A MORE REASONABLE PROOF OF COBHAM'S THEOREM

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ABSTRACT. We present a short new proof of Cobham's theorem without using Kronecker's approximation theorem, making it suitable for generalization beyond automatic sequences.

1. Introduction

In this note we give a short proof of the following celebrated theorem on automatic sequences.

Theorem 1 (Cobham). Let $a, b \in \mathbb{N}_{\geq 2}$ be multiplicatively independent (i.e. $a^m \neq b^n$ for all $m, n \in \mathbb{N}_{>0}$). A sequence $(f_x)_{x \in \mathbb{N}}$ is a- and b-automatic if and only if it is ultimately periodic.

The theorem is originally proven in [2]. To quote [4, p. 118]: "The proof is correct, long and hard. It is a challenge to find a more reasonable proof of this fine theorem.". In response, [5] suggested an easier approach that starts by showing the sequence is syndetic (i.e. the gap between any successive occurrences of any value is bounded) before proving it is ultimately periodic with a combinatorial argument. See for instance the proof in [1] (together with [8]).

Syndeticity is typically established using Kronecker's approximation theorem (see [3]), but that proved to be problematic for automatic functions on the Gaussian integers as shown in [6]. Our proof differs entirely from the classical approach and only needs a consequence of the much weaker approximation theorem of Dirichlet.

2. Preliminaries

We assume basic familiarity with formal language terminology, and briefly recall a few standard notions from automatic sequence theory. We refer to [1] for a comprehensive treatment.

Definition 2. A deterministic finite automaton with output (DFAO) is a tuple (S, D, δ, s_0, F) , where S is a finite set of states, D a finite input alphabet, $\delta \colon S \times D \to S$ a transition function, $s_0 \in S$ an initial state, and F an output function on S. On input $w \in D^*$ it outputs $F(\delta(s_0, w))$, where we extend $\delta(s_0, w) = \delta(\delta(s_0, u), v)$ for any $u, v \in D^*$ with w = uv as usual.

Definition 3. In base $b \in \mathbb{N}_{\geq 2}$, a word $w \in \mathbb{N}^*$ of length n represents the natural number $[w]_b = w_0 b^{n-1} + \ldots + w_{n-2} b + w_{n-1}$, and a language $L \subseteq \mathbb{N}^*$ represents $[L]_b = \{[w]_b \mid w \in L\}$.

Definition 4. Let $b \in \mathbb{N}_{\geq 2}$ and $\{0, 1, \ldots, b-1\} \subseteq D \subseteq \mathbb{N}$ be finite. A sequence $(f_x)_{x \in \mathbb{N}}$ is (b, D)-automatic if there is a DFAO (S, D, δ, s_0, F) such that $f_{[w]_b} = F(\delta(s_0, w))$ for all $w \in D^*$. A sequence $(f_x)_{x \in \mathbb{N}}$ is b-automatic if it is $(b, \{0, 1, \ldots, b-1\})$ -automatic.

Lemma 5. Let $b \in \mathbb{N}_{\geq 2}$ and $\{0, 1, \dots, b-1\} \subseteq D \subseteq \mathbb{N}$ be finite. A sequence $(f_x)_{x \in \mathbb{N}}$ is b-automatic if and only if it is (b, D)-automatic.

Proof. Adapt [1, Thm. 6.8.6] to use the transducer of [7, Prop. 7.1.4] for normalization on D^* .

Any two bases have relatively close powers, which follows easily from Dirichlet's approximation theorem or by mimicking its proof to avoid logarithms as follows.

Lemma 6. Let $a, b \in \mathbb{N}_{\geq 2}$ and $\epsilon \in \mathbb{R}_{> 0}$. Then there are $m, n \in \mathbb{N}_{> 0}$ such that $|a^m - b^n| \leq \epsilon b^n$.

Proof. We may assume that $a \ge b$ by taking a suitable power of a, so the sequence $(f_x)_{x \in \mathbb{N}}$ given by $a^x b^{-f_x} \in [1, b)$ for all $x \in \mathbb{N}$ is strictly increasing. By the pigeonhole principle there are natural numbers x < y such that $|a^y b^{-f_y} - a^x b^{-f_x}| \le \epsilon$, that is, $|a^{y-x} - b^{f_y-f_x}| \le \epsilon b^{f_y} a^{-x} \le \epsilon b^{f_y-f_x}$. \square

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A sequence $(f_x)_{x\in\mathbb{N}}$ has local period $p\in\mathbb{N}_{>0}$ on an interval $I\subseteq\mathbb{N}$ if $f_x=f_{x+p}$ for all $x,x+p\in I$. Local periodicity on sufficiently overlapping intervals extends to their union.

Lemma 7. Let $(f_x)_{x \in \mathbb{N}}$ have local period p on an interval I and local period q on an interval J. If $|I \cap J| \ge p + q$, then f has local period p on the interval $I \cup J$.

Proof. Pick any $x, x + p \in I \cup J$. If $x, x + p \in I$, we have $f_x = f_{x+p}$ by assumption. Otherwise, since the interval $I \cap J$ has cardinality at least p + q, we have $x, x + p \in J$ and we can pick $y, y + p \in I \cap J$ such that $y \equiv x \pmod{q}$. Then $f_x = f_y = f_{y+p} = f_{x+p}$ by local periodicity on J, I and J respectively.

3. Proof

Let $B[x;r] = \{y \in \mathbb{N} \mid |y-x| \le r\}$ be the interval centered on $x \in \mathbb{R}_{\ge 0}$ with radius $r \in [0,x]$.

Proof of theorem 1. As usual, we only prove the forward direction.

For each $c \in \{a,b\}$, f is computed by a DFAO $(S_c, D_c, \delta_c, s_{0,c}, F_c)$ in base c with digits $D_c = B[c;c]$ by lemma 5. It is easy to check that $B[c^n;c^n] \subseteq [D_c^n]_c$ for all $n \in \mathbb{N}_{>0}$. Define $L_{cs} = \{w \in D_c^* \mid \delta_c(s_{0,c},w) = s\}$ for $s \in S_c$. Then for all $w \in L_{cs}$ and $v \in D_c^*$ we have $f_{[wv]_c} = F_c(\delta_c(s_{0,c},wv)) = F_c(\delta_c(s,v))$, so for all $x,y \in [L_{cs}]_c$, $n \in \mathbb{N}$ and $z \in [D_c^n]_c$

$$f_{xc^n+z} = f_{yc^n+z}. (1)$$

We create local periods of f as follows. Let S_{∞} be the set of $s \in S_b$ for which $[L_{bs}]_b$ is infinite. Since $\{[L_{at}]_a \mid t \in S_a\}$ is a finite cover of \mathbb{N} , we can fix for each $s \in S_{\infty}$ some $t \in S_a$ and distinct $x_{st}, y_{st} \in [L_{bs}]_b \cap [L_{at}]_a$. Letting $\xi = \max\{x_{st}, y_{st} \mid s \in S_{\infty}\} + 1$, we can find $m, n \in \mathbb{N}_{>0}$ such that $\xi|a^m - b^n| \leq \frac{1}{6}b^n$ by lemma 6. In particular, we get $\frac{5}{6}b^n \leq a^m$. Since $a^m \neq b^n$ we can take $p_{st} = (x_{st} - y_{st})(a^m - b^n) \in (0, \frac{1}{6}b^n]$ for all $s \in S_{\infty}$ by swapping x_{st} and y_{st} if necessary.

We show for each $s \in S_{\infty}$ and $x \in [L_{bs}]_b$ that f has local period p_{st} on the interval $I_x = B[xb^n + b^n; \frac{2}{3}b^n]$. Pick any $z, z + p_{st} \in B[b^n; \frac{2}{3}b^n] \subseteq [D_b^n]_b$. Since

$$|z - y_{st}(a^m - b^n) - a^m| \le |z - b^n| + (y_{st} + 1)|a^m - b^n| \le \frac{5}{6}b^n \le a^m$$

we have $z - y_{st}(a^m - b^n) \in B[a^m; a^m] \subseteq [D_a^m]_a$. Hence, using (1) thrice we see as desired

$$\begin{split} f_{xb^n+z} &= f_{y_{st}b^n+z} \\ &= f_{y_{st}a^m+z-y_{st}(a^m-b^n)} \\ &= f_{x_{st}a^m+z-y_{st}(a^m-b^n)} \\ &= f_{x_{st}b^n+z+p_{st}} \\ &= f_{xb^n+z+p_{st}}. \end{split}$$

Let $x \in \mathbb{N}$ be such that $\{[L_{bs}]_b \mid s \in S_\infty\}$ covers $x + \mathbb{N}$, and fix for f a local period $p_y \leq \frac{1}{6}b^n$ on I_y for all $y \geq x$. We show that f has local period p_x on $\bigcup_{x \leq y \leq z} I_y$ for all $z \geq x$ by induction. It surely holds if z = x. Otherwise, f has local period p_x on $\bigcup_{x \leq y < z} I_y$ by induction and local period p_z on I_z , so lemma 7 proves our induction hypothesis as $\left(\bigcup_{x \leq y < z} I_y\right) \cap I_z = B[(z + \frac{1}{2})b^n; \frac{1}{6}b^n]$ has cardinality at least $\lfloor \frac{1}{3}b^n \rfloor \geq 2\lfloor \frac{1}{6}b^n \rfloor \geq p_x + p_z$.

We conclude that f has local period p_x on $\bigcup_{x \le y} I_y$, that is, f is ultimately periodic.

4. Future work

We will show that our approach extends well to prove the Cobham-Semenov theorem from [9], and to prove the Cobham-type theorem for automatic functions on vectors of imaginary quadratic integers. In particular, we will establish the conjecture for the Gaussian integers from [6].

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