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# Proof analysis in intermediate logics

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**Abstract** Using labelled formulae, a cut-free sequent calculus for intuitionistic propositional logic is presented, together with an easy cut-admissibility proof; both extend to cover, in a uniform fashion, all intermediate logics characterised by frames satisfying conditions expressible by one or more geometric implications. Each of these logics is embedded by the Gödel–McKinsey–Tarski translation into an extension of **S4**. Faithfulness of the embedding is proved in a simple and general way by constructive proof-theoretic methods, without appeal to semantics other than in the explanation of the rules.

**Keywords** Sequent calculus · Modal logic · Intermediate logic · Labelled deduction · Modal companion

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## 1 Introduction

Following Gentzen's pioneering work in the 1930s on sequent calculus, for classical and intuitionistic logic, important advances were made by Ketonen and Kanger. Ketonen [21] proposed calculi organised for root-first rather than for leaf-first proof

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construction, i.e. for the analysis of problems rather than for the synthesis of solutions; these are now better known as Kleene's calculi **G3**. Kanger [20] proposed for the classical modal logic **S5** (but the method is immediately applicable to other logics) the use of "spotted" formulae, to incorporate (in effect) the semantics; these were developed with names like 'semantic tableaux', 'labelled formulae' and 'prefixed formulae' by various authors, notably Fitch [13], Fitting [14], Kripke [23,24], Maslov [26] and Vigano [43]. Even more recently, such ideas have been studied more widely [15] (but in natural deduction settings) as 'labelled deductive systems'. We shall follow common usage by generally using 'label' despite Kanger's prior use of 'spot'; in particular cases we will also use 'world'.

The non-classical, label-free, sequent calculi (such as **G3ip** in [35]) developed from the Gentzen–Ketonen–Kleene approach generally have the feature that some of the inference rules are non-invertible; Gödel–Dummett propositional logic is an interesting exception [9,10]. In contrast, Kanger-style calculi incorporate semantic information into labels while retaining a classical basis for the inference rules themselves; despite the labelling, this basis ensures that all the inference rules are invertible, as emphasised by Maslov in the title of [26]. Subsequent developments have focused on efficient representations [14] (e.g. lists of indices) of the labels and on algorithms [44] for matching such representations; a useful survey is given by [17].

However, the underlying logical structure of the calculi can best be seen by ignoring such representations. Progress in this direction was made in [34,35,38], which showed that, in many cases, mathematical axioms can be added as inference rules to cut-free **G3**-style calculi while maintaining the admissibility of *Cut*. This is feasible for axioms that are either $\Pi_1$-formulae (i.e. universal closures of quantifier-free formulae) or, more generally [29], axioms of geometric theories (i.e. geometric implications).

Further, as developed in [30], one may use a combination of labelled formulae and of (quantifier-free) formulae expressing Kripke-style accessibility relations between labels (treated as worlds). In this fashion one may develop [30] Kanger-style cut-free calculi for a wide range of modal logics—essentially those where the axioms for the accessibility relation are geometric implications: this includes logics such as **S5** but not, in general, those where second-order notions (such as transitive closure) are used. In [30], however, it is shown how the provability logic **GL** of Gödel and Löb can be treated, despite the impossibility of expressing the finiteness condition on the accessibility relation in first-order terms.

The purpose of the present paper is to present the corresponding treatment of intermediate (also known as "super-intuitionistic") propositional logics; whereas in the modal case it is the inference rules for the modal operators that have non-trivial manipulation of labels, in the present case it is, unsurprisingly, the rules for implication that use such manipulations.

The advantage of this treatment is that, without the complexities involved with efficient representations of the labels and with matching algorithms, we obtain in a uniform way, for a wide range of intermediate logics, both elegant proof systems and results about those proof systems. As a consequence, mutual relationships between the corresponding logics can be established in a direct way. In particular, we give a straightforward proof of the faithfulness of the Gödel representations of these intermediate logics as modal logics between **S4** and **S5**, and will give (in a sequel [11]) a similar treatment using Grzegorczyk logic **Grz** in place of **S4**.

This faithfulness proof is much simpler than the standard proof-theoretic approach (e.g. [42, Sect. 9.2]). Generality and uniformity are achieved by the use of labels, which might make it appear to be a model-theoretic method in disguise. However, unlike in a model-theoretic proof, the argument is purely syntactic and constructive, with no appeal to the non-constructive features of classical logic.

## 2 Preliminaries

As noted above, in [29,34,35] a general method was presented for extending sequent calculi with rules for axiomatic theories while preserving all the structural properties of the logical calculus. We recall the general ideas of the method and the main results.

For extensions of classical predicate logic the starting point is the contraction- and cut-free sequent calculus **G3c** (cf. [35,42] for the rules). We recall (see Chapters 3 and 4 of [35] for detailed proofs) that all the rules of **G3c** are invertible and all the structural rules are admissible. Weakening and contraction are in addition *height-preserving*- (abbreviated *hp*-) admissible, that is, whenever their premisses are derivable, so also is their conclusion, with at most the same derivation height (the *height* of a derivation is its height as a tree, that is, the length of its longest branch). Moreover, the calculus enjoys hp-admissibility of substitution. Invertibility of the rules of **G3c** is also height-preserving, that is, the rules are *hp-invertible*.

These remarkable structural properties of **G3c** are maintained in extensions of the logical calculus with suitably formulated rules that represent axioms for specific theories. Universal axioms are first transformed, through the rules of **G3c**, into conjunctive normal form, that is, conjunctions of formulas of the form $P_1 \& \cdots \& P_m \supset Q_1 \vee \cdots \vee Q_n$, where the consequent is $\perp$ if $n = 0$ and all $P_i$, $Q_j$ are atomic. The universal closure of any such formula is called a *regular* formula. We abbreviate the multiset $P_1, \ldots, P_m$ as $\overline{P}$. Each conjunct is then converted into a schematic rule, called the *regular rule scheme*, of the form

$$\frac{Q_1, \overline{P}, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\; Reg \; .$$

By this method, all universal theories can be formulated as contraction- and cut-free systems of sequent calculi.

In [29], the method is extended to cover also *geometric theories*, that is, theories axiomatized by geometric implications. We recall that a *geometric formula* is a formula not containing $\supset$, $\neg$, or $\forall$ and a *geometric implication* is a sentence of the form

$$\forall \overline{z}(A \supset B)$$

where $A$ and $B$ are geometric formulas. Geometric implications can be reduced to a normal form consisting of conjunctions of formulas, called *geometric axioms*, of the form

$$\forall \overline{z}((P_1 \& \cdots \& P_m) \supset \exists \overline{x}(M_1 \vee \cdots \vee M_n))$$

where each $M_j$ is a conjunction of atomic formulas, $Q_{j_1}, \ldots, Q_{j_{k_j}}$ and $\overline{z}$ and $\overline{x}$ are sequences of bound variables. For simplicity, we assume (as in [30]) that the sequence $\overline{x}$ of bound variables has length 1 and we distribute the existential quantifier over the disjuncts, as in $\exists x_1 M_1 \vee \cdots \exists x_n M_n$.

Without loss of generality, no $x_i$ is free in any $P_j$. Note that regular formulas are geometric implications, with neither conjunctions nor existential quantifications to the right of the implication. The *geometric rule scheme* for geometric axioms takes the form

$$\frac{\overline{Q}_1(y_1/x_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n(y_n/x_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta} GRS$$

where $\overline{Q}_j$ and $\overline{P}$ indicate the multisets of atomic formulas $Q_{j_1}, \ldots, Q_{j_{k_j}}$ and $P_1, \ldots, P_m$, respectively, and the *eigenvariables* $y_1, \ldots, y_n$ of the premisses are not free in the conclusion. We use the notation $A(y/x)$ to indicate $A$ after the substitution of the variable $y$ for the variable $x$. The variables $y_i$ are, following tradition, called *eigenvariables* to emphasise their freshness.

In order to maintain admissibility of contraction in the extensions with regular or geometric rules, the formulas $P_1, \ldots, P_m$ in the antecedent of the conclusion of the scheme have (as indicated) to be repeated in the antecedent of each of the premisses. In addition, whenever an instantiation of free parameters in atoms produces a duplication (two identical atoms) in the conclusion of a rule instance, say $P_1, \ldots, P, P, \ldots, P_m, \Gamma \Rightarrow \Delta$, there is of course a corresponding duplication in each premiss. The *closure condition* imposes the requirement that the rule with the duplication $P, P$ contracted into a single $P$ is added to the system of rules. For each axiom system, there is only a bounded number of possible cases of contracted rules to be added, very often none at all, so the condition is unproblematic.

For example, with the binary predicate symbol $R$ used as an infix operator, from the geometric implication

$$\forall xy(xRy \supset ((\exists z(zRx)) \vee (\exists w(yRw))))$$

one obtains the rule scheme (in which $z$ and $w$ are eigenvariables, and so are fresh):

$$\frac{zRx, xRy, \Gamma \Rightarrow \Delta \quad xRy, yRw, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}.$$

The main result for such extensions is the following (Theorems 4 and 5 from [29]):

**Theorem 1** *The structural rules of Weakening, Contraction and Cut are admissible in all extensions of* **G3c** *with the geometric rule-scheme and satisfying the closure condition. In fact, Weakening and Contraction are hp-admissible.*

## 3 Modal logic

The above method of extension of sequent calculi can be applied not only *outside* logic for obtaining a proof-theoretical treatment of specific theories such as lattice theory [36,37], arithmetic and geometry [35] but also *inside* logic. In [30] rules expressing properties of binary relations are added to a basic labelled sequent calculus for the normal modal logic **K** in such a way that complete systems for all the modal logics characterized by geometric frame conditions are obtained. The basic labelled sequent calculus is obtained by prefixing with labels the formulas in the rules of the sequent calculus **G3cp** for classical propositional logic. As initial sequents we take those of the form $x : P, \Gamma \Rightarrow \Delta, x : P$; here, as elsewhere in this paper, $P$ is atomic. In each rule, the active and principal formulas are prefixed by the same label. This corresponds to the classical explanation of truth in Kripke semantics, flat on all the propositional logical constants. For instance, the rules for conjunction are

$$\frac{x:A, x:B, \Gamma \Rightarrow \Delta}{x:A\&B, \Gamma \Rightarrow \Delta}L\& \qquad \frac{\Gamma \Rightarrow \Delta, x:A \quad \Gamma \Rightarrow \Delta, x:B}{\Gamma \Rightarrow \Delta, x:A\&B}R\&$$

and those for implication are

$$\frac{\Gamma \Rightarrow \Delta, x:A \quad x:B, \Gamma \Rightarrow \Delta}{x:A\supset B, \Gamma \Rightarrow \Delta}L\supset \qquad \frac{x:A, \Gamma \Rightarrow \Delta, x:B}{\Gamma \Rightarrow \Delta, x:A\supset B}R\supset$$

The rules for the modal operator $\Box$ are obtained similarly from its semantical explanation in terms of possible worlds

$$x:\Box A \textit{ iff for all } y,\ xRy \textit{ implies } y:A$$

that gives the rules

$$\frac{y:A, x:\Box A, xRy, \Gamma \Rightarrow \Delta}{x:\Box A, xRy, \Gamma \Rightarrow \Delta}L\Box \qquad \frac{xRy, \Gamma \Rightarrow \Delta, y:A}{\Gamma \Rightarrow \Delta, x:\Box A}R\Box$$

with $y$ an *eigenvariable* in $R\Box$ (so $y$ is *fresh*, i.e. is not free in the conclusion).

The resulting sequent calculus, called **G3K**, gives a complete system for the basic normal modal logic **K**. Thus, a formula $A$ is valid in **K** if and only if, for some (or, indeed, any) label $x$, the sequent $\Rightarrow x : A$ is derivable. This logic is characterized by arbitrary frames; correspondingly, there are no rules for the accessibility relation. The sequent calculi for the modal logics **T, K4, KB, S4, B, S5** are obtained by adding to **G3K** the rules expressing their *frame conditions*, i.e. the properties of the accessibility relation that characterize their frames. For instance, a sequent calculus for the modal logic **S4** is obtained by adding the rules for reflexivity and transitivity of the accessibility relation

$$\frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\textit{Refl} \qquad \frac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\textit{Trans}$$

We recall from [30] the following properties of any extension **G3K*** of **G3K** with geometric rules for the frame condition:

**Theorem 2** 1. *All sequents of the form $x : A, \Gamma \Rightarrow \Delta, x : A$ are derivable, for arbitrary $A$, in* **G3K***.

2. *All sequents of the form*

$$\Rightarrow x : \Box(A \supset B) \supset (\Box A \supset \Box B)$$

*are derivable in* **G3K***.

3. *The substitution rule*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma(y/x) \Rightarrow \Delta(y/x)}\ (y/x)$$

*is hp-admissible in* **G3K***.

4. *The rules of Weakening*

$$\frac{\Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta} LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : A} RW \qquad \frac{\Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta} LW_R$$

*are hp-admissible in* **G3K***.

5. *The Necessitation rule*

$$\frac{\Rightarrow x : A}{\Rightarrow x : \Box A}\ Nec$$

*is admissible in* **G3K***.

6. *The modal axioms characterised by each frame condition are derivable in* **G3K***.
7. *All the primitive rules of* **G3K*** *are hp-invertible.*
8. *The rules of Contraction*

$$\frac{x : A, x : A, \Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta}\ L\text{-}Ctr \qquad \frac{xRy, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\ L\text{-}Ctr_R$$

$$\frac{\Gamma \Rightarrow \Delta, x : A, x : A}{\Gamma \Rightarrow \Delta, x : A}\ R\text{-}Ctr$$

*are hp-admissible in* **G3K***.

9. *The Cut rule*

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

*is admissible in* **G3K***.

**Table 1** The system **G3I**

**Initial sequents:**

$x \leqslant y, x:P, \Gamma \Rightarrow \Delta, y:P$

**Logical rules:**

$$\frac{x:A, x:B, \Gamma \Rightarrow \Delta}{x:A\&B, \Gamma \Rightarrow \Delta}\, L\& \qquad \frac{\Gamma \Rightarrow \Delta, x:A \quad \Gamma \Rightarrow \Delta, x:B}{\Gamma \Rightarrow \Delta, x:A\&B}\, R\&$$

$$\frac{x:A, \Gamma \Rightarrow \Delta \quad x:B, \Gamma \Rightarrow \Delta}{x:A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x:A, x:B}{\Gamma \Rightarrow \Delta, x:A \vee B}\, R\vee$$

$$\frac{x \leqslant y, x:A \supset B, \Gamma \Rightarrow \Delta, y:A \qquad x \leqslant y, x:A \supset B, y:B, \Gamma \Rightarrow \Delta}{x \leqslant y, x:A \supset B, \Gamma \Rightarrow \Delta}\, L\supset$$

$$\frac{}{x:\perp, \Gamma \Rightarrow \Delta}\, L\perp \qquad \frac{x \leqslant y, y:A, \Gamma \Rightarrow \Delta, y:B}{\Gamma \Rightarrow \Delta, x:A \supset B}\, R\supset$$

**Order rules:**

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, \mathit{Refl} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\, \mathit{Trans}$$

Rule $R\supset$ has the condition that $y$ is not in the conclusion.

## 4 Intuitionistic logic

We present here (Table 1) a **G3**-style labelled calculus **G3I** with internalized Kripke semantics for intuitionistic propositional logic [28]. Following the general method of Sect. 3 above, we will later extend the calculus to cover all the intermediate propositional logics characterized by frame properties expressible by means of geometric implications. Admissibility of the structural rules will be proved (in Sect. 5 below) in a uniform way for all such extensions of the calculus **G3I**.

The forcing relation $x \Vdash A$ of Kripke models is again part of the formal syntax; we represent it as above by $x : A$. The accessibility relation for intuitionistic logic is a partial order, represented as usual by $\leqslant$. In the calculus, sequents are expressions of the form $\Gamma \Rightarrow \Delta$ where $\Gamma$ and $\Delta$ are multisets of expressions $x:A$, with $x$ ranging in an infinite set $W$ (the elements of which we call *variables* or *labels* rather than "worlds") and with $A$ any formula in the language of intuitionistic propositional logic. $\Gamma$ may also contain expressions $x \leqslant y$, hereinafter called *(relational) atoms*. Negation is, as usual, a defined notion and not considered here. Recall from Sect. 3 that $P$ is restricted to range over atomic formulae.

The system **G3I** formalizes into sequent rules the inductive definition of truth in a Kripke model. As in modal logic, the intention is that a formula $A$ is valid if and only if for some (or any) label $x$ the sequent $\Rightarrow x:A$ is derivable. Standard properties such as monotonicity of the forcing relation can be obtained by means of formal derivations in the calculus:

**Lemma 1** *All sequents of the following form are derivable in* **G3I***:*

1. $x \leqslant y, x:A, \Gamma \Rightarrow \Delta, y:A$
2. $x:A, \Gamma \Rightarrow \Delta, x:A$

*Proof* By mutual induction on the structure of $A$. The implication from 1. to 2. at each step of the induction is routine by *Refl*. For atoms $P$ and $\bot$, the proof of 1. is trivial. For $A \equiv B \supset C$ we have the following derivation

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \ldots, x:B\supset C, z:B, \Gamma \Rightarrow \Delta, z:C, z:B\end{matrix} \quad \begin{matrix}\vdots\\ \ldots, x:B\supset C, z:C, z:B, \Gamma \Rightarrow \Delta, z:C\end{matrix}}{x\leqslant y,\ y\leqslant z,\ x\leqslant z,\ x:B\supset C, z:B, \Gamma \Rightarrow \Delta, z:C}\ L\supset}{x\leqslant y,\ y\leqslant z,\ x:B\supset C, z:B, \Gamma \Rightarrow \Delta, z:C}\ \mathit{Trans}}{x\leqslant y,\ x:B\supset C, \Gamma \Rightarrow \Delta, y:B\supset C}\ R\supset$$

of 1., where the premisses of $L\supset$ are derivable by the inductive hypothesis for 2. The cases where $A$ is a conjunction or disjunction are handled by the inductive hypothesis for 1. □

In the next section we will see that the usual structural rules are admissible not only in this calculus but also in the extensions with rules for geometric implications. Completeness of each such extension of the calculus will follow.

## 5 Main results

The system **G3I** can be extended with rules expressing additional properties of the partial order $\leqslant$ exactly as in Sect. 3 above. We denote by **G3I*** any extension of **G3I** with rules following the geometric rule scheme, such as the calculus for Gödel–Dummett logic with its "strongly connected" frame condition translated as the rule

$$\frac{x\leqslant y,\ x\leqslant z,\ y\leqslant z,\ \Gamma \Rightarrow \Delta \quad x\leqslant y,\ x\leqslant z,\ z\leqslant y,\ \Gamma \Rightarrow \Delta}{x\leqslant y,\ x\leqslant z,\ \Gamma \Rightarrow \Delta}\ .$$

Other examples can be seen in Sect. 6.

In this section we shall prove that all the structural rules—*Weakening*, *Contraction* and *Cut*—are admissible not only in the calculus **G3I** but also in each of its extensions **G3I*** with rules following the geometric rule scheme.

**Lemma 2** *All sequents of the form $x\leqslant y, x:A, \Gamma \Rightarrow \Delta, y:A$ are derivable in* **G3I***.

*Proof* Sequents of the form $x\leqslant y, x:A, \Gamma \Rightarrow \Delta, y:A$ are already derivable in the subsystem **G3I**. □

Next, we need an auxiliary lemma concerning admissibility of substitution in **G3I***. A similar lemma was needed in [30], owing to the presence of labels in the syntax. We define *substitution* in the obvious way as follows:

$$(x\leqslant y)(z/w) \equiv x\leqslant y \text{ if } w \neq x \text{ and } w \neq y$$
$$(x\leqslant y)(z/x) \equiv z\leqslant y \text{ if } x \neq y$$
$$(x\leqslant y)(z/y) \equiv x\leqslant z \text{ if } x \neq y$$

$$\begin{aligned}(x \leqslant x)(z/x) &\equiv z \leqslant z \\ (x : A)(z/y) &\equiv x : A \text{ if } x \neq y \\ (x : A)(z/x) &\equiv z : A\end{aligned}$$

and extend the definition to multisets componentwise. We then have:

**Lemma 3** *The substitution rule*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma(y/x) \Rightarrow \Delta(y/x)}\ (y/x)$$

*is hp-admissible in* **G3I***.

*Proof* By induction on the height $n$ of the derivation of $\Gamma \Rightarrow \Delta$.

If $n = 0$, and $(y/x)$ is not a vacuous substitution, the sequent can either be an initial sequent of the form $x \leqslant y, x : P, \Gamma' \Rightarrow \Delta', y : P$ or be of the form $\bot, \Gamma' \Rightarrow \Delta$. In each case $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is either an initial sequent of the same form or a conclusion of $L\bot$.

Suppose $n > 0$, and consider the last rule applied in the derivation. If it is a rule for & or $\vee$, apply the inductive hypothesis to the premiss(es) of the rule, and then the rule. Proceed similarly if the last rule is $L\supset$. If the last rule is $R\supset$ and $x$ is an eigenvariable of the rule then the substitution is vacuous. Else, if $y$ is not an eigenvariable, we apply the inductive hypothesis to the derivation of the premiss, and then $R\supset$.

If $y$ is the eigenvariable, we first apply the inductive hypothesis in order to replace the eigenvariable $y$ with a fresh variable $w$. By the variable condition the substitution does not affect the context, and we proceed as in the previous case.

For extensions of **G3I** with geometric rules, some care is needed in order to avoid a clash with the eigenvariables of the geometric rule scheme. The details are similar to those in [29]. □

**Proposition 1** *The rules of Weakening*

$$\frac{\Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : A}\ RW \qquad \frac{\Gamma \Rightarrow \Delta}{x \leqslant y, \Gamma \Rightarrow \Delta}\ LW_{\leqslant}$$

*are hp-admissible in* **G3I***.

*Proof* Straightforward induction on the height of the derivation of the premiss for the rules for & or $\vee$ or for $L\supset$. In case the last step is $R\supset$, the substitution lemma is applied to the premisses of the rule in order to have a fresh eigenvariable not clashing with those in $x : A$ or $x \leqslant y$. The conclusion is then obtained by applying the inductive hypothesis and the rule. An identical procedure is applied if the last step is a geometric rule and $x : A$ or $x \leqslant y$ contain some of its eigenvariables. □

In order to prove hp-admissibility of *Contraction* we need to show hp-invertibility of the rules of the sequent calculi **G3I***:

**Proposition 2** *All the rules of* **G3I*** *are hp-invertible.*

*Proof* The proof of hp-invertibility for the rules for & or $\vee$ is done exactly as for **G3c** (Theorem 3.1.1 in [35]). Rule $L\supset$ and all instances of $GRS$ (including *Refl* and *Trans*) are hp-invertible by Proposition 1.

For $R\supset$ we use induction on the height $n$ of the derivation of $\Gamma \Rightarrow \Delta, x{:}A\supset B$. If $n = 0$, it is an initial sequent or a conclusion of $L\bot$, but then $x\leqslant y, y{:}A, \Gamma \Rightarrow \Delta, y{:}B$ is also an initial sequent or a conclusion of $L\bot$ (observe that it is essential here that the initial sequents are restricted to atomic formulas).

If $n > 0$ and $\Gamma \Rightarrow \Delta, x{:}A\supset B$ is concluded by any rule $\mathcal{R}$ other than $R\supset$, we apply the inductive hypothesis to the premiss(es) $\Gamma' \Rightarrow \Delta', x{:}A\supset B$ (and perhaps also $\Gamma'' \Rightarrow \Delta'', x{:}A\supset B$) and obtain derivation(s) of height at most $n-1$ of the sequent $x\leqslant y, y{:}A, \Gamma' \Rightarrow \Delta', y{:}B$ (and perhaps also of $x\leqslant y, y{:}A, \Gamma'' \Rightarrow \Delta'', y{:}B$). By applying rule $\mathcal{R}$ we obtain a derivation of height at most $n$ of the sequent $x\leqslant y, y{:}A, \Gamma \Rightarrow \Delta, y{:}B$. If $\Gamma \Rightarrow \Delta, x{:}A\supset B$ is a conclusion of $R\supset$ with principal formula in $\Delta$, we proceed in a similar way.

If instead the principal formula is $A\supset B$, the premiss of the last step gives the conclusion, using hp-admissibility of substitution if necessary. □

We are now in a position to prove the most important structural property of our calculi besides cut-admissibility, namely hp-admissibility of the rules of contraction:

**Theorem 3** *The rules of Contraction*

$$\frac{x{:}A, x{:}A, \Gamma \Rightarrow \Delta}{x{:}A, \Gamma \Rightarrow \Delta}\ L\text{-}Ctr \qquad \frac{\Gamma \Rightarrow \Delta, x{:}A, x{:}A}{\Gamma \Rightarrow \Delta, x{:}A}\ R\text{-}Ctr$$

$$\frac{x\leqslant y, x\leqslant y, \Gamma \Rightarrow \Delta}{x\leqslant y, \Gamma \Rightarrow \Delta}\ L\text{-}Ctr_{\leqslant}$$

*are hp-admissible in* **G3I***.

*Proof* By simultaneous induction on the derivation height.

If $n = 0$ the premiss is either an initial sequent or a conclusion of $L\bot$. In each case the contracted sequent is also an initial sequent or a conclusion of $L\bot$.

If $n > 0$, consider the last step, by some rule $\mathcal{R}$, used to derive the premiss of the contraction step. If the contraction formula is not principal in it, both occurrences are found in the premiss(es) of the step, which have smaller derivation height. By the induction hypothesis, they can be contracted and the conclusion is obtained by applying rule $\mathcal{R}$ to the contracted premiss(es).

If the contraction formula is principal in it, we distinguish three cases: Either $\mathcal{R}$ is a rule in which the principal formulas appear also in the premiss (such as $L\supset$ or the rules for $\leqslant$), or it is a rule where the premisses consist of proper subformulas of the conclusion (such as the rules for & and $\vee$), or it is a rule, in fact $R\supset$, where the premisses consist of atoms $x\leqslant y$ and proper subformulas of the conclusion. In the first case contraction is applied, by induction hypothesis, to the premiss(es) of the rule. In case both contraction formulas are principal in a rule for $\leqslant$, the conclusion holds by the closure condition. This applies also to *Trans* by the presence of rule *Ref*. In the second case, contraction is reduced to contraction to smaller formulas as in the standard proof for **G3c**.

In the third case, both a subformula of the contraction formula and an atom $x \leqslant y$ are found in the premiss, for instance

$$\frac{x \leqslant y,\, y:A,\, \Gamma \Rightarrow \Delta,\, y:B,\, x:A \supset B}{\Gamma \Rightarrow \Delta,\, x:A \supset B,\, x:A \supset B}\; R\supset$$

By hp-invertibility of $R\supset$ applied to the premiss, we obtain a derivation of height at most $n-1$ of

$$x \leqslant y,\, x \leqslant y,\, y:A,\, y:A,\, \Gamma \Rightarrow \Delta,\, y:B,\, y:B$$

that yields, by induction hypothesis for all forms of contraction, a derivation of height at most $n-1$ of

$$x \leqslant y,\, y:A,\, \Gamma \Rightarrow \Delta,\, y:B$$

and the conclusion $\Gamma \Rightarrow \Delta,\, x:A \supset B$ follows in one more step by $R\supset$. □

We will let $Ctr^*$ denote repeated applications of *Contraction*. For $n > 1$, the notation $A^n$ denotes $n$ copies of the formula $A$ and $\Delta^n$ denotes $n$ copies of the multiset $\Delta$.

**Theorem 4** (Admissibility of Cut) *The Cut rule*

$$\frac{\Gamma \Rightarrow \Delta,\, x:A \quad x:A,\, \Gamma' \Rightarrow \Delta'}{\Gamma,\, \Gamma' \Rightarrow \Delta,\, \Delta'}\, Cut$$

*is admissible in* **G3I***.

*Proof* The proof has the same structure as the proof of admissibility of *Cut* for sequent calculus extended with the left rule-scheme (Theorem 6.2.3 in [35]) and for extensions with internalized Kripke semantics for modal logic [30]. In case the geometric rule-scheme is considered, the proof follows the pattern of [29]. We observe that in all the cases of permutation of cuts that may give a clash with the variable conditions in the implication rules (and in the rules for $\leqslant$ in case of geometric extensions), an appropriate substitution (Lemma 3) prior to the permutation is used.

The proof is thus by induction on the length of the cut formula, with a subinduction on the sum of the heights of the derivations of the premisses of *Cut*. We consider in detail only the case of a cut with the cut formula principal in implication rules in both premisses.

If the cut formula is $x:A \supset B$, we transform the derivation

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ x \leqslant z,\, z:A,\, \Gamma \Rightarrow \Delta,\, z:B\end{matrix}}{\Gamma \Rightarrow \Delta,\, x:A \supset B} \quad \dfrac{\begin{matrix}\vdots\\ x \leqslant y,\, x:A \supset B,\, \Gamma' \Rightarrow \Delta',\, y:A\end{matrix} \quad \begin{matrix}\vdots\\ x \leqslant y,\, x:A \supset B,\, y:B,\, \Gamma' \Rightarrow \Delta'\end{matrix}}{x \leqslant y,\, x:A \supset B,\, \Gamma' \Rightarrow \Delta'}}{x \leqslant y,\, \Gamma,\, \Gamma' \Rightarrow \Delta,\, \Delta'}\, Cut$$

into

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\(x\leqslant y)^2, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta', y:B\end{array} \quad \begin{array}{c}\vdots\\x\leqslant y, y:B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'\end{array}}{(x\leqslant y)^3, \Gamma^3, \Gamma'^2 \Rightarrow \Delta^3, \Delta'^2}\,Cut}{x\leqslant y, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Ctr^*$$

with the premisses derived by

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\\Gamma \Rightarrow \Delta, x:A\supset B\end{array} \quad \begin{array}{c}\vdots\\x\leqslant y, x:A\supset B, \Gamma' \Rightarrow \Delta', y:A\end{array}}{x\leqslant y, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', y:A}\,Cut \quad \dfrac{\begin{array}{c}\vdots\\x\leqslant z, z:A, \Gamma \Rightarrow \Delta, z:B\end{array}}{x\leqslant y, y:A, \Gamma \Rightarrow \Delta, y:B}\,(y/z)}{(x\leqslant y)^2, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta', y:B}\,Cut$$

and by

$$\dfrac{\begin{array}{c}\vdots\\\Gamma \Rightarrow \Delta, x:A\supset B\end{array} \quad \begin{array}{c}\vdots\\x\leqslant y, x:A\supset B, y:B, \Gamma' \Rightarrow \Delta'\end{array}}{x\leqslant y, y:B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut$$

The two upper cuts, on $x:A\supset B$, are of smaller derivation height; the other two are on the smaller cut formulae $y:A$ and $y:B$. □

**Corollary 1** (Admissibility of Modus Ponens) *The rule*

$$\dfrac{\Rightarrow x:A\supset B \quad \Rightarrow x:A}{\Rightarrow x:B}\,MP$$

*is admissible in* **G3I***.

*Proof* Routine use of invertibility of $R\supset$ and cut admissibility. □

Completeness of the calculus with respect to the relational semantics can as usual be proved indirectly using admissibility of Modus Ponens and derivability of the appropriate axioms. This method establishes equivalence with a given a complete Hilbert system but clearly works only when a suitable complete axiomatization is known. Alternatively, we can show completeness by showing that for any given sequent there is either a derivation or a counter-model in the given frame class. This method of establishing completeness, detailed for labelled sequent calculi in [32], parallels the method already adopted by Kripke for his modal tableau systems [24]. In the first case, a prior proof of *Cut*-admissibility is essential; in the second case, it is reassuring rather than essential.

## 6 Intermediate logics

We obtain at once labelled calculi of the form **G3I*** with admissible structural rules for a range of intermediate propositional logics, including the seven interpolable ones [5,25]: the point is simply that all these logics have, as frame conditions, geometric implications. For further details on such logics, see [5,18].

1. **Int** *Intuitionistic Logic*: As already built in above, the accessibility relation $\leqslant$ is reflexive and transitive (but we don't exploit the fact that these properties are expressed by geometric implications).
2. **Jan** *Jankov Logic or De Morgan Logic* [19]: This logic (also known [5] as **KC**, and as the "logic of weak excluded middle") is axiomatised both by $\neg A \vee \neg\neg A$ and by $\neg(A\&B) \supset (\neg A \vee \neg B)$. The relation $\leqslant$ is *directed* or *convergent*, i.e.

$$\forall xyz((x \leqslant y \;\&\; x \leqslant z) \supset \exists w(y \leqslant w \;\&\; z \leqslant w)).$$

Note in particular that this frame condition is a geometric implication but not a universal formula. Since all our frames are reflexive, we ignore the distinction between directedness and strong directedness; see [5] for a fuller discussion. The instance of the rule scheme generated by this frame condition is, with $w$ an eigenvariable and so fresh,

$$\frac{x \leqslant y, x \leqslant z, y \leqslant w, z \leqslant w, \Gamma \Rightarrow \Delta}{x \leqslant y, x \leqslant z, \Gamma \Rightarrow \Delta}\ Jan$$

3. **GD** *Gödel–Dummett Logic*:
This logic (also known as **LC**, for "linear chains") has either of the following—$(A \supset B) \vee (B \supset A)$ and $((A \supset B) \supset C) \supset (((B \supset A) \supset C) \supset C)$—as a characteristic axiom schema; see [9] for a discussion of other calculi. The accessibility relation is *strongly connected*, i.e.

$$\forall xyz((x \leqslant y \;\&\; x \leqslant z) \supset (y \leqslant z \vee z \leqslant y)).$$

The instance of the rule scheme generated by this frame condition is

$$\frac{x \leqslant y, x \leqslant z, y \leqslant z, \Gamma \Rightarrow \Delta \quad x \leqslant y, x \leqslant z, z \leqslant y, \Gamma \Rightarrow \Delta}{x \leqslant y, x \leqslant z, \Gamma \Rightarrow \Delta}\ GD$$

4. $\mathbf{Bd}_2$: The accessibility relation has *Bounded depth at most* 2, i.e.,

$$\forall xyz((x \leqslant y \leqslant z) \supset (y \leqslant x \vee z \leqslant y)).$$

This logic is axiomatised by, for example, $A \vee (A \supset (B \vee \neg B))$. The instance of the rule scheme generated by this frame condition is

$$\frac{y \leqslant x, x \leq y, y \leq z, \Gamma \Rightarrow \Delta \quad x \leq y, y \leq z, z \leqslant y, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta}\ Bd_2$$

5. **GS**: *The Greatest Semi-constructive logic* [12], also known in [2] as **GSc** and as $\mathbf{Bd}_2\mathbf{F}_2$. The accessibility relation has depth at most 2 and at most 2 final elements, i.e. satisfies both the condition (above) for $\mathbf{Bd}_2$ and (a simplification of a special case in Exercise 2.11 of [5]):

$$\forall xyz\exists v((x \leqslant v \,\&\, y \leqslant v) \vee (y \leqslant v \,\&\, z \leqslant v) \vee (x \leqslant v \,\&\, z \leqslant v)).$$

which easily enforces (under the hypothesis of reflexivity, transitivity, antisymmetry and the condition for $\mathbf{Bd}_2$) exactly the condition that, for any three elements, there is some element accessible from some two of them. (If the three elements are not all distinct, this is a triviality; so we can replace 'elements' by 'distinct elements'.) This logic is axiomatised by, for example, $(A \supset B) \vee (B \supset A) \vee ((A \supset \neg B) \& (\neg B \supset A))$ and $A \vee (A \supset (B \vee \neg B))$ together. The corresponding instantiation of the rule scheme for the first condition is given above; that for the second condition is, with $v$ an eigenvariable (and so neither in $\Gamma$ nor in $\Delta$ and distinct from all of $x$, $y$ and $z$):

$$\frac{x \leqslant v, y \leqslant v, \Gamma \Rightarrow \Delta \quad y \leqslant v, z \leqslant v, \Gamma \Rightarrow \Delta \quad x \leqslant v, z \leqslant v, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\; F_2$$

6. **Sm**: *Smetanich* logic, also known [5] as $\mathbf{LC}_2$ or **HT**, the "logic of here and there", or as Gödel's 3-valued logic. It is axiomatised by the **GD** axiom plus the $\mathbf{Bd}_2$ axiom, or, equivalently (from [5], Exercise 2.15), by $(\neg B \supset A) \supset (((A \supset B) \supset A) \supset A)$. The accessibility relation is strongly connected and has depth at most 2, i.e. the conditions

$$\forall xyz((x \leqslant y \;\&\; y \leqslant z) \supset (y \leqslant z \vee z \leqslant y))$$
$$\forall xyz((x \leqslant y \;\&\; y \leqslant z) \supset (yy \leqslant x \vee z \leqslant y)).$$

for **GD** and $\mathbf{Bd}_2$ hold. Their expression as rules is already done above.
7. **Cl** *Classical logic*: The accessibility relation is *symmetric*, i.e.

$$\forall xy(x \leqslant y \supset y \leqslant x).$$

This logic is axiomatised, of course, by $A \vee \neg A$ or by $\neg\neg A \supset A$. The corresponding instantiation of the rule scheme is useless; it is

$$\frac{x \leqslant y, y \leqslant x, \Gamma \Rightarrow \Delta}{x \leqslant y, \Gamma \Rightarrow \Delta}\; Cl$$

Several variants of these logics (see [5, p. 55]) are non-interpolable but still have geometric frame conditions, e.g. $Bd_n$ for $n > 2$ ("Bounded depth $n$") and $btw_n$ for $n > 2$ (approximately, "bounded top-width" $n$). For $n = 3$, for example, the latter's characteristic frame condition is the geometric implication

$$\forall x x_0 x_1 x_2 x_3 \left( \bigwedge_{i=1}^{3} x \leqslant x_i \supset \exists y \left( \bigvee_{i \neq j} x_i \leqslant y \;\&\; x_j \leqslant y \right) \right).$$

**KP**, Kreisel–Putnam logic, axiomatised over **Int** by the schema

$$(\neg A \supset (B \vee C)) \supset ((\neg A \supset B) \vee (\neg A \supset C)),$$

is a (non-interpolable) intermediate logic with a characteristic frame condition that is not a geometric implication. This condition (see [5, p. 55]) is, with the symbol $R$ for the accessibility relation of [5] replaced by $\leqslant$ and some negative antecedent formulae converted to positive succedent formulae,

$$\forall xyz((x \leqslant y \;\&\; x \leqslant z) \supset (y \leqslant z \;\vee\; z \leqslant y \;\vee\; \exists u(x \leqslant u \;\&\; u \leqslant y \;\&\; u \leqslant z \;\&\; F(u, y, z))))$$

where $F(u, y, z)$ abbreviates $\forall v(u \leqslant v \supset \exists w(v \leqslant w \;\&\; (y \leqslant w \vee z \leqslant w)))$; the complexity of $F(u, y, z)$ upsets the geometricity, and no obvious single alternative formula suggests itself as being both equivalent and also a geometric implication. However, let us add the ternary relation symbol $F$ and the axiom

$$\forall uvyz(u \leqslant v \;\&\; F(u, y, z) \supset (\exists w(v \leqslant w \;\&\; y \leqslant w) \vee \exists w(v \leqslant w \;\&\; z \leqslant w))).$$

This is a geometric implication, so can be converted to a schematic rule. The proof-theoretic analysis above extends to several relations and to ternary relations without difficulty. An easy argument shows that the one-way implication in this axiom is sufficient to express the equivalence implicit in the idea that $F(u, y, z)$ is an abbreviation. Detailed analysis of the scope of such situations is left for our future joint work.

## 7 Gödel translation of Int to S4

Gödel [16] defined a translation $(\cdot)^*$ from the language of intuitionistic propositional logic to the language of classical modal logic and proved by induction on derivations that his translation was *sound*, that is, $\vdash_{Int} A \rightarrow \vdash_{S4} A^*$, and conjectured faithfulness of the embedding, i.e. the converse. This was proved by McKinsey and Tarski [27], who gave a semantic proof of the implication $\nvdash_{Int} A^* \rightarrow \nvdash_{S4} A$. Dummett and Lemmon [8] proved, using the same semantic method, that $\vdash_{Int+Ax} A$ if and only if $\vdash_{S4+Ax^*} A^*$ where $A$ is any propositional formula and $Ax$ is a collection of axioms.

We consider the following variant (cf [42]) of the Gödel translation from [16]:

$$\begin{aligned} P^{\Box} &:= \Box P \\ \bot^{\Box} &:= \bot \\ (A \supset B)^{\Box} &:= \Box(A^{\Box} \supset B^{\Box}) \\ (A \& B)^{\Box} &:= A^{\Box} \& B^{\Box} \\ (A \vee B)^{\Box} &:= A^{\Box} \vee B^{\Box} \end{aligned}$$

The translation $\Gamma^{\Box}$ of a multiset $\Gamma \equiv A_1, \ldots, A_n$ is defined componentwise by

$$(A_1, \ldots, A_n)^{\Box} := A_1^{\Box}, \ldots, A_n^{\Box}$$

The same translations are applied to labelled formulae and to multisets thereof. The translation on relational atoms is the identity. The following result shows that this translation faithfully represents intermediate logics (provided the frame conditions are expressible using geometric implications) in terms of modal logics between **S4** and **S5**, as studied in [8]. Given an extension **G3I*** of **G3I** with rules for $\leq$, we denote by **G3S4*** the corresponding extension of **G3S4**. For convenience we use $\leq$ for the relation in the calculus for **S4**, despite the latter not being a partial order; this allows the translation on relational atoms to be the identity.

**Lemma 4** *If $\Gamma, \Delta$ are multisets of labelled formulas (with relational atoms also possibly in $\Gamma$) and $\Gamma', \Delta'$ are multisets of labelled atomic formulas, and* **G3S4*** $\vdash$ $\Gamma^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'$, *then* **G3I*** $\vdash$ $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

*Proof* By induction on the derivation of $\Gamma^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'$. If it is an initial sequent, then some atom $x : P$ is in $\Gamma'$ and in $\Delta'$; the conclusion then follows in **G3I*** by *Refl* from the initial sequent $x \leqslant x, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. If it is a conclusion of $L\bot$, so also is $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. If it is derived by a rule for & or for $\vee$, the inductive hypothesis applies to the premisses and then the corresponding rule in **G3I*** gives the conclusion.

If it is derived by a modal rule, the principal formula, being a translated formula, can only be of the form $\Box P$ or of the form $\Box(A^{\Box} \supset B^{\Box})$. There are thus four cases:

1. With $\Box P$ principal on the left, the step

$$\frac{x \leqslant y, y : P, x : \Box P, \Gamma''^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'}{x \leqslant y, x : \Box P, \Gamma''^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'}\ L\Box$$

is translated to the admissible **G3I*** step

$$\frac{x \leqslant y, y : P, x : P, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}{x \leqslant y, x : P, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}$$

2. With $\Box P$ principal on the right, the step (with $y$ fresh)

$$\frac{x \leqslant y, \Gamma^{\Box}, \Gamma' \Rightarrow \Delta''^{\Box}, \Delta', y : P}{\Gamma^{\Box}, \Gamma' \Rightarrow \Delta''^{\Box}, \Delta', x : \Box P}\ R\Box$$

is translated (using admissibility of substitution) to the **G3I*** steps

$$\cfrac{\cfrac{x \leqslant y, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta', y : P}{x \leqslant x, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : P}\ (x/y)}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : P}\ Refl$$

3. With $\Box(A^{\Box} \supset B^{\Box})$ principal on the left, the step

$$\frac{x \leqslant y, x : \Box(A^{\Box} \supset B^{\Box}), y : A^{\Box} \supset B^{\Box}, \Gamma''^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'}{x \leqslant y, x : \Box(A^{\Box} \supset B^{\Box}), \Gamma''^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'} \; L\Box$$

gives, by hp-invertibility of $L\supset$ in **G3S4***, derivations in **G3S4*** of the sequents

$$x \leqslant y, x : \Box(A^{\Box} \supset B^{\Box}), \Gamma''^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta', y : A^{\Box}$$

and

$$x \leqslant y, x : \Box(A^{\Box} \supset B^{\Box}), y : B^{\Box}, \Gamma''^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'$$

to which the inductive hypothesis applies. This gives us derivations in **G3I*** of the sequents

$$x \leqslant y, x : A \supset B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta', y : A \quad x \leqslant y, x : A \supset B, y : B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'$$

from which the desired conclusion

$$x \leqslant y, x : A \supset B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'$$

follows by a step of $L\supset$ in **G3I***.

4. If $\Box(A^{\Box} \supset B^{\Box})$ is principal on the right, the step is

$$\frac{x \leqslant y, \Gamma^{\Box}, \Gamma' \Rightarrow \Delta''^{\Box}, \Delta', y : A^{\Box} \supset B^{\Box}}{\Gamma^{\Box}, \Gamma' \Rightarrow \Delta''^{\Box}, \Delta', x : \Box(A^{\Box} \supset B^{\Box})} \; R\Box$$

from which, by hp-invertibility of $R\supset$ in **G3S4***, we have a derivation in **G3S4*** of

$$x \leqslant y, y : A^{\Box}, \Gamma^{\Box}, \Gamma' \Rightarrow \Delta''^{\Box}, \Delta', y : B^{\Box}$$

to which the inductive hypothesis applies. An $R\supset$ step in **G3I*** gives us the desired conclusion. □

**Theorem 5** *For any labelled sequent $\Gamma \Rightarrow \Delta$, the following are equivalent:*

1. **G3I*** $\vdash \Gamma \Rightarrow \Delta$
2. **G3S4*** $\vdash \Gamma^{\Box} \Rightarrow \Delta^{\Box}$.

*Proof* From (1) to (2) is routine, by induction on the structure of the derivation. For example, an axiom $x \leqslant y, \Gamma, x : P \Rightarrow y : P, \Delta$ translates to the **G3S4*** derivation

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{\dots, \Gamma^{\Box}, x:\Box P, z:P \Rightarrow z:P, \Delta^{\Box}}\,Ax}{x\leqslant y, y\leqslant z, x\leqslant z, \Gamma^{\Box}, x:\Box P \Rightarrow z:P, \Delta^{\Box}}\,L\Box}{x\leqslant y, y\leqslant z, \Gamma^{\Box}, x:\Box P \Rightarrow z:P, \Delta^{\Box}}\,Trans}{x\leqslant y, \Gamma^{\Box}, x:\Box P \Rightarrow y:\Box P, \Delta^{\Box}}\,R\Box \quad .
$$

Similarly, an $R\supset$-instance (with fresh $y$)

$$
\frac{x\leqslant y, \Gamma, y:A \Rightarrow y:B, \Delta}{\Gamma \Rightarrow x:A\supset B, \Delta}\,R\supset
$$

translates to the steps

$$
\cfrac{\cfrac{x\leqslant y, \Gamma^{\Box}, y:A^{\Box} \Rightarrow y:B^{\Box}, \Delta^{\Box}}{x\leqslant y, \Gamma^{\Box} \Rightarrow y:A^{\Box}\supset B^{\Box}, \Delta^{\Box}}\,R\supset}{\Gamma^{\Box} \Rightarrow x:\Box(A^{\Box}\supset B^{\Box}), \Delta^{\Box}}\,R\Box
$$

and an $L\supset$-instance is dealt with likewise. Conjunction, disjunction and absurdity are routine, as are the rules arising as instances of the geometric rule scheme.

The converse is a special case of Lemma 4. □

Observe that the translation does not affect the steps involving the rules for the accessibility relation; therefore the faithfulness of the embedding is maintained for each of the intermediate logics considered in Sect. 6 and even for those not considered here, provided the frame conditions are geometric implications.

Observe also that the admissibility of *Contraction* and *Cut* in **G3I*** may be obtained from this result (and their admissibility for extensions of **S4**), since no use is made thereof in the proof of the Theorem.

Compared with a standard proof [42] for unlabelled calculi, the above is both simple and general. The core of the above proof, that is, the erasure of all □, is reminiscent of an analogous reduction in the model-theoretic proof of faithfulness of the embedding of **Int** into **S4**. For that purpose, it is shown how a countermodel for an unprovable sequent in **Int** is turned into a countermodel for the translation of that sequent in **S4**; in particular, "it can be treated as a modal frame isomorphic to its skeleton" (see theorem 3.83 in [5]).

One may conclude, therefore, in an easy uniform fashion, the faithfulness of the embedding of each intermediate logic given in Sect. 6 as characterised by frames satisfying geometric implications into its (smallest) modal companion. Well-known modal companions are **S4** for **Int**, **S4.2** for **Jan**, **S4.3** for **GD**, **S5** for **Cl**.

## 8 Analyticity

The rules given by the above approach are not, in general, analytic, in the strong sense that each expression in a premiss is a subexpression of a rule in the conclusion: for example, in the rule for **GD**, with atoms $y\leqslant z$ and $z\leqslant y$ in one or other premiss, there is in the conclusion no expression of which these are subexpressions. More seriously, the rule *Refl* does not require the variable $x$ in the new atom $x \leqslant x$ to appear in the conclusion; several of the geometric rules illustrated above also have similar defects.

We say that a rule instance is *analytic* if and only if every variable occurring in any premiss is either an eigenvariable or occurs in the conclusion, and that a derivation is *analytic* if and only if all rule instances therein are analytic. Its conclusion is then said to be *analytically derivable*.

**Lemma 5** *The class of analytic derivations is closed under substitution of a variable $x$ for another variable, provided that $x$ occurs in the derivation's conclusion.*

*Proof* Routine. Such substitution doesn't upset the freshness of eigenvariables, since these are renamed if necessary. □

**Proposition 3** *Every sequent derivable in* **G3I**$^*$ *is analytically derivable.*

*Proof* We proceed by induction on the structure of the derivation. The base case is trivial: every initial sequent is analytically derivable, since there are no premisses. Inductively, suppose that a non-analytic step has premisses derived analytically. For each variable in any premiss that is not in the conclusion (and is not an eigenvariable), we substitute for it (in the derivations of all the premisses) any variable that is in the conclusion. Since, in any derivation, all variables in a rule's conclusion are in all its premisses, such a variable is already in the conclusion of each premiss, so Lemma 5 applies. The conclusion is unchanged: the resulting rule instance is now analytic, and the derivations of the premisses are still, by Lemma 5, analytic. So the entire derivation is now analytic. □

We now have the subterm property: see [38] for details thereof. Together with the sub-formula property of the rules so far as logical formulae (rather than relational atoms) is concerned, this ensures that the calculus (restricted to analytic derivations) is finitely generating in the sense defined by [4]: a rule is *finitely generating* iff, "given its conclusion, there is only a finite set of premises to choose from". (This applies as stated to one-premiss rules; an obvious modification extends it to multi-premiss rules.) The systems **G4ip** for **Int** described in [42] and that for **GD** from [9] are good examples of cut-free calculi where all the rules have this property, but some do not have the sub-formula property: despite the absence thereof, root-first proof search is effective. First-order logic provides a contrast: although, by convention, any formula $A(t)$ is a sub-formula of $\forall x.A(x)$, and thus the calculus **G3c** has the sub-formula property, the logic is undecidable.

We have therefore shown that our rules can, if required in the context of root-first proof search, be restricted to being analytic, and thus finitely generating.

## 9 Related work

A summary of a preliminary version of this paper appeared in Sect. 4 of the survey [31]. An extension to the first-order case presents no essential difficulties; the corresponding treatment of first-order modal logics will appear in [38]. One application of the same approach is to bi-intuitionistic logic by Pinto and Uustalu [40]. The paper [39] exploits the proof-theoretic method of [30] for modal logic in the context of conditional logics. A recent survey of the area is given in [33].

A proof-theoretical treatment of geometric implications first appeared in Alex Simpson's PhD thesis [41], in the context of natural deduction systems for intuitionistic modal logics. A detailed historical discussion and references to other literature on geometric theories can be found in the notes to Chapters 5 and 8 of [38].

Hypersequent calculi have been suggested as a powerful alternative; however, the calculi are often unsuitable for root-first proof search, with substantial non-determinism. There is interesting work in this area by Ciabattoni and others [6,7]; this gives a method for translating axioms (such as the linearity axiom $(A \supset B) \vee (B \supset A)$ of Gödel-Dummett logic **GD**) into structural rules to be added to a single-succedent calculus, the *Full Lambek* calculus **FL**; axioms are organised in a *substructural hierarchy* of complexities, with the higher complexity axioms requiring systems of higher complexity, such as hypersequents, nested sequents, higher-order sequents and beyond. (Restriction to the single-succedent calculus appears to force **GD** to require a hypersequent calculus, despite the existence of simple multi-succedent calculi [9,10].) Moreover, that particular method does not yet give a simple translation for the intermediate logic $\mathbf{Bd}_2$, and its applicability to modal logic is not yet developed.

Display calculi provide another approach in the proof theoretical investigation of non-classical logics, not using Kripke semantics (cf. the monograph [46] for extensive references). Our method is no less general: in particular, geometric implications properly cover the class of frame conditions that characterize properly displayable modal and tense logics [22].

We omit any contribution to the debate with those who see labelled calculi as impure, beyond mention of our view that no uniformly and entirely successful approach (allowing e.g. proofs of interpolation results, easy implementation, uniformity of coverage w.r.t. the intermediate logics and simple notation) has yet been achieved. A referee has suggested a reference to the work in [3], with a sequent calculus quite similar to **G3I**; this incorporates conditions on the accessibility relation as axioms, but achieving only partial admissibility of *Cut*. To achieve this in full, a transformation to another calculus is required.

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