Theories of real addition with and without a predicate for integers

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Abstract

We show that it is decidable whether or not a relation on the reals definable in the structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ can be defined in the structure $\langle \mathbb{R}, +, <, 1 \rangle$. This result is achieved by obtaining a topological characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relations in the family of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations and then by following Muchnik's approach of showing that the characterization of the relation X can be expressed in the logic of $\langle \mathbb{R}, +, <, 1, X \rangle$.

The above characterization allows us to prove that there is no intermediate structure between $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ and $\langle \mathbb{R}, +, <, 1 \rangle$. We also show that a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable if and only if its intersection with every $\langle \mathbb{R}, +, <, 1 \rangle$ -definable line is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. This gives a noneffective but simple characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relations.

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1 Introduction

Consider the structure $\langle \mathbb{R}, +, <, 1 \rangle$ of the additive ordered group of reals along with the constant 1. It is well-known that the subgroup \mathbb{Z} of integers is not first-order-definable. Add the predicate $x \in \mathbb{Z}$ resulting in the structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$. Our main result shows that given a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation it is decidable whether or not it is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

The structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ is a privileged area of application of algorithmic verification of properties of reactive and hybrid systems, where logical formalisms involving reals and arithmetic naturally appear, see e.g [1, 16, 4]. It admits quantifier elimination and is decidable as proved independently by Miller [21] and Weisfpfenning [28]. The latter's proof uses reduction to the theories of $\langle \mathbb{Z}, +, < \rangle$ and $\langle \mathbb{R}, +, <, 1 \rangle$.

There are many ways to come across the structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$, which highlights its significance. One approach is through automata. Cobham considers a fixed base r and represents integers as finite r-digit strings. A subset X of integers is r-recognizable if there exists a finite automaton accepting precisely the representations in base r of its elements. Cobham's theorem says that if X is r- and s-recognizable for two multiplicatively independent values r and s (i.e., for all i, j > 0 it holds $r^i \neq s^j$) then X is definable in Presburger arithmetic, i.e., in $\langle \mathbb{N}, + \rangle$ [13, 25]. Conversely, each Presburger-definable subset of \mathbb{N} is r-recognizable for every r. This result was extended to integer relations of arbitrary arity by Semënov [26].

Consider now recognizability of sets of reals. As early as in 1962 Büchi interprets subsets of integers as characteric functions of reals in their binary representations and shows the decidability of a structure which is essentially an extension of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$, namely $\langle \mathbb{R}_+, <, P, \mathbb{N} \rangle$ where P if the set of positive powers of 2 and \mathbb{N} the set of natural numbers [11, Thm 4]. Going one step further, Boigelot & al. [9] consider reals as infinite strings of digits and use Muller automata to speak of r-recognizable subsets and more generally of r-recognizable relations of reals. In the papers [5, 6, 8] the equivalence was proved between (1) $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definability, (2) r- and s-recognizability where the two bases have distinct primes in their factorization [8, Thm 5] and (3) r- and s-weakly recognizability for two independently multiplicative bases, [8, Thm 6] (a relation is r-weakly recognizable if it is recognized by some deterministic Muller automaton in which all states in the same

strongly connected component are either final or nonfinal). Consequently, as far as reals are concerned, definability in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ compared to recognizability or weak recognizability by automata on infinite strings can be seen as the analog of Presburger arithmetic for integers compared to recognizability by automata on finite strings.

A natural issue is to find effective characterizations of subclasses of r-recognizable relations. In the case of relations over integers, Muchnik proved that for every base $r \geq 2$ and arity $k \geq 1$, it is decidable whether a r-recognizable relation $X \subseteq \mathbb{N}^k$ is Presburger-definable [22] (see a different approach in [19] which provides a polynomial time algorithm). For relations over reals, up to our knowledge, the only known result is due to Milchior who proved that it is decidable (in linear time) whether a weakly r-recognizable subset of \mathbb{R} is definable in $\langle \mathbb{R}, +, <, 1 \rangle$ [20]. Our result provides an effective characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relations within $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations. Our approach is inspired by Muchnik's one, which consists of giving a combinatorical characterization of $\langle \mathbb{N}, + \rangle$ -definable relations that can be expressed in $\langle \mathbb{N}, + \rangle$ itself.

Our result has two interesting corollaries. The first one is that there is no intermediate structure between $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ and $\langle \mathbb{R}, +, <, 1 \rangle$, i.e., if an $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation X is not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, then \mathbb{Z} is definable in the structure $\langle \mathbb{R}, +, <, 1, X \rangle$. Along with the property that $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ is the "common" substructure of all recognizable and weakly recognizable relations this indicates that this structure is central. The second corollary is a noneffective but simple characterization: an $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable if and only if every intersection with a rational line is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. By rational we mean any line which is the intersection of hyperplanes defined by equations with rational coefficients.

The reader will be able to observe that, while our results are obviously related to automata questions, proofs do not use automata at all.

Other related works. Muchnik's approach, namely expressing in the theory of the structure a property of the structure itself, can be used in other settings. We refer the interested reader to the discussion in [27, Section 4.6] and also to [24, 2, 20] for examples of such structures. A similar method was already used in 1966, see [17] where the authors were able to express in Presburger theory whether or not a Presburger subset is the Parikh image of a context-free language.

We shall also mention a recent series of results by Hieronymi which deal with expansions of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$, and in particular with the frontier of decidability for such expansions, see, e.g., [18] and its bibliography. Finally, regarding our result that there is no intermediate structure between $\langle \mathbb{R}, +, <, 1 \rangle$ and $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$, let us mention that Conant recently proved [14] that there is no intermediate structure between $\langle \mathbb{Z}, + \rangle$ and $\langle \mathbb{Z}, +, < \rangle$. Concerning these two theories, it is decidable whether or not a $\langle \mathbb{Z}, +, < \rangle$ -definable relation is actually $\langle \mathbb{Z}, + \rangle$ -definable, see [12].

Now we give a short outline of our paper. Section 2 gathers all the basic on the two specific structures $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ and $\langle \mathbb{R}, +, <, 1 \rangle$, taking advantage of the existence of quantifier elimination which allows us to work with simpler formulas. Section 3 introduces topological notions. In particular we say that the neighborhood of a point $x \in \mathbb{R}^n$ relative to a relation $X \subseteq \mathbb{R}^n$ has strata if there exists a direction such that the intersection of all

sufficiently small neighborhoods around x with X is the trace of a union of lines parallel to the given direction. This reflects the fact that the relations we work with are defined by finite unions of regions of the spaces delimited by hyperplanes of arbitrary dimension. In Section 4 we show that when X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable all points (except finitely many which we call singular) have at least one direction which is a stratum. Section 5 studies relations between neighborhoods. In Section 6 we give a necessary and sufficient condition for a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation to be $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, namely 1) it has finitely many singular points and 2) all intersections of X with arbitrary hyperplanes parallel to n-1 axes and having rational components on the remaining axis are $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Then we show that these properties are expressible in $\langle \mathbb{R}, +, <, 1, X \rangle$. In Section 7 we show that there is no intermediate structure between $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ and $\langle \mathbb{R}, +, <, 1 \rangle$ -definable if and only if every intersection with a rational line is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

2 Preliminaries

Throughout this work we assume the vector space \mathbb{R}^n is provided with the metric L_{∞} (i.e., $|x| = \max_{1 \leq i \leq n} |x_i|$). The open ball centered at $x \in \mathbb{R}^n$ and of radius r > 0 is denoted by B(x,r). Given $x,y \in \mathbb{R}^n$ we denote by [x,y] (resp. (x,y)) the closed segment (resp. open segment) with extremities x,y. We use also notations such as [x,y) or (x,y] for half-open segments.

Let us specify our logical conventions and notations. We work within first-order predicate calculus with equality. We confuse formal symbols and their interpretations, except in subsection 6.2 where the distinction is needed. We are mainly concerned with the structures $\langle \mathbb{R}, +, <, 1 \rangle$ and $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$. In the latter structure, \mathbb{Z} should be understood as a unary predicate which is satisfied only by reals belonging to \mathbb{Z} - in other words, we deal only with one-sorted structures. Given a structure \mathcal{M} with domain D and $X \subseteq D^n$, we say that X is definable in \mathcal{M} , or \mathcal{M} -definable, if there exists a formula $\varphi(x_1, \ldots, x_n)$ in the signature of \mathcal{M} such that $\varphi(a_1, \ldots, a_n)$ holds in \mathcal{M} if and only if $(a_1, \ldots, a_n) \in X$.

The $\langle \mathbb{R}, +, <, 1 \rangle$ -theory admits quantifier elimination in the following sense, which can be interpreted geometrically as saying that a $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relation is a finite union of closed and open polyhedra.

Theorem 1. [15, Thm 1] Every formula in $\langle \mathbb{R}, +, <, 1 \rangle$ is equivalent to a Boolean combination of inequalities between linear combinations of variables with coefficients in \mathbb{Z} (or, equivalently, in \mathbb{Q}).

- **Corollary 2.** 1. A subset of \mathbb{R} is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable if and only if it is a finite union of intervals whose endpoints are rational numbers. In particular \mathbb{Z} is not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.
 - 2. For every $n \geq 1$, if $X \subseteq \mathbb{R}^n$ is nonempty and $(\mathbb{R}, +, <, 1)$ -definable then X contains an element of \mathbb{Q}^n .

Proof. (1) is an immediate consequence of Theorem 1. For (2) we proceed by induction over n. The case n=1 follows from (1). For n>1, consider the set $Y=\{x_n\mid \exists x_1,\ldots,x_{n-1}\; (x_1,\ldots,x_n)\in X\}$. The set Y is nonempty and $\langle\mathbb{R},+,<,1\rangle$ -definable by our hypothesis on X, thus by the base case of the induction Y contains a rational q. Then it suffices to apply the induction hypothesis to the (n-1)-ary relation $\{(x_1,\ldots,x_{n-1})\mid (x_1,\ldots,x_{n-1},q)\in X\}$.

In the larger structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ it is possible to separate the integer (superscript 'I') and fractional (superscript 'F') parts of the reals as follows.

Theorem 3. [10],[8, p. 7] Let $X \subseteq \mathbb{R}^n$ be definable in $(\mathbb{R}, +, <, \mathbb{Z})$. Then there exists a unique finite union

$$X = \bigcup_{k=1}^{K} (X_k^{(I)} + X_k^{(F)}) \tag{1}$$

where

- the relations $X_k^{(I)}$ are pairwise disjoint subsets of \mathbb{Z}^n and are $\langle \mathbb{Z}, +, < \rangle$ -definable
- the relations $X_k^{(F)}$ are distinct subsets of $[0,1)^n$ and are $(\mathbb{R},+,<,1)$ -definable

There is again a geometric interpretation of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations as a regular (in a precise technical way) tiling of the space by a finite number of tiles which are themselves finite unions of polyhedra. As a consequence, the restriction of a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation to a bounded subset is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, as stated in the following lemma.

Lemma 4. For every $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation $X \subseteq \mathbb{R}^n$, its restriction to a bounded domain $[a_1, b_1] \times \cdots \times [a_n, b_n]$ where the a_i 's and the b_i 's are rationals, is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

Proof. By Theorem 3 the relation X is a finite union of the form $\bigcup_{k=1}^K (X_k^{(I)} + X_k^{(F)})$ where each $X_k^{(I)} \subseteq \mathbb{Z}^n$ is $\langle \mathbb{Z}, <, + \rangle$ -definable and each $X_k^{(F)} \subseteq [0,1)^n$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Let $A_k, B_k \in \mathbb{Z}$ be such that

$$[a_1,b_1] \times \cdots \times [a_n,b_n] \subseteq [A_1,B_1] \times \cdots \times [A_n,B_n]$$

The a_i 's and the b_i 's are rational thus $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, and the relation $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable as well. Now the finite subset

$$T_k = X_k^{(I)} \cap ([A_1, B_1] \times \cdots \times [A_n, B_n]) \subseteq \mathbb{Z} \times \cdots \times \mathbb{Z}$$

is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, therefore so is the sum $T_k + X_k^{(F)}$, and also the finite union $S = \bigcup_{k=1}^K (T_k + X_k^{(F)})$. Finally the restriction $X \cap ([a_1, b_1] \times \cdots \times [a_n, b_n]) = S \cap ([a_1, b_1] \times \cdots \times [a_n, b_n])$ is also $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

By considering the restriction of the $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -relation to a ball containing all possible tiles with their closest neighbors, we get that the neighborhoods of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ - and $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relations are indistinguishable.

Lemma 5. For every $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation $X \subseteq \mathbb{R}^n$ there exists a $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relation $Y \subseteq \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ there exists $y \in \mathbb{R}^n$ and a real r > 0 such that the translation $u \mapsto u + y - x$ is a one-to-one mapping between $B(x,r) \cap X$ and $B(y,r) \cap Y$.

Proof. Let X be a $(\mathbb{R}, +, <, \mathbb{Z})$ -definable relation

$$X = \bigcup_{k=1}^{K} (X_k^{(I)} + X_k^{(F)})$$

as in expression (1). Set $C = [-1, 2]^n \subseteq \mathbb{R}^n$. Observe that the set

$$\mathcal{P} = \{ -a + ((a+C) \cap X) \mid a \in \mathbb{Z}^n \}$$

is finite and contains at most K^{3^n} elements, all of which being subsets of C. Thus there exists an integer N such that for all $P \in \mathcal{P}$ there exists $z \in B(0, N-1) \cap \mathbb{Z}^n$ such that

$$P = -z + ((z + C) \cap X)$$

We prove the statement by defining Y as the restriction of X to B(0,N). Indeed, consider an arbitrary $x \in \mathbb{R}^n$ and let x = w + t where $w_i = \lfloor x_i \rfloor$ for $i = 1, \ldots, n$. Then the translation $u \mapsto -w + u$ defines a one-to-one correspondence between $(w + C) \cap X$ and $-w + ((w + C) \cap X)$ which is some $P \in \mathcal{P}$. By definition of N there exists $z \in B(0, N-1) \cap \mathbb{Z}^n$ such that $z + P = (z + C) \cap X = (z + C) \cap Y$. Then the translation $\tau(u) = -w + z + u$ is a one-to-one correspondence between $(w + C) \cap X$ and $(z + C) \cap Y$. Since x is interior to x + C, the point $x \in \mathbb{Z}$ is interior to $x + C \in B(0, N)$, thus for sufficiently small x > 0 the ball $x \in \mathbb{Z}$ is included in x + C and the ball $x \in \mathbb{Z}$ is included in x + C. Consequently $x \in \mathbb{Z}$ defines a one-to-one mapping between $x \in \mathbb{Z}$ and $x \in \mathbb{Z}$ the ball $x \in \mathbb{Z}$ is included in $x \in \mathbb{Z}$ and the ball $x \in \mathbb{Z}$ is included in $x \in \mathbb{Z}$.

3 Strata

The aim is to decide, given $n \geq 1$ and a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation $X \subseteq \mathbb{R}^n$, whether X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Though the relations defined in the two structures have very specific properties we define properties that make sense in a setting as general as possible. The following clearly defines an equivalence relation.

Definition 6. Given $x, y \in \mathbb{R}^n$ we write $x \underset{X}{\sim} y$ or simply $x \sim y$ when X is understood, if there exists a real r > 0 such that the translation $w \mapsto w + y - x$ is a one-to-one mapping from $B(x,r) \cap X$ onto $B(y,r) \cap X$.

Example 7. Consider a closed subset of the plane delimited by a square. There are 10 equivalence classes: the set of points interior to the square, the set of points interior to its complement, the four vertices and the four open edges.

- **Definition 8.** 1. Given a non-zero vector $v \in \mathbb{R}^n$ and a point $y \in \mathbb{R}^n$ we denote by $L_v(y)$ the line passing through y in the direction v. More generally, if $X \subseteq \mathbb{R}^n$ we denote by $L_v(X)$ the set $\bigcup_{x \in X} L_v(x)$.
 - 2. A non-zero vector $v \in \mathbb{R}^n$ is an X-stratum at x (or simply a stratum when X is understood) if there exists a real r > 0 such that

$$B(x,r) \cap X = B(x,r) \cap L_v(X) \tag{2}$$

This can be seen as saying that inside the ball B(x,r), the relation X is a union of lines parallel to v.

3. The set of X-strata at x is denoted by $Str_X(x)$ or simply Str(x).

Proposition 9. For all $X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ the set $\underline{Str}(x)$ is either empty or a (vector) subspace of \mathbb{R}^n .

Proof. We start with a lemma.

Lemma 10. Let v_1, v_2 be two strata at x and let r > 0 be such that

$$B(x,r) \cap X = B(x,r) \cap L_{v_1}(X)$$
 and $B(x,r) \cap X = B(x,r) \cap L_{v_2}(X)$.

Then

$$B(x,r) \cap X = B(x,r) \cap L_{v_1+v_2}(X).$$

Proof. Let $v = v_1 + v_2$. We must prove that a point $y \in B(x,r)$ belongs to X if and only if all points of $B(x,r) \cap L_v(y)$ do.

Assume without loss of generality that $y \in X$ and consider $z = y + \lambda v \in B(x, r)$ with $\lambda \neq 0$. Let $\epsilon > 0$ be such that for every point $t \in [y, z]$ the ball $B(t, \epsilon)$ is included in B(x, r) (such a real exists because the segment [y, z] is compact). Let n be an integer such that $|\frac{1}{n}\lambda v_1| < \epsilon$. Then the points $y + \frac{i}{n}\lambda(v_1 + v_2)$ for $0 \leq i \leq n$ belong to B(x, r) because they lay in the segment [y, z] and due to the choice of ϵ the points $y + \frac{i}{n}\lambda(v_1 + v_2) + \frac{1}{n}\lambda v_1$ for $0 \leq i < n$ belong to B(x, r). Since the vectors v_1 and v_2 are strata at x, the points $y + \frac{i}{n}\lambda(v_1 + v_2)$ for $0 \leq i \leq n$ and the points $y + \frac{i}{n}\lambda(v_1 + v_2) + \frac{1}{n}\lambda v_1$ for $0 \leq i < n$ belong to X therefore so does in particular z.

Now we turn to the proof of Proposition 9. By definition v is a stratum if and only if λv is a stratum for some $\lambda \neq 0$. Thus it suffices to verify that Str(x) is closed under addition. If v_1 (resp. v_2) is a stratum then there exist $r_1, r_2 > 0$ such that

$$B(x, r_1) \cap X = B(x, r_1) \cap L_{v_1}(X)$$
 $B(x, r_2) \cap X = B(x, r_2) \cap L_{v_2}(X)$

Thus for $r \leq \min\{r_1, r_2\}$ we have

$$B(x,r) \cap X = B(x,r) \cap L_{v_1}(X)$$
 $B(x,r) \cap X = B(x,r) \cap L_{v_2}(X)$

It suffices to apply Lemma 10.

Definition 11. The <u>dimension</u> $\dim(x)$ of a point $x \in \mathbb{R}^n$ is the dimension of the subspace Str(x) if Str(x) is nonempty, or 0 otherwise.

Definition 12. Given a relation $X \subseteq \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ is X-singular, or simply singular, if Str(x) is empty, otherwise it is nonsingular.

Example 13. (Example 7 continued) Let $x \in \mathbb{R}^2$. If x belongs to the interior of the square or of its complement, then $\operatorname{Str}(x) = \mathbb{R}^2$. If x is one of the four vertices of the square then we have $\operatorname{Str}(x) = \emptyset$, i.e x is singular. Finally, if x belongs to an open edge of the square but is not a vertex, then $\operatorname{Str}(x)$ has dimension 1, and two points of opposite edges have the same one-dimensional subspace, while two points of adjacent edges have different one-dimensional subspaces.

Note that non- $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations may have no singular points. Indeed consider in the plane the set X defined as the union of vertical lines at abscissa $\frac{1}{n}$ for all positive integers n. In this case any vertical vector is a stratum at any point of the plane.

Now it can be shown that all strata at x can be defined by a common value r in expression (2).

Proposition 14. If $\underline{Str}(x) \neq \emptyset$ there exists a real r > 0 such that for every $v \in \underline{Str}(x)$ we have

$$B(x,r) \cap X = B(x,r) \cap L_v(X)$$
.

Proof. Let v_1, \ldots, v_p be a basis of the vector space Str(x). There exist $r_1, \ldots, r_p > 0$ such that

$$B(x,r_i) \cap X = B(x,r_i) \cap L_{v_i}(X)$$

Then for $r = \min\{r_1, \ldots, r_p\}$ we have

$$B(x,r) \cap X = B(x,r) \cap L_{v_i}(X)$$

Consider an arbitrary vector of Str(x), say $v = \lambda_1 v_1 + \cdots + \lambda_p v_p$. It suffices to apply Lemma 10 successively to $\lambda_1 v_1 + \lambda_2 v_2$, $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$, ..., $\lambda_1 v_1 + \cdots + \lambda_p v_p$.

Definition 15. A <u>safe radius</u> (for x) is a real r > 0 satisfying the condition of Proposition 14. Clearly if r is safe then so are all $0 < s \le r$. By convention every real is a safe radius if $Str(x) = \emptyset$.

Example 16. (Example 7 continued) For an element x of the interior of the square or the interior of its complement, let r be the (minimal) distance from x to the edges of the square. Then r is safe for x. If x is a vertex then Str(x) is empty and every r > 0 is safe for x. In all other cases r is the minimal distance of x to a vertex.

Lemma 17. If $x \sim y$ then $\underline{Str}(x) = \underline{Str}(y)$.

Proof. For some r > 0, the translation $u \mapsto u + y - x$ is a one-to-one correspondence between $B(x,r) \cap X$ and $B(y,r) \cap X$. Thus every stratum of X at x is a stratum of X at y and vice versa.

The converse of Lemma 17 is false in general. Indeed consider, e.g., $X = \{(x,y) \mid y \le 0\} \cup \{(x,y) \mid y=1\}$ in \mathbb{R}^2 . The points (0,0) and (0,1) have the same subspace of strata, namely that generated by (1,0), but $x \not\sim y$.

Now we combine the notions of strata and of safe radius.

Lemma 18. Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ and r be a safe radius for x. Then for all $y \in B(x,r)$ we have $Str(x) \subseteq Str(y)$.

Proof. Indeed, consider $v \in Str(x)$. For all s > 0 such that $B(y,s) \subseteq B(x,r)$ we have

$$B(y,s) \cap L_v(X) = B(x,r) \cap B(y,s) \cap L_v(X) = B(y,s) \cap B(x,r) \cap L_v(X) = B(y,s) \cap B(x,r) \cap X = B(y,s) \cap X$$

Example 19. (Example 7 continued) Consider a point x on an (open) edge of the square and a safe radius r for x. For every point y in B(x,r) which is not on the edge we have $Str(x) \subset Str(y) = \mathbb{R}^2$. For all other points we have Str(x) = Str(y).

We relativize the notion of singularity and strata to an affine subspace $P \subseteq \mathbb{R}^n$. The next definition should come as no surprise.

Definition 20. Given an affine subspace $P \subseteq \mathbb{R}^n$, a subset $X \subseteq P$ and a point $x \in P$, we say that a vector v parallel to P is an (X, P)-stratum for the point x if for all sufficiently small r > 0 it holds

$$P \cap X \cap B(x,r) = P \cap L_v(X) \cap B(x,r)$$

A point $x \in P$ is (X, P)-singular if it has no (X, P)-stratum. For simplicity when P is the space \mathbb{R}^n we will still stick to the previous terminology and speak of X-strata and X-singular points.

Singularity and nonsingularity do not go through restriction to affine subpaces.

Example 21. In the real plane, let $X = \{(x,y) \mid y < 0\}$ and P be the line x = 0. Then the origin is not X-singular but it is $(X \cap P, P)$ -singular. All other elements of P admit (0,1) as an $(X \cap P, P)$ -stratum thus they are not $(X \cap P, P)$ -singular. The opposite situation may occur. In the real plane, let $X = \{(x,y) \mid y < 0\} \cup P$ where $P = \{(x,y) \mid x = 0\}$. Then the origin is X-singular but it is not $(X \cap P, P)$ -singular.

4 Local properties

4.1 Local neighborhoods

In this section we recall that if $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable then the equivalence relation \sim (introduced in Definition 6) has finite index. This extends easily to the case where X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable. The claim for $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relations can be found, e.g., in [3, Thm 1] (see also [7, Section 3]) but we revisit it to some extent because of the small modifications needed to use it in our setting.

We define what we mean by "cones".

Definition 22. Given a point $x_0 \in \mathbb{R}^n$, a <u>cone</u> is an intersection of finitely many half-spaces defined by conditions of the form $u(x) - u(x_0) < 0$ or $u(x) - u(x_0) \le 0$ where u is a linear expression having rational coefficients. The point x_0 is an apex of the cone.

In particular the origin and the empty set are specific cones in our sense (on the real line they can be described respectively by $x \le 0 \land -x \le 0$ and $x < 0 \land -x < 0$). By convention, the origin is an apex of the empty set.

Let $X \subseteq \mathbb{R}^n$ be $(\mathbb{R}, +, <, 1)$ -definable. By Theorem 1 we may assume that X is defined by a formula

$$\phi(x) = \bigvee_{i \in I} C_i \text{ where } C_i = \bigwedge_{j \in J_i} u_{i,j}(x) \triangleleft_{ij} b_{ij}$$
(3)

where \triangleleft_{ij} is either the symbol " \leq " or "<" and for all $(i,j) \in I \times J_i$, each pair $(u_{i,j}(x), b_{ij})$, consists of a linear expression and a constant, both with rational coefficients.

Now we associate with ϕ a finite collection of cones.

Definition 23. Consider all formulas obtained from expression (3) by replacing in all possible ways each predicate $u_{i,j}(x) \triangleleft_{ij} b_{ij}$ by one of the three options $u_{i,j}(x) \triangleleft_{ij} 0$ or false or true. Use the routine simplifications so that the resulting formulas are reduced to false or true or are disjunctions of conjunctions with no occurrence of false or true.

Let Θ be the (finite) set of formulas thus obtained, and let us call <u>local neighborhood</u> any relation defined by some formula in Θ . In particular each formula in Θ defines a finite union of cones of which the origin is an apex.

In the terminology of [3, Thm 1] the following says that an $\mathbb{R}, +, <, 1$ -definable relation has finitely many "faces" which are our local neighborhoods.

Proposition 24. Consider an $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relation X. There exists a finite collection Θ of $\langle \mathbb{R}, +, <, 1 \rangle$ -formulas defining finite unions of cones with apex the origin such that for all $\xi \in \mathbb{R}^n$ there exist some θ in Θ and some real s > 0 such that for all $t \in \mathbb{R}^n$ we have

$$(\theta(t) \land |t| < s) \leftrightarrow (\phi(\xi + t) \land |t| < s) \tag{4}$$

Proof. We consider Θ as defined in Definition 23. For the points interior to X or to its complement it is clear that for sufficiently small |t| the predicate $\phi(\xi + t)$ has value true or false, both of which belong to Θ . We consider the remaining case and first show that there exists a formula ψ such that for sufficiently small |t| we have $\phi(\xi + t) \leftrightarrow \psi(\xi + t)$ where

$$\psi(\xi + t) = \bigvee_{h \in H} \bigwedge_{k \in K_h} u_{h,k}(\xi + t) \triangleleft_{hk} b_{hk}$$
 (5)

for some $H \subseteq I$ and $K_h \subseteq J_h$ satisfying the conditions

- 1. for all $(h,k) \in H \times K_h$ it holds $u_{h,k}(\xi) = b_{hk}$
- 2. if ξ belongs to the boundary of X then for all $h \in H$ and all $k \in K_h$ the relation symbol $\triangleleft_{h,k}$ is \leq .

3. if ξ belongs to the boundary of the complement of X then for all $h \in H$ there exists $k \in K_h$ such that the relation symbol $\triangleleft_{h,k}$ is <.

The formula ψ is obtained from ϕ (cf. expression 3) by deleting some atomic predicates and by applying the following rules:

- If for some $i \in I$ there is $j \in J_i$ such that $u_{i,j}(\xi) > b_{ij}$ then inside a small ball around ξ the conjunct $\bigwedge_{j \in J_i} u_{i,j}(x) \triangleleft_{i,j} b_{i,j}$ is false and can be cancelled out. Now consider
 - a remaining conjunct $C_h = \bigwedge_{k \in K_h} u_{h,k}(x) \triangleleft_{h,k} b_{hk}$. All occurrences $u_{h,k}(x) \triangleleft_{h,k} b_{hk}$

such that $u_{h,k}(\xi) < b_{hk}$ can be cancelled since in a neighborhood of ξ the predicate $u_{h,k}(x) < b_{hk}$ holds and the conjunct necessarily contains other atomic predicates, otherwise the conjunct, and therefore the disjunction, would be reduced to **true** which violates the hypothesis that the point is not interior of X. Thus Point 1 is satisfied.

- If for all $h \in H$ there exists $k \in K_h$ such that the relation symbol $\triangleleft_{h,k}$ is < then ξ cannot belong to X, thus the conjunct is false and can be cancelled. Point 2 is satisfied.
- If for some $h \in H$ and all $k \in K_h$ the relation symbol $\triangleleft_{h,k}$ is \leq then ξ belongs to X. Point 3 is satisfied.

In order to obtain the formula θ in Θ at ξ , we observe that by point 1 and using the linearity of the functions $u_{i,j}$ we have

$$u_{i,j}(\xi+t) \triangleleft_{i,j} b_{i,j} \leftrightarrow u_{i,j}(t) \triangleleft_{i,j} 0.$$

Corollary 25. Let $X \subseteq \mathbb{R}^n$ be $(\mathbb{R}, +, <, 1)$ -definable.

- 1. The equivalence relation \sim has finite index.
- 2. The set of (distinct) spaces Str(x) is finite when x runs over \mathbb{R}^n .
- 3. There exists a fixed finite collection C of cones (in the sense of Definition 22) such that for each \sim -class E there exists a subset $C' \subseteq C$ such that for every $x \in E$ there exists r > 0 such that

$$(x + t \in X) \wedge |t| < r \leftrightarrow (t \in \bigcup_{C \in \mathcal{C}'} C) \wedge |t| < r$$

Proof. Point 1 follows from the fact that Θ is finite and that two points x, y which are associated with the same formula θ in Proposition 24 satisfy $x \sim y$ by definition of θ . Point 2 is a straightforward consequence of Point 1 and Lemma 17. For Point 3 observe that all elements of E must be associated with equivalent formulas θ in Proposition 24 and that each formula $\theta \in \Theta$ is a disjunction of formulas which define cones.

Because of Lemma 5 we have

Corollary 26. The statements of Corollary 25 extend to the case where X is $(\mathbb{R}, +, <, \mathbb{Z})$ -definable.

Combining Corollaries 25 and 26 allows us to specify properties of singular points for $(\mathbb{R}, +, <, 1)$ - and $(\mathbb{R}, +, <, \mathbb{Z})$ -definable relations.

Proposition 27. Let $X \subseteq \mathbb{R}^n$. If X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable then it has finitely many singular points and their components are rational numbers. If X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable then it has a countable number of singular points and their components are rational numbers.

Proof. By Proposition 24, if ξ is not interior to X or its complement, for small enough r > 0 the subset X coincides on $B(\xi, r)$ with a finite nonempty union of open or closed cones of which ξ is an apex. The boundaries of these cones are hyperplanes H_1, \ldots, H_p defined by equations of the form $u_{h,k}(x) = b_{hk}$ as in formula (5). If their normals scan a subspace of dimension p < n then the space of strata has dimension at least n - p: indeed along all such directions, the expressions $u_{h,k}(x)$ are constant. Therefore a point is singular only if these normals scan the space \mathbb{R}^n . There are finitely many hyperplanes H_i , and n hyperplanes whose normals are linearly independent intersect in exactly one point, thus the number of singular points is finite and their intersections have rational components.

The fact that the set of singular points in an $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation is countable is a direct consequence of the following observation. Let $x \in [a_1, a_1 + 1] \times \cdots \times [a_n, a_n + 1]$ with $a_1, \ldots, a_n \in \mathbb{Z}$. Then x is X-singular if and only if it is Y-singular in the restriction

$$Y = X \cap ([a_1 - 1, a_1 + 2] \times \cdots \times [a_n - 1, a_n + 2])$$

because for all $0 < r < \frac{1}{2}$ we have $B(x,r) \cap X = B(x,r) \cap Y$. By Lemma 4 each set Y is $(\mathbb{R}, +, <, 1)$ -definable thus has finitely many singular points, and there is a countable number of such Y's.

4.2 Strata in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$

In Proposition 9 we proved that every non-empty set of strata at a given point is a vectorial subspace. Here we show more precisely that this subspace has a set of generators consisting of vectors with rational coefficients.

Proposition 28. Let $X \subseteq \mathbb{R}^n$ be a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation and $x \in \mathbb{R}^n$. If $Str(x) \neq \emptyset$ then there exists a set of linearly independent vectors with rational coefficients generating Str(x).

Proof. Because of Lemma 5 the collection of local neighborhoods in an $\mathbb{R}, +, <, \mathbb{Z}$ definable relation is identical to the collection of local neighborhoods in some $\mathbb{R}, +, <, 1$ definable relation, thus it suffices to treat the case of $\mathbb{R}, +, <, 1$ definable relations.

By Proposition 24 for all points $\xi \in \mathbb{R}^n$ there exists a $(\mathbb{R}, +, <, 1)$ -formula θ defining a collection of cones with apex ξ such that for some real s > 0 and for all $t \in \mathbb{R}^n$ the following condition is satisfied.

$$(\theta(t) \land |t| < s) \leftrightarrow (\phi(\xi + t) \land |t| < s) \tag{6}$$

As a first consequence we may assume without loss of generality that ξ is the origin. Given an hyperplane H defined by a linear equation u(x) = 0, we set

$$H^{\varepsilon} = \begin{cases} \{x \mid u(x) = 0\} & \text{if } \varepsilon \text{ is the symbol } = \\ \{x \mid u(x) < 0\} & \text{if } \varepsilon \text{ is the symbol } < \\ \{x \mid u(x) > 0\} & \text{if } \varepsilon \text{ is the symbol } > \end{cases}$$

Lemma 29. Let $y, z \in \mathbb{R}^n$ and let H_1, \ldots, H_q be hyperplanes in \mathbb{R}^n containing the origin. For all nonzero vectors $v \in \mathbb{R}^n$ which are parallel with no H_i , there exists $\alpha \in \mathbb{R}$ and $\varepsilon_1, \ldots, \varepsilon_q$ such that the points $y' = y + \alpha v$ and $z' = z + \alpha v$ belong to $\bigcap_{1 \le i \le q} H_i^{\varepsilon_i}$.

Proof. Since v is not parallel with H_i , for sufficiently large $\alpha_i \geq 0$ the points $y + \alpha_i v$ and $z + \alpha_i v$ belong to the same halfspace defined by H_i . It suffices to set $\alpha = \max(\alpha_i)$.

We return to the proof of Proposition 28. Expression 6 can be viewed as saying that the relation $Y = \{x \mid \theta(x)\}$ satisfies the following condition: there exists q rational hyperplanes H_1, \ldots, H_q such that Y is a finite union of subsets of the form $\bigcap_{1 \leq i \leq q} H_i^{\varepsilon_i}$ with $\varepsilon_i \in \{<,>,=\}$. Among all possible expressions defining Y and involving the hyperplanes H_1, \ldots, H_q , choose one where the minimum subset of such hyperplanes occurs. Rename if necessary the hyperplanes as H_1, \ldots, H_p with $p \leq q$. We want to show that

$$Str(x) = \bigcap_{1 \le i \le p} H_i$$

Clearly, every vector v parallel with all H_i is a stratum for all cones so that $\operatorname{Str}(x) \supseteq \bigcap_{1 \le i \le p} H_i$ holds. We prove the opposite inclusion. If $\operatorname{Str}(x) = \emptyset$ or p = 0 we are done. We assume by way of contradiction that for some vector $v \in \operatorname{Str}(x)$, the subset $J \subseteq \{1, \ldots, n\}$ of indices j such that v belongs to H_j is proper. If $J = \emptyset$, by Lemma 29 for all points $y, z \in \mathbb{R}^n$ there exist $\alpha \in \mathbb{R}$ and $\varepsilon_1, \ldots, \varepsilon_p$ such that the points $y' = y + \alpha v$ and $z' = z + \alpha v$ belong to $\bigcap_{1 \le i \le p} H_i^{\varepsilon_i}$. This implies $y' \in Y \leftrightarrow z' \in Y$, and since v is a stratum, we get

$$y \in Y \leftrightarrow y' \in Y \leftrightarrow z' \in Y \leftrightarrow z \in Y$$

thus $Y = \mathbb{R}^n$ which is defined by $\theta(x) = \text{true}$ and violates the minimality of p.

Now we deal with $J \neq \emptyset$. By possibly renaming the hyperplanes we assume $J = \{r+1,\ldots,p\}$ with $r \geq 1$. We will show that the hyperplanes H_1,\ldots,H_r are useless, i.e., that Y can be written as a finite union of subsets of the form $\bigcap_{r< i \leq p} H_i^{\varepsilon_i}$. Given a subset $A = \bigcap_{r< i \leq p} H_i^{\varepsilon_i}$ we show that for all points $y,z \in A$ we have $y \in Y \leftrightarrow z \in Y$. We apply again Lemma 29: there exist $\alpha \in \mathbb{R}$ and $\varepsilon_1,\ldots,\varepsilon_r$ such that $y' = y + \alpha v$ and $z' = z + \alpha v$ belong to $\bigcap_{1 \leq i \leq r} H_i^{\varepsilon_i}$. Since $y,z \in A$ and $v \in \bigcap_{r< i \leq p} H_i$, we get $y',z' \in A$ thus $y',z' \in \bigcap_{1 \leq i \leq p} H_i^{\varepsilon_i}$. By definition of θ we get $y' \in Y \leftrightarrow z' \in Y$, and since v is a stratum we obtain $y \in Y \leftrightarrow z \in Y$. This contradicts the minimality of p.

4.3 Application: expressing the singularity of a point in a $(\mathbb{R}, +, <, \mathbb{Z})$ -definable relation

The singularity of a point x is defined as the property that no intersection of X with a ball centered at x is a union of lines parallel with a given direction. This property is not directly expressible within $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ since the natural way would be to use multiplication on reals, which is not $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable. In order to be able to express the property, we give an alternative characterization of singularity which relies on the assumption that X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable.

Lemma 30. Given an $(\mathbb{R}, +, <, \mathbb{Z})$ -definable relation $X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ the following two conditions are equivalent:

- 1. x is singular.
- 2. for all r > 0, there exists s > 0 such that for all vectors v of norm less than s, there exist two points $y, z \in B(x, r)$ such that y = z + v and $y \in X \Leftrightarrow z \notin X$.

Observe that when X is not $\mathbb{R}, +, <, \mathbb{Z}$ -definable, the two assertions are no longer equivalent. E.g., \mathbb{Q} has only singular points but condition 2 holds for no point in \mathbb{R} .

Proof. In order to prove the equivalence of the two conditions, we write them formally

- (1) $\forall r > 0 \ \forall v \in \mathbb{R}^n \ \exists s > 0 \ \exists y, z \ (y, z \in B(x, r) \land y = sv + z \land (y \in X \Leftrightarrow z \notin X))$
- (2) $\forall r > 0 \ \exists s > 0 \ \forall v \ |v| < s \to \exists y, z \ (y, z \in B(x, r) \land y = v + z \land (y \in X \Leftrightarrow z \not\in X))$

The implication $(2) \Rightarrow (1)$ is shown by contraposition, i.e., $\neg(1) \Rightarrow \neg(2)$ and is a simple application of the rule $\exists u \forall v \psi(u, v) \rightarrow \forall v \exists u \psi(u, v)$. Indeed, if v_0 satisfies $\neg(1)$ then for every s the condition $\neg(2)$ is satisfied with any vector v colinear with v_0 of modulus less than s.

Now we prove $(1) \Rightarrow (2)$. We must prove that if the point x is singular, then for all r > 0 there exists s > 0 such that for every vector $|v| \leq s$ there exist two points y, z satisfying

$$y, z \in B(x, r) \land y = z + v \land (y \in X \Leftrightarrow z \notin X) \tag{7}$$

Let k be the number of disjuncts in the formula defining X (cf. expression (3)), which is also an upper bound on the number of cones composing the local neighborhoods at a given point. In order to simplify the notation we also assume that the point x is the origin. Also, it is clear that condition (2) is satisfied if and only if it is satisfied for r small enough which means that we may assume that the following holds

$$B(0,r) \cap X = B(0,r) \cap \mathcal{C} \tag{8}$$

where \mathcal{C} is the union of the cones at 0 as defined in Corollary 26. We claim that expression (7) holds when s is set to $\frac{r}{2k+1}$. Since 0 is singular, for every direction u there exists a line $L_u(w)$ with $w \in B(0,r) \cap \mathcal{C}$ which contains points in X and points in the complement of X, that is

$$\emptyset \subsetneq B(0,r) \cap \mathcal{C} \cap L_u(w) \subsetneq B(0,r) \cap L_u(w)$$

Because C is closed under the mappings $z \mapsto \alpha z$ for all $\alpha > 0$, for all $0 < \beta \le 1$ we have

$$\emptyset \subsetneq B(0,r) \cap \mathcal{C} \cap L_u(\beta w) \subsetneq B(0,r) \cap L_u(\beta w)$$

By choosing β small enough if necessary, it is always possible to assume that the length of $B(0,r)\cap L_u(\beta w)$ equals some $t\geq r$. The intersection of $L_u(\beta w)$ with X inside B(0,r) defines p segments, some possibly of length 0, successively included in and disjoint from the cones in \mathcal{C} , with $2\leq p\leq 2k+1$. Let x_0,x_1,\ldots,x_p be the endpoints of these segments in the order they appear along the line, with x_0 and x_p being the intersections with the frontier of the ball B(0,r). Their projections over any of the axes of \mathbb{R}^n for which the coordinate of u is maximal determines a nondecreasing sequence of reals $a_0\leq a_1\leq\cdots\leq a_p$ such that $a_p-a_0=t$. If p=2 then either $a_1-a_0\geq \frac{t}{2}\geq s$ or $a_2-a_1\geq \frac{t}{2}\geq s$ and then for all s'< s we can choose two points $y\in (x_0,x_1)$ and $z\in (x_1,x_2)$ such that |y-z|=s'. Now assume p>2. We have $a_p-a_0=t\geq r$ thus there exists $0\leq i< p$ such that $a_{i+1}-a_i\geq \frac{r}{p}\geq \frac{r}{2k+1}=s$. If i< p-1 and $x_{i+1}=x_{i+2}$, i.e., $a_{i+1}=a_{i+2}$ then for all $s'< a_{i+1}-a_i$, and hence for every s'< s, there exists a point $z\in (x_i,x_{i+1})$ such that $|x_{i+1}-z|=s'$, and we can set $y=x_{i+1}$. The case where i>0 and $x_{i-1}=x_i$ is similar. In all other cases, for all s'< s we can find $z\in (x_i,x_{i+1})$ and $y\in (x_{i+1},x_{i+2})$ such that |y-z|=s'.

5 Relations between neighborhoods

We illustrate the purpose of this section with a very simple example. We start with a cube sitting in the horizontal plane with only one face visible. The rules of the game is that we are given a finite collection of vectors such that for all 6 faces and all 12 edges it is possible to choose vectors that generate the vectorial subspace of the smallest affine subspace in which they live. Let the point at the center of the upper face move towards the observer (assuming that this direction belongs to the initial collection). It will eventually hit the upper edge of the visible face. Now let the point move to the left along the edge (this direction necessarily exists because of the assumption on the collection). The point will hit the upper left vertex. Consequently, in the trajectory the point visits three different ~-classes: that of the points on the open upper face, that of the points on the open edge and that of the upper left vertex. Here we investigate the adjacency of such equivalence classes having decreasing dimensions. Observe that another finite collection of vectors may have moved the point from the center of the upper face directly to the upper left vertex.

Since two \sim -equivalent points either have no stratum or the same subspace of strata, given a \sim -class E it makes sense to denote by $\operatorname{Str}(E)$ the empty set in the first case and the common subspace of all points in E in the latter case. Similarly, $\dim(E)$ is the common dimension of the points in E.

5.1 Compatibility

Consider the backwards trajectory on the cube as discussed above: the point passes from an \sim -equivalence class of low dimension into an \sim -equivalence class of higher dimension

along a direction that is proper to this latter class. This leads to the notion of compatibility. For technical reasons we allow a class to be compatible with itself.

Definition 31. Let E be a nonsingular \sim - class and let v be one of its strata. Given a \sim - class F, a point $y \in F$ is v-compatible with E if there exists $\epsilon > 0$ such that for all $0 < \alpha < \epsilon$ we have $y + \alpha v \in E$.

A \sim - class F is v-compatible with E if there exists a point $y \in F$ which is v-compatible with E.

Lemma 32. If the \sim - class F is v-compatible with the \sim - class E, all elements of F are v-compatible with E.

There exists at most one \sim -class E such that F is v-compatible with E.

Proof. We must show that if y and z belong to F and if y is v-compatible with E then there exists a real $\alpha > 0$ such that for all $0 < \beta < \alpha$ we have $y + \beta v \sim z + \beta v$. Indeed, by definition of \sim there exists r such that the translation $t \mapsto t + z - y$ maps $B(y, r) \cap X$ onto $B(z, r) \cap X$. For all α satisfying $|\alpha v| < r$ consider any s satisfying $|\alpha v| + s < r$. Then the above translation maps $B(y + \alpha v, s) \cap X$ onto $B(z + \alpha v, s) \cap X$, i.e., $y + \alpha v \sim z + \alpha v$.

The second claim is a consequence of the fact that a point y can only be v-compatible with a unique class.

Observe that for any nonsingular \sim -class E and one of its strata v there always exists a \sim -class v-compatible with E, namely E itself, but also that conversely there might be different classes v-compatible with E.

Example 33. Let X be the union of the two axes of the 2-dimensional plane and v = (1,1) which we assume is one of the chosen strata of the \sim -class $\{(x,y) \mid x \neq 0, y \neq 0\}$. The different classes are: the complement of X, the origin $\{0\}$ which is a singular point, the horizontal axis deprived of the origin, and the vertical axis deprived of the origin. The two latter \sim -classes are both v-compatible with the class $\mathbb{R}^2 \setminus X$.

5.2 Intersection of a line and equivalence classes

In this section we describe the intersection of a \sim -class E with a line parallel to some $v \in \text{Str}(E)$.

With the above example of the cube, a line passing through a point x on the upper face along any of the directions of Str(x) of dimension 2 intersects an open edge or a vertex at point y. In the former case $\dim(y) = 1$ and in the latter $\dim(y) = 0$, and in both cases $Str(y) \subset Str(x)$.

Lemma 34. Let $X \subseteq \mathbb{R}^n$, E, F be two \sim -classes, and $v \in \underline{Str}(E)$. Let y be an element of F which is adherent to $L_y(v) \cap E$. Then $\underline{Str}(F) \subseteq \underline{Str}(E)$.

If E, F are different, then $\underline{Str}(F) \subseteq \underline{Str}(E) \setminus \{v\}$ and therefore $\dim(F) < \dim(E)$.

Proof. If E = F then clearly Str(F) = Str(E). Thus it suffices to consider the case $F \neq E$. By hypothesis B(y,r) intersects E for every r > 0, which yields $Str(F) \subseteq Str(E)$ by Lemma 18. It remains to prove that $v \notin Str(F)$. We show that for every r > 0 we can find in B(y,r) two elements z_1, z_2 such that $z_1 \in X \leftrightarrow z_2 \notin X$ and $z_1 - z_2$ is parallel to v.

Let r be a safe radius for y. By hypothesis there exists $y' \in B(y,r) \cap L_y(v) \cap E$. Let s > 0 be a safe radius for y' such that $B(y',s) \subseteq B(y,r)$. We have $y \not\sim y'$, thus there exists u with |u| < s such that $y + u \in X \leftrightarrow y' + u \not\in X$. We set $z_1 = y + u$ and $z_2 = y' + u$. Both z_1 and z_2 belong to B(y,r) by our hypothesis on u,s and y. Moreover $z_1 - z_2 = y - y'$ and $y' \in L_v(y)$ thus $z_1 - z_2$ is parallel to v.

Lemma 35. Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ a nonsingular point and $v \in \underline{Str}(x)$. There exist $y, z \in L_v(x)$ such that $x \in (y, z)$ and every element w of (y, z) satisfies $w \sim x$.

Proof. Indeed, let r be a safe radius for x and (y, z) be an open segment on $L_v(x) \cap B(x, r)$ containing x. Let $w \in (y, z)$ and let t > 0 be any real such that $B(w, t) \subseteq B(x, r)$. We show that the translation $u \mapsto u + w - x$ defines a one-to-one correspondence from $B(x, t) \cap X$ to $B(w, t) \cap X$. Indeed, let $z' \in B(x, t) \cap X$. Since $B(x, r) \cap X$ is a union of lines parallel with v, we have $L_v(z') \cap B(w, t) \subseteq L_v(z') \cap B(x, r) \subseteq X$ implying $z' + w - x \in B(w, t) \cap X$. Conversely, for every element $u \in B(w, t) \cap X$ we have $u - w + x \in B(x, t) \cap X$.

Consequently, via Lemmas 34 and 35 we get the following.

Corollary 36. Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$, E its \sim -class and let $v \in \underline{Str}(x)$. The set $L_v(x) \cap E$ is a union of disjoint open segments (possibly infinite in one or both directions) of $L_v(x)$, i.e., of the form $(y - \alpha v, y + \beta v)$ with $0 < \alpha, \beta \leq \infty$ and $y \in E$.

If $\alpha < \infty$ (resp. $\beta < \infty$) then the point $y - \alpha v$ (resp. $y + \beta v$) belongs to a \sim -class $F \neq E$ where F is v-compatible (resp. (-v)-compatible) with E, and $\dim(F) < \dim(E)$.

6 Characterization and effectivity

6.1 Characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$

In this section we give a characterization of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relations which are $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. A <u>rational section</u> of a relation $X \subseteq \mathbb{R}^n$ is a relation of the form

$$X_c^{(i)} = X \cap (\mathbb{R}^i \times \{c\} \times \mathbb{R}^{n-i-1})$$
 for some $c \in \mathbb{Q}, \ 0 \le i < n$

Theorem 37. Let $n \geq 1$ and let $X \subseteq \mathbb{R}^n$ be $(\mathbb{R}, +, <, \mathbb{Z})$ -definable. Then X is $(\mathbb{R}, +, <, 1)$ -definable if and only if the following two conditions hold

- 1. There exist finitely many X-singular points.
- 2. Every rational section of X is $(\mathbb{R}, +, <, 1)$ -definable.

Observe that both conditions (1) and (2) are needed. Indeed, the relation $X = \mathbb{R} \times \mathbb{Z}$ is $(\mathbb{R}, +, <, \mathbb{Z})$ -definable. It has no singular point thus it satisfies condition (1), but does

not satisfy (2) since, e.g., the rational section $X_0^{(0)} = \{0\} \times \mathbb{Z}$ is not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Now, consider the relation $X = \{(x, x) \mid x \in \mathbb{Z}\}$ which is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable. It does not satisfy condition (1) since every element of X is singular, but it satisfies (2) because every rational section of X is either empty or equal to the singleton $\{(x, x)\}$ for some $x \in \mathbb{Z}$, thus is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

The necessity of point 1 follows from Proposition 27. That of point 2 results from the fact that all rational constants are $\langle \mathbb{R}, +, <, 1 \rangle$ -definable by Theorem 1, and moreover that $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relations are closed under direct product and intersection.

Now we prove that conditions 1 and 2 are sufficient. We start with some informal discussion. Since X possesses finitely many \sim -classes, Corollary 36 suggests that we prove the $\langle \mathbb{R}, +, <, 1 \rangle$ -definability of the \sim -classes by induction on their dimension. The case of classes of dimension 0 is easy to handle using condition 1. For the induction step, we use the same corollary which asserts that the intersection of a nonsingular class E with a line passing through a point x in the class and parallel to a direction of the class is a union of open segments. If the segment containing x is finite or semifinite then one of its adherent point belongs to a class E of lower dimension and we can define E relatively to E via the notion of compatibility. However the segment may be infinite and thus may intersect no other equivalence class. So we consider the canonical subspaces defined below, and will use the fact that every line has an intersection with one of these.

Formally we define

$$H_{i} = \{(x_{1}, \dots, x_{n}) \in \mathbb{R}^{n} \mid x_{i} = 0\} \quad i \in \{1, \dots, n\}$$

$$Q_{I} = \bigcap_{i \in I} H_{i}, \quad Q'_{I} = (Q_{I} \setminus \bigcup_{i \in \{1, \dots, n\} \setminus I} H_{i}) \text{ for all } \emptyset \subset I \subseteq \{1, \dots, n\}$$

$$(9)$$

In particular $Q_{\{1,\ldots,n\}} = \{0\}$ and by convention $Q_{\emptyset} = \mathbb{R}^n$. The Q_i 's are the <u>canonical subspaces</u>. The Q_i 's are not vectorial subspaces but with some abuse of language we will write $\dim(Q_I')$ to mean $\dim(Q_I) = n - |I|$. Moreover for every $I \neq \{1,\ldots,n\}$ the set Q_I' is open in Q_I , that is

$$\forall x \in Q_I' \exists \epsilon \ \forall v \ (v \in Q_I \land |v| < \epsilon \Rightarrow x + v \in Q_I')$$
(10)

(indeed, it suffices to assume $|v_j| < |x_j|$ for all $j \notin I$). Observe that point 2 of the theorem implies that for every I the intersection $X \cap Q_I$ (resp. $X \cap Q_I'$) is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Furthermore the Q_I' 's define a partition of the space. We also have a trivial but important property which is implicit in the proof of Lemma 40.

Remark 38. If $x \in Q'_I$ and v is a vector in the subspace Q_I then for all points on $y \in L_v(x)$ we have $y \in Q'_I$ for some $J \supseteq I$.

Using the canonical subspaces, the proof below can be seen as describing a trajectory starting from a point x in a \sim -class E, traveling along a stratum of E until it reaches a class of lower dimension F (by Corollary 36) or some canonical subspace. In the first case it resumes the journey from the new class F on. In the second case it is trapped in the canonical subspace: it resumes the journey by choosing one direction of the subspace until it reaches a new \sim - class or a point belonging to a proper canonical subspace. Along the

journey, either the dimension of the new class or the dimension of the canonical subspace decreases. The journey stops when the point reaches a (X, Q_I) -singular point, or the origin which is the least canonical subspace.

Definition 39. Given an $\langle \mathbb{R}, +, <, 1 \rangle$ -definable relation $X \subseteq \mathbb{R}^n$, a set of <u>principal directions</u> (for X) is any finite subset V of vectors such that for all nonsingular points $x \in \mathbb{R}^n$ and all canonical spaces Q_I , there exists a subset $V' \subseteq V$ which generates the space $Str(x) \cap Q_I$. Observe that by Proposition 28 there is no loss of generality to assume that $V \subseteq \mathbb{Q}^n$.

By Corollary 26 there exist finitely many distinct spaces $\underline{\operatorname{Str}}(x)$ when x runs over \mathbb{R}^n , thus there exists a set V of principal directions for X.

For every $I\subseteq\{1,\ldots,n\}$ and every \sim -class E we define $E^{(I)}=E\cap Q_I'$. Observe that $E=\bigcup_{I\subseteq\{1,\ldots,n\}}E^{(I)}$ which is a disjoint union. We know that X is a union of finitely many \sim -classes, see Corollary 26. Thus in order to prove that X is $\langle \mathbb{R},+,<,1\rangle$ -definable it suffices to prove that all sets $E^{(I)}$ are $\langle \mathbb{R},+,<,1\rangle$ -definable. Consider the (height) function h which assigns to every set $E^{(I)}$ the pair of integers

$$h(E^{(I)}) = (\dim(\operatorname{Str}(E) \cap Q_I), \dim(Q'_I))$$

where by convention $\dim(A) = 0$ if $A = \emptyset$. Given two \sim -classes E, F and $I, J \subseteq \{1, \ldots, n\}$ we define the (partial) ordering $F^{(J)} < E^{(I)}$ if $h(F^{(J)}) = (a', b')$ and $h(E^{(I)}) = (a, b)$ with either $(a' < a \text{ and } b' \le b)$ or $(a' \le a \text{ and } b' < b)$.

We prove by induction on (a,b) that each $E^{(I)}$ is definable from smaller classes $F^{(J)}$, i.e. that $E^{(I)}$ is definable in the expansion of $(\mathbb{R},+,<,1)$ obtained by adding a predicate for each smaller class $F^{(J)}$.

Let $h(E^{(I)})=(a,b)$. If a=0 then the elements of $E^{(I)}$ have no X-stratum in Q_I , thus they are $(X\cap Q_I,Q_I)$ -singular, see Definition 20. By point 2, $X\cap Q_I$ is $\langle \mathbb{R},+,<,1\rangle$ -definable thus it has finitely many such points and they are $\langle \mathbb{R},+,<,1\rangle$ -definable. The same holds for $E^{(I)}$ which is a finite union of such points.

If b = 0 then $Q_I = \{0\}$, thus $E^{(I)}$ is either empty or equal to the singleton $\{0\}$, and in both cases $E^{(I)}$ is $(\mathbb{R}, +, <, 1)$ -definable.

Now assume that a, b > 0. The following details a single step of the journey explained above.

Lemma 40. Let $I \subseteq \{1, ..., n\}$ and $x \in Q'_I$. Then $x \in E^{(I)}$ if and only if there exists a principal direction $v \in V \cap Str(x) \cap Q_I$, elements $y, z \in \mathbb{R}^n$, some \sim -class F and some $J \supseteq I$ such that the following holds:

- 1. $x \in (y, z)$ and $y x = \alpha v$ for some positive real α
- 2. (y, z) does not intersect any class $G^{(K)}$ such that $G^{(K)} < E^{(I)}$
- 3. (a) either $(y \in F^{(J)} \text{ where } F^{(J)} < E^{(I)} \text{ and } F \text{ is } v-\text{compatible with } E)$
 - $(b) \ \ or \ (z \in F^{(J)} \ \ where \ F^{(J)} < E^{(I)} \ \ and \ F \ \ is \ (-v)-compatible \ \ with \ E).$

Proof. We first prove that the conditions are sufficient. We assume w.l.o.g. that condition 3 holds for the case (a). By hypothesis $x \in Q_I'$ and $x \in (y, z)$, thus it suffices to prove that $(y, z) \subseteq E^{(I)}$. Set $z = y - \gamma v$ and consider the union U of all open segments $(y, y - \beta v)$, $\beta > 0$, included in E. Observe that U is nonempty since y is v-compatible with E. If U contains (y, z) we are done so we assume U = (y, t) with $t = y - \gamma' v$ and $\gamma' < \gamma$ and we let $G^{(K)}$ be the class of t. We set $h(G^{(K)}) = (a', b')$. By Lemma 34 and Corollary 36 we have $Str(G) \subseteq Str(E) \setminus \{-v\}$, and because of Remark 38 it holds $K \supseteq I$.

If $K \supseteq I$ then b' < b, which leads to the inclusions

$$Str(G) \cap Q_K \subseteq Str(E) \cap Q_K \subsetneq Str(E) \cap Q_I$$
.

This implies $a' \leq a$ and thus $G^{(K)} < E^{(I)}$.

If I = K then b = b' and we show that a' < a. The inclusion $Str(G) \subseteq Str(E) \setminus \{-v\}$ along with the fact that $v \in Q_I = Q_K$ implies $Str(G) \cap Q_K \subseteq Str(E) \cap Q_K$ which leads to a' < a. This implies again $G^{(K)} < E^{(I)}$ contradicting point 2.

Now we prove that the conditions are necessary. By hypothesis we have $a \neq 0$ thus $Str(E) \cap Q_I$ is not empty, and by Definition 39 the set V contains a basis of $Str(E) \cap Q_I$. We can choose v as any element of this basis.

By Corollary 36 and Property (10) there exists an open segment of the line $L_v(x)$ containing x and only points in $E^{(I)}$. Consider the union U of all such open segments. The hypothesis b > 0 implies $\dim(Q_I) \ge 1$ which means that $L_v(x)$ intersects some hyperplane H_j with $j \notin I$. This implies that $L_v(x)$ is not a subset of Q'_I , and a fortiori not a subset of $E^{(I)}$, hence U is not equal to $L_v(x)$. Assume without loss of generality that U has an extremity of the form $x - \alpha v$ for some $\alpha > 0$. We set $y = x - \alpha v$, and $z = x + \beta v$ where β is any positive real such that $[x, x + \beta v) \subseteq E^{(I)}$.

We prove that y, z satisfy the conditions of the lemma. Conditions (1) and (2) are easy consequences of the very definition of y and z. We show that condition (3a) holds. We set $y \in F^{(J)}$ and $h(F^{(J)}) = (a', b')$. By Remark 38 we have $I \subseteq J$ thus $b' \le b$, and by Lemma 34 we have $Str(F) \subseteq Str(E)$ thus $Str(F) \cap Q_J = Str(E) \cap Q_I$, hence $a' \le a$.

If a = a', i.e $y \in E$, then $y \notin Q_I$ by definition of y and Property (10). It follows that $I \subseteq J$ i.e. b' < b, thus $h(F^{(J)}) < h(E^{(I)})$.

If b = b', i.e I = J, then by definition of y and Corollary 36 we have $E \neq F$, and using again Lemma 34 we obtain $\operatorname{Str}(F) \subseteq \operatorname{Str}(E) \setminus \{v\}$, and this yields $v \in (\operatorname{Str}(E) \cap Q_I) \setminus (\operatorname{Str}(F) \cap Q_J)$ i.e. a' < a, which shows that $h(F^{(J)}) < h(E^{(I)})$.

We can conclude the proof of Theorem 37. By our induction hypothesis, every set $F^{(J)}$ such that $F^{(J)} < E^{(I)}$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, thus it suffices to prove that $E^{(I)}$ is definable in $\langle \mathbb{R}, +, <, 1, \mathcal{F} \rangle$ where

$$\mathcal{F} = \{F^{(J)} \mid F \text{ is } v \text{ compatible with } E \text{ for some } v \in V, \text{ and } F^{(J)} < E^{(I)}\}$$

We use the characterization of $E^{(I)}$ given by Lemma 40 and build a formula which expresses the conditions of this lemma. Set $Z = \{F^{(J)} \mid F^{(J)} < E^{(I)}\}$ and let V' be the set of vectors $v \in V$ for which there exist some class F and some subset J such that $F^{(J)} < E^{(I)}$. A

defining formula for $E^{(I)}$ is

$$\chi(x): x \in Q_I' \wedge \bigvee_{v \in V'} \chi_v(x)$$

where $\chi_v(x)$ is defined as follows. Denote by A (resp. B) the set of classes $F^{(J)}$ such that $F^{(J)} < E^{(I)}$ and F is v-compatible with E (resp. (-v)-compatible with E). Then

$$\chi_{v}(x): \quad \exists y, z, \alpha, \beta \ (\alpha < 0 \land \beta > 0 \land (y = x + \alpha v \land z = x + \beta v)$$

$$\land \forall \gamma (\alpha < \gamma < \beta \to \bigwedge_{F^{(J)} \in Z} \neg F^{(J)}(x + \gamma v)) \land$$

$$\land (\bigvee_{F^{(J)} \in A} F^{(J)}(y) \lor \bigvee_{F^{(J)} \in B} F^{(J)}(z))).$$

6.2 Decidability

In this section we prove that it is decidable whether a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. The main idea is to construct in $(\mathbb{R}, +, <, 1, X)$ a sentence ψ_n which expresses the conditions of Theorem 37, then use the $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definability of X to re-write ψ_n as a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -sentence, and conclude thanks to the decidability of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$.

In order to simplify the task of constructing ψ_n we re-formulate Theorem 37. We extend the notion of section by allowing to fix several components. A generalized section of X is a relation of the form

$$X_{s,a} = \{(x_1, \dots, x_n) \in X \mid x_{s_1} = a_{s_1}, \dots, x_{s_r} = a_{s_r}\}$$
(11)

where r > 0, $(s)_{1,\dots,r} = 1 \le s_1 < \dots < s_r \le n$ is an increasing sequence, and $a (= a_{s_1},\dots,a_{s_r})$ is a r-tuple of reals. When r = 0 we define $X_{s,a} = X$ by convention. If all elements of a are rationals then $X_{s,a}$ is called a rational generalized section of X.

Proposition 41. Let $n \geq 1$ and let $X \subseteq \mathbb{R}^n$ be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable. Then X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable if and only if every rational generalized section of X has finitely many singular points.

Proof. If X is $(\mathbb{R}, +, <, 1)$ -definable so is every rational restriction which therefore has finitely many singular points by point 1 of Theorem 37.

We show the opposite direction by decreasing induction of the number r of frozen components of the rational restriction. We use the fact that all rational restrictions are $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable. If r = n - 1 the rational generalized section is an $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable subset of \mathbb{R} with finitely many singular points which implies that it consists of finitely many intervals with rational endpoints and we are done by Corollary 2.

Fix r > 1 and assume that all rational restrictions $X_{s,a}$ as in 11 with r frozen components are $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Consider a rational generalized section $X_{t,b}$ with r-1 frozen components, say

$$(t)_{1,\dots,r-1} = 1 \le t_1 < \dots < t_{r-1} \le n$$

 $b = (b_{t_1},\dots,b_{t_{r-1}}) \quad b_i \in \mathbb{Q}, i = 1,\dots,r-1$

It has finitely many singular points by hypothesis. A rational section of $X_{t,b}$ is defined by some increasing sequence $(s)_{1,\ldots,r} = 1 \le s_1 < \cdots < s_r \le n$ and an r-tuple $a = (a_{s_1},\ldots,a_{s_r})$ of rational numbers such that for some $0 \le u \le r-1$ we have

$$s_k = t_k, \ k < u, \quad s_{k+2} = t_{k+1}, \ u \le k$$

 $a_k = b_k, \ k < u, \quad a_{k+2} = b_{k+1}, \ u \le k$

But then all $X_{t,b}$ are $(\mathbb{R}, +, <, 1)$ -definable by induction and so is $X_{s,a}$ by Theorem 37. \square

So far we did not distinguish between formal symbols and their interpretations but here we must do it if we want to avoid any confusion. Let $X_n \subseteq \mathbb{R}^n$ be a relation defined by a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -formula ϕ . In order to express that X_n is actually $\langle \mathbb{R}, +, <, 1 \rangle$ -definable we proceed as follows. Let $\{\mathcal{X}_n(x_1, \ldots, x_n) \mid n \geq 1\}$ be a collection of relational symbols. We construct a $\{+, <, 1, \mathcal{X}_n\}$ -sentence $\psi_n(\mathcal{X}_n)$ such that $\psi_n(X_n)$ holds if and only X_n is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

Proposition 42. Let $\{\mathcal{X}_n(x_1,\ldots,x_n)\mid n\geq 1\}$ denote a set of relational symbols. For every $n\geq 1$ there exists a $\{+,<,1,\mathcal{X}_n\}$ -sentence ψ_n such that for every $\{+,<,1,\mathcal{X}_n\}$ -structure $\mathcal{M}=(\mathbb{R},+,<,1,\mathcal{X}_n)$, if X_n is $\langle \mathbb{R},+,<,\mathbb{Z} \rangle$ -definable then we have $\mathcal{M}\models\psi_n$ if and only if X_n is $\langle \mathbb{R},+,<,1 \rangle$ -definable.

Proof. For each $I \subseteq \{1, ..., n\}$ we denote by \mathbb{R}^I the product of copies of \mathbb{R} indexed by I and for all nonzero reals r and all $x \in \mathbb{R}^I$ we set $B_I(x,r) = \{y \in \mathbb{R}^I \mid |x-y| < r\}$. Using Lemma 30 we can construct the following $\{+, <, 1, \mathcal{X}_n\}$ —formula which express the fact that a point x + y where $x \in \mathbb{R}^I$ and $y \in \mathbb{R}^{[n]\setminus I}$ is singular, when seen as a point of the generalized section of X_n obtained by freezing to y the components of $[n] \setminus I$.

$$\operatorname{Sing}_{n,I}(x,y,\mathcal{X}_n) \equiv \forall r \in \mathbb{R} \exists s \in]0, r[\ \forall q \in \mathbb{R}^I \ |q| < s \Rightarrow \\ \exists z \in \mathbb{R}^I ((z,z+q \in B_I(x,r)) \land (y+z \in \mathcal{X}_n \Leftrightarrow y+z+q \notin \mathcal{X}_n))$$
(12)

Now we construct a $\{+,<,1,\mathcal{X}_n\}$ -sentence ψ_n which expresses the condition of Proposi-

tion 41. Some difficulty arises from the fact that we have to express that every rational generalized section of X has finitely many singular points, but the set \mathbb{Q} is not $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable. In order to overcome this issue, we construct ψ_n in such a way that it expresses that every generalized section of X has finitely many singular points. We define ψ_n as

$$\psi_n \equiv \bigwedge_{I \subset [n]} \forall y \in \mathbb{R}^{[n] \setminus I} \ \varphi_{n,I}(y)$$

where

$$\varphi_{n,I}(y) \equiv \ \forall x \in \mathbb{R}^I \exists M > 0 \quad (\mathtt{Sing}_{n,I}(x,y,\mathcal{X}_n) \Rightarrow |x| < M)$$

The formula $\varphi_{n,I}(y)$ expresses that the generalized section of X_n obtained by freezing to y the components of $[n] \setminus I$ has finitely many singular points.

We prove first that if X_n is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable and satisfies the formula ψ_n , then it is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Consider a rational generalized section $X_{s,a}$ of X with $(s)_{1,\ldots,r} = 1 \le s_1 < \cdots < s_r \le n$ and $a = (a_1, \ldots, a_r) \in \mathbb{Q}^r$. Let $I = \{s_1, \ldots, s_r\}$. The sentence ψ_n

holds, thus in particular the formula $\varphi_{n,I}(y)$ holds when we assign the r-tuple a to the r components of y. It follows that $X_{s,a}$ has finitely many singular points, and the result follows from Proposition 41.

Conversely assume that the $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation X_n is also $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Then we show that the formula $\psi_n(X_n)$ holds. Indeed, if this were not the case, then for some $I \subseteq [n]$ the predicate $\forall y \in \mathbb{R}^{[n] \setminus I} \ \varphi_{n,I}(y)$ would be false, i.e $\varphi_{n,I}(y)$ would be false for some assignment of y. This implies that the formula $\gamma_n(y) \equiv \neg \varphi_{n,I}(y)$ (in which the only free variables are the (n-|I|) variables constituting y) defines a nonempty subset Y of $\mathbb{R}^{n-|I|}$. Now γ_n is a $\{+, <, 1, \mathcal{X}_n\}$ -formula, and X_n is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, thus Y is also $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. By Corollary $\mathbf{2}(2)$, Y contains a (n-|I|)-tuple q of rational elements. Therefore the formula $\neg \varphi_{n,I}(y)$ holds when we assign the value q to y, and this implies that there exists a rational generalized section of X which has infinitely many singular points, and by Proposition 41 this leads to a contradiction.

Theorem 43. For every $n \geq 1$ and every $(\mathbb{R}, +, <, \mathbb{Z})$ -definable relation $X \subseteq \mathbb{R}^n$, it is decidable whether X is $(\mathbb{R}, +, <, 1)$ -definable.

Proof. Assume that X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable by the formula $\phi(x)$. In Proposition 42, if we substitute $\phi(x)$ for every occurrence of $x \in \mathcal{X}_n$ in ψ_n , then we obtain a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -sentence Γ_n which holds in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ if and only if X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. The result follows from the decidability of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ [28].

Let us give a fair estimate of the complexity of the decision problem of Theorem 43. One can derive from [28, Section 5] that the known triply-exponential upper bound for the deterministic time complexity of deciding Presburger Arithmetic sentences [23] still holds for $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$. We proved that given $n \geq 1$ and a relation $X \subseteq \mathbb{R}^n$ which is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable by ϕ , the question of whether X is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable amounts to decide whether the sentence Γ_n holds in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$.

It is easy to check that the length of Γ_n is of the order of 2^n times the length of ϕ . Consequently, for fixed n, the length of Γ_n is linear with respect to the one of ϕ , thus we also get a triply-exponential upper bound for the deterministic time complexity of our decision problem.

7 No intermediate structure between $\langle \mathbb{R}, +, <, 1 \rangle$ and $\langle \mathbb{R}, +, <, 1 \rangle$

Our aim is to prove the following result.

Theorem 44. If $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable but not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable then the set \mathbb{Z} is definable in $\langle \mathbb{R}, +, <, 1, X \rangle$.

In other words, for every $X \subseteq \mathbb{R}^n$ which is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable, then $\langle \mathbb{R}, +, <, 1, X \rangle$ is inter-definable with either $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ or $\langle \mathbb{R}, +, <, 1 \rangle$.

7.1 Periodicity in \mathbb{R}

Definition 45. Consider $X \subseteq \mathbb{R}$ and $p \in \mathbb{R} \setminus \{0\}$.

Then X is <u>periodic</u> of <u>period</u> p (or <u>p-periodic</u>) if for every real x we have $x \in X \Leftrightarrow x + p \in X$.

It is <u>ultimately right p-periodic</u> if there exists $m \in \mathbb{R}$ such that for every real x with $x \geq m$, we have $x \in X \Leftrightarrow x + p \in X$. We say that p is a right ultimate period.

It is <u>ultimately left p-periodic</u> if there exists $m \in \mathbb{R}$ such that for every real x with $x \leq m$, we have $x \in X \Leftrightarrow x + p \in X$. We say that p is a left ultimate period.

Observe that the empty set is p-periodic, ultimately right p-periodic, ultimately left p-periodic for every $p \neq 0$. We apply these notions and results concerning $(\mathbb{R}, +, <, 1)$ -and $(\mathbb{R}, +, <, \mathbb{Z})$ -definable subsets of \mathbb{R} .

- **Proposition 46.** 1. $A \ \mathbb{R}, +, <, \mathbb{Z} \$ -definable set $A \subseteq \mathbb{R}$ is periodic if and only if it is of the form $p\mathbb{Z} + B$ where $p \in \mathbb{Q}$ and $B \subseteq [0,p)$ is a finite union of intervals with rational endpoints.
 - 2. For every $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable set $A \subseteq \mathbb{R}$ there exist two periodic sets $A_1, A_2 \subseteq \mathbb{R}$ and two reals m_1, m_2 such that $A \cap [m_1, +\infty) = A_1 \cap [m_1, +\infty)$ and $A \cap (-\infty, m_2] = A_2 \cap (-\infty, m_2]^{\frac{1}{2}}$.
- *Proof.* (1) Assume that A is periodic of period p. Let $B = A \cap [0, p)$. Then $A = p\mathbb{Z} + B$. Since B is $(\mathbb{R}, +, <, 1)$ -definable it is a finite union of intervals included in [0, p) with rational endpoints. The converse is trivial.
- (2) Let $A \subseteq \mathbb{R}$ be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable. We prove the existence of A_1 and m_1 (the proof for A_2 and m_2 is similar). By Theorem 3 we have

$$A = \bigcup_{k=1}^{K} (B_k + C_k) \tag{13}$$

where all $B_k \subseteq \mathbb{Z}$ are disjoint $\langle \mathbb{Z}, +, < \rangle$ -definable subsets and all $C_k \subseteq [0, 1)$ are distinct $\langle \mathbb{R}, +, <, 1 \rangle$ -definable subsets. By [25] for every k there exist two integers $n_k, p_k \geq 0$ such that

$$\forall x > n_k \ x \in B_k \leftrightarrow x + p_k \in B_k \tag{14}$$

Observe that if (14) holds for n_k, p_k then it still holds for the pair of integers $m_1, p \ge 0$ where $p = \text{lcm}\{n_j \mid 1 \le j \le K\}$ and m_1 is a sufficiently large a multiple of p. Therefore for every $k \in \{1, ..., K\}$ we have

$$\forall x \ge m_1 \ x \in B_k \leftrightarrow x + p \in B_k \tag{15}$$

This implies that there exist K disjoint sets $S_k \subseteq \{0, \ldots, p-1\}$ such that for every $k \in \{1, \ldots, K\}$ we have

¹Theorem 6.1 of [28] misses the case where the subset A has different right and left ultimate periods such as $-2\mathbb{N} \cup 3\mathbb{N}$.

$$B_k \cap [m_1, \infty) = (p\mathbb{Z} + S_k) \cap [m_1, \infty).$$

The claim of the proposition follows by setting $A_1 = p\mathbb{Z} + B$ where $B = \bigcup_{k=1}^K S_k + C_k \subseteq [0, p)$.

Lemma 47. With the notations of Proposition 46(2), a real q is a right (resp. left) ultimate period of the $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable subset A if and only if it is a period of A_1 (resp. A_2).

Proof. We only give the proof for A_1 (the proof for A_2 can be handled similarly). It follows from the equivalences

$$\forall y \ge m_1 \quad y \in A \Leftrightarrow y \in A_1 \Leftrightarrow y + q \in A_1 \Leftrightarrow y + q \in A$$
$$\forall y \ge m_1 \quad y \in A_1 \Leftrightarrow y \in A \Leftrightarrow y + q \in A \Leftrightarrow y + q \in A_1$$

Lemma 48. With the notations of Proposition 46 (2), if $A = p\mathbb{Z} + B$ with $\emptyset \subsetneq B \subsetneq [0, p]$, then the set of periods of A is a discrete cyclic subgroup of \mathbb{R} whose elements are rational. It is generated by its element of minimal positive absolute value.

Proof. The set P of periods of A is clearly a subgroup. Let us prove that P is discrete, i.e. that there cannot be arbitrarily small periods. Indeed, set $A = p\mathbb{Z} + B$ where $p \geq 0$, B is a finite union of intervals in [0,p) and $\emptyset \subsetneq B \subsetneq [0,p]$. Let q > 0 be a period of A. We consider three exclusive cases:

- 1. if $B = \{0\}$ or B = (0, p), then the condition $q = 0 + q \in A$ implies $q \ge p$.
- 2. if both B and $[0,p) \setminus B$ consist of a unique interval, then we have either B = [a,p) with a > 0 or B = [0,b) with b < p. In the first case we have $q \ge a$ and in the latter case $q \ge p b$.
- 3. if B or $[0,p) \setminus B$ consist of at least two intervals: since a set and its complement have the same periods, we can assume without loss of generality that B contains two disjoint and consecutive intervals with respective extremities a_1, b_1 and a_2, b_2 (the proof for the other case is similar). If $b_1 = a_2$ then (a_1, b_1) is right open and (a_2, b_2) is left open. Then the fact that q is a period and the equality $b_1 = a_2 \in A$ imply $b_1 q, a_2 + q \in A$ thus $q \ge \max\{b_1 a_1, b_2 a_2\}$. Now if $b_1 < a_2$ then for all $a_1 < y < b_1$ we have $y + q \in A$ which implies $q \ge a_2 b_1$.

We proved that P admits a minimal positive element, say p_0 . The fact that P is cyclic and generated by p_0 is well-known. In order to prove that $P \subseteq \mathbb{Q}$, it suffices to prove that $p_0 \in \mathbb{Q}$. Now the $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -formula

$$\phi(p) = p > 0 \land \forall x \ (x \in A \Leftrightarrow x + p \in A)$$

defines the set P^+ of positive elements of P, thus p_0 is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable as the minimal element of P^+ , and the result follows from Corollary 2(2).

As a consequence of Lemmas 47 and 48, and Proposition 46 we obtain the following result.

Lemma 49. If $A \subseteq \mathbb{R}$ is $(\mathbb{R}, +, <, \mathbb{Z})$ -definable but not $(\mathbb{R}, +, <, 1)$ -definable then A has either a minimal ultimate right or a minimal ultimate left period.

Proof. Either $A \cap (-\infty, 0]$ or $A \cap [0, \infty)$ is not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Assume that the latter case holds, and let us prove that A admits a minimal ultimate right period. By Proposition 46 there exist a periodic subset A_1 and a real m such that $A \cap [m, \infty[=A_1 \cap [m, \infty[$. With the notations of Proposition 46 we have $A_1 = p\mathbb{Z} + B$ where $B \subseteq [0, p)$. Now the set $A \cap [0, \infty)$ is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable but not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, thus by Lemma 4 it cannot have an upper bound, and the same holds for A_1 . It follows that $B \neq \emptyset$. Using a similar argument with the complement of A leads to $B \neq [0, p)$. Thus by Lemmas 47 and 48, A_1 admits a minimal period which is also a minimal ultimate right period of A.

Similarly if $A \cap (-\infty, 0]$ is not $(\mathbb{R}, +, <, 1)$ -definable then A admits a minimal ultimate left period.

Lemma 50. Let $A \subseteq \mathbb{R}$ be an ultimately right periodic set and let p > 0 be its minimal ultimate right period.

1. For all reals x, y, if

$$\forall u \ge 0 \ (x + u \in A \Leftrightarrow y + u \in A) \tag{16}$$

then p divides x - y.

2. Conversely for every z multiple of p there exist x, y satisfying (16) and x - y = z.

Proof. (1) It suffices to prove that x - y = q is an ultimate right period. We have

$$\forall v \ge x \quad v = x + (v - x) \in A \Leftrightarrow y + (v - x) \in A \Leftrightarrow v + q \in A$$

(2). Let m be such that

$$\forall t \geq m \ (t \in A \Leftrightarrow t + p \in A)$$

This condition implies that for all positive multiples q of p we have

$$\forall t \ge m \ (t \in A \Leftrightarrow t + q \in A)$$

Set x = m, y = m + q and u = t - m. The above condition is equivalent to

$$\forall u > 0 \ (x + u \in A \Leftrightarrow y + u \in A)$$

7.2 Proof of Theorem 44

We now proceed by induction on the dimension n. Assume n = 1. If X is definable in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ but not in $\langle \mathbb{R}, +, <, 1 \rangle$ then by Lemma 49, the set X has either a minimal ultimate left or a minimal ultimate right period. Assume w.l.o.g that the latter case holds, i.e that X has an ultimate right period which is a rational number, say $p = \frac{a}{h}$.

Applying Lemma 50 the subset \mathbb{Z} can be defined in $(\mathbb{R}, +, <, 1, X)$ by the formula

$$\phi_X(x) = \exists y, z \ \forall u \ge 0 \ ((y + u \in X \Leftrightarrow z + u \in X) \land ax = b(y - z))$$

Now we pass to the general case $n \geq 2$. Let $X \subseteq \mathbb{R}^n$ be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable but not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. By Theorem 37, either some rational section of X is not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, or X admits infinitely many singular points. In the first case, the result follows from the induction hypothesis and the fact that every rational section of X is definable in $\langle \mathbb{R}, +, <, 1, X \rangle$ thus also in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$.

In the second case, by Proposition 27 the set X contains a countably infinite number of singular points. The set S of singular points is $\langle \mathbb{R}, +, <, 1, X \rangle$ -definable by the formula (12) thus it is also $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable and the same holds for any projection of S over a component. Therefore some of the n projections over the n components is a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable subset of \mathbb{R} and contains a countably infinite number of singular points thus is not $\langle \mathbb{R}, +, <, 1 \rangle$ -definable, and we may apply case n = 1.

8 Yet another characterization

We provide in this section an alternative characterization of $(\mathbb{R}, +, <, 1)$ -definability for $(\mathbb{R}, +, <, \mathbb{Z})$ -definable relations.

A line in \mathbb{R}^n is <u>rational</u> if it is the intersection of hyperplanes defined by equations with rational coefficients. Every rational line is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

Theorem 51. A $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable if and only if the intersection of X with every rational line is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

Proof. The condition is necessary because every rational line is $(\mathbb{R}, +, <, 1)$ -definable and the set of $(\mathbb{R}, +, <, 1)$ -definable relations is closed under intersection.

Now we prove the converse. By Proposition 41 it suffices to prove that all rational restrictions have finitely many singular points. Since all rational restrictions of X are $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable the result will follow from the next Lemma.

Lemma 52. Let $X \subseteq \mathbb{R}^n$ be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable relation and assume the intersection of X with every rational line is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable. Then X has finitely many singular points.

Proof. We prove that if X has infinitely many singular points then there exists some rational line l such that $l \cap X$ is not a finite union of segments, which contradicts the fact that $l \cap X$ is $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

An hypercube of the form $[a_1, a_1 + 1] \times \cdots \times [a_n, a_n + 1]$ with $a_1, \ldots, a_n \in \mathbb{Z}$ is called <u>elementary</u>. All X-singular points which belong to the interior of some elementary hypercube H are $(X \cap H)$ -singular and the converse is true. Observe that a point may be X-singular without being $(X \cap H)$ -singular if it belongs to the boundary of the hypercube. In order to avoid this problem we consider an elementary cube surrounded by its $3^n - 1$ neighbours. Then a point in a elementary cube is X-singular if and only if it is singular in this enlarged hypercube.

Formally, we start by extending the notion of integer part to vectors in \mathbb{R}^n by setting for every point $x \in \mathbb{R}^n$

$$\lfloor x \rfloor = (\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$$

Set $S = \{-1, 0, 1\}^n$. For all $\sigma \in S$ and $a \in \mathbb{Z}^n$ define

$$D_{\sigma} = \overbrace{[0,1] \times \cdots \times [0,1]}^{n \text{ times}} + \sigma, \ H_{\sigma}(a) = (a + D_{\sigma}) \cap X, \ L_{\sigma}(a) = -a + H_{\sigma}(a)$$

and

$$H(a) = \bigcup_{\sigma \in \mathcal{S}} H_{\sigma}(a), \ L(a) = \bigcup_{\sigma \in \mathcal{S}} L_{\sigma}(a)$$

Because of the decomposition of Theorem 3, the set \mathcal{L} of distinct subsets L(a), called elementary neighborhood, is finite when a ranges over \mathbb{Z}^n since each $-\sigma + L_{\sigma}(a)$ is equal to some $X_k^{(F)}$. Furthermore, say that H(a) contains a singular point x if x is X-singular and $x \in H_{(0,\dots,0)}(a)$. If there exist infinitely many singular points, then there exists a fixed $L \in \mathcal{L}$ and infinitely many a's such that $a + L_{(0,\dots,0)}(a)$ contains a singular point. Furthermore, in each $L_{(0,\dots,0)}(a)$ there exist at most finitely many X-singular points. Consequently, there exists a fixed point $z \in \mathbb{Q}^n \cap [0,1] \times \cdots \times [0,1]$ and a fixed elementary neighborhood L such that for infinitely many $a \in \mathbb{Z}^n$ the point a + z is X-singular and $L_a = L$. Consider the mapping $\iota : \mathcal{S} \mapsto \{1,\dots,K\}$ such that

$$L = \bigcup_{\sigma \in \mathcal{S}} \sigma + X_{\iota(\sigma)}^{(F)}$$

The set of elements a satisfying $L_a = L$ is the intersection

$$\bigcap_{\sigma \in \mathcal{S}} -\sigma + X_{\iota(\sigma)}^{(I)} \subseteq \mathbb{Z}^n$$

which is infinite and semilinear. Now, all infinite semilinear subsets contain a subset of the form $A = u + \mathbb{N}v$ where $u, v \in \mathbb{Z}^n$ with $v \neq 0$.

For some sufficiently small positive real r, for all $w \in A$ the sets $-w + (B(w,r) \cap X)$ are identical. Consider any element $w' \in A$, and assume further that $r \in \mathbb{Q}$. Since w' is singular, $B(w',r) \cap X$ is not a union of lines parallel to any direction and in particular to the direction v. Thus the set Y_r of points $t \in \mathbb{R}^n$ such that |t| < r and $B(w',r) \cap L_v(w'+t)$ intersects both X and its complement is not empty. Now both w' and r have rational components, and X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable, thus Y_r is also $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable, and since it is bounded it is also $\langle \mathbb{R}, +, <, 1 \rangle$ -definable by Lemma 4. By Corollary 2(2), Y_r

contains an element $t \in \mathbb{Q}^n$. Now for all $w \in A$ the sets $-w + (B(w,r) \cap X)$ are identical, and |t| < r, thus for every $w \in A$ the segment $B(w,r) \cap L_v(w+t)$ intersects both X and its complement. It follows that the line $\ell = L_v(w'+t)$ (which coincides with all lines $L_v(w+t)$ for $w \in A$) is such that $l \cap X$ is not a finite union of segments, and thus by Corollary 2(1) cannot be $\langle \mathbb{R}, +, <, 1 \rangle$ -definable.

9 Conclusion

We discuss some extensions and open problems.

It is not difficult to check that the main arguments used to prove Theorems 43, 44 and 51, still hold if one replaces \mathbb{R} with \mathbb{Q} . Observe that by [21], $\langle \mathbb{Q}, +, <, \mathbb{Z} \rangle$ and $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ are elementary equivalent structures.

Is it possible to remove our assumption that X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ -definable in Theorem 37? We believe that the answer is positive and it can be formally proven in dimension 2. Note that even if one proves such a result, the question of providing an effective characterization is more complex. Indeed the sentence ψ_n of Proposition 42 expresses a variant of the criterion of Theorem 37, and we use heavily the fact that we work within $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ to ensure that this variant is actually equivalent to the criterion. In particular the construction of ψ_n relies on Lemma 30 to express that a point is X-singular. However if we consider, e.g., $X = \mathbb{Q}$ then every element x of X is singular while no element x of X satisfies the condition stated in Lemma 30.

Another question is the following. In Presburger arithmetic it is decidable whether or not a formula is equivalent to a formula in the structure without <, cf. [12]. What about the case where the structure is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$?

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