

FINDING THE INTERSECTION OF n HALF-SPACES IN TIME $O(n \log n)^*$

F. P. PREPARATA[†] and D. E. MULLER[‡]

*Coordinated Science Laboratory, University of Illinois at Urbana-Champaign,
Urbana, IL 61801, U.S.A.*

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Abstract. Given a set of n half-spaces in three-dimensional space, we develop an algorithm for finding their common intersection in time $O(n \log n)$. The intersection, if nonempty, is presented as a convex polyhedron. The algorithm is summarized as follows: (i) the half-spaces are placed in two sets depending upon whether they contain or do not contain the origin; (ii) the half-spaces in each of these sets are dualized to points, and the convex hulls of the dualized sets are obtained in time $O(n \log n)$; (iii) since the half-space intersection is nonempty if and only if these two convex hulls are disjoint, a separating plane is found, also in time $O(n \log n)$; (iv) after applying a linear spatial transformation which maps the separating plane to infinity, the convex hull of the union of the two transformed convex hulls is the transformed intersection of the half-spaces. Since the latter can be found in time $O(n)$, the overall running time of the procedure is $O(n \log n)$. A significant consequence of this result is that a three-variable linear, or convex, programming problem can be asymptotically solved faster than by the Simplex algorithm, in the worst case.

1. Introduction

To find the common intersection of a set of n three-dimensional half-spaces is an important problem in computational geometry. Perhaps the most familiar application of this problem occurs in mathematical optimization, specifically in three-variable linear and convex programming [1]. In fact, as is well-known, a linear programming problem in the variables (x, y, z) consists of extremizing a linear objective function of x , y , and z subject to a set of n linear inequalities in these variables. These inequalities define the set of the feasible solutions to the problem. It is well-known that this set is a convex region of space, which is the intersection of the half-spaces corresponding to the inequalities; it is also known that the extremum solution occurs at one vertex, or *extreme point*, of the region of the feasible solutions (*feasible region*). Thus the linear programming problem can be solved by

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[†] Also with the Departments of Electrical Engineering and of Computer Science.

[‡] Also with the Departments of Mathematics and of Computer Science.

determining the extreme points of the feasible region, and evaluating the objective function at each of them. The same considerations entirely apply to convex programming with linear constraints when the convexity of the objective function is such that its extreme value occurs at an extreme point of the feasible region.

The most successful method for solving these optimization problems is the Simplex algorithm [1]. With three variables, the Simplex algorithm involves traversing a path on the polyhedral surface which bounds the feasible region, until the extremizing vertex is reached. Since the Simplex algorithm spends time $O(n)$ in moving from one vertex to the next vertex, and in the worst case it may have to visit $O(n)$ vertices before terminating, we conclude that its running time is $O(n^2)$.

For the two-variable version of these problems, Shamos and Hoey [5] have shown that the intersection of n half-planes can be computed in time $O(n \log n)$, thereby obtaining a method for solving the two-variable linear programming problem which is faster than the Simplex algorithm in the worst-case. Their method is based on the divide-and-conquer technique, i.e., on finding in time $O(n)$ the intersection of two polygons, which are respectively the common intersections of two sets of approximately $\frac{1}{2}n$ half-planes.

Shamos [4] also suggested that, if one finds a fast algorithm for intersecting two polyhedra, then this would be used—by the divide-and-conquer technique—to obtain a fast algorithm for finding the intersection of a set of half-spaces. In turn, the latter could be applied to the linear programming problem. In a companion paper [2], we have described an algorithm for intersecting two polyhedra, whose total number of vertices is n , in time $O(n \log n)$. This method, if used as a merge technique in a divide-and-conquer approach, would yield a running time $O(n \log^2 n)$ for finding the common intersection of n half-spaces. In this paper we show, instead, that the latter problem can also be solved in time $O(n \log n)$. Our technique is crucially based on the polyhedron intersection algorithm, but not as a merge technique; rather, it uses it to transform the original problem into its dual, i.e., that of finding the convex hull of a set of n three-dimensional points, which is known to be solvable (by a divide-and-conquer technique) in time $O(n \log n)$.

The computation model adopted in the preceding algorithms was a random-access machine using real-number arithmetic. We shall also assume this model in the future discussion.

We shall now precisely formulate our problem: Finding the intersection of n three-dimensional half-spaces consists of finding the solutions to a set of n linear inequalities of the form

$$a_{i1}x + a_{i2}y + a_{i3}z + a_{i4} \geq 0 \quad (i = 1, \dots, n) \quad (1)$$

where x , y , and z are cartesian coordinates of a solution in three-space and where, for each i , the a_{i1} , a_{i2} , a_{i3} , a_{i4} are real numbers which are not all 0.

For reasons to be explained later, it is more convenient to express the points in homogeneous coordinates x_1 , x_2 , x_3 , and x_4 so that $x = x_1/x_4$, $y = x_2/x_4$, and

$z = x_3/x_4$. We obtain therefore, the related set of inequalities:

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 \geq 0 \quad (i = 1, \dots, n), \quad (2)$$

from which we can obtain the solutions to the original set. In fact, the solutions to (2) form a convex set which may be separated into three disjoint subsets each also convex. They are (i) the *positive* points, having $x_4 > 0$ which correspond to solutions of (1), (ii) the *equatorial* points, having $x_4 = 0$, and (iii) the *negative* points having $x_4 < 0$.

The set of solutions to (2) will be described by a system having the following form:

(a) A minimum subset of (2), (i.e., all redundant inequalities will be deleted), corresponding to the faces F of a generalized polyhedron \mathcal{A} , to be defined later.

(b) A minimum set V' of *extreme points* (x_1, x_2, x_3, x_4) . The set of solutions to (2) consists of the linear combinations of the elements of V' with nonnegative coefficients. The three-dimensional nonhomogeneous correspondents of the elements of V' form the set V of vertices of the generalized polyhedron \mathcal{A} .

(c) A data structure, called a *doubly-connected edge-list* (see [2]) describing the relation between V and F , that is, the cycles of edges incident on vertices in V and the cycles of edges bordering faces in F .

One advantage to treating solutions to the set (2) of inequalities is that the extreme points all lie within the range of the variables, while the extreme points of (1) may lie at infinity. A second advantage is that, as we shall see, the duality between V and F is complete in the case of (2), and we shall be able to take full advantage of it.

In the next section we shall briefly discuss some geometric notions which are instrumental to the subsequent developments.

2. Geometrical preliminaries

As is well-known, there is a convenient interpretation of homogeneous co-ordinates in E^3 if one views (x_1, x_2, x_3, x_4) as coordinates in four-dimensional space E^4 and normalizes them by multiplication by a positive constant so that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, i.e., to points on the surface of the unit hypersphere S^4 with center at the origin. The positive points lie in what we shall call the positive open hemisphere consisting of points in S^4 with $x_4 > 0$; similarly the negative points lie in the negative hemisphere and the equatorial points lie in the intersection of the hypersphere with the equatorial hyperplane $x_4 = 0$.

The three-dimensional space containing the solutions to (1) can be viewed as the hyperplane $x_4 = 1$, which is tangent to S^4 , and a projection from the origin establishes a one-to-one correspondence between points of the positive hemisphere and points of E^3 . Each of the hyperplanes of (2) passes through the origin of E^4 and

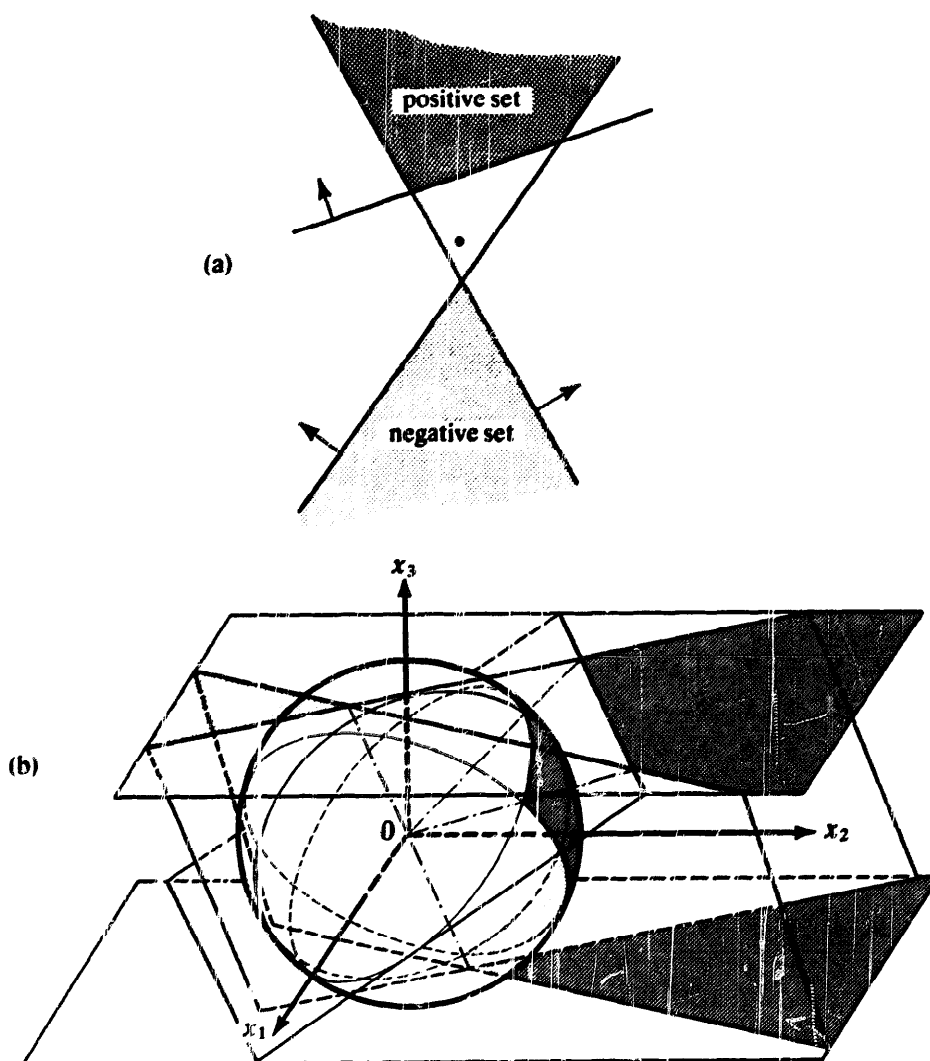


Fig. 1. An illustration of the positive and negative sets in two dimensions. The arrows indicate the half-planes defined by (1).

defines a corresponding half-space; the common intersection of these half-spaces is a (convex) cone \mathcal{C}^4 . The intersection of \mathcal{C}^4 with S^4 is a connected domain which *may* cross the equatorial hyperplane; in the latter case, points of $\mathcal{C}^4 \cap S^4$ in the positive hemisphere will be projected to points in the positive set, whereas the points of $\mathcal{C}^4 \cap S^4$ in the negative hemisphere will be projected to a set of points in the hyperplane $x_4 = -1$, which will in turn map to the negative set in $x_4 = 1$ by a symmetry with respect to the origin of E^4 (see Fig. 1(b) for an illustration in one less dimension). This combined set in the hyperplane $x_4 = 1$ may, thus, consist of two separate unbounded convex sets, the *positive* and *negative*, as illustrated two-dimensionally in Fig. 1(a). We call this the *hyperbolic* case. If all points of the solution project into the positive (or the negative) open hemisphere, the corresponding set in E^3 is bounded and we call this the *elliptic* case. The *parabolic* case occurs when there is a single, but unbounded set in E^3 , i.e., the case when

equatorial as well as positive (or negative) points occur in S^4 . In all three cases the sets of points form what we shall call a *generalized* convex polyhedron. The elliptic case alone corresponds to a conventional polyhedron.

In the sequel we shall make frequent use of an involutory transformation from points to planes and vice-versa called *dualization*. Under dualization the coordinates $(\xi_1, \xi_2, \xi_3, \xi_4)$ of a point are reinterpreted as the coefficients defining a half-space $\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 \geq 0$. Dualization is susceptible of two intuitive geometric interpretations—one in E^4 and one in E^3 . In E^4 each point on S^4 is viewed as the terminus of a (unit) vector applied to the origin and its dual is the half-space containing this vector whose boundary hyperplane passes through the origin and is orthogonal to this vector. Conversely, each half-space through the origin dualizes into the corresponding vector. In E^3 a positive point at distance l from the origin is mapped into a half-space containing the origin whose boundary plane is at distance $1/l$ from the origin in a direction opposite to the point. If the point is negative, the complementary half-space is obtained with the same boundary. Conversely, any half-space whose boundary plane does not contain the origin has a point as its dual. This point is interpreted as being positive or negative depending upon whether the half-space contains the origin.

After dualizing the system of inequalities we shall need to apply various algorithms which construct the convex hull of a set of points, and follow this operation by an inverse dualization. Since the convex hull algorithms in their published version [3] apply to the conventional or elliptic case, we shall find it convenient to first apply an invertible linear transformation of coordinates to (2) in E^4 . This transformation will have the effect in E^3 of moving the origin to a selected point and hence will enable us to reduce the dual system to the elliptic case as desired. The transformation will be described by a nonsingular 4×4 postmultiplicative matrix T . It is desirable, however, in order to avoid unnecessary coordinate manipulations, to restrict ourselves to linear transformations of E^4 which map S^4 to itself. These transformations are readily characterized. Letting the row vector ξ denote a point on S^4 , we know that $\xi \cdot \xi' = 1^1$; the image ξT of ξ must be on S^4 , whence $\xi(T) \xi(T)' = \xi T T' \xi' = 1$. Since ξ is arbitrary on S^4 , the latter holds if and only if $T \cdot T' = I_4$, the 4×4 identity matrix, i.e., $T^{-1} = T'$. This is the characteristic property of *rotations*: thus we shall only consider rotations of E^4 . It is worth noticing that rotations commute with dualization, that is, the dual of the image (under rotation) of a point coincides with the image of the dual of that point.

3. Finding the intersection

We shall now rewrite (2) in matrix form

$$Ax' \geq 0 \tag{3}$$

¹ ξ' is the transpose of ξ .

where $A = \|a_{ij}\|$ is an $n \times 4$ matrix. Let A_+ , A_0 , and A_- be the matrices formed from the rows of A for which $a_{i4} > 0$, $a_{i4} = 0$, and $a_{i4} < 0$ respectively. In a later discussion we shall show that, except in degenerate cases which can be reduced to a problem of lower dimensionality, it is possible to find a rotation R_0 of A which eliminates A_0 . Therefore, we assume that in (3) such a rotation has already been made and that either $a_{i4} > 0$ or $a_{i4} < 0$ for each i .

Each row $(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ of A_+ represents a plane in E^3 (or, equivalently, a hyperplane through the origin in E^4). We now dualize each such plane, i.e., we construct a point $(a_{i1}/a_{i4}, a_{i2}/a_{i4}, a_{i3}/a_{i4})$ in E^3 and apply the algorithm of [3] to find the convex hull $\mathcal{A}_+^{(D)}$ of these points: notice that the algorithm is directly applicable because each point has finite coordinates. The same procedure is then used to construct a convex polyhedron $\mathcal{A}_-^{(D)}$ from the matrix A_- . We may now apply to the two three-dimensional polyhedra $\mathcal{A}_+^{(D)}$ and $\mathcal{A}_-^{(D)}$ an algorithm, described in [2], which determines whether the two polyhedra intersect and, if so, finds the intersection, and if not, finds a separating plane. Actually if $\mathcal{A}_+^{(D)} \cap \mathcal{A}_-^{(D)}$ has an interior (i.e., nonzero volume), we need not explicitly construct it, since in this case the set of inequalities (3) is inconsistent and has no solution. To show this, without loss of generality let $\{(a_{j1}/a_{j4}, a_{j2}/a_{j4}, a_{j3}/a_{j4}): 1 \leq j \leq s\}$ be the set of vertices of $\mathcal{A}_+^{(D)}$. Any point (u_1, u_2, u_3) in $\mathcal{A}_+^{(D)}$ can be expressed as a convex combination of the vertices, i.e.

$$(u_1, u_2, u_3, 1) = \sum_{j=1}^s c_j \left(\frac{a_{j1}}{a_{j4}}, \frac{a_{j2}}{a_{j4}}, \frac{a_{j3}}{a_{j4}}, 1 \right); \quad (4)$$

if (u_1, u_2, u_3) is in the interior of $\mathcal{A}_+^{(D)}$, then we can choose $c_j > 0$ for all j . If $(x_1^*, x_2^*, x_3^*, x_4^*)$ is any nonzero solution of (3), then obviously

$$a_{j1}x_1^* + a_{j2}x_2^* + a_{j3}x_3^* + a_{j4}x_4^* \geq 0, \quad 1 \leq j \leq s.$$

However, not all left-sides of these s inequalities can be simultaneously 0, for, if they were, the corresponding planes of equations $a_{j1}x + a_{j2}y + a_{j3}z + a_{j4} = 0$, $1 \leq j \leq s$, would intersect in a point; but this contradicts the fact that these planes are the duals of the vertices of $\mathcal{A}_+^{(D)}$, which are not all coplanar. Suppose then that, for some m , $a_{m1}x_1^* + a_{m2}x_2^* + a_{m3}x_3^* + a_{m4}x_4^* > 0$; if we postmultiply both sides of (4) by the vector $(x_1^*, x_2^*, x_3^*, x_4^*)$, the right-side is a sum of non-negative terms of which $c_m(a_{m1}x_1^* + a_{m2}x_2^* + a_{m3}x_3^* + a_{m4}x_4^*)/a_{m4} > 0$, whence

$$u_1x_1^* + u_2x_2^* + u_3x_3^* + x_4^* > 0.$$

If (u_1, u_2, u_3) is also in the interior of $\mathcal{A}_-^{(D)}$ then it must satisfy $-u_1x_1^* - u_2x_2^* - u_3x_3^* - x_4^* > 0$ as well, so we conclude that (3) has no solution.

If $\mathcal{A}_+^{(D)} \cap \mathcal{A}_-^{(D)}$ is nonempty but has no interior, the problem reduces to one of lower dimensionality to be discussed in Section 4.

² The letter "D" as an apex is to remind us that we are dealing with the convex hull of the *dual* set of points.

Finally suppose that $\mathcal{A}_+^{(D)}$ and $\mathcal{A}_-^{(D)}$ do not intersect. In this case we find a separating plane by the method described in [2], and let $p_1x + p_2y + p_3z + p_4 = 0$ be this plane, with the signs of the coefficients so chosen that for any point (a, b, c) in $\mathcal{A}_+^{(D)}$ we have $p_1a + p_2b + p_3c + p_4 > 0$.

The separating plane just found may be extended to a hyperplane $p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 = 0$ through the origin, whose normal in E^4 is the vector (p_1, p_2, p_3, p_4) . We now wish to rotate coordinates in E^4 so that this vector points toward the origin of E^3 . In other words, we wish to find a rotation R such that $(p_1, p_2, p_3, p_4)R = (0, 0, 0, K)$, where K is a positive number (actually $K = \sqrt{p_1^2 + p_2^2 + p_3^2 + p_4^2}$). A suitable rotation is easily constructed as a product $R_1R_2R_3$, where each R_i causes p_i to go to 0. For example, we may take

$$R_1 = \begin{pmatrix} p_4/\sqrt{p_1^2 + p_4^2} & 0 & 0 & p_1/\sqrt{p_1^2 + p_4^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -p_1/\sqrt{p_1^2 + p_4^2} & 0 & 0 & p_4/\sqrt{p_1^2 + p_4^2} \end{pmatrix}.$$

The elements of the 4×4 matrix R can be computed in time bounded by a constant using the operations of arithmetic including square root.³

We claim that the fourth column of the matrix AR consists entirely of positive entries. In fact, system (3) can be rewritten as

$$Ax' = A(RR^{-1})x' = ARR'x' = AR(xR)' \geq 0. \quad (3')$$

Since the point $p = (p_1, p_2, p_3, p_4)$ satisfies $a_{i1}p_1 + a_{i2}p_2 + a_{i3}p_3 + a_{i4}p_4 > 0$ for each i , the image of p under the rotation, which is $(0, 0, 0, K)$, satisfies a corresponding set of strict inequalities using the rows of AR . In each of these inequalities the only nonzero term is the positive constant K multiplied by the fourth entry in the corresponding row of AR , thereby establishing the claim.

There is no need, however, to construct AR , since it may contain irrelevant rows which we want to suppress. Instead, for each vertex $(a_{i1}/a_{i4}, a_{i2}/a_{i4}, a_{i3}/a_{i4})$ of $\mathcal{A}_+^{(D)}$ we construct a row $(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ of a new matrix $A_+^{(m)}$ whose rows form a subset of the rows of A_+ . Similarly, for each vertex of $\mathcal{A}_-^{(D)}$ we find the corresponding row in A_- and from these rows form a new matrix $A_-^{(m)}$.

Let $A_+^* = A_+^{(m)}R$ and $A_-^* = A_-^{(m)}R$; the set of inequalities

$$[A_+^*, A_-^*]\xi' \geq 0 \quad (5)$$

is equivalent to $AR\xi' \geq 0$, with all irrelevant inequalities deleted.

Since (5) is satisfied by $(0, 0, 0, 1)$, the origin in E^3 , conceptually the intersection of the original n half-spaces can be found by: (i) dualizing the planes corresponding

³ If a more general nonsingular linear transformation T is used in place of the rotation R one can avoid the appearance of square roots. If this is done, however, it is necessary to transform vectors representing faces by the transpose of T^{-1} when vectors representing vertices are transformed by T and vice-versa.

to the rows of $[A_+^*, A_-^*]$, (ii) finding the convex hull of the set of points thus obtained, and (iii) dualizing back the convex hull. However, this sequence of operations can be considerably simplified. The duals of the planes corresponding to the rows of $A_+^{(m)}$ and $A_-^{(m)}$ are already available as the vertices of the polyhedra $\mathcal{A}_+^{(D)}$ and $\mathcal{A}_-^{(D)}$ respectively. Thus, to obtain the analogous polyhedra $\mathcal{A}_+^{*(D)}$ and $\mathcal{A}_-^{*(D)}$ for A_+^* and A_-^* respectively, we rotate the polyhedra $\mathcal{A}_+^{(D)}$ and $\mathcal{A}_-^{(D)}$ (which lie in the $x_4 = 1$ hyperplane) by R and then project them through the origin of E^4 back to the hyperplane $x_4 = 1$. In practice, this means multiplying the coordinates of each of the rotated vertices by a number which makes its fourth coordinate 1.

The two polyhedra $\mathcal{A}_+^{*(D)}$ and $\mathcal{A}_-^{*(D)}$ are disjoint because they are separated by a hyperplane through the origin of E^4 normal to the R -transform of the vector $(0, 0, 0, 1)$. We therefore may construct the convex hull of the union $\mathcal{A}_+^{*(D)} \cup \mathcal{A}_-^{*(D)}$ by a single application of the "merge" portion of the algorithm of [3] in time proportional to the total number of vertices in the two polyhedra. Let us call this resulting polyhedron $\mathcal{A}^{*(D)}$.

Now, the dual \mathcal{A}^* of the polyhedron $\mathcal{A}^{*(D)}$ represents the set of solutions to the system of inequalities

$$a_{i1}^*x + a_{i2}^*y + a_{i3}^*z + a_{i4}^* \geq 0 \quad (i = 1, \dots, n) \quad (1')$$

where the coefficients a_{ij}^* are the elements of the matrix $A^* = AR$, and we recall that all the coefficients a_{i4}^* are positive. We therefore apply the inverse rotation R^{-1} to the vertices and faces of \mathcal{A}^* . Now \mathcal{A}^* lies in the hyperplane $x_4 = 1$, so the four-dimensional rotation R^{-1} is applied to vertices of \mathcal{A}^* of the form $(v_1^*, v_2^*, v_3^*, 1)$. The resulting vertices (v_1, v_2, v_3, v_4) must be renormalized if $v_4 \neq 0$, and this is done by multiplying each of the coordinates by the positive constant $1/|v_4|$.

The result of this rotation and renormalization is a set of vertices and associated faces which we can represent by a doubly-connected edge-list and which we shall call a generalized polyhedron \mathcal{A} . The vertices of this generalized polyhedron with fourth coordinate equal to 1 are the extreme points of the intersection of the half-spaces.

In the beginning of this discussion it was stated that an initial rotation R_0 of coordinates could be found which would make each entry a_{i4} in the fourth column of the matrix A different from 0. We shall now describe how R_0 is obtained. As before, we define A_0 to be the matrix composed of rows from A having $a_{i4} = 0$. Inequalities of the set (1) corresponding to these rows appear as

$$a_{i1}x + a_{i2}y + a_{i3}z \geq 0. \quad (1'')$$

We seek to find a point $u = (u_1, u_2, u_3)$ in E^3 which strictly satisfies all these inequalities. Notice that the set of inequalities (1'') defines a convex cone \mathcal{C}^3 in E^3 ,

whose vertex is the origin. Except in degenerate cases, \mathcal{C}^3 intersects either one of planes $z = 1$ and $z = -1$, since it cannot be contained in the region $-1 \leq z \leq +1$. Hence, if a point u exists, it can be found as follows. Let $z = 1$ in (1'') and, using an algorithm due to Shamos and Hoey [5] for finding in time $O(n \log n)$ the intersection of n half-planes, find the polygon of solutions. If no such polygon exists, then set $z = -1$ and repeat the process. In one case or the other there will be a polygon P from which we can choose u as one of its interior points.

Now we augment $u = (u_1, u_2, u_3)$ to form a point $v = (u_1, u_2, u_3, M)$ in E^4 , taking $M > 0$. Clearly, v strictly satisfies all the inequalities in $A_0 v \geq 0$. As we saw in the derivation of R , we can construct a rotation R_0 such that $uR_0 = (0, 0, 0, K_0)$, and this rotation has the property that all entries in the fourth column of $A_0 R_0$ are positive.

The rotation R_0 can be made to differ very little from the identity transformation by taking M very large. Thus, we take M large enough so that no fourth column entries in A_+ or A_- change sign when the rotation R_0 is performed. Hence all entries in the fourth column of AR_0 are nonzero. As we noticed construction of u can be done in time $O(n \log n)$; the construction of a rotation R_0 , can be done in constant time, as previously discussed in a similar situation.

In conclusion we will replace the original set of inequalities (3) by

$$AR_0(xR_0) \geq 0. \quad (3'')$$

To summarize the preceding discussion, the intersection of n half-spaces, expressed by (3), can be obtained as follows.

Step 1. Find a point u in the interior of the set of solutions to (1'') and construct a corresponding rotation R_0 . Replace A by AR_0 . [This can be done in time $O(n \log n)$ using the half-plane intersection algorithm in [5].]

Step 2. Find the convex hull $\mathcal{A}_+^{(D)}$ of the set of points $(a_{i1}/a_{i4}, a_{i2}/a_{i4}, a_{i3}/a_{i4})$ for all i such that $a_{i4} > 0$. Find the convex hull $\mathcal{A}_-^{(D)}$ of the set of points $(a_{i1}/a_{i4}, a_{i2}/a_{i4}, a_{i3}/a_{i4})$ for all i such that $a_{i4} < 0$. [This can be done in time $O(n \log n)$ by using the convex hull algorithm described in [3].]

Step 3. Using the algorithm for intersecting polyhedra [2], test if $\mathcal{A}_+^{(D)} \cap \mathcal{A}_-^{(D)}$ has an interior. If so, the half-space intersection is empty; if not, find a separating plane $p_1x + p_2y + p_3z + p_4 = 0$. [This can be done in time $O(n \log n)$, according to the results reported in [2].]

Step 4. Find a rotation R of E^4 so that the separating plane is mapped to the plane at infinity [this can be done in constant time]. Find the R -transforms of the polyhedra $\mathcal{A}_+^{(D)}$ and $\mathcal{A}_-^{(D)}$ and project them through the origin back to the $x_4 = 1$ hyperplane. [This requires time $O(n)$.] Let $\mathcal{A}_+^{*(D)}$ and $\mathcal{A}_-^{*(D)}$ be the resulting polyhedra.

Step 5. Find the convex-hull $\mathcal{A}^{*(D)}$ of the union of the two polyhedra $\mathcal{A}_+^{*(D)}$ and $\mathcal{A}_-^{*(D)}$. [This is done in time $O(n)$ using the algorithm of [3].]

Step 6. Compute the $R^{-1}R_0^{-1}$ -transform of the vertices and faces of the dual \mathcal{A}^* of $\mathcal{A}^{*(D)}$. Then renormalize the transformed vertices to obtain \mathcal{A} , the generalized polyhedron which forms the intersection of the given half-spaces. [This step also requires time $O(n)$.]

In conclusion, we see that the preceding algorithm runs in time $O(n \log n)$, Steps 1, 2, and 3 being the limiting ones.

To eliminate solutions to (2) corresponding to negative points, one can augment the set (2) with the inequality $x_4 \geq 0$ before applying the algorithm. The remaining positive points project to solutions to the set (1), and if there are equatorial points which are extreme points they should be retained to facilitate the description of the set of solutions.

4. Degenerate cases

One degenerate case, which may arise in Step 3 of the main algorithm, occurs when the intersection $\mathcal{A}_+^{(D)} \cap \mathcal{A}_-^{(D)}$ is nonempty but has no interior, i.e., has zero volume. In this case, the algorithm described in [2] may be used to find a point $q = (q_1, q_2, q_3)$ in $\mathcal{A}_+^{(D)} \cap \mathcal{A}_-^{(D)}$. If $x = (x_1, x_2, x_3, x_4)$ is any solution to (3), then both $q_1x_1 + q_2x_2 + q_3x_3 + x_4 \geq 0$ and $-q_1x_1 - q_2x_2 + q_3x_3 - x_4 \geq 0$ must be satisfied. Hence, the set of solutions to (3) is constrained to the hyperplane $q_1x_1 + q_2x_2 + q_3x_3 + x_4 = 0$. Thus, we construct a rotation R^* such that $(q_1, q_2, q_3, 1)R^* = (0, 0, 0, K^*)$, where K^* is a positive constant.

Now, after R^* is applied, the set of inequalities $AR^*x' \geq 0$ imply $x_4 = 0$, so the fourth column of AR^* is superfluous. Thus, the set of inequalities $AR^*x' \geq 0$ may be replaced by a set having the form (1') whose algorithmic solution has already been described. Hence, to obtain the solution to (2) one need only rotate the result by $(R^*)^{-1}$.

Another degenerate case could arise in Step 1 of the main algorithm when there is no interior point u to the set of solutions to (1'), i.e. to $A_0x' \geq 0$. In this case, while seeking such an interior point, we either discover (a) that while no interior point exists there is a point $v = (v_1, v_2, v_3)$, with $v_3 = 1$ or $v_3 = -1$ which satisfies (1'), or (b) that the cone \mathcal{C}^3 fails to intersect either plane $z = +1$ or $z = -1$.

In case (a) the point v must lie in a plane through the origin which contains all solutions to (1'). Suppose, this plane has the equation $m_1x + m_2y + m_3z = 0$. We then choose R^* so that $(m_1, m_2, m_3, 0)R^* = (0, 0, 0, K^*)$ for some positive K^* and solve as in the previous case. The equation of the desired plane is obtained as a result of applying the algorithm due to Shamos and Hoey [5].

In case (b), the inequalities (1') imply that $z = 0$. Hence, the third column of A is superfluous. The set of inequalities $Ax' \geq 0$, given in (3), may be replaced by a set of the form (1') by deleting the third column. The solution to (1') is obtained as before.

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