Model Theory, Algebra, and Geometry
MSRI Publications
Volume **39**, 2000

Model Theory of Differential Fields

DAVID MARKER

Abstract. This article surveys the model theory of differentially closed fields, an interesting setting where one can use model-theoretic methods to obtain algebraic information. The article concludes with one example showing how this information can be used in diophantine applications.

A *differential field* is a field K equipped with a derivation $\delta : K \to K$; recall that this means that, for $x, y \in K$, we have $\delta(x+y) = \delta(x) + \delta(y)$ and $\delta(xy) = x\,\delta(y) + y\,\delta(x)$. Roughly speaking, such a field is called differentially closed when it contains enough solutions of ordinary differential equations. This setting allows one to use model-theoretic methods, and particularly dimension-theoretic ideas, to obtain interesting algebraic information.

In this lecture I give a survey of the model theory of differentially closed fields, concluding with an example — Hrushovski's proof of the Mordell–Lang conjecture in characteristic zero — showing how model-theoretic methods in this area can be used in diophantine applications. I will not give the proofs of the main theorems. Most of the material in Sections 1–3 can be found in [Marker et al. 1996], while the material in Section 4 can be found in [Hrushovski and Sokolovic $\geq$ 2001; Pillay 1996]. The primary reference on differential algebra is [Kolchin 1973], though the very readable [Kaplansky 1957] contains most of the basics needed here, as does the more recent [Magid 1994]. The book [Buium 1994] also contains an introduction to differential algebra and its connections to diophantine geometry. We refer the reader to these sources for references to the original literature.

1. Differentially Closed Fields

Throughout this article all fields will have characteristic zero. A *differential field* is a field K equipped with a derivation $\delta : K \to K$. The field of *constants* is $C = \{x \in K : \delta(x) = 0\}$.

We will study differential fields using the language $\mathcal{L} = \{+, -, \cdot, \delta, 0, 1\}$, the language of rings augmented by a unary function symbol δ. The theory of differential fields, DF, is given by the axioms for fields of characteristic zero and the axioms

$$\forall x\, \forall y\ \delta(x+y) = \delta(x) + \delta(y),$$
$$\forall x\, \forall y\ \delta(xy) = x\delta(y) + y\delta(x),$$

which assert that δ is a derivation.

If K is a differential field, we define $K\{X_1, \ldots, X_n\}$, the ring of *differential polynomials* over K, to be the following polynomial ring in infinitely many variables:

$$K\big[X_1, \ldots, X_n, \delta(X_1), \ldots, \delta(X_n), \ldots, \delta^m(X_1), \ldots, \delta^m(X_n), \ldots\big].$$

We extend δ to a derivation on $K\{X_1, \ldots, X_n\}$ by setting $\delta(\delta^n(X_i)) = \delta^{n+1}(X_i)$.

We say that K is an *existentially closed* differential field if, whenever $f_1, \ldots, f_m \in K\{X_1, \ldots, X_n\}$ and there is a differential field L extending K containing a solution to the system of differential equations $f_1 = \cdots = f_m = 0$, there is already a solution in K. Robinson gave an axiomatization of the existentially closed differential fields. Blum gave a simple axiomatization that refers only to differential polynomials in one variable.

If $f \in K\{X_1, \ldots, X_n\} \setminus K$, the *order* of f is the largest m such that $\delta^m(X_i)$ occurs in f for some i. If f is a constant, we say f has order -1.

Definition. A differential field K is *differentially closed* if, whenever $f, g \in K\{X\}$, g is nonzero and the order of f is greater than the order of g, there is $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$.

In particular, any differentially closed field is algebraically closed.

For each m and d_0 and d_1 we can write down an $\mathfrak{L}$-sentence ϕ_{m,d_0,d_1} that asserts that if f is a differential polynomial of order m and degree at most d_0 and g is a nonzero differential polynomial of order less than m and degree at most d_1, then there is a solution to $f(X) = 0$ and $g(X) \neq 0$. For example, $\phi_{2,1,1}$ is the formula

$$\begin{aligned}
&\forall a_0\, \forall a_1\, \forall a_2\, \forall a_3\, \forall b_0\, \forall b_1\, \forall b_2 \\
&\quad \big(a_3 \neq 0 \wedge (b_0 \neq 0 \vee b_1 \neq 0 \vee b_2 \neq 0) \\
&\qquad\qquad \longrightarrow \exists x\ (a_3\,\delta(\delta(x)) + a_2\,\delta(x) + a_1 x + a_0 = 0 \wedge b_2\,\delta(x) + b_1 x + b_0 \neq 0)\big).
\end{aligned}$$

The $\mathfrak{L}$-theory DCF is axiomatized by DF and the set of axioms ϕ_{m,d_0,d_1}, for all m, d_0 and d_1. The models of DCF are exactly the differentially closed fields.

It is not hard to show that if $f, g \in K\{X\}$ are as above, then there is $L \supseteq K$ containing a solution to the system $f(X) = 0$ and $Yg(X) - 1 = 0$. Indeed we could take L to be the fraction field of $K\{X\}/P$, where P is a minimal differential prime ideal with $f \in P$. Iterating this construction shows that any differential field can be extended to a differentially closed field. Thus any existentially closed field is differentially closed.

The next theorem of Blum shows that the converse holds (see [Marker et al. 1996] for the proof).

THEOREM 1.1. *The theory* DCF *has quantifier elimination and hence is model complete*.

COROLLARY 1.2. (i) DCF *is a complete theory*.
(ii) *A differential field is existentially closed if and only if it is differentially closed*.

PROOF. (i) The rational numbers with the trivial derivation form a differential subfield of any differentially closed field. If K_0 and K_1 are models of DCF and ϕ is a quantifier free sentence, then there is a quantifier free sentence ψ such that DCF $\models \phi \longleftrightarrow \psi$. But $K_i \models \psi$ if and only if $\mathbb{Q} \models \psi$. Hence $K_0 \models \phi$ if and only if $K_1 \models \phi$ and DCF is complete.

(ii) We already remarked that every existentially closed field is differentially closed. Suppose K is differentially closed. Suppose $f_1 = \cdots = f_m = 0$ is a system of polynomial differential equations solvable in an extension L of K. We can find K_1 an extension of L which is differentially closed. By model completeness K is an elementary submodel of K_1. Since there is a solution in K_1 there is a solution in K. $\square$

Pierce and Pillay [1998] have given a more geometric axiomatization of DCF. Suppose K is a differential field and $V \subseteq K^n$ is an irreducible algebraic variety defined over K. Let $I(V) \subset K[X_1, \ldots, X_n]$ be the ideal of polynomials vanishing on V and let $f_1, \ldots, f_m$ generate $I(V)$. If $f = \sum a_{i_1,\ldots,i_m} X_1^{i_1} \cdots X_m^{i_m}$, let $f^\delta = \sum \delta(a_{i_1,\ldots,i_m}) X_1^{i_1} \cdots X_m^{i_m}$. The tangent bundle $T(V)$ can be identified with the variety

$$T(V) = \left\{ (x,y) \in K^{2n} : x \in V \wedge \sum_{j=1}^{n} y_j \frac{\partial f_i}{\partial X_j}(x) = 0 \text{ for } i = 1, \ldots, m \right\}$$

We define the *first prolongation* of V to be the algebraic variety

$$V^{(1)} = \left\{ (x,y) \in K^{2n} : x \in V \wedge \sum_{j=1}^{n} y_j \frac{\partial f_i}{\partial X_j}(x) + f_i^\delta(x) = 0 \text{ for } i = 1, \ldots, m \right\}.$$

If V is defined over the constant field C, then each f_i^δ vanishes, and $V^{(1)}$ is $T(V)$. In general, for $a \in V$, the vector space $T_a(V) = \{b : (a,b) \in T(V)\}$ acts regularly on $V_a^{(1)} = \{b : (a,b) \in V^{(1)}\}$, making $V^{(1)}$ a torsor under $T(V)$. It is easy to see that $(x, \delta(x)) \in V^{(1)}$ for all $x \in V$. Thus the derivation is a section of the first prolongation.

THEOREM 1.3. *For K be a differential field, the following statements are equivalent*:

(i) *K is differentially closed*.

(ii) *K is existentially closed.*
(iii) *K is algebraically closed and for every irreducible algebraic variety $V \subseteq K^n$ if W is an irreducible subvariety of $V^{(1)}$ such that the projection of W onto V is Zariski dense in V and U is a Zariski open subset of V, then $(x, \delta(x)) \in U$ for some $x \in V$.*

Proof. We know (i) $\iff$ (ii).

(iii) $\Rightarrow$ (i) Suppose $f(X) \in K\{X\}$ has order n and $g(X)$ has lower order. Say $f(X) = p(X, \delta(X), \ldots, \delta^n(X))$ and $g = q(X, \delta(X), \ldots \delta^{n-1}(X))$, where p and q are polynomials. Without loss of generality p is irreducible. Set $V = K^n$ and

$$W = \{(x, y) \in K^{2n} : y_1 = x_2, y_2 = x_3, \ldots, y_{n-1} = x_n, p(x_1, \ldots, x_n, y_n) = 0\}.$$

It is easy to see that W is irreducible and W projects generically onto K^n. Let $U = \{(x, y) \in W : q(x) \neq 0\}$. By (iii) there is $x \in K^n$ such that $(x, \delta(x)) \in U$. Then $f(x_1) = 0$ and $g(x_1) \neq 0$.

(i) $\Rightarrow$ (iii) Let V, W and U be as in (iii). Let (x, y) be a generic point of U over K. One can show that there is a differential field L extending $K(x, y)$ with $\delta(x) = y$ (indeed we can extend δ to $K(x, y)$). We may assume L is differentially closed. Then $(x, \delta(x)) \in U$ and by model completeness we can find a solution in K. □

2. The Kolchin Topology

We say that an ideal I in $K\{X_1, \ldots, X_n\}$ is a δ-ideal if $\delta(f) \in I$ whenever $f \in I$. If $I \subset K\{X_1, \ldots, X_n\}$, let $V_\delta(I) = \{x \in K^n : f(x) = 0 \text{ for all } f \in I\}$. We can topologize K^n by taking the $V_\delta(I)$ as basic closed sets. This topology is referred to as the *Kolchin topology* or the δ-topology.

differential variety.

There are infinite ascending sequences of δ-ideals. For example,

$$\langle X^2 \rangle \subset \langle X^2, \delta(X)^2 \rangle \subset \cdots \subset \langle X^2, \, \delta(X)^2, \, \ldots, \, \delta^m(X)^2 \rangle \subset \cdots,$$

$I = \sqrt{I}$
($\subseteq$ obvious)

where $\langle f_1, \ldots, f_n \rangle$ is the δ-ideal generated by $f_1, \ldots, f_n$. But radical δ-ideals are well behaved (for a proof see [Kaplansky 1957] or [Marker et al. 1996]).

Because $I^n = (\sqrt{I})^n = \sqrt{I} = I$
($I^n \subseteq I$ because I is an ideal)

Theorem 2.1 (Ritt–Raudenbush Basis Theorem). (i) *There are no infinite ascending chains of radical differential ideals in $K\{X_1, \ldots, X_n\}$. For any radical differential ideal I there are $f_1, \ldots, f_m$ such that $I = \sqrt{\langle f_1, \ldots, f_m \rangle}$.* finitely generated
(ii) *If $I \subset K\{X_1, \ldots, X_n\}$ is a radical δ-ideal, there are distinct prime δ-ideals $P_1, \ldots, P_m$ such that $I = P_1 \cap \cdots \cap P_m$ and $P_1, \ldots, P_m$ are unique up to permutation.*

Thus the δ-topology is Noetherian and any δ-closed set is a finite union of irreducible δ-closed sets.

$\sqrt{J} = \{ f \mid \exists n \ f^n \in J \}$

$(\sqrt{J})^m = \{ f^m \mid \exists n \ f^n \in J \} = \sqrt{J}$

Theorem 2.2 (Seidenberg's Differential Nullstellensatz). *Let K be a differentially closed field. $I \mapsto V_\delta(I)$ is a one to one correspondence between* ~~*radical*~~ *δ-ideals and δ-closed sets.*

Proof. It is easy to see that $I_\delta(Y)$ is a radical δ-ideal for all $Y \subseteq K^n$. Suppose I and J are radical δ-ideals and $g \in J \setminus I$. By Theorem 2.1 there is a prime δ-ideal $P \supseteq I$ with $g \notin P$. It suffices to show there is $x \in V_\delta(P)$ with $g(x) \neq 0$. Let $P = \sqrt{\langle f_1, \ldots, f_m \rangle}$, and let L be a differentially closed field containing $K\{X_1, \ldots, X_n\}/P$. Let $x = (X_1/P, \ldots, X_n/P)$. Clearly $f(x) = 0$ for $f \in P$ and $g(x) \neq 0$. In particular,

$$L \models \exists v_1 \ldots \exists v_n \; f_1(v_1, \ldots, v_n) = \ldots = f_m(v_1, \ldots, v_n) = 0 \wedge g(v_1, \ldots, v_n) \neq 0.$$

By model completeness the same sentence is true in K. Thus there is $x \in K^n$ such that $x \in V_\delta(P) \setminus V_\delta(J)$. □

By the basis theorem every δ-closed set is definable. We say that a subset of K^n is δ-constructible if it is a finite boolean combination of δ-closed sets. The δ-constructible sets are exactly those defined by quantifier free $\mathcal{L}$-formulas. Quantifier elimination implies that the δ-constructible sets are exactly the definable sets. Thus the projection of a δ-constructible set is δ-constructible.

3. ω-Stability and Dimension

Let K be a differentially closed fields and let F be a differential subfield of K. If $p \in S_n(F)$, let $I_\delta(p) = \{f \in F\{X_1, \ldots, X_n\} : \text{"}f(x_1, \ldots, x_n) = 0\text{"} \in p\}$. The arguments for types in algebraically closed fields in [Marker 2000] work here to show that $p \mapsto I_p$ is a bijection from $S_n(F)$ onto the space of prime δ-ideals.

Corollary 3.1. DCF *is ω-stable.*

Proof. Let K and F be as above. We must show that $|S_n(F)| = |F|$. But for all p, we can find $f_1, \ldots, f_m$ such that $I_\delta(p) = \sqrt{\langle f_1, \ldots, f_m \rangle}$. Thus the number of complete n-types is equal to $|F\{X_1, \ldots, X_n\}| = |F|$. □

There is an important algebraic application of ω-stability. If F is a differential field, we say that a differentially closed $K \supseteq F$ is a *differential closure* of F if for any differentially closed $L \supseteq F$ there is a differential embedding of K into L fixing F.

This is related to a general model-theoretic notion mentioned in [Hart 2000]. A *prime model* of T over A is a model $\mathcal{M} \models T$ with $A \subseteq M$, such that if $\mathcal{N} \models T$ and $A \subset N$, then there is an elementary embedding $j : \mathcal{M} \to \mathcal{N}$ such that $j|A$ is the identity. For DCF, prime model extensions are exactly differential closures (recall that, by model completeness, all embeddings are elementary).

Theorem 3.2. *Let T be an ω-stable theory, $\mathcal{M} \models T$ and $A \subseteq M$. There is a prime model of T over A. If $\mathcal{N}_0$ and $\mathcal{N}_1$ are prime models of T over A, then $\mathcal{N}_0$ and $\mathcal{N}_1$ are isomorphic over A.*

The existence of prime models was proved by Morley and uniqueness (under less restrictive assumptions) is due to Shelah. The following corollary was later given a slightly more algebraic proof by Kolchin.

Corollary 3.3. *Every differential field F has a differential closure and any two differential closures of F are unique up to isomorphism over F.*

Differential closures need not be minimal. Let F be the differential closure of $\mathbb{Q}$. Independent results of Kolchin, Rosenlicht and Shelah show there is a nontrivial differential embedding $j : F \to F$ with $j(F)$ a proper subfield of F.

Since DCF is ω-stable, we can assign Morley rank to types and definable sets. This gives us a potentially useful notion of dimension. It is interesting to see how this corresponds to more algebraic notions of dimension.

There are two natural cardinal dimensions. Suppose $V \subseteq K^n$ is an irreducible δ-closed set. Let $K[V]$ be the *differential coordinate ring* $K\{X_1, \dots, X_n\}/I_\delta(V)$. Let$\mathrm{td}(V)$ be the transcendence degree of $K[V]$ over K. Often $\mathrm{td}(V)$ is infinite. We say that elements of a differential ring are *differentially dependent* over K if they satisfy a differential polynomial over K. Let$\mathrm{td}_\delta(V)$ be the size of a maximal differentially independent subset of $K[V]$ over V. Note that$\mathrm{td}(V)$ is finite if and only if$\mathrm{td}_\delta(V) = 0$. If V is an algebraic variety of dimension d, then$\mathrm{td}_\delta(V) = d$. Suppose W and V are proper irreducible δ-closed subsets of K with $W \subset V$. Then$\mathrm{td}_\delta(V) =$$\mathrm{td}_\delta(W) = 0$ and$\mathrm{td}(W) <$$\mathrm{td}(V)$.

There is a natural ordinal dimension that arises from the Noetherian topology. This is the analog of Krull dimension in Noetherian rings. If V is a non-empty irreducible δ-closed set, we define $\dim_\delta(V)$ as follows. If V is a point, then $\dim_\delta(V) = 0$. Otherwise

$$\dim_\delta(V) = \sup\{\dim_\delta(W) + 1 : W \subset V \text{ is irreducible, } \delta\text{-closed and nonempty}\}.$$

Since $V_\delta(X) \subset V_\delta(\delta(X)) \subset \cdots \subset V_\delta(\delta^n(X)) \subset \cdots K$, we have $\dim_\delta(K) \geq \omega$. The remarks above imply that if $V \subset K$ is δ-closed, then $\dim_\delta(V) \leq \mathrm{td}(V)$. Hence $\dim_\delta(K) = \omega$.

There are two model-theoretic notions of dimension, Morley rank and U-rank. We refer to [Hart 2000] for the definition of Morley rank but will describe U-rank in this context. If A, B and C are subsets of a differentially closed field, we say that B and C are *independent* over A if the differential field generated by $B \cup A$ and the differential field generated by $C \cup A$ are linearly disjoint over the algebraic closure of the differential field generated by A. If $B \supset A$, $p \in S_n(A)$, $q \in S_n(B)$ and $p \subset q$, we say that q is a *forking* extension of p, if for any a realizing q, a and B are dependent over A.

We define the *U-rank* of a type $p \in S_n(A)$ inductively as follows:

(i) $\mathrm{U}(p) \geq 0$.
(ii) If α is a limit ordinal, then $\mathrm{U}(p) \geq \alpha$ if and only if $\mathrm{U}(p) \geq \beta$ for all $\beta < \alpha$.

(iii) $\mathrm{U}(p) \geq \alpha + 1$ if and only if there is $B \supset A$ and $q \in S_n(B)$, q a forking extension of p and $\mathrm{U}(q) \geq \alpha$.

We write $\mathrm{U}(b/A)$ for $\mathrm{U}(\mathrm{tp}(b/A))$. If X is definable over A, we let $\mathrm{U}(X)$ be the maximum $U(b/A)$ for $b \in X$.

In algebraically closed fields we also have four notions of dimension (transcendence degree, Krull dimension, Morley rank and U-rank), all of which agree. In DCF the situation is different.

Theorem 3.4. *We have*

$$\mathrm{U}(V) \leq \mathrm{RM}(V) \leq \dim_\delta(V) \leq \omega \cdot \mathrm{td}_\delta(V)$$

and if $\mathrm{td}_\delta(V) = 0$, *then* $\dim_\delta(V) \leq \mathrm{td}(V)$.

Theorem 3.4 is a combination of results of Poizat, Johnson and Pong (see [Pong $\geq$ 2001] for details). There are examples due to Kolchin, Poizat, and Hrushovski and Scanlon showing that any of these inequalities may be strict. Although these notions may disagree, it is easy to see that U-rank is finite if and only if transcendence degree is finite. Thus finite dimensionality does not depend on which notion of dimension we choose. It is also easy to see that $\mathrm{U}(K^n) = \omega n$ so the notions of dimension agree on K^n (and on all algebraic varieties).

The following result of Pong [$\geq$ 2001] shows the usefulness of U-rank. It is part of his proof that any finite rank δ-closed set is affine.

Theorem 3.5. *Suppose* $V \subset \mathbb{P}^n$ *is* δ*-closed and* $U(V) < \omega$. *If* $H \subset \mathbb{P}^n$ *is a generic hyperplane*, *then* $H \cap V = \varnothing$.

Proof. Let $\mathcal{H}$ be the set of all hyperplanes in $\mathbb{P}^n$. Since $\mathcal{H}$ is isomorphic to $\mathbb{P}^n$, $\mathrm{U}(\mathcal{H}) = \omega n$. Similarly for any point x, the set of hyperplanes through x has U-rank $\omega(n-1)$ over x. Let $I = \{(v, H) : H \in \mathcal{H},\, v \in V \cap H\}$. Suppose $(v, H) \in I$ and $\mathrm{U}((v, H))$ is maximal. The Lascar inequality (valid in any superstable theory) asserts that

$$\mathrm{U}(v, H) \leq \mathrm{U}(H/v) \oplus \mathrm{U}(v),$$

where $\oplus$ is the symmetric sum of ordinals. In this case H is in the set of hyperplanes through v, so $\mathrm{U}(H/v) \leq \omega(n-1)$. Since $\mathrm{U}(V)$ is finite, $\mathrm{U}(v) < \omega$. Thus $\mathrm{U}(v, H) < \omega n$. But if H were a generic hyperplane $U(v, H) \geq \mathrm{U}(H) = \omega n$, a contradiction. $\square$

We conclude this section by summarizing some important results about interpretability in DCF.

Theorem 3.6 (Poizat). DCF *has elimination of imaginaries. In particular the quotient of a* δ*-constructible set by a* δ*-constructible equivalence relation is* δ*-constructible.*

THEOREM 3.7. (i) (PILLAY) *Any group interpretable in a differentially closed field K is definably isomorphic to the K-rational points of a differential algebraic group defined over K.*
(ii) (SOKOLOVIC) *Any infinite field of finite rank interpretable in a differentially closed field is definably isomorphic to the field of constants.*
(iii) (PILLAY) *Any field of infinite rank interpretable in $K \models$ DCF is definably isomorphic to K.*

4. Strongly Minimal Sets in Differentially Closed Fields

Let K be a $\aleph_0$-saturated differentially closed field. Let $X \subset K^n$ be definable. By adding parameters to the language we assume that X is defined over $\varnothing$. Recall that X is *strongly minimal* if whenever $Y \subseteq X$ is definable then either Y or $X \setminus Y$ is finite. For $A \subseteq K$ let $\mathrm{acl}^\delta(A)$ be the algebraic closure of the differential field generated by A. In DCF this is exactly the model-theoretic notion of algebraic closure. If X is strongly minimal let $\mathrm{acl}^\delta_X(A) = \mathrm{acl}^\delta(A) \cap X$. For $A \subseteq X$, let $\dim(A)$ be the maximum cardinality of an acl^δ-independent subset of $\mathrm{acl}^\delta(X)$.

We say that a strongly minimal set X is *trivial* if

$$\mathrm{acl}^\delta_X(A) = \bigcup_{a\in A} \mathrm{acl}^\delta_X(\{a\})$$

for all $A \subseteq X$. Examples of trivial strongly minimal sets are a set with no structure or the natural numbers with the successor function.

We say that a strongly minimal set X is *locally modular* if

$$\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$$

whenever A and B are finite dimensional acl^δ_X-closed subsets of X and $A \cap B \neq \varnothing$. Vector spaces are good examples of locally modular strongly minimal sets. Indeed general results of Hrushovski show that any nontrivial locally modular strongly minimal set is essentially a vector space.

To fully understand the model theory of any ω-stable theory it is essential to understand the strongly minimal sets. In differentially closed fields this is particularly fascinating because there are trivial, nontrivial locally modular and non-locally modular strongly minimal sets.

Trivial strongly minimal sets first arose in the proofs that differential closures need not be minimal. For example the solution sets to the differential equations $\delta(X) = \dfrac{X}{X+1}$ and $\delta(X) = X^3 - X^2$ are (after throwing out 0 and, in the second case, 1) sets of total indiscernibles (that is, sets with no additional structure).

There is one obvious non-locally modular strongly minimal set, C the field of constants. Hrushovski and Sokolovic proved that this is essentially the only one.

THEOREM 4.1. *If $X \subset K^n$ is strongly minimal and non-locally modular, then X is non-orthogonal to the constants.*

The proof proceeds by first showing that strongly minimal sets in differentially closed field are Zariski geometries, in the sense of [Hrushovski and Zilber 1996]. The main theorem on Zariski geometries says that non-locally modular Zariski geometries are non-orthogonal to definable fields, but the only finite rank definable field is, by Theorem 3.7, the constants.

Hrushovski and Sokolovic also showed that nontrivial locally modular strongly minimal sets arise naturally in studying abelian varieties as differential algebraic groups. In his proof of the Mordell conjecture for function fields, Manin proved the following result.

THEOREM 4.2. *If A is an abelian variety, then there is a nontrivial differential algebraic group homomorphism $\mu : A \to K^n$.*

For example, if A is the elliptic curve $y^2 = x(x-1)(x-\lambda)$, where $\delta(\lambda) = 0$, then

$$\mu(x, y) = \frac{\delta(x)}{y}.$$

If $\delta(\lambda) \neq 0$, then μ is a second order differential operator. Let $A^{\#}$ be the δ-closure of the torsion points of A. We can choose μ so that $A^{\#}$ is the kernel of μ. Building on 4.2 and further work of Buium, Hrushovski and Sokolovic showed:

THEOREM 4.3. *Suppose A is a simple abelian variety defined over K. Either*

(i) *A is isomorphic to an abelian variety B defined over the constants, or*
(ii) *$A^{\#}$ is locally modular and strongly minimal.*

Moreover, any nontrivial locally modular strongly minimal set is non-orthogonal to $A^{\#}$ for some such A, and $A^{\#}$ and $B^{\#}$ are non-orthogonal if and only if A and B are isogenous.

Thus Hrushovski and Sokolovic have completely characterized the nontrivial strongly minimal sets. Understanding the trivial ones is still a difficult open problem. We say that a strongly minimal set X is $\aleph_0$-categorical if in any model the dimension of the elements of the model in X is infinite. One open question is: In DCF is every trivial strongly minimal set $\aleph_0$-categorical? Hrushovski has proved this for strongly minimal sets of transcendence degree one.

5. Diophantine Applications

Hrushovski used Theorem 4.3 in his proof of the Mordell–Lang conjecture for function fields in characteristic zero.

THEOREM 5.1. *Suppose $K \supset k$ are algebraically closed fields of characteristic zero, A is an abelian variety defined over K such that no infinite subabelian* *variety of A is isomorphic to an abelian variety defined over k, Γ is a finite rank subgroup of A, and X is a subvariety of A such that $X \cap \Gamma$ is Zariski dense in X. Then X is a finite union of cosets of abelian subvarieties of A.*

I will sketch the ideas of the proof; for full details see [Hrushovski 1996], [Bouscaren 1998] or [Pillay 1997]. First I give the full proof in one easy case.

Theorem 5.2. *Suppose $K \supset k$ are algebraically closed fields of characteristic zero, A is a simple abelian variety defined over K that is not isomorphic to an abelian variety defined over k, Γ is the torsion points of A, and X is a proper subvariety of A. Then $X \cap \Gamma$ is finite.*

Proof. The main idea is to move to a differential field setting where we may apply model-theoretic tools. In doing so we will replace the group Γ by $A^{\#}$, a small (that is, finite-dimensional) differential algebraic group.

The first step is to define a derivation $\delta : K \to K$ such that $k = \{x \in K : \delta(x) = 0\}$. One can show that if $\hat{K}$ is the differential closure of K then the field of constants of $\hat{K}$ is still k. Thus without loss of generality we may assume that K is a differentially closed field and k is the constant field of K.

Let $A^{\#}$ be the δ-closure of the torsion points of A. It suffices to show that $A^{\#} \cap X$ is finite. Suppose not. Since $A^{\#}$ is strongly minimal, $A^{\#} \setminus X$ is finite. But then the Zariski closure of $A^{\#}$ is contained in $X \cup A^{\#} \setminus X$, which is properly contained in A. This is a contradiction since the torsion points are Zariski dense. □

The proof above does not explicitly use the fact that $A^{\#}$ is locally modular (though this does come into the proof that it is strongly minimal). Local modularity plays more of a role if we consider larger groups Γ. Suppose K, k, A and X are as above and Γ is a finite rank subgroup of A. Let $\mu : A \to K^n$ be a definable homomorphism with kernel $A^{\#}$. Since Γ has finite rank, the image of Γ under μ is contained in $V \subset K^n$ which is a finite dimensional k-vector space. Consider $G = \mu^{-1}(V)$. This is a definable finite Morley rank subgroup of A. Some analysis of this group allows us to conclude that it is 1-based (see [Hart 2000] for a definition: basically this means that all of the strongly minimal sets are locally modular). Hrushovski and Pillay showed that in an 1-based group G any definable subset of G^n is a boolean combination of cosets of definable subgroups. In particular $X \cap G$ will be a finite union of cosets of definable subgroups of G. If any of these subgroups is infinite, then its Zariski closure is an algebraic subgroup and must hence be the whole group. This would contradict the fact that the Zariski closure would be contained in X.

To prove Theorem 5.1 in general, we use the fact that every abelian variety is isogenous to a direct sum of simple abelian subvarieties together with a number of techniques from finite Morley rank group theory.

References

[Bouscaren 1998] E. Bouscaren, “Proof of the Mordell–Lang conjecture for function fields”, pp. 177–196 in *Model theory and algebraic geometry*, edited by E. Bouscaren, Lecture Notes in Math. **1696**, Springer, Berlin, 1998.

[Buium 1994] A. Buium, *Differential algebra and Diophantine geometry*, Actualités math., Hermann, Paris, 1994.

[Hart 2000] B. Hart, “Stability theory and its variants”, pp. 131–148 in *Model theory, algebra and geometry*, edited by D. Haskell et al., Math. Sci. Res. Inst. Publ. **39**, Cambridge Univ. Press, New York, 2000.

[Hrushovski 1996] E. Hrushovski, “The Mordell–Lang conjecture for function fields”, *J. Amer. Math. Soc.* **9**:3 (1996), 667–690.

[Hrushovski and Sokolovic $\geq$ 2001] E. Hrushovski and Z. Sokolovic, “Minimal subsets of differentially closed fields”. To appear in *Trans. Amer. Math. Soc.*

[Hrushovski and Zilber 1996] E. Hrushovski and B. Zilber, “Zariski geometries”, *J. Amer. Math. Soc.* **9**:1 (1996), 1–56.

[Kaplansky 1957] I. Kaplansky, *An introduction to differential algebra*, Actualités Sci. Ind. **1251**, Hermann, Paris, 1957.

[Kolchin 1973] E. R. Kolchin, *Differential algebra and algebraic groups*, Pure and Applied Mathematics **54**, Academic Press, New York, 1973.

[Magid 1994] A. R. Magid, *Lectures on differential Galois theory*, Amer. Math. Soc., Providence, RI, 1994.

[Marker 2000] D. Marker, “Introduction to model theory”, pp. 15–35 in *Model theory, algebra and geometry*, edited by D. Haskell et al., Math. Sci. Res. Inst. Publ. **39**, Cambridge Univ. Press, New York, 2000.

[Marker et al. 1996] D. Marker, M. Messmer, and A. Pillay, *Model theory of differentiable fields*, Lecture Notes in Logic **5**, Springer, Berlin, 1996.

[Pierce and Pillay 1998] D. Pierce and A. Pillay, “A note on the axioms for differentially closed fields of characteristic zero”, *J. Algebra* **204**:1 (1998), 108–115.

[Pillay 1996] A. Pillay, *Differential algebraic groups and the number of countable differentially closed fields*, Lecture Notes in Logic **5**, Springer, Berlin, 1996.

[Pillay 1997] A. Pillay, “Model theory and Diophantine geometry”, *Bull. Amer. Math. Soc.* (*N.S.*) **34**:4 (1997), 405–422. Erratum in **35**:1 (1998), 67.

[Pong $\geq$ 2001] W. Y. Pong, “Some applications of ordinal dimensions to the theory of differentially closed fields”. To appear in *J. Symb. Logic.*

David Marker
University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science (M/C 249)
851 S. Morgan St.
Chicago, IL 60613
United States
marker@math.uic.edu