Annals of Pure and Applied Logic 161 (2010) 570–598

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

Topological differential fields

Nicolas Guzy [1], Françoise Point [*,2]

Institut de Mathématique, Université de Mons, Le Pentagone, 20, Place du Parc, B-7000 Mons, Belgium

ARTICLE INFO

Article history:
Received 24 August 2004
Received in revised form 5 August 2009
Accepted 5 August 2009
Available online 17 September 2009
Communicated by Guest Editor (ESNL)

MSC:
Primary 03C10
Secondary 12L12
12J
12H05

Keywords:
Topological fields
Derivation
Model-companion
Henselization

ABSTRACT

We consider first-order theories of topological fields admitting a model-completion and their expansion to differential fields (requiring no interaction between the derivation and the other primitives of the language). We give a criterion under which the expansion still admits a model-completion which we axiomatize. It generalizes previous results due to M. Singer for ordered differential fields and of C. Michaux for valued differential fields. As a corollary, we show a transfer result for the *NIP* property. We also give a geometrical axiomatization of that model-completion. Then, for certain differential valued fields, we extend the positive answer of Hilbert's seventeenth problem and we prove an Ax–Kochen–Ershov theorem. Similarly, we consider first-order theories of topological fields admitting a model-companion and their expansion to differential fields, and under a similar criterion as before, we show that the expansion still admits a model-companion. This last result can be compared with those of M. Tressl: on one hand we are only dealing with a single derivation whereas he is dealing with several, on the other hand we are not restricting ourselves to definable expansions of the ring language, taking advantage of our topological context. We apply our results to fields endowed with several valuations (respectively several orders).

1. Introduction

We are given an inductive theory which is an expansion of a theory of fields (by relation symbols) and which has a model-completion or a model-companion. We expand the language by a new unary function symbol which satisfies the axioms of a derivation, then we consider the following question: when does the corresponding expansion of the theory retain the property of having a model-completion (respectively a model-companion)?

This kind of question has been previously considered by Singer [39] for ordered fields, and then by Michaux for valued fields [20]. In order to encompass the cases of valued and ordered fields in one setting, we will place ourselves in a general framework of topological fields and we give a criterion under which we show the existence of a model-completion (respectively model-companion). We give an axiomatization of the model-completion (respectively model-companion) of this differential expansion in the case where the topology is first-order definable.

In Section 2, we develop a rather technical topological apparatus (we wanted to translate in topological terms the conditions needed in order to perform our construction). In later sections, when we apply it to the case where the topology is first-order definable, it will have a simpler version. The main hypothesis on the classes $\mathcal{C}$ of topological fields we will work with, is called *Hypothesis* (I). It implies on the class $\mathcal{C}$ the following. Given an element K of $\mathcal{C}$, it always has an extension

* Corresponding author.
E-mail addresses: Nicolas.Guzy@umons.ac.be (N. Guzy), point@logique.jussieu.fr (F. Point).

1 Postdoctoral Researcher at the "Fonds de la Recherche Scientifique F.N.R.S.-F.R.S".
2 Senior Research Associate at the "Fonds de la Recherche Scientifique F.N.R.S.-F.R.S".

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doi:10.1016/j.apal.2009.08.001

belonging to $\mathcal{C}$ which contains the field of Laurent series $K((t))$. In the case where the class $\mathcal{C}$, viewed as a class of ordinary first-order structures, is model-complete, this implies that K, as a field, is large, since it will be existentially closed in $K((t))$.

In Section 3, we consider the differential expansions of our topological fields, where the derivation has a priori no interaction with the topology. We write down a scheme of axioms (DL) which express the property that whenever a differential polynomial has a simple zero when viewed as an ordinary polynomial (in several variables), then it has a differential zero *close* to that algebraic zero. We show that any element of an inductive class $\mathcal{C}$ of differential topological fields satisfying Hypothesis (I) can be embedded in an element of $\mathcal{C}$ satisfying (DL). This enables us to prove our transfer results on existence of model-completions and model-companions. We also prove certain properties on the elements K of $\mathcal{C}$ satisfying (DL), for instance the subfield of constants of K is dense and any finitely generated differentially separable extension of K is differentially generated by a single element.

The main results are proven in Sections 4 and 9. In these sections we assume that the topology on our structures is first-order definable.

In Section 4, we apply the machinery developed in previous sections to axiomatize the model-completion of a differential expansion of a theory with a model-completion whose class of models satisfy Hypothesis (I). We assume that we are dealing with universal theories and so we can restrict ourselves in handling 1-extensions, using Blum's criterion for model-completion. This is one advantage of working in this topological setting. Even though in the case of real-closed fields or p-adically closed fields, the topology is existentially definable in the ring language, in order to have the amalgamation property, we need to add the order (respectively the valuation) as a primitive of the language. (Recall that in a real-closed field, the positive elements are those which are squares, in a p-adically closed field, the non-zero elements of positive valuation, when $p \neq 2$, are those elements x such that there exists z with $1 + px^2 = z^2$.)

We also prove the transfer in that setting of the non-independence property (NIP), which will have the following consequence. If we can endow the definable subsets with the VC dimension, then we retain that property in the differential expansion.

In Section 5, we write down a geometrical scheme of axioms analogous (for instance) to the one, given for differentially closed fields by Pierce and Pillay [22] and for $CODF$ by Michaux and Rivière [21].

In Section 6, we apply our results to differential expansions of the following theories $ACVF$, pCF and $RCVF$, and as far as we know the last application is new.

As a byproduct in Section 7, we extend the positive answer for Hilbert's seventeenth problem for p-adically closed fields of p-rank d.

Then, in the same spirit, we prove an Ax–Kochen–Ershov type result for valued differential fields in Section 8.

In Section 9, we consider the transfer of model-companion, and we will no longer require that the theories are universally axiomatizable.

In Section 10, we apply our transfer results of model-companion to fields endowed with several orderings (respectively valuations). The first case was[3] also handled by C. Rivière who gave a geometric axiomatization, the second case however is new.

M. Tressl had introduced a more algebraic approach to this problem; he is also dealing with the case of several commuting derivations. There, the crux of the matter is to decide which finite systems of algebraic differential equations are required to have a solution (see [40]). His approach (specialized to the case of one derivation) differs from ours since he restricts himself to expansions by definition of the ring language. In order to recast his results in our framework and to use our transfer result on model-companion, we still need to consider an ordinary field as a topological field and put a topology on the field of Laurent series. By the uniqueness of model-companion, the theories we describe are the same. In Section 11, we compare both approaches. We will also examine the axiomatization given by Nicolas Guzy of the model-companion of valued fields endowed with several commuting derivations.

2. Topological fields

2.1. Preliminaries

We will always be working with $\mathcal{L}$-structures M which are expansions of commutative rings and so $\mathcal{L}$ will always denote a first-order language containing the language of rings $\mathcal{L}_{\text{rings}} := \{+, -, ., 0, 1\}$. We will say that a subset of M^n is algebraic if it is the set of solutions of a finite set of polynomial equations. We will use the following notation: for a subset E of a ring, we put $E^{\times} := E \setminus \{0\}$.

A topological $\mathcal{L}$-structure $\langle M, \tau\rangle$ is a first-order $\mathcal{L}$-structure with a Hausdorff topology τ such that every n-ary function symbol of $\mathcal{L}$ is interpreted by a continuous function M^n to M, and every m-ary relation symbol of $\mathcal{L}$ and its complement is interpreted by the union of an open subset of M^m and an algebraic set (M^n and M^m are endowed with the product topology).

Since M has in particular the underlying structure of an additive group, a fundamental system $\mathcal{V}$ of neighbourhoods of zero determines the topology: for each $m \in M$, $m + \mathcal{V}$ is a fundamental system of neighbourhoods of m (see [41, p. 18]). Now a set O is open if and only if it is a neighbourhood of each of its points, if and only if for each $m \in O$ there exists $V \in \mathcal{V}$ such that $m + V \subseteq O$ (see [41, p. 19, Theorem 3.1]). We will use the notation $\langle M, \mathcal{V}\rangle$ to denote the topological structure

[3] At the time of the first submission (summer 2004) of this article, some of the bibliography did not exist yet.

$\langle M, \tau\rangle$ with a given fundamental system $\mathcal{V}$ of neighbourhoods of zero. If $\mathcal{L} = \mathcal{L}_{\rm rings}$, then the corresponding topology on M is called a *ring topology* (see [38] p. 3).

We will mostly deal with expansions of fields; in that case, a *topological $\mathcal{L}$-field* $\langle M, \tau\rangle$ will be an $\mathcal{L}\cup\{^{-1}\}$-structure and a topological $\mathcal{L}$-structure in which, in addition, the inverse function $^{-1}$ is continuous on $M^\times$. This generalizes the well-known notion of topological fields (see [31, p. 28]). However, note that in some texts, for instance in [11, p. 120], the field topology is assumed to be non-discrete, but we will not make this assumption here.

Recall that any field K can be endowed with the *Zariski topology*. A basis for open subsets of K^n, $n \geqslant 1$, consists of the sets $\{\bar{x} \in K^n : q(\bar{x}) \neq 0\}$, where $q[\bar{X}] \in K[\bar{X}]$. The polynomial maps from K^n to K^m are continuous, where K^n and K^m are both endowed with the Zariski topology. Note that the Zariski topology on K^n is not Hausdorff and it is not the product topology ($n \geq 2$).

A classical example of topological field is a field K with an absolute value $|.|$ taking its values in the positive real numbers $\mathbb{R}^{\geq 0}$ (see [31, p. 19]). One can check that $\langle K, +, -, ., ^{-1}, 0, 1, \mathcal{V}\rangle$ is a topological $\mathcal{L}_{\rm rings}$-field with a fundamental system $\mathcal{V}$ of neighbourhoods of zero consisting of the set $\{x \in K : |x| < r\}$; with $r \in \mathbb{R}^{\geq 0}$.

J. Shafarevic and I. Kaplansky have characterized the topological fields L whose topology is given by an absolute value (see [31, Theorem 4, p. 44]), as follows. Let T be the set of nilpotent elements of L, namely those elements whose powers converge to 0 and let N be the set of elements of L neither nilpotent nor whose inverse is nilpotent. A subset B of L is *bounded* if for every neighbourhood U of 0 there exists a neighbourhood W of 0 such that $B.W \subseteq U$. A topological field is *locally bounded* if it contains a non-empty open bounded set. Then, the result of Shafarevic and Kaplansky can be stated as follows: there exists an absolute value on L which determines the topology iff T is open and $T \cup N$ is bounded iff the topology on L is locally bounded and if there exists a non-zero nilpotent element.

Note that the condition $T \cup N$ bounded implies that given a neighbourhood V of 0, every element of K can be written as $x.y^{-1}$ with $x, y \in V$ and $y \neq 0$.

Definition 2.1. Given a topological $\mathcal{L}$-field $\langle K, \mathcal{V}\rangle$ and a neighbourhood $V \in \mathcal{V}$, we will say that K is a *V-field* if every element a of K can be written as $a = x.y^{-1}$ where $x, y \in V$ and $y \neq 0$.

In the examples that we will consider, either V or its closure $\bar{V}$ will be in addition a proper *preorder*.

Definition 2.2. A subset A of K is called a preorder if $0,\ 1 \in A$, $A.A \subseteq A$, $K^\times = A^\times.A^{\times -1}$ and there exists an element $z \in A^\times$ such that $z.(A + A) \subseteq A$; a preorder is proper if $A \neq K$.

There is a one-to-one correspondence between proper preorders and proper locally bounded topologies ([38] section 4.4, Theorem 2).

In later sections, when we will give axiomatizations of existentially closed fields, in the case when K is a V-field, we may restrict ourselves to consider polynomials with coefficients in V.

We will be interested in a certain property of our structures which is best axiomatized in a monadic second-order language $\mathcal{L}_t$ introduced by McKee (see [19]); he restricts $\mathcal{L}_2$-formulas by imposing that the sub-formulas are of the form $\exists X\psi$ $(\forall X\psi)$ where X only occurs negatively in ψ (respectively positively) and X belongs to τ (see also [11]). J. Flum and M. Ziegler have shown that $\mathcal{L}_t$-sentences have the property that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases for the same topology then $\langle M, \mathcal{B}_1\rangle \models \sigma$ iff $\langle M, \mathcal{B}_2\rangle \models \sigma$, where σ is an $\mathcal{L}_t$-sentence (see [11, p. 6]).

Note that one may characterize, in the language $\mathcal{L}_t$, those topological fields with an interval topology (respectively valuation topology) (see [11, p. 123, 126]). In both cases (when v is Archimedean), there exists an absolute value which induces the same topology.

2.2. Condition *Comp*(**K**)

F. Pop extensively studied *large* fields; recall that a field K is large if every smooth integral variety defined over K which has a point in K has infinitely many points in K (see [25] and Section 11). In particular, he showed that a field K is large if and only if it is existentially closed in the field $K((t))$ of Laurent series over K (see [25, Proposition 1.1]) if and only if for all $n \in \mathbb{N}$, K is existentially closed in the field $K((t_1))\cdots((t_n))$ of iterated Laurent series over K (see [40, Proposition 5.3]).

We will consider the case where K is in addition a topological $\mathcal{L}_{\rm rings}$-field; we will show that we can endow $K((t))$ with a topology in such a way that the topology on K is the induced topology and that we can define on $K((t))$ an equivalence relation of being *infinitely close* with respect to K (see Lemma 2.10). We want to encompass the case where K has the discrete topology and $K((t))$ a valuation topology induced by the canonical valuation map sending $K^\times$ to 0 and t to 1.

Definition 2.3. Let $\mathbf{L} := \langle L, \mathcal{W}\rangle$ (respectively $\mathbf{K} := \langle K, \mathcal{V}\rangle$) be a topological $\mathcal{L}$-field, where $\mathcal{W}$ (respectively $\mathcal{V}$) is a fundamental system of neighbourhoods of zero.

Then $\mathbf{L}$ is called a *topological $\mathcal{L}$-extension* of $\mathbf{K}$ ($\mathbf{K} \subseteq \mathbf{L}$) if

(1) K is an $\mathcal{L}$-substructure of L.
(2) For all $V \in \mathcal{V}$, there exists $W \in \mathcal{W}$ such that $W \cap K = V$.

In order to define the equivalence relation, we will single out certain subsets $\mathcal{W}(K)$ of $\mathcal{W}$ satisfying *Comp*(**K**) below. (A better notation might have been *Com*(**K**, **L**), but we thought it was a bit cumbersome.) The first two requirements will be that the topology induced by $\mathcal{W}(K)$ on K (see Definition 7.1, Chapter 3 in [9]) coincides with the one given by $\mathcal{V}$. The last three requirements are compatibility conditions with respect to the language $\mathcal{L}$. Note that, in general, the topology induced by $\mathcal{W}(K)$ on L will not be Hausdorff.

Definition 2.4. Let $\mathbf{K} := \langle K, \mathcal{V}\rangle$, $\mathbf{L} := \langle L, \mathcal{W}\rangle$ be a pair of topological $\mathcal{L}$-fields with $\mathbf{K} \subseteq \mathbf{L}$. We define a relation $\sim_{\mathcal{W}(K)}$ on L with respect to a subset $\mathcal{W}(K)$ of $\mathcal{W}$ satisfying the following conditions, denoted by *Comp*($\mathbf{K}$):

(1) $\forall V \in \mathcal{V}\ \exists W \in \mathcal{W}(K)\ W \cap K = V$ (so $\mathcal{W}(K)$ is non-empty);
(2) $\forall W \in \mathcal{W}(K)\ (W \cap K) \in \mathcal{V}$;
(3) $\forall a_0, a_1 \in K\ \forall V_0, V_1 \in \mathcal{V}\ \forall W_0, W_1 \in \mathcal{W}(K)$ with $W_i \cap K = V_i$, for $i = 0, 1$,

$$[(a_0 + V_0) \cap (a_1 + V_1) = \emptyset \Rightarrow (a_0 + W_0) \cap (a_1 + W_1) = \emptyset];$$

(4) for any n-ary function symbol $f \in \mathcal{L}$ and any $V_0, \ldots, V_n \in \mathcal{V}$, $a_1, \ldots, a_n \in K$ such that $f(a_1 + V_1, \ldots, a_n + V_n) \subseteq f(a_1, \ldots, a_n) + V_0$ and any $W_0, \ldots, W_n \in \mathcal{W}(K)$ with $(W_i \cap K) = V_i$ for $i = 0, \ldots, n$, we have

$$f(a_1 + W_1, \ldots, a_n + W_n) \subseteq f(a_1, \ldots, a_n) + W_0;$$

(5) $\forall a \in K \setminus \{0\}$, for any $V_0, V_1 \in \mathcal{V}$ and $W_0, W_1 \in \mathcal{W}(K)$ with $W_i \cap K = V_i$, for $i = 0, 1$, if $0 \notin (a + V_1)$ and $(a + V_1)^{-1} \subseteq a^{-1} + V_0$ then $(a + W_1)^{-1} \subseteq a^{-1} + W_0$.

Now, let a, b be two elements of L and assume that $\mathcal{W}(K)$ satisfies *Comp*($\mathbf{K}$).

Then a and b are *infinitely close* with respect to $\mathcal{W}(K)$ ($a \sim_{\mathcal{W}(K)} b$) iff for any neighbourhood $W \in \mathcal{W}(K)$, we have $a - b \in W$.

First, we will give some examples illustrating the above definition and then we will show that whenever $\mathcal{W}(K)$ satisfies *Comp*($\mathbf{K}$), the relation $\sim_{\mathcal{W}(K)}$ is an equivalence relation (see Lemma 2.11).

We will first consider ordered fields with the interval topology and non-trivially valued fields. Both are instances of *first-order topological structures* introduced by Pillay (see [23]). Then, following the treatment of van den Dries [8], we will examine fields with several distinct topologies. First, we recall the notion of a first-order definable topology (in the particular case of field expansions).

Definition 2.5. Let K be an $\mathcal{L}$-structure expanding a field. Then K satisfies *Hypothesis* (D) if there is a $\mathcal{L}$-formula $\phi(x, \bar{y})$ such that the set of subsets of the form $\phi(K, \bar{a}) := \{x \in K :\ K \models \phi(x, \bar{a})\}$, where $\bar{a} \subseteq K$, can be chosen as a basis $\mathcal{V}$ of neighbourhoods of 0 in K in such a way that $\langle K, \mathcal{V}\rangle$ is a topological $\mathcal{L}$-field.

Note that Hypothesis (D) is preserved by elementary equivalence.

Notation 2.6. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures expanding a field, satisfying Hypothesis (D). Then, the corresponding class $\mathcal{C}_{\mathrm{top}}$ of topological $\mathcal{L}$-fields is the following:

$$\mathcal{C}_{\mathrm{top}} := \{\mathbf{K} := \langle K, \mathcal{V}\rangle :\ K \in \mathcal{C} \text{ with } \mathcal{V} \text{ as in Definition 2.5}\}.$$

Let $\langle K, \mathcal{V}\rangle \subseteq \langle L, \mathcal{W}\rangle$ be two elements of $\mathcal{C}_{\mathrm{top}}$. Let us define $\mathcal{W}(K)$ the following subset of $\mathcal{W}$:

$$\{\phi(L, \bar{a}) \text{ with } \bar{a} \subseteq K\}.$$

Assume moreover that $\mathcal{C}$ is a model-complete, elementary class. Then given two elements K and L in $\mathcal{C}$, if $K \subseteq L$, then $\mathbf{K} \subseteq \mathbf{L}$, and $\mathcal{W}(K)$ satisfies *Comp*($\mathbf{K}$) since $K \prec L$. Moreover if K is a V_K-field with $V_K := \phi(K, \bar{k}_0)$, $\bar{k}_0 \subseteq K$, then L is a V_L-field with $V_L := \phi(L, \bar{k}_0)$, since the property of being a V-field is first-order.

Notation 2.7. Let $\langle A, +, 0, <\rangle$ be an abelian totally ordered group. Then $A^{>0}$ (respectively $A^{\geq 0}$) denotes the set of elements of A which are positive (respectively non-negative).

Let $\langle K, <\rangle$ be an ordered field; let $\mathcal{L}_< := \mathcal{L}_{\mathrm{rings}} \cup \{<\}$. We endow K with the interval topology $\tau_<$, so a fundamental system $\mathcal{V}_K$ of neighbourhoods of zero is:

$$\{]-a; a[;\ a \in K^{>0}\};$$

$\mathbf{K} := \langle K, \mathcal{V}_K\rangle$ is an example of a topological $\mathcal{L}_<$-field. Let $V_K :=]-1; 1[$, then $\mathbf{K}$ is a V_K-field. Note that $\overline{V_K} = [-1; 1]$ is a preorder of K (see Definition 2.2).

Let $\langle L, <\rangle$ be an $\mathcal{L}_<$-extension of K. Endow L with the interval topology, let $\mathcal{V}_L := \{]-a; a[;\ a \in L^{>0}\}$ and set $V_L :=]-1; 1[$.

The subset $\mathcal{W}(K)$ of $\mathcal{V}_L$ that we will consider in our applications is the set of neighbourhoods of zero in L defined over K: $\left\{\{l \in L :\ -a < l < a\};\ a \in K^{>0}\right\}$. Clearly the subset $\mathcal{W}(K)$ of $\mathcal{V}_L$ satisfies *Comp*($\mathbf{K}$).

Next we consider the example of valued fields.

Notation 2.8. Let $\langle K, v\rangle$ be a valued field with v its valuation map. We denote the valuation ring, its maximal ideal, the residue field and the value group by respectively $\mathcal{O}_K$, $\mathcal{M}_K$, k_K and $v(K^\times)$. Recall that $\langle v(K^\times), +, -, <, 0\rangle$ is an abelian totally ordered group. The residue map is denoted by $\bar{\cdot} : \mathcal{O}_K \to k_K$. In the case where K is endowed with several valuations, we add a subscript to distinguish between the valuations, we get for instance: $\mathcal{O}_{K,v}$, $\mathcal{M}_{K,v}$ and $k_{K,v}$.

Let $\langle K, v\rangle$ be a valued field. We equip this field with a topology τ_v by choosing as fundamental system $\mathcal{V}_K$ of neighbourhoods of zero, the sets $\{x \in K : v(x) > v(a)\}$, with $a \in K^\times$ (see [11, p. 123]); set $\mathbf{K} := \langle K, \mathcal{V}_K\rangle$. If we let $V_K := \mathcal{O}_K$, then K is an $\mathcal{O}_K$-field and $\mathcal{O}_K$ is a preorder of K. Let $\langle L, w\rangle$ be a valued field extension of K (namely the restriction of w on K is equal to v). Endow L with the valuation topology $\mathcal{V}_L$ and L is an $\mathcal{O}_L$-field. As in the case of ordered fields, in our applications we choose the set $\mathcal{W}(K) := \left\{\{l \in L : v(l) > v(a)\}; a \in K^\times\right\}$ of neighbourhoods of zero in L defined over K as the subset of $\mathcal{V}_L$ satisfying $Comp(\mathbf{K})$.

Finally, we consider the case of topological fields with several distinct orderings or valuations. First, let us introduce the following definition.

Definition 2.9. If $\tau_1, \ldots, \tau_e$ are topologies on a set R then $\tau_1 \vee \cdots \vee \tau_e$ is by definition the least upper bound of $\{\tau_1, \ldots, \tau_e\}$ in the set of topologies on R (which is ordered by inclusion).

If R is a ring and $\tau_1, \ldots, \tau_e$ are ring topologies then $\tau_1 \vee \cdots \vee \tau_e$ is a ring topology on R and a basis of neighbourhoods of 0 is given by the sets $U_1 \cap \cdots \cap U_e$ with U_i a τ_i-neighbourhood of 0, for all $1 \leqslant i \leqslant e$ (see for instance [38] chapter 3, section 3.1).

In his thesis [8], van den Dries studied the model theory of fields equipped with e distinct total orderings (respectively valuations) in the language $\mathcal{L}_{e,<} := \mathcal{L}_{\text{rings}} \cup \{<_i; 1 \leq i \leq e\}$ (respectively $\mathcal{L}_{e,v} := \mathcal{L}_{\text{rings}} \cup \{\mathcal{D}_i; 1 \leq i \leq e\}$, where $\mathcal{D}_i$ is the linear divisibility relation associated with the valuation v_i (see also Definition 6.1). If we denote by τ_i, $1 \leq i \leq e$, then the topology corresponding to the order $<_i$ (respectively the valuation v_i), we obtain that the topology $\tau_1 \vee \cdots \vee \tau_e$ is a ring topology and that a corresponding basis of neighbourhoods of 0 is given by $U_1 \cap \cdots \cap U_e$, where $U_i := \{x \in K : -a_i <_i x <_i a_i\}$, where $a_i \in K^{>_i 0}$ (respectively $U_i := \{x \in K : v_i(x) > v_i(a_i)\}$, where $a_i \in K^\times$). If we denote by A_i the corresponding preorders, namely $[-1; 1]_i := \{l \in K : -1 \leq_i l \leq_i 1\}$ and $\mathcal{O}_{K,v_i}$, we get in both cases, that the field K is an A-field, where $A := \bigcap_{i=1}^e A_i$ provided $\tau_1 \vee \cdots \vee \tau_e$ is not the discrete topology (see [38] section 4.5, Theorem 3).

If $\mathbf{L} := \langle L, <_1, \ldots, <_e\rangle$ is an $\mathcal{L}_{e,<}$-extension of $\mathbf{K} := \langle K, <_1, \ldots, <_e\rangle$, then the fact that the corresponding $\mathcal{W}(K) := \{\bigcap_{i=1}^e \{l \in L : -a_i <_i l <_i a_i\}; a_i \in K^{>_i 0}\}$ satisfies $Comp(\mathbf{K})$ is a consequence of the quantifier-free definability of the topologies in $\mathcal{L}_{e,<}$ and the fact that $\mathbf{K}$ is a substructure of $\mathbf{L}$.

We get a similar result for the valuation case, namely extensions of the kind: $\langle K, v_1, \ldots, v_e\rangle \subseteq \langle L, v_1, \ldots, v_e\rangle$.

Now let us come back to the case of one order or one valuation and let us consider the field $L := K((t))$ of Laurent series over K.

First, we will assume that K is a totally ordered field, and we will extend the order to L by setting t to be an infinitesimally small positive element.

Second, we will endow L with the discrete valuation v defined by: $v(\sum_{i \geq i_0} \alpha_i t^i) = i_0$, where $\alpha_i \neq 0$, $i, i_0 \in \mathbb{Z}$ (so the induced topology on K is the discrete topology).

In both cases, L is a topological $\mathcal{L}_{\text{rings}}$-extension of K and we described above a fundamental system $\mathcal{V}_L$ of neighbourhoods of zero and a subset $\mathcal{W}(K)$ of $\mathcal{V}_L$ satisfying $Comp(\mathbf{K})$. Note that in the second case $\mathcal{W}(K)$ consists of only one element, namely the maximal ideal of $\mathcal{O}_L$.

Suppose now that on K we are given a fundamental system $\mathcal{V}$ of neighbourhoods of zero such that $\langle K, \tau\rangle$ is a topological $\mathcal{L}$-field. We consider the field $\langle L, v\rangle$ of Laurent series over K as a topological valued field with the discrete valuation v defined above.

We write any element $a \in L$ as $a = \sum_{i \geqslant i_0} \alpha_i . t^i$ with $i_0,\ i \in \mathbb{Z}$, $\alpha_i \in K$ and $\alpha_{i_0} \neq 0$. We define a fundamental system $\mathcal{V}_L$ of neighbourhoods of zero in L as follows. It consists of the neighbourhoods of zero either of the form

$$W_{V,0} := \{a \in K((t)) : v(a) \geqslant 0 \text{ and } \alpha_0 \in V\}, \text{ or}$$
$$W_n := \{a \in K((t)) : v(a) \geqslant n\} \text{ with } V \in \mathcal{V} \text{ and } n \in \mathbb{N} \setminus \{0\}.$$

More generally, we will consider the *iterated Laurent series field extension* $K((t_1))((t_2)) \cdots ((t_n))$ over K, for some natural number $n \geqslant 1$ with a valuation map v taking its values in the lexicographic product $\mathbb{Z}^n$ of n copies of $\langle \mathbb{Z}, +, -, <, 0, 1\rangle$. We have that $K((t_1)) \cdots ((t_n)) \cong K((\mathbb{Z}^n))$, where the domain of $K((\mathbb{Z}^n))$ is the set of formal series of the form $\sum_{\gamma \in \mathbb{Z}^n} \alpha_\gamma . t^\gamma$ with $\text{supp}(a) := \{\gamma \in \mathbb{Z}^n : \alpha_\gamma \neq 0\}$ being a well-ordered subset of $\mathbb{Z}^n$. One defines the valuation v on the non-zero elements a as follows $v(a) = \min(\text{supp}(a))$.

Then we define a fundamental system of neighbourhoods $\mathcal{W}_n$ of zero as follows:

$$W_{V,0} := \{a \in K((t_1)) \cdots ((t_n)) : \alpha_0 \in V \text{ and } v(a) \geqslant 0\},$$
$$W_\gamma := \{a \in K((t_1)) \cdots ((t_n)) : v(a) \geqslant \gamma\} \text{ with } V \in \mathcal{V} \text{ and } \gamma \in (\mathbb{Z}^n)^{\geq 0}.$$

We will denote the corresponding topological structure as $\langle K((t_1))((t_2)) \cdots ((t_n)), \mathcal{W}_n, v\rangle$.

Note that given a neighbourhood V of zero in $\mathcal{V}$, $W_{V,0}$ is the unique element of $\mathcal{W}_n$ whose trace on K is equal to V.

Lemma 2.10. *Let $\mathbf{K} := \langle K, \mathcal{V}\rangle$ be a topological $\mathcal{L}$-field which is a V_0-field for some $V_0 \in \mathcal{V}$. Let L be the field $K((t_1))((t_2)) \cdots ((t_n))$ of iterated Laurent series over K. Then $\mathbf{L} := \langle L, \mathcal{W}_n\rangle$ is a topological $\mathcal{L}_{\text{rings}}$-extension of $\mathbf{K}$, L is a $W_{V_0,0}$-field and $Comp(\mathbf{K})$ is satisfied by the following subset $\mathcal{W}(K)$ of $\mathcal{W}_n$:*

$$\{W_{V,0} : V \in \mathcal{V}\}.$$

We denote the corresponding binary relation $\sim_{\mathcal{W}(K)}$ on L by $\sim_n$. Note also that $W_{V,0} \in \mathcal{W}(K)$ with $W_{V,0} \cap K = V$ and $t_i \sim_K 0$, $1 \leqslant i \leqslant n$.

Proof. First, the fact that we have defined a field topology (see [11, p. 120]), and that $\langle K((t_1))((t_2))\cdots((t_n)), \mathcal{W}_n, v\rangle$ is a topological extension of K are standard.

Let us show that $\mathcal{W}(K)$ satisfies *Comp*($\mathbf{K}$). It is easy to check items 1, 3 of Definition 2.4. Let us check item 4. Let $V_0 \in \mathcal{V}$ be a neighbourhood of zero in K. Let $V_1, V_2 \in \mathcal{V}$ be such that $V_1 + V_2 \subseteq V_0$. If W_i is an element of $\mathcal{W}_n$ such that $W_i \cap K = V_i$ then $W_i = W_{V_i,0}$, $0 \leqslant i \leqslant 2$. So $W_{V_1,0} + W_{V_2,0} \subseteq W_{V_0,0}$ and, similar arguments for $\cdot$ and $^{-1}$ yield the result. □

Lemma 2.11. *Let* $\mathbf{K} := \langle K, \mathcal{V}\rangle$, $\mathbf{L} := \langle L, \mathcal{W}\rangle$ *be two topological* $\mathcal{L}$*-fields, with* $\mathbf{K} \subseteq \mathbf{L}$*. Assume that the subset* $\mathcal{W}(K)$ *of* $\mathcal{W}$ *satisfies Comp*($\mathbf{K}$)*. Then* $\sim_{\mathcal{W}(K)}$ *is an equivalence relation on L.*

Proof. Let $a \sim_{\mathcal{W}(K)} b$ with $a, b \in L$. Let us prove that $b \sim_{\mathcal{W}(K)} a$. Let $W \in \mathcal{W}(K)$. The continuity of $-$ on K implies that there exists a neighbourhood $V_0 \in \mathcal{V}$ such that $-V_0 \subseteq W \cap K \in \mathcal{V}$. Consider an arbitrary element W_0 in $\mathcal{W}(K)$ such that $W_0 \cap K = V_0$ and we have $a - b \in W_0$. So, by applying *Comp*($\mathbf{K}$) to the symbol $-$ in $\mathcal{L}$, we get $-W_0 \subseteq W$. We conclude that $b - a \in W$.

Let $a \sim_{\mathcal{W}(K)} b$ and $b \sim_{\mathcal{W}(K)} c$ with $a, b, c \in L$. Let us prove that $a \sim_{\mathcal{W}(K)} c$. Let $W \in \mathcal{W}(K)$. Since $+$ is continuous on K, there exist $V_0, V_1 \in \mathcal{V}$ such that $V_0 + V_1 \subseteq W \cap K$. Applying *Comp*($\mathbf{K}$) to the function symbol $+$ in $\mathcal{L}$, for any two elements $W_0, W_1 \in \mathcal{W}(K)$ such that $W_i \cap K = V_i$ $(i = 0, 1)$, we get $W_0 + W_1 \subseteq W$. Using $a \sim_{\mathcal{W}(K)} b$ and $b \sim_{\mathcal{W}(K)} c$, we deduce $a - b \in W_0$ and $b - c \in W_1$; so we get $a - c \in W$. □

Let $\mathcal{L}(K)$ be the language $\mathcal{L}$ extended with constant symbols for the elements of K. In general the equivalence relation $\sim_{\mathcal{W}(K)}$ will not be a congruence on L; however in the following lemma, we will show a weaker "continuity" property of the $\mathcal{L}(K)$-terms.

Lemma 2.12. *Let* $\mathbf{K} := \langle K, \mathcal{V}\rangle$ *be a topological* $\mathcal{L}$*-field, let* $\mathbf{L} := \langle L, \mathcal{W}\rangle$ *be a topological* $\mathcal{L}$*-extension of* $\mathbf{K}$ *satisfying Comp*($\mathbf{K}$) *with respect to a subset* $\mathcal{W}(K)$ *of* $\mathcal{W}$*. Let* $t(x_1, \ldots, x_n)$ *be an* $\mathcal{L}(K)$*-term and let* $a_1, \ldots, a_n$ *be n elements of K.*

Then for any $V_0 \in \mathcal{V}$*, there exist neighbourhoods* $V_1, \ldots, V_n$ *of zero in* $\mathcal{V}$ *such that*

(1) $t(a_1 + V_1, \ldots, a_n + V_n) \subseteq t(a_1, \ldots, a_n) + V_0$ *and,*
(2) *for any* W_i *in* $\mathcal{W}_K$ $(0 \leqslant i \leqslant n)$ *with* $W_i \cap K = V_i$*, we get*

$$t(a_1 + W_1, \ldots, a_n + W_n) \subseteq t(a_1, \ldots, a_n) + W_0.$$

Proof. Let $V_0 \in \mathcal{V}$ and n elements $a_1, \ldots, a_n$ in K. We proceed by induction on the complexity of the $\mathcal{L}(K)$-term t.

First we may assume that the $\mathcal{L}(K)$-term is of the following form

$$t(x_1, \ldots, x_n) = f(t'_1(x_1, \ldots, x_n), \ldots, t'_m(x_1, \ldots, x_n))$$

where f is an m-ary function symbol in $\mathcal{L}$. By continuity of f, there exists n neighbourhoods V'_i of zero in $\mathcal{V}$ $(1 \leqslant i \leqslant n)$ such that

$$f(t'_1(a_1, \ldots, a_n) + V'_1, \ldots, t'_m(a_1, \ldots, a_n) + V'_n) \subseteq t(a_1, \ldots, a_n) + V_0.$$

Applying the induction hypothesis to the $\mathcal{L}(K)$-terms t'_i and to the neighbourhoods V'_i, we get $V_j \in \mathcal{V}$ $(1 \leq j \leq n)$ such that

$$t'_i(a_1 + V_1, \ldots, a_n + V_n) \subseteq t'_i(a_1, \ldots, a_n) + V'_i, \quad \text{for } 1 \leq i \leq m.$$

Now, using item 4 of Definition 2.4, given any W_i, W'_j in $\mathcal{W}(K)$ such that $W_i \cap K = V_i$ and $W'_j \cap K = V'_j$ $(0 \leqslant i \leqslant n, 1 \leqslant j \leqslant n)$ and the above equality, we have that

$$\begin{aligned} f(t'_1(a_1 + W_1, \ldots, a_n + W_n), \ldots, t'_m(a_1 + W_1, \ldots, a_n + W_n)) &\subseteq f(t'_1(a_1, \ldots, a_n) + W'_1, \ldots, t'_m(a_1, \ldots, a_n) + W'_n) \\ &\subseteq t(a_1, \ldots, a_n) + W_0. \quad \square \end{aligned}$$

Lemma 2.13. *Let* $\mathbf{K} := \langle K, \mathcal{V}\rangle$ *be a topological* $\mathcal{L}$*-field, let* $\mathbf{L} := \langle L, \mathcal{W}\rangle$ *be a topological* $\mathcal{L}$*-extension of* $\mathbf{K}$ *satisfying Comp*($\mathbf{K}$) *with respect to a subset* $\mathcal{W}(K)$ *of* $\mathcal{W}$*.*

Let $f, g \in K[X_0, \ldots, X_{n+1}]$ *be such that* $f(\alpha_0, \ldots, \alpha_n) = \beta$ *and* $g(\alpha_0, \ldots, \alpha_n) = \gamma$*, for some elements* $\alpha_0, \ldots, \alpha_n, \beta, \gamma$ *in K.*

If $t_0, \ldots, t_n$ *are elements in L with* $t_i \sim_{\mathcal{W}(K)} 0$, $0 \leqslant i \leqslant n$ *then*

(1) $f(\alpha_0 + t_0, \ldots, \alpha_n + t_n) \sim_{\mathcal{W}(K)} \beta$*;*
(2) *if* $\beta \neq 0$ *then* $f(\alpha_0 + t_0, \ldots, \alpha_n + t_n) \neq 0$*, and* $g(\alpha_0 + t_0, \ldots, \alpha_n + t_n).(f(\alpha_0 + t_0, \ldots, \alpha_n + t_n))^{-1} \sim_{\mathcal{W}(K)} \gamma.\beta^{-1}$.

Proof. Let $W \in \mathcal{W}(K)$. We need to show that $f(\alpha_0 + t_0, \ldots, \alpha_n + t_n) - \beta \in W$.

First, consider the $\mathcal{L}(K)$-term $g(x_1, \ldots, x_{n+1}) := f(x_1, \ldots, x_{n+1}) - \beta$. We have $g(\alpha_1, \ldots, \alpha_{n+1}) = 0$. By Lemma 2.12, there exist $n+1$ neighbourhoods $V_0, \ldots, V_n$ in $\mathcal{W}(K)$ such that $g(\alpha_0+V_0, \ldots, \alpha_n+V_n) \subseteq W \cap K$ and, for any neighbourhood W_i of zero in $\mathcal{W}(K)$ $(0 \leqslant i \leqslant n)$ with $W_i \cap K = V_i$ we get

$$g(\alpha_0 + W_0, \ldots, \alpha_n + W_n) \subseteq W.$$

Since we have $t_i \sim_{\mathcal{W}(K)} 0$, $t_i \in W_i$ $(0 \leqslant i \leqslant n)$ and so we get $g(\alpha_0 + t_0, \ldots, \alpha_n + t_n) \in W$.

In addition, if $\beta \neq 0$ then there exists $W' \in \mathcal{W}(K)$ such that $0 \notin \beta + W'$ by item 3 in Definition 2.4 of *Comp*($\mathbf{K}$); so we get the result.

We consider the $\mathcal{L}(K)$-term $h(x_1,\ldots,x_{n+1}) := \gamma.f(x_1,\ldots,x_{n+1}) - \beta.g(x_1,\ldots,x_{n+1})$. We have $h(\alpha_0,\ldots,\alpha_n) = 0$ and we proceed as above to get that

$$h(\alpha_0+t_0,\ldots,\alpha_n+t_n) \sim_{\mathcal{W}(K)} 0.$$

Further, applying Lemma 2.12 to the $\mathcal{L}(K)$-term $h(x_1,\ldots,x_n).\beta^{-1}$, we have that $h(\alpha_0+t_0,\ldots,\alpha_n+t_n).\beta^{-1} \sim_{\mathcal{W}(K)} 0$.

It remains to show that $h(\alpha_0+t_0,\ldots,\alpha_n+t_n).\beta^{-1}.f(\alpha_0+t_0,\ldots,\alpha_n+t_n)^{-1} \sim_{\mathcal{W}(K)} 0$.

Given any $W_0 \in \mathcal{W}(K)$, we want to show that $h(\alpha_0+t_0,\ldots,\alpha_n+t_n).\beta^{-1}.f(\alpha_0+t_0,\ldots,\alpha_n+t_n)^{-1} \in W_0$. Since $\cdot$ is continuous, there exist $V_1,\ V_2 \in \mathcal{V}$ such that $V_1.(\beta^{-1}+V_2) \subseteq W_0 \cap K$. So, by item 4 of *Comp*($\mathbf{K}$), for any $W_1, W_2 \in \mathcal{W}(K)$ with $W_1 \cap K = V_1$ and $W_2 \cap K = V_2$, we have that $W_1.(\beta^{-1}+W_2) \subseteq W_0$. Since $^{-1}$ is continuous on $K^\times$, there exists $V_2' \in \mathcal{V}$ such that $(\beta+V_2')^{-1} \subseteq \beta^{-1}+V_2$. Then, using item 5 of *Comp*($\mathbf{K}$), for any W_2' with $W_2' \cap K = V_2$, we have that $(\beta+W_2')^{-1} \subseteq \beta^{-1}+W_2$. Finally, note that by (1), $f(\alpha_0+t_0,\ldots,\alpha_n+t_n) \in \beta+W_2'$. We also have that $h(\alpha_0+t_0,\ldots,\alpha_n+t_n).\beta^{-1} \in W_1$, and so $h(\alpha_0+t_0,\ldots,\alpha_n+t_n).\beta^{-1}.f(\alpha_0+t_0,\ldots,\alpha_n+t_n)^{-1} \in W_0$. □

In the following Lemma, we will consider two successive extensions of K, namely let $\mathbf{L} := \langle L, \mathcal{W}\rangle$ be an extension of $\mathbf{K} := \langle K, \mathcal{V}\rangle$ and $\mathbf{F} := \langle F, \mathcal{T}\rangle$ an extension of $\mathbf{L}$. Then, if $a, b \in F$ satisfies $a \sim_{\mathcal{T}(L)} b$ with $\mathcal{T}(L) \subseteq \mathcal{T}$ satisfying *Comp*($\mathbf{L}$), then under which conditions does $\mathcal{T}(L)$ satisfy *Comp*($\mathbf{K}$), equivalently when are a, b infinitely close with respect to $\mathbf{K}$?

Lemma 2.14. *Let $\mathbf{L} := \langle L, \mathcal{W}\rangle$ be an extension of $\mathbf{K} := \langle K, \mathcal{V}\rangle$ and $\mathbf{F} := \langle F, \mathcal{T}\rangle$ an extension of $\mathbf{L}$. Assume that $\mathcal{W}(K) \subseteq \mathcal{W}$ (respectively $\mathcal{T}(L) \subseteq \mathcal{T}$) satisfies Comp($\mathbf{K}$) (respectively Comp($\mathbf{L}$)). Let $\widetilde{\mathcal{T}}(K) := \{T \in \mathcal{T}(L) : T \cap L \in \mathcal{W}(K)\}$.*

Then $\widetilde{\mathcal{T}}(K)$ satisfies Comp($\mathbf{K}$). Moreover, for any two elements a, b in L such that $a \sim_{\mathcal{W}(K)} b$, we have $a \sim_{\widetilde{\mathcal{T}}(K)} b$.

Proof. Let us check the first condition. Let $V \in \mathcal{V}$. Then since $\mathcal{W}(K)$ satisfies *Comp*($\mathbf{K}$), there exists $W \in \mathcal{W}(K)$ such that $W \cap K = V$. Similarly, since $\mathcal{T}(L)$ satisfies *Comp*($\mathbf{L}$), there exists $T \in \mathcal{T}(L)$ such that $T \cap L = W$. So $T \in \widetilde{\mathcal{T}}(K)$ and $T \cap K = V$. Second, by definition of $\widetilde{\mathcal{T}}(K)$, for any $T \in \widetilde{\mathcal{T}}(K)$ we get $T \cap K \in \mathcal{V}$.

Let $a_1, a_2 \in K$ and $V_1, V_2 \in \mathcal{V}$ with $(a_1+V_1) \cap (a_2+V_2) = \emptyset$. Choose $T_1, T_2 \in \widetilde{\mathcal{T}}(K)$ with $T_i \cap K = V_i$, $i = 1, 2$. Since $T_i \cap L \in \mathcal{W}(K)$ and $\mathcal{W}(K)$ satisfies *Comp*($\mathbf{K}$), we get $[a_1+(T_1 \cap L)] \cap [a_2+(T_2 \cap L)] = \emptyset$ and then, by applying item 3 of *Comp*($\mathbf{L}$) to $T_i \cap L \in \mathcal{W}(K)$, we get $(a_1+T_1) \cap (a_2+T_2) = \emptyset$.

Now let us consider an n-ary function symbol f. Let $V_0, V_1, \ldots, V_n \in \mathcal{V}$ and $a_1, \ldots, a_n \in K$ with $f(a_1+V_1,\ldots,a_n+V_n) \subseteq f(a_1,\ldots,a_n)+V_0$.

Let $T_0, T_1, \ldots, T_n \in \widetilde{\mathcal{T}}(K)$ be such that $\bigwedge_{i=0}^n\ T_i \cap K = V_i$. By hypothesis $T_i \cap L \in \mathcal{W}(K)$ and we have $(T_i \cap L) \cap K = V_i$. Applying the fact that $\mathcal{W}(K)$ satisfies *Comp*($\mathbf{K}$), we get $f(a_1+(T_1 \cap L),\ldots,a_n+(T_n \cap L)) \subseteq f(a_1,\ldots,a_n)+(T_0 \cap L)$.

Since $\mathcal{T}(L)$ satisfies *Comp*($\mathbf{L}$), we have $f(a_1+T_1,\ldots,a_n+T_n) \subseteq f(a_1,\ldots,a_n)+T_0$. Finally a similar reasoning for the function symbol $^{-1}$ yields the result. □

Now let us investigate how *Comp*($\mathbf{K}$) behaves under union of chains of topological $\mathcal{L}$-fields.

Before doing that, we need to put some conditions on the elements of a chain of topological $\mathcal{L}$-fields, in order to have that the union of this chain is still a topological $\mathcal{L}$-field.

In the following, α, μ, ν, and λ range over the class of ordinal numbers. Let $\mathbf{K}_\alpha := \langle K_\alpha, \mathcal{V}_\alpha\rangle$ be a topological $\mathcal{L}$-field and $(\mathbf{K}_\alpha)_{\alpha<\lambda}$ be a sequence of such topological $\mathcal{L}$-fields, where λ is a limit ordinal, such that $(K_\alpha)_{\alpha<\lambda}$ is a chain of $\mathcal{L}$-fields (i.e., $K_\alpha \subseteq K_{\alpha+1}$ for all $\alpha < \lambda$ and $K_\mu = \bigcup_{\alpha<\mu} K_\alpha$ for every limit ordinal $\mu < \lambda$), with union $K_\lambda := \bigcup_{\alpha<\lambda} K_\alpha$. Suppose that for $\alpha < \lambda$,

(1) we are given a subset $\mathcal{W}_{\alpha+1}$ of $\mathcal{V}_{\alpha+1}$ satisfying *Comp*($\mathbf{K}_\alpha$),
(2) for any limit ordinal $\mu \le \lambda$, we let $\mathcal{V}_\mu^{\lim}$ denote the collection

$$\left\{\bigcup_{\nu\leqslant\alpha<\mu} W_\alpha :\ W_\nu \in \mathcal{V}_\nu \text{ and } W_{\alpha+1} \in \mathcal{W}_{\alpha+1} \text{ with } W_{\alpha+1} \cap K_\alpha = W_\alpha;\ \text{ for } \nu \leqslant \alpha < \mu\right\}$$

of subsets of K_μ. We say that $(\mathbf{K}_\alpha)_{\alpha<\lambda}$ is a chain of topological $\mathcal{L}$-fields (relative to the choice of $(\mathcal{W}_{\alpha+1})_{\alpha<\lambda}$) if for every $\alpha < \lambda$, $\mathbf{K}_{\alpha+1}$ is a topological $\mathcal{L}$-extension of $\mathbf{K}_\alpha$, and for every limit ordinal $\mu < \lambda$, a fundamental basis of neighbourhoods of zero in K_μ is given by $\mathcal{V}_\mu^{\lim}$.

So the topologies on K_μ given by $\mathcal{V}_\mu$ and $\mathcal{V}_\mu^{\lim}$ coincide. This has the following consequence. Let R be a n-ary relation of $\mathcal{L}$. Since $\mathbf{K}_\mu$ is a topological $\mathcal{L}$-field, the interpretation R_μ of R in K_μ is the union of an open subset O_μ of K_μ^n and an algebraic subset A_μ. On the other hand, since K_μ is the union of the topological $\mathcal{L}$-fields K_α with $\alpha < \mu$, $R_\mu = \bigcup_{\alpha<\mu} R_\alpha$, where each of the $R_\alpha = O_\alpha \cup A_\alpha$ with O_α is an open subset of K_α^n and A_α is an algebraic subset of K_α^n. So, $\bigcup_{\alpha<\mu} O_\alpha \cup \bigcup_{\alpha<\mu} A_\alpha = O_\mu \cup A_\mu$. We have a corresponding compatibility condition for the negation $\neg R$ of R; we will denote the set of these compatibility conditions for each relation symbol of $\mathcal{L}$ by $(*_R)^{\lim}$.

Example(s) 2.15. Let $\mathcal{C}$ be a model-complete elementary class of $\mathcal{L}$-structures expanding a field, satisfying Hypothesis (D) with respect to the formula $\phi(x,\bar{y})$. Let K_n, $n \in \omega$ be a chain of elements of $\mathcal{C}$, set $K_\omega := \bigcup_{n\in\omega} K_n$. Then, we define $\mathcal{W}_{n+1} := \{\phi(K_{n+1},\bar{k}) :\ \bar{k} \subseteq K_n\}$ and $\mathcal{V}_\omega^{\lim} = \{\phi(K_\omega,\bar{k}_n);\ \bar{k}_n \subseteq K_n,\ n \in \omega\} = \{\phi(K_\omega,\bar{k});\ \bar{k} \subseteq K_\omega\}$.

Lemma 2.16. *Suppose that* $(\langle K_\alpha, \mathcal{V}_\alpha\rangle)_{\alpha<\lambda}$ *is a chain of topological* $\mathcal{L}$*-fields relative to* $(\mathcal{W}_{\alpha+1})_{\alpha<\lambda}$*. Then* $\langle K_\lambda, \mathcal{V}_\lambda^{\lim}\rangle$ *is a topological* $\mathcal{L}$*-extension of* $\langle K_0, \mathcal{V}_0\rangle$.

Proof. First we show that the topology induced by $\mathcal{V}_\lambda$ on K_λ is Hausdorff. Let a, b be in K_λ. Then for some $\mu < \lambda$, we have a, $b \in K_\mu$. Since $\mathbf{K}_\mu$ is Hausdorff, there are two neighbourhoods $W_{1,\mu}, W_{2,\mu}$ in $\mathcal{V}_\mu$ such that $(a+W_{1,\mu}) \cap (b+W_{2,\mu}) = \emptyset$. Then by induction on $\alpha \geq \mu$, we assume that there exist two neighbourhoods $W_{1,\alpha}, W_{2,\alpha} \in \mathcal{W}_\alpha$ such that $(a+W_{1,\alpha}) \cap (b+W_{2,\alpha}) = \emptyset$. Then by hypothesis $Comp(\mathbf{K}_\alpha)$ (see item 3 of Definition 2.4), there exist $W_{i,\alpha+1} \in \mathcal{W}_{\alpha+1}$ such that $W_{i,\alpha+1} \cap K_\alpha = W_{i,\alpha}$, $i = 1, 2$ and $(a + W_{1,\alpha+1}) \cap (b + W_{2,\alpha+1}) = \emptyset$. For a limit ordinal $\lambda > \nu > \mu$, we let $W_{i,\nu} := \bigcup_{\alpha<\nu} W_{i,\alpha}$, $i = 1, 2$ and $W_i := \bigcup_{\mu\leqslant\alpha<\lambda} W_{i,\alpha} \in \mathcal{V}_\lambda^{\lim}$, $i = 1, 2$. By induction hypothesis, we get $(a + W_1) \cap (b + W_2) = \emptyset$.

Now we prove that the function symbols are continuous on K_λ with respect to the topology determined by $\mathcal{V}_\lambda^{\lim}$. Let f be a n-ary function symbol in $\mathcal{L}$, let $a_1, \ldots, a_n$ be in K_λ and let $V_0 \in \mathcal{V}_\lambda^{\lim}$. So there is some ordinal $\mu < \lambda$ such that $a_1, \ldots, a_n \in K_\mu$ and $V_0 := \bigcup_{\alpha\geqslant\mu} V_{0,\alpha}$ with $V_{0,\mu} \in \mathcal{V}_\mu$ and $V_{0,\alpha} \in \mathcal{W}_\alpha$ for any $\alpha > \mu$. Now by continuity of f on K_μ, there exist $V_{1,\mu}, \ldots, V_{n,\mu} \in \mathcal{V}_\mu$ with $f(a_1+V_{1,\mu}, \ldots, a_n+V_{n,\mu}) \subseteq f(a_1, \ldots, a_n)+V_{0,\mu}$. Then by induction, we assume that for all $\delta \geq \mu$ with $\delta < \alpha$, $f(a_1 + V_{1,\delta}, \ldots, a_n + V_{n,\delta}) \subseteq f(a_1, \ldots, a_n) + V_{0,\delta}$. Then either $\delta = \alpha - 1$, we choose neighbourhoods of zero $V_{1,\alpha}, \ldots, V_{n,\alpha} \in \mathcal{W}_\alpha$ such that $V_{i,\alpha} \cap K_{\alpha-1} = V_{i,\alpha-1}$, $i = 1, \ldots, n$. Or α is a limit ordinal, and we let $V_{i,\alpha} := \bigcup_{\nu<\alpha} V_{i,\nu}$, $i = 1, \ldots, n$ and $V_i := \bigcup_{\mu\leqslant\alpha<\lambda} V_{i,\alpha} \in \mathcal{V}_\lambda$, $i = 1, \ldots, n$. By applying item 4 in Definition 2.4, we get that

$$f(a_1 + V_1, \ldots, a_n + V_n) \subseteq f(a_1, \ldots, a_n) + V_0.$$

Now let us consider an element $V_0 := W_0 \in \mathcal{V}$. Since any topological $\mathcal{L}$-field $\mathbf{K}_{\alpha+1}$ satisfies $Comp(\mathbf{K}_\alpha)$, we can choose a sequence of elements $W_{\alpha+1}$ in $\mathcal{W}_{\alpha+1}$ with $W_{\alpha+1} \cap K_\alpha = W_\alpha$ and for a limit ordinal $\mu \in \lambda$, we let $W_\mu := \bigcup_{\alpha<\mu} W_\alpha \in \mathcal{V}_\mu$. So $W := \bigcup_{\alpha<\lambda} W_\alpha$ satisfies $W \cap K_0 = V_0$. □

Note that the topology we put on K_λ differs from the topology one generally puts on direct limits of topological structures. For instance, in our setting, the embeddings of the $\mathbf{K}_\alpha$'s in $\mathbf{K}_\lambda$ are not necessarily continuous.

Corollary 2.17. *The topological* $\mathcal{L}$*-field* $\langle K_\lambda, \mathcal{V}_\lambda^{\lim}\rangle$ *satisfies* $Comp(\mathbf{K_0})$ *with respect to some* $\widetilde{\mathcal{W}}_\lambda \subseteq \mathcal{V}_\lambda^{\lim}$.

Proof. Using Lemma 2.14 and induction on α, we may assume that there exists for every $\alpha < \mu$, $\widetilde{\mathcal{W}}_\alpha \subseteq \mathcal{W}_\alpha$ satisfying $Comp(\mathbf{K_0})$.

Let μ be a successor ordinal, then $\widetilde{\mathcal{W}}_\mu := \{W \in \mathcal{W}_\mu : W \cap K_{\mu-1} \in \widetilde{\mathcal{W}}_{\mu-1}\}$ satisfies $Comp(\mathbf{K_0})$ by Lemma 2.14, replacing in the proof of that Lemma K by K_0, L by $K_{\mu-1}$ and F by K_μ.

If μ is a limit ordinal, then $\widetilde{W}_\mu := \{W \in \mathcal{W}_\mu : W \cap K_\alpha \in \widetilde{W}_\alpha \ \forall\alpha < \mu\}$ also satisfies $Comp(\mathbf{K_0})$. □

2.3. Inductive topological classes

Now, we are finally coming to the definition of the (inductive) classes of topological fields we will consider in the next section. The property we will require on these classes is stated in Definition 2.21 below. It can be viewed as an attempt of translating in our topological setting the property for a (pure) field of being *large*.

Recall that an *inductive* class of $\mathcal{L}$-structures is a class closed by union of chains (see [13]). We define an *inductive class of topological* $\mathcal{L}$*-fields* as follows.

Definition 2.18. A class $\mathcal{C}$ of topological $\mathcal{L}$-fields is said to be *inductive* if it satisfies the following properties:

(1) for each element $\langle K, \mathcal{V}\rangle$ of $\mathcal{C}$, one can choose a neighbourhood $V_K \in \mathcal{V}$ such that K is a V_K-field and given any pair of elements of $\mathcal{C}$, say $\langle K_i, \mathcal{V}_i\rangle$, $1 \leq i \leq 2$, with $\langle K_1, \mathcal{V}_1\rangle \subseteq \langle K_2, \mathcal{V}_2\rangle$, then $V_{K_2} \cap K_1 = V_{K_1}$,
(2) $\mathcal{C}$ is closed under taking chains satisfying conditions (1) and (2) before Lemma 2.16, provided K_λ is equipped with the fundamental system of neighbourhoods of zero $\mathcal{V}_\lambda^{\lim}$ and satisfies $(*_R)^{\lim}$, for every n-ary relation symbol R of $\mathcal{L}$.

Example(s) 2.19. Let $\mathcal{C}$ be a model-complete elementary class of $\mathcal{L}$-structures expanding a field, satisfying Hypothesis (D) with respect to the formula $\phi(x, \bar{y})$. Assume in addition that each element K of $\mathcal{C}$ is a V_K-field with $V_K := \phi(K, \bar{c})$ where $\bar{c}$ is a tuple of constants of $\mathcal{L}$. For every chain $K_\alpha, \alpha \in On,\ \alpha < \mu, \mu \in On$, of elements of $\mathcal{C}$, set $\mathcal{W}_{\alpha+1} := \{\phi(K_{\alpha+1}, \bar{k}) :\ \bar{k} \subseteq K_\alpha\}$. Then, $\mathcal{C}_{\mathrm{top}}$ is an inductive class of topological $\mathcal{L}$-fields (see Example 2.15).

Notation 2.20. Let W be a neighbourhood of 0 in K. We will denote by $W[X_0, \ldots, X_n]$, $n \in \mathbb{N}$, the set of polynomials in $K[X_0, \ldots, X_n]$ whose coefficients are in W.

If $f(X) \in W[X]$, then $f'(X)$ denotes the formal derivative of f with respect to X.

Definition 2.21. An inductive class $\mathcal{C}$ of topological $\mathcal{L}$-fields satisfies *Hypothesis* (I) if for every element $\langle K, \mathcal{V}\rangle$ of $\mathcal{C}$ the following holds:

(1) there exists an element V_K in $\mathcal{V}$ such that K is a V_K-field,
(2) given the topological $\mathcal{L}_{\mathrm{rings}}$-extension $\mathbf{L} := \langle K((t_1)) \cdots ((t_n)), \mathcal{W}_n\rangle$ of $\mathbf{K}$ and a polynomial $f(X) \in W_{V_K,0}[X]$ (with $W_{V_K,0}$ as defined in Lemma 2.10), if we have $f(a) \sim_K 0$ and $f'^2(a) \not\sim_K 0$ for some element $a \in W_{V_K,0}$, then there exists a topological $\mathcal{L}_{\mathrm{rings}}$-extension $\langle\widetilde{L}, \mathcal{W}\rangle$ of $\mathbf{L}$ such that
   (a) $\langle\widetilde{L}, \mathcal{W}\rangle$ is a topological $\widetilde{\mathcal{L}}$-extension of $\mathbf{K}$ and belongs to $\mathcal{C}$;
   (b) there exists a subset $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ satisfying $Comp(\mathbf{K})$ with $t_i \sim_{\widetilde{\mathcal{W}}} 0$, $i = 1, \ldots, n$;
   (c) $\widetilde{L}$ is a $\widetilde{W}$-field with $\widetilde{W} \in \widetilde{\mathcal{W}}$ and;
   (d) there exists an element b of $\widetilde{W}$ with $f(b) = 0$ and $a \sim_{\widetilde{\mathcal{W}}} b$.

If $\mathcal{C}$ is an inductive class of $\mathcal{L}$-structures which are expansions of fields, then we weaken the above condition to *Hypothesis* $(I)_{\text{red}}$. We denote by $\mathbf{K}$ the structure K endowed with the discrete topology.

The class $\mathcal{C}$ satisfies *Hypothesis* $(I)_{\text{red}}$ if for every element K of $\mathcal{C}$ the following holds: given the topological $\mathcal{L}_{\text{rings}}$-extension $\mathbf{L} := \langle K((t_1))\cdots((t_n)), \mathcal{W}_n\rangle$ of $\mathbf{K}$ and a polynomial $f(X) \in W_{K,0}[X]$ (with $W_{K,0}$ as defined in Lemma 2.10), if we have $f(a) \sim_K 0$ and $f'^2(a) \not\sim_K 0$ for some element $a \in W_{K,0}$, then there exists an element $\widetilde{L} \in \mathcal{C}$ such that $\langle \widetilde{L}, \mathcal{W}\rangle$ is a topological $\mathcal{L}_{\text{rings}}$-extension of $\mathbf{L}$ and

(1) there exists a subset $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ satisfying $Comp(\mathbf{K})$ with $t_i \sim_{\widetilde{\mathcal{W}}} 0, i = 1, \ldots, n$;
(2) there exists an element b of $\widetilde{W}$ with $f(b) = 0$ and $a \sim_{\widetilde{\mathcal{W}}} b$.

Now we will revisit our previous examples of classes of topological $\mathcal{L}$-fields (note that they are inductive topological classes) and show that they satisfy *Hypothesis* (I).

Example(s) 2.22. 1. If $\langle K, <\rangle$ is an ordered field, then $\langle K, \mathcal{V}_K\rangle$ is the corresponding topological $\mathcal{L}_<$-field, and $\mathcal{C}_<$ is the class of such topological $\mathcal{L}_<$-fields. We can extend the order on the Laurent series field $K((t))$ by choosing t positive and infinitely small with respect to $K^{>0}$ for the order topology (namely $0 < t < K^{>0}$). Then, in order to show that $\mathcal{C}_<$ satisfies *Hypothesis* (I), we use the fact that any ordered field has a real closure where the intermediate value property holds. Namely, given a polynomial $f[X]$ belonging to $K[[t]][X]$ satisfying the above hypothesis, we consider the Taylor expansion of f: $f(a+h) = f(a) + h.f'(a) +$ higher-order terms and we note that for small h in K we get a change of signs.

1_e. Let $\langle K, <_1, \ldots, <_e\rangle$ be an e-fold ordered field, we consider K as a topological $\mathcal{L}_{e,<}$-field with the supremum of the corresponding topologies $\tau_{<_1} \vee \cdots \vee \tau_{<_e}$. Let $\mathcal{C}_{<,e}$ the corresponding class of topological $\mathcal{L}_{e,<}$-fields.

We equip the field of Laurent series $K((t))$ with e pairwise distinct orderings by setting $0 <_i t+(i-1) <_i K^{>_i 0}, 1 \le i \le e$. Then, in order to show that $\mathcal{C}_{<,e}$ satisfies Hypothesis (I), we use an analogous argument based on the fact proved by van den Dries that in the class of existentially closed such fields, any polynomial which changes of sign with respect to each ordering has a zero. This example will be developed in Section 10.

2. If $\langle K, v\rangle$ be a valued field, we consider K as a topological $\mathcal{L}_{\text{rings}}$-field equipped with the valuation topology. Denote by $\mathcal{C}_v$ this class of topological fields equipped with the valuation topology. Extend the valuation on $K((t))$ such that $v(t) > v(c)$ for any $c \in K^\times$. Then in order to show that $\mathcal{C}_v$ satisfies *Hypothesis* (I), we use the fact that it always has an Henselization (see [31, p. 131]) which is an immediate extension and satisfies Hensel's Lemma (or one of its equivalent form, for instance Newton's Lemma (see [31, p. 98, 100])).

2_e. Let $\langle K, v_1, \ldots, v_e\rangle$ be a valued field with e pairwise distinct valuation v_i and consider it as a topological $\mathcal{L}_{\text{rings}}$-field equipped with the supremum of the corresponding topologies $\tau_{v_1} \vee \cdots \vee \tau_{v_e}$.

We equip the field of Laurent series $K((t))$ with e pairwise distinct valuations by setting $v_i(t + (i-1)) = 1, 1 \le i \le e$. In Section 10, we will prove that the corresponding class of topological fields satisfy *Hypothesis* (I), based on the fact that a generalized Hensel's lemma holds in the class of existentially closed structures with respect to the intersection of the corresponding valuation rings $(\mathcal{O}_{v_i})$.

3. Let $\mathcal{C}$ be an inductive class closed under taking ultrapowers, consisting of $\mathcal{L}$-structures which are expansions of large fields. Then $\mathcal{C}$ satisfies Hypothesis $(I)_{\text{red}}$. Note that it follows from the definition of *large* (see [25, p. 2] and Section 2.2) that the union of a chain of large fields is a large field. Let $K \in \mathcal{C}$; let $\mathbf{K}$ be K endowed with the discrete topology. Consider $L := \langle K((t_1))\cdots((t_n)), \mathcal{W}_n\rangle$ as defined in Lemma 2.10. By that lemma, we have that $\mathcal{W}_n$ satisfies $Comp(\mathbf{K})$. Let $f[X] \in W_{K,0}[X]$.

Recall that (L, v) is a Henselian valued field and so, if we can find $a \in \mathcal{O}_L$ with $v(f(a)) > v(f'^2(a))$, then by Newton's Lemma, there exists $b \in \mathcal{O}_L$ such that $f(b) = 0$ and $v(b-a) \ge v(f(a)) - v(f'(a))$.

Since K is a large field, K is existentially closed in the field of iterated Laurent series over K ([25] Proposition 5.3) and so by Frayne's Theorem, there exists a non-principal ultrapower K^* (belonging to $\mathcal{C}$) of K such that $K \subseteq K((t_1))\cdots((t_n)) \subseteq K^*$. Finally we extend v on K^*, getting a valuation v^* (since L is Henselian v^* is unique on the algebraic closure of L in K^*). Then we define the following system of neighbourhoods of zero in K^*: $\widetilde{W_{K,0}} := \{x \in K^* : v^*(x) \ge 0 \ \& \text{ if } x \in L, \text{ then } x \in W_{K,0}\}$, $\widetilde{W_g} := \{x \in K^* : v^*(x) \ge g\}$, where $g \in v^*(K^*)^{\ge 0}$.

So (K^*, v^*) is a topological extension of (L, v) and $\mathcal{C}$ satisfies *Hypothesis* $(I)_{\text{red}}$.

Note in the above examples that the structure $\widetilde{L}$ occurring in Definition 2.21, can be viewed as a *model-theoretic* completion.

Definition 2.23. (1) Let $\langle K, \tau\rangle$ be a topological $\mathcal{L}$-field. A *Cauchy* sequence in K is a sequence $(a_\mu)_{\mu<\lambda}$ of elements of K, indexed by all ordinals less than some ordinal λ, which satisfies:

$$\forall V \in \mathcal{V} \quad \exists \lambda_0 \quad \forall \mu_1, \mu_2 \geqslant \lambda_0 \quad a_{\mu_2} - a_{\mu_1} \in V.$$

We say that $(a_\mu)_{\mu<\lambda}$ converges to $a \in K$ if for every $V \in \mathcal{V}$ there exists λ_0 such that for any $\mu \geqslant \lambda_0, a - a_\mu \in V$.

(2) The topological field $\langle K, \tau\rangle$ is *complete* if every Cauchy sequence converges to some element of K.
(3) A topological $\mathcal{L}$-extension $\langle \widehat{K}, \widehat{\tau}\rangle$ of $\langle K, \tau\rangle$ is called a *completion* of $\langle K, \tau\rangle$ if the following are satisfied:
  (a) every Cauchy sequence in $\widehat{K}$ converges to some element of $\widehat{K}$;
  (b) K is dense in $\widehat{K}$.

A special case of this construction in the case of valued fields is the following. Recall that if $\langle L, v\rangle$ is a non-trivially valued field with a discrete valuation v and with $t \in L$ such that $v(t) = 1$, then, L embeds in $k_L((t))$ and if L is complete, then this is an isomorphism.

More generally any valued field has a completion (see [28, A 4.11]) and in the case $\langle K, v\rangle$ is an Archimedean valued field (in other words v takes its values in $\mathbb{R}$ (see [31, p. 36])), this completion is Henselian (see [31, p. 98]).

In the case of an Archimedean ordered field, it has a completion and its completion is real-closed (see Theorem 1.23 in [27]).

2.4. Condition $Comp^b(K)$

In this section, we will strengthen $Comp(\mathbf{K})$ in order to show that the corresponding equivalence relation $\sim^b$ is a congruence.

Definition 2.24. Let $\langle K, \mathcal{V}\rangle \subseteq \langle L, \mathcal{W}\rangle$ be a pair of topological $\mathcal{L}$-fields. Assume that K (respectively L) is a V_K-field (respectively V_L-field) with $V_L \cap K = V_K$, $V_L \in \mathcal{W}$ and $V_K \in \mathcal{V}$.

We are going to define an equivalence relation $\sim^b_{\mathcal{W}(K)}$ on V_L with respect to a subset $\mathcal{W}(K)$ of $\mathcal{W}$ satisfying $Comp^b(\mathbf{K})$ below:

(1) items 1, 2, 3 and 5 of Definition 2.4;
(2) for any n-ary function symbol $f \in \mathcal{L}$, and any $V_0, \dots, V_n \in \mathcal{V}$ such that $f(V_1, \dots, V_n) \subseteq f(\bar{0}) + V_0$, any $W_0, \dots, W_n \in \mathcal{W}(K)$ with $(W_i \cap K) = V_i$ for $i = 0, \dots, n$ and any $a_1, \dots, a_n \in V_L$, we have

$$f(a_1 + W_1, \dots, a_n + W_n) \subseteq f(a_1, \dots, a_n) + W_0.$$

Let a, b be two elements of V_L, then a and b are infinitely close with respect to $\mathcal{W}(K)$, where $\mathcal{W}(K)$ satisfies $Comp^b(\mathbf{K})$, (denoted by $a \sim^b_{\mathcal{W}(K)} b$) iff for any neighbourhood $W \in \mathcal{W}(K)$, we have $a - b \in W$.

Let $\langle K, <\rangle \subseteq \langle L, <\rangle$ be two ordered fields with fundamental system of neighbourhoods of zero $\mathcal{V}_K$ (respectively $\mathcal{V}_L$).

Let us check that $\mathcal{V}_L$ satisfies $Comp^b(\mathbf{K})$.

Consider the binary function symbol $+$. Let $V_0 := \{x \in K : |x| < k\}$ with $k \in K_+$. Let $k_1, k_2 \in K_+$ be such that $k_1 + k_2 = k$ and set $V_i := \{x \in K : |x| < k_i\}$, $1 \leqslant i \leqslant 2$. Let $W_i \in \mathcal{V}_L$ be such that $W_i \cap K \subseteq V_i$. Now W_i is of the form $\{x \in L : |x| < \ell_i\}$ with $\ell_i \in L_+$ and $\ell_i \leqslant k_i$ (otherwise $k_i \in W_i - V_i$). Let $W_0 := \{x \in L : |x| < k\}$. Then $(W_1 + W_2) \subseteq W_0$.

Consider the binary function $\cdot$, let $a_1, b_1, a_2, b_2 \in]-1; 1[$, with $|a_1 - b_1| < \ell_1$, $|a_2 - b_2| < \ell_2$. Consider $a_1.a_2 - b_1.b_2 = a_1(a_2 - b_2) + b_2(a_1 - b_1)$. So, $|a_1.a_2 - b_1.b_2| \leqslant |a_1|.|(a_2 - b_2)| + |b_2|.|(a_1 - b_1)| \leqslant \ell_2 + \ell_1 \leqslant k_1 + k_2$. On the other hand, assuming that a_1, a_2 are positive, if we replace b_1 by $a_1 - k_1$ and b_2 by $a_2 - k_2$, we get $a_1.a_2 - b_1.b_2 = k_1.a_2 + k_2.a_1 - k_1.k_2 \leqslant k_1 + k_2 - k_1.k_2$.

Next, let us consider the example of valued fields.

Let $\langle K, v\rangle \subseteq \langle L, v\rangle$ be two valued fields with a fundamental system of neighbourhoods of zero $\mathcal{V}_K$ (respectively $\mathcal{V}_L$): $\{x \in K : v(x) > v(a)\}$ (respectively $\{x \in L : v(x) > v(a)\}$), with $a \in K^\times$.

Let us check that $\mathcal{V}_L$ satisfies $Comp^b(\mathbf{K})$.

Consider the binary function symbol $\cdot$. Let $V_0 := \{x \in K : v(x) > v(k)\}$ with $k \in \mathcal{O}_K^\times$. Let $k_1, k_2 \in \mathcal{O}_K$ be such that $v(k_1) + v(k_2) \geqslant v(k)$ and set $V_i := \{x \in K : v(x) > v(k_i)\}$ with $k_i \in \mathcal{O}_K^\times$, $1 \leqslant i \leqslant 2$. Let $W_i \in \mathcal{V}_L$ be such that $W_i \cap K \subseteq V_i$. Now W_i is of the form $\{x \in L : w(x) > w(\ell_i)\}$ with $\ell_i \in \mathcal{O}_L^\times$. So, the fact that the trace of W_i on K is included in V_i implies that $w(\ell_i) \geqslant v(k_i)$ (Suppose otherwise, then k_i would belong to W_i, a contradiction). Let $W_0 := \{x \in L : v(x) > v(k)\}$. Then $W_1.W_2 \subseteq W_0$.

We proceed similarly for the other primitives of the language.

Let $\mathcal{C}$ be a model-complete elementary class of $\mathcal{L}$-structures expanding a field and assume that any element K of $\mathcal{C}$ satisfies Hypothesis (D). Recall that the corresponding class of topological $\mathcal{L}$-fields is denoted by $\mathcal{C}_{\text{top}}$.

Let $\langle K, \mathcal{V}\rangle \subseteq \langle L, \mathcal{W}\rangle$ be two elements of $\mathcal{C}_{\text{top}}$. Recall that $\mathcal{W}(K)$ is the following subset of $\mathcal{W}$: $\{a \in L : \phi(a, \bar{k})$ with $\bar{k} \subseteq V_K\}$. Then $\mathcal{W}(K)$ satisfies $Comp^b(\mathbf{K})$ is implied by the following *condition of uniform continuity* on V_K of the functions f in $\mathcal{L}$:

let $V_0, \dots, V_n \in \mathcal{V}$ such that $f(V_1, \dots, V_n) \subseteq f(\bar{0}) + V_0$, then for any $b_1, \dots, b_n \in V_K$ we have

$$f(b_1 + V_1, \dots, b_n + V_n) \subseteq f(\bar{b}) + V_0.$$

Indeed, this follows from the fact that the elements of $\mathcal{W}(K)$ are definable subsets of L with parameters in K. So, using the model-completeness of the class $\mathcal{C}$, it is straightforward to check item 2 in Definition 2.24.

Let A be an $\mathcal{L}$-structure. Let $\mathcal{L}_f$ be the set of all function symbols in $\mathcal{L}$. By a congruence $\sim$ on a subset S of A, we mean an equivalence relation on S such that for every $f(x_1, \dots, x_n) \in \mathcal{L}_f$, for every $\bar{a}, \bar{b} \in S^n$ such that $f(\bar{a})$ and $f(\bar{b})$ belong to S, we have the following implication $\bigwedge_{i=1}^n (a_i \sim b_i) \Rightarrow (f(\bar{a}) \sim f(\bar{b}))$.

Lemma 2.25. *Let $\langle K, \mathcal{V}\rangle \subseteq \langle L, \mathcal{W}\rangle$ be two topological $\mathcal{L}$-fields. Assume that K (respectively L) is a V_K-field (respectively V_L-field) with $V_K \in \mathcal{V}$ and $V_L \in \mathcal{W}$. Moreover, assume that $\mathcal{W}(K) \subseteq \mathcal{W}$ satisfies $Comp^b(\mathbf{K})$. Then $\sim^b_{\mathcal{W}(K)}$ is an $\mathcal{L}_f(K)$-congruence on V_L.*

Proof. First, similarly as in the proof of Lemma 2.11, we get that $\sim^b_{\mathcal{W}(K)}$ is an equivalence relation.

To check that this is a congruence, we have to show that for any n-ary function symbol $f \in \mathcal{L}$ whenever $a_1 \sim^b_{W(K)} b_1, \ldots, a_n \sim^b_{W(K)} b_n$, with $a_1, \ldots, a_n, b_1, \ldots, b_n \in V_L$ and $f(\bar{a}), f(\bar{b}) \in V_L$, we have $f(\bar{a}) \sim^b_{\mathcal{W}(K)} f(\bar{b})$. Let $W_0 \in \mathcal{W}(K)$, by continuity of f, there exist $V_1, \ldots, V_n \in \mathcal{V}$ such that for any $W_1, \ldots, W_n$ with $W_i \cap K = V_i$, $1 \leq i \leq n$, we have $f(b_1 + W_1, \ldots, b_n + W_n) \subseteq f(\bar{b}) + W_0$. By assumption $a_i - b_i \in W_i$ i.e. $a_i \in b_i + W_i$, so $f(\bar{a}) - f(\bar{b}) \in W_0$.

Let $k \in K^\times$, $\cdot$ is continuous, so for any $V \in \mathcal{V}$ there exists $V' \in \mathcal{V}$ such that $V'.k \subseteq V$. Let $a \in V_L$ with $a \sim^b_{\mathcal{W}(K)} 0$. Let $W' \in \mathcal{W}(K)$ with $W' \cap K = V$ and $a \in W'$. Since $\mathcal{W}(K)$ satisfies $Comp^b(K)$, for any $W' \in \mathcal{W}(K)$ with $W' \cap K = V'$ and any $W \in \mathcal{W}(K)$ with $W \cap K = V$, then $W'.k \subseteq W$. So, $a.k \in W$. $\square$

Corollary 2.26. *Let $t(x_1, \ldots, x_n)$ be an $\mathcal{L}(K)$-term, let $a_1, b_1, \ldots, a_n, b_n \in V_L$ and assume that $a_1 \sim^b_{\mathcal{W}(K)} b_1, \ldots, a_n \sim^b_{\mathcal{W}(K)} b_n$, then*

$$t(a_1, \ldots, a_n) \sim^b_{\mathcal{W}(K)} t(b_1, \ldots, b_n).$$

Proof. An easy induction on the complexity of terms, using Definition 2.24. $\square$

3. Differential lifting

In this section, we will consider expansions of topological $\mathcal{L}$-fields to $\mathcal{L} \cup \{^{-1}, D\}$-structures, where D is a new unary function symbol which will satisfy the axioms of a derivation. We shall denote $\mathcal{L} \cup \{^{-1}, D\}$ by $\mathcal{L}_D$ and the ith iterate $D^i(x)$ of the derivation of an element x, by $x^{(i)}$.

Let us first recall some differential algebra terminology (see [15, p. 75]). In the following, $\langle R, D\rangle$ will denote a non-zero differential domain of characteristic 0 and C_R will denote the set of elements with zero derivative (i.e. the constant elements). Although in this section, we will only deal with the case when R is a field, we will directly place ourselves in a more general setting that will be used in Section 9.

Definition 3.1. Let $R\{X\}$ be the ring of differential polynomials over R in one differential indeterminate X over R. Let $f(X) \in R\{X\} \setminus R$, then we can write $f(X) = f^*(X, \ldots, X^{(n)})$ for some ordinary polynomial $f^*(X_0, \ldots, X_n) \in R[X_0, \ldots, X_n]$ and some natural number n that we choose minimal such. The variable $X^{(n)}$ is called the *leader* of f and is denoted by u_f; n is the *order* of f and $\deg_{u_f} f$ is the *degree* of f in u_f.

Suppose that f is of order $n \geqslant 1$ and degree d, then we can write f as: $f = f_d \cdot u_f^d + \cdots + f_1 \cdot u_f + f_0$ where $f_0, \ldots, f_d \in R[X, X^{(1)}, \ldots X^{(n-1)}]$, $f_d \neq 0$ and $X^{(n)} = u_f$.

The *initial* i_f of f is defined as f_d and the *separant* s_f of f is defined as $s_f = \frac{\partial f}{\partial u_f}$.

We may extend D on $R\{X\}$ as usual; we proceed by induction on the order of f. Assuming that D has been extended on elements of order $\leq n - 1$, we define $D(f) := f_d \cdot X^{(n+1)} + \cdots + f_1 \cdot X^{(n+1)} + D(f_d) \cdot X^{(n)} + \cdots + D(f_1) \cdot X^{(n)} + D(f_0)$, and we will shorten $D(f)$ by $f^{(1)}$.

Let us recall an analogue of the Euclidean division in this differential setting.

Lemma 3.2 (*See [15, Proposition 1, p. 79]*). *Let $g(X)$, $h(X)$ be non-zero differential polynomials in $R\{X\}$. Then, there exist $r(X)$, and finitely many $g_j(X) \in R\{X\}$, $j \in J \subseteq \mathbb{N}$, with the order of r smaller than or equal to the order of h and $\deg_{u_h} h > \deg_{u_h} r$ (see Definition 3.1) such that, for some natural numbers n, n',*

$$i_h^n \cdot s_h^{n'} \cdot g = \sum_{j \in J} h^{(j)} \cdot g_j + r.$$

The following result can easily be deduced from the preceding Lemma.

Lemma 3.3 (*See [15, Lemma 2, p. 167] and [40, Corollary 2.10]*). *Let P be a differential prime ideal of $R\{X\}$ with $P \cap R = \{0\}$. Let f be a non-zero differential polynomial of minimal order and then of minimal degree belonging to P. Then*

$$P = \{g \in R\{X\} \mid \exists n \in \mathbb{N} \text{ such that } (i_f \cdot s_f)^n \cdot g \in \langle f \rangle\}$$

where $\langle f \rangle$ is the differential ideal of $R\{X\}$ generated by f.

So we can define the notion of *generic* polynomial.

Definition 3.4. Let R_1 be a differential domain containing R and let a be in $R_1 \setminus R$. The set of elements of $R\{X\}$ vanishing on a is a prime differential ideal of $R\{X\}$, denoted by $I(a, R)$, whose intersection with R is $\{0\}$. If $I(a, R) = \{0\}$, then we say that a is *differentially transcendental* over R; if $I(a, R) \neq 0$, then we say that a is *differentially algebraic* over R. In this last case, let f be a non-zero element of this ideal which we choose first of minimal order and then of minimal degree; such an element f is a *generic* polynomial of a over R.

When R is a field, we get

$$I(a, R) = I(f) := \{g \in R\{X\} : {s_f}^k.g \in < f > \text{ for some } k \in \mathbb{N}\},$$

and such an element f is unique up to multiplication by a non-zero element of R (see [18, Lemma 1.4]).

Now, we are ready to write down schemes $(DL)_{\mathcal{W}}$, where $\mathcal{W} \subseteq \mathcal{V}$, of axioms in the language of $(\mathcal{L}_D)_t$ expressing the fact that if a differential polynomial while considered as an ordinary algebraic polynomial has a zero then it has a differential zero *close* (relative to $\mathcal{W}$) to this algebraic zero.

Definition 3.5. (1) A *differential topological $\mathcal{L}$-field* $\langle K, \tau\rangle$ is an $\mathcal{L}_D$-structure K satisfying the following axioms $(*)$:

$$\forall a \forall b \quad D(a+b) = D(a) + D(b), \qquad \forall a \forall b \quad D(a.b) = a.D(b) + D(a).b,$$

such that the restriction $\langle K, \tau\rangle$ is a topological $\mathcal{L}$-field.

There is no requirement of any interaction between the derivation D and the topology τ.

(2) A differential topological $\mathcal{L}$-extension is an $\mathcal{L}_D$-first-order extension which is in addition a topological $\mathcal{L}$-extension (see Definition 2.3).

(3) Let $\langle L, \mathcal{V}\rangle$ be a differential topological $\mathcal{L}$-field and assume that L is a V-field for some $V \in \mathcal{V}$. Let $\mathcal{V}_0 := \bigcup_{n\geqslant 1} \mathcal{V}_{0,n}$ where $\mathcal{V}_{0,n}$ is a subset of open neighbourhoods of $\bar{0} \in L^{n+1}$ with respect to the product topology of L^{n+1}.

Then we say that L satisfies $(DL)_{\mathcal{V}_0}$ if for every $n \geqslant 1$, for every differential polynomial $f(X) = f^*(X, X^{(1)}, \ldots, X^{(n)})$ belonging to $V\{X\}$ and for every $W \in \mathcal{V}_{0,n}$, the following implication holds:

$$(\exists \alpha_0, \ldots, \alpha_n \in V)(f^*(\alpha_0, \ldots, \alpha_n) = 0 \wedge s_f^*(\alpha_0, \ldots, \alpha_n) \neq 0) \Rightarrow$$

$$\Big((\exists z)\big(f(z) = 0 \wedge s_f(z) \neq 0 \wedge (z^{(0)} - \alpha_0, \ldots, z^{(n)} - \alpha_n) \in W\big)\Big).$$

When each $\mathcal{V}_{0,n}$ is the whole fundamental system of neighbourhoods of $\bar{0} \in L^{n+1}$ with respect to the product topology of L^{n+1}, we shall not put any subscript at (DL).

Remark 3.6. Note that we could have added in the definition of the scheme $(DL)_{\mathcal{V}_0}$ the requirement that f is irreducible (which is expressible by first-order statements in the language $\mathcal{L}_{\text{rings}}$). Anyway, if $f(c) = 0$, $s_f(c) \neq 0$ $(\star)$ for an element $c \in V$ and $f = g.h$, then one of the factors of f satisfies $(\star)$. Moreover, multiplying f by a non-zero element of V, we may assume that each of the factors have coefficients in V. Then g or h will have order n and satisfies the hypothesis $(\star)$. If we apply the scheme $(DL)_{\mathcal{V}_0}$ to one of these polynomials then f will satisfy the conclusions of the scheme $(DL)_{\mathcal{V}_0}$.

Now, we will prove a crucial lemma whose analogue for real-closed fields (respectively Henselian valued fields) have been proven by Singer in [39, p. 85] (respectively Michaux in [20, p. 34]). We will show that any element of an inductive class $\mathcal{C}$ of differential topological $\mathcal{L}$-fields satisfying *Hypothesis* (I) (see Definitions 2.18 and 2.21) can be embedded in another element of $\mathcal{C}$ satisfying a scheme (DL).

Lemma 3.7. *Let $\mathcal{C}$ be an inductive class of differential topological $\mathcal{L}$-fields satisfying Hypothesis (I). Let $\langle K, \mathcal{V}\rangle$ be a V-field in $\mathcal{C}$ and $f(X) \in V\{X\}$ be of order n. Suppose that $f^*(\alpha_0, \ldots, \alpha_n) = 0$ with $\alpha_0, \ldots, \alpha_n \in V$ and ${s_f}^*(\alpha_0, \ldots, \alpha_n) \neq 0$.*

Then there exists a differential topological $\mathcal{L}$-extension $\langle L, \widehat{\mathcal{V}}\rangle \in \mathcal{C}$ of $\langle K, \mathcal{V}\rangle$ satisfying the following properties:

(1) *there exists a subset $\mathcal{W}$ of $\widehat{\mathcal{V}}$ satisfying Comp($\mathbf{K}$),*
(2) *L is a W-field for some W in $\mathcal{W}$ such that $W \cap K = V$ and,*
(3) *there exists $z \in W$ such that $f(z) = 0$, $s_f(z) \neq 0$ and $\bigwedge_{i=0}^{n} \big(z^{(i)} \sim_{\mathcal{W}} \alpha_i\big)$.*

Proof. Let us consider the topological $\mathcal{L}_{\text{rings}}$-extension $K_1 := \langle K((t_0)) \ldots ((t_{n-1})), \mathcal{W}_n, v\rangle$ of $\langle K, \mathcal{V}\rangle$ such that K_1 is a $W_{V,0}$-field for some $W_{V,0} \in \mathcal{W}_n$ with $W_{V,0} \cap K = V$, as defined in Lemma 2.10. Note that the elements t_i's are algebraically independent over K. Let $c_i = \alpha_i + t_i$ for $i \in \{0, \ldots, n-1\}$; we have that $c_i \sim_K \alpha_i$ (see the notations in Lemma 2.10).

Assuming $f^*(\alpha_0, \ldots, \alpha_n) = 0$, ${s_f}^*(\alpha_0, \ldots, \alpha_n) \neq 0$ (so ${s_f}^{*2}(\alpha_0, \ldots, \alpha_n) = \beta \in K^\times$) and $t_0, \ldots, t_{n-1} \sim_K 0$, Lemma 2.13 yields that

$${s_f}^*(c_0, \ldots, c_{n-1}, \alpha_n)^2 \sim_K s_f^*(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n)^2 = \beta \in K^\times \quad \text{and} \quad f^*(c_0, \ldots, c_{n-1}, \alpha_n) \sim_K 0.$$

Claim: there exists $d \in V^\times$ such that

$$d \cdot f^*(c_0, \ldots, c_{n-1}, \alpha_n) \sim_K 0,\ {s_{d \cdot f}}^{*2}(c_0, \ldots, c_{n-1}, \alpha_n) \sim_K \beta' \quad \text{and}$$

$$d \cdot f^*(c_0, \ldots, c_{n-1}, X_n) \in W_{V,0}[X_n] \quad \text{for some } d \in W_{V,0} \text{ and } \beta' \in K_1^\times.$$

Proof. Write

$$f^*(X_0, \ldots, X_n) = \sum_{i=0}^{m} f_i(X_0, \ldots, X_{n-1}).X_n^i \quad \text{and} \quad s_f^*(X_0, \ldots, X_n) = \sum_{i=1}^{m} f_i(X_0, \ldots, X_{n-1}).i.X_n^{i-1}.$$

Then there exists an element $d \in W_{V,0}^\times$ such that for any $i \in \{0, \ldots, m\}$,

$$d.f_i(X_0, \ldots, X_{n-1}) \in W_{V,0}[X_0, \ldots, X_{n-1}], \qquad d \cdot f^*(c_0, \ldots, c_{n-1}, X_n) \in W_{V,0}[X_n] \quad \text{and,}$$

$$s_{d.f}^*(c_0, \ldots, c_{n-1}, \alpha_n) = \sum_{i=1}^{m} d.f_i(c_0, \ldots, c_{n-1}).i.\alpha_n^{i-1}. \quad \square$$

Now, we may apply *Hypothesis* (I). So there exists an $\mathcal{L}_{\text{rings}}$-extension $\langle L, \widehat{\mathcal{V}}\rangle \in \mathcal{C}$ of K_1 which satisfies the following properties:

(1) $\langle L, \widehat{\mathcal{V}}\rangle$ is a topological $\mathcal{L}$-extension of $\langle K, \mathcal{V}\rangle$,
(2) there exists a subset $\mathcal{W}$ of $\widehat{\mathcal{V}}$ satisfying *Comp*($\mathbf{K}$),
(3) L is a W-field for some $W \in \mathcal{W}$ with $W \cap K = V$ and,
(4) there exists an element $c_n \in W$ which satisfies $f^*(c_0, \ldots, c_{n-1}, c_n) = 0$ and $s_f{}^*(c_0, \ldots, c_n) \neq 0$ such that $c_i \sim_{\mathcal{W}} \alpha_i$, $0 \leqslant i \leqslant n$.

We first extend the derivation to the algebraic closure L_1 of K_1 in L. We can choose a transcendence basis of L over L_1 and extend D on this basis. Finally we extend again the derivation to the algebraic closure.

Since any derivation D of K_1 uniquely extends to L_1 then, in order to make L_1 a differential $\mathcal{L}$-extension of K, we only need to extend the derivation of K to K_1. This can be done by setting $D(c_0) = c_1, \ldots, D(c_{n-1}) = c_n$, the derivative $D(c_n)$ is then uniquely determined by the equation $f^*(c_0, \ldots, c_n) = 0$ since the separant s_f is non-zero at c_0.

Indeed, let $f^*(X, \ldots, X^{(n)}) = \sum_{l=0}^{d} g_l(X, \ldots, X^{(n-1)}) \cdot X^{(n)^l}$, then $D(f) = \sum_{l=0}^{d} D(g_l) \cdot X^{(n)^l} + \sum_{l>0}^{d} g_l \cdot l \cdot X^{(n)^{l-1}} \cdot X^{(n+1)}$. Now, if we evaluate this differential polynomial at c_0, since $s_f(c_0)$ is non-zero, we get that $D(c_n) := (-\sum_l{}^d D(g_l(c_0, \ldots, c_0^{(n-1)})) \cdot c_0^{(n)^l}) \cdot s_f(c_0)^{-1}$. Let $z = c_0$, by construction, we have $z^{(i)} = c_i$ for all $i \in \{0, \ldots, n\}$. As $c_i - \alpha_i = t_i \sim_{\mathcal{W}} 0$, we get $(z^{(i)} - \alpha_i) \sim_{\mathcal{W}} 0$ for all $i \in \{0, \ldots, n\}$. Let us note that if $g(X) = g^*(X, \ldots, X^{(m)})$ is a differential polynomial with $n \geqslant m$ and $g^*(\alpha_0, \ldots, \alpha_m) \neq 0$ then applying again Lemma 2.13, we have the following additional property:

$$g(z) \sim_{\mathcal{W}} g^*(\alpha_0, \ldots, \alpha_m) \quad \text{and} \quad g(z) \neq 0. \quad \square$$

Corollary 3.8. *Let $\mathcal{C}$ be an inductive elementary class of differential large fields. Let $K \in \mathcal{C}$ and let $f(X) \in K\{X\}$ be of order n. Suppose that $f^*(\alpha_0, \ldots, \alpha_n) = 0$ with $\alpha_0, \ldots, \alpha_n \in K$ and $s_f{}^*(\alpha_0, \ldots, \alpha_n) \neq 0$. Then, there exists $z \in K_1 := K((t_0)) \cdots ((t_{n-1}))$ and a derivation D on K_1 such that $f(z) = 0$, $s_f(z) \neq 0$ and $\bigwedge_{i=0}^{n} \left(z^{(i)} \sim_K \alpha_i\right)$. So, in particular given any polynomial $q(X) \in K\{X\}$ of order $\leq n$, if $q^*(\bar{\alpha}) \neq 0$, then $q(z) \neq 0$.*

Proof. Let us consider the topological $\mathcal{L}_{\text{rings}}$-extension $K_1 := \langle K((t_0)) \ldots ((t_{n-1})), \mathcal{W}_n\rangle$ of K as defined in 3. in Examples 2.22. Note that the elements t_i's are algebraically independent over K. Let $c_i = \alpha_i + t_i$ for $i \in \{0, \ldots, n-1\}$; we have that $c_i \sim_K \alpha_i$ (see the notations in Lemma 2.10).

Assuming $f^*(\alpha_0, \ldots, \alpha_n) = 0$, $s_f{}^*(\alpha_0, \ldots, \alpha_n) \neq 0$ (so $s_f{}^{*2}(\alpha_0, \ldots, \alpha_n) = \beta \in K^\times$) and $t_0, \ldots, t_{n-1} \sim_K 0$, Lemma 2.13 yields that

$$s_f{}^*(c_0, \ldots, c_{n-1}, \alpha_n)^2 \sim_K s_f^*(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n)^2 = \beta \in K^\times$$
$$\text{and} \quad f^*(c_0, \ldots, c_{n-1}, \alpha_n) \sim_K 0.$$

The field K_1 is Henselian and so there exists an element $c_n \in W$ which satisfies $f^*(c_0, \ldots, c_{n-1}, c_n) = 0$ and $s_f{}^*(c_0, \ldots, c_n) \neq 0$ such that $c_n - \alpha_n \sim_K 0$.

To extend the derivation of K to K_1, we proceed as before: we set $D(c_0) = c_1, \ldots, D(c_{n-1}) = c_n$, the derivative $D(c_n)$ is then uniquely determined by the equation $f^*(c_0, \ldots, c_n) = 0$ since the separant s_f is non-zero at c_0. $\square$

Proposition 3.9. *Let $\mathcal{C}$ be an inductive class of differential topological $\mathcal{L}$-fields, satisfying Hypothesis (I) and let $\langle K, \mathcal{V}\rangle \in \mathcal{C}$ be a V-field. Then K has a differential topological $\mathcal{L}$-extension $\langle \widehat{K}, \widehat{\mathcal{V}}\rangle \in \mathcal{C}$ satisfying the scheme (DL).*

Proof. Let $\langle f_{(\delta)}(X)\rangle_{\delta\in\lambda}$ be an enumeration of differential polynomials with order $n(\delta)$ in $V\{X\}$, for some ordinal λ. By transfinite induction we build an increasing sequence $\langle K_{(\delta)}, \mathcal{V}_{(\delta)}\rangle_{\delta\in\lambda}$ of differential topological $\mathcal{L}$-extensions of K which are $V_{(\delta)}$-fields. Set $K_{(0)} := K$ and $V_{(0)} := V$; assume that $K_{(\delta)}$ has been constructed and consider $f_{(\delta)}(X) \in V\{X\}$.

Let $\langle(\alpha_0^\beta, \ldots, \alpha_{n(\delta)}^\beta)\rangle_{\beta<\lambda(\delta)} \subseteq V$ be such that

$$f_{(\delta)}^*(\alpha_0^\beta, \ldots, \alpha_{n(\delta)}^\beta) = 0 \wedge s_{f_{(\delta)}}^*(\alpha_0^\beta, \ldots, \alpha_{n(\delta)}^\beta) \neq 0 \quad \text{for some ordinal } \lambda(\delta).$$

Using Lemma 3.7, we build an increasing sequence $\langle K_{(\delta,\beta)}, \mathcal{V}_{(\delta,\beta)}\rangle_{\beta<\lambda(\delta)}$ of differential topological $\mathcal{L}$-extensions of K which are $V_{(\delta,\beta)}$-fields. We let $\langle K_{(\delta,0)}, \mathcal{V}_{(\delta,0)}\rangle := \langle K_{(\delta)}, \mathcal{V}_{(\delta)}\rangle$ and then, we let $\langle K_{(\delta+1,0)}, \mathcal{V}_{(\delta+1,0)}\rangle := \bigcup_{\beta<\lambda(\delta)}\langle K_{(\delta,\beta)}, \mathcal{V}_{(\delta,\beta)}\rangle$ as defined in Lemma 2.16 (i.e. $\mathcal{V}_{(\delta+1,0)} = \mathcal{V}^{\lim}_{(\delta,\lambda(\delta))}$). Assume that $K_{(\delta,\beta)}$ has been constructed and consider the elements $\alpha_0^\beta, \ldots, \alpha_{n(\delta)}^\beta \in V$.

By Lemma 3.7, there exists a differential topological $\mathcal{L}$-extension $K_{(\delta,\beta+1)}$ of $\langle K_{(\delta,\beta)}, \mathcal{V}_{(\delta,\beta)}\rangle$ which satisfies the following properties:

(1) there exists a subset $\mathcal{W}_{(\delta,\beta+1)}$ of $\mathcal{V}_{(\delta,\beta+1)}$ satisfying *Comp*($\mathbf{K}_{(\delta,\beta)}$),
(2) $K_{(\delta,\beta+1)}$ is a $V_{(\delta,\beta+1)}$-field for some $V_{(\delta,\beta+1)}$ in $\mathcal{W}_{(\delta,\beta+1)}$ and $V_{(\delta,\beta+1)} \cap K_{(\delta,\beta)} = V_{(\delta,\beta)}$ and,
(3) there exists $z \in V_{(\delta,\beta+1)}$ satisfying the following

$$f_{(\delta)}(z) = 0 \quad \text{and} \quad s_{f_{(\delta)}}(z) \neq 0 \quad \text{and} \quad \bigwedge_{i=0}^{n(\delta)} \left(z^{(i)} \sim_{\mathcal{W}_{(\delta,\beta+1)}} \alpha_i^\beta\right).$$

Then we let $\langle K_1, \tau_1\rangle$ be the differential topological V_1-field $\langle \bigcup_{\delta<\lambda} K_{(\delta,0)}, \mathcal{V}_1\rangle$ in $\mathcal{C}$ where $\mathcal{V}_1$ and $\mathcal{W}_1$ are as defined in Corollary 2.17 and satisfy the following properties:

- $\langle K_1, \mathcal{V}_1\rangle$ is a topological differential $\mathcal{L}$-extension of $\langle K, \mathcal{V}\rangle$,
- $\mathcal{W}_1$ is a subset of $\mathcal{V}_1$ satisfying $Comp(\mathbf{K_0})$,
- for any natural number $n \geqslant 1$, for any differential polynomial f of order n with coefficients in V, we have if $f^*(\alpha_0, \dots, \alpha_n) = 0$ and $s_f{}^*(\alpha_0, \dots, \alpha_n) \neq 0$ for some $\alpha_0, \dots, \alpha_n \in V$ then there exists $z \in V_1$ satisfying the following

$$f(z) = 0 \quad \text{and} \quad s_f(z) \neq 0 \quad \text{and} \quad \bigwedge_{i=0}^{n} \left(z^{(i)} \sim_{\mathcal{W}_1} \alpha_i\right).$$

Let us check that K_1 is a V_1-field which satisfies the scheme $(DL)_{\mathcal{W}_1}$ with respect to differential polynomials in $V\{X\}$.

Since any element of K_1 belongs to some $K_{(\delta,0)}$ and since $K_{(\delta,0)}$ is a $V_{(\delta,0)}$-field, K_1 is a V_1-field. Let $f(X) \in V_1\{X\}$ of order n. Then it belongs to some $V_{(\delta,0)}\{X\}$. We may assume that it has an algebraic solution $\bar{\alpha}$ in $V_{(\delta,0)}$.

Let $W \in \mathcal{W}_1$, so $W \cap K_{(\mu,0)} \in \mathcal{V}_{(\mu,0)}$ for any $\mu < \lambda$ and $W \cap K_{(\alpha,0)} \in \mathcal{W}_{(\alpha,0)}$ for any $\lambda > \alpha > \mu$, by construction of $\mathcal{W}_1$ in Corollary 2.17. By construction of the extension $K_{(\delta+1,0)}$, it has a differential solution z such that $\bigwedge_{i=0}^{n}(\alpha_i - z^{(i)}) \in W \cap K_{(\delta+1,0)}$ and so, $\bigwedge_{i=0}^{n}(\alpha_i - z^{(i)}) \in W$.

We iterate this process replacing $K := K_0$ by K_1 and $V := V_0$ by V_1. So, we obtain a sequence of differential topological $\mathcal{L}$-extensions $\langle K_n, \mathcal{V}_n\rangle_{n\in\omega}$, each K_{n+1} is a V_{n+1}-field, satisfying the scheme $(DL)_{\mathcal{W}_{n+1}}$ with respect to $\mathcal{V}_n\{X\}$, where $V_{n+1} \in \mathcal{W}_{n+1} \subseteq \mathcal{V}_{n+1}$ satisfying $Comp(\mathbf{K_n})$.

Now $\langle \widehat{K}, \widehat{\mathcal{V}}\rangle$, where $\widehat{K} := \bigcup_{n\in\omega} K_n$ and $\widehat{\mathcal{V}} := \{\bigcup_{n_0\leqslant n\in\omega} V_n : V_{n_0} \in \mathcal{V}_{n_0}, n \geq n_0, V_{n+1} \in \mathcal{W}_{n+1}$ and $V_{n+1} \cap K_n = V_n, n_0 \in \omega\}$, will be the desired differential topological $\mathcal{L}$-extension of K (see Corollary 2.17).

Let $\widehat{V} := \bigcup_n V_n$. Then $\widehat{K}$ is a $\widehat{V}$-field and it satisfies the scheme $(DL)_{\widehat{\mathcal{V}}}$. Let us check this last assertion. Let $f \in \widehat{V}\{X\}$, let $W \in \widehat{\mathcal{V}}$, then for some $n_0 \in \mathbb{N}$, $f \in V_{n_0}\{X\}$ and $W = V_{n_0} \cup \bigcup_{n>n_0} W_n$ with $W_n \in \mathcal{W}_n$ and $W_{n+1} \cap K_n = W_n$, $n > n_0$. Since K_{n_0+1} satisfies $(DL)_{\mathcal{W}_{n_0+1}}$, given W_{n_0+1}, there exists a differential solution close to the algebraic solution in W_{n_0+1}. $\square$

In view of Corollary 3.8, we define a scheme $(DL)_Z$ as follows.

Definition 3.10. A differential field (K, D) satisfies $(DL)_Z$ if for any $f(X) \in K\{X\}$ of order n such that $f^*(\alpha_0, \dots, \alpha_n) = 0$ for some $\alpha_0, \dots, \alpha_n \in K$ and $s_f{}^*(\alpha_0, \dots, \alpha_n) \neq 0$. Then, there exists $z \in K$ such that $f(z) = 0$, $s_f(z) \neq 0$ and for any polynomial $q(X) \in K\{X\}$ of order $\leq n$, if $q^*(\bar{\alpha}) \neq 0$, then $q(z) \neq 0$.

Corollary 3.11. *Let $\mathcal{C}$ be an inductive elementary class of differential large fields and let $K \in \mathcal{C}$. Then K has an extension $\widehat{K} \in \mathcal{C}$ satisfying the scheme $(DL)_Z$.* $\square$

Lemma 3.12. *Let $\langle K, \mathcal{V}\rangle$ be a differential topological $\mathcal{L}$-field which is a V-field and which satisfies the scheme (DL) then*

(1) *for any non-zero natural number n, given any neighbourhood of zero W in $\mathcal{V}$ and $a \in V$, there is an element $b \in W$ such that $b^{(n)} = a$. In particular K has a non-zero derivation,*
(2) *the field of the constants C_K is dense in V,*
(3) *let $f(X_1, \dots, X_n) \in V\{X_1, \dots, X_n\}$ be a differential polynomial in n differential indeterminates vanishing on V, then f^* vanishes on V.*

Proof. To show the first assertion, we apply the scheme (DL) to the differential polynomial $f(X) := (X^{(n)} - a)$, $a \in V$, $n \in \mathbb{N} \setminus \{0\}$. Let $(0, \dots, 0, a)$ be the corresponding algebraic zero of f^*, and $s_f = 1$. Therefore, for any $W \in \mathcal{V}$, there exists $z \in W$ with $z^{(n)} = a$. When $n = 1$ and $a \neq 0$, this shows in particular that K has a non-zero derivation.

Then, to show that C_K is dense in V, we apply the scheme (DL) to the differential polynomial $X^{(1)}$. For each $a \in V$, it has an algebraic zero of the form $(a, 0)$ and $s_{X^{(1)}} = 1$. So, we obtain in each neighbourhood $W \in \mathcal{V}$ an element z with $z - a \in W$ and $z^{(1)} = 0$.

Finally, let $f(X_1, \dots, X_n)$ be a differential polynomial with coefficients in V vanishing on K. Let m_i be the order of X_i in f and set $N := \sum_{i=1}^{n} m_i$. Consider the differential polynomial $g(X) := X^{(N+1)}$. By the way of contradiction assume that f^* does not vanish on V, namely that for some $\bar{k} \subseteq V$, $f^*(\bar{k}) \neq 0$. But, $(\bar{k}, 0)$ is an algebraic solution of g^*. By the scheme (DL), for any $W \in \mathcal{V}$, there exists $c \in V$ with $\bigwedge_{i=0}^{N}(c^{(i)} - k_i) \in W$ and $c^{(N+1)} = 0$. By continuity of the ring operations and of the scalar multiplications by elements of K, we may choose W sufficiently small such that $f(c, \dots, c) \neq 0$. $\square$

In the next three statements, we will transfer the results we have for V (respectively $V\{X\}$) to K (respectively $K\{X\}$), using algebraic manipulations.

Corollary 3.13. *Under the same hypotheses as in the previous lemma, the subfield C_K of constants is dense in K.*

Proof. Let a be in $K \setminus \{0\}$. Let W be any neighbourhood of 0 in $\mathcal{V}$, then we will show that there is an element $b \in C_K$ such that $a - b \in W$. Since K is a V-field, we can write a in the form $a = \frac{u_1}{u_2}$ for some $u_1 \in V$, $u_2 \in V^{\times}$.

Choose $W_0, W_1, W_2 \in \mathcal{V}$ such that $(u_2 + W_0)^{-2}.W_2 \subseteq W$, $W_1 - W_1 \subseteq W_2$, $u_1.W_0 \subseteq W_1$ and $u_2.W_0 \subseteq W_1$. By item 2 of Lemma 3.12, there exist $c_1, c_2 \in C_K \setminus \{0\}$ such that $(c_1 - u_1)$ and $(c_2 - u_2)$ belong to W_0.

Thus we obtain $\frac{c_1}{c_2} - \frac{u_1}{u_2} = \frac{(c_1-u_1).u_2-u_1.(c_2-u_2)}{c_2.u_2} \in W$. $\square$

In the next proposition, we will show that the scheme (DL) may be extended to K itself.

Proposition 3.14. *Let $\langle K, \mathcal{V}\rangle$ be a differential topological $\mathcal{L}$-field which is a V-field and which satisfies the scheme $(DL)_{\mathcal{V}_0}$. Let $f(X) := f^*(X, \dots, X^{(n)})$ be a non-zero differential polynomial of order n in $K\{X\}$. If there exist $a_0, \dots, a_n \in K$ such that $f^*(a_0, \dots, a_n) = 0$ and $s^*_f(a_0, \dots, a_n) \neq 0$ then, for any $W \in \mathcal{V}_{0,n}$, there exists $b \in K$ with $f(b) = 0$ and $s_f(b) \neq 0$ such that $\bigwedge_{i=0}^n (b^{(i)} - a_i) \in W$.*

Proof. Let $f^*(X, \dots, X^{(n)}) = \sum_{(i_0,\dots,i_n)} c_{(i_0,\dots,i_n)}.M_{(i_0,\dots,i_n)}(X) \in K\{X\}$ with the coefficients $c_{(i_0,\dots,i_n)}$ belonging to $K^\times$ and $M_{(i_0,\dots,i_n)}(X) = X^{i_0}\dots\left(D^{(n)}(X)\right)^{i_n}$. Let $N := \max_{\bar{i}} \{\sum_{j=0}^n i_j\}$.

Since K is a V-field, for each index $\bar{i} = (i_0, \dots, i_n)$, we can write $c_{\bar{i}} := d_{1,\bar{i}}.d_{2,\bar{i}}^{-1}$ where $d_{1,\bar{i}}, d_{2,\bar{i}} \in V^\times$. Let $d' := \prod_{\bar{i}} d_{2,\bar{i}} \in V^\times$ and so, $c_{\bar{i}}.d' \in V^\times$. In the same way, we write $a_i \in K$ as $a_{i0}.a_{i1}{}^{-1}$ where $a_{i0} \in V$, $a_{i1} \in V^\times$ and we set $d'' := \prod_i a_{i1} \in V^\times$, so $a_i.d'' \in V$. By Corollary 3.13, there exists $d \in C_K^\times$ sufficiently close to $d'.d''$ (so $d \in V^\times$) such that for all $i, \bar{i}$, we get $d \cdot a_i \in V$ and $d \cdot c_{\bar{i}} \in V$. We obtain:

$$\begin{aligned} 0 &= d^{M+1} \cdot f^*(a_0, \dots, a_n) = \sum_{(i_0,\dots,i_n)} d \cdot c_{\bar{i}} \cdot d^{(N-\sum_{j=0}^n i_j)} \cdot (da_0)^{i_0} \cdots (da_n)^{i_n} \\ &= \tilde{f}^*(da_0, \dots, da_n) \quad \text{where } \tilde{f}^* \in V[X_0, \dots, X_n]. \end{aligned}$$

Moreover, we have

$$\begin{aligned} s^*_{\tilde{f}}(d.a_0, \dots, d.a_n) &= \sum_{(i_0,\dots,i_n)} d \cdot c_{\bar{i}} \cdot d^{(N-\sum_{j=0}^n i_j)} \cdot i_n \cdot (da_0)^{i_0} \cdots (da_{n-1})^{i_{n-1}} \cdots (da_n)^{i_n-1} \\ &= d^N \cdot s^*_f(a_0, \dots, a_n) \in K^\times. \end{aligned}$$

Since, for all $i \in \{0, \dots, n\}$, $d \cdot a_i \in V$, we can apply the axiom $(DL)_{\mathcal{V}_0}$. We find, for all neighbourhoods of the form $d.W$ with $W \in \mathcal{V}$, an element b of V which satisfies $\tilde{f}(b) = 0$ and $s_{\tilde{f}}(b) \neq 0$ such that $(b^{(i)} - da_i) \in d.W$. Thus we have $(d^{-1})^{M+1}\tilde{f}^*(b, b', \dots, b^{(n)}) = 0$.

It is equal to $(d^{-1})^{M+1} \cdot \sum_{(i_0,\dots,i_n)} d \cdot c_{(i_0,\dots,i_n)} \cdot d^{(M-\sum_{j=0}^n i_j)} \cdot b^{i_0} \cdots b^{(n)i_n} = f(d^{-1} \cdot b) = 0$.

In the same way, $s_f(d^{-1} \cdot b) \neq 0$ and $(d^{-1} \cdot b^{(i)} - a_i) \in W$. Now, since $d^{-1} \in C_K$, we have that $d^{-1}.b^{(i)} = (d^{-1}.b)^{(i)}$. $\square$

Corollary 3.15. *Assume that K satisfies the same hypothesis as in the above Proposition. Let $f(X_1, \dots, X_n)$ be a differential polynomial with coefficients in K. Suppose that f vanishes on K, then f^* vanishes on K.*

Proof. We argue by the way of contradiction. Let $\bar{k} \subseteq K$ be such that $f^*(\bar{k}) \neq 0$. As in the proof of the preceding proposition, we find $d \in V^\times \cap C_K$ such that $\bar{k} \cdot d \subseteq V$ and $d^N \cdot f^*(\bar{k}) = \tilde{f}^*(\bar{k}.d)$, where N is some non-zero natural number and $\tilde{f}$ is a differential polynomial with coefficients in V. By the item 3 of Lemma 3.12, there is an element $a \in K$ such that $\tilde{f}(a, \dots, a) \neq 0$, which implies that $f(a \cdot d^{-1}, \dots, a \cdot d^{-1}) \neq 0$, a contradiction. $\square$

Since no non-zero polynomial of the form f^* vanishes on a field of characteristic 0, the above corollary implies that D is independent over K (see [15, p. 97]) (D is said to be independent on a differential field K if 0 is the only differential polynomial vanishing on K). This will entail that any finitely generated differentially (separable) extension of K is differentially generated by a single element (see [15, Proposition 9]). We will use these properties in Section 9.

We will generalize the above Corollary besides atomic formulas, as follows.

Definition 3.16. Let $\phi(x_1, \dots, x_n)$ be an open $\mathcal{L}_D$-formula. We associate with $\phi(\bar{x})$ an open $\mathcal{L} \cup \{^{-1}\}$-formula $\phi^*(\bar{x}^*)$ by substituting to each term $x_j^{(i)}$, $1 \leqslant j \leqslant n$, $i \in \mathbb{N}$, appearing in ϕ a new variable $x_{j,i}$. We will call $m(j)$ the formal order x_j in ϕ if $m(j)$ is the maximum of the natural numbers i such that $x_j^{(i)}$ occurs in ϕ.

Lemma 3.17. *Let $\mathcal{C}$ be a model-complete class of differential topological $\mathcal{L}$-fields and let $\langle K, \mathcal{V}\rangle \in \mathcal{C}$ be a model of the scheme (DL). Then, given any open $\mathcal{L}_D$-formula $\phi(x_1, \dots, x_n)$ such that the corresponding $\mathcal{L} \cup \{^{-1}\}$-formula ϕ^* defines an open set in K^N, for some natural number N, if $\exists \bar{x}^*\ \phi^*(\bar{x}^*)$ is satisfied in some topological $\mathcal{L}$-extension of K belonging to $\mathcal{C}$, then $\exists \bar{x}\ \phi(\bar{x})$ is satisfied in K.*

Proof. Let $m(j)$ be the formal order x_j in ϕ and consider the formula $\phi(\bar{x}) \wedge \bigwedge_{j=1}^n x_j{}^{(m(j)+1)} = 0$. By model-completeness of $\mathcal{C}$, there exists a tuple $\bar{b}$ in K satisfying ϕ^*. Applying Proposition 3.14 to the differential polynomial $X_j{}^{(m(j)+1)} = 0$, $1 \leqslant j \leqslant n$, and to the tuple $\bar{b}_j := (b_{j,1}, \dots, b_{j,m(j)}, 0)$, we get a tuple $(a_j, a_j^{(1)}, \dots, a_j{}^{(m(j))}, 0)$ of K close to $\bar{b}_j$ for the product topology. Since ϕ^* defines an open set U in K^N, where $N = \sum_{j=1}^n (m(j) + 1)$, we may require that $\left((a_j, a_j{}^{(1)}, \dots, a_j{}^{(m(j))})\right)_{j=1}^n$ belongs to U and so we get a solution to our formula ϕ. $\square$

4. Model-completion

In this section, in order to get results about the existence of model-completions for theories of differential topological $\mathcal{L}$-fields, we will add the hypothesis that the topology is first-order definable (see Definition 2.5).

Let T be a universal $\mathcal{L}\cup\{^{-1}\}$-theory of expansions of fields of *characteristic zero*. Assume that there are no other function symbols occurring in $\mathcal{L}$ besides the ones occurring in $\mathcal{L}_{\text{rings}}$. van den Dries called such language a t-language (see chapter 3, Definitions 1.1 in [8]). Let $\mathcal{M}(T)$ be the class of models of T and assume that each element in that class satisfies *Hypothesis* (D); so the corresponding class $\mathcal{M}(T)_{\text{top}}$ is a class of topological $\mathcal{L}$-fields. Moreover, we assume that for every relation symbol $R_i\in\mathcal{L}$, we have polynomials $r_{i,k},\ s_{i,l}\in\mathbb{Z}[X_1,\dots,X_{n_i},Z_1,\dots,Z_{\ell,i}]$ such that the relation R_i (respectively its complement $\neg R_i$), with $i\in I$, is interpreted in any model K of $\mathcal{M}(T)_{\text{top}}$, as an union of an open set O_{R_i} (respectively $O_{\neg R_i}$) and the algebraic subset $\{\bar{x}\in K^{n_i}:\bigwedge_k r_{i,k}(\bar{x},\bar{a})=0\}$ of K^{n_i} (respectively $\{\bar{x}\in K^{n_i}:\bigwedge_l s_{i,l}(\bar{x},\bar{b})=0\}$ of K^{n_i}), where $\bar{a},\bar{b}$ are tuples of elements of K.

Recall that these conditions on the theory are much alike the ones used by Mathews (in [16]). In his thesis, van den Dries introduced the notion of *a t-theory* (see chapter 3, Definitions 1.1 and 1.2 in [8]). A *t-theory* T_{top} is a universal theory in a t-language $\mathcal{L}_{\text{top}}$, satisfying certain conditions slightly more restrictive than the ones above. Note that the inverse function is not, in this case, in the language. However, if one takes the fraction fields of the models of T_{top} and if one expands $\mathcal{L}_{\text{top}}$ with the inverse function, then we are in the above setting.

Assume that the theory T admits a model-completion T_c and that the corresponding class $\mathcal{M}(T_c)_{\text{top}}$ satisfies *Hypothesis* (I). Note that the class $\mathcal{M}(T_c)_{\text{top}}$ is an inductive class of topological $\mathcal{L}$-fields as in Definition 2.18 (see Section 2.3 Example 2.19).

Let T_D (respectively $T_{c,D}$) be the $\mathcal{L}_D$-theory T (respectively T_c) together with the axioms $(*)$ stating that D is a derivation (see Definition 3.5). Note that since the models of T satisfy *Hypothesis* (D), the scheme of axioms (DL) is now first-order. Let $T^*_{c,D}$ be the $\mathcal{L}_D$-theory consisting of $T_{c,D}$ together with the scheme (DL).

Theorem 4.1. *Under the above hypotheses on T and T_c, the theory $T^*_{c,D}$ is the model-completion of T_D.*

Proof. We will apply Blum's criterion for the existence of a model-completion (see [35, Theorem 17.2]).

First, let us check that any model K_0 of T_D embeds into a model of $T^*_{c,D}$. Since T has T_c as model-completion, we embed K_0 into a model K_1 of T_c. Moreover, we may always assume that K_1 is a differential extension of K_0 by extending the derivation D defined on K_0; first, to the relative algebraic closure $\widetilde{K_0}$ of K_0 and then, on a transcendence basis of K_1 over $\widetilde{K_0}$ and finally, again to the relative algebraic closure. Then, using Proposition 3.9, we embed K_1 into a model K_2 of $T_{c,D}$ which satisfies the scheme (DL).

Then, given K_0 a model of T_D, and two $\mathcal{L}_D$-extensions; on one hand a 1-extension $K_0\langle c\rangle$ which is a model of T_D and on the other hand $\widehat{K}$ a $|K_0|^+$-saturated model of $T^*_{c,D}$, we have to find an $\mathcal{L}_D$-embedding of $K_0\langle c\rangle$ into $\widehat{K}$ over K_0.

Since T_c is the model-completion of T, we can embed $K_0\langle c\rangle$ in $\widehat{K}$ over K_0, as an $\mathcal{L}$-substructure. In order to obtain an $\mathcal{L}_D$-embedding, we have to show that any set of open $\mathcal{L}_D$-formulas belonging to the 1-type $t(c,K_0)$ of c over K_0 is finitely satisfiable in $\widehat{K}$ (we use that $\widehat{K}$ is $|K_0|^+$-saturated). Note that we only need to consider conjunctions of basic $\mathcal{L}_D$-formulas of $t(c,K_0)$; namely the formula $\psi(x)=\psi_1(x)\wedge\psi_2(x)$, where

$$\begin{aligned}
\psi_1(x) &:= \bigwedge_i f_i^*(x,x^{(1)},\dots,x^{(n)})=0\\
\psi_2(x) &:= g^*(x,x^{(1)},\dots,x^{(n)})\neq 0\wedge\bigwedge_j R_j\big(p^*_{1,j}(x,x^{(1)},\dots,x^{(n)}),\dots,p^*_{n_j,j}(x,x^{(1)},\dots,x^{(n)})\big)\\
&\quad\wedge\bigwedge_k\neg R_k\big(q^*_{1,k}(x,x^{(1)},\dots,x^{(n)}),\dots,q^*_{m_k,k}(x,x^{(1)},\dots,x^{(n)})\big),
\end{aligned}$$

such that $g,f_i\in K_0\{X\}$ and $p_{r,k},q_{s,l}$ belong to the field $K_0\langle X\rangle$ of differential rational functions.

We associate to the $\mathcal{L}_D$-formula $\psi(x)$, the $\mathcal{L}\cup\{^{-1}\}$-formula, say, $\psi^*(x,x_1,\dots,x_n)$ as in Lemma 3.17 (by replacing in $x^{(i)}$ by x_i). We divide the basic sub-formulas into two sorts:

(1) the first sort concerns the differential equations and inequations which determine the differential field structure of $K_0\langle c\rangle$,

(2) the second sort concerns the formulas which contain the predicates R_i, $i\in I$.

(1) Either c is differentially algebraic over K_0, in which case we let $f(X)=f^*(X,\dots,X^{(m)})$ be a generic polynomial of c over K_0 of order m and degree d, or c is differentially transcendental over K_0, in which case we will apply Lemma 3.17.

Using the same calculation that we performed in the proof of Lemma 3.7, if $f^*(X,\dots,X^{(m)})=\sum_{l=0}^d f_l(X,\dots,X^{(m-1)})\cdot {X^{(m)}}^l$ then $D(f)=\sum_{l=0}^d D(f_l)\cdot {X^{(m)}}^l+\sum_{l>0}^d f_l\cdot l\cdot {X^{(m)}}^{l-1}.X^{(m+1)}$. Now, if we evaluate this polynomial at c, we get that $c^{(m+1)}=(-\sum_l^d f_l(c,\dots,c^{(m-1)})^{(1)}\cdot {c^{(m)}}^l)\cdot s_f(c)^{-1}$. We can iterate this process to express differential polynomials of order higher than m evaluated at c as polynomials over K_0 in $c,\dots,c^{(m)}$ and $s_f(c)^{-1}$. Since f is the generic polynomial of c, the separant $s_f(c)$ is non-zero, being of degree less than or equal to $d-1$.

We have that $K_0\langle c\rangle\models\psi^*(c,\dots,c^{(n)})$ $(n\geqslant m)$. Since $f_i(c)=0$, we get that ${s_f}^{k_i}\cdot f_i\in<f>$ for some positive integers k_i.

Define $\phi(x)$ to be the $\mathcal{L}_D$-formula $f(x) = 0 \wedge s_f(x) \neq 0 \wedge \psi_2(x)$. Since $K_0\langle c\rangle$ embeds in $\widehat{K}$ as an $\mathcal{L}$-substructure, we can find $(d_0, \ldots, d_n) \in \widehat{K}$ such that $\widehat{K} \models \phi^*(d_0, \ldots, d_n)$. Using the scheme (DL), we will show that we can find an element $z \in \widehat{K}$ such that $\widehat{K} \models \phi(z)$. Note that for such an element z, the formula $\psi_1(z)$ holds too.

Since $\widehat{K}$ satisfies the scheme (DL), we apply Proposition 3.14 to either the generic polynomial f of c in case c is differentially algebraic over K_0 or to the polynomial $X^{(n+1)}$ in case c is differentially transcendental. Note that, in both cases, the separant s_f is non-zero at c.

In the differential algebraic case, since f has order $m \leqslant n$, a priori we only get that for any neighbourhood W_{m+1} of zero in $\widehat{K}^{m+1}$, there exists z such that $(z, z^{(1)}, \ldots, z^{(m)}) - (d_0, \ldots, d_m) \in W_{m+1}$. But note that d_{m+1} is equal to $(-\sum_{l=0}{}^{d} D(g_l)^*(d_0, \ldots, d_{m-1}, d_m) \cdot d_m{}^l) \cdot s_f{}^*(d_0, \ldots, d_m)^{-1}$. So, since $\widehat{K}$ is a topological $\mathcal{L}_{\mathrm{rings}}$-field, using the continuity of the field operations (for the inverse operation away from 0), we get that if the tuple $(z, z^{(1)}, \ldots, z^{(m)})$ is close to the tuple $(d_0, d_1, \ldots, d_m)$ then so are the tuples $(z, z^{(1)}, \ldots, z^{(m+1)})$, $(d_0, d_1, \ldots, d_{m+1})$ (see Lemma 2.13).

(2) Now, we have to determine suitable neighbourhoods of $(d_0, \ldots, d_n)$ which will enforce that any element in these neighbourhoods also satisfies the second sort of formulas. Given any relation R_i or $\neg R_j$, we have to distinguish two cases: either, in case of R_i (the other case is similar) $\bigwedge_k r_{i,k}(p_{1,j}(c), \ldots, p_{n_j,j}(c)) = 0$ in which case we add the corresponding system of differential equations to $\psi_1(x)$; or the tuple $(p^*_{1,i}(\bar{d}), \ldots, p^*_{n_i,i}(\bar{d}))$ belongs to O_{R_i}. In this last case, again, we have to use the continuity of the field operations to find an open subset O_i of $\bar{d}$ in $\widehat{K}^n$ such that its image by the rational map $\bar{x} \to (p^*_{1,i}(\bar{x}), \ldots, p^*_{n_i,i}(\bar{x}))$ in $\widehat{K}^{n_i}$ goes into the open subset O_{R_i} (see Lemma 2.13). □

In the Corollary below, we generalize the transfer result of the non-independence property (NIP property) from *RCF* to *CODF* due to C. Michaux and C. Rivière (see Theorem 2.2 [21]). For a reference about the independence property, see for instance [26] Chapter 12, section 4. Note that usually one defines the *NIP* property for complete theories.

In Definition 3.16, we associated with an open $\mathcal{L}_D$-formula $\phi(\bar{x})$, an open $\mathcal{L} \cup \{^{-1}\}$-formula $\phi^*(\bar{x}^*)$.

Lemma 4.2. *Let T be a model-complete $\mathcal{L} \cup \{^{-1}\}$-theory and let $\mathcal{M} \models T$. Assume $\mathcal{M}$ can be embedded into an $\mathcal{L}_D$-structure $\mathcal{M}^*$ whose $\mathcal{L} \cup \{^{-1}\}$-reduct is a model of T and which has an open $\mathcal{L}_D$-formula $\phi(x, \bar{y})$ with the independence property.*

Then the (open) $\mathcal{L} \cup \{^{-1}\}$-formula ϕ^* has the independence property in $\mathcal{M}$.

Proof. Assume that, in the open $\mathcal{L}_D$-formula $\phi(x, \bar{y})$, with $\bar{y} = (y_1, \ldots, y_s)$, x occurs with order at most d_x and y_j, $1 \leq j \leq s$, with order at most d_y. Let $\bar{u} := (u_0, \ldots, u_{d_x})$, $\bar{v} := (v_{1,0}, \ldots, v_{0,d_y}, v_{1,1}, \ldots, v_{s,d_y})$ and let $\theta(\bar{u}, \bar{v})$ be the following $\mathcal{L}_D$-formula: $\bigwedge_{k=0}^{d_x-1}(D(u_k) = u_{k+1} \wedge \bigwedge_{j=1}^{s} \bigwedge_{\ell=0}^{d_y-1} D(v_{j,\ell}) = v_{j,\ell+1}$. So, letting $a, \bar{b}$ vary in M^*, $a^* := (a^{(0)}, \ldots, a^{(d_x)})$ and $\bar{b}^* := (\bar{b}^{(0)}, \ldots, \bar{b}^{(d_y)})$, we have that $\mathcal{M}^* \models \phi(a, \bar{b})$ iff $\mathcal{M}^* \models (\theta(a^*, \bar{b}^*)\ \&\ \phi^*(a^*, \bar{b}^*))$. By assumption on ϕ, for every $n \in \omega$, there exist a sequence of elements $a_i \in M^*$, $i \in n$, and for every subset $S \subseteq n$, tuples $\bar{b}_S \subseteq M^*$ such that

$$\mathcal{M}^* \models \phi(a_i, \bar{b}_S) \leftrightarrow i \in S.$$

Equivalently,

$$\mathcal{M}^* \models \left[\bigwedge_{j=1}^{n} \bigwedge_{S\subseteq n} \theta(a_j^*, \bar{b}_S^*)\right] \wedge \bigwedge_{S\subseteq n} \left[\bigwedge_{i\in S} \phi^*(a_i^*, \bar{b}_S^*) \wedge \bigwedge_{i\notin S} \neg\phi^*(a_i^*, \bar{b}_S^*)\right].$$

Let I_n^* be the following $\mathcal{L} \cup \{^{-1}\}$-sentence:

$$\exists \bar{u}_0 \cdots \exists \bar{u}_{n-1} \bigwedge_{S\subseteq n} \exists \bar{v}_S \left(\bigwedge_{i\in S} \phi^*(\bar{u}_i, \bar{v}_S) \wedge \bigwedge_{i\notin S} \neg\phi^*(\bar{u}_i, \bar{v}_S)\right).$$

Then $\mathcal{M}^* \models I_n^*$ and since $\mathcal{M} \subseteq \mathcal{M}^*$, $\mathcal{M}^* \models T$ and T is model-complete, then $\mathcal{M} \models I_n^*$, in other words the formula $\phi^*(\bar{u}, \bar{v})$ has the independence property in $\mathcal{M}$. □

Corollary 4.3. *Assume in addition that the theories T_c and $T^*_{c,D}$ are complete. Then, if T_c has NIP, then $T^*_{c,D}$ has NIP.*

Proof. By the way of contradiction, suppose that $T^*_{c,D}$ has the independence property. So, there is an $\mathcal{L}_D$-formula $\phi(x, \bar{y})$ witnessing that property in a model of $T^*_{c,D}$ (see Theorem 12.18 in [26]) (and since the theory $T^*_{c,D}$ is complete, in any model). Since the theory $T^*_{c,D}$ is the model-completion of a universal theory (by Theorem 4.1), it admits quantifier elimination and so we may assume that ϕ is an open $\mathcal{L}_D$-formula.

Let $\mathcal{M}$ be a model of T_c, then we may expand $\mathcal{M}$ into an $\mathcal{L}_D$-structure and embed it in a model $\mathcal{M}^*$ of $T^*_{c,D}$ by Theorem 4.1. By applying the above Lemma, the $\mathcal{L} \cup \{^{-1}\}$-formula ϕ^* has the independence property in $\mathcal{M}$, a contradiction. □

5. Geometrical formulation

Now we will introduce another scheme (*DLG*) which is a geometrical version of the scheme (*DL*). This generalizes in our topological setting, the geometrical axiomatization given by D. Pierce and A. Pillay of the differentially closed fields (see [22]). This was done before by C. Michaux and C. Rivière for *CODF* ([21]).

Notation 5.1. Let K be a field and $\tilde{K}$ its algebraic closure; let $A \subseteq \tilde{K}^n$ be an irreducible variety defined over K, we denote by $I(A(K))$ the ideal of polynomials with coefficients in K which vanish on the K-points $A(K)$ of A. Let $\tau(A) := \{(\bar{a}, \bar{b}) : \bar{a} \in A \& \sum_{i=1}^{n}(\partial_i P)(\bar{a}).b_i + P^D(\bar{a}) = 0$ for all $P(\bar{X}) \in I(A(K))\}$, where P^D is the polynomial obtained from P when one applies D on the coefficients of P and ∂_i denotes the partial derivative with respect to the i^{th} variable of P.

Definition 5.2. Under the same assumption on $\langle K, \mathcal{V}\rangle$ as before, we will say that it satisfies $(DLG)_{\mathcal{V}}$ if for any $V \in \mathcal{V}$, for any absolutely irreducible varieties A, B defined over K with $A \subseteq \tau(B)$ and if A projects generically on B and if there is a tuple $(\bar{a}, \bar{c}) \in K^{2n} \cap A$, then there is $\bar{b} \in K^n$ such that $(\bar{b}, D(\bar{b})) \in A$ and $((\bar{a}, \bar{c}) - (\bar{b}, D(\bar{b}))) \in V^{2n}$.

Proposition 5.3. *Let $\mathcal{C}$ be an inductive class of differential topological $\mathcal{L}$-fields satisfying Hypothesis (I) and let $\mathbf{K} := \langle K, \mathcal{V}\rangle \in \mathcal{C}$. Then $\mathbf{K}$ has a differential topological $\mathcal{L}$-extension $\langle \widehat{K}, \widehat{\mathcal{V}}\rangle$ satisfying the scheme $(DLG)_{\widehat{\mathcal{V}}}$.*

Proof. The proof is parallel to the proof of Proposition 3.9 which induction step is Lemma 3.7. We will use the same terminology as in [22].

We detail below the proof of the induction step, namely we show that given $\mathbf{K}$, and absolutely irreducible varieties A, B defined over K with $A \subseteq \tau(B)$ and if A projects generically on B and if there is a tuple $(\bar{a}, \bar{c}) \in K^{2n} \cap A$, then we can find a differential topological $\mathcal{L}$-extension $\langle \widehat{K}, \widehat{\mathcal{V}}\rangle \in \mathcal{C}$ of $\mathbf{K}$ satisfying $Comp(\mathbf{K})$ with respect to some $\mathcal{W}(K) \subseteq \widehat{\mathcal{V}}$ and $\bar{b} \in \widehat{K}^n$ such that $(\bar{b}, D(\bar{b})) \in A(\widehat{K})$ and $(\bar{a}, \bar{c}) \sim_{\mathcal{W}(K)} (\bar{b}, D(\bar{b}))$.

Let $\tilde{L}$ be the algebraic closure of an ω-saturated extension L of K and let $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ be a generic point on $A(\tilde{L})$. Using Noether normalisation theorem, we may suppose that $u_1, \ldots, u_d, v_1, \ldots, v_r$ are algebraically independent over K. For $d+1 \leq i \leq n$, let Q_i be the minimal polynomial of u_i over $K[u_1, \ldots u_d, v_1, \ldots, v_r]$ and for $r+1 \leq j \leq n$, let T_j be the minimal polynomial of v_i over $K[u_1, \ldots, u_d, v_1, \ldots, v_r]$. So, $\partial_i Q_i(u_i, u_1, \ldots u_d, v_1, \ldots, v_r) \neq 0$ for $d+1 \leq i \leq n$ and $\partial_j T_j(v_j, u_1, \ldots u_d, v_1, \ldots, v_r) \neq 0$ for $r+1 \leq j \leq n$.

Let $t_1, \ldots, t_d, s_1, \ldots, s_r$ be algebraically independent elements of L over K such that $t_i, s_j \sim_{\mathcal{W}(K)} 0$, $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, r\}$. We substitute for $u_1, \ldots, u_d$ (respectively $v_1, \ldots, v_r$) the elements $a_1 + t_1, \ldots, a_d + t_d$ (respectively $c_1+s_1, \ldots, c_r+s_r$). Set $\bar{a}+\bar{t} := (a_1+t_1, \ldots, a_d+t_d)$ (respectively $\bar{c}+\bar{s} := (c_1+s_1, \ldots, c_r+s_r)$) and consider the polynomials $Q_i(X, \bar{a}+\bar{t}, \bar{c}+\bar{s})$, for $d+1 \leq i \leq n$, $T_j(X, \bar{a}+\bar{t}, \bar{c}+\bar{s})$, for $r+1 \leq j \leq n$. Since, $\mathbf{K}$ satisfies *Hypothesis* (I), there is a topological $\mathcal{L}$-extension $\langle \widehat{K}, \widehat{\mathcal{V}}\rangle$ which contains zeroes of these polynomials (satisfying $Comp(\mathbf{K})$ with respect to some $\mathcal{W} \subseteq \widehat{\mathcal{V}}$). Namely there exist in $\widehat{K}$ α_i, γ_j such that $Q_i(\alpha_i, \bar{a}+\bar{t}, \bar{c}+\bar{s}) = 0$ and $\alpha_i \sim_{\mathcal{V}} a_i$, for $d+1 \leq i \leq n$, $T_j(\gamma_j, \bar{a}+\bar{t}, \bar{c}+\bar{s}) = 0$ and $\gamma_j \sim_{\mathcal{W}} c_j$, for $r+1 \leq j \leq n$. Let $\bar{\alpha} := (\bar{a}+\bar{t}, \alpha_{d+1}, \ldots, \alpha_n)$ (respectively $\bar{\gamma} := (\bar{c}+\bar{s}, \gamma_{r+1}, \ldots, \gamma_n)$). Note that since $A \subseteq \tau(B)$, $(\bar{\alpha}, \bar{\gamma}) \in \tau(B)$. There is a surjection from $\tau_{(\bar{\alpha},\bar{\gamma})}(A)$ onto $\tau_{\bar{\alpha}}(B)$ (see Lemma 1.6 in [22]). So, $\{\bar{x} \in \widehat{K}^n : (\bar{\gamma}, \bar{x}) \in \tau_{(\bar{\alpha},\bar{\gamma})}(A)\} \neq \emptyset$. Namely, we can find $\bar{x} \in \widehat{K}(\bar{\alpha}, \bar{\gamma})$ such that for each $r+1 \leq i \leq n$, let $\sum_{j=1}^{r}(\partial_j T_i)(\bar{\alpha}, \bar{\gamma}).x_j + (\partial_i T_i(\bar{\alpha}, \bar{\gamma}).x_i + T_i^D(\bar{\alpha}, \bar{\gamma}) = 0$. In order to find $\bar{x}$, we have to solve a linear system with coefficients in $K(\bar{\alpha}, \bar{\gamma})$, and so we may find a solution in $K(\bar{\alpha}, \bar{\gamma})$. Therefore, we can extend the derivation from K to $K(\bar{\alpha}, \bar{\gamma})$ by letting $D(\bar{\alpha}, \bar{\gamma}) = (\bar{\gamma}, \bar{x})$. $\square$

We could have used in the previous section this scheme (*DLG*) instead of the scheme (*DL*) in order to obtain an axiomatization of the existentially closed differential expansions of a model-complete (universal) theory. In some cases, this geometrical formulation is easier to use; in particular we expect to use it to handle fields expansions with new function symbols.

6. Applications to topological fields with an absolute value

The examples we are going to consider will be either non-Archimedean fields with the interval topology or fields with a valuation topology. In both cases, the topology will be first-order definable, and so *Hypothesis* (D) will hold. In the first case, we will add an order to the field language and in the second case, instead of adding a valuation map, we will use a *linear divisibility relation*, that we introduce below.

Definition 6.1 (*See [17, Section 4.2]*). Let A be a commutative domain. Then a *linear divisibility relation* (l.d. relation) on A is a binary relation $\mathcal{D}(.,.)$ on A such that:

$\mathcal{D}$ is transitive, $\neg\mathcal{D}(0, 1)$, compatible with $+$ and ., namely $\mathcal{D}(a, b)$ and $\mathcal{D}(a, c)$ implies that $\mathcal{D}(a, b+c)$, and for all $c \neq 0$, we have $\mathcal{D}(a, b)$ implies $\mathcal{D}(a.c, b.c)$, and either $\mathcal{D}(a, b)$ or $\mathcal{D}(b, a)$.

A *l.d. relation* $\mathcal{D}$ on the domain A induces a valuation ring $\mathcal{O}_A$ of the fraction field $F := Frac(A)$ of A:

$$\mathcal{O}_A = \left\{\frac{a}{b} : a, b \in A, \ b \neq 0, \ \mathcal{D}(b, a)\right\}.$$

The corresponding valuation $v_{\mathcal{D}}$ on $Frac(A)$ is defined by: for any $a, b \in A$,

$$v_{\mathcal{D}}(a) \leqslant v_{\mathcal{D}}(b) \iff \mathcal{D}(a, b).$$

We have a bijection between the set of l.d. relations and the set of valuation rings of $Frac(A)$. So any field with a l.d. relation can be endowed with a valuation topology.

Let $\mathcal{L}_{\mathcal{D}}$:=$\mathcal{L}_{\text{rings}} \cup \{\mathcal{D}, \underline{d}\}$, $\mathcal{L} := \mathcal{L}_{\mathcal{D}} \cup \{^{-1}\}$ and let VF_0 be the $\mathcal{L}$-theory of non-trivially valued fields of equicharacteristic zero (i.e. $\bigwedge_{n\in\mathbb{N}-\{0,1\}} \neg\mathcal{D}(n, 1)$ belongs to VF_0 and so, VF_0 is a universal $\mathcal{L}$-theory). In a non-trivially valued field K, the l.d. relation $\mathcal{D}(x, y)$ defines a set in K^2 which is the union of an open set and $\{(0, 0)\}$. Then $ACVF_0$ is the $\mathcal{L}$-theory of algebraically closed non-trivially valued fields of equicharacteristic zero and it is model-complete (see [33]), moreover using prime extensions, it is easy to see that it admits quantifier elimination in $\mathcal{L}$ (see [17, p. 83]). In this case, *Hypothesis* (I) is satisfied (see Examples 2.22) since a model of $ACVF_0$ is Henselian.

So we get the following corollary to Theorem 4.1 and to Corollary 4.3.

Corollary 6.2. *The $\mathcal{L}_D$-theory $ACVF^*_{0,D}$ is the model-completion of the $\mathcal{L}_D$-theory $VF_{0,D}$. Moreover, $ACVF^*_{0,D}$ has NIP.*

Proof. Using Corollary 4.3, a result of Bélair (see Corollaire 7.5 in [5]) and the fact that the theory of algebraically closed fields is stable, we get $ACVF^*_{0,D}$ has *NIP*. □

Let us note that a model of $ACVF^*_{0,D}$ is in particular a differentially closed field of characteristic zero.

Recall that a p-valued field K of p-rank d ($d \in \mathbb{N}$) with p a prime number, is a valued field K of characteristic 0, residue field of characteristic p and the dimension of $\mathcal{O}_K/(p)$ over the prime field $\mathbb{F}_p$ is equal to d.

Let $\mathcal{L}_{P_\omega} := \mathcal{L}_{\mathcal{D}} \cup \{^{-1}\} \cup \{P_n;\, n \in \mathbb{N} \setminus \{0, 1\}\} \cup \{c_2, \ldots, c_d\}$. The theory pCF_d of p-adically closed fields of p-rank d admits quantifier elimination in $\mathcal{L}_{P_\omega}$ and so is the model-completion of its universal part $(pCF_d)_\forall$. This last theory has been axiomatized by a set of axioms $T_{p,d}$ by Bélair (see [4, Theorem 2.4]).

In order to apply our results, we need to check that the predicate P_n defines an open subset union an algebraic set (in this case a finite set) as well as its negation in any model K of pCF_d. Let us denote by $P_n^\times$ the set of non-zero nth powers in $K^\times$. Then, one can find an integer r_n such that $\forall x \neq 0\, \forall y \neq 0\, \left[(x \in P_n^\times \wedge v(x-y) > v(x) + r_n.v(x)) \Rightarrow y \in P_n^\times\right]$ (see [4, Lemma 1.10] and also [17]). So, $P_n^\times$ is open in $K^\times$ and since it has finite index in $K^\times$, its complement is also open in $K^\times$.

Thus, we get the following corollary to Theorem 4.1 and to Corollary 4.3.

Corollary 6.3. *The $\mathcal{L}_{P_\omega,D}$-theory $(pCF_d)^*_D$ is the model-completion of $(T_{p,d})_D$. Moreover, $(pCF_d)^*_D$ has NIP.*

Proof. For the second statement, we can use Corollary 4.3, the result of Bélair [5] and the fact that the residue field of a model of pCF_d is finite, to deduce that $(pCF_d)^*_D$ has *NIP*. □

Let $\mathcal{L}_{\mathcal{D},<} := \mathcal{L}_{\mathcal{D}} \cup \{^{-1}\}$ and let OVF be the theory of non-trivially valued ordered fields, namely the $\mathcal{L}_< \cup \{^{-1}\}$-theory of ordered fields together with the theory of fields with a l.d. relation $\mathcal{D}$, and the following compatibility condition between the valuation topology and the order topology:

$$\forall a\, \forall b \quad \big(0 \leqslant a \leqslant b \Rightarrow \mathcal{D}(b, a)\big).$$

Let RVF be the $\mathcal{L}_{\mathcal{D},<}$-theory of real-closed valued fields, namely the theory OVF together with axioms of real-closed fields. Note that an $\mathcal{L}_{\mathcal{D},<}$-substructure of a model of RVF is a model of OVF.

The theory RVF is model-complete. Indeed, a real-closed valued field is a Henselian valued ordered field with a real-closed residue field and divisible abelian totally ordered group (see [7, Theorem 3]). Since the theory of real-closed field RCF and the theory of divisible totally ordered groups are model-complete and complete, the $\mathcal{L}_{\mathcal{D},<}$-theory RVF is model-complete and complete by Ax–Kochen–Ershov Theorems (see [7, Theorems A and B]) (note that the order in a real-closed field is existentially definable).

Then, we show that any $\mathcal{L}_{\mathcal{D},<}$-substructure of a model of RVF has a prime extension and so RVF is the model-completion of OVF (see [35]). Let K be a model of OVF. Let $\widetilde{K}$ be the real closure of K and let A be the convex hull of $\mathcal{O}_K$ in $\widetilde{K}$. Then A is a valuation ring of $\widetilde{K}$ and its maximal ideal $\mathcal{M}_A$ is such that $\mathcal{M}_A \cap \mathcal{O}_K = \mathcal{M}_K$ (see [3, Lemma 1.1, Lemma 1.8 and its proof]) and so $\widetilde{K}$ is an $\mathcal{L}_{\mathcal{D},<}$-extension of K, satisfying RVF (see [7, Theorems 1 and 2]).

Therefore, we get the following corollary of Theorem 4.1 and to Corollary 4.3.

Corollary 6.4. *The $\mathcal{L}_{\mathcal{D},<,D}$-theory $(RVF)^*_D$ is the model-completion of the $\mathcal{L}_{\mathcal{D},<,D}$-theory $(OVF)_D$. Moreover, RVF^*_D has NIP.*

Proof. Since the theory of real-closed fields has *NIP* (see [24]), by Corollary 4.3 and Corollary 7.5 in [5], we get that RVF^*_D has *NIP*. □

Let us note that any model of RVF^*_D is a closed ordered differential field as defined in [39].

7. Hilbert's seventeenth problem for differential *p*-adically closed fields

Let p be a prime number, let d, f be positive natural numbers. Let $\langle K, v\rangle$ be a differential p-valued field of p-rank d where $v : K \longmapsto v(K^{\times}) \cup \{\infty\}$ is a p-valuation of p-rank d and $[k_K : \mathbb{F}_p] = f$. Let π belonging to K be such that $v(\pi)$ is the least positive element of the value group.

Set $q = p^f$ and let $\gamma(X) := \frac{1}{\pi}[\frac{X^q - X}{(X^q - X)^2 - 1}]$ be the *π-adic Kochen operator*.

We have that $\gamma(K) \subseteq \mathcal{O}_K$ (see [30, Lemma 6.1]); if K is p-adically closed, then $\mathcal{O}_K = \gamma(K)$ (see [30, Theorem 6.15]).

Denote by $K\langle\overline{X}\rangle := K\langle X_1, \ldots, X_n\rangle$ the field of differential rational functions in n indeterminates. Assume now that $\langle K, v\rangle$ is a differential p-valued field of p-rank d with valuation v and derivation D. Then we can extend the valuation and the derivation on $K\langle\overline{X}\rangle$ in such a way it becomes a differential p-valued extension of K of p-rank d. Note that to check this statement it suffices to do it for the field of differential rational functions in one indeterminate X.

In Definition 3.1, we already showed how to extend D on $K\{X\}$. Then, for any elements $f(X),\ g(X) \in K\{X\}$, we define as usual $D(f/g) := (D(f).g - f.D(g))/g^2$.

To extend v to a p-valuation w of p-rank d, it suffices to check it for the ordinary polynomial ring $K[X]$ (see [30, Example 2.2]), then we go to the fraction field and then, we iterate considering first the polynomial ring $K(X)[X^{(1)}]$. Finally, we will get that $w : K\langle X\rangle^{\times} \longmapsto \mathbb{Z}^{\mathbb{N}} \times v(K^{\times})$ where $\mathbb{Z}^{\mathbb{N}}$ is the set of sequences of elements in $\mathbb{Z}$, with finite support, indexed by $\mathbb{N}$ and that w is a p-valuation of the same p-rank as v.

Now assume that $\langle K, v\rangle$ is a differential p-adically closed field of p-rank d.

Before recalling the analogue of Hilbert's seventeenth problem for p-adically closed fields of p-rank d, we need to introduce the following notation.

Let $\langle L, v\rangle$ be a p-valued extension of $\langle K, v\rangle$.

Definition 7.1 (*See [30, Chapter 6, Section 2]*). The γ-Kochen ring R_L of L over K is the subring defined by:

$$R_L = \left\{\frac{t}{1 + \pi \cdot s} : t, s \in \mathcal{O}_K[\gamma(L)] \text{ and } 1 + \pi \cdot s \neq 0\right\}.$$

The quotient field of R_L is the field generated by K and $\gamma(L) \setminus \{\infty\}$ and by Merckel's Lemma (see [30, Appendix]), $K(\gamma(L)) = L$ (see [30, Lemma 6.6]).

Theorem 7.2 (*See [30, Theorem 7.12]*). *Let K be a model of pCF_d. If $f, g \in K[X_1, \ldots, X_n]$ and f/g is integral definite (i.e. $g(\bar{a}) \neq 0$ implies $\frac{f(\bar{a})}{g(\bar{a})} \in \mathcal{O}_K$ for all $\bar{a} \subseteq K^n$), then there are $t,\ t' \in \mathcal{O}[\gamma(K(X_1, \ldots, X_n))]$ such that*

$$\frac{f}{g} = \frac{t'}{1 + \pi \cdot t}. \quad \square$$

Now, let us state and prove the differential case using the technology of Section 3 and the following result on holomorphy rings.

Theorem 7.3 (*See [30, Theorem 6.14]*). *The γ-Kochen ring R_L of L is the holomorphy ring $\bigcap_{v\in\Gamma} \mathcal{O}_v$, where Γ is the (non-empty) set of all p-valuations of p-rank d of L which extend the p-valuation of p-rank d of K.* $\square$

Theorem 7.4. *Let $\langle K, D, v\rangle$ be a model of $(pCF_d)^*_D$. Let us consider the ring $K\{\overline{X}\}$ of differential polynomials in n differential indeterminates over K. Let $f,\ g$ be two differential polynomials in $K\{\overline{X}\}$ such that $\frac{f}{g}$ is integral definite (i.e $g(\bar{a}) \neq 0$ implies $\frac{f(\bar{a})}{g(\bar{a})} \in \mathcal{O}_K$ for all $\bar{a} \subseteq K^n$). Then $\frac{f}{g}$ belongs to the γ-Kochen ring $R_{K\langle\overline{X}\rangle}$ of $K\langle\overline{X}\rangle$ over K.*

Proof. Let us assume that $\frac{f}{g} \notin R_{K\langle\bar{X}\rangle}$. Then, by Theorem 7.3, there exists one p-valuation w of p-rank d of $K\langle\overline{X}\rangle$ extending v over K such that $w(\frac{f}{g}) < 0$.

We have:

$$\langle K\langle\overline{X}\rangle, w\rangle \models \phi := \exists\bar{y}\left[w\left(\frac{f^*(\bar{y})}{g^*(\bar{y})}\right) < 0 \wedge g^*(\bar{y}) \neq 0\right]$$

where $f^*,\ g^*$ are the usual polynomials corresponding to f and g. Then, using the fact that pCF_d is the model-completion of $T_{p,d}$, we embed $K\langle\overline{X}\rangle$ in a differential p-adically closed field of p-rank d. So, since K satisfies the scheme (DL), we use the model-completeness of pCF_d and apply Lemma 3.17. So, we get a contradiction with $\frac{f}{g}$ integral definite. $\square$

8. A theorem of differential transfer of the AKE-principle

In this section, we will prove a differential analogue of Ax–Kochen–Ershov Theorem for differential Henselian valued fields satisfying the scheme (*DL*) (with respect to the valuation topology). We shall use the following "existentially closed (e.c.)" version of this theorem, which can be found in [29, Section 1].

Section 9 is devoted to the proof of a differential transfer result of model-completeness for theories of topological fields, using a stronger version of the scheme (*DL*). Note that here, we just need the (weaker) scheme (*DL*) to get Theorem 8.3.

Let $\langle K_1, v_1\rangle$ be a Henselian valued field of *equicharacteristic zero* and $\langle K_2, v_2\rangle$ a valued field extension of $\langle K_1, v_1\rangle$. Assume that

(1) $k_{K_1} \subseteq_{e.c.} k_{K_2}$ in the language of fields and,
(2) $v(K_1^\times) \subseteq_{e.c.} v(K_2^\times)$ in the language of totally ordered groups.

Then $\langle K_1, v_1\rangle \subseteq_{e.c.} \langle K_2, v_2\rangle$ in $\mathcal{L} := \mathcal{L}_{\text{fields}} \cup \{\mathcal{D}\}$ the language of fields with an l.d. relation (see Definition 6.1).

First we translate Lemma 3.13 for topological fields endowed with a valuation topology:

Lemma 8.1. *Let $\langle K, v\rangle$ be a differential valued field which satisfies the scheme (DL). Then we have $v(K^\times) = v\big(C_K^\times\big)$ i.e. the value group is the set of values of the constant field C_K.*

Proof. Let α be in $v(K^\times)$ and let c be an element of $K^\times$ of value α. We apply the scheme (*DL*) to the differential polynomial $X^{(1)}$ (its separant is 1) at the point $(c, 0)$ for the neighbourhood $\{x \in K | v(x) > \alpha\}$ of 0. So we get a constant $d \in C_K$ such that $v(c - d) > \alpha$ which implies that $v(d) = \alpha$. $\square$

Remark 8.2. Note that the hypothesis $v\big(C_K^\times\big) = v(K^\times)$ is actually used in [36] (where there is a stronger interaction between the derivation and the valuation) (see also Section 12).

Now we are ready to prove the differential version of the Ax–Kochen–Ershov Theorem for differential valued fields satisfying the scheme (*DL*).

Theorem 8.3. *Let $\langle K_1, v_1\rangle$ be a differential Henselian valued field of equicharacteristic zero which satisfies the scheme (DL) and let $\langle K_2, v_2\rangle$ be a differential valued extension of $\langle K_1, v_1\rangle$ such that*

(1) $v\big(C_{K_2}^\times\big) = v(K_2^\times)$;
(2) $k_{K_1} \subseteq_{e.c.} k_{K_2}$;
(3) $v(K_1^\times) \subseteq_{e.c.} v(K_2^\times)$;

then $\langle K_1, v_1\rangle \subseteq_{e.c.} \langle K_2, v_2\rangle$ in the language $\mathcal{L}_D$ of differential valued fields.

Proof. We closely follow the proof of the Ax–Kochen–Ershov Theorem given in [29, Appendix]. For convenience of the reader, we reproduce the proof here. As in [29], we can reduce to the following situation:

(1) $\langle \widehat{K}, \widehat{v}\rangle$ is an $\mathcal{L}_D$-elementary $|K_2|^+$-saturated extension of $\langle K_1, v_1\rangle$,
(2) $k_{K_1} \subseteq_{e.c.} k_{K_2} \subseteq k_{\widehat{K}}$,
(3) $v(K_1^\times) \subseteq v(K_2^\times) \subseteq v(\widehat{K}^\times)$ and $v(K_2^\times)/v(K_1^\times)$ is torsion-free.

Using the existential version of Frayne's Theorem, $K_1 \subseteq_{e.c.} K_2$ is equivalent to show that $\langle K_2, v_2\rangle$ $\mathcal{L}_D$-embeds into $\langle \widehat{K}, \widehat{v}\rangle$ over $\langle K_1, v_1\rangle$. We then proceed in three steps.

Note that, without loss of generality, we may assume that $\langle K_2, v_2\rangle$ is also Henselian. Otherwise we take the Henselization of K_2 inside $\widehat{K}$ and the hypotheses are still met since the Henselization is an immediate extension.

Step 1: In this step we extend the embedding of $\langle K_1, v_1\rangle$ into $\langle \widehat{K}, \widehat{v}\rangle$ (which is the identity) to a differential valued subfield of $\langle K_2, v_2\rangle$ which has residue field k_{K_2}. Suppose that $\langle K, v\rangle$ is a maximal differential valued subfield of $\langle K_2, v_2\rangle$ having value group $v(K^\times) = v(K_1^\times)$ such that $\langle K, v\rangle$ can be $\mathcal{L}_D$-embedded into $\langle \widehat{K}, \widehat{v}\rangle$. We identify $\langle K, v\rangle$ with its image and K is Henselian.

If $k_K \subsetneq k_{K_2}$, we let $0 \neq \bar{x} \in k_{K_2} \setminus k_K$, for some $x \in K_2$, and consider two cases, both leading to a contradiction to the maximality of $\langle K, v\rangle$.

<u>Case 1:</u> $\bar{x}$ is algebraic over k_K. This case is similar to Case 1 in [29]. In addition, we use that the derivation extends in a unique way to an algebraic extension since we are in characteristic zero.

<u>Case 2:</u> $\bar{x}$ is transcendental over k_K. By hypothesis and Lemma 8.1, we can choose $\tilde{x} \in C_{K_2}$ and $\widehat{x} \in C_{\widehat{K}}$ preimages of $\bar{x}$ respectively. We conclude as in [29] to obtain a contradiction.

Step 2: In this step we extend the above embedding further to a differential valued subfield of $\langle K_2, v_2\rangle$ with value group $v(K_2^\times)$. Let now $\langle K, v\rangle$ be a maximal differential valued subfield of $\langle K_2, v_2\rangle$ which $\mathcal{L}_D$-embeds into $\langle \widehat{K}, \widehat{v}\rangle$ such that $k_K = k_{K_2}$ and $v(K_2^\times)/v(K^\times)$ is torsion-free. Such differential subfield exists by Zorn's Lemma and is Henselian. We identify $\langle K, v\rangle$ with its image. Assume that $v(K_2^\times) \setminus v(K^\times)$ contains an element α. By the assumption on $v(K^\times)$, we have $v(K^\times) \cap \mathbb{Z}\alpha = \{0\}$. Since $v\big(C_{K_2}^\times\big) = v(K_2^\times)$, we can take $x \in C_{K_2}$ having value α (and x is transcendental over K) (*).

Assigning the value α to x defines a unique extension of $\mathcal{O}_K$ to the differential rational function field $K\langle x\rangle$ which coincides with the rational function field $K(x)$. From $K(x)$, we now pass to a valued algebraic extension (and so, differentially algebraic extension) K' inside K_2 such that $v(K_2^\times)/v(K'^\times)$ is torsion-free where $v(K'^\times)$ is the value group corresponding to the valuation ring $\mathcal{O}'$ of K'. This can be done in the following way. If some $\delta \in v(K_2^\times) \setminus (v(K^\times) \oplus \mathbb{Z}\alpha)$ satisfies $q.\delta \in v(K^\times) \oplus \mathbb{Z}\alpha$ for some prime $q \in \mathbb{N}$, we choose $y \in K_2$ having value δ and $a \in K\langle x\rangle$ having value $q.\delta$. Then $y^q.a^{-1}$ is a unit in $\mathcal{O}_{K_2}$. Since $k_{K_2} = k_{K\langle x\rangle}$, we find a unit $e \in K\langle x\rangle$ such that $\overline{y^q.a^{-1}.e^{-1}}$ is the unity of k_{K_2}. Since $\mathrm{char}(k_{K_2}) = 0$, Hensel's Lemma gives us a qth root of $y^q a^{-1}.e^{-1}$ in K_2. Thus, $a.e = z^q$ for some $z \in K_2$. So the value group of the differential valued field $K\langle x, z\rangle$ contains δ. Transfinite repetitions of this procedure (or simply an application of Zorn's lemma) yield an algebraic extension (so a differential extension) $\langle K', v'\rangle$ of $K\langle x\rangle$ of the desired nature. It remains to find an $\mathcal{L}_D$-embedding of $\langle K', v'\rangle$ into $\langle \widehat{K}, \widehat{v}\rangle$. Since $\langle \widehat{K}, \widehat{v}\rangle$ is $|K_2|^+$-saturated, it is sufficient to find an $\mathcal{L}_D$-embedding into $\langle \widehat{K}, \widehat{v}\rangle$ for every differential subfield of $\langle K', v'\rangle$ finitely differentially generated. It follows that $\langle K', v'\rangle$ admits an $\mathcal{L}_D$-embedding into $\langle \widehat{K}, \widehat{v}\rangle$. Therefore, we may assume that $\langle K', v'\rangle$ itself is a finitely differentially generated field extension of $\langle K, v\rangle$.

We may assume that K' is of the following form $K\langle x; x_i : i < n\rangle$ where $x \in C_{K_2}$ and is algebraically transcendental over K and the x_i's are algebraic over $K\langle x\rangle$. Since $v(K'^\times)/v(K^\times)$ is torsion-free, we have that $v(K'^\times) = v(K^\times) + \mathbb{Z}\beta$ for some $\beta \in v(K'^\times)$. We choose y in K' such that $v(y) = \beta$. As in the proof in [29], we can embed K' as an $\mathcal{L}$-structure. We are going to use this embedding to find an element $\widehat{y}$ in $\widehat{K}$ such that $K\langle y\rangle$ is isomorphic to $K\langle\widehat{y}\rangle$ as $\mathcal{L}_D$-structures. So, since $\langle K', v'\rangle$ is an immediate extension of $K\langle y\rangle$ and the derivation extends uniquely to an algebraic extension (we are in characteristic zero), the uniqueness of the Henselian closure yields an $\mathcal{L}_D$-embedding of $\langle K', v'\rangle$ into $\langle \widehat{K}, \widehat{v}\rangle$. Thus $\langle K, v\rangle$ would not be maximal with $k_{K_2} = k_K$ and $v(K_2^\times)/v(K^\times)$ being torsion-free, unless $v(K_2^\times) = v(K^\times)$. We consider the $\mathcal{L}_D$-quantifier-free type $t(y/K)$ of y over K and we want to find an element $\widehat{y}$ in $\widehat{K}$ such that $\widehat{y}$ satisfies $t(y/K)$.

Since $\widehat{K}$ is $|K_2|^+$-saturated, it is sufficient to show the finite satisfiability of this type. We easily conclude by the following argument: we use the fact that $K\langle y\rangle \subseteq K'$ which embeds in $\widehat{K}$ as a non-differential valued field and since $\langle \widehat{K}, \widehat{D}, \widehat{v}\rangle$ satisfies the scheme of axioms (DL), we finish along the lines of the proof of Theorem 4.1 by using Proposition 3.14.

<u>*Step 3*</u>: Let finally $\langle K, v\rangle$ be a maximal differential valued subfield of $\langle K_2, v_2\rangle$ such that $k_{K_2} = k_K$, $v(K_2^\times) = v(K^\times)$ and $\langle K, v\rangle$ $\mathcal{L}_D$-embeds into $\langle \widehat{K}, \widehat{v}\rangle$. We identify $\langle K, v\rangle$ with its image. Clearly $\langle K, v\rangle$ is Henselian. Using again $\mathrm{char}(k_K) = 0$, we see that K is relatively algebraically closed in K_2. Every element $x \in K_2 \setminus K$ would be transcendental over K. We consider the quantifier-free differential valued type $t(x/K)$ of x over K and we want to find an element $\widehat{x}$ in $\widehat{K}$ satisfying this type and so, we can $\mathcal{L}_D$-embed $K\langle x\rangle$ into $\widehat{K}$ sending x to $\widehat{x}$. Since $\widehat{K}$ is $|K_2|^+$-saturated, it is sufficient to show the finite satisfiability of the type $t(x/K)$. Take a finite conjunction of formulas in $t(x/K)$ and denote it by $\theta(x, \vec{k})$ for some elements $\vec{k}$ in K. Since the value groups and residue fields of $\langle K, v\rangle$ and $\langle K_2, v_2\rangle$ are the same, we obtain by the non-differential version of Ax–Kochen–Ershov Theorem that $\langle K, v\rangle \prec_{\mathcal{L}} \langle K_2, v_2\rangle$. So, since $K_2 \models \exists\vec{y}\ \theta^*(\vec{y}, \vec{k})$, we get that $K \models \exists\vec{y}\ \theta^*(\vec{y}, \vec{k})$ and so does $\widehat{K}$, where θ^* is the non-differential formula corresponding to θ. Moreover, since $\langle \widehat{K}, \widehat{v}\rangle$ satisfies the scheme of axioms (DL), we proceed as in the proof of Theorem 4.1 to find an element $\widehat{y} \in \widehat{K}$ such that $\widehat{K} \models \exists\widehat{y}\ \theta(\widehat{y}, \vec{k})$.

Thus we obtain a contradiction leading to $K = K_2$. $\square$

Remark 8.4. Thanks to the differential transfer theorem we can reprove the model-completeness of some theories of differential valued fields. Indeed we have seen that the scheme (DL) implies that the set of values of the constant field is the whole value group. For example, in Section 6, we proved that the theories $ACVF^*_{0,D}$ and RVF^*_D are model-complete. It can easily be deduced from the previous result since in both cases the theories of the residue field and of the value group are model-complete. On one hand, we know that the theory of algebraically closed fields and the theory of real-closed fields are model-complete in the language of fields and on the other hand that the theory of totally ordered divisible abelian groups is model-complete in the language of totally ordered groups.

Let T_r be a model-complete $\mathcal{L}_{\mathrm{fields}}$-theory of fields of characteristic zero and let T_g be any model-complete theory of totally ordered abelian groups (in the language of totally ordered abelian groups). We are going to build a differential Henselian valued field $\langle \widetilde{L}, D, \widetilde{v}\rangle$ such that $\widetilde{L} \models (DL)$, $k_{\widetilde{L}} \models T_r$ and $v(\widetilde{L}^\times) \models T_g$. Then, by our previous theorem, the $\mathcal{L}_D$-theory of such differential Henselian valued fields will be model-complete.

Let K be a model of T_r, let G be a model of T_g. Let $L := K((G))$ be the generalized power series field over K; L is an Henselian valued field with residue field K and with value group G (see [37, Theorems 6, 7 p. 45]). Let D be any derivation on L.

The class $\mathcal{C}$ of Henselian valued fields whose residue fields are models of T_r and value groups are models of T_g, is $\mathcal{L}$-elementary and inductive.

Let us show that we can embed L in a differential Henselian valued field $\langle \widetilde{L}, \widetilde{v}\rangle$ such that $\widetilde{L} \models (DL)$, $k_{\widetilde{L}} \models T_r$ and $v(\widetilde{L}^\times) \models T_g$. Consider a non-principal ultrapower $\widehat{L}$ of L, then $\widehat{L}$ satisfies the conclusion of Lemma 3.7 (we replace *Hypothesis* (I) by Newton's Lemma using the fact that $\widehat{L}$ is Henselian). Then, we proceed as in the proof of Proposition 3.9, we embed L in an element $\widetilde{L}$ of $\mathcal{C}$ satisfying the scheme (DL).

9. Model-companion

Let T be a model-complete $\mathcal{L} \cup \{^{-1}\}$-theory of expansions of fields of characteristic 0, satisfying *Hypothesis* (D). Assume that the corresponding class $\mathcal{M}(T)_{\mathrm{top}}$ satisfies *Hypothesis* (I) and that the language $\mathcal{L}$ satisfies the same hypothesis as in Section 4, namely $\mathcal{L}$ expands $\mathcal{L}_{\mathrm{rings}}$ only by new relation symbols R which satisfies the condition stated in the beginning of Section 4.

In this section, we will show that, under the above hypotheses, the model-companion of the $\mathcal{L}_D$-theory T_D exists. This last result will allow us to give another axiomatization of model-companions of certain theories of differential large fields that have already been handled by M. Tressl (see [40]). In Section 11, we will give the application to pseudo-finite fields. It will also allow us to obtain new results on differential fields endowed with several valuations and to give an alternative axiomatization of the model-companion in the case of differential e-fold ordered fields (see [32]).

The main technical difficulty is that instead of considering only 1-extensions, as in the case of model-completion, we will have to deal with finitely differentially generated extensions. We will use Corollary 3.15, namely that such extensions are differentially generated by a single element. We need to introduce a more general version of the scheme of axioms (DL) which deals with differential polynomials in several differential indeterminates.

Definition 9.1. Let $\langle K, D\rangle$ be a differential topological $\mathcal{L}$-field of characteristic zero satisfying all the previous hypotheses (in particular, K is a V-field).

We will say that K satisfies $(DL)_\omega$ if for any natural number $n \geqslant 1$, for any differential polynomial $f \in V\{X_1, \ldots, X_n\}$ with order N_i in X_i, $1 \leq i \leq n$, for any K-definable neighbourhood W of 0, for any $M_2, \ldots, M_n \in \mathbb{N}$,

$$\exists \bar{\alpha}^1, \ldots, \bar{\alpha}^n \in V\left[f^*(\bar{\alpha}^1, \ldots, \bar{\alpha}^n) = 0 \wedge s_f^*(\bar{\alpha}^1, \ldots, \bar{\alpha}^n) \neq 0 \wedge i_f^*(\bar{\alpha}^1, \ldots, \bar{\alpha}^n) \neq 0\right]$$

$$\Rightarrow \exists z_1, \ldots, z_n \left[f(z_1, \ldots, z_n) = 0 \wedge s_f(z_1, \ldots, z_n) \neq 0 \wedge i_f(z_1, \ldots, z_n) \neq 0\right.$$

$$\left.\wedge \bigwedge_{\substack{i=1,\, j=0,\ldots,N_1\\ i>1,\, j=0,\ldots,M_i}} (z_i^{(j)} - \alpha_j^i \in W)\right]$$

where $M_i \geq N_i$, $\bar{\alpha}^1 = (\alpha_0^1, \ldots, \alpha_{N_1}^1)$, and $\bar{\alpha}^i = (\alpha_0^i, \ldots, \alpha_{M_i}^i)$, for $2 \leq i \leq n$, s_f (resp. i_f) is the separant (resp. the initial) of f considered as a differential polynomial in X_1 over the differential polynomial ring $K\{X_2, \ldots, X_n\}$.

Note that for differential polynomials in one indeterminate the scheme $(DL)_\omega$ is equivalent to the scheme (DL). Further, note that Remark 3.6 also holds in the case of the scheme $(DL)_\omega$.

Before proving our principal result we need the analogue of Proposition 3.14 in this new setting.

Lemma 9.2. *Let K be a differential topological $\mathcal{L}$-field which is a V-field, satisfying the scheme $(DL)_\omega$. Then K satisfies the following scheme of axioms:*

for any differential polynomial $f \in K\{X_1, \ldots, X_n\}$ with order N_i in X_i, $1 \leq i \leq n$, for any K-definable neighbourhood W of 0, for any $M_2, \ldots, M_n \in \mathbb{N}$,

$$\exists \bar{\alpha}^1, \ldots, \bar{\alpha}^n \in K\left[f^*(\bar{\alpha}^1, \ldots, \bar{\alpha}^n) = 0 \wedge s_f^*(\bar{\alpha}^1, \ldots, \bar{\alpha}^n) \neq 0 \wedge i_f^*(\bar{\alpha}^1, \ldots, \bar{\alpha}^n) \neq 0\right]$$

$$\Rightarrow \exists z_1, \ldots, z_n \left[f(z_1, \ldots, z_n) = 0 \wedge s_f(z_1, \ldots, z_n) \neq 0 \wedge i_f(z_1, \ldots, z_n) \neq 0\right.$$

$$\left.\wedge \bigwedge_{\substack{i=1,\, j=0,\ldots,N_1\\ i>1,\, j=0,\ldots,M_i}} (z_i^{(j)} - \alpha_j^i \in W)\right]$$

with $M_i \geq N_i$, $2 \leq i \leq n$, keeping the same notations as in Definition 9.1.

Proof. See the proof of Proposition 3.14. $\square$

Denote by $T_D^{*,\omega}$ the $\mathcal{L}_D$-theory $T_D \cup (DL)_\omega$.

Theorem 9.3. *Let T be a model-complete $\mathcal{L} \cup \{^{-1}\}$-theory satisfying the above hypotheses. Then $T_D^{*,\omega}$ is the model-companion of the $\mathcal{L}_D$-theory T_D.*

Proof. (1) Let us prove that any model K of T_D can be $\mathcal{L}_D$-embedded into a model of $T_D^{*,\omega}$. The proof divides in two steps.

First we prove that if K is a differential topological $\mathcal{L}$-field, which is a model of T_D and satisfies *Hypothesis* (I) as a V-field and $f \in V\{X_1, \ldots, X_n\}$ (with order N_i in the variables X_i, $1 \leq i \leq n$) such that f^* has a solution $\bar{\alpha}^1, \ldots, \bar{\alpha}^n \in V$ with $s_f^*(\bar{\alpha}^1, \ldots, \bar{\alpha}^n) \neq 0 \neq i_f(\bar{\alpha}^1, \ldots, \bar{\alpha}^n)$, then there exists an $\mathcal{L}$-elementary differential topological field extension L of K such that f has a solution $\bar{z} := (z_1, \ldots, z_n)$ in L which satisfies

$$s_f(\bar{z}) \neq 0 \wedge i_f(\bar{z}) \neq 0 \wedge \bigwedge_{\substack{i=1,\, j=0,\ldots,N_1\\ i>1,\, j=0,\ldots,M_i}} (z_i^{(j)} \sim_K \bar{\alpha}_j^i).$$

The proof of this first step is similar to the proof of Lemma 3.7. First we consider an $\mathcal{L}$-elementary sufficiently saturated extension $\widehat{K}$ of K and let $\hat{V} \subseteq \widehat{K}$ be such that $\hat{V} \cap K = V$ and $\widehat{K}$ is a $\hat{V}$-field. In the neighbourhood $\hat{V}$ of $\widehat{K}$, we consider a set T of $\sum_{i=2}^n M_i + 1$ algebraically independent elements over K which are also infinitesimals with respect to K, say

$T = \{t_j^i : i = 2, \ldots, n, j = 0, \ldots, M_i\}$. We define a derivation D on the field $L := K(T)$ which extends the derivation D on K and then we extend as usual this derivation to $\widehat{K}$. To define D on L, we proceed as follows

$$D(\alpha_j^i + t_j^i) = \alpha_{j+1}^i + t_{j+1}^i \quad \text{for all } j \in \{0, \ldots, M_i - 1\} \text{ and } D(\alpha_{M_i}^i + t_{M_i}^i) = 0.$$

Now we follow the proof of Lemma 3.7 with the polynomial $d \cdot f^*(\alpha_0^1, \ldots, \alpha_{N_1-1}^1, X_{N_1}^1, \bar{\alpha}^2, \ldots, \bar{\alpha}^n)$ for some $d \in \widehat{K}$ and parameters $(\alpha_0^1, \ldots, \alpha_{N_1-1}^1, \bar{\alpha}^2, \ldots, \bar{\alpha}^n)$.

Finally, by using this first step, we can transpose the proof of Proposition 3.9 in order to show that K has an $\mathcal{L}$-elementary differential topological extension L which satisfies the scheme of axioms $(DL)_\omega$.

(2) Let $K \subseteq_{\mathcal{L}_D} K_1$ be two models of $T_D^{*,\omega}$ and let $\widehat{K}$ be a $|K_1|^+$-saturated elementary $\mathcal{L}_D$-extension of K. To show the model-completeness of the $\mathcal{L}_D$-theory $T_D^{*,\omega}$, we want to $\mathcal{L}_D$-embed K_1 into $\widehat{K}$. By the saturation hypothesis, it suffices to $\mathcal{L}_D$-embed any $\mathcal{L}_D$-substructure of K_1 which is finitely generated over K. By Corollary 3.15, we know that D is independent on K, therefore, by using [15, Theorem 4, p. 105 and Proposition 9, p. 103], we may assume that the differential field K_1 is generated by n elements, say $c_1, \ldots, c_n$, such that:

- $c_2, \ldots, c_n$ are differentially algebraically independent over K (see [15] p. 69) and,
- either c_1 is differentially algebraic over $K\{c_2, \ldots, c_n\}$, or $c_1, \ldots, c_n$ are differentially algebraically independent over K.

By hypothesis, we already have that $K \subseteq_{e.c.} K_1$ as $\mathcal{L}$-structures and we wish to show that it remains true as $\mathcal{L}_D$-structures. So we have the following $\mathcal{L}$-embedding j: $K_1 \hookrightarrow \widehat{K}$, where j is the identity on K.

In the course of this proof, for $\bar{b} := (b_1, \ldots, b_\ell) \in K^\ell$, we will abbreviate the property that for any neighbourhood $W_1 \times \cdots \times W_\ell$ of $\bar{b}$ (in K^ℓ), there exists z such that $\bigwedge_{i=0}^{\ell-1} z^{(i)} - b_{i+1} \in W_i$, by: "there exists z such that $z \sim \bar{b}$".

In order to obtain an $\mathcal{L}_D$-embedding of $K_1 := K\langle c_1, \ldots, c_n\rangle$ into $\widehat{K}$ over K, we consider the quantifier-free n-type $t(\bar{c}, K)$ of $\bar{c} = (c_1, \ldots, c_n)$ over K. It suffices to show that the type $t(\bar{c}, K)$ is finitely satisfiable in $\widehat{K}$ and then by saturation, we get the required $\mathcal{L}_D$-embedding of K_1 into $\widehat{K}$.

Let $\phi(x_1, x_2, \ldots, x_n, \bar{a})$ be an open $\mathcal{L}_D$-formula with parameters $\bar{a}$ in K which belongs to $t(\bar{c}, K)$. We may assume that the corresponding definable set in K^n consists of the union of an open set and an algebraic set, where the open set O is defined by a conjunction of basic formulas $\bigwedge_{j\in J} \phi_j(\bar{x})$ and the algebraic set is defined by $\bigwedge_{i\in I} f_i(\bar{x}, \bar{a}) = 0$.

If $c_1, \ldots, c_n$ are differentially independent over K then we may apply Lemma 3.17 directly. So we may assume that $c_2, \ldots, c_n$ are differentially independent elements over K, c_1 is differentially algebraic over $K\{c_2, \ldots, c_n\}$ and the elements $c_1, \ldots, c_n$ occur non-trivially in the equations.

By the model-completeness of the $\mathcal{L}$-theory T, we can find tuples $\bar{b}_1, \ldots, \bar{b}_n$ in $\widehat{K}$ satisfying the corresponding non-differential quantifier-free formula ϕ^* (see Definition 3.16).

In the case $n = 1$, we proceed as in the proof of Theorem 4.1, using the generic polynomial of c_1 over K and the scheme $(DL)_\omega$ associated to this polynomial.

Assume now that $n \geqslant 2$. Let h_1 be a generic polynomial of c_1 over $K\{c_2, \ldots, c_n\}$ and denote by s_{h_1} its separant and by i_{h_1} its initial (see Definition 3.1). We will need the following observation which uses the analogue of Euclid algorithm in this differential setting (see Lemma 3.2 with $R := K\{X_2, \ldots, X_n\}$).

<u>Observation:</u> Let $g(X_1, X_2, \ldots, X_n)$ be a non-zero differential polynomial in $K\{X_1, X_2, \ldots, X_n\}$ and assume that $g(c_1, \ldots, c_n) = 0$. Then, there exists $r(X_1, X_2, \ldots, X_n) \in K\{X_1, X_2, \ldots, X_n\}$, with r of order smaller than or equal to the order of h_1 with respect to X_1, with in addition $deg_{u_{h_1}} h_1 > deg_{u_{h_1}} r$ (see Definition 3.1) such that

$$i_{h_1}^{n_1} \cdot s_{f_1}^{n_1'} \cdot g = \sum_j h_1^{(j)} \cdot g_j + r$$

for some $g_j(X_1, X_2, \ldots, X_n) \in K\{X_1, X_2, \ldots, X_n\}$ and for some natural numbers n_1, n_1'.

Since $r(c_1, \ldots, c_m) = 0$ and $deg_{u_{h_1}} h_1 > deg_{u_{h_1}} r$, we get that $deg_{u_{h_1}} r = 0$. Therefore, since $c_2, \ldots, c_n$ are differentially algebraically independent over K, $r(X_1, X_2, \ldots, X_n) = 0$.

So we need to find a differential solution $\bar{z} := (z_1, \ldots, z_n) \in \widehat{K}^n$ of f such that

- $i_f(\bar{z}) \neq 0$ and $s_f(\bar{z}) \neq 0$ and,
- $z_i \sim \bar{b}_i$ for $i = 1, \ldots, n$.

If we find such a solution $\bar{z} \in \widehat{K}^n$ then by the Observation, we get $f_i(\bar{z}, \bar{a}) = 0$; and the property $\bar{z} \sim (\bar{b}_1, \ldots, \bar{b}_n)$ gives us that $\bar{z} \in O$ (we also use the continuity of polynomials to determine the open set which is convenient in the statement with $\sim$); proving that $\widehat{K} \models \phi(\bar{z}, \bar{a})$.

But we know that $h_1^*(\bar{b}_1, \ldots, \bar{b}_n) = 0$, $s_{h_1}^*(\bar{b}_1, \ldots, \bar{b}_n) \neq 0$ and $i_{h_1}^*(\bar{b}_1, \ldots, \bar{b}_n) \neq 0$; therefore since $\widehat{K} \models (DL)_\omega$, we may apply Lemma 9.2 to the polynomial h_1 and to the point $(\bar{b}_1, \ldots, \bar{b}_n)$ to find a solution $\bar{z}$ satisfying our previous requirements. □

Corollary 9.4. *Assume moreover that the theory T is complete, then the theory $T_D^{*,\omega}$ is complete. If in addition T has NIP, then $T_D^{*,\omega}$ has no open formula with IP.*

Proof. Let K, L be two models of $T_D^{*,\omega}$. Let σ be a sentence in the language $\mathcal{L}_D$. Since $T_D^{*,\omega}$ is model-complete, we may assume that σ is of the form $\exists\bar{x}\,\theta(\bar{x})$, where θ is a quantifier-free formula. Let us show that if $K \models \sigma$, then $L \models \sigma$.

Let $\bar{a} \subseteq K$ be such that $K \models \theta(\bar{a})$. Let $\mathcal{Q}$ be the $\mathcal{L}_D$-substructure generated by $\emptyset$ and let $K_1 := \mathcal{Q}\langle\bar{a}\rangle$ be the differential subfield of K (differentially) generated by $\bar{a}$. Since D is independent on K, we may assume that K_1 is generated by n elements $b_1, \ldots, b_n$ with $b_2, \ldots, b_n$ differentially algebraically independent and either b_1 is differentially algebraic over $\mathcal{Q}\{b_2, \ldots, b_n\}$ or $b_1, b_2, \ldots, b_n$ are differentially algebraically independent.

Then, we choose an $\aleph_1$-saturated elementary $\mathcal{L}_D$-extension L^* of L. We show that the type of $\bar{b}$ is finitely satisfiable in L, using the scheme $(DL)_\omega$ and the fact that T is complete. So, this type is satisfied by a tuple $\bar{c}$ of L^*. Therefore, in the differential subfield $\mathcal{Q}\langle\bar{c}\rangle$ of L^*, we can find a tuple $\bar{a}'$ such that the formula $\theta(\bar{a}')$ holds in L^*. Since $T_D^{*,\omega}$ is model-complete, $L \models \sigma$.

For the second statement, we apply Lemma 4.2 and Theorem 9.3 where we showed how to embed a model of T_D into a model of $T_D^{*,\omega}$. $\square$

10. Fields endowed with several topologies

In [8] van den Dries showed that the theory OF_e of e-fold ordered fields, namely fields equipped with e distinct orderings in the language $\mathcal{L}_{e,<} := \mathcal{L}_{\text{rings}} \cup \{<_i;\ i = 1, \ldots, e\}$ has a model-companion $\overline{OF_e}$ and that $\overline{OF_e}$ is decidable.

An e-fold ordered field $\langle K, <_1, \ldots, <_e\rangle$ is a model of $\overline{OF_e}$ if

(1) the orderings $<_i$, $1 \le i \le e$, on K are pairwise independent;
(2) for any irreducible $f(X, Y) \in K[X, Y]$ and any $a \in K$ such that $f(a, Y)$ changes sign on K with respect to each of the orderings $<_i$, there exists $(c, d) \in K^2$ such that $f(c, d) = 0$.

Another characterization of the models of $\overline{OF_e}$ is due to A. Prestel. Recall that the field K is a *PRC field* if every absolutely irreducible variety V defined over K such that every ordering of K extends to the function field of V, then V has a K-rational point. An e-fold ordered field $\langle K, <_1, \ldots, <_e\rangle$ is PRC_e if it is a *PRC* field and if $<_1, \ldots, <_e$ are its only orderings ([14] Proposition 1.3). A. Prestel identified the models of $\overline{OF_e}$ as those PRC_e fields which have no proper algebraic extensions, namely the *maximal* PRC_e *fields*.

Finally, Jarden [14] showed that models of $\overline{OF_e}$ coincide with the *van den Dries* fields. An e-fold ordered field $\langle K, <_1, \ldots, <_e\rangle$ is *a van den Dries field* if the orderings $<_i$, $1 \le i \le e$, on K are pairwise independent, if every absolutely irreducible variety V defined over K such that the ordering $<_i$, $1 \le i \le e$, of K extends to the function field of V, has a K-rational point and if the Galois group of K is isomorphic to the free product of e copies of $\mathbb{Z}/2\mathbb{Z}$ in the category of pro-2-groups.

In [32] Rivière considered the maximal PRC_e fields expanded with a derivation D with no interaction with the interval topologies. First he showed that the theory of these fields still admits a model-companion in the language $\mathcal{L}_{e,<} \cup \{D\}$ and gave a geometric axiomatization of this model-companion. Here we will first show that these class of maximal PRC_e fields can be considered in our context of topological fields as an inductive class and we will show how to deduce the Rivière's result on model-companion by establishing an alternative axiomatization.

As in subSection 2.2, we will construe an e-fold ordered field $\langle K, <_1, \ldots, <_e\rangle$ as a topological $\mathcal{L}_{e,<}$-structure with respect to the topology $\tau := \tau_{<_1} \vee \cdots \vee \tau_{<_e}$, where the $\tau_{<_i}$ are the interval topologies induced by the orderings $<_i$. The van den Dries fields form an inductive class of topological $\mathcal{L}_{e,<}$-fields. The only non-trivial point to check is the *Hypothesis* (I) (see Corollary 2.14 in [32]). One shows that given any e-fold ordered field K and the field of Laurent series $L := K((t))$, one can extend the e-orderings to L. Then given an element $f[X] \in L[X]$ and $a \in L$ such that $f(a) \sim 0$ and $f'^2(a) \not\sim 0$, we can find a zero b with $a \sim b$ in the following way. Consider the Taylor expansion of $f[X]$ and let $h \in L$ with $h \sim f(a)$. Then, $f(a + h) = f(a) + h.f'(a) + h^2.(\textit{higher order terms})$ has the same sign as the sign of h whenever $f'(a) > 0$ and h a small enough element of K. So for each order $<_i$, we have that this polynomial changes of sign in a K-neighbourhood of 0 with respect to $<_i$. By property (2) of a van den Dries field we have a zero in an extension of L satisfying $\overline{OF_e}$.

Now we will apply the results of the previous section to the expansion of these e-fold ordered fields with a derivation D which has no interaction with the interval topologies. We will apply Theorem 9.3 in order to get the following.

Theorem 10.1. *The $\mathcal{L}_{e,<,D}$-theory $\overline{OF}^{\omega}_{e,D}$ is the model-companion of the $\mathcal{L}_{e,<,D}$-theory of differential e-fold ordered fields $OF_{e,D}$ and is decidable.*

Proof. The last part of the statement follows from Corollary 9.4 and the fact that $\overline{OF_e}$ is decidable ([8] Theorem 2.1 chapter 2). $\square$

Now we will consider the general case of fields endowed with several topologies and then we will apply it to fields of characteristic zero endowed with several valuations. (We will always replace the valuation v_i by the corresponding linear divisibility relation $\mathcal{D}_i$ (see Section 6).) First we need to recall some terminology developed in Chapter 3 of [8] (see also Section 4).

Fix n an integer, $n \ge 1$ and let $T_1, \ldots, T_n$ are fixed t-theories, such that for each $1 \leqslant i \leqslant n$:

(1) T_i has a model-completion $\overline{T}_i$;
(2) for each model K of $\overline{T}_i$, $\text{char}(K) = 0$;
(3) the topology τ_K defined on K (with respect to T_i) is not discrete.

Let us note that if each T_i is chosen from among the theory of ordered domains with $\mathcal{L}_i := \mathcal{L}_<$, the theory of valued domains of characteristic zero in the language $\mathcal{L}_i := \mathcal{L}_\mathcal{D}$ or the reduct of $pCF_\forall$ (namely $(pCF_1)_\forall$) in the language $\mathcal{L}_i := \mathcal{L}_\mathcal{D} \cup \{P_n;\ n \in \mathbb{N}^\times\}$, then these assumptions on T_i are satisfied, namely the corresponding language is a t-language (see chapter 3, section 1.5 in [8]).

Theorem 1.6 of Chapter 3 in [8] was stated as follows.

Theorem 10.2. *The $\mathcal{L}$-theory $(T_1, \ldots, T_n)$ has a model-companion $\overline{(T_1, \ldots, T_n)}$ with $\mathcal{L} := \bigcup_{i=1}^n \mathcal{L}(T_i)$.*

The axiomatization of the model-companion uses the following *(l,m)-conditions*, which generalize the axiomatization of $\overline{OF_e}$. First, we need to recall what is a *t-basic formula* in a t-theory. A t-basic formula in $\bar{u}, \bar{x}$ is a quantifier-free formula of one of the following form:

(1) $R(p_1(\bar{u}, \bar{x}), \ldots, p_n(\bar{u}, \bar{x})) \ \wedge \ \bigwedge_{i=1}^n p_i(\bar{u}, \bar{x}) \neq 0$
(2) $R(p_1(\bar{u}, \bar{x}), \ldots, p_n(\bar{u}, \bar{x})) \ \wedge \ \bigwedge_{i=1}^n p_i(\bar{u}, \bar{x}) \neq 0,$

where R is a n-ary predicate and $p_1, \ldots, p_n \in \mathbb{Z}[\bar{U}, \bar{X}]$.

Definition 10.3 (*See [8] chapter 3, Definition 1.10*). Let $l, m \in \mathbb{N}$. An (l, m)-condition is a sequence

$$(\sigma_i(u), \phi_i(u, x), \theta_i(u, x, y), F(u, x, y)), \quad i = 1, \ldots, n \text{ with } u = (u_1, \ldots, u_l),\ x = (x_1, \ldots, x_m)$$

such that for each $1 \leqslant i \leqslant n$:

(1) $\sigma_i(u)$ is an open $\mathcal{L}(T_i)$-formula,
(2) $\phi_i(u, x)$ is a conjunction of t-basic $\mathcal{L}(T_i)$-formulas in (u, x),
(3) $\theta_i(u, x, y)$ is an open $\mathcal{L}(T_i)$-formula,
(4) $F(U, X, Y)$ is a polynomial in $\mathbb{Z}[U, X, Y]$, monic and of positive degree in Y,
(5) $\overline{T}_i \models \forall u(\sigma_i(u) \to \exists x \phi_i(u, x))$,
$\overline{T}_i \models \forall u \forall x[(\sigma_i(u) \wedge \phi_i(u, x)) \Rightarrow \exists y(F(u, x, y) = 0 \wedge \theta_i(u, x, y))]$.

Definition 10.4. The $\mathcal{L}$-theory $\overline{(T_1, \ldots, T_n)}$ is the theory whose models K satisfy the following

(1) K is a field,
(2) for each (l, m)-condition and each a in K^l such that $F(a, X_1, \ldots, X_m, Y)$ is irreducible and $K \models \bigwedge_{i=1}^n \sigma_i(a)$, the following holds:

$$K \models \exists x \exists y \left[F(a, x, y) = 0 \wedge \bigwedge_{i=1}^n \phi_i(a, x) \wedge \theta_i(a, x, y) \right].$$

Now we will consider specific t-theories T_i $(1 \leqslant i \leqslant n)$, having a model-completion $\overline{T}_i$. From now on, we will consider the class $\mathcal{M}(\overline{T_i})$ of models of $\overline{T_i}$ as a class of $\mathcal{L}_i \cup \{^{-1}\}$-structures. Moreover, in that expansion of the language, we will consider the corresponding class, denoted by $\mathcal{C}_i$, of topological $\mathcal{L}_i$-fields. We will assume that each $\mathcal{C}_i$ satisfies *Hypothesis* (I) and we will show that $\mathcal{M}(\overline{(T_1, \ldots, T_n)})_{\text{top}}$, considered as $\mathcal{L} \cup \{^{-1}\}$-structures, satisfies *Hypothesis* (I).

For instance, let us consider the following new example which was not treated in [12].

For each $1 \leqslant i \leqslant n$, $\overline{T}_i$ is a t-theory of Henselian valued fields of characteristic zero with elimination of quantifiers; therefore it is the model-completion of its universal theory. We already showed that the corresponding classes of topological $\mathcal{L}$-fields satisfy *Hypothesis* (I) (see Section 2). First we prove that in this case a model of $\overline{(T_1, \ldots, T_n)}$ satisfy an adapted version of Hensel's Lemma. We work in the language $\mathcal{L}_{\mathcal{D}_1} \cup \cdots \cup \mathcal{L}_{\mathcal{D}_n}$ but to make the statement more readable we will leave out the valuation symbols v_i.

Lemma 10.5. *Let $\langle K, \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ be a model of $\overline{(T_1, \ldots, T_n)}$. Let $\mathcal{O}$ be the intersection of the valuation rings $\mathcal{O}_{K,v_i}$. Then K satisfy the following property, called the multi-Hensel's Lemma:*

for any $f(X) \in \mathcal{O}[X]$, $a \in \mathcal{O}$ with $\bigwedge_{i=1}^n v_i(f(a)) > 2v_i(f'(a))$ then there exists $b \in \mathcal{O}$ with $f(b) = 0$ and $\bigwedge_{i=1}^n v_i(b - a) \geq v_i(f(a)) - v_i(f'(a))$.

Proof. Apply the axiomatization of $\overline{(T_1, \ldots, T_n)}$ to the sentence of Hensel's Lemma with respect each valuation v_i on K.

We apply the $(n + 1, 1)$ condition as follows. Set $f(X, \bar{U}) := X^n + U_{n-1}.X^{n-1} + \cdots + U_0$; let $\sigma_i(U, \bar{U}) := v_i(f(U, \bar{U})) > 2.v_i(\frac{\partial f}{\partial U}(U, \bar{U}))$, $\phi_i(U, X, \bar{U}) := \sigma_i(U, \bar{U})$ & $X = U$ and $F(\bar{U}, X, Y) := f(Y, \bar{U})$, $\theta_i(\bar{U}, U, X, Y) := v_i(Y - U) \geq v_i(f(U, \bar{U})) - v_i(\frac{\partial f}{\partial U}(U, \bar{U}))$. It is straightforward to see that these formulas satisfy the conditions imposed on a $(n + 1, 1)$ sequence (see Definition 10.3). $\square$

Let $\mathcal{C}(\overline{(T_1, \ldots, T_n)})$ be the class of models of $\overline{(T_1, \ldots, T_n)}$ considered as $\mathcal{L} \cup \{^{-1}\}$-structures. Let $\mathcal{C}(\overline{(T_1, \ldots, T_n)})_{\text{top}}$ be the corresponding class of topological $\mathcal{L}$-fields endowed with the supremum of the valuation topologies induced by the valuation v_i.

Assume now that each T_i, $1 \leq i \leq n$, is equal to $T_{p_i,1}$, with $p_1, \ldots, p_n$, n distinct prime numbers, then we can apply the results of the previous section to the expansion of these topological (valued) fields with a derivation D with no interaction with the topologies. We will call the models of $(T_{p_1,1}, \ldots, T_{p_n,1})$ n-multi-valued fields (of rank 1). We will apply Theorem 9.3 in order to get the following.

Theorem 10.6. *The $\mathcal{L}_{\mathcal{D}_1,\ldots,\mathcal{D}_n,D}$-theory $\overline{(T_{p_1,d},\ldots,T_{p_n,d})}_D^{\omega}$ is the model-companion of the $\mathcal{L}_{\mathcal{D}_1,\ldots,\mathcal{D}_n,D}$-theory of differential n-multi-valued fields $(T_{p_1,1},\ldots,T_{p_n,1})_D$ and is decidable.*

Proof. The last part of the statement follows from Corollary 9.4 and the fact that $\overline{(T_{p_1,1},\ldots,T_{p_n,1})}$ is decidable ([8] Theorem 3.1 chapter 3). □

Finally we can mix the valuations and the orderings, namely consider theories T_i's which are among OF, VF_0 or $T_{p,d}$. In order to get that the corresponding topological theory satisfies Hypothesis (I), it suffices to write down the analogous $(n, 1)$-condition for the sign change in the case of ordered fields. (However, we will not do it here.)

11. Large fields and comparison with Tressl's approach

In [40], Tressl considers differential fields of characteristic zero with finitely many commuting derivations, whose pure field theory has a model-companion and axiomatizes the model-companion of the differential theory. The main difference with our approach is that he only considers languages which are expansions by definition of $\mathcal{L}_{\text{rings}}$. He introduces a scheme (UC_m) (for uniform companion) of axioms which says that *algebraically prepared systems* of differential equations in m commuting derivations and finitely many indeterminates, have a solution (see section 3 of [40]).

The key point of Tressl's approach is the use of large fields (see subSection 2.2 and [25, p. 2]). Examples of large fields are *PAC*, *PRC* (see Section 10) and *PpC* fields (see [25, Section 3]).

We will use the terminology of Tressl (see [40] section 2). The theory (UC_m) has the following key properties.

(I) Whenever L and M are models of (UC_m) and A is a common differential subring of L and M such that L and M have the same universal theory over A as pure fields then they have the same universal theory over A as differential fields.

(II) Every differential field F which is large (as a pure field) can be extended to a model of (UC_m) and this extension is elementary in $\mathcal{L}_{\text{rings}}$.

More generally properties (I) and (II) of (UC_m) above imply that for every model-complete $\mathcal{L}_{\text{rings}}$-theory T of large fields, the $\mathcal{L}_{\text{rings}}\cup\{D_1,\ldots,D_m\}$-theory $T\cup(UC_m)$ of differential fields is model-complete. Moreover whenever T is complete, $T\cup(UC_m)$ is complete and if a definable expansion T^* of T admits quantifier elimination, then $T^*\cup(UC_m)$ admits quantifier elimination.

Now, let us deduce another axiomatization of the existentially closed differential theories of certain classes of large fields. In subSection 2.3, we show that an inductive class $\mathcal{C}$ of $\mathcal{L}$-structures closed under ultrapowers, which are expansions of large fields satisfies Hypothesis $(I)_{\text{red}}$. Recall that this condition differs from Hypothesis (I) by the fact that we no longer require the extensions to be topological extensions.

Let us consider like in [40] the case of a model-complete $\mathcal{L}_{\text{rings}}$-theory T of large fields.

First, let us define the scheme of axioms $(DL)_{\omega,Z}$ as follows.

Definition 11.1. Let K be a differential field, then K is a model of $(DL)_{\omega,Z}$ if the following scheme of axioms holds, with the same notations as in Definition 9.1:

for any two differential polynomials $f, g \in K\{X_1,\ldots,X_n\}$ with the order of f bigger than or equal to the order of g in X_1, $n \in \mathbb{N}^{\times}$,

$$\exists \bar{\alpha}^1,\ldots,\bar{\alpha}^n \in K\left[f^*(\bar{\alpha}^1,\ldots,\bar{\alpha}^n)=0 \wedge s_f^*(\bar{\alpha}^1,\ldots,\bar{\alpha}^n)\neq 0 \wedge i_f^*(\bar{\alpha}^1,\ldots,\bar{\alpha}^n)\neq 0 \wedge g^*(\bar{\alpha}^1,\ldots,\bar{\alpha}^n)\neq 0\right]$$
$$\Rightarrow \exists z_1,\ldots,z_n\left[f(z_1,\ldots,z_n)=0 \wedge s_f(z_1,\ldots,z_n)\neq 0 \wedge i_f(z_1,\ldots,z_n)\neq 0 \wedge g(z_1,\ldots,z_n)\neq 0.\right]$$

Note that when applying the results of Section 9, we no longer need the assumption that Hypothesis (D) holds since it was only used to ensure that the scheme $(DL)_{\omega}$ was first-order and also that we may replace Hypothesis (I) by Hypothesis $(I)_{\text{red}}$ since we are no longer working with topological structures.

So we can state the following analogue of Theorem 9.3.

Theorem 11.2. *Let T be a model-complete theory of large fields in $\mathcal{L}_{\text{fields}}\cup C$ (with C a set of constants). Then $T_D\cup(DL)_{\omega,Z}$ is the model-companion of T_D.* □

Let $\mathcal{L}(C)=\mathcal{L}_{\text{fields}}\cup C$ with C an infinite countable set of constants, let *PSF* be the theory of pseudo-finite fields, namely the theory *PAC* of pseudo-algebraically closed fields plus the scheme of axioms that for each n there is only one extension of degree n (the constants $\bar{c}$, where $\bar{c}$ is a n-tuple are interpreted by the coefficients of an irreducible polynomial of degree n, $n\in\mathbb{N}$). Since any model of *PSF* is large (see [25]), the class of models of *PSF* satisfies *Hypothesis* $(I)_{\text{red}}$. Moreover, in *PSF*, one has the following positive existential quantifier elimination result: any $\mathcal{L}(C)$-formula $\phi(\bar{x})$ is equivalent to a conjunction of formulas of the form $\exists T(g(\bar{c},\bar{x},T)=0)$ with $g(\bar{c},\bar{x},T)\in\mathbb{Z}[\bar{c},\bar{x},T]$ (see [6, Proposition 2.7].)

Corollary 11.3. *The theory $PSF_D\cup(DL)_{\omega,Z}$ is model-complete.* □

Note that if K is any infinite pseudo-finite field, then the theory of K has the independence property ([10]).

Now we will show how to deduce our scheme of axioms $(DL)_{\omega,Z}$ from the scheme of axioms (UC_1).

Theorem 11.4. *Any differential field $\langle K, D\rangle$ satisfying Hypothesis $(I)_{\text{red}}$ and the scheme (UC_1) satisfies the scheme $(DL)_{\omega,Z}$.*

Proof. Let us consider an instance $(f, g) \in K\{X_1, \ldots, X_n\}$ of the scheme of axioms $(DL)_{\omega,Z}$. Let L be a sufficiently saturated elementary extension of K. Then by using the proof of Theorem 9.3, we can endow L with a derivation extending the one of K such that there exists $\bar{b}, c \in L$ with $f(\bar{b}) = 0$ and $g(\bar{b}) \cdot c - 1 = 0$.

Now we consider the following prime differential ideal of $K\{X_1, \ldots, X_n, Y\}$ containing f and g:

$$I((\bar{b}, c), K) := \{h \in K\{X_1, \ldots, X_n, Y\} : h(\bar{b}, c) = 0\}.$$

Using the results in Section 2 of [40], we may consider a characteristic set $\{f_1, \ldots, f_l\}$ of $I(\bar{b}, c/K)$ such that the prime ideal $(f_1, \ldots, f_l) : H(f_1, \ldots, f_l)^\infty$ has a K-rational regular point (since $K \prec L$).

Now we apply the scheme of axioms (UC_1) to get a solution $(\bar{d}, e)$ in K to our differential system $\{f_1, \ldots, f_l\}$, which implies that $f(\bar{d}) = 0$ and $g(\bar{d}) \neq 0$. $\square$

Then, following Tressl approach but in our context of topological fields, Guzy (see [12]) considered differential valued fields equipped with m commuting derivations $D_1, \ldots, D_m$, where again there is no interaction between the valuation and the derivations.

He introduced the notion of J-algebraically prepared systems (see Definition 2.6 in [12], where J is a particular set of derivatives $(D_i^{(j)} X_k)$, and wrote an alternative scheme of axioms (UC'_m) which plays the same role as (UC_m). In particular one has that models of (UC'_m) satisfy in that valued setting the analogue of the above properties (I) and (II). He proved a differential transfer result of model-completeness and the applications of Sections 6–8 were generalized to the case of several derivations.

Here we explicit the link between the scheme of axioms (UC'_1) and the scheme $(DL)_\omega$ in the setting of valued fields.

Theorem 11.5. *If $\langle K, D, v\rangle$ is a Henselian differential valued field satisfying the scheme of axioms (UC'_1) then $\langle K, v, D\rangle$ satisfies $(DL)_\omega$.*

Proof. Let $\langle K, D\rangle$ be a model of (UC'_1) and let us consider an instance of the scheme of axioms $(DL)_\omega$ in the setting of valued fields: take f a differential polynomial in $\mathcal{O}_K\{X_1, \ldots, X_n\}$ with order N_i in X_i, $1 \leqslant i \leqslant n$, $\epsilon \in \mathcal{O}_K$, natural numbers $M_2, \ldots, M_n$ and $\overline{\alpha} := (\overline{\alpha}^1, \ldots, \overline{\alpha}^n) \subseteq \mathcal{O}_K$ such that

$$f^*(\overline{\alpha}^1, \ldots, \overline{\alpha}^n) = 0 \wedge s_f^*(\overline{\alpha}^1, \ldots, \overline{\alpha}^n) \neq 0 \wedge i_f^*(\overline{\alpha}^1, \ldots, \overline{\alpha}^n) \neq 0.$$

As in Remark 3.6 we may assume that f is irreducible over K. The proof of Theorem 9.3 shows that there exists an elementary valued field extension $\langle L, w\rangle$ of $\langle K, v\rangle$ such that we can endow L with a derivation extending the one of K with $f(\bar{b}) = 0$ and $\bar{b} - \overline{\alpha}$ is infinitely close to zero with respect to the valuation topology on K.

Now we consider the following prime differential ideal

$$I(\bar{b}, K) := \{h \in K\{X_1, \ldots, X_n\} : h(\bar{b}) = 0\}.$$

By the results in Section 2 in [40] and the construction of L, we have that f is a characteristic set of $I(\bar{b}/K)$ which is then equal to

$$\langle f\rangle : H(f)^\infty := \{g \in K\{X_1, \ldots, X_n\} : \exists n \in \mathbb{N}\, (i_f s_f)^n g \in \langle f\rangle\}.$$

Therefore $(f) : H(f)^\infty$ is a prime ideal and $\bar{\alpha}$ is a K-rational regular point of this ideal.

Now we consider Δ the set of derivatives $D^{(i)} X_j$ which appear in f and the following subset J of derivatives $D^{(i)} X_j$:

$$J := \{D^{(i)} X_1,\ D^{(j)} X_k : i = 0, \ldots, N_1; j = 0, \ldots, M_k,\ k = 2, \ldots, n\} \setminus \Delta.$$

By applying the scheme of axioms (UC'_1) to the J-algebraically prepared systems $\{f; f\}$ with respect to the tuples $\overline{\alpha}^1, \ldots \overline{\alpha}^n$ and the element ϵ, we have a solution to the instance of axioms $(DL)_\omega$. $\square$

More generally, Guzy proved that any differential Henselian valued field $\langle F, v, D_1, \ldots, D_m\rangle$ which is a model of (UC'_m) is a model of (UC_m) ([12] Proposition 3.18).

12. Concluding remarks

As we have seen with *RVF*, one can transfer model-completion results from the fraction field of an integral domain A to A itself. So, one can wonder whether one would have a "good" theory of models of *COVR* endowed with a derivation. Of course, in this new setting, one would require an interaction with the derivation and the order or the valuation (or more generally with the topology).

Differential valued fields with an interaction between the derivation and the valuation were first considered by Rosenlicht (see [34]) and more recently by Scanlon (see [36]). The study of differential valued ordered fields, generalizing Hardy fields, was undertaken by Aschenbrenner and van den Dries in [1].

In the case of a differential valued field K as introduced by Rosenlicht, provided one replaces the derivation D by a non-zero multiple aD ($a \in K^\times$), one can always assume that the valuation ring $\mathcal{O}$ is closed under D. The relationship between the valuation and the derivation for elements of $\mathcal{O}$ is as follows: for all $a, b \in \mathcal{O}$ with $b \neq 0$, $v(b) > 0$, we have $v(D(a)) > v(D(b)/b)$. In this case the non-zero constant elements of K have valuation zero.

The condition required by Scanlon is that for all field elements a, we have $v(D(a)) \geqslant v(a)$. He obtains an Ax–Kochen–Ershov type Theorem provided the subfield of constants has the same value group as the whole field and the differential residue field is differentially linearly closed. We will study the valuation rings of these differential fields in a future paper. In the framework developed by Scanlon, it is rather straightforward to obtain results similar to the ones in the present paper.

The H-fields as introduced by van den Dries and Aschenbrenner are ordered differential valued fields. Their restriction to the field language are models of *RVF*. They single out Liouville closed H-fields and show that any H-field has a Liouville closure. In this last case, the situation seems to be much more complicated; in particular they do not know whether the class of existentially closed H-fields is elementary (see [2]).

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