

Subgame-perfect Equilibria in Mean-payoff Games

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Abstract—In this paper, we provide an effective characterization of all the subgame-perfect equilibria in infinite duration games played on finite graphs with mean-payoff objectives. To this end, we introduce the notion of requirement and the notion of negotiation function. We establish that the set of plays that are supported by SPEs are exactly those that are consistent with the least fixed point of the negotiation function. Finally, we show that the negotiation function is piecewise linear and can be analyzed using the linear algebraic tool box.

I. INTRODUCTION

The notion of Nash equilibrium (NE) is one of the most important and most studied solution concepts in game theory. A profile of strategies is an NE when no rational player has an incentive to change their strategy unilaterally, i.e. while the other players keep their strategies. Thus an NE models a stable situation. Unfortunately, it is well known that, in sequential games, NEs suffer from the problem of *non-credible threats*, see e.g. [16]. In those games, some NE only exists when some players do *not* play rationally in subgames and so use non-credible threats to force the NE. This is why in sequential games, the stronger notion of *subgame-perfect equilibrium* is used instead: a profile of strategies is a subgame-perfect equilibrium (SPE) if it is an NE in all the subgames of the sequential game. Thus SPE imposes rationality even after a deviation has occurred.

In this paper, we study sequential games that are infinite duration games played on graphs with mean-payoff objectives and focus on SPEs. While NEs are guaranteed to exist in infinite duration games played on graphs with mean-payoff objectives, it is known that it is not the case for SPEs, see e.g. [17], [3]. We provide in this paper a constructive characterization of the entire set of SPEs which allows us to decide, among others, the SPE existence problem. This problem was left open in previous contributions on the subject. More precisely, our contributions are described in the next paragraphs.

Contributions. First, we introduce two important new notions that allow us to capture NEs, and more importantly SPEs in infinite duration games played on graphs with mean-payoff objectives¹: the notion of *requirement* and the notion of *negotiation function*.

A requirement λ is a function that assigns to each vertex $v \in V$ of a game graph a value in $\mathbb{R} \cup \{-\infty, +\infty\}$. The value $\lambda(v)$ represents a requirement on any play $\rho = \rho_0 \rho_1 \dots \rho_n \dots$ that traverses this vertex: if we want the player that controls the vertex v to follow ρ and to give up deviating from ρ , then the play must offer a payoff to this player that is at least $\lambda(v)$. An infinite play ρ is λ -consistent if, for each player i , the payoff of ρ for player i is larger than or equal to the largest value of λ on vertices occurring along ρ and controlled by player i .

We first establish that if λ maps a vertex v to the largest value that the player that controls v can secure against a fully adversarial coalition of the other players, i.e. if $\lambda(v)$ is the zero-sum worst-case value, then the set of plays that are λ -consistent are exactly the set of plays that are supported by an NE (Theorem 1).

As SPEs are forcing players to play rationally in all subgames, we cannot rely of the zero-sum worst-case value to characterize them. Indeed, when considering the worst-case value, we allow adversaries to play fully adversarially after a deviation and so potentially in an irrational way w.r.t. their own objective. In fact, in an SPE, a player is refrained to deviate when opposed by a coalition of *rational adversaries*. To characterize this relaxation of the notion of worst-case value, we rely on our notion of *negotiation function*.

The negotiation function operates from the set of requirements into itself. To understand the purpose of the negotiation function, let us consider its application on the requirement λ that maps every vertex v on the worst-case value as above. Now, we can naturally formulate the following question. Given v and λ , can the player that controls v improve the value that they can ensure against all the other players if only plays that are consistent with λ are proposed by the other players? In other words, can this player enforce a better value when playing against the other players if those players are not willing to give away their own worst-case value? Clearly, securing this worst-case value can be seen as a minimal goal for any *rational adversary*. So $\text{nego}(\lambda)(v)$ returns this value. But, now this reasoning can be iterated. One of the main contributions of this paper is to show that the least fixed point λ^* of the negotiation function is exactly characterizing the set of plays supported by SPEs (Theorem 2).

To turn this fixed point characterization of SPEs into algorithms, we additionally draw links between the negotiation

¹A large part of our results apply to the larger class of games with prefix independent objectives. For the sake of readability of this introduction, we focus here on mean-payoff games but the technical results in the paper are usually covering broader classes of games.

function and two classes of zero-sum games, that are called *abstract* and *concrete* negotiation games (see Theorem 3). We show that the latter can be solved effectively and allow, given λ , to compute $\text{nego}(\lambda)$ (Lemma 5). While solving concrete negotiation games allows us to compute $\text{nego}(\lambda)$ for any requirement λ , and even if the function $\text{nego}(\cdot)$ is monotone and Scott-continuous (Proposition 2), a direct application of the Kleene-Tarski fixed point theorem is not sufficient to obtain an effective algorithm to compute λ^* . Indeed, we give examples that require a transfinite number of iterations to converge to the least fixed point. To provide an algorithm to compute λ^* , we show that the function $\text{nego}(\cdot)$ is piecewise linear and we provide an effective representation of this function (Theorem 4). This effective representation can then be used to extract all its fixed points and in particular its least fixed point using linear algebraic techniques. Finally, let us note that all our results are also shown to extend to ε -SPEs, those are quantitative relaxations of SPEs.

Related works. Non-zero sum infinite duration games have attracted a large attention in recent years with applications targeting reactive synthesis problems. We refer the interested reader to the following survey papers [1], [5] and their references for the relevant literature. We detail below contributions more closely related to the work presented here.

In [4], Brihaye et al. offer a characterization of NE in quantitative games for cost-prefix-linear reward functions based on the worst-case value. The mean-payoff is cost-prefix-linear. In their paper, the authors do not consider the stronger notion of SPE which is the central solution concept studied in our paper.

In [18], Ummels proves that there always exists an SPE in games with ω -regular objectives and defines algorithms based on tree automata to decide constrained SPE problems. Strategy logics, see e.g. [10], can be used to encode the concept of SPE in the case of ω -regular objectives with application to the rational synthesis problem [13] for instance. The mean-payoff reward function is not ω -regular and so the techniques defined there cannot be used in our setting. Furthermore, as already recalled above, see e.g. [20], [3], contrary to the ω -regular case, SPEs in games with mean-payoff objectives may fail to exist.

In [3], Brihaye et al. introduce and study the notion of weak subgame-perfect equilibria which is a weakening of the classical notion of SPE. This weakening is equivalent to the original SPE concept on reward functions that are *continuous*. This is the case for example for the quantitative reachability reward function. On the contrary, the mean-payoff cost function is not *continuous* and the techniques used in [3] and generalized in [8], cannot be used to characterize SPEs for the mean-payoff reward function.

In [11], Flesch et al. show that the existence of ε -SPEs is guaranteed when the reward function is *lower-semicontinuous*, which is not the case of the mean-payoff reward function.

In [6], Bruyère et al. study secure equilibria that are a refinement of NEs. Secure equilibria are not subgame-perfect and are, as classical NEs, subject to non-credible threats in sequential games.

In [2], Brihaye et al. solve the problem of the existence of SPEs on quantitative reachability games. Their techniques rely on the property that the quantitative reachability reward function is continuous which implies that in that case weak SPEs and SPEs are equivalent. This is not the case for the mean-payoff reward function.

In [15], Noémie Meunier develops a method based on Prover-Challenger games to solve the problem of the existence of SPEs on games with a finite number of possible outcomes. This method is not applicable to the mean-payoff reward function as the number of outcomes in this case is uncountably infinite.

Structure of the paper. In Sect. 2, we introduce the necessary background. Sect. 3 defines the notion of requirement and the negotiation function. Sect. 4 contains the main technical contribution of the paper which shows that the set of plays that are supported by an SPE are those that are λ^* -consistent where λ^* is the least fixed point of the negotiation function. Sect. 5 draws a link between the negotiation function and negotiation games. Finally Sect. 6 establishes that the negotiation function is effectively piecewise linear. All the detailed proofs of our results can be found in a well identified appendix and a large number of examples are provided in the main part of the paper to illustrate the main ideas behind our new concepts and constructions.

II. BACKGROUND

In all what follows, we will use the word *game* for the infinite duration turn-based quantitative games on graphs with complete information.

Definition 1 (Game). A *game* is a tuple:

$$G = (\Pi, V, (V_i)_{i \in \Pi}, E, \mu),$$

where:

- Π is a finite set of *players*;
- (V, E) is a directed graph, whose vertices are sometimes called *states* and whose edges are sometimes called *transitions*, and in which for each $v \in V$, the set:

$$\{w \in V \mid (v, w) \in E\}$$

of the states directly accessible from v is nonempty;

- $(V_i)_{i \in \Pi}$ is a partition of V , in which V_i is the set of states *controlled* by player i ;
- $\mu : V^\omega \rightarrow \mathbb{R}^\Pi$ is an *outcome function*.

For the simplicity of writing, a transition $(v, w) \in E$ will often be written vw .

When μ is an outcome function and i is a player, μ_i denotes the i -th component of μ : if $\mu(\rho) = \bar{x} = (x_i)_{i \in \Pi}$, then $\mu_i(\rho) = x_i$. That quantity is the *payoff* of player i in ρ .

Definition 2 (Initialized game). An *initialized game* is a tuple (G, v_0) , often written $G_{\upharpoonright v_0}$, where G is a game and $v_0 \in V$ is a state called *initial state*. Moreover, the game $G_{\upharpoonright v_0}$ is *well-initialized* if any state of G is accessible from v_0 in the graph (V, E) .

Definition 3 (Play). A *play* in a game G is an infinite word $\rho = \rho_0 \rho_1 \dots \in V^\omega$ such that for all n , we have $\rho_n \rho_{n+1} \in E$. It is also a play in the initialized game $G_{\uparrow \rho_0}$. The set of plays in the game G (resp. the initialized game $G_{\uparrow v_0}$) is denoted by $\text{Plays}G$ (resp. $\text{Plays}G_{\uparrow v_0}$).

Remark. In the literature, the word *outcome* can be used to name plays, and the word *payoff* to name what we call here outcome. Here, the word *payoff* will be used to refer to outcomes, seen from the point of view of a given player – or in other words, an *outcome* will be seen as the collection of all players' payoffs.

Definition 4 (History). A *history* in a game G is a finite prefix $h_0 \dots h_n$ of a play in G . If it is nonempty, it is also a history in the initialized game $G_{\uparrow h_0}$. The set of histories in the game G (resp. the initialized game $G_{\uparrow v_0}$) is denoted by $\text{Hist}G$ (resp. $\text{Hist}G_{\uparrow v_0}$).

Definition 5 (Strategy). Let i be a player in an initialized game $G_{\uparrow v_0}$. A *strategy* for player i is a function:

$$\sigma_i : \{hv \in \text{Hist}G_{\uparrow v_0} \mid v \in V_i\} \rightarrow V$$

such that $v\sigma_i(hv)$ is an edge of (V, E) for all hv .

A play ρ is *compatible* with a strategy σ_i if and only if $\rho_{n+1} = \sigma_i(\rho_0 \dots \rho_n)$ for all n such that $\rho_n \in V_i$. A history h is compatible with a strategy σ_i if it is the prefix of a play that is compatible with σ_i .

Definition 6 (Strategy profile). Let $P \subseteq \Pi$ be a set of players in an initialized game $G_{\uparrow v_0}$. A *strategy profile* for P is a tuple $\bar{\sigma}_P = (\sigma_i)_{i \in P}$ where for each i , σ_i is a strategy for player i in $G_{\uparrow v_0}$.

A *complete* strategy profile is a strategy profile for Π , the set of all the players in the considered game; then, it is simply written $\bar{\sigma}$. A play or a history is *compatible* with a strategy profile $\bar{\sigma}_P$ if it is compatible with every strategy σ_i for $i \in P$. In a strategy profile $\bar{\sigma}_P$, the σ_i 's domains are pairwise disjoint. Therefore, we can consider $\bar{\sigma}_P$ as one function: for $hv \in \text{Hist}G_{\uparrow v_0}$ such that $v \in \bigcup_{i \in P} V_i$, we liberally write $\bar{\sigma}_P(hv)$ for $\sigma_i(hv)$ with i such that $v \in V_i$. For any complete strategy profile $\bar{\sigma}$, there exists only one play in $G_{\uparrow v_0}$ compatible with every σ_i , denoted by $\langle \bar{\sigma} \rangle_{v_0}$.

When i is a player and when the context is clear, we will often write $-i$ for the set $\Pi \setminus \{i\}$. Then, a strategy profile for all the players, except player i , will typically be written $\bar{\sigma}_{-i}$. We will often refer to $\Pi \setminus \{i\}$ as the *environment* against player i . When $\bar{\tau}_P$ and $\bar{\tau}'_Q$ are two strategy profiles with $P \cap Q = \emptyset$, $(\bar{\tau}_P, \bar{\tau}'_Q)$ denotes the strategy profile $\bar{\sigma}_{P \cup Q}$ such that $\sigma_i = \tau_i$ for $i \in P$, and $\sigma_i = \tau'_i$ for $i \in Q$.

Before moving on to SPEs, let us recall the notion of Nash equilibrium.

Definition 7 (Nash equilibrium). Let $G_{\uparrow v_0}$ be an initialized game. The strategy profile $\bar{\sigma}$ is a *Nash equilibrium*, or *NE* for

short, in $G_{\uparrow v_0}$, if and only if for each player i and for every strategy σ'_i , called *deviation* of σ_i , we have the inequality:

$$\mu_i(\langle \sigma'_i, \bar{\sigma}_{-i} \rangle_{v_0}) \leq \mu_i(\langle \bar{\sigma} \rangle_{v_0}).$$

To define SPEs, we need the notion of subgame.

Definition 8 (Subgame). Let $G = (\Pi, V, (V_i)_i, E, \mu)$ be a game, and let hv be a history in G . The *subgame* of G after hv , denoted by $G_{\uparrow hv}$, is the initialized game:

$$(\Pi, V, (V_i)_i, E, \mu_{\uparrow hv})_{\uparrow v}$$

where:

$$\mu_{\uparrow hv} : \begin{cases} vV^\omega & \rightarrow \mathbb{R}^\Pi \\ v\rho & \mapsto \mu(hv\rho) \end{cases}$$

Remark. A subgame is initialized, but defined from a game which is not: that is why $G_{\uparrow v_0}$ denotes the game G initialized in the state v_0 , which is also the subgame of G after the one-state history v_0 .

Definition 9 (Substrategy). Let $G_{\uparrow v_0}$ be an initialized game, σ_i be a strategy for some player i , and hv be a history in $G_{\uparrow v_0}$. Then, the *substrategy* of σ_i after hv , denoted by $\sigma_{i \uparrow hv}$, is the strategy in the subgame $G_{\uparrow hv}$:

$$\sigma_{i \uparrow hv} : vh' \mapsto \sigma_i(hvh').$$

Definition 10 (Subgame-perfect equilibrium). Let $G_{\uparrow v_0}$ be an initialized game. The strategy profile $\bar{\sigma}$ is a *subgame-perfect equilibrium*, or *SPE* for short, in $G_{\uparrow v_0}$, if and only if for every history h in $G_{\uparrow v_0}$, the strategy profile $\bar{\sigma}_{\uparrow h}$ is a Nash equilibrium in the subgame $G_{\uparrow h}$.

The notion of subgame-perfect equilibrium can be seen as a refinement of Nash equilibrium: it is a stronger equilibrium, which excludes players resorting to non-credible threats.

Example 1. In the game represented in Figure 1, where the square state is controlled by player \square and the round states by player \circ , if both players get the payoff 1 by reaching the state d and the payoff 0 in the other cases, there are actually two NEs: one, in blue, where \square goes to the state b and then player \circ goes to the state d , and both win, and one, in red, where player \square goes to the state c because player \circ was planning to go to the state e . However, only the blue one is an SPE, as moving from b to e is irrational for player \circ in the subgame $G_{\uparrow ab}$.

An ε -SPE is a strategy profile which is *almost* an SPE, meaning that if a player deviates after some history, they will not be able to improve their payoff by more than a quantity $\varepsilon \geq 0$.

Definition 11 (ε -SPE). Let $G_{\uparrow v_0}$ be an initialized game, and $\varepsilon \geq 0$. A strategy profile $\bar{\sigma}$ from v_0 is an ε -SPE if and only if for every history hv , for every player i , for every strategy σ'_i , we have:

$$\mu_i(\langle \bar{\sigma}_{-i \uparrow hv}, \sigma'_{i \uparrow hv} \rangle_v) \leq \mu_i(\langle \bar{\sigma}_{\uparrow hv} \rangle_v) + \varepsilon.$$

Remark. A 0-SPE is an SPE.

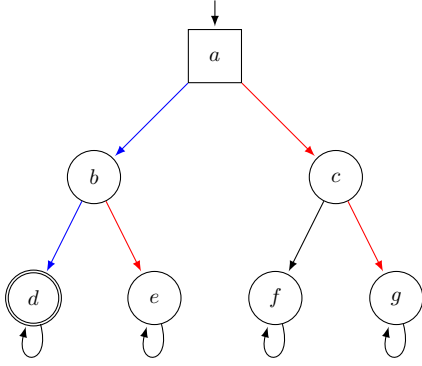


Fig. 1. A game with two NEs and one SPE

In this article, we will focus on *prefix-independent* games, and in particular *mean-payoff-inf* games.

Definition 12 (Mean-payoff-inf game). A *mean-payoff-inf game* is a game $G = (\Pi, V, (V_i)_i, E, \mu)$, where μ is defined from a function $\pi : E \rightarrow \mathbb{Q}^\Pi$, called *weight function*, by, for each player i :

$$\mu_i : \rho \mapsto \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \pi_i(\rho_k \rho_{k+1}).$$

In a mean-payoff-inf game, the weight given by the function π represents the immediate reward that each action gives to each player. The final payoff of each player is their average payoff along the play, defined as the limit inferior over n (since the limit may not be defined) of the average payoff after n steps.

Definition 13 (Prefix-independent game). A game $G = (\Pi, V, (V_i)_i, E, \mu)$ is *prefix-independent* if for every history h and for every play ρ , $\mu(h\rho) = \mu(\rho)$. We also say, in that case, that the outcome function μ is prefix-independent.

Remark. Mean-payoff-inf games are prefix-independent.

Before moving on to some examples, we recall a few classical results about two-player zero-sum games.

Definition 14 (Zero-sum game). A game $G = (\Pi, V, (V_i)_i, E, \mu)$ is *zero-sum* if for every play ρ , we have:

$$\sum_{i \in \Pi} \mu_i(\rho) = 0.$$

Definition 15 (Borelian game). A game $G = (\Pi, V, (V_i)_i, E, \mu)$ is *Borelian* if the function μ , from the set V^ω equipped with the product topology to the euclidian space \mathbb{R}^Π , is Borelian, i.e. if for any Borelian set $B \subseteq \mathbb{R}^\Pi$, the set $\mu^{-1}(B)$ is Borelian.

Proposition 1 (Determinacy of Borelian two-player games [14]). Let $G_{\uparrow v_0} = (\{1, 2\}, V, (V_i)_i, E, \mu)_{\uparrow v_0}$ be

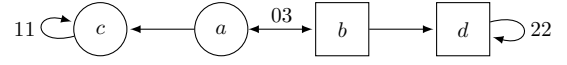


Fig. 2. A game without SPE

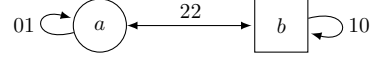


Fig. 3. A game with an infinity of SPEs

an initialized two-player zero-sum Borelian game, with a distinguished player 1. Then, we have the following equality:

$$\sup_{\sigma_1} \inf_{\sigma_2} \mu_1(\langle \bar{\sigma} \rangle_{v_0}) = \inf_{\sigma_2} \sup_{\sigma_1} \mu_1(\langle \bar{\sigma} \rangle_{v_0}).$$

Definition 16 (Value of a zero-sum game). Let:

$$G_{\uparrow v_0} = (\{1, 2\}, V, (V_i)_i, E, \mu)_{\uparrow v_0}$$

be an initialized two-player zero-sum Borelian game, with a distinguished player 1. Then, the quantity:

$$\sup_{\sigma_1} \inf_{\sigma_2} \mu_1(\langle \bar{\sigma} \rangle_{v_0}) = \inf_{\sigma_2} \sup_{\sigma_1} \mu_1(\langle \bar{\sigma} \rangle_{v_0})$$

is called *value* of $G_{\uparrow v_0}$, and denoted by $\text{val}(G_{\uparrow v_0})$.

In the two following examples, we illustrate the problem of the existence of SPEs in mean-payoff games.

Example 2. Let G be the mean-payoff-inf game of Figure 2, where for every edge, the left number is the weight for player \circ , and the right number is the weight for player \square . No weight is given for the edges ac and bd since they can be used only once, and therefore do not influence the final payoff.

As shown in [7], this game does not have any SPE, neither from the state a nor from the state b . The idea is the following:

- The only NE plays from the state b are the plays where player \square eventually leaves the cycle ab and goes to d : if they stay in the cycle ab , then player \circ would be better off leaving it, and if she does, player \square would be better off leaving it before.
- From the state a , if player \circ knows that player \square will leave, she has no incentive to do it before: there is no NE where \circ leaves the cycle and \square plans to do it if ever she does not. Therefore, there is no SPE where \circ leaves the cycle.
- But then, after a history that terminates in b , player \square has actually no incentive to leave if player \circ never plans to do it afterwards: contradiction.

Throughout the remaining of the paper, we will show how to apply our method on that example.

Example 3. Let us now study the game of Figure 3.

Using techniques from [9], we can represent the outcomes of possible plays in that game as in Figure 4 (gray and blue areas).

Following exclusively one of the three simple cycles a , ab , b of the game graph during a play yields the outcomes 01, 10 and 22, respectively. By combining those cycles with well chosen

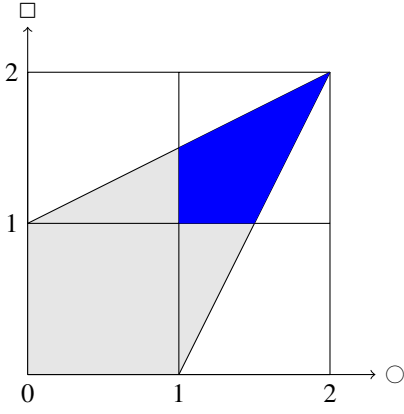


Fig. 4. The outcomes of plays and SPE plays in the game of Figure 3

frequencies, one can obtain any outcome in the convex hull of those three points.

Now, it is also possible to obtain the point 00 by using the properties of the limit inferior: it is for instance the outcome of the play:

$$a^2 b^4 a^{16} b^{256} \dots a^{2^{2^n}} b^{2^{2^{n+1}}} \dots,$$

In fact, one can construct a play that yields any outcome in the convex hull of the four points 00, 10, 01, and 22.

We claim that the outcomes of SPEs plays correspond to the entire blue area in Figure 4: there exists an SPE $\bar{\sigma}$ in $G_{\uparrow a}$ with $\langle \bar{\sigma} \rangle_a = \rho$ if and only if $\mu_{\square}(\rho), \mu_{\circ}(\rho) \geq 1$.

That statement is a direct consequence of the results we show in the remaining sections, but let us give a first intuition: a play with such an outcome necessarily uses infinitely often both states. It is an NE play because none of the players can get a better payoff by looping forever on their state, and they can both force each other to follow that play, by threatening them to loop for ever on their state whenever they can. But such a strategy profile is clearly not an SPE.

It can be transformed into an SPE as follows: when a player deviates, say player \square , then player \circ can punish him by looping on a , not forever, but a great number of times, until player \square 's mean-payoff gets very close to 1. Afterwards, both players follow again the play that was initially planned. Since that threat is temporary, it does not affect player \circ 's payoff on the long term, but it really punishes player \square if that one tries to deviate infinitely often.

For example, let us consider the play $(aaabbb)^\omega$, with outcome 11. An SPE which generates that play is the following: when player \circ has to play, by default, she loops twice on a , and when player \square has to play, he loops twice on b . But if player \circ goes to b without having looped twice on a , then player \square stays in b until \circ 's payoff passes below $1 + \frac{1}{n}$, where n is the number of times player \circ has already deviated; then and only then, he plays as it was planned. And if player \square deviates, player \circ punishes him the same way.

III. REQUIREMENTS AND NEGOTIATION

We will now see that SPEs are strategy profiles that respect some *requirements* about the payoffs, depending on the states it traverses. In this part, we develop the notions of *requirement* and *negotiation*.

A. Requirement

In the method we will develop further, we will need to analyze the players' behaviour when they have some *requirement* to satisfy. Intuitively, one can see requirements as *rationality constraints* for the players, that is, a threshold payoff value under which a player will not accept to follow a play.

In all what follows, \mathbb{R} denotes the set $\mathbb{R} \cup \{\pm\infty\}$.

Definition 17 (Requirement). A *requirement* on the game G is a function $\lambda : V \rightarrow \mathbb{R}$.

For a given state v , the quantity $\lambda(v)$ represents the minimal payoff that the player controlling v will require in a play beginning in v .

Definition 18 (λ -consistency). Let λ be a requirement on a game G . A play ρ in the game G is λ -consistent if and only if for all $n \in \mathbb{N}$, for $i \in \Pi$ such that $\rho_n \in V_i$, we have $\mu_i(\rho_n \rho_{n+1} \dots) \geq \lambda(\rho_n)$.

The set of the λ -consistent plays from a state v is denoted by $\lambda\text{Cons}(v)$.

Definition 19 (Satisfiability). A requirement λ on the initialized game $G_{\uparrow v_0}$ is *satisfiable* if and only if for each v accessible from v_0 , there exists a λ -consistent play in the game $G_{\uparrow v}$.

Definition 20 (λ -rationality). Let λ be a requirement on a mean-payoff-inf game G . Let $i \in \Pi$. A strategy profile $\bar{\sigma}_{-i}$ is λ -rational if and only if there exists a strategy σ_i such that for every history hv compatible with $\bar{\sigma}_{-i}$, the play $\langle \bar{\sigma}_{\uparrow hv} \rangle_v$ is λ -consistent. We then say that the strategy σ_i λ -rationalizes the strategy profile $\bar{\sigma}_{-i}$.

The set of λ -rational strategy profiles in $G_{\uparrow v}$ is denoted by $\lambda\text{Rat}(v)$.

Note that λ -rationality is a property of a strategy profile for all the players but one, player i . Intuitively: all players (including player i) have some requirement to satisfy. The other players than i made a coalition: they choose collectively their strategy profile, and they propose a strategy to player i , so that if player i eventually follows that strategy (i.e. possibly after a finite number of deviations), then every player has their requirements satisfied.

Remark. A requirement λ is satisfiable in $G_{\uparrow v_0}$ if and only if for some player i , there exists a λ -rational strategy profile $\bar{\sigma}_{-i}$ from v_0 .

Finally, let us define a particular requirement: the *vacuous requirement*, which requires nothing.

Definition 21 (Vacuous requirement). In any game, the *vacuous requirement*, denoted by λ_0 , is the requirement constantly equal to $-\infty$.

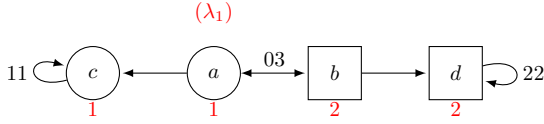


Fig. 5. The requirement λ_1 on the game of Example 2

Remark. Any play is λ_0 -consistent.

B. Negotiation

We will show that SPEs in prefix-independent games are characterized by the fixed points of a function on requirements. That function can be seen as a *negotiation*: when a player has a requirement to satisfy, another player can hope a better payoff than what they can secure in general, and therefore update their own requirement.

Definition 22 (Negotiation function). Let G be a game. The *negotiation function* is the function that transforms any requirement λ on G into a requirement $\text{nego}(\lambda)$ on G , such that for each $i \in \Pi$ and $v \in V_i$,

$$\text{nego}(\lambda)(v) = \inf_{\bar{\sigma}_{-i} \in \lambda\text{Rat}(v)} \sup_{\sigma_i} \mu_i(\langle \bar{\sigma} \rangle_v),$$

with the convention $\inf \emptyset = +\infty$.

Remark. The negotiation function has the following properties:

- The requirement λ is not satisfiable from v if and only if $\text{nego}(\lambda)(v) = +\infty$.
- The negotiation function is monotone: if $\lambda \leq \lambda'$ (for the pointwise order, i.e. if for each v , $\lambda(v) \leq \lambda'(v)$), then $\text{nego}(\lambda) \leq \text{nego}(\lambda')$.
- The negotiation function is non-decreasing: for every λ , we have $\lambda \leq \text{nego}(\lambda)$.

In the general case, the quantity $\text{nego}(\lambda)(v)$ represents the worst case value that the player controlling v can ensure, assuming that the other players play λ -rationally.

Example 4. Let us consider the game of Example 2: we represented it again in Figure 5, with the requirement $\lambda_1 = \text{nego}(\lambda_0)$, which is easy to compute since any strategy profile is λ_0 -rational: for each v , $\lambda_1(v)$ is the classical *worst-case value* or *antagonistic value* of v , i.e. the best value the player controlling v can enforce against a fully hostile environment. Let us now compute the requirement $\lambda_2 = \text{nego}(\lambda_1)$.

- From the state c , there exists exactly one λ_1 -rational strategy profile $\bar{\sigma}_{-\square} = \sigma_{\square}$, which is the empty strategy since player \square has never to choose anything. Against that strategy, the best and the only payoff player \circ can get is 1, hence $\lambda_2(c) = 1$.
- For the same reasons, $\lambda_2(d) = 2$.
- From the state b , player \circ can force \square to get the payoff 2 or less, with the strategy profile $\sigma_{\square} : h \mapsto c$. Such a strategy is λ_1 -rational, rationalized by the strategy $\sigma_{\square} : h \mapsto d$. Therefore, $\lambda_2(b) = 2$.

- Finally, from the state a , player \square can force \circ to get the payoff 2 or less, with the strategy profile $\sigma_{\square} : h \mapsto d$. Such a strategy is λ_1 -rational, rationalized by the strategy $\sigma_{\square} : h \mapsto c$. But, he can not force her to get less than the payoff 2, because she can force the access to the state b , and the only λ_1 -consistent plays starting from b are the plays with the form $(ba)^k bd^\omega$. Therefore, $\lambda_2(a) = 2$.

C. Steady negotiation

Often in what follows, we will need a game to be *with steady negotiation*, i.e. such that there always exists a worse λ -rational behaviour for the environment against a given player.

Definition 23 (Game with steady negotiation). A game G is *with steady negotiation* if and only if for every player i , for every vertex v , and for every requirement λ satisfiable from v , there exists a λ -rational strategy profile $\bar{\sigma}_{-i}$ from v such that:

$$\inf_{\bar{\sigma}_{-i} \in \lambda\text{Rat}(v_0)} \sup_{\sigma'_i} \mu_i(\langle \bar{\sigma}' \rangle_v) = \sup_{\sigma_i} \mu_i(\langle \bar{\sigma}_{-i}, \sigma_i \rangle_v).$$

Remark. In particular, when a game is with steady negotiation, the infimum in the definition of negotiation is always reached.

It will be proved in Section V that mean-payoff-inf games are with steady negotiation.

D. Link with Nash equilibria

Requirements and the negotiation function are able to capture Nash equilibria. Indeed, if λ_0 is the vacuous requirement, then $\text{nego}(\lambda_0)$ characterizes the plays that are supported by a Nash equilibrium (abbreviated by NE plays), in the following formal sense:

Theorem 1. *Let G be a game with steady negotiation. Then, a play ρ in G is an NE play if and only if ρ is $\text{nego}(\lambda_0)$ -consistent.*

Proof sketch: For a given state v , the value $\text{nego}(\lambda_0)(v)$ is defined as the best payoff that the player controlling v can ensure against any λ_0 -rational strategy profile, that is against any strategy profile: it is what is often called in the literature the *antagonistic value* or the *worst-case value* of v .

In a play that is not $\text{nego}(\lambda_0)$ -consistent, any player that does not have their requirements satisfied can deviate and ensure that requirement, and therefore has a profitable deviation.

Conversely, given a play that is $\text{nego}(\lambda_0)$ -consistent, we can construct an NE realizing that play, where all the players force each other to follow it, by threatening them to play fully adversarially against the one who would deviate. The steady negotiation property guarantees the existence of a fully adversarial strategy profile.

See Appendix A for a detailed proof. ■

Example 5. Let us consider again the game of Example 2, with the requirement λ_1 given in Figure 5. The only λ_1 -consistent plays in this game, starting from the state a , are ac^ω , and $(ab)^k d^\omega$ with $k \geq 1$. One can check that those plays are exactly the NE plays in that game.

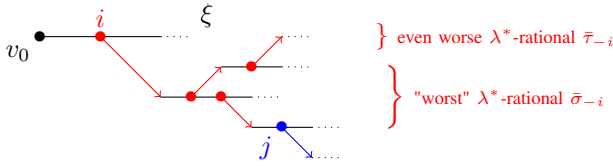


Fig. 6. The construction of the SPE $\bar{\sigma}$

In the following section, we will prove that as well as this requirement $\text{nego}(\lambda_0)$ characterizes the NEs, the requirement that is the least fixed point of the negotiation function characterizes the SPEs.

IV. LINK BETWEEN NEGOTIATION AND SPES

A. From negotiation fixed points to SPES

The notion of negotiation will enable us to find the SPEs, but also more generally the ε -SPEs, in a game. For that purpose, we need the notion of ε -fixed points of a function.

Definition 24 (ε -fixed point). Let $\varepsilon \geq 0$, let D be a finite set and let $f : \mathbb{R}^D \rightarrow \mathbb{R}^D$ be a mapping. A tuple $\bar{x} \in \mathbb{R}^D$ is a ε -fixed point of f if for each $d \in D$, if $\bar{y} = f(\bar{x})$, we have $y_d \in [x_d - \varepsilon, x_d + \varepsilon]$.

Remark. A 0-fixed point is a fixed point.

We can now prove that in games with steady negotiation, the ε -fixed points λ of the negotiation function are such that all λ -consistent plays are ε -SPEs plays.

Lemma 1. Let $G_{\uparrow v_0}$ be a well-initialized prefix-independent game with steady negotiation, and $\varepsilon \geq 0$. Let λ be an ε -fixed point of the function nego . Then, for every λ -consistent play ξ starting in v_0 , there exists an ε -SPE $\bar{\sigma}$ such that $\langle \bar{\sigma} \rangle_{v_0} = \xi$.

Proof sketch: Given a play ξ , the construction of $\bar{\sigma}$ can be represented as in Figure 6.

First, the play generated by $\bar{\sigma}$ is ξ . Then, whenever a player i deviates from that play, the other players must punish them, and stay λ -rational: they follow the λ -rational strategy profile that minimizes player i 's payoff, whose existence is guaranteed by the steady negotiation property. Player i plays a strategy that λ -rationalizes that strategy profile.

After a history where player i would have make serious mistakes, that is, choices that lower the best payoff they can ensure against a λ -rational environment, that environment must "reset" its strategy profile in order to be as hostile as it can be. Prefix-independence guarantees that those resets happen finitely many times.

We use the same construction if a second player j deviates after finitely many deviations of player i , and so on.

See Appendix B for a detailed proof. ■

B. From SPEs to negotiation fixed points

Conversely, let us prove that every ε -SPE play is λ -consistent for some ε -fixed point λ of the negotiation function.

Lemma 2. Let $G_{\uparrow v_0}$ be a well-initialized prefix-independent game, and let $\varepsilon \geq 0$. Let $\bar{\sigma}$ be an ε -SPE in $G_{\uparrow v_0}$. Then, there exists an ε -fixed point λ of the negotiation function such that for every history hv starting in v_0 , the play $\langle \bar{\sigma}_{\uparrow hv} \rangle_v$ is λ -consistent.

Proof sketch: Given an SPE $\bar{\sigma}$, we can define, for every player i and every state $v \in V_i$:

$$\lambda(v) = \inf_{hv \in \text{Hist} G_{\uparrow v_0}} \mu_i(\langle \bar{\sigma}_{\uparrow hv} \rangle_v)$$

as the lowest payoff that can be given to player i in a play starting from the state v in the strategy profile $\bar{\sigma}$. By construction, any play generated by $\bar{\sigma}$ after a finite history is λ -consistent, and the fact that $\bar{\sigma}$ is an ε -SPE implies that λ is an ε -fixed point of the negotiation function.

See Appendix C for a detailed proof. ■

C. Least ε -fixed point

Since the set of requirements, equipped with the componentwise order \leq , is a complete lattice, and since the negotiation function is monotone, Tarski's fixed point theorem states that the negotiation function has a least fixed point.

That result can be generalized to ε -fixed points:

Lemma 3. Let G be a game, and let $\varepsilon \geq 0$. The negotiation function has a least ε -fixed point.

A proof is given in Appendix D.

D. Theorem

The following theorem is an immediate consequence of the three previous lemmas, and sums up the link between the negotiation function and the SPEs, or ε -SPEs.

Theorem 2. Let $G_{\uparrow v_0}$ be an initialized prefix-independent game, and let $\varepsilon \geq 0$. Let λ^* be the least ε -fixed point of the negotiation function. Let ξ be a play starting in v_0 . If there exists an ε -SPE $\bar{\sigma}$ such that $\langle \bar{\sigma} \rangle_{v_0} = \xi$, then ξ is λ^* -consistent. The converse is true if the game G is with steady negotiation.

Proof: First, let us recall that λ^* , the least ε -fixed point of the negotiation function, exists by Lemma 3.

If $\bar{\sigma}$ is an ε -SPE, then by Lemma 2, there exists an ε -fixed point λ of the negotiation function such that all the plays generated by $\bar{\sigma}$ after some history are λ -consistent; in particular, the play ξ is λ -consistent, and therefore λ^* -consistent since $\lambda^* \leq \lambda$.

Conversely, if the game G is with steady negotiation, and if the play ξ is λ^* -consistent, then by Lemma 1, there exists an ε -SPE $\bar{\sigma}$ such that $\langle \bar{\sigma} \rangle_{v_0} = \xi$. ■

In the following sections, we will develop a method to compute the negotiation function: we will prove that in the case of mean-payoff-inf games, it is actually a piecewise linear function, which makes it feasible to compute and express the set of its ε -fixed points; and therefore, to find the least of them using classical linear algebraic techniques.

However, when one looks for a least fixed point, a usual method, under some continuity hypothesis, is to compute the

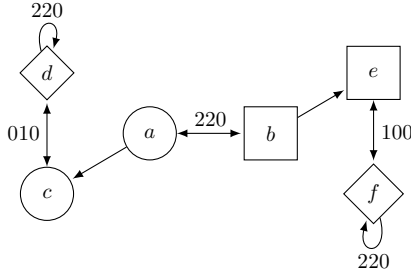


Fig. 7. A game where the negotiation function is not stationary

limit of the iterations, by successive approximations based on Kleene-Tarski theorem. We develop that possibility in the following subsection, confirming that the least fixed point of the negotiation function is indeed the limit of its iteration on the vacuous requirement λ_0 , and explaining why it can not always be used in practice.

E. An insufficient track: the negotiation sequence

We assume in that subsection that G is a game such that the negotiation function is *Scott-continuous*, i.e. such that for every non-decreasing sequence $(\lambda_n)_n$ of requirements on G , we have:

$$\text{nego} \left(\sup_n \lambda_n \right) = \sup_n \text{nego}(\lambda_n).$$

The least fixed point of the negotiation function is, then, the limit of the *negotiation sequence*, defined as the sequence $(\lambda_n)_{n \in \mathbb{N}} = (\text{nego}^n(\lambda_0))_n$.

Indeed, that sequence is non-decreasing; therefore it has a limit λ . By Scott-continuity, the equality $\lambda_{n+1} = \text{nego}(\lambda_n)$ implies, when we take the suprema over n , that λ is a fixed point of the negotiation function. If λ^* is the least fixed point of the negotiation function, then $\lambda^* \leq \lambda$; and on the other hand, $\lambda^* \geq \lambda$ by induction, since $\lambda^* \leq \lambda_0$ and if $\lambda^* \leq \lambda_n$, then $\lambda^* \leq \lambda_{n+1}$ because the negotiation function is non-decreasing. Therefore, $\lambda^* = \lambda$.

In mean-payoff-inf games, in particular, the negotiation function is Scott-continuous:

Proposition 2. *In mean-payoff-inf games, the negotiation function is Scott-continuous.*

A proof of that statement is given in Appendix I: note that it uses results that will be presented in Section V.

In many cases, the negotiation sequence is stationary, and in that case, it is possible to compute its limit: whenever a term is equal to the previous one, we know that we reached it. But actually, the negotiation sequence is not always stationary. The game of Figure 7, where for all edges, the first label is the weight for player \circ , the second one is the label for player \square , and the third one for player \diamond , is a counter-example.

For all n , we have:

$$\lambda_n(a) = \lambda_n(b) = 2 - \frac{1}{2^{n-1}}.$$

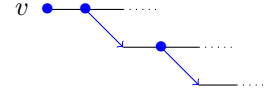


Fig. 8. The play constructed by Prover and Challenger in the abstract negotiation game

Indeed, the game is symmetric, and the lowest payoff that can be proposed to player \circ from the state a will be obtained by a combination of the cycles ef and f that has to satisfy player \square 's requirement in the state b , hence the following inductive equation:

$$\lambda_{n+1}(a) = 1 + \frac{1}{2} \lambda_n(a),$$

whose solution is the sequence proposed above. This sequence converges to 2 but never reaches it. All the details of that statement, and a similar example with only two players, are given in Example 15, in Appendix K.

V. NEGOTIATION GAMES

We have now proved that SPEs are characterized by the requirements that are fixed points of the negotiation function; but we need to know how to compute, in practice, the quantity $\text{nego}(\lambda)$ for a given requirement λ . In other words, we need an algorithm that gives, given a state v_0 controlled by a player i in the game G , and given a requirement λ , which value player i can ensure in $G_{\uparrow v_0}$ if the other players play λ -rationally. The concept of Prover-Challenger games, used for example in [15], gives us a tool for that purpose.

A. Abstract negotiation game

We first define an *abstract negotiation game*, that is conceptually simple but not directly usable for algorithmic purpose because it is defined on an uncountable infinite state space.

Here is an intuitive definition of the abstract negotiation game $\text{Abs}_{\lambda i}(G)_{\uparrow [v_0]}$ from a state v_0 , a player i and a requirement λ :

- player *Prover* proposes a λ -consistent play ρ from v_0 (or looses, if she has no play to propose);
- either:
 - player *Challenger* accepts the play and the game terminates;
 - or he chooses an edge $\rho_k \rho_{k+1}$, with $\rho_k \in V_i$, from which he can make player i deviate, using another edge $\rho_k v$ with $v \neq \rho_{k+1}$: then, the game starts again from w instead of v_0 .
- In the resulting play (either eventually accepted by Challenger, or constructed by an infinity of deviations), as represented in Figure 8, Prover wants player i 's payoff to be low, and Challenger wants it to be high.

That game gives us the basis of a method to compute $\text{nego}(\lambda)$ from λ : if α is the maximal outcome that Challenger can ensure in $\text{Abs}_{\lambda i}(G)_{\uparrow [v_0]}$, with $v_0 \in V_i$, then it is the

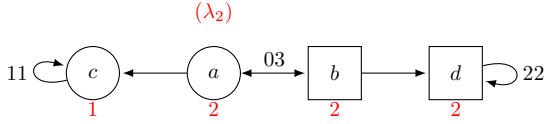


Fig. 9. The requirement λ_2 on the game of Example 2

maximal payoff that player i can guarantee in $G_{\uparrow v_0}$, against a λ -rational environment. Hence the equality:

$$\text{val}(\text{Abs}_{\lambda i}(G)_{[v_0]}) = \text{nego}(\lambda)(v_0).$$

A proof of that statement, with a complete formalization of the abstract negotiation game, are presented in Appendix E.

Example 6. Let us take the game of Example 2: in Figure 9, we wrote, in red, the requirement $\lambda_2 = \text{nego}(\lambda_1)$, computed in Section III-B. Let us use the abstract negotiation game to compute the requirement $\lambda_3 = \text{nego}(\lambda_2)$.

From the state a , Prover can propose the play abd^ω , and the only deviation Challenger can do is going to c ; he has of course no incentive to do it. Therefore, $\lambda_3(a) = 2$.

From the state b , whatever Prover proposes at first, Challenger can deviate and go to a . Then, from a , Prover cannot propose the play ac^ω , which is not λ_2 -consistent: the only possibility she has is proposing a play beginning by ab , and letting Challenger deviate once more. He can deviate infinitely often that way, and generate the play $(ba)^\omega$: therefore, $\lambda_3(b) = 3$.

The other states keep the same values.

Note that λ_3 is no longer satisfiable from a or b , and therefore that if $\lambda_4 = \text{nego}(\lambda_3)$, then $\lambda_4(a) = \lambda_4(b) = +\infty$. By the considerations on the negotiation sequence given in Section IV-E, this proves that the least fixed point of the negotiation function is not satisfiable, and therefore that there is no SPE in that game.

The interested reader will find other examples in Appendix K.

B. Concrete negotiation game

In the abstract negotiation game, Prover has to propose complete plays, on which we can make the hypothesis that they are λ -consistent. In practice, there will often be an infinity of such plays, and therefore it cannot be used directly for an algorithmic purpose. Instead, those plays can be given edge by edge, with a finite state game. Its definition is more technical, but it can be shown that it is equivalent to this abstract one.

Definition 25 (Concrete negotiation game). Let $G_{\uparrow v_0}$ be an initialized prefix-independent game, and let λ be a requirement on G , with either $\lambda(V) \subseteq \mathbb{R}$, or $\lambda = \lambda_0$.

The *concrete negotiation game* of $G_{\uparrow v_0}$ for player i is the two-player zero-sum game:

$$\text{Conc}_{\lambda i}(G)_{\uparrow s_0} = (\{\mathcal{P}, \mathcal{C}\}, S, (S_{\mathcal{P}}, S_{\mathcal{C}}), \Delta, \nu)_{\uparrow s_0},$$

that, intuitively, mimics the abstract game introduced above:

- the states controlled by Prover are:

$$S_{\mathcal{P}} = V \times 2^V$$

where the state $s = (v, M)$ represents a current state v on which Prover has to define the strategy profile, called *localization* of the state s , and M is the *memory* of s , which memorizes the states that have been traversed so far since the last deviation, in order to check the λ -consistency of the proposed play;

- the states controlled by Challenger are:

$$S_{\mathcal{C}} = E \times 2^V;$$

- there are three types of transitions: *proposals*, *acceptations* and *deviations*:

$$\Delta = \text{Prop} \cup \text{Acc} \cup \text{Dev}$$

where:

- the proposals are transitions in which Prover proposes an edge to follow in the game G :

$$\text{Prop} = \left\{ (v, M)(vw, M) \mid \begin{array}{l} vw \in E, \\ M \in 2^V \end{array} \right\};$$

- the acceptations are transitions in which Challenger accepts to follow the edge proposed by Prover (this is in particular his only possibility whenever that edge begins on a state that is not controlled by player i):

$$\text{Acc} = \left\{ (vw, M)(w, M \cup \{w\}) \mid \begin{array}{l} j \in \Pi, \\ w \in V_j \end{array} \right\}$$

(note that the memory is updated);

- the deviations are transitions in which Challenger refuses to follow the edge proposed by Prover, as he can if that edge begins in a state controlled by player i :

$$\text{Dev} = \left\{ (uv, M)(w, \{w\}) \mid \begin{array}{l} u \in V_i, \\ w \neq v, \\ uw \in E \end{array} \right\}$$

(the memory is erased, and only the new state the deviating edge leads to is memorized);

- the outcome function ν measures player i 's payoff, with a defeat condition if the constructed strategy profile is not λ -rational, that is to say if after finitely many player i 's deviations, it can generate a play which is not λ -consistent:

- $\nu_{\mathcal{C}}(\eta) = +\infty$ if there exists n such that no transition in the play $\eta_n \eta_{n+1} \dots$ is a deviation, and if there exists $j \in \Pi$ such that $\hat{\mu}_j(\eta) < 0$;
- $\nu_{\mathcal{C}}(\eta) = \hat{\mu}_*(\eta)$ otherwise;
- and $\nu_{\mathcal{P}} = -\nu_{\mathcal{C}}$;

where for each *dimension* $j \in \Pi$, $\hat{\mu}_j$ measures the difference between player j 's payoff and player j 's maximal requirement:

$$\begin{aligned} \hat{\mu}_j((\rho_0, M_0)(\rho_0 \rho'_0, M_0)(\rho_1, M_1) \dots) \\ = \mu_j(\rho) - \limsup_n \max_{v \in M_n \cap V_j} \lambda(v) \end{aligned}$$

and for the special dimension denoted by \star , $\hat{\mu}_\star$ measures player i 's payoff:

$$\hat{\mu}_\star((\rho_0, M_0)(\rho_0\rho'_0, M_0)(\rho_1, M_1)\dots) = \mu_i(\rho)$$

The dimension \star is called *main* dimension, and each $j \in \Pi$ is a *non-main* dimension;

- and finally, $s_0 = (v_0, \{v_0\})$.

Like in the abstract negotiation game, the goal of Prover is to find a λ -rational strategy profile that forces the worst possible payoff for player i , and the goal of Challenger is to find a possibly deviating strategy for player i that gives them the highest possible payoff.

In the case of mean-payoff-inf games, the function $\hat{\mu}$ is a multi-mean-payoff function, which will enable us to compute the value of the concrete negotiation game.

Remark. In the case of a mean-payoff-inf game, each function $\hat{\mu}_j$ is the mean-payoff-inf function corresponding to the weight function $\hat{\pi}_j$ defined by, for $j \in \Pi$:

$$\hat{\pi}_j((v, M)(vw, M)) = 0$$

$$\hat{\pi}_j((uv, M)(w, N)) = 2 \left(\pi_j(uw) - \max_{v_j \in M \cap V_j} \lambda(v_j) \right)$$

and:

$$\hat{\pi}_\star((v, M), (vw, M)) = 0$$

$$\hat{\pi}_\star((uv, M), (w, N)) = 2\pi_i(uw).$$

A play or a history in the concrete negotiation game has a projection in the game on which that negotiation game has been constructed, defined as follow:

Definition 26 (Projection of a history, of a play). Let G be a prefix-independent game. Let λ be a requirement and i a player, and let $\text{Conc}_{\lambda i}(G)$ be the corresponding concrete negotiation game. Let $H = (h_0, M_0)(h_0h'_0, M_0) \dots (h_nh'_n, M_n)$ be a history in $\text{Conc}_{\lambda i}(G)$: the *projection* of the history H is the history in the game G :

$$\dot{H} = h_0 \dots h_n.$$

That definition is naturally extended to plays.

Remark. For a play η where no transition is a deviation, we have that $\hat{\mu}_j(\eta) \geq 0$ for each $j \in \Pi$ if and only if η is λ -consistent.

Although the construction is technically more complex, the concrete negotiation game is equivalent to the abstract one: the only differences are that the plays proposed by Prover are proposed edge by edge, and that their λ -consistency is not written in the rules of the game but in its outcome function.

Theorem 3. Let $G|_{v_0}$ be an initialized prefix-independent Borelian game. Let λ be a requirement and i a player. Then, we have:

$$\text{val}(\text{Conc}_{\lambda i}(G)|_{s_0}) = \inf_{\bar{\sigma}_i \in \lambda\text{Rat}(v_0)} \sup_{\sigma_i} \mu_i(\langle \bar{\sigma} \rangle_{v_0}).$$

A proof can be found in Appendix F.

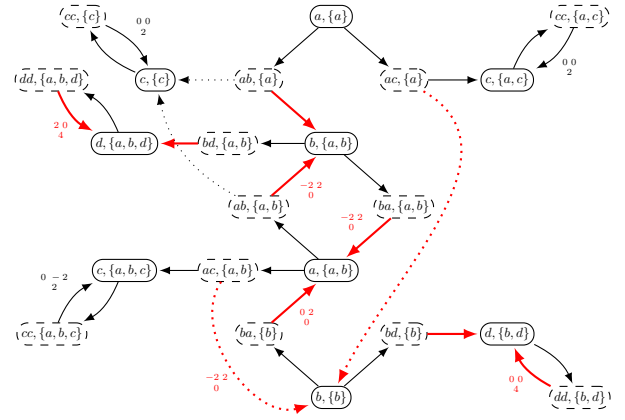


Fig. 10. A concrete negotiation game

Example 7. Let us consider again the game from Example 2.

Figure 10 represents the game $\text{Conc}_{\lambda_1 \circ}(G)$ (with $\lambda_1(a) = 1$ and $\lambda_1(b) = 2$), where the dashed states are controlled by Challenger, and the other ones by Prover.

The dotted arrows indicate the deviations, and when a transition ss' is labelled by:

$$\begin{matrix} x & y \\ z, \end{matrix}$$

x denotes $\hat{\pi}_\square(ss')$, y denotes $\hat{\pi}_\square(ss')$ and z denotes $\hat{\pi}_\star(ss')$. The transitions that are not labelled are either zero for the three coordinates, or meaningless since they cannot be used more than once.

The red arrows indicate a (memoryless) optimal strategy for Challenger. Against that strategy, the lower outcome Prover can ensure is 2.

Therefore, $\text{nego}(\lambda_1)(v_0) = 2$, in line with the abstract game in Example 6.

C. Solving the concrete negotiation game for the mean-payoff-inf case

First, let us note that mean-payoff-inf games are Borelian, and therefore satisfy the hypotheses of Theorem 3.

We now know that $\text{nego}(\lambda)(v)$, for a given requirement λ , a given player i and a given state $v \in V_i$, is the value of the concrete negotiation game $\text{Conc}_{\lambda i}(G)|_{(v, \{v\})}$. Let us now show how, in the mean-payoff-inf case, that value can be computed.

Definition 27 (Memoryless strategy). A strategy σ_i in a game G is *memoryless* if for all vertices $v \in V_i$ and for all histories h and h' , $\sigma_i(hv) = \sigma_i(h'v)$.

For any game G and any memoryless strategy σ_i , $G[\sigma_i]$ denotes the graph induced by σ_i , that is the graph (V, E') , with:

$$E' = \{vw \in E \mid v \notin V_i \text{ or } w = \sigma_i(v)\}.$$

For any finite set D and any set $X \subseteq \mathbb{R}^D$, $\text{Conv}X$ denotes the convex envelopp of X .

We can now prove that in the concrete negotiation game constructed from a mean-payoff-inf game, the player Challenger has an optimal strategy that is memoryless.

Lemma 4. *Let G be a mean-payoff-inf game, let i be a player, let λ be a requirement and let $\text{Conc}_{\lambda i}(G)$ be the corresponding concrete negotiation game. There exists a memoryless strategy τ_C such that for all states s :*

$$\inf_{\tau_P} \nu_C(\langle \bar{\tau} \rangle_s) = \text{val}(\text{Conc}_{\lambda i}(G)_{\uparrow s}),$$

i.e. that is optimal for Challenger from all state.

Proof. By [12], a player whose objective is prefix-independent and *convex*, that is, such that whenever two plays satisfy that objective, a shuffling of those two plays satisfies it too, has a memoryless optimal strategy.

For Challenger, ensuring a payoff above some value is not exactly convex; but it becomes convex if we replace the limit inferior in the definition of $\hat{\mu}_*$ by a limit superior. The optimal Challenger's strategy with regards to that slightly modified objective is also optimal in the actual concrete negotiation game.

See Appendix G for a complete proof. \square

With Lemma 4, we can now compute the value of the concrete negotiation game.

When G is a graph, $\text{SC}(G)$ denotes the set of all the simple cycles of G .

For any closed set $C \subseteq \mathbb{R}^{\Pi \cup \{\star\}}$, the quantity:

$$\min^* C = \min \{x_\star \mid \bar{x} \in C, \forall j \in \Pi, x_j \geq 0\}$$

is the \star -minimum of C : it will capture, in the concrete negotiation game, the least payoff that can be imposed on player i while keeping every player's payoff above their requirements, among a set of possible outcomes. See Figure 11 for an illustration.

For every game $G_{\uparrow v_0}$ and each player i , $\text{ML}_i(G_{\uparrow v_0})$, or $\text{ML}(G_{\uparrow v_0})$ when the context is clear, denotes the set of memoryless strategies for player i in $G_{\uparrow v_0}$.

For any graph (V, E) , $\text{SConn}(V, E)$ denotes the strongly connected components of (V, E) (considered as a subgraph of (V, E) or as a subset of V , depending on the context).

Lemma 5. *Let $G_{\uparrow v_0}$ be an initialized mean-payoff-inf game, and let $\text{Conc}_{\lambda i}(G)_{\uparrow s_0}$ be its concrete negotiation game for some λ and some i . Then, the value of the game $\text{Conc}_{\lambda i}(G)_{\uparrow s_0}$ is given by the formula:*

$$\max_{\tau_C \in \text{ML}_C(\text{Conc}_{\lambda i}(G))} \min_{\substack{K \in \text{SConn}(\text{Conc}_{\lambda i}(G)_{\uparrow \tau_C}) \\ \text{accessible from } s_0}} \text{opt}(K),$$

where $\text{opt}(K)$ is the minimal value $\nu_C(\rho)$ for ρ among the infinite paths in K .

- If K contains a deviation, then Prover can choose the simple cycle of K that minimizes player i 's payoff:

$$\text{opt}(K) = \min_{c \in \text{SC}(K)} \hat{\mu}_*(c^\omega).$$

- If K does not contain a deviation, then Prover must choose a combination of the simple cycles of K that minimizes player i 's payoff while keeping the non-main dimensions above 0:

$$\text{opt}(K) = \min^*_{c \in \text{SC}(K)} \text{Conv} \hat{\mu}(c^\omega).$$

A proof can be found in Appendix H.

Corollary 1. *For each player i and every state $v \in V_i$, the value $\text{nego}(\lambda)(v)$ can be computed with the formula given in Lemma 5 applied to the game $\text{Conc}_{\lambda i}(G)_{(v, \{v\})}$*

Moreover, another corollary of that result is that there always exists a best play that Prover can choose, i.e. Prover has an optimal strategy; by Theorem 3, this is equivalent to say that mean-payoff-inf games are games with steady negotiation.

Corollary 2. *The mean-payoff-inf games are games with steady negotiation.*

VI. ANALYSIS OF THE NEGOTIATION FUNCTION IN MEAN-PAYOFF-INF GAMES

In this section, we will show that for the case of mean-payoff-inf games, the negotiation function is a piecewise linear function from the vector space of requirements into itself, which can therefore be computed and analyzed using classical linear algebra techniques. Then, it becomes possible to search for the fixed points or the ε -fixed points of such a function, and to decide the existence or not of SPEs or ε -SPEs in the game studied.

Theorem 4. *Let $G_{\uparrow v_0}$ be an initialized mean-payoff-inf game. Let us assimilate any requirement λ on G with finite values to the tuple $\bar{\lambda} = (\lambda(v))_{v \in V}$, element of the vector space of finite dimension \mathbb{R}^V . Then, for each player i and every vertex $v_0 \in V_i$, the quantity $\text{nego}(\lambda)(v_0)$ is a piecewise linear function of $\bar{\lambda}$, which can be effectively expressed and whose ε -fixed points are computable for all ε .*

Proof sketch: By Lemma 5, the quantity $\text{nego}(\lambda)(v_0)$ is equal to:

$$\max_{\tau_C \in \text{ML}_C(\text{Conc}_{\lambda i}(G))} \min_{\substack{K \in \text{SConn}(\text{Conc}_{\lambda i}(G)_{\uparrow \tau_C}) \\ \text{accessible from } s_0}} \text{opt}(K),$$

which is defined from the concrete negotiation game $\text{Conc}_{\lambda i}(G)$, itself depending on λ . But the underlying graph of the concrete negotiation game is actually independent of λ , which appears only in the computation of the non-main dimension weights. Therefore, the indexation of the maximum and the minimum in the formula above does not depend on λ .

Then, we have to prove that the quantity $\text{opt}(K)$ is a piecewise linear function of $\bar{\lambda}$, in both of the cases given by Lemma 5: when K contains a deviation, and when it does not.

In the first case, it is trivially true, because $\text{opt}(K)$ is actually independent of λ . In the second case, $\text{opt}(K)$ is defined as the \star -minimum of the convex hull of finitely many points, i.e. of a polyhedron. A modification of λ engenders a translation of that polyhedron, and the \star -minimum of that polyhedron is

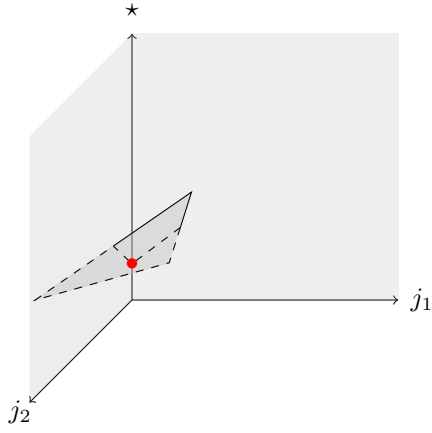


Fig. 11. The \star -minimum of a polyhedron

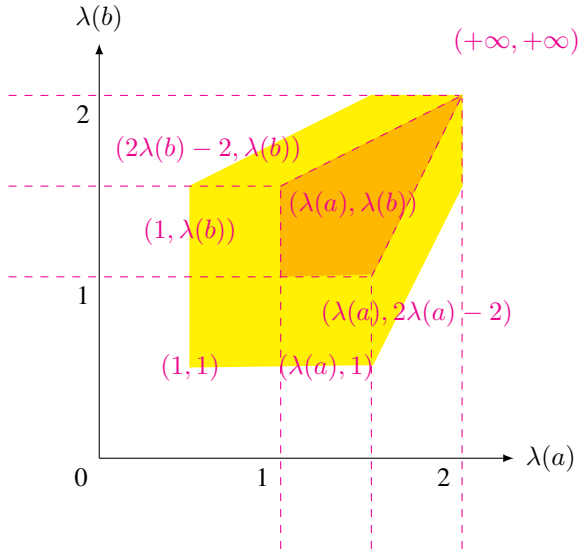


Fig. 12. The negotiation function on the game of Example 3

the minimum, along the dimension \star , of its intersection with the quadrant of the points that have positive coordinates along the non-main dimensions: that quantity evolves linearly with such translations, as illustrated in Figure 11.

See Appendix J for a detailed proof. ■

Example 8. Let G be the game of Example 3, represented in Figure 3.

Then, if a requirement λ is represented by the tuple $(\lambda(a), \lambda(b))$, then the function $\text{nego} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ can be represented by Figure 12, where in any one of the regions delimited by the dashed lines, we wrote a formula for the couple $(\text{nego}(\lambda)(a), \text{nego}(\lambda)(b))$.

The orange area indicates the fixed points of the function, and the yellow area the other $\frac{1}{2}$ -fixed points.

Remember that, by Proposition 2, the negotiation function is Scott-continuous: that means that when we are exactly on a magenta line, the good expression of nego is the one of the tile to the left (for a vertical line) or at the bottom (for a

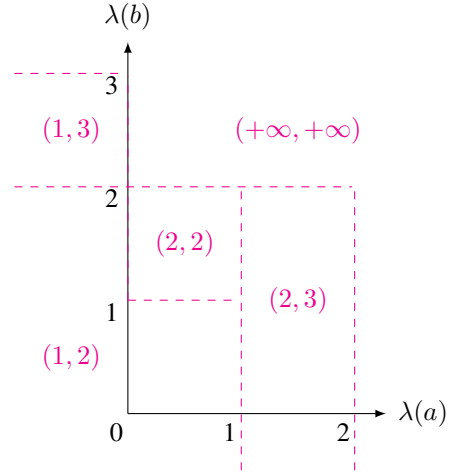


Fig. 13. The negotiation function on the game of Example 2

horizontal one).

Example 9. As a second example, let us consider the game of the Example 2, represented in Figure 2.

First, let us note that if $\lambda(c) \leq 1$ and $\lambda(d) \leq 2$, then $\text{nego}(\lambda)(c) = 1$ and $\text{nego}(\lambda)(d) = 2$. We can therefore fix $\lambda(c) = 1$ and $\lambda(d) = 2$, and represent the requirements λ by the tuples $(\lambda(a), \lambda(b))$, as in the previous example. Then, the negotiation function can be represented as in Figure 13.

One can check that there is no fixed point here, and even no $\frac{1}{2}$ -fixed point, except $(+\infty, +\infty)$.

CONCLUSION AND FUTURE WORKS

With the tools that we defined, we are now able to characterize effectively all the SPEs, and ε -SPEs, in a mean-payoff-inf game with arbitrarily many players. We do it by constructing a complete representation of the negotiation function using its associated concrete negotiation games, and then finding its least fixed point, or ε -fixed point, with classical linear algebraic tools. That algorithm also provides a solution to the (constrained) existence problem for SPEs in mean-payoff-inf games, which was left open in the literature.

If we are able to find a formula expressing the negotiation function like in Lemma 4 for other classes of prefix-independent games with steady negotiation, for example using the concrete negotiation game, then it will be also possible, by Theorem 2, to characterize the SPEs and ε -SPEs in those games.

When we look for SPEs, i.e. ε -SPEs with $\varepsilon = 0$, another method can be the computation of the limit of the negotiation sequence, using the concrete negotiation game, or the abstract one, if the game is simple enough. But such an algorithm does not always stop, since in some cases, the negotiation sequence needs transfinite number of steps to converge. Nevertheless, let us note that the sequence will always converge after finitely many iterations in games where there exists only a finite number of possible outcomes; examples of such games are liminf and limsup games, for example, or mean-payoff-inf

and mean-payoff-sup games with only disjoint cycles. An algorithmic method was already defined in [15] for such cases: an open question is then to know which alternative has the better complexity, and perhaps to define wider classes of games in which we know that the negotiation sequence will stabilize in a finite number of steps.

Finally, another open question is whether the notion of negotiation, and its link with the SPEs, can be generalized to games that are not prefix-independent, or that are not with steady negotiation.

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The following appendices are providing the detailed proofs of all our results. They are not necessary to understand our results and are meant to provide full formalization and rigorous proofs. They also provide further intuitions through additional examples for the interested reader. To improve readability, we have chosen to recall the statements that appeared in the main body of the paper before giving their detailed proofs in order to ease the work of the reader.

APPENDIX A PROOF OF THEOREM 1

Theorem 1. *Let G be a game with steady negotiation. Then, a play ρ in G is an NE play if and only if ρ is $\text{nego}(\lambda_0)$ -consistent.*

Proof:

- Let $\bar{\sigma}$ be a Nash equilibrium in $G_{\uparrow v_0}$, for some state v_0 , and let $\rho = \langle \bar{\sigma} \rangle_{v_0}$: let us prove that the play ρ is $\text{nego}(\lambda_0)$ -consistent.

Let $k \in \mathbb{N}$, let $i \in \Pi$ be such that $\rho_k \in V_i$, and let us prove that $\mu_i(\rho_k \rho_{k+1} \dots) \geq \text{nego}(\lambda_0)(\rho_k)$.

For any deviation σ'_i of $\sigma_i \upharpoonright_{\rho_0 \dots \rho_k}$, by definition of NEs, $\mu_i(\langle \bar{\sigma}_{-i} \upharpoonright_{\rho_0 \dots \rho_k}, \sigma'_i \rangle_{\rho_k}) \leq \mu_i(\rho)$. Therefore:

$$\mu_i(\rho) \geq \sup_{\sigma'_i} \mu_i(\langle \bar{\sigma}_{-i} \upharpoonright_{\rho_0 \dots \rho_k}, \sigma'_i \rangle_{\rho_k})$$

hence:

$$\mu_i(\rho) \geq \inf_{\bar{\tau}_{-i}} \sup_{\tau_i} \mu_i(\langle \bar{\tau}_{-i} \upharpoonright_{\rho_0 \dots \rho_k}, \tau_i \rangle_{\rho_k})$$

i.e.:

$$\mu_i(\rho) \geq \text{nego}(\lambda_0)(\rho_k).$$

- Let ρ be a $\text{nego}(\lambda_0)$ -consistent play from a state v_0 . Let us define a strategy profile $\bar{\sigma}$ such that $\langle \bar{\sigma} \rangle_{v_0} = \rho$, by:

- $\langle \bar{\sigma} \rangle_{v_0} = \rho$;
- for all histories of the form $\rho_0 \dots \rho_k v$ with $v \neq \rho_{k+1}$, let i be the player controlling ρ_k .

Since the game G is with steady negotiation, the infimum:

$$\inf_{\bar{\tau}_{-i} \in \lambda_0 \text{Rat}(\rho_k)} \sup_{\tau_i} \mu_i(\langle \bar{\tau}_{-i}, \tau_i \rangle_{\rho_k})$$

is a minimum. Let $\bar{\tau}_{-i}^k$ be λ_0 -rational strategy profile from ρ_k realizing that minimum, and let τ_i^k be some strategy from ρ_k such that $\tau_i^k(\rho_k) = v$. Then, we define:

$$\langle \bar{\sigma} \upharpoonright_{\rho_0 \dots \rho_k v} \rangle_v = \langle \bar{\tau}_{-i}^k, \tau_i^k \rangle_v;$$

- for every other history h , $\bar{\sigma}(h)$ is defined arbitrarily.

Let us prove that $\bar{\sigma}$ is an NE: let σ'_i be a deviation of σ_i , let $\rho' = \langle \bar{\sigma}_{-i}, \sigma'_i \rangle_{v_0}$ and let $\rho_0 \dots \rho_k$ be the longest common prefix of ρ and ρ' . Let $v = \rho_{k+1}$.

Then, we have:

$$\mu_i(\rho') \leq \sup_{\tau_i^k} \mu_i(\langle \bar{\tau}_{-i}^k, \tau_i^k \rangle_{\rho_k}) = \text{nego}(\lambda_0)(\rho_k),$$

and since ρ is λ_0 -consistent, $\text{nego}(\lambda_0)(\rho_k) \leq \mu_i(\rho)$, hence $\mu_i(\rho') \leq \mu_i(\rho)$. ■

APPENDIX B PROOF OF LEMMA 1

Lemma 1. *Let $G_{\uparrow v_0}$ be a well-initialized prefix-independent game with steady negotiation, and $\varepsilon \geq 0$. Let λ be an ε -fixed point of the function nego . Then, for every λ -consistent play ξ starting in v_0 , there exists an ε -SPE $\bar{\sigma}$ such that $\langle \bar{\sigma} \rangle_{v_0} = \xi$.*

Proof:

- *Particular case: if there exists v such that $\lambda(v) = +\infty$.*
In that case, since the game $G_{\uparrow v_0}$ is well-initialized, there is no λ -rational strategy profile from v_0 , and $\text{nego}(\lambda)(v_0) = +\infty$. Since ε is finite and since λ is an ε -fixed point of the negotiation function, it follows that $\lambda(v_0) = +\infty$: in that case, there is no λ -consistent play ξ from v_0 , and then the proof is done. Therefore, for the rest of the proof, we assume that for all v , we have $\lambda(v) \neq +\infty$. As a consequence, since λ is an ε -fixed point of the function nego , for all v , we have $\text{nego}(\lambda)(v) \neq +\infty$; and so finally, for each such v , there exists a λ -consistent play starting from v .
- *Preliminary result: a game with steady negotiation is also with subgame-steady negotiation.*

Recall that a game with steady negotiation is a game such that for every requirement λ , for every player i and for every state v , there exists a λ -rational strategy profile $\bar{\tau}^v$ such that:

$$\sup_{\tau_i^v} \mu_i(\langle \bar{\tau}^v \rangle_v) = \inf_{\bar{\tau}_{-i} \in \lambda \text{Rat}(v)} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_v)$$

is realized, i.e. there exists a worst λ -rational strategy profile against player i from the state v , with regards to player i 's payoff.

Our goal in this part of the proof is to show that a game that is with steady negotiation is also with *subgame-steady negotiation*, that is to say, for every requirement λ , for every player i and for every state v , there exists a λ -rational strategy profile $\bar{\tau}_{-i}^{v*}$ such that for every history hw starting from v compatible with $\bar{\tau}_{-i}^{v*}$, we have:

$$\sup_{\tau_i^{v*}} \mu_i(\langle \bar{\tau}_{-i}^{v*}, \tau_i \rangle_{hw}) = \inf_{\bar{\tau}_{-i} \in \lambda \text{Rat}(w)} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_w),$$

i.e. there exists a λ -rational strategy profile against player i from the state v , that is the worst with regards to player i 's payoff in any subgame, in other words a *subgame-worst* strategy profile.

Let us construct inductively the strategy profile $\bar{\tau}_{-i}^{v*}$ and the strategy τ_i^{v*} λ -rationalizing it. We define them only on histories that are compatible with $\bar{\tau}_{-i}^{v*}$, since they can be defined arbitrarily on any other histories. We proceed by assembling the strategy profiles of the form $\bar{\tau}^w$, and the histories after which we follow a new $\bar{\tau}^w$ will be called the *resets* of $\bar{\tau}_{-i}^{v*}$.

- First, $\langle \bar{\tau}^{v*} \rangle_v = \langle \bar{\tau}^v \rangle_v$: the one-state history v is then the first reset of $\bar{\tau}_{-i}^{v*}$;
- then, for every history hw from v such that h is compatible with $\bar{\tau}_{-i}^{v*}$ and ends in V_i , and such that $w \neq \tau_i^{v*}(h)$: let us write $hw = h'uh''$ so that $h'u$

is the longest reset of $\bar{\tau}_{-i}^{v*}$ among the prefixes of h , and therefore so that the strategy profile $\bar{\tau}_{|h'u}^{v*}$ has been defined as equal to $\bar{\tau}^u$ over the prefixes of h'' until w . Then, we have:

$$\sup_{\tau_i} \mu_i(\langle \bar{\tau}_{-i}^w, \tau_i \rangle_w) \leq \sup_{\tau_i} \mu_i(\langle \bar{\tau}_{-i|uh''}^u, \tau_i \rangle_w)$$

by prefix-independence of G and since by its definition, the strategy profile $\bar{\tau}_{-i}^w$ minimizes the quantity $\sup_{\tau_i} \mu_i(\langle \bar{\tau}_{-i}^w, \tau_i \rangle_w)$. Let us separate two cases.

* Suppose first that:

$$\sup_{\tau_i} \mu_i(\langle \bar{\tau}_{-i}^w, \tau_i \rangle_w) = \sup_{\tau_i} \mu_i(\langle \bar{\tau}_{-i|uh''}^u, \tau_i \rangle_w).$$

Then, $\langle \bar{\tau}_{|hw}^{v*} \rangle = \langle \bar{\tau}_{|uh''}^u \rangle_w$: the coalition of players against player i keeps following their strategy profile so that player i will have no more than the payoff they can ensure.

* Suppose now that:

$$\sup_{\tau_i} \mu_i(\langle \bar{\tau}_{-i}^w, \tau_i \rangle_w) < \sup_{\tau_i} \mu_i(\langle \bar{\tau}_{-i|uh''}^u, \tau_i \rangle_w).$$

Then, $\langle \bar{\tau}_{|hw}^{v*} \rangle = \langle \bar{\tau}^w \rangle_w$: player i has done something that lowers the payoff they can ensure, and therefore the other players have to update their strategy profile in order to enforce that new minimum.

The history hw is a reset of $\bar{\tau}_{-i}^{v*}$.

All the plays constructed are λ -consistent, hence $\bar{\tau}_{-i}^{v*}$ is indeed λ -rationalized by τ_i^{v*} .

Let us now prove that τ_i^{v*} is the subgame-worst λ -rational strategy profile against player i . Let hw be a history starting in v compatible with $\bar{\tau}_{-i}^{v*}$, let v_i be a strategy from the state w , let $\eta = \langle \bar{\tau}_{-i|hw}^{v*}, v_i \rangle_w$ and let us prove that:

$$\mu_i(\eta) \leq \inf_{\bar{\tau}_{-i} \in \lambda\text{Rat}(w)} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_w).$$

Let us consider the sequence $(\alpha_n)_{n \in \mathbb{N}}$, defined by:

$$\alpha_n = \inf_{\bar{\tau}_{-i} \in \lambda\text{Rat}(\eta_n)} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_{\eta_n}).$$

That sequence is non-increasing. Indeed, for all n :

- * If $\eta_n \in V_i$, then no action of player i can improve the payoff player i himself can secure against a λ -rational environment.
- * If $\eta_n \notin V_i$, then: $\eta_{n+1} = \bar{\tau}_{-i}^{v*}(h\eta_0 \dots \eta_n) = \bar{\tau}_{-i}^k(\eta_k \dots \eta_n)$ for some k such that, by construction of $\bar{\tau}_{-i}^{v*}$, $\alpha_k = \dots = \alpha_n$. Since the strategy profile $\bar{\tau}_{-i}^k$ is defined to realize the payoff $\alpha_k = \alpha_n$, we have $\alpha_{n+1} = \alpha_n$.

Moreover, that sequence can only take a finite number of values (at most $\text{card}V$). Therefore, it is stationary: there exists $n_0 \in \mathbb{N}$ such that $(\alpha_n)_{n \geq n_0}$ is constant, and there are no resets of $\bar{\tau}_{-i}^{v*}$ among the prefixes of η of length greater than n_0 .

Therefore, if we choose n_0 minimal (i.e., n_0 is the index of the last reset in η), then the play $\eta_{n_0}\eta_{n_0+1} \dots$ is compatible with the strategy profile $\bar{\tau}_{-i}^{\eta_{n_0}}$. Then, we have:

$$\mu_i(\eta) \leq \alpha_{n_0} \leq \alpha_0,$$

and:

$$\alpha_0 = \inf_{\bar{\tau}_{-i} \in \lambda\text{Rat}(w)} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_w),$$

which proves that $\bar{\tau}^{v*}$ is the subgame-worst λ -strategy profile against player i from the state w , and therefore that the game G is a game with subgame-steady negotiation.

• Construction of $\bar{\sigma}$.

Let $\mathcal{H}_0 = \text{Hist}G_{|v_0}$. Let us construct inductively $\bar{\sigma}$ by defining all the plays $\langle \bar{\sigma}_{|hv} \rangle_v$, for $hv \in \mathcal{H}_0$, keeping the hypothesis that at any step n , the set \mathcal{H}_n contains exactly the histories hv such that the play $\langle \bar{\sigma}_{|hv} \rangle_v$ has been defined, and that such a play is always λ -consistent: it will define a λ -rational strategy profile, and we will then prove it is an ε -SPE.

– First, $\langle \bar{\sigma} \rangle_{v_0} = \xi$, which satisfies the induction hypothesis. We remove then all the finite prefixes of ξ from \mathcal{H}_0 to obtain \mathcal{H}_1 . Note that the only history of length 1 has been removed.

– At the n -th step, with $n > 0$: let us choose $hv \in \mathcal{H}_n$ of minimal length, and therefore minimal for the prefix order: the strategy profile $\bar{\sigma}$ has been defined on all the strict prefixes of hv , but not on hv itself, and $v \neq \bar{\sigma}(h)$. Let then i be the player controlling the last state of h (which exists since all the histories of \mathcal{H}_n have length at least 2). Let $\bar{\tau}_{-i}^{v*}$ be a subgame-worst λ -rational strategy profile against player i from v , whose existence has been proved in the previous point, and let τ_i^{v*} be a strategy rationalizing it.

Then, we define $\langle \bar{\sigma}_{|hv} \rangle_v = \langle \bar{\tau}^{v*} \rangle_v$, and inductively, for every history $h'w$ starting from v and compatible with $\bar{\sigma}_{-i|hv}$ as it has been defined so far, we define $\langle \bar{\sigma}_{|hh'w} \rangle_v = \langle \bar{\tau}_{|h'w}^{v*} \rangle_w$. The strategy profile $\bar{\sigma}_{|hv}$ is then equal to $\bar{\tau}^{v*}$ on any history compatible with $\bar{\tau}_{-i}^{v*}$.

We remove all such histories from \mathcal{H}_n to obtain \mathcal{H}_{n+1} . All the plays we built are λ -consistent, which was our induction hypothesis.

Since each step removes from \mathcal{H}_n a history of minimal length, and since there are finitely many histories of any given length, we have $\bigcap_n \mathcal{H}_n = \emptyset$, and this process completely defines $\bar{\sigma}$.

• Such $\bar{\sigma}$ is an ε -SPE.

Let $h^{(0)}w \in \text{Hist}G_{|v_0}$, let $i \in \Pi$, let σ'_i be a deviation of σ_i . Let $\rho = h^{(0)}\langle \bar{\sigma}_{|h^{(0)}w} \rangle_w$ and let $\rho' = h^{(0)}\langle \sigma'_i|_{h^{(0)}w}, \bar{\sigma}_{-i|_{h^{(0)}w}} \rangle_w$. We prove that $\mu_i(\rho') \leq \mu_i(\rho) + \varepsilon$.

If ρ' is compatible with σ_i , then $\rho' = \rho$ and the proof is immediate. If it is not, we let huv denote the shortest prefix of ρ' such that $u \in V_i$ and $v \neq \sigma_i(hu)$. The transition uv can be considered as the first deviation of

player i , but note that hu can be both longer or shorter than $h^{(0)}$: player i may have already deviated in $h^{(0)}$.

Be that as it may, the history hu is a common prefix of the play ρ and ρ' , and if $\bar{\tau}_{-i}^{v*}$ denotes a subgame-worst λ -rational strategy profile against player i from the state v , and if τ_i^{v*} is a strategy λ -rationalizing it, then $\bar{\sigma}_{\upharpoonright huv}$ has been defined as equal to $\bar{\tau}^{v*}$ on any history compatible with $\bar{\sigma}_{-i\upharpoonright huv}$.

– If huv is a prefix of ρ : let $huh'w'$ be the longest common prefix of ρ and ρ' . Necessarily, $w' \in V_i$. Then, by definition of $\bar{\tau}_{-i}^{v*}$, we have:

$$\mu_i(\rho') \leq \inf_{\bar{\tau}_{-i} \in \lambda\text{Rat}(w')} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_{w'}) = \text{nego}(\lambda)(w'),$$

and since λ is an ε -fixed point of nego :

$$\mu_i(\rho') \leq \lambda(w') + \varepsilon.$$

On the other hand, the play $\langle \bar{\sigma}_{\upharpoonright h'w'} \rangle_{w'}$, which is a suffix of ρ , is λ -consistent, hence $\mu_i(\rho) \geq \lambda(w')$.

Therefore, $\mu_i(\rho') \leq \mu_i(\rho) + \varepsilon$.

– If huv is not a prefix of ρ : then, $\rho = h\langle \bar{\sigma}_{\upharpoonright hu} \rangle_u$. Since $u \in V_i$, we have:

$$\text{nego}(\lambda)(u) = \sup_{uv' \in E} \inf_{\bar{\tau}_{-i} \in \lambda\text{Rat}(v')} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_{v'}).$$

In particular, we have:

$$\inf_{\bar{\tau}_{-i} \in \lambda\text{Rat}(v)} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_v) \leq \text{nego}(\lambda)(u) \leq \lambda(u) + \varepsilon.$$

Then, for the same reason as above, we know that:

$$\mu_i(\rho') \leq \inf_{\bar{\tau}_{-i} \in \lambda\text{Rat}(v)} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_v).$$

Finally, since the suffix $\langle \bar{\sigma}_{\upharpoonright hu} \rangle_u$ of ρ is λ -consistent, we have $\mu_i(\rho) \geq \lambda(u) \geq \text{nego}(\lambda)(u) - \varepsilon \geq \mu_i(\rho')$.

The strategy profile $\bar{\sigma}$ is an ε -SPE. ■

APPENDIX C PROOF OF LEMMA 2

Lemma 2. *Let $G_{\upharpoonright v_0}$ be a well-initialized prefix-independent game, and let $\varepsilon \geq 0$. Let $\bar{\sigma}$ be an ε -SPE in $G_{\upharpoonright v_0}$. Then, there exists an ε -fixed point λ of the negotiation function such that for every history hv starting in v_0 , the play $\langle \bar{\sigma}_{\upharpoonright hv} \rangle_v$ is λ -consistent.*

Proof: Let us define the requirement λ by, for each $i \in \Pi$ and $v \in V_i$:

$$\lambda(v) = \inf_{hv \in \text{Hist}G_{\upharpoonright v_0}} \mu_i(\langle \bar{\sigma}_{\upharpoonright hv} \rangle_v).$$

Note that the set $\{\mu_i(\langle \bar{\sigma}_{\upharpoonright hv} \rangle_v) \mid hv \in \text{Hist}G_{\upharpoonright v_0}\}$ is never empty, since the game $G_{\upharpoonright v_0}$ is well-initialized.

Then, for every history hv starting in v_0 , the play $\langle \bar{\sigma}_{\upharpoonright hv} \rangle_v$ is λ -consistent. Let us prove that λ is an ε -fixed point of nego : let $i \in \Pi$, let $v \in V_i$, and let us assume towards contradiction (since the negotiation function is non-decreasing) that $\text{nego}(\lambda)(v) > \lambda(v) + \varepsilon$, that is to say:

$$\inf_{\bar{\tau}_{-i} \in \lambda\text{Rat}(v)} \sup_{\tau_i} \mu_i(\langle \bar{\tau} \rangle_v) > \inf_{hv \in \text{Hist}G_{\upharpoonright v_0}} \mu_i(\langle \bar{\sigma}_{\upharpoonright hv} \rangle_v) + \varepsilon.$$

Then, since all the plays generated by the strategy profile $\bar{\sigma}$ are λ -consistent, and therefore since any strategy profile of the form $\bar{\sigma}_{-i\upharpoonright hv}$ is λ -rational, we have:

$$\inf_{hv} \sup_{\tau_i} \mu_i(\langle \bar{\sigma}_{-i\upharpoonright hv}, \tau_i \rangle_v) > \inf_{hv} \mu_i(\langle \bar{\sigma}_{\upharpoonright hv} \rangle_v) + \varepsilon.$$

Therefore, there exists a history hv such that:

$$\sup_{\tau_i} \mu_i(\langle \bar{\sigma}_{-i\upharpoonright hv}, \tau_i \rangle_v) > \mu_i(\langle \bar{\sigma}_{\upharpoonright hv} \rangle_v) + \varepsilon,$$

which is impossible if the strategy profile $\bar{\sigma}$ is an ε -SPE. Therefore, there is no such v , and the requirement λ is an ε -fixed point of the negotiation function. ■

APPENDIX D PROOF OF LEMMA 3

Lemma 3. *Let G be a game, and let $\varepsilon \geq 0$. The negotiation function has a least ε -fixed point.*

Proof: The following proof is a generalization of a classical proof of Tarski's fixed point theorem.

Let Λ be the set of the ε -fixed points of the negotiation function. The set Λ is not empty, since it contains at least the requirement $v \mapsto +\infty$. Let λ^* be the requirement defined by:

$$\lambda^* : v \mapsto \inf_{\lambda \in \Lambda} \lambda(v).$$

For all ε -fixed point λ of the negotiation function, we have then for each v , $\lambda^*(v) \leq \lambda(v)$, and $\text{nego}(\lambda^*)(v) \leq \text{nego}(\lambda)(v)$ since nego is monotone; and therefore, $\text{nego}(\lambda^*)(v) \leq \lambda(v) + \varepsilon$.

As a consequence, we have:

$$\text{nego}(\lambda^*)(v) \leq \inf_{\lambda \in \Lambda} \lambda(v) + \varepsilon = \lambda^*(v) + \varepsilon.$$

The requirement λ^* is an ε -fixed point of the negotiation function, and is therefore the least ε -fixed point of the negotiation function. ■

APPENDIX E ABSTRACT NEGOTIATION GAME

Definition 28 (Abstract negotiation game). Let $G_{\upharpoonright v_0}$ be an initialized game, let $i \in \Pi$, and let λ be a requirement on G . The abstract negotiation game of $G_{\upharpoonright v_0}$ for player i with requirement λ is the two-player zero-sum initialized game:

$$\text{Abs}_{\lambda i}(G)_{\upharpoonright [v_0]} = (\{\mathcal{P}, \mathcal{C}\}, S, (S_{\mathcal{P}}, S_{\mathcal{C}}), \Delta, \nu)_{\upharpoonright [v_0]},$$

where:

- \mathcal{P} denotes the player *Prover* and \mathcal{C} the player *Challenger*;
- the states of $S_{\mathcal{C}}$ are written $[\rho]$, where ρ is a λ -consistent play in G ;
- the states of $S_{\mathcal{P}}$ are written $[huv]$, where huv is a history in G , with $w \in V_i$, or $[v]$ with $v \in V$, plus two additional states \top and \perp ;
- the set Δ contains the transitions of the form:
 - $[hv][v\rho]$, where $[hv] \in S_{\mathcal{P}}$ and $[v\rho] \in S_{\mathcal{C}}$ (Prover proposes a play);

- $[\rho][\rho_0 \dots \rho_n v]$, where $[\rho] \in S_C, n \in \mathbb{N}, \rho_n \in V_i$, and $v \neq \rho_{n+1}$ (Challenger makes player i deviate);
- $[\rho] \top$, where $[\rho] \in S_C$ (Challenger accepts the proposed play);
- $\top \top$ (the game is over);
- $[hv] \perp$ (Prover has no more play to propose);
- $\perp \perp$ (the game is over).
- ν is the outcome function defined by, for all $\rho^{(0)}, \rho^{(1)}, \dots, h^{(1)}v_1, h^{(2)}v_2, \dots, k, H$:

$$\begin{aligned} & \nu_C([v_0][\rho^{(0)}][h^{(1)}v_1][\rho^{(1)}] \dots [h^{(k)}v_k][\rho^{(k)}] \top^\omega) \\ &= \mu_i(h^{(1)} \dots h^{(k)} \rho^{(k)}), \\ & \nu_C([v_0][\rho^{(0)}][h^{(1)}v_1][\rho^{(1)}] \dots [h^{(n)}v_n][\rho^{(n)}] \dots) \\ &= \mu_i(h^{(1)}h^{(2)} \dots), \\ & \nu_C(H \perp^\omega) = +\infty, \end{aligned}$$

and by $\nu_P = -\nu_C$.

Remark. If the game G is Borelian, then so is the game $\text{Abs}_{\lambda i}(G)$.

Proposition 3. *Let $G_{\uparrow v_0}$ be an initialized Borelian game, let λ be a requirement on G and let $i \in \Pi$. Then, choosing Challenger as distinguished player, the value of the game $\text{Abs}_{\lambda i}(G)_{[v_0]}$ is equal to the quantity:*

$$\inf_{\bar{\sigma}_{-i} \in \lambda \text{Rat}(v_0)} \sup_{\sigma_i} \mu_i(\langle \bar{\sigma} \rangle_{v_0}).$$

Proof: Let $\alpha \in \mathbb{R}$, and let us prove that the following statements are equivalent:

- 1) there exists a strategy τ_P such that for every strategy τ_C , $\nu_C(\langle \bar{\tau} \rangle_{[v_0]}) < \alpha$;
 - 2) there exists a λ -rational strategy profile $\bar{\sigma}_{-i}$ in the game $G_{\uparrow v_0}$ such that for every strategy σ_i , we have $\mu_i(\langle \bar{\sigma} \rangle_{v_0}) < \alpha$.
- (1) implies (2).

Let τ_P be such that for every strategy τ_C , $\nu_C(\langle \bar{\tau} \rangle_{[v_0]}) < \alpha$. In what follows, any history h compatible with an already defined strategy profile $\bar{\sigma}_{-i}$ in $G_{\uparrow v_0}$ will be decomposed in:

$$h = v_0 h^{(0)} v_1 h^{(1)} \dots h^{(n-1)} v_n h^{(n)},$$

so that there exist plays $\rho^{(0)}, \dots, \rho^{(n-1)}, \eta$ and a history:

$$[v_0][\rho^{(0)}][v_1 h^{(1)} v_2] \dots [v_{n-1} h^{(n-1)} v_n][v_n h^{(n)} \eta]$$

in the game $\text{Abs}_{\lambda i}(G)$ compatible with τ_P : the existence and the unicity of that decomposition can be proved by induction. Intuitively, the history h is cut in histories which are prefixes of plays that can be proposed by Prover.

Then, let us define inductively the strategy profile $\bar{\sigma}_{-i}$ by, for every h such that $\bar{\sigma}_{-i}$ has been defined on the prefixes of h , and such that the last state of h is not controlled by player i , $\bar{\sigma}_{-i}(h) = \eta_0$ with η defined from h as higher. Let us prove that $\bar{\sigma}_{-i}$ is the desired strategy profile.

- The strategy profile $\bar{\sigma}_{-i}$ is λ -rational.

Let us define σ_i so that for every history $h v$ compatible with $\bar{\sigma}_{-i}$, the play $\langle \bar{\sigma}_{\uparrow h v} \rangle_v$ is λ -consistent.

For any history:

$$h = v_0 h^{(0)} v_1 h^{(1)} \dots h^{(n-1)} v_n h^{(n)}$$

compatible with $\bar{\sigma}_{-i}$ and ending in V_i , let $\sigma_i(h) = \eta_0$ with η corresponding to the decomposition of h , so that by induction:

$$\langle \bar{\sigma}_{\uparrow v_0 h^{(0)} v_1 h^{(1)} \dots h^{(n-1)} v_n} \rangle_{v_n} = v_n h^{(n)} \eta.$$

Let now $h v$ be a history in $G_{\uparrow v_0}$, and let us show that the play $\langle \bar{\sigma}_{\uparrow h v} \rangle_v$ is λ -consistent. If we decompose:

$$h v = v_0 h^{(0)} v_1 h^{(1)} \dots h^{(n-1)} v_n h^{(n)}$$

with the same definition of η (note that the vertex v is now included in the decomposition), then $\langle \bar{\sigma}_{\uparrow h v} \rangle_v = v \eta$, and by definition of the abstract negotiation game, $v_n h^{(n)} \eta$ is a λ -consistent play, and therefore so is $v \eta$.

- The strategy profile $\bar{\sigma}_{-i}$ keeps player i 's payoff under the value α .

Let σ_i be a strategy for player i , and let $\rho = \langle \bar{\sigma} \rangle_{v_0}$. We want to prove that $\mu_i(\rho) < \alpha$.

Let us define two finite or infinite sequences $(\rho^{(k)})_{k \in K}$ and $(h^{(k)} v_k)_{k \in K}$, where $K = \{1, \dots, n\}$ or $K = \mathbb{N} \setminus \{0\}$, by setting $h^{(0)}$ equal to the empty history, and for every $k \in K$:

$$[\rho^{(k)}] = \tau_P([v_0][\rho^{(0)}] \dots [\rho^{(k-1)}][h^{(k)} v_k])$$

and so that for every k , the history $h^{(k)} v_k$ is the shortest prefix of ρ that is not a prefix of $h^{(1)} \dots h^{(k-1)} \rho^{(k-1)}$ (or equivalently, the history $h^{(k)}$ is the longest common prefix of ρ and $h^{(1)} \dots h^{(k-1)} \rho^{(k-1)}$).

Then, the length of the longest common prefix of $h^{(1)} \dots h^{(k-1)} \rho^{(k-1)}$ and ρ increases with k , and the set K is finite if and only if there exists n such that $h^{(1)} \dots h^{(n-1)} \rho^{(n-1)} = \rho$.

In the infinite case, let:

$$\chi = [v_0][\rho^{(0)}][h^{(1)} v_1] \dots [\rho^{(k)}][h^{(k)} v_k] \dots$$

The play χ is compatible with τ_P , hence $\nu_C(\chi) < \alpha$, that is to say:

$$\mu_i(h^{(1)} h^{(2)} \dots) < \alpha,$$

ie. $\mu_i(\rho) < \alpha$.

In the finite case, let:

$$\chi = [v_0][\rho^{(0)}][h^{(1)} v_1] \dots [\rho^{(n)}] \top^\omega.$$

For the same reason, $\nu_C(\chi) < \alpha$, that is to say $\mu_i(h^{(1)} \dots h^{(n)} \rho^{(n)}) = \mu_i(\rho) < \alpha$.

- (2) implies (1).

Let $\bar{\sigma}_{-i}$ be a λ -rational strategy profile keeping player i 's payoff below α .

Then, let σ_i be a strategy λ -rationalizing $\bar{\sigma}_{-i}$. Let us define a strategy $\tau_{\mathcal{P}}$ for Prover in the abstract negotiation game.

Let $H = [v_0] [\rho^{(0)}] [h^{(1)}v_1] [\rho^{(1)}] \dots [h^{(n)}v_n]$ be a history in the abstract game, ending in $S_{\mathcal{P}}$. Then, we define:

$$\tau_{\mathcal{P}}(H) = [\langle \bar{\sigma}_{\upharpoonright h^{(1)} \dots h^{(n)} v_n} \rangle_{v_n}].$$

If H is a history ending in \top , then $\tau_{\mathcal{P}}(H) = \top$, and in the same way if H ends in \perp , then $\tau_{\mathcal{P}}(H) = \perp$.

Let us show that $\tau_{\mathcal{P}}$ is the strategy we were looking for. Let χ be a play compatible with $\tau_{\mathcal{P}}$, and let us note that the state \perp does not appear in χ . Then, the play χ can only have two forms:

– If $\chi = [v_0] [\rho^{(0)}] [h^{(1)}v_1] \dots [\rho^{(n)}] \top^\omega$, then we have:

$$\rho^{(n)} = \langle \bar{\sigma}_{\upharpoonright h^{(1)} \dots h^{(n)} v_n} \rangle_{v_n},$$

and the history $h^{(1)} \dots h^{(n)}v_n$ in the game $G_{\upharpoonright v_0}$ is compatible with $\bar{\sigma}_{-i}$. By hypothesis, we have:

$$\mu_i(h^{(1)} \dots h^{(n)}\rho^{(n)}) < \alpha,$$

hence $\nu_{\mathcal{C}}(\chi) < \alpha$.

– If $\chi = [v_0] [\rho^{(0)}] \dots [h^{(n)}v_n] [\rho^{(n)}] \dots$, then the play $\rho = h^{(1)}h^{(2)} \dots$ is compatible with $\bar{\sigma}_{-i}$, and by hypothesis $\mu_i(\rho) < \alpha$, hence $\nu_{\mathcal{C}}(\chi) < \alpha$. ■

Remark. Prover has a strategy to avoid \perp if and only if λ is satisfiable.

APPENDIX F PROOF OF THEOREM 3

Theorem 3. Let $G_{\upharpoonright v_0}$ be an initialized prefix-independent Borelian game. Let λ be a requirement and i a player.

Then, we have:

$$\text{val}(\text{Conc}_{\lambda i}(G)_{\upharpoonright s_0}) = \inf_{\bar{\sigma}_{-i} \in \lambda \text{Rat}(v_0)} \sup_{\sigma'_i} \mu_i(\langle \bar{\sigma} \rangle_{v_0}).$$

Proof: First, let us define:

$$A = \left\{ \sup_{\sigma'_i} \mu_i(\langle \bar{\sigma} \rangle_{v_0}) \mid \bar{\sigma}_{-i} \in \lambda \text{Rat}(v_0) \right\}$$

and:

$$B = \left\{ \sup_{\tau_{\mathcal{C}}} \nu_{\mathcal{C}}(\langle \bar{\tau} \rangle_{s_0}) \mid \tau_{\mathcal{P}} \right\} \setminus \{+\infty\}.$$

We prove our point if we prove that $A = B$.

- $B \subseteq A$.

Let $\tau_{\mathcal{P}}$ be a strategy such that:

$$\sup_{\tau_{\mathcal{C}}} \nu_{\mathcal{C}}(\langle \bar{\tau} \rangle_{s_0}) < +\infty,$$

and let $\bar{\sigma}$ be the strategy profile defined by:

$$\bar{\sigma}(\dot{H}) = w$$

for every history H compatible with $\tau_{\mathcal{P}}$ (by induction, the localized projection is injective on the histories compatible with $\tau_{\mathcal{P}}$) with $\tau_{\mathcal{P}}(H) = (vw, \cdot)$, and arbitrarily defined on any other histories.

- The strategy profile $\bar{\sigma}_{-i}$ is λ -rational, rationalized by the strategy σ_i . Indeed, let us assume it is not. Then, there exists a history $h = h_0 \dots h_n$ in $G_{\upharpoonright v_0}$ compatible with $\bar{\sigma}_{-i}$ such that the play $\dot{\rho} = \langle \bar{\sigma}_{\upharpoonright h} \rangle_{h_n}$ is not λ -consistent. Then, let:

$$Hs = (h_0, M_0) (h_0\bar{\sigma}(h_0), M_0) \dots (h_n, M_n)$$

be the only history in $\text{Conc}_{\lambda i}(G)_{\upharpoonright s_0}$ compatible with $\tau_{\mathcal{P}}$ such that $\dot{H} = h$.

Let $\tau_{\mathcal{C}}$ be a strategy constructing the history h , defined by:

$$\tau_{\mathcal{C}}(H_0 \dots H_{2k-1}) = H_{2k}$$

for every k , and:

$$\tau_{\mathcal{C}}(H'(vw, M)) = (w, M \cup \{w\})$$

for any other history $H'(vw, M)$.

Then, the play $\eta = \langle \bar{\tau} \rangle_{s_0}$ contains finitely many deviations (Challenger stops the deviations after having drawn the history h), and the play $\dot{\eta} = h_0 \dots h_{n-1}\dot{\rho}$ is not λ -consistent, i.e. there exists a dimension $j \in \Pi$ such that:

$$\mu_j(\dot{\eta}) - \max_{v \in M_n \cap V_j} \lambda(v) < 0$$

i.e.:

$$\hat{\mu}_j(\eta) < 0$$

and therefore $\nu_{\mathcal{C}}(\rho) = \nu_{\mathcal{C}}(\eta) = +\infty$, which is false by hypothesis.

- Now, let us prove the equality:

$$\sup_{\sigma'_i} \mu_i(\langle \bar{\sigma}_{-i}, \sigma'_i \rangle_{v_0}) = \sup_{\tau_{\mathcal{C}}} \nu_{\mathcal{C}}(\langle \bar{\tau} \rangle_{s_0}).$$

For that purpose, let us prove the equality of sets:

$$\{\mu_i(\langle \bar{\sigma}_{-i}, \sigma'_i \rangle_{v_0}) \mid \sigma'_i\} = \{\nu_{\mathcal{C}}(\langle \bar{\tau} \rangle_{s_0}) \mid \tau_{\mathcal{C}}\}.$$

- * Let $\tau_{\mathcal{C}}$ be a strategy for Challenger, and let $\rho = \langle \bar{\tau} \rangle_{s_0}$. Since $\nu_{\mathcal{C}}(\rho) \neq +\infty$ by hypothesis, we have $\nu_{\mathcal{C}}(\rho) = \hat{\mu}_*(\rho) = \mu_i(\dot{\rho})$, which is an element of the left-hand set.
- * Conversely, if σ'_i is a strategy for player i and if $\eta = \langle \bar{\sigma}_{-i}, \sigma'_i \rangle_{v_0}$, let $\tau_{\mathcal{C}}$ be a strategy such that for every k :

$$\tau_{\mathcal{C}}((\eta_0, \cdot)(\eta_0 \cdot, \cdot) \dots (\eta_k \cdot, \cdot)) = (\eta_{k+1}, \cdot),$$

i.e. a strategy forcing η .

Then, since $\nu_{\mathcal{C}}(\rho) \neq +\infty$ by hypothesis on $\tau_{\mathcal{P}}$, we have $\mu_i(\eta) = \nu_{\mathcal{C}}(\rho)$, which is an element of the right-hand set.

- $A \subseteq B$.

Let $\bar{\sigma}_{-i}$ be a λ -rational strategy profile from v_0 , rationalized by the strategy σ_i ; let us define a strategy $\tau_{\mathcal{P}}$ by, for every history H and for every $v \in V$:

$$\tau_{\mathcal{P}}(H(v, \cdot)) = (v\bar{\sigma}(\dot{H}v), \cdot).$$

Let us prove the equality:

$$\sup_{\sigma'_i} \mu_i(\langle \bar{\sigma}_{-i}, \sigma'_i \rangle_{v_0}) = \sup_{\tau_C} \nu_C(\langle \bar{\tau} \rangle_{s_0}).$$

For that purpose, let us prove the equality of sets:

$$\{\mu_i(\langle \bar{\sigma}_{-i}, \sigma'_i \rangle_{v_0}) \mid \sigma'_i\} = \{\nu_C(\langle \bar{\tau} \rangle_{s_0}) \mid \tau_C\}.$$

- Let τ_C be a strategy for Challenger, and let $\rho = \langle \bar{\tau} \rangle_{s_0}$. If $\nu_C(\rho) = +\infty$, then ρ is compatible with $\bar{\sigma}$ and not λ -consistent after finitely many steps, which is impossible. Therefore, $\nu_C(\langle \bar{\tau} \rangle_{s_0}) \neq +\infty$, and as a consequence we have $\nu_C(\rho) = \hat{\mu}_*(\rho) = \mu_i(\rho)$, which is an element of the left-hand set.
- Conversely, if σ'_i is a strategy for player i and if $\eta = \langle \bar{\sigma}_{-i}, \sigma'_i \rangle_{v_0}$, let τ_C be a strategy such that for all k :

$$\tau_C((\eta_0, \cdot)(\eta_0, \cdot) \dots (\eta_k, \cdot)) = (\eta_{k+1}, \cdot),$$

i.e. a strategy forcing η .

Then, either $\nu_C(\rho) = +\infty$, and therefore η is not λ -consistent, and is compatible with $\bar{\sigma}$ after finitely many steps, which is impossible.

Or, $\mu_i(\eta) = \nu_C(\rho)$, which is an element of the right-hand set.

APPENDIX G PROOF OF LEMMA 4

Theorem 4. *Let G be a mean-payoff-inf game, let i be a player, let λ be a requirement and let $\text{Conc}_{\lambda i}(G)$ be the corresponding concrete negotiation game. There exists a memoryless strategy τ_C such that for each state s :*

$$\inf_{\tau_P} \nu_C(\langle \bar{\tau} \rangle_s) = \text{val}(\text{Conc}_{\lambda i}(G)_{\upharpoonright s}),$$

i.e. that is optimal for Challenger from all state.

Proof: The structure of that proof is inspired from the proof of lemma 14 in [19].

Let $\alpha \in \mathbb{R}$, and let Φ be the set of the plays ρ in $\text{Conc}_{\lambda i}(G)$ such that:

- $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (-\hat{\pi}_*(\rho_k \rho_{k+1})) \geq -\alpha$;
- and either:
 - ρ contains infinitely many deviations;
 - or for each $j \in \Pi$, $\hat{\mu}_j(\rho) \geq 0$.

Note that the set of the plays ρ such that $\mu_i(\rho) \leq \alpha$ could be defined almost the same way, but with a limit superior instead of the limit inferior.

By [12], if Challenger can falsify the objective Φ , he can falsify it with a memoryless strategy, if Φ is *prefix-independent* and *convex*.

Convex objectives are defined as follows: the objective Φ is convex if for all $\rho, \eta \in \Phi$ and for any decomposition:

$$\rho_0 \dots \rho_{k_1} \dots \rho_{k_2} \dots$$

and:

$$\eta_0 \dots \eta_{\ell_1} \dots \eta_{\ell_2} \dots$$

such that:

$$\chi = \rho_0 \dots \rho_{k_1} \eta_0 \dots \eta_{\ell_1} \rho_{k_1+1} \dots \rho_{k_2} \eta_{\ell_1+1} \dots$$

is a play, we have $\chi \in \Phi$. Let then be such two plays and decomposition, and let us prove that $\chi \in \Phi$.

Let us write $\Phi = \Psi \cap (X \cup \Xi)$, where:

- Ψ is the set of the plays ρ such that:

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (-\hat{\pi}_*(\rho_k \rho_{k+1})) \geq -\alpha$$

- X is the set of the plays containing infinitely many deviations;
- Ξ is the set of the plays ρ such that for each $j \in \Pi$, $\hat{\mu}_j(\rho) \geq 0$.

As shown in [19], a mean-payoff-inf objective is convex: therefore, we can already say that $\chi \in \Psi$. Let us now prove that $\chi \in X \cup \Xi$.

- If $\rho \in X$ or $\eta \in X$.

Then, χ contains the deviations of ρ and η , hence $\chi \in X$.

- If $\rho, \eta \in \Xi$.

Then, since mean-payoff-inf objectives are convex, then $\chi \in \Xi$.

In both cases, $\chi \in X \cup \Xi$, so $\chi \in \Phi$: the objective Φ is convex.

Therefore, there exists a memoryless strategy τ_C such that for every strategy τ_P , for each state s from which Challenger has some strategy to falsify the objective Φ , we have $\langle \bar{\tau} \rangle_s \notin \Phi$.

Let s be a state from which Challenger can enforce an outcome ν_C greater than α . Then, since the limit inferior of a sequence is always lesser than or equal to its limit superior, Challenger can, from s , falsify the objective Φ . Therefore, by definition of τ_C , for every strategy τ_P , we have $\langle \bar{\tau} \rangle_s \notin \Phi$. Let us prove that $\nu_C(\langle \bar{\tau} \rangle_s) > \alpha$.

In other words, let us prove that for every infinite path ρ from s in the graph $\text{Conc}_{\lambda i}(G)[\tau_C]$, we have $\nu_C(\rho) > \alpha$. Since $\rho \notin \Phi$, we have either $\rho \notin X \cup \Xi$ or $\rho \notin \Psi$. In the first case, we have $\nu_C(\rho) = +\infty$, which ends the proof. In the second case, we have:

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{\pi}_*(\rho_k \rho_{k+1}) > \alpha.$$

We want to prove that $\nu_C(\rho) > \alpha$, that is, since we assume $\rho \in X \cup \Xi$:

$$\hat{\mu}_*(\rho) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{\pi}_*(\rho_k \rho_{k+1}) > \alpha.$$

Here, the play ρ is an infinite path in the graph $\text{Conc}_{\lambda i}(G)[\tau_C]$: by the description of the possible outcomes in a mean-payoff game given in [9], the mean-payoff-inf $\hat{\mu}_*(\rho)$ is then above or equal to the mean-payoff-inf $\hat{\mu}_*$ we get by looping on all simple cycle c of that graph accessible from the state s : intuitively, a play can be seen as a combination of those cycles. That is to say:

$$\hat{\mu}_*(\rho) \geq \min_{c \in \text{SC}(\text{Conc}_{\lambda i}(G)[\tau_C]) \text{ accessible from } s} \hat{\mu}_*(c^\omega).$$

For all such cycle, since c^ω is a play compatible with τ_C , we have:

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{\pi}_*(c_k c_{k+1}) > \alpha$$

where the indices are taken in $\mathbb{Z}/|c|\mathbb{Z}$, i.e.:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{\pi}_*(c_k c_{k+1}) > \alpha,$$

and therefore:

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{\pi}_*(c_k c_{k+1}) > \alpha,$$

that is to say:

$$\hat{\mu}_*(c^\omega) > \alpha,$$

hence $\hat{\mu}_*(\rho) > \alpha$. \blacksquare

APPENDIX H PROOF OF LEMMA 5

Lemma 5. *Let $G_{\uparrow v_0}$ be an initialized mean-payoff-inf game, and let $\text{Conc}_{\lambda i}(G)_{\uparrow s_0}$ be its concrete negotiation game for some λ and some i .*

Then, the value of the game $\text{Conc}_{\lambda i}(G)_{\uparrow s_0}$ is given by the formula:

$$\max_{\tau_C \in \text{ML}_C(\text{Conc}_{\lambda i}(G))} \min_{\substack{K \in \text{SConn}(\text{Conc}_{\lambda i}(G)[\tau_C]) \\ \text{accessible from } s_0}} \text{opt}(K),$$

where $\text{opt}(K)$ is the minimal value $\nu_C(\rho)$ for ρ among the infinite paths in K .

- If K contains a deviation, then Prover can simply choose the simple cycle of K that minimizes player i 's payoff:

$$\text{opt}(K) = \min_{c \in \text{SC}(K)} \hat{\mu}_*(c^\omega).$$

- If K does not contain a deviation, then Prover must choose a combination of the simple cycles of K that minimizes player i 's payoff while keeping the non-main dimensions above 0:

$$\text{opt}(K) = \min^*_{c \in \text{SC}(K)} \text{Conv} \hat{\mu}(c^\omega).$$

Proof: By Lemma 4, there exists a memoryless strategy τ_C which is optimal for Challenger among all his possible strategies.

It follows from Theorem 3 that the highest value player i can get against a hostile λ -rational environment is the minimal payoff of Challenger in a path in the graph $\text{Conc}_{\lambda i}(G)[\tau_C]$. For any such path ρ , there exists a strongly connected component K of $\text{Conc}_{\lambda i}(G)[\tau_C]$ accessible from s_0 such that after a finite number of steps, ρ is a play in K . The least payoff of Challenger in such a path, for a given K , is $\text{opt}(K)$; let us prove that it is given by the desired formula.

There are, then, two cases to distinguish:

- If there is at least a deviation in K .

Then, a play in K can contain infinitely many deviations. Therefore, the outcomes $\nu_C(\rho)$ of plays in K are exactly

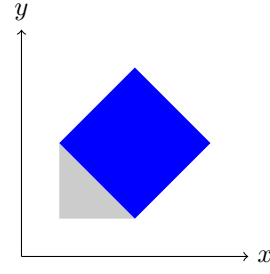


Fig. 14. An example for the operator \cdot^\perp

the mean-payoff-infs $\hat{\mu}_*(\rho)$ of plays in K , and possibly $+\infty$; and in particular, the lowest outcome Prover can get in K is the quantity:

$$\min_{c \in \text{SC}(K)} \hat{\mu}_*(c^\omega),$$

the least value of a simple cycle in K .

- If there is no deviation in K .

Let us first introduce a notation: for any finite set D and any set $X \subseteq \mathbb{R}^D$, X^\perp denotes the set:

$$X^\perp = \left\{ \left(\min_{\bar{y} \in Y} y_d \right)_{d \in D} \mid Y \subseteq X \text{ finite} \right\}.$$

For example, in \mathbb{R}^2 , if X is the blue area below, then X^\perp is the union of the blue area and the gray area in Figure 14.

Let us already note that for all $X \in \mathbb{R}^{\Pi \cup \{*\}}$,

$$\begin{aligned} & \min^* X^\perp \\ &= \min \left\{ x_* \mid \begin{array}{l} \bar{x} \in X^\perp, \\ \forall j \in \Pi, x_j \geq 0 \end{array} \right\} \\ &= \min \left\{ \min_{\bar{y} \in Y} y_* \mid \begin{array}{l} Y \subseteq X \text{ finite}, \\ \forall \bar{y} \in Y, \forall j \in \Pi, y_j \geq 0 \end{array} \right\} \\ &= \min \left\{ y_* \mid \begin{array}{l} \bar{y} \in X, \\ \forall \bar{y} \in Y, \forall j \in \Pi, y_j \geq 0 \end{array} \right\} \\ &= \min^* X. \end{aligned}$$

Then, it has been proved in [9] that the set of possible values of $\hat{\mu}(\rho)$ for all plays ρ in K is exactly the set:

$$X = \left(\text{Conv}_{c \in \text{SC}(K)} \hat{\mu}(c^\omega) \right)^\perp.$$

Since all the plays in K contain finitely many deviations (actually none), for every $\bar{x} = \hat{\mu}(\rho) \in X$, we have $\nu_C(\rho) = +\infty$ if and only if there exists $j \in \Pi$ such that $x_j < 0$. Then, the lowest outcome Prover can get in K is:

$$\min \{ x_* \mid \bar{x} \in X, \forall j \in \Pi, x_j \geq 0 \},$$

that is to say:

$$\min^* \left(\text{Conv}_{c \in \text{SC}(K)} \hat{\mu}(c^\omega) \right)^\perp,$$

i.e. $\min^*_{c \in \text{SC}(K)} \text{Conv} \hat{\mu}(c^\omega)$.

Theorem 3 enables to conclude to the desired formula. \blacksquare

APPENDIX I
PROOF OF PROPOSITION 2

Proposition 2. *In mean-payoff-inf games, the negotiation function is Scott-continuous.*

Proof: Let $(\lambda_n)_n$ be a non-decreasing sequence of requirements on a mean-payoff-inf game G , and let $\lambda = \sup_n \lambda_n$. We want to prove that $\text{nego}(\lambda) = \sup_n \text{nego}(\lambda_n)$.

Since the negotiation function is monotone, we already have $\text{nego}(\lambda) \geq \sup_n \text{nego}(\lambda_n)$. Let us prove that $\text{nego}(\lambda) \leq \sup_n \text{nego}(\lambda_n)$.

Let $\delta > 0$: we want to find n such that $\text{nego}(\lambda_n)(v) \geq \text{nego}(\lambda)(v) - \delta$ for each $v \in V$.

Let:

$$\text{Conc}_{\lambda_i}(G) \upharpoonright_{s_0} = (\{\mathcal{P}, \mathcal{C}\}, S, (S_{\mathcal{P}}, S_{\mathcal{C}}), \Delta, \nu) \upharpoonright_{s_0}$$

be the concrete negotiation game of G for λ and player i controlling v , and let:

$$\text{Conc}_{\lambda_n i}(G) \upharpoonright_{s_0} = (\{\mathcal{P}, \mathcal{C}\}, S, (S_{\mathcal{P}}, S_{\mathcal{C}}), \Delta, \nu') \upharpoonright_{s_0}$$

be the concrete negotiation game of G for some requirement λ_n in v . Let us note that both have the same underlying graph, and that the only difference are the weight functions $\hat{\pi}$ and $\hat{\pi}'$, on the non-main dimensions.

By Lemma 5, we have:

$$\text{nego}(\lambda)(v) =$$

$$\max_{\tau_{\mathcal{C}} \in \text{ML}_{\mathcal{C}}(\text{Conc}_{\lambda_i}(G) \upharpoonright_{s_0})} \min_{\substack{K \in \text{SConn}(\text{Conc}_{\lambda_i}(G) \upharpoonright_{\tau_{\mathcal{C}}}) \\ \text{accessible from } s_0}} \text{opt}(K)$$

with:

$$\text{opt}(K) = \begin{cases} \text{if } K \text{ contains a deviation :} \\ \min_{c \in \text{SC}(K)} \hat{\mu}_{\star}(c^{\omega}) \\ \text{otherwise :} \\ \min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}(c^{\omega}), \end{cases}$$

and identically:

$$\text{nego}(\lambda_n)(v) =$$

$$\max_{\tau_{\mathcal{C}} \in \text{ML}_{\mathcal{C}}(\text{Conc}_{\lambda_n i}(G) \upharpoonright_{s_0})} \min_{\substack{K \in \text{SConn}(\text{Conc}_{\lambda_i}(G) \upharpoonright_{\tau_{\mathcal{C}}}) \\ \text{accessible from } s_0}} \text{opt}'(K)$$

with:

$$\text{opt}'(K) = \begin{cases} \text{if } K \text{ contains a deviation :} \\ \min_{c \in \text{SC}(K)} \hat{\mu}_{\star}(c^{\omega}) \\ \text{otherwise :} \\ \min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}'(c^{\omega}). \end{cases}$$

Let $\tau_{\mathcal{C}}$ be a memoryless strategy for Challenger in the game $\text{Conc}_{\lambda_i}(G)_{s_0}$; it can also be considered as a memoryless strategy in the game $\text{Conc}_{\lambda_n i}(G)_{s_0}$.

Let us now define:

$$\gamma_n = \sup_{v \in V} (\lambda(v) - \lambda_n(v)).$$

Then, the sequence $(\gamma_n)_n$ is non-increasing and converges to 0. Moreover, for each transition $st \in \Delta$, we have:

$$\hat{\pi}'_j(st) \in [\hat{\pi}_j(st) - \gamma_n, \hat{\pi}_j(st)].$$

Let:

$$\Gamma_n = \left\{ \bar{x} \in \mathbb{R}^{\Pi \cup \{\star\}} \mid x_{\star} = 0 \text{ and } \forall j \in \Pi, x_j \in [0, \gamma_n] \right\}.$$

Then, let K be a strongly connected component of the graph $\text{Conc}_{\lambda_i}(G)[\tau_{\mathcal{C}}]$, without deviation, accessible from s_0 ; we have:

$$\text{Conv}_{c \in \text{SC}(K)} \hat{\mu}'(c^{\omega}) \subseteq \text{Conv}_{c \in \text{SC}(K)} \hat{\mu}(c^{\omega}) + \Gamma_n.$$

$$\text{Let } R = \left\{ \bar{x} \in \mathbb{R}^{\Pi \cup \{\star\}} \mid \forall j \in \Pi, x_j \geq 0 \right\}.$$

- If $\text{Conv}_{c \in \text{SC}(K)} \hat{\mu}(c^{\omega}) \cap R = \emptyset$, since $\text{Conv}_{c \in \text{SC}(K)} \hat{\mu}(c^{\omega})$ and R are closed sets, if γ_n is small enough, we have $\text{Conv}_{c \in \text{SC}(K)} \hat{\mu}'(c^{\omega}) \cap R = \emptyset$. Therefore, if:

$$\min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}(c^{\omega}) = +\infty,$$

then, for n great enough:

$$\min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}'(c^{\omega}) = +\infty.$$

- Otherwise, we have:

$$\min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}'(c^{\omega}) \geq$$

$$\min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}(c^{\omega}) - \gamma_n \max_{\substack{c \in \text{SC}(K) \\ d \in \text{SC}(K)}} \sum_{\substack{j \in \Pi, \\ \hat{\mu}_j(c^{\omega}) > \hat{\mu}_j(d^{\omega})}} \frac{\hat{\mu}_{\star}(c^{\omega}) - \hat{\mu}_{\star}(d^{\omega})}{\hat{\mu}_j(c^{\omega}) - \hat{\mu}_j(d^{\omega})}$$

and if γ_n is small enough, we have:

$$\min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}'(c^{\omega}) \geq \min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}(c^{\omega}) - \delta.$$

In both cases, we find that there exists γ_n small enough, i.e. n great enough, to ensure:

$$\min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}'(c^{\omega}) \geq \min_{c \in \text{SC}(K)}^{\star} \text{Conv } \hat{\mu}(c^{\omega}) - \delta.$$

We can find such n for each strongly connected component K without deviation, and there exists a finite number of such components. Moreover, when K is a strongly connected component with a deviation, the quantity:

$$\min_{c \in \text{SC}(K)} \hat{\mu}_{\star}(c^{\omega})$$

is the same in $\text{Conc}_{\lambda_i}(G)$ and in $\text{Conc}_{\lambda_n i}(G)$. Therefore, there exists $n \in \mathbb{N}$ such that:

$$\begin{aligned} & \min_{\substack{K \in \text{SConn}(\text{Conc}_{\lambda_n i}(G) \upharpoonright_{\tau_{\mathcal{C}}}) \\ \text{accessible from } s_0}} \text{opt}(K) \\ & \geq \min_{\substack{K \in \text{SConn}(\text{Conc}_{\lambda_i}(G) \upharpoonright_{\tau_{\mathcal{C}}}) \\ \text{accessible from } s_0}} \text{opt}(K) - \delta. \end{aligned}$$

We find such n for every memoryless strategy τ_C , and there exists a finite number of such strategies. Therefore, there exists $n \in \mathbb{N}$ such that:

$$\text{nego}(\lambda_n)(v) \geq \text{nego}(\lambda)(v) - \delta.$$

Finally, since there are finitely many states $v \in V$, we can conclude to the existence of $n \in \mathbb{N}$ such that for each $v \in V$, we have:

$$\text{nego}(\lambda_n)(v) \geq \text{nego}(\lambda)(v) - \delta.$$

The negotiation function is Scott-continuous. \blacksquare

APPENDIX J PROOF OF THEOREM 4

Theorem 4. *Let $G_{\uparrow v_0}$ be an initialized mean-payoff-inf game. Let us assimilate any requirement λ on G with finite values to the tuple $\lambda = (\lambda(v))_{v \in V}$, element of the vector space of finite dimension \mathbb{R}^V .*

Then, for each player i and each vertex $v_0 \in V_i$, the quantity $\text{nego}(\lambda)(v_0)$ is a piecewise linear function of λ , which can be effectively expressed and whose fixed points are computable.

Proof: By Lemma 5, we have the formula:

$$\text{nego}(\lambda)(v_0) = \max_{\tau_C \in \text{ML}_C(\text{Conc}_{\lambda_i}(G))} \text{opt}(K)$$

Let τ_C be a memoryless strategy realizing the maximum above, and let K be a strongly connected component realizing the minimum above. Let us prove that the quantity:

$$\text{opt}(K) = \begin{cases} \text{if } K \text{ contains a dev. :} & \min_{c \in \text{SC}(K)} \hat{\mu}_*(c^\omega) \\ \text{otherwise :} & \min^*_{c \in \text{SC}(K)} \text{Conv} \hat{\mu}(c^\omega) \end{cases}$$

is the desired piecewise linear function of λ .

When K contains a deviation, the quantity:

$$\min_{c \in \text{SC}(K)} \hat{\mu}_*(c^\omega)$$

is independent of λ , and the result is then immediate. Let us assume that K does not contain any deviation, and as a consequence let us prove that the quantity:

$$f(\lambda) = \min^*_{c \in \text{SC}(K)} \text{Conv} \hat{\mu}(c^\omega)$$

is an piecewise linear function of λ .

Let M be the common memory of the states of K (since K does not contain deviations). We know that for each $j \in \Pi$ and for every cycle $c \in \text{SC}(K)$, we have:

$$\hat{\mu}_j(c^\omega) = \mu_j(c^\omega) - \max_{v \in V_j \cap M} \lambda(v).$$

Let $C = \{c \mid c \in \text{SC}(K)\}$. Since there is no deviation in K , any cycle in C is a simple cycle of G . Then, the quantity $f(\lambda)$ is the minimal x_i for \bar{x} in the set:

$$X = \text{Conv}_{c \in C} \mu(c^\omega) \cap \bigcap_{v \in M} \{\bar{x} \mid x_j \geq \lambda(v) \text{ with } v \in V_j\}.$$

The set X is a polyhedron: therefore, there exists a vertex \bar{x} of that polyhedron which minimizes x_i for $\bar{x} \in X$. That vertex is the intersection between a face of the greater polyhedron

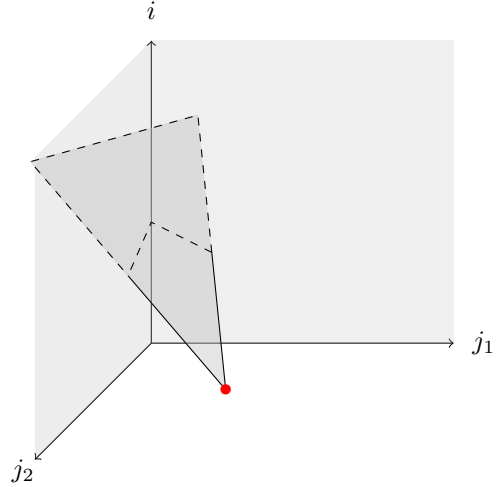


Fig. 15. The intersection between a 0-dimensional face and zero hyperplane

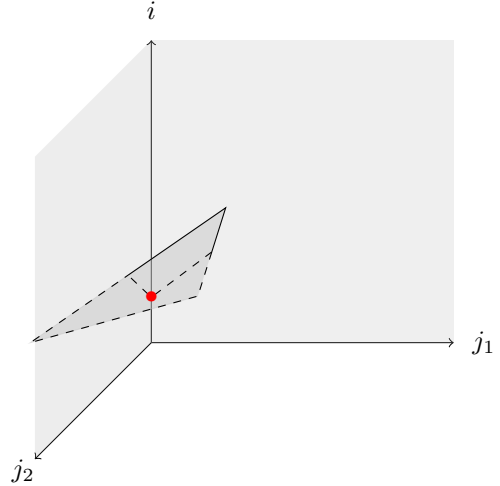


Fig. 16. The intersection between a 2-dimensional face and two hyperplanes

$P = \text{Conv}_{c \in C} \mu(c^\omega)$, and some of the hyperplanes H_v (possibly zero), defined as the hyperplanes of equation $x_j = \lambda(v)$ for $j \in \Pi$ controlling v , such that $\lambda(v) = \max_{w \in M \cap V_j} \lambda(w)$.

Example 10. With three cycles and two players against player i , each controlling one vertex v such that $\lambda(v) = 0$, the vertex \bar{x} is the red point in Figure 15 and Figure 16.

The set of vertices of the polyhedron X is included in the finite set:

$$Y = \left\{ \bar{y} \in \mathbb{R}^{\Pi \cup \{*\}} \mid \begin{array}{l} \exists W \subseteq M, \exists D \subseteq C, \\ \text{Conv}_{c \in D} \mu(c^\omega) \cap \bigcap_{w \in W} H_w = \{\bar{y}\} \\ \text{and } \forall j, \forall v \in M \cap V_j, \\ y_j \geq \lambda(v) \end{array} \right\},$$

where the tuple \bar{y} corresponding to the sets W and D is the intersection of the face of P delimited by the values of the cycles of D , and the hyperplanes H_v for $v \in W$. That set Y is itself included in X .

We have, therefore:

$$\min_{c \in C}^* \text{Conv } \mu(c^\omega) = \min_{\bar{y} \in Y} y_i.$$

Let now $\bar{y} \in Y$, and let $D \subseteq C$ and $W \subseteq M$ be the corresponding sets.

Let us choose D and W minimal, so that all player $j \in \Pi$ controls at most one state $w \in W$, and so that there exists only one decomposition:

$$\bar{y} = \sum_{c \in D} \alpha_c \mu(c^\omega)$$

with for all c , we have $0 < \alpha_c < 1$, and $\sum_c \alpha_c = 1$. Furthermore, \bar{y} is the only such solution of the system of equations:

$$\forall j \in \Pi, \forall w \in W \cap V_j, y_j = \lambda(w).$$

Therefore, $\bar{\alpha} = (\alpha_c)_{c \in D}$ is the only solution of the system:

$$\begin{cases} \sum_{c \in D} \alpha_c = 1 \\ \forall j \in \Pi, \forall w \in W \cap V_j, \sum_{c \in D} \alpha_c \mu_j(c^\omega) = \lambda(w) \\ \forall c \in D, \alpha_c > 0. \end{cases}$$

Then, if \oplus is a symbol and A_{WD} is the matrix:

$$A_{WD} = \left(\begin{cases} 1 & \text{if } w = \oplus \\ \mu_j(c^\omega) & \text{else, with } w \in V_j \end{cases} \right)_{\substack{w \in W \cup \{\oplus\}, \\ c \in D}},$$

then A_{WD} is invertible and:

$$\bar{\alpha} = A_{WD}^{-1} \left(\begin{cases} 1 & \text{if } w = \oplus \\ \lambda(w) & \text{otherwise} \end{cases} \right)_{w \in W \cup \{\oplus\}},$$

with for all $c \in D$, $\alpha_c > 0$.

Let us write:

$$\bar{\beta}_{\lambda W} = \left(\begin{cases} 1 & \text{if } j = \oplus \\ \lambda(w) & \text{otherwise} \end{cases} \right)_{w \in W \cup \{\oplus\}}.$$

We have, thus, $\bar{\alpha} = A_{WD}^{-1} \bar{\beta}_{\lambda W}$.

Let us write, for each player j , $\bar{\gamma}_D^j = (\mu_j(c^\omega))_{c \in D}$. Then, we can write:

$$\begin{aligned} y_i &= \sum_c \alpha_c \mu_i(c^\omega) \\ &= {}^t \bar{\gamma}_D^i \bar{\alpha} \\ &= {}^t \bar{\gamma}_D^i A_{WD}^{-1} \bar{\beta}_{\lambda W}. \end{aligned}$$

Finally, if we write:

$$B_W = \left(\begin{cases} 1 & \text{if } w = v \\ 0 & \text{otherwise} \end{cases} \right)_{w \in W \cup \{\oplus\}, v \in V}$$

and:

$$\delta_W = \left(\begin{cases} 1 & \text{if } w = \oplus \\ 0 & \text{otherwise} \end{cases} \right)_{w \in W \cup \{\oplus\}}$$

we have $\bar{\beta}_{\lambda W} = B_W \lambda + \delta_W$, and therefore:

$$y_i = {}^t \bar{\gamma}_D^i A_{WD}^{-1} (B_W \lambda + \delta_W).$$

Conversely, the tuple \bar{y} defined by, for each $j \in \Pi$,

$$y_j = {}^t \bar{\gamma}_D^j A_{WD}^{-1} (B_W \lambda + \delta_W)$$

for given $W \subseteq M$ and $D \subseteq C$, is an element of the set Y if and only if:

- the intersection $\bigcap_{c \in D} \text{Conv } \mu(c^\omega) \cap \bigcap_{w \in W} H_w$ is a singleton, i.e. the matrix A_{WD} is invertible (otherwise the matrix A_{WD}^{-1} is not defined);
- $\bar{y} \in \text{Conv } \mu(c^\omega)$, i.e. the tuple $\bar{\alpha} = A_{WD}^{-1} (B_W \lambda + \delta_W)$ has only non-negative coordinates (actually positive if D is minimal);
- for each player j , for each vertex $v \in M \cap V_j$, we have $y_j \geq \lambda(v)$, i.e. ${}^t \bar{\gamma}_D^j A_{WD}^{-1} (B_W \lambda + \delta_W) \geq \lambda(v)$.

Hence the formula:

$$\text{nego}(\lambda)(v_0) = \max_{\tau_C \in \text{ML}(\text{Conc}_{\lambda_0 i}(G))} \min_{\substack{K \in \text{SConn}(\text{Conc}_{\lambda_0 i}(G)[\tau_C]) \\ \text{accessible from } (v_0, \{v_0\})}} \begin{cases} \text{if } K \text{ contains a deviation : } \min_{c \in \text{SC}(K)} \hat{\mu}_*(c^\omega) \\ \text{otherwise : } \min S_K, \end{cases}$$

where S_K is the set of real numbers of the form:

$${}^t \bar{\gamma}_D^i A_{WD}^{-1} (B_W \lambda + \delta_W)$$

such that:

- $W \subseteq M_K$;
- $D \subseteq C_K$;
- the matrix A_{WD} is invertible;
- the tuple $A_{WD}^{-1} (B_W \lambda + \delta_W)$ has only positive coordinates;
- and for each $j \in \Pi$, for each $v \in M_K \cap V_j$, we have ${}^t \bar{\gamma}_D^j A_{WD}^{-1} (B_W \lambda + \delta_W) \geq \lambda(v)$.

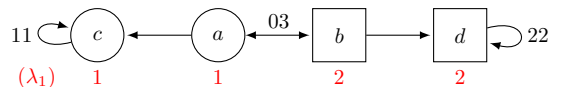
This is, indeed, the expression of a piecewise linear function. ■

APPENDIX K

SOME EXAMPLES OF NEGOTIATION SEQUENCES

We gather in this section some examples that could be interesting for the reader who would want to get a full overall view on the behaviour of the negotiation function on the mean-payoff-inf games. For all of them, we computed the negotiation sequence, as defined in Section IV-E. For some of them, we just gave the negotiation sequence; for the most important ones, we gave a complete explanation of how we computed it, using the abstract negotiation game, as defined in Appendix E.

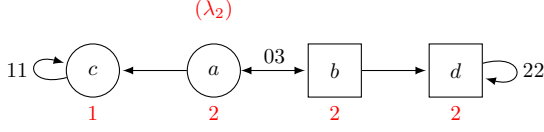
Example 11. Let us take again the game of Example 2: let us give (in red) the values of $\lambda_1 = \text{nego}(\lambda_0)$, which correspond to the antagonistic values.



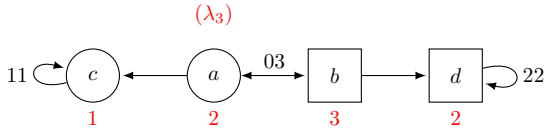
at the second step, let us execute the abstract game on the state a , with the requirement λ_1 : whatever Prover proposes at first, Challenger has the possibility to deviate and to reach the state b . Then, Prover has to propose a λ_1 -consistent play from the state b , i.e. a play in which player \bigcirc gets at least

the payoff 2: such a play necessarily ends in the state d , and gives player \square the payoff 2.

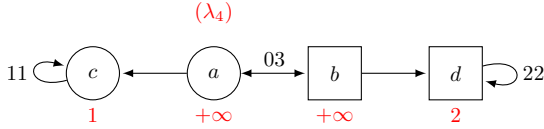
The other states keep the same values.



But then, at the third step, from the state b : whatever Prover proposes at first, Challenger can deviate to reach the state a . Then, Prover has to propose a λ_2 -consistent play from a , i.e. a play in which player \square gets at least the payoff 2: such a play necessarily end in the state d , i.e. after possibly some prefix, Prover proposes the play abd^ω . But then, Challenger can always deviate to go back to the state a ; and the play which is thus created is $(ab)^\omega$ which gives player \square the payoff 3.

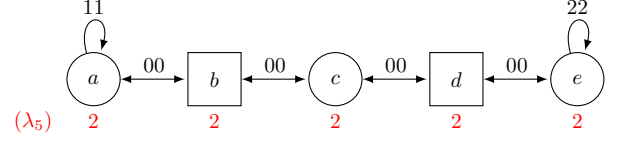
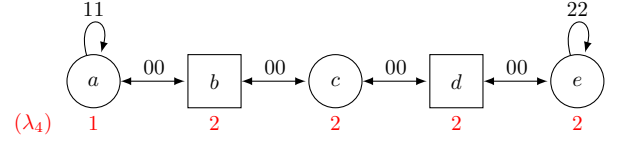
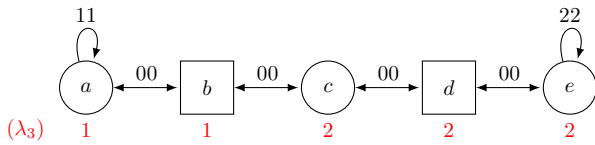
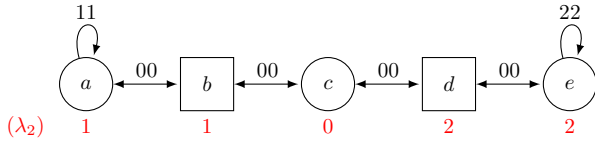
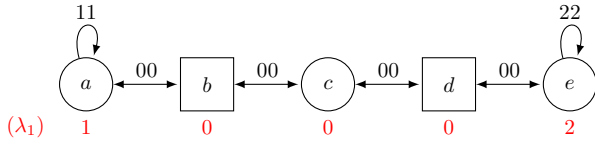


Finally, from the states a and b , there exist no λ_3 -consistent play, and therefore no λ -rational strategy profile.



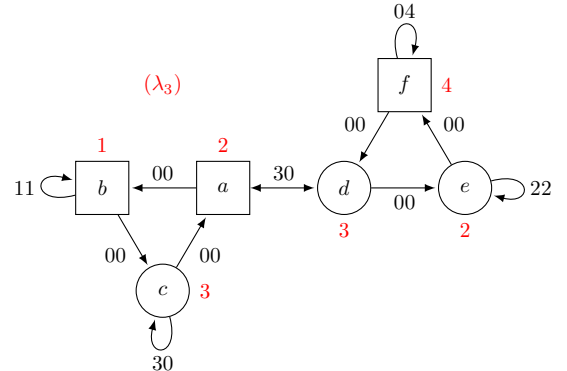
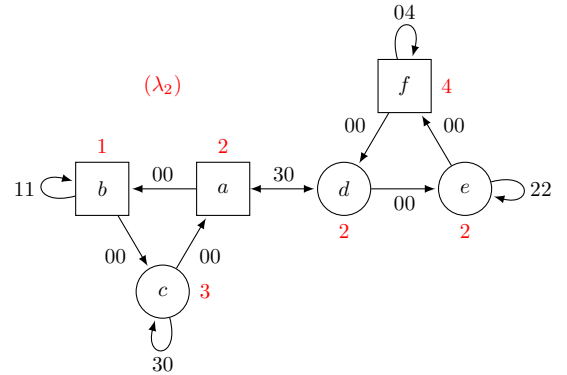
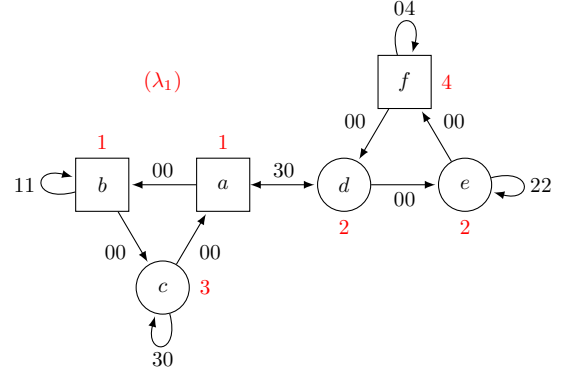
and for all $n \geq 4$, $\lambda_n = \lambda_4$.

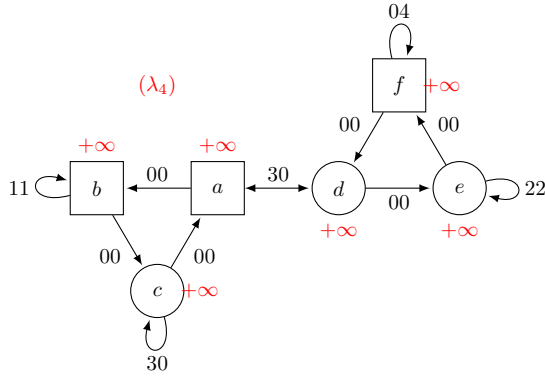
Example 12. In this example, we show a game that can be turned into a family of games, where the negotiation function needs as many steps as there are states to reach its limit: when the requirement changes in some state, it opens new possibilities from the neighbour states, and so on.



and the requirement λ_5 is a fixed point of the negotiation function.

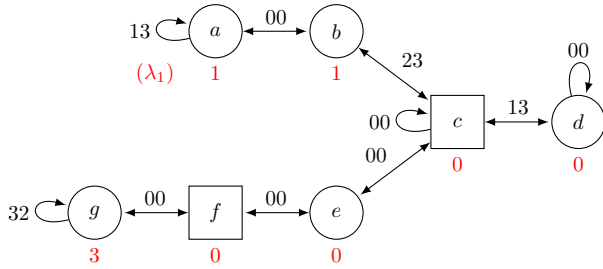
Example 13. In all the previous examples, all the games whose underlying graphs were strongly connected contained SPEs. Here is an example of game with a strongly connected underlying graph that does not contain SPEs.





Example 14. This example shows how a new requirement can emerge from the combination of several cycles.

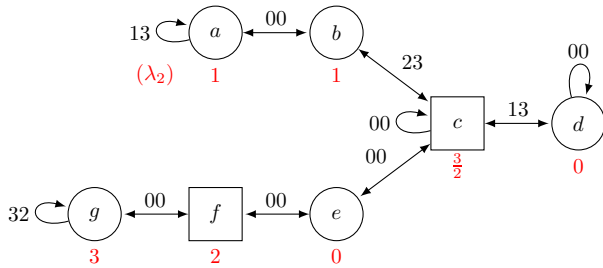
Let G be the following game:



At the first step, the requirement λ_1 captures the antagonistic values.

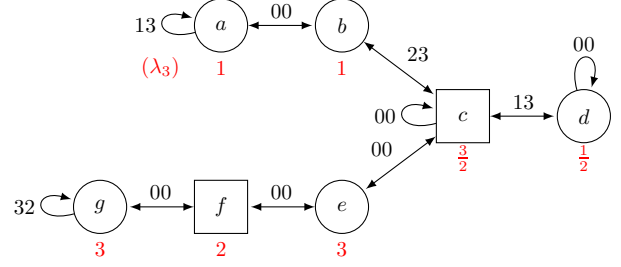
Then, from the state c , if player \square forces the access to the state b , then player \circ must get at least 1: the worst play that can be proposed to player \square is then $(babc)^\omega$, which gives player \square the payoff $\frac{3}{2}$.

From the state f , if player \square forces the access to the state g , then the worst play that can be proposed to them is g^ω .

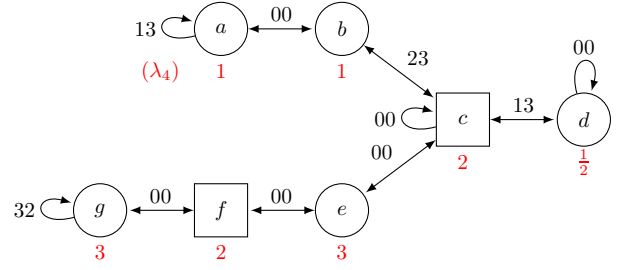


Then, from the state d , if player \circ forces the access to the state c , then player \square must get at least $\frac{3}{2}$: the worst play that can be proposed to player \circ is then $(cccd)^\omega$, which gives player \circ the payoff $\frac{1}{2}$.

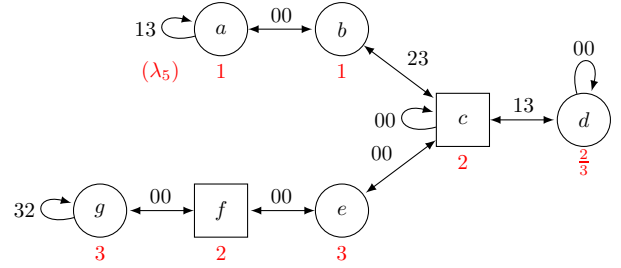
At the same time, from the state e , player \circ can now force the access to the state f : then, the worst play that can be proposed to them is fg^ω .



But then, from the state c , player \square can now force the access to the state e : then, the worst play that can be proposed to them is efg^ω .

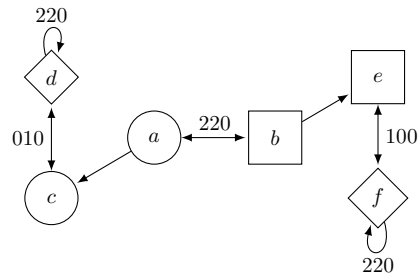


And finally, from that point, if from the state d player \circ forces the access to the state c , then player \square must have at least the payoff 2; and therefore, the worst play that can be proposed to player \circ is now $(ccd)^\omega$, which gives them the payoff $\frac{2}{3}$.



The requirement λ_5 is a fixed point of the negotiation function.

Example 15. We give here the details of the example given in Figure 15, in which the negotiation sequence was not stationary, and we provide a similar example with only two players.



For each edge, the weights are given in the following order: player \circ first, player \square second, player \diamond third. Since all the \diamond weight are equal to 0, for all $n > 0$, we have $\lambda_n(d) = \lambda_n(f) = 0$. It comes that for all $n > 0$, we also

have $\lambda_n(c) = \lambda_n(e) = 0$. Moreover, by symmetry of the game, we always have $\lambda_n(a) = \lambda_n(b)$. Therefore, to compute the negotiation sequence, it suffices to compute $\lambda_{n+1}(a)$ as a function of $\lambda_n(b)$, knowing that $\lambda_1(a) = \lambda_1(b) = 1$, and therefore that for all $n > 0$, $\lambda_n(a) = \lambda_n(b) \geq 1$.

From a , the worst play player \square could propose to player \circ would be a combination of the cycles cd and d giving her exactly 1. But then, player \circ will deviate to go to b , from which if player \square proposes plays in the strongly connected component containing c and d , then player \circ will always deviate and generate the play $(ab)^\omega$, and then get the payoff 2.

Then, in order to give her a payoff lower than 2, player \square has to go to the state e . Since player \circ does not control any state in that strongly connected component, the play he will propose will be accepted: he will, then propose the worst possible combination of the cycles ef and f for player \circ , such that he gets at least his requirement $\lambda_n(b)$. The payoff $\lambda_{n+1}(a)$ is then the maximal solution of the system:

$$\begin{cases} \lambda_{n+1}(a) = x + 2(1 - x) \\ 2(1 - x) \geq \lambda_n(b) \\ 0 \leq x \leq 1 \end{cases}$$

that is to say $\lambda_{n+1}(a) = 1 + \frac{\lambda_n(b)}{2} = 1 + \frac{\lambda_n(a)}{2}$, and by induction, for all $n > 0$:

$$\lambda_n(a) = \lambda_n(b) = 2 - \frac{1}{2^{n-1}}$$

which tends to 2 but never reaches it.

That example could let us think that we need three players to observe such a phenomena. Actually, the existence of a player \diamond for whom all the plays are equivalent was useful to build a not too complicated example, but not necessary. Here is a variant of that example with only two players, slightly less intuitive, but where the sequences $(\lambda_n(a))_n$ and $(\lambda_n(b))_n$ are the same as previously:

