Direct interpolation for modal mu-calculus

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Uniform interpolation

Definition

Craig interpolation: for every valid implication $A \rightarrow C$ there is a formula B (the interpolant) in the common language of A and C s.t.

$$A \rightarrow B \text{ is valid} \quad \& \quad B \rightarrow C \text{ is valid}$$

Uniform interpolation: B depends only on A and the 'common language'.

Modal logics are known to widely enjoy interpolation:

K, T, GL, S4, S5, ...

It is also true for the extension of K with fixpoint operators, modal μ-calculus.

Modal μ-calculus

Syntax: $\bot \mid \top \mid p \mid \overline{p} \mid A \wedge A \mid A \vee A \mid \langle \mathfrak{a} \rangle A \mid [\mathfrak{a}]A \mid \mathsf{x} \mid \mu \mathsf{x}\, A \mid \nu \mathsf{x}\, A$

where $p \in \mathsf{Prop}$, $\mathfrak{a} \in \mathsf{Act}$, and $\mathsf{x} \in \mathsf{Var}$

Complete axiomatisation:

- System K: PL $+$ $\frac{A}{[\mathfrak{a}]A}$ $+$ $[\mathfrak{a}](A \to B) \to [\mathfrak{a}]A \to [\mathfrak{a}]B$
- $A(\mu \mathsf{x} A(\mathsf{x})) \to \mu \mathsf{x} A(\mathsf{x})$
- $A(B) \to B \vdash \mu \mathsf{x} A(\mathsf{x}) \to B$

Theorem (D'Agostino and Hollenberg 2000)

Uniform interpolation holds for μ-calculus.

Techniques: (disjunctive) modal automata, bisimulation quantifiers à la Visser

Devising proof systems for interpolation

- Syntactic approach via sequent calculus: (complete) sequent calculus that admits elimination of cuts.
- Craig interpolation is often (but not always) provable via induction over the cut-free derivations.
- There is an intimate connection between interpolation and the existence of sequent calculi. (Iemhoff; Kuznets; …)

Building a proof system

Gentzen sequents: $\Gamma \Rightarrow \Delta$

Axioms:

$$p \Rightarrow p \qquad p, \neg p \Rightarrow \emptyset \qquad \emptyset \Rightarrow p, \neg p \qquad \bot \Rightarrow \emptyset \qquad \emptyset \Rightarrow \top$$

Logical rules: $\vee^l, \vee^r, \wedge^l, \wedge^r$
Modality rules:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Pi, \Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta, \Pi'}\,\mathsf{mod}^l \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Pi, \Box\Gamma \Rightarrow \Diamond\Delta, \Box A, \Pi'}\,\mathsf{mod}^r$$

Regeneration rules:

$$\frac{\Gamma, A(\nu \mathsf{x} A) \Rightarrow \Delta}{\Gamma, \nu \mathsf{x} A \Rightarrow \Delta}\,\nu \qquad \frac{\Gamma \Rightarrow \Delta, A(\mu \mathsf{x} A)}{\Gamma \Rightarrow \Delta, \mu \mathsf{x} A}\,\mu$$

Annotated sequents

Names for each variable: $N_{\mathsf{x}} = \{x_0, x_1, \dots\}$ and $N = \bigcup_{\mathsf{x}:\mathrm{Var}} N_{\mathsf{x}}$.

An annotated sequent is an expression

$$a_0 : A_1^{a_1}, \dots, A_k^{a_k} \Rightarrow A_{k+1}^{a_{k+1}}, \dots, A_n^{a_n}$$

such that $a_0, \dots, a_n \in N^*$.

Recording μs and νs:

$$\frac{ax : \Gamma, A(\mu\mathsf{x}A)^{bx} \Rightarrow \Delta}{a : \Gamma, \mu\mathsf{x}A^{b} \Rightarrow \Delta}\,\mu^{x} \qquad \frac{ax : \Gamma \Rightarrow \Delta, A(\nu\mathsf{x}A)^{bx}}{a : \Gamma \Rightarrow \Delta, \nu\mathsf{x}A^{b}}\,\nu^{x}$$

$b \leq x \in N_{\mathsf{x}}$

Also structural rules for annotation management.

Cyclic proofs

A cyclic proof is a finite tree built from these rules s.t. every leaf is either an axiom, or a successful repeat:

$$\begin{array}{c} a : \Gamma \Rightarrow \Delta \\ \vdots \\ a : \Gamma \Rightarrow \Delta \\ \vdots \end{array}$$

1. sequent is repeated (including annotations)
2. an annotation 'progresses'

One-sided version of this system is due to N. Jungteerapanich & C. Stirling.

Theorem (Stirling 2014)

The system is sound and complete for μ-calculus.

Interpolating cycles

- Given valid $\Gamma \Rightarrow \Delta$, find $B \in L$ such that $\Gamma \Rightarrow B$ and $B \Rightarrow \Delta$ are valid.
- Via induction over derivations: an interpolant for the conclusion is constructed from interpolant(s) of premise(s).
- The only non-trivial case is that of non-axiomatic leaves.
- A successful repeat on the left/right are interpolated by μ/ν formula.

Example. Successful repeat on the left:

$$
\begin{array}{c}
a : \Gamma \overset{\mathsf{y}}{\Rightarrow} \Delta \\
\vdots \\
a : \Gamma \overset{I(\mathsf{y})}{\Rightarrow} \Delta \\
\vdots
\end{array}
$$

1. Leaves $\Gamma \Rightarrow \mathsf{y}$ and $\mathsf{y} \Rightarrow \Delta$ with y fresh.
2. By IH we have formula $I(\mathsf{y})$ and proofs:

$$
\begin{array}{c}
\Gamma \Rightarrow \mathsf{y} \\
\vdots \\
\Gamma \Rightarrow I(\mathsf{y})
\end{array}
\qquad\qquad
\begin{array}{c}
\mathsf{y} \Rightarrow \Delta \\
\vdots \\
I(\mathsf{y}) \Rightarrow \Delta
\end{array}
$$

3. In this case, $\mathsf{y} = \mu \mathsf{y} I(\mathsf{y})$.

More complex interpolants

A repeated sequent may have been used for multiple non-axiomatic leaves, e.g.

$$\dfrac{\begin{array}{ccc} a : \Gamma \overset{\mathsf{y}}{\Rightarrow} \Delta & & a : \Gamma \overset{\mathsf{y}'}{\Rightarrow} \Delta \\ \vdots & & \vdots \\ & & \vdots \end{array}}{a : \Gamma \overset{I(\mathsf{y},\mathsf{y}')}{\Longrightarrow} \Delta}$$
$$\vdots$$

System of equations:

$$\begin{aligned} \mathsf{y} &=_{\mu} I(\mathsf{y}, \mathsf{y}') \\ \mathsf{y}' &=_{\sigma} I(\mathsf{y}, \mathsf{y}') \qquad \sigma \in \{\mu, \nu\} \\ &\vdots \end{aligned}$$

A subsumption ordering (given by the proof) $\mathsf{y} \sqsubset \mathsf{y}' \sqsubset \ldots$

The order is important to ensure after unraveling any infinite path has a ν-thread.

Deterministic proof search

The calculus can be used for proof search with one further structural rule:

$$u \sqsubset_a v \; \frac{a' : \Gamma \Rightarrow A^u}{a : \Gamma \Rightarrow \Delta, A^u, A^v}\,\mathsf{thin}$$

Proof search strategy:

1. Prioritise thinning and annotation management.
2. Invert logical rules
3. At modal sequents take all possible continuations
4. Stop at the first repetition

Need to know:

a) there is always a repeat along every branch.

b) there is a pruning that gives a proof if the starting sequent is valid.

Constructing the uniform interpolant

- Run proof search on $A \Rightarrow\ ?$
- We rely on the schematic property of seq. cal.
- Three general cases to check
  - disjunctions on the left, conjunctions on the right
  - modalities
  - non-axiomatic leaves

$$\dfrac{\Gamma, A \overset{I_0}{\Rightarrow} \Delta \qquad \Gamma, B \overset{I_1}{\Rightarrow} \Delta}{\Gamma, A \vee B \overset{I_0 \vee I_1}{\Rightarrow} \Delta} \vee_l \qquad\qquad \dfrac{\Gamma \overset{I}{\Rightarrow} \Delta, A \qquad \Gamma \overset{I}{\Rightarrow} \Delta, B}{\Gamma \overset{I}{\Rightarrow} \Delta, A \wedge B} \wedge_r$$

The case of mod-rule

$$\text{mod}_l \frac{\Gamma, A_1 \overset{I_1}{\Rightarrow} ? \qquad \cdots \qquad \Gamma, A_n \overset{I_n}{\Rightarrow} ? \qquad \text{OR} \qquad \Gamma \overset{I'}{\Rightarrow} ?}{\Box\Gamma, \Diamond A_1, \ldots, \Diamond A_n \Rightarrow ?} \text{mod}_r$$

By I.H. we have $\{I_i\}$ and I' so we take

$$I = (\bigwedge_i \Diamond I_i) \wedge \Box I'$$

So the interpolant is $f : \mathit{Child}(A) \longrightarrow L_\mu$

a) $f(A) \in L$;

b) $\forall B \in L$, if $D \colon A \Rightarrow B$ then $D^f : f(A) \Rightarrow B$

c) $A \Rightarrow f(A)$

Note in particular, if $A \in L$

$$A \equiv f(A)$$

Disjunctive fragment

Syntax

$A := \bot \mid \top \mid p \mid \overline{p} \mid \cancel{A \wedge A} \mid A \vee A \mid \cancel{\langle \mathfrak{a} \rangle A \mid [\mathfrak{a}] A} \mid \mathsf{x} \mid \mu \mathsf{x}\, A \mid \nu \mathsf{x}\, A \mid \nabla(\Gamma, P)$

where Γ and P are finite sets of respectively formulas and literals.

$$\nabla(\Gamma, P) \equiv \bigwedge P \wedge \bigwedge \Diamond \Gamma \wedge \Box \bigvee \Gamma$$

Theorem (Janin & Walukiewicz 1996)

Every formula of μ-calculus is equivalent to a disjunctive formula.

Proof using interpolation.

Revise the interpolant while maintaining equivalence: we make $f(\Gamma)$ disjunctive for every $\Gamma \in \mathit{Child}(A)$

$$I = (\bigwedge_{i=1}^{n} \Diamond I_i) \wedge \Box I' \text{ becomes } \begin{cases} \Box I', & \text{if } n = 0, \\ \nabla(\emptyset, \{I_1, \ldots, I_n, \top\}), & \text{if } n \neq 0. \end{cases}$$

Open problems

1. When does a (cyclic) sequent calculus entail uniform interpolation?
2. What is the form of the interpolant? Potential applications in database theory.
3. Can we apply the same arguments to other modal and temporal logics, e.g. PDL?
4. More abstractly, what about coalgebraic fixpoint logic?