Simple strategies for Banach-Mazur games and fairly correct systems

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HIGHLIGHTS of Logic, Games and Automata – Paris

Outline of the talk

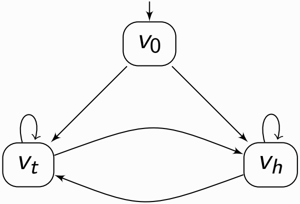
1. Motivations
2. Banach-Mazur games and large sets
3. Simple strategies for Banach-Mazur games
4. The concept of α-strategy

Motivations

Classical Model-Checking

Given a model M and a property φ, decide whether :

$$M \models \varphi, \qquad \text{i.e. } \{\rho \text{ execution of } M \mid \rho \not\models \varphi\} \text{ is } \textbf{empty}.$$



$M_{\text{coin}} \not\models \mathbf{F}\ \text{head} \quad ; \quad M_{\text{coin}} \not\models \mathbf{GF}\ \text{tails}$

Motivations

Fair Model-Checking

Given a model M and a property φ, decide whether :

$$M \models\!\!\!\approx \varphi, \quad \text{i.e. } \{\rho \text{ execution of } M \mid \rho \not\models \varphi\} \text{ is } \textbf{“very small”}.$$

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How to formalise the **fair model-checking** ?

- Via probability

$$\begin{aligned} M \mathrel{|}\!\!\approx_{\mathbb{P}} \varphi \quad & \textit{iff} \quad \mathbb{P}(\{\rho \text{ of } M \mid \rho \not\models \varphi\}) = 0 \\ & \textit{iff} \quad \mathbb{P}(\{\rho \text{ of } M \mid \rho \models \varphi\}) = 1 \end{aligned}$$

- Via topology

$$\begin{aligned} M \mathrel{|}\!\!\approx_{T} \varphi \quad & \textit{iff} \quad \{\rho \text{ of } M \mid \rho \not\models \varphi\} \ \textbf{is meagre} \\ & \textit{iff} \quad \{\rho \text{ of } M \mid \rho \models \varphi\} \ \textbf{is large} \end{aligned}$$

Motivations

A natural question

Given a model M and property φ, do we have that

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Theorem [VV06]

Given a finite system M and an ω-regular property φ, we have that

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for bounded Borel measures.

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Can we go further ?

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Banach-Mazur games

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- $G = (V, E)$ is a finite directed graph with no deadlock,
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A play $\rho = \rho_1\rho_2\rho_3\cdots$ is won by Pl. 0 wins iff $\rho \in W$.

Strategies for Banach-Mazur games

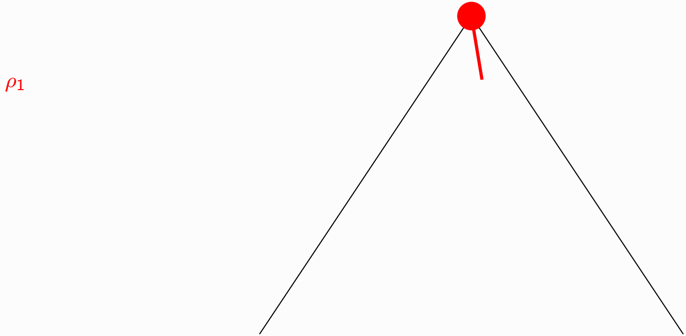
Let $G = (V, E)$ be a graph and $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game.

Definition

A strategy for Pl. 0 is a function $f : V^* \to V^*$ such that whenever

$$f(\rho) = \rho'$$

we have that ρ' prolongs ρ in G.



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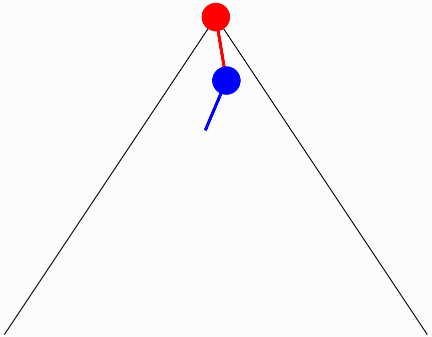

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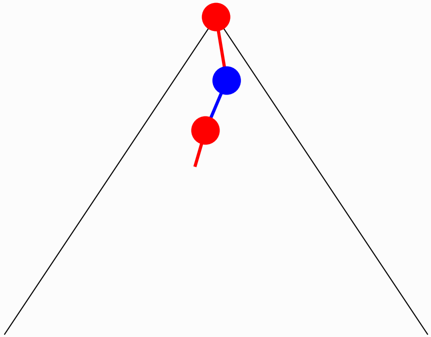

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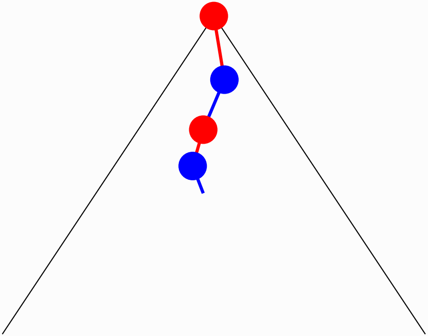

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$\rho_1\rho_2\rho_3\rho_4\rho_5$

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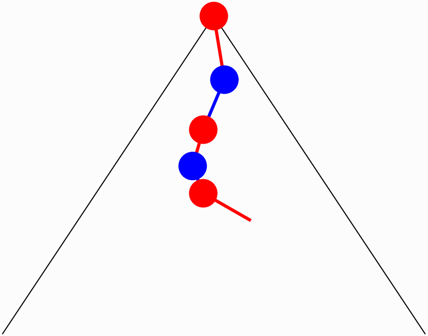

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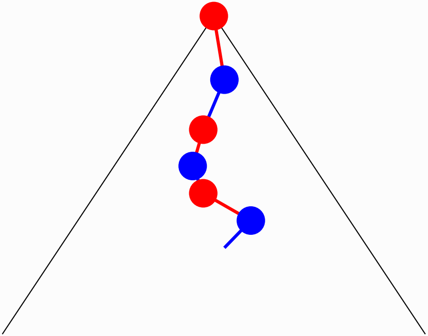

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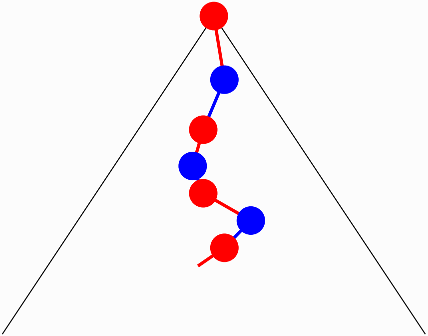

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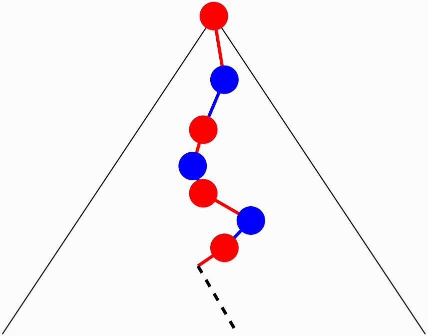

Banach-Mazur games and large sets

Let (V, E) be a graph. We consider V^ω equipped with the Cantor topology.

Definitions

A set $S \subseteq V^\omega$ is said

- *nowhere dense* if the closure of S has empty interior.
- *meagre* if it can be seen as a countable union of nowhere dense sets.
- large if S^c is meagre.

Theorem [Oxtoby57]

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph.

Pl. 0 has a winning strategy for $\mathcal{G}$ if and only if W is large.

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Back to our motivation

Theorem [VV06]

Given a finite system M and an ω-regular property φ, we have that

$$M \mathrel{|}\!\approx_{\mathbb{P}} \varphi \quad \Leftrightarrow \quad M \mathrel{|}\!\approx_{T} \varphi,$$

for bounded Borel measures.

The key ingredient to prove the above result is the following result :

Theorem [BGK03]

Given $\mathcal{G} = (G, v_0, W)$ where W is an ω-regular property, we have that

Pl. 0 has a winning strategy for $\mathcal{G}$
iff
Pl. 0 has a **positional** winning strategies for $\mathcal{G}$.

Simple strategies for Banach-Mazur game

Given $\mathcal{G} = (G, v_0, W)$, let $f : V^* \to V^*$ be a strategy for Pl. 0.

$$f(\underbrace{\rho_1 \rho_2 \cdots \rho_{2n+1}}_{\text{What is observed}}) \quad = \quad \underbrace{\rho_{2n+2}}_{\text{What is played}}$$

We say that f is

E. Grädel, S. Leßenich, *Banach-Mazur Games with Simple Winning Strategies*, CSL 2012

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- *bounded* if there is $b \geqslant 1$ such that f is b-bounded.

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- *move-counting* if it only depends on $\mathrm{Last}(\rho_{2n+1})$ and the number of moves already played.

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- *move-counting* if it only depends on $\text{Last}(\rho_{2n+1})$ and the number of moves already played.
- *length-counting* if it only depends on the $\text{Last}(\rho_{2n+1})$ and the length of the prefix already played.

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Simple strategies for Pl. 0 on finite graphs

$$\begin{tikzcd}
 & \text{Winning } \textit{positional} \text{ strategy} \arrow[d, Leftrightarrow] & \\
 & \text{Winning } \textit{finite memory} \text{ strategy} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Winning } \textit{bounded} \text{ strategy} \arrow[dr, Rightarrow] & & \text{Winning } \textit{move-counting} \text{ strategy} \arrow[dl, Rightarrow] \\
 & \text{Winning } \textit{length-counting} \text{ strategy} \arrow[d, Leftrightarrow] & \\
 & \text{Winning strategy} &
\end{tikzcd}$$

Combining simple observations and results from [BGK03], [VV06], [GL12], [BM13]

Relations with the sets of probability one

Proposition

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and $\mathbb{P}$ a reasonable probability measure.

If Pl. 0 has $\begin{cases} \text{a move-counting} \\ \text{a bounded} \end{cases}$ winning strategy for $\mathcal{G}$, then $\mathbb{P}(W) = 1$.

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There exist large **open** set of probability 1 without a positional/ bounded/ move-counting winning strategy.

We look for a new concept of "simple strategy"

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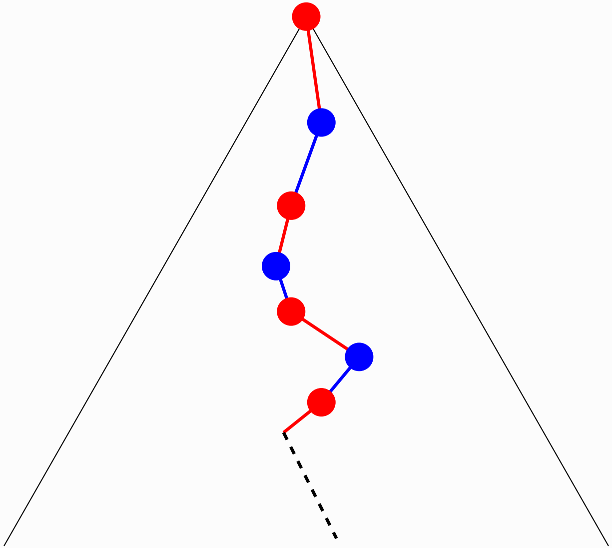

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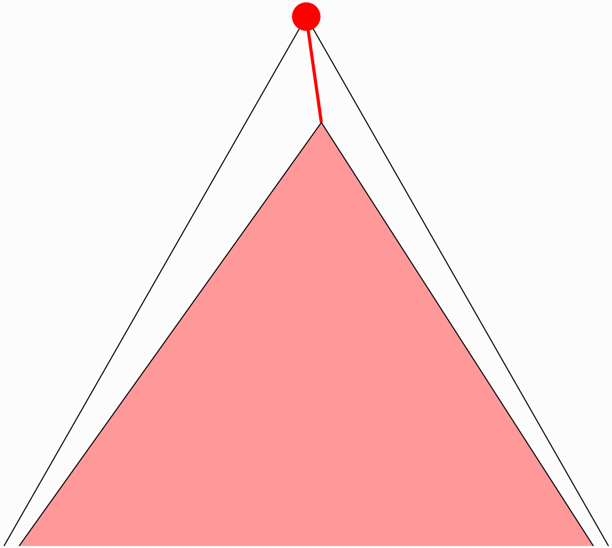

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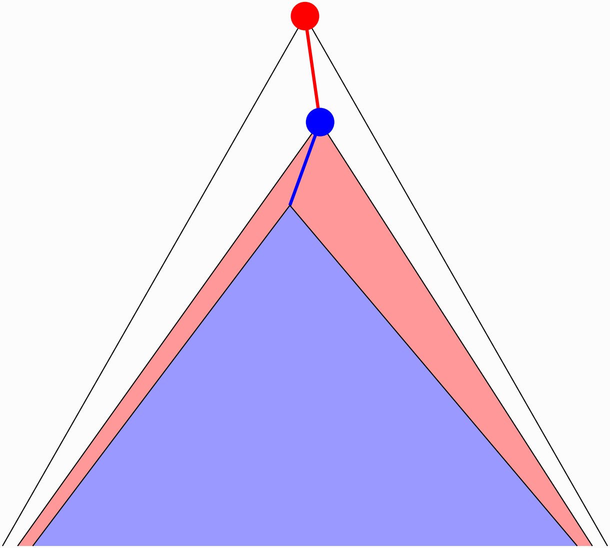

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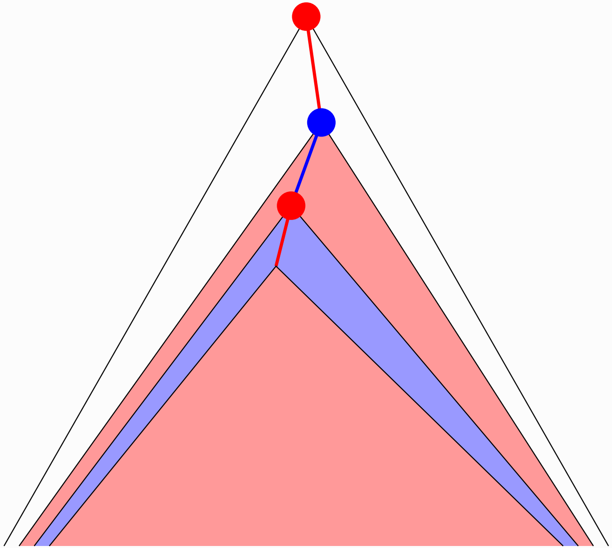

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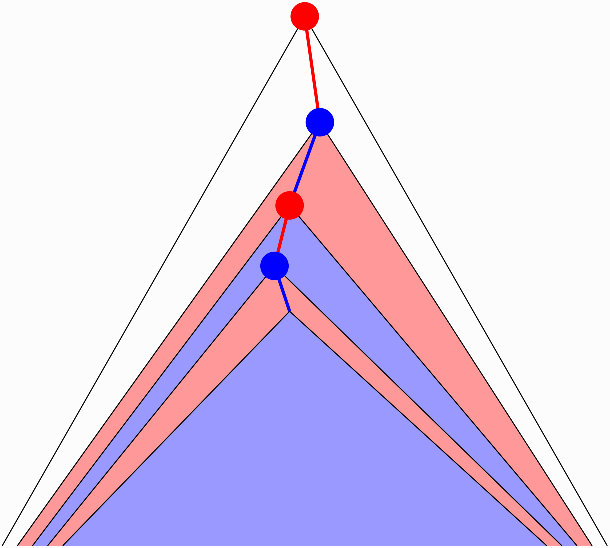

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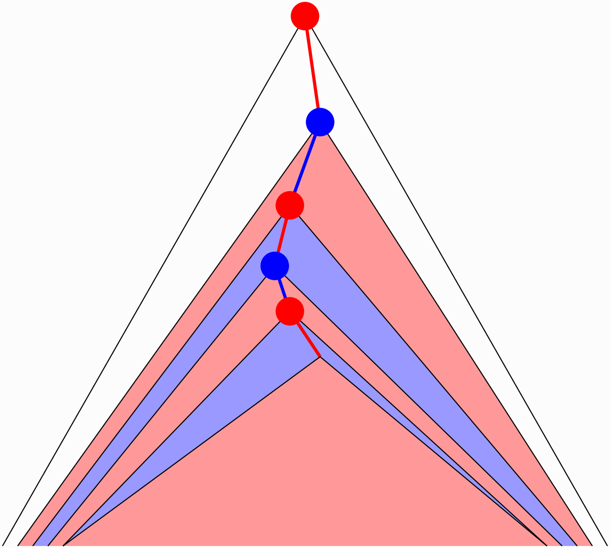

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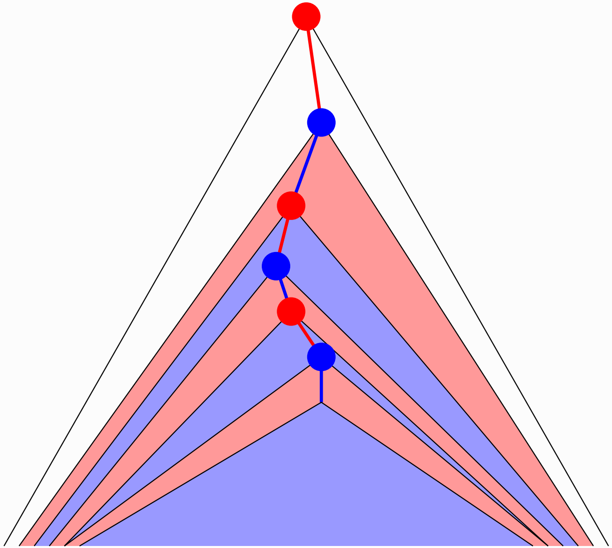

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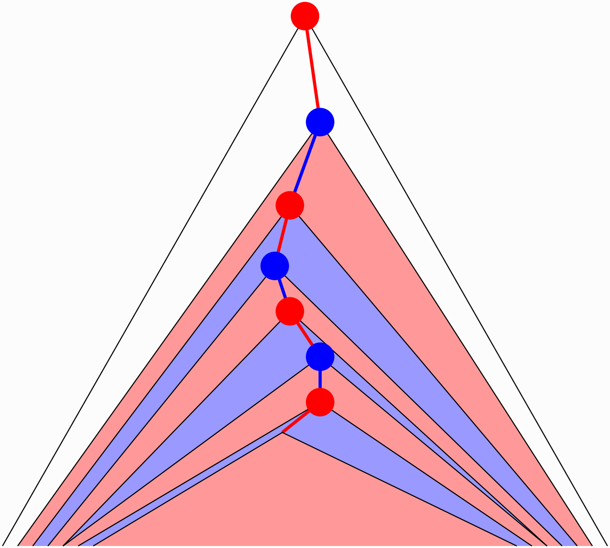

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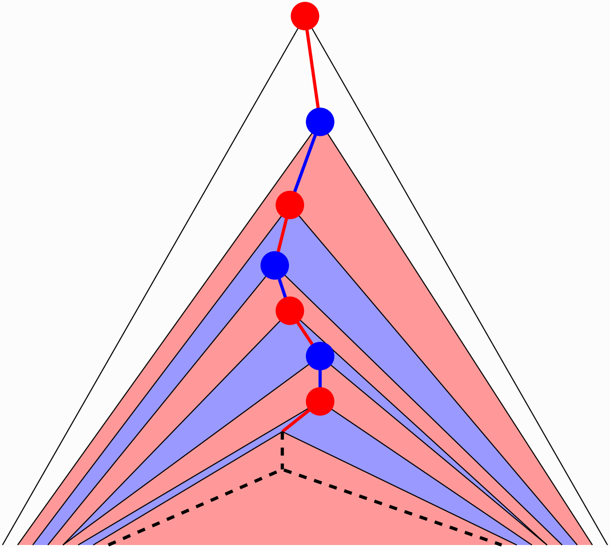

Another simple strategy for Banach-Mazur game

Given $\mathcal{G} = (G, v_0, W)$, a strategy for Pl. 0 can be seen as $f : \mathcal{O} \to \mathcal{O}$.

$$f(\underbrace{O_1 O_2 \cdots O_{2n+1}}_{\text{What is observed}}) \quad = \quad \underbrace{O_{2n+2}}_{\text{What is played}} \quad ,$$

where $O_1 \supseteq O_2 \supseteq \cdots \supseteq O_{2n+1} \supseteq O_{2n+2}$ are open sets.

Another simple strategy for Banach-Mazur game

Given $\mathcal{G} = (G, v_0, W)$, a strategy for Pl. 0 can be seen as $f : \mathcal{O} \to \mathcal{O}$.

$$f(\underbrace{O_1 O_2 \cdots O_{2n+1}}_{\text{What is observed}}) \quad = \quad \underbrace{O_{2n+2}}_{\text{What is played}} \quad ,$$

where $O_1 \supseteq O_2 \supseteq \cdots \supseteq O_{2n+1} \supseteq O_{2n+2}$ are open sets.

Assuming that G is equipped with a probability distribution on edges.

The notion of α-strategy

Given $0 < \alpha < 1$, we say that f is an α-*strategy* if and only if

$$\mathbb{P}\left(O_{2n+2} | O_{2n+1}\right) \geqslant \alpha.$$

Our results

Theorem

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and $\mathbb{P}$ a reasonable probability measure.

If Pl. 0 has a winning α-strategy for some $\alpha > 0$, then $\mathbb{P}(W) = 1$.

Our results

Theorem

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and $\mathbb{P}$ a reasonable probability measure.

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Theorem

When W is a **countable intersection of open sets**, the following assertions are equivalent :

1. $P(W) = 1$,
2. Pl. 0 has a winning α-strategy for some $\alpha > 0$,
3. Pl. 0 has a winning α-strategy for all $0 < \alpha < 1$.

Summary

$$
\begin{tikzcd}
 & \text{Winning \textit{positional} strategy} \arrow[d, Leftrightarrow] & & \\
 & \text{Winning \textit{finite memory} strategy} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
\text{Winning \textit{bounded} strategy} \arrow[dr, Rightarrow] & & \text{Winning \textit{move-counting} strategy} \arrow[dl, Rightarrow] & \\
 & \color{red}{\text{Winning α-strategy}} \arrow[rr, Leftrightarrow, red, "\text{Countable intersection}", "\text{of open sets}"'] \arrow[d, Rightarrow] & & \color{red}{\text{Probability 1}} \\
 & \text{Winning \textit{length-counting} strategy} \arrow[d, Leftrightarrow] & & \\
 & \text{Winning strategy} & &
\end{tikzcd}
$$

Thank you!!!