arXiv:1408.5805v1 [math.GR] 25 Aug 2014

GROUPS, ORDERS, AND DYNAMICS

Bertrand Deroin
CNRS and École Normale Supérieure

Andrés Navas
Universidad de Santiago de Chile

Cristóbal Rivas
Universidad de Santiago de Chile

Contents

INTRODUCTION

The theory of bi-orderable groups is a venerable subject in Algebra which has been extensively developed during the whole last century, starting from seminal works of Dedekind, Hölder, and Hilbert. Though less developed, the theory of left-orderable groups has reemerged in the recent years, mainly by the discovery of many new examples of (at least partially) left-orderable groups of geometric origin (braid groups [50]; some groups of contact diffeomorphisms [62, 74]; right-angled Artin groups [60], hence virtually the fundamental groups of closed hyperbolic 3-manifolds by the combination of several recent remarkable results [2, 9, 80, 98]). Moreover, the question of knowing whether certain groups are left-orderable or not (Garside groups [50]; lattices in higher-rank simple Lie groups [109, 173]; groups with Kazhdan's property (T) [7, 141], with the subsequent question of whether bi-orderable groups are a-(T)-menable [34]), the discovery of relevant examples of groups in the framework of orderable groups (amenable groups [115]), as well the use of orderability in the solution of long-standing problems (H. Newmann's conjecture [128]) have attracted the interest to the subject of people coming from different branches of mathematics. Furthermore, the introduction of new techniques –mainly dynamical, analytic and probabilistic– have revitalized the theory.

This is an almost self-contained monograph (containing some new results) on left-orderable groups which mostly rely on dynamical and probabilistic aspects, but also on geometric, combinatorial, analytic, and topological ones.

In Chapter 1, we review the basic definitions and treat several relevant examples, as for example solvable groups, Thompson's groups, and free groups. Also, we discuss some of the general properties of groups admitting orders with different invariance properties, as well as certain combinatorial issues that are closely related. We close the chapter with a recent result of Gromov concerning the (linear) isoperimetric profile of left-orderable groups.

In Chapter 2, we show that, besides the fact that many groups admit left-orders, in general they actually admit a lot. To better study this phenomenon, we introduce the notion of the space of left-orders associated to a left-orderable group, and we discuss some of its properties. As a concrete example, we treat the case of the free group from several points of view. Moreover, we present examples of left-orderable groups having uncountably many left-orders but whose associated spaces of left-orders contain isolated points, and we give a description of the groups admitting only finitely many left-orders.

In Chapter 3, we put in a dynamical framework some classical results of the theory, and we present some new developments via this approach. We begin with the classical Hölder Theorem characterizing group left-orders satisfying an Archimedean type property. We then pass to the theory of Conradian left-orders. We first review the classical approach of Conrad, and then we give an alternative dynamical approach leading to applications in the study of the topology of the space of left-orders. In particular, we give a complete characterization of the groups admitting finitely many Conradian left-orders, as well as a description of the space of left-orders of countable solvable groups. We close the chapter with a general decomposition of the space of left-orders of finitely-generated left-orderable groups into three canonical subsets according to their dynamical properties. We also discuss the algebraic relevance of this classification, in particular in regard with the open problem of left-orderability of lattices in higher-rank simple Lie groups.

Chapter 4 is devoted to several recent results relying on techniques with a probabilistic flavour. We begin with Morris-Witte's theorem asserting that left-orderable, amenable groups are locally indicable. We next consider actions by almost-periodic homeomorphisms, and we provide a construction of a space involving all of them which somewhat replaces the space of left-orders; using this, we give an alternative proof of Morris-Witte's theorem above. We close this chapter by considering random walks on finitely-generated left-orderable groups, showing recurrence type properties and existence of harmonic functions of dynamical origin. More importantly, we explain how probabilistic arguments provide canonical coordinates for actions on the line.

There are several classical references on the topic of (left-)orderable groups, as for example [15, 75, 108]. Quite naturally, in many sections, there is a large intersection of this monograph with these books. However, our presentation is new in many aspects. In particular, the order we have chosen for the topics is not the historical one, in opposition to most (all ?) known references of the theory. Though this may cause some minor problems of lecture (our exposition is not always "linearly ordered"), we think that, definitively, this presentation is more appropriate for our main purpose, namely, to put the classical results of the theory in a modern framework which allows integrating them with the new results as well as with some of the recent developments in the conjunction between Group Theory and Dynamics. Besides this, there are many other texts that may be considered for complementary reading, especially those concerning orders on braid groups [50] and low-dimensional topology [41]. (These two subjects are not deeply developed in this monograph.)

This text is complemented with many exercises which sometimes correspond to little results in the literature. More importantly, several open problems are spread out along the text. (A complementary list of open questions, mostly concerning classical achievements of the theory, may be found in [13]; see also [50, Chapter XVI]).

This text started growing from short Notes that the second-named author wrote for mini-courses at the Third Latinoamerican Congress of Mathematicians (2009), the Uruguayan Colloquium of Mathematics (2009), and the School Young Geometric Group Theory II (Haifa, 2013). All authors would like to express their gratitude to L. Bartholdi, D. Calegari, M. Calvez, A. Clay, Y. de Cornulier, P. Dehornoy, A. Erschler, É. Ghys, A. Glass, F. Haglund, T. Hartnick, S. Hurtado, V. Kleptsyn, T. Koberda, K. Mann, I. Marin, D. Morris-Witte, L. Paris, F. Paulin, D. Rolfsen, F. le Roux, Z. Šunić., R. Tessera and B. Wiest, for valuable discussions, comments, and suggestions, as well as E. Trucco for informatics technical support.

B. Deroin was partially supported by ANR-08-JCJC-0130-01, ANR-09-BLAN-0116 and ANR-13-BS01-0002. A. Navas and C. Rivas acknowledge the funding from the CONICYT PIA 1103 Project DySyRF. A. Navas would also like to acknowledge the support/hospitality of the ERC starting grant 257110 RaWG / Institut Henri Poincaré during the final stage of this work. C. Rivas was also partially supported by a CONICYT's Inserción Grant PAI.

To Anaïs, Raphaël,
Corita, Nachito, Rocío,
and Benito.

Notation

Because of our dynamical approach, group elements will be often denoted by the letters f, g, h (sometimes, by u, v, w as well). Nevertheless, a, b, c are used when dealing with specific groups (free groups, fundamental groups of surfaces, braid groups, etc). Following the classical notation, when dealing with braid groups, we also use the letter σ to denote elements. Groups are generally denoted by Γ, though G, H are sometimes used, as well as C (for convex subgroups), R (for nilpotent radicals) and T (for Tararin groups). In many cases, we implicitly assume that the groups we are dealing with are nontrivial; for left-orderable groups, this is equivalent to being infinite. Similarly, when we consider actions on the line, we will implicitly assume that these are actions by orientation-preserving homeomorphisms. In general, real-valued function will be denoted by ϕ, ψ, whereas group representations by Φ, Ψ. This notation is coherent in that certain functions to be constructed will appear to be representations, that is, homomorphisms into the additive group of reals.

Below, we list some notation used throughout this text:

$\langle \boldsymbol{g_1, g_2, \ldots} \rangle^+$: the semigroup generated by $g_1, g_2, \ldots$

$\boldsymbol{\Gamma_1 \star \Gamma_2}$: the (non-Abelian) free product of Γ_1 and Γ_2.

$\boldsymbol{\mathcal{LO}(\Gamma)}$: the space of left-orders of Γ.

$\boldsymbol{\mathcal{BO}(\Gamma)}$: the space of bi-orders of Γ.

$\boldsymbol{\mathcal{CO}(\Gamma)}$: the space of Conradian orders of Γ.

$\boldsymbol{C_{\preceq}(\Gamma)}$: the Conradian soul of a left-ordered group $(\Gamma, \preceq)$.

$\boldsymbol{P^+_{\preceq}}$: positive cone of a left-order $\preceq$.

$\mathbb{N} := \{1, 2, \ldots\}$.

$\mathbb{N}_{\mathbf{0}} := \{0, 1, 2, \ldots\}$.

$(\mathbb{R}, +)$: the group of reals under addition.

$\mathbb{R}^*$: the group of positive reals under multiplication.

$\mathbf{Aff}(\mathbb{R})$, $\mathbf{Aff}_+(\mathbb{R})$: the group of affine homeomorphisms of the real line and the subgroup of orientation-preserving ones, respectively.

$\mathbf{PSL(2,}\mathbb{R})$, $\widetilde{\mathbf{PSL}}\mathbf{(2,}\mathbb{R})$: the group of orientation-preserving projective homeomorphisms of the circle and the group of the lifts to the real line, respectively.

$\mathbf{Homeo}_+(\mathbb{R})$: the group of orientation-preserving homeomorphisms of the line.

F: Thompson's group of piecewise-affine, dyadic, orientation-preserving homeomorphisms of the interval.

$\mathbb{F}_{\boldsymbol{n}}$: the free group on n generators (we will implicitly assume that $n \geq 2$).

$\mathbb{B}_n$: the braid group in n strands.

$\mathbf{P}\mathbb{B}_{\boldsymbol{n}}$: the pure braid group in n strands.

$\boldsymbol{BS(1,\ell)}$: the Baumslag-Solitar group $\langle a, b \colon bab^{-1} = a^{\ell} \rangle$.

$\boldsymbol{G_{m,n}}$: the torus-knot group $\langle a, b \colon a^m = b^n \rangle$.

Chapter 1

SOME BASIC AND NOT SO BASIC FACTS

1.1 General Definitions

An order relation $\preceq$ on a group Γ is ***left-invariant*** (resp. ***right-invariant***) if for all g, h in Γ such that $g \preceq h$, one has $fg \preceq fh$ (resp. $gf \preceq hf$) for all $f \in \Gamma$. The relation is ***bi-invariant*** if it is simultaneously invariant by the left and by the right. To simplify, we will use the term ***left-order*** (resp. ***right-order***) for referring to a left-invariant total order on a group. We will say that a group Γ is ***left-orderable*** (resp. ***right-orderable***, ***bi-orderable***) if it admits a total order which is invariant by the left (resp. by the right, simultaneously by the left and right).

Example 1.1.1. Clearly, every subgroup of a left-orderable group is left-orderable. More interestingly, an arbitrary product Γ of left-orderable groups Γ_λ is left-orderable. (This also holds for bi-orderable groups with the very same proof.) Indeed, fixing an arbitrary order on the set of indices Λ and a left-order $\preceq_\lambda$ on each Γ_λ, let $\preceq$ be the associated ***lexicographic*** order. This means that $(g_\lambda) \prec (h_\lambda)$ if the smallest $\lambda \in \Lambda$ such that $g_\lambda \neq h_\lambda$ satisfies $g_\lambda \prec_\lambda h_\lambda$. It is easy to check that $\preceq$ is total and left-invariant.

1.1.1 Positive and negative cones

If $\preceq$ is an order on Γ, then $f \in \Gamma$ is said to be ***positive*** (resp. ***negative***) if $f \succ id$ (resp. $f \prec id$). Notice that if $\preceq$ is total, then every nontrivial element is either positive or negative, and $f \succ id$ if and only if $f^{-1} \prec id$. Moreover, if $\preceq$ is left-invariant and $P^+ = P^+_\preceq$ (resp. $P^- = P^-_\preceq$) denotes the set of positive (resp.

negative) elements in Γ (usually called the ***positive*** (resp. ***negative***) ***cone***), then P^+ and P^- are semigroups, and Γ is the disjoint union of P^+, P^- and $\{id\}$.

Conversely, to every decomposition of Γ as a disjoint union of semigroups P^+, P^- and $\{id\}$ such that $P^- = (P^+)^{-1} := \{f : f^{-1} \in P^+\}$, it corresponds a left-order $\preceq$ defined by $f \prec g$ whenever $f^{-1}g \in P^+$. Notice that Γ is bi-orderable exactly when these semigroups may be taken invariant by conjugacy (that is, when they are *normal* subsemigroups).

Remark 1.1.2. The characterization in terms of positive and negative cones shows immediately the following: If $\preceq$ is a left-order on a group Γ, then the ***reverse order*** $\overline{\preceq}$ defined by $g \,\bar{\succ}\, id$ if and only if $g \prec id$ is also left-invariant and total.

Remark 1.1.3. Given a left-order $\preceq$ on a group Γ, we may define an order $\preceq^*$ by letting $f \preceq^* g$ whenever $f^{-1} \succ g^{-1}$. Then the order $\preceq^*$ turns out to be *right-nvariant*. One can certainly go the other way around, producing left-orders from right-orders. As a consequence, a group is left-orderable if and only if it is right-orderable. Since our view is mostly dynamical, we prefer to work with left-orders, yet most of the literature of the subject is written for right-orders.

1.1.2 A characterization involving finite subsets

A group Γ is left-orderable if and only if for every finite family $\mathcal{G}$ of elements different from the identity, there exists a choice of (*compatible*) exponents $\eta : \mathcal{G} \to \{-1,+1\}$ such that id does not belong to the semigroup generated by the elements $g^{\eta(g)}$, $g \in \mathcal{G}$. Indeed, the necessity of the condition is clear: it suffices to fix a left-order $\preceq$ on Γ and choose each exponent $\eta(g)$ so that $g^{\eta(g)}$ becomes a positive element. Conversely, assume that for each finite family $\mathcal{G}$ of elements in Γ different from the identity there is a choice of compatible exponents $\eta : \mathcal{G} \to \{-1,+1\}$, and let $\mathcal{X}(\mathcal{G},\eta)$ denote the (nonempty and closed) subset of $\{-1,+1\}^{\Gamma\setminus\{id\}}$ formed by the functions sign satisfying

$$\mathrm{sign}(h) = +1 \quad \text{and} \quad \mathrm{sign}(h^{-1}) = -1 \qquad \text{for every } h \in \left\langle g^{\eta(g)}, g \in \mathcal{G} \right\rangle^+.$$

(Here and in what follows, given a family of group elements $\mathcal{F}$, we let $\langle \mathcal{F} \rangle^+$ be the semigroup spanned by them.) We then let $\mathcal{X}(\mathcal{G})$ be the union of all the sets of the form $\mathcal{X}(\mathcal{G},\eta)$ for some choice of compatible exponents η on $\mathcal{G}$. Notice that, if $\{\mathcal{X}_i := \mathcal{X}(\mathcal{G}_i),\ 1 \leq i \leq n\}$ is a finite family of subsets of this form, then the intersection $\mathcal{X}_1 \cap \ldots \cap \mathcal{X}_n$ contains the (nonempty) set $\mathcal{X}(\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_n)$, and it is therefore nonempty. Since $\{-,+\}^{\Gamma\setminus\{id\}}$ is compact, a direct application of the Finite Intersection Property shows that the intersection $\mathcal{X}$ of all sets of the form $\mathcal{X}(\mathcal{G})$ is (closed and) nonempty. Finally, each point in $\mathcal{X}$ corresponds in an abvious way to a left-order on Γ.

Analogously, one may show that a group is bi-orderable if and only if for every finite family $\mathcal{G}$ of elements different from the identity, there exists a choice of exponents $\eta : \mathcal{G} \to \{-1, +1\}$ such that id does not belong to the smallest semigroup which simultaneously satisfies the next two properties:

– It contains all the elements $g^{\eta(g)}$;

– For all f, g in the semigroup, both fgf^{-1} and $f^{-1}gf$ also belong to it.

We leave the proof to the reader. As a corollary, we obtain that left-orderability and bi-orderability are ***local properties***, that is, if they are satisfied by every finitely-generated subgroup of a given group, then they are satisfied by the whole group. Similarly, these are ***residual*** properties: if for every nontrivial element there is a surjective group homomorphism into a group with that property mapping the prescribed element into a nontrivial one, then the group inherits the property. (Notice that this follows more easily from Example 1.1.1.)

1.1.3 Left-orderable groups and actions on ordered spaces

If Γ is a left-orderable group, then Γ acts faithfully on a totally ordered space by order-preserving transformations. Indeed, fixing a left-order $\preceq$ on Γ, we may consider the action of Γ by left-translations on the ordered space $(\Gamma, \preceq)$. Conversely, if Γ acts on a totally ordered space $(\Omega, \leq)$ by order-preserving transformations, then we may fix an arbitrary well-order $\leqslant_{wo}$ on Ω and define a left-order $\preceq$ on Γ by letting $f \succ id$ if and only if $f(w_f) > w_f$, where $w_f = \min_{\leqslant_{wo}}\{w \colon f(w) \neq w\}$. More generally, if we also have a function $\mathrm{sign} : \Omega \to \{-, +\}$, we may associate to it the left-order $\preceq$ for which $f \succ id$ whenever $\mathrm{sign}(w_f) = +$ and $f(w_f) > w_f$, or $\mathrm{sign}(w_f) = -$ and $f(w_f) < w_f$. These left-orders will be referred to as ***dynamically-lexicographic*** ones.

Left-orders obtained from preorders. Recall that a ***preorder*** $\preceq$ on a group Γ is a reflexive and transitive relation for which both $f \preceq g$ and $g \preceq f$ may hold for different f, g. The existence of a total, left-invariant preorder is equivalent to the existence of a semigroup P (containing the identity) such that $P \cup P^{-1} = \Gamma$. Indeed, having such a P, one may declare $f \preceq g$ if and only if $f^{-1}g \in P$. Conversely, a preorder $\preceq$ as above yields the semigroup $P := \{g : g \succeq id\}$. Using the dynamical characterization of left-orders we next show that, if Γ admits sufficiently many total preorders so that different elements can be "distinguished", then it is left-orderable.

Proposition 1.1.4. *Let* Γ *be a group and* $\{P_\lambda,\, \lambda \in \Lambda\}$ *a family of subsemigroups such that:*
(i) $P_\lambda \cup P_\lambda^{-1} = \Gamma$, *for all* $\lambda \in \Lambda$;
(ii) *The intersection* $P := \bigcap_{\lambda \in \Lambda} P_\lambda$ *satisfies* $P \cap P^{-1} = \{id\}$.
Then Γ *is left-orderable.*

Proof. For each $\lambda \in \Lambda$, let $\Gamma_\lambda = P_\lambda \cap P_\lambda^{-1}$. Fix a total order on the set of indices Λ, and let Ω be the space of all cosets $g\Gamma_\lambda$, where $g \in \Gamma$ and $\lambda \in \Lambda$. Define an order $\leq$ on Ω by letting $g\Gamma_\lambda \leq h\Gamma_{\lambda'}$ if either λ is smaller than λ', or $\lambda = \lambda'$ and $g^{-1}h \in P_\lambda$ (this does not depend on the chosen representatives g, h). By property (i), this order is total. The group Γ acts on Ω by $f(g\Gamma_\lambda) = fg\Gamma_\lambda$. This action preserves $\leq$. Moreover, if f acts trivially, then f lies in Γ_λ for all λ. Hence, by property (ii) above, $f \in \bigcap_{\lambda \in \Lambda} \left(P_\lambda \cap P_\lambda^{-1}\right) = P \cap P^{-1} = \{id\}$. This shows that the action is faithful, hence Γ is left-orderable. $\square$

Exercise 1.1.5. Let $P := \{g \colon g \succeq id\}$ be the semigroup of non-positive elements of a left-invariant total preorder $\preceq$ on a group Γ. For each $h \in \Gamma$, let $P_h := \{h^{-1}gh \colon g \in P\}$.
(i) Show that each P_h induces a total preorder on Γ.
(ii) Let $H := \bigcap_{h \in \Gamma}(P_h \cap P_h^{-1})$. Show that H is a normal subgroup of Γ.
(iii) Show that Γ/H is a left-orderable quotient of Γ, which is nontrivial provided there are at least two non-equivalent for $\preceq$.
Hint. Although everything can be directly checked, a dynamical view proceeds as follows: the quotient space $\Gamma/\sim$ obtained by idendification of $\preceq$-equivalent points (*i.e.* elements f, g such that $f \preceq g \preceq f$) is totally ordered, and Γ acts on it by left-translations preserving this order; the kernel of this action corresponds to the subgroup H.

The analogue of the preceding proposition for partial left-orders does not hold. Indeed, in §1.4.1, we will see examples of torsion-free groups that are not left-orderable. Despite this, we have the next

Exercise 1.1.6. Show that a group is *torsion-free* if and only if it admits a family $\{\preceq_\lambda \colon \lambda \in \Lambda\}$ of partial, left-invariant orders such that, for each $f \neq g$, there exists $\lambda \in \Lambda$ satisfying $g \prec_\lambda f$.
Hint. If Γ acts (faithfully) on a set X and $Y \subset X$ has trivial stabilizer, then one may define a partial, left-invariant left-order $\preceq$ on Γ by letting $h \succ id$ if and only if $h(Y) \subset Y$. If Γ is torsion-free and $f \neq g$, then for $X := \Gamma$ and $Y := \{h, h^2, \ldots\}$, where $h := g^{-1}f$, this procedure yields a partial left-order for which $f \succ g$.

On group actions on the real line. For *countable* left-orderable groups, one may take the real line as the ordered space on which the group acts. (The first reference we found on this is [89]; see also [71].)

Proposition 1.1.7. *Every left-orderable countable group acts faithfully on the real line by orientation-preserving homeomorphisms.*

Proof. Let Γ be a countable group admitting a left-invariant total order $\preceq$. Choose a numbering $(g_i)_{i\geq 0}$ for the elements of Γ, put $t(g_0)=0$, and define $t(g_k)$ by induction in the following way: assuming that $t(g_0),\ldots,t(g_i)$ have been already defined, if g_{i+1} is bigger (resp. smaller) than $g_0,\ldots,g_i$ then let $t(g_{i+1})$ be $\max\{t(g_0),\ldots,t(g_i)\}+1$ (resp. $\min\{t(g_0),\ldots,t(g_i)\}-1$), and if $g_m \prec g_{i+1} \prec g_n$ for some m,n in $\{0,\ldots,i\}$ and g_j is not between g_m and g_n for any $0\leq j\leq i$ then put $t(g_{i+1}) := (t(g_m)+t(g_n))/2$.

Notice that Γ acts naturally on $t(\Gamma)$ by $g(t(g_i)) = t(gg_i)$. We leave to the reader to check that this action extends continuously to the closure of the set $t(\Gamma)$. (There is a subtle issue here that involves the choice of midpoints –and not arbitrary intermediate points– along the construction.) Finally, one can extend the action to the whole line by extending the maps g affinely to each interval of the complementary set of $t(\Gamma)$. $\square$

It is worth analyzing the preceding proof carefully. If $\preceq$ is an left-order on a countable group Γ and $(g_i)_{i\geq 0}$ is a numbering of the elements of Γ, then we will call the (associated) ***dynamical realization*** the action of Γ on $\mathbb{R}$ constructed in this proof. It is easy to see that this realization has no global fixed point unless Γ is trivial. Moreover, if f is an element of Γ whose dynamical realization has two fixed points $a<b$ (which may be equal to $\pm\infty$) and has no fixed point in $]a,b[$, then there must be some point of the form $t(g)$ inside $]a,b[$. Finally, it is not difficult to show that the dynamical realizations associated to different numberings of the elements of Γ are all topologically conjugate.[1] Therefore, we can speak of any dynamical property of the dynamical realization without referring to a particular numbering.

Remark 1.1.8. Throughout the text, if not stated otherwise, we will always assume that $t(id)=0$ in the construction of the dynamical realization.

Exercise 1.1.9. Show that for any dynamical realization of a left-order, a G_δ-dense subset of points in the real line have a free orbit.

Exercise 1.1.10. Let Γ be a countable group of orientation-preserving homeomorphisms of the real line. Using this action, produce a dynamically-lexicographic order

[1] An action $\Phi_1 : \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ is ***topologically conjugate*** (resp. ***semiconjugate***) to Φ_2 if there exists an orientation-preserving homeomorphism (resp. non-decreasing, continuous, surjective map) φ of the real line into itself such that $\varphi \circ \Phi_1(g) = \Phi_2(g) \circ \varphi$, for all $g \in \Gamma$.

$\preceq$ on Γ. Show that the original action is semi conjugate to the dynamical realization of $\preceq$. Give examples where this semiconjugacy is not a conjugacy.

Exercise 1.1.11. Let $(\Gamma, \preceq)$ be a countable left-ordered group and Γ_0 a subgroup. Show that the restriction to Γ_0 of the dynamical realization of $\preceq$ is semiconjugate to the dynamical realization of the restriction of $\preceq$ to Γ_0.

Remark 1.1.12. Notice that, to define a dynamically-lexicographic left-order on the group $\mathrm{Homeo}_+(\mathbb{R})$ of orientation-preserving homeomorphisms of the real line, it is not necessary to well-order all the points in $\mathbb{R}$: it is enough to consider a well-order on a dense set; in particular, a dense sequence suffices. Clearly, $\mathrm{Homeo}_+(\mathbb{R})$ admits uncountably many left-orders of this type. However, there are left-orders that do not arise this way; see Example 2.2.2.

Remark 1.1.13. The group $\mathcal{G}_+(\mathbb{R}, 0)$ of germs at the origin of orientation-preserving homeomorphisms of the real line is left-orderable. Perhaps the easiest way to show this is by using the characterization in terms of finite subsets above. Let $\hat{g}_1, \ldots, \hat{g}_k$ be nontrivial elements in $\mathcal{G}_+(\mathbb{R}, 0)$, and let $g_1, \ldots, g_k$ be representatives of them. Take a sequence $(x_{n,1})$ of points converging to the origin in the line so that for each n at least one of the g_i's moves $x_{n,1}$. Passing to a subsequence if necessary, we may assume that, for each $i \in \{1, \ldots, k\}$, either $g_i(x_{n,1}) > x_{n,1}$ for *all* n, or $g_i(x_{n,1}) < x_{n,1}$ for *all* n, or $g_i(x_{n,1}) = x_{n,1}$ for *all* n. In the first case we let $\eta_i := +1$, and in the second case we let $\eta_i := -1$. In the third case, η_i is still undefined. However, this may happen only for $k-1$ of the g_i's above. For these elements, we may repeat the procedure by considering another sequence $(x_{n,2})$ converging to the origin... In at most k steps, all the η_i's will be thus defined. We claim that this choice is compatible. Indeed, given an element $\hat{g} = \hat{g}_{i_1}^{\eta_{i_1}} \cdots \hat{g}_{i_\ell}^{\eta_{i_\ell}}$, the choice above implies that $g_{i_1}^{\eta_{i_1}} \cdots g_{i_\ell}^{\eta_{i_\ell}}(x_{n,1}) \geq x_{n,1}$ for all n, where the inequality is strict if some of the g_{i_j}'s moves some of (equivalently, all) the points $x_{n,1}$. If this is the case, then this implies that $\hat{g}$ cannot be the identity. If not, then we may repeat the argument with the sequence $(x_{n,2})$ instead of $(x_{n,1})$... Proceeding this way, we conclude that $\hat{g}$ is nontrivial.

A nice consequence of the claim above is that every countable group of germs at the origin of homeomorphisms of the real line admits a realization (but not necessarily an "extension"!) as a group of homeomorphisms of the interval. Notice that, conversely, $\mathrm{Homeo}_+([0,1])$ embeds into $\mathcal{G}_+(\mathbb{R}, 0)$. (This embedding is not obtained by looking at the germs of elements of $\mathrm{Homeo}_+([0,1])$ near the infinite –the homomorphism thus-obtained is not injective–, but by taking infinite copies of $\mathrm{Homeo}_+([0,1])$ in intervals accumulating the origin). Despite of this, the groups $\mathcal{G}_+(\mathbb{R}, 0)$ and $\mathrm{Homeo}_+([0,1])$ are non-isomorphic; see [118]. Actually, there is no nontrivial homomorphism from $\mathcal{G}_+(\mathbb{R}, 0)$ into $\mathrm{Homeo}_+([0,1])$. (See Example 2.2.23 for another –and simpler– example of an uncountable left-orderable group that has no nontrivial action on the real line.)

We do not know whether there is an analogue of the preceding example in higher dimensions.

Question 1.1.14. Does there exist a finitely-generated group of germs at the origin of homeomorphisms of the plane having no realization as a group of homeomorphisms of the plane ?

Notice that the results and techniques of [28] show that such a group cannot arise as a group of germs of C^1 diffeomorphisms. We should point out, however, that imposing regularity conditions for a group action may lead to very serious algebraic restrictions; see [135] for a general panorama on this topic (see also [14, 27, 137], as well as [30] for examples of a related nature).

Let us close this discussion with a nice open question raised by Calegari.

Question 1.1.15. Is the group of orientation-preserving homeomorphisms of the disc that are the identity on the boundary left-orderable ?

1.2 Some Relevant Examples

At first glance, it appears as surprising that many (classes of) torsion-free groups turn out to be left-orderable. Here we give a brief discussion on some of them.

1.2.1 Abelian and nilpotent groups

The simplest bi-orderable groups are the torsion-free, Abelian ones. Clearly, there are only two bi-orders on $\mathbb{Z}$. The case of $\mathbb{Z}^2$ is more interesting. According to [157, 162, 169], there are two different types of bi-orders on $\mathbb{Z}^2$. Bi-orders of *irrational type* are completely determined by an irrational number λ: for such an order $\preceq_\lambda$ an element (m,n) is positive if and only if $\lambda m + n$ is a positive real number. Bi-orders of *rational type* are characterized by two data, namely a pair $(x,y) \in \mathbb{Q}^2$ up to multiplication by a positive real number, and the choice of one of the two possible bi-orders on the subgroup $\{(m,n)\colon mx + ny = 0\} \sim \mathbb{Z}$. Thus, an element $(m,n) \in \mathbb{Z}^2$ is positive if and only if either $mx + ny$ is a positive real number, or $mx + ny = 0$ and (m,n) is positive with respect to the chosen bi-order on the kernel line (isomorphic to $\mathbb{Z}$). The set of left-orders on $\mathbb{Z}^2$ naturally identifies to the Cantor set (see §2.2 for more on this).

The description of all bi-orders on $\mathbb{Z}^n$ for larger n continues inductively. (A good exercise is to show all of this using the results of §3.2.3.) For a general torsion-free, Abelian group, recall that the ***rank*** is the minimal dimension of a vector space over $\mathbb{Q}$ in which the group embeds. The reader should have no problem to show in particular that a torsion-free, Abelian group of rank ≥ 2 admits uncountably many left-orders.

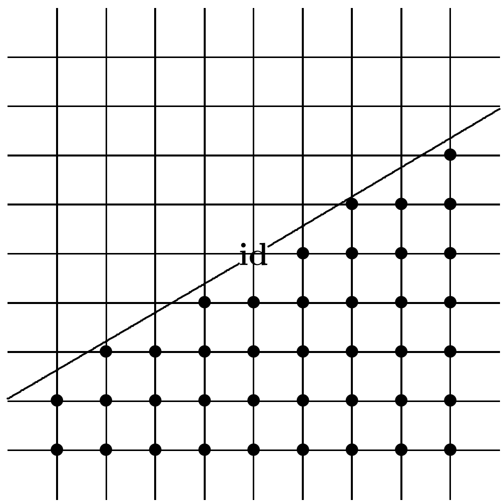

Figure 1: The positive cone of a left-order on $\mathbb{Z}^2$.

Torsion-free, nilpotent groups are also bi-orderable. Indeed, let Γ_i denote the i^{th}-term of the ***lower central series*** of a group Γ (that is, $\Gamma_1 := \Gamma$ and $\Gamma_{i+1} := [\Gamma, \Gamma_i]$), and let $H_i(\Gamma)$ be the ***isolator*** of Γ_i defined by

$$H_i(\Gamma) := \big\{ g \in \Gamma \colon g^n \in \Gamma_i \text{ for some } n \in \mathbb{N} \big\}.$$

If Γ is ***nilpotent*** (*i.e.* if $\Gamma_{k+1} = \{id\}$ for a certain k), then each $H_i(\Gamma)$ is a normal subgroup of Γ, and $H_i(\Gamma)/H_{i+1}(\Gamma)$ is a torsion-free, central subgroup of $\Gamma/H_{i+1}(\Gamma)$ (see [101] for the details). Notice that, if Γ is also torsion-free, then $H_{k+1}(\Gamma) = \{id\}$.

Let P_i be the positive cone of any left-order on (the torsion-free Abelian group) $H_i(\Gamma)/H_{i+1}(\Gamma)$, and let G_i be the set of elements in $H_i(\Gamma)$ which project to an element in P_i when taking the quotient by $H_{i+1}(\Gamma)$. Using the fact that each $H_i(\Gamma)/H_{i+1}(\Gamma)$ is *central* in $\Gamma/H_{i+1}(\Gamma)$, one may easily check that the semigroup $P := G_{k-1} \cup G_{k-2} \cup \ldots \cup G_1$ is the positive cone of a bi-order on Γ.

Example 1.2.1. The Heisenberg group

$$H = \big\langle f, g, h \colon [f,g] = h^{-1}, [f,h] = id, [g,h] = id \big\rangle$$

is a non-Abelian nilpotent group of nilpotence degree 2. It may also be seen as the group of lower triangular matrices with integer entries so that each diagonal entry equals 1 via the identifications

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Notice that the linear action of H on $\mathbb{Z}^3$ fixes the hyperplane $\{1\} \times \mathbb{Z}^2$ and preserves the lexicographic order on it. The left-orders on H induced from this restricted action (*c.f.* §1.1.3) are (total but) not bi-invariant. This example can be seen as a kind of evidence of the following nice result due to Darnel, Glass, and Rhemtulla [47]: If all the left-orders of a left-orderable group are bi-invariant, then the group is Abelian.

Both the space of left-orders and bi-orders of countable, torsion-free, nilpotent groups which are not rank-1 Abelian are homeomorphic to the Cantor set; see Theorem 3.2.20 for the former and [129] for the latter. Moreover, there is the next remarkable theorem of Malcev [122] (resp. Rhemtulla [15, Chapter 7]): Every bi-invariant (resp. left-invariant) *partial* order on a torsion-free, nilpotent group can be extended to a bi-invariant (resp. left-invariant) *total* order.

Exercise 1.2.2. Let $\bar{H}$ be the subgroup of the Heisenberg group formed by the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ z & 2y & 1 \end{pmatrix}, \quad x, y, z \text{ in } \mathbb{Z}.$$

(i) Show that the commutator subgroup is formed by the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4z & 0 & 1 \end{pmatrix}, \quad z \in \mathbb{Z}.$$

(ii) Conclude that $\bar{H}/\bar{H}'$ is isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}/4\mathbb{Z}$, hence has torsion.

Exercise 1.2.3. Show that a group Γ is residually torsion-free nilpotent if and only if $\bigcap_i H_i(\Gamma) = \{id\}$. (Since bi-orderability is a residual property, such a group is necessarily bi-orderable.)

Remark. Quite surprisingly, torsion-free, residually nilpotent groups do not necessarily satisfy this property. Actually, such a group may fail to be bi-orderable; see [8].

1.2.2 Subgroups of the affine group

Let $\mathrm{Aff}_+(\mathbb{R})$ denote the group of orientation-preserving affine homeomorphisms of the real line (the ***affine group***, for short). For each $\varepsilon \neq 0$, a partial order $\preceq_\varepsilon$ may be defined by declaring that f is positive if and only if $f(1/\varepsilon) > 1/\varepsilon$. This means that

$$P^+_{\preceq_\varepsilon} = \Big\{ f = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} \colon\ u + v\varepsilon > 1 \Big\}.$$

These orders were introduced (in a more algebraic way) by Smirnov in [163].

For a finitely-generated subgroup Γ of $\mathrm{Aff}_+(\mathbb{R})$, the corresponding action on the line has free orbits. Thus, one may choose ε so that $\preceq_\varepsilon$ is a total order. In the general case, one may "complete" the partial order into a total one (*c.f.* §1.1.3). As a consequence, non-Abelian subgroups of $\mathrm{Aff}_+(\mathbb{R})$ admit uncountably many left-orders.

As a concrete and relevant example, for each integer $\ell \geq 2$, the ***Baumslag-Solitar group*** $BS(1,\ell) = \langle g, h \colon\ hgh^{-1} = g^\ell \rangle$ embeds into the affine group by identifying g and h to $x \mapsto x+1$ and $x \mapsto \ell x$, respectively. (See Exercise 1.2.4 below.) Notice that, for an irrational $\varepsilon \neq 0$, the associate order $\preceq_\varepsilon$ is total. If one chooses a rational ε, then it may happen that $\preceq_\varepsilon$ is only a partial order. However, in this case, the stabilizer of the point $1/\varepsilon$ is isomorphic to $\mathbb{Z}$, and thus $\preceq_\varepsilon$ can be completed to a total left-order of $BS(1,\ell)$ in exactly two different ways. Notice that the reverse orders $\overline{\preceq}_\varepsilon$ may be retrieved by the same procedure but starting with the embedding $g : x \mapsto x-1$ and $h : x \mapsto \ell x$, and changing ε by $-\varepsilon$.

There is still another way to order $BS(1,\ell)$. Namely, $BS(1,\ell)$ can be thought of as the semidirect product $\mathbb{Z}[\frac{1}{\ell}] \rtimes \mathbb{Z}$ coming from the exact sequence

$$0 \longrightarrow \mathbb{Z}\Big[\frac{1}{\ell}\Big] \longrightarrow BS(1,\ell) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Using this, one may define the bi-orders $\preceq$, $\preceq'$ by letting $(\frac{m}{\ell^n}, k) \succ id$ (resp. $(\frac{m}{\ell^n}, k) \succ' id$) if and only if either $k > 0$, or $k = 0$ and $\frac{m}{\ell^n} > 0$ (resp. $k > 0$, or $k = 0$ and $\frac{m}{\ell^n} < 0$). Together with the reverse orders $\overline{\preceq}$ and $\overline{\preceq}'$, this completes the list of all bi-orders on $BS(1,\ell)$ (see §3.2.55).

A similar description applies to all non-Abelian subgroups of $\mathrm{Aff}_+(\mathbb{R})$. In the terminology of §2.2, the associated spaces of left-orders are homeomorphic to the Cantor set. The study of more general solvable left-orderable groups is more involved, yet it crucially relies on the case of affine groups. We will come back to this point in §3.3.1 and §3.3.2.

Exercise 1.2.4. Prove that the map from $BS(1,\ell) = \langle g, h \colon\ hgh^{-1} = g^\ell \rangle$ into the affine group that makes correspond g and h to $x \mapsto x+1$ and $x \mapsto \ell x$, respectively, is an embedding.
Hint. Prove that the conjugates of g commute, and then write every element of $BS(1,\ell)$ in normal form as a power of h followed by products of conjugates of g.

Exercise 1.2.5. Show that every embedding of $BS(1,\ell)$ into $\mathrm{Aff}_+(\mathbb{R})$ is obtained by letting g,h correspond, respectively, to any nontrivial translation and an homothethy of ratio ℓ.

1.2.3 Free and residually free groups

The free group $\mathbb{F}_2$ (hence every non-Abelian free group) is bi-orderable. Although this result is originally due to Vinogradov, it is in general attributed to Magnus, as it arises as a consequence of his famous "expansions". Below we sketch Magnus' construction; Vinogradov's original proof (which actually applies to free products of arbitrary bi-orderable groups) will be treated in §2.1.2.

Consider the (non-Abelian) ring $\mathbb{A} = \mathbb{Z}\langle X,Y\rangle$ formed by the formal power series with integer coefficients in two independent variables X, Y. Denoting by $o(k)$ the subset of $\mathbb{A}$ formed by the elements all of whose terms have degree at least k, one easily checks that

$$\mathbb{F} := 1 + o(1) = \big\{1 + S\colon\ S \in o(1)\big\}$$

is a subgroup (under multiplication) of $\mathbb{A}$. Moreover, if f,g are (free) generators of $\mathbb{F}_2$, the map Φ sending f (resp. g) to the element $1+X$ (resp. $1+Y$) in $\mathbb{A}$ extends in a unique way into an injective homomorphism $\Phi\colon \mathbb{F}_2 \to \mathbb{F}$. Now fix a lexicographic type order relation on $\mathbb{F}$ that is bi-invariant under multiplication by elements in $\mathbb{F}$. (Notice that this order is not invariant under multiplication by certain elements in $\mathbb{A}$.) Using this order and the homomorphism ϕ, the free group $\mathbb{F}_2$ may be easily endowed with a bi-invariant left-order.

The above technique –called the ***Magnus expansion***– actually shows that $\mathbb{F}_2$ is residually torsion-free nilpotent. Indeed, it is easy to check that $\Phi(\Gamma_i^{\mathrm{nil}})$ is contained in $1 + o(i+1)$, for every $i \geq 0$. We refer to [121] for more details on all of this.

Surface groups. Surface groups are residually free, hence bi-orderable. Actually, as we show below, these groups are fully residually free, which is a stronger property (see Remark 1.2.6 below). Recall that, if P is some group property, then a group Γ is said to be ***fully residually*** P if for every finite subset $\mathcal{G} \subset \Gamma \setminus \{id\}$, there exists a surjective group homomorphism from Γ into a group $\Gamma_{\mathcal{G}}$ satisfying P such that the image of every $g \in \mathcal{G}$ is nontrivial. Equivalently, for every finite subset $\mathcal{G} \subset \Gamma$, there is an homomorphism Φ into a group satisfying P whose restriction to $\mathcal{G}$ is injective.

Remark 1.2.6. Obviously, the direct product $\mathbb{F}_2 \times \mathbb{F}_2$ is residually free. However, it is not fully residually free, because given any f, g, h in $\mathbb{F}_2$, no homomorphism from $\mathbb{F}_2 \times \mathbb{F}_2$ to a free group maps the elements id, (f, id), (g, id), $([f, g], id)$, and (id, h), into five different ones. Indeed, as (id, h) commutes with (f, id) and (g, id), a separating homomorphism should send these three elements into some cyclic subgroup. However, if this is the case, then $([f, g], id)$ is mapped to the identity.

Below we deal with the case of surfaces of even genus (the case of odd genus easily follows from this). The following lemma, due to Baumslag, will be crucial for us. The geometric proof we give appears in [4].

Lemma 1.2.7. *Let $g_1, \ldots, g_k$ be elements in a free group $\mathbb{F}_n$, and let f be another element which does not commute with any of them. Then there exists $N \in \mathbb{N}$ such that, for every $|n_i| \geq N$, $m \in \mathbb{N}$, and $j_i \in \{1, \ldots, k\}$,*

$$g_{j_1} f^{n_1} g_{j_2} f^{n_2} \ldots g_{j_m} f^{n_m} \neq id.$$

Proof. Let a^-, a^+ be the endpoints (in the boundary at infinity $\partial \mathbb{F}_n$) of the axis $\mathrm{axis}(f)$ determined by f in the Cayley graph of $\mathbb{F}_n$. Since g_i does not commute with f, one must have $\{g_i(a^-), g_i(a^+)\} \cap \{a^-, a^+\} = \emptyset$, for every $i \in \{1, \ldots, k\}$. Let U^-, U^+ be neighborhoods in $\partial \mathbb{F}_n$ of a^- and a^+, respectively, satisfying

$$g_i(U^- \cup U^+) \cap (U^- \cup U^+) = \emptyset \quad \text{for each } \; i.$$

There exists $N \in \mathbb{N}$ such that, for all $r \geq N$,

$$f^r(\partial \mathbb{F}_n \setminus U^-) \subset U^+, \qquad f^{-r}(\partial \mathbb{F}_n \setminus U^+) \subset U^-.$$

A ping-pong type argument (see [84]) then shows the lemma. $\square$

Let $\Gamma = \Gamma_{2n}$ be the π_1 of an orientable surface S_{2n} of genus $2n$ ($n \geq 1$). Let us consider the standard presentation

$$\Gamma = \big\langle g_i, g_i', h_i, h_i', 1 \leq i \leq n : [g_1, g_1'] \cdots [g_n, g_n'] \cdot [h_n', h_n] \cdots [h_1', h_1] = id \big\rangle.$$

Following [19], let σ be the automorphism of Γ that leaves the g_i's and g_i''s fixed while sending every h_i to $f h_i f^{-1}$ and every h_i' to $f h_i' f^{-1}$, where $f := [g_1, g_1'] \cdots [g_n, g_n']$. (Geometrically, this corresponds to the Dehn twist along the closed curve obtained from a simple curve that joins the first and $2n^{th}$ vertices of the hyperbolic $4n$-gon that yields S_{2n}.) Finally, let φ be the surjective homomorphism from Γ to the free group $\mathbb{F}_{2n}$ with free generators $a_1, \ldots, a_n, a_1', \ldots, a_n'$

defined by $\varphi(g_i) = \varphi(h_i) = a_i$ and $\varphi(g_i') = \varphi(h_i') = a_i'$. We claim that the sequence of homomorphisms $\varphi \circ \sigma^k$ is ***eventually faithful***, in the sense that given $f_1, \ldots, f_m$ in Γ, there exists $N \in \mathbb{N}$ so that for all $k \geq N$, the image under $\varphi \circ \sigma^k$ of any of the f_i's is nontrivial (thus showing that Γ is fully residually free).

To show the claim above, given $g \in \Gamma \setminus \{id\}$, let us write it in the form

$$g = w_1(g_i, g_i') \cdot w_2(h_i, h_i') \cdots w_{2p-1}(g_i, g_i') \cdot w_{2p}(h_i, h_i'),$$

where each w_i is a reduced word in $2n$ letters (the first and/or the last w_i may be trivial). Up to modifying the w_{2j-1}'s, we may assume that each w_{2j} (where $1 \leq j \leq p$) is such that $w_{2j}(h_i, h_i')$ is not a power of f. Notice that the centralizer of f in Γ is the cyclic group generated by f. By regrouping several w_j's into a longer word if necessary, unless g itself is a power of f, we may also assume that $w_{2j-1}(g_i, g_i')$ is not a power of f. Let $\bar{f}$ be the image of f under φ. We have

$$\varphi \circ \sigma^k(g) = w_1 \bar{f}^k w_2 \bar{f}^k \cdots w_{2p-1} \bar{f}^k w_{2p} \bar{c}^k,$$

where $w_j = w_j(a_i, a_i')$. Since $\bar{f}$ does not commute with any of the w_j's, Lemma 1.2.7 implies that $\varphi \circ \sigma^k(g)$ is nontrivial.

1.2.4 Thompson's group F

Thompson's group F is perhaps the simplest example of a bi-orderable group that is not residually nilpotent. Recall that this is the group of the orientation-preserving, piecewise-affine homeomorphisms f of the interval $[0,1]$ such that:

– The derivative of f on each linearity interval is an integer power of 2;

– f induces a bijection of the set of dyadic rational numbers in $[0,1]$.

This group is not residually nilpotent because its commutator subgroup F' is simple (see [32]). To see that it is bi-orderable, for each nontrivial $f \in \mathrm{F}$ we denote by x_f^- (resp. x_f^+) the leftmost point x^- (resp. the rightmost point x^+) for which $Df_+(x^-) \neq 1$ (resp. $Df_-(x^+) \neq 1$), where Df_+ and Df_- stand for the corresponding lateral derivatives. One can immediately visualize four different bi-orders on (each subgroup of) F, namely the bi-order $\preceq_{x^-}^+$ (resp. $\preceq_{x^-}^-$, $\preceq_{x^+}^+$, $\preceq_{x^+}^-$) for which f is positive if and only if $Df_+(x_f^-) > 1$ (resp. $Df_+(x_f^-) < 1$, $Df_-(x_f^+) < 1$, $Df_-(x_f^+) > 1$). Although F admits many more bi-orders than these (see theorem 1.2.9 below), the case of F' is quite different. The result below is essentially due to Dlab [57] (see also [142]).

Theorem 1.2.8. *The only bi-orders on* F' *are* $\preceq_{x^-}^+$, $\preceq_{x^-}^-$, $\preceq_{x^+}^+$ *and* $\preceq_{x^+}^-$.

Remark that there are four other "exotic" bi-orders on F, namely:

– The bi-order $\preceq_{0,x^-}^{+,-}$ for which f is positive if and only if either $x_f^- = 0$ and $Df_+(0) > 1$, or $x_f^- \neq 0$ and $Df_+(x_f^-) < 1$;

– The bi-order $\preceq_{0,x^-}^{-,+}$ for which f is positive if and only if either $x_f^- = 0$ and $Df_+(0) < 1$, or $x_f^- \neq 0$ and $Df_+(x_f^-) > 1$;

– The bi-order $\preceq_{1,x^+}^{+,-}$ for which f is positive if and only if either $x_f^+ = 1$ and $Df_+(1) < 1$, or $x_f^+ \neq 1$ and $Df_-(x_f^+) > 1$;

– The bi-order $\preceq_{1,x^+}^{-,+}$ for which f is positive if and only if either $x_f^+ = 1$ and $Df_+(1) > 1$, or $x_f^+ \neq 1$ and $Df_-(x_f^+) < 1$.

Notice that, when restricted to F′, the bi-order $\preceq_{0,x^-}^{+,-}$ (resp. $\preceq_{0,x^-}^{-,+}$, $\preceq_{1,x^+}^{+,-}$, and $\preceq_{1,x^+}^{-,+}$) coincides with $\preceq_{x^-}^{-}$ (resp. $\preceq_{x^-}^{+}$, $\preceq_{x^+}^{-}$, and $\preceq_{x^+}^{+}$). Let us denote the set of the previous eight bi-orders on F by $\mathcal{BO}_{Isol}(\mathrm{F})$.

There is another natural procedure to create bi-orders on F. For this, recall the well-known (and easy to check) fact that F′ coincides with the subgroup of F formed by the elements f satisfying $Df_+(0) = Df_-(1) = 1$. Now let $\preceq_{\mathbb{Z}^2}$ be any bi-order on $\mathbb{Z}^2$, and let $\preceq_{\mathrm{F}'}$ be any bi-order on F′. It readily follows from Dlab's theorem that $\preceq_{\mathrm{F}'}$ is invariant under conjugacy by elements in F. Hence, one may define a bi-order $\preceq$ on F by declaring that $f \succ id$ if and only if either $f \notin \mathrm{F}'$ and $\big(\log_2(Df_+(0)), \log_2(Df_-(1))\big) \succ_{\mathbb{Z}^2} \big(0,0\big)$, or $f \in \mathrm{F}'$ and $f \succ_{\mathrm{F}'} id$ (see §2.1.1 for more details on this type of construction).

All possible ways of left-ordering finite-rank, Abelian groups were described in §1.2.1. Since there are only four possibilities for $\preceq_{\mathrm{F}'}$, the preceding procedure gives us four sets (which we will coherently denote by $\Lambda_{x^-}^{+}$, $\Lambda_{x^-}^{-}$, $\Lambda_{x^+}^{+}$, and $\Lambda_{x^+}^{-}$) naturally homeomorphic to the Cantor set (in the sense of §2.2) inside the set of bi-orders of F. The main result of [142] establishes that these bi-orders, together with the eight special bi-orders previously introduced, are all possible bi-orders on F. The proof is an elementary application of Conrad's theory (*c.f.* §3.2.1.)

Theorem 1.2.9. *The set of all bi-orders of* F *consists of the disjoint union of* $\mathcal{BO}_{Isol}(\mathrm{F})$ *and the sets* $\Lambda_{x^-}^{+}$, $\Lambda_{x^-}^{-}$, $\Lambda_{x^+}^{+}$, *and* $\Lambda_{x^+}^{-}$.

1.2.5 Braid groups

Perhaps the most relevant examples of left-orderable groups are the braid groups $\mathbb{B}_n$. Recall that $\mathbb{B}_n$ has a presentation of the form

$$\mathbb{B}_n = \big\langle \sigma_1, \ldots, \sigma_{n-1} \colon \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2,\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \big\rangle.$$

Following Dehornoy [49], for $i \in \{1,\ldots,n-1\}$, an element of $\mathbb{B}_n$ is said to be i-positive if it may be written as a word of the form $w_1\sigma_i w_2 \sigma_i \cdots w_k \sigma_i w_{k+1}$, where the w_i's are (perhaps trivial) words on $\sigma_{i+1}^{\pm 1},\ldots,\sigma_{n-1}^{\pm 1}$ (and σ_i appears at least once). An element in $\mathbb{B}_n$ is said to be D-positive if it is i-positive for some $i \in \{1,\ldots,n-1\}$. A remarkable result of Dehornoy establishes that the set of D-positive elements form the positive cone of a left-order order $\preceq_D$ on $\mathbb{B}_n$. In other words:

– For every nontrivial $\sigma \in \mathbb{B}_n$, either σ or σ^{-1} is i-positive for some i. (Actually, Dehornoy provides an algorithm, called ***handle reduction***, for recognizing positive elements and putting them in the normal form above.)

– If $\sigma \in \mathbb{B}_n$ is nontrivial, then σ and σ^{-1} cannot be simultaneously D-positive.

Notice that $\mathbb{B}_n$ is not bi-orderable, as it contains nontrivial elements which are conjugate to their inverses: $(\sigma_1\sigma_2\sigma_1)^{-1}(\sigma_1\sigma_2^{-1})(\sigma_1\sigma_2\sigma_1) = (\sigma_1\sigma_2^{-1})^{-1}$. Despite of this, $\preceq_D$ satisfies an important weak property of bi-invariance called ***subword property***: All conjugates of the generators σ_i are $\preceq_D$-positive.

None of the statements above is easy to prove; see for example [50]. In §2.2.3, we will give a short proof for the case of $\mathbb{B}_3$.

Pure braid groups. According to Falk and Randell [63], pure braid groups $\mathrm{P}\mathbb{B}_n$ are residually torsion-free nilpotent, hence bi-orderable. An alternative approach to the bi-orderability of $\mathrm{P}\mathbb{B}_n$ using the Magnus expansion was proposed by Rolfsen and Zhu in [103] (see also [124]). Let us point out, however, that these bi-orders are quite different from the Dehornoy left-order. Indeed, we will see in §3.2.1 that no bi-order on $\mathrm{P}\mathbb{B}_n$ can be extended to a left-order of $\mathbb{B}_n$.

For nice bi-orderable groups which are a mixture of pure braid groups and Thompson's groups, see [25].

1.3 Other Forms of Orderability

1.3.1 Lattice-orderable groups

A ***lattice-ordered group*** (or ***ℓ-ordered group***) is a partially ordered group $(\Gamma,\preceq)$ so that $\preceq$ is left and right invariant, and for each pair of group elements f, g there is a minimal (resp. maximal) element $f \vee g$ (resp. $f \wedge g$) simultaneously larger (resp. smaller) than f and g. (Notice that $f \wedge g = (f^{-1} \vee g^{-1})^{-1}$.) For instance, the group $\mathcal{A}(\Omega,\leq)$ of *all* order automorphisms of a totally ordered space $(\Omega,\leq)$ is ℓ-orderable, as one may define $f \succeq g$ whenever f,g in $\mathcal{A}(\Omega,\leq)$ satisfy

$f(w) \geq g(w)$ for *all* $\omega \in \Omega$. In this case, we have $f \vee g\,(w) = \max_{\leq}\{f(w), g(w)\}$ and $f \wedge g\,(w) = \min_{\leq}\{f(w), g(w)\}$.

Example 1.3.1. For the group $\mathrm{Homeo}_+(\mathbb{R})$, this and its reverse order (which is also an ℓ-order) are the only possible ℓ-orders; see [88].

The converse to the preceding remark is an important theorem due to Holland [89]. (The proof below is taken from [15, Chapter VII]; see also [75, Chapter 7], [76, Appendix I], and [87].)

Theorem 1.3.2. *Every ℓ-ordered group $(\Gamma, \preceq)$ acts by automorphisms of a totally ordered space $(\Omega, \leq)$ in such a way that $f \preceq g$ implies $f(w) \leq g(w)$ for all $w \in \Omega$, and $f \vee g\,(w) = \max_{\leq}\{f(w), g(w)\}$ and $f \wedge g\,(w) = \min_{\leq}\{f(w), g(w)\}$. In particular, every ℓ-orderable group is left-orderable.*

For the proof, let $(\Gamma, \preceq)$ be an ℓ-subgroup of $\mathcal{A}(\Omega, \leq)$ for a totally ordered space $(\Omega, \leq)$, and let P be the set of *non-negative* elements for the associate order. For each $w \in \Omega$, denote by P_w the semigroup $\{f \in \Gamma \colon f(w) \geq w\}$. Then:
(i) $\bigcap_{w \in \Omega} P_w = P$,
(ii) $P_w \bigcup P_w^{-1} = \Gamma$, for all $w \in \Omega$.
This turns natural the following version of Proposition 1.1.4:

Lemma 1.3.3. *Let $(\Gamma, \preceq)$ be an ℓ-ordered group with set of non-negative elements P. Assume that Γ contains a family of subsemigroups P_λ, $\lambda \in \Lambda$, satisfying* (i) *and* (ii) *above. Then the conclusion of Theorem* 1.3.2 *is satisfied.*

Proof. Proceed as in the proof of Proposition 1.1.4. Since $\preceq$ is bi-invariant, for each $f \in P$, $g \in \Gamma$, and $\lambda \in \Lambda$, we have $g^{-1} f g \in P \subset P_\lambda$. By definition, this implies that $f(g\Gamma_\lambda) \geq g\Gamma_\lambda$. Finally, if $f \notin P$, then by (ii) we have $f \in P_\lambda^{-1} \setminus P_\lambda$ for some λ, which yields $f\Gamma_\lambda < \Gamma_\lambda$. The claims concerning $f \vee g$ and $f \wedge g$ are left to the reader. $\square$

Proof of Theorem 1.3.2. Denoting by P the set of non-negative elements of $\preceq$, for each $h \in \Gamma \setminus P$ choose a maximal ℓ-subsemigroup P_h of Γ containing P but not h. We obviously have

$$\bigcap_{h \in \Gamma \setminus P} P_h = P,$$

so that condition (i) above is satisfied. The proof of condition (ii) is by contradiction. Assume throughout that for certain $h \in \Gamma \setminus P$ and $g \in \Gamma$, we have $g \notin P_h$ and $g \notin P_h^{-1}$.

Claim (i). Neither $id \wedge g$ nor $id \wedge g^{-1}$ belong to P_h.

Indeed, notice that $g = [g(id \wedge g)^{-1}](id \wedge g) = (id \vee g)(id \wedge g)$. Since $id \vee g$ belongs to $P \subset P_h$, if $id \wedge g$ were contained in P_h, then this would imply that g also belongs to P_h, contrary to our hypothesis. A similar argument applies to $id \wedge g^{-1}$.

Claim (ii). There exist n_1, n_2 in $\mathbb{N}$ and h_1, h_2 in P_h such that $[(id \wedge g)h_1]^{n_1} \preceq h$ and $[(id \wedge g^{-1})h_2]^{n_2} \preceq h$.

By the maximality of P_h, the element h belongs to the smallest ℓ-subsemigroup $\langle P_h, id \wedge g\rangle_\ell$ (resp. $\langle P_h, id \wedge g^{-1}\rangle_\ell$) containing P_h and $id \wedge g$ (resp. P_h and $id \wedge g^{-1}$). Thus, the claim follows from the following fact: For each $f \prec id$, the semigroup $\langle P_h, f\rangle_\ell$ is the set S of elements which are larger than or equal to $(f\bar{f})^n$ for some $\bar{f} \in P_h$ and some $n \in \mathbb{N}$. To show this, first notice that this set is an ℓ-semigroup. Indeed, if $(f\bar{f}_1)^{n_1} \preceq g_1$ and $(f\bar{f}_2)^{n_2} \preceq g_2$, with $\bar{f}_1, \bar{f}_2$ in P_h and n_1, n_2 in $\mathbb{N}$, then both g_1 and g_2 are larger than or equal to $(f\bar{f})^n$, where $\bar{f} := id \wedge \bar{f}_1 \wedge \bar{f}_2 \in P_h$ and $n := \max\{n_1, n_2\}$. Hence, $(f\bar{f})^{2n} \preceq g_1 g_2$ and $(f\bar{f})^n \preceq g_1 \wedge g_2$, and therefore $g_1 g_2$ and $g_1 \wedge g_2$ (as well as $g_1 \vee g_2$) belong to S. Since $f \in S$ and $P_h \subset S$, this shows that $\langle P_h, f\rangle_\ell \subset S$. Finally, we also have $S \subset \langle P_h, f\rangle_\ell$. Indeed, if $\bar{g} \succ (f\bar{f})^n$ for some $\bar{f} \in P_h$ and $n \in \mathbb{N}$, then since $(f\bar{f})^n \in \langle P_h, f\rangle_\ell$, we have $\bar{g} := \big(\bar{g}(f\bar{f})^{-n}\big)(f\bar{f})^n \in P \cdot \langle P_h, f\rangle_\ell = \langle P_h, f\rangle_\ell$. This shows the claim.

Claim (iii). Let $n := \max\{n_1, n_2\}$ and $f := id \wedge h_1 \wedge h_2$, where h_1, h_2 and n_1, n_2 are as in (ii). Then the element $\hat{f} := \big([(id \wedge g)f] \vee [(id \wedge g^{-1})f]\big)^{2n-1}$ is smaller than or equal to h.

Indeed, since f, $id \wedge g$, and $id \wedge g^{-1}$ lie in P^{-1}, we have $[(id \wedge g)f]^n \preceq h$ and $[(id \wedge g^{-1})f]^n \preceq h$. Now, as $id \wedge g$ and $id \wedge g^{-1}$ commute (their product equals $id \wedge g \wedge g^{-1}$), we easily check that $\hat{f}$ may be rewritten as

$$[(id\wedge g)f]^{2n-1} \vee \big([(id\wedge g)f]^{2n-2}[(id\wedge g^{-1})f]\big) \vee \big([(id\wedge g)f]^{2n-3}[(id\wedge g^{-1})f]^2\big) \vee \ldots \vee [(id\wedge g^{-1})f]^{2n-1}.$$

Each term of this $\vee$-product contains either $[(id \wedge g)f]^n$ or $[(id \wedge g^{-1})f]^n$ together with non-positive factors. The claim follows.

To conclude the proof, notice that from $(id \wedge g) \vee (id \wedge g^{-1}) = id$ it follows that $\hat{f} = f^{2n-1}$. Since $f \in P_h$, the same holds for $\hat{f}$. Nevertheless, as $\hat{f}^{-1}h \in P \subset P_h$, this implies that $h = \hat{f}(\hat{f}^{-1}h) \in P_h$, which is a contradiction. □

Left-orderable groups v/s ℓ-orderable groups. Let us point out that ℓ-orderability is a stronger property than left-orderability. For instance, ℓ-orderability a non-local property [75, Theorem 2.D]. A more transparent difference concerns roots of elements, as all ℓ-orderable groups satisfy the ***property of conjugate roots*** **(C.R.P.)**: Any two elements f, g satisfying $f^n = g^n$ for some $n \in \mathbb{N}$ are conjugate. Indeed, if for such f, g we let

$$h := f^{n-1} \vee f^{n-2} g \vee f^{n-3} g^2 \vee \ldots \vee g^{n-1},$$

then we have

$$fh = f^n \vee f^{n-1} g \vee f^{n-2} g^2 \vee \ldots \vee f g^{n-1} = f^{n-1} g \vee f^{n-2} g^2 \vee \ldots \vee f g^{n-1} \vee g^n = hg.$$

This property fails to be true for left-orderable groups, as shown by the next

Exercise 1.3.4. The π_1 of the Klein bottle may be presented in the form $\langle a, b \colon bab = a \rangle$. (This is nothing but the infinite dihedral group.) This group is easily seen to be left-orderable (see §2.2.3 for a discussion on this). Prove that this group is not ℓ-orderable by showing that the elements $x = ba$ and $y = a$ satisfy $x^2 = y^2$ but are not conjugate.
Remark. Despite this example, notice that every left-orderable group Γ embeds into a lattice-orderable group, namely the group of all order permutations of Γ endowed with a left-order.

Remark 1.3.5. Left-orderable groups satisfying the C.R.P. are not necessarily ℓ-orderable. Concrete examples are braid groups: in §1.2.5, we will see that these groups are left-orderable, the C.R.P. for them is shown in [77], and the fact that $\mathbb{B}_n$ is not ℓ-orderable (for $n \geq 3$) is proved in [127].

1.3.2 Locally-invariant orders and diffuse groups

Following [35] and the references therein, a partial order relation $\preceq$ on a group Γ is said to be ***locally invariant*** if for every f, g in Γ, with $g \neq id$, either $fg \succ f$ or $fg^{-1} \succ f$. Obviously, every left-order is a locally-invariant order. Examples of non left-orderable groups admitting a locally-invariant order have been recently given: see §1.4 (see also Theorem 4.1.7 for the case of amenable groups).

Exercise 1.3.6. Show that a group Γ admits a locally-invariant order if and only if there exist a partially ordered space $(\Omega, \leq)$ and a map $\varphi : \Gamma \to \Omega$ such that for every f, g in Γ, with $g \neq id$, either $\varphi(fg) > \varphi(f)$ or $\varphi(fg^{-1}) > \varphi(f)$.

Example 1.3.7. Based on [16, 51, 81], it is shown in [35] that many groups with hyperbolic properties admit locally-invariant orders. More precisely, let (X, d) be a geodesic δ-hyperbolic metric space [73] and Γ a group acting on X by isometries so that $d(x, g(x)) > 6\delta$ holds for all $x \in X$. Then fixing $x_0 \in X$, the function $g \mapsto d(x_0, g(x_0))$

satisfies the property of the preceding exercise. In particular, Γ admits a locally-invariant order.

This construction applies to many groups. In particular, if Γ is a residually finite Gromov-hyperbolic group (as for instance the π_1 of a compact hyperbolic manifold), then Γ contains a finite-index subgroup admitting a locally-invariant order. Similarly, a group acting isometrically and without inversions on a real-tree has a locally-invariant order.

At first glance, the notion of locally-invariant order may look strange. Perhaps a more clear view is provided by an equivalent formulation in terms of cones. More precisely, given a group Γ, denote by $P(\Gamma)$ the family of subsets (cones) $P \subset \Gamma$ such that $id \notin P$ and, for all $g \neq id$, at least one of the elements g, g^{-1} lies in P. A *field of cones* is a map $f \to P_f$ from Γ into $P(\Gamma)$. This field will be said to be *equivariant* if the following condition holds (see Figure 2):

$$\text{if} \quad g \in P_f \quad \text{and} \quad h \in P_{fg}, \quad \text{then} \quad gh \in P_f.$$

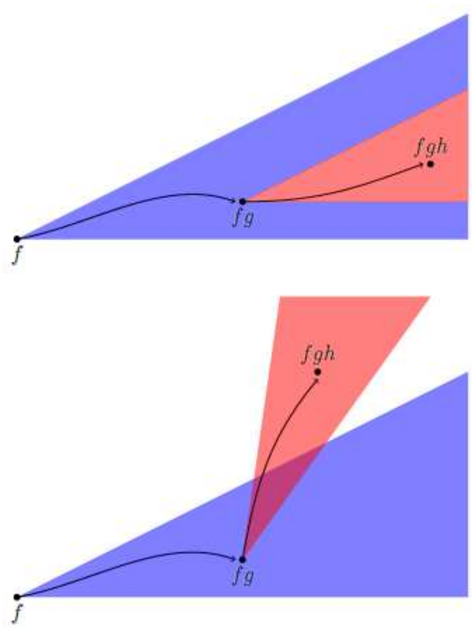

Figure 2: The cone condition (above) and its negation (below) for locally-invariant orders.

It turns out that locally-invariant orders and equivariant fields of cones are equivalent notions. Indeed, assume that $\preceq$ is a locally-invariant order on a group Γ. For each $f \in \Gamma$, define P_f by letting

$$g \in P_f \quad \text{if and only if} \quad fg \succ f.$$

By definition, each P_f belongs to $P(\Gamma)$. We claim that the field $f \to P_f$ is equivariant. Indeed, the conditions $g \in P_f$ and $h \in P_{fg}$ mean, respectively, that $fg \succ f$ and $fgh \succ fg$. Hence, by transitivity of $\preceq$, we have $fgh \succ f$, that is, $gh \in P_f$, as desired.

Conversely, let $f \mapsto P_f$ be an equivariant field of cones. Define a relation $\preceq$ on Γ by letting $f \succ g$ whenever $g^{-1}f \in P_g$. We claim that this is a locally-invariant order. To see that $\preceq$ is antisymmetric, assume $f \succ g$ and $g \succ f$. Then $g^{-1}f \in P_g$ and $f^{-1}g \in P_f$. By equivariance, this implies that $id = (g^{-1}f)(f^{-1}g) \in P_g$, which is a contradiction. To see that $\preceq$ is transitive, assume $f \succ g$ and $g \succ h$. Then $g^{-1}f \in P_g$ and $h^{-1}g \in P_h$. By equivariance, $h^{-1}f = (h^{-1}g)(g^{-1}f) \in P_h$, which means that $f \succ h$. Finally, given $f \in \Gamma$ and $g \neq id$, we have either $f^{-1}gf \in P_f$, or $f^{-1}g^{-1}f \in P_f$. In the former case, $gf \succ f$, and in the latter, $g^{-1}f \succ f$.

Exercise 1.3.8. Associated to each $\ell \in \mathbb{Z}$ there is a locally-invariant order $\preceq_\ell$ on $\mathbb{Z}$ defined by $m \succ_\ell n$ if and only if either $n > m \geq \ell$ or $n < m \leq \ell - 1$. (See Figure 3.) Show that every locally-invariant order on $\mathbb{Z}$ either is the canonical one, its reverse, or contains one of the orders $\preceq_\ell$. (Notice that we way enlarge $\preceq_\ell$ by defining non-contradictory inequalities between integers m, n such that $m > \ell > n$.)

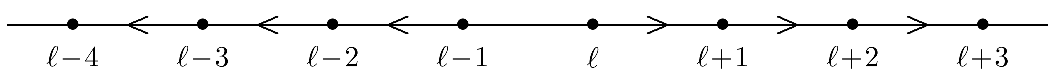

Figure 3: A locally-invariant order on $\mathbb{Z}$.

There is a closely related notion to locally-invariant orders introduced by Bowditch in [16]. Namely, given a subset A of a group Γ, an ***extremal point*** of A is a point $f \in A$ such that, if $fg \in A$ and $fg^{-1} \in A$ for some $g \in \Gamma$, then $g = id$. A group Γ is said to be ***weakly diffuse*** if every nonempty finite subset has an extremal point.

Proposition 1.3.9. *A group admits a locally-invariant order if and only if it is weakly diffuse.*

Proof. Let Γ be a group admitting a locally-invariant order $\preceq$. Given a nonempty, finite subset A of Γ, let f be a maximal element (with respect to $\preceq$) of A. We claim that f is an extremal point of A. Indeed, let $g \in \Gamma$ such that $fg \in A$ and $fg^{-1} \in A$. If g were nontrivial then we would have either $fg \succ f$ or $fg^{-1} \succ f$. However, this contradicts the maximality of $f \in A$.

For the proof of the converse implication, see Exercise 2.2.5. $\square$

Exercise 1.3.10. According to [113], every weakly diffuse group is ***diffuse***, that is, every finite subset of cardinality larger than one has at least two extremal points. Show this by contradiction.
Hint. Assume that A is a finite subset having only the identity as an extremal point. Then the same holds for A^{-1}. If A has more than one point, show that the set $B := A \cup A^{-1}$ has no extremal point.

1.4 General Properties

1.4.1 Left-orderable groups are torsion-free

Indeed, if $f \succ id$ (resp. $f \prec id$) for some left-order $\preceq$, then for all $n \in \mathbb{N}$ we have

$$f^n \succ \ldots f^2 \succ f \succ id \qquad (\text{resp. } f^n \prec \ldots \prec f^2 \prec f \prec f \prec id).$$

As we have seen in §1.2.1, the converse is true for Abelian and more generally for nilpotent groups, but does not hold for Abelian-by-finite groups: a classical relevant example is the crystallographic group

$$\Gamma = \langle f, g \colon a^2ba^2 = b, b^2ab^2 = a \rangle. \tag{1.1}$$

Here we give some properties of this group (for further details, see [16, 149] as well as [145, Chapter 13]). If we let $c := (ab)^{-1}$, then the subgroup $\langle a^2, b^2, c^2 \rangle$ is torsion-free, rank-3 Abelian, and normal. The corresponding quotient is isomorphic to the 4-Klein group. (An order-4, non-Abelian group). The crystallographic action on $\mathbb{R}^3$ is given by

$$a(x, y, x) = (x + 1, 1 - y, -z),$$

$$b(x, y, z) = (-x, y + 1, 1 - z),$$

$$c(x, y, z) = (1 - x, -y, z + 1).$$

To see that Γ is torsion-free, first notice that, since every element in the 4-Klein group has order 2, a nontrivial, finite-order element of Γ must have order 2. Now let $w \in \Gamma$ be nontrivial, say $w = a^{2i}b^{2j}c^{2k}a$ (the cases where the last factor is either b or c are similar). Then

$$w^2 \;=\; a^{2i}b^{2j}c^{2k}aa^{2i}b^{2j}c^{2k}a \;=\; a^{2i}b^{2j}c^{2k}a^{2i}b^{-2j}c^{-2k}a^2 \;=\; a^{4i+2} \;\neq\; id.$$

Finally, notice that for any choice of exponents ε, δ in $\{-1,+1\}$, the defining relations of Γ yield

$$(a^\varepsilon b^\delta)^2(b^\delta a^\varepsilon)^2 = a^\varepsilon b^{-\delta}b^{2\delta}a^\varepsilon b^{2\delta}a^\varepsilon b^\delta a^\varepsilon = a^\varepsilon b^{-\delta}a^{2\varepsilon}b^\delta a^{2\varepsilon}a^{-\varepsilon} = a^\varepsilon b^{-\delta}b^\delta a^{-\varepsilon} = id.$$

Obviously, this implies that no compatible choice of signs for a, b exists, hence Γ is not left-orderable. (For more conceptual proofs of a different nature, see either §1.4.3 or Example 3.2.11.)

Exercise 1.4.1. Consider the set G of triplets of the form (u,v,w), where each u,v,w is either an integer or of the form $\hat{m}$, with $m \in \mathbb{Z}$.
(i) Show that the rule

$$(u_1,v_1,w_1)(u_2,v_2,w_2) := (u_1 \oplus u_2, v_1 \oplus v_2, w_1 \oplus w_2),$$

where for m,n in $\mathbb{Z}$,

$$m \oplus n := m+n, \quad m \oplus \hat{n} := \widehat{m+n}, \quad \hat{m} \oplus n := \widehat{m-n}, \quad \hat{m} \oplus \hat{n} := m-n,$$

endows G with a group structure.
(ii) Show that the group Γ above identifies to the subgroup of G generated by $a := (1\hat{0}\hat{0})$ and $b := (\hat{0}1\hat{1})$.

Exercise 1.4.2. Show that every group having a locally-invariant order is torsion-free.

1.4.2 Unique roots and generalized torsion

Bi-orderable groups have a stronger property than absence of torsion, namely they have no ***generalized torsion***: If $f \neq id$, then no nontrivial product of conjugates of f is the identity. (In particular, no nontrivial element is conjugate to its inverse.) These groups also have the ***unique root property***: If $f^n = g^n$ for some integer n, then $f = g$. Once again, none of these properties characterizes bi-orderability (see [15, Example 4.3.1] and [6, 12], respectively). It seems to be unknown whether the former implies left-orderability. The latter does not: a concrete example (taken from [15, Chapter VII]) is

$$\Gamma_n \;=\; \big\langle g,h \colon fgfg^2\cdots fg^n = f^{-1}gf^{-1}g^2\cdots f^{-1}g^n = id \big\rangle, \quad \text{where } n \text{ is "large"}.$$

Example 1.4.3. As we saw in Example 1.3.4, the Klein-bottle group is left-orderable but does not satisfy the C.R.P., hence it is not bi-orderable. Another way to contradict bi-orderability consists in noticing that it has generalized torsion: $(a^{-1}ba)b = id$. Moreover, the unique root property fails: $(ba)^2 = a^2$, though $ba \neq a$.

Exercise 1.4.4. Prove that in any bi-orderable group, the following holds: If f commutes with a nontrivial power of g, then it commutes with g. Show that this is no longer true for left-orderable groups.

Exercise 1.4.5. Show that for bi-orderable groups, the normalizer of any finite subset coincides with its centralizer. Again, show that this is no longer true for left-orderable groups.

1.4.3 The Unique Product Property (U.P.P.)

A group Γ is said to have the U.P.P. if given any two finite subsets $\{g_i\}$, $\{h_j\}$, there exists $f \in \Gamma$ that may be written in a unique way as a product $g_i h_j$.

Every left-orderable group has the U.P.P. Indeed, given two finite subsets $A := \{g_1, \ldots, g_n\}$ and $B := \{h_1, \ldots, h_m\}$, let $f := g_i h_j$ be the element of AB that is maximal with respect to a fixed left-order on Γ. If f were equal to $g_{i'} h_{j'}$ for some i', j', then $h_j \succeq h_{j'}$, as otherwise $f = g_i h_j \prec g_i h_{j'}$ would contradict the maximality of f. Similarly, $h_{j'} \succeq h_j$. Thus $h_j = h_{j'}$, which yields $g_i = g_{i'}$.

Notice that the minimum element in AB has also a unique expression as above. This is coherent with a result from [165] contained in the following

Exercise 1.4.6. Show that U.P.P. implies a "double" U.P.P., in the sense that given any two finite subsets A, B such that $|A|+|B| > 2$, there exist at least *two* elements in AB which may be written in a unique way as a product ab, with $a \in A$ and $b \in B$. (Compare Exercise 1.3.10.)
<u>Hint.</u> Assume that a group Γ has the U.P.P. but only $ab \in AB$ has a unique representation in AB, and let $C := a^{-1}A$, $D := Bb^{-1}$, $E := D^{-1}C$, and $F := DC^{-1}$. Using the fact that, in CD, only $id = id \cdot id$ has a unique representation, show that, in EF, no element has unique representation.

Exercise 1.4.7. Show that a group satisfies the U.P.P. if and only if for any finite subset A there exist at least one (actually, two) elements in A^2 which may be written in a unique way as a product ab, with a, b in A.

Let us remark that groups with the U.P.P. are torsion-free. Indeed, if $f^n = id$ for some $f \neq id$, then the U.P.P. fails for the finite subsets $A = B = \{id, f, \ldots, f^{n-1}\}$. The converse to this remark is false. Indeed, as Promislow showed in [149], the crystallographic group of §1.4.1 does not satisfy the U.P.P. (see Exercise 1.4.11 below for the details; see also [152] for a different example using small cancellation techniques, and [104, 164] for more recent developments.) The following question was raised by Linnell; the recent (negative) answer from [104] will be discussed below.

Question 1.4.8. Are U.P.P.-groups left-orderable ?

Locally-invariant orders, diffuse groups, and the U.P.P. The U.P.P. is satisfied by all weakly diffuse groups (hence, by groups admitting a locally-invariant order; see Proposition 1.3.9). Indeed, given nontrivial finite subsets A, B of a weakly diffuse group Γ, let $f \in AB$ be an extremal point of AB. We claim that f may be written in a unique way as gh, with $g \in A$ and $h \in B$. Indeed, if $f = g_1 h_1 = g_2 h_2$, with g_1, g_2 in A and h_1, h_2 in B, then letting $h := h_1^{-1} h_2$ we have $fh = g_1 h_2 \in AB$ and $fh^{-1} = g_2 h_1 \in AB$. Since f is an extremal point of AB, this implies that $h = id$, which yields $h_1 = h_2$ and $g_1 = g_2$.

Below we elaborate on an example that shows that the answer to Linnell's question above is negative for a certain "large" group. (For amenable groups the situation is unclear, due to Theorem 4.1.7.) First notice that, by Example 1.3.7, isometry groups of hyperbolic metric spaces with "large displacement" admit locally-invariant orders, and hence satisfy the U.P.P. (Actually, a combination of remarkable recent results establishes that the π_1 of every closed, hyperbolic 3-manifold contains a finite-index group that is bi-orderable; see [2, 9, 60, 80, 98].) This motivates the following

Question 1.4.9. Does there exist a sequence of compact, hyperbolic 3-manifolds whose injectivity radius converges to infinity and whose π_1 are non left-orderable ? (Examples of non left-orderable 3-manifold groups appear in [29, 46].)

This question seems to have an affirmative but difficult solution. Indeed, it is not very hard to prove that, if Γ is the π_1 of a compact, hyperbolic 3-manifold with nontrivial first Betti number, then Γ is left-orderable (it is actually Conrad-orderable, in the terminology of §3.2; see [18]). A sequence of compact, hyperbolic 3-manifolds with trivial first Betti number and whose injectivity radius converges to infinite appears in [31]. However, it seems hard to adapt the methods therein to show that the π_1 of infinitely many of these manifolds are non left-orderable. Actually, an obvious difficulty comes from the fact that they are virtually orderable, as was mentioned above.

Despite the above, it was cleverly noticed by Dunfield and included in the work of Kionke and Raimbauld (see [104]) that there is an hyperbolic 3-manifold whose π_1 is known to be non left-orderable and for which a lower estimate of its injectivity radius allows applying the results described in Example 1.3.7, thus concluding that it admits a locally-invariant order, hence it satisfies the U.P.P. This yields a negative answer to Question 1.4.8 above. As Kionke and Raimbauld point out, the next question remains open:

Question 1.4.10. Does there exist a U.P.P.-group that is not weakly diffuse ?

Exercise 1.4.11. Consider the subset

$$A = B = \big\{(ba)^2, (ab)^2, a^2b, aba^{-1}, b, ab^{-1}a, b^{-1}, aba, ab^{-2}, b^2a^{-1}, a(ba)^2, bab, a, a^{-1}\big\}$$

of the crystallographic group $\Gamma = \langle a, b \colon a^2ba^2 = b, b^2ab^2 = a\rangle$ introduced in §1.4.1.

(i) Show that, via the identification of Exercise 1.4.1, this set becomes

$$(002),(00\underline{2}),(\hat{2}1\hat{1}),(\hat{2}\underline{1\hat{1}}),(\hat{0}1\hat{1}),(\hat{0}1\underline{\hat{1}}),(\hat{0}\underline{1}\hat{1}),(\hat{0}\underline{1\hat{1}}),(1\hat{2}\hat{0}),(\underline{1}\hat{2}\hat{0}),(1\hat{0}\underline{\hat{2}}),(\underline{1}\hat{0}\hat{2}),(1\hat{0}\hat{0}),(\underline{1}\hat{0}\hat{0}),$$

where $\underline{m}$ (resp. $\underline{\hat{m}}$) is written instead of $-m$ (resp. $\widehat{-m}$).

(ii) In the multiplication table below, check that the product of any two elements appears at least twice.

	(002)	$(00\underline{2})$	$(\hat{2}1\hat{1})$	$(\hat{2}\underline{1\hat{1}})$	$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1\hat{1}})$	$(1\hat{2}\hat{0})$	$(\underline{1}\hat{2}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{0})$
(002)	(004)	(000)	$(\hat{2}1\hat{3})$	$(\hat{2}\underline{1}\hat{1})$	$(\hat{0}1\hat{3})$	$(\hat{0}1\hat{1})$	$(\hat{0}\underline{1}\hat{3})$	$(\hat{0}\underline{1}\hat{1})$	$(1\hat{2}\hat{2})$	$(\underline{1}\hat{2}\hat{2})$	$(1\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(1\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{2})$
$(00\underline{2})$	(000)	$(00\underline{4})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}\underline{1\hat{3}})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}\underline{1\hat{1}})$	$(\hat{0}\underline{1\hat{3}})$	$(1\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(\underline{1}\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\underline{\hat{2}})$
$(\hat{2}1\hat{1})$	$(\hat{2}1\underline{\hat{1}})$	$(\hat{2}1\hat{3})$	(020)	(002)	(220)	(222)	(200)	(202)	$(\hat{1}\hat{3}1)$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}3)$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{3}\hat{1}1)$
$(\hat{2}\underline{1\hat{1}})$	$(\hat{2}\underline{1\hat{3}})$	$(\hat{2}\underline{1}\hat{1})$	$(00\underline{2})$	$(0\underline{2}0)$	$(20\underline{2})$	(200)	$(2\underline{22})$	$(2\underline{2}0)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{3}\underline{\hat{1}3})$	$(\hat{1}\underline{\hat{1}1})$	$(\hat{3}\underline{\hat{1}1})$
$(\hat{0}1\hat{1})$	$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\hat{3})$	$(\underline{2}20)$	$(\underline{2}02)$	(020)	(022)	(000)	(002)	$(\underline{\hat{1}}\hat{3}1)$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$
$(\hat{0}1\underline{\hat{1}})$	$(\hat{0}1\underline{\hat{3}})$	$(\hat{0}1\hat{1})$	$(\underline{2}2\underline{2})$	$(\underline{2}00)$	$(02\underline{2})$	(020)	$(00\underline{2})$	(000)	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\hat{1}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}\underline{3})$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$
$(\hat{0}\underline{1}\hat{1})$	$(\hat{0}\underline{1\hat{1}})$	$(\hat{0}\underline{1}\hat{3})$	$(\underline{2}00)$	$(\underline{22}2)$	(000)	(002)	$(0\underline{2}0)$	$(0\underline{2}2)$	$(\underline{\hat{1}}\hat{1}1)$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}\hat{1}}3)$	$(\hat{1}\underline{\hat{1}1})$	$(\underline{\hat{1}\hat{1}}1)$	$(\hat{1}\underline{\hat{1}}1)$
$(\hat{0}\underline{1\hat{1}})$	$(\hat{0}\underline{1\hat{3}})$	$(\hat{0}\underline{1}\hat{1})$	$(\underline{2}0\underline{2})$	$(\underline{22}0)$	$(00\underline{2})$	(000)	$(0\underline{22})$	$(0\underline{2}0)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\hat{1}\hat{1}\underline{1})$	$(\underline{\hat{1}\hat{1}}1)$	$(\hat{1}\underline{\hat{1}3})$	$(\underline{\hat{1}\hat{1}1})$	$(\hat{1}\underline{\hat{1}1})$
$(1\hat{2}\hat{0})$	$(1\hat{2}\underline{\hat{2}})$	$(1\hat{2}\hat{2})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{3}\hat{3}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(\hat{1}\hat{3}\underline{1})$	$(\hat{1}\hat{3}1)$	(200)	(000)	(222)	$(02\underline{2})$	(220)	(020)
$(\underline{1}\hat{2}\hat{0})$	$(\underline{1}\hat{2}\underline{\hat{2}})$	$(\underline{1}\hat{2}\hat{2})$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{3}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{3}\underline{1})$	$(\underline{\hat{1}}\hat{3}1)$	(000)	$(\underline{2}00)$	(022)	$(\underline{2}2\underline{2})$	(020)	$(\underline{2}20)$
$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\underline{\hat{4}})$	$(1\hat{0}\hat{0})$	$(\hat{3}\underline{\hat{1}3})$	$(\hat{3}\hat{1}\underline{1})$	$(\hat{1}\underline{\hat{1}3})$	$(\hat{1}\underline{\hat{1}1})$	$(\hat{1}\hat{1}\underline{3})$	$(\hat{1}\hat{1}\underline{1})$	$(2\underline{22})$	$(0\underline{22})$	(200)	$(00\underline{4})$	$(20\underline{2})$	$(00\underline{2})$
$(\underline{1}\hat{0}\hat{2})$	$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\hat{4})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}3)$	$(\underline{\hat{1}\hat{1}}1)$	$(\underline{\hat{1}\hat{1}}3)$	$(\underline{\hat{1}}\hat{1}1)$	$(\underline{\hat{1}}\hat{1}3)$	$(0\underline{2}2)$	$(\underline{22}2)$	(004)	$(\underline{2}00)$	(002)	$(\underline{2}02)$
$(1\hat{0}\hat{0})$	$(1\hat{0}\underline{\hat{2}})$	$(1\hat{0}\hat{2})$	$(\hat{3}\underline{\hat{1}1})$	$(\hat{3}\hat{1}1)$	$(\hat{1}\underline{\hat{1}1})$	$(\hat{1}\underline{\hat{1}}1)$	$(\hat{1}\hat{1}\underline{1})$	$(\hat{1}\hat{1}1)$	$(2\underline{2}0)$	$(0\underline{2}0)$	(202)	$(00\underline{2})$	(200)	(000)
$(\underline{1}\hat{0}\hat{0})$	$(\underline{1}\hat{0}\underline{\hat{2}})$	$(\underline{1}\hat{0}\hat{2})$	$(\hat{1}\underline{\hat{1}1})$	$(\hat{1}\hat{1}1)$	$(\underline{\hat{1}\hat{1}1})$	$(\underline{\hat{1}\hat{1}}1)$	$(\underline{\hat{1}}\hat{1}\underline{1})$	$(\underline{\hat{1}}\hat{1}1)$	$(0\underline{2}0)$	$(\underline{22}0)$	(002)	$(\underline{2}0\underline{2})$	(000)	$(\underline{2}00)$

Remark 1.4.12. As it was shown by Morris-Witte, finite-index subgroups of $\mathrm{SL}(3,\mathbb{Z})$ are non left-orderable (see Theorem 3.5.1). For large index, these groups are torsion-free, and it seems to be unknown whether they satisfy the U.P.P. By Exercise 1.3.6, the following question makes sense: Does there exist a norm on $\mathrm{SL}(3,\mathbb{Z})$ such that for finite but "large" index subgroups Γ one has either $\|fg\| > \|f\|$ or $\|fg^{-1}\| > \|f\|$ for every f, g in Γ, with $g \neq id$?

On Kaplansky's conjecture. A famous question due to Kaplansky (commonly referred to as the *Kaplansky conjecture*) asks whether the group algebra of a torsion-free group over a ring $\mathbb{A}$ has no zero-divisors provided $\mathbb{A}$ has no zero-divisors. (Even the case where $\mathbb{A} = \mathbb{Z}$ is open.) The restriction on the torsion is natural. Indeed,

$$f^n = id \quad \Longrightarrow \quad (f-1)(f^{n-1} + f^{n-2} + \cdots + f + 1) = f^n - 1 = 0. \tag{1.2}$$

It easily follows from the definitions that every group satisfying the U.P.P. also satisfies the conclusion of the Kaplansky conjecture. For the crystallographic group considered above, Kaplansky's conjecture is known to be true by different methods (see for instance [23, 64, 66]; see also [145, 117]).

Example 1.4.13. Consider the ***free Burnside group***

$$B(m,n) := \big\langle a_1, \ldots, a_m : W^n = id \text{ for every word } W \big\rangle.$$

It is known that for $m \geq 2$ and n odd and large-enough, $B(m,n)$ is infinite (actually, it is non-amenable; see [1]). Of course, every element in this group has finite order. However, it is still interesting to look for zero-divisors in its group algebra that are "nontrivial" (*i.e.* do not arise from an identity of the form (1.2)). For instance, according to [96], this is the case for

$$A := (1 + c + \ldots + c^{n-1})(1 - aba^{-1}), \quad B := (1-a)(1 + b + \ldots + b^{n-1}),$$

where $a := a_1$, $b := a_2$, and $c := aba^{-1}b^{-1}$. (Checking that $AB = 0$ is an easy exercise.)

1.4.4 More Combinatorial Properties

Recently, orderable groups have been considered as a natural framework to extend certain basic results of Additive Combinatorics (see [67, 132, 167] as general references). One of the most elementary ones is the inequality for product sets

$$|AB| \geq |A| + |B| - 1, \tag{1.3}$$

which holds for any finite subsets A, B of the integers (this is an easy exercise). In this regard, it is worth mentioning that this readily extends to finite subsets of left-orderable groups. Indeed, modulo multiplying B on the right by the largest possible element of type h^{-1}, where $h \in B$, we may assume that id is the smallest element of B. Then, if we order the elements in A (resp. B) in the form $g_1 \prec \ldots \prec g_n$ (resp. $id = h_1 \prec \ldots \prec h_m$), we have

$$g_1 \prec g_2 \prec \ldots \prec g_n \prec g_n h_2 \prec \ldots \prec g_n h_m.$$

Less trivially, (1.3) still holds for finite subsets of torsion-free groups, as it was proved by Kemperman in [100].

Theorem 1.4.14. *For all finite subsets A, B of a torsion-free group, it holds*

$$|AB| \geq |A| + |B| - 1.$$

Proof. To begin with, notice that the claim of the theorem trivially holds if either $|A|$ or $|B|$ equals 1. Moreover, changing A by $g^{-1}A$ and B by Bh^{-1} for $g \in A$ and $h \in B$, we reduce the general case to that where $id \in A \cap B$. Assume for a contradiction that A, B are finite subsets that do not satisfy (1.3) and for which the value of $m := |AB|$ is minimal, that of $n := |A| + |B|$ is maximal while attaining m, and that of $|A|$ is maximal while attaining both m, n (the extremal properties being realized among subsets containing id).

As $id \in A \cap B$, we also have

$$|AB| \geq |A| + |B| - |A \cap B|.$$

Hence, $|A \cap B| \geq 2$. Let H be the subsemigroup generated by $A \cap B$. We consider two different cases.

Case I. We have $Af \subset A$ for all $f \in A \cap B$.

Then, as $id \in A$, this implies $H \subset A$. Therefore, H is a finite subsemigroup of a group, hence a (finite) subgroup. As $|H| \geq 2$, this produces torsion elements.

Case II. There exists $f \in A \cap B$ such that Af is not contained in A.

Fixing such an f, let $A' := \{g \in A \colon gf \notin A\}$ and $B' := \{h \in B \colon fh \notin B\}$. There are two subcases to consider.

If $|A'| \geq |B'|$, then let $A^* := A \cup A'f$ and $B^* := B \setminus B'$. (Notice that $B \setminus B' \neq \emptyset$ since $id \notin B'$.) One easily checks that $A^*B^* \subset AB$, hence $|A^*B^*| \leq |AB|$. Moreover, $|A^*| = |A| + |A'f| = |A| + |A'|$ and $|B^*| = |B| - |B'|$, thus $|A^*| + |B^*| \geq$

$|A|+|B|$. Finally, $|A^*| > |A|$, as A' is nonempty. Therefore, by the choice of A, B, we must have
$$|A^*B^*| \geq |A^*| + |B^*| - 1,$$
hence
$$|AB| \geq |A^*B^*| \geq |A^*| + |B^*| - 1 \geq |A| + |B| - 1,$$
which is a contradiction.

If $|A'| < |B'|$, then let $A^* := A \setminus A'$ (which is nonempty as $id \notin A'$) and $B^* := B \cup fB$. Again, $A^*B^* \subset AB$, hence $|A^*B^*| \leq |AB|$. Moreover, $|A^*| = |A| - |A'|$ and $|B^*| = |B| + |B'|$ yield $|A^*| + |B^*| > |A| + |B|$. By the choice of A, B, this implies
$$|A^*B^*| \geq |A^*| + |B^*| - 1,$$
hence
$$|AB| \geq |A^*B^*| \geq |A^*| + |B^*| - 1 > |A| + |B| - 1,$$
which is again a contradiction. □

Example 1.4.15. By pursuing on the technique of proof above, Brailovsky and Freiman proved in [17] that equality arises if and only if A and B are *geometric progressions* on different sides. More precisely, there exist group elements f, g, h and non-negative integers n, m such that
$$A = \{g, gf, \ldots, gf^{n-1}\}, \qquad B = \{h, fh, \ldots, f^{m-1}h\}.$$
Showing such a claim for left-orderable groups is an straighforward exercise.

Below we present another proof of Theorem 1.4.14 following the ideas of Hamidoune [82] that is somewhat closer to the techniques of the next section. We refer to [83] for more details and furter developments, including an alternative proof of the Brailovsky-Freiman theorem above.

Another proof of Theorem 1.4.14. Given a finite subset B of a group Γ, for each finite subset $A \subset \Gamma$ we let $\partial^B A := AB \setminus A$ (compare (1.6)). Given a positive integer k, we say that a subset C is (B,k)***-critical*** if $|C| \geq k$ and
$$|\partial^B C| = \min\big\{|\partial^B A| \colon |A| \geq k\big\}.$$
We say that C is a (B,k)***-atom*** if it is a (B,k)-critical set of smallest cardinality.

Claim (i). If C is a (B,k)-atom and C' is (B,k)-critical, then either $C \subset C'$ or $|C \cap C'| \leq k-1$.

Indeed, assume C is not contained in C' and $|C \cap C'| \geq k$. Then, by definition,

$$|\partial^B C| < |\partial^B (C \cap C')|.$$

Let C_* (resp. C'_*) be the complement of $C \cup \partial^B C$ (resp. $C' \cup \partial^B C'$). On the one hand, we have

$$\begin{aligned} |\partial^B C \cap C'| + |\partial^B C \cap \partial^B C'| + |\partial^B C \cap C'_*| &= |\partial^B C| \\ &< |\partial^B (C \cap C')| \\ &\leq |C \cap \partial^B C'| + |\partial^B C \cap C'| + |\partial^B C \cap \partial^B C'|, \end{aligned}$$

hence $|\partial^B C \cap C'_*| < |C \cap \partial^B C'|$. On the other hand, we have

$$\begin{aligned} |\partial^B C' \cap C| + |\partial^B C' \cap \partial^B C| + |\partial^B C' \cap C_*| &\leq |\partial^B C'| \\ &\leq |\partial^B (C' \cap C)| \\ &\leq |C'_* \cap \partial^B C| + |\partial^B C' \cap C_*| + |\partial^B C' \cap \partial^B C|, \end{aligned}$$

hence $|\partial^B C' \cap C| \leq |C'_* \cap \partial^B C|$. These two conclusions are certainly in contradiction.

Claim (ii). If C is a (B, k)-atom and $g \neq id$, then $|C \cap gC| \leq k - 1$.

Indeed, the set gC is a (B, k)-atom as well. Moreover, we cannot have $gC \subset C$, otherwise g would be a torsion element. (If $gC \subset C$, then $gC = C$, so that g acts as a permutation of C and therefore $g^n h = h$ for all $h \in C$.)

Claim (iii). For all finite sets A, B, we have $|AB| \geq |A| + |B| - 1$.

Indeed, we may assume that B contains id. Let C be a $(B, 1)$-atom. Again, we may assume $id \in C$. If C contains another element g, then $|C \cap gC| \geq 1$, a contradiction to (ii). Hence, $C = \{id\}$. Therefore, for every (nonempty) finite subset A,

$$|AB| - |A| = |AB \setminus A| \geq |CB \setminus C| = |B \setminus \{id\}| = |B| - 1,$$

which shows the claim. □

Remark 1.4.16. It is conjectured that k-atoms have cardinality equal to k for torsion-free groups. This holds for instance for groups satisfying the U.P.P. (this is an easy exercise; see [83, Lemma 4] in case of problems).

A direct consequence of the preceding theorem is the inequality $|A^2| \geq 2|A| - 1$ for all finite subsets A of torsion-free groups. The next result from [68] improves this inequality for non-Abelian bi-orderable groups.

Theorem 1.4.17. *Let A be a finite subset of a bi-orderable group. If $|A^2| \leq 3|A| - 3$, then the subgroup generated by the elements in A is Abelian.*

Proof. The proof is by induction on $|A|$. If $|A| = 2$, say $A = \{f_1, f_2\}$, then $|A^2| = 3|A| - 3 = 2$ implies $A^2 = \{f_1^2, f_1 f_2 = f_2 f_1, f_2^2\}$, thus the group generated by f_1, f_2 is Abelian. Assume that the theorem holds for subsets of cardinality $\leq k$, and let $A := \{f_1, \ldots, f_{k+1}\}$, where $f_i \prec f_j$ holds whenever $i < j$ for a fixed bi-order $\preceq$ on the underlying group. We let i be the maximal index for which the subgroup generated by $B := \{f_1, \ldots, f_i\}$ is Abelian, and we assume that $i \leq k$. Then f_{i+1} does not belong to the subgroup generated by B. Moreover, there is $f \in B$ not commuting with f_{i+1}; we let f_j be the maximal such element. We also let $C := \{f_{i+1}, \ldots, f_{k+1}\}$. Assume throught that $|A^2| \leq 3|A| - 3$.

Claim (i). We have $|C^2| \leq 3|C| - 3$.

Indeed, using $\preceq$ (see also Exercise 1.4.5), one readily checks that

$$B^2 \cap (f_{i+1}B \cup B f_{i+1}) = \emptyset, \quad f_{i+1}B \neq B f_{i+1}, \quad C^2 \cap (B^2 \cup f_{i+1}B \cup B f_{i+1}) = \emptyset. \tag{1.4}$$

Therefore,

$$\begin{aligned} |C^2| &\leq |A^2| - |B^2| - |f_{i+1}B \cup B f_{i+1}| \\ &\leq (3|A| - 3) - (2|B| - 1) - (|B| + 1) \;=\; 3(|A| - |B|) - 3 \;=\; 3|C| - 3, \end{aligned}$$

as claimed.

By the inductive hypothesis, the group generated by C is Abelian. As f_j and $f_{i+1} \in C$ do not commute, we must have

$$C^2 \cap (f_j C \cup C f_j) = \emptyset. \tag{1.5}$$

Claim (ii). We have $B f_{i+1} \cap f_j C = \{f_j f_{i+1}\}$. In particular, $|B f_{i+1} \cup f_j C| = k$.

Indeed, assume $f_m f_{i+1} = f_j f_n$, with $f_m \in B$ and $f_n \in C$. If $f_m \prec f_j$, then $f_{i+1} \succ f_n$, which is impossible since f_{i+1} is the smallest element in C. If $f_m \succ f_j$, then f_m commutes with f_{i+1}, and so does $f_j = f_m f_{i+1} f_n^{-1}$, which is absurd.

Claim (iii). We have $B^2 \cap (f_j C \cup C f_j) = \emptyset$.

Indeed, assume $f_m f_n = f_j f_\ell$ holds for f_m, f_n in B and $f_\ell \in C$. Since $f_n \prec f_\ell$, we must have $f_m \succ f_j$. Moreover, as B generates an Abelian group, $f_m f_n = f_n f_m$, hence also $f_n \succ f_j$. Therefore, both f_m, f_n commute with f_{i+1}, and so does $f_j = f_m f_n f_\ell^{-1}$, which is absurd. This shows that $B^2 \cap f_j C = \emptyset$. That $B^2 \cap C f_j = \emptyset$ is proved similarly.

Claim (iv). We have $A^2 = B^2 \cup C^2 \cup Bf_{i+1} \cup f_jC$.

It follows from the above that

$$\begin{aligned}|B^2 \cup C^2 \cup Bf_{i+1} \cup f_jC| &= |B^2| + |C^2| + |Bf_{i+1} \cup f_jC| \\ &\geq (2i-1) + (2(k-i+1)-1) + k \;=\; 3(k+1)-3.\end{aligned}$$

By the hypothesis $|A^2| \leq 3|A| - 3$, this implies the claim.

Notice that $f_{i+1}f_j \notin B^2$ and $f_{i+1}f_j \notin C^2$, by (1.4). A contradiction is then provided by the next two claims below.

Claim (v). We have $f_{i+1}f_j \notin Bf_{i+1}$.

Indeed, assume $f_{i+1}f_j = f_mf_{i+1}$ for $f_m \in B$. If $f_m \succ f_j$, then it commutes with f_{i+1}. Thus, $f_{i+1}f_j = f_{i+1}f_m$, hence $f_j = f_m$, which is absurd. Suppose $f_m \prec f_j$. By Exercise 1.4.5, there exists $f_n \in B$ such that $f_{i+1}f_n \notin Bf_{i+1}$. Thus, necessarily, $f_n \neq f_j$. We cannot have $f_n \succ f_j$, otherwise $f_{i+1}f_n = f_nf_{i+1} \in Bf_{i+1}$, a contradiction. Therefore, $f_n \prec f_j$.

By (1.4), $f_{i+1}f_n \notin B^2 \cup C^2$. Hence, by Claim (iv), we have $f_{i+1}f_n \in f_jC$, so that there is $f_\ell \in C$ such that $f_{i+1}f_n = f_jf_\ell$. As B generates an Abelian subgroup,

$$f_jf_\ell f_j = f_{i+1}f_nf_j = f_{i+1}f_jf_n.$$

As $f_n \prec f_j$, this implies $f_jf_\ell \prec f_{i+1}f_j = f_mf_{i+1}$. However, this is impossible, since $f_j \succ f_m$ and $f_\ell \succeq f_{i+1}$.

Claim (vi). We have $f_{i+1}f_j \notin f_jC$.

Assume $f_{i+1}f_j = f_jf_m$ holds for a certain $f_m \in C$. By Exercise 1.4.5, there exists $f_n \in C$ such that $f_nf_j \notin f_jC$. As $f_{i+1}f_j \in f_jC$, it holds $f_{i+1} \prec f_n$. Moreover, by (1.5) and Claim (iii), $f_nf_j \notin B^2 \cup C^2$. Thus, by Claim (iv), we have $f_nf_j \in Bf_{i+1}$. Let $f_\ell \in B$ be such that $f_nf_j = f_\ell f_{i+1}$.

Notice that $f_\ell \neq f_j$, otherwise f_nf_j would belong to f_jC. If $f_\ell \succ f_j$, then it commutes with f_{i+1}, and so does $f_j = f_n^{-1}f_{i+1}f_\ell$, which is absurd. If $f_\ell \prec f_j$, then, as f_{i+1} and f_n commute,

$$f_{i+1}f_\ell f_{i+1} = f_{i+1}f_nf_j = f_nf_{i+1}f_j.$$

As $f_{i+1} \prec f_n$, this implies $f_\ell f_{i+1} \succ f_{i+1}f_j = f_jf_m$. However, this is impossible, since $f_\ell \prec f_j$ and $f_{i+1} \preceq f_m$. $\square$

Example 1.4.18. Following [68], let $A = A_k$ be the subset of the Baumslag-Solitar group $BS(1,2) := \langle a, b \colon aba^{-1} = b^2 \rangle$ given by $A := \{a, ab, ab^2, \ldots, ab^{k-1}\}$. Check that $|A^2| = 3|A| - 2$, yet $BS(1,2)$ is bi-orderable and non-Abelian.

Exercise 1.4.19. Let Γ be the Klein-bottle group $\langle a, b \colon aba^{-1} = b^{-1} \rangle$ (*c.f.* Example 1.3.4). Check that the set $A = A_k := \{a, ab, ab^{-1}, \ldots ab^{k-1}\}$ satisfies $|A^2| = 2|A| - 1$, yet Γ is left-orderable and non-Abelian.

Exercise 1.4.20. Using Brailovsky-Freiman's theorem (*c.f.* Example 1.4.15), prove that if A is a subset of a torsion-free group satisfying $|A^2| = 2|A| - 1$, then A generates either an Abelian subgroup or a group isomorphic to the Klein-bottle group.

1.4.5 Isoperimetry and Left-Orderable Groups

The aim of this section is to develop some ideas recently introduced by Gromov in [79]. Let us begin with the notion of isoperimetric profile, due to Vershik.

Let Γ be a finitely-generated group acting on a set X, and let $\mathcal{G}$ be a finite generating system containing id. For $Y \subset X$, its ***boundary*** (with respect to $\mathcal{G}$) is defined as

$$\partial_{\mathcal{G}} Y := \mathcal{G}Y \setminus Y, \tag{1.6}$$

where $\mathcal{G}Y := \{g(y) \colon g \in \mathcal{G},\ y \in Y\}$. The maximal function $I \colon \mathbb{N} \to \mathbb{N}$ satisfying

$$|\partial_{\mathcal{G}} Y| \geq I\big(|Y|\big)$$

for all finite $Y \subset X$ is called the ***combinatorial isoperimetric profile*** of the Γ-action, and will be denoted by $I_{(X;\Gamma,\mathcal{G})}$.

An important case arises when $X = \Gamma$ is endowed with the action by left-translations. In this case, the isoperimetric profile is denoted $I_{(\Gamma,\mathcal{G})}$. We list below some important properties.

Subaditivity. If Γ is infinite, then for all r_1, r_2 we have

$$I_{(\Gamma,\mathcal{G})}(r_1 + r_2) \ \leq\ I_{(\Gamma,\mathcal{G})}(r_1) + I_{(\Gamma,\mathcal{G})}(r_2).$$

Indeed, choose $Y_i \subset \Gamma$ such that $|Y_i| = r_i$ and $\big|\partial_{\mathcal{G}} Y_i\big| = I_{(\Gamma,\mathcal{G})}(r_i)$, with $i \in \{1, 2\}$. Since Γ is infinite, after "moving" Y_2 keeping Y_1 fixed,[2] we may assume that Y_1 and Y_2 are disjoint and $\partial_{\mathcal{G}}(Y_1 \sqcup Y_2) = \partial_{\mathcal{G}} Y_1 \sqcup \partial_{\mathcal{G}} Y_2$. This yields

$$I_{(\Gamma,\mathcal{G})}(r_1 + r_2) \ \leq\ \big|\partial_{\mathcal{G}}(Y_1 \sqcup Y_2)\big| \ =\ \big|\partial_{\mathcal{G}}(Y_1)\big| + \big|\partial_{\mathcal{G}}(Y_2)\big| \ =\ I_{(\Gamma,\mathcal{G})}(r_1) + I_{(\Gamma,\mathcal{G})}(r_2).$$

***I* is non-decreasing under extensions.** If $\Gamma \subset \Gamma_1$ are infinite groups and $\mathcal{G} \subset \mathcal{G}_1$, then for all r,

$$I_{(\Gamma_1,\mathcal{G}_1)}(r) \geq I_{(\Gamma,\mathcal{G})}(r).$$

[2]Notice that $h(\partial_{\mathcal{G}} Y) = \partial_{\mathcal{G}}(hY)$, for all $h \in \Gamma$ and all $Y \subset \Gamma$.

Indeed, any finite subset $Y \subset \Gamma_1$ may be decomposed as a disjoint union $Y = \bigsqcup_{i=1}^{k} Y_i$, where the points in each Y_i are in the same class modulo Γ. Since $\mathcal{G} \subset \Gamma$,

$$\partial_{\mathcal{G}_1} Y \supset \partial_{\mathcal{G}} Y = \bigsqcup_{i=1}^{k} \partial_{\mathcal{G}} Y_i.$$

Thus,

$$\left|\partial_{\mathcal{G}_1} Y\right| \geq \sum_{i=1}^{k} \left|\partial_{\mathcal{G}} Y_i\right| \geq \sum_{i=1}^{k} I_{(\Gamma,\mathcal{G})}\big(|Y_i|\big) \geq I_{(\Gamma,\mathcal{G})}\Big(\sum_{i=1}^{k} |Y_i|\Big) = I_{(\Gamma,\mathcal{G})}\big(|Y|\big).$$

I is non-increasing under homomorphisms. If $\Phi : \Gamma \to \underline{\Gamma}$ is a surjective group homomorphism and $\underline{\mathcal{G}} = \Phi(\mathcal{G})$, then, for all r,

$$I_{(\Gamma,\mathcal{G})}(r) \geq I_{(\underline{\Gamma},\underline{\mathcal{G}})}(r).$$

Indeed, given a finite subset $Y \subset \Gamma$, we let $\underline{Y}_m := \big\{ f \in \underline{\Gamma} \colon |\Phi^{-1}(f) \cap Y| \geq m \big\}$. Clearly, $|Y| = \sum_{m \geq 1} |\underline{Y}_m|$. If we are able to show that

$$|\partial_{\mathcal{G}} Y| \geq \sum_{m \geq 1} \left|\partial_{\underline{\mathcal{G}}} \underline{Y}_m\right|, \tag{1.7}$$

then this would yield

$$|\partial_{\mathcal{G}} Y| \;\geq\; \sum_{m \geq 1} I_{(\underline{\Gamma},\underline{\mathcal{G}})}\big(|\underline{Y}_m|\big) \;\geq\; I_{(\underline{\Gamma},\underline{\mathcal{G}})}\Big(\sum_{m \geq 1} |\underline{Y}_m|\Big) \;=\; I_{(\underline{\Gamma},\underline{\mathcal{G}})}\big(|Y|\big),$$

thus showing our claim. Now, to show (1.7), every $\underline{f}$ in $\partial_{\underline{\mathcal{G}}} \underline{Y}_m$ may be written as $\Phi(g)\underline{h}$ for some $g \in \mathcal{G}$ and $\underline{h} \in \underline{Y}_m$. By definition, $\left|\Phi^{-1}(\underline{h}) \cap Y\right| \geq m$, and $\left|\Phi^{-1}(\underline{f}) \cap Y\right| < m$. Thus, there must be some g_f in $\mathcal{G}$ and $h \in \Phi^{-1}(\underline{h})$ such that $\underline{f} = \Phi(g_f h)$, with $h \in Y$ and $g_f h \notin Y$. The correspondence $\underline{f} \mapsto g_f h$ from $\bigcup_m \partial_{\underline{\mathcal{G}}} \underline{Y}_m$ into $\partial_{\mathcal{G}} Y$ is injective, because $\Phi(g_f h) = \underline{f}$. This shows (1.7).

Suppose that Γ acts on a linear space $\mathbb{V}$. Given a subspace $D \subset \mathbb{V}$ and a finite generating set $\mathcal{G} \subset \Gamma$ containing id, we define its ***boundary*** as the quotient space

$$\partial_{\mathcal{G}} D := \mathcal{G} \cdot Y \Big/ D,$$

where $\mathcal{G} \cdot D$ is the subspace generated by $\{g(v) : g \in \mathcal{G}, v \in D\}$. Using now the notation $|\cdot|$ for the dimension of a vector space, we define the ***linear isoperimetric profile*** of the Γ-action on $\mathbb{V}$ as the maximal function I satisfying, for all finite dimensional subspaces $D \subset \mathbb{V}$,

$$|\partial_{\mathcal{G}} D| \geq I\big(|D|\big).$$

We denote this function by $I^*_{(\mathbb{V};\Gamma,\mathcal{G})}$. In the special case where $\mathbb{V}$ is the group algebra $\mathbb{R}(\Gamma)$ (viewed as the vector space of finitely-supported, real-valued functions on Γ), we simply use the notation $I^*_{(\Gamma,\mathcal{G})}$.

As is the case of $I_{(\Gamma,\mathcal{G})}$, the function $I^*(\Gamma,\mathcal{G})$ is subadditive, as well as non-increasing under group extensions. It is unclear whether it is non-increasing under group homomorphisms. However, we will see that if the target group is left-orderable, then this property holds (*c.f.* Proposition 1.4.24).

There is a simple relation between I and I^* for all finitely-generated groups.

Proposition 1.4.21. *For every finitely-generated group Γ and all $r \geq 0$,*

$$I_{(\Gamma,\mathcal{G})}(r) \;\geq\; I^*_{(\Gamma,\mathcal{G})}(r).$$

Proof. To each finite subset $Y \subset \Gamma$ we may associate the subspace $D_Y := Y^{\mathbb{R}}$ formed by all the functions whose support is contained in Y. We clearly have $|Y| = |D_Y|$ and $|\partial_{\mathcal{G}} Y| = |\partial_{\mathcal{G}} D_Y|$, which easily yields the claim. $\square$

The opposite inequality is not valid for all groups, as the following example shows.

Example 1.4.22. If Γ is a group containing a nontrivial finite subgroup Γ_0 and $\Gamma_0 \subset \mathcal{G}$, then for any finite dimension subspace $D \subset \mathbb{R}(\Gamma)$ of finitely-supported functions which are constant along the cosets of Γ_0, we have $I^*_{(\Gamma,\mathcal{G})}\big(|D|\big) < I_{(\Gamma,\mathcal{G})}\big(|D|\big)$. If Γ is infinite, this yields $I^*_{(\Gamma,\mathcal{G})}(r) < I_{(\Gamma,\mathcal{G})}(r)$, for all $r \geq 0$.

Despite the preceding example, the equivalence between I and I^* holds for left-orderable groups. (It is an open question whether this remains true for torsion-free groups; for groups with torsion, see Example 1.4.25.)

Theorem 1.4.23. *If Γ is a finitely-generated left-orderable group, then for every finite generating system $\mathcal{G}$ containing id, one has $I_{(\Gamma,\mathcal{G})} = I^*_{(\Gamma,\mathcal{G})}$.*

To show this theorem, we will use an argument which is "dual" to that of Proposition 1.4.21.

Isoperimetric Domination (ID). Let Γ be a group acting on a set X and on a vector space $\mathbb{V}$. Suppose there exists an *equivariant* map $D \mapsto Y_D$ from the Grassmanian $\mathrm{Gr}_{\mathbb{V}}$ of finite dimensional subspaces of $\mathbb{V}$ into the family of subsets of X such that:

(i) $|D| = |Y_D|$, for all $D \in \mathrm{Gr}_{\mathbb{V}}$;

(ii) $|\mathrm{span}(\bigcup_i D_i)| \geq |\bigcup_i Y_{D_i}|$, for every finite family $\{D_i\} \subset \mathrm{Gr}_{\mathbb{V}}$.

We claim that, in this case, for every finite generating set $\mathcal{G}$ containing id and all $r \geq 0$,

$$I_{(X;\Gamma,\mathcal{G})}(r) \geq I^*_{(\mathbb{V};\Gamma,\mathcal{G})}(r). \tag{1.8}$$

Indeed, taking any D so that $|D| = r$, we have $|Y_D| = r$ and

$$\begin{aligned}|\partial_{\mathcal{G}} D| = \big|\mathcal{G} \cdot D / D\big| = \Big|\mathrm{span}\big(\bigcup_{g \in \mathcal{G}} gD\big)\Big| - |D| \geq \\ \geq \Big|\bigcup_{g \in \mathcal{G}} Y_{gD}\Big| - |D| \geq \Big|\bigcup_{g \in \mathcal{G}} g(Y_D)\Big| - |Y_D| = |\partial_{\mathcal{G}} Y_D|,\end{aligned}$$

which easily yields (1.8).

ID for left-ordered groups. In view of the above discussion, in order to prove Theorem (1.4.23) it suffices to exhibit an ID from $\mathrm{Gr}_{\mathbb{R}(\Gamma)}$ to 2^Γ. The construction proceeds as follows. Fix a left-order $\preceq$ on Γ. To each finitely-supported, real-valued function φ on Γ, we may associate the minimum $g \in \Gamma$ in its support (where the *minimum* is taken with respect to $\preceq$). Denote this point by g_φ. Now, if $D \subset \mathbb{V}$ is a finitely-dimensional subspace, then the number of points g_φ which may appear for some $\varphi \in D$ is finite. In fact, a simple "passing to a triangular basis" argument using the left-order shows that the cardinality of this subset $Y_D \subset \Gamma$ equals $|D|$, so property (i) above is satisfied. Property (ii) is also easily verified, thus concluding the proof.

Proposition 1.4.24. *Let $\Phi \colon \Gamma \to \underline{\Gamma}$ be a surjective group homomorphism. If $\underline{\Gamma}$ is left-orderable, then denoting $\underline{\mathcal{G}} = \Phi(\mathcal{G})$ we have, for all $r \geq 0$,*

$$I^*_{(\Gamma,\mathcal{G})}(r) \;\geq\; I^*_{(\underline{\Gamma},\underline{\mathcal{G}})}(r).$$

Proof. Fix a left-order $\preceq$ on $\underline{\Gamma}$, and for each $\underline{g} \in \underline{\Gamma}$ denote

$$\mathbb{A}_{\prec \underline{g}} := \{\varphi \in \mathbb{A}(\Gamma) \colon g_\varphi \prec \underline{g}\}, \qquad \mathbb{A}_{\preceq \underline{g}} := \{\varphi \in \mathbb{A}(\Gamma) \colon g_\varphi \preceq \underline{g}\}.$$

Given a finitely dimensional subspace $D \subset \mathbb{A}(\Gamma)$, define $U = U_D \colon \Gamma \to \mathbb{N}_0$ by

$$U(g) := dim\Big(\mathbb{A}_{\preceq \Phi(g)} \cap D \big/ \mathbb{A}_{\prec \Phi(g)} \cap D\Big).$$

Let S_U be the subgraph of U, that is,

$$S_U := \big\{(g,n) \in \Gamma \times \mathbb{N} \colon \; U(g) \geq n\big\}.$$

Since Γ naturally acts on $\Gamma \times \mathbb{N}$ and the action is free on each level, we have

$$\begin{aligned} |\partial_{\mathcal{G}} S_U| = \sum_{m \geq 1} \big|\partial_{\mathcal{G}}\big(S_U \cap (\Gamma \times \{m\})\big)\big| &\geq \sum_{m\geq 1} I_{(\Gamma,\mathcal{G})}\big(\big|S_U \cap (\Gamma \times \{m\})\big|\big) \\ &\geq I_{(\Gamma,\mathcal{G})}\Big(\sum_{m \geq 1} |S_U \cap (\Gamma \times \{m\})|\Big) = I_{(\Gamma,\mathcal{G})}\big(|S_U|\big). \end{aligned}$$

Moreover, one easily convinces that $|S_U| = |D|$. Putting all of this together, we obtain

$$|\partial_{\mathcal{G}} D| \;=\; |\partial_{\mathcal{G}} S_U| \;\geq\; I_{(\Gamma,\mathcal{G})}\big(|S_U|\big) \;=\; I_{(\Gamma,\mathcal{G})}\big(|D|\big) \;=\; I^*_{(\Gamma,\mathcal{G})}\big(|D|\big),$$

where the last equality comes from Theorem 1.4.23. □

Example 1.4.25. Following [5], we consider the ***lamplighter group*** $\Gamma := \mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \ltimes \oplus_{i \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, where the action of $\mathbb{Z}$ consists in shifting coordinates. We view elements of Γ as pairs (t,f), where $t \in \mathbb{Z}$ and f is a finitely-supported function from $\mathbb{Z}$ into $\mathbb{Z}/2\mathbb{Z}$. As a generating set we consider $\mathcal{G} := \{id, (0,\delta_0), (\pm 1, 0)\}$, where δ_0 stands for the Dirac delta at 0. The subspaces

$$D_n := \Big\langle \sum_{supp(f) \subset \{1,\ldots,n\}} (t,f) \colon t \in \{1,\ldots,n\} \Big\rangle$$

satisfy $|D_n| = n$ and $|\partial_{\mathcal{G}} D_n| = 2$. However, every finite subset $Y \subset \Gamma$ for which $|\partial_{\mathcal{G}} Y|/|Y| \leq 2/n$ must have at least $2^{\lambda n}$ points for a certain constant $\lambda > 0$. Indeed, this follows from that the ball of radius $2n+2$ in Γ has more than 2^n points as an application of the Saloff-Coste's isoperimetric inequality [171]: If Y satisfies $|\partial_{\mathcal{G}} Y|/|Y| \leq 1/n$, then its cardinal is greater than or equal to a half of the cardinal of a ball of radius $n/2$. (See [79] for an elementary proof of this inequality.)

Remark 1.4.26. In the example above, the group Γ not only contains torsion elements but is also amenable. In this direction, let us point out that a nice theorem due to Bartholdi [5] establishes that for non-amenable groups, the linear isoperimetric profile cannot behave sublinearly along any subsequence (see [79] for an alternative proof using orderings !).

Chapter 2

A PLETHORA OF ORDERS

2.1 Producing New Left-Orders

2.1.1 Convex extensions

A subset S of a left-ordered group $(\Gamma, \preceq)$ is said to be ***convex*** (with respect to $\preceq$) if, for all $f \prec g$ in S, every element $h \in \Gamma$ satisfying $f \prec h \prec g$ belongs to S. If S is a subgroup, this is equivalent to that $g \in S$ for all $g \in \Gamma$ such that $id \prec g \prec f$ for some $f \in S$.

The family of $\preceq$-convex subgroups is linearly ordered (by inclusion). More precisely, if Γ_0 and Γ_1 are convex (with respect to $\preceq$), then either $\Gamma_0 \subset \Gamma_1$ or $\Gamma_1 \subset \Gamma_0$. Moreover, the union and the intersection of any family of convex subgroups is a convex subgroup.

Example 2.1.1. For each $g \in \Gamma$ it is usual to denote Γ_g (resp. Γ^g) the largest (resp. smallest) convex subgroup which does not (resp. does) contain g. The inclusion $\Gamma_g \subset \Gamma^g$ is referred to as the ***convex jump*** associated to g. In general, Γ_g fails to be normal in Γ^g. Normality holds for bi-orders, and in such a case the quotient Γ^g/Γ_g is Abelian (see §3.2.3 for the study of left-orders for which this holds for every g).

Example 2.1.2. It is not difficult to produce examples of group left-orders without maximal *proper* convex subgroups: consider for instance a lexicographic left-order on $\mathbb{Z}^{\mathbb{N}}$. Nevertheless, if the underlying group Γ is finitely generated, such a maximal subgroup always exists. Indeed, given a system of generators $id \prec g_1 \prec \ldots \prec g_k$, let Γ_0 be the maximal convex subgroup that does not contain g_k. Then $\Gamma_0 \subsetneq \Gamma$. Moreover, if Γ_1 is a convex subgroup containing Γ_0, then, by definition, $g_k \in \Gamma_1$. By convexity, all the g_i's belong to Γ_1, hence $\Gamma_1 = \Gamma$. Thus, Γ_0 is the maximal proper convex subgroup.

In the dynamical terms of §1.1.3, convex subgroups are characterized by the following

Proposition 2.1.3. *Let $(\Gamma, \preceq)$ be a countable left-ordered group, and let Γ_* be a convex subgroup. Then, in the dynamical realization of $\preceq$, there is a bounded, Γ_*-invariant interval I with the property that $g(I) \cap I = \emptyset$ for every $g \in \Gamma \setminus \Gamma_*$.*

Conversely, let Γ be a group acting by orientation-preserving homeomorphisms of the real line without global fixed points. Suppose that there is an interval I with the property that, for all $g \in \Gamma$, the intersection $g(I) \cap I$ either is empty or coincides with I. Then, in any dynamically-lexicographic order induced from a sequence (x_n) starting with $x_1 \in I$, the stabilizer $Stab_\Gamma(I)$ is a proper convex subgroup.

Proof. Suppose $(\Gamma, \preceq)$ is a countable left-ordered group having Γ_* as a proper convex subgroup. Consider its dynamical realization, and let $a := \inf\{h(0) \mid h \in \Gamma_*\}$ and $b := \sup\{h(0) \mid h \in \Gamma_*\}$. Then $I := (a, b)$ is a bounded interval fixed by Γ_*. Moreover, if $g \in \Gamma$ is such that $g(I) \cap I \neq \emptyset$, then there is $h \in \Gamma_*$ such that $gh(0) \in I$. By convexity, this implies that $gh \in \Gamma_*$, which yields $g \in \Gamma_*$.

Conversely, suppose that for a Γ-action on the line, there is a bounded interval I satisfying that $g(I) \cap I$ either is empty or coincides with I, for each $g \in \Gamma$. Let $\preceq$ be a dynamically-lexicographic order induced from a sequence (x_n), with $x_1 \in I$. If $g \in \Gamma$ satisfies $id \prec g \prec h$ for some $h \in Stab_\Gamma(I)$, then by definition we have $x_1 \leq g(x_1) \leq h(x_1)$, hence $g(I) = I$. Therefore, $Stab_\Gamma(I)$ is $\preceq$-convex. □

The convex extension procedure. Let Γ_* be a $\preceq$-convex subgroup of Γ, and let $\preceq_*$ be any left-order on Γ_*. The ***extension of*** $\preceq_*$ ***by*** $\preceq$ is the order relation $\preceq'$ on Γ whose positive cone is $(P^+_\preceq \setminus \Gamma_*) \cup P^+_{\preceq_*}$.

One easily checks that $\preceq'$ is also a left-invariant total order relation, and that Γ_* remains convex in Γ with respect to $\preceq'$. Moreover, the family of $\preceq'$-convex subgroups of Γ is formed by the $\preceq_*$-convex subgroups of Γ_* and the $\preceq$-convex subgroups of Γ that contain Γ_*.

Example 2.1.4. Let $(\Gamma, \preceq)$ be a left-ordered group, and Γ_* a $\preceq$-convex subgroup. The extension of (the restriction to Γ_* of) $\overline{\preceq}$ by $\preceq$ will be referred as the left-order obtained by ***flipping*** the convex subgroup Γ_*. An important case of this seemingly innocuous construction arises for braid groups; see the end of §2.2.3.

Remark 2.1.5. As we have already pointed out, convex subgroups are not necessarily normal. In case of normality, the left-order passes to the quotient. Conversely, if Γ contains a normal subgroup Γ_* such that both Γ_* and Γ/Γ_* are left-orderable, then Γ

admits a left-order for which Γ_* is convex. Indeed, letting $\preceq_*$ and $\preceq_0$ be left-orders on Γ_* and Γ/Γ_*, respectively, we may define $\preceq$ on Γ by letting $f \prec g$ if either $f\Gamma_* \prec_0 g\Gamma_*$, or $f\Gamma_* = g\Gamma_*$ and $f^{-1}g$ is $\preceq_*$-positive.

Thus, the extension of a left-orderable group by a left-orderable one is left-orderable. Using Example 1.1.1, this implies that the ***wreath product*** $\Gamma_1 \wr \Gamma_2 := \bigoplus_{\Gamma_2} \Gamma_1 \rtimes \Gamma_2$ of two left-orderable groups is left-orderable.

In dynamical terms, convex subgroups are relevant because of the following

Remark 2.1.6. Let $(\Gamma, \preceq)$ be a left-ordered group, and let Γ_* be a $\preceq$-convex subgroup. The space of left cosets $\Omega = \Gamma/\Gamma_*$ carries a natural total order $\leq$, namely $f\Gamma_* < g\Gamma_*$ if $fh_1 \prec gh_2$ for some h_1, h_2 in Γ_* (this definition is independent of the choice of h_1, h_2 in Γ_*). Moreover, the action of Γ by left-translations on Ω preserves this order. An important case (to be treated in §3.5) arises when Γ_* is the maximal proper convex subgroup (whenever it exists); see Example 2.1.2.

The preceding construction allows showing the following useful

Proposition 2.1.7. *Let Γ be a left-orderable group, and let $\{\Gamma_\lambda : \lambda \in \Lambda\}$ be a family of subgroups each of which is convex with respect to a left-order $\preceq_\lambda$. Then there exists a left-order on Γ for which the subgroup $\bigcap_\lambda \Gamma_\lambda$ is convex.*

For the proof, we need a lemma that is interesting by itself.

Lemma 2.1.8. *Let Γ be a group acting faithfully on a totally ordered space $(\Omega, \leq)$ by order-preserving transformations. Then for every $\overline{\Omega} \subset \Omega$, there is a left-order on Γ for which the stabilizer of $\overline{\Omega}$ is a convex subgroup.*

Proof. Proceed as in §1.1.3 using a well-order $\leq_{wo}$ on Ω for which $\overline{\Omega}$ is an initial segment. $\square$

Proof of Proposition 2.1.7. As we saw in Example 2.1.6, each space of cosets Γ/Γ_λ inherits a total order $\leq_\lambda$ that is preserved by the left action of Γ. Fix a well-order $\leqslant_{wo}$ on Λ, and let $\Omega := \prod_{\lambda \in \Lambda} \Gamma_\lambda \times \Gamma$ be endowed with the associate dynamically-lexicographic total order $\leq$. This means that $([g_\lambda], g) \leq ([h_\lambda], h)$ if either the smallest (according to $\leqslant_{wo}$) index λ such that $[g_\lambda] \neq [h_\lambda]$ is such that $[g_\lambda] >_\lambda [h_\lambda]$, or the classes of g_λ and h_λ (with respect to Γ_λ) are equal for every λ and $g \prec h$. The left action of Γ on Ω is faithful and preserves this order. Since the stabilizer of $([id]_\lambda)_{\lambda \in \Lambda} \times \Gamma$ coincides with $\bigcap_\lambda \Gamma_\lambda$, the proposition follows from the preceding lemma. $\square$

2.1.2 Free products

As we have seen in §1.2.3, free groups are bi-orderable. Actually, a much more general statement involving free products holds.

Theorem 2.1.9. *The free product of an arbitrary family of bi-orderable groups is bi-orderable. Moreover, given bi-orders on each of the free factors, there is a bi-order on the free product that extends these bi-orders.*

Let us point out that a similar statement holds for left-orderability. However, the proof is much simpler. Indeed, let $\Gamma = \star\Gamma_\lambda$ be a free product of left-orderable groups. Then the direct sum $\oplus_\lambda \Gamma_\lambda$ is left-orderable. Moreover, the kernel of the natural homomorphism from Γ to $\oplus_\lambda \Gamma_\lambda$ is known to be a free group. Since free groups are left-orderable, Γ itself is left-orderable. (An alternative –dynamical– argument is contained in the proof of Theorem 2.2.27.)

The statement concerning bi-orderability is more subtle. For instance, the argument above does not apply, as the bi-orders in the free kernel are not necessarily invariant under conjugacy by elements of Γ. Although we are mostly concerned with left-orders here, we will reproduce one of the two available proofs of Theorem 2.1.9 in order to give some insight in Question 2.2.36 further on. The reason is that this proof (which is the original one; see [172]) yields, for the case of free groups, bi-orders that are different from those obtained using the lower central series (*c.f.* Example 2.1.10). For an elegant alternative approach, see [11].

Bi-ordering free products via an inductive argument. Given a free product of groups $\Gamma = \star\Gamma_\lambda$, every element may be written in a unique way as a product $f = f_1 \cdots f_\ell$, where each f_i belongs to some Γ_λ and no consecutive $f_i's$ belong to the same Γ_λ. We will call the integer $\ell = \ell(f)$ the ***length*** of the element f in the free product $\star\Gamma_\lambda$.

Assume now that each Γ_λ is bi-orderable, and let $\preceq_\lambda$ be a bi-order on it. The strategy will consist in defining a normal positive cone P on Γ by looking at the lengths of the elements. Nevertheless, along this proof, we will need to consider more general free products than Γ.

<u>Elements of length 1.</u> Such an element g belongs to one of the factors Γ_λ. We then let f belong to P if and only if $f \succ_\lambda id$.

Now, to deal with elements of length larger than 1, we begin by fixing a left-order $\leq$ on Γ, and for each λ, we denote $G_\lambda = \star_{\omega > \lambda} \Gamma_\omega$.

<u>Elements of larger length.</u> Let $f := f_1 \cdots f_\ell$ be an element of length ℓ larger than 1, where $f_i \in \Gamma_{\lambda_i}$ for each i. Let $\lambda(f) := \min_\leq \{\lambda_i\}$. Then $f = g_0 h_1 g_1 h_2 \cdots h_k g_k$,

where the $h_j's$ correspond to the factors of f not lying in $\Gamma_{\lambda(f)}$, and each g_j belongs to $\Gamma_{\lambda(f)}$ (g_0 and/or g_{k+1} may be trivial). Notice that

$$\bar{f} := (h_1 \cdots h_k)^{-1} f = \prod_{i=0}^{k} (h_{i+1} \cdots h_k)^{-1} g_i (h_{i+1} \cdots h_k).$$

There are two cases:

(i) If $h_1 \dots h_k \neq id$, then since the length of $h = h_f := h_1 \cdots h_k$ is smaller than ℓ, we may assume inductively that we know whether h belongs to P or not. In the case it belongs, we let f belong to P; otherwise, we let f^{-1} belong to P.

(ii) If $h = id$, then we think of $f = \prod_{i=0}^{k} (h_{i+1} \cdots h_k)^{-1} g_i (h_{i+1} \cdots h_k)$ as an element of length smaller than ℓ in the free product $\star_{h \in G_\lambda} h \Gamma_\lambda h^{-1}$. By using the above rules inside this free product, we may determine whether f should belong to P or not.

One easily convinces that the above procedure yields a subset P of Γ that is disjoint from P^{-1} and such that $\Gamma \setminus \{id\} = P \cup P^{-1}$. We now proceed to check that P is a normal semigroup.

To see that $f_1 f_2$ belongs to P for each f_1, f_2 in P, we proceed by induction on the length of $f_1 f_2$ in an appropriate free product. More precisely, if $\ell(f_1 f_2) = 1$, then both f_1, f_2 belong to the same Γ_λ (and are $\preceq_\lambda$-positive), hence $f_1 f_2 \in P$. Otherwise, there are two cases:

– If $\lambda(f_1) < \lambda(f_2)$ (resp. $\lambda(f_1) > \lambda(f_2)$), then writing $f_1 f_2 = h_{f_1} f_2 (f_2^{-1} \bar{f}_1 f_2)$ (resp. $f_1 f_2 = f_1 h_{f_2} \bar{f}_2$) we see that $h_{f_1 f_2} = h_{f_1} f_2$ (resp. $h_{f_1 f_2} = f_1 h_{f_2}$). Since the length of $h_{f_1} f_2$ (resp. $f_1 h_{f_2}$) is (nonzero and) strictly smaller than that of $f_1 f_2$, we may argue by induction to conclude that $f_1 f_2 \in P$.

– If $\lambda(f_1) = \lambda(f_2) = \lambda$, then writing $f_1 f_2 = h_{f_1} h_{f_2} (h_{f_2}^{-1} \bar{f}_1 h_{f_2}) \bar{f}_2$ we see that $h_{f_1 f_2} = h_{f_1} h_{f_2}$. If either h_{f_1} or h_{f_2} is nontrivial, then this yield $h_{f_1} h_{f_2} \in P$, hence $f_1 f_2 \in P$. If not, then the length of $f_1 f_2 = \bar{f}_1 \bar{f}_2$ viewed in the free product $\star_{h \in G_\lambda} h \Gamma_\lambda h^{-1}$ is smaller than its length in $\star \Gamma_\lambda$, which still allows concluding by induction.

Finally, to show that P is normal, it is enough to show that $g P g^{-1} = P$ for all g of length 1. Fix $f \in P$. There are three cases:

– If $\lambda(g) < \lambda(f)$, then $h_{gfg^{-1}} = f$. Since $f \in P$, by the definition of P we must have $g f g^{-1} \in P$.

– If $\lambda(g) = \lambda(f)$, then $h_{gfg^{-1}} = h_f$. If $h_f \neq id$, then it belongs to P, hence $g f g^{-1}$ belongs to P as well. If $h_f = id$, then the length of $g f g^{-1}$ viewed in $\star_{h \in G_\lambda} h \Gamma_\lambda h^{-1}$ is smaller than its length in $\star \Gamma_\lambda$. Thus, an inductive argument applies in this case.

– If $\lambda(g) > \lambda(f)$, then $h_{gfg^{-1}} = gh_f g^{-1}$. If $h_f \neq id$, then, as its length is smaller than that of f, an inductive argument applies. If $h_f = id$, then f may be written as $f = (h_1 g_1 h_1^{-1}) \cdots (h_k g_k h_k^{-1})$, with $g_1 \ldots, g_k$ in $\Gamma_{\lambda(f)}$ and $h_1, \ldots, h_k$ in $G_{\lambda(f)}$. Therefore,

$$gfg^{-1} = \prod_{i=1}^{k} (gh_i) g_i (gh_i)^{-1}.$$

Now recall that $\leq$ was chosen to be left-invariant, hence $h_i < h_j$ holds if and only if $gh_i < gh_j$. The construction of P then shows that $f \in P$ if and only if $gfg^{-1} \in P$.

Example 2.1.10. By performing the construction above appropriately, one may obtain a bi–order on a free group $\mathbb{F}_2 = \langle f, g \rangle$ for which both f, g are positive but $f\,[f,g]$ is negative. Notice that this cannot happen for the Magnus left-order, as well as for any bi–order on $\mathbb{F}_2$ obtained via the lower central series.

Exercise 2.1.11. Show that the free product of groups with the U.P.P. has the U.P.P. (See [165] in case of problems.) Show an analogous statement for groups admitting a locally-invariant order.

2.1.3 Left-orders from bi-orders

As we already pointed out, a left-orderable group all of whose left-orders are bi-invariant is necessarily Abelian [47]. This suggests the existence of natural procedures to create left-orders starting with bi-orders on groups. Here we briefly discuss two of them.

Left-orders from the sequence of convex subgroups. Let $\{\Gamma_i\}$ be the family of convex subgroups for a bi-order $\preceq$ on a group Γ. Since $\preceq$ is bi-invariant, for every $g \in \Gamma$, each subset of the form $g\Gamma_i g^{-1}$ is also convex. Given any well-order $\leq_{wo}$ on the set of indices i, we may define a left-order $\preceq'$ on Γ as follows: Given $g \in \Gamma$, we look for the minimal (with respect to $\leq_{wo}$) i such that $g\Gamma_i g^{-1} \neq \Gamma_i$, and we let $j(i)$ so that $g\Gamma_i g^{-1} = \Gamma_{j(i)}$. If $j(i) >_{wo} i$ (resp. $j(i) <_{wo} i$), then we let $g \succ' id$ (resp. $g \prec' id$); if g fixes each Γ_i, then we let $g \succ' id$ if and only if $g \succ id$.

One easily checks that $\preceq'$ is well-defined and left-invariant. Notice that, if every convex subgroup is normal, then $\preceq'$ coincides with the original bi-order $\preceq$.

Example 2.1.12. Let us consider the bi-order $\preceq_{x^+}^{-}$ on Thompson's group F (*c.f.* §1.2.4). Let (x_i) be a numbering of all dyadic, rational numbers of $]0,1[$. Each x_i gives raise to a convex subgroup Γ_i formed by the elements g such that $g(x) = x$ for all $x \in [x_i, 1]$. Although there are more convex subgroups than these, this family is invariant under the conjugacy action. By performing the construction above, we get the left-order $\preceq$ on F for which $f \succ id$ if and only if $f(x_i) > x_i$ holds for the smallest integer i such that $f(x_i) \neq x_i$. (Compare §1.1.3.)

Combing elements with trivial conjugacy action on a certain left-order. Proposition 2.1.14 below appears in [114], yet it was already implicit in [47].

Lemma 2.1.13. *Suppose $\preceq$ is a left-order on a group Γ admitting a normal, convex subgroup Γ_*, and let $g \in \Gamma \setminus \Gamma_*$. If conjugation by g preserves the left-order on Γ_* (that is, $g(P^+_{\preceq} \cap \Gamma_*)g^{-1} = P^+_{\preceq} \cap \Gamma_*$), then there exists a left-order on the subgroup $\langle g, \Gamma_* \rangle$ that has g as minimal positive element and coincides with $\preceq$ on Γ_*.*

Proof. Since Γ_* is normal in Γ, every element in $\langle g, \Gamma_* \rangle$ may be written in a unique way in the form $g^n h$, with $n \in \mathbb{Z}$ and $h \in \Gamma_*$. Define $\preceq_*$ on $\langle g, \Gamma_* \rangle$ by letting $g^n h \succeq_* id$ if and only if either $h \in P^+_{\preceq}$ or $h = id$ and $n > 0$. Invariance of $\preceq$ under conjugacy by g shows that this is a well-defined left-order on $\langle g, \Gamma^* \rangle$. That $\preceq_*$ coincides with $\preceq$ on Γ_* follows from the definition. Finally, the fact that g is the minimal positive element of $\preceq_*$ follows from the definition. $\square$

Combined with the convex extension technique, this lemma allows producing many interesting left-orders. Invoking Example 2.1.1, this is summarized in

Proposition 2.1.14. *Let $(\Gamma, \preceq)$ be a bi-ordered group, and let $\Gamma_g \subset \Gamma^g$ be the convex jump associated to an element $g \in \Gamma$. Assume that the quotient $\Gamma^g / \langle g, \Gamma_g \rangle$ is torsion-free. Then there exists a left-order $\preceq'$ on Γ having g as minimal positive element.*

Proof. First notice that both Γ_g and Γ^g are invariant under conjugacy by g. As $\preceq$ is bi-invariant, conjugacy by g preserves the positive cone of Γ_g. Thus, we are under the hypothesis of the preceding lemma, which allows to produce a left-order on $\langle g, \Gamma_g \rangle$ having g as minimal positive element. This left-order may be extended to a left-order $\preceq_*$ on Γ^g, as $\Gamma^g / \langle g, \Gamma_g \rangle$ is assumed to be torsion-free (recall that Γ^g / Γ_g is Abelian; see Example 2.1.1). Finally, we let $\preceq'$ be the extension of $\preceq_*$ by $\preceq$. $\square$

Example 2.1.15. Given an element g in the free group $\Gamma := \mathbb{F}_n$, let $k = k(g) \in \mathbb{N}$ be such that $g \in \Gamma_k \setminus \Gamma_{k+1}$, where Γ_i denotes the i^{th}-term of the lower central series. If $g\Gamma_k$ has no nontrivial root in Γ_{k-1}/Γ_k, then we are under the hypothesis of Proposition 2.1.14 for any bi-order on $\mathbb{F}_n$ obtained from the series Γ_i. Thus, g appears as the minimal positive element for a left-order on $\mathbb{F}_n$.

2.2 The Space of Left-Orders

Following Ghys [70] and Sikora [162], given a left-orderable group Γ, we denote by $\mathcal{LO}(\Gamma)$ the set of all left-orders on Γ. This ***space of left-orders*** carries a natural (Hausdorff and totally disconnected) topology whose sub-basis is the family of sets of the form $U_{f,g} = \{\preceq\colon\ f \prec g\}$. Because of left-invariance, another sub-basis is the family of sets $V_f = \{\preceq\colon\ id \prec f\}$.

To understand the topology on $\mathcal{LO}(\Gamma)$ better, one may proceed as in §1.1.2 by identifying left-orders on Γ to certain points in $\{-1,+1\}^{\Gamma\setminus\{id\}}$. Nevertheless, to cover also the case of partial left-orders, it is better to model $\mathcal{LO}(\Gamma)$ as a subset of $\{-1,+1\}^{\Gamma\times\Gamma\setminus\Delta}$, namely the one formed by the functions φ satisfying:

– (Reflexivity) $\varphi(g,h) = +1$ if and only if $\varphi(h,g) = -1$;

– (Transitivity) if $\varphi(f,g) = \varphi(g,h) = +1$, then $\varphi(f,h) = +1$;

– (Left-invariance) $\varphi(fg,fh) = \varphi(g,h)$ for all f and $g \neq h$ in Γ.

Indeed, every left-order $\preceq$ on Γ leads to such a function $\varphi_{\preceq}$, namely $\varphi_{\preceq}(g,h) = +1$ if and only if $g \succ h$. Conversely, every φ with the above properties induces a left-order $\preceq_{\varphi}$ on Γ, namely $g \succ_{\varphi} h$ if and only if $\varphi(g,h) = +1$. Now, if we endow $\{-1,+1\}^{\Gamma\times\Gamma\setminus\Delta}$ with the product topology and the subset above with the subspace one, then the induced topology on $\mathcal{LO}(\Gamma)$ coincides with the one previously defined by prescribing the sub-basis elements. As a consequence, since $\{-1,+1\}^{\Gamma\times\Gamma\setminus\Delta}$ is a compact space and the subspace above is closed, the topological space $\mathcal{LO}(\Gamma)$ is compact.

As a matter of example, in regard to the convex extension procedure (*c.f.* §2.1.1), the reader should have no problem in showing the next

Proposition 2.2.1. Let $\preceq$ be a left-order on Γ and Γ_* a $\preceq$-convex subgroup. Then the map from $\mathcal{LO}(\Gamma_*)$ into $\mathcal{LO}(\Gamma)$ that sends $\preceq_*$ into its convex extension by $\preceq$ is a continuous injection. If, in addition, Γ_* is normal, then there is a continuous injection from $\mathcal{LO}(\Gamma_*) \times \mathcal{LO}(\Gamma/\Gamma_*)$ into $\mathcal{LO}(\Gamma)$ having $\preceq$ in its image.

Example 2.2.2. The subspace of dynamically-lexicographic left-orders on $\mathrm{Homeo}_+(\mathbb{R})$ (*c.f.* §1.1.3) is not closed inside $\mathcal{LO}(\mathrm{Homeo}_+(\mathbb{R}))$ (we do not know whether it is dense). To show this, let (y_k) be a dense sequence of real numbers, and let (x_n) be a monotone sequence converging to a point $x \in \mathbb{R}$. For each n define a sequence $(y_{n,k})_k$ by $y_{n,1} = x_n$ and $y_{n,k} = y_{k-1}$ for $k > 1$. This gives raise to a sequence of left-orders $\preceq_n$ (the sign of each point $y_{n,k}$ is chosen to be $+$). Passing to a subsequence if necessary, we may assume that $\preceq_n$ converges to a left-order $\preceq$ on $\mathrm{Homeo}_+(\mathbb{R})$. We claim that $\preceq$

is not of dynamically-lexicographic type. Indeed, let $\preceq'$ be an arbitrary dynamically-lexicographic left-order, and let x' be the first point different from x for the well-order leading to $\preceq'$ (thus, x' may be the first or the second term of this well-order). Let $f \in \mathrm{Homeo}_+(\mathbb{R})$ be such that $f(x) = x$ and $f(y) > y$ for all $y \neq x$. Let g, h be elements in $\mathrm{Homeo}_+(\mathbb{R})$ that coincide with f in a neighborhood of x and $g(x') > x' > h(x')$. By definition, the signs of g, h with respect to $\preceq'$ are different. However, since $g(y_{n,1}) = g(x_n) > x_n = y_{n,1}$ and $h(y_{n,1}) > y_{n,1}$ for all n large enough, both g, h are $\preceq_n$-positive. Passing to limits, both g, h become $\preceq$-positive, thus showing that $\preceq$ cannot coincide with $\preceq'$.

If Γ is a countable left-orderable group, then the natural topology of $\mathcal{LO}(\Gamma)$ is metrizable. Indeed, if $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots$ is a complete exhaustion of Γ by finite sets, then we can define the distance between two different left-orders $\leq$ and $\preceq$ by letting $d(\leq, \preceq) = 2^{-n}$, where n is the maximum non-negative integer such that $\leq$ and $\preceq$ coincide on $\mathcal{G}_n$. An equivalent metric d' is obtained by letting $d'(\leq, \preceq) = 2^{-n'}$, where n' is the maximum non-negative integer such that the positive cones of $\leq$ and $\preceq$ coincide on $\mathcal{G}_{n'}$, that is, $P^+_{\leq} \cap \mathcal{G}_{n'} = P^+_{\preceq} \cap \mathcal{G}_{n'}$. One easily checks that these metrics are ultrametric. Moreover, the fact that $\mathcal{LO}(\Gamma)$ is compact becomes more transparent in this case, as it follows from a Cantor diagonal type argument.

When Γ is finitely generated, one may choose $\mathcal{G}_n$ as being the ***ball of radius n*** with respect to some finite, symmetric system of generators $\mathcal{G}$ of Γ (centered at id), usually denoted by $B_n = B_n(id)$. Here, ***symmetric*** means that $g^{-1} \in \mathcal{G}$ for all $g \in \mathcal{G}$, and the ball B_n is the set of elements having word-length at most n, where the ***word-length*** of $g \in \Gamma$ is the minimum m for which g which can be written in the form $g = g_{i_1} g_{i_2} \cdots g_{i_m}$, with $g_{i_j} \in \mathcal{G}$. One easily checks that the metrics on $\mathcal{LO}(\Gamma)$ resulting from two different finite systems of generators are not only topologically but also Hölder equivalent.

Exercise 2.2.3. Given a bi-orderable group Γ, denote by $\mathcal{BO}(\Gamma)$ the ***space of bi-orders*** of Γ. Show that $\mathcal{BO}(\Gamma)$ is closed inside $\mathcal{LO}(\Gamma)$, hence compact.

Exercise 2.2.4. Given a group Γ admitting a locally-invariant order (*c.f.* §1.3.2), denote by $\mathcal{LIO}(\Gamma)$ the set of all locally-invariant orders on Γ. Consider the topology on $\mathcal{LIO}(\Gamma)$ having as a sub-basis the family of sets $U_{f,g} = \{\preceq : f \prec g\}$. Show that, endowed with this topology, $\mathcal{LIO}(\Gamma)$ is compact. Conclude that a group Γ admits a locally-invariant order if and only if each of its finitely-generated subgroups admits such an order. (Compare [35, Theorem 2.4].)
<u>Hint.</u> As a model of $\mathcal{LIO}(\Gamma)$ consider the subset of $\{-1, 0, +1\}^{\Gamma \times \Gamma \setminus \Delta}$ formed by the functions φ such that $\varphi(g, h) = +1$ if and only if $\varphi(h, g) = -1$, and such that for every

$g \neq id$ and $h \in \Gamma$ one has either $\varphi(hg, h) = +1$ or $\varphi(hg^{-1}, h) = -1$. (Two elements g, h that are incomparable for a locally invariant order will then satisfy $\varphi_{\preceq}(g, h) = 0$...)

Exercise 2.2.5. Complete the proof of Proposition 1.3.9 showing that weakly diffuse groups admit a locally-invariant order. (See [113] in case of problems.)
Hint. By a compactness type argument, it is enough to show the following: For each finite subset A of Γ, there exists a partial order $\preceq$ such that for all $f \in A$ and each nontrivial element $g \in \Gamma$ such that both fg and fg^{-1} lie in A, either $fg \succ f$ or $fg^{-1} \succ f$. To construct such a $\preceq$, proceed by induction, the case where A is a single element being evident. Now, given an arbitrary A, by the weakly diffuse property there is $h \in A$ such that for each nontrivial element $g \in \Gamma$, either $hg \notin A$ or $hg^{-1} \notin A$. By the induction hypothesis, $A \setminus \{h\}$ admits an order as requested. Extend $\preceq$ to all A by declaring h being larger than all other elements.

The group Γ acts (continuously) on $\mathcal{LO}(\Gamma)$ by conjugacy (equivalently, by right multiplication): given an order $\preceq$ with positive cone P^+ and an element $f \in \Gamma$, the image of $\preceq$ under f is the order $\preceq_f$ whose positive cone is $f\, P^+ f^{-1}$. In other words, one has $g \preceq_f h$ if and only if $f^{-1}gf \preceq f^{-1}hf$, which is equivalent to $gf \preceq hf$. Also notice that the map sending $\preceq$ to $\overline{\preceq}$ from Example 1.1.2 is a continuous involution of $\mathcal{LO}(\Gamma)$.

Example 2.2.6. It a group left-order is obtained via an action on an totally ordered space Ω, then the conjugacy action corresponds to changing the order of the comparison points. More precisely, in the notation of §1.1.3, if $\preceq$ comes from a well-order $\leq_{wo}$ on Ω, then $\preceq_f$ is obtained from the same action using the well-order $f_*(\leq_{wo})$ given by $\omega_1 \ f_*(\leq_{wo}) \ \omega_2$ whenever $f(\omega_1) \leq_{wo} f(\omega_2)$. In particular, for countable subgroups of $\mathrm{Homeo}_+(\mathbb{R})$, if $\preceq$ is induced from a dense sequence (x_n) in $\mathbb{R}$, then $\preceq_f$ is induced from the sequence $(f(x_n))$.

Remark 2.2.7. If Γ is a left-orderable group, then the whole group of automorphisms of Γ (and not only the group of internal automorphisms) acts on $\mathcal{LO}(\Gamma)$. This may be useful to study bi-orderable groups. Indeed, since the fixed points for the right action of Γ on $\mathcal{LO}(\Gamma)$ correspond to the bi-invariant left-orders, the group $Out(\Gamma)$ of outer automorphisms of Γ acts on the corresponding space of bi-orders $\mathcal{BO}(\Gamma)$. The reader is referred to [105] for some applications of this idea.

In general, the study of the dynamics of the action of Γ on $\mathcal{LO}(\Gamma)$ should reveal useful information. Let us formulate a very simple questions on this.

Question 2.2.8. For which countable left-orderable groups having an infinite space of left-orders is the action of Γ on $\mathcal{LO}(\Gamma)$ uniformly equicontinuous or distal ? The same question makes sense for *minimality*[1], or for having a dense orbit (the latter is the case of free groups, as we will see in §2.2.2; for the former, we do not know any example).

2.2.1 Finitely many or uncountably many left-orders

We now state the first nontrivial general theorem concerning the space of left-orders of left-orderable groups. This result was first obtained by Linnell [110] by elaborating on previous ideas of Smirnov, Tararin, and Zenkov. Let us point that no analogue for spaces of bi-orders holds [26]; see however §3.2.6.

Theorem 2.2.9. *If the space of left-orders of a left-orderable group is infinite, then it is uncountable.*

The starting point to show this result is the following. Let Γ be a left-orderable group and M a ***minimal subset*** of $\mathcal{LO}(\Gamma)$, that is, a nonempty, closed subset that is invariant under the conjugacy action of Γ and does not properly contain any nonempty, closed, invariant set. Since the set M' of accumulation points of M is both closed and invariant, one has either $M' = M$ or $M' = \emptyset$. In other words, either M has no isolated point, or it is finite. In the former case, a well-known result in General Topology asserts that M must be uncountable (see [86, Theorem 2-80]). In the latter case, the stabilizer of any point $\preceq$ of M is a finite-index subgroup of Γ restricted to which $\preceq$ is bi-invariant. Theorem 2.2.9 then follows from the following

Proposition 2.2.10. *Let $(\Gamma, \preceq)$ be a left-ordered group containing a finite-index subgroup Γ_0 restricted to which $\preceq$ is bi-invariant. If $\preceq$ has a neighborhood in $\mathcal{LO}(\Gamma)$ containing only countably many left-orders, then $\mathcal{LO}(\Gamma)$ is finite.*

The proof of this proposition uses results and techniques from the theory of Conradian orders. Hence, we postpone the (end of the) proof of Theorem 2.2.9 to §3.2.6.

Notice that the proof above "distinguishes" conjugate left-orders, though one would like to consider them as being "equal" in many senses (for instance, they share all dynamical features). This naturally leads to the following

Question 2.2.11. Can the space of orbits $\mathcal{LO}(\Gamma)/\Gamma$ be countable infinite for a left-orderable group Γ ?

[1] Recall that an action is said to be ***minimal*** if every orbit is dense.

In the rest of this section, we give a beautiful characterization (due to Tararin [168]) of groups having finitely many left-orders. (We will refer to them as ***Tararin groups.***) Recall that a ***rational series*** for a group Γ is a finite sequence of subgroups

$$\{id\} = \Gamma^k \lhd \Gamma^{k-1} \lhd \ldots \lhd \Gamma^0 = \Gamma$$

that is ***subnormal*** (that is, each Γ^i is normal in Γ^{i-1}, but not necessarily in Γ), and such that each quotient Γ^{i-1}/Γ^i is torsion-free rank-1 Abelian. Such a series is said to be ***normal*** if each Γ^i is normal in Γ. Notice that every group admitting a rational series is left-orderable.

Theorem 2.2.12. *Every left-orderable group having only finitely many left-orders admits a unique rational series*

$$\{id\} = \Gamma^k \lhd \Gamma^{k-1} \lhd \ldots \lhd \Gamma^0 = \Gamma.$$

This series is normal and no quotient Γ^{i-2}/Γ^i is bi-orderable. Conversely, if a group Γ admits a normal rational series such that no quotient Γ^{i-2}/Γ^i is bi-orderable, then (Γ is left-orderable and) its space of left-orders $\mathcal{LO}(\Gamma)$ is finite. In such a situation, for every left-order on Γ, the convex subgroups are exactly $\Gamma^0, \Gamma^1, \ldots, \Gamma^k$, the number of left-orders on Γ is 2^k, and each left-order is uniquely determined by the sequence of signs of any family of elements $g_i \in \Gamma^{i-1} \setminus \Gamma^i$.

Example 2.2.13. The Klein-bottle group $K_2 = \langle a, b : a^{-1}ba = b^{-1} \rangle$ admits exactly four left-orders, having as positive cones $\langle a, b\rangle^+$, $\langle a, b^{-1}\rangle^+$, $\langle a^{-1}, b\rangle^+$, and $\langle a^{-1}, b^{-1}\rangle^+$, respectively (see §2.2.3 for details). The associate rational series is $\{id\} \lhd \langle b \rangle \lhd K_2$. More generally, let us consider the group

$$K_k = \langle a_1, \ldots, a_k : a_{i+1}^{-1} a_i a_{i+1} = a_i^{-1},\ a_i a_j = a_j a_i \text{ for } |i-j| \geq 2 \rangle.$$

One can easily check (either using Theorem 2.2.12 above or by a direct computation) that K_k admits 2^k left-orders, each of which being determined by the signs of the a_i's. The corresponding rational series is

$$\{id\} \lhd \langle a_1 \rangle \lhd \langle a_1, a_2 \rangle \lhd \ldots \lhd \langle a_1, a_2, \ldots, a_k \rangle.$$

Example 2.2.14. A dynamical counterpart of having finitely many left-orders for a group is that, up to semiconjugacy, only a few actions on the real line may arise. For the case of the group K_2 above, this translates into the two items below. (See Example 2.2.23 for an application.)

Claim (i). Suppose $K_2 = \langle a, b \colon a^{-1}ba = b^{-1} \rangle$ acts on the real line and there is $x \in \mathbb{R}$ such that $x \leq a(x)$. Then b has a fixed point in $I = [x, a(x)]$.

Otherwise, changing b by its inverse if necessary, we may assume that $b(z) > z$ for all $z \in I$. In particular, $a(x) < ba(x)$, hence $x < a^{-1}ba(x) = b^{-1}(x)$. Therefore, $b(x) < x$, a contradiction.

As a consequence, every open interval I fixed by a on which a acts freely is also fixed by b. Moreover, b has infinitely many fixed points in I.

Claim (ii). For every open interval J fixed by b and containing no fixed point of b inside, we have $a(J) \cap J = \emptyset$.

Indeed, as $\langle b \rangle$ is normal in K, we have that $a(J) \cap J$ is either J or empty. But the first possibility cannot occur, since in that case b would have fixed points in J, due to Claim (i).

The proof of Theorem 2.2.12 will be divided into several parts, some of which involve notions and results contained in the beginning of the next chapter.

Lemma 2.2.15. *If a left-orderable group admits only finitely many left-orders, then all of them are Conradian.*

Proof. Let Γ be a left-orderable group whose space of left-orders is finite. For a finite-index subgroup Γ_* of Γ, the conjugacy action on $\mathcal{LO}(\Gamma)$ is trivial. This means that every left-order of Γ is bi-invariant (hence Conradian) when restricted to Γ_*. The lemma then follows from Proposition 3.2.9. $\square$

We may now proceed to show the first claim contained in Theorem 2.2.12.

Proposition 2.2.16. *Let Γ be a left-orderable group admitting only finitely many left-orders. Then, for every left-order $\preceq$ on Γ, the chain of $\preceq$-convex subgroups is a finite rational series.*

Proof. To show finiteness, let us fix $n \in \mathbb{N}$ such that the number of left-orders on Γ is strictly smaller than 2^n. Following Zenkov [175], we claim that the family of $\preceq$-convex subgroups has cardinality $\leq n$. Otherwise, if

$$\{id\} = \Gamma^0 \subsetneq \Gamma^1 \subsetneq \ldots \subsetneq \Gamma^n = \Gamma$$

is a chain of distinct $\preceq$-convex subgroups, then for each $\iota = (i_1, \ldots, i_n) \in \{-1, +1\}^n$ we may define the left-order $\preceq_\iota$ as being equal to $\preceq_n$, where $\preceq_1, \preceq_2, \ldots, \preceq_n$ are the left-orders on $\Gamma^1, \ldots, \Gamma^n$, respectively, which are inductively defined by:

– If $i_1 = 1$ (resp. $i_1 = -1$), then $\preceq_1$ is the restriction of $\preceq$ (resp. $\overline{\preceq}$) to Γ^1.

– For $n \geq k \geq 2$, if $i_k = 1$ (resp. $i_k = -1$), then $\preceq_k$ is the extension of $\preceq_{k-1}$ by the restriction of $\preceq$ (resp. $\overline{\preceq}$) to Γ^k.

Clearly, the left-orders $\preceq_\iota$ are different for different choices of ι, which shows the claim.

Now let

$$\{id\} = \Gamma^k \subsetneq \Gamma^{k-1} \subsetneq \ldots \subsetneq \Gamma^0 = \Gamma$$

be the chain of *all* $\preceq$-convex subgroups of Γ. In the terminology of Example 2.1.1, the inclusion $\Gamma^i \subsetneq \Gamma^{i-1}$ is the convex jump associated to any element in $\Gamma^{i-1} \setminus \Gamma^i$. By Theorem 3.2.27, Γ^i is normal in Γ^{i-1}, and the induced left-order on Γ^{i-1}/Γ^i is Archimedean. By Hölder's theorem (*c.f.* §3.1), this quotient is torsion-free Abelian. Finally, its rank must be 1, as otherwise it would admit uncountably many left-orders (*c.f.* §1.2.1), which would allow to produce –by convex extension (*c.f.* Proposition 2.2.1)– uncountably many left-orders on Γ. □

Proposition 2.2.17. *A left-orderable group admitting finitely many left-orders has a unique (hence normal) rational series.*

Proof. If $\{id\} = \Gamma^k \lhd \Gamma^{k-1} \lhd \ldots \lhd \Gamma^0 = \Gamma$ is a rational series for a group Γ, then for every $h \in \Gamma$, the conjugate series

$$\{id\} = h\Gamma^k h^{-1} \lhd h\Gamma^{k-1}h^{-1} \lhd \ldots \lhd h\Gamma^0 h^{-1} = \Gamma$$

is also rational. Therefore, the uniqueness of such a series implies its normality.

To show the uniqueness, let us consider two rational series

$$\{id\} = G^k \lhd G^{k-1} \lhd \ldots \lhd G^1 \lhd G^0 = \Gamma, \qquad \{id\} = H^{k'} \lhd H^{k'-1} \lhd \ldots \lhd H^1 \lhd H^0 = \Gamma,$$

where Γ is supposed to admit only finitely many left-orders. Both G^1 and H^1 are normal in Γ, and the quotients Γ/H^1 and Γ/G^1 are torsion-free Abelian. This easily implies that $G^1 \cap H^1$ is also normal in Γ and the quotient $\Gamma/(G^1 \cap H^1)$ is torsion-free Abelian. Since $G^1 \cap H^1$ is convex with respect to some left-order on Γ (*c.f.* Proposition 2.1.7), the rank of $\Gamma/(G^1 \cap H^1)$ must be 1; otherwise, this quotient would admit uncountably many left-orders, thus yielding –by convex extension– uncountably many left-orders on Γ. We conclude that, for every $g \in G^1$ (resp. $h \in H^1$), one has $g^n \in G^1 \cap H^1$ (resp. $f^n \in G^1 \cap H^1$) for some $n \in \mathbb{N}$. However, both G^1 and H^1 are stable under roots, hence $g \in H^1$ (resp. $h \in G^1$). This easily implies that $G^1 = H^1$.

Arguing similarly but with $G^1 = H^1$ instead of Γ, we obtain $G^2 = H^2$. Proceeding in this way finitely many times, we conclude that the rational series above coincide. □

The structure of the quotients Γ^{i-2}/Γ^i is given by the (proof of the) next

Proposition 2.2.18. *Let Γ be a left-orderable group having finitely many left-orders. If*

$$\{id\} = \Gamma^k \lhd \Gamma^{k-1} \lhd \ldots \lhd \Gamma^0 = \Gamma$$

is the unique rational series of Γ, then no quotient Γ^{i-2}/Γ^i is bi-orderable.

Proof. The group Γ^{i-1}/Γ^i is normal in Γ^{i-2}/Γ^i. Hence, Γ^{i-1}/Γ acts by conjugacy on the torsion-free, rank-1, Abelian group Γ^{i-1}/Γ^i. Now it is easy to see that every automorphism of a torsion-free, rank-1, Abelian group is induced by the multiplication by a real number. As a consequence, the non-Abelian group Γ^{i-2}/Γ^i embeds into $\mathrm{Aff}(\mathbb{R})$. The non bi-orderability of Γ^{i-2}/Γ^i is thus equivalent to that the image of this embedding is not contained in $\mathrm{Aff}_+(\mathbb{R})$. (This is also equivalent to that some element is conjugate to a negative power of itself.) But if this were not the case, then, according to §1.2.2, the quotient Γ^{i-2}/Γ^i (hence Γ) would admit uncountably many left-orders. $\square$

We next proceed to show the converse statements.

Proposition 2.2.19. *Let Γ be a group admitting a normal rational series*

$$\{id\} = \Gamma^k \lhd \Gamma^{k-1} \lhd \ldots \lhd \Gamma^0 = \Gamma$$

such that no quotient Γ^{i-2}/Γ^i is bi-orderable. For each $i \in \{1, \ldots, k\}$, let us choose $g_i \in \Gamma^{i-1} \setminus \Gamma^i$. Then every left-order on Γ is completely determined by the signs of the g_i's. Moreover, for any such choice of signs, there exists a left-order on Γ realizing it.

Proof. The realization of signs $\iota \in \{-1, +1\}^k$ proceeds as the proof of the first claim of Proposition 2.2.16, and we leave the details to the reader. As before, we will denote by $\preceq_\iota$ the left-order that realizes the corresponding signs.

Now let $\preceq$ be a left-order on Γ, and let $\iota = (i_1, \ldots, i_n)$ be the associate sequence of signs of the g_i's. To prove that the positive cones of $\preceq$ and $\preceq_\iota$ coincide, it suffices to show that $P^+_{\preceq_\iota} \subset P^+_{\preceq}$ (*c.f.* Exercise 2.2.38). As we saw in the proof of the preceding proposition, after changing g_i by a root if necessary, we may assume that $g_{i-1}^{-1} g_i g_{i-1} = g_i^{r_i}$ for a *negative* rational number r_i. Since $g_k^{i_k} \in \Gamma^{k-1}$ belongs to both $P^+_{\preceq_\iota}$ and $P^+_{\preceq}$, and since Γ^{k-1} is rank-1 Abelian, we have

$$P^+_{\preceq_\iota} \cap \Gamma^{k-1} \subset P^+_{\preceq}.$$

Now every element $g \in \Gamma^{k-2} \setminus \Gamma^{k-1}$ may be written as $g = g_k^{si_k} g_{k-1}^{ti_{k-1}}$ for some rational numbers s and $t \neq 0$; such an element is $\preceq_\iota$-positive if and only if $t>0$. If besides $s \geq 0$, then g is also $\preceq$-positive. Otherwise, $s < 0$, and g may be rewritten as $g = g_{k-1}^{r_i t i_{k-1}} g_k^{si_k}$, and since $r_i t > 0$, this shows that g is still $\preceq$-positive. Therefore, we have

$$P^+_{\preceq_\iota} \cap \left(\Gamma^{k-2} \setminus \Gamma^{k-1}\right) \subset P^+_{\preceq},$$

hence

$$P^+_{\preceq_\iota} \cap \Gamma^{k-2} \subset P^+_{\preceq}.$$

Proceeding in this way finitely many times, one concludes that $P^+_{\preceq_\iota} \subset P^+_{\preceq}$. $\square$

Exercise 2.2.20. Show that every Tararin group admits a unique nontrivial torsion-free Abelian quotient, namely the quotient with respect to the maximal proper convex subgroup.

Exercise 2.2.21. Let Γ be a left-orderable group for which the whole family of subgroups that are convex for some left-order on Γ is finite. Show that Γ admits only finitely many left-orders.
Remark. This result is also due to Tararin; see [108, §5.2] in case of problems.

We next provide a quite clarifying result on the dynamics of the action of a Tararin group on its space of left-orders.

Proposition 2.2.22. *The action of a Tararin group Γ on its space of left-orders $\mathcal{LO}(\Gamma)$ has two orbits. Moreover, for any two left-orders $\preceq$ and $\preceq'$ on Γ, there is $g \in \Gamma$ such that $\preceq_g$ and $\preceq'$ coincide on the maximal convex subgroup of any of its left-orders (namely Γ^1, in the notation of Theorem 2.2.12). Moreover, if we let h be any element in $\Gamma \setminus \Gamma^1$ acting on Γ^1/Γ^2 as the multiplication by a negative number (c.f. Proposition 2.2.18 and its proof), then g can be taken either in Γ^1 or in $h\Gamma^1$.*

Proof. Choose elements $g_i \in \Gamma^{i-1} \setminus \Gamma^i$, where $i \in \{2, \ldots, k\}$. In case the signs of g_2 under $\preceq$ and $\preceq'$ are the same, let $h_2 := id$; otherwise, let h_2 be the element h above. Then the sign of g_2 for $\preceq_{h_2}$ is the same as that for $\preceq'$.

In case the signs of g_3 for $\preceq_{h_2}$ and $\preceq'$ coincide, let $h_3 := id$. Otherwise, let h_3 be an element in $\Gamma^1 \setminus \Gamma^2$ acting on Γ^2/Γ^3 as the multiplication by a negative number. Then the signs of both g_2, g_3 for $\preceq_{h_3 h_2}$ and $\preceq'$ are the same.

Continuing this way, we obtain an element $g := h_k \cdots h_2$ such that the signs of all g_i's for $\preceq_g$ and $\preceq'$ coincide, where $i \in \{2, \ldots, k\}$. This certainly implies that $\preceq_g$ and $\preceq'$ are the same when restricted to Γ^1. $\square$

Example 2.2.23. Let us consider the group

$$\Gamma := \big\langle a_s, s \in \mathbb{R} \colon a_s^{-1} a_t a_s = a_t^{-1} \text{ whenever } t < s \big\rangle.$$

We claim that Γ is left-orderable but has no nontrivial action on the line. Proving that Γ is left-orderable is easy. Indeed, every $g \in \Gamma$ may be written in normal form as

$$g = a_{s_1}^{n_1} \cdots a_{s_k}^{n_k}\,, \text{ with } \; s_1 > s_2 > \ldots > s_k\,,\; n_i \neq 0.$$

We may then declare such a $g \in \Gamma$ to be positive if $n_1 > 0$, thus getting a left-order on Γ (details are left to the reader). Next, assume for a contradiction that Γ acts nontrivially on the real line. Then there is $t \in \mathbb{R}$ such that a_t acts nontrivially. Let I_t be an open interval fixed by a_t containing no fixed point of a_t. By Example 2.2.14, for each $s > t$, we have that a_s has no fixed point in the closure of I_t, and that $a_s(I_t) \cap I_t = \emptyset$. Let I_s be the minimal open interval fixed by a_s that contains I_t. Example 2.2.14 again implies that for each pair of real numbers $s_1 > s_2$ larger than t, we have $a_{s_1}(I_{s_2}) \cap I_{s_2} = \emptyset$. We thus obtain that $\{a_s(I_t)\}_{s>t}$ is an uncountable collection of disjoint open intervals, which is absurd.

2.2.2 The space of left-orders of the free group

The space of left-orders of the free group is known to be homeomorphic to the Cantor set. This is a result of McCleary essentially contained in [119], though an alternative (dynamical) proof appears in [138]. The general strategy iof [138] proceeds as follows:

– Associated to a given left-order on $\mathbb{F}_n$, let us consider the corresponding dynamical realization.

– If we perturb the generators of this realization (as homeomorphisms of the line), we still have an action of the free group, which is "in general" faithful, thus yielding a new left-order on $\mathbb{F}_n$.

– If the perturbation above is "small", then the new left-order is close to the original one.

– Finally, the perturbation can be made so that the new left-order is different from the original one, as otherwise the original action would be "structurally stable" (*i.e.* actions which are "close" to it are semiconjugate), which is easily seen to be impossible.

As we will see below, a similar but more careful argument shows that the space of left-orders of the free product of two finitely-generated left-orderable groups is a Cantor set.

More interestingly, using results from [106, 107, 119], Clay has shown the existence of a left-order on $\mathbb{F}_n$ whose orbit under the conjugacy action is dense [37]. Using this, he deduces that $\mathcal{LO}(\mathbb{F}_n)$ is homeomorphic to the Cantor set by means of the argument contained in the following

Exercise 2.2.24. Let Γ be a countable group having a left-order whose orbit under the conjugacy action is dense. Show that $\mathcal{LO}(\Gamma)$ is a Cantor set.
Hint. If there is an isolated left-order $\preceq$, then its reverse left-order $\overline{\preceq}$ is also isolated. If there is a left-order of dense orbit, this forces the existence of $g \in \Gamma$ so that $\preceq_g = \overline{\preceq}$. However, this is impossible, since the signs of g for both $\preceq$ and $\overline{\preceq}$ coincide.
Remark. For the Baumslag-Solitar group $BS(1,\ell)$, no left-order has dense orbit. However, from the description given in §1.2.3, it readily follows that, for each irrational number $\varepsilon \neq 0$, the orbit of $\preceq_\varepsilon$ under the action of the whole group of *automorphisms* is dense.

Actually, the fact that an orbit is dense is not rare but "generic" in the space of left-orders of the free group, as follows from the next (classical)

Proposition 2.2.25. *Let Γ be a countable left-orderable group. If Γ admits a dense orbit under the conjugacy action, then this is the case for the orbits of a G_δ-dense set of points in $\mathcal{LO}(\Gamma)$.*

Proof. Consider an arbitrary finite family of elements $f_1, \ldots, f_k$ in Γ for which the basic open set $V_{f_1} \cap \ldots \cap V_{f_k}$ is nonempty. (Recall that $V_f := \{\preceq : id \prec f\}$.) Let $\mathcal{LO}(\Gamma; f_1, \ldots, f_k)$ be the subset of $\mathcal{LO}(\Gamma)$ formed by the left-orders $\preceq$ for which there exists $g \in \Gamma$ such that $\preceq_g$ belongs to $V_{f_1} \cap \ldots \cap V_{f_k}$. Then the set $\mathcal{LO}(\Gamma; f_1, \ldots, f_k)$ is a union of open basic sets, hence open. Moreover, since we are assuming the existence of a dense orbit, $\mathcal{LO}(\Gamma; f_1, \ldots, f_k)$ is also dense.

Now let $\mathcal{LO}^*(\Gamma)$ be the (countable) intersection of all the sets $\mathcal{LO}(\Gamma; f_1, \ldots, f_k)$ obtained above. On the one hand, Baire's category theorem implies that $\mathcal{LO}^*(\Gamma)$ is a G_δ-dense subset of $\mathcal{LO}(\Gamma)$. On the other hand, the definition easily yields that every left-order in $\mathcal{LO}^*(\Gamma)$ has a dense orbit. $\square$

In [153], a dynamical proof of the existence of a left-order with dense orbit in $\mathcal{LO}(\mathbb{F}_n)$ is given. This proof is based on the following construction (which is closely related to the ideas of [119]):

– Choose a countable dense set $\preceq_k$ of $\mathcal{LO}(\mathbb{F}_n)$, and for each k consider the dynamical realization Φ_k of $\preceq_k$.

– Fix a sequence of positive integers $n(k)$ converging to infinity very fast, and a family of disjoint intervals $[r(k), s(k)]$ that is unbounded in both directions.

– For each k, take a conjugate copy on $[r(k), s(k)]$ of the restriction of Φ_k to $[-n(k), n(k)]$.
– Take extensions of the generators of $\mathbb{F}_n$ so that they become homeomorphisms of the line.

Roughly, the resulting action contains all the possible "finite information" of all left-orders of $\mathbb{F}_n$. By performing the construction carefully, one can ensure that there is a single orbit containing the "center" of every $[r(k), s(k)]$. It therefore yields to a new left-order $\preceq$ on $\mathbb{F}_n$, and by suitable conjugacies inside $\mathbb{F}_n$, this left-order "captures" all the information above. In concrete terms, the orbit of $\preceq$ under the conjugacy action is dense.

Example 2.2.26. For Thompson's group F, no description of *all* left-orders is available. (For bi-orders, see §1.2.4). Actually, it is unknown whether its space of left-orders is a Cantor set. This also remains open for general non-solvable groups of piecewise-affine homeomorphisms of the interval.

The case of free products. Following a short argument from [153], we next show the following

Theorem 2.2.27. *The space of left-orders of the free product of any two left-orderable groups has no isolated point.*

Proof. We assume that the factors Γ_1, Γ_2 of our free product $\Gamma := \Gamma_1 \star \Gamma_2$ are finitely generated; for the general case, see Exercise 2.2.28. Given a left-order $\preceq$ on Γ, we consider the associated dynamical realization (see Proposition 1.1.7 and the comments after it). Fix a finite system of generators of Γ, and for each $n \in \mathbb{N}$ let f_n, g_n be, respectively, the smallest and largest element (for $\preceq$) in the ball of radius n in Γ. Let φ_n an orientation-preserving homeomorphism of the real line which is the identity on $[t(f_n), t(g_n)]$. Consider the following action of Γ on the real line: for $g \in \Gamma_1$, the action is the conjugate of its $\preceq$-dynamical realization under φ; for $g \in \Gamma_2$, the action is its $\preceq$-dynamical realization. (Notice that this yields an action since Γ is a free product; however, this action may fail to be faithful). We claim that if (x_n) is a dense sequence of points starting at $x_1 := t(id)$, then the positive cone of the induced dynamical lexicographic left-order $\preceq_{\varphi_n}$ coincides with that of $\preceq$ on the ball of radius n. Indeed, by construction, an element $h \in \Gamma$ belongs to $P_{\preceq_{\varphi_n}}$ if and only if $\varphi_n^{-1} h \varphi_n(t(id)) > t(id)$. Now, since $\varphi_n|_{[t(f_n),t(g_n)]} \equiv Id$, this is equivalent to $\varphi_n^{-1} h(t(id)) > t(id)$, that is $\varphi_n^{-1}(t(h)) > t(id)$. Now, if h belongs to the ball of radius n, then $t(h)$ lies in $[t(f_n), t(g_n)]$, hence $\varphi_n^{-1}(t(h)) = t(h)$, and this is bigger than $t(id)$ if and only if h is $\preceq$-positive.

Now fix $n \in \mathbb{N}$ and let us perform the preceding construction with a map φ_n such that $\varphi_n(s_2) = s_1$ for s_1, s_2 satisfying $t(g_n) < s_1 < t(h_{1,n}) < t(h_{2,n}) < s_2$, where $h_{i,n}$ is in Γ_i. Since $t(h_{1,n}) < t(h_{2,n})$, we have $h_{1,n} \prec h_{2,n}$. Now, from $\varphi_n^{-1}(t(h_{1,n})) > \varphi_n^{-1}(s_1) = s_2 > t(h_{2,n})$ we obtain $\varphi_n^{-1} h_{1,n} \varphi_n(t(id)) > h_{2,n}(t(id))$, which by construction is equivalent to $h_{1,n} \succ_{\varphi_n} h_{2,n}$.

Although the left-order $\preceq_{\varphi_n}$ may be partial (this arises when the new action of Γ is unfaithful), it can be extended (using the convex extension procedure) to a left-order $\preceq_n$. By construction, the positive cones of $\preceq$ and $\preceq_n$ coincide on the ball of radius n, though $\preceq_n$ and $\preceq$ are different. This concludes the proof. $\square$

Exercise 2.2.28. Provide the details of the proof of the preceding theorem for non finitely-generated factors.
Hint. Use a compactness type argument.

A geometric/combinatorial proof. The fact that the space of left-orders of a free group is a Cantor set can be established by means of different arguments. Roughly, this is done in two steps:

Step I. If a left-order is an isolated point in the space of left-orders of the free group, then its positive cone must be finitely generated as a semigroup.

Step II. There is no finitely-generated positive cone in the free group.

Concerning Step I, it is not hard to see that a finitely-generated positive cone yields an isolated point in the space of left-orders (see Proposition 2.2.37 below), yet the converse is not necessarily true (*c.f.* Example 2.2.39). The issue here is that the converse can be directly established for free groups. This is due to Smith and Clay, and we reproduce their (quite involved) proof from [42] below. (It would be desirable to get more geometric/transparent arguments that apply to other groups.)

Theorem 2.2.29. *If a left-order on $\mathbb{F}_n$ is isolated in the space of left-orders, then its positive cone must be finitely generated as a semigroup.*

Proof. Let $B_N := B_N(id)$ denote the ball of radius N with respect to the canonical system of generators. Say that a subset $S \subset \mathbb{F}_n$ is *total at length* N if it is *antisymmetric* (*i.e.* $g \in S \implies g^{-1} \notin S$) and for all $g \in B_N \setminus \{id\}$, either $g \in S$ or $g^{-1} \in S$. (Notice that $id \notin S$.) The crucial point of the proof is the following

Claim (i). If $S \subset B_N$ is total at length $N-1$ and satisfies $S = \langle S \rangle^+ \cap B_N$, then for every element g of length N not lying in $S \cup S^{-1}$, the semigroup $\langle S, g\rangle^+$ remains antisymmetric.

Let us assume this for a while, and let $\preceq$ be an isolated left-order on $\mathbb{F}_n$. Let $f_1, \ldots, f_k$ be finitely many $\preceq$-positive elements such that $\preceq$ is the only left-order on $\mathbb{F}_n$ for which all these elements are positive. If $P_{\preceq}$ is not finitely generated as a semigroup, then there must exist an increasing sequence of integers N_m such that each set $S := P_{\preceq} \cap B_{N_m}$ is total at length N_m though there is $g = g_m$ of length $N_m + 1$ that is not contained in $S \cup S^{-1}$. By Claim (i), the semigroup $\langle S, g \rangle^+$ is antisymmetric. Since it is total of length N_m, using Claim (i) in an inductive way we may extend it to an antisymmetric semigroup which together with its inverse covers $\mathbb{F}_n \setminus \{id\}$, thus inducing a left-order on $\mathbb{F}_n$. Obviously, the same procedure can be carried out starting with g^{-1} instead of g. Now, if N_m is large enough so that $f_1, \ldots, f_k$ are all contained in B_{N_m}, then the procedure above would give at least two different left-orders with all these elements positive (one with g positive, the other with g negative). This is a contradiction.

Let us now proceed to the proof of Claim (i). To do this, say that a finite subset $S \subset \mathbb{F}_n$ is *stable* if for all f, g in S, the product fg lies in S whenever $\|fg\| \leq \max\{\|f\|, \|g\|\}$. (Here and in what follows, $\|\cdot\|$ stands for the word-metric on $\mathbb{F}_n$.)

Claim (ii). If $S \subset \mathbb{F}_n$ is stable and $g \in \langle S \rangle^+$ is written in the form $g = h_1 \cdots h_k$, with each $h_i \in S$ and k minimal, then for each $1 \leq i \leq k-1$,

$$\|h_{i+1} \cdots h_k\| < \|h_i h_{i+1} \cdots h_k\|.$$

Write $h_i = f_i \bar{f}_i$, $h_{i+1} = \bar{f}_i^{-1} f_{i+1}$, with no cancellation in $f_i f_{i+1} = h_i h_{i+1}$. We claim that $\|\bar{f}_i\| < \|h_i\|/2$ and $\|\bar{f}_i\| < \|h_{i+1}\|/2$. Indeed, if $\|\bar{f}_i\| \geq \|h_i\|/2$ then

$$\begin{aligned} \|h_i h_{i+1}\| = \|f_i\| + \|f_{i+1}\| &= \big(\|h_i\| - \|\bar{f}_i\|\big) + \big(\|h_{i+1}\| - \|\bar{f}_i\|\big) \\ &\leq \Big(\|h_i\| - \frac{\|h_i\|}{2}\Big) + \Big(\|h_{i+1}\| - \frac{\|h_i\|}{2}\Big) = \|h_{i+1}\|, \end{aligned}$$

which forces $h_i h_{i+1} \in S$, thus contradicting the minimality of k. The proof of the inequality $\|\bar{f}_i\| < \|h_{i+1}\|/2$ proceeds similarly.

We may hence write $h_i = \bar{f}_{i-1}^{-1} g_i \bar{f}_i$, where g_i is not the empty word and $h_i h_{i+1} = \bar{f}_{i-1}^{-1} g_i g_{i+1} \bar{f}_{i+1}$, without cancelation for all i. It follows that $h_{i+1} \cdots h_k = \bar{f}_i^{-1} g_{i+1} \cdots g_k \bar{f}_k$, without cancelation. Since $\|\bar{f}_i\| < \|h_i\| - \|\bar{f}_i\| = \|\bar{f}_{i-1}\| + \|g_i\|$, finally we have

$$\begin{aligned} \|h_{i+1} \cdots h_k\| &= \|\bar{f}_i\| + \|g_{i+1}\| + \ldots + \|g_k\| + \|\bar{f}_k\| \\ &< \|\bar{f}_{i-1}\| + \|g_i\| + \|g_{i+1}\| + \ldots + \|g_k\| + \|\bar{f}_k\| \;=\; \|h_i \cdots h_k\|. \end{aligned}$$

Claim (iii). For every subset $S \subset B_N$, the equality $S = \langle S \rangle^+ \cap B_N$ holds if and only if for each f, g in S such that $\|fg\| \leq N$, the element fg lies in S.

The forward implication is obvious. For the converse, given $g \in \langle S \rangle^+ \cap B_N$, write it in the form $g = h_1 \cdots h_k$, with each h_i in S and $k \leq N$. Since the hypothesis implies that S is stable, we may apply Claim (ii), thus yielding

$$\|h_k\| \leq \|h_{k-1}h_k\| \leq \ldots \|h_2 \cdots h_k\| \leq \|h_1 \cdots h_k\| = \|g\| \leq N.$$

Again, since S is stable, this implies $h_{k-1}h_k \in S$; hence by induction $h_{k-2}h_{k-1}h_k \in S$,..., $h_2 \cdots h_{k-1}h_k \in S$, and finally $g = h_1 \cdots h_{k-1}h_k \in S$, as claimed.

Claim (iv). If f, g are reduced words in $\mathbb{F}_n$, with $\|fg\| = N$, $\|f\| \leq N$, $\|g\| = N$, then $\|f\|$ must be even and exactly half of f must cancel in the product fg. Moreover, after cancelation, at least the right half of fg must be the same as the right half of g.

Indeed, write $f = h_1\bar{h}$ and $g = \bar{h}^{-1}h_2$, so that $fg = h_1h_2$, without cancelation. Then

$$\|h_1\| + \|\bar{h}\| \leq N, \quad \|h_2\| + \|\bar{h}\| = N, \quad \|h_1\| + \|h_2\| = N.$$

The last two eaqualities yield $\|h_1\| = \|\bar{h}\|$. Therefore, $\|f\| = 2\|h_1\|$ is even, and $\|h_1\| = \|f\|/2$, so that exactly half of f disappears in the product fg. Moreover, from the first two relations we obtain $\|h_1\| \leq \|h_2\|$, hence $\|\bar{h}\| \leq \|h_2\|$, which shows that at least the right half of g survives in the product fg.

We may finally finish the proof of Claim (i). Let us begin by letting $S_1 := S \cup \{g\}$ and, for $i > 0$,

$$S_{i+1} = S_i \bigcup \{fg \colon f, g \text{ in } S_i \text{ and } \|fg\| \leq N, fg \notin S_i\}.$$

Obviously, there must exist an index j such that $S_j = S_{j+1}$. By Claim (iii), for such a j we have $S_j = \langle S, g \rangle^+ \cap B_N$, and S_j is stable.

Assume for a contradiction that $\langle S, g \rangle$ is not antisymmetric. Since Claim (ii) easily implies that the semigroup generated by an stable set excluding id is antisymmetric, we must have $id \in S_j$, hence there is a smallest index k such that both h, h^{-1} belong to S_k for a certain element h.

Suppose that $h \in S$ and $h^{-1} \in S_k$, and write $h^{-1} = h_1h_2$, with h_1, h_2 in S_{k-1}. Either $h_1 \notin S$ or $h_2 \notin S$ (otherwise, h^{-1} would be in S). Let us consider the first case (the other is analogous). Then $h_1^{-1} = h_2h$ belongs to S_k, as $h_2 \in S_{k-1}$ and $h \in S \subset S_{k-1}$. However, $h_1^{-1} \notin S$, otherwise h_1, h_1^{-1} would be both in S_{k-1}, thus contradicting the minimality of k. Summarizing, we have that h_1, h_1^{-1} are both in S_k, though $h_1 \notin S$ and $h_1^{-1} \notin S$.

The preceding argument allows reducing the general case to that where $h \notin S$ and $h^{-1} \notin S$. Since S is total at length $N-1$, by the minimality of k, every element in $S_{k-1} \setminus S$ must have length N.

Claim (v). Every element in $S_{k-1} \setminus S$, as well as h and h^{-1}, may be written in the form $h_1 g h_2$, where h_1, h_2 lie in $S \cup \{id\}$ and have both even length, and where exactly the left (resp. right) half of h_2 (resp. h_1) cancels in the product $h_1 g h_2$ above. (Notice that this implies $\|h_1 g h_2\| = \|h_1 g\| = \|g h_2\| = N$.)

The proof is made by induction on $i \leq k$ for elements $f \in S_i \setminus S$ with $\|f\| = N$. In the case $i = 1$, such an element f corresponds to g, which is written in the desired form. For the induction step, we must consider three different cases:
– Assume f is a product $f = h_1 g h_2 \bar{h}_1 g \bar{h}_2$, with both $h_1 g h_2$ and $\bar{h}_1 g \bar{h}_2$ in $S_{i-1} \setminus S$ of length N. By Claim (iv), N must be even, and exactly the right half of $h_1 g h_2$ must cancel with the left half of $\bar{h}_1 g \bar{h}_2$ in the product. By the induction hypothesis, the former is nothing but the right half of g followed by h_1, and the later is $\bar{h}_1$ followed by the left half of g. Thus $h_1 g h_2 \bar{h}_1 g \bar{h}_2 = h_1 g \bar{h}_2$ after cancelation, so that f has the desired form.
– Suppose $f = f h_1 g h_2$, where $f \in S$, $h_1 g h_2 \in S_i \setminus S$, $\|f\| \leq N$, $\|h_1 g h_2\| = N$. By Claim (iv), exactly the right half of f cancels in $f h_1 g h_2$. If $\|f\| \leq \|h_1\|$, this implies that this cancelation happens in the product $f h_1$, so that $\|f h_1\| = \|f\| \leq N$, thus yielding $f h_1 \in S$ because S is stable. If $\|h_1\| \leq \|f\|$, then the entire left half of h_1 cancels in $f h_1$, so that $\|f h_1\| \leq \|f\| \leq N$, yielding again $f h_1 \in S$. That $f h_1$ has even length and half of it cancels in the product $f h_1 g h_2$ now follows from Claim (iv).
– Finally, the case where $f = h_1 g h_2 f$, with $f \in S$, $h_1 g h_2 \in S_i \setminus S$, $\|f\| \leq N$, $\|h_1 g h_2\| = N$, can be treated in a similar way to that of the preceding one.

To conclude the proof of Theorem 2.2.29, let us finally write $h = h_1 g h_2$ and $h^{-1} = \bar{h}_1 g \bar{h}_2$ as in Claim (v). There are two cases to consider:
– If N is even, write $g = g_1 g_2$, where $\|g_1\| = \|g_2\| = \|g\|/2$. Then

$$id = hh^{-1} = h_1 g_1 g_2 h_2 \bar{h}_1 g_1 g_2 \bar{h}_2.$$

In this product, the right half of h must cancel against the left half of h^{-1}, that is, $g_2 h_2 \bar{h}_1 g_1 = id$. Therefore, $id = h_1 g_1 g_2 \bar{h}_2 = h_1 g \bar{h}_2$. But this implies $g^{-1} = \bar{h}_2 h_1 \in \langle S \rangle^+ \cap B_N = S$, which is a contradiction.
– If N is odd, write $g = g_1 f g_2$, where f is the generator of $\mathbb{F}_n$ that appears in the central position when writing g in reduced form. Proceeding as before, we get $id = hh^{-1} = h_1 g_1 f^2 g_2 \bar{h}_2$ with no further cancelation. This is absurd. $\square$

Remark 2.2.30. There are uncountably many left-orders on $\mathbb{F}_n$ for which the canonical generators f_i are positive. Indeed, this is a direct consequence of the preceding theorem, though it can be proved directly in a much more elementary way. What is less trivial is that there are left-orders on $\mathbb{F}_n$ that extend the lexicographic order on $\langle f_1, \ldots, f_n \rangle^+$. This is proved in [166] via a concrete realization of the free group as a group of homeomorphisms of the real line. The order thus obtained is perhaps the simplest left-order that can be defined on $\mathbb{F}_n$, and can be described as follows: Letting $\varphi : \mathbb{F}_n \to \mathbb{R}$ be defined by

$$\begin{aligned}\varphi(f) \;=\; & \big|\{\text{subwords of } f \text{ of the form } f_j f_i^{-1}, j > i\}\big| \\ & \qquad - \big|\{\text{subwords of } f \text{ of the form } f_j^{-1} f_i, j > i\}\big| \\ & \qquad\qquad + \frac{1}{2}\begin{cases} 1 & \text{if } f \text{ ends with } f_1, f_2, \ldots f_n, \\ -1 & \text{if } f \text{ ends with } f_1^{-1}, f_2^{-1}, \ldots, f_n^{-1}, \\ 0 & \text{if } f \text{ is trivial,} \end{cases}\end{aligned}$$

where f is a reduced word on $f_i^{\pm 1}$, one has $f \succ id$ if and only if $\varphi(f) > 0$.

A slight modification of the method above actually provides a Cantor set of left-orders, each of which extends the lexicographic order.

Step II. above, due to Kielak [102], is of a different nature. In a much more general way, it concerns groups of fractions of finitely-generated semigroups (inside groups with infinitely many ends). Recall that given a semigroup P inside a group Γ, we say that Γ is the ***group of fractions*** of P if every element in Γ can be written in the form gh^{-1}, where both g, h lie in $P \cup \{1\}$. An illustrative example, which covers that of nilpotent groups, is given by the next

Proposition 2.2.31. *Let Γ be a group generated by a finite set of elements $f_1, \ldots, f_k$, and let P be the semigroup generated by them together with id. If Γ has no free sub-semigroup, then each of its elements may be written in the form fg^{-1} for certain f, g in P.*

Proof. We first claim that, given any f, g in P, there exist $\overline{f}$ and $\overline{g}$ such that $g^{-1}f = \bar{f}\bar{g}^{-1}$, that is $f\bar{g} = g\bar{f}$. Otherwise, we would have $fP \cap gP = \emptyset$, and this implies that the sub-semigroup generated by f and g is free. Indeed, if h_1 and h_2 are different words in positive powers of f, g, to see that $h_1 \neq h_2$ we may assume that h_1 begins with f and h_2 with g. Then the condition $fP \cap gP = \emptyset$ implies that $h_1 \neq h_2$, since $h_1 \in fP$ and $h_2 \in gP$.

Now let $h := f_1 g_1^{-1} f_2 g_2^{-1} \cdots f_k g_k^{-1}$ be an arbitrary element in Γ, where all f_i, g_i belong to P. By the discussion above, we may replace $g_{k-1}^{-1} f_k$ by $\bar{f}_k \bar{g}_{k-1}^{-1}$, thus obtaining

$$h = f_1 g_1^{-1} f_2 g_2^{-1} \cdots f_{k-1} \bar{f}_k \bar{g}_{k-1}^{-1} g_k^{-1}.$$

Now, we may replace $g_{k-2}^{-1} f_{k-1} \bar{f}_k$ by an expression of the form $\bar{f}_{k-1} \bar{g}_{k-2}^{-1}$, thus obtaining

$$h = f_1 g_1^{-1} f_2 g_2^{-1} \cdots f_{k-2} \bar{f}_{k-1} \bar{g}_{k-2}^{-1} \bar{g}_{k-1}^{-1} g_k^{-1}.$$

Repeating this argument no more than $k-1$ times, we finally get an expression of f of the form fg^{-1}, where f, g belong to P. $\square$

Exercise 2.2.32. Given $\Gamma := \mathbb{Z} \wr \mathbb{Z} = \mathbb{Z} \ltimes \oplus_{\mathbb{Z}} \mathbb{Z}$, let a be a generator of the (left) factor $\mathbb{Z}$, and let b be a generator of the 0^{th} factor $\mathbb{Z}$ in the right. Show that a, b generate Γ, though the semigroup P generated by them and id satisfies $aP \cap bP = \emptyset$.

Proposition 2.2.31 is false in a very strong way for free groups. This is the content of the next result, due to Kielak [102], which actually applies to any group having infinitely many ends.

Theorem 2.2.33. *If P is a finitely-generated, proper subsemigroup of $\mathbb{F}_n$, then $\mathbb{F}_n$ is not the group of fractions of P.*

Proof. Let P be a finitely-generated semigroup for which $\mathbb{F}_n$ is the group of fractions. Let us consider the finitely many generators of P as a system of generators of $\mathbb{F}_n$, and let us look at the corresponding Cayley graph. Since free groups have infinitely many ends, there exists a radius N such that the complement of $B_N(id)$ has at least three connected components. For simplicity, let us denote just by B the ball centered at the identity and radius N. We will show that $P = \mathbb{F}_n$.

Claim (i). One has $B(P^{-1} \cup \{id\}) = \mathbb{F}_n$.

The claim is equivalent to that $P \cup \{id\}$ intersects every ball $B_N(f) = fB$. Let us thus suppose that $(P \cup \{id\}) \cap fB = \emptyset$ for a certain $f \in \mathbb{F}_n$. Let E_0 be a connected component of the complement of fB not containing the identity. Let h be an arbitrary element in the complement of B, and let E be the connected component of $\mathbb{F}_n \setminus B$ containing h. Using the transitivity of the action of $\mathbb{F}_n$ on its space of ends, one easily convices that there exists $g \in \mathbb{F}_n$ such that $gh \in E_0$, $gB \subset E_0$, and gE does not contain B.

Write gh in the form $h_1 h_2^{-1}$, with both h_1, h_2 in $P \cup \{id\}$. Since fB does not intersect $P \cup \{id\}$, the element h_1 must lie in the connected component of $\mathbb{F}_n \setminus fB$ containing id. Starting from the point h_1, the path obtained by concatenation with h_2^{-1} must cross fB as well as gB. In particular, there is an element in P^{-1} (namely, a terminal subword of h_2^{-1}) joining some point in gB to $h_1 h_2^{-1} = gh$. Thus, there is an element of P^{-1} joining an element of B to h, which shows that h belongs to BP^{-1}.

The preceding conclusion was established for all elements $h \in \mathbb{F}_n \setminus B$. This obviously implies that $\mathbb{F}_n = B(P^{-1} \cup \{id\})$, as desired.

Claim (ii). One has $P = \mathbb{F}_n$.

Let A be a subset of minimal cardinality such that $A(P^{-1} \cup \{id\}) = \mathbb{F}_n$. We claim that A must be a singleton. Indeed, if A contains two elements $f \neq g$, then we may write $f^{-1}g = h_1 h_2^{-1}$ for certain h_1, h_2 in $P \cup \{id\}$. Hence f, g both belong to $fh_1 P^{-1}$. Therefore, letting $A' := A \cup \{fh_1\} \setminus \{f, g\}$, we still have $A'(P^{-1} \cup \{id\}) = \mathbb{F}_n$. However, this contradicts the minimality of the cardinality of A.

We thus conclude that for a certain element h, we have $h(P^{-1} \cup \{id\}) = \mathbb{F}_n$. Certainly, this implies that $P^{-1} \cup \{id\} = \mathbb{F}_n$. In particular, letting f be any nontrivial element, both f, f^{-1} must belong to P^{-1}, hence their product $ff^{-1} = id$ is also in P^{-1}. We thus obtain $P^{-1} = \mathbb{F}_n$, and taking inverses this yields $P = \mathbb{F}_n$. $\square$

Notice that the proof above still leaves open the next

Question 2.2.34. Do there exist $k > 2$ and a finitely-generated, proper subsemigroup P of $\mathbb{F}_n$ such that every element of $\mathbb{F}_n$ can be written in the form $f_1 f_2^{-1} f_3 \cdots f_k^{(-1)^{k+1}}$, with all $f_1, \ldots, f_k$ belonging to $P \cup \{id\}$?

A direct corollary of Theorem 2.2.33 is that $\mathbb{F}_n$ does not admit an order with a finitely-generated positive cone. Together with Theorem 2.2.29, this yields again that $\mathcal{LO}(\mathbb{F}_n)$ has no isolated point, hence it is a Cantor set.

So far we have seen two different ways to show that $\mathcal{LO}(\mathbb{F}_2)$ is a Cantor set. A natural problem that arises is whether these can be pursued or combined to obtain finer information of this space for different metrics of geometric origin. For instance, what is the Hausdorff dimension of this space ? In order to solve this question one would need explicit estimates on the speed of approximation of a given left-order.

Another direction of research concerns algorithmic properties for orders. Indeed, by using recursive functions (in the sense of computability theory), it is not hard to construct left-orders on F_n such that the problem of comparison of elements is undecidable, *i.e.* there is no algorithm that, on every input consisting of a pair of elements g, h in F_n, decides whether $f \prec g$ or not. This is of course not the case for higher-rank, Abelian groups, yet in both cases the spaces of left-orders are just Cantor sets.

Left-orders v/s bi-orders. Despite the good understanding of $\mathcal{LO}(\mathbb{F}_2)$, the next question remains open.

Question 2.2.35. Does there exist a partial left-order on $\mathbb{F}_2$ that cannot be extended into a total left-order ?

Let us point out that the analogous question for bi-orders has a negative answer; see for instance [108, Corollary 3.6.7] and [146]. All of this is to be compared with the case of nilpotent groups, as described at the end of Example 1.2.1. In another direction, there is a major question that remains open for bi-orders on $\mathbb{F}_n$, first raised by McCleary in [174, Page 127].

Question 2.2.36. Are the spaces of bi-orders of (finitely-generated, non-Abelian) free groups homeomorphic to the Cantor set ?

Notice that, using the lower central series, one may provide a Cantor set of "standard" bi-orders on free groups. Nevertheless, according to Example 2.1.10, there are more bi-orders on free groups than these. Actually, it is shown in [105] that, although the action of $Out(\mathbb{F}_n)$ on $\mathcal{BO}(\mathbb{F}_n)$ is faithful, its restriction to the set of standard left-orders is not (the kernel coincides with the ***Torelli subgroup***, that is, the one whose elements act trivially on homology).

2.2.3 Finitely-generated positive cones

We first recall a short argument due to Linnell showing that, if a left-order $\preceq$ on a group Γ is non-isolated in $\mathcal{LO}(\Gamma)$, then its positive cone is not finitely generated as a semigroup.

Proposition 2.2.37. *If the positive cone of a left-order $\preceq$ on a group Γ is finitely generated as a semigroup, then $\preceq$ is non-isolated in $\mathcal{LO}(\Gamma)$.*

Proof. If $g_1, \ldots, g_k$ generate $P^+_{\preceq}$, then the only left-order on Γ that coincides with $\preceq$ on any set containing these generators and the identity element is $\preceq$ itself (see the exercise below). $\square$

Exercise 2.2.38. Show that if two left-orders $\preceq$ and $\preceq'$ on the same group satisfy $P^+_{\preceq} \subset P^+_{\preceq'}$, then they coincide.

The converse to the preceding proposition is not true. For instance, the dyadic rationals admit only two left-orders, though none of them has a finitely-generated positive cone. One may easily modify this example in order to obtain a finitely-generated one (see the example below). However, the search for conditions ensuring that isolated left-orders must have a finitely-generated positive cone is interesting by itself.

Example 2.2.39. The group of presentation $\Gamma := \langle a, b : bab^{-1} = a^{-2} \rangle$ is covered by Theorem 2.2.12: it has exactly four left-orders (compare Example 2.2.13), which depend only on the signs of the generators a, b. However, the positive cone of none of these orders is finitely generated since they all contain a copy of $\mathbb{Z}[\frac{1}{2}]$ inside $\langle\langle a \rangle\rangle$ (the smallest normal subgroup containing a).

Besides the obvious case of $\mathbb{Z}$, perhaps the simplest example of a finitely generated positive cone for a group left-order occurs for the Klein-bottle group $K_2 = \langle a, b : bab = a \rangle$: one may take $\langle a, b \rangle^+$ as such a cone (see Figure 4). Actually, K_2 admits exactly four left-orders, each of which having a finitely-generated positive cone (the other cones are $\langle a, b^{-1} \rangle^+$, $\langle a^{-1}, b \rangle^+$, and $\langle a^{-1}, b^{-1} \rangle^+$).

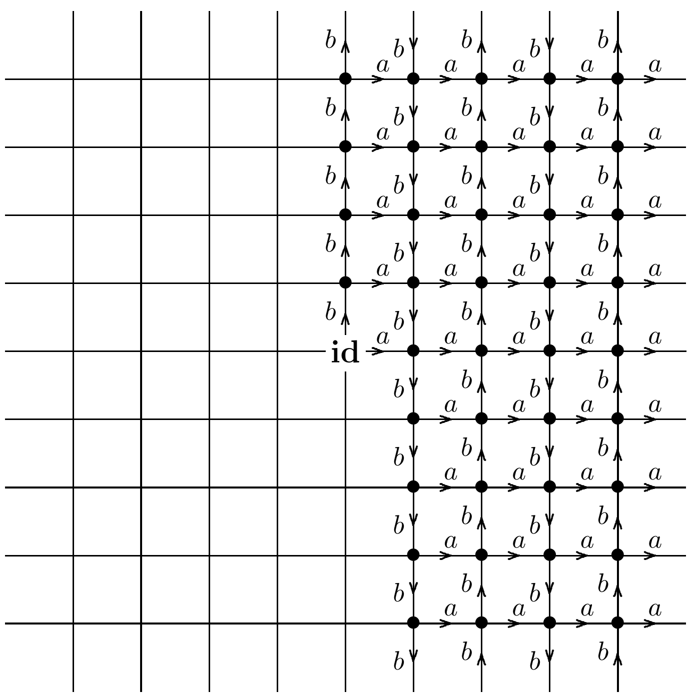

Figure 4: The positive cone $P^+ = \langle a, b \rangle^+$ on $K_2 = \langle a, b \colon a^{-1}ba = b^{-1} \rangle$.

Rather surprisingly, finitely-generated positive cones also occur on braid groups, according to a beautiful result due to Dubrovina and Dubrovin [59]

Theorem 2.2.40. *For each $n \geq 3$, the braid group $\mathbb{B}_n$ admits the decomposition*

$$\mathbb{B}_n = \left\langle a_1, \ldots, a_{n-1} \right\rangle^+ \sqcup \left\langle a_1^{-1}, \ldots, a_{n-1}^{-1} \right\rangle^+ \sqcup \{id\},$$

where $a_1 := \sigma_1 \cdots \sigma_{n-1}$, $a_2 := (\sigma_2 \cdots \sigma_{n-1})^{-1}$, $a_3 := \sigma_3 \cdots s_{n-1}$, $a_4 := (\sigma_4 \cdots \sigma_{n-1})^{-1}$, $\ldots$, *and* $a_{n-1} := \sigma_{n-1}^{(-1)^{n-1}}$.

Notice that this theorem also holds for $n = 2$, yet it is trivial in this case as $\mathbb{B}_2$ is isomorphic to $\mathbb{Z}$.

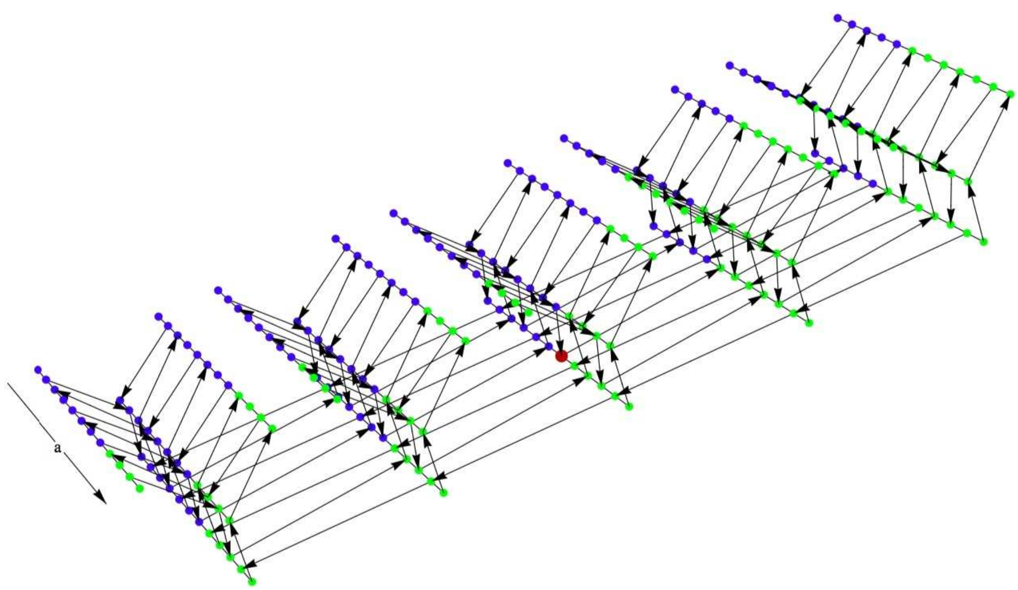

Figure 5: The DD-positive cone on $\mathbb{B}_3$.

As a particular case, for $\mathbb{B}_3$, the semigroup $P_{_{DD}} = \langle \sigma_1\sigma_2, \sigma_2^{-1} \rangle^+$ is the positive cone of a left-order $\preceq_{_{DD}}$. This can be visualized in Figure 5, where we depict the Cayley graph of $\mathbb{B}_3$ (essentially, a product of a *quasi-isometric* copy of $\mathbb{Z}^2$ by a dyadic rooted tree). Notice that for the generators $a = a_1 := \sigma_1\sigma_2$ and $b = a_2 := \sigma_2^{-1}$, the presentation of $\mathbb{B}_3$ becomes $\mathbb{B}_3 = \left\langle a, b \colon ba^2b = a \right\rangle$. Thus, in the picture above, an arrow pointing from left to right should be added to every diagonal edge of the graph. These arrows represent multiplications by a, while all arrows explicitly appearing represent multiplications by b. Starting at the identity (in red), every positive (green) element can be reached by a path that follows the direction of the arrows. Conversely, every negative (blue) element can be reached by a path starting at the identity following a direction opposite to that of the arrows. Finally, no (nontrivial) element can be reached both ways.

Quitte remarkably, this particular example was already known (and seems to be folklore, at least for a certain community) for a long time: see [58].

Retrieving the DD-order from the D-order. The proof of Theorem 2.2.40 strongly uses Dehornoy's theorem, as described in §1.2.5. To begin with, notice that it readily follows from the definition that for each $j \in \{1, \ldots, n-1\}$, the subgroup $\langle \sigma_j, \ldots, \sigma_{n-1} \rangle \sim \mathbb{B}_{n-j+1}$ of $\mathbb{B}_n$ is $\preceq_{D}$-convex.

In particular, for the case of $\mathbb{B}_3$, the cyclic subgroup $\langle \sigma_2 \rangle$ is $\preceq_{D}$-convex. One can hence define the order $\preceq_3$ on $\mathbb{B}_3$ as being the extension by $\preceq_{D}$ of the restriction to $\langle \sigma_2 \rangle$ of the reverse order $\overline{\preceq}_{D}$. (This corresponds to the flipping of $\preceq_{D}$ on $\langle \sigma_2 \rangle$, as introduced in Example 2.1.4.) We claim that the positive cone of $\preceq_3$ is generated by the elements $a_1 := \sigma_1 \sigma_2$ and $a_2 := \sigma_2^{-1}$, thus showing the theorem in this particular case. Indeed, by definition, these elements are positive with respect to $\preceq_3$. Thus, it suffices to show that for every $c \neq id$ in $\mathbb{B}_3$, either c or c^{-1} belongs to $\langle a_1, a_2 \rangle^+$. Now, if c or c^{-1} is 2-positive, then there exists an integer $m \neq 0$ such that $c = \sigma_2^m = a_2^{-m}$, and therefore $c \in \langle a_2 \rangle^+ \subset \langle a_1, a_2 \rangle^+$ if $m < 0$, and $c^{-1} \in \langle a_2 \rangle^+ \subset \langle a_1, a_2 \rangle^+$ if $m > 0$. If c is 1-positive, then for a certain choice of integers $m_1'', \ldots, m_{k''+1}''$, one has

$$c = \sigma_2^{m_1''} \sigma_1 \sigma_2^{m_2''} \sigma_1 \cdots \sigma_2^{m_{k''}''} \sigma_1 \sigma_2^{m_{k''+1}''}.$$

Using the identity $\sigma_1 = a_1 a_2$, this allows us to rewrite c in the form

$$c = a_2^{m_1'} a_1 a_2^{m_2'} a_1 \ldots a_2^{m_{k'}'} a_1 a_2^{m_{k'+1}'}$$

for some integers $m_1', \ldots, m_{k'+1}'$. Now, using several times the (easy to check) identity $a_2 a_1^2 a_2 = a_1$, one may easily express c as a product

$$c = a_2^{m_1} a_1 a_2^{m_2} a_1 \ldots a_2^{m_k} a_1 a_2^{m_{k+1}}$$

in which all the exponents m_i are non-negative. This shows that c belongs to $\langle a_1, a_2 \rangle^+$. Finally, if c^{-1} is 1-positive, then c^{-1} belongs to $\langle a_1, a_2 \rangle^+$.

The extension of the preceding argument to the general case proceeds inductively as follows. Let us see $\mathbb{B}_{n-1} = \langle \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-2} \rangle$ as a subgroup of $\mathbb{B}_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ via the homomorphism $\tilde{\sigma}_i \mapsto \sigma_{i+1}$. Then $\preceq_{n-1}$ induces an order on $\langle \sigma_2, \ldots, \sigma_{n-1} \rangle \subset \mathbb{B}_n$, which we still denote by $\preceq_{n-1}$. We then let $\preceq_n$ be the extension of $\overline{\preceq}_{n-1}$ by the D-order $\preceq_{D}$. Then, using the inductive hypothesis as well as the remarkable identities (that we leave to the reader)

$$(a_2 a_3^{-1} \cdots a_{n-1}^{(-1)^{n-1}}) a_1^{n-1} (a_2 a_3^{-1} \cdots a_{n-1}^{(-1)^{n-1}}) = a_1, \qquad (a_2 a_3^{-1} \cdots a_{n-1}^{(-1)^{n-1}})^2 = a_2^{n-1},$$

one may check as above that the positive cone of the order $\preceq_n$ coincides with the semigroup $\langle a_1, \ldots, a_{n-1} \rangle^+$, thus showing the theorem.

Exercise 2.2.41. Prove that the only convex subgroups of $\mathbb{B}_n$ for both the D-order and the DD-order are $C^0 := \mathbb{B}_n$, $C^1 := \langle a_2, \ldots, a_{n-1} \rangle = \langle \sigma_2, \ldots, \sigma_{n-1} \rangle$, $\ldots$, $C^{n-1} := \langle a_{n-1} \rangle = \langle \sigma_{n-1} \rangle$ and $C^n := \{id\}$.

Let us emphasize that assuming Theorem 2.2.40, we can follow the arguments above backwards and retrieve the Dehornoy's order on $\mathbb{B}_n$. (Details to the reader.) A more conceptual approach to this phenomenon was proposed by Ito in [95], and it is developed in the next

Exercise 2.2.42. Let $g_1, \ldots, g_k$ be finitely many generators of a group Γ. For each $i \in \{1, \ldots, k\}$, let $h_i := (g_i g_{i+1} \cdots g_k)^{(-1)^{k+1}}$, and denote by P_i the semigroup generated by $g_i, \ldots, g_k$. Assume that the following condition (called ***Property (F)*** in [95]) holds: For each $i \in \{1, \ldots, k-1\}$, both $g_i P_{i+1} g_i^{-1}$ and $g_i^{-1} P_{i+1} g_i$ are contained in the semigroup P_i^- made of the inverses of the elements in P_i.
(i) Prove that $h_1, \ldots, h_k$ generate the positive cone of a left-order on Γ if and only if the g_i's define a ***Dehornoy-like order***, which means that every nontrivial element may be written as a product of elements $g_i, \ldots, g_k$ so that g_i appears with only positive exponents, and no $g \in \Gamma$ is such that both g and g^{-1} may be written in such a way.
(ii) Referring to Theorem 2.2.40, check that $g_i := \sigma_i$ and $h_i := a_i$ satisfy property (F).

Torus-knot groups. We next give an elementary proof of that the torus-knot groups $G_{m,n} := \langle c, d : c^m = d^n \rangle$ do admit left-orders with finitely-generated positive cones. This is closely related to what was previously shown for braid groups, since for $(m,n) = (3,2)$ we retrieve the braid group $\mathbb{B}_3$ for the generating set $c \sim \sigma_1\sigma_2$ and $d \sim \sigma_1\sigma_2\sigma_1$. In this case, the positive cone given by Theorem 2.2.40 is generated by $a := c \sim \sigma_1\sigma_2$ and $b := c^{-2}d \sim \sigma_2^{-1}$, respect to which the presentation becomes $G_{3,2} = \langle a, b \colon ba^2b = a \rangle$. Also, notice that for $(m,n) = (2,2)$ we retrieve the Klein-bottle group. In this case, the generating system of the positive cone is made of $a := c$ and $b := c^{-1}d$, for which the presentation becomes $K_4 = \langle a, b \colon bab = a \rangle$.

After some computations, one easily convinces that the natural extension of this corresponds to the presentation

$$G_{m,n} = \big\langle a, b \colon (ba^{m-1})^{n-1} b = a \big\rangle,$$

where $a := c$ and $b := c^{-(m-1)}d$. The following result appears in [136] for $n = 2$ and in [95] for the general case.

Theorem 2.2.43. *For each m, n both larger than 1, the group $G_{m,n}$ can be decomposed as*

$$G_{m,n} = \langle a, b \rangle^{+} \sqcup \langle a^{-1}, b^{-1} \rangle^{+} \sqcup \{id\}.$$

Quite naturally, the proof of this theorem involves two issues:

Step I. Every nontrivial element lies in $\langle a, b \rangle^{+} \cup \langle a^{-1}, b^{-1} \rangle^{+}$.

Step II. No nontrivial element lies in both $\langle a, b \rangle^{+}$ and $\langle a^{-1}, b^{-1} \rangle^{+}$.

In what follows, we only consider the case $(m, n) \neq (2, 2)$, because $(m, n) = (2, 2)$ corresponds to the Klein's bottle group K_4 as previously described. (Some of the arguments below do not apply in this case.) For Step I, we begin with

Claim (i). The element $\Delta := a^{m}$ belongs to the center of $G_{m,n}$.

Indeed, from $(ba^{m-1})^{n-1} b = a$ it follows that $(ba^{m-1})^{n} = (a^{m-1}b)^{n} = a^{m}$. Thus,

$$b\Delta = ba^{m} = b(a^{m-1}b)^{n} = (ba^{m-1})^{n} b = a^{m} b = \Delta b.$$

Moreover, $a\Delta = a^{m+1} = \Delta a.$

A word in (positive powers of) a, b (resp. a^{-1}, b^{-1}) will be said to be positive (resp. negative). It is non-positive (resp. non-negative) if it is either trivial or negative (resp. either trivial or positive).

Claim (ii). Every element $w \in G_{m,n}$ may be written in the form $\bar{u}\Delta^{\ell}$ for some non-negative word $\bar{u}$ and $\ell \in \mathbb{Z}$.

Indeed, in any word representing w, we may rewrite the negative powers of a and b using the relations

$$a^{-1} = a^{m-1}\Delta^{-1}, \qquad b^{-1} = a^{-1}(ba^{m-1})^{n-1} = a^{m-1}\Delta^{-1}(ba^{m-1})^{n-1},$$

and then use the fact that Δ belongs to the center of $G_{m,n}$.

Notice that since $a^{m} = \Delta$, every $u \in \langle a, b \rangle^{+}$ may be written in the form

$$u = b^{s_0} a^{r_1} b^{s_1} \cdots b^{s_{k-1}} a^{r_k} \Delta^{\ell},$$

where $s_i > 0$ for $i \in \{1, \ldots, k-1\}$, $s_0 \geq 0$, $r_i \in \{1, \ldots, m-1\}$ for $i \in \{1, \ldots, k-1\}$, $r_k \geq 0$, and $\ell \geq 0$. Therefore, by Claim (ii), every $w \in G_{m,n}$ may be written as

$$w \;=\; b^{s_0} a^{r_1} b^{s_1} \cdots b^{s_{k-1}} a^{r_k} \Delta^{\ell} \;=\; b^{s_0} a^{r_1} b^{s_1} \cdots b^{s_{k-1}} a^{r_k + m\ell}, \tag{2.1}$$

where the properties of r_i, s_i above are satisfied, and $\ell \in \mathbb{Z}$. Such an expression will be said to be a ***normal form*** for w if k is minimal. (Notice that, *a priori*, these normal forms may be non-unique for a given w.)

There are two cases to consider. If $r_k + m\ell \geq 0$, then w is obviously non-negative. Step I is then concluded by the next

Claim (iii): If $r_k + m\ell < 0$, then w is negative.

The proof is by induction on the length k of the normal form. To begin with, notice that $(ba^{m-1})^{n-1}b = a$ yields $ba^{-1} = (a^{-m+1}b^{-1})^{n-1}$. Thus, for $k=1$, that is, for $w = b^{s_0}a^{r_1+m\ell}$, we have

$$w = a^{-1}\big[aba^{-1}\big]^{s_0} a^{r_1+m\ell+1} = a^{-1}\big[a(a^{-m+1}b^{-1})^{n-1}\big]^{s_0} a^{r_1+m\ell+1},$$

and this expressions is easily seen to be negative by canceling the factor a at each place, namely $a(a^{-m+1}b^{-1})^{n-1} = a^{-m+2}b^{-1}(a^{-m+1}b^{-1})^{n-2}$. Assume the claim holds up to $k-1$. Using the identity above, expression (2.1) becomes

$$w = b^{s_0}a^{r_1}\cdots b^{s_{k-1}-1}[ba^{-1}]a^{r_k+m\ell-1} = b^{s_0}a^{r_1}\cdots b^{s_{k-1}+1}[(a^{-m+1}b^{-1})^{n-1}]a^{m_k+m\ell+1}.$$

If $s_{k-1} > 1$, then writing

$$w = b^{s_0}a^{r_1}\cdots b^{s_{k-1}-2}[ba^{-1}]a^{-m+2}b^{-1}(a^{-m+1}b^{-1})^{n-2}a^{r_k+m\ell+1},$$

we see we can repeat the process changing ba^{-1} by $(a^{-m+1}b^{-1})^{n-1}$. Otherwise,

$$\begin{aligned} w &= b^{s_0}a^{r_1}\cdots a^{r_{k-1}}[(a^{-m+1}b^{-1})^{n-1}]a^{r_k+m\ell+1} \\ &= b^{s_0}a^{r_1}\cdots b^{s_{k-2}}a^{r_{k-1}-m+1}b^{-1}(a^{-m+1}b^{-1})^{n-2}a^{r_k+m\ell+1} \\ &= b^{s_0}a^{r_1}\cdots b^{s_{k-2}-1}[ba^{-1}]a^{r_{k-1}-m+2}b^{-1}(a^{-m+1}b^{-1})^{n-2}a^{r_k+m\ell+1}, \end{aligned}$$

so that in case $r_k < m-1$ (and $s_{k-2} > 0$), we may repeat the procedure. Continuing in this way, one easily convinces that, unless

$$s_{k-1} = \ldots = s_{k-(n-1)} = 1, \quad s_{k-n} > 0, \quad r_{k-1} = \ldots = r_{k-(n-1)} = m-1, \quad (2.2)$$

the expression for w above may reduced into one of the form

$$w = b^{s_0}\cdots b^{s_i}a^{-1}\overline{\omega}$$

for certain $s_i > 0$, $i < k$, and a non-positive word $\overline{\omega}$. As $i < k$, the induction hypothesis applies to $b^{s_0}\cdots b^{s_i}a^{-1}$, which is hence negative, and so does w. Assume otherwise that (2.2) holds. Then since $b(a^{m-1}b)^{n-1} = a$, replacing $(a^{m-1}b)^{n-1}$ by $b^{-1}a$ and canceling b^{-1}, we obtain a new expression for w of the form

$$w = b^{s_0}\cdots b^{s_{k-n}-1}a^{r_k+m\ell+1},$$

which contradicts the minimality of k. This closes the proof.

Step II of the proof of Theorem 2.2.43 can be established via several approaches. Here, we chose the dynamical one, based on the fact that $G_{m,n}$ embeds into $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. To see this, let us first come back to the presentation

$$G_{m,n} \,=\, \big\langle c, d \colon c^m = d^n \big\rangle,$$

which exhibits $G_{m,n}$ as a central extension of the group

$$\overline{G}_{m,n} = \big\langle \bar{c}, \bar{d} \colon \bar{c}^m = \bar{d}^n = id \big\rangle.$$

A concrete realization of $\overline{G}_{m,n}$ inside $\mathrm{PSL}(2,\mathbb{R})$ arises when identifying $\bar{c}$ to the circle rotation of angle $\frac{2\pi}{m}$, and $\bar{d}$ to an hyperbolic rotation of angle $\frac{2\pi}{n}$ centered at a point different from the origin in such a way that for a certain $p \in \mathrm{S}^1$, letting $p_0 := p, p_1 := \bar{c}(p), \ldots, p_{m-1} := \bar{c}^{m-1}(p)$ and $q_0 := p, q_1 := \bar{d}(p), \ldots, q_{n-1} := \bar{d}^{n-1}(p)$, we have that all the q_i's lie between p_0 and p_1, and $q_{n-1} = p_1$. This realization allows embedding $G_{m,n}$ into $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ by identifying $c \in G_{m,n}$ to the lifting of $\bar{c}$ to the real line given by $x \mapsto x + \frac{2\pi}{m}$, and d to the unique lifting of $\bar{d}$ to the real line satisfying $x \leq \bar{d}(x) \leq x + 2\pi$ for all $x \in \mathbb{R}$. (Actually, the arguments given so far only show that the above identifications induce a group homomorphism from $G_{m,n}$ into $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, and the injectivity follows from the arguments given below.)

The dynamics of the action of $\overline{G}_{m,n}$ on the circle is illustrated in Figure 6. Passing to the generators a, b, we have that $\bar{b} := \bar{c}^{-(m-1)} \bar{d} = \bar{c}\bar{d}$ is a parabolic Möbius transformation fixing p_1, and $\bar{a} := \bar{c}$. Using this, we next proceed to show that no element w in $\langle a, b\rangle^+ \subset G_{m,n}$ represents the identity. By taking inverses, this will imply that no element in $\langle a^{-1}, b^{-1}\rangle^+$ represents the identity, thus completing the proof.

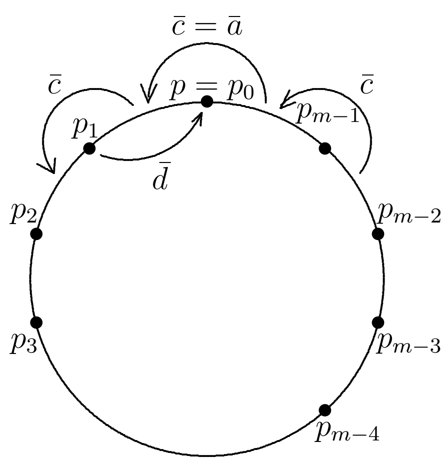

Figure 6

We begin by writing w in the form

$$w = b^{s_0} a^{r_1} b^{s_1} \cdots a^{r_k} a^{m\ell}, \quad \ell \geq 0,$$

with the corresponding restrictions on the exponents. Here, we may assume that no expression $(ba^{m-1})^{n-1}b$ appears, otherwise we may replace it by a.

Assume that w is not a power of a, and let us consider its reduction

$$\bar{w} = \bar{b}^{s_0} \bar{a}^{r_1} \bar{b}^{s_1} \cdots \bar{a}^{r_k} \in \widetilde{\mathrm{PSL}}_2(\mathbb{R}).$$

Using $\bar{b} = \bar{a}\bar{d}$ and simplifying $\bar{a}^m = id$, we may rewrite this in the form

$$\bar{w} = \bar{d}^{s'_0} \bar{a}^{r'_1} \bar{d}^{s'_1} \cdots \bar{a}^{r'_{k'}} \in \widetilde{\mathrm{PSL}}_2(\mathbb{R}),$$

with similar restrictions on the exponents r'_i, s'_i. What is crucial here is that the fact that no expression $(ba^{m-1})^{n-1}b$ appeared in the original form implies that this new expression is nontrivial, as it can be easily checked. (Indeed, no cancellation $\bar{d}^n = id$ will be performed.)

Unless $\bar{w}$ is a power of $\bar{d}\bar{a}$, we may conjugate it into either some $\bar{w}' \in \langle \bar{a}, \bar{b} \rangle^+$ beginning and finishing by $\bar{a}$ and so that all the exponents of $\bar{a}$ lie in $\{1, \ldots, m-1\}$, or to some $\bar{w}'' \in \langle \bar{a}, \bar{d} \rangle^+$ beginning and finishing with $\bar{b}$ with the same restriction on the exponents of $\bar{a}$. An easy ping-pong type argument then shows that $\bar{w}'\big(]p_0, p_1[\big) \subset]p_1, p_0[$ and $\bar{w}''\big(]p_1, p_0[\big) \subset]p_0, p_1[$, hence $\bar{w}' \neq id$ and $\bar{w}'' \neq id$.

Thus, to conclude the proof, we need to check that neither a nor da are torsion elements. That da is not torsion follows from that $\bar{d}\bar{a}$ sends $[p_0, p_1]$ into the strict subinterval $[p_0, d(p_2)]$, hence no iterate of it can equal the identity. Finally, to see that a is not torsion, just notice that it identifies to the translation by $\frac{2\pi}{m}$ in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$.

Some other examples. Searching for more examples of finitely-generated positive cones in groups with infinitely many left-orders has become a topic of much activity over the last years. The examples given above as well as the techniques used in proofs have been pursued in three directions. First, there is the close relation with Dehornoy-like orders in which the previous examples fit, as described in [136] and later in [95]. (See also Remark 3.2.45.) Second, there is an approach based on partial cyclic amalgamation, which is fully developed in [94]. This allows iterative implementation, thus establishing for instance that the groups

$$G_{m_1, m_2, \ldots, m_n} := \big\langle a_1, \ldots, a_n \colon a_1^{m_1} = a_2^{m_2} = \ldots = a_n^{m_n} \big\rangle$$

do admit finitely-generated positive cones. This approach was somewhat complemented in [93]; however, the orders constructed therein are only ensured to be isolated, and knowing whether their positive cones are finitely generated remains an interesting question. Finally, there is a more combinatorial approach starting from group presentations introduced in [48]. Roughly, in case these presentations have a *triangular form*, finitely-generated positive cones naturally appear. As a random example, we can mention that the groups

$$H_{m,n} := \big\langle a, b, c \colon a = ba^2(b^2a^2)^m c, b = c(ba^2)^n ba \big\rangle$$

fall in this category.

We do not pursue on this nice subject here; we just refer the reader to the works mentioned above for the announced results and further developments. Nevertheless, let us mention that none of these approaches has provided a new proof of Dehornoy's theorem concerning the D-order on $\mathbb{B}_n$ for $n \geq 4$. This issue seems to be beyond the scope of these methods.

Chapter 3

ORDERABLE GROUPS AS DYNAMICAL OBJECTS

3.1 Hölder's Theorem

The results of this section –essentially due to Hölder– are classical and perhaps correspond to the most beautiful founder pieces of the theory. They characterize group left-orders satisfying an Archimedean type property: the underlying ordered group must be ordered isomorphic to a subgroup of $(\mathbb{R},+)$.

Definition 3.1.1. A left-order $\preceq$ on a group Γ is said to be ***Archimedean*** if for all g,h in Γ such that $g \neq id$, there exists $n \in \mathbb{Z}$ satisfying $g^n \succ h$.

Theorem 3.1.2. *Every group endowed with an Archimedean left-order is order-isomorphic to a subgroup of* $(\mathbb{R},+)$.

Hölder proved this theorem under the extra assumption that the group is Abelian. However, his arguments work verbatim without this hypothesis but assuming that the left-order is bi-invariant. That this hypothesis is also superfluous was first remarked by Conrad in [44].

Lemma 3.1.3. *Every Archimedean left-order on a group is bi-invariant.*

Proof. Let $\preceq$ be an Archimedean left-order on a group Γ. We need to show that its positive cone is a normal semigroup.

Suppose that $g \in P_{\preceq}^{+}$ and $h \in P_{\preceq}^{-}$ are such that $hgh^{-1} \notin P_{\preceq}^{+}$. Let n be the smallest positive integer for which $h^{-1} \prec g^n$. Since $hgh^{-1} \prec id$, we have

$h^{-1} \prec g^{-1}h^{-1} \prec g^{n-1}$, which contradicts the definition of n. We thus conclude that $P_{\preceq}^{+}$ is stable under conjugacy by elements in $P_{\preceq}^{-}$.

Assume now that g, h in $P_{\preceq}^{+}$ verify $hgh^{-1} \notin P_{\preceq}^{+}$. In this case, $hg^{-1}h^{-1} \succ id$, and since $h^{-1} \in P_{\preceq}^{-}$, the first part of the proof yields $h^{-1}(hg^{-1}h^{-1})h \in P_{\preceq}^{+}$, that is, $g^{-1} \in P_{\preceq}^{+}$, which is absurd. Hence, $P_{\preceq}^{+}$ is also stable under conjugacy by elements in $P_{\preceq}^{+}$, which concludes the proof. $\square$

Exercise 3.1.4. Prove the preceding lemma by using dynamical realizations. More precisely, show that the dynamical realization of every Archimedean left-order on a countable group is a subgroup of $\mathrm{Homeo}_{+}(\mathbb{R})$ acting freely on the line (*c.f.* §1.1.3; compare Example 3.1.6).

Proof of Theorem 3.1.2. Let Γ be a group endowed with an Archimedean left-order $\preceq$. By Lemma 3.1.3, we know that $\preceq$ is bi-invariant. Let us fix a positive element $f \in \Gamma$, and for each $g \in \Gamma$ and each $p \in \mathbb{N}$, let us consider the unique integer $q = q(p)$ such that $f^{q} \preceq g^{p} \prec f^{q+1}$.

Claim (i). The sequence $\big(q(p)/p\big)$ converges to a real number as p goes to infinity.

Indeed, if $f^{q(p_1)} \preceq g^{p_1} \prec f^{q(p_1)+1}$ and $f^{q(p_2)} \preceq g^{p_2} \prec f^{q(p_2)+1}$, then the bi-invariance of $\preceq$ yields

$$f^{q(p_1)+q(p_2)} \preceq g^{p_1+p_2} \prec f^{q(p_1)+q(p_2)+2}.$$

Therefore, $q(p_1) + q(p_2) \leq q(p_1 + p_2) \leq q(p_1) + q(p_2) + 1$. The convergence of the sequence $(q(p)/p)$ then follows from Exercise 3.1.5.

Claim (ii). The map $\phi : \Gamma \to (\mathbb{R}, +)$ is a group homomorphism.

Indeed, let g_1, g_2 be arbitrary elements in Γ. Let us suppose that $g_1g_2 \preceq g_2g_1$ (the case $g_2g_1 \preceq g_1g_2$ is analogous). Since $\preceq$ is bi-invariant, if $f^{q_1} \preceq g_1^{p} \prec f^{q_1+1}$ and $f^{q_2} \preceq g_2^{p} \prec f^{q_2+1}$, then

$$f^{q_1+q_2} \preceq g_1^{p}g_2^{p} \preceq (g_1g_2)^{p} \preceq g_2^{p}g_1^{p} \prec f^{q_1+q_2+2}.$$

From these relations one concludes that

$$\phi(g_1) + \phi(g_2) = \lim_{p\to\infty} \frac{q_1+q_2}{p} \leq \phi(g_1g_2) \leq \lim_{p\to\infty} \frac{q_1+q_2+1}{p} = \phi(g_1) + \phi(g_2),$$

and therefore $\phi(g_1g_2) = \phi(g_1) + \phi(g_2)$.

Claim (iii). The homomorphism ϕ is one-to-one and order-preserving.

That ϕ is order-preserving (in the sense that if $g_1 \preceq g_2$ then $\phi(g_1) \leq \phi(g_2)$) follows from the definition. To show injectivity, first notice that $\phi(f) = 1$. Let h be an element in Γ such that $\phi(h) = 0$. Assume that $h \neq id$. Then there exists $n \in \mathbb{Z}$ such that $h^n \succeq f$. Consequently, $0 = n\phi(h) = \phi(h^n) \geq \phi(f) = 1$, which is absurd. Therefore, if $\phi(h) = 0$ then $h = id$. $\square$

Exercise 3.1.5. Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers. Assume that there exists a constant $C \in \mathbb{R}$ such that, for all m, n in $\mathbb{Z}$,

$$|a_{m+n} - a_m - a_n| \leq C. \tag{3.1}$$

Show that there exists a unique $\theta \in \mathbb{R}$ such that the sequence $\big(|a_n - n\Theta|\big)$ is bounded. Check that this number θ is equal to the limit of the sequence (a_n/n) as n goes to $\pm\infty$ (in particular, this limit exists).
Hint. For each $n \in \mathbb{N}$ let $I_n := \big[(a_n - C)/n, (a_n + C)/n\big]$. Check that I_{mn} is contained in I_n for every m, n in $\mathbb{N}$. Conclude that $I := \bigcap_{n\in\mathbb{N}} I_n$ is nonempty (any θ in I satisfies the desired property).

Example 3.1.6. Examples of groups admitting Archimedean left-orders are groups acting freely on the real line. Indeed, from such an action one may define $\preceq$ on Γ by letting $g \prec h$ if $g(x) < h(x)$ for some (equivalently, for all) $x \in \mathbb{R}$. This order relation is total, and using the fact that the action is free, one readily shows that it is Archimedean (as well as bi-invariant).

Notice that, by the proof of Theorem 3.1.2, the left-order $\preceq$ above induces an embedding ϕ of Γ into $(\mathbb{R}, +)$. If $\phi(\Gamma)$ is isomorphic to $\mathbb{Z}$, then the action of Γ is conjugate to the action by integer translations. Otherwise, unless Γ is trivial, $\phi(\Gamma)$ is dense in $(\mathrm{R}, +)$. For each point x in the line we may then define

$$\varphi(x) = \sup\big\{\phi(h) \in \mathbb{R} \colon h(0) \leq x\big\}.$$

It is easy to see that $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing map. Moreover, it satisfies $\varphi(h(x)) = \varphi(x) + \phi(h)$ for all $x \in \mathbb{R}$ and all $h \in \Gamma$. Finally, φ is continuous, as otherwise the set $\mathbb{R} \setminus \varphi(\mathbb{R})$ would be a nonempty open set invariant by the translations of $\phi(\Gamma)$, which is impossible. In summary, every free action on the line is semiconjugate to an action by translations.

3.2 The Conrad Property

3.2.1 The classical approach revisited

A left-order $\preceq$ on a group Γ is said to be ***Conradian*** (a C***-order***, for short) if for all positive elements f, g, there exists $n \in \mathbb{N}$ such that $fg^n \succ g$.

Bi-invariant left-orders are Conradian, as $n = 1$ works in the preceding inequality for bi-orders. In this direction, it is quite remarkable that one may actually take $n = 2$ in the general definition above, as the next proposition shows (the nice proof below, due to Jiménez, is taken from [97]).

Proposition 3.2.1. *If $\preceq$ is a Conradian order on a group, then $fg^2 \succ g$ holds for all positive elements f, g.*

Proof. Suppose that two positive elements f, g for a left-order $\preceq'$ on a group Γ are such that $fg^2 \preceq' g$. Then $(g^{-1}fg)g \preceq' id$, and since g is a positive element, this implies that $g^{-1}fg$ is negative, and therefore $fg \prec' g$. Now for the positive element $h := fg$ and every $n \in \mathbb{N}$, one has

$$\begin{aligned} fh^n = f(fg)^n = f(fg)^{n-2}(fg)(fg) &\prec' f(fg)^{n-2}(fg)g \\ = f(fg)^{n-2}fg^2 \preceq' f(fg)^{n-2}g &= f(fg)^{n-3}fg^2 \preceq' f(fg)^{n-3}g \preceq' \ldots \\ &\preceq' f(fg)g = ffg^2 \preceq' fg = h. \end{aligned}$$

This shows that $\preceq'$ does not satisfy the Conrad property. $\square$

The following is an easy (but important) corollary to the previous proposition, and we leave its proof to the reader. (Compare Exercise 2.2.3.)

Corollary 3.2.2. *For every left-orderable group, the subspace $\mathcal{CO}(\Gamma)$ of Conradian orders is closed inside the space of left-orders. Moreover, this subspace is invariant under the conjugacy action.*

Perhaps the most important theorem concerning C-orderable groups is the next one. The direct implication is due to Conrad [44]; the converse is due to Brodski [21], yet it was independently rediscovered by Rhemtulla and Rolfsen [151]. We postpone the proof of the first part, and for the second we offer an elementary one taken from [138]. Recall that a group is said to be ***locally indicable*** if each nontrivial finitely-generated subgroup admits a nontrivial homomorphism into $(\mathbb{R}, +)$.

Theorem 3.2.3. *A group Γ is C-orderable if and only if it is locally indicable.*

To show that local indicability implies C-orderability (the converse will be proved in §3.2.3), we will need the following lemma whose proof is left to the reader. (Compare §1.1.2.)

Lemma 3.2.4. *A group Γ is C-orderable if and only if for every finite family $\mathcal{G}$ of elements in $\Gamma \setminus \{id\}$, there exist a choice of exponents $\eta \colon \mathcal{G} \to \{-1,+1\}$ such that id does not belong to the smallest subsemigroup $\langle\langle \mathcal{G} \rangle\rangle$ satisfying:*

– It contains all the elements $g^{\eta(g)}$, with $g \in \mathcal{G}$;

– For all f, g in the semigroup, the element $g^{-1} f g^2$ also belongs to it.

Local indicability implies C-orderability. We need to check that every locally indicable group Γ satisfies the condition of the preceding lemma. Let $\{g_1, \ldots, g_k\}$ be a finite family of elements in Γ different from the identity. By hypothesis, there is a nontrivial homomorphism $\phi_1 : \langle g_1, \ldots, g_k \rangle \to (\mathbb{R}, +)$. Let $i_1, \ldots, i_{k'}$ be the indices (if any) such that $\phi_1(g_{i_j}) = 0$. Again by hypothesis, there exists a nontrivial homomorphism $\phi_2 \colon \langle g_{i_1}, \ldots, g_{i_{k'}} \rangle \to (\mathbb{R}, +)$. Letting $i'_1, \ldots, i'_{k''}$ be the indices in $\{i_1, \ldots, i_{k'}\}$ for which $\phi_2(g_{i'_j}) = 0$, we may choose a nontrivial homomorphism $\phi_3 \colon \langle g_{i'_1}, \ldots, g_{i'_{k''}} \rangle \to (\mathbb{R}, +)$... Notice that this process must finish in a finite number of steps (indeed, it stops in at most k steps). Now, for each $i \in \{1, \ldots, k\}$, choose the (unique) index $j(i)$ such that $\phi_{j(i)}$ is defined at g_i and $\phi_{j(i)}(g_i) \neq 0$, and let $\eta_i := \eta(g_i) \in \{-1,+1\}$ be so that $\phi_{j(i)}(g_i^{\eta_i}) > 0$. We claim that this choice of exponents η_i is "compatible". Indeed, for every index j and every f, g for which ϕ_j are defined, one has $\phi_j(f^{-1} g f^2) = \phi_j(f) + \phi_j(g)$. Therefore, $\phi_1(h) \geq 0$ for every $h \in \langle\langle g_1^{\eta_1}, \ldots, g_k^{\eta_k} \rangle\rangle$. Moreover, if $\phi_1(h) = 0$, then h actually belongs to $\langle\langle g_{i_1}^{\eta_{i_1}}, \ldots, g_{i_{k'}}^{\eta_{i_{k'}}} \rangle\rangle$. In this case, the preceding argument shows that $\phi_2(h) \geq 0$, with equality if and only if $h \in \langle\langle g_{i'_1}^{\eta_{i'_1}}, \ldots, g_{i'_{k''}}^{\eta_{i'_{k''}}} \rangle\rangle$... Continuing in this way, one concludes that $\phi_j(h)$ must be strictly positive for some index j. Thus, the element h cannot be equal to the identity, and this finishes the proof.

If a group Γ contains a normal subgroup Γ_* so that both Γ_* and Γ/Γ_* are locally indicable, then Γ itself is locally indicable. Equivalently, the extension of a C-orderable group by a C-orderable group is C-orderable. In a similar way we have:

Exercise 3.2.5. Let $(\Gamma, \preceq)$ be a C-ordered group, and let Γ_* be a convex subgroup. Show that for any C-order $\preceq_*$ of Γ_*, the extension of $\preceq_*$ by $\preceq$ is still Conradian. In particular, every left-order obtained from a C-left-order by flipping a convex subgroup is Conradian (*c.f.* §2.1.1).

Example 3.2.6. A remarkable theorem independently obtained by Brodski [21] and Howie [91] asserts that torsion-free, 1-relator groups are locally indicable. Also, all knot groups in $\mathbb{R}^3$ are locally indicable (see [92, Lemma 2]).

Examples of left-orderable, non C-orderable groups. Only a few examples are known. Historically, the first was exhibited (in a slightly different context) by Thurston [170], and rediscovered some years later by Bergman [10]. It corresponds to the lifting to $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the $(2,3,7)$-triangle group, and has the presentation

$$\Gamma = \big\langle f, g, h \colon f^2 = g^3 = h^7 = fgh \big\rangle.$$

Left-orderability follows from that $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$. The fact that Γ is not C-orderable is a consequence of that it has no nontrivial homomorphism into $(\mathbb{R},+)$, which may be easily deduced from the presentation above. Actually, Γ is the π_1 of an homological sphere, and this was the motivation of Thurston to deal with this group in his generalization of the famous Reeb stability theorem for codimension-1 foliations. We strongly recommend the lecture of [170] for all of this.

Below we elaborate on a different and quite important example, namely braid groups $\mathbb{B}_n$ for $n \geq 5$. Another example is the lifting $\widetilde{\mathrm{G}}$ of Thompson's group G to the real line; see [32] for more details.

Example 3.2.7. The braid groups $\mathbb{B}_3$ and $\mathbb{B}_4$ are locally indicable. For $\mathbb{B}_3$, this may be easily deduced from the exact sequence

$$0 \longrightarrow [\mathbb{B}_3, \mathbb{B}_3] \sim \mathbb{F}_2 \longrightarrow \mathbb{B}_3 \longrightarrow \mathbb{B}_3/[\mathbb{B}_3,\mathbb{B}_3] \sim \mathbb{Z} \longrightarrow 0,$$

where the isomorphism $[\mathbb{B}_3,\mathbb{B}_3] \sim \mathbb{F}_2$ may be shown by looking the action on the circle of $\mathbb{B}_3 \sim \widetilde{\mathrm{PSL}}(2,\mathbb{Z})$, and $\mathbb{B}_3/[\mathbb{B}_3,\mathbb{B}_3] \sim \mathbb{Z}$ appears by taking "total exponents". For $\mathbb{B}_4$, there is an exact sequence

$$0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{B}_4 \longrightarrow \mathbb{B}_3 \longrightarrow 0.$$

Here, the homomorphism from $\mathbb{B}_4$ to $\mathbb{B}_3$ is the one that sends σ_1 and σ_3 to σ_1, and σ_2 to σ_2. Its kernel is generated by $\sigma_1\sigma_3^{-1}$ and $\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$. To show that these elements are free generators, one may consider the homomorphism $\phi \colon \mathbb{B}_4 \to \mathrm{Aut}(\mathbb{F}_2)$ defined by $\phi(\sigma_1)(a) := a$, $\phi(\sigma_1)(b) := ab$, $\phi(\sigma_2)(a) := b^{-1}a$, $\phi(\sigma_2)(b) := b$, $\phi(\sigma_3)(a) := a$, $\phi(\sigma_3)(b) := ba$, and notice that $\phi(\sigma_1\sigma_3^{-1})$ (resp. $\phi(\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1})$) is the conjugacy by a (resp. $b^{-1}a$).

Incompatibility between bi-orders on $\mathrm{P}\mathbb{B}_n$ and left-orders on $\mathbb{B}_n$. In contrast to $\mathbb{B}_3$ and $\mathbb{B}_4$, the groups $\mathbb{B}_n$ fail to be locally indicable for $n \geq 5$. Indeed, for $n \geq 5$, the commutator subgroup $\mathbb{B}_n'$ is (finitely generated and) ***perfect*** (*i.e.* it coincides with its own commutator subgroup), as shown below.

Example 3.2.8. As is well-known (and easy to check), the commutator subgroup $\mathbb{B}_n'$ is generated by the elements of the form $s_{i,j} = \sigma_i \sigma_j^{-1}$. Also, recall that all the generators

σ_i of $\mathbb{B}_n$ are conjugate between them. Indeed, letting $\Delta := \sigma_1 \sigma_2 \cdots \sigma_{n-1}$, one readily checks that $\sigma_i \Delta = \Delta\, \sigma_{i-1}$. Now for all $i \in \{1, \ldots, n-3\}$, the equality

$$\sigma_{i,i+2} = (\sigma_i \sigma_{i+1})^{-1} \, [\sigma_{i,i+2}, \sigma_{i+1,i}] \, (\sigma_i \sigma_{i+1})$$

shows that $\sigma_{i,i+2}$ belongs to $\mathbb{B}_n''$. We will close the proof by showing that, for $n \geq 5$, the normal closure H (in $\mathbb{B}_n$) of the family of elements $\sigma_{i,i+2}$ (equivalently, of each $\sigma_{i,i+2}$) is $\mathbb{B}_n'$. To do this, notice that $\sigma_{i,j}$ and $\sigma_{i,j'}$ are conjugate whenever $\{j, j'\} \cap \{i-1, i+1\} = \emptyset$. Indeed, one may perform a conjugacy between σ_j and σ_j' as above but inside the subgroup $\mathbb{B}_{n-2}' \subset \mathbb{B}_n$ consisting of braids for which the i and $i+1$ strands remain "fixed"; such a conjugacy does not change σ_i. Therefore, $\sigma_{i,j}$ belongs to H for all $j \notin \{i-1, i+1\}$. Moreover, since for all $j \notin \{i-1, i, i+1, i+2\}$ (resp. $j \notin \{i-2, i-1, i, i+1\}$),

$$\sigma_{i,i+1} = \sigma_{i,j} \sigma_{j,i+1} \qquad (\text{resp. } \sigma_{i,i-1} = \sigma_{i,j} \sigma_{j,i-1}),$$

the elements $s_{i,i+1}$ and $s_{i,i-1}$ also belong to H. This shows that H coincides with $\mathbb{B}_n'$.

We recommend [131] for more details on this example as well as generalizations in the context of Artin groups.

A nice consequence of the example above is that the bi-orders on $\mathrm{P}\mathbb{B}_n$ do not extend to left-orders on $\mathbb{B}_n$ for any $n \geq 5$. (This fact was established, independently, in [59] and [151].) Indeed, we have the following

Proposition 3.2.9. *Let Γ_0 be a finite-index subgroup of a left-orderable group Γ. If $\preceq$ is a left-order on Γ whose restriction to Γ_0 is Conradian, then $\preceq$ is Conradian.*

Proof. Let $f \succ id$ and $g \succ id$ be elements in Γ. One has $f^m \in \Gamma_0$ and $g^n \in \Gamma_0$ for some positive n, m smaller than or equal to the index of Γ_0 in Γ. Hence, $f^m g^{2n} \succ g^n \succ g$. We claim that this implies that either $fg \succ g$ or $fg^{2n} \succ g$. Otherwise, $g^{-1} f g \prec id$ and $g^{-1} f g^{2n} \prec id$. Thus,

$$id \prec g^{-1} f^m g^{2n} = (g^{-1} f g)^{m-1} (g^{-1} f g^{2n}) \prec id,$$

which is a contradiction. □

A criterion of non left-orderability. Proposition 3.2.9 allows showing that certain "small" groups cannot be left-ordered. In concrete terms, we have the following result due to Rhemtulla [15, Chapter 7]:

Proposition 3.2.10. *Let Γ be a finitely-generated group containing a finite-index subgroup Γ_0 all of whose left-orders are Conradian. If Γ has no nontrivial homomorphism into $(\mathbb{R}, +)$, then Γ is not left-orderable.*

Indeed, if Γ were left-orderable then, by Proposition 3.2.9, every left-order on it would be Conradian. Since Γ is finitely generated, Theorem 3.2.3 would provide us with a nontrivial homomorphism into $(\mathbb{R},+)$.

Example 3.2.11. In §1.4.1, we introduced the group

$$\Gamma = \big\langle a, b \colon a^2ba^2 = b, b^2ab^2 = a \big\rangle,$$

which contains an index-4 Abelian subgroup, namely $\langle a^2, b^2, (ab)^2\rangle \sim \mathbb{Z}^3$. From the presentation, it follows that Γ admits no nontrivial homomorphism into $(\mathbb{R},+)$. Since bi-invariant left-orders are Conradian, Theorem 3.2.10 implies that Γ is not left-orderable.

3.2.2 An approach via crossings

An alternative –dynamical– approach to the theory of Conradian orders has been recently developed in [138, 143]. We begin with the definition of the notion of ***crossing***, which is the most important tool in this approach.[1]

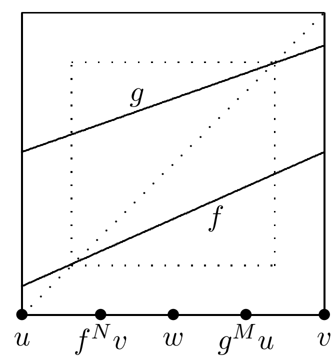

Figure 7: A reinforced crossing.

Let $\preceq$ be a left-order on a group Γ. Following [143], we say that a 5-uple $(f, g; u, v, w)$ of elements in Γ is a ***crossing*** (resp. ***reinforced crossing***) for $(\Gamma, \preceq)$ if the following conditions are satisfied:

– $u \prec w \prec v$;

– $g^n u \prec v$ and $f^n v \succ u$ for every $n \in \mathbb{N}$ (resp. $fu \succ u$ and $gv \prec v$);

– There exist M, N in $\mathbb{N}$ such that $f^N v \prec w \prec g^M u$.

[1]It should be noticed that an equivalent notion –namely that of ***overlapping elements***– was introduced by Glass in his dynamical study of lattice-orderable groups [76], though no conexion with the Conrad property is exhibited therein.

Clearly, every reinforced crossing is a crossing. Conversely, if $(f, g; u, v, w)$ is a crossing, then one easily checks that $(f^N g^M, g^M f^N; f^N w, g^M w, w)$ is a reinforced crossing.

An equivalent notion to the above ones is that of a ***resilient pair***, namely a 4-uple of group elements $(f, g; u, v)$ satisfying

$$u \prec fu \prec fv \prec gu \prec gv \prec v.$$

Indeed, if $(f, g; u, v, w)$ is a reinforced crossing, then $(f^N, g^M; u, v)$ is a resilient pair for the corresponding exponents M, N. Conversely, if $(f, g; u, v)$ is a resilient pair, then $(f^2, g; u, v, fv)$ is a reinforced crossing.

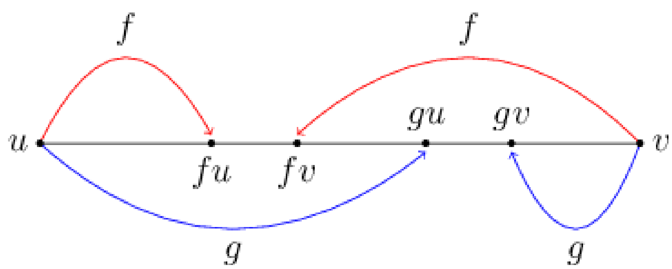

Figure 8: A resilient pair.

Theorem 3.2.12. *The left-order* $\preceq$ *is Conradian if and only if* $(\Gamma, \preceq)$ *admits no (reinforced) crossing.*

Proof. Suppose that $\preceq$ is not Conradian, and let f, g be positive elements such that $fg^n \prec g$ for every $n \in \mathbb{N}$. We claim that $(f, g; u, v, w)$ is a crossing for $(\Gamma, \preceq)$ for the choice $u := 1$, $v := f^{-1}g$, $w := g^2$. Indeed:

– From $fg^2 \prec g$ one obtains $g^2 \prec f^{-1}g$, and since $g \succ 1$, this yields $1 \prec g^2 \prec f^{-1}g$, that is, $u \prec w \prec v$;

– From $fg^n \prec g$ it follows that $g^n \prec f^{-1}g$, that is, $g^n u \prec v$ (for every $n \in \mathbb{N}$); moreover, since both f, g are positive, we have $f^{n-1}g \succ 1$, and thus $f^n(f^{-1}g) \succ 1$, that is, $f^n v \succ u$ (for every $n \in \mathbb{N}$);

– The relation $f(f^{-1}g) = g \prec g^2$ may be read as $f^N v \prec w$ for $N = 1$; finally, the relation $g^2 \prec g^3$ is $w \prec g^M u$ for $M = 3$.

Conversely, let $(f, g; u, v, w)$ be a crossing for $(\Gamma, \preceq)$ for which $f^N v \prec w \prec g^M u$ (with M, N in $\mathbb{N}$). We will prove that $\preceq$ is not Conradian by showing that, for $h := g^M f^N$ and $\bar{h} := g^M$, both elements $w^{-1}hw$ and $w^{-1}\bar{h}w$ are positive, but

$$(w^{-1}hw)(w^{-1}\bar{h}w)^n \prec w^{-1}\bar{h}w, \qquad \text{for all } n \in \mathbb{N}.$$

To do this, first notice that $gw \succ w$, as otherwise

$$w \prec g^N u \prec g^N w \prec g^{N-1} w \prec \ldots \prec gw \prec w,$$

which is absurd. Clearly, the inequality $gw \succ w$ implies $g^M w \succ w$, hence

$$w^{-1}\bar{h}w = w^{-1}g^M w \succ 1. \tag{3.2}$$

Moreover, $hw = g^M f^N w \succ g^M f^N f^N v = g^M f^{2N} v \succ g^M u \succ w$, thus

$$w^{-1}hw \succ 1. \tag{3.3}$$

Now notice that, for every $n \in \mathbb{N}$,

$$h\bar{h}^n w = hg^{Mn} w \prec hg^{Mn} g^M u = hg^{Mn+M} u \prec hv = g^M f^N v \prec g^M w = \bar{h}w.$$

After multiplying by the left by w^{-1}, the last inequality becomes

$$(w^{-1}hw)(w^{-1}\bar{h}w)^n = w^{-1}h\bar{h}^n w \prec w^{-1}\bar{h}w,$$

as we wanted to check. Together with (3.2) and (3.3), this shows that $\preceq$ is not Conradian. $\square$

Exercise 3.2.13. Using the characterization of the Conrad property in terms of resilient pairs, show that the subspace of C-left-orders is closed inside the space of left-orders of a group (*c.f.* Corollary 3.2.2).

Exercise 3.2.14. Using the notion of crossings, give an alternative proof for Proposition 3.2.9.

Hint. If $(f, g; u, v)$ is a resilient pair, then the same is true for $(f^n, g^n; u, v)$, for all $n \geq 1$.

Exercise 3.2.15. Proceed similarly with Proposition 3.2.1.
Hint. Show that, if f, g are positive elements for which $fg^2 \prec g$, then $(f, fg; id, fg, g)$ is a crossing for $M = N = 2$ (see Figure 9 below).

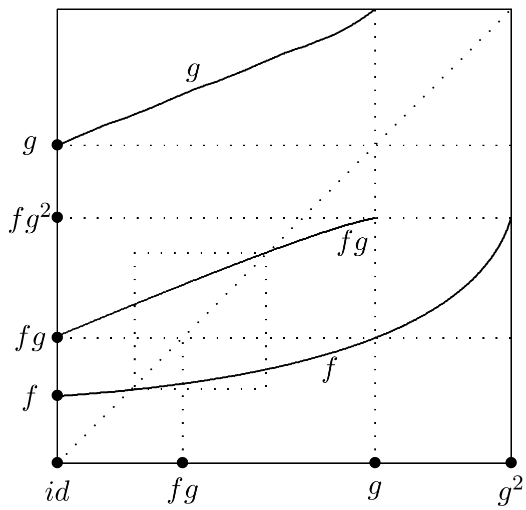

Figure 9: The $n\!=\!2$ condition.

Example 3.2.16. The Dehornoy left-order $\preceq_D$ on the braid group $\mathbb{B}_n$ (where $n \geq 3$) is not Conradian. Indeed, as we next show, $(f, g; u, v, w) := (\sigma_2^{-1}, \sigma_1, \sigma_2, \sigma_2\sigma_1, \sigma_2^{-1}\sigma_1)$ is a crossing for $\prec_D$ with $M = N = 1$ (see [144] for an alternative argument):

– It holds $\sigma_2 \prec_D \sigma_2^{-1}\sigma_1$ is $u \prec_D w$; moreover, one easily checks that $\sigma_2\sigma_1 \succ_D \sigma_1 \succ_D \sigma_2^{-1}\sigma_1$, hence $w \prec_D v$.

– For all $k > 0$, we have $g^k(u) = \sigma_1^k(\sigma_2) \prec_D \sigma_2\sigma_1 = v$, where the middle inequality follows from $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^k\sigma_2 = \sigma_1^{-1}\sigma_1\sigma_2^k\sigma_1^{-1} = \sigma_2^k\sigma_1^{-1} \prec_D 1$; analogously, for $k \in \mathbb{N}$, we have $f^k(v) = \sigma_2^{-k}(\sigma_2\sigma_1) = \sigma_2^{-(k-1)}\sigma_1 \prec_D \sigma_2\sigma_1$, where the last inequality follows from

$$\sigma_1^{-1}\sigma_2^{k-1}\sigma_2\sigma_1 = \sigma_2\sigma_1^k\sigma_2^{-1} \prec_D id.$$

– We have $f(v) = \sigma_2^{-1}(\sigma_2\sigma_1) = \sigma_1 \succ_D \sigma_2^{-1}\sigma_1 = w$ and $g(u) = \sigma_1(\sigma_2) \succ_D \sigma_1 \succ_D \sigma_2^{-1}\sigma_1 = w$.

Exercise 3.2.17. Show that the isolated left-order on the group $G_{m,n}$ constructed in §2.2.3 is not Conradian for $(m, n) \neq (2, 2)$.

Remark 3.2.18. The dynamical characterization of the Conrad property should serve as inspiration for introducing other relevant properties for group left-orders. (Compare [138, Question 3.22].) For instance, one may say that a 6-uple $(f, g; u_1, v_1, u_2, v_2)$ of elements in an ordered group $(\Gamma, \preceq)$ is a ***double resilient pair*** if both $(f, g; u_1, v_1)$ and $(g, f^{-1}; u_2, v_2)$ are resilient pairs and $u_1 \prec u_2 \prec v_1$ (see Figure 10). Finding a simpler algebraic counterpart of the property of not having a double crossing for a left-order seems to be an interesting problem.

The notion of n-resilient pair can be analogously defined. This corresponds to an $(2n+2)$-uple $(f, g; u_1, v_1, u_2, v_2, \ldots, u_n, v_n)$ such that:

– $(f, g; u_1, v_1)$, $(g, f^{-1}; u_2, v_2)$, $(f^{-1}, g^{-1}; u_3, v_3)$, $(g^{-1}, f; u_4, v_4)$, $(f, g; u_5, v_5)$, etc, are all resilient pairs,

– $u_i \prec u_{i+1} \prec v_i$, for all $i \in \{1, \ldots, n-1\}$.

An eventual affirmative answer for the question below would have interesting consequences; see Proposition 4.1.6.

Question 3.2.19. Let Γ be a left-orderable group such that no left-order admits an n-resilient pair for some (large) $n \in \mathbb{N}$. Does Γ admit a C-order ?

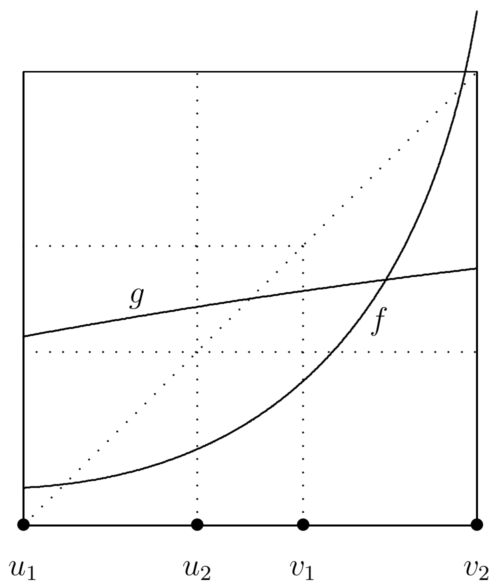

Figure 10: A double resilient pair.

Non-Conradian orders yield free subsemigroups. Let $\preceq$ be a non-Conradian order on a group Γ. Let $(f, g; u, v) \in \Gamma^4$ be a resilient pair for $\preceq$, and denote

$$A := [u, fv]_{\preceq} := \{w \colon u \preceq w \preceq fv\}, \qquad B := [gu, v]_{\preceq}.$$

Then A and B are disjoint, and for all $n \in \mathbb{N}$, we have $f^n(A \cup B) \subset A$ and $g^n(A \cup B) \subset B$. A direct application of the Positive Ping-Pong Lemma (see [84]) shows that the semigroup generated by f and g is free.

This shows in particular that all left-orders on torsion-free, virtually-nilpotent groups are Conradian, a fact first established in [116] by different methods. (This is no longer true for left-orderable polycyclic groups, even for metabelian ones;

see [15, Corollary 7.5.6].) Similarly, the equality $\mathcal{LO}(\Gamma) = \mathcal{CO}(\Gamma)$ holds for left-orderable groups Γ with subexponential growth, as for example Grigorchuk-Maki's group [78, 139] (see Exercise 4.2.7 for a precise definition). As a consequence of Proposition 3.2.53, we obtain:

Theorem 3.2.20. The space of left-orders of a countable, torsion-free, virtually-nilpotent group with infinitely many left-orders is homeomorphic to the Cantor set. The same holds for countable, left-orderable groups without free subsemigroups and having infinitely many left-orders.

Notice also that all left-orders on Tararin groups (*i.e.* groups with finitely many left-orders; see §2.2.1) are Conradian. Indeed, if Γ has finitely many left-orders, then for every $g \in \Gamma$ and every left-order $\preceq$ on Γ, the left-order $\preceq_{g^{-n}}$ must coincide with $\preceq$ for some finite n (actually, for an n smaller than or equal to the cardinality of $\mathcal{LO}(\Gamma)$). Thus, $f \succ_{g^n} id$ holds for every $\preceq$-positive element f, that is, $g^{-n} f g^n \succ id$. In particular, if $g \succ id$, then $f g^n \succ g^n \succ g$, which shows that $\preceq$ is Conradian.

Question 3.2.21. Suppose all left-orders on a finitely-generated, left-orderable group are Conradian. Must the group be residually almost-nilpotent ?

Every non-Conradian order leads to uncountably many left-orders. Using the notion of crossings, we show an straightened version of Theorem 2.2.9 in presence of non-Conradian orders.

Lemma 3.2.22. *If $\preceq$ is a non-Conradian order on a group Γ, then there exists $(f, g, h; u, v)$ in Γ^5 such that*

$$u \prec fu \prec fv \prec hu \prec hv \prec gu \prec gv \prec v.$$

Proof. Let $(\bar{f}, \bar{g}; u, v)$ be a resilient pair for $\preceq$, so that

$$u \prec \bar{f}u \prec \bar{f}v \prec \bar{g}u \prec \bar{g}v \prec v.$$

Let $f := \bar{f},\ h := \bar{g}\bar{f},\ g := \bar{g}^2$. Then:

– The inequality $fv \prec hu$ is $\bar{f}v \prec \bar{g}\bar{f}u$, which follows from

$$\bar{f}u \succ u \implies \bar{g}\bar{f}u \succ \bar{g}u \succ \bar{f}v;$$

– The inequality $hv \prec gu$ is $\bar{g}\bar{f}v \prec \bar{g}^2u$, which follows from

$$\bar{f}v \prec \bar{g}u \implies \bar{g}\bar{f}v \prec \bar{g}^2u. \qquad \square$$

Theorem 3.2.23. *If $\preceq$ is a non-Conradian order on a group Γ, then the closure of the orbit of $\preceq$ in $\mathcal{LO}(\Gamma)$ contains a Cantor set.*

Proof. Fix $(f,g,h;u,v)$ as in the previous lemma. Let I denote the closure of the subset $\{\preceq_w : u \preceq w \preceq v\}$ of $\mathcal{LO}(\Gamma)$. (Recall that $\preceq_w$ is the left-order with positive cone $w^{-1}P^+_{\preceq}w$.) Let $I^+ := \{\preceq' \in I : h \succ' id\}$ and $I^+ := \{\preceq' \in I : h \prec' id\}$. We claim that $f(I) \subset I^+$ and $g(I) \subset I^-$. Indeed, to show that $f(I) \subset I^+$, we need to check that $h(fw) \succ fw$ for all $u \preceq w \preceq v$. But this follows from

$$h(fw) \succeq h(fu) \succ hu \succ fv \succeq fw.$$

The proof of the containment $g(I) \subset I^-$ is analogous.

Denote $\Lambda := \{0,1\}^{\mathbb{N}}$, and let $h_0 := f$ and $h_1 := g$. Consider the map

$$\Lambda \to \mathcal{P}\big(\overline{orb(\preceq)}\big), \qquad \iota = (i_1, i_2, \ldots) \mapsto \bigcap_{n \geq 1} h_{i_1} h_{i_2} \cdots h_{i_n}(I) = \iota(I).$$

By the claim above, if $\iota \neq \iota'$, then $\iota(I) \cap \iota'(I) = \emptyset$. The theorem follows. $\square$

3.2.3 An extension to group actions on ordered spaces

Let Γ be a group acting by order-preserving bijections on a totally ordered space $(\Omega, \leq)$. A ***crossing*** for the action of Γ on Ω is a 5-uple $(f,g;u,v,w)$, where f,g belong to Γ and u,v,w are in Ω, such that:

– It holds $u \prec w \prec v$;

– For every $n \in \mathbb{N}$, we have $g^n u \prec v$ and $f^n v \succ u$;

– There exist M, N in $\mathbb{N}$ so that $f^N v \prec w \prec g^M u$.

Analogous definitions of *reinforced crossings* and *resilient pairs* may be given. Notice that for a left-ordered group $(\Gamma, \preceq)$, the notions of the preceding section correspond to the above ones for the action by left-translations on the ordered space $(\Gamma, \preceq)$.

For another relevant example, recall from Remark 2.1.6 that, given a $\preceq$-convex subgroup Γ_0 of a left-ordered group $(\Gamma, \preceq)$, the space of left cosets $\Omega = \Gamma/\Gamma_0$ carries a natural total order $\leq$ that is invariant by the left-translations. (Taking Γ_0 as the trivial subgroup, this reduces to the preceding example.) Whenever this action has no crossing, we will say that Γ is a ***$\preceq$-Conradian extension*** of Γ_0. Of course, this is the case of every convex subgroup Γ_0 if $\preceq$ is Conradian.

Remark 3.2.24. Let $(\Gamma, \preceq)$ be a left-ordered group, and let Γ_0 be a $\preceq$-convex subgroup. Given any left-order $\preceq_*$ on Γ_0, let $\preceq'$ be the extension of $\preceq_*$ by $\preceq$. One readily checks that Γ is a $\preceq$-Conradian extension of Γ_0 if and only if it is a $\preceq'$-Conradian extension of it.

Exercise 3.2.25. Let Γ be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$. Say that an open interval I is an ***irreducible component*** of a nontrivial element $g \in \Gamma$ if it is fixed by g and contains no fixed point inside. Equivalently, I is a connected component of the complement of the set of fixed points of g.
(i) Show that if the action of Γ is without crossings, then for any pair of different irreducible components, either one of them contains the other, or they are disjoint.
(ii) Show that the converse of (i) also holds.

For a general order-preserving action of a group Γ on a totally ordered space $(\Omega, \leq)$, the action of an element $f \in \Gamma$ is said to be ***cofinal*** if for all $x < y$ in Ω there exists $n \in \mathbb{Z}$ such that $f^n(x) > y$. Equivalently, the action of f is not cofinal if there exist $x < y$ in Ω such that $f^n(x) < y$ for every integer n. If $(\Gamma, \preceq)$ is a left-ordered group, then $f \in \Gamma$ is cofinal if it is so for the corresponding left action of Γ on itself.

Proposition 3.2.26. *Let Γ be a group acting by order-preserving bijections on a totally ordered space $(\Omega, \leq)$. If the action has no crossings, then the set of elements whose action is not cofinal forms a normal subgroup of Γ.*

Proof. Let us denote the set of elements whose action is not cofinal by Γ_0. This set is normal. Indeed, given $g \in \Gamma_0$, let $x < y$ in Ω be such that $g^n(x) < y$ for all n. For each $h \in \Gamma$ we have $g^n h^{-1}(h(x)) < y$, hence $(hgh^{-1})^n(h(x)) < h(y)$ (for all $n \in \mathbb{Z}$). Since $h(x) < h(y)$, this shows that hgh^{-1} belongs to Γ_0.

It follows immediately from the definition that Γ_0 is stable under inversion, that is, g^{-1} belongs to Γ_0 for all $g \in \Gamma_0$. The fact that Γ_0 is stable under multiplication is more subtle. For the proof, given $x \in \Omega$ and $g \in \Gamma_0$, we will denote by $I_g(x)$ the ***convex closure*** of the set $\{g^n(x) \colon n \in \mathbb{Z}\}$, that is, the set formed by the $y \in \Omega$ for which there exist m, n in $\mathbb{Z}$ so that $g^m(x) \leq y \leq g^n(x)$. Notice that $I_g(x) = I_g(x')$ for all $x' \in I_g(x)$. Moreover, $I_{g^{-1}}(x) = I_g(x)$ for all $g \in \Gamma_0$ and all $x \in \Omega$. Finally, if $g(x) = x$, then $I_g(x) = \{x\}$. We claim that if $I_g(x)$ and $I_f(y)$ are non-disjoint for some x, y in Ω and f, g in Γ_0, then one of them contains the other. Indeed, assume that there exist non-disjoint sets $I_f(y)$ and $I_g(x)$, none of which contains the other. Without loss of generality, we may assume that $I_g(x)$ contains points to the left of $I_f(y)$ (if this is not the case, just interchange the roles of f and g). Changing f and/or g by their inverses if necessary, we may assume that

$g(x) > x$ and $f(y) < y$, thus $g(x') > x'$ for all $x' \in I_g(x)$, and $f(y') < y'$ for all $y' \in I_y(f)$. Take $u \in I_g(x) \setminus I_f(y)$, $w \in I_g(x) \cap I_f(y)$, and $v \in I_f(y) \setminus I_g(x)$. Then one easily checks that $(f, g; u, v, w)$ is a crossing, which is a contradiction.

Now, let g, h be elements in Γ_0, and let $x_1 < y_1$ and $x_2 < y_2$ be points in Ω such that $g^n(x_1) < y_1$ and $h^n(x_2) < y_2$, for all $n \in \mathbb{Z}$. Set $x := \min\{x_1, x_2\}$ and $y := \max\{y_1, y_2\}$. Then $g^n(x) < y$ and $h^n(x) < y$, for all $n \in \mathbb{Z}$; in particular, y does not belong to neither $I_g(x)$ nor $I_h(x)$. Since x belongs to both sets, we have either $I_g(x) \subset I_h(x)$ or $I_h(x) \subset I_g(x)$. Both cases being analogous, let us consider only the first one. Then for all $x' \in I_g(x)$ we have $I_h(x') \subset I_g(x') = I_g(x)$. In particular, $h^{\pm 1}(x')$ belongs to $I_g(x)$ for all $x' \in I_g(x)$. Since the same holds for $g^{\pm 1}(x')$, this easily implies that $(gh)^n(x) \in I_g(x)$, for all $n \in \mathbb{Z}$. As a consequence, $(gh)^n(x) < y$ holds for all $n \in \mathbb{Z}$, thus showing that gh belongs to Γ_0. $\square$

Slightly extending Example 2.1.1, a ***convex jump*** of a left-ordered group $(\Gamma, \preceq)$ is a pair (G, H) of distinct $\preceq$-convex subgroups such that H is contained in G, and there is no $\preceq$-convex subgroup between them.

Theorem 3.2.27. *Let $(\Gamma, \preceq)$ be a left-ordered group, and let (G, H) be a convex jump in Γ. Suppose that G is a Conradian extension of H. Then H is normal in G, and the left-order induced by $\preceq$ on the quotient G/H is Archimedean.*

Proof. Let us consider the action of G on the space of cosets G/H. Each element of H fixes the coset H, hence its action is not cofinal. If we show that the action of each element in $G \setminus H$ is cofinal, then Proposition 3.2.26 will imply the normality of H in G.

Now given $f \in G \setminus H$, let G_f be the smallest convex subgroup of G containing (H and) f. We claim that G_f coincides with the set

$$S_f := \{g \in G \colon f^m \prec g \prec f^n \text{ for some } m, n \text{ in } \mathbb{Z}\}.$$

Indeed, S_f is clearly a convex subset of G containing H and contained in G_f. Thus, for showing that $G_f = S_f$, we need to show that S_f is a subgroup. To do this, first notice that, with the notation of the proof of Proposition 3.2.26, the conditions $g \in S_f$ and $I_g(H) \subset I_f(H)$ are equivalent. Therefore, for each $g \in S_f$ we have $I_{g^{-1}}(H) = I_g(H) \subset I_f(H)$, thus $g^{-1} \in S_f$. Moreover, if $\bar{g}$ is another element in S_f, then $\bar{g}gH \in \bar{g}(I_f(H)) = I_f(H)$, hence $I_{\bar{g}g}(H) \subset I_f(H)$. This means that $\bar{g}g$ belongs to S_f, thus concluding the proof that S_f and G_f coincide.

Each $f \in G \setminus H$ leads to a convex subgroup $G_f = S_f$ strictly containing H. Since (G, H) is a convex jump, we necessarily have $S_f = G$. Given $g_1 \prec g_2$ in G, choose m_1, n_2 in $\mathbb{Z}$ for which $f^{m_1} \prec g_1$ and $g_2 \prec f^{n_2}$. Then we have

$f^{n_2-m_1}g_1 \succ f^{n_2-m_1}f^{m_1} = f^{n_2} \succ g_2$, hence $f^{n_2-m_1}(g_1H) \geq g_2H$. This easily implies that the action of f is cofinal.

We have then showed that H is normal in G. The left-invariant total order on the space of cosets G/H is therefore a group left-order. Moreover, given f, g in G, with $f \notin H$, the previous argument shows that there exists $n \in \mathbb{Z}$ such that $f^n \succ g$, thus $f^nH \succeq gH$. This is nothing but the Archimedean property for the induced left-order on G/H. □

Corollary 3.2.28. *Under the hypothesis of Theorem* 3.2.27, *up to multiplication by a positive real number, there exists a unique homomorphism* $\tau : G \to (\mathbb{R}, +)$ *such that* $ker(\tau) = H$ *and* $\tau(g) > 0$ *for every positive element* $g \in G \setminus H$.

The homomorphism τ above will be referred to as the ***Conrad homomorphism*** associated to the corresponding Conradian extension (jump).

Exercise 3.2.29. Let Γ be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$. Show that the action of $g \in \Gamma$ is not cofinal if and only if g has fixed points on the line. If Γ is finitely generated and acts without crossings, show that the normal subgroup formed by the elements having fixed points has global fixed points. If the action corresponds to the dynamical realization of a left-order $\preceq$, show that this subgroup coincides with the kernel of the Conrad homomorphism associated to the convex jump with respect to the maximal proper $\preceq$-convex subgroup (*c.f.* Example 2.1.2).

C-orderability implies local indicability. Let $\preceq$ be a Conradian order on a group Γ. Let Γ_0 be a nontrivial subgroup of Γ generated by finitely many positive elements $f_1 \prec \ldots \prec f_k$. Let Γ_f (resp. Γ^f) be the largest (resp. smallest) convex subgroup which does not contain $f := f_k$ (resp. which contains f). By the corollary above, there exists a nontrivial homomorphism $\tau : \Gamma^f \to (\mathbb{R}, +)$ such that $ker(\tau) = \Gamma_f$. This shows that Γ is locally indicable.

Remark 3.2.30. The homomorphism τ produced above respects orders: if $f \preceq g$, then $\tau(f) \leq \tau(g)$. Moreover, it is trivial when restricted to the maximal convex subgroup. As commutators are mapped into zero by τ, we conclude that every element in $[\Gamma, \Gamma]$ is strictly smaller than any other element f satisfying $\tau(f) > 0$.

We close this section with the following analogue of Proposition 2.1.7.

Proposition 3.2.31. *Let* Γ *be a C-orderable group, and let* $\{\Gamma_\lambda : \lambda \in \Lambda\}$ *be a family of subgroups each of which is convex with respect to a C-left-order* $\preceq_\lambda$. *Then there exists a C-left-order on* Γ *for which the subgroup* $\bigcap_\lambda \Gamma_\lambda$ *is convex.*

The proof is based on a result concerning left-orders obtained from actions on a totally ordered space.

Proposition 3.2.32. *Let Γ be a group acting faithfully by order-preserving transformations on a totally ordered space $(\Omega, \leq)$. If the action has no crossings, then all induced left-orders on Γ are Conradian.*

Proof. Suppose that the left-order $\preceq$ on Γ induced from the action via a well-order $\leqslant_{wo}$ on Ω (*c.f.* §1.1.3) is not Conradian. Then there are $\preceq$-positive elements f, g in Γ such that $fg^n \prec g$, for every $n \in \mathbb{N}$. This easily implies $f \prec g$. Let $\bar{w} := \min_{\leqslant_{wo}}\{w_f, w_g\}$. (Recall that $w_f := \min_{\leqslant_{wo}}\{w \colon f(w) \neq w\}$, and similarly for w_g.) We claim that $(fg, fg^2; \bar{w}, g(\bar{w}), fg^2(\bar{w}))$ is a crossing for the action. Indeed:

– From $id \prec f \prec g$ we obtain $\bar{w} = w_g \leqslant_{wo} w_f$ and $g(\bar{w}) > \bar{w}$; moreover $f(\bar{w}) \geq \bar{w}$, which together with $fg^n \prec g$ yield

$$\bar{w} < fg^2(\bar{w}) < g(\bar{w}).$$

– The preceding argument actually shows that $fg^n(\bar{w}) < g(\bar{w})$, for all $n \in \mathbb{N}$. As a consequence, $fg^2fg^2(\bar{w}) < fg^3(\bar{w}) < g(\bar{w})$. A straightforward inductive argument then shows that $(fg^2)^n(\bar{w}) < g(\bar{w})$, for all $n \in \mathbb{N}$. Moreover, from $g(\bar{w}) > \bar{w}$ and $f(\bar{w}) \geq \bar{w}$, we conclude that $\bar{w} < (fg)^n(g(\bar{w}))$.

– Finally, $\bar{w} < fg^2(\bar{w})$ implies $fg^2(\bar{w}) < fg^2(fg^2(\bar{w})) = (fg^2)^2(\bar{w})$, whereas $fg^2(\bar{w}) < g(\bar{w})$ implies $(fg)^2(g(\bar{w})) = fg(fg^2(\bar{w})) < fg(g(\bar{w})) = fg^2(\bar{w})$. $\square$

The proof of Proposition 3.2.31 proceeds as that of Proposition 2.1.7. We consider the left action of Γ on $\Omega := \prod_{\lambda \in \Lambda} \Gamma/\Gamma_\lambda \times \Gamma$ endowed with the lexicographic order. The stabilizer of $([id_\lambda])_{\lambda \in \Lambda} \times \Gamma$ coincides with $\bigcap_\lambda \Gamma_\lambda$, which may be made convex for an induced left-order $\preceq$ on Γ. Now the main point is that, as the action of Γ on each Γ/Γ_λ has no crossings, the same holds for the action of Γ on Ω. By Proposition 3.2.32, the left-order $\preceq$ is Conradian, thus concluding the proof.

Exercise 3.2.33. Prove the following converse to Proposition 3.2.32: If $(\Gamma, \preceq)$ is a countable C-ordered group, then its dynamical realization is an action on the real line without crossings (*c.f.* §1.1.3).

3.2.4 The Conradian soul of a left-order

A subgroup of a left-ordered group $(\Gamma, \preceq)$ will be said to be Conradian if the restriction of $\preceq$ to it is a Conradian order.

Definition 3.2.34. The ***Conradian soul*** $C_{\preceq}(\Gamma)$ of $(\Gamma, \preceq)$ is the (unique) subgroup that is $\preceq$-convex, $\preceq$-Conradian, and that is maximal among subgroups verifying these two properties simultaneously.

Example 3.2.35. Recall from Example 3.2.7 that the commutator subgroup $[\mathbb{B}_3, \mathbb{B}_3]$ is free in $\sigma_1\sigma_2^{-1}$ and $\sigma_1^2\sigma_2^{-2}$. Denote by $\preceq$ the restriction of the Dehornoy left-order to $[\mathbb{B}_3, \mathbb{B}_3]$. As we show below, $\preceq$ has no proper convex subgroups.[2] Since, as it is easily shown, $\preceq$ is non-Conradian (compare Example 3.2.16), its Conradian soul is trivial.

Let $C \subset \mathbb{F}_2 = [\mathbb{B}_3, \mathbb{B}_3]$ be a nontrivial convex subgroup. Clearly, we may choose a 1-positive $\sigma \in \mathbb{F}_2$. If σ commutes with σ_2, then one may show that σ is of the form $\sigma = \Delta^{2p}\sigma_2^q$ for some integers p, q satisfying $3p = -q > 0$, where $\Delta = \sigma_1\sigma_2\sigma_1$. We thus have $\Delta^2 \prec \Delta^{4p}\sigma_2^{-6p} = \sigma^2$. Since Δ^2 is cofinal for the Dehornoy left-order and central, σ is cofinal as well. Since C is a convex subgroup containing C, it must coincide with $\mathbb{F}_2$.

Suppose now that σ and σ_2 do not commute. By the Subword Property (*c.f.* §1.2.5), for every $k > 0$ the braid $\sigma\sigma_2^k\sigma^{-1}$ is 1-positive, as well as $\sigma\sigma_2^k\sigma^{-1}\sigma_2^{-k}$. Next, $\sigma_2^k\sigma^{-1}\sigma_2^{-k}$ is 1-negative, so that $\sigma\sigma_2^k\sigma^{-1}\sigma_2^{-k} \prec \sigma$. By convexity, $\sigma\sigma_2^k\sigma^{-1}\sigma_2^{-k}$ must lie in C. Since $\sigma \in C$, both $\sigma_2^k\sigma^{-1}\sigma_2^{-k}$ and $\sigma_2^k\sigma\sigma_2^{-k}$ belong to C. Now σ may be represented as $\sigma_2^m\sigma_1 w$, where m is an integer, and w is a 1-positive, 1-neutral, or empty word. Choose $k > 0$ so that $m' = k + m > 0$, and set $\sigma' := \sigma_2^k\sigma\sigma_2^{-k}$. We know that σ' lies in C, and it may be represented by the 1-positive braid word $\sigma_2^\ell\sigma_1 w\sigma_2^{-k}$. We will now proceed to show that C must contain both generators of $\mathbb{F}_2$, thus $C = \mathbb{F}_2$. First notice that $\sigma_2(\sigma_1^{-1}\sigma_2^\ell\sigma_1)w\sigma_2^{-k} = \sigma_2(\sigma_2\sigma_1^\ell\sigma_2^{-1})w\sigma_2^{-k}$ is 1-positice. Therefore,

$$id \prec \sigma_2\sigma_1^{-1}\sigma_2^\ell\sigma_1 w\sigma_2^{-k} \quad \Longrightarrow \quad \sigma_1\sigma_2^{-1} \prec \sigma_2^\ell\sigma_1 w\sigma_2^{-k} = \sigma' \in C,$$

and since $id \prec \sigma_1\sigma_2^{-1}$, this implies that $\sigma_1\sigma_2^{-1} \in C$ by convexity. Concerning the second generator $\sigma_1^2\sigma_2^{-2}$, observe that

$$\sigma_2^2\sigma_1^{-1}\sigma_1^{-1}\sigma_2^\ell\sigma_1 w\sigma_2^{-k} = \sigma_2^2\sigma_1^{-1}\sigma_2\sigma_1^\ell\sigma_2^{-1}w\sigma_2^{-k} = \sigma_2^2\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{\ell-1}\sigma_2^{-1}w\sigma_2^{-k}$$

is 1-positive. Thus,

$$id \prec \sigma_2^2\sigma_1^{-2}\sigma_2^\ell\sigma_1 w\sigma_2^{-k} \quad \Longrightarrow \quad \sigma_1^2\sigma_2^{-2} \prec \sigma_2^\ell\sigma_1 w\sigma_2^{-k} = \beta' \in C,$$

and since $1 \prec \sigma_1^2\sigma_2^{-2}$, we conclude from the convexity of C that $\sigma_1^2\sigma_2^{-2} \in C$.

[2] This example is due to Clay [38]. However, the existence of a left-order on $\mathbb{F}_2$ with no proper convex subgroups also follows from the work of McCleary [119]. See also [144] for left-orders on braid groups without proper convex subgroups.

Example 3.2.36. The Conradian soul of $\preceq_D$ on $\mathbb{B}_n$ is the cyclic subgroup generated by σ_{n-1}. Indeed, this follows from the facts that the only $\preceq_n$-convex subgroups of $\mathbb{B}_n$ are $\{id\}$, $\langle \sigma_{n-1} \rangle$, $\langle \sigma_{n-2}, \sigma_{n-1} \rangle$, ... , $\langle \sigma_2, \ldots, \sigma_{n-1} \rangle$ and $\mathbb{B}_n$ itself, and that the restriction of $\preceq_D$ to $\langle \sigma_{n-2}, \sigma_{n-1} \rangle \sim \mathbb{B}_3$ is not Conradian (*c.f.* Example 3.2.16). Let us examine the case of $\mathbb{B}_3$ by denoting $a = \sigma_1 \sigma_1$ and $b = \sigma_2^{-1}$. (For a general $\mathbb{B}_n$, one uses a similar argument together with Theorem 2.2.40.) Recall that the family of $\preceq_D$-convex subgroups coincides with that of $\preceq_{DD}$-convex ones. Clearly, $\langle b \rangle$ does not properly contain any nontrivial convex subgroup. Suppose that there exists a $\preceq_{DD}$-convex subgroup B of $\mathbb{B}_3$ such that $\langle b \rangle \subsetneq B \subsetneq \mathbb{B}_3$. Let $\preceq'$, $\preceq''$, and $\preceq'''$, be the left-orders defined on $\langle b \rangle$, B, and $\mathbb{B}_3$, respectively, by:

– $\preceq'$ is the restriction of $\preceq_{DD}$ to $\langle b \rangle$;

– $\preceq''$ is the extension of $\preceq'$ by the restriction of $\overline{\preceq}_{DD}$ to B;

– $\preceq'''$ is the extension of $\preceq''$ by $\preceq_{DD}$.

The left-order $\preceq'''$ is different from $\preceq_{DD}$ (the $\preceq_{DD}$-negative elements in $B \setminus \langle b \rangle$ are $\preceq'''$-positive), but its positive cone still contains the elements a, b. Nevertheless, this is impossible, since these elements generate the positive cone of $\preceq_{DD}$.

Exercise 3.2.37. Let $\Gamma_* := C_{\preceq}(\Gamma)$ be the Conradian soul of a left-ordered group $(\Gamma, \preceq)$. Show that, for any Conradian order $\preceq_*$ on Γ_*, the extension of $\preceq_*$ by $\preceq$ has Conradian soul Γ_*.

To give a dynamical counterpart of the notion of Conradian soul in terms of crossings, we consider the set C^+ formed by the elements $h \succ id$ such that $h \preceq w$ for every crossing $(f, g; u, v, w)$ satisfying $id \preceq u$. Analogously, we let C^- be the set formed by the elements $h \prec id$ such that $w \preceq h$ for every crossing $(f, g; u, v, w)$ satisfying $v \preceq id$. Finally, we let

$$C := \{id\} \cup C^+ \cup C^-.$$

A priori, it is not clear that the set C has a nice structure (for instance, it is not at all evident that it is a subgroup). Nevertheless, we have the following

Theorem 3.2.38. *The Conradian soul of* $(\Gamma, \preceq)$ *coincides with the set* C *above.*

Before passing to the proof, we give four general lemmas on crossings for left-orders (notice that the first three lemmas still apply to crossings for actions on totally ordered spaces). The first one allows us replacing the "comparison element" w by its "images" under positive iterates of either f or g.

Lemma 3.2.39. *If $(f,g;u,v,w)$ is a crossing, then both $(f,g;u,v,g^n w)$ and $(f,g;u,v,f^n w)$ are also crossings, for every $n \in \mathbb{N}$.*

Proof. We only consider the first 5-uple (the other is analogous). Since $gw \succ w$, for every $n \in \mathbb{N}$ we have $u \prec w \prec g^n w$; moreover, $v \succ g^{M+n}u = g^n g^M u \succ g^n w$. Hence, $u \prec g^n w \prec v$. Furthermore, $f^N v \prec w \prec g^n w$. Finally, from $g^M u \succ w$ we get $g^{M+n}u \succ g^n w$. $\square$

Our second lemma allows replacing the "limiting" elements u and v by more appropriate ones.

Lemma 3.2.40. *Let $(f,g;u,v,w)$ be a crossing. If $fu \succ u$ (resp. $fu \prec u$) then $(f,g;f^n u,v,w)$ (resp. $(f,g;f^{-n}u,v,w)$) is also a crossing for every $n \geq 1$. Analogously, if $gv \prec v$ (resp. $gv \succ v$), then $(f,g;u,g^n v,w)$ (resp. $(f,g;u,g^{-n}v,w)$) is also crossing for every $n \geq 1$.*

Proof. Let us only consider the first 5-uple (the second case is analogous). Suppose that $fu \succ u$ (the case $fu \prec u$ may be treated similarly). Then $f^n u \succ u$, which yields $g^M f^n u \succ g^M u \succ w$. To show that $f^n u \prec w$, assume by contradiction that $f^n u \succeq w$. Then $f^n u \succ f^N v$ yields $u \succ f^{N-n}v$, which is absurd. $\square$

The third lemma relies on the dynamical nature of the crossing condition.

Lemma 3.2.41. *If $(f,g;u,v,w)$ is a crossing, then $(hfh^{-1},hgh^{-1};hu,hv,hw)$ is also a crossing, for every $h \in \Gamma$.*

Proof. The three conditions to be checked are nothing but the three conditions in the definition of crossing multiplied by h on the left. $\square$

A direct application of the lemma above shows that, if $(f,g;u,v,w)$ is a crossing, then the 5-uples $(f,f^n g f^{-n};f^n u,f^n v,f^n w)$ and $(g^n f g^{-n},g;g^n u,g^n v,g^n w)$ are also crossings, for every $n \in \mathbb{N}$.

Lemma 3.2.42. *If $(f,g;u,v,w)$ is a crossing and $id \preceq h_1 \prec h_2$ are elements in Γ such that $h_1 \in C$ and $h_2 \notin C$, then there exists a crossing $(\tilde{f},\tilde{g};\tilde{u},\tilde{v},\tilde{w})$ such that $h_1 \prec \tilde{u} \prec \tilde{v} \prec h_2$.*

Proof. Since $id \prec h_2 \notin C$, there must be a crossing $(f,g;u,v,w)$ such that $id \preceq u \prec w \prec h_2$. Fix $N \in \mathbb{N}$ such that $f^N v \prec w$, and consider the crossing

$$(f,\bar{g};\bar{u},\bar{v},\bar{w}) := (f,f^N g f^{-N};f^N u,f^N v,f^N w).$$

Notice that $\bar{v} = f^N v \prec w \prec h_2$. We claim that $h_1 \preceq \bar{w} = f^N w$. Indeed, if $f^N u \succ u$ then $f^n u \succ id$, and by the definition of C we must have $h_1 \preceq \bar{w}$. If $f^N u \prec u$, then we must have $fu \prec u$, thus by Lemma 3.2.40 we know that $(f, \bar{g}; u, \bar{v}, \bar{w})$ is also a crossing, which still allows concluding that $h_1 \preceq \bar{w}$.

Now, for the crossing $(f, \bar{g}; \bar{u}, \bar{v}, \bar{w})$, there exists $M \in \mathbb{N}$ such that $\bar{w} \prec \bar{g}^M \bar{u}$. Let us consider the crossing $(\bar{g}^M f \bar{g}^{-M}, \bar{g}; \bar{g}^M \bar{u}, \bar{g}^M \bar{v}, \bar{g}^M \bar{w})$. If $\bar{g}^M \bar{v} \prec \bar{v}$, then $\bar{g}^M \bar{v} \prec h_2$, and we are done. If not, then we must have $\bar{g}\bar{v} \succ \bar{v}$. By Lemma 3.2.40, $(\bar{g}^M f \bar{g}^{-M}, \bar{g}; \bar{g}^M \bar{u}, \bar{g}^M \bar{v}, \bar{w})$ is still a crossing, and since $\bar{v} \prec h_2$, this concludes the proof. $\square$

Proof of Theorem 3.2.38. The proof is divided into several steps.

Claim (i). The set C is convex.

This follows directly from the definition of C.

Claim (ii). If h belongs to C, then h^{-1} also belongs to C.

Assume that $h \in C$ is positive and h^{-1} does not belong to C. Then there exists a crossing $(f, g; u, v, w)$ such that $h^{-1} \prec w \prec v \preceq id$.

We first notice that, if $h^{-1} \preceq u$, then after conjugating by h as in Lemma 3.2.41, we get a contradiction because $(hgh^{-1}, hfh^{-1}; hu, hv, hw)$ is a crossing with $id \preceq hu$ and $hw \prec hv \preceq h$. To reduce the case $h^{-1} \succ u$ to this one, we first use Lemma 3.2.41 and consider the crossing $(g^M f g^{-M}, g; g^M u, g^M v, g^M w)$. Since $h^{-1} \prec w \prec g^M u \prec g^M w \prec g^M v$, if $g^M v \prec v$ then we are done. If not, Lemma 3.2.40 shows that $(g^M f g^{-M}, g; g^M u, g^M v, w)$ is also a crossing, which still allows concluding.

In the case where $h \in C$ is negative, we proceed similarly but we conjugate by f^N instead of g^M. Alternatively, since $id \in C$ and $id \prec h^{-1}$, if we suppose that $h^{-1} \notin C$ then Lemma 3.2.42 provides us with a crossing $(f, g; u, v, w)$ such that $id \prec u \prec w \prec v \prec h^{-1}$, which gives a contradiction after conjugating by h.

Claim (iii). If h and $\bar{h}$ belong to C, then $h\bar{h}$ also belongs to C.

First, we show that for every pair of positive elements in C, their product still belongs to C. (Notice that, by Claim (ii), the same will be true for pairs of negative elements in C.) Indeed, suppose that $h, \bar{h}$ are positive elements, with $h \in C$ but $h\bar{h} \notin C$. Then, by Lemma 3.2.42, we may produce a crossing $(f, g; u, v, w)$ such that $h \prec u \prec v \prec h\bar{h}$. After conjugating by h^{-1}, we obtain the crossing $(h^{-1}fh, h^{-1}gh; h^{-1}u, h^{-1}v, h^{-1}w)$ satisfying $id \prec h^{-1}u \prec h^{-1}w \prec \bar{h}$, which shows that $\bar{h} \notin C$.

Now, if $h \prec id \prec \bar{h}$, then $h \prec h\bar{h}$. Thus, if $h\bar{h}$ is negative, then the convexity of C yields $h\bar{h} \in C$. If $h\bar{h}$ is positive, then $\bar{h}^{-1}h^{-1}$ is negative, and since $\bar{h}^{-1} \prec \bar{h}^{-1}h^{-1}$, the convexity gives again that $\bar{h}^{-1}h^{-1}$, hence $h\bar{h}$, belongs to C. The remaining case $\bar{h} \prec id \prec h$ may be treated similarly.

Claim (iv). The subgroup C is Conradian.

In order to apply Theorem 3.2.12, we need to show that there are no crossings in C. Suppose by contradiction that $(f, g; u, v, w)$ is a crossing such that f, g, u, v, w all belong to C. If $id \preceq w$ then, by Lemma 3.2.41, we have that $(g^n f g^{-n}, g; g^n u, g^n v, g^n w)$ is a crossing. Taking $n = M$ so that $g^M u \succ w$, this contradicts the definition of C, because $id \preceq w \prec g^M u \prec g^M w \prec g^M v \in C$. The case $w \preceq id$ may be treated analogously by conjugating by powers of f instead of g.

Claim (v). The subgroup C is maximal among $\preceq$-convex, $\preceq$-Conradian subgroups.

Indeed, if H is a subgroup strictly containing C, then there is a positive element $h \in H \setminus C$. By Lemma 3.2.42, there exists a crossing $(f, g; u, v, w)$ such that $id \prec u \prec w \prec v \prec h$. If H is convex, then u, v, w belong to H. To conclude that H is not Conradian, it suffices to show that f and g belong to H.

On the one hand, since $id \prec u$, we have either $id \prec g \prec gu \prec v$ or $id \prec g^{-1} \prec g^{-1}u \prec v$. In both cases, the convexity of H implies that g belongs to H. On the other hand, if f is positive, then from $f^N \prec f^N v \prec w$ we get $f \in H$, whereas in the case of a negative f, the inequality $id \prec u$ gives $id \prec f^{-1} \prec f^{-1}u \prec v$, which still shows that $f \in H$. $\square$

3.2.5 Approximation of left-orders and the Conradian soul

The notion of Conradian soul was introduced in [138] as a tool for leading with the problem of approximating a group left-order by its conjugates. We begin with the case of trivial Conradian soul. (Compare Proposition 3.3.3 and its proof.)

Theorem 3.2.43. *If the Conradian soul of an infinite left-ordered group* $(\Gamma, \preceq)$ *is trivial, then* $\preceq$ *may be approximated by its conjugates.*

We will give two different proofs for this theorem, each of which gives some complementary information. The first one, due to Clay [39], shows that every left-order which is not approximated by its conjugates admits a nontrivial, convex, *bi-ordered* subgroup. This may also be obtained by using the method of the second proof below (which is taken from [143]) under the stronger assumption that $\preceq$ is isolated in $\mathcal{LO}(\Gamma)$. Nevertheless, though more elaborate than the first (it uses the results of the preceding section), this second proof is suitable for generalization in the case where the Conradian soul is "almost trivial" (*i.e.* it is nontrivial but admits only finitely many left-orders; see Theorem 3.2.46 below).

First proof of Theorem 3.2.43. Suppose that $\preceq$ cannot be approximated by its conjugates, and let $g_1, \ldots, g_k$ be finitely many positive elements such that the only conjugate of $\preceq$ lying in $V_{g_1} \cap \ldots \cap V_{g_k}$ is $\preceq$ itself. (Recall that V_g denotes the set of left-orders making g a positive element.) For each index $i \in \{1, \ldots, k\}$, let

$$B_i^+ = \big\{ h \in \Gamma \colon id \preceq h \preceq g_i^n \text{ for some } n \in \mathbb{N} \big\},$$

$$B_i = \big\{ h \in \Gamma \colon g_i^{-m} \preceq h \preceq g_i^n \text{ for some } m, n \text{ in } \mathbb{N} \big\}.$$

<u>Claim (i).</u> For some $j \in \{1, \ldots, k\}$ we have $h^{-1} P_{\preceq}^+ h = P_{\preceq}^+$ for every $h \in B_j^+$.

If not, then for each i there exists $h_i \in \Gamma$ such that $id \prec h_i \preceq g_i^{n_i}$ for some $n_i \in \mathbb{N}$ and $h_i^{-1} P_{\preceq}^+ h_i \neq P_{\preceq}^+$. Let $h := \min\{h_1, \ldots, h_k\}$. Then $h^{-1} P_{\preceq}^+ h \neq P_{\preceq}^+$. Moreover, $h \preceq g_i^{n_i}$ for each i, thus $h^{-1} g_i^{n_i} \succeq id$. Since h is necessarily positive, this yields $h^{-1} g_i^{n_i} h \succ id$, which implies $h^{-1} g_i h \succ id$, that is $g_i \in h^{-1} P_{\preceq}^+ h$. Since this holds for every i, by hypothesis the conjugate left-order $\preceq_{h^{-1}}$ must coincide with $\preceq$, which is a contradiction.

<u>Claim (ii).</u> All elements in B_j^+ stabilize $P_{\preceq}^+$ (under conjugacy).

Indeed, from $g_j^{-m} \preceq h \preceq g_i^n$ we obtain $id \preceq g_j^m h \preceq g_j^{m+n}$. Thus, $g_j^m h$ belongs to B_j^+. Since g_j^m also belongs to B_j^+, by Claim (i) above we have

$$(g_j^m h)^{-1} P_{\preceq}^+ (g_j^m h) = P_{\preceq}^+, \qquad g_j^{-m} P_{\preceq}^+ g_j^m = P_{\preceq}^+.$$

This easily yields $h^{-1} P_{\preceq}^+ h = P_{\preceq}^+$, which in its turn implies $h P_{\preceq}^+ h^{-1} = P_{\preceq}^+$.

<u>Claim (iii).</u> The set B_j is a $\preceq$-convex subgroup of Γ, and the restriction of $\preceq$ to it is a bi-order (hence a C-order).

The convexity of B_j as a set is obvious. Now, for each $h \in B_j$, the relations $g_j^{-m} \preceq h \preceq g_j^n$ and $h P_{\preceq}^+ h^{-1} = P_{\preceq}^+$ easily yield $g_j^m \succeq h \succeq g_j^n$, thus showing that $h^{-1} \in B_j$. Similar arguments show that $h_1 h_2$ belongs to B_i^+ for all h_1, h_2 in B_j^+, as well as the bi-invariance of the restriction of $\preceq$ to B_j^+. $\square$

Second proof of Theorem 3.2.43. Let $f_1 \prec f_2 \prec \ldots \prec f_k$ be finitely many positive elements of Γ. We need to show that there exists a conjugate of $\preceq$ that is different from $\preceq$ but for which all the f_i's are still positive.

Since $id \in C_{\preceq}(\Gamma)$ and $f_1 \notin C_{\preceq}(\Gamma)$, Theorem 3.2.38 and Lemma 3.2.42 imply that there is a crossing $(f, g; u, v, w)$ such that $id \prec u \prec v \prec f_1$. Let M, N in

$\mathbb{N}$ be such that $f^N v \prec w \prec g^M u$. We claim that $id \prec_v f_i$ and $id \prec_w f_i$ hold for all $1 \leq i \leq k$, but $g^M f^N \prec_v id$ and $g^M f^N \succ_w id$. Indeed, since $id \prec v \prec f_i$, we have $v \prec f_i \prec f_i v$, thus $id \prec v^{-1} f_i v$. By definition, this means that $f_i \succ_v id$. The inequality $f_i \succ_w id$ is proved similarly. Now notice that $g^M f^N v \prec g^M w \prec v$, hence $g^M f^N \prec_v id$. Finally, from $g^M f^N w \succ g^M u \succ w$ we deduce $g^M f^N \succ_w id$.

Now the preceding relations imply that the f_i's are still positive for both $\preceq_{v^{-1}}$ and $\preceq_{w^{-1}}$, but at least one of these left-orders is different from $\preceq$. This concludes the proof. □

We next deal with the case where the Conradian soul is nontrivial but admits finitely many left-orders (*i.e.* it is a Tararin group; see §2.2.1). It turns out that, in this case, the left-order may fail to be an accumulation point of its conjugates. A concrete example is given by the DD-left-order on $\mathbb{B}_n$. Indeed, its Conradian soul is isomorphic to $\mathbb{Z}$ (*c.f.* Example 3.2.36), though it is an isolated point of the space of braid left-orders because its positive cone is finitely generated (*c.f.* §2.2.3). Now the DD-left-order has the Dehornoy left-order $\preceq_{D}$ as a natural "associate", in the sense that the latter may be obtained from the former by successive flipings along convex jumps. For the case of B_3, this reduces to changing the left-order on the Conradian soul in the unique possible way. As shown below, $\preceq_{D}$ is an accumulation point of its conjugates. Moreover, there is a sequence of conjugates of $\preceq_{DD}$ that converges to $\preceq_{D}$ as well.

Example 3.2.44. The sequence of conjugates $\preceq_j$ of $\preceq_{D}$ by $\sigma_2^j \sigma_1^{-1}$ converges to $\preceq_{D}$ in a nontrivial way. Indeed, if $w = \sigma_2^k$ for some $k > 0$, then

$$\sigma_1^{-1} \sigma_2^j w \sigma_2^{-j} \sigma_1 = \sigma_1^{-1} \sigma_2^k \sigma_1 = \sigma_2 \sigma_1^k \sigma_2^{-1} \succ_{D} id.$$

If, on the other hand, w is a σ_1-positive word, say $w = \sigma_2^{k_1} \sigma_1 \sigma_2^{k_2} \ldots \sigma_2^{k_{\ell-1}} \sigma_1 \sigma_2^{k_\ell}$, then

$$\sigma_1^{-1} \sigma_2^j w \sigma_2^{-j} \sigma_1 = \sigma_1^{-1} \sigma_2^j \sigma_2^{k_1} \sigma_1 \sigma_2^{k_2} \ldots \sigma_2^{k_{\ell-1}} \sigma_1 \sigma_2^{k_\ell} \sigma_2^{-i} \sigma_1 = \sigma_2 \sigma_1^{j+k_1} \sigma_2^{-1} \sigma_2^{k_2} \ldots \sigma_2^{k_{\ell-1}} \sigma_1 \sigma_2^{k_\ell} \sigma_2^{-n} \sigma_1.$$

Thus, $\sigma_1 \sigma_2^{-j} w \sigma_2^j \sigma_1$ is 1-positive for sufficiently large j (namely, for $j > -k_1$). This proves the desired convergence. Finally, $\preceq_j$ is different from $\preceq_{D}$ for each positive integer j, since its smallest positive element is the conjugate of σ_2 by $\sigma_1 \sigma_2^j$, and this is different from the smallest positive element of $\preceq_{D}$, namely σ_2. We leave to the reader the task of checking that the sequence of conjugates of $\preceq_{DD}$ by $\sigma_1^{-1} \sigma_2^j$ converges to $\preceq_{D}$ as well.

Remark 3.2.45. The $\mathbb{B}_3$-case of the preceding example can be generalized as follows: For all m, n larger than 1, with $(m,n) \neq (2,2)$, the left-order $\preceq$ on $G_{m,n} = \langle a, b : (ba^{m-1})^{n-1} b = a \rangle$ with positive cone $\langle a, b \rangle^+$ given by Theorem 2.2.43 has Conradian soul $\langle b \rangle \sim \mathbb{Z}$. Flipping this order on the Conradian soul yields a left-order $\preceq'$ that is accumulated by its conjugates. Moreover, there is a sequence of conjugates of $\preceq$ that also converges to $\preceq'$. See [136] as well as [93, 94, 95] for more on this and related examples.

It turns out that the phenomenon described above for braid groups occurs for general left-ordered groups. To be more precise, let Γ be a group having a left-order $\preceq$ whose Conradian soul admits finitely many left-orders $\preceq_1, \preceq_2, \ldots, \preceq_{2^n}$, where $\preceq_1$ is the restriction of $\preceq$ to its Conradian soul. Each $\preceq_j$ induces a left-order $\preceq^j$ on Γ, namely the convex extension of $\preceq_j$ by $\preceq$. (Notice that $\preceq^1$ coincides with $\preceq$.) All the left-orders $\preceq^j$ share the same Conradian soul (*c.f.* Exercise 3.2.37). Assume throughout that $\preceq$ is not Conradian, which is equivalent to that Γ is not a Tararin group.

Theorem 3.2.46. *With the notation above, at least one of the left-orders $\preceq^j$ is an accumulation point of the set of conjugates of $\preceq$.*

Corollary 3.2.47. *At least one of the left-orders $\preceq^j$ is approximated by its conjugates.*

Proof. Assuming Theorem 3.2.46, we have that $\preceq^k$ belongs to the set of accumulation points $acc(orb(\preceq^1))$ of the orbit of $\preceq^1$ for some k in $\{1, \ldots, 2^n\}$. Theorem 3.2.46 applied to this $\preceq^k$ instead of $\preceq$ shows the existence of $k' \in \{1, \ldots, 2^n\}$ so that $\preceq^{k'} \in acc(orb(\preceq^k))$, and hence $\preceq^{k'} \in acc(orb(\preceq^1))$. If k' equals either 1 or k then we are done; if not, we continue arguing in this way... In at most 2^n steps we will find an index j such that $\preceq^j \in acc(orb(\preceq^j))$. $\square$

Theorem 3.2.46 will follow from the next

Proposition 3.2.48. *Given an arbitrary finite family $\mathcal{G}$ of $\preceq$-positive elements in Γ, there exists $h \in \Gamma$ and a positive $\bar{h} \notin C_{\preceq}(\Gamma)$ such that $id \prec h^{-1} f h \notin C_{\preceq}(\Gamma)$ for all $f \in \mathcal{G} \setminus C_{\preceq}(\Gamma)$, but $id \succ h^{-1} \bar{h} h \notin C_{\preceq}(\Gamma)$.*

Proof of Theorem 3.2.46 from Proposition 3.2.48. Let us consider the directed net formed by the finite sets $\mathcal{G}$ of $\preceq$-positive elements. For each such a $\mathcal{G}$, let $h_{\mathcal{G}}$ and $\bar{h}_{\mathcal{G}}$ be the elements in Γ provided by Proposition 3.2.48. After passing to subnets of $(h_{\mathcal{G}})$ and $(\bar{h}_{\mathcal{G}})$ if necessary, we may assume that the restrictions of $\preceq_{h_{\mathcal{G}}}$ to $C_{\preceq}(\Gamma)$ all coincide with a single $\preceq_j$. Now the properties of $h_{\mathcal{G}}$ and $\bar{h}_{\mathcal{G}}$ imply:

– $f \succ^j id$ and $f\,(\succ^j)_{h_{\mathcal{G}}}\, id$, for all $f \in \mathcal{G} \setminus C_{\preceq}(\Gamma)$;
– $\bar{h}_{\mathcal{G}} \succ id$, but $\bar{h}_{\mathcal{G}}\ (\prec^j)_{h_{\mathcal{G}}} \prec id$.
This clearly shows the theorem. $\square$

For the proof of Proposition 3.2.48 we will use some lemmas.

Lemma 3.2.49. *For every $id \prec c \notin C_{\preceq}(\Gamma)$, there is a crossing $(f, g; u, v, w)$ such that u, v, w do not belong to $C_{\preceq}(\Gamma)$ and $id \prec u \prec w \prec v \prec c$.*

Proof. By Theorem 3.2.38 and Lemma 3.2.42, for every $id \preceq s \in C_{\preceq}(\Gamma)$ there exists a crossing $(f, g; u, v, w)$ such that $s \prec u \prec w \prec v \prec c$. Clearly, v does not belong to $C_{\preceq}(\Gamma)$. The element w is also outside $C_{\preceq}(\Gamma)$, as otherwise the element $a := w^2$ would satisfy $w \prec a \in C_{\preceq}(\Gamma)$, which is absurd. Taking $M > 0$ so that $g^M u \succ w$, this gives $g^M u \notin C_{\preceq}(\Gamma)$, $g^M w \notin C_{\preceq}(\Gamma)$, and $g^M v \notin C_{\preceq}(\Gamma)$. Consider the crossing $(g^M f g^{-M}, g; g^M u, g^M v, g^M w)$. If $g^M v \prec v$, then we are done. If not, then $gv \succ v$, and Lemma 3.2.40 ensures that $(g^M f g^{-M}, g; g^M u, v, g^M w)$ is also a crossing, which still allows concluding. $\square$

Lemma 3.2.50. *Given $id \prec c \notin C_{\preceq}(\Gamma)$, there exists $id \prec a \notin C_{\preceq}(\Gamma)$ (with $a \prec c$) such that, for all $id \preceq b \preceq a$ and all $\bar{c} \succeq c$, one has $id \prec b^{-1}\bar{c}b \notin C_{\preceq}(\Gamma)$.*

Proof. Let us consider the crossing $(f, g; u, v, w)$ such that $id \prec u \prec w \prec v \prec c$ and such that u, v, w do not belong to $C_{\preceq}(\Gamma)$. We affirm that the lemma holds for $a := u$. Indeed, if $id \preceq b \preceq u$, then from $b \preceq u \prec v \prec \bar{c}$ we obtain $id \preceq b^{-1}u \prec b^{-1}v \prec b^{-1}\bar{c}$, thus the crossing $(b^{-1}fb, b^{-1}gb; b^{-1}u, b^{-1}v, b^{-1}w)$ shows that $b^{-1}\bar{c} \notin C_{\preceq}(\Gamma)$. Since $id \preceq b$, we conclude that $id \prec b^{-1}\bar{c} \preceq b^{-1}\bar{c}b$, and the convexity of S implies that $b^{-1}\bar{c}b \notin C_{\preceq}(\Gamma)$. $\square$

Lemma 3.2.51. *For every $g \in \Gamma$, the set $g\, C_{\preceq}(\Gamma)$ is convex. Moreover, for every crossing $(f, g; u, v, w)$, one has $uC_{\preceq}(\Gamma) < wC_{\preceq}(\Gamma) < vC_{\preceq}(\Gamma)$, in the sense that $uh_1 \prec wh_2 \prec vh_3$ for all h_1, h_2, h_3 in $C_{\preceq}(\Gamma)$.*

Proof. The verification of the convexity of $gC_{\preceq}(\Gamma)$ is straightforward. Suppose next that $uh_1 \succ wh_2$ for some h_1, h_2 in $C_{\preceq}(\Gamma)$. Then, since $u \prec w$, the convexity of both left classes $uC_{\preceq}(\Gamma)$ and $wC_{\preceq}(\Gamma)$ gives the equality between them. In particular, there exists $h \in C_{\preceq}(\Gamma)$ such that $uh = w$. Notice that such an h must be positive, hence $id \prec h = u^{-1}w$. But since $(u^{-1}fu, u^{-1}gu; id, u^{-1}v, u^{-1}w)$ is a crossing, this contradicts the definition of $C_{\preceq}(\Gamma)$. The proof of the fact that $wC_{\preceq}(\Gamma) \prec vC_{\preceq}(\Gamma)$ is similar. $\square$

Proof of Proposition 3.2.48. Indexing the elements of $\mathcal{G}=\{f_1,\ldots,f_r\}$ so that $f_1 \prec \ldots \prec f_r$, let k be such that $f_{k-1} \in C_{\preceq}(\Gamma)$ but $f_k \notin C_{\preceq}(\Gamma)$. Recall that, by Lemma 3.2.50, there exists $id \prec a \notin C_{\preceq}(\Gamma)$ such that, for every $id \preceq b \preceq a$, one has $id \prec b^{-1} f_{k+j} b \notin C_{\preceq}(\Gamma)$ for all $j \geq 0$. We fix a crossing $(f,g;u,v,w)$ such that $id \prec u \prec v \prec a$ and $u \notin C_{\preceq}(\Gamma)$. Notice that the conjugacy by w^{-1} yields the crossing $(w^{-1}fw, w^{-1}gw; w^{-1}u, w^{-1}v, id)$.

Case I. One has $w^{-1}v \preceq a$.

In this case, we claim that the proposition holds for the choice $h := w^{-1}v$ and $\bar{h} := w^{-1}g^{M+1}f^N w$. To show this, first notice that neither $w^{-1}gw$ nor $w^{-1}fw$ belong to $C_{\preceq}(\Gamma)$. Indeed, this follows from the convexity of $C_{\preceq}(\Gamma)$ and the inequalities $w^{-1}g^{-M}w \prec w^{-1}u \notin C_{\preceq}(\Gamma)$ and $w^{-1}f^{-N}w \succ w^{-1}v \notin C_{\preceq}(\Gamma)$. We also have $id \prec w^{-1}g^M f^N w$, hence $id \prec w^{-1}gw \prec w^{-1}g^{M+1}f^N w$, which shows that $\bar{h} \notin C_{\preceq}(\Gamma)$. Moreover, the inequality $w^{-1}g^{M+1}f^N w(w^{-1}v) \prec w^{-1}v$ can be written as $h^{-1}\bar{h}h \prec id$. Finally, Lemma 3.2.39 applied to the crossing $(w^{-1}fw, w^{-1}gw; w^{-1}u, w^{-1}v, id)$ shows that, for every $n \in \mathbb{N}$, the 5-uple $(w^{-1}fw, w^{-1}gw; w^{-1}u, w^{-1}v, w^{-1}g^{M+n}f^N w)$ is also a crossing. For $n \geq M$ we have $w^{-1}g^{M+1}f^N w(w^{-1}v) \prec w^{-1}g^{M+n}f^N w$. Since $w^{-1}g^{M+n}f^N w \prec w^{-1}v$, Lemma 3.2.51 easily implies that $w^{-1}g^{M+1}f^N w(w^{-1}v)C_{\preceq}(\Gamma) \prec w^{-1}vC_{\preceq}(\Gamma)$, which yields $h^{-1}\bar{h}h \notin C_{\preceq}(\Gamma)$.

Case II. One has $a \prec w^{-1}v$ and $w^{-1}g^m w \preceq a$, for all $m > 0$.

We claim that, in this case, the proposition holds for the choice $h := a$ and $\bar{h} := w^{-1}g^{M+1}f^N w$. This may be checked in the very same way as in Case I by noticing that, if $a \prec w^{-1}v$ but $w^{-1}g^m w \succeq a$ for all $m > 0$, then $(w^{-1}fw, w^{-1}gw; w^{-1}u, a, id)$ is a crossing.

Case III. One has $a \prec w^{-1}v$ and $w^{-1}g^m w \succ a$ for some $m > 0$. (Notice that the first condition follows from the second one.)

We claim that, in this case, the proposition holds for the choice $h := a$ and $\bar{h} := w \notin C_{\preceq}(\Gamma)$. Indeed, we have $g^m w \succ ha$ (and $w \prec ha$). Since $g^m w \prec v \prec a$, we also have $wa \prec a$, which means that $h^{-1}\bar{h}h \prec id$. Finally, from Lemmas 3.2.39 and 3.2.51, we obtain

$$waC_{\preceq}(\Gamma) \preceq g^m wC_{\preceq}(\Gamma) \prec vC_{\preceq}(\Gamma) \preceq aC_{\preceq}(\Gamma).$$

This implies that $a^{-1}waC_{\preceq}(\Gamma) \prec C_{\preceq}(\Gamma)$, which means that $h^{-1}\bar{h}h \notin C_{\preceq}(\Gamma)$. $\square$

Remark 3.2.52. In the context of Theorem 3.2.46, it is possible that one of the orders $\preceq^j$ may be not approximated by its conjugates despite being non-isolated. An illustrative example of this fact for free groups is the subject of the Appendix of [143].

3.2.6 Groups with finitely many Conradian orders

The starting point of this section is the following

Proposition 3.2.53. *Let Γ be a C-orderable group. If Γ admits a Conradian left-order having a countable neighborhood in $\mathcal{LO}(\Gamma)$, then Γ admits finitely many left-orders.*

Before showing this proposition, let us show how it leads to a

Proof of Theorem 2.2.9. We provide three different arguments (see §4.3.3 for still another one that gives supplementary information). First, as we saw in §2.2.1, the proof is reduced to showing Proposition 2.2.10. So, let $(\Gamma, \preceq)$ be a left-ordered group admitting a finite-index subgroup restricted to which $\preceq$ is bi-invariant. By Proposition 3.2.9, the left-order $\preceq$ is Conradian. By Proposition 3.2.53, if Γ admits infinitely many left-orders, then all neighborhoods of $\preceq$ in $\mathcal{LO}(\Gamma)$ are uncountable.

An alternative argument proceeds as follows. As was shown in §3.2.2, if a group admits a non Conradian order, then it has uncountably many left-orders. Assume that Γ is left-orderable and all of its left-orders are Conradian. By Proposition 3.2.53, if some of them has a countable neighborhood inside $\mathcal{LO}(\Gamma) = \mathcal{CO}(\Gamma)$ (in particular, if $\mathcal{LO}(\Gamma)$ is countable), then Γ admits only finitely many left-orders.

As a final argument, notice that Proposition 3.2.53 together with a convex extension argument (*c.f.* Section 2.1.1) show that, if Γ is a left-orderable group such that $\mathcal{LO}(\Gamma)$ has an isolated point $\preceq$, then the Conradian soul $C_{\preceq}(\Gamma)$ cannot have infinitely many left-orders. If $C_{\preceq}(\Gamma)$ is trivial (resp. if it is nontrivial and admits finitely many left-orders), then Proposition 3.2.43 (resp. Proposition 3.2.46) yields the existence of a left-order $\preceq_*$ on Γ that is accumulated by its conjugates. As we have already remarked, the closure of the orbit under the conjugacy action of such a left-order is uncountable. $\square$

Proof of Proposition 3.2.53. Let Γ be a group admitting a Conradian order $\preceq$ having a *countable* neighborhood in $\mathcal{LO}(\Gamma)$, say

$$V_{f_1} \cap \ldots \cap V_{f_k} = \big\{ \preceq' \colon \; f_i \succ' id \;\text{ for all }\; i \in \{1, \ldots, k\} \big\}.$$

<u>Claim (i).</u> The chain of $\preceq$-convex subgroups is finite.

Otherwise, there exists an infinite ascending or descending chain of convex jumps $\Gamma_{g_n} \lhd \Gamma^{g_n}$ so that $f_m \notin \Gamma^{g_n} \setminus \Gamma_{g_n}$ for every m, n. As in the proof of

Proposition 2.2.16, for each $\iota = (i_1, i_2 \ldots) \in \{-1, +1\}^{\mathbb{N}}$ let us define the left-order $\preceq_\iota$ on Γ by:

– $P^+_{\preceq_\iota} \bigcap \big(\Gamma \setminus (\Gamma^{g_n} \setminus \Gamma_{g_n})\big) = P^+_{\preceq} \bigcap \big(\Gamma \setminus (\Gamma^{g_n} \setminus \Gamma_{g_n})\big)$, for each $n \in \mathbb{N}$;

– $P_{\preceq_\iota} \cap (\Gamma^{g_n} \setminus \Gamma_{g_n})$ equals $P^+_{\preceq} \cap (\Gamma^{g_n} \setminus \Gamma_{g_n})$ (resp. $P^-_{\preceq} \cap (\Gamma^{g_n} \setminus \Gamma_{g_n})$) if $i_n = +1$ (resp. $i_n = -1$).

This yields a continuous embedding of the Cantor set $\{-1, +1\}^{\mathbb{N}}$ into $\mathcal{LO}(\Gamma)$. Moreover, since $f_m \notin \Gamma^{g_n} \setminus \Gamma_{g_n}$ for every m, n, the image of this embedding is contained in $V_{f_1} \cap \ldots \cap V_{f_k}$. This proves the claim.

Claim (ii). Denote by $\{id\} = \Gamma^k \lhd \Gamma^{k-1} \lhd \ldots \lhd \Gamma^0 = \Gamma$ the chain of *all* $\preceq$-convex subgroups. Then each quotient Γ^{i-1}/Γ^i is torsion-free, rank-1 Abelian.

If the rank of some Γ^{i-1}/Γ^i were larger than 1, then the induced left-order on the quotient would be non-isolated in the space of left-orders of Γ^{i-1}/Γ^i. This would allow to produce –by a convex extension type procedure– uncountably many left-orders on any given neighborhood of $\preceq$, which is contrary to our hypothesis.

Claim (iii). In the series above, the group Γ^{k-2} is not bi-orderable.

First notice that Γ^{k-2} is not Abelian. Otherwise, it would have rank 2. This would imply that every neighborhood of the restriction of $\preceq$ to Γ^{k-2} is uncountable, which implies –by convex extension– the same property for $\preceq$.

Now as in the case of Proposition 2.2.18, if Γ^{k-2} were bi-orderable, then it would be contained in the affine group $\mathrm{Aff}_+(\mathbb{R})$. The space of left-orders of a non-Abelian countable group inside $\mathrm{Aff}_+(\mathbb{R})$ was roughly described in §1.2.2: it is homeomorphic to the Cantor set. (See also §3.3.) In particular, no neighborhood of the restriction of $\preceq$ to Γ^{k-2} is countable, which implies –by convex extension– that the same is true for Γ. For sake of completeness, we give an explicit sequence of approximating left-orders. To do this, notice that, for some $q > 0$, the group Γ^{k-2} can be identified with the group whose elements are of the form

$$(k, a) \sim \begin{pmatrix} q^k & a \\ 0 & 1 \end{pmatrix},$$

where $a \in \Gamma^{k-1}$ and $k \in \mathbb{Z}$. Let $(k_1, a_1), \ldots, (k_n, a_n)$ be an arbitrary family of $\preceq$-positive elements indexed in such a way that $k_1 = k_2 = \ldots = k_r = 0$ and $k_{r+1} \neq 0, \ldots, k_n \neq 0$ for some $r \in \{1, \ldots, n\}$. Four cases are possible:

(i) $a_1 > 0, \ldots, a_r > 0$ and $k_{r+1} > 0, \ldots, k_n > 0$;

(ii) $a_1 < 0, \ldots, a_r < 0$ and $k_{r+1} > 0, \ldots, k_n > 0$;

(iii) $\quad a_1 > 0, \ldots, a_r > 0 \quad \text{and} \quad k_{r+1} < 0, \ldots, k_n < 0;$

(iv) $\quad a_1 < 0, \ldots, a_r < 0 \quad \text{and} \quad k_{r+1} < 0, \ldots, k_n < 0.$

As in §1.2.2, for each irrational number ε, let $\preceq_\varepsilon$ be the left-order on $\preceq_\varepsilon$ whose positive cone is

$$P^+_{\preceq_\varepsilon} = \big\{(k,a): \quad q^k + \varepsilon a > 1\big\}.$$

In case (i), for ε positive and very small, the left-order $\preceq_\varepsilon$ is different from $\preceq$ but still makes positive all the elements (k_i, a_i). The same is true in case (ii) for ε negative and near zero. In case (iii), this still holds for the order $\bar{\preceq}_\varepsilon$ when ε is negative and near zero. Finally, in case (iv), one needs to consider again the order $\bar{\preceq}_\varepsilon$ but for ε positive and small. Now letting ε vary over a Cantor set formed by irrational numbers[3] very close to 0 (and which are positive or negative according to the case), this shows that the neighborhood of (the restriction to $\preceq$ of) $\preceq$ consisting of the left-orders on Γ^{k-2} that make positive all the elements (k_i, a_i) contains a homeomorphic copy of the Cantor set.

<u>Claim (iv).</u> The series of Claim (ii) is normal (hence rational) and no quotient Γ^{i-2}/Γ^i is bi-orderable.

By Theorem 2.2.12, the group Γ^{k-2} admits a unique rational series, namely $\{id\} \lhd \Gamma^{k-1} \lhd \Gamma^{k-2}$. Since for every $h \in \Gamma^{k-3}$ the series $\{id\} \lhd h\Gamma^{k-1}h^{-1} \lhd h\Gamma^{k-2}h^{-1}$ is also rational for Γ^{k-2}, they must coincide. Hence, the rational series

$$\{id\} \lhd \Gamma^{k-1} \lhd \Gamma^{k-2} \lhd \Gamma^{k-3}$$

is normal. Moreover, proceeding as in Claim (iii) with the induced left-order on $\Gamma^{k-3}/\Gamma^{k-1}$, one readily checks that this quotient is not bi-orderable. Once again, Theorem 2.2.12 implies that the rational series of Γ^{k-3} is unique... Continuing arguing in this way, the claim follows.

We may now conclude the proof of the proposition. Indeed, we have shown that, if Γ is a group having a C-order with a countable neighborhood in $\mathcal{LO}(\Gamma)$, then Γ admits a rational series

$$\{id\} = \Gamma^k \lhd \Gamma^{k-1} \lhd \ldots \lhd \Gamma^0 = \Gamma$$

such that no quotient Γ^{i-2}/Γ^i is bi-orderable. By Theorem 2.2.12, Γ has only finitely many left-orders. $\square$

[3]Take for example the set of numbers of the form $\sum_{i \geq 1} \frac{i_k}{4^k}$, where $i_k \in \{0,1\}$, and translate it by $\sum_{j \geq 1} \frac{2}{4^{j^2}}$.

We now turn to the study of the space of Conradian orders. The next result from [155] is the analogue of Tararin's theorem describing left-orderable groups with finitely many left-orders; see §2.2.1.

Theorem 3.2.54. *If a C-orderable group Γ has only finitely many C-orders, then it has a unique (hence normal) rational series $\{id\} = \Gamma^k \lhd \Gamma^{k-1} \lhd \ldots \lhd \Gamma^0 = \Gamma$. In this series, no quotient Γ^{i-2}/Γ^i is Abelian. Conversely, if Γ is a group admitting a normal rational series $\{id\} = \Gamma^k \lhd \Gamma^{k-1} \lhd \ldots \lhd \Gamma^0 = \Gamma$ such that no quotient Γ^{i-2}/Γ^i is Abelian, then the number of C-orders on Γ is 2^k.*

Proof. The proof will be divided into four independent claims.

Claim (i). If Γ is a C-orderable group admitting only finitely many C-orders, then for every C-order $\preceq$ on Γ, the sequence of $\preceq$-convex subgroups is a rational series.

Indeed, for each convex jump $\Gamma_g \lhd \Gamma^g$, we may flip the left-order on Γ_g to produce a new left-order (*c.f.* Example 2.1.4) which is still Conradian (*c.f.* Exercise 3.2.5). If there were infinitely many $\preceq$-convex subgroups, then this would allow to produce infinitely many C-orders on Γ, contrary to our hypothesis. Let then

$$\{id\} = \Gamma^k \subsetneq \Gamma^{k-1} \subsetneq \ldots \subsetneq \Gamma^0 = \Gamma$$

be the sequence of *all* $\preceq$-convex subgroups. As in the proof of Proposition 2.2.16, Γ^i is normal in Γ^{i-1}, and Γ^{i-1}/Γ^i is torsion-free Abelian. The rank of this quotient must be 1, as otherwise it would admit uncountably many orders, which would allow to produce –by convex extension– uncountably many C-orders on Γ.

Claim (ii). If a left-orderable group admits only finitely many C-orders, then it has a unique (hence normal) rational series.

The proof is almost the same as that of Proposition 2.2.17. We just need to change the word "left-order" by "C-order" along that proof, and replace the (crucial) use of Proposition 2.1.7 by Proposition 3.2.31.

Claim (iii). If a group Γ with a normal rational series $\{id\} = \Gamma^k \lhd \Gamma \lhd \ldots \lhd \Gamma^0 = \Gamma$ admits only finitely many C-orders, then no quotient Γ^{i-2}/Γ^i is Abelian.

First notice that every group admitting a rational series is C-orderable. Actually, using the rational series above, one may produce 2^k Conradian orders on Γ. If one of the quotients Γ^{i-2}/Γ^i were Abelian, then it would have rank 2, hence it would admit uncountably many left-orders. This would allow to produce –by convex extension– uncountably many C-orders on Γ.

Claim (iv). If a group Γ has a normal rational series $\{id\} = \Gamma^k \lhd \Gamma \lhd \ldots \lhd \Gamma^0 = \Gamma$ such that no quotient Γ^{i-2}/Γ^i is Abelian, then this series coincides with that formed by the $\preceq$-convex subgroups, where $\preceq$ is any C-order on Γ. In particular, such a series is unique.

As we have already seen, the rational series above leads to 2^k Conradian left-orders. We have to prove that these are the only possible C-orders on G. To show this, let $\preceq$ be a C-order on Γ. By Claim (iii), there exist non-commuting elements $g \in \Gamma^{k-1}$ and $h \in \Gamma^{k-2} \setminus \Gamma^{k-1}$. Denote the Conrad homomorphism of the group $\langle g, h\rangle$ (endowed with the restriction of $\preceq$) by τ. Then we have $\tau(g) = \tau(hgh^{-1}) \neq 0$. Since Γ^{k-1} is rank-1 Abelian, hgh^{-1} must be equal to g^s for some rational number $s \neq 1$. Hence, $\tau(g) = s\tau(g)$, which implies that $\tau(g) = 0$. Therefore, $g^n \prec |h|$ for every $n \in \mathbb{Z}$, where $|h| := \max\{h^{-1}, h\}$. Since $\Gamma^{k-2}/\Gamma^{k-1}$ has rank 1, this actually holds for every $h \neq id$ in $\Gamma^{k-2} \setminus \Gamma^{k-1}$. Thus, Γ^{k-1} is $\preceq$-convex in Γ^{k-2}.

Repeating the argument above, though now with $\Gamma^{k-2}/\Gamma^{k-1}$ and $\Gamma^{k-3}/\Gamma^{k-1}$ instead of Γ^{k-1} and Γ^{k-2}, respectively, we see that the rational series we began with is no other than the series given by the $\preceq$-convex subgroups. Since each Γ^{i-1}/Γ^{i} is rank-1 Abelian, if we choose $g_i \in \Gamma^{i-1} \setminus \Gamma^{i}$ for each i, then any C-order on Γ is completely determined by the signs of these elements. This shows the claim, and concludes the proof of Theorem 3.2.54. $\square$

Example 3.2.55. The Baumslag-Solitar group $BS(1,\ell) = \langle a, b \colon aba^{-1} = b^{\ell}\rangle$, $\ell \geq 2$, admits the rational series

$$\{id\} \lhd b^{\mathbb{Z}[\frac{1}{\ell}]} := \langle c \colon c^{\ell^i} = b \text{ for some integer } i > 0\rangle \lhd BS(1,\ell),$$

which satisfies the conditions of Theorem 3.2.54. Therefore, it admits four C-orders –all of which are bi-invariant–, though its space of left-orders is uncountable (*c.f.* §1.2.2). See §3.3.1 for more details on this example.

Example 3.2.56. Examples of groups having exactly 2^k left-orders (hence 2^k Conradian orders) were introduced in Example 2.2.13. Namely, one may consider $K_k = \langle a_1, \ldots, a_k \mid R_k\rangle$, where the set of relations R_k is

$$a_{i+1}^{-1} a_i a_{i+1} = a_i^{-1} \quad \text{if} \quad i < k, \qquad a_i a_j = a_j a_i \quad \text{if} \quad |i-j| \geq 2.$$

The existence of groups with 2^k Conradian orders but infinitely many (hence uncountably many) left-orders is more subtle. As we have seen in the preceding example, for $n = k$ this is the case of the Baumslag-Solitar groups $BS(1,\ell)$ for $\ell \geq 2$. To construct examples for higher k having $BS(1,\ell)$ as a quotient by a normal convex subgroup, we choose an *odd* integer $\ell \geq 3$ and we let $C_n(\ell)$ be the group

$$\big\langle c, b, a_1, \ldots, a_n \mid cbc^{-1} = b^{\ell}\,,\ ca_i = a_i c\,,\ ba_n b^{-1} = a_n^{-1}\,,\ ba_i = a_i b \ \text{ if } \ i \neq n\,,\ R_n \big\rangle.$$

This corresponds to the set $\mathbb{Z} \times \mathbb{Z}[\frac{1}{3}] \times \mathbb{Z}^n$ endowed with the product rule

$$\Big(c,\, \frac{m}{\ell^k}\,, a_1, \ldots, a_n\Big) \cdot \Big(c',\, \frac{m'}{\ell^{k'}}\,, a_1', \ldots, a_n'\Big) =$$
$$= \Big(c+c',\; \ell^c \frac{m'}{\ell^{k'}} + \frac{m}{\ell^k}\,, (-1)^{a_2} a_1' + a_1, \ldots,\; (-1)^{a_n} a_{n-1}' + a_{n-1},\; (-1)^m a_n' + a_n\Big).$$

Notice that this is well-defined, as $(-1)^m = (-1)^{\bar{m}}$ whenever $m/\ell^k = \bar{m}/\ell^{\bar{k}}$ (it is here where the fact that ℓ is odd becomes important). The group $C_n(\ell)$ admits the rational series

$$\{id\} \lhd \langle a_1 \rangle \lhd \langle a_1, a_2 \rangle \lhd \ldots \lhd \langle a_1, \ldots, a_n \rangle \lhd \big\langle a_1, \ldots, a_n, b^{\mathbb{Z}[\frac{1}{\ell}]} \big\rangle \lhd C_n(\ell).$$

By Theorem 3.2.54, it admits exactly 2^{n+2} Conradian orders. However, it has $BS(1,\ell)$ as quotient by the normal convex subgroup K_n. Since $BS(1,\ell)$ admits uncountably many (left) left-orders, the same is true for $C_n(\ell)$.

We close this section with a result (also taken from [155]) to be compared with Theorem 2.2.9.

Theorem 3.2.57. *Every C-orderable group admits either finitely many or uncountably many C-orders. In the last case, none of these left-orders is isolated in the space of C-orders.*

To prove this theorem, we need the lemmas below.

Lemma 3.2.58. *If Γ is C-orderable group such that $\mathcal{CO}(\Gamma)$ has an isolated point $\preceq$, then the family of $\preceq$-convex subgroups (is finite and) is a rational series such that no quotient of the form Γ^{i-2}/Γ^i is Abelian.*

Proof. As in Claim (i) of Proposition 3.2.53, the family of $\preceq$-convex subgroups is finite, say

$$\{id\} = \Gamma^k \subsetneq \Gamma^{k-1} \subsetneq \ldots \subsetneq \Gamma^0 = \Gamma.$$

Since $\preceq$ is Conradian, Γ^i is normal in Γ^{i-1} for each i. The proofs of that Γ^{i-1}/Γ^i has rank-1 and no quotient Γ^{i-2}/Γ^i is Abelian are similar to those of Theorem 3.2.54, and we leave them to the reader. $\square$

Lemma 3.2.59. *For any C-orderable group whose space of C-orders has an isolated point $\preceq$, the rational series formed by the $\preceq$-convex subgroups is normal.*

Proof. The proof is by induction on the length k of the rational series. For $k = 1$, there is nothing to prove; for $k = 2$, the series is automatically normal. Assume that the claim of the lemma holds for k, and let

$$\{id\} = \Gamma^{k+1} \lhd \Gamma^k \lhd \ldots \lhd \Gamma^1 \lhd \Gamma^0 = \Gamma \tag{3.4}$$

be the rational series of length $k + 1$ associated to a C-order on a group Γ that is isolated in $\mathcal{CO}(\Gamma)$. Notice that the truncated chain of length k

$$\{id\} = \Gamma^{k+1} \lhd \Gamma^k \lhd \ldots \lhd \Gamma^1 \tag{3.5}$$

is a rational series for Γ^1. Moreover, this series is associated to a C-order on Γ^1 (namely the restriction of $\preceq$) that is isolated in $\mathcal{CO}(\Gamma^1)$ (otherwise, $\preceq$ would be non-isolated in $\mathcal{CO}(\Gamma)$). By the inductive hypothesis, this series is normal. By the preceding lemma, for each $i \in \{3, \ldots, k + 1\}$, the quotient Γ^{i-2}/Γ^i is non-Abelian. We are hence under the hypothesis of Theorem 3.2.54, which allows us to conclude that this is the unique rational series of Γ^1.

Now, since Γ^1 is normal in Γ, for each $h \in \Gamma$, the conjugate series

$$\{id\} = h\Gamma^{k+1}h^{-1} \lhd h\Gamma^k h^{-1} \lhd \ldots \lhd h\Gamma^1 h^{-1} = \Gamma^1$$

is also a rational series for Γ^1. By the uniqueness above, this series coincides with (3.5). Therefore, (3.4) is a *normal* rational series. □

The proof of Theorem 3.2.57 is now at hand. Indeed, the two preceding lemmas imply that, if a C-orderable group admits an isolated C-order, then it has a normal rational series satisfying the hypothesis of Theorem 3.2.54, thus it has finitely many C-orders. If, otherwise, no C-order is isolated in the space of C-orders, then this is a Hausdorff, totally disconnected, topological space without isolated points, hence uncountable ([86, Theorem 2-80]).

Exercise 3.2.60. By slightly pursuing on the arguments above, show the following analogue of Proposition 3.2.53: If a C-orderable group admits infinitely many C-orders, then every neighborhood of such a left-order in the space of Conradian orders is uncountable.

3.3 An Application: Ordering Solvable Groups

3.3.1 The space of left-orders of finite-rank solvable groups

Following [156], we will see in §3.3.2 that the space of left-orders of a countable left-orderable virtually-solvable group has no isolated point, except for the cases where it is finite that are described in §2.2.1. This result requires both algebraic and dynamical issues. As a major particular case, here we focus on finite-rank solvable groups and their finite extensions, for which the result above will follow from a rough classification of all actions on the line without global fixed points. As concrete relevant examples, we will treat the cases of the Baumslag-Solitar groups and the groups *Sol* at the end of this section.

Recall that a group Γ is said to be ***virtually finite-rank solvable*** if it contains a finite-index subgroup $\tilde{\Gamma}$ that admits a normal series

$$\{id\} = \widetilde{\Gamma}^n \lhd \widetilde{\Gamma}^1 \lhd \ldots \lhd \widetilde{\Gamma}^0 = \widetilde{\Gamma}$$

in which every quotient $\widetilde{\Gamma}^{i-1}/\widetilde{\Gamma}_i$ is finite-rank Abelian.[4] (Notice that such a group Γ is necessarily countable.) The number $\sum_i rank(\widetilde{\Gamma}^{i-1}/\widetilde{\Gamma}^i)$ is independent from both the finite-index subgroup and the normal series chosen. (In particular, we can –and we will– take $\widetilde{\Gamma}$ as being normal in Γ.) We call this number the ***rank*** of Γ. We leave to the reader the task of checking that this number strictly decreases when passing to either an infinite-index subgroup or to a quotient by an infinite subgroup. (See [158] in case of problems.)

Exercise 3.3.1. Show that every left-order on a virtually finite-rank solvable group admits a maximal proper convex subgroup (despite the fact that such a group can be non-finitely-generated).
Hint. Proceed by induction on the rank, noticing that if $G \subset H$ are distinct convex subgroups, then the rank of G is strictly smaller than that of H.

The main result of this section is

Theorem 3.3.2. *The space of left-orders of a virtually finite-rank solvable group is either finite or a Cantor set.*

The first step to show this result deals with left-orders induced from non-Abelian affine actions. (Compare Theorem 3.2.43.)

Proposition 3.3.3. *Let Γ be a subgroup of the affine group endowed with a left-order $\preceq$ induced (in a dynamically-lexicographic way) from its affine action on the real line. If Γ is non-Abelian, then $\preceq$ is an accumulation point of its set of conjugates.*

[4] In case such a series can be taken so that each Γ^{i-1}/Γ^i is cyclic, the group is said to be ***virtually polycyclic***.

Proof. First notice that, as affine homeomorphisms fix at most one point, the dynamically-lexicographic order $\preceq$ is completely determined by the first two comparison points, that we denote x_1, x_2. (In case of a single point, we let $x_2 := x_1$.)

By assumption, Γ contains both nontrivial homotheties and nontrivial translations. It follows that the translations in Γ form a subgroup with dense orbits, hence the set of points that are fixed by some nontrivial homothethy in Γ is dense in $\mathbb{R}$. Therefore, given any two distinct points in $\mathbb{R}$, there is a nontrivial homothethy whose unique fixed point lies between them. As a consequence, for any pair of comparison points y_1, y_2 such that $y_1 \neq x_1$, the induced left-order $\preceq'$ is different from $\preceq$.

We next show that y_1, y_2 may be chosen so that $\preceq'$ is close to $\preceq$. Given a finite set $\mathcal{G} \subset \Gamma$ of $\preceq$-positive elements, we write it as a disjoint union $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$, where $\mathcal{G}_1$ is the subset of elements of $\mathcal{G}$ lying in the stabilizer of x_1 in Γ. Let I denote the open interval with endpoints x_1 and x_2. On the one hand, since $\mathcal{G}_2$ is finite, there is a small neighborhood U of x_1 such that $f(x) > x$ for every $x \in U$ and every $f \in \mathcal{G}_2$. On the other hand, for every $f \in \mathcal{G}_1$, we have $f(x) > x$ for every $x \in I$. (Notice that each $f \in \mathcal{G}_1$ is an homothethy.) Thus, if we choose any y_1 in the nonempty open set $I \cap J$ (and y_2 arbitrary), then the resulting left-order $\preceq'$ is such that all elements in $\mathcal{G}$ are still $\preceq'$-positive. Finally, we can choose such a y_1 in the Γ-orbit of x_1, say $y_1 = h(x_1)$. For this choice (and letting $y_2 := h(x_2)$), we have that $\preceq'$ is the conjugate of $\preceq$ by h, as desired. $\square$

Corollary 3.3.4. *Let $(\Gamma, \preceq)$ be a countable, left-ordered group. Suppose there is a homomorphism $\Phi \colon \Gamma \to \mathrm{Aff}_+(\mathbb{R})$ with $\preceq$-convex kernel and non-Abelian image. Suppose further that the dynamical realization of $(\Gamma, \preceq)$ is semiconjugate to the action given by Φ. Then $\preceq$ is non-isolated in $\mathcal{LO}(\Gamma)$.*

Proof. By Proposition 2.2.1, it suffices to deal with the case where Φ is injective. Let φ denote the semiconjugacy assumed by hypothesis, and let Γ_0 be the stabilizer of $\varphi(0)$ in $\Phi(\Gamma)$. This is an Abelian subgroup of Γ. We claim that it is $\preceq$-convex. Indeed, if $h_1 \prec g \prec h_2$, then $h_1(0) \leq g(0) \leq h_2(0)$, thus $\varphi(h_1(0)) \leq \varphi(g(0)) \leq \varphi(h_2(0))$, and hence $\Phi(h_1)(\varphi(0)) \leq \Phi(g)(\varphi(0)) \leq \Phi(h_2)(\varphi(0))$. In particular, if h_1, h_2 lie in Γ_0, then $\Phi(g)(\varphi(0)) = \varphi(0)$, that is, g also lies in Γ_0.

If Γ_0 is trivial, then Proposition 3.3.3 directly applies, since in this case $\preceq$ coincides with the left-order induced from $\varphi(0)$ in the action given by Φ. If Γ_0 has rank 1, then the restriction of $\preceq$ to Γ_0 is completely determined by the sign of any nontrivial element therein, say $\Phi(h) \in \Gamma_0$, with $h \succ id$. As $\Phi(h)$ is a nontrivial homothethy, there exists $x \in \mathbb{R}$ such that $\Phi(h)(x) > x$. It follows that $\preceq$ coincides with the left-order induced from the action Φ using the comparison points $x_1 := \varphi(0)$ and $x_2 := x$. Therefore, Proposition 3.3.3 still allows concluding that $\preceq$ is non-isolated. Finally, the case where Γ_0 has rank > 1 is slightly different, as we cannot argue that $\preceq$ is completely induced from the affine action. However, by §1.2.1, the restriction of $\preceq$ to Γ_0 is non-isolated. Therefore, by convex extension, $\preceq$ itself is non-isolated, as desired. $\square$

To proceed with the proof of Theorem 3.3.2, we need some general results on the structure of finite-rank solvable groups. If Γ is such a group and is torsion-free, then it contains a finite-index subgroup $\tilde{\Gamma}$ whose commutator subgroup $[\tilde{\Gamma}, \tilde{\Gamma}]$ is nilpotent [150, 158]. Let R be a maximal nilpotent subgroup of $\widetilde{\Gamma}$. By maximality, R is characteristic subgroup of $\widetilde{\Gamma}$, hence normal in Γ; moreover, it is unique (see the exercise below). It is sometimes called the ***nilpotent radical*** of $\widetilde{\Gamma}$.

Exercise 3.3.5. Let Γ be a group and G, H two normal nilpotent subgroups. Show that the set $GH := \{gh \colon g \in G,\ h \in H\}$ is a nilpotent subgroup.

Theorem 3.2.20 implies Theorem 3.3.2 in the case where R has finite index in $\widetilde{\Gamma}$ (in particular, when the rank of Γ is 1). Hence, in what follows, we assume that $\widetilde{\Gamma}/R$ is infinite. We proceed by induction, thus we assume that Theorem 3.3.2 holds for every virtually finite-rank solvable group having smaller rank than that of Γ. Let $\preceq$ be a left-order on Γ. Consider its dynamical realization, and denote by $\Gamma_0 \subset R$ the set of elements in R having fixed points. Since R is normal in Γ, we have that Γ_0 is also normal in Γ. The following lemma implies that Γ_0 has a global fixed point. (Compare Exercise 3.2.29.)

Lemma 3.3.6. *Assume that a nilpotent group with finite rank acts by orientation-preserving homeomorphisms of the real line. If every element admits fixed points, then there is a global fixed point for the action.*

Proof. If the nilpotence length of the underlying group G is 0, then the group is trivial, and there is nothing to prove. We continue by induction on the nilpotent length, denoting by H the center of G. This is a finite-rank Abelian group, hence it contains a subgroup H_0 isomorphic to a certain $\mathbb{Z}^d$ such that H/H_0 is a torsion group. It follows that the (closed) set $Fix := Fix(H_0)$ of fixed points of H_0 is nonempty. Since H/H_0 is torsion, Fix coincides with the set of fixed points of H. The complement of Fix is a disjoint union $\bigsqcup_i I_i$ of open intervals I_i. Moreover, since $H \lhd G$, we have that Fix is G-invariant. In particular, the intervals in the complement of Fix are permuted by G. Furthermore, since every element of G has fixed points, we have that every element in G must fix some point in

Fix. Let us now extend in a piecewise affine way the action of G on Fix to the complementary intervals. Doing this, we obtain a new action of G on $\mathbb{R}$ which factors throughout G/H. Since this action coincides with the original one on Fix, every element of G admits fixed points. We can hence apply the induction hypothesis, thus concluding that G/H has a global fixed point in Fix, hence G has a global fixed point. $\square$

Now, since R is nilpotent, every left-order on it is Conradian (see the discussion before Theorem 3.2.20). Using Corollary 3.2.28 (more precisely, by Exercise 3.2.29), we thus obtain that Γ_0 contains $[R, R]$, and R/Γ_0 is torsion-free. Moreover, since Γ_0 is normal in Γ, we have that $Fix(\Gamma_0)$ is Γ-invariant, hence Γ_0 admits a $\mathbb{Z}$-indexed sequence of global fixed points going from $-\infty$ to ∞.

We divide the induction argument into two separated cases.

Case I. Either R/Γ_0 is trivial, or it has rank 1 and the conjugacy action of $\widetilde{\Gamma}$ on it is by multiplication by ± 1.

In this case, we have

Claim (i). The quotient $\widetilde{\Gamma}/\Gamma_0$ is Abelian.

Indeed, if R/Γ_0 is trivial, then this follows from that $[\widetilde{\Gamma}, \widetilde{\Gamma}] \subseteq R$. Otherwise, assume for a contradiction that $\widetilde{\Gamma}$ does not centralize $\tilde{\Gamma}/\Gamma_0$. As $\tilde{\Gamma}$ centralizes $\tilde{\Gamma}/R$, this means that $\tilde{\Gamma}$ does not centralize R/Γ_0. Hence, there are $f \in R \setminus \Gamma_0$ and $g \in \widetilde{\Gamma}$ such that, modulo Γ_0, one has the equality $gfg^{-1} = f^{-1}$. Now, since f acts without fixed points, changing f by f^{-1} if necessary, we can assume that $f(y) > y$ for every $y \in \mathbb{R}$. Thus, if we let x be in the set of fixed points of Γ_0, we have that $gfg^{-1}(x) = f^{-1}(x) < x$, which implies that $fg^{-1}(x) < g^{-1}(x)$, contrary to our assumption on f.

Let I be the smallest closed interval containing the origin whose endpoints are fixed by Γ_0, and let H be its stabilizer in Γ. Since Γ_0 is normal in Γ, for every $g \in \Gamma$, either $g(I)$ equals I or it is disjoint from it. By Proposition 2.1.3, this implies that H is a convex subgroup.

Claim (ii). The subgroup H has smaller rank than $\widetilde{\Gamma}$.

Indeed, on the one hand, $H \cap \widetilde{\Gamma}$ cannot be equal to $\widetilde{\Gamma}$, since the latter does not have global fixed points. On the other hand, since Γ_0 is contained in $H \cap \widetilde{\Gamma}$, Claim (i) above implies that $H \cap \widetilde{\Gamma}$ is a normal subgroup of $\widetilde{\Gamma}$ and that $\widetilde{\Gamma}/(H \cap \widetilde{\Gamma})$ is Abelian. Therefore, as the quotient $\widetilde{\Gamma}/(H \cap \widetilde{\Gamma})$ is left-orderable, it has rank > 0, thus showing the claim.

It follows by induction that the space of left-orders of H is either finite or a Cantor set. Hence, by Proposition 2.2.1, if $\preceq$ is isolated in $\mathcal{LO}(\Gamma)$, then H is a Tararin group. However, if H is a Tararin group, then every left-order on H is Conradian (*c.f.* Lemma 2.2.15). By the convexity of $H \cap \widetilde{\Gamma}$ in $\widetilde{\Gamma}$ and the fact that $\widetilde{\Gamma}/(H \cap \widetilde{\Gamma})$ is Abelian, we have that the restriction of $\preceq$ to $\widetilde{\Gamma}$ is Conradian (*c.f.* Exercise 3.2.5). Therefore, by Theorem 3.2.9, we have that $\preceq$ is a Conradian order of Γ. As a consequence, using Proposition 3.2.53 we conclude that, if $\preceq$ is isolated in $\mathcal{LO}(\Gamma)$, then Γ must be a Tararin group, as desired.

Case II. Either $rank(R/\Gamma_0) \geq 2$, or $rank(R/\Gamma_0) = 1$ and there exists $g \in \widetilde{\Gamma}$ which does not merely act on R/Γ_0 by multiplication by ± 1. (In particular, R/Γ_0 cannot be isomorphic to $\mathbb{Z}$.)

In this case, Proposition 3.5.16 and Remark 3.5.17 provides us with an R-invariant Radon[5] measure ν, associated to which there is a ***translation number homomorphism*** $\tau_\nu : R \to (\mathbb{R}, +)$ defined by $\tau_\nu(g) := \nu([x, g(x)))$ (here and below, we use the convention $\nu([x,y]) := -\nu([y,x])$ for $y < x$). Notice that this definition does not depend on the choice of the point x.

Exercise 3.3.7. Show that τ_ν coincides (up to a positive multiple) with the Conrad homomorphism on R associated to the convex jump with respect to the maximal proper convex subgroup (*c.f.* §3.2.3)

Exercise 3.3.8. Show that if G is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ with no global fixed point and whose action preserves a Radon measure for which the translation-number homomorphism has image non isomorphic to $\mathbb{Z}$, then its action is semiconjugate to an action by translations.

Exercise 3.3.9. Let G be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ preserving a Radon measure ν.
(i) Show that the kernel of τ_ν coincides with the subset G_0 made of the elements having fixed points. Moreover, show that for all x in the support $supp(\nu)$, its stabilizer in G coincides with G_0.
(ii) Conclude that if every element in G has fixed points, then there is a global fixed point for the action.

The next proposition (which is interesting in its own right) tell us that, up to multiplication by a positive constant, ν is the unique R-invariant Radon measure. It is somewhat a dynamical counterpart of Exercise 3.3.7 above.

[5] Recall that a ***Radon measure*** is a measure giving finite mass to compact sets.

Proposition 3.3.10. *Let G be a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *preserving a Radon measure ν. Then, for any other (nontrivial) G-invariant Radon measure ν', there is a positive real number κ such that $\kappa\tau_\nu = \tau_{\nu'}$. Moreover, if $\tau_\nu(G)$ is dense in $(\mathbb{R},+)$, then $\kappa\nu = \nu'$.*

Proof. It easily follows from Exercise 3.3.9 that $\tau_\nu(G)$ and $\tau_{\nu'}(G)$ are simultaneously either discrete or dense in $\mathbb{R}$. In the former case, the claim of the proposition is obvious. Below we deal with the latter case.

Fix $g \notin G_0$ and a point x that is fixed by G_0. Then, as a combination of Exercises 3.1.5 and 3.3.8, we have that for all $f \in G$,

$$\tau_\nu(f) = \tau_\nu(g) \lim_{p\to\infty} \left\{ \frac{q}{p} \colon g^q(x) \leq f^p(x) < g^{q+1}(x) \right\},$$

and the same holds changing ν by ν'. Therefore, we have $\tau_{\nu'}(g)\tau_\nu(f) = \tau_{\nu'}(f)\tau_\nu(g)$ for every $f \in G$, hence $\tau_{\nu'}$ equals $\kappa\tau_\nu$ for a certain positive κ.

Next, we claim that the supports of ν and ν' coincide. Indeed, the density of $\tau_{\nu'}(G)$ implies that ν' has no atoms and the action of G on $supp(\nu')$ is minimal (*i.e.* every orbit is dense). It follows that if there is a point $x \in supp(\nu') \setminus supp(\nu)$, then there exists $g \in G$ such that $g(x) > x$ and $\nu([x, g(x))) = 0$, contradicting the fact that $\ker(\tau_{\nu'}) = \ker(\tau_\nu)$. Therefore, $supp(\nu') \subset supp(\nu)$, and the reverse inclusion is proven analogously.

Finally, let $x < y$ be two points in the (common) supports of ν and ν', and let $g_n \in G$ be such that $g_n(x)$ converges to y. Then

$$\nu'\big([x,y]\big) = \lim_{n\to\infty} \nu'\big([x, g_n(x)]\big) = \lim_{n\to\infty} \tau_{\nu'}(g_n) = \lim_{n\to\infty} \kappa\tau_\nu(g_n) = \kappa\nu\big([x,y]\big),$$

which finishes the proof. □

Using the R-invariant Radon measure, we can describe the action of Γ up to semiconjugacy. More precisely, we have

Claim (i). There is a homomorphism $\Phi\colon \Gamma \to \mathrm{Aff}_+(\mathbb{R})$ such that $\Phi(R)$ contains nontrivial translations, and Γ_0 coincides with $\ker(\Phi) \cap R$. Moreover, the dynamical realization of $(\Gamma, \preceq)$ is semiconjugate to this affine action.

Indeed, let us continue denoting by ν an R-invariant Radon measure. Since R/Γ_0 is not isomorphic to $\mathbb{Z}$, Proposition 3.3.10 (and its proof) implies that ν is unique up to a scalar multiple, and that Γ_0 is the kernel of the translation-number homomorphism τ_ν. As R is normal in Γ, this implies that for each $g \in \Gamma$, the measure $g_*(\nu)$ is also R-invariant. Thus, for every $g \in \Gamma$, there is $\lambda_g > 0$ such that $g_*(\nu) = \lambda_g \nu$. This yields a group homomorphism $\lambda \colon \Gamma \to \mathbb{R}^*$ into the group of positive reals (with multiplication). We then define $\Phi \colon \Gamma \to \mathrm{Aff}_+(\mathbb{R})$ by

$$\Phi(g)(x) := \frac{1}{\lambda_g}\, x + \nu([0, g(0)]).$$

One can easily check that this is a homomorphism that extends τ_ν.

To show that the dynamical realization of $\preceq$ is semiconjugate to this affine action, for each $x \in \mathbb{R}$ we let $\varphi(x) := \nu([0,x])$. Then φ is a continuous, non-decreasing surjective map, and a direct computation shows that for all $g \in \Gamma$ and every $x \in \mathbb{R}$,

$$\varphi(g(x)) = \Phi(g)(\varphi(x)),$$

which shows the desired semiconjugacy.

Next, we let $I_\nu := (a,b)$, where $a := \sup\{x < 0 : x \in \mathit{supp}(\nu)\}$ and $b := \inf\{x > 0 : x \in \mathit{supp}(\nu)\}$. We also let Γ_ν be the stabilizer in Γ of I_ν. The subgroup Γ_ν is easily seen to be convex. Moreover, $\Gamma_\nu \cap R = \Gamma_0$.

Notice that the rank of Γ_ν is smaller than that of Γ. Thus, by the induction hypothesis, if Γ_ν admits infinitely many left-orders, then no left-order on it is isolated. By convex extension, we conclude that $\preceq$ is non-isolated in $\mathcal{LO}(\Gamma)$. For the other case, we have the next

Claim (ii). If Γ_ν is a Tararin group, then $\ker \Phi$ is convex.

Indeed, since $\Phi(\Gamma_\nu)$ does not contain any nontrivial translation, it can only contain homotheties centered at 0; in particular, it is Abelian. If it is trivial, then $\ker \Phi = \Gamma_\nu$, so it is convex, as desired. Assume $\Phi(\Gamma_\nu)$ is nontrivial, and let $\{id\} = \Gamma^n \lhd \Gamma^{n-1} \lhd \ldots \lhd \Gamma^0 = \Gamma_\nu$ be the series of all convex subgroups of the Tararin group Γ_ν. (Recall that Γ^{i-1}/Γ^i has rank 1 and that the action of Γ^{i-1} on Γ^{i-1}/Γ^i is by multiplication by some negative number.) By Exercise 2.2.20, Γ_ν has a unique torsion-free Abelian quotient, namely Γ_ν/Γ^1. As this must coincide with $\Phi(\Gamma)$, we conclude that $\ker \Phi$ equals Γ^1, hence it is convex.

Knowing that $\ker \Phi$ is convex, we can proceed to show that $\preceq$ is non-isolated. Indeed, either $\Gamma / \ker \Phi$ is Abelian of rank at least 2, or it is a non-Abelian subgroup of the affine group. In the former case, it has no isolated left-orders (see §1.2.1), hence -by convex extension- the left-order $\preceq$ is non-isolated in $\mathcal{LO}(\Gamma)$. In the latter case, we are under the hypothesis of Proposition 3.3.3, which yields the same conclusion. This finishes the proof of Theorem 3.3.2.

Left-orders on Baumslag-Solitar's groups. Perhaps the simplest examples of finite-rank solvable groups that are non virtually-nilpotent are the Baumslag-Solitar groups $BS(1,\ell) := \langle h, g \colon hgh^{-1} = g^{\ell} \rangle$, where $\ell > 1$. We have seen that $BS(1,\ell)$ admits only four Conradian orders (*c.f.* Example 3.2.55), yet it also admits the left-orders induced from the affine faithful actions on the line (*c.f.* §1.2.2). Below we follow the lines of the previous proof to show that, actually, these are the only possible left-orders on $B(1,\ell)$.

As in §1.2.2, we can see $B(1,\ell)$ as a semidirect product $\mathbb{Z}[\frac{1}{\ell}] \rtimes \mathbb{Z}$, where the $\mathbb{Z}$-factor acts on $\mathbb{Z}[\frac{1}{\ell}] := \big\{ \frac{k}{\ell^m} \colon k,\, m \text{ in } \mathbb{Z} \big\}$ by multiplication by ℓ. In this way, it easily follows that the nilpotent radical of $B(1,\ell)$ is $R := \mathbb{Z}[\frac{1}{\ell}]$.

Now, given a left-order on $BS(1,\ell)$, we consider its dynamical realization. Since $R = \mathbb{Z}[\frac{1}{\ell}]$ has rank one, two cases may arise.

Case I. There is a global fixed point for R.

As R is normal in $BS(1,\ell)$, the set of R-fixed points is $BS(1,\ell)$-invariant; thus, in this case, it is unbounded in both directions. In particular, the convex subgroup H stabilizing the interval that contains the origin and is enclosed by two consecutive R-fixed points must coincide with R. As a consequence, $\preceq$ is Conradian, and therefore $\preceq$ is non-isolated, since $BS(1,\ell)$ is not a Tararin group.

Case II. Every nontrivial element of R is fixed-point free.

In this case, the action of R is semiconjugate to that of a dense group of translations, thus it preserves a Radon measure ν without atoms that is unique up to a scalar factor. Moreover, h does not preserve ν, otherwise we would have $\tau_\nu(g^{\ell-1}) = \tau_\nu(g^{-1}hgh^{-1}) = 0$, contradicting the fact that $g^{\ell-1}$ acts freely. Thus, the dynamical realization of $\preceq$ is semiconjugate to (the action given by) a faithful embedding of $BS(1,\ell)$ into $\mathrm{Aff}_+(\mathbb{R})$, and the left-order $\preceq$ coincides with a left-order induced from this affine action. The fact that $\preceq$ is non-isolated in this case follows from Corollary 3.3.4.

Notice that the affine-like orders of Case II approximate the Conradian ones of Case I just by letting the first comparison point tending to either $-\infty$ or ∞.

It is worth pointing out that the description above –as well as its proof– applies not only to dynamical realizations of left-orders, but also to general (faithful) actions on the line with no global fixed point. Such an action is hence either without crossings (with R being the subgroup of elements having fixed points) or semiconjugate to an affine action. (See [155] for more details on this.)

Left-orders on Sol **groups.** Relevant examples of finite-rank solvable groups that are non virtually-polycyclic are those of the form $Sol := \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where A is an hyperbolic automorphism of $\mathbb{Z}^2$ (*i.e.* given by a matrix in $\mathrm{SL}(2,\mathbb{Z})$ with trace greater than 2, so that it has two irrational eigenvalues). Below we follow the lines of the previous proof in this particular case to get an accurate description of the space of left-orders and its subspaces of bi-invariant and Conradian orders. Actually, the methods employed yield a complete description of all faithful actions on the line with no global fixed point.

We denote by R the commutator subgroup of Sol –which coincides with the $\mathbb{Z}^2$-factor–, and we let f be the element of $\mathbb{Z}$ acting on R as A. The subgroup R is easily seen to coincide with the nilpotent radical of Sol.

Given a left-order $\preceq$ on Sol, let us consider its dynamical realization. Since A^T is $\mathbb{Q}$-irreducible and R is Abelian and finitely generated, the next three properties are equivalent:

– There is an element in R having a fixed point;

– Every element of R has a fixed point;

– There is a global fixed point for R.

Indeed, having a fixed point for $g \in R$ is equivalent to that $\tau_\nu(g) = 0$ for an R-invariant Radon measure ν (see Exercise 3.3.9), and A also acts at the level of translation numbers, as it is shown in the next

Exercise 3.3.11. Let g_1, g_2 be the canonical basis of $R = \mathbb{Z}^2$. Show that

$$\begin{pmatrix} \tau_\nu(fg_1f^{-1}) \\ \tau_\nu(fg_2f^{-1}) \end{pmatrix} = A^T \begin{pmatrix} \tau_\nu(g_1) \\ \tau_\nu(g_2) \end{pmatrix}.$$

Thus, the two cases considered in the proof of Theorem 3.3.2 fit with those considered below.

<u>Case I.</u> The subgroup R has a global fixed point.

Since Sol acts without global fixed points and R is normal in Sol, in this case the set of R-fixed points is unbounded in both directions (and Γ-invariant). As for $BS(1,\ell)$, this implies that $\preceq$ is Conradian. To see that $\preceq$ is non-isolated, one may argue by convex extension by noticing that R is convex and rank-two Abelian. Alternatively, Sol is not a Tararin group...

<u>Case II.</u> There is no global fixed point for R.

In this case, R is semiconjugate to a dense group of translations, thus it preserves a Radon measure without atoms ν that is unique up to a scalar factor. As f is hyperbolic, it cannot preserve ν: it acts as an homothethy with ratio one of the eigenvalues of A^T. Thus, the dynamical realization of $\preceq$ is semiconjugate to (the action given by) a faithful embedding of Γ into $\mathrm{Aff}_+(\mathbb{R})$. The fact that $\preceq$ is non-isolated in this case follows from Corollary 3.3.4.

Figure 11: The space of left-orders of Sol.

It follows from the previous analysis that, as it was the case for the Baumslag-Solitar groups, there are two types of left-orders on Sol:

Case I. Conradian orders.

These correspond to those left-orders for which the normal subgroup $R = \mathbb{Z}^2$ is convex. Therefore, $\mathcal{CO}(Sol)$ is made of two copies of the Cantor set $\mathcal{LO}(\mathbb{Z}^2)$, each of which corresponds to a choice of sign for f. (In Figure 11, these are represented as two "vertical dashed circles".) Observe that among all bi-orders on R, those that are invariant under conjugacy by f are those that that correspond (under Conrad's homomorphisms) to eigendirections of the matrix A^T. Since the corresponding left-orders on Sol are the bi-invariant ones, we conclude that Sol supports exactly eight bi-orders.[6]

Case II. Left-orders coming from affine actions.

These form an open set –which is locally a Cantor set– that complements the subspace of Conradian orders. These affine-like orders can be described as in §1.2.2. Notice, however, that Sol admits four embeddings into $\mathrm{Aff}_+(\mathbb{R})$. These left-orders hence appear as four "horizontal dotted lines" in Figure 11. These "lines" accumulate at the eight bi-invariant orders in $\mathcal{CO}(Sol)$, in a similar way to the approximation of the four bi-orders of Baumslag-Solitar's groups by affine-like orders previously described.

Based on all of this, it is not difficult to describe the dynamics of the conjugacy action of Sol on its space of left-orders. We leave this as an exercise to the reader.

3.3.2 The general case

In this section, we prove

[6] A classification of finitely-generated solvable groups admitting only finitely many bi-orders can be found in [15, 108].

Theorem 3.3.12. *The space of left-orders of a countable virtually-solvable group is either finite or homeomorphic to the Cantor set.*

In the preceding section, we proved this theorem for virtually finite-rank solvable groups. Our main tool in that case was the fact that those groups are virtually nilpotent-by-Abelian, with finite-rank nilpotent part. As such a nilpotent group preserves a Radon measure when acting by orientation-preserving homeomorphisms of the real line, we thus conclude that actions of virtually finite-rank solvable groups on the line *quasi-preserve* a nontrivial Radon measure[7]. Thus, the action is semiconjugate to an affine action provided the measure is non-atomic. This dynamical picture no longer arises for general solvable groups (not even for metabelian groups!), as the next classical example (due to Plante [147]) shows.

Example 3.3.13. The wreath product $\mathbb{Z} \wr \mathbb{Z} := \bigoplus_{\mathbb{Z}} \mathbb{Z} \rtimes \mathbb{Z}$ is a metabelian group having $H := \bigoplus_{\mathbb{Z}} \mathbb{Z}$ as its maximal nilpotent subgroup. We next describe an action of $\mathbb{Z} \wr \mathbb{Z}$ on the real line with the property that for every shift-invariant subgroup H, no global fixed point arises, although every element therein admits fixed points. This implies in particular that there is no quasi-invariant measure for $\mathbb{Z} \wr \mathbb{Z}$. Indeed, such a measure would be invariant by the derived subgroup $[\mathbb{Z} \wr \mathbb{Z}, \mathbb{Z} \wr \mathbb{Z}]$, which is shift-invariant. In view of the announced properties, this is in contradiction with Exercise 3.3.9.

For the construction, let f denote the homothethy $x \mapsto 2x$. Let $I_0 := [-1, 1]$, and for $i \in \mathbb{Z}$, denote $I_i := f^i(I_0)$. Let $g : I_0 \to I_0$ be a homeomorphism such that $g(-1/2) = 1/2$ and $g(x) > x$ for all $x \in (-1, 1)$. We define $g_i : I_i \to I_i$ by $g_i := f^i g f^{-i}$. Notice that this is equivalent to that $f^{-1} g_i(x) = g_{i-1} f^{-1}(x)$ holds for all $x \in I_i$, that is, $g_i f(y) = f g_{i-1}(y)$ for all $y \in I_{i-1}$. Below, we extend the definition of each g_i to the whole line in such a way that f and g_0 generate a group isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.

One easily convinces that there is a unique way to extend the maps g_i into commuting homeomorphisms of the real line. For instance, to ensure commutativity, we must necessarily have $g_{i-1}(x) := g_i^m g_{i-1} g_i^{-m}(x)$ for $x \in g_i^m(I_{i-1})$. The (proof of the uniqueness of the) extension can then be easily achieved inductively. We continue denoting by g_i the resulting homeomorphisms. We claim that $f g_i f^{-1} = g_{i+1}$ holds. Indeed, this follows from the definition for $x \in I_{i+1}$. Assume inductively that $f g_i f^{-1}(x) = g_{i+1}(x)$ holds for all $x \in I_k$ for a certain $k \geq i+1$, and let $x \in I_{k+1}$. Letting $m \in \mathbb{Z}$ be such that $x = g_{k+1}^m(y)$ for a certain $y \in I_k$, we have

$$\begin{aligned} f g_i f^{-1}(x) &= f g_i f^{-1}(g_{k+1}^m(y)) = f g_i g_k^m f^{-1}(y) \\ &= f g_k^m g_i f^{-1}(y) = g_{k+1}^m f g_i f^{-1}(y) = g_{k+1}^m g_{i+1}(y) = g_{i+1}(x), \end{aligned}$$

[7]In our context, a Radon measure ν is ***quasi-preserved*** by a group Γ if for every $g \in \Gamma$, there is a positive real number λ_g such that $g_*(\nu) = \lambda_g \, \nu$, where by definition $g_*(\nu)(X) := \nu(g^{-1}(X))$ for every measurable set X.

where the second and fourth equalities follow from the definition of g_k, the third from the commutativity between g_i and g_k, and the fifth from the induction hypothesis.

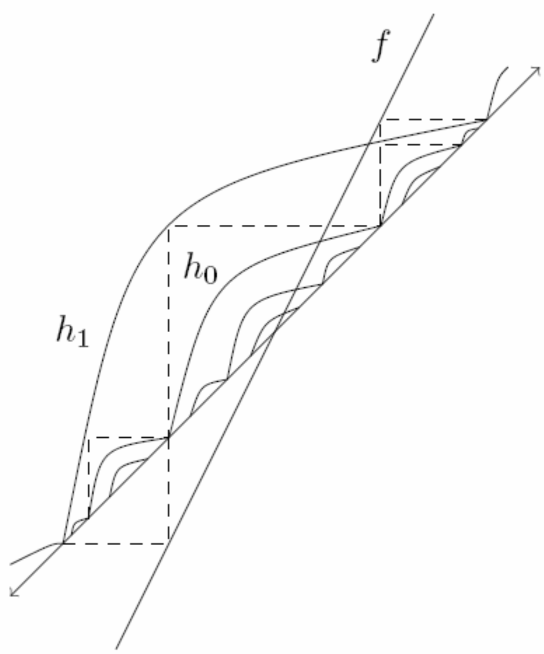

Figure 12: Plante's action of $\mathbb{Z} \wr \mathbb{Z}$.

Example 3.3.14. There exists a left-order on $\mathbb{Z} \wr \mathbb{Z}$ whose dynamical realization is semiconjugate to the action constructed in the preceding example. Indeed, using the exact sequence

$$0 \longrightarrow H \longrightarrow \mathbb{Z} \wr \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

we can produce the bi-order $\preceq'$ on $\mathbb{Z} \wr \mathbb{Z}$ as the convex extension of the lexicographic order on H by one of the two possible orders on the cyclic factor $\mathbb{Z} = \langle f \rangle$. Notice that given any element $h \in H$, we can find two elements $h_{\pm}$ in H such that $h_{-} \prec' h^n \prec' h_{+}$ holds for all $n \in \mathbb{Z}$. Moreover, H is the maximal proper $\preceq'$-convex subgroup. Using Lemma 2.1.13, we can hence produce a left-order $\preceq$ on $\mathbb{Z} \wr \mathbb{Z}$ that has f as its smallest positive element and that coincides with $\preceq'$ on H. It is not very hard to see that the dynamical realization of $\preceq$ is semiconjugate to the action constructed in Example 3.3.13 above.

To deal with this new phenomenon, we will use the machinery developed in §3.2. The price to pay is that, unlike §3.3.1, here we are not able to give a classification –up to semiconjugacy– of all actions on the real line of the involved groups. Rather, we give a rough local description of the dynamics that allows concluding Theorem 3.3.12. We start with an exercise that follows from an easy reformulation of part of the proof of Propositions 2.2.16 and 3.2.53.

Exercise 3.3.15. Show that every left-order on a group admitting infinitely many convex subgroups is non-isolated in the corresponding space of left-orders.

Due to the preceding exercise, in order to prove Theorem 3.3.12, it suffices to consider left-orders with finitely many convex subgroups. Let $\preceq$ be such an order on a group Γ, with

$$\{id\} = C_n \subsetneq C_{n-1} \subset \ldots \subsetneq C_0 = \Gamma$$

being the family of convex subgroups. One of these subgroups must coincide with the Conradian soul $G := C_{\preceq}(\Gamma)$, that is, with the maximal $\preceq$-convex subgroup restricted to which $\preceq$ is Conradian (*c.f.* §3.2.4). If G is not a Tararin group, then by Proposition 3.2.53, the restriction of $\preceq$ to G is non-isolated in $\mathcal{LO}(G)$; by convex extension, $\preceq$ is not isolated in $\mathcal{LO}(\Gamma)$. Hence, in all what follows, we assume that G is a Tararin group.

If $G = \Gamma$, then we are done: Γ admits only finitely many left-orders. If G is trivial, then $\preceq$ is non-isolated in $\mathcal{LO}(\Gamma)$, due to Theorem 3.2.43. We hence suppose that G is a nontrivial, proper subgroup of Γ, say $G = C_\ell$, with $n > \ell > 0$. We will show that the restriction of $\preceq$ to $C_{\ell-1}$ is non-isolated; by convex extension, this in turns implies that $\preceq$ is non-isolated in $\mathcal{LO}(\Gamma)$, as desired. As the claim to be shown only involves $C_{\ell-1}$, to simplify will denote this group as Γ; equivalently, we will assume that $\ell = 1$, that is, there is no convex subgroup strictly between $G = C_1$ and Γ.

We consider the dynamical realization of $\preceq$. Since G is a proper convex subgroup, it has at least one fixed point on each side of the origin. We let I_G be the smallest open interval fixed by G that contains the origin. By Proposition 2.1.3, the convexity of G immediately implies

Lemma 3.3.16. *Every element of Γ either fixes I_G or moves it into a disjoint interval. In particular, the stabilizer of I_G coincides with G.*

From now on, we assume Γ to be virtually solvable. Let $\widetilde{\Gamma}$ be a finite-index, normal, solvable subgroup of Γ. We let $\widetilde{\Gamma}^0 := \widetilde{\Gamma}$ and $\widetilde{\Gamma}^j := [\widetilde{\Gamma}^{j-1}, \widetilde{\Gamma}^{j-1}]$ be the associated derived series:

$$\{id\} = \widetilde{\Gamma}^k \lhd \widetilde{\Gamma}^{k-1} \lhd \ldots \lhd \widetilde{\Gamma}^1 \lhd \widetilde{\Gamma}^0 = \widetilde{\Gamma} \lhd \Gamma.$$

Notice that each $\widetilde{\Gamma}^j$ is normal in Γ. We let i be the minimal index such that $\widetilde{\Gamma}^i$ is contained in G. Since G is a nontrivial, proper, convex subgroup, we have that $k > i \geq 1$; see Figure 13 below.

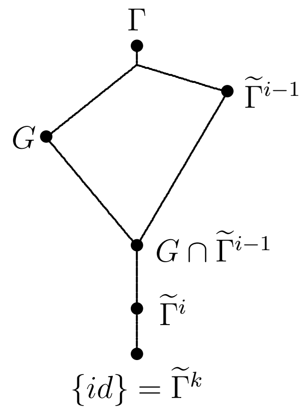

Figure 13: The groups G, $\widetilde{\Gamma}^i$ and $\widetilde{\Gamma}^{i-1}$.

The subgroup $\widetilde{\Gamma}^{i-1}$ will be of most importance in our analysis: although it is not always nilpotent, the restriction of $\preceq$ to $\widetilde{\Gamma}^{i-1}$ will be shown to be Conradian. Thus, dynamically, it will play the role played by the nilpotent radical in the finite-rank case. We like to think it as a kind of *Conradian sckeleton* of $(\Gamma, \preceq)$.

Lemma 3.3.17. *The order $\preceq$ restricted to $\widetilde{\Gamma}^{i-1}$ is Conradian.*

Proof. Indeed, by definition, the subgroup $\widetilde{\Gamma}^i = [\widetilde{\Gamma}^{i-1}, \widetilde{\Gamma}^{i-1}]$ is contained in G. Therefore, $\widetilde{\Gamma}^{i-1} \cap G$ is normal in $\widetilde{\Gamma}^{i-1}$, as well as convex therein. Moreover, as the quotient $\widetilde{\Gamma}^{i-1} / \widetilde{\Gamma}^{i-1} \cap G$ is Abelian, it only admits Conradian orders. Since $\preceq$ restricted to $\widetilde{\Gamma}^{i-1} \cap G$ is Conradian, this implies that $\preceq$ restricted to $\widetilde{\Gamma}^{i-1}$ is a convex extension of a Conradian order by a Conradian one, hence Conradian (*c.f.* Exercice 3.2.5). $\square$

Lemma 3.3.18. *The action of $\widetilde{\Gamma}^{i-1}$ has no global fixed point.*

Proof. Let I be the smallest open interval containing the origin that is fixed by $\widetilde{\Gamma}^{i-1}$. Since $\widetilde{\Gamma}^{i-1}$ is normal in Γ, the interval I is either fixed or moved disjointly by each $g \in \Gamma$. In particular, the stabilizer $Stab_{\Gamma}(I)$ of I is a convex subgroup of Γ (*c.f.* Proposition 2.1.3). Now, if I was not the whole line, then the maximality of G would imply that $Stab_{\Gamma}(I) \subseteq G$, thus yielding $\widetilde{\Gamma}^{i-1} \subset G$, which is a contradiction. $\square$

Since $\preceq$ restricted to $\widetilde{\Gamma}^{i-1}$ is Conradian, its action on the real line is without crossings (*c.f.* Exercise 3.2.33). It follows that the set of elements in $\widetilde{\Gamma}^{i-1}$ having fixed points is a normal subgroup of $\widetilde{\Gamma}^{i-1}$ (actually, of Γ); see *e.g.* Proposition 3.2.26. In particular, if $g \in \widetilde{\Gamma}^{i-1}$ does not act freely, then the set of fixed points of g accumulates at both $-\infty$ and $+\infty$. Thus, in order to prove Theorem 3.3.12, we need to analyze two cases.

Case I. The subgroup $\widetilde{\Gamma}^{i-1}$ contains elements without fixed points. (Such a case arises for instance if $\widetilde{\Gamma}^{i-1}$ has finite rank.)

We first observe that in this case $\widetilde{\Gamma}^{i-1}$ preserves a nontrivial Radon measure ν. Indeed, since the order on $\widetilde{\Gamma}^{i-1}$ is Conradian, its action on the real line is without crossings. Further, since there is $g_0 \in \widetilde{\Gamma}^{i-1}$ acting freely, Proposition 3.2.26 easily implies that $\widetilde{\Gamma}^{i-1}$ has a maximal proper convex subgroup, namely $\{g \in \widetilde{\Gamma}^{i-1} : g \text{ has a fixed point}\}$. It then follows from Proposition 3.5.16 and Remark 3.5.17 that $\widetilde{\Gamma}^{i-1}$ preserves a non-trivial Radon measure on the line.

Now, from the normality of $\widetilde{\Gamma}^{i-1}$ in Γ and Proposition 3.3.10, we have that there is a homomorphism $\lambda\colon g \mapsto \lambda_g$ from Γ into $\mathbb{R}^*$, where $\tau_{g_*\nu} = \lambda_g \tau_\nu$. The next lemma follows from the work of Plante [147].

Lemma 3.3.19. *If the homomorphism λ is trivial, then Γ preserves a Radon measure on the real line. Otherwise, Γ quasi-preserves a Radon measure which is $\widetilde{\Gamma}^{i-1}$-invariant.*

Proof. Recall that by Proposition 3.3.10, if $\tau_\nu(\widetilde{\Gamma}^{i-1})$ is dense, then ν is quasi-invariant. This occurs for instance if λ is nontrivial. Indeed, choosing $g \in \Gamma$ such that $\lambda_g < 1$, we have for every $f \in \widetilde{\Gamma}^{i-1}$,

$$\tau_\nu\big(g^{-1}fg\big) = \nu\big([g^{-1}(x), g^{-1}f(x))\big) = g_*\nu([x, f(x))) = \tau_{g_*\nu}(f) = \lambda_g \tau_\nu(f). \quad (3.6)$$

Thus we assume that $\tau_\nu(\widetilde{\Gamma}^{i-1}) \simeq \mathbb{Z}$ is not dense. In particular, we assume that λ is trivial and that $\tau_\nu(\widetilde{\Gamma}^{i-1}) \simeq \mathbb{Z}$.

We let $H := ker(\tau_\nu) = \{g \in \widetilde{\Gamma}^{i-1} \mid \tau_\nu(g) = 0\}$. We have seen that H is made of by the elements in $\widetilde{\Gamma}^{i-1}$ having fixed points (*c.f.* Exercise 3.3.9). Therefore, H is normal not only in $\widetilde{\Gamma}^{i-1}$ but also in Γ. Moreover, the condition $\tau_\nu(\widetilde{\Gamma}^{i-1}) \simeq \mathbb{Z}$ translates into that $\widetilde{\Gamma}^{i-1}/H \simeq \mathbb{Z}$. We claim that $\widetilde{\Gamma}^{i-1}/H$ is in the center of Γ/H.

Indeed, letting $f \in \widetilde{\Gamma}^{i-1}$ be a generator of $\widetilde{\Gamma}^{i-1}/H$, for each $g \in \Gamma$ we have that $g^{-1} f g = f^n h$ holds for certain $h \in H$ and $n \in \mathbb{Z}$. We need to show that $n = 1$. Now, by (3.6), we have

$$n\,\tau_\nu(f) = \tau_\nu(g^{-1} f g) = \lambda_g \tau_\nu(f) = \tau_\nu(f) \neq 0,$$

which implies $n = 1$, as desired.

Finally, the quotient group $\Gamma/\widetilde{\Gamma}^{i-1} \simeq (\Gamma/H)/(\widetilde{\Gamma}^{i-1}/H)$ acts on the compact quotient $Fix(H)/\sim$, where $x \sim f(x)$ for each $f \in \widetilde{\Gamma}^{i-1}$ and all $x \in Fix(H)$. This space is easily seen to be homeomorphic to the circle. Therefore, as Γ is amenable, it preserves a probability measure on it (see [176]). Pulling back this measure to the real line, we obtain a Γ-invariant Radon measure on $\mathbb{R}$. $\square$

We now claim that the dynamical realization of $\preceq$ is semiconjugate to a non-Abelian affine action. As in the preceding section, this will follow once we show that the homomorphism λ is nontrivial. Assume hence for a contradiction that λ is trivial. Then by the preceding Lemma, there is a Γ-invariant Radon measure ν. Moreover, as the origin is moved by every nontrivial element and Γ contains elements having fixed points (for instance, those of G), the origin does not belong to the support of ν. Let I_ν be the connected component of the complement of the support of ν containing the origin. The interval I_ν is either fixed or moved disjointly by each element of Γ, hence its stabilizer $Stab_\Gamma(I_\nu)$ is a convex subgroup of Γ. Since this subgroup contains G and since G is the maximal proper convex subgroup of Γ, we must have $Stab_\Gamma(I_\nu) = G$. Further, $Stab_\Gamma(I_\nu)$ coincides with the kernel of the translation-number homomorphism $\tau_\nu : \Gamma \to (\mathbb{R}, +)$, thus it is normal in Γ. We thus conclude that G is normal and co-Abelian in Γ. Therefore, $\preceq$ is a convex extension of a Conradian order by a Conradian one, hence it is Conradian (*c.f.* Exercise 3.2.5). However, this contradicts the fact that G is the Conradian soul of Γ.

We can finally show that $\preceq$ is non-isolated by invoking Corollary 3.3.4. Indeed, it easily follows from the construction of the dynamical realization that the kernel of the induced homomorphism from Γ into $\mathrm{Aff}_+(\mathbb{R})$ is a $\preceq$-convex subgroup, hence the hypothesis of the corollary are fulfilled.

Case II. Every element of $\widetilde{\Gamma}^{i-1}$ admits fixed points on both sides of the origin.

In this case, we will prove that the approximation scheme by conjugates developed in §3.2.5 applies. More precisely, starting from the dynamical realization of $\preceq$, we will induce a new left-order using a comparison point that is outside but very close to I_G. The main issue here is to ensure that this procedure can be made in such a way that the order restricted to G remains untouched (compare Theorem 3.2.46). Along the proof, it will actually arise that the action is somewhat similar to the one described in Example 3.3.13.

For each nontrivial element $g \in \widetilde{\Gamma}^{i-1}$, let us denote by I_g the connected component of the complement of its set of fixed points that contains the origin. It follows from Lemma 3.3.18 that the union of all the I_g's is the whole real line.

Lemma 3.3.20. *For each $f \in \Gamma$ and $g \in \widetilde{\Gamma}^{i-1}$, one of the following possibilities occurs:*

– *$f(I_g) = I_g$;*

– *$f(I_g)$ is disjoint from I_g;*

– *Up to changing f by its inverse if necessary, we have $\overline{I_g} \subset f(I_g)$.*

Proof. By Lemma 3.3.17, the order $\preceq$ restricted to $\widetilde{\Gamma}^{i-1}$ is Conradian, hence $\widetilde{\Gamma}^{i-1}$ acts without crossings (*c.f.* Exercise 3.2.33). As $\widetilde{\Gamma}^{i-1}$ is normal in Γ, the lemma easily follows. □

The next two lemmas are similar to the preceding one. The first follows from the convexity of G, and the second from the nonexistence of crossings for the action of $\widetilde{\Gamma}^{i-1}$ and the fact that $\widetilde{\Gamma}^{i-1}$ is normal in Γ. Details are left to the reader.

Lemma 3.3.21. *In the preceding lemma, if g does not belong to G, then the second possibility cannot occur for $f \in G$. In other words, for all $g \in \widetilde{\Gamma}^{i-1} \setminus G$ and each $f \in G$, either f fixes I_g or (up to changing f by f^{-1} if necessary), we have $\overline{I_g} \subset f(I_g)$.*

Lemma 3.3.22. *Let I be the intersection of all the intervals I_g, with $g \in \widetilde{\Gamma}^{i-1} \setminus G$. Then each element $f \in \Gamma$ either moves I into a disjoint interval, or up to replacing it by its inverse, we have $I \subset f(I)$.*

The next lemma is a kind of refined version of Theorem 3.2.38 knowing that G has finite rank and/or admits only finitely many left-orders, and that $\widetilde{\Gamma}^{i-1}$ is normal and $\preceq$-Conradian.

Lemma 3.3.23. *The intersection of all the intervals I_g for $g \in \widetilde{\Gamma}^{i-1} \setminus G$ coincides with I_G.*

Proof. Since $\widetilde{\Gamma}^{i-1}$ is a $\preceq$-Conradian subgroup (*c.f.* Lemma 3.3.17), its action has no crossings, which implies that the family of intervals I_g, with $g \in \widetilde{\Gamma}^{i-1} \setminus \{id\}$, is totally ordered by inclusion (*c.f.* Exercise 3.2.25). Moreover, as G is convex, for each $g \in \widetilde{\Gamma}^{i-1} \setminus G$ we have that I_g strictly contains I_G. Therefore, letting I be the intersection of all the I_g's for $g \in \widetilde{\Gamma}^{i-1} \setminus G$, we have that I is a bounded interval containing I_G.

Assume that no element $f \in \Gamma$ is such that I strictly contains $f(I)$. Then, by Corollary 3.3.22, the stabilizer of I in Γ is a proper convex subgroup of Γ; moreover, this subgroup contains G. As G is the maximal proper convex subgroup, this necessarily implies that I_G equals I.

Therefore, we are left to prove that no $f \in \Gamma$ satisfies $f(I) \subsetneq I$. Assume otherwise for a certain f. As $I = \bigcap_{g \in \widetilde{\Gamma}^{i-1} \setminus G} I_g$, there must exist $g \in \widetilde{\Gamma}^{i-1} \setminus G$ such that $I \cap f(I_g)$ is strictly contained in I. Since fgf^{-1} belongs to $\widetilde{\Gamma}^{i-1}$, by the definition of I we must have that fgf^{-1} belongs to G. Actually, the same holds for $f^n g f^{-n}$, for all $n \geq 1$.

Notice that $f^n(I_g)$ is an open interval (not necessarily containing the origin) that is fixed by $f^n g f^{-n}$ and has no fixed point inside. Moreover, as $f(I_g) \subsetneq I_g$, we have $f^{n+1}(I_g) \subsetneq f^n(I_g)$, for all $n \geq 1$. Together with the fact that the action of G has no crossings, this easily implies that $f^k g f^{-k}$ belongs to $Stab_G(f^n(I_g))$, for all $k \geq n$. As a consequence, $Stab_G(f^n(I_g))$ is an strictly decreasing sequence of convex subgroups of G for any left-order induced from a sequence starting with a point in the (nonempty) intersection of the compact intervals $\overline{f^n(I_g)}$. However, this contradicts the fact that G is a Tararin group (as well as that G has finite rank). $\square$

The next lemma follows almost directly from Proposition 3.2.48 and its proof. Actually, it is a kind of restatement of it for dynamical realizations. We leave the details to the reader.

Lemma 3.3.24. *For each x not belonging to $\overline{I_G}$, every left-order $\preceq'$ induced from the dynamical realization of $\preceq$ induced from a sequence starting with x is different from $\preceq$. Moreover, as $x \notin \overline{I_G}$ converges to any of the endpoints of I_G, we have that $\preceq'$ converges to $\preceq$ outside G.*

For the sake of concreteness, the order $\preceq'$ above may –and will– be taken as the one for which the second comparison point is the origin (so that no other comparison point is necessary). The convergence in the statement means that for any sequence x_n converging to an endpoint of I_G from outside, given $g \in \Gamma \setminus G$ we have that $g \succ id$ holds if and only if $g \succ'_n id$ holds for all large-enough n (where $\preceq'_n$ denotes the order associated to the point x_n and the origin, as before).

Therefore, to prove that $\preceq$ is non-isolated, we are left to showing that for a well-chosen sequence (x_n) as above, the left-orders $\preceq_n'$ coincide with $\preceq$ when restricted to G for large-enough n. This is achieved by the next lemma, which closes the proof of Theorem 3.3.12.

Lemma 3.3.25. *There exists a sequence of points (x_n) converging to an endpoint of I_G from outside such that the induced left-orders $\preceq_n'$ coincide with $\preceq$ on G for all n.*

To show this, we need one more general lemma for Tararin groups.

Lemma 3.3.26. *Let T be a Tararin group, with chain of convex subgroups*

$$\{id\} = T^k \lhd T^{k-1} \lhd \ldots \lhd T^1 \lhd T^0 = T,$$

and let f_T be an element in $T \setminus T^1$ acting on T^1/T^2 as the multiplication by a negative (rational) number. Suppose T is acting by orientation-preserving homeomorphisms of the line in such a way that the sets of fixed points of nontrivial elements have empty interior (as it is the case for dynamical realizations). Then, for every left-order $\preceq$ on T and each $y \in \mathbb{R}$ not fixed by f_T, there exists a point x between $f_T^{-2}(y)$ and $f_T^2(y)$ such that $\preceq$ coincides on T^1 with any left-order $\preceq'$ induced from a sequence starting with x.

Proof. As the sets of fixed points of nontrivial elements have empty interior, and since T is countable, there is a point z between $f_T^{-1}(y)$ and y whose orbit under T is free. Any such point induces –in a dynamically-lexicographic way– a left-order $\preceq^*$ on T. Since T^1 is necessarily convex for this order, there is an open interval I containing z that is fixed by T^1 and contains no other fixed point of T^1 inside. Moreover, I is moved into a disjoint interval by any nontrivial power of f_T; in particular, it contains at most one point of the orbit of y under $\langle f_T \rangle$. As a consequence, I strictly lies between $f_T^{-2}(y)$ and $f_T(y)$. Now, by Proposition 2.2.22, there exists an element g in either T^1 or $f_T T^1$ such that $\preceq$ and $\preceq_g^*$ coincide on T^1. As $\preceq_g^*$ is the dynamically-lexicographic order with comparison point $x := g(z)$, this point satisfies the conclusion of the lemma. $\square$

Proof of Lemma 3.3.25. Due to Lemma 3.3.23 (and since $\widetilde{\Gamma}^{i-1}$ acts without crossings), there exists a sequence of elements $g_n \in \widetilde{\Gamma}^{i-1} \backslash G$ such that I_{g_n} converges to I_G. As in the preceding lemma, let f_G be an element in $G \backslash G^1$ acting on G^1/G^2 as the multiplication by a negative number. According to Lemma 3.3.21, we may pass to a subsequence for which one of the two possibilities below occur.

Subcase (i). Each interval I_{g_n} is fixed by f_G.

We first claim that G must fix each interval I_{g_n}. Indeed, letting y be an endpoint of any of the I_{g_n}'s, we may induce a left-order on G from a sequence having y as its initial point. For such an order, the stabilizer of y in G is a convex subgroup of G containing f_G. It must hence coincide with G, and therefore G fixes y, as desired.

We may now let x_n be any of the endpoints of I_{g_n}, say the right one. As G fixes x_n and the second comparison point of $\preceq_n'$ is the origin, the restriction of $\preceq_n'$ to G coincides with that of $\preceq$. Moreover, since I_{g_n} converges to I_G, the points x_n converge (from outside) to the right endpoint of I_G.

Subcase (ii). For each n, we have either $\overline{I_{g_n}} \subset f_G(I_{g_n})$ or $\overline{I_{g_n}} \subset f_G^{-1}(I_{g_n})$.

Up to taking a subsequence, we may assume that for all n, we have $I_{g_n} \subsetneq I_{g_{n-1}}$ and, changing f_G by its inverse if necessary, $\overline{I_{g_n}} \subset f_G(I_{g_n})$. In case where $f_G \prec id$ (resp. $f_G \succ id$), let y_n be the left (resp. right) endpoint of I_{g_n}. Notice that y_n converges to a fixed point of f_G (hence of G). Moreover, for all $n \geq 1$, if $f_G \prec id$ (resp. $f_G \succ id$), then

$$f_G(y_n) < y_n \ \ \text{(resp. } f_G(y_n) > y_n). \tag{3.7}$$

Now, for each y_n, let us consider the point x_n provided by Lemma 3.3.26. Then $\preceq_n'$ coincides with $\preceq$ in restriction to the maximal proper convex subgroup G^1 of G. Moreover, as x_n lies between $f_G^{-2}(y_n)$ and $f_G^2(y_n)$, it has the same limit point as y_n, hence it converges to an endpoint of I_G from outside. Furthermore, (3.7) obviously holds for x_n instead of y_n. This implies that $\preceq_n'$ coincides with $\preceq$ over the whole of G, as desired. $\square$

To close this section, let us mention that it is unclear what is the most general framework in which Theorem 3.3.12 still holds. This naturally yields to the next

Question 3.3.27. Is the space of left-orders of a countable amenable group either finite or a Cantor set ? What about groups without free subgroups in two generators ?

It is very likely that the previous methods can be extended to a wide family of amenable groups, namely that of ***elementary amenable*** ones. Roughly, this is the smallest family of groups that contains all Abelian groups and that is stable under taking extensions, direct limits, quotients and subgroups (see [36, 140] for more on this). A relevant example is considered in the next

Example 3.3.28. The group $\Gamma := \mathbb{Z} \ltimes (...\mathbb{Z} \wr (\mathbb{Z} \wr ...)...)))$ in which the conjugacy action of the left factor consists in shifting (the level of) the factors in the right wreath product is obviously elementary amenable. It is somewhat a variation of Plante's example (yet it is not solvable), and was simultaneously introduced in [20] and [140]. A natural (faithful) action of this group on the line comes from identifying the generator of the left factor with te map $x \mapsto 2x$ and a generator of the 0^{th}-factor of the wreath product with a homeomorphism with support contained in $[-2, 2]$ and sending -1 to 1. It is very likely that its space of left-orders is a Cantor set, and it actually seems reasonable to trying to describe all of its actions on the line.

3.4 Verbal Properties of Left-Orders

Let $\mathcal{W}$ be the set of reduced words in two letters a, b. (This naturally identifies to the free group in two generators.) We distinguish three subsets of $\mathcal{W}$, namely $\mathcal{W}^+$, $\mathcal{W}^-$, and $\mathcal{W}^{\pm}$, the set of words involving only positive, negative, or mixed exponents in a and b, respectively. Given elements f, g in a group Γ and $W \in \mathcal{W}$, we let $W(f, g)$ be the element in Γ obtained from the expression of W by replacing a by f and b by g, respectively.

Definition 3.4.1. A left-order $\preceq$ on a group Γ satisfies the ***verbal property*** W, or it is a W***-order***, if whenever f and g are $\preceq$-positive, the element $W(f, g)$ is also $\preceq$-positive.

Notice that this defines a nontrivial property only in the case where $W \in \mathcal{W}^{\pm}$, hence in the sequel we will only consider these words.

Example 3.4.2. For $W(a, b) := b^{-1}ab$, one easily checks that the set of W-left-orders coincides with that of bi-orders.

Example 3.4.3. For $W(a, b) := b^{-1}ab^2$, Proposition 3.2.1 tells us that the set of W-orders corresponds to that of Conradian ones.

The next two questions become natural in this context.

Question 3.4.4. Does there exist a word W such that the W-orders are those that satisfy an specific and relevant algebraic property different from bi-orderability or the Conradian one ?

Question 3.4.5. Is the property of not having a double crossing (*c.f.* Example 3.2.18) for a left-order equivalent to a verbal property (or to an interesection of finitely many ones) ?

As it is easy to check, the subset of W-orders is closed inside $\mathcal{LO}(\Gamma)$, and the conjugacy action preserves this subset. The next result on free groups is only stated for two generators, though it can be easily extended to more generators.

Theorem 3.4.6. *The free group on two generators admits left-orders satisfying no verbal property $W \in \mathcal{W}^{\pm}$. Actually, this is the case of a G_δ-dense subset of $\mathcal{LO}(\mathbb{F}_2)$.*

Let us first show that the existence of a single left-order satisfying no verbal property implies that this is the case for most left-orders. Thies relies on Lemma 2.2.25, as shown by the next

Lemma 3.4.7. *Every left-order on $\mathbb{F}_2$ having a dense orbit under the conjugacy action satisfies no verbal property $W \in \mathcal{W}^{\pm}$.*

Proof. Otherwise, as the closure of such an orbit only contains W-orders, we would be in contradiction with Theorem 3.4.6. $\square$

Question 3.4.8. It is a nontrivial fact that the real-analytic homeomorphisms of the line given by $x \mapsto x+1$ and $x \mapsto x^3$ generate a free group [43]. By analyticity, a G_δ-dense subset S of points in the line have a free orbit under this action. Given a point $x \in S$, we may associate to it the left-order on F_2 defined by $f \succ g$ whenever $f(x) > g(x)$. Is the set of $x \in S$ for which the associate order satisfies no verbal property still a G_δ-dense subset of $\mathbb{R}$?

We next proceed to the proof of the first claim of Theorem 3.4.6, which is done via a very simple dynamical argument. Namely, given $W \in \mathcal{W}^{\pm}$, we will construct two increasing homeomorphisms of the real line f, g, both moving the origin to the right, such that in the action of $\mathbb{F}_2$ given by $a \to f$, $b \to g$, the homeomorphism $W(f,g)$ moves the origin to the left. Then, any dynamically-lexicographic left-order $\preceq$ associated to a sequence starting at the origin will be such that $f \succ id$, $g \succ id$, and $W(f,g) \prec id$. This is enough for our purposes except for that the action we will produce will be not necessary faithful. However, this is just a minor detail that may be solved in many ways. For instance, one can make the action faithful by perturbing it close to infinity, as in §2.2.2; alternatively, one may consider a convex-like extension of the order $\preceq$, as in §2.1.1.

Finally, the construction of the desired action is done as follows. By interchanging a and b if necessary, we may assume that the word $W = W(a,b)$ writes in the form $W = W_1 a^{-n} W_2$, where W_2 is either empty or a product of positive powers of a and b, the integer n is positive, and W_1 is arbitrary. Let us consider two local homeomorphisms defined on a right neighborhood of the real line such that $f(0) > 0$, $g(0) > 0$ and $W_2(f,g)(0) < f^n(0)$. This can be easily done by taking $f(0) \gg g(0)$ and letting g be almost flat on a very large right-neighborhood of the origin. If W_1 is empty, just extend f and g into homeomorphisms of the real line. Otherwise, write $W_1 = a^{n_k} b^{m_k} \ldots a^{n_2} b^{m_2} a^{n_1} b^{m_1}$, where all m_i, n_i are nonzero excepting perhaps n_k. The extension of f and g to a left-neighborhood of the origin depends on the signs of the exponents m_i, n_i, and is done in a constructive manner. Namely, first extend f slightly so that $f^{-n} W_2(f,g)(0)$ is defined and f has a fixed point x_1 to the left of the origin. Then extend g to a left-neighborhood of the origin so that $g^{m_1} f^{-n} W_2(f,g)(0) < x_1$ and g has a fixed point y_1 to the left of x_1. Notice that $m_1 > 0$ forces g to be right-topologically-attracting towards y_1 on an interval containg $f^{-n} W_2(f,g)(0)$, whereas $m_1 < 0$ forces right topological repulsion. Next, extend f to a left neighborhood of x_1 so that $f^{n_1} g^{m_1} f^{-n} W_2(f,g)(0) < y_1$ and f has a fixed point x_2 to the left of y_1. Again, if $n_1 > 0$, this forces right-topological-attraction towards x_2, whereas $n_1 < 0$ implies right-topological-repulsion.

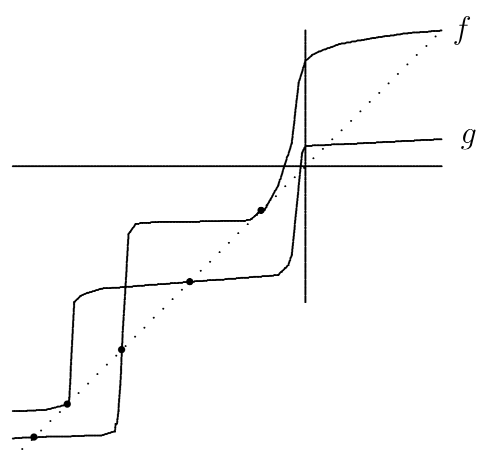

Figure 14: The case $W_1 = a^{n_2} b^{m_2} a^{n_1} b^{m_1}$, where $m_1 > 0, n_1 < 0, m_2 < 0, n_2 > 0$.

Continuing the procedure in this manner (see Figure 14 for an illustration), we get partially-defined homeomorphisms f, g for which

$$0 > f^{n_k} g^{m_k} \ldots f^{n_2} g^{m_2} f^{n_1} g^{m_1} f^{-n} W_2(f,g)(0) = W(f,g)(0).$$

Extending f, g arbitrarily into homeomorphisms of the real line, we finally obtain the desired action.

3.5 A Non Left-Orderable Group, and More

3.5.1 No left-order on finite-index subgroups of $\mathrm{SL}(n,\mathbb{Z})$

Proposition 3.2.9 gave us a simple criterium for non left-orderability of certain groups. In the same spirit, an important result due to Morris-Witte [173] establishes that finite-index subgroups of $\mathrm{SL}(n,\mathbb{Z})$ are non left-orderable for $n \geq 3$. (Notice that most of these groups are torsion-free, bacause of the classical Selberg lemma [161].)

Theorem 3.5.1. *If Γ is a finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$, with $n \geq 3$, then Γ is non left-orderable.*

Proof. Since $\mathrm{SL}(3,\mathbb{Z})$ injects into $\mathrm{SL}(n,\mathbb{Z})$ for every $n \geq 3$, it suffices to consider the case $n = 3$. Assume for a contradiction that $\preceq$ is a left-order on a finite-index subgroup Γ of $\mathrm{SL}(n,\mathbb{Z})$. Notice that for large-enough $k \in \mathbb{N}$, the following elements must belong to Γ:

$$g_1 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix}.$$

It is easy to check that for each $i \in \mathbb{Z}/6\mathbb{Z}$, the following relations hold:

$$g_i g_{i+1} = g_{i+1} g_i, \qquad [g_{i-1}, g_{i+1}] = g_i^k.$$

In particular, the group generated by g_{i-1}, g_i and g_{i+1} is nilpotent.

For $g \in \Gamma$, we define $|g| := g$ if $g \succeq id$, and $|g| := g^{-1}$ in the other case. We also write $g \gg h$ if $g \succ h^n$ for every $n \geq 1$. We claim that either $|g_{i-1}| \gg |g_i|$ or $|g_{i+1}| \gg |g_i|$. Indeed, as $\preceq$ restricted to the subgroup $\langle g_{i-1}, g_i, g_{i+1} \rangle$ is Conradian (*c.f.* Proposition 3.2.20) and a power of g_i is a commutator, this follows from Remark 3.2.30.

Assume for instance that $|g_1| \ll |g_2|$, the case where $|g_2| \ll |g_1|$ being analogous. Then we obtain $|g_1| \ll |g_2| \ll |g_3| \ll |g_4| \ll |g_5| \ll |g_6| \ll |g_1|$, which is absurd. $\square$

It follows from an important theorem due to Margulis that for $n \geq 3$, every normal subgroup of a finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$ either is finite or has finite index (see [126]). As a corollary, we obtain the following strong version of Theorem 3.5.1.

Theorem 3.5.2. *For $n \geq 3$, no torsion-free, finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$ admits a total, left-invariant preorder.*

Proof. If Γ is such a group and admits a nontrivial, total preorder, then by Exercise 1.1.5, there is a nontrivial quotient Γ/H that is left-orderable. Since Γ is torsion-free, it has no nontrivial finite subgroup. Therefore, there are only two possible cases: either H is trivial, in which case we contradict Theorem 3.5.1, or Γ/H is finite and nontrivial, which is impossible as no nontrivial finite group admits a nontrivial, left-invariant preorder. (Indeed, if $f \succ id$ for such a preorder, then $f^n \succ id$ for all $n \in \mathbb{N}$.) $\square$

In terms of semigroups, this translates into the next

Corollary 3.5.3. *If $n \geq 3$ and Γ is a torsion-free, finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$, then there is only one subsemigroup P of Γ satisfying $P \cup P^{-1} = \Gamma$, namely $P = \Gamma$.*

The results above are conjecturally true for all lattices in simple Lie groups of rank≥ 2. Although this is still open in full generality, Morris-Witte's theorem extends to the case of higher $\mathbb{Q}$-rank, as well as other non-cocompact lattices [109]. However, no proof is available for cocompact lattices.

3.5.2 A canonical decomposition of the space of left-orders

Let $(\Gamma,\preceq)$ be a finitely-generated, left-ordered group, and let Γ_0 be its maximal $\preceq$-convex subgroup (*c.f.* Example 2.1.2). There are two possibilities, namely the action of Γ on Γ/Γ_0 is or is not Conradian. In the first case, we will say that $\preceq$ is of ***type I***. The next proposition generalizes Corollary 3.2.2.

Proposition 3.5.4. *The set of left-orders of type I is closed inside $\mathcal{LO}(\Gamma)$.*

Proof. Since Γ is finitely generated, Γ/Γ' may be written as $\mathbb{Z}^k \times G$, where $k \geq 1$ and G is a finite Abelian group. Let $\preceq_n$ be a sequence of type-I left-orders on Γ converging to a left-order $\preceq$. We must show that $\preceq$ is also of type I. To do this, notice that associated to each $\preceq_n$, there is a Conrad's homomorphism

τ_n, which may be though of as defined on $\mathbb{Z}^k$. This homomorphism may be chosen *normalized*; this means that, if we denote $a_{n,i} := \tau_n(g_i)$, then the vector $(a_{n,1}, \ldots, a_{n,k})$ belongs to the k-sphere S^k, for each n.

Claim (i). The points $(a_{n,1}, \ldots, a_{n,k})$ converge to some limit $(a_1, \ldots, a_k) \in \mathrm{S}^k$.

Otherwise, there are subsequences (τ_{n_i}) and (τ_{m_i}) so that the associated vectors converge to two different points of S^k with orthogonal hyperplanes $\mathbb{H}_1, \mathbb{H}_2$, respectively. These hyperplanes divide $\mathbb{R}^k$ into four regions. Let us pick a point of integer coordinates on each of these regions, and let g_1, g_2, g_3, g_4 be elements of Γ which project to these points by the quotient $\Gamma \to \mathbb{Z}^k \times G$. For i large-enough, the value of both $\tau_{n_i}(g_j)$ and $\tau_{m_i}(g_j)$ is nonzero for each j, but the signs of these numbers must be different for some j. As Conrad's homomorphisms are non-decreasing, after passing to a subsequence of $(\preceq_{n_i})$ and $(\preceq_{m_i})$ this implies that, for some j, the element g_j will have different sign for $\preceq_{n_i}$ and $\preceq_{m_i}$. However, this is in contradiction to the convergence of $\preceq_n$.

Now, notice that the vector $(a_1, \ldots, a_k)$ gives raise to a group homomorphism $\tau \colon \mathbb{Z}^k \to \mathbb{R}$ (which may be though of as defined on Γ), namely

$$\tau(g) := \sum_{i=1}^{k} a_i n_i.$$

Claim (ii). The kernel of τ is a $\preceq$-convex subgroup of Γ.

Indeed, let $g \in \Gamma$ and $f \in \ker(\tau)$ be such that $id \preceq g \preceq f$. For each n, we have

$$0 = \tau_n(id) \leq \tau_n(g) \leq \tau_n(f).$$

As τ_n pointwise converges to τ and $\tau(f) = 0$, the inequalities above yield, after passing to the limit, $\tau(g) = 0$. Thus, g belongs to $\ker(\tau)$.

As a consequence of Claim (ii), the maximal $\preceq$-convex subgroup Γ_0 contains $\ker(\tau)$. Also, the action of Γ on Γ/Γ_0 is order-isomorphic to that on $\Gamma/\ker(\tau)/\Gamma_0/\ker(\tau)$. Since the latter is an action by translations, the former is, in particular, Conradian. Therefore, $\preceq$ is of type I. $\square$

The case where the action of Γ on Γ/Γ_0 is not Conradian is dynamically more interesting. We know by definition that there must exist a crossing for the action. The question is "how large" can be the "domain of crossing". To formalize this idea, for each $h \in \Gamma$, let us consider the "interval"

$$I(h) := \big\{\bar{h} \in \Gamma \colon \text{there exists a crossing } (f, g; u, w, v) \text{ such that } fv \prec h \prec \bar{h} \prec gu\big\}.$$

By definition, $I(h)$ is a convex subset of Γ.

Lemma 3.5.5. *If, for some $h \in \Gamma$, the set $I(h)$ is bounded from above, then it is bounded from above for all $h \in \Gamma$.*

Proof. As the notion of crossing is invariant under conjugacy, for all h_1, h_2 in Γ we have $h_1\big(I(h_2)\big) = I(h_1 h_2)$. The lemma easily follows from this. $\square$

If (the action of Γ on Γ/Γ_0 is not Conradian and) $I(h)$ is bounded from above for all $h \in \Gamma$, we will say that $\preceq$ is of ***type II***. If not, $\preceq$ will be said of ***type III***. We then have a canonical decomposition of the space of left-orders of Γ into three disjoint subsets (compare [75, Theorem 7.E]):

$$\mathcal{LO}(\Gamma) = \mathcal{LO}_I(\Gamma) \sqcup \mathcal{LO}_{II}(\Gamma) \sqcup \mathcal{LO}_{III}(\Gamma).$$

Example 3.5.6. Every Conradian order is of type I. Therefore, by Theorem 3.2.3, finitely-generated, locally-indicable groups admit left-orders of type I.

Example 3.5.7. Smirnov's left-orders $\preceq_\epsilon$ (with ε irrational; see §1.2.2) on subgroups of the affine group are prototypes of type III left-orders. However, these groups being bi-orderable, they also admit left-orders of type I, which actually arise as limits os Smirnov type orders. Moreover, the description given in 1.2.2 shows that these groups do not admit type II left-orders. As a consequence, $\mathcal{LO}_{III}(\Gamma)$ is not necessarily closed inside $\mathcal{LO}(\Gamma)$.

The last remark above may be straightened. Indeed, if we choose inside the Baumslag-Solitar group $BS(1,\ell)$ a sequence (g_n) so that $g_n^{-1}(\varepsilon)$ tends to $+\infty$, then all conjugate left-orders $(\preceq_\varepsilon)_{g_n}$ are of type III, but the limit left-order $\preceq_\infty$ is bi-invariant, hence of type I. Thus, a limit of type III left-orders in the same orbit of the conjugacy action may fail to be of type III.

Exercise 3.5.8. Let (x_i) and (y_n) be two sequences of points in $]0,1[$ so that x_i converges to the origin and $\{y_n\}$ is dense. For each $i \geq 1$, let $(z_{n,i})_n$ be the sequence having x_i as its first term and the y_n's as the next ones. Associated to this sequence there is a dynamically-lexicographical left-order $\preceq_i$ on Thompson's group F, namely, $f \succ_i id$ if and only if the smallest n for which $f(z_{n,i}) \neq z_{n,i}$ is such that $f(z_{n,i}) > z_{n,i}$ (*c.f.* §1.1.3). Show that $\preceq_i$ is of type III for all i, but any adherence point of the sequence $(\preceq_i)$ in $\mathcal{LO}(\mathrm{F})$ is of type I.

Remark 3.5.9. The subset of left-orders of type II on the free group F_2 is dense in $\mathcal{LO}(\mathbb{F}_2)$. Roughly, the proof proceeds as follows. (Compare §2.2.2.) Start with an arbitrary left-order $\preceq$ on $\mathbb{F}_2$ together with an integer $n \in \mathbb{N}$. Consider the dynamical realization of $\preceq$ as well as a very large compact subinterval I in the real line on which the dynamics captures all inequalities between elements in the ball $B_n(id)$ or radius n in $\mathbb{F}_2$. Then consider a new action of $\mathbb{F}_2$ which coincides with this dynamical realization on I and commutes with a translation of the line (of very large amplitude). This new action induces a (perhaps partial) left-order $\preceq_n$, which can be easily turned into a total one (by convex extension) which is not of type I (by adding crossings). Clearly, the left-orders $\preceq_n$ are all of type II and converge to $\preceq$.

Remark 3.5.10. A similar construction allows to produce a sequence of left-orders of type III on $\mathbb{F}_2$ that converges to an order of type II. Roughly, starting with a type II left-order $\preceq$, we consider its dynamical realization, we keep it untouched on a very large compact interval I, and outside I we perturb it by including infinitely many crossings for the generators along larger and larger domains. The new action will then induce a left-order on $\mathbb{F}_2$ which is of type III and very close to $\preceq$. We leave the details to the reader.

The main problem related to this discussion is

Question 3.5.11. Does there exist a finitely-generated, left-orderable group all of whose left-orders are of type III ?

We will come back to this question later.

Cofinal elements and the type of left-orders. Recall from §3.2.3 that an element f of a left-ordered group $(\Gamma, \preceq)$ is ***$\preceq$-cofinal*** if for any $g \in \Gamma$ there exist integers m, n such that $f^m \prec g \prec f^n$. In terms of dynamical realizations (*c.f.* §1.1.3), for countable groups, this corresponds to that f has no fixed point on the real line.

Following [40], we say that f is a ***cofinal element*** of Γ if it is $\preceq$-cofinal for every left-order $\preceq$ on Γ. The following should be clear from the discussion above.

Proposition 3.5.12. *If a finitely-generated, left-orderable group Γ has a cofinal, central element, then no left-order on Γ is of type III.*

Example 3.5.13. In §3.2, we introduced the group

$$\Gamma = \big\langle f, g, h \colon f^2 = g^3 = h^7 = fgh \big\rangle,$$

which is left-orderable but admits no nontrivial homomorphism into the reals (hence no left-order of type I). We claim that the central element $\Delta := fgh$ is cofinal. Indeed, if $\Delta = f^2 = g^3 = h^7$ has a fixed point for a dynamical realization, then this is fixed by f, g, h, hence by the whole group, which is impossible. As a consequence, every left-order on Γ is of type II.

Example 3.5.14. Recall that the center of the braid group $\mathbb{B}_n$ is generated by the square of the so-called Garside element Δ_n. Moreover, one has

$$\Delta_n^2 = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n = (\sigma_1^2\sigma_2\cdots\sigma_{n-1})^{n-1}.$$

We next reproduce the proof given in [40] of that Δ_n^2 is cofinal in $\mathbb{B}_n$.

Claim (i). If $\preceq$ is a left-order on $\mathbb{B}_n$ for which $\Delta_n \succ id$, then for any braid σ that is conjugate to either $\alpha_n := \sigma_1\sigma_2\cdots\sigma_n$ or $\beta_n := \sigma_1^2\sigma_2\cdots\sigma_n$ we have $id \prec \sigma \prec \Delta_n^2$.

Indeed, as $\sigma^k = \Delta^2$ for k equal to either n or $n-1$, we must have

$$id \prec \sigma \prec \sigma^2 \prec \sigma^3 \prec \ldots \prec \sigma^k = \Delta_n^2.$$

Claim (ii). If $\preceq$ is as above, then $\Delta_n^{-2} \prec \sigma_i \prec \Delta_n^2$, for all $i \in \{1,\ldots,n-1\}$.

Indeed, since Δ_n^2 is central, by Claim (i) we have, for all $\delta \in \mathbb{B}_n$,

$$\Delta_n^{-2} \prec \delta\alpha_n^{-1}\delta^{-1} \prec id, \qquad id \prec \delta\beta_n\delta^{-1} \prec \Delta_n^2.$$

Since $\beta_n\alpha_n^{-1} = \sigma_1$, this yields $\Delta_n^{-2} \prec \delta\sigma_1\delta^{-1} \prec \Delta_n^2$, and since all the σ_i's are conjugate between them, this shows the claim.

Claim (iii). The element Δ_n^2 is cofinal in $\mathbb{B}_n$.

Let $\preceq$ be a left-order on $\mathbb{B}_n$. Using again the fact that Δ_n^2 is central, the set $\{\sigma \in \mathbb{B}_n \colon \Delta_n^{2r} \prec \sigma \prec \Delta_n^{2s}$ for some r,s in $\mathbb{Z}\}$ is easily seen to be a subgroup of $\mathbb{B}_n$. By Claim (ii), it contains the σ_i's. Thus, it coincides with $\mathbb{B}_n$, which concludes the proof.

Remark 3.5.15. We do not know whether there exist type-III left-orders on the derived groups $\mathbb{B}_n'$ for $n \geq 5$. Notice that these groups do not admit left-orders of type I since they admit no nontrivial homomorphism into the reals (*c.f.* Example 3.2.8). By the preceding example, the restriction of any left-order on $\mathbb{B}_n$ to them is of type II.

A dynamical view. As we showed in §1.1.3 (and used many times), finitely-generated, left-ordered groups may be realized as groups of orientation-preserving homeomorphisms of the real line. In what follows, we use this to visualize the dynamical differences between orders of different type. For example, type-I left-orders are characterized as follows.

Proposition 3.5.16. *If $\preceq$ is a left-order of type I on a (finitely-generated) group Γ, then its dynamical realization preserves a Radon[8] measure on $\mathbb{R}$. Conversely, any left-order induced from a faithful action of Γ on the real line that preserves a (nontrivial) Radon measure is of type I.*

[8]Recall that a ***Radon measure*** is a measure giving finite mass to compact sets.

Proof. Let $\preceq$ be a left-order of type I on Γ, let Γ_0 be its maximal proper convex subgroup, and consider the dynamical realization of $\preceq$. By convexity, Γ_0 fixes the interval $[a,b]$, where a,b are, respectively, the infimum and the supremum of the orbit of the origin under Γ_0. Moreover, by Theorem 3.2.27 and Corollary 3.2.28, we have that Γ_0 is normal in Γ, that $\Gamma_0 = \ker(\tau_{\preceq})$, where $\tau_{\preceq} : \Gamma \to (\mathbb{R},+)$ is the Conrad homomorphism, and that the induced order on Γ/Γ_0 is Archimedean. In particular, the set $Fix(\Gamma_0)$ of global fixed points of Γ_0, is Γ-invariant (hence infinite), and the action of Γ/Γ_0 on $Fix(\Gamma_0)$ is free.

Now, if $Fix(\Gamma_0)$ is discrete (equivalently, if $\tau_{\preceq}(\Gamma) \sim \mathbb{Z}$), then for each $x \in Fix(\Gamma_0)$, the measure $\sum_{h \in \Gamma/\Gamma_0} \delta_{h(x)}$ is a Γ-invariant Radon measure. If $Fix(\Gamma_0)$ is non-discrete (equivalently, if $\tau_{\preceq}(\Gamma)$ is a dense subgroup of $\mathbb{R}$), we may proceed as in Example 3.1.6 to show that the action of Γ is semiconjugate to an action by translations that factors throughout Γ/Γ_0. Pulling back the Lebesgue measure by this semiconjugacy, we obtain a Γ-invariant Radon measure.

Conversely, assume that an action of Γ by orientation-preserving homeomorphism of the real line preserves a Radon measure ν. Then there is a translation-number homomorphism $\tau_\nu \colon \Gamma \to (\mathbb{R},+)$ defined by $\tau_\nu(g) := \nu([y, g(y)))$. (Recall that the value is independent of y, due to invariance.) We claim that $\ker(\tau_\nu)$ is a convex subgroup for any left-order induced from the action. Indeed, let $x \in \mathbb{R}$ be the first reference point for inducing such a left-order on Γ (*c.f.* §1.1.3). On the one hand, if x lies in the support of ν, then $\ker(\tau_\nu)$ coincides with the stabilizer of x, hence it is a convex subgroup. On the other hand, if x does not belong to the support of ν, let I be the connected component of the complement of the support of ν containing x. At least one endpoint of I is finite, which easily allows to show that for each $g \in \Gamma$, either $g(I) \cap I$ is empty or coincides with I. It follows from this that the stabilizer of I is a convex subgroup of Γ that coincides with $\ker(\tau_\nu)$.

Notice that for all g,h in Γ, the inequality $\tau_\nu(g) > \tau_\nu(h)$ implies $g(x) > h(x)$ for every $x \in \mathbb{R}$. It easily follows from this and the discussion above that $\ker(\tau_\nu)$ is the maximal convex subgroup. Finally, the action of Γ on $\Gamma/\ker(\tau_\nu)$ is Conradian, because it is order-isomorphic to an action by translations. $\square$

Remark 3.5.17. In the proof above, the finite-generation hypothesis was only used in the direct implication to ensure the existence of a maximal proper convex subgroup. Since this is known to exist in some other situations (see, for instance, Exercise 3.3.1), the proposition still holds in these cases.

To deal with type-II and type-III left-orders, we closely follow [55] (compare [123]). We say that the action of a subgroup Γ of $\mathrm{Homeo}_+(\mathbb{R})$ is ***locally contracting*** if for every $x \in \mathbb{R}$ there is $y > x$ such that the interval $[x,y]$ can be contracted into a point by a sequence of elements in Γ. We say that the action is ***globally contracting*** if such a sequence of contractions exists for any compact subinterval of $\mathbb{R}$. We denote by $\widetilde{\mathrm{Homeo}}_+(\mathrm{S}^1)$ the group of homeomorphisms of the line that are liftings of orientation-preserving circle homeomorphisms. The next lemma is to be compared with Example 2.1.2.

Lemma 3.5.18. *Every finitely-generated subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *preserves a nonempty minimal closed subset of the line. This set is unique in case no discrete orbit exists.*

Proof. Fix a point x_0 and a compact interval I containing x_0 as well as all its images under the (finitely many) generators of the group Γ. By obvious reasons, every closed set of the line that is invariant under the action must intersect I. Therefore, the standard argument using Zorn's lemma to detect (nonempty) minimal sets may be applied by looking at the (compact) "traces" in I of nonempty invariant closed subsets of the line. We leave the details to the reader (see [135, Proposition 2.1.12] in case of problems). To prove uniqueness, notice that for a closed invariant subset K, the set of acummulation points K' is also closed and invariant, hence $K' = K$ in case K is not a discrete orbit. Assume K is not the whole line (otherwise, the uniqueness is obvious). It is then easy to see that every connected component of the complement of K' has a sequence of images converging to any point in K'. In other words, every orbit acummulates at K, which obviously implies the uniqueness of the nonempty minimal invariant closed set. □

Theorem 3.5.19. Let Γ be a finitely-generated subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ whose action admits no global fixed point. Then one of the following mutually-exclusive possibilities occur:

(i) Γ has a discrete orbit or is semiconjugate to a minimal group of translations;

(ii) Γ is semiconjugate to a minimal, locally contracting subgroup of $\widetilde{\mathrm{Homeo}}_+(\mathrm{S}^1)$;

(iii) Γ is globally contracting.

Proof. Assume there is no discrete orbit for the action. By Lemma 3.5.18, there is a unique minimal nonempty closed Γ-invariant subset K. In case K is not the whole line, collapse each connected component of the complement of K to a point in order to semiconjugate Γ to a group $\overline{\Gamma}$ whose action is minimal. If Γ preserves a Radon measure, then after semiconjugacy, this measure becomes a $\overline{\Gamma}$-invariant Radon measure of total support and no atoms. Therefore, $\overline{\Gamma}$ (resp. Γ) is conjugate (resp. semiconjugate) to a group of translations.

Suppose next that Γ has no invariant Radon measure. Then the action of $\overline{\Gamma}$ cannot be free. Otherwise, $\overline{\Gamma}$ would be conjugate to a group of translations (*c.f.* Example 3.1.6), and the pull-back of the Lebesgue measure by the semiconjugacy would be a Γ-invariant Radon measure.

Let $\bar{g} \in \overline{\Gamma}$ be a nontrivial element having fixed points, and let $\bar{x}_0$ be a point in the boundary of $\mathrm{Fix}(\bar{g})$. Then there is a left or right neighborhood I of $\bar{x}_0$ that is contracted to $\bar{x}_0$ under iterates of either $\bar{g}$ or its inverse. By minimality, every $\bar{x}$ has a neighborhood that can be contracted to a point by elements in $\overline{\Gamma}$. Coming back to the original action, we conclude that every $x \in \mathbb{R}$ has a neighborhood that can be contracted to a point by elements in Γ. Notice that such a limit point can be chosen arbitrarily in K; in particular, it may be chosen to belong to a compact interval I that intersects every orbit (as in the proof of Lemma 3.5.18).

For each $x \in \mathbb{R}$, let $M(x) \in \mathbb{R} \cup \{+\infty\}$ be the supremum of the $y > x$ such that the interval (x, y) can be contracted to a point in I by elements of Γ. Then either $M \equiv +\infty$, in which case the group Γ is globally contracting, or $M(x)$ is finite for every $x \in \mathbb{R}$. In the last case, M induces a non-decreasing map $\overline{M} : \mathbb{R} \to \mathbb{R}$ commuting with all the elements in $\overline{\Gamma}$. Since the union of the intervals on which $\overline{M}$ is constant is invariant by $\overline{\Gamma}$, the minimality of the action implies that there is no such interval, that is, $\overline{M}$ is strictly increasing. Moreover, the interior of $\mathbb{R} \setminus \overline{M}(\mathbb{R})$ is also invariant, hence empty because the action is minimal. In other words, $\overline{M}$ is continuous. All of this shows that $\overline{M}$ induces a homeomorphism of $\mathbb{R}$ into its image. Since the image of $\overline{M}$ is $\overline{\Gamma}$-invariant, it must be the whole line. Therefore, $\overline{M}$ is a homeomorphism from the real line to itself. Observe that $\overline{M}(x) > x$ for any point x, which implies that $\overline{M}$ is conjugate to the translation $x \mapsto x + 1$. After this conjugacy, $\overline{\Gamma}$ becomes a subgroup of $\widetilde{\mathrm{Homeo}}^{+}(\mathrm{S}^1)$. $\square$

The next proposition should now be clear to the reader.

Proposition 3.5.20. *Let Γ be a finitely-generated left-orderable group, and let $\preceq$ be a left-order on it. Then $\preceq$ is of type I (resp. II, resp. III) if and only if its dynamical realization satisfies property (i) above (resp. (ii), resp. (iii)).*

Having this proposition at hand, we come back to Question 3.5.11. We claim that, in case of a negative answer, no lattice in a higher-rank simple Lie group can be left-orderable. Indeed, on the one hand, orders of type I lead to nontrivial homomorphisms into the reals in an obvious way. On the other hand, orders of type II yield (perhaps unfaithful) actions on the circle (viewed as the space of orbits of the freely-acting commuting homeomorphism) that have no invariant probability measure (otherwise, we fall into type I). Both cases are impossible for higher rank lattices: the former contradicts Kazhdan's property (T) (see [7]), and the latter contradicts a theorem of Ghys (see [72]).

Remark 3.5.21. As cited above, lattices in higher-rank simple Lie groups have ***Kazhdan's property (T)***, meaning that every affine isometric action on a Hilbert space has a fixed point (see [7]). It is unknown whether an infinite Kazhdan group may be left-orderable. A challenging case is that of lattices in $\mathrm{Sp}(n, 1)$. Indeed, orderability of such a lattice Γ would provide an affirmative answer to this and to Question 3.5.11 (due to Proposition 3.5.20 above), and non-left-orderability would yield an affirmative answer to Question 1.4.8 provided the injectivity radius of $\mathrm{Sp}(n, 1)/\Gamma$ is large enough, in virtue of Example 1.3.7 (indeed, the associated symmetric space carries an invariant negatively-curved metric).

Chapter 4

PROBABILITY AND LEFT-ORDERABLE GROUPS

4.1 Amenable Left-Orderable Groups

In this section, we discuss another nice result due to Morris-Witte [130]. The theorem below was conjectured by Linnell in [112], but it was already suggested by Thurston (see [170, page 348]).

Theorem 4.1.1. *Every amenable, left-orderable group is locally indicable.*

For the proof, we will say that a left-order $\preceq$ is ***right-recurrent*** if for every pair of elements f, h in Γ such that $f \succ id$, there exists $n \in \mathbb{N}$ satisfying $fh^n \succ h^n$. Notice that every right-recurrent order is Conradian. (The converse does not hold; see Example 4.1.4.) As subgroups of amenable groups are amenable [176], this implies that Theorem 4.1.1 follows from the next

Proposition 4.1.2. *If Γ is a finitely-generated, amenable, left-orderable group, then Γ admits a right-recurrent order.*

To prove this proposition, we will need the following weak form of the Poincaré Recurrence Theorem. We recall the proof for the reader's convenience.

Theorem 4.1.3. *If S is a measurable map that preserves a probability measure μ on a space M, then for every measurable subset A of M and μ-almost-every point $x \in A$, there exists $n \in \mathbb{N}$ such that $S^n(x)$ belongs to A.*

Proof. The set B of points in A that do not come back to A under iterates of S is $A \setminus \bigcup_{n\in\mathbb{N}} S^{-n}(A)$. One easily checks that the sets $S^{-i}(B)$, with $i \geq 1$, are two-by-two disjoint. Since S preserves μ, these sets have the same measure, and since the total mass of μ equals 1, the only possibility is that this measure equals zero. Therefore, $\mu(B) = 0$, that is, μ-almost-every point in A comes back to A under some iterate of S. $\square$

Proof of Proposition 4.1.2. Recall that one of the many characterizations of group-amenability is the existence of an invariant probability measure for every action by homeomorphisms of a compact metric space (see [176]). Therefore, if a countable left-orderable group Γ is amenable, then its action on (the compact metric space) $\mathcal{LO}(\Gamma)$ preserves a probability measure μ. We claim that μ-almost-every point in $\mathcal{LO}(\Gamma)$ is right-recurrent. To show this, for each $g \in \Gamma$, let us consider the subset V_g of $\mathcal{LO}(\Gamma)$ formed by the left-orders $\preceq$ on Γ such that $g \succ id$. By the Poincaré Recurrence Theorem, for each $f \in \Gamma$, the set $B_g(f) := V_g \setminus \bigcup_{n\in\mathbb{N}} f^{-n}(V_g)$ has null μ-measure. Therefore, the measure of $B_g := \bigcup_{f\in\Gamma} B_g(f)$ is also zero, as well as the measure of $B := \bigcup_{g\in\Gamma} B_g$. Let us consider an arbitrary element $\preceq$ in the (μ-full measure) set $\mathcal{LO}(\Gamma) \setminus B$. Given $g \succ id$ and $f \in \Gamma$, from the inclusion $B_g(f) \subset B$ we deduce that $\preceq$ does not belong to $B_g(f)$. Thus, there exists $n \in \mathbb{N}$ such that $\preceq$ belongs to $f^{-n}(V_g)$, hence $\preceq_{f^n}$ is in V_g. In other words, one has $g \succ_{f^n} id$, that is, $gf^n \succ f^n$. Since $g \succ id$ and $f \in \Gamma$ were arbitrary, this shows the right-recurrence of $\preceq$.

Example 4.1.4. Following [130, Example 4.5], we next show that there exist C-orderable groups which do not admit right-recurrent orders. This is the case of the semidirect product $\Gamma = \mathbb{F}_2 \ltimes \mathbb{Z}^2$, where $\mathbb{F}_2$ is any free subgroup of $\mathrm{SL}(2,\mathbb{Z})$ acting linearly on $\mathbb{Z}^2$. (Such a subgroup may be taken of finite-index.) Indeed, that Γ is C-orderable follows from the local indicability of both $\mathbb{F}_2$ and $\mathbb{Z}^2$. Assume throughout that $\preceq$ is a right-recurrent left-order on Γ. For a matrix $f \in \mathbb{F}_2$ and a vector $v = (m,n) \in \mathbb{Z}^2$, let us denote by $\bar{f}$ and $\bar{v}$ the corresponding elements in Γ, so that $f(v) = \bar{f}\bar{v}\bar{f}^{-1}$. Let τ be the Conrad's homomorphism associated to the restriction of $\preceq$ to $\mathbb{Z}^2$, so that we have $v \succ id$ whenever $\tau(v) > 0$, and $\tau(v) \geq 0$ for all $v \succ id$ (*c.f.* Corollary 3.2.28). Let f be a hyperbolic matrix in $\mathbb{F}_2$, with positive eigenvalues α_1, α_2 and corresponding eigenvectors v_1, v_2 in $\mathbb{R}^2$. Since v_1 and v_2 are linearly independent, we may assume that $\tau(v_1) \neq 0$. Furthermore, we may assume $\tau(v_1) > 0$ and $\alpha_1 > 1$ after replacing v_1 with $-v_1$ and/or f with f^{-1}, if necessary. Let $L : \mathbb{R}^2 \to \mathbb{R}$ be the (unique) linear functional that satisfies $L(v_1) = 1$ and $L(v_2) = 0$. Given any $v \in \mathbb{Z}^2$ such that $\tau(v) > 0$, right-recurrence provides us with an increasing sequence (n_i) such that $\bar{v}\bar{f}^{-n_i} \succ \bar{f}^{-n_i}$

for every i. This implies that $\bar{f}^{n_i}\bar{v}\bar{f}^{-n_i} \succ id$, hence $\tau(f^{n_i}(v)) \geq 0$. Since

$$\lim_{i\to\infty} \frac{\tau(f^{n_i}(v))}{\alpha_1^{n_i}} = \tau\left(\lim_{i\to\infty} \frac{f^{n_i}(v)}{\alpha_1^{n_i}}\right) = \tau(L(v)v_1) = L(v)\tau(v_1),$$

we conclude that $L(v) \geq 0$. Since v is an arbitrary element of $\mathbb{Z}^2$ satisfying $\tau(v) > 0$, this necessarily implies that $\ker(\tau) = \ker(L)$ is an eigenspace of f. But f is an arbitrary hyperbolic matrix in $\mathbb{F}_2$, and it is easy to show that there are hyperbolic matrices in $\mathbb{F}_2$ with no common eigenspace. This is a contradiction.

Remark 4.1.5. The group $\Gamma = \mathbb{F}_2 \ltimes \mathbb{Z}^2$ is particularly interesting in our context by this and many other reasons; see [137] for a discussion of this.

An extension for left-orderable groups without free subgroups ? It is unknown whether Theorem 4.1.1 extends to groups without free subgroups, that is, whether left-orderable groups not containing $\mathbb{F}_2$ are locally indicable. (See [111] for an interesting result pointing in the affirmative direction.) We next show that this would be the case if the answer to Question 3.5.11 is negative. Indeed, as type-I orders yield homomorphisms into the reals, what we need to show is that type-II orders imply the existence of free subgroups. However, as we saw at the end of §3.5.2, type-II orders yield group actions on the circle without invariant probability measures. Now, a theorem of Margulis establishes that a group admitting such an action necessarily contains a copy of $\mathbb{F}_2$ (see [125] as well as [71, 135]).

A relevant class of groups that do not contain free subgroups consists of those satisfying a nontrivial ***law*** (or ***identity***). This is a reduced word $W = W(x_1, \ldots, x_k)$ in positive and negative powers such that $W(g_1, \ldots, g_n)$ is trivial for *every* $g_1, \ldots, g_n$ in the group. For instance, Abelian groups satisfy a law, namely $W_1(x_1, x_2) := x_1 x_2 x_1^{-1} x_2^{-1}$. Nilpotent and solvable groups also satisfy group laws. Another important but less understood family is the one given by groups satisfying an ***Engel condition*** W_k^E, where

$$W_k^E(x_1, x_2) := W_1^E\big(W_{k-1}^E(x_1, x_2), x_2\big), \qquad W_1^E(x_1, x_2) := x_1 x_2 x_1^{-1} x_2^{-1}.$$

It is an open question whether left-orderable groups satisfying an Engel condition must be nilpotent. This is known to be true if the group is Conrad-orderable (see [75, Theorem 6.G]). In other words, if Γ is an Engel group having a left-order without resilient pairs, then Γ is nilpotent. In this direction, the next proposition becomes interesting, and shows the pertinence of Question 3.2.19. The (easy) proof is left to the reader. (See [133] for more on this.)

Proposition 4.1.6. *If Γ is a left-orderable group satisfying a law, then there exists $n \in \mathbb{N}$ such that no left-order on Γ admits an n-resilient pair.*

Locally-invariant orders on amenable groups. The ideas involved in the proof of Theorem 4.1.1 yield interesting results for other kind of orders on amenable groups. The next result is due to Linnell and Morris-Witte [113].

Theorem 4.1.7. *Every amenable group admitting a locally-invariant order is left-orderable (hence locally indicable).*

Proof. As for Theorem 4.1.1, we may assume that Γ is finitely generated.

First, it is not hard to extend the claim of Exercise 1.3.8 to describe the restriction of a locally-invariant order to any left coset of a cyclic subgroup: For every $f \in \Gamma$ and $g \neq id$, either

$$fg^n \prec fg^{n+1} \quad \text{for all } n \in \mathbb{Z},$$

or

$$fg^n \prec fg^{n-1} \quad \text{for all } n \in \mathbb{Z},$$

or there exists $\ell \in \mathbb{Z}$ such that

$$fg^n \prec fg^{n+1} \quad \text{for all } n \geq \ell \quad \text{and} \quad fg^n \prec fg^{n-1} \quad \text{for all } n < \ell.$$

We next argue that for amenable groups, there is a locally-invariant order for which the third possibility never arises.

Recall from Exercise 2.2.4 that the space of locally-invariant orders is a compact topological space, which is metrizable whenever Γ is countable. The group Γ acts on $\mathcal{LIO}(\Gamma)$ by left and right translations. Since Γ is amenable, both actions preserve a probability measure on $\mathcal{LIO}(\Gamma)$; let μ be a probability measure that is invariant under the right action. We leave to the reader the task of showing that a generic locally-invariant order $\preceq$ is *right-recurrent*. More precisely, there is a subset A of full μ-measure such that for every $\preceq$ in A, the following happens: If $f \prec g$, then given $h \in \Gamma$, the set of integers n such that $fh^n \prec gh^n$ is unbounded in both directions. Since in the third case above this property fails, we conclude that a generic locally-invariant order is either the canonical one or its reverse whenever restricted to a left-coset of a cyclic subgroup.

We next show that every $\preceq$ in A is a left-order. Indeed, by the definition of locally-invariant order, for every $g \neq id$ we have either $g \succ id$ or $g^{-1} \succ id$. Both inequalities cannot hold simultaneously, otherwise the restriction of $\preceq$ to the cyclic subgroup $\langle g \rangle$ wouldn't be neither the canonical order nor its reverse. Therefore, the positive cone $P := \{g : g \succ id\}$ is disjoint from its inverse, and their union covers $\Gamma \setminus \{id\}$.

It remains to show that P is a semigroup. Assume for a contradiction that g, h in P are such that $gh \notin P$. Then $gh \prec id$. Thus $g \succ id \succ gh$, hence using the property of a locally-invariant order, one easily checks that, necessarily, $gh^2 \prec gh$. More generally,

$$g \succ gh \succ gh^2 \succ gh^3 \succ \ldots .$$

Therefore, $gh^n \prec id$, for all $n \geq 1$. However, due to the right-recurrence of $\preceq$, there is some $n \in \mathbb{N}$ such that $gh^n \succ h^n \succeq h \succ id$. This is a contradiction. $\square$

The theorem above makes natural the following

Question 4.1.8. Does there exist an amenable U.P.P. group that is not left-orderable ? (See §1.4.3.)

4.2 Almost-Periodicity

In this section, we develop the notion of almost-periodicity for group actions on the real line. A left-orderable group being given, the set of such actions equipped with the compact-open topology can be used as a substitute to the space of left-orders. As an example, we will show how this yields an alternative proof of Theorem 4.1.1 which does not rely on the theory of Conradian orders.

4.2.1 Almost-periodic representations

The group of orientation-preserving homeomorphisms of the real line is equipped with the compact-open topology, which turns it a topological group. A homeomorphism $h \in \mathrm{Homeo}_+(\mathbb{R})$ is said to be ***almost-periodic*** if the set

$$\{\tau_s^{-1} \circ h \circ \tau_s \mid s \in \mathbb{R}\}$$

is relatively compact in $\mathrm{Homeo}_+(\mathbb{R})$, where $\tau_s(t) := s + t$. The set of almost-periodic, orientation-preserving homeomorphisms of the line is denoted $APH_+(\mathbb{R})$.

Example 4.2.1. Certainly, every homeomorphism that is periodic (*i.e.* that commutes with a nontrivial translation) is almost-periodic. An example of an almost-periodic homeomorphism that is non-periodic is given by

$$\varphi(t) := t + \frac{1}{3}\big(\sin(t) + \sin(\sqrt{2}t)\big). \tag{4.1}$$

The fact that φ is almost-periodic can be checked directly, yet a more conceptual argument proceeds as follows: Consider the diffeomorphism of the 2-torus defined by

$$F(x,y) = \left(x + \frac{1}{3}\big(\sin(x) + \sin(y)\big), y + \frac{\sqrt{2}}{3}\big(\sin(x) + \sin(y)\big) \right).$$

This diffeomorphism preserves the orbits of the (irrational) flow $S_{\mathbb{R}} = \{S_s\}_{s \in \mathbb{R}}$ generated by the linear vector field $\big(\frac{\partial}{\partial x}, \sqrt{2}\frac{\partial}{\partial y}\big)$. Moreover, we have $F(S_t(x_0)) = S_{\varphi(t)}(x_0)$ for every $t \in \mathbb{R}$, where $x_0 := (0,0)$. From this and Proposition 4.2.3 below, it follows that φ is almost-periodic.

Lemma 4.2.2. *The subset $APH_+(\mathbb{R})$ is a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$.

Proof. This is a consequence of the continuity of the composition and inverse operations on $\mathrm{Homeo}_+(\mathbb{R})$ with respect to the compact-open topology. $\square$

A group action on the real line whose image is contained in $APH_+(\mathbb{R})$ will be said to be ***almost-periodic***. There are several ways to construct faithful almost-periodic actions of a given left-orderable, countable group Γ on the line. The simplest one consists in considering a faithful action on the interval and then to extend it to the whole line so that it commutes with the translation $t \mapsto t+1$. This somewhat trivial construction shows that $APH_+(\mathbb{R})$ contains a copy of every left-orderable, countable group. Nevertheless, at this point we should stress that, in order to carry out a study that also involves actions that appear as limits of conjugates of a given one, we are forced to consider actions that may be unfaithful. This is the reason why we use the notation $\Phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ in what follows.

Starting with an almost-periodic action of a group Γ on the real line, we next provide a compact one-dimensional foliated space together with a Γ-action on it that preserves the leaves. It is this construction which gives interest to considering almost-periodic actions.

Proposition 4.2.3. *Let Γ be a finitely-generated group and $\Phi_0\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ an action by orientation-preserving homeomorphisms of the line. Then Φ_0 is almost-periodic if and only if there exists a topological flow $S_{\mathbb{R}} = \{S_s\}_{s\in\mathbb{R}}$ acting freely on a compact space X, an action of Γ on X by homeomorphisms preserving every S-orbit together with its orientation, and a point $x_0 \in X$, such that for every $g \in \Gamma$ and every $t \in \mathbb{R}$,*

$$g\big(S_t(x_0)\big) = S_{\Phi_0(g)(t)}(x_0). \tag{4.2}$$

Moreover, the flow can be taken so that the S-orbit of x_0 is dense in X.

Proof. Let us first show that if there is a compact space X together with a flow $S_{\mathbb{R}}$ and a Γ-action verifying (4.2), then the action Φ_0 is almost-periodic. Indeed, for each $x \in X$, we can lift the Γ-action on X to an action $\Phi^x : \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ verifying

$$g\big(S_t(x)\big) = S_{\Phi^x(g)(t)}(x),$$

which is well-defined since the flow $S_{\mathbb{R}}$ is free. Moreover, as the Γ-action on X is by homeomorphisms, for every $g \in \Gamma$, the map $x \mapsto \Phi^x(g)$ from X into $\mathrm{Homeo}_+(\mathbb{R})$ is continuous. Hence, the set of elements $\Phi^x(g)$, where $x \in X$, is compact. Now, for every s, t in $\mathbb{R}$ and every $x \in X$, we have

$$g\big(S_t(S_s(x))\big) = g\big(S_{t+s}(x)\big) = S_{\Phi^x(g)(t+s)}(x) = S_{\Phi^x(g)(t+s)-s}(S_s(x)),$$

which yields

$$\Phi^{S_s(x)}(g) = \tau_{-s} \circ \Phi^x(g) \circ \tau_s.$$

Therefore, for every $g \in \Gamma$, the conjugates of $\Phi^x(g)$ by the translations τ_s stay in a compact set, which proves that Φ^x is almost-periodic for every $x \in X$. In particular, for $x = x_0$, we deduce that $\Phi_0 = \Phi^{x_0}$ is almost-periodic.

Conversely, let us start with an almost-periodic action Φ_0, and let us provide the compact space X together with the flow $S = S_{\mathbb{R}}$ and the Γ-action verifying (4.2). Denote by $APA_+(\Gamma)$ the set of almost-periodic actions of Γ on the real line. This can be seen as a closed subset of $APH_+(\mathbb{R})^{\mathcal{G}}$, where $\mathcal{G}$ is a finite generating set of Γ. Define the ***translation flow*** $S_{\mathbb{R}}$ of conjugacies by translations on $APA_+(\Gamma)$, namely

$$S_s(\Phi)(g) := \tau_{-s} \circ \Phi(g) \circ \tau_s, \tag{4.3}$$

where $\Phi \in APA_+(\Gamma)$ and $g \in \Gamma$. This is a topological flow acting on $APA_+(\Gamma)$. Denote by X the closure of the S-orbit of Φ_0. This is a compact S-invariant subset of $APA_+(\Gamma)$, since Φ_0 is almost-periodic.

We claim that the formula

$$g(\Phi) := \tau_{-\Phi(g)(0)} \circ \Phi \circ \tau_{\Phi(g)(0)} \tag{4.4}$$

defines an action of Γ on $APA_+(\Gamma)$. One can verify this by a tedious computation, but a more conceptual argument proceeds as follows. Consider the actions of $\mathbb{R}$ and Γ on the product space $APA_+(\Gamma) \times \mathbb{R}$ given by

$$s(\Phi, t) := \big(T_s(\Phi), t - s\big) \quad \text{and} \quad g(\Phi, t) = \big(\Phi, \Phi(g)(t)\big).$$

A point in $APA_+(\Gamma) \times \mathbb{R}$ can be thought of as an almost-periodic action of Γ together with a marker. The action of the reals on $APA_+(\Gamma) \times \mathbb{R}$ corresponds to translating the marker while conjugating the almost-periodic action by the same translation. The Γ-action on $APA_+(\Gamma) \times \mathbb{R}$ corresponds to acting on the marker using the action of the first coordinate while leaving the almost-periodic representation unchanged. An easy computation shows that these two actions commute. Hence, there is a natural action of Γ on the quotient of $APA_+(\Gamma) \times \mathbb{R}$ by $\mathbb{R}$, which naturally identifies with $APA_+(\Gamma)$ via the embedding

$$\Phi \in APA_+(\Gamma) \mapsto (\Phi, 0) \in APA_+(\Gamma) \times \mathbb{R}.$$

The action of Γ on $APA_+(\Gamma)$ induced by this idenfication is given by the formula (4.4).

A priori, there is no reason to expect for the flow $S_{\mathbb{R}}$ on X to be free. However, it is possible to change it into a free one by the following procedure: Let Y be any compact space endowed with a topological flow $S_{\mathbb{R}}$ acting freely. (For instance, the toral flow of Example 4.2.1; see also Exercise 4.2.4 below.) Consider the space $\tilde{X} := X \times Y$ together with the Γ-action on it defined as

$$g : (\Phi, y) \mapsto \big(g(\Phi), S_{\Phi(g)(0)}(y)\big)$$

and the (diagonal) flow

$$s : (\Phi, y) \mapsto \big(S_s(\Phi), S_s(y)\big).$$

Then all the properties above are still satisfied, and moreover the flow on $\tilde{X}$ is now free. Hence, we can (and we will) assume that the flow $S_{\mathbb{R}}$ on X is free.

Equation (4.2) is obvious from the construction, as well as the fact that X can be taken as being the closure of a single point. This closes the proof. $\square$

Exercise 4.2.4. A ***Delone set*** D in $\mathbb{R}$ is a subset that is discrete and almost dense in a uniform way. More concretely, there exist positive constants ε, δ such that $|x-y| \geq \varepsilon$ for all $x \neq y$ in D, and for all $z \in \mathbb{R}$ there is $x \in D$ such that $|x - y| \leq \delta$. Two Delone sets are *close* if they coincide over a very large interval centered at the origin (this induces a topology that is metrizable). A Delone set is repetitive if for all $r > 0$ there is $R = R(r) > 0$ such that for every pair of intervals I, J of length r, R, respectively, there is a translated copy of $D \cap I$ contained in $D \cap J$.

Assume that D_0 is a repetitive yet non-periodic Delone set in $\mathbb{R}$ (it is easy to build such sets). Show that the natural translation flow $S_t : D \mapsto D + t$, restricted to the closure of the orbit of D_0, is a minimal flow.

4.2.2 A bi-Lipschitz conjugacy theorem

We denote by $\mathrm{BiLip}_+(\mathbb{R})$ the group of orientation-preserving, bi-Lipschitz homeomorphisms of the real line. For every $h \in \mathrm{BiLip}_+(\mathbb{R})$, we let $L(h)$ be its ***bi-Lipschitz constant***, that is, the minimum of the numbers $L \geq 1$ such that

$$L^{-1}|y-x| \leq |h(y)-h(x)| \leq L|y-x| \quad \text{for all } x, y \text{ in } \mathbb{R}. \tag{4.5}$$

We equip $\mathrm{BiLip}_+(\mathbb{R})$ with the topology of uniform convergence on compact sets.

Theorem 4.2.5. *Every finitely-generated group of homeomorphisms of the real line is topologically conjugate to a group of bi-Lipschitz homeomorphisms.*

Proof. Let $\nu = \lambda(t)dt$ be a probability measure on $\mathbb{R}$ with a smooth, positive density λ such that for $|t|$ large enough, we have $\lambda(t) = 1/t^2$. The following observation will be central in what follows: If for some constant $L \geq 1$, a homeomorphism h of the real line satisfies

$$h_*(\nu) \leq L\nu \quad \text{and} \quad (h^{-1})_*(\nu) \leq L\nu, \tag{4.6}$$

then h is Lipschitz. To prove this fact, first notice that $\nu\big([t,+\infty)\big) = 1/t$ for all large-enough positive numbers t (and similarly, $\nu\big((-\infty,t]\big) = 1/|t|$ if $|t|$ is large-enough and t is negative). Thus, the left-side inequality in (4.6) shows that $|h(t)| \leq L|t|$ holds for $|t|$ large enough. The density of $(h^{-1})_*(\nu)$ being given by $Dh(t)\lambda(h(t))$, the right-side inequality in (4.6) yields $Dh(t) \leq L\lambda(t)\big/\lambda(h(t))$ for almost-every t. Thus, up to sets of zero Lebesgue measure, the derivative Dh is bounded on every compact interval, and for $|t|$ large enough, we have

$$Dh(t) \leq \frac{L\lambda(t)}{\lambda(h(t))} = \frac{L|h(t)|^2}{|t|^2} \leq L^3.$$

This proves that Dh is a.e. bounded, hence h is Lipschitz, with Lipschitz constant at most L^3.

Next, let Γ be a finitely-generated subgroup of $\mathrm{Homeo}_+(\mathbb{R})$, and let $\mathcal{G}$ be a finite, symmetric system of generators of Γ. Let $\phi \in \mathcal{L}^1(\Gamma)$ be a function taking positive values such that, for every $h \in \mathcal{G}$, there is a constant L_h satisfying $\phi(hg) \leq L_h \phi(g)$ for all $g \in \Gamma$. For instance, one can take $\phi(g) = \kappa^{\|g\|}$, where $\|g\|$ is the word-length of g with respect to $\mathcal{G}$ and κ a small-enough positive number. (For $\kappa < 1/|\mathcal{G}|$, one can ensure that ϕ belongs to $\mathcal{L}^1(\Gamma)$; see also Excercise 4.2.7

below.) Let us normalize the function ϕ so that $\sum_{g\in\Gamma}\phi(g)=1$, and let us define the probability measure ν_0 on $\mathbb{R}$ by letting

$$\nu_0 := \sum_{g\in\Gamma} \phi(g)\, g_*(\nu).$$

Notice that for each $h \in \Gamma$, we have

$$h_*(\nu_0) = \sum_{g\in\Gamma} \phi(g)\,(hg)_*(\nu) \leq L_h \nu_0.$$

The measure ν_0 has no atoms and is of full support (*i.e.* the ν_0-measure of every nonempty open set is positive). Thus, there exists a homeomorphism φ of the real line sending ν_0 into ν. For each $h \in \Gamma$, we have

$$(\varphi \circ h \circ \varphi^{-1})_*(\nu) = \varphi_* h_*(\nu_0) \leq L_h\, \varphi_*(\nu_0) = L_h\, \nu.$$

From the discussion at the beginning of the proof, we deduce that the conjugate of Γ by φ is contained in $\mathrm{BiLip}_+(\mathbb{R})$. $\square$

The proof above was taken from [53]. In §4.3, we will give a more conceptual (yet quite elaborate) proof based on probabilistic arguments. For analogous results for transverse pseudo-groups of codimension-one foliations or groups acting on the circle, see [52, Proposition 2.5] and [54, Théorème D]. We point out that a conjugacy into a group of C^1 diffeomorphisms is, in general, impossible; see [14, 137].

Exercise 4.2.6. Let Γ be a subgroup of $\mathrm{BiLip}_+([0,1])$, and let $\varphi\colon [0,1] \to \mathbb{R}$ be an orientation-preserving homeomorphism such that $\varphi(x) = -1/x$ for x close to zero, and $\varphi(x) = 1/(1-x)$ for x close to 1. Check that the conjugate of Γ by φ is a subgroup of $\mathrm{BiLip}_+(\mathbb{R})$.

Exercise 4.2.7. Assume that a subgroup Γ of $\mathrm{Homeo}_+(\mathbb{R})$ has ***subexponential growth***, that is, the number of elements in the ball of radius n with respect to a finite generating system growth subexponentially in n. (This condition does not depend on the generating set.) Show that, for every $L>1$, it is possible to simultaneously conjugate the generators of Γ into L-bi-Lipschitz homeomorphisms.
Hint. In the proof above, the (positive) function $\phi \in \mathcal{L}^1(\Gamma)$ can be taken so that $\phi(hg) \leq L^{1/3}\phi(g)$ for every $h \in \mathcal{G}$ and every $g \in \Gamma$. See [134] for more on this.

We should stress that there is no analogue of Theorem 4.2.5 in higher dimension, even for actions of (infinite) cyclic groups; see [85].

Exercise 4.2.8. Let D be a Delone subset of $\mathbb{R}$ (*c.f.* Exercise 4.2.4). Show that there exists a bi-Lipschitz homeomorphism of the real line that sends D onto $\mathbb{Z}$. (Again, there is no two-dimensional analogue of this fact; see [24, 120] as well as [45].)

4.2.3 Actions almost having fixed points

Let Γ be a finitely-generated group with finite generating set $\mathcal{G}$, and let Φ be an almost-periodic action of Γ on $\mathbb{R}$. (Recall that we do not assume Φ to be faithful.) We say that Φ ***almost has fixed points*** if

$$\inf_{t \in \mathbb{R}} \sup_{g \in \mathcal{G}} \big| \Phi(g)(t) - t \big| = 0.$$

An equivalent way to think about this property is that the action of Γ on the compact space constructed in Proposition 4.2.3 has a global fixed point in the closure of the S-orbit of Φ_0.

It is not obvious how to construct almost-periodic actions that do not almost have fixed points. (Consider, for instance, the case of affine groups.) This is the goal of the next

Theorem 4.2.9. *All actions of finitely-generated groups by orientation-preserving homeomorphisms of the real line are topologically conjugate to almost-periodic actions. Moreover, if the original action has no fixed point, then there is such a conjugate that does not almost have fixed points.*

To prove this result, let Γ be a finitely-generated group provided with a finite, symmetric system of generators $\mathcal{G}$. Given constants $L > 1$ and $D > D' > 0$, we denote $R = R(\Gamma, \mathcal{G}, L, D, D')$ the set of representations $\Phi\colon \Gamma \to \mathrm{BiLip}_+(\mathbb{R})$ such that every $g \in \mathcal{G}$ satisfies $L(\Phi(g)) \leq L$ and

$$t - D \leq \min_{g \in \mathcal{G}} \Phi(g)(t) \leq t - D' \leq t + D' \leq \max_{g \in \mathcal{G}} \Phi(g)(t) \leq t + D \qquad (4.7)$$

for all $t \in \mathbb{R}$. This set can be seen as a closed subset of $\mathrm{BiLip}_+(\mathbb{R})^{\mathcal{G}}$, and as such is equipped with the product topology. Relations (4.5) and (4.7) imply that R is compact, by Arzela-Ascoli's theorem. Moreover, the same relations show that the translation flow $S_{\mathbb{R}}$ defined by (4.3) preserves R. Hence, every element of R is an almost-periodic action of Γ, and (4.7) shows that, moreover, such an element does not almost have fixed points.

Lemma 4.2.10. *Let $\Phi_0 : \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ be a faithful action without global fixed points of a finitely-generated group Γ. Then there are constants $L > 1$ and $D > D' > 0$, as well as a finite, symmetric generating system of Γ, such that the corresponding set R contains a representation that is conjugate to Φ_0.*

Proof. By Theorem 4.2.5, it is enough to prove the statement in the case where Γ is a subgroup of $\mathrm{BiLip}_+(\mathbb{R})$. We see Γ as being contained in $\mathrm{Homeo}_+(\mathbb{R})$ via Φ_0. Let $\mathcal{G}$ be a finite, symmetric generating set of Γ, and let L be a constant such that every $g \in \mathcal{G}$ is L-bi-Lipschitz. Let $(t_n)_{n\in\mathbb{Z}}$ be the sequence of points in $\mathbb{R}$ defined by $t_0 := 0$ and $t_{n+1} := \max_{g\in\mathcal{G}} g(t_n)$. Equivalently, $t_{n-1} = \min_{g\in\mathcal{G}} g(t_n)$, as $\mathcal{G}$ is symmetric. Since Γ has no fixed point on the real line, we have

$$\lim_{n\to\pm\infty} t_n = \pm\infty.$$

Let φ be the homeomorphism of the real line that sends t_n to n and that is affine on each interval $[t_n, t_{n+1}]$. We claim that the action of Γ defined by $\Phi(g) := \varphi \circ g \circ \varphi^{-1}$ belongs to $R(\Gamma, \overline{\mathcal{G}}, L^6, 1, 4)$ for the generating set $\overline{\mathcal{G}} := \mathcal{G} \cup \mathcal{G}^2$.

To prove this, we first notice that the distortion of the sequence t_n is uniformly bounded. In concrete terms, if for each $n \in \mathbb{Z}$ we denote $\delta_n := t_{n+1} - t_n$, then

$$L^{-1}\delta_{n+1} \leq \delta_n \leq L\,\delta_{n+1}. \tag{4.8}$$

Indeed, let $g_n \in \mathcal{G}$ be such that $t_{n+1} = g_n(t_n)$. By definition, $g_n(t_{n+1}) \leq t_{n+2}$, and since g_n is an L-bi-Lipschitz map, we have

$$t_{n+2} - t_{n+1} \geq g_n(t_{n+1}) - g_n(t_n) \geq L^{-1}(t_{n+1} - t_n),$$

which yields the right-side inequality in (4.8). (The left-side inequality is obtained analogously.) Notice that, by construction, for every all w, z in $[t_n, t_{n+1}]$,

$$\frac{|z-w|}{\delta_n} \leq \big|\varphi(z) - \varphi(w)\big| \leq \frac{|z-w|}{\delta_n}. \tag{4.9}$$

We next claim that for every $g \in \mathcal{G}$, we have $L(\Phi(g)) \leq L^3$. To show this, it suffices to prove that each such $\Phi(g)$ is Lipschitz on every interval $[n, n+1]$, with Lipschitz constant smaller than or equal to L^3. To check this, consider two arbitrary points x, y in $[n, n+1]$, and define $w := \varphi^{-1}(x)$ and $z := \varphi^{-1}(y)$. Then w, z both belong to $[t_n, t_{n+1}]$, which in virtue of (4.8) and (4.9) yields

$$\big|\Phi(g)(y) - \Phi(g)(x)\big| = \big|\varphi(g(z)) - \varphi(g(w))\big| \leq \frac{L\,\big|g(z) - g(w)\big|}{\delta_n} \leq \frac{L^2\,\big|z-w\big|}{\delta_n} \leq L^3|y-x|,$$

as desired.

By construction, for every generator $g \in \mathcal{G}$ and all $x \in \mathbb{R}$,

$$x - 2 \leq \Phi(g)(x) - x \leq x + 2.$$

Indeed, the integer points just after and before x are moved a distance less than or equal to 1 by $\Phi(g)$. Moreover, as for every $n \in \mathbb{Z}$ we have $\Phi(g_{n+1}g_n)(n) = n+2$, taking n being the integer part of x, this yields $\Phi(g_{n+1}g_n)(x) \geq x + 1$. We have hence proved that Φ belongs to $R(\Gamma, \overline{\mathcal{G}}, L^6, 1, 4)$. $\square$

Theorem 4.2.9 immediately follows from the preceding lemma in the case where Γ has no global fixed point. In the case where such a point exists, we replace Γ by the free product $\Gamma \star \mathbb{Z}$ (or a quotient of it) and we extend Φ_0 so that the generator the $\mathbb{Z}$-factor is mapped to a nontrivial translation. This new group has no fixed point, so that the preceding lemma applies to it, hence to Γ.

A substitute to the space of left-orders. The previous construction allows replacing the space of left-orders of a given left-orderable group by an object provided with a flow which is more natural when dealing with dynamical realizations.

Corollary 4.2.11. *Let Γ be a finitely-generated, left-orderable group. Then there exists a compact space X, a free flow $S_{\mathbb{R}}$ on X, and an action of Γ on X without global fixed points which preserves the S-orbits together with their orientations.*

Proof. Since Γ is finitely-generated and left-orderable, it admits a faithful action by orientation-preserving homeomorphisms of the real line without global fixed point (*c.f.* §1.1.3). By Theorem 4.2.9, this action is conjugate to an almost-periodic action Φ_0 that does not almost have fixed points. Consider the space X constructed in (the proof of) Proposition 4.2.3 together with the free flow $S = S_{\mathbb{R}}$ and the Γ-action on it. Because Φ_0 does not almost have fixed points and $S_{\mathbb{R}}(\Phi_0)$ is dense in X, there is no fixed point for the Γ-action on X. Moreover, Γ stabilizes every S-orbit, and preserves the orientation on each of them. $\square$

4.2.4 Local indicability of amenable left-orderable groups revisited

Based on the previous construction, and following [53], we next give an alternative proof of Theorem 4.1.1. Let Γ be a finitely-generated, left-orderable group, and let X be a compact space equipped with a free action $S_{\mathbb{R}}$ and an action of

Γ, as described by Corollary 4.2.11. If Γ is amenable, then there exists a probability measure μ on X that is invariant by Γ. Consider the conditional measures of μ along the orbits of the translation flow $S_{\mathbb{R}}$. These are Radon measures on μ-almost every S-orbit that are well-defined up to multiplication by a positive constant. We denote by μ_l this Radon measure on a S-orbit l. More precisely, in a flow box $[0,1] \times \Lambda$ where the flow S is given by the formula $S_s(t,l) = (t+s,\lambda)$, we disintegrate the measure μ as

$$\mu(dt, dl) = \mu_l(dt)\ \overline{\mu}(d\lambda)$$

where $\overline{\mu}$ is the image of μ under the projection $[0,1] \times \Lambda \to \Lambda$ into the transversal Λ and the measures μ_l are measures on the unit interval; see [159, Section 3]. The measures μ_l depend on the flow box; however, they are well-defined up to a positive constant on almost-every orbit.

Because Γ preserves μ and is countable, for μ-almost-every S-orbit l in X, the measure μ_l is nonzero, and every $g \in \Gamma$ multiplies it by a certain factor:

$$g_*(\mu_l) = c_l(g)\mu_l, \quad \text{where} \quad c_l(g) > 0.$$

If μ_l is not preserved by Γ, then the map $g \mapsto \log c_l(g)$ is a nontrivial homomorphism from Γ into $(\mathbb{R}, +)$. (See also Remark 4.2.12 below.) Otherwise, μ_l is preserved by Γ. If μ_l has an atom, then its orbit must be discrete, and Γ acts by translations along this orbit, thus giving rise to a nontrivial homomorphism into the integers. If μ_l has no atom, then the Γ-action on l is semiconjugate to an action by translations, which induces a nontrivial homomorphism into the reals.

Remark 4.2.12. It seems that the the condition $\mu_l = c_l(g)\mu_l$ for a function c_l that is not identically equal to 1 cannot arise in the proof above. Actually, this can be proven under the extra hypothesis that there is no discrete orbit for the action of Γ on l (which is identified to the real line). Indeed, in this case, it is not hard (yet not straightforward) to prove that this action is semiconjugate to that of an affine group. We claim that the image group must be Abelian for a generic leaf. To see this, assume the image group is non-Abelian. Then there is a resilient pair $u < f(u) < f(v) < g(u) < g(v) < v$ for the action on l after this semiconjugacy (where the order is given by the orientation of l). With no loss of generality, we may assume that the origin 0 belongs to the region of crossing; let Φ be the associated representation. Besides, there is an element $h \in \Gamma$ whose inverse (and all its iterates) sends $[u,v]$ into a disjoint interval (hence to a region where no crossing for f,g arises; *c.f.* §3.2.2). As a consequence, for all $n \in \mathbb{N}$, the conjugate representations $h^n(\Phi)$ remain outside a certain neighborhood of Φ. However, this is in contradiction with Poincaré's recurrence, hence cannot arise for a μ-generic leaf l. (We refer to Examples 4.3.16 and 4.3.18 for another application of this idea.)

4.3 Random Walks on Left-Orderable Groups

In what follows, we provide a more conceptual proof of the existence of almost-periodic actions for left-orderable groups based on probabilistic arguments. Along this section, Γ will denote a finitely-generated group and ρ a probability measure on Γ whose support is finite, generates Γ, and is ***symmetric***, in the sense that $\rho(g) = \rho(g^{-1})$ for all $g \in \Gamma$.

We start with an emphasis on a particular type of actions, namely, those for which the Lebesgue measure is stationary, *i.e.* invariant in mean. These actions are called ρ-harmonic. Their basic properties are listed in the next two subsections. In particular, it is proven that they are almost-periodic. The main result is proven in the third and fourth subsections. It states that, under suitable conjugacies, all actions on $\mathbb{R}$ become harmonic, and the conjugacy is unique up to post-composition with an affine map.

4.3.1 Harmonic actions and Derriennic's property

Let Γ be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ having no global fixed point in the line. The action of Γ is said to be ***ρ-harmonic*** (or just harmonic, if the probability ρ is clear from the context) if the Lebesgue measure is ***stationary***, that is, if for every x, y in $\mathbb{R}$,

$$y - x = \int_\Gamma \big(g(y) - g(x)\big)\, d\rho(g) = \sum_{g \in \mathcal{G}} \big(g(y) - g(x)\big)\, \rho(g). \qquad (4.10)$$

Actually, no group action on the line satisfying this property can have a global fixed point; see Exercise 4.3.8.

Obviously, ρ-harmonic actions include those that satisfy, for every $x \in \mathbb{R}$,

$$x = \int_\Gamma g(x)\, d\rho(g).$$

This will be called the ***Derriennic property***, as it corresponds to a weak form of a property studied by Derriennic in [56] in the more general context of Markov processes on the line (not necessarily coming from a group action). Quite suprisingly, all ρ-harmonic actions do satisfy this property.

Proposition 4.3.1. *Every ρ-harmonic action has the Derriennic property.*

For the proof, we need the following

Lemma 4.3.2. *For all $h \in \mathrm{Homeo}_+(\mathbb{R})$ and each compact interval $[a,b]$, we have*

$$\int_a^b \big[(h(x)-x)+(h^{-1}(x)-x)\big]\,dx = \Delta^h(b)-\Delta^h(a), \tag{4.11}$$

where $\Delta^h(x)$ is the non-signed area of the region depicted in Figure 15:

$$\Delta^h(c) := \begin{cases} \int_{h^{-1}(c)}^{c} \big[h(s)-c\big]ds, & \text{if } h(c)\geq c, \\ \int_{h(c)}^{c} \big[h^{-1}(s)-c\big]ds, & \text{if } h(c)\leq c. \end{cases}$$

Proof. Denoting $|A|$ the Lebesgue measure of a subset $A\subset\mathbb{R}^2$, we have that $\int_a^b (h(x)-x)\,dx$ equals

$$\big|\big\{(x,y)\colon a<x<b,\, x<y<h(x)\big\}\big| \,-\, \big|\big\{(x,y)\colon a<x<b,\, h(x)<y<x\big\}\big|,$$

which may be rewritten as

$$\big|\big\{(x,y)\colon a<x<b,\, b<y<h(x)\big\}\big| + \big|\big\{(x,y)\colon a<x<b,\, a<y<b,\, x<y<h(x)\big\}\big|$$
$$-\big|\big\{(x,y)\colon a<x<b,\, h(x)<y<a\big\}\big| - \big|\big\{(x,y)\colon a<x<b,\, a<y<b,\, h(x)<y<x\big\}\big|.$$

A similar equality holds for h^{-1}. Now, in the sum

$$\int_a^b (h(x)-x)\,dx + \int_a^b (h^{-1}(x)-x)\,dx,$$

the corresponding second and fourth terms above cancel each other. Indeed, these terms involve all couples $(x,y)\in[a,b]^2$, and we have $x<y<h(x)$ if and only if $h^{-1}(y)<x<y$. Therefore, the second term for h is exactly the negative of the fourth term for h^{-1}, and viceversa.

As a consequence, the value of

$$\int_a^b \big[(h(x)-x)+(h^{-1}(x)-x)\big]\,dx$$

equals

$$\big|\big\{(x,y)\colon a<x<b,\; b<y<h(x)\big\}\big| \,+\, \big|\big\{(x,y)\colon a<x<b,\; b<y<h^{-1}(x)\big\}\big|$$
$$-\,\big|\big\{(x,y)\colon a<x<b,\, h(x)<y<a\big\}\big| \,-\, \big|\big\{(x,y)\colon a<x<b,\; h^{-1}(x)<y<a\big\}\big|,$$

and one can easily check that the expressions above and below are equal to $\Delta^h(b)$ and $\Delta^h(a)$, respectively. This proves the desired equality. $\square$

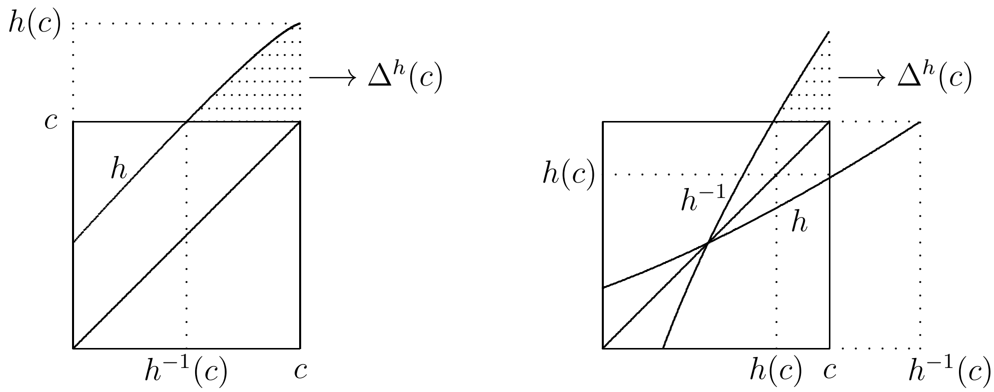

Figure 15: The definition of $\Delta^h(c)$ in the two possible cases.

Proof of Proposition 4.3.1. First notice that, by ρ-harmonicity, the value of

$$\int_\Gamma \big(g(x) - x\big) d\rho(g)$$

is independent of x. We call it the ***drift*** of the action and denote it by $Dr(\Gamma, \rho)$. The statement to be proved is hence equivalent to the vanishing of the drift. To show this, we integrate (4.11) over Γ and use the symmetry of ρ to obtain, for all $a < b$,

$$2(b-a)Dr(\Gamma, \rho) = \int_\Gamma \big(\Delta^g(b) - \Delta^g(a)\big)\, d\rho(g).$$

Denoting now $\Delta_\rho(c) := \int_\Gamma \Delta^g(c)\, d\rho(g)$, this yields

$$2(b-a)Dr(\Gamma, \rho) = \Delta_\rho(b) - \Delta_\rho(a).$$

The last equality shows that Δ_ρ is an affine function. On the other hand, Δ_ρ is an average of non-negative functions, thus it is non-negative. Therefore, Δ_ρ must be constant, which implies that $Dr(\Gamma, \rho) = 0$, as desired. $\square$

The next proposition shows the relevance of the Derriennic property in the study of the smoothness of a group action.

Proposition 4.3.3. *If the Γ-action is ρ-harmonic, then every $g \in \Gamma$ is Lipschitz. Moreover, the displacement function $x \mapsto g(x) - x$ is uniformly bounded in x. In particular, every ρ-harmonic action is almost-periodic.*

Proof. It suffices to prove the lemma for the elements of the support of ρ. Indeed, this is obvious for the Lipschitz property, and for the boundedness of the displacements, this is a consequence of the relation

$$gh(x) - x = \big(gh(x) - h(x)\big) + \big(h(x) - x\big).$$

Now, for every element g in the support of ρ and every $x < y$, we have

$$\rho(g)\big[g(y) - g(x)\big] \leq \int_{\Gamma} \big[g(y) - g(x)\big]\, d\rho(g) = y - x.$$

Hence,

$$g(y) - g(x) \leq \frac{y - x}{\rho(g)},$$

proving that g has Lipschitz constant at most $1/\rho(g)$. To show that the displacements are bounded, we will show that the value of $\Delta^g(x)$ is comparable, up to a multiplicative constant, to $[g(x) - x]^2$. In concrete terms, set $L := \max\{1/\rho(h) : \rho(h) > 0\}$. Then g^{-1} has Lipschitz constant at most L. Letting $x \in \mathbb{R}$ be such that $g(x) \geq x$, for each $y \in \mathbb{R}$ such that $g^{-1}(x) \leq y \leq x$, we have

$$g(y) - x \geq \frac{1}{L}\big(y - g^{-1}(x)\big).$$

Therefore,

$$\int_{g^{-1}(x)}^{x} \big[g(y) - x\big]\, dy \geq \frac{1}{L} \int_{g^{-1}(x)}^{x} \big[y - g^{-1}(x)\big]\, dy = \frac{\big[x - g^{-1}(x)\big]^2}{2L}.$$

Since $g(x) - x \leq L\big(x - g^{-1}(x)\big)$, using the notation of the preceding proposition, this implies that

$$\Delta^g(x) \geq \frac{\big[g(x) - x\big]^2}{2L^3},$$

hence

$$\Delta_\rho(x) = \int_{\Gamma} \Delta^g(x)\, d\rho(g) \geq \frac{\big[g(x) - x\big]^2}{2L^3}.$$

However, by (the proof of) Proposition 4.3.1, the function $\Delta_\rho(\cdot)$ is constant. Thus, the value of $|g(x) - x|$ is bounded from above by a constant, as we wanted to show. Analogous arguments apply in the case where $g(x) \leq x$. $\square$

Exercise 4.3.4. If Γ is a free Abelian group, show that every ρ-harmonic action of Γ on the real line is an action by translations.
Hint. Observe that the functions $g \mapsto g(x)$ are ρ-harmonic and Lipschitz for every $x \in \mathbb{R}$, and use the fact that Lipschitz harmonic functions on an Abelian group are linear. (Here, $\phi\colon \Gamma \to \mathbb{R}$ is said to be harmonic if $\phi(g) = \sum_{h \in \Gamma} \phi(gh)\rho(h)$ holds for all $g \in \Gamma$.)

4.3.2 Properties of stationary measures

As we have seen, very nice properties hold for actions for which the Lebesgue measure is stationary. Although for most actions this is certainly not the case, in §4.3.3, we will see that, under very mild hypothesis, there is always a stationary measure, which is unique up to multiplication by a positive constant and can be transformed into the Lebesgue one by a semiconjugacy. We will also show several general properties of stationary measures that actually will be crucial for the proof of this fact. Throughout this section, Γ will continue denoting a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ having no global fixed point.

Following [55], let us introduce the Markov process on the line defined by

$$X_x^n = g_n \cdots g_1(x),$$

where (g_n) is a family of independent random variables with law ρ. (For a general introduction to the theory of Markov processes, we refer the reader to the very nice book [61].) The transition probabilities of this process are

$$\rho_X(x,y) := \sum_{y=g(x)} \rho(g).$$

The associated Markov operator $P = P_X$ acting on the space of bounded continous functions $C_b(\mathbb{R})$ is given by

$$P\phi(x) = \mathbb{E}(\phi(X_1^x)) = \int_\Gamma \phi(gx)\, d\rho(g). \tag{4.12}$$

The iterates of this operator correspond to the operators associated to the convolutions of ρ. More precisely, we have $P_\rho^n = P_{\rho^{*n}}$, where the ***convolution*** of two probabilities ρ_1, ρ_2 on Γ is

$$\rho_1 * \rho_2(h) := \sum_{fg=h} \rho_1(f)\,\rho_2(g),$$

and $\rho^{*n} := \rho * \rho * \cdots * \rho$ (n times).

We will still denote by P the dual action on the space of Radon measures on the line. Such a measure is said to be ***stationary*** if it is P-invariant, that is, $P\nu = \nu$. Equivalently,

$$\nu = \sum_{g \in \Gamma} g_*(\nu)\, \rho(g).$$

By definition, an action is harmonic if and only if the Lebesgue measure is stationary.

Lemma 4.3.5. *Every nonzero stationary Radon measure ν on the real line is bi-infinite (i.e. $\nu(x,\infty) = \infty$ and $\nu(-\infty,x) = \infty$, for all $x \in \mathbb{R}$).*

Proof. Suppose that there exists $x \in \mathbb{R}$ such that $\nu(x,\infty) < \infty$. Since we are assuming that the Γ-action has no global fixed point on $\mathbb{R}$, for every $y \in \mathbb{R}$, there is an element $g \in \Gamma$ such that $g(x) < y$. As the support of ρ generates Γ, we can choose $n > 0$ such that $p^{\star n}(g^{-1}) > 0$. Then

$$\nu(y,\infty) \leq \nu(g(x),\infty) \leq \frac{\nu(x,\infty)}{\rho^{\star n}(g^{-1})} < \infty.$$

This shows that $\nu(y,\infty) < \infty$ holds for all $y \in \mathbb{R}$.

Now let $\phi \colon \mathbb{R} \to (0,\infty)$ be the function defined by $\phi(x) := \nu(x,\infty)$. Since ρ is symmetric, this function is harmonic; in other words, we have $P\phi = \phi$, where $P\phi$ is defined by (4.12). (This definition still makes sense yet ϕ is not necessarily continuous.) Fix a real number A satisfying $0 < L < \nu(-\infty,\infty)$, and let $\psi := \max\{0, L - \phi\}$. The function ψ is ***subharmonic***, which means that $\psi \leq P\psi$. Moreover, it vanishes on a neighborhood of $-\infty$ and is bounded on a neighborhood of ∞. This implies that ψ is ν-integrable, and since $\int P\psi\, d\nu = \int \psi\, d\nu$, the function ψ must be ν-a.e P-invariant. Now a classical lemma from [69] asserts that a measurable function which is in $\mathcal{L}^1(\nu)$ and P-invariant must be a.e. Γ-invariant (see Exercise 4.3.6 for a schema of proof). Thus, ψ is constant on almost-every orbit. However, this is impossible, since every orbit intersects every neighborhood of $-\infty$ (where ψ vanishes) and of ∞ (where ψ is positive). This contradiction establishes the lemma. $\square$

Exercise 4.3.6. Let Γ be a countable group and ρ a measure on Γ whose support generates Γ. Assume that Γ acts on a probability space (X,ν) by measurable maps and that ν is ρ-stationary, meaning that

$$\nu = \int_\Gamma g_*(\nu)\, d\rho(g).$$

Prove that any function $\phi \in \mathcal{L}^1(X,\nu)$ that is a.e. P-invariant is a.e. Γ-invariant.
Hint. Pick a constant $L \in \mathbb{R}$ and consider the function $\psi := \max\{\phi, L\}$. Observe that ψ belongs to $L^1(\nu)$ and satisfies $\psi \leq P\psi$, and deduce that ψ is ρ-harmonic on almost-every Γ-orbit. Conclude by ranging L along all rational numbers.

Exercise 4.3.7. Let Γ be a finitely-generated subgroup of $\mathrm{Homeo}_+(\mathbb{R})$, and let ρ be a symmetric probability measure on it with generating support. Prove that for all $x \in \mathbb{R}$, every compact interval I, and almost-every sequence $(g_n) \in \Gamma^{\mathbb{N}}$, the set of integers n for which X_x^n belongs to I has density zero, that is,

$$\lim_{k\to\infty} \frac{1}{k}\big|\{n \in \{1,\ldots,k\} \colon X_x^n \in I\}\big| = 0.$$

Hint. Let ν_k be the measure on the line defined by

$$\nu_k(I) := \frac{1}{k}\sum_{n=1}^{k} \rho^{\star n}\big(\{h \colon h(x) \in I\}\big).$$

Assuming that the zero-density above doesn't hold, show that, up to a subsequence, ν_k converges to a nonzero, finite, stationary measure, thus contradicting Lemma 4.3.5.

Exercise 4.3.8. Prove that no nontrivial group action on the line satisfying (4.10) for all x, y in $\mathbb{R}$ can have a global fixed point.
Hint. By definition, the Lebesgue measure is P-invariant for a ρ-harmonic action. Apply Lemma 4.3.5 to the restriction of the action to a connected component of the set of global fixed points.

We next study the case when the Γ-action admits discrete orbits. Obviously, such an orbit supports an invariant Radon measure, namely the counting measure; in particular, this measure is P-invariant. The next two lemmas prove that in the case there are discrete orbits, all P-invariant Radon measures lie in the convex closure of the set of counting measures along discrete orbits.

Lemma 4.3.9. *Let ν be a stationary measure on the line. If there is a discrete orbit, then ν is supported on the union of discrete orbits and is totally invariant.*

Proof. If there is a discrete orbit $\mathcal{O}$, then Γ acts on it by translating its points. Thus, the normal subgroup Γ_* formed by the elements acting trivially on $\mathcal{O}$ is recurrent, by Polya's classical theorem [148] (see also Corollary 4.3.15 for an alternative proof of this fact). Let ρ_* be the (symmetric) measure on Γ_* obtained by ***balayage*** of ρ to Γ_*; more precisely, we consider the random walk on Γ with initial distribution ρ and we stop it at the first moment where it visits

Γ_*, thus yielding a random variable with values in Γ_* whose distribution is ρ_*. Observe that the restriction of ν to each connected component C of $\mathbb{R} \setminus \mathcal{O}$ is a finite measure that is invariant for the Markov process induced by ρ_* on C. It follows from Lemma 4.3.5 that this measure is supported on $\mathrm{Fix}(\Gamma_*) \cap \overline{C}$, the set of global fixed points for the group Γ_* contained in the closure of C. (Notice that Lemma4.3.5 dod not use that the support of ρ was finite.) As a consequence, Γ acts by "integer translations" on the support of ν, which consists of discrete orbits. To see that ν is invariant, notice that for each atom $x \in \mathbb{R}$, the function $g \mapsto \nu(g(x))$ viewed as a function defined on $\Gamma/\Gamma_* \sim \mathbb{Z}$ is harmonic, hence constant. $\square$

Exercise 4.3.10. Assume that the Γ-action is harmonic and admits a discrete orbit. Prove that Γ is contained in the (cyclic) group generated by a translation of the line.

Lemma 4.3.11. *Let ν be a stationary measure on the real line. If the atomic part of ν is nontrivial, then it is supported on a union of discrete orbits.*

Proof. The proof uses the recurrence property that will be proven in §4.3.3. Let $x \in \mathbb{R}$ be a point such that $\nu(x) > 0$. Let $\mathcal{O}$ be the orbit of x endowed with the discrete topology, and let $\overline{\nu}$ be the measure on $\mathcal{O}$ defined by $\overline{\nu}(y) := \nu(y)$. We claim that $\overline{\nu}$ is an invariant measure for the Markov process induced by ρ on $\mathcal{O}$. To see this, consider an arbitrary function $\phi \colon \mathcal{O} \to \mathbb{R}$ with finite support. Let $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a family of functions with support a union of intervals of length ε centered at the points of the support of ϕ. Then,

$$\int_{\mathcal{O}} \phi \, d\overline{\nu} = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \phi_\varepsilon \, d\nu = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \phi_\varepsilon \, dP\nu = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} P\phi_\varepsilon \, d\nu = \int_{\mathcal{O}} P\phi \, d\overline{\nu},$$

which is sufficient to establish our claim.

Let $K = [A, B]$ be an interval such that for any element g of the support of ρ, we have $g(A) < B$. (Recall that ρ is assumed to be finitely supported.) Let I be an arbitrary compact interval containing K, and let $\mathcal{O}_I := I \cap \mathcal{O}$. We want to show that $\mathcal{O}_I$ is finite. To do this, first observe that $\mathcal{O}_I$ is a recurrent subset of $\mathcal{O}$, by Corollary 4.3.15. Let Y be the Markov process on $\mathcal{O}_I$ defined by the first return of the original Markov process X to $\mathcal{O}_I$. The process Y is symmetric, because X is symmetric. Moreover, the restriction of $\overline{\nu}$ to $\mathcal{O}_I$ is stationary. Now, since $\sum_{y \in \mathcal{O}_I} \overline{\nu}(y) < \infty$, there must be an atom $y \in \mathcal{O}_I$ for which $\overline{\nu}(y)$ is maximal. The P_Y-invariance of $\overline{\nu}$ and the symmetry of the transition probabilities ρ_Y yield

$$\sum_{z \in \mathcal{O}} \rho_Y(y, z) \overline{\nu}(z) = \sum_{z \in \mathcal{O}_I} \rho_Y(z, y) \overline{\nu}(z) = \overline{\nu}(y).$$

The maximum principle now implies that $\overline{\nu}(z) = \overline{\nu}(y)$. Thus, all the atoms of $\overline{\nu}$ contained in $\mathcal{O}$ have the same mass, hence there is only a finite number of them. As this holds for an arbitrary compact interval I containing K, this shows that $\mathcal{O}$ is discrete. $\square$

Next, we consider the case where Γ has no discrete orbits. Recall that in this situation, there is a unique nonempty minimal invariant closed set for the action, that we denote $\mathcal{M}$; see Lemma 3.5.18.

Lemma 4.3.12. *Assume that Γ does not have any discrete orbit on the real line. Then any stationary measure is supported on the minimal set $\mathcal{M}$.*

Proof. Let ν be a stationary measure on the real line. Then ν is quasi-invariant by Γ, because for all h in the support of ρ, we have

$$h_*(\nu) \leq \frac{1}{\rho(h)} \sum_{g \in \Gamma} g_*(\nu)\, \rho(g) = \frac{\nu}{\rho(h)}.$$

Therefore, the support of ν is a closed Γ-invariant subset of the line, hence it contains $\mathcal{M}$. Thus, it suffices to show that ν does not charge any connected component of the complement $\mathcal{M}^c$.

Assume $\mathcal{M}^c$ is nonempty, and collapse each of its connected components to a point, thus obtaining a topological line carrying a Γ-action for which all orbits are dense. The stationary measure ν can be pushed to a stationary measure $\overline{\nu}$ for this new action. If a component of $\mathcal{M}^c$ has a positive ν-measure, then $\overline{\nu}$ has atoms. Lemma 4.3.11 then implies that the new Γ-action cannot be minimal, which is a contradiction. $\square$

Exercise 4.3.13. Prove that if the action of Γ is harmonic, then Γ either is a group of translations or acts minimally on the real line.

4.3.3 Recurrence

As in previous sections, we continue considering a finitely-supported, symmetric probability measure ρ on a group Γ acting on the real line without global fixed points. We also assume that the support of ρ generates Γ. We start with an *oscillation* result claiming that almost-every random orbit escapes to the infinity in both directions.

Proposition 4.3.14. *For every $x \in \mathbb{R}$, almost surely we have*

$$\limsup_{n\to\infty} X_x^n = \infty \quad \text{and} \quad \liminf_{n\to\infty} X_x^n = -\infty.$$

Proof. Denote $\mathbb{P} := \rho^{\mathbb{N}}$, and given points L and x on the real line, let

$$p_L(x) := \mathbb{P}\Big[\limsup_{n\to\infty} X_x^n > L\Big].$$

Since Γ acts by orientation-preserving homeomorphisms, for all $x \leq y$, we have

$$\Big\{(g_n) \in \Gamma^{\mathbb{N}} \colon \limsup_{n\to\infty} X_x^n > L\Big\} \subset \Big\{(g_n) \in \Gamma^{\mathbb{N}} \colon \limsup_{n\to\infty} X_y^n > L\Big\}.$$

In particular, $p_L(x) \leq p_L(y)$, that is, p_L is non-decreasing. Moreover, since p_L is the probability of the tail event

$$\Big[\limsup_{n\to\infty} X_x^n > L\Big]$$

and X is a Markov chain, p_L is a *harmonic* function, that is, for every $x \in \mathbb{R}$ and every integer $n \geq 0$,

$$p_L(x) = \sum_{g\in\Gamma} p_L\big(g(x)\big)\rho^{\star n}(g) = \mathbb{E}\big(p_L(X_x^n)\big).$$

Now, we would like to see p_L as the distribution function of a *finite* measure on the line. However, this is only possible when p_L is continuous on the right, which is a priori not necessarily the case. We are hence led to consider the right-continuous function

$$\overline{p_L}(x) := \lim_{y\to x,\ y>x} p_L(y).$$

This function is still non-decreasing. Therefore, there exists a finite measure ν on $\mathbb{R}$ such that for all $x < y$,

$$\nu(x,y] \ = \ \overline{p_L}(y) - \overline{p_L}(x).$$

Since p_L is harmonic and Γ acts by homeomorphisms, $\overline{p_L}$ is also harmonic. Since ρ is symmetric, this yields that ν is P-invariant. Now recall that Lemma 4.3.5 implies that any P-invariant finite measure identically vanishes (see also [54, Proposition 5.7]). Therefore, $\overline{p_L}$ is constant, hence in particular its value does not depend on the starting point x. The 0-1 law then allows to conclude that (for any fixed L) either $p_L \equiv 0$ or $p_L \equiv 1$.

Let us now show that p_L identically equals to 1 for each L. To do this, fix any $x_0 > L$. As for any $g \in \mathrm{Homeo}_+(\mathbb{R})$, we have either $g(x_0) \geq x_0$ or $g^{-1}(x_0) \geq x_0$, the symmetry of ρ yields that $X^n_{x_0} \geq x_0$ holds with probability at least $1/2$, for all $n \in \mathbb{N}$. It is then easy to see that

$$p_L = p_L(x_0) \geq \mathbb{P}\Big[\limsup_{n\to\infty} X^n_{x_0} \geq x_0\Big] \geq 1/2.$$

As we have already shown that p_L equals 0 or 1, this implies that p_L is identically equal to 1.

The latter means that for every $x \in \mathbb{R}$,

$$\limsup_{n\to\infty} X^n_x = \infty$$

holds almost surely. Analogously, for every $x \in \mathbb{R}$, almost surely we have

$$\liminf_{n\to\infty} X^n_x = -\infty.$$

This completes the proof of the proposition. □

We are now ready to prove the main result of this section, namely the *recurrence* of the Markov process.

Corollary 4.3.15. *There exists a compact interval K such that, for every $x \in \mathbb{R}$, almost surely the sequence (X^n_x) intersects K infinite many times.*

Proof. Consider a closed interval K as in the proof of Lemma 4.3.11, that is, $K = [A, B]$, where $A < B$ are such that for every g of the support of ρ, we have $g(A) < B$. (Recall that ρ is finitely supported.) By Proposition 4.3.14, for every $x \in \mathbb{R}$, almost surely the sequence (X^n_x) will pass from $(-\infty, A]$ to $[B, +\infty)$ infinite many times. Now the desired conclusion follows from the observation that the choice of A and B imply that every time this happens, (X^n_x) must cross the interval K. □

On left-orders that are generic with respect to a stationary measure. Given a finitely-supported probability on a left-orderable group Γ with generating support, we can also consider stationary probability measures for the action of Γ

on its space of left-orders (see Example 4.3.6). By this, we mean a probability measure μ on $\mathcal{LO}(\Gamma)$ such that

$$\mu = \sum_{g \in \Gamma} g_*(\mu)\rho(g). \tag{4.13}$$

Since $\mathcal{LO}(\Gamma)$ is compact, such a probability measure μ always exists. (This follows from a direct application of either Kakutani's fixed point theorem or the Bogoliubov-Krylov procedure.) It seems quite interesting to study the relation of μ with the algebraic properties of Γ as well as its dependence on ρ. We give below two examples on this.

Example 4.3.16. We next give still another proof of Theorem 2.2.9 for finitely-generated groups. To do this, fix ρ and μ as above. We can assume that μ is *ergodic*, in the sense that it cannot be written as a nontrivial convex combination of two different stationary probability measures. We have two possibilities:

Case (i). The measure μ has an atom.

If $\preceq$ is an atom of maximal μ-measure, then (4.13) easily implies that its orbit must be finite. (Actually, by ergodicity, this orbit coincides with the support of μ.) In particular, $\preceq$ is right-reccurrent, hence Conradian. Thus, if Γ has infinitely many left-orders, then Proposition 3.2.53 implies that $\mathcal{LO}(\Gamma)$ is uncountable, as desired.

Case (ii). The measure μ is non-atomic.

By ergodicity, for almost-every $\big(\preceq, (g_n)\big)$ in $\mathcal{LO}(\Gamma) \times \Gamma^{\mathbb{N}}$ (endowed with the measure $\mu \times \rho^{\mathbb{N}}$) the sequence $(\preceq_{\sigma^n(\omega)}, \sigma^n(\omega))$ is dense in $supp(\mu) \times \Gamma^{\mathbb{N}}$, where σ stands for the shift $\sigma((g_n)) := (g_{n+1})$. Let us fix such a pair $\big(\preceq, (g_n)\big)$, and let U_k be a sequence of open subsets of positive ρ-measure in $\mathcal{LO}(\Gamma)$, none of which containing $\preceq$, but which do converge to $\preceq$. For each k, there exists $n(k) \in \mathbb{N}$ such that $\preceq_{\sigma^{n(k)}(\omega)}$ belongs to U_k. Hence, $\preceq_{\sigma^{n(k)}(\omega)}$ converges to $\preceq$, with $\preceq_{\sigma^{n(k)}(\omega)}$ being distinct from $\preceq$ for all k. Therefore, the closure of the orbit of $\preceq$ under the action of Γ is a totally disconnected compact metric space with no isolated point, that is, a Cantor set. In particular, $\mathcal{LO}(\Gamma)$ is uncountable.

Remark 4.3.17. Notice that the approximation by conjugates in the example above is essentially different from that of §3.2.5. Indeed, the conjugating elements therein are positive but "small' (outside the Conradian soul). In the proof above, the conjugating elements are "random", and hence by Exercise 4.3.7, they are mostly "near the infinite" (in any direction), despite the recurrence of the associated random walk on the line (in the case where ρ is symmetric).

Example 4.3.18. According to Example 3.2.55, for each integer $\ell \geq 2$, the Baumslag-Solitar group $B(1,\ell) = \langle a, b : aba^{-1} = b^{\ell} \rangle$ admits four Conradian orders, which are actually bi-invariant and come from the exact sequence

$$0 \longrightarrow \mathbb{Z}\Big[\frac{1}{\ell}\Big] \longrightarrow B(1,\ell) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

We claim that, although $\mathcal{LO}(B(1,\ell))$ is a Cantor set (*c.f.* §3.3.1), for every symmetric probability distribution ρ on $B(1,\ell)$ as above, every stationary probability measure μ on $\mathcal{LO}(B(1,\ell))$ is supported on these four points. Indeed, we proved in §3.3.1 that for every left-order $\preceq$ on $BS(1,\ell)$ that is not bi-invariant, the associated dynamical realization is semiconjugate to a non-Abelian subgroup of the affine group. In particular, there exist elements whose sets of fixed points are bounded and for which $-\infty$ and $+\infty$ are topologically-repelling fixed points. Let g be such an element (actually, such a g can be taken as the image of a), and denote by $Fix(g)$ its set of fixed points. Let f_1, f_2 be in the realization of $B(1,\ell)$ so that f_1 (resp. f_2) sends the leftmost (resp. the rightmost) fixed point of g to the right (resp. left) of $0 = t(id)$. Denote $g_1 := f_1 g f_1^{-1}$ and $g_2 := f_2 g f_2^{-1}$. If we identify elements in $BS(1,\ell)$ with their realizations, we have $g_1 \succ id$ and $g_2 \prec id$. Moreover, $hg_ih^{-1} \prec id$ holds for both $i=1$ and $i=2$ provided h is sufficiently large (say, larger than a certain element h_+). Similarly, $hg_ih^{-1} \succ id$ holds for $i = 1$ and $i = 2$ provided h is smaller than a certain element h_-; see Figure 16 below.

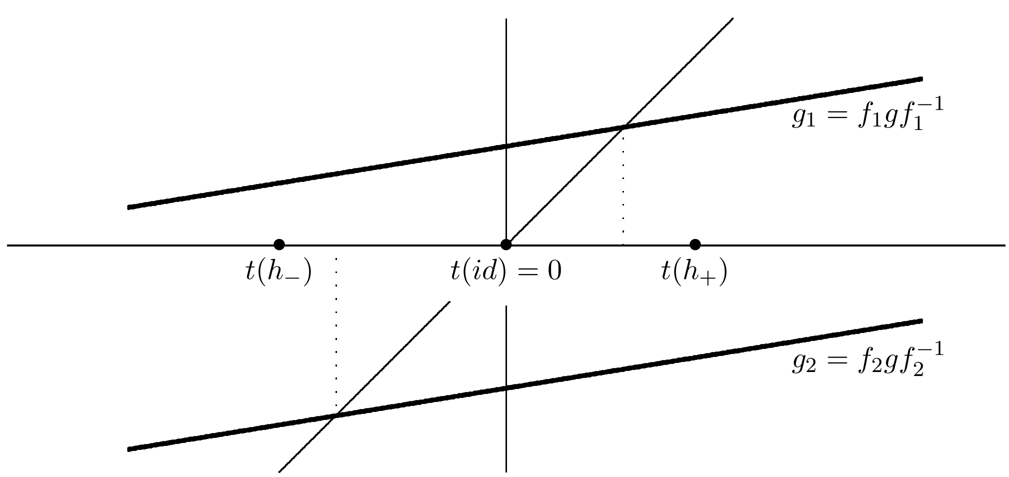

Figure 16: The elements g_1, g_2, h_- and h_+.

Assume μ is an stationary probability measure on $\mathcal{LO}(BS(1,\ell))$ that is not fully supported on the four bi-orders. Then any ergodic component of this measure outside these bi-orders is still stationary, and supported on the complement of the bi-orders.

For simplicity, we still denote this measure by μ. Let $\preceq$ a point in the support of μ. If we perform the construction of the elements g_1, g_2, h_-, h_+ above, then the measure of the open neighborhood $V_{g_1} \cap V_{g_2^{-1}} = \big\{ \preceq' \colon\ g_1 \succ' id,\ g_2 \prec' id \big\}$ of $\preceq$ must be positive, say equal to $\kappa > 0$. A direct application of the ergodic theorem then shows that, for a generic random path $(h_n) \in B(1,\ell)^{\mathbb{N}}$, the set of integers n for which $\preceq_{X^n}$ lies in $V_{g_1} \cap V_{g_2^{-1}}$, where $X^n := h_n \cdots h_1$, has density κ. Nevertheless, among these integers n, with density 1 we have either $X^n \prec h_-$ or $X^n \succ h_+$ (see Exercise 4.3.7), thus providing a contradiction.

4.3.4 Existence of stationary measures

Using the recurrence result of the preceding section, we can now establish the existence of a P-invariant Radon measure via a quite long but standard argument.

Theorem 4.3.19. *Let* Γ *be a finitely-generated subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *endowed with a symmetric probability measure whose support generates* Γ*. Then there exists a (nonzero) ρ-stationary measure on the real line.*

Proof. Fix a continuous compactly-supported function $\xi \colon \mathbb{R} \to [0,1]$ such that $\xi \equiv 1$ on K. For any initial point x, let us stop the process X_x^n at a *random* stopping time T chosen in a Markovian way so that, for all $n \in \mathbb{N}$,

$$\mathbb{P}\big[T = n+1 \mid T \geq n\big] = \xi(X_x^{n+1}).$$

(Here, $T = T(\omega)$, where $\omega = (g_i)_{i \in \mathbb{N}}$.) In other words, after each step of the initial random walk arriving to a point $y = X_x^{n+1}$, we stop with probability $\xi(y)$, and we continue the compositions with probability $1 - \xi(y)$.

Denote by Y_x the random stopping point X_x^T, and consider its distribution ρ_x (notice that T is almost-surely finite since the process X_x^n almost surely visits K and $\xi \equiv 1$ on K). Due to the continuity of ξ, the measure ρ_x on $\mathbb{R}$ depends continuously (in the weak topology) on x. Therefore, the corresponding diffusion operator P_ξ defined by

$$P_\xi(\phi)(x) = \mathbb{E}\big(\phi(Y_x)\big) = \int_{\mathbb{R}} \phi(y)\, d\rho_x(y)$$

acts on the space of continuous bounded functions on $\mathbb{R}$, and hence it acts by duality on the space of probability measures on $\mathbb{R}$. Notice that for any such probability measure, its image under P_ξ is supported on $\hat{K} := \mathrm{supp}(\xi)$. Thus, by applying the Bogolyubov-Krylov procedure of time averaging (and extracting a convergent subsequence), we see that there exists a P_ξ-invariant probability measure ν_0.

To construct a Radon measure that is stationary for the initial process, we proceed as follows. For each point $x \in \mathbb{R}$, let us take the sum of the Dirac measures supported in its random trajectory before the stopping time T. In other words, we consider the "random measure"

$$m_x(\omega) := \sum_{j=0}^{T(w)-1} \delta_{X_x^j}$$

and its expectation

$$m_x := \mathbb{E}\big(m_x(\omega)\big) = \mathbb{E}\left(\sum_{j=0}^{T(w)-1} \delta_{X_x^j}\right)$$

as a measure on $\mathbb{R}$. Finally, we integrate m_x with respect to the measure ν_0 on x, thus yielding a Radon measure $\nu := \int m_x \, d\nu_0(x)$ on $\mathbb{R}$. Formally speaking, for any compactly supported function ϕ, we have

$$\int_{\mathbf{R}} \phi \, d\nu = \int_{\mathbb{R}} \mathbb{E}\left(\sum_{j=0}^{T(w)-1} \phi(X_x^j)\right) d\nu_0(x). \tag{4.14}$$

Notice that the right-side expression in (4.14) is well-defined and finite. Indeed, there exist $N \in \mathbb{N}$ and $p_0 > 0$ such that with probability at least p_0 a trajectory starting at any point of $\mathrm{supp}(\phi)$ hits K in at most N steps. Therefore, the distribution of the measure $m_x(w)$ on $\mathrm{supp}(\phi)$ (*i.e.* the number of steps that are spent in $\mathrm{supp}(\phi)$ until the stopping time) has an exponentially decreasing tail. Thus, its expectation is finite and bounded uniformly on $x \in \mathrm{supp}(\phi)$, which implies the finiteness of the integral.

Next, let us check that the measure ν is P-invariant. To do this, let us rewrite the measure ν as follows. First, notice that, by definition, we have

$$m_x = \sum_{n \geq 0} \sum_{g_1, \ldots, g_n \in G} \left[\prod_{j=1}^{n} \rho(g_j) \prod_{j=1}^{n} \big[1 - \xi(g_j \cdots g_1(x))\big]\right] \delta_{g_n \cdots g_1(x)}.$$

Thus,

$$\begin{aligned}
P(m_x) &= \sum_{g \in G} \rho(g)\, g_*(m_x) \\
&= \sum_{g \in G} \rho(g)\; g_* \Bigg(\sum_{n \geq 0} \sum_{g_1,\ldots,g_n \in G} \Bigg[\prod_{j=1}^{n} \rho(g_j) \prod_{j=1}^{n} \big[1 - \xi(g_j \cdots g_1(x))\big] \Bigg] \delta_{g_n \cdots g_1(x)} \Bigg) \\
&= \sum_{n \geq 0} \sum_{g_1,\ldots,g_n, g \in G} \Bigg(\rho(g) \prod_{j=1}^{n} p(g_j) \Bigg) \prod_{j=1}^{n} \big[1 - \xi(g_j \cdots g_1(x))\big]\; g_*\big(\delta_{g_n \cdots g_1(x)}\big) \\
&= \sum_{n \geq 0} \sum_{g_1,\ldots,g_n, g_{n+1} \in G} \Bigg(\prod_{j=1}^{n+1} \rho(g_j) \Bigg) \prod_{j=1}^{(n+1)-1} \big[1 - \xi(g_j \cdots g_1(x))\big]\; \delta_{g_{n+1} g_n \cdots g_1(x)}.
\end{aligned}$$

As before, the last expression equals the expectation of the random measure $\sum_{j=1}^{T(\omega)} \delta_{X_x^j}$. In this sum, we are counting the stopping time, but not the initial one. Therefore,

$$Pm_x = m_x - \delta_x + \mathbb{E}(\delta_{Y_x}).$$

By integrating with respect to ν_0, this yields

$$\begin{aligned}
P\nu = P\Big(\int_{\mathbb{R}} m_x \, d\nu_0(x) \Big) = \int_{\mathbb{R}} P(m_x)\, d\nu_0(x) = \\
\int_{\mathbb{R}} m_x \, d\nu_0(x) - \int_{\mathbb{R}} \delta_x \, d\nu_0(x) + \int_{\mathbb{R}} \mathbb{E}(\delta_{Y^x})\, d\nu_0(x) = \nu - \nu_0 + P_\xi(\nu_0).
\end{aligned}$$

Since ν_0 is P_ξ-invariant, we finally obtain $P\nu = \nu$, as we wanted to show. □

Theorem 4.3.20. *Every minimal action $\Phi : \Gamma \to Homeo_+(\mathbb{R})$ is topologically conjugate to a ρ-harmonic action.*

Proof. Take a P-invariant Radon measure ν whose existence follows from Theorem 4.3.19. By Lemma 4.3.5, it is bi-infinite. By Lemma 4.3.11, it has no atoms. Finally, by Lemma 4.3.12, it has total support. As a consequence, there exists a homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi_*(\nu)$ is the Lebesgue measure. The conjugate action $\varphi \circ \Phi \circ \varphi^{-1}$ is then ρ-harmonic. □

When the action of Γ admits discrete orbits, we know that every stationary Radon measure must be Γ-invariant. However, two such measures may be supported on different orbits. We next establish the uniqueness (up to a scalar factor) of the stationary measure in the case there is no discrete orbit. Recall that, in this case, there exists a unique nonempty, closed, minimal Γ-invariant set $\mathcal{M}$ (see Lemma 3.5.18).

Proposition 4.3.21. *Assume that there is no discrete orbit for the Γ-action on the line. Then the P-invariant Radon measure ν is unique up to a scalar factor.*

We begin the proof by some reductions. First, we can assume that the action is minimal, since stationary measures are supported on $\mathcal{M}$ (*c.f.* Lemma 4.3.12) and the action semiconjugates into a minimal one. Moreover, after changing coordinates (by a topological conjugacy), we may assume that the Lebesgue measure is stationary, that is, that the action is ρ-harmonic.

Recall that a P-invariant measure is said to be ***ergodic*** if every Γ-invariant measurable subset of the line either has measure 0 or its complement has measure 0. Every P-invariant measure decomposes as an integral of ergodic measures. Thus, to prove Theorem 4.3.21, it suffices to show that there exists a unique ergodic harmonic measure, up to multiplication by a constant.

Lemma 4.3.22. *Assume that the action of Γ is minimal and ρ-stationary. Let ν be an ergodic P-invariant measure. Then for all continuous functions ϕ, ψ with compact support, with $\phi \geq 0$ and $\phi \equiv 1$ on the recurrence interval K given by Corollary 4.3.15, and for every $x \in \mathbb{R}$, it almost surely holds*

$$\frac{S_k\psi(x,\omega)}{S_k\phi(x,\omega)} \longrightarrow \frac{\int \psi \, d\nu}{\int \phi \, d\nu} \tag{4.15}$$

as n tends to infinity, where $S_k\psi(x,\omega) := \psi(X_x^0) + \psi(X_x^1) + \ldots + \psi(X_x^{k-1})$ (and similarly for $S_k\phi$).

For the proof, we will apply Hopf's ratio ergodic theorem [90] (see also [99]) to the system $(\mathbb{R}^{\mathbb{N}_0}, \sigma, \hat{\nu})$, where σ is the shift operator $\sigma(X^n)_n = (X^{n+1})_n$, and $\hat{\nu}$ is the image of the measure $\nu \times \rho^{\mathbb{N}}$ under the map

$$\big(x, \omega = (g_n)\big) \mapsto (X_x^0 = x, X_x^1, \ldots, X_x^n, \ldots).$$

We leave as an exercise to the reader to verify that $\hat{\nu}$ is invariant by σ. (Actually, this is nothing but a reformulation of the fact that ν is P-invariant).

We claim that the system $(\mathbb{R}^{\mathbb{N}_0}, \sigma, \hat{\nu})$ is ***ergodic***, that is, every measurable σ-invariant subset A of $\mathbb{R}^{\mathbb{N}_0}$ has either zero or full $\hat{\nu}$-measure. Indeed, for such an

A, and for a fixed $x \in \mathbb{R}$, let $p_A(x)$ be the probability that the sequence $(X_x^n)_{n \geq 0}$ belongs to A. The thus defined function $p_A \colon \mathbb{R} \to [0,1]$ is measurable. Since A is σ-invariant, the property of belonging to A does only depend on the tail of the sequence, hence the function p_A is P-invariant. (This can be easily checked by the reader.) We claim that this function is indeed constant. To prove this, notice that we cannot directly apply Exercise 4.3.6, because the function p_A has no reason to belong to $\mathcal{L}^1(\mathbb{R}, \nu)$. To overcome this difficulty, let us consider a compact interval I containing the recurrence interval K. Given a point $x \in I$, we denote by $Y_x^1, \ldots, Y_x^m \ldots$ the points of the sequence $X_x^1, \ldots, X_x^n, \ldots$ that belong to I. As we are assuming that the Lebesgue measure is P-stationary, the Markov process Y on I leaves invariant the restriction of the Lebesgue measure on I. Moreover, the restriction of the function p_A is still harmonic for the Markov process Y, namely we have $p_A(x) = \mathbb{E}(p_A(Y_x^1))$ for every $x \in I$. The Lebesgue measure of I being finite, an easy extension of Exercise 4.3.6 for Markov processes shows that p_A is almost-surely constant on I. As this is true for every compact interval I containing K, we conclude that p_A is almost everywhere constant, as was claimed. Now, the $0-1$ law shows that this constant is either 0 or 1, thus showing that A has measure 0 or its complementary has measure 0. This concludes the proof that the system $(\mathbb{R}^n, \sigma, \hat{\nu})$ is ergodic.

Next, let $\phi \colon \mathbb{R} \to \mathbb{R}$ be a non-negative function with compact support such that $\phi \equiv 1$ on the recurrence interval K. Then, letting $\widehat{\phi}(x, (X^n)_{n \geq 1}) := \phi(x)$, the function $\widehat{\phi}$ belongs to $L^1(\mathbb{R}^{\mathbb{N}_0}, \widehat{\nu})$, and the recurrence property implies that for $\hat{\nu}$-almost-every $(x, (X^n))$, we have

$$\sum_{k \geq 0} \widehat{\phi}\big(\sigma^k(x, (X^n)_n)\big) = \infty.$$

Hopf's ratio ergodic theorem then states that for every function $\widehat{\psi} \in \mathcal{L}^1(\mathbb{R}^{\mathbb{N}_0}, \widehat{\nu})$, almost surely we have the convergence

$$\frac{\widehat{\psi} + \widehat{\psi} \circ \sigma + \ldots + \widehat{\psi} \circ \sigma^{k-1}}{\widehat{\phi} + \widehat{\phi} \circ \sigma + \ldots + \widehat{\phi} \circ \sigma^{k-1}} \longrightarrow \frac{\int \widehat{\psi} d\hat{\nu}}{\int \widehat{\phi} d\hat{\nu}}.$$

Applying this to a function of the form $\widehat{\psi}(x, (X^n)_n) := \psi(x)$, where $\psi \colon \mathbb{R} \to \mathbb{R}$ is continuous with compact support, and noticing that

$$\int \widehat{\phi}\, d\hat{\nu} = \int \phi\, d\nu \qquad \text{and} \qquad \widehat{\phi} + \widehat{\phi} \circ \sigma + \ldots + \widehat{\phi} \circ \sigma^{k-1}\big(x, (X^n)_n\big) = S_k\phi(x, \omega)$$

(and similarly for ψ), we conclude that (4.15) holds for ν-almost-every $x \in \mathbb{R}$.

The difficulty now is to extend (4.15) to *every* $x \in \mathbb{R}$. This will follow from the following contraction property for ρ-harmonic actions:

Lemma 4.3.23. *For any fixed number $0 < p < 1$ and all x, y, with probability at least p we have*

$$\lim_{n\to\infty} |X_x^n - X_y^n| \leq \frac{|x-y|}{1-p}.$$

Proof. Assume $y < x$ for simplicity. Since ν is P-invariant, the sequence of random variables $\omega \mapsto X_x^n - X_y^n$ is a *positive martingale*. In particular, for every integer $n \geq 1$, we have

$$\mathbb{E}(X_x^n - X_y^n) = x - y.$$

By the Martingale Convergence Theorem, the sequence $X_x^n - X_y^n$ almost surely converges to a non-negative random variable $v(x,y)$. By Fatou's inequality, we have

$$\mathbb{E}(v(x,y)) \leq \lim_{n\to\infty} \mathbb{E}(X_x^n - X_y^n) = x - y.$$

The lemma then follows from Chebyshev's inequality. $\square$

Let now $y \in \mathbb{R}$ and the functions ϕ, ψ as in the hypothesis of Lemma 4.3.22 be fixed. We claim that, for any $m \geq 1$, with probability at least $1 - 1/m$ we have

$$\limsup_{k\to\infty} \left| \frac{S_k\psi(y,\omega)}{S_k\phi(y,\omega)} - \frac{\int \psi \, d\nu}{\int \phi \, d\nu} \right| \leq \frac{1}{m}. \tag{4.16}$$

Once this established, it will obviously implies that (4.15) holds almost surely at all points, as desired.

To show (4.16), notice that since ν has total support, one can find x that is *generic*, so that (4.16) holds at x for every m. Moreover, given $\varepsilon > 0$, the point x may be chosen sufficiently close to y so that $|x - y| \leq \varepsilon$. By Lemma 4.3.23, with probability at least $1 - 1/m$ we have for all k sufficiently large, say $k \geq k_0(\omega)$,

$$|X_y^k - X_x^k| \leq m\varepsilon. \tag{4.17}$$

Next, as we already know that (with probability 1)

$$\lim_{k\to\infty} \frac{S_k\psi(x,\omega)}{S_k\phi(x,\omega)} = \frac{\int \psi \, d\nu}{\int \phi \, d\nu},$$

instead of estimating the difference in (4.16), it suffices to obtain estimates of the "relative errors"

$$\limsup_{k\to\infty}\left|\frac{S_k\psi(y,\omega)-S_k\psi(x,\omega)}{S_k\phi(x,\omega)}\right|\leq\delta_1(\varepsilon) \tag{4.18}$$

and

$$\limsup_{k\to\infty}\left|\frac{S_k\phi(y,\omega)-S_k\phi(x,\omega)}{S_k\phi(x,\omega)}\right|\leq\delta_2(\varepsilon), \tag{4.19}$$

in such a way that $\delta_1(\varepsilon)\to 0$ and $\delta_2(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Since the estimate (4.19) for ϕ is a particular case of the estimate (4.18), we will only check (4.18). Now, (4.17) implies that $|S_k\psi(y,\omega)-S_k\psi(x,\omega)|$ is at most

$$\begin{aligned}&\mathrm{mod}(m\varepsilon,\psi)\,\mathrm{card}\big\{k_0(\omega)\leq j\leq k\colon \text{either } X_x^j \text{ or } X_y^j \text{ is in } \operatorname{supp}\psi\big\}+2k_0(\omega)\max|\psi|\\ &\qquad\leq \mathrm{mod}(m\varepsilon,v)\;\mathrm{card}\{j\leq k\mid X_x^j\in U_{m\varepsilon}(\operatorname{supp}\psi)\}+\mathrm{const}(\omega).\end{aligned}$$

Here, $\mathrm{mod}(\cdot,\psi)$ stands for the modulus of continuity of ψ with respect to the distance d on the variable, and $U_{m\varepsilon}(\operatorname{supp}\psi)$ denotes the $m\varepsilon$-neighborhood of the support of ψ, again with respect to d.

Let ξ be a continuous function satisfying $0\leq\xi\leq 1$ and that is equal to 1 on $U_{m\varepsilon}(\operatorname{supp}\psi)$ and to 0 outside $U_{(m+1)\varepsilon}(\operatorname{supp}\psi)$. We have

$$\mathrm{card}\big\{j\leq k\colon X_x^j\in U_{m\varepsilon}(\operatorname{supp}\psi)\big\}\leq S_k\xi(x,\omega).$$

Thus,

$$\begin{aligned}\left|\frac{S_k\psi(y,\omega)-S_k\psi(x,\omega)}{S_k\phi(x,\omega)}\right| \quad&\leq\quad \frac{\mathrm{const}(\omega)+\mathrm{mod}(m\varepsilon,\psi)\,S_k\xi(x,\omega)}{S_k\phi(x,\omega)}\\ &\longrightarrow\quad \mathrm{mod}(m\varepsilon,\psi)\,\frac{\int\xi\,d\nu}{\int\phi\,d\nu}\quad=:\quad\delta_1(\varepsilon).\end{aligned}$$

Here, we have applied the fact that, by our choice of x, equality (4.15) holds with ξ in the numerator and ϕ in the denominator. Since $\mathrm{mod}(m\varepsilon,\psi)$ tends to 0 as $\varepsilon\to 0$ and the quotient

$$\frac{\int\xi\,d\nu}{\int\phi\,d\nu}\leq\frac{\nu(U_{(m+1)\varepsilon}(\operatorname{supp}\phi))}{\int\phi\,d\nu}$$

remains bounded, this yields $\delta_1(\varepsilon)\to 0$ as $\varepsilon\to 0$. $\square$

Having Lemma 4.3.22 at hand, it is now easy to finish the proof of Proposition 4.3.21. Indeed, given any two ergodic P-invariant Radon measures ν_1, ν_2, for each $x \in \mathcal{M}$ and every compactly supported, real-valued function ψ, almost surely we have

$$\frac{S_k \psi(x,\omega)}{S_k \phi(x,\omega)} \longrightarrow \frac{\int \psi \, d\nu_i}{\int \phi \, d\nu_i},$$

where $i \in \{1,2\}$. Thus, $\int \psi \, d\nu_1 = \lambda \int \psi \, d\nu_2$, with $\lambda := \int \phi \, d\nu_1 / \int \phi \, d\nu_2$. This proves that $\nu_1 = \lambda \nu_2$, and concludes the proof of Proposition 4.3.21.

Exercise 4.3.24. Show that the condition on ϕ in Lemma 4.3.22 can be relaxed to $\phi \geq 0$ and $\phi \not\equiv 0$.

We close with the next

Theorem 4.3.25. *The conjugacy of a minimal action to a ρ-harmonic one is unique up to post-composition by an affine map.*

Proof. Given a minimal action $\Phi \colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ and two homeomorphisms $\varphi_i \colon \mathbb{R} \to \mathbb{R}$ such that each $\varphi_i \circ \Phi \circ \varphi_i^{-1}$ is ρ-harmonic, the images of the Lebesgue measure by $\varphi_1^{-1}, \varphi_2^{-1}$ are ρ-stationary for Φ, hence they differ by multiplication by a constant. Therefore, $\varphi_2 \circ \varphi_1^{-1}$ sends the Lebesgue measure into a multiple of itself, which means that $\varphi_2 \circ \varphi_1^{-1}$ is an affine map. $\square$

Bibliography

[1] S. Adyan. Random walks on free periodic groups. *Izv. Akad. Nauk SSSR Ser. Mat.* **46** (1982), 1139-1149.

[2] I. Agol, with an appendix by I. Agol, D. Groves & J. Manning. The virtual Haken conjecture. *Doc. Math.* **18** (2013), 1045-1087.

[3] M. Babillot, P. Bougerol & L. Elie The random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case. *Annals of Probability* **25** (1997), 478-493.

[4] J. Barlev & T. Gelander. Compactifications and algebraic completions of limit groups. *J. Anal. Math.* **112** (2010), 261-287.

[5] L. Bartholdi. On amenability of group algebras, I. *Israel J. Math.* **168** (2008), 153-165.

[6] G. Baumslag. Some aspects of groups with unique roots. *Acta Math.* **104** (1960), 217-303.

[7] B. Becka, P. de la Harpe & A. Valette. *Kazhdan's Property (T).* New Mathematical Monographs, Cambridge (2008).

[8] P. Bellingeri, S. Gervais & J. Guaschi. Lower central series for Artin Tits and surface braid groups. *J. Algebra* **319** (2008), 1409-1427.

[9] N. Bergeron & D.T. Wise. A boundary criterion for cubulation. *Amer. J. Math.* **134** (2012), no. **3**, 843-859.

[10] G. Bergman. Right-orderable groups which are not locally indicable. *Pac. J. Math.* **147** (1991), 243-248.

[11] G. Bergman. Ordering coproducts of groups and semigroups. *J. Algebra* **133** (1990), 313-339.

[12] V. Bludov. An example of an unordered group with strictly isolated identity element. *Algebra and Logic* **11**, 341-349 (1972).

[13] V. Bludov, A. Glass, V. Kopytov & N. Medvedev. Unsolved problems in ordered and orderable groups. Preprint (2009), arXiv:0906.2621.

[14] C. Bonatti, I. Monteverde, A. Navas & C. Rivas. Rigidity for C^1 actions on the interval arising from hyperbolicity I: solvable groups. Preprint (2013), arXiv:1309.5277.

[15] R. Botto-Mura & A. Rhemtulla. *Orderable groups.* Lecture Notes in Pure and Applied Mathematics, Vol. **27**. Marcel Dekker, New York-Basel (1977).

[16] B. H. Bowditch. A variation on the unique product property. *J. London Math. Soc.* **62** (2000), 813-826.

[17] L. Brailovsky & G. Freiman. On a product of finite subsets in a torsion-free group. *J. Algebra* **130** (1990), 462-476.

[18] S. Boyer, D. Rolfsen & B. Wiest. Ordering three-manifold groups. *Ann. Inst. Fourier (Grenoble)* **55** (2005), 243-288.

[19] E. Breuillard, T. Gelander & P. Storm. Dense embeddings of surface groups. *Geometry and Topology* **10**, 1373-1389 (2006).

[20] M. Brin. Elementary amenable subgroups of R. Thompson's group F. *Internat. J. Algebra Comput.* **15** (2005), 619-642.

[21] S. Brodskii. Equation over groups, and groups with one defining relation. *Sibirsk. Mat. Zh.* **25** (1984), 84-103. Translation to English in *Siberian Math. Journal* **25** (1984), 235-251.

[22] S. Brofferio. How a centered rnadom walk on the affine group goes to infinity. *Ann. Inst. Henri Poincaré Probab. Stat.* **39** (2003), 371-384.

[23] K. Brown. On zero divisors in group rings. *Bull. London Math. Soc.* **8** (1976), 251-256.

[24] D. Burago & B. Kleiner. Separated nets in Euclidean space and Jacobians of bi-Lipschitz maps. *Geometric and Functional Analysis (GAFA)* **8** (1998), 273-282.

[25] J. Burillo & J. González-Meneses. Bi-orders on pure braided Thompson's groups. *Q. J. Math.* **59** (2008), 1-14.

[26] R. Buttsworth. A family of groups with a countable infinite number of full orders. *Bull. Austr. Math. Soc.* **12** (1971), 97-104.

[27] D. Calegari. Nonsmoothable, locally indicable group actions on the interval. *Algebr. Geom. Topol.* **8** (2008), no. **1**, 609-613.

[28] D. Calegari. Circular groups, planar groups, and the Euler class. *Geometry & Topology Monographs*, Volume **7**: Proceedings of the Casson Fest (2004), 431-491.

[29] D. Calegari & N. Dunfield. Laminations and groups of homeomorphisms of the circle. *Invent. Math.* **152** (2003), 149-204.

[30] D. Calegari & D. Rolfsen. Groups of PL homeomorphisms of cubes. Preprint (2014), arXiv:1401.0570.

[31] F. Calegari & N. Dunfield. Automorphic forms and rational homology 3-spheres. *Geometry and Topology* **10** (2006), 295-329.

[32] J. Cannon, W. Floyd & W. Parry. Introductory notes on Richard Thompson's groups. *L'Enseignement Mathématique* **42** (1996), 215-256.

[33] W. Carter. New examples of torsion-free non-unique product groups. *J. Group Theory* **17** (2014), no .**3**, 445-464.

[34] P.-A. Cherix, P. Jolissaint, A. Valette, M. Cowling & P. Julg *Groups with the Haagerup Property. Gromov's a-T-menability.* Progress in Mathematics, Volume **197**, Birhäuser (2001).

[35] I.M. Chiswell. Locally-invariant orders on groups. *Int. Journal Algebra and Computation* **16** (2006), 1161-1180.

[36] C. Chou. Elementary amenable groups. *Illinois J. Math.* **24** (1980), 396-407.

[37] A. Clay. Free lattice-ordered groups and the space of left orderings. *Monatsh. Math.* **167** (2012), no. **3-4**, 417-430.

[38] A. Clay. Exotic left-orderings of the free groups from the Dehornoy ordering. *Bull. Aust. Math. Soc.* **84** (2011), no. **1**, 103-110.

[39] A. Clay. Isolated points in the space of left orderings of a group. *Groups Geom. Dyn.* **4** (2010), no. **3**, 517-532.

[40] A. Clay. Cofinal elements in left orderings of the braid groups. Preprint (2009).

[41] A. Clay & D. Rolfsen. *Ordered Groups and Topology.* In press (2014).

[42] A. Clay & L. Smith. Corrigendum to: "On ordering free groups" [J. Symbolic Comput. 40 (2005) 12851290] *J. Symbolic Comput.* **44** (2009), no. **10**, 15291532.

[43] S. Cohen & A. Glass. Free groups from fields. *J. London Math. Soc.* **55** (1997), 309-319.

[44] P. Conrad. Right-ordered groups. *Mich. Math. Journal* **6** (1959), 267-275.

[45] M. I. Cortez & A. Navas. Some examples of non-rectifiable, repetitive Delone sets. Preprint (2014), arXiv:1401.7927.

[46] M. Dąbkowski, J. Przytycki & A. Togha. Non-left-orderable 3-manifold groups. *Canad. Math. Bull.* **48** (2005), 32-40.

[47] M. Darnel, A. Glass & A. Rhemtulla. Groups in which every right order is two sided. *Arch. Math.* **53** (1989), 538-542.

[48] P. Dehornoy. Monoids of O-type, subword reversing, and ordered groups. *J. Group Theory* **17** (2014), no. **3**, 465-524.

[49] P. Dehornoy. *Braids and self-distributivity.* Progress in Math. **192**, Birkhäuser (2000).

[50] P. Dehornoy, I. Dynnikov, D. Rolfsen & B. Wiest. *Ordering Braids.* Math. Surveys and Monographs **148** (2008).

[51] T. Delzant. Sur l'anneau d'un groupe hyperbolique. *C. R. Acad. Sci. Paris Série. I Math.* **324** (1997), 381384.

[52] B. Deroin. Hypersurfaces Levi-plates immergées dans les surfaces complexes de courbure positive. *Ann. Sci. École Norm. Sup.* **38** (2005), no. **1**, 57-75.

[53] B. Deroin. Almost-periodic actions on the real line. *Enseign. Math.* **59** (2013), no. **1-2**, 183-194.

[54] B. Deroin, V. Kleptsyn & A. Navas. Sur la dynamique unidimensionnelle en régularité intermédiaire. *Acta Math.* **199** (2007), 199-262.

[55] B. Deroin, V. Kleptsyn, A. Navas & K. Parwani. Symmetric random walks on $\mathrm{Homeo}_+(\mathbf{R})$. *Annals of Probability* **41** (2013), 2069-2087.

[56] Y. Derriennic. Une condition suffisante de récurrence pour des chaines de Markov sur la droite. In *Probability Measures on Groups VII Oberwolfach. Lecture Notes in Math.* **1064** (1983), 49-55.

[57] V. Dlab. On a family of simple ordered groups. *J. Austr. Math. Soc.* **8** (1968), 591-608.

[58] N. Dubrovin. An example of a chain primitive ring with nilpotent elements. *Sbornik Mathematics* **120** (1983), no. **3**, 441-447.

[59] T. Dubrovina & N. Dubrovin. On braid groups. *Sbornik Mathematics* **192** (2001), no. **5**, 693-703.

[60] G. Duchamp & J.-Y. Thibon. Simple orderings for free partially commutative groups. *Internat. J. Algebra Comput.* **2** (1992), no. **3**, 351-355.

[61] E. B. Dynkin & A.A. Yushkevich. Markov processes: Theorems and problems. *Plenum Press*, New York 1969 x+237 pp.

[62] Y. Eliashberg & L. Polterovich. Partially ordered groups and geometry of contact transformations. *Geometric and Functional Analysis (GAFA)* **10** (2000), 1448-1476.

[63] M. Falk & R. Randell. Pure braid groups and products of free groups. *Contemp. Math.* **78** (1988). 217-228.

[64] D. Farkas & R. Snider. K_0 and Noetherian group rings. *J. Algebra* **4** (1976), no. **2**, 192-198.

[65] S. Fenley, R. Feres & K. Parwani. Harmonic functions on $\mathbb{R}$-covered foliations. *Ergodic Theory and Dynam. Systems* **29** (2009), 1141-1161.

[66] E. Formanek. The zero divisor question for supersolvable groups. *Bulletin of the Australian Mathematical Society* **9** (1973), no. **1**, 69-71.

[67] G. Freiman. *Foundations of a Structural Theory of Set Addition.* American Math. Society (1973).

[68] G. Freiman, M. Herzog, P. Longobardi & M. Maj. Small doubling in ordered groups. Preprint (2013).

[69] L. Garnett. Foliations, the ergodic theorem, and Brownian motion. *J. Funct. Anal.* **51** (1983), 285-311.

[70] É. Ghys. Personal communication (2002).

[71] É. Ghys. Groups acting on the circle. *L'Enseignement Mathématique* **47** (2001), 329-407.

[72] É. Ghys. Actions de réseaux sur le cercle. *Inventiones Math.* **137** (1999), 199-231.

[73] É. Ghys & P. de la Harpe. *Sur les groupes hyperboliques d'après M. Gromov.* Progr. in Math. **83**, Birkhäuser, Boston (1990).

[74] E. Giroux. Sur la géométrie et la dynamique des transformations de contact, d'après Y. Eliashberg, L. Polterovich et al. Séminaire Bourbaki, exposé 1004, Mars 2009.

[75] A. Glass. *Partially ordered groups.* Series in Algebra **7**, Cambridge Univ. Press (1999).

[76] A. Glass. *Ordered permutation groups.* London Math. Soc. Lecture Note Series **55**, Cambridge Univ. Press (1981).

[77] J. González-Meneses. The n^{th} root of a braid is unique up to conjugacy. *Algebr. Geom. Topol.* **3** (2003), 1103-1118.

[78] R. Grigorchuk & A. Machi. On a group of intermediate growth that acts on a line by homeomorphisms. *Mat. Zametki* **53** (1993), 46-63. Translation to English in *Math. Notes* **53** (1993), 146-157.

[79] M. Gromov. Entropy and isoperimetry for linear and non-linear group actions. *Groups, Geometry, and Dynamics* **2** (2008), 499-593.

[80] F. Haglund & D. T. Wise. Special cube complexes. *Geometric and Functional Analysis (GAFA)* **17** (2008), 1551-1620.

[81] S. Hair. *New Methods for Finding Non-Left-Orderable and Unique Product Groups.* Master Thesis, Virginia Polytechnic Institute and State University (2003).

[82] Y. Hamidoune. An isoperimetric method in additive theory. *J. Algebra* **179** (1996), 622-630.

[83] Y. Hamidoune, A. Lladó & O. Serra. On subsets with small product in torsion-free groups. *Combinatorica* **18**, no. **4** (1998), 529-540.

[84] P. de la Harpe. *Topics in Geometric Group Theory.* Chicago Lectures in Mathematics (2000).

[85] J. Harrison. Unsmoothable diffeomorphisms on higher dimensional manifolds. *Proc. of the AMS* **73** (1979), 249-255.

[86] J. G. Hocking & G. S. Young. *Topology.* Addison-Wesley Publishing Co., Inc., Reading, Mass.-London (1961).

[87] W.C. Holland (ed). *Ordered Groups and Infinite Permutation Groups.* Math. and its Applications, Kluwer Acad. Publ. (1996).

[88] W.C. Holland. Partial orders on the group of automorphisms of the real line. Proceedings of the Malcev Conference, Novosibirsk. In *Contemporary Math.* **131** (1992), 197-207.

[89] W. C. Holland. The lattice-ordered group of automorphisms of an ordered set. *Michigan Math. J.* **10** (1963), 399-408.

[90] E. Hopf. *Ergodentheorie.* Ergebnisse der Mathematik, Springer-Verlag, Berlin, **5**, (1937).

[91] J. Howie. On locally indicable groups. *Math. Z.* **180** (1982), 445-461.

[92] J. Howie & H. Short. The band-sum problem. *J. London Math. Soc.* **31** (1985), 571-576.

[93] T. Ito. Isolated left orderings on amalgamated free products. Preprint (2014), arXiv:1405.1163.

[94] T. Ito. Construction of isolated left orderings via partially central cyclic amalgamation. To appear in *Tohoku Math. Journal*, arXiv:1107.0545.

[95] T. Ito. Dehornoy-like left orderings and isolated left orderings. *J. Algebra* **374** (2013), 42-58.

[96] S. Ivanov & R. Mikhailov. On zero divisors in group rings of groups with torsion. To appear in *Canadian Math. Bull.*, arXiv:1209.1443.

[97] L. Jiménez. *Grupos Ordenables: Estructura Algebraica y Dinámica.* Master Thesis, Univ. de Chile (2008).

[98] J. Kahn & V. Markovic. Immersing almost geodesic surfaces in a closed hyperbolic three manifold. *Annals of Math.* **175** (2012), 1127-1190.

[99] T. Kamae & M. Keane. A simple proof of the ratio ergodic theorem. *Osaka J. Math.* **34** (1997), 653–657.

[100] J.H.B. Kemperman. On complexes in a semigroup. *Indag. Math.* **18** (1956), 247-254.

[101] E.I. Khukhro. *Nilpotent groups and their automorphisms.* W. de Gruyter, New York (1993).

[102] D. Kielak. Groups with infinitely many ends are not fraction groups. Preprint (2013), arXiv:1303.1656.

[103] D. Kim & D. Rolfsen. An ordering for groups of pure braids and fibre-type hyperplane arrangements. *Canadian J. Math.* **55** (2002), 822-838.

[104] S. Kionke & J. Raimbault. On geometric aspects of diffuse groups. Preprint (2014).

[105] T. Koberda. Faithful actions of automorphisms on the space of orderings of a group. *New York J. Math.* **17** (2011), 783-798.

[106] V. Kopytov. Free lattice-ordered groups. *Algebra i Logika* **18** (1979), 426-441.

[107] V. Kopytov. Free lattice-ordered groups. *Sibirsk. Mat. Zh.* **24** (1983), 120-124.

[108] V. Kopitov & N. Medvedev. *Right Ordered Groups.* Siberian School of Algebra and Logic, Plenum Publ. Corp., New York (1996).

[109] L. Lifschitz & D. Morris-Witte. Bounded generation and lattices that cannot act on the line. *Pure and Applied Mathematics Quarterly* **4** (2008), 99-126.

[110] P. Linnell. The space of left orders of a group is either finite or uncountable. *Bull. London Math. Society* **43** (2011), no. **1**, 200-202.

[111] P. Linnell. Left ordered groups with no non-abelian free subgroups. *J. Group Theory* **4** (2001), 153-168.

[112] P. Linnell. Left ordered amenable and locally indicable groups. *J. London Math. Society* **60** (1999), no. **1**, 133-142.

[113] P. Linnell & D. Morris-Witte. Amenable groups with a locally-invariant order. Preprint (2012).

[114] P. Linnell, A. Rhemtulla & D. Rolfsen. Discretely ordered groups. *Algebra and Number Theory* **3** (2009), no. **7**, 797-807.

[115] Y. Lodha & J. Tatch Moore. A geometric solution to the von Neumann-Day problem for finitely presented groups. Preprint (2014), arXiv:1308.4250

[116] P. Longobardi, M. Maj & H. Rhemtulla. Groups with no free subsemigroups. *Trans. Amer Math. Soc.* **347** (1995), 1419-1427.

[117] W. Lück. *L^2-invariants: Theory and Applications to Geometry and K-theory.* Springer-Verlag (2002).

[118] K. Mann. Left-orderable groups that don't act on the line. Preprint (2014), arXiv:1406.5575.

[119] S. McCleary. Free lattice-ordered groups represented as o-2 transitive ℓ-permutation groups. *Trans. Amer Math. Soc.* **290** (1985), 81-100.

[120] C.T. McMullen. Lipschitz maps and nets in Euclidean space. *Geometric and Functional Analysis (GAFA)* **8** (1998), 304-314.

[121] W. Magnus, A. Karras & D. Solitar. *Combinatorial group theory.* Interscience, New York (1966).

[122] A.I. Malcev. On partially ordered nilpotent groups. *Algebra i Logika* **2** (1962), 5-9.

[123] A.V. Malyutin. Classification of group actions on the line and the circle. *Algebra i Analiz.* **19** (2007), 156-182. Translation to English in *Petersburg Math. J.* **19** (2008), 279-296

[124] I. Marin. On the residual nilpotence of pure Artin groups. *J. Group Theory* **9** (2006), 483-485.

[125] G. Margulis. Free subgroups of the homeomorphism group of the circle. *C. R. Acad. Sci. Paris Sér. I Math.* **331** (2000), 669-674.

[126] G. Margulis. *Discrete subgroups of semisimple Lie groups.* Springer-Verlag (1991).

[127] N.Ya. Medvedev. Orders on braid groups. *Algebra and Logic* **42** (2003), 177-180.

[128] I. Mineyev. Submultiplicativity and the Hanna Neumann Conjecture. *Annals of Math.* **175** (2012), 393-414.

[129] D. Morris-Witte. The space of bi-orders on a nilpotent group. *New York Journal of Math.* **18** (2012), 261-273.

[130] D. Morris-Witte. Amenable groups that act on the line. *Algebr. Geom. Topol.* **6** (2006), 2509-2518.

[131] J. Mulholland & D. Rolfsen. Local indicability and commutator subgroups of Artin groups. Preprint (2008), arXiv:math/0606116.

[132] M. B. Nathanson. *Additive Number Theory: Inverse Problems and the Geometry of Sumsets.* Grad. Text in Math., Springer Verlag (1996).

[133] A. Navas. Groups, orders, and laws. To appear in *L'Enseignement Mathématique*, arxiv:1405.0912.

[134] A. Navas. Sur les rapprochements par conjugaison en dimension 1 et classe C^1. *Compositio Math.* **150** (2014), no. **07**, 1183-1195.

[135] A. Navas. *Groups of Circle Diffeomorphisms.* Chicago Lect. Series in Math. (2011).

[136] A. Navas. A remarkable family of left-orderable groups: central extensions of Hecke groups. *J. Algebra* **328** (2011), 31-42.

[137] A. Navas. A finitely generated, locally indicable group with no faithful action by C^1 diffeomorphisms of the interval. *Geometry and Topology* **14** (2010), 573-584.

[138] A. Navas. On the dynamics of left-orderable groups. *Ann. Inst. Fourier (Grenoble)* **60** (2010), 1685-1740.

[139] A. Navas. Growth of groups and diffeomorphisms of the interval. *Geometric and Functional Analysis (GAFA)* **18** (2008), 988-1028.

[140] A. Navas. Quelques groupes moyennables de groupes de difféomorphismes de l'intervalle. *Bull. Soc. Mex. Mat.* **10** (2004), 219-244.

[141] A. Navas. Actions de groupes de Kazhdan sur le cercle. *Ann. Sci. École Norm. Sup.* **35** (2002), no. **5**, 749-758.

[142] A. Navas & C. Rivas. Describing all bi-orders on Thompson's group F. *Groups, Geometry, and Dynamics* **4** (2010), 163-177.

[143] A. Navas & C. Rivas, with an Appendix by A. Clay. A new characterization of Conrad's property for groups orderings, with applications. *Algebr. Geom. Topol.* **9** (2009), 2079-2100.

[144] A. Navas & B. Wiest. Nielsen-Thurston orders and the space of braid orders. *Bull. London Math. Society* **43** (2011), 901-911.

[145] D. Passman. *The Algebraic Structure of Group Rings.* John Wiley & Sons (1977).

[146] K. Pierce. Extensions of right orders in solvable groups. *J. London Math. Soc.* **11** (1975), no. **1**, 81-87.

[147] J.F. Plante. On solvable groups acting on the real line. *Trans. Amer. Math. Soc.* **278** (1983), 401-414.

[148] G. Polya. Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irfahrt im Strassennetz. *Math. Ann.* **84** (1921), no. **1-2**, 149-160.

[149] S.D. Promislow. A simple example of a torsion-free nonunique product group. *Bull. London Math. Soc.* **20** (1988), 302-304.

[150] M. Raghunathan. *Discrete Subgroups of Lie Groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **68**, Springer-Verlag, New York-Heidelberg (1972).

[151] A. Rhemtulla & D. Rolfsen. Local indicability in ordered groups: braids and elementary amenable groups. *Proc. Amer. Math. Soc.* **130** (2002), 2569-2577.

[152] E. Rips & Y. Segev. torsion-free groups without unique product property. *J. Algebra* **108** (1987), 116-126.

[153] C. Rivas. Left-orderings on free products of groups. *J. Algebra* **350** (2012), 318-329.

[154] C. Rivas. On groups with finitely many Conradian orderings. *Comm. Algebra* **40** (2012), no. **7**, 2596-2612.

[155] C. Rivas. On spaces of Conradian group orderings. *J. Group Theory* **13** (2010), 337-353.

[156] C. Rivas & R. Tessera. On the space of left-orderings of virtually solvable groups. Preprint (2013), arXiv:1209.3251.

[157] T. Robbiano. Terms orderings on the polynomial ring. In: *Eurocal'85, Vol. II, Springer LNCS* **204** (1985), 513-517.

[158] D.J.S Robinson. *Finiteness condition and generalized solvable groups 1-2.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol **63**, Springer-Verlag, New York-Heidelberg (1972).

[159] V.A. Rohlin. On the fundamental ideas of measure theory. *Amer. Math. Soc. Translations* **71** (1952), 1-54. Translated from *Math. Sbornik NS* **25** (1949), 107-150.

[160] D. Rolfsen & J. Zhu. Braids, ordered groups and zero divisors. *J. Knot Theory and Ramifications* **7** (1998), 837-841.

[161] A. Selberg. On discontinuous groups in higher-dimensional symmetric spaces. In *Contributions to Function Theory*, Tata Institute of Fundamental Research, Bombay (1960), 147-164.

[162] A. Sikora. Topology on the spaces of left-orders of groups. *Bull. London Math. Soc.* **36** (2004), 519-526.

[163] D. Smirnov. Right orderable groups. *Algebra i Logica* **5** (1966), 41-69.

[164] M. Steenbock. Rips-Segev torsion-free groups without unique product. Preprint (2013), arXiv:1307.0981.

[165] A. Strojnowski. A note on u.p. groups. *Comm. Algebra* **3** (1980), 231-234.

[166] Z. Šunić. Explicit left orders on free groups extending the lexicographic order on free monoids. *C. R. Math. Acad. Sci. Paris* **351** (2013), no. **13-14**, 507-511.

[167] T. Tao & V. H. Hu. *Additive Combinatorics.* Cambridge Studies in Advances Mathematics **105**, Cambridge Univ. Press (2009).

[168] V. Tararin. On groups having a finite number of orders. Dep. Viniti (Report), Moscow (1991).

[169] H. Teh. Construction of orders in abelian groups. *Proc. Camb. Phil. Soc.* **57** (1961), 467-482.

[170] W. Thurston. A generalization of the Reeb stability theorem. *Topology* **13** (1974), 347-352.

[171] N. Varopoulos, L. Saloff-Coste & T. Coulhon. *Analysis and Geometry on Groups.* Cambridge Tracts in Mathematics, vol. **100**, Cambridge University Press, Cambridge (1992).

[172] A.A. Vinogradov. On the free product of ordered groups. *Sbornik Mathematics* **25** (1949), 163-168.

[173] D. Witte. Arithmetic groups of higher $\mathbb{Q}$-rank cannot act on 1-manifolds. *Proc. Amer. Math. Soc.* **122** (1994), 333-340.

[174] S. Wolfenstein (ed.) *Algebra and Order.* Proceedings of the First International Symposium on Ordered Algebraic Structures, Luminy, Marseille (1984). Research and Exposition in Mathematics **14**, Heidemann Verlag, Berlin (1986).

[175] A. Zenkov. On groups with an infinite set of right orders. *Sibirsk. Mat. Zh.* **38** (1997), 90-92. Translation to English in *Siberian Math. Journal* **38** (1997), 76-77.

[176] R. Zimmer. *Ergodic Theory of Semisimple Groups.* Monographs in Mathematics, Birkhäuser (1984).