# REASONING ABOUT STRATEGIES ON COLLAPSIBLE PUSHDOWN ARENAS WITH IMPERFECT INFORMATION

BASTIEN MAUBERT, ANIELLO MURANO, AND OLIVIER SERRE

Università degli Studi di Napoli Federico II, Naples, Italy

Università degli Studi di Napoli Federico II, Naples, Italy

Université de Paris, IRIF, CNRS, France

ABSTRACT. Strategy Logic with imperfect information  $(SL_{iR})$  is a very expressive logic designed to express complex properties of strategic abilities in distributed systems. Previous work on  $SL_{iR}$  focused on *finite* systems, and showed that the model-checking problem is decidable when information on the control states of the system is hierarchical among the players or components of the system, meaning that the players or components can be totally ordered according to their respective knowledge of the state. We show that moving from finite to infinite systems generated by collapsible (higher-order) pushdown systems preserves decidability, under the natural restriction that the stack content is visible.

The proof follows the same lines as in the case of finite systems, but requires to use (collapsible) alternating pushdown tree automata. Such automata are undecidable, but semi-alternating pushdown tree automata were introduced and proved decidable, to study a strategic problem on pushdown systems with two players. In order to tackle multiple players with hierarchical information, we refine further these automata: we define direction-guided (collapsible) pushdown tree automata, and show that they are stable under projection, nondeterminisation and narrowing. For the latter operation, used to deal with imperfect information, stability holds under some assumption that is satisfied when used for systems with visible stack. We then use these automata to prove our main result.

#### 1. Introduction

Logics for strategic reasoning, such as Alternating-time Temporal Logic (ATL) [AHK02] and Strategy Logic (SL) [CHP10, MMPV14], are powerful languages to specify complex synthesis problems for distributed systems and verify strategic abilities in multi-agent systems. Strategy Logic in particular is very expressive: it can express the existence of distributed strategies satisfying important game-theoretic solution concepts such as Nash equilibria or subgame-perfect equilibria; and since model-checking algorithms for SL can usually provide witnesses of distributed strategies when they exist, such algorithms constitute generic solutions for a range of synthesis problems such as distributed synthesis [PR90, KV01, FS05] or rational synthesis [FKL10, CFGR16, KPV16, FGR18].

Key words and phrases: strategic reasoning; infinite-state systems; higher-order pushdown automata; imperfect information; model checking; distributed systems; hierarchical information.

Most works on such logics have focused on finite-state systems, but in the recent years a line of work has considered the model-checking problem for ATL and SL on a class of infinite systems that plays an important role in program verification, namely those generated by pushdown systems. Pushdown systems are finite-state transition systems equipped with a stack. Because these systems can capture the flow of procedure calls and returns in programs [JM77], many problems in formal methods that were initially concerned with finite-state systems have been studied and solved on such infinite systems: model checking temporal logics [BEM97, FWW97, EKS03], solving reachability and parity games [Wal01, Ser03, Cac03, Ser04, PV04, HO09], module checking [BMP10, ALM+13], and more recently model checking of logics for strategic reasoning [MP15, CSW16a, CSW16b, CSW17].

Two of these works [ALM<sup>+</sup>13, CSW17] consider pushdown systems with imperfect information, i.e. systems where players or components may not observe perfectly the state of the system. Imperfect information plays an important role in game theory and distributed systems, but it usually increases greatly the complexity of analysing such systems: already for finite-state systems, multiplayer reachability games are undecidable when no assumption is made on the relative information of the players [PR79]. To retrieve decidability, a common restriction is to consider systems with hierarchical information, i.e. where the players can be totally ordered according to how well they observe the system. This restriction has been used to establish results on multiplayer games [PRA02, BMvdB18] and distributed synthesis [PR90, KV01, FS05], and more recently on the model-checking problem for SL<sub>iR</sub>. an extension of Strategy Logic to the imperfect-information setting [BMM<sup>+</sup>17]. This result states that the model-checking problem for  $SL_{iR}$  is decidable as long as strategies quantified deeper in the formula observe the system better than those higher up in the syntactic tree. We show that this result can be extended to infinite arenas generated by collapsible pushdown systems, as long as the stack is visible to all players, who thus have imperfect information only on the control states. This higher-order extension of pushdown system permits to capture higher-order procedure calls (see e.g. [HMOS17, BCHS12, BCHS13]), a feature embraced by many modern day programming languages such as C++, Haskell, OCaML, Javascript, Python, or Scala.

We first consider the simpler case of pushdown systems. We extend the approach followed in [BMM $^+$ 17], which consists in reducing the model-checking problem for  $SL_{iR}$  to that of  $QCTL_{iR}^*$ , an intermediary, low-level logic introduced in [BMM $^+$ 17] as an imperfect-information extension of  $QCTL^*$  [LM14], which itself extends  $CTL^*$  with second-order quantification on atomic propositions. In [BMM $^+$ 17],  $QCTL_{iR}^*$  is evaluated on finite compound Kripke structures, which are Kripke structures whose states are tuples of local states, and the second-order quantifiers are parameterised by an indication of which components of states they can observe. We introduce pushdown compound Kripke structures, which are compound Kripke structures equipped with a stack, and we show that the model-checking problem for  $QCTL_{iR}^*$  on such structures is decidable for the hierarchical fragment of  $QCTL_{iR}^*$ , where innermost quantifiers observe better than outermost ones.

To prove this, we generalise the automata construction from [BMM<sup>+</sup>17]. Instead of alternating tree automata we naturally use alternating pushdown tree automata (APTA), introduced in [KPV02]. The emptiness problem for these automata being undecidable [ALM<sup>+</sup>13], we actually resort to the subclass of semi-alternating pushdown tree automata (SPTA). These automata were introduced in [ALM<sup>+</sup>13] to solve the module-checking problem on pushdown systems with imperfect information. The idea is the following: in the automata constructions considered, the stack of an automaton is always used to simulate that of the

(unfolding of the) pushdown system that it reads as input. In addition, the operations on the system's stack are coded in the directions of the input tree. It follows that the content of an automaton's stack is determined by the node it visits, and thus all copies of an automaton that visit a same node in the input tree share the same stack content, unlike general alternating pushdown tree automata. SPTAs were introduced to exploit this property and obtain a simulation procedure (elimination of alternation), and thus a decidable class of APTAs. However, these automata are not closed under an operation that is central in the approach from [BMM<sup>+</sup>17] that we generalise: the narrowing operation.

Narrowing is an operation on tree automata that was introduced by Kupferman and Vardi to deal with imperfect information in the automata approach to LTL synthesis [KV99, KV01]. Intuitively, if a tree automaton works on  $X \times Y$ -trees (i.e. trees where nodes are words over  $X \times Y$ ), its narrowing to X is an automaton that works on X-trees and can thus guess a strategy that observes only X. We generalise this operation to pushdown tree automata. This presents no difficulty, but it turns out that SPTAs are not closed under narrowing: if an SPTA sends two copies of itself in two directions (x,y) and (x,y') with different operations on the stack (which is possible in an SPTA if  $y \neq y'$ ), in its narrowing to X these two copies take the same direction x and thus arrive in the same node with two different stack contents. To solve this problem, we identify a subclass of semi-alternating pushdown tree automata that is stable under narrowing, and we prove that it is also stable under simulation and projection (the latter is trivial), the two other main operations involved in the automata construction.

The idea is the following: in SPTAs the operation on the stack can depend on the direction taken in the input tree. We observe that actually, since the automata we build work on unfoldings of pushdown systems whose stack operations are coded as part of the directions, these stack operations are determined by a specific component of the directions. And moreover, because the stack is visible, this component coding stack operations is never erased by the narrowing operations we perform. We say that an APTA working on  $X \times Y$ -trees is X-guided if stack operations are determined by the X component of the direction taken, and we will use the fact that if an automaton working on  $X \times Y \times Z$ -trees is X-guided, then its narrowing to  $X \times Y$  is also X-guided.

In the higher-order case, we follow the same road map. The main technical difficulty arises when defining regular labelling functions, which are tools to describe the atomic propositions satisfied in a given configuration of the collapsible pushdown system. For that we follow the approach from [BCOS10] which, in particular, permits to rely on a closure property of the model of alternating collapsible pushdown automata to solve most of the technical difficulties.

**Related work.** Pushdown systems with imperfect information and visible stack were considered in [ALM $^+$ 13], where it is proved that module checking is undecidable if the stack is not visible. This is also the case of the model-checking problem for  $SL_{iR}$ , as it subsumes module checking. The only existing work on logics for strategic reasoning on pushdown systems with imperfect information is [CSW17]. The logics it considers are incomparable to  $SL_{iR}$ : they involve epistemic operators, but are based on ATL instead of the richer SL; also, while we work in the setting of perfect recall, they consider memoryless players, which makes it possible to make less restrictive assumptions on the visibility of the stack while retaining decidability.

Plan. We start in Section 2 by defining QCTL\*\*<sub>R</sub> and pushdown compound Kripke structures. Section 3 contains the main conceptual novelty of this work, which is the introduction of direction-guided pushdown automata, and the proof that they are stable under projection, simulation and narrowing. In Section 4 we use these automata to extend the automata construction from [BMM+17] to the case of pushdown systems, and obtain our decidability result for QCTL\*\*<sub>iR</sub> model checking (Theorem 2.8). We then apply this result to Strategy Logic with imperfect information. In Section 5 we recall its syntax, define its semantics on pushdown game arenas, and we show how the hierarchy-preserving reduction from SL\*<sub>iR</sub> to QCTL\*\*<sub>iR</sub> can be extended to the pushdown setting, which entails our main result on pushdown arenas (Theorem 5.6). Finally, in Section 6 we show how to generalise this result to a much more general case in which pushdown arenas are replaced with collapsible pushdown arenas while preserving decidability (Theorem 6.18).

#### 2. QCTL\* WITH IMPERFECT INFORMATION

We start by recalling the syntax and semantics of  $QCTL_{iR}^*$ . The definitions are as in [BMM<sup>+</sup>17], except that the models are now *pushdown* compound Kripke structures instead of finite ones.

**Preliminaries.** As usual we write  $A^*$  (resp.  $A^+$ ,  $A^\omega$ ) for the set of finite (resp. finite nonempty, infinite) words over some finite alphabet A. The *length* of a finite word  $w = w_0w_1 \dots w_n$  is |w| := n + 1, and  $\operatorname{last}(w)$  is the last letter. Given a finite (resp. infinite) word w and  $0 \le i < |w|$  (resp.  $i \in \mathbb{N}$ ), we let  $w_i$  be the letter at position i in w,  $w_{\le i}$  is the prefix of w that ends at position i and  $w_{\ge i}$  is the suffix of w that starts at position i. The domain of a mapping f is written  $\operatorname{dom}(f)$ , for a relation  $R \subseteq A \times B$  and  $a \in A$ ,  $R(a) := \{b \in B \mid (a, b) \in R\}$ , and for  $n \in \mathbb{N}$  we let  $[n] := \{i \in \mathbb{N} : 1 \le i \le n\}$ .

2.1. QCTL<sub>iR</sub> Syntax. For the rest of the paper we fix a finite set of **atomic propositions** AP, and some natural number  $n \in \mathbb{N}$  which parameterises the logic QCTL<sub>iR</sub>, and which is the number of components in states of the models. We also let  $\{L_i\}_{i \in [n]}$  be a family of n disjoint sets of **local states**. In QCTL<sub>iR</sub> each quantifier on atomic propositions is parameterised by a set of indices that represents which components of each state the quantifier observes; it thus defines the "observation" of that quantifier. Accordingly, a set  $\mathbf{o} \subseteq [n]$  is called a **concrete observation** (to distinguish it from observation symbols o used in  $\mathsf{SL}_{iR}$ , see Section 5).

**Definition 2.1.** The syntax of  $QCTL_{iR}^*$  is defined by the following grammar:

$$\varphi := p \mid \neg \varphi \mid \varphi \lor \varphi \mid \mathbf{E}\psi \mid \exists^{\mathbf{o}} p. \varphi$$
$$\psi := \varphi \mid \neg \psi \mid \psi \lor \psi \mid \mathbf{X}\psi \mid \psi \mathbf{U}\psi$$

where  $p \in AP$  and  $\mathbf{o} \subseteq [n]$ .

Formulas of type  $\varphi$  are state formulas, those of type  $\psi$  are path formulas, and QCTL<sub>iR</sub> consists of all the state formulas defined by the grammar.

The set of quantified propositions  $AP_{\exists}(\varphi) \subseteq AP$  of a  $QCTL_{iR}^*$  formula  $\varphi$  is the set of atomic propositions p such that  $\varphi$  has a subformula of the form  $\exists^{\mathbf{o}} p. \varphi'$ . We also define the set of free propositions  $AP_f(\varphi) \subseteq AP$  as the set of atomic propositions that have an occurrence which is not under the scope of any quantifier of the form  $\exists^{\mathbf{o}} p$ . Without loss

of generality we will assume that  $AP_{\exists}(\varphi) \cap AP_f(\varphi)$  is empty and that each  $p \in AP_{\exists}(\varphi)$  is quantified at most once in  $\varphi$ .

2.2. Compound Kripke structures. Compound Kripke structures [BMM+17] are Kripke structures where states are tuples  $s = (l_1, \ldots, l_n)$  in which the  $l_i$  are local states. A concrete observation  $\mathbf{o} \subseteq [n]$  indicates the indices of the local states observed by a propositional quantifier. Unlike [BMM<sup>+</sup>17], here we define the semantics of QCTL\* on potentially infinite structures, that will be generated first by finite-state pushdown compound Kripke structures that we introduce in Section 2.4, and later by (higher-order) collapsible pushdown compound Kripke structures in Section 6.

**Definition 2.2.** A compound Kripke structure, or CKS, over local states  $\{L_i\}_{i\in[n]}$  is a tuple  $K = (S, R, \ell, s_{\iota})$  where

- S ⊆ ∏<sub>i∈[n]</sub> L<sub>i</sub> is a set of states,
  R ⊆ S × S is a left-total transition relation,
- $\ell: S \to 2^{AP}$  is a labelling function and
- $s_t \in S$  is an initial state.

A path in K is an infinite sequence of states  $\lambda = s_0 s_1 \dots$  such that  $s_0 = s_i$  and for all  $i \in \mathbb{N}, (s_i, s_{i+1}) \in R$ . A partial path is a finite non-empty prefix of a path.

2.3. QCTL<sub>iR</sub> semantics. QCTL<sub>iR</sub> is interpreted on infinite trees, which represent unfoldings of CKSs. Let X be a (possibly infinite) set of directions. An X-tree  $\tau$  is a set of words  $\tau \subseteq X^+$  such that (1) there exists  $r \in X$ , called the root of  $\tau$ , such that each  $u \in \tau$  starts with r; (2) if  $u \cdot x \in \tau$  and  $u \cdot x \neq r$ , then  $u \in \tau$ ; and (3) if  $u \in \tau$  then there exists  $x \in X$ such that  $u \cdot x \in \tau$ .

The elements of a tree  $\tau$  are called *nodes*. A path in  $\tau$  is an infinite sequence of nodes  $\mu = u_0 u_1 \dots$  such that for all  $i \in \mathbb{N}$ ,  $u_{i+1} = u_i \cdot x$  for some  $x \in X$ , and Paths(u) is the set of paths that start in node u. An X-tree  $\tau$  is complete if for every  $u \in \tau$  and  $x \in X$ ,  $u \cdot x \in \tau$ . An AP-labelled X-tree, or (AP, X)-tree for short, is a pair  $t = (\tau, \ell)$ , where  $\tau$  is an X-tree called the domain of t and  $\ell: \tau \to 2^{AP}$  is a labelling. A pointed labelled tree is a pair (t, u)where u is a node of t.

Let  $p \in AP$  and  $\tau$  a tree. A p-labelling for  $\tau$  is a mapping  $\ell_p : \tau \to \{0,1\}$  that indicates in which nodes p holds, and for a labelled tree  $t = (\tau, \ell)$ , the p-labelling of t is the p-labelling  $u\mapsto 1$  if  $p\in\ell(u)$ , 0 otherwise. The composition of a labelled tree  $t=(\tau,\ell)$  with a p-labelling  $\ell_p$  for  $\tau$  is defined as  $t \otimes \ell_p := (\tau, \ell')$ , where  $\ell'(u) = \ell(u) \cup \{p\}$  if  $\ell_p(u) = 1$ , and  $\ell(u) \setminus \{p\}$ otherwise. A p-labelling for a labelled tree  $t = (\tau, \ell)$  is a p-labelling for its domain  $\tau$ .

Let X and Y be two sets, and let  $(x,y) \in X \times Y$ . The X-narrowing of (x,y) is  $(x,y)\downarrow_X:=x$ . This definition extends naturally to words and trees over  $X\times Y$ . For  $I\subseteq [n]$ , we let  $L_I := \prod_{i \in I} L_i$  if  $I \neq \emptyset$  and  $L_\emptyset := \{0\}$ , where **0** is a special symbol. For  $I, J \subseteq [n]$  and  $z=(l_i)_{i\in I}\in L_I$ , we also let

$$z\downarrow_J := z\downarrow_{L_{I\cap J}} \in L_{I\cap J},$$

where z is seen as a pair  $z = (x, y) \in L_{I \cap J} \times L_{I \setminus J}$ , i.e. we apply the above definition with  $X = L_{I \cap J}$  and  $Y = L_{I \setminus J}^{-1}$ . We extend this definition to words and trees.

To define the semantics of quantifier  $\exists^{\mathbf{o}} p$  we need to define what it means for a plabelling of a tree to be **o**-uniform. For  $\mathbf{o} \subseteq [n]$  and  $I \subseteq [n]$ , two tuples  $x, x' \in L_I$  are

<sup>&</sup>lt;sup>1</sup>Since sets  $L_i$  are disjoint, the ordering of local states in z is indifferent and thus this is well defined.

**o-indistinguishable**, written  $x \approx_{\mathbf{o}} x'$ , if  $x \downarrow_{I \cap \mathbf{o}} = x' \downarrow_{I \cap \mathbf{o}}$ . Two words  $u = u_0 \dots u_i$  and  $u' = u'_0 \dots u'_i$  over alphabet  $L_I$  are **o-indistinguishable**, written  $u \approx_{\mathbf{o}} u'$ , if i = j and for all  $k \in \{0, \dots, i\}$  we have  $u_k \approx_{\mathbf{o}} u'_k$ . Finally, a *p*-labelling  $\ell_p$  for an  $L_I$ -tree  $\tau$  is **o-uniform** if for all  $u, u' \in \tau$ ,  $u \approx_{\mathbf{o}} u'$  implies  $\ell_p(u) = \ell_p(u')$ .

**Definition 2.3.** We define by induction the satisfaction relation  $\models$  of  $QCTL_{iR}^*$ . Let  $I \subseteq [n]$ , let  $t = (\tau, \ell)$  be an AP-labelled  $L_I$ -tree, u a node and  $\mu$  a path in  $\tau$ :

```
if p \in \ell(u)
 t, u \models p
 t, u \models \neg \varphi
                               if t, u \not\models \varphi
t, u \models \varphi \lor \varphi' if t, u \models \varphi or t, u \models \varphi'
t, u \models \mathbf{E}\psi
                               if \exists \mu \in Paths(u) \text{ s.t. } t, \mu \models \psi
t, u \models \exists^{\mathbf{o}} p. \varphi
                           if \exists \ell_p a o-uniform p-labelling for t such that t \otimes \ell_p, u \models \varphi
t, \mu \models \varphit, \mu \models \neg \psi
                              if t, \mu_0 \models \varphi
                               if t, \mu \not\models \psi
t, \mu \models \neg \psi
t, \mu \models \psi \lor \psi' if t, \mu \models \psi or t, \mu \models \psi'
                              if t, \mu_{>1} \models \psi
t, \mu \models \mathbf{X}\psi
t, \mu \models \psi \mathbf{U} \psi' if \exists i \geq 0 \text{ s.t. } t, \mu \geq i \models \psi' \text{ and } \forall j \text{ s.t. } 0 \leq j < i, \ t, \mu > i \models \psi
```

Let  $K = (S, R, \ell, s_{\iota})$  be a compound Kripke structure over AP. The **tree-unfolding** of K is the (AP, S)-tree  $t_{K} := (\tau, \ell')$ , where  $\tau$  is the set of all partial paths in K, and for every  $u \in \tau$ ,  $\ell'(u) := \ell(\text{last}(u))$ . We write  $K \models \varphi$  if  $t_K, s_\iota \models \varphi$ .

2.4. Pushdown compound Kripke structures. We now focus on infinite compound Kripke structures generated by pushdown compound Kripke structures, which are compound Kripke structures equipped with a (visible) stack.

**Definition 2.4.** A pushdown compound Kripke structure, or PCKS, over local states  $\{L_i\}_{i\in[n]}$ is a tuple  $PK = (\Gamma, S, R, \ell, s_{\iota})$  where

- $\Gamma$  is a finite stack alphabet together with a bottom symbol  $\flat \notin \Gamma$ , and we let  $\Gamma_{\flat} = \Gamma \cup \{\flat\}$ ;

- S ⊆ ∏<sub>i∈[n]</sub> L<sub>i</sub> is a finite set of states;
  R ⊆ S × Γ<sub>b</sub> × S × Γ<sub>b</sub>\* is a transition relation;
  ℓ : S × Γ\* · b → 2<sup>AP</sup> is a regular labelling function (defined below);
- $s_t \in S$  is an initial state.

We require that the bottom symbol can never be removed nor pushed: for any  $s \in S$ one has  $R(s, \flat) \subseteq S \times \Gamma^* \cdot \flat$  (the bottom symbol is never removed), and for every  $\gamma \in \Gamma$ ,  $R(s,\gamma) \subseteq S \times \Gamma^*$  (the bottom symbol is never pushed).

A regular labelling function is given as a set of finite word automata  $\mathcal{B}^p_s$  over alphabet  $\Gamma$ , one for each  $p \in AP$  and each  $s \in S$ . They define the labelling function that maps to each state  $s \in S$  and stack content  $w \in \Gamma^* \cdot b$  the set  $\ell(s, w)$  of all atoms p such that w belongs to  $\mathcal{L}(\mathcal{B}_s^p)$ , the language accepted by  $\mathcal{B}_s^p$ .

Remark 2.5. Because of the definition of regular labelling function, whether an atomic proposition holds in a configuration depends not only on the control state but on the whole content of the stack. We believe that it is important to be able to express properties about the whole stack content, as the latter reflects the recursive calls of a system.

The choice of restricting to regular properties is for decidability issues. However it is already expressive, as for instance, it permits to capture all sets of configurations that one can define in popular logics such as the monadic second order logic or the modal  $\mu$ -calculus. Such regular labelling functions were used for instance in [EKS03].

A configuration is a pair  $c = \langle s, w \rangle \in S \times (\Gamma^* \cdot \flat)$  where s is the current state and w the current content of the stack. From configuration  $\langle s, \gamma \cdot w \rangle$  the system can move to a configuration  $\langle s', w' \cdot w \rangle$  if  $(s, \gamma, s', w') \in R$ , which we write  $\langle s, \gamma \cdot w \rangle \hookrightarrow \langle s', w' \cdot w \rangle$ . We assume that for every configuration  $\langle s, w \rangle$  there exists at least one configuration  $\langle s', w' \rangle$  such that  $\langle s, w \rangle \hookrightarrow \langle s', w' \rangle$ . A **path** in PK is an infinite sequence of configurations  $\lambda = c_0 c_1 \dots$ such that  $c_0 = \langle s_t, \flat \rangle$  and for all  $i \in \mathbb{N}$ ,  $c_i \hookrightarrow c_{i+1}$ . A **partial path** is a finite non-empty prefix of a path. We let  $Paths^{\omega}(PK)$  (resp.  $Paths^{*}(PK)$ ) be the set of all paths (resp. partial paths) in PK.

**Definition 2.6.** A PCKS  $PK = (\Gamma, S, R, \ell, s_{\iota})$  over  $\{L_i\}_{i \in [n]}$  generates an infinite CKS  $\mathsf{K}_{\mathsf{PK}} = (S', R', \ell', s'_{\iota}) \text{ over } \{L_i\}_{i \in [n+1]}, \text{ where }$ 

- $L_{n+1} = \Gamma^* \cdot \flat$ ,  $S' = S \times \Gamma^* \cdot \flat$ ,
- $(s', w') \in R'(s, w)$  if  $(s, w) \hookrightarrow (s', w')$ ,
- $\ell' = \ell$  and
- $s'_{\iota} = (s_{\iota}, \flat).$

We write  $PK \models \varphi$  if  $K_{PK} \models \varphi_{n+1}$ , where  $\varphi_{n+1}$  is obtained from  $\varphi$  by replacing each concrete observation  $\mathbf{o} \subseteq [n]$  with  $\mathbf{o}' = \mathbf{o} \cup \{n+1\}$ . This reflects the fact that the stack content is visible to all quantifiers in  $\varphi$ .

There is a reduction from the model-checking problem for MSO with equal-level predicate on the infinite binary tree to the model-checking problem for QCTL\*<sub>iR</sub>, so that this problem is undecidable already on finite compound Kripke structures [BMM+17]. However it is proved in [BMM<sup>+</sup>17] that the problem is decidable for the fragment of hierarchical formulas. We now recall this notion, and then we generalise this result to the case of pushdown compound Kripke structures.

**Definition 2.7.** A QCTL<sub>iR</sub> formula  $\varphi$  is *hierarchical* if for every subformula  $\varphi_1 = \exists^{\mathbf{o}_1} p_1. \varphi_1'$ of  $\varphi$  and subformula  $\varphi_2 = \exists^{\mathbf{o}_2} p_2 . \varphi_2'$  of  $\varphi_1'$ , we have  $\mathbf{o}_1 \subseteq \mathbf{o}_2$ .

A formula is thus hierarchical if innermost propositional quantifiers observe at least as much as outermost ones. We let  $QCTL_{iR,\subset}^*$  be the set of hierarchical  $QCTL_{iR}^*$  formulas.

**Theorem 2.8.** Model checking QCTL $_{iR,\subset}^*$  on pushdown compound Kripke structures is decidable.

Before proving this result in Section 4, we introduce a new subclass of alternating pushdown tree automata, and show that it is stable under the operations that are required for the automata construction on which the proof relies.

#### 3. A Subclass of Pushdown tree automata

In this section we present the class of direction-guided pushdown tree automata, a subclass of alternating pushdown tree automata that is decidable and stable under the operations needed to generalise the construction from [BMM<sup>+</sup>17] to the case of pushdown systems.

3.1. Alternating pushdown tree automata. We recall alternating pushdown parity tree automata [LLS84, KPV02]. Because it is sufficient for our needs and simplifies definitions, we assume that all input trees are complete trees.

For a set Z,  $\mathbb{B}^+(Z)$  is the set of formulas built from the elements of Z as atomic propositions using the connectives  $\vee$  and  $\wedge$ , and with  $\top, \bot \in \mathbb{B}^+(Z)$ . For AP a finite set of atomic propositions and X a finite set of directions, an **alternating pushdown tree automaton (APTA) on** (AP, X)-trees is a tuple  $\mathcal{A} = (\Gamma, Q, \delta, q_\iota, C)$  where  $\Gamma$  is a finite stack alphabet with a special bottom symbol  $\flat$ , Q is a finite set of states,  $q_\iota \in Q$  is an initial state,  $\delta: Q \times 2^{\mathrm{AP}} \times \Gamma \to \mathbb{B}^+(X \times Q \times \Gamma^*)$  is a transition function (that never pushes nor removes the bottom symbol  $\flat$ ), and  $C: Q \to \mathbb{N}$  is a colouring function. Atoms in  $\mathbb{B}^+(X \times Q \times \Gamma^*)$  are written between brackets, such as [x, q, w]. A **nondeterministic pushdown tree automaton (NPTA)** is an alternating pushdown tree automaton  $\mathcal{N} = (\Gamma, Q, \delta, q_\iota, C)$  such that for every  $q \in Q$ ,  $a \in 2^{\mathrm{AP}}$  and  $\gamma \in \Gamma$ ,  $\delta(q, a, \gamma)$  is written in disjunctive normal form and for every direction  $x \in X$ , each disjunct contains exactly one element of  $\{x\} \times Q \times \Gamma^*$ .

We define acceptance of a tree by an APTA in a given initial node and a given initial stack content via a two-player (Eve and Adam) turn-based perfect-information parity game. Due to space constraints, we do not give a formal definition of parity games but we refer the reader to, e.g., [Zie98, GTW02] for definitions and classical concepts such as strategies and winning positions. Let  $\mathcal{A} = (\Gamma, Q, \delta, q_{\iota}, C)$  be an APTA over (AP, X)-trees, let  $t = (\tau, \ell)$  be such a tree, let  $u_{\iota} \in \tau$  be a starting node and let  $w_{\iota} \in \Gamma^*$  be an initial stack content. We define the parity game  $\mathcal{G}(\mathcal{A}, t, u_{\iota}, w_{\iota})$  whose set of positions is  $\tau \times Q \times \Gamma^* \times \mathbb{B}^+(X \times Q \times \Gamma^*)$ , and the initial position is  $v_{\iota} = (u_{\iota}, q_{\iota}, w_{\iota}, \delta(q_{\iota}, \gamma, \ell(u_{\iota})))$ , where  $\gamma \in \Gamma$  is the top symbol of  $w_{\iota}$ . A position  $(u, q, w, \alpha)$  belongs to Eve if  $\alpha$  is of the form  $\alpha_1 \vee \alpha_2$ , otherwise it belongs to Adam (note that if  $\alpha$  is of the form [x, q', w'] then there is no choice to be made). Moves in  $\mathcal{G}(\mathcal{A}, t, u_{\iota}, w_{\iota})$  are defined by the following rules:

$$(u, q, w, \alpha_1 \dagger \alpha_2) \to (u, q, w, \alpha_i)$$
 where  $\dagger \in \{ \lor, \land \}$  and  $i \in \{1, 2\}$ ,  $(u, q, \gamma \cdot w, [x, q', w']) \to (u \cdot x, q', w' \cdot w, \delta(q', \ell(u \cdot x), \gamma'))$  where  $\gamma'$  is the top of  $w' \cdot w$ 

*i.e.* Eve resolves existential/disjunctive choices in the formula while Adam resolves universal/conjunctive choices.

Positions of the form  $(u, q, w, \top)$  and  $(u, q, w, \bot)$  are deadlocks, winning for Eve and Adam respectively. Finally, the colouring function (used to define the parity condition) is  $C'(u, q, w, \alpha) = C(q)$ .

A pointed tree (t, u) is **accepted** by  $\mathcal{A}$  with initial stack content w if Eve has a winning strategy in  $\mathcal{G}(\mathcal{A}, t, u, w)$ , *i.e.* she has a way of playing such that whatever the choices of Adam are, the resulting play either ends up in a winning deadlock for her or is such that the largest colour visited infinitely often is even. We also let  $\mathcal{L}(\mathcal{A}, w)$  be the set of pointed trees accepted by  $\mathcal{A}$  with initial stack content w.

Finally, classic ATAs and NTAs (without pushdown store) are obtained by removing from the above definitions all components referring to the pushdown store. For instance, an ATA is a tuple  $\mathcal{A} = (Q, \delta, q_{\iota}, C)$  with a transition function of type  $\delta : Q \times 2^{AP} \to \mathbb{B}^+(X \times Q)$ .

3.2. Direction-guided pushdown tree automata. We recall how semi-alternating pushdown tree automata are defined by constraining the behaviour of the stack in APTAs [ALM<sup>+</sup>13],

and then we constrain it further by letting stack operations depend only on precise components of directions taken in the input tree. We call the resulting class of automata direction-quided pushdown tree automata.

A semi-alternating pushdown tree automaton (SPTA) is an APTA such that for all  $q_1, q_2 \in Q$ ,  $\gamma \in \Gamma$  and  $a \in 2^{AP}$ , if  $[x, q'_1, w_1]$  appears in  $\delta(q_1, a, \gamma)$  and  $[x, q'_2, w_2]$  appears in  $\delta(q_2, a, \gamma)$ , then  $w_1 = w_2$ : whenever two copies of the automaton, possibly in different states, read the same input with the same symbol on the top of the stack, and move in the same direction, they must push the same thing on the stack. The transition function of an SPTA can be split into a state transition function  $\delta_Q : Q \times 2^{AP} \times \Gamma \to \mathbb{B}^+(X \times Q)$  and a stack update function  $\delta_{\Gamma} : 2^{AP} \times \Gamma \times X \to \Gamma^*$  such that for all  $(q, a, \gamma) \in Q \times 2^{AP} \times \Gamma$  we have  $\delta(q, a, \gamma) = \delta_Q(q, a, \gamma)$ , in which each [x, q'] is replaced with  $[x, q', \delta_{\Gamma}(a, \gamma, x)]$  (see [ALM+13] for details).

We now refine this definition to capture semi-alternating automata whose stack operations do not depend on the whole directions, but only on precise components of the directions.

**Definition 3.1.** An APTA  $\mathcal{A} = (\Gamma, Q, \delta, q_{\iota}, C)$  over  $X \times Y$ -trees has an X-guided stack, or simply is X-guided, if there exists a function  $\delta_{\Gamma} : 2^{AP} \times \Gamma \times X \to \Gamma^*$  such that for all  $(q, a, \gamma) \in Q \times 2^{AP} \times \Gamma$ , all atoms appearing in  $\delta(q, a, \gamma)$  are of the form  $[(x, y), q', \delta_{\Gamma}(a, \gamma, x)]$ .

Note that X-guided APTAs are semi-alternating.

We will need three operations on tree automata: projection, to guess valuations of atomic propositions, simulation, because projection is defined only for nondeterministic automata, and narrowing, to deal with imperfect information by hiding components of directions. It was established in  $[ALM^+13]$  that SPTAs can be nondeterminised, and we show that the projection and narrowing operations on classic tree automata can be easily extended to pushdown tree automata. We then notice that the simulation procedure presented in  $[ALM^+13]$  preserves X-guidedness, and so does projection, as well as narrowing if the X component is not erased by the operation.

3.3. **Projection.** Projection is defined in [Rab69] for classic nondeterministic tree automata. The construction is simple: the automaton projected on atom  $p \in AP$  guesses, in every node of its input, a valuation for p in this node, and proceeds accordingly. This construction is correct because nondeterministic automata only visit each node at most once. The construction and the proof of correctness are indifferent to the pushdown aspect, so that we have the following result.

**Proposition 3.2.** Given an NPTA  $\mathcal{N}$  and  $p \in AP$ , one can build an NPTA  $\mathcal{N} \downarrow p$  such that for every pointed tree (t, u) and initial stack content  $w_t \in \Gamma^* \cdot \flat$ ,

$$(t,u) \in \mathcal{L}(\mathcal{N} \downarrow p_{-p}, w_{\iota})$$
 iff  $\exists \ell_p \ a \ p$ -labelling for  $t \ s.t. \ (t \otimes \ell_p, u) \in \mathcal{L}(\mathcal{N}, w_{\iota})$ 

*Proof.* Let  $\mathcal{N} = (\Gamma, Q, \delta, q_{\iota}, C)$  be a nondeterministic pushdown tree automaton, and let  $\mathcal{N} \Downarrow_{-p} = (\Gamma, Q, \delta', q_{\iota}, C)$  where for all  $(q, a, \gamma) \in Q \times 2^{AP \setminus \{p\}} \times \Gamma$ ,

$$\delta'(q, a, \gamma) = \delta(q, a \setminus \{p\}, \gamma) \vee \delta(q, a \cup \{p\}, \gamma).$$

To see that this construction is correct, fix a pointed tree (t, u), and assume first that there exists a p-labelling  $\ell_p$  for t such that  $(t \otimes \ell_p, u) \in \mathcal{L}(\mathcal{N}, w_\iota)$ , i.e. Eve has a winning strategy  $\sigma$  in the acceptance game  $\mathcal{G}(\mathcal{N}, t \otimes \ell_p, u, w_\iota)$ . Observe that  $\mathcal{G}(\mathcal{N} \downarrow p, t, u, w_\iota)$  is essentially the same game, except that Eve has additional choices to make: everytime a new node v is reached, Eve has to choose between  $\delta(q, a \setminus \{p\}, \gamma)$  and  $\delta(q, a \cup \{p\}, \gamma)$ . A

winning strategy for Eve in this game is obtained by letting her choose  $\delta(q, a \cup \{p\}, \gamma)$  if  $\ell_p(u) = 1$ ,  $\delta(q, a \setminus \{p\}, \gamma)$  otherwise, and all her remaining choices follow  $\sigma$ . In other words, Eve guesses the p-labelling  $\ell_p$  and otherwise behaves as in  $\mathcal{G}(\mathcal{N}, t \otimes \ell_p, u, w_t)$ .

Now assume that  $\mathcal{N} \downarrow_{-p}$  accepts  $(t = (\tau, \ell), u)$ , and let  $\sigma$  be a winning strategy for Eve in  $\mathcal{G}(\mathcal{N} \downarrow_{-p}, t, u, w_{\iota})$ . Since  $\mathcal{N}$  is nondeterministic, by construction  $\mathcal{N} \downarrow_{-p}$  is also nondeterministic and thus each node of t is visited exactly once in the outcomes of  $\sigma$ . More precisely, for each node  $v \in t$  that is below u, there is a unique position of the form  $(v, q, \gamma \cdot w, \delta'(q, \ell(v), \gamma))$  that can be reached while Eve follows strategy  $\sigma$ . In addition, by definition of  $\mathcal{N} \downarrow_{-p}$ , we have that

$$\delta'(q, \ell(v), \gamma) = \delta(q, \ell(v) \setminus \{p\}, \gamma) \vee \delta(q, \ell(v) \cup \{p\}, \gamma).$$

We can thus define the p-labelling

$$\ell_p: v \mapsto \begin{cases} 0 & \text{if } \sigma \text{ chooses the first disjunct,} \\ 1 & \text{otherwise.} \end{cases}$$

It is then not hard to see that  $\sigma$  induces a winning strategy for Eve in  $\mathcal{G}(\mathcal{N}, t \otimes \ell_p, u, w_\iota)$ .  $\square$ 

3.4. **Simulation.** It is proved in [ALM<sup>+</sup>13] that, unlike alternating pushdown tree automata, semi-alternating ones can be nondeterminised.

**Theorem 3.3** ([ALM<sup>+</sup>13]). Given an SPTA  $\mathcal{A}$ , one can build an NPTA  $\mathcal{N}$  such that for every initial stack content  $w \in \Gamma^* \cdot \flat$ ,

$$\mathcal{L}(\mathcal{N}, w) = \mathcal{L}(\mathcal{A}, w).$$

We observe that the construction in [ALM $^+$ 13] for the simulation of SPTAs preserves X-guidedness, and thus we can refine the above result as follows:

**Proposition 3.4.** Given an X-guided SPTA  $\mathcal{A}$ , one can build an X-guided NPTA  $\mathcal{N}$  such that for every initial stack content  $w \in \Gamma^* \cdot \flat$ ,  $\mathcal{L}(\mathcal{N}, w) = \mathcal{L}(\mathcal{A}, w)$ .

Proof. Let  $\mathcal{A} = (\Gamma, Q, \delta, q_{\iota}, C)$  be an SPTA over X-trees, and let  $\delta_{\Gamma} : 2^{\text{AP}} \times \Gamma \times X \to \Gamma^*$  be its stack update function. The construction from [ALM+13] goes as follows. First, they observe that an SPTA  $\mathcal{A}$  induces, for every input tree t, a decorated version t' of t where the label of each node is enriched with the top symbol of the automaton'stack when it visits that node. This is well-defined because the automaton is semi-alternating. One can then build a classic ATA  $\tilde{\mathcal{A}}$  (without pushdown stack) such that  $\mathcal{A}$  accepts t if and only if  $\tilde{\mathcal{A}}$  accepts t'. With a classic simulation procedure, one then obtains an NTA  $\tilde{\mathcal{N}} = (Q, \delta', q'_{\iota}, C')$  equivalent to  $\tilde{\mathcal{A}}$ . It remains to define the NPTA  $\mathcal{N} = (\Gamma, Q', \delta'', q'_{\iota}, C')$  where  $\delta''(q, a, \gamma)$  is obtained from  $\delta'(q, (a, \gamma))$  by replacing every [x, q'] with  $[x, q', \delta_{\Gamma}(a, \gamma, x)]$ .

Now, if the initial automaton  $\mathcal{A}$  works on  $X \times Y$ -trees and is X-guided, by definition its stack update function  $\delta_{\Gamma}$  does not depend on the Y-components. By the above construction, it is also the case of the final NPTA:  $\delta''(q, a, \gamma)$  is now obtained from  $\delta'(q, (a, \gamma))$  by replacing every [(x, y), q'] with  $[(x, y), q', \delta_{\Gamma}(a, \gamma, x)]$ .

Finally, one can see that all the above arguments generalise easily to the case of automata starting in a given node u with a given initial stack content w.

3.5. Narrowing. For the last operation, we first recall the widening operation on trees, defined in [KV99]: given two sets of directions X and Y, for every X-tree  $\tau$  with root  $x \in X$  and every  $y \in Y$  we define the Y-widening of  $\tau$  rooted in (x, y) as the  $X \times Y$ -tree

$$\tau \uparrow_{y}^{X \times Y} := \{ u \in (x, y) \cdot (X \times Y)^* \mid u \downarrow_{X} \in \tau \}.$$

Also, for an (AP, X)-tree  $t=(\tau,\ell)$  and an element  $y\in Y,$  we let

$$t\!\uparrow_y^{X\times Y}:=(\tau\!\uparrow_y^{X\times Y},\ell'), \text{ where } \ell'(u):=\ell(u\!\downarrow_X).$$

We may write simply  $\tau \uparrow^{X \times Y}$  and  $t \uparrow^{X \times Y}$  when the choice of y does not matter or is understood. In particular, when referring to *pointed* widenings of trees such as  $(t \uparrow^{X \times Y}, u)$ , the choice of the root is determined by u: more precisely, y is taken to be the Y-component of the first direction in u.

We now generalise the narrowing operation [BMM<sup>+</sup>17] to the case of SPTAs. The idea behind this narrowing operation is that, if one just observes X, uniform p-labellings on  $X \times Y$ -trees can be obtained by choosing the labellings on X-trees, and then lifting them to  $X \times Y$ -trees.

The construction and proof of correctness are straightforwardly adapted from those in [KV99] for ATAs.

**Theorem 3.5** (Narrowing). Given an APTA  $\mathcal{A}$  on  $X \times Y$ -trees, one can build an APTA  $\mathcal{A} \downarrow_X$  on X-trees such that for every pointed (AP, X)-tree (t, u), every  $u' \in (X \times Y)^+$  such that  $u' \downarrow_X = u$ , and every initial stack content  $w_\iota \in \Gamma^* \cdot \flat$ ,

$$(t, u) \in \mathcal{L}(\mathcal{A}\downarrow_X, w_t)$$
 iff  $(t\uparrow^{X\times Y}, u') \in \mathcal{L}(\mathcal{A}, w_t)$ .

*Proof.* For a formula  $\alpha \in \mathbb{B}^+((X \times Y) \times Q \times \Gamma^*)$ , we let  $\alpha \downarrow_X \in \mathbb{B}^+(X \times Q \times \Gamma^*)$  be the formula obtained from  $\alpha$  by replacing each atom of the form [(x,y),q,w] with atom [x,q,w]. We define the automaton  $\mathcal{A}\downarrow_X = (\Gamma,Q,\delta',q_\iota,C)$  where for every  $q \in Q$ ,  $a \in 2^{\mathrm{AP}}$  and  $\gamma \in \Gamma$ ,  $\delta'(q,a,\gamma) := \delta(q,a,\gamma)\downarrow_X$ . We now prove that this construction is correct.

Let (t,u) be a pointed (AP,X)-tree, let  $u' \in (X \times Y)^+$  be such that  $u' \downarrow_X = u$ , and let  $w_\iota \in \Gamma^* \cdot \flat$ . First, assume that  $(t \uparrow^{X \times Y}, u') \in \mathcal{L}(A, w_\iota)$ . Let  $\sigma$  be a winning strategy for Eve in the acceptance game  $\mathcal{G}(A, t \uparrow^{X \times Y}, u', w_\iota)$ . By projecting on X nodes and formulas in positions of a play  $\lambda$  in this game, *i.e.* by replacing each position  $(v, q, w, \alpha)$  with  $(v \downarrow_X, q, w, \alpha \downarrow_X)$ , we obtain a play  $\lambda \downarrow_X$  in  $\mathcal{G}(A \downarrow_X, t, u, w_\iota)$ . Applying this projection to the set of outcomes of  $\sigma$ , we obtain a set of plays Out in  $\mathcal{G}(A \downarrow_X, t, u, w_\iota)$  that is the set of all outcomes of some strategy  $\sigma'$  for Eve (there are actually infinitely many such  $\sigma'$ , which differ only on partial plays that are not prefixes of plays in Out). And because the sequences of states, and thus of colours, are the same in the projected and original plays,  $\sigma'$  is winning for Eve in  $\mathcal{G}(A \downarrow_X, t, u, w_\iota)$ .

Now assume that  $(t,u) \in \mathcal{L}(\mathcal{A}\downarrow_X, w_\iota)$  and we show that  $(t\uparrow^{X\times Y}, u') \in \mathcal{L}(\mathcal{A}, w_\iota)$ . There exists a winning strategy  $\sigma$  for Eve in  $\mathcal{G} = \mathcal{G}(\mathcal{A}\downarrow_X, t, u, w_\iota)$ , from which we define a winning strategy  $\sigma'$  for Eve in  $\mathcal{G}' = \mathcal{G}(\mathcal{A}, t\uparrow^{X\times Y}, u', w_\iota)$ . Let  $\lambda'$  be a partial play in  $\mathcal{G}'$  in which it is Eve's turn to play, *i.e.*  $\lambda'$  is of the form  $\lambda'' \cdot (v', q', w', \alpha'_1 \vee \alpha'_2)$ . Its projection on X is thus of the form  $\lambda' \downarrow_X = \lambda'' \downarrow_X \cdot (v, q, w, \alpha_1 \vee \alpha_2)$ , where  $v = v' \downarrow_X, q' = q, w' = w$  and  $\alpha_i = \alpha'_i \downarrow_X$ . We let  $\sigma'(\lambda') := (v', q', w', \alpha'_i)$ , where i is such that  $\sigma(\lambda' \downarrow_X) = (v, q, w, \alpha_i)$ . Using the fact that a node v' in  $t\uparrow^{X\times Y}$  is labelled as  $v' \downarrow_X$  in t, one can check that  $\sigma'$  generates the same sequences of states of the automaton as  $\sigma$ , and is thus winning for Eve.

It then follows directly that:

**Proposition 3.6.** If an APTA  $\mathcal{A}$  over  $X \times Y \times Z$ -trees is X-guided, then so is  $\mathcal{A} \downarrow_{X \times Y}$ .

Indeed the stack update function  $\delta_{\Gamma}: 2^{AP} \times \Gamma \times X \to \Gamma^*$  of  $\mathcal{A}$  is also that of  $\mathcal{A} \downarrow_{X \times Y}$ .

## 4. Model Checking Hierarchical QCTL\*

Before presenting our automata construction we introduce succinct unfoldings, which allow us to work with trees over a *finite* set of directions.

4.1. Succinct unfoldings. The semantics of QCTL\*<sub>iR</sub> on a finite PCKS PK =  $(\Gamma, S, R, \ell, s_{\iota})$  is defined via the unfolding of the infinite CKS K<sub>PK</sub>, which is a tree over the infinite set of directions  $S \times \Gamma^* \cdot \flat$ . In this tree each node contains the entire content of the stack. It is however enough to record only the operations made on the stack in each node: then, starting from the root and the initial stack  $\flat$ , one can reconstruct the stack content at each node by following the unique path from the root to this node, and applying the successive operations on the stack. By doing so we obtain a tree over the finite set of directions  $S \times \Pi_{PK}$ , where

$$\Pi_{\mathsf{PK}} = \{ w \mid (s, \gamma, s', w) \in R \text{ for some } s, \gamma \text{ and } s' \} \cup \{ \varepsilon \}$$

(we require that  $\Pi_{PK}$  always contain the empty word).

First, for a partial path  $\lambda = \langle s_{\iota}, \flat \rangle \langle s_1, w_1 \rangle \dots \langle s_n, w_n \rangle$  in PK, we define its **succinct** representation

$$\pi(\lambda) = (s_{\iota}, \flat)(s_1, w'_1) \dots (s_n, w'_n) \in (S \times \Pi_{\mathsf{PK}})^*$$

where for  $i \ge 0$ ,  $w'_{i+1}$  is such that  $w_{i+1} = w'_{i+1} \cdot w''_i$ , with  $w_i = \gamma \cdot w''_i$ ; that is,  $w'_{i+1}$  is what has been pushed on the stack at step i+1.

If  $\pi = (s_{\iota}, \flat)(s_1, w'_1) \dots (s_n, w'_n) \in (S \times \Pi_{PK})^*$  is a succinct representation, we can reconstruct the unique partial path  $\lambda(\pi) = \langle s_{\iota}, \flat \rangle \langle s_1, w_1 \rangle \dots \langle s_n, w_n \rangle$  such that  $\pi(\lambda(\pi)) = \pi$ . We also let  $w_{\pi} := w_n$  denote the stack content after  $\pi$ .

**Definition 4.1.** Let  $\mathsf{PK} = (\Gamma, S, R, \ell, s_{\iota})$  be a PCKS. Its *succinct unfolding* is the  $(\mathsf{AP}, S \times \Pi_{\mathsf{PK}})$ -tree  $t_{\mathsf{PK}} := (\tau, \ell')$  where  $\tau = \{\pi(\lambda) \mid \lambda \in \mathsf{Paths}^*(\mathsf{PK})\}$  and for each  $\pi \in \tau$  ending in  $(s_n, w'_n), \ell'(\pi) = \ell(s_n, w_{\pi})$ .

The following is a direct consequence of the semantics of  $QCTL_{iR}^*$  (recall that  $\varphi_{n+1}$  is obtained from  $\varphi$  by replacing each concrete observation  $\mathbf{o} \subseteq [n]$  with  $\mathbf{o}' = \mathbf{o} \cup \{n+1\}$ ):

**Lemma 4.2.** For every PCKS PK over  $\{L_i\}_{i\in[n]}$  and every QCTL\* formula  $\varphi$ , PK  $\models \varphi$  iff  $t_{PK} \models \varphi_{n+1}$ .

Note that succinct unfoldings were implicitly used in [ALM<sup>+</sup>13].

4.2. **Automata construction.** We generalise the automata construction from [BMM<sup>+</sup>17] to the case of pushdown compound Kripke structures. The main novelties are, first, that we use direction-guided pushdown tree automata instead of classic alternating automata, relying on the fact that they are stable under the necessary operations as proved in Section 3, and second, that we have to deal with regular labellings for atomic propositions.

For the rest of this section we fix a PCKS PK and a formula  $\Phi \in \mathsf{QCTL}^*_{\mathrm{IR},\subseteq}$ . States in PK are elements of  $\prod_{i=1}^n L_i$  and concrete observations in  $\Phi$  are subsets of [n]. But according to Lemma 4.2, we will in fact consider the succinct unfolding of PK which is a tree over directions  $S \times \Pi_{\mathsf{PK}}$ , where  $\Pi_{\mathsf{PK}} = L_{n+1}$  captures stack operations, and with formula  $\Phi_{n+1}$  in

which each concrete observation  $\mathbf{o}$  has been replaced with  $\mathbf{o} \cup \{n+1\}$ , as the stack is visible. More precisely, for each subformula  $\varphi$  of  $\Phi_{n+1}$  we will build an automaton that works on  $X_{\varphi}$ -trees, where  $X_{\varphi}$  is defined as follows:

**Definition 4.3.** For every  $\varphi$ , let  $I_{\varphi} := \bigcap_{\mathbf{o} \in \mathrm{Obs}(\varphi)} \mathbf{o}$ , where  $\mathrm{Obs}(\varphi)$  is the set of concrete observations that occur in  $\varphi$ , with the intersection over the empty set defined as [n+1]. We then let  $X_{\varphi} := S_{\varphi} \times \Pi_{\mathsf{PK}}$ , where  $S_{\varphi} = \{s \downarrow_{I_{\varphi}} | s \in S\}$ .

We assumed free atoms to be disjoint from quantified ones in  $QCTL_{iR}^*$  formulas, *i.e.*  $AP_{\exists}(\Phi) \cap AP_f(\Phi) = \emptyset$ . We can thus assume that the PCKS PK is labelled over  $AP_f(\Phi)$ , while the input trees of our automata will be labelled over  $AP_{\exists}(\Phi)$ . To merge the labels for quantified propositions carried by the (complete) input tree, with those for free propositions carried by PCKS K, we use the merge operation from [BMM<sup>+</sup>17].

**Definition 4.4.** Let  $t = (\tau, \ell)$  be a complete (AP, X)-tree and  $t' = (\tau', \ell')$  an (AP', X)-tree with same root as t, where  $AP \cap AP' = \emptyset$ . The **merge** of t and t' is the  $(AP \cup AP', X)$ -tree

$$t \wedge t' := (\tau \cap \tau' = \tau', \ell''), \text{ where } \ell''(u) = \ell(u) \cup \ell'(u).$$

We now describe our automata construction to inductively build automata for subformulas of  $\Phi_{n+1}$  and the fixed PCKS PK =  $(\Gamma, S, R, \ell_{PK}, s_{\iota})$ . The construction is very similar to that in [BMM<sup>+</sup>17] for QCTL\*<sub>iR,⊆</sub> on finite systems: it builds on that for CTL\* [KVW00], and in addition it uses narrowing, nondeterminisation and projection to guess uniform labellings for quantified propositions. Also, in order not to lose information on the model while hiding components with the narrowing operation, the model is encoded in the automata instead of being given as input, and the input tree is only used to carry labellings for quantified propositions. We only give here the cases that contain significant differences from [BMM<sup>+</sup>17]: atomic proposition, and second-order quantification.

For atomic propositions we have to deal with regular labellings. To evaluate an atomic proposition p in a node u of the input tree, in state s of PK and with stack content w, automaton  $\mathcal{A}_s^p$  will simply read the p-labelling of u if p is quantified; otherwise it will simulate  $\mathcal{B}_s^p$  (which represents the regular labelling for p in state s) on the stack content by popping symbol after symbol, feeding them to  $\mathcal{B}_s^p$  while following an arbitrary direction in the input tree, and it will accept if  $\mathcal{B}_s^p$  is in an accepting state when the stack is empty.

For  $\varphi = \exists^{\mathbf{o}} p. \varphi'$ , we will first use the induction hypothesis to build a  $\Pi_{\mathsf{PK}}$ -guided automaton for  $\varphi'$ . Then we will use the results established in Section 3 to, first, narrow it to make it observe only  $\mathbf{o}$ , then nondeterminise it, which is possible because  $\Pi_{\mathsf{PK}}$ -guided automata are semi-alternating, and finally project it over p, while remaining  $\Pi_{\mathsf{PK}}$ -guided.

**Lemma 4.5.** For every subformula  $\varphi$  of  $\Phi_{n+1}$  and state  $s \in S$ , one can build an APTA  $\mathcal{A}_s^{\varphi}$  on  $(AP_{\exists}(\Phi), X_{\varphi})$ -trees with  $\Pi_{\mathsf{PK}}$ -guided stack and such that, for every  $(AP_{\exists}(\Phi), X_{\varphi})$ -tree t rooted in  $(s_t \downarrow_{I_{\varphi}}, \flat)$  and every partial path  $\lambda \in \mathsf{Paths}^*(\mathsf{PK})$  ending in  $\langle s, w \rangle$ , it holds that

*Proof.* The proof is by induction on  $\varphi$ . The automata we build will be  $\Pi_{PK}$ -guided, and more precisely the stack update function  $\delta_{\Gamma}: 2^{AP} \times \Gamma \times \Pi_{PK} \to \Gamma^*$  will be the mapping  $(a, \gamma, w) \mapsto w$ .

 $\varphi = p$ : Recall that automaton  $\mathcal{B}_s^p$  accepts the language  $\{w \in \Gamma^* \cdot \flat \mid p \in \ell(v, w)\}$ . From  $\mathcal{B}_s^p$  we can easily build an APTA  $\mathcal{A}'$  over  $(S \times \Pi_{PK})$ -trees (by Definition 4.3,  $X_{\varphi} = S \times \Pi_{PK}$ ) that, when started in a node u with a stack content w, simulates  $\mathcal{B}_s^p$  on the stack by popping

it until reaching b, going down in the input tree in direction  $(s, \varepsilon)$  for some arbitrary state  $s \in S$ . Automaton  $\mathcal{A}'$  accepts if  $\mathcal{B}_s^p$  is in an accepting state when the bottom of the stack is reached.

Writing  $\mathcal{A}' = (\Gamma, Q, \delta, q_{\iota}, C)$ , we define

$$\mathcal{A}_s^p = (\Gamma, Q \cup \{q_t'\}, \delta', q_t', C'),$$

where  $q'_{\iota}$  is a fresh initial state, C' extends C by assigning some insignificant colour to  $q'_{\iota}$ , and  $\delta'$  extends  $\delta$  by letting, for each  $\gamma \in \Gamma$  and  $a \in 2^{\text{AP}\exists}$ ,

$$\delta'(q'_{\iota}, \gamma, a) = \begin{cases} \top & \text{if } p \in AP_{\exists} \text{ and } p \in a \\ \bot & \text{if } p \in AP_{\exists} \text{ and } p \notin a \\ \delta(q_{\iota}, \gamma, a) & \text{otherwise, } i.e. \text{ if } p \in AP_{f} \end{cases}$$

 $\varphi = \neg \varphi'$ : We obtain  $\mathcal{A}_s^{\varphi}$  by complementing  $\mathcal{A}_s^{\varphi'}$ .

 $\varphi = \varphi_1 \vee \varphi_2$ : We first build  $\mathcal{A}_s^{\varphi_1}$  and  $\mathcal{A}_s^{\varphi_2}$ . Each  $\mathcal{A}_s^{\varphi_i}$  works on  $X_{\varphi_i}$ -trees, and  $I_{\varphi} = \varphi_1 \vee \varphi_2$  $I_{\varphi_1} \cap I_{\varphi_2}$ , so that by definition  $X_{\varphi} = S_{I_{\varphi_1} \cap I_{\varphi_2} \cap [n]} \times \Pi_{PK}$ . Thus we first narrow down each  $\mathcal{A}_s^{\varphi_i}$ so that they both work on  $X_{\varphi}$ -trees: for  $i \in \{1,2\}$ , we let  $\mathcal{A}_i := \mathcal{A}_s^{\varphi_i} \downarrow_{I_{\varphi}} = (\Gamma, Q^i, \delta^i, q_t^i, C^i)$ . Letting  $q_t$  be a fresh initial state we define  $\mathcal{A}_s^{\varphi} := (\Gamma, \{q_t\} \cup Q^1 \cup Q^2, \delta, q_t, C)$ , where  $\delta$  and C agree with  $\delta^i$  and  $C^i$ , respectively, on states from  $Q^i$ , and  $\delta(q_t, \gamma, a) = \delta^1(q_t^1, \gamma, a) \vee \delta^2(q_t^2, \gamma, a)$ . The colour of  $q_{\iota}$  does not matter.

 $\varphi = \mathbf{E}\psi : \text{Let } \max(\psi) = \{\varphi_1, \dots, \varphi_k\}$  be the set of maximal state sub-formulas of  $\psi$ . In a first step we see these maximal state sub-formulas as atomic propositions, we see the formula  $\psi$  as an LTL formula, and we build a nondeterministic parity word automaton  $\mathcal{B}^{\psi} = (Q^{\psi}, \Delta^{\psi}, q_{\iota}^{\psi}, C^{\psi})$  over alphabet  $2^{\max(\psi)}$  that accepts exactly the models of  $\psi$  [VW94].<sup>2</sup> We define the APTA  $\mathcal{A}$  that, given as input a  $(\max(\psi), X_{\varphi})$ -tree t, nondeterministically guesses a path  $\mu$  in  $t_{PK}$  and simulates  $\mathcal{B}^{\psi}$  on it, assuming that the labels it reads while following  $\mu \downarrow_{I_{\varphi}}$  in its input correctly represent the truth value of formulas in  $\max(\psi)$  along  $\mu$ . We define  $\mathcal{A} := (\Gamma, Q, \delta, q_{\iota}, C)$ , where

- $Q = Q^{\psi} \times S$ ,
- $\bullet \ q_{\iota} = (q_{\iota}^{\psi}, s),$
- for each  $(q^{\psi}, s') \in Q$ ,  $C(q^{\psi}, s') = C^{\psi}(q^{\psi})$ , and for each  $(q^{\psi}, s') \in Q$  and  $a \in 2^{\max(\psi)}$ ,

$$\delta((q^{\psi}, s'), \gamma, a) = \bigvee_{q' \in \Delta^{\psi}(q^{\psi}, a)} \bigvee_{(s'', w) \text{ s.t. } (s', \gamma, s'', w) \in R} [(s'' \downarrow_{I_{\varphi}}, w), (q', s''), w].$$

Intuitively,  $\mathcal{A}$  reads the current label a in its input and the top symbol of the stack  $\gamma$ . It then chooses nondeterministically which transition to take in  $\mathcal{B}^{\psi}$ , and it chooses also a possible transition  $(s', \gamma, s'', w)$  in PK. Then it moves in the input tree in direction  $(s'' \downarrow_{I_{c}}, w)$ , sending there a state that records the new current state in  $\mathcal{B}^{\psi}$  and the new current state in PK, and it pushes w on the stack of the automaton.

In general it is not possible to define a  $\max(\psi)$ -labelling of t that faithfully represents the truth values of formulas in  $\max(\psi)$ , because a node in t may correspond to different nodes in  $t_{PK}$  that have same projection on  $X_{\varphi}$  but satisfy different formulas of  $\max(\psi)$ .

<sup>&</sup>lt;sup>2</sup>Note that, as usual for nondeterministic word automata, we take the transition function of type  $\Delta^{\psi}: Q^{\psi} \times 2^{\max(\psi)} \to 2^{Q^{\psi}}$ . Note also that these automata use two colours, as actually Büchi automata are enough.

However this is not a problem because different copies of the final automaton (defined below) that visit the same node can guess different labellings, depending on the actual state of PK.

From  $\mathcal{A}$  we build automaton  $\mathcal{A}_s^{\varphi}$  over  $X_{\varphi}$ -trees labelled with atomic propositions in  $AP_{\exists}$ . In each node it visits,  $\mathcal{A}_s^{\varphi}$  guesses which formulas of  $\max(\psi)$  hold in this node with the current state of PK and current stack content, it simulates  $\mathcal{A}$  accordingly, and checks that the guess it made is correct. If the path being guessed in  $t_{\mathsf{PK}}$  is currently in node  $\pi$  ending with state s' and stack content  $w_{\pi}$ , and  $\mathcal{A}_s^{\varphi}$  guesses that  $\varphi_i$  holds, it launches a copy of automaton  $\mathcal{A}_{s'}^{\varphi_i}$  from node  $\pi \downarrow_{I_{\varphi}}$  in its input t, with the current stack content  $w_{\pi}$ .

For each  $s' \in S$  state of PK, and each  $\varphi_i \in \max(\psi)$ , we first build  $\mathcal{A}_{s'}^{\varphi_i}$  which works on  $X_{\varphi_i}$ -trees. We narrow down these automata to work on  $I_{\varphi} = \cap_{i=1}^k I_{\varphi_i}$ : let  $\mathcal{A}_{s'}^i := \mathcal{A}_{s'}^{\varphi_i} \downarrow_{I_{\varphi}} = (\Gamma, Q_{s'}^i, \delta_{s'}^i, q_{s'}^i, \overline{C_{s'}^i})$ . We also let  $\overline{\mathcal{A}_{s'}^i} = (\Gamma, \overline{Q_{s'}^i}, \overline{\delta_{s'}^i}, \overline{q_{s'}^i}, \overline{C_{s'}^i})$  be the dualisation of  $\mathcal{A}_{s'}^i$ , and we assume that all the state sets are pairwise disjoint. We define the APTA

$$\mathcal{A}_{s}^{\varphi} = (\Gamma, Q \cup \bigcup_{i,s'} Q_{s'}^{i} \cup \overline{Q_{s'}^{i}}, \delta', q_{\iota}, C'),$$

where the colours of states remain unchanged, and  $\delta'$  is defined as follows. For states in  $Q_{s'}^i$  (resp.  $\overline{Q_{s'}^i}$ ),  $\delta'$  agrees with  $\delta_{s'}^i$  (resp.  $\overline{\delta_{s'}^i}$ ), and for  $(q^{\psi}, s') \in Q$ ,  $\gamma \in \Gamma$  and  $a \in 2^{AP_{\exists}}$  we let

$$\delta'((q^{\psi},s'),\gamma,a) = \bigvee_{a' \in 2^{\max(\psi)}} \left( \delta\left((q^{\psi},s'),\gamma,a'\right) \wedge \bigwedge_{\varphi_i \in a'} \delta^i_{s'}(q^i_{s'},\gamma,a) \right. \wedge \bigwedge_{\varphi_i \notin a'} \overline{\delta^i_{s'}}(\overline{q^i_{s'}},\gamma,a) \right).$$

 $\varphi = \exists^{\mathbf{o}} p. \varphi'$ : First, we build automaton  $\mathcal{A}_s^{\varphi'}$  that works on  $X_{\varphi'}$ -trees; since  $\varphi$  is hierarchical, we have that  $I_{\varphi} = \mathbf{o} \subseteq I_{\varphi'}$  and we can narrow down  $\mathcal{A}_s^{\varphi'}$  to work on  $X_{\varphi}$ -trees: we let  $\mathcal{A}_1 := \mathcal{A}_s^{\varphi'} \downarrow_{\mathbf{o}}$ . By induction hypothesis,  $\mathcal{A}_s^{\varphi'}$  is  $\Pi_{\mathsf{PK}}$ -guided. By definition of  $\Phi_{n+1}$  we have  $n+1 \in \mathbf{o}$ , and thus  $\mathcal{A}_1$  is also  $\Pi_{\mathsf{PK}}$ -guided, by Proposition 3.6. Now, by Theorem 3.3 we can nondeterminise  $\mathcal{A}_1$ , getting  $\mathcal{A}_2$ , which by Theorem 3.2 we can project with respect to p, obtaining  $\mathcal{A}_s^{\varphi} := \mathcal{A}_2 \Downarrow_{-p}$ .

Correctness. In the following, for  $J \subseteq I \subseteq [n+1]$ , for every  $(AP, L_J)$ -tree t with root  $x \in L_J$ , and every  $y \in L_{I \setminus J}$ , we note  $t \uparrow_y^I$  for  $t \uparrow_y^{L_I}$  (recall that  $L_I = \prod_{i \in I} L_i$ , and that  $L_{n+1} = \prod_{PK}$ ). Let  $t = (\tau, \ell)$  be a complete  $(AP_{\exists}(\Phi), X_{\varphi})$ -tree rooted in  $(s \downarrow_{I_{\varphi}}, \flat)$ , let  $\lambda \in Paths^*(PK)$  be some partial path ending in  $\langle s, w \rangle$ , and let  $\pi = \pi(\lambda)$ .

 $\varphi = p$ : First, note that  $I_p = [n+1]$ , so that  $s\downarrow_{I_{\varphi}} = s$ , t is rooted in  $(s, \flat)$ ,  $\pi\downarrow_{I_p} = \pi$  and  $t\uparrow^{[n+1]} = t$ . Let us consider first the case where  $p \in AP_f$ : by definition of  $\mathcal{A}^p_s$ , we have  $(t,\pi) \in \mathcal{L}(\mathcal{A}^p_s, w)$  iff w is accepted by  $\mathcal{B}^p_s$ , i.e. iff  $p \in \ell_{PK}(\langle s, w \rangle)$ ; also, by definition of the merge, we have that  $t \wedge t_{PK}$ ,  $\pi \models p$  iff  $p \in \ell_{PK}(\langle s, w \rangle)$ , which concludes this case. Now if  $p \in AP_\exists$ , by definition of  $\mathcal{A}^p_s$ , we have that  $(t,\pi) \in \mathcal{L}(\mathcal{A}^p_s, w)$  iff node  $\pi$  is labelled with p in t. On the other hand, by definition of the merge, we have  $t \wedge t_{PK}$ ,  $\pi \models p$  iff  $\pi$  is labelled with p in t, and we are done.

 $\varphi = \neg \varphi'$ : This case is trivial. We only remark that the dualisation of a  $\Pi_{PK}$ -guided APTA is also  $\Pi_{PK}$ -guided.

 $\varphi_1 \vee \varphi_2$ : For  $i \in \{1, 2\}$  we have  $\mathcal{A}_i = \mathcal{A}_s^{\varphi_i} \downarrow_{I_o}$ , so by Theorem 3.5 we have that

$$(t, \pi \downarrow_{I_{\varphi}}) \in \mathcal{L}(\mathcal{A}_i, w) \text{ iff } (t \uparrow^{I_{\varphi_i}}, \pi \downarrow_{I_{\varphi_i}}) \in \mathcal{L}(\mathcal{A}_s^{\varphi_i}, w).$$

By induction hypothesis the latter holds iff

$$t \uparrow^{I_{\varphi_i}} \uparrow^{[n+1]} \wedge t_{\mathsf{PK}}, \pi \models \varphi_i,$$

and thus

$$(t, \pi \downarrow_{I_{\omega}}) \in \mathcal{L}(\mathcal{A}_i, w) \text{ iff } t \uparrow^{[n+1]} \mathbb{M} t_{\mathsf{PK}}, \pi \models \varphi_i.$$

We conclude by noting that  $\mathcal{L}(\mathcal{A}_s^{\varphi}, w) = \mathcal{L}(\mathcal{A}_1, w) \cup \mathcal{L}(\mathcal{A}_2, w)$ .

 $\varphi = \mathbf{E}\psi$ : Suppose that  $t\uparrow^{[n+1]} \mathbb{M}$   $t_{\mathsf{PK}}, \pi \models \mathbf{E}\psi$ . By definition, there exists an infinite path  $\mu'$  that starts at node  $\pi$  of  $t\uparrow^{[n+1]} \mathbb{M}$   $t_{\mathsf{PK}}$  such that  $t\uparrow^{[n+1]} \mathbb{M}$   $t_{\mathsf{PK}}, \mu' \models \psi$ , and by definition of succinct unfoldings and merge operation,  $\mu'$  corresponds to a unique infinite path  $\lambda^{\omega} \in \mathsf{Paths}^{\omega}(\mathsf{PK})$  that extends  $\lambda$ . Again, let  $\max(\psi)$  be the set of maximal state subformulas of  $\varphi$ , and let  $w(\mu')$  be the infinite word over  $2^{\max(\psi)}$  that agrees with  $\mu'$  on the state formulas in  $\max(\psi)$ , *i.e.* for each node  $\mu'_k$  of  $\mu'$  and formula  $\varphi_i \in \max(\psi)$ , it holds that

$$\varphi_i \in w(\mu')_k$$
 iff  $t \uparrow^{[n+1]} \mathbb{M} t_{\mathsf{PK}}, \mu'_k \models \varphi_i$ .

By definition,  $\mathcal{B}^{\psi}$  has an accepting execution on  $w(\mu')$ . To show that  $(t, \pi\downarrow_{I_{\varphi}}) \in \mathcal{L}(\mathcal{A}_{s}^{\varphi}, w)$  we show that Eve can win the acceptance game  $\mathcal{G}(\mathcal{A}_{s}^{\varphi}, t, \pi\downarrow_{I_{\varphi}}, w)$ . In this game, Eve can guess the continuation  $\lambda^{\omega}$  of  $\lambda$ , or equivalently the path  $\mu'$  in  $t\uparrow^{[n+1]}$   $\wedge$   $t_{\mathsf{PK}}$ , while the automaton follows  $\mu = \mu'\downarrow_{I_{\varphi}}$  in its input t, and she can also guess the corresponding word  $w(\mu')$  on  $2^{\max(\psi)}$  and an accepting execution of  $\mathcal{B}^{\psi}$  on  $w(\mu')$ . Let  $\pi' \in t\uparrow^{[n+1]} \wedge t_{\mathsf{PK}}$  be a node along  $\mu'$ , let (s', w') be its last direction and let  $\pi'' = \pi'\downarrow_{I_{\varphi}} \in t$ . Assume that in node  $\pi''$  of the input tree, in a state  $(q^{\psi}, s') \in Q$ , Adam challenges Eve on some  $\varphi_i \in \max(\psi)$  that she assumes to be true in  $\pi'$ , i.e. Adam chooses the conjunct  $\delta_{s'}^i(q_{s'}^i, \gamma, a)$ , where  $\gamma$  is the top of the current stack content and a is the label of  $\pi''$ . Note that in the evaluation game this means that Adam moves to position  $(\pi'', (q^{\psi}, s'), w_{\pi'}, \delta_{s'}^i(q_{s'}^i, \gamma, a))$ . We want to show that Eve wins from this position. To do so we first show that  $(t, \pi'') \in \mathcal{L}(\mathcal{A}_{s'}^i, w_{\pi'})$ .

First, recall that  $\mathcal{A}_{s'}^i = \mathcal{A}_{s'}^{\varphi_i} \downarrow_{I_{\varphi}}$ . By Theorem 3.5, it holds that  $(t, \pi'') \in \mathcal{L}(\mathcal{A}_{s'}^i, w_{\pi'})$  iff  $(t \uparrow^{I_{\varphi_i}}, \pi' \downarrow_{I_{\varphi_i}}) \in \mathcal{L}(\mathcal{A}_{s'}^{\varphi_i}, w_{\pi'})$ . Next, by applying the induction hypothesis we get that

$$(t\uparrow^{I_{\varphi_i}}, \pi'\downarrow_{I_{\varphi_i}}) \in \mathcal{L}(\mathcal{A}_{s'}^{\varphi_i}, w_{\pi'}) \text{ iff } t\uparrow^{I_{\varphi_i}}\uparrow^{[n+1]} \wedge h t_{\mathsf{PK}}, \pi' \models \varphi_i,$$

i.e. iff  $t \uparrow^{[n+1]} \land \land t_{PK}, \pi' \models \varphi_i$ , which holds because we have assumed that Eve guesses w correctly.

Eve thus has a winning strategy from the initial position of  $\mathcal{G}(\mathcal{A}_{s'}^i, t, \pi'', w_{\pi'})$ , the acceptance game of  $\mathcal{A}_{s'}^i$  on  $(t, \pi'')$  with initial stack content  $w_{\pi'}$ . This initial position is

$$(\pi'', q_{s'}^i, w_{\pi'}, \delta_{s'}^i(q_{s'}^i, \gamma, a)).$$

Since this position and position

$$(\pi'', (q^{\psi}, s'), w_{\pi'}, \delta^i_{s'}(q^i_{s'}, \gamma, a))$$

in  $\mathcal{G}(\mathcal{A}_s^{\varphi}, t, \pi \downarrow_{I_{\varphi}}, w_{\pi})$  contain the same node  $\pi'$ , stack content  $w_{\pi'}$  and transition formula  $\delta_{s'}^i(q_{s'}^i, \gamma, a)$ , a winning strategy in one of these positions<sup>3</sup> is also a winning strategy in the other, and therefore Eve wins Adam's challenge. With a similar argument, we get that also when Adam challenges Eve on some  $\varphi_i$  assumed not to be true in node  $\pi'$ , Eve wins the challenge, which concludes this direction.

For the other direction, assume that  $(t, \pi \downarrow_{I_{\varphi}}) \in \mathcal{L}(\mathcal{A}_{s}^{\varphi}, w)$ , *i.e.* Eve wins the evaluation game  $\mathcal{G}(\mathcal{A}_{s}^{\varphi}, t, \pi \downarrow_{I_{\varphi}}, w)$ . A winning strategy for Eve describes a path  $\mu$  in  $t_{\mathsf{PK}}$  starting in node  $\pi$ , which is also a path in  $t \uparrow^{[n+1]} \mathcal{M} t_{\mathsf{PK}}$ . This winning strategy also defines an infinite word  $w(\mu)$  over  $2^{\max(\psi)}$  such that  $w(\mu)$  agrees with  $\mu$  on the formulas in  $\max(\psi)$ ,

<sup>&</sup>lt;sup>3</sup>Recall that positional strategies are sufficient in parity games [Zie98].

and it also describes an accepting run of  $\mathcal{B}^{\psi}$  on w. Hence  $t \uparrow^{[n+1]} \mathbb{M} t_{\mathsf{PK}}, \mu \models \psi$ , and  $t \uparrow^{[n+1]} \mathbb{M} t_{\mathsf{PK}}, \pi \models \varphi$ .

 $\varphi = \exists^{\mathbf{o}} \mathbf{p}. \varphi'$ : First, by definition we have  $I_{\varphi} = \mathbf{o} \cap I_{\varphi'}$ . Because  $\varphi$  is hierarchical we have that  $\mathbf{o} \subseteq \mathbf{o}'$  for every  $\mathbf{o}'$  that occurs in  $\varphi'$ , and thus  $\mathbf{o} \subseteq I_{\varphi'}$ . It follows that  $I_{\varphi} = \mathbf{o}$ .

Next, since  $\mathcal{A}_s^{\varphi} = \mathcal{A}_2 \Downarrow_{-p}$ , by Theorem 3.2 we have that

$$(t, \pi \downarrow_{I_{\varphi}}) \in \mathcal{L}(\mathcal{A}_{s}^{\varphi}, w_{\pi})$$
 iff  $\exists \ell_{p} \text{ a } p$ -labelling for  $t$  such that  $(t \otimes \ell_{p}, \pi \downarrow_{I_{\varphi}}) \in \mathcal{L}(\mathcal{A}_{2}, w)$ .

By Theorem 3.3 for simulation,  $\mathcal{L}(\mathcal{A}_2, w) = \mathcal{L}(\mathcal{A}_1, w)$ , and since  $\mathcal{A}_1 = \mathcal{A}_s^{\varphi'} \downarrow_{\mathbf{o}} = \mathcal{A}_s^{\varphi'} \downarrow_{I_{\varphi}}$  we get by Theorem 3.5 that

$$(t \otimes \ell_p, \pi \downarrow_{I_{\varphi'}}) \in \mathcal{L}(\mathcal{A}_2) \quad \text{iff} \quad ((t \otimes \ell_p) \uparrow^{I_{\varphi'}}, \pi \downarrow_{I_{\varphi'}}) \in \mathcal{L}(\mathcal{A}_s^{\varphi'}).$$

By induction hypothesis,

$$((t\otimes \ell_p)\!\uparrow^{I_{\varphi'}},\pi\!\downarrow_{I_{\varphi'}})\in \mathcal{L}(\mathcal{A}_s^{\varphi'})\quad \text{iff}\quad (t\otimes \ell_p)\!\uparrow^{I_{\varphi'}}\!\!\uparrow^{[n+1]} \mathbb{M}\ t_{\mathsf{PK}},\pi\models\varphi'.$$

The three equivalences above plus the fact that  $(t \otimes \ell_p) \uparrow^{I_{\varphi'}} \uparrow^{[n+1]} = (t \otimes \ell_p) \uparrow^{[n+1]}$  imply that

$$(t, \pi \downarrow_{I_{\varphi}}) \in \mathcal{L}(\mathcal{A}_{s}^{\varphi}, w)$$

$$\exists \ell_p \text{ a } p\text{-labelling for } t \text{ s.t. } (t \otimes \ell_p) \uparrow^{[n+1]} \mathbb{M} t_{\mathsf{PK}}, \pi \models \varphi'.$$

We now prove the following which, together with the latter equivalence, concludes the proof:

$$\exists \ell_p \text{ a } p\text{-labelling for } t \text{ s.t. } (t \otimes \ell_p) \uparrow^{[n+1]} \land h t_{\mathsf{PK}}, \pi \models \varphi'$$
 iff 
$$t \uparrow^{[n+1]} \land h t_{\mathsf{PK}}, \pi \models \exists^{\mathbf{o}} p. \varphi'$$
 (4.1)

Assume that there exists a p-labelling  $\ell_p$  for t such that  $(t \otimes \ell_p) \uparrow^{[n+1]} \land \land t_{PK}, \pi \models \varphi'$ . Let  $\ell'_p$  be the p-labelling of  $(t \otimes \ell_p) \uparrow^{[n+1]} \land \land t_{PK}$ . By definition of the merge,  $\ell'_p$  is equal to the p-labelling of  $(t \otimes \ell_p) \uparrow^{[n+1]}$ ; therefore

$$(t \otimes \ell_p) \uparrow^{[n+1]} \wedge t_{\mathsf{PK}} = (t \uparrow^{[n+1]} \wedge t_{\mathsf{PK}}) \otimes \ell'_p,$$

and  $\ell_p'$  is  $I_{\varphi}$ -uniform, *i.e.* **o**-uniform (by definition of the widening). This concludes this direction.

Now assume that  $t \uparrow^{[n+1]} \wedge \!\!\! h t_{\mathsf{PK}}, \pi \models \exists^{\mathbf{o}} p. \varphi'$ : there exists a **o**-uniform p-labelling  $\ell'_p$  for  $t \uparrow^{[n+1]} \wedge \!\!\! h t_{\mathsf{PK}}$  such that  $(t \uparrow^{[n+1]} \wedge \!\!\! h t_{\mathsf{PK}}) \otimes \ell'_p, u \models \varphi'$ . We define a p-labelling  $\ell_p$  for t such that  $(t \otimes \ell_p) \uparrow^{[n+1]} \wedge \!\!\! h t_{\mathsf{PK}}, \pi \models \varphi'$ . First, let us write  $t' = t \uparrow^{[n+1]} \wedge \!\!\! h t_{\mathsf{PK}} = (\tau', \ell')$ . For each node u of t, let

$$\ell_p(u) = \begin{cases} \ell_p'(u') & \text{if there exists } u' \in \tau' \text{ such that } u' \downarrow_{\mathbf{o}} = u, \\ 0 & \text{otherwise.} \end{cases}$$

This is well defined because  $\ell_p'$  is **o**-uniform in p, so that if two nodes u', v' project on u, i.e.  $u' \approx_{\mathbf{o}} v'$ , we have  $\ell_p'(u') = \ell_p'(v')$ . In case there is no  $u' \in \tau'$  such that  $u' \downarrow_{I_{\varphi}} = u$ , the value of  $\ell_p(u)$  has no impact on  $(t \otimes \ell_p) \uparrow^{[n+1]} \mathbb{M}$   $t_{\mathsf{PK}}$ . Finally,  $(t \otimes \ell_p) \uparrow^{[n+1]} \mathbb{M}$   $t_{\mathsf{PK}} = (t \uparrow^{[n+1]} \mathbb{M}) \otimes \ell_p'$ , hence the result.

4.3. **Proof of Theorem 2.8.** We now prove Theorem 2.8. Let PK be a PCKS with initial state  $s_{\iota}$  and  $\Phi \in \mathsf{QCTL}^*_{\mathsf{iR},\subseteq}$ . For readability let us also write  $\Phi' = \Phi_{n+1}$ . Applying Lemma 4.5 to  $\Phi'$  and state  $s_{\iota}$ , we can construct an APTA  $\mathcal{A}^{\Phi'}_{s_{\iota}}$  with  $\Pi_{\mathsf{PK}}$ -guided stack such that for every  $(\mathsf{AP}_{\exists}(\Phi), X_{\Phi'})$ -tree t rooted in  $(s_{\iota} \downarrow_{I_{\Phi'}}, \flat)$ , every partial path  $\lambda$  in Paths\*(PK) ending in  $\langle s_{\iota}, w \rangle$ , it holds that

$$(t, \pi(\lambda)\downarrow_{I_{\Phi'}}) \in \mathcal{L}(\mathcal{A}_{s_t}^{\Phi'}, w) \text{ iff } t \uparrow^{S \times \Pi_{\mathsf{PK}}} \wedge \!\! \wedge t_{\mathsf{PK}}, \pi(\lambda) \models \Phi'.$$

Let  $\tau$  be the full  $X_{\Phi'}$ -tree rooted in  $(s_{\iota}\downarrow_{I_{\Phi'}}, \flat)$ , and let  $t=(\tau,\ell_{\emptyset})$ , where  $\ell_{\emptyset}$  is the empty labelling. Clearly, we have that  $t\uparrow^{S\times\Pi_{\mathsf{PK}}} \wedge \!\!\! \wedge t_{\mathsf{PK}} = t_{\mathsf{PK}}$ , and because t is rooted in  $(s_{\iota}\downarrow_{I_{\Phi'}}, \flat)$ , applying the above equivalence to t and  $\lambda = \langle s_{\iota}, \flat \rangle$ , we get that  $(t, (s_{\iota}\downarrow_{I_{\Phi'}}, \flat)) \in \mathcal{L}(\mathcal{A}_{s_{\iota}}^{\Phi'}, \flat)$  iff  $t_{\mathsf{PK}} \models \Phi'$ .

Since, by Lemma 4.2,  $t_{\mathsf{PK}} \models \Phi'$  holds iff  $\mathsf{PK} \models \Phi$ , it only remains to check whether tree t, which is regular<sup>4</sup>, is accepted by  $\mathcal{A}_{s_{\iota}}^{\Phi'}$ . This can be done by taking the product of  $\mathcal{A}_{s_{\iota}}^{\Phi'}$  with a finite Kripke structure representing t and checking for emptiness, which is decidable for semi-alternating pushdown tree automata [ALM<sup>+</sup>13].

## 5. **SL** WITH IMPERFECT INFORMATION

We recall the syntax and semantics of Strategy Logic with imperfect information ( $SL_{iR}$ ). The semantics is defined as in [BMM<sup>+</sup>17] on concurrent game arenas with imperfect information, except that we allow for infinite ones. We then define the subclass of infinite arenas generated by pushdown arenas with imperfect information on control states, on which we study the model-checking problem for  $SL_{iR}$ .

5.1. **Syntax.** For the rest of the section we fix a finite set of **agents** or **players** Ag, a finite set of **observation symbols** or simply **observations** Obs and a finite set of **variables** Var. Observations represent observational powers for the players.

**Definition 5.1.** The syntax of  $SL_{iR}$  is defined by the following grammar:

$$\varphi := p \mid \neg \varphi \mid \varphi \vee \varphi \mid \langle \langle x \rangle \rangle^o \varphi \mid (a, x) \varphi \mid \mathbf{E} \psi$$
 State formulas 
$$\psi := \varphi \mid \neg \psi \mid \psi \vee \psi \mid \mathbf{X} \psi \mid \psi \mathbf{U} \psi$$
 Path formulas

where  $p \in AP$ ,  $x \in Var$ ,  $o \in Obs$  and  $a \in Ag$ .

Boolean and temporal operators have their usual meaning. Strategy quantification  $\langle\!\langle x \rangle\!\rangle^o \varphi$  reads as "there exists a strategy x that takes decisions based on observational power o such that  $\varphi$  holds". Binding  $(a,x)\varphi$  reads as "when agent a plays strategy x,  $\varphi$  holds", and finally,  $\mathbf{E}\psi$  reads as " $\psi$  holds in some outcome of the strategies currently used by the players".  $\mathsf{SL}_{iR}$  consists of all state formulas.

For  $\varphi \in \mathsf{SL}_{iR}$ , we let  $free(\varphi)$  be the set of variables that appear free in  $\varphi$ , *i.e.* that appear out of the scope of a strategy quantifier. A formula  $\varphi$  is a **sentence** if  $free(\varphi)$  is empty.

<sup>&</sup>lt;sup>4</sup>A tree is regular if it has only finitely many distinct infinite subtrees; equivalently if it can be obtained by unfolding a finite labelled Kripke structure.

5.2. **Semantics.**  $SL_{\rm iR}$  formulas are evaluated on (possibly infinite) concurrent game arenas with interpretations for observation symbols.

**Definition 5.2.** A *concurrent game arena* (or CGA) is a tuple  $\mathcal{G} = (Ac, V, \Delta, \ell, v_{\iota}, \mathcal{O})$  where

- Ac is a finite set of actions,
- V is a set of positions,
- $\Delta: V \times Ac^{Ag} \to V$  is a transition function,
- $\ell: V \to 2^{AP}$  is a labelling function,
- $v_{\iota} \in V$  is an initial position, and
- $\mathcal{O}: \text{Obs} \to 2^{V \times V}$  is an observation interpretation.

For  $o \in \text{Obs}$ ,  $\mathcal{O}(o)$  is an equivalence relation on positions, that we may write  $\sim_o$ . It represents what a player using a strategy with observation o can see:  $\sim_o$ -equivalent positions are indistinguishable to a player using a strategy associated with observation o.

In a position  $v \in V$ , each player a chooses an action  $\alpha_a \in Ac$ , and the game proceeds to position  $\Delta(v, \boldsymbol{\alpha})$ , where  $\boldsymbol{\alpha} \in Ac^{Ag}$  stands for the **joint action**  $(\alpha_a)_{a \in Ag}$ . Given a joint action  $\boldsymbol{\alpha} = (\alpha_a)_{a \in Ag}$  and  $a \in Ag$ , we let  $\boldsymbol{\alpha}(a)$  denote  $\alpha_a$ . A **play** is an infinite word  $\lambda = v_0 \boldsymbol{\alpha}_0 v_1 \boldsymbol{\alpha}_1 v_2 \dots$  such that  $v_0 = v_t$  and for every  $i \geq 0$ ,  $\Delta(v_i, \boldsymbol{\alpha}_i) = v_{i+1}$ . A finite prefix of a play ending in a position is a **partial play**, and we let Plays\* be the set of partial plays. For each observation o we define the equivalence relation  $\sim_o$  on partial plays as follows:  $v_0 \boldsymbol{\alpha}_0 v_1 \boldsymbol{\alpha}_1 v_2 \dots v_k \sim_o v_0' \boldsymbol{\alpha}_0' v_1' \boldsymbol{\alpha}_1' v_2' \dots v_{k'}'$  if k = k', and  $v_i \sim_o v_i'$  for every  $i \in \{0, \dots, k\}$ .

A strategy is a function  $\sigma: \operatorname{Plays}^* \to \operatorname{Ac}$  that maps each partial play to an action. For  $o \in \operatorname{Obs}$ , an o-strategy is a strategy  $\sigma$  such that  $\sigma(\lambda) = \sigma(\lambda')$  whenever  $\lambda \sim_o \lambda'$ . We let  $\operatorname{Str}_o$  be the set of all o-strategies. An  $\operatorname{assignment}$  is a partial function  $\chi: \operatorname{Ag} \cup \operatorname{Var} \to \operatorname{Str}$ , assigning to each player and variable in its domain a strategy. For an assignment  $\chi$ , a player a and a strategy  $\sigma$ ,  $\chi[a \mapsto \sigma]$  is the assignment of domain  $\operatorname{dom}(\chi) \cup \{a\}$  that maps a to  $\sigma$  and is equal to  $\chi$  on the rest of its domain, and  $\chi[x \mapsto \sigma]$  is defined similarly, where x is a variable. In addition, given a formula  $\varphi \in \operatorname{SL}_{iR}$ , an assignment is  $\operatorname{variable-complete}$  for  $\varphi$  if its domain contains all free variables of  $\varphi$ .

For an assignment  $\chi$  and a partial play  $\lambda$ , we let  $\operatorname{Out}(\chi, \lambda)$  be the set of plays that extend  $\lambda$  by letting each player a follow strategy  $\chi(a)$ . Formally, if  $\lambda = v_0 \boldsymbol{\alpha}_0 v_1 \dots \boldsymbol{\alpha}_{k-1} v_k$ , then  $\operatorname{Out}(\chi, \lambda)$  is the set of plays of the form  $\lambda \cdot \boldsymbol{\alpha}_k v_{k+1} \boldsymbol{\alpha}_{k+1} v_{k+2} \dots$  such that for all  $i \geq 0$  and all  $a \in \operatorname{dom}(\chi) \cap \operatorname{Ag}, \ \boldsymbol{\alpha}_{k+i}(a) = \chi(a)(\lambda \cdot \boldsymbol{\alpha}_k v_{k+1} \dots \boldsymbol{\alpha}_{k+i-1} v_{k+i})$  and  $v_{k+i+1} = \Delta(v_{k+i}, \boldsymbol{\alpha}_{k+i})$ .

**Definition 5.3.** The semantics of a state (resp. path) formula is defined on a CGA  $\mathcal{G}$ , an assignment  $\chi$  that is variable-complete for  $\varphi$ , and a partial play  $\lambda$  (resp. an infinite play  $\lambda'$ 

and an index  $i \in \mathbb{N}$ ). The inductive definition is as follows:

```
p \in \ell(\operatorname{last}(\lambda))
\mathcal{G}, \chi, \lambda \models p
\mathcal{G}, \chi, \lambda \models \neg \varphi
                                                               if \mathcal{G}, \chi, \lambda \not\models \varphi
\mathcal{G}, \chi, \lambda \models \varphi \vee \varphi'
                                                               if \mathcal{G}, \chi, \lambda \models \varphi or \mathcal{G}, \chi, \lambda \models \varphi'
\mathcal{G}, \chi, \lambda \models \langle\!\langle x \rangle\!\rangle^o \varphi
                                                               if \exists \sigma \in Str_o \text{ s.t. } \mathcal{G}, \chi[x \mapsto \sigma], \lambda \models \varphi
\mathcal{G}, \chi, \lambda \models (a, x)\varphi
                                                              if \mathcal{G}, \chi[a \mapsto \chi(x)], \lambda \models \varphi

\mathcal{G}, \chi, \lambda \models \mathbf{E}\psi 

\mathcal{G}, \chi, \lambda', i \models \psi 

\mathcal{G}, \chi, \lambda', i \models \neg \psi

                                                              if \exists \lambda' \in \text{Out}(\chi, \lambda) such that \mathcal{G}, \chi, \lambda', |\lambda| - 1 \models \psi
                                                              if \mathcal{G}, \chi, \lambda'_{\leq i} \models \psi
                                                             if \mathcal{G}, \chi, \lambda^{\overline{i}}, i \not\models \psi
\mathcal{G}, \chi, \lambda', i \models \psi \lor \psi' if \mathcal{G}, \chi, \lambda', i \models \psi or \mathcal{G}, \chi, \lambda', i \models \psi'
\mathcal{G}, \chi, \lambda', i \models \mathbf{X}\psi
                                                               if \mathcal{G}, \chi, \lambda', i+1 \models \psi
                                                              if \exists j \geq i \text{ s.t. } \mathcal{G}, \chi, \lambda', j \models \psi' \text{ and } \forall k \text{ s.t. } i \leq k < j, \mathcal{G}, \chi, \lambda', k \models \psi
\mathcal{G}, \chi, \lambda', i \models \psi \mathbf{U} \psi'
```

A sentence  $\varphi$  can be evaluated in the empty assignment  $\emptyset$ . Given a sentence  $\varphi$  and a CGA  $\mathcal{G}$  with initial position  $v_{\iota}$ , we write  $\mathcal{G} \models \varphi$  if  $\mathcal{G}, \emptyset, v_{\iota} \models \varphi$ .

5.3. Pushdown game arenas. We introduce Pushdown Game Arenas with Visible Stack, a variant of Epistemic Pushdown Game Structures (EPGS) defined in [CSW17], themselves an imperfect-information generalisation of the Pushdown Game Structures from [MP15]. While in EPGS players have imperfect information both on the control states and the stack, in Pushdown Game Arenas with Visible Stack, the stack is perfectly observed by all players. Another minor difference is that while in EPGS, observational equivalence relations are associated to players, in our models they are associated to observation symbols.

**Definition 5.4.** A *Pushdown Game Arena with Visible Stack*, or PGA<sub>VS</sub>, is a tuple  $\mathcal{PG} = (Ac, \Gamma, V, \mathcal{T}, \ell, v_{\iota}, \mathcal{O})$  where

- Ac is a finite set of actions,
- $\Gamma$  is a finite stack alphabet together with a bottom symbol  $\flat \notin \Gamma$  and we let  $\Gamma_{\flat} = \Gamma \cup \{\flat\}$ ,
- V is a finite set of control states,
- $\mathcal{T}: V \times \Gamma_{\flat} \times Ac^{Ag} \to V \times \Gamma_{\flat}^*$  is a transition function,
- $\ell: V \times \Gamma^* \cdot \flat \to 2^{AP}$  is a regular labelling function,
- $v_{\iota} \in V$  is an initial control state, and
- $\mathcal{O}: \text{Obs} \to 2^{V \times V}$  is an observation interpretation.

As in Definition 2.4, we require that the bottom symbol never be removed or pushed: for any  $v \in V$  and  $\boldsymbol{\alpha} \in \operatorname{Ac^{Ag}}$ , one has  $\mathcal{T}(v, \flat, \boldsymbol{\alpha}) \in V \times \Gamma^* \cdot \flat$  (the bottom symbol is never removed), and for every  $\gamma \in \Gamma$ ,  $\mathcal{T}(v, \gamma, \boldsymbol{\alpha}) \in V \times \Gamma^*$  (the bottom symbol is never pushed).

For  $o \in \text{Obs}$ ,  $\mathcal{O}(o)$  is an equivalence relation on control states, that we may write  $\sim_o$ . Also, by regular labelling function, we mean that for each  $p \in \text{AP}$  and  $v \in V$ , the set  $\{w \in \Gamma^* \cdot \flat \mid p \in \ell(v, w)\}$  forms a regular language [EKS03].

A **configuration** is a pair  $\langle v, w \rangle \in V \times (\Gamma^* \cdot \flat)$  where v represents the current control state and w the current content of the stack. When the players choose a joint move  $\alpha \in \operatorname{Ac^{Ag}}$  in a configuration  $\langle v, \gamma \cdot w \rangle$  the system moves to configuration  $\langle v', w' \cdot w \rangle$ , where  $\langle v', w' \rangle = \mathcal{T}(v, \gamma, \alpha)$ ; we denote this by  $\langle v, \gamma \cdot w \rangle \stackrel{\alpha}{\hookrightarrow} \langle v', w' \cdot w \rangle$ .

A PGA<sub>VS</sub>  $\mathcal{PG} = (Ac, \Gamma, V, \mathcal{T}, \ell, v_{\iota}, \mathcal{O})$  induces an infinite CGA  $\mathcal{G}_{\mathcal{PG}} = (Ac, V', \Delta, \ell', v_{\iota}, \mathcal{O}')$  where

• 
$$V' = V \times (\Gamma^* \cdot \flat)$$
,

- $\Delta(\langle v, \gamma \cdot w \rangle, \boldsymbol{\alpha}) = \langle v', w' \cdot w \rangle$  if  $\langle v, \gamma \cdot w \rangle \stackrel{\boldsymbol{\alpha}}{\hookrightarrow} \langle v', w' \cdot w \rangle$ ,
- $\ell' = \ell$ .
- $v'_{\iota} = \langle v_{\iota}, \flat \rangle$ ,
- $(\langle v, w \rangle, \langle v', w' \rangle) \in \mathcal{O}'(o)$  if w = w' and  $(v, v') \in \mathcal{O}(o)$ .

Plays and partial plays of  $\mathcal{PG}$  are those of  $\mathcal{G}_{\mathcal{PG}}$ . For an  $\mathsf{SL}_{iR}$  sentence  $\varphi$ , we write  $\mathcal{PG} \models \varphi$  if  $\mathcal{G}_{\mathcal{PG}} \models \varphi$ .

5.4. Model checking hierarchical instances. We study the model-checking problem for  $SL_{iR}$  evaluated on pushdown game arenas with visible stack. This problem is clearly undecidable as it captures multiplayer games with imperfect information (see for instance [PR79, PR90]). We generalise a result from [BMM<sup>+</sup>17], which shows that model-checking  $SL_{iR}$  on finite CGAs is decidable for so-called *hierarchical instances*, *i.e.* when each strategy quantifier in a formula is associated to an observation finer than those associated to strategy quantifiers higher up in the syntactic tree.

Given an *instance*  $(\mathcal{PG}, \Phi)$ , where  $\mathcal{PG}$  is a  $PGA_{VS}$  and  $\Phi$  is an  $SL_{iR}$  sentence, the model-checking problem consists in deciding whether  $\mathcal{PG} \models \Phi$ .

**Definition 5.5.** An instance  $(\mathcal{PG}, \Phi)$  is *hierarchical* if for every subformula  $\varphi_1 = \langle \langle y \rangle \rangle^{o_1} \varphi_1'$  of  $\Phi$  and subformula  $\varphi_2 = \langle \langle x \rangle \rangle^{o_2} \varphi_2'$  of  $\varphi_1'$ , it holds that  $\mathcal{O}(o_2) \subseteq \mathcal{O}(o_1)$ .

The rest of this section is dedicated to the proof of the following result:

**Theorem 5.6.** Model checking  $SL_{\mathrm{iR}}$  on pushdown game arenas with visible stack is decidable for hierarchical instances.

We adapt the reduction from [BMM<sup>+</sup>17] to transform hierarchical instances of  $SL_{iR}$  on  $PGA_{VS}$  into hierarchical instances of  $QCTL_{iR}^*$  on PCKS. Let  $(\mathcal{PG}, \Phi)$  be a hierarchical instance of the model-checking problem for  $SL_{iR}$ , and assume without loss of generality that each strategy variable is quantified at most once in  $\Phi$ .

**Model transformation.** We first define the PCKS  $PK_{\mathcal{PG}}$ . Let  $Obs = \{o_1, \ldots, o_n\}$ , and let  $\mathcal{PG} = (Ac, \Gamma, V, \mathcal{T}, \ell, v_\iota, \mathcal{O})$ . For  $i \in [n]$ , define the local states  $L_i := \{[v]_{o_i} \mid v \in V\}$ , where  $[v]_o$  is the equivalence class of v for relation  $\mathcal{O}(o)$ . For each control state  $v \in V$  and joint move  $\alpha \in Ac^{Ag}$ , we define  $s_{v,\alpha} := ([v]_{o_1}, \ldots, [v]_{o_n}, v, \alpha)$ . Each tuple  $s_{v,\alpha} \in \prod_{i \in [n]} L_i \times V \times Ac^{Ag}$  contains the equivalence class of v for each observation  $o_i \in Obs$ ; we include the exact control state v of  $\mathcal{PG}$  because it is needed to define the dynamics, and we also include the last joint action played to make it possible to check that players follow their strategies.

Let  $AP_{\alpha} = \{p_{\alpha} \mid \alpha \in Ac^{Ag}\}$  be a set of fresh atomic propositions. Define the PCKS  $PK_{\mathcal{PG}} = (\Gamma, S, R, \ell', s_{\iota})$  over  $AP \cup AP_{\alpha}$ , where

- $S = \{s_{v,\alpha} \mid v \in V \text{ and } \alpha \in Ac^{Ag}\},\$
- $R = \{(s_{v,\alpha}, \gamma, s_{v',\alpha'}, w') \mid \mathcal{T}(v, \gamma, \alpha') = (v', w')\},\$
- $\ell'(\langle s_{v,\alpha}, w \rangle) = \ell(\langle v, w \rangle) \cup \{p_{\alpha}\}, \text{ and }$
- $s_{\iota} = s_{v_{\iota}, \boldsymbol{\alpha}_{\iota}}$  for some arbitrary  $\boldsymbol{\alpha}_{\iota} \in \operatorname{Ac}^{\operatorname{Ag}}$ .

The labelling  $\ell'$  is regular because  $\ell$  is regular for atoms in AP, and the truth value of atoms in AP<sub> $\alpha$ </sub> is determined by the control state only.

For every partial play  $\lambda = \langle v_{\iota}, \flat \rangle \alpha_{0} \langle v_{1}, w_{1} \rangle \dots \langle v_{k}, w_{k} \rangle$  in  $\mathcal{PG}$ , define the partial path  $\lambda' = \langle s_{\iota}, \flat \rangle \langle s_{1}, w_{1} \rangle \dots \langle s_{k}, w_{k} \rangle$  in  $\mathsf{PK}_{\mathcal{PG}}$  where  $s_{i} = s_{v_{i}, \alpha_{i-1}}$ , for each  $i \in [k]$ . The mapping  $\lambda \mapsto \lambda'$  puts in bijection partial plays of  $\mathcal{G}_{\mathcal{PG}}$  with partial paths of  $\mathsf{PK}_{\mathcal{PG}}$ .

**Formula transformation.** We now describe how to transform an  $\mathsf{SL}_{iR}$  formula  $\varphi$  and a partial function  $f: \mathsf{Ag} \to \mathsf{Var}$  into a  $\mathsf{QCTL}^*_{iR}$  formula  $(\varphi)^f_s$  (that will also depend on  $\mathcal{PG}$ ). Suppose that  $\mathsf{Ac} = \{\alpha_1, \ldots, \alpha_l\}$ , and define  $(\varphi)^f_s$  and  $(\psi)^f_p$  by mutual induction on state and path formulas.

Base, boolean and temporal cases are as follows:

$$(p)_{s}^{f} := p \qquad (\varphi)_{p}^{f} := (\varphi)_{s}^{f}$$

$$(\neg \varphi)_{s}^{f} := \neg (\varphi)_{s}^{f} \qquad (\neg \psi)_{p}^{f} := \neg (\psi)_{p}^{f}$$

$$(\varphi_{1} \lor \varphi_{2})_{s}^{f} := (\varphi_{1})_{s}^{f} \lor (\varphi_{2})_{s}^{f} \qquad (\psi_{1} \lor \psi_{2})_{p}^{f} := (\psi_{1})_{p}^{f} \lor (\psi_{2})_{p}^{f}$$

$$(\mathbf{X}\psi)_{p}^{f} := \mathbf{X}(\psi)_{p}^{f} \qquad (\psi_{1}\mathbf{U}\psi_{2})_{p}^{f} := (\psi_{1})_{p}^{f}\mathbf{U}(\psi_{2})_{p}^{f}.$$

For the strategy quantifier we let

$$(\langle\!\langle x \rangle\!\rangle^o \varphi)_s^f := \exists^{\widetilde{o}} p_{\alpha_1}^x \dots \exists^{\widetilde{o}} p_{\alpha_r}^x \varphi_{\operatorname{str}}(x) \wedge (\varphi)_s^f,$$

where  $\widetilde{o_i} := \{j \mid \mathcal{O}(o_i) \subseteq \mathcal{O}(o_j)\}$  and  $\varphi_{\text{str}}(x)$ , which checks that atoms  $p_{\alpha}^x$  indeed code for a strategy, is defined as

$$\varphi_{\operatorname{str}}(x) := \mathbf{AG} \bigvee_{\alpha \in \operatorname{Ac}} (p_{\alpha}^{x} \wedge \bigwedge_{\alpha' \neq \alpha} \neg p_{\alpha'}^{x}).$$
Let  $((a, x)\varphi)_{s}^{f} := (\varphi)_{s}^{f[a \mapsto x]}$ , and  $(\mathbf{E}\psi)_{s}^{f} := \mathbf{E} (\psi_{\operatorname{out}}^{f} \wedge (\psi)_{p}^{f})$ , where 
$$\psi_{\operatorname{out}}^{f} := \mathbf{G} \bigvee_{\alpha \in \operatorname{Ac^{\operatorname{Ag}}}} \bigwedge_{a \in \operatorname{dom}(f)} p_{\alpha(a)}^{f(a)} \wedge \mathbf{X} p_{\alpha}$$

Formula  $\psi_{\text{out}}^f$  holds on a path if and only if each player a in dom(f) follows the strategy coded by atoms  $p_{\alpha}^{f(a)}$ .

The correctness of the translation is stated by the following lemma:

## Lemma 5.7.

$$\mathcal{PG} \models \Phi \text{ if, and only if, } \mathsf{PK}_{\mathcal{PG}} \models (\Phi)_s^{\emptyset}.$$

To establish this lemma we need a few additional definitions. Given a strategy  $\sigma$  and a strategy variable x we let  $\ell_{\sigma}^x := \{\ell_{p_{\alpha}^x} \mid \alpha \in Ac\}$  be the family of  $p_{\alpha}^x$ -labellings for tree  $t_{\mathsf{PK}_{\mathcal{PG}}}$  defined as follows: for each finite play  $\lambda$  in  $\mathcal{PG}$  and  $\alpha \in Ac$ , we let  $\ell_{p_{\alpha}^x}(u_{\lambda}) := 1$  if  $\alpha = \sigma(\lambda)$ , 0 otherwise. For a labelled tree t with same domain as  $t_{\mathsf{PK}_{\mathcal{PG}}}$  we write  $t \otimes \ell_{\sigma}^x$  for  $t \otimes \ell_{p_{\alpha_1}^x} \otimes \ldots \otimes \ell_{p_{\alpha_n}^x}$ .

Given a partial play  $\lambda$  in  $\mathcal{PG}$ , we define the node  $u_{\lambda} = \pi(\lambda') \in t_{\mathsf{PK}_{\mathcal{PG}}}$ : it is the succinct representation of  $\lambda'$ , the finite path of  $\mathsf{PK}_{\mathcal{PG}}$  that corresponds to  $\lambda$ . Also, given an infinite play  $\lambda$  and a point  $i \in \mathbb{N}$ , we let  $\mu_{\lambda,i}$  be the infinite path in  $t_{\mathsf{PK}_{\mathcal{PG}}}$  that starts in node  $u_{\lambda \leq i}$  and is defined as  $\mu_{\lambda,i} := u_{\lambda \leq i} u_{\lambda \leq i+1} u_{\lambda \leq i+2} \dots$ 

and is defined as  $\mu_{\lambda,i} := u_{\lambda \leq i} u_{\lambda \leq i+1} u_{\lambda \leq i+2} \dots$ Finally, for an assignment  $\chi$  and a partial function  $f : Ag \to Var$ , we say that f is compatible with  $\chi$  if  $dom(f) = dom(\chi) \cap Ag$  and for all  $a \in dom(f)$ ,  $\chi(a) = \chi(f(a))$ .

Lemma 4.2 is now obtained by applying the following result to sentence  $\Phi$ ,  $\lambda = v_{\iota}$ , the empty assignment and the empty function  $\emptyset$ :

**Proposition 5.8.** For every state subformula  $\varphi$  and path subformula  $\psi$  of  $\Phi$ , partial play  $\lambda$ , play  $\lambda'$ , point  $i \in \mathbb{N}$ , for every assignment  $\chi$  variable-complete for  $\varphi$  (resp.  $\psi$ ) and partial

function  $f : Ag \rightarrow Var$  compatible with  $\chi$ , assuming also that no  $x_i$  in  $dom(\chi) \cap Var = \{x_1, \ldots, x_k\}$  is quantified in  $\varphi$  or  $\psi$ , we have

$$\mathcal{PG}, \chi, \lambda \models \varphi \quad iff \quad t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell_{\chi(x_1)}^{x_1} \otimes \ldots \otimes \ell_{\chi(x_k)}^{x_k}, u_{\lambda} \models (\varphi)_s^f$$

and

$$\mathcal{PG}, \chi, \lambda', i \models \psi$$
 iff  $t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell_{\chi(x_1)}^{x_1} \otimes \ldots \otimes \ell_{\chi(x_k)}^{x_k}, \mu_{\lambda', i} \models (\psi)_p^f$ 

*Proof.* The proof is by induction on  $\varphi$ . We detail the cases for binding, strategy quantification and outcome quantification, the others follow simply by definition of  $\mathsf{PK}_{\mathcal{PG}}$  for atomic propositions and induction hypothesis for remaining cases.

For  $\varphi = (a, x)\varphi'$ , we have  $\mathcal{PG}, \chi, \lambda \models (a, x)\varphi'$  iff  $\mathcal{PG}, \chi[a \mapsto \chi(x)], \lambda \models \varphi'$ . The result follows by using the induction hypothesis with assignment  $\chi[a \mapsto x]$  and function  $f[a \mapsto x]$ . This is possible because  $f[a \mapsto x]$  is compatible with  $\chi[a \mapsto x]$ : indeed  $dom(\chi[a \mapsto x]) \cap Ag$  is equal to  $(dom(\chi) \cap Ag) \cup \{a\}$  which, by assumption, is equal to  $dom(f) \cup \{a\} = dom(f[a \mapsto x])$ . Also by assumption, for all  $a' \in dom(f), \chi(a') = \chi(f(a'))$ , and by definition

$$\chi[a \mapsto \chi(x)](a) = \chi(x) = \chi(f[a \mapsto x](a)).$$

For  $\varphi = \langle \langle x \rangle \rangle^o \varphi'$ , assume first that  $\mathcal{PG}, \chi, \lambda \models \langle \langle x \rangle \rangle^o \varphi'$ . There exists an o-uniform strategy  $\sigma$  such that

$$\mathcal{PG}, \chi[x \mapsto \sigma], \lambda \models \varphi'.$$

Since f is compatible with  $\chi$ , it is also compatible with assignment  $\chi' = \chi[x \mapsto \sigma]$ . By assumption, no variable in  $\{x_1, \ldots, x_k\}$  is quantified in  $\varphi$ , so that  $x \neq x_i$  for all i and thus  $\chi'(x_i) = \chi(x_i)$  for all i; and because no strategy variable is quantified twice in a same formula, x is not quantified in  $\varphi'$ , so that no variable in  $\{x_1, \ldots, x_k, x\}$  is quantified in  $\varphi'$ . By induction hypothesis

$$t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell_{\chi'(x_1)}^{x_1} \otimes \ldots \otimes \ell_{\chi'(x_k)}^{x_k} \otimes \ell_{\chi'(x)}^{x}, u_{\lambda} \models (\varphi')_s^f.$$

Because  $\sigma$  is o-uniform, each  $\ell_{p_{\alpha}^x} \in \ell_{\sigma}^x = \ell_{\gamma(x)}^x$  is  $\widetilde{o}$ -uniform, and it follows that

$$t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell^{x_1}_{\chi'(x_1)} \otimes \ldots \otimes \ell^{x_k}_{\chi'(x_k)}, u_{\lambda} \models \exists^{\tilde{o}} p^x_{\alpha_1} \ldots \exists^{\tilde{o}} p^x_{\alpha_l}.\varphi_{\mathsf{str}}(x) \wedge (\varphi')^f_s.$$

Finally, since  $\chi'(x_i) = \chi(x_i)$  for all i, we conclude that

$$t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell_{\chi(x_1)}^{x_1} \otimes \ldots \otimes \ell_{\chi(x_k)}^{x_k}, u_{\lambda} \models (\langle\!\langle x \rangle\!\rangle^o \varphi')_s^f.$$

For the other direction, assume that

$$t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell_{\chi(x_1)}^{x_1} \otimes \ldots \otimes \ell_{\chi(x_k)}^{x_k}, u_{\lambda} \models (\varphi)_s^f,$$

and recall that  $(\varphi)_s^f = \exists^{\widetilde{o}} p_{\alpha_1}^x \dots \exists^{\widetilde{o}} p_{\alpha_l}^x \cdot \varphi_{\operatorname{str}}(x) \wedge (\varphi')_s^f$ . Write  $t = t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell_{\chi(x_1)}^{x_1} \otimes \dots \otimes \ell_{\chi(x_k)}^{x_k}$ . There exist  $\widetilde{o}$ -uniform  $\ell_{p_{\alpha}^x}$ -labellings such that

$$t \otimes \ell_{p_{\alpha_1}^x} \otimes \ldots \otimes \ell_{p_{\alpha_l}^x} \models \varphi_{\text{str}}(x) \wedge (\varphi')_s^f.$$

By  $\varphi_{\text{str}}(x)$ , these labellings code for a strategy  $\sigma$ , and because they are  $\tilde{o}$ -uniform,  $\sigma$  is o-uniform. Let  $\chi' = \chi[x \mapsto \sigma]$ . For all  $1 \leq i \leq k$ , by assumption  $x \neq x_i$ , and thus  $\chi'(x_i) = \chi(x_i)$ . The above can thus be rewritten

$$t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell^{x_1}_{\chi'(x_1)} \otimes \ldots \otimes \ell^{x_k}_{\chi'(x_k)} \otimes \ell^{x}_{\chi'(x)} \models \varphi_{\mathrm{str}}(x) \wedge (\varphi')^f_s.$$

By induction hypothesis we have  $\mathcal{PG}, \chi[x \mapsto \sigma], \lambda \models \varphi'$ , hence  $\mathcal{PG}, \chi, \lambda \models \langle \langle x \rangle \rangle^o \varphi'$ .

For  $\varphi = \mathbf{E}\psi$ , assume first that  $\mathcal{PG}, \chi, \lambda \models \mathbf{E}\psi$ . There exists an infinite play  $\lambda' \in \text{Out}(\chi, \lambda)$  s.t.  $\mathcal{PG}, \chi, \lambda', |\lambda| - 1 \models \psi$ . By induction hypothesis,

$$t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell_{\chi(x_1)}^{x_1} \otimes \ldots \otimes \ell_{\chi(x_k)}^{x_k}, \mu_{\lambda',|\lambda|-1} \models (\psi)_p^f.$$

Since  $\lambda'$  is an outcome of  $\chi$ , each agent  $a \in dom(\chi) \cap Ag$  follows strategy  $\chi(a)$  in  $\lambda'$ . Because  $dom(\chi) \cap Ag = dom(f)$  and for all  $a \in dom(f)$ ,  $\chi(a) = \chi(f(a))$ , each agent  $a \in dom(f)$  follows the strategy  $\chi(f(a))$ , which is coded by atoms  $p_{\alpha}^{f(a)}$  in the translation of  $\Phi$ . Therefore  $\mu_{\lambda',|\lambda|-1}$  also satisfies  $\psi_{\text{out}}^{\chi}$ , hence  $t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell_{\chi(x_1)}^{x_1} \otimes \ldots \otimes \ell_{\chi(x_k)}^{x_k}$ ,  $\mu_{\lambda',|\lambda|-1} \models \psi_{\text{out}}^{\chi} \wedge (\psi)_p^f$ , and we are done.

For the other direction, assume that

$$t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell^{x_1}_{\chi(x_1)} \otimes \ldots \otimes \ell^{x_k}_{\chi(x_k)}, u_{\lambda} \models \mathbf{E}(\psi^f_{\mathrm{out}} \wedge (\psi)^f_p).$$

There exists a path  $\mu$  in  $t_{\mathsf{PK}_{\mathcal{PG}}} \otimes \ell_{\chi(x_1)}^{x_1} \otimes \ldots \otimes \ell_{\chi(x_k)}^{x_k}$  starting in node  $u_{\lambda}$  that satisfies both  $\psi_{\mathrm{out}}^f$  and  $(\psi)_p^f$ . By construction of  $\mathsf{PK}_{\mathcal{PG}}$  and definition of succinct unfoldings, there exists an infinite play  $\lambda'$  such that  $\lambda'_{\leq |\lambda|-1} = \lambda$  and  $\mu = \mu_{\lambda',|\lambda|-1}$ . By induction hypothesis,  $\mathcal{PG}, \chi, \lambda', |\lambda| - 1 \models \psi$ . Because  $\mu_{\lambda',|\lambda|-1}$  satisfies  $\psi_{\mathrm{out}}^f$ ,  $dom(\chi) \cap \mathsf{Ag} = dom(f)$ , and for all  $a \in dom(f), \chi(a) = \chi(f(a))$ , it is also the case that  $\lambda' \in \mathsf{Out}(\chi, \lambda)$ , hence  $\mathcal{PG}, \chi, \lambda \models \mathsf{E}\psi$ .  $\square$ 

To complete the proof of Theorem 5.6 it remains to check that  $(\Phi)_s^{\emptyset}$  is a hierarchical QCTL\*\* formula, which is the case because  $\Phi$  is hierarchical in  $\mathcal{PG}$  and for every two observations  $o_i$  and  $o_j$  in Obs such that  $\mathcal{O}(o_i) \subseteq \mathcal{O}(o_j)$ , by definition of  $\widetilde{o_k}$  we have that  $\widetilde{o_i} \subseteq \widetilde{o_j}$ .

#### 6. Higher-order extension

We have shown so far that the techniques developed for *finite* concurrent game arenas with imperfect information in  $[BMM^+17]$  can be extended and adapted to deal with the case of infinite concurrent game arenas defined by *pushdown* systems when the stack is visible. In particular we proved in Theorem 5.6 that the model-checking problem for  $SL_{iR}$  on pushdown game arenas with visible stack is decidable for hierarchical instances. Moving from finite structures to infinite structures (in our case defined by pushdown systems) is of interest for instance when dealing with system verification as it permits to capture richer classes, in particular those coming from programs making use of recursion.

A natural line of research is to go beyond pushdown systems, and a natural candidate here is to move to the higher-order setting, *i.e.* to consider higher-order pushdown systems or even collapsible pushdown systems [HMOS17]. These are very natural models in particular regarding application for programs using higher-order functions.

We first briefly discuss the global road map.

- The decidability proof for SL<sub>iR</sub> will again go through a reduction to model checking hierarchical QCTL<sub>iR</sub> and this is where most technicalities are coming.
- We give in sections 6.1.1 and 6.1.2 definition of higher-order stacks and stacks with links.
- Next in Section 6.1.3, we adapt Definition 2.4 and introduce higher-order pushdown compound Kripke structures and collapsible pushdown compound Kripke structures. The main technicality here is to introduce a suitable notion of regular labelling functions.

- In Section 6.2 we explain (in Section 6.2.1) how to generalise to higher-order the definitions of pushdown tree automata from Section 3 and we also adapt the results concerning projection, simulation and narrowing (in Section 6.2.2).
- In Section 6.3, we adapt the notion of succinct unfolding to handle higher-order. This, together with a closure property for alternating collapsible pushdown tree automata, permits to establish decidability of model checking hierarchical QCTL\*\* on collapsible pushdown compound Kripke structures.
- Finally, in Section 6.4, we prove that the model-checking problem for  $SL_{iR}$  on collapsible pushdown game arenas with visible stack is decidable for hierarchical instances.
- 6.1. Higher-order and collapsible pushdown compound Kripke structures. We explain how to adapt the definitions from Section 2.4 to deal with higher-order stacks (possibly with links).
- 6.1.1. Higher-order stacks and their operations. Fix a finite stack alphabet  $\Gamma$  and a distinguished bottom symbol  $\flat \notin \Gamma$ .

**Definition 6.1.** An order-1 stack is a word  $ba_1 \dots a_\ell \in b \cdot \Gamma^*$  which is denoted  $[ba_1 \dots a_\ell]_1$ . An *order-k stack* (or a *k-stack*), for k > 1, is a non-empty sequence  $w_1, \dots, w_\ell$  of order-(k-1) stacks which is written  $[w_1 \dots w_\ell]_k$ .

For convenience, we may sometimes see an element  $a \in \Gamma$  as an order-0 stack, denoted  $[a]_0$ . We define  $\flat_h$ , the **empty** h-**stack**, as:  $\flat_0 = \flat$  and  $\flat_{h+1} = [\flat_h]$ . We denote by  $\operatorname{Stacks}_k$  the set of all order-k stacks and  $\operatorname{Stacks} = \bigcup_{k \geq 1} \operatorname{Stacks}_k$  the set of all higher-order stacks. The height of the stack w, denoted |w|, is simply the length of the sequence. We denote by  $\operatorname{ord}(w)$  the order of the stack w.

In addition to the operations  $push_1^a$  and  $pop_1$  that respectively pushes and pops a symbol in the topmost order-1 stack, one needs extra operations to deal with the higher-order stacks: the  $pop_k$  operation removes the topmost order-k stack, while the  $push_k$  duplicates it.

For an order-n stack  $w = [w_1 \dots w_\ell]_n$  and an order-k stack w' with  $0 \le k < n$ , we define  $w +\!\!\!\!+ w'$  as the order-n stack obtained by pushing w' on top of w:

$$w + w' = \begin{cases} [w_1 \dots w_\ell w']_n & \text{if } k = n - 1, \\ [w_1 \dots (w_\ell + + w')]_n & \text{otherwise.} \end{cases}$$

We first define the (partial) operations  $pop_i$  and  $top_i$  with  $i \geq 1$ :  $top_i(w)$  returns the top (i-1)-stack of w, and  $pop_i(w)$  returns w with its top (i-1)-stack removed. Formally, for an order-n stack  $[w_1 \cdots w_{\ell+1}]_n$  with  $\ell \geq 0$ ,

$$top_i(w) = \begin{cases} w_{\ell+1} & \text{if } i = n \\ top_i(w_{\ell+1}) & \text{if } i < n \end{cases}$$

$$pop_i(w) = \begin{cases} [w_1 \cdots w_{\ell}]_n & \text{if } i = n \text{ and } \ell \ge 1 \\ [w_1 \cdots w_{\ell} pop_i(w_{\ell+1})] & \text{if } i < n \end{cases}$$

By abuse of notation, we let  $top_{ord(w)+1}(w) = w$ . Note that  $pop_i(w)$  is defined if and only if the height of  $top_{i+1}(w)$  is strictly greater than 1. For example  $pop_2([[b \ a \ b]_1]_2)$  is undefined.

We now introduce the operations  $push_i$  with  $i \geq 2$  that duplicates the top (i-1)-stack of a given stack. More precisely, for an order-n stack w and for  $2 \leq i \leq n$ , we let  $push_i(s) = w + top_i(w)$ .

The last operation,  $push_1^a$  pushes the symbol  $a \in \Gamma$  on top of the top 1-stack. More precisely, for an order-n stack w and for a symbol  $a \in \Gamma$ , we let  $push_1^a(w) = w + [a]_0$ .

## **Example 6.2.** Let w be the following 3-stack of height 2:

$$[[[bbaac]_1[bbcc]_1[bcba]_1]_2[[bbaa]_1[bbab]_1]_2]_3$$

Then  $top_3(w)$  is the 2-stack

$$[[baa]_1[bbab]_1]_2$$

and  $pop_3(w)$  is the stack

$$w' = \llbracket \llbracket [bbaac]_1 \llbracket bbb]_1 \llbracket bcba]_1 \rrbracket_2 \rrbracket_3$$

Note that  $pop_3(pop_3(w))$  is undefined. Then  $push_2(w')$  is the stack

$$[[bbaac]_1[bbb]_1[bcba]_1[bcba]_1]_2]_3$$

and

$$push_1^c(w') = [[[bbaac]_1[bbb]_1[bcbac]_1]_2]_3$$

6.1.2. Stacks with links and their operations. We now define a richer structure of higher-order stacks where we allow links. Intuitively, a stack with links is a higher-order stack in which any symbol may have a link that points to an internal stack below it. This link may be used later to collapse part of the stack.

Order-k stacks with links are order-k stacks with a richer stack alphabet. Indeed, each symbol in the stack can be either an element  $a \in \Gamma$  (*i.e.* it is not the source of a link) or an element  $(a, \ell, h) \in \Gamma \times \{2, \dots, k\} \times \mathbb{N}$  (*i.e.* it is the source of an  $\ell$ -link pointing to the h-th  $(\ell - 1)$ -stack inside the topmost  $\ell$ -stack below the source of the link). Formally, order-k stacks with links over alphabet  $\Gamma$  are defined as order-k stacks  $\Gamma$  over alphabet  $\Gamma \cup \Gamma \times \{2, \dots, k\} \times \mathbb{N}$ .

## **Example 6.3.** Stack w below is an order-3 stack with links:

$$[[[bb]_1[bbc(c,2,1)]_1]_2[[ba]_1[bbc]_1[bb(a,2,1)(b,3,1)]_1]_2]_3.$$

To improve readability when displaying n-stacks in examples, we shall explicitly draw the links rather than using stack symbols in  $\Gamma \times \{2, \dots, k\} \times \mathbb{N}$ . For instance, we represent w as follows:

In addition to the previous operations  $pop_i$ ,  $push_i$  and  $push_1^a$ , we introduce two extra operations: one to create links, and the other to collapse the stack by following a link. Link creation is made when pushing a new stack symbol, and the target of an  $\ell$ -link is always the  $(\ell-1)$ -stack below the topmost one. Note that due to possible subsequent copies links can

<sup>&</sup>lt;sup>5</sup>Note that we therefore slightly generalise our previous definition as we implicitly use an infinite stack alphabet, but this does not introduce any technical change in the definition.

point to arbitrarily deep stacks. Formally, we define  $push_1^{a,\ell}(w) = push_1^{(a,\ell,h)}$  where we let  $h = |top_{\ell}(w)| - 1$  and require that h > 1.

The collapse operation is defined only when the topmost symbol is the source of an  $\ell$ -link, and results in truncating the topmost  $\ell$ -stack to only keep the component below the target of the link. Formally, if  $top_1(w) = (a, \ell, h)$  and  $w = w' + [w''_1 \cdots w''_k]_{\ell}$  with k > h we let collapse $(w) = w' + [w_1'' \cdots w_h'']_{\ell}$ .

For any k, we let  $\operatorname{Op}_k(\Gamma)$  denote the set of all operations over order-k stacks with links.

**Example 6.4.** Let  $w = [[[b a]_1]_2 [[b]_1 [b a]_1]_2]_3$ . We have

$$\begin{array}{rcl} push_1^{b,2}(w) & = & \left[\left[\left\lfloor \flat a\right\rfloor_1\right\rfloor_2\left[\left\lfloor \flat\right\rfloor_1\left\lfloor \flat a b\right\rfloor_1\right]_2\right]_3 \\ \operatorname{collapse}\left(push_1^{b,2}(w)\right) & = & \left[\left[\left\lfloor \flat a\right\rfloor_1\right]_2\left[\left\lfloor \flat\right\rfloor_1\right]_2\right]_3 \\ \underbrace{push_1^{c,3}(push_1^{b,2}(w))}_{\theta} & = & \left[\left[\left\lfloor \flat a\right\rfloor_1\right]_2\left[\left\lfloor \flat\right\rfloor_1\left[\left\lfloor \flat a b c\right\rfloor_1\right]_2\right]_3. \end{array}$$

Then  $push_2(\theta)$  and  $push_3(\theta)$  are respectively

$$[[[ \flat a]_1]_2 [[ \flat ]_1 [ \flat abc]_1 [ \flat abc]_1]_2]_3 \text{ and}$$

$$[[[ \flat a]_1]_2 [[ \flat ]_1 [ \flat abc]_1]_2 [[ \flat ]_1 [ \flat abc]_1]_2]_3.$$

We have

collapse 
$$(push_2(\theta))$$
 = collapse  $(push_3(\theta))$   
= collapse  $(\theta)$   
=  $[[[ b a]_1]_2]_3$ .

6.1.3. Higher-order and collapsible pushdown compound Kripke structures. We are now ready to generalise Definition 2.4 (pushdown compound Kripke structures) to higher-order. Note that pushdown compound Kripke structures will correspond to order-1 collapsible pushdown compound Kripke structures.

**Definition 6.5.** An order-k collapsible pushdown compound Kripke structure or CPCKS, over local states  $\{L_i\}_{i\in[n]}$ , is a tuple  $\mathsf{CoK} = (\Gamma, S, R, \ell, s_{\iota})$  where

- $\Gamma$  is a finite stack alphabet together with a bottom symbol  $\flat \notin \Gamma$ , and we let  $\Gamma_{\flat} = \Gamma \cup \{\flat\}$ ;
- $S \subseteq \prod_{i \in [n]} L_i$  is a finite set of states;
- $R \subseteq S \times \Gamma_{\flat} \times S \times \operatorname{Op}_{k}(\Gamma)$  is a transition relation;  $\ell : S \times \operatorname{Stacks}_{k} \to 2^{\operatorname{AP}}$  is a regular labelling function (defined below);
- $s_{\iota} \in S$  is an initial state.

A higher-order pushdown compound Kripke structure (HOPCKS) is a collapsible pushdown compound Kripke structure that never uses the collapse operation.

A configuration is a pair  $c = \langle s, w \rangle \in S \times \text{Stacks}_k$  where s is the current state and  $w \in \text{Stacks}_k$  the current stack with links; we call  $\langle s_t, \flat_k \rangle$  the *initial configuration*.

From configuration  $\langle s, w \rangle$  with  $top_1(w) = \gamma$  the system can move to  $\langle s', op(w) \rangle$  if  $(s, \gamma, s', op) \in R$ , which we write  $\langle s, w \rangle \hookrightarrow \langle s', op(w) \rangle$ . We assume that for every configuration  $\langle s, w \rangle$  there exists at least one configuration  $\langle s', w' \rangle$  such that  $\langle s, w \rangle \hookrightarrow \langle s', w' \rangle$ .

A **path** in CoK is an infinite sequence of configurations  $\lambda = c_0 c_1 \dots$  such that  $c_0$  is the initial configuration and for all  $i \in \mathbb{N}$ ,  $c_i \hookrightarrow c_{i+1}$ . A **partial path** is a finite non-empty prefix of a path. We let Paths<sup>\omega</sup>(CoK) (resp. Paths\*(CoK)) be the set of all paths (resp. partial paths) in CoK.

Regular labelling functions. We need to adapt the concept of regular labelling functions to the higher-order setting. In the case of pushdown compound Kripke structures, recall that the criterion used was whether the stack content belongs to a regular language. Equivalently, one could have used an MSO-logic formula or a  $\mu$ -calculus formula on words (as these frameworks are equivalent to finite-state automata when defining sets of words). In the higher-order case, we take a similar approach, *i.e.* we consider a model of automata working on higher-order stacks (resp. stacks with links) that is equivalent with the  $\mu$ -calculus when defining sets of higher-order stacks (resp. stack with links). Note that it is not equivalent with MSO-logic, which is in fact undecidable over collapsible pushdown Kripke structures. We start by first giving the definition for higher-order pushdown compound Kripke structures and then move to collapsible pushdown compound Kripke structures.

In the (simpler) case of higher-order pushdown compound Kripke structures, a **regular labelling function** is given as a set of finite word automata  $\mathcal{B}_s^p$  over alphabet  $\Gamma \cup \{[,]\}$ , one for each proposition  $p \in AP$  and each state  $s \in S$ . They define the labelling function that maps to each state  $s \in S$  and higher-order stack content  $w \in (\Gamma_b \cup \{[,]\})^*$  the set  $\ell(s, w)$  of all atoms p such that  $w \in \mathcal{L}(\mathcal{B}_s^p)$ . In other words, one reads the higher-order stack in a bottom-up fashion to determine which atoms are satisfied in the current configuration. We refer the reader to  $[CHM^+08]$  for related work on this notion of regular sets of higher-order stacks (without links).

In the general case of collapsible pushdown compound Kripke structures, regular labelling functions are defined using a richer model of automata introduced in [BCOS10, Section 3], that we recall here. Note that if one considers stacks without links, this model corresponds to the previous one.

Let w be an order-k collapsible stack. We first associate with  $w = w_1, \dots, w_\ell$  a well-bracketed word of bracket-depth  $k, \widetilde{w} \in (\Gamma_b \cup \{[,]\})^*$ , defined as follows:

$$\widetilde{w} := \begin{cases} [\widetilde{w_1} \cdots \widetilde{w_\ell}] & \text{if } k \geq 1 \\ w & \text{if } k = 0 \ (i.e. \ w \in \Gamma_{\flat}) \end{cases}$$

In order to reflect the link structure, we define a partial function  $\operatorname{target}(w):\{1,\cdots,|\widetilde{w}|\} \to \{1,\cdots,|\widetilde{w}|\}$  that assigns to every position in  $\{1,\cdots,|\widetilde{w}|\}$  the index of the end of the stack targeted by the corresponding link (if it exists; indeed this is only defined if the symbol at this position is in  $\Gamma \times \{2,\cdots,k\} \times \mathbb{N}$ ). Thus with w is associated the pair  $\langle \widetilde{w}, \operatorname{target}(w) \rangle$ ; and with a set W of stacks is associated the set  $\widetilde{W} = \{\langle \widetilde{w}, \operatorname{target}(w) \rangle \mid w \in W\}$ .

## Example 6.6. Let

$$w = [[[ [ \flat \alpha]_1 ]_2 ]_2 [[ \flat ]_1 [ \flat a \beta \gamma]_1 ]_2 ]_3$$

Then

$$\widetilde{w} = [[[b \alpha]][[b][b \alpha \beta \gamma]]]$$

target(15) = 7, target(16) = 11, and target(i) is undefined for all other  $i \in \{1, ..., |\widetilde{w}| = 19\}$ .

We consider deterministic finite automata working on such representations of collapsible stacks. The automaton reads the word  $\widetilde{w}$  from left to right. On reading a letter that does not have a link (i.e. target is undefined on its index) the automaton updates its state according to the current state and the letter; on reading a letter that has a link, the automaton updates its state according to the current state, the letter and the state it was in after processing the targeted position. A run is accepting if it ends in a final state.

Formally, such an automaton is a tuple  $\langle Q, A, q_{\iota}, F, \delta \rangle$  where Q is a finite set of states, A is a finite input alphabet,  $q_{\iota} \in Q$  is the initial state,  $F \subseteq Q$  is a set of final states and  $\delta : (Q \times A) \cup (Q \times A \times Q) \to Q$  is a transition function. With a pair  $\langle u, \tau \rangle$  where  $u = a_1 \cdots a_n \in A^*$  and  $\tau$  is a partial map from  $\{1, \dots n\} \to \{1, \dots n\}$ , we associate a unique run  $r = r_0 \cdots r_n$  as follows:

- $r_0 = q_\iota$ ;
- for all  $0 \le i < n$ ,

$$r_{i+1} = \begin{cases} \delta(r_i, a_{i+1}) & \text{if } i+1 \notin dom(\tau), \\ \delta(r_i, a_{i+1}, r_{\tau(i+1)}) & \text{otherwise.} \end{cases}$$

The run is accepting just if  $r_n \in F$ , and the pair  $(u, \tau)$  is accepted just if the associated run is accepting.

A regular labelling function is given as a set of such automata  $\mathcal{B}^p_s$  over alphabet  $\Gamma_{\flat} \cup \{[,]\}$ , one for each  $p \in AP$  and each  $s \in S$ . They define the labelling function that maps to each state  $s \in S$  and stack with link w the set  $\ell(s, w)$  of all atoms p such that  $\langle \widetilde{w}, \operatorname{target}(w) \rangle$  is accepted by  $\mathcal{B}^p_s$ .

Associated compound Kripke structure. An order-k CPCKS  $\mathsf{CoK} = (\Gamma, S, R, \ell, s_{\iota})$  over local states  $\{L_i\}_{i \in [n]}$  generates an infinite CKS  $\mathsf{K}_{\mathsf{CoK}} = (S', R', \ell', s'_{\iota})$  over  $\{L_i\}_{i \in [n+1]}$ , where

- $L_{n+1} = \operatorname{Stacks}_k$ ,
- $S' = S \times \text{Stacks}_k$ ,
- $(s', w') \in R'(s, w)$  if  $(s, w) \hookrightarrow (s', w')$ ,
- $\ell' = \ell$  and
- $\bullet \ s'_{\iota} = (s_{\iota}, \flat_k).$

We write  $\mathsf{CoK} \models \varphi$  if  $\mathsf{K}_{\mathsf{CoK}} \models \varphi_{n+1}$ , where  $\varphi_{n+1}$  is obtained from  $\varphi$  by replacing each concrete observation  $\mathbf{o} \subseteq [n]$  with  $\mathbf{o}' = \mathbf{o} \cup \{n+1\}$ , to reflect the fact that the stack is always visible.

#### 6.2. Collapsible pushdown tree automata.

6.2.1. Definitions. We explain how to generalise to higher-order the definitions of pushdown tree automata from Section 3. The idea is simple: we now work with stacks with links instead of usual stacks and the operation performed on the stack depends on the current  $top_1$  element. Formally, this leads to the following definition.

For AP a finite set of atomic propositions and X a finite set of directions, an order-k collapsible alternating pushdown tree automaton (CAPTA) on (AP, X)-trees is a tuple  $\mathcal{A} = (\Gamma, Q, \delta, q_{\iota}, C)$  where  $\Gamma$  is a finite stack alphabet with a special bottom symbol  $\flat$ , Q is a finite set of states,  $q_{\iota} \in Q$  is an initial state,  $\delta : Q \times 2^{\mathrm{AP}} \times \Gamma \to \mathbb{B}^+(X \times Q \times \mathrm{Op}_k)$  is a transition function, and  $C : Q \to \mathbb{N}$  is a colouring function.

Acceptance of a tree by a CAPTA is again defined as a parity game, the only difference being that now the game we obtain is played on a richer underlying arena. While in the case of APTA we had pushdown games, we now obtain collapsible pushdown games (see [HMOS08] for more results on this). Note that such games are decidable, hence acceptance of a regular tree by a CAPTA is decidable as well.

A collapsible nondeterministic pushdown tree automaton (CNPTA) is a collapsible alternating pushdown tree automaton  $\mathcal{N} = (\Gamma, Q, \delta, q_{\iota}, C)$  such that for every  $q \in Q$ ,  $a \in 2^{\text{AP}}$  and  $\gamma \in \Gamma$ ,  $\delta(q, a, \gamma)$  is written in disjunctive normal form and for every direction  $x \in X$ , each disjunct contains exactly one element of  $\{x\} \times Q \times \text{Op}_k$ .

The restrictions leading respectively to semi-alternating collapsible pushdown tree automata and X-guided stack alternating collapsible pushdown tree automata are essentially the same as in Section 3.2 except that now the requirement is that the stack operation is the same when going in the same direction (previously, we were requiring that the same content was pushed on the stack). Formally, we have the following definition (generalising Definition 3.1):

**Definition 6.7.** An order-k CAPTA  $\mathcal{A} = (\Gamma, Q, \delta, q_{\iota}, C)$  over  $X \times Y$ -trees has an X-guided stack, or simply is X-guided, if there exists a function  $\delta_{\Gamma} : 2^{\text{AP}} \times \Gamma \times X \to \text{Op}_k$  such that for all  $(q, a, \gamma) \in Q \times 2^{\text{AP}} \times \Gamma$ , all atoms appearing in  $\delta(q, a, \gamma)$  are of the form  $[(x, y), q', \delta_{\Gamma}(a, \gamma, x)]$ .

6.2.2. *Projection, simulation and narrowing.* Following the same proof as for Proposition 3.2 we have the following generalisation to higher-order.

**Proposition 6.8.** Given an CNPTA  $\mathcal{N}$  and  $p \in AP$ , one can build an CNPTA  $\mathcal{N} \downarrow_{-p}$  such that for every pointed tree (t, u) and initial k-stack  $w_t \in Stacks_k$ ,

$$(t, u) \in \mathcal{L}(\mathcal{N} \downarrow p, w_t)$$
 iff  
 $\exists \ell_p \ a \ p\text{-labelling for } t \ s.t. \ (t \otimes \ell_p, u) \in \mathcal{L}(\mathcal{N}, w_t).$ 

Now, moving to simulation, one easily generalises the proof in [ALM<sup>+</sup>13] to higher-order.

**Theorem 6.9.** Given a semi-alternating CAPTA  $\mathcal{A}$ , one can build an CNPTA  $\mathcal{N}$  such that for every initial k-stack  $w \in \operatorname{Stacks}_k$ ,  $\mathcal{L}(\mathcal{N}, w) = \mathcal{L}(\mathcal{A}, w)$ .

*Proof.* The key idea in the proof in the pushdown case is to remark that it is sufficient to do a subset construction on the set of states as the stack is the same when moving down in the same direction in the tree. Here, the same approach is also working as the stack operation (hence the stack with links) is the same when moving down in the same direction in the

As the previous construction also preserves X-guidedness, we can refine the above result as follows:

**Proposition 6.10.** Given an X-guided CAPTA  $\mathcal{A}$ , one can build an X-guided CNPTA  $\mathcal{N}$  such that for every initial k-stack  $w \in \text{Stacks}_k$ ,  $\mathcal{L}(\mathcal{N}, w) = \mathcal{L}(\mathcal{A}, w)$ .

Finally, narrowing directly extends to higher-order.

**Theorem 6.11.** Given a CAPTA A on  $X \times Y$ -trees, one can build a CAPTA  $A \downarrow_X$  on X-trees such that for every pointed (AP, X)-tree (t, u), every  $u' \in (X \times Y)^+$  such that  $u' \downarrow_X = u$ , and every initial k-stack  $w \in \text{Stacks}_k$ ,

$$(t, u) \in \mathcal{L}(\mathcal{A}\downarrow_X, w_t) \text{ iff } (t\uparrow^Y, u') \in \mathcal{L}(\mathcal{A}, w_t).$$

**Proposition 6.12.** If a CAPTA  $\mathcal{A}$  over  $X \times Y \times Z$ -trees is X-guided, then so is  $\mathcal{A} \downarrow_{X \times Y}$ .

6.3. Model checking hierarchical QCTL\*\* on collapsible pushdown compound **Kripke structures.** We now describe how one establishes an extension to higher-order of Theorem 2.8.

**Theorem 6.13.** Model checking  $QCTL_{iR,\subseteq}^*$  on collapsible pushdown compound Kripke structures is decidable.

6.3.1. Succinct unfoldings. The first notion that needs to be generalised is the one of succinct unfoldings. Recall that the idea was to consider trees over a finite set of directions, to later use tree automata. The trick was to choose as set of directions for these trees the set of all possible finite words that the pushdown system to be model-checked could push on the stack. Here, as we have to handle stacks with links, the stack can be deeply modified by a single transition and the set of all those possible modifications is no longer finite. However, a simple solution consists in choosing as set of directions the set of all possible higher-order stack operations used by the collapsible pushdown system.

Hence, it is enough to record only the operations made on the stack in each node: then, starting from the root and the initial stack content  $\flat_k$ , one can reconstruct the stack content at each node by following the unique path from the root to this node, and applying the successive operations on the stack. By doing so we obtain a tree over the finite set of directions  $S \times \Pi_{\mathsf{CoK}}$ , where

$$\Pi_{\mathsf{CoK}} = \{ op \mid (s, \gamma, s', op) \in R \text{ for some } s, \gamma \text{ and } s' \}$$

Note that we no longer keep a neutral direction (the empty word in the pushdown setting): it was previously useful in the proof of Lemma 4.5 (subcase  $\varphi = p$ ) but in the higher-order setting we will need a more involved tool as the simple trick of destroying the stack content through direction  $\varepsilon$  will no longer be sufficient.

Definitions of the succinct representation of a partial path as well as the succinct unfolding (Definition 4.1) are adapted to CPCKS in the straightforward way (we keep the same notations).

The following result is then proved as Lemma 4.2.

**Lemma 6.14.** For every CPCKS CoK over  $\{L_i\}_{i\in[n]}$  and every QCTL\* formula  $\varphi$ , CoK  $\models \varphi$  iff  $t_{\mathsf{CoK}} \models \varphi_{n+1}$ .

6.3.2. *Proof of Theorem 6.13*. The proof of Theorem 6.13 follows the same lines as the one of Theorem 2.8, *i.e.* it relies deeply on an inductive construction of a tree automaton working on the succinct representation (Lemma 4.5 in the pushdown setting).

In the higher-order setting this leads to the following statement (the only difference is now that we replaced APTA by CAPTA and consider a collapsible pushdown Kripke compound structure  $CoK = (\Gamma, S, R, \ell, s_{\ell})$ ).

**Lemma 6.15.** For every subformula  $\varphi$  of  $\Phi_{n+1}$  and state  $s \in S$ , one can build a CAPTA  $\mathcal{A}_s^{\varphi}$  on  $(AP_{\exists}(\Phi), X_{\varphi})$ -trees with  $\Pi_{\mathsf{CoK}}$ -guided stack and such that for every  $(AP_{\exists}(\Phi), X_{\varphi})$ -tree t rooted in  $(s_\iota \downarrow_{I_\omega}, \flat_k)$ , every partial path  $\lambda \in Paths^*(\mathsf{CoK})$  ending in  $\langle s, w \rangle$ , it holds that

*Proof.* As for Lemma 4.5 the proof is by induction on  $\varphi$ . The only case that differs from the pushdown case is the base case where  $\varphi = p$  as it requires to handle regular labelling functions (which are now richer than in the pushdown case).

In the pushdown case, to check for a formula  $\varphi = p$  we followed the dummy direction  $(s, \varepsilon)$  to destroy letter by letter the current stack content while simulating on the fly  $\mathcal{B}_s^p$ . In the setting without links, a similar trick would work: one would read letter by letter the higher order stack performing  $pop_1$  operations, or  $pop_{i+1}$  operations when getting to an empty topmost *i*-stack which can be handled thanks to a small change of the model where one can test whether the topmost-*i* stack is empty (this can be simulated by the present model); see e.g. [Car06, Fra06]. However for stacks with links this approach no longer works as one also needs to follow the links and this would require to destroy the stack.

The solution in the general case of CAPTA is to anticipate these tests and to enrich the automaton so that it has in its control states an extra component that, for every  $\mathcal{B}_s^p$ , gives the state reached in  $\mathcal{B}_s^p$  after processing the current stack content. In the pushdown automaton it is an easy exercise how to compute such an enriched version (when pushing some content one simply simulates  $\mathcal{B}_s^p$  on the new symbols added, and pushes each symbol together with its corresponding state of  $\mathcal{B}_s^p$ ; popping is then for free). In the case of higher-order pushdown [CHM<sup>+</sup>08] and collapsible pushdown [BCOS10] it is a highly non-trivial result (we rephrase it here for CAPTA but the proof ingredients are the same).

**Theorem 6.16.** [BCOS10, Theorem 3] Given an order-k CAPTA  $\mathcal{A}$  with a state-set Q and an automaton  $\mathcal{B}$  (that takes as input stacks with links over the same alphabet as  $\mathcal{A}$ ), there exist an order-k CAPTA  $\mathcal{A}[\mathcal{B}]$  with state-set Q', a subset  $F \subseteq Q'$  and a mapping  $\chi : Q' \to Q$  such that:

- (i)  $\mathcal{A}$  and  $\mathcal{A}[\mathcal{B}]$  accept the same trees.
- (ii) for every configuration  $\langle q, w \rangle$  of  $\mathcal{A}[\mathcal{B}]$ , the corresponding configuration<sup>6</sup> of  $\mathcal{A}$  has state  $\chi(q)$  and its stack content is accepted by  $\mathcal{B}$  if and only if  $q \in F$ .

Now applying the construction from Theorem 6.16 at every step in the inductive construction gives the base case  $\varphi = p$  for free.

The rest of the proof is similar to the one of Lemma 4.5.

<sup>&</sup>lt;sup>6</sup>More precisely,  $\mathcal{A}[\mathcal{B}]$  works on a stack alphabet and set of control states that extend those of  $\mathcal{A}$ , and configurations of  $\mathcal{A}$  are obtained from configurations of  $\mathcal{A}[\mathcal{B}]$  by forgetting the extra components from the control state and from the stack symbols. Hence, one should think of  $\mathcal{A}[\mathcal{B}]$  as a version of  $\mathcal{A}$  with extra information stored both in the control states and in the stack symbols, and this information is precisely used to check whether the current stack content is accepted by  $\mathcal{B}$ .

6.4. Model checking hierarchical instances of SL<sub>iB</sub> on collapsible pushdown games arenas with visible stacks. Regarding  $SL_{iR}$  we need to generalise the notion of pushdown game arena with visible stack to higher-order.

Definition 6.17. An order-k Collapsible Pushdown Game Arena with Visible Stack, or CPGA<sub>VS</sub> for short, is a tuple  $\mathcal{CG} = (Ac, \Gamma, V, \mathcal{T}, \ell, v_{\iota}, \mathcal{O})$  where

- Ac is a finite set of actions,
- $\Gamma$  is a finite stack alphabet together with a bottom symbol  $\flat \notin \Gamma$  and we let  $\Gamma_{\flat} = \Gamma \cup \{\flat\}$ ,
- V is a finite set of control states,
- T: V × Γ<sub>b</sub> × Ac<sup>Ag</sup> → V × Op<sub>k</sub> is a transition function,
  ℓ: V × Stacks<sub>k</sub> → 2<sup>AP</sup> is a regular labelling function (as defined in Section 6.1.3),
- $v_i \in V$  is an initial control state, and
- $\mathcal{O}: \text{Obs} \to 2^{V \times V}$  is an observation interpretation.

A configuration is a pair  $\langle v, w \rangle$  where  $v \in V$  represents the current control state and w is the current content of the stack with links. When the players choose a joint move  $\alpha \in Ac^{Ag}$  in a configuration  $\langle v, w \rangle$  the system moves to configuration  $\langle v', op(w) \rangle$ , where  $\langle v', op \rangle = \mathcal{T}(v, top_1(w), \boldsymbol{\alpha});$  we denote this by  $\langle v, w \rangle \stackrel{\boldsymbol{\alpha}}{\hookrightarrow} \langle v', op(w) \rangle.$ 

A CPGA<sub>VS</sub>  $\mathcal{CG} = (Ac, \Gamma, V, \mathcal{T}, \ell, v_{\iota}, \mathcal{O})$  induces an infinite CGA  $\mathcal{G}_{\mathcal{CG}} = (Ac, V', \Delta, \ell', v_{\iota}, \mathcal{O}')$ where

- $V' = V \times \text{Stacks}_k$ ,
- $\Delta(\langle v, \gamma \cdot w \rangle, \boldsymbol{\alpha}) = \langle v', w' \cdot w \rangle$  if  $\langle v, \gamma \cdot w \rangle \stackrel{\boldsymbol{\alpha}}{\hookrightarrow} \langle v', w' \cdot w \rangle$ ,
- $\ell' = \ell$ ,
- $v'_{\iota} = \langle v_{\iota}, [\flat_k] \rangle$ ,
- $(\langle v, w \rangle, \langle v', w' \rangle) \in \mathcal{O}'(o)$  if w = w' and  $(v, v') \in \mathcal{O}(o)$ .

We call plays and partial plays of  $\mathcal{CG}$  those of  $\mathcal{G_{CG}}$ . For an  $\mathsf{SL}_{iR}$  sentence  $\varphi$ , we write  $\mathcal{CG} \models \varphi$ if  $\mathcal{G}_{\mathcal{CG}} \models \varphi$ .

Decidability of hierarchical instances of  $SL_{iR}$  on collapsible pushdown game arenas with visible stack follows the same line as in the pushdown setting. Indeed, the reduction to QCTL\*\* works the same (the only point to check is that the labellings obtained in the reduction are regular ones but this is immediate).

**Theorem 6.18.** The model-checking problem for  $SL_{iR}$  on collapsible pushdown game arenas with visible stack is decidable for hierarchical instances.

#### 7. Conclusion

We proved that we can model check Strategy Logic with imperfect information on collapsible pushdown game arenas when the stack is visible and information hierarchical. This implies that, on such infinite systems and for LTL objectives, one can decide the existence of Nash equilibria or solve a variety of synthesis problems such as distributed synthesis, rational synthesis or assume-guarantee synthesis, all easily expressible in Strategy Logic.

Strategy Logic is also known to be decidable, with elementary complexity, on imperfectinformation arenas where actions are public [BLMR17]. One interesting future work would be to extend also this result to the pushdown setting.

Another possible continuation of this work is to consider synthesis rather than modelchecking. More specifically, to start with a collapsible pushdown system with controllable and uncontrollable actions and a specification in Strategy Logic with imperfect information, and ask for a controller that restricts the system so that the specification is satisfied. Of course, due to the non-elementary underlying complexity of the model-checking problem, one should restrict to sub-classes (of system and/or formulas) to hope for tractable results.

#### References

- [AHK02] Rajeev Alur, Thomas A. Henzinger, and Orna Kupferman. Alternating-time temporal logic. Journal of the ACM, 49(5):672–713, 2002.
- [ALM<sup>+</sup>13] Benjamin Aminof, Axel Legay, Aniello Murano, Olivier Serre, and Moshe Y Vardi. Pushdown module checking with imperfect information. *Information and Computation*, 223:1–17, 2013.
- [BCHS12] Christopher H. Broadbent, Arnaud Carayol, Matthew Hague, and Olivier Serre. A saturation method for collapsible pushdown systems. In *Proceedings of the 39th International Colloquium on Automata, Languages, and Programming*, volume 7392 of *Lecture Notes in Computer Science*, pages 165–176. Springer, 2012.
- [BCHS13] Christopher H. Broadbent, Arnaud Carayol, Matthew Hague, and Olivier Serre. C-shore: a collapsible approach to higher-order verification. In Proceedings of the 18th ACM SIGPLAN International Conference on Functional Programming, pages 13–24. ACM, 2013.
- [BCOS10] Christopher H. Broadbent, Arnaud Carayol, C.-H. Luke Ong, and Olivier Serre. Recursion schemes and logical reflection. In *Proceedings of the 25th Annual IEEE Symposium on Logic in Computer Science*, pages 120–129. IEEE Computer Society, 2010.
- [BEM97] Ahmed Bouajjani, Javier Esparza, and Oded Maler. Reachability analysis of pushdown automata: Application to model-checking. In *Proceedings of the 8th International Conference on Concurrency Theory*, volume 1243 of *Lecture Notes in Computer Science*, pages 135–150. Springer, 1997.
- [BLMR17] Francesco Belardinelli, Alessio Lomuscio, Aniello Murano, and Sasha Rubin. Verification of broadcasting multi-agent systems against an epistemic strategy logic. In Proceedings of the 26th International Joint Conference on Artificial Intelligence, volume 17, pages 91–97, 2017.
- [BMM+17] Raphaël Berthon, Bastien Maubert, Aniello Murano, Sasha Rubin, and Moshe Y. Vardi. Strategy logic with imperfect information. In Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, pages 1–12. IEEE Computer Society, 2017.
- [BMP10] Laura Bozzelli, Aniello Murano, and Adriano Peron. Pushdown module checking. Formal Methods in System Design, 36(1):65–95, 2010.
- [BMvdB18] Dietmar Berwanger, Anup Basil Mathew, and Marie van den Bogaard. Hierarchical information and the synthesis of distributed strategies. *Acta Informatica*, 55(8):669–701, 2018.
- [Cac03] Thierry Cachat. Games on pushdown graphs and extensions. PhD thesis, Bibliothek der RWTH Aachen, 2003.
- [Car06] Arnaud Carayol. Automates infinis, logiques et langages. PhD thesis, Université de Rennes 1, 2006.
- [CFGR16] Rodica Condurache, Emmanuel Filiot, Raffaella Gentilini, and Jean-François Raskin. The complexity of rational synthesis. In Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming, volume 55 of LIPIcs, pages 121:1–121:15. Schloss Dagstuhl -Leibniz-Zentrum fuer Informatik, 2016.
- [CHM<sup>+</sup>08] Arnaud Carayol, Matthew Hague, Antoine Meyer, C.-H. Luke Ong, and Olivier Serre. Winning regions of higher-order pushdown games. In *Proceedings of the 23rd Annual IEEE Symposium* on Logic in Computer Science, pages 193–204. IEEE Computer Society, 2008.
- [CHP10] Krishnendu Chatterjee, Thomas A Henzinger, and Nir Piterman. Strategy logic. Information and Computation, 208, 2010.
- [CSW16a] Taolue Chen, Fu Song, and Zhilin Wu. Global model checking on pushdown multi-agent systems. In Proceedings of the 30th AAAI Conference on Artificial Intelligence, pages 2459–2465. AAAI Press, 2016.
- [CSW16b] Taolue Chen, Fu Song, and Zhilin Wu. Verifying pushdown multi-agent systems against strategy logics. In Proceedings of the 25th International Joint Conference on Artificial Intelligence, pages 180–186. IJCAI/AAAI Press, 2016.
- [CSW17] Taolue Chen, Fu Song, and Zhilin Wu. Model checking pushdown epistemic game structures. In Proceedings of Formal Methods and Software Engineering 19th International Conference on Formal Engineering Methods, volume 10610 of Lecture Notes in Computer Science, pages 36–53. Springer, 2017.
- [EKS03] Javier Esparza, Antonín Kucera, and Stefan Schwoon. Model checking LTL with regular valuations for pushdown systems. *Information and Computation*, 186(2):355–376, 2003.

- [FGR18] Emmanuel Filiot, Raffaella Gentilini, and Jean-François Raskin. Rational synthesis under imperfect information. In Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, pages 422–431. ACM, IEEE Computer Society, 2018.
- [FKL10] Dana Fisman, Orna Kupferman, and Yoad Lustig. Rational synthesis. In *Proceedings of the 16th International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, pages 190–204. Springer, 2010.
- [Fra06] Séverine Fratani. Automates à piles de piles . . . de piles. PhD thesis, Université de Bordeaux, 2006.
- [FS05] Bernd Finkbeiner and Sven Schewe. Uniform distributed synthesis. In *Proceedings of the 20th IEEE Symposium on Logic in Computer Science*, pages 321–330. IEEE Computer Society, 2005.
- [FWW97] Alain Finkel, Bernard Willems, and Pierre Wolper. A direct symbolic approach to model checking pushdown systems. *Electronic Notes in Theoretical Computer Science*, 9:27–37, 1997.
- [GTW02] Erich Grädel, Wolfgang Thomas, and Thomas Wilke, editors. Automata, Logics, and Infinite Games: A Guide to Current Research, volume 2500 of Lecture Notes in Computer Science. Springer, 2002.
- [HMOS08] Matthew Hague, Andrzej S. Murawski, C.-H. Luke Ong, and Olivier Serre. Collapsible pushdown automata and recursion schemes. In Proceedings of the 23rd Annual IEEE Symposium on Logic in Computer Science, pages 452–461. IEEE Computer Society, 2008.
- [HMOS17] Matthew Hague, Andrzej S. Murawski, C.-H. Luke Ong, and Olivier Serre. Collapsible pushdown automata and recursion schemes. ACM Transactions on Computational Logic, 18(3):25:1–25:42, 2017.
- [HO09] Matthew Hague and C-H Luke Ong. Winning regions of pushdown parity games: A saturation method. In *International Conference on Concurrency Theory*, pages 384–398. Springer, 2009.
- [JM77] Neil D Jones and Steven S Muchnick. Even simple programs are hard to analyze. Journal of the ACM, 24(2):338–350, 1977.
- [KPV02] Orna Kupferman, Nir Piterman, and Moshe Y Vardi. Pushdown specifications. In *Proceedings of the 9th International Conference on Logic for Programming Artificial Intelligence and Reasoning*, volume 2514 of *Lecture Notes in Computer Science*, pages 262–277. Springer, 2002.
- [KPV16] Orna Kupferman, Giuseppe Perelli, and Moshe Y. Vardi. Synthesis with rational environments.

  Annals of Mathematics and Artificial Intelligence, 78(1):3–20, 2016.
- [KV99] Orna Kupferman and Moshe Y. Vardi. Church's problem revisited. *Bulletin of Symbolic Logic*, 5(2):245–263, 1999.
- [KV01] Orna Kupferman and Moshe Y. Vardi. Synthesizing distributed systems. In *Proceedings of the* 16th Annual IEEE Symposium on Logic in Computer Science, pages 389–398. IEEE Computer Society, 2001.
- [KVW00] Orna Kupferman, Moshe Y. Vardi, and Pierre Wolper. An automata-theoretic approach to branching-time model checking. *Journal of the ACM*, 47(2):312–360, 2000.
- [LLS84] Richard E Ladner, Richard J Lipton, and Larry J Stockmeyer. Alternating pushdown and stack automata. SIAM Journal on Computing, 13(1):135–155, 1984.
- [LM14] François Laroussinie and Nicolas Markey. Quantified CTL: expressiveness and complexity. Logical Methods in Computer Science, 10(4), 2014.
- [MMPV14] Fabio Mogavero, Aniello Murano, Giuseppe Perelli, and Moshe Y. Vardi. Reasoning about strategies: On the model-checking problem. *ACM Transactions on Computational Logic*, 15(4):34:1–34:47, 2014.
- [MP15] Aniello Murano and Giuseppe Perelli. Pushdown multi-agent system verification. In Proceedings of the 24th International Joint Conference on Artificial Intelligence. IJCAI/AAAI Press, 2015.
- [PR79] Gary L. Peterson and John H. Reif. Multiple-person alternation. In Proceedings of the 20th Annual Symposium on Foundations of Computer Science, pages 348–363. IEEE Computer Society, 1979.
- [PR90] Amir Pnueli and Roni Rosner. Distributed reactive systems are hard to synthesize. In *Proceedings* of the 31st Annual Symposium on Foundations of Computer Science, pages 746–757. IEEE Computer Society, 1990.
- [PRA02] Gary Peterson, John Reif, and Salman Azhar. Decision algorithms for multiplayer noncooperative games of incomplete information. *Computers & Mathematics with Applications*, 43(1):179–206, 2002.

- [PV04] Nir Piterman and Moshe Y Vardi. Global model-checking of infinite-state systems. In *Proceedings* of the 16th International Conference on Computer Aided Verification, volume 3114 of Lecture Notes in Computer Science, pages 387–400. Springer, 2004.
- [Rab69] Michael O. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of the American Mathematical Society*, 141:1–35, 1969.
- [Ser03] Olivier Serre. Note on winning positions on pushdown games with  $\omega$ -regular conditions. Information Processing Letters, 85(6):285–291, 2003.
- [Ser04] Olivier Serre. Contribution à l'étude des jeux sur des graphes de processus à pile. PhD thesis, Université Paris 7, 2004.
- [VW94] Moshe Y. Vardi and Pierre Wolper. Reasoning about infinite computations. *Information and Computation*, 115(1):1–37, 1994.
- [Wal01] Igor Walukiewicz. Pushdown processes: Games and model-checking. *Information and computation*, 164(2):234–263, 2001.
- [Zie98] Wieslaw Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theoretical Computer Science*, 200(1-2):135–183, 1998.