# BOUNDED ALGOL-LIKE LANGUAGES(1)

BY

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**Introduction.** ALGOL-like languages (hereafter called "definable sets") were introduced in [4] as a generalization of the constituent parts arising in the international programming language ALGOL. It is known [4] that definable sets are identical to the context free languages introduced by Chomsky [2] in his study of natural languages such as English, French, etc. The mathematics of these sets has been studied by those interested in either programming or natural languages. This paper introduces a special family of definable sets, studies their structure, and shows that certain questions about them are recursively solvable.

Let $\theta$ be a finitely generated free semi-group (with identity) and let $X$ be a subset of $\theta$. $X$ is said to be bounded if there exists a finite set of words $w_1, \cdots, w_n$ in $\theta$ such that for every word $w$ in $X$ there exist nonnegative integers $i_1, \cdots, i_n$ such that $w = w_1^{i_1} \cdots w_n^{i_n}$. The special family of definable sets considered in this paper is the family of bounded definable sets. It will be shown that this family is more tractable than the family of definable sets. Because of this fact it seems reasonable to expect that bounded definable sets will play an important role in studying arbitrary definable sets. (Bounded definable sets have already been of some value. For example, the first pair of definable sets whose intersection was shown not to be a definable set was a pair of bounded definable sets [1], [11]. The first known "inherently ambiguous" definable set was bounded [8]. Recent work has applied bounded definable sets in a proof of the recursive unsolvability of identifying inherently ambiguous definable sets.)

The paper is divided into six sections. §1 summarizes some of the basic terminology and results about definable sets and introduces the concept of bounded definable sets. §2 contains a structure theorem (Theorem 2.1) which describes (1) the bounded definable subsets expressible by means of words $w_1$ and $w_2$, and (2) shows how more complicated bounded definable sets are built from these inductively. §3 contains a theorem (Theorem 3.1) characterizing bounded definable sets as the family of sets obtained from the finite sets by applying three "elementary operations." §4 is devoted to a proof that a generalized sequential machine (a frequently used model for a computer) transforms a bounded definable set into a bounded definable set. §5 contains a decision procedure for determining

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of an arbitrary definable set whether it is bounded. §6 studies certain sets of lattice points in $n$-space which arise in the theory of bounded sets and gives a decision procedure for determining of arbitrary definable sets $L_1$ and $L_2$, one of which is bounded, whether $L_1 \subseteq L_2$ and whether $L_2 \subseteq L_1$. (The same problems with the boundedness condition removed are known to be recursively unsolvable [1].) This theorem (Theorem 6.3) follows from results (Theorems 6.1 and 6.2) about sets of lattice points which have independent mathematical interest when interpreted as results in the theory of semi-groups and as results about the set of nonnegative integral solutions of linear equations with integral coefficients. The results of this section can be used to give a proof of the decidability of Boolean relations between sets defined by modified Presburger formulas [10].

1. **Basic concepts.** We now present a brief description of the main terms and concepts to be used. Further details, as well as motivation for these ideas, are in the principal references [1], [4], [6]. With the exception of bounded sets, all of the material in this section is already in the literature.

Let $\Sigma$ denote an alphabet, i.e., a finite nonempty set of symbols. Let $\theta(\Sigma)$, or $\theta$ when $\Sigma$ is understood, be the set of all words of elements from $\Sigma$, including the empty word $\varepsilon$. (If $\Sigma = \{a_1, \cdots, a_r\}$, we write $\theta(a_1, \cdots, a_r)$ instead of $\theta(\{a_1, \cdots, a_r\})$.) We are interested in certain subsets of $\theta(\Sigma)$ called "definable sets."

For each word $x$, let $|x|$ denote the length of $x$.

Consider functions $f(\xi_1, \cdots, \xi_n)$ constructed from a finite number of set variables $\xi_1, \cdots, \xi_n$, each $\xi_i$ ranging over $2^\theta$ ( = all subsets of $\theta$), and a finite number of subsets of $\theta$ (called *coefficients*); using the operations of "+" (addition or set union)(2) and "$\cdot$" (multiplication or complex product )(3) a finite number of times. Since multiplication is distributive over addition, each of these may be regarded as in polynomial form, i.e., $f = \sum_1^s \pi_i$, where each $\pi_i$, called a *term*, is a product of set variables and constant. Furthermore, if all the coefficients are finite sets, then it may be assumed that each constant is an element of $\Sigma \cup \{\varepsilon\}$. If all the coefficients are finite sets, then $f$ is said to be a *standard* function.

Let $f_1, \cdots, f_n$ be a sequence of $n$ standard functions of $(\xi_1, \cdots, \xi_n)$ each. Then $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$(4) is called an *n-tuple standard* function. An $n$-tuple standard function $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$ is said to be an *n-tuple sequentially*

(2) Both + and $\cup$ are used to denote set union.

(3) Let $A_1, \cdots, A_m$ be a sequence of sets of words. The (*complex*) *product* $A_1 \cdot A_2 \cdot \cdots \cdot A_m$, or $A_1 \cdots A_m$ for short, is the set of words $\{x_1 \cdots x_m / \text{each } x_i \text{ in } A_i\}$, $x_1 \cdots x_m$ being the word formed from the concatenation of the words $x_i$ in the given order. If one or more of the $A_i$, say $A_{j(1)}, \cdots, A_{j(r)}$ consist of just a single word, say $a_{j(1)}, \cdots, a_{j(r)}$ respectively; then $a_{j(i)}$ is written instead of $A_{j(i)}$ at each occurrence. For example, $aA$ is written instead of $\{a\}A$, and $\varepsilon$ instead of $\{\varepsilon\}$.

(4) $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$ is the mapping of $(2^\theta)^n$ (Cartesian product of $2^\theta$ taken $n$ times) into $(2^\theta)^n$ defined by $f(\xi_1, \cdots, \xi_n) = (f_1(\xi_1, \cdots, \xi_n), \cdots, f_n(\xi_1, \cdots, \xi_n))$.

*standard* function if $f_i = f_i(\xi_1, \cdots, \xi_i)$ for $1 \leqq i \leqq n$, i.e., $f_i$ is a function of only the first $i$ variables.

We now define the subsets of $\theta$ which form the "definable (sequentially definable)" sets.

A subset $L$ of $\theta(\Sigma)$ is said to be $\Sigma$-*definable* (*sequentially* $\Sigma$-*definable*) or definable (sequentially definable) when $\Sigma$ is understood, if for some $n$ there exists an $n$-tuple standard (sequentially standard) function $f$ such that one of the coordinates of the minimal fixed point (abbreviated "mfp") of $f$(5) is $L$.

The definable sets are identical to the context free languages of Chomsky [4, Theorem 2](6). Thus we may cite and use results from the literature on either definable sets or context free languages. A number of these are indicated below. Others appear in the text.

(1) The finite union and finite product of definable sets are definable [1, p. 149].

(2) The mfp of $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$, each $f_i$ a polynomial, is $(\alpha_1, \cdots, \alpha_n)$, where for each $i$ $(1 \leqq i \leqq n)$ and $k \geqq 0$, $\alpha_i^{(0)} = f_i(\phi, \cdots, \phi)$, $\alpha_i^{(k+1)} = f_i(\alpha_1^{(k)}, \cdots, \alpha_n^{(k)})$, and $\alpha_i = \bigcup_{k=0}^{\infty} \alpha_i^{(k)}$ [4, Theorem 1].

(3) Each definable (sequentially definable) set is the last coordinate in the mfp of some $n$-tuple standard (sequentially standard) function for some $n$ [4].

(4) If $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$, where each $f_i$ is a polynomial (polynomial in $\xi_1, \cdots, \xi_i$) with definable (sequentially definable) coefficients, then each coordinate in the mfp of $f$ is definable (sequentially definable) [4, Theorem C].

(5) If $L$ is definable, then so is $L - \{\varepsilon\}$. If $L$ is definable and does not contain $\varepsilon$, then an $n$-tuple standard function $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$ can be found so that no term of any $f_i$ is $\varepsilon$ and $L$ is the last coordinate in the mfp of $f$ [1, Lemma 4.1].

(6) Each definable (sequentially definable) set is the last coordinate of an $n$-tuple standard (sequentially standard) function $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$ with the following properties:

(a) If $(\alpha_1, \cdots, \alpha_n)$ is the mfp of $f$, then $\alpha_i \neq \phi$ for $1 \leqq i \leqq n-1$.

(b) For each $i \neq n$, $\xi_n$ depends on $\xi_i$(7).

If $\xi_n$ depends on $\xi_i$, $\alpha_n \neq \phi$, and (a) prevails, then there exist words $w$ and $y$ in $\theta$ such that $w\alpha_i y \subseteq \alpha_n$ [1, p.158].

Another family of subsets of $\theta$ which plays a prominent part in our investigation is the family of "regular sets." This family may be characterized as the smallest family of subsets of $\theta$ which contains the finite sets and is closed under the operations of $+, \cdot$, and $*$ (8) [9, Theorem 14].

(5) Let $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$, where the $f_i$ are polynomials. $a = (a_1, \cdots, a_n)$ is said to be a *fixed point* of $f(\xi_1, \cdots, \xi_n)$ if $f(a) = a$. In addition, if $a \subseteq \beta$ for each $n$-tuple of sets $\beta = (\beta_1, \cdots, \beta_n)$ such that $f(\beta) = \beta$, then $a$ is said to be a *minimal fixed point* (of $f$). Each $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$ has one and only one minimal fixed point.

(6) Context free languages are defined in §5.

(7) Let $W_1 = \{\xi_n\}$ and $W_{k+1} = W_k \cup \{\xi_i$/for some $\xi_j$ in $W_k$, $\xi_j$ occurs in a term of $f_i\}$. If $\xi_i$ is in $W_n$, then $\xi_n$ is said to *depend* on $\xi_i$.

(8) If $A$ is a subset of $\theta$, then $A^* = \bigcup_0^{\infty} A^i$, where $A^0 = \{\varepsilon\}$ and $A^{k+1} = A^k A$ for each $k$

Regular sets are (sequentially) definable sets [3, Theorem 1]. Furthermore, if $L$ is definable and $R$ is regular, then $L \cap R$ is definable [1, Theorem 8.1].

We now introduce the notion of a bounded set.

DEFINITION. A subset $X$ of $\theta$ is said to be *bounded* if there exist words $w_1, \cdots, w_r$ (in $\theta$) such that $X \subseteq w_1^* \cdots w_r^*$.

We summarize some elementary facts about bounded sets in the following lemma.

LEMMA 1.1. (a) *The finite product of bounded sets is bounded.*

(b) *The finite union of bounded sets is bounded.*

(c) *If $X$ is bounded and $Y$ is a set of subwords of wards in $X$, then $Y$ is bounded. In particular, a subset of a bounded set is bounded.*

The proofs are obvious and are omitted.

Finally we shall need the concepts of linear and semi-linear sets in the sense of Parikh [8].

Let $N$ denote the nonnegative integers and let $N^n$ be the Cartesian product of $N$ with itself $n$ times. For elements $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $N^n$, let $x + y = (x_1 + y_1, \cdots, x_n + y_n)$ and $c(x_1, \cdots, x_n) = (cx_1, \cdots, cx_n)$, $c$ in $N$. A subset $A$ of $N^n$ is said to be *linear* if there exist elements $v, v_1, \cdots, v_m$ in $N^n$ such that

$$A = \{x / x = v + k_1 v_1 + \cdots + k_m v_m, \text{ each } k_i \text{ in } N\}.$$

A subset $A$ of $N^n$ is said to be *semi-linear* if it is a finite union of linear sets.

Note that the empty set is semi-linear, being the union of zero linear sets.

Now a linear set is a coset of a semi-group in $N^n$. Thus any results on linear and semi-linear sets may be interpreted as results about cosets of special semi-groups.

The finite union of semi-linear sets is semi-linear.

Our interest in semi-linear sets stems from the following result.

PARIKH'S THEOREM [8, THEOREM 2]. *Let $\Sigma = \{a_i / 1 \leqq i \leqq n\}$ and let $\psi_n$ be the mapping of $\theta(\Sigma)$ into $N^n$ defined as follows: $\psi_n(\varepsilon) = (0, \cdots, 0)$, $\psi_n(a_i) = (z_{i1}, \cdots, z_{in})$ where $z_{ij} = 0$ for $j \neq i$ and $z_{ii} = 1$, and*

$$\psi_n(x_1 \cdots x_k) = \sum_{j=1}^{k} \psi_n(x_j)$$

*for each word $x_1 \cdots x_k$ in $\theta(\Sigma) - \varepsilon$, each $x_i$ in $\Sigma$. If $L$ is a definable set, then $\psi_n(L)$ is semi-linear.*

## 2. Structure.

We now consider the structure of bounded definable sets. The main result, a culmination of six lemmas, indicates how bounded definable sets are constructed from "simpler" bounded definable sets.

NOTATION. Let $Z$ be a subset of $\theta(\Sigma)$. If $X$ and $Y$ are subsets of $x^*$ and $y^*$ respectively, $x$ and $y$ in $\theta(\Sigma)$, then we write

$$(X, Y)^*Z = \bigcup_{k \geqq 0} X^k Z Y^k.$$

Inductively, for $Z$ a subset of $\theta(\Sigma)$, $X_i$ and $Y_i$ subsets of $x^*$ and $y^*$ respectively $(1 \leqq i \leqq n)$, $x$ and $y$ in $\theta(\Sigma)$, we write

$$(X_n, Y_n) \cdots (X_1, Y_1)^*Z = (X_n, Y_n)^*((X_{n-1}, Y_{n-1}) \cdots (X_1, Y_1)^*Z).$$

LEMMA 2.1. $(X_n, Y_n) \cdots (X_1, Y_1)^*Z = \bigcup_{k_i \geqq 0; 1 \leqq i \leqq n}^{\infty} X_n^{k_n} \cdots X_1^{k_1} Z Y_1^{k_1} \cdots Y_n^{k_n}$.

The proof is obvious and so is omitted.

We now consider definable subsets of $a^*b^*$.

LEMMA 2.2. *Let $\Sigma = \{a, b\}$. Each definable subset of $a^*b^*$ is the finite union of sets of the form*

$$(*) \qquad (x_m, y_m) \cdots (x_1, y_1)^* z,$$

*where each $x_i$ is in $a^*$, each $y_i$ is in $b^*$, and $z$ is in $a^*b^*$; and each finite union of sets of the form (*) is a definable subset of $a^*b^*$.*

**Proof.** Suppose that $L$ is a definable subset of $a^*b^*$. Let $N$ be the set of nonnegative integers and $\psi_2$ the mapping of $a^*b^*$ into $N^2$ defined in Parikh's Theorem. Let $\sigma$ be the mapping of $N^2$ into $a^*b^*$ defined by $\sigma(x, y) = a^x b^y$. Clearly $\psi_2$ and $\sigma$ are inverse functions (over $N^2$ and $a^*b^*$). If $U$ is a linear subset of $N^2$,

$$U = \left\{ u / u = (u_0(a), u_0(b)) + \sum_{i=1}^{m} k_i(u_i(a), u_i(b)), \text{each } k_i \text{ in } N \right\},$$

then

$$(**) \qquad \sigma(U) = (a^{u_m(a)}, b^{u_m(b)}) \cdots (a^{u_1(a)}, b^{u_1(b)})^* a^{u_0(a)} b^{u_0(b)}.$$

By Parikh's Theorem, $\psi_2(L)$ is the finite union of linear subsets of $N^2$. Thus $L = \sigma\psi_2(L)$ is the finite union of sets of the form (**), thus the finite union of sets of the form (*).

Now suppose that $L$ is the finite union of sets of the form (*). To show that $L$ is definable, it suffices to show that each set $M = (x_m, y_m) \cdots (x_1, y_1)^* z$ is a definable subset of $a^*b^*$. However, this follows from the fact that $M$ is the mfp of $f(\xi) = x_m \xi y_m + \cdots + x_1 \xi y_1 + z$. Q.E.D.

COROLLARY. *The subset $X$ of $a^*b^*$ is definable if and only if the subset $\{(m, n)/ a^m b^n \text{ in } X\}$ of $N^2$ is semi-linear.*

Lemma 2.2 indicates one way to generate the definable subsets of $a^*b^*$ from "simple" subsets of $a^*$, $b^*$, and $a^*b^*$. Another possibility is suggested by noting that $\bigcup_0^\infty A(a^p)^k(b^q)^k B$ is a definable subset of $a^*b^*$ for every definable subset $A$

of $a^*$, $B$ of $b^*$, and non negative integers $p$ and $q$. However, the following example shows that not every definable subset of $a^*b^*$ is a finite union of sets of that form.

EXAMPLE. Consider the set $X = \{a^{i+2j}b^{i+4j}/ i,j \geqq 0\}$. $X$ is definable being the mfp of $f = a\xi b + a^2\xi b^4 + \varepsilon$. We shall show that $X$ is not the finite union of sets of the form

$$(1) \qquad \bigcup_{k \geqq 0}^{\infty} A(a^p)^k(b^q)^k B,$$

where $A$ and $B$ are definable subsets of $a^*$ and $b^*$ respectively.

For each $m \geqq 0$ and $n \geqq 0$ let $I_m = \{k/ a^k b^m \text{ in } X\}$ and $J_n = \{k/ a^n b^k \text{ in } X\}$. For each $m$, $I_m$ has exactly $[m/4] + 1$ elements, namely, the number of pairs of nonnegative integers $(i,j)$ such that $i + 4j = m$. Similarly each $J_n$ has exactly $[n/2] + 1$ elements. Therefore the set $X$ contains no subset of form (1) where $A$ or $B$ is infinite. If $X$ is a finite union of sets of form (1), where $A$ and $B$ are both finite, then $X$ is a finite union of sets of the form

$$(2) \qquad \bigcup_{k \geqq 0}^{\infty} a^r(a^p)^k(b^q)^k b^s.$$

It thus suffices to show that $X$ cannot be a finite union of sets of form (2). Now $X$ cannot contain a set of form (2), where $p > 0$ and $q = 0$. For otherwise, $I_s$ would contain all the nonnegative integers $r + kp$ and thus be infinite, a contradiction. Similarly $X$ cannot contain a set of form (2), where $p = 0$ and $q > 0$. Hence we need only prove that $X$ is not a finite union of sets of form (2) where, in each set, either $p = q = 0$ or both $p > 0$ and $q > 0$. For a given $m > 0$ each set in (2) with either $p = q = 0$ or both $p > 0$ and $q > 0$ can contain at most one element of $X$ of the form $a^k b^m$. Suppose that $X$ were the union of $t$ such sets. Then each $I_m$ would contain at most $t$ elements. This, however, contradicts the fact that the number of elements in $I_m$ becomes unbounded as $m$ becomes large.

We now consider the structure of definable subsets of $a_1^* \cdots a_n^*$, $n \geqq 3$, $a_i \neq a_j$ for $i \neq j$.

LEMMA 2.3. *Let* $\Sigma = \{a_i/ 1 \leqq i \leqq n\}$. *Let* $D$ *be a definable subset of* $a_1^* a_n^*$ *and* $E$ *be a definable subset of* $a_1^* \cdots a_n^*$. *Then* $\bigcup_{a_1^i a_n^j \text{ in } D} a_1^i E a_n^j$ *is a definable subset of* $a_1^* \cdots a_n^*$.

Using Lemma 2.2 the proof is straightforward and is omitted.

LEMMA 2.4. *Let* $g(\xi_1, \cdots, \xi_n) = (g_1, \cdots, g_n)$, *where, for* $1 \leqq i \leqq n$,

$$g_i = \sum_{j=1}^{n} \sum_{t=1}^{t(i,j)} A_{ijt}\xi_j B_{ijt} + \sum_{t=1}^{s(i)} G_{it},$$

*each* $A_{ijt}$, $B_{ijt}$, *and* $G_{it}$ *being subsets of* $\theta(\Sigma)$. *Let* $(\alpha_1, \cdots, \alpha_n)$ *be the mfp of* $g(\xi_1, \cdots, \xi_n)$. *Then, for* $1 \leqq k \leqq n$,

$$\alpha_k = \bigcup_{\substack{r \geqq 2 \\ i_r = k}} \bigcup_{\text{all } i_j, t_j} A_{i_r i_{r-1} t_r} \cdots A_{i_2 i_1 t_2} G_{i_1 t_1} B_{i_2 i_1 t_2} \cdots B_{i_r i_{r-1} t_r} \cup \bigcup_{t=1}^{s(k)} G_{kt}.$$

The proof is straightforward and is omitted.

LEMMA 2.5. *Let* $\Sigma = \{a_i / 1 \leqq i \leqq n\}$, $n \geqq 3$. *Each definable subset* $L$ *of* $a_1^* \cdots a_n^*$ *is the finite union of sets of the following form*:

(1) $$L(D,E,F) = \{a_1^i x y a_n^j / a_1^i a_n^j \text{ in } D, x \text{ in } E, y \text{ in } F\},$$

*where* $D$, $E$, *and* $F$ *are definable subsets of* $a_1^* a_n^*$, $a_1^* \cdots a_q^*$, $a_q^* \cdots a_n^*$ *respectively, and* $1 < q < n$. *Conversely, each finite union of sets of form* (1) *is a definable subset of* $a_1^* \cdots a_n^*$.

**Proof.** By Lemma 2.3, each finite union of sets of form (1) is a definable subset of $a_1^* \cdots a_n^*$. It thus suffices to show that each definable subset $L$ of $a_1^* \cdots a_n^*$ is the finite union of sets of form (1).

Let $L$ be a definable subset of $a_1^* \cdots a_n^*$. Then $L$ is the last coordinate in the mfp $(\alpha_1, \cdots, \alpha_k)$ of $f(\xi_1, \cdots, \xi_k) = (f_1, \cdots, f_k)$. Suppose that $a_n$ occurs in no word of $L$. Then $L \subseteq a_1^* \cdots a_{n-1}^*$. Thus $L(\varepsilon, L, \varepsilon)$ satisfies (1) and $L = L(\varepsilon, L, \varepsilon)$. A similar result holds if $a_1$ occurs in no word of $L$. Thus we may assume that

(2) $a_1$ occurs in some word of $L$ and $a_n$ occurs in some word of $L$.

We may also assume that

(3) $L$ does not contain $\varepsilon$.

(For if it does, then we could consider the definable set $L - \{\varepsilon\}$.) Thus we may assume that

(4) Each term in each $f_i$ is a product of letters and variables [1, Lemma 4.1].

Finally we may assume that

(5) $\xi_k$ depends on each variable $\xi_j$, $j \leqq k$,

so that

(6) $\alpha_i \subseteq a_1^* \cdots a_n^*$ for each $i$.

We shall use the following terminology in this proof. A subset $X$ of $a_1^* \cdots a_n^* - \{\varepsilon\}$ is said to be of type $a_p a_q$, $p \leqq q$, if

(7) There is some word in $X$ which contains $a_p$ and some word containing $a_q$.

If $h(\xi_1, \cdots, \xi_u) = (h_1, \cdots, h_u)$, each $h_i$ an arbitrary polynomial, and $(\beta_1, \cdots, \beta_u)$ is the mfp of $h$, then the variable $\xi_i$ is said to be of type $a_p a_q$ if $\beta_i$ is of type $a_p a_q$.

By a change in notation if necessary, we may assume that for some integer $v$, $\{\xi_i / v \leqq i \leqq k\}$ is the set of all variables of type $a_1 a_n$. By (2), $\xi_k$ is of type $a_1 a_n$. For $j \geqq v$ let

(8) $g_j(\xi_v, \cdots, \xi_k) = f_j(\alpha_1, \cdots, \alpha_{v-1}, \xi_v, \cdots, \xi_k)$.

By Lemma 3 of [4], $(\alpha_v, \cdots, \alpha_k)$ is the mfp of $g(\xi_v, \cdots, \xi_k) = (g_v, \cdots, g_k)$. Obviously $g(\xi_v, \cdots, \xi_k)$ satisfies (5), (6), and

(9) Each $\xi_1$ is a variable of type $a_1 a_n$.

Let $H$ be any constant term in $g_i(\xi_v, \cdots, \xi_k)$. By (4) and (8), $H$ is a product of letters and sets $\alpha_j$, $j \leqq v-1$, say $H = x_1 \cdots x_z$. By (6), $H \subseteq a_1^* \cdots a_n^*$. Suppose there is an $m$ such that $x_m$ is in $a_2^* \cdots a_n^*$ or $x_m$ is a set not of type $a_1 a_p$, $p = 1$. Let $u$ be the smallest such $i$. Let $E = \{\varepsilon\}$ if $u = 1$, $E = x_1 \ldots x_{u-1}$ if $u > 1$, and $F = x_u \cdots x_z$. If no such $m$ exists, let $E = x_1 \cdots x_z$ and $F = \{\varepsilon\}$. Since no $\alpha_j$, $j \leqq v-1$, is of type $a_1 a_n$,

(10) $H = EF$, where $E$ is a definable subset of $a_1^* \cdots a_q^*$ and $F$ is a definable subset of $a_q^* \cdots a_n^*$, with $1 < q < n$.

Since each variable in $g$ is of type $a_1 a_n$, every nonconstant term in $g_h$ is of the form $A\xi_j B$, where $A$ and $B$ are sets. Suppose that $A\xi_j B$ is a term in $g_h$. Since $\xi_j$ is of type $a_1 a_n$,

(11) $A$ is a definable subset of $a_1^*$ and $B$ is a definable subset of $a_n^*$. Thus each term in $g_h$ is of the form $EF$ of (10) or $A\xi_j B$ of (11). In other words, for each $i$

(12) $g_i = \sum_{j,t} A_{ijt}\xi_j B_{ijt} + \sum_t E_{it} F_{it}$,

where $A_{ijt}$, $B_{ijt}$, $E_{it}$, $F_{it}$ are definable subsets of $a_1^*$, $a_1^* \cdots a_q^*$, $a_q^* \cdots a_n^*$ respectively, with $1 < q < n$. The index $t$ refers to the various terms which contain $\xi_j$ and the various constant terms. From Lemma 2.4, it follows that $L = \alpha_k$ is the finite union of sets of the form

$$(13) \qquad M = \bigcup_{n_1 \geqq 0 - n_{r \ldots 1} \geqq 0} A_{i_r i_{r-1} t_r}^{n_{r-1}} \cdots A_{i_2 i_1 t_2}^{n_1} E_{i_1 t_1} F_{i_1 t_1} B_{i_2 i_1 t_2}^{n_1} \cdots B_{i_r i_{r-1} t_r}^{n_{r-1}},$$

where $r \geqq 2$, $i_r = k$, and no $ijt$ occurs more than once as a subscript of some $A$ in each summand of (13). Let $D$ be the set

$$(14) \qquad D = (A_{i_r i_{r-1} t_r}, B_{i_r i_{r-1} t}) \cdots (A_{i_2 i_1 t_2}, B_{i_2 i_1 t_2})^* \varepsilon.$$

Then $D$ is a definable subset of $a_1^* a_n^*$, being the mfp of $\sum_{j=2}^r A_{i_j i_{j-1} t_j} \xi B_{i_j i_{j-1} t_j} + \varepsilon$, and $M = L(D, E_{i_1 t_1}, F_{i_1 t_1})$. Q.E.D.

LEMMA 2.6. *Let $w_1, \cdots, w_n$ be words and $a_1, \cdots, a_n$ distinct symbols. If $W$ is a definable subset of $w_1^* \cdots w_n^*$, then*

$$\{a_1^{k_1} \cdots a_n^{k_n} / w_1^{k_1} \cdots w_n^{k_n} \text{ in } W\}$$

*is a definable subset of $a_1^* \cdots a_n^*$.*

**Proof.** Let $S$ be the one state gsm[9] which maps each $a_i$ into $w_i$. Then $W' = \{v / S(v) \text{ in } W\}$[10] is a definable subset of $\theta(a_1, \cdots, a_n)$ by Theorem 3.4 of [6] since $W$ is definable. Let $Y = W' \cap a_1^* \cdots a_n^*$. Then

$$Y = \{a_1^{k_1} \cdots a_n^{k_n} / w_1^{k_1} \cdots w_n^{k_n} \text{ in } W\}$$

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(9) A *generalized sequential machine* (gsm) $S$ is a 6-tuple $(K, \Sigma, \Delta, \delta, \lambda, p_1)$ where (i) $K$ is a finite nonempty set (of "states"); (ii) $\Sigma$ is a finite nonempty set (of "inputs"); (iii) $\Delta$ is a finite nonempty set (of "outputs"); (iv) $\delta$ is a mapping of $K \times \Sigma$ into $K$ (the "next state" function); (v) $\lambda$ is a mapping of $K \times \Sigma$ into $\theta(\Delta)$ (the "output" function): and (vi) $p_1$ is an element of $K$ (the "start" state).

(10) Extend $\delta$ and $\lambda$ to $K \times \theta(\Sigma)$ as follows. Let $\delta(q, \varepsilon) = q$ and $\lambda(q, \varepsilon) = \varepsilon$. For each word $u_1 \cdots u_{k+1}$, each $u_i$ in $\Sigma$, let $\delta(q, u_1 \cdots u_{k+1}) = \delta[\delta(q, u_1 \cdots u_k), u_{k+1}]$ and $\lambda(q, u_1 \cdots u_{k+1}) = \lambda(q, u_1 \cdots u_k)\, \lambda[\delta(q, u_1 \cdots u_k), u_{k+1}]$. For each word $v$ in $\theta(\Sigma)$, let $S(r) = \lambda(p_1, v)$.

and $S(Y) = W$[11]. Now the intersection of a definable set and a regular set is definable. Since $a_1^* \cdots a_n^*$ is regular, $Y$ is definable.

We are now ready to prove our main structure result.

THEOREM 2.1. *Let $w_1$ and $w_2$ be words. Each definable subset of $w_1^* w_2^*$ is the finite union of sets of the form*

$$(1) \qquad (x_m, y_m) \cdots (x_1, y_1)^* z,$$

*where $x_i$ is in $w_1^*$, $y_i$ is in $w_2^*$, and $z$ is in $w_1^* w_2^*$; and each finite union of sets of the form* (1) *is a definable subset of $w_1^* w_2^*$.*

(b) *Let $w_1, \cdots, w_n$, $n \geqq 3$, be words. Each definable subset of $w_1^* \cdots w_n^*$ is the finite union of sets of the following form*:

$$(2) \qquad L(D,E,F) = \bigcup_{a_1^i a_n^j \text{ in } D} w_1^i E F w_n^j,$$

*where $D$, $E$, and $F$ are definable subsets of $a_1^* a_n^*$ $(a_1 \neq a_n)$, $w_1^* \cdots w_q^*$, $w_q^* \cdots w_n^*$ respectively, and $1 < q < n$. Conversely, each finite union of sets of form* (2) *is a definable subset of $w_1^* \cdots w_n^*$.*

**Proof.** (a) Let $W$ be a definable subset of $w_1^* w_2^*$, $w_1$ and $w_2$ words. Let $a_1$ and $a_2$ be two distinct symbols. Let $S$ be the one state gsm which maps $a_i$ into $w_i$, $i = 1,2$. The machine operation here commutes with union and product. Let

$$Y = \{a_1^{k_1} a_2^{k_2} / w_1^{k_1} w_2^{k_2} \text{ in } W\}.$$

By Lemma 2.6, $Y$ is a definable subset of $a_1^* a_2^*$. By Lemma 2.2, $Y$ is the finite union of sets of the form $(x_m, y_m) \cdots (x_1, y_1)^* z$, where $x_i$ is in $a_1^*$, $y_i$ is in $a_2^*$, and $z$ is in $a_1^* a_1^*$. Thus $W = S(Y)$ is the finite union of sets of the form $S((x_m, y_m) \cdots (x_1, y_1)^* z)$. By Lemma 2.1,

$$(3) \qquad (x_m, y_m) \cdots (x_1, y_1)^* z = \bigcup_{\text{each } k_i \geqq 0}^{\infty} x_m^{k_m} \cdots x_1^{k_1} z y_1^{k_1} \cdots y_m^{k_m}.$$

Thus

$$\begin{aligned} S((x_m, y_m) \cdots (x_1, y_1)^* z) &= S\left( \bigcup_{\text{each } k_i \geqq 0} x_m^{k_m} \cdots x_1^{k_1} z y_1^{k_1} \cdots y_m^{k_m} \right) \\ &= \bigcup_{\text{each } k_i \geqq 0}^{\infty} S(x_m)^{k_m} \cdots S(x_1)^{k_1} S(z) S(y_1)^{k_1} \cdots S(y_m)^{k_m} \\ &= (S(x_m), S(y_m)) \cdots (S(x_1), S(y_1))^* S(z), \end{aligned}$$

the last equality occurring since each $S(x_i)$ is in $w_1^*$ and each $S(y_i)$ is in $w_2^*$. Thus $W$ is the finite union of sets satisfying (1).

[11] If $T$ is a mapping of words and $E$ is a set of words, then $T(E) = \{T(w)/w \text{ in } E\}$.

Each set of the form (1) is the mfp of $f(\xi) = \sum x_i \xi y_i + z$. Thus the finite union of sets of the form (1) is a definable subset of $w_1^* w_2^*$.

(b) Let $W$ be a definable subset of $w_1^* \cdots w_n^*$, $n \geqq 3$, each $w_i$ a word. Let $a_1, \cdots, a_n$ be $n$ distinct symbols. Let $S$ be the one state gsm which maps each $a_i$ into $w_i$. The machine operation $S$ again commutes with union and product. By Lemma 2.6,

$$Y = \{a_1^{k_1} \cdots a_n^{k_n} / w_1^{k_1} \cdots w_n^{k_n} \text{ in } W\}$$

is a definable subset of $a_1^* \cdots a_n^*$. By Lemma 2.5, $Y$ is the finite union of sets of the form

$$M(D, E', F') = \bigcup_{a_1^i a_n^j \text{ in } D} a_1^i E' F' a_n^j, \tag{4}$$

where $D, E', F'$ are definable subsets of $a_1^* a_n^*$, $a_1^* \cdots a_q^*$, $a_q^* \cdots a_n^*$ respectively, and $1 < q < n$. Then $S(Y) = W$ is the finite union of sets of the form

$$S(M(D, E', F')) = \bigcup_{a_1^i a_n^j \text{ in } D} w_1^i S(E') S(F') w_n^j.$$

Since $E'$ and $F'$ are definable subsets of $a_1^* \cdots a_q^*$ and $a_q^* \cdots a_n^*$ respectively, $E = S(E')$ and $F = S(F')$ are definable subsets of $w_1^* \cdots w_q^*$ and $w_q^* \cdots w_n^*$ respectively(12). Then $S(M(D, E', F')) = L(D, E, F)$ satisfies (2). Therefore $W$ is the finite union of sets of the form (2).

Suppose that $L(D, E, F) = \bigcup_{a_1^i a_n^j \text{ in } D} w_1^i E F w_n^j$ satisfies (2). Let

$$E' = \{a_1^{k_1} \cdots a_q^{k_q} / w_1^{k_1} \cdots w_q^{k_q} \text{ in } E\}$$

and

$$F' = \{a_q^{k_q} \cdots a_n^{k_n} / w_q^{k_q} \cdots w_n^{k_n} \text{ in } F\}.$$

By Lemma 2.6, $E'$ and $F'$ are definable subsets of $a_1^* \cdots a_q^*$ and $a_q^* \cdots a_n^*$ respectively. By Lemma 2.4, $\bigcup_{a_1^i a_n^j \text{ in } D} a_1^i E' F' a_n^j$ is a definable subset of $a_1^* \cdots a_n^*$. Again let $S$ be the one state machine which maps each $a_i$ into $w_i$. Then

$$S\left( \bigcup_{a_1^i a_n^j \text{ in } D} a_1^i E' F' a_n^j \right) = L(D, E, F)$$

is definable. Thus the finite union of sets of form (2) is definable. Q.E.D.

3. **Applications.** We now present two applications of the structure results. The first application is to characterizing the bounded definable sets. The second is to showing that certain subsets of $a^* b^* c^*$ are not definable.

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(12) It is known that if $S$ is a gsm and $L$ is a definable set, then $S(L)$ is definable [6, Theorem 3.1]. We shall use this result frequently.

LEMMA 3.1. *Let* $\Sigma = \{a_i/1 \leqq i \leqq n\}$. *Each definable subset of* $a_1^* \cdots a_n^*$ *is obtained from finite sets by a finite number of applications of the following operations*:

(a) *The union of two sets.*

(b) *The product of two sets.*

(c) $(x, y)^*Z$, *where* $x$ *and* $y$ *are words.*

**Proof.** The lemma is true for $n = 1$ since, by Corollary 2 of Theorem 4 of [4], every definable subset of $a_1^*$ is regular. The lemma is true for $n = 2$ by Lemma 2.2. Suppose the lemma is true for $k \geqq 2$. Let $L$ be a definable subset of $a_1^* \cdots a_{k+1}^*$. By Lemma 2.5, $L$ is the finite union of sets of the form $\bigcup_{a_1^i a_{k+1}^j \text{ in } D} a_1^i E F a_{k+1}^j$, where for some $1 < q < k+1$, $E$ is a definable subset of $a_1^* \cdots a_q^*$, $F$ is a definable subset of $a_q^* \cdots a_{k+1}^*$, and $D$ is a definable subset of $a_1^* a_{k+1}^*$. By induction, $E$ and $F$ are obtained from finite sets using (a), (b), and (c) a finite number of times. By Lemma 2.2, $D = (x_m, y_m) \cdots (x_1, y_1)^* a_1^s a_{k+1}^t$ where $x_i$ is in $a_1^*$ and $y_i$ is in $a_{k+1}^*$ for each $i$. Then

$$\bigcup_{a_1^i a_{k+1}^j \text{ in } D} a_1^i E F a_{k+1}^j = (x_m, y_m) \cdots (x_1, y_1)^* a_1^s E F a_{k+1}^t.$$

Thus $L$ is obtained from finite sets using (a), (b), and (c) a finite number of times. By induction, the lemma is true for all $n \geqq 1$.

We now characterize the family of bounded definable sets.

THEOREM 3.1. *The family of bounded definable sets is the smallest family of sets containing all finite sets and closed with respect to the following operations*:

(a) *Finite union.*

(b) *Finite product.*

(c) $(x, y)^*Z$, *where* $x$ *and* $y$ *are words.*

**Proof.** If $A_1, \cdots, A_r$ are bounded definable sets, then so are $\bigcup_1^r A_i$ and $A_1 \cdots A_r$. Suppose that $Z$ is a bounded definable set and $x$, $y$ are words. Then $(x, y)^*Z = \bigcup_{n \geqq 0} x^n Z y^n$ is bounded. $(x, y)^*Z$ is also definable. For it is the mfp of $f(\xi) = x\xi y + Z$. Since every finite set is a bounded definable set, it follows that every set built from finite sets by a finite number of operations of type (a), (b), or (c) is a bounded definable set.

Now let $A$ be a bounded definable set, i.e., $A$ is a definable subset of $w_1^* \cdots w_r^*$ for some words $w_1, \cdots, w_r$. Let $a_1, \cdots, a_r$ be $r$ distinct symbols. By Lemma 2.6,

$$B = \{a_1^{k_1} \cdots a_r^{k_r} / w_1^{k_1} \cdots w_r^{k_r} \text{ in } A\}$$

is a definable subset of $a_1^* \cdots a_r^*$. By Lemma 3.1, $B$ is obtained from finite sets by a finite number of operations of type (a), (b), or (c). In other words, $B$ is obtained by using a sequence of operations $T_1, \cdots, T_m$, each $T_i$ of type (a), (b), or (c).

Let $S$ be the one state gsm which maps each $a_i$ into $w_i$. The machine operation $S$ here commutes with union and product. Also,

$$\begin{aligned} S((x,y)^*Z) &= S\left(\bigcup_{n\geqq 0} x^n Z y^n\right) \\ &= \bigcup_{n\geqq 0} S(x^n Z y^n) \\ &= \bigcup_{n\geqq 0} S(x)^n S(Z) S(y)^n \\ &= ((S(x), S(y))^* S(Z)). \end{aligned}$$

Thus $A$ is also built up from finite sets using the sequence $T_1, \cdots, T_m$. Q.E.D.

It follows from Theorem 3.1 that for any bounded definable set $L$ there exists a finite sequence $F_0, \cdots, F_m$ of sets such that

(1) $F_0$ is a finite family of finite sets.

(2) $F_{i+1}$ is obtained from $F_i$ by adjoining to $F_i$ one set which is either (i) the union of two sets in $F_i$, (ii) the product of two sets in $F_i$, or (iii) the set $(x,y)^*C$ for some set $C$ in $F_i$ and some words $x$, $y$.

(3) $L$ is in $F_m$.

By induction, each $F_i$ contains only sequentially definable sets. Thus we get

COROLLARY 1. *Each bounded definable set is sequentially definable.*

COROLLARY 2. *Let $a_1, \cdots, a_n$ be $n \geqq 2$ symbols. Then $\theta(a_1, \cdots, a_n)$ is not bounded.*

**Proof.** It suffices to show that $\theta(a_1, a_2)$ is not bounded. Therefore suppose that $\theta(a_1, a_2)$ is bounded. By Lemma 1.1, each definable subset of $\theta(a_1, a_2)$ is bounded. By Corollary 1, each bounded definable subset of $\theta(a_1, a_2)$ is sequentially definable. But there exist definable subsets of $\theta(a_1, a_2)$ which are not sequentially definable [5, Theorem 2.1]. Thus $\theta(a_1, a_2)$ is not bounded([13]).

COROLLARY 3. *Each bounded definable set is the nth coordinate of some n-tuple standard function $f(\xi_1, \cdots, \xi_n) = (f_1, \cdots, f_n)$ whose variables form a partially ordered set with the following properties*:

(a) *$\xi_i \geqq \xi_j$ if and only if $\xi_i$ depends on $\xi_j$.*

(b) *If there is a $\xi_k$ such that $\xi_1 \geqq \xi_k$ and $\xi_j \geqq \xi_k$, then $\xi_i \geqq \xi_j$ or $\xi_j \geqq \xi_i$.*

(c) *Each $f_i$ has one of the four forms* :

(i) $f_i = A$, *where $A$ is a finite set*;

(ii) $f_i = \xi_j + \xi_k$, *where $\xi_i > \xi_j$, $\xi_i > \xi_k$, and $\xi_j$ is incomparable with $\xi_k$*;

(iii) $f_i = \xi_j \xi_k$, *where $\xi_i > \xi_j$, $\xi_i > \xi_k$, and $\xi_j$ is incomparable with $\xi_k$*;

(iv) $f_i = x\xi_i y + \xi_j$, *where $x$, $y$ are words in $\theta(\Sigma)$ and $\xi_i > \xi_j$.*

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(13) There are proofs of Corollary 2 that do not depend on the existence of a definable set which is not sequentially definable.

**Proof.** The corollary results from Theorem 3.1 by induction. The finite sets are the mfp of functions of type (i). Suppose that $A$ and $B$ are bounded definable sets, with $A$ the $n$th coordinate of $f(\xi_1,\cdots,\xi_n)=(f_1,\cdots,f_n)$, $B$ the $m$th coordinate of $g(v_1,\cdots,v_m)=(g_1,\cdots,g_m)$, the $\xi_i$ and $v_j$ satisfying the conclusion of the corollary. Then $A\cup B$ is the $(m+n+1)$st coordinate of

$$h(\xi_1,\cdots,\xi_n,\ v_1,\cdots,v_m,\ \xi_{n+m+1})=(f_1,\cdots,f_n,\ g_1,\cdots,g_m,f_{n+1}),$$

where $f_{n+1}=\xi_n+v_m$. The variables in $h(\xi_1,\cdots,\xi_n,\ v_1,\cdots,v_m,\ \xi_{n+1})$ are partially ordered as follows: $\xi_i$ and $\xi_j$, $i,j\leqq n$ are partially ordered as in $f(\xi_1,\cdots,\xi_n)$. $v_i$ and $v_j$, $i,j\leqq m$, are partially ordered as in $g(v_1,\cdots,v_m)$. For $i\leqq n$ and $j\leqq m$, $\xi_i$ is incomparable with $\xi_j$, $\xi_{n+1}>\xi_i$, and $\xi_{n+1}>v_j$. A similar procedure holds for $AB$. $(x,y)^*A$ is the $(n+1)$st coordinate in the mfp of

$$h(\xi_1,\cdots,\xi_{n+1})=(f_1,\cdots,f_{n+1}),$$

where $f_{n+1}=x\xi_{n+1}y+\xi_n$. The variables in $h(\xi_1,\cdots,\xi_{n+1})$ are partially ordered as above except that there are no variables $v_i$, $i\leqq m$.

REMARK. It is readily seen that the converse to the corollary is also true.

We briefly consider the smallest family $J$ of subsets of $\theta(a_1,\cdots,a_n)$ which contains the finite sets and is closed with respect to the following operations:

(a) Finite union.

(b) Finite product.

(c) Double star, i.e., the operation which maps $(A,B,C)$ into $\bigcup_{n\geqq 0}^{\infty}A^nBC^n$. This is the family of sets obtained by letting $x$ and $y$ in (c) of Theorem 3.1 be *sets* instead of words. By Theorem 3.1, $J$ contains the bounded definable sets. The converse is not true. For consider the definable set $\theta(a,b)$. By Corollary 2 of Theorem 3.1, $\theta(a,b)$ is not bounded. However $\theta(a,b)=\bigcup_{n\geqq 0}\{a,b\}^n\{\varepsilon\}\{\varepsilon\}^n$ and so is in $J$. (In fact, $J$ contains every regular set. For $J$ contains the finite sets and is closed under union, product, and star.) Now each set in $J$ is sequentially definable. For the finite sets are sequentially definable and each of the operations (a), (b), and (c) preserves sequential definability. Unfortunately (from the point of view of generating the sequentially definable sets by a finite number of particularly "simple" operations), there exist sequentially definable sets which are not in $J$ as shown by the following example.

EXAMPLE. Let $M=\{wcw^R/w \text{ in } \theta(a,b)-\varepsilon\}$ (14). $M$ is sequentially definable being the mfp of $f(\xi)=a\xi a+b\xi b+aca+bcb$. By a long and complicated argument it can be shown that $M$ is not in $J$.

For our second application of the structure results, we shall show that certain subsets of $a^*b^*c^*$ are not definable.

THEOREM 3.2. *Let $\Sigma=\{a,b,c\}$. There is no definable subset of $\{a^ib^jc^k/i\leqq j,k\leqq j\}$ which intersects $\{a^nb^nc^n/n\geqq 0\}$ infinitely often.*

(14) If $w=w_1\ldots w_k$, each $w_i$ in $\Sigma$, then $w^R=w_k\ldots w_1$.

**Proof.** Suppose that $L$ is a definable subset of

$$M = \{a^i b^j c^k / i \leqq j,\ k \leqq j\}$$

which intersects $\{a^n b^n c^n / n \geqq 0\}$ infinitely often. By Lemma 2.5, $L = \bigcup_1^r L(D_i, E_i, F_i)$, where $D_i, E_i, F_i$ are definable subsets of $a^*c^*$, $a^*b^*$, and $b^*c^*$ respectively. Since $L \cap \{a^n b^n c^n / n \geqq 0\}$ is infinite, there exists an integer $s$ such that

(1) $$L(D_s, E_s, F_s) \cap \{a^n b^n c^n / n \geqq 0\} \text{ is infinite.}$$

By definition, $L(D_s, E_s, F_s) = \bigcup_{a^i c^j \text{ in } D_s} a^i E_s F_s c^j$. Let $I$ be the set of $i$ for which there exists an integer $j(i)$ such that $a^i E_s F_s c^{j(i)} \subseteq L(D_s, E_s, F_s)$, $a^i c^{j(i)}$ in $D_s$. Suppose that $I$ is infinite. Let $a^{m_0} b^{m_1}$ and $b^{m_2} c^{m_3}$ be specific words in $E_s$ and $F_s$ respectively. Then there is some $i$ in $I$ for which $i + m_0 > m_1 + m_2$ and $a^{i+m_0} b^{m_1+m_2} c^{m_3+j(i)}$ is in $L$. This contradicts the fact that $L \subseteq M$. Therefore $I$ is finite. Similarly, the set $J$ of those integers $j$ for which there exists an integer $i(j)$ such that $a^{i(j)} E_s F_s c^j \subseteq L(D_s, E_s, F_s)$, $a^{i(j)} c^j$ in $D_s$, is finite. Thus $L(D_s, E_s, F_s)$ is the finite union of sets of the form $a^i E_s F_s c^j$, $a^i c^j$ in $D_s$, namely

(2) $$L(D_s, E_s, F_s) = \bigcup_{\text{some } (i,j) \text{ in } I \times J} a^i E_s F_s c^j.$$

Hence there exist integers $i_0$ and $j_0$ such that

(3) $$a^{i_0} E_s F_s c^{j_0} \subseteq L(D_s, E_s, F_s),\ a^{i_0} c^{j_0} \text{ in } D_s,$$

and

(4) $$a^{i_0} E_s F_s c^{j_0} \cap \{a^n b^n c^n / n \geqq 0\} \text{ is infinite.}$$

It follows from (4) that for each $i \geqq 1$, there exist words $w_i = a^{\alpha(i)} b^{\beta(i)}$ in $E_s$ and $y_i = b^{\gamma(i)} c^{\delta(i)}$ in $F_s$ such that

(5) $$i_0 + \alpha(i) = \beta(i) + \gamma(i) = \delta(i) + j_0,$$

with $i_0 + \alpha(i) < i_0 + \alpha(j)$ when $i < j$. Thus $\{\alpha(i)\}_{i \geqq 1}$ is an increasing infinite sequence, i.e., $\alpha(i) < \alpha(j)$ when $i < j$. Since $a^{i_0} E_s F_s c^{j_0} \subseteq M$,

(6) $$i_0 + \alpha(i) \leqq \beta(i) + \gamma(j),$$

and

(7) $$j_0 + \delta(j) \leqq \beta(i) + \gamma(j)$$

for all $i$ and $j$.

Two cases arise:

($\alpha$) There exists an increasing infinite sequence $\{i_j\}_{j \geqq 1}$ such that for each $i_j$

(8) $$\beta(i_j) \leqq 3/4[i_0 + \alpha(i_j)].$$

Since the $\alpha(i)$ are increasing, there exists $i_n$ so that $i_0 + \alpha(i_n) > 4\gamma(i_1)$. Then

$$(9) \qquad i_0 + \alpha(i_n) = 3/4[i_0 + \alpha(i_n)] + 1/4[i_0 + \alpha(i_n)] > \beta(i_n) + \gamma(i_1).$$

But (9) contradicts (6).

($\beta$) There exists an increasing infinite sequence $\{i_j\}_{j \geqq 0}$ so that for each $i_j$,

$$(10) \qquad \beta(i_j) \geqq 3/4[i_0 + \alpha(i_j)],$$

whence

$$(11) \qquad \gamma(i_j) \leqq 1/4[i_0 + \alpha(i_j)] = 1/4[j_0 + \delta(i_j)].$$

There exists $i_n$ so that

$$(12) \qquad j_0 + \delta(i_n) = i_0 + \alpha(i_n) > 4/3[j_0 + \delta(i_1)].$$

From (5), it follows that

$$(13) \qquad \beta(i_1) \leqq j_0 + \delta(i_1).$$

Then

$$\begin{aligned} \beta(i_1) + \gamma(i_n) &\leqq j_0 + \delta(i_1) + 1/4[j_0 + \delta(i_n)], \text{ by (13) and (11)}, \\ (14) \qquad &< 3/4[j_0 + \delta(i_n)] + 1/4[j_0 + \delta(i_n)], \text{ by (12)}, \\ &= j_0 + \delta(i_n), \end{aligned}$$

which contradicts (7).

REMARK. By a similar method we can show that there is no definable subset of $\{a^i b^j c^k / i \geqq j \geqq k\}$ which intersects $\{a^n b^n c^n / n \geqq 0\}$ infinitely often.

The statements in both the theorem and the remark remain true, of course, if each occurence of $c$ is replaced by $a$. In this form the remark implies the result, proved in the appendix of [6] by a different method, that $\{a^i b^j a^k / i \geqq j \geqq k\}$ is not definable.

4. **Machine mapping.** As noted earlier, a gsm maps a definable set into a definable set. It will now be shown that a gsm maps a bounded definable set into a bounded definable set.

LEMMA 4.1. *If $S$ is a gsm and $a$ is in $\Sigma$, then $S(a^*)$ is bounded.*

**Proof.** Let $S = (K, \Sigma, \Delta, \delta, \lambda, q_0)$. For $i \geqq 1$ let $q_i = \delta(q_{i-1}, a)$. Thus $q_i = \delta(q_0, a^i)$ for each $i$. Since there are only a finite number of states there is a smallest integer, say $k$, such that $q_k = q_i$ for an infinite number of $i$. Then there is a smallest integer $p > 0$ such that $q_k = q_{k+p}$. Let $x = \lambda(q_0, a^{k-1})$ and $y_j = \lambda(q_k, a^j)$ for $0 \leqq j \leqq p$. It is easily seen that

$$S(a^*) = \{S(a^0), \cdots, S(a^{k-1})\} \cup \bigcup_{j=0}^{p-1} x y_p^* y_j.$$

Therefore $S(a^*)$ is bounded.

LEMMA 4.2. *For each gsm S and each sequence* $a_1, \cdots, a_n$ *of elements of* $\Sigma$, $S(a_1^* \cdots a_n^*)$ *is bounded.*

**Proof.** By Lemma 4.1, the result is true for $n = 1$. Suppose that the lemma is true for $k \geqq 1$. Let $S_1 = (K, \Sigma, \Delta, \delta, \lambda, q_1)$ be a gsm with states $q_1, \cdots, q_m$. For $1 \leqq i \leqq m$ let $S_i$ be the gsm $(K, \Sigma, \Delta, \delta, \lambda, q_i)$. By induction, $S_1(a_1^*)$ and $S_i(a_2^* \cdots a_{k+1}^*)$ are bounded for each $i$. Therefore the set

$$Y = \bigcup_1^m S_i(a_2^* \cdots a_{k+1}^*)$$

is bounded. Then $S_1(a_1^*)Y$ is bounded. Since $S_1(a_1^* \cdots a_{k+1}^*)$ is a subset of $S_1(a_1^*)Y$, $S_1(a_1^* \cdots a_{k+1}^*)$ is bounded. Hence the result.

LEMMA 4.3. $S(w_1^* \cdots w_n^*)$ *is bounded for each gsm S and all words* $w_1, \cdots, w_n$.

**Proof.** Let $S = (K, \Sigma, \Delta, \delta, \lambda, q_1)$ and $w_1, \cdots, w_n$ be words. Let $a_1, \cdots, a_n$ be $n$ symbols and $\Sigma' = \{a_i / 1 \leqq i \leqq n\}$. Let $S'$ be the one state gsm $(\{p_1\}, \Sigma', \Sigma, \delta', \lambda', p_1)$ which maps each $a_i$ into $w_i$. Consider the composite machine $T$ of $S'$ and $S$. That is, consider $T = (p_1 \times K, \Sigma', \Delta, \delta_T, \lambda_T, (p_1, q_1))$ where

$$\delta_T((p_1, q), a_i) = (p_1, \delta_S(q, w_i))$$

and

$$\lambda_T((p_1, q), a_i) = \lambda_S(q, w_i)$$

for each $q$ in $K$ and $a_i$ in $\Sigma'$. Clearly $T(a_1^* \cdots a_n^*) = S(w_1^* \cdots w_n^*)$. By Lemma 4.2, therefore, $S(w_1^* \cdots w_n^*)$ is bounded.

COROLLARY. $S(X)$ *is bounded for each bounded set X and gsm S.*

**Proof.** Let $X \subseteq w_1^* \cdots w_n^*$. By Lemma 4.3, $S(w_1^* \cdots w_n^*)$ is bounded. Since $S(X) \subseteq S(w_1^* \cdots w_n^*)$, $S(X)$ is bounded.

From the corollary we get

THEOREM 4.1. $S(L)$ *is a bounded definable set for each bounded definable set L and each gsm S.*

5. **Recognition.** We now consider the problem of determining of a given definable set whether or not it is bounded. We shall show that there is a decision procedure. We shall also give a reasonably simple characterization of bounded definable sets.

We first prove three lemmas concerning the commutativity and noncommutativity of words.

LEMMA 5.1. *Let u and v be words in* $\theta(\Sigma)$. *Then the following statements are equivalent.*

(a) *u and v commute, i.e.,* $uv = vu$.

(b) $u^p = v^q$ *for some* $p, q \geqq 1$.

(c) $u = w^r$, $v = w^s$ *for some word* $w$ *and some* $r, s \geqq 1$.

**Proof.** Suppose that $uv = vu$. Let $p = |v|$ and $q = |u|$. Since $uv = vu$, $u^p v^q = v^q u^p$. Then $u^p$ and $v^q$ are both words of length $pq$ which are initial subwords of the same word. Thus $u^p = v^q$, i.e., (a) implies (b).

Suppose that $u^p = v^q$ for some $p, q \geqq 1$. Let $d$ be the greatest common divisor of $|u|$ and $|v|$. Then $|u| = dr$ and $|v| = ds$, with $r$ and $s$ relatively prime. Let $u^p = v^q = w_1 \cdots w_{rp}$, where each $w_i$ is of length $d$. Let $1 < g \leqq rp$. To prove (b) implies (c) it suffices to show that $w_g = w_1$. For since $g$ is arbitrary, it will follow that $u = (w_1)^r$ and $v = (w_1)^s$. As $r$ and $s$ are relatively prime, there exist integers $k_1$ and $k_2$ so that $1 = k_1 r + k_2 s$. We may assume that $k_1 \geqq 0$ and $k_2 \leqq 0$. (For otherwise, reverse the roles of $u$ and $v$.) Then

$$g - 1 = (g - 1)k_1 r + (g - 1)k_2 s.$$

Denote by $h$ the positive integer $(g - 1)(k_1 r - k_2 s)$. Let $u^{ph} = v^{qh} = w_1 \cdots w_{rph}$, each $w_i$ of length $d$. For $1 \leqq j \leqq ph$ and $1 \leqq i \leqq r$, $w_i = w_{i+(j-1)r}$ since $u^j = u^{j-1} w_1 \cdots w_r$. Similarly $w_i = w_{i+(j-1)s}$ for $1 \leqq j \leqq qh$ and $1 \leqq i \leqq s$. Thus $w_i = w_j$ if $i \equiv j \bmod r$ or $i \equiv j \bmod s$. Then

$$\begin{aligned} w_1 &= w_{1+(g-1)k_1 r} \\ &= w_{1+(g-1)k_1 r + (g-1)k_2 s} \\ &= w_g. \end{aligned}$$

Finally, suppose that $u = w^r$ and $v = w^s$ for some word $w$ and some $r, s \geqq 1$. Then $uv = vu = w^{r+s}$. Thus (c) implies (a).

LEMMA 5.2. *If $U$ is a commutative subset of $\theta(\Sigma)$, i.e., $uv = vu$ for each two words in $U$, then there is some word $u$ in $\theta(\Sigma)$ such that $U \subseteq u^*$.*

**Proof.** The lemma is true if $U = \phi$ or if $U = \{\varepsilon\}$. Therefore let $u_1 \neq \varepsilon$ be a word in $U$. Let $w$ be a subword of $u_1$ of smallest length so that $u_1$ is a power of $w$. By Lemma 5.1, $w$ commutes with each word $u$ in $U$. Let $u$ be an arbitrary word in $U$. Since $uw = wu$, $u$ and $w$ are both powers of some word $w_1$. Then $u_1$ is a power of $w_1$ and $w_1$ is a subword of $w$. Thus $w_1$ is a subword of $u_1$. By the minimality of $w$, $w = w_1$. Thus $u$ is a power of $w$, i.e., each word in $U$ is a power of $w$.

LEMMA 5.3. *Let $u$ and $v$ be two words such that $uv \neq vu$. Let $X$ be a set with the property that each word in $\{u, v\}^*$ is a subword of some word in $X$. Then $X$ is not bounded.*

**Proof.** Let $y = u^{|v|}$ and $z = v^{|u|}$. Then $y$ and $z$ are of the same length. Since $uv \neq vu$, $y \neq z$ by Lemma 5.1. Let $y = y_1 \cdots y_r$ and $z = z_1 \cdots z_r$, each $y_i$ and $z_i$

in $\Sigma$. Since $y \neq z$, there is a smallest integer $k$ such that $y_k \neq z_k$ and $y_i = z_i$ for $i < k$. Let $a_1$ and $a_2$ be two symbols. Denote by $S$ the gsm $(K,\Sigma,\{a_1,a_2\},\delta,\lambda,q_1)$ defined as follows. The states of $S$ are $q_1,\cdots,q_r$. For $x$ in $\Sigma$, $\delta(q_i,x)=q_{i+1}$ $(1 \leqq i < r)$ and $\delta(q_r,x) = q_1$. $\lambda(q_k,y_k) = a_1$, $\lambda(q_k,z_k) = a_2$, and $\lambda = \varepsilon$ otherwise. Then $S(\{y,z\}^*) = \theta(a_1,a_2)$. Thus $S(\{u,v\}^*) = \theta(a_1,a_2)$.

Suppose that $\{u,v\}^*$ is bounded. By Theorem 4.1, $S(\{u,v\}^*) = \theta(a_1,a_2)$ is bounded. But by Corollary 2 of Theorem 3.1, $\theta(a_1,a_2)$ is not bounded. Thus $\{u,v\}^*$ is not bounded. Since each word in $\{u,v\}^*$ is a subword of a word in $X$, $X$ is not bounded.

In the remainder of this section we shall be considering definable sets as defined by productions. From this point of view they are called "context free languages" in the literature [2].

A *grammar* $G$ is a 4-tuple $(V,P,\Sigma,\sigma)$ where $V$ is a finite set, $\Sigma$ is a subset of $V$, $\sigma$ is an element of $V - \Sigma$, and $P$ is a finite set of ordered pairs of the form $(\xi, w)$ with $\xi$ in $V - \Sigma$ and $w$ in $\theta(V)$. $P$ is called the set of *productions* in $G$ and an element $(\xi, w)$ in $P$ is denoted by $\xi \to w$. The elements of $V - \Sigma$ are called *variables*. If $y,z$ are in $\theta(V)$, we write $y \Rightarrow z$ if $y = u\xi v$, $z = uwv$, and $\xi \to w$. We write $y \Rightarrow^* z$ if either $y = z$ or if there exists a sequence of words $z_0,\cdots,z_r$, called a *derivation* of $y \Rightarrow^* z$, such that $y = z_0$, $z_r = y$, and $z_i \Rightarrow z_{i+1}$ for each $i$. Denote by $L(G)$ the set of words $\{w/\sigma \Rightarrow^* w,\ w \text{ in } \theta(\Sigma)\}$.

It is known [4] that the family of $L(G)$ is exactly the family of definable sets. The correspondence is as follows. Let $G = (V,P,\Sigma,\sigma)$ be a grammar with $V - \Sigma = \{\xi_1,\cdots,\xi_n = \sigma\}$. For each $i$ let $f_i = \sum_{\xi_i \to w} w$. Then $f(\xi_1,\cdots,\xi_n) = (f_1,\cdots,f_n)$ is an $n$-tuple standard function whose mfp has $L(G)$ as its last coordinate. Conversely, suppose that $f(\xi_1,\cdots,\xi_n) = (f_1,\cdots,f_n)$ is an $n$-tuple standard function, each term of each $f_i$ being a word in $\theta(V)$, where $V = \Sigma \cup \{\xi_1,\cdots,\xi_n\}$. Let $\sigma = \xi_n$ and $P$ consist of all productions $\xi_i \to w$, where $w$ is a term in $f_i$. Then $L(G)$ is the last coordinate in the mfp of $f(\xi_1,\cdots,\xi_n)$, where $G = (V,P,\Sigma,\sigma)$.

In the sequel *we shall assume that* $G = (V,P,\Sigma,\sigma)$, *that* $\sigma$ *depends on each variable in* $G$(15), *and that* $W_\xi = \{w/\xi \Rightarrow^* w,\ w \text{ in } \theta(\Sigma)\}$ *is nonempty for each variable* $\xi \neq \sigma$. This is no loss of generality. For if $W_\xi = \phi$ for some variable $\xi \neq \sigma$, or if $\sigma$ does not depend on the variable $\xi$, then $L(G) = L(G')$, where $G' = (V - \{\xi\}, P', \Sigma, \sigma)$ and $P'$ consists of all productions in $P$ which do not involve $\xi$. Furthermore, it is easily seen from the definition of dependency that we can effectively decide whether or not $\sigma$ depends on a given variable. By Theorem 5.2 of [1] it is also decidable whether or not $W_\xi = \phi$ for a variable $\xi$.

LEMMA 5.4. *Suppose that* $y_1 = w_1 \cdots w_r$, *each* $w_i$ *in* $\theta(V)$, *and that*

$$(*) \qquad y_1,\cdots,y_k$$

(15) $\sigma$ depends of a variable $\xi$ if $\sigma$ depends on $\xi$ in the corresponding $n$-tuple standard function.

*is a derivation of* $y_1 \Rightarrow^* y_k$. *Then there exist words* $w_1', \cdots, w_r'$ *in* $\theta(V)$ *such that* $y_k = w_1' \cdots w_r'$ *and* $w_i \Rightarrow^* w_i'$ *for each* $i$. *Furthermore, for each* $i$ *there exists a derivation of* $w_i \Rightarrow^* w_i'$ *which involves only productions arising in* (*).

The proof of the lemma is obvious and is omitted.

We now resume our presentation of the recognition problem of bounded definable sets.

NOTATION. For each grammar $G$ and variable $\xi$ let

$$Y_\xi(G) = \{u/u \text{ in } \theta(\Sigma),\ \xi \Rightarrow^* u\xi v \text{ for some } v \text{ in } \theta(\Sigma)\}$$

and

$$Z_\xi(G) = \{v/v \text{ in } \theta(\Sigma),\ \xi \Rightarrow^* u\xi v \text{ for some } u \text{ in } \theta(\Sigma)\}.$$

LEMMA 5.5. *If* $L(G)$ *is nonempty and bounded, then* $Y_\xi(G)$ *and* $Z_\xi(G)$ *are both commutative for each variable* $\xi$.

**Proof.** For each variable $\xi$, let $W_\xi = \{w/w \text{ in } \theta(\Sigma),\ \xi \Rightarrow^* w\}$. Since $\sigma$ depends on $\xi$, there exist $u$, $v$ in $\theta(\Sigma)$ such that $uW_\xi v \subseteq L(G)$. Thus $W_\xi$ is nonempty and bounded. Let $w_0$ be a specific word in $W_\xi$.

Consider the set $Y_\xi(G)$. Suppose that there exist words $u_1, u_2$ in $Y_\xi(G)$ so that $u_1u_2 \neq u_2u_1$. There exist $v_1, v_2$ in $\theta(\Sigma)$ so that $\xi \Rightarrow^* u_1\xi v_1$ and $\xi \Rightarrow^* u_2\xi v_2$. By iteration of $\xi \Rightarrow^* u_1\xi v_1$ and $\xi \Rightarrow^* u_2\xi v_2$, it is easily seen that for each $w$ in $\{u_1,u_2\}^* - \varepsilon$ there exists $w'$ in $\theta(\Sigma)$ so that $\xi \Rightarrow^* w\xi w'$, thus $\xi \Rightarrow^* ww_0w'$. Thus $\{u_1,u_2\}^* - \varepsilon \subseteq Y_\xi(G)$. Clearly $\varepsilon$ is also in $Y_\xi(G)$. Thus each word in $\{u_1,u_2\}^*$ is a subword of some word in $W_\xi$. By Lemma 5.3, $W_\xi$ is not bounded. This is a contradiction. Therefore $u_1u_2 = u_2u_1$ for every two words $u_1, u_2$ in $Y_\xi(G)$, i.e., $Y_\xi(G)$ is a commutative set.

A similar argument shows that $Z_\xi(G)$ is commutative.

The necessary condition of Lemma 5.5 is also sufficient.

LEMMA 5.6. *If* $Y_\xi(G)$ *and* $Z_\xi(G)$ *are both commutative for each variable* $\xi$, *then* $L(G)$ *is bounded.*

**Proof.** We shall prove the lemma by induction on the number of variables. First suppose that $\sigma$ is the only variable. By hypothesis, $Y_\sigma(G)$ is a commutative set. Thus, by Lemma 5.2, $Y_\sigma(G) \subseteq u^*$ for some word $u$ in $\theta(\Sigma)$. Similarly $Z_\sigma(G) \subseteq v^*$ for some word $v$ in $\theta(\Sigma)$. Let $w_1, \cdots, w_t$ be the finite number of words in $\theta(\Sigma)$ for which $\sigma \to w_i$ is in $G$. Let $y$ be any word in $L(G)$ and let

$$\sigma = y_1, \cdots, y_r = y$$

be any derivation of $y$. Then

$$\sigma \Rightarrow^* y_{r-1} = u_1\sigma v_1 \Rightarrow u_1w_iv_1 = y_r$$

for some $u_1, v_1$ in $\theta(\Sigma)$ and some $w_i$. Since $u_1$ is in $Y_\sigma(G) \subseteq u^*$ and $v_1$ is in $Z_\sigma(G) \subseteq v^*$,

$$L(G) \subseteq \bigcup_{i=1}^{t} u^* w_i v^*.$$

Therefore $L(G)$ is bounded.

Suppose that $G$ has $n$ variables, $n > 1$, and that the lemma is true for all grammars with fewer than $n$ variables. For each variable $\xi \neq \sigma$, let $G_\xi$ be the grammar $(V - \{\sigma\}, P_\xi, \Sigma, \xi)$, where $P_\xi$ is the set of productions $v \to w$ in $P$ which do not involve $\sigma$(16). Now any derivation in $G_\xi$ is also a derivation in $G$. Thus the sets $Y_v(G_\xi)$ and $Z_v(G_\xi)$, $v$ a variable in $G_\xi$, are contained in $Y_v(G)$ and $Z_v(G)$ respectively. By assumption, $Y_v(G)$ and $Z_v(G)$ are both commutative. Therefore $Y_v(G_\xi)$ and $Z_v(G_\xi)$ are both commutative. By the induction hypothesis, $L(G_\xi)$ is bounded.

Let $\Gamma$ be the set of those words $\gamma$ in $\theta(V - \{\sigma\})$ such that $\sigma \to \gamma$ is in $G$. Thus $\Gamma$ is finite. For each element $x$ in $V - \{\sigma\}$, let $L_x = \{x\}$ if $x$ is in $\Sigma$ and $L_x = L(G_x)$ if $x$ is a variable. For $\gamma$ in $\Gamma$, let $L_\gamma = \{\varepsilon\}$ if $\gamma = \varepsilon$, and $L_\gamma = L_{x_1} \cdots L_{x_t}$ if $\gamma = x_1 \cdots x_t$, each $x_i$ in $V - \{\sigma\}$. Since each $L_x$ is bounded, $L_\gamma$ is bounded.

Consider $L(G)$. By assumption, $Y_\sigma(G)$ and $Z_\sigma(G)$ are both commutative. By Lemma 5.2, $Y_\sigma(G)$ and $Z_\sigma(G)$ are bounded. Thus

$$M = \bigcup_{\gamma \text{ in } \Gamma} Y_\sigma(G) L_\gamma Z_\sigma(G)$$

is bounded. To complete the induction, it suffices to show that $L(G) \subseteq M$. For then $L(G)$ is bounded. Thus let $w$ be any word in $L(G)$ and

$$\sigma = y_1, \cdots, y_m = w$$

be a derivation of $\sigma \Rightarrow^* w$. Let $p$ be the largest integer such that $\sigma$ occurs in $y_p$. Then $y_p = v_1 \sigma v_2$ and $y_{p+1} = v_1 \gamma v_2$, where $v_1$, $v_2$, and $\gamma$ are in $\theta(V - \{\sigma\})$ and $\sigma \to \gamma$. Then

$$v_1 \gamma v_2, y_{p+2}, \cdots, y_m = w$$

is a derivation of $v_1 \gamma v_2 \Rightarrow^* w$ involving no production with an occurence of $\sigma$. By Lemma 5.4, $w = uw'v$, where $v_1 \Rightarrow^* u$, $v_2 \Rightarrow^* v$, and $\gamma \Rightarrow^* w'$. Furthermore, there is a derivation of $\gamma \Rightarrow^* w'$ involving no production with an occurrence of $\sigma$. Since $\sigma \Rightarrow^* v_1 \sigma v_2 \Rightarrow^* u \sigma v$, $u$ is in $Y_\sigma(G)$ and $v$ is in $Z_\sigma(G)$. If $\gamma = \varepsilon$, then $w' = \varepsilon$ and $w$ is in $M$. Suppose that $\gamma \neq \varepsilon$, say $\gamma = z_1 \cdots z_t$, each $z_i$ in $V - \{\sigma\}$. By Lemma 5.4, there exist words $w_1, \cdots, w_t$ so that $w' = w_1 \cdots w_t$ and there is a derivation, involving no production with an occurence of $\sigma$, of $z_i \Rightarrow^* w_i$ for each $i$. Thus each $w_i$ is in $L(G_{z_i})$, so that $w'$ is in $Y_\sigma(G) L_\gamma Z_\sigma(G) \subseteq M$. Thus $L(G) \subseteq M$. Q.E.D.

By Lemmas 5.5 and 5.6 we get

(16) It may happen that $\xi$ does not depend on all the variables in $G_\xi$ or that there are variables $v \neq \xi$ such that $\{w/v \Rightarrow^* w \text{ in } G_\xi,\ w \text{ in } \theta(\Sigma)\} = .\phi$ In either case, we may effectively remove the "superfluous" variables and call the resulting grammar $G_\xi$.

**Theorem** 5.1. *A necessary and sufficient condition that $L(G) \neq \phi$ be bounded is that $Y_\xi(G)$ and $Z_\xi(G)$ both be commutative for each variable $\xi$.*

We need two additional lemmas in order to obtain a decision procedure for deciding of a given definable set whether or not it is bounded.

**Lemma** 5.7. *For each variable $\xi$, $Y_\xi(G)$ and $Z_\xi(G)$ are definable sets and are effectively determined.*

**Proof.** Let $\xi$ be a variable. It suffices to consider only $Y_\xi(G)$. Let $\xi'$ be a symbol not in $V$. Let $G_\xi = (V, P, \Sigma, \xi)$ and $G' = (V \cup \{\xi'\}, P', \Sigma \cup \{\xi'\}, \xi)$, where $P' = P \cup \{\xi \to \xi'\}$. Then

$$L(G') = L(G_\xi) \cup \{u\xi'v/\xi \Rightarrow^* u\xi'v \text{ in } G', u \text{ and } v \text{ in } \theta(\Sigma \cup \{\xi'\})\}.$$

Let $L_1 = L(G') \cap [\theta(\Sigma)\xi'\theta(\Sigma)]$. Then

$$L_1 = \{u\xi'v/\xi \Rightarrow^* u\xi v \text{ in } G, u \text{ and } v \text{ in } \theta(\Sigma)\}.$$

Since $\theta(\Sigma)\xi'\theta(\Sigma)$ is regular, by Theorem 8.1 of [1] $L_1$ is definable and *effectively calculable from* $L(G')$. Let $S$ be the gsm $(\{p_1, p_2\}, \Sigma \cup \{\xi'\}, \Sigma, \delta, \lambda, p_1)$ defined by $\delta(p_1, \xi') = \delta(p_2, \xi') = p_2$, $\lambda(p_1, \xi') = \lambda(p_2, \xi') = \varepsilon$, $\delta(p_1, x) = p_1$, $\lambda(p_1, x) = x$, $\delta(p_2, x) = p_2$, and $\lambda(p_2, x) = \varepsilon$, $x$ in $\Sigma$. From Theorem 3.1 of [6], $S(L_1)$ is definable and *effectively calculable from* $L_2$. But

$$\begin{aligned} S(L_1) &= \{u/\xi \Rightarrow^* u\xi v \text{ in } G, u \text{ and } v \text{ in } \theta(\Sigma)\} \\ &= Y_\xi(G). \end{aligned}$$

Hence the result.

**Lemma** 5.8. *Let $U$ be a given definable set. It is solvable to determine whether $U$ is commutative; and if $U$ is commutative then a word $u$ in $\theta(\Sigma)$ can be effectively found so that $U \subseteq u^*$.*

**Proof.** We can decide if $U$ is empty or not. Clearly we need only treat the case when $U$ is nonempty. By §§4 and 5 of [1], we can effectively determine $U - \{\varepsilon\}$, test $U - \{\varepsilon\}$ for emptiness, and if $U - \{\varepsilon\}$ is nonempty find a word $w$ in $U - \{\varepsilon\}$. (If $U - \{\varepsilon\}$ is empty, then $U = \{\varepsilon\}$.) Let $w_1, \cdots, w_s$ be the non-$\varepsilon$ initial subwords of $w$. By Lemma 5.2, $U$ is commutative if and only if $U \subseteq u^*$ for some word $u$ in $\theta(\Sigma)$. Clearly each such word $u$ is an initial subword of each non-$\varepsilon$ word in $U$. Thus $U$ is commutative if and only if $U \subseteq w_i^*$ for some $i$. To complete the proof it suffices to show that for each $i$, $U \subseteq w_i^*$ is decidable.

Now $w_i^*$ is regular. Thus $U - w_i^*$ is definable and is effectively determined. Thus it can be decided whether or not $U - w_i^*$ is empty. The lemma then follows from the fact that $U \subseteq w_i^*$ if and only if $U - w_i^*$ is empty.

THEOREM 5.2. (a) *It is decidable whether or not a given* $L(G)$ *is bounded.* (b) *If* $L(G)$ *is bounded, then words* $w_1, \cdots, w_t$ *in* $\theta(\Sigma)$ *can be effectively found so that* $L(G) \subseteq w_1^* \cdots w_t^*$.

**Proof.** (a) By Theorem 5.1, $L(G)$ is bounded if and only if $L(G) = \phi$ or $Y_\xi(G)$ and $Z_\xi(G)$ are both commutative for each variable $\xi$. It is decidable whether $L(G) = \phi$. Suppose that $L(G) \neq \phi$. By Lemma 5.7, $Y_\xi(G)$ and $Z_\xi(G)$ can be effectively found. By Lemma 5.8, it is decidable whether or not $Y_\xi(G)$ and $Z_\xi(G)$ are both commutative. Thus it is decidable whether or not $L(G)$ is bounded.

(b) Suppose that $G$ contains just one variable. By Lemmas 5.7 and 5.8, words $u$ and $v$ can be effectively found so that $Y_\sigma(G) \subseteq u^*$ and $Z_\sigma(G) \subseteq v^*$. By the proof of Lemma 5.6, $L(G) \subseteq \bigcup_{i=1}^m u^* z_i v^*$, where $z_1, \cdots, z_m$ are all the words $z$ such that $\sigma \rightarrow z$. From this, the required words $w_1, \cdots, w_t$ can readily be found.

Suppose that (b) is true for all grammars with fewer than $n > 1$ variables and that $G$ has $n$ variables. Let $\Gamma$, $L(G_x)$, and $L_\gamma$ be the same as in Lemma 5.6. By the proof of Lemma 5.6,

$$L(G) \subseteq \bigcup_{\gamma \text{ in } \Gamma} Y_\sigma(G) L_\gamma Z_\sigma(G) = M.$$

Now $\Gamma$, each $L(G_x)$, and each $L_\gamma$ are effectively determined. By Lemmas 5.7 and 5.8, words $u$ and $v$ in $\theta(\Sigma)$ can be effectively found so that $Y_\sigma(G) \subseteq u^*$ and $Z_\sigma(G) \subseteq v^*$. By induction, (b) is true for each $L(G_x)$, thus for each $L_\gamma$. Thus (b) is true for $M$ and hence for $L(G)$.

6. **Intersection and complement of semi-linear sets.** By Theorem 6.3 of [1], it is recursively unsolvable to determine of arbitrary definable sets $L_1$ and $L_2$ whether (a) $L_1 \subseteq L_2$, and whether (b) $L_1 = L_2$. We shall see that (a) and (b) are solvable when the languages are bounded. These results are consequences of certain theorems involving intersection and complement of semi-linear sets which we shall prove in this section.

In general, the intersection and the complement of definable sets are not definable. The classical example is $X = \{a^n b^n c^i / n, i \geqq 0\}$ and $Y = \{a^i b^n c^n / n, i \geqq 0\}$. (These two sets are also bounded.) It is known that $X \cap Y$ is not definable [11]. Since the union of definable sets is definable, the complement of a definable set is not necessarily definable. For if it were, then the intersection of definable sets would be definable. The complement of a bounded set with respect to $\theta = \theta(a_1, \cdots, a_n)$, $n \geqq 2$, is *never* bounded. For if the complement $\theta - X$ of some bounded set $X$ were bounded, then $\theta = X + (\theta - X)$ would be bounded, contradicting Corollary 2 of Theorem 3.1. However, it will follow from our results on the intersection and complement of semi-linear sets that the intersection and difference of definable subsets of $w_1^* w_2^*$ are definable.

NOTATION. For elements $u=(u_1,\cdots,u_n)$ and $v=(v_1,\cdots,v_n)$ in $N^n$, we write $u \leqq v$ if $u_i \leqq v_i$ for each $i$.

The set $N^n$ is partially ordered by the relation $\leqq$. The following is a well-known result about $N^n$ [7, p. 168], and is included here (together with its corollary) as background material only.

LEMMA 6.1. *Each set of pairwise incomparable elements of $N^n$ is finite.*

Since the set of minimal elements of a subset of $N^n$ is a set of pairwise incomparable elements, we get

COROLLARY. *Each subset of $N^n$ has only a finite number of minimal elements.*

In the sequel we shall prove a number of results about semi-linear sets and definable sets. The proof of all the lemmas and theorems are effective(17). Except for Lemmas 6.4, 6.5, Theorems 6.1, 6.2, and 6.3, we shall omit reference to the effectivity. Lemmas 6.4, 6.5, and Theorem 6.3 involve the concept of effectivity. Theorems 6.1 and 6.2, while meaningful without explicitly stating the effectivity, are important enough facts to warrant the stating of the effectivity.

In order to treat semi-linear sets, we introduce the following concepts.

DEFINITION. Given subsets $C, P$ of $N^n$, let $L(C;P)$ denote the set of all $x$ in $N^n$ which can be represented in the form

$$x = x_0 + x_1 + \cdots + x_m,$$

with $x_0$ in $C$ and $x_1,\cdots,x_m$ a (possibly empty) sequence of elements of $P$. $C$ is called the set of *constants* and $P$ the set of *periods* of $L(C;P)$.

The set $L(C;P)$ may also be described as the set of all words $x$ in $N^n$ of the form $x = x_0 + \sum_1^m k_i x_i$, with $x_0$ in $C$, $x_1,\cdots,x_m$ elements of $P$, and each $k_i$ a non-negative integer.

NOTATION. For $x=(x_1,x_n)$ in $N^n$ and $y=(y_1,\cdots,y_m)$ in $N^m$ let $x \times y$ denote the element in $N^{n+m}$ defined by $(x \times y)_i$(18) $= x_i$ for $1 \leqq i \leqq n$ and $(x \times y)_i = y_{i-n}$ for $i > n$. For $X \subseteq N^n$ and $Y \subseteq N^m$ let $X \times Y$ denote the set

$$X \times Y = \{x \times y / x \text{ in } X,\ y \text{ in } Y\}.$$

Let $0^n$ denote the element in $N^n$ defined by $(0^n)_i = 0$ for each $1 \leqq i \leqq n$.

The next lemma summarizes some of the basic properties of the set $L(C;P)$.

LEMMA 6.2. (a) *If $C_1$, $C_2$, and $P$ are subsets of $N^n$, then*

$$L(C_1 \cup C_2; P) = L(C_1;P) \cup L(C_2;P).$$

(17) The proofs of Lemma 6.1 and its corollary are noneffective. Lemma 6.5 serves as its replacement in those instances where the minimal elements of certain subsets of $N^n$ are needed.

(18) If $u$ is an element of $N^k$, then $(u)_i$ denotes the $i$th coordinate of $u$.

(b) *If $C$, $P_1$, and $P_2$ are subsets of $N^n$, then*

$$L(L(C;P_1);P_2) = L(C;P_1 \cup P_2).$$

(c) *If $C_1, P_1$ are subsets of $N^n$ and $C_2$, $P_2$ subsets of $N^m$, then*

$$L(C_1;P_1) \times L(C_2;P_2) = L(C_1 \times C_2;(P_1 \times 0^m) \cup (0^n \times P_2)).$$

The proof is straightforward and is omitted.

Note that a set $X$ is linear if and only if there exist a finite set $P$ and a set $C$ containing just one element such that $X = L(C;P)$. It follows from (a) of Lemma 6.2 that if $C$ and $P$ are finite sets with $C$ nonempty then $L(C;P)$ is semi-linear.

From (a) and (b) of Lemma 6.2 there immediately follows

Corollary 1. *If $C$ is a semi-linear subset of $N^n$ and $P$ is a finite subset of $N^n$, then $L(C;P)$ is a semi-linear subset of $N^n$.*

From (c) of Lemma 6.2 there follows

Corollary 2. *If $X$ and $Y$ are semi-linear subsets of $N^n$ and $N^m$ respectively, then $X \times Y$ is a semi-linear subset of $N^{n+m}$.*

We need one more lemma (Lemma 6.3) to show that the intersection of two semi-linear subsets of $N^n$ is semi-linear. To show that the intersection can be effectively calculated, we need two additional lemmas (Lemmas 6.4 and 6.5).

Lemma 6.3. *Let $\tau$ be a linear function of $N^n$ into $N^m$*(19)*. If $X$ is a linear subset of $N^n$, then $\tau(X)$ is a linear subset of $N^m$. If $X$ is a semi-linear subset of $N^n$, then $\tau(X)$ is a semi-linear subset of $N^m$.*

**Proof.** Since a function commutes with union, it suffices to show the lemma for a linear set. The proof here follows from the fact that $\tau(x_0 + \sum x_i) = \tau(x_0) + \sum \tau(x_i)$, i.e., $\tau(L(\{x_0\};P)) = L(\{\tau(x_0)\};\tau(P))$.

Definition. Given $u_i$ $(1 \leqq i \leqq p)$ and $v_j$ $(1 \leqq j \leqq q)$ in $N^n$, and an $n$-tuple $w$ of integers; an element $(a_1, \cdots, a_p, b_1, \cdots, b_q)$ such that

$$(*) \qquad w = \sum^p a_i u_i - \sum^q b_j v_j$$

is said to be a *positive solution* of (*) if $(a_1, \cdots, a_p, b_1, \cdots, b_q)$ is in $N^{p+q} - \{0^{p+q}\}$.

Lemma 6.4. *It is solvable to determine for arbitrary $u_i$ $(1 \leqq i \leqq p)$ and $v_j$ $(1 \leqq j \leqq q)$ in $N^n$, and an arbitrary $n$-tuple $w$ of integers whether*

(1) *there exists a positive solution to $w = \sum^p a_i u_i - \sum^q b_j v_j$*(20).

(19) A function $\tau$ of $N^n$ into $N^m$ is said to be *linear* if $\tau(x+y) = \tau(x) + \tau(y)$ for all $x, y$ in $N^n$.

(20) If there exists a positive solution then a particular solution can be effectively found. This is done by effectively enumerating all elements of $N^{p+q} - \{0^{p+q}\}$ and testing each tuple until one is found which satisfies (1).

**Proof.** The lemma is proved by induction on $p+q$. If $p+q=1$, then clearly given $w$ and $u_1$ (or $w$ and $v_1$) it is solvable to determine whether a positive integer $a_1$ (or $b_1$) exists such that $w = a_1u_1$ (or $w = b_1v_1$).

Assume the lemma is true if $p+q<m$, where $m>1$. Now suppose that $p+q=m$. First assume that $u_i$ and $v_j$ are independent([21]). (This assumption can be effectively verified.) It can be effectively determined if $w$ is dependent on the $u_i$ and $v_j$. If it is not, then (1) has no solution. If it is, then rational numbers $r_i$ $(1 \leqq i \leqq p)$ and $s_j$ $(1 \leqq j \leqq q)$ can be effectively found so that

$$w = \sum^{p} r_iu_i + \sum^{q} s_jv_j. \tag{2}$$

Since the $u_i$ and $v_j$ are independent, the $r_i$ and $s_j$ are unique. Thus (1) has a positive solution if and only if $r_i$ and $-s_j$ are nonnegative integers with one of the $r_i$ or $-s_j$ positive.

Next assume that $u_i$ and $v_j$ are dependent. Then subsets $I \subseteq \{1,\cdots,p\}$, $J \subseteq \{1,\cdots,q\}$, and vectors $(r_1,\cdots,r_p)$ in $N^p$, $(s_1,\cdots,s_q)$ in $N^q$ can be effectively found such that

$$\sum_{i \text{ in } I} r_iu_i - \sum_{j \text{ in } J} s_jv_j = \sum_{i \text{ not in } I} r_iu_i - \sum_{j \text{ not in } J} s_jv_j \tag{3}$$

and

(4) for some $i$ in $I$ or $j$ in $J$ either $r_i$ or $s_j$ is positive.

Now (1) has a positive solution if and only if (1) has a positive solution such that either $a_i \leqq r_i$ for some $i$ in $I$ or $b_j \leqq s_j$ for some $j$ in $J$. For if $w = \sum a_iu_i - \sum b_jv_j$ is a solution in non-negative integers with $a_i > r_i$ and $b_j > s_j$ for all $i, j$; then (by (3) and (4))

$$\begin{aligned} w = &\sum_{i \text{ in } I} (a_i - r_i)u_i + \sum_{i \text{ not in } I} (a_i + r_i)u_i \\ &+ \sum_{j \text{ in } J} (b_j - s_j)v_j + \sum_{j \text{ not in } J} (b_j + s_j)v_j \end{aligned} \tag{5}$$

is a positive solution to (1) with the following property:

(6) Either the coefficient of $u_i$ is less than $a_i$ for some $i$ in $I$ or the coefficient of $v_j$ is less than $b_j$ for some $j$ in $J$.

Continuing in this way we ultimately obtain a positive solution to (1) of the desired type.

By induction, for each $i(0)$ in $I$ and each integer $k$ $(1 \leqq k \leqq r_{i(0)})$ it is solvable whether the equation

$$w - ku_{i(0)} = \sum_{i \neq i(0)} a_iu_i - \sum b_jv_j \tag{$a, i(0), k$}$$

([21]) $N^n$ is a subset of the vector space consisting of all $n$-tuples, with rational coordinates, over the field of rationals. Independence, linear combination, etc., is with respect to this underlying vector space.

has a positive solution, thus any solution in nonnegative integers. Similarly, for each $j(0)$ in $J$ and each $k'$ $(1 \leqq k' \leqq s_{j(0)})$, it is solvable whether the equation

$$(b,j(0),k') \qquad w + k'v_{j(0)} = \sum a_i u_i - \sum_{j \neq j(0)} b_j v_j$$

has a positive solution, thus any solution in nonnegative integers. Therefore (1) is effectively reduced to determining if there is a solution in nonnegative integers to one of the equations $(a,i(0),k)$ or $(b,j(0),k')$. (1) has an affirmative answer if and only if one of the $(a,i(0),k)$ or $(b,j(0),k')$ equations has a solution in nonnegative integers.

LEMMA 6.5. *Let $u_i$ $(1 \leqq i \leqq p)$ and $v_j$ $(1 \leqq j \leqq q)$ be in $N^n$, and let $w$ be a fixed n-tuple of integers. Then it is solvable to determine all minimal positive solutions to the equation*

$$(1) \qquad w = \sum^{p} a_i u_i - \sum^{q} b_j v_j.$$

**Proof.** The lemma is proved by induction on $p+q$. If $p+q=1$, then the lemma is obviously true. Suppose that the lemma is true if $p+q<m$, where $m>1$. Consider the case where $p+q=m$. By Lemma 6.4, it is solvable to determine if there exists a positive solution to (1). If there is no positive solution, then we are through. Suppose that there is a positive solution. As noted in footnote 20, we may effectively find a positive solution, thus a minimal one, say $\bar{v} = (\bar{a}_1, \cdots, \bar{a}_p, \bar{b}_1, \cdots, \bar{b}_q)$. For each integer $i(0)$, $1 \leqq i(0) \leqq p$ and $\bar{a}_{i(0)} > 0$, and each integer $k$, $0 \leqq k < \bar{a}_{i(0)}$, it follows from the induction hypothesis that it is solvable to determine all the minimal positive solutions to the equation

$$(a,i(0),k) \qquad w - ku_{i(0)} = \sum_{i \neq i(0)} a_i u_i - \sum b_j v_j.$$

Similarly, for each integer $j(0)$, $1 \leqq j(0) \leqq q$ and $\bar{b}_{j(0)} > 0$, and each integer $k'$, $0 \leqq k' < \bar{b}_{j(0)}$, it is solvable to determine all the minimal positive solutions to the equation

$$(b,j(0),k') \qquad w + k'v_{j(0)} = \sum a_i u_i - \sum_{j \neq j(0)} b_j v_j.$$

If $(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_p, b_1, \cdots, b_q)$ is a minimal positive solution to the equation $(a,i,k)$, then

$$(a_1, \cdots, a_{i-1}, k, a_{i+1}, \cdots, a_p, b_1, \cdots, b_q)$$

is called an *extended* minimal positive solution to $(a,i,k)$. Extended minimal positive solutions to $(b,j,k')$ are defined analogously.

Let $v' = (a'_1, \cdots, a'_p, b'_1, \cdots, b'_q)$ be any minimal positive solution of (1). Since $v'$ and $\bar{v}$ are incomparable, either (*) $a'_i < \bar{a}_i$ for some $i$ with $\bar{a}_i > 0$ or (**) $b'_j < \bar{b}_j$ for some $j$ with $\bar{b}_j > 0$. If (*) holds, let $v'(a,i)$ be the $(p+q-1)$-tuple obtained by deleting the $i$th coordinate of $v'$. Then $v'(a,i)$ is a minimal positive solution of the equation $(a,i,a'_i)$ and $v'$ is an extended minimal positive solution. If (**)

holds, it follows similarly that $v'$ is an extended minimal positive solution. Therefore, the minimal positive solutions of (1) are the minimal elements in the finite set consisting of $\bar{v}$ and all extended minimal positive solutions of the $(a,i,k)$ and $(b,j,k')$ equations. Thus it is solvable to determine all the minimal positive solutions of (1).

THEOREM 6.1. *Let $X$ and $X'$ be semi-linear subsets of $N^n$. Then $X \cap X'$ is a semi-linear subset of $N^n$ and is effectively calculable from $X$ and $X'$.*

**Proof.** Since intersection is distributive over union, it suffices to prove that $X \cap X'$ is semi-linear whenever $X$ and $X'$ are linear. Let $X$ have the single constant $x_0$ and the periods $x_1, \cdots, x_p$. Let $X'$ have the single constant $x_0'$ and the periods $x_1', \cdots, x_q'$. Let $y = (y_1, \cdots, y_p)$ and $z = (z_1, \cdots, z_q)$ represent typical elements of $N^p$ and $N^q$ respectively. Denote by $A$ and $B$ the subsets of $N^{p+q}$ defined by

$$A = \left\{ y \times z / x_0 + \sum_1^p y_i x_i = x_0' + \sum_1^q z_i x_i' \right\}$$

and

$$B = \left\{ y \times z / \sum_1^p y_i x_i = \sum_1^q z_i x_i' \right\}.$$

Let $\tau$ be the mapping of $N^{p+q}$ into $N^n$ defined by $\tau(y \times z) = \sum_1^p y_i x_i$. Then $\tau$ is a linear function and $X \cap X' = \{x_0 + u / u \text{ in } \tau(A)\}$. It suffices to show that $A$ is a semi-linear subset of $N^{p+q}$. For by Lemma 6.3, $\tau(A)$, whence $\{x_0 + u / u \text{ in } \tau(A)\}$, is semi-linear.

Let $C$ and $P$ be the set of minimal elements of $A$ and $B - 0^{p+q}$ respectively. By Lemmas 6.1 and 6.5, $C$ and $P$ are both finite and effectively calculable. Thus $L(C;P)$ is a semi-linear subset of $N^{p+q}$. We shall prove the theorem by showing that $A = L(C;P)$.

It is obvious that $L(C;P) \subseteq A$. To see the reverse inclusion, assume that $y \times z$ is in $A$. There exists $y' \times z'$ in $C$ such that $y' \times z' \leqq y \times z$. Let $y'' \times z''$ be the element in $N^{p+q}$ defined by $(y'' \times z'')_i = (y \times z)_i - (y' \times z')_i$ for each $i$. Then $y \times z = y' \times z' + y'' \times z''$. Furthermore

$$\begin{aligned}
\sum_1^p y_i'' x_i &= \sum_1^p (y_i - y_i') x_i \\
&= \sum_1^p y_i x_i - \sum_1^p y_i' x_i \\
&= (x_0' - x_0) + \sum_1^q z_i x_i' - \left[ (x_0' - x_0) + \sum_1^q z_i' x_i' \right] \\
&= \sum_1^q (z_i - z_i') x_i' \\
&= \sum_1^q z_i'' x_i'.
\end{aligned}$$

Thus $y'' \times z''$ is in $B$. It thus suffices to show that each element in $B$ is a sum of (zero or more) elements of $P$.

Now $0^{p+q}$ is in $B$ and is the sum of (zero) elements in $P$. Suppose that each element $y \times z$ in $B$ such that $\sum^p y_i + \sum^q z_i \leqq k$ is the sum of elements of $P$. Let $y \times z$ in $B$ be such that $\sum^p y_i + \sum^q z_i \leqq k+1$. By induction, we may assume that $\sum y_i + \sum z_i = k+1$. There exists $y' \times z'$ in $P$ such that $y' \times z' \leqq y \times z$. Then there exists $y'' \times z''$ in $B$ so that $y \times z = y' \times z' + y'' \times z''$. Since $y' \times z' \neq 0^{p+q}$, $\sum^p y_i'' + \sum^q z_i'' \leqq k$. By induction, $y'' \times z''$ is the sum of elements of $P$. Thus $y \times z$ is the sum of elements of $P$.

Corollary 1. *If $L_1$ and $L_2$ are definable subsets of $a^*b^*$, $a$ and $b$ in $\Sigma$, then $L_1 \cap L_2$ is definable.*

**Proof.** Let $Y = \{(y_1, y_2)/a^{y_1}b^{y_2} \text{ in } L_1\}$ and $Z = \{(z_1, z_2)/a^{z_1}b^{z_2} \text{ in } L_2\}$. Then $Y$ and $Z$ are both semi-linear. Thus $Y \cap Z$ is semi-linear. Then

$$L_1 \cap L_2 = \{a^u b^v/(u, v) \text{ in } Y \cap Z\}$$

and is definable.

Corollary 2. *If $L_1$ is a definable subset of $w_1^* w_2^*$ and $L_2$ is definable, then $L_1 \cap L_2$ is definable.*

**Proof.** Let $L_3 = L_2 \cap w_1^* w_2^*$. Then $L_3$ is definable since $w_1^* w_2^*$ is regular. Also, $L_1 \cap L_2 = L_1 \cap w_1^* w_2^* \cap L_2 = L_1 \cap L_3$. Let $a_1$ and $a_2$ be two symbols. Let $S$ be the one state gsm which maps each $a_i$ into $w_i$. For $k = 1, 3$ let

$$L_k' = \{a_1^i a_2^j / w_1^i w_2^j \text{ in } L_k\}.$$

By Lemma 2.6, $L_1'$ and $L_3'$ are definable subsets of $a_1^* a_2^*$. By Corollary 1, $L_1' \cap L_3'$ is definable. Since $L_1 \cap L_3 = S(L_1' \cap L_3')$, $L_1 \cap L_3 = L_1 \cap L_2$ is definable.

Corollary 3. *Let $\tau$ be a linear function of $N^n$ into $N^m$. If $Y$ is a semi-linear subset of $N^m$, then $\tau^{-1}(Y)$[22] is a semi-linear subset of $N^n$.*

**Proof.** Let $\mu$ be the mapping of $N^n$ into $N^n \times N^m$[23] defined by $\mu(x) = x \times \tau(x)$. Then $\mu$ is linear since $\mu(x + x') = (x + x') \times \tau(x \times x') = (x + x') \times (\tau(x) + \tau(x')) = (x \times \tau(x)) + (x' \times \tau(x')) = \mu(x) + \mu(x')$. By Lemma 6.3, $\mu(N^n)$ is a semi-linear subset of $N^n \times N^m$. Since $Y$ is a semi-linear subset of $N^m$, $N^n \times Y$ is a semi-linear subset of $N^n \times N^m$ by Corollary 2 of Lemma 6.2. By Theorem 6.1, $\mu(N^n) \cap (N^n \times Y)$ is a semi-linear subset of $N^n \times N^m$. Let $\pi$ be the mapping of $N^n \times N^m$ into $N^n$ defined by $\pi(x \times y) = x$. Then $\pi$ is a linear function and

$$\pi(\mu(N^n) \cap (N^n \times Y)) = \tau^{-1}(Y).$$

---

(22) If $f$ is a function of $E_1$ into $E_2$ and $E_3 \subseteq E_2$, then $f^{-1}(E_3) = \{x/f(x) \text{ in } E_3\}$.

(23) We write $N^n \times N^m$ instead of $N^{n+m}$ to indicate that for an element $x \times y$ in $N^{n+m}$, $x$ is to be in $N^n$ and $y$ in $N^m$.

By Lemma 6.3, $\pi(\mu(N^n) \cap (N^n \times Y))$, thus $\tau^{-1}(Y)$, is semi-linear.

We now consider the difference of two semi-linear sets. In particular, we shall show that the difference of two semi-linear sets is semi-linear. To do this, though, we shall need a number of preliminary results.

Lemma 6.6. *Every semi-linear set is a finite union of linear sets, each of which has linearly independent periods.*

**Proof.** It suffices to show that

(1) every linear set is a finite union of linear sets, each of which has linearly independent periods.

Obviously (1) is true if there is just one period. Suppose that (1) is true for each linear set with at most $m-1$ periods ($m \geqq 2$). Let $X$ be a linear set with constant $x_0$ and periods $x_1, \cdots, x_m$. Suppose that $x_1, \cdots, x_m$ are dependent. Then we can relabel the $x_1, \cdots, x_m$ so that for some $1 \leqq k < m$, there exist nonnegative integers $a_i$ ($1 \leqq i \leqq m$) such that $\sum_1^k a_i x_i = \sum_{i>k} a_i x_i$. For each $j > k$, let $C_j$ be the finite set $C_j = \{x_0 + i x_j / 0 \leqq i \leqq a_j - 1\}$ if $a_j \geqq 2$ and let $C_j = \{x_0\}$ if $a_j = 0$ or 1. For each $j > k$, let $P_j$ be the finite set

$$P_j = \{x_1, x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_m\}.$$

Then $Z_j = L(C_j; P_j)$ is semi-linear. We shall show that $X = \bigcup_{j>k} Z_j$.

Clearly $C_j \subseteq X$. Since $P_j$ consists of certain periods of $X$, $L(C_j;\ P_j) \subseteq X$. Thus $\bigcup_{j>k} Z_j \subseteq X$. To see the reverse inclusion, let $y$ be an element of $X$. Then $y = x_0 + \sum_1^m b_i x_i$ for nonnegative integers $b_i$. Suppose that $b_j \geqq a_j$ for each $j > k$. Then

$$\begin{aligned} y &= x_0 + \sum_{i \leqq k} b_i x_i + \sum_{i>k} b_i x_i + \sum_{i \leqq k} a_i x_i - \sum_{i>k} a_i x_i \\ &= x_0 + \sum_{i \leqq k} (b_i + a_i) x_i + \sum_{i>k} (b_i - a_i) x_i. \end{aligned} \tag{2}$$

Note that each coefficient of $x_i$ in (2) is a nonnegative integer. Thus, without loss of generality, we may assume that $y = x_0 + \sum b_i x_i$, each $b_i$ a nonnegative integer, so that $0 \leqq b_j < a_j$ for some $j > k$. Then

$$y = x_0 + b_j x_j + \sum_{i \neq j} b_i x_i$$

is in $Z_j$ since $x_0 + b_j x_j$ is in $C_j$ and the $x_i$, $i \neq j$, are in $P_j$. Thus $X \subseteq \bigcup_{j>k} Z_j$, so that $X = \bigcup_{j>k} Z_j$.

Since $X = \bigcup_{j>k} Z_j$, $X$ is a finite union of linear sets $Z_j$, each having the $x_i$, $i \neq j$, as periods. Thus each $Z_j$ has fewer than $m$ periods. By induction, each $Z_j$ satisfies (1). Consequently $X$ satisfies (1).

LEMMA 6.7. *Let* $\{x_i/1 \leqq i \leqq n\}$ *be an independent set of elements of* $N^n$.

(a) *There exists a positive integer* $k(0)$ *with the following property: For each element* $y$ *in* $N^n$ *there is a sequence* $k, a_1, \cdots, a_n$ *of (not necessarily positive) integers such that* $1 \leqq k \leqq k(0)$ *and* $ky = \sum a_i x_i$.

(b) *For each element* $y$ *in* $N^n$ *let* $k_y$ *denote the smallest positive integer for which there exist (not necessarily positive) integers* $a_1, \cdots, a_n$ *so that* $ky = \sum a_i x_i$. *Let* $k_y y = \sum a_i^y x_i$. *If* $k$ *is any positive integer for which there exist integers* $a_1, \cdots, a_n$ *so that* $ky = \sum a_i x_i$, *then there exists a positive integer* $p$ *so that* $a_i = p a_i^y$ *for each* $i$ *and* $k = pk_y$.

**Proof.** (a) Since $\{x_1, \cdots, x_n\}$ is an independent set of $n$ elements of $N^n$, each vector in the underlying vector space is a linear ( = rational in this case) combination of the $x_i$. Thus each $y$ in $N^n$ is a rational combination of the $x_i$, say $y = \sum_1^n (a_i/b_i) x_i$, each $a_i$ and $b_i$ integral, and each $b_i > 0$. Letting $k = b_1 \cdots b_n \neq 0$, we get

$$ky = \sum_1^n a_i x_i, \text{ where } k > 0 \text{ and each } k, a_i \text{ is integral.} \tag{1}$$

We may assume that

$$k, a_1, \cdots, a_n \text{ are relatively prime.} \tag{2}$$

For if $k, a_1, \cdots, a_n$ are not relatively prime, we can factor out their greatest common divisor. To prove the lemma it suffices to show that there are only a finite number of such $k$. For then we can let $k(0)$ be the maximum of the $k$.

Let $k, a_1, \cdots, a_n$ satisfy (1) and (2). Let $y = (y_1, \cdots, y_n)$ and $x_i = (x_{i1}, \cdots, x_{in})$ for each $i$. Then (1) becomes the system of $n$ equations

$$\begin{aligned} ky_1 &= \sum_1^n a_i x_{i1} \\ &\cdots\cdots\cdots\cdots \\ ky_n &= \sum_1^n a_i x_{in}. \end{aligned} \tag{3}$$

Let $\Delta$ be the determinant

$$\Delta = \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ & \cdots & \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}.$$

Since the system of equations (3) has a solution for the $a_i$, it follows from elementary determinant theory that for each $i$, $a_i = k\Delta_i/\Delta$, $\Delta_i$ being an appropriate determinant and of integral value here. Note that $\Delta$ is not zero since the $x_i$ are independent. Thus $k$ divides each of the integers $a_i \Delta$. Let $k_1$ be the greatest common divisor of $k$ and $\Delta$. Then $k = k_1k_2$. Since $k = k_1k_2$ divides each $a_i\Delta$, $k_2$ divides $k$ and each $a_i$. Since $k, a_1,\cdots,a_n$ are relatively prime, $k_2 = 1$. Thus $k = k_1$ is a divisor of $\Delta$. Since there are only a finite number of divisors of $\Delta$, there are only a finite number of $k$.

(b) Since $k \geqq k_y$, there exist integers $p \geqq 1$ and $0 \leqq r < k_y$ so that $k = k_yp + r$. Then $ry = ky - k_ypy = \sum_1^n ax_i - \sum_1^n pa_i^y x_i = \sum_1^n (a_i - pa_i^y)x_i$. Since $r < k_y$, $r = 0$ by the minimality of $k_y$. Since the $x_i$ are independent, $a_i - pa_i^y = 0$ for each $i$, i.e., $a_i = pa_i^y$ for each $i$.

LEMMA 6.8. *Let $X$ be a linear subset of $N^n$, with constant $0^n$ and independent periods. Then $N^n - X$ is semi-linear.*

**Proof.** Let $x_1,\cdots,x_{j(0)}(j(0) \leqq n)$ be the independent periods of $X$. By elementary vector-space theory, we can adjoin some $n - j(0)$ of the unit vectors $\varepsilon_1^n,\cdots,\varepsilon_n^n$[24], call them $x_{j(0)+1},\cdots,x_n$, so that $x_1,\cdots,x_n$ are independent. By Lemma 6.7, there is an integer $k(0) \geqq 1$ with the following property: For each element $y$ in $N^n$ there exists $1 \leqq k \leqq k(0)$ and (not necessarily positive) integers $a_i$ such that $ky = \sum_1^n a_ix_i$. Let $k_y$ and $a_i^y$ have the same significance as in (b) of Lemma 6.7.

For $1 \leqq k \leqq k(0)$ and each (possibly empty) subset $I$ of $\{1,2,\cdots,n\}$, let $\tau_{kI}$ be the function of $N^n \times N^n$ into $N^n \times N^n$ defined by

$$\tau_{kI}(y \times (a_1,\cdots,a_n)) = \left(ky + \sum_{i \text{ in } I} a_ix_i\right) \times \left(\sum_{i \text{ not in } I} a_ix_i\right) \tag{1}$$

for each $y, (a_1,\cdots,a_n)$ in $N^n$. Denote by $K$ the set

$$K = \{y \times y/y \text{ in } N^n\}. \tag{2}$$

Since $K = \mu(N^n)$, where $\mu$ is the linear function of $N^n$ into $N^n \times N^n$ defined by $\mu(y) = y \times y$, $K$ is semi-linear by Lemma 6.3. By Corollary 3 of Theorem 6.1 $\tau_{kI}^{-1}(K)$ is a semi-linear subset of $N^n \times N^n$. $\tau_{kI}^{-1}(K)$ is the set of all $y \times (a_1,\cdots,a_n)$ in $N^n \times N^n$ for which

$$ky + \sum_{i \text{ in } I} a_ix_i = \sum_{i \text{ not in } I} a_ix_i. \tag{3}$$

Let $A_I$ be the set of all $y \times (a_1,\cdots,a_n)$ in $N^n \times N^n$ such that $a_i > 0$ for $i$ in $I$. (If $I = \phi$, then $A_I = N^n \times N^n$.) Then

$$A_I = L(\{c_0\}, \{\varepsilon_1^{2n},\cdots,\varepsilon_{2n}^{2n}\}), \tag{4}$$

where $(c_0)_i = 1$ for $i$ in $I$ and $(c_0)_i = 0$ for $i$ not in $I$. Thus $A_I$ is semi-linear (actually linear). By Theorem 6.1, $A_I \cap \tau_{kI}^{-1}(K)$ is semi-linear. Note that $A_I \cap \tau_{kI}^{-1}(K)$ is the set of all $y \times (a_1,\cdots,a_n)$ in $N^n \times N^n$ satisfying (3) and such that $a_i > 0$ for all $i$ in $I$. Let $\pi$ be the mapping of $N^n \times N^n$ into $N^n$ defined by $\pi(y \times (a_1,\cdots,a_n)) = y$. Then $\pi$ is linear. Therefore $\pi(A_I \cap \tau_{kI}^{-1}(K))$ is semi-linear. Thus

(24) **The unit vector $\varepsilon_i^n$ is defined by $(\varepsilon_i^n)_i = 1$ and $(\varepsilon_i^n)_j = 0$ for $i \neq j$.**

$$(5) \qquad G_1 = \bigcup_{1 \leqq k \leqq k(0);\, I \neq \phi} \pi(A_I \cap \tau_{kI}^{-1}(K)) \text{ is semi-linear.}$$

$G_1$ is the set of all $y$ in $N^n$ such that if some multiple of $y$ is a linear combination, with integral coefficients, of the $x_i$, i.e., $ky = \sum_1^n a_i x_i$, at least one of the coefficients is negative. (This follows from $(b)$ of Lemma 6.7.) Let $y$ be an element of $X$, i.e., $y = \sum_1^{j(0)} a_i x_i$, with $(a_1, \cdots, a_{j(0)})$ in $N^{j(0)}$. By (b) of Lemma 6.7, if $ky = \sum_1^n b_i x_i$, $1 \leqq k \leqq k(0)$, each $b_i$ integral, then each $b_i \geqq 0$. Thus $y$ is not in $G_1$. Hence $X \subseteq N^n - G_1$, so that $G_1 \subseteq N^n - X$. Let $H_1$ be the set of all elements $y$ in $N^n$ such that $k_y y = \sum_1^n a_i^y x_i$, each $a_i^y \geqq 0$. Clearly $H_1 = N^n - G_1$. Therefore $X \subseteq H_1$. Then $N^n - X = G_1 + (H_1 - X)$. To complete the proof of the lemma, it suffices to show that $H_1 - X$ is semi-linear.

For $1 \leqq i \leqq n$, let $D_i$ be the set of all $y \times (a_1, \cdots, a_n)$ in $N^n \times N^n$ such that $a_i > 0$. Since $D_i = L(\{\varepsilon_{n+i}^{2n}\}; \{\varepsilon_1^{2n}, \cdots, \varepsilon_{2n}^{2n}\})$, $D_i$ is semi-linear. Thus, by Theorem 6.1, $D_i \cap A_I \cap \tau_{kI}^{-1}(K)$ is semi-linear for each subset $I$ of $\{1, \cdots, n\} - \{i\}$. Therefore

$$(6) \qquad G_2 = \bigcup_{i > j(0)} \bigcup_{I \subseteq \{1, \ldots, n\} - \{i\};\, 1 \leqq k \leqq k(0)} \pi(D_i \cap A_I \cap \tau_{kI}^{-1}(K)) \text{ is semi-linear.}$$

$G_2$ is the set of all $y$ in $N^n$ such that $k_y y = \sum_1^n a_i^y x_i$, with some $a_i^y > 0$ for some $i > j(0)$. Clearly $G_2 \subseteq N^n - X$. Thus $N - X = G_1 + G_2 + (H_2 - X)$, where $H_2 = H_1 - G_2$. To complete the proof of the lemma, it suffices to show that $H_2 - X$ is semi-linear.

For $1 \leqq k \leqq k(0)$ and $1 \leqq j \leqq j(0)$, let $B_{kj}$ be the set of all $y \times (a_1, \cdots, a_{j(0)})$ in $N^n \times N^{j(0)}$ such that $ky = \sum_1^{j(0)} a_i x_i$ and $a_j$ is not divisible by $k$. We shall show that $B_{kj}$ is semi-linear. Let $E_k$ be the set of all $y \times (a_1, \cdots, a_{j(0)})$ in $N^n \times N^{j(0)}$ such that $ky = \sum_1^{j(0)} a_i x_i$. By Lemma 6.5, it is solvable to determine the set $P_k$ consisting of all (the finite number of) minimal elements of $E_k - \{0^{n+j(0)}\}$. Since $E_k = L(\{0^{n+j(0)}\}; P_k)$, $E_k$ is semi-linear (and effectively calculable). Let $F_{kj}$ be the set of all $y \times (a_1, \cdots, a_{j(0)})$ in $N^n \times N^{j(0)}$ such that $a_j$ is not divisible by $k$. Then $F_{kj} = L(C_{kj}; P_{kj})$, where

$$C_{kj} = \{0^{n+j-1} \times u \times 0^{j(0)-j} / u = 1, \cdots, k-1\}$$

and

$$P_{kj} = \{\varepsilon_1^{n+j(0)}, \cdots, \varepsilon_{n+j-1}^{n+j(0)}, k\varepsilon_{n+j}^{n+j(0)}, \varepsilon_{n+j+1}^{n+j(0)}, \cdots, \varepsilon_{j(0)}^{n+j(0)}\}.$$

Thus $F_{kj}$ is semi-linear. Since $B_{kj} = E_k \cap F_{kj}$, $B_{kj}$ is semi-linear.

Let $\bar{\pi}$ be the function of $N^n \times N^{j(0)}$ into $N^n$ defined by $\bar{\pi}(y \times (a_1, \cdots, a_{n(0)})) = y$. Since $\bar{\pi}$ is linear and $B_{kj}$ is semi-linear, $\bar{\pi}(B_{kj})$ is semi-linear. We shall show that $H_2 - X = \bigcup_{1 \leqq j \leqq j(0);\, 1 \leqq k \leqq k(0)} \bar{\pi}(B_{kj})$, thereby proving that $H_2 - X$ is semi-linear. Now $\bigcup_{k,j} \bar{\pi}(B_{kj})$ is the set of all $y$ in $H_2$ such that $ky = \sum_1^{j(0)} a_i x_i$ for some $1 \leqq k \leqq k(0)$ and $a_j$ is not divisible by $k$ for some $i \leqq j \leqq j(0)$. Let $y$ be an element in $\bigcup_{k,j} \bar{\pi}(B_{kj})$, say in $\bar{\pi}(B_{kj})$. Then $y = \sum_1^{j(0)} (a_i/k) x_i$, each $a_i \geqq 0$, and one of the $a_i/k$ nonintegral. By the independence of the $x_i$, $y$ cannot be a linear combination of $x_i$, with integral coefficients. Thus $y$ is not in $X$, i.e., $y$ is in $H_2 - X$. Finally, suppose that $y$ is an element in $H_2 - X$. Since $y$ is in $H_2$, $k_y y = \sum_1^{j(0)} a_i^y x_i$ with each $a_i^y$ integral and nonnegative. If $k_y = 1$, then $y$ is in $X$. Thus $k_y > 1$. By the minimality of $k_y$, the integers $k_y$, $a_1, \cdots, a_{j(0)}$ are relatively prime. Thus some $a_i$, say $a_{i(0)}$ is not divisible by $ky$. Then $y$ is in $\bar{\pi}(B_{k_y i(0)})$. Hence $y$ is in $\bigcup_{k,j} \bar{\pi}(B_{kj})$. Therefore $H_2 - X = \bigcup_{k,j} \bar{\pi}(B_{kj})$. Q.E.D.

Lemma 6.9. *If $X$ is a linear subset of $N^n$ with independent periods, then $N^n - X$ is a semi-linear subset of $N^n$.*

**Proof.** Suppose that $X$ has constant $x_0$ and periods $x_1, \cdots, x_j$ $(j \leqq n)$. For each $i$ such that $(x_0)_i > 0$ let

$$C_i = \{(u_1, \cdots, u_n)/u_j = 0 \text{ for } j \neq i, 0 \leqq u_i < (x_0)_i\}$$

and

$$P_i = \{\varepsilon_1^n, \cdots, \varepsilon_{i-1}^n, \varepsilon_{i+1}^n, \cdots, \varepsilon_n^n\}.$$

Then $L(C_i; P_i)$ is semi-linear. Thus

$$G = \bigcup_{(x_0)_i > 0} L(C; P_i)$$

is semi-linear. $G$ is the set of all $y$ in $N^n$ such that $x_0 \leqq y$ is false. Thus $G \subseteq N_n - X$. Let $Y = \{y/y \text{ in } N^n, x_0 \leqq y\}$. Since $Y = N^n - G$, $N^n - X = G + (N^n - X - G) = G + (Y - X)$. To prove the lemma it suffices to show that $Y - X$ is semi-linear.

Let $f$ be the one to one function of $N^n$ onto $Y$ defined by $f(y) = y + x_0$. For all subsets $C$, $P$ of $N^n$, $f(L(C;P)) = L(f(C);P)$. Thus a subset $Z$ of $N^n$ is semi-linear if and only if $f(Z)$ is semi-linear. Thus $Y - X$ is semi-linear if and only if

$$\begin{aligned} f^{-1}(Y - X) &= f^{-1}(Y) - f^{-1}(X) \\ &= N^n - f^{-1}(X) \end{aligned}$$

is semi-linear. Now $f^{-1}(X) = L(\{0^n\}; \{x_1, \cdots, x_j\})$. Thus $f^{-1}(X)$ is linear. By Lemma 6.8, $N^n - f^{-1}(X)$ is semi-linear. Then $f^{-1}(Y - X)$, thus $Y - X$, is semi-linear.

We are now ready to prove our second main result about semi-linear sets.

Theorem 6.2. *If $X$ and $Y$ are semi-linear subsets of $N^n$, then $X - Y$ is also a semi-linear subset of $N^n$ and is effectively calculable from $X$ and $Y$.*

**Proof.** By Lemma 6.6, $Y = \bigcup_1^m Z_j$, where each $Z_j$ is a linear set with independent periods. By Lemma 6.9, each $N^n - Z_j$ is semi-linear. Then

$$\begin{aligned} X - Y &= X \cap (N^n - Y) \\ &= X \cap \bigcap_1^m (N^n - Z_j) \end{aligned}$$

is semi-linear by Theorem 6.1.

Since $N^n$ is semi-linear we get

COROLLARY 1. *If $Y$ is a semi-linear subset of $N^n$, then $N^n - Y$ is semi-linear.*

COROLLARY 2. *If $L_1$ is a definable set and $L_2$ is a definable subset of $a_1^*a_2^*$ ($a_1, a_2$ in $\Sigma$), then $L_1 - L_2$ is definable.*

**Proof.** Since $L_2 \subseteq a_1^*a_2^*$, $L_1 - L_2 = (L_1 - a_1^*a_2^*) + (a_1^*a_2^* - L_2)$. Since $L_1$ is definable and $a_1^*a_2^*$ is regular, $L_1 - a_1^*a_2^*$ is definable. It thus suffices to show that $a_1^*a_2^* - L_2$ is definable. Now a subset $X$ of $a_1^*a_2^*$ is definable if and only if

$$Y(X) = \{(m,n)/a^m b^n \text{ in } X\}$$

is semi-linear. Thus $Y(L_2)$ is semi-linear. By Corollary 1, $N^2 - Y(L_2)$ is semi-linear. Since $N^2 - Y(L_2) = Y(a_1^*a_2^* - L_2)$, $a_1^*a_2^* - L_2$ is definable.

COROLLARY. 3. *If $L_1$ is definable and $L_2$ is a definable subset of $w_1^*w_2^*$ ($w_1, w_2$ in $\theta(\Sigma)$), then $L_1 - L_2$ is definable.*

**Proof.** $L_1 - L_2 = (L_1 - w_1^*w_2^*) + (w_1^*w_2^* - L_2)$. Since $w_1^*w_2^*$ is regular, $L_1 - w_1^*w_2^*$ is definable. Using the familiar argument involving a one state gsm and Corollary 2, it readily follows that $w_1^*w_2^* - L_2$ is definable. Thus $L_1 - L_2$ is definable.

COROLLARY 4. *If $L_2$ is definable and $L_1$ is a definable subset of $w_1^*w_2^*$ ($w_1, w_2$ in $\theta(\Sigma)$), then $L_1 - L_2$ is definable.*

**Proof.** Since $w_1^*w_2^*$ is regular, $L_2 \cap w_1^*w_2^*$ is a definable subset of $w_1^*w_2^*$. By Corollary 3, $L_1 - (L_2 \cap w_1^*w_2^*) = L_1 - L_2$ is definable.

We now come to the main results of this section as regards bounded definable sets.

THEOREM 6.3. *If $L_1, L_2$ are definable sets and one of them is bounded, then it is solvable whether* (a) $L_1 \subseteq L_2$, *and whether* (b) $L_2 \subseteq L_1$.

**Proof.** By Theorem 5.2 we can determine which of the sets $L_1$ or $L_2$ is bounded. By a change of notation if necessary we may assume that $L_1$ is bounded. By. Theorem 5.2, we can determine words $w_1, \cdots, w_n$ so that $L_1 \subseteq w_1^* \cdots w_n^*$.

(a) Since $w_1^* \cdots w_n^*$ is regular, $L_3 = L_2 \cap w_1^* \cdots w_n^*$ is definable, thus bounded definable. Now $L_1 \subseteq L_2$ if and only if $L_1 \subseteq L_3$. Let $a_1, \cdots, a_n$ be $n$ distinct symbols. For $i = 1, 3$, let

$$M_i = \{a_1^{i_1} \cdots a_n^{i_n}/w_1^{i_1} \cdots w_n^{i_n} \text{ in } L_i\}.$$

By Lemma 2.6, $M_1$ and $M_3$ are definable sets. For $i = 1, 3$, let

$$\begin{aligned} S_i &= \{(i_1, \cdots, i_n)/a^{i_1} \cdots a^{i_n} \text{ in } M_i\} \\ &= \{(i_1, \cdots, i_n)/w^{i_1} \cdots w^{i_n} \text{ in } M_i\}. \end{aligned}$$

By Parikh's Theorem, $S_1$ and $S_3$ are semi-linear. Now $L_1 \subseteq L_3$ if and only if $S_1 \subseteq S_3$. $S_1 \subseteq S_3$ if and only if $S_1 - S_3$ is empty. By Theorem 6.2, $S_1 - S_3$ is semi-linear (and effectively calculable). Thus it is solvable to determine if $S_1 - S_3$ is empty, whence (a).

(b) Since $w_1^* \cdots w_n^*$ is regular, $L_2 - w_1^* \cdots w_n^*$ is definable. Then it is solvable whether $L_2 - w_1^* \cdots w_n^*$ is empty. If $L_2 - w_1^* \cdots w_n^*$ is nonempty, then $L_2 \subseteq L_1$ is false. Suppose that $L_2 - w_1^* \cdots w_n^*$ is empty. Then $L_2 \subseteq w_1^* \cdots w_n^*$. Thus $L_2$ is bounded definable. By (a), it is solvable whether $L_2 \subseteq L_1$.

COROLLARY. *If $L_1, L_2$ are definable sets and one of them is bounded, then it is solvable whether $L_1 = L_2$.*

The proof follows from Theorem 6.3 and the fact that $L_1 = L_2$ if and only if $L_1 \subseteq L_2$ and $L_2 \subseteq L_1$.

The results of this section can also be proved by using the theorem of Presburger, as extended by Robinson and Zakon [10, Theorem 4.4], which implies:

"There is a decision procedure for Boolean relations (i.e., equality and inclusion) between subsets of ordered $n$-tuples of $N$ defined by an expression having $n$ free variables built by universal quantification, existential quantification, conjunction, disjunction, and negation from a finite number of linear (homogeneous and inhomogeneous) equalities, inequalities, and congruences."

It is true that the semi-linear subsets of $N^n$ are the same as the subsets of $N^n$ in the above theorem. (It is easy to see that any semi-linear subset of $N^n$ can be defined by an expression of the type specified in the theorem, but the converse seems to involve parts of Theorems 6.1 and 6.2 or similar results.) This fact, together with the theorem above, imply the results of this section. However it does not appear to be substantially simpler to use this method rather than the arguments presented in the section (because most of Theorem 6.1 and 6.2 would still be needed). The method we have adopted has the merit of being self-contained. and in the spirit of the rest of the paper, and provides another proof of the above theorem.

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