The Inclusion Problem for Unambiguous Rational Trace Languages*

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Abstract. Given a class $\mathcal{C}$ of languages, the Inclusion Problem for $\mathcal{C}$ consists of deciding whether for $L_1, L_2 \in \mathcal{C}$ we have $L_1 \subseteq L_2$.
In this work we prove that the Inclusion Problem is decidable for the class of unambiguous rational trace languages that are subsets of the monoid $(((a_1^\star \cdot b_1^\star) \times c_1^\star) \cdot ((a_2^\star \cdot b_2^\star) \times c_2^\star)) \times c_3^\star$.

1 Introduction

Trace languages have been introduced by Mazurkiewicz [15] as a model of the behaviour of concurrent processes. More precisely, trace languages are subsets of free partially commutative monoids for which several decision problems have been studied. In particular, given a class $\mathcal{C}$ of trace languages the Inclusion Problem for $\mathcal{C}$ consists of deciding whether for $L_1, L_2 \in \mathcal{C}$ we have $L_1 \subseteq L_2$. The Equivalence Problem for $\mathcal{C}$ ($L_1 = L_2$?) can be defined similarly; moreover, it is immediate to see that, for every class $\mathcal{C}$, Equivalence for $\mathcal{C}$ is reducible to Inclusion for $\mathcal{C}$ since $L_1 = L_2$ iff $L_1 \subseteq L_2$ and $L_2 \subseteq L_1$. Therefore, if Inclusion is decidable then Equivalence is decidable, and if a class $\mathcal{C}$ is closed under union (or intersection) then both problems are either decidable or undecidable since $L_1 \subseteq L_2$ if and only if $L_1 \cup L_2 = L_2$ ($L_1 \cap L_2 = L_1$).

In this paper we deal with the class of unambiguous rational trace languages $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$, a particular subclass of the class of rational trace languages $\mathrm{Rat}(\Sigma, C)$ that has been widely studied and for which many results are known. In particular, using a technique due to Ibarra ([12]), in [1], [11] it is shown that Equivalence for $\mathrm{Rat}(\Sigma, C)$ is undecidable when $\Sigma = \{a, b, c\}$ and $C = {}_{a}\overset{b}{\wedge}{}_{c}$. On the other hand, Inclusion turns out to be decidable when C is transitive ([5]). We also recall that Equivalence for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ is decidable for any commutation relation C ([19]), while, for the same class, Inclusion is undecidable if C is the relation ${}^{a}_{d}\square^{b}_{c}$ ([4]). Last but not least, in [7] it is shown that Inclusion is decidable for the class $\mathrm{Rat}_{\mathrm{Fin}}(\Sigma, C)$ of finitely ambiguous rational trace languages over an alphabet $\Sigma = A \,\dot\cup\, B$ with $C = (A \times \Sigma \cup \Sigma \times A) \setminus I$.

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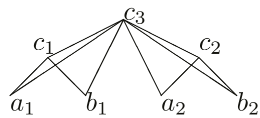
The main result we present here is that Inclusion is decidable for the class $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ with alphabet $\Sigma = \{a_1, b_1, c_1, a_2, b_2, c_2, c_3\}$ and commutation relation C given by



We follow a technique similar to that used in [8] to show that Equivalence is decidable for a class $\mathcal{C}$ of recursive languages that is c-holonomic and c-closed under intersection (i.e. the elements of $\mathcal{C}$ admit holonomic generating functions, have finite computable specifications, their intersection is in $\mathcal{C}$ and has a specification computed in a finite time). More precisely, given two trace languages $L_1, L_2 \in \mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$, we reduce the problem of deciding whether $L_1 \subseteq L_2$ to the problem of verifying an equality between generating functions:

$$\phi_{L_1}(z) = \phi_{L_1 \cap L_2}(z) \ .$$

By showing that these generating functions are holonomic, it turns out that Inclusion for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ is reduced to Equivalence for holonomic functions, a problem that is well known to be decidable (see, for instance, [20]).

2 Preliminaries

In this section we recall some basic definitions and results about trace languages and formal series.

2.1 Monoids and Languages

Let M_1, M_2 be two submonoids of a monoid M. The free product of M_1 and M_2 is defined as the monoid $M_1 \cdot M_2 = \{m \in M \mid m = m_1 m_1' m_2 m_2' \cdots m_k m_k', m_i \in M_1, m_i' \in M_2\}$. The direct product of monoids is denoted by $\times$ and defined as $M_1 \times M_2 = \{(m_1, m_2) \mid m_1 \in M_1, m_2 \in M_2\}$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a finite alphabet and let $\Sigma^\star$ be the free monoid generated by Σ, that is, the monoid $\sigma_1^\star \cdot \sigma_2^\star \cdot \ldots \cdot \sigma_n^\star$. The elements of $\Sigma^\star$ are called words. If $w = \sigma_{i_1} \cdots \sigma_{i_n} \in \Sigma^\star$ its length is $|w| = n$. A language is simply a subset of $\Sigma^\star$.
A commutation relation on Σ is an irreflexive and symmetric relation $C \subseteq \Sigma \times \Sigma$. We denote by $F(\Sigma, C)$ the free partially commutative (f.p.c.) monoid $\Sigma^\star / \rho_C$ where $\rho_C \subseteq \Sigma^\star \times \Sigma^\star$ is the congruence generated by C. We call *trace* an element of a f.p.c. monoid. A trace can be interpreted as an equivalence set of words: given a string $w \in \Sigma^\star$ we denote by $[w]_{\rho_C}$ the equivalence class of w (i.e. the trace generated by w). A *trace language* is a subset of $F(\Sigma, C)$; given a language $L \subseteq \Sigma^\star$ and a commutation relation C, the trace language generated by L is

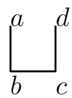
$$[L]_{\rho_C} = \{[w]_{\rho_C} \mid w \in L\} \subseteq F(\Sigma, C) \ .$$

When $C = \Sigma \times \Sigma$, we denote by Σ^c the free commutative monoid generated by Σ. An element $\underline{\sigma}^{\underline{a}} = \sigma_1^{a_1} \cdots \sigma_n^{a_n}$ of Σ^c is called monomial and the product of monomials becomes $\underline{\sigma}^{\underline{a}} \cdot \underline{\sigma}^{\underline{b}} = \sigma_1^{a_1+b_1} \cdots \sigma_n^{a_n+b_n}$.

We define here a class $\mathcal{M}$ of monoids that is of particular interest for the cases we consider later on.

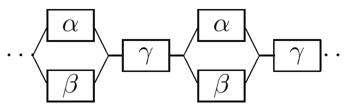
Definition 1. *$\mathcal{M}$ is the smallest class of monoids containing the monoids $\sigma^\star$ ($\sigma \in \Sigma$) and closed under $\cdot$ and $\times$.*

It is easily shown that all the elements of $\mathcal{M}$ are f.p.c. monoids and that there exist f.p.c. monoids that are not in $\mathcal{M}$ as, for example, the f.p.c. monoid defined by the commutation relation



The elements of a monoid $m \in \mathcal{M}$ are called *serial/parallel* traces since they can be obtained as serial/parallel composition of basic components belonging to free noncommutative monoids.

Example 1. Let $\Sigma = \{a, b, c, d, e\}$. An element of the monoid $((a^\star \cdot d^\star) \times (b^\star \cdot c^\star)) \cdot e^\star$ has a structure of type



where α belongs to the monoid $a^\star \cdot d^\star$, β belongs to $b^\star \cdot c^\star$ and $\gamma \in e^\star$.

2.2 Formal Series and Rational Languages

Let $\mathbb{Q}$ be the field of rational numbers. A *formal series* ψ on a monoid M with coefficients in $\mathbb{Q}$ is a function $\psi : M \mapsto \mathbb{Q}$. We denote by (ψ, w) the coefficient in $\mathbb{Q}$ associated with w by ψ, and we encode ψ by the formal sum $\sum_{w \in M} (\psi, w) w$. The *support* of a series ψ is the set $\mathrm{Supp}(\psi) = \{m \in M \mid (\psi, m) \neq 0\}$. The sum and the Cauchy product of formal series are defined respectively as

$$(\phi + \psi, w) = (\phi, w) + (\psi, w), \qquad (\phi \cdot \psi, w) = \sum_{xy=w} (\phi, x) \cdot (\psi, y) \ .$$

We also consider the Hadamard product of two series, defined as

$$(\phi \odot \psi, w) = (\phi, w) \cdot (\psi, w) \ .$$

Let Σ be a finite alphabet, we denote by $\mathbb{Q}\langle\langle \Sigma \rangle\rangle$ the ring of formal series on the free monoid $\Sigma^\star$ having values in $\mathbb{Q}$. We also indicate by $\mathbb{Q}\langle \Sigma \rangle$ the ring of polynomials, that is, the ring of series in $\mathbb{Q}\langle\langle \Sigma \rangle\rangle$ having finite support.

Definition 2. *A series $\phi \in \mathbb{Q}\langle\langle \Sigma \rangle\rangle$ is called* rational *if it is an element of the rational closure of $\mathbb{Q}\langle \Sigma \rangle$ (operations of sum, Cauchy product, external product of $\mathbb{Q}$ on $\mathbb{Q}\langle\langle \Sigma \rangle\rangle$, star).*

Definition 3. *A series $\phi \in \mathbb{Q}\langle\langle \Sigma \rangle\rangle$ is said recognizable if and only if there exist an integer $n \geq 1$, a morphism of monoids $\mu : \Sigma^\star \mapsto \mathbb{Q}^{n\times n}$, a row vector $\lambda \in \mathbb{Q}^{1\times n}$ and a column vector $\gamma \in \mathbb{Q}^{n\times 1}$ such that for all $w \in \Sigma^\star$ $(\phi, w) = \lambda\mu w\gamma$. The triple $\langle \lambda, \mu, \gamma \rangle$ is called the linear representation of ϕ.*

Theorem 1 ([17]). *A formal series is recognizable if and only if it is rational.*

We refer to [3] for a detailed introduction to the class of rational formal series. When we consider series on a f.p.c. monoid $\mathrm{F}(\Sigma, C)$, we say that $\phi : \mathrm{F}(\Sigma, C) \mapsto \mathbb{Q}$ is rational if and only if there exists a linear representation $\langle \lambda, \mu, \gamma \rangle$ such that

$$(\phi, t) = \sum_{\substack{w\in\Sigma^\star \\ [w]_C = t}} \lambda\mu w\gamma \ .$$

Given a subset $L \subseteq M$, the *characteristic series* of L is the formal series $\chi_L : M \mapsto \{0, 1\}$ defined as

$$(\chi_L, m) = \begin{cases} 1 & m \in L, \\ 0 & \text{otherwise} \end{cases} .$$

Given a formal series ϕ, the generating function associated with ϕ is the function

$$f_\phi(z) = \sum_{m\in M} (\phi, m) z^{|m|} \ .$$

In particular, the *generating function* of L is the generating function associated with the characteristic series of L, that is,

$$f_L(z) = f_{\chi_L}(z) = \sum_{m\in M} (\chi_L, m) z^{|m|} = \sum_{n\geq 0} c_n z^n \ ,$$

where $c_n = \sharp\{m \in L \mid |m| = n\}$.

We also consider formal series on the free commutative monoid Σ^c. A formal series in commutative variables is a function $\psi : \Sigma^c \mapsto \mathbb{Q}$,

$$\psi(\sigma_1, \dots, \sigma_n) = \sum_{\underline{\sigma}^{\underline{\imath}}\in\Sigma^c} \psi_(\underline{\sigma}^{\underline{\imath}})\underline{\sigma}^{\underline{\imath}} = \sum_{\underline{\imath}\geq 0} \psi_(\underline{\sigma}^{\underline{\imath}})\underline{\sigma}^{\underline{\imath}} \ ,$$

where $\psi(\underline{\sigma}^{\underline{\imath}})$ indicates the coefficient in ψ of the monomial $\underline{\sigma}^{\underline{\imath}}$. In the rest of the paper, we use the operator $[\underline{\sigma}^{\underline{\imath}}]$ applied to $\psi(\sigma_1, \dots, \sigma_n)$, $[\underline{\sigma}^{\underline{\imath}}]\psi(\sigma_1, \dots, \sigma_n)$, in order to extract the coefficient $\psi_(\underline{\sigma}^{\underline{\imath}})$.
The set of formal series in commutative variables Σ with coefficients in $\mathbb{Q}$ is denoted by $\mathbb{Q}[[\Sigma]]$. On $\mathbb{Q}[[\Sigma]]$ we consider the usual operations of sum $(+)$, Cauchy product $(\cdot)$ and

- Partial derivative: $(\partial_i \phi)(\sigma_1^{a_1} \cdots \sigma_i^{a_i} \cdots \sigma_n^{a_n}) = (a_i+1)\phi(\sigma_1^{a_1} \cdots \sigma_i^{a_i+1} \cdots \sigma_n^{a_n})$,
- Primitive diagonal: if $p \neq q$ then $(\Delta_{pq}(\phi))(\sigma_1^{i_1} \cdots \sigma_{q-1}^{i_{q-1}} \sigma_{q+1}^{i_{q+1}} \cdots \sigma_n^{i_n}) = \phi(\sigma_1^{i_1} \cdots \sigma_{q-1}^{i_{q-1}} \sigma_q^{i_p} \sigma_{q+1}^{i_{q+1}} \cdots \sigma_n^{i_n})$,
- Substitution: $\phi(\psi_1(\tau_1, \ldots, \tau_m), \ldots, \psi_n(\tau_1, \ldots, \tau_m))$.

Some interesting subclasses of $\mathbb{Q}[[\Sigma]]$ are the class of commutative polynomials $\mathbb{Q}[\Sigma]$ and the class of rational formal series $\mathbb{Q}[[\Sigma]]_r$. A series $\phi \in \mathbb{Q}[[\Sigma]]_r$ can be thought as the power series expansion of a function P/Q with $P, Q \in \mathbb{Q}[\Sigma]$ and $Q(0) = 1$. It is known that $\mathbb{Q}[[\Sigma]]_r$ is not closed with respect to the Hadamard product and primitive diagonal (see, for instance, [10]), and so it is quite natural to look for an extension of $\mathbb{Q}[[\Sigma]]_r$ that is closed with respect to these operations: this extension consists of the class of holonomic series $\mathbb{Q}[[\Sigma]]_h$, formally defined as follows.

Definition 4. *A formal series $\phi \in \mathbb{Q}[[\Sigma]]$ is said to be holonomic iff there exist some polynomials $p_{ij} \in \mathbb{Q}[\Sigma]$, $1 \leq i \leq n$, $\quad 0 \leq j \leq d_i$, $p_{id_i} \neq 0$, such that*

$$\sum_{j=0}^{d_i} p_{ij} \partial_i^j \phi = 0, \quad 1 \leq i \leq n \ .$$

As a matter of fact, holonomic series are power series expansions of suitable functions that belong to the class of holonomic functions. This class was first introduced by I.N. Bernstein in the '1970 ([2]) and deeply investigated by Stanley, Lipshitz, Zeilberger et al. (see [9], [13], [14], [18] and [20]).
The closure properties of $\mathbb{Q}[[\Sigma]]_h$ are summarized in the following theorem.

Theorem 2. *The class $\mathbb{Q}[[\Sigma]]_h$ is closed under the operations of sum, Cauchy product, Hadamard product, primitive diagonal, substitution with algebraic series.*

Proof. See, for instance, [14]. □

At last, we recall that $\mathbb{Q}[[\Sigma]]_h$ properly contains the class $\mathbb{Q}[[\Sigma]]_a$ of algebraic formal series (see [16] for a definition of algebraic series). So, we have

$$\mathbb{Q}[[\Sigma]]_r \subset \mathbb{Q}[[\Sigma]]_a \subset \mathbb{Q}[[\Sigma]]_h \ .$$

A language $L \subseteq \Sigma^\star$ is said to be *rational* if and only if it is the support of a rational series on $\Sigma^\star$. In the case of trace languages we have the following:

Definition 5. *A trace language $L \in F(\Sigma, C)$ is rational if and only if there exists a rational series ϕ on $F(\Sigma, C)$ such that $L = Supp(\phi)$.*

Definition 6. *A trace language $L \in F(\Sigma, C)$ is said unambiguous rational if and only if there exists a rational series $\phi : F(\Sigma, C) \mapsto \{0, 1\}$ such that $L = Supp(\phi)$.*

We denote by $\mathrm{Rat}(\Sigma, C)$ the set of rational trace languages on $F(\Sigma, C)$ and by $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ the set of unambiguous rational trace languages.

3 The Inclusion Problem for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$

We formally define the Inclusion Problem for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ as follows.

Problem (Inclusion for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$). *Given two linear representations*

$$\langle \lambda_1, \mu_1, \gamma_1 \rangle , \quad \langle \lambda_2, \mu_2, \gamma_2 \rangle$$

defining two unambiguous rational trace languages L_1, L_2, decide whether

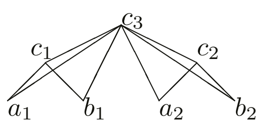
$$L_1 \subseteq L_2 \ .$$

We approach the Inclusion Problem by means of generating functions. In fact, given two subsets of a monoid, $L_1, L_2 \subseteq M$, it is immediate to note that $L_1 \subseteq L_2$ if and only if $L_1 \cap L_2 = L_1$, that is, if and only if $f_{\chi_{L_1} \odot \chi_{L_2}}(z) = f_{\chi_{L_1}}(z)$. Therefore, a natural way to solve Inclusion for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ leads to the following steps:

Step 1: Given $L_1, L_2 \in \mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ compute $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$ and $f_{\chi_{L_1}}(z)$.
Step 2: Given $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$, $f_{\chi_{L_1}}(z)$, decide whether $f_{\chi_{L_1} \odot \chi_{L_2}}(z) = f_{\chi_{L_1}}(z)$.

By showing that for a particular commutation relation C Step 1 and Step 2 are decidable, we prove the following:

Theorem 3. *The Inclusion Problem for $Rat_U(\Sigma, C)$ is decidable when C is given by*



Observe that the previous commutation relation can be obtained by considering a complete binary tree of height 2 (with symbols associated with nodes) and setting that two nodes a, b commute if and only if a is in the subtree rooted at b or vice versa.

3.1 Step 1: Computing $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$ and $f_{\chi_{L_1}}(z)$

In this section we show two examples that illustrate how to compute the generating function $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$ when $L_1, L_2 \in \mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ are subsets of particular monoids that belong to $\mathcal{M}$ (see Definition 1). Recall that the characteristic series $f_{\chi_L}(z)$ of $L \in \mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ is rational and so it admits a linear representation.

Example 2. Let $L_1, L_2 \in \mathrm{Rat}_{\mathrm{U}}(\{a, b\}, \emptyset)$ be defined by two rational characteristic series χ_{L_1}, χ_{L_2} (on the free monoid $a^\star \cdot b^\star$) with linear representations $\langle \lambda_1, \mu_1, \gamma_1 \rangle$, $\langle \lambda_2, \mu_2, \gamma_2 \rangle$ respectively. Given $w = x_1 \cdots x_n$, $x_i \in \{a, b\}$, we have

$$(\chi_{L_1}, x_1 \cdots x_n) = \lambda_1 \mu_1(x_1) \cdots \mu_1(x_n) \gamma_1 \ ,$$
$$(\chi_{L_2}, x_1 \cdots x_n) = \lambda_2 \mu_2(x_1) \cdots \mu_2(x_n) \gamma_2$$

and, by a well-known result (see, for instance, chapter 2 in [16]),

$$(\chi_{L_1} \odot \chi_{L_2}, x_1 \cdots x_n) = \lambda_1 \otimes \lambda_2 \cdot \mu_1(x_1) \otimes \mu_2(x_1) \cdots \mu_1(x_n) \otimes \mu_2(x_n) \cdot \gamma_1 \otimes \gamma_2 \ ,$$

where $\otimes$ is the usual Kronecker product of matrices.
Then, the generating function associated with $\chi_{L_1} \odot \chi_{L_2}$ is

$$\begin{aligned} f_{\chi_{L_1} \odot \chi_{L_2}}(z) &= \sum_{w \in \{a,b\}^\star} (\chi_{L_1} \odot \chi_{L_2}, w) z^{|w|} = \sum_{n \geq 0} z^n \sum_{\substack{w \in \{a,b\}^\star \\ |w|=n}} (\chi_{L_1} \odot \chi_{L_2}, w) = \\ &= \sum_{n \geq 0} z^n \lambda_1 \otimes \lambda_2 \cdot (\mu_1(a) \otimes \mu_2(a) + \mu_1(b) \otimes \mu_2(b))^n \cdot \gamma_1 \otimes \gamma_2 = \\ &= \lambda_1 \otimes \lambda_2 \, (I - z(\mu_1(a) \otimes \mu_2(a) + \mu_1(b) \otimes \mu_2(b)))^{-1} \cdot \gamma_1 \otimes \gamma_2 \end{aligned}$$

and belongs to $\mathbb{Q}[[z]]_r$.

The next example deals with unambiguous rational trace languages that are subsets of the monoid $(a^\star \cdot b^\star) \times c^\star \in \mathcal{M}$.

Example 3. Let $L_1, L_2 \in \mathrm{Rat}_{\mathrm{U}}(\{a,b,c\}, {}_a\overset{c}{\wedge}_b)$ be defined by the rational characteristic series χ_{L_1}, χ_{L_2} with linear representations $\langle \lambda_1, \mu_1, \gamma_1 \rangle$, $\langle \lambda_2, \mu_2, \gamma_2 \rangle$ respectively. We define the matrices

$$A_\sigma(c) = \mu_1(\sigma) \cdot \sum_{k \geq 0} c^k \mu_1(c)^k = \mu_1(\sigma) \cdot (I - c\mu_1(c))^{-1} \quad (\sigma \in \{a,b\}) \ ,$$

$$A'_\sigma(c) = \mu_2(\sigma) \cdot \sum_{k \geq 0} c^k \mu_2(c)^k = \mu_2(\sigma) \cdot (I - c\mu_2(c))^{-1} \ ,$$

and the row vectors

$$\Lambda_1(c) = \lambda_1 \sum_{k \geq 0} c^k \mu_1(c)^k = \lambda_1 \cdot (I - c\mu_1(c))^{-1} \ ,$$

$$\Lambda_2(c) = \lambda_2 \sum_{k \geq 0} c^k \mu_2(c)^k = \lambda_2 \cdot (I - c\mu_2(c))^{-1} \ .$$

Note that the entries in $A_\sigma(c), A'_\sigma(c), \Lambda_1(c)$ and $\Lambda_2(c)$ belong to $\mathbb{Q}[[c]]_r$. Therefore, for a trace $x_1 \cdots x_n c^k$, $x_i \in \{a,b\}$, we have

$$(\chi_{L_1}, x_1 \cdots x_n c^k) = [c^k] \Lambda_1(c) A_{x_1}(c) \cdots A_{x_n}(c) \gamma_1 \ ,$$

$$(\chi_{L_2}, x_1 \cdots x_n c^k) = [c^k] \Lambda_2(c) A'_{x_1}(c) \cdots A'_{x_n}(c) \gamma_2 \ .$$

By setting

$$A''_\sigma(c_1, c_2) = A_\sigma(c_1) \otimes A'_\sigma(c_2), \quad \Upsilon(c_1, c_2) = \Lambda_1(c_1) \otimes \Lambda_2(c_2), \quad \Gamma = \gamma_1 \otimes \gamma_2 \ ,$$

we have

$$(\chi_{L_1} \odot \chi_{L_2}, x_1 \cdots x_n c^k) = [c_1^k c_2^k](\Upsilon(c_1, c_2) \cdot A''_{x_1}(c_1, c_2) \cdots A''_{x_n}(c_1, c_2) \cdot \Gamma) \ .$$

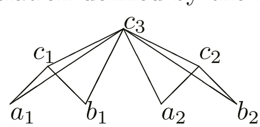
So, the generating function associated with $\chi_{L_1} \odot \chi_{L_2}$ is

$$\begin{aligned} f_{\chi_{L_1} \odot \chi_{L_2}}(z) &= \sum_{\substack{w \in \{a,b\}^\star \\ k \geq 0}} (\chi_{L_1} \odot \chi_{L_2}, wc^k) z^{|w|+k} = \\ &= \sum_{\substack{n \geq 0 \\ k \geq 0}} z^n z^k [c_1^k c_2^k] \sum_{\substack{w \in \{a,b\}^\star \\ |w|=n}} (\chi_{L_1} \odot \chi_{L_2}, w) = \\ &= \sum_{k \geq 0} z^k [c_1^k c_2^k] \sum_{\substack{n \geq 0 \\ x_1 \cdots x_n \in \{a,b\}^n}} z^n \Upsilon(c_1, c_2) \cdot A''_{x_1}(c_1, c_2) \cdots A''_{x_n}(c_1, c_2) \cdot \Gamma = \\ &= \sum_{k \geq 0} z^k [c_1^k c_2^k] \Upsilon(c_1, c_2)(I - (A''_a(c_1, c_2) + A''_b(c_1, c_2))z)^{-1} \Gamma = \\ &= F(z, z) \ , \end{aligned}$$

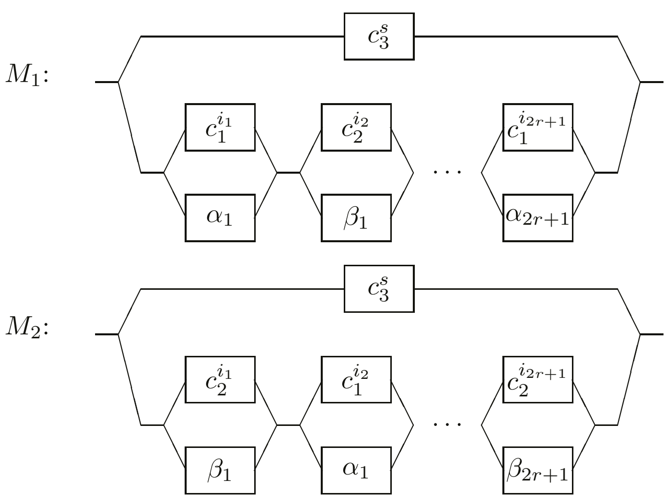
where $F(c_1, z) = \Delta_{c_1 c_2}(\Upsilon(c_1, c_2)(I - (A''_a(c_1, c_2) + A''_b(c_1, c_2))z)^{-1}\Gamma)$ is the diagonal of a rational function. So, the generating function $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$ turns out to be algebraic (see [14]).

3.2 Rational Languages in $(((a_1^\star \cdot b_1^\star) \times c_1^\star) \cdot ((a_2^\star \cdot b_2^\star) \times c_2^\star)) \times c_3^\star$

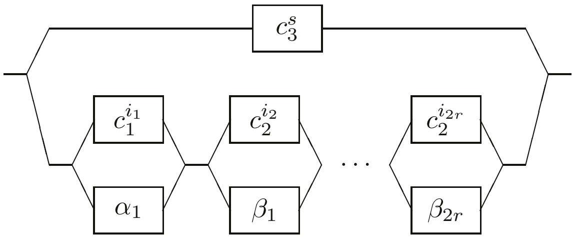
Let C be the commutation relation defined by the following diagram:



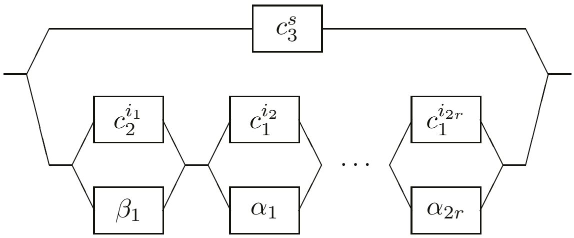
Then, it is easy to observe that there exist four classes $M_1, \ldots, M_4$ that describe the structure of all traces, as shown in the following block-diagrams ($\alpha_i \in \{a_1, b_1\}^\star, \beta_i \in \{a_2, b_2\}^\star$),



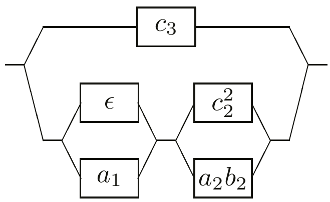
M_3:



M_4:



For instance, the trace $[a_1c_3a_2c_2b_2c_2]_{\rho_C}$ belongs to M_3 since its structure is



We now consider trace languages in $\mathrm{Rat}_{\mathrm{U}}(\{a_1, b_1, c_1, a_2, b_2, c_2, c_3\}, C)$ and we prove the following:

Theorem 4. *Let $L_1, L_2 \in Rat_U(\{a_1, b_1, c_1, a_2, b_2, c_2, c_3\}, C)$. Then, the generating function $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$ is holonomic.*

Proof. Let $L_1, L_2 \in \mathrm{Rat}_{\mathrm{U}}(\{a_1, b_1, c_1, a_2, b_2, c_2, c_3\}, C)$ be defined by the rational characteristic series χ_{L_1}, χ_{L_2} with linear representations $\langle \lambda_1, \mu_1, \gamma_1 \rangle$, $\langle \lambda_2, \mu_2, \gamma_2 \rangle$ respectively. Let $\Lambda = \lambda_1 \otimes \lambda_2$ and $\Gamma = \gamma_1 \otimes \gamma_2$. For $\sigma \in \{a_1, b_1, a_2, b_2\}$ and $i = 1, 2$ we define the matrices of rational entries

$$\begin{array}{ll}
C_i(c_i, c_3) & = \sum_{k \geq 0} (c_i\mu_1(c_i) + c_3\mu_1(c_3))^k = (I - (c_i\mu_1(c_i) + c_3\mu_1(c_3))^{-1} \ , \\
C'_i(c_i, c_3) & = \sum_{k \geq 0} (c_i\mu_2(c_i) + c_3\mu_2(c_3))^k = (I - (c_i\mu_2(c_i) + c_3\mu_2(c_3))^{-1} \ , \\
C''_i(c_i, c_3, c'_i, c'_3) & = C_i(c_i, c_3) \otimes C'_i(c'_i, c'_3) \ , \\
A_\sigma(c_i, c_3) & = \mu_1(\sigma) C_i(c_i, c_3) \ , \\
A'_\sigma(c_i, c_3) & = \mu_2(\sigma) C'_i(c_i, c_3), \\
A''_\sigma(c_i, c_3, c'_i, c'_3) & = A_\sigma(c_i, c_3) \otimes A'_\sigma(c'_i, c'_3)
\end{array}$$

and

$$T_1(z, c_1, c_3, c'_1, c'_3) = (I - z(A''_{a_1}(c_1, c_3, c'_1, c'_3) + A''_{b_1}(c_1, c_3, c'_1, c'_3)))^{-1} - I \ ,$$

$$T_2(z, c_2, c_3, c'_2, c'_3) = (I - z(A''_{a_2}(c_2, c_3, c'_2, c'_3) + A''_{b_2}(c_2, c_3, c'_2, c'_3)))^{-1} - I \ .$$

Then, we define the matrices of algebraic entries

$$\xi_1 = \sum_{n\geq 0}(\Delta_{c_1,c_1'}T_1(\Delta_{c_2,c_2'}C_2'' + \Delta_{c_2,c_2'}T_2) + \Delta_{c_1,c_1'}C_1''(\Delta_{c_2,c_2'}T_2 + \Delta_{c_2,c_2'}C_2''))^n \ ,$$

$$\xi_2 = \sum_{n\geq 0}(\Delta_{c_2,c_2'}T_2(\Delta_{c_1,c_1'}C_1'' + \Delta_{c_1,c_1'}T_1) + \Delta_{c_2,c_2'}C_2''(\Delta_{c_1,c_1'}T_1 + \Delta_{c_1,c_1'}C_1''))^n \ .$$

By considering the partition of traces in 4 classes $M_1, \dots, M_4$, we observe that the generating function $f_{\chi_{L_1}\odot\chi_{L_2}}(z)$ can be written as

$$f_{\chi_{L_1}\odot\chi_{L_2}}(z) = f^{(1)}_{\chi_{L_1}\odot\chi_{L_2}}(z) + f^{(2)}_{\chi_{L_1}\odot\chi_{L_2}}(z) + f^{(3)}_{\chi_{L_1}\odot\chi_{L_2}}(z) + f^{(4)}_{\chi_{L_1}\odot\chi_{L_2}}(z)$$

where $f^{(i)}_{\chi_{L_1}\odot\chi_{L_2}}(z) = \sum_{w\in\{M_i\}}(\chi_{L_1}\odot\chi_{L_2}, w)z^{|w|}$. By noting that $f^{(i)}_{\chi_{L_1}\odot\chi_{L_2}}(z) = F_i(z,z,z,z)$ with

$$\begin{aligned}
F_1(c_1,c_2,c_3,z) &= \Delta_{c_3,c_3'}(\Lambda\xi_1\Gamma) \ ,\\
F_2(c_1,c_2,c_3,z) &= \Delta_{c_3,c_3'}(\Lambda(\Delta_{c_2,c_2'}T_2)\xi_1\Gamma) \ ,\\
F_3(c_1,c_2,c_3,z) &= \Delta_{c_3,c_3'}(\Lambda(\Delta_{c_1,c_1'}T_1)\xi_2\Gamma) \ ,\\
F_4(c_1,c_2,c_3,z) &= \Delta_{c_3,c_3'}(\Lambda(\Delta_{c_2,c_2'}T_2)T_1\xi_2)\Gamma) \ ,
\end{aligned}$$

we obtain that the functions $f^{(i)}_{\chi_{L_1}\odot\chi_{L_2}}(z)$ are holonomic: in fact, the entries in T_1,T_2, ξ_1, ξ_2 are holonomic and we know that the class of holonomic functions is closed under the operations of diagonal, product and substitution with algebraic functions. At last, we conclude that $f_{\chi_{L_1}\odot\chi_{L_2}}(z)$ is holonomic since it is the sum of holonomic functions. □

3.3 Step 2: $f_{\chi_{L_1}\odot\chi_{L_2}}(z) = f_{\chi_{L_1}}(z)$?

Given two unambiguous rational trace languages $L_1, L_2 \subseteq (((a_1^\star \cdot b_1^\star) \times c_1^\star) \cdot ((a_2^\star \cdot b_2^\star) \times c_2^\star)) \times c_3^\star$, we know that the generating functions $f_{\chi_{L_1}}(z)$, $f_{\chi_{L_2}}(z)$ and $f_{L1\cap L_2}(z) = f_{\chi_{L_1}\odot\chi_{L_2}}(z)$, are holonomic. So, starting from the rational characteristic series χ_{L_1}, χ_{L_2}, it is possible to compute three linear differential equations with polynomial coefficients satisfied by the functions $f_{\chi_{L_1}}(z)$, $f_{\chi_{L_2}}(z)$ and $f_{\chi_{L_1}\odot\chi_{L_2}}(z)$ respectively (see [20] for details on computing with holonomic functions). Useful packages for these computations under Maple may be found in the Chyzak's Mgfun Project page (http://algo.inria.fr/chyzak/mgfun.html).

Hence, the problem of verifying whether $f_{\chi_{L_1}\odot\chi_{L_2}}(z) = f_{\chi_{L_1}}(z)$ is reduced to the problem of testing that the holonomic function $f_{\chi_{L_1}\odot\chi_{L_2}}(z) - f_{\chi_{L_1}}(z)$ is the zero function. So, we first compute the linear differential equation

$$\sum_{i=0}^{d} q_i(z)D^i g(z) = 0$$

satisfied by $g(z) = f_{\chi_{L_1}\odot\chi_{L_2}}(z) - f_{\chi_{L_1}}(z) = \sum_{n\geq 0} c_n z^n$. Then, by noting that $zDg(z)$ corresponds to $\{nc_n\}$ and $z^k g(z)$ to $\{c_{n-k}\}$, we find a linear recurrence equation with polynomial coefficients satisfied by $\{c_n\}$, $\sum_{i=0}^{e} p_i(n)c_{n-i} = 0$. At last, $f_{\chi_{L_1} \odot \chi_{L_2}}(z) - f_{\chi_{L_1}} = 0$ if and only if $c_i = 0$, $i = 0, \ldots e$, and $\{0\}$ satisfies the recurrence equation.

3.4 Conclusions

We conclude with two remarks. We have shown that Inclusion is decidable for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ with respect to a particular commutation relation associated with a complete binary tree of height 2. Thus, what happens if we consider commutation relations associated with complete binary trees of arbitrary height (two nodes commute if and only if one is in the subtree of the other)? Note that traces in such languages have a serial/parallel structure so it naturally rises a more general question: is it possible to extend Theorem 4 to unambiguos rational trace languages with commutation relations as those associated with monoids in $\mathcal{M}$ (see Definition 1)?
Another interesting topic is that of studying the complexity of Inclusion for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$, a problem closely related to the complexity of computing with holonomic functions, for which several results are known (see, for instance, [6]).

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