Discrete Mathematics 21 (1978) 219-221.

NOTE

ON GENERALISED CATALAN NUMBERS

A. D. SANDS
The University, Dundee DDI 4HN, Scotland

Received 13 July 1976
Revised 18 March 1977

The Catalan number C_n is defined to be $\binom{2n}{n}/(n+1)$. One of its occurrences is as the number of ways of bracketing a product of $n+1$ terms taken from a set with binary operation. In this note the corresponding result for a set with a k-ary operation is considered. A combinatorial proof is given which does not involve generating functions or inversion formulae. The result is further generalised to obtain a simpler proof of a formula of Erdelyi and Etherington [2], interpreted here as a result concerning a set with several k_i-ary operations.

For a set with a non-associative binary operation the proof that the Catalan number C_n is the number of ways in which a product of $n+1$ terms may be bracketed is usually given [1, p. 52–53; 3, p. 18–19] using recurrence formulae, generating functions and inversion theorems. More direct proofs have been given by Silberger [4] and Singmaster [5]. Here we consider first a set E with a k-ary operation. To avoid brackets we use the backwards Polish notation, where the occurrence of μ indicates that the previous k terms are to be multiplied. Thus for $k=3$ and 5 terms we have the following possibilities:

$$abcde\mu\mu = ab(cde); \quad abcd\mu e\mu = a(bcd)e;$$
$$abc\mu de\mu = (abc)de.$$

Clearly not every sequence of μ's and elements of E can be represented by a product. Those which do correspond to products will be termed "representable" sequences. The result is obtained by finding the proportion of sequences which are representable.

Proposition. *A sequence involving $r\mu$'s and $n+1$ members of E is representable if and only if $n=(k-1)r$ and the number of members of E preceding any point in the sequence exceeds $k-1$ times the number of μ's preceding this point.*

Proof. This is clear when $r=1$ and is then a routine proof by induction on r.

Lemma. *Given any sequence involving $r\mu$'s and $(k-1)r+1$ members of E then exactly one of the $kr+1$ sequences obtained from it by cyclic permutation is representable.*

Proof. Suppose that two such sequences are representable and let them be

$$x_1, \ldots, x_m, \ldots, x_{kr+1} \quad \text{and} \quad x_{m+1}, \ldots, x_{kr+1}, x_1, \ldots, x_m.$$

Then from the first sequence we deduce that the number of members of E in $x_1, \ldots, x_m$ exceeds $(k-1)$ (number of μ's here) and from the second sequence that the number of members of E in $x_{m+1}, \ldots, x_{kr+1}$ exceeds $(k-1)$ (numbers of μ's here). Together these imply $(k-1)r+1 \geqslant (k-1)r+2$. Thus there cannot be two representable sequences.

Now suppose that none of these sequences is representable. Then, starting from x_1, there exists a least $m_1 > 0$ such that the number of members of E in $x_1, \ldots, x_{m_1}$ does not exceed $(k-1)$ (number of μ's here). Going through the sequence cyclically there exists $m_2 > 0$ such that the same result holds for $x_{m_1+1}, \ldots, x_{m_1+m_2}$. When this process is repeated cyclically then eventually we must use the same breakpoint in the sequence for a second time. Then counting the number of members of E and the number of μ's between these two occurrences of this breakpoint we find that $(k-1)r+1 \leqslant (k-1)r$.

This contradiction completes the proof of the lemma.

It follows that the number of representable sequences is $1/(kr+1)$ times the number of sequences involving $(k-1)r+1$ members of E in order and $r\mu$'s. This gives the generalised Catalan number as

$$((kr+1)!\,/((k-1)r+1)!\,r!\,)/(kr+1) = \binom{n+r}{r}\bigg/(n+1),$$

where $n=(k-1)r$. When $k=2$ we have $n=r$ and so obtain the usual Catalan number C_n.

The first occurrence of the Catalan numbers was obtained by Euler, who was counting the subdivisions of a convex polygon into triangles using chords. Erdelyi and Etherington [2] considered a generalisation of this to a subdivision into polygons with the restriction that polygons with a given number of sides should be used a fixed number of times. The corresponding problem is to consider a set E with k_i-ary operations μ_i, $i=1, \ldots, q$ and ask for the number of representable sequences involving elements of E and μ_i's where μ_i is to be used exactly r_i times.

As before it is a routine proof by induction to show that such a sequence with $n+1$ members of E is representable if and only if $n=\sum(k_i-1)r_i$ and preceding any point in the sequence the number of members of E exceeds $\sum(k_i-1)$ (number of occurrences of μ_i here). The Lemma then generalises, with exactly the same ideas involved in the proof, to show that with $n=\sum(k_i-1)r_i$, exactly one representable sequence is obtained among the cyclic permutations of a given sequence. This gives the required number of representable sequences to be

$$\left(n+\sum r_i\right)!/(n+1)!\,r_1!\ldots r_q!\,.$$

This agrees with formula (8) of [2] when the appropriate changes in notation are made, including the fact that a sequence with $n+1$ terms corresponds to a polygon with $n+2$ sides.

References

[1] L. Comtet, Advanced Combinatorics (Reidel, Dordrecht, 1974).
[2] A. Erdelyi and I.H.M. Etherington, Some problems of non-associative combinations (2). Edinburgh Math. Notes 32 (1940) vii–xii.
[3] N. Jacobson, Lectures in abstract algebra, Vol. 1 (Van Nostrand, New York, 1951).
[4] D.M. Silberger, Occurrences of the integer $(2n-2)!/n!(n-1)!$, Comment. Math. Prace Mat. 13 (1969) 92–96.
[5] D. Singmaster, An elementary evaluation of the Catalan numbers, Amer. Math. Month, (to appear).