12 Computations in Orthogonal Rewriting Systems, II

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Nonambiguous linear term rewriting systems were introduced and studied in the first part of this paper (chapter 11). In particular, it was shown that the space of derivations is a pushout category, and a standardization theorem was shown, leading to the existence of a correct *call by name* computation rule. Finally, *call by need* computations were defined. However, this does not lead in general to an effectively computable computation rule.

We shall in this part attack the problem of effective call-by-need computation rules.

The first problem we address is the definition of sequential interpreters for our systems. Using a sequentiality criterion derived from the semantic notion of sequentiality in concrete data types [11], we characterize an important subclass of our systems which admit a correct sequential strategy of interpretation. Intuitively, Σ is sequential iff it is possible to compute in call by need without looking ahead. That is, for any term M not in normal form, there exists a place in M which we need to compute in order to get to the normal form of M (if any), and furthermore, we can determine this place without looking at the subparts of M which are not computed yet. Unfortunately, it is undecidable in general whether Σ is sequential or not.

We finally consider a restriction of sequential systems which allows us to define an effective sequential strategy. Intuitively, a system Σ is strongly sequential if the computation effected to determine the needed positions is independent of the right hand sides of Σ. For a strongly sequential system Σ it is possible to represent the left-hand sides of Σ in a trie structure which we call matching dag. The computation of the needed subpart, together with the actual pattern-matching of the left-hand sides, is driven by the matching dag. We show that this computation is linear in the size of the term we compute on, extending to trees the technique of the Knuth-Morris-Pratt string-pattern-matching algorithm. Our technique uses a new notion of top-down tree automaton with a markers stack.

Strong sequentiality is decidable and is easily verified in many important subcases, under which fall, for instance, combinatory logic, primitive recursive schemes and recursive programming languages with constructors such as HOPE [5] and ML.

4 Sequential and Strongly Sequential Systems

4.1 A Definition of Sequentiality

Obviously, some rewriting systems require parallel evaluation strategies in order to compute normal forms. Take, for instance, the well-known example of the *parallel or*, defined by "true or $x \to$ true," "x or true $\to$ true," where we use an infix notation for the binary operator *or*. For computing any term $T = T_1$ or T_2, one needs a parallel evaluation of T_1 and T_2, since it is not generally decidable whether T_1 or T_2 produces the value *true*. Furthermore, in case both T_1 and T_2 evaluate to *true*, one easily shows

that needed redexes (in the sense of section 3) could not exist. This is because the above system is ambiguous. Fortunately, the nonambiguity condition prevents us from considering such systems. And we showed (in section 3) that call-by-need evaluation strategies exist in our systems. Hence, in order to get a normal form, one has just to contract needed redexes.

However, even if we know the existence of these critical needed redexes, there still remains to find them in any given term. Unfortunately, this can also demand some parallelism, as illustrated in the following example inspired from Berry [2] and which is obtained by a slight modification of *parallel or*. Consider any system Σ containing the three rules

$$\{F(A, B, x) \to C, F(x, A, B) \to C, F(B, x, A) \to C\}.$$

Note first that these rules are indeed nonambiguous, since none of these three left-hand sides can be unified with any other one. We know from section 3 that in any term $T = F(T_1, T_2, T_3)$ there is some needed redex. But as for the previous example of the *parallel or*, it is impossible to avoid parallel evaluation, since one cannot in general decide whether T_1, T_2, T_3 evaluate to A or B.

Thus, the nonambiguity condition is not sufficient for insuring a sequential evaluator. It seems necessary to consider the way in which terms are traversed in order to find some needed redex. We shall restrict our attention to top-down searches and develop some useful notations for prefixes of terms.

We represent prefixes by Ω-terms, i.e., by terms where a new nullary function symbol Ω can occur. Intuitively, Ω means the part of the term which has not been yet traversed. Let T_Ω be the set of these Ω-terms. Let us also consider the prefix ordering $\preceq$ on T_Ω obviously defined as

$\Omega \preceq M$ for any $M \in T_\Omega$,

$F(M_1, M_2, \ldots, M_n) \preceq F(N_1, \ldots, N_n)$ if $M_i \preceq N_i$ for $1 \leq i \leq n$,

$x \preceq x$ for any variable x.

All the previously defined operations on terms are obviously extended to Ω-terms. One just adds a very few new notions.

If $M \preceq P$ and $N \preceq P$, then M and N are said to be *compatible*, written $M \uparrow N$. The greatest lower bound (glb) of two Ω-terms M and N, written $M \sqcap N$, and the least upper bound (lub) of two compatible Ω-terms M and N, written $M \sqcup N$, are defined as follows:

$$F(M_1, \ldots, M_n) \sqcap F(N_1, \ldots, N_n) = F(M_1 \sqcap N_1, \ldots, M_n \sqcap N_n),$$

$$x \sqcap x = x,$$

$$M \sqcap N = \Omega \text{ in all other cases}$$

$$\Omega \sqcup M = M \sqcup \Omega = M,$$

$$F(M_1, \ldots, M_n) \sqcup F(N_1, \ldots, N_n) = F(M_1 \sqcup N_1, \ldots, M_n \sqcup N_n),$$

$$x \sqcup x = x$$

It will be convenient to write M_Ω for $\sigma(M)$, where σ is the substitution such that $\sigma(x) = \Omega$ for all variables x. We define a predicate *nf* on Ω-terms by $nf(M)$ iff M has a normal form, i.e., $\exists N \in \mathcal{NF}$ such that $M \xrightarrow{*} N$. Note that we require N to be a term, i.e., without Ωs. An Ω-term N such that $\mathcal{R}(N) = \varnothing$ will be called an Ω-normal form. Suppose that the set $B = \{\text{tt}, \text{ff}\}$ of boolean values is ordered by $\text{ff} \sqsubseteq \text{tt}$. Then it is straightforward to check that nf is a monotonic predicate on the set of Ω-terms:

LEMMA 4.1 If $M \preceq N$, then $\text{nf}(M) \sqsubseteq \text{nf}(N)$.

Proof First one shows that if $M \xrightarrow{*} N$ and $M \preceq M'$, there is an N' such that $M' \xrightarrow{*} N'$ and $N \preceq N'$. This is because our systems are linear. Second, if N is in normal form and $N \preceq N'$, then N' is also in normal form, since then $N = N'$. ■

For the time being, we forget the problem of defining sequential evaluators and show the relevance for our systems of a notion of stability defined in Berry [2]. In [2] it is claimed in an axiomatic way that this notion is median between sequential and parallel functions, and a stable model of the lambda-calculus is constructed. (Note that the problem is still open for sequential models.) We show that the predicate *nf* is stable, i.e., conditionally multiplicative, in our systems. For this purpose, consider temporarily a new kind of rewriting rules, the ω-rules, associated with any system Σ and defined as follows: $M \to_\omega N$ iff, for some $u \in \mathcal{O}(M)$ and redex scheme α in Σ, one has $M/u \uparrow \alpha_\Omega$ but not $\alpha_\Omega \preceq M/u$ and $N = M[u \leftarrow \Omega]$ (i.e., M/u is compatible with a redex scheme but is not a redex).

LEMMA 4.2 $(\to \cup \to_\omega)^*$ is confluent.

Proof We know already that $\xrightarrow{*}$ is confluent. Now if we temporarily write $M \xrightarrow{\varepsilon} N$ for $M \to N$ or $M = N$, and similarly for $M \xrightarrow{\varepsilon}_\omega N$, the nonambiguity condition of our system enables us to prove the lemma via the two properties summarized in the two diagrams of figure 9. ■

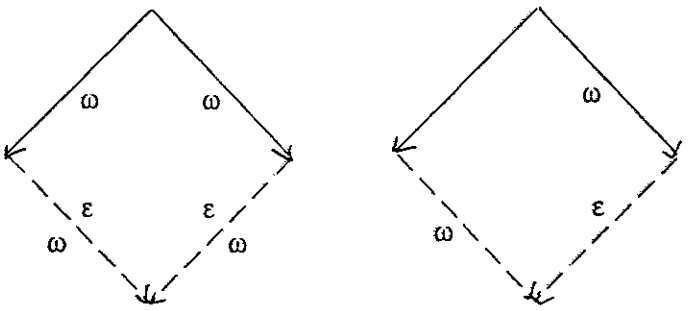


Figure 9
Church-Rosser diagrams for ω-rules.

The ω-rules will be also useful for the semantics of our systems. (Note that they correspond exactly to the ones defined by Wadsworth [23] in the λ-calculus.) Finally, let $\wedge$ also denote the trivial glb operation in truth values domain B (i.e., the usual *and*). We now prove Berry's stability property: nf is conditionally multiplicative.

LEMMA 4.3 If $M \uparrow N$, then $\mathrm{nf}(M \sqcap N) = \mathrm{nf}(M) \wedge \mathrm{nf}(N)$.

Proof By monotonicity, one easily has $\mathrm{nf}(M \sqcap N) \sqsubseteq \mathrm{nf}(M) \wedge \mathrm{nf}(N)$. Conversely, the only problem is when $\mathrm{nf}(M) \wedge \mathrm{nf}(N) = \mathrm{tt}$, i.e., when M and N have a normal form. By monotonicity, we know that $\mathrm{nf}(M \sqcup N) = \mathrm{tt}$. We work by induction on the length $d(M \sqcup N)$ of any outside-in derivation A from $(M \sqcup N)$ to its normal form. Let $A = A_1 A_2$, with $A_1 : (M \sqcup N) \xrightarrow{u} P$. Then $u \in \mathscr{E}(M \sqcup N)$. We claim that $u \in \mathscr{R}(M \sqcap N)$. Otherwise we have 3 cases.

Case 1: $u = u_1 u_2$, with $(M \sqcap N)/u_1 = \Omega$ and $u \in \mathscr{R}(M)$. Then since

$M \preceq (M \sqcup N)[u \leftarrow \Omega]$,

we know by monotonicity that $(M \sqcup N)[u \leftarrow \Omega]$ has a normal form. This contradicts $u \in \mathscr{E}(M \sqcup N)$

Case 2: $u = u_1 u_2$, with $(M \sqcap N)/u_1 = \Omega$ and $u \in \mathscr{R}(N)$. Same as in case 1.

Case 3: $u \in \mathscr{O}(M \sqcap N)$, but $u \notin \mathscr{R}(M \sqcap N)$. This means that if α is the redex scheme such that $(M \sqcup N)/u = \sigma\alpha$, one has $M/u \preceq (M \sqcup N)/u \uparrow \alpha_\Omega$, and similarly $N/u \uparrow \alpha_\Omega$. Now since $u \notin \mathscr{R}(M \sqcap N)$, one cannot have $\alpha_\Omega \preceq M/u$ and $\alpha_\Omega \preceq N/u$. Assume, for instance, that not $\alpha_\Omega \preceq M/u$. Then $M \to_\omega M[u \leftarrow \Omega]$. By the previous confluence property, $M[u \leftarrow \Omega]$ has a normal form. Let $M_1 = M[u \leftarrow \Omega]$. We conclude that $(M_1 \sqcup N)[u \leftarrow \Omega]$ has a normal form by reasoning as in the first two cases, and by monotonicity, $(M \sqcup N)[u \leftarrow \Omega]$ has a normal form. This contradicts $u \in \mathscr{E}(M \sqcup N)$.

Thus $u \in \mathscr{R}(M \sqcap N)$. Now, let $M \xrightarrow{u} M'$ and $N \xrightarrow{u} N'$. Since $\mathrm{nf}(M) = \mathrm{nf}(N) = \mathrm{tt}$, one has also that $\mathrm{nf}(M') = \mathrm{nf}(N') = \mathrm{tt}$. Furthermore, $M \sqcup N \xrightarrow{u} M' \sqcup N'$. Finally, by induction we know that $M' \sqcap N'$ has a normal form, and thus $\mathrm{nf}(M \sqcap N) = \mathrm{tt}$. ■

COROLLARY For any Ω-term M having a normal form there is a (unique) minimum prefix N of M having a normal form.

This is not true in general in ambiguous systems. Consider again the *parallel or* example. Let M = true or true. Then, both "true or Ω" and "Ω or true" have a normal form, but not "Ω or Ω." Now we can also consider the already mentioned example of Berry [2], which is linear and nonambiguous. Let $T = F(T_1, T_2, T_3)$. We know by the previous result that there is a minimum prefix of T having a normal form if $\mathrm{nf}(T) = \mathrm{tt}$. But nothing ensures us that for all Ts of this form, we need to compute always a fixed argument of F. Here this is false, since $\mathrm{nf}(F(\Omega, \Omega, \Omega)) = \mathrm{ff}$, but $\mathrm{nf}(F(A, B, \Omega)) = \mathrm{tt}$, $\mathrm{nf}(F(\Omega, A, B)) = \mathrm{tt}$, and $\mathrm{nf}(F(B, \Omega, A)) = \mathrm{tt}$. (Note that this does not contradict our last result, since these last three terms are not compatible with respect to the prefix ordering.) Using our new notations for prefixes of terms, we can state what is going wrong in the previous example. In $M = F(\Omega, \Omega, \Omega)$ it is possible to get normal forms

by increasing some Ωs. But there is not a uniform one to consider. One can always increase the two other Ωs in order to get a normal form.

For short, let a sequential system be a term rewriting system which has a sequential evaluator. (Thus the two previously considered systems are not sequential.) In a sequential system it seems required that in any Ω-term in Ω-normal form (having at least one Ω occurrence) there is an occurrence of Ω which it is necessary to increase in order to make the nf predicate true. This means, when we forget Ω-terms, that for all terms not in normal form there is a redex occurrence u such that if we consider redexes as black boxes which might produce any term, it is impossible to get a normal form without computing the redex of occurrence u. In other words, there is always one needed redex occurrence which can be found without any look-ahead computation.

Technically, we want the predicate *M has a normal form* to be sequential in the sense of Kahn-Plotkin [17]. So let us first recall what a sequential predicate is in their sense.

DEFINITION 4.4 Let P be a monotonic predicate on T_Ω (with the truth values domain B ordered as before). An occurrence u of M is said to be an *index* of P in M iff $M/u = \Omega$ and $\forall N$ such that $N \succeq M$, $P(N) = \text{tt}$ implies $N/u \neq \Omega$. Then P is *sequential* at M iff whenever $P(M) = \text{ff}$ and $\exists N \succeq M$ such that $P(N) = \text{tt}$, it follows that there exists an index of P in M. Finally, P is said to be sequential iff it is sequential at every M in T_Ω.

Thus, if P is a sequential predicate and if there is an Ω in M and $P(M) = \text{ff}$, there is a critical Ω in M, namely an index, to increase in order to make the predicate true.

DEFINITION 4.5 Σ is a sequential system iff nf is a sequential predicate.

Note that this definition is independent of the linearity and nonambiguity assumptions. However, when these assumptions are true, it is possible to show that sequentiality is only required at Ω-normal forms. So let $\omega'(M)$ be M, where all (outermost) redexes are replaced by the constant Ω.

LEMMA 4.6 Σ is a sequential system iff the predicate $\text{nf}(M)$ is sequential at any Ω-normal form M (i.e., such that $M = \omega'(M)$).

Proof By induction on $n = \text{Min}\{d(N) | M \preceq N, \text{nf}(N) = \text{tt}\}$. Let $n = 0$. Then there is N in normal form such that $M \preceq N$. Thus $M = \omega'(M)$ and this case is trivial. Let $n > 0$ and u be an index of nf in $\omega'(M)$. We have two possibilities. First $M/u = \Omega = \omega'(M)/u$. As $\omega'(M) \preceq M$, the occurrence u is also an index in M. Second, $M/u \neq \Omega$, which means M/u is a redex. Then $\text{nf}(N[u \leftarrow \Omega]) = \text{ff}$ for all N such that $M \preceq N$, and $\text{nf}(N) = \text{tt}$. Thus $u \in \mathcal{N}(N)$. Let $M \xrightarrow{u}_k M'$ and $N \xrightarrow{u}_k N'$. Then $d(N') < d(N)$, by lemma 3.34. Therefore, by induction there is an index v' of nf in M'. Now let v be the occurrence of M defined by $v = v'$ if u and v' are disjoint, $v = uwu'$ if $v' = uw'u'$, where $\alpha_k/w = \beta_k/w' \in \mathcal{V}$. One then easily checks that v is an index of nf in M. ∎

This last lemma is not true for ambiguous systems. Take

$$\Sigma = \{F(A, B, x) \to K, F(B, x, A) \to K, C \to A, C \to B\}$$

and consider $M = F(C, \Omega, \Omega)$. Then there is no index in M, but Σ is sequential at any Ω-normal form.

Going back to the evaluation strategy problem, an easy terminating strategy can be designed: for any term M, contract a redex at occurrence u which is an index of nf in $\omega'(M)$. (Then $u \in \mathcal{N}(M)$ if M has a normal form.) But, in general, these indexes are not computable. For instance, take Σ containing the following four rules:

$F(G(A, x), B) \to K$

$F(G(x, A), C) \to K$

$F(D, x) \to K$

$G(E, E) \to \cdots$

Then if $M = F(G(\Omega, \Omega), \Omega)$, the occurrence $u = 2 \in \mathcal{O}(M)$ is an index iff $G(E, E)$ cannot derive to D, which is not decidable, since Σ can have many other rules. Therefore, neither indexes, nor sequentiality are decidable. In order to make these two questions effective, we consider more restricted systems. The idea is to forget right hand sides of rules and to make sequentiality be determined only from the left hand sides.

4.2 Strongly Sequential Systems

In order to forget right hand sides of rules, we introduce a new derivation relation; the intuition behind it is to permit any redex to produce any term.

NOTATION Let M and N be two Ω-terms. We say that M possibly reduces to N, and write $M \to' N$, iff $N = M[u \leftarrow P]$ for some redex occurrence $u \in \mathcal{R}(M)$ and Ω-term P. Furthermore, let $\mathrm{nf}'(M) = \mathrm{tt}$ iff $M \xrightarrow{*}{}' N$ for some term N in normal form.

DEFINITION 4.7 Σ is a *strongly sequential system* iff the predicate nf′ is sequential at any M in Ω-normal form. We denote by $\mathcal{I}(M)$ the set of indexes of nf′ in M.

The restriction to Ω-normal forms is necessary here because lemma 4.6 is no longer true for strong sequentiality. (Take $\Sigma = \{F(A, B, x) \to K, F(B, x, A) \to K\}$ and consider $M = F(R, \Omega, \Omega)$, where R is a redex. Then $\omega'(M) = F(\Omega, \Omega, \Omega)$ has an index for nf′, but M has not.) However, since $\mathrm{nf}'(M) = \mathrm{ff}$ iff M is in Ω-normal form and $\mathrm{nf}(M)$ implies $\mathrm{nf}'(M)$ for any M, a strongly sequential system is sequential by lemma 4.6.

DEFINITION 4.8 A redex occurrence $u \in \mathcal{R}(M)$ is strongly needed iff $u \in \mathcal{I}(\omega'(M))$.

Clearly, if $u \in \mathcal{R}(M)$ is strongly needed, then u is needed, i.e., $u \in \mathcal{N}(M)$. We want to show that strongly needed redex occurrences in M can be found effectively and that the strong sequentiality of any system Σ can be decided. A priori, this is not too straightforward. Consider for instance Σ having the following redex schemes:

$F(G(A, x), F(B, y)) \to \cdots$

$F(G(x, A), F(C, y)) \to \cdots$

$G(D, D) \rightarrow$

Then the smallest Ω-term without indexes for nf′ is

$M = F(G(\Omega, \Omega), F(G(\Omega, \Omega), \Omega))$.

(Notice that M is not a prefix of any redex scheme.)

4.3 Decision Procedures for Strongly Needed Redexes

We show that strongly needed redexes can be computed with the help of some *direct approximation* function. (An alternative characterization will be also given in the next section). This function will represent for any term the maximum prefix which is guaranteed to remain after any derivation. Before introducing this function, we need to introduce some notation.

If E is any set of Ω-terms, we write $M \preceq E$ iff $\exists N \in E$ such that $M \preceq N$. Similarly for $M \succeq E$ and $M \uparrow E$. Finally, with $\text{red}_\Omega = \{\alpha_\Omega | \alpha \in \text{Red}\}$, we write $M \uparrow$ for $M \uparrow \text{red}_\Omega$.

DEFINITION 4.9 For any Ω-term M, we define its *direct approximation* $\omega(M)$ and *internal direct approximation* $\bar{\omega}(M)$ by

$\omega(x) = \bar{\omega}(x) = x$,

$\omega(\Omega) = \bar{\omega}(\Omega) = \Omega$,

$\bar{\omega}(F(M_1, M_2, \ldots, M_n)) = F(\omega(M_1), \omega(M_2), \ldots, \omega(M_n))$,

$\omega(F(M_1, M_2, \ldots, M_n)) = \Omega$ if $\bar{\omega}(F(M_1, M_2, \ldots, M_n)) \uparrow$,

$= \bar{\omega}(F(M_1, M_2, \ldots, M_n))$ otherwise.

Remark We can see $\omega(M)$ as the $\rightarrow_\omega$-normal form of $\omega'(M)$. It follows that $\omega(M) = \bigsqcap \{N | M \overset{*}{\rightarrow}' N\}$. Intuitively, $\omega(M)$ is the maximum prefix of M guaranteed to exist in every N which can be derived from M, whatever could be the right-hand sides of the rules in Σ. Our terminology is consistent with [23] for λ-calculus and [18, 20] for recursive program schemes.

We note that in our TRSs, subparts of redex schemes are their own direct approximation:

LEMMA 4.10 Let $\alpha \in \text{red}$. For every nonempty u in $\mathcal{O}(\alpha)$, we have $\omega(\alpha_\Omega/u) = \alpha_\Omega/u$.

Proof Induction on the size of α_Ω/u. If $\alpha_\Omega/u = \Omega$, it is trivial. Otherwise, by induction hypothesis we get $\bar{\omega}(\alpha_\Omega/u) = \alpha_\Omega/u$. By nonambiguity, we cannot have $\alpha_\Omega/u \uparrow$, and thus $\omega(\alpha_\Omega/u) = \alpha_\Omega/u$. ∎

It follows easily from its definition that ω is monotonic increasing, i.e., $M \preceq N$ implies $\omega(M) \preceq \omega(N)$, idempotent, i.e., $\omega(M) = \omega^2(M)$, and a projection, i.e., $\omega(M) \preceq M$. More important, if $M \overset{*}{\rightarrow} N$, then $\omega(M) \preceq \omega(N)$. Namely, approximation is increasing with derivation (In fact, this is true also with $\overset{*}{\rightarrow}'$). Finally, an Ω-term M is said to be *ω-maximum* iff for every N such that $N \succeq M$ we have $\omega(N) = \omega(M)$.

Now we connect strongly sequential systems and the direct approximation function by showing that $\mathrm{nf}'(M) = \mathrm{tt}$ if M is ω-maximum.

NOTATION Let $x \in \mathscr{V}$. For any $M \in \mathscr{T}_\Omega$ we define M_x as M in which all Ωs are replaced by xs, i.e., $M_x = M[u \leftarrow x | M/u = \Omega]$.

LEMMA 4.11 $M \xrightarrow{*}{}' N$ iff $\omega(M_x) \preceq N_x$ for some $x \notin \mathscr{V}(M) \cup \mathscr{V}(N)$.

Proof By induction on the size of M. The substitution of variables for the Ωs is a technical trick. ■

LEMMA 4.12 Let $x \in \mathscr{V}$ and $u \in \mathscr{O}(M)$ such that $M/u = \Omega$. Then $\omega(M) = \omega(M[u \leftarrow x])$ iff $\omega(M) = \omega(M[u \leftarrow N])$ for all Ω-terms N.

Proof Obvious, once one remarks that $M[u \leftarrow x] \uparrow$ implies $M[u \leftarrow N] \uparrow$ for every N. ■

LEMMA 4.13 We have $\mathrm{nf}'(M) = \mathrm{tt}$ iff M is ω-maximum iff $\omega(M) = \omega(M_x)$, where x is any variable.

Proof Let $\mathrm{nf}'(M) = \mathrm{tt}$. Then $M \xrightarrow{*}{}' N$ for some normal form N. Thus $\omega(M_y) \preceq N_y$ by lemma 4.11 if $y \notin \mathscr{V}(M) \cup \mathscr{V}(N)$. But $N_y = N$ as N is a normal form (without Ω's). Thus $y \notin \mathscr{V}(\omega(M_y))$ and $\omega(M_y) \preceq M$. By monotonicity one gets $\omega(\omega(M_y)) \preceq \omega(M)$. Hence $\omega(M_y) \preceq \omega(M)$, since ω is idempotent. Finally, $\omega(M) = \omega(M_y)$ because $M \preceq M_y$. By lemma 4.12, the Ω-term M is ω-maximum. Conversely, let M be ω-maximum. Then $\omega(M) = \omega(M_y)$ for some $y \notin \mathscr{V}(M)$. Thus $\omega(M_y) \preceq M \preceq M_z$ for some $z \neq y$. Therefore, $M \xrightarrow{*}{}' M_z$, since $y \notin \mathscr{V}(M) \cup \mathscr{V}(M_z)$ by lemma 4.11, and $\mathrm{nf}'(M) = \mathrm{tt}$. Now M is ω-maximum iff $\omega(M) = \omega(M_x)$, by lemma 4.12. ■

Therefore, one can easily decide if $\mathrm{nf}'(M) = \mathrm{tt}$ by testing $\omega(M_x) = \omega(M)$. Similarly, an index of the nf′ predicate can also be effectively found with the help of the following lemma.

LEMMA 4.14 Let $x \in \mathscr{V}$ and $u \in \mathscr{O}(M)$ such that $M/u = \Omega$. Then $u \in \mathscr{S}(M)$ iff $\omega(M) \neq \omega(M[u \leftarrow x])$ iff $u \in \mathscr{O}(\omega(M[u \leftarrow x]))$.

Proof Let $u \notin \mathscr{S}(M)$. That is, there is some $N \succeq M$ such that $\omega(N) = \omega(N_x)$ and $N/u = \Omega$. Then $M[u \leftarrow x] \preceq N_x$, and $\omega(M[u \leftarrow x] \preceq \omega(N_x) = \omega(N)$. Without loss of generality, by lemma 4.12 we may assume $x \notin \mathscr{V}(N)$. Thus $u \notin \mathscr{O}(\omega(M[u \leftarrow x]))$, and $\omega(M[u \leftarrow x]) \preceq M$, which implies that $\omega(M[u \leftarrow x]) \preceq \omega(M)$. Therefore, $\omega(M) = \omega(M[u \leftarrow x])$. For the converse, assume $\omega(M) = \omega(M[u \leftarrow x])$. This implies that Σ is not empty. Let α be any redex scheme, and consider

$$N = M[v \leftarrow \alpha | M/v = \Omega \text{ and } v \neq u].$$

We have $N \succeq M$ and $\omega(N_x) = \omega(M[u \leftarrow x]) = \omega(M) \preceq \omega(N)$. Thus $\omega(N_x) = \omega(N)$, and since $N/u = \Omega$, we have $u \notin \mathscr{S}(M)$. Finally, $\omega(M) = \omega(M[u \leftarrow x])$ implies $u \notin \mathscr{O}(\omega(M[u \leftarrow x]))$, which concludes the proof. ■

In order to find a strongly needed redex occurrence u in any given term M, one has just to test $u \in \mathcal{O}(\omega(M[u \leftarrow x]))$ for some redex occurrence u and any given variable x. Thus a strongly needed redex is a redex which may increase, after its reduction, the direct approximation.

Let us now give a few properties of indexes needed in the following.

LEMMA 4.15 If $uv \in \mathscr{I}(M)$ and $\omega(M/u) = \Omega$, then $u \in \mathscr{I}(M[u \leftarrow \Omega])$.

Proof Since $\omega(M[uv \leftarrow x]) \neq \omega(M) = \omega(M[u \leftarrow \Omega])$, $\omega(M[u \leftarrow x]) \neq \omega(M[u \leftarrow \Omega])$, by lemma 4.12 with $N = M/u[v \leftarrow x]$. ■

LEMMA 4.16 If $uv \in \mathscr{I}(M)$, then $v \in \mathscr{I}(M/u)$.

Proof Let $uv \in \mathscr{I}(M)$ and $v \notin \mathscr{I}(M/u)$. Then $\omega(M/u) = \omega(M/u[v \leftarrow x])$, by lemma 4.14, and therefore

$$\omega(M) = \omega(M[u \leftarrow \omega(M/u)] = \omega(M[u \leftarrow \omega(M/u[v \leftarrow x])]) = \omega(M[uv \leftarrow x]),$$

which contradicts $uv \in \mathscr{I}(M)$. ■

LEMMA 4.17 If $\omega(M/u) = \Omega$, then for every v such that $v|u$, we have $v \in \mathscr{I}(M)$ equivalent to $v \in \mathscr{I}(M[u \leftarrow \Omega])$.

Proof Obvious, since $\omega(M/u) = \Omega$ implies

$$\omega(M) = \omega(M[u \leftarrow \Omega])$$

$$\omega(M[v \leftarrow x]) = \omega(M[u \leftarrow \Omega][v \leftarrow x]).$$ ■

We are now left with the problem of deciding whether a TRS Σ is strongly sequential or not. Since this property is closely related to the organization of matching of redexes, we shall first consider this problem.

5 Deterministic Automata for Strongly Sequential Systems

In this section we show how the previous definitions permit to derive algorithms for computing strongly needed redexes. First we consider matching of redexes and see that this operation is sequential in case of strongly sequential systems.

5.1 Sequential Sets, Directions

DEFINITION 5.1 Let E be any set of Ω-terms. The predicate *match* is defined by $\text{match}(M, E) = \text{tt}$ iff $M \geq E$. We say that E is *sequential* iff $\text{match}(M, E)$ is sequential at every M. The set $\text{Dir}(M, E)$ of the indexes of this predicate in M is called the set of *directions* from M to E. As a particular case we shall consider the predicate $\text{isredex}(M) = \text{match}(M, \text{red}_\Omega)$ and will abbreviate its set of indexes $\text{Dir}(M, \text{red}_\Omega)$ as $\text{Dir}(M)$.

We have, of course, $\mathscr{I}(M) \subseteq \text{Dir}(M)$, and we shall show below that if Σ is strongly sequential, isredex is also sequential. Furthermore, the decidability of sequentiality of isredex is straightforward.

NOTATION We write $M \# N$ iff M and N are incompatible, i.e., not $M \uparrow N$, and also $M \# E$ iff not $M \uparrow N$, i.e., $M \# N$ for all N in E, and $M \#$ iff $M \# \mathrm{red}_\Omega$.

LEMMA 5.2 Let M be an Ω-term and E be a set of Ω-terms. Then the following are equivalent:

$u \in \mathrm{Dir}(M, E)$

$M/u = \Omega$, and for all $N \in E$ such that $N \uparrow M$, one has $u \in \mathcal{O}(N)$ and $N/u \neq \Omega$

$M/u = \Omega$ and $M[u \leftarrow x] \# E$, where x is not a variable of any Ω-term in E

Proof Obvious ■

LEMMA 5.3 If $N \in E$ and $N \uparrow M$, then $\mathrm{Dir}(M, E) = \mathrm{Dir}(M \sqcap N, E')$, where $E' = \{P \in E | P \uparrow M\}$.

Proof Suppose $u \in \mathrm{Dir}(M, E)$. Then $M/u = \Omega$. But, since $N \in E$ and $N \uparrow M$, lemma 5.2 says that $u \in \mathcal{O}(N)$. Thus $u \in \mathcal{O}(M \sqcap N)$ and $(M \sqcap N)/u = \Omega$. Now if $P \in E'$ and $P \uparrow (M \sqcap N)$, then $P \uparrow M$ and $u \in \mathcal{O}(P)$, with $P/u \neq \Omega$. Thus $u \in \mathrm{Dir}(M \sqcap N, E')$. Conversely, since $(M \sqcap N)/u = \Omega$ and $N \in E'$, with $N \uparrow (M \sqcap N)$, one has $u \in \mathcal{O}(N)$ and $N/u \neq \Omega$. Thus $M/u = \Omega$ because $N \uparrow M$. Now take any $P \in E$ such that $P \uparrow M$. Then $P \in E'$ and $P \uparrow (M \sqcap N)$. Thus $u \in \mathcal{O}(P)$ and $P/u \neq \Omega$. That is, $u \in \mathrm{Dir}(M, E)$. ■

LEMMA 5.4 For any finite E, one can decide if E is sequential.

Proof One just checks that $\mathrm{match}(M, E)$ is sequential at every M such that $M \preceq E$. ■

LEMMA 5.5 If Σ is a strongly sequential system, then isredex is a sequential predicate.

Proof If E is sequential, then $\mathrm{match}(M, E)$ is sequential at every M and in particular at every Ω-term prefix of some element of E. Conversely, E is sequential iff $\mathrm{Dir}(M, E) \neq \varnothing$ for all M not in normal form such that $\mathrm{match}(M, E) = \mathrm{ff}$. Now if $M \# E$, then

$$\mathrm{Dir}(M, E) = \{u \in \mathcal{O}(M) | M/u = \Omega\} \neq \varnothing,$$

since M is not in normal form. Assume that $M \uparrow N$ for some $N \in E$. Then $\mathrm{Dir}(M, E) \supseteq \mathrm{Dir}(M \sqcap N, E)$, by lemma 5.3. But $M \sqcap N \preceq N$ and $\mathrm{match}(M \sqcap N, E) = \mathrm{ff}$, since $\mathrm{match}(M, E) = \mathrm{ff}$. Thus $\mathrm{Dir}(M \sqcap N, E) \neq \varnothing$, which implies $\mathrm{Dir}(M, E) \neq \varnothing$.

Now let Σ be strongly sequential. Then nf′ is sequential at every M in Ω-normal form (i.e., $\omega'(M) = M$), by definition. Thus isredex is sequential at every Ω-normal form M. But since Σ is not ambiguous, every N such that $N \preceq \mathrm{red}_\Omega$ is in Ω-normal form. Thus $\mathrm{isredex}(M) = \mathrm{match}(M, \mathrm{red}_\Omega)$ is sequential at every M according to the first part of the proof. ■

Intuitively, a set E of Ω-terms is sequential iff we can factor the matching of any term against E. As in Curien [6], for the general case of sequential functions, this factorization can be achieved here by a tree gathering all possible patterns. For instance, take

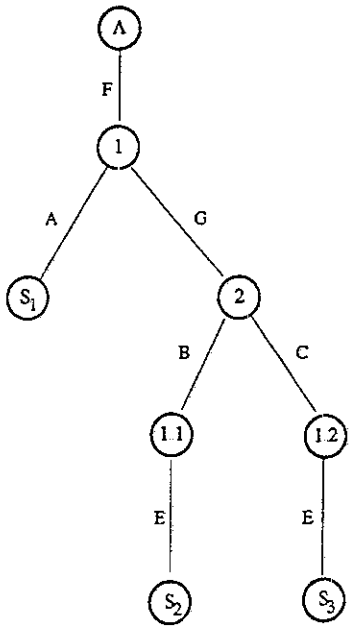


Figure 10

$$E = \{F(A, \Omega), F(G(E, \Omega), B), F(G(\Omega, E), C)\},$$

which is sequential. The corresponding *matching tree* is shown in figure 10, where a node u corresponds to a query of the function symbol at occurrence u in the term M, branches from u are the different successful alternatives for this function symbol query, and leaves are conventionally written as success nodes S_i. These matching trees generalize the trie structures used to store dictionaries of strings.

This tree can easily be built up from Ω-terms M such that $M \preceq E$ by considering the $\mathrm{Dir}(M, E)$ sets. We shall see in the next section how, in the case of strongly sequential systems, we may construct matching dags which permit to factorize local and global matches, yielding an efficient (linear) match algorithm.

5.2 Matching Dags

For the time being, we forget strong sequentiality. We only define matching dags and shall show in the next section that these dags exist for a rewriting system iff it is strongly sequential.

NOTATION We shall write $M \prec \mathrm{red}_\Omega$ for $M \preceq \mathrm{red}_\Omega$ and $M \notin \mathrm{red}_\Omega$, and $M\uparrow_+$ for $M\uparrow$ and $M \neq \Omega$.

HYPOTHESIS We assume the existence of a function Q mapping every Ω-term M such that $M \prec \mathrm{red}_\Omega$ into a non-empty set of its directions, $\varnothing \neq Q(M) \subseteq \mathrm{Dir}(M)$, and such that $\forall u \in Q(M), \forall N \prec \mathrm{red}_\Omega$, if $M[u \leftarrow N] \prec \mathrm{red}_\Omega$, then

$\exists v\ uv \in Q(M[u \leftarrow N])$ (Q_1)

$\forall v\ uv \in Q(M[u \leftarrow N])$ implies $v \in Q(N)$ (Q_2)

As previously stated, the existence of such a mapping will be shown in the case of strong sequentiality. Intuitively, an occurrence in $Q(M)$ is a direction for matching redexes and is also a direction for internal matches (by condition Q_2). Furthermore, it is possible to find some (strongly needed) redex without any dangling partial matches (condition Q_1).

CONSTRUCTION To construct a matching dag associated with Q and verifying Q_1 and Q_2, let $\text{red} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. We consider $\text{Occ} = \bigcup \{\bar{\mathcal{O}}(\alpha_i) | i \leq n\}$ and a set $\text{Succ} = \{S_1, S_2, \ldots, S_n\}$ of success tokens disjoint from Occ. We construct a graph whose nodes are all

a pairs $\langle M, v\rangle$ such that $M \prec \text{red}_\Omega$, $v \in Q(M)$ and $\forall u\ M/u\uparrow_+ \Rightarrow u \prec v$,
b pairs $\langle \alpha_{i\Omega}, S_i\rangle$, with $1 \leq i \leq n$ (the success nodes)

Let $\langle M, v\rangle$ be a node, F a function symbol in $\mathcal{F}$, and $M' = M[v \rightarrow F\vec{\Omega}]$. If $\langle M', *\rangle$ is a node, where $*$ denotes any element of Occ $\cup$ Succ, we draw an arc: $\langle M, v\rangle \xrightarrow{F} \langle M', *\rangle$. It is clear that our graphs are directed acyclic. We shall therefore call them *matching dags*.

DEFINITION 5.6 The node $\langle M, *\rangle$ is said to be *accessible* iff either it is the origin node $\langle \Omega, \Lambda\rangle$ or there is some accessible node $\langle M_1, v_1\rangle$ and $F \in \mathcal{F}$ such that $\langle M_1, v_1\rangle \xrightarrow{F} \langle M, *\rangle$

LEMMA 5.7 If $\langle M, v\rangle$ is accessible, then for every u such that $M/u\uparrow_+$ the pair $\langle M[u \leftarrow \Omega], u\rangle$ is an accessible node and there exists in the matching dag a path

$$\langle M[u \leftarrow \Omega], u\rangle = \langle M[u \leftarrow N_1], uw_1\rangle \xrightarrow{F_1} \langle M[u \leftarrow N_2], uw_2\rangle \xrightarrow{F_2} \cdots \xrightarrow{F_{k-1}} \langle M[u \leftarrow N_k], uw_k\rangle = \langle M, v\rangle.$$

Furthermore, for every such path there exists in the matching dag an identically labeled path

$$\langle \Omega, \Lambda\rangle = \langle N_1, w_1\rangle \xrightarrow{F_1} \langle N_2, w_2\rangle \xrightarrow{F_2} \cdots \xrightarrow{F_{k-1}} \langle N_k, w_k\rangle = \langle M/u, v/u\rangle$$

Proof Induction on the definition of accessible. If $M = \langle \Omega, \Lambda\rangle$, the proof is trivial. Otherwise, assume $\langle M, v\rangle$ is an accessible node satisfying the condition above such that $\langle M, v\rangle \xrightarrow{F} \langle M', v'\rangle$. Now let u be such that $M'/u\uparrow_+$. Since $\langle M', v'\rangle$ is a node, we have $u \prec v'$. Similarly, as $\langle M, v\rangle$ is a node, the case $u | v$ is impossible, and therefore $u \preceq v$. $M'[u \leftarrow \Omega] = M[u \leftarrow \Omega]$, and $M'/u = M/u[v/u \leftarrow F\vec{\Omega}]$, and thus $M/u\uparrow$.

Case 1: $u = v$. Then $\langle M'[u \leftarrow \Omega], u\rangle = \langle M, v\rangle$ is accessible by hypothesis. Also $M'/u = F\vec{\Omega}$ and $M'/u\uparrow$ implies $M'/u \preceq \text{red}_\Omega$.

Case 2: Otherwise, we get $M/u\uparrow_+$, and by the induction hypothesis we get $\langle M'[u \leftarrow \Omega], u\rangle = \langle M[u \leftarrow \Omega], u\rangle$ accessible, and $\langle M/u, v/u\rangle$ is a node. In particular $M/u \prec \text{red}_\Omega$, and $v/u \in Q(M/u) \subseteq \text{Dir}(M/u)$. Therefore in this case too $M'/u \preceq \text{red}_\Omega$.

In both cases we have shown that $\langle M'[u \leftarrow \Omega], u\rangle$ is an accessible node, and the existence of the first path for $\langle M', v'\rangle$ follows, since $\forall j \leq k\ M[u \leftarrow N_j] = M'[u \leftarrow N_j]$, assuming the existence of the path for $\langle M, v\rangle$, as in the statement of the lemma. Let us now show that the corresponding path for $\langle M/u, v/u\rangle$ can be extended. We have shown above that $M'/u \preceq \mathrm{red}_\Omega$. But since $\langle M', v'\rangle$ is a node, $M' \prec \mathrm{red}_\Omega$. We easily get $M'/u \prec \mathrm{red}_\Omega$, because either $u = \Omega$ or else we use the nonambiguity condition. Also, for all u', $M'/uu'\uparrow_+$ implies $u' \prec v'/u$. Finally, using $u \in Q(M[u \leftarrow \Omega]) = Q(M'[u \leftarrow \Omega])$, we get by condition Q_2 that $v'/u \in Q(M', u)$, and therefore $\langle M/u, v/u\rangle \xrightarrow{F} \langle M'/u, v'/u\rangle$.

■

LEMMA 5.8 Suppose $\langle M, v\rangle$ is an accessible node, and $N = M[v \leftarrow F\vec{\Omega}]$. If $N \uparrow \mathrm{red}_\Omega$, then $N \preceq \mathrm{red}_\Omega$.

Proof Suppose that $N \uparrow \alpha_\Omega$. $M \prec N \uparrow \alpha_\Omega$ and $v \in \mathrm{Dir}(M)$, so $\alpha_\Omega(v) = F$. If $\bar{\mathcal{O}}(N) \subset \bar{\mathcal{O}}(\alpha)$, then clearly $N \prec \alpha_\Omega$. So assume $\exists u \in \bar{\mathcal{O}}(N)$ with $u \notin \bar{\mathcal{O}}(\alpha)$. Let $u' \in \bar{\mathcal{O}}(N)$ such that $\alpha_\Omega/u' = \Omega$. Suppose $N(u') = H$. Since $\langle M, v\rangle$ is accessible, we thus have a path in the matching dag (notice that $u' \in \bar{\mathcal{O}}(M)$):

$$\begin{aligned}\langle \Omega, \Lambda\rangle &= \langle N_1, w_1\rangle \rightarrow \cdots \rightarrow \langle N_k, w_k\rangle \\ &= \langle N_k, u'\rangle \xrightarrow{H} \langle N_{k+1}, w_{k+1}\rangle \rightarrow \cdots \rightarrow \langle N_l, w_l\rangle \\ &= \langle M, v\rangle.\end{aligned}$$

Hence $N_k \prec M \prec N \uparrow \alpha_\Omega$ and $u' \in \mathrm{Dir}(N_k)$. So $\alpha_\Omega/u' \neq \Omega$. A contradiction. ■

COROLLARY When we remove inaccessible nodes from the matching dag, the terminal nodes are exactly all the success nodes.

This allows us to get rid of inaccessible nodes, which we shall assume from now on.

We use our matching dags to store the redex schemes of Σ. The search for a strongly needed redex is driven by the matching dag, in which are factored local and global matches. We need some extra information to know how to continue the search when the global match has failed but some local ones may be still successful. The idea is similar to the one used in the extension to multiple patterns of the Knuth-Morris-Pratt string-pattern-matching algorithm [13].

DEFINITION 5.9 Let $\langle M, v\rangle$ be a nonsuccess accessible node different form $\langle \Omega, \Lambda\rangle$, and u be the minimum non-null prefix of v such that $M/v\uparrow$. We define $\mathrm{Fail}(\langle M, v\rangle)$ as $\langle M/u, v/u\rangle$, which is an accessible node by lemma 5.7.

We shall indicate in our matching dags $w' = \mathrm{Fail}(w)$ by a dotted arc going from w to w'. Also, we shall drop the first component of the nodes, which is redundant since it is uniquely determined from a path accessing it. This first component is only convenient during the construction and in proofs concerning our dags.

Example Let

$\Sigma = \{F(F(x, A), B) \rightarrow F(D, x), F(C, F(D, x)) \rightarrow F(x, A)\}$.

The matching dag associated with red_Σ is given in figure 11.

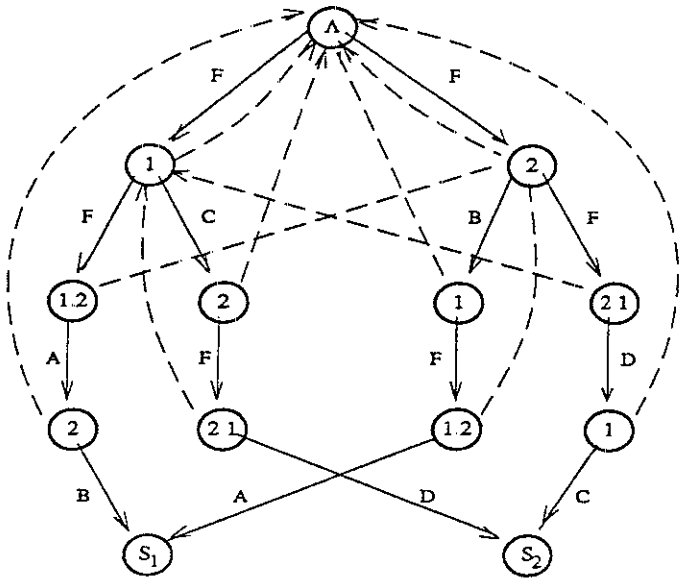


Figure 11

We shall defer for the moment the computation of sets Q verifying our hypothesis. However, the next lemma shows us that they must be chosen among indexes.

LEMMA 5.10 Let M, u and v be such that $\langle M/u, v\rangle$ is an accessible node, and $\forall w\ M/w\uparrow_+ \Rightarrow w \geq u$. Then $uv \in \mathscr{I}(M)$.

Proof First of all we remark that for any $w \in \mathscr{O}(M)$ such that $w|uv$, we have $\omega(M/w) = M/w$, by lemma 4.10. Now for any w such that $w \preceq uv$, let $M_w = M[uv \leftarrow x]/w$. We show that $\omega(M_w) = M_w$, by induction on $|uv/w|$. For $w = uv$, this is obvious. Otherwise, we get $\bar{\omega}(M_w) = M_w$, from above and the induction hypothesis. Let us show that $M_w\#$.

Case 1: $M/w\#$. Then obviously $M_w\#$.

Case 2: Otherwise, $M/w\uparrow_+$. By hypothesis, $w \geq u$. By lemma 5.7, $w/u \in Q(M[w \leftarrow \Omega]/u)$ and $M/w \prec \mathrm{red}_\Omega$. Using property Q_2, we get $uv/w \in \mathrm{Dir}(M/w)$, and thus $M_w\#$.

In all cases we get $\omega(M_w) = M_w$. When $w = \Lambda$ this shows that

$$\omega(M[uv \leftarrow x]) = M[uv \leftarrow x] \neq \omega(M) \preceq M[u \leftarrow \Omega],$$

and therefore $uv \in \mathscr{I}(M)$. ■

COROLLARY If $\langle M, v\rangle$ is accessible, we have $v \in \mathscr{I}(M)$.

Assuming the existence of a matching dag, we shall now give an algorithm that computes, for any term M not in normal form, a strongly needed redex occurrence of M. We assume, without loss of generality, that M has no variables (Otherwise, treat them as constants). We present algorithm A annotated with assertions, $\{\ldots\}$.

ALGORITHM A

begin $\{M \in \mathscr{T}, \mathscr{V}(M) \neq \varnothing\}$
  $u \leftarrow \Lambda$; $P \leftarrow \Omega$; $v \leftarrow \Lambda$; $w \leftarrow \langle \Omega, \Lambda \rangle$;
getsymbol: $\{w = \langle P/u, v \rangle, w \text{ accessible}, uv \in \mathscr{I}(P), P \prec M,$
    $P[u \leftarrow \Omega] \preceq \omega'(M), \forall u'\, P/u'\uparrow_+ \Rightarrow u' \succeq u\}$
    Let F be the head symbol of M at uv; $P^- \leftarrow P$; $P \leftarrow P[uv \leftarrow F\vec{\Omega}]$;
search: $\{w = \langle P^-/u, v \rangle, w \text{ accessible}, uv \in \mathscr{I}(P^-), P \preceq M,$
  $P[u \leftarrow \Omega] \preceq \omega'(M), \forall u'\, P/u'\uparrow_+ \Rightarrow u' \succeq u\}$
  By cases on the matching dag at node w:
  case 1: $w \xrightarrow{F} w' = \langle \alpha_{i\Omega}, S_i \rangle$: $\{u \in \mathscr{I}(P[u \leftarrow \Omega]), P/u = \alpha_{i\Omega}\}$
    exit with answer "redex of type i at u"
  case 2: $w \xrightarrow{F} w' = \langle P', v' \rangle$: $\{uv' \in \mathscr{I}(P), P/u = P'\}$
    $v \leftarrow v'$; $w \leftarrow w'$; go to getsymbol;
  otherwise $\{P/u\#\}$
    case 3: $w \neq \langle \Omega, \Lambda \}$:
      Let $w' = \text{Fail}(w) = \langle v', P' \rangle$;
      Let u' such that $v = u'v'$;
      $\{u' \text{ min such that } \Lambda \neq u' \preceq v \text{ and } P^-/uu'\uparrow, P' = P^-/uu'\}$
      $u \leftarrow uu'$; $v \leftarrow v'$; $w \leftarrow w'$; go to *search*;
    case 4: otherwise: $\{w = \langle \Omega, \Lambda \rangle, P^-/u = \Omega, v = \Lambda, \nexists u'\, P/u'\uparrow_+\}$
      choose u such that $P/u = \Omega$;
      if no such u then $\{P = M, \omega'(M) = M\}$
      exit with answer "normal form"
      else go to getsymbol;
end

THEOREM 5.11: MATCHING DAG THEOREM Let M be a term without variables. If M is in normal form, algorithm A applied to M terminates with answer "normal form." Otherwise, it terminates with answer "redex of type i at u" with $u \in \mathscr{I}(\omega'(M))$ and $M/u \succeq \alpha_{i\Omega}$.

Proof We use Floyd's method to prove the weak correctness of algorithm A, annotated with its assertions. We leave it to the reader to show the invariance of these assertions, using lemmas 4.15, 5.7, 5.8 and 5.10. The termination of the algorithm is easily established, since at getsymbol, $|M| - |P|$ decreases, and at search, $|v|$ decreases, with $|M| - |P|$ constant. When A terminates at exit "normal form," we get $\omega'(M) = M$, i.e., $M \in \mathscr{NF}$. When it terminates at exit "redex of type i at u," we get $M/u \succeq P/u = \alpha_\Omega$. Therefore, $\omega'(M)/u = \Omega$. Furthermore, $P[u \leftarrow \Omega] \preceq \omega'(M)$, and $u \in \mathscr{I}(P[u \leftarrow \Omega])$. Therefore, by definition of index we get $u \in \mathscr{I}(\omega'(M))$. ∎

Now that we have proved the correctness of our algorithm, we give a more readable version of it. We remove the redundant first component of the node in the matching dag and implement P by marking the occurrences explored in M. Also, variable u' now stands for uv of algorithm A, and we use the notation $u' \div v$ to denote u such that $u' = uv$.

In the following algorithm u is the place in M where we are trying to match the root of a redex, u' is the descendant of u where we are pursuing the match, w is pointing to the node of the matching dag which represents the current state of matching.

ALGORITHM B

begin Input: term M, with $\mathscr{V}(M) = \emptyset$.
  $u \leftarrow \Lambda$; $w \leftarrow \Lambda$;
init: $u' \leftarrow u$;
getsymbol: let F be the head symbol of M at u';
    mark occurrence u' in M;
search: By cases on the matching dag at node w:
    case 1: $w \xrightarrow{F} S_i$: exit "redex of type i at u"
    case 2: $w \xrightarrow{F} w' = v'$: $u' \leftarrow uv'$; $w \leftarrow w'$; go to getsymbol;
    otherwise
      case 3: $w \neq \Lambda$: let $w' = \mathrm{Fail}(w) = v'$;
        $u \leftarrow u' \div v'$; $w \leftarrow w'$; go to search;
      case 4: otherwise
        if all occurrences in M marked then exit "normal form"
        else choose u unmarked and go to init;
end.

Algorithm B is nondeterministic for two reasons. The first one comes from the choice of u in case 4. It is natural to implement this choice by taking for u the first unmarked occurrence in M, say in preorder. By keeping a pointer (Free) to the last chosen occurrence, one pass in M will suffice for all the necessary choices. We assume below the existence of the function Searchfree(Free), which returns the first unmarked occurrence of M following Free in preorder and the special value Done if there is none.

The second cause of nondeterminism is that there may be several nodes w' in the matching dag satisfying case 2. This can be easily eliminated by making our matching dags deterministic, keeping only one arc labeled F from any node w. This does not mean that we can restrict $Q(M)$ to be a singleton, since the extremity of a deleted arc may be reacheable through a failure arc. For instance, in figure 11, we may delete one of the two arcs issued from the origin node, but we must keep all the nodes.

When the matching dag is made deterministic, it is natural to store it in a double array Next(w, F), indexed firstly by the nodes of the dag, and secondly by the function symbols in $\mathscr{F}$. This way the computation of which case to choose at search is constant, i.e., independent of the size of red.

For a given Σ algorithm B is linear in the size of M, since an occurrence of M is marked at every passage at getsymbol, and Searchfree makes one pass in M. However, the computations done at search depend on the length of v', i.e., on red, and algorithm B has a running time $O(h \times |M|)$, where h is the maximum height of a redex scheme. We shall now describe one more refinement, in which we get rid of this h factor.

We need pointer u' only to read the occurrence in M at getsymbol. The new value of u' updated at case 2 may be at some distance from the old value, but in any case it

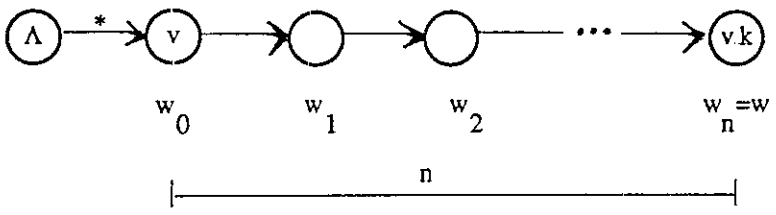


Figure 12

is the son of some node in M that has been already explored. If we keep in a stack all accesses of M at getsymbol, we shall be able to get directly at this father in a way that depends only on the path chosen in the matching dag. We can precompute the corresponding displacement in the matching dag as follows.

Let $v \cdot k$ be the label of node w, and let w_0 be the node with label v on some access path leading to w. See figure 12. We shall keep as information for node w the *virtual address* $\langle -n, k\rangle$, indicating that a query corresponding to node w is effected by accessing the kth son of the node of M accessed at time $-n$. If we keep these accesses in a stack Display with current index Top, we have direct access to this node through Display(Top $- n$).

We must now make sure that our virtual addressing mechanism is correct, independently of the path chosen to access w. The only difficulty comes from the sharing in the dag. More precisely, we may have two paths of different lengths leading from a node labeled with v to node w. Whenever this is the case, we shall duplicate the common nodes of these two paths so as to associate a unique virtual address with every node.

After this unsharing has been effected, we compute the fail arcs. More precisely, let $w = \langle M, v\rangle$ and let u be the minimum nonempty prefix of v such that $M/u\uparrow$. Let w_1, F_1, w_2, F_2, ..., w_k be the sequence whose existence is given by lemma 5.7. Lemma 5.8 tells us that by following this sequence from the origin, we arrive at some node w'. We must precisely take Fail(w) $= w'$, and we shall be sure in this way that our virtual addressing will be correct, even when backtracking through the fail arcs. In particular, note that w and Fail(w) have the same virtual address.

Example Let us consider the system Σ

$G(F(A, x)) \rightarrow \cdots$

$F(F(x, H(A)), B) \rightarrow \cdots$

$G(F(F(x, H(y)), C)) \rightarrow \cdots$

with corresponding matching dag shown in figure 13. The node marked X has two distinct virtual addresses, and there is no way to get rid of the conflicting paths by making the dag deterministic. Figure 14 shows the corresponding *virtual matching dag*, after duplication of node X, computation of virtual addresses, computation of fail arcs and suppression of nondeterminism.

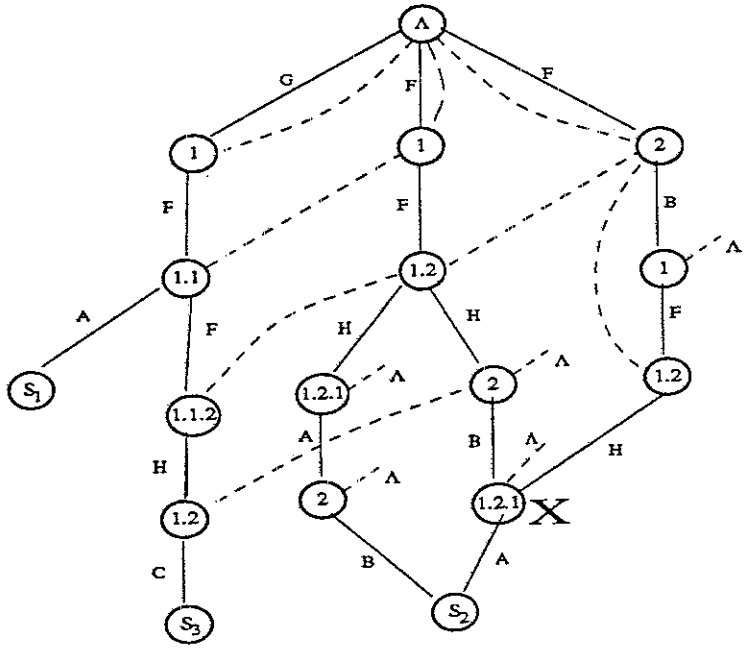


Figure 13

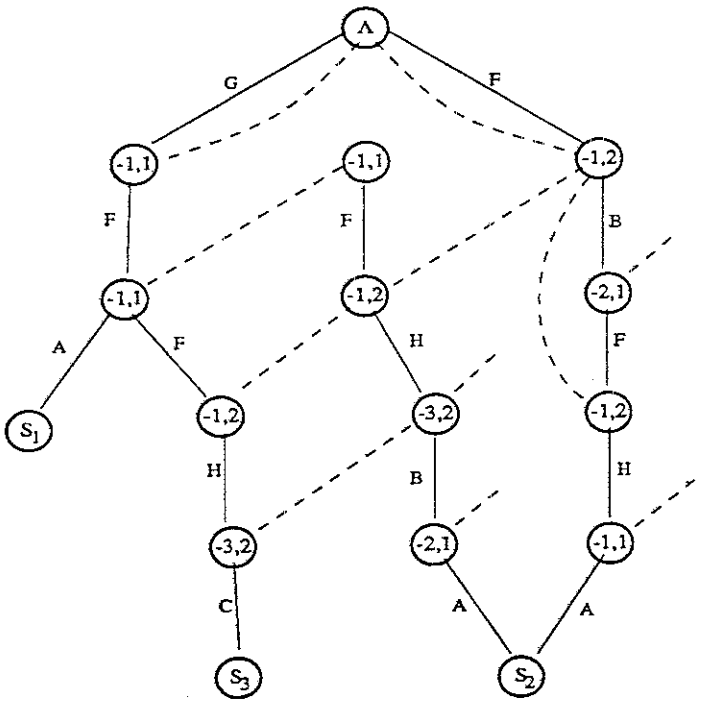


Figure 14

We are now ready to present the next refinement of our algorithm. We assume that the nonsuccess nodes of the dag are coded with successive integers, with the origin coded 0. For w a dag address and F a function symbol, Next(w, F) contains the address of its F-successor, if it exists, or S_i, or 0 to indicate failure. For $w \neq 0$, Node(w) contains a virtual address $\langle -n, k\rangle$. Finally, for every success node S_i, we keep in Size(i) the size of $\text{red}_{i\Omega}$ (i.e., the length of any path leading from the origin in the dag to S_i); this is used as virtual address of the redex in case of success. We use pointer P (represented in algorithm B by occurrence u') to access term M, and $M(P)$ denotes the top function symbol of M at P.

ALGORITHM C

```
begin Free ← origin of M;
init: Top ← 0; w ← 0; P ← Free;
getsymbol: F ← M(P); Mark P;
           Display(Top) ← P; Top ← Top + 1;
search: w' ← Next(w, F); by cases on w':
        S_i: exit "redex of type i at Display(Top − Size(i))"
        0: if w ≠ 0 then w ← Fail(w); go to search;
           else Free ← Searchfree(Free);
              if Free = Done then exit "normal form"
              else go to init;
        otherwise: ⟨−n, k⟩ ← Node(w');
           P ← kth son at Display(Top − n);
           w ← w'; go to getsymbol;
end
```

Algorithm C is a generalization to trees of the extension of the Knuth-Morris-Pratt fast string-matching algorithm to several patterns. It is straightforward to show that C has a running time of $O(|M|)$, independently of red.

Our final refinement consists of getting rid of the inner loop at search by iterating sequences of global failures at compile time. The array Next is now computed as follows. Let us consider the chain of Fail arcs issued from node w in the dag: $w_1 = w$, $w_{i+1} = \text{Fail}(w_i)$ for $1 \leq i \leq p$, $w_p = 0$. Let $F \in \mathscr{F}$. There are two cases:

- For some i, $1 \leq i \leq p$, there is an arc $w_i \xrightarrow{F} w_i'$. We set Next$(w, F) \leftarrow w_m'$ for m the minimum such i.
- Otherwise, Next$(w, F) \leftarrow 0$, indicating failure of all local and global searches.

We can now get rid of Fail(w) altogether, and the final version of our algorithm is given below. This corresponds to the technique of [13] for elimination of failure transitions.

ALGORITHM FINDREDEX

```
begin Free ← origin of M;
init: Top ← 0; w ← 0; P ← Free;
getsymbol: F ← M(P); Mark P;
```

```
Display(Top) ← P; Top ← Top + 1;
w' ← Next(w, F);
By cases on w':
S_i: exit "redex of type i at Display(Top − Size(i))"
0: Free ← Searchfree(Free)
   if Free = Done then exit "normal form"
   else go to init;
otherwise:
   ⟨−n, k⟩ ← Node(w');
   P ← kth son at Display(Top − n);
   w ← w'; go to getsymbol;
end
```

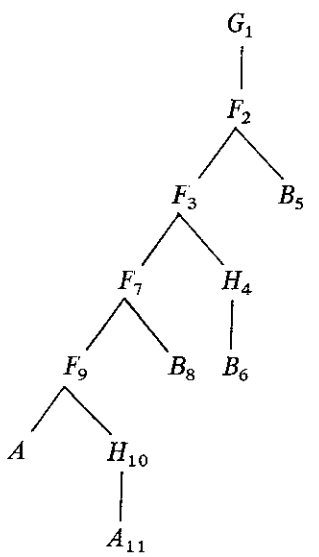
Note that more nodes may now be garbage-collected as inaccessible. The resulting structure Next is no longer a dag but rather a finite automaton transition graph. It is not clear whether this graph should be implemented as lists rather than arrays. This space-time trade-off depends on the structure of the particular Σ.

Example With Σ as above, we construct from figure 14 the information given in table 1. With input term

$M = G(F(F(F(F(A, H(A)), B), H(B)), B))$,

algorithm findredex will explore M in the order shown by the numbers in (1) and will stop with answer "redex of type 2 at x," with x the address of the node labeled 7.

(1)



The only nonconstant space used by findredex (apart from the mark bits in M) is the stack Display. This stack is popped at init, i.e., every time all current searches fail. Actually, we could do much better, since Display is accessed only at a maximum distance of h from its top, where h the maximum size of a redex scheme. We could

Table 1

		Next					
w	Node(w)	F	G	H	A	B	C
0		2	1	0	0	0	0
1	$\langle -1, 1\rangle$	3	1	0	0	0	0
2	$\langle -1, 2\rangle$	2	1	0	0	4	0
3	$\langle -1, 1\rangle$	5	1	0	S_1	0	0
4	$\langle -2, 1\rangle$	6	1	0	0	0	0
5	$\langle -1, 2\rangle$	2	1	7	0	4	0
6	$\langle -1, 2\rangle$	2	1	9	0	4	0
7	$\langle -3, 2\rangle$	2	1	0	0	8	S_3
8	$\langle -2, 1\rangle$	2	1	0	S_2	0	0
9	$\langle -1, 1\rangle$	2	1	0	S_2	0	0

Notes: Size(1) = 3. Size(2) = Size(3) = 5.

therefore implement Display as a circular ring of length h. The instruction Top $\leftarrow$ Top + 1 would become Top $\leftarrow$ (Top + 1) mod h, and our algorithm would work in constant space (plus one bit for each node in M for the marks). However, this might not be desirable in practice, since if we want to implement a lazy interpreter using algorithm findredex, we should be able to dynamically restart it after effecting one step of reduction, and in that case we would need to keep the whole Display.

Note that in order to effect a reduction step, we must have unambiguous addresses for the variable occurrences in the redex scheme. For instance, in the example above we would distinguish between the two possible successes s_2, since variable x has virtual address $-3, 1$ in one case and $-2, 1$ in the other. This unsharing problem could be simplified by completely developing our dags as trees. Actually, a slightly more general construction would be to directly construct matching trees with nodes $w = \langle M, v\rangle$ such that $v \in \mathcal{Q}(w)$, i.e., have Q depend on the access path to w and not only on M.

We summarize this section with the following theorem:

THEOREM 5.12: FAST TREE PATTERN MATCHING THEOREM Let Σ be such that there exists a matching dag for red_Σ. For any term M not in normal form, the algorithm findredex will find a strongly needed redex occurrence of M in time $O(M)$.

5.3 Deciding Strong Sequentiality

We shall now show that strong sequentiality is equivalent to the conditions Q_1 and Q_2 of section 5.2.

DEFINITION 5.13 Let M be an Ω-term. We define its set $\mathcal{T}(M)$ of *increasing indexes* as

$$\mathcal{T}(M) = \{u \in \mathcal{O}(M) | \forall N\ \omega(N) = \Omega\ \&\ \omega'(N) = N \Rightarrow \exists v \geq u\ v \in \mathcal{O}(M[u \leftarrow N])\}$$

Intuitively, an increasing index can be increased into an index whenever we replace it by a term which may become a redex after some reductions.

Example With

$\Sigma = \{F(G(A, x), B) \to \cdots, F(G(x, A), C) \to \cdots, G(E, E) \to \cdots\}$,

the first occurrence of Ω in $F(\Omega, \Omega)$ is a nonincreasing index

LEMMA 5.14 Let Σ be strongly sequential. Then for any M in Ω-normal form we have $\mathscr{T}(M) \neq \varnothing$.

Proof Assume Σ strongly sequential and $M = \omega'(M)$. Let $\mathscr{I}(M) = \{u_1, u_2, \ldots, u_n\}$. Assume that $\mathscr{T}(M) = \varnothing$, i.e., for every $i \leq n$ there exists N_i such that $\omega(N_i) = \Omega$, $\omega'(N_i) = N_i$, and $\nexists v \geq u_i$, $v \in \mathscr{I}(M[u_i \leftarrow N_i])$. Now consider

$P = M[u_i \leftarrow N_i | 1 \leq i \leq n]$

We first show $\omega'(P) = P$. Since M and the N_is are in Ω-normal form, the only possible case of a redex in P is when there is some α in red and u in $\bar{\mathscr{O}}(\alpha)$ such that $\alpha_\Omega/u \preceq N_k$ for some $k \leq n$. But then $\omega(\alpha_\Omega/u) \preceq \omega(N_k) = \Omega$, i.e., $\omega(\alpha_\Omega/u) = \Omega$, which is impossible by lemma 4.10. Thus $\omega'(P) = P$.

Σ being strongly sequential, there exists v in $\mathscr{I}(P)$. There are two cases:

Case 1: $\exists k \leq n$, $u_k \preceq v$. Then by repeated application of lemma 4.17 we get $v \in \mathscr{I}(M[u_k \leftarrow N_k])$, contrary to the hypothesis on N_k.

Case 2: $\forall k \leq n$, $u_k | v$. Then by repeated application of lemma 4.17 we get $v \in \mathscr{I}(M)$, which is impossible.

This completes the proof that $\mathscr{T}(M) \neq \varnothing$ ∎

Let us now extend lemma 4.16 to $\mathscr{T}$:

LEMMA 5.15 $uv \in \mathscr{T}(M) \Rightarrow v \in \mathscr{T}(M/u)$

Proof Assume that $uv \in \mathscr{T}(M)$ and $v \notin \mathscr{T}(M/u)$. That is, there exists N such that $\omega'(N) = N$, $\omega(N) = \Omega$, and $\nexists v' \geq v$, $v' \in \mathscr{I}(M/u[v \leftarrow N])$. Now let $P = M[uv \leftarrow N]$. Since $uv \in \mathscr{T}(M)$, there exists $w \geq uv$ such that $w \in \mathscr{I}(P)$. By lemma 4.16, $w/u \in \mathscr{I}(M/u[v \leftarrow N])$, a contradiction. ∎

COROLLARY If Σ is strongly sequential, $\mathscr{T}(M)$ satisfies Q_2.

Proof $M \prec \mathrm{red}_\Omega$ implies $\omega'(M) = M$ and $\mathscr{T}(M) \subseteq \mathscr{I}(M) \subseteq \mathrm{Dir}(M)$ ∎

An increasing index may, by definition, be increased into an index. Actually, it may even be increased into an increasing index:

LEMMA 5.16 Let N be such that $\omega(N) = \Omega$ and $\omega'(N) = N$. For every $u \in \mathscr{T}(M)$, there exists $v \geq u$ such that $v \in \mathscr{T}(M[u \leftarrow N])$

Proof Assume $u \in \mathscr{T}(M)$, $\omega(N) = \Omega$, $\omega'(N) = N$, and $\nexists v \geq u$ $v \in \mathscr{T}(M[u \leftarrow N])$. Let $\{u_1, \ldots, u_n\} = \{u_i \in \mathscr{I}(M[u \leftarrow N]) | u_i \geq u\}$. For every $i \leq n$, $u_i \notin \mathscr{T}(M[u \leftarrow N])$, by hypothesis; that is, there exists N_i such that $\omega(N_i) = \Omega$, $\omega'(N_i) = N_i$ and $\nexists w \geq u_i$, $w \in \mathscr{I}(M[u \leftarrow N[v_i \leftarrow N_i]])$, with $v_i = u_i/u$. Consider now $P = N[v_i \leftarrow N_i | 1 \leq i \leq n]$.

We get $\omega(P) = \Omega$ and $\omega'(P) = P$, like in the proof of lemma 5.14, and from $u \in \mathscr{T}(M)$ we know that there exists $w \geq u$ such that $w \in \mathscr{I}(M[u \leftarrow P])$. From above and repeated application of lemma 4.17 we must have $w|u_i$ for all $i \leq n$. Using lemma 4.17 again we get $w \in \mathscr{I}(M[u \leftarrow N])$, which is impossible by definition of the u_is. ∎

COROLLARY If Σ is strongly sequential, $\mathscr{T}(M)$ satisfies Q_1.

THEOREM 5.17: DECIDABILITY OF STRONG SEQUENTIALITY Σ is strongly sequential iff there exists a matching dag for red_Σ.

Proof If Σ is strongly sequential, we have just seen that $\mathscr{T}(M)$ satisfies Q_1 and Q_2. For the converse, consider a matching dag as constructed in 5.2. Let M be any Ω-term such that $\omega'(M) = M$ and $\omega(M_x) \neq \omega(M)$. We shall show that there exists an index in M by processing M with algorithm A slightly modified as follows. There are two more cases at getsymbol for the symbol read. If it is a variable, do as in "otherwise" (i.e., failure of global match). If it is an Ω, stop. We know from the assertions given for algorithm A, which are all still valid except that $P \prec M$ at getsymbol should become $P \preceq M$, that in this last case we have $uv \in \mathscr{I}(P)$. Since $M/uv = \Omega$, it follows that $uv \in \mathscr{I}(M)$, by definition of index. Note that this is the only way we may stop, since an exit at "normal form" is impossible by condition $\omega(M_x) \neq \omega(M)$, which implies $\exists w \in M$, $M/w = \Omega$, and an exit at "redex at u" is impossible by condition $\omega'(M) = M$. Therefore, $\mathscr{I}(M) \neq \varnothing$, which concludes the proof that Σ is strongly sequential. ∎

Note that the statement of the theorem gives effectively a decision procedure: to test whether Σ is strongly sequential, try constructing a matching dag as defined in 5.2. We may restrict conditions Q_1 and Q_2 to accessible nodes, and in that case, $\mathscr{T}$ is actually the maximum solution for Q. This is the solution we obtain if we start with $Q(M) = \text{Dir}(M)$, suppressing progressively elements with Q_2 and checking Q_1. In practice we might want to generate a smaller solution, starting with $Q(M) = \varnothing$ and adding an element with Q_1, checking Q_2. In either case, several passes may be needed, and we do not know of an efficient algorithm (i.e., linear in the size of red) to build a matching dag for a strongly sequential Σ. Once a dag is found verifying Q_1 and Q_2, we effect the necessary unsharing and compute the fail arcs. (This process could be simplified by completely developing our dags as trees). We then make the dag deterministic, complete the graph by iterating failures, garbage-collect inaccessible nodes, and finally get an automaton table such as the one given in table 1. Some ingenuity may be needed in this process if we want to minimize the size of the automaton. Of course, this is done once and for all for a given Σ, and therefore we do not care very much about the cost of these operations. In practical terms, this is the cost of building a compiler for the programming language defined by Σ. However, as pointed out by one of the referees, the size of the tables can blow up exponentially. Fortunately, this problem disappears in the practically relevant cases of the next two sections.

6 Applications

6.1 Recursive Functions with Constructors

We now consider a special case of our rewriting systems, corresponding to HOPE programs [5]. We assume that the set $\mathscr{F}$ of function symbols is partitioned into two set $\mathscr{F}_R$ and $\mathscr{F}_C$. $\mathscr{F}_R$ is the set of recursive functions symbols and $\mathscr{F}_C$ the set of constructors. Then any redex scheme $\alpha \in \text{red}$ is of the form $F(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $F \in \mathscr{F}_R$ and, for every i, the term $\alpha_i \in \mathscr{M}(\mathscr{F}_C, \mathscr{V})$ contains only constructor function symbols and variables. These particular systems will be called *systems with constructors*.

LEMMA 6.1 A system Σ with constructors is strongly sequential iff the set red_Ω is sequential.

Proof It is easy to check that for every $M \prec \text{red}_\Omega$ the set $\text{Dir}(M)$ of directions of M satisfies Q_1 and Q_2. ■

6.2 Simple Systems

In the general case of strongly sequential systems, it is not easy to find a corresponding matching dag. We consider now a restricted case (which often meets in TRS) when it is possible to factor the matching for all subexpressions of redexes. We call these systems *simple systems*.

DEFINITION 6.2 Let $\text{red}^*_\Omega = \{\alpha_\Omega/u | \alpha \in \text{red}, u \in \bar{\mathscr{O}}(\alpha)\}$. Then Σ is a *sequential simple system* iff all subsets of red^*_Ω are sequential sets.

The set red is a simple forest in the sense of Hoffman and O'Donnell [7], since if α and β are two compatible Ω-terms of red^*_Ω, they must be comparable in order to make the subset $\{\alpha, \beta\}$ sequential. But the algorithm defined in [7, appendix C] does not work if non-sequential simple systems are considered. In the terminology of [7], the subsumption graph (i.e., the covering relation of $\preceq$ in red^*_Ω) is a tree where at each node there is an occurrence of Ω in the corresponding Ω-term where the different branches are differenciated.

Let us define $\text{Dir}^*(M) = \text{Dir}(M, \text{red}^*_\Omega)$, $\mathscr{I}^*(\Omega) = \{\Lambda\}$, and

$$\mathscr{I}^*(M) = \{uv | u \in \text{Dir}^*(\bar{\omega}(M)), v \in \mathscr{I}^*(M/u)\}$$

when $M \neq \Omega$

LEMMA 6.3 Let $\omega(M) = \Omega$. Then $\mathscr{I}^*(M) \subseteq \mathscr{I}(M)$.

Proof Let $u \in \mathscr{I}^*(M)$, and let x be any variable. We prove first by induction that $\omega(M[u \leftarrow x]) \# \text{red}^*_\Omega$. If $M = \Omega$, then $u = \Lambda$ and $\omega(M[u \leftarrow x]) = x \# \text{red}^*_\Omega$, since $\Omega \notin \text{red}^*_\Omega$. Now suppose $M \neq \Omega$ and $\omega(M) = \Omega$. Then $u = vw$ with $v \in \text{Dir}^*(\bar{\omega}(M))$ and $w \in \mathscr{I}^*(M/v)$. Let $M_v = \omega((M/v)[w \leftarrow x])$. By induction hypothesis $M_v \# \text{red}^*_\Omega$. Thus $M_v \# \text{red}_\Omega$ and $\bar{\omega}(M[u \leftarrow x]) = \bar{\omega}(M)[v \leftarrow M_v]$. Now suppose that there is $\alpha \in \text{red}^*_\Omega$ such that $\alpha \uparrow \bar{\omega}(M[u \leftarrow x])$. Since $\bar{\omega}(M) \preceq \bar{\omega}(M[u \leftarrow x])$, we get $\alpha \uparrow \bar{\omega}(M)$. But

$v \in \mathrm{Dir}^*(\bar{\omega}(M))$. Thus $v \in \mathcal{O}(\alpha)$ and $\alpha/v \neq \Omega$. Therefore $\alpha/v \in \mathrm{red}_\Omega^*$ and $\alpha/v \uparrow M_v$. A contradiction. Hence $\bar{\omega}(M[u - x]) \# \mathrm{red}_\Omega^*$. In particular, $\bar{\omega}(M[u \leftarrow x]) \# \mathrm{red}_\Omega$, which implies $\omega(M[u \leftarrow x]) = \bar{\omega}(M[u \leftarrow x])$ and $\omega(M[u \leftarrow x]) \# \mathrm{red}_\Omega^*$. Now since $\omega(M) = \Omega$ and $\omega(M[u \leftarrow x]) \neq \Omega$, the occurrence u is in $\mathcal{I}(M)$, by lemma 4.14. ∎

LEMMA 6.4 Any sequential simple system is strongly sequential.

Proof Assume that Σ is a sequential simple system. We show first by induction that $\mathcal{I}^*(M) \neq \varnothing$ for every M such that $\omega(M) = \Omega$ and $\omega'(M) = M$. If $M = \Omega$, then $\mathcal{I}^*(M) = \{\Lambda\} \neq \varnothing$. Suppose now that $M \neq \Omega$, $\omega(M) = \Omega$, and $\omega'(M) = M$. Then $\bar{\omega}(M)\uparrow_+$. Thus because of the nonambiguity condition and since $\omega'(M) = M$, one cannot have $\alpha \preceq \bar{\omega}(M)$ for some $\alpha \in \mathrm{red}_\Omega^*$. Therefore $\mathrm{Dir}^*(\bar{\omega}(M)) \neq \varnothing$ and by induction $\mathcal{I}^*(M) \neq \varnothing$. Now if $M \prec \mathrm{red}_\Omega$, then $\omega(M) = \Omega$ and $\omega'(M) = M$ (again by nonambiguity). Thus $\mathcal{I}^*(M) \neq \varnothing$, and $\mathcal{I}^*(M)$ obviously satisfies Q_1 and Q_2. ∎

The matching dag of a sequential simple system can be built up easily from $\mathcal{I}^*(M)$ by induction on the size of M as follows. Suppose $\langle N, v\rangle$ is a node of the matching dag, $M = N[v \leftarrow F\vec{\Omega}]$ and $M \prec \mathrm{red}_\Omega$. Then search, along the chain of failure arcs starting at $\langle N, v\rangle$, the first node $\langle N', v'\rangle$ with an outgoing arc labelled by F. If there is no such node, then choose any $u \in \mathrm{Dir}^*(M)$, create node $\langle M, u\rangle$ with an F-arc from $\langle N, v\rangle$ to $\langle M, u\rangle$ and a failure arc from $\langle M, u\rangle$ to $\langle \Omega, \Lambda\rangle$. Otherwise, let $v = wv'$ and $\langle M', u'\rangle$ be the extremity of the F-arc outgoing from $\langle N', v'\rangle$. Then, taking $u = wu'$, create node $\langle M, u\rangle$ with an F-arc from $\langle N, v\rangle$ to $\langle M, u\rangle$ and a failure arc from $\langle M, u\rangle$ to $\langle M', u'\rangle$. Thus the matching dag (which is then a tree) can be efficiently built up in one top-down pass starting from the origin (not bottom-up as in [7])

Example Let Σ be such that

$$\mathrm{red} = \{F(A, G(x, y, C)), F(B, G(D, x, C)), G(E, x, D)\}.$$

Then the two possible matching dags associated to the sequential simple system Σ are given in figure 15.

Finally, O'Donnell [19] gave a sufficient condition for insuring the termination of the leftmost outermost derivation sequence. This condition is as follows. Remember that when $u, v \in \mathcal{O}(M)$, then u is said to be before v in preorder, written $u \leq_M v$, iff either $u \preceq v$ or $u|v$ with u to the left of v. Let Σ be a *left system* iff

$$\forall \alpha \in \mathrm{red}_\Sigma,\ \forall u, v \in \mathcal{O}(\alpha),\ \alpha/u \in \mathscr{V}\ \&\ u \leq_\alpha v \Rightarrow \alpha/v \in \mathscr{V}.$$

Thus left systems are characterised by redex schemes in which after a variable there could be only variables in preorder. One can easily prove that left systems are particular sequential simple systems and that they correspond exactly to systems where the leftmost outermost redex of any term is strongly needed. One nice example of left systems, given in [19], is the case of combinatory logic i.e., when

$$\Sigma = \{A(I, x) \to x, A(A(K, x), y) \to x, A(A(A(S, x), y), z) \to A(A(x, z), A(y, z))\}.$$

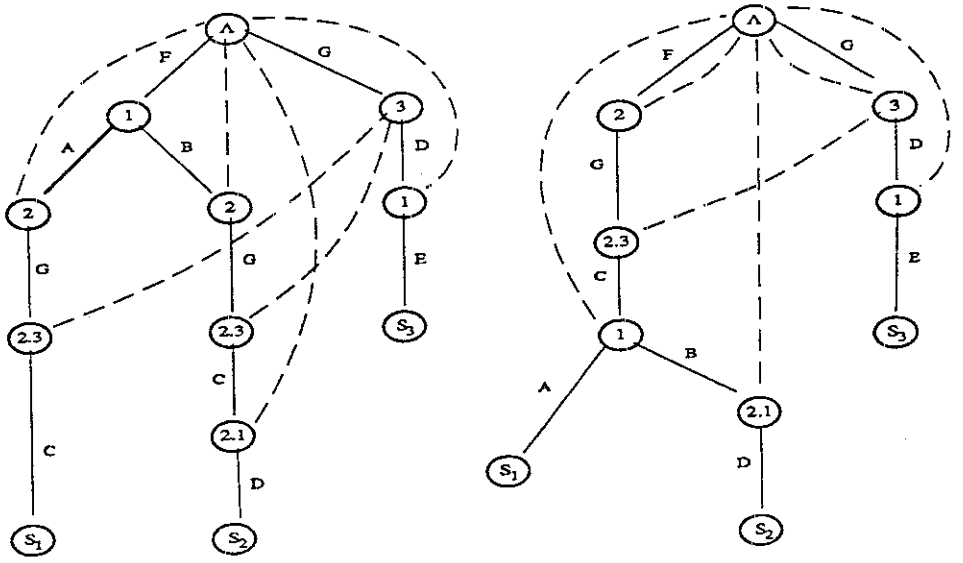


Figure 15

6.3 An Example

The left-linear non-overlapping term rewriting systems define a rather general family of applicative programming languages. The restriction to strongly sequential programs permits to define efficient interpreters for these languages. Important subcases are recursive equations with if-then-else expressions [16, 3], HOPE [5], and combinatory logic, as shown above. Several examples of such "programming with equations" are given in [8], including the definitions for a LISP and a LUCID interpreters.

We shall illustrate our methods on a small example of call-by-need computations in a highly functional programming language. This example defines the set of prime numbers, Primes, implemented as an infinite stream and computed along the lines suggested by [10]. First, we generalize the above coding of combinatory terms to arbitrary curried operators, using the following abbreviation:

$fM_1M_2 \ldots M_N$ stands for $\mathrm{App}(\ldots \mathrm{App}(\mathrm{App}(f, M_1), M_2), \ldots M_n)$.

Second, we assume given a minimum set of predefined arithmetic operations, that is a constant $\underline{n}$ for every integer n, and the standard operations $+$ and $\times$.

6.3.1 General combinators We use an infix "·" for list construction, and an infix "∘" for function composition.

$\mathrm{hd}(x \cdot y) = x$

$\mathrm{tl}(x \cdot y) = y$

$(f \circ g)x = f(g(x))$

$\mathrm{map}\ f\ (x \cdot y) = (f\ x) \cdot (\mathrm{map}\ f\ y)$

Note that *all* our function symbols are (zero-ary) constants, with the sole exception of the binary App implicit from our notation.

6.3.2 Arithmetic operators

$N_2 = \underline{2} : \text{map}\ (+\underline{1})\ N_2$

$\text{mult}\ x = \text{map}\ (\times x)\ N_2$

$\text{filter}(\underline{n} : x)(\underline{m} : y) = \underline{n} : \text{filter}\ x\ (\underline{m} : y) \quad \text{if } n \neq m$

$\text{filter}(\underline{n} : x)(\underline{n} : y) = \text{filter}\ x\ y$

These equations should present no problem: N_2 is the stream of integers starting from 2, mult x is the stream of all multiples of x, and filter x y removes elements of y from x, assuming x is a sorted stream, and y a sub-stream of x. The following function removes from a stream all multiples of its first element:

$\text{remult}(x : y) = \text{filter}\ y\ (\text{mult}\ x)$

6.3.3 Computing primes It is now easy to compute the various stages of Erathostenes' sieve in a stream of streams:

$\text{Sieve} = N_2 : ((\text{map} \circ \text{map})\ \text{remult}\ \text{Sieve})$

At every stage, one new prime number is produced, i.e.

$\text{Primes} = \text{map hd Sieve}$

We leave it to the reader to check that it is possible to use the program rules to evaluate, say, hd(tl Primes) to $\underline{3}$. In doing so, he should convince himself that the correct computation strategy is not quite obvious, since non-terminating computations lurk everywhere.

Actually, it is not hard to show that this example is strongly sequential (it falls under the conditions of simple systems). A part of the matching dag, showing all non-trivial fail arcs, can be seen in figure 16.

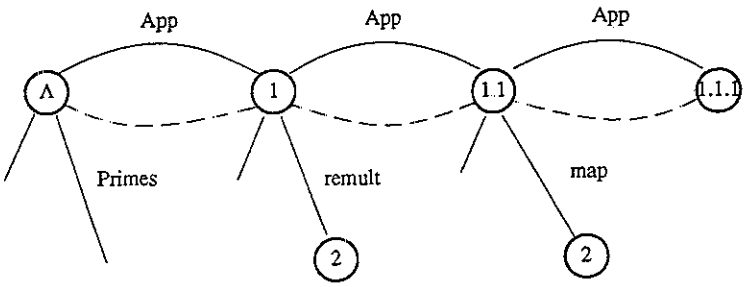


Figure 16

7 Conclusion

In our paper we established some mathematical foundations for computing with the call-by-need technique in TRS. As usual, optimal evaluation strategies are expected by adding some sharing mechanism. Duplications of subterms can be avoided by implementing terms in dags. However, the dags formalism need to be neatly defined. Another issue is to generalize propositions of our paper to more complicated rewriting systems. For instance, does one really need the nonambiguity condition? Can we treat systems with bound variables, such as the lambda calculus? Or with imperative features? Moreover, for the semantics of our systems, is it possible to get some full abstraction result? For which class of functions? Finally, it seems possible to consider the general problem of tree pattern matching along the lines of this paper and thus to define sets of patterns where a linear top-down matching algorithm could exist. However, this problem differs from finding strongly needed redexes, since then one has not to consider replacement of matching subterms.

Addendum

The results of this paper were obtained in 1979. Since then, extensions appeared in [4] for a non-deterministic TRS and in [17] for parallelism. A sophisticated implementation has been done by Laville [14] for ML. Efficient compilation of a TRS are also in Strandh [22]. Corrections and improvements in some of the proofs are reported in [12]. We especially thank Aart Middeldorp for giving us a correct proof of lemma 5.8.

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