JOURNAL OF ALGEBRA 64, 37-45 (1980)

A Krohn-Rhodes Theorem for Categories

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Communicated by Saunders Mac Lane

Received December 8, 1978

1. INTRODUCTION

Recently, W. Nico [5] gave a general construction for functors between small categories which generalizes the notion of the kernel of a group homomorphism. Tilson's "derived semigroup" of a homomorphism (Eilenberg [1]) is in the monoid case essentially a special case of Nico's concept. I say "essentially" because Nico's construction yields in general a category, not a monoid, even when the functor goes between monoids; the derived semigroup is obtained from the Nico category by adjoining a zero which is to be the product of arrows which do not compose.

This situation suggests that the theory of group complexity of Rhodes, Tilson, et al., has its proper setting in finite categories instead of in finite semigroups. The first step in developing a theory of group complexity for categories has to be the formulation and proof of a generalization of the Krohn-Rhodes Theorem. That is accomplished in this paper; after some preliminaries the main Theorem is stated and proved in Section 5, and a definition of the group complexity of a finite category is suggested there. Except for the proof of Proposition 5.1, the paper is completely selfcontained and can be read by anyone with a basic knowledge of categories and functors.

2. THE WREATH PRODUCT

Let $\mathbf{B}$ and $\mathbf{C}$ be small categories and $G: \mathbf{C} \to \mathbf{Set}$ a functor. $\mathbf{B}wr^G\mathbf{C}$ (or $\mathbf{B}wr\mathbf{C}$ if G is clear from context), the *wreath product* of $\mathbf{B}$ by $\mathbf{C}$ induced by G, is a category defined as follows. An object of $\mathbf{B}wr^G\mathbf{C}$ is a pair (P, c) with c an object of $\mathbf{C}$ and $P: cG \to |\mathbf{B}|$ ($|\mathbf{B}|$ is the set of objects of $\mathbf{B}$) a function. It is useful to think of P as a set of objects of $\mathbf{B}$ indexed by the set cG. An arrow $(\lambda, f): (P, c) \to (Q, d)$ has $f: c \to d$ an arrow of $\mathbf{C}$ and $\lambda: cG \to \mathbf{B}$ a function with the property that for each $x \in cG$, $x\lambda: xP \to x(fG \circ Q)$. In order to define

0021-8693/80/050037-09$02.00/0

composition of arrows, a definition is needed. For $\lambda, \lambda' : cG \to \mathbf{B}$ such that for all $x \in cG$, $\mathrm{cod}(x\lambda) = \mathrm{dom}(x\lambda')$, define $\lambda * \lambda'$ by

$$x \cdot (\lambda * \lambda') = x\lambda \circ x\lambda'. \tag{2.1}$$

Then for $(\lambda, f) : (P, c) \to (Q, d)$ and $(\mu, g) : (Q, d) \to (R, e)$, set

$$(\lambda, f) \circ (\mu, g) = (\lambda * (fG \circ \mu), f \circ g); \tag{2.2}$$

since $x\lambda : xP \to x \cdot (fG \circ Q) = \mathrm{dom}(x \cdot fG)\mu$, the condition for defining $\lambda * (fG \circ \mu)$ is satisfied. If each cG is regarded as a discrete category, then P, Q and R are functors, $\lambda : P \to fG \circ Q$, $fG \circ \mu : fG \circ Q \to (f \circ g)(G \circ R)$ are natural transformations, and $*$ is the usual horizontal composition of natural transformations. This observation is the basis for the more general definition of wreath product for **Cat**-valued functors given by Kelly [3] (who calls it the composite) and Wells [7].

If $F : \mathbf{B} \to \mathbf{Set}$ is also a functor, there is an induced functor $FwrG : \mathbf{B}wr^G\mathbf{C} \to \mathbf{Set}$ defined as follows.

$$(P, c) \cdot FwrG = \{(m, x) \mid x \in cG, m \in x \cdot PF\} \tag{2.3}$$

for any object (P, c) of $\mathbf{B}wr^G\mathbf{C}$, and if $(m, x) \in (P, c) \cdot FwrG$ and $(\lambda, f) : (P, c) \to (Q, d)$ in $\mathbf{B}wr^G\mathbf{C}$, then

$$(m, x) \cdot [(\lambda, f) \cdot FwrG] = (m \cdot (x \cdot \lambda F), x \cdot fG). \tag{2.4}$$

Here, $\lambda F : cG \to \mathbf{Set}$ is the set-function-valued function such that for $x \in G$,

$$x \cdot \lambda F = (x\lambda)F : xPF \to [x \cdot (fG \circ Q)]F. \tag{2.5}$$

The wreath product is associative in the following precise sense:

PROPOSITION 2.1. *Let* $F : \mathbf{B} \to \mathbf{Set}$, $G : \mathbf{C} \to \mathbf{Set}$, $H : \mathbf{D} \to \mathbf{Set}$ *be functors. Then there is a category isomorphism I making this diagram commute:*

$$\begin{tikzcd} \mathbf{B}wr^{GwrH}(\mathbf{C}wr^H\mathbf{D}) \arrow[rr, "I"] \arrow[dr, "Fwr(GwrH)"'] & & (\mathbf{B}wr^G\mathbf{C})\, wr^H\mathbf{D} \arrow[dl, "(FwrG)wrH"] \\ & \mathbf{Set} & \end{tikzcd}$$

Proof. For $(\lambda, (\mu, g)) : (P, (Q, d)) \to (P', (Q', d'))$ in $\mathbf{B}wr(\mathbf{C}wr\mathbf{D})$, I shall define a function $T_{\lambda,\mu}$ with domain dH; its value at $y \in dH$ will be a function from $(yQ)G$ to the arrows of Bwr^GC, as follows:

$$x \cdot yT_{\lambda,\mu} = ((x, y)\lambda, y\mu) : (P, yQ) \to (P', y \cdot (y\mu H \circ Q')). \tag{2.6}$$

Then set

$$(\lambda, (\mu, g))I = (T_{\lambda,\mu}, g). \tag{2.7}$$

A verification that I fulfills the claims of the Proposition is painful but straightforward.

Via the second projection, $\mathbf{B}wr^G\mathbf{C}$ is the discrete split normal fibration corresponding to the functor $\mathbf{Set}((-)G, \mid \mathbf{B} \mid)$ where $\mid \mathbf{B} \mid$ is the set of objects of $\mathbf{B}$. This statement reduces in the case that $\mathbf{B}$ and $\mathbf{C}$ are semigroups to the well-known fact (see Wells [6]) that the wreath product is a certain semidirect product. If $\mathbf{B}$ and $\mathbf{C}$ are groupoids the construction is both more and less general than that of Houghton [2]. His untwisted wreath product $\mathbf{B}wr^G\mathbf{C}$ of groupoids is in my terminology the wreath product obtained when $G\colon \mathbf{C} \to \mathbf{Set}$ is the "global hom functor" defined by setting $cG = \bigcup \mathbf{C}(-, c)$ (union over all objects of $\mathbf{C}$) for c an object, and on arrows by composition. In Wells [7] I propose a general definition for the twisted wreath product with respect to $\mathbf{Cat}$-valued functors.

3. Holonomy Groups

Let $F\colon \mathbf{C} \to \mathbf{Set}$ be a finite-set-valued functor where $\mathbf{C}$ is a finite category. There is no harm in assuming that F is *separated*, that is, that $cF \cap dF$ is empty if c and d are distinct objects of $\mathbf{C}$. In fact, every set-valued functor is naturally equivalent to a separated one. Define a functor $\mathcal{O}_F\colon \mathbf{C} \to \mathbf{Set}$ as follows: if c is an object of $\mathbf{C}$, $c\mathcal{O}_F$ is the subset of the powerset of cF consisting of all singletons $\{x\}$ for $x \in cF$ and all sets $\mathrm{im}(fF)$ for all arrows $f\colon b \to c$ of $\mathbf{C}$ (a fortiori, $cF \in c\mathcal{O}_F$). If $g\colon c \to d$ in $\mathbf{C}$, $g\mathcal{O}_F\colon c\mathcal{O}_F \to d\mathcal{O}_F$ is the image function induced by gF.

Suppose $A \in c\mathcal{O}_F$, $B \in d\mathcal{O}_F$. Following Eilenberg [1], $B \leqslant A$ means that there is some arrow $f\colon c \to d$ for which $B \subset A \cdot f\mathcal{O}_F$. The relation $\leqslant$ is a preorder (transitive and reflexive relation, called a quasiorder by some). Like any preorder, $\leqslant$ induces an equivalence relation $\sim$ defined by requiring $A \sim B$ if $A \leqslant B$ and $B \leqslant A$.

Proposition 3.1 (Eilenberg [1, p. 44]). *If $A \in c\mathcal{O}_F$, $B \in d\mathcal{O}_F$, $A \sim B$ and $f\colon c \to d$ has the property that $B \subset A \cdot f\mathcal{O}_F$, then $B = A \cdot f\mathcal{O}_F$ and there is an arrow $\bar{f}\colon d \to c$ of $\mathbf{C}$ such that $A = B \cdot \bar{f}\mathcal{O}_F$, $(f \circ \bar{f})F$ restricted to A is id_A, and $(\bar{f} \circ f)F$ restricted to B is id_B.*

Proof. By definition there is an arrow $g\colon d \to c$ such that $A \subset B \cdot g\mathcal{O}_F$. Since $B \subset Af\mathcal{O}_F \subset B \cdot gf\mathcal{O}_F$ and $\#(B \cdot gf\mathcal{O}_F) \leqslant \#B$ (where $\#$ denotes cardinality), one has $B = A \cdot f\mathcal{O}_F = B \cdot gf\mathcal{O}_F$ and analogously $A = B \cdot g\mathcal{O}_F = A \cdot fg\mathcal{O}_F$. Thus $(f \circ g)F$ restricted to A is a permutation of A, so for some integer $n > 1$, $(f \circ g)^n F$ restricted to A is id_A. Define $\bar{f} = g \circ (f \circ g)^{n-1}$ and the Proposition follows.

It follows from Proposition 3.1 that if $A \sim B$ then $\#A = \#B$.

For any object c of $\mathbf{C}$ and set $A \in c\mathcal{O}\!\mathcal{l}_F$, let $\mathscr{B}_A$ denote the set of proper subsets B of A which are in $c\mathcal{O}\!\mathcal{l}_F$ and are maximal in that respect; so $\mathscr{B}_A = \{B \mid B \subset A, B \neq A, B \in c\mathcal{O}\!\mathcal{l}_F$, and $(C \subset A, C \neq A, B \subset C, C \in c\mathcal{O}\!\mathcal{l}_F) \Rightarrow B = C\}$. Note that $A = \bigcup_{B \in \mathscr{B}_A} B$ since every singleton is in $c\mathcal{O}\!\mathcal{l}_F$. If $f: c \to c$ in $\mathbf{C}$ is an arrow such that $A \cdot \mathcal{O}\!\mathcal{l}_F = A$, then by Proposition 3.1 f induces a permutation of A and therefore of $\mathscr{B}_A$. The set of permutations of $\mathscr{B}_A$ obtained in this way is a (not necessarily transitive) permutation group, denoted $\mathscr{H}_A$ and called the ***holonomy group*** of A. Observe for later use that $\mathscr{H}_A$ is a quotient of a submonoid of the $\mathbf{C}$-endomorphism monoid of c. It is straightforward to see that if $A \sim B$, then $\mathscr{H}_A \cong \mathscr{H}_B$.

Let $A_1, A_2, \ldots, A_N$ be a set of representatives of those equivalence classes (mod $\sim$) whose members have cardinality *greater* than one, indexed in such a way that if $i < j$ then $\#A_i \leqslant \#A_j$. Following Eilenberg [1, II.7], define the ***height*** Ah of an element A of $c\mathcal{O}\!\mathcal{l}_F$ (c any object of $\mathbf{C}$) as follows: (a) A singleton has height 0. (b) If $A \sim A_i$ then $Ah = i$. Observe that the function h respects the equivalence relation $\sim$ and the preorder $\leqslant$ (if $A \leqslant B$ then $Ah \leqslant Bh$). The height function will be the basis for the inductive proof of Proposition 5.2.

A functor $F': \mathbf{C} \to \mathbf{Set}$ is a *subfunctor* of $F: \mathbf{C} \to \mathbf{Set}$ if for each arrow $f: b \to c$ of $\mathbf{C}$, $bF' \subset bF$ and $fF' = fF \mid bF'$ (the vertical line denotes restriction). A natural transformation between $\mathbf{Set}$-valued functors is *surjective* if each component map is surjective.

4. Covering

A functor $F: \mathbf{C} \to \mathbf{D}$ which is surjective on arrows *lifts composition* if for all $t: w \to x$, $u: x \to y$, $v: y \to z$ of $\mathbf{D}$, if $g: c \to d$ in $\mathbf{C}$ and $gF = u$, then there are $f: b \to c$ and $h: d \to e$ in $\mathbf{C}$ with $fF = t$, $hF = v$. (I called this *triangle-reflecting* in Wells [7] but that is a misuse of the word "reflect.")

Now let $F: \mathbf{C} \to \mathbf{Set}$, $G: \mathbf{D} \to \mathbf{Set}$ be functors. Then G *covers* F if there are

(a) a subcategory $\mathbf{D}' \subset \mathbf{D}$,

(b) a functor $G_0: \mathbf{D}' \to \mathbf{Set}$,

(c) a functor $H: \mathbf{D}' \to \mathbf{C}$, and

(d) a natural transformation $\theta: G_0 \to H \circ F$

for which

$$G_0 \text{ is a subfunctor of } G \mid \mathbf{D}', \tag{4.1}$$

$$H \text{ lifts composition (hence is surjective on arrows), and} \tag{4.2}$$

$$\theta \text{ is surjective.} \tag{4.3}$$

It follows that if $f\colon c \to c'$ is any arrow of $\mathbf{C}$, then there is an arrow $g\colon d \to d'$ of $\mathbf{D}$ such that $gH = f$ for which

$$\begin{array}{ccc} dG_0 & \xrightarrow{gG_0} & d'G_0 \\ \Big\downarrow{\scriptstyle d\theta} & & \Big\downarrow{\scriptstyle d'\theta} \\ cF & \xrightarrow{fF} & c'F \end{array} \tag{4.4}$$

commutes. Moreover, if $f'\colon c' \to c''$ is an arrow of $\mathbf{C}$, then there is $g'\colon d' \to d''$ of $\mathbf{D}$ (starting from the same d'!) with $g'H = f'$, and similarly for an arrow going into c. It is this sense that G simulates F. In Rhodes' language, *F divides G.*

As an aid to proving the Main Theorem, it is necessary to introduce a weaker notion of covering. Given $F\colon \mathbf{C} \to \mathbf{Set}$, $G\colon \mathbf{D} \to \mathbf{Set}$, *$G$ weakly covers F* if there is a subcategory $\mathbf{D}'' \subset \mathbf{D}$, a surjective function K from the objects of $\mathbf{D}''$ onto the objects of $\mathbf{C}$, and for each object d of $\mathbf{D}$ a function $d\bar{\theta}\colon dG \to dK\mathfrak{A}_F$ satisfying this condition:

(C) If $f\colon c \to c_0$ is an arrow of $\mathbf{C}$, and $dK = c$, $d_0K = c_0$, then there are objects e, e_0 and arrows $g\colon d \to e_0$, $g'\colon e \to d_0$ of $\mathbf{D}''$ such that

$$eK = c, \qquad e_0K = c_0 \tag{4.5}$$

$$\bigcup_{x \in dG} x \cdot d\bar{\theta} = cF \tag{4.6}$$

$$(x \cdot d\bar{\theta})(f\mathfrak{A}_F) \subset x \cdot gG \cdot e_0\bar{\theta} \qquad (\text{all } x \in dG), \qquad \text{and} \tag{4.7}$$

$$(y \cdot e\bar{\theta})(f\mathfrak{A}_F) \subset y \cdot g'G \cdot d_0\bar{\theta} \qquad (\text{all } y \in eG). \tag{4.7$'$}$$

G weakly covers F with *height i* if the function $\bar{\theta}$ in the preceding definition satisfies the requirement that for $x \in dG$ the set $x \cdot d\bar{\theta}$ is of height $\leqslant i$.

If $\mathbf{C}$ is a category, $\mathbf{C}\#$ denotes the category (preorder) with the same objects as $\mathbf{C}$, and

$$\mathrm{Hom}_{\mathbf{C}\#}(c, c') = \begin{cases} \text{singleton if } \mathrm{Hom}_{\mathbf{C}}(c, c') \neq \varnothing \\ \varnothing \quad \text{otherwise.} \end{cases}$$

Let $J\colon \mathbf{C}\# \to \mathbf{Set}$ be the unique functor which on objects takes c to $\{c\}$.

PROPOSITION 4.1. *Let $\mathbf{C}$ be a finite category and $F\colon \mathbf{C} \to \mathbf{Set}$ a finite-set-valued functor. Let N be the maximum value of the height function for F. Then $J\colon \mathbf{C}\# \to \mathbf{Set}$ weakly covers F with height N.*

Proof. In the notation of the definition of "weakly covers," take $\mathbf{D} = \mathbf{D}'' = \mathbf{C}\#$, K the identity function, $G = J$, $c\bar{\theta}\colon cJ \to c\mathfrak{A}_F$ the function taking $\{c\}$ to $\{cF\}$. Then (4.5)–(4.7′) follow easily (take $e = c$, $e_0 = c_0$ in (4.5)).

PROPOSITION 4.2. *Let* $\mathbf{C}$ *be a finite category and* $F: \mathbf{C} \to \mathbf{Set}$ *a faithful finite-set-valued functor. If* $G: \mathbf{D} \to \mathbf{Set}$ *weakly covers* F *with height* 0, *then* G *covers* F.

Proof. By hypothesis there is a subcategory $\mathbf{D}'' \subset \mathbf{D}$, a surjective function K from the objects of $\mathbf{D}''$ to the objects of $\mathbf{C}$, and for each object d of $\mathbf{D}''$ a function $d\bar{\theta}: dG \to dK\mathcal{O}_F$ that satisfy (4.5) through (4.7′). Let $\mathbf{D}'$ be the subcategory of $\mathbf{D}$ whose objects are the objects of $\mathbf{D}''$ and whose arrows consist of all the g, g' given by the definitions of "weakly covers with height 0" for all arrows f of $\mathbf{C}$. (It is easy to check that $\mathbf{D}'$ is closed under composition and has the requisite identity arrows.) For each object d of $\mathbf{D}'$ and each $x \in dG$, $x \cdot d\bar{\theta} = \{x \cdot d\theta\}$. Then $d\theta$ is surjective by (4.6). Define a functor $H: \mathbf{D}' \to \mathbf{C}$ as follows. For each object d, $dH = dK$. If $g: d \to d'$ in $\mathbf{D}'$, suppose $dH = c$, $d'H = c'$. By construction there is an arrow $f: c \to c'$ such that for all $x \in dG$,

$$(x \cdot d\bar{\theta})(f\mathcal{O}_F) \subset x \cdot gG \cdot d'\bar{\theta};$$

but since $d\bar{\theta}$ and $d'\bar{\theta}$ are singleton-valued, this means

$$(x \cdot d\theta) fF = x \cdot gG \cdot d'\theta. \tag{4.8}$$

There cannot be another f making (4.8) true because F is faithful and $d\theta$ is surjective. Let $gH = f$. Then if $g': d' \to d''$ is in D' with $g'H = f'$, we have

$$\begin{aligned}(x \cdot d\theta) fF \circ f'F &= [x \cdot gG \cdot d'\theta] f'F \\ &= x \cdot gg'G \cdot d''\theta \\ &= (x \cdot d\theta) \cdot ff'F\end{aligned}$$

so again because $d\theta$ is surjective and F is faithful H must preserve composition. H preserves identity arrows by a similar argument. Thus H is a functor. It follows from (4.8) that θ is a surjective natural transformation from $G \mid \mathbf{D}'$ to $H \circ F$. Taking $G_0 = G \mid \mathbf{D}'$, I have already verified (4.1) and (4.3). The assumption (C) forces H to lift composition, so (4.2) is true. This proves Proposition 4.2.

5. THE MAIN THEOREM

A functor $F: \mathbf{C} \to \mathbf{Set}$ is a *constant-function functor*, or *c.f. functor* for short, if for every arrow $f: c \to d$ of $\mathbf{C}$, fF is a constant function.

THEOREM. *Let* C *be a finite category and* $F: \mathbf{C} \to \mathbf{Set}$ *a faithful finite-set-valued functor. Then for some integer* n *there are categories* $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_n$ *and functors* $F_i: \mathbf{D}_i \to \mathbf{Set}$ *such that*

(i) *$F_1 wr F_2 wr \cdots wr F_n$ covers F, and*

(ii) *for each i, one of the following two possiblities hold:*

(a) *D_i is a finite category with no more objects than $\mathbf{C}$ has and F_i is a c.f. functor, or*

(b) *$\mathbf{D}_i$ is a group which is a homomorphic image of a submonoid of the endomorphism monoid of some object of $\mathbf{C}$, and F_i is the right regular representation of $\mathbf{D}_i$.*

Note. The Theorem is stated in a form intended to suggest the possibility of defining the "group complexity" of a finite category $\mathbf{C}$: that would presumably be the least number of groups appearing in any iterated wreath product $\mathbf{W}$ of finite categories $\mathbf{D}_1, \ldots, \mathbf{D}_n$ and functors $F_i: \mathbf{D}_i \to \mathbf{Set}$ where (a) for each i either F_i is a c.f. functor or $\mathbf{D}_i$ is a finite group and (b) there is a subcategory $\mathbf{D} \subset \mathbf{W}$ and a composition lifting functor $H: \mathbf{D} \to \mathbf{C}$. The iterated wreath product is well-defined by Proposition 2.1.

Proof of the Theorem. The theorem follows immediately from Propositions 5.1 and 5.2 below and Proposition 2.1. Some definitions are needed. If X is a set, $\bar{X}$ denotes the monoid consisting of the identity function and all constant transformations of X. If G is a permutation group on X, $\bar{G} = G \cup \bar{X}$ is a monoid of transformations of X.

PROPOSITION 5.1. *Let G be a permutation group on a set X. Let R be the action of $\bar{G}$ on X, κ the action of $\bar{X}$ on X, and ρ the right regular representation of G. Then R is covered by $\kappa wr \rho$: $\bar{X} wr G \to \mathbf{Set}$.*

Proof. Follows immediately from "Method II" of Meyer and Thompson [4]. Recent expositions are in Eilenberg [1, II, Cor. 3.2] and Wells [6, Theorem 13.2].

PROPOSITION 5.2. *Let $F: \mathbf{C} \to \mathbf{Set}$ be a faithful finite-set-valued functor, $\mathbf{C}$ a finite category. Let $A_1, \ldots, A_n$ be representatives of equivalence classes* (Mod $\sim$) *and let $\mathscr{H}_i = \mathscr{H}_{A_i}$ be the corresponding holonomy group. Then F is covered by*

$$R_1 wr R_2 wr \cdots wr R_N wr J: \mathscr{H}_1 wr \mathscr{H}_2 wr \cdots wr \mathscr{H}_N wr \mathbf{C}\# \to \mathbf{Set},$$

where R_i is the action of $\mathscr{H}_i$ on $\mathscr{B}_{A_i}$.

Proof. Suppose $F: \mathbf{C} \to \mathbf{Set}$ satisfies the hypotheses of the Proposition, and suppose $G: \mathbf{D} \to \mathbf{Set}$ weakly covers F with height i. I shall construct a functor $\bar{G}: \mathscr{H}_i wr^G \mathbf{D} \to \mathbf{Set}$ which weakly covers F with height $i - 1$. The Proposition then follows from Propositions 4.1 and 4.2.

Since $\mathscr{H}_i$ is a category with only one object, an object of $\mathscr{H}_i wr \mathbf{D}$ may be identified with an object of $\mathbf{D}$; however I shall write such an object d as d^i

when I regard it as an object of $\mathcal{H}_i wr\mathbf{D}$. An arrow $(\lambda, f)\colon d^i \to e^i$ has $f\colon d \to e$ in $\mathbf{D}$ and $\lambda\colon dG \to \mathcal{H}_i$ any function.

It is given that $G\colon \mathbf{D} \to \mathbf{Set}$ weakly covers F with height i. Thus there is a subcategory $\mathbf{D}'' \subset \mathbf{D}$, a function κ from the objects of $\mathbf{D}''$ to the objects of $\mathbf{C}$, and functions $d\bar{\theta}\colon dG \to dK\mathcal{O}_F$ satisfying (4.5) through (4.7′). To prove the Proposition it is necessary to construct a subcategory $\mathbf{E}$ of $\mathcal{H}_i wr_G\mathbf{D}$, a functor $\bar{G}\colon \mathcal{H}_i wr_G\mathbf{D} \to \mathbf{Set}$, a function L from the objects of $\mathbf{E}$ to the objects of $\mathbf{C}$, and arrows $d^i\psi\colon d^i\bar{G} \to d^iL\mathcal{O}_F$ satisfying the analogs of (4.5)–(4.7′) with all $d^i\psi$ of height $<i$. To start with, take the objects of $\mathbf{E}$ to be the objects of $\mathbf{D}''$ (via the identification of objects of $\mathcal{H}_i wr_G\mathbf{D}$ with objects of $\mathbf{D}$) and then take L to be K. $\bar{G}$ will be the restriction of $R_i wrG$ to $\mathbf{E}$. It is necessary to define ψ and the arrows of $\mathbf{E}$. I shall write $\mathcal{B}_i$ for $\mathcal{B}_{A_i}$.

Now, $d^iR_i wrG = \mathcal{B}_i \times dG$. Thus we must define $d^i\psi\colon \mathcal{B}_i \times dG \to dK\mathcal{O}_F$. (Remember $d\bar{\theta}\colon dG \to dK\mathcal{O}_F$.) A typical element of $\mathcal{B}_i \times dG$ is (B, x) where $x \in dG$ and $B \subset A_i$. For this $x \in dG$, if $x \cdot d\bar{\theta}$ has height i, then $x \cdot d\bar{\theta} \sim A_i$. Suppose that $A_i \in a_i\mathcal{O}_F$. Then by Proposition 3.1, there is an arrow $\bar{u}\colon a_i \to d$ of $\mathbf{D}$ for which $A_i \cdot u\mathcal{O}_F = x \cdot d\bar{\theta}$. This sets the stage for the definition of ψ:

$$(B, x)\, d^i\psi = \begin{cases} x \cdot d\bar{\theta} & \text{if } x \cdot d\bar{\theta} \text{ has height } <i \\ B \cdot \bar{u}\mathcal{O} & \text{if } x \cdot d\bar{\theta} \text{ has height } =i. \end{cases} \tag{5.1}$$

Observe that $(B, x) \cdot d^i\psi$ has height $<i$. Also, since $\bigcup_{B\in\mathcal{B}_i} B = A_i$ and $\bar{u}\mathcal{O}_F$ is bijective, $\bigcup_{B\in\mathcal{B}_i} (B \cdot \bar{u}\mathcal{O}_F) = x \cdot d\bar{\theta}$ whenever $x \cdot d\bar{\theta}$ has height i; therefore, $\bigcup_{(B,x)\in d^iG} (B, x) \cdot d^i\psi = \bigcup_{x\in dG} x \cdot d\bar{\theta} = cF$, so that (4.6) is satisfied for $\theta = \psi$.

Let $f\colon c \to c_0$ in $\mathbf{C}$. Suppose $dK = c$, $d_0K = c_0$, and $g\colon d \to e_0$, $g'\colon e \to d_0$ are arrows of $\mathbf{D}''$ for which (4.5), (4.7) and (4.7′) are satisfied. I shall construct arrows $(\lambda, g)\colon d^i \to e_0{}^i$, $(\lambda', g')\colon e^i \to d_0{}^i$ for which

$$(B, x) \cdot d^i\psi \cdot f\mathcal{O}_F \subset (B, x) \cdot (\lambda, g)\, R_i wrG \cdot e_0{}^i\psi \tag{5.2}$$

and

$$(B_0, y) \cdot e^i\psi \cdot f\mathcal{O}_F \subset (B_0, y) \cdot (\lambda', g')\, R_i wrG \cdot d_0{}^i\psi \tag{5.3}$$

for $(B, x) \in d^iR_i wrG$, $(B_0, y) \in e^iR_i wrG)$. The arrows of $\mathbf{E}$ will be all arrows (λ, g) and (λ', g') satisfying (5.2) and (5.3). Observe that

$$(B, x)(\lambda, g)\, R_i wrG = (B \cdot x\lambda, x \cdot gG). \tag{5.4}$$

Case 1. If $x \cdot d\bar{\theta}$ has height $<i$ and $x \cdot gG \cdot e_0\bar{\theta}$ also has height $<i$, then the left side of (5.2) is $(x \cdot d\bar{\theta})(f\mathcal{O}_F)$ and the right side is $x \cdot gG \cdot e_0\bar{\theta}$, so (5.2) holds by (4.7), for any choice of $x\lambda \in \mathcal{H}_i$.

Case 2. Let $x \cdot d\bar{\theta}$ have height $<i$ and $x \cdot gG \cdot e_0\bar{\theta}$ have height $=i$. By Proposition 3.1 there is an arrow $v\colon c_0 \to a_i$ (where a_i is the object for which $A_i \in a_i\mathcal{O}_F$) such that $x \cdot gG \cdot e_0\bar{\theta} \cdot v\mathcal{O}_F = A_i$. The set $x \cdot d\bar{\theta} \cdot f\mathcal{O}_F \cdot v\mathcal{O}_F$ is a

subset of A_i by (4.7); it is a proper subset because $v\alpha_F$ is a bijection and $x \cdot d\bar{\theta} \cdot f\mathcal{Cl}_F$ is a proper subset of $x \cdot gG \cdot e_0\bar{\theta}$ (because its height is less). Therefore there is an element B_0 of $\mathscr{B}_i$ which contains $x \cdot d\bar{\theta} \cdot f\mathcal{Cl}_F \cdot v\mathcal{Cl}_F$. Let $x\lambda$ be the constant function from $\mathscr{B}_i$ to $\mathscr{B}_i$ with value B_0, and let $\bar{v}$: $a_i \to c_0$ be the "inverse" to v given by Proposition 3.1. Then for any $B \in \mathscr{B}_i$, $(B, x) \cdot d^i\psi \cdot f\mathcal{Cl}_F = x \cdot d\bar{\theta} \cdot f\mathcal{Cl}_F = x \cdot d\theta \cdot f\mathcal{Cl}_F \cdot v\mathcal{Cl}_F \cdot \bar{v}\mathcal{Cl}_F \subset B_0 \cdot \bar{v}\mathcal{Cl}_F = (B, x)(\lambda, g) \cdot R_i wrG \cdot e_0{}^i\psi$, the last equality by (5.4) and (5.1). Thus (5.2) holds in this case.

Case 3. $xd\bar{\theta}$ has height i; by (4.7) and the assumption that G weakly covers F with height i, $x \cdot gG \cdot e_0\bar{\theta}$ also has height i and moreover $xd\bar{\theta} = x \cdot gG \cdot e_0\bar{\theta}$. Let u: $c \to a_i$ be an arrow taking $x \cdot d\bar{\theta}$ to A_i, $\bar{u}$: $a_i \to c$ its "inverse" as in Proposition 3.1, and v be as in Case 2. Let $x\lambda$ be the element of $\mathscr{H}_i$ induced by $\bar{u}fv$ (it is easy to check that $A_i \cdot \bar{u}fv\mathcal{Cl}_F = A_i$). Then for any $B \in \mathscr{B}_i$, $(B, x) \cdot d^i\psi \cdot f\mathcal{Cl}_F = B \cdot \bar{u}\mathcal{Cl}_F \cdot f\mathcal{Cl}_F = B \cdot \bar{u}fv\bar{v}\mathcal{Cl}_F = B \cdot x\lambda \cdot \bar{v}\mathcal{Cl}_F = (B \cdot x\lambda, x \cdot gG) \cdot e_0{}^i\psi$, as required. This verifies (5.2), and (5.3) is verified analogously.

ACKNOWLEDGMENT

The author is grateful to S. Eilenberg for suggesting the use of holonomy groups in proving this theorem.

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