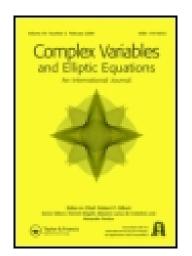
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Differentially Transcendental Formal Power Series

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We prove that a formal power series in 1/x, whose coefficients are in a field extension of \mathbb{Q} and are algebraically independent over \mathbb{Q} , is differentially transcendental (i.e. not differentially algebraic) over this field extension. This is stated without proof in [2]. This result provides a source of functions analytic at ∞ that are not differentially algebraic over \mathbb{R} . Such functions are of particular interest, because their germs belong to Hardy fields, but not to the class E of [1]—the intersection of all maximal Hardy fields.

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Suppose F is a field extension of the field of rational numbers Q. Let x be an indeterminate and let u = 1/x. Let F[[u]] = F[[1/x]] denote the ring of formal power series in u with coefficients in F. Then its field of quotients F((u)) is a differential field with at least two possible derivations: formal differentiation with respect to x and y denoted by y and y.

DEFINITION 1 (see e.g. §VI.1 [7]) Suppose F is an extension field of K and $S \subseteq F$.

- (i) S is algebraically dependent over K if a nonzero polynomial in finitely many variables with coefficients in K is annulled by elements of S.
- (ii) S is a transcendence basis over K means that S is algebraically independent (i.e. not algebraically dependent) over K and is maximal with respect to this property (i.e. F is algebraic over K(S)).
- (iii) A transcendence basis has unique cardinality (Theorem VI.1.9 [7]) which is called the transcendence degree of F over K and is denoted by $\operatorname{tr.deg}_K F$.

DEFINITION 2 (see e.g. §1.6 [8]) Suppose F is a field, K is a differential field, $F \subseteq K$, and $f \in K$. To say that f is differentially algebraic over F means that $\{f, f', f'', \ldots\}$ is algebraically dependent over K, i.e. f is a root of a nonzero differential polynomial with coefficients in F.

PROPOSITION 1 (Proposition 7.4, [2]) If the set $\{a_i \in F, i = 0, 1, ...\}$ is algebraically independent over \mathbf{Q} then

$$f = \sum_{i=0}^{\infty} a_i x^{-i} \in F\left(\left(\frac{1}{x}\right)\right)$$

is not differentially algebraic over F with respect to D_x .

Proof Suppose f is a root of a non-zero differential polynomial p over F with respect to D_x . Let $b_j \in F$ $(j = 0, 1, ..., \overline{j})$ be the coefficients of p. Then f is differentially algebraic with respect to D_x over K, where $K = Q(\{b_i\})$, so

$$\operatorname{tr.deg}_K K(f, D_x f, D_x^2 f, \ldots) < \infty.$$

Since tr.deg_O $K < \infty$, we have

$$\operatorname{tr.deg}_Q K(f,D_xf,D_x^2f,\ldots) = \operatorname{tr.deg}_Q K + \operatorname{tr.deg}_K K(f,D_xf,D_x^2f,\ldots) < \infty.$$

Therefore

$$\operatorname{tr.deg}_{Q} \mathbf{Q}(f, D_{x}f, D_{x}^{2}f, \ldots) = \operatorname{tr.deg}_{Q} K(f, D_{x}f, D_{x}^{2}f, \ldots)$$

- tr. deg_{$$Q(f,D_xf,D_x^2f,...)$$} $K(f,D-xf,D_x^2f,...)<\infty$.

Note that

$$D_x f = D_u f\left(-\frac{1}{x^2}\right) = D_u F(-u^2),$$

$$D_x^2 f = (D_u^2 f(-u^2) + D_u f(-2u))(-u^2)$$

and so on, so $\mathbf{Q}(x, f, D_x f, D_x^2 f, ...) = \mathbf{Q}(u, f, D_u f, D_u^2 f, ...)$.

Now since tr. $\deg_{\mathcal{O}} \mathbf{Q}(u) = 1$ and tr. $\deg_{\mathcal{O}} \mathbf{Q}(f, D_x f, D_x^2 f, \dots) < \infty$, we have

$$\operatorname{tr.deg}_Q \mathbf{Q}(u,f,D_uf,D_u^2f,\ldots) < \infty \qquad \text{ and } \qquad \operatorname{tr.deg}_Q \mathbf{Q}(f,D_uf,D_u^2f,\ldots) < \infty.$$

Therefore, f satisfies some non-zero differential polynomial q over \mathbf{Q} with respect to D_u . Since the field \mathbf{Q} is infinite, there exist $t_k \in \mathbf{Q}$ (k = 0, 1, ...k) such that $q(t_0, t_1, ...t_{\overline{k}}) \neq 0$. Let

$$g(u) = \sum_{k=0}^{\infty} \frac{t_k}{k!} u^k.$$

Then $q(g, D_u g, ...) \neq 0$, because it is non-zero when evaluated at u = 0 (it equals $q(t_k)$).

If h is a formal sum $\sum_{i=0}^{\infty} y_i u^i$ with y_i indeterminate, then $q(h, D_u h, ...)$ is a power series in u, whose coefficients are polynomials in y_i over \mathbf{Z} . Since evaluating y_i at $(t_0, t_1, ..., t_{\overline{k}}, 0, 0 \cdots)$ produces a non-zero answer, one of the above coefficients is a non-zero polynomial in y_i over \mathbf{Q} . Let r denote this polynomial. Evaluating y_i at a_i gives $q(f, D_u f, ...) = 0$, so, in particular, $r(a_i) = 0$, which contradicts the assumption that a_i are algebraically independent over \mathbf{Q} .

The value of this result lies in providing a class of functions analytic at ∞ that are differentially transcendental over **R**. The germs of such functions necessarily belong to Hardy fields (see the definition below), but not to Boshernitzan's class E[1]—the intersection of all maximal Hardy fields.¹

 $^{^{1}}E$ is an extension of Hardy's class L of logarithmico-exponential functions [5] and is the maximal scale for functions in Hardy fields (functions of *regular growth*).

DEFINITION 3 (see [4]) A differential field of continuous germs² of real functions at $+\infty$, where the derivation is ordinary differentiation, is called a Hardy field.

COROLLARY 1 Suppose f is a real function that is analytic at ∞ and the set of coefficients of its Taylor series at ∞ is algebraically independent over \mathbf{Q} . Then

- (i) f is differentially transcendental over **R**,
- (ii) $\mathbf{R}(f, f', \cdots)$ is a Hardy field,
- (iii) $f \notin E$, where E is the intersection of all maximal Hardy fields.

Proof (i) is a special case of Proposition 1 with $F = \mathbf{R}$ and the series actually convergent. (ii) is a consequence of the fact that the zeros of an analytic function are necessarily isolated and, thus, cannot have ∞ as an accumulation point. This means that every nonzero element of $\mathbf{R}[f, f' \cdots]$, i.e. a nonzero differential polynomial of f, is invertible in a (punctured) neighborhood of ∞ (see Theorem 7.1[1]). (iii) is a consequence of the fact that E is a differentially algebraic extension of \mathbf{R} (Theorem 14.4 [2]).

The few other known examples of classes of functions satisfying the conditions of Corollary 1 include

- (i) Euler's Γ-function, which is not differentially algebraic over **R** by Hölder's theorem [6] and generates a Hardy field [9];
- (ii) Functions represented by a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^x},$$

where $a_n \in \mathbf{R}$ are subpolynomial in n and the set of all prime divisors of n in the support of a_n ³ is infinite, e.g. the Riemann ζ -function on a positive half-line [9];

(iii) The function

$$\sum_{n=1}^{\infty} \frac{1}{e_n(x)},$$

where $e_1(x) = e^x$ and $e_n(x) = e^{e_{n-1}(x)}$ for n > 1 [9];

- (iv) Certain fractional iterates of e^x [3];
- (v) Certain ultimately 4 C^{∞} transexponential solutions of two difference equations: $f(x+1) = e^{f(x)}$ and $f(x+1) = e^{f(x)} 1$ [3].

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²A germ is an equivalence class of functions, where two functions are equivalent, exactly when they agree in a neighborhood of the point of interest, in our case $+\infty$.

³The support of a_n is the set of all n such that $a_n \neq 0$.

⁴A property is said to hold *ultimately*, exactly when it holds in a neighborhood of $+\infty$.

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