Is the generic algorithm for first-order model-checking automatic structures optimal ?

Antoine Durand-Gasselin Peter Habermehl

LIAFA, Université Paris Diderot

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Contents

First-Order Logics

Interpreted relational structure

the domain

a set of symbols with arity

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for $P_i \in \mathcal{S}$, $P_i^{\mathcal{A}} \subseteq D^{ar_{P_i}}$

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First-order model-checking problem over $\mathcal{A}$

Input: First-order formula over some signature $\mathcal{S}$

Output: Whether the formula is satisfiable in the structure

Some structures can be presented with automata

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Theorem [Hodgson, 1982, Khoussainov and Nerode, 1994]

For automatic structures, any FO relation is synchronously regular

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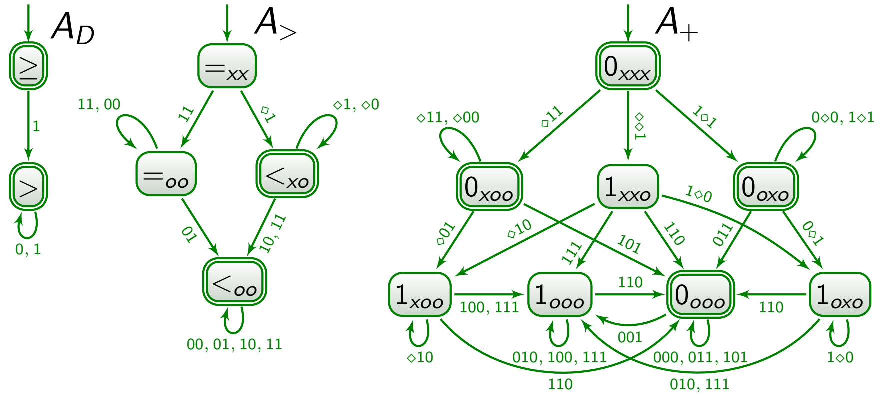
Example: Automatic Presentation of Presburger Arithmetic



Example: $(18, 170, 188) \in L(A_+)$
encoded by $(\diamond 11, \diamond 00, \diamond 11, 101, 011, 001, 110, 000)$

◇	◇	◇	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	1	1	1	1	0	0

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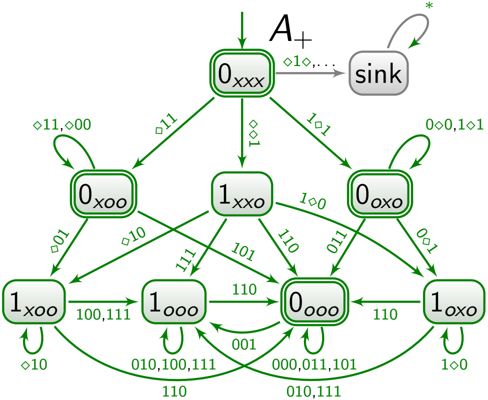
- Existential quantification corresponds to language projection:
  From a language over $(\Sigma \cup \{\diamond\})^r$ we need to accept accept a language over $(\Sigma \cup \{\diamond\})^{r-1}$, erasing the track corresponding to the quantified variable.

Building $A_{\exists y.x+y=z}$



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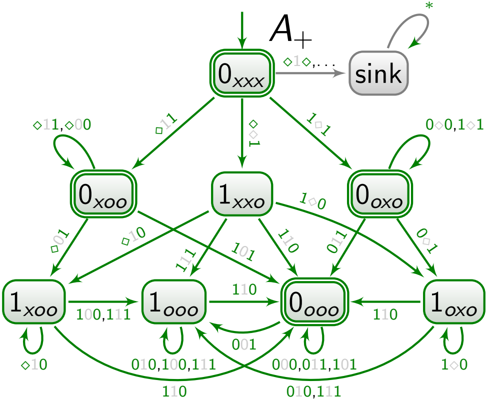

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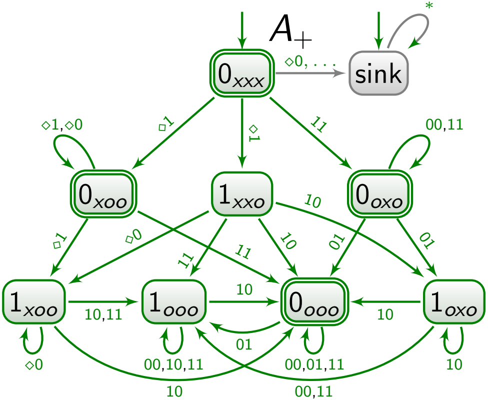

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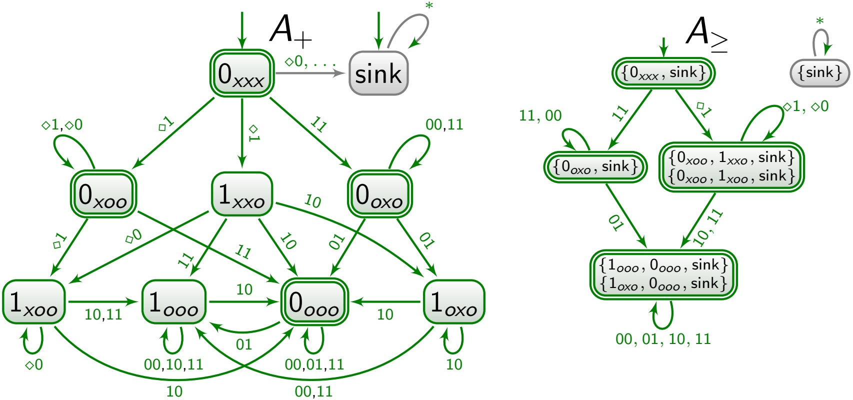

Is this generic algorithm optimal ?

Model-Checking through inductive automaton construction

- Model-checking φ is reduced to checking emptiness of A_φ
- What is the complexity of this automaton construction?

We will characterize the *deterministic time complexity* of this construction.

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A naive analysis

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- The complexity of the inductive automaton construction is non-elementary

Indeed a quantifier alternation corresponds to a language projection and a complementation: it may lead to an exponential blow-up.

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Yes, but what if we fix the structure ?

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Given an automatic presentation, we define its saturated structure:

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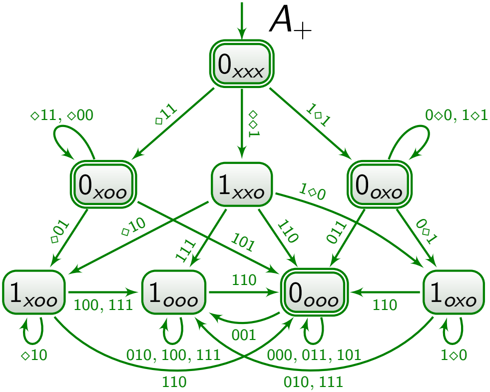

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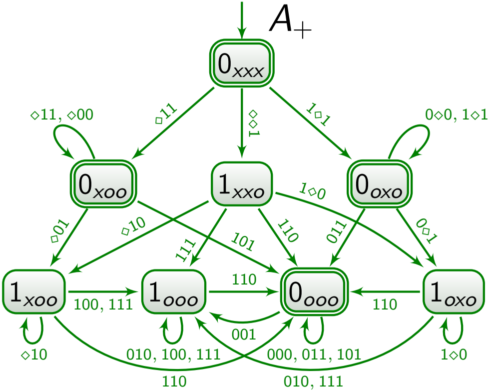

$0_{xxx} = \{(\varepsilon, \varepsilon, \varepsilon)\}$
$0_{xoo} = \{(\varepsilon, w, w) | w \in 1\Sigma^*\}$
$1_{xoo} = \{(0, x, x + 1)\}$
$0_{ooo} = \{(x, y, x + y)\}$
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Remark that words of the form $0\Sigma^*$ are elements in the domain of the saturated structure but are not in any relation.

The saturated structure exhibits the complexity

Theorem: [Durand-Gasselin and Habermehl, 2012]

The deterministic time complexity of this automaton construction is the same as for model-checking the saturated structure

Remark

The first-order theory of the saturated structure of that Presburger Arithmetic presentation is no harder than Presburger Arithmetic

Corrolary

The inductive construction of an automaton accepting solutions of a Presburger formula is in 3EXPTIME

This algorithm is optimal in this case □

Bonus: the automaton has at most a triple exponential number of states

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Thank you for your attention !

References

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