Annals of Mathematics
Vol 63, No. 2, March, 1956
Printed in U.S.A.

THE REPRESENTATION OF RELATION ALGEBRAS, II

By Roger C. Lyndon

(Received July 30, 1954)

(Revised August 18, 1955)

Introduction

This paper completes, in a sense, the investigation undertaken in our earlier paper RRA;[1] in particular, we correct an error in that paper, which was pointed out by Professor Alfred Tarski and Mr. Dana Scott.[2]

Tarski[3] has recently shown, as a consequence of a general theorem, that, contrary to the erroneous Theorem IV of RRA, *there exists a set of axioms in the form of equations, which is necessary and sufficient for a relation algebra to be isomorphic to a proper algebra of relations.* His theorem provides no means of constructing such a set of axioms. By a refinement of the method of RRA, *an explicit set of equational axioms for representable relation algebras is here obtained.*

The chief new tool is a polarization operator which expresses that a property ϕ of elements x of $\mathfrak{A}$, a Boolean algebra with operators,[4] is *divisible*. Divisible properties are dual to local properties, and are such that ϕ holds on x only if ϕ holds on some member of every finite covering of x. By this device, assertions of the existence of certain maximal dual ideals in $\mathfrak{A}$ are expressed by universal sentences in the elementary language of $\mathfrak{A}$. A second crucial fact, implicit in much of our argument, is that the Stone-Jónsson-Tarski completion $\bar{\mathfrak{A}}$ of $\mathfrak{A}$ is compact under the topology defined by $\mathfrak{A}$.[5] For $\mathfrak{A}$ countable and simple, a countable Boolean representation of a Boolean algebra dense in $\bar{\mathfrak{A}}$ is contructed in such a way that it yields a representation of $\mathfrak{A}$ as relation algebra. The condition of simplicity is removed by a device of Tarski, and that of countability by a theorem of Henkin.[6]

In an Appendix we itemize the erroneous statements in RRA, and formulate a proper reinterpretation of the parts thus affected. In this connection we reproduce an example, due to Tarski,[2] of *a complete and atomistic algebra of sets in which not every element of a set in the algebra belongs to an atom.*

---

[1] Lyndon: [3] in the bibliography following the present paper.

[2] Private communication. Tarski recognized that Theorem IV of RRA was at fault. Scott located the dubious argument. Tarski provided the example invalidating this argument.

[3] Tarski [6], Theorem 2.4. Herein a *relation algebra* is understood as in Tarski [5], or, in the present notation, as defined by the axioms A1–5 of RRA. A *proper relation algebra* is one whose elements are relations, that is, sets of ordered pairs. The *elementary language* comprises the first order calculus with equality and the operations of relation algebra. A *universal sentence* is one whose only quantifiers are initial universal quantifiers.

[4] Jónsson-Tarski [2].

[5] Jónsson-Tarski [2], Part I, §2.

[6] Henkin [1], p. 417, second paragraph.

1. Divisible predicates

Let $\mathfrak{A}$ be a Boolean algebra with operators, or, for the present application, a relation algebra. Let $\phi(x, y, \cdots)$ be a predicate defined for variable elements $x, y, \cdots$ from $\mathfrak{A}$; we shall ordinarily write ϕ or $\phi(x)$, suppressing mention of those free variables with which we are not explicitly concerned. Predicates that are expressible in the elementary language of the algebra $\mathfrak{A}$ will be of special importance; however, we do not impose this assumption in general.

The *polars* of a predicate $\phi(x)$ with respect to x are defined as follows:

$$(\Delta^m x \mid y)\phi. \equiv (y_1, \cdots, y_m)[y \subset y_1 \cup \cdots \cup y_m . \\ \rightarrow . \phi(y_1) \vee \cdots \vee \phi(y_m)], \qquad 1 \leqq m < \omega,$$

$$(\Delta^\omega x \mid y)\phi. \equiv \bigwedge\nolimits_{1 \leqq m < \omega} (\Delta^m x \mid y), \qquad \text{an infinite conjunction.}$$

(A suitable tacit convention is assumed regarding the introduction of new variables.) We abbreviate $\phi^m(x)$ for $(\Delta^m x \mid x)\phi$, and write $\phi^0(x)$ for $\phi(x)$ itself. Observe that if $\phi(x)$ is expressible in the elementary language of $\mathfrak{A}$, then so is $\phi^m(x)$ for each finite m, but not necessarily $\phi^\omega(x)$.

The first polar $\phi^1(x)$ asserts that ϕ is increasing on x, in the sense that $x \subset y$ implies $\phi(y)$. If, for all x, $\phi(x) \Leftrightarrow \phi^1(x)$, we call ϕ *increasing in the variable* x.

The ω^{th} polar $\phi^\omega(x)$ asserts that ϕ is divisible on x in the sense that, for any finite covering $x \subset y_1 \cup \cdots \cup y_m$, some member y_k of the covering satisfies ϕ. If, for all x, $\phi(x) \Leftrightarrow \phi^\omega(x)$, we call ϕ *divisible in the variable* x.

LEMMA 1.1. (i) *If ϕ does not contain x, then $\phi^m(x) \Leftrightarrow \phi(x)$ for all m,*

$$(\phi(x) \;\&\; \psi(x))^m \Leftrightarrow . \phi^m(x) \;\&\; \psi^m(x),$$

$$(\phi(x) \rightarrow \psi(x))^m \Leftrightarrow . \phi^m(x) \rightarrow \psi^m(x);$$

(ii) *If $0 \leqq m \leqq n \leqq \omega$, then $\phi^n(x) \Rightarrow \psi^m(x)$;*

(iii) *If $\phi(x) \Rightarrow \psi(x)$, then $\phi^m(x) \Rightarrow \psi^m(x)$;*

(iv) $$(\phi^\omega)^\omega(x) \Leftrightarrow \phi^\omega(x);$$

(v) *ϕ is divisible in x, that is $\phi(x) \Leftrightarrow \phi^\omega(x)$, if and only if $\phi(x) \Leftrightarrow \phi^2(x)$.*

PROOF. (i, ii, iii) are immediate from the definitions. *For* (iv), that $\phi^{\omega\omega} \Rightarrow \phi^\omega$ follows from (i). For the converse, suppose $\phi^{\omega\omega}(x)$ failed. Then for some covering $x \subset y_1 \cup \cdots \cup y_m$, each $\phi^\omega(y_k)$ fails, and there exist coverings $y_k \subset y_{k1} \cup \cdots \cup y_{km_k}$ such that all $\phi(y_{kj})$ fail. Since the y_{kj} cover x, this contradicts $\phi^\omega(x)$. *For* (v), using (i), it remains only to show that $\phi \Rightarrow \phi^2$ implies $\phi \Rightarrow \phi^\omega$, that is, implies $\phi \Rightarrow \phi^m$ for all m. By induction on m, separating a covering $x \subset x_1 \cup \cdots \cup x_{m+1}$ into a covering $x \subset y \cup x_{m+1}$, where $y = x_1 \cup \cdots \cup x_m$, $\phi^2(x)$ gives $\phi(x_{m+1})$ or $\phi^m(y)$, hence $\phi(x_k)$ for some k.

LEMMA 1.2. *Let the set F of predicates be convergent in the sense that if ϕ_1 and ϕ_2 are any two members of the set, then there is a third ϕ such that $\phi(x) \Rightarrow \phi_1(x) \;\&\; \phi_2(x)$. Then, for all m, $0 \leqq m \leqq \omega$,*

$$\bigwedge\nolimits_F (\Delta^m x \mid y)\phi \Leftrightarrow (\Delta^m x \mid y) \bigwedge\nolimits_F \phi.$$

PROOF. Since $\bigwedge\phi \Rightarrow \phi$, the implication in one direction follows by (iii). For the converse, suppose $(\Delta^m x \mid y)\bigwedge\phi$ fails. Then $y \subset y_1 \cup \cdots \cup y_n$, $n \leqq m$, such that $\bigwedge\phi(y_k)$ fails for each y_k, and hence some $\phi_k(y_k)$ fails for each y_k. There exists then ϕ such that $\phi(x) \Rightarrow \phi_k(x)$ for all $k = 1, 2, \cdots, n$, hence $\phi(y_k)$ fails for all y_k, $(\Delta^m x \mid y)\phi$ fails, and hence $\bigwedge(\Delta^m x \mid y)\phi$ fails.

A *dual ideal* in $\mathfrak{A}$ is a subset K of $\mathfrak{A}$ with the properties that $x, y \in K \Rightarrow x \cap y \in K$ and $x \in K \Rightarrow x \cup y \in K$. A *prime* in $\mathfrak{A}$ is a maximal dual ideal.

LEMMA 1.3. *If $\phi(x)$ is divisible and holds for every x in a dual ideal $K \neq \mathfrak{A}$, then there exists a prime Z such that $\phi(z)$ holds for every z in Z.*

PROOF. Since ϕ is divisible, $\phi \Leftrightarrow \phi^2$ by (v). If N is the set of all x in $\mathfrak{A}$ for which $\phi(x)$ fails, this gives that $x, y \in N .\&. z \subset x \cup y . \Rightarrow . z \in N$, whence N is an ideal in $\mathfrak{A}$ as Boolean ring.[7] K is a multiplicative system disjoint from N, and $0 \notin K$ since $K \neq \mathfrak{A}$. If M is a ring-ideal maximal among those containing N and disjoint from K, then M is a maximal ring-ideal, and its complement Z will be a prime with the required property.

COROLLARY 1.4. *If $K \neq \mathfrak{A}$ is a dual ideal and $\phi^\omega(x)$ holds for all x in K, then there exists a prime Z such that $\phi^\omega(z)$ holds for all z in Z.*

PROOF. In Lemma 1.3, replace ϕ by ϕ^ω, which is divisible by (iv).

Henceforth we assume that $\mathfrak{A}$ is a relation algebra.[8]

LEMMA 1.5. *If Z is a prime, and K any dual ideal, then the property $\phi(x) \equiv Kx \subset Z$ is divisible.*

PROOF. Assuming $\phi(x)$, we must verify $\phi^2(x)$, in accordance with (v). If $\phi^2(x)$ failed, then $x \subset y \cup z$ where both $Ky \subset Z$ and $Kz \subset Z$ fail, hence for certain a and b in K, neither ay nor bz is in Z. For $c = a \cap b$, this gives neither cy nor cz in Z. However, $\phi(x)$ gives cx in Z, and, since $cx \subset cy \cup cz$, this contradicts the assumption that Z was prime.

COROLLARY 1.6. *If Z is a prime and $xy \in Z$, then there exist primes X and Y such that $x \in X$, $y \in Y$ and $XY \subset Z$.*

PROOF. Two applications of Lemma 1.5, first with $K = (x)$ to obtain Y, and second, in symmetrically reversed form, with $K = Y$ to obtain X.

Note that if Z is a prime, then $Z^\perp$, the set of converses of elements of Z, is also a prime.

LEMMA 1.7. *For all primes X, Y, Z,*

$$XY \subset Z^\perp \Leftrightarrow YZ \subset X^\perp \Leftrightarrow ZX \subset Y^\perp.$$

PROOF. If $XY \subset Z^\perp$, then $xyz \cap I \neq 0$ for all x in X, y in Y, z in Z. Conversely, this last implies that all xy meet all $z^\perp$ in $Z^\perp$, hence that $XY \subset Z^\perp$. The condition thus obtained is clearly invariant under cyclic permutation.

[7] Terminology and details of proof may be found in McCoy [4], Lemma 2 on p. 105. Many of the results of this section could be given more directly in terms of sets N instead of properties ϕ, but this formulation is less suited to our purpose.

[8] These special results are implicit in [2].

LEMMA 1.8. *The primes containing I constitute a set of orthogonal idempotents, in the sense that*

(i) $$XX = X,$$

(ii) $$0 \in XY \quad for \quad X \neq Y.$$

PROOF. Note that, for X a prime containing I, X contains x if and only if X contains $x \cap I$. Moreover,

$$(x \cap I)(y \cap I) = x \cap y \cap I.$$

To show that $XX \subset X$, observe that if x, y are in X, then $x \cap y \cap I = (x \cap I)(y \cap I) \subset xy$ implies xy is in X. Further, $(x \cap I)(x \cap I) = x \cap I \subset x$ implies that $X \subset XX$. If $X \neq Y$, there exist x in X and y in Y with $x \cap y = 0$, whence

$$0 = x \cap y \cap I = (x \cap I)(y \cap I) \in XY.$$

LEMMA 1.9. *For every prime Z, there exists a prime X such that I is in X and $ZX = Z$.*

PROOF. By Lemma 1.5, the property $\phi(x) \equiv .Zx \subset Z$ is divisible. Since $\phi(x)$ holds for all x in the principal dual ideal of all x containing I, it follows from Lemma 1.3 that $ZX \subset Z$ for some prime X containing I. If z is in Z, and $x \cap I$ is in X, then $z(x \cap I) \subset z$, which shows that $Z \subset ZX$.

LEMMA 1.10. *Suppose that $0 \notin ZW^{\perp}$, and that $ZX = Z$, $WY = W$, where X and Y contain I; then $X = Y$.*

PROOF. First, $Y^{\perp} = Y$, which gives $W^{\perp} = YW^{\perp}$. Then $0 \notin ZW^{\perp} = ZXYW^{\perp}$ implies $0 \notin XY$, whence by Lemma 1.8, $X = Y$.

COROLLARY 1.11. *For every prime Z, there exists a unique prime X such that I is in X and $ZX = Z$.*

PROOF. By Lemma 1.9, such X exists; by Lemma 1.10, with $W = Z$, this X is unique.

2. The axioms $\mathfrak{U}$

A set of axioms will now be defined, of which it will be shown, on the one hand, that they are equivalent to a set of universal sentences in the elementary language of relation algebra, and, on the other hand, that they are necessary and sufficient for a simple relation algebra to be representable.

These axioms will contain variables x_{jk} bearing double subscripts. We adopt the tacit convention that, always, $x_{kj} = x_{jk}^{\perp}$ and $x_{jj} = x_{jk}x_{\kappa j} \cap I$. We introduce the abbreviation

$$\delta_{hc}(a, b) = \cap x_{hj}x_{jc}$$

where j runs through the set of integers $\{1, \cdots, h - 1\} \cup \{a, b\}$.

A set of operators will be defined, transforming predicates into new predicates. Let $1 \leqq a < b < c < \omega$, and $h \in \{1, \cdots, c-1\} - \{a, b\}$. We define, for $1 \leqq m \leqq \omega$ and every predicate ϕ:

$$E^m_{hc}(a, b)\phi \equiv (\Delta^m x_{hc} \mid \delta_{hc}(a, b))\ (x_{hc} \neq 0 \cdot \& \phi),$$

$$A_2(a, b)\phi \equiv (x_{12})\ (x_{12} \neq 0 \cdot \rightarrow \phi),$$

$$A_c(a, b)\phi \equiv (x_{ac}, x_{bc})\ (x_{ab} \subset x_{ac}x_{cb} \cdot \rightarrow \phi),$$

$$W^m_c(a, b)\phi \equiv A_c(a, b)E^m_{1c}(a, b) \cdots E^m_{hc}(a, b) \cdots E^m_{c-1,c}(a, b)\phi,$$

where, in the last definition, h ranges in order through the set

$$\{1, \cdots, c-1\} - \{a, b\}.$$

Next, let $\alpha(c)$ and $\beta(c)$ be integer valued functions, defined for all $c \geqq 3$, and satisfying $1 \leqq \alpha(c) < \beta(c) < c$. We write $\delta_{hc}(\alpha, \beta)$ for $\delta_{hc}(\alpha(c), \beta(c))$, and analogously for E^m_{hc}, A_c, and W^m_c. Let T be the tautology $0 = 0$. We define, for $1 \leqq r \leqq n$,

$$W^m_{r,n}(\alpha, \beta)\phi \equiv W^m_{r+1}(\alpha, \beta) \cdots W^m_n(\alpha, \beta)\phi,$$

$$U^m_n(\alpha, \beta) \equiv W^m_{1,n}(\alpha, \beta)T.$$

It is understood that an empty product of operators, such as $W^m_{1,1}(\alpha, \beta)$, is the identity operator. We shall regard the operator A_2 as an irregular case of the operator A_c; all assertions below proved for A_c go through with trivial modifications for A_2. Finally, we shall omit the indices $m, a, b, c, h, n, \alpha, \beta$ whenever this can be done without serious ambiguity.

We define $\mathfrak{U}$ to be the set of all the sentences $U^m_n(\alpha, \beta)$ for $1 \leqq m < \omega$, and $\mathfrak{U}^\omega$ to be the set of all sentences $U^\omega_n(\alpha, \beta)$.

Example. We exemplify the sentences $\mathfrak{U}$ by giving the family of sentences, depending on the parameter m, that correspond to the condition C4 of RRA.[9] As in RRA, we employ the abbreviation $x^{a \cdots m}_{uv} = x_{ua}x_{av} \cap \cdots \cap x_{um}x_{mv}$; we distinguish indices arising from polarization by enclosing them in parentheses. In accordance with the definition, these sentences take the form

$$(x_{12})\ (x_{12} \neq 0. \rightarrow (x_{13}, x_{23})\ (x_{12} \subset x_{13}x_{32}. \rightarrow (x_{14}, x_{24})\ (x_{12} \subset x_{14}x_{42}.$$
$$\rightarrow (\Delta^m x_{34} \mid x^{12}_{34})\ (x_{31} \neq 0. \&$$
$$(x_{35}, x_{45})\ (x_{34} \subset x_{35}x_{54}. \rightarrow (\Delta^m x_{15} \mid x^{34}_{15})\ (x_{15} \neq 0. \& (\Delta^m x_{25} \mid x^{134}_{25})\ (x_{25} \neq 0))))))).$$

Proceeding from inside out, by elementary logic and the axioms of relation algebra, this sentence reduces to the following universal sentence:

$$x^{12}_{34} \subset \bigcup{}^m_1 x^{(k)}_{35} x^{(k)}_{54}. \rightarrow. x^{34}_{12} \subset \bigcup{}^m_1 x^{34(k)}_{15} x^{34(k)}_{52}.$$

A sentence, which may contain free variables, will be called *universal* if it contains no quantifiers other than an initial sequence of universal quantifiers Two sentences will be called *equivalent* if $\phi \Leftrightarrow \psi$ follows from the axioms of relation algebra.

[9] RRA, p. 712; the case for $m = 1$ of the universal sentence obtained below appears in RRA, p. 713.

THEOREM 2.1. *Every sentence in the set* $\mathfrak{U}$ *is equivalent to a universal sentence in the elementary language of relation algebra.*

PROOF. It is clear that T is a universal sentence, and that, if ϕ is a universal sentence, then $A_c\phi$ is equivalent to a universal sentence. Consider a universal sentence

$$\phi(x) \equiv (u_1, u_2, \cdots)\theta(x; u_1, u_2, \cdots)$$

where θ contains no quantifiers. Then $E\phi$ becomes, after change of variables,

$$(y_1, \cdots, y_m)[\delta \subset \textstyle\bigcup y_k. \rightarrow \bigvee_1^m(y_k \neq 0.\&(u_{k1}, u_{k2}, \cdots)\theta(y_k; u_{k1}, u_{k2}, \cdots))],$$

which is equivalent to the universal sentence

$$(y_1, \cdots, y_m; u_{11}, u_{12}, \cdots; \cdots; u_{m1}, u_{m2}, \cdots)$$
$$\cdot[\delta \subset \textstyle\bigcup y_k. \rightarrow \bigvee_1^m(y_k \neq 0.\&\theta(y_k; u_{k1}, u_{k2}, \cdots))].$$

It now follows by induction that if ϕ is equivalent to a universal sentence, then $W_c^m(a, b)\phi$ is also, and hence, by a second induction, beginning with T, that every sentence $U_n^m(\alpha, \beta)$ in $\mathfrak{U}$ is equivalent to a universal sentence.

We define $\overline{\mathfrak{U}}$ to be the set of universal sentences thus associated with the members of $\mathfrak{U}$.

LEMMA 2.2. *If F is a convergent set of predicates, in the sense of Lemma* 1.2, *then, for all the operators E, A, W,*

$$\textstyle\bigwedge_F E\phi \Leftrightarrow E\bigwedge_F\phi; \qquad \bigwedge_F A\phi \Leftrightarrow A\bigwedge_F\phi; \qquad \bigwedge_F W\phi \Leftrightarrow W\bigwedge_F\phi.$$

PROOF. From the form of the operator $E_{hc}^m(a, b)$, m finite, $\bigwedge E\phi \Leftrightarrow E\bigwedge\phi$ follows directly from Lemma 1.2, and the passage to the case $m = \omega$ is immediate. For the operators A the assertion is obvious, and the assertion for W now follows by induction.

THEOREM 2.3. *The set* $\mathfrak{U}$ *is equivalent to the set* $\mathfrak{U}^\omega$.

PROOF. For α, β, n fixed, consider the family of sentences $U^m \equiv U_n^m(\alpha, \beta)$, depending on the parameter m, $1 \leqq m \leqq \omega$. Let ϕ_m be a convergent set, with $\phi_\omega \Leftrightarrow \bigwedge\phi_m$, and let $\phi_m' \equiv A_c\phi_m$; then clearly $\phi_\omega' \Leftrightarrow \bigwedge\phi_m'$. Next, let $\phi_m' \equiv E_{hc}^m\phi_m$. (Throughout, conjunction restricted to indices $< \omega$.) Using Lemma 2.2 we have

$$\phi_\omega' \Leftrightarrow E^\omega\phi_\omega \Leftrightarrow \textstyle\bigwedge_n E^n \bigwedge_m \phi_m \Leftrightarrow \bigwedge_{n,m}\bigwedge E^n\phi_m;$$

since $E^p\phi_p$ implies $E^n\phi_m$ for $n, m \leqq p$, and is implied by $E^n\phi_m$ for $p \leqq n, m$, this gives

$$\phi_\omega' \Leftrightarrow \textstyle\bigwedge_p E^p\phi_p \Leftrightarrow \bigwedge_p\phi_p'.$$

Now an induction, beginning with T, yields $U^\omega \Leftrightarrow \bigwedge_m U^m$.

We conclude with the observation that $\delta_{hc}(a, b)$ is increasing in the sense that if we form δ' by replacing any x_{jk} by an x_{jk}' such that $x_{jk} \subset x_{jk}'$, then $\delta \subset \delta'$. It now follows easily that if ϕ is a predicate increasing in all of its variables, then $E_{hc}^m(a, b)\phi$ has the same property; the analogous remark is obvious for $A_c\phi$, and hence for the $W_{rn}^m(\alpha, \beta)\phi$ and for any predicate obtained from these by finite or infinite conjunction.

3. Necessity

THEOREM 3.1. *The sentences of the set* $\mathfrak{U}$ *are all valid in any proper relation algebra.*

PROOF. Let $\mathfrak{A}$ be a proper relation algebra, whose elements x are sets of ordered pairs pq of "points" p, q from a space Σ. We consider a fixed sentence $U = U^m_n(\alpha, \beta)$, $m < \omega$, from $\mathfrak{U}$, and will show that it is valid in $\mathfrak{A}$.

With each operator appearing in the definition of U, we associate an index set J of ordered pairs jk of integers.

Define $J_c = \{jk: j, k < c\}$, and with A_c, $c \geqq 2$, associate the set J_c. With each $E^m_{hc}(a, b)$ we associate the set

$$J_{hc}(a, b) = J_c \cup \{jc, cj: j = a, b \quad \text{or} \quad 1 \leqq j < h\}.$$

With each operator we associate also a hypothesis H: that for each jk in J an element x_{jk} has been selected in $\mathfrak{A}$, and for all j, k thus appearing points p_j, p_k have been found in Σ such that p_jp_k is in x_{jk}.

Let H be associated with an operator E yielding

$$\phi \equiv E^m_{hc}\phi' \equiv (y_1, \cdots, y_m)[\delta \subset \bigcup y_k . \rightarrow \bigvee(x_{hc} \neq 0.\&\phi'(y_k))].$$

From H we conclude that p_hp_c is in δ, and from $\delta \subset \bigcup y_k$ that p_hp_c is in some $y_k \neq 0$. Writing x_{hc} for this y_k, the implication $H \Rightarrow \phi$ evidently will follow from $H' \Rightarrow \phi'$. Again, if H is associated with an operator A yielding $\phi \equiv A_c\phi' \equiv (x_{ac}, x_{bc})\ (x_{ab} \subset x_{ac}x_{cb} . \rightarrow \phi')$, from H and $x_{ab} \subset x_{ac}x_{cb}$ we conclude that there exists p_c such that p_ap_c is in x_{ac} and p_bp_c is in x_{bc}, thus H'; then $H \Rightarrow \phi$ will follow from $H' \Rightarrow \phi'$.

If ϕ' is T, then $H' \Rightarrow \phi'$ is vacuously true; hence, by induction, we obtain $H \Rightarrow U$, which, under our conventions regarding double subscripts, reduces to $A^m_2U_3$, equivalent to U. This establishes U.

4. The conditions $\mathfrak{U}^*$

We parallel the sentences $\mathfrak{U}$ with conditions $\mathfrak{U}^*$ on the primes of $\mathfrak{A}$. These sentences will contain variables X_{jk}, which are understood to range over the primes of $\mathfrak{A}$; thus $(X_{jk})\phi$ is to be read as $(X_{jk})\ (X_{jk}$ a prime. $\rightarrow \phi)$, and $(\exists X_{jk})\phi$ as $(\exists X_{jk})\ (X_{jk}$ a prime. $\&\ \phi)$. We adopt the convention that $X_{kj} = X^{\smile}_{jk}$, and that X_{kk} is that prime, whose existence and uniqueness is provided by Lemma 1.8, such that $I \in X_{kk}$ and $X_{jk}X_{kk} = X_{jk}$. With the same conditions on the indices as in §2, we define

$$\delta^*_{hc}(a, b) \equiv \bigcap X_{hj}X_{jc}$$

$$E^*_{hc}(a, b)\phi \equiv (\exists X_{hc})\ (\delta^*_{hc}(a, b) \subset X_{hc} .\ \&\ \phi)$$

$$A^*_2(a, b)\phi \equiv (X_{12})\phi$$

$$A^*_c(a, b)\phi \equiv (X_{ac}, X_{bc})\ (X_{ac}X_{cb} \subset X_{ac} . \rightarrow \phi)$$

$$W^*_c(a, b)\phi \equiv A^*_c(a, b)E^*_{1c}(a, b) \cdots E^*_{c-1,c}(a, b)\phi$$

$$W^*_{r,n}(\alpha, \beta)\phi \equiv W^*_{r+1}(\alpha, \beta) \cdots W^*_n(\alpha, \beta)\phi.$$

For each n, define

$$\Theta_n \equiv \bigwedge W^{\omega}_{n,m}(\alpha, \beta)T,$$

the conjunction ranging over all α, β and all $m > n$. Define operators

$$\Omega_n\phi \equiv \bigwedge\nolimits_{1 \leqq a<b<n} W^{\omega}_{n}(a, b)\phi$$

$$\Omega_{r,n}\phi \equiv \Omega_{r+1} \cdots \Omega_n\phi.$$

The definitions of Θ^*_n, Ω^*_n, and $\Omega^*_{r,n}$ are exactly parallel.

LEMMA 4.1. *Let n and α', β', $m' > n$, α'', β'', $m'' > n$ be given. Then there exist α, β, and m such that*

$$W^{\omega}_{n,m}(\alpha, \beta)T \Rightarrow. W^{\omega}_{n,m'}(\alpha', \beta')T \;\&\; W^{\omega}_{n,m''}(\alpha'', \beta'')T.$$

PROOF. Take $m = m' + m'' - n$, and $\alpha(c) = \alpha'(c)$ for $1 \leqq c \leqq m'$, $\alpha(c) = \alpha''(c - m' + n)$ for $m' < c \leqq m$, with $\beta(c)$ analogous. Then $W \equiv W^{\omega}_{n,m}(\alpha, \beta)T$ is of the form $W^{\omega}_{n,m'}(\alpha', \beta')S$. Since $S \Rightarrow T$, $W \Rightarrow W'$. Let R be obtained from W'' by replacing every index j, for which $n < j$, by $j + m'$. Then S will differ from R only in that, first, S contains operators E_{hc}, $n < c \leqq m'$, that are absent from R, and, second, certain δ_{hc} of R are replaced in S by $\delta_{hc} \cap \theta_{hc}$ where $\theta_{hc} = \bigcap x_{hj}x_{jc}$, $n < j \leqq m$. If S' is obtained by setting $x_{jk} = V$ for $n < j \leqq m$, it follows that R, and so W'', is a consequence of S'. Finally, since $E\phi(x) \Rightarrow \phi(V)$ and $A\phi(x) \Rightarrow \phi(V)$, it results that $W \equiv W^{\omega}_{n,m'}(\alpha', \beta')S$ implies S', and hence W''.

LEMMA 4.2. $\Theta_{n-1} \Leftrightarrow \Omega_n\Theta_n$.

PROOF. First, for all $1 \leqq a < b < n$,

$$W^{\omega}_{n}(a, b)\Theta_n \Leftrightarrow A_nE_3 \cdots E_{n-1} \bigwedge W^{\omega}_{n,m}(\alpha', \beta')T.$$

By Lemma 4.1, the sentences $W^{\omega}_{n,m}(\alpha', \beta')T$ form a convergent set, whence by Lemma 2.2 we obtain

$$W^{\omega}_{n}(a, b)\Theta_n \Leftrightarrow \bigwedge W^{\omega}_{n-1,m}(\alpha, \beta)T$$

where the conjunction is over all $m \geqq n$ and all α, β such that $\alpha(n) = a$, $\beta(n) = b$. Admitting $m = n - 1$ involves only adjoining trivial terms T to the conjunction. Taking the conjunction of the last equivalence over all a, b satisfying $1 \leqq a < b < n$ gives now

$$\Omega_n\Theta_n \Leftrightarrow \bigwedge_{\substack{1 \leqq a<b<n \\ \alpha(n)=a, \beta(n)=b}} W^{\omega}_{n-1,m}(\alpha, \beta)T \Leftrightarrow \Theta_{n-1}.$$

If J is an index set, associated with some operator in accordance with §3, we denote by x_J a set $\{x_{jk}\}$ of elements of $\mathfrak{A}$ indexed by the pairs jk in J; similarly, X_J denotes a set of primes indexed by J. We write $x_J \subset x'_J$ if each $x_{jk} \subset x'_{jk}$; $x_J \cap x'_J$ for the set $\{x_{jk} \cap x'_{jk}\}$; and $x_J \in X_J$ if each $x_{jk} \in X_{jk}$.

LEMMA 4.3. *Let x be one of the set of variables x_J, let $\psi(x_J, y, z)$ be increasing in x_J, and $\phi \equiv (y, z)(x \subset yz. \to \psi)$. Let*

$$\phi^* \equiv (Y, Z)(YZ \subset X. \;\&\; .x_J \in X_J,\, y \in Y,\, z \in Z. \to \psi).$$

Then $x_J \in X_J \Rightarrow \phi$ implies ϕ^*.

PROOF. Assume that $x_J \in X_J \Rightarrow \phi$, that $YZ \subset X$, and that $x_J \in X_J$, $y \in Y$ and $z \in Z$. If x is the component of x_J at X, then x and yz are in X, hence $x' = x \cap yz \in X$. Form $x_J' \in X_J$ by replacing x by x'. Then $x' \subset yz$, and ϕ gives

$$\psi(x_J', y, z).$$

Since $x_J \subset x_J'$, and ψ is increasing, this gives $\psi(x_J, y, z)$ as required.

LEMMA 4.4. *Let $\psi(x_J, y)$ be increasing in x_J, let $\delta(x_J)$ be increasing in x_J, and let $\phi \equiv (\Delta^\omega y \mid \delta)(y \neq 0. \,\&\, \psi)$. Let*

$$\phi^* \equiv (\exists Y)(\delta \in Y\colon \& \,{:}x_J \in X_J, y \in Y. \rightarrow \psi).$$

Then $x_J \in X_J \Rightarrow \phi$ implies ϕ^*.

PROOF. Since δ is increasing, x_J and x_J' in X_J implies that $x_J \cap X_J' \in X_J$ and $\delta(x_J \cap x_J') \subset \delta(x_J) \cap \delta(x_J')$. Therefore the set K of all z such that $\delta(x_J) \subset z$ for some $x_J \in X_J$ is a dual ideal. Assuming that $x_J \in X_J \Rightarrow \phi$, from

$$(\Delta^1 y \mid \delta)(y \neq 0. \,\&\, \psi)$$

we conclude that $\delta \subset y$ implies $y \neq 0$ and $\psi(x_J, y)$. The first gives $K \neq \mathfrak{A}$. Second, let $x_J \in X_J$ and z in K, hence $\delta(x_J') \subset z$ for some x_J' in X_J. For $x_J'' = x_J \cap x_J' \in X_J$, we have $\delta(x_J'') \subset \delta(x_J') \subset z$, hence, by the above, $\psi(x_J'', z)$. Since ψ is increasing in x_J, this gives $\psi(x_J, z)$. Thus we have shown that $K \neq \mathfrak{A}$ is a dual ideal, and that every z in K has the property $\gamma(z)$ that $\psi(x_J, z)$ holds for all $x_J \in X_J$.

We show that $\gamma^\omega(z)$ holds for all z in K. Suppose $z \in K$, with $\delta(x_{J0}) \subset z$, $x_{J0} \in X_J$, and yet $\gamma^\omega(z)$ failed. Then $z \subset y_1 \cup \cdots \cup y_m$ such that $\gamma(y_k)$ fails for each y_k. Then there exist $x_{J1}, \cdots, x_{Jm}$ in X_J such that each $\psi(x_{Jk}, y_k)$ fails. Taking $x_J = x_{J0} \cap x_{J1} \cap \cdots \cap x_{Jm} \in X_J$, each $\psi(x_J, y_k)$ fails. Since $\delta(x_J) \subset \delta(x_{J0}) \subset z \subset y_1 \cup \cdots \cup y_m$, this contradicts ϕ.

It now follows by Lemma 1.3 that there exists a prime Y containing K and such that $\gamma^\omega(y)$, and therefore $\gamma(y)$, holds for all y in Y. Since $\delta \in K \subset Y$, this establishes ϕ^*.

COROLLARY 4.5. *Let ϕ be increasing in all its arguments.*

If (i): $$x_{J_n} \in X_{J_n}. \Rightarrow W_n^\omega(a, b)\phi,$$

then (ii): $$W_*^n(a, b)[x_{J_{n+1}} \in X_{J_{n+1}}. \Rightarrow \phi].$$

COROLLARY 4.6. *Let ϕ be increasing in all its arguments.*

If (iii): $$x_{J_n} \in X_{J_n}. \Rightarrow \Omega_n \phi,$$

then (iv): $$\Omega_n^*[x_{J_{n+1}} \in X_{J_{n+1}}. \Rightarrow \phi].$$

PROOF OF 4.6. Since Ω_n is a conjunction of conditions $W_n^\omega(a, b)$, (iii) implies a conjunction of implications (i) and hence (ii), which is equivalent to (iv).

For each $n < \omega$ define

$$\tilde{\Theta}_n \equiv [x_{J_n} \in X_{J_n}. \rightarrow \Theta_n];$$

for $n = 1$ the hypothesis, $x_{11} \in X_{11}$, is trivial, whence $\tilde{\Theta}_1$ is equivalent to Θ_1, the conjunction of the conditions in $\mathfrak{U}^\omega$. Thus, by virtue of Theorem 2.3,

LEMMA 4.7. *The conditions* $\mathfrak{U}$ *imply the condition* $\tilde{\Theta}_1$.

LEMMA 4.8. $\tilde{\Theta}_{n-1} \Rightarrow \Omega_n^* \tilde{\Theta}_n$.

PROOF.

$$\begin{aligned}\tilde{\Theta}_{n-1} &\Leftrightarrow [X_{J_{n-1}} \in X_{J_{n-1}} \rightarrow \Omega_n \Theta_n] \\ &\Leftrightarrow [x_{J_{n-1}} \in X_{J_{n-1}} \rightarrow \textstyle\bigwedge_{1 \leqq a < h < n} W_n^\omega(a, b)\Theta_n] \\ &\Leftrightarrow \textstyle\bigwedge_{1 \leqq a < h < n} [x_{J_{n-1}} \in X_{J_{n-1}} \rightarrow W_n^\omega(a, b)\Theta_n].\end{aligned}$$

Applying Corollary 4.6 to the expression in brackets gives

$$\begin{aligned}\tilde{\Theta}_{n-1} &\Rightarrow \textstyle\bigwedge_{1 \leqq a < h < n} W_n^*(a, b)[x_{J_n} \in X_{J_n} \rightarrow \Theta_n] \\ &\Leftrightarrow \Omega_n^* \tilde{\Theta}_n .\end{aligned}$$

5. Sufficiency

Let $\alpha(c, X_{J_c})$ and $\beta(c, X_{J_c})$ be functions depending, for each c, on the choice of a set of primes X_{jk} for all $j, k < c$, and satisfying $1 \leqq \alpha(c, X_{J_c}) < \beta(c, X_{J_c}) < c$. Let π, ρ be functions of the same arguments whose values are primes:

$$\pi(c, X_{J_c}) = \bar{X}_{\alpha(c)c}, \qquad \rho(c, X_{J_c}) = \bar{X}_{\beta(c)c}.$$

We assume that α, β, π, ρ are so related that, for each c, if

$$(P_c) \qquad X_{ij}X_{jk} \subset X_{ik} \qquad \text{for all } i, j, k < c,$$

then

$$(Q_c) \qquad \bar{X}_{\alpha(c)c}\bar{X}_{c\beta(c)} \subset X_{\alpha(c)\beta(c)} .$$

Given α, β, π, ρ, and a prime $\bar{X}_{12}$, we are interested in the existence of functions σ_n assuming values $\sigma_n(jk) = X_{jk}$ for all $j, k < n < \omega$, such that P_c holds for all $c < n$, and further, for all $c < n$

$$(R_c) \qquad X_{12} = \bar{X}_{12}, \qquad X_{\alpha(c)c} = \bar{X}_{\alpha(c)c}, \qquad X_{\beta(c)c} = \bar{X}_{\beta(c)c} .$$

LEMMA 5.1. *Let* α, β, π, ρ, *and* $\bar{X}_{12}$ *be given such that* P_c *implies* Q_c *for all* $c < \omega$. *Then the axioms* $\mathfrak{U}$ *imply the existence of* σ_ω *whose values* $\sigma(jk) = X_{jk}$ *satisfy* P_c *and* R_c *for all* $c < \omega$.

PROOF. Suppose σ_n given with values satisfying P_n, and that this set of values, X_{J_n}, satisfies $\Omega_n^*\phi$. Expanding, this asserts that for all a, b, satisfying $1 \leqq a < b < n$, and for all X_{an}, X_{bn} satisfying $X_{an}X_{nb} \subset X_{ab}$,

$$(\exists X_{1n})(\delta_{1n}^* \subset X_{1n}. \;\&\; \cdots \;\&\; (\exists X_{n-1,n})(\delta_{n-1,n}^* \subset X_{n-1,n} \;\&\; \phi) \;\cdots\;).$$

Taking

$$a = \alpha(n), \qquad b = \beta(n), \qquad X_{an} = \bar{X}_{\alpha(c)c}, \qquad X_{bn} = \bar{X}_{\beta(c)c},$$

this implies the possibility of extending σ_n to a σ_{n+1} by choosing values

$$X_{1n}, \cdots, X_{n-1,n}$$

which satisfy P_{n+1} by virtue of the clauses $\delta^*_{hn} \subset X_{hn}$, and such that the new set of values, $X_{J_{n+1}}$, will satisfy ϕ.

We now apply this result inductively. The axioms $\mathfrak{U}$ imply $\tilde{\Theta}_1$ by Lemma 4.7. By Lemma 4.8, this implies $\Omega^*_2\tilde{\Theta}$, which asserts the possibility of choosing σ_3 such that X_{J_3} satisfies $\tilde{\Theta}_2$. Inductively, suppose σ_n determined such that P_c and R_c hold for all $c < n$, and such that X_{J_n} satisfies $\tilde{\Theta}_{n-1}$. By Lemma 4.7, this implies that X_{J_n} satisfies $\Omega^*_n\tilde{\Theta}_n$, hence, by the argument above, σ_n can be extended to σ_{n+1} satisfying P_c and R_c for all $c < n + 1$, and such that $X_{J_{n+1}}$ satisfies $\tilde{\Theta}_n$.

This establishes the existence of a sequence of functions σ_n for all $3 \leqq n < \omega$, such that σ_n is an extension of σ_m whenever $m \leqq n$, and all satisfying the conditions P_c and R_c. It follows that the union σ_ω of these functions σ_n fulfils the requirements of the lemma.

Now let $\mathfrak{A}$ be a simple relation algebra satisfying the axioms $\mathfrak{U}$. We assume that $\mathfrak{A}$ is countable, either finite or infinite, and that the elements of $\mathfrak{A}$ are indexed by the positive integers: $x_1, x_2, \cdots$. Apart from the trivial case $I = V$, we choose $\bar{X}_{12}$ as any prime of $\mathfrak{A}$ not containing I.

We next define functions α, β, π, ρ such that, for all c, P_c implies Q_c. For any c, suppose then that X_{jk} are given, $j, k < c$, satisfying P_c. If, for all a, b, p, q such that

(i) $1 \leqq a < b < c, \qquad x_p \cap I = 0, \qquad x_q \cap I = 0, \qquad x_px_q \in X_{ab}$

(ii) there exists an $h < c$ such that $x_p \in X_{ah}$, $x_q \in X_{hb}$,

we choose $\alpha(c)$, $\beta(c)$, $\bar{X}_{\alpha(c)c}$, $\bar{X}_{\beta(c)c}$ arbitrarily, subject only to Q_c. Otherwise there exists a quadruple a, b, p, q satisfying (i) for which (ii) fails; choosing one for which $a + b + p + q$ is a minimum, we define $\alpha(c) = a$, $\beta(c) = b$, and, in accordance with Corollary 1.6, $\bar{X}_{ac}$, X_{bc} such that $x_p \in \bar{X}_{ac}$, $x_q \in \bar{X}_{cb}$ and

$$\bar{X}_{ac}\bar{X}_{cb} \subset X_{ab}.$$

It follows by Lemma 5.1 that there exists a function $\sigma(jk) = X_{jk}$ defined for all $1 \leqq j, k < \omega$ satisfying P_c and R_c for all c. In view of the construction of α, β, π, ρ, we have that σ satisfies the two conditions

$$(P) \qquad X_{ij}X_{jk} \subset X_{ik} \qquad \text{for all } i, j, k < \omega$$

(S) if $xy \in X_{ab}$, for any $a, b < \omega$, there exists $c < \omega$ such that $x \in X_{ac}$, and $y \in X_{cb}$.

LEMMA 5.2. *If $\mathfrak{A}$ is a countable simple relation algebra satisfying the axioms $\mathfrak{U}$, then $\mathfrak{A}$ is isomorphic to a proper algebra of relations.*

PROOF. Let Σ be the set of positive integers, $\Sigma \times \Sigma$ the set of all ordered pairs of positive integers, and $\mathfrak{B}$ the algebra of all subsets of $\Sigma \times \Sigma$. We use the function σ constructed above, with properties P and S, to define an isomorphism of $\mathfrak{A}$ onto a subalgebra of $\mathfrak{B}$. For each x in $\mathfrak{A}$, let $r(x)$ in $\mathfrak{B}$ be defined by setting

$$r(x) = \{jk: x \in \sigma(jk)\}.$$

It is immediate that r defines a homomorphism of $\mathfrak{A}$ as Boolean algebra onto a subset of $\mathfrak{B}$. From the conventions governing the X_{jk} it is immediate that $r(x^{\perp}) = r(x)^{\perp}$, and $r(I) = \{jj\}$, the identity element of $\mathfrak{B}$. To verify that $r(x)r(y) \subset r(xy)$, suppose that $ij \in r(x)$, $jk \in r(y)$; then $x \in X_{ij}$, $y \in X_{jk}$, and, by P, $xy \in X_{ij}X_{jk} \subset X_{ik}$, whence $ik \in r(xy)$. To verify that, conversely, $r(xy) \subset r(x)r(y)$, suppose that $ik \in r(xy)$, whence $xy \in X_{ik}$; by S there exists j such that $x \in X_{ij}$, $y \in X_{jk}$, whence $ij \in r(x)$, $jk \in r(y)$, and thus $ik \in r(x)r(y)$. This establishes that r defines a homomorphism of $\mathfrak{A}$ as relation algebra into a subalgebra of $\mathfrak{B}$; since $\mathfrak{A}$ is simple and r is clearly non-trivial, r is in fact an isomorphism.

To remove the conditions of countability and simplicity, contained in Lemma 5.2, we follow a line of reasoning due to Tarski[10]. Let S be any universal sentence in the elementary language of relation algebra; with S we correlate an equation E, obtained from S by successively replacing parts

$$\begin{array}{lll} x = y & \text{by} & x\bar{y} \cup \bar{x}y = 0 \\ x = 0 \;\&\; y = 0 & \text{by} & x \cup y \quad\;\;\, = 0 \\ x \neq 0 & \text{by} & \overline{VxV} \quad\;\, = 0. \end{array}$$

Then S and E are equivalent for simple algebras. By Theorem 2.1, we were able to replace the set $\mathfrak{U}$ of axioms by an equivalent set $\overline{\mathfrak{U}}$ of universal sentences; we now define $\mathfrak{E}$ to be the set of equations correlated by the above scheme with the sentences of the set $\overline{\mathfrak{U}}$.

LEMMA 5.3. *If a relation algebra $\mathfrak{A}$ is isomorphic to a proper algebra of relations, then $\mathfrak{A}$ satisfies the equations $\mathfrak{E}$.*

PROOF. We may suppose $\mathfrak{A}$ itself is an algebra of relations over some domain D. The universal relation V of $\mathfrak{A}$ is then an equivalence relation over the domain D, and as such partitions D into classes D_ν. The restriction of elements of $\mathfrak{A}$ to the domain $D_\nu : \rho_\nu(x) = x \cap (D_\nu \times D_\nu)$, defines a homomorphism of $\mathfrak{A}$ onto a simple algebra $\mathfrak{A}_\nu$ of relations over the domain D_ν. It follows that $\mathfrak{A}$ is isomorphic to a subdirect sum of the simple algebras $\mathfrak{A}_\nu$. By Theorem 3.1, each $\mathfrak{A}_\nu$ satisfies the conditions $\mathfrak{U}$, and therefore, since the $\mathfrak{A}_\nu$ are simple, the conditions $\mathfrak{E}$. Since the $\mathfrak{E}$ are equations, it follows that the subdirect sum $\mathfrak{A}$ of the $\mathfrak{A}_\nu$ satisfies the conditions $\mathfrak{E}$.

LEMMA 5.4. *If a countable relation algebra $\mathfrak{A}$ satisfies the equations $\mathfrak{E}$, then $\mathfrak{A}$ is isomorphic to a proper algebra of relations.*

PROOF. The lattice of ideals J in the relation algebra $\mathfrak{A}$ is isomorphic to the lattice of ideals K in the Boolean algebra $V\mathfrak{A}V$, under the correspondence $J \rightarrow K = VJV$, $K \rightarrow J = [K]$. It follows that the intersection of all maximal ideals in $\mathfrak{A}$ is 0, whence $\mathfrak{A}$ is isomorphic to a subdirect sum of simple algebras $\mathfrak{A}_\nu$. Since $\mathfrak{A}$ is countable and satisfies the equations $\mathfrak{E}$, each $\mathfrak{A}_\nu$, as a homomorphic image of $\mathfrak{A}$, is countable and satisfies the equations $\mathfrak{E}$. Since each $\mathfrak{A}_\nu$ is representable, by Lemma 5.2, the subdirect sum $\mathfrak{A}$ is representable.

[10] Tarski [6], Theorem 2.4. The first step of the argument that follows was given in Tarski [5]. It should be noted that for a non-simple proper relation algebra it cannot be required that the universal element V contain all ordered couples of elements from the domain.

LEMMA 5.5. *If a relation algebra $\mathfrak{A}$ satisfies the equations $\mathfrak{E}$, then $\mathfrak{A}$ is isomorphic to a proper algebra of relations.*

PROOF. A result of Henkin[6] asserts, for the case at hand: If $\mathfrak{Q}$, $\mathfrak{R}$ are arithmetical classes, and every finitely generated algebra in $\mathfrak{Q}$ can be embedded isomorphically in an algebra in $\mathfrak{R}$, then every algebra in $\mathfrak{Q}$ can be so embedded. To apply this, let $\mathfrak{Q}$ be the class of relation algebras satisfying the equations $\mathfrak{E}$, and let $\mathfrak{R}$ be the set of algebras isomorphic to proper atomistic algebras of relations in which each atom is a single ordered couple; the set $\mathfrak{R}$ is an arithmetical class, by a result of Jónsson and Tarski.[11] Every finitely generated algebra in $\mathfrak{Q}$ is countable, hence by Lemma 5.4 is isomorphic to a subalgebra of an algebra in $\mathfrak{R}$; it follows then that every algebra in $\mathfrak{Q}$ can be embedded in an algebra in $\mathfrak{R}$, and hence is isomorphic to a proper algebra of relations.

Combining Lemmas 5.3 with 5.5 gives our final result.

THEOREM 5.6. *A relation algebra $\mathfrak{A}$ is isomorphic to a proper algebra of relations if and only if it satisfies the set $\mathfrak{E}$ of equations.*

Appendix

We begin with Tarski's complete atomistic Boolean algebra of sets, in which not every element of one of the sets belongs to an atom.[2] Let $\mathfrak{A}$ be the Boolean algebra of all subsets of an infinite set S. Since S is infinite, $\mathfrak{A}$ contains a prime ideal $\mathfrak{p}$ which is not principal. Form S' by adjoining to S some element p not already in S. Let $\mathfrak{A}'$ consist of all the elements x of $\mathfrak{A}$ such that x is in $\mathfrak{p}$, together with all elements $x \cup \{p\}$ for x not in $\mathfrak{p}$. $\mathfrak{A}'$ is now easily seen to be a Boolean algebra of sets with the desired properties.

The crucial error in RRA occurs on p. 713, §6, in the sentence beginning "But then A . . .". The argument here is that the ordered couples $p_j p_n$ must belong to some atoms X_{jn}, an argument invalidated by Tarksi's example.

Part I of RRA, insofar as it deals with finite algebras (essentially, omitting §6), is unaffected. Theorem II stands, although superseded by the present paper, and Theorem III, exhibiting a non-representable finite algebra, retains its significance. Before passing to a reinterpretation of Part II, we list the minimum changes necessary to correct RRA.

p. 713: THEOREM I: *for* "complete relation(al) algebra" *read* "finite relation algebra".
p. 723 LEMMA: *for* "C5" *read* "C4";
— COROLLARY: *delete entire corollary.*
p. 726 First COROLLARY, and LEMMA: *delete* "unrepresentable";
— Second COROLLARY: *for* "C5" *read* "C4", and *delete* "which is necessary . . . representable";
— THEOREM IV: *delete entire theorem.*
p. 727 First three items of §15 become erroneous or irrelevant: *delete them.*
p. 729 Footnote 19, first paragraph: *for* "contrast between the results" *read* "similarity between the aims";
— Same footnote, second paragraph: *delete entirely.*

[11] Jónsson-Tarski [2], Part II, Theorem 4.31.

To restore its proper significance to the reasoning of Part II of RRA, we follow Scott[12] in defining a relation algebra to be *strongly representable* if it is isomorphic to a proper algebra of relations in which the universal element is the set-theoretic union of atoms. With this modification, the reasoning of §6 and Part II goes through, and establishes, in place of Theorem IV, the result that: *the class of strongly representable atomistic relation algebras is not definable by universal sentences*. An important application of this is to relation algebras that are complete as Boolean algebras, where we do not obtain any axioms ensuring that the operations of infinite union and intersection will, in a representation, receive their natural set-theoretical interpretation. Indeed, it does not seem assured that a representation of $\mathfrak{A}$ can always be extended to a strong representation of its Stone-Jónsson-Tarski completion $\bar{\mathfrak{A}}$.

We cannot conclude without remarking on the status of what is perhaps the most important outstanding problem in the theory of relation algebra: when an algebra $\mathfrak{A}$, satisfying the condition $\mathfrak{J}$ of having no proper zero-divisors, is isomorphic to an algebra whose elements are subsets (Frobenius complexes) of a group. Tarski[6] is able to show that these "complex algebras" are definable by universal sentences, or indeed by equations together with the condition for simplicity, or with $\mathfrak{J}$, which implies simplicity. But it is not known whether every representable algebra (that is, satisfying the conditions $\bar{\mathfrak{U}}$) which also satisfies $\mathfrak{J}$ is isomorphic to a complex algebra.[13]

UNIVERSITY OF MICHIGAN

BIBLIOGRAPHY

1. L. HENKIN, *Some interconnections between modern algebra and mathematical logic*, Trans. Amer. Math. Soc., vol. 74 (1953), pp. 410–427.
2. B. JÓNSSON and A. TARSKI, *Boolean algebras with operators*, I, II, Amer. J. Math., vol. 73 (1951), pp. 891–939; vol. 74 (1952), pp. 127–162.
3. R. C. LYNDON, *The representation of relational algebras*, Ann. of Math., vol. 51 (1950), pp. 707–729.
4. N. H. MCCOY, Rings and Ideals, Carus Mathematical Monographs, Buffalo, 1947.
5. A. TARSKI, *On the calculus of relations*, J. Symbolic Logic, vol. 6 (1941), pp. 73–89.
6. A. TARSKI, *Contributions to the theory of models*, Proc. Koninkl. Nederl. Akad. van Wetensch., Ser. A, vol. 57 (1954), pp. 572–588; vol. 58 (1955), pp. 56–64.

[12] The present formulation was suggested by Scott, who has also observed that, by way of contrast, every atomistic Boolean algebra is strongly representable.

[13] We have obtained the following very limited experimental confirmation of this conjecture. Every relation algebra without zero divisors that is of order not exceeding 8 (there are 13 such) is commutative and isomorphic to a complex algebra of either the additive rationals or a cyclic group of order not exceeding 13.