal{C}\}$  is a transduction of  $\mathcal{C}$ . Then, for every class  $\mathcal{D}$  we have  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{D}$  if and only if there exists some integer  $r$  with  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{B}_r^{\mathcal{D}}$ .*

**Proof.** Obviously, if there exists some integer  $r$  with  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{B}_r^{\mathcal{D}}$ , then  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{D}$ .

Now assume  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{D}$ . As  $\mathcal{C}' \equiv_{\text{FO}} \mathcal{C}$  we deduce from Lemma 13 that there is a transduction of  $\mathcal{C}'$  in  $\mathcal{D}$  that is the composition of a copy operation  $\mathcal{C}$  and an immersive transduction  $\mathcal{T}$ . Let  $\mathcal{D}' = \mathcal{C}(\mathcal{D})$ . According to Lemma 10, there is an integer  $r$  such that  $\mathcal{C} \sqsubseteq_{\text{FO}}^{\circ} \mathcal{B}_r^{\mathcal{D}'}$  thus, as  $\mathcal{B}_r^{\mathcal{D}'} = \mathcal{C}(\mathcal{B}_r^{\mathcal{D}})$ , we have  $\mathcal{C} \sqsubseteq_{\text{FO}} \mathcal{B}_r^{\mathcal{D}}$ .  $\blacktriangleleft$

We now prove that the local versions of monadic dependence, straightness, and stability are equivalent to their non local versions.

**Proof of Theorem 6.** The class  $\{n(G \oplus K_1) \mid n \in \mathbb{N}, G \in \mathcal{G}\}$  is obviously a transduction of  $\mathcal{G}$ . Hence, according to Lemma 19, a class  $\mathcal{C}$  is locally monadically dependent if and only if it is monadically dependent. The class  $\{n(G \oplus K_1) \mid n \in \mathbb{N}, G \in \mathcal{TP}\}$  is an easy transduction of  $\mathcal{TP}$ . Hence, according to Lemma 19, a class  $\mathcal{C}$  is locally monadically straight if and only if it is monadically straight. The class  $\{n(G \oplus K_1) \mid n \in \mathbb{N}, G \in \mathcal{H}\}$  is an easy transduction of  $\mathcal{H}$ . Hence, according to Lemma 19, a class  $\mathcal{C}$  is locally monadically stable if and only if it is monadically stable.  $\blacktriangleleft$

► **Example 20.** Although the class of unit interval graphs has unbounded clique-width, every proper hereditary subclass of unit interval graphs has bounded clique-width [19]. This is in particular the case of the class of unit interval graphs with bounded radius. As classes with bounded clique-width are monadically dependent, the class of unit interval graphs is locally monadically dependent, hence monadically dependent.

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## A Transduction of the monotone closure in a class with bounded star coloring number

Recall that a *star coloring* of a graph  $G$  is a proper coloring of  $V(G)$  such that any two color classes induce a star forest (or, equivalently, such that no path of length four is 2-colored), and that the star chromatic number  $\chi_{\text{st}}(G)$  of  $G$  is the minimum number of colors in a star coloring of  $G$  [1]. The star chromatic number is bounded on classes of graphs excluding a minor [21] and, more generally, on every bounded expansion class [22].

► **Lemma 21.** *Let  $\mathcal{C}$  be a class with bounded star chromatic number. Then there is an immersive transduction from  $\mathcal{C}$  onto its monotone closure.*

**Proof.** We consider the immersive transduction  $\mathsf{T}$  with  $\Sigma_{\mathsf{T}} = \{M_1, \dots, M_C, N_1, \dots, N_C, X\}$  and  $\mathsf{l}_{\mathsf{T}} = (\nu(x), \eta(x, y))$ , where  $C = \max_{G \in \mathcal{C}} \chi_{\text{st}}(G)$ ,  $\nu(x) = X(x)$ , and

$$\eta(x, y) = E(x, y) \wedge \left( \bigvee_{i=1}^C M_i(x) \wedge N_i(y) \right) \wedge \left( \bigvee_{i=1}^C M_i(y) \wedge N_i(x) \right).$$

Let us prove that  $\mathsf{T}$  is a transduction from  $\mathcal{C}$  onto its monotone closure. Let  $H$  be a graph in the monotone closure of  $\mathcal{C}$  and let  $G \in \mathcal{C}$  be a supergraph of  $H$  (we identify  $H$  with a subgraph of  $G$ ). We consider an arbitrary star coloring  $\gamma: V(G) \rightarrow [C]$  and define the  $\Sigma_{\mathsf{T}}$ -expansion  $G^+$  of  $G$  as follows:

$$\begin{aligned} X(G) &= V(H), \\ M_i(G) &= \{v \in V(G) \mid \gamma(v) = i\} && (\text{for } 1 \leq i \leq C), \\ N_i(G) &= \{v \in V(G) \mid (\exists u \in N_H(v)) \gamma(u) = i\} && (\text{for } 1 \leq i \leq C). \end{aligned}$$

According to the definitions of  $\nu$  and as  $\models \eta(x, y) \rightarrow E(x, y)$ , it is clear that  $\mathsf{l}_{\mathsf{T}}(G^+)$  is a subgraph of  $G[V(H)]$ . According to our definitions of  $M_i$  and  $N_i$ , it is also clear that  $H$  is a subgraph of  $\mathsf{l}_{\mathsf{T}}(G^+)$ . Thus, in order to prove that  $H = \mathsf{l}_{\mathsf{T}}(G^+)$ , we have only to prove that  $\mathsf{l}_{\mathsf{T}}(G^+)$  contains no more edges than  $H$ . Assume towards a contradiction that  $u, v \in V(H)$  are adjacent in  $\mathsf{l}_{\mathsf{T}}(G^+)$  and not in  $H$ . According to the definition of  $M_i$  and  $N_i$  it follows that  $u$  has a neighbor  $v'$  in  $H$  with  $\gamma(v') = \gamma(v)$  and that  $v$  has a neighbor  $u'$  in  $H$  with  $\gamma(u') = \gamma(u)$ . It follows that the path  $v', u, v, u'$  of  $G$  is 2-colored by  $\gamma$ , contradicting the hypothesis that  $\gamma$  is a star coloring. ◀



## B

 Monadic stability, monadic dependence, and their local variants

The notions of stability and dependence are two of the most important notions in classification theory [28]. The notions of *monadic stability* and *monadic dependence* were studied by Baldwin and Shelah in [2] and naturally fit into the framework of transductions. After recalling their definitions we prove that a class of graphs is monadically stable (resp. dependent) if and only if it is locally monadically stable (resp. dependent). In fact, more generally, a class of graphs is stable (resp. dependent) if and only if it is locally stable (resp. dependent). The proof of this more general fact is based on a standard Ramsey argument, while the monadic cases illustrate nicely the use of our new normal form.

► **Definition 22.** Let  $\mathcal{C}$  be a class of  $\Sigma$ -expanded graphs. A formula  $\varphi(\bar{x}, \bar{y})$  with  $|\bar{x}| = |\bar{y}| = k$  has the  $n$ -order property over  $\mathcal{C}$  if there exists  $G \in \mathcal{C}$  and  $\bar{a}_1, \dots, \bar{a}_n \in V(G)^k$  such that  $G \models \varphi(\bar{a}_i, \bar{a}_j)$  if and only if  $i < j$ . The formula  $\varphi$  has the order property if it has the  $n$ -order property for all  $n \geq 1$ .

► **Definition 23.** A class  $\mathcal{C}$  of  $\Sigma$ -expanded graphs is *stable* if no  $\Sigma$ -formula  $\varphi(\bar{x}, \bar{y})$  with  $|\bar{x}| = |\bar{y}| = k$  has the order property over  $\mathcal{C}$ .

► **Definition 24.** A class  $\mathcal{C}$  of  $\Sigma$ -expanded graphs is *monadically stable* if every monadic expansion of  $\mathcal{C}$  is stable.

In model theory we usually do not deal with classes of graphs (or structures), but rather with theories. However, we can naturally associate with every first-order theory  $T$  the class  $\mathcal{C}_T$  of its (finite and infinite) models. Then the definition of the  $n$ -order property, order property and stability naturally extends from classes of graphs to theories, e.g., a theory  $T$  is stable if  $\mathcal{C}_T$  is stable. By compactness, if  $T$  is a theory, then if a formula  $\varphi$  has the order property, then there exist  $(\bar{a}_i)_{i \in \mathbb{N}}$  in a model  $G$  of  $T$  such that  $G \models \varphi(\bar{a}_i, \bar{a}_j)$  if and only if  $i < j$  (and this is the original definition from model theory). However, this is not relevant for our work.

► **Theorem 25** ([2]). If a theory  $T$  has the order property, then there is a first-order formula  $\psi(x, y)$  with only two free variables and with monadic parameters that has the order property (for an appropriate interpretation of the monadic parameters).

The graph  $H_n$  with vertices  $a_1, \dots, a_n, b_1, \dots, b_n$  and edges  $\{a_i, b_j\}$  for  $i \leq j$  is called a *half-graph* of length  $n$  [9]. Observe that for a graph  $G$  and a formula  $\varphi(\bar{x}, \bar{y})$ , if we have  $\bar{a}_1, \dots, \bar{a}_n \in V(G)^{|\bar{x}|}$  and  $\bar{b}_1, \dots, \bar{b}_n \in V(G)^{|\bar{y}|}$  such that  $G \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$ , then the formula  $\psi(\bar{x}_1 \bar{y}_1, \bar{x}_2 \bar{y}_2) := \varphi(\bar{x}_1, \bar{y}_2) \wedge \neg \varphi(\bar{x}_2, \bar{y}_2)$  has the  $n$ -order property as witnessed by  $\bar{a}_1 \bar{b}_1, \dots, \bar{a}_n \bar{b}_n$ . For monadic stability the situation is even simpler. For a graph  $G$  and a formula  $\varphi(x, y)$  with two free variables, if we have  $a_1, \dots, a_n \in V(G)$  and  $b_1, \dots, b_n \in V(G)$  such that  $G \models \varphi(a_i, b_j) \Leftrightarrow i \leq j$ , we can mark the  $b_i$  with a monadic predicate  $B$ . Then  $\psi(x, y) := \forall z (B(z) \rightarrow (\varphi(z, y) \rightarrow \varphi(z, x)))$  has the  $n$ -order property as witnessed by  $a_1, \dots, a_n$ . Hence, orders and half-graphs can be used interchangeably.

► **Corollary 26.** For a class of finite graphs  $\mathcal{C}$ , the following are equivalent:

1.  $\mathcal{C}$  is monadically stable,
2. there is no transduction from  $\mathcal{C}$  onto the class  $\mathcal{H}$  of half-graphs.

► **Remark 27.** One could wonder how Theorem 25, which concerns theories, implies Corollary 26, which concerns (possibly non-definable) classes of finite graphs. In particular, the use of a monadic expansion makes it difficult to apply arguments based on ultra-products and Łoś's theorem. The proof of Theorem 25 is based on Ramsey arguments, like the extraction of order-indiscernible sequences.

► **Definition 28.** Let  $\mathcal{C}$  be a class of  $\Sigma$ -expanded graphs. A formula  $\varphi(\bar{x}, \bar{y})$  has the  $n$ -independence property over  $\mathcal{C}$  if there exists  $G \in \mathcal{C}$  and  $\bar{a}_1, \dots, \bar{a}_n \in V(G)^{|\bar{x}|}$  and  $\bar{b}_J \in V(G)^{|\bar{y}|}$  for  $J \subseteq [n]$  such that  $G \models \varphi(\bar{a}_i, \bar{b}_J)$  if and only if  $i \in J$ . The formula  $\varphi$  has the independence property if it has the  $n$ -independence property for all  $n \in \mathbb{N}$ .

► **Definition 29.** A class  $\mathcal{C}$  of  $\Sigma$ -expanded graphs is dependent if no formula  $\varphi(\bar{x}, \bar{y})$  has the independence property over  $\mathcal{C}$ .

► **Definition 30.** A class  $\mathcal{C}$  of  $\Sigma$ -expanded graphs is monadically dependent if every monadic expansion of  $\mathcal{C}$  is dependent.

While it is well known that for dependence it is sufficient to consider formulas  $\varphi(\bar{x}; \bar{y})$  with  $|\bar{y}| = 1$ , it is not possible in general to reduce further to formulas  $\varphi(x, y)$  with only two free variables. However, as mentioned before, this is possible when we consider monadic dependence.

► **Theorem 31** ([2]). If a theory  $T$  has the independence property then there is a first-order formula  $\psi(x, y)$  with monadic parameters that has the independence property (for an appropriate interpretation of the monadic parameters).

► **Corollary 32.** Let  $\mathcal{C}$  be a class of graphs. Then the following are equivalent:

1.  $\mathcal{C}$  is monadically dependent,
2. there is no transduction from  $\mathcal{C}$  onto the class  $\mathcal{G}$  of all finite graphs.

For  $F$  be a graph with vertex set  $[n]$  and  $r \in \mathbb{N}$  define

$$\zeta_{r,F}(x_1, \dots, x_n) := \left( \bigwedge_{\{i,j\} \in E(F)} \text{dist}(x_i, x_j) \leq r \right) \wedge \left( \bigwedge_{\{i,j\} \notin E(F)} \text{dist}(x_i, x_j) > r \right).$$

► **Lemma 33.** If a class of graphs is independent, then there is a strongly local formula that is independent.

**Proof.** We rewrite the formula  $\varphi(\bar{x}, y)$  in Gaifman's normal form as a Boolean combination of sentences and  $r$ -local formulas. By an easy argument, we reduce to the case where the formula is  $r$ -local. By a standard Ramsey argument we can select a subsequence such that the every subset of size  $p$  of the  $\bar{a}_i$  is order-indiscernible for formulas with quantifier rank  $r$ . Let  $b$  be a vertex adjacent to all  $\bar{a}_i$  with even  $i$  and to none with odd  $i$ . We rewrite  $\varphi(\bar{x}, y)$  as a Boolean combination of strongly local formulas. We can then extract a subsequence such that for some of these formula  $\psi$ ,  $\psi(\bar{a}_i, b)$  alternates between true (for even indices) and false (for even ones). Using order-indiscernibility, we find all the needed  $b_J$  (by reducing the global size from  $p$  to  $p/3$ ). Thus we found a strongly local independent formula. ◀

► **Corollary 34.** A class of graphs is dependent if and only if it is locally dependent; a class of graphs is monadically dependent if and only if it is locally monadically dependent.

► **Lemma 35.** If a class of graphs is unstable, then there is a strongly local formula that is unstable.

**Proof.** By applying Gaifman's Theorem, we rewrite the formula as a Boolean combination of sentences and  $r$ -local formulas. By an easy argument, we reduce to the case where the formula is  $r$ -local.

By a standard Ramsey argument, if  $n > f(p)$  we can find  $1 \leq i_1 < \dots < i_p \leq n$  such that  $G \models \zeta_{2r,F}(\bar{a}_{i_j}, \bar{a}_{i_k}, \bar{a}_{i_\ell})$  for every  $1 \leq j < k < \ell \leq p$ . (It follows that  $F$  has a symmetric structure.) We define  $\bar{b}_j = \bar{a}_{i_j}$ . We consider the connected components of  $F$ . Let  $m = |\bar{a}_i|$ . If  $\alpha$  and  $\beta + m$  are adjacent in  $F$  (for some  $1 \leq \alpha, \beta \leq m$  then  $\alpha + m$  is adjacent to  $\beta + 2m$

and  $\alpha$  is adjacent to  $\beta + 2m$  hence  $\alpha, \alpha + m, \alpha + 2m, \beta, \beta + m$ , and  $\beta + 2m$  belong to the same connected component of  $F$ . The partition of  $[3m]$  into connected components consists of parts of the form  $A \cup (A + m) \cup (A + 2m)$  and of triples of parts of the form  $A, (A + m)$  and  $(A + 2m)$  (for some  $A \subseteq [m]$ ). It follows that  $\varphi(\bar{x}, \bar{y})$  is equivalent, on the tuples  $\bar{b}_i$  to a Boolean combination of strongly local formulas involving the same indices for  $\bar{x}$  and  $\bar{y}$  for parts of the first kind, or involving only some indices of  $\bar{x}$  or of  $\bar{y}$  in the second case. By Ramsey argument, we can assume that the satisfaction of these formulas only depend on the order of the  $\bar{b}_i$ . At least one of these is thus antisymmetric. It follows that there exists a strongly local formula  $\psi(\bar{x}', \bar{y}')$  with the order property (with  $|\bar{x}'| \leq |\bar{x}|$  and  $|\bar{y}'| \leq |\bar{y}|$ ). ◀

► **Corollary 36.** *A class of graphs is stable if and only if it is locally stable; a class of graphs is monadically stable if and only if it is locally monadically stable.*