# On probabilistic stable event structures

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### 1. Introduction

Verification has been a core concern of theoretical computer science for several decades. As most real systems are in some sense distributed or concurrent, one has to decide how to model concurrency. Broadly, there are two schools: the interleaving approach imposes a global 'clock' on the system, and says that independent concurrent events occur in some order, but that order is arbitrary – encapsulated by the CCS interleaving law a|b=a.b+b.a. 'True concurrency', on the other hand, represents concurrency explicitly in the semantics of the system, as exemplified by Petri nets. There are many arguments about the merits, ranging from the practical – e.g. that true concurrent models avoid the exponential state explosion arising from arbitrary interleaving – to the philosophical – one should understand concurrency in its own right.

Another extension of classical computation is probability, whether used to model genuine randomness, or uncertainty. Probabilistic models have in the last two or three decades also been the subject of intensive research, and by now there are many widely used tools for practical verification of probabilistic systems, based on a range of different theoretical models. A small selection of established models and systems might be [16, 15, 22, 14, 26, 18, 5, 3]. Such work adopts the 'interleaving' approach to concurrency – that is, from the true concurrency proponent's point of view, it ignores concurrency.

The topic of this paper is the combination of probability with true concurrency. As well as the core interleaving vs. true concurrency distinction, there is another crucial difference between the two in the probabilistic framework: in probability, one cares about the probabilistic (in)dependence of events, and in true concurrency this relation is (at least) correlated with, and ideally derived from, the concurrency relation. To put it another way, temporal stochastic processes and models capturing concurrency through nondeterminism such as [12, 6, 16, 17, 7] have a global state corresponding to a global time. In a distributed system, however, this is neither feasible nor natural. Thus, in true concurrency approaches there is no notion of global time or state, but rather local ones. In other words, the local components have their own local states and act in their own local time until they communicate together. This results in a highly desirable match between concurrency and probability, so that concurrent choices can be made probabilistically independent.

There is some work, quite recently, on probabilistic true concurrency models [23, 9, 20, 21, 4, 1, 2], but this is still at the foundational semantic stage, and there are very few no true concurrent probabilistic temporal logics – a recent example in the setting of distributed Markov chains is in [10].

Therefore, in [8] the first author aimed to develop a probabilistic temporal logic suitable for application to Petri nets, which are one of the best known true concurrent models, even though often used with a forced interleaving semantics. As part of this programme, it was necessary first to extend lower-level models, which is the work reported here.

#### 1.1. Related work

Event structures [24], and extensions thereof, are our basic model of concurrency. While they are a very 'low level' model, unsuited for direct modelling of systems, they model directly concurrency and causality – causality being (roughly) the converse of concurrency.

To our knowledge, the first probabilistic model for event structures was given in [13], which defines probabilistic extended bundle event structures. As with the stable event structures we discuss later, these allow different possible causes for an event, only one of which can be the cause in a run. However, they capture choice by means of groups of events which are mutually in conflict and enabled at the same time, called *clusters*. Clusters can be determined statically, and maintain the concept of choice internal to the system.

A domain theoretic view, closely related to the probabilistic powerdomains of [11, 19], was taken by Varacca, Volzer and Winskel [21], who define continuous valuations on the domain of configurations. They then define *non-leaking valuations with independence* on confusion-free event structures. These turn out to coincide with distributed probabilities as defined by [1].

Randomised Petri nets along with their corresponding probabilistic branching processes defined in [23] focus on free-choice conflicts only and therefore do not deal with confusion.

Finally, the most recent probabilistic event structures defined by Winskel [25] extend existing notions of probabilistic event structures in order to make them suitable for dealing with certain interactions between strategies.

The main hurdle in defining a probabilistic concurrent system with partial order semantics is to find units of choice so that concurrent units are probabilistically independent. Abbes and Benveniste in [1] define distributed probabilities taking branching cells as units of choice and show that in probabilistic event structures not only does concurrency match probabilistic independence, but also that this cannot be achieved at a grain finer than that of branching cells. Furthermore, they show how finite configurations can be decomposed into branching cells dynamically, where maximal configurations of the branching cells enforce all the conflicts within the cell to be resolved. Local probabilities are assigned to each branching cell and these can be extended to a limiting probability measure on the space of maximal configurations. The only constraint required is that of local finiteness which can be viewed as bounded confusion and which is defined later.

#### 1.2. Summary

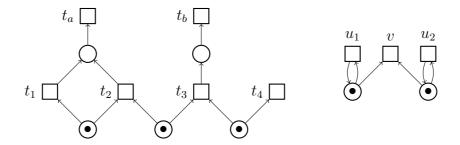
In this article, we first explain the motivation for studying probabilistic stable event structures. Then, after the necessary preliminaries, we define *conflict-driven* stable event structures, a superclass of event structures allowing enough 'confusion' to model Petri nets (the ultimate aim of this work), but constrained enough to allow a manageable probabilistic semantics. To extent the concepts of [1] to this setting, we need to constrain them to a certain sub-class of *jump-free* (stable) event structures. We then show that probabilistic jump-free stable event structures can be defined analogously to probabilistic event structures.

# 2. Motivation

As we said above, our original aim was probabilistic logic for Petri nets, and this work forms part of the route to there. So before diving in to the unfortunately but inevitably highly technical development of probabilistic stable event structures, let us explain the key issues in the more widely understood formalism of Petri nets.

Consider the following two safe nets:

The left net demonstrates several instances of confusion, where the occurrence of one event changes the possibilities elsewhere in the net. Now suppose that we add probabilities. There



are several ways to do this, but they will all boil down to assigning probabilities to the choices between say  $t_1$  or  $t_2$ , and  $t_2$  or  $t_3$ , and so on. However, because of confusion, the various choices are not independent. For example, whether a choice arises between  $t_a$  and  $t_b$  depends on whether  $t_b$  was enabled by  $t_1$  or  $t_2$ . Worse, an event arbitrarily far away may be the source of such confusion: the firing of  $t_4$  disables  $t_3$ , thereby stopping it from conflicting with  $t_2$ . Worse still, this confusion then propagates to affect the choice between  $t_a$  and  $t_b$ , which superficially have nothing to do with each other. In the absence of independence, it is hard to see how to give a useful probabilistic semantics that reflects concurrency. Thus, it is equally hard to extend probabilistic temporal logics (making statements such as 'transition t fires with probability > 0.5 in all futures') to nets.

The right net is perhaps even more pathological:  $u_1$  and  $u_2$  can fire away independently, but the malign presence of v waiting to disable them both at some time means that they cannot be treated as truly independent.

This problem, which more generally we might characterize as 'the problem of negative causality', has been troubling the community for decades, and we have not solved it. Instead we looked for ways to analyse differing levels of confusion, so that at least we could define manageable classes of nets. In the course of this, the first author [8] first developed a new notion of 'compact unfolding' of Petri net, which allows some degree of backward conflict to be maintained without being unfolded out. It turns out that just as standard net unfoldings give rise to an event structure semantics for Petri nets, compact unfoldings give rise to a stable event structure semantics. Hence we needed to extend the existing work on probabilistic semantics from event structures to stable event structures.

## 3. Preliminaries

Owing to the lengthy formal definitions needed for the work we build on, we shall present most of the preliminaries in summary form, referring to [8] or standard texts for formalities.

## 3.1. Event structures and their variants

The event structures of Winskel [24] model concurrency by describing *consistency* and *enabling* relations between *events*. They are a low-level model, as each event corresponds to an occurrence of a transition or action in higher-level models such as Petri nets or process calculi. Thus, they are suited to foundational investigations in concurrency theory.

For historical reasons, the terminology is a little confusing, and has also changed during the development of the subject. Here we shall use *general event structures* for the original and most general formulation; *stable event structures* impose the restriction that an event has a unique set of causing events; and plain *event structures* are those generated by binary *causality* and *conflict* relations on events, rather than more general predicates. Owing to our very frequent use of these long terms, we will abbreviate them.

**Definition 1.** A general event structure (GES)  $\mathcal{E}$  is a set of events E equipped with a nonempty subset-closed consistency predicate  $Con \subseteq \wp_{fin}(E)$  on finite sets of events, and an enabling relation  $\vdash \subseteq Con \times E$ , monotone in the first argument. A configuration of  $\mathcal{E}$  is a (possibly infinite) subset  $x \subseteq E$  that is finitely consistent, and such that every  $e \in x$  has a finite enabling set in x. The set of all configurations of  $\mathcal{E}$  is represented by  $\mathcal{V}(\mathcal{E})$  or  $\mathcal{V}_{\mathcal{E}}$  or  $\mathcal{V}$  when no confusion arises.

**Definition 2.** Let  $(P, \sqsubseteq)$  be a partial order. Then  $S \subseteq P$  is compatible  $(S \uparrow)$  iff  $\exists p \in P. \forall s \in S. \ s \sqsubseteq p.$  A subset is finitely compatible  $(S \uparrow^{fin})$  iff  $\forall S_0 \subseteq_{fin} S. \ S_0 \uparrow.$ 

The configurations  $\mathcal{V} = \mathcal{V}(\mathcal{E})$  of  $\mathcal{E}$  form a family of subsets of E that is finite-complete  $(\mathcal{A} \subseteq \mathcal{V} \& \mathcal{A} \uparrow^{fin} \Rightarrow \bigcup \mathcal{A} \in \mathcal{V})$ , finitely based  $(\forall u \in \mathcal{V}. \forall e \in u. \exists v \in \mathcal{V}. (v \text{ is finite } \& e \in v \& v \subseteq u))$ , and coincidence-free  $(\forall u \in \mathcal{V}. \forall e, e' \in u. e \neq e' \Rightarrow (\exists v \in \mathcal{V}. v \subseteq u \& (e \in v \Leftrightarrow e' \notin v)))$ . It can be shown that given a family  $\mathcal{V} \subseteq 2^E$  that is finite-complete, finitely based and coincidence-free,  $\mathcal{V}$  is the set of configurations of the GES  $\mathcal{E} = (E, Con, \vdash)$  given by  $x \in Con \Leftrightarrow_{def} x$  is finite  $\& \exists u \in \mathcal{V}. x \subseteq u$  and  $x \vdash e \Leftrightarrow_{def} x \in Con \& \exists u \in \mathcal{V}. e \in u \& u \subseteq x \cup \{e\}.$ 

General event structures are one of the most general classes of event structures; they allow for an event to have different causes, which is a desirable property. However, problems arise when dealing with configurations in which an event does not have a unique cause, in that its different causes can occur at the same time. Such configurations can be excluded by applying a *stability* constraint, leading to the definition of stable event structures. It is worth noting that for stable event structures there is no global partial order of causal dependency on events, but each configuration has its own local partial order of causal dependency.

**Definition 3.** A stable event structure (SES), is an GES such that mutually consistent enabling sets are closed under intersection, so that any set enabling e has a unique minimal enabling subset, the 'causes' of e.

Non-empty compatible configurations of an SES are closed under intersection, and such a stable family of configurations induces an SES.

Given a stable family V of configurations, and  $e, e' \in u \in V$  define  $e \leq_u e'$  if for all  $v : V \ni v \subseteq u$  we have  $e' \in v$  implies  $e \in v$  (i.e. e is a necessary member of the u-history of e'). Let  $[e]_u$  be the least configuration  $v \subseteq u$  containing e (the 'causal history' of e in u).

An alternative approach to stability is to define  $\leq$  in the structure as a global partial order of causal dependency on events:

**Definition 4.** A prime event structure (PES)  $\mathcal{E}$  on E comprises C on as before, and a partial order  $\leq$  on E (the causality relation) such that the down-closure  $\lceil e \rceil$  of e under  $\leq$  is finite, singleton events are consistent, C on is closed under subsets and under adding causes of events. The set of configurations of a PES is the set of  $\leq$ -down-closed consistent configurations, and is a stable family of configurations for  $\mathcal{E}$  [24].

Thus a PES is itself a SES with the naturally induced enabling relation. Less obviously, a SES can be translated into a PES by extending and renaming the events, so that each event is augmented with the history of how it occurred in a configuration. Then, even though the events and as such the configurations are different, the domains of configurations of both event structures are isomorphic as partial orders [24]. This translation is indeed right adjoint to the embedding of PES in SES, in the categorical framework that Winskel sets up.

Finally, generating the consistency relation from a binary conflict relation gives

**Definition 5.** An event structure (ES)  $\mathcal{E}$  on E comprises a causality relation  $\leq$  as above, and a binary, symmetric and irreflexive conflict relation # on E which is  $is \leq$ -upwards-closed (i.e. events inherit the conflicts of their causes).

A configuration of  $\mathcal{E}$  is a  $\leq$ -downwards-closed and conflict-free subset of E. Write  $\lceil e \rceil$  as above, and  $\lceil x \rceil$  for  $\bigcup_{e \in x} \lceil e \rceil$ .

## 3.2. Probabilistic Event Structures

This subsection is a (severe) summarization of the concepts introduced by [1]. The abstract definition is conceptually fairly simple.

A probability measure on a space  $\Omega$  is a function from a suitable collection (technically a  $\sigma$ -algebra) of subsets of  $\Omega$  to [0,1], satisfying the appropriate behaviour for probabilities (the measure of a union of disjoint sets is the sum of the measures).

Now consider an ES, and let  $\Omega$  be the space of its maximal configurations (finite or infinite, perhaps uncountably infinite) where the system has run to completion. There is a  $\sigma$ -algebra on  $\Omega$  comprising the sets  $\{\omega \in \Omega : \omega \supseteq v\}$  for each configuration v – that is, each set is a full subtree of the configuration tree rooted at v. Call this set S(v) (the shadow of v). The intuition is that the probability of v is equal to the sum of the probabilities of all the maximal configurations  $\omega$  reachable from v – except that not all such configurations are probabilistically independent, so the usual probability calculations for non-independent events have to be made.

**Definition 6.** A probabilistic event structure (PrES) is a pair  $(\mathcal{E}, \mathbb{P})$ , where  $\mathbb{P}$  is a probability measure on the space of maximal configurations of  $\mathcal{E}$  with the  $\sigma$ -algebra of shadows.

The likelihood function on configurations of  $\mathcal{E}$  is the function  $p(v) = \mathbb{P}(S(v))$ .

This definition is very abstract, and hard to calculate with or implement via a process calculus. Much of the work of [1] is in giving an alternative operational presentation, which we now introduce via a long series of notions. In the rest of this section,  $\mathcal{E} = (E, \leq, \#)$  is an ES, e, e' etc. range over E, and u, v etc. range over  $\mathcal{V}(\mathcal{E})$ .

**Definition 7.**  $P \subseteq E$  is a prefix if  $P = \lceil P \rceil$  (closed under causes). If  $F \subseteq E$ , then  $(F, \leq \upharpoonright F, \# \upharpoonright (F \times F))$  is an ES (which is a sub-ES of  $\mathcal{E}$ ). We use F to also refer to the sub-ES induced by F, when no confusion arises.

Hence, configurations are conflict-free prefixes. v is maximal iff it is  $a \subseteq$ -maximal configuration. We denote the maximal configurations of  $\mathcal{E}$  by  $\Omega(\mathcal{E})$  or just  $\Omega$ .

The future  $\mathcal{E}^v = (E^v, \leq^v, \#^v)$  of v, where  $E^v = \{e \notin v : \lceil e \rceil \cup v \in \mathcal{V}\}$ , is the ES generated by events that can happen after v. Given  $u \in \mathcal{V}$  and  $v \in \mathcal{V}^u$ , the concatenation  $u \oplus v$  is just  $u \cup v$ ; and given configurations  $v \subseteq u$ , the subtraction  $u \oplus v$  is just  $u \setminus v$ .

In a probabilistic event structure, the probability reflects the notion of choice which arises whenever a conflict is encountered for the first time. Thus, we consider the 'first-hand' conflicts, i.e. conflicts which are not inherited, as constituents of units of choice. Then we consider those prefixes which contain all immediate conflicts, and maximal configurations within them, which resolve all choices that can be made. The final step is to close under concatenation to give a notion of 'R-stopped configuration', which provides a probabilistically independent unit.

**Definition 8.** Events e and e' are in immediate conflict  $e\#_{\mu}e'$  iff  $\#\cap(\lceil e\rceil\times\lceil e'\rceil)=\{(e,e')\}$  (i.e. their conflict is not from conflicting causes).

A prefix B of  $\mathcal{E}$  is a stopping prefix if it is  $\#_{\mu}$ -closed. Given a set  $X \subseteq E$ ,  $X^*$  is the closure of X under  $\#_{\mu}$  and  $\lceil - \rceil$  – it is the minimal stopping prefix containing X.

v is B-stopped if v is maximal in  $\mathcal{V}(B)$ ; and v is stopped if it is  $v^*$ -stopped.

v is R-stopped if there is a sequence  $\varnothing \subseteq v_1 \subseteq \cdots \subseteq v_n \ldots$  such that  $v = \bigcup v_n$  and each  $v_{n+1} \ominus v_n$  is finite and stopped in  $\mathcal{E}^{v_n}$ .

The sequence is called a valid decomposition of v and if  $v = v_N$  for some N, then v is finite R-stopped. The set of R-stopped configurations is written  $W(\mathcal{E})$ .

Thus R-stopped configurations are the end of a sequence of steps, each of which resolves all possible choices before it. It remains to decompose these steps into their concurrent (i.e. independent) components.

**Definition 9.**  $\mathcal{E}$  is pre-regular, if for every finite v, there are finitely many events enabled by v.

 $\mathcal{E}$  is locally finite if every  $e \in E$  is in some finite stopping prefix.

[1] theorem 3.12 shows that then all maximal configurations are R-stopped. Henceforth all ES are locally finite.

**Definition 10.** If  $\mathcal{X} \subset \mathcal{V}$ , write  $\overline{\mathcal{X}}$  for the subset of its finite configurations.

An initial stopping prefix is a minimal non-empty stopping prefix.

A branching cell enabled by a finite R-stopped v is an initial stopping prefix of  $\mathcal{E}^v$ . The set of such branching cells is notated  $\delta_{\mathcal{E}}(v)$ .

It can be shown that v, with valid decomposition  $(v_n)$ , has a covering  $\Delta_{\mathcal{E}}(v)$ , a sequence  $(c_n)$  of branching cells such that  $v_n$  enables  $c_{n+1}$  and  $v_{n+1} \setminus v_n$  is maximal in  $c_{n+1}$ . That is,  $(c_n)$  gives a sequence of maximal independent steps to reach v.

The set of all branching cells of  $\mathcal{E}$  is denoted by  $\mathcal{C}(\mathcal{E})$  and the set of maximal configurations of a branching cell c is denoted by  $\Omega_c$ .

It is important to note that while branching cells decomposing a configuration are disjoint, in general, different branching cells of an ES may overlap. This is because not all of the events that can potentially constitute a branching cell are enabled at every configuration. Thus, configurations determine their corresponding branching cells and in this way we say that decomposition through branching cells is dynamic.

Now the operational definition of PrES's is given by equipping each branching cell of an ES with a probability for its maximal configurations; [1] show (highly non-trivially) that the operational and abstract definitions give the same class of structures.

**Definition 11.** A locally finite ES  $\mathcal{E}$  is locally randomised if each branching cell  $c \in \mathcal{C}$  is equipped with a local transition probability  $q_c$  on the set  $\Omega_C$  of configurations that can be chosen within the cell. The likelihood function  $p: \overline{\mathcal{W}} \to [0,1]$  is defined as:

$$\forall v \in \overline{\mathcal{W}}, \ p(v) = \prod_{c \in \Delta(v)} q_c(v \cap c)$$

where  $\Delta(v)$  denotes the covering of v in  $\mathcal{E}$ .

p induces a probability measure  $\mathbb{P}_B$  on the (countable) space B of stopping prefixes of  $\mathcal{E}$ ; and the full probability measure  $\mathbb{P}$  can be derived from  $\mathbb{P}_B$  via a construction called the distributed product and the Prokhorov extension theorem. Hence the measure  $\mathbb{P}$  is called the distributed product of the branching probabilities  $\{q_c : c \in C\}$ .

Further, this definition of likelihood function also applies to the space of R-stopped configurations of  $\mathcal{E}$ , and so [1] obtains a definition of probabilistic event structures in which local choice probabilities are attached either to branching cells or to R-stopped configurations.

## 3.3. Categorical notions

It is convenient to have to hand a few items from Winskel's categorical toolkit:

**Definition 12.** [24] Let  $\mathcal{E}_0 = (E_0, Con_0, \vdash_0)$  and  $\mathcal{E}_1 = (E_1, Con_1, \vdash_1)$  be two SES. A morphism from  $\mathcal{E}_0$  to  $\mathcal{E}_1$  is a partial function  $\theta : E_0 \to E_1$  on events satisfying:

- 1.  $X \in Con_0 \Rightarrow \theta.X \in Con_1$
- 2.  $\{e, e'\} \in Con_0 \& \theta(e) = \theta(e') \Rightarrow e = e'$
- 3.  $X \vdash_0 e \& \theta(e)$  is defined  $\Rightarrow \theta.X \vdash_1 \theta(e)$

A morphism is synchronous if it is a total function.

As remarked earlier, there is an inclusion I from PES to SES, with an adjoint functor back. We will need this functor:

**Definition 13.** Given a SES  $\mathcal{E}_0 = (E, Con, \vdash)$ , let  $\Theta(\mathcal{E}_0)$  be the PES  $\mathcal{E}_1 = (P, Con_P, \leq)$ , with isomorphic domain of configurations, defined as follows.

- 1.  $P = \{ [e]_x \mid e \in x \in \mathcal{V}(\mathcal{E}) \}.$
- 2.  $p' \le p \Leftrightarrow p' \subseteq p$ .
- 3.  $X \in Con_P \Leftrightarrow X \subseteq_{fin} P \& X \uparrow$ .

 $\Theta$  maps morphisms thus: if  $\theta \colon \mathcal{E} \to \mathcal{E}'$  (induced by an event map  $\theta \colon E \to E'$ ), then  $\Theta(\theta)$  is  $\{ [e]_x \mid e \in x \in \mathcal{V}(\mathcal{E}) \} \mapsto \{ [\theta(e)]_{x'} \mid \theta(e) \in x' \in \mathcal{V}(\mathcal{E}') \}.$ 

For a SES  $\mathcal{E}_0$  we refer to  $\Theta(\mathcal{E}_0)$  as its associated PES.

The counit of the adjunction is  $\theta \colon \mathcal{E}_0 \mapsto \theta_{\mathcal{E}_0}$ , where  $\theta_{\mathcal{E}_0} \colon \Theta(\mathcal{E}_0) \to \mathcal{E}_0$  is the synchronous morphism (of SES) given by  $\theta_{\mathcal{E}_0}(p) = e$  for  $p = \lceil e \rceil_x \in P$ ,  $e \in E \& x \in \mathcal{V}(\mathcal{E}_0)$ .

## 4. Conflict-driven (Stable or Prime) Event Structures

The difficulty in extending probabilistic notions to rich concurrent structures such as SES lies in the consistency relation, and ensuring that causal (in)dependence matches appropriately with probabilistic (in)dependence. As we have just seen, it is already quite technically intricate for the relatively simple case of event structures with a binary conflict relation. Our contribution here is to develop the framework further to allow more sophisticated consistency relations. In particular, we will define a class of SES which is sufficient to give a low-level semantics for Petri nets, in which consistency arises not just out of immediate conflicts, but out of the history of previous conflicts; and moreover there is the possibility of *confusion*, where the execution of *prima facie* concurrent events is interfered with by other events.

In the definition of SES, the consistency predicate Con is required to satisfy only one condition. Namely,  $Y \subseteq X \& X \in Con \Rightarrow Y \in Con$ . This does not necessarily have to fit with the configurations of the SES. Consider the following example.

**Example 1.** Let  $\mathcal{E} = (E, Con, \vdash)$  be a SES, where  $E = \{e_1, e_2, e_3, e_4\}$ ,  $\{e_1\} \vdash e_2, \{e_3\} \vdash e_4$ ,  $\{e_1, e_3\} \notin Con$  and  $\{e_2, e_4\} \in Con$ . It is easy to see that even though  $e_2$  and  $e_4$  are consistent, they can never appear together in a configuration. Therefore, the consistency predicate is not sensible with respect to the configurations. (In terms introduced just below, the consistency relation here is only recording the immediate conflicts, not the inherited conflicts.)

We therefore define *sensible* SES and PES as follows.

**Definition 14.** Let  $\mathcal{E}$  be a SES or PES with the consistency relation Con. We say  $\mathcal{E}$  is sensible iff  $\forall X \in Con \Leftrightarrow \exists v \in \mathcal{V}(\mathcal{E}). X \subseteq v.$  If  $\mathcal{E}$  is not sensible, it can be made so by pruning Con of the unreachable consistent sets.

So far we have described how the consistency predicate in sensible structures relates to the configurations of that structure. We now define the notions of conflict and immediate conflict.

**Definition 15.** Two events e and e' of a SES are in conflict under a finite configuration v, represented by  $e\#_v e'$  iff  $(\{e\} \cup \{e'\} \cup v) \notin Con$ . Then two events are in conflict, represented by e#e', iff  $\forall v \in \mathcal{V}$ .  $e\#_v e'$ .

Define immediate conflict between two events e and e' of a SES under configuration v:

$$e\#_{\mu,v}e' \Leftrightarrow_{def} v \vdash e \& v \vdash e' \& \#_v \cap (\lceil e \rceil_v \times \lceil e' \rceil_v) = \{(e,e')\}.$$

and define  $e \#_{\mu} e'$  iff  $\forall v, v' \in \mathcal{V}$ .  $v \vdash e \& v' \vdash e' \Rightarrow \exists v'' \subseteq v \cup v'$ .  $e \#_{\mu,v''} e'$ .

**Definition 16.** For a set  $X \subseteq_{fin} E$ , let  ${}_*X$  be the set of the sets consisting of exactly one history  $\lceil e \rceil_v$  for each event e in X and configuration  $v \ni e$ . (That is, for every configuration,  ${}_*X$  contains a single choice among all the histories that can have produced each event in X.)

We now define a special class of SES, namely, conflict-driven SES. Originally, their definition arose through considering unfoldings of Petri nets, and so in [8] they are called 'net-driven', but here we abstract away from the net derivation.

**Definition 17.** A SES  $\mathcal{E} = (E, Con, \vdash)$  is called conflict-driven iff it satisfies the following.

- 1.  $\mathcal{E}$  is sensible.
- 2.  $\forall X \subseteq_{fin} E. X \notin Con \Rightarrow \forall T \in {}_{*}X. \exists e_1, e_2 \in \bigcup T. e_1 \#_{\mu} e_2$
- 3.  $\forall e, e' \in E, v \in \mathcal{V}. \ e\#_{u,v}e' \Rightarrow e\#e'$

Note that from 2 and 3 it follows that  $\forall X \subseteq E. \ X \notin Con \Rightarrow \forall T \in {}_{*}X. \ \exists e_1, e_2 \in \mathsf{I} \ \mathsf{J} T. \ e_1 \# e_2$ 

As mentioned before, the first characteristic describes that the consistency predicate is in line with configurations and the second one implies that the source of inconsistency is a conflict in the past. The last constraint describes the persistence of conflicts (originally because immediate conflict in a Petri net is a cause of later conflict).

We can now show that for the associated PES of a conflict-driven SES, the consistency predicate can be generated from a binary conflict relation:

**Theorem 1.** Let  $\mathcal{E}_0 = (E, Con, \vdash)$  be a conflict-driven SES and let  $\mathcal{E}_1 = \Theta(\mathcal{E}_0) = (P, Con_P, \leq)$  be its associated PES. Then, we have:

$$X \in Con_P \Leftrightarrow X \subseteq_{fin} P \& \forall p, p' \in X. \neg (p \# p')$$

where p # p' iff  $\{p, p'\} \notin Con_P$ .

If a PES does have a consistency relation generable from a binary conflict relation, it can be seen as an ES via an inclusion mapping  $\hat{I}$ . Thus a conflict-driven SES generates an ES.

**Definition 18.** Given a conflict-driven SES  $\mathcal{E}$ , we denote by  $\hat{\mathcal{E}}$  or  $\hat{\Theta}(\mathcal{E})$  the ES  $\hat{I}(\Theta(\mathcal{E}))$ , and we refer to  $\hat{\mathcal{E}}$  as the associated ES of  $\mathcal{E}$ . Similarly we write  $\tilde{\theta}_{\mathcal{E}}$  for the adjunct morphism  $\hat{I}(\Theta(\mathcal{E})) \to \Theta(\mathcal{E}) \to \mathcal{E}$ .

## 5. Probabilistic Jump-free Stable Event Structures

We now consider adjoining probabilities to SES. First we present the definition of concepts analogous to those of probabilistic event structures. Then our aim is to derive an isomorphism between the events of branching cells of conflict-driven SES and their associated ES; we find that such isomorphisms exist if the SES are *jump-free*, as we shall define.

We assume that the SES in this section are conflict-driven unless stated otherwise.

## 5.1. Branching Cells on Stable Event Structures

We now define branching cells for SES, in a similar manner and show that in general, unlike the branching cells of ES, they do not form the units of choice.

**Definition 19.** A subset  $P \subseteq E$  is called a prefix of a SES  $\mathcal{E}$  iff  $\forall e \in P$ .  $\exists X \subseteq P$ .  $X \vdash e$ .

Let  $\mathcal{E} = (E, Con, \vdash)$  be a SES and let F be a prefix of E. Then  $(F, \{X \cap F : X \in Con\}, \{(X \cap F, e \in F) : X \vdash e\})$  is a SES (which is a sub-SES of  $\mathcal{E}$ ). We use F also for the sub-SES induced by F, when no confusion arises. Configurations can then be viewed as consistent prefixes as before. Concepts of compatibility of configuration and maximal configurations are defined as for ES and we represent the set of maximal configurations of SES  $\mathcal{E}$  by  $\Omega(\mathcal{E})$ .

In this setting, the future of a configuration v of  $\mathcal{E}$  is  $\mathcal{E}^v = (E^v, Con^v, \vdash^v)$  where  $E^v = \{e \notin v : \{e\} \cup v \in Con\}$  and  $Con^v, \vdash^v$  are the natural restrictions to  $E^v$ . (Note that the future of v includes all events that might happen, both those directly enabled by v and those completely independent.)

The notions of B-stopped and stopped configurations for SES are similar to those of ES. However, unlike ES, given X a subset of events, a canonical stopping prefix including X cannot be derived. This is because in the definition of prefix for SES an event can have different sets of events enabling it. Thus, these notions are defined as follows, recalling the definition of  $\#_{\mu,v}$  (definition 15).

**Definition 20.** A prefix B of a SES  $\mathcal{E}$  is called a stopping prefix if it is  $\#_{\mu,v}$ -closed in the following sense:

$$\forall v \subseteq B. \ e \in v \ \& \ \exists e' \in E. \ e \#_{\mu,v} e' \Rightarrow e' \in B$$

A configuration v of  $\mathcal{E}$  is called B-stopped if v is a maximal configuration of B; v is called stopped if there is a stopping prefix B such that v is B-stopped.

Stopping prefixes of a conflict-driven SES have a close relation with the stopping prefixes of their associated ES. To expand this further, we first observe the relation between  $\#_{\mu,v}$  of a SES and  $\#_{\mu}$  of its corresponding ES by the following immediate lemma:

**Lemma 1.** Given a SES 
$$\mathcal{E}$$
 and  $\hat{\mathcal{E}} = \hat{\Theta}(\mathcal{E})$ , then  $e \#_{\mu,\nu} e' \Leftrightarrow \lceil e \rceil_{\nu} \#_{\mu} \lceil e' \rceil_{\nu}$ 

Using this lemma we can describe the relation between SES stopping prefixes and ES stopping prefixes as follows.

**Proposition 1.** Given a SES  $\mathcal{E}$ , then B is a stopping prefix of  $\mathcal{E}$  iff  $\hat{\Theta}(B)$  is a stopping prefix of  $\hat{\mathcal{E}}$ .

**Proof 1.** It is easy to verify that B is a prefix iff  $\hat{B} = \hat{\Theta}(B)$  is a prefix as well (follows from the definition of  $\hat{\Theta}$  and  $\tilde{\theta}_{\mathcal{E}}$  (definitions 13, 18) being a morphism). Also, from lemma 1 it follows that B is  $\#_{\mu,v}$ -closed (for  $v \subseteq B$ ) iff  $\hat{B}$  is  $\#_{\mu}$ -closed.

The following are the appropriate adaptations of the ES definitions:

**Definition 21.** A configuration v of SES  $\mathcal{E}$  is R-stopped if there is a non-decreasing sequence of configurations  $(v_n)$  for  $0 \le n < N \le \infty$  such that:

- 1.  $v_0 = \emptyset$  and  $v = \bigcup_{0 \le n \le N} v_n$ , and
- 2.  $\forall n \geq 0, \ n+1 < N \Rightarrow v_{n+1} \ominus v_n \text{ is finite stopped in } \mathcal{E}^{v_n}$ .

The sequence is called a valid decomposition of v and if  $N < \infty$  then v is said to be finite R-stopped. The set of R-stopped configurations of an SES  $\mathcal{E}$  is denoted  $\mathcal{W}(\mathcal{E})$ .

- $\mathcal{E}$  is pre-regular, if for every finite configuration v of  $\mathcal{E}$ , the set  $\{e \in E \mid v \oplus \{e\}\}$  is finite.
- $\mathcal{E}$  is locally finite if for every  $e \in E$ , there is a finite stopping prefix of  $\mathcal{E}$  containing e.
- As before, an initial stopping prefix is a minimal non-empty stopping prefix.

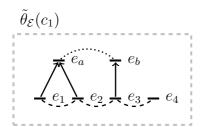
A branching cell of  $\mathcal{E}$  and configuration  $v \in \overline{\mathcal{W}}(\mathcal{E})$  is an initial stopping prefix of  $\mathcal{E}^v$ . The set of all branching cells of  $\mathcal{E}$  is denoted by  $\mathcal{C}(\mathcal{E})$  and the set of maximal configurations of a branching cell c is denoted by  $\Omega_c$ .

The branching cells which are initial stopping prefixes of  $\mathcal{E}^v$  are called the branching cells enabled by v, and denoted by  $\delta_{\mathcal{E}}(v)$  or  $\delta(v)$  if no confusion arises.

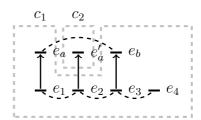
### 5.2. Probabilistic Event Structures and Stable Event Structures

We are now ready to add probability to SES. This would be fairly straightforward, if the branching cells of conflict-driven SES and their associated ES were isomorphic. However, the following example shows why this is not the case.

**Example 2.** Consider the below SES  $\mathcal{E}$ , corresponding to the Petri net at the beginning of the paper, and its associated ES  $\hat{\mathcal{E}}$ , where the dashed curved lines represent immediate conflicts. The dotted curved line shows immediate conflict under a particular configuration; in this case  $e_a \#_{\mu,\{e_1\}} e_b$ .



A (conflict-driven) stable event structure  $\mathcal{E}$ 



The corresponding ES  $\hat{\mathcal{E}} = \hat{\Theta}(\mathcal{E})$ .

As it can be seen from the figures,  $\hat{\mathcal{E}}$  has two branching cells  $(c_1 \text{ and } c_2)$ , while  $\mathcal{E}$  (which  $= \theta_{\mathcal{E}}(\mathcal{E})$ ) has only one. To see this, let us construct the branching cells of  $\hat{\mathcal{E}}$ . Consider the configuration  $v_1 = \{e_2, e_4, ...\}$ . Having event  $e_2$  implies that events  $e_1, e_3, e_4, e_b$  and  $e_a$  must be added to the branching cell because of  $\#_{\mu}$ -closure. Thus,  $e'_a$  is not in this branching cell, but in the next branching cell, consisting of  $e'_a$  only. However, in the SES, both  $e_a$  and  $e'_a$  of  $\hat{\mathcal{E}}$  are represented by event  $e_a$  of  $\mathcal{E}$ . Therefore, it is not possible to cover  $\mathcal{E}$  in any manner that is consistent with the covering for  $\hat{\mathcal{E}}$ .

Although in this example  $c_2$  does not reflect any choice being made, more complex examples exist where all branching cells have a choice to make. Therefore, it is not possible to resolve this at the probabilistic level, e.g. by trying to combine a number of branching cells with respect to their probabilities.

In order to achieve isomorphic branching cells and avoid the above situation (and more complex problems), we consider a class of SES (and their associated ES) which forbid these cases, namely, those structures which are *jump-free*.

**Definition 22.** An ES is jump-free iff

$$\forall e, e'. \ e < e' \Rightarrow \not \equiv e_1, \dots, e_k. k > 1 \& e \#_{\mu} e_1, e_i \#_{\mu} e_{i+1} \& e_k \#_{\mu} e'$$

for  $1 \le i \le k$ . A SES is jump-free iff

$$\forall e, e'. e <_v e' \Rightarrow \not \equiv e_1, \dots, e_k. k > 1 \& e \#_{\mu, v_0} e_1, e_i \#_{\mu, v_i} e_{i+1}, e_k \#_{\mu, v_k} e'$$

for 
$$1 \le i \le k-1 \& v_i \subseteq v$$
.

For example, the SES in example 2 is not jump-free as either of the chains of events  $e_3, e_4, e_b$  and  $e_1, \ldots, e_b$  break jump-freeness.

Jump-free ES are simpler to deal with, as, unlike ES, they are flat in the following sense.

**Proposition 2.** The branching cells of jump-free ES (as initial stopping prefixes) consist of initial events only (and the converse holds also).

**Proof 2.** Suppose c has non-initial events and let  $e \in c$  be such that  $\exists e_0 \in c$ .  $e_0 < e \& \nexists e_1 \in c$ .  $e < e_1$ . Then there exists an initial event e' s.t.  $\nexists e_0 \in c$ .  $e_0 < e'$  & e' < e. Noting that c is an initial stopping prefix and therefore,  $\nexists c'$ .  $c' \subset c$ , then in the formation of  $\{e'\}^*$ , e can only be added to achieve the closure of  $\#_{\mu}$ . Therefore, there must be a chain of events  $e_1, \ldots, e_k$  s.t.  $e' \#_{\mu} e_1, e_i \#_{\mu} e_{i+1}$  &  $e_k \#_{\mu} e$ . Observe that k > 1 as otherwise  $\neg (e \#_{\mu} e_1)$ . Therefore, above chain forms a jump which is a contradiction and as such c only consists of non-initial events.

The converse follows immediately from branching cells being closed under  $\#_{\mu}$ .

The same holds for SES:

**Proposition 3.** The branching cells of jump-free SES (as initial stopping prefixes) consist of initial events only.

**Proof 3.** The proof follows a similar reasoning to that of ES, noting that for all the initial events in  $\mathcal{E}^v$ ,  $\#_{\mu}$  is resolved meaning  $\forall v' \in \mathcal{V}(\mathcal{E}^v)$ .  $e\#_{\mu,v'}e' \Rightarrow e\#_{\mu}e'$ .

Recalling that the configurations of a SES and its associated ES are isomorphic, we show in the following theorem that the branching cells of a jump-free conflict-driven SES and its associated ES are isomorphic.

**Theorem 2.** Given a jump-free conflict-driven SES  $\mathcal{E}$  and its associated ES  $\hat{\mathcal{E}} = \hat{\Theta}(\mathcal{E})$ ,  $\hat{\mathcal{C}}$ , the set of branching cells of  $\hat{\mathcal{E}}$ , is isomorphic to  $\mathcal{C}$ , the set of branching cells of  $\mathcal{E}$ .

The most important consequence of theorem 2 is that the covering of any configuration in a SES is exactly the same as that of its corresponding configuration in its associated ES. That is because the configurations in the future of two isomorphic configurations are also isomorphic and therefore, two isomorphic configurations have isomorphic coverings. Therefore, all the probabilistic properties of branching cells of ES are applicable to those of SES, and as such, all the probabilistic machinery described in section 3.2 for ES can be applied to conflict-driven jump-free SES. Thus, for example, the likelihood function for a SES  $\mathcal{E}$ ,  $p:\overline{\mathcal{W}} \to \mathbb{R}$  is defined as:

$$\forall v \in \overline{\mathcal{W}}, \ p(v) = \prod_{c \in \Delta(v)} q_c(v \cap c)$$

#### 6. Conclusion

We have introduced a new class of SES called conflict-driven SES, which includes those SES that arise from Petri nets under the first author's 'compact unfoldings'.

We then proceeded to extend the results of [1] to SES, finding that this is not possible in general, but that it is possible for our new class of 'jump-free' ES and SES for which the branching cells consist of events which are not causally related. We then proved that for such stable event structures and their associated event structures, the branching cells are isomorphic. Thus, probabilistic jump-free SES were defined in a similar manner to probabilistic event structures of [1].

The jump-free notion can be translated back to Petri nets, where it means, more or less, that confusion is allowed provided that it is not directly propagated forward by the causality relation. While the jump-free structures are a larger class than the free-choice structures, they are still far from complete: both the example nets in our motivation section have jumps. We hope in future work to find weakenings of the jump-free constraint.

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# Appendix

**Proof 4 (Proof of Theorem 1).** ( $\Rightarrow$ ) follows from the definition of consistency relation. More precisely, if  $\exists p, p' \in X$ . p # p' then it follows that p and p' are not consistent (by definition of #) and therefore, they are not compatible as configurations of  $\mathcal{E}_0$ . Thus,  $X \notin Con_P$  which is a contradiction.

 $(\Leftarrow)$  follows from the definition of conflict-driven SES. More specifically, suppose by contradiction that there is a finite set of events X s.t.  $\forall p, p' \in X$ .  $\neg(p \# p')$  and  $X \notin Con_P$ . Let  $X = \{p_i \mid p_i = \lceil e_i \rceil_{v_i} \& v_i \in \mathcal{V}(\mathcal{E}_0)\}$ . Since  $X \notin Con_P$  this implies that  $p_i$  as configurations of  $\mathcal{E}_0$  are not compatible, in other words, if we let  $\bar{X} = \bigcup \lceil e_i \rceil_{v_i}$ , then  $\bar{X} \notin Con$ . Now since  $\mathcal{E}_0$  is conflict-driven, then for any  $T \in {}_*\bar{X}$ .  $\exists e_1, e_2 \in \bigcup T$ .  $e_1 \# e_2$ . For such  $e_1, e_2$ , suppose  $p_1, p_2 \in X$ .  $p_1 = \lceil e \rceil_{v_1} \& e_1 \in p_1 \& p_2 = \lceil e' \rceil_{v_2} \& e_2 \in p_2$ . Then  $p_1$  and  $p_2$  are not compatible as configurations of  $\mathcal{E}_0$ , therefore  $\{p_1, p_2\} \notin Con_P$ , which contradicts  $\forall p, p' \in X$ .  $\neg(p \# p')$ . Therefore, X must be consistent, i.e.  $X \in Con_P$ .

The proof of theorem 2 requires a few trivial lemmas.

**Lemma 2.** The following facts are immediate from the relevant definitions: PES are sensible.

Let  $\mathcal{E}_0$  be a sensible SES and let  $(P, Con_P, \leq) = \Theta(\mathcal{E})$ . Then

$$e \leq_x e' \Leftrightarrow \lceil e \rceil_x \subseteq \lceil e' \rceil_x$$

$$X \in Con \Leftrightarrow \forall v \in \mathcal{V}(\mathcal{E}_0) \ s.t. \ X \subseteq v. \{ [e]_v \mid e \in X \} \in Con_P$$

**Lemma 3.** Let  $\mathcal{E}$  be a conflict-driven SES with associated ES  $\hat{\mathcal{E}}$ . Then we have

$$e \#_{\mu,v} e' \Leftrightarrow \lceil e \rceil_v \#_{\mu} \lceil e' \rceil_v.$$

**Proof 5.** Follows trivially from lemma 2 and definitions of  $\#_{\mu,\nu}$  and  $\#_{\mu}$ .

**Lemma 4.** Let  $\mathcal{E}$  be a jump-free conflict-driven SES. Then  $\hat{\mathcal{E}}$  is jump-free.

**Proof 6.** Follows trivially from lemmas 2 and 3.

We now establish the desired isomorphism of branching cells. First:

**Lemma 5.** Let  $\mathcal{E}$  be a conflict-driven SES. Then, for  $p \neq p'$  if  $\Theta_{\mathcal{E}}(p) = \Theta_{\mathcal{E}}(p') \Rightarrow p \# p' \& \neg (p \#_{\mu} p')$ , where  $\Theta_{\mathcal{E}}$  is the mapping from the events of  $\mathcal{E}$  to the events of  $\Theta(\mathcal{E})$  (definition 13).

**Proof 7.** Suppose  $\Theta_{\mathcal{E}}(p) = \Theta_{\mathcal{E}}(p') = e$ , then  $p = \lceil e \rceil_v$ ,  $p' = \lceil e \rceil_{v'}$ . Let  $v_0$  be the subset of v s.t.  $v_0 \vdash_{min} e$  and similarly, let  $v_1$  be the subset of v' s.t.  $v_1 \vdash_{min} e'$ . By the stability axiom it follows that  $v_0 \cup v_1 \notin Con$ . It is then obvious that p # p'. Since  $\mathcal{E}$  is conflict-driven, it follows that  $\exists e_0 \in v_0, e_1 \in v_1. e_0 \# e_1$ , and therefore,  $\lceil e_0 \rceil_v \# \lceil e_1 \rceil_{v'}$  and since  $\lceil e_0 \rceil_v < \lceil e \rceil_v$  and  $\lceil e_0 \rceil_{v'} < \lceil e \rceil_{v'}$  it follows that  $\neg (p \#_u p')$ .

**Proof 8 (Proof of Theorem 2).** Let  $\vartheta = \tilde{\theta}_{\mathcal{E}}$  (definitions 13, 18). First consider configuration v of  $\hat{\mathcal{E}}$  and  $v' = \vartheta(v)$  of  $\mathcal{E}$ . Since configurations of  $\hat{\mathcal{E}}$  and  $\mathcal{E}$  are isomorphic, it is clear that  $\exists e \in E \setminus v. \ v \cup \{e\} \in \mathcal{V}(\mathcal{E}) \Leftrightarrow \exists e' \in E \setminus v'. \ v' \cup \{e'\} \in \mathcal{V}(\mathcal{E})$ . In other words, every initial event in future of v has an associated initial event in future of v' and vice versa. Let  $\hat{\mathcal{E}}_0^v$  and  $\mathcal{E}_0^{v'}$  represent the initial events of each structure, respectively. Then we show that  $\vartheta$  yields a bijection between  $\hat{\mathcal{E}}_0^v$  and  $\mathcal{E}_0^{v'}$ .

Suppose  $e, e' \in \hat{\mathcal{E}}_0^v$  and  $\vartheta(e) = \vartheta(e') = e''$ . By lemma 5 it follows that e # e' and  $\neg e \#_{\mu} e'$ , i.e. there is a conflict in their past. But this is a contradiction as they are both initial events in future of a configuration which is conflict-free. As shown above, every initial event in future of v' has an associated event in future of v, therefore,  $\vartheta$  (applied to initial events  $\hat{\mathcal{E}}_v^v$ ) is onto the initial events of  $\mathcal{E}_v^v$ . It then follows that  $\vartheta$  yields a bijection between the events of  $\hat{\mathcal{E}}_0^v$  and  $\mathcal{E}_0^{v'}$ , making them isomorphic.

Furthermore, as we are dealing with the initial events that can occur in future of v' i.e. immediately after v', the immediate conflict relation among the events of  $\mathcal{E}_0^{v'}$  is resolved, in the sense that it does not depend on any configuration in  $\mathcal{E}_0^{v'}$ , and therefore, is obviously compatible with the immediate conflict relation of  $\hat{\mathcal{E}}_0^v$ . Thus, the branching cells of  $\hat{\mathcal{E}}_0^v$  and  $\mathcal{E}_0^{v'}$  are isomorphic, which implies the branching cells of  $\hat{\mathcal{E}}$  and those of  $\mathcal{E}$  are isomorphic.