Noncommutative Partial Derivative

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Abstract

We introduce the axiomatic definition of the point-derivative for noncommutative algebras and present the counterparts of the ordinary multivariable chain rule and Clairaut's Theorem in the context of partial pointderivatives.

Key Words: Hausdorff derivative, cyclic derivative and point-derivative.

Among various attempts of extending the notion of the ordinary derivative to noncommutative polynomials, there are two noteworthy extensions of the ordinary derivative. One is the Hausdorff derivative, and the other one is the cyclic derivative introduced in [2]. Some well-known properties of the ordinary derivative can be naturally extended to the cyclic derivative for formal series in a single variable, but any significant properties of cyclic derivative for formal series in several variables has not been obtained since this problem was mentioned in [2]. In particular, there is not any counterpart of the ordinary multi-variable chain rule and Clairaut's Theorem in the context of both Hausdorff derivatives and cyclic derivatives.

In this paper, we introduce the axiomatic definition of the point-derivative for noncommutative algebras, explain how to define the point-derivative in the algebra of noncommutative formal power series, and present the counterparts of the ordinary multi-variable chain rule and Clairaut's Theorem in the context of partial point-derivatives.

Throughout this paper, we use the same notations and terminologies as the ones in [2]. In particular, K denotes a field of characteristic zero and all algebras are over the field K and have the identity.

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1 The Notion of Point-derivative

Let \mathcal{A} be a K-algebra. Recall that a K-linear map $d: \mathcal{A} \to \mathcal{A}$ is called a derivation on \mathcal{A} if d(ab) = ad(b) + d(a)b for $a, b \in \mathcal{A}$. The axiomatic definition of the point-derivative for noncommutative algebras is given in the following

Definition 1.1 Let \mathcal{A} be a K-algebra and let $End(\mathcal{A})$ be the set of all K-linear maps from \mathcal{A} to \mathcal{A} . A K-linear map $\mathcal{D}: \mathcal{A} \to End(\mathcal{A})$ is called a **point-derivation** on \mathcal{A} if

- (i) $\mathcal{D}_{\beta} := \mathcal{D}(\beta) \in End(\mathcal{A})$ is a derivation on \mathcal{A} for all $\beta \in \mathcal{A}$,
- (ii) $\mathcal{D}_z = z \mathcal{D}_1$ for all z in the center of \mathcal{A} ,

where 1 is the identity of the algebra \mathcal{A} . For $f \in \mathcal{A}$ and $\beta \in \mathcal{A}$, $\mathcal{D}_{\beta}(f) \in \mathcal{A}$ is called the **point-derivative depending on** β of f or β -derivative of f.

If \mathcal{A} is a commutative algebra, then a derivation d on \mathcal{A} produces naturally a point-derivation \mathcal{D} on \mathcal{A} as follows:

$$\mathcal{D}(\beta) := \beta d \text{ for } \beta \in \mathcal{A},$$

where $\beta d \in End(\mathcal{A})$ is defined by $(\beta d)(a) := \beta \cdot d(a)$ for $a \in \mathcal{A}$.

We now explain how to introduce point-derivative in noncommutative formal power series. Let A be the alphabet set which has a distinguished letter, denoted by x and called the variable, and an infinite supply of other letters a, b, \ldots called constants. A word w is an element of the free monoid M generated by A, and the degree of a word w is the number of occurrences of the letter x in w. The empty word, which is denoted by 1, is the identity element of free monoid M. Following [2], we use $K\{\{a, b, \ldots, x\}\}$ to denote the algebra of noncommutative formal power series in a single variable x over the field K.

Let $\beta \in K\{\{a,b,\ldots,x\}\}$ be a fixed formal power series. For a given word $w=c_1x^{i_1}c_2x^{i_2}\cdots c_nx^{i_n}c_{n+1}\in M$, where $c_1,\,c_2,\,\cdots,\,c_n,\,c_{n+1}$ are the words of degree 0 and $i_1,\,i_2,\,\cdots,\,i_n$ are nonnegative integers, we define a map $\frac{d}{d_\beta x}$ from M to $K\{\{a,b,\ldots,x\}\}$ as follows:

$$\frac{d}{d_{S}x}(w) := 0$$
 if $i_1 = i_2 = \dots = i_n = 0$

and

$$\frac{d}{d_{\beta}x}(w) := \sum_{k=1}^{n} c_1 x^{i_1} \cdots c_{k-1} x^{i_{k-1}} c_k \cdot \frac{d}{d_{\beta}x}(x^{i_k}) \cdot c_{k+1} x^{i_{k+1}} \cdots c_n x^{i_n} c_{n+1}$$
if $i_1 i_2 \cdots i_n > 0$,

where $c_1 x^{i_1} \cdots c_{k-1} x^{i_{k-1}} := 1$ for k = 1, $c_{k+1} x^{i_{k+1}} \cdots c_n x^{i_n} c_{n+1} := 1$ for k = n and $\frac{d}{d_{\beta} x}(x^{i_k})$ is defined by

$$\frac{d}{d_{\beta}x}(x^{i_k}) = \begin{cases} \beta & \text{if } i_k = 1\\ x^{i_k - 1}\beta + x^{i_k - 2}\beta x + \dots + x\beta x^{i_k - 2} + \beta x^{i_k - 1} & \text{if } i_k > 1 \end{cases}.$$

After extending the map $\frac{d}{d_{\beta}x}: M \to K\{\{a,b,\ldots,x\}\}$ linearly and continuously, we get a K-linear map $\frac{d}{d_{\beta}x}: K\{\{a,b,\ldots,x\}\} \to K\{\{a,b,\ldots,x\}\}$, which is called the **point-derivative operator depending on** β . Clearly, The map: $\beta \to \frac{d}{d_{\beta}x}$ is a point-derivation on $K\{\{a,b,\ldots,x\}\}$. The β -derivative $\frac{d}{d_{\beta}x}(f(x))$ of a formal power series $f(x) \in K\{\{a,b,\ldots,x\}\}$ is also denoted by $\frac{df}{d_{\beta}x}$ or $f'_{\beta}(x)$. Note that the 1-derivative $\frac{df}{d_{1}x} = f'_{1}(x)$ of the formal power series f(x) is the Hausdorff derivative H < f > 0 of f(x).

Like the Hausdorff derivative and cyclic derivative, we have the following characterization of the point-derivative operator $\frac{d}{d_{\beta}x}$.

Proposition 1.1 Let $\beta \in K\{\{a,b,\ldots,x\}\}$ be a fixed formal power series. The point–derivative operator $\frac{d}{d_{\beta}x}: K\{\{a,b,\ldots,x\}\} \to K\{\{a,b,\ldots,x\}\}$ is the unique K-linear continuous operator on $K\{\{a,b,\ldots,x\}\}$ satisfying

(i)
$$\frac{d}{d_{\beta}x}(a) = 0$$
 for any constant a ,

(ii)
$$\frac{d}{d_{\beta}x}(x) = \beta$$
,

(iii)
$$\frac{d}{d_{\beta}x}(fg) = \frac{d}{d_{\beta}x}(f) \cdot g + f \cdot \frac{d}{d_{\beta}x}(g) \text{ for } f, g \in K\{\{a, b, \dots, x\}\}.$$

2 Partial Point-derivatives

For convenience, we use $K\{\{a,b,\ldots,||x,y\}\}$ to denote the algebra of noncommutative formal power series in two noncommutative variables x and y. Let $f(x,y) \in K\{\{a,b,\ldots,||x,y\}\}$ and $\beta = \beta(x,y) \in K\{\{a,b,\ldots,||x,y\}\}$ be formal power series in two noncommutative variables x and y. If we keep y constant, then f(x,y) and $\beta = \beta(x,y)$ are formal power series of x and its β -derivative with respect to the variable x is denoted by $\frac{\partial f(x,y)}{\partial_{\beta}x}$ or $\frac{\partial f}{\partial_{\beta}x}$, which is called the

partial β -derivative of f(x,y) with respect to the variable x. If we keep x constant, then f(x,y) and $\beta = \beta(x,y)$ are formal power series of y and its β -derivative with respect to the variable y is denoted by $\frac{\partial f(x,y)}{\partial \beta y}$ or $\frac{\partial f}{\partial \beta y}$, which is called the partial β -derivative of f(x,y) with respect to the variable y.

Let $\beta=\beta(x,y),\ \gamma=\gamma(x,y)$ and f=f(x,y) be formal power series in noncommutative variables x and y. We define the second partial $(\gamma y,\ \beta x)$ -derivative $\frac{\partial^2 f}{\partial \gamma y \partial_\beta x}$ of f(x,y) and the second partial $(\beta x,\ \gamma y)$ -derivative $\frac{\partial^2 f}{\partial \beta x \partial_\gamma y}$ of f(x,y) by

$$\frac{\partial^2 f}{\partial_\gamma y \partial_\beta x} := \frac{\partial}{\partial_\gamma y} \left(\frac{\partial f}{\partial_\beta x} \right) \quad \text{and} \quad \frac{\partial^2 f}{\partial_\beta x \partial_\gamma y} := \frac{\partial}{\partial_\beta x} \left(\frac{\partial f}{\partial_\gamma y} \right).$$

The following proposition extends multi-variable chain rule and Clairaut's Theorem to partial point-derivatives for formal power series containing x and y.

Proposition 2.1 If f = f(x,y), $\beta = \beta(x,y)$, $\gamma = \gamma(x,y)$, u = u(x,y) and v = v(x,y) are formal power series in two noncommutative variables x and y, then

$$\frac{\partial \big(f(u,v)\big)}{\partial_{\beta}x} = \frac{\partial \big(f(u,v)\big)}{\partial_{\frac{\partial u}{\partial \Rightarrow x}}u} + \frac{\partial \big(f(u,v)\big)}{\partial_{\frac{\partial v}{\partial \Rightarrow x}}v}.$$

and

$$\frac{\partial^2 f}{\partial_\gamma y \partial_\beta x} - \frac{\partial^2 f}{\partial_\beta x \partial_\gamma y} = \frac{\partial f}{\partial_{\left(\frac{\partial \beta}{\partial_\gamma y}\right)} x} - \frac{\partial f}{\partial_{\left(\frac{\partial \gamma}{\partial_\beta x}\right)} y}.$$

The proof of the proposition above follows from a direct computation.

By [1], the ordinary partial derivatives is a useful tool in the study of commutative arithmetic circuits. Hence, one problem, which is worthy to study it in future, is to determine if the partial point-derivatives can be used to study non-commutative arithmetic circuits.

References

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