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A Structure Exploiting Algorithm for Approximate Robust Optimal Control with Application to Power Generating Kites

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Abstract— **In the present paper we discuss numerical solution strategies for robust optimal control problems. Here, our motivation is to optimize the dynamic open-loop behaviour of a kite, which is affected by unknown wind turbulence. After reviewing some existing strategies for robust optimal control, we specialize on a particular approach which uses a Lyapunov differential equation in order to approximate the influence of the uncertainties on the state of the dynamic system and to guarantee stability of periodic systems. We propose strategies to exploit the structure of the corresponding formulation and provide an open-source implementation of the algorithm. The practical advantage of this implementation is illustrated by applying it to a robustness and stability optimization problem for a power generating kite system. Here, the model equations and system parameters are inspired from a real-world application.**

I. INTRODUCTION

In this paper, we are interested in uncertain optimal control problems of the form

$$\begin{aligned} &\underset{x(\cdot),u(\cdot),p,T}{\text{minimize}} \; J(u(\cdot),p,T) \\ &\text{s.t.:} \left\{ \begin{array}{lcll} \dot{x}(t) & = & f(x(t),u(t),p,w(t)), & \forall t \in [0,T] \\ 0 & \geq & h(x(t),u(t),p,T) & \forall t \in [0,T] \\ x(0) & = & B_0 w_0, & \end{array} \right. \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}^{n_u}$ are the state and the control variables, respectively. Moreover, the parameter $p \in \mathbb{R}^{n_p}$ and the end time $T \in \mathbb{R}_{++}$ might be optimized, too. The functions f and h are assumed to be differentiable in their arguments. Note that the objective functional J is in our formulation only allowed to depend on u, p, and T. This can always be achieved by reformulating the problem using slack variables if necessary. The matrix $B_0 \in \mathbb{R}^{n_x \times n_{w_0}}$ is given. Finally, $w_0 \in \mathbb{R}^{n_{w_0}}$ and $w(t) \in \mathbb{R}^{n_w}$ are assumed to be uncertain, i.e., the vector w_0 and the function w are only known to be contained in a common uncertainty set $(w_0, w) \in Z$.

A. Power Generating Kite Systems

Our motivation to consider uncertain optimal control problems of the form (1) is that this type of problems occurs in the context of kite control. Here, we are particularly interested in kite systems which can be used for wind power generation. The idea of using kites for power generation has originally been proposed by Loyd [25]. During the last years kite power generating devices have attracted more and more research and we refer to [10], [23], [32], [34] and the references therein for an overview. For a system with a single kite and one fixed generator on the ground the principle is very simple: the kite pulls as strong as possible on its cable slowly driving the generator while flying fast in a crosswind direction. To achieve a periodic power-generating cycle the kite is depowered periodically by changing the angle of attack and retracted easily while the tension in the cable is low.

The main challenge for optimal control of kites is that the dynamics are affected by wind turbulences which can not be predicted in advance. Moreover, it is difficult to find precise kite models and thus additional model-plant mismatches might be encountered, too. Obviously, we can design a feedback controller to compensate these uncertainties. However, we might also try to improve the inherent robustness and stability properties of the open-loop controlled system. This is the focus of this article. Note that open-loop robust and stable orbits are also useful in combination with feedback control as tracking an inherently open-loop stable orbit is typically easier than tracking an unstable trajectory.

A necessary requirement for the robustness of open-loop controlled periodic systems is the stability of these systems. Most existing stability optimization techniques are either based on the optimization of the asymptotical decay rate of the system, the optimization of the so called pseudo-spectral abscissa, or on the smoothed spectral abscissa or radius. Taking robustness into account, there is a well developed field for the robust optimal control of linear systems. Robust approaches for non-linear systems often employ linear approximations [13], [18], [30].

B. Contributions and Overview

The paper starts in Section II with a brief review of some existing approaches in robust optimization. Here, we concentrate on several variants to compute first order approximations with respect to the uncertainty. In this context, we highlight a particular approximation strategy which is based on Lyapunov differential equations. The main contributions of this paper can be summarized as follows:

- In Section III we present a numerical structure exploitation technique which accelerates existing code for approximate robust optimal control based on Lyapunov differential equations. The corresponding implementation of the structure exploiting algorithm is freely available in form of a new feature within the open source software `ACADO Toolkit` [20].

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- In Section IV we present some new robust and open-loop stable orbits for a power generation kite system. The formulation of the problem and the chosen model parameters for the kite are inspired from a real-world setup which is shown in Figure 1.

Fig. 1. In-door setup built in K.U. Leuven

The paper concludes in Section V with a summary and an outlook on how the presented techniques and results might become relevant for the realization and control of kite power generation.

II. EXISTING APPROACHES IN ROBUST OPTIMIZATION

The main numerical question in the field of worst-case robust optimization is how to check numerically whether semi-infinite constraints of the form

$$\forall z \in Z: \quad F(y,z) \leq 0$$

are satisfied for a given y and for a given uncertainty set Z. Alternatively, if Z is compact and if F is a continuous scalar valued function in z, the above feasibility problem can be written in the form $V(y) \leq 0$, where V is the optimal value of the parametric optimization problem

$$V(y) := \max_{z \in Z} F(y,z) .$$

Note that Ben-Tal and Nemirovski [3], [4] have analyzed many cases in which it is possible to find numerically tractable expressions for the function V. Here, a commonly proposed assumption is that F is concave in z while the set Z is convex. However, there exist also other cases in which it is possible to solve the maximization problem globally. For example, if F is polynomial in z and Z is an algebraic set, the theory of positive polynomials and convex LMI-reformulations can be applied for which we refer to the work of Lasserre [24] and Parillo [33]. However, for the case that F is a general nonlinear function without a particular structure, the problem becomes much harder [15].

In the work of Nagy and Braatz [30], [31] but also in [12] it is suggested to not solve the exact robust counterpart problem, but to consider a first order Tailor expansion instead (assume $0 \in Z$ and F differentiable):

$$V(y) \approx \max_{z \in Z} F(y,0) + \frac{\partial F(y,0)}{\partial z} z .$$

Clearly, this approach is questionable as there is in general neither a guarantee of conservatism nor an upper bound on the approximation error available. However, in this paper we simply accept this linearization approach which can be justified as a practical heuristic if the uncertainty set Z is sufficiently small. The focus of this paper is not about the approximation accuracy, but about how to efficiently compute the derivative $\frac{\partial F(y,0)}{\partial z}$ which is needed in the linearization.

In order to understand why computing derivatives can be a problem, we have to recall that in our case an evaluation of the function F requires us to solve a differential equation. More precisely, if $\xi[t,u(\cdot),p,w,w_0]$ denotes the solution of the differential equation

$$\forall \tau \in [0,t]: \quad \dot{x}(\tau) = f(x(\tau),u(\tau),p,w(\tau)) \quad x(0) = B_0 w_0 ,$$

then the semi-infinite inequalities of our interest can be written as

$$\forall z \in Z: \quad F_i(t,y,z) := h_i(\xi[t,u(\cdot),p,w,w_0],u(t),p,T) \leq 0 .$$

Here, we define $y := (u(\cdot),p,T)$ and $z := (w_0, w(\cdot))$. In order to simplify the following consideration, we work from now on with the following assumption:

Assumption 1: We assume that w is L_2-integrable while the uncertainty set Z is of the form

$$Z := \left\{ z \,\middle|\, z^\top z := w_0^\top B_0 B_0^T w_0 + \int_{-\infty}^{\infty} w(\tau)^\top w(\tau)\, \mathrm{d}\tau \leq \gamma^2 \right\} .$$

The above assumption implies that the approximation robust counterpart problem associated with (1) may formally be written as

$$\min_y J(y) \;\text{ s.t. }\; F_i(t,y,0) + \gamma \left\| \frac{\partial F_i(t,y,0)}{\partial z} \right\|_2 \leq 0 , \qquad (2)$$

where the constraints have to be satisfied for all $t \in [0,T]$ and all $i \in \{1,\ldots,n_h\}$.

A. Linear Approximations Based on Forward or Backward Automatic Differentiation

One way to compute the required sensitivities is to directly use forward differentiation (cf. [30], [31]). If we define for all $t \in [0,T]$ a linear operator $H(t) : (w_0, w(\cdot)) \mapsto \Delta x(t)$ by:

$$\forall t \in [0,T] : \Delta\dot{x}(t) = A(t)\Delta x(t) + B(t)w(t) \quad \Delta x(0) = B_0 w_0 ,$$

then the required robustness margins can be written as

$$\left\| \frac{\partial F_i(t,y,0)}{\partial z} \right\|_2 = \| C_i(t) H(t) \|_2 .$$

Here, we use the short hands

$$\begin{array}{lclclcl} A(t) & := & \frac{\partial f(x,u,p,0)}{\partial x}, & & B(t) & := & \frac{\partial f(x,u,p,0)}{\partial w} \\ C(t) & := & \frac{\partial h(x,u,p,T)}{\partial x} . & & & & \end{array}$$

In order to compute $H(t)$ numerically, we have to discretize the input function w such that $H(t)$ can be represented in form of a matrix. Such a direct computation of $H(t)$ corresponds to a forward differentiation. Note that the cost of computing $H(t)$ depends on the number N_w of discretization intervals which are used in order to discretize the function w and on the dimension of n_{w_0} of initial uncertainties, i.e., the dimension of the vector w_0.

TABLE I

THE NUMBER OF VARIATIONAL DIFFERENTIAL EQUATIONS FOR THE THREE LINEARIZATION MODES.

Method	Association	# of ODEs
Forward Differentiation	Input $\hat{=}$ Uncertainties	$n_w N_w + n_{w_0}$
Backward Differentiation	Output $\hat{=}$ Constraints	$n_h N_h$
Lyapunov ODE	State $\hat{=}$ Storage	$\frac{1}{2} n_x(n_x+1)$

Alternatively, we can compute the terms $C_i(t)H(t)$ by adjoint differentiation. This adjoint approach has in this context originally been proposed in [12]. In this case, the number of required adjoint differential equations depends mainly on the number N_h of time discretization intervals which are used to discretize the path constraints as well as on the dimension n_h of the constraint function h.

B. Linear Approximations Based on Time-Varying Lyapunov Differential Equations

In order to overcome the numerical limitations of computing forward or backward directions of the operator $H(t)$, it is helpful to recognize that the term of our interest can also be written as

$$\left\| \frac{\partial F_i(t,y,0)}{\partial z} \right\|_2 = \|C_i(t)H(t)\|_2 = \sqrt{C_i(t)P(t)C_i(t)^T}\,, \quad (3)$$

if we define $P(t) := H(t)^* H(t)$. Here, $H(t)^*$ denotes the adjoint operator of $H(t)$. Note that $P(t)$ is an $n_x \times n_x$ matrix which satisfies for all $t \in [0,T]$ a Lyapunov differential equation [7], [22], [26] of the form

$$\begin{aligned} \dot{P}(t) &= A(t)P(t) + P(t)A(t)^\top + B(t)B(t)^\top, \\ P(0) &= B_0 B_0^\top \,. \end{aligned} \quad (4)$$

The number of differential equations which are needed to compute P is $\frac{n_x(n_x+1)}{2}$. The main point is that this number depends neither on the number N_w of discretization intervals which are used to discretize w nor on the number N_h of discretization intervals which are used to discretize the path constraints. Thus, if we have only a moderate number of states, the computation of robustness margins via Lyapunov differential equations can be cheaper than a direct computation via automatic differentiation in forward or backward mode. This approach has for example been proposed in [18], [19].

Table I summarizes the complexity of the three discussed strategies: forward differentiation, backward differentiation, and computation via the Lyapunov ODE (4). The Lyapunov ODE is the best option, if we have

$$\frac{n_x(n_x+1)}{2} \ll n_w N_w + n_{w_0} \quad \text{and} \quad \frac{n_x(n_x+1)}{2} \ll n_h N_h \,.$$

This condition is satisfied in most of the applications as we typically prefer to choose large N_h and N_w aiming at an accurate discretization.

C. Periodic Lyapunov Differential Equations

In many applications it is important to deal with periodic orbits. In this case, the initial value conditions in the uncertain optimal control problem (1) are replaced by periodic state constraints of the form $x(0) = x(T)$. A robust counterpart for this optimal control problem using linear approximations based on Lyapunov differential equations also involves periodic conditions for the Lyapunov matrix P(t), i.e. $P(0) = P(T)$. Note that only the Lyapunov approach is able to deal with this case, i.e. periodic orbits with L_2 bounded perturbations, because due to the infinite time horizon we have both infinitely many disturbance inputs and constraint outputs so that neither forward nor adjoint sensitivities can be used.

Using the following lemma, we can guarantee existence and uniqueness of the periodic solution for the Lyapunov differential equation, provided some assumptions on the dynamic system are satisfied:

Lemma 1 (Lyapunov Lemma, cf. ([9])): The periodic Lyapunov differential equation admits a unique T-periodic and positive definite solution $P(t) \succ 0$ if and only if the monodromy matrix $X := Y(T,0)$ is asymptotically stable (all eigenvalues are contained in the open unit disc) and the reachability Grammian matrix $Q(T)$ is positive definite. Here, $Y : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ is the fundamental solution obtained as

$$\frac{\partial Y(t,\tau)}{\partial t} = A(t)Y(t,\tau) \quad \text{with} \quad Y(\tau,\tau) = 1 \quad \text{for all}\, t,\tau \in \mathbb{R}.$$

The reachability Grammian matrix $Q(T) \in \mathbb{R}^{n_x \times n_x}$ is defined as

$$Q(T) := \int_0^T Y(T,\tau)B(\tau)B(\tau)^\top Y(T,\tau)^\top d\tau.$$

III. STRUCTURE EXPLOITATION FOR LYAPUNOV DIFFERENTIAL EQUATIONS

In this section, we concentrate on structure exploiting numerical algorithms which can solve the approximate robust counterpart problem (2). Here, we specialize on the Lyapunov differential equation based linearization approach, i.e., we write the approximate robust counterpart problem in the form[1]

$$\begin{aligned} &\min_{x(\cdot),u(\cdot),p,T} && J(u(\cdot),p,T) \\ &\text{s.t.} && \\ &\forall t \in [0,T]: && \dot{x}(t) = f(x(t),u(t),p,0), \\ &\forall t \in [0,T]: && \dot{P}(t) = A(t)P(t) + P(t)A(t)^\top + B(t)B(t)^\top, \\ &\forall t \in [0,T]: && 0 \geq h_i(x(t),u(t),p,T) + \gamma\sqrt{C_i(t)P(t)C_i(t)^\top}\,, \\ & && x(0) = 0, \qquad P(0) = B_0 B_0^\top \,. \end{aligned} \quad (5)$$

Clearly, the above optimal control problem has a particular structure. Due to the high practical relevance of the formulation (5), it seems worthwhile to work out the details of this structure. Here, our aim is an efficient open source

[1]If the matrix $P(t)$ is not strictly positive definite or if vector $C_i(t)^T$ can become 0, the square-root term in the robustified constraint might be evaluated at a non-differentiable point. In this case it is advisable to add a small regularization term within the numerical implementation.

code implementation which can deal with the above type of optimal control problem. Note that a blind application of generic optimal control tools leads quickly to computational limits, as the number of independent Lyapunov differential states increases quadratically with the dimension n_x.

A. Exploitation of Algebraic Properties

Let us exploit the linear algebra properties of the Lyapunov differential equation. Besides symmetry, Lyapunov equations have some inherent linear algebra structure which can for example be exploited in implicit integrators or also if additional periodicity constraints are present. In order to understand this aspect, we consider the implicit Euler method for Lyapunov differential equations. Here, an implicit step (with stepsize h) has the form

$$P^{+} = P + h\left[A^{+}P^{+} + P^{+}\left(A^{+}\right)^{\top} + B^{+}\left(B^{+}\right)^{\top}\right].$$

In order to find the next iterate P^{+}, we summarize this equation in form of a static Lyapunov equation

$$\bar{A}P^{+} + P^{+}\bar{A}^{\top} \;=\; \bar{Q} \tag{6}$$

with $\bar{A} := \frac{1}{2}I - hA^{+}$ and $\bar{Q} := P + hB^{+}\left(B^{+}\right)^{\top}$. At first view, we might expect that the cost of solving this equation with respect to the unknown $P^{+} \in \mathbb{R}^{n_x \times n_x}$ should be $O(n_x^6)$ as the number of unknown matrix coefficients is already $O(n_x^2)$. However, there exist linear algebra routines [1], [2], [5], [21] which can solve the static Lyapunov equation (6) with complexity $O(n_x^3)$. This comment also transfers to higher order implicit integrators where the structure of the Lyapunov equation can be exploited in an analogous way.

B. Exploitation of Sensitivity Structure

In this paper we are interested in direct optimal control algorithms based on collocation [6] or multiple shooting [8]. The following notation is tailored to a multiple shooting algorithm, but the ideas also transfer to collocation or even indirect approaches. Here, we choose a time discretization $0 = t_0 < t_1 < \ldots < t_N$ and an associated piecewise constant control discretization

$$u(t) \approx \sum_{i=1}^{N} \hat{u}_i I_{[t_i,t_{i+1})}(t)\;,$$

where $I_{[a,b)}(t)$ is equal to one if $t \in [a,b)$ and equal to zero otherwise. Now, we regard the differential equation of our interest on one of the time intervals $[t_i, t_{i+1}]$:

$$\begin{array}{rclrcl} \dot{x} & = & f & x(t_i) & = & x_i \\ \dot{P} & = & AP + PA^{\top} + BB^{\top} & P(t_i) & = & P_i\;. \end{array} \tag{7}$$

Here, we use the short hand $f := f(x(\tau), u_i, p, 0)$. Similarly, the matrices $A = \partial_x f$ and $B = \partial_w f$ will in general depend on $x(\tau)$, u_i, and p. As we are aiming at a derivative based optimization algorithm, we need at least the first order derivatives of $x(t_{i+1})$ and $P(t_{i+1})$ with respect to P_i, and $r_i^{\top} := \left(x_i^{\top}, u_i^{\top}, p^{\top}\right)$. As the differential equation for x does not depend on P, we compute $Y_{x,r} := (Y_{x,x}, Y_{x,u}, Y_{x,p}) := \partial_r x$ via the standard variational differential equation

$$\dot{Y}_{x,r} \;=\; AY_{x,r} + (0, \partial_u f, \partial_p f) \quad Y_{x,r}(t_i) \;=\; (I,0,0)\;.$$

Here, we can use that $A = \partial_x f$ has to be computed anyhow, as this derivative is needed in the right-hand side of the Lyapunov differential equation.

Fortunately, if we have computed $Y_{x,x}$, we have an explicit expression for the derivative

$$Y_{P,P} \;:=\; \partial_{P_i} P \;=\; Y_{x,x} \otimes Y_{x,x}\;,$$

where $\otimes$ denotes the Kronecker product. However, note that in an efficient implementation the matrix $Y_{P,P}$ should not be stored in form of a 4-dimensional tensor, but in form of the above Kronecker product (the matrix $Y_{x,x}$ must be stored anyhow). The remaining tensor $Y_{P,r} := \partial_r P$ can again be computed by propagating a variational ODE of the form

$$\begin{aligned} \dot{Y}_{P,r} &= AY_{P,r} + Y_{P,r}A^{\top} + (\partial_r A)P + P(\partial_r A)^{\top} + \partial_r(BB^{\top}) \\ Y_{P,r}(t_i) &= 0\;. \end{aligned}$$

In this context, we can additionally exploit that A is already known as well as the fact that $Y_{P,r}$ is symmetric. However, as we need the terms $\partial_r A$ and $\partial_r(BB^{\top})$, the computation of some second order derivatives of f cannot be avoided.

As soon as the derivatives of the states x and P can efficiently be computed, we proceed with a standard optimal control algorithm. For the results in this paper, we have extended an existing multiple-shooting SQP code by exploiting the sensitivity structure which is outlined above. Here, the main idea is to exchange the existing integrator with a tailored structure exploiting Lyapunov integrator. For this aim, we use the concept of internal automatic differentiation [8], i.e., we use the same time discretization grid for the nominal integration and the associated variational ODEs for $Y_{x,r}$ and $Y_{P,r}$. The corresponding code is open-source and freely available as an add-on module to the `ACADO Toolkit` [20].

IV. ROBUST OPTIMAL CONTROL OF POWER GENERATING KITES

This section is about an application of the proposed structure exploiting robust optimal control algorithm to a kite system. For this aim we briefly introduce a dynamic kite model, which has originally been proposed in [11], [14] and was further developed in [17], [19], [27], [28], [29]. Moreover, we present nominally optimized periodic orbits in Section IV-B while Section IV-C discusses robustly optimized open-loop stable periodic orbits. In [19] the authors have already shown that such open-loop stable orbits exist in simulation for a large power generating kite system with almost 2 km line length and a kite with 500 m^2 wing area. In this paper, we regard a much smaller scale kite which is currently built by our team. The main result of this section is that open-loop stable orbits exist also for small scale kite systems which is especially astonishing as the nominally optimized orbit turns out to be unstable for our model.

A. The Dynamic Kite Model

In this paper we employ the existing kite model from [16], [17], but we analyze the system for a different set of parameters which are inspired from the existing real-world

TABLE II

PARAMETERS OF THE KITE MODEL

Dimension	Description [unit]	Value
A_k	wing surface area [m^2]	2.5
c_D	aerodynamic drag coefficient [-]	0.2
ρ	air density [kg/m^3]	1.23
m	kite mass [kg]	1
r	tether length [m]	100
w	wind velocity [m/s]	10.0
ϕ_l	lower bound for ϕ	-0.8
ϕ_u	upper bound for ϕ	0.8
θ_l	lower bound for θ	0.4
θ_u	upper bound for θ	1.45
ψ_l	lower bound for ψ	-0.045
ψ_u	upper bound for ψ	0.045
$C_{L\mathrm{l}}$	lower bound for C_L	0.6
$C_{L\mathrm{u}}$	upper bound for C_L	1.0

kite setup. Table II summarizes the choice. Here, the position of the kite is given in spherical coordinates (r, ϕ, θ), where r is the cable length. Note that $\phi = 0$ corresponds to the downwind direction. If we could fly at $\theta = 0$ the kite would be at its zenith position while the position with $\theta = \frac{\pi}{2}$ indicates that the kite touches the ground. The controls are the kite's roll rate $\dot{\psi}$ and the change of the lift coefficient $\dot{C}_L$, summarized as $u := (\dot{\psi}, \dot{C}_L)$. Our point model of the kite includes the physical equations for the lift and drag force (including cable drag), and gravitational forces. Here, the kite's orientation is computed under the assumption that the kite's main axis is always in line with the effective wind vector w_e, i.e., a possible side slip is neglected. The details of the model equations can be found in [17].

B. Nominal Optimization of Periodic Kite Orbits

The optimal control problem of our interest has the form:

$$
\begin{array}{ll}
\displaystyle\min_{x(\cdot),u(\cdot),p,T} & J(x(\cdot),u(\cdot),p,T) \\
\text{subject to:} & \\
\forall t \in [0,T]: & \dot{x}(t) = f(x(\cdot),u(\cdot),w(\cdot)), \\
\forall t \in [0,T]: & 0 \geq h(x(\cdot),u(\cdot),w(\cdot)), \\
 & x(0) = x(T),
\end{array}
\tag{8}
$$

where $w(t)$ is assumed to be an unknown L_2-bounded uncertainty function.

Our objective is to maximize the norm of the effective wind vector, i.e., we choose

$$
J := \frac{1}{T}\int_0^T \|w_\mathrm{e}(t)\|_2^2 \,\mathrm{d}t \,.
$$

Note that this is equivalent to a maximization of the power at a small on-board generator with constant drag. Here, $f(x(t),u(t),w(t))$ is the right-hand side of the kite model. The function $h(x(t),u(t),w(t))$ is used to impose upper and lower bounds on the inputs and states (cf. Table II). The periodicity of the kite trajectory is guaranteed by the constraint of the form $x(0) = x(T)$.

We first solve the periodic optimal control problem (1) for no disturbances, i.e., with a constant wind. For this aim, we employ the optimal control software `ACADO Toolkit` [20]. A locally optimal solution is shown in Figure 2. Here, the final time is optimized, too, finding $T = 10.372\,$s. However, the spectral radius of the associated monodromy matrix is $2.2321 > 1$, and thus, the power optimal trajectory is unstable. In order to illustrate this aspect, we can simulate the dynamic system with the optimized open-loop control applying a small wind turbulence. In Figure 2 it is shown that the kite crashes during such a simulation.

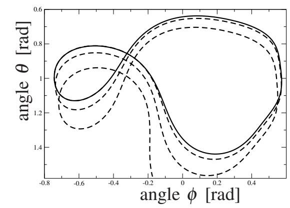

Fig. 2. Unstable kite trajectory: nominal solution (solid line) and simulated solution in presence of small wind turbulences (dashed line)

C. Robust Optimization of Kite Orbits

In order to robustify our nominal periodic solution with respect to wind turbulences, we add some robustness aspects according to the linear approximation based on the Lyapunov differential equation, cf. Section II-B. Here, we propose to maximize the confidence level γ while the value of the original objective is required to satisfy a constraint of the form $J \geq 0.8 J_\mathrm{nominal}$, where J_nominal is the optimal value of the nominally optimal solution from above. This constraint makes sure that we do not pay more than 20% of optimality. The state of the Lyapunov equation is required to satisfy $P(0) = P(T) \succ 0$ such that open-loop stability can be guaranteed.

For the above kite model it was possible to find an open-loop stable and robust solution. The maximal possible confidence level is $\gamma = 1.16$. The optimal cycle duration is $T = 13.83$ seconds. The spectral radius of the associated monodromy matrix is $\rho(X) = 0.41 < 1$. The optimized robust and open-loop stable trajectory is shown in Fig. 3.

For solving the robustified optimal control problem, we have used an SQP method which needed in our case approximately $5\,$s per SQP iteration. Running the same algorithm with the improved code which uses the structure exploiting techniques from Section III, only $1.5\,$s per SQP iteration are needed. Thus, the proposed structure exploiting robust optimal control algorithm reduces the computation time by a factor 3.3.

V. CONCLUSIONS AND OUTLOOK

In this paper we have developed a structure exploitation strategy for an approximate robust optimal control approach. After reviewing different existing linearization techniques, the first main development has been summarized in Section III, where the particular structure of Lyapunov differential equations in the context of a robust optimal control algorithm has been exploited. A second contribution has been presented in Section IV, where we computed an optimal robust solution for a realistic small scale setup, which can in principle be built using the experimental facilities currently developed at KU Leuven. Moreover, we have shown how the computation time required for solving the approximate robust optimal control (5) can be reduced by applying the structure exploitation techniques from Section III.

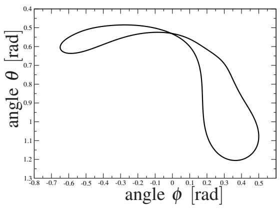

Fig. 3. An open loop stable and robust kite trajectory in the ϕ - θ plane

Future research might investigate low rank approximations of the state of the Lyapunov differential equation which could be beneficial for large scale systems hoping for further improvements in terms of computation time. Moreover, the method will be applied to system models which closely reflect the experimental device at KU Leuven.

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