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Complexity and Nicety of Fluted Logic

Abstract. Fluted Logic is essentially first-order predicate logic deprived of variables. The lack of variables results in reduced expressiveness. Nevertheless, many logical problems that can be stated in natural language, such as the famous Schubert's Steamroller, can be rendered in fluted logic. Further evidence of the expressiveness of fluted logic is its close relation to description logics. Already it has been shown that fluted logic is decidable and has the finite-model property. This paper shows that fluted logic has the exponential-model property and that deciding satisfiability is NEXPTIME-complete. It is shown further that fluted logic is 'nice', that is, it shares with first-order predicate logic the interpolation property and model preservation properties.

Keywords: fluted logic, first-order fragment, complexity, interpolation property, preservation properties.

1. Introduction

Fluted Logic (FL) is essentially first-order predicate logic (FO) deprived of variables. It lacks the capabilities to

1. permute the arguments of a predicate,
2. identify or equate the arguments of a predicate, and
3. add vacuous arguments to the arguments of a predicate.

Put differently, a fluted formula is a first-order formula in which the order of the arguments of each predicate is precisely the order of the enclosing quantifier scopes. Thus in FL variables have no essential role and simply could be eliminated.

In spite of its reduced capabilities, FL retains a significant part of the expressiveness of FO. Many logical problems that can be stated in natural language, such as the famous Schubert's Steamroller (Stickel [19]), can be rendered in FL. FL also has a close relation to description logics such as $\mathcal{ALC}$, to modal logic, and to path logic (e.g., see Ohlbach and Schmidt [12], Hustadt and Schmidt [10], Schmidt [16]).

A concise characterization of the expressiveness of fluted logic is not known, but the following should be obvious. Fluted formulas cannot directly

Presented by **Robert Goldblatt**; *Received* April 11, 2001

Studia Logica **71**: 177–198, 2002.

express reflexivity, symmetry, or transitivity of relations. Therefore they can describe a path in terms of properties of its points, but can neither require nor proscribe cycles on the path. Hence fluted logic cannot differentiate between finite and infinite models.

It has been shown (Purdy [13, 14]) that FL is decidable and that it has the finite-model property. A sublogic of FO that is decidable and has the finite-model property is said to be 'tame'. If it still retains certain desirable features of FO, viz., the interpolation property and the model preservation properties, it will be said to be 'nice'. (Andréka, Kurucz, Németi, and Sain [2] define the term 'nice' differently. The use of the term in this paper is consistent with Andréka, van Benthem, and Németi [1], who speak of 'nice properties' that a sublogic may or may not have.)

This paper shows that fluted logic has the exponential-model property and that deciding satisfiability is NEXPTIME-complete.

In contrast to more familiar fragments of FO, the syntax of FL is extremely simple. Taking advantage of the fact that variables are dispensable in FL, the set of formulas $\mathrm{fml}(L)$ of FL for a set of predicates L can be defined:

1. if $R \in L$, then $R \in \mathrm{fml}(L)$;
2. if $\phi, \psi \in \mathrm{fml}(L)$, then $\neg\phi, \phi \wedge \psi, \phi \vee \psi, \phi \rightarrow \psi \in \mathrm{fml}(L)$;
3. if $\phi \in \mathrm{fml}(L)$, then $\exists\phi, \forall\phi \in \mathrm{fml}(L)$.

As an example of this syntax, consider a bipartite graph with edge relation R, whose parts (even and odd) are nonempty. This graph can be described: $\exists E, \exists O, \forall(E \rightarrow (\exists(R \wedge O) \wedge \neg\exists(R \wedge E))), \forall(O \rightarrow (\exists(R \wedge E) \wedge \neg\exists(R \wedge O)))$.

Also in contrast to more familiar fragments of FO, fluted logic is nice, which will be shown in this paper.

2. Preliminaries

We assume the usual definition of first-order predicate logic with connectives $\neg, \wedge, \vee$, and $\rightarrow$, and quantifiers $\exists$ and $\forall$. The set of predicate symbols will be those that occur in some given finite set of formulas called *premises*. The finite set of predicate symbols will be referred to as the *lexicon*. If L is a lexicon and $R \in L$, then $\mathrm{ar}(R)$ denotes the arity of R.

A subformula is *prime* if it is atomic or of the form $\exists x\zeta$ or $\forall x\zeta$.

An *L-structure* $\mathcal{A}$ consists of a set A, the *domain*, and a mapping that assigns to each $R \in L$ a subset $R^{\mathcal{A}} \subseteq A^{\mathrm{ar}(R)}$. The notions of satisfaction and truth are the standard ones. If ψ is a formula over L with free variables among $\{x_1, \ldots, x_k\}$, and ψ is satisfied in $\mathcal{A}$ by the assignment of values to variables $\{x_i \mapsto a_i\}_{1\leq i\leq k}$, we write $\mathcal{A}, a_1 \cdots a_k \models \psi$. If ψ is a sentence and ψ is true in $\mathcal{A}$, we write $\mathcal{A}, \varepsilon \models \psi$ or simply $\mathcal{A} \models \psi$.

Fluted logic, a fragment of FO, is now defined. In this definition and throughout the paper variables will be retained to facilitate understanding and to point up the relation of FL to FO. Let L be a lexicon. Let $X_m := \{x_1, \ldots, x_m\}$ be an ordered set of m variables where $m \geq 0$. An *atomic fluted formula of L over X_m* is $Rx_{m-n+1} \cdots x_m$, where $R \in L$ and $\mathrm{ar}(R) = n \leq m$. The set of all atomic fluted formulas of L over X_m will be denoted $\mathrm{Af}_L(X_m)$. Define $\mathrm{Af}_L(X_0) := \{\top\}$.

A *fluted formula of L over X_m* is defined inductively. The *quantifier rank (*qr*)* is defined simultaneously.

1. An atomic fluted formula of L over X_m is a fluted formula of L over X_m with quantifier rank 0.
2. If ψ is a fluted formula of L over X_m with quantifier rank r, then $\neg\psi$ is a fluted formula of L over X_m with quantifier rank r.
3. If ψ and ϕ are fluted formulas of L over X_m with quantifier ranks r_ϕ and r_ψ, respectively, then $\psi \wedge \phi$, $\psi \vee \phi$, and $\psi \rightarrow \phi$ are fluted formulas of L over X_m with quantifier rank $\max(r_\phi, r_\psi)$.
4. If ψ is a fluted formula of L over X_{m+1} with quantifier rank r, then $\exists x_{m+1}\psi$ and $\forall x_{m+1}\psi$ are fluted formulas of L over X_m with quantifier rank $r + 1$.

The fluted formulas just defined will be referred to as *standard* fluted formulas. In addition, any first-order formula that can be transformed into a standard fluted formula by a consistent renaming of variables (both bound and free) is defined to be a fluted formula. No other formula is a fluted formula. Note that ψ is a standard fluted formula *over X_m* iff its free variables are exactly $\{x_i, x_{i+1}, \ldots, x_m\}$ for some $i, 1 \leq i \leq m+1$, and its bound variables are exactly $\{x_{m+1}, \ldots, x_{m+r}\}$ for $r = \mathrm{qr}(\psi)$.

The semantics of the fluted formulas of L coincides with the usual (Tarskian) semantics of first-order predicate logic. Also the semantic consequence relation is that of first-order logic, defined: $\psi \models \phi :\Leftrightarrow$ for all L-structures $\mathcal{A}$, for all assignments $\boldsymbol{a} \in A^\omega$, $\mathcal{A}, \boldsymbol{a} \models \psi$ only if $\mathcal{A}, \boldsymbol{a} \models \phi$.

Let θ be a subformula of formula ϕ. The *polarity (positive or negative)* of θ is defined as follows.

1. If $\phi = \theta$, then θ is positive in ϕ.
2. If $\phi = \neg\psi$ and θ is positive (negative) in ϕ, then θ is negative (positive) in ψ.

3. If $\phi = \psi \wedge \rho$ or $\phi = \psi \vee \rho$ and θ is positive (negative) in ϕ, then if θ is a subformula of ψ, θ is positive (negative) in ψ, and if θ is a subformula of ρ, θ is positive (negative) in ρ.
4. If $\phi = \psi \rightarrow \rho$ and θ is positive (negative) in ϕ, then if θ is a subformula of ψ, θ is negative (positive) in ψ, and if θ is a subformula of ρ, θ is positive (negative) in ρ.
5. If $\phi = \exists x\psi$ or $\phi = \forall x\psi$ and θ is positive (negative) in ϕ, then θ is positive (negative) in ψ.

An important principle in fluted logic is the *Principle of Monotonicity*, embodied in the following theorem.

THEOREM 1 (The Principle of Monotonicity). *Let θ be a positive (respectively, negative) subformula of fluted formula ϕ, let $\theta \rightarrow \rho$ (respectively, $\rho \rightarrow \theta$) be a fluted formula, and let ϕ' be obtained from ϕ by substituting ρ for θ in ϕ. Then from $\theta \rightarrow \rho$ (respectively, $\rho \rightarrow \theta$), $\phi \rightarrow \phi'$ can be inferred.*

PROOF. A semantic proof will be given. While it is not difficult to provide an axiomatization for FL, it will not be needed in this paper. The reader interested in a syntactic proof of the Principle of Monotonicity can consult Andrews [3], Theorem 2105, Substitutivity of Implication. The proof in [3] is for FO, but its restriction to FL is obvious.

It will be proved that $\models \theta \rightarrow \rho$ (respectively, $\models \rho \rightarrow \theta$) implies $\models \phi \rightarrow \phi'$. We proceed by induction on the complexity of ϕ. Let $\mathcal{A}$ be an arbitrary L-structure and $\boldsymbol{a}$ be an arbitrary element of A^ω.

Case 1: $\phi = \theta$. Hence θ is positive in ϕ. Then $\models \theta \rightarrow \rho$ can be rewritten $\models \phi \rightarrow \phi'$.

Case 2: $\phi = \neg\psi$. Hence θ is negative (respectively, positive) in ψ if θ is positive (respectively, negative) in ϕ. Then by the inductive hypothesis, $\models \theta \rightarrow \rho$ (respectively, $\models \rho \rightarrow \theta$) implies $\models \psi' \rightarrow \psi$. By the definition of the semantics of fluted logic and propositional calculus, $\models \neg\psi \rightarrow \neg\psi'$.

Case 3: $\phi = \psi \wedge \xi$ or $\phi = \psi \vee \xi$. The argument is similar to that of case 2.

Case 4: $\phi = \psi \rightarrow \xi$. The argument is similar to that of case 2.

Case 5: $\phi = \exists x_m\psi$ or $\phi = \forall x_m\psi$. Hence θ is positive (negative) in ψ if θ is positive (negative) in ϕ. By definition of the semantics of fluted logic, $\mathcal{A}, \boldsymbol{a} \models \exists x_m\psi$ iff for some $a \in A$, $\mathcal{A}, \boldsymbol{a}[a/a_m] \models \psi$. By the inductive hypothesis, $\models \theta \rightarrow \rho$ (respectively, $\models \rho \rightarrow \theta$) implies $\models \psi \rightarrow \psi'$. Again by definition of the semantics of fluted logic and propositional calculus, $\mathcal{A}, \boldsymbol{a} \models \exists x_m\psi$ only if $\mathcal{A}, \boldsymbol{a} \models \exists x_m\psi'$. Hence $\models \exists x_m\psi \rightarrow \exists x_m\psi'$. The argument for $\forall x_m\psi$ is similar. ∎

COROLLARY 2. *Let θ be a positive (respectively, negative) subformula of fluted formula ϕ, and let ϕ' be obtained from ϕ by substituting $\top$ (respectively, $\bot$) for θ in ϕ. Then from ϕ, ϕ' can be inferred.*

3. Fluted constituents

A constituent is a generalization to FO of the well-known *minterm* of Boolean logic. A minterm is also called a *minimal conjunction* because it is an atom in the lattice corresponding to the particular Boolean logic. Just as Boolean logic can be couched in terms of minimal conjunctions, so first-order logic can be couched in terms of constituents. In Boolean logic it is proved that any Boolean formula is equivalent to a disjunction of minimal conjunctions. Similarly, in FO it is proved that any first-order formula is equivalent to a disjunction of constituents. With this foundation, one can go on to prove most if not all of the theorems of first-order logic, often quite perspicuously. A clear and concise review of constituent theory applied to FO is given in Rantala [15]. The reader is directed to that source for background.

Constituents are especially useful in fluted logic since they have a simple representation as labeled trees. In this form, deciding consistency of a constituent is simple, requiring only inspection.

This section reviews the main results of Hintikka's constituent theory applied to fluted logic.

Let Φ be any set of prime formulas. A conjunction in which for each $\rho \in \Phi$ either ρ or $\neg\rho$ (but not both) occurs as a conjunct is a minimal conjunction over Φ. The set of minimal conjunctions over Φ will be denoted $\Delta\Phi$. It is well-known from Boolean logic that if $\Delta\Phi = \{\theta_1, \dots, \theta_l\}$, and ψ is any Boolean combination of formulas of Φ, then the following are tautologies.

1. $\neg(\theta_i \wedge \theta_j)$, for $i \neq j$,
2. $\theta_1 \vee \cdots \vee \theta_l$,
3. either $\theta_i \to \psi$ or $\theta_i \to \neg\psi$, for $1 \leq i \leq l$.

Of particular interest is $\Phi = \mathrm{Af}_L(X_m)$, since this forms the basis case in the definition of fluted constituents.

The results for Boolean logic can be extended to quantificational fluted logic. First define the following operations on sets of formulas. Let Θ be a set of formulas.

$$\begin{aligned} \neg\Theta &:= \{\neg\theta : \theta \in \Theta\} \\ \exists x\Theta &:= \{\exists x\theta : \theta \in \Theta\} \\ \forall x\Theta &:= \{\forall x\theta : \theta \in \Theta\} \end{aligned}$$

Now fluted constituents are defined inductively as follows.

Basis $\Gamma_L^{(0)}(X_m) := \Delta\,\mathrm{Af}_L(X_m)$

Induction $\Gamma_L^{(i+1)}(X_m) := \{\theta \wedge \bigwedge \exists x_{m+1} \Theta \wedge \forall x_{m+1} \bigvee \Theta : (\theta \in \Delta\,\mathrm{Af}_L(X_m)) \wedge (\emptyset \neq \Theta \subseteq \Gamma_L^{(i)}(X_{m+1}))\}$

A formula $\phi \in \Gamma_L^{(h)}(X_m)$ is a *fluted constituent of* L *of height* h *over the variables* X_m. If $m = 0$, then ϕ is a *fluted constituent sentence.* As defined here, height is synonymous with quantifier rank. Height is used to suggest a tree representation of the constituent. (Note that in the literature relating to constituent theory, the term 'depth' is so used.) If $\Phi \subseteq \Gamma_L^{(n)}(X_k)$, define $\sim \Phi := \Gamma_L^{(n)}(X_k) - \Phi$.

Now the main results of constituent theory, applied to fluted logic, can be given.

THEOREM 3. 1. (Incompatibility Property) *If* ϕ *and* ψ *are fluted constituents of* L *of height* h *over the variables* X_m*, and* $\phi \neq \psi$*, then* $\phi \wedge \psi$ *is inconsistent.*

2. (Exhaustiveness Property) *The disjunction of all fluted constituents of* L *of height* h *over the variables* X_m *is logically valid.*

PROOF. It suffices to observe that $\theta \wedge \bigwedge \exists x_{m+1} \Theta \wedge \forall x_{m+1} \bigvee \Theta$ in the definition of $\Gamma_L^{(i+1)}(X_m)$ is equivalent to $\theta \wedge \bigwedge \exists x_{m+1} \Theta \wedge \bigwedge \neg \exists x_{m+1} (\Gamma_L^{(i)}(X_{m+1}) - \Theta)$, and then to show by induction that this is a minimal conjunction over a set of prime formulas. ∎

It follows from the Exhaustiveness Property that if $\mathcal{A}$ is any L-structure, $\mathcal{A} \models \bigvee \Gamma_L^{(n)}(X_0)$. Further, because of the Incompatibility Property, $\mathcal{A} \models \phi$ for exactly one constituent $\phi \in \Gamma_L^{(n)}(X_0)$. Thus there exists a many-one relation from L-structures to $\Gamma_L^{(n)}(X_0)$ for each n. This observation can be generalized to $\Gamma_L^{(n)}(X_k)$. It can be shown that consistent countable sets of constituents $\{\phi^{(n)} : \phi^{(n)} \in \Gamma_L^{(n)}(X_k) \wedge n \in \omega\}$ correspond to *k-types* [15].

It was remarked that any first-order formula is equivalent to a disjunction of first-order constituents. The disjunction of constituents to which a formula ψ is logically equivalent is called a *distributive normal form of* ψ. The analogous situation holds in fluted logic. The constituents in a distributive normal form of ψ are a subset of the constituents of some lexicon L and height h. In this case the distributive normal form of ψ is denoted $\bigvee \Gamma_L^{(h)}(X_k)(\psi)$, or simply $\bigvee \Gamma_\psi$ when no confusion can result. The disjuncts are called the *constituents of* ψ *of lexicon* L *and height* h. Notice that if ψ is a fluted formula over X_k, with quantifier rank h and lexicon L, then ψ has constituents $\Gamma_{L'}^{(n)}(X_k)(\psi)$ for every $n \geq h$ and $L' \supseteq L$.

We next present a construction of a distributive normal form. First some notation is defined. If $\phi \in \Gamma_L^{(n)}(X_k)$ and $\phi = \theta \wedge \exists x_{k+1}\zeta_1 \wedge \cdots \wedge \exists x_{k+1}\zeta_r \wedge \forall x_{k+1}(\zeta_1 \vee \cdots \vee \zeta_r)$, where $\theta = \pm\rho_1 \wedge \cdots \wedge \pm\rho_q$, then $\pm\rho_1, \ldots, \pm\rho_q, \exists x_{k+1}\zeta_1, \ldots, \exists x_{k+1}\zeta_r$ as well as their conjunctions are said to *occur positively in the main conjunction of ϕ*. Define

$$(\Gamma_L^{(n)}(X_k) : \xi) \quad := \quad \{\phi \in \Gamma_L^{(n)}(X_k) : \xi \text{ occurs positively in the main conjunction of } \phi\}.$$

Now the construction of a distributive normal form can be presented. Let ψ be a standard fluted formula over X_m with $\mathrm{qr}(\psi) = r$. The set of constituents $\Gamma_\psi \subseteq \Gamma_L^{(n)}(X_m)$, where $n \geq r$, is constructed recursively as follows. (χ is an arbitrary subformula.)

1. If χ is atomic in the scope of quantifiers $Qx_{m+1}, \ldots, Qx_{m+k}$, then $\Gamma_\chi := (\Gamma_L^{(n-k)}(X_{m+k}) : \chi)$.
2. If $\chi = \neg\xi$, where $\Gamma_\xi \subseteq \Gamma_L^{(n-k)}(X_{m+k})$, then $\Gamma_\chi := {\sim}\Gamma_\xi$.
3. If $\chi = \xi_1 \wedge \cdots \wedge \xi_q$, where $\Gamma_{\xi_i} \subseteq \Gamma_L^{(n-k)}(X_{m+k})$ for $1 \leq i \leq q$, then $\Gamma_\chi := \Gamma_{\xi_1} \cap \cdots \cap \Gamma_{\xi_q}$.
4. If $\chi = \xi_1 \vee \cdots \vee \xi_q$, where $\Gamma_{\xi_i} \subseteq \Gamma_L^{(n-k)}(X_{m+k})$ for $1 \leq i \leq q$, then $\Gamma_\chi := \Gamma_{\xi_1} \cup \cdots \cup \Gamma_{\xi_q}$.
5. If $\chi = \exists x_{m+k+1}\xi$, where $\Gamma_\xi \subseteq \Gamma_L^{(n-k-1)}(X_{m+k+1})$, then $\Gamma_\chi := \bigcup_{\phi \in \Gamma_\xi} (\Gamma_L^{(n-k)}(X_{m+k}) : \exists x_{m+k+1}\phi)$.
6. If $\chi = \forall x_{m+k+1}\xi$, where $\Gamma_\xi \subseteq \Gamma_L^{(n-k-1)}(X_{m+k+1})$, then $\Gamma_\chi := {\sim} \bigcup_{\phi \in {\sim}\Gamma_\xi} (\Gamma_L^{(n-k)}(X_{m+k}) : \exists x_{m+k+1}\phi)$.

To make the notation still more concise, define

$$(\Gamma_L^{(n-k)}(X_{m+k}) : \exists x_{m+k+1}\Gamma_\xi) := \bigcup_{\phi \in \Gamma_\xi} (\Gamma_L^{(n-k)}(X_{m+k}) : \exists x_{m+k+1}\phi)$$

and

$${\sim}(\Gamma_L^{(n-k)}(X_{m+k}) : \exists x_{m+k+1} {\sim}\Gamma_\xi) := {\sim} \bigcup_{\phi \in {\sim}\Gamma_\xi} (\Gamma_L^{(n-k)}(X_{m+k}) : \exists x_{m+k+1}\phi).$$

Example 1. Let $\psi = \exists x_1 Px_1 \wedge \forall x_1(\neg Px_1 \vee \exists x_2(\neg Px_2 \wedge Rx_1x_2))$. Let $n = r = 2$, and $L = \{P, R\}$. Then construction of Γ_ψ starts out as follows.

1. $\chi_1 = Rx_1x_2$. $\Gamma_{\chi_1} = (\Gamma_L^{(0)}(X_2) : Rx_1x_2) = \{Px_2 \wedge Rx_1x_2, \neg Px_2 \wedge Rx_1x_2\}$.
2. $\chi_2 = Px_2$. $\Gamma_{\chi_2} = (\Gamma_L^{(0)}(X_2) : Px_2) = \{Px_2 \wedge Rx_1x_2, Px_2 \wedge \neg Rx_1x_2\}$.
3. $\chi_3 = \neg Px_2$. $\Gamma_{\chi_3} = {\sim}\Gamma_{\chi_2} = \{\neg Px_2 \wedge Rx_1x_2, \neg Px_2 \wedge \neg Rx_1x_2\}$.
4. $\chi_4 = \neg Px_2 \wedge Rx_1x_2$. $\Gamma_{\chi_4} = \Gamma_{\chi_3} \cap \Gamma_{\chi_1} = \{\neg Px_2 \wedge Rx_1x_2\}$.
5. $\chi_5 = \exists x_2(\neg Px_2 \wedge Rx_1x_2)$. $\Gamma_{\chi_5} = (\Gamma_L^{(1)}(X_1) : \exists x_2 \Gamma_{\chi_4})$. Γ_{χ_5} has 16 members, in all of which $\exists x_2(\neg Px_2 \wedge Rx_1x_2)$ occurs positively in the main conjunction. For example, $Px_1 \wedge \exists x_2(\neg Px_2 \wedge Rx_1x_2) \wedge \exists x_2(Px_2 \wedge \neg Rx_1x_2) \wedge \forall x_2(\neg Px_2 \wedge Rx_1x_2 \vee Px_2 \wedge \neg Rx_1x_2)$ is a member. Because of the large size of this and subsequent sets of constituents, they will not be enumerated.
6. $\chi_6 = Px_1$. $\Gamma_{\chi_6} = (\Gamma_L^{(1)}(X_1) : Px_1)$. This set has 15 members, in all of which Px_1 occurs positively in the main conjunction.
7. $\chi_7 = \neg Px_1$. $\Gamma_{\chi_7} = {\sim}\Gamma_{\chi_6}$. This set also has 15 members.
8. $\chi_8 = \neg Px_1 \vee \exists x_2(\neg Px_2 \wedge Rx_1x_2)$. $\Gamma_{\chi_8} = \Gamma_{\chi_7} \cup \Gamma_{\chi_5}$. This set has 8 members in which Px_1 occurs positively in the main conjunction and 15 members in which $\neg Px_1$ occurs positively in the main conjunction.
9. $\chi_9 = \neg\chi_8$. $\Gamma_{\chi_9} = {\sim}\Gamma_{\chi_8} = {\sim}\Gamma_{\chi_7} \cap {\sim}\Gamma_{\chi_5}$. This set has 7 members, in all of which Px_1 occurs positively in the main conjunction, and in none of which $\exists x_2(\neg Px_2 \wedge Rx_1x_2)$ occurs positively in the main conjunction.

At this point, sets of constituents grow rapidly. $\Gamma_L^{(2)}(X_0)$ contains $2^{2(2^4-1)}-1$ members. This should make it clear that the use of disjunctive normal forms is not a feasible technique in computational logic. The construction finally yields at completion

$$(\Gamma_L^{(2)}() : \exists x_1(\Gamma_L^{(1)}(x_1) : Px_1)) \cap {\sim}(\Gamma_L^{(2)}() : \exists x_1((\Gamma_L^{(1)}(x_1) : Px_1) \cap {\sim}(\Gamma_L^{(1)}(x_1) : \exists x_2({\sim}(\Gamma_L^{(0)}(x_1x_2) : Px_2) \cap (\Gamma_L^{(0)}(x_1x_2) : Rx_1x_2))))).$$

Note that the expression for Γ_ψ can be read (and constructed) from innermost to outermost subformulas *or* from outermost to innermost subformulas.

THEOREM 4. *Let ψ be a standard fluted formula of L over X_m, where $\mathrm{qr}(\psi) = r$. Then for every $n \geq r$, ψ is logically equivalent to the disjunction of constituents $\Gamma_\psi \subseteq \Gamma_L^{(n)}(X_m)$ resulting from the above construction.*

PROOF. The proof is by induction on the complexity of ψ.

Case 1: ψ is atomic in the scope of quantifiers $Qx_{m+1}, \ldots, Qx_{m+k}$. Then $\Gamma_\psi = (\Gamma^{(n-k)}(X_{m+k}) : \psi)$. Since in Boolean logic $(\chi_1 \wedge \theta \wedge \chi_2) \vee (\chi_1 \wedge \neg\theta \wedge \chi_2) \leftrightarrow (\chi_1 \wedge \chi_2)$, it follows that $\bigvee \Gamma_\psi \leftrightarrow \psi$.

Case 2: $\psi = \neg\xi$, where $\Gamma_\xi \subseteq \Gamma_L^{(n-k)}(X_{m+k})$. By the inductive hypothesis, $\xi \leftrightarrow \bigvee \Gamma_\xi$. Then by Theorem 3, $\neg\xi \leftrightarrow \bigvee \sim \Gamma_\xi$.

Case 3: $\psi = \xi_1 \wedge \cdots \wedge \xi_q$, where $\Gamma_{\xi_i} \subseteq \Gamma_L^{(n-k)}(X_{m+k})$. By the inductive hypothesis, $\xi_i \leftrightarrow \bigvee \Gamma_{\xi_i}$. Then by the distributivity of $\wedge$ over $\vee$ and the Incompatibility Property, $\xi_1 \wedge \cdots \wedge \xi_q \leftrightarrow \bigvee(\Gamma_{\xi_1} \cap \cdots \cap \Gamma_{\xi_q})$.

Case 4: $\psi = \xi_1 \vee \cdots \vee \xi_q$. The argument is similar to that of case 3.

Case 5: $\psi = \exists x_{m+k+1}\xi$, where $\Gamma_\xi \subseteq \Gamma_L^{(n-k-1)}(X_{m+k+1})$. By the inductive hypothesis, $\xi \leftrightarrow \bigvee \Gamma_\xi$. Since $\exists x_{m+k+1}(\phi_1 \vee \cdots \vee \phi_l) \models \exists x_{m+k+1}\phi_1 \vee \cdots \vee \exists x_{m+k+1}\phi_l$, $\exists x_{m+k+1}\xi \leftrightarrow \bigvee_{\phi \in \Gamma_\xi} \exists x_{m+k+1}\phi$. By Boolean logic as in case 1, for $\phi \in \Gamma_\xi$, $\exists x_{m+k+1}\phi \leftrightarrow \bigvee(\Gamma_L^{(n-k)}(X_{m+k}) : \exists x_{m+k+1}\phi)$. Hence $\psi \leftrightarrow \bigvee_{\phi \in \Gamma_\xi}(\Gamma_L^{(n-k)}(X_{m+k}) : \exists x_{m+k+1}\phi) = \bigvee \Gamma_\psi$.

Case 6: $\psi = \forall x_{m+k+1}\xi$. The argument is similar to that of case 5. ∎

COROLLARY 5. *Let ψ be a standard fluted formula of L over X_m, where $\mathrm{qr}(\psi) = r$. Let ϕ be a constituent of L of height $n \geq r$ over X_m. Then either $\phi \to \psi$ or $\phi \to \neg\psi$ is logically valid.*

PROOF. The corollary follows immediately from Theorem 4 and the Incompatibility Property of constituents. ∎

A fluted constituent sentence has a convenient representation as a tree. This representation is described next. Let $\boldsymbol{P}^*$ be the set of finite strings over $\boldsymbol{P}$, the positive integers. String concatenation is denoted by juxtaposition. The empty string is ε.

A subset $\mathcal{T} \subseteq \boldsymbol{P}^*$ is a *tree domain* if

1. $\varepsilon \in \mathcal{T}$, and
2. if $\sigma i \in \mathcal{T}$, where $\sigma \in \boldsymbol{P}^*$ and $i \in \boldsymbol{P}$, then
   (a) $\sigma j \in \mathcal{T}$ for $0 < j < i$, and
   (b) $\sigma \in \mathcal{T}$.

Define the *height of* $\sigma \in \mathcal{T}$, $\mathrm{h}(\sigma) :=$ the length of string σ. For all $\sigma, \tau \in \boldsymbol{P}^*, i \in \boldsymbol{P}$, if $\sigma i \tau \in \mathcal{T}$ then $\sigma i \tau$ is a *descendant of* σ and σi is an *immediate descendant of* σ. Define $\mathrm{w}(\sigma) :=$ the number of immediate descendants of σ. Thus $\sigma 1$, $\sigma 2$, $\ldots$, $\sigma\,\mathrm{w}(\sigma)$ are the immediate descendants of σ. If $\mathrm{w}(\sigma) = 0$, then σ is *terminal* in $\mathcal{T}$. If all terminal elements of $\mathcal{T}$ have the same height, then $\mathcal{T}$ is *balanced*. In this case, $\mathrm{h}(\mathcal{T}) := \mathrm{h}(\sigma)$, where

σ is any terminal element in $\mathcal{T}$. If $0 < \mathrm{h}(\sigma) < \mathrm{h}(\mathcal{T})$, then σ is *internal* in $\mathcal{T}$. An element σ together with all of its descendants is defined to be the *subtree rooted on* σ, and is denoted $(\sigma]$.

Let $\mathcal{T}$ be a balanced tree domain. A *labeled tree domain* $\mathcal{T}_L$ is defined to be $\mathcal{T}$ with a formula $\theta_\sigma \in \Delta\,\mathrm{Af}_L(X_{\mathrm{h}(\sigma)})$ associated with each $\sigma \in \mathcal{T}$. The labeled subtree of $\mathcal{T}_L$ rooted on σ will be denoted $(\theta_\sigma]$. The subtree $(\theta_\sigma]$ is given the following interpretation.

1. If σ is terminal, then $(\theta_\sigma]$ denotes θ_σ.
2. If σ is nonterminal with height k, then $(\theta_\sigma]$ denotes $\theta_\sigma \wedge \exists x_{k+1}(\theta_{\sigma 1}] \wedge \cdots \wedge \exists x_{k+1}(\theta_{\sigma\,\mathrm{w}(\sigma)}] \wedge \forall x_{k+1}((\theta_{\sigma 1}] \vee \cdots \vee (\theta_{\sigma\,\mathrm{w}(\sigma)}])$.

Thus the formula denoted by $(\theta_\sigma]$ is a fluted constituent of L of height $\mathrm{h}(\mathcal{T}) - \mathrm{h}(\sigma)$ over the variables $X_{\mathrm{h}(\sigma)}$. If $\mathrm{h}(\sigma) = 0$, the formula denoted by $(\theta_\sigma]$ is a fluted constituent sentence. If $\theta_\varepsilon = \neg\top$, then $\mathcal{T}_L$ is *trivial.*

In the sequel, all tree domains will be nontrivial labeled balanced tree domains. Moreover, $(\theta_\sigma]$ will not be distinguished from the formula it denotes.

There is an easy test for inconsistency of constituents, based on omission of variables. If ϕ is a fluted constituent, then $\phi^{[-k]}$ is defined to be ϕ with the last k variables omitted, and $\phi_{[-k]}$ is defined to be ϕ with the first k variables omitted. Here omission of a variable is accomplished by removing all atomic formulas in which that variable occurs, as well as the quantifier, if any, associated with that variable, and any connectives that thereby become idle. Semantically, the L- structures that satisfy a consistent constituent ϕ form a subclass of the L-structures that satisfy $\phi^{[-1]}$. The same result holds for $\phi_{[-1]}$. But as the following theorem shows, the superclass is the same in both cases.

THEOREM 6. *A constituent sentence ϕ is inconsistent unless $\phi^{[-1]}$ and $\phi_{[-1]}$ are equivalent.*

PROOF. Since ϕ is a constituent sentence, $\phi \to \phi^{[-1]}$ and $\phi \to \phi_{[-1]}$ by the Principle of Monotonicity. Hence $\phi \to (\phi^{[-1]} \wedge \phi_{[-1]})$. Moreover, $\phi^{[-1]}$ and $\phi_{[-1]}$ are constituent sentences of the same height. It follows from the Incompatibility Property that either $\phi^{[-1]}$ and $\phi_{[-1]}$ are equivalent (i.e., identical up to possible repetition of constituents, order of conjunction and disjunction, and change of variable), or ϕ is inconsistent. Hence the theorem follows. ∎

Note that deciding whether ϕ satisfies this condition requires only inspection of (the tree representation of) ϕ. For this reason, if a constituent fails to satisfy the condition of Theorem 6, the constituent is said to be *trivially inconsistent* (cf. Hintikka [7, 8]). Of course Theorem 6 holds for FO as well as FL. But in contrast to the state of affairs in FO, in FL the converse of Theorem 6 holds ([13]). That is, the condition of Theorem 6 is both necessary and sufficient for consistency. To prove the converse, it is shown that the tree representation of a fluted constituent sentence satisfying the condition of Theorem 6 can be used to build a model of that constituent. A similar method is used with the tree domain defined in the next section to build a model of a consistent fluted sentence.

Let $\hat{\Gamma}_L^{(n)}(X_k)$ be the consistent constituents in $\Gamma_L^{(n)}(X_k)$. Then from the Incompatibility Property, it follows that $\bigvee \hat{\Gamma}_L^{(n)}(X_k)(\phi) \to \bigvee \hat{\Gamma}_L^{(n)}(X_k)(\psi)$ iff $\hat{\Gamma}_L^{(n)}(X_k)(\phi) \subseteq \hat{\Gamma}_L^{(n)}(X_k)(\psi)$.

The Principle of Monotonicity immediately yields another omission result. If $\phi \in \Gamma_L^{(n)}(X_k)$ and $R \in L$, then if R is omitted from ϕ, the result is implied by ϕ. Here omission of R is accomplished by removing all atomic formulas in which R occurs as well as any connectives that thereby become idle. It must be observed that the empty conjunction is equivalent to $\top$ and the empty disjunction is equivalent to $\bot$. For $L' \subseteq L$, define $\Gamma_{L\uparrow L'}^{(n)}(X_k)(\phi)$ to result from omitting $L - L'$ from $\Gamma_L^{(n)}(X_k)(\phi)$.

LEMMA 7 (Separation Lemma). *Let ϕ and ψ be incompatible fluted formulas over X_k, i.e., $\phi \to \neg\psi$. Let $r := \max(\mathrm{qr}(\phi), \mathrm{qr}(\psi))$, $L := L_\phi \cup L_\psi$, and $L' := L_\phi \cap L_\psi$. Then $\hat{\Gamma}_{L\uparrow L'}^{(r)}(X_k)(\phi) \cap \hat{\Gamma}_{L\uparrow L'}^{(n)}(X_k)(\psi) = \emptyset$.*

PROOF. It follows from Theorem 4 and the above omission result that $\phi \leftrightarrow \bigvee \hat{\Gamma}_L^{(r)}(X_k)(\phi) \to \bigvee \hat{\Gamma}_{L\uparrow L'}^{(r)}(X_k)(\phi)$. Similarly, $\psi \leftrightarrow \bigvee \hat{\Gamma}_L^{(r)}(X_k)(\psi) \to \bigvee \hat{\Gamma}_{L\uparrow L'}^{(r)}(X_k)(\psi)$. But $\phi \wedge \psi \to \bot$. Hence $\hat{\Gamma}_{L\uparrow L'}^{(r)}(X_k)(\phi) \cap \hat{\Gamma}_{L\uparrow L'}^{(r)}(X_k)(\psi) = \emptyset$. ■

4. Exponential model property

Let ϕ be a standard fluted sentence with $\mathrm{qr}(\phi) = r$. Define $\mathrm{sub}(\phi)$ to be the set of subformulas of ϕ.

Let ψ be a fluted formula such that the variables occurring in ψ (bound and free) are $\{x_l, \ldots, x_m\}$. If $l > 1$, define $\psi^\dagger := \psi[x_{l-1}/x_l, \ldots, x_{m-1}/x_m]$; otherwise, $\psi^\dagger := \top$. Define $\psi^\ddagger := \psi[x_{l+1}/x_l, \ldots, x_{m+1}/x_m]$. Further, $\top^\dagger := \top^\ddagger := \top$.

Define $\mathrm{cl}(\phi)$ as follows.

1. $\mathrm{sub}(\phi) \subseteq \mathrm{cl}(\phi)$.
2. If $\psi \in \mathrm{cl}(\phi)$, then $\neg\psi \in \mathrm{cl}(\phi)$.

3. If $\psi \in \mathrm{cl}(\phi)$, then $\psi^\dagger \in \mathrm{cl}(\phi)$.
4. If $\psi \in \mathrm{cl}(\phi)$ is a fluted formula over X_k where $k < r$, then $\psi^\ddagger \in \mathrm{cl}(\phi)$.

If $\psi \in \mathrm{cl}(\phi)$, θ is a subformula of ψ, and $\theta \leftrightarrow \rho$ is a tautology in propositional logic, then ψ and $\psi[\rho/\theta]$ are considered aliases. Similarly, if $\psi \in \mathrm{cl}(\phi)$ and $\exists x\theta$ is a subformula of ψ, then ψ and $\psi[\neg\forall x\neg\theta/\exists x\theta]$ are considered aliases. (Cf. Smullyan's *unified notation* in Smullyan [18].)

For $0 \le k \le r$, let $\mathrm{cl}_k(\phi)$ be all and only formulas of $\mathrm{cl}(\phi)$ over X_k. For $0 \le k \le r$, let $\mathrm{at}_k(\phi)$ be the set of all maximal FL-consistent subsets of $\mathrm{cl}_k(\phi)$. Define $\mathrm{at}(\phi) := \bigcup_{0\le k\le r} \mathrm{at}_k(\phi)$.

Using elements of $\mathrm{at}(\phi)$ as labels, a labeled tree domain similar to a fluted constituent will be constructed. The operation $\chi_{[-1]}$ on constituents was defined in terms of its effect on atomic subformulas. Now an operation $A_{[[-1]]}$ will be defined on atoms to have a similar effect on prime subformulas.

Let $A \in \mathrm{at}(\phi)$ and $\psi \in A$. Define $\psi_{[[-1]]}$ as follows.

1. If ψ is a positive (respectively, negative) prime formula containing an occurrence of x_1, $\psi_{[[-1]]} := \top$ (respectively, $\psi_{[[-1]]} := \bot$).
2. Otherwise, ψ is a Boolean combination of prime subformulas: for every positive (respectively, negative) prime subformula θ in ψ containing an occurrence of x_1, $\psi_{[[-1]]} := \psi[\top/\theta]$ (respectively, $\psi_{[[-1]]} := \psi[\bot/\theta]$).

Define $A_{[[-1]]} := \{\psi_{[[-1]]} : \psi \in A\}$.

The following are simple consequences of the definition. Notice that since $\cdot_{[[-1]]}$ can only effectively eliminate a term or factor in a Boolean combination, $\phi_{[[-1]]}$ is always a subformula of ϕ.

1. If $\psi \in \mathrm{cl}_{k+1}(\phi)$, then $\psi_{[[-1]]} \in \mathrm{cl}_{k+1}(\phi)$ and so $(\psi_{[[-1]]})^\dagger \in \mathrm{cl}_k(\phi)$.
2. $((\psi^\ddagger)_{[[-1]]})^\dagger = (\psi^\ddagger)^\dagger = \psi$.
3. $((\exists x_{k+1}\theta^\ddagger)_{[[-1]]})^\dagger = \exists x_k\theta$.
4. $\psi \to \psi_{[[-1]]}$.

Let $A \in \mathrm{at}_k(\phi)$. Then $A^\ddagger$ is consistent (though not maximal) and $A^\ddagger \subseteq \mathrm{cl}_{k+1}(\phi)$. Hence there exists $B \in \mathrm{at}_{k+1}(\phi)$ such that $A^\ddagger \subseteq B$. $(B_{[[-1]]})^\dagger \subseteq \mathrm{cl}_k(\phi)$ and it is easily shown by a simple *reductio* proof that $(B_{[[-1]]})^\dagger \in \mathrm{at}_k(\phi)$. Since $A \subseteq (B_{[[-1]]})^\dagger$, it follows that $A = (B_{[[-1]]})^\dagger$. So $A^\ddagger \subseteq B$ iff $A = (B_{[[-1]]})^\dagger$.

Let ϕ be a consistent fluted sentence with $\mathrm{qr}(\phi) = r$. Define the structure $\mathcal{A} = (\mathrm{at}(\phi), <, \searrow)$ by induction on height of the elements of $\mathcal{A}$ as follows.

The elements of $\mathcal{A}$ will be named by their indexes. As before, if $\sigma \in \boldsymbol{P}^*$ is an element of $\mathcal{A}$, then $\mathrm{h}(\sigma) :=$ length of the string σ. Initially, for all elements σ, the number of immediate descendants $\mathrm{w}(\sigma) := 0$.

Basis. $\phi \in A_\varepsilon$, and for each $A \in \mathrm{at}_1(\phi)$ not already considered such that

1. $(A_{[[-1]]})^\dagger = A_\varepsilon$, and
2. for each $\psi \in A$: if $(\exists x_1 \psi) \in \mathrm{cl}_0(\psi)$, then $(\exists x_1 \psi) \in A_\varepsilon$ (hence if $(\neg\exists x_1 \psi) \in A_\varepsilon$, then $(\neg\psi) \in A$)

increment $\mathrm{w}(\varepsilon)$ and define

1. $A_{\mathrm{w}(\varepsilon)} := A$
2. $\varepsilon < \mathrm{w}(\varepsilon)$
3. $\mathrm{w}(\varepsilon) \searrow \varepsilon$

Induction. Suppose that $\mathcal{A}$ has been defined up to height $k < r$ with $\mathrm{h}(\sigma) = k$, $A_\sigma \in \mathrm{at}_k(\phi)$, and $\sigma \searrow \tau$. For each $A \in \mathrm{at}_{k+1}(\phi)$ not already considered such that

1. $(A_{[[-1]]})^\dagger = A_{\tau j}$ for some $1 \le j \le \mathrm{w}(\tau)$, and
2. for each $\psi \in A$: if $(\exists x_{k+1} \psi) \in \mathrm{cl}_k(\psi)$, then $(\exists x_{k+1} \psi) \in A_\sigma$ (hence if $(\neg\exists x_{k+1}\psi) \in A_\sigma$, then $(\neg\psi) \in A$)

increment $\mathrm{w}(\sigma)$ and define

1. $A_{\sigma\,\mathrm{w}(\sigma)} := A$
2. $\sigma < \sigma\,\mathrm{w}(\sigma)$
3. $\sigma\,\mathrm{w}(\sigma) \searrow \tau j$ for each j such that $1 \le j \le \mathrm{w}(\tau)$ and $((A_{\sigma\,\mathrm{w}(\sigma)})_{[[-1]]})^\dagger = A_{\tau j}$

The structure $\mathcal{A}$ just defined is a labeled tree domain of height r ordered by $<$, having an additional relation $\searrow$. By construction, A_σ uniquely determines $(A_\sigma]$, and $i \ne j \to A_{\sigma i} \ne A_{\sigma j}$.

If $\mathrm{h}(\sigma) = k \wedge (k \ne r \to \forall\rho((\rho = (\exists x_{k+1}\psi) \wedge \rho \in A_\sigma) \to \exists i(1 \le i \le \mathrm{w}(\sigma) \wedge \psi \in A_{\sigma i}))$, then we say σ *is demand-satisfied.* $\mathcal{A}$ *is demand-satisfied* if each of its elements is demand-satisfied. If $\mathrm{h}(\sigma) = k \wedge (k \ne 0 \to \exists\tau(\sigma \searrow \tau)) \wedge (k \ne r \to \exists\nu(\nu \searrow \sigma))$, then we say σ *is consistent.* $\mathcal{A}$ *is consistent* if each of its elements is consistent.

Observe that $\mathrm{card}(\mathrm{sub}(\phi)) \le |\phi|$, where $|\phi|$ is the length of the string ϕ. With only items 1 and 2 in the definition of $\mathrm{cl}(\phi)$, $\mathrm{card}(\mathrm{cl}(\phi)) \le 2|\phi|$. When items 3 and 4 are added, $\mathrm{card}(\mathrm{cl}(\phi)) \le 2r|\phi|$. Using the fact that $r < |\phi|/2$, one sees that $\mathrm{card}(\mathrm{cl}(\phi)) < |\phi|^2$. Hence $\mathit{card}(\mathrm{at}(\phi)) < 2^{|\phi|^2}$. It follows that $\mathrm{card}(\mathcal{A}) \le \mathrm{card}(\mathrm{at}(\phi))^{r+1} < (2^{|\phi|^2})^{r+1} < 2^{|\phi|^3}$.

LEMMA 8. *$\mathcal{A}$ is demand-satisfied and consistent.*

PROOF. By construction of $\mathcal{A}$, for all $\sigma \in \mathcal{A}$,

1. if $\sigma \searrow \tau$ then for each i such that $1 \leq i \leq \mathrm{w}(\sigma)$, there exists j such that $1 \leq j \leq \mathrm{w}(\tau)$ and $(A_{\sigma i})_{[[-1]]})^{\dagger} = A_{\tau j}$;
2. for each i such that $1 \leq i \leq \mathrm{w}(\sigma)$, for each $\psi \in A_{\sigma i}$, if $\exists x_{k+1}\psi \in \mathrm{cl}_k(\phi)$ then $\exists x_{k+1}\psi \in A_\sigma$.

It is to be proved that for all $\sigma \in \mathcal{A}$,

1. if $\sigma \searrow \tau$ then for each j such that $1 \leq j \leq \mathrm{w}(\tau)$ there exists an i such that $1 \leq i \leq \mathrm{w}(\sigma)$ and $(A_{\sigma i})_{[[-1]]})^{\dagger} = A_{\tau j}$ (i.e., $\mathcal{A}$ is consistent);
2. if $\exists x_{k+1}\psi \in A_\sigma$ then for some i, $1 \leq i \leq \mathrm{w}(\sigma)$ and $\psi \in A_{\sigma i}$ (i.e., $\mathcal{A}$ is demand-satisfied).

The proof proceeds by induction on the height of σ.

1. Let $1 \leq j \leq \mathrm{w}(\tau)$. Then $A^{\ddagger}_{\tau j}$ is consistent and so there exist atoms $A \in \mathrm{at}_{k+1}(\phi)$ such that $A^{\ddagger}_{\tau j} \subseteq A$ and so $(A_{[[-1]]})^{\dagger} = A_{\tau j}$. Therefore by definition of the construction of $\mathcal{A}$, for some i such that $1 \leq i \leq \mathrm{w}(\sigma)$, $A_{\sigma i} = A$ unless there exists $\psi^{\ddagger} \in A^{\ddagger}_{\tau j}$ such that $\neg\exists x_{k+1}\psi^{\ddagger} \in A_\sigma$. But in this event, by construction of $\mathcal{A}$, $\neg\exists x_k \psi \in A_\tau$. Hence, for $1 \leq j \leq \mathrm{w}(\tau), \psi \notin A_{\tau j}$, resulting in a contradiction. This completes the proof of 1.

2. Let $(\exists x_{k+1}\psi) \in A_\sigma$. Then $(\neg\exists x_{k+1}\psi) \notin A_\sigma$ and so there exist i such that $1 \leq i \leq \mathrm{w}(\sigma)$ and $\psi \in A_{\sigma i}$, unless for all j such that $1 \leq j \leq \mathrm{w}(\tau)$, $\neg(\psi_{[[-1]]})^{\dagger} \in A_{\tau j}$. But in this event, by the inductive hypothesis, $(\exists x_k((\psi_{[[-1]]})^{\dagger})) \notin A_\tau$. Hence by 1., $(\exists x_{k+1}\psi) \notin A_\sigma$, which is a contradiction. This completes the proof of 2. and of the lemma. ∎

A constituent can be used to construct a model ([14]). In the same way the structure $\mathcal{A}$ can be used to construct a model. This is undertaken next.

Define a function $\mathrm{n}\colon \mathcal{A} \to \boldsymbol{P}$ by induction on $d = \mathrm{h}(\mathcal{A}) - \mathrm{h}(\sigma i)$:

Basis $d = 0$ (σi is terminal): $\mathrm{n}(\sigma i) := 1$

Induction $0 < d < \mathrm{h}(\mathcal{A})$: if $\sigma i \searrow \nu$, then $\mathrm{n}(\nu) := \sum_{\sigma l \searrow \nu} \mathrm{n}(\sigma l)$

n is a total function since $\mathcal{A}$ is consistent. Observe that

$$\mathrm{n}(\varepsilon) = \sum_{i \searrow \varepsilon} \mathrm{n}(i) = \sum_{1 \leq i \leq \mathrm{w}(\varepsilon)} \mathrm{n}(i) < \mathrm{card}(\mathcal{A}) < 2^{|\phi|^3}.$$

Let $D := \{0, 1, \ldots, \mathrm{n}(\varepsilon) - 1\}$. Elements of D will be denoted by $a, b, c, \ldots$, and strings in D^* by $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$. Define $V_\sigma^{[-1]} := \{\boldsymbol{a} : \exists b \in D(\boldsymbol{a}b \in V_\sigma)\}$ and $V_{\sigma[-1]} := \{\boldsymbol{a} : \exists b \in D(b\boldsymbol{a} \in V_\sigma)\}$.

Now with each element $\sigma \in \mathcal{A}$, associate a set $V_\sigma \subseteq D^*$. These sets will be used to define a model of ϕ. They are computed by induction on the height of elements in $\mathcal{A}$.

Basis. At height 0, $V_\varepsilon := \{\varepsilon\}$. At height 1, $V_1 := \{0, \ldots, \mathrm{n}(1) - 1\}$, $V_2 := \{\mathrm{n}(1), \ldots, \mathrm{n}(1) + \mathrm{n}(2) - 1\}, \ldots, V_{\mathrm{w}(\varepsilon)} := \{\mathrm{n}(1) + \cdots + \mathrm{n}(\mathrm{w}(\varepsilon) - 1), \ldots, \mathrm{n}(\varepsilon) - 1\}$.

Induction. Assume that for all σ of height $\leq k$, V_σ has been defined. For $1 \leq i \leq \mathrm{w}(\sigma)$, define $V_{\sigma i}$ as follows. Let $\sigma i \searrow \tau j$. For each $\boldsymbol{a} \in V_\sigma$, assign $\mathrm{n}(\sigma i)$ elements not already assigned from $(\{\boldsymbol{a}\} \times D) \cap (D \times V_{\tau j})$ to $V_{\sigma i}$. Since $\mathrm{n}(\tau j) = \sum_{\sigma l \searrow \tau j} \mathrm{n}(\sigma l)$, there are just enough elements to make the $V_{\sigma l}$ disjoint.

By construction, the sets just defined have the properties

1. for all nonterminal $\sigma \in \mathcal{A}$ and for all i such that $1 \leq i \leq \mathrm{w}(\sigma)$: $V_\sigma = V_{\sigma i}^{[-1]}$;
2. for all nonterminal $\sigma \in \mathcal{A} : V_\sigma \times D = \bigcup_{1 \leq j \leq \mathrm{w}(\sigma)} V_{\sigma j}$;
3. for all $\sigma, \tau \in \mathcal{A} : \sigma \neq \tau$ implies $V_\sigma \cap V_\tau = \emptyset$;
4. for all $\sigma \in \mathcal{A} : \sigma \searrow \nu$ implies $V_{\sigma[-1]} \subseteq V_\nu$.

LEMMA 9. *Define L-structure $\mathcal{M}$ with domain D as follows. For each $R \in L$ such that $\mathrm{ar}(R) = n$: $a_{m-n+1} \cdots a_m \in R^{\mathcal{M}}$ iff for some $\sigma \in \mathcal{A}$, $Rx_{m-n+1} \cdots x_m \in A_\sigma$ and $a_1 \cdots a_{m-n+1} \cdots a_m \in V_\sigma$. If $\mathcal{M}$ is so defined, then $\mathcal{M} \models \phi$.*

PROOF. It must be shown that

1. $\mathcal{M}$ is well-defined, i.e., if ψ has free variables $x_i, \ldots, x_k$, $\boldsymbol{a} \in \boldsymbol{P}^{k-i+1}$, $\boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{P}^*$, then $\mathcal{M}, \boldsymbol{ba} \models \psi$ iff $\mathcal{M}, \boldsymbol{ca} \models \psi$;
2. $\mathcal{M} \models \phi$.

1. Suppose the contrary, i.e., suppose $\mathcal{M}, \boldsymbol{ba} \models \psi$ and $\mathcal{M}, \boldsymbol{ca} \models \neg\psi$. Then there exist $\tau, \nu \in \mathcal{A}$ such that $\mathrm{h}(\tau) = k + j$ and $\mathrm{h}(\nu) = k + l$ for $j, l \geq 0$, and such that $\boldsymbol{ba} \in V_\tau, \boldsymbol{ca} \in V_\nu, \psi \in A_\tau$, and $\neg\psi \in A_\nu$. By properties 3 and 4 of the sets V_σ, for some σ such that $\boldsymbol{a} \in V_\sigma$, $\tau \searrow^j \sigma$ and $\nu \searrow^l \sigma$. But this entails that $\psi, \neg\psi \in A_\sigma$, which is a contradiction.

2. The following will be proved. For all $\sigma \in \mathcal{A}$, for all $\psi \in A_\sigma$ such that $\mathrm{qr}(\psi) \leq r - \mathrm{h}(\sigma)$, for all $\boldsymbol{a} \in V_\sigma$: $\mathcal{M}, \boldsymbol{a} \models \psi$. In particular, $\mathcal{M} \models A_\varepsilon$. Hence $\mathcal{M} \models \phi$.

Proof is by induction on $d = r - \mathrm{h}(\sigma)$. In view of the properties of atoms, it suffices to consider only prime ψ.

Basis. $d = 0$ ($\mathrm{h}(\sigma) = r$, i.e., σ is terminal). So $\psi \in A_\sigma$ such that $\mathrm{qr}(\psi) \leq 0$ is (ψ quantifier-free). Then for all $\boldsymbol{a} \in V_\sigma : \mathcal{M}, \boldsymbol{a} \models \psi$ by definition of $\mathcal{M}$.

Induction. $0 < d \leq r$ ($0 \leq \mathrm{h}(\sigma) < r$). Let $\mathrm{h}(\sigma) = r - d = k$. Let $\psi \in A_\sigma$ such that $\mathrm{qr}(\psi) \leq r - \mathrm{h}(\sigma)$.

Case 1: $\psi = Rx_i \cdots x_k$. Let $\boldsymbol{a} \in V_\sigma$. $\mathcal{M}, \boldsymbol{a} \models \psi$ by definition of $\mathcal{M}$.

Case 2: $\psi = \exists x_{k+1}\theta$, where $\mathrm{qr}(\theta) \leq r - \mathrm{h}(\sigma) - 1$. Since $\mathcal{A}$ is demand-satisfied, $\exists i(1 \leq i \leq \mathrm{w}(\sigma) \wedge \theta \in A_{\sigma i})$. By the inductive hypothesis, for all $\boldsymbol{ab} \in V_{\sigma i} : \mathcal{M}, \boldsymbol{ab} \models \theta$. Since $V_\sigma = V_{\sigma i}^{[-1]}$, $\boldsymbol{ab} \in V_{\sigma i}$ implies $\boldsymbol{a} \in V_\sigma$. By the semantics of fluted logic, if $\mathcal{M}, \boldsymbol{ab} \models \theta$, then $\mathcal{M}, \boldsymbol{a} \models \exists x_{k+1}\theta$.

Case 3: $\psi = \forall x_{k+1}\theta$, where $\mathrm{qr}(\theta) \leq r - \mathrm{h}(\sigma) - 1$. The argument is similar to that for case 2. ■

The following corollary is immediate.

COROLLARY 10. *If ϕ is a fluted sentence, then ϕ has a model of cardinality exponential in $|\phi|$.*

This result cannot be improved upon. An example of a fluted sentence having an exponential model is provided by applying the standard translation (see Section 5) to ϕ_m^K in the proof of Theorem 6.5 in Halpern and Moses ([6]).

5. Complexity of fluted logic

It has long been known that polymodal logic can be embedded into FO. For background and further references, see van Benthem [4]. The *standard translation* ι is defined as follows.

1. $\iota(p) := Px$
2. $\iota(\neg\phi) := \neg\iota(\phi)$
3. $\iota(\phi \wedge \psi) := \iota(\phi) \wedge \iota(\psi)$
4. $\iota(\langle R_i \rangle \phi) := \exists y(R_i xy \wedge \iota(\phi)[y/x])$

where $\langle R_i \rangle$ are the I-indexed modal operators and R_i are the associated accessibility relations.

Boolean modal logic extends polymodal logic by providing a modal operator for each Boolean combination of the accessibility relations. The standard translation is extended accordingly. Here R_i (respectively R_j) denotes either an initially given accessibility function or a Boolean combination of accessibility functions.

5. $\iota(\langle\neg R_i\rangle\phi) := \exists y(\neg R_i xy \wedge \iota(\phi)[y/x])$
6. $\iota(\langle R_i \vee R_j\rangle\phi) := \exists y((R_i xy \vee R_j xy) \wedge \iota(\phi)[y/x])$
7. $\iota(\langle R_i \wedge R_j\rangle\phi) := \exists y(R_i xy \wedge R_j xy \wedge \iota(\phi)[y/x])$

Since the FO formulas of the definientia are all fluted, Boolean modal logic can be embedded into FL as well. It follows easily by induction that a Boolean modal sentence ϕ is satisfiable in a model

$$\mathcal{M} = (S, \{R_i : i \in I\}, F)$$

iff the fluted sentence $\exists x\iota(\phi)$ is satisfiable in the model

$$\mathcal{A} = (A, \cdot^{\mathcal{A}})$$

where $A = S$, $R_i^{\mathcal{A}} = R_i$, and $P^{\mathcal{A}} = F(p)$.

The complexity of Boolean modal logic has been investigated by Lutz and Sattler ([11]). The result relevant to the present discussion is given by the following theorem.

THEOREM 11. *The satisfiability problem for Boolean modal logic is NEXPTIME-complete.*

PROOF. See Theorem 23 in [11]. ■

Since the standard translation is polynomial time and preserves satisfiability, we have the following corollary.

COROLLARY 12. *The satisfiability problem for FL is NEXPTIME-hard.*

To show that the satisfiability problem for FL is in NEXPTIME, it suffices to show that one can guess an L_ϕ-structure $\mathcal{M}$ and decide $\mathcal{M} \models \phi$ in EXPTIME. In Section 4 it was shown that if a fluted sentence ϕ is consistent, it has a model whose domain has cardinality $< 2^{|\phi|^3}$. Let $\mathrm{card}(L_\phi) = s < |\phi|$, $\mathrm{ar}(L_\phi) = t < |\phi|$, and $\mathrm{card}(M) = u < 2^{|\phi|^3}$. Then the L_ϕ-structure can be specified by u^t strings of the form $\boldsymbol{a} : \theta$, where $\boldsymbol{a} \in M^t$ and $\theta \in \Delta\,\mathrm{Af}_L(X_t)$, signifying $\mathcal{M}, \boldsymbol{a} \models \theta$. Only those specifications that satisfy the consistency condition:

$$\boldsymbol{a}_{i[-k]} = \boldsymbol{a}_{j[-k]} \rightarrow \theta_{i[-k]} = \theta_{j[-k]} \text{ for } 0 \leq k < t$$

will count as admissible.

Since each string $\boldsymbol{a} : \theta$ has length $t + s$, the specification has length $u^t(t+s) < (2^{t|\phi|^3})(2|\phi|) \leq |\phi|2^{|\phi|^4}$. $\mathcal{M} \models \phi$ can be decided by scanning the specification at most once for each subformula of ϕ. That is, $\mathcal{M} \models \phi$ can be decided in time at most $|\phi|^2 2^{|\phi|^4}$, which is $2^{\mathcal{O}(|\phi|^4)}$. This agrees with the known fact that $\mathcal{M} \models \phi$ can be decided in time polynomial in $|\phi||\mathcal{M}|^{\mathrm{width}(\phi)}$, where $\mathrm{width}(\phi)$ is the arity of $\mathrm{sub}(\phi)$.

In view of Theorem 11 and Corollary 12, the above considerations yield the following theorem.

THEOREM 13. *The satisfiability problem for FL is NEXPTIME-complete.*

6. Interpolation property

The following theorem states that fluted logic has the interpolation property.

THEOREM 14 (Interpolation Theorem). *Let ϕ, ψ be fluted sentences with lexicons L_ϕ, L_ψ, respectively, such that $\phi \rightarrow \psi$ is valid. If $L_\phi \cap L_\psi \neq \emptyset$, then there exists a* fluted *sentence θ with lexicon $L_\phi \cap L_\psi$ such that $\phi \rightarrow \theta$ and $\theta \rightarrow \psi$ are valid. If $L_\phi \cap L_\psi = \emptyset$, then $\neg\phi$ is valid or ψ is valid.*

PROOF. Let $r := \max(\mathrm{qr}(\phi), \mathrm{qr}(\psi))$, $L := L_\phi \cup L_\psi$, and $L' := L_\phi \cap L_\psi$. Suppose $L' \neq \emptyset$. Then

$$\bigvee \hat{\Gamma}^{(r)}_{L_\phi}()(\phi) \leftrightarrow \bigvee \hat{\Gamma}^{(r)}_{L}()(\phi) \rightarrow \bigvee \hat{\Gamma}^{(r)}_{L \uparrow L'}()(\phi)$$

Similarly,

$$\bigvee \hat{\Gamma}^{(r)}_{L_\psi}()(\neg\psi) \leftrightarrow \bigvee \hat{\Gamma}^{(r)}_{L}()(\neg\psi) \rightarrow \bigvee \hat{\Gamma}^{(r)}_{L \uparrow L'}()(\neg\psi)$$

Define $\theta := \bigvee \hat{\Gamma}^{(r)}_{L \uparrow L'}()(\phi)$. Thus $\phi \rightarrow \theta$. Since ϕ and $\neg\psi$ are incompatible, it follows from the Separation Lemma that $\hat{\Gamma}^{(r)}_{L \uparrow L'}()(\phi)$ and $\hat{\Gamma}^{(r)}_{L \uparrow L'}()(\neg\psi)$ are disjoint. Therefore, $\neg\psi \rightarrow \neg\theta$, and so $\theta \rightarrow \psi$.

Suppose $L' = \emptyset$. Then $\bigvee \hat{\Gamma}^{(r)}_{L_\phi}()(\phi) \leftrightarrow \bigvee \hat{\Gamma}^{(r)}_{L}()(\phi) \rightarrow \bigvee \hat{\Gamma}^{(r)}_{L \uparrow L_\psi}()(\phi)$. Since $\phi \rightarrow \psi$ is valid, $\hat{\Gamma}^{(r)}_{L}()(\phi) \subseteq \hat{\Gamma}^{(r)}_{L}()(\psi)$. So $\hat{\Gamma}^{(r)}_{L \uparrow L_\psi}()(\phi) \subseteq \hat{\Gamma}^{(r)}_{L \uparrow L_\psi}()(\psi) = \hat{\Gamma}^{(r)}_{L_\psi}()(\psi)$. But $L_\phi \cap L_\psi = \emptyset$. Thus either (1) $\hat{\Gamma}^{(r)}_{L_\phi}()(\phi) = \emptyset$, i.e., $\neg\phi$ is valid; or (2) $\hat{\Gamma}^{(r)}_{L_\phi}()(\phi) \neq \emptyset$, in which case $\hat{\Gamma}^{(r)}_{L \uparrow L_\psi}()(\phi) = \hat{\Gamma}^{(r)}_{L_\psi}() \subseteq \hat{\Gamma}^{(r)}_{L_\psi}()(\psi)$, i.e., ψ is valid. This completes the proof of the theorem. ∎

7. Preservation properties

Let $\chi, \chi' \in \Gamma_L^{(n)}(X_k)$. If there exists a tree embedding $\chi \to \chi'$ such that $\sigma \mapsto \sigma'$ implies $\theta_\sigma = \theta'_{\sigma'}$, then we say $\chi \subset \chi'$ or $\chi' \supset \chi$. Thus $(\Gamma_L^{(n)}(X_k), \subset)$ and $(\hat{\Gamma}_L^{(n)}(X_k), \subset)$ are posets.

Let $\boldsymbol{M}_L$ be the L-models. If $\mathcal{M}, \mathcal{M}' \in \boldsymbol{M}_L$, $M \subseteq M'$, and for all $R \in L$, $R^{\mathcal{M}'} \cap M^{\mathrm{ar}(R)} = R^{\mathcal{M}}$, then we say $\mathcal{M} \subset \mathcal{M}'$ ($\mathcal{M}$ is a submodel of $\mathcal{M}'$) or $\mathcal{M}' \supset \mathcal{M}$ ($\mathcal{M}'$ is an extension of $\mathcal{M}$). Thus $(\boldsymbol{M}_L, \subset)$ is a poset.

THEOREM 15. *μ: $\boldsymbol{M}_L \to \hat{\Gamma}_L^{(n)}()$, defined $\mu(\mathcal{M}) = \chi$ iff $\mathcal{M} \models \chi$, is a poset homomorphism. Moreover, this homomorphism is onto.*

PROOF. Let $\mathcal{M} \subset \mathcal{M}', \mathcal{M} \models \chi, \mathcal{M}' \models \chi'$ where $\chi, \chi' \in \hat{\Gamma}_L^{(n)}()$. It must be proved that $\chi \subset \chi'$. Let $\mathcal{T}_L, \mathcal{T}'_L$ be the tree representations of χ, χ', respectively. The elements and labels of $\mathcal{T}'_L$ will be represented by primed symbols, and those of $\mathcal{T}_L$ by unprimed symbols. Define an embedding ι : $\chi \to \chi'$ inductively as follows.

basis: Define $\varepsilon \mapsto \varepsilon'$. Observe that $\theta_\varepsilon = \top$ and $\theta'_{\varepsilon'} = \top$. Moreover, $\mathcal{M}, \varepsilon \models \mathcal{T}_L$ and $\mathcal{M}', \varepsilon \models \mathcal{T}'_L$.

induction: Suppose $\nu \mapsto \nu'$, $\mathcal{M}, \sigma \models (\theta_\nu]$, and $\mathcal{M}', \sigma \models (\theta'_{\nu'}]$. Let $1 \leq j \leq \mathrm{w}(\nu)$. Then for some $a \in M$, $\mathcal{M}, \sigma a \models (\theta_{\nu j}]$, and in particular, $\mathcal{M}, \sigma a \models \theta_{\nu j}$. Moreover, $M \subseteq M'$, and for all $a \in M'$ there exists j' such that $1 \leq j' \leq \mathrm{w}(\nu')$ and $\mathcal{M}', \sigma a \models (\theta'_{\nu' j'}]$. In particular, $\mathcal{M}', \sigma a \models \theta'_{\nu' j'}$. Since $\mathcal{M} \subset \mathcal{M}'$, $\mathcal{M}, \sigma a \models \theta_{\nu j} \wedge \theta'_{\nu' j'}$. By the Incompatibility Property, $\theta_{\nu j} = \theta'_{\nu' j'}$. Accordingly, define $\nu j \mapsto \nu' j'$.

Thus ι so defined is a tree embedding $\chi \to \chi'$ such that $\nu \mapsto \nu'$ implies $\theta_\nu = \theta'_{\nu'}$, i.e., $\chi \subset \chi'$. The mapping μ is total since for every $\mathcal{M} \in \boldsymbol{M}_L$, $\mathcal{M} \models \bigvee \hat{\Gamma}_L^{(n)}()$. Therefore, μ : $\boldsymbol{M}_L \to \hat{\Gamma}_L^{(n)}()$ is a poset homomorphism. Moreover, μ is onto since every $\chi \in \hat{\Gamma}_L^{(n)}()$ is consistent and so there exists $\mathcal{M} \in \boldsymbol{M}_L$ such that $\mathcal{M} \models \chi$. This completes the proof of the theorem. ∎

A fluted sentence is *existential* (respectively, *universal*) if it is logically equivalent to a negation normal form using only existential (respectively, universal) quantification. Let $\boldsymbol{M}_L(\phi)$ be the L-models of ϕ.

THEOREM 16. *Let ϕ be a consistent fluted sentence such that* $\mathrm{qr}(\phi) = n$. *The following are equivalent.*

1. *ϕ is existential.*

2. *$M_L(\phi)$ is closed under $\supset$ (extension),*

3. *$\hat{\Gamma}^{(n)}_L()(\phi)$ is closed under $\supset$ (extension).*

PROOF. "$1 \Rightarrow 2$" This follows directly from the definitions of $\models$ and $\supset$.

"$2 \Rightarrow 3$" Let $\chi \in \hat{\Gamma}^{(n)}_L()(\phi)$. Let $\mathcal{M} \models \chi$, so $\mathcal{M} \in M_L(\phi)$. By Theorem 15, $M_L(\phi)$ closed under $\supset$ implies $\hat{\Gamma}^{(n)}_L()(\phi)$ closed under $\supset$.

"$\neg 1 \Rightarrow \neg 3$" If $\chi \subset \chi'$, then χ' must make *all* the existential claims that χ does, and can also make arbitrary additional existential claims. Now suppose that ϕ is not existential, and consider the first universal quantifier encountered by the construction of the distributive normal form of ϕ (Section 3). $\forall x\xi$ denies existence to every $\chi \notin \Gamma_\xi$. That is, it prohibits arbitrary additional existential claims. Hence the constituents of $\forall x\xi$ cannot be closed under $\supset$. ■

The following is an immediate corollary of Theorem 16.

COROLLARY 17. *Let ϕ be a consistent fluted sentence such that* $\mathrm{qr}(\phi) = n$. *The following are equivalent.*

1. *ϕ is universal.*
2. *$M_L(\phi)$ is closed under $\subset$ (submodel).*

8. Discussion

Extended modal logics and various description logics have become important in computer science. Because these logics can be embedded in sufficiently expressive fragments or sublogics of first-order predicate logic, the latter have become the subject of increased interest. Interest has focused on fragments that are 'tame' (i.e., are decidable and have the finite-model property) yet still 'nice' (i.e., have the interpolation and model preservation properties).

One of the more obvious candidates, FO^2, has been found to be in the same complexity class as FL (Grädel, Kolaitis, and Vardi [5]), but not nice ([1]). The guarded fragment has been shown to be tame ([1]), but also is not nice ([9]). FL on the other hand is both tame and nice. Note too that FL allows the relational atoms to be negated, while the guarded fragment does not. This means that certain non-classical logics can be embedded in FL but not in the guarded fragment (Schmidt and Hustadt [17]).

In view of the potential of FL for interpretation of important non-classical logics, investigation of procedures for reasoning in FL is called for. A beginning has been made: a resolution-based decision procedure for satisfiability in FL was presented recently by Schmidt and Hustadt ([17]).

Acknowledgment. I am indebted to Renate A. Schmidt for pointing out errors in previous versions of this paper and for directing my attention to several related papers. I am also indebted to the anonymous referee, whose comments and criticism led to numerous and significant improvements in the paper.

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