BOUNDS FOR SOLUTION OF LINEAR DIOPHANTINE EQUATIONS

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Abstract. Given $A \in \mathbf{Z}^{m\times n}$, $rankA = m$, $b \in \mathbf{Z}^m$. Let d be the maximum of absolute values of the $m \times m$ minors of the matrix $(A\,b)$, $M = \{x \in \mathbf{Z}^n | Ax = b,\ x \geq 0\}$. It is shown that if $M \neq \emptyset$, then there exists $x^0 = (x_1^0, ..., x_n^0) \in M$, such that $x_i^0 \leq d$ $(i = 1, 2, .., n)$.

Introduction

Let $A(m\times n)$ be a matrix of rank m with integer elements, b be an integer vector, d be the maximum of the absolute values of the $m \times m$ minors of $(A\,b)$, M be the set of nonnegative integer solutions for the system $Ax = b$, $N = \{1, ..., n\}$.

In [1] the conjecture was made that the following theorem is true:

Theorem 0.1. *If M is nonempty, then there exists $x^0 \in M$ such that $x_i^0 \leq d, i \in N$.*

This conjecture was considered in [1-3], howerever, full and strict answer had not given. The complete proof of the theorem was given in [4]. See also [5]. In this paper I state the translation of the proof from [4].

Notation. Let H denotes the matrix which rows are the lattice basis for integer solutions of system $Ax = 0$;

$h_1, ..., h_n$ be columns of H;

$a_1, ..., a_n$ be columns of A;

x^1 be any vector of M;

M^1 be the set of integer solutions of the system $H^T y + x^1 \geq 0$.

The proof of Theorem 1

Without loss of generality, assume that g.c.d of all $m \times m$ minors of A is equal to 1. It is clear that the relation $x = H^T y + x^1$ determines one-to-one mapping between M and M^1. We should use next result from [6]:

Lemma 0.2. . *Let $I \subseteq N, |\ I\ | = m, A^1$ be the matrix that consists the columns of matrix H with index from $N \setminus I$. Then $|\det A^1 | = |\det H^1 |$.*

The theorem can be proved by induction on n. Case $n = m$ is obvious. Assume that theorem is true for $n \leq n_0$ and prove it is true for $n = n_0 + 1$.

Key words: integer solution, diophantine equation .

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1. Suppose first that from $u^T A \geq 0$ it follows that $u = 0$. According to Minkowsky-Farkas' theorem the cone

$$Ax = 0, x \geq 0$$

has dimension $n - m$. Then cone $H^T y \geq 0$ has dimension $n - m$ as well. Suppose without loss of generality that equation $h_1^T y = 0$ determines $(n - m-1)$ dimensional face of cone and $x_1^1 = max(-h_1^T y)$ for subject $H^T y + x^1 \geq 0$. Let s be g.c.d. of components of vector h_1 and r be minimal nonnegative integer number such that $x_1^1 - r$ is divided to s. Then the set M^1 is described by system

$$\begin{array}{l} h_1^T y + x_1^1 - r \geq 0 \\ \mathrm{h_i^T y + x_i^1 \geq 0, i \in \{2, 3, .., n\}} \end{array}$$

and there is $y^0 \in M^1$, such that $h_1^T y^0 + x_1^1 - r = 0$, hence the system $a_2 x_2 + ... + a_n x_n = b - r a_1$ has a nonnegative integer solution. From the lemma it follows that s is divisor of each minor of extended matrix of the system. Therefore there exists a matrix D with determinant s such that vectors $a_i^1 = D^{-1} a_i, i \in \{2, 3, .., n\}$ and $b^1 = D^{-1}(b - r a_1)$ have integer components. Since $r \leq s - 1$,maximal absolute value of minors with rank m is not more than d and by induction the system has solution $(x_2^0, x_3^0, ..., x_n^0)$, which components do not exceed d. As x^0 it is possible to choose $(r, x_2^0, x_3^0, ..., x_n^0)$

2. Suppose now that there exists $u \neq 0$ such that $u^T A \geq 0$. If $u^T A > 0$ then M is bounded, hence, the theorem is true. Otherwise, we can do an unimodular transformation for rows of matrix (Ab), permute columns and get the matrix $\left(\begin{array}{lll} \mathrm{a_1^1 ... a_v^1} & \mathrm{a_{v+1}^1 ... a_n^1} & \mathrm{b^1} \\ \mathrm{0 ... 0} & \mathrm{a_{v+1}^2 ... a_n^2} & \mathrm{b^2} \end{array} \right)$ where submatrix $(a_1^1 ... a_v^1)$ has rank k and consists of k rows and columns $(a_{v+1}^2 ... a_n^2)$ have positive last component. Set $b^3 = b^1 - a_{v+1}^1 x_{v+1}^1 - ... - -a_n^1 x_n^1$ and consider the system

$$a_1^1 x_1^1 + ... + a_v^1 x_v^1 = b^3. \tag{1}$$

We will prove that if d^1 is maximal absolute value of $k \times k$ minors of this system, then d^1 does not exceed d.

Let $k \geq 2$. Suppose without loss of generality that $d^1 = abs(det(a_1^1 \; ... a_{k-1} \; b^3))$. Show that it is possible to choose the set $I = \{i_1, ..., i_j\}$, where $j = n - m$, so that determinants

$$det \left(\begin{array}{lll} \mathrm{a_1^1 ... a_{k-1}^1} & \mathrm{a_{i_1}^1 ... a_{i_j}^1} & \mathrm{b^3} \\ \mathrm{0 ... 0} & \mathrm{a_{i_1}^2 ... a_{i_j}^2} & \mathrm{0} \end{array} \right), det \left(\begin{array}{lll} \mathrm{a_1^1 ... a_{k-1}^1} & \mathrm{a_{i_1}^1 ... a_{i_j}^1} & \mathrm{a_i^1} \\ \mathrm{0 ... 0} & \mathrm{a_{i_1}^2 ... a_{i_j}^2} & \mathrm{a_i^2} \end{array} \right), i \notin I$$

are either nonpositive or nonnegative.

Define $\lambda \neq 0$ so that $\lambda^t a_i^1, i \in \{1, ..., k-1\}$ and $\lambda^T b^3 > 0$. Further find a vertex $\mu = (\mu_1, ..., \mu_n)$ of the set of solution of the system $u^T a_i^2 + \lambda^T a_i^1 \geq 0, i \in \{v+1, ..., n\}$. As the set I we choose the set of numbers of linear indepedent inequalities wich are equalities on μ. Let $\lambda_e \neq 0$,then replace e-th row of each determinant by linear combination of rows with coefficients

$\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_{m-k}$. Decomposing each determinant on the e-th row we get the sequence of numbers $Z\lambda_e\lambda^T b^3, Z\lambda_e(\lambda^T a_i^1 + \mu^T a_i^2), i \notin I$ where Z is value of algebraic supplement. Obviosly, all numbers are either nonpositive or nonnegative.

Now we have

$$d \geq |det \begin{pmatrix} \mathrm{a}_1^1 ... \mathrm{a}_{\mathrm{k}-1}^1 \ \mathrm{a}_{\mathrm{i}_1}^1 ... \mathrm{a}_{\mathrm{i}_\mathrm{j}}^1 \ \mathrm{b}^1 \\ 0 \ ... \ 0 \quad \mathrm{a}_{\mathrm{i}_1}^2 ... \mathrm{a}_{\mathrm{i}_\mathrm{j}}^2 \ \mathrm{b}^2 \end{pmatrix}| = |\sum_{i \in I} det \begin{pmatrix} \mathrm{a}_1^1 ... \mathrm{a}_{\mathrm{k}-1}^1 \ \mathrm{a}_{\mathrm{i}_1}^1 ... \mathrm{a}_{\mathrm{i}_\mathrm{j}}^1 \ \mathrm{a}_\mathrm{i}^1 \\ 0 \ ... \ 0 \quad \mathrm{a}_{\mathrm{i}_1}^2 ... \mathrm{a}_{\mathrm{i}_\mathrm{j}}^2 \ \mathrm{a}_\mathrm{i}^2 \end{pmatrix} x_i^1| =$$

$$|det \begin{pmatrix} \mathrm{a}_1^1 ... \mathrm{a}_{\mathrm{k}-1}^1 \ \mathrm{a}_{\mathrm{i}_1}^1 ... \mathrm{a}_{\mathrm{i}_\mathrm{j}}^1 \ \mathrm{b}^3 \\ 0 \ ... \ 0 \quad \mathrm{a}_{\mathrm{i}_1}^2 ... \mathrm{a}_{\mathrm{i}_\mathrm{j}}^2 \ 0 \end{pmatrix} x_i^1| \geq |det \begin{pmatrix} \mathrm{a}_1^1 ... \mathrm{a}_{\mathrm{k}-1}^1 \ \mathrm{a}_{\mathrm{i}_1}^1 ... \mathrm{a}_{\mathrm{i}_\mathrm{j}}^1 \ \mathrm{b}^3 \\ 0 \ ... \ 0 \quad \mathrm{a}_{\mathrm{i}_1}^2 ... \mathrm{a}_{\mathrm{i}_\mathrm{j}}^2 \ 0 \end{pmatrix} x_i^1| \geq d^1$$

If $k = 1$, then put $\lambda_1 = \begin{cases} 1 \text{ for } \mathrm{b}^3 > 0, \\ -1 \text{ for } \mathrm{b}^3 \leq 0, \ \mathrm{e} = 1 \end{cases}$ and repeat reasoning.

By induction hypothesis the system (1) has solution $(x_1^0, ..., x_v^0)$, and $x_i^0 \leq d$ for $i \in \{1, 2, ..., v\}$. Since x_i^1 for $i \in \{v+1, ..., n\}$ is bounded from above by $\max\{x_i | \, Ax = b, \ x \geq 0\}$, hence, it does not exceed d. So we may take $x^0 = (x_1^0, ..., x_v^0, x_{v+1}^1, ..., x_n^1)$. The proof is completed.

A little modification of the above proof allow to prove that there exists vertex x of $conv(M)$ such that $x_i \leq d$ for all $i \in N$.

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