Existence and Uniqueness Theorems for Formal Power Series Solutions of Analytic Differential Systems

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Abstract

We present Existence and Uniqueness Theorems for formal power series solutions of analytic systems of PDE in a certain form. This form can be obtained by a finite number of differentiations and eliminations of the original system, and allows its formal power series solutions to be computed in an algorithmic fashion.

The resulting reduced involutive form (`rif'` form) produced by our `rif'` algorithm is a generalization of the classical form of Riquier and Janet, and that of Cauchy–Kovalevskaya. We weaken the assumption of linearity in the highest derivatives in those approaches to allow for systems which are nonlinear in their highest derivatives.

A new formal development of Riquier's theory is given, with proofs, modeled after those in Gröbner Basis Theory. For the nonlinear theory, the concept of relative Riquier Bases is introduced. This allows for the easy extension of ideas from the linear to the nonlinear theory. The essential idea is that an arbitrary nonlinear system can be written (after differentiation if necessary), as a system which is linear in its highest derivatives, and a constraint system, which is nonlinear in its highest derivatives. Our theorems are applied to several examples.

1 Introduction

The Cauchy–Kovalevskaya Theorem [11] gives an Existence and Uniqueness Theorem for analytic solutions to systems of analytic PDE in a certain form. For example, it applies to a PDE of form $u_{yy} = F(u, u_x, u_y, u_{xx}, u_{xy})$ subject to initial conditions $u(x, y^0) = f(x), u_y(x, y^0) = g(x)$ where f, g, F are analytic functions of their arguments. Riquier loosened the restriction of Cauchy–Kovalevskaya's form, which required that it has the same number of unknown functions, as equations. For example, the classical Riquier Existence and Uniqueness Theorem [16] can, under suitable conditions, be applied to the system $u_{yy} = G(u, u_x, u_y)$, $u_{xy} = H(u, u_x, u_y)$, for analytic G, H, while the Cauchy–Kovalevskaya Theorem [11] can not. The Riquier-Janet Theory retained the restriction that each PDE is written in solved form for one of its highest derivatives.

In this paper we weaken the solved form requirements to obtain Existence and Uniqueness Theorems for formal power series solutions of analytic differential systems that are nonlinear in their highest derivatives.

The theoretical development of this paper parallels that of Gröbner Basis Theory. In Section 2 we introduce notation, rankings and give our definition of (differential) reduction. In Section 3 we give a new development of essential aspects of the Riquier Janet Theory. Our approach avoids a notoriously complicated process for identifying integrability conditions in the classical Riquier–Janet Theory. That process has been an obstacle to its widespread application. In Section 4, we give a proof of a Formal Riquier Existence Theorem for formal power series solutions. In Section 5, we describe our theoretical development of the nonlinear theory which hinges on our concept of Relative Riquier Bases. This allows techniques from the linear theory to be easily extended to the nonlinear case. Applications are discussed in Section 6.

For polynomially nonlinear differential systems, alternative methods for constructing formal power series solutions have been developed based on the work of Kolchin [10] and Seidenberg[21]. For such systems, using a result of Rosenfeld[17], Boulier et al [3] obtained an algorithmic solution for determining radical differential ideal membership. The differential elimination algorithms of these and related approaches are based on extending Ritt's concept of a (differential) characteristic set [3, 12].

Applications of differential elimination algorithms are given in [20, 7, 13, 5, 4, 9, 8, 3].

Our paper is based on results in Rust's Ph.D thesis [18] which is available on the web.[1]

2 Preliminaries

2.1 Functions and Derivatives

In theoretical arguments we represent differential equations as functions, much as in algebra polynomials have long since largely replaced polynomial equations. Thus an equation that is customarily written in the form LHS = RHS is instead represented by the function LHS − RHS.

Let n be a positive integer, representing the number of dependent variables $u = (u^1, ..., u^n)$ and let m be a

[1] C. Rust's PhD thesis, on which this paper is based, is available at http://www.cecm.sfu.ca/~rust or http://www.cecm.sfu.ca/~reid

nonnegative integer, representing the number of independent variables $x = (x_1, ..., x_m)$ in a system of PDE. Let $\mathbb{N} := \{0, 1, 2, ...\}$ denote the natural numbers including 0 and let $\mathbb{N}_n := \{1, ..., n\}$. Let $\Delta = \{\delta^i_\alpha | i \in \mathbb{N}_n, \alpha \in \mathbb{N}^m\}$ be a set of indeterminates, with $\alpha = (a_1, a_2, ..., a_m)$ corresponding to the various derivatives of the dependent variables with respect to the independent variables:

$$\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2} \cdots \left(\frac{\partial}{\partial x_m}\right)^{a_m} u^i(x) \longleftrightarrow \delta^i_\alpha$$

Let $\tilde{\Delta} := \{x_1, ..., x_m\} \cup \Delta$. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let f be an $\mathbb{F}$-analytic function of $\tilde{\Delta}$.[2] One should think of f as representing an $\mathbb{F}$-analytic differential equation. *We always insist that functions depend on only finitely many of the indicated variables.* We also insist that the domain is an open subset of the cartesian product $\mathbb{F}^{\tilde{\Delta}_0}$, where $\tilde{\Delta}_0$ is the finite subset of $\tilde{\Delta}$ on which the function depends. For $i \in \mathbb{N}_n$, the *total derivative* of f with respect to x_i is defined as:

$$D_i f := \frac{\partial f}{\partial x_i} + \sum_{\delta \in \Delta} \frac{\partial f}{\partial \delta} D_i \delta,$$

where $D_i \delta^j_{(a_1,...,a_m)} := \delta^j_{(a_1,...,a_j+1,...,a_m)}$. For $\alpha = (a_1, ..., a_m) \in \mathbb{N}^m$, define $D_\alpha f := (D_1)^{a_1} \cdots (D_m)^{a_m} f$.

Let x^0 be a point in $\mathbb{F}^m$ and let $u(x) = (u^1(x), ..., u^n(x))$ be a vector of formal power series in $\mathbb{F}[[x - x^0]]^n$. We write $D_\alpha u^i(x)$ to denote the usual (term by term) partial derivative of $u^i(x)$ with respect to $\alpha = (a_1, ..., a_m) \in \mathbb{N}^m$:

$$D_\alpha u^i(x) = (D_1)^{a_1} \cdots (D_m)^{a_m} u^i(x).$$

For $v(x) \in \mathbb{F}[[x - x^0]]$, $v(x^0)$ denotes the constant term of $v(x)$. If f is analytic at the point $\left(x^0, (D_\alpha u^i(x^0))_{\delta^i_\alpha \in \Delta}\right)$, let $f[u](x)$ denote the formal power series at x^0 given by

$$f[u](x) := f\left(x, (D_\alpha u^i(x))_{\delta^i_\alpha \in \Delta}\right).$$

In the above equation, the subscript "$\delta^i_\alpha \in \Delta$" indicates that $D_\alpha u^i(x)$ is to be substituted in the argument of f corresponding to δ^i_α, for each $\delta^i_\alpha \in \Delta$.

2.2 Rankings and Highest Derivatives

A *positive ranking* $\leq$ of Δ is a total order of Δ which is compatible with differentiation

$$\delta^i_\alpha < \delta^j_\beta \Longrightarrow \delta^i_{\alpha+\gamma} < \delta^j_{\beta+\gamma}$$

and which is a well-ordering

$$\delta^i_0 \leq \delta^i_\alpha,$$

for $\alpha, \beta, \gamma \in \mathbb{N}^m$, $i, j \in \mathbb{N}_n$. *For the rest of this paper, let a positive ranking $\leq$ be fixed.*

For f a function of $\tilde{\Delta}$, we write $\text{HD} f$ for the highest derivative (highest with respect to $\leq$) which actually occurs in f.

[2] It is also possible to give a treatment where the PDE are assumed to be formal power series, but certain restrictions are necessary, which we will discuss in future work

Note that $\text{HD} f$ is undefined if f depends only on $\{x_1, ..., x_m\}$ and does not depend on Δ. In this case, we insist as a matter of notational convenience that $\text{HD} f \leq \text{HD} g$ for arbitrary g. For the rest of this paragraph, let f be an analytic function of $\tilde{\Delta}$. We say that f is *$\leq$-monic* if f has the form $f = \text{HD} f + F$, with $\text{HD} F < \text{HD} f$. Following the terminology of [15], we say that f is *$\leq$-leading linear* if there exists an analytic function g of $\tilde{\Delta}$ with $\text{HD} g < \text{HD} f$ such that f/g is $\leq$-monic. We say that g (which is easily seen to be unique) is the *leading coefficient* of f, denoted $\text{HC} f$. We say that f is *$\leq$-leading nonlinear* if $\text{HD} f$ is well-defined, but f is not $\leq$-leading linear.

2.3 Reductions

For this subsection, let a finite set $\mathcal{M}$ of $\leq$-monic analytic functions of $\tilde{\Delta}$ be fixed. Let g, h be functions of $\tilde{\Delta}$. We say that h is a *one step reduction* of g if there exist $f \in \mathcal{M}$ and $\alpha \in \mathbb{N}^m$ such that, with $\delta_0 := \text{HD} D_\alpha f$, h can be given by substituting $\delta_0 - D_\alpha f$ for δ_0 in g:

$$h = g(x, (\delta)_{\delta \neq \delta_0}, (\delta_0 - D_\alpha f)_{\delta = \delta_0}).$$

This is denoted $g \longmapsto h$ or, if we want to be more specific, $g \overset{(\alpha,f)}{\longmapsto}_{\mathcal{M}} h$. See Example 1 for an explicit calculation.

We say that $u(x) \in \mathbb{F}[[x - x^0]]^n$ (for some $x^0 \in \mathbb{F}^m$) is a *solution* to a system of analytic PDE if $f[u](x)$ is well-defined and $f[u](x) = 0$ for all f in the system. *Throughout this paper, all solutions to differential equations are assumed to be formal power series solutions unless otherwise specified.* Suppose that $u(x) \in \mathbb{F}[[x - x^0]]^n$ is a solution to $\mathcal{M}$. Clearly $D_\alpha f[u](x) = 0$ for all $\alpha \in \mathbb{N}^m$ and $f \in \mathcal{M}$. Therefore for g, h analytic, if h is a one step reduction of g then $h[u](x)$ is well-defined if and only if $g[u](x)$ is well-defined and in this case $g[u](x) = h[u](x)$.

We say that g *reduces* to h if h can be obtained from g by a finite chain of one step reductions. That is, g reduces to h if there exists a positive integer k and $h_1, ..., h_k$ such that

$$g = h_1 \longmapsto h_2 \longmapsto \ldots \longmapsto h_k = h.$$

We write $g \longrightarrow h$ or $g \overset{\mu}{\longrightarrow}_{\mathcal{M}} h$ where μ is of the form

$$\mu = ((\alpha_1, f_1), ..., (\alpha_{k-1}, f_{k-1})) \qquad (1)$$

with $h_i \overset{(\alpha_i, f_i)}{\longmapsto}_{\mathcal{M}} h_{i+1}$ for $i \in \mathbb{N}_{k-1}$. We also write $h = \text{red}(g, \mu)$. For $u(x)$ a solution to $\mathcal{M}$, we again have for g, h analytic that $h[u](x)$ is well-defined if and only if $g[u](x)$ is well-defined and in this case $g[u](x) = h[u](x)$. Therefore, u is a solution to $\mathcal{M} \cup \{g\}$ iff u is a solution to $\mathcal{M} \cup \{h\}$.

We say that g *completely reduces* to h if g reduces to h and h reduces to h' implies that $h = h'$. By a standard Dickson's Lemma argument (see Section 4.4 of [18]) any infinite chain

$$h_0 \longmapsto h_1 \longmapsto \ldots$$

is eventually constant.

Example 1 (Non-uniqueness of complete reductions) *Let $m = 1$, $n = 2$. Let $\mathcal{M} = \{u_x - u, u_y - u + y^2\}$ and $g = u_{xy}$. Reducing g first by $\frac{\partial}{\partial y}(u_x - u)$ and then by $u_y - u + y^2$ we obtain: $g \longmapsto u_y \longmapsto u - y^2$, which cannot be reduced any further. If instead we reduce g by $\frac{\partial}{\partial x}(u_y - u + y^2)$ and then by $u_x - u$ we obtain: $g \longmapsto u_x \longmapsto u$, which again* *cannot be reduced further, illustrating the non-uniqueness of complete reductions. Note that the above computations are independent of the ranking $\leq$, since the only comparisons used were $u_x > u$ and $u_y > u$ which are true for any positive ranking.*

Following traditional terminology, we define the *principal derivatives* of $\mathcal{M}$ to be those derivatives that may be obtained by differentiating the highest derivative of an element of $\mathcal{M}$:

$$\operatorname{Prin}\mathcal{M} \quad := \quad \{\delta \in \Delta \mid \text{ there exist } f \in \mathcal{M} \text{ and } \alpha \in \mathbb{N}^n \text{ with } \delta = D_\alpha \,\mathrm{HD}\, f\}.$$

The *parametric derivatives* of $\mathcal{M}$, which we denote $\operatorname{Par}\mathcal{M}$, are those derivatives that are not principal:

$$\operatorname{Par}\mathcal{M} := \Delta \setminus \operatorname{Prin}\mathcal{M}.$$

Note that a reduction h of g is a complete reduction iff h depends on $\{x\} \cup \operatorname{Par}\mathcal{M}$ only.

3 Leading Linear Theory

In this section, we give a new development of the essential aspects of the Riquier–Janet Theory, motivated by algebraic Gröbner basis theory. *For this paper, fix a non-empty open subset U of $\mathbb{F}^{\tilde{\Delta}}$ and a finite set $\mathcal{M}$ of $\leq$-monic analytic functions on U which are polynomial in* $\operatorname{Prin}\mathcal{M}$. The condition that $\mathcal{M}$ be polynomial in $\operatorname{Prin}\mathcal{M}$ is not an onerous restriction. In particular, orthonomic systems - and therefore the usual standard forms [16, 14, 20] – satisfy this condition. Note that the class of analytic functions on U that are polynomial in $\operatorname{Prin}\mathcal{M}$ has the useful properties that it is closed under algebraic operations, differentiation and reductions by $\mathcal{M}$. This property is used in the proof of the Formal Riquier Existence Theorem to show that the Initial Data map is well-defined.

3.1 Riquier Bases

In this section, we introduce Riquier bases (cf. Reid's "standard forms" [14] and Schwarz's "Janet bases" [20]), which are the differential analogs of Gröbner bases. We prove the uniqueness of complete reductions with respect to a Riquier basis, which is the analog of the uniqueness of normal forms with respect to a Gröbner basis. This uniqueness is the key to the proof of our version of the Riquier Existence Theorem for formal power series solutions to analytic systems of PDE in Riquier basis form (Theorem 2).

Lemma 1 *Let f, f' be elements of $\mathcal{M}$ and g an analytic function on U that is polynomial in* $\operatorname{Prin}\mathcal{M}$. *Let α, β be elements of $\mathbb{N}^n$ and let μ be as in Section 2.3. Let h and k denote the one step reductions: $h = \operatorname{red}(g, (\alpha, f))$ and $k = \operatorname{red}(g, (\beta, f'))$. Then:*

1. *If $\mathrm{HD}\, D_\alpha f < \mathrm{HD}\, D_\beta f'$ then*

$$\operatorname{red}(h, ((\beta, f'), (\alpha, f))) = \operatorname{red}(k, (\alpha, f)).$$

In particular, there exists an analytic function l of U that is polynomial in $\operatorname{Prin}\mathcal{M}$, *such that $h \longrightarrow l$ and $k \longrightarrow l$.*

2. *If $D_\alpha f - D_\beta f' \xrightarrow{\mu} 0$, then*

$$\operatorname{red}(h, \mu) = \operatorname{red}(k, \mu).$$

In particular, there exists l such that $h \longrightarrow l$ and $k \longrightarrow l$.

Proof: Let $\delta_1 = \mathrm{HD}\, D_\alpha f$ and $\delta_2 = \mathrm{HD}\, D_\beta f'$. If $\delta_1 < \delta_2$, we have:

$$\begin{aligned}
&\operatorname{red}(k, D_\alpha f)\\
&\quad = g(x, (\delta)_{\delta \neq \delta_1, \delta_2}, (\delta_1 - D_\alpha f)_{\delta=\delta_1},\\
&\qquad (\delta_2 - D_\beta f'(x, (\delta)_{\delta \neq \delta_1}, (\delta_1 - D_\alpha f)_{\delta = \delta_1})_{\delta=\delta_2})\\
&\quad = \operatorname{red}(h, ((\beta, f'), (\alpha, f))).
\end{aligned}$$

If $\delta_1 = \delta_2$ and $D_\alpha f - D_\beta f' \xrightarrow{\mu} 0$, then:

$$\begin{aligned}
&\operatorname{red}(h, \mu)\\
&= g\left(x, (\operatorname{red}(\delta, \mu))_{\delta \neq \delta_1}, (\operatorname{red}(\delta - D_\alpha f, \mu))_{\delta=\delta_1}\right)\\
&= g\left(x, (\operatorname{red}(\delta, \mu))_{\delta \neq \delta_1}, (\operatorname{red}(\delta, \mu) - \operatorname{red}(D_\alpha f, \mu))_{\delta=\delta_1}\right)\\
&= g\left(x, (\operatorname{red}(\delta, \mu))_{\delta \neq \delta_1}, (\operatorname{red}(\delta, \mu) - \operatorname{red}(D_\beta f'; \mu))_{\delta=\delta_1}\right)\\
&= \operatorname{red}(k, \mu).
\end{aligned}$$

■

Lemma 2 *Fix $\delta \in \Delta$. Suppose that for all $\alpha, \alpha' \in \mathbb{N}^n$ and $f, f' \in \mathcal{M}$ with $\mathrm{HD}\, D_\alpha f = \mathrm{HD}\, D_{\alpha'} f' \leq \delta$, we have $D_\alpha f - D_{\alpha'} f' \longrightarrow 0$. Let g be an analytic function on U that is polynomial in* $\operatorname{Prin}\mathcal{M}$ *with $\mathrm{HD}\, g \leq \delta$. Suppose $g \longrightarrow h$ and $g \longrightarrow k$. Then there exists l with $h \longrightarrow l$ and $k \longrightarrow l$. In particular, g has a unique complete reduction.*

This lemma follows from the previous lemma. The proof [18] is the virtually the same as in the algebraic case (for example see Theorem 4.75, pp. 176-7 of [1]).

The analog of the Gröbner basis in the differential context is given by the following

Definition 1 *We say that $\mathcal{M}$ is a* Riquier basis *if for all $\alpha, \alpha' \in \mathbb{N}^n$ and $f, f' \in \mathcal{M}$ with $\mathrm{HD}\, D_\alpha f = \mathrm{HD}\, D_{\alpha'} f'$, the integrability condition $D_\alpha f - D_{\alpha'} f'$ can be reduced to* 0.

A "standard form" in the sense of Reid [14] is shown in [18] to be a Riquier basis $\mathcal{S}$ such that each $f \in \mathcal{S}$ is completely reduced with respect to $\mathcal{S} \setminus \{f\}$. This more restrictive notion corresponds to a reduced Gröbner basis in the algebraic theory. We therefore use the term *reduced Riquier basis* for this.

An immediate consequence of Lemma 2 is:

Theorem 1 *Suppose that $\mathcal{M}$ is a Riquier basis and g is an analytic function on U that is polynomial in* $\operatorname{Prin}\mathcal{M}$. *Then g has a unique complete reduction.*

We denote the complete reduction of g by $\operatorname{red}(g, \mathcal{M})$. The uniqueness of complete reductions is the key idea underlying the formal Riquier Existence Theorem treated in the next section.

4 The Formal Riquier Existence Theorem

In this section, we give the proof of the formal Riquier Existence Theorem (Theorem 2), which gives an Existence and Uniqueness result for formal power series solutions to systems of analytic PDE in Riquier basis form.

A *specification of initial data* for $\mathcal{M}$ is a map

$$\text{ID} : \{x\} \cup \operatorname{Par}\mathcal{M} \longrightarrow \mathbb{F}.$$

Note that $\{x\} \cup \operatorname{Par}\mathcal{M}$ may be infinite. For $x^0 \in \mathbb{F}^m$, we say that ID is a specification at x^0 if $\text{ID}(x) = x^0$. (Note that we are abusing notation slightly. In reality, the domain of ID is $\{x_1, ..., x_m\} \cup \operatorname{Par}\mathcal{M}$ and we should write $(\text{ID}(x_1), ..., \text{ID}(x_m)) = x^0$ not $\text{ID}(x) = x^0$.) For g a function of $\tilde{\Delta}$, let $\text{ID}(g)$ be the function of the principal derivatives obtained from g by evaluating x and the parametric derivatives using ID:

$$\text{ID}(g)((\delta)_{\delta\notin\operatorname{Par}\mathcal{M}}) := g\left(\text{ID}(x), (\text{ID}(\delta))_{\delta\in\operatorname{Par}\mathcal{M}}, (\delta)_{\delta\notin\operatorname{Par}\mathcal{M}}\right).$$

Example 2 *Let $m = 2$ and $n = 1$. Let $\mathcal{M} = \{u_{xx}, u_{yy} - xe^u\}$. Then $\operatorname{Par}\mathcal{M} = \{u, u_x, u_y, u_{xy}\}$. Let ID be given by $\text{ID}(x) = \text{ID}(y) = \text{ID}(u) = 0$, $\text{ID}(u_x) = 1$, $\text{ID}(u_y) = 2$ and $\text{ID}(u_{xy}) = 3$. Then $\text{ID}\left(u_{xyy} + xu_x + u_y^2\right) = u_{xyy} + 4$.*

Lemma 3 *Let $\mathcal{G}$ be a finite set of functions of $\tilde{\Delta}$. Then there exists μ such that $\operatorname{red}(g, \mu)$ is a complete reduction of g for all $g \in \mathcal{M}$.*

The easy proof of this lemma is given in [18].

In particular, if $\mathcal{M}$ is a Riquier basis and $\mathcal{G}$ is a finite set of analytic functions of $\tilde{\Delta}$ that are polynomial in $\operatorname{Prin}\mathcal{M}$, then there exists μ such that $\operatorname{red}(g, \mu) = \operatorname{red}(g, \mathcal{M})$ for all $g \in G$.

Theorem 2 (Formal Riquier Existence Theorem) *Suppose that $\mathcal{M}$ us a Riquier basis. Fix $x^0 \in \mathbb{F}^m$. Let ID be a specification of initial data for $\mathcal{M}$ at x^0 such that $\text{ID}(f)$ is well-defined for all $f \in \mathcal{M}$. Then there is a unique solution $u(x) \in \mathbb{F}[[x - x^0]]^n$ to $\mathcal{M}$ at x^0 such that $D_\alpha u^i(x^0) = \text{ID}(\delta^i_\alpha)$ for all $\delta^i_\alpha \in \operatorname{Par}\mathcal{M}$. Furthermore, every solution to $\mathcal{M}$ at x^0 may be obtained in this way for some ID.*

Proof: Let ID be defined as in the statement of the theorem. Define $u(x) \in \mathbb{F}[[x - x^0]]^n$ by

$$D_\alpha u^i(x^0) := \text{ID}\left(\operatorname{red}(\delta^i_\alpha, \mathcal{M})\right) \tag{2}$$

for $i \in \mathbb{N}_n$ and $\alpha \in \mathbb{N}^m$. We must verify that $u(x)$ is a well-defined solution to $\mathcal{M}$. Note that $\operatorname{red}(\delta^i_\alpha, \mathcal{M})$ depends only on the parametric derivatives and so $\text{ID}(\operatorname{red}(\delta^i_\alpha, \mathcal{M}))$ is an element of $\mathbb{F}$, so long as it is well-defined.

We must first check that $\text{ID}(\operatorname{red}(\delta^i_\alpha, \mathcal{M}))$ is well-defined. Let g be an analytic function of $\tilde{\Delta}$ such that $\text{ID}(g)$ is well-defined. Recall that any partial derivative of g is defined wherever g is. It follows that for h any partial or total derivative of g, $\text{ID}(h)$ is also well-defined. Suppose that g is polynomial in the principal derivatives. Take $\alpha \in \mathbb{N}^m$ and $f \in \mathcal{M}$. Let δ' be the highest derivative of $D_\alpha f$. We have:

$$\operatorname{red}(g, (\alpha, f)) = \sum_{k=0}^{\deg_{\delta'} g} \frac{\partial^k g}{\partial \delta'^k}(-D_\alpha f)^k$$

and so $\text{ID}(\operatorname{red}(g, (\alpha, f)))$ is also well-defined. This result, coupled with our assumption that $\mathcal{G}$ is polynomial in $\operatorname{Prin}\mathcal{M}$, shows that $\text{ID}(\operatorname{red}(\delta^i_\alpha, \mathcal{M}))$ is well-defined as required.

Clearly, $u(x)$ satisfies $D_\alpha u^i(x^0) = \text{ID}(\delta^i_\alpha)$ for all $\delta^i_\alpha \in \operatorname{Par}\mathcal{M}$. To verify that $u(x)$ is a solution to $\mathcal{M}$, it suffices to verify that $(D_\beta f[u])(x^0) = 0$ for all $f \in \mathcal{M}$ and $\beta \in \mathbb{N}^m$. Fix f, β. We have

$$\begin{aligned}
(D_\beta f[u])(x^0) &= (D_\beta f)\left(x^0, (D_\alpha u^i(x^0))\right) \\
&= (D_\beta f)\left(x^0, (\text{ID}(\operatorname{red}(\delta, \mathcal{M})))\right) \\
&= (D_\beta f)\left(\text{ID}(x), (\text{ID}(\operatorname{red}(\delta, \mathcal{M})))\right) \\
&= \text{ID}(D_\beta f\left(x, (\operatorname{red}(\delta, \mathcal{M}))\right))
\end{aligned}$$

where the permutation of ID follows from its properties as an evaluation map. Let Δ_0 be the finite subset of Δ on which $D_\beta f$ depends. By the lemma above there exists μ such that $\operatorname{red}(\delta, \mathcal{M})$=$\operatorname{red}(\delta, \mu)$ for all $\delta \in \Delta_0$. Therefore, $(D_\beta f[u])(x^0)$ $= \text{ID}(D_\beta f(x, (\operatorname{red}(\delta, \mu)))) = \text{ID}(\operatorname{red}(D_\beta f(x, (\delta)), \mu))$.
Note that $\operatorname{red}(D_\beta f(x, (\delta)), \mu)$ depends only on the parametric derivatives (and x). Therefore $\operatorname{red}(D_\beta f(x, (\delta)), \mu)$ is a complete reduction of $D_\beta f(x, (\delta))$ and we have

$$\begin{aligned}
\operatorname{red}(D_\beta f(x, (\delta)), \mu) &= \operatorname{red}(D_\beta f(x, (\delta)), \mathcal{M}) \\
&= \operatorname{red}(D_\beta f(x, (\delta)), (\beta, f)) \\
&= 0.
\end{aligned}$$

Thus $(D_\beta f[u])(x^0) = 0$, as required. This completes the proof of the existence part of the theorem. To prove the uniqueness part, we need only observe that u must satisfy Equation 2 for all $i \in \mathbb{N}_n$ and $\alpha \in \mathbb{N}^m$ and that this condition uniquely defines u. ■

4.1 Sufficient Finite Sets of Integrability Conditions

Our definition of a Riquier basis requires that all the infinitely many integrability conditions reduce to 0. As in the algebraic case, it turns out that it suffices to check a finite set of integrability conditions (see Theorem 3 and Corollary 1, below). The sequence of lemmas that follow is directed towards proving this.

Lemma 4 *Suppose $h \longrightarrow 0$ and $k \longrightarrow 0$. Suppose further that for all $\alpha, \alpha' \in \mathbb{N}^m$ and $f, f' \in \mathcal{M}$ with $\text{HD}\, D_\alpha f = \text{HD}\, D_{\alpha'} f' \leq \text{HD}\, k$, we have $D_\alpha f - D_{\alpha'} f' \longrightarrow 0$. Then $h + k \longrightarrow 0$.*

Proof: Say $h \xrightarrow{\mu} 0$. Let $l = \operatorname{red}(k, \mu)$. Since $k \longrightarrow l$, by Lemma 2 there exists j with $0 \longrightarrow j$ and $l \longrightarrow j$. Since $0 \longrightarrow j$, we have $j = 0$ and hence $l \longrightarrow 0$, say $0 = \operatorname{red}(l, \nu)$. Then we have: $\operatorname{red}(h + k, (\mu, \nu)) = \operatorname{red}(h, (\mu, \nu)) + \operatorname{red}(k, (\mu, \nu)) = \operatorname{red}(0, \nu) + \operatorname{red}(l, \nu) = 0$. Therefore, $h + k \longrightarrow 0$, as required. ■

Example 3 *Let $m = n = 1$ and let $\mathcal{M} = \{u - x, u\}$. Let $g = (u - x)(u_x - 1) + uu_x$. Observe that $g \xrightarrow{u-x} xu_x \xrightarrow{\frac{\partial u}{\partial x}} 0$ and so $g \longrightarrow 0$. However, one computes that the only complete reduction of $\frac{\partial g}{\partial x} = (u_x - 1)^2 + (u - x)u_{xx} + u_x^2 + uu_{xx}$ is 1 and so $\frac{\partial g}{\partial x} \not\longrightarrow 0$.*

Thus in general $g \longrightarrow 0$ does not imply $D_i g \longrightarrow 0$. Note that the analogous implication always holds in the algebraic case: if a polynomial reduces to zero then any multiple of the polynomial reduces to zero. In Lemma 6 we will give a condition on $\mathcal{M}$ that guarantees $g \longrightarrow 0 \implies D_i g \longrightarrow 0$.

Lemma 5 *Take $\alpha \in \mathbb{N}^m$, $f \in \mathcal{M}$, $i \in \mathbb{N}_n$ and g an analytic function of $\tilde{\Delta}$. Let δ_1 be the highest derivative of $D_\alpha f$. Let δ_0 be given by $\delta_1 = D_i\delta_0$, if this is well-defined. Let g be an analytic function on U that is polynomial in* $\mathrm{Prin}\,\mathcal{M}$. *Then:*

$$\begin{aligned}\mathrm{red}(D_i g, (\alpha, f)) &= D_i \,\mathrm{red}(g, (\alpha, f)) \\ &\quad + \mathrm{red}\left(\frac{\partial g}{\partial \delta_1}, (\alpha, f)\right) D_i D_\alpha f \\ &\quad - \mathrm{red}\left(\frac{\partial g}{\partial \delta_0}, (\alpha, f)\right) D_\alpha f.\end{aligned}$$

If δ_0 is not well-defined, then the last term is omitted in the above formula.

The proof [18] is by direct computation.

Lemma 6 *Let g be an analytic function on U that is polynomial in* $\mathrm{Prin}\,\mathcal{M}$ *such that $g \longrightarrow 0$. Fix $i \in \mathbb{N}_m$. Suppose that for all $\alpha, \alpha' \in \mathbb{N}^m$ and $f, f' \in \mathcal{M}$ with $\mathrm{HD}\, D_\alpha f = \mathrm{HD}\, D_{\alpha'} f' < \mathrm{HD}\, D_i g$, $D_\alpha f - D_{\alpha'} f' \longrightarrow 0$. Then $D_i g \longrightarrow 0$.*

Proof: By induction on the length of the minimal chain required to reduce g to 0, we may assume that there exists an analytic function h of $\tilde{\Delta}$ with $g \longmapsto h \longrightarrow 0$ and $D_i h \longrightarrow 0$, say $h = \mathrm{red}(g, (\alpha_h, f_h))$. If $g = 0$ the result is clear. Thus we may assume $h \neq g$. By the previous lemma, we have an expression of the form

$$\mathrm{red}(D_i g, (\alpha_h, f_h)) = D_i h + k D_i D_{\alpha_h} f_h + l D_{\alpha_h} f_h, \qquad (3)$$

with k and l analytic functions of $\tilde{\Delta}$ satisfying $\mathrm{HD}\, k \leq \mathrm{HD}\, g$ and $\mathrm{HD}\, l \leq \mathrm{HD}\, g$. Furthermore, either $\mathrm{HD}(D_i h) < \mathrm{HD}(D_i g)$ or $\mathrm{HD}(D_i D_\alpha f) < \mathrm{HD}(D_i g)$. Thus at least two of the three summands in (3) have highest derivative strictly less than $\mathrm{HD}(D_i g)$. Therefore by (two applications of) Lemma 4, we have $\mathrm{red}(D_i g, (\alpha_h, f_h)) \longrightarrow 0$ and so $D_i g \longrightarrow 0$. ∎

The *least common multiple* of $\alpha = (a_1, ..., a_m)$ and $\beta = (b_1, ..., b_m) \in \mathbb{N}^m$ is defined by $\mathrm{LCM}(\alpha, \beta) := (\max(a_1, b_1), ..., \max(a_m, b_m))$. We say that α *divides* β if $a_i \leq b_i$ for all $i \in \mathbb{N}_m$; in this case obviously $\mathrm{LCM}(\alpha, \beta) = \beta$. For $\delta_\alpha^j, \delta_\beta^j \in \Delta$ we define $\mathrm{LCM}(\delta_\alpha^j, \delta_\beta^j) = \delta^j_{\mathrm{LCM}(\alpha,\beta)}$. The differential analog of an S-polynomial is given in the following:

Definition 2 *Take $f, f' \in \mathcal{M}$. Let their highest derivatives be δ_α^i and $\delta_{\alpha'}^{i'}$, respectively. Let β be the least common multiple of α and α'. Then if $i = i'$, define the* minimal integrability condition *of f and f' to be*

$$\mathrm{IC}(f, f') := D_{\beta-\alpha} f - D_{\beta-\alpha'} f'.$$

If $i \neq i'$, then $\mathrm{IC}(f, f')$ is said to be undefined.

Note that, in the notation of the definition above, if $\mathrm{IC}(f, f')$ is well-defined then $\mathrm{HD}\, D_{\beta-\alpha} f = \mathrm{HD}\, D_{\beta-\alpha'} f'$ and β is minimal such that this holds, in that at least one of the coordinates of $\beta - \alpha$ or of $\beta - \alpha'$ is 0. In particular, the highest derivatives cancel in the expression for $\mathrm{IC}(f, f')$, that is $\mathrm{HD}(\mathrm{IC}(f, f')) < \mathrm{HD}\, D_{\beta-\alpha} f$.

The proof [18] of the following theorem uses a straightforward transfinite induction.

Theorem 3 *Suppose that for each pair $(f, f') \in \mathcal{M}^2$ with $\mathrm{IC}(f, f')$ well-defined there exists an expansion of $\mathrm{IC}(f, f')$ of the form*

$$\mathrm{IC}(f, f') = \sum_{(b,b') \in B_{f,f'}} D_{\alpha_{f,f',b,b'}} \mathrm{IC}(b, b'),$$

where $B_{f,f'}$ is a subset of $\mathcal{M}^2$ such that for each $(b, b') \in B_{f,f'}$, $\mathrm{IC}(b, b') \longrightarrow 0$ and $D_{\alpha_{f,f',b,b'}} \mathrm{LCM}(\mathrm{HD}\, b, \mathrm{HD}\, b') \leq \mathrm{LCM}(\mathrm{HD}\, f, \mathrm{HD}\, f')$. Then $\mathcal{M}$ is a Riquier basis.

No doubt the most natural special case is the following:

Corollary 1 *Suppose that $\mathrm{IC}(f, f') \longrightarrow 0$ for all $f, f' \in \mathcal{M}$ with $\mathrm{IC}(f, f')$ well-defined. Then $\mathcal{M}$ is a Riquier basis.*

Proof: This follows immediately from Theorem 3, by setting $B_{f,f'} = \{(f, f')\}$ for $f, f' \in \mathcal{M}$ with $\mathrm{IC}(f, f')$ well-defined and using the trivial expansion for $\mathrm{IC}(f, f')$ as $\mathrm{IC}(f, f') = \mathrm{IC}(f, f')$. ∎

Although the result of Corollary 1 is widely believed to be true, it appears that it has never been proved before for $m > 2$, as noted by Reid et al. [15], although for $m \leq 2$ and $\mathcal{M}$ reduced, it is known [14]. This result gives a very natural finite set of integrability conditions that it is sufficient to check reduce to 0. However, this set of integrability conditions is in general far from minimal. As shown in [18], a well known criterion due to Buchberger in the algebraic case (see e.g. [1], p. 223) goes through in the differential case. Also see the preprint [2] for a related result in the case of differential polynomials.

5 Leading Nonlinear Theory

The Janet algorithm, and consequently the Riquier Existence Theorem, can only be successfully applied to systems that are linear or almost linear in some sense. The `rif` algorithm of Reid et al. [15] manipulates nonlinear analytic systems into variations of what Reid et al. call rif form (reduced involutive form), which is a generalization of Riquier Basis form discussed earlier. We state an Existence and Uniqueness Theorem for systems in a related form, called rif′ form, output by Rust's `rif′` algorithm. A sketch of aspects of the proof by Rust [18] is also given. For polynomially nonlinear systems, it also applies to the output of Reid's `rif` algorithm.

A ranking $\leq$ is said to be *sequential* if for all $\delta \in \Delta$, the set $\{\delta' \in \Delta | \delta' \leq \delta\}$ is finite. For $\alpha = (a_1, ..., a_m) \in \mathbb{N}^m$, let $|\alpha|$ denote $a_1 + ... + a_m$. *For the rest of this paper, we insist that the fixed ranking $\leq$ be sequential and we let $\mathcal{N}$ be a finite set of analytic functions on U that depend only on $\{x\} \cup \mathrm{Par}\,\mathcal{M}$.* For $\mathcal{S}$ a finite set of analytic functions of $\tilde{\Delta}$, let $\mathrm{L}(\mathcal{S})$ denote the leading linear elements of $\mathcal{S}$. Let $\mathrm{N}(\mathcal{S})$ denote the set of leading nonlinear elements of $\mathcal{S}$. (The terms "leading linear" and "leading non-linear" were defined in Subsection 2.2.)

5.1 Relative Riquier Bases and rif′ Form

Roughly speaking, we say that $\mathcal{M}$ is a *Riquier basis relative to $\mathcal{N}$* if $\mathcal{M}$ "looks like" a Riquier basis so far as the algebraic solutions to $\mathcal{N}$ are concerned. We define what it means for $\mathcal{M}$ to be a Riquier basis relative to $\mathcal{N}$ and mimic the development of the case where the system is linear in its highest derivatives.

Definition 3 *For η an analytic function on U, we say that η is a* special consequence *of $\mathcal{N}$ if η lies in the ideal generated by $\mathcal{N}$ in the ring of analytic functions on U that depend on $\{x\} \cup \mathrm{Par}\,\mathcal{M}$ only. That is, η is a special consequence of $\mathcal{N}$ if it admits an expansion of the form*

$$\eta = \sum_{i=1}^{k} h_i g_i \qquad (4)$$

with $g_1, \ldots, g_k \in \mathcal{N}$ and $h_1, \ldots, h_k$ analytic functions on U that depend on $\{x\} \cup \operatorname{Par} \mathcal{M}$ only. We say that $\mathcal{M}$ is a Riquier basis relative to *$\mathcal{N}$ on U if for all $\alpha, \alpha' \in \mathbb{N}^m$ and $f, f' \in \mathcal{M}$ with $\mathrm{HD}\, D_\alpha f = \mathrm{HD}\, D_{\alpha'} f'$, the integrability condition $D_\alpha f - D_{\alpha'} f'$ can be reduced to a special consequence of $\mathcal{N}$.*

Definition 4 *We say that $(\mathcal{M}, \mathcal{N})$ (or, by abuse, $\mathcal{M} \cup \mathcal{N}$) is in* rif′ form *on U if $\mathcal{M}$ is a Riquier basis relative to $\mathcal{N}$ on U and for all $i \in \mathbb{N}_m$ and $g \in \mathcal{N}$, $D_i g$ can be reduced to a special consequence of $\mathcal{N}$ on U.*

The following lemma generalizes Lemma 1. Its proof, given in [18], is similar to that of Lemma 1.

Lemma 7 *Let g be an analytic functions on U that is polynomial in $\operatorname{Prin} \mathcal{M}$. Fix $f, f' \in \mathcal{M}$, $\alpha, \beta \in \mathbb{N}^m$ and μ as in Section 2.3. Let h and k denote the one step reductions: $h = \operatorname{red}(g, (\alpha, f))$ and $k = \operatorname{red}(g, (\beta, f'))$. Then:*

1. *If $\mathrm{HD}\, D_\alpha f < \mathrm{HD}\, D_\beta f'$ then*

$$\operatorname{red}(h, ((\beta, f'), (\alpha, f))) = \operatorname{red}(k, (\alpha, f)).$$

2. *Suppose that $D_\alpha f - D_\beta f' \xrightarrow{\mu} \eta$ for some special consequence η of $\mathcal{N}$ and chain of one step reductions μ 1 Then*

$$\operatorname{red}(h, \mu) - \operatorname{red}(k, \mu) = h' \eta$$

for some analytic function h' on U with $\mathrm{HD}\, h' \leq \mathrm{HD}\, g$.

The following lemma generalizes Lemma 2. It is proved in much the same way as in the algebraic case.

Lemma 8 *Let g be an analytic function on U that is polynomial in $\operatorname{Prin} \mathcal{M}$. Let $\delta \in \Delta$ be the highest principal derivative that occurs in g. Suppose that for all $\alpha, \alpha' \in \mathbb{N}^m$ and $f, f' \in \mathcal{M}$ with $\mathrm{HD}\, D_\alpha f = \mathrm{HD}\, D_{\alpha'} f' \leq \delta$, $D_\alpha f - D_{\alpha'} f'$ reduces to a special consequence of $\mathcal{N}$. Suppose $g \longrightarrow h$ and $g \longrightarrow k$. Then there exist l, l' with $h \longrightarrow l$ and $k \longrightarrow l'$ such that $l - l'$ is a special consequence of $\mathcal{N}$.*

An immediate consequence of Lemma 8 is:

Theorem 4 *Suppose that $\mathcal{M}$ is a Riquier basis relative to $\mathcal{N}$ and g is an analytic function on U, polynomial in $\operatorname{Prin} \mathcal{M}$. Then the difference of any two complete reductions of g is a special consequence of $\mathcal{N}$. In particular, any two complete reductions of g agree on the algebraic solutions to $\mathcal{N}$.*

For g an analytic function on U polynomial in $\operatorname{Prin} \mathcal{M}$, let $\operatorname{redmod}(g, \mathcal{M})$ denote a particular choice of complete reduction of g with respect to $\mathcal{M}$. Let ID be a specification of initial data for $\mathcal{M}$ that satisfies $\mathrm{ID}(\eta) = 0$ for all $\eta \in \mathcal{N}$. Theorem 4 implies that $\operatorname{redmod}(g, \mathcal{M})$ is uniquely defined up to a special consequence of $\mathcal{N}$. In particular, $\mathrm{ID}(\operatorname{redmod}(g, \mathcal{M}))$ is independent of the choice of complete reduction. Note that if 1 lies in the ideal generated by $\mathcal{N}$ in an appropriate ring of analytic functions, then no such ID exists.

Theorem 5 *Suppose that $\mathcal{M}$ is a Riquier basis relative to $\mathcal{N}$. Fix $x^0 \in \mathbb{F}^m$. Let ID be a specification of initial data for $\mathcal{M}$ at $x^0 \in \mathbb{F}^m$ with $\mathrm{ID} \in U$ and $\mathrm{ID}(\eta) = 0$ for all $\eta \in \mathcal{N}$. Then there exists a unique solution $u(x) \in \mathbb{F}[[x - x^0]]^n$ to $\mathcal{M}$ such that $D_\alpha u^i(x^0) = \mathrm{ID}(\operatorname{redmod}(\delta^i_\alpha, \mathcal{M}))$ for all $\alpha \in \mathbb{N}^m$ and $i \in \mathbb{N}_n$.*

Proof: The uniqueness property is clear. Define $u(x) \in \mathbb{F}[[x - x^0]]^n$ by $D_\alpha u^i(x^0) = \mathrm{ID}(\operatorname{redmod}(\delta^i_\alpha, \mathcal{M}))$ for $\alpha \in \mathbb{N}^m$ and $i \in \mathbb{N}_n$. We must verify that $u(x)$ is a solution to $\mathcal{M}$. It suffices to show that $D_\beta f[u](x^0) = 0$ for all $f \in \mathcal{M}$ and $\beta \in \mathbb{N}^m$. Fix f, β. We have

$$\begin{aligned} D_\beta f[u](x^0) &= (D_\beta f)\left(x^0, (D u^i_\alpha(x^0))\right) \\ &= (D_\beta f)\left(x^0, (\mathrm{ID}(\operatorname{redmod}(\delta, \mathcal{M})))\right) \\ &= \mathrm{ID}(D_\beta f\,(x, (\operatorname{redmod}(\delta, \mathcal{M})))). \end{aligned}$$

Let Δ_0 be the finite subset of Δ on which $D_\beta f$ depends. By Lemma 3 there exists μ such that $\operatorname{redmod}(\delta, \mathcal{M}) = \operatorname{red}(\delta, \mu)$ for all $\delta \in \Delta_0$. Therefore,

$$\begin{aligned} (D_\beta f[u])(x^0) &= \mathrm{ID}(D_\beta f\,(x, (\operatorname{red}(\delta, \mu)))) \\ &= \mathrm{ID}(\operatorname{red}(D_\beta f(x, (\delta)), \mu)). \end{aligned}$$

Note that $\operatorname{red}(D_\beta f(x, (\delta)), \mu)$ depends only on the parametric derivatives (and x). Therefore $\operatorname{red}(D_\beta f(x, (\delta)), \mu)$ is a complete reduction of $D_\beta f(x, (\delta))$ and we have

$$\begin{aligned} \mathrm{ID}(\operatorname{red}(D_\beta f(x, (\delta)), \mu)) &= \mathrm{ID}(\operatorname{redmod}(D_\beta f(x, (\delta)), \mathcal{M})) \\ &= \mathrm{ID}(\operatorname{red}(D_\beta f(x, (\delta)), (\beta, f))) \\ &= \mathrm{ID}(0) \\ &= 0. \end{aligned}$$

Thus $(D_\beta f[u])(x^0) = 0$, as required. ■

Just as in the leading linear case, the theory becomes most technical when it is shown that only a finite subset of integrability conditions have to be satisfied. We omit the full discussion which is given in [18]. Its result is the analog of Corollary 1:

Corollary 2 *Suppose that for each $(f, f') \in \mathcal{M}^2$ with $\mathrm{IC}(f, f')$ well-defined, $\mathrm{IC}(f, f')$ can be reduced to a special consequence of $\mathcal{N}$. Suppose also that for all $(f, f') \in \mathcal{M}^2$ with $\mathrm{IC}(f, f')$ well-defined and $\alpha, \alpha' \in \mathbb{N}^m$ such that $\mathrm{HD}\, D_\alpha f = \mathrm{HD}\, D_{\alpha'} f' \leq \max_{i \in \mathbb{N}_m, \eta \in \mathcal{N}} (D_i\, \mathrm{HD}\, \eta)$, $D_\alpha f - D_{\alpha'} f'$ can be reduced to a special consequence of $\mathcal{N}$. Finally, suppose that $D_i \eta$ can be reduced to a special consequence of $\mathcal{N}$ for all $\eta \in \mathcal{N}$. Then $(\mathcal{M}, \mathcal{N})$ is in rif′ form.*

5.2 A Nonlinear Existence and Uniqueness Theorem

We state an Existence and Uniqueness Theorem for systems of analytic PDE in rif′ form.

Let $\mathcal{S}$ be a finite set of analytic functions of $\tilde{\Delta}$ in rif′ form on an open subset U of $\{x\} \cup \operatorname{Par}(\mathrm{L}(\mathcal{S}))$. Let ID be a specification of initial data for $\mathrm{L}(\mathcal{S})$ with $\mathrm{ID} \in U$. Clearly, for ID to correspond to a formal solution $u(x)$ of $\mathcal{S}$ it is necessary that $\mathrm{ID}(g) = 0$ for all $g \in \mathrm{N}(\mathcal{S})$. This turns out to be sufficient also:

Theorem 6 *Let $\mathcal{S}$ be a finite set of analytic functions of $\tilde{\Delta}$. Let $\mathcal{L}$ denote $L(\mathcal{S})$ and $\mathcal{N}$ denote $N(\mathcal{S})$. Let U be an open subset of $\{x\} \cup \operatorname{Par} \mathcal{L}$. Suppose that $\mathcal{S}$ is in rif′ form on U and that ID is a specification of initial data for $\mathcal{L}$ that lies in U. Suppose further that $\mathrm{ID}(g) = 0$ for all $g \in \mathcal{N}$. Let $u(x) \in \mathbb{F}[[x - x^0]]^n$ be the solution to $\mathcal{L}$ that corresponds to ID via Theorem 5. Then $u(x)$ is a solution to $\mathcal{S}$.*

The proof, which is given in [18], uses induction to show that for all $g \in \mathcal{N}$ and $\alpha \in \mathbb{N}^m$, there exists an expansion $\mathrm{redmod}(D_\alpha g, \mathcal{L}) = \sum_{g_i \in \mathcal{N}} h_i g_i$ with $\mathrm{ID}(h_i)$ well-defined. The other main step in the proof is that since $u(x)$ is a solution to $\mathcal{L}$, we have that $D_\alpha g[u](x) = \mathrm{redmod}(D_\alpha g, \mathcal{L})[u](x)$. Thus,

$$
\begin{aligned}
D_\alpha g[u](x^0) &= \mathrm{redmod}(D_\alpha g, \mathcal{L})[u](x^0) \\
&= \mathrm{ID}(\mathrm{redmod}(D_\alpha g, \mathcal{L})) = \sum_{g_i \in \mathcal{N}} \mathrm{ID}(h_i)\,\mathrm{ID}(g_i) \\
&= 0.
\end{aligned}
$$

6 Applications

6.1 Existence Uniqueness Theorems

Consider a system of real analytic or complex analytic PDE of the form

$$u_y = f(u_x, v),\ v_x = g(v_y, u). \tag{5}$$

With respect to a ranking that satisfies $u_y > u_x, v$ and $v_x > v_y, u$, and only with respect to such a ranking, this system is a Riquier basis. There do exist rankings that satisfy these conditions [19].

Here, the parametric derivatives are $\{u, u_x, u_{xx}, \ldots, v, v_y, v_{yy}, \ldots\}$. Fix a point (x^0, y^0) in the domains of f and g. Ascribing values to the parametric derivatives at the point (x^0, y^0) uniquely determines a formal power series solution to the system at (x^0, y^0), and every such solution may be obtained in this way. Equivalently, one may specify $u(x, y^0)$, $v(x^0, y)$. A discussion of Existence Uniqueness Theory for analytic solutions will appear elsewhere.

This example, although artificial, also illustrates that the theorems apply to systems which are not necessarily polynomially nonlinear. Such non-polynomial systems arise in applications. For example in [15], an example of a frictionless bead sliding under gravity on wire with shape $\Psi(x, y) = 0$ is given. The shape of a physical wire is not necessary polynomial, although possibly it might be approximated by some function in a sufficiently complicated differential-algebraic extension.

6.2 Application to Non-classical Symmetries

In the previous subsection, the system studied was already in the form of a Riquier basis (or more generally rif′-form). However, in many applications, such as the one described in this section, the system is not in such a form, and must be manipulated to such a form. The algorithms for this purpose (see [15, 18]), which are not described in detail here, have been implemented by Wittkopf in the symbolic language Maple. They essentially compute a sequence of approximations to an Existence and Uniqueness theorem for a system of PDE. At each step integrability conditions are calculated, any which are not satisfied are appended to the system. Simplification is performed, and the process repeated until no new conditions arise.

During the course of a calculation, the leading derivatives of leading linear equations must be solved for. When the coefficients of these leading derivatives contain as yet unknown functions, a decision must be made as to whether the coefficient (or *pivot* or *separant*) is identically zero, or non-zero. This results in a binary tree, with each branch corresponding to a possible rif′-form for the system. This binary tree is automatically generated by Wittkopf's Maple package for the example problem below.

This example is concerned with the application of Bluman and Cole's so-called non-classical method which is a generalization of Lie's symmetry method. In particular this non-classical method is applied to the following system of coupled nonlinear Schrödinger (CNLS) equations, which describes transverse effects in nonlinear optical systems

$$
\begin{aligned}
\mathrm{i}\Psi_t &+ \nabla^2\Psi + \left(|\Psi|^2 + |\Phi|^2\right)\Psi = 0, \\
\mathrm{i}\Phi_t &+ \nabla^2\Phi + \left(|\Psi|^2 + |\Phi|^2\right)\Phi = 0.
\end{aligned}
$$

Nonclassical symmetry vector fields of form

$$\xi 1\partial_x + \xi 2\partial_y + \xi 3\partial_z + \partial_t + \phi 1\partial_p + \phi 2\partial_q + \phi 3\partial_r + \phi 4\partial_s$$

of this system have been studied in detail by Mansfield, Reid and Clarkson [6]. Even though such vector fields do not generate a classical Lie group they lead to reductions in the number of variables of the CNLS equations.

The application of the non-classical method yields a polynomially nonlinear system of 856 equations, in 7 dependent $(\phi 1, \phi 2, \phi 3, \phi 4, \xi 1, \xi 2, \xi 3)$ and 8 independent variables (x, y, z, t, p, q, r, s). In a prior calculation, certain dependencies were discovered allowing the ϕ variables to be expressed in terms of some addtional variables, $g1, g2, g3, g4$ which depend upon (x, y, z, t) only:

$$\phi k = gk1\ p + gk2\ q + gk3\ r + gk4\ s + gk,\quad k = 1, 2, 3, 4.$$

Application of rif to this system, under the ranking described below, on a PII 333MHz machine required 120 CPU sec., consuming 4M of memory, and resulted in 3 consistent cases, of which only the following case corresponded to a genuine non-classical symmetry:

$$
\begin{aligned}
&\mathit{case1}: \\
&\quad \mathit{Solved} = [\phi 1 = -g41\ s + g44\ p - g43\ q + g42\ r, \\
&\quad \phi 2 = -g42\ s + g43\ p + g44\ q - g41\ r, \\
&\quad \phi 3 = -g43\ s - g42\ p + g41\ q + g44\ r, \\
&\quad \phi 4 = g44\ s + g41\ p + g42\ q + g43\ r,\ \xi 1_{t,t} = 4\,g44\,\xi 1_t, \\
&\quad \xi 2_{t,t} = 4\,g44\,\xi 2_t,\ \xi 3_{t,t} = 4\,g44\,\xi 3_t,\ \xi 1_x = -g44, \\
&\quad \xi 2_x = 0,\ \xi 3_x = 0,\ g41_x = -\xi 1\,g42,\ g42_x = \xi 1\,g41, \\
&\quad g43_x = \frac{1}{2}\xi 1_t - g44\,\xi 1,\ g44_x = 0,\ \xi 1_y = 0,\ \xi 2_y = -g44, \\
&\quad \xi 3_y = 0,\ g41_y = -\xi 2\,g42,\ g42_y = \xi 2\,g41, \\
&\quad g43_y = -g44\,\xi 2 + \frac{1}{2}\xi 2_t,\ g44_y = 0,\ \xi 1_z = 0,\ \xi 2_z = 0, \\
&\quad \xi 3_z = -g44,\ g41_z = -\xi 3\,g42,\ g42_z = \xi 3\,g41, \\
&\quad g43_z = \frac{1}{2}\xi 3_t - g44\,\xi 3,\ g44_z = 0, \\
&\quad g41_t = 3\,g41\,g44 - (-\xi 1^2 + 2\,g43 - \xi 3^2 - \xi 2^2)\,g42, \\
&\quad g42_t = -(\xi 1^2 + \xi 2^2 + \xi 3^2 - 2\,g43)\,g41 + 3\,g42\,g44, \\
&\quad g43_t = 2\,g44\,g43,\ g44_t = 2\,g44^2,\ \xi 1_p = 0,\ \xi 2_p = 0, \\
&\quad \xi 3_p = 0,\ \xi 1_q = 0,\ \xi 2_q = 0,\ \xi 3_q = 0,\ \xi 1_r = 0,\ \xi 2_r = 0, \\
&\quad \xi 3_r = 0,\ \xi 1_s = 0,\ \xi 2_s = 0,\ \xi 3_s = 0,\ g11 = g44, \\
&\quad g12 = -g43,\ g13 = g42,\ g14 = -g41,\ g21 = g43, \\
&\quad g22 = g44,\ g23 = -g41,\ g24 = -g42,\ g31 = -g42, \\
&\quad g32 = g41,\ g33 = g44,\ g34 = -g43,\ g1 = 0,\ g2 = 0, \\
&\quad g3 = 0,\ g4 = 0] \\
&\quad \mathit{PolyRed} = [g41^2 + g42^2 = 0] \\
&\quad \mathit{Pivots} = [g41 + g32 \neq 0]
\end{aligned}
$$

The ranking used in the reduction was to first eliminate all ϕ dependent variables in favor of g's, then by total order of derivative, then lexicographically by derivative ($\partial_x \succ \partial_y \succ \partial_z \succ \partial_t \succ \partial_p \succ \partial_q \succ \partial_r \succ \partial_s$) and finally lexicographically on dependent variable ($\xi 1 \succ \xi 2 \succ \xi 3 \succ g11 \succ g12 \succ g13 \succ g14 \succ g21 \succ \ldots \succ g24 \succ g31 \succ \ldots \succ g34 \succ g41 \succ \ldots \succ g44 \succ g1 \succ g2 \succ g3 \succ g4$).

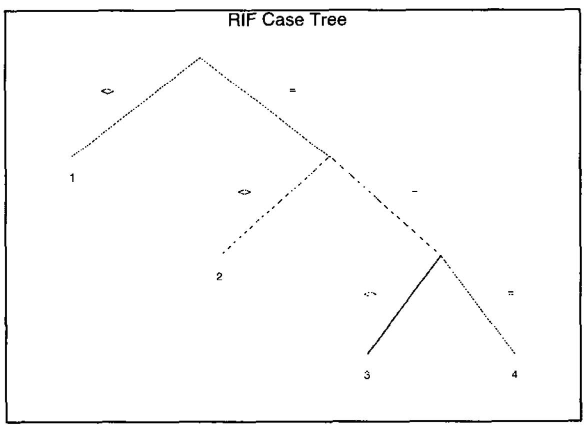
Here the *Solved* equations are leading linear in, and solved for their highest ranked derivatives, the *PolyRed* are leading nonlinear in their highest ranked derivatives, and the *Pivots* represent any non-trivial divisions (case splittings) that the algorithm performed to obtain the given system. Any equations in *PolyRed* are only algebraic constraints as all differential consequences of these equations have been accounted for.

Calculation of the initial data gives

$$\begin{aligned}
&\mathit{case1_initdata}: \\
&\quad \mathit{Infinite} = [] \\
&\quad \mathit{Finite} = [g41 = a1,\ g42 = a2,\ g43 = a3,\ g44 = a4, \\
&\quad \xi 1 = a5,\ \xi 1_t = a6,\ \xi 2 = a7,\ \xi 2_t = a8,\ \xi 3 = a9, \\
&\quad \xi 3_t = a10]
\end{aligned}$$

where the a constants represent the values of the corresponding derivatives at a point $(x_0, y_0, z_0, t_0, p_0, q_0, r_0, s_0)$. This yields a 10 dimensional variety, but the nonlinear constraint must also be considered. When we enforce $a1^2 + a2^2 = 0$ we are left with a 9 dimensional variety. This variety describes the data required to determine a unique (formal power series) solution for the input system.

The tree of cases which was automatically generated by Wittkopf's package is



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