INVARIANTS

By Hermann Weyl

The *theory of invariants* came into existence about the middle of the nineteenth century somewhat like Minerva: a grown-up virgin, mailed in the shining armor of algebra, she sprang forth from Cayley's Jovian head. Her Athens over which she ruled and which she served as a tutelary and beneficent goddess was *projective geometry*. From the beginning she was dedicated to the proposition that all projective coördinate systems are created equal. Indeed, at that time the viewpoint of projective invariance was the one universally accepted in geometry. The rise of projective geometry had first been brought about by truly geometric stimuli, the study of conic sections, the theory of perspective and by the development of descriptive geometry, and the so-called synthetic direction of Steiner and von Staudt has confirmed the fertility of the projective attitude with respect to pure geometry.

However, its gaining such immense preponderance was, if I am not mistaken, due to algebraic rather than geometric reasons: namely, to the fact that the group of projectivities is expressed by the simplest of all continuous groups, the group of all homogeneous linear transformations. Plücker in the preface of his first work (*Analytisch-geometrische Entwicklungen*, vol. 1, 1828) openly espoused the ascendency of algebra, or, as he said, analysis, over geometry. So that perhaps one had better speak of geometric algebra than of algebraic geometry, namely, of an algebra which, in establishing its theorems and in the search for the proofs thereof, uses geometric terms and is guided by geometric intuition. The modern evolution, as far as it does not point its needle toward topology, has on the whole been marked by a trend of algebraization, notwithstanding the undeniable merits of the great school of Italian geometers.

The dictatorial rule of the projective idea in geometry was first successfully broken by the German astronomer and geometer Möbius. One is forced to realize that the group of all homogeneous linear transformations is not the only one worthy of consideration and capable of serving as the group of automorphisms in a geometric space. Möbius does not yet possess the general idea of a group; however, his notion of *Verwandtschaft* meets the same purpose in each special case he considers. The universal group theoretic interpretation was first

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No explicit references to the literature were given in the address; they can readily be supplied from the author's book *The Classical Groups, their Invariants and Representations*, Princeton, 1939. For the general foundations of the theory of invariants compare in particular v.d. Waerden, Mathematische Annalen, vol. 113(1936), pp. 14–35.

promulgated in plain words by Felix Klein in his famous *Erlanger Programm* 1872, which is the classical document of the democratic platform in geometry yielding equal rights to each and every imaginable group. The adaptation of his standpoint to the study of invariants has been somewhat slow. Before I discuss the main problems of the theory of invariants I find it convenient to rephrase Klein's fundamental idea in slightly modernized and hence slightly more abstract terms.

One wishes to associate with the points P of a space numerical (i.e., reproducible) symbols x as their *coördinates*. In general this is possible in a conceptual manner, without pointing out the individual points with my finger, only with respect to a *frame of reference*, e.g., in Euclidean geometry with respect to an arbitrarily assumed Cartesian set of axes. Transition from one frame $\mathfrak{f}$ to another equally admissible one is accomplished by means of a one-to-one correspondence S in the domain of symbols x. One has to deal, therefore, with 4 kinds of objects: a set of symbols or cöordinates x, a group $\mathfrak{G}$ of transformations S of this set into itself, and further points P and frames $\mathfrak{f}$. Their connection is to be described thus: A point P relative to a frame $\mathfrak{f}$ determines a coördinate $x = (P, \mathfrak{f})$. Any two frames $\mathfrak{f}, \mathfrak{f}'$ determine a transformation S of our group $\mathfrak{G}$, such that the coördinate $x' = (P, \mathfrak{f}')$ of an arbitrary point P arises from its coördinate $x = (P, \mathfrak{f})$ by S, $x' = Sx$. With two given frames $\mathfrak{f}, \mathfrak{f}'$ the equation

$$(1) \qquad (P, \mathfrak{f}) = (P', \mathfrak{f}')$$

defines an *automorphism* $P \rightleftarrows P'$ of the space. If the same S carries $\mathfrak{f}, \mathfrak{f}'$ into $\mathfrak{g}, \mathfrak{g}'$ respectively, one has, along with (1),

$$(P, \mathfrak{g}) = (P', \mathfrak{g}').$$

This shows that the automorphisms of our space form a group isomorphic with $\mathfrak{G}$; however, the isomorphic correspondence between the two groups depends on the frame of reference and is hence determined up to an arbitrary inner automorphism of the group. If in studying a given group $\mathfrak{G}$ of transformations $x' = Sx$ in a domain of symbols x one wishes to make use of a geometric nomenclature, it is quite fitting to *invent* a point space with its equally admissible frames to which the above scheme applies.

However, in one regard the scheme is still incomplete. Not only are the *points* of the space to be submitted to symbolical representation, but as has been emphasized by Plücker, other geometric entities also, e.g., the straight lines, may serve as spatial elements. Nay, in physics all sorts of physical quantities, velocities, forces, field strengths, electronic spins, etc., should be fixed by numerical symbols relative to a frame of reference. The law according to which the transformation S depends on the transition $\mathfrak{f} \rightarrow \mathfrak{f}'$ will then be determined by the type of the quantity in question, and will differ for points, lines, velocities, spins, etc. Only the elements s of the *abstract* group are tied up with the transitions in a manner independent of the type of quantity under consideration. After this correction, Klein's axiomatics looks as follows. (In its description I use the language of physicists: instead of several points, I speak of *a* point which may assume several positions, or rather of a quantity, e.g., the electromagnetic field strength, capable of several values.)

A. *The "symbolic" part* (dealing with group elements and coördinates).

(1) Let there be given a set γ of elements called *group elements.* Each pair s, t of group elements shall give rise to a composite element ts. There shall be a unit element e satisfying $es = se = s$ and an inverse s^{-1} for each group element s: $s^{-1}s = ss^{-1} = e$. (The associative law is not explicitly required.)

(2) Let there be given a set of elements called coördinates x and a realization $\mathfrak{A}$: $s \to S$ of the group γ by means of one-to-one correspondences $x \to x' = Sx$ within that set.

B. *The "geometric" part* (dealing with frames and quantities).

(1) Any two *frames* $\mathfrak{f}$, $\mathfrak{f}'$ determine a group element s, called the *transition* from $\mathfrak{f}$ to $\mathfrak{f}'$. Vice versa, a group element s "carries" a frame $\mathfrak{f}$ into a uniquely determined frame $\mathfrak{f}' = s\mathfrak{f}$ such that the transition $(\mathfrak{f} \to \mathfrak{f}') = s$. The transition $\mathfrak{f} \to \mathfrak{f}$ is the unit element e, the transition $\mathfrak{f}' \to \mathfrak{f}$ the inverse element. If s, t are the transitions $\mathfrak{f} \to \mathfrak{f}'$, $\mathfrak{f}' \to \mathfrak{f}''$ respectively, then the composite ts is the transition $\mathfrak{f} \to \mathfrak{f}''$.

(2) A quantity q of the type $\mathfrak{A}$ is capable of different values. Relative to an arbitrarily fixed frame $\mathfrak{f}$ each value of q determines a coördinate x such that $q \to x$ is a one-to-one mapping of the possible values of q on the set of coördinates. The coördinate x' corresponding to the same arbitrary value q in any other frame $\mathfrak{f}'$ is linked to x by the transformation $x' = Sx$ associated with the transition $(\mathfrak{f} \to \mathfrak{f}') = s$ by the given realization $\mathfrak{A}$.

(1) refers to the space, (2) to a special quantity therein.

This sounds fairly general and abstract. As algebraists we are interested almost exclusively in the case where the realization of the group is a *representation* $s \to A(s)$ by linear transformations $A(s)$ in an n-dimensional vector space and where the coördinate is therefore a *k-vector,* i.e., any n-tuple of numbers $(x_1, \cdots, x_n)$. By "number" we mean here a number in an arbitrarily given field k. With this limitation we repeat once more our definition of a quantity:

A quantity q of type $\mathfrak{A}$ is characterized by a representation of γ in k, $s \to A(s)$, of a certain degree n. Each value of q relative to a frame $\mathfrak{f}$ determines a k-vector $(x_1, \cdots, x_n)$ such that under the transition s to another frame $\mathfrak{f}'$ the components x_i of q transform according to $A(s)$. [The representation $s \to 1$ of degree 1 is called the identical representation. A quantity of this type is a *scalar.*]

For the purposes of differential geometry this set-up is also of basic importance, though its does not tell the whole story. Here the procedure consists in associating with each point of the non-homogeneous "differential" manifold M a homogeneous Klein space of fixed type $\mathfrak{G}$ and in establishing transitions between these Klein spaces by moving around in M. For example, in a recent review of E. Cartan's method of répères mobiles in the Bulletin of the American Mathematical Society, I was able to show the adequacy of the axiomatic foundation as given here for his treatment of manifolds M, that are embedded in a Klein space, by means of differential invariants. But I shall not enter into this subject here, my sole concern at present being algebraic invariants.

I denote by $\mathrm{P} = \mathrm{P}_n$ the "space" of n-dimensional k-vectors $(x_1, \cdots, x_n)$. A change of the vector basis in P transmutes $\mathfrak{A}$ into an equivalent representation $\mathfrak{A}'$. $\mathfrak{A}$ or the corresponding type of quantities is *reducible*, provided P has a linear subspace P' invariant under all transformations $A(s)$ of the group $\mathfrak{A}$ which is neither the total P nor contains only the vector 0. By appropriate choice of the vector basis one then may split off a part of the components x_i such that these transform only among themselves. *Decomposition* occurs if P can be decomposed into two complementary invariant subspaces $\mathrm{P}_1 + \mathrm{P}_2$. This means that, relative to a suitable vector basis, the components break up into two classes

$$x_1, \cdots, x_l \mid y_1, \cdots, y_m \qquad (l + m = n), \tag{2}$$

the members of each transforming among themselves. The corresponding quantity consists of the juxtaposition of two quantities x and y which vary independently of each other. Thus one may look upon the electromagnetic four-potential together with the field strength as *one* quantity of $4 + 6 = 10$ components; but everybody will agree that this is a very artificial union. Looking from the other direction one will try and wish to *decompose every quantity into independent irreducible* ("primitive") *constituents.* For most groups, indeed for all which will engage our attention here, this is in fact possible. But the demonstration by algebraic means of the theorem of full reducibility is one of the chief goals of the theory.

Juxtaposition was defined thus: If the variables $x_1, \cdots, x_l$ are subject to the substitution A, and $y_1, \cdots, y_m$ to the substitution B, then the row (2) undergoes the substitution $A \dotplus B$. Another process of great importance is $\times$-multiplication: under the conditions just described, the lm products $x_i y_k$ undergo the substitution $A \times B$ which one calls the Kronecker product. Hence one may add and multiply representations $\mathfrak{A}$: $s \to A(s)$ and $\mathfrak{B}$: $s \to B(s)$ of the same group:

$$\mathfrak{A} + \mathfrak{B}: s \to A(s) \dotplus B(s); \qquad \mathfrak{A} \times \mathfrak{B}: s \to A(s) \times B(s);$$

or what is the same, one may add and $\times$-multiply quantities. In performing the second process, the representation spaces P and P' of l and m dimensions over which the vectors x and y range, give rise to an lm-dimensional space PP' which contains the vector $z = x \times y$ with the components

$$z_{ik} = x_i y_k .$$

In studying linear forms in PP' one often finds it convenient to replace the most general vector z with lm independent components z_{ik} by the vector $x \times y$ with x_i and y_k as independent variables. This procedure is called the *symbolic method* in the theory of invariants. One of the most important problems for quantities is to decompose the product of two primitive quantities (or of two irreducible representations) into its irreducible constituents. Special cases will soon occupy us.

After all these preliminaries I shall finally say what an *invariant* is. I begin with the notion of a *vector invariant* which presupposes that we are given a group Γ of linear transformations A in an n-dimensional vector space P. Suppose we are given a form $f(x, y, \cdots)$, i.e., a homogeneous polynomial of certain degrees $\mu, \nu, \cdots$ in the components of each argument vector $x, y, \cdots$ which vary in P. The cogredient transformations

$$x' = Ax, \qquad y' = Ay, \qquad \cdots$$

change f into a new form $f' = Af$ defined by

$$f'(x', y', \cdots) = f(x, y, \cdots).$$

f is an (absolute) vector invariant with respect to the group Γ if $f = Af$ for all transformations A in Γ. A simple generalization of this elementary concept will introduce contravariant argument vectors $\xi, \eta, \cdots$ which undergo the transformation contragredient to A while the covariant arguments $x, y, \cdots$ are transformed by A.

To this elementary notion I oppose the *general notion of invariants* resting upon a given *abstract* group $\gamma = \{s\}$ and a series of given representations of γ,

$$\mathfrak{A}\colon s \rightarrow A(s), \qquad \mathfrak{B}\colon s \rightarrow B(s), \qquad \cdots$$

of degrees $m, n, \cdots$, respectively. A function $\varphi(\mathfrak{x}, \mathfrak{y}, \cdots)$ depending on an arbitrary quantity $\mathfrak{x}$ of type $\mathfrak{A}$, another quantity $\mathfrak{y}$ of type $\mathfrak{B}$, $\cdots$ will be expressed by a certain function F of the numerical vectors

$$x = (x_1, \cdots, x_m), \qquad y = (y_1, \cdots, y_n),$$

in terms of a given frame of reference $\mathfrak{f}$, and will be expressed by a certain function $F' = sF$ in terms of another frame $\mathfrak{f}'$ into which $\mathfrak{f}$ changes by the group element s. If $F' = F$ for all s, then φ is an invariant. If we make use of the numerical vectors and the given representations only, invariance may be simply stated by the equation

$$F(A(s)x, B(s)y, \cdots) = F(x, y, \cdots),$$

holding for all s in γ. Again, we limit ourselves to the case where F is a polynomial homogeneous in the components of each vector. Another way of expressing the same thing would be to say that *an invariant is a scalar depending on variable quantities of given types* $\mathfrak{A}, \mathfrak{B}, \cdots$.

One speaks of a *relative invariant* if $F' = \lambda \cdot F$ where the multiplier $\lambda = \lambda(s)$ depends on s only. $s \rightarrow \lambda(s)$ is then necessarily a representation of degree 1. More generally, a *covariant* is a quantity of a certain type $\mathfrak{H}\colon s \rightarrow H(s)$, depending on variable quantities $x, y, \cdots$ of given types $\mathfrak{A}, \mathfrak{B}, \cdots$.

After having fixed the concepts, we can now turn to the fundamental theorems concerning invariants. The *first main theorem* maintains that the invariants for a given group γ and a given set of its representations $\mathfrak{A}, \mathfrak{B}, \cdots$ have a *finite integrity basis*; i.e., one can pick out a finite number among them in terms of which all these invariants are expressible in an integral rational manner. We do not know whether the proposition holds good for any group γ and representations $\mathfrak{A}$, $\mathfrak{B}$, $\cdots$. One has been able to prove it, however, in the most important cases, in particular for finite groups γ. After one has ascertained a finite complete set of basic invariants $J_1(x, y, \cdots), \cdots, J_h(x, y, \cdots)$, the second task is to survey all existing algebraic *relations* among them. A relation is a polynomial $R(t_1, \cdots, t_h)$ of h variables $t_1, \cdots, t_h$ which is turned identically into zero by the substitution

$$t_1 = J_1(x, y, \cdots), \cdots, t_h = J_h(x, y, \cdots).$$

The *second main theorem* states that one can find a finite number of relations of which all relations are algebraic consequences. This is merely a special case of Hilbert's universal proposition about the finiteness of an ideal basis for any ideal of polynomials. Indeed the relations form an ideal in the ring of all polynomials of $t_1, \cdots, t_h$. Thus the second main theorem is settled once for all and we shall pay little further attention to it. To be sure, in each single case the problem remains actually to ascertain an ideal basis of the relations.

I give two examples from the elementary domain of vector invariants. Let us deal with invariant forms depending on an arbitrary number of vectors $x^{(1)}, x^{(2)}, \cdots$ in the same space; invariance refers to a given group Γ of linear transformations in that space. If Γ is the group of all unimodular transformations, one gets an integrity basis by forming from the given argument vectors in all possible combinations the determinants $[xy \cdots z]$ of the components of n vectors $x, y, \cdots, z$. If contravariant arguments $\xi^{(1)}, \xi^{(2)}, \cdots$ are admitted, one must add the following two types

$$(\xi x) = \xi_1 x_1 + \cdots + \xi_n x_n$$

and $[\xi \eta \cdots \zeta]$. On the other hand, if Γ is the group of all orthogonal transformations, then the scalar products

$$(xy) \qquad \left\{\begin{matrix} x = x^{(1)}, x^{(2)}, \cdots \\ y = x^{(1)}, x^{(2)}, \cdots \end{matrix}\right\}$$

of the argument vectors constitute an integrity basis for invariants. Surprisingly enough the last result holds good even when the underlying number field is any field of characteristic zero in the sense of abstract algebra. Since the construction of suitable Cartesian coördinate systems to which the proofs resort depends on laying off a given segment on a given line, one would have expected the result to be restricted to *"Pythagorean" fields.* As one knows, a field is called *real* (Artin-Schreier) provided a square sum never vanishes unless each term vanishes. I name a real field Pythagorean if the square sum of two numbers is always a square. All relations between scalar products are in the case of the orthogonal group consequences of the relations of the following type:

$$\begin{vmatrix} (xx) & (xx') & \cdots & (xx^{(n)}) \\ \cdots & \cdots & \cdots & \cdots \\ (x^{(n)}x) & (x^{(n)}x') & \cdots & (x^{(n)}x^{(n)}) \end{vmatrix} = 0.$$

(Second main theorem for orthogonal vector invariants.)

By means of his theorem about polynomial ideals Hilbert had reduced the general proof of the first main theorem to the construction of a linear operator ω working on polynomials $F(x, y, \cdots)$ and having the following two properties:

$$\omega(1) = 1, \qquad \omega(F \cdot J) = \omega(F) \cdot J \tag{3}$$

whenever J is an invariant. If γ is a compact Lie group, one can follow a procedure inaugurated by Adolf Hurwitz and define an *invariant measure of volumes* on γ by means of which one is able to form the average $\mathfrak{M}_s\{\psi(s)\}$ of any continuous function $\psi(s)$ on γ. One sees at once that

$$\omega(F) = \mathfrak{M}_s(sF)$$

is a process of the desired nature. By this topological method which necessarily presupposes the continuum K of all real numbers as reference field, one succeeds in proving the first fundamental theorem for any compact Lie group. I mention the instance of the real orthogonal group in K. By the same method I. Schur succeeded in carrying over from finite to compact Lie groups Frobenius' theory of group representations, in particular, the orthogonality relations for the representing matrices and their characters, while the speaker, together with F. Peter, established the completeness relation.

A. Haar freed the definition of the volume measure of the awkward differentiability conditions imposed by the Lie nature of the group. H. Bohr's theory of almost periodic functions could be interpreted as the simplest example of a similar theory for open, non-compact groups, namely, for the group of translations of a straight line. With the theory of compact groups and Bohr's example of a non-compact group before his eyes, von Neumann established the theory of almost periodic representations, their orthogonality and completeness, for any group whatsoever. Hence the first main theorem for invariants is proved for each group as long as we restrict ourselves to quantities $x, y, \cdots$ as arguments whose types are described by almost periodic representations.

All this sounds as if we could rest as God did after the sixth day of creation, finding that it was very good! But now enters the snake into the paradise. Let us once more envisage the classical case of the group L^r of all real unimodular transformations A in n dimensions. Not one of the representations with which the classic theory of invariants deals, not even the representation $A \to A$, is almost periodic! Thus the "almost periodic" theory fails just in the most important and natural cases. Nevertheless it has been possible to make the theory of compact groups fruitful for all semi-simple Lie groups by what I have called the *unitarian trick*. For the group L^r it consists in first extending L^r to embrace all unimodular transformations with *complex* coefficients and then limiting oneself within this wider group L^c to the *unitary* operations. By following Lie's fundamental suggestion and going back to the infinitesimal elements of a group, one linearizes and thereby algebraizes all problems concerning structure, representations and invariants of a group; and then such reality restrictions as the two encountered above, either to real coefficients or to the unitary subgroup, become irrelevant. Hence each of these subgroups can stand for the other, and one of them, namely, the unitary subgroup, is compact and thus accessible to the integration method. In the linkage between the infinitesimal and the total group a topological element is involved; but I shall not dwell here on this subtle point. Anyway I have been able to show that the unitarian trick is effective with all semi-simple Lie groups, and thus not only to confirm by a combination of the infinitesimal and integral methods the results derived in a purely infinitesimal manner by E. Cartan for the irreducible representations of the semi-simple groups, but also to supplement them by the theorem of full reducibility and explicit formulas for their characters. At the same time *the first main theorem for invariants was thus secured for all semi-simple groups.*

The problem naturally puts itself: to corroborate by direct and explicit algebraic construction these results first obtained in a transcendental way. If one succeeds, one may hope at the same time to remove the bond by which the topological approach ties these results to the field K of real numbers and to extend them to any field in the abstract algebraic sense, at least to any field of characteristic zero. This is a goal at which I have aimed for many years, though not at all with the necessary persistence and singleness of purpose. So many other mathematical things have diverted my interests, and the whirlwind of political events has had a most disturbing effect on my concentration. However, younger men came to my aid, above all Richard Brauer, to whom I owe the most essential link in the chain of the algebraic theory. At present I have come to a certain end, or at least to a certain halting point, from which it seemed profitable to look back upon the track so far pursued, and this is what I tried to do in my recent book *The Classical Groups, their Invariants and Representations.* The most important simple groups in the field of all complex numbers are: the group $L(n)$ of all (non-singular or merely of all unimodular) linear transformations in n dimensions, the group $O(n)$ of all (or all proper) orthogonal transformations in n dimensions, and the group $Sp(n)$ of all linear transformations in $n = 2\nu$ dimensions leaving invariant a non-degenerate skew-symmetric bilinear form. The last I have christened the symplectic group. These are even the only ones, apart from 5 quite singular exceptional groups. I shall deal exclusively with these groups $L(n)$, $O(n)$, $Sp(n)$. For their investigation a finite group, the group of all permutations, must be drawn in, and one could also include the alternating group of permutations. These groups are in my mind when I speak of *classical groups.* We are first engaged in algebraically constructing the *possible types of quantities* under their reign.

Again we start with the universal linear group $L(n)$, an arbitrary element of which we denote by A:

$$x'_i = \sum a(ik)x_k \qquad (i, k = 1, \cdots, n).$$

You all know what a tensor of rank r is. It has n^r components $t(i_1 i_2 \cdots i_r)$ labeled by r indices $i_1, i_2, \cdots, i_r$ ranging from 1 to n; under the influence of the transformation A of the coördinates in the underlying vector space these components are transformed according to the substitution

$$\Pi_r(A) = A \times A \times \cdots \times A \qquad (r \text{ factors})$$

or more explicitly

$$t'(i_1 \cdots i_r) = \sum_{k_1, \cdots, k_r} a(i_1 k_1) \cdots a(i_r k_r) \cdot t(k_1 \cdots k_r). \tag{4}$$

The generic tensor of rank r is the quantity arising by r-fold $\times$-multiplication of the quantity vector. But the space P^r of all tensors is not irreducible under the group $\Pi_r(L)$ consisting of the substitutions $\Pi_r(A)$ which are induced in tensor space by the elements A of $L(n)$, whereas the words symmetric tensor, skew-symmetric tensor, indicate irreducible quantities. The tensor space P^r must therefore be split into irreducible invariant parts by imposing symmetry conditions upon the tensors. The possibility of doing so is based on the fact that one can perform an arbitrary permutation p on the r indices or arguments $i_1, \cdots, i_r$, whereby t changes into another tensor pt. In this way enters the group π_r of permutations p of r figures $1, \cdots, r$. Associating the transition $t \to pt$ with p defines a representation of π_r by linear transformations in P^r. But why is it that these permutation operators are of importance for the decomposition of tensor space into invariant subspaces? One understands this if one replaces the group $\Pi_r(L)$ of the substitutions (4) to which the tensor space is submitted by its *enveloping algebra,* containing all those substitutions which can be gained by linearly combining any finite number of substitutions of the group $\Pi_r(L)$. It is easily seen that the enveloping algebra consists of all linear substitutions $t \to Ht$ commuting with the permutations $p: t \to pt$. The group π_r of permutations may also conveniently be replaced by the enveloping algebra, i.e., by the corresponding group ring whose elements

$$\sum_p \alpha(p) \cdot p \qquad [\alpha(p) \text{ numbers}]$$

may be interpreted as "symmetry operators" working on tensors.

The general situation under which our problem is naturally to be subsumed is now this: Instead of the tensor space we consider an arbitrary vector space P whose vectors are called t; there is given a finite group $\gamma = \{p\}$ and a representation of γ in P representing the abstract group element p by a linear substitution $t \to pt$. We are interested in the algebra $\mathfrak{A}$ of linear operators $t \to Ht$ commuting with all operators $t \to pt$ of γ. The regular representation $\mathfrak{r}$ of a finite group γ or of its group ring $(\gamma) = \rho$ has ρ itself as its representation space, representing any element a of ρ by the transformation $x \to ax$ of ρ into itself. By a well-known theorem due to Maschke the regular representation of γ is fully reducible; this holds good in any field, unless it is of a prime characteristic dividing the order of γ. We take into account only fields of characteristic zero. A thoroughly elementary method permits establishment of a complete parallelism between the subspaces of ρ invariant under $\mathfrak{r}$ on the one side and the subspaces of P invariant under the algebra $\mathfrak{A}$ on the other side. The parallelism is faithful with respect to addition and the relation of being contained for invariant subspaces, and also with respect to equivalence under their respective operator algebras $\mathfrak{r}$ and $\mathfrak{A}$.

For the symmetric group π_r one knows how to carry out the decomposition into irreducible invariant subspaces by means of the symmetry operators which were invented by A. Young and later, under the leadership of E. Wigner, have found such surprising applications in quantum mechanics. Let us attach the word *quantics*, originally coined by Cayley, to the quantities which one prepares in this way from the material of tensors under the rule of the full linear group $L(n)$. The domain of quantics is closed with respect to the two most important operations: (1) $\times$-multiplication of two quantics followed by decomposition into irreducible constituents, (2) transition from a representation to its contragredient. Each Young operator and hence each quantic[1] is characterized by a partition of the rank number r into n integral summands

$$r = r_1 + r_2 + \cdots + r_n \quad (r_1 \geqq r_2 \geqq \cdots \geqq r_n \geqq 0).$$

We represent this partition by a symmetry diagram whose rows have the lengths $r_1, r_2, \cdots, r_n$. Example:

$$\begin{array}{ll} r_1 = 7 & \bigcirc\bigcirc\bigcirc\bigcirc\bigcirc\bigcirc\bigcirc \\ r_2 = 5 & \bigcirc\bigcirc\bigcirc\bigcirc\bigcirc \\ r_3 = 5 & \bigcirc\bigcirc\bigcirc\bigcirc\bigcirc \\ r_4 = 2 & \bigcirc\bigcirc \\ r_5 = 1 & \bigcirc \end{array}$$

If one wishes to employ a similar method for the orthogonal and the symplectic groups one has first to get hold by a simple description of the enveloping algebra of the substitutions $\Pi_r(A)$ induced in tensor space by the elements A of these more limited groups $O(n)$ and $Sp(n)$. The problem is not as trivial by far as in the former case of the full linear group $L(n)$, and R. Brauer succeeded in solving it only by resorting to the general theory of matric algebras. If one is given an algebra $\mathfrak{A}$ of linear substitutions or matrices A in a certain vector space P, then the matrices B commuting with all matrices A of the set $\mathfrak{A}$ form in their turn an algebra $\mathfrak{B}$ which I call *the commutator algebra* of $\mathfrak{A}$. The key principle asserts that if $\mathfrak{A}$ is fully reducible, the commutator algebra of the commutator algebra of $\mathfrak{A}$ is not larger than $\mathfrak{A}$ as one might expect, but coincides with $\mathfrak{A}$. This principle holds in any field. It is the crowning result of a theory of matric algebras based on this fundamental advice due to I. Schur: along with a given matric algebra, always consider its commutator algebra. Unable to refer to any other place, I had to incorporate in my book this theory which has become a central issue in the whole non-commutative algebra. Perhaps many a reader will find such a concrete treatment in terms of matrices more easily accessible than the abstract handling of semi-simple rings.

[1] We disregard here a slight modification necessary to work the process (2).

If one replaces the group $\Pi_r(O)$ induced by the orthogonal group $O(n)$ in tensor space by its enveloping algebra thus determined, one succeeds again in decomposing the most general tensor into quantics with respect to the orthogonal group. The primitive quantics to which one finds oneself reduced differ from those for the full linear group in two essential regards: (1) The Young symmetry operator is applied not to an arbitrary tensor, but to the most general tensor whose $\frac{1}{2}r(r - 1)$ traces vanish; the (12)-trace of a tensor $\mathfrak{t}(i_1\, i_2\, i_3 \cdots i_r)$ being given by

$$\mathfrak{t}_{12}(i_3 \cdots i_r) = \sum_i \mathfrak{t}(i\, i\, i_3 \cdots i_r)$$

(process of *Verjüngung*). (2) While all symmetry diagrams whose first column had a length $\leqq n$ were admitted for the full linear group, only those occur here whose first two columns have a total length $\leqq n$. Similar results obtain for the symplectic group which in many regards is easier to handle than the orthogonal group.

The algebraic concept of invariants which we adopt for the classical groups is that of a scalar depending on one or more independent variable *quantics*. Arbitrary *forms* such as occurred as arguments in the classical theory of invariants are identical with arbitrary symmetric tensors, and under the reign of $L(n)$ these are quantics which correspond to a symmetry diagram of one row (the length of the row being the order of the form). Let us first stick again to $L(n)$! In all textbooks on our subject one is told a lot about the so-called *symbolic method* which reduces *form invariants* to *vector invariants*. On the basis of the above analysis one shows readily that the method still works for invariants of variable quantics of any type. However, since the number of argument vectors to be introduced depends on the degree of the invariant under consideration with respect to the variable quantics, the reduction to vector invariants is by no means sufficient without further resources for a proof of the first main theorem. Certainly the importance of the symbolic method whose formal elegance nobody will deny has been greatly exaggerated. I consider it one of the more glaring examples of the power of tradition and inertia in mathematics that the elementary textbooks on invariants up to this day deal almost exclusively with this method and its applications. Frequently quite different approaches, e.g., irrational methods, lead much faster to the goal. Hilbert's general proof dealing with form invariants of the group $L(n)$ is based on his general proposition about a finite ideal basis for polynomial ideals and employs as the above mentioned process ω with the properties (3) Cayley's purely algebraic Ω-process. The method goes through in any field of characteristic zero even if quantics instead of forms are the arguments.

The projective geometers were able to cope with affine and metric geometry by adjoining some entities given once for all which they called the *absolute*: the plane at infinity and the absolute involution. In the same line of ideas relativity theory has found it convenient to treat the metric vector space as an affine vector space in which a positive quadratic form is appointed as metric ground form. Is this standpoint justified as far as invariants are concerned? Let variable forms $u, v, \cdots$ be our arguments in the invariants $J(u, v, \cdots)$ which we envisage. Then the question means whether any *orthogonal* invariant $J(u, v, \cdots)$ arises by the Cartesian specialization $g_{ik} = \delta_{ik}$ from a L-invariant $J(g_{ik}; u, v, \cdots)$ involving besides the arguments $u, v, \cdots$ a variable covariant quadratic form

$$\sum_{i,k} g_{ik}\xi_i\xi_k. \tag{5}$$

That it can answer this question in the affirmative goes to the credit of the symbolic method and is its highest triumph. Indeed, each orthogonal vector invariant is expressible in terms of the scalar products (xy) of the vectorial arguments, and if we replace the scalar product

$$x_1y_1 + \cdots + x_ny_n \tag{6}$$

by the L-invariant formation

$$-\begin{vmatrix} g_{11} & \cdots & g_{1n} & x_1 \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & \cdots & g_{nn} & x_n \\ y_1 & \cdots & y_n & 0 \end{vmatrix}$$

which depends on (5) besides the two vectors x and y and changes into (6) by the specialization $g_{ik} = \delta_{ik}$, then we attain our ends, first for vector invariants and then, owing to the symbolic treatment, for form invariants. The procedure remains applicable even for arbitrary quantics, although the quantics for the group L split into more primitive quantics under the orthogonal subgroup O. In this purely algebraic way based on the adjunction argument we master the orthogonal and the symplectic invariants. This procedure has even stood the test in certain special cases where the statement of full reducibility breaks down.

In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain. As to the decompositions into invariant subspaces whose algebraic construction has here been indicated, one would like to know in some explicit way with which *multiplicity* each of the inequivalent irreducible constituents occurs. This question is answered most readily if one replaces the representations by their *characters*. Explicit formulas for the characters are much more easily obtained by the *integral-topological approach*. The identification of the characters thus derived with the algebraically constructed representations to which they correspond causes a little headache; however, if this has been remedied one has seized upon a result which by its very nature is independent of the nature of the reference field, and that in spite of the fact that the topological method operates in the field K of all real numbers. Thus we are face to face with a peculiar application of analysis to purely algebraic problems whose stage is set in an arbitrary field. The multiplicities just mentioned yield at the same time formulas for the explicit enumeration of invariants and covariants. This field has recently been tilled with high success by Professor Murnaghan.

I must forego giving examples of such enumerating formulas. Instead I prefer to mention a by-product of the algebraic investigation. In the n^2-dimensional space of all matrices $A = \| a_{ik} \|$ the equations

$$(7) \qquad \sum_k a_{ik} a_{jk} - \delta_{ij} = 0, \qquad \det (a_{ik}) - 1 = 0$$

define a certain algebraic manifold, the proper orthogonal group O^+. *This manifold is irreducible*, or, what is the same, the ideal of all polynomials $\Phi(a_{ik})$ vanishing on $O^+(n)$ is a prime ideal. The polynomials constituting the left members of the equations (7) form an ideal basis for our ideal, and Cayley's rational parametrization of orthogonal substitutions A,

$$A = (E - S)(E + S)^{-1},$$

in terms of an arbitrary skew-symmetric matrix S yields a generic zero (allgemeine Nullstelle) of the ideal, provided the elements s_{ik} $(i < k)$ of the skew matrix $S = \| s_{ik} \|$ are treated as indeterminates. All this holds good in any field of characteristic zero.

I hope my sketch has shown how closely the investigation of the invariants of a group is tied up with the ascertainment of its representations. This connection with the general theory of representations and of matric algebras has carried new life-blood into the older theory of invariants which thus has joined the modern forward movement of algebra and now participates in its general conceptual structure. I feel bound to add a personal confession. In my youth I was almost exclusively active in the field of analysis; the differential equations and expansions of mathematical physics were the mathematical things with which I was on the most intimate footing. I have never succeeded in completely assimilating the abstract algebraic way of reasoning, and constantly feel the necessity of translating each step into a more concrete analytic form. But for that reason I am perhaps fitter to act as intermediary between old and new than the younger generation which is swayed by the abstract axiomatic approach, both in topology and algebra.

In closing I should like to point out a few lines of probable *further advance.* First, one naturally wishes to do all things also in a field of prime characteristic. Secondly, it is desirable to find *all* inequivalent irreducible representations in an arbitrarily given field; it is doubtful enough that they are exhausted by the quantics, though these form a class of quantities algebraically closed in a certain sense. If one replaces a continuous group by its infinitesimal elements, one has to deal with a *Lie algebra* and one will ask for its representations and invariants. The classical groups together with the 5 exceptional groups mentioned above yield all the simple Lie-algebras in the field of complex numbers, or in any algebraically closed field. However, this does not remain true in an arbitrary field. In this question Landherr, A. A. Albert, Jacobson and Zassenhaus have recently made much headway. So I am confident that in a few years a younger algebraist will be able to write a similar book dealing comprehensively with the representations and invariants of all semi-simple Lie algebras in an arbitrary field.

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