Algebra-based Loop Synthesis

Andreas Humenberger

TU Wien, Austria ahumenbe@forsyte.at

Laura Kovács

TU Wien, Austria Chalmers University of Technology, Sweden lkovacs@forsyte.at

- Abstract

We present an algorithm for synthesizing program loops satisfying a given polynomial loop invariant. The class of loops we consider can be modeled by a system of algebraic recurrence equations with constant coefficients. We turn the task of loop synthesis into a polynomial constraint problem by precisely characterizing the set of all loops satisfying the given invariant. We prove soundness of our approach, as well as its completeness with respect to an a priori fixed upper bound on the number of program variables. Our work has applications towards program verification, as well as generating number sequences from algebraic relations. We implemented our work in the tool Absynth and report on our initial experiments with loop synthesis.

2012 ACM Subject Classification Theory of computation \rightarrow Program analysis; Theory of computation \rightarrow Invariants; Computing methodologies \rightarrow Symbolic and algebraic manipulation

Keywords and phrases loop synthesis, invariants, recurrence equations, non-linear arithmetic

1 Introduction

The classical setting of program synthesis has been to synthesize programs from proofs of logical specifications that relate the inputs and the outputs of the program [19]. This traditional view of program synthesis has been refined to the setting of syntax-guided synthesis (SyGuS) [2]. In addition to logical specifications, SyGuS approaches consider further constraints on the program template to be synthesized, thus limiting the search space of possible solutions [10, 13, 8, 20].

One of the main challenges in synthesis remains however to reason about program loops – for example by answering the question whether there exists a loop satisfying a given loop invariant and synthesizing a loop with respect to a given invariant. We refer to this task of synthesis as *loop synthesis*, which can be considered as the reverse problem of loop invariant generation: rather than generating invariants summarizing a given loop as in [22, 12, 16], we synthesize loops whose functional behavior is captured by a given invariant.

Motivating Example. We motivate the use of loop synthesis by considering the program snippet of Figure 1a. The loop in Figure 1a is a variant of one of the examples from the online tutorial¹ of the Dafny verification framework [18]: the given program is not partially correct with respect to the pre-condition $N \geq 0$ and post-condition $c = N^3$ and the task is to revise/repair Figure 1a into a partially correct program using the invariant $n \leq N \wedge c = n^3 \wedge k = 3n^2 + 3n + 1 \wedge m = 6n + 6$.

Our work introduces an algorithmic approach to loop synthesis by relying on algebraic recurrence equations and constraint solving over polynomials. In particular, using our approach we automatically synthesize Figures 1b and 1c by using the given non-linear poly-

https://rise4fun.com/Dafny/

- (a) Faulty loop
- (b) Synthesized loop
- (c) Synthesized loop

Figure 1 Program repair via loop synthesis. Figures 1b and 1c are revised versions of Figure 1a such that $c = n^3 \wedge k = 3n^2 + 3n + 1 \wedge m = 6n + 6$ is an invariant of Figures 1b-1c.

nomial equalities $c = n^3 \wedge k = 3n^2 + 3n + 1 \wedge m = 6n + 6$ as input invariant to our loop synthesis task. While we do not synthesize loop guards, we note that we synthesize loops such that the given invariant holds for an arbitrary (and thus unbounded) number of loop iterations. Both synthesized programs, with the loop guard n < N as in Figure 1a, revise Figure 1a into a partially correct program with respect to the given requirements.

Algebra-based Loop Synthesis. Following the SyGuS setting, we consider additional requirements on the loop to be synthesized: we impose syntactic requirements on the form of loop expressions and guards. The imposed requirements allow us to reduce the synthesis task to the problem of generating linear recurrences with constant coefficients, called C-finite recurrences [15]. As such, we define our loop synthesis task as follows:

- ▶ **Problem** (Loop Synthesis). Given a polynomial p(x) over a set x of variables, generate a loop \mathcal{L} with program variables x such that
 - (i) p(x) = 0 is an invariant of \mathcal{L} , and
 - (ii) each program variable in \mathcal{L} induces a C-finite number sequence.

Our approach to synthesis is conceptually different than other SyGuS-based methods, such as [10, 8, 20]: rather than iteratively refining both the input and the solution space of synthesized programs, we take polynomial relations describing a potentially infinite set of input values and precisely capture not just one loop, but the set of all loops (i) whose invariant is given by our input polynomial and (ii) whose variables induce C-finite number sequences. That is, any instance of this set yields a loop that is partially correct by construction. Figures 1b and 1c depict two solutions of our loop synthesis task for the invariant $c = n^3 \wedge k = 3n^2 + 3n + 1 \wedge m = 6n + 6$.

The main steps of our approach are as follows. (i) Let p(x) be a polynomial over variables x and let $s \ge 0$ be an upper bound on the number of program variables occurring in the loop. If not specified, s is considered to be the number of variables from x. (ii) We use syntactic constraints over the loop body to be synthesized and define a loop template, as given by our programming model (7). Our programming model imposes that the functional behavior of the synthesized loops can be modeled by a system of C-finite recurrences (Section 3). (iii) By using the invariant property of p(x) = 0 for the loops to the synthesized, we construct a polynomial constraint problem (PCP) characterizing the set of all loops satisfying (7) for which p(x) = 0 is a loop invariant (Section 4). Our approach combines symbolic computation techniques over algebraic recurrence equations with polynomial constraint solving. We prove

that our approach to loop synthesis is both sound and complete. By completeness we mean, that if there is a loop \mathcal{L} with at most s variables satisfying the invariant p(x) = 0 such that the loop body meets our C-finite syntactic requirements, then \mathcal{L} is synthesized by our method (Theorem 15). Moving beyond this a priori fixed bound s, that is, deriving an upper bound on the number of program variables from the invariant, is an interesting but hard mathematical challenge, with connections to the inverse problem of difference Galois theory [25].

We finally note that our work is not restricted to specifications given by a single polynomial equality invariant. Rather, the invariant given as input to our synthesis approach can be conjunctions of polynomial equalities – as also shown in Figure 1.

Beyond Loop Synthesis. Our work has potential applications beyond loop synthesis – such as in generating number sequences from algebraic relations and program optimizations.

■ Generating number sequences. Our approach provides a partial solution to an open mathematical problem: given a polynomial relation among number sequences, e.g.

$$f(n)^{4} + 2f(n)^{3}f(n+1) - f(n)^{2}f(n+1)^{2} - 2f(n)f(n+1)^{3} + f(n+1)^{4} = 1,$$
 (1)

synthesize algebraic recurrences defining these sequences. There exists no complete method for solving this challenge, but we give a complete approach in the C-finite setting parameterized by an a priori bound s on the order of the recurrences. For the above given relation among f(n) and f(n+1), our approach generates the C-finite recurrence equation f(n+2) = f(n+1) + f(n) which induces the Fibonacci sequence.

■ Program optimizations. Given a polynomial invariant, our approach generates a PCP such that any solution to this PCP yields a loop satisfying the given invariant. By using additional constraints encoding a cost function on the loops to be synthesized, our method can be extended to synthesize loops that are optimal with respect to the considered costs, for example synthesizing loops that use only addition in variable updates. Consider for example Figures 1b-1c: the loop body of Figure 1b uses only addition, whereas Figure 1c implements also multiplications by constants.

Contributions. In summary, this paper makes the following contributions.

- We propose an automated procedure for synthesizing loops that are partially correct with respect to a given polynomial loop invariant (Section 4). By exploiting properties of C-finite sequences, we construct a PCP which precisely captures all solutions of our loop synthesis task. We are not aware of other approaches synthesizing loops from (non-linear) polynomial invariants.
- We prove that our approach to loop synthesis is sound and complete (Theorem 15). That is, if there is a loop whose invariant is captured by our given specification, our approach synthesizes this loop. To this end, we consider completeness modulo an a priori fixed upper bound s on the number of loop variables.
- We implemented our approach in the new open-source framework Absynth. We evaluated our work on a number of academic examples and considered measures for handling the solution space of loops to be synthesized (Section 5).

2 Preliminaries

Let \mathbb{K} be a computable field with characteristic zero. We also assume \mathbb{K} to be algebraically closed, that is, every non-constant polynomial in $\mathbb{K}[x]$ has at least one root in \mathbb{K} . The

algebraic closure \mathbb{Q} of the field of rational numbers \mathbb{Q} is such a field; \mathbb{Q} is called the field of algebraic numbers.

Let $\mathbb{K}[x_1,\ldots,x_n]$ denote the multivariate polynomial ring with variables x_1,\ldots,x_n . For a list x_1, \ldots, x_n , we write \boldsymbol{x} if the number of variables is known from the context or irrelevant. As K is algebraically closed, every polynomial $p \in K[x]$ of degree r has exactly r roots. Therefore, the following theorem follows immediately:

Theorem 1. The zero polynomial is the only polynomial in $\mathbb{K}[x]$ having infinitely many roots.

2.1 Polynomial Constraint Problem (PCP)

A polynomial constraint F is a constraint of the form $p \bowtie 0$ where p is a polynomial in $\mathbb{K}[x]$ and $\bowtie \in \{<, \leq, =, \neq, \geq, >\}$. A clause is then a disjunction $C = F_1 \vee \cdots \vee F_m$ of polynomial constraints. A unit clause is a special clause consisting of a single disjunct (i.e. m=1). A polynomial constraint problem (PCP) is then given by a set of clauses C. We say that a variable assignment $\sigma: \{x_1, \ldots, x_n\} \to \mathbb{K}$ satisfies a polynomial constraint $p \bowtie 0$ if $p(\sigma(x_1),\ldots,\sigma(x_n))\bowtie 0$ holds. Furthermore, σ satisfies a clause $F_1\vee\cdots\vee F_m$ if for some i, F_i is satisfied by σ . Finally, σ satisfies a clause set – and is therefore a solution of the PCP – if every clause within the set is satisfied by σ . We write $\mathcal{C} \sqsubset \mathbb{K}[x]$ to indicate that all polynomials in the clause set \mathcal{C} are contained in $\mathbb{K}[x]$. For a matrix M with entries m_1, \ldots, m_s we define the clause set $\operatorname{cstr}(M)$ to be $\{m_1 = 0, \ldots, m_s = 0\}$.

Number Sequences and Recurrence Relations

A sequence $(x(n))_{n=0}^{\infty}$ is called *C-finite* if it satisfies a linear recurrence with constant coefficients, also known as C-finite recurrence [15]. Let $c_0, \ldots, c_{r-1} \in \mathbb{K}$ and $c_0 \neq 0$, then

$$x(n+r) + c_{r-1}x(n+r-1) + \dots + c_1x(n+1) + c_0x(n) = 0$$
(2)

is a C-finite recurrence of order r. The order of a sequence is defined by the order of the recurrence it satisfies. We refer to a recurrence of order r also as an r-order recurrence, for example as a first-order recurrence when r=1 or a second-order recurrence when r=2. A recurrence of order r and r initial values define a sequence, and different initial values lead to different sequences. For simplicity, we write $(x(n))_{n=0}^{\infty} = 0$ for $(x(n))_{n=0}^{\infty} = (0)_{n=0}^{\infty}$

Example 2. Let $a \in \mathbb{K}$. The constant sequence $(a)_{n=0}^{\infty}$ satisfies a first-order recurrence equation x(n+1) = x(n) with x(0) = a. The geometric sequence $(a^n)_{n=0}^{\infty}$ satisfies x(n+1) = ax(n) with x(0) = 1. The sequence $(n)_{n=0}^{\infty}$ satisfies a second-order recurrence x(n+2) = 2x(n+1) - x(n) with x(0) = 0 and x(1) = 1.

From the closure properties of C-finite sequences [15], the product and the sum of C-finite sequences are also C-finite. Moreover, we also have the following properties:

- ▶ Theorem 3 ([15]). Let $(u(n))_{n=0}^{\infty}$ and $(v(n))_{n=0}^{\infty}$ be C-finite sequences of order r and s, respectively. Then:
- 1. $(u(n) + v(n))_{n=0}^{\infty}$ is C-finite of order at most r + s, and 2. $(u(n) \cdot v(n))_{n=0}^{\infty}$ is C-finite of order at most rs.
- ▶ Theorem 4 ([15]). Let $\omega_1, \ldots, \omega_t \in \mathbb{K}$ be pairwise distinct and $p_1, \ldots, p_t \in \mathbb{K}[x]$. The sequence $(p_1(n)\omega_1^n + \cdots + p_t(n)\omega_t^n)_{n=0}^{\infty}$ is the zero sequence if and only if the sequences $(p_1(n))_{n=0}^{\infty}, \ldots, (p_t(n))_{n=0}^{\infty} \text{ are zero.}$

- ▶ **Theorem 5** ([15]). Let $p = c_0 + c_1 x + \cdots + c_k x^k \in \mathbb{K}[x]$. Then $(p(n))_{n=0}^{\infty} = 0$ if and only if $c_0 = \cdots = c_k = 0$.
- ▶ **Theorem 6** ([15]). Let $(u)_{n=0}^{\infty}$ be a sequence satisfying a C-finite recurrence of order r. Then, u(n) = 0 for all $n \in \mathbb{N}$ if and only if u(n) = 0 for $n \in \{0, ..., r-1\}$.

We define a system of C-finite recurrences of order r and size s to be of the form

$$X_{n+r} + C_{r-1}X_{n+r-1} + \dots + C_1X_{n+1} + C_0X_n = 0$$

where $X_n = (x_1(n) \cdots x_s(n))^{\mathsf{T}}$ and $C_i \in \mathbb{K}^{s \times s}$. Every C-finite recurrence system can be transformed into a first-order system of recurrences by increasing the size such that we get

$$X_{n+1} = BX_n$$
 where B is invertible. (3)

The closed form solution of a C-finite recurrence system (3) is determined by the roots $\omega_1, \ldots, \omega_t$ of the characteristic polynomial of B, or equivalently by the eigenvalues $\omega_1, \ldots, \omega_t$ of B. We recall that the characteristic polynomial χ_B of the matrix B is defined as $\chi_B(\omega) = \det(\omega I - B)$, where det denotes the (matrix) determinant and I the identity matrix. Let m_1, \ldots, m_t respectively denote the multiplicities of the roots $\omega_1, \ldots, \omega_t$ of χ_B . The closed form of (3) is then given by

$$X_n = \sum_{i=1}^t \sum_{j=1}^{m_i} C_{ij} \omega_i^n n^{j-1} \quad \text{with } C_{ij} \in \mathbb{K}^{s \times 1}.$$

$$\tag{4}$$

However, not every choice of the C_{ij} gives rise to a solution. For obtaining a solution, we substitute the general form (4) into the original system (3) and compare coefficients. The following example illustrates the procedure for computing closed form solutions.

▶ **Example 7.** The most well-known C-finite sequence is the Fibonacci sequence satisfying a recurrence of order 2 which corresponds to the following first-order recurrence system:

The eigenvalues of B are given by $\omega_{1,2} = \frac{1}{2}(1 \pm \sqrt{5})$ with multiplicities $m_1 = m_2 = 1$. Therefore, the general solution for the recurrence system is of the form

$$\begin{pmatrix} f(n) \\ g(n) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \omega_1^n + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \omega_2^n.$$
 (6)

By substituting (6) into (5), we get the following constraints over the coefficients:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \omega_1^{n+1} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \omega_2^{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \omega_1^n + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \omega_2^n \end{pmatrix}$$

Bringing everything to one side yields:

$$\begin{pmatrix} c_1\omega_1 - c_1 - c_2 \\ c_2\omega_1 - c_1 \end{pmatrix} \omega_1^n + \begin{pmatrix} d_1\omega_2 - d_1 - d_2 \\ d_2\omega_2 - d_1 \end{pmatrix} \omega_2^n = 0$$

For the above equation to hold, the coefficients of the ω_i^n have to be 0. That is, the following linear system determines c_1, c_2 and d_1, d_2 :

$$\begin{pmatrix} \omega_1 - 1 & -1 & 0 & 0 \\ -1 & \omega_1 & 0 & 0 \\ 0 & 0 & \omega_2 - 1 & -1 \\ 0 & 0 & -1 & \omega_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{pmatrix} = 0$$

6 Algebra-based Loop Synthesis

The solution space is generated by $(\omega_1, 1, 0, 0)$ and $(0, 0, \omega_2, 1)$. The solution space of the C-finite recurrence system hence consists of linear combinations of

$$\begin{pmatrix} \omega_1 \\ 1 \end{pmatrix} \omega_1^n$$
 and $\begin{pmatrix} \omega_2 \\ 1 \end{pmatrix} \omega_2^n$.

That is, by solving the linear system

$$\begin{pmatrix} f(0) \\ g(0) \end{pmatrix} = E \begin{pmatrix} \omega_1 \\ 1 \end{pmatrix} \omega_1^0 + F \begin{pmatrix} \omega_2 \\ 1 \end{pmatrix} \omega_2^0$$
$$\begin{pmatrix} f(1) \\ g(1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(0) \\ g(0) \end{pmatrix} = E \begin{pmatrix} \omega_1 \\ 1 \end{pmatrix} \omega_1^1 + F \begin{pmatrix} \omega_2 \\ 1 \end{pmatrix} \omega_2^1$$

for $E, F \in \mathbb{K}^{2 \times 1}$ with f(0) = 1 and g(0) = 0, we get closed forms for (5):

$$f(n) = \frac{5 + \sqrt{5}}{5(1 + \sqrt{5})}\omega_1^{n+1} - \frac{1}{\sqrt{5}}\omega_2^{n+1} \text{ and } g(n) = \frac{1}{\sqrt{5}}\omega_1^n - \frac{1}{\sqrt{5}}\omega_2^n$$

Then f(n) represents the Fibonacci sequence starting at 1 and g(n) starts at 0. Solving for E and F with symbolic f(0) and g(0) yields a parameterized closed form, where the entries of E and F are linear functions in the symbolic initial values.

3 Our Programming Model

Given a polynomial relation $p(x_1, \ldots, x_s) = 0$, our loop synthesis procedure generates a first-order C-finite recurrence system of the form (3) with $X_n = (x_1(n) \cdots x_s(n))^{\mathsf{T}}$, such that $p(x_1(n), \ldots, x_s(n)) = 0$ holds for all $n \in \mathbb{N}$. It is not hard to argue that every first-order C-finite recurrence system corresponds to a loop with simultaneous variable assignments of the following form:

$$(x_1, \dots, x_s) \leftarrow (a_1, \dots, a_s)$$
while $true \ \mathbf{do}$

$$(x_1, \dots, x_s) \leftarrow (p_1(x_1, \dots, x_s), \dots, p_s(x_1, \dots, x_s))$$
end
$$(7)$$

The program variables x_1, \ldots, x_s are numeric, a_1, \ldots, a_s are (symbolic) constants in \mathbb{K} and $p_1, \ldots, p_s \in \mathbb{K}[x_1, \ldots, x_s]$. For every loop variable x_i , we denote by $x_i(n)$ the value of x_i at the *n*th loop iteration. That is, we view loop variables x_i as sequences $(x_i(n))_{n=0}^{\infty}$.

We call a loop (7) parameterized if at least one of a_1, \ldots, a_s is symbolic, and non-parameterized otherwise.

▶ Remark 8. While the output of our synthesis procedure is basically an affine program, we note that C-finite recurrence systems capture a larger class of programs. E.g. the program:

$$(x,y) \leftarrow (0,0)$$
; while true do $(x,y) \leftarrow (x+y^2,y+1)$ end

can be modeled by a C-finite recurrence system of order 4, which can be turned into an equivalent first-order system of size 6. That is, in order to synthesize a program which induces the sequences $(x(n))_{n=0}^{\infty}$ and $(y(n))_{n=0}^{\infty}$ we have to consider a recurrence system of size 6.

Example 9. The recurrence system (5) in Example 7 corresponds to the following loop:

$$(f,g) \leftarrow (1,0); \text{ while } true \text{ do } (f,g) \leftarrow (f+g,f) \text{ end}$$

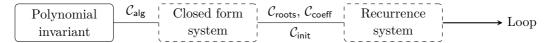


Figure 2 Overview of the PCP describing loop synthesis

Algebraic relations and loop invariants. Let p be a polynomial in $\mathbb{K}[z_1,\ldots,z_s]$ and let $(x_1(n))_{n=0}^{\infty},\ldots,(x_s(n))_{n=0}^{\infty}$ be number sequences. We call p an algebraic relation for the given sequences if $p(x_1(n),\ldots,x_s(n))=0$ for all $n\in\mathbb{N}$. Moreover, p is an algebraic relation for a system of recurrences if it is an algebraic relation for the corresponding sequences. It is immediate that for every algebraic relation p of a recurrence system, p=0 is a loop invariant for the corresponding loop (7); that is, p=0 holds before and after every loop iteration.

4 Algebra-based Loop Synthesis

We now present our approach for synthesizing loops satisfying a given polynomial property (invariant). We transform the loop synthesis problem into a PCP as described in Section 4.1. In Section 4.2, we introduce the clause sets of our PCP which precisely describe the solutions for the synthesis of loops, in particular to non-parameterized loops. We extend this approach in Section 4.3 to parameterized loops.

4.1 Setting and Overview of Our Method

Given a constraint p = 0 with $p \in \mathbb{K}[x_1, \ldots, x_s, y_1, \ldots, y_s]$, we aim to synthesize a system of C-finite recurrences such that p is an algebraic relation thereof. Intuitively, the values of loop variables x_1, \ldots, x_s are described by the number sequences $x_1(n), \ldots, x_s(n)$ for arbitrary n, and y_1, \ldots, y_s correspond to the initial values $x_1(0), \ldots, x_s(0)$. That is, we have a polynomial relation p among loop variables x_i and their initial values y_i , for which we synthesize a loop (7) such that p = 0 is a loop invariant of loop (7).

▶ Remark 10. Our approach is not limited to invariants describing the relationship between program variables among a single loop iteration. Instead, it naturally extends to relations among different loop iterations. For instance, by considering the relation in equation (1), we synthesize a loop computing the Fibonacci sequence.

The key step in our work comes with precisely capturing the solution space for our loop synthesis problem as a PCP. Our PCP is divided into the clause sets C_{roots} , C_{coeff} , C_{init} and C_{alg} , as illustrated in Figure 2 and explained next. Our PCP implicitly describes a first-order C-finite recurrence system and its corresponding closed form system. The one-to-one correspondence between these two systems is captured by the clause sets C_{roots} , C_{coeff} and C_{init} . Intuitively, these constraints mimic the procedure for computing the closed form of a recurrence system (see [15]). The clause set C_{alg} interacts between the closed form system and the polynomial constraint p=0, and ensures that p is an algebraic relation of the system. Furthermore, the recurrence system is represented by the matrix B and the vector A of initial values where both consist of symbolic entries. Then a solution of our PCP – which assigns values to those symbolic entries – yields a desired synthesized loop.

In what follows we only consider a unit constraint p=0 as input to our loop synthesis procedure. However, our approach naturally extends to conjunctions of polynomial equality constraints.

4.2 Synthesizing Non-Parameterized Loops

We now present our work for synthesizing loops, in particular non-parameterized loops (7). That is, we aim at computing concrete initial values for all program variables. Our implicit representation of the recurrence system is thus of the form

$$X_{n+1} = BX_n \qquad X_0 = A \tag{8}$$

where $B \in \mathbb{K}^{s \times s}$ is invertible and $A \in \mathbb{K}^{s \times 1}$, both containing symbolic entries.

As described in Section 2.2, the closed form of (8) is determined by the eigenvalues ω_i of B which we thus need to synthesize. Note that B may contain both symbolic and concrete values. Let us denote the symbolic entries of B by \mathbf{b} . Since \mathbb{K} is algebraically closed we know that B has s (not necessarily distinct) eigenvalues. We therefore fix a set of distinct symbolic eigenvalues $\omega_1, \ldots, \omega_t$ together with their multiplicities m_1, \ldots, m_t with $m_i > 0$ for $i = 1, \ldots, t$ such that $\sum_{i=1}^t m_i = s$. We call m_1, \ldots, m_t an integer partition of s. We next define the clause sets of our PCP.

Root constraints C_{roots} . The clause set C_{roots} imposes that B is invertible and ensures that $\omega_1, \ldots, \omega_t$ are distinct symbolic eigenvalues with multiplicities m_1, \ldots, m_t . Note that B is invertible if and only if all eigenvalues ω_i are non-zero. Furthermore, since \mathbb{K} is algebraically closed, every polynomial f(z) can be written as the product of linear factors of the form $z - \omega$, with $\omega \in \mathbb{K}$, such that $f(\omega) = 0$. Therefore, the equation

$$\chi_B(z) = (z - \omega_1)^{m_1} \cdots (z - \omega_t)^{m_t}$$

holds for all $z \in \mathbb{K}$, where $\chi_B(z) \in \mathbb{K}[\boldsymbol{\omega}, \boldsymbol{b}, z]$. Bringing everything to one side, we get

$$q_0 + q_1 z + \dots + q_d z^d = 0,$$

implying that the $q_i \in \mathbb{K}[\boldsymbol{\omega}, \boldsymbol{b}]$ have to be zero. The clause set characterizing the eigenvalues ω_i of B is then

$$\mathcal{C}_{\mathsf{roots}} = \{q_0 = 0, \dots, q_d = 0\} \cup \bigcup_{\substack{i, j = 1, \dots, t \\ i \neq j}} \{\omega_i \neq \omega_j\} \cup \bigcup_{i = 1, \dots, t} \{\omega_i \neq 0\}.$$

Coefficient constraints C_{coeff} . The fixed symbolic roots/eigenvalues $\omega_1, \ldots, \omega_t$ with multiplicities m_1, \ldots, m_t induce the general closed form solution

$$X_n = \sum_{i=1}^t \sum_{j=1}^{m_i} C_{ij} \omega_i^n n^{j-1}$$
(9)

where the $C_{ij} \in \mathbb{K}^{s \times 1}$ are column vectors containing symbolic entries. As stated in Section 2.2, not every choice of the C_{ij} gives rise to a valid solution. Instead, C_{ij} have to obey certain conditions which are determined by substituting into the original recurrence system of (8):

$$X_{n+1} = \sum_{i=1}^{t} \sum_{j=1}^{m_i} C_{ij} \omega_i^{n+1} (n+1)^{j-1} = \sum_{i=1}^{t} \sum_{j=1}^{m_i} \left(\sum_{k=j}^{m_i} {k-1 \choose j-1} C_{ik} \omega_i \right) \omega_i^n n^{j-1}$$
$$= B \left(\sum_{i=1}^{t} \sum_{j=1}^{m_i} C_{ij} \omega_i^n n^{j-1} \right) = BX_n$$

Bringing everything to one side yields $X_{n+1} - BX_n = 0$ and thus

$$\sum_{i=1}^{t} \sum_{j=1}^{m_i} \underbrace{\left(\left(\sum_{k=j}^{m_i} \binom{k-1}{j-1} C_{ik} \omega_i \right) - B C_{ij} \right)}_{D_{ij}} \omega_i^n n^{j-1} = 0.$$
(10)

Equation (10) holds for all $n \in \mathbb{N}$. By Theorem 5 we then have $D_{ij} = 0$ for all i, j and define

$$C_{\mathsf{coeff}} = \bigcup_{i=1}^t \bigcup_{j=1}^{m_i} \operatorname{cstr}(D_{ij}).$$

Initial values constraints C_{init} . The constraints C_{init} describe properties of initial values $x_1(0), \ldots, x_s(0)$. We enforce that (9) equals $B^n X_0$, for $n = 0, \ldots, d-1$, where d is the degree of the characteristic polynomial χ_B of B, by

$$C_{\text{init}} = \operatorname{cstr}(M_0) \cup \cdots \cup \operatorname{cstr}(M_{d-1})$$

where $M_i = X_i - B^i X_0$, with X_0 as in (8) and X_i being the right-hand side of (9) where n is replaced by i.

Algebraic relation constraints C_{alg} . The constraints C_{alg} are defined to ensure that p is an algebraic relation among the $x_i(n)$. Using (9), the closed forms of the $x_i(n)$ are expressed as

$$x_i(n) = p_{i,1}\omega_1^n + \dots + p_{i,t}\omega_t^n$$

where the $p_{i,j}$ are polynomials in $\mathbb{K}[n, \mathbf{c}]$. By substituting the closed forms and the initial values into the polynomial p, we get

$$p' = p(x_1(n), \dots, x_s(n), x_1(0), \dots, x_s(0)) = q_0 + nq_1 + n^2q_2 + \dots + n^kq_k$$
(11)

where the q_i are of the form

$$w_{i,1}^n u_{i,1} + \dots + w_{i,\ell}^n u_{i,\ell} \tag{12}$$

with $u_{i,1}, \ldots, u_{i,\ell} \in \mathbb{K}[\boldsymbol{a}, \boldsymbol{c}]$ and $w_{i,1}, \ldots, w_{i,\ell}$ being monomials in $\mathbb{K}[\boldsymbol{\omega}]$.

▶ Proposition 11. Let
$$p$$
 be of the form (11). Then $(p(n))_{n=0}^{\infty} = 0$ if and only if $(q_i(n))_{n=0}^{\infty} = 0$ for $i = 0, ..., k$.

Proof. One direction is obvious and for the other assume p(n) = 0. By rearranging p we get $p_1(n)w_1^n + \dots + p_\ell(n)w_\ell^n$. Let $\tilde{\omega}_1, \dots, \tilde{\omega}_t \in \mathbb{K}$ be such that $\tilde{p} = p_1(n)\tilde{w}_1^n + \dots + p_\ell(n)\tilde{w}_\ell^n = 0$ with $\tilde{w}_i = w_i(\tilde{\boldsymbol{\omega}})$. Note that the \tilde{w}_i are not necessarily distinct. However, consider v_1, \dots, v_r to be the pairwise distinct elements of the \tilde{w}_i . Then we can write \tilde{p} as $\sum_{i=1}^r v_i^n(p_{i,0} + np_{i,1} + \dots + n^k p_{i,k})$. By Theorems 4 and 5 we get that the $p_{i,j}$ have to be 0. Therefore, also $v_i^n p_{i,j} = 0$ for all i, j. Then, for each $j = 0, \dots, k$, we have $v_1^n p_{1,j} + \dots + v_r^n p_{1,j} = 0 = q_j$.

As p is an algebraic relation, we have that p' should be 0 for all $n \in \mathbb{N}$. Proposition 11 then implies that the q_i have to be 0 for all $n \in \mathbb{N}$.

▶ Lemma 12. Let q be of the form (12). Then q = 0 for all $n \in \mathbb{N}$ if and only if q = 0 for $n \in \{0, \dots, \ell - 1\}$.

Proof. The proof follows from Theorem 6 and from the fact that q satisfies a C-finite recurrence of order l. To be more precise, the $u_{i,j}$ and $w_{i,j}^n$ satisfy a first-order C-finite recurrence: as $u_{i,j}$ is constant it satisfies a recurrence of the form x(n+1) = x(n), and $w_{i,j}^n$ satisfies $x(n+1) = w_i x(n)$. Then, by Theorem 3 we get that $w_{i,j}^n u_{i,j}$ is C-finite of order at most 1, and q is C-finite of order at most ℓ .

Even though the q_i contain exponential terms in n, it follows from Lemma 12 that the solutions for the q_i being 0 for all $n \in \mathbb{N}$ can be described as a finite set of polynomial equality constraints: Let Q_i^j denote the polynomial constraint $w_{i,1}^j u_{i,1} + \cdots + w_{i,\ell}^j u_{i,\ell} = 0$ for q_i of the form (12), and let $C_i = \{Q_i^0, \dots, Q_i^{\ell-1}\}$ be the associated clause set. Then the clause set ensuring that p is indeed an algebraic relation is given by

$$C_{\mathsf{alg}} = C_0 \cup \cdots \cup C_k$$
.

▶ Remark 13. Observe that Theorem 6 can be applied to (11) directly, as p' satisfies a C-finite recurrence. Then by the closure properties of C-finite recurrences, the upper bound on the order of the recurrence which p' satisfies is given by $r = \sum_{i=0}^k 2^i \ell$. That is, by Theorem 6, we would need to consider p' with $n = 0, \ldots, r - 1$, which yields a non-linear system with a degree of at least r - 1. Note that r depends on 2^i , which stems from the fact that n = 0 satisfies a recurrence of order 2, and n^i satisfies therefore a recurrence of order at most n = 0. Thankfully, Proposition 11 allows us to only consider the coefficients of the n^i and therefore lower the size of our constraints.

Having defined the clause sets C_{roots} , C_{coeff} , C_{init} and C_{alg} , we define our PCP as the union of these four clause sets. Note that the matrix B, the vector A, the polynomial p and the multiplicities of the symbolic roots $\mathbf{m} = m_1, \ldots, m_t$ uniquely define the clauses discussed above. We hence define our PCP to be the clause set $C_{AB}^p(\mathbf{m})$ as follows:

$$\mathcal{C}_{AB}^{p}(\boldsymbol{m}) = \mathcal{C}_{\mathsf{roots}} \cup \mathcal{C}_{\mathsf{init}} \cup \mathcal{C}_{\mathsf{coeff}} \cup \mathcal{C}_{\mathsf{alg}} \tag{13}$$

Recall that \boldsymbol{a} and \boldsymbol{b} are the symbolic entries in the matrices A and B in (8), \boldsymbol{c} are the symbolic entries in the C_{ij} in (9), and $\boldsymbol{\omega}$ are the symbolic eigenvalues of B. We then have $C_{\text{roots}} \sqsubset \mathbb{K}[\boldsymbol{\omega}, \boldsymbol{b}]$, $C_{\text{coeff}} \sqsubset \mathbb{K}[\boldsymbol{\omega}, \boldsymbol{b}, \boldsymbol{c}]$, $C_{\text{init}} \sqsubset \mathbb{K}[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ and $C_{\text{coeff}} \sqsubset \mathbb{K}[\boldsymbol{\omega}, \boldsymbol{c}]$. Hence $C_{AB}^p(\boldsymbol{m}) \sqsubset \mathbb{K}[\boldsymbol{\omega}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$.

It is not difficult to see that the constraints in $\mathcal{C}_{\mathsf{alg}}$ determine the size of our PCP. As such, the degree and the number of terms in the invariant have a direct impact on the size and the maximum degree of the polynomials in our PCP. Which might not be obvious is that the number of distinct symbolic roots influences the size and the maximum degree of our PCP. The more distinct roots are considered the higher is the number of terms in (12), and therefore more instances of (12) have to be added to our PCP.

Let $p \in \mathbb{K}[x_1, \dots, x_s, y_1, \dots, y_s]$, $B \in \mathbb{K}^{s \times s}$ and $A \in \mathbb{K}^{s \times 1}$, and let m_1, \dots, m_t be an integer partition of $\deg_{\omega}(\chi_B(\omega))$. We then get the following theorem:

▶ Theorem 14. The mapping $\sigma : \{\omega, a, b, c\} \to \mathbb{K}$ is a solution of $\mathcal{C}_{AB}^p(m)$ if and only if $p(\boldsymbol{x}, x_1(0), \dots, x_s(0))$ is an algebraic relation for $X_{n+1} = \sigma(B)X_n$ with $X_0 = \sigma(A)$, and the eigenvalues of $\sigma(B)$ are given by $\sigma(\omega_1), \dots, \sigma(\omega_t)$ with multiplicities m_1, \dots, m_t .

From Theorem 14, we then get Algorithm 1 for synthesizing the C-finite recurrence representation of a non-parameterized loop (7): the function $\mathtt{IntPartitions}(s)$ returns the set of all integer partitions of an integer s; and $\mathtt{Solve}(\mathcal{C})$ returns whether the clause set \mathcal{C} is satisfiable and a model σ if so. We note that the growth of the number of integer partitions

```
Input: A polynomial p \in \mathbb{K}[x_1, \dots, x_s, y_1, \dots, y_s].

Output: A vector A \in \mathbb{K}^{s \times 1} and a matrix B \in \mathbb{K}^{s \times s} s.t. p is an algebraic relation of X_{n+1} = BX_n and X_0 = A, if such A and B exist.

A \leftarrow (a_i) \in \mathbb{K}^{s \times 1} // symbolic vector B \leftarrow (b_{ij}) \in \mathbb{K}^{s \times s} // symbolic matrix for m_1, \dots, m_t \in \text{IntPartitions}(s) do

\begin{bmatrix} \text{sat}, \sigma \leftarrow \text{Solve}(\mathcal{C}^p_{AB}(m_1, \dots, m_t)) \\ \text{if sat then return } \sigma(A), \sigma(B) \end{bmatrix} end
```

Algorithm 1 Synthesis of a non-parameterized C-finite recurrence system

is subexponential, and so is the complexity Algorithm 1. A more precise complexity analysis of Algorithm 1 is an interesting future work.

Finally, based on Theorem 14 and on the property that the number of integer partitions of a given integer is finite, we obtain the following result:

▶ Theorem 15. Algorithm 1 is sound, and complete w.r.t. recurrence systems of size s. ◀

The completeness in Theorem 15 is relative to systems of size s which is a consequence of the fact that we synthesize first-order recurrence systems. That is, there exists a recurrence system of order > 1 and size s with an algebraic relation $p \in \mathbb{K}[x_1, \ldots, x_s]$, but there exists no first-order system of size s where p is an algebraic relation.

The precise characterization of non-parameterized loops by non-parameterized C-finite recurrence systems implies soundness and completeness for non-parameterized loops from Theorem 15.

▶ **Example 16.** We showcase our procedure in Algorithm 1 by synthesizing a loop for the invariant x = 2y. That is, the polynomial constraint is given by $p = x - 2y \in \mathbb{K}[x, y]$ and we want to find a recurrence system of the following form:

$$\begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x(n) \\ y(n) \end{pmatrix} \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$(14)$$

The characteristic polynomial of B is then given by $\chi_B(\omega) = \omega^2 - b_{11}\omega - b_{22}\omega - b_{12}b_{21} + b_{11}b_{22}$ where its roots define the closed form system. Since we cannot determine the actual roots of $\chi_B(\omega)$ we have to fix a set of symbolic roots. The characteristic polynomial has two – not necessarily distinct – roots: Either $\chi_B(\omega)$ has two distinct roots ω_1, ω_2 with multiplicities $m_1 = m_2 = 1$, or a single root ω_1 with multiplicity $m_1 = 2$. Let us consider the latter case. The first clause set we define is $\mathcal{C}_{\text{roots}}$ for ensuring that B is invertible (i.e. ω_1 is nonzero), and that ω_1 is indeed a root of the characteristic polynomial with multiplicity 2. That is, $\chi_B(\omega) = (\omega - \omega_1)^2$ has to hold for all $\omega \in \mathbb{K}$, and bringing everything to one side yields

$$(b_{11} + b_{22} - 2\omega_1)\omega + b_{12}b_{21} - b_{11}b_{22} + \omega_1^2 = 0.$$

We then get the following clause set:

$$\mathcal{C}_{\mathsf{roots}} = \{b_{11} + b_{22} - 2\omega_1 = 0, b_{12}b_{21} - b_{11}b_{22} + \omega_1^2 = 0, \omega_1 \neq 0\}$$

As we fixed the symbolic roots, the general closed form system is of the form

$$\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \omega_1^n + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \omega_1^n n$$
 (15)

By substituting into the recurrence system we get:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \omega_1^{n+1} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \omega_1^{n+1} (n+1) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \omega_1^n + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \omega_1^n n \right)$$

By further simplifications and re-ordering of terms we then obtain:

$$0 = \begin{pmatrix} c_1\omega_1 + d_1\omega_1 - b_{11}c_1 - b_{12}c_2 \\ c_2\omega_1 + d_2\omega_1 - b_{21}c_1 - b_{22}c_2 \end{pmatrix} \omega_1^n + \begin{pmatrix} d_1\omega_1 - b_{11}d_1 - b_{12}d_2 \\ d_2\omega_1 - b_{21}d_1 - b_{22}d_2 \end{pmatrix} \omega_1^n n$$

Since this equation has to hold for $n \in \mathbb{N}$ we get the following clause set:

$$\mathcal{C}_{\text{coeff}} = \{ c_1 \omega_1 + d_1 \omega_1 - b_{11} c_1 - b_{12} c_2 = 0, c_2 \omega_1 + d_2 \omega_1 - b_{21} c_1 - b_{22} c_2 = 0, d_1 \omega_1 - b_{11} d_1 - b_{12} d_2 = 0, d_2 \omega_1 - b_{21} d_1 - b_{22} d_2 = 0 \}$$

For defining the relationship between the closed forms and the initial values, we set (15) with n = i to be equal to the i^{th} unrolling of (14) for i = 0, 1:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \omega_1 + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \omega_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

The resulting constraints for defining the initial values are then given by

$$\mathcal{C}_{\text{init}} = \{c_1 - a_1 = 0, c_1\omega_1 + d_1\omega_1 - b_{11}a_1 - b_{12}a_2 = 0, c_2 - a_2 = 0, c_2\omega_1 + d_2\omega_1 - b_{21}a_1 - b_{22}a_2 = 0\}.$$

Eventually, we want to restrict the solutions such that x - 2y = 0 is an algebraic relation for our recurrence system. That is, by substituting the closed forms into x(n) - 2y(n) = 0 we get

$$0 = x(n) - 2y(n) = c_1\omega_1^n + d_1\omega_1^n n - 2(c_2\omega_1^n + d_2\omega_1^n n) = \underbrace{(c_1 - 2c_2)\,\omega_1^n}_{q_0} + \underbrace{((d_1 - 2d_2)\,\omega_1^n)}_{q_1} n$$

where q_0 and q_1 have to be 0 since the above equation has to hold for all $n \in \mathbb{N}$. Then, by applying Lemma 12 to q_0 and q_1 , we get the following clauses:

$$\mathcal{C}_{\mathsf{alg}} = \{c_1 - 2c_2 = 0, d_1 - 2d_2 = 0\}$$

Our PCP is then the union of C_{roots} , C_{coeff} , C_{init} and C_{alg} . Two possible solutions for our PCP, and therefore of the synthesis problem, are given by the following loops:

$$(x,y) \leftarrow (2,1)$$
 $(x,y) \leftarrow (2,1)$ while $true \ \mathbf{do} \ (x,y) \leftarrow (x+2,y+1) \ \mathbf{end}$ while $true \ \mathbf{do} \ (x,y) \leftarrow (2x,2y) \ \mathbf{end}$

Note that both loops above have mutually independent updates. Yet, the second one induces geometric sequences and requires handling exponentials of 2^n .

4.3 Synthesizing Parameterized Loops

We now extend the loop synthesis approach from Section 4.2 to an algorithmic approach synthesizing parameterized loops, that is, loops which satisfy a loop invariant for arbitrary input values. Let us first consider the following example motivating the synthesis problem of parameterized loops.

▶ Example 17. We are interested to synthesize a loop implementing Euclidean division over $x,y \in \mathbb{K}$. Following the problem specification of $[17]^2$, a synthesized loop performing Euclidean division satisfies the polynomial invariant $p = \bar{x} - \bar{y}q - r = 0$, where \bar{x} and \bar{y} denote the initial values of x and y before the loop. It is clear, that the synthesized loop should be parameterized with respect to \bar{x} and \bar{y} . With this setting, input to our synthesis approach is the invariant $p = \bar{x} - \bar{y}q - r = 0$. A recurrence system performing Euclidean division and therefore satisfying the algebraic relation $\bar{x} - \bar{y}q - r$ is then given by $X_{n+1} = BX_n$ and $X_0 = A$ with a corresponding closed form system $X_n = A + Cn$ where:

$$X_{n} = \begin{pmatrix} x(n) \\ r(n) \\ q(n) \\ y(n) \\ t(n) \end{pmatrix} \quad A = \begin{pmatrix} \bar{x} \\ \bar{x} \\ 0 \\ \bar{y} \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 \\ -\bar{y} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Here, the auxiliary variable t plays the role of the constant 1, and x and y induce constant sequences. When compared to non-parameterized C-finite systems/loops, note that the coefficients in the above closed forms, as well as the initial values of variables, are functions in the parameters \bar{x} and \bar{y} .

Example 17 illustrates that the parameterization has the effect that we have to consider parameterized closed forms and initial values. For non-parameterized loops we have that the coefficients in the closed forms are constants, whereas for parameterized systems we have that the coefficients are functions in the parameters – the symbolic initial values of the sequences. In fact, we have linear functions since the coefficients are obtained by solving a linear system (see Example 7).

As already mentioned, the parameters are a subset of the symbolic initial values of the sequences. Therefore, let $I = \{k_1, \ldots, k_r\}$ be a subset of the indices $\{1, \ldots, s\}$. We then define $\bar{X} = (\bar{x}_{k_1} \cdots \bar{x}_{k_r} \ 1)^{\mathsf{T}}$ where $\bar{x}_{k_1}, \ldots, \bar{x}_{k_r}$ denote the parameters. Then, instead of (8), we get

$$X_{n+1} = BX_n \qquad X_0 = A\bar{X} \tag{16}$$

as the implicit representation of our recurrence system where the entries of $A \in \mathbb{K}^{s \times r+1}$ are defined as

$$a_{ij} = \begin{cases} 1 & i = k_j \\ a_{ij} \text{ symbolic} & i \notin I \\ 0 & \text{otherwise} \end{cases}$$

and, as before, we have $B \in \mathbb{K}^{s \times s}$. Intuitively, the complex looking construction of A makes sure that we have $x_i(0) = \bar{x}_i$ for $i \in I$.

▶ **Example 18.** For the vector $X_0 = (x_1(0) \ x_2(0) \ x_3(0))^\mathsf{T}$, the set $I = \{1, 3\}$ and therefore $\bar{X} = (\bar{x}_1 \ \bar{x}_3 \ 1)^\mathsf{T}$, we get the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 1 & 0 \end{pmatrix}$$

² for $x, y \in \mathbb{K}$ we want to compute $q, r \in \mathbb{K}$ such that x = yq + r holds

Thus, $x_1(0)$ and $x_3(0)$ are set to \bar{x}_1 and \bar{x}_3 respectively, and $x_2(0)$ is a linear function in \bar{x}_1 and \bar{x}_3 .

In addition to the change in the representation of the initial values, we also have a change in the closed forms. That is, instead of (9) we get

$$X_{n} = \sum_{i=1}^{t} \sum_{i=1}^{m_{i}} C_{ij} \bar{X} \omega_{i}^{n} n^{j-1}$$

as the general form for the closed form system with $C_{ij} \in \mathbb{K}^{s \times r+1}$. Then C_{roots} , C_{init} , C_{coeff} and C_{alg} are defined analogously to Section 4.2, and similar to the non-parameterized case we define $C_{AB}^p(\boldsymbol{m}, \bar{\boldsymbol{x}})$ as the union of those clause sets. The polynomials in $C_{AB}^p(\boldsymbol{m}, \bar{\boldsymbol{x}})$ are then in $\mathbb{K}[\boldsymbol{\omega}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \bar{\boldsymbol{x}}]$. Then, for each $\boldsymbol{\omega}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{K}$ satisfying the clause set for all $\bar{\boldsymbol{x}} \in \mathbb{K}$ gives rise to the desired parameterized loop, that is, we have to solve an $\exists \forall$ problem. However, since all constraints containing $\bar{\boldsymbol{x}}$ are polynomial equality constraints, we apply Theorem 1: Let $p \in \mathbb{K}[\boldsymbol{\omega}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \bar{\boldsymbol{x}}]$ be a polynomial such that $p = p_1q_1 + \cdots + p_kq_k$ with $p_i \in \mathbb{K}[\bar{\boldsymbol{x}}]$ and q_i monomials in $\mathbb{K}[\boldsymbol{\omega}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$. Then, Theorem 1 implies that the q_i have to be 0.

We therefore define the following operator $\operatorname{split}_{\boldsymbol{x}}(p)$ for collecting the coefficients of all monomials in \boldsymbol{x} in the polynomial p: Let p be of the form $q_0 + q_1 x + \cdots + q_k x^k$, P a clause and let \mathcal{C} be a clause set, then:

$$\operatorname{split}_{\boldsymbol{y},x}(p) = \begin{cases} \{q_0 = 0, \dots, q_k = 0\} & \text{if } \boldsymbol{y} \text{ is empty} \\ \operatorname{split}_{\boldsymbol{y}}(q_0) \cup \dots \cup \operatorname{split}_{\boldsymbol{y}}(q_k) & \text{otherwise} \end{cases}$$

$$\operatorname{split}_{\boldsymbol{y}}(P) = \begin{cases} \operatorname{split}_{\boldsymbol{y}}(p) & \text{if } P \text{ is a unit clause } p = 0 \\ \{P\} & \text{otherwise} \end{cases}$$

$$\operatorname{split}_{\boldsymbol{y}}(\mathcal{C}) = \bigcup_{P \in \mathcal{C}} \operatorname{split}_{\boldsymbol{y}}(P)$$

We then have $\operatorname{split}_{\bar{\boldsymbol{x}}}(\mathcal{C}_{AB}^p(\boldsymbol{m},\bar{\boldsymbol{x}})) \sqsubset \mathbb{K}[\boldsymbol{\omega},\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$. Moreover, for $p \in \mathbb{K}[x_1,\ldots,x_s,y_1,\ldots,y_s]$, matrices A,B and \bar{X} as in (16), and an integer partition m_1,\ldots,m_t of $\deg_{\omega}(\chi_B(\omega))$ we get the following theorem:

▶ Theorem 19. The map $\sigma : \{\omega, a, b, c\} \to \mathbb{K}$ is a solution of split_{\bar{x}} $(\mathcal{C}_{AB}^p(m, \bar{x}))$ if and only if $p(x, x_1(0), \ldots, x_s(0))$ is an algebraic relation for $X_{n+1} = \sigma(B)X_n$ with $X_0 = \sigma(A)\bar{X}$, and $\sigma(\omega_1), \ldots, \sigma(\omega_t)$ are the eigenvalues of $\sigma(B)$ with multiplicities m_1, \ldots, m_t .

Theorem 19 gives rise to an algorithm analogous to Algorithm 1. Furthermore, we get an analogous soundness and completeness result as in Theorem 15 which implies soundness and completeness for parameterized loops.

▶ Example 20. We illustrate the construction of the constraint problem for Example 17. For reasons of brevity, we consider a simplified system where the variables r and x are merged. The new invariant is then $\bar{r} = \bar{y}q + r$ and the parameters are given by \bar{r} and \bar{y} . That is, we consider a recurrence system of size 4 with sequences y, q and r, and t for the constant 1. As a consequence we have that the characteristic polynomial B is of degree 4, and we fix the symbolic root ω_1 with multiplicity 4. For simplicity, we only show how to construct the clause set $\mathcal{C}_{\mathsf{alg}}$.

With the symbolic roots fixed we get the following template for the closed form system: Let $X_n = (r(n) \ q(n) \ y(n) \ t(n))^{\mathsf{T}}$ and $V = (\bar{r} \ \bar{y} \ 1)^{\mathsf{T}}$, and let $C, D, E, F \in \mathbb{K}^{4\times 3}$ be symbolic matrices. Then the closed form is given by

$$X_n = (CV + DVn + EVn^2 + FVn^3) \omega_1^n$$

and for the initial values we get

$$X_0 = \begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 1 & 0 \\ a_{41} & a_{42} & a_{43} \end{pmatrix} V.$$

By substituting the closed forms into the invariant r(0) - y(0)q(n) - r(n) = 0 and rearranging we get:

$$0 = \bar{r} - \left(c_{21}\bar{r}\bar{y} - c_{22}\bar{y}^2 - c_{23}\bar{y} - c_{11}\bar{r} - c_{12}\bar{y} - c_{13}\right)\omega_1^n$$

$$- \left(d_{21}\bar{r}\bar{y} + d_{22}\bar{y}^2 + d_{23}\bar{y} - d_{11}\bar{r} - d_{12}\bar{y} - d_{13}\right)\omega_1^n n$$

$$- \left(e_{21}\bar{r}\bar{y} + e_{22}\bar{y}^2 + e_{23}\bar{y} - e_{11}\bar{r} - e_{12}\bar{y} - e_{13}\right)\omega_1^n n^2$$

$$- \left(f_{21}\bar{r}\bar{y} + f_{22}\bar{y}^2 + f_{23}\bar{y} - f_{11}\bar{r} - f_{12}\bar{y} - f_{13}\right)\omega_1^n n^3$$

Since the above equation should hold for all $n \in \mathbb{N}$ we get:

$$(\bar{r}) 1^{n} - (c_{21}\bar{r}\bar{y} - c_{22}\bar{y}^{2} - c_{23}\bar{y} - c_{11}\bar{r} - c_{12}\bar{y} - c_{13}) \omega_{1}^{n} = 0$$

$$(d_{21}\bar{r}\bar{y} + d_{22}\bar{y}^{2} + d_{23}\bar{y} - d_{11}\bar{r} - d_{12}\bar{y} - d_{13}) \omega_{1}^{n} = 0$$

$$(e_{21}\bar{r}\bar{y} + e_{22}\bar{y}^{2} + e_{23}\bar{y} - e_{11}\bar{r} - e_{12}\bar{y} - e_{13}) \omega_{1}^{n} = 0$$

$$(f_{21}\bar{r}\bar{y} + f_{22}\bar{y}^{2} + f_{23}\bar{y} - f_{11}\bar{r} - f_{12}\bar{y} - f_{13}) \omega_{1}^{n} = 0$$

Then, by applying Lemma 12, we get:

$$\bar{r} - \left(c_{21}\bar{r}\bar{y} - c_{22}\bar{y}^2 - c_{23}\bar{y} - c_{11}\bar{r} - c_{12}\bar{y} - c_{13}\right) = 0$$

$$\bar{r} - \left(c_{21}\bar{r}\bar{y} - c_{22}\bar{y}^2 - c_{23}\bar{y} - c_{11}\bar{r} - c_{12}\bar{y} - c_{13}\right)\omega_1 = 0$$

$$d_{21}\bar{r}\bar{y} + d_{22}\bar{y}^2 + d_{23}\bar{y} - d_{11}\bar{r} - d_{12}\bar{y} - d_{13} = 0$$

$$e_{21}\bar{r}\bar{y} + e_{22}\bar{y}^2 + e_{23}\bar{y} - e_{11}\bar{r} - e_{12}\bar{y} - e_{13} = 0$$

$$f_{21}\bar{r}\bar{y} + f_{22}\bar{y}^2 + f_{23}\bar{y} - f_{11}\bar{r} - f_{12}\bar{y} - f_{13} = 0$$

Finally, by applying the operator $\mathrm{split}_{\bar{u},\bar{r}}$, we get the following constraints for $\mathcal{C}_{\mathsf{alg}}$:

$$c_{21} = 1 - c_{11} = c_{22} = c_{23} + c_{12} = c_{13} = 0$$

$$\omega_1 c_{21} = 1 - \omega_1 c_{11} = \omega_1 c_{22} = \omega_1 (c_{23} + c_{12}) = \omega_1 c_{13} = 0$$

$$d_{21} = d_{11} = d_{22} = d_{23} + d_{12} = d_{13} = 0$$

$$e_{21} = e_{11} = e_{22} = e_{23} + e_{12} = e_{13} = 0$$

$$f_{21} = f_{11} = f_{22} = f_{23} + f_{12} = f_{13} = 0$$

5 Implementation and Experiments

Our approach to algebra-based loop synthesis is implemented in the tool Absynth which is available at https://github.com/ahumenberger/Absynth.jl. Inputs to Absynth are conjunctions of polynomial equality constraints, representing a loop invariant. As a result, Absynth derives a program that is partially correct with respect to the given invariant.

Loop synthesis in Absynth is reduced to solving PCPs. These PCPs are expressed in the quantifier-free fragment of non-linear real arithmetic (QF_NRA). We used Absynth in conjunction with the SMT solvers Yices [7] and Z3 [6] for solving the PCPs and therefore synthesizing loops. For instance, the loops depicted in Figures 1b and 1c, and in Example 16 are synthesized automatically using Absynth.

4

16 Algebra-based Loop Synthesis

Optimizing and Exploring the Search Space. Absynth implements additional constraints to restrict the search space of solutions to loop synthesis. Namely, Absynth (i) avoids trivial loops/solutions and (ii) restricts the shape of B to be triangular or unitriangular. The latter allows Absynth to synthesize loops whose loop variables are not mutually dependent on each other. We note that such a pattern is a very common programming paradigm – all benchmarks from Table 1 in Appendix A.1 satisfy such a pattern. Yet, as a consequence of restricting the shape of B, the order of the variables in the recurrence system matters. That is, we have to consider all possible variable permutations for ensuring completeness w.r.t. (uni)triangular matrices.

Absynth however supports an iterative approach for exploring the solution space. One can start with a small recurrence system and a triangular/unitriangular matrix B, and then stepwise increase the size/generality of the system. Our initial results from Table 1 in Appendix A.1 demonstrate the practical use of our approach to loop synthesis: all examples could be solved in reasonable time.

6 Related Work

Synthesis. To the best of our knowledge, existing synthesis approaches are restricted to linear invariants, see e.g. [24], whereas our work supports loop synthesis from non-linear polynomial properties. In the setting of counterexample-guided synthesis – CEGIS [3, 23, 8, 20], input-output examples satisfying a specification S are used to synthesize a candidate program P that is consistent with the given inputs. Correctness of the candidate program Pwith respect to S is then checked using verification approaches, in particular using SMTbased reasoning. If verification fails, a counterexample is generated as an input to P that violates S. This counterexample is then used in conjunction with the previous set of inputoutputs to revise synthesis and generate a new candidate program P. Unlike these methods, input specifications to our approach are relational (invariant) properties describing all, potentially infinite input-output examples of interest. Hence, we do not rely on interactive refinement of our input but work with a precise characterization of the set of input-output values of the program to be synthesized. Similarly to sketches [23, 20], we consider loop templates restricting the search for solutions to synthesis. Yet, our templates support non-linear arithmetic (and hence multiplication), which is not yet the case in [20, 8]. We precisely characterize the set of all programs satisfying our input specification, and as such, our approach does not exploit learning to refine program candidates. On the other hand, our programming model is more restricted than [20, 8] in various aspects: we only handle simple loops and only consider numeric data types and operations.

The programming by example approach of [9] learns programs from input-output examples and relies on lightweight interaction to refine the specification of programs to be specified. The approach has further been extended in [14] with machine learning, allowing to learn programs from just one (or even none) input-output example by using a simple supervised learning setup. Program synthesis from input-output examples is shown to be successful for recursive programs [1], yet synthesizing loops and handling non-linear arithmetic is not yet supported by this line of research. Our work does not learn programs from observed input-output examples, but uses loop invariants to fully characterize the intended behavior of the program to be synthesized. Our technique precisely characterizes the solution space of loops to be synthesized by a system of algebraic recurrences, and hence we do not rely on statistical models supporting machine learning.

A related approach to our work is tackled in [5], where a fixed-point implementation

for an approximated real-valued polynomial specification is presented, by combining genetic programming [21] with abstract interpretation [4] to estimate and refine the (floating-point) error bound of the inferred fixed-point implementation. While the underlying abstract interpreter is precise for linear expressions, precision of the synthesis is lost in the presence of non-linear arithmetic. Unlike [5], we consider polynomial specification in the abstract algebra of real-closed fields and do not address challenges rising from machine reals.

Algebraic Reasoning. When compared to works on generating polynomial invariants [22, 12, 16, 11], the only common aspect between these works and our synthesis method is the use of linear recurrences to capture the functional behavior of program loops. Yet, our work is conceptually different than [22, 12, 16, 11], as we reverse engineer invariant generation and do not rely on the ideal structure/Zariski closure of polynomial invariants. We do not use ideal theory nor Gröbner bases computation to generate invariants from loops; rather, we generate loops from invariants by formulating and solving PCPs.

7 Conclusions

We proposed a syntax-guided synthesis procedure for synthesizing loops from a given polynomial loop invariant. We consider loop templates and use reasoning over recurrence equations modeling the loop behavior. The key ingredient of our work comes with translating the loop synthesis problem into a polynomial constraint problem and showing that this constraint problem precisely captures all solutions to the loop synthesis problem. We implemented our work and evaluated on a number of academic examples. Understanding and encoding the best optimization measures for loop synthesis is an interesting line for future work.

- References -

- Aws Albarghouthi, Sumit Gulwani, and Zachary Kincaid. Recursive Program Synthesis. In *CAV*, pages 934–950, 2013. doi:10.1007/978-3-642-39799-8_67.
- 2 Rajeev Alur, Rastislav Bodík, Eric Dallal, Dana Fisman, Pranav Garg, Garvit Juniwal, Hadas Kress-Gazit, P. Madhusudan, Milo M. K. Martin, Mukund Raghothaman, Shambwaditya Saha, Sanjit A. Seshia, Rishabh Singh, Armando Solar-Lezama, Emina Torlak, and Abhishek Udupa. Syntax-Guided Synthesis. In *Dependable Software Systems Engineering*, volume 40, pages 1–25. IOS Press, 2015. doi:10.3233/978-1-61499-495-4-1.
- 3 Rajeev Alur, Rishabh Singh, Dana Fisman, and Armando Solar-Lezama. Search-Based Program Synthesis. *Commun. ACM*, 61(12):84–93, 2018. URL: https://doi.org/10.1145/3208071.
- 4 Patrick Cousot and Radhia Cousot. Abstract Interpretation: A Unified Lattice Model for Static Analysis of Programs by Construction or Approximation of Fixpoints. In *POPL*, pages 238–252, 1977. doi:10.1145/512950.512973.
- 5 Eva Darulova, Viktor Kuncak, Rupak Majumdar, and Indranil Saha. Synthesis of fixed-point programs. In *EMSOFT*, pages 22:1–22:10, 2013. doi:10.1109/EMSOFT.2013.6658600.
- 6 Leonardo De Moura and Nikolaj Bjørner. Z3: An efficient SMT solver. In TACAS, pages 337–340, 2008. doi:10.1007/978-3-540-78800-3_24.
- 7 Bruno Dutertre. Yices 2.2. In CAV, pages 737–744, 2014. doi:10.1007/978-3-319-08867-9_49.
- Yu Feng, Ruben Martins, Osbert Bastani, and Isil Dillig. Program Synthesis Using Conflict-Driven Learning. In *PLDI*, pages 420–435, 2018. doi:10.1145/3192366.3192382.
- 9 Sumit Gulwani. Automating String Processing in Spreadsheets using Input-Output Examples. In POPL, pages 317–330, 2011. doi:10.1145/1926385.1926423.

- Sumit Gulwani. Programming by examples: Applications, algorithms, and ambiguity resolution. In *IJCAR*, pages 9–14, 2016. doi:10.1007/978-3-319-40229-1_2.
- Ehud Hrushovski, Joël Ouaknine, Amaury Pouly, and James Worrell. Polynomial Invariants for Affine Programs. In *LICS*, pages 530–539. ACM, 2018. doi:10.1145/3209108.3209142.
- Andreas Humenberger, Maximilian Jaroschek, and Laura Kovács. Automated Generation of Non-Linear Loop Invariants Utilizing Hypergeometric Sequences. In *ISSAC*, pages 221–228. ACM, 2017. doi:10.1145/3087604.3087623.
- Susmit Jha, Sumit Gulwani, Sanjit A. Seshia, and Ashish Tiwari. Oracle-Guided Component-Based Program Synthesis. In *ICSE*, pages 215–224, 2010. doi:10.1145/1806799.1806833.
- Ashwin Kalyan, Abhishek Mohta, Oleksandr Polozov, Dhruv Batra, Prateek Jain, and Sumit Gulwani. Neural-Guided Deductive Search for Real-Time Program Synthesis from Examples. In ICLR, 2018.
- Manuel Kauers and Peter Paule. The Concrete Tetrahedron Symbolic Sums, Recurrence Equations, Generating Functions, Asymptotic Estimates. Texts & Monographs in Symbolic Computation. Springer, 2011. doi:10.1007/978-3-7091-0445-3.
- Zachary Kincaid, John Cyphert, Jason Breck, and Thomas W. Reps. Non-Linear Reasoning for Invariant Synthesis. PACMPL, 2(POPL):54:1-54:33, 2018. doi:10.1145/3158142.
- 17 Donald Ervin Knuth. The art of computer programming. Addison-Wesley, 1997.
- 18 K. Rustan M. Leino. Accessible Software Verification with Dafny. IEEE Software, 34(6):94–97, 2017. doi:10.1109/MS.2017.4121212.
- Zohar Manna and Richard J. Waldinger. A Deductive Approach to Program Synthesis. ACM Trans. Program. Lang. Syst., 2(1):90–121, 1980. doi:10.1145/357084.357090.
- 20 Maxwell Nye, Luke Hewitt, Joshua Tenenbaum, and Armando Solar-Lezama. Learning to Infer Program Sketches. In *ICML*, pages 4861–4870, 2019. URL: http://proceedings.mlr.press/v97/nye19a.html.
- 21 Riccardo Poli, William B. Langdon, and Nicholas Freitag McPhee. A Field Guide to Genetic Programming. lulu.com, 2008. URL: http://www.gp-field-guide.org.uk/.
- Enric Rodríguez-Carbonell and Deepak Kapur. Generating All Polynomial Invariants in Simple Loops. J. Symb. Comput., 42(4):443–476, 2007. doi:10.1016/j.jsc.2007.01.002.
- 23 Armando Solar-Lezama. The Sketching Approach to Program Synthesis. In APLAS, pages 4-13, 2009. doi:10.1007/978-3-642-10672-9_3.
- Saurabh Srivastava, Sumit Gulwani, and Jeffrey S. Foster. From Program Verification to Program Synthesis. In *POPL*, pages 313–326, 2010. doi:10.1145/1706299.1706337.
- Marius Van der Put and Michael F Singer. Galois Theory of Difference Equations. Springer, 1997. doi:10.1007/BFb0096118.

| Instance | s | I | D | С | Yices | | | Z3 | | | | Z3* | | |
|---------------------|---|---|----|-----|-------|------|------|-----|-----|-----|---|-----|-----|-----|
| | | | | | UN | UP | FU | UN | UP | FU | U | N | UP | FU |
| add1* | 5 | 1 | 5 | 173 | 932 | 921 | - | 117 | - | - | 6 | 22 | 726 | _ |
| add2* | 5 | 1 | 5 | 173 | 959 | 861 | - | 115 | - | - | 2 | 22 | 109 | - |
| cubes | 5 | 3 | 6 | 94 | - | _ | - | 116 | 114 | - | 1 | 8 | 496 | 575 |
| double1 | 3 | 1 | 4 | 29 | 114 | 112 | 3882 | 113 | 111 | 113 |] | .3 | 21 | 63 |
| double2 | 3 | 1 | 3 | 24 | 110 | 106 | 1665 | 115 | 106 | 115 |] | .3 | 18 | 40 |
| ${\tt eucliddiv}^*$ | 5 | 1 | 5 | 185 | 213 | 537 | - | 114 | 115 | - |] | 9 | 73 | - |
| ${\tt intcbrt}^*$ | 5 | 2 | 12 | 262 | - | _ | - | 117 | 116 | - | 2 | 22 | 83 | 469 |
| intsqrt1 | 4 | 2 | 6 | 53 | - | - | - | 113 | 108 | 114 |] | .5 | 19 | - |
| ${\tt intsqrt2}^*$ | 4 | 1 | 6 | 104 | 105 | 1164 | - | 113 | 111 | 115 |] | 5 | 27 | 37 |
| petter1 | 3 | 1 | 4 | 29 | 112 | 116 | - | 114 | 113 | 113 |] | 5 | 18 | 32 |
| square | 3 | 1 | 4 | 29 | 112 | 112 | - | 112 | 114 | 117 |] | .3 | 17 | 26 |
| dblsquare | 3 | 1 | 4 | 30 | 109 | 105 | - | 105 | 105 | 110 | 1 | 2 | 17 | 26 |
| sum1 | 4 | 2 | 6 | 53 | 617 | - | _ | 108 | 112 | 113 | 1 | 7 | 24 | 99 |
| sum2 | 5 | 3 | 6 | 82 | - | - | - | 220 | 112 | - | 4 | 20 | 516 | - |

- s size of the recurrence system
- I number of polynomial invariants
- D maximum monomial degree of constraints
- c number of constraints
- **Table 1** Benchmark results in milliseconds
- * parameterized system
- timeout (60 seconds)

A Appendix

A.1 Examples and Experiments

Table 1 summarizes our experimental results. The experiments were performed on a machine with a 2.9 GHz Intel Core i5 and 16 GB LPDDR3 RAM, and for each instance a timeout of 60 seconds was set. The results are given in milliseconds, and only include the time needed for solving the constraint problem as the time needed for constructing the constraints is neglectable. We used the SMT solvers Yices [7] (version 2.6.1) and Z3 [6] (version 4.8.6) to conduct our experiments. In Table 1, the columns Yices and Z3 correspond to the results where the respective solver is called as an external program with and SMTLIB 2.0 file as input; column Z3* shows the results where our improved, direct interface (C++ API) was used to call Z3.

Our benchmark set consists of invariants for loops from the invariant generation literature. Note that the benchmarks cubes and double2 in Table 1 are those from Figure 1 and Example 16, respectively. A further presentation of a selected set of our benchmarks is given in Appendix A.2.

Our work supports an iterative approach for exploring the solution space of loops to be synthesized. One can start with a small recurrence system and a triangular/unitriangular matrix B, and then stepwise increase the size/generality of the system. The columns UN and UP in Table 1 show the results where the coefficient matrix B is restricted to be upper unitriangular and upper triangular respectively. FU indicates that no restriction on B was set.

Note that the running time of Algorithm 1 heavily depends on the order of which the

Figure 3 Example eucliddiv with input x0 == y0*q+r

```
# square
# square
                                            # square
                    # Solver: Yices
# Original loop
                                            # Solver: Z3
                                            a, b = 1/16, -1/4
a, b = 0, 0
                    a, b = 0, 0
while true
                   while true
                                            while true
 a = a + 2b + 1
                    a = a - 2b + 1
                                            a = a + 2b + 1
 b = b + 1
                      b = b - 1
                                             b = b + 1
end
                     end
                                            end
```

Figure 4 Example square with input a == b^2

integer partitions and the variable permutations are traversed. Therefore, in order to get comparable results, we fixed the integer partition and the variable permutation. That is, for each instance, we enforced that B has just a single eigenvalue, and we fixed a variable ordering where we know that there exists a solution with a unitriangular matrix B. Hence, there exists at least one solution which all cases – UN, UP and FU – have in common. Furthermore, for each instance we added constraints for avoiding trivial solutions, i.e. loops inducing constant sequences.

A.2 Examples of Synthesized Loops

We took loops from the invariant generation literature and computed their invariants. Our benchmark set consists of these generated invariants. For each example in Figures 3-7, we first list the original loop and then give the first loop synthesized by our work in combination with Yices and Z3 respectively.

Observe that in most cases our work was able to derive the original loop – apart from the initial values – with either Z3 or Yices.

```
# sum1
                          # sum1
                                                        # sum1
                     # Sum1 # Sum1

# Solver: Yices # Solver: Z3

a, b, c = 1/2, 1/4, 2 a, b, c = -5/8, 25/64, -1/4
# Original loop
a, b, c = 0, 0, 1
while true
                         while true
                                                       while true
                           a = a - 1/2

b = b - 1/2 c + 3/4
                                                         a = a + 1
 a = a + 1
                                                         b = b + c
 b = b + c
                           c = c - 1
 c = c + 2
                                                        c = c + 2
end
                          end
                                                        end
```

Figure 5 Example sum1 with input 1+2a == c && 4b == (c-1)^2

Figure 6 Example intsqrt2 with input a0+r == r^2+2y

Figure 7 Example intcbrt with input 1/4+3r^2 == s && 1+4a0+6r^2 == 3r+4r^3+4x