

VASS Coverability in ExpSpace

Fix a d dimensional Vector Addition System with States (VASS) with state space Q . For two configurations $x, y \in Q \times \mathbb{N}^d$ where $x = (q_x, v_x)$ and $y = (q_y, v_y)$ we say that x is *bigger* than y , denoted $x \succeq y$, if they have the same state ($q_x = q_y$) and v_x is coordinatewise bigger or equal v_y . For two configurations source s and target t , both in $Q \times \mathbb{N}^d$, a path $s \rightarrow t'$ is a *covering path from s to t* if $t' \succeq t$. The **Coverability Problem** asks for two given configurations $s, t \in Q \times \mathbb{N}^d$ whether there exists a covering path from s to t .

We present here a proof that Coverability Problem is in exponential space, a small modification of the original proof by Charles Rackoff from 1978 (*The Covering and Boundedness Problems for Vector Addition Systems*). It is clearly enough to show that for any two configurations s, t if there is a covering path from s to t then there is such of at most doubly exponential length, which is achieved by the Lemma below. Then algorithm for coverability problem simply guesses such a path and it can verify in exponential space that it is indeed a correct guess.

Let *norm* of a vector be maximal absolute value of any of its entries and *norm* of a VASS be maximal norm of a transition of this VASS. *Norm* of a configuration is the norm of its vector. *Length* of a path is the number of transitions in it.

Lemma 1. *Consider a VASS, which has a norm bounded by M and two its configurations $s, t \in Q \times \mathbb{N}^d$ such that norm of t is bounded by M (note that norm of s can be arbitrary). If there is a covering path from s to t then there is such of length at most $(M+1)^{4d^{d-1}}$.*

Proof. We proceed by induction on d . For $d = 1$ it is immediate to show that any covering path, which does not repeat a configuration has length at most $M+1$.

For the induction step assume that ρ is a covering path $s \xrightarrow{\rho} t'$ in a $d+1$ dimensional VASS. Let $B_d = (M+1)^{4d^{d-1}}$. We distinguish two cases:

1. norm of every configuration on ρ is bounded by $(M+1) \cdot B_d$;
2. norm of some u on ρ exceeds $(M+1) \cdot B_d$.

Of course we can assume that no configuration on ρ appears more than once, otherwise we unpump ρ . Thus in case 1 length of ρ is bounded by $C = ((M+1) \cdot B_d)^{d+1}$, we estimate C from above later.

In case 2 length of ρ might be long, but we show that we can find another covering path ρ' , which is short enough. Let u be the first configuration on ρ with norm exceeding $(M+1) \cdot B_d$. Let $s \xrightarrow{\rho_1} u \xrightarrow{\rho_2} t'$. Clearly length of ρ_1 is bounded by C , similarly as in case 1. Now we aim at modifying ρ_2 to obtain $u \xrightarrow{\rho_3} t''$ for some $t'' \succeq t$ such that ρ_3 is short too. Some coordinate in u is at least $(M+1) \cdot B_d$, assume without loss of generality that it is the last, $d+1$ -th coordinate. We ignore for a moment this last coordinate in a whole VASS and by induction assumption we get that there is a path π of length at most B_d such that $u_d \xrightarrow{\pi} t''_d$. Here $u_d, t_d \in Q \times \mathbb{N}^d$ are obtained from $u, t \in Q \times \mathbb{N}^{d+1}$ by removing the last coordinate

π_d : sequence of transitions obtained by removing the last coordinate of transitions of the original VASS.

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using the fact that the size of the source does not matter

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and $t''_d \in Q \times \mathbb{N}^d$ is some configuration fulfilling $t''_d \succeq t_d$. Then we remind ourselves that coordinate $d+1$ exists, let $u \xrightarrow{\pi} t''$. It is clear by above reasoning that for all coordinates $i \in \{1, \dots, d\}$ we have $t''[i] \leq t[i]$, but what happens on that last coordinate? By our assumption we have that $u[d+1] \geq (M+1) \cdot B_d$. Every of at most B_d transitions in π can decrease the last coordinate by at most M . So $t''[d+1] \geq (M+1) \cdot B_d - B_d \cdot M = B_d \geq M \geq t[d+1]$. Thus indeed $t'' \preceq t$ and path $s \xrightarrow{\rho_1} u \xrightarrow{\rho_3} t''$ is a covering path from s to t . Length of $\rho' = \rho_1 \rho_3$ is at most $C + B_d$.

In order to finish the argument we have to show that $C + B_d \leq B_{d+1} = (M+1)^{(4(d+1))^d}$. We perform very rough estimations:

$$\begin{aligned} C + B_d &\leq (M+1) \cdot C = (M+1) \cdot \left((M+1) \cdot B_d \right)^{d+1} = (M+1) \cdot \left((M+1) \cdot (M+1)^{(4d)^{d-1}} \right)^{d+1} \\ &= (M+1)^{((4d)^{d-1}+1)(d+1)+1} \leq (M+1)^{4 \cdot (4d)^{d-1} \cdot (d+1)} \leq (M+1)^{(4(d+1))^d} = B_{d+1}. \end{aligned}$$

This finishes the proof. \square