Accepted Manuscript

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PII: S0377-0427(18)30431-X
DOI: https://doi.org/10.1016/j.cam.2018.07.011
Reference: CAM 11797

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 10 May 2016
Revised date : 24 June 2018

Please cite this article as: G. Groza, M. Razzaghi, Approximation of solutions of polynomial partial differential equations in two independent variables, *Journal of Computational and Applied Mathematics* (2018), https://doi.org/10.1016/j.cam.2018.07.011

Approximation of solutions of polynomial partial differential equations in two independent variables

Ghiocel Groza[a,*], Mohsen Razzaghi[b]

[a] Technical University of Civil Eng. of Bucharest, Department of Mathematics and Computer Science, 020396, Romania
[b] Mississippi State University, Department of Mathematics and Statistics, Mississippi State, MS 397622, USA

Abstract

A numerical method for solving polynomial partial differential equations in two independent variables, defined in the paper, is presented. The technique is based on polynomial approximation. Properties and the operational matrices for partial derivatives for a polynomial in two variables are presented first. These properties are then used to reduce the solution of partial differential equations in two independent variables to a system of algebraic equations. Five illustrative examples are presented to prove the effectiveness of the present method. Results show that the numerical scheme is very convenient for solving polynomial partial differential equations.

Mathematics Subject Classification (2010): 35G20, 41A10, 41A58

Keywords: polynomial partial differential equation, polynomial approximation, analytic functions, distance in a metric space

1. Introduction

Many important models in physical, biological or other sciences are based on partial differential equations (PDE). The nonlinear PDE are in a central position because they govern a large area of complex phenomena of motion, reaction, diffusion, equilibrium, conservation, and more (see [1] and [2]). By using Stone-Weierstrass Theorem many of these PDE's can be reduced to polynomial partial differential equations (see Eqs. (5), (6) and Example 5).

The problem of finding exact solutions to partial differential equations has been deeply studied in the literature. However, there is not a general method to be followed when handling a specific equation [3]. The authors in [3] present a procedure for solving first-order autonomous algebraic

*Corresponding author
Email addresses: grozag@utcb.ro (Ghiocel Groza), razzaghi@math.msstate.edu (Mohsen Razzaghi)

partial differential equations in an arbitrary number of variables. The exact solutions for Fisher, Burger-Fisher, Benjamin-Bona-Mahony-Burgers and Modified Benjamin-Bona-Mahony are obtained in [4] by using $\left(\frac{G'}{G}\right)$-expansion method.

Numerical solutions of the ordinary differential equations (ODE) by using Taylor series method have been investigated by many authors (see, for example, [5]-[10] and references therein). However, there are few references on the solution of the partial differential equations (PDE) by using Taylor series method (see [11]-[13]). One advantage of the method of using Taylor series or the polynomial approximation is that a differentiable approximate solution is obtained, which can be replaced into the equation and the initial or boundary conditions. In this manner, the accuracy of solution can be evaluated directly and the problem is reduced to that of solving a system of algebraic equations.

Partial differential equations (PDE) arise in connection with various physical and geometrical problems in which the functions involved depend on two or more independent variables, on time t and on one or several space variables. The numerical solution of PDE has been of great importance in recent years (see, for example, [14]-[17]). Other approximation methods for partial differential equations based on series of functions can be obtained by Adomian's decomposition method [18], radial basis collocation method [19], homotopy perturbation method [20], or variational iteration method [21].

In the present paper, we introduce a new computational method to solve the initial boundary-value problems for PDE in two independent variables. The method consists of reducing the PDE to a set of algebraic equations by approximating the solution of the PDE and some of its derivative by a polynomial with unknown coefficients and its corresponding derivatives. The operational matrices for partial derivatives of a polynomial in two variables are given. These matrices are then used to evaluate the coefficients of the required polynomial. If the solutions are analytic, this method is easier to apply (see Remark 1). As compared to variational iteration method [21], radial basis collocation method [19], homotopy perturbation method [20], or Adomian's decomposition method [18], in the present method, only algebraic methods are used to approximate the solutions. Further, the present method is more general than that [13].

The outline of this paper is the following: In Section 2 we introduce the basic notations and definitions required for our subsequent development. Section 3 is devoted to the derivation of operational matrices for partial derivatives of a polynomial in 2 variables. In Section 4, we apply the proposed numerical method to approximate the solutions of polynomial PDE in two independent variables. In this section, the error estimates for the present method are also included. In Section 5, we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme by considering five numerical examples.

2. Polynomial partial differential equations in two independent variables

We first introduce some efficient notations for polynomials in two variables. We shall denote the pair of non-negative integers (i_1, i_2) by the multi-index $\mathbf{i}$. Then we put $i_1 = \mathrm{pr}_1(\mathbf{i})$ and $i_2 = \mathrm{pr}_2(\mathbf{i})$. The sum $i_1 + i_2$ will be denoted by $|\mathbf{i}|$. We order the set $\mathbb{N}^2$ of pairs of non-negative integers by grouping the elements $\mathbf{i}$ having distinct $|\mathbf{i}|$ and then using a lexicographic order. Thus

$$\mathbf{i} = (i_1, i_2) < \mathbf{i}' = (i_1', i_2') \text{ if either } |\mathbf{i}| < |\mathbf{i}'| \text{ or } |\mathbf{i}| = |\mathbf{i}'| \text{ and } i_2 < i_2'. \tag{1}$$

Let P be a polynomial in two variables x_1, x_2 with real coefficients. Thus, for $\mathbf{i} = (i_1, i_2)$, we write $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} x_2^{i_2}$ and $D^{\mathbf{i}} P = \frac{\partial^{|\mathbf{i}|} P}{\partial x_1^{i_1} \partial x_2^{i_2}}$. The terms of the polynomials can be ordered by means of Eq. (1) with respect to exponents of the unknowns. Hence we can write

$$P(\mathbf{x}) = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{m}} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}, \tag{2}$$

where $\mathbf{0} = (0, 0)$, $c_{\mathbf{m}} = c_{(m_1, m_2)} \neq 0$, and $\mathbf{m}$ is the greatest index $\mathbf{i}$ with respect to the order defined by Eq. (1) such that $c_{\mathbf{i}} \neq 0$. Then $\mathbf{m} = \mathbf{d}_P$ is called the degree of P.

More generally, for a fixed N,

$$\begin{gathered} \mathbf{i} = (i_1, i_2, ..., i_N) < \mathbf{i}' = (i_1', i_2', ..., i_N') \text{ if either } |\mathbf{i}| < |\mathbf{i}'| \text{ or } |\mathbf{i}| = |\mathbf{i}'|, \\ \text{and there exists } s \leq N \text{ such that } i_j = i_j', \text{ for } j = 1, 2, ..., s-1 \text{ and } i_s < i_s', \end{gathered} \tag{3}$$

where $|\mathbf{i}| = i_1 + i_2 + ... + i_N$. If Q is a polynomial in N variables $z_1, ..., z_N$ with real coefficients, then for $\mathbf{i} = (i_1, i_2, ..., i_N)$, we write $\mathbf{z}^{\mathbf{i}} = z_1^{i_1} ... z_N^{i_N}$, and

$$Q(\mathbf{z}) = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{d}_Q} q_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}, \tag{4}$$

where $\mathbf{0} = (0, 0, ..., 0)$, $q_{\mathbf{i}} = q_{(i_1, i_2, ..., i_N)}$, and the degree of Q, denoted by $\mathbf{d}_Q$, is the greatest index $\mathbf{i}$ with respect to the order defined by Eq. (3) such that $q_{\mathbf{i}} \neq 0$. The ordinary degree of the polynomial Q is $\deg(Q) = |\mathbf{d}_Q|$.

Let Ω be an open set in the real two-dimensional space $\mathbb{R}^2$. For every non-negative integer k we denote by $\mathcal{C}^k(\Omega)$ the set of all functions u defined in Ω, whose all partial derivatives of order less or equal to k exist and are continuous. If $\overline{\Omega}$ is the closure of Ω, we denote by $\mathcal{C}^k(\overline{\Omega})$ the set of all functions $u \in \mathcal{C}^k(\Omega)$ whose all partial derivatives of order less or equal to k have continuous extensions to $\overline{\Omega}$.

Consider Ω a domain in $\mathbb{R}^2$ whose boundary Γ is a piecewise smooth curve and $u \in \mathcal{C}^k(\overline{\Omega})$ an unknown function. If Q is a polynomial of the form (4), the PDE

$$Q\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1^2}, ..., \frac{\partial u^{|\mathbf{n}|}}{\partial x_1^{n_1} \partial x_2^{n_2}}\right) = f(x_1, x_2),\ (x_1, x_2) \in \Omega, \tag{5}$$

where $\mathbf{n} = (n_1, n_2)$ is the greatest index with respect to the order given in Eq. (1), $f \in \mathcal{C}(\Omega)$, is called a polynomial partial differential equation of order $|\mathbf{n}|$ in two independent variables.

Suppose that $\Gamma = \bigcup_{r=1}^{N'_\Gamma} \Gamma'_r$, with Γ'_r a smooth curve, for every r, and u satisfies

$$Q_j\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1^2}, ..., \frac{\partial u^{|\mathbf{n}^{(j)}|}}{\partial x_1^{n_1^{(j)}} \partial x_2^{n_2^{(j)}}}\right) = f_j(x_1, x_2),\ (x_1, x_2) \in \Gamma_j, \tag{6}$$

where, for every j, there exists $r = r(j)$ such that $\Gamma_j = \Gamma'_r$, Q_j are polynomials, $f_j \in \mathcal{C}(\Omega \cup \Gamma_j)$, $|\mathbf{n}^{(j)}| = |(n_1^{(j)}, n_2^{(j)})| < |\mathbf{n}|$, $j \le N_\Gamma$. If the problem from Eq. (5) with the boundary conditions in Eq. (6) has a unique solution in $\mathcal{C}^{|\mathbf{n}|}(\Omega) \cap \bigcap_{j=1}^{N_\Gamma} \mathcal{C}^{|\mathbf{n}^{(j)}|}(\Omega \cup \Gamma_j)$ we shall approximate the solution u by a polynomial P in two variables.

Now, let Ω be a bounded domain and $u, v \in \mathcal{C}^{|\mathbf{n}|}(\overline{\Omega})$. We define the real number

$$\begin{aligned} &d(u,v) = \lambda_0 \cdot \\ &\left\| Q\left(x_1, x_2, u, \tfrac{\partial u}{\partial x_1}, \tfrac{\partial u}{\partial x_2}, \tfrac{\partial^2 u}{\partial x_1^2}, ..., \tfrac{\partial^{|\mathbf{n}|} u}{\partial x_1^{n_1} \partial x_2^{n_2}}\right) - Q\left(x_1, x_2, v, \tfrac{\partial v}{\partial x_1}, \tfrac{\partial v}{\partial x_2}, \tfrac{\partial^2 v}{\partial x_1^2}, ..., \tfrac{\partial^{|\mathbf{n}|} v}{\partial x_1^{n_1} \partial x_2^{n_2}}\right)\right\|_{\overline{\Omega},\infty} \\ &+ \sum_{j=1}^{N_\Gamma} \lambda_j \left\| Q_j\left(x_1, x_2, u, \tfrac{\partial u}{\partial x_1}, ..., \tfrac{\partial^{|\mathbf{n}^{(j)}|} u}{\partial x_1^{n_1^{(j)}} \partial x_2^{n_2^{(j)}}}\right) - Q_j\left(x_1, x_2, v, \tfrac{\partial v}{\partial x_1}, ..., \tfrac{\partial^{|\mathbf{n}^{(j)}|} v}{\partial x_1^{n_1^{(j)}} \partial x_2^{n_2^{(j)}}}\right)\right\|_{\overline{\Gamma_j},\infty}, \end{aligned} \tag{7}$$

where $\lambda_j \ge 0$, $\sum_{j=0}^{N_\Gamma} \lambda_j = 1$ and $\| \ \|_{\overline{\Omega},\infty}$ is the supremum norm. If, for every f, the solution of Eq. (5), satisfying the boundary conditions in Eq. (6) is uniquely determined, then it is easy to prove that d is a metric defined on $\mathcal{C}^{|\mathbf{n}|}(\overline{\Omega})$.

Usually, in Eq. (7), we may take $\lambda_j = \frac{1}{N_\Gamma + 1}$, for every $j = 0, 1, ..., N_\Gamma$. Sometimes, in solving real problems, the accuracy of defining Eq. (5) and some of the boundary conditions in Eq. (6) is not the same. In this case, it is necessary to choose different values of λ_j to obtain a useful estimation of the errors.

3. Properties of polynomials in two variables

In this section, some properties of polynomials in two variables are given.

3.1. Position of terms in a polynomial

Let P be a polynomial in two variables written in the form of Eq. (2) with respect to the order defined by Eq. (1). Thus

$$\begin{aligned} P(x_1, x_2) = c_{(0,0)} + c_{(1,0)} x_1 + c_{(0,1)} x_2 + c_{(2,0)} x_1^2 + c_{(1,1)} x_1 x_2 + c_{(0,2)} x_2^2 \\ + c_{(3,0)} x_1^3 + c_{(2,1)} x_1^2 x_2 + c_{(1,2)} x_1 x_2^2 + c_{(0,3)} x_2^3 + ... \ . \end{aligned} \tag{8}$$

For our subsequent development, we need to find the position of a term in Eq. (8). To this end, first we write $P(x_1, x_2)$ in Eq. (8) as

$$P(x_1,x_2) = a_1 + a_2x_1 + a_3x_2 + a_4x_1^2 + a_5x_1x_2 + a_6x_2^2 + a_7x_1^3 + a_8x_1^2x_2 + a_9x_1x_2^2 + a_{10}x_2^3 + \dots \ . \quad (9)$$

Then we describe (see [13]) a one-to-one mapping γ from $\mathbb{N}^2$ in Eq. (8) to $\mathbb{N}^*$ in Eq. (9), which keeps the order, and its inverse $\gamma^{-1} : \mathbb{N}^* \to \mathbb{N}^2$.

In order to define γ we remark that for a given m for which $i_1+i_2 = m$, there exist $m+1$ elements $\mathbf{i} \in \mathbb{N}^2$. Hence, for a fixed positive integer k we have $e(k)$ elements $\mathbf{i} \in \mathbb{N}^2$ such that $i_1 + i_2 \leq k$, where

$$e(k) := \sum_{m=0}^{k} (m+1) = \frac{(k+1)(k+2)}{2}. \quad (10)$$

By using Eqs. (8) and (10), we obtain the position $\gamma(i_1, i_2)$ of the term $c_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$ as

$$\gamma(i_1,i_2) = e(i_1+i_2-1) + i_2 + 1 = \frac{(i_1+i_2)(i_1+i_2+1)}{2} + i_2 + 1. \quad (11)$$

For example, $\gamma(2,1) = 8$, which implies that the term $c_{(2,1)}x_1^2x_2$ is located in the eighth term in Eq. (8).

To define the inverse of γ we recall that a triangular number has the form $\frac{n(n+1)}{2}$, where $n \in \mathbb{N}$. The following result holds:

Proposition 1. *Let $j = \gamma(\mathbf{m}) \in \mathbb{N}^*$, where $\mathbf{m} = (m_1, m_2) \in \mathbb{N}^2$. Then $\gamma^{-1}(j) = (\gamma_1^{-1}(j), \gamma_2^{-1}(j))$ $= (m_1, m_2)$, where*

$$m_1 = \begin{cases} \left[\frac{\sqrt{8j+1}-1}{2}\right] - j + 1 + \frac{1}{2}\left[\frac{\sqrt{8j+1}-1}{2}\right]\left[\frac{\sqrt{8j+1}+1}{2}\right], & \textit{if } j \textit{ is not a triangular number}, \\ \left[\frac{\sqrt{8j+1}-3}{2}\right] - j + 1 + \frac{1}{2}\left[\frac{\sqrt{8j+1}-3}{2}\right]\left[\frac{\sqrt{8j+1}-1}{2}\right], & \textit{if } j \textit{ is a triangular number}. \end{cases} \quad (12)$$

$$m_2 = \begin{cases} j - 1 - \frac{1}{2}\left[\frac{\sqrt{8j+1}-1}{2}\right]\left[\frac{\sqrt{8j+1}+1}{2}\right], & \textit{if } j \textit{ is not a triangular number}, \\ j - 1 - \frac{1}{2}\left[\frac{\sqrt{8j+1}-3}{2}\right]\left[\frac{\sqrt{8j+1}-1}{2}\right], & \textit{if } j \textit{ is a triangular number}. \end{cases} \quad (13)$$

Proof. By Eq. (11), for each $j = \gamma(\mathbf{m}) = \frac{(m_1+m_2)(m_1+m_2+1)}{2} + m_2 + 1 \in \mathbb{N}^*$, the greatest triangular number $\frac{n(n+1)}{2}$ less than or equal to j is either $\frac{(m_1+m_2)(m_1+m_2+1)}{2}$, for $m_1 \neq 0$ or $\frac{(m_1+m_2+1)(m_1+m_2+2)}{2}$, for $m_1 = 0$. Since n is the greatest nonnegative integer such that $\frac{n(n+1)}{2} \leq j$ it follows that

$$n = \left[\frac{-1+\sqrt{8j+1}}{2}\right], \quad (14)$$

where $[x]$ denotes the largest integer not greater than x, and

$$n = \begin{cases} m_1 + m_2, & \text{if } j \text{ is not a triangular number} \\ m_1 + m_2 + 1, & \text{if } j \text{ is a triangular number.} \end{cases}$$

Thus, by Eqs. (11) and (14), we get

$$n = \begin{cases} m_1 + m_2 = \left[\frac{\sqrt{8j+1}-1}{2}\right], & \text{if } j \text{ is not a triangular number,} \\ \\ m_1 + m_2 + 1 = \left[\frac{\sqrt{8j+1}-3}{2}\right], & \text{if } j \text{ is a triangular number,} \end{cases}$$

$$m_2 + 1 = \begin{cases} j - \frac{1}{2}\left[\frac{\sqrt{8j+1}-1}{2}\right]\left[\frac{\sqrt{8j+1}+1}{2}\right], & \text{if } j \text{ is not a triangular number,} \\ \\ j - \frac{1}{2}\left[\frac{\sqrt{8j+1}-3}{2}\right]\left[\frac{\sqrt{8j+1}-1}{2}\right], & \text{if } j \text{ is a triangular number.} \end{cases}$$

Hence, for $j \in \mathbb{N}^*$, we obtain $\gamma^{-1}(j) = (\gamma_1^{-1}(j), \gamma_2^{-1}(j)) = (m_1, m_2)$, where m_1 and m_2 are given by Eqs. (12) and (13). $\square$

For example, by Proposition 1, since 8 is not a triangular number, it follows that $\gamma^{-1}(8) = \left(\left[\frac{\sqrt{65}-1}{2}\right] - 8 + 1 + \frac{1}{2}\left[\frac{\sqrt{65}-1}{2}\right]\left[\frac{\sqrt{65}+1}{2}\right], 8 - 1 - \frac{1}{2}\left[\frac{\sqrt{65}-1}{2}\right]\left[\frac{\sqrt{65}+1}{2}\right]\right) = (3-7+6, 7-6) = (2,1)$, that is $f_{(2,1)}x_1^2x_2$ is the eighth term in Eq. (8). Similarly, because 10 is a triangular number we get $\gamma^{-1}(10) = \left(\left[\frac{\sqrt{81}-3}{2}\right] - 10 + 1 + \frac{1}{2}\left[\frac{\sqrt{81}-3}{2}\right]\left[\frac{\sqrt{81}-1}{2}\right],\ 10 - 1 - \frac{1}{2}\left[\frac{\sqrt{81}-3}{2}\right]\left[\frac{\sqrt{81}-1}{2}\right]\right) = (3-9+6, 9-6) = (0,3)$ and $f_{(0,3)}x_2^3$ is the tenth term in Eq. (8).

3.2. Operational matrices for partial derivatives

Let P be a polynomial given by Eq. (2). We put

$$C_{\mathbf{m}} = [c_{(0,0)}, c_{(1,0)}, c_{(0,1)}, ..., c_{(m_1,m_2)}], \tag{15}$$

$$T_{\mathbf{m}}(\mathbf{x}) = [1, x_1, x_2, ..., {x_1}^{m_1}{x_2}^{m_2}], \tag{16}$$

where $C_{\mathbf{m}}$ and $T_{\mathbf{m}}(\mathbf{x})$ are $1 \times \gamma(\mathbf{m})$ matrices. Then by Eq. (2) we have

$$P(\mathbf{x}) = C_{\mathbf{m}} T_{\mathbf{m}}^T(\mathbf{x}). \tag{17}$$

Consider the sequence

$$(0,0),\ (1,0),\ (0,1),\ (2,0),\ (1,1),\ (0,2),\ ... \tag{18}$$

of elements of $\mathbb{N}^2$ ordered with respect to the relation defined in Eq. (1). If a is a fixed nonnegative integer and $k \in \mathbb{N}^*$, we denote

$$p^{(a,*)}(k) \text{ (resp. } p^{(*,a)}(k)), \tag{19}$$

the number of elements of the form (a,b), (resp. (b,a)), where b is an arbitrary nonnegative integer, through the first k terms of Eq. (18), respectively.

Lemma 1. *The following identities hold:*

$$p^{(a,*)}(k) = \begin{cases} \left[\frac{\sqrt{8k+8a+1}-2a-1}{2}\right], \text{ if } k \geq \frac{a(a+1)}{2}+1, \\ 0, \text{ otherwise}, \end{cases} \tag{20}$$

$$p^{(*,a)}(k) = \begin{cases} \left[\frac{\sqrt{8k-8a-7}-2a+1}{2}\right], \text{ if } k \geq \frac{a(a+3)}{2}+1, \\ 0, \text{ otherwise}. \end{cases} \tag{21}$$

Proof. The sequence elements of the form (a,b) in Eq. (18), where b is an arbitrary nonnegative integer, is

$$(a,0),\ (a,1),\ (a,2),\ \ldots\ .$$

Thus, by Eq. (11), the position of the nth element of this sequence in Eq. (18) is given by

$$\gamma(a,n-1) = \frac{(a+n-1)(a+n)}{2} + n = \frac{n^2+(2a+1)n+a^2-a}{2}.$$

If $\frac{n^2+(2a+1)n+a^2-a}{2} \leq k$, it follows that

$$\frac{-\sqrt{8k+8a+1}-2a-1}{2} \leq n \leq \frac{\sqrt{8k+8a+1}-2a-1}{2}.$$

Because n is a positive integer it follows that $\frac{\sqrt{8k+8a+1}-2a-1}{2} \geq 1$, which implies $k \geq \frac{a(a+1)}{2}+1$. Hence we obtain Eq. (20).

The proof of Eq. (21) is similar. $\square$

To find the operational matrices for partial derivatives we use the following result:

Theorem 1. *Let $\mathbf{n} = (n_1,n_2) \in \mathbb{N}^2$. If P is a polynomial given by Eq. (17), then*

$$\frac{\partial^{n_1+n_2}P}{\partial x_1^{n_1}\partial x_2^{n_2}} = C_{\mathbf{m}} D^{\mathbf{n},\mathbf{m}} T^T_{\gamma^{-1}(\theta(\mathbf{n},\mathbf{m}))}(\mathbf{x}), \tag{22}$$

where

$$\theta(\mathbf{n},\mathbf{m}) = \gamma(\mathbf{m}) - \left(\sum_{i=0}^{n_1-1} p^{(i,*)}(\gamma(\mathbf{m})) + \sum_{j=0}^{n_2-1} p^{(*,j)}(\gamma(\mathbf{m}))\right) + n_1 n_2, \tag{23}$$

and $D^{\mathbf{n},\mathbf{m}}$ are $\gamma(\mathbf{m}) \times \theta(\mathbf{n},\mathbf{m})$ matrices whose (i,j)th elements $d^{\mathbf{n},\mathbf{m}}_{i,j}$ are given by

$$d^{\mathbf{n},\mathbf{m}}_{i,j} = \prod_{r=0}^{n_1-1}(\mathrm{pr}_1(\gamma^{-1}(j)+\mathbf{n})-r)\prod_{s=0}^{n_2-1}(\mathrm{pr}_2(\gamma^{-1}(j)+\mathbf{n})-s)\delta_{i,\gamma(\gamma^{-1}(j)+\mathbf{n})}. \tag{24}$$

Here δ_{ij} denotes the Kronecker delta function, that is

$$\delta_{ij} = \begin{cases} 1, & \text{for} \quad i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We write $c_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$ as

$$c_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} = c_{\mathbf{i}} x_1^{\mathrm{pr}_1(\mathbf{i})} x_2^{\mathrm{pr}_2(\mathbf{i})}.$$

Then it follows that

$$\frac{\partial^{n_1+n_2} P}{\partial x_1^{n_1} \partial x_2^{n_2}}(\mathbf{x}) = \sum_{\mathbf{i}=\mathbf{n}}^{\mathbf{m}} \prod_{r=0}^{n_1-1} (i_1 - r) \prod_{s=0}^{n_2-1} (i_2 - s) c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}-\mathbf{n}}. \tag{25}$$

For example, if $\mathbf{n} = (1, 0)$, we have

$$\frac{\partial P^{1+0}}{\partial x_1^1 \partial x_2^0}(\mathbf{x}) = \frac{\partial P}{\partial x_1}(\mathbf{x}) = \sum_{\mathbf{i}=(1,0)}^{\mathbf{m}} i_1 c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}-(1,0)}$$

$$= c_{(1,0)} + 2c_{(2,0)}x_1 + c_{(1,1)}x_2 + 3c_{(3,0)}x_1^2 + \ldots .$$

The number of terms in Eq. (25) depends on $c_{\mathbf{i}}$, $\mathbf{i} = \mathbf{0}, \ldots, \mathbf{m}$, different from zero for which $i_1 \geq n_1$ and $i_2 \geq n_2$. The number of terms having $i_1 \geq n_1$ and $i_2 \geq n_2$, denoted by $\theta(\mathbf{n}, \mathbf{m})$, is equal to the number of the terms $(i_1, i_2) \leq \mathbf{m}$ in the sequence in Eq. (18) such that $i_1 \geq n_1$ and $i_2 \geq n_2$. If $n_2 = 0$, all the terms which do not have this form can be written as $(a, 0)$, where $a \leq n_1 - 1$. Thus, by Eq. (19), it follows that the number of these terms is equal to $\sum\limits_{i=0}^{n_1-1} p^{(i,*)}(\gamma(\mathbf{m}))$. Hence we obtain that Eq. (23) holds for $n_2 = 0$. Similarly we can verify Eq. (23) in the case when $n_1 = 0$. If $n_1 n_2 \neq 0$, by Inclusion-Exclusion Principle (see [22], p. 64), it follows that

$$|\{(a,b) : a \leq n_1 - 1\} \cup \{(a,b) : b \leq n_2 - 1\}| = |\{(a,b) : a \leq n_1 - 1\}| + |\{(a,b) : b \leq n_2 - 1\}|$$

$$-|\{(a,b) : a \leq n_1 - 1 \text{ and } b \leq n_2 - 1\}| = \sum_{i=0}^{n_1-1} p^{(i,*)}(\gamma(\mathbf{m})) + \sum_{j=0}^{n_2-1} p^{(*,j)}(\gamma(\mathbf{m})) - n_1 n_2,$$

where, for a finite set A we denoted by $|A|$ the number of elements of A. Hence it follows Eq. (23).

Because the sequence of the terms having $i_1 \geq n_1$ and $i_2 \geq n_2$ can be written as

$$\gamma^{-1}(1) + \mathbf{n}, \gamma^{-1}(2) + \mathbf{n}, \gamma^{-1}(3) + \mathbf{n}, ...,$$

by Eq. (25), we obtain Eq. (22). $\square$

3.3. Operational matrices for the product

Suppose

$$P_1(\mathbf{x}) = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{m}_1} c_{\mathbf{i}}^{(1)} \mathbf{x}^{\mathbf{i}} = C_{\mathbf{m}_1}^{(1)} T_{\mathbf{m}_1}^T(\mathbf{x}), \tag{26}$$

$$P_2(\mathbf{x}) = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{m}_2} c_{\mathbf{i}}^{(2)} \mathbf{x}^{\mathbf{i}} = C_{\mathbf{m}_2}^{(2)} T_{\mathbf{m}_2}^T(\mathbf{x}), \tag{27}$$

where $\mathbf{m}_r = (m_{1,r}, m_{2,r})$,

$$C_{\mathbf{m}_r}^{(r)} = [c_{(0,0)}^{(r)}, c_{(1,0)}^{(r)}, ..., c_{(m_{1,r},m_{2,r})}^{(r)}],\ r = 1, 2, \tag{28}$$

and $T_{\mathbf{m}_r}(\mathbf{x})$ is given by

$$T_{\mathbf{m}_r}(\mathbf{x}) = [1, x_1, x_2, x_1^2, x_1x_2, ..., x_1^{m_{1,r}} x_2^{m_{2,r}}]. \tag{29}$$

Let $\mathbb{R}^\infty$ be the real vector space of all sequences with real terms. By adding infinitely many terms equal to zeros, we may suppose that $C_{\mathbf{m}_r}^{(r)} \in \mathbb{R}^\infty$. Hence we identify $C_{\mathbf{m}_r}^{(r)} \in \mathbb{R}^{\gamma(\mathbf{m}_r)}$ and $C_{\mathbf{m}_r}^{(r)} \in \mathbb{R}^\infty$ such that, for example, we may consider $[c_{(0,0)}, c_{(1,0)}, c_{(0,1)}] = [c_{(0,0)}, c_{(1,0)}, c_{(0,1)}, 0, 0, ...] = [c_{(0,0)}, c_{(1,0)}, c_{(0,1)}, 0]$. Thus we can compute the sum of $C_{\mathbf{m}_1}^{(1)}$ and $C_{\mathbf{m}_2}^{(2)}$ by considering the componentwise addition in $\mathbb{R}^\infty$. Moreover we define the following composition law for two vectors:

$$C_{\mathbf{m}_1}^{(1)} \circ C_{\mathbf{m}_2}^{(2)} := C_{\mathbf{m}_1}^{(1)} \Pi(C_{\mathbf{m}_2}^{(2)}) \in \mathbb{R}^\infty, \tag{30}$$

where $\Pi(C_{\mathbf{m}_2}^{(2)})$ is a $\gamma(\mathbf{m}_1) \times \gamma(\mathbf{m}_1 + \mathbf{m}_2)$ matrix whose (i,j)th elements $\pi_{ij}^{(2)}$ are given by

$$\pi_{i,j}^{(2)} = \begin{cases} c_{\gamma^{-1}(k)}^{(2)}, \text{ if } \exists\, k \in \{1, 2, ..., \gamma(\mathbf{m}_2)\} \text{ such that } \gamma^{-1}(j) = \gamma^{-1}(i) + \gamma^{-1}(k), \\ \qquad 0, \text{ otherwise}, \end{cases} \tag{31}$$

and $\mathbf{m}_1 + \mathbf{m}_2$ is the componentwise addition of $\mathbf{m}_1$ and $\mathbf{m}_2$ in $\mathbb{N}^2$.

Consider, for example, $C_{(2,0)}^{(1)} = [c_{(0,0)}^{(1)}, c_{(1,0)}^{(1)}, c_{(0,1)}^{(1)}, c_{(2,0)}^{(1)}]$, and $C_{(0,1)}^{(2)} = [c_{(0,0)}^{(2)}, c_{(1,0)}^{(2)}, c_{(0,1)}^{(2)}]$. We compute $C_{\mathbf{m}_3}^{(3)} = C_{(2,0)}^{(1)} + C_{(2,0)}^{(1)} \circ C_{(0,1)}^{(2)}$.

In this case $\mathbf{m}_1 = (2,0)$, $\mathbf{m}_2 = (0,1)$, $\gamma(\mathbf{m}_1) = 4$, $\gamma(\mathbf{m}_2) = 3$ and $\mathbf{m}_3 = \mathbf{m}_1 + \mathbf{m}_2 = (2,1)$. For $j = 1$, by Eq. (31), since $\gamma^{-1}(j) = (0,0)$, we find $\pi_{1,1}^{(2)} = c_{(0,0)}^{(2)}$, $\pi_{i,1}^{(2)} = 0$, $i = 2,3,4$. If $j = 2$, since $\gamma^{-1}(j) = (1,0)$, we find $\pi_{1,2}^{(2)} = c_{(1,0)}^{(2)}$, $\pi_{2,2}^{(2)} = c_{(0,0)}^{(2)}$, $\pi_{i,2}^{(2)} = 0$, $i = 3,4$. In this manner we find

$$\Pi(C_{(0,1)}^{(2)}) = \begin{pmatrix} c_{(0,0)}^{(2)} & c_{(1,0)}^{(2)} & c_{(0,1)}^{(2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{(0,0)}^{(2)} & 0 & c_{(1,0)}^{(2)} & c_{(0,1)}^{(2)} & 0 & 0 & 0 \\ 0 & 0 & c_{(0,0)}^{(2)} & 0 & c_{(1,0)}^{(2)} & c_{(0,1)}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & c_{(0,0)}^{(2)} & 0 & 0 & c_{(1,0)}^{(2)} & c_{(0,1)}^{(2)} \end{pmatrix},$$

$$C^{(1)}_{(2,0)} \circ C^{(2)}_{(0,1)} = [c^{(1)}_{(0,0)}c^{(2)}_{(0,0)}, c^{(1)}_{(0,0)}c^{(2)}_{(1,0)} + c^{(1)}_{(1,0)}c^{(2)}_{(0,0)}, c^{(1)}_{(0,0)}c^{(2)}_{(0,1)} + c^{(1)}_{(0,1)}c^{(2)}_{(0,0)}, c^{(1)}_{(1,0)}c^{(2)}_{(1,0)}$$
$$+c^{(1)}_{(2,0)}c^{(2)}_{(0,0)}, c^{(1)}_{(1,0)}c^{(2)}_{(0,1)} + c^{(1)}_{(0,1)}c^{(2)}_{(1,0)}, c^{(1)}_{(0,1)}c^{(2)}_{(0,1)}, c^{(1)}_{(2,0)}c^{(2)}_{(1,0)}, c^{(1)}_{(2,0)}c^{(2)}_{(0,1)}],$$

and

$$C^{(3)}_{(2,1)} = [c^{(1)}_{(0,0)} + c^{(1)}_{(0,0)}c^{(2)}_{(0,0)}, c^{(1)}_{(1,0)} + c^{(1)}_{(0,0)}c^{(2)}_{(1,0)} + c^{(1)}_{(1,0)}c^{(2)}_{(0,0)}, c^{(1)}_{(0,1)} + c^{(1)}_{(0,0)}c^{(2)}_{(0,1)} + c^{(1)}_{(0,1)}c^{(2)}_{(0,0)},$$
$$c^{(1)}_{(2,0)} + c^{(1)}_{(1,0)}c^{(2)}_{(1,0)} + c^{(1)}_{(2,0)}c^{(2)}_{(0,0)}, c^{(1)}_{(1,0)}c^{(2)}_{(0,1)} + c^{(1)}_{(0,1)}c^{(2)}_{(1,0)}, c^{(1)}_{(0,1)}c^{(2)}_{(0,1)}, c^{(1)}_{(2,0)}c^{(2)}_{(1,0)}, c^{(1)}_{(2,0)}c^{(2)}_{(0,1)}].$$

Lemma 2. *The composition law defined by Eq. (30) on the subset of elements of the form in Eq. (28) from $\mathbb{R}^\infty$ is commutative and associative. Moreover if $P_3(\mathbf{x}) = P_1(\mathbf{x})P_2(\mathbf{x})$, then*

$$P_3(\mathbf{x}) = (C^{(1)}_{\mathbf{m}_1} \circ C^{(2)}_{\mathbf{m}_2})T_{\mathbf{m}_1+\mathbf{m}_2}(\mathbf{x}). \tag{32}$$

Proof. We define the linear mapping Φ from the real vector space of polynomial $\mathbb{R}[\mathbf{x}]$ to $\mathbb{R}^\infty$ given by

$$\Phi(P_1(\mathbf{x})) := C^{(1)}_{\mathbf{m}_1}, \tag{33}$$

where $P_1(\mathbf{x})$ and $C^{(1)}_{\mathbf{m}_1}$ are defined by Eq. (26). If $P_3(\mathbf{x}) = P_1(\mathbf{x})P_2(\mathbf{x})$, then by Eq. (30) we get $\Phi(P_3(\mathbf{x})) = C^{(1)}_{\mathbf{m}_1} \circ C^{(2)}_{\mathbf{m}_2}$. Hence, because the polynomial multiplication is commutative and associative, it follows the lemma. □

Now, based on these results, for a fixed t, we may denote

$$C^{(1)\circ t}_{\mathbf{m}_1} = \underbrace{C^{(1)}_{\mathbf{m}_1} \circ C^{(1)}_{\mathbf{m}_1} \circ \cdots \circ C^{(1)}_{\mathbf{m}_1}}_{t \text{ times}}.$$

By Lemma 2, by induction, it follows the following result:

Theorem 2. *Let Q be a polynomial given by Eq. (4) with $\mathbf{m} = \mathbf{d}_Q$. If P_i are polynomials such that $P_i(\mathbf{x}) = C^{(i)}_{\mathbf{m}_i}T^T_{\mathbf{m}_i}(\mathbf{x})$, $i = 1, ..., N$, then, for every $\mathbf{n} \geq \gamma(\mathbf{d}_Q)\mathbf{m} = (\gamma(\mathbf{d}_Q)m_1, \gamma(\mathbf{d}_Q)m_2)$, with $\mathbf{m} = \max\{\mathbf{m}_i,\ i = 1, 2, ..., N\}$, we can write in $\mathbb{R}^\infty$*

$$Q(P_1(\mathbf{x}), ..., P_N(\mathbf{x})) = G_\mathbf{n}T^T_\mathbf{n}(\mathbf{x}) \tag{34}$$

where $G_\mathbf{n}$ is a $1 \times \gamma(\mathbf{n})$ matrix given by

$$G_\mathbf{n} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{d}_Q} q_\mathbf{i}\mathbf{Z}^{\circ\mathbf{i}}, \tag{35}$$

with $\mathbf{Z} = (Z_{1,\mathbf{m}_1}, Z_{2,\mathbf{m}_2}, ..., Z_{N,\mathbf{m}_N})$, $Z_{j,\mathbf{m}_j} = C^{(j)}_{\mathbf{m}_j}$, $\mathbf{Z}^{\circ\mathbf{i}} = Z^{\circ i_1}_{1,\mathbf{m}_1} \circ ... \circ Z^{\circ i_N}_{N,\mathbf{m}_N}$.

3.4. Operational matrices for particular evaluations

Let P be a polynomial given by Eq. (17). If $x_1 = a$ is a constant, then it follows that

$$P(a, x_2) = C_{\mathbf{m}} S_{\mathbf{m}}^{(1)}(a) T_{\mathbf{m}}^{(2)}(x_2), \tag{36}$$

where $S_{\mathbf{m}}^{(1)}(a)$ are $\gamma(\mathbf{m}) \times (\gamma(\mathbf{m}) - \theta((1,0), \mathbf{m}))$ matrices whose (i,j)th elements are given by

$$s_{i,j}^{\mathbf{m},1,a} = \delta_{j-1,\gamma_2^{-1}(i)} a^{\gamma_1^{-1}(i)}, \tag{37}$$

where we define $0^0 = 1$, and

$$T_{\mathbf{m}}^{(2)}(x_2) = [1, x_2, x_2^2, ..., x_2^{\gamma(\mathbf{m})-\theta((1,0),\mathbf{m})}].$$

Similarly, we get

$$P(x_1, a) = C_{\mathbf{m}} S_{\mathbf{m}}^{(2)}(a) T_{\mathbf{m}}^{(1)}(x_1), \tag{38}$$

where $S_{\mathbf{m}}^{(2)}(a)$ are $\gamma(\mathbf{m}) \times (\gamma(\mathbf{m}) - \theta((0,1), \mathbf{m}))$ matrices whose (i,j)th elements are given by

$$s_{i,j}^{\mathbf{m},2,a} = \delta_{j-1,\gamma_1^{-1}(i)} a^{\gamma_2^{-1}(i)}, \tag{39}$$

and

$$T_{\mathbf{m}}^{(1)}(x_1) = [1, x_1, x_1^2, ..., x_1^{\gamma(\mathbf{m})-\theta((0,1),\mathbf{m})}].$$

4. Polynomial approximation of solution for polynomial partial differential equations

4.1. Description of the method

Consider Ω a bounded domain in $\mathbb{R}^2$ with boundary Γ which is a piecewise smooth curve. Suppose that the problem from Eq. (5) with the boundary conditions in Eq. (6) has an unique solution $u \in \mathcal{C}^{|\mathbf{n}|}(\overline{\Omega})$.

Approximation theorem of Weierstrass admits an extension to continuous functions of several variables simultaneously with their partial derivatives. Thus, for example, in the case used in this paper, when Ω is the interior of a rectangle $[a, b] \times [c, d]$, the following result holds (see, for example, [23], p. 68 and [24], Section 3.1 for the general case):

Theorem 3. *A function $f(x_1, x_2)$ which, along with its partial derivatives up to the k-th order, is continuous in $\overline{\Omega}$, may be uniformly approximated by polynomials $P(x_1, x_2)$ in such a way that the derivatives of f up to the k-th order are also approximated uniformly by the corresponding derivatives of the polynomials.*

By using this theorem, for every $\varepsilon > 0$, there exists a polynomial $P_{\mathbf{m}}$ given by Eq. (2), depending on $\varepsilon > 0$, such that, for every $\mathbf{j} \le \mathbf{n}$,

$$\|D^{\mathbf{j}}u(x_1, x_2) - D^{\mathbf{j}}P_{\mathbf{m}}(x_1, x_2)\|_{\overline{\Omega},\infty} < \varepsilon. \tag{40}$$

Similarly we find the polynomials $R_{\mathbf{m}^{(j)}}$, $j = 0, 1, ..., N_\Gamma$ such that

$$\|f(x_1, x_2) - R_{\mathbf{m}^{(0)}}(x_1, x_2)\|_{\overline{\Omega},\infty} < \varepsilon, \tag{41}$$

and

$$\|f_j(x_1, x_2) - R_{\mathbf{m}^{(j)}}(x_1, x_2)\|_{\overline{\Gamma}_j,\infty} < \varepsilon,\ j = 1, ..., N_\Gamma. \tag{42}$$

Then, by Eq. (17), for $j = 0, 1, ..., N_\Gamma$, we may write

$$R_{\mathbf{m}^{(j)}}(\mathbf{x}) = C^{(j)}_{\mathbf{m}^{(j)}} T^T_{\mathbf{m}^{(j)}}(\mathbf{x}). \tag{43}$$

In order to approximate the solution u of the problem from Eq. (5) with the boundary conditions in Eq. (6) we replace u, its partial derivatives, and f, in Eq. (5) with $P_{\mathbf{m}}$, its partial derivatives, and $R_{\mathbf{m}^{(0)}}$, respectively. If $Q(\mathbf{z})$ has the form in Eq. (4), thus, by Eqs. (17), (22) and (35) we get, in $\mathbb{R}^\infty$,

$$Q\left(x_1, x_2, P_{\mathbf{m}}, \frac{\partial P_{\mathbf{m}}}{\partial x_1}, \frac{\partial P_{\mathbf{m}}}{\partial x_2}, \frac{\partial^2 P_{\mathbf{m}}}{\partial x_1^2}, ..., \frac{\partial P_{\mathbf{m}}^{|\mathbf{n}|}}{\partial x_1^{n_1} \partial x_2^{n_2}}\right) = G_{\mathbf{m}'} T^T_{\mathbf{m}'}(\mathbf{x}), \tag{44}$$

where

$$G_{\mathbf{m}'} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{d}_Q} q_{\mathbf{i}} \mathbf{Z}^{\circ \mathbf{i}}, \tag{45}$$

$\mathbf{Z} = (Z_{1,(1,0)}, Z_{2,(0,1)}, ..., Z_{N,\gamma^{-1}(\theta(\mathbf{n},\mathbf{m}))})$, $N = \gamma(\mathbf{n}) + 2$, $Z_{1,(1,0)} = [0, 1]$, $Z_{2,(0,1)} = [0, 0, 1]$, $Z_{3,\mathbf{m}} = C_{\mathbf{m}}$, $Z_{4,\gamma^{-1}(\theta((1,0),\mathbf{m}))} = C_{\mathbf{m}} D^{(1,0),\mathbf{m}}$, $Z_{5,\gamma^{-1}(\theta((0,1),\mathbf{m}))} = C_{\mathbf{m}} D^{(0,1),\mathbf{m}}$, $Z_{6,\gamma^{-1}(\theta((2,0),\mathbf{m})} = C_{\mathbf{m}} D^{(2,0),\mathbf{m}}$, ..., $Z_{N,\gamma^{-1}(\theta(\mathbf{n},\mathbf{m}))} = C_{\mathbf{m}} D^{\mathbf{n},\mathbf{m}}$, and $\mathbf{Z}^{\circ \mathbf{i}} = Z^{\circ i_1}_{1,(1,0)} \circ ... \circ Z^{\circ i_N}_{N,\gamma^{-1}(\theta(\mathbf{n},\mathbf{m}))}$ (see Section 3.3). We may consider Eq. (44) in $\mathbb{R}^{\gamma(\mathbf{m}')}$ and $\mathbf{m}' \ge \mathbf{m}$, where $\mathbf{m}$ is large enough (see Section 4.2). By using Eqs. (5) and (44) we find

$$G_{\mathbf{m}'} = C_{\mathbf{m}'^{(0)}}, \tag{46}$$

where $C_{\mathbf{m}'^{(0)}}$ is equal to $C_{\mathbf{m}^{(0)}}$ in $\mathbb{R}^\infty$. Thus Eq. (46) is an algebraic system of equations with respect to the unknowns $c_{\mathbf{j}}$. In Eq. (46) a necessary condition is

$$\mathbf{m}' \ge \alpha_0(\mathbf{m}), \tag{47}$$

with $\alpha_0(\mathbf{m})$ defined by Eq. (52), $G_{\mathbf{m}'} = (g_{(0,0)}, g_{(1,0)}, ..., g_{(m_1', m_2')})$, and $C^{(0)}_{\mathbf{m}'}$ is obtained from $C^{(0)}_{\mathbf{m}^{(0)}}$ by adding $\gamma(\mathbf{m}') - \gamma(\mathbf{m}^{(0)})$ elements equal to zero (see Example 1 for the details).

Assume $\Omega = [a,b] \times [c,d]$. Then by Eqs. (6), (30), (36) and (38) we obtain, for every $j = 1, 2, ..., N_\Gamma$, the algebraic system of equations

$$H_{j,\mathbf{m}'} = C^{(j)}_{\mathbf{m}'} \tag{48}$$

with respect to the unknowns $c_{\mathbf{j}}$. For example, if $\Gamma_1 = \{(a, x_2) : x_2 \in [c,d]\}$, $Q_1(\mathbf{z}) = \sum\limits_{\mathbf{i}=\mathbf{0}}^{\mathbf{d}_{Q_1}} q_{1,\mathbf{i}}\mathbf{z}^{\mathbf{i}}$, by Eqs. (17), (22) and (35) we get, in $\mathbb{R}^\infty$,

$$Q_1\left(x_1, x_2, P_{\mathbf{m}}, \frac{\partial P_{\mathbf{m}}}{\partial x_1}, \frac{\partial P_{\mathbf{m}}}{\partial x_2}, \frac{\partial^2 P_{\mathbf{m}}}{\partial x_1^2}, ..., \frac{\partial P^{|\mathbf{n}|}_{\mathbf{m}}}{\partial x_1^{n_1}\partial x_2^{n_2}}\right) = G_{1,\mathbf{m}'}T^T_{\mathbf{m}'}(\mathbf{x}), \tag{49}$$

where

$$G_{1,\mathbf{m}'} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{d}_{Q_1}} q_{1,\mathbf{i}}\mathbf{Z}_1^{\circ\mathbf{i}},$$

$\mathbf{Z}_1 = (Z_{1,1,(1,0)}, Z_{1,2,(0,1)}, ..., Z_{1,N_1,\gamma^{-1}(\theta(\mathbf{n}^{(1)},\mathbf{m}))})$, $N_1 = \gamma(\mathbf{n}^{(1)})+2$, $Z_{1,1,(1,0)} = [1,0]$, $Z_{1,2,(0,1)} = [0,0,1]$, $Z_{1,3,\mathbf{m}} = C_{\mathbf{m}}$, $Z_{1,4,\gamma^{-1}(\theta((1,0),\mathbf{m}))} = C_{\mathbf{m}}D^{(1,0),\mathbf{m}}$, $Z_{1,5,\gamma^{-1}(\theta((0,1),\mathbf{m}))} = C_{\mathbf{m}}D^{(0,1),\mathbf{m}}$, $Z_{1,6,\gamma^{-1}(\theta((2,0),\mathbf{m}))} = C_{\mathbf{m}}D^{(2,0),\mathbf{m}}$, ..., $Z_{1,N_1,\gamma^{-1}(\theta(\mathbf{n}^{(1)},\mathbf{m}))} = C_{\mathbf{m}}D^{\mathbf{n}^{(1)},\mathbf{m}}$, $\mathbf{Z}^{\circ\mathbf{i}} = Z^{\circ i_1}_{1,(1,0)} \circ ... \circ Z^{\circ i_{N_1}}_{N_1,\gamma^{-1}(\theta(\mathbf{n}^{(1)},\mathbf{m}))}$. Then, by Eqs. (6), (36) and (49), it follows that

$$G_{1,\mathbf{m}'}S^{(1)}_{\mathbf{m}'}(a)T^{(2)T}_{\mathbf{m}'}(x_2) = C^{(1)}_{\mathbf{m}^{(1)}}S^{(1)}_{\mathbf{m}^{(1)}}(a)T^{(2)T}_{\mathbf{m}^{(1)}}(x_2). \tag{50}$$

Hence we get

$$H_{1,\mathbf{m}'} = C^{(1)}_{\mathbf{m}'}S^{(1)}_{\mathbf{m}'}(a),$$

where $H_{1,\mathbf{m}'} = G_{1,\mathbf{m}'}S^{(1)}_{\mathbf{m}'}(a)$.

If $\mathbf{g}_{1,\mathbf{m}'} = \mathbf{d}_{G_{1,\mathbf{m}'}}$, then by Section 3.4 a necessary condition is

$$\alpha_1(\mathbf{g}_{1,\mathbf{m}'}) \geq \alpha_1(\mathbf{m}^{(1)}) = \deg(f_1(a, x_2)), \tag{51}$$

where $\alpha_1(\mathbf{m}) = \gamma(\mathbf{m}) - \theta((1,0),\mathbf{m})$ (see Eq. (54)).

Remark 1. Suppose

$$G_{\mathbf{m}'} = (g^{\mathbf{m}}_{(0,0)}, ..., g^{\mathbf{m}}_{(m_1',m_2')}),\ H_{j,\mathbf{m}'} = (h^{(j,\mathbf{m})}_{(0,0)}, ..., h^{(j,\mathbf{m})}_{(m_1',m_2')}).$$

We must find an approximate solution of the system which contains the equations which follows from Eqs. (46) and (48). Thus it is enough to find $c_{\mathbf{i}}$ such that

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}'} (g^{\mathbf{m}}_{\mathbf{k}})^2 + \sum_{j=1}^{N_\Gamma}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}'} \left(h^{(j,\mathbf{m})}_{\mathbf{k}}\right)^2$$

is small enough.

When the boundary value problem has an analytic solution $u = u(x_1, x_2)$, then $P_{\mathbf{m}}$ may be taken a partial sum of u and, for every $j = 0, 1, ..., N_\Gamma$, there exist maximum values $\mathbf{k}_j(\mathbf{m}) \leq \mathbf{m}'$, such that, for all $\mathbf{p} \geq \mathbf{m}$, it follows that $g_{\mathbf{k}}^{(\mathbf{p})} = g_{\mathbf{k}}^{(\mathbf{m})}$, for all $\mathbf{k} \leq \mathbf{k}_0(\mathbf{m})$ and $h_{\mathbf{k}}^{(j,\mathbf{p})} = h_{\mathbf{k}}^{(j,\mathbf{m})}$, for all $\mathbf{k} \leq \mathbf{k}_j(\mathbf{m})$, $j = 1, ..., N_\Gamma$. Thus the system which follows from Eqs. (46) and (48) can be replaced by

$$\begin{cases} g_{\mathbf{k}}^{(\mathbf{m})} = \mathbf{0}, & \mathbf{k} \leq \mathbf{k}_0(\mathbf{m}) \\ h_{\mathbf{k}}^{(j,\mathbf{m})} = \mathbf{0}, & \mathbf{k} \leq \mathbf{k}_j(\mathbf{m}),\ j = 1, ..., N_\Gamma \end{cases},$$

and $c_{\mathbf{i}}$, $\mathbf{i} \leq \mathbf{m}$ are the solutions of this system. Then by Eq. (2) we get the polynomial $P_{\mathbf{m}}$ which approximates the solution of the BVP problem from Eq. (5) with the boundary conditions in Eq. (6).

4.2. Necessary conditions for the choice of $\mathbf{m}$

We must choose $\mathbf{m}$ large enough to obtain a suitable approximation of the function $u(x_1, x_2)$. By Eq. (45) it follows that a necessary condition is

$$\alpha_0(\mathbf{m}) = \max\{i_1(1,0) + i_2(0,1) + i_3\mathbf{m} + ... + i_N\gamma^{-1}(\theta(\mathbf{n}, \mathbf{m})) : \mathbf{i} \leq \mathbf{d}_Q,\ \mathbf{q}_{\mathbf{i}} \neq \mathbf{0}\} \geq \mathbf{m}^{(0)}, \qquad (52)$$

with respect to the order defined on $\mathbb{N}^2$ by Eq. (1).

Similarly, if $j = 1, 2, ..., N_\Gamma$,

$$G_{j,\mathbf{m}} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{d}_{Q_j}} q_{j,\mathbf{i}} \mathbf{Z}_j^{\circ \mathbf{i}},$$

where $\mathbf{Z}_j = (Z_{j,1,\mathbf{m}_{j,1}}, ..., Z_{j,N_j,\mathbf{m}_{j,N_j}})$. If $\mathbf{g}_{j,\mathbf{m}'} = \mathbf{d}_{G_{j,\mathbf{m}'}}$, then by Section 3.4 and Eq. (51) we obtain that

$$\alpha_j(\mathbf{g}_{j,\mathbf{m}'}) \geq \alpha_j(\mathbf{m}^{(j)}), \qquad (53)$$

where

$$\alpha_j(\mathbf{m}) = \begin{cases} \gamma(\mathbf{m}) - \theta((1,0), \mathbf{m}),\ \Gamma_j = \{(K, x_2), x_2 \in [c,d]\},\ K \in \{a, b\} \\ \gamma(\mathbf{m}) - \theta((0,1), \mathbf{m}),\ \Gamma_j = \{(x_1, K), x_1 \in [a,b]\},\ K \in \{c, d\}. \end{cases} \qquad (54)$$

4.3. Convergence and error estimate

By using the notations from Eqs. (5) and (6), we denote for simplicity

$$Q(\mathbf{u}) = Q\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1^2}, ..., \frac{\partial u^{|\mathbf{n}|}}{\partial x_1^{n_1} \partial x_2^{n_2}}\right).$$

Suppose that the problem from Eq. (5) with the boundary conditions in Eq. (6) has a unique solution u, for every f with f_j, given by Eq. (6), $j = 1, 2, ..., N_\Gamma$. If $\varepsilon > 0$ is fixed, we choose $R_{\mathbf{m}^{(j)}}$,

$j = 0, 1, ..., N_\Gamma$ such Eqs. (41) and (42) hold for $\frac{\varepsilon}{2}$. By Theorem 3, for every $\varepsilon' > 0$ there exist polynomials $V_{\varepsilon'}$ such that $\|D^{\mathbf{k}}u - D^{\mathbf{k}}V_{\varepsilon'}\|_{\overline{\Omega},\infty} < \varepsilon'$, for every $\mathbf{k} \leq \max\{\mathbf{n}, \mathbf{n}^{(j)},\ j = 1, 2, ..., N_\Gamma\}$. Hence, if u is the solution of the problem from Eq. (5) with the boundary conditions in Eq. (6), there exist polynomials $W_{\varepsilon'}$, for ε' small enough, such that $\|Q(\mathbf{u}) - Q(\mathbf{W}_{\varepsilon'})\|_{\overline{\Omega},\infty} < \frac{\varepsilon}{2}$, and $\|Q_j(\mathbf{u}) - Q_j(\mathbf{W}_{\varepsilon'})\|_{\overline{\Gamma_j},\infty} < \frac{\varepsilon}{2}$, for $j = 1, 2, ..., N_\Gamma$. Thus, by Eqs. (5), (6), (41) and (42), it follows that we can choose a polynomial $P_{\mathbf{m}}$ such that

$$\|Q(\mathbf{P_m}) - R_{\mathbf{m}^{(0)}}\|_{\overline{\Omega},\infty} < \varepsilon, \|Q_j(\mathbf{P_m}) - R_{\mathbf{m}^{(j)}}\|_{\overline{\Gamma_j},\infty} < \varepsilon, j = 1, 2, ..., N_\Gamma. \tag{55}$$

Now, if we find a polynomial satisfying Eq. (55), which is the goal of the presented method, then

$$\begin{aligned} d(u, P_{\mathbf{m}}) = \lambda_0 \|Q(\mathbf{u}) - Q(\mathbf{P_m})\|_{\overline{\Omega},\infty} &+ \sum_{j=1}^{N_\Gamma} \lambda_j \|Q_j(\mathbf{u}) - Q_j(\mathbf{P_m})\|_{\overline{\Gamma_j},\infty} \\ &\leq Est < 2\varepsilon, \end{aligned}$$

where

$$\begin{aligned} Est = \lambda_0 &\left(\|f - R_{\mathbf{m}^{(0)}}\|_{\overline{\Omega},\infty} + \|R_{\mathbf{m}^{(0)}} - Q(\mathbf{P_m})\|_{\overline{\Omega},\infty}\right) \\ + \sum_{j=1}^{N_\Gamma} \lambda_j &\left(\|f_j - R_{\mathbf{m}^{(j)}}\|_{\overline{\Gamma_j},\infty} + \|R_{\mathbf{m}^{(j)}} - Q_j(\mathbf{P_m})\|_{\overline{\Gamma_j},\infty}\right). \end{aligned} \tag{56}$$

Thus $P_{\mathbf{m}}$ is an approximation of u and the error can be estimated by evaluation of the expression Est.

5. Applications

In this section five examples are given to prove the performance and efficiency of the method for solving polynomial partial differential equations in two independent variables.

5.1. Example 1

Consider the following partial differential equation from [25]:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \left(\frac{\partial u}{\partial x_2}\right)^2 = 2x_2 + x_1^4,\ (x_1, x_2) \in [0,1] \times [0,1], \tag{57}$$

with the boundary conditions

$$u(x_1, 0) = \alpha x_1,\ u(1, x_2) = \alpha + x_2,\ u(x_1, 1) = \alpha x_1 + x_1^2,\ u(0, x_2) = 0, \tag{58}$$

where $\alpha \in \mathbb{R}$ is a constant.

The exact solution for the problem from Eq. (57) with the boundary conditions in Eq. (58) is $u(x_1, x_2) = \alpha x_1 + x_1^2 x_2$.

Here the polynomial $Q = Q(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) = z_5^2 + z_6 + z_8$, $f(x_1, x_2) = x_1^4 + 2x_2$, $N = 8$, $\mathbf{d}_Q = (0, 0, 0, 0, 2, 0, 0, 0)$ and $\deg(Q) = 2$.

By the boundary conditions we see that $N'_\Gamma = N_\Gamma = 4$, $a = c = 0$, $b = d = 1$, $\Gamma_1 = \{(x_1, 0) : x_1 \in [0, 1]\}$, $\Gamma_2 = \{(1, x_2) : x_2 \in [0, 1]\}$, $\Gamma_3 = \{(x_1, 1) : x_1 \in [0, 1]\}$, $\Gamma_4 = \{(0, x_2) : x_2 \in [0, 1]\}$, $Q_i(z_1, z_2, z_3) = z_3$, $i = 1, 2, 3, 4$.

If we take, for example, $\mathbf{m} = (2, 1)$, $C_{(2,1)} = [c_{(0,0)}, c_{(1,0)}, c_{(0,1)}, c_{(2,0)}, c_{(1,1)}, c_{(0,2)}, c_{(3,0)}, c_{(2,1)}]$, and by Eqs. (11), (21), (23) and Proposition 1 we obtain $\gamma(2, 1) = 8$, $\theta((0, 1), (2, 1)) = 8 - p^{(*,0)}(8) = 8 - [\frac{\sqrt{57}+1}{2}] = 8 - 4 = 4$, $\gamma^{-1}(\theta((0, 1), (2, 1))) = \gamma^{-1}(4) = (2, 0)$. Similarly we get $\gamma^{-1}(\theta((2, 0), (2, 1))) = \gamma^{-1}(3) = (0, 1)$, $\gamma^{-1}(\theta((0, 2), (2, 1))) = (0, 0)$. Hence, by Eq. (52) it follows that $\alpha_0(\mathbf{m}) = \max\{2(2, 0), (0, 1), (0, 0)\} = (4, 0)$. Since $\mathbf{m}^{(0)} = (4, 0)$, by choosing $\mathbf{m}' = (4, 0)$ we find that Eqs. (47) and (52) hold.

Since $\mathbf{g}_{j,\mathbf{m}'} = \mathbf{m}'$, by Eq. (54) $\alpha_1(\mathbf{m}') = \alpha_3(\mathbf{m}') = \gamma(4, 0) - \theta((0, 1), (4, 0)) = 11 - 11 + p^{(*,0)}(11) = 5$, $\alpha_1(\mathbf{m}^{(1)}) = \deg(f_1(x_1, 0)) \leq 1$, $\alpha_3(\mathbf{m}^{(3)}) = \deg(f_3(x_1, 1)) = 2$ and Eq. (53) holds for $j = 1, 3$. Similarly $\alpha_2(\mathbf{m}') = \alpha_4(\mathbf{m}') = \gamma(4, 0) - \theta((1, 0), (4, 0)) = 11 - 11 + p^{(0,*)}(11) = 4$, $\alpha_2(\mathbf{m}^{(2)}) = \deg(f_2(1, x_2)) = 1$, $\alpha_4(\mathbf{m}^{(4)}) = \deg(f_4(0, x_2)) = -\infty$ and Eq. (53) holds for $j = 2, 4$.

By Eq. (45) we get

$$G_{\mathbf{m}'} = Z^{\circ 2}_{5,\gamma^{-1}(\theta((0,1),(2,1)))} + Z_{6,\gamma^{-1}(\theta((2,0),(2,1)))} + Z_{8,\gamma^{-1}(\theta((0,2),(2,1)))},$$

where $Z_{5,\gamma^{-1}(\theta((0,1),(2,1)))} = Z_{5,(2,0)} = C_{(2,1)} D^{(0,1),(2,1)}$, $Z_{6,\gamma^{-1}(\theta((2,0),(2,1)))} = Z_{6,(0,1)} = C_{(2,1)} D^{(2,0),(2,1)}$ and $Z_{8,\gamma^{-1}(\theta((0,2),(2,1)))} = Z_{8,(0,0)} = C_{(2,1)} D^{(0,2),(2,1)}$. Then, by Eq. (24), we find

$$D^{(0,1),(2,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Z_{5,(2,0)} = C_{(2,1)} D^{(0,1),(2,1)} = (c_{(0,1)}, c_{(1,1)}, 2c_{(0,2)}, c_{(2,1)}),$$

To find $Z^{\circ 2}_{5,(2,0)} = Z_{5,(2,0)} \circ Z_{5,(2,0)}$, by using Eq. (31) we calculate

$$\Pi(Z_{5,(2,0)}) = \begin{pmatrix} c_{(0,1)} & c_{(1,1)} & 2c_{(0,2)} & c_{(2,1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{(0,1)} & 0 & c_{(1,1)} & 2c_{(0,2)} & 0 & c_{(2,1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{(0,1)} & 0 & c_{(1,1)} & 2c_{(0,2)} & 0 & c_{(2,1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{(0,1)} & 0 & 0 & c_{(1,1)} & 2c_{(0,2)} & 0 & 0 & c_{(2,1)} \end{pmatrix}.$$

Thus

$$Z^{\circ 2}_{5,(2,0)} = [c^2_{(0,1)}, 2c_{(0,1)}c_{(1,1)}, 4c_{(0,1)}c_{(0,2)}, 2c_{(0,1)}c_{(2,1)}$$

$$+c^2_{(1,1)}, 4c_{(1,1)}c_{(0,2)}, 4c^2_{(0,2)}, 2c_{(1,1)}c_{(2,1)}, 4c_{(0,2)}c_{(2,1)}, 0, 0, c^2_{(2,1)}].$$

Similarly we get

$$Z_{6,(0,1)} = C_{(2,1)}D^{(2,0),(2,1)} = C_{(2,1)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} = [2c_{(2,0)}, 6c_{(3,0)}, 2c_{(2,1)}],$$

$$Z_{8,(0,0)} = C_{(2,1)}D^{(0,2),(2,1)} = C_{(2,1)} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} = [2c_{(0,2)}].$$

Hence, because $C^{(0)}_{\mathbf{m}^{(0)}} = [0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 1]$, the system in Eq. (46) has the form

$$\Big[2c_{(2,0)} + 2c_{(0,2)} + c^2_{(0,1)}, 6c_{(3,0)} + 2c_{(0,1)}c_{(1,1)}, 2c_{(2,1)} + 4c_{(0,1)}c_{(0,2)}, 2c_{(0,1)}c_{(2,1)} + c^2_{(1,1)},$$

$$4c_{(1,1)}c_{(0,2)}, 4c^2_{(0,2)}, 2c_{(1,1)}c_{(2,1)}, 4c_{(0,2)}c_{(2,1)}, c^2_{(2,1)}\Big] = [0, 0, 2, 0, 0, 0, 0, 0, 1].$$

From the boundary condition $u(0, x_2) = 0$, that is $Q_4(x_1, x_2, u(x_1, x_2)) = 0$, for $(x_1, x_2) \in \Gamma_4$, by using Eq. (36), it follows that

$$G_{4,\mathbf{m}'} = C_{\mathbf{m}},\ H_{4,\mathbf{m}'} = C_{\mathbf{m}}S^{(1)}_{\mathbf{m}}(0).$$

Since $\mathbf{m} = (2,1)$, $\gamma(\mathbf{m}) = 8$, $\theta((1,0),(2,1)) = 5$, by Eq. (37), we get

$$S^{(1)}_{(2,1)}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$H_{4,\mathbf{m}'} = [c_{(0,0)}, c_{(0,1)}, c_{(0,2)}].$$

Thus the system in Eq. (48), for $j = 4$, has the form

$$[c_{(0,0)}, c_{(0,1)}, c_{(0,2)}] = [0, 0, 0].$$

Similarly, from the boundary condition $u(1, x_2) = \alpha + x_2$, it follows that

$$G_{2,\mathbf{m}'} = C_{\mathbf{m}},\ H_{2,\mathbf{m}'} = C_{\mathbf{m}} S^{(1)}_{\mathbf{m}}(1),$$

$$S^{(1)}_{(2,1)}(1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ H_{2,\mathbf{m}'} = [c_{(0,0)} + c_{(1,0)} + c_{(2,0)}, c_{(0,1)} + c_{(1,1)} + c_{(2,1)}, c_{(0,2)}],$$

the system in Eq. (48), for $j = 2$, has the form

$$[c_{(0,0)} + c_{(1,0)} + c_{(2,0)}, c_{(0,1)} + c_{(1,1)} + c_{(2,1)}, c_{(0,2)}] = [\alpha, 1, 0].$$

Then, by $u(x_1, 0) = \alpha x_1$ and $u(x_1, 1) = \alpha x_1 + x_1^2$, by using Eq. (38), we get

$$[c_{(0,0)}, c_{(1,0)}, c_{(2,0)}, c_{(3,0)}] = [0, \alpha, 0, 0]$$

$$[c_{(0,0)} + c_{(0,1)} + c_{(0,2)}, c_{(1,0)} + c_{(1,1)}, c_{(2,0)} + c_{(2,1)}, c_{(3,0)}] = [0, \alpha, 1, 0],$$

respectively. By solving the system we find $u(x_1, x_2) = \alpha x_1 + x_1^2 x_2$, which is the exact solution of the problem.

5.2. Example 2

Consider the following nonlinear equation from [26]:

$$u(x_1,x_2)\frac{\partial^2 u}{\partial x_1^2}(x_1,x_2) - \frac{\partial^2 u}{\partial x_2^2}(x_1,x_2) + \left(\frac{\partial u}{\partial x_1}(x_1,x_2)\right)^2 = 0,\ (x_1,x_2) \in (0,+\infty)\times(0,+\infty), \quad (59)$$

subject to the initial conditions

$$u(x_1,0) = x_1^2,\ \frac{\partial u}{\partial x_2}(x_1,0) = -2x_1^2. \quad (60)$$

The exact solution of this problem is $u(x_1,x_2) = \left(\frac{x_1}{x_2+1}\right)^2$.

As in [26] we shall approximate the solution for $x_1, x_2 \in [0,1]$. In this case $Q = Q(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) = z_3 z_6 + z_4^2 - z_8$, $f(x_1,x_2) = 0$, $\mathbf{d}_Q = (0,0,1,0,0,1,0,0)$ and $\deg(Q) = 2$. By Eq. (52) we may choose, for example, $\mathbf{m} = (2,1)$ and by Eq. (47), $\mathbf{m}' = (2,2)$. Then, by Eq. (45), it follows that

$$G_{\mathbf{m}'} = Z_{3,(2,1)} \circ Z_{6,\gamma^{-1}(\theta((2,0),(2,1)))} + Z^{\circ 2}_{4,\gamma^{-1}(\theta((1,0),(2,1)))} - Z_{8,\gamma^{-1}(\theta((0,2),(2,1)))},$$

where, as in Example 1, $Z_{3,(2,1)} = C_{(2,1)}$, $Z_{6,\gamma^{-1}(\theta((2,0),(2,1)))} = Z_{6,(0,1)} = C_{(2,1)}D^{(2,0),(2,1)} = [2c_{(2,0)}, 6c_{(3,0)}, 2c_{(2,1)}]$, $Z_{4,\gamma^{-1}(\theta((1,0),(2,1)))} = Z_{4,(1,1)} = C_{(2,1)}D^{(1,0),(2,1)} = [c_{(1,0)}, 2c_{(2,0)}, c_{(1,1)}, 3c_{(3,0)}, 2c_{(2,1)}]$ and $Z_{8,\gamma^{-1}(\theta((0,2),(2,1)))} = Z_{8,(0,0)} = C_{(2,1)}D^{(0,2),(2,1)} = [2c_{(0,2)}]$. Then, by Eq. (31),

$$\Pi(Z_{6,(0,1)})$$

$$= \begin{pmatrix}
2c_{(2,0)} & 6c_{(3,0)} & 2c_{(2,1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2c_{(2,0)} & 0 & 6c_{(3,0)} & 2c_{(2,1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2c_{(2,0)} & 0 & 6c_{(3,0)} & 2c_{(2,1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2c_{(2,0)} & 0 & 0 & 6c_{(3,0)} & 2c_{(2,1)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2c_{(2,0)} & 0 & 0 & 6c_{(3,0)} & 2c_{(2,1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2c_{(2,0)} & 0 & 0 & 6c_{(3,0)} & 2c_{(2,1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2c_{(2,0)} & 0 & 0 & 0 & 6c_{(3,0)} & 2c_{(2,1)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_{(2,0)} & 0 & 0 & 0 & 6c_{(3,0)} & 2c_{(2,1)}
\end{pmatrix}.$$

Hence we get $Z_{3,(2,1)} \circ Z_{6,(0,1)} = [2c_{(0,0)}c_{(2,0)}, 6c_{(0,0)}c_{(3,0)} + 2c_{(1,0)}c_{(2,0)},\ 2c_{(0,0)}c_{(2,1)} + 2c_{(0,1)}c_{(2,0)},$ $6c_{(1,0)}c_{(3,0)} + 2c^2_{(2,0)}, 2c_{(1,0)}c_{(2,1)} + 6c_{(3,0)}c_{(0,1)} + 2c_{(1,1)}c_{(2,0)}, 2c_{(0,1)}c_{(2,1)} + 2c_{(2,0)}c_{(0,2)},\ 8c_{(2,0)}c_{(3,0)},$

$4c_{(2,0)}c_{(2,1)} + 6c_{(1,1)}c_{(3,0)}, 2c_{(1,1)}c_{(2,1)} + 6c_{(0,2)}c_{(3,0)}, 2c_{(0,2)}c_{(2,1)},\ 6c^2_{(3,0)}, 8c_{(3,0)}c_{(2,1)},\ 2c^2_{(2,1)}]$

Similarly it follows

$$\Pi(Z_{4,(1,1)})$$

$$= \begin{pmatrix} c_{(1,0)} & 2c_{(2,0)} & c_{(1,1)} & 3c_{(3,0)} & 2c_{(2,1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{(1,0)} & 0 & 2c_{(2,0)} & c_{(1,1)} & 0 & 3c_{(3,0)} & 2c_{(2,1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{(1,0)} & 0 & 2c_{(2,0)} & c_{(1,1)} & 0 & 3c_{(3,0)} & 2c_{(2,1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{(1,0)} & 0 & 0 & 2c_{(2,0)} & c_{(1,1)} & 0 & 0 & 3c_{(3,0)} & 2c_{(2,1)} & 0 \\ 0 & 0 & 0 & 0 & c_{(1,0)} & 0 & 0 & 2c_{(2,0)} & c_{(1,1)} & 0 & 0 & 3c_{(3,0)} & 2c_{(2,1)} \end{pmatrix}$$

and $Z^{\circ 2}_{4,(1,1)} = [c^2_{(1,0)}, 4c_{(1,0)}c_{(2,0)}, 2c_{(1,0)}c_{(1,1)}, 6c_{(1,0)}c_{(3,0)} + 4c^2_{(2,0)}, 4c_{(1,0)}c_{(2,1)} + 4c_{(2,0)}c_{(1,1)},\ c^2_{(1,1)},$ $12c_{(2,0)}\ c_{(3,0)},\ 8c_{(2,0)}c_{(2,1)} + 6c_{(1,1)}c_{(3,0)}, 4c_{(1,1)}c_{(2,1)}, 0, 9c^2_{(3,0)}, 12c_{(3,0)}c_{(1,1)}, 4c^2_{(2,1)}]$. Thus we get the system in Eq. (46) .

Since we consider $x_1, x_2 \in [0,1]$ it follows that $N'_\Gamma = 4$, $a = c = 0$, $b = d = 1$, $\Gamma'_1 = \{(x_1, 0) : x_1 \in [0,1]\}$, $\Gamma'_2 = \{(1, x_2) : x_2 \in [0,1]\}$, $\Gamma'_3 = \{(x_1, 1) : x_1 \in [0,1]\}$, $\Gamma'_4 = \{(0, x_2) : x_2 \in [0,1]\}$. By the boundary conditions we see that $N_\Gamma = 2$, $\Gamma_1 = \Gamma_2 = \Gamma'_1$, $Q_1(z_1, z_2, z_3) = z_3$, $Q_2(z_1, z_2, z_3, z_4) = z_5$, $f_1(x_1, x_2) = x_1^2$, $f_2(x_1, x_2) = -2x_1^2$.

From the boundary condition $u(x_1, 0) = x_1^2$, that is $Q_1(x_1, x_2, u(x_1, x_2)) = x_1^2$, for $(x_1, x_2) \in \Gamma_1$, by using Eq. (36), it follows that

$$G_{1,\mathbf{m}'} = C_{\mathbf{m}},\ \mathbf{g}_{1,\mathbf{m}'} = \mathbf{m}',\ H_{1,\mathbf{m}'} = C_{\mathbf{m}} S^{(2)}_{\mathbf{m}}(0).$$

Since $\mathbf{m} = (2,1)$, $\gamma(\mathbf{m}) = 8$, $\theta((0,1),(2,1)) = 8 - p^{(*,0)}(8) = 4$, by Eq. (37), we get

$$S^{(2)}_{(2,1)}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$H_{1,\mathbf{m}'} = [c_{(0,0)}, c_{(1,0)}, c_{(2,0)}, c_{(3,0)}].$$

Thus the system in Eq. (48), for $j = 1$, has the form

$$[c_{(0,0)}, c_{(1,0)}, c_{(2,0)}, c_{(3,0)}] = [0, 0, 1, 0].$$

Similarly, from the boundary condition $\frac{\partial u}{\partial x_2}(x_1, 0) = -2x_1^2$, we get

$$G_{2,\mathbf{m}'} = C_{\mathbf{m}} D^{(1,0),\mathbf{m}}, \mathbf{g}_{2,\mathbf{m}'} = \gamma^{-1}(\theta((1,0), \mathbf{m}')),$$

$$H_{2,\mathbf{m}'} = C_{\mathbf{m}} D^{(0,1),\mathbf{m}} S_{\mathbf{m}}^{(2)}(0),\ H_{2,\mathbf{m}'} = [c_{(0,1)}, c_{(1,1)}, c_{(2,1)}].$$

Thus the system in Eq. (48), for $j = 2$, has the form

$$[c_{(0,1)}, c_{(1,1)}, c_{(2,1)}] = [0, 0, -2].$$

By solving the system we find $P_{(2,1)}(x_1, x_2) = x_1^2 - 2x_1^2 x_2$, which approximates the solution of the BVP problem.

To estimate the error we compute Est by using Eq. (56) with $\lambda_0 = \lambda_1 = \lambda_2 = \frac{1}{3}$. Since $f(x_1, x_2) = R_{\mathbf{m}^{(0)}}(x_1, x_2) = 0$, $f_1(x_1) = R_{\mathbf{m}^{(1)}}(x_1) = P_{(2,1)}(x_1, 0)$, $f_2(x_1) = R_{\mathbf{m}^{(2)}}(x_1) = \frac{\partial P_{(2,1)}}{\partial x_2}(x_1, 0)$ we get $Est = \frac{1}{3}\|Q(P_{(2,1)})\|_{\bar{\Omega},\infty}$. As in [25], for numerical approximation we consider $\bar{\Omega} = [0.25, 1] \times [0.2, 0.6]$. Then we get the estimation $d(u, P_{(2,1)}) \le Est < \frac{2.16}{3} = 0.72$.

Similarly, by choosing $\mathbf{m} = (2, 22)$, we find the estimation $d(u, P_{(2,22)}) \le Est < \frac{0.158}{3} = 0.0527$. In this case the fourth column of the Table 1 contains the approximate solution for $\mathbf{m} = (2, 22)$ obtained by the present method.

Table 1

x_2	x_1	GDTM [26]	present method	u exact
0.2	0.25	0.043403	0.043403	0.043403
	0.50	0.173611	0.173611	0.173611
	0.75	0.390625	0.390625	0.390625
	1.00	0.694444	0.694444	0.694444
0.4	0.25	0.031888	0.031888	0.031888
	0.50	0.127551	0.127551	0.127551
	0.75	0.286990	0.286990	0.286990
	1.00	0.510204	0.510204	0.510204
0.6	0.25	0.024433	0.024421	0.024414
	0.50	0.097730	0.097685	0.097656
	0.75	0.219893	0.219792	0.219727
	1.00	0.390921	0.390742	0.390625

For this example we compare our findings with the numerical results in [26] by using the generalized differential transform method (GDTM), which have been shown to be superior or comparable to those in [27] by using variational iteration method (VIM) and Adomian decomposition method (ADM). Table 1 shows the approximate solutions for the problem from Eq. (59) with the initial conditions in Eq. (60) in some points for x_1 and x_2 given in [26] together with the results in [26] and the exact solution.

From Table 1, we can see that the approximate solutions based on our method have less errors than the solution in [26].

5.3. Example 3

The nonlinear diffusion equation is considered (see [20] and [28]):

$$u(x,t)^2 \frac{\partial^2 u}{\partial x^2}(x,t) + 2u(x,t)\left(\frac{\partial u}{\partial x}(x,t)\right)^2 - \frac{\partial u}{\partial t}(x,t) = 0,\ (x,t) \in [0,1] \times [0,\infty), \tag{61}$$

subject to the initial condition

$$u(x,0) = \frac{x+h}{2\sqrt{c}}, \tag{62}$$

where $c > 0$ and h is an arbitrary constant.

The exact solution of this problem is $u(x,t) = \frac{x+h}{2\sqrt{c-t}}$.

Table 2

x_1	x_2	HPM [20]	PSS [28]	present method	u exact
0.6	0.0	.25298221	.25298221	.25298221	.25298221
	0.2	.25555062	.25555061	.25555063	.25555063
	0.4	.25819884	.25819871	.25819889	.25819889
	0.6	.26093095	.26093028	.26093121	.26093123
	0.8	.26375124	.26374914	.26375210	.26375219
	1.0	.26666420	.26665906	.26666637	.26666667
0.8	0.0	.28460499	.28460499	.28460499	.28460499
	0.2	.28749445	.28749444	.28749445	.28749445
	0.4	.29047369	.29047354	.29047375	.29047375
	0.6	.29354732	.29354657	.29354761	.29354763
	0.8	.29672015	.29671778	.29672109	.29672121
	1.0	.29999723	.29999145	.29999962	.30000000
1.0	0.0	.31622777	.31622777	.31622777	.31622777
	0.2	.31943828	.31943827	.31943828	.31943828
	0.4	.32274855	.32274838	.32274861	.32274861
	0.6	.32616368	.32616285	.32616400	.32616404
	0.8	.32968905	.32968642	.32969009	.32969024
	1.0	.33333025	.33332383	.33333287	.33333333

Here, by taking $x_1 = x$ and $x_2 = t \in [0,1]$, the polynomial $Q = Q(z_1, z_2, z_3, z_4, z_5, z_6) = z_3^2 z_6 + 2z_3 z_4^2 - z_5$, $f(x_1, x_2) = 0$, $N = 6$, $\mathbf{d}_Q = (0,0,2,0,0,1)$ and $\deg(Q) = 3$.

By the boundary conditions we see that $N'_\Gamma = 4; N_\Gamma = 1$, $a = c = 0$, $b = d = 1$, $\Gamma'_1 = \{(x_1, 0) : x_1 \in [0,1]\}$, $\Gamma'_2 = \{(1, x_2) : x_2 \in [0,1]\}$, $\Gamma'_3 = \{(x_1, 1) : x_1 \in [0,1]\}$, $\Gamma'_4 = \{(0, x_2) : x_2 \in [0,1]\}$, $\Gamma_1 = \Gamma'_1$ and $Q_1(z_1, z_2, z_3) = z_3$. In Table 2 we compare the solution obtained by the present method for $\mathbf{m} = (0,5)$ with the results obtained by Power Series Solutions (PSS) in [28] and by homotopy perturbation method (HPM) in [20], also with a polynomial degree five. By using an ordinary personal computer, the CPU time for calculation of the solution for our method was 18 s (seconds). As compared to the previous two methods, our method has higher accuracy, is more general, and can be applied to larger classes of PDE.

5.4. Example 4

Consider the following variable coefficient fourth-order partial parabolic differential equation given in [29]

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u}{\partial x^4}(x,t) = 0, \quad (x,t) \in (0,1) \times (0,\infty), \tag{63}$$

subject to the initial conditions

$$u(x,0) = x - \sin x,$$
$$\frac{\partial u}{\partial t}(x,0) = -(x - \sin x), \tag{64}$$

and the boundary conditions

$$u(0,t) = 0,\ u(1,t) = e^{-t}(1 - \sin 1),$$
$$\frac{\partial^2 u}{\partial x^2}(0,t) = 0,\ \frac{\partial^2 u}{\partial x^2}(1,t) = e^{-t}\sin 1. \tag{65}$$

The problem arises in the study of the transverse vibrations of a uniform flexible beam.

The exact solution of this problem is $u(x,t) = (x - \sin x)e^{-t}$.

We apply the method to the equation (63) multiplied by $\sin x$. Let $\mathrm{Sin}_r(x) = \frac{x}{1!} - \frac{x^3}{3!} + ... + (-1)^r \frac{x^{2r+1}}{(2r+1)!}$ be a partial sum of the Taylor series of $\sin x$. Then we get $Q = Q(z_1, z_2, z_3, ..., z_{13}) = \mathrm{Sin}_r(z_1) z_8 + (z_1 - \mathrm{Sin}_r(z_1)) z_{13}$ (see (44)), $f(x,t) = 0$. We may choose, for example, $\mathbf{m} = (10,4) = \mathbf{m}'$ and by Eq. (45), it follows that

$$G_{\mathbf{m}'} = \left(\frac{1}{1!}Sh^{(1)}_{-1} - \frac{1}{3!}Sh^{(1)}_{-3} + ... + (-1)^r \frac{1}{(2r+1)!}Sh^{(1)}_{-2r-1}\right) Z_{8,\gamma^{-1}(\theta((0,2),(10,4)))}$$
$$+ \left(\frac{1}{3!}Sh^{(1)}_{-3} - ... - (-1)^r \frac{1}{(2r+1)!}Sh^{(1)}_{-2r-1}\right) Z_{13,\gamma^{-1}(\theta((4,0),(10,4)))},$$

where $Z_{8,\gamma^{-1}(\theta((0,2),(10,4)))} = C_{(10,4)}D^{(0,2),(10,4)}$, $Z_{13,\gamma^{-1}(\theta((4,0),(10,4)))} = C_{(10,4)}D^{(4,0),(10,4)}$, $Sh^{(1)}_{-k}$ $C_{(10,4)} = [c_{(-k,0)}, c_{(1-k,0)}, c_{(-k,1)}, ..., c_{(i-k,j)}, ..., c_{10-k,4}]$, (i.e. a shift operator) with $c_{(r,s)} = 0$, if either $r < 0$ or $s < 0$.

Since we consider $x, t \in [0,1]$, as in Example 2, $N'_\Gamma = 4$, but here $N_\Gamma = 6$, $\Gamma_1 = \Gamma_2 = \Gamma'_1$, $\Gamma_3 = \Gamma_5 = \Gamma'_4$, $\Gamma_4 = \Gamma_6 = \Gamma'_2$. By using $r = 6$ we find $d(u, P_{(10,4)}) < Est < 0.34 \cdot 10^{-5}$. In Table 3 we give the approximate solution $P_{(10,4)}$ for different t and x together with their errors. From Table 3 it is seen that the method gives a high accuracy which increases by increasing $\mathbf{m}$.

Table 3

t	x	$P_{(10,4)}$	$\lvert u - P_{(10,4)}\rvert$
0.25	0.00	0.00000000	0.00000000
	0.25	0.00202180	$0.44913508 \cdot 10^{-17}$
	0.50	0.01602341	$0.24041692 \cdot 10^{-13}$
	0.75	0.05323979	$0.12611381 \cdot 10^{-11}$
	1.00	0.12346252	$0.18872058 \cdot 10^{-10}$
0.50	0.00	0.00000000	0.00000000
	0.25	0.00157458	$0.40678907 \cdot 10^{-14}$
	0.50	0.01247904	$0.75241940 \cdot 10^{-13}$
	0.75	0.04146319	$0.18575565 \cdot 10^{-10}$
	1.00	0.09615271	$0.39530290 \cdot 10^{-9}$
0.75	0.00	0.00000000	0.00000000
	0.25	0.00122628	$0.13018202 \cdot 10^{-11}$
	0.50	0.00971869	$0.10504769 \cdot 10^{-11}$
	0.75	0.03229156	$0.22444964 \cdot 10^{-10}$
	1.00	0.07488380	$0.13598700 \cdot 10^{-8}$
1.00	0.00	0.00000000	0.00000000
	0.25	0.00095503	$0.41880651 \cdot 10^{-10}$
	0.50	0.00756892	$0.65137773 \cdot 10^{-10}$
	0.75	0.02514869	$0.99423784 \cdot 10^{-10}$
	1.00	0.05831956	$0.12863481 \cdot 10^{-8}$

In [29] the solution of this problem is found by an interesting method based on Adomian decomposition method. Thus by using the initial conditions (64) is taken $u_0(x,t) = (x - \sin x)(1-t)$, and by the recurrence relation

$$u_{k+1}(x,t) = -\int_0^t\int_0^t \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u_k}{\partial x^4}\,dtdt, \tag{66}$$

are computed $u_{k+1}(x,t)$, $k \geq 0$. Then

$$u(x,t) \approx \tilde{u}_n(x,t) = \sum_{k=0}^{n} u_k(x,t),$$

the accuracy increases by increasing n. For example, for $n = 3$, $|u(x, 0.25) - \tilde{u}_3(x, 0.25)|$, for $x = 0.00, 0.25, 0.50, 0.75, 1.00$, are equal to 0.00000000, $0.95582810 \cdot 10^{-12}$, $0.75752464 \cdot 10^{-11}$, $0.25169710 \cdot 10^{-10}$, $0.58368301 \cdot 10^{-10}$, respectively. By comparing with the results from Table 3 it follows that it is necessary to take $n \geq 4$ to obtain higher accuracy than that which we obtained by taken $\mathbf{m} = (10, 4)$. Thus the accuracy depends on $\mathbf{m}$ and n, but the computation by using the method from the present paper use only the solutions of linear system of equations instead to compute a double integral and a partial derivative of order 4 as in (66). Moreover, the method from this paper is more general because Adomian decomposition method requires a suitable expansion into a series of the solution of the initial boundary-value problem for partial differential equations.

We note that the solution $u(x, t)$ of this problem describes the deflection of the beam. Since it is used to compute, by using derivatives of u, it is suitable to obtain a smooth function for computing the bending moment and the shear force which is an advantage of this method compared with finite element methods. In fact finite element methods approximate the weak solution by functions belonging to a suitable Sobolev space where along a boundary of finite elements the functions are not necessary smooth. When the integral surface has, for example, saddle points (which often appear in the vibration problems) the accuracy of solutions obtained by finite element methods depend of a very fine mesh chosen in a suitable form. We do not have this problem in the present method. Another advantage of the present method is the possibility to estimate the error of the approximate solution (by means of Est), when the exact solution it is not known, by the uniform norm.

Remark 2. When the coefficients are analytic functions the existence and the uniqueness of an analytic solution for an initial value problem is guaranteed by the Cauchy-Kowalevski theorem. There exist similar results for some boundary value problems. If the coefficients have singular points the hypotheses of these theorems are not fulfilled and we can study the problem by means of the systems containing equations of the form (46) and (48).

To illustrate the method we consider the partial differential equation

$$\frac{\partial^2 u}{\partial x_1^2}(x_1, x_2) + \frac{\partial^2 u}{\partial x_2^2}(x_1, x_2) + \frac{\alpha x_1}{x_2^2}\frac{\partial u}{\partial x_1}(x_1, x_2) + \frac{\beta}{x_2}\frac{\partial u}{\partial x_2}(x_1, x_2) + \frac{\gamma}{x_2^2} u(x_1, x_2) = f(x_1, x_2), \quad (67)$$

where $(x_1, x_2) \in (0, 1) \times (0, 1)$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $f(x_1, x_2) = \sum_{\mathbf{i} \geq \mathbf{0}} d_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ is an analytic function on $[0, 1] \times [0, 1]$. In this case the coefficients have an infinite number of singular points $S(x_1, 0)$, $x_1 \in [0, 1]$.

Assume, for the moment, that the equation (67) has an analytic solution $u(x_1, x_2)$ of the form

$$u(x_1, x_2) = \sum_{\mathbf{i} \geq \mathbf{0}} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} c_{(k-j,j)} x_1^{k-j} x_2^{j}. \quad (68)$$

By multiplying the equation (67) by x_2^2 and by replacing u from (68) into (67) we get the system of equations

$$(k-j+1)(k-j+2)c_{(k-j+2,j-2)} + E(k-j,j)c_{(k-j,j)} = d_{(k-j,j-2)}, \tag{69}$$

where $k = 0, 1, ..., j = 0, 1, ..., k$, $E(k-j,j) = j(j-1) + \alpha(k-j) + \beta j + \gamma$ and $c_{(i_1,i_2)} = 0$, for $i_1 < 0$ or $i_2 < 0$.

If $j = 0, 1$, by (69), it follows that

$$E(k-j,j)c_{(k-j,j)} = 0, \; j = 0, 1. \tag{70}$$

There are two cases to consider. (A) $E(i_1,i_2) \neq 0$, for every $(i_1,i_2) \in \mathbb{N} \times \mathbb{N}$. (B) There exist pairs $(i_1,i_2) \in \mathbb{N} \times \mathbb{N}$ such that $E(i_1,i_2) = 0$.

In case A, by taking, for example $\alpha = 0$, $\beta = 1$ and $\gamma = 2$, by (70), we get $c_{(i_1,0)} = c_{(i_1,1)} = 0$, for every i_1. Then, for $f(x_1,x_2) = x_1 e^{x_2}$, by (69), for every $i_2 \geq 2$, it follows that

$$c_{(i_1,i_2)} = \frac{d_{(i_1,i_2-2)} - (i_1+1)(i_1+2)c_{(i_1+2,i_2-2)}}{E(i_1,i_2)} = \frac{d_{(i_1,i_2-2)}}{i_2^2+2} = \begin{cases} 0 & \text{if } i_1 \neq 1 \\ \frac{1}{(i_2-2)!(i_2^2+2)} & \text{if } i_1 = 1. \end{cases} \tag{71}$$

Suppose that

$$u(0,x_2) = g_0(x_2), \; \frac{\partial u}{\partial x_1}(0,x_2) = g_1(x_2), \tag{72}$$

where

$$g_k(x_2) = \sum_{j \geq 0} d_j^{(k)} x_2^j, \; k = 0, 1, \tag{73}$$

are analytic functions on $[0,1]$. Then, by (71)-(73), it follows that the initial value problem (67), (72) has a unique solution only if,

$$d_{i_2}^{(0)} = 0, \; d_{i_2}^{(1)} = \begin{cases} 0 & \text{if } i_2 \leq 1 \\ \frac{1}{(i_2-2)!(i_2^2+2)} & \text{if } i_2 > 1. \end{cases} \tag{74}$$

Thus in the case A, additional requirements (i. e. the conditions of the type (74)) on the functions g_0 and g_1 are necessary for the existence and the uniqueness of an analytic solution of the initial value problem. If these conditions are fulfilled and the series from (68) converges, then the solution can be approximated as in the usual case. For example, by choosing $\mathbf{m} = (0, 15)$, we get the estimation $d(u, P_{\mathbf{m}}) \leq Est < 2 \cdot 10^{-9}$.

Similarly, if we consider the same equation but the initial conditions are replaced by the boundary conditions

$$u(0,x_2) = h_1(x_2), \; u(1,x_2) = h_2(x_2), \; u(x_1,0) = h_3(x_1), \;\; u(x_1,1) = h_4(x_1), \tag{75}$$

where $h_k = \sum\limits_{j \geq 0} e_j^{(k)} x_2^j$, $k = 1, 2$, $h_k = \sum\limits_{j \geq 0} e_j^{(k)} x_1^j$, $k = 3, 4$, are analytic functions, by (71) and (75) are obtained additional requirements on the functions h_k, $k = 1, 2, 3, 4$. Thus, for example, by (71) and the first condition from (75) it follows that $e_j^{(1)} = 0$, for every j. Hence, when the series from (68) converges, we obtain sufficient conditions which guarantee the existence and the uniqueness of an analytic solution of the boundary value problem (71), (75).

In the case B denote $SI := \{(i_1, i_2) \in \mathbb{N} \times \mathbb{N} : E(i_1, i_2) = 0\}$ the set of these special pairs of indexes. Consider, as an example, $\alpha = -3$, $\beta = -4$ and $\gamma = 6$. Then $E(k-j, j) = j(j-1) - 3(k-j) - 4j + 6$ and $E(k-j, j) = 0$ implies $k - j = 2 - 2j + \frac{j(j+1)}{3}$. Hence it follows that

$$SI = \left\{ (i_1, i_2) \in \mathbb{N} \times \mathbb{N} : i_2 \equiv 0 \pmod 3 \text{ or } i_2 \equiv 2 \pmod 3,\ i_1 = 2 - 2i_2 + \frac{i_2(i_2+1)}{3} \right\}.$$

By choosing $i_2 = 0, 2, 3, 5, 6, 8, 9, ...$ we get

$$SI = \{(2,0), (0,2), (0,3), (2,5), (4,6), (10,8), (14,9), ...\}. \tag{76}$$

Then, by (70), we find $c_{(i_1,0)} = 0$, for every $i_1 \neq 2$, $c_{(2,0)} \in \mathbb{R}$ (i.e. $c_{(2,0)}$ is an arbitrary real number) and $c_{(i_1,1)} = 0$, for every i_1. If $j = i_2 = 2$ and $k - j = i_1 \neq 0$, by (69), it follows that

$$c_{(i_1,2)} = \frac{d_{(i_1,0)} - (i_1+1)(i_1+2)c_{(i_1+2,0)}}{E(i_1,2)} = -\frac{d_{(i_1,0)}}{3i_1}$$

and, for $i_1 = 0$, we get $c_{(2,0)} = \frac{d_{(0,0)}}{2}$, $c_{(0,2)} \in \mathbb{R}$.

Similarly, if $i_2 = 3$, we get, for $i_1 \neq 0$,

$$c_{(i_1,3)} = \frac{d_{(i_1,1)} - (i_1+1)(i_1+2)c_{(i_1+2,1)}}{E(i_1,3)} = -\frac{d_{(i_1,1)}}{3i_1}, \tag{77}$$

and $c_{(2,1)} = \frac{d_{(0,1)}}{2}$, $c_{(0,3)} \in \mathbb{R}$. Since $c_{(2,1)} = 0$ it follows $d_{(0,1)} = 0$, that is an additional requirement on f. Then, for every i_1,

$$c_{(i_1,4)} = \frac{d_{(i_1,2)} - (i_1+1)(i_1+2)c_{(i_1+2,2)}}{E(i_1,4)}, \tag{78}$$

$$c_{(i_1,5)} = \frac{d_{(i_1,3)} - (i_1+1)(i_1+2)c_{(i_1+2,3)}}{E(i_1,5)}, \text{ if } i_1 \neq 2,$$

and $c_{(4,3)} = \frac{d_{(2,3)}}{12}$, $c_{(2,5)} \in \mathbb{R}$. Comparing with (77) we get the additional requirement $d_{(4,1)} = -d_{(2,3)}$. Generally, when $i_2 \geq 2$, for every $i_2 \equiv 1 \pmod 3$, that is when $(i_1, i_2) \notin SI$, for every i_1, we get

$$c_{(i_1,i_2)} = \frac{d_{(i_1,i_2-2)} - (i_1+1)(i_1+2)c_{(i_1+2,i_2-2)}}{E(i_1,i_2)},$$

and, for $i_2 \equiv 0 \pmod 3$ or $i_2 \equiv 2 \pmod 3$, we obtain

$$c_{(i_1,i_2)} = \frac{d_{(i_1,i_2-2)} - (i_1+1)(i_1+2)c_{(i_1+2,i_2-2)}}{E(i_1,i_2)}, \text{ if } (i_1,i_2) \notin SI,$$

and $c_{(i_1+2,i_2-2)} = \frac{d_{(i_1,i_2-2)}}{(i_1+1)(i_1+2)}$, $c_{(i_1,i_2)} \in \mathbb{R}$, when $(i_1, i_2) \in SI$. The last case imposes additional requirements on f, because $c_{(i_1+2,i_2-2)}$ was determined in the case when the second index was equal to $i_2 - 2$. Thus, if we assume, for example $f(x_1, x_2) = d_{(0,0)} + d_{(1,0)}x_1 + d_{(0,1)}x_2 + d_{(2,0)}x_1^2 + d_{(1,1)}x_1x_2 + d_{(0,2)}x_2^2$, we get the additional requirement $d_{(0,1)} = 0$. By considering the initial value problem (67), (72), as in the case A, it follows that $d^{(0)}_{i_2} = c_{(0,i_2)}$, $d^{(1)}_{i_2} = c_{(1,i_2)}$. In this case the solution of the problem is approximated as in the case when the coefficients have not singular points. Thus, for $f(x_1, x_2) = 5 - 9x_1 + 6x_1^2 + 3x_1x_2 + 2x_2^2$, $g_0(x_2) = 2x_2^2 - x_2^3 + 2x_2^4$, $g_1(x_2) = 3x_2^2 - x_2^3$, we get the exact solution $u(x_1, x_2) = 2.5x_1^2 + 2x_2^2 + 3x_1x_2^2 - x_2^3 - x_1^2x_2^2 - x_1x_2^3 + 2x_2^4$.

5.5. Example 5

Consider the following one-dimensional nonlinear undamped Sine-Gordon equation (see [15]):

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + \sin(u(x,t)) = 0, \quad (x,t) \in [-1,1] \times [0,\infty), \tag{79}$$

subject to the initial conditions

$$\begin{aligned} u(x,0) &= 0, \\ \frac{\partial u}{\partial t}(x,0) &= 4\mathrm{sech}(x), \end{aligned} \tag{80}$$

and the boundary conditions

$$\begin{aligned} u(-1,t) &= 4\tan^{-1}(\mathrm{sech}(-1)t), \\ u(1,t) &= 4\tan^{-1}(\mathrm{sech}(1)t). \end{aligned} \tag{81}$$

The exact solution of this problem is $u(x,t) = 4\tan^{-1}(\mathrm{sech}(x)t)$.

This problem is reduced to a polynomial PDE by replacing, as in Example 4, $\sin(u)$ by a suitable partial sum $Sin_r(u)$ of its Taylor series. Then $Q = Q(z_1, z_2, z_3, ..., z_8) = z_8 - z_6 + \mathrm{Sin}_r(z_3)$. By considering $t \in [0,1]$, we obtain $N_\Gamma = 4$, $\Gamma_1 = \Gamma_2 = \{(x,0) : x \in [-1,1]\}$, $\Gamma_3 = \{(-1,t) : t \in [0,1]\}$, $\Gamma_4 = \{(1,t) : t \in [0,1]\}$. We may choose, for example, $\mathbf{m} = (19,2) = \mathbf{m}'$ and by Eq. (45), it follows that

$$G_{\mathbf{m}'} = Z_{8,\gamma^{-1}(\theta((0,2),(19,2)))} - Z_{6,\gamma^{-1}(\theta((2,0),(19,2)))} + Z_{3,(19,2)} - \frac{1}{3!}Z^{\circ 3}_{3,(19,2)} + ... + (-1)^r \frac{1}{(2r+1)!}Z^{\circ 2r+1}_{3,(19,2)},$$

where $Z_{6,\gamma^{-1}(\theta((2,0),(19,2)))} = C_{(19,2)}D^{(2,0),(19,2)}$, $Z_{8,\gamma^{-1}(\theta((0,2),(19,2)))} = C_{(19,2)}D^{(0,2),(19,2)}$, $Z_{3,(19,2)} = C_{(19,2)}$. Then, by applying the method described in this paper, we obtain an approximation of the solution of the problem (79)-(81). The accuracy is higher for larger $\mathbf{m}$. For $\mathbf{m} = (19,2)$, $r = 3$ and $t = 0.25$ we found L_2-error $1.40 \cdot 10^{-5}$ compared with $3.91 \cdot 10^{-5}$ obtained in [15] by using radial basis functions and collocation method. The possibility to estimate the error of the approximate solution by the uniform norm, when the exact solution is not known, is an advantage of the present method, which in the case of analytic solution (as in this example) does not use any collocation points.

6. Conclusion

In this paper, polynomial approximations of solutions of polynomial partial differential equations in two independent variables are given. The solution of partial differential equations has been reduced to a problem of solving a system of algebraic equations. Five examples are given to demonstrate the validity and applicability of the present method.

Acknowledgment

The authors wish to express their sincere thanks to anonymous referees for their valuable suggestions that improved the final form of the paper.

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