

Beyond the Existential Theory of the Reals

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Abstract

We show that completeness at higher levels of the theory of the reals is a robust notion (under changing the signature and bounding the domain of the quantifiers). This mends recognized gaps in the hierarchy, and leads to stronger completeness results for various computational problems. We exhibit several families of complete problems which can be used for future completeness results in the real hierarchy. As an application we answer an open question by Jungeblut, Kleist, and Miltzow on the complexity of two semialgebraic sets having Hausdorff distance 0, and sharpen some results by Bürgisser and Cucker.

Keywords: existential theory of the real numbers, theory of the reals, real hierarchy, computational complexity, semialgebraic sets, Hausdorff distance.

1 Introduction

Three recent papers have captured the complexity of computational problems at the second level of the theory of the real numbers: Jungeblut, Kleist and Miltzow [20] show that computing the Hausdorff distance of two semialgebraic sets is as hard as the $\forall\exists_{<}$ -fragment of the theory of the reals (with $<$ as the only relation, no negation); D’Costa, Lefaucheux, Neumann, Ouaknine, and Worrell [14] study a problem in linear dynamical system, whose complement is as also as hard as the $\forall\exists_{<}$ -fragment; and Dobbins, Kleist, Miltzow and Rzażewski [15, 16] showed that a version of area universality is as hard as the $\forall\exists$ -fragment (with no restriction on relations).

The different restrictions on which relations are allowed suggest the possibility that there are distinct sublevels of complexity at the second level of the theory of the reals. We show in Section 2 that this is not the case: there

is only one such level of complexity. Previously, such a result was known only for the first level of the hierarchy [33], justifying the complexity class $\exists\mathbb{R}$, which corresponds to the existential theory of the reals. The robustness results of this paper justify the *real (complexity) hierarchy* corresponding to all finite levels of the theory of the reals. We identify several (very restricted) families of complete problems for the real hierarchy. These problems should be useful in future hardness proofs.

As a first step towards testing that claim, we round of the paper in Section 3 by looking at the computational complexity of various problems about semialgebraic sets, extending and sharpening relevant work by Bürgisser and Cucker [12]. The description of some of these problems require exotic (non-standard) quantifiers, which we show can be eliminated in nearly all the natural problems considered by Bürgisser and Cucker.

Before we can state our results formally, we introduce the theory of the reals in some more depth, describe the encoding of computational problems, and discuss how our work relates to the Blum-Shub-Smale model of real computation.

The Theory of the Reals

The *theory of the reals* is the set of all sentences (that is, no free variables) true over the real numbers. It includes statements such as $(\exists x) x^2 > 0$, $(\forall y)(\exists x) x^2 = y \vee x^2 = -y$, and $(\exists x)(\forall y_1, y_2)(\exists z) x + y_1 = y_1 + x \wedge y_2 + z = x$. Tarski [39] showed that the theory of the reals, and thereby analytic geometry, is decidable.

The theory of the reals is highly expressive. Many natural problems in computational geometry, graph drawing and other areas can be expressed in small fragments of the theory, most prominently, the existential fragment, in which only existential quantifiers are allowed. Within the past decade it has become increasingly evident that the existential fragment not only expresses many of these problems, but captures their computational, and sometimes algebraic, complexity precisely. This led to the introduction of a complexity class, $\exists\mathbb{R}$ (we read this as “exists \mathbb{R} ”), the complexity of deciding truth in the *existential theory of the reals*. This is analogous to \mathbf{NP} , which captures the complexity of deciding truth of existentially quantified statements over $\{0, 1\}$. Recent examples of problems shown to be $\exists\mathbb{R}$ -complete include the art gallery problem [2], polygon coverings [1], continuous constraint satisfaction problems [26], Nash equilibria [6], and training neural networks [7]. For a very enjoyable and thorough introduction to the existential theory of the reals, see Matoušek [25]; for a (partial) list of $\exists\mathbb{R}$ -complete problems,

see [40].

Just as in the case of **NP**, attention has extended beyond the existential fragment. Many computational problems very naturally involve quantifier alternation. Take, as an example, the notion of Hausdorff distance. Given two sets $A, B \subseteq \mathbb{R}^n$, the *directed Hausdorff distance from A to B* is $\sup_{x \in A} d(x, B)$, where $d(x, B) = \inf_{y \in B} \|x - y\|$. The *Hausdorff distance*, $d_H(A, B)$, of two sets A and B is the minimum of the directed Hausdorff distances from A to B and from B to A . If A and B are semialgebraic, membership in A and B is encoded by Boolean formulas $\varphi(x)$ and $\psi(y)$ in the quantifier-free fragment, so $d_H(A, B) \leq r$ can be expressed as

$$(\forall \varepsilon > 0)(\forall a, b)(\exists a', b') \\ (\varphi(a) \wedge \psi(b)) \rightarrow (\psi(b') \wedge \varphi(a') \wedge \max(\|a - b'\|, \|b - a'\|) < r + \varepsilon),$$

which belongs to the $\forall\exists$ -fragment of the theory of the reals. But is it as hard as deciding truth in that fragment?

That question turns out to be somewhat subtle. It is straightforward to define complexity classes $\Sigma_k \mathbb{R}$ and $\Pi_k \mathbb{R}$ that correspond to deciding the truth in the $\exists\forall \dots$ and $\forall\exists \dots$ ($k - 1$ quantifier alternations) fragments of the theory, but showing hardness for these classes has run into issues. What happened?

The general theory of the reals allows a large signature, including function symbols $\{0, 1, +, \cdot\}$ and relation symbols $\{<, \leq, =, >, \geq\}$.¹ Clearly, the expressiveness of the language does not change if we drop $>$ and \geq (though we will keep using them, since they can easily be expressed by exchanging lhs and rhs of the inequality and flipping the sign); but what happens with $=$, $<$ and \leq ? To make the question non-trivial, we have to disallow negation; because of trichotomy, $x < y \vee x = y \vee y < x$, that is not an onerous restriction.

There are seven non-trivial combinations of $\{<, \leq, =\}$, but many of these are easily seen to be equivalent (in terms of expressiveness). For example, once we have $<$ and $=$, we have \leq , and once we have \leq , we have $=$. It follows that there are only four signatures of interest: $\{<\}$, $\{\leq\}$, $\{=\}$ and $\{<, \leq, =\}$.

When studying the existential theory of the reals in [33], we identified only two of these: $\exists \mathbb{R}_{<}$, corresponding to $\{<\}$ and $\exists \mathbb{R}$, corresponding to $\{<, \leq, =\}$. There is a reason for that: with an existential quantifier we can define $a < b$ as $(\exists x) ax^2 + 1 \leq bx^2$, and $a \leq b$ as $(\exists x) a + x^2 = b$, so

¹We do allow abbreviations, e.g. we write powers such as x^4 , but we understand that to be shorthand for $x \cdot x \cdot x \cdot x$.

in the existential theory, only signatures $\{<\}$ and $\{<, \leq, =\}$ are of interest. Since $\exists\mathbb{R}_{<} = \exists\mathbb{R}$, as we showed in [33], $\exists\mathbb{R}$ is very robust as a complexity class: Problems identified earlier as $\exists\mathbb{R}_{<}$ -complete, such as rectilinear crossing number [8] and segment intersection graphs [22], and problems identified as $\exists\mathbb{R}$ -complete, such as the pseudoline stretchability problem [27, 34] are all computationally equivalent (even if they differ algebraically).

As researchers began exploring the second level of the theory, similar gaps became apparent. D’Costa, Lefauchaux, Neumann, Ouaknine, and Worrell [14] suggest three variants of each class: $\Sigma_k\mathbb{R}$, $\Sigma_k^<\mathbb{R}$, and $\Sigma_k^{\leq}\mathbb{R}$. They observe that $\Sigma_k\mathbb{R} = \Sigma_k^<\mathbb{R}$ for $k = 2$, since any equality $a = b$ can be replaced with $(\forall\varepsilon)(a - b)^2 < \varepsilon^2$, and the universal quantifier can be added to the final, universal, block of quantifiers; clearly the same argument applies for all even $k \geq 2$, and to $\Pi_k\mathbb{R}$ if $k \geq 1$ is odd, e.g. for $\forall\mathbb{R}$. D’Costa *et al.* did not consider the variant $\Sigma_k^=\mathbb{R}$.

In this terminology, D’Costa, Lefauchaux, Neumann, Ouaknine, and Worrell [14] showed that the compact escape problem is complete for $\Sigma_2^<\mathbb{R}$ (which implies that the complement of the problem is $\Pi_2^<\mathbb{R}$ -complete). Dobbins, Kleist, Miltzow and Rzażewski [15, 16] showed that a variant of area universality is $\Pi_2\mathbb{R}$ -complete. And Jungeblut, Kleist and Miltzow [20] showed that the Hausdorff distance of two semialgebraic sets is $\Pi_2^<\mathbb{R}$ -complete. Earlier, Bürgisser and Cucker [12] showed that testing whether a rational function is surjective is $\Pi_2\mathbb{R}$ -complete.

Our main contribution in this paper is to show that all these variants, with one exception, are the same, not just at the second level, but at all finite levels.

Theorem 1.1. *We have $\Sigma_k\mathbb{R} = \Sigma_k^<\mathbb{R}$ and $\Pi_k\mathbb{R} = \Pi_k^<\mathbb{R}$ for all $k \geq 1$.*

It follows that computing the Hausdorff distance of two semialgebraic sets is $\Pi_2\mathbb{R}$ -complete [20]. Theorem 1.1 resolves an open question by D’Costa, Lefauchaux, Neumann, Ouaknine, and Worrell [14].

Corollary 1.2. *We have $\Sigma_k\mathbb{R} = \Sigma_k^{\leq}\mathbb{R}$ and $\Pi_k\mathbb{R} = \Pi_k^{\leq}\mathbb{R}$ for all $k \geq 1$.*

The corollary implies that the compact escape problem, as defined in [14], is $\Sigma_2\mathbb{R}$ -complete.

With an eye towards future hardness reductions, we also show that the problems remain hard for a bounded universe. Specifically, we consider $I = (-1, 1)$, the *bounded open* case and $I = [-1, 1]$ the *bounded closed* case. We will use prefixes *bo-* and *bc-* to distinguish these variants of the complexity classes.

Corollary 1.3. *We have $\Sigma_k \mathbb{R} = \text{bc-}\Sigma_k^< \mathbb{R} = \text{bo-}\Sigma_k^< \mathbb{R}$ and $\Pi_k \mathbb{R} = \text{bc-}\Pi_k^< \mathbb{R} = \text{bo-}\Pi_k^< \mathbb{R}$ for all $k \geq 1$.*

This result generalizes earlier work by the authors [33] for $k = 1$, and D’Costa, Lefauchaux, Neumann, Ouaknine, and Worrell [14] for $k = 2$.

Theorem 1.1, as well as Corollaries 1.2 and 1.3 can be sharpened by exhibiting very restricted families of sentences which are $\Sigma_k \mathbb{R}$ -complete. This is done in Propositions 2.1, 2.6, and 2.9; Figures 1 and 2 summarize the families of complete problems.

Remark 1.4 (Equality). Noticeably absent are $\Sigma_k^- \mathbb{R}$ and $\Pi_k^- \mathbb{R}$ because they behave anomalously. If the final quantifier block is existential we can define \leq and $<$ using $a \leq b \Leftrightarrow (\exists x) b = a + x^2$ and $a < b \Leftrightarrow (\exists x, x') b = a + x^2 \wedge xx' = 1$. So $\Sigma_k^- \mathbb{R} = \Sigma_k \mathbb{R}$ for all odd k and $\Pi_k^- \mathbb{R} = \Pi_k \mathbb{R}$ for all even k ; this was first shown by Dobbins, Kleist, Miltzow and Rzażewski [16, Theorem 2.2].

On the other hand, if the final quantifier block is universal, equality is weak. Using the Schwartz-Zippel lemma one can easily show that $\Pi_1^- \mathbb{R} \subseteq \text{coRP}$, so each language in $\Pi_1^- \mathbb{R}$ can be decided in randomized polynomial time, and it is quite possible that $\Pi_1^- \mathbb{R} = \text{P}$. The complexity of $\Pi_1^- \mathbb{R}$ is closely related to polynomial identity testing, a notoriously difficult field, see [35] for a paper investigating a similar model.

It may be true that $\Sigma_k^- \mathbb{R} = \Sigma_{k-1} \mathbb{R}$ for even $k \geq 2$ and $\Pi_k^- \mathbb{R} = \Pi_{k-1} \mathbb{R}$ for odd $k \geq 3$, and this may be easier to settle than the case of $\Pi_1^- \mathbb{R}$ by making use of the additional quantifiers. \blacktriangleleft

Remark 1.5 (Addition only). Sontag [37] showed that if we do not allow multiplication, the resulting complexity classes are exactly the levels of the polynomial-time hierarchy. So anybody conjecturing $\text{NP} = \exists \mathbb{R}$ or $\Sigma_k^{\text{P}} = \Sigma_k \mathbb{R}$ is conjecturing that multiplication can be eliminated at these levels. \blacktriangleleft

A Word on Encodings

Since we are working within a logical fragment, some natural objects require encoding. The theory of the reals allows only two constants: 0 and 1. This means that any number has to be written based on these; computing the number in binary using Horner’s scheme, this can be done efficiently, e.g. $13 = (1 + 1) \cdot ((1 + 1) \cdot (1 + 1) \cdot (1 + 1) + (1 + 1)) + 1$. So we will write integer constants, since we know that they can be removed without increasing the encoding length significantly. We have to be more careful with polynomials. For us, a (multivariate) polynomial is a sum of monomial terms,

such as $107x_1^3x_3^5 - 93x_2x_3^4$. In formulas we allow arithmetical terms, like $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, but in a polynomial representation, we would have to multiply out the terms, which can lead to an exponential blow-up in the encoding length. With the help of existential quantifiers, such a blow-up can be avoided using a standard Tseitin-reduction. We will use the following lemma, which is based on Lemma 3.2 in [33].

Lemma 1.6. *For every Boolean formula $\varphi(x_1, \dots, x_i)$ in real variables x_1, \dots, x_i there is a polynomial $f : \mathbb{R}^{i+j} \rightarrow \mathbb{R}$ of degree at most 4 such that*

$$\varphi(x_1, \dots, x_i) \Leftrightarrow (\exists y_1, \dots, y_j) f(x_1, \dots, x_i, y_1, \dots, y_j) = 0$$

for all x_1, \dots, x_i . The parameter j is polynomial in the length of φ .

We study computational problems over different types of sets defined by conditions on polynomials. A set S is

- *algebraic* if $S = \{x : f(x) = 0\}$ for some polynomial f ,
- *basic semialgebraic* if $S = \{x : f(x) = 0, g_1(x) \geq 0, \dots, g_n(x) \geq 0, h_1(x) > 0, \dots, h_m(x) > 0\}$ for some polynomials f, g_i , and h_j , and
- *semialgebraic* if it is the Boolean combination of basic semialgebraic sets, equivalently, if there is a Boolean formula $\varphi(x)$ consisting of conditions on polynomials such that $S = \{x : \varphi(x)\}$.

We specify algebraic and basic semialgebraic sets by their polynomials. A semialgebraic set is given by its defining formula φ . Lemma 1.6 implies that every semialgebraic set can be viewed as the projection of an algebraic set, and semialgebraic sets are sometimes specified that way: $S = \{x : (\exists y) f(x, y) = 0\}$, though we will not do so here. Semialgebraic sets can also be specified as $S = \{x : C(x) = 1\}$, where C is an *algebraic circuit*, see [12, Section 2] for a precise definition. Circuits are a very succinct way of encoding a semialgebraic set central to the BSS-model, which we discuss next. It is unlikely that algebraic circuits can be turned into formulas without significant overhead, or the addition of existential quantifiers.

The Blum-Shub-Smale Model

There is a notion of Turing-machine computations over real (and complex) numbers introduced by Blum, Shub, and Smale, known as the *BSS-model* [9]; Turing-machines in this model can have real constants, and perform addition, multiplication and comparisons of arbitrary real numbers in a single step.

In the BSS-model, a real Turing machine takes as an input a tuple of real numbers and accepts or rejects if it terminates. A real Turing machine accepts a subset of $\mathbb{R}^* = \bigcup_k \mathbb{R}^k$. In this way, we can extend computation over $\{0, 1\}$ to \mathbb{R} , or any ring.

As an example, consider the problem $4\text{-FEAS}_{\mathbb{R}}$ which asks whether a polynomial of degree at most 4 is *feasible*, that is, has a zero. A polynomial of degree at most 4 can be written $f(x) = \sum_{\alpha} c_{\alpha} x^{\alpha}$, where α ranges over $\{0, 1, 2, 3, 4\}^n$, and each α has weight at most 4. In the BSS-model we then encode f as the sequence c_{α} of its $O(n^4)$ coefficients.

In analogy with the discrete classes, Blum, Shub and Smale introduced real equivalents of \mathbf{P} and \mathbf{NP} , known as $\mathbf{P}_{\mathbb{R}}$ and $\mathbf{NP}_{\mathbb{R}}$. Given a polynomial f , a real Turing-machine with non-determinism can guess $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and verify that $f(x) = 0$. This shows that $4\text{-FEAS}_{\mathbb{R}} \in \mathbf{NP}_{\mathbb{R}}$. Blum, Shub, and Smale [10, Section 6] proved an analogue of Cook's theorem for $\mathbf{NP}_{\mathbb{R}}$ by showing that the $4\text{-FEAS}_{\mathbb{R}}$ -problem is $\mathbf{NP}_{\mathbb{R}}$ -complete. Since then, these classes, with their own, real " $\mathbf{P}_{\mathbb{R}} = \mathbf{NP}_{\mathbb{R}}?$ "-problem have been studied in depth.

From the real world, we can return to discrete computational problems in two steps: we first restrict the real Turing-machines so they are only allowed to work with real constants 0 and 1. We write $\mathbf{P}_{\mathbb{R}}^0$ and $\mathbf{NP}_{\mathbb{R}}^0$ for the resulting restricted classes. These classes are still classes of real numbers, so in a second step we discretize a class \mathcal{C} , by intersecting it with $\{0, 1\}^* = \bigcup_k \{0, 1\}^k$, resulting in the *Boolean part*, $\text{BP}(\mathcal{C})$, of \mathcal{C} . We also write $\text{BP}^0(\mathcal{C})$ for $\text{BP}(\mathcal{C}^0)$, that is, applying both discretization steps to the class \mathcal{C} at once.

Bürgisser and Cucker [11, Proposition 8.5] showed that the discrete version of the $4\text{-FEAS}_{\mathbb{R}}$ -problem: decide the feasibility of a polynomial of degree at most 4 with integer coefficients, is $\text{BP}(\mathbf{NP}_{\mathbb{R}}^0)$ -complete. Since the same problem is $\exists\mathbb{R}$ -complete [33, Theorem 4.1], this implies that $\text{BP}(\mathbf{NP}_{\mathbb{R}}^0) = \exists\mathbb{R}$, so the discretization of $\mathbf{NP}_{\mathbb{R}}$ corresponds exactly to $\exists\mathbb{R}$.

Blum, Cucker, Shub, and Smale [9, Section 21.4] also introduce the notion of a (*real*) *polynomial hierarchy*, and they identify a family of complete problems (see comment after Proposition 2.1). Bürgisser and Cucker [12, Section 9] introduced a discrete version of that hierarchy, e.g. writing $\text{BP}^0(\forall\exists)$ for the second level of that hierarchy. They identify a complete problem for each level, which is essentially the same problem we are working with, see Proposition 2.1 below. This shows that the discretized real polynomial hierarchy, and the hierarchy we introduce below are (extensionally) the same. In particular $\text{BP}^0(\forall\exists) = \mathbf{\Pi}_2\mathbb{R}$.

Intensionally, though, our two approaches differ. Bürgisser and Cucker derive the discretized problems from their real counterparts in the real Tur-

ing machine model. We start with discrete computational problems and focus on determining their complexity through their logical structure. This gives us a more fine-grained view, closer to descriptive complexity, which can distinguish variant complexity classes like $\Pi_k^<\mathbb{R}$, $\Pi_k\mathbb{R}$, and $\text{bc-}\Pi_k\mathbb{R}$, while proving that they are equal. Section 3 contains a more detailed discussion of results from [12].

Basu and Zell [5] prove a real analogue of Toda’s theorem for the BSS polynomial hierarchy by showing that computing the Betti numbers of a semialgebraic set is hard for that hierarchy.

2 A Hierarchy of Real Complexity

In Section 2.1 we define complexity classes corresponding to various fragments of the theory of the reals, establishing a *real (complexity) hierarchy*, in analogy with the polynomial-time hierarchy. Our main contribution is to show that the levels of the real hierarchy are robust under certain definitional modifications. We base our results on an effective Łojasiewicz inequality due to Solernó [36], as explained in Section 2.2. In Section 2.3 we extend our results to bounded quantification.

2.1 Leveling the Theory of the Reals

We assume sentences are in *prenex* form, that is,

$$(Q_1x_1)(Q_2x_2)\cdots(Q_ix_i)\varphi(x_1,\dots,x_i),$$

where $Q \in \{\exists, \forall\}$ and φ is a quantifier-free Boolean formula (without negation) over atomic predicates of the form $t(x_1, \dots, x_i) \bowtie 0$, where $\bowtie \in \{<, \leq, =, \geq, >\}$ and $t(x_1, \dots, x_i)$ is an arithmetical term in variables x_1, \dots, x_i ; φ is known as the *matrix* of the sentence. We call a sentence *strict* if it does not contain the predicates \leq , $=$, and \geq (and this is the reason we do not allow negation, with it, these predicates can be simulated).

We say a sentence is in Σ_k (Π_k) if $Q_1 = \exists$ ($Q_1 = \forall$) and there are $k - 1$ quantifier alternations. We let $\Sigma_k\text{-TR}$ ($\Sigma_k^<\text{-TR}$) be the set of all true (strict) sentences of the form Σ_k , and, similarly for $\Pi_k\text{-TR}$. Taking these as complete problems, we define a hierarchy of complexity classes: $\Sigma_k\mathbb{R}$ ($\Sigma_k^<\mathbb{R}$) is the set of problems that polynomial-time reduce to $\Sigma_k\text{-TR}$ ($\Sigma_k^<\text{-TR}$) and similarly for $\Pi_k\mathbb{R}$ and $\Pi_k^<\mathbb{R}$.

For small, finite k , there are alternative names for these classes, including $\Sigma_1\mathbb{R} = \exists\mathbb{R}$, $\Pi_1\mathbb{R} = \forall\mathbb{R}$, $\Pi_2\mathbb{R} = \forall\exists\mathbb{R}$, $\Sigma_2\mathbb{R} = \exists\forall\mathbb{R}$; for the strict versions we have $\exists_<\mathbb{R}$, $\forall\exists_<\mathbb{R}$ and so on.

In terms of traditional complexity, Renegar [29, 30, 31] showed that $\Sigma_k\mathbb{R}$ and $\Pi_k\mathbb{R}$ can be decided in time $O(2^{2^{O(k)}n^k})$, where n is the length of the formula. So $\Sigma_k\mathbb{R}$ and $\Pi_k\mathbb{R}$ lie in **EXP** for any fixed k (they even lie in **PSPACE**, see [4, Remark 13.10]).

By definition, $\Sigma_k^<\mathbb{R} \subseteq \Sigma_k\mathbb{R}$ and $\Pi_k^<\mathbb{R} \subseteq \Pi_k\mathbb{R}$. In Section 2.2 we will show that equality holds for all k . This closes the gap between two possible definitions of the k -th level of the real hierarchy. We take the investigation further, by sharpening the form of the complete problems for Σ_k -TR and Π_k -TR, restricting the matrix of the sentences. Let Σ_k^R -POLY (and Π_k^R -POLY) be the special case of Σ_k -TR (Π_k -TR) in which the matrix of the sentences has the form $\varphi(x_1, \dots, x_i) = f(x_1, \dots, x_i) \bowtie 0$ for a polynomial $f \in \mathbb{Z}(x_1, \dots, x_i)$ and $\bowtie \in \{=, <, \leq\}$. If we specify $\bowtie = \#$, we assume \bowtie to be $=$ if the final quantifier block of the sentence is existential, and $<$ otherwise.

$\Sigma_k^\#$ -POLY and $\Pi_k^\#$ -POLY are complete for $\Sigma_k\mathbb{R}$ and $\Pi_k\mathbb{R}$. This was first shown by Bürgisser and Cucker [12, Section 9], where the problem (with \neq in place of $<$) is called **STANDARD $^{\mathbb{Z}}(Q_1 \cdots Q_k)$** ; the same result was also observed in [14]. In the BSS-model, the result goes even farther back, to Blum, Cucker, Shub, and Smale [9, Section 21.4]; they avoid the degree increase to 8 by allowing \neq as a comparison operator.

Proposition 2.1. *$\Sigma_k^\#$ -POLY is $\Sigma_k\mathbb{R}$ -complete, and $\Pi_k^\#$ -POLY is $\Pi_k\mathbb{R}$ -complete for all $k \geq 1$. We can assume that the degree of the polynomial in the matrix is at most 8 (at most 4 if the final quantifier block is existential).*

Proof. If the final quantifier block is existential, we use a Tseitin reduction, Lemma 1.6, to find a polynomial $f \in \mathbb{Z}(x_1, \dots, x_i, y_1, \dots, y_j)$ such that $\varphi(x_1, \dots, x_i)$ is equivalent to $(\exists y_1, \dots, y_j) f(x_1, \dots, x_i, y_1, \dots, y_j) = 0$. The additional existential quantifier can be absorbed by the final quantifier block of the original sentence, since it is existential as well.

Otherwise, the final quantifier block is universal. We apply the Tseitin reduction to $\neg\varphi(x_1, \dots, x_i)$ to get a polynomial $f \in \mathbb{Z}(x_1, \dots, x_i, y_1, \dots, y_j)$ such that $\varphi(x_1, \dots, x_i)$ is equivalent to $(\forall y_1, \dots, y_j) f(x_1, \dots, x_i, y_1, \dots, y_j) \neq 0$. Since $f \neq 0$ is equivalent to $f^2 > 0$ the result follows in this case as well. Computing f^2 can at most square the encoding length and double the degree. \square

Proposition 2.1 directly implies Theorem 1.1 for those cases in which the final quantifier block is universal, for example, $\forall_{<}\mathbb{R} = \forall\mathbb{R}$ and $\exists\forall_{<}\mathbb{R} = \exists\forall\mathbb{R}$.

2.2 Bridging the Gap

We work with an effective Łojasiewicz inequality due to Solernó [36, Theorem 3]. Write $[n]$ for $\{1, \dots, n\}$.

Theorem 2.2 (Solernó). *There exists a positive integer constant c such that the following is true. Let $\Omega = \prod_{j \in [n]} [s_j, t_j] \subseteq \mathbb{R}^n$. Let $f, g \in \mathbb{R}[x_1, \dots, x_n]$ be polynomials of degree at most D in each variable. Assume that $f(x_1, \dots, x_n) = 0$ implies $g(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in \Omega$. Then there exists $\alpha > 0$ (dependent on f, g , and Ω) such that*

$$|f(x_1, \dots, x_n)| \geq \alpha |g(x_1, \dots, x_n)|^L$$

for all $x_1, \dots, x_n \in \Omega$, where $L = D^{n^c}$. If $f, g \in \mathbb{Z}[x_1, \dots, x_n]$, then we can choose $\alpha = 2^{-\ell D^{cn^2}}$, where ℓ is the number of bits in the longest coefficient of f and g .

Remark 2.3. The reader familiar with the proof that $\exists \mathbb{R} = \exists \mathbb{R}_{<}$ in [33] may remember that it was based on an explicit lower bound on positive polynomials by Jeronimo and Perrucci [19]. Why are we taking recourse to Solernó's much older, less explicit bound [36] here? The reason is that Solernó's paper is one of the last that gives bounds for polynomials with real coefficients; more recent bounds tend to assume integer coefficients, and express bounds in terms of the bit-length of the coefficients, rather than—as Solernó did—their size. While it is possible that these newer, more explicit bounds can be extended to real coefficients, this would probably require reproving these results carefully. Since we do not require explicit bounds for our results, Solernó's effective Łojasiewicz inequality is fully sufficient for our purposes. ◀

Theorem 2.2 allows us to turn an $\exists =$ into a $\forall \exists_{<}$ statement.

Theorem 2.4. *Let $f(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$ be of degree at most D in each variable. Then the sentence*

$$\Phi = (\exists x_1, \dots, x_n) f(x_1, \dots, x_n) = 0,$$

is equivalent to

$$\Psi = (\forall z > 0)(\exists y \in (0, z))(\exists x_1, \dots, x_n) \left(|f(x_1, \dots, x_n)| < y^H \right) \wedge \bigwedge_{j \in [n]} (|y x_j| < 1),$$

where $H = nD^{n^c} + (D+1)n$.

Proof. Assume that Φ is true. Let $x_1, \dots, x_n \in \mathbb{R}$ be values such that $f(x_1, \dots, x_n) = 0$. For sufficiently small $y' > 0$ we have $|y'x_j| < 1$ for all $j \in [n]$. For every $z > 0$ we can then take $y = \min\{z/2, y'\}$ to satisfy Ψ .

Assume that Ψ is true and that Φ is false. Since polynomials are continuous we can assume that $f(x_1, \dots, x_n) > 0$ for all $x_1, \dots, x_n \in \mathbb{R}^n$. Let y_1, y_2, \dots be a sequence of positive reals in $(0, 1]$ such that $y_{k+1} < y_k/2$. Corresponding to these we choose $x_1^{(k)}, \dots, x_n^{(k)}$ that satisfies Ψ for y_k , that is,

$$\left(|f(x_1^{(k)}, \dots, x_n^{(k)})| < y_k^H\right) \wedge \bigwedge_{j \in [n]} \left(|y_k x_j^{(k)}| < 1\right). \quad (1)$$

Then $(x_1^{(k)}, \dots, x_n^{(k)})_{k=1}^\infty$, as an infinite sequence in the closed space $[-\infty, \infty]^n$, contains a convergent subsequence; abusing notation we will use the same sequence for the convergent subsequence. Let $(z_1, \dots, z_n) \in [-\infty, \infty]^n$ be the limit of the sequence.

We next apply a transformation that removes infinite limit points. To that end, let S be the set of indices $i \in [n]$ for which z_i is finite, $T = 1 + \max_{i \in S} |z_i|$, and $\Omega = \prod_{j \in [n]} [s_j, t_j]$ where $s_j = -T, t_j = T$ for $j \in S$, and $s_j = 0, t_j = T$, for $j \notin S$. Define

$$\sigma_j(x) = \begin{cases} x & \text{if } z_j \in [-T, T], \\ 1/x & \text{if } z_j = \infty, \\ -1/x & \text{if } z_j = -\infty, \end{cases}$$

for $j \in [n]$,

$$g(x_1, \dots, x_n) = \prod_{j \in [n] \setminus S} x_j,$$

and

$$W(x_1, \dots, x_n) = (g(x_1, \dots, x_n))^D f(\sigma_1(x_1), \dots, \sigma_n(x_n)).$$

Then $W(x_1, \dots, x_n)$ is a polynomial of degree at most D in each variable. Since f is positive, it follows that $W(x_1, \dots, x_n) = 0$ implies $g(x_1, \dots, x_n) = 0$ for all (x_1, \dots, x_n) . By Theorem 2.2, there exists $\alpha > 0$ such that for all $(y_1, \dots, y_n) \in \Omega$ we have

$$W(y_1, \dots, y_n) \geq \alpha (g(y_1, \dots, y_n))^L, \quad (2)$$

where $L = D^{n^c}$.

Let $(\hat{z}_1, \dots, \hat{z}_n) = (\sigma_1(z_1), \dots, \sigma_n(z_n)) \in \Omega$. Note that

$$\left(\sigma_1(x_1^{(k)}), \dots, \sigma_n(x_n^{(k)})\right)_{k=1}^\infty$$

converges to $(\hat{z}_1, \dots, \hat{z}_n)$ and for all sufficiently large k we have

$$(\sigma_1(x_1^{(k)}), \dots, \sigma_n(x_n^{(k)})) \in \Omega.$$

Using $|x_j^{(k)}| < 1/y_k$ we obtain

$$g(\sigma_1(x_1^{(k)}), \dots, \sigma_n(x_n^{(k)})) \geq y_k^{n-|S|} \geq y_k^n. \quad (3)$$

We have

$$\begin{aligned} W(\sigma_1(x_1^{(k)}), \dots, \sigma_n(x_n^{(k)})) \left(g(\sigma_1(x_1^{(k)}), \dots, \sigma_n(x_n^{(k)})) \right)^{-D} = \\ f(x_1^{(k)}, \dots, x_n^{(k)}) < y_k^H. \end{aligned} \quad (4)$$

Using (2), with $y_i = \sigma_i(x_i^{(k)})$, in (4) we obtain

$$\alpha \left(g(\sigma_1(x_1^{(k)}), \dots, \sigma_n(x_n^{(k)})) \right)^{L-D} < y_k^H.$$

Using (3) in the last equation we obtain

$$\alpha y_k^{n(L-D)} < y_k^H.$$

which is false for sufficiently large k , since $H - n(L - D) \geq 1$. \square

We first use Theorem 2.4 to prove Theorem 1.1; we then show how to sharpen the result in Proposition 2.6.

Proof of Theorem 1.1. Proposition 2.1 implies that $\Sigma_k^\#$ -POLY and $\Pi_k^\#$ -POLY are complete for $\Sigma_{\mathbf{k}}\mathbb{R}$ and $\Pi_{\mathbf{k}}\mathbb{R}$. If the last quantifier block is universal, then the matrix is strict, so the result follows in this case.

This leaves us with the case that the last quantifier block is existential. For $k = 1$ the result $\Sigma_1\mathbb{R} = \Sigma_1^<\mathbb{R}$ was proved in [33], hence in all remaining cases there is at least one quantifier alternation, and the final block of quantifiers is existential.

The complete problems $\Sigma_k^\#$ -POLY and $\Pi_k^\#$ -POLY in these cases have a matrix of the form $g(x_1, \dots, x_i) = 0$. Without loss of generality, let us assume that the variables existentially quantified in the final block are x_1, \dots, x_n for some $n < i$. Fix arbitrary values for the remaining variables $x_{n+1}, \dots, x_i \in \mathbb{R}$. Let $f(x_1, \dots, x_n)$ be the polynomial resulting from fixing x_{n+1}, \dots, x_i in $g(x_1, \dots, x_i)$. Theorem 2.4 implies that

$$(\exists x_1, \dots, x_n) f(x_1, \dots, x_n) = 0$$

is equivalent to the strict formula

$$(\forall z > 0)(\exists y \in (0, z))(\exists x_1, \dots, x_n) \left(|f(x_1, \dots, x_n)| < y^H \right) \wedge \bigwedge_{j \in [n]} \left(|yx_j| < 1 \right),$$

where $H = nD^{n^c} + (D + 1)n$. But then

$$(\exists x_1, \dots, x_n) g(x_1, \dots, x_i) = 0$$

is equivalent to

$$(\forall z > 0)(\exists y \in (0, z))(\exists x_1, \dots, x_n) \left(|g(x_1, \dots, x_i)| < y^H \right) \wedge \bigwedge_{j \in [n]} \left(|yx_j| < 1 \right),$$

for the fixed values of x_{n+1}, \dots, x_i . Since those values were chosen arbitrarily, and nothing in the two formulas depends on those values, we conclude that the two formulas are equivalent for all values of $x_{n+1}, \dots, x_i \in \mathbb{R}$. We can therefore replace the first formula with the second without affecting the truth of the full sentence. Since the final two quantifier blocks of our formula were of the type $\forall \exists$ this does not change the type of the formula, but the resulting formula is now strict. \square

The complete problems produced by the previous proof contain Boolean matrices which require extra work to transform into polynomial inequalities. The following lemma helps reduce the degree of the polynomial by using repeated squaring.

Lemma 2.5. *The inequality*

$$\sum_{k \in [m]} (y_k - y_{k-1}^2 - 1)^2 < 1$$

is feasible (has a solution), and in any solution, $y_m > y_0^{2^m}$.

Proof. The inequality implies that $y_i > y_{i-1}^2$, from which the claim follows inductively. \square

Proposition 2.6. $\Sigma_k^<$ -POLY is $\Sigma_k \mathbb{R}$ -complete, and $\Pi_k^<$ -POLY is $\Pi_k \mathbb{R}$ -complete for all $k \geq 1$. We can assume that the degree of the polynomial in the matrix is at most 9 (at most 4 if the final quantifier block is existential).

Proof. This follows immediately from Proposition 2.1 if the last quantifier block is universal, so we can assume that the last quantifier block is existential. For $k = 1 \dots$ fill in details ...

For $k \geq 2$, we follow the proof of Theorem 1.1. We start with a matrix of the form $g(x_1, \dots, x_i) = 0$ of degree at most 4, and argue that

$$(\exists x_1, \dots, x_n) g(x_1, \dots, x_i) = 0$$

is equivalent to

$$(\forall z > 0)(\exists y \in (0, z))(\exists x_1, \dots, x_n) \left(|g(x_1, \dots, x_i)| < y^H \right) \wedge \bigwedge_{j \in [n]} (|yx_j| < 1), \quad (5)$$

for all x_{n+1}, \dots, x_i . We need to reduce the unbounded degree term, y^H , and combine the multiple conditions into a single inequality. Let $m = \lceil \log_2(H) \rceil + 1$, so that $2^m > H$. Then Equation 5 is equivalent to

$$(\forall z > 0)(\exists y_0 \in (0, z))(\exists y_1, \dots, y_m)(\exists x_1, \dots, x_n) \sum_{k \in [m]} (y_k - y_{k-1}^2 - 1)^2 + y_m (g(x_1, \dots, x_i))^2 + \sum_{j \in [n]} x_j^2 y_m^2 < 1. \quad (6)$$

The matrix has degree at most 9. As in Theorem 1.1, we can then replace $(\exists x_1, \dots, x_n) g(x_1, \dots, x_i) = 0$ with the strict condition in Equation 6. \square

Corollary 1.2 follows: For a given complexity class \mathcal{C} we write $\text{co-}\mathcal{C}$ for the class of all problems whose complement belongs to \mathcal{C} . By definition, $\text{co-}\Sigma_k \mathbb{R} = \Pi_k \mathbb{R}$, and vice versa, for all $k \geq 1$.

Proof of Corollary 1.2. We have $\text{co-}\Pi_k^{\leq} \mathbb{R} \subseteq \Sigma_k^{\leq} \mathbb{R}$ by the definition of complementation. Since $\Pi_k^{\leq} \mathbb{R} = \Pi_k \mathbb{R}$, by Theorem 1.1, we know that $\text{co-}\Pi_k^{\leq} \mathbb{R} = \text{co-}\Pi_k \mathbb{R} = \Sigma_k \mathbb{R}$, and $\Sigma_k \mathbb{R} \subseteq \Sigma_k^{\leq} \mathbb{R}$ which implies $\Sigma_k^{\leq} \mathbb{R} = \Sigma_k \mathbb{R}$. An analogous argument works for $\Pi_k^{\leq} \mathbb{R}$. \square

Jungeblut, Kleist and Miltzow [20] observed that $\text{co-}\forall\exists_{<} \mathbb{R} = \exists\forall_{\leq} \mathbb{R}$; the corollary allows us to conclude that both classes equal $\Sigma_2 \mathbb{R}$.

2.3 Bounding the Universe

We say a Σ_k - or Π_k -formula is *bounded*, and write $\text{bc-}\Sigma_k$ and $\text{bc-}\Pi_k$ if the quantifiers range over a (closed) bounded set; the set is arbitrary, but to

make it concrete we restrict each variable in a bounded sentence to the interval $[-1, 1]$. We write $\text{bc-}\Sigma_k\mathbb{R}$ and $\text{bc-}\Pi_k\mathbb{R}$ for the bounded versions of $\Sigma_k\mathbb{R}$ and $\Pi_k\mathbb{R}$, and $\text{bc-}\Sigma_k\text{-POLY}$ and $\text{bc-}\Pi_k\text{-POLY}$ for the bounded forms of $\Sigma_k\text{-POLY}$ and $\Pi_k\text{-POLY}$. It follows from the arguments in [33] that $\Sigma_1\mathbb{R} = \text{bc-}\Sigma_1\mathbb{R}$, that is, $\exists\mathbb{R} = \text{bc-}\exists\mathbb{R}$. D’Costa, Lefaucheux, Neumann, Ouaknine, and Worrell [14, Theorem 3] showed that $\Sigma_2^{\leq}\mathbb{R} = \text{bc-}\Sigma_2^{\leq}\mathbb{R}$, and Jungeblut, Kleist and Miltzow [20, Theorem 12] implies $\Pi_2^{\leq}\mathbb{R} \subseteq \text{bo-}\Pi_2\mathbb{R}$.

We also consider *bounded open* variants in which quantification is over the open interval $(-1, 1)$. For the bounded open variants we use the prefix bo- in place of bc- .

Theorem 2.4 has a version for bounded quantification. Differently from the unbounded case, we directly establish a bounded-degree version. To that end, we want to replace the term $|y|^H$ which occurs in the bound with a fixed-degree computation. In the unbounded case we used Lemma 2.5 to achieve that goal, but that result required variables with unbounded values, which are not allowed in bounded quantification. Instead we need to compute doubly-exponentially small values. At first glance that appears hard using just strict inequalities, but the following lemma shows that it is possible.

Lemma 2.7. *Consider the inequality*

$$400 \sum_{k \in [m]} (y_k - y_{k-1}^2)^2 < y_m^2, \quad (7)$$

where $y_0 \in (0, 1/2)$. The inequality (7) is feasible with $y_i \in [-1, 1]$ for all $i \in [m]$. For any solution $y_1, \dots, y_m \in [-1, 1]$ of the inequality (7) we have

$$0 < y_m^2 \leq y_0^{(5/6)2^m}. \quad (8)$$

Proof. To show feasibility take $y_k = y_0^{2^k}$. Note that this makes each term on LHS of (7) equal to zero and hence (7) is satisfied.

To argue (8) suppose we have a solution y_1, \dots, y_m of (7). From (7) it follows that for every $k \in [m]$ we have

$$y_{k-1}^2 - \frac{y_n}{20} < y_k < y_{k-1}^2 + \frac{y_m}{20}.$$

Suppose there exists $k \in \{1, \dots, m\}$ such that $y_k < y_m/2$. Then

$$y_{k+1} < y_k^2 + \frac{y_m}{20} < \frac{y_m}{2} \left(\frac{y_m}{2} + \frac{1}{10} \right) \leq \frac{y_m}{2},$$

which, by induction, implies $y_m < y_m/2$, a contradiction. Thus $y_k \geq y_m/2$ for all $k \in [m]$. Hence,

$$y_{k-1}^2 < y_k + \frac{y_m}{20} \leq y_k \left(1 + \frac{1}{10}\right) \quad \text{and} \quad y_k \left(1 - \frac{1}{10}\right) \leq y_k - \frac{y_m}{20} < y_{k-1}^2, \quad (9)$$

for all $k \in [m]$. Letting $x_k = \log_{y_0} y_k$ and using $\log_{y_0}(9/10) \leq -\log_2(9/10) \leq 1/6$, $\log_{y_0}(11/10) \geq -\log_2(11/10) \geq -1/6$ from (9) we obtain that for $k \in [m]$ we have

$$|x_k - 2x_{k-1}| \leq 1/6;$$

note that $x_0 = 1$. By induction we now have for $k \in [m]$

$$x_k \in [(5/6)y_0^k, (7/6)y_0^k].$$

which implies (8). \square

We can now establish the bounded equivalent of Theorem 2.4.

Corollary 2.8. *Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial of degree d , and let $I = (-1, 1)$ or $I = [-1, 1]$. Then there is a polynomial $g \in \mathbb{R}(x_1, \dots, x_n, y, y_1, \dots, y_n)$ of degree at most $2d$ such that*

$$\Phi = (\exists x_1, \dots, x_n \in I) f(x_1, \dots, x_n) = 0,$$

is equivalent to

$$\Psi = (\forall y \in I)(\exists y_1, \dots, y_m \in I)(\exists x_1, \dots, x_n \in I) g(x_1, \dots, x_n, y, y_1, \dots, y_m) > 0.$$

If $f \in \mathbb{Z}[x_1, \dots, x_n]$ then there is a polynomial $g \in \mathbb{R}(x_1, \dots, x_n, y, y_1, \dots, y_n)$ of degree at most $2d$ such that Φ is equivalent to

$$\Psi = (\exists y_0, y_1, \dots, y_m \in I)(\exists x_1, \dots, x_n \in I) g(x_1, \dots, x_n, y_0, y_1, \dots, y_m) > 0.$$

Proof. Following the proof of Theorem 2.4, we obtain that

$$\Phi = (\exists x_1, \dots, x_n \in I) f(x_1, \dots, x_n) = 0,$$

is equivalent to

$$(\forall y \in I)(\exists x_1, \dots, x_n \in I) \left(|f(x_1, \dots, x_n)| < |y|^H \right), \quad (10)$$

where $H = nD^{n^c} + (D+1)n$. We can drop the $|yx_i| < 1$ requirements which occur in the original result, since x_i is bounded, and we need to replace y with $|y|$, since we allow negative y in I .

Pick the smallest $m \in \mathbb{N}$ for which $(5/6)2^m > H$. Using Lemma 2.7 then shows that Equation 10 is equivalent to

$$(\forall y \in I)(\exists y_1, \dots, y_m \in I)(\exists x_1, \dots, x_n \in I) \\ 2y_0^2 + 400 \sum_{k \in [m]} (y_k - y_{k-1}^2)^2 + (f(x_1, \dots, x_n))^2 < y_m^2. \quad (11)$$

We can then define g as $y_m^2 - (2y_0^2 + 400 \sum_{k \in [m]} (y_k - y_{k-1}^2)^2 + (f(x_1, \dots, x_n))^2)$, and g has double the degree of f .

If $f \in \mathbb{Z}[x_1, \dots, x_n]$ we apply Solerno's inequality, Theorem 2.2 with $g = 1$. If $f(x_1, \dots, x_n) \neq 0$ for all $x_1, \dots, x_n \in \Omega$, the inequality tells us that

$$|f(x_1, \dots, x_n)| \geq \alpha |g(x_1, \dots, x_n)|^L = \alpha = 2^{-\ell D^{cn^2}},$$

where ℓ is the number of bits in the longest coefficient of f . We pick the smallest $m \in \mathbb{N}$ so that $(5/6)2^m > \ell D^{cn^2}$. Then Φ is equivalent to

$$(\exists y_0 \in I)(\exists y_1, \dots, y_m \in I)(\exists x_1, \dots, x_n \in I) \\ 2y_0^2 + 400 \sum_{k \in [m]} (y_k - y_{k-1}^2)^2 + (f(x_1, \dots, x_n))^2 < y_m^2. \quad (12)$$

as long as we choose $0 < y_0 < 1/2$, which is forced by $2y_0^2 < y_m^2 < 1$. As before, we define g as $y_m^2 - (2y_0^2 + 400 \sum_{k \in [m]} (y_k - y_{k-1}^2)^2 + (f(x_1, \dots, x_n))^2)$. \square

Corollary 1.3 is an immediate consequence of the following proposition.

Proposition 2.9. *bo- $\Sigma_k^<$ -POLY and bc- $\Sigma_k^<$ -POLY are $\Sigma_k \mathbb{R}$ -complete, and bo- $\Pi_k^<$ -POLY and bc- $\Pi_k^<$ -POLY are $\Pi_k \mathbb{R}$ -complete for all $k \geq 1$. We can assume that the degree of the polynomial in the matrix is at most 16 (at most 8 if the final quantifier block is existential).*

Proof. By Proposition 2.1 we can assume that we are starting with a sentence of the form

$$(Q_1 x_1)(Q_2 x_2) \cdots (Q_i x_i) \varphi(x_1, \dots, x_i),$$

where $\varphi(x_1, \dots, x_i)$ is of the form $f(x_1, \dots, x_i) \bowtie 0$, where $\bowtie \in \{=, <\}$ depends on which class we are working with.

It is easier to deal with the bounded open case first, and then show how to modify the construction to obtain the bounded close case.

If the final quantifier block is existential, Proposition 2.1 implies that the formula's matrix $\varphi(x_1, \dots, x_i)$ has the form $f(x_1, \dots, x_i) = 0$, where f is

a polynomial of degree at most 4. Using the bijection $z \rightarrow \frac{z}{(1-z^2)}$ between $(-1, 1)$ and \mathbb{R} we replace each x_j in f with the term $\frac{z_j}{(1-z_j^2)}$. To get rid of the rational terms, we multiply the whole expression by $\prod_{\ell \in [i]} (1 - z_\ell^2)^4$. After cancellation, this leaves us with a sum of the original monomials of f each multiplied by a product of terms of the form $(1 - z_\ell^2)^d$ for some $\ell \in [i]$ and $d \in [4]$. Replace each of the terms $(1 - z_\ell^2)^d$ with a new variable $z_{\ell,d}$. This gives us a polynomial $g(z_1, \dots, z_i, z_{1,1}, \dots, z_{1,4}, \dots, z_{i,1}, \dots, z_{i,4})$.

Note that the $z_i \in (-1, 1)$ and, therefore, the $z_{\ell,d} \in (0, 1)$. It follows that $f(x_1, \dots, x_i) = 0$ if and only if for the z_ℓ that satisfy $x_\ell = \frac{z_\ell}{1-z_\ell^2}$ we have

$$(\exists z_{1,1}, \dots, z_{i,4} \in (-1, 1)) \\ g(z_1, \dots, z_{i,4}) + \sum_{\ell \in [i], d \in [4]} (z_{\ell,d} - (1 - z_\ell^2)^d)^2 = 0$$

Since $d \leq 4$, the matrix of this expression can be multiplied out to give a polynomial $h(z_1, \dots, z_{i,4})$ which has polynomial size in f .

The original sentence

$$(Q_1 x_1)(Q_2 x_2) \cdots (Q_i x_i) f(x_1, \dots, x_i) = 0,$$

is then equivalent to

$$(Q_1 z_1 \in (-1, 1))(Q_2 z_2 \in (-1, 1)) \cdots (Q_i z_i \in (-1, 1)) \\ (\exists z_{1,1}, \dots, z_{i,4} \in (-1, 1)) h(z_1, \dots, z_i, z_{1,1}, \dots, z_{1,4}) = 0.$$

We then apply Corollary 2.8 to the function resulting from h by restricting it to the final block of existentially quantified variables. If that is all variables, so $k = 1$, then we apply the second, integer, version of Corollary 2.8 to get an existentially quantified, bounded inequality. If there is quantified alternation, we apply the first, real, version of Corollary 2.8. As in Theorem 1.1, we can absorb the $\forall \exists$ quantifiers in the final two quantifier blocks.

To modify the argument to apply to $[-1, 1]$, we add constraints to the matrix $h(x_1, \dots, x_i, z_1, \dots, z_{1,4}) = 0$ to ensure that the $z_i^2 \leq r^2$ for some new variable r ; for that we add some slack variables z'_i . That is, we work with

$$(Q_1 z_1 \in [-1, 1])(Q_2 z_2 \in [-1, 1]) \cdots (Q_i z_i \in [-1, 1]) \\ (\exists z_{1,1}, \dots, z_{i,4} \in [-1, 1]) \\ (\exists z'_1, \dots, z'_i \in [-1, 1]) \\ h(z_1, \dots, z_i, z_{1,1}, \dots, z_{1,4}) + \sum_{\ell=1}^i (z_\ell^2 + (z'_\ell)^2 - r^2)^2 = 0.$$

In a solution, we have that $z_i^2 + (z'_i)^2 \leq r^2$, so $z_i^2 \leq r^2$. We can now apply Corollary 2.8 to $h = 0$ to get a strict condition.

This leaves us with the case that the final quantifier block is universal. Let Θ denote the sentence in this case. It follows from the proof of Proposition 2.1 that the matrix $\varphi(x_1, \dots, x_i)$ of Θ has the form $f(x_1, \dots, x_i) > 0$, where f is a non-negative polynomial of degree at most 8. By negating Θ we can then proceed as in the first case, since we can negate $f(x_1, \dots, x_i) > 0$ as $f(x_1, \dots, x_i) = 0$, because f is non-negative. Following the first case up to and excluding the last step (the application of Corollary 2.8), we build a sentence equivalent to $\neg\Theta$, with the same quantifier structure as $\neg\Theta$ and a matrix of the form $h(x_1, \dots, x_i, z_1, \dots, z_{1,4}) = 0$. Negating that sentence, and negating the matrix condition as $h^2(x_1, \dots, x_i, z_1, \dots, z_{1,4}) > 0$ then gives us a strict, bounded sentence with the same quantifier structure as the original sentence Θ . \square

The first part of the proof of Proposition 2.9 shows that the equality variants are also hard for that level, we simply stop the proof before applying Corollary 2.8.

Corollary 2.10. *bo- Σ_k^- -POLY and bc- Σ_k^- -POLY are $\Sigma_k\mathbb{R}$ -complete for all odd $k \geq 1$, and bo- Π_k^- -POLY and bc- Π_k^- -POLY are $\Pi_k\mathbb{R}$ -complete for even $k \geq 1$. We can assume that the degree of the polynomial in the matrix is at most 4.*

3 Properties of Semi-Algebraic Sets And Exotic Quantifiers

Testing properties of semialgebraic sets, such as being convex or closed, very naturally leads to problems captured by the real hierarchy. The complexity of such problems was first studied in the real polynomial hierarchy based on the BSS-model, going back to the original papers introducing the BSS-model, see [10, 9].

Bürgisser and Cucker [12, Section 9] showed that many of these results, often with the same proofs, carry over to the real hierarchy, the discrete version of the BSS-hierarchy. We sharpen some of these results using the ideas from the previous section, and add some new results.

Various properties defy classification using standard quantifiers, so Bürgisser and Cucker made use of three “exotic” quantifiers, H , \forall^* and \exists^* . We will define those quantifiers below, and show that in the real hierarchy the

power of these quantifiers can sometimes be explained using standard quantifiers.

3.1 First-Level Problems

We start with a simple example: testing the convexity of a semialgebraic set. The complexity of this problem was first determined in a paper by Cucker and Rosselló [13].

Theorem 3.1 (Cucker, Rosselló [13]). *Testing the convexity of a (bounded) semialgebraic set is $\forall\mathbb{R}$ -complete.*

Proof. A semialgebraic set $S = \{x : \varphi(x)\}$ is convex if

$$(\forall x, y)(\forall t \in [0, 1]) (\varphi(x) \wedge \varphi(y) \rightarrow \varphi(tx + (1 - ty))),$$

so testing convexity belongs to $\forall\mathbb{R}$.

By Corollary 2.10, $\text{bo-}\Pi_1^>\text{-POLY}$ is $\forall\mathbb{R}$ -complete. So we can assume we are given a sentence $\Phi = (\forall x \in (-1, 1)^n) f(x) > 0$. We define two semialgebraic sets,

$$T = \{x \in [-1, 1]^n : x_i \in \{-1, 1\} \text{ for some } i \in [n]\},$$

the bounding box of $[-1, 1]^n$, and with that

$$S = \{x \in (-1, 1)^n : f(x) > 0\} \cup T.$$

Then Φ is true if and only if S is convex. □

Let's look at a more challenging problem: how hard is it to test whether a semialgebraic set is unbounded? The problem lies in $\Pi_2\mathbb{R}$, since S is unbounded if for every $d > 0$ there is an $x \in S$ with $\|x\| > d$, but is hard for $\Pi_2\mathbb{R}$?

The universal quantifier can be eliminated, because it only affects a single variable, and the quantified condition is monotone in that variable. This was proved by Bürgisser and Cucker [12, Theorem 9.2], and rediscovered in [33, Lemma 4.1] (and used earlier in [32]). For Bürgisser and Cucker this is their first exotic quantifier, the infinitesimal quantifier H , where $(H\varepsilon) \varphi(\varepsilon, x)$ is defined to mean

$$(\exists \varepsilon' > 0)(\forall \varepsilon \in (0, \varepsilon')) \varphi(\varepsilon, x).$$

In plain English $H\varepsilon$ can be read as “for all sufficiently small values of $\varepsilon > 0$ ”.

The unboundedness of a semialgebraic set $S = \{x : \varphi(x)\}$ is then equivalent to

$$(H\varepsilon)(\exists x) \varphi(x) \wedge \varepsilon\|x\| > 1.$$

Since the H quantifier can be eliminated, the problem lies in $\exists\mathbb{R}$.

Theorem 3.2 (Bürgisser, Cucker [12, Proposition 6.4, Corollary 9.4]). *Testing whether a semialgebraic set is unbounded is $\exists\mathbb{R}$ -complete.*

Proof. We already saw membership. For hardness, we can start with the sentence $\Phi = (\exists x \in (-1, 1)^n) f(x) = 0$, by Corollary 2.10. Then $S = \{(x, y) \in \mathbb{R}^{n+1} : f(x) = 0\}$ is unbounded if and only if Φ is true. \square

The proof shows the slightly stronger result that testing unboundedness of an algebraic set is $\exists\mathbb{R}$ -complete.

A trickier problem is testing whether a semialgebraic set S is dense in another set T . Let us assume that T is the ambient space \mathbb{R}^n , we will discuss more general choices of T in the next section. The denseness of $S = \{x : \phi(x)\}$ in \mathbb{R}^n can be expressed as

$$(\forall x)(\forall \varepsilon > 0)(\exists x') \|x - x'\| < \varepsilon \wedge \phi(x'),$$

which is of the form $\forall\exists$. Very surprisingly, Koiran [21] showed that the existential quantifier can be eliminated in this case, and the problem lies in $\forall\mathbb{R}$. Compared to that, proving $\forall\mathbb{R}$ -hardness is relatively easy. The following result was proved for semialgebraic sets defined by algebraic circuits rather than formulas by Bürgisser and Cucker [12, Proposition 5.2, Corollary 9.3(ii)]. They discuss the difficulty of moving from circuits to formulas for this problem in their Remark 5.4(ii).

Theorem 3.3. *Testing whether a semialgebraic set $S \subseteq \mathbb{R}^n$ is dense in \mathbb{R}^n is $\forall\mathbb{R}$ -complete.*

As the proof will show, the ambient space \mathbb{R}^n is not special, it can easily be replaced with other semialgebraic sets such as $[-1, 1]^n$.

Proof. Membership in $\forall\mathbb{R}$ is non-trivial and follows from work by Koiran [21], also see [12, Corollary 5.3]. To prove hardness, we work with an instance $\Phi = (\forall x \in [-1, 1]^n) f(x) > 0$ given by Proposition 2.9. If Φ is true, then Theorem 2.2, Solerno's inequality, implies, just as in the proof of 2.8, that

$$|f(x)| \geq 2^{-\ell D^{cn^2}}$$

for all $x \in [-1, 1]^n$, where D is the degree of f and ℓ bounds the number of bits in every coefficient of f . With $m = \lceil \ell D^{cn^2} \rceil + 1$ define the semialgebraic set

$$T = \{y \in [-1, 1]^m : 1/4 \leq y_1 \leq 1/2, \\ y_1^2/2 \leq y_2 \leq y_1^2, \dots, y_{m-1}^2/2 \leq y_m \leq y_{m-1}^2\}.$$

For $y \in T$ we have $0 < 4^{-\ell D^{cn^2}} < y_m < 2^{-\ell D^{cn^2}}$, so $f(x) > 0$ implies that $f(x) > y_m$ for all $y \in T$. We can then define

$$S = \{x \in [-1, 1]^n, y \in [-1, 1]^m : f(x) \geq y_m \wedge y \in T\} \\ \cup (\mathbb{R}^{n+m} - [-1, 1]^n \times ([-1, 1]^m - T)).$$

By definition of T and S , the truth of Φ implies that $S = \mathbb{R}^{n+m}$, in particular, S is dense in \mathbb{R}^{n+m} . Assume then that Φ is false, so there is an $x \in [-1, 1]^n$ for which $f(x) = 0$. In this case $(x, y) \notin S$ for any $y \in T$. Since f is continuous, we can pick a $\varepsilon > 0$ such that $|f(x')| < 4^{-\ell D^{cn^2}}$ for all x' with $\|x - x'\| < \varepsilon$. Since $y_m > 4^{-\ell D^{cn^2}}$, this implies that $(x', y) \notin S$ for all such x' and all $y \in T$, so S is not dense in \mathbb{R}^{n+m} . \square

Following Bürgisser and Cucker, we can capture the complexity of the denseness problem using the quantifier \forall^* , where $(\forall^* x) \varphi(x)$ is defined as

$$(\forall x)(\forall \varepsilon > 0)(\exists x') \|x - x'\| < \varepsilon \wedge \varphi(x').$$

The \forall^* quantifier can be read as “for almost all”, or “for a dense set of”. The meaning of the dual quantifier \exists^* in $(\exists^* x) \varphi(x)$ is defined by

$$(\exists x)(\exists \varepsilon > 0)(\forall x') \|x - x'\| < \varepsilon \rightarrow \varphi(x').$$

So $\forall^* = \neg \exists^* \neg$.

Using \forall^* the denseness of the semialgebraic set $S = \{x : \varphi(x)\}$ can be written simply as $(\forall^* x) \phi(x)$. Bürgisser and Cucker introduce the classes $\text{BP}^0(\exists^*)$ and $\text{BP}^0(\forall^*)$ for problems of this type, but there is one subtle difference to our approach: since they base their approach on the BSS machine model, they need to allow algebraic circuits to define semialgebraic sets whenever the final quantifier block is exotic, that is \exists^* , \forall^* , or H . So the complete problem for $\text{BP}^0(\forall^*)$ is of the form $(\forall^* x) C(x)$, where C is an algebraic circuit, see [12, Section 2]. This leads to stronger upper bound results, but weaker lower bounds. At the first level, the exotic classes collapse to their non-exotic counterparts.

Corollary 3.4 (Bürgisser and Cucker [12, Corollary 9.3(ii)]). $\text{BP}^0(\exists^*) = \text{BP}^0(\exists) = \exists\mathbb{R}$ and $\text{BP}^0(\forall^*) = \text{BP}^0(\forall) = \forall\mathbb{R}$.

Bürgisser and Cucker show several additional, non-trivial relationships between the exotic and the standard quantifiers [12, Corollary 6.2, Corollary 8.3, Proposition 8.4].

If the final quantifier is H , we can easily remove it using quantifier elimination.

Lemma 3.5. *Let Φ be the sentence $(Q_1x_1) \cdots (Q_ix_i)(H\varepsilon) \varphi(x_1, \dots, x_i, \varepsilon)$, where $Q_i \in \{\exists, \forall, \exists^*, \forall^*, H\}$. We can construct, in polynomial time, an equivalent sentence $\Psi = (Q_1x_1) \cdots (Q_ix_i) \psi(x_1, \dots, x_i)$.*

Proof. Consider $\varphi(x_1, \dots, x_i, \varepsilon)$ as a formula in the single variable ε . Using the definition of H we can rewrite the sentence $(H\varepsilon)\varphi(x_1, \dots, x_i, \varepsilon)$ as $(\exists\varepsilon' > 0)(\forall\varepsilon \in (0, \varepsilon')) \varphi(x_1, \dots, x_i, \varepsilon)$, and then apply quantifier elimination (e.g. in the form of Theorem 1 in [36]) twice to eliminate the two quantifiers. This gives us, in polynomial time, a formula $\psi(x_1, \dots, x_i)$ equivalent to $(H\varepsilon) \varphi(x_1, \dots, x_i, \varepsilon)$. Reintroducing the quantifiers Q_i, Q_{i-1}, \dots, Q_1 , in this order, we obtain that Φ is equivalent to $\Psi = (Q_1x_1) \cdots (Q_ix_i) \psi(x_1, \dots, x_i)$. \square

Applying the lemma repeatedly, allows us to remove a fixed number of final H quantifiers. We would like to conclude that $\text{BP}^0(\exists^*H) = \text{BP}^0(\exists H) = \exists\mathbb{R}$ and $\text{BP}^0(\forall^*H) = \text{BP}^0(\forall H) = \forall\mathbb{R}$, but we cannot, because the complete problems for any of the H classes involve algebraic circuits; to rewrite these as formulas, we would have to add existential quantifiers after the H , leading to formulas of the form $\exists^*H\exists$. We cannot then apply quantifier elimination to H without dealing with the final block of existential quantifiers first.

What can we do? Bürgisser and Cucker studied the complexity of the $\text{LocSupp}_{\mathbb{R}}$ problem: they showed that testing whether a given hyperplane locally supports a semialgebraic set (given by an algebraic circuit) is complete for $\text{BP}^0(\exists^*H)$. We can prove an analogue for semialgebraic sets defined by formulas.

A hyperplane *locally supports* a set T if there is a point on the hyperplane so that a neighborhood of that point in the hyperplane belongs to T , and T , at least close to the point, lies entirely on one side of the hyperplane. For example, a square in \mathbb{R}^2 has four supporting hyperplanes (lines); a line just passing through a corner is not considered locally supporting in this definition, since its intersection with the square is not two-dimensional.

Corollary 3.6. *Given a hyperplane and a bounded semialgebraic set, testing whether the hyperplane locally supports the semialgebraic set is $\exists\mathbb{R}$ -complete.*

Proof. Let us assume the hyperplane S is given by a normal vector v such that $S = \{x : v \cdot x = 0\}$, and $T = \{x : \varphi(x)\}$. Then S locally supports T if and only if

$$(\exists^* x)(H\varepsilon) [\varphi(x) \wedge (\varepsilon > 0 \leftrightarrow \varphi(x + \varepsilon n))].$$

Using Lemma 3.5 we can eliminate H , so we obtain a formula $\psi(x)$ such that S locally supports T if and only if $(\exists^* x) \psi(x)$, but then the problem lies in $\exists\mathbb{R}$ by Corollary 3.4.

To show $\exists\mathbb{R}$ -hardness, we can work with the sentence

$$\Phi = (\exists x \in (-1, 1)^i) f(x) > 0,$$

by Proposition 2.9. Let $S = \{(x, y) \in \mathbb{R}^i \times \mathbb{R} : y = 0\}$ and $T = \{(x, y) \in \mathbb{R}^i \times \mathbb{R} : f(x) > 0 \wedge y \geq 0\}$. Then S is a hyperplane, and T a bounded semialgebraic set. S touches T if $S \cap T$ is i -dimensional. But that is the case if and only if there is an $x \in \mathbb{R}^i$ such that $f(x) > 0$, since if $f(x) > 0$ is true for some x , it is true for a neighborhood of x , but continuity. \square

For a list of further problems complete for $\exists\mathbb{R}$ and $\forall\mathbb{R}$ see [12, Corollary 9.4].

3.2 Second-Level Problems

Bürgisser and Cucker [12, Proposition 4.1, Table 1] showed that testing whether a piecewise rational function given by an algebraic circuit is surjective is $\Pi_2\mathbb{R}$ -complete. Below we include a proof that extends their result to polynomials (of bounded degree).

Theorem 3.7. *Testing whether a polynomial $g : A \rightarrow B$ is surjective is $\Pi_2\mathbb{R}$ -complete, even if g is a polynomial of degree at most 8, and A and B are Cartesian products of \mathbb{R} , and $[-1, 1]$.*

Proof. By Corollary 2.10, $\text{bc-}\Pi_2^-\text{-POLY}$ is $\Pi_2\mathbb{R}$ -complete. So we can assume we are given a sentence $\Phi = (\forall x \in [-1, 1]^n)(\exists y \in [-1, 1]^m) f(x, y) = 0$ with an explicit polynomial f of degree at most 4. Define a new polynomial

$$g : \begin{array}{ccc} [-1, 1]^n \times [-1, 1]^m \times \mathbb{R} \times \mathbb{R} & \rightarrow & [-1, 1]^n \times \mathbb{R} \\ (x, y, a, b) & \mapsto & (x, a - (af(x, y))^2 - b^2) \end{array}$$

Then g is surjective if and only if Φ is true: If Φ is true, then for every $x \in [-1, 1]^n$ there is a $y \in [-1, 1]^m$ such that $f(x, y) = 0$ and $g(x, y, a, 0) = (x, a)$,

which shows that g is surjective. If Φ is false, then there is an $x \in [-1, 1]^n$ such that $(f(x, y))^2 > 0$ for all $y \in [-1, 1]^m$. By compactness, there is a ε with $(f(x, y))^2 > \varepsilon > 0$ for all $y \in [-1, 1]^m$. Then $a - (af(x, y))^2 - b^2 < a - (a\varepsilon)^2 - b^2$, which has a finite upper bound for $(a, b) \in \mathbb{R}^2$, so g is not surjective. \square

There are various ways to measure how close two semialgebraic sets are. For example, testing equality of two semialgebraic sets is $\forall\mathbb{R}$ -complete [20, Theorem 3], and the same is true for containment: $S \subseteq T$. What happens if we replace equality with denseness? In the previous section we looked at the problem of deciding whether a set S contained in a set T is dense in T (for $T = \mathbb{R}^n$). Writing \overline{S} for the closure of the set S we can express this as $\overline{S} = T$.

What about $S \subseteq \overline{T}$ instead? If $S = \{x : \varphi(x)\}$ and $T = \{x : \psi(x)\}$, then $S \subseteq \overline{T}$ is equivalent to

$$(\forall^* x)(\forall^* \varepsilon)(\exists y) \varphi(x) \wedge \psi(y) \wedge \|x - y\| < \varepsilon,$$

and Bürgisser and Cucker [12, Proposition 5.5] show completeness of testing $S \subseteq \overline{T}$ for problems of the type $\forall^*\exists$. In the discrete version, their result implies completeness for the class $\text{BP}^0(\forall^*\exists)$. Clearly, $\text{BP}^0(\forall^*\exists) \subseteq \text{BP}^0(\forall\exists)$.² We argue below that the two classes are the same. As was the case for the denseness problem at the first level, the quantifier \forall^* does not differ from \forall in computational power at the second level.

Our proof that testing $S \subseteq \overline{T}$ is $\Pi_2\mathbb{R}$ -complete is closely based on [12, Proposition 5.5].

Theorem 3.8. *Testing whether $S \subseteq \overline{T}$ for two semialgebraic sets S and T is $\Pi_2\mathbb{R}$ -complete, even if S is of the form $[-1, 1]^p \times \{0\}^q$.*

Proof. Since the \forall^* -quantifier can be replaced by $\forall\exists$, membership in $\Pi_2\mathbb{R}$ easily follows. For the other direction, we use the closed case of Corollary 2.10. So we are given a sentence

$$\Phi = (\forall x \in [-1, 1]^n)(\exists y \in [-1, 1]^m) f(x, y) = 0.$$

Write $f(x, y) = \sum_{\alpha} f_{\alpha}(x)y^{\alpha}$. Following [12] we define

$$f'(x, y, y_0) = \sum_{\alpha} f_{\alpha}(x)y_0^{d-|\alpha|}y^{\alpha} = y_0^d \sum_{\alpha} f_{\alpha}(x)(y/y_0)^{\alpha} = y_0^d f(x, y/y_0),$$

²At the second level we do not need Koiran's result, we can argue directly: \forall^* can be rewritten as $\forall\exists$, and the additional existential quantifiers merged with the existing ones.

the homogenization of f , where d is the largest total degree of y in f .

Define the sets

$$\begin{aligned} S &= [-1, 1]^n \times \{0\}^m \times \{0\}, \text{ and} \\ T &= \{(x, y, y_0) \in [-1, 1]^n \times [-1, 1]^m \times [0, 1] : \\ &\quad f'(x, y, y_0) = 0 \wedge y_0 \neq 0 \wedge \|y\| \leq y_0\}. \end{aligned}$$

We claim that Φ is true if and only if $S \subseteq \overline{T}$.

Suppose Φ is true, and let $(x, 0, 0) \in S$. Since Φ is true, there is a y such that $f(x, y) = 0$. We can then find a point (x, y', y_0) in T arbitrarily close to $(x, 0, 0)$ by choosing y_0 arbitrarily small (but bigger than 0), and setting $y' = y_0 y$. Then

$$f'(x, y', y_0) = y_0^d f(x, y) = 0,$$

and $\|y'\| \leq y_0 \|y\| \leq y_0$, so $(x, 0, 0)$ lies in \overline{T} .

For the other direction, the difference between the BSS-model, and the discrete model comes into play. Suppose $S \subseteq \overline{T}$, and we are given $x \in [-1, 1]^n$. We have to show that there is a $y \in [-1, 1]^m$ such that $f(x, y) = 0$.

For a contradiction assume that $f(x, y) \neq 0$ for all $y \in [-1, 1]^m$; without loss of generality, $f(x, y) > 0$ for all $y \in [-1, 1]^m$ (if not, replace f with $-f$, we use the fact that f is continuous). Since $[-1, 1]^m$ is compact, this implies that the minimum of $f(x, y)$ over all $y \in [-1, 1]^m$ is greater than some $\delta > 0$. Since f is continuous (as a polynomial), this remains true in a sufficiently small neighborhood of x . But then

$$\begin{aligned} f'(x', y', y_0) &= y_0^d \sum_{\alpha} f_{\alpha}(x) (y'/y_0)^{\alpha} \\ &\geq y_0^d \delta. \end{aligned}$$

for all x' sufficiently close to x , and all y' with $\|y'\| \leq y_0$. In particular, $f'(x', y', y_0) > 0$ for $y_0 \neq 0$, so $(x', y', y_0) \notin T$. Hence, $(x, 0, 0)$ does not belong to \overline{T} . This contradicts $(x, 0, 0) \in S \subseteq \overline{T}$. \square

The sets S and T in Theorem 3.8 are basic semialgebraic sets. We do not know whether the hardness result can be extended to S and T being algebraic.

Corollary 3.9. $\text{BP}^0(\forall^* \exists) = \text{BP}^0(\forall \exists) = \Pi_2 \mathbb{R}$.

The corollary also settles the complexity of the problems $\text{ERD}_{\mathbb{R}}$, $\text{LERD}_{\mathbb{R}}$, and $\text{ImageEDense}_{\mathbb{R}}$ mentioned in [12], they are all $\Pi_2 \mathbb{R}$ -complete. With the exception of $\text{LocSupp}_{\mathbb{R}}$, these were the last remaining problems from [12]

whose complexity in terms of the real hierarchy remained open because they included exotic quantifiers. We have to exclude $\text{LocSupp}_{\mathbb{R}}$, since the complexity of that problem remains open for semialgebraic sets defined by circuits.

As a second consequence of Theorem 3.8, we can strengthen the results of Jungeblut, Kleist and Miltzow [20] that the Hausdorff distance problem is $\Pi_2\mathbb{R}$ -complete: it remains hard even for distance 0. This is mentioned as an open problem in the paper.

Corollary 3.10. *Given two semialgebraic sets S and T testing whether their (directed) Hausdorff distance is 0 is $\Pi_2\mathbb{R}$ -complete.*

Proof. The directed Hausdorff distance of S and T is 0 if and only if $S \subseteq \overline{T}$, so hardness of the directed case follows from Theorem 3.8.

The directed case reduces to the undirected case as follows: $S \subseteq \overline{T}$ is equivalent to $S \cup T \subseteq \overline{T}$. Since $\overline{T} \subseteq \overline{S \cup T}$ this implies that $S \subseteq \overline{T}$ if and only if $S \cup T$ and T have the same closure: $\overline{S \cup T} = \overline{T}$, which is equivalent to $S \cup T$ and T having Hausdorff distance 0. \square

4 Conclusion and Open Problems

In this paper we (re)introduced the real hierarchy, a hierarchy of complexity classes based on the theory of the reals, and justified the definition by showing that, just like $\exists\mathbb{R}$, it is robust under changes in the definition. We identified several, very restricted families of problems complete for $\Sigma_k\mathbb{R}$, see Figure 1, and $\Pi_k\mathbb{R}$, see Figure 2.

$\Sigma_k\mathbb{R}$ -complete	k even	k odd
$\Sigma_k^<\text{-POLY}$	degree 8 (P2.1)	degree 9 (P2.6)
$\Sigma_k^=\text{-POLY}$	na	degree 4 (P2.1)
$\Sigma_k^{\leq}\text{-POLY}$	degree 9 (P2.6 + C1.2)	degree 8 (P2.1 + C1.2)
$\text{bo-}\Sigma_k^<\text{-POLY}$	degree 16 (C2.8)	degree 8 (C2.8)
$\text{bc-}\Sigma_k^<\text{-POLY}$	degree 16 (C2.8)	degree 8 (C2.8)
$\text{bo-}\Sigma_k^=\text{-POLY}$	na	degree 4 (C2.10)
$\text{bc-}\Sigma_k^=\text{-POLY}$	na	degree 4 (C2.10)

Figure 1: Degree bounds and references for complete problems for $\Sigma_k\mathbb{R}$.

We hope that these problems will be useful in future investigations at higher levels of the real hierarchy. Are there any natural candidates for complete problems at higher levels? Dobbins, Kleist, Miltzow and Rzążewski [16]

$\Pi_k \mathbb{R}$ -complete	k even	k odd
$\Pi_k^<$ -POLY	degree 9 (P2.6)	degree 8 (P2.1)
$\Pi_k^=$ -POLY	degree 4 (P2.1)	na
Π_k^{\leq} -POLY	degree 8 (P2.1 + C1.2)	degree 9 (P2.6+ C1.2)
bo- $\Pi_k^<$ -POLY	degree 8 (C2.8)	degree 16 (C2.8)
bc- $\Pi_k^<$ -POLY	degree 8 (C2.8)	degree 16 (C2.8)
bo- $\Pi_k^=$ -POLY	degree 4 (C2.10)	na
bc- $\Pi_k^=$ -POLY	degree 4 (C2.10)	na

Figure 2: Degree bounds and references for complete problems for $\Pi_k \mathbb{R}$.

explore this question in depth for $\Sigma_2 \mathbb{R}$ and $\Pi_2 \mathbb{R}$. They identify three mechanisms that often lead to a jump in complexity: universal extension, imprecision, and robustness. In a *universal extension* variant, we ask whether any given partial solution can be extended to a full solution. E.g. starting with the \mathbf{NP} -complete graph coloring, the problem 2-COLORING EXTENSION, whether any partial 2-coloring of a given set of vertices can be extended to a 3-coloring of the graph is $\mathbf{coNP}^{\mathbf{NP}}$ -complete [38]; so the complexity makes a \mathbf{coNP} -jump. Dobbins *et al.* discuss a universal extension variant of the art gallery problem, and also mention some other candidates. In an *imprecision* variant each real parameter is replaced by a (metric) neighborhood. E.g. the coordinates of the art gallery may be given to within some precision bound. Can all the resulting art galleries be guarded by at most k guards [16]? The *robustness* variant applies the precision bound to the solution set: Given an art gallery are there k guards that guard the whole gallery, even if they are perturbed (each within some precision bound) [16]? We refer to the paper by Dobbins *et al.* [16] for more details, discussion, and further candidates.

As we saw, other natural problems arise when studying properties of semialgebraic sets, such as the Hausdorff distance of two semialgebraic sets. The paper by Bürgisser and Cucker [12] includes a pretty comprehensive look at computational questions about semialgebraic sets, and it leaves some interesting open questions. For example, they were able to show that testing whether a basic semialgebraic set is closed is $\forall \mathbb{R}$ -complete, but it is open whether membership in $\forall \mathbb{R}$ still holds for semialgebraic sets, see [12, Theorem 6.15]. A problem of practical interest [23] is testing whether a semialgebraic set contains an isolated point. The problem is $\forall \mathbb{R}$ -hard [12, Corollary 9.4], but the best known upper bound is $\Sigma_2 \mathbb{R}$. Some further properties to consider: A set is *star-shaped* if it contains a point that can “see” all other points in the set; by definition, testing whether a semialgebraic set is star-

shaped belongs to $\Pi_2\mathbb{R}$, but is it $\Pi_2\mathbb{R}$ -complete? Is testing the radius (or diameter) of a semialgebraic set $\Sigma_2\mathbb{R}$ -complete?

We can also consider replacing families of geometric objects with parameterized families of semialgebraic sets. In this context one could study the complexity of computing the Vapnik-Červonenkis dimension of the family $\{x \in \mathbb{R}^n : (\exists y \in \mathbb{R}^m) \varphi(x, y)\}$. The problem clearly lies in $\Sigma_3\mathbb{R}$. Is it $\Sigma_3\mathbb{R}$ -complete? This appears unlikely, since the universal quantifier is not really real, but Boolean, quantifying over subsets, so the problem lies in a hybrid discrete/real complexity class, another topic that deserves attention.³

Other problems related to semialgebraic sets, like testing connectivity, or counting connected components likely do not lie at a finite level of the real hierarchy, since they cannot be defined in first-order logic, but it would be interesting to find out for which levels they are hard, following Basu and Zell [5].

Finally, we need a deeper understanding of the power of Bürgisser and Cucker’s exotic quantifiers in the discrete setting. We conjecture that H does not affect the complexity at all, while \forall^* and \exists^* can be replaced with their non-exotic counterparts (at least in most situations). We have only shown this for \forall^* at the second level, and we saw that H can be eliminated as a first and last quantifier, but what if H occurs in the middle? We are not aware of any natural problems that would require such an H -quantifier, with the exception of the $\text{LocSupp}_{\mathbb{R}}$ -problem which has this flavor if we use existential quantifiers to replace the algebraic circuit defining the semialgebraic set with a formula.

Let us move on to some more structural, complexity-theoretic questions. Renegar’s quantifier elimination result implies that formulas with $O(\log n)$ quantifier alternations can be decided in **PSPACE**. Is it possible that the logarithmic fragment of the theory of the reals exhausts all of **PSPACE**? We would also like to see some oracle separations. The theory of the reals can be relativized in various (standard) ways. Are there oracles that separate **NP** from $\exists\mathbb{R}$? Or $\exists\mathbb{R}$ from **PSPACE**? Because $\mathbf{P} \subseteq \exists\mathbb{R} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP}$, any oracle that collapses **PSPACE** to **P** separates $\exists\mathbb{R}$ from **EXP**. On the other hand, since $\mathbf{NP} \subseteq \exists\mathbb{R}$, any oracle that separates **P** and **NP** also separates **P** and $\exists\mathbb{R}$, but these few facts seem to be the extent of our knowledge. In particular, we do not know whether the real hierarchy is proper (relative to an oracle).

Are there interesting problems in $\exists\mathbb{R} \cap \forall\mathbb{R}$? One such problem is semidef-

³We’d like to write $\exists\mathbb{R}^{\text{coNP}^{\exists\mathbb{R}}}$ for this class, but this notation suggests an oracle model for $\exists\mathbb{R}$, and the details of that would still need to be worked out.

inite programming; we can phrase the decision version of semidefinite programming as follows [24]: Given symmetric $n \times n$ matrices A_1, \dots, A_m, B are there $x_1, \dots, x_m \in \mathbb{R}$ such that $\sum_{i \in [m]} x_i A_i - B$ is positive semidefinite? Ramana [28] showed that the problem lies in $\mathbf{NP}_{\mathbb{R}} \cap \mathbf{coNP}_{\mathbb{R}}$, that is the intersection of the real versions of \mathbf{NP} and \mathbf{coNP} in the BSS-model, and it is not known whether the problem lies in $\mathbf{NP} \cap \mathbf{coNP}$. Using Ramana’s duality result for semidefinite programming it is easy to show that the problem also lies in $\exists\mathbb{R} \cap \forall\mathbb{R}$. One reason this is interesting is that the sum of square roots problem, that is, deciding the truth of sentences of the form $\sum_{i \in [n]} \sqrt{a_i} \leq k$, is a special case of semidefinite programming (e.g. [18]), and is one of the few problems in $\exists\mathbb{R}$ for which a better complexity bound than \mathbf{PSPACE} is known; it is located in the counting hierarchy [3].

Finally, we should mention that Erickson, van der Hoog, and Miltzow [17] defined a RAM-machine model for $\exists\mathbb{R}$ which goes beyond the BSS-model to capture integer operations; this model was extended to $\Pi_2\mathbb{R}$ by Dobbins, Kleist, Miltzow and Rzażewski [16], and it appears likely that it can be generalized to arbitrary levels, maybe even including an oracle mechanism.

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