Regular languages of infinite trees of low Borel complexity

H. Michalewski

Faculty of Mathematics, Informatics and Mechanics
University of Warsaw

Highlights 2013, Paris

The hierarchy of regular languages of trees stretches from the bottom of the Borel hierarchy up to Δ^1_2–level of the projective hierarchy.

Easy Algorithm.

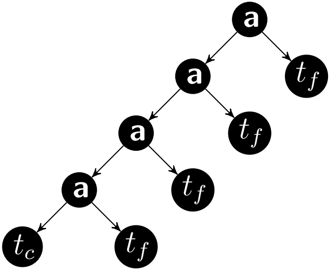
Decide whether a given regular language of trees is open.

Serious algorithm.

(M. Bojańczyk, T. Place) Decide whether a given regular language of trees is a Boolean combination of open sets.

Example 1.

(borrowed from M. Bojańczyk, T. Place)



Tree t_c consists only of letter c.
Tree t_f either contains no b or all labels, except of finitely many, are c.

Example 2.

(borrowed from M. Bojańczyk, T. Place) Language of infinite words

$$L = \{w \in \{a, b\}^\omega : \exists_n^\infty w_n = a\}.$$

Characterization 1.

(a cutting game used by M. Bojańczyk, T. Place) L is a finite combination of open sets if and only if there exists n such that II has a winning strategy in n moves in the following game

Player I plays a tree $t_0 \in L$

Player II plays a cut C_0 of t_0

Player I plays a tree $t_1 \notin L$ which agrees with t_0 up to C_0

Player II plays a cut C_1 of t_n

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Player I looses if he cannot make a legal move.

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The equations of the first type are not satisfied by the language from Example 1.

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Relaxation.

Remove all equations of the first type.

Result.

A bigger family of languages.

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Alternative characterizations of $\Delta_{0,2}$:

Characterization 1. Player II wins the infinite cutting game.

Characterization 2. L is of Wadge degree smaller than ω_1.

Characterization 3. Neither L or its complement contains a closed copy of the rationals numbers $\mathbb{Q}$.

None of these characterizations is effective, hence we have to resort to algebraic methods.

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Reusing the proof of M. Bojańczyk and T. Place we obtain

Theorem

(joint work with A. Facchini) A regular language L *is in* $\Delta_{0,2}$ *if and only if* L *satisfies the equations of the second type.*

Well known problems:

Find effective characterization of G_δ regular langauges.
Find effective characterization of Borel regular languages.

Less known problems

Problem 1. Characterize in effective way regular languages of Wadge degrees ω^2,ω^3,....

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