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INTUITIONISTIC TENSE AND MODAL LOGIC

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§1. Introduction. In this article we shall construct intuitionistic analogues to the main systems of classical tense logic. Since each classical modal logic can be gotten from some tense logic by one of the definitions

(i) $\Box p \equiv p \wedge Gp \wedge Hp$, $\Diamond p \equiv p \vee Fp \vee Pp$; or,

(ii) $\Box p \equiv p \wedge Gp$, $\Diamond p \equiv p \vee Fp$

(see [5]), we shall find that our intuitionistic tense logics give us analogues to the classical modal logics as well.

We shall not here discuss the philosophical issues raised by our logics. Readers interested in the intuitionistic view of time and modality should see [2] for a detailed discussion.

In §2 we define the Kripke models for IK_t, the intuitionistic analogue to Lemmon's system K_t. We then prove the completeness and decidability of this system (§§3–5). Finally, we extend our results to other sorts of tense logic and to modal logic.

§2. Basic definitions. In the *language* of IK_t we have: sentence-letters p, q, r, etc.; the (intuitionistic) connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$; and unary operators P ("it was the case"), F (it will be the case"), H ("it has always been the case") and G ("it will always be the case"). *Formulas* are defined inductively: all sentence-letters are formulas; if X is a formula, so are $\neg X$, PX, FX, HX, and GX; if X and Y are formulas, so are $X \wedge Y$, $X \vee Y$, and $X \rightarrow Y$. We shall see that, in contrast to classical tense logic, F and P cannot be defined in terms of G and H.

The *semantics* has the following intuitive motivation. As in the semantics for intuitionistic logic we have a partially-ordered set of "states-of-knowledge," which we think of as belonging to a tense-logician who is studying a set of times. Within each state-of-knowledge there is a set of times and a temporal ordering. As the tense-logician moves to a greater state-of-knowledge, he retains all the information he had in lesser states-of-knowledge. Formally:

DEFINITION. An *intuitionistic tense structure* Γ is an ordered quintuple

$$\langle \Gamma, \leq, \{T_\gamma\}_{\gamma \in \Gamma}, \{\mu_\gamma\}_{\gamma \in \Gamma}, \{R_t^\gamma\}_{\gamma \in \Gamma, t \in T_\gamma} \rangle$$

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such that $(\Gamma, \leq)$ is a partially-ordered set (the "states-of-knowledge"), T_γ is a set (the set of times known at state-of-knowledge γ), and u_γ is a binary relation on T_γ (the temporal ordering of T_γ as it is understood at state-of-knowledge γ). We require that $\gamma \leq \varphi$ imply $T_\gamma \subseteq T_\varphi$ and $u_\gamma \subseteq u_\varphi$—that is, if we advance in knowledge we retain what we know about times and their temporal ordering. Observe that we do not require $T_\gamma = T_\varphi$. (Later we shall construct a system in which the set of times is unchanging.) Finally, each R_t^γ is a relation on formulas such that:

1. $R_t^\gamma(p)$ and $\gamma \leq \varphi$ imply $R_t^\varphi(p)$ for atomic p;
2. $R_t^\gamma(X \wedge Y)$ iff $R_t^\gamma(X)$ and $R_t^\gamma(Y)$;
3. $R_t^\gamma(X \vee Y)$ iff $R_t^\gamma(X)$ or $R_t^\gamma(Y)$;
4. $R_t^\gamma(\neg X)$ iff $(\forall \varphi\colon \varphi \geq \gamma)$ not $R_t^\varphi(X)$;
5. $R_t^\gamma(X \to Y)$ iff $(\forall \varphi\colon \varphi \geq \gamma)(R_t^\varphi(X)$ implies $R_t^\varphi(Y))$;
6. $R_t^\gamma(PX)$ iff $(\exists t')(t' u_\gamma t$ and $R_{t'}^\gamma(X))$;
7. $R_t^\gamma(FX)$ iff $(\exists t')(t u_\gamma t'$ and $R_{t'}^\gamma(X))$;
8. $R_t^\gamma(HX)$ iff $(\forall \varphi \geq \gamma)(\forall t' \in T_\varphi)(t' u_\varphi t$ implies $R_{t'}^\varphi(X))$;
9. $R_t^\gamma(GX)$ iff $(\forall \varphi \geq \gamma)(\forall t' \in T_\varphi)(t u_\varphi t'$ implies $R_{t'}^\varphi(X))$.

Rules 4 and 5 are explained by standard Kripke-model arguments; see e.g. [1]. Rules 8 and 9 require comment. To know that GX holds at time t it is not enough to know that X holds at all future times t' contained in the current state-of-knowledge γ; rather, one must know that X holds at every future time that can *ever* be contained in a greater state-of-knowledge $\varphi \geq \gamma$. For otherwise we could have $R_t^\gamma(GX)$ without having $R_t^\varphi(GX)$.

It is easy to prove that, for any formula X, if $R_t^\gamma(X)$ and $\gamma \leq \varphi$ then $R_t^\varphi(X)$.

We say that X *holds* in Γ ($\Gamma \vDash X$) iff for all $\gamma \in \Gamma$ and $t \in T_\gamma$ we have $R_t^\gamma(X)$. We say X *is valid* ($\vDash X$) iff $\Gamma \vDash X$ for every model Γ. One can easily check that the laws of intuitionistic logic are valid.

Note that if we are given all the ingredients of a model, but with the $\{R_t^\gamma\}$ defined only on atomic formulas, then we can extend the $\{R_t^\gamma\}$ in one and only one way to end up with a model.

Finally, observe that we distinguish between *states-of-knowledge* (which our logician is in) and *times* (which he is studying). We argue in [2] that intuitionism must draw such a distinction and that consequently our semantics is the "right" semantics for an intuitionistic tense-logic.

§3. Completeness. We shall use the following Gentzen-style formulation of intuitionistic logic (where $\mathscr{A}$ and $\mathscr{B}$ are finite sets of formulas):

(G1) $\dfrac{\mathscr{A}, X \wedge Y \vdash \mathscr{B}}{\mathscr{A}, X, Y \vdash \mathscr{B}}$ (double line) (G2) $\dfrac{\mathscr{A} \vdash \mathscr{B}, X \vee Y}{\mathscr{A} \vdash \mathscr{B}, X, Y}$ (double line)

(G3) $\dfrac{\mathscr{A}, X \vdash}{\mathscr{A} \vdash \neg X}$ (G3′) $X, \neg X \vdash$

(G4) $\dfrac{X, \mathscr{A} \vdash Y}{\mathscr{A} \vdash X \to Y}$ (G4′) $X, X \to Y \vdash Y$

(G5) $\dfrac{\mathscr{A} \vdash \mathscr{B}}{\mathscr{A}', \mathscr{A} \vdash \mathscr{B}, \mathscr{B}'}$ (G6) $\dfrac{\mathscr{A}, X \vdash \mathscr{B};\ \mathscr{A} \vdash \mathscr{B}, X}{\mathscr{A} \vdash \mathscr{B}}$

(G7) $\mathscr{A} \vdash \mathscr{A}$

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(A single line means we are entitled to deduce from top to bottom; a double line, that we are entitled to deduce in both directions.)

If $\mathscr{A} = \{A_1, \ldots, A_n\}$ and $\mathscr{B} = \{B_1, \ldots, B_m\}$ then $\mathscr{A} \vdash \mathscr{B}$ may be regarded as an abbreviation for $\vdash A_1 \wedge \cdots \wedge A_n . \rightarrow . B_1 \vee \cdots \vee B_m$.

We add the following tense-logical rules:

(G8) $\dfrac{\mathscr{A} \vdash X}{G\mathscr{A} \vdash GX}$ (G8′) $\dfrac{\mathscr{A} \vdash X}{H\mathscr{A} \vdash HX}$

(G9) $\dfrac{\mathscr{A}, X \vdash \mathscr{B}}{G\mathscr{A}, FX \vdash F\mathscr{B}}$ (G9′) $\dfrac{\mathscr{A}, X \vdash \mathscr{B}}{H\mathscr{A}, PX \vdash P\mathscr{B}}$

(G10a) $\vdash FHX \rightarrow X$ (G10a′) $\vdash PGX \rightarrow X$

(G10b) $\vdash X \rightarrow GPX$ (G10b′) $\vdash X \rightarrow HFX$

(G11a) $\vdash (FX \rightarrow GY) \rightarrow G(X \rightarrow Y)$ (G11a′) $\vdash (PX \rightarrow HY) \rightarrow H(X \rightarrow Y)$

(G11b) $\vdash F(X \rightarrow Y) \rightarrow (GX \rightarrow FY)$ (G11b′) $\vdash P(X \rightarrow Y) \rightarrow (HX \rightarrow PY)$

In the above rules, $G\mathscr{A}$ is $\{GA: A \in \mathscr{A}\}$ and $F\mathscr{B}$ is $\{FB: B \in \mathscr{B}\}$; similarly for $H\mathscr{A}$ and $P\mathscr{B}$.

We define a pair of sets $(\mathscr{A}, \mathscr{B})$ to be *consistent* if for no finite subsets $\mathscr{A}_0 \subseteq \mathscr{A}$ and $\mathscr{B}_0 \subseteq \mathscr{B}$ do we have $\mathscr{A}_0 \vdash \mathscr{B}_0$. By (G6) and (G7) we have:

THEOREM 1. *If* $(\mathscr{A}, \mathscr{B})$ *is consistent, then there exists a consistent pair* $(\mathscr{A}', \mathscr{B}')$ *such that:*

(i) $\mathscr{A} \subseteq \mathscr{A}'$ *and* $\mathscr{B} \subseteq \mathscr{B}'$;

(ii) $\mathscr{A}' \cap \mathscr{B}' = \varnothing$; *and*

(iii) *for every formula* X, $X \in \mathscr{A}'$ or $X \in \mathscr{B}'$.

We call a pair satisfying (ii) and (iii) *maximal consistent*. Any maximal consistent pair $(\mathscr{A}, \mathscr{B})$ can be represented by a *valuation*

$$v: \{\text{formulas}\} \rightarrow \{0, 1\}$$

such that $v(X) = 1$ iff $X \in \mathscr{A}$. Valuations are partially-ordered by the definition

$$v \leq w \quad \text{iff} \quad (\forall X)(v(X) = 1 \text{ implies } w(X) = 1).$$

We define two further relations, R and S, on valuations as follows:

$$vRw \quad \text{iff} \quad (\forall X)(v(GX) = 1 \text{ implies } w(X) = 1 \,.\&.\, w(X) = 1 \text{ implies } v(FX) = 1),$$
$$vSw \quad \text{iff} \quad (\forall X)(w(HX) = 1 \text{ implies } v(X) = 1 \,.\&.\, v(X) = 1 \text{ implies } w(PX) = 1).$$

Let C_{IK_t} be the set of all IK_t valuations.

LEMMA 1. *For all* v, w *in* C_{IK_t}, *if* vRw *then* vSw.

PROOF. If $w(HX) = 1$ then, since vRw, $v(FHX) = 1$. So by (G10a), $v(X) = 1$. If $v(X) = 1$, then, by (G10b), $v(GPX) = 1$. So $w(PX) = 1$. QED.

LEMMA 2. *For all* v, w, *if* vSw *then* vRw.

PROOF. Same as Lemma 1, but using rules (G10a′, b′). QED.

Lemmas 1 and 2 show that vRw iff vSw.

LEMMA 3. *For any valuation* v *and formula* X, $v(FX) = 1$ *iff* $(\exists w)(vRw$ *and* $w(X) = 1)$.

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PROOF. Right-to-left is trivial. So assume $v(FX) = 1$. Consider the pair $(\{A: v(GA) = 1\} \cup \{X\}; \{B: v(FB) = 0\})$. This pair is consistent by rule (G9): for if not then we would have finite $\mathscr{A}, \mathscr{B}$ such that $\mathscr{A}, X \vdash \mathscr{B}$. But then $G\mathscr{A}$, $FX \vdash F\mathscr{B}$, which is impossible since v is consistent.

Using Theorem 1, extend the pair to a maximal consistent pair with valuation w. We claim vRw. Clearly, if $v(GA) = 1$ then $w(A) = 1$. And if $w(B) = 1$ then by construction if $v(FB) = 0$ then $w(B) = 0$; so $v(FB) = 1$. QED.

Similarly we have

LEMMA 4. *For any valuation v and formula X, $v(PX) = 1$ iff $(\exists w)(wRv$ and $w(X) = 1)$.*

LEMMA 5. *For any valuation v and formula X, $v(GX) = 1$ iff*

$$(\forall w)(\forall w')(v \leq w \text{ and } wRw'. \text{ implies } w'(X) = 1).$$

PROOF. Right-to-left is trivial. So assume $v(GX) = 0$. Let the maximal consistent pair associated with v be $(\mathscr{A}; \mathscr{B}, GX)$. Consider the pair $(\{C: v(GC) = 1\}; \{X\})$. This pair is consistent by the rule (G8). Extend it to a maximal consistent pair $(\mathscr{C}, \mathscr{D})$ represented by the valuation w'. We must show that there exists w such that $v \leq w$ and wRw'.

So consider the pair $(\mathscr{A} \cup \{FC: w'(C) = 1\}; \{GD: w'(D) = 0\})$. We claim that this pair is consistent. If it is not, then for some finite $\mathscr{A}_0 \subseteq \mathscr{A}$ we have $\mathscr{A}_0$, $FC \vdash GD$, where $w'(C) = 1$ and $w'(D) = 0$. But then, by rule (G4), $\mathscr{A}_0 \vdash FC \rightarrow GD$. But then the formula $FC \rightarrow GD$ belongs to $\mathscr{A}$. So by rule (G11a), $G(C \rightarrow D)$ also belongs to $\mathscr{A}$. So $C \rightarrow D$ belongs to $\mathscr{C}$. So $w'(D) = 1$. Contradiction. Hence the pair is consistent. Extend it to a maximal consistent pair represented by the valuation w. Clearly, $v \leq w$ and wRw'. QED.

Similarly we have

LEMMA 6. *For any valuation v and formula X, $v(HX) = 1$ iff*

$$(\forall w)(\forall w')(v \leq w \text{ and } w'Rw. \text{ implies } w'(X) = 1).$$

LEMMA 7. *If vRv' and $v \leq w$, then $(\exists w')(wRw'$ and $v' \leq w')$.*

PROOF. Suppose $\mathscr{A} = \{A: v'(A) = 1\}$. Consider the pair $(\mathscr{A} \cup \{X: w(GX) = 1\}; \{Y: w(FY) = 0\})$. We claim this is consistent. If not, then we have A, $X \vdash Y$ for formulas A, X, Y with $v'(A) = 1$, $w(GX) = 1$ and $w(FY) = 0$. But then $A \vdash X \rightarrow Y$; so $v'(X \rightarrow Y) = 1$. So $v(F(X \rightarrow Y)) = 1$. So, by rule (G11b), $v(GX \rightarrow FY) = 1$. So $w(GX \rightarrow FY) = 1$. Contradiction. QED.

LEMMA 8. *If $v'Rv$ and $v \leq w$, then $(\exists w')(w'Rw$ and $v' \leq w')$.*

PROOF. Similar to that of Lemma 7, but using rule (G11a). QED.

We now have the tools to prove the strong completeness of IK_t; in other words,

THEOREM 3. *For every IK_t-valuation v there exists an intuitionistic tense structure*

$$\langle \Gamma, \leq, \{T_\gamma\}_{\gamma \in \Gamma}, \{\mu_\gamma\}_{\gamma \in \Gamma}, \{R_t^\gamma\}_{t \in T_\gamma} \rangle,$$

a state-of-knowledge $\gamma \in \Gamma$, and a time $t \in T_\gamma$ such that, for every formula X of IK_t,

$$R_t^\gamma(X) \quad \textit{iff} \quad v(X) = 1.$$

PROOF. Our proof will use Gabbay's technique of *selective filtration*; see [3]. The idea is to start with the valuation v and from it construct the tense structure

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inductively, using the lemmas proved above. We think of v as representing a time in a state-of-knowledge γ_0, and of $\{X: v(X) = 1\}$ as the set of formulas holding at that time. We pick an enumeration of the formulas of IK_t and use this enumeration to construct our model. Thus, at stage n of our construction we look at formula X_n. If, for example, X_n is FY and $v(FY) = 1$, then we must add a time t to T_{γ_0} with $vu_{\gamma_0}t$ and $R_t^{\gamma_0}(Y)$. Similarly, if X_n is GY and $v(GY) = 0$ then we must add a new state-of-knowledge $\gamma_1 \geq \gamma_0$ and a new time t in T_{γ_1} with $vu_{\gamma_1}t$ and not $R_t^{\gamma_1}(Y)$. At each stage n of the construction we have a "finite approximation" Δ_n to the final model Δ that will satisfy v. Each Δ_n is a set of ordered triples $\langle \gamma_p, t_q, w\rangle$, where p and q are natural numbers and w is a valuation. Intuitively, γ_p is a state-of-knowledge, t_q is a time in T_{γ_p}, and $\{X: w(X) = 1\}$ is the set of formulas such that $R_{t_q}^{\gamma_p}(X)$. The model Δ will be the union of the sets Δ_n.

The intuitive idea behind the construction is thus quite simple, although the details are complex.

Fix an enumeration $X_1, X_2, X_3, \ldots$ of the formulas of IK_t.

Stage 0. $\langle \gamma_0, t_0, v\rangle \in \Delta_0$.

Stage 1. Consider the formula X_1. There are eight cases.

Case 1. X_1 is FY for some Y. If $v(FY) = 1$, then by Lemma 3 there is a valuation w such that wRw and $w(Y) = 1$. Set $\Delta_1 = \{\langle \gamma_0, t_1, w\rangle\} \cup \Delta_0$ and set $t_0 u_{\gamma_0} t_1$. If $v(FY) = 0$, then set $\Delta_0 = \Delta_1$.

Case 2. X_1 is PY for some Y. This case is similar to Case 1, only using Lemma 4 instead.

Case 3. X_1 is GY for some Y. If $v(GY) = 1$, set $\Delta_1 = \Delta_0$. If $v(GY) = 0$, then by Lemma 5 there exist valuations w, w' such that $v \leq w$ and wRw' and $w'(Y) = 1$. Set $\Delta_1 = \Delta_0 \cup \{\langle \gamma_1, t_0, w\rangle, \langle \gamma_1, t_1, w'\rangle\}$. Set $t_0 u_{\gamma_1} t_1$.

Case 4. X_1 is HY for some Y. This case is similar to Case 3, but using Lemma 6 instead.

Cases 5 *through* 8. X_1 is $\neg Y$, $Y \to Z$, $Y \wedge Z$, $Y \vee Z$. If $v(\neg Y) = 0$ ($v(Y \to Z) = 0$) then by the ordinary intuitionistic completeness proof there is an IK_t-valuation w with $v \leq w$ and $w(Y) = 1$ ($w(X) = 1$ and $w(Z) = 0$). In these two cases, set $\Delta_1 = \Delta_0 \cup \{\langle \gamma_1, t_0, w\rangle\}$ and set $\gamma_0 \leq \gamma_1$. In the other cases, set $\Delta_1 = \Delta_0$.

Stage n. Δ_{n-1} is a finite set of ordered triples $\{\langle \gamma_p, t_q, w\rangle\}_{0 \leq p \leq b}^{0 \leq q \leq a}$ with a partial ordering on the γ_p's and a temporal ordering u_{γ_p} of the t_q's. We consider the formulas $X_1, \ldots, X_n$. There are, as in Stage 1, eight cases.

Case 1. If any of these formulas X_i is FY for some Y, and if in any $\langle \gamma_p, t_q, w\rangle \in \Delta_{n-1}$ we have $w(FY) = 1$ but no $\langle \gamma_p, t', w'\rangle$ with $t_q u_{\gamma_p} t'$ and $w'(Y) = 1$, then (using Lemma 3) add such a $\langle \gamma_p, t', w'\rangle$ to Δ_{n-1} with wRw' and $t_q u_{\gamma_p} t'$. Do this for all the formulas $X_1, \ldots, X_n$ and all the elements of Δ_{n-1}. Whenever we add such an ordered triple $\langle \gamma_p, t', w'\rangle$ to Δ_{n-1} we must, for every γ_s such that $\gamma_s \geq \gamma_p$, add a triple $\langle \gamma_s, t', x\rangle$ with $x \geq w'$ and with yRx (where y is the valuation associated to γ_s and t_q). Lemma 7 lets us do this, and insures that if $\gamma_p \in \gamma_s$ then $T_{\gamma_p} \subseteq T_{\gamma_s}$. Finally, set $t_q u_{\gamma_s} t'$.

Case 2. If any of the formulas $X_1, \ldots, X_n$ is PY for some Y, we proceed similarly to Case 1.

Case 3. Suppose one of the formulas $X_1, \ldots, X_n$ is GY for some Y and that for some $\langle \gamma_p, t_q, z\rangle$ in Δ_{n-1} we have $z(GY) = 0$ and that, for all $\langle \gamma_p, t, x\rangle$ with $t_q u_{\gamma_p} t$, $x(Y) = 1$. Then using Lemma 5 we find valuations w, w' such that $z \leq w$, wRw', and $w'(Y)$

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$= 0$. Intuitively, we are advancing to a new state-of-knowledge ($z \leq w$) and adding a new time (w') at which Y fails. So formally we pick a new γ_r and t_s, with $\gamma_p \leq \gamma_r$ and $t_q u_{\gamma_r} t_s$. We must now insure that $T_{\gamma_p} \subseteq T_{\gamma_r}$ and $u_{\gamma_p} \subseteq u_{\gamma_r}$. We add $\langle \gamma_r, t_q, w \rangle$ and $\langle \gamma_r, t_s, w' \rangle$ to Δ_{n-1}, and set $t_q u_{\gamma_r} t_s$. Now suppose $t_q u_{\gamma_p} t_u$ ($t_u u_{\gamma_p} t_q$). Using Lemma 7 (Lemma 8) we find an appropriate valuation x with wRx (xRw). We put $\langle \gamma_r, t_u, x \rangle$ into Δ_n with $t_q u_{\gamma_r} t_u$ ($t_u u_{\gamma_r} t_q$). We then move on to the untreated immediate successors and predecessors of t_u, and so on, until we have exhausted the set $\{t : \langle \gamma_p, t, \cdot \rangle \in \Delta_{n-1}\}$. Notice that this set is a connected tree under the relation u_{γ_p}, so we can be sure of completely reproducing the tree at the γ_s level.

Cases 5 *through* 8. If X_1 or $\cdots$ or X_n is $HY, \neg Y, Y \to Z, Y \wedge Z, Y \vee Z$ we act as in Case 3 or as in Step 1, Cases 5 through 8.

We set $\Delta = \bigcup_{n=1}^{\infty} \Delta_n$. Notice that if $\langle \gamma, t, w \rangle$ and $\langle \gamma, t', x \rangle$ are in Δ and $t u_\gamma t'$, then wRx.

The sought-after intuitionistic tense structure is defined from Δ in the obvious way: $R_t^\gamma(X)$ iff $\langle \gamma, t, w \rangle \in \Delta$ and $w(X) = 1$. It is a trivial matter to check that this yields a model; and by construction $R_{t_0}^{\gamma_0}(X)$ iff $v(X) = 1$. QED.

Axioms. From the Gentzen-style formulation of IK_t we can easily derive an axiomatization.

AXIOMS.

(1) All axioms of the intuitionistic sentential calculus;

(2)	$G(X \to Y) \to (GX \to GY)$,	(2′)	$H(X \to Y) \to (HX \to HY)$;
(3)	$G(X \wedge Y) \leftrightarrow GX \wedge GY$,	(3′)	$H(X \wedge Y) \leftrightarrow (HX \wedge HY)$;
(4)	$F(X \vee Y) \leftrightarrow FX \vee FY$,	(4′)	$P(X \vee Y) \leftrightarrow (PX \vee PY)$;
(5)	$G(X \to Y) \to (FX \to FY)$,	(5′)	$H(X \to Y) \to (PX \to PY)$;
(6)	$GX \wedge FY \to F(X \wedge Y)$,	(6′)	$HX \wedge PY \to P(X \wedge Y)$;
(7)	$G\neg X \to \neg FX$,	(7′)	$H\neg X \to \neg PX$;
(8)	$FHX \to X$,	(8′)	$PGX \to X$;
(9)	$X \to GPX$,	(9′)	$X \to HFX$;
(10)	$(FX \to GY) \to G(X \to Y)$,	(10′)	$(PX \to HY) \to H(X \to Y)$;
(11)	$F(X \to Y) \to (GX \to FY)$,	(11′)	$P(X \to Y) \to (HX \to PY)$.

RULES OF INFERENCE. (R1) *modus ponens*;
(R2) $\vdash X$ implies $\vdash GX$;
(R3) $\vdash X$ implies $\vdash HX$.

THEOREM 4. *The above axioms yield the Gentzen rules for IK_t.*

PROOF. The only nontrivial cases are the rules (G8) and (G9). We first prove (G8). If $\vdash A_1 \wedge \cdots \wedge A_n . \to X$, then, by (R2), $\vdash G(A_1 \wedge \cdots \wedge A_n . \to X)$. By *modus ponens* and axiom 2, $\vdash G(A_1 \wedge \cdots \wedge A_n) \to GX$. By axiom 3, $\vdash GA_1 \wedge \cdots \wedge GA_n . \to GX$. If $\mathscr{A} = \varnothing$ then (G8) is just (R2).

We now prove (G9). If $\vdash A_1 \wedge \cdots \wedge A_n \wedge X . \to . B_1 \vee \cdots \vee B_m$ then by (R2) and axiom 5 and *modus ponens*

$$\vdash F(A_1 \wedge \cdots \wedge A_n \wedge X) \to F(B_1 \vee \cdots \vee B_m).$$

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By axiom 6,

$$\vdash G(A_1 \wedge \cdots \wedge A_n) \wedge FX .\rightarrow. F(B_1 \vee \cdots \vee B_m).$$

By axioms 3 and 4,

$$\vdash GA_1 \wedge \cdots \wedge GA_n \wedge FX .\rightarrow. FB_1 \vee \cdots \vee FB_m.$$

If $\mathscr{A} = \varnothing$ then

$$\vdash X \rightarrow. B_1 \vee \cdots \vee B_m.$$

By (R2), $\vdash G(X \rightarrow. B_1 \vee \cdots \vee B_m)$. So by axiom 2, $\vdash FX \rightarrow F(B_1 \vee \cdots \vee B_m)$. If $\mathscr{B} = \varnothing$ then $\vdash A_1 \wedge \cdots \wedge A_n . \rightarrow X$. By (R2) and axiom 2, $\vdash G(A_1 \wedge \cdots \wedge A_n) \rightarrow G(\neg X)$. By axioms 4 and 3, $\vdash GA_1 \wedge \cdots \wedge GA_n . \rightarrow \neg FX$.

The rules (G8′) and (G9′) are proved similarly. QED.

Unchanging times. In the semantics for IK_t we require that $\gamma \leq \varphi$ imply $T_\gamma \subseteq T_\varphi$ and $u_\gamma \subseteq u_\varphi$. Thus, as we advance in knowledge it is possible for us to discover more times and more facts about the temporal ordering. Let us now consider what happens if we require times and the temporal ordering to be unchanging—that is, for all γ, φ we have $T_\gamma = T_\varphi$ and $u_\gamma = u_\varphi$. We claim that this new system can be axiomatized by replacing the Gentzen rules (G8) and (G8′) by the rules

$$(\text{G8}^*) \quad \frac{\mathscr{A} \vdash \mathscr{B}, X}{G\mathscr{A} \vdash F\mathscr{B}, GX}, \qquad (\text{G8}'^*) \quad \frac{\mathscr{A} \vdash \mathscr{B}, X}{H\mathscr{A} \vdash P\mathscr{B}, HX}.$$

To prove this, let $C_{IK_t^*}$ be the set of all IK_t^*-valuations; define R, S, and $\leq$ as before. All the lemmas we proved for C_{IK_t} hold for $C_{IK_t^*}$ as well. In addition:

LEMMA 9. *In $C_{IK_t^*}$, if $v \leq v'$ and $v'Rw'$, then, for some w, vRw and $w \leq w'$.*

PROOF. Consider the pair $(\{A : v(GA) = 1\}; \{B : v(FB) = 0\} \cup \{X : w'(X) = 0\})$. It is consistent by (G8*) and the observation that for any formula X, if $w'(X) = 0$ then $v'(GX) = 0$, hence $v(GX) = 0$. Extend the pair to a maximal consistent w. Since $w'(X) = 0$ implies $w(X) = 0$, we conclude that $w \leq w'$. QED.

LEMMA 10. *If $v \leq v'$ and $w'Rv'$ then, for some w, wRv and $w \leq w'$.*

PROOF. The same, but using (G8′*) and the fact that $R = S$. QED.

The completeness theorem for IK_t^* is the same as for IK_t except that in each Δ_n we require the stock of times and the temporal ordering to be the same at each state-of-knowledge; Lemmas 9 and 10 allow us to do this.

The rules (G8*) and (G8′*) are closely related to the intuitionistically unacceptable quantifier rule in the predicate calculus

$$\frac{A \vdash B \vee C}{(\forall x)A \vdash (\exists x)B \vee (\forall x)C}$$

which is characteristic for Kripke models with constant domains [1].

If we set up a system with unchanging times but changing temporal relations, then we get the same tense logic as IK_t. For every IK_t model can be modified in the following way: if $\gamma \leq \varphi$ and $t \in T_\gamma$ but $t \notin T_\gamma$, put $t \in T_\gamma$ but leave t temporally unconnected to all other times in T_γ. If a formula X is valid in the modified tense structure, then it is valid in the original tense structure. Hence, if X is valid for

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unchanging times it is valid for IK_t; the converse is obvious.

Similarly, if we set up a tense system with changing times but unchanging temporal relations, i.e. one in which

$$(*) \qquad (\forall\gamma)(\forall\varphi \geq \gamma)((t_1 \in T_\gamma \text{ and } t_2 \in T_\gamma) \Rightarrow (t_1 u_\gamma t_2 \text{ iff } t_1 u_\varphi t_2)),$$

then we get the same tense logic as IK_t. This fact can be established by observing that in the completeness proof for IK_t the model Δ satisfies $(*)$, so that IK_t is complete for the class of models satisfying $(*)$.

§4. Decidability. We use the filtration techniques of Lemmon and Scott [4]. (An alternative technique, that of reducing the decision problem to that for $S2S$, is mentioned by Rabin ([8], p. 621).)

DEFINITION. A *decidability model* for IK_t is an ordered quadruple

$$\langle \Gamma \leq U, \{R_v\}_{v \in \Gamma} \rangle$$

such that $(\Gamma, \leq)$ is a partially-ordered set, U is a relation on Γ satisfying

$$vUw \text{ and } v \leq v' \quad \text{imply} \quad (\exists w')(v'Uw' \text{ and } w \leq w'),$$

$$wUv \text{ and } v \leq v' \quad \text{imply} \quad (\exists w')(w'Uv' \text{ and } w \leq w'),$$

and, finally, each R_v is a relation on formulas of IK_t such that:

i) $R_v(p)$ and $v \leq w$ imply $R_w(p)$ for atomic p;
ii) $R_v(X \wedge Y)$ iff $R_v(X)$ or $R_v(Y)$;
iii) $R_v(X \vee Y)$ iff $R_v(X)$ or $R_v(Y)$;
iv) $R_v(\neg X)$ iff $(\forall w: w \geq v)$ (not $R_w(X)$);
v) $R_v(X \rightarrow Y)$ iff $(\forall w: w \geq v)(R_w(X)$ implies $R_w(Y))$;
vi) $R_v(FX)$ iff $(\exists w)(vUw$ and $R_w(X))$;
vii) $R_v(PX)$ iff $(\exists w)(wUv$ and $R_w(X))$;
viii) $R_v(GX)$ iff $(\forall w: w \geq v)(\forall w': wUw')(R_{w'}(X))$;
ix) $R_v(HX)$ iff $(\forall w: w \geq v)(\forall w': w'Uw)(R_{w'}(X))$.

The definition of validity is as before. Lemmas 1–8 show that C_{IK_t} is a decidability model for IK_t. Since, for any formula X of IK_t, $\vdash X$ iff $v(X) = 1$ for every v in C_{IK_t}, we shall call C_{IK_t} the *canonical decidability model* for IK_t. It is easy to check, using the axiomatization, that every theorem of IK_t is valid in the decidability modelling. Thus:

THEOREM 4. *IK_t is complete with respect to the class of decidability models.*

Now let X be a formula of IK_t. We shall show that if X is not valid then there exists a finite decidability model in which X fails to hold. Form the closure under the intuitionistic connectives $\wedge, \vee, \neg$, and $\rightarrow$ of the set of all subformulas of X. Call the resulting set $\Phi(X)$. Note that $\Phi(X)$ modulo intuitionistic equivalence is the free Heyting algebra generated by the set of subformulas of X. If the cardinality of this set is n, then the cardinality of the algebra is at most 2^{2^n} (see [6, p. 61]; [1, §5.2]). We now define an equivalence relation $\sim$ on C_{IK_t} as follows:

DEFINITION. For v, w in C_{IK_t}, $v \sim w$ iff for all $Y \in \Phi(X)$, $v(Y) = w(Y)$.

We let v^* be the equivalence class of v under $\sim$; we let C_{IK_t}/X be the set of all such v^*.

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DEFINITION. For v^*, w^* in C_{IK_t}/X,

$$\begin{aligned} v^*Rw^* &\quad \text{iff} \quad (\exists v_1 \in v^*)(\exists w_1 \in w^*)(v_1 R w_1); \\ v^*Sw^* &\quad \text{iff} \quad (\exists v_1 \in v^*)(\exists w_1 \in w^*)(v_1 S w_1); \\ v^* \leq w^* &\quad \text{iff} \quad (\exists v_1 \in v^*)(\exists w_1 \in w^*)(v_1 \leq w_1). \end{aligned}$$

To show that these definitions turn C_{IK_t}/X into a decidability model we must show four things:

(i) v^*Rw^* iff v^*Sw^*;

(ii) $\leq$ is a partial ordering;

(iii) if v^*Rw^* and $v^* \leq v'^*$, then, for some w'^*, $v'^*Rw'^*$ and $w^* \leq w'^*$;

(iv) if w^*Rv^* and $v^* \leq v'^*$, then, for some w'^*, $w'^*Rv'^*$ and $w^* \leq w'^*$.

The proof of (i) is trivial. The proofs of (ii)–(iv) require a lemma.

LEMMA 11. *If $v_1 \leq x_1$ and $v_1 \sim v_2$, then there exists x_2 such that $v_2 \leq x_2$ and $x_1 \sim x_2$.*

PROOF. Consider the ordered pair

$$(\{Y: v_2(Y) = 1\} \cup \{Z \in \Phi(X): x_1(Z) = 1\}; \{W \in \Phi(X): x_1(W) = 0\}).$$

We claim it is consistent. If it is not, then there exist formulas Y, Z, W with $v_2(Y) = 1$, $x_1(Z) = 1$, $x_1(W) = 0$, $Z \in \Phi(X)$, and $W \in \Phi(X)$, and such that $Y, Z \vdash W$. But then $Y \vdash Z \to W$. So $v_2(Z \to W) = 1$. Since $Z \to W \in \Phi(X)$ and since $v_2 \sim v_1$, $v_1(Z \to W) = 1$. So $x_1(Z - W) = 1$. But $x_1(Z) = 1$; so $x_1(W) = 1$. Contradiction. Thus the pair is consistent. Extend it to the maximal consistent x_2 which we were seeking. QED.

We now show (ii). Reflexivity is trivial. Transitivity: if $v^* \leq w^* \leq x^*$ then $v \sim v_1 \leq w_1 \sim w_2 \leq x_1 \sim x$. By Lemma 11, there exists x_2 with $x_1 \sim x_2$ and $x_2 \geq w_1$. Thus $x_2 \geq v_1$. So $v^* \leq x^*$. Anti-symmetry: if $v^* \leq w^* \leq v^*$ then $v \sim v_1 \leq w_1 \sim w \sim w_2 \leq v_2 \sim v$. Thus for any $Y \in \Phi(X)$, $v(Y) = w(Y)$, so that $v^* = w^*$.

As for (iii), if $x^* \geq v^*Rw^*$ we have $x \sim x_1 \geq v_2 \sim v \sim v_1 R w_1 \sim w$. By Lemma 11, there exists $x_2 \geq v_1$ with $x_2 \sim x_1$. Thus, by Lemma 7, there exists w_2 such that $w_2 \geq w_1$ and $x_2 R w_2$. The proof of (iv) is similar to that of (iii) but using Lemma 8.

Now that we have proved (i)–(iv), if we want to make C_{IK_t}/X into a decidability model we need only say which atomic formulas hold at each v^*. We can then extend to a decidability model by induction. Thus we define

$$\begin{aligned} v^*(p) = 1 &\quad \text{iff } v(p) = 1 \text{ and } p \in \Phi(X); \\ v^*(p) = 0 &\quad \text{otherwise.} \end{aligned}$$

We extend to a model. Then

THEOREM 5. *For all $Y \in \Phi(X)$ and for all $v \in C_{IK_t}$ we have $v^*(Y) = v(Y)$.*

PROOF. For atomic Y this is the definition. The cases of $\wedge$ and $\vee$ are trivial. If Y is $\neg Z$ and $v^*(Y) = 0$ then, for some $w^* \geq v^*$, $w^*(Z) = 1$. Since $w^* \geq v^*$, there exist $v_1 \sim v$ and $w_1 \sim w$ with $v_1 \leq w_1$. By the induction hypothesis $w(Z) = 1$; thus $w(\neg Z) = 0$; thus $w_1(\neg Z) = 0$; thus $v_1(\neg Z) = v(\neg Z) = 0$. The case where $v^*(\neg Z) = 1$ is trivial. The case of $\to$ is similar to that of $\neg$.

If Y is GZ and $v^*(GZ) = 0$, then

$$(\exists w^* \geq v^*)(\exists x^*)(w^*Rx^* \text{ and } x^*(Z) = 0).$$

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Thus there exist v_1, w_1, w_2, x_1 such that $v \sim v_1 \le w_1 \sim w_2 R x_1 \sim x$. By the induction hypothesis, $x(Z) = 0$. So $w_1(GZ) = 0$. So $v(GZ) = 0$. The case where $v^*(GZ) = 1$ and the case where Y is FZ are trivial. The cases of H and P are exactly like the cases of F and G. QED.

It follows from this theorem that IK_t has the finite model property; thus it is decidable.

§5. Transitive, reflexive, and serial time. If we wish to make our temporal ordering *transitive* we add the axioms

$$(12) \quad \vdash FFX \to FX, \qquad (12') \quad \vdash GX \to GGX$$

to the axioms for IK_t. It is clear that these new axioms are valid in all intuitionistic tense structures satisfying

$$(\forall \gamma \in \Gamma)(\forall x, y, z \in T_\gamma)(xu_\gamma y \text{ and } yu_\gamma z \text{ imply } xu_\gamma z).$$

And conversely if C_{IK_c} is the set of all valuations for the new system (which we call IK_c), then it is easy to check that its temporal ordering R (defined as for C_{IK_t}) is transitive: if vRw and wRx then $v(GX) = 1$ implies (by 12′) $v(GGX) = 1$ implies $w(GX) = 1$ implies $x(X) = 1$. And if $x(X) = 1$ then $w(FX) = 1$, so $v(FFX) = 1$, so, by 12, $v(FX) = 1$. So vRx.

The selective filtration completeness proof for IK_t can therefore be converted into a completeness proof for IK_c: the only difference is that we require the temporal ordering to be transitive at each stage of the construction. It follows that IK_c is complete for the class of transitive intuitionistic tense structures.

Similarly, if we wish to make the temporal ordering *reflexive* we add the axioms

$$(13) \quad \vdash X \to FX, \qquad (13') \quad \vdash GX \to X$$

to IK_c. If we wish to make the ordering *serial* we add

$$(14) \quad \vdash F(X \to X).$$

We call these new systems IK_tS4 and IK_tD respectively.

Decidability. We shall now show that the systems IK_c, IK_tS4, and IK_tD are decidable. We use the method of filtration described in §4.

Let S be any of the above systems of tense logic. We define S-decidability models as before, except that we require U to be transitive, serial, or reflexive as the case may require. Let C_S be the set of S-valuations. Then the same proof as before shows that C_S is the canonical decidability model, that S is complete for the class of S-decidability models, and that

$$\vdash_S X \quad \text{iff} \quad \vDash X \quad \text{iff} \quad \vDash_D X \quad \text{iff} \quad C_S \vDash X.$$

Let X be any formula of S. Define $F(X)$, $\sim$, C_S/X, $\le$, and R as before. For any sentence letter p in $F(X)$ set $v^*(p) = v(p)$ and extend to an S-decidability model. We can then show by an easy induction that

THEOREM 1. *For all Y in $F(X)$ and for all v in C_S we have $v^*(Y) = v(Y)$.*

Now, if S is one of the systems IK_tS4 or IK_tD then R is reflexive or serial respectively, and we are done. But if S is IK_c then R need not be transitive. Therefore we let R^* be the transitive closure of R.

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LEMMA 1. *Suppose* $v^*R\,w^*$ *and* $v^*(GY) = 1$ *for* GY *in* $F(X)$. *Then* $w^*(GY) = 1$.

PROOF. Since $GY \in F(X)$, by Theorem 1, $v^*(GY) = v_0(GY)$ for any $v_0 \in v^*$. Since v^*Rw^*, there exist valuations v_1, w_1 with $v_1 \in v^*$ and $w_1 \in w^*$ and $v_1 R w_1$. Now $v_1(GY) = 1$ implies $v_1(GGY) = 1$ implies $w_1(GY) = 1$ implies $w^*(GY) = 1$. QED.

Similarly we can show that $w^*(FY) = 1$ implies $v^*(FY) = 1$. Thus by induction one can show that, for FY and GY in $F(X)$, $v^*R^*w^*$ and $v^*(GY) = 1$ imply $w^*(GY) = 1$; $v^*R^*w^*$ and $w^*(FY) = 1$ imply $v^*(FY) = 1$. Hence, from the point of view of the sentences in $F(X)$ the models obtained from R and R^* are the same. Therefore, IK_c is decidable.

§6. Linear time. We now wish to make our time-series linear. We accordingly form the system IK_1 by adding to the axioms of IK_c:

$$(15) \quad \vdash FX \wedge FY \rightarrow F(X \wedge Y) \vee F(FX \wedge Y) \vee F(X \wedge FY),$$

$$(15') \quad \vdash PX \wedge PY \rightarrow P(X \wedge Y) \vee P(PX \wedge Y) \vee P(X \wedge PY),$$

$$(16) \quad \vdash PFX \rightarrow FX \vee X \vee PX,$$

$$(16') \quad \vdash FPX \rightarrow FX \vee X \vee PX.$$

It is clear that 15 and 15′ are valid in all intuitionistic tense structures whose temporal ordering satisfies

$$(\forall \gamma)(\forall t, t' \in T_\gamma)(t = t' \text{ or } t u_\gamma t' \text{ or } t' u_\gamma t).$$

Notice that we do not require the temporal ordering to be *strictly* linear (i.e. irreflexive). Our completeness proof will show that IK_1 is in fact complete for the class of strictly linear tense structures, thus proving that the two classes of models are tense-logically indistinguishable.

Before giving the details of our construction we explain it intuitively. If vRw and $v(FX) = 1$ then by axiom 15 we can show that one of the pairs

i) $(\{Y: w(Y) = 1\} \cup \{X\}; \{Z: v(FZ) = 0\})$,

ii) $(\{Y: w(Y) = 1\} \cup \{FX\}; \{Z: v(FZ) = 0\})$,

iii) $(\{FY: w(Y) = 1\} \cup \{X\} \cup \{w: v(GW) = 1\}; \{Z: v(FZ) = 0\})$

is consistent. For if none of them is consistent then there exist formulas Y, Z, and W with $w(Y) = 1$, $v(FZ) = 0$, $v(GW) = 1$, and

1) $Y, X \vdash Z$,

2) $Y, FX \vdash Z$,

3) $FY, X, W \vdash Z$.

Because vRw and $w(Y) = 1$, $v(FY) = 1$. Thus $v(FY \wedge FX) = 1$. By 15, then, $v(F(Y \wedge X)) = 1$ or $v(F(Y \wedge FX)) = 1$ or $v(F(FY \wedge X)) = 1$. But none of these cases can hold. For if $v(F(Y \wedge X)) = 1$ then (since by 1 and necessitation $v(G(Y \wedge X \vdash Z)) = 1$) $v(FZ) = 1$, against the hypothesis. Similar arguments apply to the other two cases. Thus one of the pairs (i)–(iii) is consistent.

Classically we could conclude from this that $w(X) = 1$ or $w(FX) = 1$ or $(\exists w')(vRw'Rw$ and $w'(X) = 1)$. But intuitionistically the pairs (i)–(iii) need not be maximally consistent. So we can only conclude that

$$(\exists w')(w' \geq w \text{ and } vRw' \text{ and } w'(X) = 1), \text{ or}$$

$$(\exists w')(w' \geq w \text{ and } vRw' \text{ and } w'(FX) = 1), \text{ or}$$

$$(\exists w')(\exists w_0)(w' \geq w \text{ and } vRw_0Rw' \text{ and } w_0(X) = 1).$$

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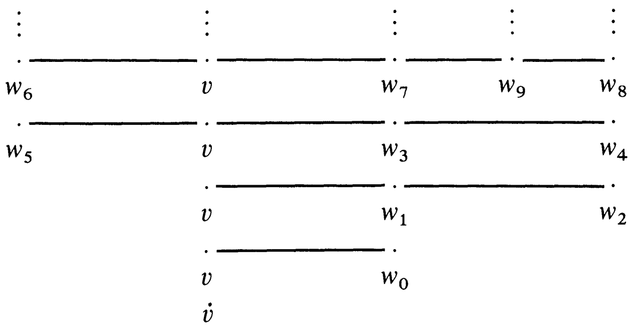
To prove this, note that, for example, if (iii) is consistent then it can be extended to a maximal consistent pair represented by the valuation w_0. If $v(GW) = 1$ then $w_0(W) = 1$, and if $v(FZ) = 0$ then $w_0(Z) = 0$. So vRw_0. Now consider the pair

$$(\{Y: w(Y) = 1\} \cup \{Z: w_0(GZ) = 1\}; \{W: w_0(FW) = 0\}).$$

This pair is consistent because if $w(Y) = 1$ then $w_0(FY) = 1$. Extend it to a maximally consistent pair with valuation w'. Clearly w_0Rw' and $w \in w'$.

Similarly, if wRv and $w(FX) = 1$, then using (16) we can show that either $v(FX) = 1$ (in which case $(\exists w_0)(wRvRw_0$ and $w_0(X) = 1))$ or $v(X) = 1$ or $(\exists w')(\exists w_0)(w \leq w'$ and $w'Rw_0Rv$ and $w_0(X) = 1)$.

Now, at each stage of our construction we have a finite set of valuations linearly ordered by R. We look at a formula FX_i; if it appears in one of these valuations then we shall in an obvious way add a new valuation to our set in a linear order, using axiom 15. But as in the above example we may have to replace some w in our set by $w' \geq w$. So pictorially our construction looks like this:



The inclusion relations give us a directed set; we take the direct limit to get T_{y_0}.

The formal construction is similar to the completeness proof for IK_t. Suppose v is a valuation in C_{IK_1}. Let $\{FX_i\}_{i=1}^{\infty}, \{PY_i\}_{i=1}^{\infty}$ be an enumeration of all formulas of this form. We begin by constructing the set T_{y_0} in stages, as follows.

Set $T_0 = \{v\}$.

Suppose that at stage n we have constructed a finite set

$$T_n = \{\langle t_{-j}, v_{-j}\rangle, \langle t_{1-j}, v_{1-j}\rangle, \ldots, \langle t_0, v\rangle, \ldots, \langle t_k, v_k\rangle\}$$

such that $t_{-j}, \ldots, t_0, \ldots, t_k$ are strictly linearly ordered by u_{y_0} and such that $-j \leq x \leq y \leq k$ implies $t_x u_{y_0} t_y$ and $v_x R v_y$. We now show how to construct T_{n+1} by a sub-induction.

Set $D_{n+1}^0 = T_n$. Consider the formula FX_1. If it appears in some valuation v_p in T_n and if X_1 appears in no v_y, $y \geq p$, then by axiom 15 and the remarks made above we can show that there exist ordered pairs

$$\langle t_{-j}, v'_{-j}\rangle, \ldots, \langle t_0, v\rangle, \ldots, \langle t_k, v'_k\rangle, \langle t_{k+1}, w\rangle$$

such that, for $-j \leq x \leq y \leq k$,

i) $v_y \leq v'_y$,
ii) $v'_x R v'_y$,
iii) $v_p R w_1$ and $w_1(X_1) = 1$, and
iv) the resulting set of ordered pairs can be strictly linearly ordered by u

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Put this set of ordered pairs in its linear order and call the result D^1_{n+1}. In the same way, form D^2_{n+1} from D^1_{n+1} using FX_2, and so on, up through FX_{n+1}. Then do the same thing for PY_1 through PY_{n+1}. The result is T_{n+1}.

The sets $\{T_n\}_{n=0}^{\infty}$ form a directed set in an obvious way. Take the direct limit; call it $T_{\gamma 0}$. We note that

a) $v \in T_0$;

b) $T_{\gamma 0}$ is linearly ordered; and,

c) if FX_i appears in any ordered pair $\langle t_j, v_j \rangle$ in $T_{\gamma 0}$, then there is an ordered pair $\langle t_k, v_k \rangle$ in $T_{\gamma 0}$ with $v_k(X_i) = 1$ and $t_j u_{\gamma 0} t_k$.

This takes care of the formulas FX_i and PY_i. Suppose now that for some formula X and valuation $w \in T_{\gamma 0}$ we have $w(GX) = 0$. We know from the properties of the canonical model that there exist valuations w' and y such that $w' \geq w$ and $w'Ry$ and $y(X) = 0$. We now construct a set of valuations $T_{\gamma'}$ linearly ordered by R such that $T_{\gamma 0} \subseteq T_{\gamma'}$. The construction is much the same as for $T_{\gamma 0}$: we set $T_0 = \{w\}$; at stage n using axiom 15 we add to T_{n-1} an extension of the nth valuation added to $T_{\gamma 0}$, and we check that new valuations are added to accomodate any new formulas of the form FX or PX that many appear. We repeat this construction for every valuation v and formula X with $v(GX) = 0$. We then do the cases where $v(\neg X) = 0$ and $v(X \to Y) = 0$. In this way, for any consistent theory v in the canonical model C_{IK_1} we construct a model C_v in which the temporal ordering is linear and in which $v(X) = 1$ implies $C_v \models X$. So IK_1 is complete for the class of linear intuitionistic tense structures.

If we wish to make our temporal ordering both linear and *dense* we add to IK_1 the axioms

$$(16) \quad \vdash FX \to FFX, \qquad (16') \quad \vdash PX \to PPX,$$
$$(17) \quad \vdash GGX \to GX, \qquad (17') \quad \vdash HHX \to HX.$$

If we wish to make the ordering a dense linear ordering without endpoints we add the axioms

$$(18) \quad \vdash F(X \to X), \qquad (18') \quad \vdash P(X \to X).$$

Smoryński [7] has shown that the first-order intuitionistic theory of dense linear orderings without endpoints is undecidable. A similar result holds for intuitionistic linear tense logic. Consider the following mapping of formulas of the intuitionistic monadic predicate calculus into tense logic:

$$\varphi(Px) = p, \quad \varphi(Qx) = q, \quad \varphi(Rx) = r, \quad \text{etc.};$$
$$\varphi(X \wedge Y) = \varphi(X) \wedge \varphi(Y); \qquad \varphi(X \vee Y) = \varphi(X) \vee \varphi(Y);$$
$$\varphi(\neg X) = \neg \varphi(X); \qquad \varphi(X \to Y) = \varphi(X) \to \varphi(Y);$$
$$\varphi(\forall x X) = H\varphi(X) \wedge \varphi(X) \wedge G\varphi(X); \qquad \varphi(\exists x X) = P\varphi(X) \vee \varphi(X) \vee F\varphi(X).$$

Then X is valid in the intuitionistic monadic predicate calculus iff $\varphi(X)$ is valid in intuitionistic linear tense logic, as can be shown using the Kripke models. But since the intuitionistic monadic predicate calculus is undecidable, so is linear intuitionistic tense logic.[2]

[2] I owe this argument to an anonymous referee for this JOURNAL.

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§7. Modal logics. As we noted at the outset, the future-tense fragments of our tense logics give us intuitionistic analogues to the classical modal logics. It is easy to define intuitionistic modal structures, and, as one would expect, the addition of axioms

$$\Box X \to X, \qquad X \to \Diamond X$$

gives us reflexive accessibility relations between possible worlds, while

$$\Diamond \Box X \to X, \qquad X \to \Box \Diamond X$$

gives us symmetric accessibility relations. We omit details.

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