Graphs and Combinatorics (2006) 22:161–172
Digital Object Identifier (DOI) 10.1007/s00373-005-0627-y

Graphs and Combinatorics

Directed Path-width and Monotonicity in Digraph Searching

János Barát*

Department of Mathematics, Technical University of Denmark, B.303. 2800 Lyngby, Denmark. E-mail: barat@math.u-szeged.hu

Abstract. Directed path-width was defined by Reed, Thomas and Seymour around 1995. The author and P. Hajnal defined a cops-and-robber game on digraphs in 2000. We prove that the two notions are closely related and for any digraph D, the corresponding graph parameters differ by at most one. The result is achieved using the mixed-search technique developed by Bienstock and Seymour. A search is called monotone, in which the robber's territory never increases. We show that there is a mixed-search of D with k cops if and only if there is a monotone mixed-search with k cops. For our cops-and-robber game we get a slightly weaker result: the monotonicity can be guaranteed by using at most one extra cop.

Key words. Search games, Path-width, Cops-and-robber games, Monotonicity, Directed graph

1. Introduction

After the pioneering work of Robertson and Seymour, the path- and tree-width of a graph became well-known and standard concepts. We refer to [6] or [10] for the definitions. The challenge to generalize the notions of the graph minors project to digraphs has attracted several people in the last decade [8], [9]. In this paper, we focus on two notions: directed path-width and a cops-and-robber game variant. The latter concept is also known as *node search games*. There is a thorough overview given about them in [5]. Let us only recall that monotonicity of such a game means that the robber's territory is a non-increasing set. Proving the monotonicity of a game means that allowing recontamination of some vertices does not help the cops. This monotonicity may play a decisive role in proving NP-hardness of a game.

Graphs and digraphs in this paper are simple, that is without loops and multiple edges, except when otherwise stated. The directed edge (arc) with tail u and head v is denoted by (u, v).

There are several possible ways to define cops-and-robber games on directed graphs; see [2], [8] and Section 6 of this note. We concentrate on the following:

Send offprint requests to: János Barát

* On leave from Bolyai Institute, University of Szeged, Hungary. This research has been supported by a Marie Curie Fellowship of the European Community under contract number HPMF-CT-2002-01868 and by OTKA Grants F.030737 and T.34475.

Definition 1. *Let a directed graph D be given. The robber stands on a vertex, and he can run at infinite speed to any other vertex along the directed edges in the indicated direction. He is not permitted to run through a cop, however. The cops stand also on the vertices, and move by helicopters from vertex to vertex. Assume in this version, that the robber is invisible. The cops* capture *the robber, if a cop lands on a vertex where the robber stands and can not move anywhere i.e. all out-neighbors are also occupied by cops. The goal is to decide how many cops are necessary to capture the robber. Denote this minimum by* $\overline{cn}(D)$. *Here cn stands for cop number, overline for the invisible case.*

Instead of the invisible robber, it is very convenient to think of an infection in the graph, which contaminates along cop-free paths. In that language the infected area corresponds to the robber's territory, i.e. where the robber can potentially be, but we do not see where he actually is. The visible robber version of the above game is defined analogously. The minimum number of necessary cops in that case is denoted by $cn(D)$.

Johnson, Robertson, Seymour and Thomas defined directed tree-width in [8] and they also studied a game in connection with the new parameter, see Definition 8. Later it turned out that their game is not monotone. The first counterexample was recently constructed by Adler [1]. In contrast to this, we conjecture that the game in Definition 1 has the monotonicity property. Our Theorem 2 in Section 4 proves a slight weakening of this. The main building bricks of the proof are developed in Section 3. We use the mixed-search approach of Bienstock and Seymour [3], which incorporates node search and edge search simultaneously. We prove that k cops can mixed-search D if and only if the same number of cops can do it in a monotone way. An even more general setting was invented by Fomin and Thilikos [7]. In their frame a so called *connectivity function* is used to get a min-max theorem, through which the monotonicity result arises for a large class of games. Similar to that, our proof uses the submodularity of a function δ. However, δ is not symmetric as it is required from a connectivity function.

Many graph-theoretic parameters have been characterized by their obstructing analogue. We address the question: what property implies large directed path-width? Using the blockage ideas from [4], we show one obstruction class for small directed path-width. Here we use the method investigated by Bienstock, Robertson, Seymour and Thomas for undirected graphs.

Finally, we compare the above game to the one of [8]. We show in Lemma 9 that a visible robber has really different escape chance in the two games. We exhibit a class of graphs, whose *cn* parameter tends to infinity, but following the rules of the game in [8] two cops are enough to catch the robber. However, distinguishing the two different cop numbers in case of an invisible robber remains unsolved. As well as the possible monotonicity of the visible version of our game.

2. Directed path-width

Let us look at Definition 1 from the cops' point of view. We observe that one cop is enough to capture the invisible robber in D if and only if D is acyclic. Also sources and sinks do not play any important role, so we may assume that the minimum out- and in-degree is at least one. If D is a directed circuit with at least two vertices, then $\overline{cn}(D) = 2$. Generalizing this, we notice that k cops do not suffice if $\delta^+(D) \geq k$. To give a positive result for the cops — similarly to the undirected case — linearly ordered bags produce a strategy. This is probably the way how the following notion of directed path-width came up in a joint work of Reed, Seymour and Thomas:

Definition 2. [11] *Let D be a directed graph. A* directed path-decomposition *(DPD) is a sequence of subsets of vertices $W_1, W_2, \ldots, W_k$ such that*

(i) $\cup_{i=1}^{k} W_i = V(D)$, *and*
(ii) *if $i < j < k$, then $W_i \cap W_k$ is a subset of W_j, and*
(iii) *an edge either has both endvertices in the same W_i or has its tail in W_i and head in W_j, where $i < j$.*

The width *of a DPD is the maximum size of a W_i minus one. The* directed path-width (dpw) *of a digraph D is the minimum width over all possible DPD's.*

$$dpw = \min_{\{W_i\} \text{ is a } DPD} \left(\max_{1 \leq i \leq k} (|W_i| - 1) \right)$$

The next proposition justifies the name directed path-width. For the sake of completeness, we recall here the undirected notion:

Definition 3. *Let G be a graph. A* path-decomposition *of G is a sequence of subsets of vertices $W_1, W_2, \ldots, W_k$ such that*

(i) $\cup_{i=1}^{k} W_i = V(G)$, *and*
(ii) *if $i < j < k$, then $W_i \cap W_k$ is a subset of W_j, and*
(iii) *every edge has both endvertices in some W_i.*

The width *of a decomposition is the maximum size of a W_i minus one. The* path-width *of G is the minimum width over all possible path-decompositions.*

Lemma 1. *Let G be a graph, and let D be the digraph obtained from G by replacing every edge by two anti-parallel arcs. Then the path-width of G is equal to the directed path-width of D.*

Proof. Assume first that a path-decomposition of G is given. Every edge of G is in some W_i by definition. Hence, if we make the replacing to get D keeping the W_i's unchanged, we get a directed path-decomposition as well with the same width.

Assume now that a *DPD* of D is given. Suppose there is an arc (u, v), $u \in W_i$ and $v \in W_j$ such that $i > j$. But also (v, u) is an arc by assumption contradicting (iii) of Definition 2. Hence, every arc is contained in some W_i, so the W_i's give a path-decomposition of G with the same width. □

However, the directed path-width of a digraph is not proportional to the path-width of the underlying graph. Consider the transitive orientation T_n of the complete graph on n vertices. The path-width of K_n is $n-1$, while the directed path-width of T_n is only 0.

Notice, that if $dpw(D) \leq k-1$, then k cops are enough to capture an invisible robber in D. The cops simply occupy the sets W_i of an optimal DPD in order. Hence $dpw(D) \leq k-1$ implies $\overline{cn}(D) \leq k$. The opposite implication will be discussed next.

3. Monotonicity of mixed-search in directed graphs

We show a monotone graph searching result for directed graphs. The proof of Theorem 1 follows the proof of (2.4) in [3] by Bienstock and Seymour for undirected graphs. We give the appropriate definitions for directed graphs, and write the differences in *emphasized text*. Then, one has to check that the proofs are valid for our settings.

Consider the game described in Definition 1. We would like to show that if k cops can capture the invisible robber, then they can do it in a monotone way too.

First we describe a slightly more general game, which is easier to work with. Observe that in our terminology a *search* clears all the edges, while a *capture* clears all the vertices of a digraph. We note here that in the description below at most one edge can be cleared at each step.

Definition 4. *A* mixed-search *in a directed graph D is a sequence of pairs*

$$(A_0, Z_0), \dots, (A_n, Z_n)$$

(intuitively Z_i is the set of vertices occupied by the cops immediately before the $(i+1)_{st}$ step, and A_i is the set of clear edges) such that

(I) $0 \leq i \leq n$, $A_i \subseteq E(D)$, $Z_i \subseteq V(D)$,

(II) $0 \leq i \leq n$, *any vertex which is* a head of *an edge in* $E(D) \setminus A_i$ *and* tail of *an edge in A_i is in Z_i,*

(III) $A_0 = \emptyset$, $A_n = E(D)$,

(IV) *(List of possible steps) for* $1 \leq i \leq n$, *either*

 (a) *(placing new cops)* $Z_i \supseteq Z_{i-1}$, *and* $A_i = A_{i-1}$, *or*

 (b) *(removing cops)* $Z_i \subset Z_{i-1}$, *and A_i is the set of edges f such that every directed path containing an edge of* $E(D) \setminus A_{i-1}$ before f in order, *has an internal vertex in Z_i (note that* $A_i \subseteq A_{i-1}$*) or*

 (c) *(*tail *searching e)* $Z_i = Z_{i-1}$ *and* $A_i = A_{i-1} \cup \{e\}$ *for some edge* $e = (u, v) \in E(D) \setminus A_{i-1}$ *with* $u \in Z_{i-1}$, *or*

 (d) (*sliding*) $Z_i = (Z_{i-1} \setminus \{u\}) \cup \{v\}$ *for some* $u \in Z_{i-1}$ *and* $v \in V(D) \setminus Z_{i-1}$ *and there is an edge* $e = (v, u) \in E(D)$ *such that every other edge with head u belongs to* A_{i-1}, *and* $A_i = A_{i-1} \cup \{e\}$, *or*

 (e) *(*extension*)* $Z_i = Z_{i-1}$, *and* $A_i = A_{i-1} \cup \{e\}$ *for some edge* $e = (u, v) \in E(D) \setminus A_{i-1}$, *where* $v \in V(D) \setminus Z_i$ *and every (possibly 0) edge with head u belongs to* A_{i-1}.

If $|Z_i| \leq k$ for $1 \leq i \leq n$, then we say that there is a *mixed-search with k cops*, denoted by $ms(D) \leq k$.

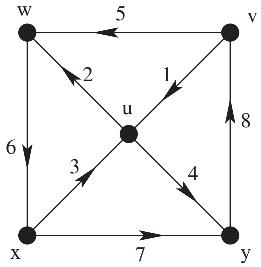

Fig. 1.

Example 1. We give a mixed-search of the graph in Fig. 1 with 2 cops.
$(\emptyset, \emptyset) \xrightarrow{(a)} (\emptyset, \{u, v\}) \xrightarrow{(c)} (1, \{u, v\}) \xrightarrow{(e)} (\{1, 2\}, \{u, v\}) \xrightarrow{(e)} (\{1, 2, 4\}, \{u, v\}) \xrightarrow{(e)} (\{1, 2, 4, 5\}, \{u, v\}) \xrightarrow{(d)} (\{1, 2, 4, 5, 8\}, \{u, y\}) \xrightarrow{(d)} (\{1, 2, 4, 5, 7, 8\}, \{u, x\}) \xrightarrow{(c)} (\{1, 2, 3, 4, 5, 7, 8\}, \{u, x\}) \xrightarrow{(d)} (\{1, 2, 3, 4, 5, 6, 7, 8\}, \{u, w\})$

The game described in Definition 1 is a version of mixed-search, where (d) and (e) are not allowed. Hence if $\overline{cn}(D) \leq k$, then $ms(D) \leq k$.

Example 2. We give a mixed-search of the graph in Fig. 1 without (d) and (e) using 3 cops.
$(\emptyset, \emptyset) \xrightarrow{(a)} (\emptyset, u) \xrightarrow{(c)} (2, u) \xrightarrow{(c)} (\{2, 4\}, u) \xrightarrow{(a)} (\{2, 4\}, \{u, v\}) \xrightarrow{(c)} (\{1, 2, 4\}, \{u, v\}) \xrightarrow{(c)} (\{1, 2, 4, 5\}, \{u, v\}) \xrightarrow{(a)} (\{1, 2, 4, 5\}, \{u, v, y\}) \xrightarrow{(c)} (\{1, 2, 4, 5, 8\}, \{u, v, y\}) \xrightarrow{(b)} (\{1, 2, 4, 5, 8\}, \{u, y\}) \xrightarrow{(a)} (\{1, 2, 4, 5, 8\}, \{u, x, y\}) \xrightarrow{(c)} (\{1, 2, 4, 5, 7, 8\}, \{u, x, y\}) \xrightarrow{(c)} (\{1, 2, 3, 4, 5, 7, 8\}, \{u, x, y\}) \xrightarrow{(b)} (\{1, 2, 3, 4, 5, 7, 8\}, \{u, x\}) \xrightarrow{(a)} (\{1, 2, 3, 4, 5, 7, 8\}, \{u, x, w\}) \xrightarrow{(c)} (\{1, 2, 3, 4, 5, 6, 7, 8\}, \{u, x, w\})$

A mixed-search of D is called *monotone*, if every edge of D is cleared exactly once. This is the same as saying that the cleared edges form a monotone increasing set.

If $X \subseteq E(D)$, let $\delta(X)$ be the set of vertices, which are both the tail of an edge in X and the head of an edge in $E(D) \setminus X$. We call these vertices *dangerous*.

Lemma 2. *The parameter* $|\delta|$ *satisfies the submodular inequality, i.e.* $|\delta(X \cap Y)| + |\delta(X \cup Y)| \leq |\delta(X)| + |\delta(Y)|$ *for any vertex sets* X *and* Y.

Proof. We have to prove that every dangerous vertex counted on the left-hand side is also counted at least as many times on the right-hand side.

– If $v \in \delta(X \cap Y)$, then by definition there is an edge e with head v and not in $X \cap Y$, and also an edge f with tail v and in $X \cap Y$. Hence $f \in X$, $f \in Y$, but $e \notin X$ or $e \notin Y$.

– If $v \in \delta(X \cup Y)$, then with similar notation $e \notin X$, $e \notin Y$, and $f \in X$ or $f \in Y$. Hence if a vertex v is counted on the left-hand side, then it is also counted on the right-hand side.
– If $v \in \delta(X \cap Y)$ and $v \in \delta(X \cup Y)$, then $v \in \delta(X)$ and $v \in \delta(Y)$ too. So if v is counted twice on the left-hand side , then it is also counted twice on the right-hand side. □

With the directed notions given above, we use (2.1)–(2.4) of [3] to achieve the monotonicity result.

Definition 5. *A* raid *in D is a sequence* $(X_0, X_1, \dots, X_n)$ *of subsets of* $E(D)$ *such that* $X_0 = \emptyset$, $X_n = E(D)$, *and* $|X_i \setminus X_{i-1}| \leq 1$, *for* $1 \leq i \leq n$ *(i.e. at most one new clear edge).*
The raid uses at most k cops if $|\delta(X_i)| \leq k$ *for* $0 \leq i \leq n$.

Lemma 3. *If* $ms(D) \leq k$, *then there is a raid in D using at most k cops.*

Proof. Let $(A_0, Z_0), \dots, (A_n, Z_n)$ be a mixed-search in D with each $|Z_i| \leq k$. Then each $\delta(A_i) \subseteq Z_i$, hence each $|\delta(A_i)| \leq k$, and also $|A_i \setminus A_{i-1}| \leq 1$ by definition, so $(A_0, \dots, A_n)$ is a raid using at most k cops. □

Definition 6. *A raid is* progressive *if* $X_0 \subset \cdots \subset X_n$. *Hence there is always a new clear edge and* $|X_i \setminus X_{i-1}| = 1$.

Lemma 4. *Suppose there is a raid in D using at most k cops. Then there is a progressive raid in D using at most k cops.*

Proof. Choose a raid $X_0, \dots, X_n$ with at most k cops such that

(1) $\sum_{i=0}^{n} |\delta(X_i)|$ is minimum, and subject to (1),
(2) $\sum_{i=0}^{n} |X_i|$ is minimum.

We are going to show that $X_0 \subseteq \cdots \subseteq X_n$ is progressive.

(3) $|X_j \setminus X_{j-1}| = 1$, for $1 \leq j \leq n$.

For $|X_j \setminus X_{j-1}| \leq 1$, and if $|X_j \setminus X_{j-1}| = 0$, then $X_j \subseteq X_{j-1}$, and $(X_0, \dots, X_{j-1}, X_{j+1}, \dots, X_n)$ is a raid with at most k cops, contradicting (1)–(2).

(4) $|\delta(X_{j-1} \cup X_j)| \geq |\delta(X_j)|$.

For otherwise $|\delta(X_{j-1} \cup X_j)| < k$, hence $(X_0, \dots, X_{j-1}, X_{j-1} \cup X_j, X_{j+1}, \dots, X_n)$ is a raid with at most k cops, contradicting (1).

(5) $X_{j-1} \subseteq X_j$.

From the submodularity

$$|\delta(X_{j-1} \cap X_j)| + |\delta(X_{j-1} \cup X_j)| \leq |\delta(X_{j-1})| + |\delta(X_j)|$$

From (4) it follows that $|\delta(X_{j-1} \cap X_j)| \le |\delta(X_{j-1})|$. Hence $(X_0, \dots, X_{j-2}, X_{j-1} \cap X_j, X_j, \dots, X_n)$ is a raid with at most k cops. From (2) $|X_{j-1} \cap X_j| \ge |X_{j-1}| \Rightarrow X_{j-1} \subseteq X_j$. □

Lemma 5. *Let* $(X_0, X_1, \dots, X_n)$ *be a progressive raid with at most k cops, and for* $1 \le j \le n$ *let* $X_j \setminus X_{j-1} = \{e_j\}$. *Then there is a monotone mixed-search of D using at most k cops such that the edges of D are cleared in the order* $e_1, \dots, e_n$.

Proof. We construct the monotone mixed-search inductively. Suppose that $1 \le j \le n$, and we have cleared the edges $e_1, \dots, e_{j-1}$ in order such that no other edges have been cleared yet. Let H be the set of all vertices $v \in V(D)$ such that every edge having v as its head is in X_{j-1}, i.e. the non-dangerous vertices. Certainly each vertex in $\delta(X_{j-1})$ is currently occupied by a cop. Remove all other cops. Since $e_j \notin X_{j-1}$, its head is not in H. Let N be the set of ends of e_j.

If $|N \cup \delta(X_{j-1})| \le k$, we may place a new cop on the tail of e_j and do tail searching.

So assume $|N \cup \delta(X_{j-1})| > k$. Since $|\delta(X_{j-1})| \le k$, $N \not\subseteq \delta(X_{j-1})$. Choose $v \in N \setminus \delta(X_{j-1})$. Assume that v is the tail of e_j. There are two possibilities.
First that v is a head of an edge not in X_{j-1}. In that case $v \in \delta(X_j)$. Hence $\delta(X_{j-1}) \not\subseteq \delta(X_j)$. Choose $u \in \delta(X_{j-1}) \setminus \delta(X_j)$. Then $u \in N$, $(v, u) = e_j$ and u is not the head of any other edge in $E(D) \setminus X_{j-1}$ except e_j. Hence we can slide on e_j.
Secondly that all edges with head v are in X_{j-1}. Then e_j can be cleared by extension.

Assume now that v is the head of e_j. Then the tail u is either in H, or in $\delta(X_{j-1})$. In the former case e_j can be declared clear by extension and in the latter case by tail searching. □

Summarizing the previous claims, we get

Theorem 1. *If there is a mixed-search of D with at most k cops, then there is a monotone mixed-search of D with at most k cops.*

4. Equivalence of *dpw* and $\overline{cn}$

Our goal was to deduce a result extending Theorem 1, which includes directed path-width. First we have to transform edge-monotonicity to vertex-monotonicity. The next result is almost trivial, and can be proved in various ways.

Lemma 6. *If there exists a monotone mixed-search* (*i.e. no edge is cleared twice*) *of D with at most k cops avoiding sliding and extension, then there is also a monotone capture* (*i.e. where no vertex is revisited by the cops*) *of D with at most k cops.*

Proof. A simple greedy argument works here. Let us consider a search S satisfying the assumptions. Monotonicity of the search implies that the cops are never removed from a dangerous vertex. Let us consider a vertex v and edges $e_1, \dots, e_t$ with tail v. Assume that among them e_1 is the first edge to become clear in S at step i. Let us tail search all the other edges $e_2, \dots, e_t$ immediately after step i shifting the rest of the steps back, yielding a modified monotone search S'. Suppose that in S all cops were removed from v in a later step j. Since e_1 was not recontaminated in S the same holds for $e_2, \dots, e_t$ in S'. Notice that if new cops were placed on v after step j in S, that can be ignored in S'.

Doing a similar rearrangement for each vertex yields a search, which is also non-vertex-revisiting, so a monotone capture. □

We show that dpw and $\overline{cn}$ can differ by at most one.

Theorem 2. *For $k \geq 1$ the following downward implications apply:*

(0) *There is a monotone capture of an invisible robber in D with at most k cops.*
(i) *The directed path-width of D is at most $k - 1$.*
(ii) *There is a capture of an invisible robber in D with at most k cops.*
(iii) *There is a monotone capture of an invisible robber in D with at most $k + 1$ cops.*

We have already mentioned that $\overline{cn}(D) \leq k$ is equivalent to a mixed-search with at most k cops avoiding sliding and extension. Hence we keep the mixed-search language when appropriate and always show that sliding and extension were not used.

Proof. (0)⇒(i). Assume there is a monotone capture with at most k cops in D. If we simulate the moving of the cops, every vertex is occupied precisely once. The cop-moves can be arranged in such a way, that in every move one cop takes off and he lands immediately. Let the set of vertices occupied by the cops after the i^{th} move be called W_i.

We notice that the series W_i is a *DPD* having width at most k.

(i)⇒(ii). By assumption there exists a *DPD* where the W_i's have size at most k. First the cops occupy W_1. All the edges induced by W_1 can be tail searched as well as those having their tail in W_1. Then the cops on $W_1 \setminus W_2$ take off and the cops not in $W_1 \cap W_2$ fly to $W_2 \setminus W_1$. Now the edges with tail in W_2 can be tail searched. Continuing this all edges will be cleared. We did not use (d) and (e), hence this is a capture with at most k cops. Notice that we actually proved the stronger implication (i)⇒(0).

(ii)⇒(iii). If D is a directed graph, let D^d denote the directed graph, where every edge of D is duplicated. We have to show that $\overline{cn}(D) \leq ms(D^d) + 1$. Observe that in any mixed-search of D^d the first of the duplicated edges may not be cleared by sliding. So every edge of D^d must be cleared by either tail searching or extension. When one of them is cleared by tail searching, then also the other one can be done immediately after. The extension steps might need an extra cop to handle. If all k cops are occupying a dangerous vertex when some extension steps are applied, then we replace those steps by tail searching with an extra cop. The edges cleared by extension between two other type of clearance form acyclic graphs, hence one extra cop is sufficient.

Theorem 1 says that $ms(D^d) \leq K$ implies the existence of a monotone mixed-search M of D^d with the same number of cops. Hence, we may use the previous argument for the monotone search, yielding M' avoiding both sliding and extension with an extra cop. Then by Lemma 6, M' can be seen as a monotone capture in D^d, and M' also yields a monotone capture in D itself. □

We strongly believe that (ii)⇒(0) can be proved with a more technical argument. Hence Definition 1 and Definition 2 lead to exactly the same parameter. That would also imply the monotonicity of our game as conjectured in the introduction.

5. Blockages in directed graphs

In the previous section we described the winning strategy from the cops' point of view. We may also ask the opposite question; what is favourable for the robber? Is there something he can look for to ensure his escape? The notion of a blockage was introduced by Bienstock et al. in [4] as obstructions for having small path-width. Thomas asked whether $dpw(D) \leq k-1$ implies (equivalent to?) the non-existence of a blockage of order k. As in the previous section the main task here is to define the concepts for directed graphs such that the undirected theory goes through.

Let $X \subseteq V(D)$. The attachment $att(X) = \{x \in X : \exists y \in V(D) \setminus X$ such that $(y,x) \in E(D)\}$, and $\alpha(X) = |att(X)|$. The complement $X^c = (V(D) \setminus X) \cup att(X)$. The subset $Y \subset V(D)$ is a complement of X if $X^c \subseteq Y$. The sets X and Y are complementary if $X^c \subseteq Y$ or $Y^c \subseteq X$ or both.

Observe that $(X^c)^c \subseteq X$ is not always true. If X and Y are complementary and $|X \cap Y| \leq k$, then $\alpha(X) \leq k$ or $\alpha(Y) \leq k$.

Definition 7. *Let $k \geq 0$ be an integer. A* blockage *(in D, of order k) is a set $\mathcal{B}$ such that*

(i) *each $X \in \mathcal{B}$ is a subset of $V(D)$ with $\alpha(X) \leq k$,*
(ii) *if $X \in \mathcal{B}$ and $Y \subseteq X$ and $\alpha(Y) \leq k$, then $Y \in \mathcal{B}$,*
(iii) *if X_1 and X_2 are complementary and $|X_1 \cap X_2| \leq k$, then $\mathcal{B}$ contains exactly one of X_1, X_2.*

We call these criteria the blockage axioms.

Considering axiom (iii), it can happen that $\alpha(X_1) \leq k$ but $\alpha(X_2) > k$ or vice versa. In such a case axiom (i) determines which one of X_1, X_2 is the set in $\mathcal{B}$.

Lemma 7. *Let $\mathcal{B}$ be a blockage of order k in D, let $X \in \mathcal{B}$, and let $Y \subseteq V$ with $\alpha(Y) \leq k$ and $|(Y \setminus X) \cup att(X)| \leq k$. Then $Y \in \mathcal{B}$.*

Proof. Since X and X^c are complementary, $|X \cap X^c| = \alpha(X) \leq k$, and $X \in \mathcal{B}$, axiom (iii) implies that $X^c \notin \mathcal{B}$. Since $att(X \cup Y) \subseteq att(X) \cup (V \setminus X) = X^c$, the sets $X \cup Y$ and X^c are complementary. Also $|(X \cup Y) \cap X^c| = |(Y \setminus X) \cup att(X)| \leq k$, hence axiom (iii) implies that $X \cup Y \in \mathcal{B}$. Now $\alpha(Y) \leq k$, so $Y \in \mathcal{B}$ by axiom (ii). □

With the above concepts [4] yields one implication concerning blockages and directed path-width.

Lemma 8. *If the directed path-width of D is at most $k-1$, then there is no blockage of order k in D.*

Proof. Assume to the contrary that there is a blockage $\mathcal{B}$ of order k in D. By assumption $dpw(D) < k$. Let $(W_1, \dots, W_m)$ be a *DPD*, where each $|W_i| \leq k$. Since $\emptyset$ and $V(D)$ are complementary and $\emptyset \subset V$, it follows from axioms (ii) and (iii) that $\emptyset \in \mathcal{B}$. From Lemma 7 $W_1 \in \mathcal{B}$ too. For $1 \leq i \leq k$, let $X_i := W_1 \cup \cdots \cup W_i$, and choose i maximum with $X_i \in \mathcal{B}$. Now $i \neq m$, because $V \notin \mathcal{B}$. Moreover, $att(X_i) \subseteq W_{i+1}$ by the definition of DPD. So $|(X_{i+1} \setminus X_i) \cup att(X_i)| \leq |W_{i+1}| \leq k$. By Lemma 7 $X_{i+1} \in \mathcal{B}$, contrary to the maximality of i. □

Most likely blockages do not fully characterize directed path-width. We informally mention here a related conjecture of Thomas without specifying what a 'minor' is.

Conjecture 1. If the directed path-width of D is large, then it has a big cylindrical grid minor or a big binary tree minor with each edge replaced by two anti-parallel edges.

6. Related questions

Let us consider another version of cops-and-robber games defined in [8].

Definition 8. *Let a directed graph D be given. The cops are either standing on a vertex or in a helicopter (temporarily removed from the game). The robber stands on a vertex of D, and can at any time with infinite speed run to another vertex* in the same strong component of $D \setminus Z$, *where Z is the set of vertices occupied by the cops. In other words, the robber can only move from a to b, if there is also a cop-free directed path from b to a. The goal is to decide how many cops are necessary to capture the robber. Denote this minimum by* $cn^*(D)$ *if the robber is visible, and* $\overline{cn}^*(D)$ *if the robber is invisible.*

In the aforementioned [8] only the visible case was considered. Observe the following immediate connections between the parameters.

$$cn(D) \leq \overline{cn}(D) \text{ and } cn^*(D) \leq \overline{cn}^*(D)$$
$$cn(D) \geq cn^*(D) \text{ and } \overline{cn}(D) \geq \overline{cn}^*(D)$$

We show that cn^* and cn behave differently. We recall a construction of Thomassen from [12]. It was used to prove that for each natural number k, there exists a digraph with no even cycle such that each vertex has outdegree k.

The digraph D_k is constructed recursively. Clearly D_1 exists, let it be a directed cycle of length 3.

Suppose that D_k is given. Then D_{k+1} is obtained by adding, for each vertex x of D_k, a set A_x of $k+1$ vertices and a vertex x' such that each vertex of A_x dominates x and the out-neighbours of x in D_k. Also let x' dominate the vertices of A_x and let x dominate x'. Observe that each vertex of D_{k+1} has outdegree $k+1$.

Lemma 9. *Let* D_k *be the digraph defined above. Then* $cn^*(D_k) = 2$ *while* $cn(D_k) > k$.

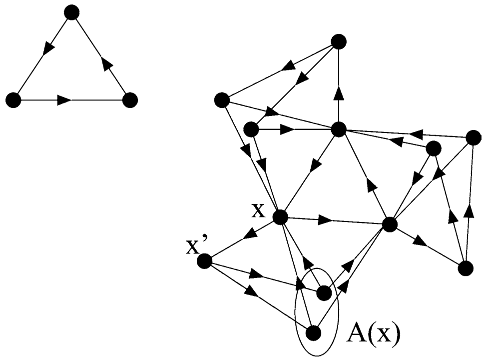

Fig. 2. The graphs D_1 and D_2

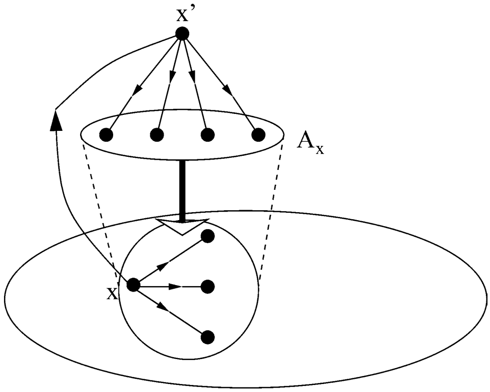

Fig. 3. The recursive step for vertex x

Proof. The latter statement can be seen from Definition 1. Every vertex has outdegree k, hence the robber has always a choice where to move when a cop is approaching his vertex, unless there are more than k cops.

The former claim is shown by induction. Assume we have proved it for some k. Let us consider the game on D_{k+1}. However, the cops will look at D_k instead and mimic the game there. The recursive step of the construction blows up each vertex of D_k. Let us consider the opposite operation and contract $A_x \cup x'$ to x for all x to get D_k from D_{k+1}. Notice that any robber move on D_{k+1} remains a valid robber move after this contraction, but now on D_k. The two cops catch the robber on D_k by assumption. When they claim the robber to be captured in D_k at vertex x, probably now the robber is not in x but in $A_x \cup x'$. However leaving one cop on x, the other cop catches the robber on $A_x \cup x'$ since all directed cycles leaving A_x contain x. □

As we have seen $\delta^+(D) \geq k$ implies $cn(D) > k$. The opposite implication is not true, however. Let $D_{k,s}$ be the Cartesian product of two directed cycles C_k and C_s. Then the outdegree is two at each vertex, but $cn(D) > \min(k, s)$. To prove this, we may repeat the argument which worked in the undirected case for the $k \times s$ grid [6]. So let $D_{k,s}$, $2 \leq k \leq s$ be drawn as a grid with k rows and s columns. Assume there are at most k cops. Then the robber can not be captured immediately, just placing the cops on the graph. So the cops have to move. However, when a cop moves, then there are at most $k-1$ cops on $D_{k,s}$. Hence leaving at least one row cop-free. We tell the robber to occupy that row. It is easy to see that the robber can follow this advice and escapes with this strategy.

To compare the parameters for the invisible robber, that is $\overline{cn}(D)$ and $\overline{cn}^*(D)$ is an open problem. To give some hint, we conjecture that $\overline{cn}(D) \leq C \cdot \overline{cn}^*(D)$, where the constant C might be 2.

Acknowledgements. We would like to thank Robin Thomas for helpful correspondence, Carsten Thomassen for fruitful discussions and providing us the construction in [12] and Péter Hajnal for 'playing' cops-and-robber games with us. We are also grateful the referees, who have pointed out numerous inaccuracies and a relevant recent reference. Their suggestions made the presentation considerably more transparent.

References

1. Adler, I.: A note on monotonicity in digraph searching. Manuscript, 2003
2. Barát, J.: Width-type graph parameters (PhD-Thesis). University of Szeged, Hungary, 2001
3. Bienstock, D., Seymour, P.: Monotonicity in Graph Searching. J. Algo. **12**, 239–245 (1991)
4. Bienstock, D., Robertson, N., Seymour, P., Thomas, R.: Quickly Excluding a Forest. J. Combin. Theory Ser. B **52**, 274–283 (1991)
5. Dendris, N.D., Kirousis, L.M., Thilikos, D.M.: Fugitive-search games on graphs and related parameters. Theoret. Comput. Sci. **172**, 233–254 (1997)
6. Diestel, R.: Graph Theory. Springer, 1997; xiv+289 pp. and second edition (Springer 2000) xiv+313 pp.
7. Fomin, F.V., Thilikos, D.M.: On the monotonicity of games generated by symmetric submodular functions. Discrete Appl. Math. **131**, 323–335 (2003)
8. Johnson, T., Robertson, N., Seymour, P.D., Thomas, R.: Directed Tree-Width. J. Combin. Theory Ser. B **82**, 138–154 (2001)
9. Reed, B.: Introducing Directed Tree Width, 6th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1999), 8 pp. (electronic), Electron. Notes Discrete Math. **3**, Elsevier, Amsterdam, 1999
10. Thomas, R.: Tree-decomposition of graphs. Class notes, 1996, http://www.math.gatech.edu/~thomas/tree.ps
11. Thomas, R.: Personal communication
12. Thomassen, C.: Even Cycles in Directed Graphs. Europ. J. Combinatorics **6**, 85–89 (1985)

Received: December 15, 2003
Final version received: March 7, 2005