The Pseudo-Skolem Problem is Decidable

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Abstract

We study fundamental decision problems on linear dynamical systems in discrete time. We focus on *pseudo-orbits*, the collection of trajectories of the dynamical system for which there is an arbitrarily small perturbation at each step. Pseudo-orbits are generalizations of orbits in the topological theory of dynamical systems. We study the pseudo-orbit problem, whether a state belongs to the pseudo-orbit of another state, and the pseudo-Skolem problem, whether a hyperplane is reachable by an ϵ-pseudo-orbit for every ϵ. These problems are analogous to the well-studied orbit problem and Skolem problem on unperturbed dynamical systems. Our main results show that the pseudo-orbit problem is decidable in polynomial time and, surprisingly, the Skolem problem on pseudo-orbits is also decidable. The former extends the seminal result of Kannan and Lipton from orbits to pseudo-orbits. The latter is in contrast to the Skolem problem for linear dynamical systems, which remains open for proper orbits.

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1 Introduction

A (discrete-time) linear dynamical system in m dimensions is defined by a linear map $x \mapsto Ax$ for an $m \times m$ rational matrix A. The map specifies how an individual state (a real-valued vector in m dimensions) evolves over time; a trajectory starting from a state s is given by the sequence $(s, As, A^2s, \ldots)$. Linear dynamical systems are fundamental models in many different domains of science and engineering, and the computability and complexity of decision problems for linear dynamical systems are of both theoretical and practical interest.

The *orbit* of a point s is the smallest transitive set containing s and closed under the dynamic map. The *orbit problem* for linear dynamical systems asks, given s and t, if t is in the orbit of s [11]. In a seminal paper, Kannan and Lipton [12] showed that the orbit problem can be decided in polynomial time. However, a natural generalization of the orbit problem, the *Skolem problem*, in which we ask whether the orbit of a given state s intersects a given hyperplane, turns out to be notoriously difficult and remains open after many decades [20, 16]. A breakthrough occurred in the mid-1980s, when Mignotte *et al.* [14] and Vereshchagin [21] independently showed decidability in dimension 4 or less. These deep results make essential use of Baker's theorem on linear forms in logarithms (which earned Baker the Fields Medal in 1970), as well as a p-adic analogue of Baker's theorem due to van der Poorten. Unfortunately, little progress on that front has since been recorded.

The orbit and Skolem problems are defined on the exact dynamics of the linear system. In dynamical systems theory, one is often interested in "rough" dynamics of a system—in topological terms, we wish to study closed sets containing the orbit. Orbits arising from linear dynamics are usually not closed sets. Indeed, the orbit of the dynamics $x \mapsto \frac{1}{2}x$

does not contain the limit point 0. One way to retain closure is through *pseudo-orbits* [8], a concept going back several decades. A pseudo-orbit generalizes the orbit by allowing arbitrarily small imprecisions throughout the dynamics. For a precision $\epsilon > 0$, we say t is in the ϵ-pseudo-orbit of s if there is a sequence of points $(s = s_0, s_1, \ldots, s_n = t)$ with $n > 0$ such that $\|As_i - s_{i+1}\| < \epsilon$ for each $i \in \{0, \ldots, n-1\}$. That is, an ϵ-pseudo-orbit contains the sequence of points that would be an orbit if each state were known only up to precision ϵ. Finally, t is in the pseudo-orbit of s if it is in the ϵ-pseudo-orbit of s *for all* $\epsilon > 0$.

One can provide a computational analogue of pseudo-orbits (see [17]). Alice is simulating the trajectory of a dynamical system but in every iteration, her computation has a rounding error ϵ. An infinitely powerful adversary, Bob, rounds Alice's result in an arbitrary fashion to a new state within a distance of ϵ of the actual outcome. A state t is pseudo-reachable from s iff Bob can fool Alice into believing that t is reachable in the simulation no matter how accurate her simulation is.

We can formulate analogous decision problems on pseudo-orbits. The *pseudo-orbit problem* asks, given a linear dynamical system and two states s and t, whether t is in the pseudo-orbit of s. The *hyperplane pseudo-reachability* (or pseudo-Skolem) problem asks, given a linear dynamical system, an initial state s, and a hyperplane, if there is an ϵ-pseudo orbit from s that intersects the hyperplane for every $\epsilon > 0$.

In this paper, we study decision problems for pseudo-orbits of linear dynamical systems. We show that **the pseudo-orbit problem is decidable in polynomial time**. More surprisingly, we show that **the Skolem problem is decidable in full generality on pseudo-orbits**!

We proceed in two steps. First, we generalize Kannan and Lipton's analysis to show that the pseudo-orbit problem can be decided in polynomial time. Our proof involves a careful examination of the eigenvalues of the matrix A, similar to Kannan and Lipton's proof. More generally, we show that pseudo-reachability to a bounded semi-algebraic set is decidable.

Next, we consider the hyperplane pseudo-reachability (a.k.a. pseudo-Skolem) problem. Our proof again proceeds by a case analysis on the eigenvalues of A. The most interesting case is when there is an eigenvalue of modulus greater than 1. We analyze a series whose terms are polynomial-exponential functions of $n \in \mathbb{N}$ associated with the dynamics. We show that the infimum of this sum can be effectively computed. The proof of effective computability uses tools from Diophantine approximation as well as a reduction to the theory of reals.

We show that the dynamics pseudo-reaches the hyperplane in case the infimum of this sum is 0. If the infimum is non-zero, we prove that we can find an effective bound N such that the dynamics pseudo-reaches the hyperplane iff, for sufficiently small ϵ, it pseudo-reaches the hyperplane within N steps.

Putting everything together, we conclude that the pseudo-Skolem problem is decidable.

Other related work. The study of pseudo-orbits go back to Anosov, Bowen, and Conley [1, 3, 8]. Conley [8] formulated the fundamental theorem of dynamical systems: the iteration of any continuous, possibly non-linear, map on a compact metric space decomposes the space into a chain-recurrent part (the pseudo-orbit analogue of a period orbit) and a gradient-like part. Our results imply that deciding if a state is chain recurrent is decidable for linear systems.

In linear systems theory, *controllability* is a fundamental property of linear systems [19]. Controllability states that the system can be controlled from any point to any other point. However, this may require unboundedly large control actions. A pseudo-orbit can be seen as a stronger notion, where we ask if the dynamics can be controlled from a starting point to an ending point no matter how small the control input is: if a state belongs to the pseudo-orbit, then for every ϵ, there is a sequence of control inputs each bounded in norm by ϵ that steers the system to that state.

2 Linear Dynamical Systems

Notation. The sets of natural numbers (including zero), rational numbers, real numbers, and algebraic numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\overline{\mathbb{Q}}$, respectively. We assume a standard representation of algebraic numbers in terms of their defining polynomials, by which we can perform arithmetic operations and test equality in polynomial time in their representation (see, e.g., [6]).

For any column vector $x = [x_1, x_2, \ldots, x_m]^\top \in \mathbb{R}^m$, we use the notations $\|x\|_2 := \sqrt{x^\top x}$ and $\|x\|_\infty := \max_i |x_i|$ to indicate respectively the two norm and infinity norm of x. For any matrix $A = [a_{ij}]_{i,j} \in \mathbb{R}^{m\times m}$, we define $\|A\|_2$ and $\|A\|_\infty$ to indicate respectively the (induced) two norm and infinity norm of A. Note that $\|Ax\|_2 \le \|A\|_2 \|x\|_2$ and $\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty$ for all $x \in \mathbb{R}^m$. We write $\mathbf{0} \in \mathbb{R}^m$ for the zero vector and $\mathbf{1} \in \mathbb{R}^m$ for the all-ones vector. We denote by $\rho(A)$ the spectral radius of a matrix A, which is the largest absolute value of the eigenvalues of A. For any $A \in \mathbb{R}^{m\times m}$ and any $\gamma > \rho(A)$, recall that there is a constant $c > 0$ such that $\|A^n\|_2 \le c\gamma^n$ for all $n \in \mathbb{N}$.

Discrete-Time Linear Dynamical Systems. An m-dimensional discrete-time linear dynamical system is specified by an $m \times m$ matrix A of rational numbers. The *trajectory* determined by an initial state $x_0 \in \mathbb{R}^m$ is the sequence $(x_n)_{n\ge 0}$ given by

$$x_{n+1} = Ax_n, \quad (n \in \mathbb{N}).$$

We call the set $\mathcal{O}(A, x_0) := \{x_n \mid n \in \mathbb{N}\}$ the *orbit* of x_0.

For any $\epsilon > 0$, an ϵ-perturbed linear dynamical system has state trajectories $(x_n)_{n\ge 0}$ such that

$$x_{n+1} = Ax_n + d_n, \quad (n \in \mathbb{N}),$$

where A is as before and $d_n \in [-\epsilon, \epsilon]^m$ for all n. For an initial state $x_0 \in \mathbb{R}^m$, we define the *ϵ-pseudo-orbit* $\widetilde{\mathcal{O}}_\epsilon(A, x_0)$ of the dynamics as the set of states reachable in the perturbed dynamics. More formally, define

- for $n = 0$, $\widetilde{\mathcal{O}}_\epsilon^{(n)}(A, x_0) := \{x_0\}$,
- for all $n \in \mathbb{N}$, $\widetilde{\mathcal{O}}_\epsilon^{(n+1)}(A, x_0) := \{Ax + d \in \mathbb{R}^m \mid x \in \widetilde{\mathcal{O}}_\epsilon^{(n)}(A, x_0), d \in [-\epsilon, \epsilon]^m\}$, and
- $\widetilde{\mathcal{O}}_\epsilon(A, x_0) := \bigcup_{n\ge 0} \widetilde{\mathcal{O}}_\epsilon^{(n)}(A, x_0)$.

Finally, we define the *pseudo-orbit* $\widetilde{\mathcal{O}}(A, x_0) := \bigcap_{\epsilon>0} \widetilde{\mathcal{O}}_\epsilon(A, x_0)$ as the intersection of all the ϵ-pseudo orbits of x_0, for all $\epsilon > 0$. Clearly, $\mathcal{O}(A, x) \subseteq \widetilde{\mathcal{O}}(A, x)$ for any A and x.

We will make use of the following characterization, which follows directly from the definition: Any $t \in \widetilde{\mathcal{O}}_\epsilon(A, s)$ is of the form $t = A^n s + \sum_{i=0}^{n-1} A^i d_{n-i-1}$ for some $n \in \mathbb{N}$ and some sequence of perturbations d_i with $\|d_i\|_\infty \le \epsilon$.

We also need the following properties of $\widetilde{\mathcal{O}}_\epsilon(A, x)$ and $\widetilde{\mathcal{O}}(A, x)$.

▷ Claim 1 (Transitivity). For every A and $\epsilon > 0$, and for states $s, t, u \in \mathbb{R}^m$, if $t \in \widetilde{\mathcal{O}}_\epsilon(A, s)$ and $u \in \widetilde{\mathcal{O}}_\epsilon(A, t)$ then $u \in \widetilde{\mathcal{O}}_\epsilon(A, s)$. If $t \in \widetilde{\mathcal{O}}(A, s)$ and $u \in \widetilde{\mathcal{O}}(A, t)$, then $u \in \widetilde{\mathcal{O}}(A, s)$.

▷ Claim 2 (Closure). For every A, $\epsilon > 0$, and state $s \in \mathbb{R}^m$, the sets $\widetilde{\mathcal{O}}_\epsilon(A, s)$ and $\widetilde{\mathcal{O}}(A, s)$ are closed sets.

▶ **Problem 3** (Orbit problem). *Given $A \in \mathbb{Q}^{m\times m}$ and $s, t \in \mathbb{Q}^m$, decide whether $t \in \mathcal{O}(A, s)$.*

A celebrated result of Kannan and Lipton [12] shows that the Orbit Problem is decidable in polynomial time.

▶ **Theorem 4.** *[12] The orbit problem is decidable in polynomial time.*

In this paper, we study the following problems.

▶ **Problem 5** (Pseudo-orbit problem). *Given $A \in \mathbb{Q}^{m\times m}$ and $s, t \in \mathbb{Q}^m$, decide whether $t \in \widetilde{\mathcal{O}}(A, s)$.*

▶ **Problem 6** (Hyperplane pseudo-reachability problem). *Given $A \in \mathbb{Q}^{m\times m}$, $s \in \mathbb{Q}^m$, and a hyperplane $c^T \cdot x = v$ for $c, v \in \mathbb{Q}^m$, decide whether $\widetilde{\mathcal{O}}_\epsilon(A, s)$ intersects the hyperplane for all $\epsilon > 0$.*

The following summarizes our main theorem.

▶ **Theorem 7** (Main Theorem).

1. *The pseudo-orbit problem is decidable in polynomial time.*
2. *The hyperplane pseudo-reachability problem is decidable.*

The rest of the paper is devoted to the proof of this theorem.

2.1 Preliminaries

First we establish that pseudo-orbits can be translated with change of bases.

▶ **Proposition 8.** *For matrices $A, B, Q \in \mathbb{R}^{m\times m}$ with $A = QBQ^{-1}$ and for any $x \in \mathbb{R}^m$, we have $Q\,\widetilde{\mathcal{O}}_{\gamma_2}(B, Q^{-1}x) \subseteq \widetilde{\mathcal{O}}_\epsilon(A, x) \subseteq Q\,\widetilde{\mathcal{O}}_{\gamma_1}(B, Q^{-1}x)$, where $\gamma_1 = \epsilon \left\|Q^{-1}\right\|_\infty$ and $\gamma_2 = \epsilon / \|Q\|_\infty$. Moreover, $\widetilde{\mathcal{O}}(A, x) = Q\,\widetilde{\mathcal{O}}(B, Q^{-1}x)$.*

We will use Proposition 8 with matrix A represented using the *Jordan canonical form.*
Jordan Decomposition. For a given rational square matrix A one can compute *change of basis matrix* Q and *Jordan normal form* J so that $A = QJQ^{-1}$ and $J = \mathrm{diag}(J_1, J_2, \cdots, J_z)$ with J_i representing the i^{th} Jordan block taking the following form

$$J_i = \begin{bmatrix} \Lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \Lambda_i & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \Lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \Lambda_i \end{bmatrix}, \tag{1}$$

where Λ_i denotes the i^{th} eigenvalue of A. The size of J_i is equal to the multiplicity of the eigenvalue Λ_i and is denoted by $\kappa(\Lambda_i)$.

Real Jordan form. For any $A \in \mathbb{R}^{n\times n}$ having complex eigenvalues, matrices Q and J in the Jordan normal form could have complex entries. In this case, the complex eigenvalues form complex conjugate pairs and give a *real Jordan form*: there are *real* matrices Q and J such that $A = QJQ^{-1}$ and $J = \mathrm{diag}(J_1, J_2, \cdots, J_z)$. The matrix J_i represents the i^{th} real Jordan block corresponding to either a real eigenvalue Λ_i or a complex pair $\Lambda_i = a_i \pm jb_i$. It is equal to (1) for real Λ_i and has the following form for the complex pair $\Lambda_i = a_i \pm jb_i$,

$$J_i = \begin{bmatrix} \Lambda_i & I_{2\times 2} & 0_{2\times 2} & \dots & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \Lambda_i & I_{2\times 2} & \dots & 0_{2\times 2} & 0_{2\times 2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \dots & \Lambda_i & I_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \dots & 0_{2\times 2} & \Lambda_i \end{bmatrix}, \tag{2}$$

where with abuse of notation, we have indicated $\Lambda_i = \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$. $I_{2\times 2}$ and $0_{2\times 2}$ denote identity and fully zero matrices of size 2 by 2.

The real Jordan normal form and the change of basis matrices Q and Q^{-1} can be computed in polynomial time (see [5] and also Appendix E).

Computing matrix powers. If $A = QJQ^{-1}$, then we have $A^n = QJ^nQ^{-1}$ for $n \in \mathbb{N}$, where $J^n = \operatorname{diag}(J_1^n, J_2^n, \dots, J_z^n)$ and

$$J_i^n = \begin{bmatrix} \Lambda_i^n & n\Lambda_i^{n-1} & \binom{n}{2}\Lambda_i^{n-1} & \cdots & \binom{n}{k-1}\Lambda_i^{n-k+1} \\ 0 & \Lambda_i^n & n\Lambda_i^{n-1} & \cdots & \binom{n}{k-2}\Lambda_i^{n-k+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n\Lambda_i^{n-1} \\ 0 & 0 & 0 & \cdots & \Lambda_i^n \end{bmatrix}$$

3 The pseudo-orbit problem is decidable in polynomial time

In this section, we show that Problem 5 is decidable in polynomial time. Fix a matrix A and let J be the real Jordan form for A. Proposition 8 shows that $\widetilde{\mathcal{O}}(A, x)$ can be obtained from the pseudo-orbit $\widetilde{\mathcal{O}}(J, x)$. Our proof involves a case analysis on the modulus of the eigenvalues of J. We first consider the cases where J is a single block, i.e.,

$$J = \begin{bmatrix} \Lambda & I & & \\ & \Lambda & \ddots & \\ & & \ddots & I \\ & & & \Lambda \end{bmatrix} \text{ with } \Lambda = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ or } \Lambda = [r] \text{ and } I = [1], \tag{3}$$

with real matrix entries $a, b, r \in \mathbb{R}$.

We shall case split on the spectral radius $\rho(J)$, which is the absolute value of the unique eigenvalue of the Jordan block J. We consider three cases: $\rho(J) < 1$, $\rho(J) = 1$ and $\rho(J) > 1$. The following lemma will be useful in relating the first and third cases. Its proof is simply by reversing time.

▶ **Lemma 9** (Reversibility Lemma). *For any invertible matrix $A \in \mathbb{R}^{m\times m}$, $x \in \widetilde{\mathcal{O}}_\epsilon(A, s)$ implies $s \in \widetilde{\mathcal{O}}_\gamma(A^{-1}, x)$ with $\gamma = \epsilon \left\|A^{-1}\right\|_\infty$. Moreover,*

$$x \in \widetilde{\mathcal{O}}(A, s) \iff s \in \widetilde{\mathcal{O}}(A^{-1}, x). \tag{4}$$

▶ **Lemma 10** (Eigenvalues inside the unit circle). *Let $J \in \mathbb{R}^{m\times m}$ be a Jordan block of the form (3) with $\rho(J) < 1$. For every $s \in \mathbb{R}^m$,*

$$\widetilde{\mathcal{O}}(J, s) = \mathcal{O}(J, s) \cup \{\mathbf{0}\} = \overline{\mathcal{O}(J, s)},$$

where $\overline{\mathcal{O}(J, s)}$ denotes the closure of the orbit.

Proof. We prove the lemma by showing there is a constant $C > 0$ satisfying

$$\overline{\mathcal{O}(J,s)} \stackrel{*}{=} \mathcal{O}(J,s) \cup \{\mathbf{0}\} \stackrel{**}{\subseteq} \widetilde{\mathcal{O}}(J,s) \stackrel{\dagger}{\subseteq} \bigcap_{\epsilon>0} \bigcup_{z\in\mathcal{O}(J,s)} \mathcal{B}(z, C\epsilon) \stackrel{\S}{\subseteq} \overline{\mathcal{O}(J,s)}, \tag{5}$$

where $\mathcal{B}(z,\epsilon) := \{y \in \mathbb{R}^m \mid \|z-y\|_2 \leq \epsilon\}$ is the closed ball with respect to two norm with center z and radius ϵ. It is easy to see that equality (*) holds since all the eigenvalues of J are inside the unit circle, $\lim_{n\to\infty} J^n = 0$, and 0 is the only limiting point of any state trajectory. It is also easy to see that inclusion (**) is correct. Note that for any $\epsilon > 0$, $\mathcal{O}(J,s) \subseteq \widetilde{\mathcal{O}}_\epsilon(J,s)$ and the set $\widetilde{\mathcal{O}}_\epsilon(J,s)$ is closed by definition. Taking intersection over $\epsilon > 0$, we get $\mathcal{O}(J,s) \subseteq \widetilde{\mathcal{O}}(J,s)$ with $\widetilde{\mathcal{O}}(J,s)$ being a closed set. Therefore, $\overline{\mathcal{O}(J,s)} \subseteq \widetilde{\mathcal{O}}(J,s)$.

We now choose a value of C which allows us to prove inclusion (†). First pick γ such that $\rho(J) < \gamma < 1$. Next choose c_1 to be a constant (which is guaranteed to exist) satisfying $\|J^n\|_2 \leq c_1\gamma^n$ for all $n \in \mathbb{N}$, and finally set $C := c_1 m/(1-\gamma)$.

Finally, the inclusion § is proved by noting that the complement of $\overline{\mathcal{O}(J,s)}$ is an open set. So for any state y in the complement, there is a neighborhood, say of size θ, that does not intesect the closure. If $C\epsilon < \theta$, then $y \notin \mathcal{B}(z, C\epsilon)$ for all $z \in \mathcal{O}(J,s)$. ◀

▶ **Lemma 11** (Eigenvalues outside the unit circle). *Let $J \in \mathbb{R}^{m\times m}$ be a Jordan block of the form* (3) *with $\rho(J) > 1$. For every $s \in \mathbb{R}^m$, we have $\widetilde{\mathcal{O}}(J,\mathbf{0}) = \mathbb{R}^m$ and $\widetilde{\mathcal{O}}(J,s) = \mathcal{O}(J,s)$ if $s \neq \mathbf{0}$.*

Proof. In this case, J is invertible and all eigenvalues of J^{-1} are inside the unit circle. We apply the Reversibility Lemma 9 and Lemma 10.

$$x \in \widetilde{\mathcal{O}}(J,s) \iff s \in \widetilde{\mathcal{O}}(J^{-1},x) \iff s \in \mathcal{O}(J^{-1},x) \cup \{\mathbf{0}\} \iff s = \mathbf{0} \text{ or } x \in \mathcal{O}(J,s).$$

Therefore, any x is in $\widetilde{\mathcal{O}}(J,s)$ if $s = 0$, and $\widetilde{\mathcal{O}}(J,s) = \mathcal{O}(J,s)$ for $s \neq \mathbf{0}$. ◀

▶ **Lemma 12** (Eigenvalues on the unit circle). *Let $J \in \mathbb{R}^{m\times m}$ be a Jordan block of the form* (3) *with $\rho(J) = 1$. For every $s \in \mathbb{R}^m$, we have $\widetilde{\mathcal{O}}(J,s) = \mathbb{R}^m$.*

Proof. (Sketch.) The key idea of the proof is to show that $\mathbf{0} \in \widetilde{\mathcal{O}}(A,s)$ for any s and for any A having the eigenvalues on the unit circle. Once we show this, we know that $s \in \widetilde{\mathcal{O}}(A^{-1},\mathbf{0})$ is true for any s and any matrix A due to the Reversibility lemma. Stated for the inverse of A and any x, we get $x \in \widetilde{\mathcal{O}}(A,\mathbf{0})$. Since pseudo-orbits are transitive, we have $x \in \widetilde{\mathcal{O}}(A,s)$ for any x and s, which is the intended result.

We show $\mathbf{0} \in \widetilde{\mathcal{O}}(A,s)$ equivalently by replacing A with its Jordan form J and doing induction on the structure of J. The proof has two stages. The first stage is to show that $\mathbf{0} \in \widetilde{\mathcal{O}}(J,s)$ for all s when J has a single block of simple eigenvalues. The second stage is to show that we can sequentially increase the multiplicity of eigenvalues and multiple blocks. ◀

We now consider the general case where J has multiple blocks.

▶ **Definition 13.** *Let $J \in \mathbb{R}^{m\times m}$ be a real Jordan block matrix and $s \in \mathbb{R}^m$. We define*

$$\Delta(J,s) := \begin{cases} \mathbb{R}^m & \textit{if } \rho(J) = 1 \textit{ or, } \rho(J) > 1 \textit{ and } s = \mathbf{0}, \\ \{\mathbf{0}\} & \textit{if } \rho(J) < 1, \\ \emptyset & \textit{otherwise.} \end{cases}$$

The following lemma states that certain points in the pseudo-orbit of real Jordan blocks are ϵ-pseudo reachable exactly at any sufficiently large time step, for every $\epsilon > 0$. The lemma provides the flexibility to "synchronize" reaching parts of the state for different Jordan blocks.

▶ **Lemma 14** (Synchronization Lemma). *Let $J \in \mathbb{R}^{m\times m}$ be a Jordan block with eigenvalue λ. For $s \in \mathbb{R}^m$, $t \in \Delta(J, s)$ if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $N > N_\epsilon$, there exists an ϵ-pseudo-orbit $(x_i)_{i\in\mathbb{N}}$ of s under J such that $x_N = t$.*

Proof.
- $|\lambda| < 1$ and $\Delta(J, s) = \{\mathbf{0}\}$. By Lemma 10, $\mathbf{0} \in \widetilde{\mathcal{O}}(J, s)$ and hence for every $\epsilon > 0$, there exists N_ϵ such that $t = \mathbf{0}$ can be ϵ-pseudo reached at time N_ϵ. Now simply observe that once an ϵ-pseudo-orbit reaches $\mathbf{0}$, it can remain there forever by setting all future perturbations to zero. To prove the other direction, suppose $t \neq \mathbf{0}$. There must exist a time bound T such that for sufficiently small ϵ, all ϵ-pseudo-orbits of s after time T are contained in $\mathcal{B}(\mathbf{0}, \frac{\|t\|_2}{2})$. Hence for sufficiently small ϵ no N_ϵ with the the specified property can exist.
- $|\lambda| = 1$ and $\Delta(J, s) = \mathbb{R}^m$. In the proof of Lemma 12, for every $t \in \mathbb{R}^m$ and $\epsilon > 0$ we construct an ϵ-pseudo-orbit from s that visits $\mathbf{0}$ followed by t. Let N_ϵ be the number of steps required to ϵ-reach t. We can postpone visiting t to any time step $N > N_\epsilon$ by simplying waiting at the point $\mathbf{0}$ for $N - N_\epsilon$ steps.
- $|\lambda| > 1$, $s = \mathbf{0}$ and $\Delta(J, s) = \mathbb{R}^m$. Similarly to the case above, in Lemma 11 for each ϵ we construct an ϵ-pseudo-orbit that visits t at time N_ϵ, and reaching t can be delayed arbitrarily by spending a necessary number of steps at $\mathbf{0}$ at the beginning.
- $|\lambda| > 1$, $s \neq \mathbf{0}$ and $\Delta(J, s) = \emptyset$. Let $t \in \mathbb{R}^m$. In this case, observe that there must exist a time bound T such that for sufficiently small ϵ, all ϵ-pseudo-orbits of s after time T are contained outside $\mathcal{B}(\mathbf{0}, 2\,\|t\|_2)$. Hence for sufficiently small ϵ no N_ϵ with the the specified property can exist. ◀

There are two modes of pseudo reachability: via orbit, or at larger and larger time steps for smaller ϵ.

▶ **Lemma 15.** *Let $A \in \mathbb{R}^{m\times m}$ and $s, t \in \mathbb{R}^m$. If there exists N such that for every ϵ, t is ϵ-pseudo-reachable from s within the first N steps, then $t \in \mathcal{O}(A, s)$.*

Proof. Suppose such N exists. By continuity of the map $x \mapsto Ax$, for every $\delta > 0$ there exists $\epsilon > 0$ such that for every $\epsilon' < \epsilon$ and ϵ'-pseudo-orbit $(x_i)_{i\in\mathbb{N}}$, $\left\|x_i - A^i s\right\|_2 < \delta$ for $0 \leq i < N$. Hence the intersection of the first N elements of all ϵ-pseudo-orbits is exactly $\{s, As, \ldots, A^{N-1}s\}$. ◀

▶ **Lemma 16.** *For $J = \mathrm{diag}(J_1, \ldots, J_l)$ in real Jordan normal form and $s \in \mathbb{R}^m$,*

$$\widetilde{\mathcal{O}}(J, s) = \mathcal{O}(J, s) \cup \Pi_{i=1}^{l} \Delta(J_i, s_i).$$

Proof. Suppose $t = (t_1, \ldots, t_l) \in \Pi_{i=1}^{l} \Delta(J_i, s_i)$. That is, for every ϵ and $1 \leq i \leq l$ there exists an ϵ-pseudo-orbit $(x^i_j)_{j\in\mathbb{N}}$ of s_i under J_i that reaches t_i. By Lemma 14, for every ϵ there exist ϵ-pseudo-orbits $(y^i_j)_{j\in\mathbb{N}}$ of $s_1, \cdots, s_l$ that reach $t_1, \ldots, t_l$, respectively, at the same time N. That is, $y^i_N = t_i$ for $1 \leq i \leq m$. Hence $(y^1_i, \ldots, y^z_i)_{i\in\mathbb{N}}$ is an ϵ-pseudo-orbit of s under J that reaches t.

Now suppose $t \in \widetilde{\mathcal{O}}(J, s) \setminus O(J, s)$. We prove, by a case analysis on J_i, that $t_i \in \Delta(J_i, s_i)$ for $1 \leq i \leq l$.

- $\rho(J_i) < 1$. Since s is not in the orbit, we can find a sequence $N_1 < N_2 < \cdots$ of time steps and $\epsilon_1 > \epsilon_2 > \cdots$ such that t is ϵ_i-reachable from s at time N_i. In particular, t_i is ϵ_i reachable from s_i at time N_i. But this means that $t_i \notin \mathcal{O}(s_i, J_i)$ and hence $t_i \in \widetilde{\mathcal{O}}(s_i, J_i) \setminus \mathcal{O}(s_i, J_i) = \{\mathbf{0}\} = \Delta(J_i, s_i)$.

- $\rho(J_i) = 1$. Since in this case $\Delta(J_i, s_i) = \mathbb{R}^{\kappa(i)}$, trivially $t \in \Delta(J_i, s_i)$.
- $\rho(J_i) > 1$ and $s_i = \mathbf{0}$. Since in this case $\Delta(J_i, s_i) = \mathbb{R}^{\kappa(i)}$, trivially $t \in \Delta(J_i, s_i)$.
- $\rho(J_i) > 1$ and $s_i \neq \mathbf{0}$. This case cannot arise: $t_i \in \widetilde{\mathcal{O}}(J_i, s_i) \setminus \mathcal{O}(J_i, s_i)$. ◀

Proof. (of Theorem 7(1)). We now put everything together to show the pseudo-orbit problem is decidable in polynomial time. Given $A \in \mathbb{Q}^{m\times m}$, and $s, t \in \mathbb{Q}^m$, we compute (in polynomial time) matrices $Q, J, Q^{-1} \in (\mathbb{R} \cap \overline{\mathbb{Q}})^{m\times m}$ such that $A = QJQ^{-1}$ and J is in real Jordan normal form [5]. Then, we compute $t' = Q^{-1}t$ and $s' = Q^{-1}s$, and by Proposition 8 we have that $t \in \widetilde{\mathcal{O}}(A, s)$ if and only if $t' \in \widetilde{\mathcal{O}}(J, s')$. It remains to decide whether $t' \in \widetilde{\mathcal{O}}(J, s')$. For this we use the characterization described in Lemma 16. To decide whether $t' \in \mathcal{O}(J, s')$, observe that $Q^{-1}t \in \mathcal{O}(J, Q^{-1}s) \iff t \in \mathcal{O}(A, s)$, and whether $t \in \mathcal{O}(A, s)$ is an instance of the Orbit Problem and can be decided in polynomial time.[1] Finally, it remains to check whether $t_i \in \Delta(J_i, s_i)$ for each block J_i, which can be done easily given the simplicity of $\Delta(J_i, s_i)$. ◀

We end the section with an application of Theorem 7(1). A set S is *pseudo-reachable* from s under A if for every $\epsilon > 0$, there exists a point $x_\epsilon \in S$ that is ϵ-pseudo-reachable from s under A. An *algebraic* set is the set of zeros of a collection of polynomials. A *semialgebraic* set is a union of algebraic sets and projections of algebraic sets. We show that we can decide if a bounded semialgebraic set is pseudo-reachable, by reducing the problem to the pseudo-orbit problem.

▶ **Theorem 17.** *Given $A \in \mathbb{Q}^{m\times m}$, $x_0 \in \mathbb{Q}^m$, and a bounded semialgebraic set S, it is decidable if S is pseudo-reachable from x_0 under A.*

4 Hyperplane pseudo-reachability is decidable

In this section, we prove Theorem 7(2). First we consider the case where we are given:
- a hyperplane $H = \{x \in \mathbb{R}^m : c^\top x = v\}$ with $(c, v) \in (\mathbb{R} \cap \overline{\mathbb{Q}})^m \times (\mathbb{R} \cap \overline{\mathbb{Q}})$,
- $J = \operatorname{diag}(J_1, \ldots, J_z) \in (\mathbb{R} \cap \overline{\mathbb{Q}})^{m\times m}$ in real Jordan normal form, and
- a starting point $x_0 \in (\mathbb{R} \cap \overline{\mathbb{Q}})^m$.

We want to decide if for every $\epsilon > 0$ there exists an ϵ-pseudo-orbit $(x_i)_{i\in\mathbb{N}}$ of x_0 under J that *hits* the hyperplane, i.e. $c^\top x_N - v = 0$ for some $N \in \mathbb{N}$.

A block J_i is *relevant* with respect to hyperplane $H = \{x : c^\top x = v\}$ if the coefficients of c at the coordinates corresponding to J_i are not all 0. Intuitively, dimensions corresponding to blocks that are not relevant can simply be omitted from the analysis as they do not play a role in determining whether a point is in H or not. *Relevant eigenvalues* of J are the eigenvalues of relevant blocks. The *relevant spectral radius*, written $\rho_H(J)$, is the largest modulus of all relevant eigenvalues. Our proof is based on a case analysis on the relevant spectral radius of J. We shall see that the proof is simple when the relevant spectral radius is ≤ 1 but requires more technical ideas when it is > 1.

▶ **Lemma 18** (Case $\rho_H(J) \leq 1$). *Fix a matrix J in real Jordan normal form, a starting state s, and a hyperplane $H = \{x : c^\top x = v\}$.*

1. *If $\rho_H(J) = 1$, then H is pseudo-reachable.*

[1] Technically, [12] consider the orbit problem for rational inputs and we require the orbit problem where the input can contain algebraic numbers. However, a polynomial time algorithm is still possible.

2. *If $\rho_H(J) < 1$ and $\mathbf{0} \in H$ then H is pseudo-reachable. If $\rho_H(J) < 1$ and $\mathbf{0} \notin H$, there exists an effectively computable time bound N such that H is pseudo-reachable if and only if there exists $1 \le i \le N$ such that $J^i x_0 \in H$ (that is, H is reachable from x_0 under J within N steps).*

The proof of (1) uses Lemmas 10 and 12 to construct ϵ-pseudo-orbits that can be composed using Lemma 14 to reach the hyperplane. When $\rho_H(J) < 1$, the first case follows because the origin is pseudo-reachable from s (Lemma 10). In the second case, we obtain the threshold from Lemma 28 that shows existence of computable N such that for sufficiently small ϵ after time N all ϵ-pseudo-orbits are trapped in a small ball around the origin.

We now consider the case $\rho_H(J) > 1$. The main ideas of our proof are as follows:

1. A point x_n in the ϵ-pseudo orbit belongs to the hyperplane (c, v) if $c^\top x_n - v = 0$. In particular, $c^\top x_n - v$ can be written as a sum over exponential polynomials in eigenvalues of different sizes.
2. We factor out the scaling factor corresponding to the top eigenvalues, leaving a sum over normalized eigenvalues, together with a sum over disturbances (of order ϵ) and additional terms which go to zero with large n.
3. We relate hyperplane pseudo-reachability to the limit inferior of the sum over normalized eigenvalues. If the limit is zero, we show the hyperplane is pseudo-reachable. If the limit is positive, we show there is an effective bound N such that if the hyperplane is pseudo-reachable, it is reachable within N steps.
4. We apply results from Diophantine approximation and the theory of reals to compute the limit inferior of the sum over normalized eigenvalues.

Fix $J = \mathrm{diag}(J_1, \ldots, J_l) \in (\mathbb{R} \cap \overline{\mathbb{Q}})^{m \times m}$, a starting point $x_0 \in (\mathbb{R} \cap \overline{\mathbb{Q}})^m$, and a hyperplane $H = \{x \in \mathbb{R}^m \mid c^\top x = v\}$ with $c, v \in (\mathbb{R} \cap \overline{\mathbb{Q}})^m$. We assume without loss of generality that all blocks are relevant.

Step 1: Analysing $c^\top x_n - v$. Let $L = \rho_H(J) > 1$ be the largest modulus of relevant eigenvalues of J and suppose the blocks are arranged in non-increasing order of the modulus of eigenvalues. In particular, let $t \le l$ be such that the first t blocks (t for "top") have $\rho(J_1) = \cdots = \rho(J_t) = L > 1$. We call the eigenvalues of these blocks the *top eigenvalues*. The remaining blocks satisfy $L > \rho(J_{t+1}) \ge \cdots \ge \rho(J_l)$.

Let $(d_i)_{i \in \mathbb{N}}$ be a sequence of perturbations and $(x_i)_{i \in \mathbb{N}}$ the resulting pseudo-orbit. We have

$$c^\top x_n - v = c^\top \left(J^n x_0 + \sum_{k=0}^{n-1} J^k d_{n-k-1} \right) - v = \sum_{i=1}^{l} \left(c^i J_i^n x_0^i + c^i \sum_{k=0}^{n-1} J_i^k d_{n-k-1}^i \right) - v,$$

where for all n and $1 \le i \le l$, c^i, x_n^i d_n^i are projections of $c^\top$, x_n and d_n, respectively, onto the coordinates governed by J_i. Observe that c^i is a row vector for every i.

Step 2: Normalized sum. We define a normalized version of this sum by factoring out L^n (the size of the top eigenvalues) and n^D, where we define D in such a way that we normalize polynomials in n that appear in the sum. Observe that for $1 \le i \le t$ (the top eigenvalues),

$$c^i J_i^n = \begin{cases} \left[p_1^i(n)\lambda^n + \overline{p_1^i(n)\lambda^n} \quad \cdots \quad p_{2\kappa(i)}^i(n)\lambda^n + \overline{p_{2\kappa(i)}^i(n)\lambda^n} \right] & \text{if } J_i \text{ has eigenvalues } \lambda, \overline{\lambda} \\ \left[p_1^i(n)\rho^n \quad \cdots \quad p_{\kappa(i)}^i(n)\rho^n \right] & \text{if } J_i \text{ has a single eigenvalue } \rho \end{cases}$$

for polynomials $p_1^i, \ldots, p_{\kappa(i)}^i$ (with algebraic coefficients) where $\kappa(i)$ is the multiplicity of the block J_i.

We define D to be the largest number such that the monomial n^D appears with a non-zero coefficient in at least one of $c^i J_i^n$ for $1 \leq i \leq t$. (Note that if all entries of c are non-zero $D+1$ is equal to the largest multiplicity of a top eigenvalue block of J, as can be seen from the description of powers of a Jordan block in Section 2.)

We can now define

$$f(n) := \frac{c^\top \cdot x_n - v}{L^n n^D} \quad = \sum_{i=1}^{l} \left(c^i \frac{J_i^n}{L^n n^D} x_0^i + c^i \sum_{k=0}^{n-1} \frac{J_i^k}{L^n n^D} d_{n-k-1}^i \right) - \frac{v}{L^n n^D}$$

For notational convenience we define vector-valued functions $g^i(n) := c^i \frac{J_i^n}{L^n n^D}$ for $1 \leq i \leq l$. The following technical lemma summarizes the relevant properties of these scaled terms.

▶ **Lemma 19** (Normalization Lemma).

1. *For $1 \leq i \leq t$ (top eigenvalues), $\left\|g^i(n)\right\|_\infty = O(1)$ (with respect to n).*
2. *For $t+1 \leq i \leq l$ (non-top eigenvalues), $\lim_{n\to\infty} \left\|g^i(n)\right\|_\infty = 0$.*
3. *There exists $1 \leq j \leq t$ and effectively computable $N \in \mathbb{N}$ and $C > 0$ such that $n > N \implies \left\|g^j(n)\right\|_\infty > C$.*

Step 3: Conditions for reachability and non-reachability. Now we are ready to attack our original problem. Going back, H is ϵ-pseudo-reachable if and only if $f(n) = 0$ for some disturbance sequence $(d_i)_{i\in\mathbb{N}}$ with $d_i \in [-\epsilon, \epsilon]^m$ for all i. We analyze how $f(n)$ can be brought to 0 in this way.

▶ **Lemma 20.** *Let*

$$D = \liminf_{n\to\infty} \left| \sum_{i=1}^{t} g^i(n) x_0^i \right|. \tag{6}$$

If $D = 0$, then H is pseudo-reachable. If $D > 0$, there exists a computable time bound N such that H is pseudo-reachable if and only if it is reachable (in the standard sense) within the first N steps.

Step 4: Analyzing $\liminf_{n\to\infty} |\sum_{i=1}^t g^i(n)x_0^i|$. Consider a single term $g^i(n)x_0^i$. Writing $x_0^i = \begin{bmatrix} X_0 & X_1 & \cdots & X_z \end{bmatrix}^\top$, where $X_1, \ldots, X_z \in \mathbb{R}$, we have

$$g^i(n)x_0^i = \sum_{r=1}^{z} \left(\frac{p_r^i(n)}{n^D} \left(\frac{\lambda}{L}\right)^n + \frac{\overline{p_r^i(n)}}{n^D} \left(\frac{\overline{\lambda}}{L}\right)^n \right) X_z.$$

Let $\gamma_i = \frac{\lambda}{L}$. Note that $|\gamma_i| = 1$. By the construction of n^D, none of the polynomials have a term of degree higher than D. Therefore, we can absorb the constants X_r and the monomial n^D into the polynomials, sum the terms up, and write them as polynomials in $\frac{1}{n}$. That is,

$$g^i(n)x_0^i = q^i(1/n)\gamma_i^n + \overline{q^i(1/n)}\overline{\gamma_i}^n$$

for suitable polynomials q^i with algebraic coefficients. Thus

$$\liminf_{n\to\infty} \left| \sum_{i=1}^{t} g^i(n)x_n^i \right| = \liminf_{n\to\infty} \left| \sum_{i=1}^{t} q^i(1/n)\gamma_i^n + \overline{q^i(1/n)}\overline{\gamma_i}^n \right|$$

We defer the proof of the following lemma, which requires tools from Diophantine analysis and the theory of reals, to the next section.

▶ **Lemma 21.** *Let $\gamma_1, \ldots, \gamma_t$ be algebraic numbers with modulus 1. Let $q^1, \ldots, q^t$ be polynomials with algebraic coefficients. The quantity*

$$\liminf_{n\to\infty} \left| \sum_{i=1}^{t} q^i(1/n)\gamma_i^n + \overline{q^i(1/n)}\overline{\gamma_i}^n \right|$$

can be effectively computed. If it is greater than zero, there is an effectively computable N satisfying the requirement of Lemma 20.

Proof of Theorem 7(2). We are now ready to aggregate our case analysis into the proof the pseudo-reachability in hyperplanes is decidable. Given $A \in \mathbb{Q}^{m\times m}$, $x_0 \in \mathbb{Q}^m$ and $H = \{x : c^\top \cdot x = 0\}$, we first convert A to real Jordan normal form as described in Section 2 to obtain $J = Q^{-1}AQ$. We then perform a coordinate transform on x_0 and H to obtain $H' = \{x : c^\top Qx = 0\}$ and $x_0' = Q^{-1}x_0$. The original problem is now equivalent to pseudo-reachability of H' from x_0' under J.

Next, we remove dimensions from $x_0', c^\top Q$ and J that do not correspond to relevant blocks and determine the relevant spectral radius $\rho_H(J)$ of J. If $\rho_H(J) = 1$ then H' is reachable by Lemma 18(1). If $\rho_H(J) < 1$, then by Lemma 18(2), H' is pseudo-reachable if and only if $\mathbf{0} \in H'$ or $x_0', Jx_0', \ldots, J^N x_0'$ hits H', where N is the computable bound in the Lemma.

Finally, we consider the case where $\rho_H(J) > 1$. Let $J_1, \ldots, J_t$ be the blocks of J with $\rho(J) = \rho_H(J)$ and $c^1, \ldots, c^t$, $x_0^1, \ldots, x_0^t$ be the corresponding coordinates of $c^\top Q$ and x_0', respectively. Finally, compute the value of $\liminf_{n\to\infty} \left|\sum_{i=1}^{t} g^i(n)x_n^i\right|$ using Lemma 21 and use Lemma 20 to either immediately conclude reachability or to compute the bound N and determine reachability by checking if $x_0', Jx_0', \ldots, J^N x_0'$ hits H'.

5 Proof of Lemma 21

We now prove a generalization of Lemma 21. Let $\lambda_1, \ldots, \lambda_m$ be algebraic numbers of modulus 1 and let $p_1, \ldots, p_m$ be polynomials with algebraic coefficients. Let n range over the natural numbers. We show how to effectively determine the value of $\liminf_{n\to\infty} |\sum_{j=1}^{m} p_j(1/n)\lambda_j^n|$. Moreover, if the value is strictly greater than 0, we show we can find an explicit bound Δ and $N \in \mathbb{N}$ such that for all $n > N$, we have $|\sum_{j=1}^{m} p_j(1/n)\lambda_j^n| > \Delta$. Lemma 21 follows as a special case.

We require some technical machinery from the theory of Diophantine approximations. We need the following theorem of Masser [13]. A proof can be found in [4] or [9].

▶ **Theorem 22** ([13]). *Let $m \in \mathbb{N}$ be fixed and let $\lambda_1, \ldots, \lambda_m$ be complex algebraic numbers each of modulus 1. Consider the free abelian group*

$$L = \{(v_1, \ldots, v_m) \in \mathbb{Z}^m : \lambda_1^{v_1}\lambda_2^{v_2}\ldots\lambda_m^{v_m} = 1\}.$$

L has a basis $\{\vec{\ell_1}, \ldots, \vec{\ell_p}\} \subseteq \mathbb{Z}^m$ (with $p \le m$), where the entries of each of the $\vec{\ell_j}$ are all polynomially bounded in the total description length of $\lambda_1, \ldots, \lambda_m$. Moreover, such a basis can be can also computed in time polynomial in the total description length.

Let L be as described in Theorem 22 above and suppose we have computed a basis $\{\vec{\ell_1}, \ldots, \vec{\ell_p}\} \subseteq \mathbb{Z}^m$. For each $j \in \{1, \ldots, p\}$, let $\vec{\ell_j} = (\ell_{j,1}, \ldots, \ell_{j,m})$. Now we define a set

$$\begin{aligned} T := \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_1| = \cdots = |z_m| = 1 \text{ and} \\ \text{for each } j \in \{1, \ldots, p\},\ z_1^{\ell_{j,1}} \ldots z_m^{\ell_{j,m}} = 1\} \end{aligned} \tag{7}$$

Notice that $|z| = 1 \iff \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 - 1 = 0$, and the $\ell_{j,k}$ are fixed integers, and thus the conditions above can be written as polynomials in the real and imaginary parts of z. Thus T is an algebraic set.

We now state a version of Kronecker's theorem on simultaneous Diophantine approximation. A derivation of this version of the theorem from the standard version ([10] Chap 23) can be found in [15].

▶ **Theorem 23** (Kronecker's theorem, density version). *Let T be defined from $\lambda_1, \ldots, \lambda_m$ as in (7). Then $\{(\lambda_1^n, \ldots, \lambda_m^n) : n \in \mathbb{N}\}$ is a dense subset of T.*

Theorem 23 enables us to compute the lim inf by minimizing a function over a compact algebraic set:

▶ **Theorem 24.** *Let $\lambda_1, \ldots, \lambda_m$ be complex numbers of modulus 1. Let $p_1, \ldots, p_m$ be polynomials (with algebraic coefficients) with constant terms $c_1, \ldots, c_m$ respectively. Let $\mathbf{z} = (z_1, \ldots, z_m)$ and $\mathbf{c} = (c_1, \ldots, c_m)$. We have that*

$$\liminf_{n\to\infty} \left| \sum_{j=1}^{m} p_j(1/n)\lambda_j^n \right| = \liminf_{n\to\infty} \left| \sum_{j=1}^{m} c_j\lambda_j^n \right| = \inf_{\mathbf{z}\in T} |\mathbf{c}^\top \cdot \mathbf{z}| = \min_{\mathbf{z}\in T} |\mathbf{c}^\top \cdot \mathbf{z}|,$$

where T is the algebraic set computed in (7) as the closure of $\{(\lambda_1^n, \ldots, \lambda_m^n) : n \in \mathbb{N}\}$.

To prove the theorem, we need the following lemma shows that we can replace the polynomials by their constant terms.

▶ **Lemma 25.** *Let $\lambda_1, \ldots, \lambda_m$ be complex numbers of modulus 1. Let $p_1, \ldots, p_m$ be polynomials (with algebraic coefficients) with constant terms $c_1, \ldots, c_m$ respectively. Then*

$$\liminf_{n\to\infty} \left| \sum_{j=1}^{m} p_j(1/n)\lambda_j^n \right| = \liminf_{n\to\infty} \left| \sum_{j=1}^{m} c_j\lambda_j^n \right|.$$

Proof. (of Theorem 24). The first equality follows from Lemma 25 and the second follows from Theorem 23. The third equality holds because the function $\mathbf{z} \mapsto |\mathbf{c}^\top \cdot \mathbf{z}|$ is continuous and T is compact. ◀

Now, since T is an algebraic set, the minimum $\min_{\mathbf{z}\in T} |\mathbf{c}^\top \cdot \mathbf{z}|$ can be expressed in the theory of reals with addition and multiplication (we omit the standard encoding of absolute values):

$$\exists \mathbf{z} \in T. v = |\mathbf{c}^\top \cdot \mathbf{z}| \wedge \forall \mathbf{z}' \in T. v \leq |\mathbf{c}^\top \cdot \mathbf{z}'|$$

Therefore, by Tarski's theorem [18, 2, 7], we can find the unique v that attains the minimum.

Suppose the minimum v is some number $B > 0$. In this case, we require a bound $\Delta \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $|\sum_{j=1}^{m} p_j(1/n)\lambda_j^n| > B$ for all $n > N$. By emulating the proof of Lemma 25, we can find a bound N such that for all $n > N$, we have $|\sum_{j=1}^{m} p_j(1/n)\lambda_j^n| > B/2$. The required bounds are $\Delta = B/2$ and this N.

This concludes the proof of Lemma 21 and therefore also Theorem 7.

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A Proof of Proposition 8

We want to show

$$Q\,\widetilde{\mathcal{O}}_{\gamma_2}(B, Q^{-1}x) \subseteq \widetilde{\mathcal{O}}_{\epsilon}(A, x) \subseteq Q\,\widetilde{\mathcal{O}}_{\gamma_1}(B, Q^{-1}x), \tag{8}$$

where $\gamma_1 = \epsilon \left\|Q^{-1}\right\|_\infty$ and $\gamma_2 = \epsilon / \left\|Q\right\|_\infty$.

Take any $y \in \widetilde{\mathcal{O}}_{\epsilon}(A, x)$. We show that $Q^{-1}y \in \widetilde{\mathcal{O}}_{\gamma_1}(B, Q^{-1}x)$ to get the right-hand side of (8). Since $y \in \widetilde{\mathcal{O}}_{\epsilon}(A, x)$, there is a state trajectory $(x_0, x_1, \ldots)$ and a sequence $(d_0, d_1, \ldots)$ such that $x_0 = x$, $x_{n+1} = Ax_n + d_n$, $d_n \in [-\epsilon, \epsilon]^m$ for all $n \in \mathbb{N}$, and y appears in the state trajectory. We construct a new state trajectory $(y_0, y_1, \ldots)$ and the sequence $(\bar{d}_0, \bar{d}_1, \ldots)$ with the transformation $x_n = Qy_n$ and $d_n = Q\bar{d}_n$. Then we have $y_{n+1} = Q^{-1}AQy_n + Q^{-1}d_n = By_n + \bar{d}_n$. Note that $\|\bar{d}_n\|_\infty = \|Q^{-1}d_n\|_\infty \le \|Q^{-1}\|_\infty \|d_n\|_\infty \le \gamma_1$. Since y appears in the state trajectory $(x_0, x_1, \ldots)$, $Q^{-1}y$ appears in the state trajectory $(y_0, y_1, \ldots)$ with $y_0 = Q^{-1}x_0 = Q^{-1}x$. Therefore, $Q^{-1}y \in \widetilde{\mathcal{O}}_{\gamma_1}(B, Q^{-1}x)$ which results in $y \in Q\widetilde{\mathcal{O}}_{\gamma_1}(B, Q^{-1}x)$.

To prove the left-hand side of (8), We invoke the right-hand side by replacing (A, B, Q, x, ϵ) with $(B, A, Q^{-1}, Q^{-1}x, \gamma_2)$. This gives $\widetilde{\mathcal{O}}_{\gamma_2}(B, Q^{-1}x) \subseteq Q^{-1}\,\widetilde{\mathcal{O}}_{\gamma_1'}(A, x)$ with $\gamma_1' = \gamma_2 \|Q\|_\infty$. Setting $\gamma_1' = \epsilon$ proves the left-hand side of (8).

To prove that $\widetilde{\mathcal{O}}(A, x) = Q\widetilde{\mathcal{O}}(B, Q^{-1}x)$, we take intersection of all the sides in (8) over $\epsilon > 0$:

$$\bigcap_{\epsilon>0} Q\,\widetilde{\mathcal{O}}_{\gamma_2}(B, Q^{-1}x) \subseteq \bigcap_{\epsilon>0} \widetilde{\mathcal{O}}_\epsilon(A, x) \subseteq \bigcap_{\epsilon>0} Q\,\widetilde{\mathcal{O}}_{\gamma_1}(B, Q^{-1}x).$$

Due to the linear relation between γ_1 and γ_2 with ϵ, we get

$$Q\,\widetilde{\mathcal{O}}(B, Q^{-1}x) \subseteq \widetilde{\mathcal{O}}(A, x) \subseteq Q\,\widetilde{\mathcal{O}}(B, Q^{-1}x) \quad \Rightarrow \quad \widetilde{\mathcal{O}}(A, x) = Q\,\widetilde{\mathcal{O}}(B, Q^{-1}x).$$

B Proofs from Section 3

B.1 Proof of Lemma 9

Any $t \in \widetilde{\mathcal{O}}_\epsilon(A, s)$ is of the form $t = A^n s + \sum_{i=0}^{n-1} A^i d_{n-i-1}$ for some $n \in \mathbb{N}$ and some d_i with $\|d_i\|_\infty \le \epsilon$. This means $s = A^{-n}t + \sum_{i=0}^{n-1} A^{-i} d'_{n-i-1}$ with $d'_{n-1-i} = A^{-1}d_i$. Since $\|d'_{n-1-i}\|_\infty \le \|A^{-1}\|_\infty \epsilon$, we get $s \in \widetilde{\mathcal{O}}_\gamma(A^{-1}, t)$. To get (4), notice that

$$t \in \widetilde{\mathcal{O}}(A, s) \Rightarrow t \in \bigcap_{\epsilon>0} \widetilde{\mathcal{O}}_\epsilon(A, s) \Rightarrow s \in \bigcap_{\gamma>0} \widetilde{\mathcal{O}}_\gamma(A^{-1}, t) \Rightarrow s \in \widetilde{\mathcal{O}}(A^{-1}, t).$$

Applying the same argument to the matrix A^{-1} will give the other side of (4).

B.2 Proof of Lemma 10

We prove the lemma by showing there is a constant $C > 0$ satisfying

$$\overline{\mathcal{O}(J, s)} \overset{*}{=} \mathcal{O}(J, s) \cup \{\mathbf{0}\} \overset{**}{\subseteq} \widetilde{\mathcal{O}}(J, s) \overset{\dagger}{\subseteq} \bigcap_{\epsilon>0} \bigcup_{z \in \mathcal{O}(J,s)} \mathcal{B}(z, C\epsilon) \overset{\S}{\subseteq} \overline{\mathcal{O}(J, s)}, \tag{9}$$

where $\mathcal{B}(z, \epsilon) := \{y \in \mathbb{R}^m \mid \|z - y\|_2 \le \epsilon\}$ is the closed ball with respect to two norm with center z and radius ϵ. It is easy to see that equality (*) holds since all the eigenvalues of J are inside the unit circle, $\lim_{n\to\infty} J^n = 0$, and 0 is the only limiting point of any state trajectory.

It is also easy to see that inclusion (**) is correct. Note that for any $\epsilon > 0$, $\mathcal{O}(J, s) \subseteq \widetilde{\mathcal{O}}_\epsilon(J, s)$ and the set $\widetilde{\mathcal{O}}_\epsilon(J, s)$ is closed by definition. Taking intersection over $\epsilon > 0$, we get $\mathcal{O}(J, s) \subseteq \widetilde{\mathcal{O}}(J, s)$ with $\widetilde{\mathcal{O}}(J, s)$ being a closed set. Therefore, $\overline{\mathcal{O}(J, s)} \subseteq \widetilde{\mathcal{O}}(J, s)$.

We now choose a value of C which allows us to prove inclusion (†). First pick γ such that $\rho(J) < \gamma < 1$. Next choose c_1 to be a constant (which is guaranteed to exist) satisfying $\|J^n\|_2 \le c_1\gamma^n$ for all $n \in \mathbb{N}$, and finally set $C := c_1 m/(1 - \gamma)$. We show that $\widetilde{\mathcal{O}}_\epsilon(J, s) \subseteq \bigcup_{z \in \mathcal{O}(J,s)} \mathcal{B}(z, C\epsilon)$ for any $\epsilon > 0$. Take any $x \in \widetilde{\mathcal{O}}_\epsilon(J, s)$. Then there is a sequence

$(d_0, d_1, \ldots)$ and $n \in \mathbb{N}$ such that $\|d_i\|_\infty \le \epsilon$ and $x = J^n s + \sum_{i=0}^{n-1} J^i d_{n-i-1}$. Now

$$\|x - J^n s\|_2 = \left\| \sum_{i=0}^{n-1} J^i d_{n-i-1} \right\|_2 \le \sum_{i=0}^{n-1} \left\|J^i\right\|_2 \|d_{n-i-1}\|_2 \le \sum_{i=0}^{n-1} c_1 \gamma^i m\epsilon \le \frac{c_1 m\epsilon}{1-\gamma} = C\epsilon,$$

We then get $x \in \mathcal{B}(z, C\epsilon)$ for $z := J^n s \in \mathcal{O}(J, s)$.

The inclusion § can be proven by taking an arbitrary point $y \notin \overline{\mathcal{O}(J,s)}$ and showing that there is an $\epsilon > 0$ for which $y \notin \mathcal{B}(z, C\epsilon)$ for all $z \in \mathcal{O}(J, s)$. Note that the complement of $\overline{\mathcal{O}(J,s)}$ is an open set, which means there is a $\theta > 0$ such that $\mathcal{B}(y, \theta) \cap \overline{\mathcal{O}(J,s)} = \emptyset$. Taking ϵ such that $C\epsilon < \theta$ will give the intended result.

B.3 Proof of Lemma 12

The key part of the proof is to show that $\mathbf{0} \in \widetilde{\mathcal{O}}(A, s)$ for any s and for any A having the eigenvalues on the unit circle. Once we show this, we know that $s \in \widetilde{\mathcal{O}}(A^{-1}, \mathbf{0})$ is true for any s and any matrix A due to the Reversibility lemma. Stated for the inverse of A and any x, we get $x \in \widetilde{\mathcal{O}}(A, \mathbf{0})$. Since pseudo-orbits are transitive, we have $x \in \widetilde{\mathcal{O}}(A, s)$ for any x and s, which is the intended result.

We show $\mathbf{0} \in \widetilde{\mathcal{O}}(A, s)$ equivalently by replacing A with its Jordan form J and doing induction on the structure of J. The proof has two stages. The first stage is to show that $\mathbf{0} \in \widetilde{\mathcal{O}}(J, s)$ for all s when J has a single block simple eigenvalues. The second stage is to show that we can sequentially increase the multiplicity of eigenvalues and multiple blocks.

Base case: Suppose $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a^2 + b^2 = 1$ or $J = r$ with $|r| = 1$. Observe that the multiplication by J does not increase the two norm of a vector. Hence setting

$$d_n = \begin{cases} -\epsilon \cdot \frac{Jx_n}{\|Jx_n\|_2} & \text{if } ||Jx_n||_\infty > \epsilon, \\ -Jx_n & \text{otherwise,} \end{cases}$$

we obtain the ϵ-pseudo-orbit $(x_0 = s, x_1, x_2, \ldots, x_m, \mathbf{0}, \mathbf{0}, \ldots)$ from any s where $\|x_k\|_2 = \|x_{k-1}\|_2 - \epsilon$ for $k \le m$, which gives $\mathbf{0} \in \widetilde{\mathcal{O}}(J, s)$.

Inductive case: We show that if $\mathbf{0} \in \widetilde{\mathcal{O}}(J_1, s_1)$ and $\mathbf{0} \in \widetilde{\mathcal{O}}(J_2, s_2)$ for all s_1 and s_2 of appropriate dimensions, we also have $\mathbf{0} \in \widetilde{\mathcal{O}}(J, s)$ with $J = \begin{bmatrix} J_1 & B \\ 0 & J_2 \end{bmatrix}$ for any B and any s with appropriate dimensions. Let us partition any state $x = (x^1, x^2)$ according to the dimensions of J_1 and J_2. Let $\epsilon > 0$ and $s = (s^1, s^2)$. By the assumption, there exist ϵ-perturbations $(d_0^2, d_1^2, \ldots, d_{N-1}^2)$ that bring s^2 to $\mathbf{0}$ under J_2. Let $d_n = (\mathbf{0}, d_n^2)$ for $0 \le n < N$ be a sequence of ϵ-perturbations for the linear system with mapping J. We obtain the sequence $(x_0 = s, x_1, \ldots, x_N)$ with $x_N^2 = \mathbf{0}$: the ϵ-perturbations $d_0, \ldots, d_{N-1}$ have brought the second coordinate to $\mathbf{0}$. By the assumption, we also have $\mathbf{0} \in \widetilde{\mathcal{O}}_\epsilon(J_1, x_N^1)$, which gives ϵ-perturbations $(d_0^1, \ldots, d_M^1)$ that bring x_N^1 to $\mathbf{0}$ under J_1. Let us expand the sequence of perturbations for the linear system J with $d_{n+N} = (d_n^1, \mathbf{0})$ for $0 \le n \le M$. It is easy to see that $(d_0, \ldots, d_M)$ bring the system from s to $\mathbf{0}$ due to the structure of J that is upper triangular.

B.4 Proof of Theorem 17

Recall that a set S is pseudo-reachable from s under A if for every $\epsilon > 0$, there exists a point $x_\epsilon \in S$ that is ϵ-pseudo-reachable from s under A. In this section, we show that pseudo-reachability in bounded semialgebraic sets is decidable.

We need the following lemma that shows that deciding pseudo-rechability in a given bounded set S reduces to checking whether $\overline{S} \cap \widetilde{\mathcal{O}}(J,s) = \emptyset$, allowing us to restrict our attention to compact sets and the existence of a pseudo-reachable point in a set as opposed to pseudo-reachability of the set as a whole.

▶ **Lemma 26.** *Let S be a bounded set. S is pseudo-reachable from s under A if and only if there exists $x \in \overline{S}$ that is pseudo-reachable from s under A.*

Proof. Suppose S is pseudo-reachable. Let $(\epsilon_i)_{i\in\mathbb{N}}$ be a sequence of positive numbers with $\lim_{\epsilon\to 0} = 0$, and $(x_i)_{i\in\mathbb{N}}$ be a sequence of elements of S such that x_i is ϵ_i-pseudo-reachable for all $i \geq 0$. By the Bolzano–Weierstrass theorem, boundedness of S implies that $(x_i)_{i\in\mathbb{N}}$ must have a limit point x in $\overline{S}$. To argue that x is pseudo-reachable, let $\epsilon > 0$. Since x is the limit point of $(x_i)_{i\in\mathbb{N}}$, there must exist an $\frac{\epsilon}{2}$-pseudo-orbit $(y_i)_{i\in\mathbb{N}}$ containing a point y_N such that $\|x - y_N\|_\infty < \frac{\epsilon}{2}$. Therefore, x is ϵ-pseudo-reachable from s via the sequence $s, y_1, \ldots, y_{N-1}, x$.

Now suppose $x \in \overline{S}$ is pseudo-reachable. To argue that S is pseudo-reachable, let $\epsilon > 0$. Since $x \in \overline{S}$, there must exist a point $x' \in S$ such that $\|x' - x\|_\infty < \frac{\epsilon}{2}$. Since x is $\frac{\epsilon}{2}$-pseudo-reachable, x' must be ϵ-pseudo-reachable. ◀

Now we are ready to prove the main theorem.

▶ **Theorem 27.** *Given a bounded semialgebraic set S, it is decidable whether S is pseudo-reachable from x_0 under A.*

Proof. It suffices to consider $A = J$ for J in real Jordan normal form (see Proposition 8) and S that is closed. Let $J_1, \ldots, J_t$ be all the blocks of J with spectral radius > 1, $J_{t+1}, \ldots, J_{t'}$ all the blocks of with spectral radius $= 1$ and $J_{t'+1}, \ldots, J_l$ all the blocks of with spectral radius < 1. Let M be an upper bound on the ℓ_2-norm of all vectors in S. We show how to decide whether there exists a point $x \in S$ that is also in $\widetilde{\mathcal{O}}(J,s)$.

1. Suppose J has a block J_i with an eigenvalue of modulus greater than 1 such that $s^i \neq \mathbf{0}$. Then $\widetilde{\mathcal{O}}(J,s) = \mathcal{O}(J,s)$ (Lemma 11). By Lemma 28, there exists a computable N such that for all $n > N$, $\left\|J_i^n s^i\right\|_2 > M$ (observe that orbit itself is a pseudo-orbit), and therefore we only need to check whether any of the first N points in orbit of s under J belong to the set S.
2. Let $J_c = \mathrm{diag}(J_{t'+1}, \ldots, J_l)$. If for all $x \in S$, the projection x^c of x onto the coordinates governed by J_c is not $\mathbf{0}$, then using our characterization of the pseudo-orbit we can conclude that $x \in \widetilde{\mathcal{O}}(J,s)S$ only if $x^c \in \mathcal{O}(J_c, s^c)$. Now observe that because S is compact, it must be the case that $\inf_{x\in S} \|x^c\| > 0$ (since by assumption $\|x^c\|$ is never 0). Therefore, using Lemma 28 we can compute a time bound N such that for $n > N$ and sufficiently small ϵ, $\|x_n^c\| < \inf_{x\in S} \|x^c\|$ and hence S can only be reached within the first N steps. It then remains to check whether any of $s, Js, \ldots, J^{N-1}s$ belongs to S.
3. Suppose 1 and 2 are not the case. Assuming S is not empty, it must then contain a point whose projection onto $J_c = \mathbf{0}$. In other words S must contain a point x whose projections onto blocks with spectral radius < 1 are all $\mathbf{0}$. From our characterization of the pseudo-orbit we can see that $x_i \in \Delta(J_i, s_i)$ for every $1 \leq i \leq m$ and hence $x \in \widetilde{\mathcal{O}}(J,s)$.

◀

C Proofs from Section 4

C.1 Proof of Lemma 18

Let $\rho_H(J) = 1$. We write $J = diag(J_h, J_r)$, where $\rho_H(J_h) = 1$ and $\rho_H(J_r) < 1$ and correspondingly set $s = (s_h, s_r)$, $c = (c_h, c_r)$. Note that $c_h \neq 0$ by the relevance of at least one of eigenvalues of modulus 1.

By Lemma 10 we know $\mathbf{0} \in \widetilde{\mathcal{O}}(J_r, s_r)$. By Lemma 12, we can select y such that $c_h^\top y - v = 0$ and $y \in \widetilde{\mathcal{O}}(J_h, s_h)$. Therefore, invoking Lemma 14, for every $\epsilon > 0$ we can construct ϵ-pseudo-orbits $(x_n^h)_{n\in\mathbb{N}}$ and $(x_n^r)_{n\in\mathbb{N}}$ such that at some point $N \in \mathbb{N}$ we have $x_N^h = y$ and $x_N^r = \mathbf{0}$, which implies $c^\top x_N - v = c_h^\top y + c_r^\top \mathbf{0} - v = \mathbf{0}$ as desired.

Now suppose $\rho_H(J) < 1$.

Case 1: $v = 0$. Since the origin is pseudo-reachable from s (Lemma 10), for any given ϵ one can construct an ϵ-pseudo-orbit $(x_n)_{n\in\mathbb{N}}$ such that at some time-step $N \in \mathbb{N}$ we have $c^\top x_N = 0$.

Case 2: $v \neq 0$. Using the reasoning of Lemma 14, and setting $\delta = |v|/(2\|c\|_2)$, we can find $\epsilon > 0$ and horizon $N \in \mathbb{N}$ after which the ϵ-pseudo-orbit is trapped in $\mathcal{B}(0, \delta)$. Thus, the hyperplane cannot be pseudo-reached after time N, as the hyperplane does not intersect with $\mathcal{B}(0, \delta)$. It remains to check if the hyperplane is pseudo-reachable at any of the first N time-steps. In fact, for a bounded time interval, a hyperplane is pseudo-reachable iff it is reachable. This is because the effect of finitely many disturbance terms $(d_0, \ldots, d_{N-1})$ can be made arbitrarily small for small enough ϵ. Therefore, decidability in this case only requires checking if the bounded orbit $(y_n)_{0\le n\le N}$ hits the hyperplane before the time horizon N, that is, if there exists a time-step $0 \le n \le N$ such that $c^\top y_n - v = 0$, which is clearly decidable.

▶ **Lemma 28.** **1.** *Let $A \in \mathbb{R}^{m\times m}$ and $s \in \mathbb{R}^m$. If $\rho(A) < 1$, then for every $\delta > 0$ there exists an effectively computable $N \in \mathbb{N}$ and $\epsilon > 0$ such that after time N, all ϵ-pseudo-orbits are contained inside the ball $\mathcal{B}(\mathbf{0}, \delta)$.*

2. *Suppose $J = diag(J_1, \ldots, J_z)$ is a Jordan normal form with one block J_i associated to eigenvalues outside the unit circle. for any $s = (s^1, \ldots, s^z)$ with $s^i \neq \mathbf{0}$ and for every $\delta > 0$, there exists an effectively computable $N \in \mathbb{N}$ and $\epsilon > 0$ such that after time N, all ϵ-pseudo-orbits of J are contained outside the ball $\mathcal{B}(\mathbf{0}, \delta)$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ denote an ϵ-pseudo-orbit starting from s with a sequence of disturbances $(d_n)_{n\in\mathbb{N}}$. Suppose $\rho(A) < 1$ and let $\gamma \in (\rho(A), 1)$. There is a constant $c > 0$ satisfying $\|A^n\|_2 \le c\gamma^n$ for all n. Then we get

$$\|x_n\|_2 = \left\| A^n s + \sum_{k=0}^{n-1} A^k d_{n-k-1} \right\|_2 \le \|A^n\|_2 \|s\|_2 + \sum_{k=0}^{n-1} \left\|A^k\right\|_2 \|d_{n-k-1}\|_2$$

$$\le c\gamma^n \|s\|_2 + \sum_{k=0}^{n-1} m\,\epsilon\, c\gamma^k \le c\gamma^n \|s\|_2 + \frac{m\,\epsilon\, c}{1-\gamma}.$$

Taking $\epsilon = \delta(1-\gamma)/(2mc)$ and N with $\gamma^N \|s\|_2 \le \delta/(2c)$ gives the intended result.

For the second part of the lemma, take an ϵ-pseudo-orbits of J as $(s = x_0, x_1, x_2, \ldots)$ with $x_{n+1} = Jx_n + d_n$. We have $\|x_n\|_2 \ge \left\|x_n^i\right\|_2$ where the states x_n are partitioned according to the structure of J and x_n^i is the one associated with J_i. Note that $(s^i = x_0^i, x_1^i, x_2^i, \ldots)$

satisfy $x^i_{n+1} = J_i x^i_n + d^i_n$. There is a constant $c > 0$ satisfying $\left\|J_i^{-n}\right\|_2 \leq c\gamma^n$ for all n with some $\gamma \in (\rho(J_i^{-1}), 1)$. We can write

$$x^i_n = J_i^n s^i + \sum_{k=0}^{n-1} J_i^k d^i_{n-k-1} \quad \Rightarrow \quad s^i = J_i^{-n} x^i_n - \sum_{k=0}^{n-1} J_i^{k-n} d^i_{n-k-1}$$

$$\Rightarrow \left\|s^i\right\|_2 \leq \left\|J_i^{-n}\right\|_2 \left\|x^i_n\right\|_2 + \sum_{k=0}^{n-1} \left\|J_i^{k-n}\right\|_2 \left\|d^i_{n-k-1}\right\|_2 \leq c\gamma^n \left\|x^i_n\right\|_2 + \frac{c\, m\, \epsilon}{1-\gamma}$$

$$\Rightarrow \left\|x^i_n\right\|_2 \geq \frac{1}{c\gamma^n}\left[\left\|s^i\right\|_2 - \frac{c\, m\, \epsilon}{1-\gamma}\right].$$

It is sufficient to take $\epsilon = \frac{(1-\gamma)}{2cm}\left\|s^i\right\|_2 > 0$ and N sufficiently large such that $2\delta c\gamma^N < \left\|s^i\right\|_2$. This forces x^i_n (thus also x_n) to move outside the ball $\mathcal{B}(\mathbf{0}, \delta)$ for all $n > N$. ◀

C.2 Proof of Lemma 19

Proof. We address each point individually.

1: For $1 \leq i \leq t$ let J_i have eigenvalues λ and $\overline{\lambda}$ (the case where J_i has a single real eigenvalue is similar but simpler) and observe that

$$g^i(n) = \left[\frac{p^i_1(n)}{n^D}\left(\frac{\lambda}{L}\right)^n + \overline{\frac{p^i_1(n)}{n^D}}\left(\frac{\overline{\lambda}}{L}\right)^n \quad \cdots \quad \frac{p^i_{2\kappa(i)}(n)}{n^D}\left(\frac{\lambda}{L}\right)^n + \overline{\frac{p^i_{2\kappa(i)}(n)}{n^D}}\left(\frac{\overline{\lambda}}{L}\right)^n\right].$$

By the definition of top eigenvalues, $|\lambda| = L$ and thus $\frac{\lambda}{L}$ and $\frac{\overline{\lambda}}{L}$ have modulus 1. By construction of n^D, the polynomials $p^i_1(n), \ldots, p^i_{2\kappa(i)}$ all have degree at most D and hence the terms $\frac{p^i_1(n)}{n^D}, \ldots, \frac{p^i_{2\kappa(i)}(n)}{n^D}$ are bounded from above by a constant.

2: For $t+1 \leq i \leq l$ let J_i have eigenvalues λ and $\overline{\lambda}$ and observe that

$$g^i(n) = \left[\frac{p^i_1(n)}{n^D}\left(\frac{\lambda}{L}\right)^n + \overline{\frac{p^i_1(n)}{n^D}}\left(\frac{\overline{\lambda}}{L}\right)^n \quad \cdots \quad \frac{p^i_{\kappa(i)}(n)}{n^D}\left(\frac{\lambda}{L}\right)^n + \overline{\frac{p^i_{\kappa(i)}(n)}{n^D}}\left(\frac{\overline{\lambda}}{L}\right)^n\right].$$

By construction $|\lambda| < L$ and thus $\gamma := \frac{\lambda}{L}$ and $\overline{\gamma}$ have moduli $\|\gamma\|, \|\overline{\gamma}\| > 1$. The polynomials $p^i_1(n), \ldots, p^i_{2\kappa(i)}(n)$ may not be bounded (since n^D was constructed only considering top eigenvalues). However, it is clear (as in the proof of Lemma 10) that there must exist a constant $c > 0$ such that each entry of $g^i(n)$ is dominated for all $n \in N$ by $c\gamma^n$ (which goes to zero as n tends to infinity).

3: Observe that by construction of n^D, there must exist a top eigenvalue block J_j $(1 \leq j \leq t)$ for which at least one polynomial in $c^j J_j^n$ has degree D. Let $r > D$ be the multiplicity of the block J_j, which has the form of the real Jordan form with a single block (Eq. (3)) with sub-blocks Λ. One can write

$$c^j J_j^n = \begin{bmatrix} c^j_r & c^j_{r-1} & \cdots & c^j_0 \end{bmatrix} \begin{pmatrix} \Lambda^n & n\Lambda^{n-1} & \binom{n}{2}\Lambda^{n-1} & \cdots & \binom{n}{r-1}\Lambda^{n-r+1} \\ 0 & \Lambda^n & n\Lambda^{n-1} & \cdots & \binom{n}{r-2}\Lambda^{n-r+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n\Lambda^{n-1} \\ 0 & 0 & 0 & \cdots & \Lambda^n \end{pmatrix}, \quad (10)$$

where c_k^j for $1 \leq k \leq r$ corresponds to a row vector of size two or one, respectively, when Λ is a 2×2 or 1×1 matrix. Analyzing this product, we see that $c_r^j, \ldots, c_{D+1}^j = \mathbf{0}$, $c_D^j \neq \mathbf{0}$ and the single entry of $c^j J_j^n$ whose polynomial component has degree D is exactly $c_D^j \binom{n}{D} \Lambda^{n-D}$.

We define $\hat{\Lambda} := \Lambda/L$. Note that $\left\|\hat{\Lambda}\right\|_2 = 1$. Now observe that for this block J_j, we have

$$g^j(n) = \frac{1}{L^n n^D} c^j J_i^n = c_D^j \frac{1}{D!} \hat{\Lambda}^n + \frac{1}{n}(O(1))$$

Therefore, there exists sufficiently large N such that for all $n \in \mathbb{N}$,

$$n > N \implies \left\| \frac{1}{L^n n^D} c^j J_j^n \right\|_\infty > \frac{1}{2} \left\| c_D^j \frac{1}{D!} \hat{\Lambda}^n \right\|_\infty > \frac{\left\| c_D^j \right\|_2}{4D!}.$$

Thus we have shown Point 3 with $C = \frac{\|c_D^j\|_2}{4D!}$. ◀

C.3 Proof of Lemma 20

Proof. Suppose $D = 0$. Take an arbitrary $\epsilon > 0$. We argue that H is ϵ-pseudo-reachable. Recall that

$$\begin{aligned} f(n) &= \sum_{i=1}^{l} \left(c^i \frac{J_i^n}{L^n n^D} x_0^i + c^i \sum_{k=0}^{n-1} \frac{J_i^k}{L^n n^D} d_{n-k-1}^i \right) - \frac{v}{L^n n^D} \\ &= \sum_{i=1}^{l} \left(g^i(n) x_0^i + c^i \sum_{k=0}^{n-1} \frac{J_i^k}{L^n n^D} d_{n-k-1}^i \right) - \frac{v}{L^n n^D}. \end{aligned}$$

Let I be such that $\left\|g^I(n)\right\|_\infty > C$, for $C > 0$ and sufficiently large n (Point 3 of the Normalization Lemma). We construct a pseudo-orbit with all perturbations set to zero except d_0^i and obtain

$$f(n) = c^I \frac{J_I^{n-1}}{L^n n^D} d_0^I + \sum_{i=1}^{t} g^i(n) x_0^i + \sum_{i=t+1}^{l} c^i \frac{J_i^n}{L^n n^D} x_0^i - \frac{v}{L^n n^D}.$$

Intuitively, we will use the term $c^I \frac{J_I^{n-1}}{L^n n^D} d_0^I$ to cancel out the remaining summands above, but we have to argue that this can be done using a disturbance of size at most ϵ. Moreover, observe that $c^I \frac{J_I^{n-1}}{L^n n^D}$ is very close to $g^I(n)$. Formally, we first find N large enough such that

- $\left\|g^I(N)\right\|_\infty > C$,
- $\left\|\sum_{i=t+1}^{l} c^i \frac{J_i^N}{L^N N^D} x_0^i - \frac{v}{L^N N^D}\right\|_\infty < \frac{C^2}{\|J_i\|_\infty} \frac{\epsilon}{2}$ (possible because for $t+1 \leq i \leq l$, $\rho(J_i) < 1$ and $L > 1$), and
- $\left\|\sum_{i=1}^{t} g^i(N) x_0^i\right\|_\infty < \frac{C^2}{\|J_i\|_\infty} \frac{\epsilon}{2}$ (possible because $\liminf_{n \to \infty} \left\|\sum_{i=1}^{t} g^i(n) x_n^i\right\| = 0$).

Finally, we determine the value of d_0^i. Without loss of generality, assume that $g^I(N)$ is of the form $\begin{bmatrix} C' & \cdots \end{bmatrix}$ where $|C'| > C$, that is the first entry of $g^I(N)$ is large. We then observe that $c^I \frac{J_I^{N-1}}{L^N N^D} d_0^I = g^I(N) J_I^{-1} d_0^I$ and set

$$d_0^I = J_I \cdot \left[-\frac{1}{C'} \left(\sum_{i=1}^{t} g^i(N) x_0^i + \sum_{i=t+1}^{l} c^i \frac{J_i^N}{L^N N^D} x_0^i - \frac{v}{L^N N^D} \right) \quad 0 \quad 0 \quad \cdots \quad 0 \right]^\top$$

to obtain

$$c^I \frac{J_I^{N-1}}{L^N N^D} d_0^I = -\left(\sum_{i=1}^{t} g^i(N) x_0^i + \sum_{i=t+1}^{l} c^i \frac{J_i^N}{L^N N^D} x_0^i - \frac{v}{L^N N^D} \right)$$

and hence $f(N) = 0$.

Now suppose $D > 0$. Recall

$$f(n) = \sum_{i=1}^{l} \left(g^i(n)x_0^i + c^i \sum_{k=0}^{n-1} \frac{J_i^k}{L^n n^D} d_{n-k-1}^i \right) - \frac{v}{L^n n^D}$$
$$= \sum_{i=1}^{t} g^i(n)x_0^i + \sum_{i=t+1}^{l} c^i \frac{J_i^n}{L^n n^D} x_0^i + \sum_{i=1}^{l} c^i \sum_{k=0}^{n-1} \frac{J_i^k}{L^n n^D} d_{n-k-1}^i - \frac{v}{L^n n^D}.$$

In this case we shall construct a time bound N after which for all sufficiently small value of ϵ, the term $\sum_{i=1}^{t} g^i(n)x_0^i$ will dominate the other summands. Let $\Delta > 0$ be a lower bound on $\liminf_{n\to\infty} |\sum_{i=1}^{t} g^i(n)x_0^i| > 0$. We shall see how to obtain such a bound effectively later (Lemma 21). We compute N with the following properties.

- For all $n > N$, $\left|\sum_{i=1}^{t} g^i(n)x_0^i\right| > \Delta$. Possible because $\liminf_{n\to\infty} \left|\sum_{i=1}^{t} g^i(n)x_n^i\right| > \Delta$.
- For all $n > N$, $\left|\sum_{i=t+1}^{l} c^i \frac{J_i^n}{L^n n^d} x_0^i\right|, \left|\frac{v}{L^n n^d}\right| \ll \Delta$. The former is possible because for $t+1 \le i \le l$, $\rho(J_i) < L$.
- For sufficiently small ϵ, for all $n > N$, $\left|c^i \sum_{k=0}^{n-1} \frac{J_i^k}{L^n n^d} d_{n-k-1}^i\right| \ll \Delta$ for $1 \le i \le l$. To see that this is always possible, observe that

$$\left|c^i \sum_{k=0}^{n-1} \frac{J_i^k}{L^n n^d} d_{n-k-1}^i\right| \le \sum_{k=0}^{n-1} \left\| c^i \frac{J_i^k}{L^n n^d} \right\|_\infty M\epsilon \text{ (where fixed } M \text{ bounds the matrix dimension)}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left\| c^i \frac{J_i^k}{L^n n^d} \right\|_\infty \le \lim_{n\to\infty} \sum_{k=0}^{n} \left\| c^i \frac{1}{L^{n-k}} \frac{J_i^k}{L^k k^d} \right\|_\infty = \lim_{n\to\infty} \sum_{k=0}^{n} \left\| \frac{1}{L^{n-k}} g^i(k) \right\|_\infty.$$

Recalling Point 1 of the Normalization Lemma, $\|g^i(n)\|_\infty = O(1)$ and hence

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left\| \frac{1}{L^{n-k}} g^i(k) \right\|_\infty = O(1),$$

by bounding the sum $\sum_{k=0}^{n} \left\| \frac{1}{L^{n-k}} g^i(k) \right\|_\infty$ from above by a geometric sequence. Therefore, $\sum_{k=0}^{n-1} \left\| c^i \frac{J_i^k}{L^n n^d} \right\|_\infty M\epsilon$ can be made $\ll \Delta$ by choosing ϵ to be sufficiently small.

Once we have chosen N, by the properties above we will have that for all $n > N$, for sufficiently small ϵ,

$$|f(n)| \ge \left|\sum_{i=1}^{t} g^i(n)x_0^i\right| - \left|\sum_{i=t+1}^{l} c^i \frac{J_i^n}{L^n n^d} x_0^i + \sum_{i=1}^{l} c^i \sum_{k=0}^{n-1} \frac{J_i^k}{L^n n^d} d_{n-k-1}^i - \frac{v}{L^n n^d}\right| > 0.$$

Therefore, H is pseudo-reachable if and only if for every $\epsilon > 0$, H is ϵ-pseudo-reachable within the first N steps. By Lemma 15, this is the case if and only if H is reachable within the first N steps. ◀

D Proofs from Section 5

D.1 Proof of Lemma 25

We can write $p_j(1/n)$ as $c_j + \sum_{i=1}^{d_j} c_{(j,i)} \frac{1}{n^i}$, where c_j is the constant term, $c_{(j,i)}$ are the other coefficients, and d_j is the degree. Define $A_i = \sum_{i=1}^{d_j} |c_{(j,i)}|$ and observe that

$$|p_j(1/n) - c_j| < |\sum_{i=1}^{d_j} c_{(j,i)} \frac{1}{n^i}| < \frac{\sum_{i=1}^{d_j} |c_{(j,i)}|}{n} = \frac{A_j}{n}$$

Thus for any ϵ, setting $N_j(\epsilon) = \lceil A_j/\epsilon \rceil$ ensures that

$$n > N_j(\epsilon) \implies |p_j(1/n) - c_j| < \epsilon.$$

Define $N(\epsilon) = \max_{j \in \{1,\ldots,m\}} N_j(\epsilon/m)$.

▷ Claim 29. Let S_n be defined as $|\sum_{j=1}^m c_j \lambda_j^n|$. For all $\epsilon > 0$,

$$S_n - \epsilon \le |\sum_{j=1}^m p_j(1/n)\lambda_j^n| \le S_n + \epsilon$$

Taking the limit inferior of each term gives us the desired result.

Proof. (of Claim 29) We write

$$|\sum_{j=1}^m p_j(1/n)\lambda_j^n| = |\sum_{j=1}^m (c_j + p_j(1/n) - c_j)\lambda_j^n|,$$

which gives us

$$S_n - |\sum_{j=1}^m (p_j(1/n) - c_j)\lambda_j^n| \le |\sum_{j=1}^m p_j(1/n)\lambda_j^n| \le S_n + |\sum_{j=1}^m (p_j(1/n) - c_j)\lambda_j^n|$$

and thus

$$S_n - \sum_{j=1}^m |(p_j(1/n) - c_j)\lambda_j^n| \le |\sum_{j=1}^m p_j(1/n)\lambda_j^n| \le S_n + \sum_{j=1}^m |(p_j(1/n) - c_j)\lambda_j^n|,$$

by elementary properties of sums of absolute values. Observing that λ_js have absolute value 1, we can reduce the proposition above to

$$S_n - \sum_{j=1}^m |(p_j(1/n) - c_j)| \le |\sum_{j=1}^m p_j(1/n)\lambda_j^n| \le S_n + \sum_{j=1}^m |(p_j(1/n) - c_j)|.$$

Now setting $n > N(\epsilon) = \max_{j \in \{1,\ldots,m\}} N_j(\epsilon/m)\}$, we have $|(p_j(1/n) - c_j)| < \epsilon/m$ for all j, which gives us

$$S_n - \epsilon \le \left|\sum_{j=1}^m p_j(1/n)\lambda_j^n\right| \le S_n + \epsilon$$

◁

E Computing real JNF in polynomial time

We discuss how to compute the the real Jordan normal form of A in polynomial time. First compute, in polynomial time, the (complex) Jordan normal form J' and matrices T, T^{-1} such that $A = TJ'T^{-1}$ using the algorithm from [5].

Computing J: Suppose, without loss of generality, that

$$J' = \mathrm{diag}(J'_1, J'_2, \ldots, J'_{2k-1}, J'_{2k}, J'_{2k+1}, \ldots, J'_{2k+z})$$

where for $1 \le j \le k$, the Jordan blocks J_{2j-1} and J_{2j} have the same dimension and have conjugate eigenvalues $\lambda_j = a_j + b_j i$ and $\overline{\lambda} = a_j - b_j i$, respectively. The blocks $J'_{2k+1}, \ldots, J'_{2k+z}$, on the other hand, have real eigenvalues. J is obtained by replacing, for each $1 \le j \le k$, $\mathrm{diag}(J'_{2j-1}, J'_{2j})$ with a real Jordan block of the same dimension with $\Lambda = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and keeping the blocks $J'_{2k+1}, \ldots, J'_{2k+z}$ unchanged.

Computing P: Let $\kappa(j)$ denote the multiplicity of the Jordan block J'_i for $1 \leq i \leq 2k+z$, and $v_1^1, \ldots, v_{\kappa(1)}^1, \ldots, v_1^{2k}, \ldots, v_{\kappa(2k)}^{2k}, \ldots, v_1^{2k+z}, \ldots, v_{\kappa(2k+z)}^{2k+z} \in \overline{\mathbb{Q}}^m$ be the columns of T. It will be the case that for all $1 \leq j \leq k$ and l, $v_l^{2j-1} = \overline{v_l^{2j}}$ in the sense that $v_l^{2j-1} = x_l^j + y_l^j i$ and $v_l^{2j} = x_l^j - y_l^j i$ for vectors $x_l^j, y_l^j \in \mathbb{R}^m$. Moreover, for $j > 2m$, $v_l^{2j} \in \mathbb{R}^m$. Finally, columns of P are obtained from columns of T as follows. For $1 \leq j \leq k$ and all l, replace v_l^{2j-1} with x_l^j and v_l^{2j} with y_l^j and keep v_l^{2k+z} for all l and $m > 0$ unchanged, in the same way the proof of existence of real Jordan normal form proceeds.

Computing P^{-1}: Summarizing the construction above, P is obtained from T by replacing columns $x+yi$ and $x-yi$, $x, y \in \mathbb{R}^m$ by x and y, respectively. Since $x = \frac{1}{2}(x+yi) + \frac{1}{2}(x-yi)$ and $y = -\frac{1}{2}i(x+yi) + \frac{1}{2}i(x-yi)$, this construction is linear and we can write $P = T \cdots A$ for some $A \in \mathbb{C}^{m \times m}$ with entries in $\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}i, -\frac{1}{2}i, 1, 0\}$. Moreover, the linear transformation is clearly invertible: $x + yi = 1 \cdot x + iy$ and $x - yi = 1 \cdot x - (-i)y$, and hence $A^{-1} \in \mathbb{C}^{m \times m}$ with entries in $\{1, i, -i\}$. Finally, compute P^{-1} via $P = TA \implies P^{-1} = A^{-1}T^{-1}$, observing that we already know how to compute T^{-1} in polynomial time.