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Overt choice

Matthew de Brecht
Graduate School of Human and Environmental Studies, Kyoto University, Japan
matthew@i.h.kyoto-u.ac.jp

Arno Pauly
Department of Computer Science, Swansea University, UK
Arno.M.Pauly@gmail.com

Matthias Schröder
Department of Mathematics, Technical University Darmstadt, Germany
Matthias.Schroeder@cca-net.de

Abstract. We introduce and study the notion of *overt choice* for countably-based spaces and for CoPolish spaces. Overt choice is the task of producing a point in a closed set specified by what open sets intersect it. We show that the question of whether overt choice is continuous for a given space is related to topological completeness notions such as the Choquet-property; and to whether variants of Michael's selection theorem hold for that space. For spaces where overt choice is discontinuous it is interesting to explore the resulting Weihrauch degrees, which in turn are related to whether or not the space is Fréchet–Urysohn.

Keywords: Weihrauch reducibility, computable topology, quasi-Polish space

1. Introduction

Let us assume that we have the ability to recognize, given an open predicate, that there exists a solution satisfying that predicate. Under which conditions does this suffice to actually obtain a solution? This idea is formalized in the notion of *overt choice*. In this paper, we investigate overt choice under the assumption that the set of solutions is a topologically closed set, although we will often omit the word "closed" for simplicity.

On the one hand, studying overt choice is a contribution to (computable) topology. Overt choice for a given space being computable/continuous is a completeness notion, which in the metric case coincides with being Polish. Understanding in more generality for what spaces overt choice is continuous will aid us in extending results and constructions from Polish spaces to more general classes of spaces. On the other hand, the degrees of non-computability of choice principles have turned out to be an extremely useful scaffolding structure in the Weihrauch lattice. Studying the Weihrauch degrees of overt choice in spaces where this is not computable reveals more about hitherto unexplored regions of the Weihrauch lattice.

Overtness is the often overlooked dual notion to compactness. A subset of a space is overt, if the set of open subsets intersecting it is itself an open subset of the corresponding hyperspace. Equivalently, if existential quantification over the set preserves open predicates. Since in classical topology, arbitrary unions of open sets are open, overtness becomes trivial. In constructive or synthetic topology, however, it is a core concept. Even classically, though, we can make sense of the space $\mathcal{V}(\mathbf{X})$ of (topologically closed) overt subsets of a given space $\mathbf{X}$. The space $\mathcal{V}(\mathbf{X})$ is isomorphic to the hyperspace of closed sets with the positive information topology (equivalently, the sequentialization of the lower Vietoris or lower Fell topology). Overt choice for $\mathbf{X}$, which we denote $\mathrm{VC}_{\mathbf{X}}$, is just the task of producing an element of a given non-empty set $A \in \mathcal{V}(\mathbf{X})$.

Since the lower-semicontinuous closed-valued functions into $\mathbf{X}$ are equivalent to the continuous functions into $\mathcal{V}(\mathbf{X})$, the continuity of $\mathrm{VC}_{\mathbf{X}}$ gives rise to variants of Michael's selection theorem [23]. We can thus view the question of what spaces make overt choice continuous as asking about for which spaces Michael's selection theorem holds.

Our contributions. We generalize the known result that overt choice is computable for computable Polish spaces to computable quasi-Polish spaces (Theorem 20). Since the latter notion is not yet fully established, we first investigate a few candidate definitions for effectivizing the notion of a quasi-Polish space, and show that the candidate definitions fall into two equivalence classes, which we then dub *precomputably quasi-Polish* (Definition 14) and *computably quasi-Polish* (Definition 17).

As a partial converse, we show that for countably-based T_1-spaces the continuity of overt choice is equivalent to being quasi-Polish (Corollary 23). In Section 5 we then explore overt choice for several canonic examples of countably-based yet not quasi-Polish spaces, and study the Weihrauch degrees of overt choice for these.

Besides countably-based spaces, we also investigate CoPolish spaces (Section 6). We see that overt choice is continuous for a CoPolish space iff that space is actually countably-based. Moreover, the topological Weihrauch degree of overt choice on a CoPolish space is always comparable with LPO, and whether it is above or strictly below LPO tells us whether the space has the Fréchet–Urysohn property. These results are summarized in Corollary 69.

2. Background on represented spaces and Weihrauch degrees

2.1. Represented spaces and synthetic topology

The formal setting for our investigation will be the category of represented spaces [25], which is commonly used in computable analysis. It constitutes a model for synthetic topology in the sense of Escardó [15]. We will contend ourselves with giving a very brief account of the essential notions for our purposes, and refer to [25] for more details and context. In the area of computable analysis, most of the following were first obtained in [26].

Definition 1. A *represented space* is a pair (X, δ) of a set X and a partial surjection $\delta :\subseteq \mathbb{N}^\mathbb{N} \to X$. We commonly write $\mathbf{X}$, $\mathbf{Y}$, etc, for represented spaces (X, δ_X), (Y, δ_Y). A (multivalued) function between represented spaces is a (multivalued) function between the underlying sets. A partial function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a *realizer* of $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ (written $F \vdash f$) if $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \operatorname{dom}(f \circ \delta_X)$. We call $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ *computable* respectively *continuous* if it has a computable respectively continuous realizer.

By the grace of the UTM-theorem, the category of represented spaces and continuous functions is cartesian-closed, i.e. we have a function space construction – that even makes all the expected operations computable. We denote the space of continuous functions from $\mathbf{X}$ to $\mathbf{Y}$ by $\mathcal{C}(\mathbf{X}, \mathbf{Y})$. A special represented space of significant relevance is Sierpiński space Σ, having the two elements $\top$ and $\bot$ and represented via $\delta_\Sigma : \mathbb{N}^\mathbb{N} \to \{\top, \bot\}$ where $\delta_\Sigma^{-1}(\{\bot\}) = \{0^\omega\}$.

We obtain a space $\mathcal{O}(\mathbf{X})$ of subsets of a given space $\mathbf{X}$ by identifying $U \subseteq \mathbf{X}$ with its characteristic function $\chi_U : \mathbf{X} \to \Sigma$. The elements of $\mathcal{O}(\mathbf{X})$ are called *open* sets, which is justified in particular by noting that $\mathcal{O}(\mathbf{X})$ is the final topology induced along $\delta_\mathbf{X}$ by the subspace topology on $\operatorname{dom}(\delta_\mathbf{X})$. The space $\mathcal{A}(\mathbf{X})$ of closed subsets is obtained by identifying a set $A \in \mathcal{A}(\mathbf{X})$ with its complement $(X \setminus A) \in \mathcal{O}(\mathbf{X})$.

The space $\mathcal{O}(\mathbb{N})$ has a particularly nice characterization: Its elements are all subsets of $\mathbb{N}$, and they are represented as enumerations. The topology on $\mathcal{O}(\mathbb{N})$ is thus the Scott topology. We then define the notion of an effective countable basis:

Definition 2. $\mathbf{X}$ is effectively countably-based if there is some computable function $B : \mathbb{N} \to \mathcal{O}(\mathbf{X})$ such that the computable function $U \mapsto \bigcup_{n\in U} B(n) : \mathcal{O}(\mathbb{N}) \to \mathcal{O}(\mathbf{X})$ has a computable multi-valued inverse.

For any subset $A \subseteq \mathbf{X}$, the set $U_A = \{U \in \mathcal{O}(\mathbf{X}) \mid U \cap A \neq \emptyset\}$ is an open subset of $\mathcal{O}(\mathbf{X})$, hence an element of $\mathcal{O}(\mathcal{O}(\mathbf{X}))$. The corresponding characteristic function $\chi_{U_A} : \mathcal{O}(\mathbf{X}) \to \Sigma$ is continuous and preserves finite joins between the lattices $\mathcal{O}(\mathbf{X})$ and Σ (i.e., $\chi_{U_A}(\emptyset) = \bot$ and $\chi_{U_A}(U \cup V) = \chi_{U_A}(U) \vee \chi_{U_A}(V)$). Conversely, if $\chi : \mathcal{O}(\mathbf{X}) \to \Sigma$ is continuous and preserves finite joins, then by defining $A = X \setminus \bigcup\{U \in \mathcal{O}(\mathbf{X}) \mid \chi(U) = \bot\}$, we see that χ is the characteristic function of the open subset $\{U \in \mathcal{O}(\mathbf{X}) \mid U \cap A \neq \emptyset\}$ of $\mathcal{O}(\mathbf{X})$.

We define $\mathcal{V}(\mathbf{X})$ to be the subspace of $\mathcal{O}(\mathcal{O}(\mathbf{X}))$ of join preserving functions in the above sense. Since any $A \subseteq \mathbf{X}$ determines an element $\{U \in \mathcal{O}(\mathbf{X}) \mid U \cap A \neq \emptyset\}$ in $\mathcal{V}(\mathbf{X})$ which encodes the information about which open sets intersect A, it is convenient to think of $\mathcal{V}(\mathbf{X})$ as the *space of overt subsets* of $\mathbf{X}$. However, this does not characterize subsets of $\mathbf{X}$ uniquely: For sets $A, B \subseteq \mathbf{X}$ we have that $\{U \in \mathcal{O}(\mathbf{X}) \mid U \cap A \neq \emptyset\} = \{U \in \mathcal{O}(\mathbf{X}) \mid U \cap B \neq \emptyset\}$ if and only if A and B have equal closures in $\mathbf{X}$.

To avoid this ambiguity, we adopt the convention that the elements of $\mathcal{V}(\mathbf{X})$ are encoding topologically closed subsets of $\mathbf{X}$. Under this convention, the space $\mathcal{V}(\mathbf{X})$ is isomorphic to the space of closed subsets of $\mathbf{X}$ with the positive information topology (equivalently, the sequentialization of the lower Vietoris or lower Fell topology). We will, however, sometimes simply refer to the elements of $\mathcal{V}(\mathbf{X})$ as "overt" sets, with the implicit understanding that they are topologically closed.

If $\mathbf{X}$ is effectively countably-based via some $(B_n)_{n\in\mathbb{N}}$, we can conceive of $A \in \mathcal{V}(\mathbf{X})$ as being represented via $\{n \in \mathbb{N} \mid B_n \cap A \neq \emptyset\} \in \mathcal{O}(\mathbb{N})$.

The map $x \mapsto \overline{\{x\}} : \mathbf{X} \to \mathcal{V}(\mathbf{X})$ is computable for every represented space $\mathbf{X}$. If it is a computable embedding (i.e. has a computable partial inverse), we call $\mathbf{X}$ *computably admissible*. Computably admissible spaces are those that can be understood as topological spaces with $\mathcal{O}(\mathbf{X})$ serving as the topology. Any space of the form $\mathcal{O}(\mathbf{X})$ and subspace thereof is computably admissible, so in particular is $\mathcal{V}(\mathbf{X})$.

2.2. Weihrauch degrees

Weihrauch reducibility is a preorder between multivalued functions on represented spaces. It is a many-one reducibility which captures the idea of when f is solvable using computable means and a single application of another principle g. Inspired by earlier work by Weihrauch [32,33] it was promoted as a setting for computable metamathematics in [5,6,16]. A recent survey and introduction is found in [7], to which we refer for further reading.

Definition 3. Let $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $g :\subseteq \mathbf{U} \rightrightarrows \mathbf{V}$ be multi-valued functions between represented spaces. We say that f is Weihrauch reducible to g ($f \leqslant_{\mathrm{W}} g$) if there are computable functions $H, K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that whenever $G \vdash g$, then $K(\langle \mathrm{id}, GH\rangle) \vdash f$. If we can even choose H, K such that $KGH \vdash f$ whenever $G \vdash g$, we have a strong Weihrauch reduction ($f \leqslant_{\mathrm{sW}} g$).

We write $\leqslant^{\mathrm{t}}_{\mathrm{W}}$ respectively $\leqslant^{\mathrm{t}}_{\mathrm{sW}}$ for the relativized versions (equivalently, the versions where *computable* is replaced by *continuous*). With $f \equiv_{\mathrm{W}} g$ we abbreviate $f \leqslant_{\mathrm{W}} g \wedge g \leqslant_{\mathrm{W}} f$, with $f <_{\mathrm{W}} g$ we abbreviate $f \leqslant_{\mathrm{W}} g \wedge g \nleqslant_{\mathrm{W}} f$, and $f|_{\mathrm{W}} g$ stands in for $f \nleqslant_{\mathrm{W}} g \wedge g \nleqslant_{\mathrm{W}} f$.

The equivalence classes for $\leqslant_{\mathrm{W}}$ are the Weihrauch degrees, which form a distributive lattice. The cartesian product of multivalued functions induces an operation $\times$ on the Weihrauch degrees. We also use the closure operator $\widehat{\ }$ which is induced by the lifting of $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ to $\widehat{f} :\subseteq \mathbf{X}^\mathbb{N} \rightrightarrows \mathbf{Y}^\mathbb{N}$. The operation $f \star g$ captures the idea of first making one call to g, and then one call to f. As such, $f \star g$ is the maximal Weihrauch degree arising as a composition $f' \circ g'$ where $f' \leqslant_{\mathrm{W}} f$ and $g' \leqslant_{\mathrm{W}} g$. A formal construction is found in [9].

Many computational tasks that have been classified in the Weihrauch lattice turned out to be equivalent to a *closed choice* principle parameterized by some represented space:

Definition 4. For represented space $\mathbf{X}$, let its closed choice $\mathrm{C}_\mathbf{X} :\subseteq \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbf{X}$ be defined by $A \in \mathrm{dom}(\mathrm{C}_\mathbf{X})$ iff $A \neq \emptyset$ and $x \in \mathrm{C}_\mathbf{X}(A)$ iff $x \in A$.

The topological Weihrauch degree of $\mathrm{C}_\mathbf{X}$ reflects topological properties of $\mathbf{X}$. For example, for any uncountable compact metric space $\mathbf{X}$ we find that $\mathrm{C}_\mathbf{X} \equiv^{\mathrm{t}}_{\mathrm{W}} \mathrm{C}_{\{0,1\}^\mathbb{N}}$. This is discussed further in [4]. The principle $\mathrm{C}_\mathbb{N}$ has the more intuitive characterization of finding a natural number not occurring in an enumeration (that does not exhaust all natural numbers). Both $\mathrm{C}_{\{0,1\}^\mathbb{N}}$ and $\mathrm{C}_{\mathbb{N}^\mathbb{N}}$ are about finding infinite paths through ill-founded trees. For $\mathrm{C}_{\{0,1\}^\mathbb{N}}$, the tree is binary, whereas for $\mathrm{C}_{\mathbb{N}^\mathbb{N}}$ the tree can be countably-branching.

3. Fundamentals on overt choice

As mentioned above, *overt choice* is the task of finding a point in a given overt set. A priori, this is an ill-specified task, as overt sets do not uniquely determine an actual set of points. Our convention that elements of $\mathcal{V}(\mathbf{X})$ are topologically closed does ensure the well-definedness of overt choice. Consequently, it might be more accurate to speak of *closed overt choice*. To keep notation simple, we omit the reminder of our convention in the following.

Definition 5. For represented space $\mathbf{X}$, let its overt choice $\mathrm{VC}_\mathbf{X} :\subseteq \mathcal{V}(\mathbf{X}) \rightrightarrows \mathbf{X}$ be defined by $A \in \mathrm{dom}(\mathrm{VC}_\mathbf{X})$ iff $A \neq \emptyset$ and $x \in \mathrm{VC}_\mathbf{X}(A)$ iff $x \in A$.

Note the similarity between the definitions of closed choice (Definition 4) and overt choice (Definition 5). Given the importance of the former in the study of Weihrauch degrees, this is an argument in favour of exploring the latter notion, too.

We shall observe some basic properties of how overt choice for various spaces is related, similar to the investigation for closed choice in [4].

Proposition 6. *Let $s : \mathbf{X} \to \mathbf{Y}$ be an effectively open computable surjection. Then* $\mathrm{VC}_{\mathbf{Y}} \leqslant_{\mathrm{W}} \mathrm{VC}_{\mathbf{X}}$.

Proof. As s is effectively open, we can compute $s^{-1}(A) \in \mathcal{V}(\mathbf{X})$ from $A \in \mathcal{V}(\mathbf{Y})$. Then $\mathrm{VC}_{\mathbf{X}}$ can be used to obtain some $x \in s^{-1}(A)$. Computability of s then lets us compute $s(x) \in A$. □

Corollary 7. *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are computably isomorphic, then* $\mathrm{VC}_{\mathbf{X}} \equiv_{\mathrm{W}} \mathrm{VC}_{\mathbf{Y}}$.

Proposition 8. *Let* $\mathbf{X}$ *be a closed subspace of* $\mathbf{Y}$*. Then* $\mathrm{VC}_{\mathbf{X}} \leqslant_{\mathrm{W}} \mathrm{VC}_{\mathbf{Y}}$.

Proof. The map $\overline{\mathrm{id}} : \mathcal{V}(\mathbf{X}) \to \mathcal{V}(\mathbf{Y})$ is computable for any subspace $\mathbf{X}$ of $\mathbf{Y}$: To see this, we just need to note that $U \mapsto U \cap X : \mathcal{O}(\mathbf{Y}) \to \mathcal{O}(\mathbf{X})$ is computable, and that for $A \subseteq \mathbf{X}$ we have that $U \cap A \neq \emptyset$ iff $(U \cap X) \cap A \neq \emptyset$. If $\mathbf{X}$ is closed, then any closed subset of $\mathbf{X}$ is already closed as a subset of $\mathbf{Y}$, and we thus even have $\overline{\mathrm{id}} : \mathcal{V}(\mathbf{X}) \to \mathcal{V}(\mathbf{Y})$ being computable. This yields the claim. □

The requirement of the subspace being closed is necessary for the previous proposition to hold. To see this, note that we can adjoin a computable bottom element to an arbitrary represented space, in such a way that the original space is computably open inside the resulting space, and that the only open set containing the bottom element is the entire space. Since every non-empty closed subset of the resulting space contains the bottom element, closed overt choice becomes trivially computable.

Lemma 9. *The map* $\times : \mathcal{V}(\mathbf{X}) \times \mathcal{V}(\mathbf{Y}) \to \mathcal{V}(\mathbf{X} \times \mathbf{Y})$ *defined by* $(A, B) \mapsto A \times B$ *is computable.*

Proof. For all $x \in \mathbf{X}$ and all $W \in \mathcal{O}(\mathbf{X} \times \mathbf{Y})$ the set $V_{x,W} := \{y \in \mathbf{Y} \mid (x, y) \in W\}$ is open and the map $(x, W) \mapsto V_{x,W}$ is computable. By composition, it follows that

$$(W, B) \mapsto U_{W,B} := \{x \in \mathbf{X} \mid B \cap V_{x,W} \neq \emptyset\} : \mathcal{O}(\mathbf{X} \times \mathbf{Y}) \times \mathcal{V}(\mathbf{Y}) \to \mathcal{O}(\mathbf{X})$$

is well-defined and computable. For all $A \in \mathcal{V}(\mathbf{X})$, we have $A \cap U_{W,B} \neq \emptyset \iff (A \times B) \cap W \neq \emptyset$. Therefore the function $\mathcal{V}(\mathbf{X}) \times \mathcal{V}(\mathbf{Y}) \times \mathcal{O}(\mathbf{X} \times \mathbf{Y}) \to \Sigma$ mapping (A, B, W) to $\top$ iff W intersects $A \times B$ is computable. □

We conclude:

Corollary 10. $\mathrm{VC}_{\mathbf{X}} \times \mathrm{VC}_{\mathbf{Y}} \leqslant_{\mathrm{sW}} \mathrm{VC}_{\mathbf{X} \times \mathbf{Y}}$.

4. Overt choice for quasi-Polish spaces

Quasi-Polish spaces were introduced in [12] as a suitable setting for descriptive set theory. They generalize both Polish spaces, which form the traditional hunting grounds of descriptive set theory, as well as ω-continuous domains which had been the focus of previous work to extend descriptive set theory (e.g. [29,30]). Essentially, they are complete countably-based spaces.

4.1. Defining computable quasi-Polish spaces

An attractive feature of the class of quasi-Polish spaces is the multitude of very different yet equivalent definitions for it, as demonstrated in [12]. This makes the task of identifying the *correct* definition of a computable quasi-Polish space challenging, however: It does not suffice to effectivize one definition, but one needs to check to what extent the classically equivalent definitions remain equivalent in the computable setting, and in case they are not all equivalent, to choose which one is the most suitable definition. While we do not explore effectivizations of all characterizations of quasi-Polish spaces here, we exhibit two classes of definitions equivalent up to computable isomorphism, and propose those as *precomputable quasi-Polish spaces* and *computable quasi-Polish spaces*.

The task of effectivizing the definition of a quasi-Polish space has already been considered by V. Selivanov [28] and by M. Korovina and O. Kudinov [22]. We discuss the relationship between the various proposals in the remark after Definition 17 below.

Definition 11. Given a transitive binary relation $\prec$ on $\mathbb{N}$, we say that $I \subseteq \mathbb{N}$ is a rounded ideal for $\prec$, iff the following are satisfied:

(1) $I \neq \emptyset$
(2) $y \in I \wedge x \prec y \Rightarrow x \in I$
(3) $x, y \in I \Rightarrow \exists z \in I.\, x \prec z \wedge y \prec z$

Let $\mathrm{RI}(\prec) \subseteq \mathcal{O}(\mathbb{N})$ be the space of rounded ideals of $\prec$ equipped with the subspace topology.[1]

For $(\mathbb{N}, \prec)$, let the extendability predicate $E \subseteq \mathbb{N}$ be defined as $n \in E$ iff there exists a rounded ideal $I \ni n$. This is equivalent to the existence of an infinite increasing chain in $(\mathbb{N}, \prec)$ containing n (by which we mean an order-preserving function $\phi : (\omega, <) \to (\mathbb{N}, \prec)$, not necessarily injective).

Recall that a representation δ of a represented space $\mathbf{X}$ is *effectively fiber-overt*, if $x \mapsto \overline{\delta^{-1}(\{x\})} : \mathbf{X} \to \mathcal{V}(\mathbb{N}^\mathbb{N})$ is computable. This notion is studied in [9,21]. It is closely related to the representation being effectively open.

Theorem 12. *The following are equivalent for a represented space* $\mathbf{X}$*:*

(1) *There exists a c.e. transitive relation* $\prec \subseteq \mathbb{N} \times \mathbb{N}$ *such that* $\mathbf{X} \cong \mathrm{RI}(\prec)$.
(2) $\mathbf{X}$ *is computably isomorphic to a* Π^0_2*-subspace of* $\mathcal{O}(\mathbb{N})$.
(3) $\mathbf{X}$ *admits an effectively fiber-overt computably admissible representation* δ *such that* $\mathrm{dom}(\delta) \subseteq \mathbb{N}^\mathbb{N}$ *is* Π^0_2.

Before proving Theorem 12, we provide a technical lemma:

Lemma 13. *For effectively fiber-overt and computably admissible* δ*, the map* $x \mapsto \{w \in \mathbb{N}^* \mid w\mathbb{N}^\mathbb{N} \cap \delta^{-1}(x) \neq \emptyset\} : \mathbf{X} \to \mathcal{O}(\mathbb{N}^*)$ *is an embedding.*

Proof. By definition of effective fiber-overtness, from $x \in \mathbf{X}$ we can compute $\overline{\delta^{-1}(x)} \in \mathcal{V}(\mathbb{N}^\mathbb{N})$. That lets us semi-decide whether $w\mathbb{N}^\mathbb{N} \cap \delta^{-1}(x) \neq \emptyset$, and thus ensures computability of the map under investigation by currying.

Next, we observe that $\{w \in \mathbb{N}^* \mid w\mathbb{N}^\mathbb{N} \cap \delta^{-1}(x) \neq \emptyset\} \mapsto \{U \in \mathcal{O}(\mathbf{X}) \mid x \in U\} :\subseteq \mathcal{O}(\mathbb{N}^*) \to \mathcal{O}(\mathcal{O}(\mathbf{X}))$ is computable in general: Given $U \in \mathcal{O}(\mathbf{X})$, we can compute $\delta^{-1}(U) \in \mathcal{O}(\mathbb{N}^\mathbb{N})$. We then have that $x \in U$ iff there exists some w with $w\mathbb{N}^\mathbb{N} \cap \delta^{-1}(x) \neq \emptyset$ such that $\delta^{-1}(U)$ accepts its input upon reading w.

Computable admissibility then lets us compute $x \in \mathbf{X}$ from $\{U \in \mathcal{O}(\mathbf{X}) \mid x \in U\} \in \mathcal{O}(\mathcal{O}(\mathbf{X}))$, and thus ensures that we are dealing with a computable embedding. □

Proof of Theorem 12. If $\prec$ is c.e., then the property of being a rounded ideal is Π^0_2. Hence, 1 implies 2. To see that 2. implies 3., we first observe that $\mathcal{O}(\mathbb{N})$ is computably admissible and admits an effectively fiber-overt representation. Then we just take the post-restriction of the standard representation of $\mathcal{O}(\mathbb{N})$ to the relevant subspace, and copy

[1]Note that if $\prec$ is actually a (reflexive) partial order, then $\mathrm{RI}(\prec)$ is the set of all ideals of $\prec$ in the usual sense. On the other hand, if $\prec$ is anti-reflexive, then a rounded ideal of $\prec$ will not have a maximal element.

via the isomorphism to $\mathbf{X}$. This preserves effective fiber-overtness and computable admissibility. The non-trivial step is the implication from 3. to 1.

Let $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denote the space of finite subsets of $\mathbb{N}$ (given as unordered tuples). We construct a c.e. transitive relation $\prec$ on $\mathcal{P}_{\text{fin}}(\mathbb{N}) \times \mathbb{N}$, but the translation to $\mathbb{N}$ is straight-forward.

A computable realizer of fiber-overtness will, given a name $p \in \operatorname{dom}(\delta)$ and some $w \in \mathbb{N}^*$ confirm if there is some name q extending w with $\delta(p) = \delta(q)$, if this is the case. By observing when this realizer provides its confirmations we obtain a computable function $f : \mathbb{N}^* \times \mathbb{N} \to \mathcal{P}_{\text{fin}}(\mathbb{N}^*)$ such that

(1) $w \in f(u, n)$ implies that if u can be extended to some $p \in \operatorname{dom}(\delta)$, then w can be extended to some $q \in \operatorname{dom}(\delta)$ satisfying $\delta(p) = \delta(q)$
(2) If $p, q \in \operatorname{dom}(\delta)$ and $\delta(p) = \delta(q)$, then for any prefix w of q there is a prefix u of p and $n \in \mathbb{N}$ such that $w \in f(u, n)$
(3) $u \in f(u, n)$
(4) $f(u, n)$ is closed under prefixes
(5) $f(u, n) \subseteq f(u', n)$ whenever u' extends u
(6) $f(u, m) \subseteq f(u, n)$ whenever $m \leqslant n$

Since $\operatorname{dom}(\delta)$ is a Π^0_2-subset of $\mathbb{N}^{\mathbb{N}}$, we can understand it to be given via a computable function $\lambda : \mathbb{N}^* \to \mathbb{N}$ which is order-preserving (prefix-order on $\mathbb{N}^*$, standard order on $\mathbb{N}$) such that $p \in \operatorname{dom}(\delta)$ iff $\{\lambda(p_{\leqslant n}) \mid n \in \mathbb{N}\}$ is unbounded.

Now we define $\prec$ on $\mathcal{P}_{\text{fin}}(\mathbb{N}) \times \mathbb{N}$ as $(A, n) \prec (B, m)$ iff the following all hold:

(1) $B \neq \emptyset$
(2) $n < m$
(3) $n < \lambda(w)$ for each $w \in B$
(4) $A \subseteq f(u, m)$ for each $u \in B$
(5) Each $w \in \bigcup_{u \in A} f(u, n)$ has an extension $w' \in B$

One easily checks that $\prec$ is a c.e. transitive relation.

For effectively fiber-overt and computably admissible δ, the map $x \mapsto \{w \in \mathbb{N}^* \mid w\mathbb{N}^{\mathbb{N}} \cap \delta^{-1}(x) \neq \emptyset\} : \mathbf{X} \to \mathcal{O}(\mathbb{N}^*)$ is an embedding (see Lemma 13). It is straight-forward to see that the map $U \mapsto \{A \in \mathcal{P}_{\text{fin}}(\mathbb{N}^*) \mid \forall w \in A.\ w \in U\} : \mathcal{O}(\mathbb{N}^*) \to \mathcal{O}(\mathcal{P}_{\text{fin}}(\mathbb{N}^*))$ is an embedding, too. To conclude our proof, we show that the range of the composition of these two embeddings coincides with the rounded ideals of $\prec$.

Let $U_x := \{w \in \mathbb{N}^* \mid w\mathbb{N}^{\mathbb{N}} \cap \delta^{-1}(x) \neq \emptyset\}$ and $F_x = \{(A, n) \in \mathcal{P}_{\text{fin}}(\mathbb{N}^*) \times \mathbb{N} \mid A \subseteq U_x\}$. First, we shall see that F_x is indeed a rounded ideal for $\prec$. Clearly F_x is non-empty. If $(B, m) \in F_x$ and $(A, n) \prec (B, m)$, then every $u \in A$ has an extension $u' \in B \subseteq U_x$, hence $(A, n) \in F_x$. Now consider any pair $(A, n), (B, m) \in F_x$. If A and B are both empty, we can choose any $w \in U_x$ with $n + m < \lambda(w)$, and get $(\{w\}, n + m + 1)$ as a joint $\prec$-upper bound for A and B in F_x. If A or B is non-empty, then we construct a joint $\prec$-upper bound (C, r) in F_x by defining C as a finite set of suitable extensions w' to each $w \in \bigcup_{u \in A \cup B} f(u, n + m)$. To see how this can be done, note that if $w \in f(u, n + m)$ for some $u \in A \cup B$, then since there is a name p of x extending u there must exist a name q of x extending w. So we can choose any prefix w' of q long enough that $n + m < \lambda(w')$ and $A \cup B \subseteq f(w', r_w)$ for large enough r_w. Taking the set C of prefixes w' chosen in this way and letting $r \in \mathbb{N}$ be larger than all the corresponding r_w, we get a joint $\prec$-upper bound for (A, n) and (B, m) satisfying $(C, r) \in F_x$. Therefore, $F_x \in \mathrm{RI}(\prec)$.

It remains to argue that any rounded ideal F for $\prec$ is of the form F_x. Given a rounded ideal F, consider the set $N \subseteq \mathbb{N}^{\mathbb{N}}$ consisting of those p for which there exists a cofinal $\prec$-chain $(A_i, n_i)_{i \in \mathbb{N}}$ in F where each A_i contains some prefix of p. Since F is a non-empty countable ideal, it is clear that at least one cofinal $\prec$-chain exists in F. Given such a cofinal chain $(A_i, n_i)_{i \in \mathbb{N}}$, we can assume w.l.o.g. that each A_i is non-empty. Also note that $(n_i)_{i \in \mathbb{N}}$ is a strictly increasing chain, and each $w \in A_i$ has an extension $w' \in A_{i+1}$ satisfying $n_i < \lambda(w')$. It follows that $\emptyset \neq N \subseteq \operatorname{dom}(\delta)$.

Consider $p, q \in N$. Then arbitrarily long prefixes of p and q appear in finite sets occurring in F, and these have common upper bounds. The requirement that $(A, n) \prec (B, m)$ and $u \in B$ implies $A \subseteq f(u, m)$ then tells us that every prefix w of p can be extended to some q_w with $\delta(q_w) = \delta(q)$, and every prefix u of q can be extended to some p_u with $\delta(p_u) = \delta(p)$. This shows that for any $U \in \mathcal{O}(\mathbf{X})$ we have that $\delta(p) \in U \Leftrightarrow \delta(q) \in U$. As admissibility implies being T_0, we conclude that $\delta(p) = \delta(q)$. It follows that $\{\delta(p) \mid p \in N\}$ is some singleton $\{x\}$. Let $(A, n) \in F$ and $w \in A$. By definition of cofinality, there is some $(B, m) \succ (A, n)$ occurring in a cofinal chain. We know that any $v \in B$ is extendible to a name for x, and that $w \in f(v, m)$, hence w is extendible to a name of x, too. This shows $F \subseteq F_x$.

To show that $F_x \subseteq F$, assume $(A, n) \in F_x$ and fix any $p \in N$. Since $A \subseteq U_x$ and $\delta(p) = x$, for each $w \in A$ there is a prefix v of p such that $w \in f(v, m)$ for some m. Since $p \in N$, using the monotonicity of f and the fact that A is finite, it follows that there is some $(B, m) \in F$ with $n < m$ such that B contains a prefix v of p long enough to satisfy $A \subseteq f(v, m)$. Next let (B', m') be any immediate $\prec$-successor of (B, m) in F, and note that every $w \in A$ has an extension $w' \in B'$. Finally, let (B'', m'') be any immediate $\prec$-successor of (B', m') in F. Clearly $B'' \neq \emptyset$ and $n < m''$ and $n < \lambda(w)$ for each $w \in B''$. Furthermore, $A \subseteq f(u, m'')$ for each $u \in B''$, because each $w \in A$ has an extension $w' \in B'$, $B' \subseteq f(u, m'')$, and $f(u, m'')$ is closed under prefixes. Finally, if $w \in f(u, n)$ for some $u \in A$, then $w \in f(u', n)$ for any extension $u' \in B$ of u, hence w has an extension w' in B''. Therefore, $(A, n) \prec (B'', m'') \in F$.

We conclude that $F = F_x$, and that $\mathbf{X}$ is computably isomorphic to $\mathrm{RI}(\prec)$. □

Definition 14. We call a space $\mathbf{X}$ satisfying the equivalent criteria of Theorem 12 a *precomputable quasi-Polish space*.

Theorem 15. *The following are equivalent for a represented space* $\mathbf{X}$*:*

(1) $\mathbf{X}$ *is precomputably quasi-Polish and effectively separable (i.e. admits a computable dense sequence).*
(2) $\mathbf{X}$ *is precomputably quasi-Polish and computably overt.*
(3) $\mathbf{X}$ *admits an effectively fiber-overt computably admissible total representation* δ*.*
(4) *There exists a c.e. transitive relation* $\prec \subseteq \mathbb{N} \times \mathbb{N}$ *such that* $\mathbf{X} \cong \mathrm{RI}(\prec)$*, such that* $\prec$ *has a c.e. extendability predicate* E*.*

Again, we state and prove a lemma before proceeding to the proof of the theorem:

Lemma 16. *From non-empty* $A \in \underset{\sim}{\mathbf{\Pi}}^0_2(\mathbb{N}^\mathbb{N})$ *and* $\overline{A} \in \mathcal{V}(\mathbb{N}^\mathbb{N})$ *we can compute* $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *with* $f[\mathbb{N}^\mathbb{N}] = A$ *together with* $f[\cdot] : \mathcal{O}(\mathbb{N}^\mathbb{N}) \to \mathcal{O}(\mathbb{N}^\mathbb{N})$ *and* $i : \mathbf{A} \to \mathbb{N}^\mathbb{N}$ *with* $f \circ i = \mathrm{id}_A$.

Proof. We take the Π^0_2-information about A via some monotone function $\lambda : \mathbb{N}^* \to \mathbb{N}$ such that $p \in A$ iff $\{\lambda(p_{\leqslant n}) \mid n \in \mathbb{N}\}$ is unbounded. We take the overt information about A as an enumeration of all $w \in \mathbb{N}^*$ that are extendible to an element of A. Call $w \in \mathbb{N}^*$ *productive*, if $\lambda(w) > \lambda(w_{\leqslant(|w|-1)})$ and w is enumerated as extending to a member of A. Let the empty word ε be productive by convention. Clearly, we can enumerate all productive words, and each productive word has some productive extensions.

We construct f as the limit of a monotone function $F : \mathbb{N}^* \to \mathbb{N}^*$, which we define in turn by induction on the length of the input. The range of F will be exactly the productive words. Set $F(\varepsilon) = \varepsilon$. If $F(w) = u$, then we search for productive extensions of u. We can enumerate those as $(u_n)_{n \in \mathbb{N}}$ (in particular, there are some). We then extend $F(wn) = u_n$. It is straight-forward to check that the construction of F gives the desired properties to f. □

Proof of Theorem 15.

1. $\Rightarrow$ 2. Effective separability trivially implies computable overtness.
2. $\Rightarrow$ 3. From Theorem 12 we obtain an effectively fiber-overt computably admissible representation δ' with Π^0_2-domain. As the preimage of computable overt $\mathbf{X}$ under the effectively fiber-overt δ', we see that we can also obtain $\overline{\mathrm{dom}(\delta')} \in \mathcal{V}(\mathbb{N}^\mathbb{N})$ (since overt unions of overt sets are overt, see e.g. [24, page 27]). We apply Lemma 16 to obtain a computable, computably invertible and effectively open $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\delta = \delta' \circ f$ is the desired total representation: Since f is computable and computably invertible, δ is equivalent to δ', and in particular is computably admissible again. That δ is effectively fiber-overt follows from δ' and f being effectively open.

3. $\Rightarrow$ 1. Using Theorem 12 and that spaces with total representations inherit effective separability from $\mathbb{N}^{\mathbb{N}}$.
2. $\Leftrightarrow$ 4. A space of the form $\mathrm{RI}(\prec)$ is computably overt iff the extendability predicate for $\prec$ is computably enumerable. □

Definition 17. We call a space $\mathbf{X}$ satisfying the equivalent criteria of Theorem 15 a *computable quasi-Polish space*.

Remark 18. Criterion 2 of Theorem 12 is proposed as the definition of a "computable quasi-Polish space" by M. Korovina and O. Kudinov in [22, Definition 7]. We prefer to include computable overtness in the definition of *computable quasi-Polish* (see Theorem 15 and Definition 17) as this property is often useful for applications, and present in all natural examples. This also mirrors the terminology for metric spaces, where a computable Polish space is by definition computably overt, rather than being merely computably completely metrizable. Therefore, our notion of computable quasi-Polish essentially corresponds to an "effectively enumerable computable quasi-Polish space" in the terminology of [22].

V. Selivanov [28] suggested the possibility of using an effective version of a *convergent approximation space* [2], which is closely related to Criterion 4 of Theorem 15.

In an independent work, Hoyrup, Rojas, Stull and Selivanov [18,19] have also proposed to define computable quasi-Polish spaces in a way equivalent to ours, and proved similar results to our Theorem 15. They subsequently show that finding a characterization of computable quasi-Polish spaces in terms of quasi-metrics seems difficult.

Proposition 19. *For a computable metric space $\mathbf{X}$ the following are equivalent:*

(1) *$\mathbf{X}$ is computably Polish.*
(2) *$\mathbf{X}$ is precomputably quasi-Polish.*
(3) *$\mathbf{X}$ is computably quasi-Polish.*

Proof.

1. $\Rightarrow$ 2. We work with a complete computable metric. The Cauchy representation of a computable metric space is computably admissible and effectively fiber-overt. Being a Cauchy sequence is a Π^0_2-property. We thus have Condition 3 of Theorem 12.
2. $\Rightarrow$ 3. A computable metric space is by definition effectively separable, hence Condition 1 of Theorem 15 applies.
3. $\Rightarrow$ 1. Any computable metric space is a subspace of its completion, which is a computable Polish space and hence a computable quasi-Polish space. If a computable quasi-Polish is a subspace of a computable quasi-Polish space, it is a Π^0_2-subspace. Any effective separable Π^0_2-subspace of a computable Polish space is computably Polish. □

4.2. Computability of overt choice

We first show that overt choice is computable for every precomputable quasi-Polish space. We then prove a partial converse, that if overt choice is computable (even with respect to some oracle) for a countably based T_1-space $\mathbf{X}$, then $\mathbf{X}$ is quasi-Polish (but not necessarily precomputable quasi-Polish).

Theorem 20. *Overt choice $\mathrm{VC}_{\mathbf{X}}$ is computable for every precomputable quasi-Polish space $\mathbf{X}$.*

Proof. Let $\prec\subseteq \mathbb{N}\times\mathbb{N}$ be a c.e. transitive relation such that $\mathbf{X}\cong \mathrm{RI}(\prec)$. For $n\in\mathbb{N}$ we write $\uparrow n$ for the basic open subset of $\mathrm{RI}(\prec)$ consisting of all rounded ideals that contain n. Clearly $\mathcal{V}(\mathrm{RI}(\prec))\cong\mathcal{V}(\mathbf{X})$, so it suffices to consider overt choice for $\mathrm{RI}(\prec)$.

Given a presentation for some non-empty closed $A\in\mathcal{V}(\mathrm{RI}(\prec))$, we construct a $\prec$-ascending chain $(n_i)_{i\in\mathbb{N}}$ such that A intersects each basic open $\uparrow n_i$. To construct the chain, first choose any $n_0\in\mathbb{N}$ such that A has non-empty intersection with $\uparrow n_0$. Once n_i has been decided, choose any $n_{i+1}\in\mathbb{N}$ such that A intersects $\uparrow n_{i+1}$ and $n_i\prec n_{i+1}$.

Such a chain can be computed from a presentation of A because $\prec$ is a c.e. relation and it can be semi-decided whether A intersects a given basic open set.

Finally, we can enumerate the set $I = \{n \in \mathbb{N} \mid (\exists i \in \mathbb{N})\ n \prec n_i\}$, which is the rounded ideal generated by the sequence $(n_i)_{i\in\mathbb{N}}$. For any $n \in \mathbb{N}$ with $I \in \uparrow n$, there is $i \in \mathbb{N}$ with $n \prec n_i \in I$, hence $I \in \uparrow n_i \subseteq \uparrow n$. Therefore, $I \in A$ because every basic open containing I intersects A. □

We obtain the following corollary, which generalizes the corresponding theorem for computable Polish spaces from [10]:

Corollary 21. *Let $\mathbf{X}$ be a precomputable quasi-Polish space. The computable map $(a_i)_{i\in\mathbb{N}} \mapsto \mathrm{cl}\{a_i \mid i \in \mathbb{N}\} : \mathbf{X}^{\mathbb{N}} \to \mathcal{V}(\mathbf{X}) \setminus \{\emptyset\}$ has a computable multi-valued inverse.*

Proof. Given non-empty $A \in \mathcal{V}(\mathbf{X})$ we can enumerate all basic open sets U_i having a non-empty intersection with A. We can then compute $\mathrm{cl}(U_i \cap A) \in \mathcal{V}(\mathbf{X})$, and use $\mathrm{VC}_{\mathbf{X}}$ to extract a point. The resulting sequence is dense in A. □

4.3. Continuity of overt choice as a completeness notion

We next prove a partial converse to Theorem 20 using a game theoretic characterization of quasi-Polish spaces.

Given a non-empty space $\mathbf{X}$, the *convergent strong Choquet game* [12,14] is played as follows. Player I first plays a pair (U_0, x_0) with $U_0 \in \mathcal{O}(\mathbf{X})$ and $x_0 \in U_0$. Player II must respond with an open set V_0 such that $x_0 \in V_0 \subseteq U_0$. Player I then responds with a pair (U_1, x_1) with U_1 open and $x_1 \in U_1 \subseteq V_0$, then Player II must play an open V_1 with $x_1 \in V_1 \subseteq U_1$, and so on. Player II wins the game if and only if the sequence of opens $(V_i)_{i\in\mathbb{N}}$ is a neighborhood basis for a unique point in $\mathbf{X}$. It was shown in [12] that a non-empty countably based T_0-space $\mathbf{X}$ is quasi-Polish iff Player II has a winning strategy (see also [11], which fills a gap in the original proof).

Theorem 22. *If $\mathbf{X}$ is a countably based T_1-space and $\mathrm{VC}_{\mathbf{X}}$ is continuous, then $\mathbf{X}$ is quasi-Polish.*

Proof. Let R be a continuous realizer for $\mathrm{VC}_{\mathbf{X}}$. We show that R can be used to define a winning strategy for Player II in the convergent strong Choquet game for $\mathbf{X}$. The basic idea of Player II's strategy is to present to R the closure of the sequence of elements $(x_0, x_1, \dots, x_i)$ played by Player I, and the V_i played by Player II will correspond to the output of R.

Fix a countable basis $(B_k)_{k\in\mathbb{N}}$ for $\mathbf{X}$. A valid input to R consists of an enumeration of all the B_k that intersect some non-empty closed $A \subseteq \mathbf{X}$. We can assume the output of R will be a decreasing sequence of basic opens forming a neighborhood basis for some $x \in A$.

At each round i, Player II will keep track of a finite set A_i (which is closed because $\mathbf{X}$ is T_1). Set $A_0 = \emptyset$. (In round i, the set A_i will actually be the finite set of elements that have been played by Player I that are distinct from the element x_i played that round.)

At round i, Player I plays (U_i, x_i). Up until now, Player II's strategy will have guaranteed that the following all hold at each round i:

(1) $x_i \notin A_i$,
(2) $U_i \cap A_i = \emptyset$,
(3) the information fed to R until now is consistent with a presentation for $A_i \cup \{x_i\}$,
(4) the output of R until now is consistent with a presentation of x_i,
(5) the output of R until now can only be extended to a presentation of an element in $\mathbf{X} \setminus A_i$.

(This is trivial for $i = 0$, before any information is fed to the realizer R.)

Player II chooses V_i as follows. There are two cases:

Case 1) Either $i = 0$ or $x_i = x_{i-1}$. Then define $A_{i+1} = A_i$.
Case 2) $x_i \neq x_{i-1}$. Then define $A_{i+1} = A_i \cup \{x_{i-1}\}$.

In either case, extend the presentation being fed to R so that it is a presentation of $A_i \cup \{x_i\}$ (this is possible by item 3 above). Player II makes sure the presentation is extended enough so that it includes every basic open B_k that intersects $A_i \cup \{x_i\}$ for each $k \leqslant i$ (this is to guarantee that as i goes to infinity we are actually giving R a valid presentation).

Since R is being fed a presentation of $A_i \cup \{x_i\}$, items 1 and 5 above force the output of R to be a presentation of x_i. So R must eventually output a basic open W with $x_i \in W \subseteq (U_i \setminus A_{i+1})$. At this point we pause our execution of R and do not feed it any more information. Define V_i to be the intersection of W with all of the basic opens B_k that contain x_i that have been fed as input to R so far. Player II plays V_i, and the game continues to round $i + 1$.

We check that the items 1)–5) still hold in round $i+1$. Since we must have $x_{i+1} \in U_{i+1} \subseteq V_i$, and $V_i \cap A_{i+1} = \emptyset$, items 1 and 2 hold. Similarly, items 4 and 5 hold because $x_{i+1} \in U_{i+1} \subseteq W$ and $W \cap A_{i+1} = \emptyset$, where W is the last basic open that was outputted by R. Concerning item 3, the only subtlety is if Case 2) held in round i and $x_{i+1} \neq x_i$. So $x_i \notin A_{i+1}$ but so far R has been presented with information for $A_{i+1} \cup \{x_i\}$. However, item 3 still holds in round $i + 1$ because x_{i+1} will be in V_i which is a subset of the intersection of all the basic opens containing x_i that have so far been fed to R. Therefore, any basic open B_k that has been fed as input to R either intersects A_{i+1} or else contains V_i, hence contains x_{i+1}. Therefore, the presentation given to R is consistent with a presentation for $A_{i+1} \cup \{x_{i+1}\}$. This shows that Player II's strategy is well-defined.

Finally, we must show that this strategy is winning. It suffices to show that the information fed to R is a valid presentation of the overt closed set A defined as the closure of the infinite sequence $(x_i)_{i\in\mathbb{N}}$, because then the sequence V_i will be an open neighborhood basis of the point in A chosen by R.

Clearly, every basic open that was presented to R intersects A. So to prove that the presentation given to R is valid, it only remains to check that if B_k is a basic open that intersects A, then B_k was included in the presentation to R at some time. But if B_k intersects A, then B_k contains some x_i, and so for round $i' > \max\{i, k\}$ we have that B_k intersects $A_{i'} \cup \{x_{i'}\}$, and so B_k was included in the presentation to R in round i'. □

Corollary 23. *A countably-based T_1-space $\mathbf{X}$ is quasi-Polish if and only if $\mathrm{VC}_{\mathbf{X}}$ is continuous.*

Proof. Combine Theorem 22 with the relativization of Theorem 20. □

A classical result by E. Michael [23] states that if X is a zero-dimensional metrizable space and Y is a complete metric space, then every lower semi-continuous function from X to the non-empty closed subsets of Y admits a continuous selection. It was then shown in [31] that the completeness of Y is necessary. If X is a separable zero-dimensional metrizable space and Y is a QCB_0-space, then any lower semi-continuous function F from X to the closed subsets of Y can be viewed as a continuous function from X to $\mathcal{V}(Y)$. Since there exists a continuous reduction of F to an admissible representation of $\mathcal{V}(Y)$, it is clear that if overt choice on Y is continuous then F has a continuous selection. Conversely, a continuous solution to overt choice on Y is equivalent to the existence of a continuous selection for the admissible representation of $\mathcal{V}(Y)$. Therefore, we can view Corollary 23 as an extension of these classical selection results to the case that Y is a countably based (possibly non-metrizable) T_1-space.

Note that a computable version of Corollary 23 does not hold, because the computability of $\mathrm{VC}_{\mathbf{X}}$ does not imply that $\mathbf{X}$ is even a precomputable quasi-Polish space. A trivial counter example is the singleton space $\{p\}$ where $p \in \{0, 1\}^{\mathbb{N}}$ is chosen such that $p \not\leq_{\mathrm{M}} A$ for any non-empty Π^0_2-set A. Then p can be computed from any non-empty $A \in \mathcal{V}(\{p\})$, even though $\{p\}$ is not precomputably quasi-Polish as it violates Condition 3 of Theorem 12 by design.

5. Other countably-based spaces

In this section, we study overt choice on countably-based spaces that are not quasi-Polish. We do not know of relevant examples in this class where overt choice would be computable, and as such our focus is on investigating the Weihrauch degrees of overt choice of such spaces. First we gather some auxiliary on Weihrauch reducibility in Section 5.1. The main content of this section is spread over Section 5.2 where we provide results pertaining to spaces fulfilling various general conditions, and Section 5.3, where we consider overt choice for two specific spaces, namely the Euclidean rationals $\mathbb{Q}$ and the space S_0 from [13]. Plenty of questions are left open, and we state some of those in Section 5.4.

5.1. Some auxiliary results

We consider multivalued functions $f :\subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathbf{X}$. Call such f *uncomputable everywhere* if for every finite $A \subseteq \mathbb{N}$ the restriction $f|_{\{U \in \mathcal{O}(\mathbb{N}) \mid A \subseteq U\}}$ is uncomputable.

Proposition 24. *Let $f :\subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathbf{X}$ be uncomputable everywhere, and satisfy $\mathbb{N} \in \operatorname{dom}(f)$. Then $f \nleq_W \mathrm{C}_\mathbb{N}$.*

Proof. Assume that $f \leqslant_W \mathrm{C}_\mathbb{N}$ via some computable $K, H :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. The semantics for H mean that if p is some enumeration of the input to f, then $H(p)$ is some enumeration of the input to $\mathrm{C}_\mathbb{N}$. We will construct some $p \in \operatorname{dom}(H)$ with $\operatorname{ran}(p) = \mathbb{N}$ and $\operatorname{ran}(H(p)) = \mathbb{N}$, i.e. p is a name for some valid input to f, but $H(p)$ is not a name for some valid input to $\mathrm{C}_\mathbb{N}$, and thus derive a contradiction.

Let $h :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ be a word function for H. Assume that $\forall p \in \operatorname{dom}(H)\ 0 \notin \operatorname{ran}(H(p))$. Then 0 is a valid output to $\mathrm{C}_\mathbb{N}$ on any set represented by $H(p)$, which would imply computability of f. Thus, there is some finite prefix $w_0 \in \mathbb{N}^*$ such that $0 \in B_0 := \operatorname{ran}(h(w_0))$. As before, assuming that $\forall p \in \operatorname{dom}(H) \cap w_0 0\mathbb{N}^\mathbb{N}\ 1 \notin \operatorname{ran}(H(p))$ leads to a contradiction of f being uncomputable everywhere, so we can extend to some $w_0 0 w_1$ such that $1 \in B_1 := \operatorname{ran}(h(w_0 0 w_1))$, and so on. Let $p := w_0 0 w_1 1 w_2 2 \ldots$. This p is the desired contradictory input to H. □

Recall that a space $\mathbf{X}$ is computably Hausdorff, iff $x \mapsto \{x\} : \mathbf{X} \to \mathcal{A}(\mathbf{X})$ is well-defined and computable.

Proposition 25. *Let $\mathbf{Y}$ be a computable Hausdorff space. If $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ satisfies $f \leqslant_W \mathrm{C}_{\{0,1\}^\mathbb{N}} \star g$, then also $f \leqslant_W \mathrm{C}_\mathbf{Y} \star g$.*

Proof. The reduction $f \leqslant_W \mathrm{C}_{\{0,1\}^\mathbb{N}} \star g$ provides us with, for any $x \in \operatorname{dom}(f)$, a tree $T_x \subseteq \{0,1\}^*$ and a continuous function $K_x : [T_x] \to \mathbf{Y}$ such that $K_x([T_x]) \subseteq f(x)$. We can compute T_x and K_x jointly with a single application of g, which also yields all the other information from g we may desire. From T_x and K_x we can compute $K_x([T_x]) \in \mathcal{A}(\mathbf{Y})$ by noting that $y \notin K_x([T_x])$ iff $K_x^{-1}(\mathbf{Y} \setminus \{y\}) \supseteq T_x$. The argument follows. □

Proposition 26. $\mathrm{C}_\mathbb{Q} \equiv_W \mathrm{C}_\mathbb{N}$

Proof. That $\mathrm{C}_\mathbb{N} \leqslant_W \mathrm{C}_\mathbb{Q}$ follows from $\mathbb{N}$ embedding as a computably closed subspace into $\mathbb{Q}$ by [4, Corollary 4.3], and that $\mathrm{C}_\mathbb{Q} \leqslant_W \mathrm{C}_\mathbb{N}$ follows from the existence of a computable surjection $s : \mathbb{N} \to \mathbb{Q}$ and [4, Proposition 3.7]. □

Corollary 27. *If $f :\subseteq \mathbf{X} \rightrightarrows \mathbb{Q}$ satisfies $f \leqslant_W \mathrm{C}_\mathbb{R}$, then already $f \leqslant_W \mathrm{C}_\mathbb{N}$.*

Proof. As shown in [4] we have $\mathrm{C}_\mathbb{R} \equiv_W \mathrm{C}_{\{0,1\}^\mathbb{N}} \star \mathrm{C}_\mathbb{N}$. We can thus apply Proposition 25 to conclude that $f \leqslant_W \mathrm{C}_\mathbb{Q} \star \mathrm{C}_\mathbb{N}$, and then use Proposition 26 together with $\mathrm{C}_\mathbb{N} \equiv_W \mathrm{C}_\mathbb{N} \star \mathrm{C}_\mathbb{N}$ from [4]. □

5.2. General observations

Our first result shows that overt choice for countably-based spaces is never able to provide non-computable discrete information:

Proposition 28. *Let $\mathbf{X}$ be effectively countably-based. If $f :\subseteq \{0,1\}^\mathbb{N} \rightrightarrows \mathbb{N} \leqslant_W \mathrm{VC}_\mathbf{X}$, then f is computable.*

Proof. We pick a standard countable basis $(U_n)_{n \in \mathbb{N}}$ for $\mathbf{X}$, and assume $\mathbf{X}$ to be represented by the associated standard representation. Consider the computable outer reduction witness $K :\subseteq \{0,1\}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}$. This gives rise to a computable sequence $(n_i, w_i, k_i)_{i \in \mathbb{N}}$ (where $n_i, k_i \in \mathbb{N}$ and $w_i \in \{0,1\}^*$) such that $K(p,q)$ can return n iff $\exists i \in \mathbb{N}$ $n = n_i \wedge \delta(q) \in U_{k_i} \wedge w_i \prec p$. Let $H :\subseteq \{0,1\}^\mathbb{N} \to \mathcal{V}(\mathbf{X})$ be the inner reduction witness. Given $p \in \operatorname{dom}(f)$, start testing for all $i \in \mathbb{N}$ in parallel whether U_{k_i} intersects $H(p) \in \mathcal{V}(\mathbf{X})$ and $w_i \prec p$. This has to be true for at least one $i \in \mathbb{N}$, and once we have found a suitable candidate, we can return n_i as a correct output to $f(p)$. □

Corollary 29. *For effectively countably-based* $\mathbf{X}$*, its overt choice* $\mathrm{VC}_{\mathbf{X}}$ *is not* ω*-discriminative (in the sense of [8]).*

Besides the degrees of problems inspired by computability theory (which would often return Turing degrees as outputs), the investigations of specific Weihrauch degrees so far have not yet encountered non-computable yet not ω-discriminative degrees. We thus see that for countably-based spaces with non-computable overt choice principles we find ourselves in an unexplored region of the Weihrauch lattice. We can go even further for sufficiently homogeneous spaces:

Corollary 30. *Let* $\mathbf{X}$ *be effectively countably-based such that every non-empty open subset contains a copy of* $\mathbf{X}$*. If* $\mathrm{VC}_{\mathbf{X}}$ *is non-computable, then* $\mathrm{VC}_{\mathbf{X}} |_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$.

Proof. We can identify $A \in \mathcal{V}(\mathbf{X})$ with $\{n \in \mathbb{N} \mid I_n \cap A \neq \emptyset\} \in \mathcal{O}(\mathbb{N})$ for some canonical basis $(I_n)_{n\in\mathbb{N}}$ of $\mathbf{X}$. Since $\mathbf{X}$ by assumption embeds into any of its non-trivial basic open sets, we find that the result map $\mathrm{VC}_{\mathbf{X}} :\subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathbf{X}$ is even non-computable everywhere, and hence Proposition 24 lets us conclude $\mathrm{VC}_{\mathbf{X}} \nleq_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$. That $\mathrm{C}_{\mathbb{N}} \nleq_{\mathrm{W}} \mathrm{VC}_{\mathbf{X}}$ follows from Proposition 28. □

We can actually obtain some upper bounds for overt choice on countably-based spaces. Recall that $\Pi^0_2\mathrm{C}_{\mathbf{X}}$ takes as input a non-empty $\underset{\sim}{\mathbf{\Pi}}^0_2$-subset of $\mathbf{X}$ (coded in the usual way via an appropriate Borel code), and outputs an element of that set.

Proposition 31. *Let* $s : \mathbf{X} \to \mathbf{Y}$ *be a computable surjection, and* $\mathbf{Y}$ *be effectively countably-based. Then* $\mathrm{VC}_{\mathbf{Y}} \leqslant_{\mathrm{W}} \Pi^0_2\mathrm{C}_{\mathbf{X}}$.

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be an effective countable basis of $\mathbf{Y}$. Given $A \in \mathcal{V}(\mathbf{Y})$ we can compute $\{x \in \mathbf{X} \mid \forall n \in \mathbb{N}\ s(x) \in U_n \Rightarrow U_n \cap A \neq \emptyset\} \in \underset{\sim}{\mathbf{\Pi}}^0_2(\mathbf{X})$, apply $\Pi^0_2\mathrm{C}_{\mathbf{X}}$ to obtain an element x_0 of that set, and then notice that $s(x_0) \in A$. □

Corollary 32. *Let* $\mathbf{X}$ *be a* $\underset{\sim}{\mathbf{\Sigma}}^1_1$*-subspace of* $\mathcal{O}(\mathbb{N})$*. Then relative to some oracle it holds that* $\mathrm{VC}_{\mathbf{X}} \leqslant_{\mathrm{W}} \mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$.

Proof. A $\underset{\sim}{\mathbf{\Sigma}}^1_1$-subspace is the range of a continuous surjection from $\mathbb{N}^{\mathbb{N}}$. This is computable relative to some oracle, and the relativization of Proposition 31 then gives $\mathrm{VC}_{\mathbf{X}} \leqslant_{\mathrm{W}} \Pi^0_2\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$. That $\Pi^0_2\mathrm{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$ is straight-forward, it was observed e.g. in [20]. □

Corollary 33. *Let* $\mathbf{X}$ *be effectively countable and effectively countably-based. Then* $\mathrm{VC}_{\mathbf{X}} \leqslant_{\mathrm{W}} \Pi^0_2\mathrm{C}_{\mathbb{N}}$.

5.3. Overt choice for specific spaces

We consider overt choice for two specific countably-based yet not quasi-Polish spaces. The first space is $\mathbb{Q}$, seen as a subspace of $\mathbb{R}$. The second specimen is the space S_0 defined as follows:

Definition 34. The underlying set of S_0 is $\mathbb{N}^*$, the set of finite sequences of natural numbers (including the empty word ε). The topology is generated by the sets $\{\{u \in \mathbb{N}^* \mid u \not\succeq w\} \mid w \in \mathbb{N}^*\}$. This means membership in a basic open set provides the knowledge of finitely many words which are not a prefix of the given point.

Our choice of studying these specific spaces is not completely arbitrary: They both belong to the four canonic counter-examples for being quasi-Polish (meaning that a coanalytic subspace of $\mathcal{O}(\mathbb{N})$ is either quasi-Polish or contains a $\underset{\sim}{\mathbf{\Pi}}^0_2$ copy of one of the canonic counter-examples) [13]. Given our interest in the continuity of overt choice as a completeness notion, this makes these spaces high priority targets for classification.

Since $\mathbb{Q}$ (respectively S_0) is computably isomorphic to $\mathbb{Q} \times \mathbb{Q}$ (respectively to $S_0 \times S_0$), we can conclude the following from Corollary 10:

Corollary 35. $\mathrm{VC}_{\mathbb{Q}} \equiv_{\mathrm{W}} \mathrm{VC}^*_{\mathbb{Q}}$ *and* $\mathrm{VC}_{S_0} \equiv_{\mathrm{W}} \mathrm{VC}^*_{S_0}$.

Overt choice on $\mathbb{Q}$ was already studied by Brattka in [3] (without having the modern terminology available) and shown to be uncomputable. Together with the result from [10] that overt choice on Polish spaces is continuous, and the Hurewicz dichotomy stating that a coanalytic separable metric space is either Polish or has a copy of $\mathbb{Q}$ as a closed subspace, it already follows that:

Corollary 36 ([2]). *For a coanalytic separable metric space* $\mathbf{X}$ *we find that* $\mathrm{VC}_{\mathbf{X}}$ *is continuous iff* $\mathbf{X}$ *is Polish.*

The starting point of our investigation of the degree of $\mathrm{VC}_{\mathbb{Q}}$ will be to introduce a somewhat more accessible problem defined on trees. We say that a tree $T \subseteq \{0,1\}^*$ has eventually constant paths everywhere, if for each $w \in T$ there is $u \succ w$ and $b \in \{0,1\}$ such that for all $n \in \mathbb{N}$ we have $ub^n \in T$. In words, every vertex in the tree can be extended to a path that eventually goes always right or always left. The principle ECP has to find such an eventually constant path from an enumeration of the tree.

Definition 37. Let $\mathrm{ECP} :\subseteq \mathcal{O}(\{0,1\}^*) \rightrightarrows \{0,1\}^{\mathbb{N}}$ be defined by $T \in \mathrm{dom}(\mathrm{ECP})$ iff $T \neq \emptyset$ has eventually constant paths everywhere, and $p \in \mathrm{ECP}(T)$ if $p = w0^\omega$ or $p = w1^\omega$ for some $w \in \{0,1\}$ and $\forall n \in \mathbb{N}\ p_{\leqslant n} \in T$.

Proposition 38. ECP *is not computable.*

Proof. We describe a strategy how to construct an input on which a putative algorithm fails. Start by enumerating longer and longer prefixes of 01^ω. If the algorithm does not eventually output 01, then continuing to output all prefixes of 01^ω makes the algorithm fail. If the algorithm output 01 at the moment where the longest prefix enumerated so far is 01^{k_0}, then we enumerate all 0^n, as well as longer and longer prefixes of $01^{k_0}0^\omega$. The algorithm has to output $01^{k_0}0$ eventually. At that moment, we enumerate all prefixes of 01^ω, and start enumerating prefixes of $01^{k_0}0^{k_1}1^\omega$, where $01^{k_0}0^{k_1}$ is the longest prefix of $01^{k_0}0^\omega$ we enumerated so far. Continuing this process will force the algorithm to either deviate at some stage and fail, or to output a sequence with infinitely many alternations between 0 and 1, and hence fail, too. □

Proposition 39. $\mathrm{ECP} \equiv_{\mathrm{W}} \mathrm{VC}_{\mathbb{Q}}$.

Proof. We construct a translation between overt subsets of $[0,1]\cap\mathbb{Q}$ and enumerations of suitable trees. Let $(q_n)_{n\in\mathbb{N}}$ be some standard enumeration of $[0,1]\cap\mathbb{Q}$. We construct a basis $(B_w)_{w\in\{0,1\}^*}$ of $[0,1]\cap\mathbb{Q}$ as follows: $B_\varepsilon := [0,1]\cap\mathbb{Q}$, $B_0 := [0,\frac{\pi}{5}]\cap\mathbb{Q}$, $B_1 := [\frac{\pi}{5},1]\cap\mathbb{Q}$. Once $B_{wc} = (a,b)\cap\mathbb{Q}$ is defined, pick some irrational $\tau \in (\frac{2}{3}a+\frac{1}{3}b, \frac{1}{3}a+\frac{2}{3}b)$. Of the intervals $(a,\tau)\cap\mathbb{Q}$ and $(\tau,b)\cap\mathbb{Q}$ one contains the least rational (w.r.t. $(q_n)_{n\in\mathbb{N}}$). If it is $(a,\tau)\cap\mathbb{Q}$, then $B_{wcc} = (a,\tau)\cap\mathbb{Q}$ and $B_{wc\overline{c}} = (\tau,b)\cap\mathbb{Q}$ (here $\overline{c}$ denotes the complementary bit to c), otherwise the intervals are assigned in reversed roles.

The neighborhood filter of some $q \in \mathbb{Q}\cap[0,1]$ now corresponds to an eventually constant element $p \in \{0,1\}^{\mathbb{N}}$, and an overt set in $\mathcal{V}(\mathbb{Q}\cap[0,1])$ corresponds to an enumeration of a tree where each vertex can be extended into an eventually constant path. This constitutes the desired equivalence. □

Corollary 40. $\mathrm{VC}_{\mathbb{Q}} |_{\mathrm{W}} \mathrm{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathrm{C}_{\mathbb{Q}}$.

Proof. By Propositions 38, 39 $\mathrm{VC}_{\mathbb{Q}}$ is not computable, and we can thus apply Corollary 30 to conclude $\mathrm{VC}_{\mathbb{Q}} |_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$. That $\mathrm{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathrm{C}_{\mathbb{Q}}$ is Proposition 26. □

Corollary 41. $\mathrm{VC}_{\mathbb{Q}} \not\leq_{\mathrm{W}} \mathrm{C}_{\mathbb{R}}$.

Proof. Combine Corollary 40 and Corollary 27. □

[2] Note that Corollary 23 above shows that the restriction to coanalytic spaces is not actually necessary here.

Before moving on from $\mathrm{VC}_{\mathbb{Q}}$ to VC_{S_0} we shall examine the content of Brattka's proof from [3] that $\mathrm{VC}_{\mathbb{Q}}$ is non-computable. From his construction we extract the following definition:

Definition 42. Let HitSparse $:\subseteq \mathcal{A}(\mathbb{N}) \times \mathbb{N}^{\mathbb{N}} \to \mathcal{O}(\mathbb{N})$ be defined as follows:

- $(A, f) \in \mathrm{dom}(\mathrm{HitSparse})$ if A is infinite, and
- $U \in \mathrm{HitSparse}(A, f)$ if $U \cap A \neq \emptyset$ and $\forall n \in \mathbb{N}\ |[n, f(n)] \cap U| \leqslant 1$

The intuition is that we are trying to solve the usual discrete choice $\mathrm{C}_{\mathbb{N}}$, but are allowed to make infinitely many guesses. These guesses, however, have to be sparse – to make up for that, we assume that there are actually infinitely many correct solutions (on its own this requirement has no impact on the degree of $\mathrm{C}_{\mathbb{N}}$.

Theorem 43 (Brattka [3])**.**

(1) HitSparse *is not computable.*
(2) HitSparse $\leqslant_{\mathrm{W}} \mathrm{VC}_{\mathbb{Q}}$

Since trivially HitSparse $\leqslant_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$, from Corollary 40 it follows that:

Corollary 44. HitSparse $<_{\mathrm{W}} \mathrm{VC}_{\mathbb{Q}}$

To make VC_{S_0} more accessible, we again introduce a problem on trees. This time, we need a new represented space $\mathcal{S}\{0,1\}^*$ of finite sequences via the representation $\delta_{\mathcal{S}}$ defined inductively as $\delta_{\mathcal{S}}(0^\omega) = \varepsilon$, $\delta_{\mathcal{S}}(00p) = \delta_{\mathcal{S}}(11p) = \delta_{\mathcal{S}}(p)$, $\delta_{\mathcal{S}}(10p) = 0\delta_{\mathcal{S}}(p)$ and $\delta_{\mathcal{S}}(01p) = 1\delta_{\mathcal{S}}(p)$. Intuitively, if we are given $w \in \mathcal{S}\{0,1\}^*$ we never know for sure that we have seen the end of the finite sequence, for it can always be extended again.

Definition 45. Let FSL $:\subseteq \mathcal{O}(\{0,1\}^*) \rightrightarrows \mathcal{S}\{0,1\}^*$ be defined by $T \in \mathrm{dom}(\mathrm{FSL})$ if T is a non-empty tree such that there exists a leaf below any vertex, and $w \in \mathrm{FSL}(T)$ if w is a leaf of T.

Proposition 46. FSL *is non-computable.*

Proof. We describe how to diagonalize against a hypothetical algorithm solving FSL. The argument is essentially the same as in Proposition 38. We start off with the input $\{\varepsilon, 0\}$. The algorithm needs at some point to commit to output a leaf extending 0. At this point, we add 1 and 00 to the tree. Since the new leaves are 1 and 00, and the algorithm can no longer output 1, it needs to commit to 00 eventually. At that point, we add 01 and 000 to the tree, and so on. Either the algorithm will at some point fail to commit to an extension of the current output, and thus output an internal vertex, or it will commit infinitely often, and thereby not output a vertex at all. □

Proposition 47. FSL $\leqslant_{\mathrm{W}} \mathrm{VC}_{S_0}$

Proof. We construct some $A \in \mathcal{V}(S_0)$ from the tree $T \in \mathrm{dom}(\mathrm{FSL})$ by iteratively updating a partial mapping $\phi :\subseteq \{0,1\}^* \to \mathbb{N}^*$ such that if L is the set of leaves of our current approximation to T, then our current approximation to A is consistent with $A = \phi[L]$. If we learn at some point that w is not actually a leaf of T (because it has some extension $wi \in T$), we will have given a finite amount of information about A yet. In particular, there is some $N \in \mathbb{N}$ such that no mentioning of $\phi(w)N$ and $\phi(w)(N+1)$ has been given so far. This ensures that if we update our assumption that $\phi(w) \in A$ to either $\phi(w)N \in A$ or $\phi(w)(N+1) \in A$ this is consistent with all information given so far. We can thus set $\phi(w0) = \phi(w)N$ and $\phi(w1) = \phi(w)(N+1)$ without compromising our construction. The promise that there is a leaf below any vertex in T ensures that any candidate put into A will have a surviving candidate below it, which provides the well-definedness of A.

Let us assume that we are given some $u = \phi(w) \in A$ by VC_{S_0}. If we knew $\phi(w) \in \mathbb{N}^*$, we could obviously reconstruct $w \in \{0,1\}^*$ and complete the reduction. However, we only know $\phi(w) \in S_0$, but only need $w \in \mathcal{S}\{0,1\}^*$. In particular, we can wait with extending our current candidate w' for w until we learn that w indeed has an extension in T. But at that moment, we know the values of $\phi(w'0)$ and $\phi(w'1)$. Since at least one of them is not $\phi(w)$, we will eventually learn about $\phi(w)$ that it is either not below $\phi(w'0)$ or not below $\phi(w'1)$. But that answer then tells us how we should extend w' to obtain a longer prefix of w. □

Corollary 48. $\mathrm{FSL} <_{\mathrm{W}} \mathrm{VC}_{S_0}$

Proof. The reduction is Proposition 47. It is easy to see that $\mathrm{FSL} \leqslant_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$, as we can just guess a potential leaf and the time it will be enumerated into the tree. By Proposition 28 then $\mathrm{VC}_{S_0} \leqslant_{\mathrm{W}} \mathrm{FSL} \leqslant_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$ would imply that VC_{S_0} is computable, and thus also that FSL is computable by Proposition 47. But that contradicts Proposition 46. □

5.4. Open questions

We have presented our results on overt choice for countably-based non-quasi-Polish spaces not with the intention of concluding their investigation, but in the hope to spark further interest. We do not even dare to list a comprehensive list of open questions that we deem worthy of future work, but instead only list some prototypical questions.

If we consider upper bounds for overt choice amongst the usual Weihrauch degrees used for calibration in the literature, we see a huge gap between our negative result ruling out $\mathrm{C}_{\mathbb{N}}$ (Proposition 28) and the positive answer providing $\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$ as upper bound for a large class of spaces (Corollary 32). We thus ask whether this can be tightened. On the upper end of that gap, an initial question would be whether $\mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}$ might suffice.[3]

Open Question 49. Is there an effectively analytic effectively countably-based space $\mathbf{X}$ with $\mathrm{VC}_{\mathbf{X}} \not\leq_{\mathrm{W}} \mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}$?

On the lower end of the gap, comparing $\mathrm{VC}_{\mathbb{Q}}$ and VC_{S_0} with degrees such lim and Sort. Between $\mathrm{C}_{\mathbb{R}}$ not being an upper bound (Corollary 41) and $\Pi^0_2\mathrm{C}_{\mathbb{N}}$ serving as such (Corollary 33), these would seem to be the next suitable candidates:

Open Question 50. Are $\mathrm{VC}_{\mathbb{Q}}$ and/or VC_{S_0} reducible to lim, or even to Sort?

Open Question 51. How are $\mathrm{VC}_{\mathbb{Q}}$ and VC_{S_0} related?

It seems very desirable to study overt choice for a broader range of countably-based non-quasi-Polish spaces than just two examples (however well the choice of these is motivated). Another simple and natural example would be the space $\mathbb{N}_{\mathrm{cof}}$ of integers with the cofinite topology (this is essentially the subspace $\{\{n\} \in \mathcal{A}(\mathbb{N}) \mid n \in \mathbb{N}\} \subseteq \mathcal{A}(\mathbb{N})$. This space is the typical example of a T_1 non-T_2-space. As such, we know from Theorem 22 below that $\mathrm{VC}_{\mathbb{N}_{\mathrm{cof}}}$ is discontinuous – but without a concrete proof giving us a meaningful lower bound in the Weihrauch lattice.

Open Question 52. What else can we say about $\mathrm{VC}_{\mathbb{N}_{\mathrm{cof}}}$?

6. Overt choice for CoPolish spaces

6.1. Background on CoPolish spaces

In general, non-countably based spaces are often very difficult to understand (see e.g. [17]). A nice class of not-necessarily countably-based topological spaces is formed by the class of CoPolish spaces. They play a role in Type-2-Complexity Theory [27] by allowing simple complexity. Concrete examples of CoPolish spaces relevant for analysis include the space of polynomials over the reals, the space of analytic functions and the space of compactly-supported continuous real functions.

Definition 53 ([27]). A *CoPolish space* $\mathbf{X}$ is the direct limit of an increasing sequence of compact metrisable subspaces $\mathbf{X}_k$.

Any CoPolish space is a Hausdorff normal qcb-space. We present a characterization of CoPolish spaces.

Proposition 54 ([27]). *Let* $\mathbf{X}$ *be a Hausdorff qcb-space. Then the following are equivalent:*

[3] While a number of intermediate (between $\mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}$ and $\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$) principles were studied in [20], $\mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}$ still seems like a reasonable step down from $\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$.

(1) $\mathbf{X}$ *is a CoPolish space.*
(2) *The space* $\mathcal{O}(\mathbf{X})$ *of open subsets of* $\mathbf{X}$ *equipped with the Scott-topology is a quasi-Polish space and* $\mathbf{X}$ *is regular.*
(3) $\mathbf{X}$ *has an admissible representation with a locally compact domain.*
(4) $\mathbf{X}$ *has a countable pseudobase consisting of compact subsets.*

In the realm of countably-based Hausdorff spaces, Copolishness is just local compactness.

Lemma 55. *A countably-based Hausdorff space is CoPolish if, and only if, if it is locally compact.*

Proof. Let $\mathbf{X}$ be a locally compact Hausdorff space with countable basis $\mathcal{B}$. Then the countable subfamily $\mathcal{B}'$ of basic open sets whose closure is compact forms a basis as well. The family $\mathcal{K}$ of the closures of all sets in $\mathcal{B}'$ is then a countable pseudobase for $\mathbf{X}$ consisting of compact sets.

Conversely, let $\mathbf{X}$ be the direct limit of a increasing sequence of compact metrisable subspaces $\mathbf{X}_m$. Let x be a point in $\mathbf{X}$ with countable neighbourhood basis $\{B_i \mid i \in \mathbb{N}\}$. Assume for contradiction $\bigcap_{i=0}^{n} B_i \nsubseteq \mathbf{X}_n$ for all $n \in \mathbb{N}$. Then for every n there exists some $y_n \in \bigcap_{i=0}^{n} B_i \setminus \mathbf{X}_n$. Clearly $(y_n)_n$ converges to x, so there is some m such that $\{x, y_n \mid n \in \mathbb{N}\} \subseteq \mathbf{X}_m$, a contradiction.

We conclude that $\mathbf{X}_m$ is a compact neighbourhood of x. By Hausdorffness this implies that $\mathbf{X}$ is locally compact. $\square$

CoPolish spaces can be separated into three classes, the countably-based ones, the non-countably-based Fréchet–Urysohn ones and the non-Fréchet–Urysohn ones.

6.2. Fréchet–Urysohn spaces

A topological space X is called *Fréchet–Urysohn*, if the closure of any subset M is equal to the set of all limits of sequences in M. Any countably-based space and any metrisable space is a Fréchet–Urysohn space. We present an example of a Fréchet–Urysohn CoPolish space $\mathbf{T}_{\min}$ that does not have a countable base. In Lemma 58 we will see that $\mathbf{T}_{\min}$ is a minimal such space.

Example 56. The underlying set of $\mathbf{T}_{\min}$ is $\mathbb{N}^2 \cup \{\infty\}$. A basis $\mathcal{B}_{\mathbf{T}_{\min}}$ of the topology is given by the sets

$$\{(a,b)\} \quad \text{and} \quad U_\ell := \{\infty\} \cup \{(a,b) \in \mathbb{N}^2 \mid b \geqslant \ell(a)\}$$

for all $(a,b) \in \mathbb{N}^2$ and $\ell \in \mathbb{N}^{\mathbb{N}}$. Clearly, $\mathbf{T}_{\min}$ is the direct limit of the compact subspaces $\mathbf{X}_m$ that have $\{\infty\} \cup \{(a,b) \mid a \leqslant m\}$ as their respective underlying sets. So $\mathbf{T}_{\min}$ is CoPolish. It is Fréchet–Urysohn, because it is sequential and has only one point that does not form an open singleton. A computably admissible representation $\delta_{\mathbf{T}_{\min}}$ for $\mathbf{T}_{\min}$ has $\{m0^\omega, m0^b(a+1)0^\omega \mid a, b, m \in \mathbb{N}, a \leqslant m\}$ as its locally compact domain. It maps $m0^\omega$ to ∞ and $m0^b(a+1)0^\omega$ to (a,b).

Overt choice for $\mathbf{T}_{\min}$ is not computable, because all-or-co-unique-choice on the natural numbers, denoted by $\mathrm{ACC}_{\mathbb{N}}$ in [7], is Weihrauch-reducible to $\mathrm{VC}_{\mathbf{T}_{\min}}$.

Proposition 57. $\mathrm{ACC}_{\mathbb{N}} \leqslant_{\mathrm{sW}} \mathrm{VC}_{\mathbf{T}_{\min}}$.

Proof. $\mathrm{ACC}_{\mathbb{N}}$ is the problem of finding an element in a given set A in the family $\{\mathbb{N}, \mathbb{N} \setminus \{i\} \mid i \in \mathbb{N}\} \subseteq \mathcal{A}(\mathbb{N})$. A computably admissible representation ψ of this family is given by

$$\psi(0^\omega) := \mathbb{N} \quad \text{and} \quad \psi(0^j(i+1)0^\omega) := \mathbb{N} \setminus \{i\}.$$

We define the preprocessor $K\colon \operatorname{dom}(\psi) \to \mathcal{V}(\mathbf{T}_{\min})$ to map 0^ω to the set $\{\infty\}$ and $0^j(i+1)0^\omega$ to the closed set $\{(i+1, j), (0, i)\}$. To prove that K is computable, it suffices to show that for the basic sets $B \in \mathcal{B}_{\mathbf{T}_{\min}}$ (see Example 56) the test $K(r) \cap B \neq \emptyset$ is semi-decidable. For all $r \in \operatorname{dom}(\psi)$ and $a, b \in \mathbb{N}$ we have

$$K(r) \cap \{(a,b)\} \neq \emptyset \iff \bigl(a > 0 \wedge r = 0^b a 0^\omega\bigr) \vee \bigl(a = 0 \wedge \exists j \in \mathbb{N}.\, r = 0^j(b+1)0^\omega\bigr).$$

Obviously, the right-hand side is semi-decidable in $(r, (a, b)) \in \operatorname{dom}(\psi) \times \mathbb{N}^2$. Now we define a function $k\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}$ by

$$k(\ell) := \max\bigl\{0, \ell(i+1) \mid 0 \leqslant i < \ell(0)\bigr\}.$$

The closed set $K(r)$ intersects U_ℓ if, and only if, one of the following holds:

(1) $0^{k(\ell)}$ is a prefix of r;
(2) there are $j < k(\ell)$ and $i \in \mathbb{N}$ such that $r = 0^j(i+1)0^\omega$ and $j \geqslant \ell(i+1) \vee i \geqslant \ell(0)$.

Note that if $0^{k(\ell)}$ is a prefix of $r = 0^j(i+1)0^\omega$, then $i < \ell(0)$ implies $(i+1, j) \in K(r) \cap U_\ell$ and $i \geqslant \ell(0)$ implies $(0, i) \in K(r) \cap U_\ell$. Since k is computable, (1) and (2) are semi-decidable in $(r, \ell) \in \operatorname{dom}(\psi) \times \mathbb{N}^\mathbb{N}$. We conclude that K is computable.

Let the computable postprocessor $H\colon \operatorname{dom}(\delta_{\mathbf{T}_{\min}}) \to \mathbb{N}$ be defined by

$$H\bigl(m0^\omega\bigr) := m \quad \text{and} \quad H\bigl(m0^b(a+1)0^\omega\bigr) := \begin{cases} m+1 & \text{if } b = m \\ m & \text{otherwise.} \end{cases}$$

Let $p \in \operatorname{dom}(\psi)$ and $A := \psi(p)$. If a realizer G of overt choice applied to (a name of) $K(p)$ returns $m0^\omega$, then the input set A is $\mathbb{N}$, so that m is a legitimate result. If G returns $m0^b(a+1)0^\omega$, then A is either $\mathbb{N} \setminus \{a-1\}$ or $\mathbb{N} \setminus \{b\}$. As $a \leqslant m$, we have $H(m0^b(a+1)0^\omega) \in A$ as required. □

We list a few properties of Fréchet–Urysohn CoPolish spaces that will be instrumental to understand the complexity of their overt choice principles.

Lemma 58. *Let $\mathbf{X}$ be a Fréchet–Urysohn CoPolish space.*

(1) *The subspace $\mathbf{X}_\omega$ of the points in $\mathbf{X}$ that have a countable neighbourhood base is open.*
(2) *The complement $\mathbf{X}_{\mathrm{nc}} := \mathbf{X} \setminus \mathbf{X}_\omega$ forms a closed and discrete subspace of $\mathbf{X}$.*
(3) *If $\mathbf{X}_{\mathrm{nc}} \neq \emptyset$, then $\mathbf{T}_{\min}$ embeds into $\mathbf{X}$ as a closed subspace.*

Proof. Let $(\mathbf{X}_m)_m$ an increasing sequence of compact metrisable subspaces such that $\mathbf{X}$ is the direct limit of $(\mathbf{X}_m)_m$.

(1) In the proof of Lemma 55 we have seen that for any point x with a countable neighbourhood base there is some m such that x is in the interior $\operatorname{int}(\mathbf{X}_m)$ of some $\mathbf{X}_m$. Since $\mathbf{X}_m$ has a countable base, any point in the interior of $\mathbf{X}_m$ has a countable neighbourhood base in $\mathbf{X}$, namely the one in the subspace $\operatorname{int}(\mathbf{X}_m)$. Hence $\mathbf{X}_\omega = \bigcup_m \operatorname{int}(\mathbf{X}_m)$ is open.
(2) For any point $x \in \mathbf{X}_{\mathrm{nc}}$ and any $m \in \mathbb{N}$ there is a sequence $(y_i)_i$ outside $\mathbf{X}_m$ which converges to x, as otherwise x were in the interior of $\mathbf{X}_m$ due to the Fréchet–Urysohn property, which would imply $x \in \mathbf{X}_\omega$ by the discussion in item (1).
Assume that there exists an injective sequence $(x_n)_n$ in $\mathbf{X}_{\mathrm{nc}}$ that converges to some point $x_\infty \in \mathbf{X}_{\mathrm{nc}}$. W.l.o.g. $x_\infty \notin \{x_n \mid n \in \mathbb{N}\}$. Set $m_{-1} := \min\{i \in \mathbb{N} \mid \{x_\infty, x_n \mid n \in \mathbb{N}\} \subseteq \mathbf{X}_i\}$. By the above observation we can construct an increasing sequence $(m_a)_a$ of natural numbers strictly above m_{-1} and a double sequence $(y_{a,b})_{a,b}$ such that $(y_{a,b})_b$ converges to x_a and $y_{a,b} \in \mathbf{X}_{m_a} \setminus \mathbf{X}_{m_{a-1}}$ for every a, b. Obviously, x_∞ is in the closure of $\{y_{a,b} \mid a, b \in \mathbb{N}\}$. So there are functions $s, t\colon \mathbb{N} \to \mathbb{N}$ such that $(y_{s(i),t(i)})_i$ converges to x_∞ by the

Fréchet–Urysohn property. Then s is bounded, because any convergent sequence in $\mathbf{X}$ is contained in some subspace $\mathbf{X}_m$. But then there is a subsequence of $(y_{s(i),t(i)})_i$ converging either to some $x_n \neq x_\infty$ or to some $y_{a,b} \neq x_\infty$. This contradicts the Hausdorff property.
We conclude that in $\mathbf{X}_{\mathrm{nc}}$ all converging sequences are eventually constant. Since $\mathbf{X}_{\mathrm{nc}}$ is sequential by being a closed subspace of a qcb-space and Hausdorff, $\mathbf{X}_{\mathrm{nc}}$ is discrete. Discrete qcb-spaces are at most countable because of the existence of a countable pseudobase.

(3) Choose some point $x \in \mathbf{X}_{\mathrm{nc}}$. Set $m_{-1} := \min\{i \mid x \in \mathbf{X}_i\}$. In a similar way as in the proof of (2), we construct an increasing sequence $(m_a)_a$ of natural numbers strictly above m_{-1} and a double sequence $(y_{a,b})_{a,b}$ such that $(y_{a,b})_b$ converges (now) to x and $y_{a,b} \in \mathbf{X}_{m_a} \setminus \mathbf{X}_{m_{a-1}}$ for every a, b. Since $\mathbf{X}$ is Hausdorff and $y_{a,b} \neq x$, for all a the sequence $(y_{a,b})_b$ contains an injective subsequence $(z_{a,j})_j$. We define $e\colon \mathbf{T}_{\min} \to \mathbf{X}$ by $e(\infty) := x$ and $e(a, b) := z_{a,b}$. By construction e is injective and continuous. Moreover if $(t_n)_n$ is an injective sequence in $\mathbf{T}_{\min}$ such that $(e(t_n))_n$ converges to some point y in $\mathbf{X}$, then $y = x$ by the Hausdorffness of $\mathbf{X}$ and again by the fact that any convergent sequence is contained in some $\mathbf{X}_m$. So the image of e is closed and e reflects converging sequences (meaning that $(t_n)_n$ converges to t_∞, whenever $(e(t_n))_n$ converges to $e(t_\infty)$). Therefore $\mathbf{T}_{\min}$ embeds topologically into $\mathbf{X}$ as a closed subspace. □

Theorem 59. *Let $\mathbf{X}$ be a Fréchet–Urysohn CoPolish space. Then overt choice for $\mathbf{X}$ is continuous if, and only if, $\mathbf{X}$ is countably-based.*

Proof. If $\mathbf{X}$ has a countable base, then $\mathbf{X}$ is locally compact and therefore a Polish space. Hence overt choice for $\mathbf{X}$ is continuous by Corollary 23.

If $\mathbf{X}$ is not countably-based, then $\mathbf{X}$ is not first-countable by [26, Proposition 3.3.1], thus $\mathbf{X}_{\mathrm{nc}} \neq \emptyset$. Therefore $\mathbf{T}_{\min}$ embeds topologically into $\mathbf{X}$ as a closed subspace by Lemma 58. Since $\mathrm{VC}_{\mathbf{T}_{\min}}$ is discontinuous by Proposition 57, $\mathrm{VC}_{\mathbf{X}}$ is discontinuous as well. □

Overt choice for Fréchet–Urysohn CoPolish spaces turns out to have LPO as an upper bound in the topological Weihrauch lattice. Remember that $\mathrm{LPO}\colon \mathbb{N}^{\mathbb{N}} \to \{0, 1\}$ is defined by $\mathrm{LPO}(r) = 1 :\Longleftrightarrow \exists k \in \mathbb{N}.\, r(k) = 0$.

Theorem 60. *Let $\mathbf{X}$ be a Fréchet–Urysohn CoPolish space. Then $\mathrm{VC}_{\mathbf{X}} \leqslant^{\mathrm{t}}_{\mathrm{W}} \mathrm{LPO}$.*

Proof. Given a positive name p of a non-empty closed set A, we first use LPO to decide whether or not A intersects the open set $\mathbf{X}_\omega$.

(1) If it does, we proceed as follows. By being an open subspace of a CoPolish space, $\mathbf{X}_\omega$ is CoPolish as well, because those elements of a countable compact pseudobase that are contained in $\mathbf{X}_\omega$ form a countable compact pseudobase for $\mathbf{X}_\omega$. Since $\mathbf{X}_\omega$ is first-countable, it has a countable base by [26, Proposition 3.3.1]. Therefore $\mathbf{X}_\omega$ is locally compact and Polish. Since $\mathcal{O}(\mathbf{X}_\omega)$ is a retract of $\mathcal{O}(\mathbf{X})$, we can continuously convert the given name of A into a positive name of the closed subset $A \cap \mathbf{X}_\omega$ in the space $\mathbf{X}_\omega$. Now we can apply the algorithm implicitly provided by Corollary 23 to obtain an element of $A \cap \mathbf{X}_\omega$.

(2) Now we consider the case $A \subseteq \mathbf{X}_{\mathrm{nc}}$. Since $\mathbf{X}_{\mathrm{nc}}$ is a discrete qcb-space, it is countable. So there are elements z_i with $\{z_i \mid i \in \mathbb{N}\} = \mathbf{X}_{\mathrm{nc}}$. By Lemma 58 the sets $W_i := \{z_i\} \cup \mathbf{X}_\omega$ are open. By dovetailing we systematically search for a set W_i that intersects A. Once we have found one, we output the corresponding element z_i. □

Remark 61. If we require that the subspace $\mathbf{X}_\omega$ of $\mathbf{X}$ is computable equivalent to a computable Polish space, the set $\mathbf{X}_\omega$ is computably open in $\mathbf{X}$ and the elements of $\mathbf{X}_{\mathrm{nc}}$ form a computable sequence, then we have $\mathrm{VC}_{\mathbf{X}} \leqslant_{\mathrm{W}} \mathrm{LPO}$.

We proceed to show that the reduction in Theorem 60 is strict by revealing the weakness of $\mathrm{VC}_{\mathbf{X}}$ for Fréchet–Urysohn spaces $\mathbf{X}$. Again we use a technical lemma:

Lemma 62. *Let $\mathbf{X}$ be a admissibly represented Fréchet–Urysohn space, and let $(A_n)_{n\leqslant\infty}$ be a sequence of non-empty closed subsets. Then $(A_n)_n$ converges to A_∞ in $\mathcal{V}(\mathbf{X})$ if, and only if, for any $x_\infty \in A_\infty$ and any strictly increasing function $\varphi\colon \mathbb{N} \to \mathbb{N}$ there is a sequence $(x_n)_n$ converging to x_∞ and a strictly increasing function $\xi\colon \mathbb{N} \to \mathbb{N}$ with $x_n \in A_{\varphi\xi(n)}$ for all $n \in \mathbb{N}$.*

Proof. The backward direction is obvious. For the forward direction, let $(A_n)_n$ be a sequence of non-empty closed sets converging in $\mathcal{V}(\mathbf{X})$ to A_∞ and let $x_\infty \in A_\infty$. It suffices to consider $\varphi = \mathrm{id}_{\mathbb{N}}$. If x_∞ is contained in A_n for infinitely many n, then we simply choose $x_n = x_\infty$ and ξ as the strictly increasing function with $\mathrm{range}(\xi) = \{n \in \mathbb{N} \mid x_\infty \in A_n\}$.

Otherwise there is some n_0 with $x_\infty \notin \bigcup_{i \geqslant n_0} A_i$. Since $(A_n)_{n \geqslant n_0}$ converges to A_∞ with respect to the lower fell topology, x_∞ is in the closure of $\bigcup_{i \geqslant n_0} A_i$. By the Fréchet–Urysohn property, there is a sequence in $(y_m)_m \in \bigcup_{i \geqslant n_0} A_i$ converging to x_∞. As the closed set $\bigcup_{i=n_0}^{n} A_i$ does not contain x_∞, it contains y_n for finitely many n's. So we have a strictly increasing sequence $(m_n)_n$ such that $y_m \notin \bigcup_{i=n_0}^{n} A_i$ for all $m \geqslant m_n$. We inductively define $x_0 := y_0$, $\xi(0) := \min\{i \geqslant n_0 \mid x_0 \in A_i\}$, $x_{k+1} := y_{m_{\xi(k)}}$ and $\xi(k+1) := \min\{i \geqslant n_0 \mid x_{k+1} \in A_i\}$. Clearly $\xi(k+1) > \xi(k)$ and thus $m_{\xi(k+1)} > m_{\xi(k)}$. So $(x_k)_k$ converges to x_∞. □

Recall that ACC_m is the problem of finding an element in a given closed set A in the family $\{M, M \setminus \{i\} \mid i \in \mathbb{N}\}$, where $M = \{0, \dots, m-1\}$ (see [7]). Note that $\mathrm{ACC}_2 \equiv_{\mathrm{W}} \mathrm{LLPO}$. A computably admissible representation ψ of this family is given by

$$\psi\big(0^\omega\big) := M \quad \text{and} \quad \psi\big(0^j(i+1)0^\omega\big) := M \setminus \{i\}.$$

Theorem 63. *Let $\mathbf{X}$ be an admissibly represented Fréchet–Urysohn space and let $m \geqslant 2$. Then $\mathrm{ACC}_m \nleqslant_{\mathrm{W}} \mathrm{VC}_{\mathbf{X}}$.*

Proof. Assume there were continuous functions $K\colon \mathrm{dom}(\psi) \to \mathcal{V}(\mathbf{X})$ and $H\colon \mathrm{dom}(\psi) \times \mathrm{dom}(\delta_{\mathbf{X}}) \to \mathbb{N}$ witnessing $\mathrm{ACC}_m \leqslant_{\mathrm{W}} \mathrm{VC}_{\mathbf{X}}$.

We choose some $x_\infty \in K(0^\omega)$. Let $a \in M = \{0, \dots, m-1\}$. Since $K(0^n(a+1)0^\omega)$ converges to $K(0^\omega)$, by the above lemma there is a strictly increasing function $\xi_a\colon \mathbb{N} \to \mathbb{N}$ and a sequence $(y_{a,n})_n$ converging to x_∞ such that $y_{a,n} \in K(0^{\xi_a(n)}(a+1)0^\omega)$. The sequence $(x_n)_n := (y_{n \bmod m, n \operatorname{div} m})_n$ converges to x_∞ as well, because M is finite. Since $\delta_{\mathbf{X}}$ is admissible, there is a sequence $(s_n)_n$ converging to some name s_∞ of x_∞ such that $\delta_{\mathbf{X}}(s_n) = x_n$. Now we consider $b := H(0^\omega, s_\infty)$. For almost all n we have $H(0^{\xi_b(n)}(b+1)0^\omega, s_{mn+b}) = b$ and $\delta_{\mathbf{X}}(s_{mn+b}) = y_{b,n} \in K(0^{\xi_b(n)}(b+1)0^\omega)$, contradicting $b \notin \psi(0^{\xi_b(n)}(b+1)0^\omega) = M \setminus \{b\}$. □

6.3. Non-Fréchet–Urysohn spaces

Now we turn our attention to non-Fréchet–Urysohn T_1-spaces. First we show that overt choice for them is above LPO in the continuous Weihrauch lattice.

Theorem 64. *Let $\mathbf{Y}$ be an admissibly represented space such that its topology is T_1, but not Fréchet–Urysohn. Then* $\mathrm{LPO} \leqslant_{\mathrm{W}}^{\mathrm{t}} \mathrm{VC}_{\mathbf{Y}}$.

Proof. We choose a subset M such that on the one hand there is some y in the closure of M, but on the other hand y is not the limit of any sequence in M. By [26, Proposition 3.3.1], M equipped with the subsequential topology contains a dense sequence $(z_i)_i$. For $n \in \mathbb{N}$ we define the closed set A_n by $\{z_i \mid i \leqslant n\}$. Then $(A_n)_n$ converges to $A_\infty := \{y\}$ in the lower Fell topology. Since the standard positive representation of closed is admissible w.r.t. the lower Fell topology by [26, Proposition 4.4.5], there is a sequence $(p_n)_n$ of names for the A_n's converging to some name p_∞ of A_∞. We define a continuous function $K\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$K(r) := \begin{cases} p_\infty & \text{if } r \text{ does not contain } 0 \\ p_m & \text{if } m := \min\{k \in \mathbb{N} \mid r(k) = 0\} \text{ exists.} \end{cases}$$

The singleton $\{y\}$ is sequentially open in the subspace of $\mathbf{Y}$ with underlying set $\{y, z_i \mid i \in \mathbb{N}\}$, because no sequence in $\{z_i \mid i \in \mathbb{N}\}$ converges to y. By the T_1-property, $\{y\}$ is even clopen. So there is continuous function $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \{0,1\}$ such that

$$\delta_{\mathbf{Y}}(r) = y \implies H(r) = 1 \quad \text{and} \quad \delta_{\mathbf{Y}}(r) \in \{z_i \mid i \in \mathbb{N}\} \implies H(r) = 0.$$

Clearly, for any realizer G of overt choice for $\mathbf{Y}$ the function HGK is a realizer for LPO. Hence $\mathrm{LPO} \leqslant_{\mathrm{sW}}^{\mathrm{t}} \mathrm{VC}_{\mathbf{Y}}$. □

We present an example of a CoPolish non-Fréchet–Urysohn space $\mathbf{S}_{\min}$ for which overt choice is Weihrauch equivalent to LPO.

Example 65. We choose $\{(\infty,\infty)\} \cup (\mathbb{N} \times \{\infty\}) \cup \mathbb{N}^2$ as the underlying set of $\mathbf{S}_{\min}$. The topology of $\mathbf{S}_{\min}$ is induced by the basis consisting of the sets

- $\{(a,b)\}$,
- $\{(a,\infty)\} \cup \{(a,j) \mid j \geqslant b\}$,
- $\{(\infty,\infty)\} \cup \{(i,\infty) \mid i \geqslant a\} \cup \{(i,j) \mid i \geqslant a, j \geqslant \ell(i)\}$

for all $a,b \in \mathbb{N}$, $\ell \in \mathbb{N}^{\mathbb{N}}$. The space is not Fréchet–Urysohn, because (∞,∞) belongs to the closure of $\mathbb{N}^2$, but fails to be a limit of any sequence in $\mathbb{N}^2$. An admissible representation $\delta_{\mathbf{S}_{\min}}$ of $\mathbf{S}_{\min}$ has $\{i0^\omega, i0^j1^\omega \mid i,j \in \mathbb{N}\}$ as its locally compact domain and is defined by

$$\delta_{\mathbf{S}_{\min}}(00^\omega) := (\infty,\infty), \qquad \delta_{\mathbf{S}_{\min}}(00^a1^\omega) := \delta_{\mathbf{S}_{\min}}\big((a+1)0^\omega\big) := (a,\infty), \qquad \delta_{\mathbf{S}_{\min}}\big((a+1)0^b1^\omega\big) := (a,b)$$

for all $a,b \in \mathbb{N}$. A proof of admissibility can be found in [26, Example 2.3.15]. The space $\mathbf{S}_{\min}$ is the direct limit of its compact subspaces $\mathbf{X}_m = \{(\infty,\infty),(a,\infty),(i,b) \mid a,b \in \mathbb{N}, i \leqslant m\}$. Hence $\mathbf{S}_{\min}$ is CoPolish.

The space $\mathbf{S}_{\min}$ is even a minimal non-Fréchet–Urysohn CoPolish space.

Proposition 66. *Any CoPolish space that is not Fréchet–Urysohn contains a copy of* $\mathbf{S}_{\min}$ *as a closed subspace.*

Proof. Let $\mathbf{X}$ be a non Fréchet–Urysohn CoPolish space. Let $\mathbf{X}_m$ be an increasing sequence of compact metrisable subspaces such that $\mathbf{X}$ is the direct limit of $(\mathbf{X}_m)_m$. As $\mathbf{X}$ is a non-Fréchet–Urysohn sequential space, there are a double sequence $(z_{s,t})_{s,t}$, a sequence $(y_s)_s$ and a point x such that

(1) $\lim_{s\to\infty} y_s = x$;
(2) $\lim_{t\to\infty} z_{s,t} = y_s$ for all $s \in \mathbb{N}$;
(3) no sequence in $\{z_{s,t} \mid s,t \in \mathbb{N}\}$ converges to x.

Since $\mathbf{X}$ is Hausdorff, we can additionally assume that

4. $j \mapsto y_j$ and $t \mapsto z_{s,t}$ are injective for all $s \in \mathbb{N}$.

Now the problem arises that the set of these points might not be closed in $\mathbf{X}$. We inductively construct strictly increasing sequences $(m_a)_{a\geqslant -1}$, $(s_a)_{a\geqslant -1}$ and $(t_a)_{a\geqslant -1}$ as follows:

- "$a = -1$": Set $s_{-1} := t_{-1} := 0$. As every convergent sequence of $\mathbf{X}$ lies in some subspace $\mathbf{X}_m$, the number $m_{-1} := \min\{m \in \mathbb{N} \mid \{x, y_j \mid j \in \mathbb{N}\} \subseteq \mathbf{X}_m\}$ exists.
- "$a-1 \to a$": Since $\mathbf{X}_{m_{a-1}}$ is Fréchet–Urysohn by being metrisable and no sequence in $\{z_{s,t} \mid s > s_{a-1}, t \in \mathbb{N}\}$ converges to x, there are some $s_a > s_{a-1}$ and $t_a > t_{a-1}$ such that $\{z_{s_a,t} \mid t \geqslant t_a\} \cap \mathbf{X}_{m_{a-1}} = \emptyset$. Set $m_a := \min\{m > m_{a-1} \mid \{y_{s_a}, z_{s_a,t} \mid t \geqslant t_a\} \subseteq \mathbf{X}_m\}$.

We define $e\colon \mathbf{S}_{\min} \to \mathbf{X}$ by

$$e(\infty,\infty) := x, \qquad e(a,\infty) := y_{s_a}, \qquad e(a,b) := z_{s_a,t_a+b}.$$

By the construction, e is injective and sequentially continuous. Now let $(v_n)_n$ be a sequence in $\mathbf{S}_{\min}$ such that $(e(v_n))_n$ converges to some $w \in \mathbf{X}$. Then there is some $m \in \mathbb{N}$ such that $\{e(v_n) \mid n \in \mathbb{N}\} \subseteq \mathbf{X}_m$. We define $k := \min\{a \in \mathbb{N} \mid m_a \geqslant m\}$ and observe

$$\big\{e(v_n) \mid n \in \mathbb{N}\big\} \subseteq A := \{x, y_{s_j}, z_{s_a,t_a+j} \mid a \leqslant k, j \in \mathbb{N}\} \subseteq e[\mathbf{S}_{\min}].$$

Since A is closed and $\mathbf{X}$ is Hausdorff, we have $w \in e[\mathbf{S}_{\min}]$. Hence $e[\mathbf{S}_{\min}]$ is closed. Moreover, as $\mathbf{X}$ is Hausdorff, there exists a unique $v_\infty \in \mathbf{S}_{\min}$ with $e(v_\infty) = w$. By the construction of e the sequence $(v_n)_n$ converges to

v_∞ in $\mathbf{S}_{\min}$. Therefore e reflects convergent sequences. As $e[\mathbf{S}_{\min}]$ is closed and $\mathbf{S}_{\min}$, $\mathbf{X}$ are sequential, e is an homeomorphic embedding of $\mathbf{S}_{\min}$ into $\mathbf{X}$. □

Now we show that overt choice for $\mathbf{S}_{\min}$ is Weihrauch-equivalent to LPO.

Theorem 67. $\mathrm{VC}_{\mathbf{S}_{\min}} \equiv_{\mathrm{W}} \mathrm{LPO}$.

Proof. For the direction $\mathrm{LPO} \leqslant_{\mathrm{W}} \mathrm{VC}_{\mathbf{S}_{\min}}$ we effectivize the proof of Theorem 64. We choose a computable pairing function $\langle\cdot,\cdot\rangle$ and define $A_n := \{(a,b) \mid \langle a,b\rangle \leqslant n\}$. Clearly, the sequence $(A_n)_n$ converges effectively to the singleton $\{(\infty,\infty)\}$ in the space of closed sets with the positive representation. So we have a computable sequence $(p_n)_n$ of names of the A_n's which converges computably to a name p_∞ of $\{(\infty,\infty)\}$. Therefore the preprocessor K defined like in the proof of Theorem 64 is computable. The computable postprocessor $H\colon \mathrm{dom}(\delta_{\mathbf{S}_{\min}}) \to \{0,1\}$ can be defined by $H(s) = 1 :\Longleftrightarrow s(0) = 0$. Clearly, K and H witness $\mathrm{LPO} \leqslant_{\mathrm{sW}} \mathrm{VC}(\mathbf{S}_{\min})$.

To show $\mathrm{VC}_{\mathbf{S}_{\min}} \leqslant_{\mathrm{W}} \mathrm{LPO}$, we first employ LPO to decide whether or not the computably open set $\mathbf{X}_\omega := \mathbf{S}_{\min} \setminus \{(\infty,\infty)\}$ intersects the given non-empty closed set A. If not, then we output the element (∞,∞). In the positive case we employ the fact that the space $\mathbf{X}_\omega$ forms a computable Polish space. Moreover, we can compute a name of $A \cap \mathbf{X}_\omega$ in the positive representation for the closed subsets of $\mathbf{X}_\omega$. Hence the algorithm from Theorem 20 computes for us an element of $A \cap \mathbf{X}_\omega \neq \emptyset$. □

6.4. Upper bound for overt choice for coPolish spaces

Consider the map $\mathrm{isFull}_{\mathbb{N}^\mathbb{N}} : \mathcal{O}(\mathbb{N}^\mathbb{N}) \to \mathbb{S}$ mapping $\mathbb{N}^\mathbb{N}$ to $\top$ and any $U \neq \mathbb{N}^\mathbb{N}$ to $\bot$. Essentially, the discontinuity of this map expresses how far Baire space is from being compact. Concretely, $\mathrm{isFull}_{\mathbb{N}^\mathbb{N}}$ can be seen as mapping well-founded trees to $\top$ and ill-founded trees to $\bot$, which suggests another perspective: Let the space $\Sigma_{\Pi^1_1}$ have the two elements $\top$ and $\bot$, with $p \in \mathbb{N}^\mathbb{N}$ being a name for $\top$ iff p codes a well-founded tree, and a name for $\bot$ iff it codes an ill-founded tree. The map $\mathrm{id} : \Sigma_{\Pi^1_1} \to \Sigma$ essentially lets us treat a single Π^1_1-set as an open set. It trivially holds that $(\mathrm{id} : \Sigma_{\Pi^1_1} \to \Sigma) \equiv_{\mathrm{W}} \mathrm{isFull}_{\mathbb{N}^\mathbb{N}}$. We find that $\mathrm{isFull}_{\mathbb{N}^\mathbb{N}} \leqslant_{\mathrm{W}} \mathrm{TC}_{\mathbb{N}^\mathbb{N}}$ for the principle $\mathrm{TC}_{\mathbb{N}^\mathbb{N}}$ introduced and studied in [20], and that $\Pi^1_1\mathrm{CA} \equiv_{\mathrm{W}} \lim \star \widehat{\mathrm{isFull}_{\mathbb{N}^\mathbb{N}}}$. In particular, it holds that $\mathrm{isFull}_{\mathbb{N}^\mathbb{N}} \nleq_{\mathrm{W}} \mathrm{C}_{\mathbb{N}^\mathbb{N}}$.

Theorem 68. *Let* $\mathbf{X}$ *be coPolish. Then* $\mathrm{VC}_{\mathbf{X}} \leqslant^{\mathrm{t}}_{\mathrm{W}} \widehat{\mathrm{isFull}_{\mathbb{N}^\mathbb{N}}}$.

Proof. We use the characterization of $\mathbf{X}$ as being a direct limit of a sequence of compact Polish spaces $\mathbf{K}_0 \hookrightarrow \mathbf{K}_1 \hookrightarrow \cdots$. Given some basic open set σ of $\mathbf{K}_\ell$ and $f \in \mathbb{N}^\mathbb{N}$, we inductively define $U^0(\sigma, f) = \sigma$ and $U^{n+1}(\sigma, f) = \{x \in \mathbf{K}_{\ell+n+1} \mid d(x, U^n(\sigma, f)) < 2^{-f(n)}\}$. Then let $U(\sigma, f) \subseteq \mathbf{X}$ be the corresponding direct limit. Note that $U(\sigma, f)$ is open in $\mathbf{X}$, and that the sets of the form $U(\sigma, f)$ form a basis of $\mathbf{X}$.

Now given some $A \in \mathcal{V}(\mathbf{X})$, we find that $\sigma \cap A \neq \emptyset$ iff $\forall f \in \mathbb{N}^\mathbb{N}\ U(\sigma, f) \cap A \neq \emptyset$. Since $U(\sigma, f) \cap A \neq \emptyset$ is an open property, we will recognize that it holds true for some f based on some finite prefix of f. From A and σ we can thus construct a tree T such that the paths through T are exactly those f with $A \cap U(\sigma, f) = \emptyset$. Thus, we can use $\widehat{\mathrm{isFull}_{\mathbb{N}^\mathbb{N}}}$ to obtain a list of all basic open sets in any $\mathbf{K}_i$ intersecting A. Once we have identified a $\mathbf{K}_i$ with $\mathbf{K}_i \cap A \neq \emptyset$, this lets us obtain $\mathbf{K}_i \cap A \in \mathcal{V}(\mathbf{K}_i)$, from which we can compute a point $x \in \mathbf{K}_i \cap A$ since overt choice for Polish spaces is computable. We then translate $x \in \mathbf{K}_i$ into $x \in \mathbf{X}$. □

6.5. Summary

We obtain the following corollary to Theorems 60, 63, 64 and 68. It shows that the topological Weihrauch degree of overt choice for a CoPolish space characterizes whether or not the space is countably-based, and whether or not the space has the Fréchet–Urysohn property:

Corollary 69. *For a CoPolish space* $\mathbf{X}$ *exactly one of the following cases holds:*

(1) $\mathbf{X}$ *is Polish and* $\mathrm{VC}_{\mathbf{X}}$ *is continuous.*
(2) $\mathbf{X}$ *is not countably-based Fréchet–Urysohn, and* $\mathrm{ACC}_{\mathbb{N}} \leqslant^{\mathrm{t}}_{\mathrm{W}} \mathrm{VC}_{\mathbf{X}} <^{\mathrm{t}}_{\mathrm{W}} \mathrm{LPO}$.
(3) $\mathbf{X}$ *is not countably-based not Fréchet–Urysohn, and* $\mathrm{LPO} \leqslant^{\mathrm{t}}_{\mathrm{W}} \mathrm{VC}_{\mathbf{X}} \leqslant^{\mathrm{t}}_{\mathrm{W}} \widehat{\mathrm{isFull}_{\mathbb{N}^{\mathbb{N}}}}$.

How much the Fréchet–Urysohn property fails for a sequential space can be characterized by the ordinal invariant σ defined in [1]. σ specifies how many times you need to iterate sequential closures to get the closure of an arbitrary subset. We wonder whether a more precise classification of overt choice for CoPolish spaces might be achievable depending on σ. Note that the interval of the Weihrauch lattice we know $\mathrm{VC}_{\mathbf{X}}$ to fall into in this case contains the Baire hierarchy.

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References

[1] A.V. Arhangel'skii and S.P. Franklin, Ordinal invariants for topological spaces, *Michigan Math. J.* **15**(3) (1968), 313–320. doi:10.1307/mmj/1029000034.
[2] V. Becher and S. Grigorieff, Borel and Hausdorff hierarchies in topological spaces of Choquet games and their effectivization, *Mathematical Structures in Computer Science* **25**(7) (2015), 1490–1519. doi:10.1017/S096012951300025X.
[3] V. Brattka, Random numbers and an incomplete immune recursive set, in: *Automata, Languages and Programming, 29th International Colloquium, ICALP 2002, Malaga, Spain, July 8–13, 2002, Proceedings*, 2002, pp. 950–961. doi:10.1007/3-540-45465-9_81.
[4] V. Brattka, M. de Brecht and A. Pauly, Closed choice and a uniform low basis theorem, *Annals of Pure and Applied Logic* **163**(8) (2012), 968–1008. doi:10.1016/j.apal.2011.12.020.
[5] V. Brattka and G. Gherardi, Weihrauch degrees, omniscience principles and weak computability, *Journal of Symbolic Logic* **76** (2011), 143–176, arXiv:0905.4679. doi:10.2178/jsl/1294170993.
[6] V. Brattka and G. Gherardi, Effective choice and boundedness principles in computable analysis, *Bulletin of Symbolic Logic* **17** (2011), 73–117, arXiv:0905.4685. doi:10.2178/bsl/1294186663.
[7] V. Brattka, G. Gherardi and A. Pauly, Weihrauch complexity in computable analysis, arXiv:1707.03202, 2017.
[8] V. Brattka, M. Hendtlass and A. Kreuzer, On the uniform computational content of computability theory, *Theory of Computing Systems* **61**(4) (2017), 1376–1426. doi:10.1007/s00224-017-9798-1.
[9] V. Brattka and A. Pauly, On the algebraic structure of Weihrauch degrees, *Logical Methods in Computer Science* **14**(4) (2018), http://arxiv.org/abs/1604.08348. doi:10.23638/LMCS-14(4:4)2018.
[10] V. Brattka and G. Presser, Computability on subsets of metric spaces, *Theoretical Computer Science* **305**(1–3) (2003), 43–76. doi:10.1016/S0304-3975(02)00693-X.
[11] R. Chen, Notes on quasi-Polish spaces, arXiv:1809.07440, 2018.
[12] M. de Brecht, Quasi-Polish spaces, *Annals of Pure and Applied Logic* **164**(3) (2013), 354–381.
[13] M. de Brecht, A generalization of a theorem of Hurewicz for quasi-Polish spaces, *Logical Methods in Computer Science* **14**(1) (2018). doi:10.23638/LMCS-14(1:13)2018.
[14] F. Dorias and C. Mummert, Stationary and convergent strategies in Choquet games, *Fundamenta Mathematicae* **209** (2019), 59–79. doi:10.4064/fm209-1-5.
[15] M. Escardó, Synthetic topology of datatypes and classical spaces, *Electronic Notes in Theoretical Computer Science* **87** (2004).

[16] G. Gherardi and A. Marcone, How incomputable is the separable Hahn–Banach theorem?, *Notre Dame Journal of Formal Logic* **50**(4) (2009), 393–425. doi:10.1215/00294527-2009-018.
[17] M. Hoyrup, Results in descriptive set theory on some represented spaces, arXiv:1712.03680, 2017.
[18] M. Hoyrup, C. Rojas, V. Selivanov and D.M. Stull, Computability on quasi-Polish spaces, arXiv:1903.02979, 2019.
[19] M. Hoyrup, C. Rojas, V.L. Selivanov and D.M. Stull, Computability on quasi-Polish spaces, in: *Descriptional Complexity of Formal Systems (DCFS) 2019, Proceedings*, 2019, pp. 171–183. doi:10.1007/978-3-030-23247-4_13.
[20] T. Kihara, A. Marcone and A. Pauly, Searching for an analogue of ATR in the Weihrauch lattice, arXiv:1812.01549, 2018.
[21] T. Kihara and A. Pauly, Point degree spectra of represented spaces, arXiv:1405.6866, 2014.
[22] M. Korovina and O. Kudinov, On higher effective descriptive set theory, in: *Computability in Europe 2017 – Unveiling Dynamics and Complexity*, J. Kari, F. Manea and I. Petre, eds, LNCS, Springer, 2017, pp. 282–291. doi:10.1007/978-3-319-58741-7_27.
[23] E. Michael, Selected selection theorems, *The American Mathematical Monthly* **63**(4) (1956), 233–238. doi:10.2307/2310346.
[24] E. Neumann, Universal envelopes of discontinuous functions, PhD thesis, Aston University, 2018.
[25] A. Pauly, On the topological aspects of the theory of represented spaces, *Computability* **5**(2) (2016), 159–180, http://arxiv.org/abs/1204.3763. doi:10.3233/COM-150049.
[26] M. Schröder, Admissible representations for continuous computations, PhD thesis, FernUniversität Hagen, 2002.
[27] M. Schröder, Spaces allowing type-2 complexity theory revisited, *Mathematical Logic Quarterly* **50**(4/5) (2004), 443–459. doi:10.1002/malq.200310111.
[28] V. Selivanov, Towards the effective descriptive set theory, in: *Evolving Computability: CiE 2015*, A. Beckmann, V. Mitrana and M. Soskova, eds, Springer, 2015, pp. 324–333. doi:10.1007/978-3-319-20028-6_33.
[29] V.L. Selivanov, Difference hierarchy in φ-spaces, *Algebra and Logic* **43**(4) (2004), 238–248. doi:10.1023/B:ALLO.0000035115.44324.5d.
[30] V.L. Selivanov, Towards a descriptive set theory for domain-like structures, *Theoretical Computer Science* **365**(3) (2006), 258–282. doi:10.1016/j.tcs.2006.07.053.
[31] J. van Mill, J. Pelant and R. Pol, Selections that characterize topological completeness, *Fundamenta Mathematicae* **149**(2) (1996), 127–141, https://eudml.org/doc/212112.
[32] K. Weihrauch, *The Degrees of Discontinuity of Some Translators Between Representations of the Real Numbers*, Informatik Berichte, Vol. 129, FernUniversität Hagen, Hagen, 1992.
[33] K. Weihrauch, *The TTE-Interpretation of Three Hierarchies of Omniscience Principles*, Informatik Berichte, Vol. 130, FernUniversität Hagen, Hagen, 1992.