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On better-quasi-ordering transfinite sequences

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Abstract. Let Q be a quasi-ordered set, i.e. a set on which a reflexive and transitive relation $\leqslant$ is defined. If, for every finite sequence $q_1, q_2, \ldots$ of elements of Q, there exist i and j such that $i < j$ and $q_i \leqslant q_j$, then we call Q well-quasi-ordered. For any ordinal number α, the set of all ordinal numbers less than α is called an initial set. A function from an initial set into Q is called a transfinite sequence on Q. If $f: I_1 \to Q$, $g: I_2 \to Q$ are transfinite sequences on Q, the statement $f \leqslant g$ means that there is a one-to-one order-preserving function $\phi: I_1 \to I_2$ such that $f(\alpha) \leqslant g(\phi(\alpha))$ for every $\alpha \in I_1$. Milner has conjectured in (3) that, if Q is well ordered, then any set of transfinite sequences on Q is well-quasi-ordered under the quasi-ordering just defined. In this paper, we define so-called 'better-quasi-ordered sets', which are well-quasi-ordered sets of a particularly 'well-behaved' kind, and we prove that any set of transfinite sequences on a better-quasi-ordered set is better-quasi-ordered. Milner's conjecture follows *a fortiori*, since every well ordered set is better-quasi-ordered and every better-quasi-ordered set is well-quasi-ordered.

1. *Introduction.* This paper could be regarded as the fifth in a series, its predecessors being (4), (5), (6) and (7). In particular, the arguments used here are reminiscent of those of (6). The present paper assumes certain results from (6) and (7), but is self-contained to the extent that it contains definitions of all terminology and notation used and full statements of results quoted from elsewhere. It should be mentioned that certain concepts defined in (6) and (7) are here re-defined a little differently since, for instance, the set of non-negative integers is sometimes used here as an indexing set where the set of positive integers was used previously. Thus, strictly speaking, proofs given in (6) and (7) of certain results require minor changes in order to show that these results remain true when the language in which they are formulated is re-interpreted in the sense of the present paper. However, in cases where the necessary changes in the proofs are trivial and formal, we shall not hesitate to quote such results without further discussion.

In this paper, a *function* is understood to be a set f of ordered pairs such that, if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. The empty set (considered as an empty set of ordered pairs) is a function according to this definition. If f is a function and $(x, y) \in f$, we write $y = f(x)$. The *domain* $\mathrm{D}f$ of f is the set of all elements x such that $(x, y) \in f$ for some y and the *image* of f, denoted by $\mathrm{Im}\, f$ or $\bar{f}$, is $\{f(x): x \in \mathrm{D}f\}$. If $\mathrm{D}f = A$ and $\mathrm{Im}\, f \subseteq B$, we say that f is a function *from A into B* and write $f: A \to B$. If $S \subseteq \mathrm{D}f$, the expression $f|S$ denotes the function $\{(x, f(x)): x \in S\}$. For every ordinal number α, O_α will denote the set of all ordinal numbers less than α, and sets of the form O_α (where α is an ordinal number) will be called *initial sets*. For the purposes of this paper, a *sequence* is a function whose domain is an initial set. A sequence with domain O_α has *length* α and will be called an *α-sequence*. A sequence of finite length is a *finite* sequence. The sequence of length 0 will be denoted by $\square$. In fact, this sequence is according to our definitions simply the empty set and could therefore be denoted by $\varnothing$; but the use of $\square$ for the empty set when it is being considered as a sequence makes the discussion clearer. The length of a sequence s will be denoted by $l(s)$. If S is a set of sequences, the statement that a sequence s is the *longest* element of S will mean that $s \in S$ and $l(s) > l(s')$ for every $s' \in S - \{s\}$, and the statement that s is the *shortest* element of S means that $s \in S$ and $l(s) < l(s')$ for every $s' \in S - \{s\}$. A sequence s such that $\mathrm{Im}\, s \subseteq B$ is a sequence *on* the set B. If s is a sequence and $\theta \in \mathbf{D}s$, we call $s(\theta)$ the *θth term* of s. A set of ordinal numbers will be called an *ordinal set*. An *ordinal sequence* is a sequence whose terms are ordinal numbers. An ordinal sequence s is *ascending* if $s(\theta) < s(\phi)$ whenever θ, $\phi \in \mathbf{D}s$ and $\theta < \phi$. Since in particular ordinal sequences of length 0 or 1 satisfy this condition vacuously, they are automatically ascending. [Note, however, that in (7) the term 'ascending' was reserved for sequences of non-zero length.] If X is an ordinal set and α is an ordinal number, $A_\alpha(X)$ will denote the set of all ascending α-sequences on X and $\tilde{A}(X)$ will denote the set of all ascending finite sequences on X. We shall write $\tilde{A}(X) - \{\square\} = A(X)$. [Our notation agrees with that of (7) but not with that of (6), where $A(X)$ was used for the set of all ascending finite sequences on X including $\square$. Such spaces of ascending sequences as those now being considered have been previously used in connexion with the study of well-quasi-ordered sets by Dr J. B. Kruskal.] A *subsequence* of a sequence s is a sequence of the form $s \circ f$, i.e. $\{(\theta, s(f(\theta))) : \theta \in \mathbf{D}f\}$, where f is an ascending sequence on $\mathbf{D}s$. In particular, $\square$ is a subsequence of every sequence s since $\square = s \circ \square$. A subsequence of s other than s itself is a *strict subsequence* of s. If n is a positive integer, an n-sequence s may be denoted by $[s(0), s(1), \ldots, s(n-1)]$ and an ω-sequence t may be denoted by $[t(0), t(1), \ldots]$: in other words, we shall sometimes denote a sequence by a list of its terms in the appropriate order enclosed in square brackets. If s, t are sequences, st will denote the sequence v of length $l(s) + l(t)$ such that $v(\theta) = s(\theta)$ for every $\theta \in \mathbf{D}s$ and $v(l(s) + \phi) = t(\phi)$ for every $\phi \in \mathbf{D}t$. For instance, if $s = [3, 4, 2, 2]$ and $t = [4, 1, 6]$ then $st = [3, 4, 2, 2, 4, 1, 6]$. Note that in this case $s[1]$ would denote $[3, 4, 2, 2, 1]$, and should not be confused with $s(1)$ which denotes the 'first' term of s, i.e. 4. (3 is the '0th' term of s.) Note also that $s\square = \square s = s$ for any sequence s. If s, t, u are sequences, stu denotes $s(tu) = (st)u$. If s, t are sequences such that $t = sw$ for some sequence w, we call s a *left-segment* of t and t an *extension* of s and write $s \prec t$. If $s \prec t$ and $s \neq t$, we call s a *strict left-segment* of t and write $s \prec\prec t$. A sequence s is a *right-segment* of a sequence t if $t = ws$ for some sequence w, and s is a *segment* of a sequence u if $u = xsy$ for some sequences x, y. The *last term* $s(l(s) - 1)$ of a finite sequence $s \neq \square$ is denoted by $\lambda(s)$. If s is a sequence different from $\square$, ${}_*s$ and, if s is finite, s_* will denote the sequences defined by the relations $s = [s(0)]({}_*s)$, $s = (s_*)[\lambda(s)]$, i.e. intuitively, the sequences obtained from s by omitting its 0th and last terms respectively. The set of all non-negative integers will be denoted by N and the set of all positive integers by P.

(Thus $N = O_\omega$.) If s, $u \in \tilde{A}(N)$, the statement $s \lhd u$ means that there exists a $t \in A(N)$ such that $s \prec\prec t$ and $u = {}_*t$. For example, $[1, 2, 6] \lhd [2, 6, 7, 9, 10]$. If $B \subseteq \tilde{A}(N)$, $\bar{B}$ will denote $\bigcup\{\bar{s}: s \in B\}$, i.e. the set of all non-negative integers which are terms of one or more members of B. A *block* is an infinite subset B of $A(N)$ such that every element of $A_\omega(\bar{B})$ has a left-segment which belongs to B. (Since a subset B of $A(N)$ is clearly infinite if and only if $\bar{B}$ is infinite, this definition of 'block' is the same as that given in (7) except that we now define it to be a subset of $A(N)$ and not $A(P)$.) If S is a set, an *S-pattern* is a function from a block into S.

A *quasi-ordered* (*qo*) set is a set on which a reflexive and transitive relation is defined: this reflexive and transitive relation is usually denoted by $\leqslant$. The symbol Q will always denote a qo set. If $q_1, q_2 \in Q$ and $q_1 \leqslant q_2$, we shall say that q_1 *anticipates* q_2. An ω-sequence s on Q is *good* if there exist $i, j \in N$ such that $i < j$ and $s(i) \leqslant s(j)$, and is *bad* if not. A *well-quasi-ordered* (*wqo*) set is a qo set Q such that every ω-sequence on Q is good. A Q-pattern f is *good* if there exist $s, t \in \mathbf{D}f$ such that $s \lhd t$ and $f(s) \leqslant f(t)$, and is *bad* otherwise. A *better-quasi-ordered* (*bqo*) set is a qo set Q such that every Q-pattern is good. It is a fairly easy exercise to prove the following lemma: this lemma is essentially a combination of Lemmas 1 and 27 of (7), where a detailed proof may be found.

LEMMA 1. *Every well ordered set is bqo, and every bqo set is wqo.*

We quasi-order any set S of sequences on Q by the rule that, if $f, g \in S$, then '$f \leqslant g$' means that there exists a subsequence f' of g such that $l(f') = l(f)$ and $f(\theta) \leqslant f'(\theta)$ for every $\theta \in \mathbf{D}f$. In particular, when we take Q to be an ordinal set with the usual ordering, this defines a quasi-ordering on any set of ordinal sequences, and Dr E. C. Milner has conjectured that, under this quasi-ordering, any set of ordinal sequences is wqo. In (3), Milner proves the truth of this conjecture for ordinal sequences of length less than ω^3. The truth of Milner's conjecture in general is, in view of Lemma 1, a consequence of the following theorem, which it is the purpose of this paper to prove.

THEOREM. *Any set of sequences on a bqo set is bqo.*

The foregoing method of quasi-ordering a set of sequences on a qo set was introduced by Higman in (2) and certain of its properties have been examined in (1), (2), (3) (6) and (8) and in unpublished work of Dr J. B. Kruskal. Bqo sets have been previously considered in (7), where the above theorem was mentioned as a conjecture.

2. *Preliminary lemmas.*

Definitions. Let M denote an infinite subset of N. If $s \in \tilde{A}(M)$, M/s will denote the set of those elements of M which are greater than every element of $\bar{s}$. (Thus $M/\square = M$.) We shall tend for clarity to refer to a set of sets as a *collection* of sets. An *M-admissible* collection of sets (or, in the terminology of (6), 'admissible class of subsets of M') is a collection $\mathfrak{C}$ of infinite subsets of M such that (i) every infinite subset of M contains a member of $\mathfrak{C}$ and (ii) every infinite subset of a member of $\mathfrak{C}$ belongs to $\mathfrak{C}$. An *M-admissible* function is a function $\mathfrak{F}$ with domain $\tilde{A}(M)$ such that, for every $s \in \tilde{A}(M)$, $\mathfrak{F}(s)$ is an (M/s)-admissible collection of sets. Let ϕ denote a function from $\tilde{A}(M)$ into an ordinal set. We call ϕ *dwindling* if $\phi(s) \geqslant \phi(t)$ whenever s, $t \in \tilde{A}(M)$ and $s \prec t$. If L is an infinite subset of M, an element s of $\tilde{A}(L)$ is *ϕ-extensible in L* if there exists a

$t \in \tilde{A}(L)$ such that $s \prec t$ and $\phi(s) > \phi(t)$; and s is *ϕ-inextensible in L* if it is not ϕ-extensible in L. (We note that, if $s \prec t \in \tilde{A}(L)$ and $\phi(s) > \phi(t)$, then in fact $s \prec\prec t$ and therefore $t \in A(L)$.) An element s of $\tilde{A}(L)$ is *strongly ϕ-extensible in L* if it is ϕ-extensible in every infinite subset of L which contains $\bar{s}$. An element s of $\tilde{A}(L)$ is *ϕ-objectionable in L* if there exists a $u \in A_\omega(L/s)$ such that every strict left-segment of su is ϕ-extensible in L. If no such u exists, s is *ϕ-unobjectionable in L.* We shall call s *strongly ϕ-objectionable in L* if it is ϕ-objectionable in every infinite subset of L which contains $\bar{s}$.

Lemma 2. *If M is an infinite subset of N and $\mathfrak{F}$ is an M-admissible function, then there exists an infinite subset L of M such that $L/s \in \mathfrak{F}(s)$ for every $s \in \tilde{A}(L)$.*

This is, in essentials, Lemma 2 of (6).

Lemma 3. *Let K, L, M be infinite subsets of N such that $K \subseteq L \subseteq M$ and let ϕ be a function from $\tilde{A}(M)$ into an ordinal set. If $s \in \tilde{A}(K)$ and s is ϕ-extensible in K, then s is ϕ-extensible in L.*

The proof is trivial and is left to the reader.

Lemma 4. *Let M be an infinite subset of N and L, L' be infinite subsets of M. Let $s \in \tilde{A}(L \cap L')$ and let $L/s = L'/s$. Let ϕ be a function from $\tilde{A}(M)$ into an ordinal set. Then*

(i) *if s is ϕ-extensible in L then s is ϕ-extensible in L';*

(ii) *if s is strongly ϕ-extensible in L then s is strongly ϕ-extensible in L'.*

Proof. (i) Suppose that s is ϕ-extensible in L. Then we can select a $t \in \tilde{A}(L)$ such that $s \prec t$ and $\phi(s) > \phi(t)$. Since $s \prec t \in \tilde{A}(L)$, it follows that

$$\bar{t} \subseteq \bar{s} \cup (L/s) = \bar{s} \cup (L'/s) \subseteq L'$$

and therefore $t \in \tilde{A}(L')$. Since $t \in \tilde{A}(L')$ and $s \prec t$ and $\phi(s) > \phi(t)$, we have proved that s is ϕ-extensible in L'.

(ii) Suppose that s is strongly ϕ-extensible in L. If K is an infinite subset of L' which contains $\bar{s}$, then

$$\bar{s} \cup (K/s) \subseteq \bar{s} \cup (L'/s) = \bar{s} \cup (L/s) \subseteq L$$

and therefore s, being strongly ϕ-extensible in L, is ϕ-extensible in $\bar{s} \cup (K/s)$ and hence, by Lemma 3, is ϕ-extensible in K. Hence s is strongly ϕ-extensible in L'.

Lemma 5. *Let M be an infinite subset of N and ϕ be a function from $\tilde{A}(M)$ into an ordinal set. Let $s \in \tilde{A}(M)$ and $\mathfrak{C}$ be the collection of all infinite subsets K of M/s such that s is either ϕ-inextensible in $K \cup \bar{s}$ or strongly ϕ-extensible in $K \cup \bar{s}$. Then $\mathfrak{C}$ is an (M/s)-admissible collection of sets.*

Proof. Let L be an infinite subset of M/s. If L has an infinite subset K such that s is ϕ-inextensible in $K \cup \bar{s}$, then K is a member of $\mathfrak{C}$ contained in L. If L has no such subset, then s is ϕ-extensible in $K \cup \bar{s}$ for every infinite subset K of L. Since $L \subseteq M/s$, this is equivalent to saying that s is ϕ-extensible in every infinite subset of $L \cup \bar{s}$ which contains $\bar{s}$. Therefore s is strongly ϕ-extensible in $L \cup \bar{s}$, and so $L \in \mathfrak{C}$. Therefore, once again, L contains a member of $\mathfrak{C}$. This argument proves that every infinite subset of M/s contains a member of $\mathfrak{C}$.

Let I be an infinite subset of a member J of $\mathfrak{C}$. Then s is either ϕ-inextensible in $J \cup \bar{s}$ or strongly ϕ-extensible in $J \cup \bar{s}$. If s is ϕ-inextensible in $J \cup \bar{s}$, then by Lemma 3 it is ϕ-inextensible in $I \cup \bar{s}$ and therefore $I \in \mathfrak{C}$. If s is strongly ϕ-extensible in $J \cup \bar{s}$, then s is ϕ-extensible in every infinite subset of $J \cup \bar{s}$ which contains $\bar{s}$ and therefore *a fortiori* s is ϕ-extensible in every infinite subset of $I \cup \bar{s}$ which contains $\bar{s}$ and therefore s is strongly ϕ-extensible in $I \cup \bar{s}$ and therefore $I \in \mathfrak{C}$. This argument proves that every infinite subset of a member of $\mathfrak{C}$ is itself a member of $\mathfrak{C}$.

LEMMA 6. *Let M be an infinite subset of N and ϕ be a function from $\tilde{A}(M)$ into an ordinal set. Then there exists an infinite subset L of M such that every element of $\tilde{A}(L)$ is either ϕ-inextensible in L or strongly ϕ-extensible in L.*

Proof. Let $\mathfrak{F}$ be the function with domain $\tilde{A}(M)$ defined by the rule that, for every $s \in \tilde{A}(M)$, $\mathfrak{F}(s)$ is the collection of all infinite subsets K of M/s such that s is either ϕ-inextensible in $K \cup \bar{s}$ or strongly ϕ-extensible in $K \cup \bar{s}$. Then $\mathfrak{F}$ is an M-admissible function by Lemma 5. Therefore by Lemma 2 we can select an infinite subset L of M such that $L/s \in \mathfrak{C}(s)$ for every $s \in \tilde{A}(L)$. If $s \in \tilde{A}(L)$, then $L/s \in \mathfrak{F}(s)$ and therefore s is either ϕ-inextensible in $\bar{s} \cup (L/s)$ or strongly ϕ-extensible in $\bar{s} \cup (L/s)$ and therefore, by Lemma 4, s is either ϕ-inextensible in L or strongly ϕ-extensible in L.

LEMMA 7. *Let K, L, M be infinite subsets of N such that $K \subseteq L \subseteq M$ and let ϕ be a function from $\tilde{A}(M)$ into an ordinal set. If $s \in \tilde{A}(K)$ and s is ϕ-objectionable in K, then s is ϕ-objectionable in L.*

Proof. Since s is ϕ-objectionable in K, we can select a $u \in A_\omega(K/s)$ such that every strict left-segment of su is ϕ-extensible in K and therefore also, by Lemma 3, ϕ-extensible in L. Since $u \in A_\omega(K/s) \subseteq A_\omega(L/s)$ and every strict left-segment of su is ϕ-extensible in L, s is ϕ-objectionable in L.

LEMMA 8. *Let M be an infinite subset of N and L, L' be infinite subsets of M. Let $s \in \tilde{A}(L \cap L')$ and let $L/s = L'/s$. Let ϕ be a function from $\tilde{A}(M)$ into an ordinal set. Then*

(i) *if s is strongly ϕ-objectionable in L then s is strongly ϕ-objectionable in L';*
(ii) *if ϕ is dwindling and s is ϕ-objectionable in L then s is ϕ-objectionable in L'.*

Proof. The proof of Lemma 8(i) is similar to the proof of Lemma 4(ii), except that Lemma 7 is used in place of Lemma 3.

Now assume that ϕ is dwindling and that s is ϕ-objectionable in L. Then we can select a $u \in A_\omega(L/s) = A_\omega(L'/s)$ such that every strict left-segment of su is ϕ-extensible in L. Let $t \prec\prec su$. Select a $\hat{t}$ such that $s \prec \hat{t}$, $t \prec \hat{t}$ and $\hat{t} \prec\prec su$. Then $\hat{t}$ is ϕ-extensible in L and therefore there exists $w \in \tilde{A}(L)$ such that $\hat{t} \prec w$ and $\phi(\hat{t}) > \phi(w)$. Moreover, $\phi(t) \geqslant \phi(\hat{t})$ since ϕ is dwindling. Since $s \prec \hat{t} \prec w \in \tilde{A}(L)$, it follows that

$$\overline{w} \subseteq \bar{s} \cup (L/s) = \bar{s} \cup (L'/s) \subseteq L'$$

and therefore $w \in \tilde{A}(L')$. Since $w \in \tilde{A}(L')$ and $t \prec \hat{t} \prec w$ and $\phi(t) \geqslant \phi(\hat{t}) > \phi(w)$, we have proved that t is ϕ-extensible in L'. Since this argument is valid for every strict left-segment t of su, it follows that s is ϕ-objectionable in L'.

Lemma 9. *Let M be an infinite subset of N and ϕ be a function from $\tilde{A}(M)$ into an ordinal set. Let $s \in \tilde{A}(M)$ and $\mathfrak{C}$ be the collection of all infinite subsets K of M/s such that s is either ϕ-unobjectionable in $K \cup \bar{s}$ or strongly ϕ-objectionable in $K \cup \bar{s}$. Then $\mathfrak{C}$ is an (M/s)-admissible collection of sets.*

The proof is similar to that of Lemma 5, but uses Lemma 7 in place of Lemma 3.

Lemma 10. *Let M be an infinite subset of N and ϕ be a dwindling function from $\tilde{A}(M)$ into an ordinal set. Then there exists an infinite subset L of M such that every element of $\tilde{A}(L)$ is either ϕ-unobjectionable in L or strongly ϕ-objectionable in L.*

The proof is similar to that of Lemma 6, but uses Lemmas 8 and 9 in place of Lemmas 4 and 5 respectively.

Lemma 11. *Let L, M be infinite subsets of N such that $L \subseteq M$ and ϕ be a function from $\tilde{A}(M)$ into an ordinal set. Then there exists an element of $\tilde{A}(L)$ which is ϕ-unobjectionable in L.*

Proof. Let s be an element of $\tilde{A}(L)$ such that $\phi(s)$ is the smallest ordinal number in the set $\phi(\tilde{A}(L))$. Then there is no $s' \in \tilde{A}(L)$ such that $\phi(s) > \phi(s')$; therefore s is ϕ-inextensible in L. If $u \in A_\omega(L/s)$, the facts that $s \prec\prec su$ and s is ϕ-inextensible in L preclude the possibility of every strict left-segment of su being ϕ-extensible in L. Therefore s is ϕ-unobjectionable in L.

Lemma 12. *Let M be an infinite subset of N and ϕ be a dwindling function from $\tilde{A}(M)$ into an ordinal set. Then there exists an infinite subset K of M such that every element of $A_\omega(K)$ has a strict left-segment which is ϕ-inextensible in K.*

Proof. By Lemma 10, we can select an infinite subset L of M such that every element of $\tilde{A}(L)$ is either ϕ-unobjectionable in L or strongly ϕ-objectionable in L. If $s \in \tilde{A}(L)$, let $J(s)$ be the set of all elements x of L/s such that $s[x]$ is ϕ-objectionable in L. Let $\mathfrak{F}$ be the function with domain $\tilde{A}(L)$ defined by the rule that, for every $s \in \tilde{A}(L)$, $\mathfrak{F}(s)$ is the collection of all infinite subsets of L/s which are contained in either $J(s)$ or $(L/s) - J(s)$. Then $\mathfrak{F}$ is clearly an L-admissible function. Therefore by Lemma 2 we can select an infinite subset K of L such that $K/s \in \mathfrak{F}(s)$ for every $s \in \tilde{A}(K)$. To complete the proof of Lemma 12, we require the following subsidiary lemma concerning this set K.

Lemma 12A. *If $s \in A(K)$ and s_* is ϕ-objectionable in K, then s is ϕ-objectionable in K.*

Proof. Since s_* is by hypothesis ϕ-objectionable in K we can select a $u \in A_\omega(K/s_*)$ such that every strict left-segment of $(s_*)\, u$ is ϕ-extensible in K. Therefore, by Lemma 3, every strict left-segment of $(s_*)\, u$ is ϕ-extensible in L, i.e. every strict left-segment of $(s_*)\,[u(0)]\,({}_*u)$ is ϕ-extensible in L. Therefore $(s_*)\,[u(0)]$ is ϕ-objectionable in L, and hence $u(0) \in J(s_*)$. But since $s_* \in \tilde{A}(K)$, it follows from the definition of K that K/s_* belongs to $\mathfrak{F}(s_*)$ and so is contained in either $J(s_*)$ or $(L/s_*) - J(s_*)$. Since the element $u(0)$ of K/s_* belongs to $J(s_*)$, we conclude that $K/s_* \subseteq J(s_*)$. Therefore $\lambda(s) \in J(s_*)$, i.e. s is ϕ-objectionable in L. It follows by the definition of L that s is strongly ϕ-objectionable in L and therefore is ϕ-objectionable in K.

Lemma 12A implies that, if $s \in A(K)$ and s is ϕ-unobjectionable in K, then s_* is ϕ unobjectionable in K. If any $t \in \tilde{A}(K)$ is ϕ-unobjectionable in K, $l(t)$ applications of this consequence of Lemma 12A tell us that $\square$ is ϕ-unobjectionable in K. But by Lemma 11 some element of $\tilde{A}(K)$ is ϕ-unobjectionable in K. Therefore $\square$ is indeed ϕ-unobjectionable in K. By the definition of ϕ-objectionability in K, this implies that there is no $u \in A_\omega(K)$ such that every strict left-segment of u is ϕ-extensible in K; and Lemma 12 is proved.

Corollary 12A. *Let L, M be infinite subsets of N such that $L \subseteq M$ and ϕ be a dwindling function from $\tilde{A}(M)$ into an ordinal set. Then there exists an infinite subset K of L such that every element of $A_\omega(K)$ has a strict left-segment which is ϕ-inextensible in K.*

Proof. Let $\phi' = \phi|\tilde{A}(L)$. Then ϕ' is dwindling since ϕ is dwindling. Therefore, by Lemma 12, there exists an infinite subset K of L such that every element of $A_\omega(K)$ has a strict left-segment which is ϕ'-inextensible in K and therefore also ϕ-inextensible in K.

Lemma 13. *If B is a block, then there exist $s, t \in B$ such that $s \lhd t$.*

Proof. Select an element of minimum length in B and denote it by s. Since B is a block, it is an infinite subset of $A(N)$; therefore $\bar{B}$ is an infinite subset of N and so we can select a $u \in A_\omega(\bar{B})$ such that $s \prec\prec u$. Since B is a block and ${}_*u \in A_\omega(\bar{B})$, we can select a $t \in B$ such that $t \prec\prec {}_*u$. Since s is of minimum length in B, $l(s) \leqslant l(t)$ and therefore $s \lhd t$.

Definitions. A subset T of $A(N)$ is *thin* if there do not exist $s, t \in T$ such that $s \prec\prec t$. A *barrier* is a block B such that no strict subsequence of any element of B is itself an element of B. (Thus a barrier is certainly thin.) A Q-pattern f is *perfect* if $f(s) \leqslant f(t)$ for *every* pair s, t of elements of Df such that $s \lhd t$. A *Q-array* is a function from a barrier into Q: thus a Q-array is a special kind of Q-pattern.

We shall subsequently require the following lemmas, which are essentially Lemma 4 of (6) and Lemmas 14, 19, 20 and 21 of (7).

Lemma 14. *Let M be an infinite subset of N and T be a thin subset of $\tilde{A}(M)$ and T_1 be a subset of T. Then there exists an infinite subset L of M such that $T \cap \tilde{A}(L)$ is contained either in T_1 or in $T - T_1$.*

Lemma 15. *If B is a barrier and L is an infinite subset of $\bar{B}$, then $B \cap A(L)$ is a barrier.*

Lemma 16. *If f is a Q-array, Df contains a barrier B such that $f|B$ is either bad or perfect.*

Lemma 17. *Every block contains a barrier.*

Lemma 18. *Q is bqo if and only if every Q-array is good.*

Any subset S of a qo set Q will itself be regarded as a qo set by using the obvious convention that, for any two elements q_1, q_2 of S, $q_1 \leqslant q_2$ in the quasi-ordering on S if and only if $q_1 \leqslant q_2$ in the quasi-ordering on Q. This convention is used in our next lemma.

Lemma 19. *If $q \in Q$ and $Q-\{q\}$ is bqo, then Q is bqo.*

Proof. Let f be a Q-array and let $\mathbf{D}f = B$. Then, by Lemma 14, there exists an infinite subset L of $\bar{B}$ such that either $B \cap \tilde{A}(L) \subseteq f^{-1}(\{q\})$ or

$$B \cap \tilde{A}(L) \subseteq B - f^{-1}(\{q\}) = f^{-1}(Q-\{q\}).$$

By Lemma 15, $B \cap A(L)$ is a block and therefore we can by Lemma 13 select $s, t \in B \cap A(L)$ such that $s \lhd t$. [Since B is a barrier and therefore a block and therefore a subset of $A(N)$, there is in fact no difference between $B \cap A(L)$ and $B \cap \tilde{A}(L)$.] If $B \cap \tilde{A}(L) \subseteq f^{-1}(\{q\})$, then $f(s) \leqslant f(t)$ since $f(s) = q = f(t)$ and therefore f is good. If $B \cap \tilde{A}(L) \subseteq f^{-1}(Q-\{q\})$, then since $B \cap A(L)$ is a block we infer that $f|(B \cap A(L))$ is a $(Q-\{q\})$-pattern and so must be good since $Q-\{q\}$ is bqo; this implies that f is once again good. We have thus shown that every Q-array is good, whence Q is bqo by Lemma 18.

3. *Ascending sequences of ascending sequences.* A sequence σ on $A(N)$ is *ascending* if $\sigma(i) \lhd \sigma(i+1)$ whenever $i, i+1$ both belong to $\mathbf{D}\sigma$. In particular, all sequences on $A(N)$ of length 0 or 1 are ascending, since they vacuously satisfy the condition just stated. Let C be a subset of $A(N)$ and let $n \in P$. Then $\tilde{A}(C)$ will denote the set of all ascending finite sequences on C and $A(C)$ will denote the set of all ascending finite sequences on C of non-zero length and $A_n(C)$ will denote the set of all ascending sequences on C of length n. If $s_0, s_1, \ldots, s_{n-1} \in A(N)$, then $s_0 \otimes s_1 \otimes \ldots \otimes s_{n-1}$ will denote the unique element s of $A(N)$ such that $\bar{s} = \bar{s}_0 \cup \bar{s}_1 \cup \ldots \cup \bar{s}_{n-1}$. For instance,

$$[1,5] \otimes [7,8,9] \otimes [3,5,8] = [1,3,5,7,8,9].$$

If $\sigma = [s_0, s_1, \ldots, s_{n-1}] \in A(A(N))$, we define $a(\sigma)$ to be $s_0 \otimes s_1 \otimes \ldots \otimes s_{n-1}$. By defining $a(\square)$ to be $\square$, we render $a(\sigma)$ defined for every $\sigma \in \tilde{A}(A(N))$. The ith term of $a(\sigma)$ will be denoted by $a(\sigma, i)$. If B is a barrier, B^{Δ} will denote $\{a(\sigma): \sigma \in A(B)\}$ and $\tilde{B}^{\Delta}$ will denote $\{a(\sigma): \sigma \in \tilde{A}(B)\}$.

Lemma 20. *Let B be a barrier and let $\sigma = [s_0, s_1, \ldots, s_{n-1}] \in A(B)$. Then s_i is a segment of $a(\sigma)$ and $s_i(0) = a(\sigma, i)$ for $i = 0, 1, \ldots, n-1$ and*

$$a(\sigma) = [s_0(0), s_1(0), \ldots, s_{n-2}(0)]\, s_{n-1},$$

where $[s_0(0), s_1(0), \ldots, s_{n-2}(0)]$ is interpreted as $\square$ if $n = 1$.

Proof. We note first that, since B is a block, $B \subseteq A(N)$ and therefore none of $s_0, s_1, \ldots, s_{n-1}$ is $\square$. Let $s_0 = [p_0, p_1, \ldots, p_{\phi(0)}]$. Then $p_0 < p_1 < \ldots < p_{\phi(0)}$ since s_0 is ascending. Moreover, if $n > 1$, then $s_0 \lhd s_1$ and so there exists a $u_0 \in A(N)$ such that $s_0 \prec\prec u_0$ and $s_1 = {}_*u_0$. Since $s_0 \prec\prec u_0 \in A(N)$, we can write $u_0 = [p_0, p_1, \ldots, p_{\phi(1)}]$, where $p_0 < p_1 < \ldots < p_{\phi(1)}$ and $\phi(0) < \phi(1)$. Since $s_1 = {}_*u_0$, it follows that

$$s_1 = [p_1, p_2, \ldots, p_{\phi(1)}].$$

Now, if $n > 2$, then $s_1 \lhd s_2$ and so a similar argument shows that we can write

$$s_2 = [p_2, p_3, \ldots, p_{\phi(2)}],$$

where $\phi(1) < \phi(2)$ and $p_1 < p_2 < \ldots < p_{\phi(2)}$. Similarly, if $n > 3$, we can write

$$s_3 = [p_3, p_4, \ldots, p_{\phi(3)}], \quad \text{where} \quad \phi(2) < \phi(3), \quad p_2 < p_3 < \ldots < p_{\phi(3)};$$

and so on. Thus we can find non-negative integers

$$\phi(0), \phi(1), \ldots, \phi(n-1), \quad p_0, p_1, \ldots, p_{\phi(n-1)}$$

such that

$$\phi(0) < \phi(1) < \ldots < \phi(n-1), \tag{1}$$

$$p_0 < p_1 < \ldots < p_{\phi(n-1)}, \tag{2}$$

$$s_i = [p_i, p_{i+1}, \ldots, p_{\phi(i)}] \quad (i = 0, 1, \ldots, n-1). \tag{3}$$

From (1), (2) and (3) it follows that

$$s_0 \otimes s_1 \otimes \ldots \otimes s_{n-1} = [p_0, p_1, \ldots, p_{\phi(n-1)}],$$

i.e.

$$a(\sigma) = [p_0, p_1, \ldots, p_{\phi(n-1)}]. \tag{4}$$

It follows from (1), (3) and (4) that s_i is a segment of $a(\sigma)$ and $s_i(0) = a(\sigma, i)$ for $i = 0, 1, \ldots, n-1$ and that $a(\sigma) = [s_0(0), s_1(0), \ldots, s_{n-2}(0)]\, s_{n-1}$.

LEMMA 21. *Let B be a barrier and let $\sigma, \tau \in \tilde{A}(B)$. Then $a(\sigma) \prec a(\tau)$ if and only if $\sigma \prec \tau$.*

Proof. Since $\square \notin B$ (by the definition of a block), $a(\sigma) = \square$ if and only if $\sigma = \square$, and $a(\tau) = \square$ if and only if $\tau = \square$. Hence (i) if $\sigma = \square$, then $\sigma \prec \tau$ and $a(\sigma) \prec a(\tau)$, and (ii) if $\tau = \square \neq \sigma$, then $\sigma \not\prec \tau$ and $a(\sigma) \not\prec a(\tau)$. We may therefore henceforward assume that neither of σ, τ is $\square$.

Let $\sigma = [s_0, s_1, \ldots, s_{m-1}]$, $\tau = [t_0, t_1, \ldots, t_{n-1}]$. Assume, first, that $a(\sigma) \prec a(\tau)$. By Lemma 20, s_i is a segment of $a(\sigma)$ such that $s_i(0) = a(\sigma, i)$ and t_j is a segment of $a(\tau)$ such that $t_j(0) = a(\tau, j)$, Since $a(\sigma) \prec a(\tau)$, it follows that, for $i < m$ and $j < n$, s_i and t_j are segments of $a(\tau)$ such that $s_i(0) = a(\tau, i)$ and $t_j(0) = a(\tau, j)$. Moreover, by Lemma 20, t_{n-1} is the right-segment of $a(\tau)$ with 0th term $a(\tau, n-1)$, so that, if m were greater than n, then s_{m-1}, being a segment of $a(\tau)$ with 0th term $a(\tau, m-1)$, would be a strict subsequence of t_{n-1}, which is impossible since $s_{m-1}, t_{n-1} \in B$. Therefore $m \leqslant n$. Moreover, for each $i < m$, s_i and t_i are segments of $a(\tau)$ with 0th term $a(\tau, i)$, and therefore either $s_i \prec t_i$ or $t_i \prec s_i$, but neither of s_i, t_i can be a strict subsequence of the other because they both belong to B. It follows that $s_i = t_i$ for $i = 0, 1, \ldots, m-1$. Therefore $\sigma \prec \tau$.

We must now prove that if $\sigma \prec \tau$ then $a(\sigma) \prec a(\tau)$. Assume that $\sigma \prec \tau$. Then $m \leqslant n$ and $s_i = t_i$ for $i = 0, \ldots, m-1$. Therefore, by Lemma 20,

$$a(\sigma) = [t_0(0), t_1(0), \ldots, t_{m-2}(0)]\, t_{m-1},$$

$$a(\tau) = [t_0(0), t_1(0), \ldots, t_{n-2}(0)]\, t_{n-1}$$

and t_{m-1} is a segment of $a(\tau)$ with 0th term $t_{m-1}(0)$. From these facts it is clear that $a(\sigma) \prec a(\tau)$.

COROLLARY 21A. *If B is a barrier and $\sigma, \tau \in \tilde{A}(B)$ and $a(\sigma) = a(\tau)$, then $\sigma = \tau$.*

Proof. Since $a(\sigma) \prec a(\tau)$ and $a(\tau) \prec a(\sigma)$, it follows by Lemma 21 that $\sigma \prec \tau$ and $\tau \prec \sigma$.

COROLLARY 21B. *If B is a barrier and $\sigma, \tau \in \tilde{A}(B)$, then $a(\sigma) \prec\prec a(\tau)$ if and only if $\sigma \prec\prec \tau$.*

Proof. By Lemma 21, ($a(\sigma) \prec a(\tau)$ and $a(\tau) \not\prec a(\sigma)$) if and only if ($\sigma \prec \tau$ and $\tau \not\prec \sigma$).

Lemma 22. *If B is a barrier and $\sigma \in A(B)$ and $l(\sigma) \geqslant 2$, then $a({}_*\sigma) = {}_*a(\sigma)$.*

Proof. Let $\sigma = [s_0, s_1, \ldots, s_{n-1}]$. Then $n = l(\sigma) \geqslant 2$ and by Lemma 20

$$a(\sigma) = [s_0(0), s_1(0), \ldots, s_{n-2}(0)]\, s_{n-1},$$

$$a({}_*\sigma) = [s_1(0), s_2(0), \ldots, s_{n-2}(0)]\, s_{n-1},$$

from which we see that $a({}_*\sigma) = {}_*a(\sigma)$.

Lemma 23. *Let B be a barrier and ρ, τ be elements of $A(B)$ such that $l(\rho) \geqslant 2$ and ${}_*\rho \prec\prec \tau$. Then $a(\rho) \lhd a(\tau)$.*

Proof. The hypotheses of Lemma 23 imply that there exists a $\sigma \in A(B)$ such that $\rho \prec\prec \sigma$ and $\tau = {}_*\sigma$ and $l(\sigma) \geqslant 3$. Since $a(\rho) \prec\prec a(\sigma)$ by Corollary 21B and

$$a(\tau) = {}_*a(\sigma)$$

by Lemma 22 and $a(\sigma) \in A(N)$, it follows that $a(\rho) \lhd a(\tau)$.

Definition. If u is an ω-sequence and $n \in N$, ${}_{[n]}u$ will denote the ω-sequence

$$[u(n), u(n+1), u(n+2), u(n+3), \ldots].$$

Lemma 24. *If B is a barrier and $u \in A_\omega(\bar{B})$, then infinitely many left-segments of u belong to B^Δ.*

Proof. For every non-negative integer i, ${}_{[i]}u \in A_\omega(\bar{B})$ and therefore (since B is a block) we can select an $s_i \in B$ such that $s_i \prec\prec {}_{[i]}u$. Since $s_i \prec\prec {}_{[i]}u$ and $s_{i+1} \prec\prec {}_{[i+1]}u$, it follows that either s_{i+1} is a strict subsequence of s_i or $s_i \lhd s_{i+1}$. However, s_{i+1} cannot be a strict subsequence of s_i since $s_i, s_{i+1} \in B$. Therefore $s_i \lhd s_{i+1}$ for every $i \in N$. Therefore, for every $n \in P$, $[s_0, s_1, \ldots, s_{n-1}] \in A(B)$ and hence $s_0 \otimes s_1 \otimes \ldots \otimes s_{n-1} \in B^\Delta$. Moreover, if $n \in P$, then by Lemma 20,

$$\begin{aligned} s_0 \otimes s_1 \otimes \ldots \otimes s_{n-1} &= [s_0(0), s_1(0), \ldots, s_{n-2}(0)]\, s_{n-1} \\ &= [u(0), u(1), \ldots, u(n-2)]\, s_{n-1}, \end{aligned}$$

which is a strict left-segment of u since $s_{n-1} \prec\prec {}_{[n-1]}u$. The length of this strict left-segment of u is at least n. Thus we have proved that, for every positive integer n, u has a strict left-segment of length not less than n which belongs to B^Δ; and Lemma 24 is proved.

Definitions. Let B be a barrier, ψ be a function from $A(B)$ into an ordinal set and L be an infinite subset of $\bar{B}$. We shall call ψ *dwindling* if $\psi(\sigma) \geqslant \psi(\tau)$ whenever σ, $\tau \in A(B)$ and $\sigma \prec \tau$. An element σ of $A(B \cap A(L))$ is *ψ-extensible in L* if there exists a $\tau \in A(B \cap A(L))$ such that $\sigma \prec \tau$ and $\psi(\sigma) > \psi(\tau)$, and σ is *ψ-inextensible in L* if it is not ψ-extensible in L. An element σ of $A(B \cap A(L))$ is *strongly ψ-extensible in L* if it is ψ-extensible in every infinite subset K of L such that $\sigma \in A(A(K))$. We shall let $\tilde{\psi}$ denote the function with domain $\tilde{A}(B)$ defined by the rule that $\tilde{\psi}(\sigma) = \psi(\sigma)$ for every $\sigma \in A(B)$ and $\tilde{\psi}(\square)$ is the least ordinal number greater than all elements of $\operatorname{Im}\psi$. If $s \in \tilde{A}(\bar{B})$, s^B will denote the longest left-segment of s which belongs to $\bar{B}^\Delta$: note that at least one left-segment of s, namely $\square$, belongs to $\tilde{B}^\Delta$. If $s \in \tilde{B}^\Delta$, there is by

Corollary 21 A a unique $\sigma \in \tilde{A}(B)$ such that $a(\sigma) = s$: we shall write $\sigma = \alpha_B(s)$. We shall let ψ_B denote the function with domain $\tilde{A}(\bar{B})$ defined by writing $\psi_B(s) = \tilde{\psi}(\alpha_B(s^B))$ for every $s \in \tilde{A}(\bar{B})$.

LEMMA 25. *If B is a barrier and ψ is a dwindling function from $A(B)$ into an ordinal set, then ψ_B is dwindling.*

Proof. Let s, t be elements of $\tilde{A}(\bar{B})$ such that $s \prec t$. Since $s^B \prec s \prec t$ and $s^B \in \tilde{B}^\Delta$, it follows from the definition of t^B that $s^B \prec t^B$. Therefore by Lemma 21

$$\alpha_B(s^B) \prec \alpha_B(t^B).$$

Therefore, since ψ is dwindling, $\psi(\alpha_B(s^B)) \geqslant \psi(\alpha_B(t^B))$ unless $\alpha_B(s^B) = \square$. From this fact and the definition of $\tilde{\psi}$, it is obvious that $\tilde{\psi}(\alpha_B(s^B)) \geqslant \tilde{\psi}(\alpha_B(t^B))$ whether or not $\alpha_B(s^B) = \square$; therefore $\psi_B(s) \geqslant \psi_B(t)$.

LEMMA 26. *Let B be a barrier, L be an infinite subset of $\bar{B}$ and ψ be a function from $A(B)$ into an ordinal set. Let $\sigma \in A(B \cap A(L))$. Then σ is ψ-extensible in L if and only if $a(\sigma)$ is ψ_B-extensible in L.*

Proof. Write $a(\sigma) = s$. First, assume that σ is ψ-extensible in L. Then there is a $\tau \in A(B \cap A(L))$ such that $\sigma \prec \tau$ and $\psi(\sigma) > \psi(\tau)$. Let $a(\tau) = t$. Since $\sigma \in \tilde{A}(B)$, it follows that $s \in \tilde{B}^\Delta$ and therefore $s^B = s$. Hence

$$\psi_B(s) = \tilde{\psi}(\alpha_B(s^B)) = \tilde{\psi}(\alpha_B(s)) = \tilde{\psi}(\sigma) = \psi(\sigma),$$

and similarly $\psi_B(t) = \psi(\tau)$. Since $\psi(\sigma) > \psi(\tau)$, we infer that $\psi_B(s) > \psi_B(t)$. Furthermore, $s \prec t$ by Lemma 21, and $t \in A(L)$ since $\tau \in A(B \cap A(L))$. Hence s is ψ_B-extensible in L.

Conversely, let us assume that s is ψ_B-extensible in L. Then there exists an $x \in \tilde{A}(L)$ such that $s \prec x$ and $\psi_B(s) > \psi_B(x)$. Let $\xi = \alpha_B(x^B)$. Since $s \prec x$ and $s = a(\sigma) \in \tilde{B}^\Delta$, it follows that $s \prec x^B$ and therefore by Lemma 21 $\sigma \prec \xi$. Since $\sigma \in A(B)$ and $\sigma \prec \xi$, ξ cannot be $\square$ and therefore belongs to $A(B)$, and therefore since $a(\xi) = x^B \prec x \in \tilde{A}(L)$, ξ must belong to $A(B \cap A(L))$. Moreover $s^B = s$ since $s \in \tilde{B}^\Delta$, and therefore

$$\tilde{\psi}(\sigma) = \tilde{\psi}(\alpha_B(s)) = \tilde{\psi}(\alpha_B(s^B)) = \psi_B(s) > \psi_B(x) = \tilde{\psi}(\alpha_B(x^B)) = \tilde{\psi}(\xi).$$

Since $\sigma, \xi \in A(B)$, this gives $\psi(\sigma) = \tilde{\psi}(\sigma) > \tilde{\psi}(\xi) = \psi(\xi)$, which, since

$$\sigma \prec \xi \in A(B \cap A(L)),$$

implies that σ is ψ-extensible in L.

LEMMA 27. *Let B be a barrier and ψ be a function from $A(B)$ into an ordinal set. Then there exists an infinite subset L of $\bar{B}$ such that every element of $A(B \cap A(L))$ is either ψ-inextensible in L or strongly ψ-extensible in L.*

Proof. By Lemma 6 we can select an infinite subset L of $\bar{B}$ such that every element of $\tilde{A}(L)$ is either ψ_B-inextensible in L or strongly ψ_B-extensible in L. Let

$$\sigma \in A(B \cap A(L)).$$

Then $a(\sigma)$ is either ψ_B-inextensible in L or strongly ψ_B-extensible in L. If $a(\sigma)$ is ψ_B-inextensible in L then by Lemma 26 σ is ψ-inextensible in L. Now suppose that

$a(\sigma)$ is strongly ψ_B-extensible in L. Then, for every infinite subset K of L such that $\sigma \in A(A(K))$, we have $\overline{a(\sigma)} \subseteq K$ and therefore $a(\sigma)$, being strongly ψ_B-extensible in L, must be ψ_B-extensible in K so that by Lemma 26 σ is ψ-extensible in K. Hence σ is strongly ψ-extensible in L.

Lemma 28. *Let B be a barrier and L be an infinite subset of $\bar{B}$ and ψ be a dwindling function from $A(B)$ into an ordinal set. Then there exists an infinite subset K of L such that, for every $u \in A_\omega(K)$, there exists a $\sigma \in A(B \cap A(K))$ such that $a(\sigma) \prec\prec u$ and σ is ψ-inextensible in K.*

Proof. By Lemma 25 ψ_B is dwindling. Therefore, by Corollary 12A, we can select an infinite subset K of L such that every element of $A_\omega(K)$ has a strict left-segment which is ψ_B-inextensible in K.

Let $u \in A_\omega(K)$. Then we can select a strict left-segment x of u which is ψ_B-inextensible in K. By Lemma 24 we can select an $s \in B^\Delta$ such that $x \prec s \prec\prec u$. Since $s \in B^\Delta$, we can write $s = a(\sigma)$ where $\sigma \in A(B)$. Then $a(\sigma) \prec\prec u$. Since

$$\sigma \in A(B) \quad \text{and} \quad a(\sigma) \prec\prec u \in A_\omega(K),$$

clearly $\sigma \in A(B \cap A(K))$. Moreover, if $t \in A(K)$ and $s \prec t$, then $x \prec s \prec t$ and therefore the facts that ψ_B is dwindling and that x is ψ_B-inextensible in K imply respectively that $\psi_B(x) \geqslant \psi_B(s) \geqslant \psi_B(t)$ and that $\psi_B(x) \ngtr \psi_B(t)$, whence it follows that

$$\psi_B(s) = \psi_B(t).$$

Hence s is ψ_B-inextensible in K and consequently, by Lemma 26, σ is ψ-inextensible in K.

4. *Proof of the Theorem.* Throughout section 4, we will suppose that Q is a bqo set and Λ is a set of sequences on Q which is not bqo. We will ultimately show that this leads to a contradiction, thereby proving our Theorem. We begin by observing that, since Λ is assumed not to be bqo, we can by Lemma 18 select a bad Λ-array f. Let $\mathrm{D}f = B$. For any $s \in B$, let f_s denote $f(s)$. Let α be an ordinal number such that $\omega_\alpha > l(f_s)$ for every $s \in B$. Let $Q^* = Q \cup \{\infty\}$, where ∞ denotes some element which does not belong to Q, and let Q^* be quasi-ordered by the rule that, for all elements q_1^*, q_2^* of Q^*, q_1^* anticipates q_2^* in the quasi-ordering of Q^* if and only if either (i) $q_1^*, q_2^* \in Q$ and q_1^* anticipates q_2^* in the quasi-ordering of Q or (ii) $q_2^* = \infty$. For every $s \in B$, let g_s be the extension of f_s such that $l(g_s) = \omega_\alpha$ and $g_s(\theta) = \infty$ whenever $l(f_s) \leqslant \theta < \omega_\alpha$. Then the g_s are sequences on Q^*. Let O denote O_{ω_α}, which is the domain of every g_s.

Lemma 29. *Let $s, t \in B$. Then there is a unique function $\mu: O \to O$ such that, for every $\theta \in O$, $\mu(\theta)$ is the smallest element ϕ of O which satisfies the conditions*

$$g_s(\theta) \leqslant g_t(\phi), \quad \mu(O_\theta) \subseteq O_\phi.$$

Proof. We first establish the existence of at least one function μ satisfying the conditions of the lemma. This may be done by defining $\mu(\theta)$ by transfinite induction. Suppose, therefore, that $\theta \in O$ and that we have defined $\mu(\xi)$ for every $\xi < \theta$ in such a way that, for every $\xi < \theta$, $\mu(\xi)$ is the least element χ of O such that $g_s(\xi) \leqslant g_t(\chi)$ and $\mu(O_\xi) \subseteq O_\chi$. Then we must prove that we can define $\mu(\theta)$ to be the least $\phi \in O$ such that

$g_s(\theta) \leqslant g_t(\phi)$ and $\mu(O_\theta) \subseteq O_\phi$. To prove this, it is only necessary to show that there exists a $\phi \in O$ which satisfies these two conditions.

Since $l(f_t) < \omega_\alpha$, it follows that $|\mathbf{D}f_t| < \aleph_\alpha$. Since $\theta < \omega_\alpha$, it follows that $|O_\theta| < \aleph_\alpha$ and therefore $|\mu(O_\theta)| < \aleph_\alpha$. But $|O| = \aleph_\alpha$. Therefore the set $O - ((\mathbf{D}f_t) \cup (\mu(O_\theta)))$ is non-empty: let ϕ be an element of this set. Since $\phi \in O - (\mathbf{D}f_t)$, it follows that $g_t(\phi) = \infty$ and therefore $g_s(\theta) \leqslant g_t(\phi)$. Suppose that $\xi \in O_\theta$ and $\mu(\eta) \in O_\phi$ for every $\eta < \xi$. Then $\mu(O_\xi) \subseteq O_\phi$. Since $\mu(O_\xi) \subseteq O_\phi$ and $g_s(\xi) \leqslant \infty = g_t(\phi)$, and since $\mu(\xi)$ is the least $\chi \in O$ such that $g_s(\xi) \leqslant g_t(\chi)$ and $\mu(O_\xi) \subseteq O_\chi$, we may infer that $\mu(\xi) \leqslant \phi$. But $\mu(\xi) \neq \phi$ since $\xi \in O_\theta$ and $\phi \notin \mu(O_\theta)$. Therefore $\mu(\xi) \in O_\phi$. This discussion establishes by transfinite induction on ξ that $\mu(\xi) \in O_\phi$ for every $\xi \in O_\theta$. Therefore $\mu(O_\theta) \subseteq O_\phi$.

We have thus proved that there exists a $\phi \in O$ satisfying the two required conditions, thereby vindicating our definition by transfinite induction of $\mu(\theta)$.

Finally, suppose that μ_1 and μ_2 are functions from O into O such that, for every $\theta \in O$, $\mu_1(\theta)$ is the smallest $\phi \in O$ such that $g_s(\theta) \leqslant g_t(\phi)$ and $\mu_1(O_\theta) \subseteq O_\phi$ and $\mu_2(\theta)$ is the smallest $\phi \in O$ such that $g_s(\theta) \leqslant g_t(\phi)$ and $\mu_2(O_\theta) \subseteq O_\phi$. Then, if $\theta \in O$ and

$$\mu_1(\theta') = \mu_2(\theta')$$

for every $\theta' < \theta$, we have $\mu_1(O_\theta) = \mu_2(O_\theta)$ and therefore $\mu_1(\theta) = \mu_2(\theta)$ by the hypotheses just made about μ_1 and μ_2. This establishes by transfinite induction on θ that

$$\mu_1(\theta) = \mu_2(\theta)$$

for every $\theta \in O$, which shows that there is only one function μ satisfying the requirements of the lemma.

Definition. Let $s, t \in B$. Then $\mu_{s,t}$ will denote the unique function μ from O into O such that, for every $\theta \in O$, $\mu(\theta)$ is the smallest element ϕ of O which satisfies the conditions $g_s(\theta) \leqslant g_t(\phi)$, $\mu(\theta_\theta) \subseteq O_\phi$. We note that $\mu_{s,t}$, being a function from O into O, is an ordinal sequence.

LEMMA 30. *For all $s, t \in B$, the ordinal sequence $\mu_{s,t}$ is ascending.*

Proof. For every $\theta \in O$, $\mu_{s,t}(\theta)$ is by the preceding definition an ordinal ϕ such that $\mu_{s,t}(O_\theta) \subseteq O_\phi$, which implies that $\mu_{s,t}(\theta') < \phi = \mu_{s,t}(\theta)$ for every $\theta' < \theta$.

COROLLARY 30A. $|\mu_{s,t}(O)| = \aleph_\alpha$ *for all $s, t \in B$.*

Proof. This is so since, by Lemma 30, $\mu_{s,t}$ is one-to-one.

LEMMA 31. *If $s, t \in B$ and $s \lhd t$, then $\mu_{s,t}(\theta) \geqslant l(f_t)$ for some $\theta \in \mathbf{D}f_s$.*

Proof. Let $\mu_{s,t}|\mathbf{D}f_s = \nu$. Suppose that $\mu_{s,t}(\theta) < l(f_t)$ for every $\theta \in \mathbf{D}f_s$. Then

$$\nu(\mathbf{D}f_s) \subseteq \mathbf{D}f_t$$

and therefore, in view of Lemma 30, ν is an ascending sequence on $\mathbf{D}f_t$ and therefore $f_t \circ \nu$ is a subsequence of f_t. Moreover, if $\theta \in \mathbf{D}f_s$, then $\nu(\theta) = \mu_{s,t}(\theta)$, which is the smallest $\phi \in O$ such that $g_s(\theta) \leqslant g_t(\phi)$ and $\mu_{s,t}(O_\theta) \subseteq O_\phi$. Therefore $g_s(\theta) \leqslant g_t(\nu(\theta))$ for every $\theta \in \mathbf{D}f_s$, i.e. by the definitions of g_s, g_t and the fact that $\nu(\mathbf{D}f_s) \subseteq \mathbf{D}f_t$,

$$f_s(\theta) \leqslant f_t(\nu(\theta)) = (f_t \circ \nu)(\theta) \quad (\theta \in \mathbf{D}f_s).$$

Since $f_t \circ \nu$ is a subsequence of f_t, this implies that $f_s \leqslant f_t$, which contradicts the hypothesis that f is bad. Because this contradiction is obtained by supposing that

$$\mu_{s,t}(\theta) < l(f_t)$$

for every $\theta \in \mathrm{D}f_s$, Lemma 31 is proved.

We shall now define a function $\psi: A(B) \to O$. First, for every $s \in B$, we define $\psi([s])$ to be $l(f_s)$ (which is less than ω_α and so belongs to O). Now suppose that $n \geqslant 2$ and that $\psi(\sigma)$ has been defined and belongs to O for every $\sigma \in A_{n-1}(B)$. If $\tau \in A_n(B)$, then ${}_*\tau \in A_{n-1}(B)$ and therefore $\psi({}_*\tau)$ has been defined and belongs to O. Since $\psi({}_*\tau) \in O$, the set of ordinals less than $\psi({}_*\tau)$ has cardinal number less than $\aleph_\alpha$ and therefore, by Corollary 30A, cannot contain $\mu_{\tau(0),\tau(1)}(O)$. We may therefore define $\psi(\tau)$ to be the smallest $\theta \in O$ such that $\mu_{\tau(0),\tau(1)}(\theta) \geqslant \psi({}_*\tau)$. In this way, an element $\psi(\tau)$ of O is defined for every $\tau \in A_n(B)$. Thus, using induction on n, we define $\psi(\tau)$ for every $\tau \in A(B)$.

Lemma 32. *If $\sigma, \tau \in A(B)$ and $\sigma \prec \tau$ and $l(\sigma) \geqslant 2$ and $\psi({}_*\sigma) \geqslant \psi({}_*\tau)$, then $\psi(\sigma) \geqslant \psi(\tau)$.*

Proof. By definition, $\psi(\sigma)$ is the smallest $\theta \in O$ such that $\mu_{\sigma(0),\sigma(1)}(\theta) \geqslant \psi({}_*\sigma)$. Therefore $\mu_{\sigma(0),\sigma(1)}(\psi(\sigma)) \geqslant \psi({}_*\sigma)$. But $\sigma(0) = \tau(0)$ and $\sigma(1) = \tau(1)$ since $\sigma \prec \tau$. Hence

$$\mu_{\tau(0),\tau(1)}(\psi(\sigma)) = \mu_{\sigma(0),\sigma(1)}(\psi(\sigma)) \geqslant \psi({}_*\sigma) \geqslant \psi({}_*\tau).$$

Therefore $\psi(\sigma) \geqslant \psi(\tau)$ by the definition of $\psi(\tau)$.

Lemma 33. *If $s, t \in B$ and $s \lhd t$, then $\psi([s]) > \psi([s,t])$.*

Proof. By Lemma 31, $\mu_{s,t}(\theta) \geqslant l(f_t) = \psi([t])$ for some $\theta < l(f_s) = \psi([s])$. Therefore the smallest θ such that $\mu_{s,t}(\theta) \geqslant \psi([t])$ is less than $\psi([s])$, i.e. $\psi([s,t]) < \psi([s])$.

Lemma 34. *ψ is dwindling.*

Proof. Let $\mathscr{P}_n$ denote the proposition that $\psi(\sigma) \geqslant \psi(\tau)$ whenever $\sigma \in A(B)$, $\tau \in A_n(B)$ and $\sigma \prec \tau$. Then $\mathscr{P}_1$ is true since the hypotheses $\sigma \in A(B)$, $\tau \in A_1(B)$, $\sigma \prec \tau$ imply that $\sigma = \tau$. Now suppose that $n \geqslant 2$ and assume that $\mathscr{P}_{n-1}$ is true. Let σ, τ be such that $\sigma \in A(B)$, $\tau \in A_n(B)$ and $\sigma \prec \tau$. If $l(\sigma) \geqslant 2$, then $\psi({}_*\sigma) \geqslant \psi({}_*\tau)$ by our inductive hypothesis $\mathscr{P}_{n-1}$ and therefore $\psi(\sigma) \geqslant \psi(\tau)$ by Lemma 32. If $l(\sigma) = 1$, then $\sigma = [\sigma(0)]$ and $\sigma(0) = \tau(0) \lhd \tau(1)$ since $\sigma \prec \tau \in A_n(B)$ and therefore

$$\psi(\sigma) = \psi([\tau(0)]) > \psi([\tau(0), \tau(1)])$$

by Lemma 33. But $\psi([\tau(1)]) \geqslant \psi({}_*\tau)$ by our inductive hypothesis $\mathscr{P}_{n-1}$ and therefore $\psi([\tau(0), \tau(1)]) \geqslant \psi(\tau)$ by Lemma 32. Hence $\psi(\sigma) > \psi(\tau)$ when $l(\sigma) = 1$. This proves that $\mathscr{P}_{n-1}$ implies $\mathscr{P}_n$. Hence, by induction, $\mathscr{P}_n$ is true for all $n \in P$ and Lemma 34 is proved.

We can now, by Lemma 27, select an infinite subset L of $\bar{B}$ such that every element of $A(B \cap A(L))$ is either ψ-inextensible in L or strongly ψ-extensible in L. We can further, by Lemmas 28 and 34, select an infinite subset K of L such that, for every $u \in A_\omega(K)$, there exists a $\sigma \in A(B \cap A(K))$ such that $a(\sigma) \prec\prec u$ and σ is ψ-inextensible in K. Let Θ be the set of all elements of $A(B \cap A(K))$ which are ψ-inextensible in K. Let Γ be the set of those elements of Θ which have no strict left-segment belonging to Θ and let $C = \{a(\sigma): \sigma \in \Gamma\}$.

LEMMA 35. *$l(\sigma) \geqslant 2$ for every $\sigma \in \Theta$.*

Proof. Let $\tau \in A_1(B \cap A(K))$. Then $\tau = [t]$ for some $t \in B \cap A(K)$. Choose a $u \in A_\omega(K)$ such that $t \prec\prec u$. Since B is a block and ${}_*u \in A_\omega(K) \subseteq A_\omega(\bar{B})$, there exists a $w \in B$ such that $w \prec\prec {}_*u$. Moreover, w is not a strict subsequence of t since $t, w \in B$ and B is a barrier. Since $t \prec\prec u$ and $w \prec\prec {}_*u$ and w is not a strict subsequence of t and u is ascending, it follows that $t \lhd w$. Furthermore, t, w belong to $B \cap A(K)$. Therefore $[t, w] \in A(B \cap A(K))$. Since $\psi([t]) > \psi([t, w])$ by Lemma 33, it follows that $[t] = \tau$ is ψ-extensible in K. This argument establishes that any element of $A_1(B \cap A(K))$ is ψ-extensible in K, thereby proving Lemma 35.

LEMMA 36. *C is a block.*

Proof. We note first that, since $\Gamma \subseteq A(B \cap A(K)) \subseteq A(A(K))$, the sequence $\square$ cannot belong to C and so $C \subseteq A(N)$.

Let $u \in A_\omega(K)$. Then by the definitions of K and Θ we can select a $\sigma \in \Theta$ such that $a(\sigma) \prec\prec u$. If ρ is the shortest left-segment of σ which belongs to Θ, then $\rho \in \Gamma$ and therefore $a(\rho) \in C$, and furthermore by Lemma 21 $a(\rho) \prec a(\sigma) \prec\prec u$. Hence every element of $A_\omega(K)$ has a strict left-segment which belongs to C. From this fact and the facts that K is an infinite set and that $\square \notin C$, it is easily seen that C is an infinite set and that $K \subseteq \bar{C}$. But $\bar{C} \subseteq K$ since $\Gamma \subseteq A(B \cap A(K))$, and therefore $\bar{C} = K$. Hence $A_\omega(\bar{C}) = A_\omega(K)$, each of whose elements has a strict left-segment belonging to C; and we have completed the proof that C is a block.

By Lemmas 17 and 36, we can select a barrier D which is contained in C. Since $D \subseteq C$, every element s of D is equal to $a(\sigma)$ for some $\sigma \in A(B \cap A(K))$, and by Corollary 21A σ is uniquely determined by s. We may therefore define a function $h: D \to Q^*$ by the rule that $h(a(\sigma)) = g_{\sigma(0)}(\psi(\sigma))$ for every $\sigma \in A(B \cap A(K))$ such that $a(\sigma) \in D$. Since h is a Q^*-array, we can by Lemma 16 select a barrier $E \subseteq D$ such that $h|E$ is either bad or perfect. However, since $h|E$ is a Q^*-pattern and Q^* is by Lemma 19 bqo, $h|E$ cannot be bad and must therefore be perfect. Let Γ_E denote $\Gamma \cap A(A(\bar{E}))$.

LEMMA 37. *$C \cap A(\bar{E}) = E$.*

Proof. Suppose that $s_1, s_2 \in C$ and $s_1 \prec s_2$. Then we can select $\sigma_1, \sigma_2 \in \Gamma$ such that $a(\sigma_1) = s_1$, $a(\sigma_2) = s_2$. Since $\Gamma \subseteq A(B)$, it follows from Lemma 21 that $\sigma_1 \prec \sigma_2$. But, since $\sigma_1 \in \Gamma \subseteq \Theta$ and $\sigma_2 \in \Gamma$, it follows from the definition of Γ that σ_1 cannot be a strict left-segment of σ_2. Therefore $\sigma_1 = \sigma_2$ and therefore $s_1 = s_2$. We have thus shown that, if $s_1, s_2 \in C$ and $s_1 \prec s_2$, then $s_1 = s_2$.

Let $s \in C \cap A(\bar{E})$. Since E is an infinite set, $\bar{E}$ is also infinite and thus we can select a $u \in A_\omega(\bar{E})$ such that $s \prec\prec u$. Since E is a barrier, u has a (strict) left-segment t (say) which belongs to E. Since s, t are strict left-segments of u, one of them is a left-segment of the other and therefore, since they both belong to C, they must be equal in view of what was said in the preceding paragraph. Hence $s \in E$. We have thus shown that $C \cap A(\bar{E}) \subseteq E$. Since it is obvious that $E \subseteq C \cap A(\bar{E})$, Lemma 37 is proved.

Corollary 37 A. $E = \{a(\sigma);\ \sigma \in \Gamma_E\}$.

Proof. Let $t \in E$. Since $E \subseteq D \subseteq C$, it follows that $t = a(\tau)$ for some $\tau \in \Gamma$. Since $\tau \in \Gamma \subseteq A(B \cap A(K))$, τ is an ascending sequence of non-zero length and all terms of τ are ascending sequences of non-zero length. Since $a(\tau) = t \in E$, each term of τ belongs to $\bar{E}$. Hence $\tau \in A(A(\bar{E}))$. Therefore $\tau \in \Gamma_E$ and so $t \in \{a(\sigma);\ \sigma \in \Gamma_E\}$.

Let $t' \in \{a(\sigma) : \sigma \in \Gamma_E\}$. Then $t' = a(\tau')$ for some $\tau' \in \Gamma_E = \Gamma \cap A(A(\bar{E}))$. Now $t' \in C$ since $\tau' \in \Gamma$ and $t' \in A(\bar{E})$ since $\tau' \in A(A(\bar{E}))$ and therefore by Lemma 37 $t' \in E$.

Lemma 38. *If $\sigma \in A(B \cap A(\bar{E}))$ and σ is ψ-extensible in $\bar{E}$, then there exists a*

$$\tau \in A(B \cap A(\bar{E}))$$

such that $\sigma \prec\prec \tau$ and $\psi(\tau_*) > \psi(\tau)$ and $a(\tau) \in E$.

Proof. Let Σ be the set of all extensions of σ which belong to $A(B \cap A(\bar{E}))$ and let δ be the least element of $\psi(\Sigma)$. Choose a $\tau \in \Sigma$ such that $\psi(\tau) = \delta$ and such that, subject to this condition, $l(\tau)$ is as small as possible. Then $\tau \in A(B \cap A(\bar{E}))$.

Since σ is ψ-extensible in $\bar{E}$, we can certainly find a $\sigma' \in A(B \cap A(\bar{E}))$ such that $\sigma \prec \sigma'$ and $\psi(\sigma) > \psi(\sigma')$; this implies that $\sigma' \in \Sigma$ and hence that

$$\psi(\sigma) > \psi(\sigma') \geqslant \delta = \psi(\tau).$$

Therefore $\sigma \neq \tau$ and so, since $\tau \in \Sigma$, it follows that $\sigma \prec\prec \tau$. Therefore $\sigma \prec \tau_*$ and therefore $\tau_* \in \Sigma$. Since $\tau_* \in \Sigma$ and $l(\tau_*) < l(\tau)$, it follows from the definitions of δ and τ that $\psi(\tau_*) > \delta$. Therefore $\psi(\tau_*) > \psi(\tau)$. If $\tau \prec \tau' \in A(B \cap A(\bar{E}))$, then $\tau' \in \Sigma$ (since $\sigma \prec \tau \prec \tau'$) and therefore $\psi(\tau') \geqslant \delta = \psi(\tau)$ (which in fact by Lemma 34 implies that $\psi(\tau') = \psi(\tau)$). Hence τ is ψ-inextensible in $\bar{E}$. But $\bar{E}$ is an infinite set since E is an infinite set, and

$$\bar{E} \subseteq \bar{D} \subseteq \bar{C} = \bigcup \{\overline{a(\sigma)} : \sigma \in \Gamma\} \subseteq \bigcup \{\overline{a(\sigma)} : \sigma \in \Theta\} \subseteq K \subseteq L, \tag{5}$$

from which we also observe that

$$\tau \in \Sigma \subseteq A(B \cap A(\bar{E})) \subseteq A(B \cap A(K)) \subseteq A(B \cap A(L)).$$

Since τ belongs to $A(B \cap A(L))$ and is ψ-inextensible in the infinite subset $\bar{E}$ of L, it is an element of $A(B \cap A(L))$ which is not strongly ψ-extensible in L and must therefore, by the definition of L, be ψ-inextensible in L. Since $K \subseteq L$, this clearly implies that τ is ψ-inextensible in K and so belongs to Θ. Moreover, if $\Box \neq \rho \prec\prec \tau$, then $\rho \prec \tau^*$ and therefore by Lemma 34 $\psi(\rho) \geqslant \psi(\tau_*) > \psi(\tau)$ and therefore ρ is ψ-extensible in K and therefore $\rho \notin \Theta$. Hence no strict left-segment of τ belongs to Θ, and therefore $\tau \in \Gamma$ and therefore $a(\tau)$ belongs to C. Moreover, $a(\tau) \in A(\bar{E})$ since $\tau \in A(B \cap A(\bar{E}))$. Therefore $a(\tau) \in E$ by Lemma 37. Lemma 38 is now proved.

We can now complete the proof of the Theorem.

Since E is a block $E \neq \varnothing$ and therefore by Corollary 37 A $\Gamma_E \neq \varnothing$. Let ρ be an element of minimum length in Γ_E. Then $\rho \in \Gamma_E \subseteq \Gamma \subseteq \Theta$ and therefore $l(\rho) \geqslant 2$ by Lemma 35. Since $l(\rho) \geqslant 2$ and

$$\rho \in \Gamma_E = \Gamma \cap A(A(\bar{E})) \subseteq A(B) \cap A(A(\bar{E})) = A(B \cap A(\bar{E}))$$

and $\bar{E} \subseteq K$ by (5), we may conclude that

$$_*\rho \in A(B \cap A(\bar{E})) \subseteq A(B \cap A(K)) \subseteq A(B \cap A(L)).$$

If any left-segment π of $_*\rho$ belonged to Γ, then π would be different from $\square$ since $\Gamma \subseteq \Theta \subseteq A(B)$ and π would be a subsequence of the element ρ of $A(A(\bar{E}))$, and hence π would belong to $A(A(\bar{E}))$ and therefore to Γ_E, which would contradict the minimality condition on $l(\rho)$. Therefore no left-segment of $_*\rho$ belongs to Γ and hence $_*\rho \notin \Theta$. Therefore $_*\rho$ is ψ-extensible in K and hence, since $K \subseteq L$, $_*\rho$ is ψ-extensible in L. But the definition of L implies that $_*\rho$ is either ψ-inextensible in L or strongly ψ-extensible in L. Therefore $_*\rho$ is strongly ψ-extensible in L, and hence (in view of (5)) is ψ-extensible in $\bar{E}$. Therefore, by Lemma 38, we can select a $\tau \in A(B \cap A(\bar{E}))$ such that $_*\rho \prec\prec \tau$ and $\psi(\tau_*) > \psi(\tau)$ and $a(\tau) \in E$. Since $\rho, \tau \in A(B \cap A(\bar{E}))$ and $l(\rho) \geqslant 2$ and $_*\rho \prec\prec \tau$, it follows that there is a $\sigma \in A(B \cap A(\bar{E}))$ such that $\rho \prec\prec \sigma$ and $\tau = {_*\sigma}$ and $l(\sigma) \geqslant 3$. Since $\rho \in \Gamma_E \subseteq \Theta$, ρ is ψ-inextensible in K. Since $\rho \prec\prec \sigma$ and

$$\rho, \sigma \in A(B \cap A(\bar{E})) \subseteq A(B \cap A(K))$$

and ρ is ψ-inextensible in K and by Lemma 34 ψ is dwindling, it follows that

$$\psi(\rho) = \psi(\sigma).$$

Since $l(\sigma) \geqslant 3$, $\psi(\sigma)$ is by definition the least $\theta \in O$ such that $\mu_{\sigma(0),\sigma(1)}(\theta) \geqslant \psi(_*\sigma) = \psi(\tau)$. Therefore $\psi(\rho)$ is the least $\theta \in O$ such that $\mu_{\sigma(0),\sigma(1)}(\theta) \geqslant \psi(\tau)$. Therefore

$$\mu_{\sigma(0),\sigma(1)}(\theta) < \psi(\tau)$$

for every $\theta < \psi(\rho)$, i.e.
$$\mu_{\sigma(0),\sigma(1)}(O_{\psi(\rho)}) \subseteq O_{\psi(\tau)}. \tag{6}$$

Moreover $a(\rho) \in E$ by Corollary 37A and $a(\tau) \in E$ by the definition of τ and $a(\rho) \lhd a(\tau)$ by Lemma 23 and therefore, since $h|E$ is perfect, $h(a(\rho)) \leqslant h(a(\tau))$, i.e. (by the definition of h) $g_{\rho(0)}(\psi(\rho)) \leqslant g_{\tau(0)}(\psi(\tau))$, i.e.

$$g_{\sigma(0)}(\psi(\rho)) \leqslant g_{\sigma(1)}(\psi(\tau)). \tag{7}$$

From (6), (7) and the definition of $\mu_{\sigma(0),\sigma(1)}$ it follows that $\mu_{\sigma(0),\sigma(1)}(\psi(\rho)) \leqslant \psi(\tau)$. This inequality may be written as $\mu_{\rho(0),\rho(1)}(\psi(\rho)) \leqslant \psi(\tau)$ because $l(\rho) \geqslant 2$ and $\rho \prec\prec \sigma$. But, since $l(\rho) \geqslant 2$, it follows from the definition of $\psi(\rho)$ that $\mu_{\rho(0),\rho(1)}(\psi(\rho)) \geqslant \psi(_*\rho)$. Moreover, since $_*\rho \prec\prec \tau$, it follows that $_*\rho \prec \tau_*$ and therefore $\psi(_*\rho) \geqslant \psi(\tau_*)$ by Lemma 34. We thus obtain the contradiction

$$\psi(\tau_*) > \psi(\tau) \geqslant \mu_{\rho(0),\rho(1)}(\psi(\rho)) \geqslant \psi(_*\rho) \geqslant \psi(\tau_*),$$

which serves to establish our Theorem.

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