Fairly Correct Systems: Beyond ω-regularity

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Abstract. In 2006, Varacca and Völzer proved that on finite graphs, ω-regular large sets coincide with ω-regular sets of probability 1, by using the existence of positional strategies in the related Banach-Mazur games. Motivated by this result, we try to extend it to other classes of sets by means of various notions of simple strategy introduced in a recent paper of Grädel and Leßenich. We derive that the existence of a move-counting winning strategy implies that the winning condition has probability 1. In particular, we observe that the above result of Varacca and Völzer extends to the ωS-regular sets introduced by Bojanczyk and Colcombet in 2006. Then, we introduce a generalisation of the classical Banach-Mazur game and in particular, a probabilistic version whose goal is to characterise sets of probability 1 (as classical Banach-Mazur games characterise large sets). We obtain a determinacy result for these games, when the winning set is a countable intersection of open sets.

1 Introduction

Systems (automatically) controlled by computer programs abound in our everyday life. Clearly enough, it is of a capital importance to know whether the programs governing these systems are *correct*. Over the last thirty years, formal methods for verifying computerised systems have been developed for validating the adequation of the systems against their requirements. Model checking is one such approach: it consists first in modelling the system under study (for instance by a automaton), and then in applying algorithms for comparing the behaviours of that model against a specification (modelled for instance by a logical formula). Model checking has now reached maturity, through the development of efficient symbolic techniques, state-of-the-art tool support, and numerous successful applications to various areas.

As argued in [8]: *'Sometimes, a model of a concurrent or reactive system does not satisfy a desired linear-time temporal specification but the runs violating the specification seem to be artificial and rare'*. As a naive example of this phenomenon, consider a coin flipped an infinite number of times. Classical verification will assure that the property stating *"one day, we will observe a head"* is false, since there exists a unique execution of the system violating the property. In some situation, for instance when modeling non-critical systems, one could prefer to know whether the system is *fairly correct*. Roughly speaking, a system is fairly correct against a property if the set of executions of the system violating the property is *"very small"*; or equivalently if the set of executions of the system satisfying the property is *"very big"*. A first natural notion of fairly correct system is related to probability: *almost-sure correctness*. A system is almost-surely correct against a property if the set of executions of the system satisfying the property has probability one. Another interesting notion of fairly correct system is related to topology: *large correctness*. A system is largely correct against a property if the set of executions of the system satisfying the property is *large* (in the topological sense). There exists a lovely characterisation of *large sets* by means of the *Banach-Mazur games*. In [7], it has been shown that a set W is large if and only if a player has a winning strategy in the related Banach-Mazur game.

Although, the two notions of *fairly correct systems* does not coincide in general, in [8], the authors proved (amongst others) the following result: when considering ω-regular properties on finite systems, the *almost-sure correctness* and the *large correctness* coincide, for bounded Borel measures on finite-state systems. Motivated by this very nice result, we intend to extend it to a larger class of specifications. The key ingredient to prove the previously mentioned result of [8] is that when considering ω-regular properties, *positional* strategies are sufficient in order to win the related Banach-Mazur game [1]. For this reason, we investigate *simple strategies* in Banach-Mazur games, inspired by the recent work [4] where infinite graphs are studied.

Our contributions. In this paper, we first compare various notions of simple strategies on finite graphs. From this, we derive that the existence of a move-counting winning strategy implies that the winning condition has probability 1. As a byproduct, we get that the almost-sure correctness and the large correctness coincide on finite graphs, when considering the ωS-regular sets, introduced in [2]. Then, we introduce a generalisation of the classical Banach-Mazur game and in particular, a probabilistic version whose goal is to characterise sets of probability 1 (as classical Banach-Mazur games characterise large sets). We obtain a determinacy result for these games, when the winning set is a countable intersection of open sets. Due to the lack of the space, the complete proofs are given in the Appendix.

2 Banach-Mazur Games on finite graphs

Let $(X, \mathcal{T})$ be a topological space. A notion of topological "bigness" is given by large sets. A subset $W \subset X$ is said to be *nowhere dense* if the closure of W has empty interior. A subset $W \subset X$ is said to be *meagre* if it can be expressed as the union of countably many nowhere dense sets and a subset $W \subset X$ is said to be large if W^c is meagre. In particular, we remark that a countable intersection of large sets is still large and if $W \subset X$ is large, then any set $Y \supset W$ is large.

If $G = (V, E)$ is a finite directed graph and $v_0 \in V$, then the space of infinite paths in G from v_0, noted $\mathrm{Paths}(G, v_0)$, can be endowed with the complete metric

$$d((\sigma_n)_n, (\rho_n)_n) = 2^{-k} \quad \text{where} \quad k = \min\{n \geq 0 : \sigma_n \neq \rho_n\} \tag{1}$$

with the conventions that $\min \emptyset = \infty$ and $2^{-\infty} = 0$. We can therefore study the large subsets of the metric space $(\text{Paths}(G, v_0), d)$. Banach-Mazur games allow us to characterise large subsets of this metric space through the existence of winning strategies.

Definition 1. *A* Banach-Mazur game $\mathcal{G}$ *on a finite graph is a triplet* (G, v_0, W) *where* $G = (V, E)$ *is a finite directed graph where every vertex has a successor,* $v_0 \in V$ *is the initial state,* W *is a subset of the infinite paths in* G *starting in* v_0*.*

A Banach-Mazur game $\mathcal{G} = (G, v_0, W)$ on a finite graph is a two-player game where Pl. 0 and Pl. 1 alternate in choosing a finite path as follows: Pl. 1 begins with choosing a finite path π_1 starting in v_0; Pl.0 then prolongs π_1 by choosing another finite path π_2 and so on. A play of $\mathcal{G}$ is thus an infinite path in G and we say that Pl. 0 wins if this path belongs to W; while Pl. 1 wins if this path does not belong to W. The set W is called the winning condition. It is important to remark that, in general, in the literature, Pl. 0 begins to play Banach-Mazur games but in this paper, we always consider that Pl. 1 begins in order to bring out the notion of large set (rather than meagre set). The main result about Banach-Mazur games can then be stated as follows:

Theorem 1 ([7]). *Let* $\mathcal{G} = (G, v_0, W)$ *be a Banach-Mazur game on a finite graph. Pl. 0 has a winning strategy for* $\mathcal{G}$ *if and only if* W *is large.*

3 Simple strategies in Banach-Mazur games

In a Banach-Mazur game (G, v_0, W) on a finite graph, a strategy for Pl. 0 is given by a function f defined on $\text{FinPaths}(G, v_0)$, the set of finite paths of G starting from v_0, such that for any $\pi \in \text{FinPaths}(G, v_0)$, we have $f(\pi) \in \text{FinPaths}(G, \text{last}(\pi))$. However, we can imagine some restrictions on the strategies of Pl. 0:

1. A strategy f is said to be *positional* if it only depends on the current vertex i.e f is a function defined on V such that for any $v \in V$, $f(v) \in \text{FinPaths}(G, v)$.
2. A strategy f is said to be *finite-memory* if it only depends on the current vertex and a finite memory $\texttt{memory}$ i.e $\texttt{memory}(\text{FinPaths}(G, v_0))$ is a finite set and f is a function on $V \times \texttt{memory}(\text{FinPaths}(G, v_0))$ such that for any $v \in V$, any $m \in \texttt{memory}(\text{FinPaths}(G, v_0))$, $f(v, m) \in \text{FinPaths}(G, v)$ and a play ρ is according to f if after a prefix π, the moves of Pl. 0 is given by $f(\text{last}(\pi), \texttt{memory}(\pi))$.
3. A strategy f is said to be *b-bounded* if for any $\pi \in \text{FinPaths}(G, v_0)$, $f(\pi)$ has length less than b and a strategy is said to be *bounded* if there is $b \geq 1$ such that f is b-bounded.
4. A strategy f is said to be *move-counting* if it only depends on the current vertex and the number of moves already played i.e. f is a function defined on $V \times \mathbb{N}$ such that for any $v \in V$, any $n \in \mathbb{N}$, $f(v, n) \in \text{FinPaths}(G, v)$ and a play ρ is according to f if the moves of Pl. 0 are generated by $f(\cdot, 1), f(\cdot, 2), f(\cdot, 3), \cdots$.

5. A strategy f is said to be *length-counting* if it only depends on the current vertex and the length of the prefix already played i.e. f is a function defined on $V \times \mathbb{N}$ such that for any $v \in V$, any $n \in \mathbb{N}$, $f(v, n) \in \mathrm{FinPaths}(G, v)$ and a play ρ is according to f if after a prefix π, the move of Pl. 0 is given by $f(\mathrm{last}(\pi), |\pi|)$.

The notions of positional and finite memory strategies are classical, bounded strategies are present in [8], move-counting and length-counting strategies have been introduced in [4]. We first remark that, by definition, the existence of a positional winning strategy implies the existence of finite-memory/ move-counting/ length-counting winning strategies. Moreover, since G is a finite graph , a positional strategy is always bounded. In [3], it is proved that the existence of a finite-memory winning strategy implies the existence of a positional winning strategy.

Proposition 1 ([3]). *A Banach-Mazur game $\mathcal{G} = (G, v_0, W)$ is determined via a finite-memory winning strategy is in fact positionally determined.*

Using the ideas of the proof of the above proposition, we can also show that the existence of a winning strategy implies the existence of a length-counting winning strategy.

Proposition 2. *Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph. Pl. 0 has a length-counting winning strategy if and only if Pl. 0 has a winning strategy.*

On the other side, the notions of move-counting winning strategy and bounded winning strategy are incomparable.

*Example 1 (**Set with a move-counting winning strategy and without bounded winning strategy**).* We consider the complete graph $G_{0,1}$ on $\{0, 1\}$. Let W be the set of any sequences $(\sigma_n)_{n \geq 1}$ in $\{0, 1\}^\omega$ with $\sigma_1 = 0$ such that $(\sigma_n)_{n \geq 1}$ contains a finite sequence of 1 strictly longer than the initial finite sequence of 0. In other words, $(\sigma_n)_{n \geq 1} \in W$ if $\sigma_1 = 0$ and there exist $j, k \geq 1$ such that $\sigma_j = 1$ and $\sigma_{k+1} = \cdots = \sigma_{k+j} = 1$. Let $\mathcal{G} = (G_{0,1}, 0, W)$. The strategy $f(\cdot, n) = 1 \ldots 1$ with n occurrences of 1 is a move-counting winning strategy for Pl 0 for the game $\mathcal{G}$. On the other hand, there does not exist a bounded winning strategy for Pl. 0 for the game $\mathcal{G}$ because Pl. 1 can start by playing the word compounded of n zeros (with n as large as desired) and then, always play 0.

*Example 2 (**Set with a bounded winning strategy and without move-counting winning strategy**).* We consider the complete graph $G_{0,1}$ on $\{0, 1\}$. Let $(\pi_n)_{n \geq 0}$ be an enumeration of $\mathrm{FinPaths}(G)$ with $\pi_0 = 0$. We let W be the set of any sequences in $\{0, 1\}^\omega$ starting by 0 except the sequence $\rho = \pi_0 \pi_1 \pi_2 \ldots$. Let $\mathcal{G} = (G_{0,1}, 0, W)$. It is obvious that Pl. 0 has a 1-bounded winning strategy for $\mathcal{G}$ but we can also prove that Pl. 0 has no move-counting winning strategy. Indeed, if h is a move-counting strategy of Pl. 0, then Pl. 1 can start by playing a prefix π of ρ so that $\pi h(\mathrm{last}(\pi), 1)$ is a prefix of ρ. Afterwards, Pl. 1 complete the path of ρ such that Pl. 0 plays according to ρ and so on.

We remark that the sets W, considered in these examples, are *open* sets i.e. sets on a low level of the Borel hierarchy. Moreover, by Proposition 2, there also exist length-counting winning strategies for these two examples. The relations between the simple strategies are thus completely characterised and are summarised in Figure 1. The latter Figure also contains other simple strategies which will be discussed later.

4 Link with the sets of probability 1

Let $G = (V, E)$ be a finite directed graph. We can easily define a probability measure P, on the set of infinite paths in G, in giving a weight $w_e > 0$ at each edge $e \in E$ and in considering that for any $v, v' \in V$, $p_w(v, v') = 0$ if $(v, v') \notin E$ and $p_w(v, v') = \frac{w_{(v,v')}}{\sum_{e' \text{ enabled from v}} w_{e'}}$ else, where $p_w(v, v')$ denotes the probability of taking edge (v, v') from state v. Given $v_1 \cdots v_n \in \mathrm{FinPaths}(G, v_1)$, we denote by $\mathrm{Cyl}(v_1 \cdots v_n)$ the cylinder generated by $v_1 \cdots v_n$ and defined by $\mathrm{Cyl}(v_1 \cdots v_n) = \{\rho \in \mathrm{Paths}(G, v_1) \mid v_1 \cdots v_n \text{ is a prefix of } \rho\}$.

Definition 2. *Let $G = (V, E)$ be a finite directed graph and $w = (w_e)_{e \in E}$ a family of positive weights. We define the probability measure P_w by the relation*

$$P_w(\mathrm{Cyl}(v_1 \cdots v_n)) = p_w(v_1, v_2) \cdots p_w(v_{n-1}, v_n). \tag{2}$$

and we say that such a probability measure is reasonable*.*

We are interested in characterising the sets W of probability 1 and their link with the different notions of simple winning strategy. We remark that, in general, Banach-Mazur games do not characterise sets of probability 1. In other words, the notions of large sets and sets of probability 1 do not coincide in general on a finite graph. Indeed, there exist some large sets of probability 0. We present here an example of such sets:

*Example 3 (**Large set of probability** 0).* We consider the complete graph $G_{0,1,2}$ on $\{0, 1, 2\}$ and the set $W = \{(w_i w_i^R)_i : w_i \in \{0, 1, 2\}^*\}$, where for any finite word $\sigma \in \{0, 1, 2\}^*$ given by $\sigma = \sigma(1) \cdots \sigma(n)$ with $\sigma(i) \in \{0, 1, 2\}$, we let $\sigma^R = \sigma(n) \cdots \sigma(1)$. It is obvious that Pl. 0 has a winning strategy for the Banach-Mazur game $(G, 2, W)$ and thus that W is large. On the other hand, if P is the reasonable probability given by the weights $w_e = 1$ for any $e \in E$, then we can verify that $P(W) = 0$.

For certain families of sets, we can however have an equivalence between the notion of large set and the notion of set of probability 1. It is the case for the family of sets W representing ω-regular properties on finite graphs (see [8]). In order to prove this equivalence for ω-regular sets, Varacca and Völzer have in fact used the fact that for these sets, the Banach-Mazur game is positionally determined ([1]) and that the existence of a positional winning strategy implies $P(W) = 1$. This latter assertion follows from the fact that every positional strategy is bounded and that, by Borel-Cantelli lemma, the set of plays played according with a bounded strategy is a set of probability 1. Nevertheless, if W does not represent an ω-regular properties, it is possible that W is a large set of probability 1 and that there is no positional winning strategy for Pl. 0 and even no bounded or move-counting winning strategy.

Example 4 ***(Large set of probability*** 1 ***without positional/ bounded/ move-counting winning strategy).*** We consider the complete graph $G_{0,1}$ on $\{0,1\}$ and the reasonable probability P given by $w_e = 1$ for any $e \in E$. Let $a_n = \sum_{k=1}^{n} k$. We let $W = \{(\sigma_k)_{k\geq 1} \in \{0,1\}^\omega : \sigma_1 = 0 \text{ and } \sigma_{a_n} = 1 \text{ for some } n \geq 1\}$ and $\mathcal{G} = (G_{0,1}, 0, W)$. Since Pl. 0 has a winning strategy for $\mathcal{G}$, we deduce that W is a large set. We can also show that $P(W) = 1$. On the other hand, there does not exist any positional (resp. bounded) winning strategy f for Pl. 0. Indeed, if f is a positional (resp. bounded) strategy for Pl. 0 such that $f(0)$ (resp. $f(\pi)$ for any π) has length less than n, then Pl. 1 has just to start by playing a_n zeros so that Pl. 1 does not reach the index a_{n+1} and afterwards to complete the sequence by a finite number of zeros to reach the next index a_k, and so on. Moreover, in the same way, one can see that there is no move-counting winning strategy for Pl. 0.

However, we can show that the existence of a move-counting winning strategy for Pl. 0 implies $P(W) = 1$. The key idea is to realise that given h a move-counting winning strategy, the strategy $h(\cdot, n)$ is positional.

Proposition 3. *Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and P a reasonable probability measure. If Pl. 0 has a move-counting winning strategy for $\mathcal{G}$, then $P(W) = 1$.*

Let us notice that Example 4 implies that the existence of a move-counting winning strategy is strictly stronger than the assertion W is large and $P(W) = 1$. However, if W is a countable intersection of ω-regular sets, then the existence of a winning strategy for Pl. 0 implies the existence of a move-counting winning strategy for Pl. 0 and in particular, the notion of large sets and the notion of sets of probability 1 coincide for the countable intersections of ω-regular sets. Given $W = \cap_{n\geq 1} W_n$ a large set, where the W_n's are ω-regular sets. Since each W_n is large, we know by [1], that there exists a positional winning strategy f_n for each W_n. A move-counting winning strategy for W can then be built from the f_n's.

Proposition 4. *Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph where W is a countable intersection of ω-regular sets W_n. If Pl. 0 has a winning strategy, then Pl. 0 has a move-counting winning strategy.*

Remark 1. We cannot extend this result to countable unions of ω-regular sets because the set of countable unions of ω-regular sets contains the open sets and Example 2 exhibit a Banach-Mazur game where W is an open set and Pl. 0 has a winning strategy but no move-counting winning strategy.

Corollary 1. *Let W be a countable intersection of ω-regular sets on a finite graph and P a reasonable probability measure. The set W is large if and only if W is a set of probability* 1.

As a consequence of the above corollary, we have that if W is a ωS-regular sets, as defined in [2], the set W is large if and only if W is a set of probability 1. Indeed, it is shown in [6] that ωS-regular sets are countable intersection of ω-regular sets. Moreover, the following example show that, unlike the case of ω-regular sets, positional strategies are not sufficient for ωS-regular sets.

*Example 5 (**ωS-regular set with a move-counting strategy and without positional/ bounded strategy**).* We consider the complete graph $G_{0,1}$ on $\{0,1\}$ and the set W corresponding to the ωS-regular expression $((0^*1)^*0^S1)^\omega$, which corresponds to the language of words where the number of consecutive 0 is unbounded. The move-counting strategy which consists in playing n consecutive 0's at the nth step is winning for Pl. 0. However, clearly enough Pl. 0 does not have a positional (nor bounded) winning strategy for W.

Example 3 shows that Corollary 1 does not extend to ω-context-free sets. Another notion of simple strategies, natural in view of Example 3, is the notion of last-move strategy. A strategy f for Pl. 0 is said to be *last-move* if it only depends on the last move of Pl. 1 i.e. for any $v \in V$, for any $\pi \in \mathrm{FinPaths}(G, v)$, $f(\pi) \in \mathrm{FinPaths}(G, \mathrm{last}(\pi))$ and a play ρ is according to f if after the moves $\pi_0, \pi_1, \pi_2, \ldots, \pi_{2n}$, the move π_{2n+1} of Pl. 0 is given by $f(\pi_{2n})$. It is obvious that there exists a last-move winning strategy for Pl. 0 in the game described in Example 3. This Banach-Mazur game allows also us to see that the existence of a last-move winning strategy does not imply in general the existence of a move-counting winning strategy or a bounded winning strategy.

*Example 6 (**Set with a last-move winning strategy and without move-counting/ bounded winning strategy**).* We consider the complete graph $G_{0,1,2}$ on $\{0,1,2\}$ and the set $W = \{(w_i w_i^R)_i : w_i \in \{0,1,2\}^*\}$ defined in Example 3. If h is a move-counting strategy for Pl. 0 and if we denote by w_n the finite word $h(0,n)$ then, in order to win, Pl. 1 can play at each step the word $2(10)^{k_n}$ with $2k_n > |w_n|$ and $k_n > 2k_{n-1}$. Indeed, it is not difficult to see that the first symmetry of the run $2(10)^{k_1}w_1 2(10)^{k_2}w_2 2(10)^{k_3}w_3 \cdots$ can only appear on the moves of Pl. 0 but if we suppose that the first symmetry appears in the word w_n, then the letter 2 played by Pl. 0 in the previous move can not match with another letter 2 because $2k_n > |w_n|$ and $k_{n+1} > 2k_n$. Let $b \geq 1$. We also remark that Pl. 0 has no b-bounded winning strategy for the game $(G, 2, W)$ because it suffices that Pl. 1 plays at each time the word $2(10)^b$ in order to win.

The notion of last-move winning strategy is in fact incomparable with the notion of move-counting winning strategy and the notion of bounded winning strategy. Indeed, on the complete graph $G_{0,1}$ on $\{0,1\}$, if we denote by W the set of runs in $G_{0,1}$ such that for any $n \geq 1$, the word 1^n appear, then Pl. 0 has a move-counting winning strategy for the game $(G_{0,1}, 0, W)$ but no last-move winning strategy. In the same way, if we denote by W the set of no periodic runs on $G_{0,1}$ then Pl. 0 has a 1-bounded winning strategy for the game $(G_{0,1}, 0, W)$ but no last-move winning strategy. We also remark in view of Example 3 that the existence of a last-move winning strategy for W does not imply that W has probability 1.

5 Generalised Banach-Mazur games

Since the notions of bounded strategy and move-counting strategy are too strong to characterise the sets of probability 1, we search a strictly weaker notion of strategy such that the existence of such a winning strategy implies $P(W) = 1$. To this end, we introduce a new type of Banach-Mazur games:

Definition 3. *A* generalised Banach-Mazur game $\mathcal{G}$ *on a finite graph is a tuple* $(G, v_0, \phi_0, \phi_1, W)$ *where* $G = (V, E)$ *is a finite directed graph where every vertex has a successor,* $v_0 \in V$ *is the initial state,* $W \subset \mathrm{Paths}(G, v_0)$*, and* ϕ_i *is a map on* $\mathrm{FinPaths}(G, v_0)$ *such that for any* $\pi \in \mathrm{FinPaths}(G, v_0)$*,*

$$\phi_i(\pi) \subset \mathcal{P}\big(\mathrm{FinPaths}(G, \mathrm{last}(\pi))\big) \textit{ and } \phi_i(\pi) \neq \emptyset.$$

A generalised Banach-Mazur game $\mathcal{G} = (G, v_0, \phi_0, \phi_1, W)$ on a finite graph is a two-player game where Pl. 0 and Pl. 1 alternate in choosing a *set of finite paths* as follows: Pl. 1 begins with choosing a set of finite paths $\Pi_1 \in \phi_1(v_0)$; Pl.0 selects a finite path $\pi_1 \in \Pi_1$ and chooses a set of finite paths $\Pi_2 \in \phi_0(\pi_1)$; Pl 1. then selects $\pi_2 \in \Pi_2$ and proposes a set $\Pi_3 \in \phi_1(\pi_1\pi_2)$ and so on. A play of $\mathcal{G}$ is thus an infinite path $\pi_1\pi_2\pi_3\dots$ in G and we say that Pl. 0 wins if this path belongs to W; while Pl. 1 wins if this path does not belong to W.

We remark that if we let $\phi_{\mathrm{ball}}(\pi) := \{\{\pi'\} : \pi' \in \mathrm{FinPaths}(G, \mathrm{last}(\pi))\}$ for any $\pi \in \mathrm{FinPaths}(G, v_0)$, then the generalised Banach-Mazur game given by $(G, v_0, \phi_{\mathrm{ball}}, \phi_{\mathrm{ball}}, W)$ coincides with the classical Banach-Mazur game (G, v_0, W). We also obtain a game similar to the classical Banach-Mazur if we consider the function $\phi(\pi) = \mathcal{P}(\mathrm{FinPaths}(G, \mathrm{last}(\pi)))$. On the other hand, if we consider $\phi(\pi) := \{\{\pi'\} : \pi' \in \mathrm{FinPaths}(G, \mathrm{last}(\pi)), |\pi'| = 1\}$, we obtain the classical games on graphs such as the one studied in [5].

We are interested in defining a map ϕ_0 such that Pl. 0 has a winning strategy for $(G, v_0, \phi_0, \phi_{\mathrm{ball}}, W)$ if and only if $P(W) = 1$. To this end, we notice that we can restrict actions of Pl. 0 by forcing each set in $\phi_0(\pi)$ to be "big" in some sense. The idea to characterise $P(W) = 1$ is therefore to force Pl. 0 to play with finite sets of finite paths of conditional probability bigger than α for some $\alpha > 0$.

Definition 4. *Let* $\mathcal{G} = (G, v_0, W)$ *be a Banach-Mazur game on a finite graph,* P *a reasonable probability measure and* $\alpha > 0$*. An* α-strategy *of Pl. 0 for* $\mathcal{G}$ *is a strategy of Pl. 0 for the generalised Banach-Mazur game* $\mathcal{G}_\alpha = (G, v_0, \phi_\alpha, \phi_{ball}, W)$ *where*

$$\phi_\alpha(\pi) = \Big\{\Pi \subset \mathrm{FinPaths}(G, \mathrm{last}(\pi)) : P\Big(\bigcup_{\pi' \in \Pi} \mathrm{Cyl}(\pi\pi')\Big|\mathrm{Cyl}(\pi)\Big) \geq \alpha \textit{ and } \Pi \textit{ is finite}\Big\}.$$

We notice that every bounded strategy can be seen as an α-strategy for some $\alpha > 0$, since for any $N \geq 1$, there exists $\alpha > 0$ such that for any π of length less than N, we have $P(\{\pi\}) \geq \alpha$. We can also show that the existence of a move-counting winning strategy for Pl. 0 implies the existence of a winning α-strategy for Pl. 0 for any $0 < \alpha < 1$.

Proposition 5. *Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph. If Pl. 0 has a move-counting winning strategy, then Pl. 0 has a winning α-strategy for any $0 < \alpha < 1$.*

Moreover, the existence of a winning α-strategy for some $\alpha > 0$ still implies $P(W) = 1$.

Theorem 2. *Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and P a reasonable probability measure. If Pl. 0 has a winning α-strategy for some $\alpha > 0$ then $P(W) = 1$.*

The proof of the above theorem is based on a slicing of the infinite plays into finite words whose probability to be played according to the α-strategy is bigger than α. We conclude by applying some kind of Borel-Cantelli Lemma.

If W is a countable intersection of open sets, we can prove the converse of the previous theorem and so obtain a characterisation of sets of probability 1.

Theorem 3. *Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph where W is a countable intersection of open sets and P a reasonable probability measure. Then the following assertions are equivalent:*

1. *$P(W) = 1$,*
2. *Pl. 0 has a winning α-strategy for some $\alpha > 0$,*
3. *Pl. 0 has a winning α-strategy for any $0 < \alpha < 1$.*

We only need to prove 1. $\Rightarrow$ 3. Let us notice that if W is an open set of probability 1, one can find a finite union of cylinders included in W whose probability is bigger than α. The desired result then follows.

Remark 2. We cannot hope generalise the latter result to any set W because the existence of a winning α-strategy implies the existence of a winning strategy and we know that there exists some meagre set W of probability 1 (see Example 3). However, we can wonder whether the existence of a winning α-strategy is equivalent to the fact that W is a large set of probability 1.

When W is a countable intersection of open sets, we remark that the generalised Banach-Mazur game $\mathcal{G}_\alpha = (G, v_0, \phi_\alpha, \phi_{\text{ball}}, W)$ is in fact determined.

Theorem 4. *Let $\mathcal{G}_\alpha$ be the generalised Banach-Mazur game given by $\mathcal{G}_\alpha = (G, v_0, \phi_\alpha, \phi_{ball}, W)$ where W is a countable intersection of open sets and P a reasonable probability measure. Then the following assertions are equivalent:*

1. *$P(W) < 1$,*

2. Pl. 1 has a winning strategy for $\mathcal{G}_\alpha$ for some $\alpha > 0$,
3. Pl. 1 has a winning strategy for $\mathcal{G}_\alpha$ for any $0 < \alpha < 1$.

Thanks to Theorem 3, it suffices to prove 1. ⇒ 3. W.l.o.g. we can assume that W is open. As $P(W) < 1$, one can prove that Pl. 1 has a move π_1 forcing Pl. 0 to propose him a finite path π_2 such that $P(W|\pi_1\pi_2) \leq P(W)$. By iterating this reasoning, Pl. 1 will prevent the final play to be in W. Indeed, as W is an open set, if a play $(\pi_i)_{i\geq 1}$ is winning for Pl. 0, then we have that $P(W|\pi_1\pi_2\cdots\pi_n) = 1$, for some n.

Corollary 2. *Let $0 < \alpha < 1$. The generalised Banach-Mazur game $\mathcal{G}_\alpha = (G, v_0, \phi_\alpha, \phi_{ball}, W)$ is determined when W is a countable intersection of open sets. More precisely, Pl. 0 has a winning strategy for $\mathcal{G}_\alpha$ if and only if $P(W) = 1$, and Pl. 1 has a winning strategy for $\mathcal{G}_\alpha$ if and only if $P(W) < 1$.*

Since the notion of winning α-strategy is weaker than the notions of bounded winning strategy and move-counting winning strategy, we deduce from Example 1 and Example 2 that the notion of winning α-strategy is even strictly weaker than these notions. On the other hand, we know that there exists a Banach-Mazur game for which Pl. 0 has a bounded winning strategy and no last-move winning strategy. The existence of a winning α-strategy does not thus imply in general the existence of a last-move winning strategy. Conversely, if we consider the game $(G_{0,1}, 0, W)$ described in Example 3, Pl. 0 has a last-move winning strategy but no winning α-strategy (as $P(W) = 0$). The notion of α-strategy is thus incomparable with the notion of last-move strategy.

6 More on simple strategies

We finish this section by considering the crossings between the different notions of simple strategy and the notion of bounded strategy i.e. the bounded length-counting strategies, the bounded move-counting strategies and the bounded last-move strategies. Obviously, each notion of bounded strategies implies their no bounded counterpart. We start by noticing that the existence of a bounded move-counting winning strategy is equivalent to the existence of a positional winning strategy.

Proposition 6. *Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph. Pl. 0 has a bounded move-counting winning strategy if and only if Pl. 0 has a positional winning strategy.*

The other notions of bounded strategy are not equivalent to any other notion of simple strategy.

*Example 7 (**Set with a bounded length-counting winning strategy and without positional winning strategy**).* Let $G_{0,1}$ be the complete graph on $\{0,1\}$, (ρ_n) an enumeration of finite words in $\{0,1\}$ and $\rho_{\text{target}} = 0\rho_1\rho_2\cdots$. We consider the set $W = \{\sigma \in \{0,1\}^\omega : \#\{i \geq 1 : \sigma(i) = \rho_{\text{target}}(i)\} = \infty\}$. It is evident that Pl. 0 has a bounded length-counting winning strategy for the game $(G_{0,1}, 0, W)$. However, Pl. 0 has no positional winning strategy. Indeed, if f is a positional strategy such that $f(0) = a(1) \cdots a(k)$, then Pl. 1 can play according to the strategy h defined by $h(\sigma(1) \cdots \sigma(n)) = \sigma(n+1) \cdots \sigma(N)\ 0$ such that for any $n+1 \leq i \leq N$, $\sigma(i) \neq \rho_{\text{target}}(i)$, $\rho_{\text{target}}(N+1) \neq 0$ and for any $1 \leq i \leq k$, $a(i) \neq \rho_{\text{target}}(N+i+1)$.

Example 8 ***(Set with a bounded last-move winning strategy and without positional winning strategy)****.* Let $G_{0,1,2}$ be the complete graph on $\{0,1,2\}$. We consider a function $\phi : \{0,1,2\}^* \to \{0,1\}$ such that for any $\pi \in \{0,1,2\}^*$, any $n \geq 1$, any $\sigma(1), \ldots, \sigma(n) \in \{0,1,2\}$, there exists $k \geq 1$ such that $\phi(\pi 2^k) \neq \sigma(1)$ and for any $1 \leq i \leq n-1$, $\phi(\pi 2^k \sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$. Such a function exists because the set $\{0,1\}^*$ is countable. Let $W = \{(\pi_i \phi(\pi_i))_i : \pi_i \in \{0,1,2\}^*\}$. It is obvious that Pl. 0 has a 1-bounded last-move winning strategy for the game $(G_{0,1,2}, 2, W)$. Nevertheless, Pl. 0 has no positional winning strategy. Indeed, if f is a positional strategy such that $f(2) = a(1) \ldots a(n)$, then Pl. 1 can play according to the strategy h defined by $h(\pi) = 2^k$ such that $\phi(\pi 2^k) \neq a(1)$ and for any $1 \leq i \leq n-1$, $\phi(\pi 2^k a(1) \cdots a(i)) \neq a(i+1)$.

Example 9 ***(Set with a bounded winning strategy and without bounded length-counting winning strategy)****.* Let $G_{0,1,2,3}$ be the complete graph on $\{0,1,2,3\}$ and $n_k = \sum_{i=1}^{k} 3i$. We consider a function $\phi : \{0,1,2,3\}^* \to \{0,1\}$ such that for any $k \geq 1$, any $\pi \in \{0,1,2,3\}^*$ of length n_k and any $\sigma(1), \ldots, \sigma(k) \in \{0,1,2,3\}$, there exists $\tau \in \{2,3\}^*$ of length $2k$ such that $\phi(\pi\tau) \neq \sigma(1)$ and for any $1 \leq i \leq n-1$, $\phi(\pi\tau\sigma(1) \cdots \sigma(i)) \neq \sigma(i+1)$. Such a function exists because the cardinality of $\{2,3\}^{2k}$ is equal to the cardinality of $\{0,1,2,3\}^k$ and the length of $\pi\tau\sigma(1) \cdots \sigma(k) < n_{k+1}$. Let W be the set of runs ρ such that $\#\{n \geq 1 : \phi(\rho(1) \ldots \rho(n)) = \rho(n+1)\} = \infty$. Pl. 0 has a bounded winning strategy for W but no bounded length-counting winning strategy.

The relations between the different notions of simple strategy on a finite graph can be summarised as depicted in Figure 1.

We draw attention to the fact that the situation is very different in the case of infinite graphs. For example, a positional strategy can be unbounded, the notion of length-counting winning strategy is not equivalent to the notion of winning strategy (except if the graph is finitely branching), and the notion of bounded move-counting winning strategy for Pl. 0 is not equivalent to the notion of positional winning strategy.

Example 10 ***(Set on an infinite graph with a bounded move-counting winning strategy and without positional winning strategy)****.* We consider the complete graph $G_{\mathbb{N}}$ on $\mathbb{N}$ and the game $\mathcal{G} = (G_{\mathbb{N}}, 0, W)$ where $W = \{(\sigma_k) \subset \mathbb{N} : \forall\ n \geq 1,\ \exists\ k \geq 1,\ (\sigma_k, \sigma_{k+1}) = (n, n+1)\}$. Pl. 0 has a bounded move-counting winning strategy given by $h(v,n) = n\ n+1$ but no positional winning strategy.

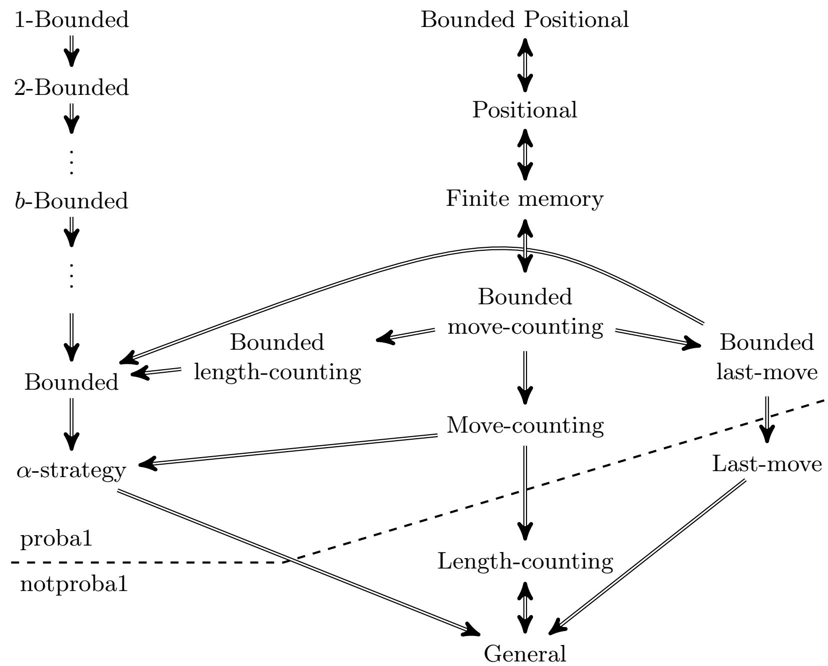

Fig. 1. Winning strategies for Player 0 in finite graphs.

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A Appendix

A.1 Proof of Proposition 2

Proof. Let f be a winning strategy for Pl. 0. Since G is a finite graph, for any $n \geq 0$ and any $v \in V$, we can consider an enumeration $\pi_1, \ldots, \pi_m$ of finite paths in $\mathrm{FinPaths}(G, v_0)$ of length n such that $\mathrm{last}(\pi_i) = v$. We then let

$$h(v,n) = f\big(\pi_1\big) f\big(\pi_2 f(\pi_1)\big) f\big(\pi_3 f(\pi_1) f(\pi_2 f(\pi_1))\big) \ldots f\big(\pi_n f(\pi_1) f(\pi_2 f(\pi_1)) \cdots\big).$$

If ρ is a play played according to h, then ρ is a play where the strategy f is applied infinitely often. Each play played according to h can thus be seen as a play played according to f and we deduce that the strategy h is a length-counting winning strategy. □

A.2 Details for Example 3

We consider the complete graph $G_{0,1,2}$ on $\{0,1,2\}$ and the set $W = \{(w_i w_i^R)_i : w_i \in \{0,1,2\}^*\}$, where for any finite word $\sigma \in \{0,1,2\}^*$ given by $\sigma = \sigma(1) \cdots \sigma(n)$ with $\sigma(i) \in \{0,1,2\}$, we let $\sigma^R = \sigma(n) \cdots \sigma(1)$. It is obvious that Pl. 0 has a winning strategy for the Banach-Mazur game $(G, 2, W)$ and thus that W is large. On the other hand, if P is the reasonable probability given by the weights $w_e = 1$ for any $e \in E$, then we can verify that $P(W) = 0$. Indeed, we have

$$\begin{aligned}
P(W) &\leq \sum_{n=1}^{\infty} P(\{w \in W : w(1) \cdots w(n) = w(2n) \cdots w(n+1)\}) \\
&= \sum_{n=1}^{\infty} P(W) \cdot P(\{w \in \mathrm{Paths}(G_{0,1,2}) : w(1) \cdots w(n) = w(2n) \cdots w(n+1)\}) \\
&\leq \sum_{n=1}^{\infty} \frac{P(W)}{3^n} = \frac{1}{2} P(W).
\end{aligned}$$

A.3 Details for Example 4

We consider the complete graph $G_{0,1}$ on $\{0,1\}$ and the reasonable probability P given by $w_e = 1$ for any $e \in E$. Let $a_n = \sum_{k=1}^{n} k$. We let $W = \{(\sigma_k)_{k \geq 1} \in \{0,1\}^\omega : \sigma_1 = 0 \text{ and } \sigma_{a_n} = 1 \text{ for some } n \geq 1\}$ and $\mathcal{G} = (G_{0,1}, 0, W)$. Since Pl. 0 has a winning strategy for $\mathcal{G}$, we deduce that W is a large set. We can also compute that $P(W) = 1$ because if we denote by A_n, $n > 1$, the set

$$A_n := \{(\sigma_k)_{k \geq 1} \in \{0,1\}^\omega : \sigma_{a_n} = 1 \text{ and } \sigma_{a_k} = 0 \text{ for any } k < n\},$$

we have:

$$W = \dot{\bigcup}_{n>1} A_n \quad \text{and} \quad P(A_n) = \frac{1}{2^{n-1}}.$$

On the other hand, there does not exist any positional (resp. bounded) winning strategy f for Pl. 0. Indeed, if f is a positional (resp. bounded) strategy for

Pl. 0 such that $f(0)$ (resp. $f(\pi)$ for any π) has length less than n, then Pl. 1 has just to start by playing a_n zeros so that Pl. 1 does not reach the index a_{n+1} and afterwards to complete the sequence by a finite number of zeros to reach the next index a_k, and so on. Moreover, there does not exist any move-counting winning strategy h for Pl. 0 because Pl. 1 can start by playing a_n zeros so that $|h(0,1)| \leq n$ and because, at each step k, Pl. 1 can complete the sequence by a finite number of zeros to reach a new index a_n such that $|h(0,k)| \leq n$.

A.4 Proof of Proposition 3

Proof. Let h be a move-counting winning strategy of $Pl.$ 0. We denote by f_n the strategy $h(\cdot, n)$. Each set

$$M_n := \{\rho \in \text{Paths}(G, v_0) \ : \ \rho \text{ is a play played according to } f_n\}$$

has probability 1 since f_n is a positional winning strategy for the Banach-Mazur game (G, v_0, M_n). Moreover, if ρ is a play played according to f_n for each $n \geq 1$, then ρ is a play played according to h. In other words, since h is a winning strategy, we get $\bigcap_n M_n \subset W$. Therefore, as $P(M_n) = 1$, we conclude that $P(W) = 1$. □

A.5 Proof of Proposition 4

Proof. Let $W = \bigcap_{n \geq 1} W_n$ where W_n is an ω-regular set and f a winning strategy for Pl. 0 for $\mathcal{G}$. For any $n \geq 1$, the strategy f is a winning strategy for the Banach-Mazur game (G, v_0, W_n). Thanks to [1], we therefore know that for any $n \geq 1$, there exists a positional winning strategy $\tilde{f}_n$ for Pl. 0 for (G, v_0, W_n).

Let $\phi : \mathbb{N} \to \mathbb{N}$ such that for any $k \geq 1$, $\{n \in \mathbb{N} : \phi(n) = k\}$ is an infinite set. We consider the move-counting strategy $h(v, n) = \tilde{f}_{\phi(n)}(v)$. This strategy is winning because each play ρ played according to h consists in a play played according to $\tilde{f}_n$ for any n and thus

$$\begin{aligned}
&\{\rho \in \text{Paths}(G, v_0) \ : \ \rho \text{ is a play played according to } h\} \\
&\quad \subseteq \bigcap_n \{\rho \in \text{Paths}(G, v_0) \ : \ \rho \text{ is a play played according to } \tilde{f}_n\} \\
&\quad \subseteq \bigcap_n W_n = W.
\end{aligned}$$

□

A.6 Proof of Proposition 5

Proof. Let P be a reasonable probability measure, h a move-counting winning strategy for Pl. 0 and $0 < \alpha < 1$. We denote by g_n the positional strategy defined by

$$g_n(v) = h(v,1) \ h\big(\text{last}(h(v,1)), 2\big) \ \cdots \ h\big(\text{last}(h(v,1) \ h(\text{last}(h(v,1)), 2) \cdots), n\big).$$

Since g_n is a positional strategy, we know that each set

$$M_n := \{\rho \in \mathrm{Paths}(G, v_0) \ : \ \rho \text{ is a play played according to } g_n\}$$

has probability 1. In particular, for any $\pi_0 \in \mathrm{FinPaths}(G, v_0)$, we deduce that $P(M_n|\mathrm{Cyl}(\pi_0)) = 1$. Since

$$M_n \cap \mathrm{Cyl}(\pi_0) \subseteq \bigcup_{\pi \in \mathrm{FinPaths}(G, \mathrm{last}(\pi_0))} \mathrm{Cyl}\big(\pi_0 \pi g_n(\mathrm{last}(\pi))\big),$$

we deduce that for any $n \geq 1$, any $\pi_0 \in \mathrm{FinPaths}(G, v_0)$, there exists a finite subset $\Pi_n(\pi_0) \subset \mathrm{FinPaths}(G, \mathrm{last}(\pi_0))$ such that

$$P\Big(\bigcup_{\pi \in \Pi_n(\pi_0)} \mathrm{Cyl}\big(\pi_0 \pi g_n(\mathrm{last}(\pi))\big) \Big| \mathrm{Cyl}(\pi_0) \Big) \geq \alpha.$$

We denote by $\Pi'_n(\pi_0)$ the set $\{\pi g_n(\mathrm{last}(\pi)) : \pi \in \Pi_n(\pi_0)\}$ and we let

$$f(\pi_0) := \Pi'_{n+1}(\pi_0),$$

where $n \geq 0$ is the biggest integer such that π_0 can be seen as a finite path where Pl. 0 has played n times according to h. The above-defined strategy f is therefore a winning α-strategy for Pl. 0 because each play played according to f is a play played according to h.

A.7 Proof of Theorem 2

Proof. Let f be a winning α-strategy. We consider an increasing sequence $(a_n)_{n\geq 1}$ such that for any $n \geq 1$, any π of length a_n, each $\pi' \in f(\pi)$ has length less than $a_{n+1} - a_n$; this is possible because for any π, $f(\pi)$ is a finite set by definition of α-strategy. Without loss of generality, we can even suppose that for any $n \geq 1$, any π of length a_n, each $\pi' \in f(\pi)$ has exactly length $a_{n+1} - a_n$. We therefore let

$$A := \{(\sigma_k)_{k\geq 1} \in \mathrm{Path}(G, v_0) : \#\{n : (\sigma_k)_{a_n+1 \leq k \leq a_{n+1}} \in f((\sigma_k)_{1\leq k \leq a_n})\} = \infty\}.$$

In other words, $(\sigma_k)_{k\geq 1} \in A$ if (σ_k) can be seen as a play played according to f on an infinite number of indices a_n. Since f is a winning strategy, A is included in W and it thus suffices to prove that $P(A) = 1$. However, we notice that for any $m \geq 1$, any $n \geq m + 1$, if we let

$$B_{m,n} = \{(\sigma_k)_{k\geq 1} \in \mathrm{Path}(G, v_0) : (\sigma_k)_{a_j+1\leq k\leq a_{j+1}} \notin f((\sigma_k)_{1\leq k\leq a_j}),\ \forall m \leq j \leq n\},$$

then $P(B_{m,n}) \leq (1 - \alpha)^{n-m}$ as f is an α-strategy. We therefore deduce that for any $m \geq 1$,

$$P\Big(\bigcap_{n=m+1}^{\infty} B_{m,n} \Big) = 0$$

and since $A^c = \bigcup_{m\geq 1} \bigcap_{n=m+1}^{\infty} B_{m,n}$, we conclude that $P(A) = 1$. ◻

A.8 Proof of Theorem 3

Proof. We have already proved 2. $\Rightarrow$ 1., and 3. $\Rightarrow$ 2. is obvious.
1. $\Rightarrow$ 3. Let $0 < \alpha < 1$. Let $W = \bigcap_{n=1}^{\infty} W_n$ where W_n is an open set. Since $P(W) = 1$, we deduce that for any $n \geq 1$, $P(W_n) = 1$. We can therefore define a winning α-strategy f of Pl. 0 as follows: if $\mathrm{Cyl}(\pi) \subset \bigcap_{k=1}^{n-1} W_k$ and $\mathrm{Cyl}(\pi) \not\subset W_n$, we let $f(\pi)$ be a finite set $\Pi \subset \mathrm{FinPaths}(G, \mathrm{last}(\pi))$ such that $P\Big(\bigcup_{\pi' \in \Pi} \mathrm{Cyl}(\pi\pi') | \mathrm{Cyl}(\pi)\Big) \geq \alpha$ and for any $\pi' \in \Pi$, $\mathrm{Cyl}(\pi\pi') \subset W_n$. The previous construction is possible because W_n is an open set of probability 1 and it concludes the proof. $\square$

A.9 Proof of Theorem 4

Proof. We deduce from Theorem 3 that 2. $\Rightarrow$ 1. because $\mathcal{G}_\alpha$ is a zero-sum game, and 3. $\Rightarrow$ 2. is obvious.
1. $\Rightarrow$ 3. Let $W = \cap_{n=1}^{\infty} W_n$ with $P(W) < 1$ and W_n open. We know that there exists $n \geq 1$ such that $P(W_n) < 1$. It then suffices to prove that Pl. 1 has a winning strategy for the generalised Banach-Mazur game $(G, v_0, \phi_\alpha, \phi_{\mathrm{ball}}, W_n)$ for any $0 < \alpha < 1$. Without loss of generality, we can thus assume that W is an open set. We recall that W is open if and only if it is a countable union of cylinder.

Since any strategy of Pl. 1 is winning if $W = \emptyset$, we suppose that $W \neq \emptyset$. Let $0 < \alpha < 1$. We let

$$I_W := \inf\{P(W|\pi) : \pi \in \mathrm{FinPaths}(G, v_0)\},$$

where $P(W|\pi) = P(W \cap \mathrm{Cyl}(\pi))/P(\mathrm{Cyl}(\pi))$. We first remark that $I_W < P(W)$. Indeed, since W is a non-empty union of cylinder, there exists $\pi \in \mathrm{FinPaths}(G, v_0)$ such that $P(W|\pi) = 1$; however, if $P(W) \leq I_W$ then for any $\pi \in \mathrm{FinPaths}(G, v_0)$,

$$P(W|\pi) = P(W) < 1.$$

Therefore, there exists $\pi_1 \in \mathrm{FinPaths}(G, v_0)$ such that

$$I_W + \frac{1}{\alpha}(P(W|\pi_1) - I_W) < P(W). \tag{3}$$

We consider that the first move of Pl. 1 is given by π_1 and that Pl. 0 then chooses a finite set $\Pi_2 \in \phi_\alpha(\pi_1)$. Let us notice that we can suppose that every finite path $\pi \in \Pi_2$ has same length N. We therefore consider that Pl. 1 selects $\pi_2 \in \Pi_2$ such that $P(W|\pi_1\pi_2) = \min_{\pi \in \Pi_2} P(W|\pi_1\pi)$. We affirm that

$$P(W|\pi_1\pi_2) \leq P(W) < 1. \tag{4}$$

If this inequality is proven, we obtain a winning strategy for Pl. 1 by repeating the above method. Indeed, as W is an open set and thus a countable union of cylinders, if $P(W|\pi_1 \cdots \pi_n)$ is not ultimately equal to 1, then the infinite path $\pi_1\pi_2\pi_3 \cdots \notin W$ and we obtain the desired result.

It remains to prove (4) but if we suppose that (4) is not satisfied then

$$P(W|\pi_1) = \sum_{\pi \in \Pi_2} P(W|\pi_1\pi)P(\pi_1\pi|\pi_1) + \sum_{\substack{\pi \in \mathrm{FinPaths}(G, \mathrm{last}(\pi_1)) \setminus \Pi_2 \\ \pi \text{ has length } N}} P(W|\pi_1\pi)P(\pi_1\pi|\pi_1)$$

$$\geq P(W)\alpha + (1-\alpha)I_W$$

and thus

$$P(W) \leq I_W + \frac{1}{\alpha}(P(W|\pi_1) - I_W)$$

which is a contradiction with (3). □

A.10 Proof of Proposition 6

Proof. Let h be a bounded move-counting winning strategy for Pl. 0. We denote by $C_1, \dots, C_N$ the bottom strongly connected components (BSCC) of G. Let $1 \leq i \leq N$. Since h is a bounded strategy and G is finite, there exist some finite paths $w_1^{(i)}, \dots, w_{k_i}^{(i)} \subset C_i$ such that for any $v \in C_i$, for any $n \geq 1$,

$$h(v, n) \in \{w_1^{(i)}, \dots, w_{k_i}^{(i)}\}.$$

Let $v \in V$. If $v \in C_i$, we let $f(v) = \sigma_0 w_1^{(i)} \sigma_1 w_2^{(i)} \sigma_2 \dots w_{k_i}^{(i)}$ where σ_l are finite paths in C_i such that $f(v)$ is a finite path in C_i starting from v. If $v \notin \bigcup_i C_i$, we let $f(v) = \sigma_v$ where σ_v starts from v and leads in a BSCC of G. The positional strategy f is therefore winning as each play ρ according to f can be seen as a play played according to h. □