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# An Equational Axiomatization of Dynamic Negation and Relational Composition

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**Abstract.** We consider algebras on binary relations with two main operators: relational composition and dynamic negation. Relational composition has its standard interpretation, while dynamic negation is an operator familiar to students of Dynamic Predicate Logic (DPL) (Groenendijk and Stokhof, 1991): given a relation $R$ its dynamic negation $\sim R$ is a *test* that contains precisely those pairs $(s, s)$ for which $s$ is not in the domain of $R$. These two operators comprise precisely the propositional part of DPL.

This paper contains a finite equational axiomatization for these *dynamic relation algebras*. The completeness result uses techniques from modal logic. We also look at the variety generated by the class of dynamic relation algebras and note that there exist nonrepresentable algebras in this variety, ones which cannot be construed as spaces of relations. These results are also proved for an extension to a signature containing atomic tests and union.

**Key words:** Dynamic Predicate Logic, relation algebra, modal logic, dynamic logic, finite axiomatization, bisimulation, unraveling, variety, representability

## 1. Introduction

In the tradition of dynamic semantics in the style of Dynamic Predicate Logic (DPL) (Groenendijk and Stokhof, 1991), formulas of first order logic are interpreted in a nonstandard way, not on assignments, but on *pairs* of assignments. The interpretation of a formula thus becomes a *binary relation* on assignments. The four main ingredients of this approach are the following. First there are the constants $P(x_1, \ldots, x_n)$ and $\exists x$. The first is a *test*: it defines a subset of the diagonal identity relation, containing all pairs of assignments $(f, f)$ for which $\langle f(x_1), \ldots, f(x_n)\rangle$ is in the interpretation of $P$. $\exists x$ is not a test, but a *random assignment* to $x$: it holds between assignments $f$ and $g$ if $f$ and $g$ disagree on at most $x$. Next, we have two operators on relations, $\sim$ (dynamic negation$^\star$) and ; (composition). The unary $\sim$ yields again a test. Given a relation $R$, $\sim R$ holds between two assignments $f$ and $g$ iff $f = g$ and $f$ is not in the domain of $R$. Negation in DPL is interpreted by $\sim$. The binary operator ; is simply relational composition. Given two relations $R$ and

$^\star$ In (van Benthem, 1993) this operation is referred to as *strong* negation. We choose to refer to the operation as "dynamic negation" because "strong negation" is a well-known concept from intuitionistic logic (see van Dalen, 1986: 301).

$R'$, $R;R'$ holds between two assignments $f$ and $g$ iff there is a third, $h$, such that $fRh$ and $hR'g$. It is used as the interpretation of conjunction in DPL.

In this paper, we study the operations $\sim$ and ; on their own, as operations on arbitrary relations, not just on relations between assignments. The *empty* relation, denoted by $\perp$, is definable in this fragment as $(\sim x); x$. Nevertheless, for expository's sake, we include $\perp$ in our signature.

Besides the fact that this is a natural fragment of DPL, much in the same way as propositional logic is a natural fragment of predicate logic, there is an additional motivation for studying it. Consider algebraic terms $t_1$ and $t_2$, constructed using $\perp$, $\sim$ and ;. Then $t_1$ and $t_2$ are equal for any possible uniform choice of relations assigned to the variables in these terms iff they are always equal whenever we substitute the variables in $t_1$ and $t_2$ uniformly to interpretations of DPL-formulas. In other words, equality in the general case corresponds exactly to *schematic equality* in DPL. This interesting result can be derived from the results in (Visser, 1995).

This paper will axiomatize these equalities, in an algebraic way, i.e. by giving a finite set of axioms, from which all other valid equalities can be derived simply by means of equational logic. Its main tools are from polymodal logic, namely the completeness theorem for minimal modal logic and an unraveling technique.

An important preliminary for the result in the present paper can be found in (Blackburn and Venema, 1995). The main objective of the latter paper is to study the logic of *dynamic implication* $\Rightarrow$, which in our system can be defined by means of the equation $t \Rightarrow t' := \sim(t; \sim t')$. The first part of (Blackburn and Venema, 1995) studies the logic of $\Rightarrow$ in combination with the booleans, here interpreted as the usual set-theoretic operations on relations. An axiomatization is given of what in the present setting corresponds to $\models t = \top$ (i.e. the interpretation of $t$ is always the full square $S \times S$, when we are considering relations on $S$). This is an interesting and beautiful result, which establishes a connection between the static tradition (the booleans) and the dynamic one ($\Rightarrow$). This very strength can also be viewed as a weakness, as it does perhaps not do much justice to DPL, where $\sim$ and ; were presented as *alternatives* to the booleans. The second part of (Blackburn and Venema, 1995) addresses this issue and studies a proper fragment of our system, namely the $\{\Rightarrow, \perp\}$-fragment. $\sim t$ can be defined in this setting as $t \Rightarrow \perp$, but composition cannot be defined, hence this is really a *proper* fragment of $\{\perp, \sim, ;\}$. A tableau system is presented, axiomatizing for which $t$ (in this restricted language) $\models t = \text{ID}$ holds, where ID is interpreted as the identity relation and defined as $\sim \perp$. The present paper may be viewed as an attempt to extend the results of (Blackburn and Venema, 1995) and to axiomatize the fragment where the important ; is also present.

## 2. Preliminary Definitions

Let $\mathcal{T}$ be the set of *terms* constructed from an infinite set $\mathcal{V}$ of variables, a constant $\perp$, the unary operation $\sim$ and the binary ;. The binding strength of $\sim$ is assumed greater than ;. We have already seen some abbreviations of terms. We repeat these here and add an extra one:

$$\begin{array}{lcr} \mathrm{ID} & := & \sim\perp \\ t \Rightarrow t' & := & \sim(t;\sim t') \\ t \vee t' & := & \sim(\sim t;\sim t'). \end{array}$$

Both $\Rightarrow$ and $\vee$ are discussed in (Groenendijk and Stokhof, 1991) as natural interpretations of implication and disjunction in natural language, the main motivation for dynamic semantics.

A *dynamic relation algebra* is an algebra for the signature $\{\perp, \sim, ;\}$, of the following form. Its domain is the powerset of a *square* (see Venema, 1991, for this terminology). That is, it is of the form $\wp(S \times S)$ for some set $S$. The elements are thus binary relations on $S$. $\perp$ is interpreted as the empty relation $\emptyset$, $\sim$ as:

$$\sim R := \{(s, s) \mid s \in S \text{ and } \neg\exists t.(sRt)\}$$

and ; simply as relational composition. So a dynamic relation algebra is determined completely by a set $S$. The class of all dynamic relation algebras is denoted by $\mathsf{DRA}$. We write $\mathsf{DRA} \models t_1 = t_2$, or just $\models t_1 = t_2$, when the equation is valid (under all assignments to variables) in all dynamic relation algebras.

A $\mathsf{DRA}$ of special importance to DPL is what we may call an *assignment* $\mathsf{DRA}$. This is a $\mathsf{DRA}$ over a set $S = D^V$, where $D$ is any set and $V$ is a set of variables. $S$ is thus the full set of assignments over $D$ and the corresponding $\mathsf{DRA}$ has as its domain all relations between such assignments. It follows from the results in (Visser, 1995) that the equations valid in all assignment $\mathsf{DRA}$s are precisely those that are valid in the class of *all* $\mathsf{DRA}$s. Thus, any axiomatization we give of the latter is automatically one for the former.

The reader is warned that our definition of a dynamic relation algebra is not what is commonly referred to as a "relation algebra" in the literature (originating in Tarski, 1941): these more standard definitions also include the booleans (relation algebras are thus boolean algebras) and a *converse* operator. Dynamic negation can be defined in ordinary relation algebra by means of the equation:

$$\sim R := \mathrm{ID} \cap -(R;\top).$$

(The identity relation *is* present in ordinary relation algebra.) Our quest can thus also be situated within the field of relation algebra: we study a small nonboolean fragment.

Let us consider what relations are denoted by our abbreviations. We have already remarked that ID is interpreted as $\{(s, s), \mid s \in S\}$, the identity-relation on $S$, also known as the *diagonal* on $S$. $\Rightarrow$ and $\vee$ are more interesting:

Table I. **AX.**

| | | |
|---|---|---|
| **A1:** | $\sim x; x = \bot$ | (falsum definition) |
| **A2:** | $x; \bot = \bot$ | (falsum right) |
| **A3:** | $\text{ID}; x = x$ | (identity left) |
| **A4:** | $x; (y; z) = (x; y); z$ | (associativity) |
| **A5:** | $\sim x; \sim y = \sim y; \sim x$ | (test permutation) |
| **A6:** | $x = (\sim\sim x); x$ | (domain test) |
| **A7:** | $\sim\sim (\sim x; \sim y) = \sim x; \sim y$ | (test composition) |
| **A8:** | $\sim (x; y); x = (\sim (x; y); x); \sim y$ | (modus ponens) |
| **A9:** | $\sim (x; (y \vee z)) = \sim ((x; y) \vee (x; z))$ | (distribution) |

– $R_1 \Rightarrow R_2 := \{(s, s) \mid s \in S$ and $\forall u(sR_1u \to \exists v.uR_2v)\}$. In words, $R_1 \Rightarrow R_2$ is a test, that succeeds only at states such that after every execution of $R_1$ from these states, we are able to continue with an $R_2$-execution.
– $R_1 \vee R_2 := \{(s, s) \mid s \in S$ and $\exists u.(sR_1u) \vee \exists v.(sR_2v)\}$. Thus $R_1 \vee R_2$ is again a test, that succeeds when we can proceed with either an $R_1$- or an $R_2$-step.

Table I contains a finite set of axioms, **AX**, that completely axiomatizes equational validity in dynamic relation algebras. We write $\vdash t_1 = t_2$ if this equation is derivable from the equations in **AX** and the rules of equational logic. The main theorem of this paper will state that $\models t_1 = t_2$ iff $\vdash t_1 = t_2$.

The result suggests that the dynamic way of looking at the step from propositional logic to predicate logic behaves better than in the static view. In the static view, first order logic is arrived at, algebraically, by moving from boolean algebras, via relation algebras, to cylindric algebras. In the dynamic camp this same sequence of steps would be as follows: we start again from boolean algebras, as an intermediate step we get dynamic relation algebras (that this is an important intermediate step, comparable to the role of relation algebras in the static view, follows from the results in Visser, 1995), and finally standard algebras for DPL (that is, algebras over relations between assignments, with standard interpretations for the DPL-constants). The advantage of the dynamic view here is that the intermediate step is a class of algebras that is finitely axiomatizable (the contribution of this paper), while that of the static view is not (cf. Monk, 1969).

*Remark 2.1.* Some more needs to be said about the first step above, from boolean algebras to the intermediate setting. In the static view, this step is clear: relation algebras *are* boolean algebras, in an expanded signature. The step from boolean algebras to dynamic relation algebras is of a different nature. Roughly, boolean algebras live inside relation algebras, by means of an embedding. To be precise, any powerset boolean algebra $(\wp(S), -, \cap, \emptyset)$ can be embedded into the dynamic relation algebra $(\wp(S \times S), \sim, ;, \emptyset)$ via the function $\iota : A \mapsto \{(s, s) \mid s \in A\}$. Applying dynamic negation to any $R \subseteq S \times S$ gives you $\sim R = \iota(S - \mathit{dom}(R))$, i.e. an element in the range of $\iota$. This implies that when $t_1 = t_2$ is a valid equation in the class of boolean algebras (in the signature given above), then $t_1^\circ = t_2^\circ$ must be a valid equation in DRA, where the translation $\circ$ is given by:

$$\begin{array}{lcr} x^\circ & := & \sim x \\ 0^\circ & := & \bot \\ (-t)^\circ & := & \sim t^\circ \\ (t_1 \cap t_2)^\circ & := & t_1^\circ; t_2^\circ. \end{array}$$

This immediately gives us the axioms **A5** (via the valid boolean law $x \cap y = y \cap x$) and **A7** (via $- - (x \cap y) = x \cap y$). □

It is trivial to verify that **AX** is sound for DRA. Let us dwell on the axioms a while longer.

- **A1** is obviously valid. Order is important here, as $R; \sim R$ need *not* always be the empty relation.
- A *test* is any subset of the identity relation. Thus for any $R, \sim R$ is a test. Tests can also be characterized as those relations $R$ for which $R = \sim\sim R$. $\sim\sim R$ tests whether an element is in the domain of $R$: it contains exactly those pairs $(s, s)$ for which $s \in$ *dom*$(R)$.

  Quite a few of our axioms are about the behaviour of tests:

  - **A5** says that tests permute.
  - **A6** says that to do an $x$-step from $s$ to $t$ is the same as first *checking* (or testing) that $s$ is in the domain of $x$ and *then* taking the $x$-step to $t$.
  - **A7** states that composition of two tests again yields a test. This axiom can also be viewed as the familiar De Morgan law: when we apply our definition for dynamic disjunction, it states that $\sim (x \vee y) = \sim x; \sim y$.
  - Note that *dom*$(R_1 \vee R_2)$ is simply *dom*$(R_1 \cup R_2)$. So **A9** is actually an embodiment of the fact that $R_0; (R_1 \cup R_2) = (R_0; R_1) \cup (R_0; R_2)$ is always the case.
- **A8** is called "modus ponens" because of the following instance of it, which will be its main use: $(x \Rightarrow y); x = (x \Rightarrow y); x; \sim\sim y$.

We list a few useful consequences of **AX**. In the proofs we treat ; as an associative operation, which is justified by **A4**. These derivable laws are given names, so that we may refer to them later.

- Identity right: $x; \text{ID} = x$.

$$\begin{array}{rcll} x;\mathrm{ID} & = & \mathrm{ID};x;\mathrm{ID} & \textbf{(A3)} \\ & = & \sim(x;\bot);x;\sim\bot & \textbf{(A2)} \\ & = & \sim(x;\bot);x & \textbf{(A8)} \\ & = & \mathrm{ID};x & \textbf{(A2)} \\ & = & x & \textbf{(A3)} \end{array}$$

– Triple negation law: $\sim\sim\sim x = \sim x$.

$$\begin{array}{rcll} \sim\sim\sim x & = & \sim\sim(\sim\bot;\sim x) & \textbf{(A3)} \\ & = & \sim\bot;\sim x & \textbf{(A7)} \\ & = & \sim x & \textbf{(A3)} \end{array}$$

– Test idempotency: $\sim x;\sim x = \sim x$.

$$\begin{array}{rcll} \sim x;\sim x & = & \sim\sim\sim x;\sim x & \text{(triple negation)} \\ & = & \sim x & \textbf{(A6)} \end{array}$$

So $x \vee x = \sim\sim x$ is provable, as $x \vee x$ is defined as $\sim(\sim x;\sim x)$.

– Range test: $\sim(x;y) = \sim(x;\sim\sim y)$.

$$\begin{array}{rcll} \sim(x;y) & = & \sim\sim\sim(x;y) & \text{(triple negation)} \\ & = & \sim((x;y)\vee(x;y)) & \text{(idempotency)} \\ & = & \sim(x;(y\vee y)) & \textbf{(A9)} \\ & = & \sim(x;\sim\sim y) & \text{(idempotency)} \end{array}$$

– Double negation law for falsum: $\sim\sim\bot = \bot$.

$$\begin{array}{rcll} \sim\sim\bot & = & \sim\sim(\sim x;x) & \textbf{(A1)} \\ & = & \sim\sim(\sim x;\sim\sim x) & \text{(range test)} \\ & = & \sim x;\sim\sim x & \textbf{(A7)} \\ & = & \sim\sim x;\sim x & \textbf{(A5)} \\ & = & \bot & \textbf{(A1)} \end{array}$$

– Falsum left: $\bot;x = \bot$.

$$\begin{array}{rcll} \bot;x & = & ((\bot;x)\Rightarrow\bot);\bot;x & \textbf{(A2)} \\ & = & ((\bot;x)\Rightarrow\bot);\bot;x;\sim\sim\bot & \textbf{(A8)} \\ & = & ((\bot;x)\Rightarrow\bot);\bot;x;\bot & \text{(double negation law for } \bot) \\ & = & \bot & \textbf{(A2)} \end{array}$$

Finally, we mention a few surprising (from the static logician's point of view) invalid equations: $x;x = x$ and $x \Rightarrow x = \mathrm{ID}$.

*Remark 2.2.* Throughout this paper, the terms "equal," "equivalent" and the like will be used frequently, although sometimes in different senses. Either they will be used in a *semantic* way, where "equal" means "equal under all assignments in all dynamic relation algebras," or in a *syntactic* way, where "equal" means "provably equal in **AX**." We hope the sense intended will always be clear from the context. □

## 3. Modal Techniques

Before we prove the completeness theorem, let us focus on our main techniques, which originate in modal logic.

Define MOD to be the set of all modal formulas constructed from $\perp$ (as a modal formula, not as a $\mathcal{T}$-term), unary modalities $\langle x\rangle$ (for $x \in \mathcal{V}$) and the booleans $\wedge$ and $\neg$. Standard abbreviations are used for other connectives: $\top := \neg\perp$, $\phi \to \psi := \neg(\phi \wedge \neg\psi)$, $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$, $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$ and $[x]\phi := \neg\langle x\rangle\neg\phi$.

A dynamic relation algebra determined by a set $S$ together with a valuation to variables $\sigma : \mathcal{V} \to \wp(S \times S)$ can easily be seen as a Kripke model* $\mathcal{M} = (S, \xrightarrow{x})_{x\in\mathcal{V}}$, with $\xrightarrow{x} := \sigma(x)$ the accessibility relation for a modal diamond $\langle x\rangle$. We will write $(S, \sigma)$ when we have this model in mind. The modal satisfaction relation $(S, \sigma), s \Vdash \phi$, for $s \in S$ and $\phi$ a modal formula is defined as usual. We omit $(S, \sigma)$ when it is clear from the context.

Besides diamonds $\langle x\rangle$ for each variable, we could introduce diamonds $\langle t\rangle$ for all the other $\mathcal{T}$-terms as well. In a model $(S, \sigma)$ such a diamond would have $\sigma(t)$ as its accessibility relation, where $\sigma$ is here viewed as the obvious extension to $\mathcal{T}$-terms. This would not really give us any new modal formulas though, as $\langle\perp\rangle\phi$ is equivalent to $\perp$, $\langle\sim t\rangle\phi$ to $\phi \wedge [t]\perp$ and $\langle t; t'\rangle\phi$ to $\langle t\rangle\langle t'\rangle\phi$. So formulas that contain these new diamonds can be viewed as abbreviations for modal formulas only containing diamonds $\langle x\rangle$.

This immediately gives us decidability of the problem $\models t_1 = t_2$, because this is true iff for some proposition letter $p$, $\langle t_1\rangle p$ is equivalent (in minimal polymodal logic) to $\langle t_2\rangle p$, which is decidable. By the same reasoning we get the finite model property for nonvalid equations: any nonvalid equation $t_1 = t_2$ can be falsified in a finite dynamic relation algebra.

A *bisimulation* between two Kripke models $(S, \sigma)$ and $(T, \tau)$ is a relation $Z \subseteq S \times T$ such that the following two *zigzag-conditions* hold:

**Zig**: $aZb$ and $(a, a') \in \sigma(x)$ imply the existence of a $b' \in T$ with $a'Zb'$ and $(b, b') \in \tau(x)$.

**Zag**: Vice versa: $aZb$ and $(b, b') \in \tau(x)$ imply the existence of an $a' \in S$ such that $a'Zb'$ and $(a, a') \in \sigma(x)$.

We write $Z : (S, \sigma)\underline{\leftrightarrow}(T, \tau)$ when $Z$ satisfies these constraints. A *full* bisimulation is a bisimulation $Z : (S, \sigma)\underline{\leftrightarrow}(T, \tau)$ such that the domain of $Z$ is $S$ and its range is

* We will not be concerned with truth of modal formulas under all valuations to proposition letters, so we refrain from calling this a *frame*.

$T$. We say that two models are *bisimilar* when there is a full bisimulation between them.

The operations $\perp$, $\sim$ and ; are *safe for bisimulation* (van Benthem, 1993): if $Z : (S, \sigma)\underline{\leftrightarrow}(T, \tau)$ then the bisimulation clauses extend to the interpretation of any $\mathcal{T}$-term $t$:

**Zig**: $aZb$ and $(a, a') \in \sigma(t)$ imply the existence of a $b' \in T$ with $a'Zb'$ and $(b, b') \in \tau(t)$.

**Zag**: Vice versa: $aZb$ and $(b, b') \in \tau(t)$ imply that there is an $a' \in S$ such that $a'Zb'$ and $(a, a') \in \sigma(t)$.

In particular, when $(S, \sigma)$ and $(T, \sigma)$ are bisimilar then $\sigma(t)$ is nonempty iff $\tau(t)$ is also nonempty.

Examples of operations that are not safe are boolean intersection, complement and $\top$ (interpreted as $S \times S$).

A model $(S, \sigma)$ is called *unraveled* when it satisfies the following:

1. The relation $\bigcup_{x\in\mathcal{V}} \sigma(x)$ is well founded: there is no infinite decreasing sequence $s_0 \stackrel{x_0}{\leftarrow} s_1 \stackrel{x_1}{\leftarrow} \ldots$.
2. For any $s \in S$ there is at most one pair $(s', x)$ such that $(s', s) \in \sigma(x)$. So if $s_1 \stackrel{x}{\rightarrow} s$ and $s_2 \stackrel{y}{\rightarrow} s$ then $s_1 = s_2$ and $x = y$.

These conditions ensure that $(S, \sigma)$ behaves like a tree, locally.

Any model is bisimilar to an unraveled one. This is achieved as follows. Let $(S, \sigma)$ be any model. Let $S^\Diamond$ consist of all pairs $(s_0 \ldots s_n, x_1 \ldots x_n)$ with $s_0, \ldots, s_n \in S$ and $x_1, \ldots, x_n \in \mathcal{V}$ such that $s_0 \stackrel{x_1}{\rightarrow} s_1 \stackrel{x_2}{\rightarrow} \ldots \stackrel{x_n}{\rightarrow} s_n$ in $(S, \sigma)$. We also define an assignment $\tau : \mathcal{V} \rightarrow \wp(S^\Diamond \times S^\Diamond)$. $\tau(x)$ contains precisely all pairs

$$((s_0 \ldots s_n, x_1 \ldots x_n), (s_0 \ldots s_{n+1}, x_1 \ldots x_n x))$$

in $S^\Diamond \times S^\Diamond$ such that $s_n \stackrel{x}{\rightarrow} s_{n+1}$ (which is actually a superfluous remark, because otherwise the second pair would not be in $S^\Diamond$ in the first place). Now $(S^\Diamond, \tau)$ is unraveled and the function $f : S^\Diamond \rightarrow S$ sending $(s_0 \ldots s_n, x_1 \ldots x_n)$ to $s_n$ is a full bisimulation. (It is a functional bisimulation, or a *zigzagmorphism*, also known in the modal literature as a *p-morphism*.) What is of most importance to this paper is that if $\sigma(t)$ is nonempty in some model $(S, \sigma)$, we now know that there must exist an unraveled model $(T, \tau)$ where $\tau(t)$ is also nonempty. It is in fact the only use we make of the unraveling technique, and of the notion of bisimulation.

## 4. The Completeness Theorem

We will prove that the axiom-system **AX** displayed in Table I is sound and complete with respect to equational validity in dynamic relation algebras.

*Modal tests* $\phi?$, where $\phi$ is a modal formula (as in the previous section), are defined as follows:

$$\bot? := \bot \qquad (\langle x\rangle\phi)? := \sim\sim (x;\phi?)$$
$$(\phi\wedge\psi)? := \phi?;\psi? \qquad (\neg\phi)? := \sim (\phi?)$$

Note that the following are now trivially derivable in **AX** (mostly just by definition, but in the last $[x]$-case by using the triple negation law):

$$\top? = \mathrm{ID} \qquad (\phi\to\psi)? = \phi?\Rightarrow\psi?$$
$$(\phi\leftrightarrow\psi)? = (\phi?\Rightarrow\psi?);(\psi?\Rightarrow\phi?)$$
$$(\phi\vee\psi)? = \phi?\vee\psi? \qquad ([x]\phi)? = x\Rightarrow\phi?$$

LEMMA 4.1. *A modal test $\phi?$ is provably equal to $\sim\sim \phi?$ (i.e. $\vdash \phi? = \sim\sim \phi?$).*

*Proof*. This is proved by induction on $\phi$. For the base case, we need the double negation law for $\bot$. The induction step for $\wedge$ uses **A5**, while those for $\neg$ and $\langle x\rangle$ use the triple negation law. □

LEMMA 4.2. *Any term $\sim t$ of $\mathcal{T}$ is provably equivalent (in* **AX***) to $\phi?$ for some modal formula $\phi$.*

*Proof*. To prove this we prove that, for any term $t$ and any modal formula $\phi$, $\sim\sim (t;\phi?)$ is provably equivalent to $(\langle t\rangle\phi)?$, where the modal formula $\langle t\rangle\phi$ is defined as in Section 1.

The result then follows:

$$\begin{aligned}\sim t &= \sim\sim\sim t && \text{(triple negation)}\\ &= \sim\sim ((\sim t);\mathrm{ID}) && \text{(identity right)}\\ &= (\langle\sim t\rangle\top)? && \text{(to be proved)}\end{aligned}$$

We proceed by induction on $t$:

- If $t$ is a variable $x$, the statement is true by definition.
- If $t=\bot$, observe that: $\vdash\sim\sim (\bot;\phi?) =\sim\sim \bot$. By the double negation law for falsum, the latter is equivalent to $\bot$, which is the definition of $\bot?$.
- The negation case:

$$\begin{aligned}\sim\sim (\sim t;\phi?) &= \phi?;\sim t && (\textbf{A5}, \textbf{A7})\\ &= \phi?;(t\Rightarrow\bot)\\ &= \phi?;[t]\bot && \text{(induction hypothesis on } t)\\ &= (\langle\sim t\rangle\phi)?\end{aligned}$$

- The composition case:

$$\begin{aligned}
&\sim\sim (t_1; t_2; \phi?) \\
&= \sim\sim (t_1; \sim\sim (t_2; \phi?)) && \text{(range test)} \\
&= \sim\sim (t_1; (\langle t_2\rangle\phi)?) && \text{(induction hypothesis on } t_2\text{)} \\
&= (\langle t_1\rangle\langle t_2\rangle\phi)? && \text{(induction hypothesis on } t_1\text{)} \\
&= (\langle t_1; t_2\rangle\phi)?
\end{aligned}$$

□

It can easily be seen that for all modal formulas $\phi$:

$$\models \phi \quad \text{iff} \quad \models \phi? = \text{ID},$$

where $\models$ on the left is the usual modal validity in the class of all polymodal Kripke models, while the one on the right is validity in dynamic relation algebras. We assume that the reader will not confuse these two different validity relations (even though they are denoted by the same symbol $\models$) as modal validity only applies to modal formulas, while relational validity only to equations between $\mathcal{T}$-terms.

We already have an axiomatization for validity of modal formulas:

**Minimal modal logic axioms:**

| | |
|---|---|
| 1. $\phi \to (\psi \to \phi)$ | (K) |
| 2. $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$ | (S) |
| 3. $(\phi_1 \wedge \phi_2) \to \phi_i$ | ($i$-th projection, for $i \in \{1, 2\}$) |
| 4. $\phi \to (\psi \to (\phi \wedge \psi))$ | ($\wedge$-introduction) |
| 5. $\neg\neg\phi \to \phi$ | (double negation) |
| 6. $\perp \to \phi$ | (falsum law) |
| 7. $\neg\phi \leftrightarrow (\phi \to \perp)$ | (negation definition) |
| 8. $\langle x\rangle(\phi \vee \psi) \leftrightarrow (\langle x\rangle\phi \vee \langle x\rangle\psi)$ | (distribution) |

**Rules:**

| | |
|---|---|
| 1. $\phi, \phi \to \psi/\psi$ | (modus ponens) |
| 2. $\phi/[x]\phi$ | (necessitation) |
| 3. $\phi \leftrightarrow \psi/\chi[\phi] \leftrightarrow \chi[\psi]$ | (substitution of equivalents) |

LEMMA 4.3. *If a modal formula $\phi$ is derivable in the above system then $\vdash \phi? = \text{ID}$. (So if $\models \phi$ then $\vdash \phi? = \text{ID}$, by the completeness theorem for minimal modal logic.)*

First we prove another useful lemma:

LEMMA 4.4. $\vdash (\phi \leftrightarrow \psi)? = \text{ID}$ iff $\vdash \phi? = \psi?$.

*Proof.* From left to right, assume $\vdash (\phi \leftrightarrow \psi)? = \text{ID}$. Then:

$$
\begin{aligned}
\phi? &= \textsc{id}; \phi? && (\mathbf{A3})\\
&= (\phi? \Rightarrow \psi?); (\psi? \Rightarrow \phi?); \phi? && (\text{assumption})\\
&= (\psi? \Rightarrow \phi?); (\phi? \Rightarrow \psi?); \phi? && (\mathbf{A5})\\
&= (\psi? \Rightarrow \phi?); (\phi? \Rightarrow \psi?); \phi?; \psi? && (\mathbf{A8})\\
&= (\phi? \Rightarrow \psi?); (\psi? \Rightarrow \phi?); \psi?; \phi? && (\mathbf{A5})\\
&= (\phi? \Rightarrow \psi?); (\psi? \Rightarrow \phi?); \psi? && (\mathbf{A8})\\
&= \psi? && (\text{assumption}, \mathbf{A3})
\end{aligned}
$$

Note that the use in this proof of **A5** is justified, as all modal tests are equivalent to dynamically negated terms, by Lemma 4.1.

From right to left, we need to show that $\phi? \Rightarrow \phi? = \textsc{id}$, which we leave to the reader (use Lemma 4.1). □

*Proof of Lemma 4.3.* We first show that for all modal axioms $\phi$, $\phi? = \textsc{id}$ is provable.

1. K-axiom:

$$
\begin{aligned}
\phi? \Rightarrow (\psi? \Rightarrow \phi?) &= \sim (\phi?; \sim\sim (\psi?; \sim \phi?)) && (\text{definition of} \Rightarrow)\\
&= \sim (\phi?; \psi?; \sim \phi?) && (\text{range test})\\
&= \sim (\psi?; \sim \phi?; \phi?) && (\mathbf{A5})\\
&= \sim (\psi?; \bot) && (\mathbf{A1})\\
&= \sim \bot && (\mathbf{A2})\\
&= \textsc{id}
\end{aligned}
$$

2. S-axiom:

$$
\begin{aligned}
&(\phi? \Rightarrow (\psi? \Rightarrow \chi?)) \Rightarrow ((\phi? \Rightarrow \psi?) \Rightarrow (\phi? \Rightarrow \chi?))\\
&= \sim ((\phi? \Rightarrow (\psi? \Rightarrow \chi?)); (\phi? \Rightarrow \psi?); \phi?; \sim \chi?)\\
&= \sim ((\phi? \Rightarrow (\psi? \Rightarrow \chi?)); (\phi? \Rightarrow \psi?); \phi?; \psi?; \sim \chi?)\\
&= \sim ((\phi? \Rightarrow (\psi? \Rightarrow \chi?)); \phi?; \psi?; \sim \chi?; (\phi? \Rightarrow \psi?))\\
&= \sim ((\phi? \Rightarrow (\psi? \Rightarrow \chi?)); \phi?; (\psi? \Rightarrow \chi?); \psi?; \chi?; \sim \chi?; (\phi? \Rightarrow \psi?))\\
&= \sim ((\phi? \Rightarrow (\psi? \Rightarrow \chi?)); \phi?; (\psi? \Rightarrow \chi?); \psi?; (\phi? \Rightarrow \psi?); \underline{\sim \chi?; \chi?})\\
&= \textsc{id}
\end{aligned}
$$

   where these equivalences are derivable by means of range test, **A8**, **A5**, **A8**, **A5** and **A1–2** respectively.
3. Projection axioms. As $(\phi_1?; \phi_2?) \Rightarrow \phi_i?$ is the same as $\sim (\phi_1?; \phi_2?; \sim \phi_i?)$, simply apply **A5** and the $\bot$-axioms to arrive at $\textsc{id}$.
4. Using range test, $\phi? \Rightarrow (\psi? \Rightarrow (\phi?; \psi?))$ is equivalent to $\sim (\phi?; \psi?; \sim (\phi?; \psi?))$. With **A1** and **A5** this is equal to $\textsc{id}$.
5. Double negation axiom: by Lemma 4.4, it suffices to show $\vdash \sim\sim \phi? = \phi?$. But this was already shown in Lemma 4.1.
6. Falsum law: by falsum left, $\vdash \bot \Rightarrow \phi? = \sim (\bot; \sim \phi?) = \textsc{id}$.
7. Negation definition: use the right identity law to show $\vdash \phi? \Rightarrow \bot = \sim (\phi?; \sim \bot) = \sim \phi?$.

8. Distribution. First observe that double dynamic negation distributes over disjunction:

$$\begin{aligned} \sim\sim (t \vee t') &= \sim\sim\sim (\sim t; \sim t') && \text{(definition of } \vee\text{)} \\ &= \sim (\sim\sim\sim t; \sim\sim\sim t') && \text{(triple negation)} \\ &= \sim\sim t \vee \sim\sim t' && \text{(definition of } \vee\text{)} \end{aligned}$$

Now:

$$\begin{aligned} &(\langle x \rangle (\phi \vee \psi))? \\ &= \sim\sim (x; (\phi? \vee \psi?)) && \text{(definition of ?)} \\ &= \sim\sim ((x; \phi?) \vee (x; \psi?)) && (\mathbf{A9}) \\ &= \sim\sim (x; \phi?) \vee \sim\sim (x; \psi?) && (\sim\sim \text{ distributes over } \vee) \\ &= (\langle x \rangle \phi \vee \langle x \rangle \psi)? && \text{(definition of ?)} \end{aligned}$$

What remains is to show closure under the rules.

1. Modus ponens. If $\vdash \phi? = \textsc{id}$ and $\vdash \phi? \Rightarrow \psi? = \textsc{id}$, substitution gives us $\vdash \textsc{id} \Rightarrow \psi? = \textsc{id}$. But $\vdash \textsc{id} \Rightarrow \psi? = \sim(\textsc{id}; \sim \psi?) = \sim\sim \psi?$. The latter term is provably equal, by Lemma 4.1, to $\psi?$, so we are done.
2. Necessitation. Suppose $\phi? = \textsc{id}$ has been proved. Then:

$$\begin{aligned} \sim (x; \sim \phi?) &= \sim (x; \sim \textsc{id}) && \text{(assumption)} \\ &= \sim (x; \sim\sim \bot) && (\textsc{id}\text{-definition}) \\ &= \sim (x; \bot) && \text{(range test)} \\ &= \sim \bot && (\mathbf{A2}) \\ &= \textsc{id} && (\textsc{id}\text{-definition}) \end{aligned}$$

3. Substitution of equivalents: by Lemma 4.4 this rule may be dealt with by means of the usual equational substitution rules. □

Let us call a term $t$ *empty* if $\models t = \bot$, i.e. if its interpretation in any dynamic relation algebra is always empty. We call a term $t$ *nonempty* if in *some* dynamic relation algebra and for *some* assignment to variables $t$ is interpreted as a nonempty relation. In other words, a term is nonempty if it is not empty.

LEMMA 4.5. *If $t$ is empty then* $\vdash t = \bot$.

*Proof.* If $\models t = \bot$ then also $\models \sim\sim t = \bot$, because the domain of the empty relation is of course empty. $\sim\sim t$ is equivalent in **AX** to a modal test $\phi?$, by Lemma 4.2. Thus $\models \phi? = \bot$, and hence $\models \phi \leftrightarrow \bot$. By Lemmas 4.4 and 4.3 this gives us $\vdash \phi? = \bot$ and thus $\vdash \sim\sim t = \bot$. By **A6**, $\vdash t = \sim\sim t; t$, so $\vdash t = \bot; t = \bot$. □

LEMMA 4.6. *Any $\mathcal{T}$-term $t$ is provably equivalent to a "path-term," i.e. a formula of the form*

$$\phi_0?; x_1; \phi_1?; \ldots; x_n; \phi_n?,$$

*where $n$ is any natural number (possibly $0$, in which case a path-term is just a modal test).*

*Proof.* Any variable $x$ is equal to $\mathrm{ID}; x; \mathrm{ID}$, which is the same as $\top?; x; \top?$ and thus a path-term. If $\phi_0?; x_1; \phi_1?; \ldots; x_n; \phi_n?$ and $\psi_0?; y_1; \psi_1?; \ldots; y_m; \psi_m?$ are two path-terms then

$$\phi_0?; x_1; \phi_1?; \ldots; x_n; (\phi_n \wedge \psi_0)?; y_1; \psi_1?; \ldots; y_m; \psi_m?$$

is equal to their composition. We have already seen that any dynamically negated term is equivalent to a modal test, and thus to a path-term. The same remark applies to $\perp$. □

LEMMA 4.7. *If $t_1 := \phi_0?; x_1; \phi_1?; \ldots; x_n; \phi_n?$ and $t_2 := \psi_0?; y_1; \psi_1?; \ldots; y_m; \psi_m?$ are two semantically equal nonempty path-terms, then $n = m$, $x_1 = y_1, \ldots, x_n = y_n$.*

*Proof.* $t_1$ is nonempty, so there must be a dynamic relation algebra $(\wp(S \times S), \sim, ;, \emptyset)$ and an assignment to variables $\sigma$ such that $\sigma(t_1)$ is nonempty. Then, in the associated polymodal Kripke frame $\mathcal{M} = (S, \xrightarrow{x})_{x \in \mathcal{V}}$ (with $\xrightarrow{x} = \sigma(x)$) there are points $s_0, \ldots, s_n$ with $s_0 \xrightarrow{x_1} s_1 \ldots s_{n-1} \xrightarrow{x_n} s_n$ and $s_i \Vdash \phi_i$, for all $i \leq n$ (i.e. $(s_0, s_n) \in \sigma(t_1)$). By the remarks in Section 3 we may in fact assume that $\mathcal{M}$ is unraveled. Then $s_0 \xrightarrow{x_1} s_1 \xrightarrow{x_2} \ldots \xrightarrow{x_n} s_n$ is the *unique* path from $s_0$ to $s_n$. By assumption $\sigma(t_1) = \sigma(t_2)$, so there will also be a $t_2$-path from $s_0$ to $s_n$. Together these imply the desired conclusions. □

Lemmas 4.5 and 4.7 together *almost* give us the completeness theorem. For suppose $\models t_1 = t_2$. We may assume that they are path-terms. If these path-terms are empty, by Lemma 4.5 they will both be provably equivalent to $\perp$, hence $\vdash t_1 = t_2$. And if $t_1$ and $t_2$ are nonempty then by Lemma 4.7 they will be of the same shape. If we could now show that $\vdash \phi_i? = \psi_i?$ for each $i \leq n$ (where we assume that $t_1$ and $t_2$ are as in Lemma 4.7), $\vdash t_1 = t_2$ would follow. This may not be the case though. Two prime examples are the following:

**Up**: Clearly $\models ([x]\phi)?; x; \psi? = ([x]\phi)?; x; (\phi \wedge \psi)?$, while $\psi$ need not imply $\phi$. We need to be able to move $\phi$ up the path. This is precisely the use of our "modus ponens" axiom **A8**, which equates $(x \Rightarrow \phi?); x; \psi?$ to $(x \Rightarrow \phi?); x; \phi?; \psi?$ (which is provably equal to $([x]\phi)?; x; (\phi \wedge \psi)?$).

**Down**: Sometimes information may be required to move *down* a path as well. An example is the semantic equality $\phi?; x; \psi? = (\phi \wedge \langle x \rangle \psi)?; x; \psi?$. Now we can use the "domain test" axiom **A6**, which gives us that $\vdash \phi?; x; \psi? = \phi?; \sim\sim(x; \psi?); x; \psi?$, which is equal to $(\phi \wedge \langle x \rangle \psi)?; x; \psi?$.

These two tricks (the use of **A6** and **A8**) turn out to suffice, as we will see.

The *degree* $d(\phi)$ of a modal formula $\phi$ is defined as usual:

$$\begin{array}{lll} d(\bot) & = & 0 \\ d(\langle x\rangle\phi) & = & d(\phi)+1 \\ d(\phi\wedge\psi) & = & \max(d(\phi),d(\psi)) \\ d(\neg\phi) & = & d(\phi). \end{array}$$

Intuitively, the degree of a formula tells you how deep down the model the formula can *see*, how much of the model is relevant to it. The degree of a path-term $\phi_0?;x_1;\phi_1?;\ldots;x_n;\phi_n?$ is defined as $d(\phi_0\wedge\langle x_1\rangle(\phi_1\wedge\ldots\langle x_n\rangle(\phi_n)\ldots))$.

Given a finite set of variables $X\subset\mathcal{V}$ and a certain $n\in\mathbb{N}$, we may define $\text{MOD}_n(X)$, the set of all modal formulas $\phi$ of degree at most $n$ and that only contain modalities $\langle x\rangle$ with $x\in X$. There are, up to equivalence, only finitely many formulas in $\text{MOD}_n(X)$. Thus there exists a finite set $\text{FIN}_n(X)\subset\text{MOD}_n(X)$ such that if $\phi\in\text{MOD}_n(X)$ then there is a formula $\psi\in\text{FIN}_n(X)$ equivalent to $\phi$.

Whenever $s$ is a point in a Kripke model (having relations $\xrightarrow{x}$ for each $x\in X$) the *total* $n$*-description* of $s$ is defined as:

$$\bigwedge\{\phi\in\text{FIN}_n(X)\mid s\Vdash\phi\}.$$

As there are only finitely many subsets of $\text{FIN}_n(X)$, there are only finitely many total descriptions.

LEMMA 4.8 (Normal Form Lemma). *Let* $t:=\phi_0?;x_1;\phi_1?;\ldots;x_n;\phi_n?$ *be a path-term whose degree is at most* $m$ *and whose variables occur among the finite set* $X$*. Then* $t$ *is provably equivalent to a path-term of the form* $\chi_0?;x_1;\chi_1?;\ldots;x_n;\chi_n?$*, where*

1. *Each* $\chi_i$ *is a disjunction of total* $(m-i)$*-descriptions.*
2. *For each* $i<n$*:* $\chi_i\models\langle x_{i+1}\rangle\chi_{i+1}$.
3. *For each* $i<n$*, whenever* $\psi\in\text{MOD}_{(m-i)-1}(X)$ *and* $\chi_i\models[x_{i+1}]\psi$*, then* $\chi_{i+1}\models\psi$.

*Proof.* Define:

$$\begin{array}{ll} \mu_n:=\phi_n & \pi_n:=\phi_n? \\ \mu_i:=\phi_i\wedge\langle x_{i+1}\rangle\mu_{i+1} & \pi_i:=\phi_i?;x_{i+1};\pi_{i+1}\quad(\text{if } i<n). \end{array}$$

Note that $\pi_0=t$. Clearly $\models\sim\sim\pi_i=\mu_i?$. Furthermore, as $\mu_i\models\phi_i$, $\models\sim\sim\pi_i;\phi_i?=\mu_i?$. So $\vdash\sim\sim\pi_i;\phi_i?=\mu_i?$, by Lemma 4.3. Thus for any $i<n$: $\vdash\pi_i=\sim\sim\pi_i;\pi_i=\sim\sim\pi_i;\phi_i?;x_{i+1};\pi_{i+1}=\mu_i?;x_{i+1};\pi_{i+1}$, by **A6**. Using this fact we get $\vdash t=\mu_0?;x_1;\mu_1?;\ldots;x_n;\mu_n?$. So we have already fulfilled the second of our desired results: for $i<n$, $\mu_i\models\langle x_{i+1}\rangle\mu_{i+1}$, by definition. The others need not yet be satisfied, so we cannot stop here.

We prove, by induction on $i \leq n$, the following: there are sets $D_j$ of total $(m-j)$-descriptions (with $j \leq i$) with:

$$\vdash t = (\bigvee D_0)?; x_1; (\bigvee D_1)?; \ldots ; x_i; (\bigvee D_i)?; x_{i+1}; \mu_{i+1}?; \ldots ; x_n; \mu_n?$$

such that moreover:

1. $\bigvee D_j \models \langle x_{j+1} \rangle \bigvee D_{j+1}$ for every $j < i$.
2. $\bigvee D_i \models \langle x_{i+1} \rangle \mu_{i+1}$ if $i < n$.
3. If $\phi \in \text{MOD}_{(m-j)-1}(X)$ (for each $j < i$) and $\bigvee D_j \models [x_{j+1}]\phi$ then $\bigvee D_{j+1} \models \phi$.

First of all note that $\mu_0$ has a degree of at most $m$, otherwise $t$ would have a higher degree than $m$. So $D_0$ is readily picked, since any modal formula of depth at most $m$ is equivalent to a disjunction of total $m$-descriptions. Thus $\vdash \mu_0? = (\bigvee D_0)?$, hence $\vdash t = (\bigvee D_0)?; x_1; \mu_1?; \ldots ; x_n; \mu_n?$. The desired results obviously hold: we only need to check the second item and as $\mu_0 \models \langle x_1 \rangle \mu_1$, this does not change when we change to an equivalent formula.

So suppose we have found an appropriate term for $i$ and we are looking for one for $i+1$. Let $N$ (for "*n*ecessary") be the unique minimal set of total $((m-i)-1)$-descriptions such that $\bigvee D_i \models [x_{i+1}] \bigvee N$. This set must exist. For the disjunction of the set $A$ of *all* total $((m-i)-1)$-descriptions is a tautology, hence $\bigvee D_i \models [x_{i+1}] \bigvee A$ holds. Furthermore, if $N_1$ and $N_2$ are two minimal sets satisfying the requirement then $\bigvee D_i \models [x_{i+1}]((\bigvee N_1) \wedge (\bigvee N_2))$. Because $N_1$ and $N_2$ are sets of total descriptions (and hence *different* elements of $N_1$ and $N_2$ are incompatible), $(\bigvee N_1) \wedge (\bigvee N_2)$ is equivalent to $\bigvee (N_1 \cap N_2)$. Using the assumption that $N_1$ and $N_2$ are minimal, we must conclude that $N_1 = N_1 \cap N_2 = N_2$. Thus uniqueness is guaranteed.

Choose $D_{i+1} := \{\delta \in N \mid \delta \models \mu_{i+1}\}$. By Lemma 4.3 and the fact that $\bigvee D_i$ implies $[x_{i+1}] \bigvee N$:

$$\vdash (\bigvee D_i)?; ([x_{i+1}] \bigvee N)? = (\bigvee D_i)?$$

Thus $(\bigvee D_i)?; x_{i+1}; \mu_{i+1}?$ is equal in **AX** to:

$$(\bigvee D_i)?;\ (x_{i+1} \Rightarrow (\bigvee N)?)\ ; x_{i+1}; \mu_{i+1}?$$

By **A8**, this equals:

$$(\bigvee D_i)?;\ (x_{i+1} \Rightarrow (\bigvee N)?)\ ; x_{i+1};\ (\bigvee N)?\ ; \mu_{i+1}?$$

But $(\bigvee N) \wedge \mu_{i+1}$ is modally equivalent to $\bigvee D_{i+1}$, hence the above term is provably equal to:

$$(\bigvee D_i)?; x_{i+1}; (\bigvee D_{i+1})?$$

So:

$$\vdash t = (\bigvee D_0)?; x_1; (\bigvee D_1)?; \ldots ; x_i; (\bigvee D_{i+1})?; x_{i+2}; \mu_{i+2}?; \ldots ; x_n; \mu_n?$$

as desired.

What remains is to verify the desired results:

1. We have to show $\bigvee D_i \models \langle x_{i+1} \rangle \bigvee D_{i+1}$. By the induction hypothesis we have $\bigvee D_i \models \langle x_{i+1} \rangle \mu_{i+1}$. By definition of $N$, we have $\bigvee D_i \models [x_{i+1}] \bigvee N$. Combining these two statements we get that $\bigvee D_i \models \langle x_{i+1} \rangle ((\bigvee N) \wedge \mu_{i+1})$. As $\bigvee D_{i+1}$ is equivalent to $(\bigvee N) \wedge \mu_{i+1}$, we are done.
2. Suppose that $i + 1 < n$. Then we need to show that $\bigvee D_{i+1} \models \langle x_{i+2} \rangle \mu_{i+2}$. Each $\delta \in D_{i+1}$ implies $\mu_{i+1}$, hence $\bigvee D_{i+1} \models \mu_{i+1}$, which by definition implies $\langle x_{i+2} \rangle \mu_{i+2}$.
3. Let $\psi$ be a modal formula of degree at most $(m - i) - 1$ and suppose $\bigvee D_i \models [x_{i+1}]\psi$. If $\bigvee N \not\models \psi$ then for some $\delta \in N$, $\delta \not\models \psi$ (and as $\delta$ is a total description, $\delta \models \neg\psi$). But then $\bigvee D_i \models [x_{i+1}] \bigvee (N - \{\delta\})$, which contradicts the minimality of $N$. So $\bigvee N$ *does* imply $\psi$. As $D_{i+1} \subseteq N$, $\bigvee D_{i+1}$ also implies $\psi$, which is what we had to prove.

The desired formula for the proof is thus

$$(\bigvee D_0)?; x_1; (\bigvee D_1)?; \ldots ; x_n; (\bigvee D_n)?$$

$\square$

A path-term that satisfies the constraints of the above lemma is said to be in *normal form*. We have shown that any term can be proved equal to a normal form.

THEOREM 4.9 (Completeness theorem). $\models t_1 = t_2$ *iff* $\vdash t_1 = t_2$.

*Proof.* Soundness is trivial to verify. For completeness, suppose $\models t_1 = t_2$. By Lemmas 4.6 and 4.8, we may assume that these terms are in normal form (with respect to $m = \max(d(t_1), d(t_2))$ and an $X$ containing the variables of $t_1$ and $t_2$). For reference, suppose:

$$t_1 := \phi_0?; x_1; \phi_1?; \ldots ; x_n; \phi_n?$$

$$t_2 := \psi_0?; y_1; \psi_1?; \ldots ; y_k; \psi_k?$$

We may furthermore assume that $t_1$ and $t_2$ are nonempty, otherwise we have already seen that $\vdash t_1 = t_2$ (Lemma 4.5). But if they are nonempty, by Lemma 4.7 they must have the same shape: $n = k$, $x_1 = y_1, \ldots , x_n = y_n$. We will show that $\phi_i$ is semantically equivalent to $\psi_i$ for each $i \leq n$, hence $\vdash \phi_i? = \psi_i?$ for such $i$ (by Lemmas 4.3 and 4.4). This is of course sufficient to show that $\vdash t_1 = t_2$.

Suppose $\phi_i \not\models \psi_i$. $\phi_i$ is the disjunction of a set $D_i$ of total $(m - i)$-descriptions, so for some $\delta \in D_i$: $\delta \not\models \psi_i$. As $\psi_i$ has degree $m - i$, $\delta$ in fact implies $\neg\psi_i$. We will show that this $\delta$ must occur at the $i$-th step of a $t_1$-path in some Kripke model.

Suppose that there exists no path (in any model) of the form $s_0 \xrightarrow{x_1} s_1 \xrightarrow{x_2} \ldots s_{i-1} \xrightarrow{x_i} s_i$ with $s_j \Vdash \phi_j$ for each $j < i$ and $s_i \Vdash \delta$. Then:

$$\phi_0 \wedge \langle x_1\rangle(\phi_1 \wedge \ldots \langle x_{i-1}\rangle(\phi_{i-1} \wedge \langle x_i\rangle\delta)\ldots)$$

must be a contradiction, and its negation:

$$\phi_0 \to [x_1](\phi_1 \to \ldots [x_{i-1}](\phi_{i-1} \to [x_i]\neg\delta)\ldots)$$

is valid. Now $\phi := \phi_1 \to [x_2](\phi_2 \to \ldots [x_{i-1}](\phi_{i-1} \to [x_i]\neg\delta)\ldots)$ is of depth $m-1$ and $\phi_0 \models [x_1]\phi$. So, by the fact that $t_1$ is in normal form, $\phi_1 \models \phi$ and thus $\phi_1 \models [x_2](\phi_2 \to \ldots [x_{i-1}](\phi_{i-1} \to [x_i]\neg\delta)\ldots)$. Repeating this argument, we arrive at the conclusion that $\phi_i \models \neg\delta$. But then $\delta \models \neg\delta$, which is contradictory with the fact that $\delta$ is a total description and thus consistent.

So we have a path of the required form. It may still not be of the desired length (if $i \neq n$), but as $\phi_i$ implies $\langle x_{i+1}\rangle(\phi_{i+1} \wedge \ldots \langle x_n\rangle\phi_n \ldots)$ (due to the normal form of $t_1$), we can extend the path to a longer path $s_0 \xrightarrow{x_1} s_1 \xrightarrow{x_2} \ldots\ldots \xrightarrow{x_{n-1}} s_n$ with $s_j \models \phi_j$ for each $j \leq n$ and in particular $s_i \models \delta$. So this is a $t_1$-path. We may assume that it occurs in an unraveled model (see Section 3). By the assumption that $\models t_1 = t_2$, this same path must also be a $t_2$-path. But then $s_i \Vdash \psi_i$, which is in contradiction with the assumption that $\delta \models \neg\psi_i$.

We have thus proved that $\phi_i \models \psi_i$. The proof that $\psi_i \models \phi_i$ is entirely analogous. $\square$

## 5. Atomic Tests and Union

The operations $\sim$ and ; are not just of interest to dynamic semantics, but also to computer science, as far as the study of Propositional Dynamic Logic (PDL) (Harel, 1984) is concerned. Van Benthem (1993) shows that the first order definable operations that are safe for bisimulation are precisely the ones that can be constructed from the following repertoire: atomic tests $p$?, the basic binary relations, dynamic negation, relational composition and union $\cup$. These are precisely the first order definable PDL-programs.*

Our technique naturally generalizes to this broader language. Dynamic relation algebras are the same as before, except for the extra binary operation $\cup$ (simply interpreted as union of relations) and possibly infinitely many constants $p$? which have to be interpreted as some subset of the diagonal.

The valid equations in this setting are axiomatized by **AX** in addition to the axioms displayed in Table II (let us call this enlarged set of equations **BX**). **A5**, **A7** and **A9** may now be omitted, as they become derivable in the rest of the system.

We sketch a proof of the completeness theorem. First of all, any term $t$ can be proved equal to a union of path-terms. As an example, consider four path-terms

* Dynamic negation is usually not present in presentations of PDL, but it is implicitly present, as modal tests $\phi$? for any modal $\phi$ are considered PDL-programs and $\sim t$ is equivalent to $([t]\bot)$?.

Table II. Extra axioms for $p?$ and $\cup$.

| | | |
|---|---|---|
| **B1:** | $p? = \sim\sim p?$ | (test) |
| **B2:** | $x \cup (y \cup z) = (x \cup y) \cup z$ | (associativity) |
| **B3:** | $x \cup y = y \cup x$ | (commutativity) |
| **B4:** | $x \cup x = x$ | (idempotency) |
| **B5:** | $(x \cup y); z = (x; z) \cup (y; z)$ | (left-distribution) |
| **B6:** | $x; (y \cup z) = (x; y) \cup (x; z)$ | (right-distribution) |
| **B7:** | $x \cup \perp = x$ | ($\cup$-unit) |
| **B8:** | $\sim (x \cup y) = \sim x; \sim y$ | (De Morgan) |
| **B9:** | $\sim x \cup \sim y = \sim\sim (\sim x \cup \sim y)$ | (test union) |

$t_1, \ldots, t_4$. Then $(t_1 \cup t_2); (t_3 \cup t_4)$ is equivalent to $(t_1; t_3) \cup (t_1; t_4) \cup (t_1; t_3) \cup (t_1; t_4)$, using only **B5** and **B6**. The latter term is a union of path-terms, as desired.

Furthermore, we can make sure that the path-terms occurring in these unions only contain modal tests $\phi?$ where $\phi$ is a total description. For instance, suppose we have a path-term $t := \phi_1?; x; \phi_2?$ and an $m \geq d(t)$. Then $\phi_1$ is equivalent to some disjunction $\bigvee D_1$ of total $m$-descriptions and $\phi_2$ to a disjunction $\bigvee D_2$ of total $(m-1)$-descriptions. By **B9**, $(\bigvee D_i)? = \bigcup\{\delta? \mid \delta \in D_i\}$, so using **B5** and **B6** again we may derive that:

$$t = \bigcup\{\delta_1?; x; \delta_2? \mid \delta_1 \in D_1, \delta_2 \in D_2\}.$$

Now if $\models t_1 = t_2$, $\vdash t_1 = \bigcup_{i \in I} \pi_i$ and $\vdash t_2 = \bigcup_{j \in J} \rho_j$, where the $\pi_i$ and $\rho_j$ are path-terms in the restricted format described above, then if $\pi_i$ is a nonempty path-term, it must be actually equal to some $\rho_j$ and vice versa (we again use unraveling here). So using **B7** to remove empty paths, and **B2–4** to order the unions as we please, $\vdash t_1 = t_2$. We see that the completeness proof is even simpler when we extend the language with union: the subtlety of the Normal Form Lemma 4.8 proves unnecessary.

## 6. Representable Dynamic Relation Algebras

We have proved that the variety generated by $\mathsf{DRA}$ (i.e. $\mathbf{V}(\mathsf{DRA})$) is the same as the set of algebras that satisfy the equations of **AX**. An interesting question is what $\mathbf{V}(\mathsf{DRA})$ actually looks like. A similar question may be asked of dynamic relation algebras with union.

By Birkhoff's theorem (see, for instance, Burris and Sankappanavar, 1981), $\mathbf{V}(\mathsf{DRA}) = \mathbf{HSP}(\mathsf{DRA})$. In words: every algebra in the variety is a homomorphic image of a subalgebra of products of $\mathsf{DRA}$s. It is well known that for standard relation algebra, this general statement can be sharpened somewhat: $\mathbf{V}(\mathsf{RA}) = \mathbf{SP}(\mathsf{RA})$, i.e. closure under homomorphic images may be left out. This means that algebras that satisfy the equations valid in all relation algebras are *representable* as relations: the objects are (modulo isomorphism of course) sequences of relations $(R_i)_{i \in I}$ on which the operations are computed componentwise.

Both for DRAs with and without union, the following hold:

1. Representable algebras are subalgebras of DRAs: this is because $\mathbf{P}(\mathsf{DRA}) \subseteq \mathbf{S}(\mathsf{DRA})$.
2. Alas, there exist nonrepresentable algebras in $\mathbf{V}(\mathsf{DRA})$ (contrary to the situation in classical relation algebra).

First we prove these statements for DRAs without union.

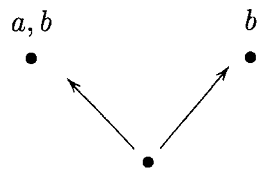
1. $\prod_{i\in I} \mathsf{DRA}(S_i)$ can be embedded into $\mathsf{DRA}(\biguplus_{i\in I} S_i)$ (where $\biguplus$ denotes disjoint union), via the embedding:

   $$\iota : (R_i)_{i\in I} \mapsto \biguplus_{i\in I} R_i.$$

2. Consider the following *quasi-equation* $Q$:

   $$(\sim\sim x \,; y = \sim\sim x \wedge \sim x \,; y = \sim x) \to y = \mathrm{ID}.$$

   A quasi-equation holds in an algebra if its universal closure holds in the algebra. It can be verified that $Q$ holds in all dynamic relation algebras. Quasi-equations are preserved under subalgebras and products, so $\mathbf{SP}(\mathsf{DRA}) \models Q$. Therefore, if we could find an algebra in $\mathbf{V}(\mathsf{DRA})$ that does not satisfy $Q$ we would be done.
   To construct such an algebra, we first observe that *Heyting algebras* (van Dalen, 1986) are in $\mathbf{V}(\mathsf{DRA})$ when we interpret ; by conjunction and $\sim$ by negation. This is proved by simply checking that such algebras satisfy the equations in **AX**. But $Q$ is not satisfied in all Heyting algebras. An example is given by the following intuitionistic Kripke structure:

   In the corresponding Heyting algebra $\neg\neg a \wedge b = \neg\neg a$ (the set of worlds where $\neg\neg a \wedge b$ holds is the same as the set of worlds where $\neg\neg a$ holds: just the upper left world), $\neg a \wedge b = \neg a$ but $b \neq \neg\bot$, as $b$ does not hold in the bottom world. So there are unrepresentable algebras in $\mathbf{V}(\mathsf{DRA})$: **H** is necessary in the equation $\mathbf{V}(\mathsf{DRA}) = \mathbf{HSP}(\mathsf{DRA})$.

Next, we consider the case of DRAs *with* union. For the rest of this section DRA denotes dynamic relation algebras of this richer signature.

1. The same trick as above works when we add union.
2. Heyting Algebras can be made into algebras of the right signature by interpreting $\sim$,; and $\cup$ by $\neg$, $\wedge$ and $\vee$ respectively. This does not always give us an algebra satisfying the equations in **BX**: the troublesome equation is **B9**. But a Heyting algebra based on a *linear* Kripke structure, i.e. one where the partial order satisfies $\forall x, y(x \leq y \vee y \leq x)$, *does* satisfy **B9**, so such algebras are in the variety $\mathbf{V}(\mathsf{DRA})$.

   Now consider the following linear Kripke structure:

   In the corresponding Heyting algebra $a \subseteq \text{ID}$ (where $x \subseteq y$ is defined as $x \cup y = y$) while $a \neq \sim\sim a$. So if this algebra were representable, we would have a direct contradiction with the fact that $x \subseteq \text{ID} \rightarrow x = \ \sim\sim x$ is a quasi-equation valid in all $\mathsf{DRA}$s.

## 7. Further Research

In this paper we have shown that the $\{\bot, \sim, ;\}$-fragment may be axiomatized by essentially coding modal logic in the calculus, plus a few extra tricks to deal with proper paths. The notion of unraveling was especially useful in proving the completeness theorem. These same methods were used to prove a completeness theorem for an extended fragment that includes union and atomic tests. We list a few possible extensions that the results in this paper may lead to.

Extending our axiom systems to larger fragments including intersection $t \cap t'$ and iteration $t^*$ call for different methods, intersection because it is not safe for bisimulation and our unraveling arguments will fail, and iteration because of its infinitary nature. For iteration we should not expect a finite equational axiomatization, due to Redko's (1964) result (see also Conway, 1971). Extending the set of operations with converse or a constant for the universal relation should not be difficult: to get completeness theorems for these we should consult axiomatizations for temporal modal logic (see, for instance, Goldblatt, 1992) and modal logic with a universal modality (Blackburn, 1993; Gargov and Goranko, 1993) respectively.

An alternative road to axiomatizing validity in our fragment is to design a sequent calculus for these systems, where a sequent $\phi_1, \ldots, \phi_n \vdash \psi$ corresponds to $\models ((\phi_1; \ldots; \phi_n) \Rightarrow \psi) = \text{ID}$, i.e. to the usual notion of validity in DPL (discussed in Groenendijk and Stokhof, 1991). The work of (Groeneveld, 1995) may be relevant to such a calculus.

Of most interest however is to extend the result to real DPL-equality, and axiomatize which DPL-formulas are equal as relations. Such an axiomatization would have to have at least the following as extra axioms:

$$\begin{array}{lcl} P(x_1,\ldots,x_n) & = & \sim\sim P(x_1,\ldots,x_n) \\ \exists x;\exists y & = & \exists y;\exists x \\ \exists x;\exists x & = & \exists x \\ \sim\exists x & = & \perp, \end{array}$$

where the variables in these equations are not algebraic variables: $P(x_1,\ldots,x_n)$ and $\exists x$ are viewed as constants. By the result in (Visser, 1995), only axioms that contain these constants need to be added. As an intermediate step we could consider adding constants for equivalence relations to DRAs, and then finding an axiomatization: note that all constants of DPL denote equivalence relations themselves.

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