Note on Undecidability of Bisimilarity for Second-Order Pushdown Processes

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Abstract

Broadbent and Göller (FSTTCS 2012) proved the undecidability of bisimulation equivalence for processes generated by ε-free second-order pushdown automata. We add a few remarks concerning the used proof technique, called Defender's forcing, and the related undecidability proof for first-order pushdown automata with ε-transitions (Jančar and Srba, JACM 2008).

Language equivalence of pushdown automata (PDA) is a well-known problem in computer science community. There are standard textbook proofs showing the undecidability even for ε-free PDA, i.e. for PDA that have no ε-transitions; such PDA are sometimes called *real-time* PDA. The decidability question of language equivalence for *deterministic* PDA (DPDA) was a famous long-standing open problem. It was positively answered by Oyamaguchi [3] for ε-free DPDA and later by Sénizergues [4] for the whole class of DPDA.

Besides their role of language acceptors, PDA can be also viewed as generators of (infinite) labelled transition systems; in this context it is natural to study another fundamental equivalence, namely *bisimulation equivalence*, also called *bisimilarity*. This equivalence is finer than language equivalence, but the two equivalences in principle coincide on deterministic systems.

Sénizergues [5] showed an involved proof of the decidability of bisimilarity for (nondeterministic) ε-free PDA but also for PDA in which ε-transitions are deterministic, popping and do not collide with ordinary input a-transitions. It turned out that a small relaxation, allowing for nondeterministic popping ε-transitions, already leads to undecidability [2]. In [2], the authors of this note also explicitly describe a general proof technique called *Defender's forcing*. It is a simple, yet powerful, idea related to the bisimulation game played between Attacker and Defender; it was used, sometimes implicitly, also in context of other hardness results for bisimilarity on various classes of infinite state systems.

The classical PDA, to which we have been referring so far, are the *first-order* PDA in the hierarchy of *higher-order* PDA that were introduced in connection with higher-order recursion schemes already in 1970s. The decidability question for equivalence of deterministic nth-order PDA, where $n \geq 2$, seems to be open so far. A step towards a solution was made by Stirling [6] who showed the decidability for a subclass of ε-free second-order DPDA.

Recently Broadbent and Göller [1] noted that the results in [2], or anywhere else in the literature, do not answer the decidability question for bisimilarity of ε-free second-order PDA.

They used the above mentioned technique of Defender's forcing to show that this problem is also undecidable. This result helps to further clarify the (un)decidability border, now in another direction: a mild use of second-order operations (on a stack of stacks) is sufficient to establish undecidability without using ε-transitions (that are necessary in the first-order undecidability proof [2]).

The authors of [1] concentrate on giving a complete self-contained technical construction yielding the undecidability proof, however, they do not discuss in detail its relation to the constructions in [2]. Here, in Section 2, we try to concisely present the idea of the relevant first-order proof from [2], and then, in Section 3, we highlight the idea in [1] that makes it possible to replace the use of ε-transitions in the undecidability proof with second-order operations.

We hope that this note may help to popularize the Defender's forcing technique, and that it might be found useful by other researchers tackling further open problems in the area.

1 Definitions

A *labelled transition system* (LTS) is a (possibly infinite) directed multigraph with action-labelled edges. By a triple $s \xrightarrow{a} s'$, called a *transition*, or an *a-transition*, we denote that there is an edge from node s to node s' labelled with a; we also refer to the nodes as to the *states*. A *symmetric* binary relation R on the set of states is a *bisimulation* if for any $(s, t) \in R$ and any transition $s \xrightarrow{a} s'$ there is a transition $t \xrightarrow{a} t'$ (with the same label a) such that $(s', t') \in R$. Two states s and t are *bisimilar*, written $s \sim t$, if there is a bisimulation containing (s, t).

Bisimilarity is often presented in terms of a two-player game between Attacker (he) and Defender (she). In the current game position, that is a pair of states (s_1, s_2) in an LTS, Attacker chooses a transition $s_j \xrightarrow{a} s'_j$ (for $j \in \{1, 2\}$) and Defender then chooses a transition $s_{3-j} \xrightarrow{a} s'_{3-j}$; the pair (s'_1, s'_2) becomes the new current position. If one player gets stuck then the other player wins; an infinite play is a win of Defender. It is easy to verify that s, t are bisimilar iff Defender has a winning strategy when starting from the position (s, t).

An *ε-free second-order pushdown system* is a tuple $(Q, \Gamma, \mathcal{A}ct, \Delta)$ consisting of four finite nonempty sets: Q contains the *control states*, Γ the *stack symbols*, $\mathcal{A}ct$ the *actions* (corresponding to classical input letters), and Δ the *rules* of the following three types:

$$pX \xrightarrow{a} q\alpha, \; pX \xrightarrow{a} (q, \text{PUSH}), \; pX \xrightarrow{a} (q, \text{POP}), \tag{1}$$

where $p, q \in Q$, $X \in \Gamma$, $a \in \mathcal{A}ct$, and $\alpha \in \Gamma^*$. The LTS generated by $(Q, \Gamma, \mathcal{A}ct, \Delta)$ has the set $Q \times (\Gamma^+)^*$ as the set of states; a state is written in the form $q[\delta_1][\delta_2] \cdots [\delta_n]$ where q is a control state and δ_i is a nonempty sequence of stack symbols (for $i = 1, 2, \ldots, n$). By ε we denote the empty sequence; hence $[\delta_1][\delta_2] \cdots [\delta_n] = \varepsilon$ when $n = 0$. The transitions in the generated LTS are induced by the rules from Δ as follows:

- the rule $pX \xrightarrow{a} q\alpha$ implies $p[X\gamma][\delta_2][\delta_3] \ldots [\delta_n] \xrightarrow{a} q[\alpha\gamma][\delta_2][\delta_3] \ldots [\delta_n]$ if $\alpha\gamma \neq \varepsilon$, and $p[X\gamma][\delta_2][\delta_3] \ldots [\delta_n] \xrightarrow{a} q[\delta_2][\delta_3] \ldots [\delta_n]$ if $\alpha\gamma = \varepsilon$;

- the rule $pX \xrightarrow{a} (q, \text{PUSH})$ implies $p[X\gamma][\delta_2][\delta_3] \ldots [\delta_n] \xrightarrow{a} q[X\gamma][X\gamma][\delta_2][\delta_3] \ldots [\delta_n]$;

- the rule $pX \xrightarrow{a} (q, \text{POP})$ implies $p[X\gamma][\delta_2][\delta_3] \ldots [\delta_n] \xrightarrow{a} q[\delta_2][\delta_3] \ldots [\delta_n]$.

We remark that the definitions of second-order pushdown systems in the literature vary in details that are insignificant for us. If we restrict the rules to the type $pX \xrightarrow{a} q\alpha$ then we get *ε-free first-order pushdown systems.* In this paper we do not introduce ε-rules (of the types (1) with $a = \varepsilon$); their restricted use in our paper is handled by a remark at the respective place.

2   Undecidability of bisimilarity for PDA with ε-transitions

In this section, we briefly explain a result from [2], namely the undecidability of bisimilarity for (normal, i.e. first-order) pushdown systems with popping ε-rules (of the type $pX \xrightarrow{\varepsilon} q$). The text closely follows the beginning of Section 5.1 from [2], though it is a bit modified, concentrating on illustrating the ideas.

The undecidability result is achieved by a reduction from the following variant of Post's Correspondence Problem (PCP). As usual, by a *word u over an alphabet* we mean a finite sequence of letters; $|u|$ denotes the length of u.

Definition 2.1 *A PCP-instance INST is a nonempty sequence* $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$ *of pairs of nonempty words over the alphabet* $\{A, B\}$ *where* $|u_i| \leq |v_i|$ *for all* $i \in \{1, 2, \ldots, n\}$. *An* infinite initial solution *of INST, a* solution *of INST for short, is an infinite sequence of indices* $i_1, i_2, i_3, \ldots$ *from the set* $\{1, 2, \ldots, n\}$ *such that* $i_1 = 1$ *and the infinite words* $u_{i_1} u_{i_2} u_{i_3} \cdots$ *and* $v_{i_1} v_{i_2} v_{i_3} \cdots$ *are equal. A finite sequence* $i_1, i_2, \ldots, i_\ell$ *is a* partial solution *of INST if* $i_1 = 1$ *and* $u_{i_1} u_{i_2} \cdots u_{i_\ell}$ *is a prefix of* $v_{i_1} v_{i_2} \cdots v_{i_\ell}$.
The problem inf-PCP *asks if there is a solution for a given INST.*

The next proposition can be shown by standard arguments, related to simulations of *nonterminating* Turing machine computations; the respective reduction easily guarantees our technical condition $|u_i| \leq |v_i|$ (see also [2]).

Proposition 2.2 *Problem inf-PCP is undecidable; more precisely, inf-PCP is* Π_1^0*-complete.*

We now consider a fixed instance INST of inf-PCP, i.e. $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$ as above. Let us imagine the following game, played between Attacker (he) and Defender (she); this game is more abstract, it will be only later implemented as the bisimulation game.

Starting with the one-element sequence i_1, where $i_1 = 1$, Attacker repeatedly asks Defender to prolong the current sequence $i_\ell i_{\ell-1} \ldots i_1$ by one $i_{\ell+1} \in \{1, 2, \ldots, n\}$ (of her choice), to get $i_{\ell+1} i_\ell \ldots i_1$. (We use prolongations to the left, to ease the later implementation by a pushdown system.) Attacker can thus ask Defender indefinitely, in which case the play is a win for Defender, or he can eventually decide to switch to checking whether the current sequence represents a partial solution, i.e., whether $u_{i_1} u_{i_2} \ldots u_{i_\ell}$ is a prefix of $v_{i_1} v_{i_2} \ldots v_{i_\ell}$; the negative case is a win for Attacker, the positive case is a win for Defender. In another formulation, the checking phase finds out whether $(u_{i_\ell})^R (u_{i_{\ell-1}})^R \ldots (u_{i_1})^R$ is equal to a suffix of $(v_{i_\ell})^R (v_{i_{\ell-1}})^R \ldots (v_{i_1})^R$, where w^R denotes the reverse of w. It is obvious that

$$\text{INST has a solution iff Defender has a winning strategy.} \qquad (2)$$

With an eye to the later implementation of the game by pushdown rules, we formulate an intermediate version of the game as follows. (In fact, this intermediate game replaces the arguments given in [2] to justify the rules of Fig. 1.)

3

- (*Generating phase*)
  The game starts with a pair $(q_0\,i_1, q_0'\,i_1)$ where $i_1 = 1$ and q_0, q_0' are auxiliary symbols that we can call "control states". Attacker repeatedly asks Defender to prolong both sequences in the current pair $(q_0\,i_\ell i_{\ell-1}\ldots i_1, q_0'\,i_\ell i_{\ell-1}\ldots i_1)$ by some $i_{\ell+1} \in \{1, 2, \ldots, n\}$, thus creating the next current pair $(q_0\,i_{\ell+1}i_\ell\ldots i_1, q_0'\,i_{\ell+1}i_\ell\ldots i_1)$.

- (*Switching phase*)
  For any current pair

$$(q_0\,i_\ell i_{\ell-1}\ldots i_1, q_0'\,i_\ell i_{\ell-1}\ldots i_1) \tag{3}$$

  Attacker can decide to switch (to the verification): the control state in the left-hand sequence changes to q_u; in the right-hand side sequence the control state changes to q_v but before that Defender can erase a chosen prefix $i_\ell i_{\ell-1}\ldots i_{\ell-k}$ and replace $i_{\ell-k-1}$ with a suffix w of $(v_{\ell-k-1})^R$; we thus get

$$(q_u\,i_\ell i_{\ell-1}\ldots i_1, q_v\,w\,i_m i_{m-1}\ldots i_1) \text{ where } m < \ell \text{ and } w \text{ is a suffix of } v_{i_{m+1}}. \tag{4}$$

- (*Verification phase*)
  Here the play is completely determined, verifying (step by step) that $(u_{i_\ell})^R(u_{i_{\ell-1}})^R\ldots(u_{i_1})^R$ is equal to $w\,(v_{i_m})^R(v_{i_{m-1}})^R\ldots(v_{i_1})^R$. The control state q_u signals that i_j is interpreted as $(u_{i_j})^R$, and q_v signals that i_j is interpreted as $(v_{i_j})^R$. If a mismatch is encountered then Attacker wins, otherwise Defender wins.

Property (2) obviously holds for the above (intermediate) game as well. We now show that this game is implemented as the bisimulation game in the LTS generated by the pushdown system in Fig. 1, starting in the position $(q_0 I_1\bot, q_0' I_1\bot)$. We use the symbol I_i instead of i; the "bottom-of-the-stack" symbol $\bot$ is used for technical reasons.

Any position $(p\gamma, p\gamma)$ in the bisimulation game is trivially winning for Defender. To avoid this "equality-win", when starting from the position $(q_0 I_1\bot, q_0' I_1\bot)$, Attacker obviously must not use the framed rule $q_0 \xrightarrow{g} p_i$ (for any $i \in \{1, 2, \ldots, n\}$), nor $q_0' \xrightarrow{g} p_i$ which would allow Defender to choose the framed rule to install equality. The frames just highlight the use of Defender's forcing; the rules are constructed so that Attacker must ensure that neither him nor Defender ever uses a framed rule.

In the first round of the game, Attacker is thus forced to use either $q_0 \xrightarrow{g} t$ (g for "generating") or $q_0 \xrightarrow{s} q_u$ (s for "switching"). In the first case Defender uses $q_0' \xrightarrow{g} p_k$ for some (freely chosen) $k \in \{1, 2, \ldots, n\}$; the current position becomes $(t I_1\bot, p_k I_1\bot)$. Attacker is now forced to use $p_k \xrightarrow{a_k} q_0' I_k$ or $t \xrightarrow{a_k} q_0 I_k$, since using an a_j-transition for $j \neq k$ allows Defender to install equality. After Defender's response, the current position is $(q_0 I_k I_1\bot, q_0' I_k I_1\bot)$ where k has been chosen by Defender. We can thus see that the rules (G1) implement the generating phase. As long as Attacker chooses g, the play goes through longer and longer pairs

$$(\,q_0\,I_{i_\ell}I_{i_{\ell-1}}\ldots I_{i_1}\bot\,,\ q_0'\,I_{i_\ell}I_{i_{\ell-1}}\ldots I_{i_1}\bot\,).$$

Since any infinite play is a win of Defender, Attacker needs to enter the switching phase eventually, by using $q_0 \xrightarrow{s} q_u$ from (S1). The rules $q_0(I^*)I_i \xrightarrow{s} q_v w$, $q_0'(I^*)I_i \xrightarrow{s} q_v w$ constitute the only place where ε-transitions enter the stage. These rules stand for the following family of rules given in (S1-τ) in [2] (where i ranges over $\{1, 2, \ldots, n\}$):

$$q_0 \xrightarrow{s} z,\ q_0' \xrightarrow{s} z,\ zI_i \xrightarrow{\varepsilon} z,\ zI_i \xrightarrow{\varepsilon} q_v w \text{ (for all suffixes } w \text{ of } v_i^R) . \tag{5}$$

(G1) rules:
$$q_0 \xrightarrow{g} t$$

$$\boxed{q_0 \xrightarrow{g} p_i} \qquad q_0' \xrightarrow{g} p_i$$

$$t \xrightarrow{a_i} q_0 I_i \qquad p_i \xrightarrow{a_i} q_0' I_i$$

$$\boxed{p_i \xrightarrow{a_j} q_0 I_j} \quad \text{where } i \neq j$$

(S1) rules:
$$q_0 \xrightarrow{s} q_u$$

$$\boxed{q_0(I^*)I_i \xrightarrow{s} q_v w} \qquad q_0'(I^*)I_i \xrightarrow{s} q_v w \quad \text{for all suffixes } w \text{ of } v_i^R$$

(V1) rules:
$$q_u I_i \xrightarrow{h(u_i^R)} q_u \, tail(u_i^R) \qquad q_v I_i \xrightarrow{h(v_i^R)} q_v \, tail(v_i^R)$$

$$q_u A \xrightarrow{a} q_u \qquad\qquad q_v A \xrightarrow{a} q_v$$

$$q_u B \xrightarrow{b} q_u \qquad\qquad q_v B \xrightarrow{b} q_v$$

Notation. A rule $p \xrightarrow{a} q\alpha$ replaces the family $pX \xrightarrow{a} q\alpha X$ for all stack symbols X. By $head(w)$ we denote the first symbol of w; $tail(w)$ is the rest of w, and thus $w = head(w)tail(w)$. By $h(w)$ (head-action) we mean a if $head(w) = A$, and b if $head(w) = B$. Subscripts i, j range over $\{1, 2, \ldots, n\}$; thus the rule $q_0 \xrightarrow{g} p_i$ stands for the n rules $q_0 \xrightarrow{g} p_1$, $q_0 \xrightarrow{g} p_2$, $\ldots$, $q_0 \xrightarrow{g} p_n$, the rule $p_i \xrightarrow{a_j} q_0 I_j$, $i \neq j$, stands for $n \cdot (n-1)$ rules like $p_1 \xrightarrow{a_2} q_0 I_2$, $p_8 \xrightarrow{a_5} q_0 I_5$, etc. (Rules with (I^*) in (S1) are explained in the text.)

Figure 1: Rules from [2], showing undecidability in the first-order case

We note that the last rule $zI_i \xrightarrow{\varepsilon} q_v w$ could be made ε-popping by remembering w in the control state but we prefer the given form for simplicity. The ε-rules generate ε-transitions in the respective *fine-grained* LTS. Nevertheless we refer to the ε-free LTS where $s \xrightarrow{a} s'$ iff $s \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} s'' \xrightarrow{a} s''' \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} s'$ in the fine-grained LTS.

It is thus clear that Attacker is indeed forced to start the switching phase by choosing the rule $q_0 \xrightarrow{s} q_u$ and performing the transition $q_0 I_{i_\ell} I_{i_{\ell-1}} \ldots I_{i_1} \bot \xrightarrow{s} q_u I_{i_\ell} I_{i_{\ell-1}} \ldots I_{i_1} \bot$. Defender then chooses m and w, and the corresponding transition $q_0' I_{i_\ell} I_{i_{\ell-1}} \ldots I_{i_1} \bot \xrightarrow{s} q_v w I_{i_m} I_{i_{m-1}} \ldots I_{i_1} \bot$ (where w is a suffix of $(v_{i_{m+1}})^R$). The next current pair thus becomes

$$(\, q_u \, I_{i_\ell} I_{i_{\ell-1}} \ldots I_{i_1} \bot \,, \; q_v \, w \, I_{i_m} I_{i_{m-1}} \ldots I_{i_1} \bot \,).$$

Rules (5) also allow us to choose $q_0' I_{i_\ell} I_{i_{\ell-1}} \ldots I_{i_1} \bot \xrightarrow{s} z I_{i_{m+1}} I_{i_{m-1}} \ldots I_{i_1} \bot$; but once we understand the verification phase, we can easily check that this is of no help for Defender. The verification phase is implemented by the rules (V1). Defender can no longer threaten with installing equality but this is not needed anymore; this phase is completely determined, giving no real choice to any of the players. It is obvious that Defender wins iff

$$(u_{i_\ell})^R (u_{i_{\ell-1}})^R \ldots (u_{i_1})^R = w \, (v_{i_m})^R (v_{i_{m-1}})^R \ldots (v_{i_1})^R \, .$$

Since the described bisimulation game closely mimics our previous (intermediate) game, it is easy to check that it also has Property (2).

3 Second-Order Pushdown Systems

The "first-order" proof in Section 2 (captured by the rules in Fig. 1) trivially shows the undecidability of bisimilarity for second-order pushdown systems when ε-transitions are allowed. When we explore the decidability question for ε-*free* second-order pushdown systems then it is natural to ask whether we can implement the switching phase (captured by (S1)) without using ε-rules, when we have second-order PUSH and POP at our disposal. So in terms of our intermediate game, we want to implement the switching from (3) to (4). Without ε-transitions we cannot implement erasing a prefix of $i_\ell i_{\ell-1} \cdots i_1$ (in the right-hand side string) in one move. A natural idea is to shorten the right-hand side string step-by-step while Defender should decide when to finish. But it is not clear how to implement this in the "first-order" bisimulation game since Defender loses the possibility of threatening with equality during such a step-by-step process. (Sénizergues's decidability result [5] shows that such an implementation is indeed impossible in the first-order case.)

The idea (i.e., the crucial point in the undecidability proof in [1]) can be explained as follows. When Attacker wants to switch at the position $(q_0\, i_\ell i_{\ell-1} \ldots i_1,\ q_0'\, i_\ell i_{\ell-1} \ldots i_1)$ then the stacks are doubled (using PUSH), and the next position becomes

$$(r\,[i_\ell i_{\ell-1} \ldots i_1][i_\ell i_{\ell-1} \ldots i_1],\ r'\,[i_\ell i_{\ell-1} \ldots i_1][i_\ell i_{\ell-1} \ldots i_1])\ . \tag{6}$$

Now the top stacks are being synchronously shortened, the play going through positions

$$(r\,[i_{m+1} i_m \ldots i_1][i_\ell i_{\ell-1} \ldots i_1],\ r'\,[i_{m+1} i_m \ldots i_1][i_\ell i_{\ell-1} \ldots i_1])$$

for decreasing m. During this process Defender can threaten with equality, so it is possible to implement that it is Defender who decides when the process should stop, forcing POP on the left-hand side (with entering q_u) and choosing a suffix w of $(v_{i_{m+1}})^R$ on the right-hand side; the reached position is then

$$(q_u\,[i_\ell i_{\ell-1} \ldots i_1],\ q_v\,[w\, i_m i_{m-1} \ldots i_1][i_\ell i_{\ell-1} \ldots i_1])\ . \tag{7}$$

The bottom stack on the right-hand side is now superfluous; it only served for the previous threatening with equality. The verification phase is the same as previously (with no choice for any player). Implementing the described switching via the second-order rules is now a routine, once we understand the Defender's forcing technique. We just replace the rules (S1) in Fig. 1 with (S1-2nd) in Fig. 2 (where i ranges over $\{1, 2, \ldots, n\}$).
Now if Attacker chooses to switch (by action s) then a position corresponding to (6) is reached (where i_j is replaced with I_{i_j}, and $\perp$ is added). By Defender's forcing, Attacker must now use the rule $r \xrightarrow{c} q$ and Defender decides whether to enter the control state q' (meaning that she wishes to erase a further symbol I_i from the top stacks, by the rules $qI_i \xrightarrow{c_1} r$ and $q'I_i \xrightarrow{c_1} r'$) or the control state q'' (meaning that she wishes to enter the verification phase, by the rules $q \xrightarrow{c_2} p$ and $q'' \xrightarrow{c_2} p'$). In the next round Attacker must follow these choices otherwise he will lose (by the framed rules). The rule $q\perp \xrightarrow{h} q$ forces Defender to choose the second option (entering q'') before the top stacks are emptied (otherwise she loses). Finally, once a position

$$(p\,[I_{i_{m+1}} I_{i_m} \ldots I_{i_1} \perp][I_{i_\ell} I_{i_{\ell-1}} \ldots I_{i_1} \perp],\ p'\,[I_{i_{m+1}} I_{i_m} \ldots I_{i_1} \perp][I_{i_\ell} I_{i_{\ell-1}} \ldots I_{i_1} \perp])$$

is reached, the last application of Defender's forcing results in an analogue of (7):

$$(q_u\,[I_{i_\ell} I_{i_{\ell-1}} \ldots I_{i_1} \perp],\ q_v\,[w\, I_{i_m} I_{i_{m-1}} \ldots I_{i_1} \perp][I_{i_\ell} I_{i_{\ell-1}} \ldots I_{i_1} \perp])\ .$$

(S1-2nd) rules:
$$q_0 \xrightarrow{s} (r, \text{PUSH}) \qquad\qquad q'_0 \xrightarrow{s} (r', \text{PUSH})$$
$$r \xrightarrow{c} q \qquad\qquad\qquad r' \xrightarrow{c} q', \, r' \xrightarrow{c} q''$$
$$\boxed{r \xrightarrow{c} q', \, r \xrightarrow{c} q''}$$
$$qI_i \xrightarrow{c_1} r \qquad\qquad\qquad q'I_i \xrightarrow{c_1} r'$$
$$\boxed{q''I_i \xrightarrow{c_1} r}$$
$$q \xrightarrow{c_2} p \qquad\qquad\qquad q'' \xrightarrow{c_2} p'$$
$$\boxed{q' \xrightarrow{c_2} p}$$
$$q\bot \xrightarrow{h} q$$
$$p \xrightarrow{d} (q_u, \text{POP})$$
$$\boxed{pI_i \xrightarrow{d} q_v w} \qquad\qquad p'I_i \xrightarrow{d} q_v w \quad \text{(for each suffix } w \text{ of } (v_i)^R)$$

Figure 2: A replacement of (S1) to show undecidability for ε-free second-order PDA

3.1 Normedness

Bisimilarity problems like those we discuss here are often simpler when restricted to *normed systems*; in our case, a *state s* in the LTS generated by a pushdown system is *normed* if from each state that is reachable from s we can reach a state where the stack is empty. But restricting to the normed case does not affect the undecidability here. The states $q_0\, I_1\bot$, $q'_0\, I_1\bot$ in the system defined by Fig. 1 are normed, if we view the states $q_u\bot$, $q_v\bot$ as having the empty stack; otherwise we can add the rules $q_u\bot \xrightarrow{e} q_u$, $q_v\bot \xrightarrow{e} q_v$. (In [2], there are used the rules $q_u\bot \xrightarrow{e} \varepsilon$, $q_v\bot \xrightarrow{e} \varepsilon$ in the context of prefix-rewrite system definition.)

The authors of [1] are also interested in normedness for higher-order PDA as a natural extension of normedness for first-order PDA. We can note that after replacing (S1) in Fig. 1 with (S1-2nd) in Fig. 2, the system is not normed anymore (exemplified by the state $q_v\,[\bot][I_{i_\ell} I_{i_{\ell-1}}...I_{i_1}\bot]$). In [1] a triple copy of the stack is used to handle the specific normedness definition there. Another possibility is to start from $(q_0\,[I_1\bot][\bot], q'_0\,[I_1\bot][\bot])$ and add a new control state q_{POP} and the rules $q_u \xrightarrow{f} q_{\text{POP}}$ and $x \xrightarrow{f} (q_{\text{POP}}, \text{POP})$ for all control states x where $x \neq q_u$, including the rule $q_{\text{POP}} \xrightarrow{f} (q_{\text{POP}}, \text{POP})$.

Additional comments

As already mentioned, the undecidability result for ε-free second-order pushdown systems in [1] clarifies the (un)decidability border in another direction than the undecidability result for first-order pushdown systems with ε-transitions in [2]. The border can be surely explored further. For example it seems that we cannot avoid using several control states in the above undecidability proofs (though we can surely decrease their number by extending the stack alphabet). Hence (normed) second-order simple grammars, studied in [6], are a possible target for exploring.

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