Aperiodic Weighted Automata and Weighted First-Order Logic

Manfred Droste

Institut für Informatik, Universität Leipzig droste@informatik.uni-leipzig.de

Paul Gastin 0



LSV, ENS Paris-Saclay, CNRS, Université Paris-Saclay paul.gastin@ens-paris-saclay.fr

Abstract

By fundamental results of Schützenberger, McNaughton and Papert from the 1970s, the classes of first-order definable and aperiodic languages coincide. Here, we extend this equivalence to a quantitative setting. For this, weighted automata form a general and widely studied model. We define a suitable notion of a weighted first-order logic. Then we show that this weighted first-order logic and aperiodic polynomially ambiguous weighted automata have the same expressive power. Moreover, we obtain such equivalence results for suitable weighted sublogics and finitely ambiguous or unambiguous aperiodic weighted automata. Our results hold for general weight structures, including all semirings, average computations of costs, bounded lattices, and others.

2012 ACM Subject Classification Theory of computation → Quantitative automata; Theory of computation \rightarrow Logic and verification

Keywords and phrases Weighted automata, weighted logic, aperiodic automata, first-order logic, unambiguous, finitely ambiguous, polynomially ambiguous

Related Version An extended abstract of the paper appeared at MFCS'19. https://doi.org/10.4230/LIPIcs.MFCS.2019.76

Funding Manfred Droste: Partly supported by a visiting professorship at ENS Paris-Saclay Paul Gastin: Partly supported by the DFG Research Training Group QuantLA.

1 Introduction

Fundamental results of Schützenberger, McNaughton and Papert established that aperiodic, star-free and first-order definable languages, respectively, coincide [39, 31]. In this paper, we develop such an equivalence in a quantitative setting, i.e., for suitable notions of aperiodic weighted automata and weighted first-order logic.

Already Schützenberger [38] investigated weighted automata and characterized their behaviors as rational formal power series. Weighted automata can be viewed as classical finite automata in which the transitions are equipped with weights. These weights could model, e.g., the cost, reward or probability of executing a transition. The wide flexibility of this automaton model soon led to a wealth of extensions and applications, cf. [37, 27, 2, 35, 14] for monographs and surveys. Whereas traditionally weights are taken from a semiring, recently, motivated by practical examples, also average and discounted computations of weights were considered, cf. [8, 7].

In the boolean setting, the seminal Büchi-Elgot-Trakhtenbrot theorem [6, 20, 40] established the expressive equivalence of finite automata and monadic second-order logic (MSO). A weighted monadic second-order logic with the same expressive power as weighted automata was developed in [12, 13]. This led to various extensions to weighted automata and weighted logics on trees [18], infinite words [17], timed words [33], pictures [21], graphs [10], nested words [11], and data words [1], but also for more complicated weight structures including

2 Aperiodic Weighted Automata and Weighted First-Order Logic

average and discounted calculations [15] or multi-weights [16]. Recently, in [22], weighted MSO logic was revisited with a more structured syntax, called core-wMSO, and shown to be expressively equivalent to weighted automata, while permitting a uniform approach to semirings and these more complicated weight structures.

Here, we consider the first-order fragment wFO of this weighted logic. It extends the full classical boolean first-order logic quantitatively by adding weight constants and if-thenelse applications, followed by a first-order (universal) product and then further if-then-else applications, finite sums, or first-order (existential) sums. We will show that its expressive power leads to aperiodic weighted automata which, moreover, are polynomially ambiguous. Natural subsets of connectives will correspond to unambiguous or finitely ambiguous aperiodic weighted automata. These various levels of ambiguity are well-known from classical automata theory [23, 41, 25, 24].

Following the approach of [22], we take an arbitrary set R of weights. A path in a weighted automaton over R then has the sequence of weights of its transitions as its value. The abstract semantics of the weighted automaton is defined as the function mapping each non-empty word to the multiset of weight sequences of the successful paths executing the given word. Correspondingly, we will define the abstract semantics of wFO sentences also as functions mapping non-empty words to multisets of sequences of weights. Our main result will be the following.

- ▶ **Theorem 1.** Let Σ be an alphabet and R a set of weights. Then the following classes of weighted automata and weighted first-order logics are expressively equivalent:
- 1. Aperiodic polynomially ambiguous weighted automata (wA) and wFO sentences,
- 2. Aperiodic finitely ambiguous wA and wFO sentences without first-order sums,
- 3. Aperiodic unambiguous wA and wFO sentences without binary or first-order sums.

Note that these characterizations only need aperiodicity of the underlying input automaton and hold without any restrictions on the weights. The above result applies not only to the abstract semantics. As immediate consequence, we obtain corresponding expressive equivalence results for classical weighted automata over arbitrary (even non-commutative) semirings, or with average or discounted calculations of weights, or bounded lattices as in multi-valued logics. All our constructions are effective. In fact, given a wFO sentence and deterministic aperiodic automata for its boolean subformulas, we can construct an equivalent aperiodic weighted automaton of exponential size. We give typical examples for our constructions. We also show that the class of arbitrary aperiodic weighted automata and its subclasses of polynomially resp. finitely ambiguous or unambiguous weighted automata form a proper hierarchy for each of the following semirings: natural numbers $\mathbb{N}_{+,\times}$, the max-plus-semiring $\mathbb{N}_{\max,+}$ and the min-plus semiring $\mathbb{N}_{\min,+}$. Results are summarized in the Table 1. The second column lists existing characterizations for general weighted automata. The last column contains our results concerning aperiodic weighted automata. Examples referred to in the table separate the classes.

It should be noticed that standard constructions used to establish equivalence between automata and MSO logic cannot be applied. Indeed, starting from an automaton \mathcal{A} , one usually constructs an *existential* MSO sentence where the existential set quantifications are used to guess an accepting run and the easy first-order kernel is used to check that this guess indeed defines an accepting run. Here, we cannot use quantifications $\exists X$ over set variables X, or their weighted equivalent \sum_X . Instead, we take advantage of the fine structure of possible paths of polynomially ambiguous automata, namely the fact that it must be unambiguous on strongly connected components (SCC-unambiguous), as employed for different goals already

ambiguity	general WA	aperiodic WA
exponential	wMSO: [12, 13]	Ex. 26 for $\mathbb{N}_{+,\times}$, Ex. 27 for $\mathbb{N}_{\max,+}$
	Ex. 28 for $\mathbb{N}_{\min,+}$: [29]	Ex. 28 for $\mathbb{N}_{\min,+}$
	wMSO without \sum_X : [26]	wFO: Thm. 13, Thm. 24
polynomial		Ex. 30 for $\mathbb{N}_{+,\times}$, Ex. 31 for $\mathbb{N}_{\max,+}$
	Ex. 33 for $\mathbb{N}_{\min,+}$: [24, 29]	Ex. 33 for $\mathbb{N}_{\min,+}$
	wMSO without \sum_{X}, \sum_{x} : [26]	wFO without \sum_{x} : Cor. 11(2), Thm. 24
finite		Ex. 34 for $\mathbb{N}_{+,\times}$
	Ex.35: $\mathbb{N}_{\max,+}$ [25], $\mathbb{N}_{\min,+}$ [29]	Ex. 35 for $\mathbb{N}_{\max,+}$ and $\mathbb{N}_{\min,+}$
unambiguous	wMSO without $\sum_{X}, \sum_{x}, +$: [26]	wFO without \sum_{x} , +: Cor. 11(1), Thm. 24

Table 1 Summary of our main results.

in [23, 41]. We first give a new construction of a wFO sentence without sums starting from an aperiodic and unambiguous automaton. Then, we extend the construction to polynomially ambiguous aperiodic automata using first-order sums \sum_x to guess positions where the run switches between the unambiguous SCCs. For part 2 of Theorem 1, we also prove that for each aperiodic finitely ambiguous weighted automaton we can construct finitely many aperiodic unambiguous weighted automata whose disjoint union has the same semantics.

Again, for the implication from weighted formulas to weighted automata, we cannot simply use standard constructions which crucially rely on the fact that functions defined by weighted automata are closed under morphic images. This was used to handle first-oder sums \sum_x and second-order sums \sum_X , but also in the more involved proof for the first-order product \prod_x applied to finitely valued weighted automata. But it is well-known that aperiodic languages are not closed under morphic images. Handling the first-order product \prod_x requires a completely new and highly non-trivial proof preserving aperiodicity properties.

Related work. In [26], polynomially ambiguous, finitely ambiguous and unambiguous weighted automata (without assuming aperiodicity) over commutative semirings were shown to be expressively equivalent to suitable fragments of weighted monadic second order logic. This was further extended in [32] to cover polynomial degrees and weighted tree automata.

A hierarchy of these classes of weighted automata (again without assuming aperiodicity) over the max-plus semiring was described in [25]. As a consequence of pumping lemmas for weighted automata, a similar hierarchy was obtained in [29] for the min-plus semiring.

We note that in [13, 19], an equivalence result for full weighted first-order logic was given, but only for very particular classes of semirings or strong bimonoids as weight structures.

A characterization of the full weighted first-order logic with transitive closure by weighted pebble automata was obtained in [5]. An equivalence result for fragments of weighted first-order logic, weighted LTL and weighted counter-free automata over the max-plus semiring with discounting was given in [28]. Various further equivalences to boolean first-order definability of languages were described in the survey [9]. Due to its possible applications for quantitative verification questions, it remains a challenging problem to develop a weighted linear temporal logic for general classes of semirings with sufficiently large expressive power.

2 Preliminaries

A non-deterministic automaton is a tuple $\mathcal{A} = (Q, \Sigma, \Delta)$ where Q is a finite set of states, Σ is a finite alphabet, and $\Delta \subseteq Q \times \Sigma \times Q$ is the set of transitions. The automaton \mathcal{A} is complete if $\Delta(q, a) \neq \emptyset$ for all $q \in Q$ and $a \in \Sigma$. A run ρ of \mathcal{A} is a nonempty sequence of

transitions $\delta_1 = (p_1, a_1, q_1)$, $\delta_2 = (p_2, a_2, q_2)$, ..., $\delta_n = (p_n, a_n, q_n)$ such that $q_i = p_{i+1}$ for all $1 \le i < n$. We say that ρ is a run from state p_1 to state q_n and that ρ reads, or has label, the word $a_1 a_2 \cdots a_n \in \Sigma^+$. We denote by $\mathcal{L}(\mathcal{A}_{p,q}) \subseteq \Sigma^*$ the set of labels of runs of \mathcal{A} from p to q. When p = q, we include the empty word ε in $\mathcal{L}(\mathcal{A}_{p,q})$ and say that ε labels the empty run from p to p.

An automaton with accepting conditions is a tuple $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ where (Q, Σ, Δ) is a non-deterministic automaton, $I, F \subseteq Q$ are the sets of initial and final states respectively. The language defined by the automaton is $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_{I,F}) = \bigcup_{p \in I, q \in F} \mathcal{L}(\mathcal{A}_{p,q})$. Subsequently, we also consider automata with several accepting sets F, G, \ldots so that the same automaton may define several languages $\mathcal{L}(\mathcal{A}_{I,F}), \mathcal{L}(\mathcal{A}_{I,G}), \ldots$ An automaton $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ is deterministic if $I = \{\iota\}$ is a singleton and the set Δ of transitions is a partial function: for all $(p, a) \in Q \times \Sigma$ there is at most one state $q \in Q$ such that $(p, a, q) \in \Delta$.

Next, we consider degrees of ambiguity of automata. A run in an automaton is successful, if it leads from an initial to a final state. The automaton $\mathcal A$ is called polynomially ambiguous if there is a polynomial p such that for each $w \in \Sigma^+$ the number of successful paths in $\mathcal A$ for w is at most p(|w|). Then, $\mathcal A$ is finitely ambiguous if p can be taken to be a constant. Further, for an integer $k \geq 1$, $\mathcal A$ is k-ambiguous if p = k, and unambiguous means 1-ambiguous. Notice that k-ambiguous implies (k+1)-ambiguous. An automaton $\mathcal A$ is at most exponentially ambiguous.

A non-deterministic automaton $\mathcal{A} = (Q, \Sigma, \Delta)$ is aperiodic if there exists an integer $m \geq 1$, called aperiodicity index, such that for all states $p, q \in Q$ and all words $u \in \Sigma^+$, we have $u^m \in \mathcal{L}(\mathcal{A}_{p,q})$ iff $u^{m+1} \in \mathcal{L}(\mathcal{A}_{p,q})$. In other words, the non-deterministic automaton \mathcal{A} is aperiodic iff its transition monoid $\text{Tr}(\mathcal{A})$ is aperiodic. It is well-known that aperiodic languages coincide with first-order definable languages, cf. [39, 31, 9].

The syntax of first-order logic is given in Section 5 (FO). The semantics is defined by structural induction on the formula and requires an interpretation of the free variables. Let $\mathcal{V} = \{y_1, \ldots, y_n\}$ be a finite set of first-order variables. Given a nonempty word $u \in \Sigma^+$, we let $\mathsf{pos}(u) = \{1, \ldots, |u|\}$ be the set of positions of u. A valuation or interpretation is a map $\sigma \colon \mathcal{V} \to \mathsf{pos}(u)$ assigning positions of u to variables in \mathcal{V} . For a first-order formula φ having free variables contained in \mathcal{V} , we write $u, \sigma \models \varphi$ when the word u satisfies φ under the interpretation defined by σ . When φ is a *sentence*, the valuation σ is not needed and we simply write $u \models \varphi$.

We extend the classical semantics by defining when the empty word ε satisfies a sentence. We have $\varepsilon \models \top$ and if $\forall x \psi$ is a sentence then $\varepsilon \models \forall x \psi$. The semantics $\varepsilon \models \varphi$ is extended to all sentences φ since they are boolean combinations of the basic cases above. Notice that if φ has free variables then $\varepsilon \models \varphi$ is not defined. When φ is a sentence we denote by $\mathcal{L}(\varphi) \subseteq \Sigma^*$ the set of words satisfying φ . Notice that $\mathcal{L}(\forall x \bot) = \{\varepsilon\}$ where $\bot = \neg \top$.

▶ Theorem 2 ([39, 31, 9]). Let \mathcal{A} be an aperiodic non-deterministic automaton. For all states p, q of \mathcal{A} we can construct a first-order sentence $\varphi_{p,q}$ such that $\mathcal{L}(\mathcal{A}_{p,q}) = \mathcal{L}(\varphi_{p,q})$.

For the converse of Theorem 2, we need a stronger statement to deal with formulas having free variables. As usual, we encode a pair (u, σ) where $u \in \Sigma^+$ is a nonempty word and $\sigma \colon \mathcal{V} \to \mathsf{pos}(u)$ is a valuation by a word \overline{u} over the extended alphabet $\Sigma_{\mathcal{V}} = \Sigma \times \{0, 1\}^{\mathcal{V}}$. A word \overline{u} over $\Sigma_{\mathcal{V}}$ is a valid encoding if for each variable $y \in \mathcal{V}$, its projection on the y-component belongs to 0^*10^* . Throughout the paper, we identify a valid word \overline{u} with its encoded pair (u, σ) .

▶ **Theorem 3** ([39, 31, 9]). For each FO-formula φ having free variables contained in \mathcal{V} , we can build a deterministic, complete and aperiodic automaton $\mathcal{A}_{\varphi,\mathcal{V}} = (Q, \Sigma_{\mathcal{V}}, \Delta, \iota, F, G)$ over

the extended alphabet $\Sigma_{\mathcal{V}}$ such that for all words $\overline{u} \in \Sigma_{\mathcal{V}}^+$ we have:

- $\Delta(\iota, \overline{u}) \in F \text{ iff } \overline{u} \text{ is a valid encoding of a pair } (u, \sigma) \text{ with } (u, \sigma) \models \varphi,$
- $\Delta(\iota, \overline{u}) \in G \text{ iff } \overline{u} \text{ is a valid encoding of a pair } (u, \sigma) \text{ with } (u, \sigma) \models \neg \varphi,$
- $\Delta(\iota, \overline{u}) \notin F \cup G$ otherwise, i.e., iff \overline{u} is not a valid encoding of a pair (u, σ) .

Given $u \in \Sigma^+$ and integers k, ℓ , we denote by $u[k, \ell]$ the factor of u between positions k and ℓ . By convention $u[k, \ell] = \varepsilon$ is the empty word when $\ell < k$ or $\ell = 0$ or k > |u|.

We will apply the equivalence of Theorem 2 to prefixes, infixes or suffixes of words. Towards this, we use the classical *relativization* of sentences. Let φ be a first-order sentence and let $x, y \in \mathcal{V}$ be first-order variables. We define below the relativizations $\varphi^{< x}$, $\varphi^{(x,y)}$ and $\varphi^{>y}$ so that for all words $u \in \Sigma^+$, and all positions $i, j \in \mathsf{pos}(u) = \{1, \ldots, |u|\}$ we have

$$\begin{array}{lll} u,x\mapsto i\models\varphi^{< x} & \text{iff} & u[1,i-1]\models\varphi\\ \\ u,x\mapsto i,y\mapsto j\models\varphi^{(x,y)} & \text{iff} & u[i+1,j-1]\models\varphi\\ \\ u,x\mapsto j\models\varphi^{> x} & \text{iff} & u[j+1,|u|]\models\varphi \end{array}$$

Notice that, when i=1 or $j \leq i+1$ or j=|u|, the relativization is on the empty word, this is why we had to define when $\varepsilon \models \psi$ for sentences ψ . The relativization is defined by structural induction on the formulas as follows:

The relativizations $\varphi^{(x,y)}$ and $\varphi^{>x}$ are defined similarly. Notice that when φ is a sentence, i.e., a boolean combination of formulas of the form \top or $\forall z\psi$, then the above equivalences hold even when i=1 for $\varphi^{<x}$, or when i=|u| for $\varphi^{>x}$, or when $j\leq i+1$ for $\varphi^{(x,y)}$.

3 Weighted Automata

Given a set X, we let $\mathbb{N}\langle X \rangle$ be the collection of all finite multisets over X, i.e., all functions $f \colon X \to \mathbb{N}$ such that $f(x) \neq 0$ only for finitely many $x \in X$. The multiset union $f \uplus g$ of two multisets $f, g \in \mathbb{N}\langle X \rangle$ is defined by pointwise addition of functions: $(f \uplus g)(x) = f(x) + g(x)$ for each $x \in X$.

For a set R of weights, an R-weighted automaton over Σ is a tuple $\mathcal{A} = (Q, \Sigma, \Delta, \mathsf{wt})$ where (Q, Σ, Δ) is a non-deterministic automaton and $\mathsf{wt} \colon \Delta \to \mathsf{R}$ assigns a weight to every transition. The weight sequence of a run $\rho = \delta_1 \delta_2 \cdots \delta_n$ is $\mathsf{wt}(\rho) = \mathsf{wt}(\delta_1) \mathsf{wt}(\delta_2) \cdots \mathsf{wt}(\delta_n) \in \mathsf{R}^+$. The abstract semantics of \mathcal{A} from state p to state q is the map $\{|\mathcal{A}_{p,q}|\} \colon \Sigma^+ \to \mathbb{N} \setminus \mathsf{R}^+ \setminus \mathsf{which}$ assigns to a word $u \in \Sigma^+$ the multiset of weight sequences of runs from p to q with label u:

$$\{|\mathcal{A}_{p,q}|\}(u) = \{\{\mathsf{wt}(\rho) \mid \rho \text{ is a run from } p \text{ to } q \text{ with label } u\}\}.$$

Notice that $\{|\mathcal{A}_{p,q}|\}(u) = \emptyset$ is the empty multiset when there are no runs of \mathcal{A} from p to q with label u, i.e., when $u \notin \mathcal{L}(\mathcal{A}_{p,q})$. When we consider a weighted automaton $\mathcal{A} = (Q, \Sigma, \Delta, \mathsf{wt}, I, F)$ with initial and final sets of states, for all $u \in \Sigma^+$ the semantics $\{|\mathcal{A}|\}$ is defined as the multiset union: $\{|\mathcal{A}|\}(u) = \biguplus_{p \in I, q \in F} \{|\mathcal{A}_{p,q}|\}(u)$. Hence, $\{|\mathcal{A}|\}$ assigns to every word $u \in \Sigma^+$ the multiset of all weight sequences of accepting runs of \mathcal{A} reading u. The support of \mathcal{A} is the set of words $u \in \Sigma^+$ such that $\{|\mathcal{A}|\}(u) \neq \emptyset$, i.e., $\mathsf{supp}(\mathcal{A}) = \mathcal{L}(\mathcal{A})$.

For instance, consider the weighted automaton \mathcal{A} of Figure 1. We have $\operatorname{supp}(\mathcal{A}) = a^+a(a+b)^*b^+$. Moreover, consider $w = a^m(ba)^nb^p$ with m > 1 and p > 0. We have $w \in \operatorname{supp}(\mathcal{A})$ and

$$\{|\mathcal{A}|\}(w) = \{\{2^{k-1} \cdot 1 \cdot 3^{m-k-1} \cdot 5 \cdot (3 \cdot 5)^n \cdot 5^{\ell-1} \cdot 1 \cdot 2^{p-\ell} \mid 1 \le k < m \text{ and } 1 \le \ell \le p\}\}.$$

Figure 1 A weighted automaton, which is both aperiodic and polynomially ambiguous.

A concrete semantics over semirings, or valuation monoids, or valuation structures can be obtained from the abstract semantics defined above by applying the suitable aggregation operator $\operatorname{\mathsf{aggr}} \colon \mathbb{N}\langle \mathbb{R}^+ \rangle \to S$ as explained in [22]. For convenience, we include a short outline.

A semiring is a structure $(S, +, \times, 0, 1)$ where (S, +, 0) is a commutative monoid, $(S, \times, 1)$ is a monoid, multiplication distributes over addition, and $0 \times s = s \times 0 = 0$ for each $s \in S$. If the multiplication is commutative, we say that S is commutative. If the addition is idempotent, the semiring is called *idempotent*. Important examples of semirings include:

- the natural numbers $\mathbb{N}_{+,\times} = (\mathbb{N}, +, \times, 0, 1)$ with the usual addition and multiplication;
- the Boolean semiring $\mathcal{B} = (\{0,1\}, \vee, \wedge, 0, 1);$
- the min-plus (or tropical) semiring $\mathbb{N}_{\min,+} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$;
- the max-plus (or arctical) semiring $\mathbb{N}_{\max,+} = ((\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0);$
- the semiring of languages $(\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ where \cdot denotes concatenation of languages;
- the semiring of multisets of sequences $(\mathbb{N}\langle \mathbb{R}^* \rangle, \uplus, \cdot, \emptyset, \{\{\varepsilon\}\})$. Here, \cdot denotes the concatenation of multisets (Cauchy product), cf. [22].

Let $(S, +, \times, 0, 1)$ be a semiring and $\mathcal{A} = (Q, \Sigma, \Delta, \mathsf{wt})$ be an S-weighted automaton over Σ . The value of a run $\rho = \delta_1 \delta_2 \cdots \delta_n$ is then defined as $\mathsf{val}(\rho) = \mathsf{wt}(\delta_1) \times \mathsf{wt}(\delta_2) \times \cdots \times \mathsf{wt}(\delta_n)$. The concrete semantics of \mathcal{A} is the function $[\![\mathcal{A}]\!] \colon \Sigma^+ \to S$ given by $[\![\mathcal{A}]\!](w) = \sum_{\rho} val(\rho)$ where the sum is taken over all successful paths ρ executing the word w.

Let us define the aggregation function $\operatorname{\mathsf{aggr}}_{\mathsf{sp}} \colon \mathbb{N}\langle \mathsf{R}^+ \rangle \to S$ by letting $\operatorname{\mathsf{aggr}}_{\mathsf{sp}}(f)$ be the sum over all sequences $s_1 s_2 \cdots s_k$ in the multiset f of the products $s_1 \times s_2 \times \cdots \times s_k$ in S. It follows that the concrete semantics of \mathcal{A} is the composition of the aggregation function and the abstract semantics of \mathcal{A} , i.e., $[\![\mathcal{A}]\!](w) = \operatorname{\mathsf{aggr}}_{\mathsf{sp}}(\{\![\mathcal{A}]\!\}(w))$ for all $w \in \Sigma^+$. Also, the abstract semantics $\{\![\mathcal{A}]\!\}$ conicides with the concrete semantics of \mathcal{A} over the semiring of multisets of sequences over S, i.e., $\{\![\mathcal{A}]\!\} = [\![\mathcal{A}]\!]$ (since the aggregation function is the identity function).

As another example, assume the weights of \mathcal{A} are taken in $\mathbb{R}_{\geq 0} \cup \{-\infty\}$), the weight of a run ρ is computed as the average $\operatorname{avg}(\rho) = (\operatorname{wt}(\delta_1) + \cdots + \operatorname{wt}(\delta_n))/n$ of the weights in ρ , and the concrete semantics of \mathcal{A} is defined for $w \in \Sigma^+$ by $[\![\mathcal{A}]\!](w) = \max_{\rho} \operatorname{avg}(\rho)$ where the maximum is taken over all successful runs ρ executing w, cf. [7, 8, 15]. In this case, we define the aggregation $\operatorname{aggr}_{\mathsf{ma}}(M)$ of a multiset M by taking the maximum of all averages of sequences in M. Again, we obtain $[\![\mathcal{A}]\!](w) = \operatorname{aggr}_{\mathsf{ma}}(\{\![\mathcal{A}]\!\}(w))$ for all $w \in \Sigma^+$. See [22] for further discussion and examples.

Now, consider the natural semiring $(\mathbb{N}, +, \times, 0, 1)$ and the sum-product aggregation operator $\mathsf{aggr}_{\mathsf{sp}}$. We continue the example above with the automaton \mathcal{A} of Figure 1 and the word $w = a^m (ba)^n b^p$ with m > 1 and p > 0. The concrete semantics is given by

$$[\![\mathcal{A}]\!](w) = \mathsf{aggr}_{\mathsf{sp}}(\{\![\mathcal{A}]\!\}(w)) = \sum_{1 \leq k < m} \sum_{1 \leq \ell \leq p} 2^{k-1+p-\ell} 3^{m-k-1+n} 5^{n+\ell} \,.$$

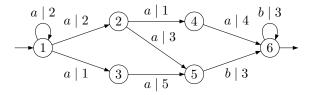


Figure 2 A 3-ambiguous weighted automaton.

4 Finitely ambiguous Weighted Automata

In this section, we investigate finitely ambiguous weighted automata. It was shown in [25] that over the max-plus semiring $\mathbb{N}_{\max,+}$ they are expressively equivalent to finite disjoint unions of unambiguous weighted automata. Moreover, it was proved in [36] that a K-valued rational transducer can be decomposed into K unambiguous transducers. In particular this implies that a K-ambiguous weighted automaton can be decomposed into K unambiguous weighted automata. Here we show that the same holds for aperiodic weighted automata.

▶ Theorem 4. Let $K \geq 1$. Given an aperiodic K-ambiguous weighted automaton A, we can construct aperiodic unambiguous weighted automata $\mathcal{B}_1, \ldots, \mathcal{B}_K$ such that $\{|\mathcal{A}|\} = \{|\mathcal{B}_1| \uplus \cdots \uplus \mathcal{B}_K|\} = \{|\mathcal{B}_1|\} \uplus \cdots \uplus \{|\mathcal{B}_K|\}$.

We give below a direct and simple construction which works for arbitrary (possibly non-aperiodic) K-ambiguous weighted automata. Then, we show that our construction preserves aperiodicity. Our proof is based on lexicographic ordering of runs. The proof of [36] uses lexicographic coverings. It would be interesting to see whether this proof also preserves aperiodicity and to compare the complexity of the constructions.

We first explain our construction on an example. Consider the 3-ambiguous weighted automaton \mathcal{A} of Figure 2 over the alphabet $\Sigma = \{a,b\}$ and the semiring $\mathbb{N}_{+,\times}$ of natural numbers. Clearly, the support of \mathcal{A} is $a^*(a^3+a^2b)b^*$ and $[\![\mathcal{A}]\!](a^na^3bb^p)=2^n\cdot(2\cdot 1\cdot 4\cdot 3+2\cdot 2\cdot 3\cdot 3+2\cdot 1\cdot 5\cdot 3)\cdot 3^p$ for $n,p\geq 0$. We construct in the proof an automaton $\mathcal{A}_{\geq 3}$ which checks that \mathcal{A} has 3 accepting runs on a given word. Hence, we will have $\mathcal{L}(\mathcal{A}_{\geq 3})=a^*a^3bb^*$. To do so, $\mathcal{A}_{\geq 3}$ will run three copies of \mathcal{A} , make sure that the three runs are lexicographically ordered (to be unambiguous) and accept if the three runs are accepting and pairwise distinct. The set of states is $Q'=Q^3\times\{0,1\}^2$ where $Q=\{1,\ldots,6\}$ is the set of states of \mathcal{A} . The initial state is (1,1,1,0,0) and the booleans turn to 1 when the runs differ. The accepting state is (6,6,6,1,1). The unique accepting run of $\mathcal{A}_{\geq 3}$ on the word a^3b^2 is

$$(1,1,1,0,0) \xrightarrow{a} (1,1,2,0,1) \xrightarrow{a} (2,3,4,1,1) \xrightarrow{a} (5,5,6,1,1) \xrightarrow{b} (6,6,6,1,1) \xrightarrow{b} (6,6,6,1,1) .$$

Proof. First, let $\mathcal{A} = (Q, \Sigma, \Delta, I, F, \mathsf{wt})$ be an arbitrary weighted automaton. We may assume that \mathcal{A} has a single initial state q_0 . For $k \geq 1$, we construct an automaton $\mathcal{A}_{\geq k} = (Q', \Sigma, \Delta', I', F')$ which accepts the set of words $w = a_1 a_2 \cdots a_n \in \Sigma^+$ having at least k accepting runs in \mathcal{A} . Moreover, if \mathcal{A} is aperiodic then so is $\mathcal{A}_{\geq k}$.

Fix a strict total order \prec on Q. We write \preceq for the induced lexicographic order on Q^+ and \prec for the strict order. A run of \mathcal{A} on w induces a sequence of states $\rho = q_0q_1q_2\cdots q_n \in Q^+$ with $(q_{i-1}, a_i, q_i) \in \Delta$ for all $1 \leq i \leq n$. Overloading our terminology, such a sequence is also called a run below. Runs of \mathcal{A} on w are lexicographically ordered. For $0 \leq j \leq n$, we denote by $\rho[j] = q_0q_1q_2\cdots q_j$ the prefix of length j of ρ .

The idea is that $\mathcal{A}_{\geq k}$ will guess k runs $\rho^1 \preceq \rho^2 \preceq \cdots \preceq \rho^k$ of \mathcal{A} on w. For $1 \leq \ell \leq k$, we let $\rho^\ell = q_0^\ell q_1^\ell q_2^\ell \cdots q_n^\ell$. Now, after reading the prefix $w[j] = a_1 a_2 \cdots a_j$, the state of $\mathcal{A}_{\geq k}$ will

consist of the tuple (q_j^1,\ldots,q_j^k) of states reached by the prefixes $\rho^1[j] \preceq \cdots \preceq \rho^k[j]$ together with a bit vector (c_j^1,\ldots,c_j^{k-1}) such that for all $1 \leq \ell < k, \ c_j^\ell = 1$ iff $\rho^\ell[j] \prec \rho^{\ell+1}[j]$. The automaton $\mathcal{A}_{\geq k}$ will accept if all states $q_n^\ell \in F$ are final in \mathcal{A} and the bit vector contains only 1's. This ensures that $\rho^1 \prec \rho^2 \prec \cdots \prec \rho^k$ are distinct accepting runs for w in \mathcal{A} .

We turn now to the formal definition of $\mathcal{A}_{\geq k}$. Let $Q' = Q^k \times \{0,1\}^{k-1}$, $I' = \{q_0\}^k \times \{0\}^{k-1}$ and $F' = F^k \times \{1\}^{k-1}$. We write tuples with superscripts: $(\overline{q}, \overline{c}) \in Q'$ with $\overline{q} = (q^1, \dots, q^k)$ and $\overline{c} = (c^1, \dots, c^{k-1})$. Now, $((\overline{q}, \overline{c}), a, (\overline{q}', \overline{c}'))$ is a transition of $\mathcal{A}_{\geq k}$ if the following conditions hold:

- $(q^{\ell}, a, q'^{\ell}) \in \Delta$ for all $1 \le \ell \le k$ (the k runs are non-deterministically guessed),
- and the bit vector is deterministically updated as follows: for $1 \le \ell < k$ we have either $(c^{\ell} = 0, q'^{\ell} = q'^{\ell+1} \text{ and } c'^{\ell} = 0)$, or $((c^{\ell} = 1 \text{ or } q'^{\ell} \prec q'^{\ell+1}) \text{ and } c'^{\ell} = 1)$. Notice that $c^{\ell} = 0$ and $q'^{\ell+1} \prec q'^{\ell}$ is not allowed.

When k = 1 then the accessible part of A_1 is equal to A. We will now state formally the main properties of $A_{>k}$.

 \triangleright Claim 5. For each $w \in \Sigma^+$, there is a bijection between the accepting runs $\overline{\rho}$ of $A_{\geq k}$ on w and the tuples (ρ^1, \ldots, ρ^k) of accepting runs of A on w such that $\rho^1 \prec \cdots \prec \rho^k$.

Proof. Consider a word $w=a_1a_2\cdots a_n\in \Sigma^+$ and a run $\overline{\rho}$ of $\mathcal{A}_{\geq k}$ on w starting from its initial state. Write $(\overline{q}_j,\overline{c}_j)$ the jth state of $\overline{\rho}$. For $1\leq \ell\leq k$, let ρ^ℓ be the projection of $\overline{\rho}$ on the ℓ th component: $\rho^\ell=q_0^\ell q_1^\ell q_2^\ell\cdots q_n^\ell$. Clearly, ρ^ℓ is a run of \mathcal{A} on w. Moreover, we can easily check by induction on $0\leq j\leq n$ that for all $1<\ell\leq k$ we have $\rho^\ell[j]=\rho^{\ell+1}[j]$ if $c_j^\ell=0$ and $\rho^\ell[j]\prec \rho^{\ell+1}[j]$ if $c_j^\ell=1$. We deduce that if $\overline{\rho}$ is accepting in $\mathcal{A}_{\geq k}$ then each ρ^ℓ is accepting in \mathcal{A} and $\rho^1\prec\cdots\prec\rho^k$. Therefore, every word accepted by $\mathcal{A}_{\geq k}$ admits at least k accepting runs in \mathcal{A} .

Conversely, assume that $w \in \Sigma^+$ has at least k accepting runs $\rho^1 \prec \cdots \prec \rho^k$ in \mathcal{A} . We can easily construct an accepting run $\overline{\rho}$ of $\mathcal{A}_{\geq k}$ on w such that the ℓ th projection of $\overline{\rho}$ is ρ^ℓ for each $1 \leq \ell \leq k$. We deduce that $\mathcal{A}_{\geq k}$ accepts exactly the set of words $w \in \Sigma^+$ having at least k accepting runs in \mathcal{A} .

We deduce from Claim 5 that if \mathcal{A} is k-ambiguous then $\mathcal{A}_{\geq k}$ is unambiguous and accepts exactly the words accepted by \mathcal{A} with ambiguity k.

 \triangleright Claim 6. If \mathcal{A} is aperiodic with index m then $\mathcal{A}_{\geq k}$ is aperiodic with index k(m+1).

Proof. Consider a word $w \in \Sigma^+$ and a run $\overline{\rho}$ of $\mathcal{A}_{\geq k}$ reading $w^{k(m+1)}$. The sequence of bit vectors along $\overline{\rho}$ is monotone component-wise. Hence, its value can change at most k-1 times. We deduce that we can write $\overline{\rho} = \overline{\rho}' \overline{\rho}'' \overline{\rho}'''$ where $\overline{\rho}''$ reads w^{m+1} with the bit vector unchanged. Let $(\overline{p}, \overline{c})$ and $(\overline{q}, \overline{c})$ be the source and target states of ρ'' . The projections ρ^1, \ldots, ρ^k of $\overline{\rho}''$ are runs reading w^{m+1} in \mathcal{A} . Since \mathcal{A} is aperiodic with index m, we find runs $\sigma^1, \ldots, \sigma^k$ reading w^m in \mathcal{A} from states p^1, \ldots, p^k to q^1, \ldots, q^k respectively. We may assume that for all $1 \leq \ell < k$ we have $\sigma^\ell = \sigma^{\ell+1}$ if $\rho^\ell = \rho^{\ell+1}$. Consider the run $\overline{\sigma}$ of $\mathcal{A}_{\geq k}$ starting from $(\overline{p}, \overline{c})$ whose projections are $\sigma^1, \ldots, \sigma^k$. It reaches a state $(\overline{q}', \overline{c}')$. Clearly, we have $\overline{q}' = \overline{q}$. We show that $\overline{c}' = \overline{c}$. Let $1 \leq \ell < k$. If $c^\ell = 1$ then $c'^\ell = 1$ by definition of $\mathcal{A}_{\geq k}$. If $c^\ell = 0$ then $\rho^\ell = \rho^{\ell+1}$ by definition of $\mathcal{A}_{\geq k}$. We deduce that $\sigma^\ell = \sigma^{\ell+1}$ and $c'^\ell = 0$. Finally, we conclude that $\overline{\rho}' \overline{\sigma} \overline{\rho}'''$ is a run of $\mathcal{A}_{\geq k}$ reading $w^{k(m+1)-1}$ with the same source (resp. target) state as $\overline{\rho}$.

By choosing runs $\sigma^1, \ldots, \sigma^k$ reading the word w^{m+2} instead of w^m , we otain a run of $\mathcal{A}_{>k}$ reading $w^{k(m+1)+1}$ with the same source (resp. target) state as $\overline{\rho}$.

Now, let $\mathcal{A}_{\leq k}$ be the minimal automaton for the complement of the language accepted by $\mathcal{A}_{\geq k+1}$. Notice that $\mathcal{A}_{\leq k}$ is deterministic, complete. Moreover, it is aperiodic if \mathcal{A} is aperiodic.

For each $1 \leq \ell \leq k$, define the weighted automaton $\mathcal{A}_{\geq k}^{\ell} = (\mathcal{A}_{\geq k}, \mathsf{wt}^{\ell})$ where the weight function corresponds to the ℓ th path computed by $\mathcal{A}_{\geq k}$. More precisely, we set $\mathsf{wt}^{\ell}((\overline{q}, \overline{c}), a, (\overline{q}', \overline{c}')) = \mathsf{wt}(q^{\ell}, a, q'^{\ell})$. Finally, let $\mathcal{A}_{k}^{\ell} = \mathcal{A}_{\leq k} \times \mathcal{A}_{\geq k}^{\ell}$. It is not difficult to see that \mathcal{A}_{k}^{ℓ} has the following properties.

ightharpoonup Claim 7. The automaton \mathcal{A}_k^{ℓ} is unambiguous. A word $w \in \Sigma^+$ is in the support of \mathcal{A}_k^{ℓ} iff it admits exactly k accepting runs $\rho^1 \prec \ldots \prec \rho^k$ in \mathcal{A} . Moreover, in this case, $\{|\mathcal{A}_k^{\ell}|\}(w) = \{\{\mathsf{wt}(\rho^{\ell})\}\}$. Also, if \mathcal{A} is aperiodic then so is \mathcal{A}_k^{ℓ} .

Finally, to conclude the proof of Theorem 4, we define for each $1 \leq \ell \leq K$ the weighted automaton $\mathcal{B}_{\ell} = \mathcal{A}_{\ell}^{\ell} \uplus \cdots \uplus \mathcal{A}_{K}^{\ell}$. Since the automata $(\mathcal{A}_{k}^{\ell})_{\ell \leq k \leq K}$ have pairwise disjoint supports, we deduce that \mathcal{B}_{ℓ} is unambiguous. Moreover, using Claim 7 we can easily show that $\{|\mathcal{A}|\} = \{|\mathcal{B}_{1} \uplus \cdots \uplus \mathcal{B}_{K}|\}$.

5 Weighted First-Order Logic

In this section, we define the syntax and semantics of our weighted first-order logic. In [12, 13], weighted MSO used the classical syntax of MSO logic; only the semantics over a semiring was changed to use sums for disjunction and existential quantifications, and products for conjunctions and universal quantifications. The possibility to express boolean properties in wMSO was obtained via so-called unambiguous formulae. To improve readability, a more structured syntax was later used [3, 15, 26], separating a boolean MSO layer with classical boolean semantics from the higher level of weighted formulas using $\operatorname{products}\left(\prod_X, \prod_x \operatorname{corresponding} \operatorname{to} \forall X, \forall x\right)$ and sums $(\sum_X, \sum_x \operatorname{corresponding} \operatorname{to} \exists X, \exists x)$ with quantitative semantics. As shown in [12, 13], in general, to retain equivalence with weighted automata, wMSO has to be restricted. Products \prod_X over set variables are disallowed, and first-order products \prod_x must be restricted to finitely valued series where the pre-image of each value is recognizable. This basically means that first-order products cannot be nested or applied after first-order or second-order sums $\sum_x \operatorname{or} \sum_X$. This motivated the equivalent and even more structured syntax of core-wMSO introduced in [22].

As in Section 3, we consider a set R of weights. The syntax of wFO is obtained from core-wMSO by removing set variables, set quantifications and set sums. In addition to the classical boolean first-order logic (FO), it has two weighted layers. Step formulas defined in (step-wFO) consist of constants and if-then-else applications, where the conditions are formulated in boolean first-order logic. Finally, wFO builds on this by performing products of step formulas and then applying if-then-else, finite sums, or existential sums.

$$\varphi ::= \top \mid P_a(x) \mid x \le y \mid \neg \varphi \mid \varphi \land \varphi \mid \forall x \varphi \tag{FO}$$

$$\Psi ::= r \mid \varphi ? \Psi : \Psi \tag{step-wFO}$$

$$\Phi ::= \mathbf{0} \mid \prod_{x} \Psi \mid \varphi ? \Phi : \Phi \mid \Phi + \Phi \mid \sum_{x} \Phi$$
 (wFO)

with $a \in \Sigma$, $r \in \mathbb{R}$ and x, y first-order variables.

The semantics of step-wFO formulas is defined inductively. As above, let $u \in \Sigma^+$ be a nonempty word and $\sigma \colon \mathcal{V} \to \mathsf{pos}(u) = \{1, \dots, |u|\}$ be a valuation. For step-wFO formulas whose free variables are contained in \mathcal{V} , we define the \mathcal{V} -semantics as

$$\llbracket r \rrbracket_{\mathcal{V}}(u,\sigma) = r \qquad \qquad \llbracket \varphi \, ? \, \Psi_1 : \Psi_2 \rrbracket_{\mathcal{V}}(u,\sigma) = \begin{cases} \llbracket \Psi_1 \rrbracket_{\mathcal{V}}(u,\sigma) & \text{if } u,\sigma \models \varphi \\ \llbracket \Psi_2 \rrbracket_{\mathcal{V}}(u,\sigma) & \text{otherwise.} \end{cases}$$

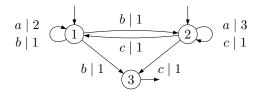


Figure 3 A weighted automaton, which is both aperiodic and unambiguous.

Notice that the semantics of a step-wFO formula is always a single weight from R.

For wFO formulas Φ whose free variables are contained in \mathcal{V} , we define the \mathcal{V} -semantics $\{\![\Phi]\!\}_{\mathcal{V}} \colon \Sigma_{\mathcal{V}}^+ \to \mathbb{N}\langle \mathbb{R}^+ \rangle$. First, we let $\{\![\Phi]\!\}_{\mathcal{V}}(\overline{u}) = \emptyset$ be the empty multiset when $\overline{u} \in \Sigma_{\mathcal{V}}^+$ is not a valid encoding of a pair (u, σ) . Assume now that $\overline{u} = (u, \sigma)$ is a valid encoding of a nonempty word $u \in \Sigma^+$ and a valuation $\sigma \colon \mathcal{V} \to \mathsf{pos}(u)$. The semantics of wFO formulas is also defined inductively: $\{\![0]\!\}_{\mathcal{V}}(u, \sigma) = \emptyset$ is the empty multiset, and

$$\begin{split} \{ \prod_x \Psi \}_{\mathcal{V}}(u,\sigma) &= \{ \{r_1 r_2 \cdots r_{|u|} \} \} \text{ where } r_i = \llbracket \Psi \rrbracket_{\mathcal{V} \cup \{x\}}(u,\sigma[x \mapsto i]) \text{ for } 1 \leq i \leq |u| \\ \{ [\varphi ? \Phi_1 : \Phi_2] \}_{\mathcal{V}}(u,\sigma) &= \begin{cases} \{ \Phi_1 \}_{\mathcal{V}}(u,\sigma) & \text{if } u,\sigma \models \varphi \\ \{ [\Phi_2] \}_{\mathcal{V}}(u,\sigma) & \text{otherwise} \end{cases} \\ \{ [\Phi_1 + \Phi_2] \}_{\mathcal{V}}(u,\sigma) &= \{ [\Phi_1] \}_{\mathcal{V}}(u,\sigma) \uplus \{ [\Phi_2] \}_{\mathcal{V}}(u,\sigma) \\ \{ [\sum_x \Phi] \}_{\mathcal{V}}(u,\sigma) &= \biguplus_{i \in \mathsf{pos}(u)} \{ [\Phi] \}_{\mathcal{V} \cup \{x\}}(u,\sigma[x \mapsto i]) \,. \end{split}$$

The semantics of the product (first line), is a singleton multiset which consists of a weight sequence whose length is |u|. We deduce that all weight sequences in a multiset $\{|\Phi|\}_{\mathcal{V}}(u,\sigma)$ have the same length and $\{|\Phi|\}_{\mathcal{V}}(u,\sigma) \in \mathbb{N}\langle \mathbb{R}^{|u|}\rangle$. We simply write $[\![\Psi]\!]$ and $\{\![\Phi]\!]$ when the set \mathcal{V} of variables is clear from the context.

As explained in Section 3, applying an aggregation function allows to recover the semantics $\llbracket \Phi \rrbracket$ over semirings such as $\mathbb{N}_{+,\times}$, $\mathbb{N}_{\max,+}$, etc. For instance, consider the function $f \colon \{a,b\}^+ \to \mathbb{N}$ which assign to a word $w \in \{a,b\}^+$ the length of the maximal a-block, i.e., f(w) = n if a^n is a factor of w but a^{n+1} is not. Over $\mathbb{N}_{\max,+}$, we have $f = \llbracket \Phi \rrbracket$ where

$$\Phi = \sum_{y,z} (\forall u (y \le u \le z) \to P_a(u)) ? (\prod_x (y \le x \le z) ? 1 : 0) : (\prod_x 0).$$

We refer to [13] for further examples of quantitative specifications in weighted logic.

6 From Weighted Automata to Weighted FO

We say that a non-deterministic automaton $\mathcal{A} = (Q, \Sigma, \Delta)$ is unambiguous from state p to state q if for all words $u \in \Sigma^+$, there is at most one run of \mathcal{A} from p to q with label u.

▶ **Theorem 8.** Let \mathcal{A} be an aperiodic weighted automaton which is unambiguous from p to q. We can construct a wFO sentence $\Phi_{p,q} = \varphi_{p,q} ? \prod_x \Psi_{p,q} : \mathbf{0}$ where $\varphi_{p,q}$ is a first-order sentence and $\Psi_{p,q}(x)$ is a step-wFO formula with a single free variable x such that $\{|\mathcal{A}_{p,q}|\} = \{|\Phi_{p,q}|\}$.

Before proving Theorem 8, we start with an example. The automaton \mathcal{A} of Figure 3 is unambiguous and it accepts the language $\mathcal{L}(\mathcal{A}) = (a^*b + a^*c)^+ = (a+b+c)^*(b+c)$. We define a wFO sentence $\Phi_{1,3} = \varphi_{1,3} ? \prod_x \Psi_{1,3}(x) : \mathbf{0}$ as follows. The FO sentence $\varphi_{1,3}$ checks that \mathcal{A} has a run from state 1 to state 3 on the input word w, i.e., that $w \in a^*b(a^*b + a^*c)^*$:

$$\varphi_{1,3} = \exists y (P_b(y) \land \forall z (z < y \implies P_a(z))) \land \exists y (\neg P_a(y) \land \forall z (z < y))$$

When this is the case, the step-wFO formula $\Psi_{1,3}(x)$ computes the weight of the transition taken at a position x in the input word:

$$\Psi_{1,3}(x) = (P_b(x) \lor P_c(x)) ? 1 : \exists y (x < y \land P_b(y) \land \forall z (x < z < y \implies P_a(z))) ? 2 : 3.$$

Notice that the same formula $\Psi = \Psi_{2,3} = \Psi_{1,3}$ also allows to compute the sequence of weights for the accepting runs starting in state 2. Therefore, \mathcal{A} is equivalent to the wFO sentence

$$\Phi = \exists y \left(\neg P_a(y) \land \forall z \left(z \leq y \right) \right) ? \prod_x \Psi(x) : \mathbf{0}.$$

Proof of Theorem 8. Let $\mathcal{A} = (Q, \Sigma, \Delta, \mathsf{wt})$ be the aperiodic weighted automaton. By Theorem 2, for every pair of states $r, s \in Q$ there is a first-order sentence $\varphi_{r,s}$ such that $\mathcal{L}(\mathcal{A}_{r,s}) = \mathcal{L}(\varphi_{r,s})$. This gives in particular the first-order sentence $\varphi_{p,q}$ which is used in $\Phi_{p,q}$.

 \triangleright Claim 9. We can construct a step-wFO formula $\Psi_{p,q}(x)$ such that for each word $u \in \mathcal{L}(\mathcal{A}_{p,q})$ and each position $1 \leq i \leq |u|$ in the word u, we have $\llbracket \Psi_{p,q} \rrbracket (u, x \mapsto i) = \mathsf{wt}(\delta)$ where δ is the ith transition of the unique run ρ of \mathcal{A} from p to q with label u.

Before proving this claim, let us show how we can deduce the statement of Theorem 8. Clearly, if a word $u \in \Sigma^+$ is not in $\mathcal{L}(\mathcal{A}_{p,q})$ then we have $\{|\mathcal{A}_{p,q}|\}(u) = \emptyset = \{|\Phi_{p,q}|\}(u)$. Consider now a word $u = a_1 a_2 \cdots a_n \in \mathcal{L}(\mathcal{A}_{p,q})$ and the unique run $\rho = \delta_1 \delta_2 \cdots \delta_n$ of \mathcal{A} from p to q with label u. We have $\{|\mathcal{A}_{p,q}|\}(u) = \{\{\mathsf{wt}(\delta_1)\mathsf{wt}(\delta_2)\cdots\mathsf{wt}(\delta_n)\}\} = \{|\prod_x \Psi_{p,q}|\}(u)$ where the second equality follows from Claim 9. We deduce that $\{|\mathcal{A}_{p,q}|\} = \{|\Phi_{p,q}|\}$.

We turn now to the proof of Claim 9. Let $\delta = (r, a, s) \in \Delta$ be a transition of \mathcal{A} . We define the FO-formula with one free variable $\varphi_{\delta}(x) = \varphi_{p,r}^{< x} \wedge P_a(x) \wedge \varphi_{s,q}^{> x}$.

 \triangleright Claim 10. For each word $u \in \Sigma^+$ and position $1 \le i \le |u|$, we have $u, x \mapsto i \models \varphi_\delta$ iff $u \in \mathcal{L}(\mathcal{A}_{p,q})$ and δ is the *i*th transition of the unique run of \mathcal{A} from p to q with label u.

Indeed, assume that $u, x \mapsto i \models \varphi_{\delta}$. Then, $u[1, i-1] \models \varphi_{p,r}$ and there is a run ρ' of \mathcal{A} from p to r with label u[1, i-1]. Notice that if i=1 then p=r and ρ' is the empty run. Similarly, from $u[i+1, |u|] \models \varphi_{s,q}$ we deduce that there is a run ρ'' of \mathcal{A} from s to q with label u[i+1, |u|]. Finally, $u, x \mapsto i \models P_a(x)$ means that the ith letter of u is a. We deduce that $\rho = \rho' \delta \rho''$ is a run of \mathcal{A} from p to q with label u, hence $u \in \mathcal{L}(\mathcal{A}_{p,q})$. Moreover, ρ is the unique such run since \mathcal{A} is unambiguous from p to q. Now, δ is the ith transition of ρ , which concludes one direction of the proof. Conversely, assume that $u \in \mathcal{L}(\mathcal{A}_{p,q})$ and δ is the ith transition of the unique run of \mathcal{A} from p to q with label u. Then, $u[1, i-1] \models \varphi_{p,r}$, $u[i+1, |u|] \models \varphi_{s,q}$, and the ith letter of u is a. Therefore, $u, x \mapsto i \models \varphi_{\delta}$. This concludes the proof of Claim 10.

Now, choose an arbitrary enumeration $\delta^1, \delta^2, \dots, \delta^k$ of the transitions in Δ and define the step-wFO formula with one free variable

$$\Psi_{p,q}(x) = \varphi_{\delta^1}(x) ? \operatorname{wt}(\delta^1) : \varphi_{\delta^2}(x) ? \operatorname{wt}(\delta^2) : \cdots \varphi_{\delta^k}(x) ? \operatorname{wt}(\delta^k) : \operatorname{wt}(\delta^k).$$

We show that this formula satisfies the property of Claim 9. Consider a word $u \in \mathcal{L}(\mathcal{A}_{p,q})$ and a position $1 \leq i \leq |u|$. Let δ be the *i*th transition of the unique run of \mathcal{A} from p to q with label u. By Claim 10, we have $u, x \mapsto i \models \varphi_{\delta^j}$ iff $\delta^j = \delta$. Therefore, $\llbracket \Psi_{p,q} \rrbracket (u, x \mapsto i) = \mathsf{wt}(\delta)$, which concludes the proof of Claim 9.

► Corollary 11.

1. Let \mathcal{A} be an aperiodic and unambiguous weighted automaton. We can construct a wFO sentence Φ which does not use any \sum_{x} operator or + operator, and such that $\{|\mathcal{A}|\} = \{|\Phi|\}$.

2. Let \mathcal{A} be an aperiodic and finitely ambiguous weighted automaton. We can construct a wFO sentence Φ which does not use any \sum_x operator, and such that $\{|\mathcal{A}|\} = \{|\Phi|\}$.

Proof. 1. Since \mathcal{A} is unambiguous, it is also unambiguous from p to q for all $p \in I$ and $q \in F$. Therefore, a first attempt is the formula $\Phi' = \sum_{p \in I, q \in F} \Phi_{p,q}$ where the wFO sentences $\Phi_{p,q}$ are given by Theorem 8. We have $\{|\mathcal{A}|\} = \{|\Phi'|\}$ and the formula Φ' does not use any \sum_x operator, but it does use some + operator. One should notice that, since \mathcal{A} is unambiguous, for any word $u \in \Sigma^+$ at most one of the $(\{|\Phi_{p,q}|\}(u))_{p \in I, q \in F}$ is nonempty. Therefore, if $(p_1, q_1), (p_2, q_2), \ldots, (p_m, q_m)$ is an enumeration of $I \times F$ then we define

$$\Phi = \varphi_{p_1,q_1} ? \Phi_{p_1,q_1} : \varphi_{p_2,q_2} ? \Phi_{p_2,q_2} : \cdots \varphi_{p_m,q_m} ? \Phi_{p_m,q_m} : \mathbf{0}.$$

We have $\{A\} = \{\Phi\}$ and the formula Φ does not use any \sum_x or + operator. Notice that in the formula above, we may replace Φ_{p_i,q_i} by $\prod_x \Psi_{p_i,q_i}(x)$ as given by Theorem 8.

Alternatively, by a standard construction adding a new initial and a new final state and appropriate transitions, we can obtain an aperiodic weighted automaton \mathcal{A}' with a single initial and a single final state such that $\{|\mathcal{A}'|\} = \{|\mathcal{A}|\}$ and, moreover, \mathcal{A}' becomes unambiguous because \mathcal{A} is unambiguous. Then apply Theorem 8 to \mathcal{A}' .

2. Immediate by Theorem 4 and part 1 above.

Let $\mathcal{A}=(Q,\Sigma,\Delta)$ be a non-deterministic automaton. Two states $p,q\in Q$ are in the same strongly connected component (SCC), denoted $p\approx q$, if p=q or there exist a run of \mathcal{A} from p to q and also a run of \mathcal{A} from q to p. Notice that \approx is an equivalence relation on Q. We denote by [p] the strongly connected component of state p, i.e., the equivalence class of p under \approx .

The automaton \mathcal{A} is SCC-unambiguous if it is unambiguous on each strongly connected component, i.e., \mathcal{A} is unambiguous from p to q for all p,q such that $p \approx q$. Notice that a trimmed (all states are reachable and co-reachable) and unambiguous automaton is SCC-unambiguous.

For instance, the automaton \mathcal{A} of Figure 1 has three strongly connected components: $\{1\}, \{2,3\}$ and $\{4\}$. It is not unambiguous from 1 to 4, but it is SCC-unambiguous.

- ▶ Proposition 12 ([34, 23] and [41] Theorem 4.1). Let $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ be a trimmed non-deterministic automaton. Then \mathcal{A} is polynomially ambiguous if and only if \mathcal{A} is SCC-unambiguous.
- ▶ Theorem 13. Let \mathcal{A} be an aperiodic weighted automaton which is SCC-unambiguous. For each pair of states p and q, we can construct a wFO sentence $\Phi_{p,q}$ such that $\{|\mathcal{A}_{p,q}|\} = \{|\Phi_{p,q}|\}$. Moreover, we can construct a wFO sentence Φ such that $\{|\mathcal{A}|\} = \{|\Phi|\}$.

Before starting the proof of Theorem 13, we give for the weighted automaton \mathcal{A} of Figure 1 the equivalent wFO formula $\Phi_{1,4} = \sum_{y_1} \sum_{y_2} \varphi(y_1,y_2) ? \prod_x \Psi(x,y_1,y_2) : \mathbf{0}$ where φ and Ψ are defined below. When reading a word $w \in \operatorname{supp}(\mathcal{A})$, the automaton makes two non-deterministic choices corresponding to the positions y_1 and y_2 at which the transitions switching between the strongly connected components are taken, i.e., transition from state 1 to state 2 is taken at position y_1 , and transition from state 3 to state 4 is taken at position y_2 . Since the automaton is SCC-unambiguous, given the input word and these two positions, the run is uniquely determined. We use the FO formula $\varphi(y_1,y_2)$ to check that it is possible to take the switching transitions at positions y_1 and y_2 :

$$\varphi(y_1, y_2) = y_1 < y_2 \land \forall z (z \le y_1 \to P_a(z)) \land P_a(y_1 + 1) \land \forall z (y_2 \le z \to P_b(z)).$$

When this is the case, the step-wFO formula $\Psi(x, y_1, y_2)$ computes the weight of the transition taken at a position x in the input word:

$$\Psi(x, y_1, y_2) = (x < y_1 \lor y_2 < x)?2: (x = y_1 \lor x = y_2)?1: P_a(x+1)?3:5.$$

With these definitions, we obtain $\{|A|\} = \{|\Phi_{1,4}|\}.$

Proof of Theorem 13. Let $\mathcal{A} = (Q, \Sigma, \Delta, \mathsf{wt})$ be the aperiodic weighted automaton which is SCC-unambiguous. Let $p, q \in Q$ be a pair of states of \mathcal{A} . Assume first that $p \approx q$ are in the same strongly connected component. Then \mathcal{A} is unambiguous from p to q and we obtain the formula $\Phi_{p,q}$ directly by Theorem 8. So we assume below that $p \not\approx q$ are not in the same SCC.

Consider a word $u \in \mathcal{L}(\mathcal{A}_{p,q})$. Let ρ be a run from p to q with label u. This run starts in the SCC of p and ends in the SCC of q. So it uses some transitions linking different SCCs. More precisely, we can uniquely split the run as $\rho = \rho_0 \delta_1 \rho_1 \delta_2 \rho_2 \cdots \delta_m \rho_m$ with $m \geq 1$ such that each subrun ρ_i stays in some SCC and each transition $\delta_i = (p_i, a_i, q_i)$ switches to a different SCC:

$$p \approx p_1 \not\approx q_1 \approx p_2 \not\approx q_2 \approx p_3 \dots \approx p_m \not\approx q_m \approx q$$
. (1)

This motivates the following definition. A sequence of *switching* transitions from p to q is a tuple $\overline{\delta} = (\delta_1, \dots, \delta_m)$ with $m \geq 1$ satisfying (1), where $\delta_i = (p_i, a_i, q_i)$ for $1 \leq i \leq m$. A $\overline{\delta}$ -run from p to q is a run from p to q using exactly the sequence of switching transitions $\overline{\delta}$, i.e., a run of the form $\rho = \rho_0 \delta_1 \rho_1 \cdots \delta_m \rho_m$. Notice that each subrun ρ_i must stay in some SCC of \mathcal{A} .

 \triangleright Claim 14. For each sequence $\bar{\delta}$ of switching transitions from p to q, we can construct a wFO sentence $\Phi_{p,\bar{\delta},q}$ such that for all $u \in \Sigma^+$ we have

$$\{\Phi_{n\,\overline{\delta}\,a}\}(u) = \{\{\mathsf{wt}(\rho) \mid \rho \text{ is a } \overline{\delta}\text{-run from } p \text{ to } q \text{ with label } u\}\}. \tag{2}$$

Proof. During the proof of Claim 14, we fix the sequence $\overline{\delta} = (\delta_1, \dots, \delta_m)$ of switching transitions from p to q, with $m \ge 1$ and $\delta_i = (p_i, a_i, q_i)$ for $1 \le i \le m$.

By Theorem 2, for every pair of states $r, s \in Q$ there is a first-order sentence $\varphi_{r,s}$ such that $\mathcal{L}(\mathcal{A}_{r,s}) = \mathcal{L}(\varphi_{r,s})$. We will use these formulas and also their relativizations $\varphi_{r,s}^{< y}$, $\varphi_{r,s}^{(y,z)}$ and $\varphi_{r,s}^{>z}$.

We define the FO formula φ with free variables $\mathcal{V} = \{y_1, \dots, y_m\}$ by

$$\varphi = y_1 < y_2 < \dots < y_m \land \bigwedge_{1 \le i \le m} P_{a_i}(y_i) \land \varphi_{p,p_1}^{< y_1} \land \bigwedge_{1 \le i < m} \varphi_{q_i,p_{i+1}}^{(y_i,y_{i+1})} \land \varphi_{q_m,q}^{> y_m}.$$

Now, we fix a word $u \in \Sigma^+$.

ightharpoonup Claim 15. There is a bijection between the valuations $\sigma \colon \mathcal{V} \to \mathsf{pos}(u) = \{1, \dots, |u|\}$ such that $u, \sigma \models \varphi$ and the $\overline{\delta}$ -runs ρ from p to q with label u.

Proof. First, let $\sigma \colon \mathcal{V} \to \mathsf{pos}(u)$ be such that $u, \sigma \models \varphi$. We have $\sigma(y_1) < \sigma(y_2) < \dots < \sigma(y_m)$. Since $u, \sigma \models \varphi_{p,p_1}^{< y_1}$, there is a (possibly empty) run $\rho_0(\sigma)$ from p to p_1 reading the prefix $u_0 = u[1, \sigma(y_1) - 1]$ of u. Notice that such a run is unique since $p \approx p_1$ and \mathcal{A} is SCC-unambiguous. Similarly, for all $1 \leq i < m$, $u, \sigma \models \varphi_{q_i, p_{i+1}}^{(y_i, y_{i+1})}$ implies that there is a unique run $\rho_i(\sigma)$ from q_i to p_{i+1} reading the factor $u_i = u[\sigma(y_i) + 1, \sigma(y_{i+1}) - 1]$ of u. Also, $u, \sigma \models \varphi_{q_m, q}^{> y_m}$

implies that there is a unique run $\rho_m(\sigma)$ from q_m to q reading the suffix $u_m = u[\sigma(y_m) + 1, |u|]$ of u. Now, since $u, \sigma \models \bigwedge_{1 \leq i \leq m} P_{a_i}(y_i)$, we deduce that $u = u_0 a_1 u_1 a_2 \cdots a_m u_m$ and that $\rho(\sigma) = \rho_0(\sigma) \delta_1 \rho_1(\sigma) \cdots \delta_m \rho_m(\sigma)$ is a $\bar{\delta}$ -run of A from p to q with label u.

Conversely, let $\rho = \rho_0 \delta_1 \rho_1 \cdots \delta_m \rho_m$ be a $\overline{\delta}$ -run of \mathcal{A} from p to q with label u. Define the valuation $\sigma \colon \mathcal{V} \to \mathsf{pos}(u)$ so that the switching transitions $\overline{\delta}$ along this run are taken at positions $\sigma(y_1) < \sigma(y_2) < \cdots < \sigma(y_m)$. We can easily check that $u, \sigma \models \varphi$ and that $\rho = \rho(\sigma)$. This concludes the proof of Claim 15.

Let $\delta = (r, a, s) \in \Delta$ be a transition such that $q_i \approx r \approx s \approx p_{i+1}$ for some $1 \leq i < m$. Define the FO formula

$$\varphi_{\delta} = y_i < x < y_{i+1} \land \varphi_{q_i,r}^{(y_i,x)} \land P_a(x) \land \varphi_{s,p_{i+1}}^{(x,y_{i+1})}.$$

It is not difficult to see that for all valuations $\sigma \colon \mathcal{V} \cup \{x\} \to \mathsf{pos}(u)$ we have $u, \sigma \models \varphi_{\delta}$ iff the factor $v = u[\sigma(y_i) + 1, \sigma(y_{i+1}) - 1]$ of u is such that $v \in \mathcal{L}(\mathcal{A}_{q_i, p_{i+1}})$ and the unique run of \mathcal{A} from q_i to p_{i+1} with label v takes transition δ on position $\sigma(x) - \sigma(y_i)$. This is similar to Claim 10.

Now, if $\delta=(r,a,s)\in \Delta$ is a transition such that $p\approx r\approx s\approx p_1$, then we define the FO formula

$$\varphi_{\delta} = x < y_1 \wedge \varphi_{p,r}^{< x} \wedge P_a(x) \wedge \varphi_{s,p_1}^{(x,y_1)}$$
.

Then, $u, \sigma \models \varphi_{\delta}$ iff the prefix $v = u[1, \sigma(y_1) - 1]$ of u is such that $v \in \mathcal{L}(\mathcal{A}_{p,p_1})$ and the unique run of \mathcal{A} from p to p_1 with label v takes transition δ on position $\sigma(x)$.

Next, if $\delta=(r,a,s)\in\Delta$ is a transition such that $q_m\approx r\approx s\approx q$, then we define the FO formula

$$\varphi_{\delta} = y_m < x \wedge \varphi_{q_m,r}^{(y_m,x)} \wedge P_a(x) \wedge \varphi_{s,q}^{>x}$$
.

Then, $u, \sigma \models \varphi_{\delta}$ iff the suffix $v = u[\sigma(y_m) + 1, |u|]$ of u is such that $v \in \mathcal{L}(\mathcal{A}_{q_m,q})$ and the unique run of \mathcal{A} from q_m to q with label v takes transition δ on position $\sigma(x) - \sigma(y_m)$.

Finally, for a switching transition δ_i of δ we let $\varphi_{\delta_i} = (x = y_i)$ and for all other transitions $\delta = (r, a, s) \in \Delta \setminus \{\delta_1, \dots, \delta_m\}$ such that r, s are not both in the strongly connected component of one of the states p_1, p_2, \dots, p_m, q then we let $\varphi_{\delta} = \mathsf{false}$.

As in the proof of Theorem 8, we choose an arbitrary enumeration $\delta^1, \delta^2, \dots, \delta^k$ of the transitions in Δ and define the step-wFO formula with free variables $\mathcal{V} \cup \{x\}$

$$\Psi = \varphi_{\delta^1} ? \mathsf{wt}(\delta^1) : \varphi_{\delta^2} ? \mathsf{wt}(\delta^2) : \cdots \varphi_{\delta^k} ? \mathsf{wt}(\delta^k) : \mathsf{wt}(\delta^k).$$

Finally, the wFO sentence for Claim 14 is defined by

$$\Phi_{p,\overline{\delta},q} = \sum_{y_1} \sum_{y_2} \cdots \sum_{y_m} \left(\varphi \, ? \prod_x \Psi : \mathbf{0} \right)$$
 .

We prove now that Equation (2) holds. By definition, $\{\!\!\{ \Phi_{p,\overline{\delta},q} \}\!\!\} (u)$ is the (multiset) union over all valuations $\sigma \colon \mathcal{V} \to \mathsf{pos}(u)$ of $\{\!\!\{ \varphi \,?\, \prod_x \Psi : \mathbf{0} \}\!\!\} (u,\sigma)$. By Claim 15, there is a bijection between the valuations $\sigma \colon \mathcal{V} \to \mathsf{pos}(u)$ such that $u,\sigma \models \varphi$ and the $\overline{\delta}$ -runs from p to q with label u. Therefore, it remains to show that for all valuations $\sigma \colon \mathcal{V} \to \mathsf{pos}(u)$ such that $u,\sigma \models \varphi$ with associated $\overline{\delta}$ -run ρ we have

$$\{\{\mathsf{wt}(\rho)\}\} = \{\prod_{x} \Psi | \{u, \sigma\} .$$

Let $i \in \mathsf{pos}(u)$ and let δ be the ith transition of ρ . From the definitions above, we deduce easily that $u, \sigma[x \mapsto i] \models \varphi_{\delta^j}$ iff $\delta^j = \delta$. Therefore, $\llbracket \Psi \rrbracket (u, \sigma[x \mapsto i]) = \mathsf{wt}(\delta)$. The announced equality $\{\{\mathsf{wt}(\rho)\}\} = \{\{\prod_x \Psi\}\}(u, \sigma)$ follows. This concludes the proof of Claim 14.

To conclude the proof of the first part of Theorem 13, we define

$$\Phi_{p,q} = \sum_{\overline{\delta}} \Phi_{p,\overline{\delta},q}$$

where the sum ranges over all sequences $\bar{\delta}$ of switching transitions from p to q. Recall that we have assumed that $p \not\approx q$ are not in the same SCC of \mathcal{A} . Therefore, each run from p to q should go through some sequence of switching transitions. More precisely, given a word $u \in \Sigma^+$, the runs of \mathcal{A} from p to q with label u can be partitionned according to the sequence $\bar{\delta}$ of switching transitions that they use. Therefore, $\{|\mathcal{A}_{p,q}|\}(u)$ is the multiset union over all sequences $\bar{\delta}$ of switching transitions from p to q of the multisets $\{\{\mathsf{wt}(\rho) \mid \rho \text{ is a } \bar{\delta}\text{-run from } p \text{ to } q \text{ with label } u\}\}$. Using Claim 14, we deduce that $\{|\mathcal{A}_{p,q}|\}(u) = \{|\Phi_{p,q}|\}(u)$.

Finally, consider a weighted automaton with acceptance conditions $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ which is aperiodic and SCC-unambiguous. We set $\Phi = \sum_{p \in I, q \in F} \Phi_{p,q}$ where for each pair of states $(p,q) \in I \times F$, the formula $\Phi_{p,q}$ is defined as above.

7 From Weighted FO to Weighted Automata

Let $\mathcal{A} = (Q, \Sigma, \Delta)$ and $\mathcal{A}' = (Q', \Sigma, \Delta')$ be two non-deterministic automata over the same alphabet Σ . Assuming that $Q \cap Q' = \emptyset$, we define their disjoint union as $\mathcal{A} \uplus \mathcal{A}' = (Q \uplus Q', \Sigma, \Delta \uplus \Delta')$ and their product as $\mathcal{A} \times \mathcal{A}' = (Q \times Q', \Sigma, \Delta'')$ where $\Delta'' = \{((p, p'), a, (q, q')) \mid (p, a, p') \in \Delta \text{ and } (p', a, q') \in \Delta'\}$.

- **▶ Lemma 16.** *The following holds.*
- 1. If A and A' are aperiodic, then $A \uplus A'$ and $A \times A'$ are also aperiodic.
- **2.** If A and A' are SCC-unambiguous, then $A \uplus A'$ and $A \times A'$ are also SCC-unambiguous.

Now let φ be an FO-formula with free variables contained in the finite set \mathcal{V} , and let $\mathcal{A}_{\varphi,\mathcal{V}} = (Q, \Sigma_{\mathcal{V}}, \Delta, \iota, F, G)$ be the deterministic, complete, trim and aperiodic automaton given by Theorem 3. For i = 1, 2, let $\mathcal{A}_i = (Q_i, \Sigma_{\mathcal{V}}, \Delta_i, \mathsf{wt}_i, I_i, F_i)$ be two weighted automata over $\Sigma_{\mathcal{V}}$ with $Q_1 \cap Q_2 = \emptyset$. We define the weighted automaton $\mathcal{A}' = (Q', \Sigma_{\mathcal{V}}, \Delta', \mathsf{wt}', I', F')$ by letting

- $Q' = Q \times Q_1 \uplus Q \times Q_2, \ I' = \{\iota\} \times I_1 \uplus \{\iota\} \times I_2, \ F' = F \times F_1 \uplus G \times F_2,$
- $\Delta' = \{((p, p'), a, (q, q')) \mid (p, a, q) \in \Delta \text{ and } (p', a, q') \in \Delta_1 \cup \Delta_2\}, \text{ and } \text{wt'}((p, p'), a, (q, q')) = \text{wt}_i(p', a, q') \text{ if } (p', a, q') \in \Delta_i \text{ for } i = 1, 2.$

Then we have:

▶ Lemma 17. For each $\overline{u} \in \Sigma_{\mathcal{V}}^+$, we have

$$\{|\mathcal{A}'|\}(\overline{u}) = \begin{cases} \{|\mathcal{A}_1|\}(\overline{u}), & \text{if } \overline{u} \text{ is valid and } \overline{u} \models \varphi, \\ \{|\mathcal{A}_2|\}(\overline{u}), & \text{if } \overline{u} \text{ is valid and } \overline{u} \not\models \varphi, \\ \emptyset, & \text{if } \overline{u} \text{ is not valid.} \end{cases}$$

Moreover, if A_1 and A_2 are aperiodic (resp. unambiguous, SCC-unambiguous) then so is A'.

Proof. The first part is immediate by the construction of \mathcal{A}' and Theorem 3. For the final statement, we can argue as for Lemma 16; for the unambiguity part observe that the sets F and G of $\mathcal{A}_{\varphi,\mathcal{V}}$ are disjoint.

Let \mathcal{V} be a finite set of first-order variables and let $\mathcal{V}' = \mathcal{V} \cup \{y\}$ where $y \notin \mathcal{V}$. Given a word $\overline{w} \in \Sigma_{\mathcal{V}}^+$ and a position $i \in \mathsf{pos}(w)$, we denote by $(\overline{w}, y \mapsto i)$ the word over $\Sigma_{\mathcal{V}'}$ whose projection on $\Sigma_{\mathcal{V}}$ is \overline{w} and projection on the y-component is $0^{i-1}10^{|w|-i}$, i.e., has a unique 1 on position i. Given a function $A \colon \Sigma_{\mathcal{V}'}^+ \to \mathbb{N}\langle X \rangle$, we define the function $\sum_y A \colon \Sigma_{\mathcal{V}}^+ \to \mathbb{N}\langle X \rangle$ for $\overline{w} \in \Sigma_{\mathcal{V}}^+$ by

$$(\sum_{y} A)(\overline{w}) = \biguplus_{i \in \mathsf{pos}(w)} A(\overline{w}, y \mapsto i) .$$

- ▶ **Lemma 18.** Let \mathcal{A} be a weighted automaton over $\Sigma_{\mathcal{V}'}$. We can construct a weighted automaton \mathcal{A}' over $\Sigma_{\mathcal{V}}$ such that $\{|\mathcal{A}'|\} = \sum_{u} \{|\mathcal{A}|\}$. Moreover,
- **1.** If A is aperiodic then A' is also aperiodic.
- **2.** If A is SCC-unambiguous then A' is also SCC-unambiguous.

Proof. Let $\mathcal{A} = (Q, \Sigma_{\mathcal{V}'}, \Delta, \mathsf{wt}, I, F)$. We construct $\mathcal{A}' = (Q', \Sigma_{\mathcal{V}}, \Delta', \mathsf{wt}', I', F')$ as follows: $Q' = Q \times \{0, 1\}, \ I' = I \times \{0\}, \ F' = F \times \{1\}$ and for $\overline{a} \in \Sigma_{\mathcal{V}}$ the transitions and weights are given by:

- If $\delta = (p, (\overline{a}, 0), q) \in \Delta$ then $\delta^0 = ((p, 0), \overline{a}, (q, 0)) \in \Delta'$, $\delta^1 = ((p, 1), \overline{a}, (q, 1)) \in \Delta'$ and $\mathsf{wt}'(\delta^0) = \mathsf{wt}'(\delta^1) = \mathsf{wt}(\delta)$.
- If $\delta = (p, (\overline{a}, 1), q) \in \Delta$ then $\delta' = ((p, 0), \overline{a}, (q, 1)) \in \Delta'$ and $\mathsf{wt}'(\delta') = \mathsf{wt}(\delta)$.
- \triangleright Claim 19. We have $\{|\mathcal{A}'|\} = \sum_{u} \{|\mathcal{A}|\}.$

Proof. Consider a word $\overline{w} \in \Sigma_{\mathcal{V}}^+$ and let $i \in \mathsf{pos}(w)$. It is easy to see that there is a bijection between the accepting runs ρ of \mathcal{A} on $(\overline{w}, y \mapsto i)$ and the accepting runs ρ' of \mathcal{A}' on \overline{w} and switching from $Q \times \{0\}$ to $Q \times \{1\}$ on the *i*th transition. Moreover, this bijection preserves the weight sequences: $\mathsf{wt}'(\rho') = \mathsf{wt}(\rho)$. We deduce easily that $\{|\mathcal{A}'|\}(\overline{w}) = (\sum_{y} \{|\mathcal{A}|\})(\overline{w})$.

 \triangleright Claim 20. If \mathcal{A} is aperiodic then \mathcal{A}' is also aperiodic.

Proof. Assume that m is an aperiodicity index of \mathcal{A} . We claim that m'=2m is an aperiodicity index of \mathcal{A}' . Let $\overline{w} \in \Sigma_{\mathcal{V}}^+$, let $k \geq m'$ and let ρ' be a run of \mathcal{A}' reading \overline{w}^k from some state (p,b) to some state (r,c). We distinguish two cases. Either there is a prefix ρ'_1 of ρ' reading \overline{w}^m and staying in $Q \times \{0\}$, i.e., ρ'_1 goes from (p,b) = (p,0) to some (q,0). We deduce that there is a run ρ_1 of \mathcal{A} from p to q and reading $(\overline{w},0)^m$ (recall that we denote by $(\overline{w},0)$ the word over $\Sigma_{\mathcal{V}'}$ whose projection on $\Sigma_{\mathcal{V}}$ is \overline{w} and projection on the last component belongs to 0^+). Since m is an aperiodicity index of \mathcal{A} there is another run ρ_2 of \mathcal{A} from p to q reading $(\overline{w},0)^{m+1}$. We obtain a run ρ'_2 of \mathcal{A}' from (p,0) to (q,0) reading \overline{w}^{m+1} . Now, replacing the prefix ρ'_1 of ρ' by ρ'_2 we obtain a new run ρ'' of \mathcal{A}' reading \overline{w}^{m} from some state (p,0) = (p,b) to (r,c). In the second case, there is a suffix ρ'_1 of ρ' reading \overline{w}^{m} from some state (q,1) to (r,c) = (r,1). We construct as above another run ρ'_2 from (q,1) to (r,c) reading \overline{w}^{m+1} . Replacing the suffix ρ'_1 of ρ' by ρ'_2 , we obtain the run ρ'' from (p,b) to (r,c) reading \overline{w}^{k+1} . Finally, when k > m' = 2m, a similar argument allows to construct a run ρ'' from (p,b) to (r,c) reading \overline{w}^{k-1} .

 \triangleright Claim 21. If \mathcal{A} is SCC-unambiguous then \mathcal{A}' is also SCC-unambiguous.

Proof. Let $\overline{w} \in \Sigma_{\mathcal{V}}^+$ and let $(p, b) \approx' (q, c)$ be two states of Q' which are in the same SCC of \mathcal{A}' . Then, b = c and $p \approx q$ are in the same SCC of \mathcal{A} . Since b = c, there is a bijection between the runs of \mathcal{A}' from (p, b) to (q, c) reading \overline{w} and the runs of \mathcal{A} from p to q reading $(\overline{w}, 0)$. Since \mathcal{A} is SCC-unambiguous and $p \approx q$, there is at most one run of \mathcal{A} from p to q reading $(\overline{w}, 0)$. Hence, there is at most one run of \mathcal{A}' from (p, b) to (q, c) reading \overline{w} .

•

We turn now to one of our main results: given a step-wFO formula Ψ , we can construct a weighted automaton for $\prod_x \Psi$ which is both aperiodic and unambiguous.

When weights are uninterpreted, a weighted automaton $\mathcal{A}=(Q,\Sigma,\Delta,\operatorname{wt},I,F)$ is a letter-to-letter transducer from its input alphabet Σ to the output alphabet R. If in addition the input automaton is unambiguous, then we have a functional transducer. In the following lemma, we will construct such functional transducers using the boolean output alphabet $\mathbb{B}=\{0,1\}.$

- ▶ Lemma 22. Let $\mathcal{V} = \{y_1, \dots, y_m\}$. Given an FO formula φ with free variables contained in $\mathcal{V}' = \mathcal{V} \cup \{x\}$, we can construct a transducer $\mathcal{B}_{\varphi,\mathcal{V}}$ from $\Sigma_{\mathcal{V}}$ to \mathbb{B} which is aperiodic and unambiguous and such that for all words $\overline{w} \in \Sigma_{\mathcal{V}}^+$
- 1. there is a (unique) accepting run of $\mathcal{B}_{\varphi,\mathcal{V}}$ on the input word \overline{w} iff it is a valid encoding of a pair (w,σ) where $w \in \Sigma^+$ and $\sigma \colon \mathcal{V} \to \mathsf{pos}(w)$ is a valuation,
- **2.** and in this case, for all $1 \le i \le |w|$, the ith bit of the output is 1 iff $w, \sigma[x \mapsto i] \models \varphi$.

Proof. Notice that $\Sigma_{\mathcal{V}'} = \Sigma_{\mathcal{V}} \times \mathbb{B}$ so letters in $\Sigma_{\mathcal{V}'}$ are of the form $(\overline{a}, 0)$ or $(\overline{a}, 1)$ where $\overline{a} \in \Sigma_{\mathcal{V}}$. Abusing the notations, when $\overline{v} \in \Sigma_{\mathcal{V}}^*$, we write $(\overline{v}, 0)$ to denote the word over $\Sigma_{\mathcal{V}'}$ whose projection on $\Sigma_{\mathcal{V}}$ is \overline{v} and projection on the *x*-component consists of 0's only.

Consider the deterministic, complete and aperiodic automaton $\mathcal{A}_{\varphi,\mathcal{V}'}=(Q,\Sigma_{\mathcal{V}'},\Delta,\iota,F,G)$ associated with φ by Theorem 3. We also denote by Δ the extension of the transition function to subsets of Q. So we see the deterministic and complete transition relation both as a total function $\Delta\colon Q\times\Sigma_{\mathcal{V}'}\to Q$ and $\Delta\colon 2^Q\times\Sigma_{\mathcal{V}'}\to 2^Q$.

We construct now the transducer $\mathcal{B}_{\varphi,\mathcal{V}}=(Q',\Sigma_{\mathcal{V}},\Delta',\operatorname{wt},I',F')$. The set of states is $Q'=Q\times 2^Q\times 2^Q\times \mathbb{B}$. The unique initial state is $\iota'=(\iota,\emptyset,\emptyset,0)$. The set of final states is $F'=(Q\times 2^F\times 2^G\times \mathbb{B})\setminus\{\iota'\}$. Then, we define the following transitions:

- $\delta = ((p, X, Y, b), \overline{a}, (p', X', Y', 1)) \in \Delta' \text{ is a transition with weight } \mathsf{wt}(\delta) = 1 \text{ if }$ $p' = \Delta(p, (\overline{a}, 0)), \ X' = \Delta(X, (\overline{a}, 0)) \cup \{\Delta(p, (\overline{a}, 1))\} \text{ and } Y' = \Delta(Y, (\overline{a}, 0)),$
- $\delta = ((p, X, Y, b), \overline{a}, (p', X', Y', 0)) \in \Delta' \text{ is a transition with weight } \mathsf{wt}(\delta) = 0 \text{ if }$ $p' = \Delta(p, (\overline{a}, 0)), \ X' = \Delta(X, (\overline{a}, 0)) \text{ and } Y' = \Delta(Y, (\overline{a}, 0)) \cup \{\Delta(p, (\overline{a}, 1))\}.$

Notice that, whenever we read a new input letter $\overline{a} \in \Sigma_{\mathcal{V}}$, there is a non-deterministic choice. In the first case above, we guess that formula φ will hold on the input word when the valuation is extended by assigning x to the current position, whereas in the second case we guess that φ will not hold. The guess corresponds to the output of the transition, as required by the second condition of Lemma 22. Now, we have to check that the guess is correct. For this, the first component of $\mathcal{B}_{\varphi,\mathcal{V}}$ computes the state $p = \Delta(\iota,(\overline{u},0))$ reached by $\mathcal{A}_{\varphi,\mathcal{V}'}$ after reading $(\overline{u},0)$ where $\overline{u} \in \Sigma_{\mathcal{V}}^*$ is the current prefix of the input word. When reading the current letter $\overline{a} \in \Sigma_{\mathcal{V}}$, the transducer adds the state $\Delta(p,(\overline{a},1)) = \Delta(\iota,(\overline{u},0)(\overline{a},1))$ either to the "positive" X-component or to the "negative" Y-component of its state, depending on its guess as explained above. Then, the transducer continues reading the suffix $\overline{v} \in \Sigma_{\mathcal{V}}^*$ of the input word. It updates the X (resp. Y)-component so that it contains the state $q = \Delta(\iota,(\overline{u},0)(\overline{a},1)(\overline{v},0))$ at the end of the run. Now, the acceptance condition allows us to check that the guess was correct.

- 1. If $\overline{w} = \overline{uav}$ is not a valid encoding of a pair (w, σ) with $w \in \Sigma^+$ and $\sigma \colon \mathcal{V} \to \mathsf{pos}(w)$ then $q \notin F \cup G$ and the run of the transducer is not accepting. Otherwise, let $i \in \mathsf{pos}(w)$ be the position where the guess was made.
- 2. If the guess was positive then q belongs to the X-component and the accepting condition implies $q \in F$, which means by definition of $\mathcal{A}_{\varphi,\mathcal{V}'}$ that $w, \sigma[x \mapsto i] \models \varphi$.
- **3.** If the guess was negative then q belongs to the Y-component and the accepting condition implies $q \in G$, which means by definition of $\mathcal{A}_{\varphi,\mathcal{V}'}$ that $w, \sigma[x \mapsto i] \not\models \varphi$.

We continue the proof with several remarks.

First, since the automaton $\mathcal{A}_{\varphi,\mathcal{V}'}$ is complete, after reading a nonempty input word $\overline{w} \in \Sigma_{\mathcal{V}}^+$ the transducer cannot be back in its initial state $\iota' = (\iota, \emptyset, \emptyset, 0)$. This is because the second and third components of the state cannot both be empty. Since $\iota' \notin F'$, the support of the transducer consists of nonempty words only.

Second, consider a run of the transducer on some input word $\overline{w} \in \Sigma_{\mathcal{V}}^+$ from its initial state ι' to some state (p, X, Y, b). As explained above, one can check that $X \cup Y \subseteq F \cup G$ iff \overline{w} is a valid encoding of a pair (w, σ) . Therefore, the support of the transducer consists of valid encodings only.

Now, consider a valid encoding \overline{w} of a pair (w,σ) and consider a run ρ of $\mathcal{B}_{\varphi,\mathcal{V}}$ on \overline{w} from ι' to some state (p,X,Y,b). This run is entirely determined by the sequence of guesses made at every position of the input word. As explained above, one can check that all guesses are correct iff $X \subseteq F$ and $Y \subseteq G$. Therefore, $\mathcal{B}_{\varphi,\mathcal{V}}$ admits a unique accepting run on \overline{w} . This shows that the support of $\mathcal{B}_{\varphi,\mathcal{V}}$ is exactly the set of valid encodings, that this transducer is unambiguous, and that the last condition of the lemma holds, i.e., the *i*th bit of the output is 1 iff $w, \sigma[x \mapsto i] \models \varphi$.

To complete the proof, it remains to show that $\mathcal{B}_{\varphi,\mathcal{V}}$ is aperiodic. Let $m \geq 1$ be an aperiodicity index of $\mathcal{A}_{\varphi,\mathcal{V}}$. We claim that m' = 2m + 2|Q| is an aperiodicity index of $\mathcal{B}_{\varphi,\mathcal{V}}$. Let $\alpha = (p, X, Y, b)$ and $\alpha' = (p', X', Y', b')$ be two states of $\mathcal{B}_{\varphi,\mathcal{V}}$ and let $\overline{w} \in \Sigma_{\mathcal{V}}^+$ be a nonempty word.

Assume first that there is a run ρ of $\mathcal{B}_{\varphi,\mathcal{V}}$ from α to α' reading the input word \overline{w}^k with $k \geq 2m+1$. We show that there is another run of $\mathcal{B}_{\varphi,\mathcal{V}}$ from α to α' reading the input word \overline{w}^{k+1} . We split ρ in three parts: $\rho = \rho_1 \rho_2 \rho_3$ where ρ_1 reads the prefix \overline{w}^m , ρ_2 reads \overline{w} and ρ_3 reads the suffix \overline{w}^{k-m-1} . Consider the intermediary states $\alpha_i = (q_i, X_i, Y_i, b_i)$ reached after ρ_i $(1 \leq i \leq 3)$: $\alpha \xrightarrow{\rho_1} \alpha_1 \xrightarrow{\rho_2} \alpha_2 \xrightarrow{\rho_3} \alpha_3 = \alpha'$. Since $\mathcal{A}_{\varphi,\mathcal{V}'}$ is deterministic with aperiodicity index m we obtain $\Delta(p, (\overline{w}, 0)^m) = \Delta(p, (\overline{w}, 0)^{m+1}) = \Delta(p, (\overline{w}, 0)^k)$. Therefore, $q_1 = q_2 = q_3 = p'$.

Notice that, by definition of the transitions of $\mathcal{B}_{\varphi,\mathcal{V}}$, a run is entirely determined by its starting state, its input word, and the sequence of choices which is indicated in the fourth component of the states. Let ρ_2' be the run starting from α_2 , reading \overline{w} and following the same sequence of choices as ρ_2 . Let $\alpha_2' = (q_2', X_2', Y_2', b_2')$ be the state reached after ρ_2' . Let also ρ_3' be the run starting from α_2' , reading \overline{w}^{k-m-1} and following the same sequence of choices as ρ_3 . Let $\alpha_3' = (q_3', X_3', Y_3', b_3')$ be the state reached after ρ_3' . Thus, we obtain a run $\rho' = \alpha \xrightarrow{\rho_1} \alpha_1 \xrightarrow{\rho_2} \alpha_2 \xrightarrow{\rho_2'} \alpha_2' \xrightarrow{\rho_3'} \alpha_3'$ reading the input word \overline{w}^{k+1} . It remains to show that $\alpha_3' = \alpha_3$. As above, we have $q_3' = \Delta(p, (\overline{w}, 0)^{k+1}) = \Delta(p, (\overline{w}, 0)^k) = q_3$. Also, b_3' stores the last choice of ρ_3' , which is the same as the last choice of ρ_3 stored in b_3 and we get $b_3' = b_3$. It remains to show that $X_3' = X_3$ and $Y_3' = Y_3$. To this end, we introduce yet another variant of the runs ρ_2 and ρ_3 . Let ρ_2'' be the run starting from $(p', \emptyset, \emptyset, 0)$, reading \overline{w} and following the same sequence of choices as ρ_2 . Let $\alpha_2'' = (q_2'', X_2'', Y_2'', b_2'')$ be the state reached after ρ_2'' . It is easy to see that $q_2'' = q_2 = p'$ and $b_2'' = b_2$. Moreover, we have

$$X_{2} = X_{2}'' \cup \Delta(X_{1}, (\overline{w}, 0))$$

$$X_{2}' = X_{2}'' \cup \Delta(X_{2}, (\overline{w}, 0))$$

$$Y_{2} = Y_{2}'' \cup \Delta(Y_{1}, (\overline{w}, 0))$$

$$Y_{2}' = Y_{2}'' \cup \Delta(Y_{2}, (\overline{w}, 0)) .$$

Similarly, let ρ_3'' be the run starting from $(p', \emptyset, \emptyset, 0)$, reading \overline{w}^{k-m-1} and following the same sequence of choices as ρ_3 . Let $\alpha_3'' = (p', X_3'', Y_3'', b_3)$ be the state reached after ρ_3'' . We have

$$X_3 = X_3'' \cup \Delta(X_2, (\overline{w}, 0)^{k-m-1})$$

$$X_3' = X_3'' \cup \Delta(X_2', (\overline{w}, 0)^{k-m-1})$$

$$Y_3 = Y_3'' \cup \Delta(Y_2, (\overline{w}, 0)^{k-m-1})$$

$$Y_3' = Y_3'' \cup \Delta(Y_2', (\overline{w}, 0)^{k-m-1}) .$$

Notice that $k-m-1 \geq m$, hence we get $\Delta(X_2,(\overline{w},0)^{k-m-1}) = \Delta(X_2,(\overline{w},0)^{k-m})$ from the aperiodicity of $\mathcal{A}_{\varphi,\mathcal{V}'}$. Finally, using $X_2'' \subseteq X_2$, we obtain $\Delta(X_2',(\overline{w},0)^{k-m-1}) = \Delta(X_2,(\overline{w},0)^{k-m-1})$ and $X_3' = X_3$. Similarly, we prove that $Y_3' = Y_3$.

Conversely, we assume that there is a run ρ of $\mathcal{B}_{\varphi,\mathcal{V}}$ from α to α' reading the input word \overline{w}^k with k > m' = 2m + 2|Q|. We show that there is another run ρ' of $\mathcal{B}_{\varphi,\mathcal{V}}$ from α to α' reading the input word \overline{w}^{k-1} . We split ρ in 2|Q|+3 parts: $\rho = \rho_0 \rho_1 \cdots \rho_{2|Q|+1} \rho_{2|Q|+2}$ where ρ_0 reads the prefix $\overline{w}^{k-2|Q|-m-1}$, each ρ_i with $1 \le i \le 2|Q|+1$ reads \overline{w} , and $\rho_{2|Q|+2}$ reads the suffix \overline{w}^m . Consider the intermediary states $\alpha_i = (q_i, X_i, Y_i, b_i)$ reached after ρ_i $(0 \le i \le 2|Q|+2)$. We have

$$\alpha \xrightarrow{\rho_0} \alpha_0 \xrightarrow{\rho_1} \alpha_1 \cdots \alpha_{2|Q|+1} \xrightarrow{\rho_{2|Q|+2}} \alpha_{2|Q|+2} = \alpha'$$
.

Since $k-2|Q|-m-1 \ge m$ and $\mathcal{A}_{\varphi,\mathcal{V}'}$ is deterministic with aperiodicity index m, we deduce that $q_0 = q_1 = \cdots = q_{2|Q|+1} = q_{2|Q|+2} = p'$. As in the previous part of the aperiodicity proof, for each $1 \le i \le 2|Q|+2$, we consider the run ρ'_i starting form $(p',\emptyset,\emptyset,0)$, reading the same input word as ρ_i and making the same sequence of choices as ρ_i . Let $\alpha'_i = (p', X'_i, Y'_i, b_i)$ be the state reached after ρ'_i $(1 \le i \le 2|Q|+2)$. We have, for all $1 \le i \le 2|Q|+1$:

$$X_{i} = X'_{i} \cup \Delta(X_{i-1}, (\overline{w}, 0))$$

$$X_{2|Q|+2} = X'_{2|Q|+2} \cup \Delta(X_{2|Q|+1}, (\overline{w}, 0)^{m})$$

$$Y_{i} = Y'_{i} \cup \Delta(Y_{i-1}, (\overline{w}, 0))$$

$$Y_{2|Q|+2} = Y'_{2|Q|+2} \cup \Delta(Y_{2|Q|+1}, (\overline{w}, 0)^{m}) .$$

The states in $X' = X_{2|Q|+2}$ and $Y' = Y_{2|Q|+2}$ originate from the initial sets X_0 and Y_0 and from the sets X'_i and Y'_i created by the subruns ρ_i $(1 \le i \le 2|Q|+2)$. Intuitively, there is at least one index $1 \le i \le 2|Q|+1$ such that the contribution of ρ_i is subsumed by other subruns (formal proof below). Removing the subrun ρ_i yields the desired run ρ' of $\mathcal{B}_{\varphi,\mathcal{V}}$ from α to α' reading the input word \overline{w}^{k-1} (formal proof below).

For $0 \le i \le 2|Q|+1$, we let $k_i = 2|Q|+1-i+m$. For $1 \le i \le 2|Q|+2$, we define by descending induction on i the contributions X_i'' and Y_i'' to $X' = X_{2|Q|+2}$ and $Y' = Y_{2|Q|+2}$ which originate from subruns ρ_j with $j \ge i$:

$$\begin{split} X_{2|Q|+2}'' &= X_{2|Q|+2}' \\ Y_{2|Q|+2}'' &= Y_{2|Q|+2}' \\ \end{split} \qquad \qquad X_i'' &= X_{i+1}'' \cup \Delta(X_i', (\overline{w}, 0)^{k_i}) \\ Y_i'' &= Y_{i+1}'' \cup \Delta(Y_i', (\overline{w}, 0)^{k_i}) \,. \end{split}$$

We deduce easily that for all $1 \le i \le 2|Q| + 2$ we have

$$X_{2|Q|+2} = X_i'' \cup \Delta(X_{i-1}, (\overline{w}, 0)^{k_{i-1}}) \qquad Y_{2|Q|+2} = Y_i'' \cup \Delta(Y_{i-1}, (\overline{w}, 0)^{k_{i-1}}).$$

Let $1 \leq i \leq 2|Q|+1$ be such that $X_i'' = X_{i+1}''$ and $Y_i'' = Y_{i+1}''$. Using the monotonicity of the sequences, it is easy to see that such an index i must exist. We show that we can remove the subrun ρ_i . Let ρ'' be the run from α_{i-1} (and not α_i) which reads \overline{w}^{k_i} and makes the same sequence of choices as $\rho_{i+1}\cdots\rho_{2|Q|+2}$. Let $\alpha''=(q'',X'',Y'',b'')$ be the state reached after ρ'' . It is easy to see that $q''=q_{2|Q|+2}=p'$ and $b''=b_{2|Q|+2}=b'$. We show that $X''=X_{2|Q|+2}=X'$. Since ρ'' makes the same sequence of choices as $\rho_{i+1}\cdots\rho_{2|Q|+2}$, we see that the contribution to X'' coming from ρ'' is exactly X_{i+1}'' . Therefore,

$$X'' = X''_{i+1} \cup \Delta(X_{i-1}, (\overline{w}, 0)^{k_i}) = X''_i \cup \Delta(X_{i-1}, (\overline{w}, 0)^{k_{i-1}}) = X_{2|Q|+2} = X'$$

where the second equality follows from the hypothesis $X_i'' = X_{i+1}''$ and the aperiodicity of $\mathcal{A}_{\varphi,\mathcal{V}'}$ with index m since $k_{i-1} = k_i + 1 > m$. Similarly, we can prove that Y'' = Y' and we obtain $\alpha'' = \alpha'$. Therefore, $\rho' = \rho_0 \cdots \rho_{i-1} \rho''$ is the desired run of $\mathcal{B}_{\varphi,\mathcal{V}}$ from α to α' reading the input word \overline{w}^{k-1} . This concludes the proof of aperiodicity of $\mathcal{B}_{\varphi,\mathcal{V}}$ with index m' = 2|Q| + 2m.

▶ Theorem 23. Let $\mathcal{V} = \{y_1, \dots, y_m\}$. Given a step-wFO formula Ψ with free variables contained in $\mathcal{V}' = \mathcal{V} \cup \{x\}$, we can construct a weighted automaton $\mathcal{A}_{\Psi,\mathcal{V}}$ over $\Sigma_{\mathcal{V}}$ which is aperiodic and unambiguous and which is equivalent to $\prod_x \Psi$, i.e., such that $\{|\mathcal{A}_{\Psi,\mathcal{V}}|\}(\overline{w}) = \{\prod_x \Psi\}_{\mathcal{V}}(\overline{w})$ for all words $\overline{w} \in \Sigma_{\mathcal{V}}^+$.

Proof. In case $\Psi = r$ is an atomic step-wFO formula, we replace it with the equivalent T ? r : r step-wFO formula. Let $\varphi_1, \ldots, \varphi_k$ be the FO formulas occurring in Ψ . By the above remark, we have $k \geq 1$. Consider the aperiodic and unambiguous transducers $\mathcal{B}_1, \ldots, \mathcal{B}_k$ given by Lemma 22. For $1 \leq i \leq k$, we let $\mathcal{B}_i = (Q_i, \Sigma_{\mathcal{V}}, \Delta_i, \operatorname{wt}_i, I_i, F_i)$. The weighted automaton $\mathcal{A}_{\Psi,\mathcal{V}} = (Q, \Sigma_{\mathcal{V}}, \Delta, \operatorname{wt}_i, I, F)$ is essentially a cartesian product of the transducers \mathcal{B}_i . More precisely, we let $Q = \prod_{i=1}^k Q_i, I = \prod_{i=1}^k I_i, F = \prod_{i=1}^k F_i$, and

$$\Delta = \{((p_1, \dots, p_k), \overline{a}, (q_1, \dots, q_k)) \mid (p_i, \overline{a}, q_i) \in \Delta_i \text{ for all } 1 \le i \le k\}.$$

Since the transducers \mathcal{B}_i are all aperiodic and unambiguous, we deduce by Lemma 16 that $\mathcal{A}_{\Psi,\mathcal{V}}$ is also aperiodic and unambiguous. It remains to define the weight function wt.

Given a bit vector $\bar{b} = (b_1, \dots, b_k) \in \mathbb{B}^k$ of size k, we define $\Psi(\bar{b})$ as the weight from R resulting from the step-wFO formula Ψ when the FO conditions $\varphi_1, \dots, \varphi_k$ evaluate to \bar{b} . Formally, the definition is by structural induction on the step-wFO formula:

$$r(\overline{b}) = r \qquad \qquad (\varphi_i \, ? \, \Psi_1 : \Psi_2)(\overline{b}) = \begin{cases} \Psi_1(\overline{b}) & \text{if } b_i = 1 \\ \Psi_2(\overline{b}) & \text{if } b_i = 0 \, . \end{cases}$$

Consider a transition $\delta = ((p_1, \dots, p_k), \overline{a}, (q_1, \dots, q_k)) \in \Delta$ and let $\delta_i = (p_i, \overline{a}, q_i)$ for $1 \le i \le k$. Let $\overline{b} = (b_1, \dots, b_k) \in \mathbb{B}^k$ where $b_i = \mathsf{wt}(\delta_i) \in \mathbb{B}$ for all $1 \le i \le k$. We define $\mathsf{wt}(\delta) = \Psi(\overline{b})$.

Let $\overline{w} \in \Sigma_{\mathcal{V}}^+$. If \overline{w} is not a valid encoding of a pair (w,σ) then $\{\|\prod_x \Psi\|\}_{\mathcal{V}}(\overline{w}) = \emptyset$ by definition. Moreover, $\{\|\mathcal{A}_{\Psi,\mathcal{V}}\|\}_{(\overline{w})} = \emptyset$ since by Lemma 22, \overline{w} is not in the support of \mathcal{B}_1 . We assume below that \overline{w} is a valid encoding of a pair (w,σ) where $w \in \Sigma^+$ and $\sigma \colon \mathcal{V} \to \mathsf{pos}(w)$ is a valuation. Then, each transducer \mathcal{B}_i admits a unique accepting run ρ_i reading the input word \overline{w} . These result in the unique accepting run ρ of $\mathcal{A}_{\Psi,\mathcal{V}}$ reading \overline{w} . The projections of ρ on $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are ρ_1, \ldots, ρ_k . Let $j \in \mathsf{pos}(w) = \{1, \ldots, |w|\}$ be a position in \overline{w} and let δ^j be the j-th transition of ρ . For $1 \leq i \leq k$, we denote by δ_i^j the projection of δ^j on \mathcal{B}_i and we let $b_i^j = \mathsf{wt}(\delta_i^j)$. By Lemma 22, we get $b_i^j = 1$ iff $w, \sigma[x \mapsto j] \models \varphi_i$. Finally, let $\overline{b}^j = (b_1^j, \ldots, b_k^j)$. From the above, we deduce that $\|\Psi\|_{\mathcal{V} \cup \{x\}}(w, \sigma[x \mapsto j]) = \Psi(\overline{b}^j) = \mathsf{wt}(\delta^j)$. Putting things together, we have

$$\{|\mathcal{A}_{\Psi,\mathcal{V}}|\}(w,\sigma)=\{\{\mathsf{wt}(\rho)\}\}=\{\{\mathsf{wt}(\delta^1)\cdots\mathsf{wt}(\delta^{|w|}\}\}=\{|\prod_x\!\Psi|\}_{\mathcal{V}}(w,\sigma)\,. \qquad \blacktriangleleft$$

▶ Theorem 24. Let Φ be a wFO sentence. We can construct an aperiodic SCC-unambiguous weighted automaton \mathcal{A} such that $\{|\mathcal{A}|\} = \{|\Phi|\}$. Moreover, if Φ does not contain the sum operations + and \sum_x , then \mathcal{A} can be chosen to be unambiguous. If Φ does not contain the sum operation \sum_x , we can construct \mathcal{A} as a finite union of unambiguous weighted automata.

Proof. We proceed by structural induction on Φ . For $\Phi = \mathbf{0}$ this is trivial. For $\Phi = \prod_x \Psi$ with a step-wFO formula Ψ , we obtain an aperiodic unambiguous weighted automaton \mathcal{A} by Theorem 23. For formulas $\varphi ? \Phi_1 : \Phi_2$, $\Phi_1 + \Phi_2$ and $\sum_x \Phi$, we apply Lemmas 17, 16 and 18, respectively.

In the proof of Theorem 24, we may obtain the final statement also as a consequence of the preceding one by the following observations which could be of independent interest. Let φ be an FO-formula and Φ_1 , Φ_2 two wFO formulas, each with free variables contained in \mathcal{V} . Then,

$$\{ |\varphi? \Phi_1 : \Phi_2 \}_{\mathcal{V}} = \{ |\varphi? \Phi_1 : \mathbf{0} + \neg \varphi? \Phi_2 : \mathbf{0} \}_{\mathcal{V}},$$

$$\{ |\varphi? \Phi_1 + \Phi_2 : \mathbf{0} \}_{\mathcal{V}} = \{ |\varphi? \Phi_1 : \mathbf{0} + \varphi? \Phi_2 : \mathbf{0} \}_{\mathcal{V}}.$$

Hence, given a wFO sentence Φ not containing the sum operation \sum_x , we can rewrite Φ as a sum of $\mathbf{0}$, $\prod_x \Psi$ and if-then-else sentences of the form $\varphi ? \Phi' : \mathbf{0}$ where Φ' does not contain the sum operations + or \sum_x .

Proof of Thm 1. Immediate by Theorem 13, Theorem 4, Corollary 11 and Theorem 24. ◀

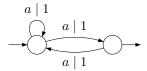
8 Examples

In this section, we give examples separating the classes of finitely, polynomially and exponentially ambiguous aperiodic weighted automata for several weight structures including the semiring of natural numbers $\mathbb{N}_{+,\times}$, the max-plus semiring $\mathbb{N}_{\max,+}$ and the min-plus semiring $\mathbb{N}_{\min,+}$.

- ▶ **Example 25.** Let Σ be any alphabet, R a set of weights, and $\mathcal{A} = (Q, \Sigma, \Delta, \mathsf{wt}, I, F)$ any (possibly aperiodic) weighted automaton over Σ and R which is not polynomially ambiguous.
- 1. Since the size of the multisets $\{|\mathcal{A}|\}(w)$ is not polynomially bounded with respect to |w|, there can be no polynomially ambiguous weighted automaton \mathcal{B} with $\{|\mathcal{A}|\} = \{|\mathcal{B}|\}$.
- 2. Assume that $|\Delta| \leq |R|$ and all transitions of \mathcal{A} have different weights, and consider \mathcal{A} as a weighted automaton over the semiring $(\mathcal{P}_{fin}(R^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, or, equivalently, as a non-deterministic transducer outputting the weights of the transitions. Again, there can be no polynomially ambiguous weighted automaton \mathcal{B} with $||\mathcal{A}|| = ||\mathcal{B}||$.
- 3. For each $q \in Q$ and $a \in \Sigma$, the transitions $\delta = (q, a, p) \in \Delta$ $(p \in Q)$ are enumerated as $\delta_1, \ldots, \delta_m$ where m is the degree of non-determinism for $q \in Q$ and $a \in \Sigma$. Then put $\mathsf{wt}(\delta_i) = i$, and let R comprise all these numbers. In comparison to 2., $|\mathsf{R}|$ might be considerably smaller than $|\Delta|$. But, again, over the semiring $(\mathcal{P}_{fin}(\mathsf{R}^*), \cup, \emptyset, \{\varepsilon\})$ there is no polynomially ambiguous weighted automaton equivalent to \mathcal{A} .

This shows that for suitable idempotent semirings and also for non-deterministic transducers, there are aperiodic weighted automata for which there is no equivalent polynomially ambiguous weighted automaton. Next we show that this is also the case for the semiring of natural numbers $\mathbb{N}_{+,\times}$, the max-plus semiring $\mathbb{N}_{\max,+}$ and the min-plus semiring $\mathbb{N}_{\min,+}$.

▶ **Example 26.** Let $\Sigma = \{a\}$ and consider the automaton \mathcal{A} below over the semiring $\mathbb{N}_{+,\times}$ of natural numbers.



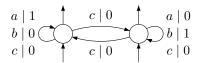
Note that the weighted automaton computes the sequence $(F_n)_{n\geq 0}$ of Fibonacci numbers $0, 1, 1, 2, 3, 5, \cdots$. More precisely, for any $n \in \mathbb{N}$, we have $[\![A]\!](a^n) = F_n$.

Clearly, \mathcal{A} is exponentially ambiguous and aperiodic with index 2. In [30], it was shown that the Fibonacci numbers cannot be computed by copyless cost-register automata. Here, we prove that there is no aperiodic polynomially ambiguous weighted automaton $\mathcal{B} = (Q, \Sigma, \Delta, \mathsf{wt}, I, F)$ with $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$. Suppose there was such a trimmed automaton \mathcal{B} .

First, consider any loop $q \xrightarrow{a^k} q$ with $k \geq 1$ of \mathcal{B} . Since \mathcal{B} is aperiodic and SCC-unambiguous, hence unambiguous on the component containing q, as in Example 27, it follows that $(q, a, q) \in \Delta$. Next, we claim $\alpha = \mathsf{wt}(q, a, q) = 1$. Indeed, suppose that $\alpha \geq 2$. Choose $m, \ell \geq 2$ minimal such that there is a path reading a^m from I to q and a path for a^ℓ from q to F. Considering, for $n \geq m + \ell$, the path $\rho_n \colon I \xrightarrow{a^m} q \xrightarrow{a^{n-m-\ell}} q \xrightarrow{a^\ell} F$, we obtain $[\![\mathcal{B}]\!](a^n) \geq \mathsf{wt}(\rho_n) \geq 2^{n-m-\ell}$. Since $F_n = o(2^n)$, for n large enough, we get $F_n < 2^{-m-\ell} \cdot 2^n$, a contradition.

So, in \mathcal{B} all loops have weight 1. Hence there exists $K \in \mathbb{N}$ such that $\mathsf{wt}(\rho) \leq K$ for all paths ρ in \mathcal{B} . Consequently, if \mathcal{B} is polynomially ambiguous of degree d, we have $[\![\mathcal{B}]\!](a^n) \leq O(n^d)$ for $n \in \mathbb{N}$. This yields a contradiction since $F_n \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$ grows exponentially.

▶ Example 27. Let $\Sigma = \{a, b, c\}$ and consider the function $f_{\max} \colon \Sigma^* \to \mathbb{N}$ defined as follows. For a word $w = w_0 c w_1 c \dots c w_n$ with $w_0, \dots, w_n \in \{a, b\}^*$, we let $f_{\max}(w) = \sum_{i=0}^n \max\{|w_i|_a, |w_i|_b\}$. Over the max-plus semiring $\mathbb{N}_{\max,+}$, this function is realized by the automaton \mathcal{A} below.



Notice that \mathcal{A} is aperiodic and not polynomially ambiguous. We show that f_{max} cannot be realized over the max-plus semiring by a polynomially ambiguous and aperiodic weighted automaton.

Notice that a similar automaton was considered in [25], the only difference being that c-transitions have weight 1. It was shown that the corresponding series cannot be realized over $\mathbb{N}_{\max,+}$ by a finitely ambiguous weighted automaton, be it aperiodic or not. Here we want to separate exponentially ambiguous from polynomially ambiguous. We prove this separation for aperiodic automata which makes some of the arguments in the proof simpler (essentially we have self-loops instead of cycles). The separation also holds if we drop aperiodicity.

Towards a contradiction, assume that there was a polynomially ambiguous and aperiodic weighted automaton $\mathcal{B} = (Q, \Sigma, \Delta, \mathsf{wt}, I, F)$ which realizes the function f_{\max} . We assume \mathcal{B} to be trimmed. We start with some easy remarks.

- 1. If there is a cycle $p \xrightarrow{u^k} p$ in \mathcal{B} with $u \in \Sigma^+$ and $k \ge 1$ then $p \xrightarrow{u} p$. Let $m \ge 1$ be the aperiodicity index of \mathcal{B} . For $\ell k \ge m$ we have $u^{\ell k}, u^{\ell k + 1} \in \mathcal{L}(\mathcal{B}_{p,p})$. Since \mathcal{B} is polynomially ambiguous, these cycles lie in some SCC which is unambiguous. If the cycle around p reading $u^{\ell k}$ is not a prefix of the cycle reading $u^{\ell k + 1}$ then we have two different cycles reading $u^{\ell k (\ell k + 1)}$, a contradiction. Therefore, the cycle reading $u^{\ell k + 1}$ is $p \xrightarrow{u^{\ell k}} p \xrightarrow{u} p$.
- 2. Consider a looping transition $\delta = (p, v, p)$ in \mathcal{B} with $v \in \Sigma$. Then, $\mathsf{wt}(\delta) \in \{0, 1\}$. Since \mathcal{B} is trimmed, there is an accepting run $p_1 \stackrel{u}{\longrightarrow} p \stackrel{w}{\longrightarrow} p_2$ with $|uw| \leq 2|Q|$. We deduce that for all $\ell \geq 0$ there is an accepting run reading $uv^{\ell}w$ with weight at least $\mathsf{wt}(\delta) \cdot \ell$. Since $f_{\max}(uv^{\ell}w) \leq \ell + |uw|$, we deduce that $\mathsf{wt}(\delta) \in \{0, 1\}$.
- **3.** If there is a path $p \xrightarrow{a} p \xrightarrow{v} q \xrightarrow{b} q$ in \mathcal{B} with $v \in \{a,b\}^*$, then one of the two looping transitions has weight zero: $\mathsf{wt}(p,a,p) = 0$ or $\mathsf{wt}(q,b,q) = 0$. Since \mathcal{B} is trimmed, there are two runs $p_1 \xrightarrow{u} p$ and $q \xrightarrow{w} p_2$ with $p_1 \in I$ initial, $p_2 \in F$ final and $|uw| \leq 2|Q|$. We deduce that for all $\ell \geq 0$ there is an accepting run reading

 $ua^{\ell}vb^{\ell}w$ with weight at least $\ell \cdot (\mathsf{wt}(p,a,p) + \mathsf{wt}(q,b,q))$. Since $f_{\max}(ua^{\ell}vb^{\ell}w) \leq \ell + |uvw|$, we deduce that $\mathsf{wt}(p,a,p) + \mathsf{wt}(q,b,q) \leq 1$.

Let n = |Q| be the number of states in B. We show below that for each $k \ge 1$, the word $w_k = a^n n^n (ca^n b^n)^{k-1}$ admits at least 2^k accepting runs in \mathcal{B} . This implies that \mathcal{B} is not polynomially ambiguous, a contradiction.

Let $M = \max(\mathsf{wt}(\Delta))$ be the maximal weight used in \mathcal{B} . Notice that $M \geq 1$. Fix $k \geq 1$ and let $N \geq 2knM$. Define $u_0 = a^Nb^n$ and $u_1 = a^nb^N$. For each word $x = x_1 \cdots x_k \in \{0, 1\}^k$, define $w_x = u_{x_1}cu_{x_2}c\cdots cu_{x_k}$ and consider an accepting run ρ_x of \mathcal{B} reading w_x and realizing $f_{\max}(w_x) = kN$. For each $1 \leq j \leq k$, we focus on the subrun ρ_x^j of ρ_x reading u_{x_j} .

Assume that $x_j = 0$. Using the remarks above, we deduce that the prefix of ρ_x^j reading a^N is of the form

$$p_1 \xrightarrow{a^{\ell_1}} p_1 \xrightarrow{a} p_2 \xrightarrow{a^{\ell_2}} p_2 \xrightarrow{a} \cdots \xrightarrow{a} p_m \xrightarrow{a^{\ell_m}} p_m$$
 (3)

where p_1, \ldots, p_m are pairwise distinct and $N=m-1+\ell_1+\cdots+\ell_m$. Since looping a-transitions have weights in $\{0,1\}$, we deduce that $\operatorname{wt}(\rho_x^j) \leq N + (2n-1)M$. We claim that in ρ_x^j , some a-loop has weight 1. If this is not the case, then $\operatorname{wt}(\rho_x^j) \leq (2n-1)M$. We deduce that $\operatorname{wt}(\rho_x) \leq (k-1)(N+(2n-1)M)+(2n-1)M+(k-1)M=(k-1)N+(2nk-1)M$, but $\operatorname{wt}(\rho_x) = kN = f_{\max}(w_x)$, a contradiction with $N \geq 2knM$. Let (p_i, a, p_i) be some a-loop of weight 1 in ρ_x^j . We replace the prefix of ρ_x^j reading a^N with

$$p_1 \xrightarrow{a^{i-1}} p_i \xrightarrow{a^{n-m+1}} p_i \xrightarrow{a^{m-i}} p_m$$

to obtain a run $\hat{\rho}_x^j$ reading $a^n b^n$. The suffix of ρ_x^j reading b^n has a form similar to (3), having at least one b-loop since n = |Q|. From the third remark above, all b-loops in ρ_x^j have weight 0. We deduce that $\hat{\rho}_x^j$ has one a-loop with weight 1 but all its b-loops have weight 0.

We proceed similarly when $x_j = 1$ defining a run $\hat{\rho}_x^j$ reading $a^n b^n$ where all a-loops have weight 0 and one b-loop has weight 1. Now, consider the run $\hat{\rho}_x$ obtained from ρ_x by replacing ρ_x^j with $\hat{\rho}_x^j$ for each $1 \leq j \leq k$. We see that $\hat{\rho}_x$ is an accepting run for w_k . Also, if $x, y \in \{0, 1\}^k$ are different then $\hat{\rho}_x \neq \hat{\rho}_y$. Therefore, \mathcal{B} has at least 2^k accepting runs reading w_k , which concludes the proof.

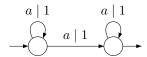
▶ Example 28. Let $\Sigma = \{a, b, c\}$ and consider the function $f_{\min} \colon \Sigma^* \to \mathbb{N}$ defined as follows. For a word $w = w_0 c w_1 c \dots c w_n$ with $w_0, \dots, w_n \in \{a, b\}^*$, we let $f_{\min}(w) = \sum_{i=0}^n \min\{|w_i|_a, |w_i|_b\}$. Over the min-plus semiring $\mathbb{N}_{\min,+}$, this function is realized by the automaton \mathcal{A} depicted in Example 27 which is aperiodic and not polynomially ambiguous. It was shown in [29], that in the min-plus semiring there is no polynomially ambiguous weighted automaton \mathcal{B} with $\|\mathcal{A}\| = \|\mathcal{B}\|$.

Next we wish to show that aperiodic polynomially ambiguous weighted automata are strictly more expressive than aperiodic finitely ambiguous weighted automata.

▶ Example 29. Let Σ be any alphabet, R a set of weights and \mathcal{A} an aperiodic polynomially ambiguous weighted automaton which is not finitely ambiguous. We may argue as in Example 25 to show that there is no finitely ambiguous weighted automaton \mathcal{B} with $\{A\} = \{B\}$, respectively, under the assumptions of Example 25, with $\{A\} = \{B\}$ for the idempotent semiring $(\mathcal{P}_{fin}(\mathsf{R}^*), \cup, \cdot, \emptyset, \{\varepsilon\})$.

We show that this is also the case for the semiring of natural numbers $\mathbb{N}_{+,\times}$, the max-plus semiring $\mathbb{N}_{\max,+}$ and the min-plus semiring $\mathbb{N}_{\min,+}$.

Example 30. Consider the following automaton \mathcal{A} over $\Sigma = \{a\}$ and the semiring $\mathbb{N}_{+,\times}$.



Clearly, $[\![A]\!](a^n) = n$ for each n > 0, and A is aperiodic and polynomially (even linearly) ambiguous. But A is not equivalent to any finitely ambiguous weighted automaton.

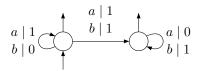
Towards a contradiction, suppose there was a trimmed finitely ambiguous weighted automaton \mathcal{B} with $\llbracket \mathcal{B} \rrbracket = \llbracket \mathcal{A} \rrbracket$.

▶ Remark. Let $q \xrightarrow{a^m} q$ be a loop in \mathcal{B} with weight α , where $m \geq 1$. Then $\alpha = 1$.

Indeed, choose a path in \mathcal{B} from I to q with label u and a path from q to F with label v. Then $[\![\mathcal{B}]\!](ua^{mn}v) \geq \alpha^n$, for each $n \in \mathbb{N}$. On the other hand, $f(ua^{mn}v) = |uv| + m \cdot n$. Hence $\alpha \geq 2$ is impossible, showing $\alpha = 1$.

Consequently, in paths of \mathcal{B} we may remove all loops without changing the weight. Hence there is $C \in \mathbb{N}$ such that $\mathsf{wt}(\rho) \leq C$ for each run ρ of \mathcal{B} . Since \mathcal{B} is finitely ambiguous, it follows that $\{\llbracket \mathcal{B} \rrbracket(w) \mid w \in \Sigma^*\}$ is bounded. This contradicts $\llbracket \mathcal{B} \rrbracket = \llbracket \mathcal{A} \rrbracket$.

Example 31. Consider the following automaton \mathcal{A} over $\Sigma = \{a, b\}$ and $\mathbb{N}_{\max, +}$.



Note that \mathcal{A} is almost identical to the automaton of Example 33-2, used for $\mathbb{N}_{\min,+}$ in [29]. Now for $f = [\![\mathcal{A}]\!]$ we have $f(w) = \max\{|u|_a + |v|_b \mid w = uv\}$ for each $w \in \Sigma^+$. Clearly, \mathcal{A} is aperiodic and polynomially ambiguous. Now, we show that no aperiodic finitely ambiguous weighted automaton is equivalent to \mathcal{A} over $\mathbb{N}_{\max,+}$.

Suppose there was a trimmed weighted automaton $\mathcal{B} = (Q, \Sigma, \Delta, \mathsf{wt}, I, F)$ both aperiodic and finitely ambiguous, and with $\llbracket \mathcal{B} \rrbracket = f$. We make the following observations on the structure of \mathcal{B} .

▶ Remark 1. If \mathcal{B} contains a loop $q \xrightarrow{a^k} q$ for some $q \in Q$ and $k \geq 1$, then $t = (q, a, q) \in \Delta$, and the loop is a sequence of this transition t.

This follows from the fact that \mathcal{B} is aperiodic and unambiguous on the strong component containing q (as in Example 27).

- ▶ Remark 2. \mathcal{B} cannot contain a path of the form $p \xrightarrow{a} p \xrightarrow{a^k} q \xrightarrow{a} q$ with $p \neq q$. Indeed, otherwise the word a^{n+k} would have at least n+1 different paths from p to q. Since \mathcal{B} is trimmed, this contradicts the finite ambiguity of \mathcal{B} .
- ▶ Remark 3. If $(q, a, q) \in T$ and $\alpha = \mathsf{wt}(q, a, q)$, then $\alpha \in \{0, 1\}$. Indeed, let u be the label of a path from I to q and v the label of a path from q to F. Let $w_n = ua^n v$. Then $f(w_n) \leq |uv| + n$, and $[\![\mathcal{B}]\!](w_n) \geq \alpha \cdot n$ for each $n \in \mathbb{N}$. This shows that $\alpha \leq 1$.
- Remark 4. \mathcal{B} cannot contain a path of the form $p \xrightarrow{b|1} p \xrightarrow{v} q \xrightarrow{a|1} q$ with $v \in \Sigma^*$. Indeed, otherwise let u be a label of a path from I to p and w the label of a path from q to F. Consider $w_n = ub^n va^n w$ $(n \in \mathbb{N})$. Then $f(w_n) \leq |uvw| + n$ but $[\![\mathcal{B}]\!](w_n) \geq 2n$, a contradiction for n > |uvw|.

▶ Lemma 32. Let $m \ge |Q|$ and $u, v \in \Sigma^*$. Then \mathcal{B} contains an accepting path for the word ua^mb^mv of the form

with $k_1, k_2, k_3, k_4 < |Q|$.

Proof. Let $n \ge m$ and $w_n = ua^nb^nv$. Then $f(w_n) \ge 2n$. Consider a path ρ for w_n in \mathcal{B} with $\mathsf{wt}(\rho) = f(w_n)$. The subpath of ρ realizing a^n must contain at least one a-loop, and by Remarks 1 and 2 it contains exactly one a-loop which is a power of a single transition.

Hence ρ has the form

$$\begin{array}{c|c}
a \mid \alpha & b \mid \beta \\
\hline
 & a^{k_1} \\
\hline
 & p \\
\end{array}$$

$$\begin{array}{c|c}
a^{k_2}b^{k_3} \\
\hline
 & q \\
\end{array}$$

$$\begin{array}{c|c}
b^{k_4}v \\
\hline
 & f \\
\end{array}$$

with $k_1, k_2, k_3, k_4 < |Q|$, and where the transition (p, a, p) is taken $n - k_1 - k_2$ times and the transition (q, b, q) is taken $n - k_3 - k_4$ times in ρ .

By Remark 3, we have $\alpha, \beta \in \{0, 1\}$. Let $\rho_1 \rho_2 \rho_3$ be the path obtained from ρ by deleting the loops at p and at q: $\rho_1 = i \xrightarrow{ua^{k_1}} p$, $\rho_2 = p \xrightarrow{a^{k_2}b^{k_3}} q$, and $\rho_3 = q \xrightarrow{b^{k_4}v} f$. Let $c = \mathsf{wt}(\rho_1 \rho_2 \rho_3)$. Then $\mathsf{wt}(\rho) \leq c + n \cdot \alpha + n \cdot \beta$.

But $\operatorname{wt}(\rho) = f(w_n) \geq 2n$. Since $u, v \in \Sigma^*$ are fixed, there are only finitely many values $c = \operatorname{wt}(\rho_1 \rho_2 \rho_3) \in \mathbb{N}$ which can arise in \mathcal{B} as above with $i, p, q, f \in Q$ and $k_1, k_2, k_3, k_4 < |Q|$,. By choosing n larger than their maximum, we obtain a path for $w_n = ua^n b^n v$ as above and now for this path it follows that $\alpha = \beta = 1$. By reducing the number of loops taken at p and at q, we obtain an accepting path of the prescribed form for $w_m = ua^m b^m v$, proving the lemma.

Now, let $m \ge |Q|$ and consider the word $w_K = (b^m a^m)^K$ $(K \in \mathbb{N})$. For all 0 < k < K we can write $w_K = u_k a^m b^m v_k$ with $u_k = (b^m a^m)^{k-1} b^m$ and $v_k = a^m (b^m a^m)^{K-k-1}$. We apply Lemma 32 to the word $u_k a^m b^m v_k$ and obtain a path ρ_k of the form

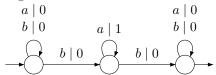
We claim that if 0 < k < k' < K, then $\rho_k \neq \rho_{k'}$. Indeed, if $\rho_k = \rho_{k'}$, we see that the path $\rho_{k'}$ must have the form

contradicting Remark 4. Therefore \mathcal{B} contains at least K-1 accepting paths for w_K ($K \in \mathbb{N}$). This contradicts \mathcal{B} being finitely ambiguous.

We just note that by similar arguments and further analysing the weights of loops, it can be shown that \mathcal{A} is not equivalent to any finitely ambiguous weighted automaton, even if it is not aperiodic.

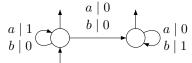
▶ Example 33. Let $\Sigma = \{a, b\}$.

1. Consider the following weighted automaton \mathcal{A} over Σ and $\mathbb{N}_{\min,+}$ from [24, p.558]:



Here $[\![\mathcal{A}]\!](w)$ is the least $\ell \geq 0$ such that $ba^{\ell}b$ is a factor of w. If w does not admit a factor of this form, than $[\![\mathcal{A}]\!](w) = \infty$. Clearly, \mathcal{A} is SCC-unambiguous and aperiodic, but, as shown in [24, Proposition 3.2], \mathcal{A} is not equivalent to any finitely ambiguous weighted automaton.

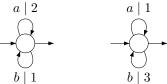
2. Consider the following weighted automaton \mathcal{A} over Σ and $\mathbb{N}_{\min,+}$ from [29]:



Then $[\![\mathcal{A}]\!](w) = \min\{|u|_a + |v|_b \mid w = uv\}$. Clearly, \mathcal{A} is aperiodic and polynomially ambiguous. As shown in [29, Example 15], as a consequence of a pumping lemma, \mathcal{A} is not equivalent to any finitely ambiguous weighted automaton.

Finally, we wish to show that aperiodic finitely ambiguous weighted automata are strictly more expressive than aperiodic unambiguous weighted automata. Clearly, this can be derived for the idempotent semiring $(\mathcal{P}_{fin}(R^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ as in Examples 25 and 29. We show that this is also the case for the semirings $\mathbb{N}_{+,\times}$, $\mathbb{N}_{max,+}$ and $\mathbb{N}_{min,+}$.

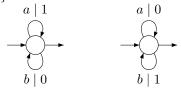
▶ **Example 34.** Let $\Sigma = \{a, b\}$ and consider the automaton \mathcal{A} below over the semiring $\mathbb{N}_{+,\times}$ of natural numbers.



Clearly, \mathcal{A} is a periodic and 2-ambiguous, and $[\![\mathcal{A}]\!](w) = 2^{|w|_a} + 3^{|w|_b}$ for each $w \in \Sigma^*$. We show that no unambiguous weighted automaton is equivalent to \mathcal{A} .

Suppose there was an unambiguous weighted automaton $\mathcal B$ with n states and with $[\![\mathcal B]\!] = [\![\mathcal A]\!]$. Consider $w = a^{n+2}b$. There is a unique successful path in $\mathcal B$ for w, having weight $[\![\mathcal B]\!](w) = [\![\mathcal A]\!](w) = 2^{n+2} + 3$. Then this path contains an a-loop ρ of length $m \le n$ and with $\operatorname{wt}(\rho) = C \in \mathbb N$. We have $[\![\mathcal A]\!](a^{n+m+2}b) = 2^{n+m+2} + 3$ and $[\![\mathcal B]\!](a^{n+m+2}b) = C \cdot [\![\mathcal B]\!](w) = C \cdot (2^{n+2} + 3)$. So $2^{n+m+2} + 3 = C \cdot (2^{n+2} + 3)$. Then $C < 2^m$. But $(2^m - 1) \cdot (2^{n+2} + 3) < 2^{n+m+2} + 3$ as $m \le n$, a contradiction.

Example 35. Let $\Sigma = \{a, b\}$ and consider the automaton \mathcal{A} below with weights in \mathbb{N} .



Clearly, \mathcal{A} is aperiodic and 2-ambiguous. Over the semiring $\mathbb{N}_{\max,+}$ we have $[\![\mathcal{A}]\!]_{\max}(w) = \max\{|w|_a,|w|_b\}$, and over the semiring $\mathbb{N}_{\min,+}$ we have $[\![\mathcal{A}]\!]_{\min}(w) = \min\{|w|_a,|w|_b\}$. As shown in [25, p.255], resp. [29, Example 8], in both cases there is no unambiguous weighted automaton equivalent to \mathcal{A} .

9 Conclusion

We introduced a model of aperiodic weighted automata and showed that a suitable concept of weighted first order logic and two natural sublogics have the same expressive power as polynomially ambiguous, finitely ambiguous, resp. unambigous aperiodic weighted automata. For the three semirings $\mathbb{N}_{+,\times}$, $\mathbb{N}_{\max,+}$ and $\mathbb{N}_{\min,+}$ we showed that the hierarchies of these automata classes and thereby of the corresponding logics are strict.

Our main theorem generalizes to the weighted setting a classical result of automata theory. A challenging open problem is to obtain similar results for suitable weighted linear temporal logics. Another interesting problem is to characterize wFO with unrestricted weighted products, possibly using *aperiodic* restrictions of the pebble weighted automata studied in [4, 26, 5].

Decidability problems for wFO or equivalently for weighted aperiodic automata are also open and very interesting. For instance, given a wMSO sentence, is there an equivalent wFO sentence? Decidability may indeed depend on the specific semiring.

References

- 1 Parvaneh Babari, Manfred Droste, and Vitaly Perevoshchikov. Weighted register automata and weighted logic on data words. *Theor. Comput. Sci.*, 744:3–21, 2018.
- 2 Jean Berstel and Christophe Reutenauer. Rational Series and their Languages. Springer, 1988.
- 3 Benedikt Bollig and Paul Gastin. Weighted versus probabilistic logics. In Volker Diekert and Dirk Nowotka, editors, *International Conference on Developments in Language Theory* (DLT'09), volume 5583 of Lecture Notes in Computer Science, pages 18–38. Springer, 2009.
- 4 Benedikt Bollig, Paul Gastin, Benjamin Monmege, and Marc Zeitoun. Pebble weighted automata and transitive closure logics. In *International Colloquium on Automata, Languages and Programming (ICALP'10)*, volume 6199 of *Lecture Notes in Computer Science*, pages 587–598. Springer, 2010.
- 5 Benedikt Bollig, Paul Gastin, Benjamin Monmege, and Marc Zeitoun. Pebble weighted automata and weighted logics. ACM Transactions on Computational Logic, 15(2):1–35, 2014.
- 6 J. Richard Büchi. Weak second-order arithmetic and finite automata. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 6:66–92, 1960.
- 7 Krishnendu Chatterjee, Laurent Doyen, and Thomas A. Henzinger. Expressiveness and closure properties for quantitative languages. *Logical Methods in Computer Science*, 6(3), 2010.
- 8 Krishnendu Chatterjee, Laurent Doyen, and Thomas A. Henzinger. Quantitative languages. *ACM Trans. Comput. Log.*, 11(4):23:1–23:38, 2010.
- 9 Volker Diekert and Paul Gastin. First-order definable languages. In Jörg Flum, Erich Grädel, and Thomas Wilke, editors, *Logic and Automata: History and Perspectives*, volume 2 of *Texts in Logic and Games*, pages 261–306. Amsterdam University Press, 2008.
- 10 Manfred Droste and Stefan Dück. Weighted automata and logics on graphs. In *Mathematical Foundations of Computer Science (MFCS'15)*, volume 9234 of *Lecture Notes in Computer Science*, pages 192–204. Springer, 2015.
- Manfred Droste and Stefan Dück. Weighted automata and logics for infinite nested words. Inf. Comput., 253:448–466, 2017.
- 12 Manfred Droste and Paul Gastin. Weighted automata and weighted logics. In *International Colloquium on Automata, Languages and Programming (ICALP'05)*, volume 3580 of *Lecture Notes in Computer Science*, pages 513–525. Springer, 2005.
- Manfred Droste and Paul Gastin. Weighted automata and weighted logics. *Theor. Comput. Sci.*, 380(1-2):69–86, 2007.
- 14 Manfred Droste, Werner Kuich, and Heiko Vogler, editors. Handbook of Weighted Automata. Springer Berlin Heidelberg, 2009.

- Manfred Droste and Ingmar Meinecke. Weighted automata and weighted MSO logics for average and long-time behaviors. Inf. Comput., 220:44-59, 2012.
- 16 Manfred Droste and Vitaly Perevoshchikov. Multi-weighted automata and MSO logic. Theory Comput. Syst., 59(2):231–261, 2016.
- 17 Manfred Droste and George Rahonis. Weighted automata and weighted logics on infinite words. Izvestiya VUZ. Matematika, 54:26-45, 2010.
- 18 Manfred Droste and Heiko Vogler. Weighted tree automata and weighted logics. Theor. Comput. Sci., 366(3):228-247, 2006.
- 19 Manfred Droste and Heiko Vogler. Weighted automata and multi-valued logics over arbitrary bounded lattices. Theor. Comput. Sci., 418:14–36, 2012.
- Calvin C. Elgot. Decision problems of finite automata design and related arithmetics. Trans-20 actions of the American Mathematical Society, 98:21-52, 1961.
- 21 In a Fichtner. Weighted picture automata and weighted logics. Theory Comput. Syst., 48(1):48– 78, 2011.
- Paul Gastin and Benjamin Monmege. A unifying survey on weighted logics and weighted 22 automata. Soft Computing, 22(4):1047–1065, Dec 2018.
- Oscar H Ibarra and Bala Ravikumar. On sparseness, ambiguity and other decision problems for acceptors and transducers. In Symposium on Theoretical Aspects of Computer Science (STACS'86), volume 210 of Lecture Notes in Computer Science, pages 171–179. Springer, 1986.
- 24 Daniel Kirsten. A Burnside approach to the termination of Mohri's algorithm for polynomially ambiguous min-plus-automata. RAIRO - Theoretical Informatics and Applications, 42(3):553-581, Jun 2008.
- Ines Klimann, Sylvain Lombardy, Jean Mairesse, and Christophe Prieur. Deciding unambiguity and sequentiality from a finitely ambiguous max-plus automaton. Theoretical Computer Science, 327(3):349–373, Nov 2004.
- Stephan Kreutzer and Cristian Riveros. Quantitative monadic second-order logic. In Symposium on Logic in Computer Science (LICS'13), pages 113-122. IEEE, June 2013.
- 27 Werner Kuich and Arto Salomaa. Semirings, Automata, Languages. Springer Berlin Heidelberg,
- Eleni Mandrali and George Rahonis. On weighted first-order logics with discounting. Acta28 Informatica, 51(2):61–106, Jan 2014.
- Filip Mazowiecki and Cristian Riveros. Pumping lemmas for weighted automata. In Symposium on Theoretical Aspects of Computer Science (STACS'18), volume 96 of Leibniz International Proceedings in Informatics (LIPIcs), pages 50:1-50:14. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.
- 30 Filip Mazowiecki and Cristian Riveros. Copyless cost-register automata: Structure, expressiveness, and closure properties. Journal of Computer and System Sciences, 100:1–29, Mar
- Robert McNaughton and Seymour Papert. Counter-Free Automata. The MIT Press, Cambridge, Mass., 1971.
- 32 Erik Paul. On finite and polynomial ambiguity of weighted tree automata. In International Conference on Developments in Language Theory (DLT'16), volume 9840 of Lecture Notes in Computer Science, pages 368–379. Springer, 2016.
- 33 Karin Quaas. MSO logics for weighted timed automata. Formal Methods in System Design, 38(3):193-222, 2011.
- 34 Christophe Reutenauer. Propriétés arithmétiques et topologiques de séries rationnelles en variables non commutatives. PhD thesis, Université Paris VI, 1977.
- 35 Jacques Sakarovitch. Elements of Automata Theory. Cambridge University Press, 2009.
- Jacques Sakarovitch and Rodrigo de Souza. Lexicographic decomposition of k-valued transducers. Theory of Computing Systems, 47(3):758–785, Apr 2009.
- 37 Arto Salomaa and Matti Soittola. Automata-Theoretic Aspects of Formal Power Series. Springer, 1978.

- Marcel Paul Schützenberger. On the definition of a family of automata. *Information and Control*, 4(2-3):245–270, Sep 1961.
- Marcel Paul Schützenberger. On finite monoids having only trivial subgroups. *Information and Control*, 8(2):190–194, 1965.
- **40** Boris A. Trakhtenbrot. Finite automata and logic of monadic predicates. *Doklady Akademii Nauk SSSR*, 149:326–329, 1961.
- 41 Andreas Weber and Helmut Seidl. On the degree of ambiguity of finite automata. *Theoretical Computer Science*, 88(2):325–349, Oct 1991.