Hypercomplex Numbers, Lie Groups, and the Creation of Group Representation Theory

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The original motivation behind this paper came from my interest[1] in the early history of group representation theory and the role played therein by the work of W. BURNSIDE and T. MOLIEN. Historical comments in works on representation theory or on the history of mathematics indicated the need for research in this area. BURNSIDE, in his widely-read *Theory of Groups of Finite Order* [1911: 269n], claimed he had "obtained independently" the chief results in FROBENIUS' early papers on characters and representations. Many authors have repeated BURNSIDE's claim verbatim[2], although it turns out to be misleading as will be seen in Section 4. What MOLIEN did is even less clear. Perhaps following the lead of BURNSIDE [1911: 269n], many authors make no mention of MOLIEN in connection with the creation of group representation theory.[3] Others link him with the theory but fail to indicate the nature of his contribution and, in referring to MOLIEN's work, omit what must be considered his most important papers from the standpoint of associating him with its actual creation.[4]

In order to obtain an accurate historical picture of what BURNSIDE and MOLIEN did, why and how they did it, and how what they did fits into the history of group representation theory, it is necessary to treat certain aspects of the history of hypercomplex number systems (*i.e.* associative algebras with identity over the complex field) and LIE's theory of groups which were involved in the discovery of the basic structure theorems usually associated—in the context

[1] Most of the research for this paper was done while the author was a Fellow of the American Council of Learned Societies at the Eidg. Technische Hochschule, Zürich, during 1969–70. I wish to thank Professors B. L. VAN DER WAERDEN and B. ECKMANN for arranging a location for me at the Forschungsinstitut für Mathematik.

[2] *E.g.* WEYL [1928: 277n.3; 1939: 29], BURROW [1965: v], WUSSING [1969: 216].

[3] *E.g.* CAJORI [1919], BURROW [1965], CURTIS & REINER [1962], BOERNER [1967], BELL [1945].

[4] *E.g.* WEYL [1928: 277n.3; 1939: 29], WUSSING [1969: 109, 154, 216].

of algebras over any field—with the name of WEDDERBURN [1908]. Consequently the first three sections can be regarded as a contribution to the history of the WEDDERBURN structure theorems.

Section 1 treats the developments that led to an interest in the relation between matrices and hypercomplex numbers and which (through POINCARÉ) stirred up interest in the connections between hypercomplex numbers and LIE's theory of groups. Also of particular importance is the recognition of the special class of hypercomplex systems which we shall term complete matrix algebras. By this we mean systems whose elements can be expressed in the form $\sum_{i,j=1}^{n} a_{ij} e_{ij}$ where the n^2 basis elements e_{ij} multiply according to the rule $e_{ij} e_{kl} = \delta_{jk} e_{il}$. Interest in complete matrix algebras and recognition that the (complex) quaternions are of this type was a necessary preliminary to, and instrumental in, the discovery of the important role they play in the general structure theory of hypercomplex systems. Section 2 focuses upon the closely related developments in LIE's theory of groups and the theory of hypercomplex numbers that formed the immediate background to, and motivation for, the contributions of MOLIEN and E. CARTAN, which are then discussed in Section 3. In Section 4 the work of MOLIEN and BURNSIDE on the representation theory of finite groups is considered in the light of the developments discussed in the previous sections.

These considerations make it clear that, although FROBENIUS probably would not have created the theory of characters and group representations without the fillip of DEDEKIND's letters[5], other developments within the mathematics of the nineteenth century would have led—in cerain cases did lead—to its creation. Multiple discoveries (in the sense of R. K. MERTON [1961]) are certainly not novelties in the history of science, but the creation of representation theory has not been studied as an instance of such. Furthermore, the nature of the multiple discovery in the case of representation theory is especially interesting because a variety of mathematical lines of thought were leading to it. MASCHKE's discovery of the complete reducibility of finite groups of linear transformations (Section 5) is a further case in point. And, by virtue of the diversity of approaches to representation theory involved in its multiple discovery, a mere duplication of FROBENIUS' work did not occur. Representation theory as it is nowadays used and presented reflects this diversity. In particular, MOLIEN'S approach to the theory (via the structure and representation theory of algebras, which is then specialized to group algebras) was again taken up by EMMY NOETHER in her influential paper "Hypercomplexe Grössen und Darstellungstheorie" [1929].

1. Hypercomplex Numbers and Matrices

In a series of seven memoirs published in CRELLE's journal [1855], ARTHUR CAYLEY (1821–95) had introduced the notion of an n-by-n matrix as a notational convenience for expressing systems of linear equations and bilinear and quadratic forms. For his purposes at the time it sufficed to define the product of two matrices, but shortly thereafter, in "A Memoir on the Theory of Matrices" [1858], CAYLEY also defined what it meant to add matrices and to multiply them by

[5] See HAWKINS [1971].

scalars. He showed, in effect, that they form a hypercomplex system, although he did not use such terminology.[6]

The notion of an n-dimensional hypercomplex system—a natural extension of the ideas of Sir WILLIAM ROWAN HAMILTON (1805–65) on algebraic couples, triplets, and quaternions—was, however, familiar to CAYLEY.[7] Indeed, in a paper in which CAYLEY introduced the notion of an abstract group, he also observed [1854: 46–7] that what is now called the group algebra of a finite group forms a system of "complex quantities" analogous in many ways to HAMILTON's quaternions. But in his memoir on matrices, CAYLEY did not bring out the connection between the algebra of matrices and hypercomplex numbers; he did not express his matrices in the form $\sum_{i=1}^{n} x_i e_i$ of a hypercomplex system. He did, however, mention in passing [1858: 33] that if L and M are 2-by-2 matrices satisfying $LM=-ML$ and $L^2=M^2=-I$, "then putting $N=LM$, we obtain

$$L^2=-I, \quad M^2=-I, \quad N^2=-I,$$
$$L=MN=-NM, \quad M=NL=-NL \text{ [sic]}, \quad N=LM=-ML,$$

which is a system of relations precisely similar to that in the theory of quaternions." CAYLEY did not bother to give examples of L and M but instead derived conditions on the entries of L and M which imply $LM=-ML$ [1858: 31]—conditions which later proved to be suggestive to J. J. SYLVESTER.

CAYLEY's memoir [1858] does not appear to have become widely known in the years immediately following its publication, nor was there widespread interest in the general theory of hypercomplex numbers. The relationship between the two theories thus remained unexplored. The work of BENJAMIN PEIRCE (1809–80) and his son, CHARLES SAUNDERS PEIRCE (1839–1914), during the 1870's, however, helped to make the relationship clear. In 1870 B. PEIRCE, who was Professor of Astronomy and Mathematics at Harvard, had 100 copies of his work *Linear Associative Algebra* [1870] lithographed and distributed among his friends. In this memoir PEIRCE concerned himself with the problem of determining all possible associative algebras over the complex field. (PEIRCE's algebras did not have to possess an identity element.) To aid him in his search, PEIRCE introduced a number of important ideas—*e.g.* the concepts of nilpotent and idempotent elements—which need not be discussed here. Although his methods were not foolproof, he was able to determine 162 algebras of dimension 6 or less. Among them he included the 4-dimensional algebra with basis i, j, k, l, with multiplication corresponding to that of $e_{11}, e_{12}, e_{21}, e_{22}$; and he noted [1870: 132] that this algebra "is a form of quaternions."

A word of explanation is necessary here. HAMILTON was primarily concerned with real quaternions, although he did briefly consider complex quaternions—

[6] Independently of CAYLEY, E. LAGUERRE [1867] and G. FROBENIUS [1878] studied what amounts to the algebra of matrices and also indicated the connection with hypercomplex numbers. FROBENIUS' paper is discussed in Section 2.

[7] See, *e.g.*, papers 11, 20, 21, 46 in CAYLEY [1889a]. HAMILTON discoursed on n-dimensional hypercomplex numbers in [1848] and in the preface to his *Lectures on Quaternions* [1853]. For the early history of hypercomplex systems, see CARTAN [1908] and CROWE [1967].

"biquaternions" as he termed them [1853: 664–65]. PEIRCE's remark reflects his recognition [1870: 105, 105n] that his algebra was simply the complex quaternions expressed in terms of a different basis. PEIRCE realized more than that. He wrote to a friend in 1870 that in *Linear Associative Algebra*, "Hamilton's quaternions appear ... in a very strange form with which a very curious philosophy is connected as I shall show in some subsequent memoir. This form leads by simple induction to a natural class of algebras, of which quaternions is the simplest, and which I shall hereafter treat under the name of quaternionoidal" (GINSBURG [1934: 280]). The quaternionoidal, or complete, matrix algebras to which PEIRCE referred were closely related to his son's work in logic. As C. S. PEIRCE later explained [1883: 413]:

> While my father was making his investigations of multiple algebra I was, in my own humble way, studying the logic of relatives and an algebraic notation for it; and in the ninth volume of the *Memoirs of the American Academy of Arts and Sciences*, appeared my first paper [1873] upon the subject. In this memoir I was led, from logical considerations that are patent to those who read it, to endeavor to put the general expression of any linear associative algebra into a certain form ....

PEIRCE was referring to the concluding section of his memoir (presented in 1870). There he considered symbols $u_1, u_2, u_3, \ldots$ representing mutually disjoint classes of individuals, pairs of which determine elementary relations:

$$\begin{array}{cccc} u_1\colon u_1 & u_1\colon u_2 & u_1\colon u_3 & \ldots \\ u_2\colon u_1 & u_2\colon u_2 & u_2\colon u_3 & \ldots \\ u_3\colon u_1 & u_3\colon u_2 & u_3\colon u_3 & \ldots \end{array}$$

The logical multiplication of these "dual relatives" (as he later termed them) was simply $(u_i\colon u_j)(u_k\colon u_l)=\delta_{jk}(u_i\colon u_l)$. PEIRCE [1873: 76–8] gave the following simple example, which helps explain his ideas. Consider all the individuals in a school. They are assumed divided into two disjoint classes: those who teach (u_1) and those who are taught (u_2). These two "universal extremes" determine 4 elementary relatives: colleague ($u_1\colon u_1$), teacher ($u_1\colon u_2$), pupil ($u_2\colon u_1$), and schoolmate ($u_2\colon u_2$). Assuming that every teacher teaches every pupil, then $(u_1\colon u_2)(u_2\colon u_2)=(u_1\colon u_2)$ means: "The teachers of a person are that person's teachers." Similarly the equation $(u_2\colon u_1)(u_2\colon u_2)=0$, means: "There are no pupils of a person's schoolmates." The symbols $(u_i\colon u_j)$, $i, j=1, 2$, of course, multiply like B. PEIRCE's basis for complex quaternions. B. PEIRCE was impressed with his son's notation for quaternionoidal or quadrate algebras and called attention to it in a paper presented to the American Academy in 1875.

It was not until early in 1882 that C. S. PEIRCE read CAYLEY's memoir on matrices (C. S. PEIRCE [1933: 186]). During the 1870's neither of the PEIRCE's seems to have been acquainted with the algebra of matrices. The relationship between quadrate algebras and CAYLEY's matrices was brought to attention by J. J. SYLVESTER (1818–97), who also created an interest in matrix theory in America and Europe. SYLVESTER had been a professor of mathematics at the Johns Hopkins University since 1876. According to SYLVESTER [1884c: 209]:

... I gave the first course of lectures ever delivered on Multinomial Quantity [that is, matrix algebra], in 1881, at the Johns Hopkins University. Much as I owe in the way of fruitful suggestion to Cayley's immortal memoir [1858], the idea of subjecting matrices to the additive process and of their consequent amenability to the laws of functional operation was not taken from it, but occurred to me independently before I had seen the memoir or was acquainted with its contents; and indeed forced itself upon my attention as a means of giving simplicity and generality to my formula for the powers of [characteristic] roots of matrices, published in the *Comptes Rendus* ... [1882a]. My memoir [1881] on Tchebycheff's method concerning the totality of prime numbers within certain limits, was the indirect cause of turning my attention to the subject, as (through the systems of difference-equations therein employed to contract Tchebycheff's limits) I was led to the discovery of the properties of the latent [*i.e.* characteristic] roots of matrices, and had made considerable progress in developing the theory of matrices considered as quantities, when on writing to Prof. Cayley upon the subject he referred me to the memoir in question ...."

Upon reading CAYLEY'S memoir, SYLVESTER became interested in CAYLEY'S characterization of the matrices satisfying $uv = -vu$. In the course of his lectures on matrix algebra in 1881, he discovered that $uv = -vu$ and $u^2 = v^2 = -1$ if and only if $\det(z + yu + xv) = x^2 + y^2 + z^2$. For 3-by-3 matrices SYLVESTER thought he had obtained an analogous result [1882b]: $uv = \varrho vu$ and $u^3 = v^3 = -1$ (ϱ a primitive cube root of unity) if and only if $\det(z + yu + xv) = x^3 + y^3 + z^3$.[8] SYLVESTER also provided an example of matrices, u, v satisfying these conditions, *viz.*

$$u = \begin{pmatrix} 0 & 0 & 1 \\ \varrho & 0 & 0 \\ 0 & \varrho^2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 & 1 \\ \varrho^2 & 0 & 0 \\ 0 & \varrho & 0 \end{pmatrix}.$$

These matrices led him to the discovery that [1882b: 648–9] "there will be a system of Nonions (precisely analogous to the known system of quaternions) represented by the 9 matrices [$1, u, v, uv, u^2, v^2, u^2v, uv^2, u^2v^2$] ...."

SYLVESTER'S papers on matrices led C. S. PEIRCE to the realization that SYLVESTER "appears to have come, by a line of approach totally different from mine, upon a system which coincides, in some of its main features, with the Algebra of Relatives ..." [1933: 414]. SYLVESTER was also the founder and editor-in-chief of the *American Journal of Mathematics*, and in 1881 B. PEIRCE'S *Linear Associative Algebra* and later paper (1875) calling attention to quadrate algebras were published in that journal, with footnotes and addenda by C. S. PEIRCE, who was then a lecturer in logic at Johns Hopkins. It is in this form that PEIRCE'S original memoir became known, and the additions were important because they stressed the matrix representation of algebras. Already in 1870 C. S. PEIRCE had begun putting his father's algebras in "relative form"—*i.e.* expressing the basis elements as linear combinations of dual relatives $(u_i : u_j)$—

[8] Later SYLVESTER discovered [1884a: 124] this result is not correct; the condition $(uv)^3 = 1$ must be included.

and had conjectured [1873: 80–1] "upon reasonable inductive evidence, that all such algebras can be interpreted on the principles of the present notation in the same way ... ." For the publication of his father's memoir in the *American Journal*, PEIRCE systematically added footnotes to each of his father's algebras in which he exhibited a representation in relative form; and in one of the addenda, "On the Relative Forms of the Algebras" [1881: 221–25], he proved that any associative algebra can be put in relative form.[9]

While PEIRCE learned of the matrix interpretation of his ideas through SYLVESTER'S work, SYLVESTER in turn thus learned from PEIRCE the "method by which a matrix is robbed of its areal dimensions and represented as a linear sum ... ." SYLVESTER made this remark in a paper in the *American Journal of Mathematics* [1884: 220] and followed it with a detailed discussion of the manner in which 2-by-2 and 3-by-3 matrices can be expressed as linear combinations of the e_{ij} units. (SYLVESTER did not use the e_{ij} notation.)

Many of the above-discussed ideas on matrices, hypercomplex numbers, and their relation were called to the attention of continental mathematicians when SYLVESTER published two notes [1883, 1884b] on nonions in the *Comptes Rendus* of the Académie des Sciences. SYLVESTER pointed out explicitly that HAMILTON'S quaternions i, j, k can be represented by matrices, that, conversely, every 2-by-2 matrix is a linear combination of $1, i, j, k$, and that similarly, any 3-by-3 matrix can be expressed as a linear combination of the 9 nonions. The work of SYLVESTER and the PEIRCES, with which continental mathematicians became acquainted, thus called attention to the matrix viewpoint, to the fact that hypercomplex numbers can be represented by matrices, and to the fact that certain hypercomplex systems, including the quaternions and nonions, were only disguised forms of complete matrix algebras.

2. Hypercomplex Numbers and Lie Groups

SYLVESTER'S numerous notes in the *Comptes Rendus*, including those on nonions, brought the notions of matrices and hypercomplex numbers to the attention of continental mathematicians.[10] The response was immediate. EDUARD WEYR (1852–1903), a Czechoslovakian mathematician, published several notes on quaternions [1884] and matrix theory [1885a, 1885b] in the *Comptes Rendus* and also pointed out elsewhere [1887] that every hypercomplex system could be represented as matrices (the matrices of the regular representation). WEYR'S observation, of course, had already been made by C. S. PEIRCE. It had also been made in 1884 by HENRI POINCARÉ (1854–1912) and in a manner which stirred up interest in hypercomplex numbers because it related them to the theory of transformation groups. Discontinuous transformation groups were being studied at the time by POINCARÉ and FELIX KLEIN (1849–1925), while SOPHUS LIE

[9] The relative form used by C. S. PEIRCE in many of the footnotes was that given by the regular representation. Likewise, the relative form used in his general proof is that obtained from the regular representation of the algebra obtained by formally adjoining an identity (to rule out $ax = 0$ for all x and $a \neq 0$), a procedure B. PEIRCE had indicated [1870: 118].

[10] During the period 1882–84, 13 papers on quaternions and matrices—some in several installments—were published by SYLVESTER in the *Comptes Rendus*. See SYLVESTER [1909, 1912].

(1842–99) in Christiania (now Oslo), who had just visited Paris in 1882, was devoting all his energy to the creation of a far-reaching theory of continuous groups. In the *Comptes Rendus* [1884: 740] POINCARÉ observed that:

> The remarkable works of M. Sylvester on matrices have recently drawn attention again to complex numbers analogous to Hamilton's quaternions. The problem of complex numbers is easily reduced to the following:
>
> *To find all the continuous groups of linear substitutions in n variables, the coefficients of which are linear functions of n arbitrary parameters.*

In other words, every element $u=\sum_{i=1}^{n} u_i e_i$ of a hypercomplex system $\mathscr{H}$ determines a linear transformation u_R: $x \to xu$. If $\mathscr{H}$ is defined by structural constants a_{ij}^k where

$$e_i e_j = \sum_{k=1}^{n} a_{ij}^k e_k \qquad (i, j = 1, \dots n),$$

then the coordinate equations of u_R are

$$x'_k = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} a_{ij}^k u_j\right] x_i,$$

so that they are linear and homogeneous in both variables and parameters. It was undoubtedly SYLVESTER's matrix representation of quaternions and nonions that prompted POINCARÉ's remark. In fact, the examples that POINCARÉ gave of "bilinear groups" (as we shall call them following E. CARTAN) were presented in matrix form.

In his note POINCARÉ introduced the characteristic polynomial of an element $u \in \mathscr{H}$ (namely that of u_R) and posited some results about noncommutative systems which hinted at the possibility of using this polynomial to get at the structure of $\mathscr{H}$, but he became absorbed in other work and did not develop his ideas further (POINCARÉ [1934: 167]). POINCARÉ's note, however, made an impression upon LIE. LIE had worked in relative isolation in Christiania, but in 1886 he accepted a position at the University of Leipzig in order to make his work better known and to have students. At Leipzig LIE repeatedly invited his listeners to apply the theorems of his theory of groups to hypercomplex systems (SCHEFFERS [1891: 387]). Interest in hypercomplex numbers in the mid-1880's had also increased as a result of a series of papers on commutative hypercomplex systems in the Göttingen *Nachrichten*, including papers by such prominent mathematicians as WEIERSTRASS and DEDEKIND.[11] LIE's first student at Leipzig, GEORG SCHEFFERS (1866–1945), had spent some time studying hypercomplex numbers, possibly as a result of the *Nachrichten* papers. And so in his seminar LIE assigned SCHEFFERS the problem of investigating POINCARÉ's bilinear groups, emphasizing that his earlier calculations of all projective groups in two variables, made in September of 1884, made possible the determination

[11] WEIERSTRASS [1884], DEDEKIND [1885, 1887], HÖLDER [1886], PETERSEN [1887]. The papers of WEIERSTRASS and DEDEKIND also played a role in FROBENIUS' creation of the theory of group characters. See HAWKINS [1971].

of the bilinear groups in 3 variables (SCHEFFERS [1889a: 290n]). Using LIE'S work, SCHEFFERS obtained all 5 3-dimensional hypercomplex systems [1889a]. In subsequent papers [1889b, 1891], however, he dropped the approach of his first paper for more direct methods.

EDUARD STUDY (1862–1930) also focused attention upon the connection between hypercomplex numbers and LIE'S theory of groups.[12] STUDY had studied at Leipzig and was a friend of FRIEDRICH ENGEL (1861–1941), LIE'S right-hand man and amanuensis for the composition of his monumental *Theorie der Transformationsgruppen* [1888, 1890, 1893a]. Study was also a student of GRASSMANN'S *Ausdehnungslehre* (1861), but his interest in hypercomplex numbers also derived in part from the above-mentioned papers in the Göttingen *Nachrichten*. In his first paper on hypercomplex numbers [1889a], STUDY developed some ideas in WEIERSTRASS' paper [1884] into a procedure for determining all 4-dimensional systems and certain n-dimensional systems. He also called attention in this paper to the work of the PEIRCES [1881]. In his second paper on hypercomplex numbers [1889b], STUDY turned to their connection with transformation groups and made POINCARÉ'S observation more precise. It is easily seen that the bilinear group determined by a hypercomplex system is simply transitive. STUDY proved the converse: In any LIE group of linear transformations which is simply transitive the parameters may always be chosen so that the group is bilinear.[13]

STUDY'S work also led to the introduction of the minimal polynomial of a hypercomplex system [1889a: 241]. As B. PEIRCE and WEIERSTRASS had already observed, in an n-dimensional system the powers $u^0=1, u, u^2, u^3, \ldots$ of $u\in\mathscr{H}$ cannot be linearly independent. Thus, associated with $\mathscr{H}$ is an integer k, $2\leqq k\leqq n$, such that: (1) For all $u\in\mathscr{H}$, u^k is a linear combination of $1, u, u^2, \ldots u^{k-1}$; (2) There exists a $u\in\mathscr{H}$ such that $1, u, u^2, \ldots u^{k-1}$ are linearly independent. In other words, for every $u\in\mathscr{H}$ there exist numbers $a_i=a_i(u)$ such that $u^k+a_1u^{k-1}+\cdots+a_1u+a_0 1=0$; the polynomial $q_u(\omega)=\omega^k+a_1\omega^{k-1}+\cdots+a_1\omega+a_0$ is the minimal polynomial of the transformation u_R for u defined in (2) above. STUDY did not mention minimal polynomials of transformations or matrices—his introduction of k represented a generalization of WEIERSTRASS' work, which had involved systems for which $k=n$—, but this notion had been introduced by FROBENIUS [1878] and also by SYLVESTER [1884b] in his discussion of matrices and nonions in the *Comptes Rendus*.

SCHEFFERS papers reflected a greater interest in the structure of $q_u(\omega)$—*i.e.* the nature of its factorization over $\mathscr{C}(u_1, \ldots u_n)$—and its relation to the structure of $\mathscr{H}$. In his first paper [1889a] he considered $\Delta(u)=\det u_R$. (When $\mathscr{H}$ is a group algebra, $\Delta(u)$ is, of course, the group determinant considered by DEDEKIND and FROBENIUS; see HAWKINS [1971].) In general, if $\mathscr{H}$ is n-dimensional and

[12] STUDY did not actually read POINCARÉ'S note [1884] until after he had done his own work and apparently had been led to consider the connection between LIE groups and hypercomplex numbers independently (ENGEL [1931: 147]).

[13] LIE [1888: 213] defined a transformation group in variables $x_1, \ldots x_n$ to be transitive "if in the space $(x_1 \ldots x_n)$ there exists an n-fold extended region within which every point is carried over into any other by at least one transformation of the group." It is simply transitive if in addition the number of essential parameters is also n. For the bilinear group determined by $\mathscr{H}$, the n-dimensional region is the open set of all $u\in\mathscr{H}$ such that $\det u_R\neq 0$.

$u = \sum_{i=1}^{n} u_i e_i$, $\Delta(u)$ is a homogeneous function of degree n. SCHEFFERS called attention to the fact that, for $n=3$, $\Delta(u)$ always factors into linear homogeneous factors in the u_i. The same, he noted, is true for $n=4$ *with one exception*, namely the quaternions for which $\Delta(u) = (u_1^2 + u_2^2 + u_3^2 + u_4^2)^2$, $u = u_1 + u_2 i + u_3 j + u_4 k$, so that $\Delta(u)$ is the square of an irreducible *second-degree* factor.

In his next paper [1889b], SCHEFFERS switched from consideration of $\Delta(u)$ to that of $q_u(\omega)$, influenced in this regard by STUDY'S paper. By introducing the distinction between what he (later) termed quaternion and nonquaternion systems[14] SCHEFFERS developed methods that enabled him to determine the 5-dimensional hypercomplex systems. For each such system SCHEFFERS also computed $q_u(\omega)$ [1889b: 446–57]. Of course, over the complex field $\mathscr{C}$—*i.e.* for each fixed $u \in \mathscr{H}$—$q_u(\omega)$ factors into first-degree factors, and in each case SCHEFFERS obtained this factorization. What his tables clearly showed, however, was that, except for the one system which contained the quaternions as a subsystem, $q_u(\omega)$ factors into linear factors over $\mathscr{C}(u_1, \dots u_n)$. The determination of all systems of dimension less than or equal to 5 made it possible to see the unique position of the quaternions with respect to the structure of $q_u(\omega)$. In fact, in his next paper [1891] SCHEFFERS actually computed and factored $q_u(\omega)$ for every "irreducible" system—see below for the definition—for dimension $n \leqq 5$, and his work showed that, except for systems involving the quaternions, $q_u(\omega)$ factors into linear factors over $\mathscr{C}(u_1, \dots u_n)$. (MOLIEN did not read this paper until he had done his own work.)

On the more theoretical side, SCHEFFERS also showed [1889b: 438–9] that every nonquaternion system—systems he hoped to characterize as precisely those systems not containing the quaternions as a subsystem—have a basis $e_1, \dots e_n$ such that $e_i e_j$ is a linear conbination of $e_1, \dots e_k$, where k is the smaller of i and j. In this basis the matrix of u_R factors into linear factors. It follows immediately that the characteristic polynomial $p_u(\omega)$ of u_R also has a linear factorization over $\mathscr{C}(u_1, \dots u_n)$. The same is true of $q_u(\omega)$, as SCHEFFERS indicated later [1891: 316].

The quaternions and nonions were regarded as specially important in the papers of STUDY and SCHEFFERS. As SCHEFFERS explained [1889b: 401], his quaternion systems "might well be singled out as by far the most interesting. Among them belong the *quaternions* of Hamilton and the analogous system of *nonions* that Sylvester constructed a few years ago." STUDY pointed out [1889b: 220], as had B. PEIRCE, that the (complex) quaternion system is simply the complete matrix algebra $\sum_{i,j=1}^{2} a_{ij} e_{ij}$. STUDY also introduced the general n^2-dimensional complete matrix algebra and appears to have been the first to have used the e_{ij} notation. He was probably familiar with the quadrate algebras of the PEIRCES since he had read their paper [1881]—although how thoroughly is not clear. SYLVESTER'S notes in the *Comptes Rendus*, which showed that the quaternions and nonions were complete matrix algebras, would also naturally suggest

[14] These notions are defined in Section 3 where other aspects of SCHEFFERS' work are discussed in relation to CARTAN'S.

this general class of systems, although in STUDY's case it is not clear whether he had read SYLVESTER's papers yet. Once the connection between LIE groups and hypercomplex numbers had been realized, however, consideration of complete matrix algebras also followed from that of the general linear group. In fact, in *Theorie der Transformationsgruppen* [1888: 557], LIE had written the elements of the associated n^2-dimensional LIE algebra in the suggestive form $\sum b_{ik} x_i p_k$, where $x_i p_k = x_i \partial/\partial x_k$. (*Cf.* the discussion of LIE algebras and hypercomplex systems presented below.) Matrices as hypercomplex numbers was also suggested by the parameter multiplication of the general projective group (LIE [1888: 554]), which was simply the matrix multiplication of the coefficients defining the projective transformations. STUDY's interest in hypercomplex numbers and LIE groups derived from the fact that one could abbreviate the parameter relations occurring in $T_u T_v = T_w$ in an r-parameter LIE transformation group as $uv = w$, which, since u, v, w are r-tuples of parameters, naturally leads to the consideration of when $uv = w$ represents multiplication in a hypercomplex system. For the general projective group the hypercomplex system is a complete matrix algebra. Finally, complete matrix algebras and the e_{ij} basis were suggested by a paper by FROBENIUS [1878] on bilinear forms which STUDY regarded highly and referred to often. FROBENIUS explicitly related bilinear forms and hypercomplex systems in the final section of his paper, entitled "Complexe Zahlen" [1878: 59]:

> From the algorithm for the composition of forms, i.e. of systems of n^2 quantities arranged by n rows and n columns, countless other algorithms can be derived. Several independent forms $E, E_1, \ldots E_m$ form a *form system* if the product of any two of them is composed linearly out of the forms of the system. Then if $A = \sum a_x E_x$ and $B = \sum b_x E_x$, AB can be brought into the form $\sum c_x E_x$.

These remarks by FROBENIUS suggest considering the system of all forms $F = \sum a_{ij} x_i y_j$ with the obvious basis $e_{ij} = x_i y_j$. (FROBENIUS also identified the quaternions $1, i, j, k$ with forms in two variables [1878: 62].[15] In short, complete matrix algebras were naturally suggested by a number of different considerations. Study explicitly made the identification $e_{ij} = x_i y_j$ and noted the connection with the general projective group.

Besides singling out the complete matrix algebras and indicating that the quaternions were this type of a system, STUDY also noted that for the n^2-dimensional system $k = n$. These systems thus have the property that their dimension is the square of the degree k of the minimal polynomial—a property that was to become important in MOLIEN's analysis of the structure of simple systems. Although STUDY did not explicitly mention it, the minimal polynomial $q_u(\omega)$ is just $\det[\omega - (u_{ij})]$, where $u = \sum_{i,j=1}^{n} u_{ij} e_{ij}$, so that $q_u(\omega)$ is irreducible. Such considerations also indicate without difficulty that the characteristic poly-

[15] STUDY also referred to a paper by C. STEPHANOS [1883: 354–5] in which the real quaternions are identified with certain bilinear forms in two variables and complex coefficients. The identification was already implicit in a note by CAYLEY [1879]. According to STEPHANOS, KLEIN had devoted one or two meetings of his seminar at Munich in 1880 to similar considerations.

nomial $p_u(\omega)$ of u_R is $[q_u(\omega)]^n$. These properties of $q_u(\omega)$ were also to play an important role in MOLIEN'S work.

It is interesting that at the time MOLIEN was introducing and analyzing the notion of a simple system, STUDY and SCHEFFERS [1891: 317] introduced the somewhat analagous notion of an irreducible system. $\mathscr{H}$ is reducible if there exist subsystems $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$.[16] Otherwise it is said to be irreducible. SCHEFFERS proved [1891: 318; 1893] that any system has a unique decomposition as a direct sum of irreducible systems so that the problem of determining all systems can be reduced to that of finding all irreducible systems. STUDY also pointed out to SCHEFFERS that if $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$, then $q_u(\omega)$ is the product of the corresponding minimal polynomials of $\mathscr{H}_1$ and $\mathscr{H}_2$ (SCHEFFERS [1891: 320, 320n], STUDY [1891: 44]). Thus if $\mathscr{H}$ is reducible so is its minimal polynomial. It is obvious, however, that the converse fails to hold: of the 48 irreducible systems of degree at most 5 computed, together with their minimal polynomials, by SCHEFFERS in [1891], only the minimal polynomial of the quaternions is irreducible. But SCHEFFERS and STUDY failed to explore this peculiarity of the quaternions and, more generally, of complete matrix algebras.

The close relationship between hypercomplex systems and LIE'S theory of groups was also expressed in another manner during the late 1880's. Associated with an n-dimensional LIE transformation group G is its "infinitesimal group" or LIE algebra $\mathscr{L}$. G determines n linearly independent "infinitesimal transformations" $e_1, \dots e_n$ and constants c_{ij}^k which define the infinitesimal group or LIE algebra $\mathscr{L}$ with multiplication defined by

$$[e_i, e_j]=\sum_{k=1}^{n} c_{ij}^k e_k$$

and satisfying $[x, y]=-[y, x]$ and the JACOBI identity

$$[[x, y], z]+[[y, z], x]+[[z, x], y]=0.$$

Although LIE'S own notation for, and treatment of, infinitesimal groups did not emphasize the close connection with the purely algebraic theory of hypercomplex numbers, he recognized this viewpoint; and it was stressed by ENGEL [1886], as well as by W. KILLING, whose work is discussed below.

In developing the theory of transformation groups, LIE was particularly interested in its application to differential equations. LIE found that the concept of a simple group plays a role in the theory of differential equations akin to that played by simple groups in GALOIS' theory of equations. Hence simple groups were of special interest to LIE. In view of the above-mentioned (local) correspondence between LIE groups and algebras, simplicity can be stated in terms of $\mathscr{L}$ (LIE [1888: 264]): $\mathscr{L}$ is simple if it has no "invariant subgroups;" a subgroup (*i.e.* subalgebra) $\mathscr{L}_1$ of $\mathscr{L}$ is invariant if $[x, y]\in\mathscr{L}_1$ whenever x or y is in $\mathscr{L}_1$. In modern terminology, $\mathscr{L}$ is simple if it contains no proper ideals. In view of the analogy between LIE algebras and hypercomplex numbers, it might have been rather natural to consider "simple" hypercomplex systems, *i.e.* systems with no proper two-sided ideals.

[16] That is, $\mathscr{H}_1\cap\mathscr{H}_2=(0)$ and $h_1h_2=0$ for all $h_1\in\mathscr{H}_1$ and $h_2\in\mathscr{H}_2$. STUDY suggested this notion to SCHEFFERS (SCHEFFERS [1891: 317n]).

In the case of the bilinear group determined by a hypercomplex system $\mathscr{H}$, the infinitesimal group $\mathscr{L}$ can be identified with $\mathscr{H}$ with $[x, y] = xy - yx$. This was pointed out by LIE [1889: 327], STUDY [1889b: 193], and SCHEFFERS [1889a: 295]. If $\mathscr{L}$ denotes the LIE algebra thus defined by $\mathscr{H}$, then $\mathscr{L}$ is never simple because the identity of $\mathscr{H}$ generates a proper invariant subalgebra. There was another LIE algebra associated with $\mathscr{H}$, however, that could be simple. As STUDY pointed out, it is always possible to choose a basis $e_0, e_1, \dots e_{n-1}$ for $\mathscr{H}$ such that $tr(e_i)_R = 0$ for $i > 0$. Then $e_i e_k - e_k e_i$ is a linear combination of $e_1, \dots e_{n-1}$ so that $e_1, \dots e_{n-1}$ form a basis for an $(n-1)$-dimensional LIE algebra, $\mathscr{L}_{n-1}$. LIE [1889: 326] also called attention to $\mathscr{L}_{n-1}$ which, he observed, could be regarded as a subalgebra of the LIE algebra of the special linear group. As LIE had proved, the latter algebra is simple. Probably LIE had this fact in mind when he made the following remark [1889: 326]: "I wish to direct attention to *all systems of complex numbers for which the group G_{n-1}* [i.e. $\mathscr{L}_{n-1}$] *is simple*; among them is the HAMILTONIAN quaternion system. According to my older investigations, there is no such system for $n = 5, 6, 7, 8$." For $n = 9$ SYLVESTER'S nonions have this property. Both the nonions and the quaternions have this property because they are complete matrix algebras and hence the associated $\mathscr{L}_{n-1}$ consists of all matrices of trace zero—the LIE algebra of the special linear group. Here, then, was a problem for someone interested in hypercomplex systems. It turned out, as MOLIEN discovered, that $\mathscr{L}_{n-1}$ is simple if and only if $\mathscr{H}$ is simple (as a hypercomplex system) and that $\mathscr{H}$ is simple if and only if it is a complete matrix algebra so that $\mathscr{L}_{n-1}$ is simple only when it corresponds to the special linear group.

At the same time that LIE formulated his problem concerning $\mathscr{L}_{n-1}$, WILHELM KILLING (1847–1923) was in the process of publishing a series of groundbreaking papers under the title "Die Zusammensetzung der stetigen endlichen Transformationsgruppen" [1888, 1889a, 1889b, 1890]. KILLING, a professor at the Lyceum Hosianum in Braunsberg (now Braniewo, Poland), had been led through his work on the foundations of geometry to LIE'S notion of an infinitesimal group or LIE algebra. Even before KLEIN called KILLING'S attention to LIE'S work, KILLING had posed to himself the problem of investigating the possible structures an n-dimensional LIE algebra could have. This was a problem that LIE had never seriously considered because its purely algebraic nature did not appeal to him (ENGEL [1930: 147]). That the problem was indeed a worthwhile one was demonstrated brilliantly by KILLING in the above-mentioned series of papers.

Central to KILLING'S approach was consideration of the characteristic polynomial of a LIE algebra $\mathscr{L}$, *i.e.* the characteristic polynomial of the linear transformation ad u: $x \to [x, u]$. By writing the equations for this transformation in terms of the structural constants c_{ij}^k and coordinates $u_1, \dots u_n$ of $u \in \mathscr{L}$, one sees that this polynomial, which we denote by $k_u(\omega)$, has the form

$$k_u(\omega) = \omega^n + \psi_1(u)\,\omega^{n-1} + \cdots + \psi_{n-1}(u)\,\omega,$$

where $\psi_i(u)$ is a homogeneous function of $u_1, \dots u_n$ of degree i. The equation $k_u(\omega) = 0$ had already been introduced by LIE, but KILLING went much further in relating it to the structure of $\mathscr{L}$. For example, KILLING'S analysis is based upon classifying algebras $\mathscr{L}$ according to their rank—the number of functionally independent coefficients $\psi_i(u)$ (see KILLING [1888: 266]). Also algebras of rank zero—those for which $k_u(\omega)=\omega^n$ (i.e. nilpotent algebras)—play an important role. In short, KILLING's work demonstrated how useful considerations involving the characteristic polynomial could be for the structure theory of algebras.

In the second paper [1889a] KILLING brought his analysis of simple algebras to a conclusion with the startling result that, beside the 4 general types of simple groups indicated by LIE, there were only 6 other possibilities.[17] KILLING's result had a definite bearing upon LIE's problem concerning hypercomplex systems $\mathscr{H}$ such that $\mathscr{L}_{n-1}$ is simple. As LIE had noted, $\mathscr{L}_{n-1}$ could be regarded as a subalgebra of the LIE algebra of the special linear group—one of the 4 general types of simple algebra indicated by LIE. In view of KILLING's result it would seem plausible that when $\mathscr{L}_{n-1}$ is simple, it must actually coincide with the algebra of the special linear group, which would mean that $\mathscr{H}$ must be a complete matrix algebra. Thus, KILLING's result made it appear likely that the hypercomplex systems singled out by LIE were simply the complete matrix algebras. At the same time, the emphasis upon simple algebras—and KILLING's success in determining their structure—could have easily suggested the analogous problem of determining the structure of all simple hypercomplex systems. Furthermore, once it is observed that complete matrix algebras are simple (an easy enough observation to make), it would be natural to suspect that the systems $\mathscr{H}$ called to attention by LIE, the complete matrix algebras, and simple systems are one and the same class of systems. These implications of the work of LIE and KILLING were, we suggest, grasped by MOLIEN and to a large extent motivated his work, which established, among other things, the identity of these 3 types of hypercomplex system. To a certain extent, the same can be said in connection with CARTAN's work.

Having disposed of simple algebras, KILLING turned in the third installment [1889b] to the larger class of algebras which are their own primary subgroup, i.e. for which $\mathscr{L}'=\mathscr{L}$. (In LIE's terminology, borrowed from CANTOR's theory of sets, $\mathscr{L}$ is said to be perfect.) "The result of my investigations," KILLING explained in the introduction [1889b: 57], "is surprisingly simple in that it shows how all composite groups which satisfy the stated condition are easily and naturally reduced to simple groups." To formulate his result succinctly, KILLING introduced the notion of semisimplicity. A perfect group can have the property that $\mathscr{L}$ has a basis $e_1, \dots e_{m_1}, e_{m_1+1}, \dots e_{m_2}, e_{m_2+1}, \dots$ such that $\mathscr{L}_1 = \operatorname{span}\{e_1, \dots e_{m_1}\}$; $\mathscr{L}_2 = \operatorname{span}\{e_{m_1+1}, \dots e_{m_2}\}, \dots$ are simple and commute in the sense that for $i \neq j$ $[x, y]=0$ for all $x \in \mathscr{L}_i$ and $y \in \mathscr{L}_j$. Such groups, KILLING observed [1889b: 74], "have many properties in common with the simple groups. For want of a better name, let such a group be termed *semisimple*." With this definition KILLING could formulate his main result as follows [1889b: 107]: "Every group which is its own primary subgroup can be composed out of a simple or semisimple group and an invariant subgroup of rank zero."

No one questioned the importance of KILLING's results, but their validity was another matter. LIE and ENGEL, for example, at first regarded them as

[17] Actually there are only five, as CARTAN [1893c: 411n] pointed out.

hypotheses because KILLING's arguments, especially those applying to nonsimple algebras, were not rigorous.[18] Despite these flaws, KILLING's work was brilliant, and in 1900 he became the second mathematician to receive the LOBACHEVSKY prize of the University of Kazan (the first having gone to LIE in 1897). The strong influence exerted by KILLING's work on the development of hypercomplex number theory will be clear from the following section.

3. The Structure Theorems of Molien and Cartan

THEODOR MOLIEN (1861–1941) and ÉLIE CARTAN (1869–1951) made essentially the same discoveries about the structure theory of hypercomplex systems—discoveries that went far beyond the work of STUDY and SCHEFFERS. Of special importance from our perspective are the introduction of the notions of simple and semisimple systems, the characterization of the former as complete matrix algebras, and the discovery of necessary and sufficient conditions for semisimplicity in terms of the nonsingularity of certain bilinear or quadratic forms. Although both MOLIEN and CARTAN received a good deal of inspiration from the same developments—the work of LIE and KILLING on LIE groups and algebras, the work of STUDY and SCHEFFERS on hypercomplex numbers—their methods were entirely different, and it appears that they were led to the same results independently of each other.

MOLIEN was born in Riga, Latvia, and educated at the University of Yurev (or Dorpat) in Estonia. After completing his formal education there in 1883, he spent a brief period (1884–85) at Leipzig, where KLEIN was then a professor, and published a paper [1885] on elliptic functions. Upon returning to Dorpat as a *Dozent* in 1885, however, it was to the theory of hypercomplex systems that he turned for the subject of his doctoral thesis.

MOLIEN's thesis, which documents his familiarity with the literature discussed in Sections 1 and 2, was submitted in 1892 and published in the *Mathematische Annalen* [1893a]. In many ways it resembled the series of papers on LIE algebras that KILLING had recently published in the *Annalen*. Both authors were relatively unknown and unacclaimed as mathematicians, and both made brilliant discoveries concerning the structure of the systems they were studying. Contemporary mathematicians recognized the importance of their work, but criticized some of the proofs and questioned some of the more general theorems. Many of the main results of both authors were more rigorously established by CARTAN and, in MOLIEN's case, also by FROBENIUS [1903b, 1903c, 1903d]. Both authors (MOLIEN and KILLING) contributed little else of comparable significance to mathematics. MOLIEN remained at Dorpat as *Docent* until 1901 when he became professor of mathematics at the University of Tomsk in Siberia. Nevertheless what MOLIEN did accomplish is highly significant and interesting, from both a mathematical and an historical standpoint, and deserves wider recognition. His thesis alone contained the fundamentals of the structure and representation theory of algebras; and in Section 4 we shall see how MOLIEN was led by his work on algebras to many of the basic theorems of group representation theory.

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[18] Relations between LIE and KILLING were far from cordial. See ENGEL [1930: 148] and LIE's devastating remarks [1893a: 769–70].

In his thesis MOLIEN considered a hypercomplex system $\mathscr{H}$ with basis $e_1, \dots e_n$ and multiplication constants a_{ij}^k. If $x = \sum_{k=1}^{n} x_k e_k$ and $u = \sum_{l=1}^{n} u_l e_l$ and $x' = xu$, then the coordinates x_i' of x' are

$$x_i' = \sum_{k,l=1}^{n} a_{kl}^i x_k u_l, \quad i = 1, \dots n. \tag{3.1}$$

Equation (3.1) can be regarded as defining $\mathscr{H}$, and this was the viewpoint taken by MOLIEN. Suppose that for some integer r, $1 \leqq r < n$, there is a basis $\{e_i\}$ for $\mathscr{H}$ such that the r bilinear forms x_i', $1 \leqq i \leqq r$ depend on only the first r variables, *viz.*

$$x_i' = \sum_{k,l=1}^{r} a_{kl}^i x_k u_l. \tag{3.2}$$

Then (3.2) defines an r-dimensional system $\mathscr{H}^*$ which is said to *accompany* (*begleiten*) $\mathscr{H}$. A system $\mathscr{H}$ which has no accompanying systems is said to be *primitive* (*ursprünglich*). These were MOLIEN's definitions [1893a: 93]. Note that if $\{e_i\}$ is the basis of $\mathscr{H}$ which determines $\mathscr{H}^*$, the subspace $\mathscr{I}$ spanned by $e_{r+1}, \dots e_n$ is a two-sided ideal in $\mathscr{H}$, and $\mathscr{H}^*$ is isomorphic to $\mathscr{H}/\mathscr{I}$. Thus $\mathscr{H}$ is primitive if and only if $\mathscr{H}$ is simple. Although it seems likely that MOLIEN was led to consider the notion of primitivity by analogy with the notion of a simple LIE algebra, he nowhere indicated this. Consequently in discussing his work we shall use the term "primitive" rather than "simple."

MOLIEN began [1893a: 93–6] by considering the relationship between systems $\mathscr{H}_1^*$ and $\mathscr{H}_2^*$ which accompany $\mathscr{H}$. If bases for $\mathscr{H}$, $\mathscr{H}_1^*$, and $\mathscr{H}_2^*$ are chosen, let $x_1, \dots x_n$ denote the parameters of $h \in \mathscr{H}$ in this basis while $y_1, \dots y_r$ and $z_1, \dots z_s$ denote the parameters of the corresponding elements $h_i^* \in \mathscr{H}_i^*$ determined by the homomorphisms implicit in the definition of the $\mathscr{H}_i^*$ (and yielding the isomorphisms $\mathscr{H}_i^* \cong \mathscr{H}/\mathscr{I}_i$). Then each y_i and z_j is a linear form in $x_1, \dots x_n$ (*i.e.* a linear functional on $\mathscr{H}$). Of particular interest is the case in which the parameters $y_1, \dots y_r, z_1, \dots z_s$ (as functionals) form a linearly independent set. MOLIEN determined necessary and sufficient conditions for this to be the case [1893a: Sätze 3–4] which imply, among other things, that the parameters of two non-isomorphic primitive systems are always linearly independent in the above sense. Furthermore he showed [1893a: Satz 5] that if $\mathscr{H}_1^*, \dots \mathscr{H}_k^*$ accompany $\mathscr{H}$ and if the parameters of every two of them are linearly independent, then the set of all the parameters is linearly independent. The significance of these results is this: If $\mathscr{H}_1^*, \dots \mathscr{H}_k^*$ denotes a maximal sequence of primitive systems accompanying $\mathscr{H}$ such that the set of their parameters is linearly independent, then it must include every primitive system $\mathscr{H}^*$ accompanying $\mathscr{H}$.[19] For if $\mathscr{H}^*$ is not isomorphic to any of the $\mathscr{H}_i^*$, the parameters of each pair $\mathscr{H}^*$, $\mathscr{H}_i^*$ are linearly independent, and so (by Satz 5) the parameters of the $\mathscr{H}_i^*$ and $\mathscr{H}^*$ are independent, contradicting the maximality assumption.

Moreover, because the parameters of the $\mathscr{H}_i^*$ are linearly independent, a basis for $\mathscr{H}$ can be chosen such that, if $n_i = \dim \mathscr{H}_i^*$ and $r = n_1 + \cdots + n_k$, the first n_1

[19] It may, of course, include a system several times—*i.e.*, several of the $\mathscr{H}_i^*$ may be isomorphic.

parameters of $h \in \mathscr{H}$ are the parameters of the corresponding $h_1^* \in \mathscr{H}_1^*$, the next n_2 parameters of h are the parameters of $h_2^* \in \mathscr{H}_2^*$, and so on.[20] Thus the defining equation (3.1) of $\mathscr{H}$ becomes in this basis:

$$
(3.3) \qquad
\begin{aligned}
x_i' &= \sum_{k,l=1}^{n_1} a_{kl}^i x_k u_l, && i=1,\dots n_1\\
x_i' &= \sum_{k,l=n_1+1}^{n_2} a_{kl}^i x_k u_l, && i=n_1+1,\dots n_2\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
x_i' &= \sum_{k,l=1}^{n} a_{kl}^i x_k u_l, && i=r+1,\dots n.
\end{aligned}
$$

The first n_1 equations define $\mathscr{H}_1^*$, the next n_2 $\mathscr{H}_2^*$, and so on. Translating (3.3) into familiar terms, $\mathscr{I} = \operatorname{span}\{e_{r+1},\dots e_n\}$ is a two-sided ideal and $\mathscr{H}/\mathscr{I} \cong \mathscr{H}_1^* \oplus \cdots \oplus \mathscr{H}_k^*$. MOLIEN was able to obtain a sharper version of this result, but first it was necessary to determine the nature of primitive systems.

The characteristic and minimal polynomials played a central role in MOLIEN's approach to primitive systems. Unlike SCHEFFERS, MOLIEN applied CAYLEY's Theorem to $u_R: x \to xu$ to obtain $p_u(u)=0$, where $p_u(\omega)$ is the characteristic polynomial of u_R. The minimal polynomial could then be defined [1893a: 113] as that polynomial $q_u(\omega)$ over $\mathscr{C}(u_1,\dots u_n)$ of minimal degree and leading coefficient 1 such that $q_u(u)=0$. (In MOLIEN's terminology, which indicated his reading of KILLING [1888], $q_u(\omega)=0$ is called the rank equation of $\mathscr{H}$.) It then followed easily that $q_u(\omega)$ divides $p_u(\omega)$. If

$$
(3.4) \qquad p_u(\omega) = \det(u_R - \omega) = \omega^n - f_1(u)\,\omega^{n-1} + \cdots \pm f_n(u),
$$

then, as KILLING noted for $k_u(\omega)$, $f_i(u)$ is a homogeneous polynomial of degree i. In particular, $f_1(uv) = f_1(vu)$ since $f_1(u) = \operatorname{tr} u_R$. We shall see that this trace function played an important role in MOLIEN's work. These properties of the coefficients $f_i(u)$ and many others were proved by MOLIEN [1893a: 109–12] to be true of the coefficients of any divisor of $p_u(\omega)$ and hence of the coefficients of $q_u(\omega)$. It is not difficult to see that if $\mathscr{H}^*$ accompanies $\mathscr{H}$, then $p_{u^*}^*(\omega)$ divides $p_u(\omega)$ (the parameters u_i of u and u^* are those in (3.2)) and $q_{u^*}^*(\omega)$ divides $q_u(\omega)$. Thus the structure of $\mathscr{H}$ in terms of accompanying systems is reflected in that of $p_u(\omega)$ and $q_u(\omega)$.

Consideration of $p_u(\omega)$ and $q_u(\omega)$ was, of course, in itself not a novelty. But MOLIEN introduced ideas of his own that enabled him to go further in understanding the relation between these polynomials and the structure of $\mathscr{H}$. Any linear form $f(u) = \sum_{i=1}^{n} \alpha_i u_i$ determines a bilinear form which reflects the

[20] The basis for $\mathscr{H}$ is chosen as follows. Extend the linearly independent set of parameters to a basis for the dual space $\hat{\mathscr{H}}$ of $\mathscr{H}$. The desired basis for $\mathscr{H}$ is simply the dual of the above basis for $\hat{\mathscr{H}}$.

multiplicative structure of $\mathscr{H}$, namely the form

$$\varphi(x, u) = \sum_{i=1}^{n} \alpha_i (xu)_i = \sum_{i,k,l} \alpha_i a^i_{kl} x_k u_l.$$

MOLIEN [1893a: 97] called such a form a form with polarity property provided $\varphi(x, u) = \varphi(u, x)$, *i.e.*, provided it was symmetric. He also pointed out that every system possesses such a form, namely the form

$$\varphi = f_1(xu) = \sum_{i,k,l,s} a^i_{si} a^i_{kl} x_k u_l, \tag{3.5}$$

where $f_1(u)$ is the coefficient of $p_u(\omega)$ defined in (3.4). (Equivalently, $\varphi = \operatorname{tr}(xu)_R$.) MOLIEN observed [1893a: 98–9] that if φ is a form with polarity property and if the rank of its matrix is $r \neq 0$, then the change of basis that puts φ in the form $\varphi(x, u) = \sum_{k,l=1}^{r} \alpha_{kl} x_k u_l$ also puts the forms $x'_i = (xu)_i$, $i = 1, \ldots r$, in a similar form—that is, (3.2) holds. The form φ thus determines an accompanying system $\mathscr{H}^*$ or, as MOLIEN put it, φ generates $\mathscr{H}^*$. It follows immediately that if $\mathscr{H}$ is primitive, then the form $f_1(uv) = \operatorname{tr}(uv)_R$ must be nonsingular. More generally, MOLIEN discovered that $\mathscr{H}$ is semisimple if and only if $f_1(uv)$ is nonsingular, as will be seen below.

The above considerations indicate that not only do accompanying systems determine factors of the minimal and characteristic polynomials, but, conversely, if $r_u(\omega) = \omega^k - r_1(u)\omega^{k-1} + \cdots$ is a divisor of one of these polynomials, then $r_1(uv)$ is a form with polarity property which therefore generates an accompanying system. This relationship enabled MOLIEN [1893a: 118–19] to prove, for example, that if the minimal polynomial $q_u(\omega) = \omega^m - h_1(u)\omega^{m-1} + \cdots$ of $\mathscr{H}$ is irreducible and the form $h_1(uv)$ is nonsingular, then $\mathscr{H}$ is primitive; and, conversely, if $\mathscr{H}$ is primitive then its minimal polynomial is irreducible. Another result of MOLIEN's is especially important:

Theorem 3.1. *If $\mathscr{H}$ is primitive, then its characteristic polynomial is a power of its minimal polynomial.*

Theorem 3.1 is, of course true for complete matrix algebras and, in fact, in a stronger form: the power to which the minimal polynomial is raised to give the characteristic polynomial is equal to the degree of the minimal polynomial. An immediate consequence of the stronger version of Theorem 3.1 is that $\dim \mathscr{H} = m^2$, $m = \deg q_u(\omega)$. Once this is known about $\mathscr{H}$, it is not difficult to deduce that $\mathscr{H}$ must be a complete matrix algebra. (See below.) To obtain this stronger version of Theorem 3.1, MOLIEN introduced considerations of a different sort, suggested by KILLING's work on LIE algebras.

If $\mathscr{L}$ is the LIE algebra associated with $\mathscr{H}$ (as in § 2), then the "Killing equation" $k_u(\omega)$, as MOLIEN termed it, is the characteristic equation of $\operatorname{ad} u$: $x \to [x, u] = xu - ux$. For the complete matrix algebra of n-by-n matrices, MOLIEN discovered the following theorem [1893a: 105], which was inspired by some remarks in KILLING's paper [1888]: "That equation whose roots ϱ are the totality of differences of the roots of the equation ... [$\det(b_{ik} - \delta_{ik}\omega)$] are obtained by elimination of the quantities c_{ik} from the equations

$$\varrho c_{ik} = \sum_l c_{il} b_{lk} - \sum_l b_{il} c_{lk} \qquad (i, k, l = 1 \dots n)\text{''}.^{21}$$

In other words:

Theorem 3.2. *If $\mathscr{H}$ is the complete matrix algebra of n-by-n matrices $u = (u_{ij})$, then*

$$k_u(\omega) = \prod_{i,j=1}^{n} (\omega - \omega_i + \omega_j),$$

where $\det(\delta_{ij}\omega - u_{ij}) = \prod_{i=1}^{n} (\omega - \omega_i)$.

MOLIEN did not phrase Theorem 3.2 as we have, but it will be clear that it was this characterization of complete matrix algebras that guided his analysis of primitive system.

Theorem 3.2 applies to a primitive system $\mathscr{H}$ because $\mathscr{H}$ can be identified with the matrix algebra given by its regular representation. Using this viewpoint, MOLIEN concluded [1893a: 115] that the roots of the KILLING equation $k_u(\omega)$ of $\mathscr{H}$ are differences of the roots ω_i of its characteristic polynomial $p_u(\omega)$. But the roots of $p_u(\omega)$ are all roots of $q_u(\omega)$ so that the roots of $k_u(\omega)$ are actually differences of roots of $q_u(\omega)$. For primitive systems MOLIEN was able to strengthen this result and obtain, using Theorem 3.1, an analog of Theorem 3.1 for $k_u(\omega)$ [1893a: 120]:

Theorem 3.3. *If $\mathscr{H}$ is primitive, then*

$$k_u(\omega) = \prod_{i,j=1}^{m} (\omega - \omega_i + \omega_j)^s,$$

where $\omega_1, \dots \omega_m$ are the roots of $q_u(\omega)$.

For complete matrix algebras, of course, $s = 1$ by Theorem 3.2; and when $s = 1$ in Theorem 3.3, clearly $\dim \mathscr{H} = \deg k_u(\omega) = m^2$—*i.e.*, Theorem 3.1 is sharpened to assert $p_u(\omega) = q_u(\omega)^m$.

MOLIEN's proof that $s = 1$ is based upon consideration of the order of $\omega = 0$ as a root of $k_u(\omega)$ and was undoubtedly inspired by KILLING's paper. For in the introduction [1888: 254–5] KILLING stressed the utility of the following theorem: "If $[k_u(\omega) = 0]$ ... contains in general m vanishing roots ... $m - 1$ additional transformations can be determined which are independent of one another and of the transformation $[u]$ ... and are permutable with one another and the given

[21] According to MOLIEN [1893a: 87], this theorem "goes, I believe, back to Killing," and he referred especially to pp. 271 *ff.* of [1888]. These pages contain a discussion of the general projective and (complex) orthogonal groups. The theorem is not actually stated by KILLING, but one can see how KILLING's treatment of these groups might have suggested it. The LIE algebras of these groups can be identified with matrix algebras, and KILLING indicated how the rank l and invariants of the adjoint group can be determined by considering determinants of smaller dimension than that defining $k_u(\omega)$ which are defined in terms of the matrix coefficients of u itself. Particularly suggestive is the discussion of the orthogonal group in $2l + 1$ variables [1888: 274–5]. The LIE algebra can be identified with all skew-symmetric l-by-l matrics (η_{ij}), and KILLING expressed the roots of $k_u(\omega)$ as differences of the roots of the characteristic polynomial of (η_{ij}).

$[u]$." Notice the similarity with the proposition posited by MOLIEN as Satz 19 [1893a: 115]: "If a number system is generated by a form with polarity property, then its KILLING equation possesses as many vanishing roots as there are linearly independent numbers which are permutable with a generally chosen number of the system." It is interesting that, although these theorems played an important role in the work of their respective authors, neither was provided with an adequate proof. Many of KILLING's arguments relative to nonsimple algebras were, in fact, vitiated by the failure of his theorem for algebras as general as he assumed (CARTAN [1893a: 785]). And MOLIEN's proof of Satz 19 was based upon an assumption as formidible to prove as Satz 19 itself. FROBENIUS, who first indicated this [1903b: 408–9], was able to verify the assumption for primitive systems by applying another of MOLIEN's theorems. For MOLIEN's characterization of simple systems it is naturally only the special case of Satz 19 that is needed.

Some notation will be helpful for clarifying the meaning and consequences of Satz 19 for primitive systems. For $u\in\mathscr{H}$ let $\mathscr{N}_u$ denote the set of all $x\in\mathscr{H}$ such that $xu-ux=0$. $\mathscr{N}_u$ is thus the space of all eigenvectors x corresponding to the eigenvalue 0 of $\operatorname{ad} u$. Let $z_0(u)$ denote the order of 0 as a root of $k_u(\omega)$. In general $\dim\mathscr{N}_u\leqq z_0(u)$, and the inequality can be strict even for primitive $\mathscr{H}$. By a "generally chosen" $u\in\mathscr{H}$, MOLIEN meant a u such that $z_0(u)=z_0$, where z_0 denotes the degree of 0 as a root of $k_u(\omega)$ over $\mathscr{C}(u_1,\dots u_n)$. When $\mathscr{H}$ is primitive, $q_u(\omega)$ is irreducible over $\mathscr{C}(u_1,\dots u_n)$ so that its roots are distinct and nonzero. Hence $z_0=ms$ by Theorem 3.3. Satz 19 then asserts that for $\mathscr{H}$ primitive and u "generally chosen",

$$ms=z_0=z_0(u)=\dim\mathscr{N}_u. \tag{3.6}$$

On the other hand, it is clear that $\mathscr{N}_u$ contains all powers of u, so that $\dim\mathscr{N}_u\leqq m(u)$, where $1, u,\dots u^{m(u)-1}$ are linearly independent but not $1, u,\dots u^{m(u)}$. MOLIEN proved [1893a: 122] that when $\mathscr{H}$ is primitive these powers of u span $\mathscr{N}_u$. Thus when $\mathscr{H}$ is primitive

$$\dim\mathscr{N}_u=m(u)\leqq m=\deg q_u(\omega). \tag{3.7}$$

Together, (3.6) and (3.7) imply that $ms\leqq m$ so that $s=1$ in Theorem 3.3.

Theorem 3.3 with the refinement $s=1$ thus implies that if $\mathscr{H}$ is primitive $\dim\mathscr{H}=m^2$ and $p_u(\omega)=q_u(\omega)^m$. MOLIEN was now easily able to conclude that $\mathscr{H}$ is the m^2-dimensional complete matrix algebra as follows [1893a: 124–5]. Let ω_0 denote a root of $q_u(\omega)$. Then ω_0 is an eigenvalue for u_R. Let $x^{(1)},\dots x^{(r)}$ be a basis for the space of eigenvectors corresponding to ω_0. Clearly for any $v\in\mathscr{H}$, $vx^{(i)}$ is also an eigenvector for u_R with eigenvalue ω_0. Thus v determines coefficients v_{ki} such that

$$v\,x^{(i)}=\sum_{k=1}^{r}v_{ki}\,x^{(k)}.$$

Furthermore direct computation of $vwx^{(i)}$ for any $v, w\in\mathscr{H}$ shows that these coefficients satisfy

$$(vw)_{ki}=\sum_{j=1}^{r}v_{kj}w_{ji}\qquad\text{for all } v, w\in\mathscr{H}. \tag{3.8}$$

(In more familiar terms, the space spanned by the $x^{(i)}$ is a left ideal of $\mathscr{H}$, and $v \to (v_{ki})$ is the representation of $\mathscr{H}$ induced by this ideal.) Because $q_u(\omega)$ is irreducible over $\mathscr{C}(u_1, \dots u_n)$, it follows that u exists so that $\omega_0 \neq 0$ has multiplicity 1 as a root of $q_u(\omega)$. For such a u, $r \leqq m$, since ω_0 is a root of multiplicity m of $p_u(\omega) = q_u(\omega)^m$. That $r = m$ and that $\mathscr{H}$ is the m^2-dimensional complete matrix algebra then follows easily from (3.8) and the primitivity of $\mathscr{H}$.

Thus the defining equations of a primitive system can be put in the form

$$x'_{ik} = \sum_{j=1}^{m} x_{ij} u_{jk}, \qquad i, k = 1, \dots m.$$

MOLIEN termed this the normal form of the equations of a primitive system.

The ultimate objective of MOLIEN'S thesis was to obtain a normal form for any hypercomplex system which would make the structure of the system more evident. His approach to the more general normalization problem involved a new strategy since "the need very quickly arises for a broader interpretation of the system of product equations of a number system [(3.1)]; a significant simplification is achieved if now at the outset a more general problem is posed, which arises from the theory of transformation groups" [1893a: 125]. POINCARÉ had suggested that the study of hypercomplex numbers be reduced to the study of bilinear groups, and STUDY had pointed out that if one limits oneself to simply transitive bilinear groups, the two notions are equivalent. MOLIEN, however, proposed to drop the condition of simple transitivity and to study bilinear groups of transformations

$$T_u\colon x'_i = \sum_{k=1}^{m} b_{ik}(u)\, x_k, \qquad i = 1, \dots m, \tag{3.9}$$

where $u = (u_1, \dots u_n)$ represents the parameters of an element u of an n-dimensional hypercomplex system $\mathscr{H}$ and $T_u T_v = T_{uv}$, uv denoting the product of u and v as members of $\mathscr{H}$. The group (semigroup in modern terms) $\{T_u\}$ was also assumed to satisfy the standard conditions that the parameters u_i are essential,—*i.e.*, that the linear forms $b_{ik}(u)$ cannot be expressed, by a linear change of parameter, as forms in less than n parameters—and that $\det(T_u)$ is not identically zero as a function of the u_i. Such a group is said to belong to the system $\mathscr{H}$. It is easily seen that the bilinearity of T_u and the above conditions imply that $T_{\alpha u + \beta v} = \alpha T_u + \beta T_v$, that $T_e = I$ (e = the identity of $\mathscr{H}$), and that the correspondence $u \to T_u$ is one-to-one. Thus $u \to T_u$ is a faithful representation of $\mathscr{H}$ of degree m. A nonfaithful representation $u \to T_u$ can be considered as a group belonging to the accompanying system $\mathscr{H}^* \cong \mathscr{H}/\mathscr{I}$, where $\mathscr{I}$ is the kernel of $u \to T_u$. By means of such considerations, MOLIEN actually did include nonfaithful representations in his study.

MOLIEN'S objective was to obtain a normal form for the equations (3.9) of any group $\{T_u\}$ belonging to $\mathscr{H}$, which, by virtue of the isomorphism $u \to T_u$, would enable him to define a normal form for $\mathscr{H}$ itself. In the process of obtaining the normal form for $\{T_u\}$ MOLIEN discovered many of the basic theorems on the representation of hypercomplex systems. The key concept employed in his analysis of $\{T_u\}$ was that of an accompanying group, the analog of an accompanying hypercomplex system. If by a linear change of variables x_k the matrix of equations $(b_{ik}(u))$ defining T_u can be put in the form

$$\begin{pmatrix} A(u) & 0 \\ B(u) & C(u) \end{pmatrix}$$

for all $u \in \mathscr{H}$, then the group $\{T_u^*\}$ defined by the matrices $A(u)$ was said to accompany the group $\{T_u\}$. A group with no nontrivial accompanying groups was said to be primitive. (These are, of course, irreducible representations of $\mathscr{H}$.) The parameter equations $x'_{ik} = \sum_{j=1}^{m} x_{ij} u_{jk}$ of a primitive system $\mathscr{H}$ afford an example of a primitive group belonging to $\mathscr{H}$. That all primitive groups are of this form was one of the consequences of MOLIEN's work.

A word is in order concerning the use of matrix notation. In the initial presentation of his theory of the group $\{T_u\}$ MOLIEN used the customary notation of linear substitutions as in (3.9); but he also reformulated his major results in the notation of matrices [1893a: 148–56]. The matrix viewpoint was still not popular on the Continent, and MOLIEN was the first to illustrate its usefulness in representation theory.

MOLIEN also made use of LIE's distinction between transitive and intransitive groups. (See footnote 13.) If $\{T_u\}$ is transitive, then the x'_k (in the notation of (3.9)), considered as linear forms in $u_1, \dots u_n$, are not linearly dependent for all values of $x_1, \dots x_k$. (It is easily seen that $\{T_u\}$ is transitive if and only if an $x = (x_1, \dots x_m)$ exists such that for every $x' \neq 0$, $T_u(x) = x'$ for some $u \in \mathscr{H}$. Irreducible representations are always transitive and intransitive representations are always reducible.) MOLIEN established two theorems of fundamental significance for the further development of his theory. The first [1893a: Satz 35] states that every intransitive group belonging to $\mathscr{H}$ is accompanied by a transitive group. The second [1893a: Satz 36] asserts that every transitive group belonging to $\mathscr{H}$ can be realized as a group accompanying the parameter group u_R: $x \to xu$, *i.e.*, is contained in the regular representation of $\mathscr{H}$.

The advantage afforded by these theorems was that [1893a: 134] "for the further investigation of groups, those properties of their parameter-groups can be drawn upon which are now known through the investigation of number systems." MOLIEN had in mind especially the normal form obtained for primitive systems and the form (3.3) for a general system. Using these results in conjunction with the above-mentioned theorems, he obtained the following [1893a: 134–43]: The variables x_i and parameters u_j of $\{T_u\}$ can be chosen so that the matrix of equations defining T_u has the form

$$(3.10)\qquad \begin{pmatrix} s_{11}(u) & & & & \\ s_{21}(u) & s_{22}(u) & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{d1}(u) & \cdot & \cdot & \cdot & s_{dd}(u) \end{pmatrix}$$

where the $s_{ij}(u)$ are matrices—MOLIEN called them "elementary matrices"; moreover, the diagonal matrices are primitive groups $s_{ii}(u)=(u_{jk}^{(i)})$ belonging to primitive systems accompanying $\mathscr{H}$, so that the normal form parameters of two diagonal elementary matrices are either identical or linearly independent (in the sense discussed in the context of (3.3)). MOLIEN further proved [1893a: 144–6] that the form (3.10) for $\{T_u\}$ can be obtained in such a manner that it contains the group defined by

$$\begin{pmatrix} s_{11}(u) & & & \\ & s_{22}(u) & & \\ & & \ddots & \\ & & & s_{dd}(u) \end{pmatrix} \tag{3.11}$$

as a subgroup. That is [1893a: 128], $\mathscr{H}$ contains a subalgebra $\mathscr{K}$ such that in a suitably chosen basis the parameters of those $u\in\mathscr{H}$ which are in $\mathscr{K}$ are precisely the distinct parameters $u_{jk}^{(i)}$ occurring in the elementary diagonal matrices. $\mathscr{K}$ is, of course, a direct sum of complete matrix algebras and, hence, semisimple. When T_u is in the form (3.10) with (3.11) as a subgroup, it is said to be in normal form.

Using the properties of the coefficients of the nondiagonal elementary matrices, MOLIEN extended the basis for the subsystem $\mathscr{K}$ to all of $\mathscr{H}$, obtaining what he called a normal form for $\mathscr{H}$. For our purposes it is not necessary to describe this form in any detail; in its essentials it is similar to the normal form derived with a greater degree of detail and clarity by CARTAN [1898: 50–2]. But there is one immediate consequence of the above results that bears upon MOLIEN's subsequent papers on group representations. We have already indicated the important role played in MOLIEN's work by the bilinear form $f_1(uv)=\operatorname{tr}(uv)_R$. When $\mathscr{H}$ is put in its normal form, it becomes evident that $f_1(uv)=\sum_{i=1}^{d}\operatorname{tr} s_{ii}(u)$. Now if $s_{p_jp_j}(u)$, $j=1,\ldots k$ are the distinct elementary diagonal matrices, the rank of $f_1(uv)$ is clearly $\sum_{j=1}^{k} n_j^2=r$, where n_j denotes the degree of the matrix $s_{p_jp_j}(u)$. Also r is the dimension of $\mathscr{K}$, and $\mathscr{K}$ is the system generated by $f_1(uv)$ (in the sense defined following (3.5)). Thus the following theorem is an immediate consequence of the results in MOLIEN's thesis: If r is the rank of $f_1(uv)$, then $f_1(uv)$ generates an accompanying system of dimension r. Furthermore: "The accompanying number system thus defined decomposes by suitable choice of basis into primitive number systems which are strictly independent of one another, and this can occur in only one manner." MOLIEN never mentioned this theorem explicitly in [1893a]. The above quotation is from a later paper by MOLIEN [1897a: 266] in which he derived the basics of group representation theory, and there the theorem is simply stated as one of the theorems on hypercomplex systems to be applied to group algebras. The point is this: the above theorem, which implies in particular that $\mathscr{H}$ is semisimple if and only if $f_1(uv)$ is nonsingular, was an immediate consequence of the work in MOLIEN's thesis and provided him with a criterion for semisimplicity easily applicable to group algebras; it undoubtedly encouraged him, once he thought to consider group algebras, to pursue the matter further, since $f_1(uv)$ is always nonsingular for a group algebra.

One further comment is in order before turning to the work of CARTAN. The nonsingularity of the matrix of coefficients of $f_1(uv)$ was the condition under which DEDEKIND had proved [1885] that a commutative system is semisimple (see HAWKINS [1971: § 5]), and MOLIEN was familiar with the contents of DEDEKIND'S paper.

MOLIEN submitted his thesis to the *Annalen* in August of 1891. At that time KILLING'S work, especially his results on nonsimple LIE algebras, was still relegated by LIE to the limbo between mere hypotheses and established results. Some gaps in KILLING'S arguments had been filled in by ENGEL and one of his students, but the task was first taken on in full by CARTAN. CARTAN was one of several students from the École Normale Supérieure to work with LIE, his interest in continuous groups having been stimulated by a fellow student, A. TRESSE, who had recently returned from Leipzig. CARTAN completed his studies at the École Normale in 1891, and two years later he was announcing [1893 a, 1893 b] his success in verifying most of KILLING'S major results and in discovering some theorems of his own as well.

It will be recalled from Section 2 that the characteristic equation

$$k_u(\omega) = \omega^n - \psi_1(u)\omega^{n-1} + \psi_2(u)\omega^{n-2} - \cdots \pm \psi_{n-1}(u)\omega$$

of the linear transformation ad u: $x \to [x, u]$ played an important role in KILLING'S work. In the course of reworking KILLING'S analysis, CARTAN discovered the utility of considering the quadratic form $\psi_2(u)$. CARTAN [1893 c] was able to confirm KILLING'S results on the structure of simple algebras by making use of the theorem that if $\mathscr{L}$ is perfect, then ψ_2 does not vanish identically on $\mathscr{L}$ [1893 c: 400]. In his doctoral thesis [1894], CARTAN presented his conclusions relating to KILLING'S work on nonsimple algebras. Here again the form ψ_2 played an important role. CARTAN showed [1894: 47] that $\mathscr{L}$ is integrable (*i.e.*, solvable) if and only if ψ_2 vanishes on $\mathscr{L}'$, a result which proved useful in his treatment of semisimple algebras. Unlike KILLING, CARTAN defined $\mathscr{L}$ to be semisimple if it contains no invariant integrable subalgebra [1894: 51]. Making use of his criterion for integrability, CARTAN then proved [1894: 52–3] that $\mathscr{L}$ is semisimple if and only if ψ_2 is nonsingular and that this is the case if and only if $\mathscr{L}$ is the direct sum of simple algebras (in accordance with KILLING'S definition of semisimplicity). CARTAN also proved [1894: 97] that any algebra $\mathscr{L}$ contains a largest invariant integrable subalgebra $\mathscr{R}$ and that $\mathscr{L}/\mathscr{R}$ is semisimple.

When CARTAN was busily reworking KILLING'S results he was well aware of the close connection between hypercomplex numbers and LIE groups and algebras. In fact, in his thesis [1894: 121–3] he had used hypercomplex numbers (without an identity) in his treatment of certain LIE algebras. The following year CARTAN published a note in the *Comptes Rendus*, in which he proposed "to indicate some very general theorems to which the latest results found on the structure of finite groups lead" [1895: 545]. It is not surprising that CARTAN here considered, among other things, the implications of his thesis for bilinear groups and hypercomplex systems.

By virtue of the results in his thesis, a nonintegrable bilinear group G has a LIE algebra $\mathscr{L}$ with invariant integrable subalgebra $\mathscr{R}$ and semisimple homomorphic image $\mathscr{L}/\mathscr{R}$. CARTAN was able to sharpen this to the proposition that G is composed out of an invariant integrable subgroup and simple subgroups isomorphic to the special linear groups in n_i variables. Moreover:

> If, in particular, those among these groups are considered which are *simply transitive* and which are so intimately connected with systems of complex numbers ..., *one can*, keeping the notation of the previous theorem, *find* n_1^2 *variables which transform among themselves in the manner of the parameters of the general linear homogeneous group in* n_1 *variables, and likewise* n_2^2 *other variables transformed among themselves in an analogous manner, and so on ....*
>
> By transferring that to the theory of complex numbers, the following theorem can be stated:
>
> *If a system of complex numbers is associated with a nonintegrable simply transitive group such that the largest semisimple group to which it is isomorphic* [i.e., homomorphic] *is composed of simple subgroups with* m^2-1, $n^2-1, \ldots$ *parameters, a first system of* m^2 *units* e_{ik} $(i, k=1, 2, \ldots m)$ *can be found which determine ... a system of complex numbers with the law of multiplication*
>
> $$(1) \qquad e_{\alpha\beta}\, e_{\gamma\delta}=\varepsilon_{\beta\gamma}\, e_{\alpha\delta}\ (\varepsilon_{\beta\gamma}=1 \text{ if } \beta=\gamma,\ \varepsilon_{\beta\gamma}=0 \text{ if } \beta\neq\gamma);$$
>
> *then a second system of* n^2 *units, independent of the first and possessing analogous properties, and so on* [1895: 546–7].

The above quotation clearly indicates that CARTAN'S work on LIE algebras had provided him with a good deal of insight into the structure of hypercomplex systems! Apparently he was not familiar at this time with MOLIEN'S work[22], but he was acquainted with that of SCHEFFERS, who had become a *Privatdozent* at Leipzig in 1891. SCHEFFERS' work played an important role in shaping the direction of CARTAN'S interests in hypercomplex numbers. Much of SCHEFFERS' work was inspired by ENGEL'S theorem [1887: 95–99] that a LIE algebra is nonintegrable if and only if it contains elements e_1, e_2, e_3 satisfying:

$$(3.12) \qquad [e_1, e_2]=e_1, \quad [e_1, e_3]=2e_2, \ [e_2, \quad e_3]=e_3.$$

SCHEFFERS pointed out [1889b: 403] that (3.12) can be rewritten, by considering linear combinations of the e_i, as:

$$[e_1', e_2']=2e_3' \qquad [e_2', e_3']=2e_1' \qquad [e_3', e_1']=2e_2'.$$

In this form it is easy to see that the e_i' can be identified with the quaternions, i, j, k under LIE multiplication. SCHEFFERS thus proposed to divide hypercomplex systems into two categories, those which are integrable (as a LIE algebra with $[x, y]=xy-yx$) and those which are not, and to prove [1889b: 405–9] that any system in the second category contains the complex quaternions $a_0+a_1 i+$

[22] The first mention of MOLIEN in CARTAN'S publications is in the *Encyclopédie* article on hypercomplex numbers he wrote [1908] based on STUDY'S German version [1898]. STUDY discussed MOLIEN'S work, and in [1908] CARTAN fully acknowledged MOLIEN'S accomplishments.

$a_2 j + a_3 k$ (a_i complex) as a subsystem. In a subsequent paper [1891], SCHEFFERS adopted the name "quaternion systems" for systems in the second category, even though he had to admit [1891: 361–2] that his original proof that every such system does in fact contain the quaternions was defective. The best he could do in [1891] was to confirm his theorem for all quaternion systems of dimension less than 9 by enumerating them. As CARTAN pointed out [1895: 547], SCHEFFERS' problem is resolved by the above-quoted results, which, in fact, were probably inspired by it.[23]

SCHEFFERS had more success analyzing the structure of nonquaternion systems, *i.e.*, systems that are integrable as LIE algebras, and here, too, his work engaged CARTAN's interest. Briefly, SCHEFFERS discovered[24] that a basis $e_1, \dots e_r$, $\eta_1, \dots \eta_s$ can be chosen for a nonquaternion system $\mathscr{H}$ such that: $e_i e_j = \delta_{ij} e_i$; each η_i has a definite "character" (k, l), which means that $e_k \eta_i = \eta_i e_l = \eta_i$ and that $e_p \eta_i = \eta_i e_q = 0$ for all $p \neq k$ and $q \neq l$; and the minimal polynomial $q_u(\omega)$ has the form

$$q_u(\omega) = \prod_{i=1}^{r} (\omega - x_i)^{k_i}, \qquad \text{where } u = \sum_{i=1}^{r} x_i e_i + \sum_{i=1}^{s} \xi_i \eta_i.$$

To put it more suggestively, SCHEFFERS showed that

$$\mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_r + \mathscr{R},$$

where $\mathscr{H}_i$ denotes the simple system spanned by e_i and $\mathscr{R} = \operatorname{span}\{\eta_1, \dots \eta_s\}$ has the property that for all $u \in \mathscr{R}$, $q_u(\omega) = \omega^k$, which implies that $p_u(\omega) = \omega^n$ since (as SCHEFFERS proved) $q_u(\omega)$ and $p_u(\omega)$ have the same roots.

SCHEFFERS' results on nonquaternion systems were thus analogous to CARTAN's in [1895] for quaternion systems. CARTAN proposed to himself the problem of reestablishing and extending his and SCHEFFERS' results by methods that did not rely upon theorems from the theory of LIE groups and LIE algebras (CARTAN [1931: 371]). His conclusions were announced in the *Comptes Rendus* [1897a, 1897b] and then appeared with detailed proofs in [1898]. CARTAN's approach which combined ideas of SCHEFFERS—particularly his notion of the character of an element—with considerations involving the characteristic polynomial $p_u(\omega)$, led him to the important discovery, also presented in MOLIEN's paper [1893a: 86, 155–6], that the structure of any quaternion system is similar to, and determined by, that of an associated nonquaternion system (see CARTAN [1898: 50–1]).

CARTAN's main theorem implied in particular that any system $\mathscr{H}$ is composed of an invariant "pseudo null" system and a semisimple system. The terms "simple" and "semisimple" were first used by CARTAN in the context of hypercomplex systems in order to simplify the statement of his results. (That a simple

[23] CARTAN actually made a careless slip, for he wrote [1895: 547], referring to the last theorem in the quotation: "That is a generalization of a theorem stated by M. Scheffers but which is only true when one of the numbers $m, n, \dots$ is equal to 2." As he later pointed out [1898: 47], any complete matrix algebra of dimension $m^2 > 4$ contains the quaternions as a subalgebra.

[24] See SCHEFFERS [1889b: 405; 1891: 307–17, 327–30]. In the above account, we follow CARTAN's notation; the e's and η's have the opposite meaning in SCHEFFERS' paper [1891].

system is a complete matrix algebra was also a consequence of his main theorem.) Likewise an invariant pseudo null system was defined to be a two-sided ideal of elements $u \in \mathscr{H}$ such that $p_u(\omega) = \omega^n$—the analog of KILLING's notion of an algebra of rank 0 (nilpotent algebra). Finally, CARTAN's main theorem made it clear that the coefficient $f_2(u)$ of the characteristic polynomial $p_u(\omega)$ played a role in the structure theory of $\mathscr{H}$ similar to that played by $\psi_2(u)$ in the theory of LIE algebras: $\mathscr{H}$ is semisimple if and only if the quadratic form $f_2(u)$ is nonsingular. Thus CARTAN's work also provided a criterion for semisimplicity easily applicable to group algebras: a straightforward computation shows that the discriminant of $f_2(u)$ for any n-dimensional group algebra is $(-1)^{n-1}[\frac{1}{2}(n-1)n]^n$ so that all group algebras are semisimple.

4. Group Algebras and Group Representations

Reflecting upon the mathematics of the nineteenth century, J. PIERPONT wrote [1904: 144]: "the group concept, hardly noticeable at the beginning of the century, has at its close become one of the fundamental and most fruitful notions in the whole range of our science." Although somewhat exaggerated, PIERPONT's statement accurately reflected the growing interest and activity in the theory of groups, both finite and infinite, towards the end of the century. It was also at that time that the theory of hypercomplex numbers first achieved a degree of maturity through the structure theorems of MOLIEN and CARTAN. Hypercomplex numbers provided a link between the theories of finite and continuous groups, the link being made through the notion of the group algebra of a finite group. It was through this link and within the context of the growing interest in group theory that MOLIEN and BURNSIDE were led to the representation theory of finite groups.

Before turning to their work, there is one aspect of the increasing interest in group theory that requires further mention. No one did more to promote the view that group-theoretic ideas are fundamental in mathematics than FELIX KLEIN.[25] Among other things, KLEIN's work in the 1870's and 1880's focused interest upon finite groups of linear transformations or of projective transformations with complex coefficients. One of KLEIN's projects was the development of a general program of mathematical investigation that would generalize GALOIS's theory of equations. His views were widely circulated through the publication of his *Vorlesungen über das Ikosaeder* [1884]. In the *Vorlesungen* KLEIN showed how the general quintic equation could be reduced to that of the icosahedron equation because the GALOIS group of the former is (after adjunction of the square root of the discriminant) the alternating group A_5 which is isomorphic to the collineation group of 60 linear homogeneous substitutions in two variables associated with the latter. The quintic equation and its solution through the icosahedron equation served as the paradigm for KLEIN's general program, which centered on the "form-problem" (*Formenproblem*) associated with a finite group of linear homogeneous substitutions. Briefly, if the elements of G are substitutions in variables $x_1, \ldots x_n$, the form-problem for G was that of calculating the x_i from the forms (polynomials) in $x_1, \ldots x_n$ left invariant by the

[25] For a broader discussion of KLEIN's work on group theory, see WUSSING [1969].

substitutions of G. The basic problem of GALOIS theory—to determine the roots $x_1, \dots x_n$ of $f(x)$ by radical extensions of the field of numbers left invariant by the GALOIS group of $f(x)$—was thus a special type of form-problem.

Closely connected with the form-problem in KLEIN'S general program was another which involved the reduction of the form-problem for one group to another, associated with a group that is a homomorphic image—"isomorpihc" in KLEIN'S terminology—of the first.

> The formulation of this problem has a certain importance, because we obtain therewith a general program for the further development of the theory of equations. Among form-problems ... with isomorphic groups we already designated above as the simplest the one which possesses the smallest number of variables. Given an equation $f(x)=0$, we first of all seek to determine the smallest number of variables for which a group of linear substitutions can be constructed that is isomorphic with the Galois group of $f(x)=0$. Then we would set up ... the form-problem belonging to this group and seek to reduce the solution of $f(x)=0$ to this form-problem. [1884: 125–6]

This "normal problem" as he later termed it, had first been formulated by KLEIN, together with the form-problem, in [1879]. It clearly focused attention on the possible representations of a finite group as a group (possibly collinear) of linear transformations and invited investigation of the properties of such representations, particularly as to their degrees. In his *Lectures on Mathematics* [1894], presented at the mathematics colloguium held in Evanston, Illinois, the summer of 1893, KLEIN specifically singled out this part of his general program as worthy of further research [1894: 74]: "A first problem I wish to propose is as follows. In recent years many groups of permutations of 6, 7, 8, 9, ... letters have been made known. The problem would be to determine the minimum number of variables with which isomorphic groups of linear substitutions can be formed."

KLEIN himself and the mathematicians directly associated with him in the execution of his program were concerned with representations by collineation groups and with the representation of specific groups rather than with the creation of a general theory of such representations.[26] But it is not difficult to imagine how suggestive their work might appear, especially to someone aware of the developments taking place in LIE'S theory of groups and the theory of hypercomplex systems. Indeed, in the paper in which BURNSIDE arrived at the basics of representation theory, he pointed out [1898c: 547] that his work "obviously has a bearing on the question of the smallest number of variables in which [a finite group] g can be represented as a group of linear substitutions, *i.e.*, on what Prof. KLEIN calls the degree of the normal problem connected with g." Likewise, it seems probable that MOLIEN, who studied at Leipzig while KLEIN was a professor there—in fact the year *Vorlesungen über das Ikosaeder* appeared—, was familiar with KLEIN'S ideas. In fact, in his first paper on group representations [1897a], MOLIEN showed how all finite groups of linear transformations are built up out

[26] For references to the literature on the normal problem, see KLEIN [1894] and WIMAN [1899]. The general representation theory of finite groups by collineation groups was studied by FROBENIUS' student I. SCHUR [1904, 1907], who related it to the ordinary theory of group representations.

of irreducible representations and in the second [1897b] he concentrated on the properties of the degrees of the latter.

MOLIEN's work on group representations was submitted to the scientific society of the University of Yurev and acknowledged at the meetings of April 24 and September 25, 1897. MOLIEN was still a *Privatdozent* at the University and had published nothing since his thesis. In the first note he explained that he wished to communicate "some general theorems relating to the representability of a given discrete group in the form of a homogeneous linear substitution group," which are derived from the theory of hypercomplex numbers.

MOLIEN began with the observation that any finite group $G=\{S_1, \dots S_n\}$ determines a system $\mathscr{H}$ of hypercomplex numbers $x=x_1 S_1+\cdots+x_n S_n$ with multiplication determined by G:

$$S_i S_j=\sum_{k=1}^{n} a_{ij}^k S_k, \tag{4.1}$$

where $a_{ij}^k=1$ if $S_k=S_i S_j$ and $a_{ij}^k=0$ otherwise. It was in this paper that MOLIEN stated the criterion for semisimplicity implicit in his thesis (and discussed in Section 3). In the case of the group algebra $\mathscr{H}$ defined by (4.1) the associated bilinear form $f_1(uv)$ is easy to compute (see (3.5)) and seen to have discriminant $\pm n^n$; and MOLIEN was thus able to conclude that $\mathscr{H}$ is a direct sum of complete matrix algebras so that the product equations for $x'=ux$ can be put in the form

$$\begin{aligned} x'_{ik}&=\sum_{j=1}^{\nu_1} u_{ij}\, x_{jk} \qquad (i, k=1, \dots \nu_1)\\ \bar{x}'_{ik}&=\sum_{j=1}^{\nu_2} \bar{u}_{ij}\, \bar{x}_{jk} \qquad (i, k=1, \dots \nu_2)\\ &\dots\dots\dots\dots\dots \end{aligned} \tag{4.2}$$

where $n=\nu_1^2+\nu_2^2+\cdots$.

In terms of the parameters $u_1, \dots u_n$ of $u=u_1 S_1+\cdots+u_n S_n$ (4.2) can be written as

$$\begin{aligned} x'_{ik}&=\sum_{i=1}^{\nu_1} b_{ij}(u)\, x_{jk} \qquad (i, k=1, \dots \nu_1)\\ \bar{x}'_{ik}&=\sum_{j=1}^{\nu_2} \bar{b}_{ij}(u)\, \bar{x}_{jk} \qquad (i, k=1, \dots \nu_2)\\ &\dots\dots\dots\dots\dots \end{aligned} \tag{4.3}$$

where $b_{ij}(u)$, $\bar{b}_{ij}(u), \dots$ are linear forms in $u_1, \dots u_n$. Then, since the index k "has no influence on the coefficients" [1897a: 268], MOLIEN considered the system of equations in $\nu_1+\nu_2+\cdots$ variables

$$\begin{aligned} x'_i&=\sum_{j=1}^{\nu_1} b_{ij}(u)\, x_j \qquad (i=1, \dots \nu_1)\\ \bar{x}'_i&=\sum_{j=1}^{\nu_2} \bar{b}_{ij}(u)\, x_j \qquad (i=1, \dots \nu_2)\\ &\dots\dots\dots\dots\dots \end{aligned} \tag{4.4}$$

He observed that each subsystem in (4.4) yields a finite group of linear transformations T_k by setting $u_i=\delta_{ik}$, $k=1, \dots n$ in the subsystem equations and that $S_k \to T_k$ is a homomorphism. These are, of course, the irreducible representations of G.

The main point of MOLIEN's first communication concerned the following question [1897a: 270]: "If a discrete group is already given in the form of a linear substitution group, what is its relation to the systems of equations [(4.4)] considered by us?" In other words, given a faithful representation $S_k \to T_k$ of G, what is its relation to the representations defined by (4.4)? To answer this question he introduced the associated continuous group $T_u = \sum_{k=1}^{n} u_k T_k$. The group $\{T_u\}$ is easily seen to be a group belonging to the group algebra $\mathscr{H}$ in the sense defined in MOLIEN's thesis. The answer—already implicit in his thesis [1893a: Satz 40]—that MOLIEN gave was that, by a linear change of variables, the matrix of equations defining T_u can be put in the form

$$\begin{pmatrix} A(u) & & \\ & B(u) & \\ & & \ddots \end{pmatrix}$$

where each of $A(u)$, $B(u)$, ... represents the matrix of equations of a subsystem of (4.4). Thus: "From the given 'structure' of a discontinuous finite group, all linear groups of substitutions with the same structure can be obtained" [1897a: 276].

It was in the second note that MOLIEN went beyond what was, more or less, an immediate consequence of the results in his thesis. Having shown in his first note "how a given substitution group can be decomposed into its irreducible components", he proposed in the second to consider "only the properties of the irreducible groups ..." [1897b: 277]. MOLIEN was primarily interested in what could be said about the numbers ν_i, the number of variables occurring in the irreducible groups of (4.4). His main result was that the ν_i divide n, the order of G. In the course of proving this result, MOLIEN obtained further basic theorems of group representation theory, including the orthogonality relations for characters.

Let ϱ denote the number of irreducible subsystems in (4.4), *i.e.*, the number of inequivalent irreducible representations of G. The characteristic polynomial of the p^{th} system is, recalling the relation between (4.3) and (4.4), the minimal polynomial $q_u^{(p)}(\omega) = \omega^{\nu_p} - f_1(u)\,\omega^{\nu_p - 1} + \cdots$ of the corresponding complete matrix algebra. The linear forms $f_1^{(p)}(u)$ and bilinear forms $f_1^{(p)}(uv)$ had played an important role in MOLIEN's thesis, and it is not surprising to find him again putting them to good use. Let

$$f_1^{(p)}(u) = \nu_{p1} u_1 + \cdots + \nu_{pn} u_n,$$

where $u = \sum_{i=1}^{n} u_i S_i$ and $S_1 = 1$, the identity of G. Then $f_1^{(p)}(S_k) = \nu_{pk}$ is the coefficient of $\omega^{\nu_\varrho - 1}$ of the characteristic polynomial of the transformation $T_k^{(p)}$ obtained from the p^{th} subsystem in (4.4) by setting $u_i = \delta_{ik}$. In particular, $\nu_{p1} = \nu_p$, the degree of the p^{th} irreducible representation and the number that interested MOLIEN. Also since the transformation $T_k^{(p)}$ belongs to a finite group, it followed from the "known theorem" that the characteristic roots of $T_k^{(p)}$ are roots of unity that the ν_{pk} are algebraic integers.[27] (In more familiar terms, $\nu_{pk} = \chi^{(p)}(S_k)$, where $\chi^{(p)}$ denotes the character of the p^{th} irreducible representation.) As MOLIEN explained, the advantage of working with the coefficients $f_1^{(p)}(u)$ is that they remain invariant under linear change of variables.

All of MOLIEN's main theorems derived from considering the relation between the trace functions $f_1^{(p)}(u)$ corresponding to the complete matrix algebras into which the group algebra $\mathscr{H}$ decomposes and the bilinear form

$$\Omega = \sum_{s,t,k,l} a_{kt}^s \, a_{sl}^t \, v_k \, u_l \tag{4.5}$$

associated with $\mathscr{H}$. Ω is the trace function of $x \to v x u$. MOLIEN had not considered this form in his thesis, although STUDY [1889b] had considered the group of transformations $x \to v x u$ determined by a hypercomplex system. The choice of Ω by MOLIEN may have been motivated by the fact that for a ν^2-dimensional complete matrix algebra it is easily seen that $\Omega = f_1(v) f_1(u)$, where $f_1(u)$ is the coefficient of $\omega^{\nu-1}$ in $q_u(\omega) = \det[\omega - (u_{ij})]$. Since the group algebra is a direct sum of complete matrix algebras, one obtains

$$\Omega = \sum_{p=1}^{\varrho} f_1^{(p)}(v) f_1^{(p)}(u). \tag{4.6}$$

Ω can also be computed directly from (4.5) using the multiplication constants a_{ij}^k corresponding to the group basis S_k. The result of this straightforward group-theoretic computation is the following [1897b: 282–3]: If the group elements are arranged so that $S_1 = 1$, and $S_1, \dots S_\sigma$ are representatives of the σ conjugate classes $C_1, \dots C_\sigma$ of G, and if n_l denotes the order of the normalizer of S_l, then

$$\Omega = \sum_{l=1}^{\sigma} n_l C_l'(v) C_l(u), \tag{4.7}$$

where $C_l'(v) = \sum_{S_k \in C_l'} v_k$, $C_l(u) = \sum_{S_k \in C_l} u_k$, and C_l' is the conjugate class of inverses of elements in C_l.

The first consequence MOLIEN derived from equating (4.6) and (4.7) was that $\varrho = \sigma$: the number of irreducible representations derived from (4.4) is equal to the number of conjugate classes of G[28]. Moreover, by definition

$$f_1^{(p)}(u) = \sum_{l=1}^{\sigma} \nu_{pl} C_l(u), \tag{4.8}$$

[27] C. JORDAN, for example, had pointed out [1878: 112–13] that if a transformation is a member of a finite subgroup of the general linear group, then its canonical form is diagonal and the diagonal entries are roots of unity. JORDAN's remark played a role in MASCHKE's work (Section 5). It should be noted that interest in finite groups of linear transformations was also stimulated by JORDAN's numerous publications in this area during the late 1870's and 1880's.

[28] MOLIEN's proof that $\varrho = \sigma$ is typically obscure, but it follows easily from (4.6) and (4.7). The linear forms $f_1^{(p)}(u)$ are linearly independent, as are the forms $C_l(u)$. By appropriately specializing the values of v in (4.6) and (4.7), one can show that each $f_1^{(p)}(u)$ is a linear combination of the $C_l(u)$'s and that, conversely, each $C_l(u)$ is a linear combination of the $f_1^{(p)}(u)$'s. Hence $\varrho = \sigma$.

and, similarly,

$$f_1^{(p)}(v) = \sum_{l=1}^{\sigma} v'_{pl} C'_l(v), \tag{4.9}$$

where $v'_{pl} = f_1^{(p)}(S_l^{-1})$. Making the substitutions (4.8) and (4.9) in (4.6) and equating the result with (4.7) then yields immediately the following relations on the coefficients v'_{pl} and v_{pk}: $\sum_{p=1}^{\sigma} v'_{pl} v_{pk} = n_l \delta_{kl}$. Setting $m_l = n/n_l$, MOLIEN also wrote this as

$$m_l \sum_{p=1}^{\sigma} v'_{pl} v_{pk} = n\,\delta_{kl}, \tag{4.10}$$

and, taking transposes, obtained "the further important formula"

$$\sum_{l=1}^{\sigma} m_l\, v'_{pl}\, v_{ql} = n\,\delta_{pq}. \tag{4.11}$$

Equations (4.10) and (4.11) are, of course, the two fundamental orthogonality relations for the irreducible characters of G! Using (4.11), MOLIEN then proved that v_k divides n in much the same way as FROBENIUS had a year earlier, namely by proving that $m_l v_{kl}/v_k$ is an algebraic integer.

FROBENIUS' first paper on the matrix representation of groups was presented for publication at the November 18, 1897 meeting of the Berlin Academy. Shortly before he wrote it up for publication, FROBENIUS learned of MOLIEN's work through STUDY (FROBENIUS [1897: 945]). The question naturally arises: When did MOLIEN become familiar with FROBENIUS' work, especially FROBENIUS' papers on group determinants and characters [1896b, 1896c]? The answer appears to be that MOLIEN did not known of FROBENIUS' work until after he had composed his two notes [1897a, 1897b]. As MOLIEN himself explained in a paper submitted through FROBENIUS at the December 16, 1897 meeting of the Berlin Academy [1898: 1152]:

> In two notes [1897a, 1897b] ... I have drawn certain conclusions about the properties of substitution groups from the general theory of number systems with noncommutative units.
>
> In terms of content, my observations prove to be closely related to the works of Herr Frobenius [1896b, 1896c, 1897] .... In my first note, however, I placed the main emphasis on the reversibility of the theorems. The investigations of Herr Frobenius have just now come to my knowledge through the good offices of the author.

Considering the results in MOLIEN's thesis and the suggestiveness of the work of KLEIN and others on finite groups of linear transformations, MOLIEN's independence of FROBENIUS is not surprising. Even MOLIEN's discovery of the orthogonality relations for characters and their utility is not altogether surprising in view of the fundamental role played by the trace function $f_1(u)$ in his thesis and his interest in the properties of the degrees v_k of the irreducible representations.

When WILLIAM BURNSIDE (1852–1927) was elected a fellow of the Royal Society in 1893, it was on the basis of his work in mathematical physics and complex function theory. The following year, however, BURNSIDE's "Notes on the Theory of Groups of Finite Order" began to appear in the *Proceedings* of the London Mathematical Society. They marked the beginning of a new phase in his career, one in which he became engrossed in the theory of finite groups. BURNSIDE'S interests in group theory were actually not confined exclusively to finite groups but included the full range of research on finite and infinite groups that was taking place on the Continent. His first papers with a group-theoretic flavor had dealt with automorphic functions (1892), and he was also one of the few British mathematicians familiar with LIE'S theory of groups (BURNSIDE [1895a: 42n, 55]).

In 1898 BURNSIDE was able to combine his interests in finite and continuous groups, and published a paper "On the Continuous Group that is defined by any given Group of Finite Order" [1898a]. One of his main sources of information on continuous groups appears to have been LIE'S *Vorlesungen über continuirliche Gruppen* [1893b]. LIE'S lectures had been prepared for publication by SCHEFFERS and contained, as one would expect, a detailed discussion of hypercomplex systems and STUDY'S work on their relation to LIE groups. Two simple examples of the simply transitive, bilinear groups that correspond to such systems are:

$$T_y': \begin{array}{l} x_1' = x_1 y_1 + x_2 y_2 \\ x_2' = x_1 y_2 + x_2 y_1 \end{array} \quad \text{and} \quad T_y': \begin{array}{l} x_1' = x_1 y_1 + x_2 y_2 + x_3 y_3 \\ x_2' = x_1 y_3 + x_2 y_1 + x_3 y_2 \\ x_3' = x_1 y_2 + x_2 y_3 + x_3 y_1 . \end{array}$$

These groups are obtained by permuting the parameters y_i of each line of the above equations by the members, respectively, of the cyclic groups generated by the permutations (12) and (123). According to BURNSIDE [1898a: 208], "it was, in fact, a consideration of these simple cases that suggested to me the form for a system of transformations of n variables, which is given by any finite group of order n." The continuous n-parameter group G which BURNSIDE associated with a finite group g of order n was the group of transformations y_R: $x \to xy$ determined by the group algebra $\mathscr{H}$ of g. BURNSIDE did not speak of hypercomplex numbers *per se*. He wrote down the equations for y_R that come from setting $x = \sum_{i=1}^{n} x_i S_i$ and $y = \sum_{i=1}^{n} y_i S_i^{-1}$, where $g = \{S_1, \dots S_n\}$. He may have been influenced in the use of the inverses by FROBENIUS. (See below.)

To study the structure of G, BURNSIDE considered the associated LIE algebra $\mathscr{L}$ and showed, in essence, that $\mathscr{L}$ can be identified with the group algebra $\mathscr{H}$ with $[x, y] = xy - yx$ [1898a: 211]. Next he determined the "self-conjugate" members of $\mathscr{L}$, *i.e.*, those $z \in \mathscr{H}$ such that $zx = xz$ for all $x \in \mathscr{H}$. These elements form the center $\mathscr{Z}$ of $\mathscr{H}$, and he proved that $z = \sum_{i=1}^{n} z_i S_i^{-1}$ is self-conjugate if and only if the z_i corresponding to conjugate group elements are equal so that these elements form a subalgebra of $\mathscr{H}$ of dimension r, r being the number of conjugate classes of g. Similar considerations showed that the derived algebra $\mathscr{L}'$ has dimension $n - r$ and that $\mathscr{L} = \mathscr{Z} \oplus \mathscr{L}'$. BURNSIDE was familiar with CARTAN'S thesis [1894], and he applied CARTAN'S criterion for semisimplicity to conclude that $\mathscr{L}'$ is semisimple. Thus G is integrable if and only if g is ABELIAN.

The structure of G had thus been fairly well described by BURNSIDE. One question remaining was: Into what type of simple algebras does $\mathscr{L}'$ decompose (as it must by CARTAN's theorems in [1894]) ? He did not consider it, but contented himself with actually computing the simple algebras into which $\mathscr{L}'$ decomposes when $g = D_p$, the dihedral group of order $2p$ (p an odd prime). He showed that in this case $\mathscr{L}'$ is the direct sum of $\frac{1}{2}(p-1)$ 3-dimensional simple algebras, but he did not conclude from this anything about the structure of G itself.

The above-described paper was submitted on January 10, 1898. Six months later, BURNSIDE submitted another with the same title in which he proposed [1898c: 546] to bring the analysis of G "to a definite conclusion" by proving that G is a "direct product" of r general linear groups (equivalently: $\mathscr{H}$ is the direct sum of r complete matrix algebras). In what follows we shall briefly indicate the manner in which BURNSIDE's second paper [1898c] grew out of his first and another paper of his [1898b] under the guidance of ideas and theorems from the papers of FROBENIUS [1896a, 1896b, 1896c] and also MOLIEN [1893a].

BURNSIDE was already familiar with FROBENIUS' work when he published [1898a].[29] A few years earlier he had published two papers only to find that FROBENIUS had completely anticipated him (BURNSIDE [1895b: 191–2]). As a result, he undoubtedly read the *Sitzungsberichte* of the Berlin Academy—where FROBENIUS was publishing all his work—as soon as it appeared. Thus he commented [1898a: 207]: "The matrix of n^2 coefficients in any transformation [of G] ... has formed the subject of investigation of Herr Frobenius [1896c] ...; and from a result given by him [1896c: 1346] ... it follows at once that such a system of transformations have the group property." But, as he justifiably added: "The main subject of Herr Frobenius' memoir is, however, entirely different from that of the present paper, and he makes no reference to the continuous group which is connected with the matrix."

One wonders if BURNSIDE would have written those words if he had read FROBENIUS' paper [1897] on matrix representations of groups, but he did not see it until after he had written both [1898a] and [1898c]. Nevertheless even when he wrote [1898a] he was aware of further connections between his and FROBENIUS' work. He observed [1898a: 222] that the subgroup H of G corresponding to the algebra $\mathscr{Z}$, i.e., $H = \{a_R : a \in \mathscr{Z}\}$, can be expressed in terms of parameters $a_1, \ldots a_r$ where $a = a_1 z_1 + \cdots + a_r z_r$, $z_i = \sum_{S_j \in C_i} S_j^{-1}$, and $C_1, \ldots C_r$ are the conjugate classes of g. In terms of these parameters, the parameter equations corresponding to $a'_R a_R = a''_R$ are

$$a''_s = \sum_{t,k=1}^{r} \gamma_{sk^{-1}t}\, a'_t a_k \qquad s = 1, \ldots r. \tag{4.12}$$

Now BURNSIDE recognized that $\gamma_{sk^{-1}t}$ was simply FROBENIUS' $a_{s'k't'} = h_{sk't'}/h_s$ and, hence, that (4.12), which defines $\mathscr{Z}$, played a fundamental role in FROBENIUS' work. (See HAWKINS [1971: Thm. 6.1].)

BURNSIDE's interest in STUDY's results (as expounded in LIE's *Vorlesungen* [1893b]) not only prompted him to compose [1898a] but also another paper

[29] A familiarity with FROBENIUS' papers as discussed in Hawkins [1971] is assumed in what follows.

"On Linear Homogeneous Continuous Groups whose Operations Are Permutable" [1898b], submitted on March 7, 1898. Two such groups in the same number of variables are said to be conjugate if they are isomorphic by an inner automorphism ϕ_S: $A \to S^{-1} A S$ of the general linear group in which they are contained. BURNSIDE's objective was to determine the "conjugate classes" into which these groups divide. His method of attacking the problem was based upon the theory of elementary divisors and CAMILLE JORDAN's canonical form for the equations of a linear transformation. WEIERSTRASS, who had developed the theory of elementary divisors [1868], had proved that two transformations are congruent (or similar) if and only if they have the same elementary divisors. The elementary divisors of a congruence class of transformations can be read off immediately from the canonical form introduced by JORDAN (for fields of finite characteristic) in his *Traité des substitutions* [1870: 114–26], as E. NETTO had pointed out in a paper [1893] that BURNSIDE read.

BURNSIDE's idea was to determine the classification of the above-described groups by looking at their elementary divisors, after putting the transformations in a canonical form induced by putting a fixed member of the group in its JORDAN form. Using this tactic, he showed that the transformations T_y of an ABELIAN linear group G can be defined by equations with coefficient matrix

$$\begin{pmatrix} a_1 I_1 & & & & \\ * & \cdot & & & \\ * & * & \cdot & & \\ * & * & * & \cdot & \\ * & * & * & * & a_s I_s \end{pmatrix} \tag{4.13}$$

where I_i is the p_i-by-p_i identity matrix and the a_i's need not be distinct [1898b: Section 9]. This form will be seen to play a significant role in BURNSIDE's second paper [1898c] on group representations.

Using (4.13), BURNSIDE had some success determining the conjugate classes for transitive groups. He concluded his paper by turning briefly to intransitive groups and considered "the most extreme case of an intransitive group," namely the case in which $G = \{T_y\}$ is bilinear and the first elementary divisor of

$$\det(T_y + \lambda) = \prod_{i=1}^{s} (a_i + \lambda)^{m_i} \qquad (a_i = a_i(y) \text{ distinct})$$

is $\prod_{i=1}^{s} (a_i + \lambda)$. (That is, the greatest common divisor of the $(n-1)$-by-$(n-1)$ minors of $p(\lambda) = \det(T_y + \lambda)$ is $\prod_{i=1}^{s} (a_i + \lambda)^{m_i - 1}$.) In this case, as he showed, the canonical form (4.13) simplifies to

$$\begin{pmatrix} a_1 I_1 & & \\ & \ddots & \\ & & a_s I_s \end{pmatrix} \tag{4.14}$$

where the a_i are now distinct (as linear functions in y).

Consideration of this extreme type of intransitive group may have been partly motivated by the ABELIAN group H which BURNSIDE had defined in

[1898a]. As noted above, he was well aware of the important role played by H (via (4.12)) in FROBENIUS' work. If the parameters y_i of $H=\{y_R: y\in\mathscr{X}\}$ are those determined by $y=y_1S_1^{-1}+\cdots+y_nS_n^{-1}$, then $y_i=y_j$ if S_i and S_j are congruent; and hence

$$\Theta^*(y_1,\dots y_n)=\det(y_R),$$

where Θ^* is FROBENIUS' special group determinant. Furthermore, FROBENIUS' calculations concerning Θ^* made it clear that H is an intransitive group of the above type. (*Cf.* BURNSIDE [1898c: 557] and FROBENIUS [1896b: 1005; 1896c: 1357].) Thus H has the canonical form (4.14); and, since the dimension of H is the s of (4.14) and the r of (4.12), it follows that $r=s$. FROBENIUS' theorem that the number of distinct linear factors in Θ^* is equal to r, the number of conjugate classes of g, is thereby proved.

In summary, the results in [1898b], when applied to the H of [1898a], yielded FROBENIUS' major results about Θ^*. BURNSIDE explicitly pointed this out in [1898c], but it is clear he knew much of the above while composing [1898b]. (See, *e.g.* [1898b: 332n].) His success at reproving FROBENIUS' theorems on Θ^* undoubtedly encouraged him to investigate the connection between FROBENIUS' results for the general group determinant Θ and the group G of [1898a]. BURNSIDE discovered that the variables and parameters of G may be chosen so that the equations of G have coefficient matrix

$$\begin{pmatrix} (u_{ij}^{(1)}) & & & \\ & (u_{ij}^{(2)}) & & \\ & & \ddots & \\ & & & (u_{ij}^{(r)}) \end{pmatrix}. \tag{4.15}$$

Thus G is a "direct product" of r general linear groups G_i in μ_i variables, where $n=\mu_1^2+\cdots+\mu_r^2$. (4.15) is, of course, the analog of FROBENIUS' theorem that $\Theta=\prod_{i=1}^{r}\Phi_i^{\mu_i}$; it is also similar to considerations in MOLIEN's thesis [1893a], which BURNSIDE had read by the time he wrote [1898c].

BURNSIDE deduced (4.15) by using the form (4.14) for H. Since the transformations in H commute with all transformations in G, the corresponding form for the members T_y of G must be

$$\begin{pmatrix} N_1(y) & & & \\ & N_2(y) & & \\ & & \ddots & \\ & & & N_r(y) \end{pmatrix}.$$

The groups defined by the $N_i(y)$ turn out to be the desired G_i. The idea of using the decomposition of H by (4.14) to induce a decomposition of G may have been inspired by FROBENIUS' paper; for FROBENIUS proved that, in passing from the general group determinant Θ to the special determinant Θ^*, each irreducible factor $\Phi_i(y)$ of the former corresponds to a distinct linear factor $a_i(y)$ of the latter: $\Phi_i(y)^*=a_i(y)^{\mu_i}$ (in an appropriate ordering). Thus the $N_i(y)$ would correspond to the Φ_i. BURNSIDE's proof that the groups defined by the $N_i(y)$ can be put in the form described in (4.15) relied heavily for inspiration on the results in MOLIEN'S thesis [1893a] on the characterization of simple hypercomplex systems (BURNSIDE [1897c: 547]).[30]

KLEIN'S normal problem, to which BURNSIDE referred [1898c: 547], and FROBENIUS' group determinant Θ undoubtedly prompted BURNSIDE to specialize his results about G to the underlying finite group g. Each group G_i of (4.15) defines a finite group g_i of linear transformations homomorphic to g (by specializing the parameters in (4.15) so as to obtain $(S_i)_R$, $i=1, \dots n$). "Now g, and the groups of smaller order with which g is [homomorphic] ... can, in general, be represented as groups of linear substitutions in a variety of ways, and the question arises as to which of these different modes of representation occur among the groups g_i" [1898c: 562]. This question was certainly inspired by KLEIN'S normal problem. Suppose $g \to g'$ is a representation of g by a group g' of linear transformations on m variables. BURNSIDE considered the continuous group G' determined by g' and showed that it is the direct product of G_i's from (4.15). Specialized to g', this showed that g' is equivalent to a direct sum of representations g_i. In particular, "if it is impossible, by choosing new variables, to divide the variables into sets such that those of each set transform among themselves by every operation of g', then m must be one of the integers μ_s ..." [1898c: 563].

In this manner BURNSIDE deduced the basic theorems of group representation theory. Later, in a historical footnote [1911: 269n], BURNSIDE wrote that in the papers [1898a] and [1898c] he had "obtained independently the chief results of Prof. Frobenius' earlier memoirs." In what sense is this the case? On the basis of the above discussion, we suggest the following interpretation. Inspired by the work of STUDY, BURNSIDE hit upon the relation between g and G. He was familiar with FROBENIUS' papers on the group determinant and may have been influenced by them to see the connection between g and G, although probably not. When he wrote [1898a], he probably did not envision the continuation [1898c] but was encouraged by his results in [1898b] and their relation to FROBENIUS' theorems on Θ^* to hope that the correspondence between H and Θ^* would carry over to one between Θ and G. In this connection the results in the papers of FROBENIUS and MOLIEN proved very suggestive.

BURNSIDE thus obtained his results "independently" in the sense that his approach was different and he proved all his theorems without invoking, as proved, theorems from the papers of FROBENIUS or MOLIEN, and without knowing of the papers of FROBENIUS [1897] and MOLIEN [1897a] on the matrix representation of groups. And it is quite conceivable that someone, probably BURNSIDE himself, would have eventually discovered the results in [1898c], even without FROBENIUS' papers. Indeed they were already suggested by the theorems posited in CARTAN'S note [1895] discussed in Section 3. Also the structure theorems of MOLIEN [1893a] and CARTAN [1898], considered in the light of BURNSIDE'S paper [1898a], would have eventually inspired someone to make BURNSIDE'S discoveries. But FROBENIUS did write his papers; BURNSIDE did read them; and, as we have indicated, the inspiration BURNSIDE received from them played a considerable role in the actual discovery and proof of the theorems presented in [1898c].

[30] BURNSIDE'S proof was incomplete, as FROBENIUS [1903: 401] pointed out.

The work of MOLIEN and BURNSIDE thus represented two possible approaches to the creation of group representation theory via the group algebra. MOLIEN'S consisted in applying the structure theorems for hypercomplex systems directly to the group algebra. Here MOLIEN'S or CARTAN'S criterion for semisimplicity (that $f_1(uv)$ or $f_2(u)$ be nonsingular) played an important role. BURNSIDE'S approach involved looking at the LIE group defined by the group algebra or, equivalently, looking at the group algebra as a LIE algebra $\mathscr{L}$. Here CARTAN'S criterion for semisimplicity (that $\psi_2(u)$ be nonsingular) is important because it shows that $\mathscr{L}'$ is semisimple and, hence, a direct sum of simple algebras. BURNSIDE himself did not follow through on this line of thought due to the influence of FROBENIUS' work. But considerations such as those in CARTAN'S note [1895], or a reading of that note, could have indicated that the simple components of $\mathscr{L}'$ are made up of all matrices of a given size with zero trace—a fact that could also have led to the realization that the group algebra is a sum of complete matrix algebras and hence to group representation theory.

It is of interest to note in passing that ALFRED YOUNG was led, independently of FROBENIUS, BURNSIDE and MOLIEN, to study the structure of the group algebra of the symmetric group S_n [1901, 1902]. His work on the theory of invariants turned his interest to equations of the form

$$\sum_{i=1}^{n!} \lambda_i \sigma_i F(x_1, \ldots x_n) = 0,$$

where F is an unknown form (or polynomial) in the x_i, $\sigma_i \in S_n$, and $\sigma_i F(x_1, \ldots x_n)$ denotes the form obtained from F by permuting the x_i according to σ_i. YOUNG showed that this equation can be replaced by the "substitutional equation" $ax=0$, where a and x are "substitutional expressions", *i.e.*, expressions of the form $\sum_{i=1}^{n!} \lambda_i \sigma_i$, and ax denotes the product of a and x as members of the group algebra of S_n.[31]

5. Maschke's Discovery of Complete Reducibility

Although he obtained his doctorate at Göttingen in 1880, and hence before KLEIN became a professor there, HEINRICH MASCHKE (1853–1908) was a student of KLEIN'S insofar as his mathematical activity was concerned. After receiving his doctorate, MASCHKE taught in a Gymnasium in Berlin until he returned to Göttingen in 1886 on a leave of absence. Under KLEIN'S influence, he soon became a contributor to the general research program of KLEIN'S form-problem (discussed in Section 4) and published a number of papers dealing with the determination of the invariants for particular groups of linear transformations and also with the linear groups themselves. (See BOLZA [1908, 1909].) Realizing he had no future as a mathematician in a Gymnasium and little chance of obtaining a university position in Germany, he immigrated to the United States

[31] YOUNG'S work, which reflects no acquaintance with the theory of hypercomplex numbers, was related to group representation theory by FROBENIUS [1903a: 349–50]. For a treatment of YOUNG'S contributions (including his later papers) and their significance in the representation theory of S_n, see RUTHERFORD [1948].

in 1891. In 1892 he joined his friend OSKAR BOLZA, another student of KLEIN'S, and E. H. MOORE to form the Mathematics Department of the recently-established University of Chicago.

At Chicago, MASCHKE continued to do work connected with KLEIN'S program, but his papers that led to the discovery of the complete reducibility theorem involved a generality not characteristic of the KLEINIAN program and that may have reflected MOORE'S influence. MOORE was the acting (and, after 1896, permanent) chairman of the Department and its guiding spirit. His work, which focused on group theory in the mid- and late-1890's, was characterized by its generality. In 1896 KLEIN'S influential paper [1876] on the determination of all binary linear groups led MOORE to the discovery that any finite group G of linear transformations possesses a "universal invariant". That is, if G is a group of order r in n variables and H is any positive definite HERMITIAN form, *e.g.*, $H=\sum_{i=1}^{n} x_i \bar{x}_i$, and if $H_1, \ldots H_r$ denote the forms into which H is transformed by the members of G, then $\varphi=\sum_{i=1}^{r} H_i$ is a positive definite HERMITIAN form that is invariant under the transformations of G.[32] For groups in two or three variables the existence of an invariant form had been proved by VALANTINER [1889: 102, 210, 138, 218] by a complicated analysis based on the canonical form for a linear transformation of finite order, which JORDAN had indicated [1878: 112–3]. (See Section 4, footnote 28.) This apparently prompted MOORE to publish a paper [1898] in which, conversely, the JORDAN form for a transformation of finite order is established by applying the universal invariant theorem.

Interest in the JORDAN form was not limited to MOORE. MASCHKE also submitted a paper [1898a] that appeared alongside MOORE'S and in which the JORDAN form for a transformation of finite order was established using ideas from a paper [1887] by LIPSCHITZ—a paper also inspired by JORDAN'S observation in [1878]. An immediate consequence of the JORDAN form is that for any finite group G of linear transformations, the trace of T is a sum of roots of unity for all $T \in G$. In a subsequent paper [1898b], MASCHKE proposed to investigate the algebraic nature of the coefficients comprising the matrix of the equations of the transformations in an arbitrary finite group G—a problem which, by virtue of its generality, stood in sharp contrast to his earlier work. Starting from the result about the traces of $T \in G$, he was able to prove that, for any finite group G of transformations satisfying a certain condition (given below), a linear change of variables is possible so that the corresponding matrix of coefficients consists entirely of cyclotomic numbers, *i.e.*, rational functions with rational coefficients of roots of unity. The condition G had to satisfy was that some $T_0 \in G$ have distinct characteristic roots. It was in the course of proving this theorem that MASCHKE was led to the complete reducibility theorem.

[32] This result was discovered independently by A. LOEWY [1896], who devised the same proof but did not publish it. PICARD [1887] had established the theorem for groups in two and three variables, except that for one type of group he gave a form invariant up to a constant factor, thereby prompting LOEWY'S remarks. A weaker version of the theorem was also implicit in a paper by L. FUCHS [1896a] on differential equations. (See FUCHS [1896b] and LOEWY [1898: 561n].)

MASCHKE began in a manner similar to BURNSIDE in [1898b]: Consider the group G expressed in the variables which put T_0 in its JORDAN form. He then first considered the special case in which the corresponding matrices for $T \in G$ have no "identically zero" (necessarily nondiagonal) entry, *i.e.*, no entry that is zero for every $T \in G$. Having proved the theorem in this special case, he next sought to reduce the general case to the special one in the following manner. Let G be a group in n variables and assume without loss of generality that the last $n-r$ entries of the first row of the matrices of $T \in G$ are identically zero. It then followed by a theorem due to VALENTINER [1889: 93–95] that the matrices of $T \in G$ must actually have the form

$$\begin{pmatrix} Q_1 & 0 \\ R & Q_2 \end{pmatrix} \tag{5.1}$$

where Q_1 and Q_2 are, respectively, r-by-r and $(n-r)$-by-$(n-r)$ matrices. MASCHKE's idea was to prove that when (5.1) is possible, then it is also possible by a further change of variables to put the matrices for $T \in G$ in the form

$$\begin{pmatrix} Q_1' & 0 \\ 0 & Q_2' \end{pmatrix}. \tag{5.2}$$

It would then follow that a change of variables is possible so that the matrices of $T \in G$ have the form

$$\begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_s \end{pmatrix} \tag{5.3}$$

where the matrices Q_i have no identically zero nondiagonal entry so that the proof for the special case applies to them. To go from (5.1) to (5.2), MASCHKE made use of MOORE's theorem that a positive definite HERMITIAN form exists which is an invariant for G (see below).

MASCHKE had made use of the assumption that $T_0 \in G$ has distinct characteristic roots throughout his paper, including the part in which he proceeded from (5.1) to (5.2). Shortly thereafter, he showed [1899] that this part, at least, could be freed from the restriction concerning T_0. Again, he made use of MOORE's universal invariant φ. Assuming the transformations of G are in the form (5.1), he proved that a linear change of variables was possible so that the matrices of $T \in G$ retain the general form (5.1) while φ takes the form $\varphi = \sum_{i=1}^{n} x_i \bar{x}_i$. In other words, MASCHKE proved that any representation of a group is equivalent to one by unitary matrices. The fact that φ is invariant under $T \in G$ then easily establishes that $R=0$ in (5.1), *i.e.*, that the matrices of $T \in G$ can be put in the form (5.2).[33]

[33] It is somewhat misleading to speak, as do CURTIS & REINER [1962: 420], of "Maschke's idea of 'averaging' over a finite group" in connection with the complete reducibility theorem. Association with the names of MOORE and LOEWY would be more appropriate. That all invariants of a finite group of linear transformations are obtainable by such an averaging process was, however, also pointed out by HURWITZ [1897: 71], who was primarily concerned with the extension of the averaging process to continuous groups. For the historical significance of HURWITZ' paper, see BOURBAKI [1969: 290].

MASCHKE'S discovery of complete reducibility was to a certain extent fortuitous. It was not an inevitable outcome of the theory of finite groups of linear transformations (as of 1898) to the same extent that the work of MOLIEN and BURNSIDE can be regarded as an inevitable consequence of developments in the theories of hypercomplex systems and LIE groups and algebras. Nevertheless, MASCHKE'S work further illustrates the fact that the creation of group representation theory was linked with a broad spectrum of late-nineteenth century mathematical thought. MOLIEN'S discovery and application of the orthogonality relations for group characters shows that even character theory was not exclusively the product of FROBENIUS' genius and the number-theoretic tradition (GAUSS, DIRICHLET, DEDEKIND) out of which the notion of a character had evolved.

In view of the developments discussed here and in HAWKINS [1971], it is clear that no less than four lines of mathematical investigation were leading to group representation theory: (1) the structure theory of hypercomplex numbers (MOLIEN, CARTAN); (2) the theory of LIE groups and algebras (CARTAN, BURNSIDE); (3) KLEIN'S form-problem (MASCHKE); (4) number theory and group determinants (DEDEKIND, FROBENIUS). Of course, these developments were interconnected. KILLING'S work was vital in both (1) and (2); FROBENIUS' paper on bilinear forms [1878] was important in (1) and (4); KLEIN'S normal-problem, an important part of (3), was of relevance to both (1) and (2), as we suggested; DEDEKIND'S paper on commutative hypercomplex numbers [1885] helped stimulate the developments in (1) and was vital in (4); and so on. These developments were, however, essentially distinct. Furthermore, it is of interest to note that (4) was perhaps the least inevitable of all. If DEDEKIND had not decided to introduce and study group determinants—a subject really outside his main interests in algebraic number theory—or if he had not decided to communicate his ideas on group determinants to FROBENIUS, it is unlikely that FROBENIUS would have created the theory of group characters and representations.

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