A Factorization for Formal Laurent
Series and Lattice Path Enumeration

IRA M. GESSEL[*]

IBM T. J. Watson Research Center, Yorktown Heights, New York 10598[†]

Communicated by the Managing Editors

If $f = \sum_{n=-\infty}^{\infty} a_n t^n$ is a formal Laurent series with certain restrictions on the a_n, then $f = f_- f_0 f_+$, where f_- contains only negative powers of t, f_+ contains only positive powers of t, and f_0 is independent of t. Applications include Lagrange's formula for series reversion, the problem of counting lattice paths below a diagonal, and a theorem of Furstenberg that the diagonal of a rational power series in two variables is algebraic.

1. INTRODUCTION

Given a Laurent series $f(x) = \sum_{n=-\infty}^{\infty} a_n x^n$ convergent in some annulus, the coefficients a_n can often be evaluated by complex variable theory, $a_n = 1/(2\pi i) \oint (f(x)/x^{n+1}) \, dx$, where the circle of integration lies within the annulus of convergence. If $f(x)$ contains no singularities other than poles inside the annulus, then by the residue theorem a_n can be expressed in terms of the poles of $f(x)$. For example, it can be shown by this method that if $\sum_{m,n=0}^{\infty} a_{m,n} x^m y^n$ is a rational function of x and y, then $\sum_{n=0}^{\infty} a_{n,n} z^n$ is an algebraic function [9; 12; 22, Theorem 5.3].

In this paper, we develop a formal power series method for proving such theorems. An important tool which we introduce here is the formal power series analog of the Laurent series for a function analytic in an annulus. One of our basic results, Theorem 4.1, is that any formal Laurent series $\sum_{n=-\infty}^{\infty} a_n t^n$, with suitable restrictions on the a_n, has a unique expression in the form

$$A \cdot \left[1 + \sum_{m=1}^{\infty} b_m t^{-m} \right] \left[1 + \sum_{n=1}^{\infty} c_n t^n \right],$$

where A is independent of t.

A factorization of this form in which $a_n = 0$ for $n < -1$ yields Lagrange's formula for reversion of series. (For an analytic proof see [25, pp. 132–133], and for a formal power series proof and further references see [6, pp. 148–153].) A similar factorization solves the problem of counting lattice paths in the plane which stay below the diagonal $x = y$ with an arbitrary set of steps, the simplest nontrivial case of which is the well-known ballot problem [2; 6, pp. 21–23, 80; 23, Chap. 1]. This seems to be the first application of Laurent series (as opposed to formal power series) to a specific enumeration problem.

Furstenberg's theorem on the diagonal of a rational power series is proved by partial-fraction expansion of a factorization into linear factors. This gives a new purely formal method for proving that certain series are algebraic.

2. Formal Laurent Series

We would like to study formal power series of the form

$$f(x, y) = \sum_{m,n=0}^{\infty} a_{mn} x^m y^n \tag{1}$$

by grouping together those terms for which $m - n$ is constant. This is most easily accomplished by introducing the new variable t and substituting t for x and y/t for y in (1), getting

$$g(t, y) = f\left(t, \frac{y}{t}\right) = \sum_{m,n=0}^{\infty} a_{mn} t^{m-n} y^n$$

$$= \sum_{l=-\infty}^{\infty} t^l \sum_{n=\max\{-l,0\}}^{\infty} a_{n+l,n} y^n. \tag{2}$$

The transformation is easily reversed, since $f(x, y) = g(x, xy)$.

When dealing with formal series such as (2) which contain arbitrary positive and negative exponents, we must insure that expressions such as $(\sum_{n=-\infty}^{\infty} t^n)^2$, which require that an infinite sum of scalars be evaluated, do not arise. For simplicity we take for our ring of scalars the complex numbers $\mathbb{C}$, although any algebraically closed field of characteristic zero would suffice. Then the series that we study will for the most part lie in the ring $\mathbb{C}[[t, y/t]]$, which is the ring of formal power series in the two variables t and y/t. However, in order to give meaning to an equation such as $t(y/t)^2 = t^{-1}y^2$, we must extend our ring to include negative powers of t. To do this, we first consider the ring $\mathbb{C}((t))$ of formal Laurent series in t, that is, the ring of all formal sums of the form $\sum_{n=-k}^{\infty} a_n t^n$ for some k, multiplied term by term as

usual. $\mathbb{C}((t))$ is the field of fractions of $\mathbb{C}[[t]]$. Then the ring $\mathbb{C}((t))[[y]]$ of formal power series in y with coefficients in $\mathbb{C}((t))$ will serve as our extension of $\mathbb{C}[[t, (y/t)]]$. It is easily seen that $\mathbb{C}((t))[[y]]$ consists of those series $\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} a_{mn} t^m y^n$ such that for fixed n, a_{mn} is nonzero for only finitely many negative values of m. Intuitively, we may think of y as being "infinitesimally small" compared to t, so that y/t^k is "small enough" that $\sum_{n=0}^{\infty} (y/t^k)^n$ converges for all k.

We use the notation $\langle t^i \rangle f$ to denote the coefficient of t^i in f; $\langle t^i y^j \rangle f$ is defined similarly.

We shall use without comment some of the basic properties of formal power series such as infinite summation and functional composition. For a rigorous and detailed development of the theory of formal power series in many variables, see [24].

3. LAGRANGE'S INVERSION FORMULA

2.1. THEOREM. *Let $G(t)$ be any element of $\mathbb{C}[[t]]$. Then the equation $f(y) = yG[f(y)]$ has a unique solution in $\mathbb{C}[[y]]$, and*

(i) $$\langle y^n \rangle [f(y)]^k = \frac{k}{n} \langle t^{n-k} \rangle [G(t)]^n \qquad for\ n, k > 0,$$

(ii) $$\langle y^n \rangle \frac{[f(y)]^k}{1 - yG'[f(y)]} = \langle t^{n-k} \rangle [G(t)]^n \qquad for\ n, k \geqslant 0.$$

Proof. We first observe that the equation $f(y) = yG[f(y)]$ yields a recurrence for the coefficients of $f(y)$ which shows that they exist and are unique. (Note that if $G(0) = 0$, then $f(y) = 0$.)

We now claim that (i) and (ii) are equivalent to

(i') $$\langle y^n t^{-k} \rangle \log \left[1 - \frac{f(y)}{t}\right]^{-1} = \langle y^n t^{-k} \rangle \log \left[1 - \frac{y}{t} G(t)\right]^{-1}$$
$$for\ n, k > 0$$

(ii') $$\langle y^n t^{-k} \rangle \left[1 - \frac{f(y)}{t}\right]^{-1} \cdot \{1 - yG'[f(y)]\}^{-1}$$
$$= \langle y^n t^{-k} \rangle \left[1 - \frac{y}{t} G(t)\right]^{-1} \qquad for\ n, k \geqslant 0.$$

To show the equivalence of (i) and (i') we have

$$\langle y^n t^{-k} \rangle \log \left[1 - \frac{f(y)}{t}\right]^{-1} = \langle y^n t^{-k} \rangle \sum_{j=1}^{\infty} \frac{1}{j} \left[\frac{f(y)}{t}\right]^j$$

$$= \frac{1}{k} \langle y^n \rangle [f(y)]^k,$$

and

$$\langle y^n t^{-k}\rangle \log \left[1 - \frac{y}{t} G(t)\right]^{-1}$$

$$= \langle y^n t^{-k}\rangle \sum_{j=1}^{\infty} \frac{1}{j}\left[\frac{y}{t} G(t)\right]^{j} = \frac{1}{n}\langle t^{-k}\rangle\left[\frac{G(t)}{t}\right]^{n}$$

$$= \frac{1}{n}\langle t^{n-k}\rangle[G(t)]^{n}.$$

The equivalence of (ii) and (ii') is similar.

Then to prove the theorem, we need only separate from $[1 - (y/t)G(t)]^{-1}$ and its logarithm those terms which contain nonpositive (or negative) powers of t.

It is easily verified that we may expand $t - y\, G(t)$ as a Taylor series in $t - f(y)$; thus $t - yG(t) = \{f(y) - yG[f(y)]\} + [t - f(y)]\{1 - yG'[f(y)]\} + [t - f(y)]^2 R$, where $R \in \mathbb{C}[[t, y]]$. Then since $f(y) - yG[f(y)] = 0$, we have,

$$\left[1 - \frac{y}{t} G(t)\right]^{-1} = \frac{t}{t - yG(t)}$$

$$= \frac{t}{[t - f(y)]\{1 - yG'[f(y)]\} + [t - f(y)]^2\, R}$$

$$= \left[1 - \frac{f(y)}{t}\right]^{-1} \cdot \{\{1 - yG'[f)y)]\} + [t - f(y)]\, R\}^{-1}.$$

Thus,

$$\log \left[1 - \frac{y}{t} G(t)\right]^{-1}$$

$$= \log \left[1 - \frac{f(y)}{t}\right]^{-1} + \log\{1 - yG'[f(y)] + [t - f(y)]\, R\}^{-1},$$

and the second term on the right clearly contains no negative powers of t on expansion. This proves (i').

For (ii'), we have

$$\left[1 - \frac{y}{t} G(t)\right]^{-1}$$

$$= \left[1 - \frac{f(y)}{t}\right]^{-1} \cdot \{1 - yG'[f(y)]\}^{-1} \cdot \left\{1 + \frac{[t - f(y)]\, R}{1 - yG'[f(y)]}\right\}^{-1}$$

$$= \left[1 - \frac{f(y)}{t}\right]^{-1} \cdot \{1 - yG'[f(y)]\}^{-1} + t \sum_{n=1}^{\infty} \frac{(-R)^n\, [t - f(y)]^{n-1}}{\{1 - yG'[f(y)]\}^{n+1}},$$

and the sum on the right contains only positive powers of t. This completes the proof.

4. The Multiplicative Decomposition

In our proof of Lagrange's inversion formula we utilized the existence of a factorization of $[1 - (y/t)G(t)]^{-1}$ into a factor involving only negative powers of t and a factor involving only positive powers. In this section we show that such a factorization exists for an arbitrary Laurent series (with suitable restrictions) and we give an application to the problem of counting lattice paths.

4.1. THEOREM. *Let f be an element of $\mathbb{C}[[t, y/t]]$ with constant term 1. Then f has a unique decomposition $f = f_- f_0 f_+$, where f_-, f_0, and f_+ are in $\mathbb{C}[[t, y/t]]$, f_- is of the form $1 + \sum_{i>0, j>0} a_{ij} y^i t^{-j}$, f_0 is of the form $1 + \sum_{i>0} a_i y^i$, and f_+ is of the form $1 + \sum_{i \geqslant 0, j>0} a_{ij} y^i t^j$.*

Proof. Let $\log f = \sum_{i,j} b_{ij} y^i t^j$. Then $f_- = \exp(\sum_{i>0, j<0} b_{ij} y^i t^j)$, $f_0 = \exp(\sum_{i=1}^{\infty} b_{i0} y^i)$, and $f_+ = \exp(\sum_{i \geqslant 0, j>0} b_{ij} y^i t^j)$. Uniqueness is clear.

The following variant of Lagrange inversion gives us more information about an important special case of the multiplicative decomposition.

4.2. THEOREM. *Let $h(t, y) = \sum_{i=0}^{\infty} a_i t^i$, where $a_0, a_1 \in y\mathbb{C}[[y]]$ and $a_i \in \mathbb{C}[[y]]$. Then the equation $\alpha = h(\alpha, y)$ has a unique solution α in $y\mathbb{C}[[y]]$. Let $f = [1 - t^{-1} h(t, y)]^{-1}$. Then $f_- = [1 - t^{-1}\alpha]^{-1}$ and $f_0 = \alpha/a_0$. For $k > 0$,*

$$\alpha^k = \sum_{n=1}^{\infty} \frac{k}{n} \langle t^{n-k} \rangle [h(t, y)]^n.$$

Proof. The existence and uniqueness of α are straightforward. The conditions on a_0 and a_1 insure that f is in $\mathbb{C}[[t, y/t]]$ with constant term 1. As in the proof of Theorem 2.1, we have $t - h(t, y) = [\alpha - h(\alpha, y)] + (t - \alpha)[1 - h'(\alpha, y)] + (t - \alpha)^2 R = (t - \alpha)S$, where $h'(t, y) = \partial h(t, y)/\partial t$, R is in $\mathbb{C}[[t, y]]$, and S is in $\mathbb{C}[[t, y]]$ with constant term 1. Thus $f = t/(t-h) = (1 - t^{-1}\alpha)^{-1} S^{-1}$, so $f_- = (1 - t^{-1}\alpha)^{-1}$.

Since $f_0 = (f_0 f_+)|_{t=0}$, we have

$$f_0 = \frac{f}{f_-} \bigg|_{t=0} = \frac{t - \alpha}{t - h(t, y)} \bigg|_{t=0} = \frac{\alpha}{h(0, y)} = \frac{\alpha}{a_0}.$$

Finally,

$$\frac{\alpha^k}{k} = \langle t^{-k} \rangle \log f_- = \langle t^{-k} \rangle \log f$$

$$= \sum_{n=0}^{\infty} \langle t^{-k} \rangle \frac{[t^{-1} h(t, y)]^n}{n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n} \langle t^{n-k} \rangle [h(t, y)]^n.$$

We now give a combinatorial interpretation to the decomposition in terms of lattice paths in the plane. (Related decompositions are described in [8, p. 383; 10].)

Let S be a set of ordered pairs of nonnegative integers, not both zero. We call the elements of S *steps*. A *path* is a finite sequence of steps: we think of the path $(a_1, b_1), (a_2, b_2),..., (a_n, b_n)$ as starting at $(0, 0)$, then proceeding to (a_1, b_1), then to $(a_1 + a_2, b_1 + b_2)$, and ending at $(a_1 + a_2 + \cdots + a_n, b_1 + b_2 + \cdots + b_n)$. The *points* of the path are $(a_1, b_1), (a_1 + a_2, b_1 + b_2),..., (a_1 + a_2 + \cdots + a_n, b_1 + b_2 + \cdots + b_n)$, the last of which is the *endpoint*. The *height* of the point (a, b) is $a - b$, and the height of a path is the height of its endpoint. We allow the empty path, which has no steps and no points, and height zero.

Now if P is a set of paths, we define the *counting series* (or generating function) for P to be $\Gamma(P) = \Gamma(P)(t, y) = \sum_{\pi \in P} t^{e_1(\pi) - e_2(\pi)} y^{e_2(\pi)}$, where $(e_1(\pi), e_2(\pi))$ is the endpoint of π. Note that $\Gamma(P)(x, xy) = \sum_{\pi \in P} x^{e_1(\pi)} y^{e_2(\pi)}$, which might seem a more natural choice for a counting series for P; however, both contain exactly the same information, and our definition will be more convenient for our purposes.

Given two paths σ_1 and σ_2, we define their product $\sigma_1\sigma_2$ to be the path whose steps are those of σ_1 followed by those of σ_2. If $\pi = \sigma_1\sigma_2$, then we call σ_1 a *head* of π. We write $\Gamma(\pi)$ for $\Gamma(\{\pi\})$; then it is easily seen that $\Gamma(\sigma_1\sigma_2) = \Gamma(\sigma_1)\Gamma(\sigma_2)$.

Now let S^* be the set of paths all of whose steps lie in S. Then we have $\Gamma(S^*) = \sum_{n=0}^{\infty} [\Gamma(S)]^n = [1 - \Gamma(S)]^{-1}$, since each term in $[\Gamma(S)]^n$ corresponds to a path of n steps.

We now define three classes of paths: A *minus-path* is either the empty path or a path the height of whose endpoint is negative and less than that of any other point. A *zero-path* is a path of height zero all of whose points have nonnegative height. A *plus-path* is a path all of whose points have positive height.

Note that the empty path, but no other path, belongs to all three classes. The path $(a_1, b_1) \cdots (a_n, b_n)$ is a minus-path if and only if $(b_n, a_n)(b_{n-1}, a_{n-1}) \cdots (b_1, a_1)$ is a plus-path; thus the theories of minus- and plus-paths are identical.

4.3. LEMMA.   *Let π be a path. Then π has a unique factorization $\pi_-\pi_0\pi_+$, where π_- is a minus-path, π_0 is a zero-path, and π_+ is a plus-path.*

Proof.   We describe π_-, π_0, and π_+, and leave to the reader the verification that they have the desired properties. Of the heads of π of minimal height, let π_- be the shortest. Then if $\pi = \pi_-\sigma$, let $\sigma = \pi_0\pi_+$ where π_0 is the longest head of σ of height zero.

4.4. THEOREM.   *Let S be a set of steps, and let S_-, S_0, and S_+ be the sets of*

minus-, zero-, and *plus-paths* with steps in S. Then $\Gamma(S_-) = [\Gamma(S^*)]_-$, $\Gamma(S_0) = [\Gamma(S^*)]_0$, and $\Gamma(S_+) = [\Gamma(S^*)]_+$.

Proof. From Lemma 4.3 it follows that $\Gamma(S^*) = \Gamma(S_-)\Gamma(S_0)\Gamma(S_+)$. The theorem then follows from Theorem 4.1.

We now give some examples in which $\Gamma(S_-)$, $\Gamma(S_0)$, and $\Gamma(S_+)$ can be computed by solving a quadratic equation. The numerical results we need are given in the following lemma.

4.5. LEMMA. *Let* $f = (1 - t - t^{-1}y - z)^{-1}$, where $z \in y\mathbb{C}[[y]]$. Then

$$f_- = \left\{1 - t^{-1} \frac{1 - z - [(1 - z)^2 - 4y]^{1/2}}{2}\right\}^{-1}$$

$$= 1 + \sum_{k=1}^{\infty} \sum_{n,j=0}^{\infty} t^{-k}y^{n+k}z^j \frac{k}{2n + j + k} \binom{2n + j + k}{n + k, n, j};$$

$$f_0 = \frac{1 - z - [(1 - z)^2 - 4y]^{1/2}}{2y}$$

$$= \sum_{n,j=0}^{\infty} y^n z^j \frac{1}{2n + j + 1} \binom{2n + j + 1}{n + 1, n, j};$$

$$f_+ = \left\{1 - t \frac{1 - z - [(1 - z)^2 - 4y]^{1/2}}{2y}\right\}^{-1}$$

$$= 1 + \sum_{k=1}^{\infty} \sum_{n,j=0}^{\infty} t^k y^n z^j \frac{k}{2n + j + k} \binom{2n + j + k}{n + k, n, j}.$$

Proof. The formulas for f_- and f_0 come from Theorem 4.2. Note that $f(t, y) = f(t^{-1}y, y)$. Thus $f(t, y) = f_-(t^{-1}y, y)f_0(t^{-1}y, y)f_+(t^{-1}y, y)$. By uniqueness of the decomposition, $f_-(t^{-1}y, y) = f_+(t, y)$.

EXAMPLE 1. Let S be the set $\{(1, 0), (0, 1)\}$. Then $\Gamma(S^*) = [1 - t^{-1}y - t]^{-1}$. By Lemma 4.5 we have

$$\Gamma(S_0) = \frac{1 - (1 - 4y)^{1/2}}{2y} = \sum_{n=0}^{\infty} \frac{1}{2n + 1} \binom{2n + 1}{n},$$

$$\Gamma(S_+) = \left[1 - t \frac{1 - (1 - 4y)^{1/2}}{2y}\right]^{-1}$$

$$= 1 + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} t^k y^n \frac{k}{2n + k} \binom{2n + k}{n}.$$

The numbers

$$\frac{1}{2n + 1} \binom{2n + 1}{n} = \frac{1}{n + 1} \binom{2n}{n}$$

are the *Catalan numbers* and the numbers

$$\frac{k}{2n+k}\binom{2n+k}{n}$$

are the *ballot numbers*. Many interesting properties of these numbers are described in [6, 19], and further references can be found in Gould's bibliography [14].

EXAMPLE 2.   Let $S = \{(1, 0), (0, 1), (1, 1)\}$. Here we have $\Gamma(S^*) = [1 - t^{-1}y - y - t]^{-1}$. By Lemma 4.4 we have

$$\Gamma(S_0) = \frac{1 - y - (1 - 6y + y^2)^{1/2}}{2y}$$

$$= \sum_{n,j=0}^{\infty} y^{n+j} \frac{1}{2n+j+1}\binom{2n+j+1}{n+1, n, j}$$

$$= \sum_{n=0}^{\infty} y^n \sum_{j=0}^{n} \frac{1}{2n-j+1}\binom{2n-j+1}{n-j+1, n-j, j},$$

$$\Gamma(S_+) = \left[1 - t\frac{1 - y - (1 - 6y + y^2)^{1/2}}{2y}\right]^{-1}$$

$$= 1 + \sum_{k=1}^{\infty} \sum_{n,j=0}^{\infty} t^k y^{n+j} \frac{k}{2n+j+k}\binom{2n+j+k}{n+k, n, j}$$

$$= 1 + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} t^k y^n \sum_{j=0}^{n} \frac{k}{2n-j+k}\binom{2n-j+k}{n-j+k, n-j, j}.$$

We remark that there is an alternate expression

$$\Gamma(S_+) = \sum_{k=0}^{\infty} t^k + \sum_{k,n=1}^{\infty} t^k y^n \sum_{j=1}^{n} 2^j \frac{k}{j}\binom{n-1}{j-1}\binom{n+k-1}{j-1},$$

which can be obtained from the representation

$$\Gamma(S^*) = (1 - t^{-1}y - y - t)^{-1} = [(1 - t^{-1}y)(1 - t) - 2y]^{-1}$$

$$= [(1 - t^{-1}y)(1 - t)]^{-1}\left[1 - \frac{2y}{(1 - t^{-1}y)(1 - t)}\right]^{-1}.$$

For work on this problem see [11] and the references cited there. Other

combinatorial interpretations and asymptotic expressions for the coefficients of

$$\Gamma(S_0) = \frac{1 - y - (1 - 6y + y^2)^{1/2}}{2y}$$

can be found in [15, p. 239, exercises 11 and 12, solution, p. 534; 16; 21]. The polynomials $H_n(z)$ defined by

$$\sum_{n=0}^{\infty} H_n(z) y^n = \frac{1 + y - [1 - 2(1 + 2z) y + y^2]^{1/2}}{2(1 + z)}$$

have been considered by Riordan [19, pp. 150–151]; for $n > 0$, $\langle y^n \rangle \, \Gamma(S_0) = 2H_n(1)$. The coefficients of $\Gamma(S_+)$ have been studied by Rogers [20], who gives further combinatorial interpretations and references.

EXAMPLE 3.   Let $S = \{(2, 0), (1, 1), (0, 2)\}$. Then $\Gamma(S^*) = [1-t^{-2}y^2-y-t^2]^{-1}$. Substituting y^2 for y, t^2 for t, and y for z in Lemma 4.5, we get

$$\Gamma(S_0) = \frac{1 - y - (1 - 2y - 3y^2)^{1/2}}{2y^2}$$

$$= \sum_{n,j=0}^{\infty} y^{2n+j} \frac{1}{2n + j + 1} \binom{2n + j + 1}{n + 1, n, j}$$

$$= \sum_{n=0}^{\infty} y^n \sum_{m \leqslant n/2} \frac{1}{n + 1} \binom{n + 1}{m + 1, m, n - 2m},$$

$$\Gamma(S_+) = \left[1 - t^2 \frac{1 - y - (1 - 2y - 3y^2)^{1/2}}{2y^2} \right]^{-1}$$

$$= 1 + \sum_{k=1}^{\infty} \sum_{n,j=0}^{\infty} t^{2k} y^{2n+j} \frac{k}{2n + j + k} \binom{2n + j + k}{n + k, n, j}$$

$$= 1 + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} t^{2k} y^n \sum_{m \leqslant n/2} \frac{k}{n + k} \binom{n + k}{m + k, m, n - 2m}.$$

The coefficients of $\Gamma(S_0)$ are sometimes called *Motzkin numbers* [17]. These numbers, and more generally the coefficients of $\Gamma(S_+)$, have been studied by Donaghey and Shapiro [7].

EXAMPLE 4.   Let S consist of all possible steps. Then

$$\Gamma(S) = [(1 - t)(1 - t^{-1}y)]^{-1} - 1,$$

so

$$\Gamma(S^*) = \{2 - [(1 - t)(1 - t^{-1}y)]^{-1}\}^{-1} = \frac{(1 - t)(1 - t^{-1}y)}{1 - 2t - 2t^{-1}y + 2y}.$$

Thus, substituting $2t$ for t, $4y$ for y, and $-2y$ for z in Lemma 4.5, we get

$$\Gamma(S_0) = \frac{1 + 2y - (1 - 12y + 4y^2)^{1/2}}{8y}$$

$$= \sum_{n,j=0}^{\infty} (4y)^n (-2y)^j \frac{1}{2n + j + 1} \binom{2n + j + 1}{n + 1, n, j}$$

$$= \sum_{n=0}^{\infty} y^n \sum_{j=0}^{n} (-1)^j \frac{2^{2n-j}}{2n - j + 1} \binom{2n - j + 1}{n - j + 1, n - j, j},$$

$$\Gamma(S_+) = (1 - t)\left[1 - t\frac{1 + 2y - (1 - 12y + 4y^2)^{1/2}}{4y}\right]^{-1}.$$

(Here we have applied Lemma 4.4 to $f = (1 - 2t - 2t^{-1}y + 2y)^{-1}$; then the factor $1 - t$ of $\Gamma(S^*)$ goes with f_+.)

A more detailed study of the factorization described by Lemma 4.3 in the case in which no step has height less than -1 (i.e., the case to which Theorem 4.2 applies) is given in [13].

5. ALGEBRAIC SERIES

In the last section we saw instances in which f_-, f_0, and f_+ can be expressed as rational functions of square roots of polynomials. In this section we show that if f is rational, then f_-, f_0, and f_+ are algebraic.

A formal power series f in $\mathbb{C}[[t, y/t]]$ is *rational* if it is a quotient of polynomials. It is *algebraic* (over $\mathbb{C}[t, y]$) if there exist polynomials p_0, p_1, ..., p_n in $\mathbb{C}[t, y]$, not all zero, such that $\sum_{i=0}^{n} p_i f^i = 0$. A series g in $\mathbb{C}[[y]]$, or more generally, in $\mathbb{C}[[y^{1/r}]]$, is algebraic (over $\mathbb{C}[y]$) if there exist polynomials q_0, ..., q_m in $\mathbb{C}[y]$, not all zero, such that $\sum_{i=0}^{m} q_i g^i = 0$. Algebraic series in one variable are especially significant computationally, since if $\sum_{n=0}^{\infty} a_n y^n$ is algebraic, then the a_n satisfy a recursion of the form $\sum_{i=0}^{k} p_i(n) a_{n-i} = 0$ for sufficiently large n, where the p_i are polynomials and $p_0(n)$ is not identically zero [5; 22, Theorem 5.1].

We first observe that $\mathbb{C}[[y]]$ can be embedded in its field of fractions, $\mathbb{C}((y))$, which in turn can be embedded in its algebraic closure. (All the rings we consider can be embedded in the algebraic closure of $\mathbb{C}((t))((y))$.)

Now let $y^{1/r}$ be an rth root of y. We recall that by the Newton–Puiseux theorem [4, pp. 373–396] every element of the algebraic closure of $\mathbb{C}((y))$ is in $\mathbb{C}((y^{1/r}))$ for some integer $r > 0$.

5.1. THEOREM. *Let f be an element of $\mathbb{C}[t, y/t]$ of the form $\sum_{i=-m}^{n} a_i t^i$, where $a_{-m} a_n \neq 0$, $a_i \in \mathbb{C}[y]$, and a_0 has constant term 1. Then*

$$f = \gamma \left[\prod_{i=1}^{m} (1 - t^{-1}\alpha_i) \right] \left[\prod_{j=1}^{n} (1 - t\beta_j^{-1}) \right],$$

where α_i, β_j, and γ are algebraic over $\mathbb{C}[y]$, γ is in $\mathbb{C}[[y]]$ and has constant term 1, and for some integer $r > 0$, α_i is in $y^{1/r}\mathbb{C}[[y^{1/r}]]$ and β_j is in $\mathbb{C}[[y^{1/r}]]$.

Proof. We have $t^m f = \sum_{i=0}^{m+n} a_{i-m} t^i = a_n \prod_{k=1}^{m+n} (t - \xi_k)$, where ξ_k is in the algebraic closure of $\mathbb{C}[y]$, hence is in $\mathbb{C}((y^{1/r}))$ for some r. Now let us rename the ξ_k as $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$, where $\alpha_i \in y^{1/r} \mathbb{C}[[y^{1/r}]]$ and $\beta_j \notin y^{1/r}\mathbb{C}[[y^{1/r}]]$ (so that $\beta_j^{-1} \in \mathbb{C}[[y^{1/r}]]$). Now $t - \alpha_i = t(1 - t^{-1}\alpha_i)$ and $t - \beta_j = -\beta_j(1 - t\beta_j^{-1})$, so

$$t^m f = a_n t^p \prod_{j=1}^{q} (-\beta_j) \prod_{i=1}^{p} (1 - t^{-1}\alpha_i) \prod_{j=1}^{q} (1 - t\beta_j^{-1}). \qquad (3)$$

Note that $t^{-1}\alpha_i$ and $t\beta_j^{-1}$ are in $\mathbb{C}[[t, y^{1/r}/t]]$. Then since f and the second and third products on the rights of (3) have constant term 1, so does $a_n t^{p-m} \prod_{j=1}^{q} (-\beta_j)$. Since a_n and β_j do not involve t, we must have $p = m$, whence $q = n$. Set $\gamma = a_n \prod_{j=1}^{q} (-\beta_j)$. It remains only to show that γ is in $\mathbb{C}[[y]]$. It is easily seen that the uniqueness of f_-, f_0, and f_+ given by Theorem 4.1 remains if we allow f_-, f_0, and f_+ to be in $\mathbb{C}[[t, y^{1/r}/t]]$, rather than $\mathbb{C}[[t, y/t]]$. Thus

$$f_- = \prod_{i=1}^{m} (1 - t^{-1}\alpha_i), \quad f_0 = \gamma, \quad \text{and} \quad f_+ = \prod_{j=1}^{n} (1 - t\beta_j^{-1}).$$

We would now like to extend Theorem 5.1 to rational power series. To do this we use the following lemma.

5.2. LEMMA. *Let f and g be polynomials in $\mathbb{C}[x, y]$ with no common factor such that f/g is in $\mathbb{C}[[x, y]]$. Then g has nonzero constant term.*

Proof. Since f and g have no common factor, there exist polynomials p and q in $\mathbb{C}[x, y]$ such that $pg + qf$ is a polynomial in y. Setting $h = f/g$, we have $g(p + qh) = y^m u$ for some integer $m \geqslant 0$, where u is a polynomial in y with nonzero constant term, hence is a unit in $\mathbb{C}[[x, y]]$. Since $\mathbb{C}[[x, y]]$ is a unique factorization domain [3, p. 45], it follows that g is a power of y times a unit of $\mathbb{C}[[x, y]]$. Symmetrically, we see that g is a power of x times a unit of $\mathbb{C}[[x, y]]$. Thus g is a unit of $\mathbb{C}[[x, y]]$.

5.3. COROLLARY. *Let f be an element of $\mathbb{C}[[t, y/t]]$ with constant term 1 which is a quotient of polynomials. Then f_-, f_0, f_+ are algebraic.*

Proof. By the lemma, f can be expressed in the form g/h, where g and h are of a form to which Theorem 5.1 applies.

5.4. COROLLARY. *Let S be a set of steps such that $\Gamma(S)$ is rational. (For example, we may take S to be finite.) Then $\Gamma(S_-)$, $\Gamma(S_0)$, $\Gamma(S_+)$, $\langle t^i \rangle \Gamma(S_-)$, and $\langle t^i \rangle \Gamma(S_+)$ are algebraic.*

We now consider a lattice path problem which illustrates some of these ideas. Let our set S of steps be $\{(m - 1, 0), (0, 1)\}$. Note that by a change of scale, problems involving these steps can be transformed into problems involving the steps $(1, 0)$ and $(0, 1)$ in which the role of the diagonal $x = y$ is replaced by the diagonal $y = (m - 1)x$.

Here we have $\Gamma(S) = t^{m-1} + t^{-1}y$, so $\Gamma(S^*) = (1 - t^{m-1} - t^{-1}y)^{-1} = f$. By Theorem 5.1, we have

$$f = \frac{\gamma}{(1 - \alpha t^{-1}) \prod_{j=1}^{m-1} (1 - t\beta_j^{-1})} \, ,$$

where

$$\Gamma(S_-) = f_- = (1 - \alpha t^{-1})^{-1}, \; \Gamma(S_0) = f_0 = \gamma,$$

and

$$\Gamma(S_+) = f_+ = \left[\prod_{j=1}^{m-1} (1 - t\beta_j^{-1}) \right]^{-1}.$$

By Theorem 4.2, α satisfies $\alpha = y + \alpha^m$, and for $k > 0$

$$\alpha^k = \sum_{j=0}^{\infty} \frac{k}{mj + k} \binom{mj + k}{j} y^{(m-1)j+k}.$$

Thus

$$\Gamma(S_-) = \sum_{k=0}^{\infty} t^{-k}\alpha^k = 1 + \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{k}{mj + k} \binom{mj + k}{j} y^{(m-1)j+k} t^{-k}$$

and

$$\Gamma(S_0) = \sum_{j=0}^{\infty} \frac{1}{mj + 1} \binom{mj + 1}{j} y^{(m-1)j}.$$

This result is well known; Takács [23] attributes it to Aeppli [1] in 1923.

To compute the coefficients of $f_+ = \Gamma(S_+)$ is somewhat harder. We can use Lagrange's formula to derive formulas for the powers of each β_i^{-1}, but this does not suffice to give the coefficients of $[\prod_{j=1}^{m-1} (1 - t\beta_j^{-1})]^{-1}$. Instead we use another approach.

The idea is that we know f_0, f_-, and $f = f_- f_0 f_+$, so we compute f_+ as $f/(f_- f_0)$. We have $f_- f_0 = (\alpha/y)(1 - t^{-1}\alpha)^{-1}$, so $(f_- f_0)^{-1} = \alpha^{-1}y - t^{-1}y$. Since $\alpha^{-1}y = 1 - \alpha^{m-1}$, we obtain

$$\alpha^{-1}y = 1 - \sum_{j=0}^{\infty} \frac{m-1}{mj+m-1} \binom{mj+m-1}{j} y^{(m-1)j+m-1}$$

$$= 1 - \sum_{j=1}^{\infty} \frac{m-1}{mj-1} \binom{mj-1}{j-1} y^{(m-1)j}$$

$$= 1 - \sum_{j=1}^{\infty} \frac{1}{mj-1} \binom{mj-1}{j} y^{(m-1)j}.$$

Then since $f^{-1} = 1 - t^{-1}y - t^{m-1}$, we have

$$(f_- f_0)^{-1} = f^{-1} + t^{m-1} - \sum_{j=1}^{\infty} \frac{1}{mj-1} \binom{mj-1}{j} y^{(m-1)j},$$

so

$$f_+ = f(f_- f_0)^{-1}$$
$$= 1 + (1 - t^{-1}y - t^{m-1})^{-1} \left[t^{m-1} - \sum_{j=1}^{\infty} \frac{1}{mj-1} \binom{mj-1}{j} y^{(m-1)j} \right].$$

Using the expansion

$$(1 - t^{-1}y - t^{m-1})^{-1} = \sum_{k=-\infty}^{\infty} \sum_{i=0}^{\infty} y^{(m-1)i-k} t^k \binom{mi-k}{i},$$

we get

$$f_+ = 1 + \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} y^{(m-1)i-k} t^k \left\{ \binom{mi-k-1}{i-1} \right.$$

$$\left. - \sum_{j=1}^{\infty} \frac{1}{mj-1} \binom{mj-1}{j} \binom{m(i-j)-k}{i-j} \right\}.$$

Here we use the convention that $\binom{n}{j} = 0$ if $n < 0$ or $j < 0$.

We may restate our result as follows:

5.5. THEOREM. *The number of lattice paths with steps $(1, 0)$ and $(0, 1)$ from the origin to (i, j), with every point below the line $y = (m - 1)x$, is*

$$\binom{i+j-1}{i-1} - \sum_{r=1}^{\infty} \frac{1}{mr-1} \binom{mr-1}{r} \binom{i+j-mr}{i-r}.$$

For work on this and related problems, and further references, see [11, 26]. For the case $m = 3$, we can express f_+ in another form. We have

$$f = \frac{1}{1 - t^{-1}y - t^2} = \frac{1}{1 - t^{-1}\alpha} \cdot \frac{\alpha}{y} \cdot \frac{1}{1 - (\alpha^2/y)\, t - (\alpha/y)\, t^2},$$

as can be verified directly, using the fact that $\alpha^3 - \alpha + y = 0$. Thus

$$f_+ = \left(1 - \frac{\alpha^2}{y} t - \frac{\alpha}{y} t^2\right)^{-1}$$

$$= \sum_{j=0}^{\infty} \left(\frac{\alpha}{y} t\right)^j (\alpha + t)^j = \sum_{j,m=0}^{\infty} \left(\frac{\alpha}{y} t\right)^j t^m \alpha^{j-m} \binom{j}{m}$$

$$= \sum_{j,k=0}^{\infty} t^k y^{-j} \alpha^{3j-k} \binom{j}{k-j}.$$

Now by Lagrange's formula,

$$\alpha^l = \sum_{m=0}^{\infty} \frac{l}{3m+l} \binom{3m+l}{m} y^{2m+l} \qquad \text{for } l > 0,$$

so

$$f_+ = 1 + \sum_{j,k=1}^{\infty} t^k \binom{j}{k-j} \sum_{m > k/3-j} \frac{3j-k}{3(m+j)-k} \binom{3(m+j)-k}{m} y^{2(m+j)-k}$$

$$= 1 + \sum_{1 \leqslant k < 2n}^{\infty} t^k y^{2n-k} \sum_{j=0}^{\infty} \frac{3j-k}{3n-k} \binom{3n-k}{n-j} \binom{j}{k-j}.$$

In other words, the quantity evaluated in Theorem 5.5 has for $m = 3$ the value

$$\sum_{s=0}^{\infty} \frac{3s-2i+j}{i+j} \binom{i+j}{i-s} \binom{s}{2i-j-s} = \sum_{r=0}^{\infty} \frac{i+j-3r}{i+j} \binom{i+j}{r} \binom{i-r}{j-2r}.$$

Our approach to the lattice path problem has been to demonstrate that Lemma 4.3 gives all the combinatorial information needed to derive an explicit formula for the solution. However, many of the formulas derived here have more direct combinatorial interpretations, which I hope to discuss elsewhere.

6. The Diagonal of a Rational Series

The following theorem was first proved analytically by Furstenburg [12], using the calculus of residues. (See also Stanley [22, Theorem 5.3].) A related

theorem was proved similarly by Pólya [18]. In our proof the use of residues is replaced by a partial fraction decomposition.

A completely different proof, using formal power series in noncommuting variables, was found by Fliess [9].

6.1. THEOREM.   *Let* $f(x, y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$ *be a rational power series in* $\mathbb{C}[[x, y]]$. *Then* $D(y) = \sum_{i=0}^{\infty} a_{ii} y^i$ *is algebraic.*

Proof.   Let $g = f(t, y/t) \in \mathbb{C}[[t, y/t]]$, so that $D(y) = \langle t^0 \rangle g$. Then $g = P/Q$ where P and Q are polynomials in t and y. Then by the Newton–Puiseux theorem and partial fraction decomposition we have

$$g = \sum_{k=1}^{l} \frac{s_k(t)}{(t - \xi_k)^{e_k}},$$

for some positive integers e_k, where for some r, each ξ_k is in $\mathbb{C}((y^{1/r}))$ and $s_k(t) \in \mathbb{C}((y^{1/r}))[t]$. Rename the ξ_k as $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n$, where, as in the proof of Theorem 5.1, $\alpha_i \in y^{1/r} \mathbb{C}[[y^{1/r}]]$ and $\beta_{j-1} \in \mathbb{C}[[y^{1/r}]]$. Then

$$g = \sum_{i=1}^{m} \frac{p_i(t)}{(1 - t^{-1}\alpha_i)^{c_i}} + \sum_{j=1}^{n} \frac{q_j(t)}{(1 - t\beta_j^{-1})^{d_j}} \tag{4}$$

for some positive integers c_i and d_j, with $p_i(t)$ and $q_j(t)$ in $\mathbb{C}((y^{1/r}))[t^{-1}, t]$.

If we consider g as an element of $\mathbb{C}((t))((y^{1/r}))$ then its constant term in t is well defined. Thus $D(y)$ is equal to the constant term of the right side of (4). (Note that $\mathbb{C}((y^{1/r}))[t^{-1}, t]$ is contained in $\mathbb{C}((t))((y^{1/r}))$.) Since the α_i and β_j are algebraic and each $p_i(t)$ and $q_j(t)$ is a polynomial in t and t^{-1} with coefficients in $\mathbb{C}((y^{1/r}))$ which are algebraic, it follows that $D(y)$ is a finite sum of terms of the form $\gamma\alpha^a$ and $\delta\beta^b$ with α, β, γ, and δ algebraic. Thus $D(y)$ is algebraic.

As the simplest nontrivial example of Theorem 6.1, take

$$f(x, y) = (1 - x - y)^{-1} = \sum_{i,j=0}^{\infty} \binom{i+j}{i} x^i y^j.$$

Then $g = (1 - t - y/t)^{-1} = -t/(t^2 - t + y)$. We have $t^2 - t + y = (t - \alpha)(t - \beta)$, where $\alpha = [1 - (1 - 4y)^{1/2}]/2$ and $\beta = [1 + (1 - 4y)^{1/2}]/2$, and a partial fraction expansion of g is

$$g = \frac{1}{\beta - \alpha}\left[\frac{t}{t - \alpha} - \frac{t}{t - \beta}\right].$$

Note that α has constant term zero, but β does not. Then using $\beta - \alpha = (1 - 4y)^{1/2}$ and $\beta^{-1} = \alpha/y$ we have

$$g = (1 - 4y)^{-1/2} \left[\frac{1}{1 - t^{-1}\alpha} + \frac{t\beta^{-1}}{1 - t\beta^{-1}} \right]$$

$$= (1 - 4y)^{-1/2} \left[\frac{1}{1 - t^{-1}\alpha} + \frac{t(\alpha/y)}{1 - t(\alpha/y)} \right]$$

$$= (1 - 4y)^{-1/2} \left[\sum_{m=0}^{\infty} t^{-m}\alpha^m + \sum_{n=1}^{\infty} t^n(\alpha/y)^n \right].$$

Thus $\langle t^0 \rangle g = (1 - 4y)^{-1/2}$.

Note that we cannot write

$$\frac{t}{t - \alpha} = -\frac{t\alpha^{-1}}{1 - t\alpha^{-1}} = -\sum_{n=1}^{\infty} (t\alpha^{-1})^n$$

or

$$\frac{t}{t - \beta} = \frac{1}{1 - t^{-1}\beta} = \sum_{m=0}^{\infty} t^{-m}\beta^m,$$

since these infinite sums do not exist in $\mathbb{C}((t))[[y]]$ (or in any of its extensions).

REFERENCES

1. A. AEPPLI, A propos de l'interprétation géométrique du problème du scrutin, *Enseignement Math.* **23** (1923), 328–329.
2. D. E. BARTON AND C. L. MALLOWS, Some aspects of the random sequence, *Ann. Math. Statist.* **36** (1965), 236–260.
3. L. BERS, "Introduction to Several Complex Variables," Courant Institute, New York, 1964.
4. G. CHRYSTAL, "Textbook of Algebra," Part 2, Dover, New York, 1961.
5. L. COMTET, Calcul pratique des coefficients de Taylor d'une fonction algébrique, *Enseignement Math.* (2) **10** (1964), 267–270.
6. L. COMTET, "Advanced Combinatorics," Reidel, Boston, 1974.
7. R. DONAGHEY AND L. W. SHAPIRO, Motzkin numbers, *J. Combinatorial Theory Ser. A* **23** (1977), 291–301.
8. W. FELLER, "An Introduction to Probability Theory and Its Applications," Vol. II, Wiley, New York, 1966.
9. M. FLIESS, Sur divers produits de séries formelles, *Bull. Soc. Math. France* **102** (1974), 181–191.
10. D. FOATA AND M.-P. SCHÜTZENBERGER, On the principle of equivalence of Sparre Anderson, *Math. Scand.* **28** (1971), 308–316.
11. R. D. FRAY AND D. P. ROSELLE, Weighted lattice paths, *Pacific J. Math.* **37** (1971), 85–96.

12. H. FURSTENBERG, Algebraic functions over finite fields, *J. Algebra* **7** (1967), 271–277.
13. I. GESSEL, A noncommutative generalization and *q*-analog of the Lagrange inversion formula, *Trans. Amer. Math. Soc.* **257** (1980), 455–482.
14. H. W. GOULD, "Research Bibliography of Two Special Number Sequences," rev. ed., Combinatorial Research Institute, Morgantown, W. Va., 1977.
15. D. E. KNUTH, "The Art of Computer Programming," Vol. I, "Fundamental Algorithms," 2nd ed., Addison-Wesley, Reading, Mass., 1973.
16. J. S. LEW, Polynomial enumeration of multidimensional lattices, *Math. Systems Theory* **12** (1979), 253–270.
17. TH. MOTZKIN, Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products, *Bull. Amer. Math. Soc.* **54** (1948), 352–360.
18. G. PÓLYA, Sur les séries entières, dont la somme est une fonction algébrique, *Enseignement Math.* **22** (1921–22), 38–47.
19. J. RIORDAN, "Combinatorial Identities," Wiley, New York, 1968.
20. D. G. ROGERS, A Schröder triangle: Three combinatorial problems, *in* "Combinatorial Mathematics V, Melbourne, 1976" (C. H. C. Little, Ed.), pp. 175–196, Lecture Notes in Mathematics No. 622, Springer-Verlag, Berlin, 1977.
21. D. G. ROGERS AND L. W. SHAPIRO, Some correspondences involving the Schröder numbers and relations, *in* "Combinatorial Mathematics Proceedings, Canberra, 1977," (D. A. Holton and J. Seberry, Eds.), pp. 267–274, Lecture Notes in Mathematics No. 686, Springer-Verlag, Berlin, 1978.
22. R. P. STANLEY, Generating functions, *in* "MAA Studies in Combinatorics" (G.-C. Rota, Ed.), pp. 100–141, Math. Assoc. Amer., Washington, D.C., 1978.
23. L. TAKÁCS, "Combinatorial Methods in the Theory of Stochastic Processes," Wiley, New York, 1967.
24. W. T. TUTTE, On elementary calculus and the Good formula, *J. Combinatorial Theory Ser. B* **18** (1975), 97–137.
25. E. T. WHITTAKER AND G. N. WATSON, "A Course of Modern Analysis," 4th ed., Cambridge Univ. Press, Cambridge, 1963.
26. S. ZAKS, Generating *k*-ary trees lexicographically, to appear.