

# Decisiveness of Stochastic Systems and its Application to Hybrid Models<sup>★</sup>

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**Abstract.** In [3], Abdulla et al. introduced the concept of decisiveness, an interesting tool for lifting good properties of finite Markov chains to denumerable ones. Later [10], this concept was extended to more general stochastic transition systems (STSs), allowing the design of various verification algorithms for large classes of (infinite) STSs. We further improve the understanding and utility of decisiveness in two ways.

First, we provide a general criterion for proving decisiveness of general STSs. This criterion, which is very natural but whose proof is rather technical, (strictly) generalizes all known criteria from the literature.

Second, we focus on stochastic hybrid systems (SHSs), a stochastic extension of hybrid systems. We establish the decisiveness of a large class of SHSs and, under a few classical hypotheses from mathematical logic, we show how to decide reachability problems in this class, even though they are undecidable for general SHSs. This provides a decidable stochastic extension of o-minimal hybrid systems [15,20,30].

## 1 Introduction

**Hybrid and stochastic models.** Various kinds of mathematical models have been proposed to represent real-life systems. In this article, we focus on models combining *hybrid* and *stochastic* aspects. We outline the main features of these models to motivate our approach.

The idea of *hybrid systems* originates from the urge to study systems subject to both *discrete* and *continuous* phenomena, such as digital computer systems interacting with analog data. These systems are transition systems with two kinds of transitions: continuous transitions, where some continuous variables evolve over time (e.g., according to a differential equation), and discrete transitions, where the system changes modes and variables can be reset. Much of the research about hybrid systems focuses on *non-deterministic* hybrid systems, i.e., systems modeling uncertainty by considering all possible behaviors (e.g., different possibilities for discrete transitions at a given time, arbitrary long time between discrete transitions). A typical question concerns the *safety* of such systems—if a system can reach an undesirable state, it is said to be *unsafe*; if not, it is *safe*. If there is even a single behavior that does not satisfy the properties that the system should verify, then the whole system is deemed inadequate—such a specification is called *qualitative*. However, this is limiting for two reasons. First, it does not take into account that some behaviors are more likely to occur than others. Second, risks cannot necessarily be avoided, and it is unrealistic to prevent undesirable outcomes altogether. Therefore, we want to make probabilities an integral part of our models, in order to be able to *quantify* the probability that they behave according to the specification. We thus consider the class of *stochastic hybrid systems* (SHSs, for short), hybrid systems in which a stochastic semantics replaces non-determinism.

**Goals.** Our interest lies in the formal analysis of continuous-time SHSs, and more specifically in *reachability* questions, i.e., concerning the likelihood that some set of states is reached in a system. The questions we

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seek to answer are both of the qualitative kind (is some region of the state space *almost surely* reached, i.e., reached with probability 1?) and of the quantitative kind (what is the probability that some part of the state space is eventually reached?). Such questions are crucial, as verifying that a system works safely often reduces to verifying that some undesirable state of the system is never reached (or reached with a very low probability), or that some desirable state is to be reached with high probability [8]. We want to give algorithms that decide, for an SHS  $\mathcal{H}$  and reachability property  $P$ , whether  $P$  is satisfied in  $\mathcal{H}$ . Such an endeavor faces multiple challenges; a first obvious one being that even for rather restricted classes of non-deterministic hybrid systems, reachability problems are undecidable [22,23]. We want to define and consider a *class* of SHSs for which *some* reachability problems are decidable.

**Methods and contributions.** Our methodology consists of two main steps. In a first step, we follow the approach of Bertrand et al. [10]: we study general *stochastic transition systems* (STSs) through the *decisiveness* concept (Section 2). The class of STSs is a very versatile class of systems encompassing many well-known families of stochastic systems, such as Markov chains, but also stochastic systems with a continuous state-space such as generalized semi-Markov processes, stochastic timed automata, stochastic Petri nets, and stochastic hybrid systems. Decisiveness was introduced in [3] to study Markov chains, and extended to STSs in [10]. An STS is said to be *decisive* with respect to a set of states  $B$  if executions of the system almost surely reach either this set  $B$  or a state from which  $B$  is unreachable. Decisive STSs benefit from many useful properties that make possible the design of some verification algorithms related to reachability properties. Our first contribution is to provide a **criterion to check decisiveness of STSs** (Proposition 11), which generalizes the decisiveness criteria from [3,10]. This generalization was mentioned as an open problem in [10].

In a second step, we focus on *stochastic hybrid systems*, which we introduce in Section 3. Our contributions regarding SHSs are split in three parts.

First, we show, by extending undecidability results about non-deterministic hybrid systems [22,23], that **reachability problems are undecidable for general SHSs** (Proposition 25).

Second, we aim to use the decisiveness idea to get closer to the decidability frontier. Albeit desirable, the decisiveness of a class of SHSs is not sufficient to handle algorithmic questions about each SHS, as we need an effective way to apprehend their uncountable state space. In this regard, an often-used technique is to consider a *finite abstraction* of the system, that is, a finite partition of the state space that preserves the properties to be verified (a well-known example is the region graph for timed automata [5]). To find such an abstraction of SHSs, we borrow ideas from [15,30]: we consider SHSs with *strong resets* (Section 4), a syntactic condition that decouples their continuous behavior from their discrete behavior. We show that **SHSs with adequately placed strong resets** (at least one per cycle of their discrete graph) *(i) have a finite abstraction* (Proposition 32), and *(ii) are decisive* (Proposition 30), which can be proved using our new criterion.

Third, in Section 5, we show, under the hypotheses of the previous section, how to effectively compute a finite abstraction and use it to perform a reachability analysis of the original system. The way we proceed is by assuming that the components of our systems are definable in an *o-minimal structure*. The main difficulty here lies in the fact that “*a satisfactory theory of measure and integration seems to be lacking in the o-minimal context*” [9]. In particular, in an o-minimal structure, the primitive of a definable function is in general not definable in the same o-minimal structure, which complicates definability questions regarding probabilities. We therefore slightly restrict the possible probability distributions that can be used, using properties of o-minimal structures to keep our class as large as possible. When the **theory** of the structure is **decidable** (as is the case for the ordered field of real numbers [42]), the **reachability problems then become decidable**. This provides a stochastic extension to the theory of o-minimal hybrid systems [30]. We study both **qualitative** (Section 5.1) and **quantitative** (Section 5.2) problems. Due to space constraints, some proofs and technical details are deferred to the appendix.

**Related work.** Our results combine previous work on stochastic systems and on hybrid systems. About stochastic systems, we build on the work of [3,10]. Our work also takes inspiration from research about *stochastic timed automata*, a subclass of stochastic hybrid systems which already combines stochastic and

timed aspects; model checking of stochastic timed automata has been studied in [7,11,12] and considered in the context of the decisiveness property in [16]. Fundamental results about the decidability of the reachability problem for hybrid systems can be found in [5,22,23]. The class of *o-minimal hybrid systems*, of which we introduce a stochastic extension, has been studied in [15,20,30].

The literature about SHSs often follows a more practical or numerical approach. A first introduction to the model was provided in [27]. An extensive review of the underlying theory and of many applications of SHSs is provided in the book [17], and a review of different possible semantics for this model is provided in [32]. Applications of SHSs are numerous: a few examples are air traffic management [39,40], communication networks [24], biochemical processes [31,41]. The software tool UPPAAL [18] implements a model of SHS similar to the one studied in this article and uses numerical methods to compute reachability probabilities, through numerical solving of differential equations and Monte Carlo simulation. Reachability problems have also been considered in an alternative semantics with *discrete time*; a numerical approach is for instance provided in [1,2].

**Notations.** We write  $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$  for the set of non-negative real numbers. To emphasize that a union is disjoint, we denote it by  $\uplus$  instead of  $\cup$ . Let  $(\Omega, \Sigma)$  be a measurable space. We write  $\text{Dist}(\Omega, \Sigma)$  (or  $\text{Dist}(\Omega)$  if there is no ambiguity) for the set of *probability distributions* over  $(\Omega, \Sigma)$ . The complement of a set  $A \in \Sigma$  is denoted by  $A^c = \Omega \setminus A$ . For  $A \in \Sigma$  a measurable set, we say that two probability distributions  $\mu, \nu \in \text{Dist}(\Omega)$  are *qualitatively equivalent on A* (or *equivalent on A*) if for each  $B \in \Sigma$ , if  $B \subseteq A$ , then  $\mu(B) > 0$  if and only if  $\nu(B) > 0$ . For  $s \in \Omega$ , we denote by  $\delta_s$  the Dirac distribution centered on  $s$ .

## 2 Decisiveness of Stochastic Transition Systems

In this section, we define *stochastic transition systems* (STSs, for short) as in [10]. We then describe the concept of *decisiveness*, first defined in the specific case of Markov chains [3], and then extended to STSs [10]. Decisive stochastic systems benefit from “nice” properties making their qualitative and quantitative analysis more accessible. A first contribution of our work is a new decisiveness criterion (Proposition 11), which generalizes existing criteria from the literature [3,10]. It is an intuitive criterion, which was conjectured in [10] but could not be proved. We finish with a brief subsection on the notion of *abstraction* between STSs, which will be useful to apply our results to stochastic hybrid systems. We will use STSs in Section 3 to define the semantics of *stochastic hybrid systems*, a model of STSs to which the newly developed techniques will apply.

### 2.1 Stochastic Transition Systems [10, Section 2]

**Definition 1 (Stochastic transition system).** A stochastic transition system (STS) is a tuple  $\mathcal{T} = (S, \Sigma, \kappa)$  consisting of a measurable space of states  $(S, \Sigma)$ , and a function  $\kappa: S \times \Sigma \rightarrow [0, 1]$  such that for every  $s \in S$ ,  $\kappa(s, \cdot)$  is a probability measure and for every  $A \in \Sigma$ ,  $\kappa(\cdot, A)$  is a measurable function.

The second condition on  $\kappa$  implies in particular that for a measurable set  $B \in \Sigma$ , the set

$$\text{Pre}^{\mathcal{T}}(B) = \{s \in S \mid \kappa(s, B) > 0\}$$

is measurable.

**Measuring runs.** We interpret STSs as systems generating executions, with a probability measure over these executions. We fix an STS  $\mathcal{T} = (S, \Sigma, \kappa)$ . From a state  $s \in S$ , a probabilistic transition is performed according to distribution  $\kappa(s, \cdot)$ , and the system resumes from one of the successor states; this process generates random sequences of states. To formally provide a probabilistic semantics to STSs, we define a probability measure over the set of *runs* of STSs. A *run* of  $\mathcal{T}$  is an infinite sequence  $\rho = s_0 s_1 s_2 \dots$  of states. We write  $\text{Runs}(\mathcal{T})$  for the set of runs of  $\mathcal{T}$ .

In order to get a probability measure over  $\text{Runs}(\mathcal{T})$ , we equip this set with a  $\sigma$ -algebra. For every finite sequence  $(A_i)_{0 \leq i \leq n} \in \Sigma^{n+1}$ , we define the *cylinder*

$$\text{Cyl}(A_0, A_1, \dots, A_n) = \{\rho = s_0 s_1 \dots s_n \dots \in \text{Runs}(\mathcal{T}) \mid \forall 0 \leq i \leq n, s_i \in A_i\}.$$

Given an initial distribution  $\mu \in \text{Dist}(S)$ , we define in a classical way (see below) a pre-measure on the set of all cylinders. This pre-measure is then lifted to a unique probability measure  $\text{Prob}_\mu^\mathcal{T}$  on the  $\sigma$ -algebra generated by all the cylinders, using Caratheodory's extension theorem.

We initialize it for  $A_0 \in \Sigma$  with  $\text{Prob}_\mu^\mathcal{T}(\text{Cyl}(A_0)) = \mu(A_0)$ : this is the probability to be in  $A_0$ , starting with probability distribution  $\mu$ . For every finite sequence  $(A_i)_{0 \leq i \leq n} \in \Sigma^{n+1}$ , we then set inductively:

$$\text{Prob}_\mu^\mathcal{T}(\text{Cyl}(A_0, A_1, \dots, A_n)) = \int_{s_0 \in A_0} \text{Prob}_{\kappa(s_0, \cdot)}^\mathcal{T}(\text{Cyl}(A_1, \dots, A_n)) d\mu(s_0).$$

The intuition is that we split the probability on the left-hand side into all the possible ways to perform the first step in  $A_0$ , according to the initial distribution  $\mu$ .

**Expressing properties of runs.** To express properties of runs of  $\mathcal{T}$ , we use standard notations taken from LTL [38]. In particular, if  $B, B' \in \Sigma$ , we write  $\mathbf{F} B$  (resp.  $\mathbf{F}_{\leq n}$ ,  $B' \mathbf{U} B$ ,  $\mathbf{G} \mathbf{F} B$ ) for the set of runs that visit  $B$  at some point (resp. visit  $B$  in less than  $n$  steps, stay in  $B'$  until a first visit to  $B$ , visit  $B$  infinitely often). We postpone more formal definitions to Appendix A. We will be especially interested in two kinds of reachability problems.

**Definition 2 (Qualitative and quantitative reachability).** *Let  $B \in \Sigma$  be a measurable set of target states, and  $\mu \in \text{Dist}(S)$  be an initial distribution. The qualitative reachability problems consist in deciding whether  $\text{Prob}_\mu^\mathcal{T}(\mathbf{F} B) = 1$ , and whether  $\text{Prob}_\mu^\mathcal{T}(\mathbf{F} B) = 0$ . The quantitative reachability problem consists in deciding, given  $\epsilon, p \in \mathbb{Q}$  with  $\epsilon > 0$ , whether  $|\text{Prob}_\mu^\mathcal{T}(\mathbf{F} B) - p| < \epsilon$ .*

**Transforming probability distributions.** Another useful way to reflect on STSs is as transformers of probability distributions on  $(S, \Sigma)$ .

**Definition 3 (STS as a transformer).** *For  $\mu \in \text{Dist}(S)$ , its transformation through  $\mathcal{T}$  is the probability distribution  $\Omega_\mathcal{T}(\mu) \in \text{Dist}(S)$  defined for  $A \in \Sigma$  by*

$$\Omega_\mathcal{T}(\mu)(A) = \int_{s \in S} \kappa(s, A) d\mu(s) = \text{Prob}_\mu^\mathcal{T}(\text{Cyl}(S, A)).$$

The meaning of function  $\Omega_\mathcal{T}$  can be interpreted as follows:  $\Omega_\mathcal{T}(\mu)(A)$  is the probability to reach  $A$  in one step, from the initial distribution  $\mu$ .

**Definition 4 (Conditional distributions).** *For  $\mu \in \text{Dist}(S)$  and  $A \in \Sigma$  such that  $\mu(A) > 0$ , the conditional probability distribution of  $\mu$  given  $A$  is denoted as  $\mu_A$  and is such that for  $B \in \Sigma$ ,  $\mu_A(B) = \mu(A \cap B) / \mu(A)$ .*

*Let  $\mathcal{A} = (A_i)_{0 \leq i \leq n} \in \Sigma^{n+1}$ . We define a distribution  $\mu_\mathcal{A} = \mu_{A_0, \dots, A_n}$  by induction:  $\mu_{A_0}$  is defined as above, and for  $1 \leq j \leq n$ :*

$$\mu_{A_0, \dots, A_j} = \Omega_\mathcal{T}(\mu_{A_0, \dots, A_{j-1}})_{A_j}.$$

*To be well-defined, we require in addition that  $\mu(A_0) > 0$  and that for all  $0 \leq j \leq n-1$ ,  $\Omega_\mathcal{T}(\mu_{A_0, \dots, A_j})(A_{j+1}) > 0$ .*

The intuition is that  $\mu_\mathcal{A}$  is the conditional distribution on  $A_n$  after normalizing and restricting at each step  $i$  the distribution  $\mu$  to the set  $A_i$ .

We connect the two interpretations of the semantics of an STS: as an object generating a measure on infinite runs, and as a transformer of probability distributions. The next result originates from [10, Lemma 5].

**Lemma 5.** Let  $\mu \in \text{Dist}(S)$  be an initial distribution and  $(A_i)_{0 \leq i \leq n}$  be a sequence of measurable sets of states. For  $0 \leq j \leq n$ , we denote by  $\mu_j$  the conditional distribution  $\mu_{A_0, \dots, A_j}$ , which we assume to be well-defined. Then for every  $0 \leq j \leq n$ , we have

$$\begin{aligned} \text{Prob}_\mu^\mathcal{T}(\text{Cyl}(A_0, \dots, A_n)) &= \text{Prob}_\mu^\mathcal{T}(\text{Cyl}(A_0, \dots, A_j)) \cdot \text{Prob}_{\Omega_\mathcal{T}(\mu_j)}^\mathcal{T}(\text{Cyl}(A_{j+1}, \dots, A_n)) \\ &= \text{Prob}_\mu^\mathcal{T}(\text{Cyl}(A_0, \dots, A_j)) \cdot \text{Prob}_{\mu_j}^\mathcal{T}(\text{Cyl}(A_j, \dots, A_n)). \end{aligned}$$

**Attractors.** We will be particularly interested in the existence of *attractors* for STSs.

**Definition 6 (Attractor).** A set  $A \in \Sigma$  is an attractor for  $\mathcal{T}$  if for every  $\mu \in \text{Dist}(S)$ ,  $\text{Prob}_\mu^\mathcal{T}(\mathbf{F} A) = 1$ .

While  $S$  is always an attractor for  $\mathcal{T}$ , we will later search for attractors with more interesting properties. The definition of *attractor* actually implies a seemingly stronger statement: an attractor is almost surely visited *infinitely often* from any initial distribution.

**Lemma 7 ([10, Lemma 19]).** Let  $A$  be an attractor for  $\mathcal{T}$ . Then, for every initial distribution  $\mu \in \text{Dist}(A)$ ,  $\text{Prob}_\mu^\mathcal{T}(\mathbf{G} \mathbf{F} A) = 1$ .

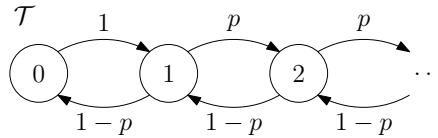
## 2.2 Decisiveness

Before introducing decisiveness, we give the definition of an *avoid-set*: for  $B \in \Sigma$ , its *avoid-set* is written as  $\tilde{B} = \{s \in S \mid \text{Prob}_{\delta_s}^\mathcal{T}(\mathbf{F} B) = 0\}$ . The avoid-set  $\tilde{B}$  corresponds to the set of states from which executions almost surely stay out of  $B$  *ad infinitum*. One can show that the set  $\tilde{B}$  belongs to the  $\sigma$ -algebra  $\Sigma$  [10, Lemma 14]. We can now define the concept of decisiveness as in [10].

**Definition 8 (Decisiveness).** Let  $B \in \Sigma$  be a measurable set. We say that  $\mathcal{T}$  is decisive w.r.t.  $B$  if for every  $\mu \in \text{Dist}(S)$ ,  $\text{Prob}_\mu^\mathcal{T}(\mathbf{F} B \vee \mathbf{F} \tilde{B}) = 1$ .

Intuitively, the decisiveness property states that, almost surely, either  $B$  will eventually be visited, or states from which  $B$  can no longer be reached will eventually be visited.

*Example 9 (Random walk).* We consider the random walk  $\mathcal{T}$  from Figure 1. We want to find out whether  $\mathcal{T}$  is decisive w.r.t.  $B = \{0\}$ . We assume that the initial distribution is given by  $\delta_1$  (the Dirac distribution at 1). By the theory on random walks, we know that if  $\frac{1}{2} < p < 1$ , the walk will almost surely diverge to  $\infty$ . This entails that  $\text{Prob}_{\delta_1}^\mathcal{T}(\mathbf{F} B) < 1$  and  $\text{Prob}_{\delta_1}^\mathcal{T}(\mathbf{G} \mathbf{F} B) = 0$ . Moreover, since  $p < 1$ , there is a path with positive probability from every state to 0, so  $\tilde{B} = \emptyset$ . Therefore,  $\text{Prob}_{\delta_1}^\mathcal{T}(\mathbf{F} B \vee \mathbf{F} \tilde{B}) = \text{Prob}_{\delta_1}^\mathcal{T}(\mathbf{F} B) < 1$ , which means that  $\mathcal{T}$  is not decisive w.r.t.  $B$ . If  $p \leq \frac{1}{2}$ , we have that  $\text{Prob}_{\delta_1}^\mathcal{T}(\mathbf{F} B) = 1$ . Hence, in this case, STS  $\mathcal{T}$  is decisive w.r.t.  $B$ .



**Fig. 1.** Random walk on  $\mathbb{N}$ .

A major interest of the decisiveness concept lies in the design of simple procedures for the qualitative and quantitative analysis of stochastic systems. Indeed, as exposed in [3,10], it allows for instance to compute, under some effectiveness hypotheses, arbitrary close approximations of the probabilities of various properties,

like reachability, repeated reachability, and even arbitrary  $\omega$ -regular properties. We refer to [10, Section 7] for more details, but briefly recall the approximation scheme for reachability properties.

Let  $B \in \Sigma$  be a measurable set and  $\mu \in \text{Dist}(S)$  be an initial distribution. To compute an approximation of  $\text{Prob}_\mu^\mathcal{T}(\mathbf{F} B)$ , we define two sequences  $(p_n^{\text{Yes}})_{n \in \mathbb{N}}$  and  $(p_n^{\text{No}})_{n \in \mathbb{N}}$  such that for  $n \in \mathbb{N}$ ,

$$p_n^{\text{Yes}} = \text{Prob}_\mu^\mathcal{T}(\mathbf{F}_{\leq n} B) \text{ and } p_n^{\text{No}} = \text{Prob}_\mu^\mathcal{T}(B^c \mathbf{U}_{\leq n} \tilde{B}).$$

These sequences are non-decreasing and converge respectively to  $\text{Prob}_\mu^\mathcal{T}(\mathbf{F} B)$  and  $\text{Prob}_\mu^\mathcal{T}(B^c \mathbf{U} \tilde{B})$ . Observe moreover that for all  $n \in \mathbb{N}$ , we have that

$$p_n^{\text{Yes}} \leq \text{Prob}_\mu^\mathcal{T}(\mathbf{F} B) \leq 1 - p_n^{\text{No}}.$$

The main idea behind decisiveness of STSs lies in the following property [3,10]: if  $\mathcal{T}$  is decisive w.r.t.  $B$ , then

$$\lim_{n \rightarrow \infty} p_n^{\text{Yes}} + p_n^{\text{No}} = 1.$$

Therefore, for any given  $\epsilon > 0$ , for some  $n$  sufficiently large,

$$p_n^{\text{Yes}} \leq \text{Prob}_\mu^\mathcal{T}(\mathbf{F} B) \leq p_n^{\text{Yes}} + \epsilon.$$

In situations where  $p_n^{\text{Yes}}$  and  $p_n^{\text{No}}$  can be effectively approximated arbitrarily closely and  $\mathcal{T}$  is decisive w.r.t.  $B$ , we can thus approximate  $\text{Prob}_\mu^\mathcal{T}(\mathbf{F} B)$  up to any desired error bound.

### 2.3 A New Criterion for Decisiveness

Our goal is to provide new sufficient conditions for decisiveness of STSs. To this end, we expose the following crucial lemma. For  $\mathcal{A} = (A_i)_{0 \leq i \leq n} \in \Sigma^{n+1}$ , we denote by  $\phi_{\mathcal{A}}$  the LTL formula  $A_0 \wedge \mathbf{X} A_1 \wedge \dots \wedge \mathbf{X}^n A_n$ , where  $\mathbf{X}$  is the standard “next” modality, and  $\mathbf{X}^n$  its  $n^{\text{th}}$  iterate. The missing proofs for this section are provided in Appendix B.

**Lemma 10.** *Let  $B \in \Sigma$ , and  $\mathcal{A} = (A_i)_{0 \leq i \leq n} \in \Sigma^{n+1}$ . Suppose that there is  $p > 0$  such that for all  $\nu \in \text{Dist}(S)$  with  $\nu_{\mathcal{A}}$  well-defined, we have  $\text{Prob}_{\nu_{\mathcal{A}}}^\mathcal{T}(\mathbf{F} B) \geq p$ . Then for any  $\mu \in \text{Dist}(S)$ ,*

$$\text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} \phi_{\mathcal{A}}) = 0.$$

This result seems rather intuitive: if we go infinitely often through the sequence  $\mathcal{A}$  in order, and after every passage through  $\mathcal{A}$  we have a probability bounded from below to reach  $B$ , then the probability to stay in  $B^c$  forever is 0. An equivalent statement for Markov chains has been used without proof in [3, Lemmas 3.4 & 3.7]. A proof of a weaker version is given as part of the proof of [10, Proposition 36], where it is said that this general case was not known to be true or false; we reuse many insights from this proof and expand on it to show the most general statement.

The main technical difficulty arising to prove this lemma lies in the number of steps needed after each passage through  $\mathcal{A}$  to guarantee a good probability to visit  $B$ . It may be that given any positive bound  $q < p$ , the minimal number of steps  $k$  needed to obtain  $\text{Prob}_{\nu_{\mathcal{A}}}^\mathcal{T}(\mathbf{F}_{\leq k} B) \geq q$  is dependent on  $\nu$ , with no uniform upper bound. In the proof, we thus need to use different values for this number of steps after each passage through  $\mathcal{A}$ , depending on which sets of states were visited at the beginning of each run. The weaker version in [10, Proposition 36] actually assumed that there was a uniform upper bound  $k$  such that for all  $\nu \in \text{Dist}(S)$ ,  $\text{Prob}_{\nu_{\mathcal{A}}}^\mathcal{T}(\mathbf{F}_{\leq k} B) \geq p$  to obtain a similar conclusion. We have removed the need for this constraint.

**Main decisiveness criterion.** We can now state our main contribution to decisiveness.



**Proposition 11 (Decisiveness criterion).** *Let  $B \in \Sigma$  and  $m \in \mathbb{N} \cup \{\infty\}$ . For every  $0 \leq j < m$ , let  $n_j \in \mathbb{N}$  and  $\mathcal{A}_j = (A_i^{(j)})_{0 \leq i \leq n_j} \in \Sigma^{n_j+1}$ . We assume that for all  $\nu \in \text{Dist}(S)$ ,*

$$\text{Prob}_\nu^\mathcal{T} \left( \bigvee_{0 \leq j < m} \mathbf{GF} \phi_{\mathcal{A}_j} \right) = 1.$$

*Assume that there exists  $p > 0$  such that for all  $0 \leq j < m$ , for all  $\nu \in \text{Dist}(S)$  such that  $\nu_{\mathcal{A}_j}$  is well-defined, either  $\nu_{\mathcal{A}_j}(\tilde{B}) = 1$  or*

$$\text{Prob}_{(\nu_{\mathcal{A}_j})_{(\tilde{B})^c}}^\mathcal{T}(\mathbf{F} B) \geq p.$$

*Then  $\mathcal{T}$  is decisive w.r.t.  $B$ .*

In order to provide an intuitive understanding of this proposition, we instantiate its statement with  $m = 1$ ,  $n_0 = 0$ .

**Corollary 12.** *Let  $B \in \Sigma$  be a measurable set, and  $A \in \Sigma$  be an attractor for  $\mathcal{T}$ . We denote  $A' = A \cap (\tilde{B})^c$  the set of states of  $A$  from which  $B$  is reachable with a positive probability. Assume that there exists  $p > 0$  such that for all  $\nu \in \text{Dist}(A')$ ,  $\text{Prob}_\nu^\mathcal{T}(\mathbf{F} B) \geq p$ . Then  $\mathcal{T}$  is decisive w.r.t.  $B$ .*

With probability 1, every run visits attractor  $A$  infinitely often (Lemma 7), but the hypotheses imply a dichotomy between runs. Some runs will reach a state of  $A$  from which  $B$  is almost surely non-reachable, and will end up in  $\tilde{B}$ . The other runs will go infinitely often through states of  $A$  such that the probability of reaching  $B$  is lower bounded by  $p$  (i.e., states of  $A'$ ), and will almost surely visit  $B$  by Lemma 10. This almost-sure dichotomy between runs is required to show decisiveness. In the more general statement of Proposition 11, instead of a simple attractor  $A$ , we assume that we visit infinitely often some finite sequences of sets of states. This allows us to have a weaker assumption on the probability lower bound  $p$ ; it is enough to obtain this lower bound from more specific conditional distributions.

This criterion strictly generalizes those used in the literature: [3, Lemmas 3.4 & 3.7] and [10, Propositions 36 & 37] (see proofs page 26). The criterion in [3, Lemma 3.4] assumes the existence of a finite attractor; the criteria in [10, Propositions 36 & 37] assume some finiteness property in an abstraction (see next section), which we do not. In [3, Lemma 3.7], a similar kind of property as ours is required from all the states of the STSs, not only from some specific distributions.

## 2.4 Abstractions of Stochastic Transitions Systems

Decisiveness and abstractions are deeply intertwined concepts, so we briefly recall this notion [10] and related properties. We let  $\mathcal{T}_1 = (S_1, \Sigma_1, \kappa_1)$  and  $\mathcal{T}_2 = (S_2, \Sigma_2, \kappa_2)$  be two STSs, and  $\alpha: (S_1, \Sigma_1) \rightarrow (S_2, \Sigma_2)$  be a measurable function. We say that a set  $B \in \Sigma_1$  is  $\alpha$ -closed if  $B = \alpha^{-1}(\alpha(B))$ . To mean that  $B$  is  $\alpha$ -closed, we also say that  $\alpha$  is *compatible* with  $B$ . Following [13, 19], we define a natural way to transfer measures from  $(S_1, \Sigma_1)$  to  $(S_2, \Sigma_2)$  through  $\alpha$ : the *pushforward* of  $\alpha$  is the function  $\alpha_\#: \text{Dist}(S_1) \rightarrow \text{Dist}(S_2)$  such that  $\alpha_\#(\mu)(B) = \mu(\alpha^{-1}(B))$  for every  $\mu \in \text{Dist}(S_1)$  and for every  $B \in \Sigma_2$ .

**Definition 13 ( $\alpha$ -abstraction).** *STS  $\mathcal{T}_2$  is an  $\alpha$ -abstraction of  $\mathcal{T}_1$  if for all  $\mu \in \text{Dist}(S_1)$ ,*

$$\alpha_\#(\Omega_{\mathcal{T}_1}(\mu)) \text{ is qualitatively equivalent to } \Omega_{\mathcal{T}_2}(\alpha_\#(\mu)).$$

Informally, the two STSs have the same “qualitative” single steps. Later, one may speak of *abstraction* instead of  $\alpha$ -abstraction if  $\alpha$  is clear in the context.

Notice that  $\alpha$  induces a natural equivalence relation  $\sim_\alpha$  on  $S$ : for  $s, s' \in S$ ,  $s \sim_\alpha s'$  if and only if  $\alpha(s) = \alpha(s')$ . Using this, we provide an equivalent definition of  $\alpha$ -abstraction.

**Lemma 14.** *There is an  $\alpha$ -abstraction of  $\mathcal{T}_1$  if and only if for all  $P \in S/\sim_\alpha$ ,  $\text{Pre}^{\mathcal{T}_1}(P)$  is  $\alpha$ -closed.*

This lemma suggests that abstractions of  $\mathcal{T}_1$  can be seen as partitions that are stable w.r.t. the function  $\text{Pre}^{\mathcal{T}_1}$ , to which we only need to add stochastic transitions between the adequate pieces of the partition. This also indicates that to obtain an  $\alpha$ -abstraction from a finite partition, we can apply a procedure very similar to the classical bisimulation algorithm for non-deterministic transition systems, adapting slightly the meaning of operator  $\text{Pre}^{\mathcal{T}}$  to our stochastic setting. The procedure is given in Procedure 1. We start with an initial (finite) partition  $\mathcal{P}_{init}$  of the state space, of which we want a compatible abstraction, i.e., an abstraction compatible with every set of states in  $\mathcal{P}_{init}$ .

---

**Procedure 1** Abstraction refinement procedure.

---

**Inputs:** an STS  $\mathcal{T}$ , and an initial finite partition  $\mathcal{P}_{init}$  of its state space.

**Output:** the coarsest finite abstraction compatible with  $\mathcal{P}_{init}$ , if it exists.

---

```

 $\mathcal{P} \leftarrow \mathcal{P}_{init}$ 
while  $\exists P, P' \in \mathcal{P}$  such that  $P' \cap \text{Pre}^{\mathcal{T}}(P) \neq \emptyset \wedge P' \setminus \text{Pre}^{\mathcal{T}}(P) \neq \emptyset$  do
   $P_1 \leftarrow P' \cap \text{Pre}^{\mathcal{T}}(P), P_2 \leftarrow P' \setminus \text{Pre}^{\mathcal{T}}(P)$ 
   $\mathcal{P} \leftarrow (\mathcal{P} \setminus \{P'\}) \cup \{P_1, P_2\}$ 
end while
return  $\mathcal{P}$ 

```

---

**Lemma 15.** *Procedure 1 terminates if and only if there exists a finite abstraction of  $\mathcal{T}_1$  compatible with  $\mathcal{P}_{init}$ . In this case, it returns the coarsest such partition.*

The objective behind the notion of abstraction is that by finding an  $\alpha$ -abstraction  $\mathcal{T}_2$  which is somehow simpler than  $\mathcal{T}_1$  (for example, with a smaller state space), we should be able to use  $\mathcal{T}_2$  (with initial distribution  $\alpha_{\#}(\mu)$ ) to analyze some properties of  $\mathcal{T}_1$  (with initial distribution  $\mu$ ). To do so, we need to know which properties are preserved through an  $\alpha$ -abstraction. As a first observation, positive probability of reachability properties is preserved. Stronger conditions are required to study almost-sure reachability properties of an STS through its  $\alpha$ -abstraction. We select a definition and two key results of [10] about that matter which will be useful in the subsequent sections.

**Definition 16 (Sound  $\alpha$ -abstraction).** *We say that  $\mathcal{T}_2$  is a sound  $\alpha$ -abstraction of  $\mathcal{T}_1$  if for all  $B \in \Sigma_2$ ,  $\text{Prob}_{\alpha_{\#}(\mu)}^{\mathcal{T}_2}(\mathbf{F} B) = 1$  implies  $\text{Prob}_{\mu}^{\mathcal{T}_1}(\mathbf{F} \alpha^{-1}(B)) = 1$ .*

Sound abstractions preserve almost-sure reachability properties from  $\mathcal{T}_2$  to  $\mathcal{T}_1$ . Soundness of the abstraction is a sufficient condition to lift decisiveness of the abstraction to the original system: for every  $B \in \Sigma_2$ , if  $\mathcal{T}_2$  is decisive w.r.t.  $B$ , then  $\mathcal{T}_1$  is decisive w.r.t.  $\alpha^{-1}(B)$  [10, Proposition 33]. We also have that some decisiveness property is sufficient to guarantee soundness of the abstraction.

**Proposition 17 ([10, Prop. 40]).** *If  $\mathcal{T}_1$  is decisive w.r.t. every  $\alpha$ -closed set, then  $\mathcal{T}_2$  is a sound  $\alpha$ -abstraction of  $\mathcal{T}_1$ .*

We can now formulate the most important result from this section, linking the qualitative reachability of an STS and its abstraction under soundness and decisiveness hypotheses.

**Proposition 18 ([16, Prop. 6.1.9]).** *Let  $B \in \Sigma_2$  be a measurable set of  $\mathcal{T}_2$ . If  $\mathcal{T}_2$  is a sound  $\alpha$ -abstraction of  $\mathcal{T}_1$  and is decisive w.r.t.  $B$ , then for every  $\mu \in \text{Dist}(S_1)$ ,  $\text{Prob}_{\mu}^{\mathcal{T}_1}(\mathbf{F} \alpha^{-1}(B)) = 1$  if and only if  $\text{Prob}_{\alpha_{\#}(\mu)}^{\mathcal{T}_2}(\mathbf{F} B) = 1$ .*

### 3 Stochastic Hybrid Systems

We choose to restrict our attention to a stochastic extension of the well-studied *hybrid systems*, a large class of timed transition systems.



A *hybrid system* is a dynamical system combining discrete and continuous transitions. It can be defined as a non-deterministic automaton with a finite number of continuous variables, whose evolution is described via an infinite transition system. Hybrid systems have been widely studied since their introduction in the 1990s (e.g., [4,21]). They are effectively used to model various time-dependent reactive systems; systems that need to take into account both continuous factors (e.g., speed, heat, time, distance) and discrete factors (e.g., events, instructions) are ubiquitous.

It is worthwhile to add *stochasticity* to hybrid systems, as this permits a more fine-grained analysis by distinguishing scenarios that are likely to happen and scenarios that are not. If we cannot completely prevent an undesirable outcome from happening, it is still beneficial to have the ability to quantify its probability.

We define hybrid systems and give them fully stochastic semantics, yielding the class of *stochastic hybrid systems* (SHSs). We then prove in Section 3.3 the undecidability of reachability problems for SHSs, thereby showing the interest of finding large decidable subclasses. This undecidability result is not surprising, as these problems are already undecidable in non-deterministic hybrid systems, but a proof in the stochastic context allows to reason about the quantitative problem as well.

### 3.1 Hybrid Systems

We proceed with the definition of a (non-deterministic) *hybrid system*.

**Definition 19 (Hybrid system).** A hybrid system (*HS*) is a tuple  $\mathcal{H} = (L, X, \mathcal{A}, E, \gamma, \mathcal{I}, \mathcal{G}, \mathcal{R})$  where:  $L$  is a finite set of locations (discrete states);  $X = \{x_1, \dots, x_n\}$  is a finite set of continuous variables;  $\mathcal{A}$  is a finite alphabet of events;  $E \subseteq L \times \mathcal{A} \times L$  is a finite set of edges; for each  $\ell \in L$ ,  $\gamma(\ell): \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$  is a continuous function describing the dynamics in location  $\ell$ ;  $\mathcal{I}$  assigns to each location a subset of  $\mathbb{R}^n$  called invariant;  $\mathcal{G}$  assigns to each edge a subset of  $\mathbb{R}^n$  called guard;  $\mathcal{R}$  assigns to each edge  $e$  and valuation  $\mathbf{v} \in \mathbb{R}^n$  a subset  $\mathcal{R}(e)(\mathbf{v})$  of  $\mathbb{R}^n$  called reset. For  $\ell \in L$ ,  $e \in E$ , we usually denote  $\gamma(\ell)$  and  $\mathcal{R}(e)$  by  $\gamma_\ell$  and  $\mathcal{R}_e$ .

We denote the number of variables  $|X|$  by  $n$ . We denote by  $\mathbb{R}^X$  the set of valuations that map variables from  $X$  to real numbers. In what follows, we treat elements of  $\mathbb{R}^X$  as elements of  $\mathbb{R}^n$  through the bijection  $\mathbf{v} \mapsto (\mathbf{v}(x_1), \dots, \mathbf{v}(x_n))$ .

We now give the semantics of hybrid systems. Given a hybrid system  $\mathcal{H}$ , we define  $S_{\mathcal{H}} = L \times \mathbb{R}^n$  as the states of the hybrid system. We distinguish two kinds of transitions between states:

- there is a *switch-transition*  $(\ell, \mathbf{v}) \xrightarrow{a} (\ell', \mathbf{v}')$  if there exists  $e = (\ell, a, \ell') \in E$  such that  $\mathbf{v} \in \mathcal{I}(\ell) \cap \mathcal{G}(e)$ ,  $\mathbf{v}' \in \mathcal{R}_e(\mathbf{v}) \cap \mathcal{I}(\ell')$ ;
- there is a *delay-transition*  $(\ell, \mathbf{v}) \xrightarrow{\tau} (\ell, \mathbf{v}')$  if there exists  $\tau \in \mathbb{R}^+$  such that for all  $0 \leq \tau' \leq \tau$ ,  $\gamma_\ell(\mathbf{v}, \tau') \in \mathcal{I}(\ell)$  and  $\mathbf{v}' = \gamma_\ell(\mathbf{v}, \tau)$ .

Informally, a switch-transition  $(\ell, \mathbf{v}) \xrightarrow{a} (\ell', \mathbf{v}')$  means that an edge  $e = (\ell, a, \ell')$  can be taken without violating any constraint: the value  $\mathbf{v}$  of the continuous variables is an element of the invariant  $\mathcal{I}(\ell)$  and of the guard  $\mathcal{G}(e)$ , and there is a possible reset  $\mathbf{v}'$  of the variables which is an element of the invariant  $\mathcal{I}(\ell')$ . A delay-transition  $(\ell, \mathbf{v}) \xrightarrow{\tau} (\ell, \mathbf{v}')$  means that some time  $\tau$  elapses without changing the discrete location of the system—the only constraint is that all the values taken by the continuous variables during this time are in the invariant  $\mathcal{I}(\ell)$ .

Given  $s = (\ell, \mathbf{v}) \in L \times \mathbb{R}^n$  a state of the hybrid system, and  $\tau \in \mathbb{R}^+$ , we denote by  $s + \tau = (\ell, \gamma_\ell(\mathbf{v}, \tau))$  the new state after time  $\tau$  has elapsed, without changing the location;  $\tau$  is then referred to as a *delay*.

We only consider *mixed transitions* in runs, i.e., transitions that consist of a delay-transition (some time elapses) followed by a switch-transition (an edge is taken and the location changes). A mixed transition is denoted by  $(\ell, \mathbf{v}) \xrightarrow{\tau, a} (\ell', \mathbf{v}')$  if and only if there exists  $\mathbf{v}'' \in \mathbb{R}^n$  such that  $(\ell, \mathbf{v}) \xrightarrow{\tau} (\ell, \mathbf{v}'') \xrightarrow{a} (\ell', \mathbf{v}')$ .

We usually assume that there is a bijection between the edges  $E$  and the alphabet of events  $\mathcal{A}$ , and we omit mentioning this alphabet. If  $e = (\ell, a, \ell') \in E$ , we can thus denote  $\xrightarrow{e}$  (resp.  $\xrightarrow{\tau, e}$ ) for switch-transitions (resp. mixed transitions) instead of  $\xrightarrow{a}$  (resp.  $\xrightarrow{\tau, a}$ ), if there is no ambiguity.

**Definition 20 (Run of an HS).** A run of an HS is an infinite sequence

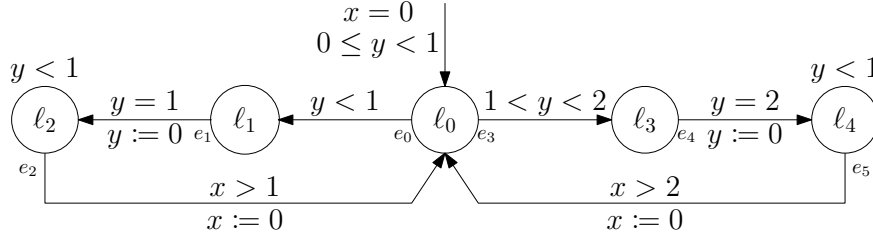
$$(\ell_0, \mathbf{v}_0) \xrightarrow{\tau_0, e_0} (\ell_1, \mathbf{v}_1) \xrightarrow{\tau_1, e_1} (\ell_2, \mathbf{v}_2) \xrightarrow{\tau_2, e_2} \dots$$

of elements in  $L \times \mathbb{R}^n$  such that for all  $i \geq 0$ ,  $(\ell_i, \mathbf{v}_i) \xrightarrow{\tau_i, e_i} (\ell_{i+1}, \mathbf{v}_{i+1})$  is a mixed transition.

*Example 21.* We provide in Figure 2 an example of a hybrid system. This example was first studied in [7]. There are two continuous variables ( $x$  and  $y$ ) and five locations, each of them equipped with the same simple dynamics:  $\dot{x} = \dot{y} = 1$  (i.e.,  $\gamma_\ell((x, y), \tau) = (x + \tau, y + \tau)$  for every location  $\ell \in L$ ). Locations  $\ell_2$  and  $\ell_4$  have the same invariant, which is  $\{(x, y) \mid y < 1\}$ ; the other invariants are simply  $\mathbb{R}^2$ . Guards are written next to the edge to which they are related: for instance,  $\mathcal{G}(e_4) = \{(x, y) \mid y = 2\}$ . The notation “ $x := 0$ ” is used to denote a deterministic reset (in this case, the value of  $x$  is reset to 0 after taking the edge). For instance,  $R_{e_1}(x, y) = \{(x, 0)\}$  (the value of  $x$  is preserved and  $y$  is reset to 0). If nothing else is written next to an edge  $e$ , it means that there is no reset on  $e$ , i.e., that  $R_e(\mathbf{v}) = \{\mathbf{v}\}$  for all  $\mathbf{v} \in \mathbb{R}^n$ . An example of the beginning of a run of this system can be

$$\begin{aligned} (\ell_0, (0, 0)) &\xrightarrow{0.4, e_0} (\ell_1, (0.4, 0.4)) \xrightarrow{0.6, e_1} (\ell_2, (1, 0)) \xrightarrow{0.2, e_2} (\ell_0, (0, 0.2)) \\ &\xrightarrow{1.5, e_3} (\ell_3, (1.5, 1.7)) \xrightarrow{0.3, e_4} (\ell_4, (1.8, 0)) \xrightarrow{0.8, e_5} (\ell_0, (0, 0.8)) \dots \end{aligned}$$

Due to his fairly simple dynamics, guards and resets, this hybrid system actually belongs to the class of *timed automata* [5].



**Fig. 2.** Example of a hybrid system with two continuous variables. Each location is equipped with the dynamics  $\dot{x} = \dot{y} = 1$ .

We now give more vocabulary to refer to hybrid systems. If there is a switch-transition  $(\ell, \mathbf{v}) \xrightarrow{e} (\ell', \mathbf{v}')$ , we say that edge  $e$  is *enabled* at state  $(\ell, \mathbf{v})$ . An edge  $e$  is enabled if it can be taken with no delay from state  $(\ell, \mathbf{v})$ . Given a state  $s = (\ell, \mathbf{v})$  and an edge  $e = (\ell, a, \ell')$  of  $\mathcal{H}$ , we define  $I(s, e) = \{\tau \in \mathbb{R}^+ \mid s \xrightarrow{\tau, e} s'\}$  as the set of delays after which edge  $e$  is enabled from  $s$ , and  $I(s) = \bigcup_e I(s, e)$  as the set of delays after which any edge is enabled from  $s$ . For instance, in the hybrid system from Example 21, for  $s = (\ell_0, (0, 0.2))$ , the set  $I(s, e_0) = [0, 0.8)$ , and  $I(s) = [0, 1.8) \setminus \{0.8\}$ .

We say that a state  $s \in L \times \mathbb{R}^n$  is *non-blocking* if  $I(s) \neq \emptyset$ . In the sequel, we only consider hybrid systems such that all states are non-blocking, thereby justifying why considering solely mixed transitions is doable—such a transition is available from any state.

### 3.2 Probabilistic Semantics for Hybrid Systems

We expand on the definition of a hybrid system by replacing the non-deterministic aspects of the definition with stochasticity. *Stochastic hybrid systems* will be our main focus of attention in the rest of the paper.

**Definition 22 (Stochastic hybrid system).** A stochastic hybrid system (SHS) is defined as a tuple  $\mathcal{H} = (\mathcal{H}', \mu_L, \eta_R, \theta)$ , where:

- $\mathcal{H}' = (L, X, \mathcal{A}, E, \gamma, \mathcal{I}, \mathcal{G}, \mathcal{R})$  is a hybrid system, which is referred to as the underlying hybrid system of  $\mathcal{H}$ . We require guards, invariants and resets to be Borel sets.
- $\mu_L: L \times \mathbb{R}^n \rightarrow \text{Dist}(\mathbb{R}^+)$  associates a probability distribution on the time delay in  $\mathbb{R}^+$  (equipped with the classical Borel  $\sigma$ -algebra) before leaving a location. Given  $s \in L \times \mathbb{R}^n$  (a state of  $\mathcal{H}$ ), the distribution  $\mu_L(s)$  will also be denoted by  $\mu_s$ . We require that for every  $s \in L \times \mathbb{R}^n$ ,  $\mu_s(\mathbb{R}^+) = \mu_s(I(s)) = 1$ , i.e., the probability that at least one edge is enabled after a delay is 1.
- $\eta_{\mathcal{R}}$  associates to each edge  $e$  and valuation  $\mathbf{v}$  a probability distribution on the set  $\mathcal{R}_e(\mathbf{v}) \subseteq \mathbb{R}^n$ . Given  $e \in E$  and  $\mathbf{v} \in \mathbb{R}^n$ , the distribution  $\eta_{\mathcal{R}}(e)(\mathbf{v})$  will also be denoted by  $\eta_e(\mathbf{v})$ .
- $\theta: L \times \mathbb{R}^n \rightarrow \text{Dist}(E)$  is a function assigning to each state of  $\mathcal{H}$  a probability distribution on the edges. We require that  $\theta(s)(e) > 0$  if and only if edge  $e$  is enabled at  $s$ . For  $s \in L \times \mathbb{R}$ , we denote  $\theta(s)$  by  $\theta_s$ . This distribution is only defined for states for which at least one edge is enabled.

*Remark 23.* The term “stochastic hybrid system” is used for a wide variety of stochastic extensions of hybrid systems throughout the literature. In this work, we consider stochastic delays, stochastic resets, a stochastic edge choice, and initial states given by a probability distribution. The way this probabilistic semantics is added on top of hybrid systems is very similar to how *timed automata* are converted to *stochastic timed automata* in [7, 11, 12]. The major difference is that in the case of timed automata, resets are deterministic, hence do not require stochasticity. Notice that although dynamics appear to be deterministic, the model is powerful enough to emulate stochastic dynamics by assuming that extra variables are solely used to influence the continuous flow of the other variables. These variables can be chosen stochastically in each location through the reset mechanism. This is for example sufficient to consider a stochastic extension of the *rectangular automata* [22], whose variables evolve according to slopes inside an interval (such as  $\dot{x} \in [1, 4]$ ); the actual value of each slope would then depend on the value of one of these extra variables. Our model is similar to the one of the software tool UPPAAL [18], showing the practical relevance of our work.

When referring to an SHS, we make in particular use of the same terminology as for hybrid systems (e.g., runs, enabled edges, allowed delays  $I(\cdot)$ ) to describe its underlying hybrid system.

In order to apply the theory developed in Section 2, we give the semantics of an SHS  $\mathcal{H}$  as an STS  $\mathcal{T}_{\mathcal{H}} = (S_{\mathcal{H}}, \Sigma_{\mathcal{H}}, \kappa_{\mathcal{H}})$ . The set  $S_{\mathcal{H}}$  is the set  $L \times \mathbb{R}^n$  of states of  $\mathcal{H}$ , and  $\Sigma_{\mathcal{H}}$  is the  $\sigma$ -algebra product between  $2^L$  and the Borel  $\sigma$ -algebra on  $\mathbb{R}^n$ . To define  $\kappa_{\mathcal{H}}$ , we first explain briefly the role of each probability distribution in the definition of SHS. Starting from a state  $s = (\ell, \mathbf{v})$ , a delay  $\tau$  is chosen randomly, according to the distribution  $\mu_s$ . From the state  $s + \tau = (\ell, \mathbf{v}')$ , an edge  $e = (\ell, a, \ell')$  (enabled in  $s + \tau$ ) is selected, following the distribution  $\theta_{s+\tau}$  (such an edge is almost surely available, as  $\mu_s(I(s)) = 1$  by hypothesis). The next state will be in location  $\ell'$ , and the values of the continuous variables are stochastically reset according to the distribution  $\eta_e(\mathbf{v}')$ . We can thus define  $\kappa_{\mathcal{H}}$  as follows: for  $s = (\ell, \mathbf{v}) \in S_{\mathcal{H}}$ ,  $B \in \Sigma_{\mathcal{H}}$ ,

$$\kappa_{\mathcal{H}}(s, B) = \int_{\tau \in \mathbb{R}^+} \sum_{e=(\ell, a, \ell') \in E} \left( \theta_{s+\tau}(e) \cdot \int_{\mathbf{v}'' \in \mathbb{R}^n} \mathbf{1}_B(\ell', \mathbf{v}'') d(\eta_e(\gamma_{\ell}(\mathbf{v}, \tau)))(\mathbf{v}'') \right) d\mu_s(\tau)$$

where  $\mathbf{1}_B$  is the characteristic function of  $B$ . It gives the probability to hit set  $B \subseteq S_{\mathcal{H}}$  from state  $s$  in one step (representing a mixed transition). The function  $\kappa_{\mathcal{H}}(s, \cdot)$  defines a probability distribution for all  $s \in S_{\mathcal{H}}$ .

**Definition 24 (STS induced by an SHS).** For an SHS  $\mathcal{H}$ , we define  $\mathcal{T}_{\mathcal{H}} = (S_{\mathcal{H}}, \Sigma_{\mathcal{H}}, \kappa_{\mathcal{H}})$  as the STS induced by  $\mathcal{H}$ .

Thanks to the stochasticity of our models, we can reason about both *qualitative* and *quantitative* reachability problems, as defined in Definition 2.

### 3.3 Undecidability of Reachability for Stochastic Hybrid Systems

We now show that qualitative and quantitative reachability problems for SHSs are undecidable, even for SHSs with relatively simple features. This demonstrates the significance of establishing results about the decisiveness

of classes of SHSs (done in Section 4). Along with classical effectiveness assumptions (in Section 5), these decisiveness results will be sufficient to guarantee the decidability of reachability problems for these classes.

Reachability problems have been extensively studied for non-deterministic hybrid systems, and some of the undecidability proofs [22,23] can be translated almost directly to our stochastic setting. A classical method to show undecidability consists in encoding every computation of a Turing-complete model as an execution of a hybrid system. The undecidability proof of [22] builds for every Turing-machine  $M$  a hybrid system with one accepting run that encodes the halting computation of  $M$ . This proof is not “robust” in the sense that a slight perturbation of this accepting run does not encode an execution of  $M$ . As argued in [23], this means that undecidability might stem from the perfect (unrealistic) accuracy required to process such an execution, and not from the very nature of hybrid systems. This is why in [23], the authors establish an undecidability result for *robust* hybrid systems, i.e., hybrid systems such that if they accept some run  $\rho$ , have to accept all runs “close enough” to  $\rho$ . This provides a more convincing argument that hybrid systems are intrinsically undecidable.

In our stochastic framework, to obtain a similar idea of “robustness”, our goal is to prove undecidability even when constrained to purely continuous distributions on time delays from any state, and very simple guards, resets, and dynamics. This requires a distinct proof from [22]. The result from [23] is much closer to the one that we want to achieve, and we take inspiration from its proof to show the undecidability of SHSs. The proof consists of reducing the *halting problem for two-counter machines* to deciding whether a measurable set in an SHS is reached with probability 1. It is provided in Appendix C. We obtain the following result.

**Proposition 25.** *The qualitative reachability problems and the approximate quantitative reachability problem are undecidable for stochastic hybrid systems with purely continuous distributions on time delays, guards that are linear comparisons of variables and constants, and using positive integer slopes for the flow of the continuous variables. The approximate quantitative problem is moreover undecidable for any fixed precision  $\epsilon < \frac{1}{2}$ .*

Although the proof is centered on showing the undecidability of qualitative reachability problems, we get as a by-product the undecidability of the approximation. Indeed, as the systems used throughout the proof reach a target set  $B$  with a probability that is either 0 or 1, the ability to approximate  $\text{Prob}_\mu^{\mathcal{H}}(\mathbf{F} B)$  with  $\epsilon < \frac{1}{2}$  would be sufficient to solve the qualitative problem. As the proof shows that these qualitative problems are undecidable, we obtain that the approximate quantitative problem is also undecidable. The proof therefore also shows that deciding whether a state lies in  $\bar{B}$  is already undecidable. By considering a stochastic framework rather than a non-deterministic one as in [23], we thus obtain a seemingly more powerful result with a similar proof.

## 4 Properties of Cycle-Reset Stochastic Hybrid Systems

The literature about non-deterministic hybrid systems suggests that to obtain subclasses for which the reachability problem becomes decidable, one must set sharp restrictions on the continuous flow of the variables, and/or on the discrete transitions (via the *reset* mechanism). In this decidable spectrum lies for instance the *rectangular initialized automata*, which are quite permissive toward the continuous evolution of the variables, but need strong hypotheses about what is allowed in the discrete transitions [22]. Our approach lies at one end of this spectrum: we will simply restrict the discrete behavior by considering *strong resets*, i.e., resets that forget about the previous values of the variables, decoupling the discrete behavior from the continuous behavior. We show that one strong reset per cycle of the graph is sufficient to obtain our results, and we name this property *cycle-reset*. This point of view has already been studied in [15,20,30] for non-deterministic hybrid systems.

**Definition 26 (Strong reset).** *Given  $\mathcal{H}$  a stochastic hybrid system and  $e$  an edge of  $\mathcal{H}$ , we say that  $e$  has a strong reset (or is strongly reset) if  $\mathcal{R}_e$  and  $\eta_e$  are constant functions.*

If an edge  $e$  is strongly reset, it stochastically resets all the continuous variables when it is taken, and the stochastic reset does not depend on their values. It means that for any  $\mathbf{v}, \mathbf{v}' \in \mathcal{G}(e)$ , the distributions  $\eta_e(\mathbf{v})$  and  $\eta_e(\mathbf{v}')$  are equal. If an edge is strongly reset, we denote its reset distribution by  $\eta_e^*$  (which equals  $\eta_e(\mathbf{v})$  for all  $\mathbf{v} \in \mathbb{R}^n$ ).

Let  $\mathcal{H} = (L, \mathcal{A}, E, \dots)$  be an SHS. We denote by

$$\begin{aligned} C^{\mathcal{H}} = \{ (e_0, e_1, \dots, e_m) \in E^{m+1} \mid & \forall 0 \leq i \leq m, e_i = (\ell_i, a_i, \ell'_i), \\ & \ell'_m = \ell_0, \forall 0 \leq i < m, \ell'_i = \ell_{i+1}, \\ & \text{and } \forall 0 \leq i < j \leq m, e_i \neq e_j \} \end{aligned}$$

the set of *simple cycles* of  $\mathcal{H}$ .

**Definition 27 (Cycle-reset SHSs).** *We say that an SHS  $\mathcal{H}$  is cycle-reset if for every simple cycle  $(e_0, e_1, \dots, e_m) \in C^{\mathcal{H}}$ , there exists  $0 \leq i \leq m$  such that  $e_i$  is strongly reset.*

We show two independent and very convenient results of cycle-reset SHSs: such SHSs are *decisive w.r.t. any measurable set* (the proof of this statement relies on the decisiveness criterion from Proposition 11), and *admit a finite abstraction*. The proofs for this section are provided in Appendix D.

*Remark 28.* It is interesting to notice that properties similar to “one strong reset per cycle” are given in various places throughout the literature about timed and hybrid systems. In [11], the authors perform the quantitative analysis of stochastic timed automata with only one continuous variable, assuming that any bounded cycle in an abstraction contains a reset of the continuous variable. In [20], the class of *relaxed o-minimal (non-deterministic) hybrid systems*, with one strong reset per cycle, is shown to admit a finite bisimulation, making the reachability problem decidable in cases where this bisimulation is effectively computable.

#### 4.1 Decisiveness

We motivate this section with an example of a simple non-decisive SHS, which we will use to show that our decisiveness result is tight.

*Example 29.* We add a stochastic layer to the hybrid system of Example 21, pictured in Figure 2. The distributions on the time delays in locations  $\ell_0, \ell_2$  and  $\ell_4$  are uniform distributions on the interval of allowed delays. For instance, at state  $s = (\ell_0, (x, y))$ , the distribution  $\mu_s$  follows a uniform distribution  $\mathcal{U}(0, 2 - y)$ . In locations  $\ell_1$  and  $\ell_3$ , the distributions on the delays are Dirac distributions, since the sets of allowed delays are singletons. As all resets are deterministic, they are simply modeled as Dirac distributions and since at most one edge is enabled in each state, the distributions  $\theta_s$  are also Dirac distributions.

It is proved in [12, Section 6.2.2] that this SHS is not decisive w.r.t.  $B = \{\ell_2\} \times \mathbb{R}^2$ . The reason is that from a state  $s = (\ell_0, (0, y))$  with  $0 \leq y < 1$ , at each subsequent passage through location  $\ell_0$ , the value of  $y$  increases but stays bounded from above by 1, which decreases the probability to take edge  $e_0$  (and thus reach  $B$ ). Let  $\mu$  be the Dirac distribution at  $(\ell_0, (0, 0))$ . As we will go infinitely often from  $\mu$  through  $\{\ell_0\} \times \{0\} \times [0, 1)$ , we will never reach a state in  $\tilde{B}$ . However, it is proved that  $\text{Prob}_{\mu}^{\mathcal{T}_{\mathcal{H}}}(\mathbf{G} B^c) > 0$ . The proof is quite technical and we do not recall it here. This implies in particular that

$$\text{Prob}_{\mu}^{\mathcal{T}_{\mathcal{H}}}(\mathbf{F} B \vee \mathbf{F} \tilde{B}) = \text{Prob}_{\mu}^{\mathcal{T}_{\mathcal{H}}}(\mathbf{F} B) = 1 - \text{Prob}_{\mu}^{\mathcal{T}_{\mathcal{H}}}(\mathbf{G} B^c) < 1,$$

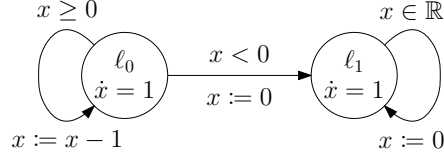
which means that  $\mathcal{T}_{\mathcal{H}}$  is not decisive w.r.t.  $B$  from  $\mu$ .

**Proposition 30.** *Every cycle-reset SHS is decisive w.r.t. any measurable set.*

Placing (at least) one strong reset per simple cycle is an easy syntactic way to guarantee that almost surely, infinitely many strong resets are performed, which is the actual sufficient property used in the proof. As there are only finitely many edges, we can find a probability lower bound  $p$  on the probability to reach  $B$  after any strong reset, as required in the criterion of Proposition 11. Notice that as shown in Example 29, having independent flows for each variable and resetting each variable once in each cycle is not sufficient to obtain decisiveness; variables need to be reset *on the same discrete transition* in each cycle.

## 4.2 Existence of a Finite Abstraction

We show that cycle-reset SHSs admit a finite  $\alpha$ -abstraction. We first give a simple example showing that without one strong reset per cycle, some simple systems do not admit a finite  $\alpha$ -abstraction compatible with the locations.



**Fig. 3.** The time delays are given by exponential distributions from any state; resets are Dirac distributions. The smallest abstraction compatible with  $\{\ell_1\} \times \mathbb{R}$  is denumerable.

*Example 31.* Consider the SHS of Figure 3. The self-loop edge of  $\ell_0$  is the only edge not being strongly reset. We assume that we want to have an abstraction compatible with  $s^* := \{\ell_1\} \times \mathbb{R}$ . Using Procedure 1, we have to split  $\{\ell_0\} \times \mathbb{R}^n$  in  $s_0 := \text{Pre}^{\mathcal{H}}(s^*) = \{\ell_0\} \times (-\infty, 0)$  and  $\{\ell_0\} \times [0, +\infty)$ , as all the states of  $s_0$  can reach  $s^*$  with a positive probability in one step, but none of the states of  $\{\ell_0\} \times [0, +\infty)$  can. Then,  $\{\ell_0\} \times [0, +\infty)$  must also be split into  $s_1 := \{\ell_0\} \times [0, 1)$  and  $\{\ell_0\} \times [1, +\infty)$  because the states of  $s_1$  can all reach  $s_0$  with a positive probability in one step, but none of the states of  $\{\ell_0\} \times [1, +\infty)$  can. By iterating this argument, we find that the smallest  $\alpha$ -abstraction compatible with  $\{\ell_1\} \times \mathbb{R}$  is denumerable, and the partition that it induces is composed of  $s^*$ ,  $s_0$  and  $s_i = \{\ell_0\} \times [i-1, i)$  for  $i \geq 1$ . The underlying hybrid system actually belongs to the class of *updatable timed automata* [14], and the abstraction almost coincides with the *region graph* of the automaton.

The cycle-reset assumption is sufficient to guarantee the existence of a finite abstraction, as formulated in the next proposition. Its proof consists of showing that Procedure 1 terminates under cycle-reset assumption. With our decisiveness result, we can even show that this abstraction is *sound*.

**Proposition 32.** *Let  $\mathcal{H}$  be an SHS, and  $B \in \Sigma_{\mathcal{H}}$ . If  $\mathcal{H}$  is cycle-reset, it has a finite and sound abstraction compatible with  $B$  and with the locations.*

## 5 Reachability Analysis in Cycle-Reset Stochastic Hybrid Systems

Our goal in this section is to perform a qualitative and quantitative analysis of cycle-reset stochastic hybrid systems, using the properties established in the previous section. A first hurdle to circumvent is that arbitrary stochastic hybrid systems are very difficult to apprehend algorithmically: for instance, the continuous evolution of their variables may be defined by solutions of systems of differential equations, which we do not know how to solve in general. To make the problem more accessible, we follow the approach of [15,30] for non-deterministic hybrid systems by assuming that some key components of our systems are definable in a mathematical structure. This restricts the syntax of our SHSs in such a way that, as we will show with a few extra hypotheses, qualitative and quantitative reachability problems become decidable.

We adapt this point of view to the stochastic framework. In Section 5.1, we formulate simple decidability assumptions under which the finite  $\alpha$ -abstraction from Section 4.2 is computable, which makes qualitative reachability problems decidable. We identify a large class of SHSs satisfying these hypotheses, namely *o-minimal SHSs defined in a decidable theory*. In Section 5.2, we also identify sufficient hypotheses for the approximate quantitative problem to be decidable, in the form of a finite set of probabilities that have to be approximately computable.



### 5.1 Qualitative Reachability Analysis

We assume that the reader is familiar with the following model-theoretic and logical terms: first-order formula, structure, definability of sets, functions and elements, theory, decidability of a theory. We refer to [25] for an introduction to these concepts. In what follows, by *definable*, we mean definable without parameters.

**Definition 33 (Definable SHS).** *Given a structure  $\mathcal{M}$ , an SHS  $\mathcal{H}$  is said to be defined in  $\mathcal{M}$  if for every location  $\ell \in L$ ,  $\gamma_\ell$  is a function definable in  $\mathcal{M}$  and  $\mathcal{I}(\ell)$  is a set definable in  $\mathcal{M}$ , and for every edge  $e \in E$ , the set  $\mathcal{G}(e)$  is definable and there exists a first-order formula  $\psi_e(\mathbf{x}, \mathbf{y})$  such that  $\mathbf{v}' \in \mathcal{R}_e(\mathbf{v})$  if and only if  $\psi_e(\mathbf{v}, \mathbf{v}')$  is true.*

Note that we require that the *flow* of the dynamical system in each location is definable in  $\mathcal{M}$ , and not that it is the solution to a definable system of differential equations. We fix  $\mathcal{M} = \langle \mathbb{R}, <, \dots \rangle$  a structure, and  $\mathcal{H}$  an SHS. We make three assumptions:

- (H1)  $\mathcal{H}$  is defined in  $\mathcal{M}$ ,
- (H2) the theory of  $\mathcal{M}$  is decidable,
- (H3) for every  $\ell \in L$  and definable  $D \subseteq \mathbb{R}^n$ ,  $\text{Pre}^{\mathcal{T}_\mathcal{H}}(\{\ell\} \times D)$  is definable.

These hypotheses give us an automatic way to handle many questions about  $\mathcal{H}$ . For instance, for  $s = (\ell, \mathbf{v}) \in S_\mathcal{H}$  and  $e = (\ell, a, \ell') \in E$ , the set  $I(s, e)$  (and thus  $I(s)$ ) is definable since  $\mathcal{I}(\ell)$ ,  $\gamma_\ell(\cdot, \cdot)$  and  $\mathcal{G}(e)$  are definable by (H1), and

$$I(s, e) = \{\tau \mid \tau \geq 0 \wedge \forall \tau' (0 \leq \tau' \leq \tau \implies \gamma_\ell(s, \tau') \in \mathcal{I}(\ell)) \wedge \gamma_\ell(s, \tau) \in \mathcal{G}(e)\}.$$

The main issue when lifting such considerations to the stochastic setting is that the definability of probabilities and measures is not guaranteed. For instance, the function  $x \mapsto \frac{1}{x}$  is definable in  $\langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle$ , but its integral is not: for  $t \geq 1$ ,

$$\int_1^t \frac{1}{x} dx = \log(t).$$

Therefore, using probability measures given by a definable probability density function is not sufficient to write arbitrary first-order formulae about actual probabilities, so  $\text{Pre}^{\mathcal{T}_\mathcal{H}}(\{\ell\} \times D)$  is not necessarily definable, even for  $D$  definable. To compensate, we make assumption (H3), which amounts to assuming that distributions  $\mu_s$  for  $s \in S_\mathcal{H}$  and  $\eta_e(\mathbf{v})$  for  $e \in E$  and  $\mathbf{v} \in \mathbb{R}^n$  still have some properties that are sufficient to decide whether their evaluation on definable sets is positive. We summarize in the following proposition what the previous hypotheses entail.

**Proposition 34.** *We assume that  $\mathcal{H}$  is a cycle-reset SHS which satisfies hypotheses (H1), (H2), and (H3). Let  $B \in S_\mathcal{H}$  be a measurable and definable set of states and  $\mu \in \text{Dist}(S_\mathcal{H})$  be an initial distribution such that for every location  $\ell$  and definable set  $D \subseteq \mathbb{R}^n$ , we can decide whether  $\mu(\{\ell\} \times D) > 0$ . Then we can decide whether  $\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F} B) = 1$  and whether  $\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F} B) = 0$ .*

In particular, this implies that we can decide whether a definable state is in  $\tilde{B}$ , which is not the case in general, as shown through Proposition 25.

*Proof.* Using that  $\mathcal{H}$  is cycle-reset, we know by Proposition 32 that there exists a sound  $\alpha$ -abstraction of  $\mathcal{H}$  compatible with  $B$ , which is a finite STS (that is, a finite Markov chain)  $\mathcal{T}_\mathcal{H}^* = (\mathcal{P}_\mathcal{H}, 2^{\mathcal{P}_\mathcal{H}}, \kappa_\mathcal{H})$ . Partition  $\mathcal{P}_\mathcal{H}$  is definable by (H3) and Procedure 1, and Procedure 1 is an algorithm thanks to hypothesis (H2).

By hypothesis on the initial distribution  $\mu$ , we can decide for which sets  $P \in \mathcal{P}_\mathcal{H}$  we have  $\mu(P) > 0$ . Therefore, we can compute an initial distribution  $\mu' \in \text{Dist}(\mathcal{P}_\mathcal{H})$  which is qualitatively equivalent to  $\alpha_\#(\mu)$ . By the properties of an abstraction, it holds that

$$\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F} B) = 0 \iff \text{Prob}_{\alpha_\#(\mu)}^{\mathcal{T}_\mathcal{H}^*}(\mathbf{F} \alpha(B)) = 0 \iff \text{Prob}_{\mu'}^{\mathcal{T}_\mathcal{H}^*}(\mathbf{F} \alpha(B)) = 0,$$

which can be decided for a finite Markov chain. Similarly, by Proposition 18, as  $\mathcal{T}_\mathcal{H}^*$  is a sound  $\alpha$ -abstraction and is decisive w.r.t.  $\alpha(B)$  (since its state space is finite), it holds that

$$\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F} B) = 1 \iff \text{Prob}_{\alpha_\#(\mu)}^{\mathcal{T}_\mathcal{H}^*}(\mathbf{F} \alpha(B)) = 1 \iff \text{Prob}_{\mu'}^{\mathcal{T}_\mathcal{H}^*}(\mathbf{F} \alpha(B)) = 1,$$

which can also be decided. □

**O-minimal SHSs.** We identify a large subclass of SHSs satisfying hypotheses **(H1)**, **(H2)**, and **(H3)**. This subclass consists of the *stochastic o-minimal hybrid systems*, i.e., SHSs defined in an *o-minimal structure* (introduced in [43,37]), with additional assumptions on the probability distributions.

**Definition 35.** A totally ordered structure  $\mathcal{M} = \langle M, <, \dots \rangle$  is o-minimal if every definable subset of  $M$  is a finite union of points and open intervals (possibly unbounded).

In other words, the definable subsets of  $M$  are exactly the ones that are definable with parameters in  $\langle M, < \rangle$ . Some well-known structures are o-minimal: the ordered additive group of rationals  $\langle \mathbb{Q}, <, +, 0 \rangle$ , the ordered additive group of reals  $\mathbb{R}_{\text{lin}} = \langle \mathbb{R}, <, +, 0, 1 \rangle$ , the ordered field of reals  $\mathbb{R}_{\text{alg}} = \langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle$ , the ordered field of reals with the exponential function  $\mathbb{R}_{\text{exp}} = \langle \mathbb{R}, <, +, \cdot, 0, 1, e^x \rangle$  [44].

There is no general result about the decidability of the theories of o-minimal structures. A well-known case is the Tarski-Seidenberg theorem, which asserts that there exists a quantifier-elimination algorithm for sentences in the first-order language of real closed fields [42]. This result implies the decidability of the theory of  $\mathbb{R}_{\text{alg}}$ . However, it is not known whether the theory of  $\mathbb{R}_{\text{exp}}$  is decidable. Its decidability is implied by *Schanuel's conjecture*, a famous unsolved problem in transcendental number theory [34].

The o-minimality of  $\mathcal{M}$  implies that definable subsets of  $M^n$  have a very “nice” structure, described notably by the *cell decomposition theorem* [29]. This implies in particular that every subset of  $\mathbb{R}^n$  definable in an o-minimal structure belongs to the  $\sigma$ -algebra of Borel sets of  $\mathbb{R}^n$  [28, Proposition 1.1]. We have in addition the following result.

**Lemma 36 ([28, Remark 2.1]).** Let  $\lambda_n$  be the Lebesgue measure on  $\mathbb{R}^n$ . Let  $\mathcal{M}$  be an o-minimal structure. If  $A \subseteq \mathbb{R}^n$  is definable in  $\mathcal{M}$ , then  $\lambda_n(A) > 0$  if and only if  $A^\circ \neq \emptyset$ , where  $A^\circ$  denotes the interior of  $A$ .

*Remark 37.* The definition of non-deterministic *o-minimal hybrid system* in the literature usually assumes that all edges are strongly reset. *O-minimal hybrid systems* were first introduced in [30], and further studied notably in [15]. The strong reset hypothesis was relaxed in [20] to “one strong reset per cycle”. The main result showing the interest of such hybrid systems is that they admit a finite abstraction (called *time-abstract bisimulation* in this case), which is computable when the underlying theory is decidable. This finite abstraction does not necessarily extend to a “stochastic” abstraction (as defined in Section 2.4), as there may be transitions that have probability 0 to happen, and the corresponding sets of states may not be definable without hypothesis **(H3)**.

Let  $\mathcal{M} = \langle \mathbb{R}, <, +, \dots \rangle$  be an o-minimal structure whose theory is decidable, such as  $\mathbb{R}_{\text{alg}}$ . Let  $\mathcal{H} = (\mathcal{H}', \mu_L, \eta_R, \theta)$  be a cycle-reset stochastic o-minimal hybrid system defined in  $\mathcal{M}$ . It therefore satisfies hypotheses **(H1)** and **(H2)**. We still lack some hypotheses about the definability of the probability distributions to obtain the definability of the finite abstraction. Let  $\mu \in \text{Dist}(S_{\mathcal{H}})$  be an initial distribution. We make the following assumptions, which we denote by **(†)**:

- for all  $s = (\ell, \mathbf{v}) \in L \times \mathbb{R}^n$ , if  $I(s)$  is finite, then  $\mu_s$  is equivalent to the uniform discrete distribution on  $I(s)$ ; if  $I(s)$  is infinite, then  $\mu_s$  is equivalent to the Lebesgue measure on  $I(s)$ ;
- for each  $\ell \in L$ , the distribution  $\mu_{\{\ell\} \times \mathbb{R}^n}$  is either equivalent to the discrete measure on some finite definable support  $D_\ell$ , or equivalent to the Lebesgue measure on a definable support  $D_\ell$ ;
- for  $e \in E$ ,  $\mathbf{v} \in \mathbb{R}^n$ , we require that  $\mathcal{R}_e(\mathbf{v})$  is either finite or has non-zero Lebesgue measure  $\lambda_n$ ;  $\eta_e(\mathbf{v})$  is respectively either equivalent to the discrete measure on  $\mathcal{R}_e(\mathbf{v})$  or equivalent to the Lebesgue measure on  $\mathcal{R}_e(\mathbf{v})$ .

The first requirement, about the distribution on the time delays, is a standard assumption in the case of stochastic timed automata [7,11,12]. Notice that as  $I(s) \subseteq \mathbb{R}^+$  is definable in an o-minimal structure, it is infinite if and only if it contains an interval with non-empty interior, hence if and only if  $\lambda_1(I(s)) > 0$  (by Lemma 36). We formulate a similar requirement on the initial distribution and on the reset distributions, and we restrict their support to be either finite or to have non-zero Lebesgue measure. This postulate is quite natural and easily satisfied: for instance, exponential distributions (resp. uniform distributions  $\mathcal{U}(a, b)$ ) are equivalent to the Lebesgue measure on  $\mathbb{R}^+$  (resp.  $[a, b]$ ).

If a distribution  $\nu$  is discrete with finite definable support  $T$ , for all definable sets  $D$ , we can express that  $\nu(D) > 0$  as a first-order formula  $(\exists s \in T, s \in D)$ . If  $\nu$  is equivalent to the Lebesgue measure on a definable set  $T$  with  $\nu(T) = 1$ , for all definable sets  $D \subseteq \mathbb{R}^n$ ,

$$\begin{aligned} \nu(D) > 0 &\iff \nu(T \cap D) > 0 \\ &\iff \lambda_n(T \cap D) > 0 \\ &\iff (T \cap D)^\circ \neq \emptyset \quad (\text{by Lemma 36}) \\ &\iff \exists \mathbf{x} \in T \cap D, \exists r > 0 \wedge (\forall \mathbf{y}, \|\mathbf{x} - \mathbf{y}\| < r \implies \mathbf{y} \in T \cap D), \end{aligned}$$

where  $\|\mathbf{z}\| = \sum_{i=1}^n |z_i|$  is a definable function, as we have assumed that  $+$  is in  $\mathcal{M}$ . The same reasoning can be applied to distinguish whether the supports of the distributions are finite or have positive Lebesgue measure.

*Remark 38.* We could also consider distributions that are a linear combination of both a discrete distribution and a distribution equivalent to the Lebesgue measure. This would require for each occurring distribution to have distinct first-order formulae to define the finite support and the continuous support. We choose to omit this generalization in order not to complicate the notations.

Thanks to these hypotheses, we show that we obtain **(H3)**. Let  $e = (\ell, a, \ell') \in E$  be an edge, and  $D \subseteq \mathbb{R}^n$  be a definable set. The set of states in  $\{\ell\} \times \mathbb{R}^n$  that can reach  $D$  through  $e$  *without delay* with a positive probability is given by

$$D' = \{\ell\} \times \{\mathbf{v}' \in \mathbb{R}^n \mid \mathbf{v}' \in \mathcal{G}(e), \eta_e(\mathbf{v}')(D \cap \mathcal{R}_e(\mathbf{v}')) > 0\}.$$

Therefore, we have that

$$\text{Pre}^{\mathcal{T}_H}(\{\ell'\} \times D) = \bigcup_{(\ell, a, \ell') \in E} \{\ell\} \times \{\mathbf{v} \in \mathbb{R}^n \mid \mu_{(\ell, \mathbf{v})}(\gamma_\ell(\mathbf{v}, \cdot)^{-1}(D')) > 0\}$$

is definable. By Proposition 34, we conclude that the qualitative reachability problem is decidable for stochastic o-minimal hybrid systems satisfying **(†)**. We summarize these ideas in the next proposition.

**Proposition 39.** *Let  $\mathcal{H}$  be a stochastic o-minimal hybrid system defined in a structure whose theory is decidable. Let  $B \in \Sigma_{\mathcal{H}}$  be a definable set and  $\mu \in \text{Dist}(S_{\mathcal{H}})$  be an initial distribution. We assume that assumption **(†)** holds. Then one can decide whether  $\text{Prob}_\mu^{\mathcal{T}_H}(\mathbf{F} B) = 1$  and whether  $\text{Prob}_\mu^{\mathcal{T}_H}(\mathbf{F} B) = 0$ .*

In particular, we can decide the qualitative reachability problems for SHSs defined in  $\mathbb{R}_{\text{alg}}$  and satisfying **(†)**. Assuming Schanuel's conjecture [34], we could extend this result to  $\mathbb{R}_{\text{exp}}$ .

## 5.2 Approximate Quantitative Reachability Analysis

In this section, we show under strengthened numerical hypotheses that we can solve the quantitative reachability problem in cycle-reset SHSs. Let  $\mathcal{H}$  be a cycle-reset SHS,  $B \in \Sigma_{\mathcal{H}}$  and  $\mu \in \text{Dist}(S_{\mathcal{H}})$ . Our goal is to apply the approximation scheme described in Section 2.2 in order to approximate  $\text{Prob}_\mu^{\mathcal{T}_H}(\mathbf{F} B)$ . To do so, we remind that we require that  $\mathcal{T}_H$  is decisive w.r.t.  $B$ , which is implied by the cycle-reset hypothesis (Proposition 30), and the ability to compute for all  $m \in \mathbb{N}$ , an arbitrarily close approximation of

$$p_m^{\text{Yes}} = \text{Prob}_\mu^{\mathcal{T}}(\mathbf{F}_{\leq m} B) \text{ and } p_m^{\text{No}} = \text{Prob}_\mu^{\mathcal{T}}(B^c \mathbf{U}_{\leq m} \tilde{B}).$$

Notice that

$$\begin{aligned} p_m^{\text{Yes}} &= \sum_{j=0}^m \text{Prob}_\mu^{\mathcal{T}_H}(\text{Cyl}(\underbrace{B^c, \dots, B^c}_j \text{ times}, B)) \\ &= \sum_{j=0}^m \sum_{(\ell_0, \dots, \ell_j) \in L^{j+1}} \text{Prob}_\mu^{\mathcal{T}_H}(\text{Cyl}(\ell_0 \cap B^c, \dots, \ell_{j-1} \cap B^c, \ell_j \cap B)), \end{aligned}$$

where we write  $\ell_i$  as a shorthand for  $\{\ell_i\} \times \mathbb{R}^n$ . To compute  $p_m^{\text{Yes}}$ , it is thus sufficient to be able to compute, for every  $0 \leq j \leq m$  and for every path  $(\ell_0, \dots, \ell_j)$  of the graph  $(L, E)$ , the probability

$$\text{Prob}_{\mu}^{\mathcal{T}_{\mathcal{H}}}(\text{Cyl}(\ell_0 \cap B^c, \dots, \ell_{j-1} \cap B^c, \ell_j \cap B)).$$

Using the cycle-reset hypothesis, we show that we can express this probability as the product of probabilities of paths with bounded length  $b \in \mathbb{N}$ , where  $b$  is the length of the longest path without encountering a strong reset. If  $j$  is greater than  $b$ , we know that there is necessarily a smallest index  $i \leq b$  for which all edges  $(\ell_i, a, \ell_{i+1})$  are strongly reset. We have

$$\begin{aligned} \text{Prob}_{\mu}^{\mathcal{T}_{\mathcal{H}}}(\text{Cyl}(\ell_0 \cap B^c, \dots, \ell_{j-1} \cap B^c, \ell_j \cap B)) \\ = \text{Prob}_{\mu}^{\mathcal{T}_{\mathcal{H}}}(\text{Cyl}(\ell_0 \cap B^c, \dots, \ell_i \cap B^c)) \cdot \text{Prob}_{\mu_i}^{\mathcal{T}_{\mathcal{H}}}(\text{Cyl}(\ell_i \cap B^c, \dots, \ell_{j-1} \cap B^c, \ell_j \cap B)), \end{aligned}$$

where  $\mu_i = \mu_{\ell_0 \cap B^c, \dots, \ell_i \cap B^c}$ , using Lemma 5. As there may be multiple strongly reset edges between  $\ell_i$  and  $\ell_{i+1}$ , we can rewrite the second factor as

$$\sum_{e=(\ell_i, a, \ell_{i+1})} p_e(\mu_i) \cdot \text{Prob}_{\eta_e^*}^{\mathcal{T}_{\mathcal{H}}}(\text{Cyl}(\ell_{i+1} \cap B^c, \dots, \ell_{j-1} \cap B^c, \ell_j \cap B)),$$

where

$$p_e(\mu_i) = \int_{s \in S_{\mathcal{H}}} \int_{\tau \in \mathbb{R}^+} \theta_{s+\tau}(e) d\mu_s(\tau) d\mu_i(s)$$

is the probability to take edge  $e$  in one step from distribution  $\mu_i$ . We can then iterate this reasoning for each probability  $\text{Prob}_{\eta_e^*}^{\mathcal{T}_{\mathcal{H}}}(\text{Cyl}(\ell_{i+1} \cap B^c, \dots, \ell_{j-1} \cap B^c, \ell_j \cap B))$ ; notice that this value does not depend on the initial distribution. A similar formula can be obtained to compute  $p_m^{\text{No}}$ , by replacing the final occurrence of  $B$  by  $\tilde{B}$ .

We write

$$L^{\text{NoSR}} = \{(\ell_0, \dots, \ell_j) \in L^{j+1} \mid j \in \mathbb{N}, \forall 0 \leq i \leq j-1, \exists e = (\ell_i, a, \ell_{i+1}) \in E \text{ such that } e \text{ is non-strongly reset}\}$$

for the set of paths of the underlying graph of  $\mathcal{H}$  such that strong resets can be avoided. This set is finite by the cycle-reset hypothesis. To approximate  $\text{Prob}_{\mu}^{\mathcal{T}_{\mathcal{H}}}(\mathbf{F} B)$ , it is sufficient to be able to approximate arbitrarily closely

- for all paths  $(\ell_0, \dots, \ell_j) \in L^{\text{NoSR}}$ , distributions  $\nu \in \{\mu\} \cup \{\eta_{e'}^* \mid e' \in E \text{ strongly reset}\}$ , the probability

$$\text{Prob}_{\nu}^{\mathcal{T}_{\mathcal{H}}}(\text{Cyl}(\ell_0 \cap B^c, \dots, \ell_{j-1} \cap B^c, \ell_j \cap B))$$

and the same probability replacing the final occurrence of  $B$  by  $\tilde{B}$ ;

- for all edges  $e = (\ell, a, \ell') \in E$  strongly reset, paths  $(\ell_0, \dots, \ell_{j-1}, \ell) \in L^{\text{NoSR}}$ , and distributions  $\nu \in \{\mu\} \cup \{\eta_{e'}^* \mid e' \in E \text{ strongly reset}\}$ , the probability

$$p_e(\nu_{\ell_0 \cap B^c, \dots, \ell_{j-1} \cap B^c, \ell \cap B^c}).$$

Without strong resets (but still decisiveness w.r.t.  $B$ ), a similar scheme may work. However, this would require a way to compute probabilities involving  $\mu_{C_0, \dots, C_n}$  with arbitrarily long sequences  $(C_i)_{0 \leq i \leq n}$ . With strong resets, we can compute a finite number of probabilities and assemble them to compute  $p_m^{\text{Yes}}$  and  $p_m^{\text{No}}$  for arbitrarily large values of  $m$ .

*Remark 40.* It is undecidable in general to decide whether some elements of the state space are in  $\tilde{B}$  (by-product of Proposition 25). However, the definition of  $\tilde{B}$  is given by a qualitative reachability property: using the work done in the previous section, and under the hypotheses of Proposition 34, we can obtain a first-order formula defining  $\tilde{B}$ . These hypotheses can thus help compute the probabilities involving  $\tilde{B}$  required for the approximate quantitative problem.

## 6 Conclusion

**Summary.** This article presented in Section 2 how to solve reachability problems in stochastic transition systems (STSs) via the *decisiveness* notion, introduced in [3,10]. We notably solved in Lemma 10 a question that was left open in [10] about the almost-sure reachability of a set of states in the presence of an attractor. This allowed formulating a general sufficient condition for decisiveness in Proposition 11, which encompasses known decisiveness criteria from the literature.

From Section 3 onward, we focused our attention on *hybrid* models. We considered a stochastic extension of the classical *hybrid systems*, called *stochastic hybrid systems* (SHSs), and gave them a semantics as STSs in order to apply the theory developed in Section 2. We showed that the qualitative and quantitative reachability problems are undecidable even for reasonably well-behaved SHSs. This result is not surprising, as reachability problems are already undecidable for simple classes of non-deterministic hybrid systems [22,23]. We then showed in Section 4 that SHSs with one *strong reset* per cycle (*cycle-reset*) are decidable with respect to any measurable set, using our new decisiveness criterion, and admit a finite abstraction.

We identified in Section 5.1 reasonable assumptions leading to the effective computability of this abstraction. These assumptions pertain to the definability of the different components of the SHSs (resets, guards, invariants, dynamics, specific properties of distributions) in a mathematical structure, and the decidability of first-order formulae in this structure. Combined with the decisiveness results from Section 4, the finite abstraction can be used to decide qualitative reachability problems. We proved that *o-minimal SHSs*, which are SHSs defined in an *o-minimal* structure, satisfy these hypotheses. When the theory of the *o-minimal* structure is decidable (which is for instance the case of  $\mathbb{R}_{\text{alg}} = \langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle$ ), the nice properties of the definable sets allow deciding for a large class of measures if definable sets have positive measure. This is sufficient to compute the finite abstraction, which can then be used to decide qualitative reachability problems. We ended the article (Section 5.2) with sufficient numerical assumptions to solve approximate quantitative reachability problems in *cycle-reset* SHSs.

**Possible extensions and future work.** We identify some possible applications and extensions of our results.

A first direction of study is to find other classes of decisive stochastic systems that can be encompassed by our decisiveness criterion (Proposition 11). In that respect, a good candidate is the class of *stochastic regenerative Petri Nets* [26,36]. An application of decisiveness results to *stochastic Petri nets* was briefly discussed in [10, Section 8.3], but under severe constraints; we may be able to relax part of these constraints with the generalized criterion.

In [10, Sections 6 & 7], the authors show that we can reduce the verification of a large class of properties ( *$\omega$ -regular properties*) of STSs to the verification of reachability properties, under decisiveness assumptions. This generalization should be transferable to our work with stochastic hybrid systems.

In Section 5, we circumvent the issue of the definability of measures and their integrals by using a specific property of the *o-minimal* structures (namely, that the Lebesgue measure of a definable set is positive if and only if the interior of that set is non-empty, which is definable as a first-order formula). However, more powerful results exist about the compatibility of *o-minimal* structures and measure theory [28]. Some *o-minimal* structures are closed under integration with respect to a given measure (then called *tame* measure). This consideration may help extend our results about stochastic *o-minimal* hybrid systems to a larger class that is less restrictive with respect to probability distributions. It could also help for the quantitative problem, as approximations of probabilities may be definable.

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## A Expressing Properties of Runs

To express properties of runs of an STS  $\mathcal{T} = (S, \Sigma, \kappa)$ , we use a notation very similar to LTL for transition systems [38], with each formula characterizing a measurable set of runs. We define a language  $\mathcal{L}_{S, \Sigma}$  of formulae. Its syntax is defined by the following grammar:

$$\phi ::= B \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid \neg \phi_1 \mid \phi_1 \mathbf{U}_{\bowtie k} \phi_2,$$

where  $B \in \Sigma$ ,  $\phi_1, \phi_2 \in \mathcal{L}_{S, \Sigma}$ ,  $\bowtie \in \{\leq, \geq, =\}$ , and  $k \in \mathbb{N}$ . Let  $\rho = s_0 s_1 s_2 \dots$  be an run of  $\mathcal{T}$ . We denote by  $\rho_{\geq i} = s_i s_{i+1} s_{i+2} \dots$  the run starting at the  $i^{\text{th}}$  step of  $\rho$ . For each kind of formula  $\phi$ , we define when  $\rho$  satisfies formula  $\phi$ , denoted by  $\rho \models \phi$ :

$$\begin{aligned} \rho \models B & \iff s_0 \in B, \\ \rho \models \phi_1 \vee \phi_2 & \iff \rho \models \phi_1 \text{ or } \rho \models \phi_2, \\ \rho \models \phi_1 \wedge \phi_2 & \iff \rho \models \phi_1 \text{ and } \rho \models \phi_2, \\ \rho \models \neg \phi_1 & \iff \rho \not\models \phi_1, \\ \rho \models \phi_1 \mathbf{U}_{\bowtie k} \phi_2 & \iff \exists i \in \mathbb{N}, i \bowtie k, \text{ s.t. } \forall 0 \leq j < i, \rho_{\geq j} \models \phi_1 \text{ and } \rho_{\geq i} \models \phi_2. \end{aligned}$$

For  $\phi \in \mathcal{L}_{S, \Sigma}$  a formula, we write  $\text{Ev}_{\mathcal{T}}(\phi) = \{\rho \in \text{Runs}(\mathcal{T}) \mid \rho \models \phi\}$  for the set of runs of  $\mathcal{T}$  satisfying  $\phi$ . A standard result states that for any formula  $\phi \in \mathcal{L}_{S, \Sigma}$ , the set of runs  $\text{Ev}_{\mathcal{T}}(\phi)$  is measurable.

For an initial distribution  $\mu$ , we therefore have that  $\text{Prob}_{\mu}^{\mathcal{T}}(\text{Ev}_{\mathcal{T}}(\phi))$  is well-defined. In the article, we write  $\text{Prob}_{\mu}^{\mathcal{T}}(\phi)$  instead of  $\text{Prob}_{\mu}^{\mathcal{T}}(\text{Ev}_{\mathcal{T}}(\phi))$  for brevity. As is standard, we also denote  $\phi_1 \mathbf{U} \phi_2$  for  $\phi_1 \mathbf{U}_{\geq 0} \phi_2$ ,  $\mathbf{F}_{\leq n} \phi$  for  $S \mathbf{U}_{\leq n} \phi$ ,  $\mathbf{F} \phi$  for  $S \mathbf{U}_{\geq 0} \phi$ ,  $\mathbf{G} \phi$  for  $\neg \mathbf{F} \neg \phi$ ,  $\mathbf{X} \phi$  for  $S \mathbf{U}_{=1} \phi$ , and  $\mathbf{X}^n \phi$  for  $S \mathbf{U}_{=n} \phi$ .

## B Technical Results and Proofs of Section 2.3

We now restate and prove the technical lemma from Section 2.

**Lemma 10.** *Let  $B \in \Sigma$ , and  $\mathcal{A} = (A_i)_{0 \leq i \leq n} \in \Sigma^{n+1}$ . Suppose that there is  $p > 0$  such that for all  $\nu \in \text{Dist}(S)$  with  $\nu_{\mathcal{A}}$  well-defined, we have  $\text{Prob}_{\nu_{\mathcal{A}}}^{\mathcal{T}}(\mathbf{F} B) \geq p$ . Then for any  $\mu \in \text{Dist}(S)$ ,*

$$\text{Prob}_{\mu}^{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} \phi_{\mathcal{A}}) = 0.$$

*Proof.* We prove the lemma for  $n = 0$  (with  $\mathcal{A} = A \in \Sigma$ ), and explain afterwards how to extend the proof to obtain the general statement. We want to prove that for all  $\mu \in \text{Dist}(S)$ ,

$$\text{Prob}_{\mu}^{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A) = 0.$$

Let  $\mu \in \text{Dist}(S)$ . We assume w.l.o.g. that  $A$  is a subset of  $B^c$ . If not, we notice that

$$\text{Ev}_{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A) = \text{Ev}_{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} (A \cap B^c))$$

and we simply replace  $A$  by  $A \cap B^c$  in the rest of the proof.

We introduce some useful notations. For a set  $C \in \Sigma$  and  $j \in \mathbb{N}$ , we write  $C_{[j]}$  for the finite sequence  $C, \dots, C$  where  $C$  occurs exactly  $j$  times. Observe that

$$\text{Ev}_{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A) = \bigcap_{n \in \mathbb{N}} \bigcup_{(j_0, j_1, \dots, j_n) \in \mathbb{N}^{n+1}} \text{Cyl}(B_{[j_0]}^c, A, B_{[j_1]}^c, A, \dots, B_{[j_{n-1}]}^c, A, B_{[j_n]}^c).$$

Notice that for any  $\nu \in \text{Dist}(S)$  and any sequence  $(C_i)_{0 \leq i \leq n} \in \Sigma^{n+1}$ ,  $\nu_{C_0, \dots, C_n, A}$  (defined as the conditional distribution of Definition 4) is a distribution on  $A$ , so we know by hypothesis that  $\text{Prob}_{\nu_{C_0, \dots, C_n, A}}^{\mathcal{T}}(\mathbf{F} B) \geq p$ . Hence there exists a smallest integer  $k_{C_0, \dots, C_n, A}^{\nu} \in \mathbb{N}$  such that

$$\text{Prob}_{\nu_{C_0, \dots, C_n, A}}^{\mathcal{T}}(\mathbf{F}_{\leq k_{C_0, \dots, C_n, A}^{\nu}} B) \geq \frac{p}{2}.$$

By taking the complement, we obtain

$$\text{Prob}_{\nu_{C_0, \dots, C_n, A}}^{\mathcal{T}}(\text{Cyl}(B_{[k_{C_0, \dots, C_n, A}^\nu]}^c)) = \text{Prob}_{\nu_{C_0, \dots, C_n, A}}^{\mathcal{T}}(\text{Cyl}(A, B_{[k_{C_0, \dots, C_n, A}^\nu]}^c)) \leq 1 - \frac{p}{2},$$

where the first equality holds because  $\nu_{C_0, \dots, C_n, A}(A) = 1$  and  $A$  is a subset of  $B^c$ .

We write  $\mathbb{N}^+ = \bigcup_{n \geq 1} \mathbb{N}^n$ . We now define, for each  $\nu \in \text{Dist}(S)$  and  $w \in \mathbb{N}^+$ , a sequence of sets  $\mathcal{B}_\nu(w)$  by induction on the length of  $w$ . The base case, for  $w = (j_0)$ , is given by

$$\mathcal{B}_\nu(j_0) = (B^c \setminus A)_{[j_0]}, A.$$

The induction case, for  $w = (j_0, \dots, j_n)$  with  $n \geq 1$ , is

$$\begin{aligned} \mathcal{B}_\nu(j_0, \dots, j_n) &= \mathcal{B}_\nu(j_0, \dots, j_{n-1}), B_{[k_{\mathcal{B}_\nu(j_0, \dots, j_{n-1})}^\nu]}^c, (B^c \setminus A)_{[j_n]}, A \\ &= (B^c \setminus A)_{[j_0]}, A, B_{[k_{\mathcal{B}_\nu(j_0)}^\nu]}^c, (B^c \setminus A)_{[j_1]}, \dots, A, B_{[k_{\mathcal{B}_\nu(j_0, \dots, j_{n-1})}^\nu]}^c, (B^c \setminus A)_{[j_n]}, A. \end{aligned}$$

We have in particular that

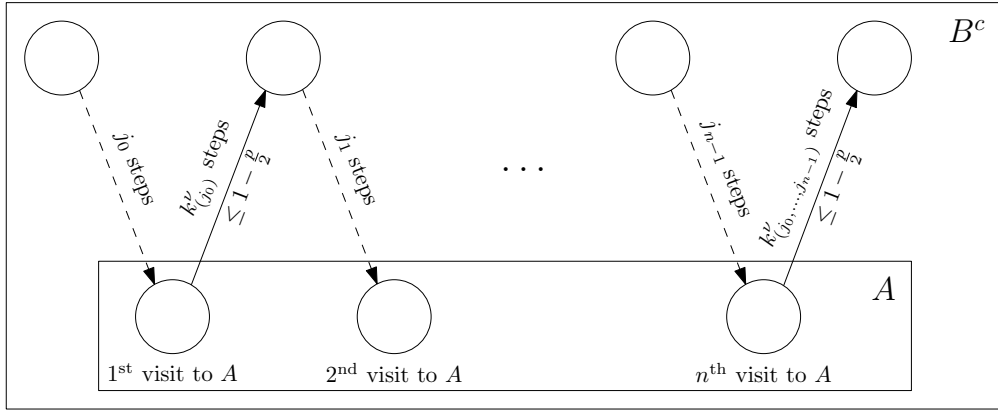
$$\text{Prob}_{\nu_{\mathcal{B}_\nu(w)}}^{\mathcal{T}}(\text{Cyl}(A, B_{[k_{\mathcal{B}_\nu(w)}^\nu]}^c)) \leq 1 - \frac{p}{2}. \quad (1)$$

We assume that the conditional distributions  $\nu_{\mathcal{B}_\nu(w)}$  are well-defined; if some of them are not, this means that the probability of the corresponding cylinders  $\text{Cyl}(\mathcal{B}_\nu(w))$  is zero from  $\nu$ , and this does not affect the following results. In what follows, we denote for convenience  $k_{(j_0, \dots, j_n)}^\nu = k_{\mathcal{B}_\nu(j_0, \dots, j_n)}^\nu$ .

For  $n \geq 1$ , we define the following measurable set of runs:

$$\mathcal{C}_\nu(n) = \bigcup_{(j_0, j_1, \dots, j_{n-1}) \in \mathbb{N}^n} \text{Cyl}(\mathcal{B}_\nu(j_0, \dots, j_{n-1}), B_{[k_{(j_0, \dots, j_{n-1})}^\nu]}^c).$$

The idea is that we ensure that enough sets  $B^c$  are inserted between each forced passage through  $A$  to guarantee the probability upper bound given by Equation (1). We then stay in  $B^c \setminus A$  until  $A$  is reached again. For  $(j_0, \dots, j_{n-1})$  fixed, we depict such a cylinder and where we obtain probability bounds on Figure 4. As all runs of  $\text{Ev}_{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A)$  are always in  $B^c$ , the large rectangle represents this set, while the small one represents  $A \subseteq B^c$ . The runs consist of at least  $n$  forced visits to  $A$ , the  $m^{\text{th}}$  of these visits being followed by at least  $k_{(j_0, \dots, j_{m-1})}^\nu$  steps in  $B^c$ . The dashed arrows represent a number of steps in  $B^c \setminus A$  that can be arbitrarily large (but bounded). The non-dashed arrows represent a number of steps in  $B^c$  that is fixed by the beginning of the cylinder.



**Fig. 4.** Representation of the cylinder  $\text{Cyl}(\mathcal{B}_\nu(j_0, \dots, j_{n-1}), B_{[k_{(j_0, \dots, j_{n-1})}^\nu]}^c) \subseteq \mathcal{C}_\nu(n)$ .

The proof is split in two claims.

*Claim.* For all  $\nu \in \text{Dist}(S)$ , for all  $n \geq 1$ ,  $\text{Ev}_{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A) \subseteq \mathcal{C}_{\nu}(n)$ .

*Proof.* Let  $\nu \in \text{Dist}(S)$ . Let  $\rho \in \text{Ev}_{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A)$ ; we want to show that for all  $n \geq 1$ ,  $\rho \in \mathcal{C}_{\nu}(n)$ . We proceed by induction on  $n$ .

First, we consider the case  $n = 1$ . We know that  $\rho$  will eventually go through  $A$ . Assume the first visit of  $\rho$  in  $A$  happens at step  $j_0$ . Then clearly,

$$\rho \in \text{Cyl}((B^c \setminus A)_{[j_0]}, A).$$

As  $\rho \in \text{Ev}_{\mathcal{T}}(\mathbf{G} B^c)$ , we can append arbitrarily many  $B^c$ 's to that cylinder while keeping  $\rho$  as its element. In particular,

$$\rho \in \text{Cyl}((B^c \setminus A)_{[j_0]}, A, B_{[k_{(j_0)}^{\nu}]}^c) \subseteq \mathcal{C}_{\nu}(1).$$

Now, assume that  $\rho \in \mathcal{C}_{\nu}(n)$  for some  $n$ , i.e., there exists  $(j_0, \dots, j_{n-1}) \in \mathbb{N}^n$  such that

$$\rho \in \text{Cyl}(\mathcal{B}_{\nu}(j_0, \dots, j_{n-1}), B_{[k_{(j_0, \dots, j_{n-1})}^{\nu}]}^c).$$

We show that  $\rho \in \mathcal{C}_{\nu}(n+1)$ . The argument is similar to the one for the base case: since  $\rho \in \text{Ev}_{\mathcal{T}}(\mathbf{G} \mathbf{F} A)$ , we know that  $\rho$  will go through  $A$  again. Assume the next visit of  $\rho$  in  $A$  happens after  $j_n + 1$  more steps. Therefore,

$$\rho \in \text{Cyl}(\mathcal{B}_{\nu}(j_0, \dots, j_{n-1}), B_{[k_{(j_0, \dots, j_{n-1})}^{\nu}]}^c, (B^c \setminus A)_{[j_n]}, A) = \text{Cyl}(\mathcal{B}_{\nu}(j_0, \dots, j_n)).$$

As  $\rho \in \text{Ev}_{\mathcal{T}}(\mathbf{G} B^c)$ , clearly

$$\rho \in \text{Cyl}(\mathcal{B}_{\nu}(j_0, \dots, j_n), B_{[k_{(j_0, \dots, j_n)}^{\nu}]}^c) \subseteq \mathcal{C}_{\nu}(n+1).$$

□

Thanks to the previous claim, we can bound  $\text{Prob}_{\mu}^{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A)$  by

$$\text{Prob}_{\mu}^{\mathcal{T}}(\mathcal{C}_{\mu}(n))$$

for any  $n$ . The following claim will prove that  $\lim_{n \rightarrow \infty} \text{Prob}_{\mu}^{\mathcal{T}}(\mathcal{C}_{\mu}(n)) = 0$ , thereby showing that  $\text{Prob}_{\mu}^{\mathcal{T}}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A) = 0$  and ending the proof.

*Claim.* For each  $n \geq 1$  and for each  $\nu \in \text{Dist}(S)$ ,

$$\text{Prob}_{\nu}^{\mathcal{T}}(\mathcal{C}_{\nu}(n)) \leq \left(1 - \frac{p}{2}\right)^n.$$

*Proof.* We proceed by induction on  $n$ . First, we fix  $n = 1$  and  $\nu \in \text{Dist}(S)$ . It corresponds to the two first arrows on Figure 4. We show that

$$\text{Prob}_{\nu}^{\mathcal{T}}\left(\bigcup_{j_0 \in \mathbb{N}} \text{Cyl}(\mathcal{B}_{\nu}(j_0), B_{[k_{(j_0)}^{\nu}]}^c)\right) \leq 1 - \frac{p}{2}.$$

This union of cylinders over  $j_0 \in \mathbb{N}$  is disjoint: two runs in two different cylinders cannot be equal since they do not reach  $A$  for the first time at the same step.

Using Lemma 5 to split the first two arrows of Figure 4, we obtain that

$$\begin{aligned} \text{Prob}_{\nu}^{\mathcal{T}}\left(\biguplus_{j_0 \in \mathbb{N}} \text{Cyl}(\mathcal{B}_{\nu}(j_0), B_{[k_{(j_0)}^{\nu}]}^c)\right) &= \sum_{j_0 \in \mathbb{N}} \text{Prob}_{\nu}^{\mathcal{T}}(\text{Cyl}((B^c \setminus A)_{[j_0]}, A, B_{[k_{(j_0)}^{\nu}]}^c)) \\ &= \sum_{j_0 \in \mathbb{N}} \text{Prob}_{\nu}^{\mathcal{T}}(\text{Cyl}((B^c \setminus A)_{[j_0]}, A)) \cdot \underbrace{\text{Prob}_{\nu_{\mathcal{B}_{\nu}(j_0)}}^{\mathcal{T}}(\text{Cyl}(A, B_{[k_{(j_0)}^{\nu}]}^c))}_{\leq 1 - \frac{p}{2} \text{ by (1)}} \\ &\leq \underbrace{\left(1 - \frac{p}{2}\right)}_{\leq 1 \text{ since the union is disjoint}} \cdot \text{Prob}_{\nu}^{\mathcal{T}}\left(\biguplus_{j_0 \in \mathbb{N}} \text{Cyl}((B^c \setminus A)_{[j_0]}, A)\right) \leq 1 - \frac{p}{2}, \end{aligned}$$

which proves the case  $n = 1$ .

Now fix  $n \geq 1$  and assume that for each  $\nu \in \text{Dist}(S)$ ,

$$\text{Prob}_\nu^\mathcal{T} \left( \bigcup_{(j_0, \dots, j_{n-1}) \in \mathbb{N}^n} \text{Cyl}(\mathcal{B}_\nu(j_0, \dots, j_{n-1}), B_{[k_{(j_0, \dots, j_{n-1})}^\nu]}^c) \right) \leq \left(1 - \frac{p}{2}\right)^n \quad (2)$$

holds. We want to show that (2) is still satisfied for  $n + 1$ , i.e., that

$$\text{Prob}_\nu^\mathcal{T} \left( \bigcup_{(j_0, \dots, j_n) \in \mathbb{N}^{n+1}} \text{Cyl}(\mathcal{B}_\nu(j_0, \dots, j_n), B_{[k_{(j_0, \dots, j_n)}^\nu]}^c) \right) \leq \left(1 - \frac{p}{2}\right)^{n+1}.$$

We again decompose the scheme of Figure 4 according to the length of the first arrow. The proof is similar to the case  $n = 1$  as once you go through the first two arrows on the figure, the induction hypothesis can be applied. What happens beforehand is the same behavior as for  $n = 1$ . The first union over  $j_0 \in \mathbb{N}$  is again disjoint for the same reason. Therefore, using Lemma 5, it holds that

$$\begin{aligned} \text{Prob}_\nu^\mathcal{T} \left( \biguplus_{j_0 \in \mathbb{N}} \bigcup_{(j_1, \dots, j_n) \in \mathbb{N}^n} \text{Cyl}(\mathcal{B}_\nu(j_0, \dots, j_n), B_{[k_{(j_0, \dots, j_n)}^\nu]}^c) \right) = \\ \sum_{j_0 \in \mathbb{N}} \alpha_{j_0} \cdot \text{Prob}_{\nu_{j_0}^*}^\mathcal{T} \left( \bigcup_{(j_1, \dots, j_n) \in \mathbb{N}^n} \text{Cyl}((B^c \setminus A)_{[j_1]}, A, B_{[k_{(j_0, j_1)}^\nu]}^c, \dots, (B^c \setminus A)_{[j_n]}, A, B_{[k_{(j_0, \dots, j_n)}^\nu]}^c) \right) \end{aligned} \quad (3)$$

where for each  $j_0 \in \mathbb{N}$ ,

$$\alpha_{j_0} = \text{Prob}_\nu^\mathcal{T}(\text{Cyl}((B^c \setminus A)_{[j_0]}, A, B_{[k_{(j_0)}^\nu]}^c)) \quad \text{and} \quad \nu_{j_0}^* = \Omega_{\mathcal{T}}(\nu_{\mathcal{B}_\nu(j_0), B_{[k_{(j_0)}^\nu]}^c}).$$

The key element to apply the induction hypothesis is to notice that for all  $1 \leq m \leq n$ ,

$$k_{(j_0, j_1, \dots, j_m)}^\nu = k_{(j_1, \dots, j_m)}^{\nu_{j_0}^*}.$$

Therefore,

$$\begin{aligned} \text{Prob}_{\nu_{j_0}^*}^\mathcal{T} \left( \bigcup_{(j_1, \dots, j_n) \in \mathbb{N}^n} \text{Cyl}((B^c \setminus A)_{[j_1]}, A, B_{[k_{(j_0, j_1)}^\nu]}^c, \dots, (B^c \setminus A)_{[j_n]}, A, B_{[k_{(j_0, \dots, j_n)}^\nu]}^c) \right) \\ = \text{Prob}_{\nu_{j_0}^*}^\mathcal{T} \left( \bigcup_{(j_1, \dots, j_n) \in \mathbb{N}^n} \text{Cyl}((B^c \setminus A)_{[j_1]}, A, B_{[k_{(j_1)}^{\nu_{j_0}^*}]}^c, \dots, (B^c \setminus A)_{[j_n]}, A, B_{[k_{(j_1, \dots, j_n)}^{\nu_{j_0}^*}]}^c) \right) \\ = \text{Prob}_{\nu_{j_0}^*}^\mathcal{T} \left( \bigcup_{(j_1, \dots, j_n) \in \mathbb{N}^n} \text{Cyl}(\mathcal{B}_{\nu_{j_0}^*}(j_1, \dots, j_n), B_{[k_{(j_1, \dots, j_n)}^{\nu_{j_0}^*}]}^c) \right) \\ \leq \left(1 - \frac{p}{2}\right)^n \end{aligned}$$

where the last inequality is given by the induction hypothesis (2). Also, we have proved in the case  $n = 1$  that  $\sum_{j_0 \in \mathbb{N}} \alpha_{j_0} \leq 1 - \frac{p}{2}$ . We obtain that expression (3) is indeed less than  $(1 - \frac{p}{2})^{n+1}$ , as required.  $\square$

By combining the two claims, we conclude by taking the limit over  $n$  that

$$\text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A) \leq \lim_{n \rightarrow \infty} \text{Prob}_\mu^\mathcal{T}(\mathcal{C}_\mu(n)) \leq \lim_{n \rightarrow \infty} \left(1 - \frac{p}{2}\right)^n = 0,$$

which settles that  $\text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} A) = 0$ .

To prove the statement for  $\mathcal{A} = (A_i)_{0 \leq i \leq n}$ , we first assume w.l.o.g. that every  $A_i$  is a subset of  $B^c$ . We then replace the occurrences of  $A$  by  $\mathcal{A} = A_0, \dots, A_n$  (in cylinders) or  $\phi_{\mathcal{A}}$  (in LTL formulae). The expression  $(B^c \setminus \mathcal{A})_{[j]}$  then refers to all the states in  $B^c$  before the first passage through  $A_0, \dots, A_n$  in order.  $\square$

Using this lemma, we can prove our new decisiveness criterion.

**Proposition 11 (Decisiveness criterion).** *Let  $B \in \Sigma$  and  $m \in \mathbb{N} \cup \{\infty\}$ . For every  $0 \leq j < m$ , let  $n_j \in \mathbb{N}$  and  $\mathcal{A}_j = (A_i^{(j)})_{0 \leq i \leq n_j} \in \Sigma^{n_j+1}$ . We assume that for all  $\nu \in \text{Dist}(S)$ ,*

$$\text{Prob}_\nu^\mathcal{T} \left( \bigvee_{0 \leq j < m} \mathbf{G} \mathbf{F} \phi_{\mathcal{A}_j} \right) = 1.$$

*Assume that there exists  $p > 0$  such that for all  $0 \leq j < m$ , for all  $\nu \in \text{Dist}(S)$  such that  $\nu_{\mathcal{A}_j}$  is well-defined, either  $\nu_{\mathcal{A}_j}(\tilde{B}) = 1$  or*

$$\text{Prob}_{(\nu_{\mathcal{A}_j})_{(\tilde{B})^c}}^\mathcal{T}(\mathbf{F} B) \geq p.$$

*Then  $\mathcal{T}$  is decisive w.r.t.  $B$ .*

*Proof.* We want to prove that  $\mathcal{T}$  is decisive w.r.t.  $B$ , i.e., that for all  $\mu \in \text{Dist}(S)$ ,  $\text{Prob}_\mu^\mathcal{T}(\mathbf{F} B \vee \mathbf{F} \tilde{B}) = 1$ . Let  $\mu \in \text{Dist}(S)$ . We show equivalently that

$$\text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G}(\tilde{B})^c) = 0.$$

We write  $\mathcal{A}'_j$  for the sequence  $A_0^{(j)}, \dots, A_{n_j-1}^{(j)}, A_{n_j}^{(j)} \cap (\tilde{B})^c$ . We have that

$$\begin{aligned} \text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G}(\tilde{B})^c) &= \text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G}(\tilde{B})^c \wedge \bigvee_{0 \leq j < m} \mathbf{G} \mathbf{F} \phi_{\mathcal{A}_j}) \\ &= \text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G}(\tilde{B})^c \wedge \bigvee_{0 \leq j < m} \mathbf{G} \mathbf{F} \phi_{\mathcal{A}'_j}) \\ &\leq \sum_{0 \leq j < m} \text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} \phi_{\mathcal{A}'_j}). \end{aligned}$$

Let  $0 \leq j < m$ , let  $\nu \in \text{Dist}(S)$  with  $\nu_{\mathcal{A}'_j}$  well-defined. Notice that  $\nu_{\mathcal{A}'_j} = (\nu_{\mathcal{A}_j})_{(\tilde{B})^c}$ . As it holds that  $\text{Prob}_{\nu_{\mathcal{A}'_j}}^\mathcal{T}(\mathbf{F} B) \geq p$ , we obtain by Lemma 10 that

$$\text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G} \mathbf{F} \phi_{\mathcal{A}'_j}) = 0.$$

Hence  $\text{Prob}_\mu^\mathcal{T}(\mathbf{G} B^c \wedge \mathbf{G}(\tilde{B})^c) = 0$ . □

The next two propositions focus on proving a claim that was stated throughout the article, which is that the decisiveness criterion from Proposition 11 generalizes those from [3,10].

**Proposition 41.** *The criterion provided in Proposition 11 generalizes those found in [3, Lemmas 3.4 & 3.7].*

*Proof.* The result of [3, Lemma 3.4] states that if there exists a *finite* attractor  $A$  for an STS  $\mathcal{T} = (S, \Sigma, \kappa)$ , then  $\mathcal{T}$  is decisive w.r.t. any measurable set  $B$ . For  $B \in \Sigma$ , using Corollary 12 and writing  $A' = A \cap (\tilde{B})^c$ , we can thus simply take

$$p = \min_{s \in A'} \text{Prob}_{\delta_s}^\mathcal{T}(\mathbf{F} B) > 0,$$

or  $p = 1$  if  $A'$  is empty.

The result of [3, Lemma 3.7] states that *globally coarse* Markov chains [3, Definition 3.5] are decisive. We say that an STS  $\mathcal{T} = (S, \Sigma, \kappa)$  is *globally coarse* w.r.t.  $B \in \Sigma$  if there exists  $p > 0$  such that for all  $s \in S$ , either  $\text{Prob}_{\delta_s}^\mathcal{T}(\mathbf{F} B) \geq p$ , or  $\text{Prob}_{\delta_s}^\mathcal{T}(\mathbf{F} B) = 0$ . To prove the decisiveness of a globally coarse STS, we can simply apply Corollary 12 with  $A = S$  as the attractor. □

**Proposition 42.** *The criterion provided in Proposition 11 generalizes the one found in [10, Proposition 36].*



*Proof.* We state the result of [10, Proposition 36].

*Claim.* Let  $\mathcal{T}_2$  be a countable Markov chain such that  $\mathcal{T}_2$  is an  $\alpha$ -abstraction of  $\mathcal{T}_1$ .

1. Assume that there is a finite set  $A_2 = \{s_1, \dots, s_n\} \subseteq S_2$  such that  $A_2$  is an attractor for  $\mathcal{T}_2$  and  $A_1 = \alpha^{-1}(A_2)$  is an attractor for  $\mathcal{T}_1$ .
2. Assume moreover that for every  $\alpha$ -closed set  $B$  in  $\Sigma_1$ , there exist  $p > 0$  and  $k \in \mathbb{N}$  such that for every  $1 \leq i \leq n$ :
  - for every  $\mu \in \text{Dist}(\alpha^{-1}(s_i))$ ,  $\text{Prob}_\mu^{\mathcal{T}_1}(\mathbf{F}_{\leq k} B) \geq p$ , or
  - for every  $\mu \in \text{Dist}(\alpha^{-1}(s_i))$ ,  $\text{Prob}_\mu^{\mathcal{T}_1}(\mathbf{F} B) = 0$ .

Then  $\mathcal{T}_1$  is decisive w.r.t. every  $\alpha$ -closed set.

We prove that these hypotheses satisfy our criterion. Let  $B$  be an  $\alpha$ -closed set in  $\Sigma_1$ . We use Corollary 12, using  $A = A_1$  as an attractor for  $\mathcal{T}_1$ . We consider

$$A'_2 = \{s_i \in A_2 \mid \forall \mu \in \text{Dist}(\alpha^{-1}(s_i)), \text{Prob}_\mu^{\mathcal{T}_1}(\mathbf{F}_{\leq k} B) \geq p\}.$$

Writing  $A'_1 = A_1 \cap (\tilde{B})^c$ , notice that  $A'_1 = \bigcup_{s_i \in A'_2} \alpha^{-1}(s_i)$ . Hence for all distributions  $\mu \in \text{Dist}(A'_1)$ , we have that  $\text{Prob}_\mu^{\mathcal{T}_1}(\mathbf{F} B) \geq p$ , so  $\mathcal{T}_1$  is decisive w.r.t.  $B$ .  $\square$

The criterion from [10, Proposition 37] can also be recovered. We omit to prove it, as this would require to use properties that are established in the proof of the proposition itself.

## C Proofs of Section 3

To prove the undecidability of reachability problems for SHSs, we reduce the halting problem for two-counter machines to qualitative reachability problems for SHSs. We first recall the definition of a two-counter machine.

**Definition 43 (Two-counter machine).** A two-counter machine  $M$  is a triple  $(\{b_0, \dots, b_m\}, C, D)$  where  $\{b_0, \dots, b_m\}$  are instructions and  $C, D$  are two counters ranging over the natural numbers. Each instruction  $b_i$ ,  $0 \leq i \leq m$ , is of one of the following three types:

- an increment (resp. decrement) instruction increments (resp. decrements) a specific counter by one, and then jumps to an instruction  $b_{k_i}$ ;
- a conditional jump instruction branches to one or another instruction based upon whether a specific counter has currently value 0;
- one halting instruction  $b_m$  terminates the machine execution.

We assume that before every decrement instruction, a test is done to verify that the counter being decremented is not zero; if it is, then its value is unchanged. A *configuration* of a two-counter machine is a triple  $q = (i, c, d)$  where  $i$  is the number of the current instruction, and  $c$  and  $d$  specify respectively the values of counters  $C$  and  $D$  before executing  $b_i$ . An *execution* is a finite or infinite sequence of configurations  $\rho = q_0 q_1 q_2 \dots$  such that  $q_0 = (0, 0, 0)$  is the initial configuration, and for  $i \geq 0$ , if  $q_j = (i, c, d)$ ,  $q_{j+1}$  is the new configuration after executing instruction  $b_i$ . We denote by  $\text{len}(\rho) \in \mathbb{N} \cup \{\infty\}$  the length of the execution  $\rho$ . If an execution is finite, then its last instruction has to be the halting instruction  $b_m$ . The problem of deciding whether the unique execution of a two-counter machine ends with a halting instruction is undecidable [35].

**Lemma 44.** Let  $M = (\{b_0, \dots, b_m\}, C, D)$  be a two-counter machine. There is an SHS  $\mathcal{H}$ , a measurable set  $B \in \Sigma_{\mathcal{H}}$  and an initial distribution  $\mu \in \text{Dist}(S_{\mathcal{H}})$  such that  $\text{Prob}_\mu^{\mathcal{T}_{\mathcal{H}}}(\mathbf{F} B) = 1$  if and only if  $M$  halts and  $\text{Prob}_\mu^{\mathcal{T}_{\mathcal{H}}}(\mathbf{F} B) = 0$  if and only if  $M$  does not halt. Moreover, the probability distributions on time delays of  $\mathcal{H}$  are all purely continuous distributions, guards are defined by linear comparisons of variables and constants, and dynamics are positive integer slopes.

*Proof.* We define an SHS  $\mathcal{H}$  such that for each instruction  $b_i$  of  $M$ , there is a location  $\ell_i^*$ . To encode the value of counter  $C$  (resp.  $D$ ), we use four variables  $x_1, \dots, x_4$  (resp.  $y_1, \dots, y_4$ ). We will also use four extra variables  $z_1, \dots, z_4$ , for a total of twelve continuous variables ( $|X| = 12$ ). We explain how we encode the value of counter  $C$  in our SHS  $\mathcal{H}$ ; the same method is used for counter  $D$ . If  $C = n$ , we will require that

$$\frac{1}{2^{n+1}} < x_2 - x_1 < x_4 - x_3 < \frac{1}{2^n} \quad (4)$$

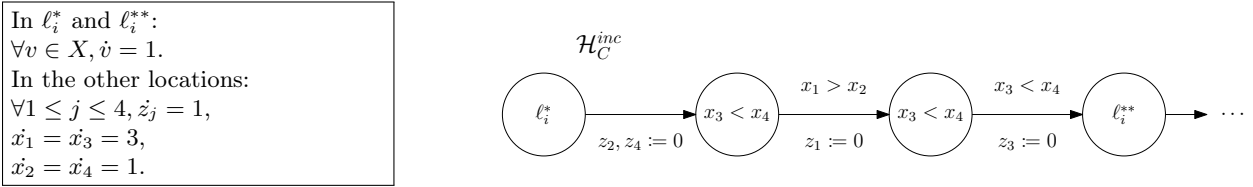
almost surely in some key locations of  $\mathcal{H}$ .

We show how we perform each type of instruction with our encoding. We only show the instructions concerning counter  $C$ . Notice that we can always preserve the encoding of the value of a counter by setting the evolution rate of all the variables used for its encoding to 1. Let  $0 \leq i \leq m$  be the index of an instruction of  $M$ .

- If  $b_i$  increments  $C$ : the SHS  $\mathcal{H}_C^{inc}$  incrementing counter  $C$  is shown in Figure 5. Notice that if  $x_1^*, \dots, x_4^* \in \mathbb{R}$  are the values of  $x_1, \dots, x_4$  when entering  $\ell_i^*$  and encode that  $C = n$  as in Equation (4), then in location  $\ell_i^{**}$ , almost surely,

$$\frac{1}{2^{n+2}} < z_2 - z_1 < z_4 - z_3 < \frac{1}{2^{n+1}}.$$

Indeed, if we set  $d = x_2^* - x_1^*$ , then the time spent in the second location of  $\mathcal{H}_C^{inc}$  is at least  $\frac{d}{2} > \frac{1}{2^{n+2}}$ , as we need to wait until  $x_1 > x_2$ . Hence in  $\ell_i^{**}$ ,  $\frac{1}{2^{n+2}} < z_2 - z_1$ . Clearly, we also have that  $z_2 - z_1 < z_4 - z_3$ . If we set  $d' = x_4^* - x_3^*$ , then the sum of the times spent in the two central locations is  $z_4 - z_3 < \frac{d'}{2} < \frac{1}{2^{n+1}}$  as  $x_3$  has not yet become greater than  $x_4$ .

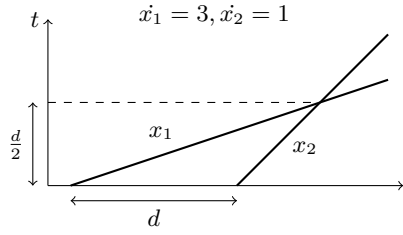


**Fig. 5.** SHS  $\mathcal{H}_C^{inc}$  incrementing the value of  $C$ . The distribution on the time delay in  $\ell_i^*$  is any exponential distribution, and in the next two locations it is a uniform distribution on the times after which the outgoing edge is enabled.

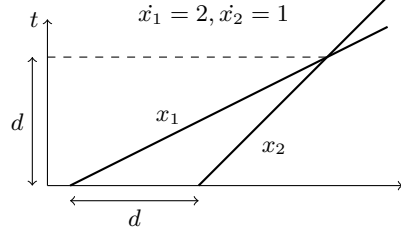
A visual representation of the time it takes for  $x_1$  to become larger than  $x_2$  in the second location is shown in Figure 6a. Therefore the  $z_i$  variables satisfy almost surely the requirement for a counter to be equal to  $n + 1$  in location  $\ell_i^{**}$ . To recover the right encoding using the  $x_i$  variables, we simply have to append to  $\mathcal{H}_C^{inc}$  the exact same SHS where for  $1 \leq j \leq 4$ ,  $x_j$  and  $z_j$  have been swapped and  $\dot{z}_1 = \dot{z}_3 = 2$  instead of 3.

- If  $b_i$  decrements  $C$ : we use an SHS  $\mathcal{H}_C^{dec}$  which is almost the same SHS as  $\mathcal{H}_C^{inc}$  in Figure 5, but we modify the evolution rates in the two central locations: for  $1 \leq j \leq 4$ ,  $\dot{z}_j = 2$ ,  $\dot{x}_1 = \dot{x}_3 = 2$ ,  $\dot{x}_2 = \dot{x}_4 = 1$ . A visual representation of the time it takes for  $x_1$  to overtake  $x_2$  in the second location is shown in Figure 6b.
- If  $b_i$  tests whether  $C = 0$ : the part of the SHS simulating this instruction is shown in Figure 7. No matter at what time the transition is taken, only one of the edges is enabled. Notice that  $C = 0$  in a location  $\ell_i^*$  if and only if  $\frac{1}{2} < x_2 - x_1 < 1$  almost surely.
- If  $b_i = b_m$  is the halting instruction: we model it using a single location  $\ell_m^*$  with a self-loop edge with no guard or invariant.

We set  $B = \ell_m^* \times \mathbb{R}^{12}$ . The initial distribution  $\mu$  assigns  $x_1 := 0$ ,  $x_2 := \frac{5}{8}$ ,  $x_3 := 0$ ,  $x_4 := \frac{7}{8}$  in location  $\ell_0^*$  ( $z_1, \dots, z_4$  can take any value).

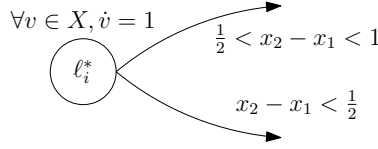


(a) Time taken by  $x_1$  to overtake  $x_2$  in the second location of  $\mathcal{H}_C^{inc}$ .



(b) Time taken by  $x_1$  to overtake  $x_2$  in the second location of  $\mathcal{H}_C^{dec}$ .

**Fig. 6.** Time taken by  $x_1$  to overtake  $x_2$  given their evolution rate. The initial difference between  $x_2$  and  $x_1$  is  $d$ . Remember that for decrementing, for  $1 \leq j \leq 4$ ,  $\dot{z}_j = 2$ .



**Fig. 7.** SHS testing whether  $C = 0$ . The distribution on the time delay is any exponential distribution.

We now prove that  $M$  halts if and only if  $\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F} B) = 1$ . Let  $\rho = (i_0, c_0, d_0)(i_1, c_1, d_1) \dots$  be the execution of  $M$ . For  $c, d \in \mathbb{N}$ , let

$$\mathcal{V}_{c,d} = \{\mathbf{v} \in \mathbb{R}^{12} \mid \mathbf{v} \text{ encodes the value } c \text{ (resp. } d) \text{ for counter } C \text{ (resp. } D)\}.$$

For an instruction  $b_i$  of  $M$ , we denote by  $m_i$  the number of locations used in  $\mathcal{H}$  to represent it. We can prove by induction that for all  $0 \leq k < \text{len}(\rho)$ ,

$$\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\text{Cyl}(\ell_{i_0}^* \times \mathcal{V}_{c_0,d_0}, \underbrace{S_{\mathcal{H}}, \dots, S_{\mathcal{H}}}_{m_{i_0}-1 \text{ times}}, \ell_{i_1}^* \times \mathcal{V}_{c_1,d_1}, \underbrace{S_{\mathcal{H}}, \dots, S_{\mathcal{H}}}_{m_{i_1}-1 \text{ times}}, \dots, \ell_{i_k}^* \times \mathcal{V}_{c_k,d_k})) = 1.$$

By definition of  $\mu$ , we have  $\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\text{Cyl}(\ell_{i_0}^* \times \mathcal{V}_{c_0,d_0})) = 1$ . The induction step is proved by construction of  $\mathcal{H}$ ; we showed that every instruction could be simulated to preserve almost surely our encoding.

Therefore, if  $M$  halts with counter values  $c$  and  $d$ , we get that

$$\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F} B) \geq \text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F}(\ell_m^* \times \mathcal{V}_{c,d})) = 1.$$

If  $M$  does not halt, then location  $\ell_m^*$  is almost surely never reached. Thus  $\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F} B) = 0$ .  $\square$

As a consequence of this lemma, we can easily prove Proposition 25. We recall its statement.

**Proposition 25.** *The qualitative reachability problems and the approximate quantitative reachability problem are undecidable for stochastic hybrid systems with purely continuous distributions on time delays, guards that are linear comparisons of variables and constants, and using positive integer slopes for the flow of the continuous variables. The approximate quantitative problem is moreover undecidable for any fixed precision  $\epsilon < \frac{1}{2}$ .*

*Proof.* In the previous lemma, we have reduced the halting problem for two-counter machines to the qualitative reachability problem for stochastic hybrid systems. As this halting problem is undecidable [35], using the notations of the previous lemma, we have proved that there is no algorithm to decide if  $\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F} B) = 1$  or  $\text{Prob}_\mu^{\mathcal{T}_\mathcal{H}}(\mathbf{F} B) = 0$ . Moreover, if we could approximate the probability of eventually reaching  $B$ , since

$\text{Prob}_\mu^{\mathcal{T}_H}(\mathbf{F} B)$  is either 0 or 1, we could decide if  $\text{Prob}_\mu^{\mathcal{T}_H}(\mathbf{F} B)$  is greater or less than  $\frac{1}{2}$ , and thus decide whether  $M$  halts or not. This means that the approximate quantitative reachability problem is also undecidable for any fixed  $\epsilon < \frac{1}{2}$ .  $\square$

*Remark 45.* Note that the SHS  $\mathcal{H}$  that we have built in the proof of Lemma 44 is decisive w.r.t.  $B$  from  $\mu$ . We have indeed that either  $\text{Prob}_\mu^{\mathcal{T}_H}(\mathbf{F} B) = 1$ , or  $B$  is almost surely non-reachable from  $\mu$  (which means that  $\mu(\tilde{B}) = 1$ ). In both cases, we have  $\text{Prob}_\mu^{\mathcal{T}_H}(\mathbf{F} B \vee \mathbf{F} \tilde{B}) = 1$ . The key element preventing us from using the procedure to approximate  $\text{Prob}_\mu^{\mathcal{T}_H}(\mathbf{F} B)$  described in Section 2.2 is the computation of  $\tilde{B}$ . Deciding whether the initial distribution lies in  $\tilde{B}$  is indeed equivalent to deciding whether  $M$  halts. This shows that decisiveness, albeit a desirable property to get closer to the decidability frontier, is not sufficient to make any reachability problem decidable, even for very simple (uniform and exponential) distributions, evolution rates, and resets.

## D Proofs of Section 4

We show the two aforementioned properties pertaining to cycle-reset SHSs: such SHSs are decisive w.r.t. any measurable set, and admit a finite abstraction.

**Proposition 30.** *Every cycle-reset SHS is decisive w.r.t. any measurable set.*

*Proof.* Let  $\mathcal{H}$  be a cycle-reset SHS, and  $B \in \Sigma_{\mathcal{H}}$  be a measurable set. We define

$$\begin{aligned} L^{\text{SR}} &= \{(\ell, \ell') \in L^2 \mid \exists e' = (\ell, a, \ell') \in E \wedge \forall e = (\ell, a, \ell') \in E, e \text{ is strongly reset}\}, \\ E^{\text{SR}} &= \{e = (\ell, a, \ell') \in E \mid (\ell, \ell') \in L^{\text{SR}}\}. \end{aligned}$$

We show that  $\mathcal{T}_H$  is strongly decisive w.r.t.  $B$ , using the criterion from Proposition 11. In this proof, for  $\ell$  a location of  $\mathcal{H}$ , we abusively write  $\ell$  instead of  $\{\ell\} \times \mathbb{R}^n$  in LTL formulae and in conditional distributions. Since every infinite run goes infinitely often through at least one cycle, and  $\mathcal{H}$  is cycle-reset, it holds that for all  $\nu \in \text{Dist}(S_{\mathcal{H}})$ ,

$$\text{Prob}_\nu^{\mathcal{T}_H} \left( \bigvee_{(\ell, \ell') \in L^{\text{SR}}} \mathbf{G} \mathbf{F} (\ell \wedge \mathbf{X} \ell') \right) = 1.$$

We set

$$p = \min_{e \in E^{\text{SR}}, \eta_e^*(\tilde{B}) < 1} \text{Prob}_{(\eta_e^*(\tilde{B}))^c}^{\mathcal{T}_H}(\mathbf{F} B) > 0$$

if there is  $e \in E^{\text{SR}}$  such that  $\eta_e^*(\tilde{B}) < 1$ , and  $p = 1$  otherwise. Let  $\nu \in \text{Dist}(S_{\mathcal{H}})$ ,  $(\ell, \ell') \in L^{\text{SR}}$ . We have that  $\nu_{\ell, \ell'}$  is a linear combination of strong reset distributions. If  $\nu_{\ell, \ell'}(\tilde{B}) < 1$ , we then have that

$$\text{Prob}_{(\nu_{\ell, \ell'})_{(\tilde{B})^c}}^{\mathcal{T}_H}(\mathbf{F} B) \geq p.$$

We thus satisfy the hypotheses of Proposition 11, and  $\mathcal{T}_H$  is decisive w.r.t.  $B$ .  $\square$

**Proposition 32.** *Let  $\mathcal{H}$  be an SHS, and  $B \in \Sigma_{\mathcal{H}}$ . If  $\mathcal{H}$  is cycle-reset, it has a finite and sound abstraction compatible with  $B$  and with the locations.*

*Proof.* For  $e = (\ell, a, \ell') \in E$  and  $D \subseteq \mathbb{R}^n$ , we write  $\text{Pre}_e^{\mathcal{T}_H}(\{\ell'\} \times D)$  for the set of states of  $\{\ell\} \times \mathbb{R}^n$  that have a positive probability to reach  $D$  through edge  $e$ .<sup>4</sup> Note that  $\text{Pre}^{\mathcal{T}_H}(\{\ell'\} \times D) = \bigcup_{e=(\ell, a, \ell')} \text{Pre}_e^{\mathcal{T}_H}(\{\ell'\} \times D)$ .

<sup>4</sup> Formally, the set  $\text{Pre}_e^{\mathcal{T}_H}(\{\ell'\} \times D)$  can be expressed as

$$\{s = (\ell, \mathbf{v}) \in \{\ell\} \times \mathbb{R}^n \mid \int_{\tau \in \mathbb{R}^+} \left( \theta_{s+\tau}(e) \cdot \int_{\mathbf{v}' \in \mathbb{R}^n} \mathbf{1}_D(\mathbf{v}') d(\eta_e(\gamma_\ell(\mathbf{v}, \tau)))(\mathbf{v}') \right) d\mu_s(\tau) > 0\}.$$

The fact that we consider transitions combining both a continuous evolution and a discrete step is crucial; every transition has to go through an edge.

We show that Procedure 1 terminates, with

$$\mathcal{P}_{init} = \bigcup_{\ell \in L} \{B \cap (\{\ell\} \times \mathbb{R}^n), B^c \cap (\{\ell\} \times \mathbb{R}^n)\} \setminus \{\emptyset\}.$$

Now let  $e = (\ell, a, \ell')$  be a *strongly reset* edge. Notice that for any  $D \subseteq \mathbb{R}^n$ ,

$$\text{Pre}_e^{\mathcal{T}_H}(\{\ell'\} \times D) = \begin{cases} \text{Pre}_e^{\mathcal{T}_H}(\{\ell'\} \times \mathbb{R}^n) & \text{if } \eta_e^*(D) > 0, \\ \emptyset & \text{if } \eta_e^*(D) = 0. \end{cases}$$

Indeed, states that can take edge  $e$  with a positive probability will always reach the same parts of the partition of  $\{\ell'\} \times \mathbb{R}^n$ . Hence, edge  $e$  induces one refinement in the states of location  $\ell$ , but no matter how the partition of  $\{\ell'\} \times \mathbb{R}^n$  is refined, it will not induce an extra refinement of the partition of  $\{\ell\} \times \mathbb{R}^n$ , thanks to the strong reset. This may not be the case without strong reset, as shown in Example 31.

The procedure propagates each extra refinement following the edges backwards, but no extra refinement is propagated past strongly reset edges. As there is at least one strong reset per cycle, this implies that for each refinement, the number of iterations of the procedure is finite. Hence, it terminates and produces a finite abstraction. As cycle-reset SHSs are decisive w.r.t. any measurable set, we immediately obtain by Proposition 17 that this abstraction is sound.  $\square$