LOGARITHMIC-EXPONENTIAL POWER SERIES

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Abstract

We use generalized power series to construct algebraically a nonstandard model of the theory of the real field with exponentiation. This model enables us to show the undefinability of the zeta function and certain non-elementary and improper integrals. We also use this model to answer a question of Hardy by showing that the compositional inverse to the function $(\log x)(\log\log x)$ is not asymptotic as $x \to +\infty$ to a composition of semialgebraic functions, log and exp.

1. *Introduction and preliminaries*

Let $\mathbb{R}\{X_1, \ldots, X_m\}$ denote the ring of all real power series in $X_1, \ldots, X_m$ that converge in a neighborhood of I^m, where $I = [-1, 1]$. For $f \in \mathbb{R}\{X_1, \ldots, X_m\}$ we let $\tilde{f}\colon \mathbb{R}^m \to \mathbb{R}$ be given by:

$$\tilde{f}(x) = \begin{cases} f(x), & \text{for } x \in I^m, \\ 0, & \text{for } x \notin I^m. \end{cases}$$

We call the maps $\tilde{f}$ *restricted analytic functions*. Let $\mathcal{L}_{\mathrm{an}}$ be the language of ordered rings $\{<, 0, 1, +, -, \cdot\}$ augmented by a new function symbol for each function $\tilde{f}$. We let $\mathbb{R}_{\mathrm{an}}$ be the reals with its natural $\mathcal{L}_{\mathrm{an}}$-structure. Let $\mathcal{L}_{\mathrm{an,exp}} = \mathcal{L}_{\mathrm{an}} \cup \{\exp\}$, where exp is a new unary function symbol, and let $\mathbb{R}_{\mathrm{an,exp}}$ be $\mathbb{R}_{\mathrm{an}}$ with exponentiation. Let T_{an} and $T_{\mathrm{an,exp}}$ denote the first order theories of $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\mathrm{an,exp}}$ respectively.

The theory $T_{\mathrm{an,exp}}$ was studied extensively in [**6**]. In this paper we shall show that generalized power series can be used to give an algebraic construction of nonstandard models of $T_{\mathrm{an,exp}}$. These models have surprising applications. In §4 we shall show that the compositional inverse to $x \mapsto (\log x)(\log\log x)$ is not asymptotic to a composition of semialgebraic functions, log and exp. This answers a question of Hardy posed in [**9**]. In §5 we shall show that functions on $(0, +\infty)$, including $\Gamma(x)$, $\int_0^x e^{t^2}\,dt$, and $\int_0^\infty e^{-t}(t+x)^{-1}\,dt$, are not definable in $\mathbb{R}_{\mathrm{an,exp}}$. By different methods we shall also show that the Riemann zeta function restricted to $(1, +\infty)$ is not definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Our arguments are motivated by the following theorem from [**6**].

THEOREM 1.1. *The theory $T_{\mathrm{an,exp}}$ is* o*-minimal and is axiomatized by T_{an} and the following axioms;*

(E1) $\exp(x+y) = \exp(x)\exp(y)$,

(E2) $x < y \to \exp(x) < \exp(y)$,

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(E3) $x > 0 \to \exists y \exp(y) = x$,
(E4$_n$) $x > n^2 \to \exp(x) > x^n$ *for each natural number* $n > 0$;
(E5) $-1 \leqslant x \leqslant 1 \to \exp(x) = e(x)$; *where* e *is the function symbol of* $\mathscr{L}_{\mathrm{an}}$ *representing the restricted analytic function arising from the power series* $\sum (1/n!) X^n \in \mathbb{R}\{X\}$.

Thus, to build a nonstandard model of $T_{\mathrm{an,exp}}$, one need only build a nonstandard model of T_{an} on which there is an exponential function satisfying (E1)–(E5). While building models of T_{an} may, at first, seem a daunting task, in **[6]** we showed that generalized power series fields naturally yield nonstandard models of T_{an}.

(1.2) Let Γ be an ordered abelian group. We consider formal sums

$$f = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma,$$

where $a_\gamma \in \mathbb{R}$. Let $\operatorname{supp} f = \{\gamma \in \Gamma : a_\gamma \neq 0\}$ and let

$$\mathbb{R}((t^\Gamma)) = \{f = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma : \operatorname{supp} f \text{ is well ordered}\}.$$

The usual definitions of $+$ and $\cdot$ make $\mathbb{R}((t^\Gamma))$ into a field, which is real closed if Γ is divisible (see [**18**, **16**]). We order $\mathbb{R}((t^\Gamma))$ by setting $f = \sum a_\gamma t^\gamma > 0$ if and only if $f \neq 0$ and $a_\delta > 0$, where $\delta = \min(\operatorname{supp} f)$. We equip $\mathbb{R}((t^\Gamma))$ with the valuation $v : \mathbb{R}((t^\Gamma))^\times \to \Gamma$ given by $v(f) = \min(\operatorname{supp} f)$. The infinitesimals of the ordered field $\mathbb{R}((t^\Gamma))$ are exactly the elements of the maximal ideal $\{x : v(x) > 0\}$ of the valuation ring $\{x : v(x) \geqslant 0\}$.

The following lemma will be useful in analyzing generalized power series fields.

LEMMA 1.3 [**14**]. *Let G be an ordered abelian group and let $S \subset G$ be well ordered with $s > 0$ for all $s \in S$. Then for all $g \in G$, the set*

$$S_g = \{(h_1, \ldots, h_n) : n \in \mathbb{N},\ h_i \in S,\ \sum h_i = g\}$$

is finite and $\{g : S_g \neq \varnothing\}$ is well ordered.

(1.4). There is a natural way to equip $\mathbb{R}((t^\Gamma))$ with an $\mathscr{L}_{\mathrm{an}}$-structure. Let $X = (X_1, \ldots, X_n)$ and let $g \in \mathbb{R}[[X]]$, say

$$g(X) = \sum_{\nu \in \mathbb{N}^m} a_\nu X^\nu.$$

Let $\mu^n = \{x \in \mathbb{R}((t^\Gamma))^n : v(x_i) > 0 \text{ for } i = 1, \ldots, n\}$. Define $\hat{g} : \mu^n \to \mathbb{R}((t^\Gamma))$ by

$$\hat{g}(x) = \sum a_\nu x^\nu.$$

It follows from Lemma 1.3 that this last sum is a well defined element of $\mathbb{R}((t^\Gamma))$ (for $n = 1$ this is proved in [**8**]).

Let $f \in \mathbb{R}\{X_1, \ldots, X_n\}$. We wish to interpret $\tilde{f}$ on $\mathbb{R}((t^\Gamma))^n$. For $z \in \mathbb{R}((t^\Gamma))^n$ with $-1 \leqslant z_i \leqslant 1$ for all i, there is some $a \in [-1, 1]^n \cap \mathbb{R}^n$ such that $v(z_i - a_i) > 0$ for all i.

Take the convergent power series $g_a(X_1, \ldots, X_n)$ over $\mathbb{R}$ such that for some $\epsilon > 0$, if $\|x-a\| < \epsilon$, then $f(x) = g_a(x-a)$. We define

$$\tilde{f}(z) = \widehat{g_a}(z-a).$$

In this way we equip $\mathbb{R}((t^\Gamma))$ with a natural $\mathscr{L}_{\mathrm{an}}$-structure.

THEOREM 1.5 [**6**]. *Let Γ_0 and Γ_1 be divisible ordered abelian groups with $\Gamma_0 \subseteq \Gamma_1$. Consider $\mathbb{R}((t^{\Gamma_i}))$ with its natural $\mathscr{L}_{\mathrm{an}}$-structure. Then $\mathbb{R}((t^{\Gamma_i}) \models T_{\mathrm{an}}$ and $\mathbb{R}_{\mathrm{an}} \preccurlyeq \mathbb{R}((t^{\Gamma_0})) \preccurlyeq \mathbb{R}((t^{\Gamma_1}))$.*

We shall construct a nonstandard model $\mathbb{R}((t))^{\mathrm{LE}}$ of $T_{\mathrm{an,\,exp}}$. We begin by constructing $\mathbb{R}((t))^{\mathrm{E}}$, which will be a model of T_{an}, (E1), (E2), (E4) and (E5). It will be built as a union $\bigcup_{n\in\mathbb{N}} \mathbb{R}((t^{\Gamma_n}))$, where $\Gamma_0 \subset \Gamma_1 \subset \cdots$ is an increasing chain of divisible ordered abelian groups. Theorem 1.5 will ensure that $\mathbb{R}((t))^{\mathrm{E}} \models T_{\mathrm{an}}$ and we shall define an exponential E so that (E1), (E2), (E4), and (E5) are satisfied. This step of the construction is due to Dahn [**3**]. The second step of the construction is to extend $\mathbb{R}((t))^{\mathrm{E}}$ to $\mathbb{R}((t))^{\mathrm{LE}}$ by adding logarithms. Our methods here considerably simplify earlier work of Dahn and Göring [**4**]. This construction is carried out in §2.

In §3 we prove some technical results about truncations of series. These results are crucial for the applications in §4 and §5. In §4 we answer Hardy's question, while in §5 we prove the undefinability of certain natural integrals and the zeta function on $(1, +\infty)$.

In his work on the finiteness of limit cycles for plane polynomial vector fields, Ecalle [**7**] introduced a field of logarithmic–exponential series which he calls the 'trigèbre $\mathbb{R}[[[x]]]$ des transséries'. His field is a proper subfield of $\mathbb{R}((t))^{\mathrm{LE}}$. We shall briefly discuss the relationship between $\mathbb{R}((t))^{\mathrm{LE}}$ and $\mathbb{R}[[[x]]]$ at the end of §5.

The exponential field $\mathbb{R}((t))^{\mathrm{LE}}$ has many interesting properties which we intend to explore in future papers.

2. *Logarithmic–exponential series*

In this section we construct the field $\mathbb{R}((t))^{\mathrm{LE}}$ of logarithmic–exponential series. We start by constructing the smaller field $\mathbb{R}((t))^{\mathrm{E}}$ of exponential series. We call a map $E: A \to F$ from an ordered additive group into an ordered field an *exponential* if $E(x+y) = E(x)E(y)$ and $E(x) > 0$ for all $x, y \in A$. Given subsets X, Y of a linearly ordered set A and $a \in A$, we write $X < Y$ if $x < y$ for all $x \in X$ and $y \in Y$, and $X < a$ (or $a > X$) if $x < a$ for all $x \in X$.

(2.1) *Dahn's construction of* $\mathbb{R}((t))^{\mathrm{E}}$.

We build a chain $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots$ of ordered abelian groups and, at the same time, a corresponding chain $K_0 \subset K_1 \subset K_2 \subset \cdots$ of ordered fields, such that for each n we have $K_n = \mathbb{R}((t^{\Gamma_n}))$ and K_n is equipped with an exponential $E_n: K_n \to K_{n+1}$ with E_{n+1} extending E_n. Then we set $\Gamma = \bigcup \Gamma_n$, $\mathbb{R}((t))^{\mathrm{E}} = \bigcup K_n$, and $E = \bigcup E_n$.

To start, we let $\Gamma_{-1} = \{0\}$, $K_{-1} = \mathbb{R}$, $\Gamma_0 = \mathbb{R}$, and $K_0 = \mathbb{R}((t^{\Gamma_0}))$ with exponential $E_{-1}: K_{-1} \to K_0$ given by $E_{-1}(r) = \mathrm{e}^r$.

Let $n > 0$ and assume inductively that we have constructed ordered abelian groups $\Gamma_{n-1} \subset \Gamma_n$ with Γ_{n-1} convex in Γ_n, and corresponding ordered fields $K_{n-1} \subset K_n$, where $K_i = \mathbb{R}((t^{\Gamma_i}))$, together with an exponential $E_{n-1}: K_{n-1} \to K_n$. (We have just done this for $n = 0$.) Then we let $O_n = \{x \in K_n : v(x) \geqslant \gamma$ for some $\gamma \in \Gamma_{n-1}\}$, and $\mathscr{M}_n = \{x \in K_n : v(x) > \Gamma_{n-1}\}$, and we note that O_n is a valuation ring of K_n with maximal ideal $\mathscr{M}_n$ and that $O_n = K_{n-1} \oplus \mathscr{M}_n$. We extend E_{n-1} to an exponential $\hat{E}_n : O_n \to K_n$ by

$$\hat{E}_n(x) = E_{n-1}(r) \sum_{i=0}^{\infty} \frac{\alpha^i}{i!},$$

where $x = r + \alpha$, $r \in K_{n-1}$, $\alpha \in \mathscr{M}_n$.

Let $J_n = \{x \in K_n : \operatorname{supp} x < \Gamma_{n-1}\}$ so that $K_n = J_n \oplus O_n$ as K_{n-1}-linear spaces. We now make the additional inductive assumption that Γ_n is given as an ordered $\mathbb{R}$-linear subspace of K_{n-1}. (Strange as this assumption may look at first sight, note that it is actually satisfied for $n = 0$, where $\Gamma_0 = \mathbb{R} = K_{-1}$.) Then we let $\Gamma_{n+1} = J_n \oplus \Gamma_n \subseteq K_n$, ordered as an $\mathbb{R}$-linear subspace of K_n, so Γ_n is convex in Γ_{n+1}. Finally, put $K_{n+1} = \mathbb{R}((t^{\Gamma_{n+1}}))$, and extend $\hat{E}_n$ to the exponential $E_n : K_n \to K_{n+1}$ given by

$$E_n(x) = t^{-a}\, \hat{E}_n(b) \quad \text{for } x = a + b \quad \text{with } a \in J_n, b \in O_n.$$

Clearly the inductive assumptions carry over from stage n to stage $n+1$. This finishes the inductive construction.

This gives the field $\mathbb{R}((t))^{\mathrm{E}}$ with exponential $E : \mathbb{R}((t))^{\mathrm{E}} \to \mathbb{R}((t))^{\mathrm{E}}$ as indicated above. We consider $\mathbb{R}((t))^{\mathrm{E}}$ as an ordered and valued subfield of $\mathbb{R}((t^{\Gamma}))$, where $\Gamma = \bigcup \Gamma_n$. Our construction gives $\Gamma = J \oplus \mathbb{R}$, where

$$J = J_0 \oplus J_1 \oplus \cdots = \{x \in \mathbb{R}((t))^{\mathrm{E}} : \operatorname{supp} x < 0\}.$$

Note also the canonical decomposition $\mathbb{R}((t))^{\mathrm{E}} = J \oplus \mathbb{R} \oplus \mathscr{M}$, where

$$\mathscr{M} = \{x \in K : v(x) > 0\}$$

is the maximal ideal of the valuation ring $\mathbb{R} \oplus \mathscr{M}$ of $\mathbb{R}((t))^{\mathrm{E}}$. If $x = \alpha + r + \epsilon$, where $\alpha \in J$, $r \in \mathbb{R}$ and $\epsilon \in \mathscr{M}$, then

$$E(x) = t^{-\alpha}\, \mathrm{e}^r \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!}.$$

We view the rational function field $\mathbb{R}(t)$ as a subfield of $\mathbb{R}((t))^{\mathrm{E}}$ by identifying t with the element $t^1 \in \mathbb{R}((t^{\mathbb{R}})) = K_0$.

It is clear from the construction that $\mathbb{R}((t))^{\mathrm{E}}$ satisfies (E1), (E2), (E4) and (E5). Moreover, by Theorem 1.5, $\mathbb{R}((t))^{\mathrm{E}}$ is an $\mathscr{L}_{\mathrm{an}}$-elementary extension of each K_n.

(2.2) *The construction of* $\mathbb{R}((t))^{\mathrm{LE}}$.

It is easy to see that, for instance, the element $x = t^{-1} \in \mathbb{R}((t))^{\mathrm{E}}$ has no logarithm in $\mathbb{R}((t))^{\mathrm{E}}$, that is, $x \neq E(y)$ for all $y \in \mathbb{R}((t))^{\mathrm{E}}$. To remove this defect we extend $\mathbb{R}((t))^{\mathrm{E}}$ to an ordered exponential field $\mathbb{R}((t))^{\mathrm{LE}}$ in which every positive element does have a logarithm. We keep the notation of (2.1).

We begin by inductively defining ordered group embeddings $\Psi_n : \Gamma_n \to \Gamma_{n+1}$ and corresponding ordered field embeddings $\Phi_n : K_n \to K_{n+1}$ given by

$$\Phi_n\Big(\sum_{\gamma \in \Gamma_n} a_\gamma t^\gamma\Big) = \sum_{\gamma \in \Gamma_n} a_\gamma t^{\Psi_n(\gamma)} \tag{$*$}$$

with $\Psi_n \subset \Psi_{n+1}$ and hence $\Phi_n \subset \Phi_{n+1}$. This gives an ordered field embedding $\Phi = \bigcup \Phi_n : \mathbb{R}((t))^{\mathrm{E}} \to \mathbb{R}((t))^{\mathrm{E}}$. Let $x = t^{-1} \in \mathbb{R}((t))^{\mathrm{E}}$. We can think of Φ as 'substituting $E(x)$ for x' in every series $f(t) = f(1/x) \in \mathbb{R}((t))^{\mathrm{E}}$. This idea motivates our construction.

Define $\Psi_0: \mathbb{R} = \Gamma_0 \to \Gamma_1$ by $\Psi_0(r) = rx$, and define Φ_0 by $(*)$ above. Clearly Ψ_0 is an ordered group embedding, Φ_0 is an ordered field embedding, and $\Phi_0(J_0) \subseteq J_1$.

Assume inductively that we have an ordered group embedding $\Psi_n: \Gamma_n \to \Gamma_{n+1}$ and a corresponding ordered field embedding $\Phi_n: K_n \to K_{n+1}$ given by $(*)$ above, and that $\Phi_n(J_n) \subseteq J_{n+1}$. Using $\Gamma_{n+1} = J_n \oplus \Gamma_n$, we extend Ψ_n to $\Psi_{n+1}: \Gamma_{n+1} \to \Gamma_{n+2}$ by setting $\Psi_{n+1}(\gamma) = \Phi_n(\alpha) + \Psi_n(\beta)$ for $\gamma = \alpha + \beta$ with $\alpha \in J_n$, $\beta \in \Gamma_n$. Since Γ_n is convex in Γ_{n+1}, it follows easily that Ψ_{n+1} is again an ordered group embedding. Then $(*)$ gives the corresponding ordered field embedding $\Phi_{n+1}: K_{n+1} \to K_{n+2}$. To show that the inductive assumptions carry over it remains to establish the following inclusion.

(2.3) $\Phi_{n+1}(J_{n+1}) \subseteq J_{n+2}$.

To see that, let $a \in J_{n+1}$ and $\gamma \in \operatorname{supp} a$. By the definition of J_{n+2} and $(*)$ it suffices to show that $\Psi_{n+1}(\gamma) < \Gamma_{n+1}$. Since $\gamma \in \Gamma_{n+1}$ and $\gamma < \Gamma_n$, we have $\gamma = \alpha + \beta$ for $\alpha \in J_n$, $\beta \in \Gamma_n$, $\alpha < 0$. Then $\Psi_{n+1}(\gamma) = \Phi_n(\alpha) + \Psi_n(\beta)$ with $\Psi_n(\beta) \in \Gamma_{n+1}$, and $\Phi_n(\alpha) \in J_{n+1}$ with $\Phi_n(\alpha) < 0$ (using $\Phi_n(J_n) \subseteq J_{n+1}$). Since Γ_{n+1} is convex in $\Gamma_{n+2} = J_{n+2} \oplus \Gamma_{n+1}$, it follows that $\Psi_{n+1}(\gamma) < \Gamma_{n+1}$, as desired.

This finishes the construction of the Ψ_n and Φ_n and gives us the ordered group embedding $\Psi = \bigcup \Psi_n: \Gamma \to \Gamma$ and the ordered field embedding $\Phi = \bigcup \Phi_n: \mathbb{R}((t))^{\mathrm{E}} \to \mathbb{R}((t))^{\mathrm{E}}$. Note that $\Phi(x) = t^{-x} = E(x)$ for $x = t^{-1}$. We also remark that an easy induction shows that $\Psi(\alpha) = \Phi(\alpha)$ for $\alpha \in J$.

The next lemmas will show that Φ is an $\mathscr{L}_{\mathrm{an,exp}}$-embedding of $\mathbb{R}((t))^{\mathrm{E}}$ into itself such that every positive element of $\Phi(\mathbb{R}((t))^{\mathrm{E}})$ has a logarithm in $\mathbb{R}((t))^{\mathrm{E}}$.

LEMMA 2.4. *The embedding Φ is an $\mathscr{L}_{\mathrm{an}}$-embedding.*

Proof. It follows easily from the definitions that if $\sigma: G \to H$ is an ordered abelian group embedding, then the map $\hat{\sigma}: \mathbb{R}((t^G)) \to \mathbb{R}((t^H))$ given by $\hat{\sigma}(\sum a_g t^g) = \sum a_g t^{\sigma(g)}$ is an $\mathscr{L}_{\mathrm{an}}$-embedding. Thus each Φ_n is an $\mathscr{L}_{\mathrm{an}}$-embedding.

LEMMA 2.5. *The embedding Φ is a $\mathscr{L}_{\mathrm{an,exp}}$-embedding.*

Proof. We claim that $\Phi(E(z)) = E(\Phi(z))$ for all $z \in \mathbb{R}((t))^{\mathrm{E}}$. This is clear for $z \in O_0$ by Lemma 2.4.

First, we suppose that the claim is true for all $z \in O_n$. Let $z \in K_n$ be such that $z = \alpha + \beta$, where $\alpha \in J_n$ and $\beta \in O_n$; then $E(\alpha+\beta) = t^{-\alpha} E(\beta)$ and

$$\Phi(E(z)) = t^{-\Psi(\alpha)}\,\Phi(E(\beta)) = E(\Phi(\alpha) + \Phi(\beta)),$$

since $\Psi(\alpha) = \Phi(\alpha) \in J_{n+1}$ and $\Phi(\beta) \in O_{n+1}$.

Next, suppose that the claim is true for all $z \in K_n$. Let $z \in O_{n+1}$ be such that $z = \alpha + \beta$, where $\alpha \in K_n$ and $v(\beta) > \Gamma_n$; then

$$\Phi_n(E(\alpha+\beta)) = \Phi(E(\alpha))\,\Phi\left(\sum \frac{\beta^i}{i!}\right) = E(\Phi(\alpha))\,E(\Phi(\beta)),$$

by Lemma 2.4 and the induction hypothesis.

LEMMA 2.6. *Every positive element z of $\Phi(\mathbb{R}((t))^{\mathrm{E}})$ has a logarithm in $\mathbb{R}((t))^{\mathrm{E}}$; that is, $z = E(y)$ for some $y \in \mathbb{R}((t))^{\mathrm{E}}$.*

Proof. Let $z \in \Phi(\mathbb{R}((t))^{\mathrm{E}})$ with $z > 0$; then $z = at^{\Psi(\alpha)+rx}(1+\epsilon)$, where $a \in \mathbb{R}$, $a > 0$, $\alpha \in J$, $r \in \mathbb{R}$, and $v(\epsilon) > 0$. It is easy to see that z has a logarithm

$$\log a - \Psi(\alpha) - rx + \log(1+\epsilon)$$

in $\mathbb{R}((t))^{\mathrm{E}}$, where $\log(1+\epsilon)$ is obtained by applying the power series for $\log(1+X)$ to the infinitesimal ϵ.

(2.7) We construct an increasing chain of $\mathscr{L}_{\mathrm{an,exp}}$-structures $L_0 \subset L_1 \subset L_2 \subset \cdots$ and isomorphisms η_i from L_i onto $\mathbb{R}((t))^{\mathrm{E}}$. Let $L_0 = \mathbb{R}((t))^{\mathrm{E}}$ and let η_0 be the identity. Given L_i and η_i, take $L_{i+1} \supset L_i$ and an $\mathscr{L}_{\mathrm{an,exp}}$-isomorphism $\eta_{i+1}\colon L_{i+1} \to \mathbb{R}((t))^{\mathrm{E}}$ such that $\eta_{i+1}(z) = \Phi(\eta_i(z))$ for all $z \in L_i$. So L_i sits inside L_{i+1} as $\Phi(\mathbb{R}((t))^{\mathrm{E}})$ sits in $\mathbb{R}((t))^{\mathrm{E}}$.

Let $\mathbb{R}((t))^{\mathrm{LE}} = \bigcup L_i$.

COROLLARY 2.8. $\mathbb{R}((t))^{\mathrm{LE}} \models T_{\mathrm{an,exp}}$.

Proof. By Lemma 2.5 each L_i is an $\mathscr{L}_{\mathrm{an,exp}}$-substructure of L_{i+1}. By Proposition 2.9 of [**6**], L_{i+1} is an $\mathscr{L}_{\mathrm{an}}$-elementary extension of $\mathbb{R}_{\mathrm{an}}$. By Lemma 2.6 every positive element of L_i has a logarithm in L_{i+1}. Thus $\mathbb{R}((t))^{\mathrm{LE}} \models$ (E1)–(E5). Hence, by Theorem 1.1, $\mathbb{R}((t))^{\mathrm{LE}} \models T_{\mathrm{an,exp}}$.

For each $n \in \mathbb{N}$, we define $e_n\colon \mathbb{R} \to \mathbb{R}$ by $e_0(z) = z$ and $e_{n+1}(z) = \exp(e_n(z))$.

(2.9) Let $l_0 = x = t^{-1}$, $l_{n+1} = \log l_n$, and $t_n = l_n^{-1}$. The isomorphism η_n from L_n onto $\mathbb{R}((t))^{\mathrm{E}}$ maps t_n to t. Thus we can view L_n as a field of generalized power series in the variable t_n. To be precise, we view L_n as a subfield of $\mathbb{R}((t_n^{\Gamma}))$, in such a way that η_n is a restriction of the isomorphism from $\mathbb{R}((t_n^{\Gamma}))$ to $\mathbb{R}((t^{\Gamma}))$ given by $\sum a_\gamma t_n^\gamma \mapsto \sum a_\gamma t^\gamma$.

Since the sequence $x, E(x), E(E(x)), \ldots$ is cofinal in $\mathbb{R}((t))^{\mathrm{E}}$, the sequence $l_n, l_{n-1}, \ldots, l_1, x, E(x), E(E(x)), \ldots$ is cofinal in L_n. Thus $x, E(x), E(E(x)), \ldots$ is cofinal in $\mathbb{R}((t))^{\mathrm{LE}}$.

COROLLARY 2.10. *If $f\colon \mathbb{R} \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$, then there are $M, n \in \mathbb{N}$ such that $f(z) < e_n(z)$ for $z > M$.*

Proof. Since $T_{\mathrm{an,exp}}$ is o-minimal, it suffices to show that $f(z) < e_n(z)$ for some $z > \mathbb{R}$ in $\mathbb{R}((t))^{\mathrm{E}}$. Since $x, E(x), E(E(x)), \ldots$ is cofinal in $\mathbb{R}((t))^{\mathrm{LE}}$, then $f(x) < e_n(x)$ for some $n \in \mathbb{N}$.

For functions definable in $\mathbb{R}_{\mathrm{exp}}$ this was shown in [**5**, Proposition 9.2] by another method. Actually, Proposition 9.10 of [**5**] states Corollary 2.10, but the proof indicated there is incorrect.

A strengthening of Corollary 2.10 will be given in [**11**].

(2.11). All of the constructions and results of this chapter could be carried out starting with an arbitrary $k \models T_{\mathrm{an,exp}}$ instead of $\mathbb{R}$. The resulting structure $k((t))^{\mathrm{LE}}$ would be an elementary extension of k in which the sequence $x, E(x), E(E(x)), \ldots$ is cofinal. It follows that any definable $f\colon k \to k$ is bounded by an iterated exponential. A simple compactness argument then shows that if $f\colon \mathbb{R}^{m+1} \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$, then there is an $n \in \mathbb{N}$ such that for all $a \in \mathbb{R}^m$ we have $f(a, x) < e_n(x)$ for sufficiently large x.

(2.12) Finally, we add the word of warning that the monomial t^{γ} with $\gamma \in \Gamma$, is not, in general, equal to $E(\gamma \log t)$.

3. *Truncation*

We work in a field $K = \mathbb{R}((t^G))$ of generalized power series, where G is a divisible ordered abelian group. Thus K is real closed. Let $x \in K$, $x = \sum a_g t^g$. We say that $y \in K$ is a *truncation* of x if there is an $h \in G$ such that $y = \sum_{g<h} a_g t^g$. We write $y \trianglelefteq x$ if y is a truncation of x and $y \triangleleft x$ if $y \trianglelefteq x$ and $y \neq x$.

If $F \subseteq K$ we say that F is *truncation closed* if $y \in F$ whenever $x \in F$ and $y \trianglelefteq x$.

Mourgues and Ressayre introduced the notion of truncation in studying models of arithmetic associated to real closed fields. Ressayre [**17**] then gave a stunning application of this method to the study of exponentiation. While it later turned out that truncation is not actually necessary to prove Ressayre's results on exponentiation (see for example [**6**]), truncation is essential for the arguments given later in §4 and §5. Throughout this section F will denote a subfield of K containing $\mathbb{R}$.

The following results from [**13**] will be of great use here.

LEMMA 3.1. (a) *Suppose that F is truncation closed, $x \in K$, and $y \in F$ for all $y \triangleleft x$. Then $F(x)$ is truncation closed.*

(b) *If F is truncation closed, then the real closure of F in K is also truncation closed.*

Our first goal will be to extend Lemma 3.1(b) to allow closures under more functions. Two settings will be of particular interest.

(3.2.) DEFINITION. The *restricted* LE-*closure* of F is the smallest among the real closed subfields R of K containing F such that $\exp x \in R$ and $\log(1+x) \in R$ for infinitesimal $x \in R$.

The T_{an}-*closure* of F is the smallest among the real closed subfields R of K containing F such that $f(x_1, \dots, x_n) \in R$ for every convergent power series $f(X_1, \dots, X_n)$ and all infinitesimal $x_1, \dots, x_n \in R$. This is also the smallest $\mathscr{L}_{\mathrm{an}}$-elementary submodel of K containing F.

We shall show that if F is truncation closed then the restricted LE-closure and the T_{an}-closure are closed under truncation. We begin by giving a more general setting for the proof.

Let $\mathscr{F}_n$ be a subset of $\mathbb{R}[[X_1, \dots, X_n]]$ for each n, such that the subring $\mathbb{R}[X_1, \dots, X_n, \mathscr{F}_n]$ is closed under $\partial/\partial X_i$ for $1 \leqslant i \leqslant n$. Let $F = \bigcup \mathscr{F}_n$.

For our applications we deal with $\mathscr{F}_{\mathrm{LE}} = \{1/(1+X), \exp X, \log(1+X)\}$ and $\mathscr{F}_{\mathrm{an}} = \{f(X_1, \dots, X_n) : f$ converges on some neighborhood of the origin, $n \in \mathbb{N}\}$.

We say that $F \subseteq K$ is $\mathscr{F}$-*closed* if F is real closed and $f(x_1, \dots, x_n) \in F$ whenever $n \in \mathbb{N}$, $f \in \mathscr{F}_n$ and $x_1, \dots, x_n \in F$ are infinitesimal. The $\mathscr{F}$-*closure* $\hat{F}$ of F is the smallest $\mathscr{F}$-closed subfield of K containing F. Clearly the restricted LE-closure and the T_{an}-closure of F are the $\mathscr{F}_{\mathrm{LE}}$ and the $\mathscr{F}_{\mathrm{an}}$-closures, respectively.

We shall show that if F is truncation closed, then the $\mathscr{F}$-closure $\hat{F}$ of F is truncation closed. This type of result was proved independently by Mourgues in [**12**].

LEMMA 3.3. *If F is truncation closed, then the $\mathscr{F}$-closure $\hat{F}$ of F is truncation closed.*

Proof. By Zorn's lemma there is a maximal truncation closed F^* such that $F \subseteq F^* \subseteq \hat{F}$. By Lemma 3.1(b), F^* is real closed. We claim that F^* is closed under application of power series in $\mathscr{F}$ to infinitesimals. Suppose not. Let n be the least integer such that there are $f \in \mathscr{F}_n$ and infinitesimal $x_1, \ldots, x_n \in F^*$ with $f(x_1, \ldots, x_n) \notin F^*$. For notational simplicity we shall assume that $n = 2$; the general case is similar.

Using the fact that supports are well ordered, we can find $f \in \mathscr{F}_2$ and infinitesimal x and y in F^* such that:

(i) $f(x, y) \notin F^*$,

(ii) for all $x' \lhd x, g \in \mathscr{F}_2$ and infinitesimal $y' \in F^*$, we have $g(x', y') \in F^*$, and

(iii) for all $y' \lhd y$ and $g \in \mathscr{F}_2$, we have $g(x, y') \in F^*$.

We shall show that $z \in F^*$ for all $z \lhd f(x, y)$. Thus, by Lemma 3.1(a), $F^*(f(x, y))$ is closed under truncation, contradicting the minimality of F^*.

Let $z \lhd f(x, y)$. Since one of x and y must be nonzero, we may take $\gamma \in \operatorname{supp} x \cup \operatorname{supp} y$ and a natural number M such that $M\gamma > \operatorname{supp} z$. We shall assume that $\gamma \in \operatorname{supp} y$; the other case is similar. Let $y = y_0 + \epsilon$, where y_0, $\epsilon \in F^*$, $v(\epsilon) = \gamma$ and $\operatorname{supp} y_0 < \gamma$ (note that we allow the possibility that $y_0 = 0$).

Consider the Taylor expansion

$$f(x, y) = f(x + 0, y_0 + \epsilon) = \sum_{n=0}^{\infty} \frac{\partial^n f}{\partial Y^n}(x, y_0) \frac{\epsilon^n}{n!}.$$

From $v((\partial^n f / \partial Y^n)(x, y_0)) \geqslant 0$, for $n \geqslant M$ we obtain:

$$v\left(\frac{\partial^n f}{\partial Y^n}(x, y_0) \frac{\epsilon^n}{n!}\right) \geqslant n\gamma \geqslant M\gamma > \operatorname{supp} z.$$

Thus z is a truncation of w, where

$$w = \sum_{n=0}^{M-1} \frac{\partial^n f}{\partial Y^n}(x, y_0) \frac{\epsilon^n}{n!}.$$

Since $y_0 \lhd y$, assumption (iii) guarantees that $(\partial^n f / \partial Y^n)(x, y_0) \in F^*$ and $w \in F^*$. Since F^* is closed under truncation, $z \in F^*$ as desired.

We next consider subfields of the exponential series field $\mathbb{R}((t))^{\mathrm{E}}$. Here we consider $\mathbb{R}((t))^{\mathrm{E}}$ as a truncation closed subfield of $\mathbb{R}((t^{\Gamma}))$, where $\Gamma = \bigcup \Gamma_n$, as in (2.1).

LEMMA 3.4. *Let $\mathscr{F}$ be as above with $\exp(X) \in \mathscr{F}$. Let $F \subseteq \mathbb{R}((t))^{\mathrm{E}}$ be truncation closed with $\mathbb{R} \subseteq F$, and let $\hat{F}$ be the smallest $\mathscr{F}$-closed subfield of $\mathbb{R}((t))^{\mathrm{E}}$ containing F and closed under E. Then $\hat{F}$ is truncation closed.*

Proof. Suppose that $F^* \subseteq \hat{F}$ is maximal such that $F \subseteq F^*$ and F^* is closed under truncation. By Lemma 3.3, F^* is $\mathscr{F}$-closed. Suppose that $x \in F^*$ and $E(x) \notin F^*$. Let $x = \alpha + r + \epsilon$, where α, $\epsilon \in F^*$, $r \in \mathbb{R}$, $\operatorname{supp} \alpha < 0$, and $v(\epsilon) > 0$. Then $E(x) = t^{-\alpha} \mathrm{e}^r \sum \epsilon^n / n!$. Since F^* is $\mathscr{F}$-closed, $\sum \epsilon^n / n! \in F^*$. Thus $E(x) \in F^*(t^{-\alpha})$, which, by Lemma 3.1(a), is truncation closed. Since $F^*(t^{\alpha}) \subseteq \hat{F}$, this contradicts the maximality of F^*.

We shall also need the following additional information.

LEMMA 3.5. *Let $\mathscr{F}$, F, and $\hat{F}$ be as in Lemma 3.4. Suppose that $\gamma \in F$, whenever $\gamma \in \Gamma$ and $t^{\gamma} \in F$. Then $\hat{F}$ also has this property.*

Proof. We start with an observation about truncation closed subfields L of $\mathbb{R}((t))^{\mathrm{E}}$ containing $\mathbb{R}$. Since the $\mathscr{F}$-closure $\hat{L}$ of L is also truncation closed with value group the divisible hull of the value group of L, it follows that if L has the property that $\gamma \in L$ whenever $\gamma \in \Gamma$ and $t^{\gamma} \in L$, then $\hat{L}$ has this property as well.

The argument from Lemma 3.4 shows that we can build $\hat{F}$ as the union of a chain $(F_{\alpha}: \alpha < \lambda)$, where

F_0 is the $\mathscr{F}$-closure of F,

$F_{\alpha} = \bigcup_{\beta < \alpha} F_{\beta}$ for α a nonzero limit ordinal, and

$F_{\alpha+1}$ is the $\mathscr{F}$-closure of $F_{\alpha}(t^{\gamma})$ for some $\gamma \in \Gamma \cap F_{\alpha}$.

It then follows from the above remark that $\gamma \in \hat{F}$ whenever $\gamma \in \Gamma$ and $t^{\gamma} \in \hat{F}$.

(3.6) We next consider subfields of $\mathbb{R}((t))^{\mathrm{LE}}$. We say that a subfield $F \subseteq \mathbb{R}((t))^{\mathrm{LE}}$ is truncation closed if $\eta_i(F \cap L_i)$ is truncation closed in $\mathbb{R}((t))^{\mathrm{E}}$ for all i, where η_i is as in (2.7). For $y, z \in \mathbb{R}((t))^{\mathrm{LE}}$ we write $y \trianglelefteq z$ if $\eta_i(y) \trianglelefteq \eta_i(z)$, where i is the least integer such that $y, z \in L_i$. Note that then $\eta_j(y) \trianglelefteq \eta_j(z)$ for all j with $y, z \in L_j$. (To see this, use the fact that for $y, z \in \mathbb{R}((t))^{\mathrm{E}}$ we have $y \trianglelefteq z$ if and only if $\Phi(y) \trianglelefteq \Phi(z)$, which is easily established.) With this notation it is now clear that a field $F \subseteq \mathbb{R}((t))^{\mathrm{LE}}$ is truncation closed if $y \in F$ whenever $y \trianglelefteq z$ and $z \in F$.

We also say that a subfield F of $\mathbb{R}((t))^{\mathrm{LE}}$ is $\mathscr{F}$-closed if $\eta_i(F \cap L_i)$ is $\mathscr{F}$-closed in $\mathbb{R}((t))^{\mathrm{E}}$ for all i. For subfields of $\mathbb{R}((t))^{\mathrm{E}}$ this agrees with the notion of being $\mathscr{F}$-closed given above. It now follows easily that a subfield of L_n is $\mathscr{F}$-closed if and only if $\eta_n(F)$ is $\mathscr{F}$-closed. In particular, L_n itself is $\mathscr{F}$-closed.

LEMMA 3.7. *Suppose that $\mathscr{F}_{\mathrm{LE}} \subseteq \mathscr{F}$. Let $\hat{F}$ be the smallest $\mathscr{F}$-closed subfield of $\mathbb{R}((t))^{\mathrm{LE}}$ containing $\mathbb{R}(t)$ that is closed under* log *and E. Then $\hat{F}$ is closed under truncation.*

Proof. Let t_n be as in (2.9). Let F_n be the smallest $\mathscr{F}$-closed subfield of L_n that contains $\mathbb{R}(t_n)$ and is also closed under E. Note that $\eta_n(F_n) = F_0$. By Lemma 3.4 and 3.5, F_n is closed under truncation. Since $F_n \subset F_{n+1}$ for all n, the union of the F_n is $\mathscr{F}$-closed, closed under E, and truncation closed. Hence it suffices to show that $\hat{F} = \bigcup F_n$. For this it is enough to show that each positive element of F_n has a logarithm in F_{n+1}; applying η_n, it actually suffices to show this for $n = 0$.

Let $z \in F_0$ with $z > 0$. Write $z = at^{\alpha+r}(1+\epsilon)$, where $a \in \mathbb{R}$, $a > 0$, $r \in \mathbb{R}$, $\alpha \in J$ and $v(\epsilon) > 0$. Since $t^{\alpha} = E(-\alpha)$ this gives $\log a = \log a - \alpha + r \log t + \log(1+\epsilon)$. Since F_0 is truncation closed we have $t^{\alpha+r} \in F_0$. Hence $\alpha + r \in F_0$, by Lemma 3.5, and thus $\alpha \in F_0$. Moreover, $\log t = -1/t_1$, so $\log z \in F_1$ as desired.

(3.8) Let H_{LE} be the smallest $\mathscr{F}_{\mathrm{LE}}$-closed subfield of $\mathbb{R}((t))^{\mathrm{LE}}$ containing $\mathbb{R}(t)$ and closed under log and exp. Similarly let $H_{\mathrm{an,exp}}$ be the smallest $\mathscr{F}_{\mathrm{an}}$-closed subfield of K containing $\mathbb{R}(t)$ and closed under log and exp. Clearly $H_{\mathrm{an,exp}}$ is the smallest $\mathscr{L}_{\mathrm{an,exp}}$-elementary submodel of $\mathbb{R}((t))^{\mathrm{LE}}$ containing $\mathbb{R}(t)$.

COROLLARY 3.9. *The fields H_{LE} and $H_{\mathrm{an,exp}}$ are closed under truncation.*

(3.10) We let $\mathscr{G}$ be the ring of germs at $+\infty$ of (not necessarily continuous) real valued functions defined, at least, on some interval $(a, +\infty)$ with $a \in \mathbb{R}$ (where a may vary with f). We shall not usually distinguish notationally between a function and its germ. We use the term 'ultimately' to abbreviate 'for all sufficiently large real numbers'.

A subring A of $\mathscr{G}$ is called a *$\mathscr{G}$-domain* if for each $f \in A$ one of the following holds: ultimately $f(x) > 0$, ultimately $f(x) < 0$, ultimately $f(x) = 0$. If A is a $\mathscr{G}$-domain, then A is an integral domain and A has a natural ordering given by $f > 0$ if and only if ultimately $f(x) > 0$. A $\mathscr{G}$-field is a $\mathscr{G}$-domain that happens to be a field. If A is a $\mathscr{G}$-domain then A has a (unique) fraction field in $\mathscr{G}$, and this fraction field is a $\mathscr{G}$-field.

We identify $\mathbb{R}$ with the $\mathscr{G}$-field of germs of constant functions. If we view x as the identity function, then $\mathbb{R}[x]$ is a $\mathscr{G}$-domain and its fraction field $\mathbb{R}(x)$ is a $\mathscr{G}$-field.

The field of germs at $+\infty$ of the functions $f\colon \mathbb{R} \to \mathbb{R}$ definable in $\mathbb{R}_{\mathrm{an,exp}}$ form a $\mathscr{G}$-field, denoted by $H(\mathbb{R}_{\mathrm{an,exp}})$. The field LE of *logarithmic–exponential* functions is the smallest real closed subfield of $H(\mathbb{R}_{\mathrm{an,exp}})$ that contains $\mathbb{R}(x)$, and is closed under exp and taking logarithms of positive elements.

Since $\mathbb{R}_{\mathrm{an,exp}}$ is an elementary submodel of $H_{\mathrm{an,exp}}$, each germ $f \in H(\mathbb{R}_{\mathrm{an,exp}})$ naturally defines a function on the set of positive infinite elements of $H_{\mathrm{an,exp}}$. As in [**6**, §5], there is a well-defined $\mathscr{L}_{\mathrm{an,exp}}$-isomorphism $f \mapsto f(x)$ from $H(\mathbb{R}_{\mathrm{an,exp}})$ onto $H_{\mathrm{an,exp}}$. This isomorphism sends the germ of the identity function to the element $x = t^{-1}$ of $\mathbb{R}((t))^{\mathrm{LE}}$, so the clash of notation is no problem here. This isomorphism clearly maps LE onto H_{LE}.

(3.11) A *Hardy field* is a $\mathscr{G}$-field K consisting of germs at $+\infty$ of $\mathscr{C}^1$-functions $f\colon (a, +\infty) \to \mathbb{R}$ such that the germ at $+\infty$ of f' also belongs to K. Taking derivatives induces a derivation on K.

Since the field of germs of functions $f\colon \mathbb{R} \to \mathbb{R}$, definable in a fixed o-minimal expansion of $\mathbb{R}$, is always a Hardy field, $H(\mathbb{R}_{\mathrm{an,exp}})$ is a Hardy field. It is a well-known fact, essentially due to Hardy, that LE is also a Hardy field.

4. *Solution to a problem of Hardy*

In this section we deal with germs at $+\infty$ of real valued functions defined, at least, on some interval $(a, +\infty)$ with $a \in \mathbb{R}$, but, for simplicity, we refer to these objects simply as functions. Note that if such a function f is ultimately continuous, strictly increasing and $f(x) \to +\infty$ as $x \to +\infty$, then the compositional inverse of f exists in this sense, and also has these properties. Let $i(x)$ be the compositional inverse of the function $x \log x$. Then $\mathrm{e}^{i(x)}$ is the inverse of the function $(\log x)(\log\log x)$. We shall prove that $\mathrm{e}^{i(x)}$ is not asymptotic to a logarithmic–exponential function. This was conjectured by Hardy in [**9**] and proved for $\mathrm{e}^{\mathrm{e}^{i(x)}}$, instead of $\mathrm{e}^{i(x)}$, by Shackell in [**21**].

(4.1) We first show that $i(x)$ is asymptotic to $x/\log x$. Let $x = y \log y$. Then

$$\lim_{x\to\infty} \frac{i(x)\log x}{x} = \lim_{y\to\infty} \frac{y(\log y + \log\log y)}{y \log y} = \lim_{y\to\infty}\left(1 + \frac{\log\log y}{\log y}\right) = 1.$$

We next argue that $i(x)$ itself is not a logarithmic–exponential function.

(4.2) Definition. Let K and k be differential fields of characteristic zero with K a finitely generated extension of k containing no new constants. We say that $K|k$ is a *Liouville extension* if there are differential fields $k \subseteq K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$, with $K_0|k$ algebraic, and nonzero elements $u_1, \ldots, u_n \in K$ such that

(i) either $u_i' \in K_{i-1}$ or $u_i'/u_i \in K_{i-1}$, and

(ii) K_i is an algebraic extension of $K_{i-1}(u_i)$. (In this definition there is no loss of generality in also assuming that u_i is transcendental over K_{i-1} for $i = 1, \dots, n$.)

If $f(x)$ is a logarithmic–exponential function, then $f(x)$ belongs to a Hardy field that is a Liouville extension of the Hardy field $\mathbb{R}(x)$. We shall show that the function $i(x)$ is not contained in any Hardy field K which is a Liouville extension of $\mathbb{R}(x)$. We use the following theorem of Rosenlicht [**19**].

THEOREM 4.3. *Let k be a differential field of characteristic zero and let K be a differential field extension with the same constants. Suppose that $u_1, \dots, u_n$, $v_1, \dots, v_n \in K$ such that $u_i \neq 0$ and $u_i'/u_i + v_i' \in k$ for $i = 1, \dots, n$. Then either*

(i) *the transcendence degree of $k(u_1, \dots, u_n, v_1, \dots, v_n)$ over k is at least n, or*

(ii) (a) *there are constants $c_1, \dots, c_n$, not all zero, such that $\sum c_i v_i$ is algebraic over k, and*

(b) *there are integers $r_1, \dots, r_n$, not all zero, such that $\prod u_i^{r_i}$ is algebraic over k.*

LEMMA 4.4. *Let $k \subseteq K \subseteq L$ be differential fields of characteristic zero with the same constants. Let $u \in K$ be transcendental over k such that K is algebraic over $k(u)$ and either $u' \in k$ or $u'/u \in k$. Let $x, y \in L$ such that $y \neq 0$, $xy \in k$, $y' = yx'$ and x is transcendental over k. Then x is transcendental over K.*

Proof. *Case* 1: $u' \in k$. Then $y'/y - x' = 0$ and $1'/1 + u' \in k$. Thus, by Theorem 4.3, either $k(u, x, y)$ has transcendence degree at least two or there are constants c, d (not both zero) such that $cu + dx$ is algebraic over k. We claim that the latter is impossible. Since u is transcendental over k, we must have $d \neq 0$. Taking $(cu + dx)'$ we see that x' is algebraic over k. Since L contains no new constants, $x' \neq 0$. If $xy = \alpha \in k$, then, since $xy' = \alpha x'$, we have $y = (\alpha' - \alpha x')/x'$ and y is algebraic over k, a contradiction.

Thus $k(u, x, y)$ has transcendence degree at least two. Since $k(u, x, y) \subseteq K(x)$ and K is algebraic over $k(u)$, it follows that x is transcendental over K.

Case 2: $u'/u \in k$. Then $y'/y - x' = 0$ and $u'/u + 1' \in k$. Thus by Theorem 4.3 either $k(u, x, y)$ has transcendence degree at least two or there are integers r and s (not both zero) such that $y^r u^s$ is algebraic over k. Again we show that the latter case is impossible. Since u is transcendental over k, we must have $r \neq 0$. Thus there is a nonzero β, algebraic over k, and a rational number q such that $y = \beta u^q$. Hence

$$y' = \beta' u^q + q\beta u^{q-1} u'.$$

Since $u' = \gamma u$ for some $\gamma \in k$ and $y' = x'y$, we have

$$x'y = u^q(\beta' + q\beta\gamma) \quad \text{and} \quad x' = \frac{\beta' + q\beta\gamma}{\beta}.$$

Hence x' is algebraic over k. Arguing as in Case 1, we see that x must be transcendental over K.

COROLLARY 4.5. *Let K be a Hardy field which is a Liouville extension of the Hardy field $\mathbb{R}(x)$. Then $i(x) \notin K$.*

Proof. Suppose that $i(x) \in K$. Take a Hardy field extension L of K that contains the function $g(x) = \log i(x)$. Note that since $x \log x$ is the compositional inverse to $i(x)$, we have $x = i(x) \log i(x) = i(x) g(x)$.

Let $\mathbb{R}(x) \subseteq K_0 \subseteq \cdots \subseteq K_n = K$ be as in (4.2). Since $g(x)$ is asymptotic to $\log x - \log\log x$, it is not algebraic over $\mathbb{R}(x)$. We may now inductively apply Lemma 4.4 (with $g(x)$ as x and $i(x)$ as y) to show that $g(x)$ is transcendental over K. Since $i(x)g(x) = x \in K_0$, it follows that $i(x)$ is transcendental over K.

COROLLARY 4.6. *The function* $i(x)$ *is not a logarithmic–exponential function.*

We next give a more careful analysis of $i(x)$.

(4.7) Let $y = i(x)$. Let $\epsilon(x) = (i(x)\log x)/x - 1$. Then

$$y = \frac{x}{\log x}(1+\epsilon(x)), \quad \text{where} \quad \lim_{x\to\infty} \epsilon(x) = 0.$$

Thus

$$x = \frac{x}{\log x}(1+\epsilon(x))(\log x - \log\log x + \log(1+\epsilon(x))).$$

Let

$$F(u, v, \epsilon) = (1+\epsilon)(1-u+v\log(1+\epsilon)) - 1.$$

Then F is analytic at $(0,0,0)$, $F(0,0,0) = 0$ and $(\partial F/\partial\epsilon)(0,0,0) = 1$. Thus by the implicit function theorem there is an analytic function f defined on an open $U \subset \mathbb{R}^2$ such that $(0,0) \in U$, $f(0,0) = 0$ and $F(u, v, f(u,v)) = 0$. Letting $u(x) = (\log\log x)/\log x$ and $v(x) = 1/\log x$, we see that $\epsilon(x) = f(u(x), v(x))$ for large enough x.

THEOREM 4.8. *Let* $g(x)$ *be a logarithmic–exponential function. Then* $e^{i(x)}$ *is not asymptotic to* $g(x)$.

Proof. Let $h(x) = \log g(x)$. If $\lim_{x\to\infty}(e^{i(x)}/g(x)) = 1$, then $\lim_{x\to\infty} i(x) - h(x) = 0$.

The functions i and h belong to the Hardy field $H(\mathbb{R}_{\mathrm{an,exp}})$. We now identity this Hardy field with $H_{\mathrm{an,exp}} \subset \mathbb{R}((t))^{\mathrm{LE}}$ via the embedding in (3.10). Let $x = t^{-1}$. Then $v(i(x) - h(x)) > 0$ since $H_{\mathrm{an,exp}}$ is an elementary extension of $\mathbb{R}$.

We claim that $i(x) \in H_{\mathrm{an,exp}} \cap L_2$, and, when viewing L_2 as a field of power series in t_2, we have $\mathrm{supp}(i(x)) < 0$.

We first note that f is given near $(0,0)$ by a convergent power series

$$f(X, Y) = \sum b_{i,j} X^i Y^j \in \mathbb{R}[[X, Y]],$$

and, as explained in (1.4), we can substitute infinitesimals from generalized power series fields like L_n for the variables X and Y. Then, by (4.7), we have

$$i(x) = \frac{x}{\log x}\left(1 + f\left(\frac{\log\log x}{\log x}, \quad \frac{1}{\log x}\right)\right).$$

Clearly $i(x) \in L_2$.

Let η_2 be the isomorphism from L_2 onto $\mathbb{R}((t))^{\mathrm{E}}$ given in (2.7). Then, since $\eta_2(x) = E(E(x))$, we have

$$\eta_2(i(x)) = \frac{E(E(x))}{E(x)}\left(1 + f\left(\frac{x}{E(x)}, \quad \frac{1}{E(x)}\right)\right).$$

Note that in $\mathbb{R}((t))^{\mathrm{E}}$ we have

$$\frac{1}{E(x)} = t^{t^{-1}}, \quad \frac{x}{E(x)} = t^{-1+t^{-1}} \quad \text{and} \quad \frac{E(E(x))}{E(x)} = t^{-t^{-t^{-1}}+t^{-1}}.$$

Thus

$$\operatorname{supp} \eta_2((x)) \subseteq \{-t^{-t^{-1}} + (m+n+1)\, t^{-1} - m \colon m, n \in \mathbb{N}\}.$$

Since $t^{-t^{-1}} = E(x)$ and $t^{-1} = x$, the support of $\eta_2(i(x))$ is negative. Thus, since η_2 is, in particular, a valued field isomorphism, the support of $i(x)$ is negative.

Since $\operatorname{supp}(i(x)) < 0$ and $v(i(x) - h(x)) > 0$, it follows that $i(x) \trianglelefteq h(x)$. By Corollary 3.9, H_{LE} is truncation closed, thus $i(x) \in H_{\mathrm{LE}}$. Thus there is a logarithmic–exponential function s such that $s(x) = i(x)$ in $\mathbb{R}((t))^{\mathrm{LE}}$. Since $\mathbb{R}((t))^{\mathrm{LE}}$ is an elementary extension of $\mathbb{R}_{\mathrm{an,exp}}$, we have $i(z) = s(z)$ for all suitably large z, contradicting Corollary 4.6.

5. *Undefinable functions*

Our goal in this section is to show that several natural functions are not definable in $\mathbb{R}_{\mathrm{an,exp}}$. In **[5]** van den Dries and Miller showed that any real analytic function on an open interval definable in $\mathbb{R}_{\mathrm{exp}}$ is differentially algebraic. Hölder (see, for example, **[20]**) proved that the Gamma function is not differentially algebraic. Thus the restriction of the Gamma function to any interval is not definable in $\mathbb{R}_{\mathrm{exp}}$. Here we shall prove, by a quite different argument, that the Gamma function on $(0, +\infty)$ is not definable in $\mathbb{R}_{\mathrm{an,exp}}$. We do this by showing that definable functions do not have divergent power series developments. We make this precise below. By different arguments we shall show that the restriction of the zeta function to $(1, +\infty)$ is not definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Let F be the smallest $\mathscr{F}_{\mathrm{an}}$-closed subfield of $\mathbb{R}((t))^{\mathrm{E}}$ containing $\mathbb{R}(t)$ that is closed under E. By Lemma 3.4, F is closed under truncation.

LEMMA 5.1. *Let $g \in \Gamma$ with $g > 0$, and suppose that $\sum_{n=0}^{\infty} a_n t^{ng} \in F$. Then $\sum_{n=0}^{\infty} a_n X^n$ converges near* 0.

Proof. As in the proof of Lemma 3.5, we see that F is the union of a chain of subfields $(F_\alpha \colon \alpha < \lambda)$, where

(i) $F_0 = \mathbb{R}$,

(ii) $F_{\alpha+1}$ is the smallest $\mathscr{L}_{\mathrm{an}}$-elementary submodel of $\mathbb{R}((t))^{\mathrm{E}}$ containing $F_\alpha(t^{g_\alpha})$, where $g_\alpha \in \Gamma \setminus v(F_\alpha^\times)$, and

(iii) if α is a nonzero limit ordinal, then $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$.

The case $w \in F_0$ being trivial, we assume that $w = \sum a_n t^{ng} \in F_{\alpha+1} \setminus F_\alpha$. Then $g = h + q g_\alpha$ for some $h \in v(F_\alpha^\times)$, $q \in \mathbb{Q}$. By the exchange lemma for o-minimal theories **[15]**, $F_{\alpha+1}$ is also the smallest $\mathscr{L}_{\mathrm{an}}$-elementary submodel of $\mathbb{R}((t))^{\mathrm{E}}$ containing $F_\alpha(w)$. By transfinite induction one shows easily that $F_\alpha \subseteq \mathbb{R}((t^{v(F_\alpha^\times)}))$ (inside the field $\mathbb{R}((t^\Gamma))$). If $g \in v(F_\alpha^\times)$, then $w \in \mathbb{R}((t^{v(F_\alpha^\times)}))$, hence $F_{\alpha+1} \subseteq \mathbb{R}((t^{v(F_\alpha^\times)}))$, contradicting $g_{\alpha+1} \notin v(F_\alpha^\times)$. Hence $g \notin v(F_\alpha^\times)$. Since $F_{\alpha+1}$ is truncation closed, $t^g \in F_{\alpha+1}$, and $F_{\alpha+1}$ is the smallest $\mathscr{L}_{\mathrm{an}}$-elementary submodel of $\mathbb{R}((t))^{\mathrm{E}}$ containing $F_\alpha(t^g)$.

From the construction of F_α it follows that there is a minimal $k \in \mathbb{N}$ such that we can find $g_1, \ldots, g_k \in v(F_\alpha^\times)$ where $g, g_1, \ldots, g_k$ are linearly independent over $\mathbb{Q}$ and w is in the smallest $\mathscr{L}_{\mathrm{an}}$-elementary submodel of $\mathbb{R}((t))^{\mathrm{E}}$ containing $\mathbb{R}(t^g, t^{g_1}, \ldots, t^{g_k})$. We claim w is in the smallest $\mathscr{L}_{\mathrm{an}}$-elementary submodel of $\mathbb{R}((t))^{\mathrm{E}}$ containing $\mathbb{R}(t^g)$. This is clear if $k = 0$. If $k > 0$, then, by the exchange lemma, t^{g_k} is in the smallest $\mathscr{L}_{\text{an}}$-elementary submodel of $\mathbb{R}((t))^{\text{E}}$ containing $\mathbb{R}(w, t^g, t^{g_1}, \dots, t^{g_{k-1}})$. But this model has value group $\mathbb{Q}g + \mathbb{Q}g_1 + \dots + \mathbb{Q}g_{k-1}$, a contradiction.

Thus it suffices to show that if $\sum a_n t^{gn}$ is in the smallest $\mathscr{L}_{\text{an}}$-elementary submodel of $\mathbb{R}((t))^{\text{E}}$ containing $\mathbb{R}(t^g)$, then $\sum a_n X^n$ is convergent. Let $\mathbb{R}\langle t^g \rangle$ be the field of convergent Puiseux series in t^g, that is, the field of all $\sum_{i \in \mathbb{Z}} c_i t^{ig/d}$, where d is a positive integer, $c_i = 0$ for all i less than some $i_0 \in \mathbb{Z}$, and $\sum_{i \geqslant 0} c_i X^i$ converges near 0. Then $\mathbb{R}\langle t^g \rangle$ is a real closed field such that if $f(X_1, \dots, X_m)$ is a power series over $\mathbb{R}$ converging near 0, and $x_1, \dots, x_m$ are infinitesimals in $\mathbb{R}\langle t^g \rangle$, then $f(x_1, \dots, x_m) \in \mathbb{R}\langle t^g \rangle$. Thus $\mathbb{R}\langle t^g \rangle \models T_{\text{an}}$. Since $\sum a_n t^{ng} \in \mathbb{R}\langle t^g \rangle$, the series $\sum a_n X^n$ converges.

In (3.8) we defined $H_{\text{an, exp}}$, which is the smallest $\mathscr{L}_{\text{an, exp}}$-elementary submodel of $\mathbb{R}((t))^{\text{E}}$ containing $\mathbb{R}(t)$.

Corollary 5.2. *Suppose that* $\sum a_n t^n \in H_{\text{an, exp}}$. *Then* $\sum a_n X^n$ *converges on a neighborhood of* 0.

Proof. As in the proof of Lemma 3.7, $H_{\text{an, exp}}$ is the smallest $\mathscr{L}_{\text{an}}$-elementary submodel of $\mathbb{R}((t))^{\text{E}}$ containing $\mathbb{R}(t, l_1, l_2, \dots)$ which is closed under E. Thus $H_{\text{an, exp}} = \bigcup F_i$, where F_i is the smallest $\mathscr{L}_{\text{an}}$-elementary submodel of $\mathbb{R}((t))^{\text{E}}$ containing $\mathbb{R}(l_i)$ and closed under E. The isomorphism η_i from L_i onto $\mathbb{R}((t))^{\text{E}}$ maps F_i to F of Lemma 5.1. This isomorphism takes $\sum a_n t^n$ to $\sum a_n t^{gn}$ for some positive $g \in \Gamma$. Thus, by Lemma 5.1, $\sum a_n X^n$ converges.

Corollary 5.3. *Suppose that* $f\colon \mathbb{R} \to \mathbb{R}$ *is definable in* $\mathbb{R}_{\text{an, exp}}$ *and* $\mathscr{C}^\infty$ *on an open neighborhood of* 0. *Let* $\sum a_n X^n$ *be the Taylor series at* 0. *Then* $\sum a_n X^n$ *converges on a neighborhood of* 0.

Proof. Consider the element $f(t) \in H_{\text{an, exp}}$. By Taylor's theorem

$$\left| f(t) - \sum_{n=0}^{m} a_n t^n \right| < t^m$$

for all $m \in \mathbb{N}$. Thus $\sum a_n t^n \trianglelefteq f(t)$.

But $f(t)$ is in $H_{\text{an, exp}}$ and, by Corollary 3.9, $H_{\text{an, exp}}$ is truncation closed. Thus $\sum a_n t^n \in H_{\text{an, exp}}$ and, by Corollary 5.2, $\sum a_n X^n$ converges.

We shall use a generalization of this idea to prove the undefinability of some naturally arising integrals.

(5.4) Definition. Let $f_0, f_1, f_2, \dots$ be ultimately positive real valued functions, with each f_n defined, at least, on some interval $(a_n, +\infty)$ for $a_n \in \mathbb{R}$. Assume in addition that $\lim_{x \to +\infty} f_{n+1}(x)/f_n(x) = 0$ for all n.

Then we say that $f\colon (r, +\infty) \to \mathbb{R}$ has an *asymptotic expansion*

$$\sum_{n=0}^{\infty} c_n f_n$$

(with all $c_n \in \mathbb{R}$) if for each m there is a positive constant C such that ultimately $|f(x) - \sum_{n=0}^{m} c_n f_n(x)| < C f_{m+1}(x)$.

We write this as $f(x) \sim \sum c_n f_n(x)$.

COROLLARY 5.5. *Suppose that* $f\colon (r,\infty)\to\mathbb{R}$ *is definable in* $\mathbb{R}_{\text{an, exp}}$ *and has an asymptotic expansion* $f(x)\sim\sum_{n=0}^{\infty}a_n x^{-n}$. *Then* $\sum a_n X^n$ *converges on a neighborhood of* 0.

Proof. In $\mathbb{R}((t))^{\mathrm{LE}}$ we have

$$\left|f(t^{-1})-\sum_{n=0}^{m}a_n t^n\right|<t^m$$

for all m. Hence $\sum a_n t^n \trianglelefteq f(t^{-1})$. By Corollaries 3.9 and 5.2, $\sum a_n X^n$ is a convergent series.

We shall need the following asymptotic expansions. (See, for example, [**1**, **2** or **9**].)

(5.6) Recall that $\Gamma(x)=\int_0^\infty \mathrm{e}^{-t}t^{x-1}\,dt$ for $x>0$. We have Stirling's asymptotic expansion for $\log\Gamma(x)$, namely,

$$\log\Gamma(x)\sim(x-\tfrac{1}{2})\log x-x+\tfrac{1}{2}\log 2\pi+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k-1)(2k)}x^{-1-2k},$$

where B_k is the kth Bernoulli number.

(5.7) We have

$$\mathrm{e}^{x^2}\int_x^\infty \mathrm{e}^{-t^2}\,dt\sim\sum_{n=0}^{\infty}(-1)^n a_{2n+1}x^{-1-2n}$$

where $a_1=\frac{1}{2}$ and $a_{2n+1}=(1\cdot 3\cdots(2n-1))/2^{n+1}$ for $n\geqslant 1$.

(5.8) $$\mathrm{e}^{x}\int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,dt\sim\sum_{n=0}^{\infty}(-1)^n n!x^{-1-n}.$$

(5.9) $$\mathrm{e}^{-x}\int_1^x \frac{\mathrm{e}^{t}}{t}\,dt\sim\sum_{n=0}^{\infty}n!x^{-1-n}.$$

(5.10) We have

$$\mathrm{e}^{-x^2}\int_0^x \mathrm{e}^{t^2}\,dt\sim\sum_{n=0}^{\infty}a_{2n+1}x^{-1-2n},$$

where a_n is as in (5.7).

THEOREM 5.11. *The following functions on* $(0,+\infty)$ *are not definable in* $\mathbb{R}_{\text{an, exp}}$:

(i) $\Gamma(x)$,
(ii) the error function $\int_0^x \mathrm{e}^{-t^2}\,dt$,
(iii) the logarithmic integral $\int_x^\infty \mathrm{e}^{-t}t^{-1}\,dt$,
(iv) $\int_0^\infty \mathrm{e}^{-t}(t+x)^{-1}\,dt$,
(v) $\int_0^\infty \mathrm{e}^{-t}(1+xt)^{-1}\,dt$,
(vi) $\int_0^x \mathrm{e}^{t^2}\,dt$,
(vii) $\int_0^x \mathrm{e}^{\mathrm{e}^t}\,dt$.

Proof. If these functions were definable, then simple modifications, and the asymptotic expansions above would lead to a divergent series contradicting Corollary 5.5.

For (ii) note that

$$\int_0^x \mathrm{e}^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} - \int_x^\infty \mathrm{e}^{-t^2}\,dt.$$

For (iv) and (v) note that by a change of variables

$$\int_0^\infty \frac{\mathrm{e}^{-t}}{t+x}\,dt = e^x \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,dt,$$

while

$$\int_0^\infty \frac{\mathrm{e}^{-t}}{1+xt}\,dt = \frac{1}{x}\int_0^\infty \frac{\mathrm{e}^{-t}}{t+1/x}\,dt.$$

For (vii) note that

$$\int_1^x \frac{\mathrm{e}^t}{t}\,dt = \int_0^{\log x} \mathrm{e}^{\mathrm{e}^u}\,du.$$

(5.12) While the above functions are not definable in $\mathbb{R}_{\mathrm{an,\,exp}}$, for (i), (ii), (vi) or (vii), the restriction to the interval $(0, a)$ is definable in $\mathbb{R}_{\mathrm{an}}$ for each positive $a \in \mathbb{R}$. For

$$f(x) = \int_0^\infty \frac{\mathrm{e}^{-t}}{1+xt}\,dt, \quad \text{where } x > 0,$$

it is shown in [**10**, pp. 26–27] that there is an analytic function $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \frac{1}{x} e^{1/x} \log x + h\left(\frac{1}{x}\right)$$

for all $x > 0$. Thus the restriction of f to $(1, +\infty)$ is definable in $\mathbb{R}_{\mathrm{an,\,exp}}$.

Finally, we show that the restriction of the zeta function to $(1, +\infty)$ is not definable in $\mathbb{R}_{\mathrm{an,\,exp}}$. This does not come from divergence but relies on the following fact.

PROPOSITION 5.13. *Let* $y \in \mathbb{R}((t))^{\mathrm{E}} \cap H_{\mathrm{an,\,exp}}$ *and view* y *as an element of* $\mathbb{R}((t^\Gamma))$. *Then* $\operatorname{supp} y$ *is contained in a finitely generated subgroup of* Γ.

Proof. Let H be the union of the subfields $\mathbb{R}((t^G))$ of $\mathbb{R}((t))^{\mathrm{E}}$ with G ranging over all finitely generated subgroups of Γ. Clearly H is a subfield of $\mathbb{R}((t))^{\mathrm{E}}$. We first show that H is $\mathscr{F}_{\mathrm{an}}$-closed. This reduces to showing that H is real closed, which, in view of [**6**, Corollary 2.6] amounts to showing that H is closed under taking nth roots of positive elements for each $n > 0$. Let $0 < h \in H$, say $h \in \mathbb{R}((t^G))$, let G be a finitely generated subgroup of Γ, and write $h = at^g(1+\epsilon)$, where $0 < a \in \mathbb{R}$, $g \in G$, and $\epsilon \in \mathbb{R}((t^G))$ is infinitesimal. Then for each $n > 0$ we have

$$h^{1/n} = a^{1/n}\, t^{g/n}(1+\delta)$$

for some infinitesimal $\delta \in \mathbb{R}((t^G))$, which shows that $h^{1/n} \in \mathbb{R}((t^{G/n})) \subset H$, as desired.

Next we show H is closed under E. Let $h \in H$ and write $h = \alpha + a + \epsilon$, where $\operatorname{supp} \alpha < 0$, $a \in \mathbb{R}$ and $\epsilon \in H$ is infinitesimal. Then $\alpha \in \Gamma$ and $E(h) = t^{-\alpha} e^a(1+\delta)$ for some infinitesimal $\delta \in H$. Clearly $E(h) \in H$.

It follows, in particular, that $F \subseteq H$, where F is the field introduced at the beginning of this section. Hence we are done if our element y lies in F. We reduce to this case as follows: as in the proof of Corollary 5.2, take $k \in \mathbb{N}$ such that $\eta_k(y) \in F$. The restriction of η_k to $\mathbb{R}((t))^{\mathrm{E}}$ is Φ^k (the kth iterate of Φ), and Φ^k is given by an ordered group embedding $\sigma\colon \Gamma \to \Gamma$, that is, $\Phi^k(\Sigma a_\gamma t^\gamma) = \Sigma a_\gamma t^{\sigma(\gamma)}$. The support of $\eta_k(y) = \Phi^k(y)$ is contained in a finitely generated subgroup G of Γ, hence $\operatorname{supp} y$ is contained in $\sigma^{-1}(G)$, which is also finitely generated.

Corollary 5.14. *The restriction of the Riemann zeta function to* $(1, +\infty)$ *is not definable in* $\mathbb{R}_{\mathrm{an,exp}}$.

Proof. Recall that the zeta function is given for $z > 1$ by the convergent series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \mathrm{e}^{-z\log n}.$$

Let $x = t^{-1} \in \mathbb{R}((t))^{\mathrm{E}}$, so $E(-x\log n) = t^{x\log n}$. Let $\hat{\zeta}$ be the formal series

$$\sum_{n=1}^{\infty} t^{x\log n} \in \mathbb{R}((t))^{\mathrm{E}}.$$

Suppose that the restriction of ζ to $(1, +\infty)$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$. Then $\zeta(x)$ is a well defined element of $H_{\mathrm{an,exp}}$ with

$$|\zeta(x) - \hat{\zeta}| < t^{x\log m}$$

for all positive integers m. Since $H_{\mathrm{an,exp}}$ is truncation closed, $\hat{\zeta} \in H_{\mathrm{an,exp}}$. Clearly, $\operatorname{supp}\hat{\zeta} = \{x\log n\colon n = 1, 2, \ldots\}$. If $p_1, \ldots, p_k$ are distinct prime numbers, then $x\log p_1, \ldots, x\log p_k$ are linearly independent over $\mathbb{Q}$. Thus $\operatorname{supp}\hat{\zeta}$ cannot be contained in any finitely generated subgroup of Γ, contradicting Proposition 5.13.

We remark that as the zeta function has a pole at 1, its restriction to any interval $(1, a)$ with $1 < a \in \mathbb{R}$ is definable in $\mathbb{R}_{\mathrm{an}}$.

It may be of some interest to relate the material here to Ecalle's construction **[7]** of the 'trigèbre $\mathbb{R}[[[x]]]$ des transséries'. Ecalle's construction of $\mathbb{R}[[[x]]]$ is a bit informal, but it is clearly in the same spirit as our construction of $\mathbb{R}((t))^{\mathrm{LE}}$, and in a later paper we intend to identify $\mathbb{R}[[[x]]]$ with an $\mathscr{L}_{\mathrm{an,exp}}$-elementary submodel of $\mathbb{R}((t))^{\mathrm{LE}}$. Ecalle lists (mostly without proof) many interesting properties of $\mathbb{R}[[[x]]]$, which also hold for $\mathbb{R}((t))^{\mathrm{LE}}$, as we plan to show in later papers. The formal counterpart $\hat{\zeta}$ of the zeta function does not belong to $\mathbb{R}[[[x]]]$, so $\mathbb{R}[[[x]]]$ is in fact properly contained in $\mathbb{R}((t))^{\mathrm{LE}}$.

Ecalle also introduces the subfield of $\mathbb{R}[[[x]]]$ consisting of the 'transséries convergentes', which should be exactly our field $H_{\mathrm{an,exp}}$. As Ecalle emphasizes, the field of 'transséries convergentes' is far too small for his purpose, it is not even closed under formal integration. The remarkable thing is that Ecalle manages to introduce a much larger subfield of $\mathbb{R}[[[x]]]$, the algebra of 'transséries accéléro-sommables', which is stable under exponentiation, differentiation, composition and their inverses (taking logarithms, formal integration, and compositional inverse) and whose members are the formal counterparts of the germs at $+\infty$ of the so called 'fonctions analysables'. It is clear from the information provided in **[7]** that each function listed in our Theorem 5.11 is 'analysable', except perhaps for $\Gamma(x)$, where we are not sure, not having understood Ecalle sufficiently well. It would be of great interest to obtain an o-minimal expansion of the real field where all of the 'fonctions analysables' are definable. Work is currently underway to construct an o-minimal expansion of $\mathbb{R}_{an, exp}$ in which all functions listed in Theorem 5.11 are definable.

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