Reachability in Petri Nets with Inhibitor Arcs

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Abstract

We define 2 operators on relations over natural numbers such that they generalize the operators '+' and '*' and show that the membership and emptiness problem of relations constructed from finite relations with these operators and ∪ is decidable. This generalizes Presburger arithmetics and allows to decide the reachability problem for those Petri nets where inhibitor arcs occur only in some restricted way. Especially the reachability problem is decidable for Petri nets with only one inhibitor arc, which solves an open problem in [11]. Furthermore we describe the corresponding automaton having a decidable emptiness problem.

Keywords: Petri nets, inhibitor arcs, reachability, monotone transitive closure, counter automata

1 Introduction

The decidability of the reachability problem in Petri nets without inhibitor arcs is proved in [14] and later in [12] and [13]. On the other hand, the reachability problem is undecidable for Petri nets with two inhibitor arcs which follows from [15]. An open problem in [11] was the reachability problem for Petri nets with one inhibitor arc.

An important method is the use of semilinear sets which are defined using the operators +, *, ∪ over finite sets of vectors (multisets). Semilinear sets are the solutions of Presburger formula, where Presburger arithmetic is the first order logic over the natural numbers and the addition. Presburger arithmetic is decidable and semilinear sets are closed under ∩ and complement [8],[5].

But a reachability relation for a Petri net is in general not semilinear. For that reason the basic idea of this paper is to replace + and * by suitable operations $\circ_Q$ and $*_Q$ which are able to express a reachability relation as the sequence of relations (like the transitive closure used in [9] to characterize NL with first order logic or more generally in [1]).

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But the transitive closure over first order logic over natural numbers with the addition immediately becomes undecidable. For that reason the important principle of monotonicity in the reachability relation of Petri nets is combined with the idea of the transitive closure. That means the operator $*_Q$ is a monotone transitive closure (see Corollary 3.6). We consider the following three steps:

(i) One application of $*_Q$ already allows us to express the reachability problem in a Petri net without inhibitor arcs (Lemma 2.1).

(ii) A second application of $*_Q$ (containing the first one in a nested way) allows us to express the reachability problem in a Petri net with one inhibitor arc (Lemma 2.4).

(iii) Arbitrary nested applications of $*_Q$ allow us to express the reachability problem in a Petri net for which there exists an ordering of the places such that a place has an inhibitor arc to all those transitions which have an inhibitor arc from a preceding place (Theorem 5.1).

In Section 3 we use expressions consisting of the operators $\cup$, $\circ_Q$ and $*_Q$ on sets of multisets in a special form (Lemmata 3.1 and 3.5 show that we can bring every such expression in this form), which models the idea of a nested Petri net: The firing behavior of a complex (nested) transition is linked to firing sequences in inner Petri nets by a semilinear relation (unlike in the structured nets in [2]). The connection between these inner Petri nets corresponds to the chain of vector addition systems used in [12] and it is described by the same semilinear relation. The main difference to the structure of the proofs in [12] and [13] is that states are not anymore necessary since their function is instead fulfilled (Section 4.4) by the nestedness of expressions (like regular expressions replace a finite automaton).

Furthermore we define a condition (normal form $\mathcal{T}$ corresponding to the property Θ in [12]), which allows to check the emptiness of the expressed set of multisets, we define a size of the expressions leading to a Noetherian order and construct an algorithm in Section 4 which finds an equivalent expression fulfilling condition $\mathcal{T}$. Each step of the algorithm constructs an equivalent expression which is smaller with respect to the defined size. Some kind of pumping property (Lemma 4.3) ensures the existence of arbitrary high firing sequences if condition $\mathcal{T}$ is fulfilled.

The most general result (see Step 3 above) is described in Section 5 This allows to decide the expressed reachability problem. Sections 6 and 7 describe the conclusions for emptiness problems for automata.

The contents of this paper mainly corresponds to Chapter 5 in the habilitation-thesis [17], which also contains the proofs which had to be left out here.

1.1   Multisets

For the sake of a flexible description, we use multi-sets instead of vectors. A *multiset* over B is a function in $\mathbb{N}^B$.

We might write a multiset $\mathbf{f} \in \mathbb{N}^B$ as a set $\{b \mapsto f(b) \mid b \in B\}$, as a table

$$\left[\begin{matrix} b_1, & b_2, & \dots, & b_n \\ f(b_1), & f(b_2), & \dots, & f(b_n) \end{matrix}\right] \text{ or as an } n\text{-ary vector } \begin{pmatrix} \mathbf{f}(b_1) \\ \mathbf{f}(b_2) \\ \vdots \\ \mathbf{f}(b_n) \end{pmatrix}. \text{ For the latter, we have to}$$

assume an ordering on $B = \{b_1, b_2, ..., b_n\}$ (without relevance to the contents), and in the first two descriptions, we only need to write those b's with $f(b) > 0$. Although we do not a priori limit the size of B, we only use multisets for a finite B in this paper. For multisets, we use the variables $\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{m}, \mathbf{n}, \mathbf{r}, \mathbf{s}, \mathbf{x}, \mathbf{y}$, and for sets of multisets, we use the capitals $\mathbf{E}, \mathbf{L}, \mathbf{M}, \mathbf{N}, \mathbf{R}$ and $\mathbf{I}d$ (the latter will denote the identity for the operator $\circ_Q$ to be defined).

For $A \subseteq B$, we regard functions in $\mathbb{N}^A \subseteq \mathbb{N}^B$ as extended to zero for undefined values. This allows us to add any two multisets $\mathbf{f} \in \mathbb{N}^A$ and $\mathbf{g} \in \mathbb{N}^B$ and obtain a multiset in $(\mathbf{f} + \mathbf{g}) \in \mathbb{N}^{A \cup B}$ with $(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x)$ in the same way as we would add the corresponding vectors assuming an ordering on $A \cup B$. The neutral element for addition is $\emptyset$ with $\emptyset(x) = 0$ for all x. It holds $\mathbb{N}^A \cap \mathbb{N}^B = \mathbb{N}^{A \cap B}$.

The restriction $\mathbf{f}|_A$ of a multi-set $\mathbf{f} \in \mathbb{N}^B$ to A is

$$\mathbf{f}|_A (b) := \mathbf{f}(b) \text{ if } b \in A \text{ else } \mathbf{f}|_A (b) := 0.$$

This means $\mathbf{f}|_A := \{b \mapsto \mathbf{f}(b) \mid b \in A\}$. The complement operator is $\mathbf{f}|_{\overline{A}} := \{b \mapsto \mathbf{f}(b) \mid b \notin A\}$, thus $\mathbf{f} = \mathbf{f}|_A + \mathbf{f}|_{\overline{A}}$.

For a finite set $\mathbf{M} = \{\mathbf{m}_1, ..., \mathbf{m}_k\} \subseteq \mathbb{N}^A$ of multi-sets,

$$\mathbf{M}^* := \{a_1\mathbf{m}_1 + ... + a_k\mathbf{m}_k \mid \forall i \le k \; a_i \in \mathbb{N}\}$$

is the set of all linear combinations generated by $\mathbf{M}$. More generally, by $\mathbf{M}^0 := \{\emptyset\}$ and $\mathbf{M}^{i+1} := \mathbf{M}^i + \mathbf{M}$, we can define $\mathbf{M}^* := \bigcup_i \mathbf{M}^i$.

1.1.1 New operator on multisets

For an unambiguous [2] and injective binary relation Q, we define the operator $\circ_Q$ on two sets of Multisets $\mathbf{M}$ and $\mathbf{N}$ as

$$\mathbf{N} \circ_Q \mathbf{M} := \left\{ \mathbf{n}|_{\overline{\pi_1(Q)}} + \mathbf{m}|_{\overline{\pi_2(Q)}} \;\middle|\; \mathbf{n} \in \mathbf{N}, \mathbf{m} \in \mathbf{M}, \forall (a, b) \in Q \; \mathbf{n}(a) = \mathbf{m}(b) \right\}.$$

This means if $\mathbf{n}$ and $\mathbf{m}$ "match" according to Q, then the values for an $a \in \pi_1(Q) = \{a | (a, b) \in Q\}$ in $\mathbf{n}$ and the values for a $b \in \pi_2(Q) = \{b | (a, b) \in Q\}$ in $\mathbf{m}$ are "used up against each other" and the rest is added. For example,

$$\left\{ \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} \right\} \circ_{\{(b_1, b_2)\}} \left\{ \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} \right\}$$

or

$$\left\{ \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} \right\} \circ_{\{(b_3, b_3)\}} \left\{ \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 11 \\ 9 \end{pmatrix}, \begin{pmatrix} 9 \\ 7 \end{pmatrix} \right\}$$

The latter example shows that the dimension is necessarily reduced (b_3 is used up on both sides) if $\pi_1(Q) \cap \pi_2(Q)$ is not empty. We will later need $\hat{\circ}_A := \circ_{\{(a,a)|a \in A\}}$

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[2] A binary Q is unambiguous if Q^{-1} is injective.

to get $\mathbf{N}\hat{\circ}_A\mathbf{M} = \big\{\, \mathbf{n}|_{\overline{A}} + \mathbf{m}|_{\overline{A}} \,\big|\, \mathbf{n} \in \mathbf{N}, \mathbf{m} \in \mathbf{M}, \forall a \in A\ \mathbf{n}(a) = \mathbf{m}(a) \big\}$. For example $\circ_{\{(b_3,b_3)\}} = \hat{\circ}_{\{b_3\}}$.

If $\pi_1(Q)$ and $\pi_2(Q)$ are disjoint, we define $\mathbf{Id}_Q := \{\{a \mapsto 1, b \mapsto 1\} \mid (a,b) \in Q\}^*$ which is the neutral element for $\circ_Q$. Obviously, it holds $\mathbf{N}\circ_\emptyset \mathbf{M} = \mathbf{N} + \mathbf{M}$ which makes $+$ with the neutral element $\mathbf{Id}_\emptyset = \{\emptyset\}$ a special case of the $\circ_Q$ operator.

Furthermore, for Q with $\pi_1(Q)$ and $\pi_2(Q)$ disjoint, we define $*_Q^0(\mathbf{M}) := \mathbf{Id}_Q$, $*_Q^{i+1}(\mathbf{M}) := *_Q^i(\mathbf{M})\circ_Q(\mathbf{M} + \mathbf{Id}_Q)$ and $*_Q(\mathbf{M}) := \bigcup_i *_Q^i(\mathbf{M})$. Again, $*_\emptyset(\mathbf{M}) = \mathbf{M}^*$ is a special case. It is easy to see that $*_Q(\mathbf{M})$ is the closure of $\mathbf{M} \cup \mathbf{Id}_Q$ under $\circ_Q$ and the addition $\circ_\emptyset$:

If $\mathbf{f}, \mathbf{g} \in *_Q(\mathbf{M})$ then there are $i,j \in \mathbb{N}$ with $\mathbf{f} \in *_Q^i(\mathbf{M})$ and $\mathbf{g} \in *_Q^j(\mathbf{M})$ thus $\{\mathbf{f}\}\circ_Q\{\mathbf{g}\} \subseteq *_Q^i(\mathbf{M})\circ_Q *_Q^j(\mathbf{M}) = *_Q^{i+j}(\mathbf{M}) \subseteq *_Q(\mathbf{M})$.

Let $\mathbf{f}' = \mathbf{f} + \{a \mapsto \mathbf{g}(b), b \mapsto \mathbf{g}(b) \mid (a,b) \in Q\} \in \mathbf{f} + \mathbf{Id}_Q$ and $\mathbf{g}' = \mathbf{g} + \{a \mapsto \mathbf{f}(a), b \mapsto \mathbf{f}(a) \mid (a,b) \in Q\} \in \mathbf{g} + \mathbf{Id}_Q$. Then $\mathbf{f} + \mathbf{g} = \mathbf{f}'|_{\overline{\pi_1(Q)}} + \mathbf{g}'|_{\overline{\pi_2(Q)}} \in \{\mathbf{f}'\}\circ_Q\{\mathbf{g}'\} \subseteq *_Q^i(\mathbf{M})\circ_Q *_Q^j(\mathbf{M}) = *_Q^{i+j}(\mathbf{M}) \subseteq *_Q(\mathbf{M})$ according to the definition of $\circ_Q$. Clearly $*_Q(\mathbf{M}) = *_Q(\mathbf{M}) + \mathbf{Id}_Q$.

For example, for $i > 0$ we have

$$*_{\{(b_1,b_2)\}}^i \left(\left\{\begin{pmatrix}2\\5\\1\end{pmatrix}\right\}\right) = \left\{\begin{pmatrix}2+j\\2+3i+j\\i\end{pmatrix}\,\middle|\, j \geq 0\right\} \text{ leading to}$$

$$*_{\{(b_1,b_2)\}}^{i+1} \left(\left\{\begin{pmatrix}2\\5\\1\end{pmatrix}\right\}\right) = *_{\{(b_1,b_2)\}}^i \left(\left\{\begin{pmatrix}2\\5\\1\end{pmatrix}\right\}\right) \circ_{\{(b_1,b_2)\}} \left\{\begin{pmatrix}2+k\\5+k\\1\end{pmatrix}\,\middle|\, k \geq 0\right\}$$

$$= \left\{\begin{pmatrix}2+j\\5+3i+j\\i+1\end{pmatrix}\,\middle|\, j \geq 0\right\}$$

(by "matching" with $k = 3i+j$) and

$$*_{\{(b_1,b_2)\}} \left(\left\{\begin{pmatrix}2\\5\\1\end{pmatrix}\right\}\right) = \left\{\begin{pmatrix}j\\j\\0\end{pmatrix}, \begin{pmatrix}2+j\\2+3i+j\\i\end{pmatrix}\,\middle|\, i,j \geq 0\right\}$$

Remark: Adding $\mathbf{Id}_Q$ is a crucial point: It corresponds to the monotonicity in Petri nets. Without this, deciding emptiness for the expressions would become undecidable.

1.2　Properties of the new operators

Obviously, it holds $\mathbf{N}\circ_Q\mathbf{M} = \mathbf{M}\circ_{Q^{-1}}\mathbf{N}$. Furthermore, we can express the intersection of $\mathbf{N}, \mathbf{M} \subseteq \mathbb{N}^A$ by $\mathbf{N}\circ_{Q'}\mathbf{L}\circ_{Q''}\mathbf{M} = \mathbf{N} \cap \mathbf{M}$ with $Q' := \{(a,a') \mid a \in A\}$, $Q'' := \{(a'',a) \mid a \in A\}$ and $\mathbf{L} := \{\{a \mapsto 1, a' \mapsto 1, a'' \mapsto 1\} \mid a \in A\}^*$.

Note here that, in general, $\mathbf{N}\circ_{Q'}\mathbf{L}\circ_{Q''}\mathbf{M}$ can only be written without brackets because $\pi_1(Q'')\cup(\{a \mid \exists \mathbf{f} \in \mathbf{M}, \mathbf{f}(a) > 0\}\setminus\pi_2(Q''))$ and $\pi_2(Q')\cup(\{a \mid \exists \mathbf{f} \in \mathbf{N}, \mathbf{f}(a) > 0\} \setminus \pi_1(Q'))$ are disjoint. If, additionally, $\pi_2(Q'')$ and $\{a \mid \exists \mathbf{f} \in \mathbf{N}, \mathbf{f}(a) > 0\}$ are disjoint and $\{a \mid \exists \mathbf{f} \in \mathbf{M}, \mathbf{f}(a) > 0\}$ and $\pi_1(Q'))$ are disjoint, then $\mathbf{N}\circ_{Q'}\mathbf{L}\circ_{Q''}\mathbf{M} = \mathbf{L}\circ_{Q'^{-1}\cup Q''}(\mathbf{M} + \mathbf{N})$.

1.3 Semilinearity

The class of semilinear sets is the smallest class of sets of multisets containing all finite sets of multisets and being closed under $\cup, +$ and $*$. The semilinear sets are also closed under $\cap$, as shown in [8] and [5]. As a normal form to express semilinear sets, we will use the union of linear sets of the form $\mathbf{c} + \Gamma^*$ for a constant $\mathbf{c} \in \mathbb{N}^A$ and periods $\Gamma \subseteq \mathbb{N}^A$.

The operator $\circ_Q$ preserves semilinearity: Assume $\mathbf{N}$ and $\mathbf{M}$ are semilinear sets over A, then

$$\mathbf{N}' := \{\mathbf{f}' \mid \exists \mathbf{f} \in \mathbf{N} \forall a \in \pi_1(Q) \; \mathbf{f}'(a') = \mathbf{f}(a) \wedge \mathbf{f}'(a) = 0 \wedge \forall a \notin \pi_1(Q) \; \mathbf{f}'(a) = \mathbf{f}(a)\},$$

$$\mathbf{M}' := \{\mathbf{f}' \mid \exists \mathbf{f} \in \mathbf{M} \forall a \in \pi_2(Q) \; \mathbf{f}'(a') = \mathbf{f}(a) \wedge \mathbf{f}'(a) = 0 \wedge \forall a \notin \pi_2(Q) \; \mathbf{f}'(a) = \mathbf{f}(a)\},$$

$$\mathbf{E}'_Q := \{\{a' \mapsto 1, b' \mapsto 1\}, \{c \mapsto 1\} \mid (a, b) \in Q, c \in A\}^*$$

$$= \{\mathbf{f} \mid \forall (a, b) \in Q \; \mathbf{f}(a') = \mathbf{f}(b')\}$$

are as well semilinear sets over the set $A \cup \pi_1(Q)' \cup \pi_1(Q)'$ which is extended by new elements. Thus, $\mathbf{N} \circ_Q \mathbf{M} = ((\mathbf{N}' + \mathbf{M}') \cap \mathbf{E}'_Q)|_{\overline{\pi_1(Q)' \cup \pi_1(Q)'}}$ is semilinear. Since the closure under $\cap$ is effective, this will allow a decision algorithm to continue with the representation $\bigcup_{j=1}^{l} L_j$ by linear sets L_j.

On the other hand, $*_Q$ does not preserve semilinearity:

Let $\mathbf{M} := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}^*$, then $*_{\{(b_3, b_2)\}}(\mathbf{M}) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| c \leq b2^a \right\}$ is not semilinear.

2 The reachability relation for Petri nets

2.1 The reachability relation for Petri nets without inhibitor arcs

We describe a *Petri net* as the triple $N = (P, T, W)$ with the places P, the transitions T and the weight function $W \in \mathbb{N}^{P \times T \cup T \times P}$. A transition $t \in T$ can fire from a marking $\mathbf{m} \in \mathbb{N}^P$ to a marking $\mathbf{m}' \in \mathbb{N}^P$, denoted by $\mathbf{m}[t\rangle\mathbf{m}'$, if

$$\mathbf{m} - W(., t) = \mathbf{m}' - W(t, .) \in \mathbb{N}^P.$$

A *firing sequence* $w = t_1...t_n \in T^*$ can fire from $\mathbf{m}_0$ to $\mathbf{m}_n$, denoted by $\mathbf{m}_0[w\rangle\mathbf{m}_n$, if $\mathbf{m}_1, ...\mathbf{m}_{n-1}$ exist with $\mathbf{m}_0[t_1\rangle\mathbf{m}_1[t_2\rangle...[t_n\rangle\mathbf{m}_n$. The *reachability problem* is to decide for a given net N with start- and end markings $\mathbf{m}_0, \mathbf{m}_e \in \mathbb{N}^P$, if there is a $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$.

Let $P^+ := \{p^+ \mid p \in P\}$ and $P^- := \{p^- \mid p \in P\}$ be copies of the places. For any multiset, $\mathbf{m}$ we define the corresponding copies $\mathbf{m}^- := \{p^- \mapsto \mathbf{m}(p) \mid p \in P\}$ and $\mathbf{m}^+ := \{p^+ \mapsto \mathbf{m}(p) \mid p \in P\}$. Then, we can define the *reachability relation* for

a transition t as

$$\mathbf{R}(t) := \left\{ \mathbf{m}^- + \mathbf{m}'^+ \,\middle|\, \mathbf{m}[t\rangle\mathbf{m}' \right\}$$
$$= \left\{ \mathbf{r} \in \mathbb{N}^{P^+ \cup P^-} \,\middle|\, \forall p \in P \; \mathbf{r}(p^-) - W(p,t) = \mathbf{r}(p^+) - W(t,p) \in \mathbb{N} \right\}$$

and the reachability relation for a set of transitions T as $\mathbf{R}(T) := \bigcup_{t \in T} \mathbf{R}(t)$.

The important property of monotonicity means that whenever $\mathbf{m}[w\rangle\mathbf{m}'$, then also $(\mathbf{m}+\mathbf{n})[w\rangle(\mathbf{m}'+\mathbf{n})$ for any $\mathbf{n} \in \mathbb{N}^P$. This corresponds to adding $\mathbf{Id}_P := \mathbf{Id}_{\hat{P}}$ with $\hat{P} := \{(p^+, p^-) \mid p \in P\}$ and $\mathbf{R}(t)$ can be written as the linear set $\mathbf{R}(t) = \mathbf{c}_t + \mathbf{Id}_P$ using $\mathbf{c}_t$ with $\mathbf{c}_t(p^-) := W(p,t)$ and $\mathbf{c}_t(p^+) := W(t,p)$ for all $p \in P$. The reachability relation for the concatenation of two firing sequences is described by $\mathbf{R}(w_1 w_2) = \mathbf{R}(w_1) \circ_P \mathbf{R}(w_2)$ with $\circ_P := \circ_{\hat{P}}$ and the iteration is done by $*_P := *_{\hat{P}}$. We define the reachability relation of the Petri net N as $\mathbf{R}(N) := \mathbf{R}(T^*) := *_P(\mathbf{R}(T))$. The reachability problem formulates as $(\mathbf{m}_0^- + \mathbf{m}_e^+) \in \mathbf{R}(N)$.

Lemma 2.1 *There is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$ in N if and only if*

$$\mathbf{m}_0^+ \circ_P \mathbf{R}(N) \circ_P \mathbf{m}_e^- = (\mathbf{m}_0^- + \mathbf{m}_e^+)\hat{\circ}_{P^+ \cup P^-}\mathbf{R}(N) = \{\emptyset\}.$$

(If there is no $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$, we have $(\mathbf{m}_0^- + \mathbf{m}_e^+)\hat{\circ}_{P^+ \cup P^-}\mathbf{R}(N) = \emptyset$.)

Proof. There is a firing sequence $t_1...t_n \in T^*$ with $\mathbf{m}_0[t_1\rangle\mathbf{m}_1[t_2\rangle...[t_n\rangle\mathbf{m}_n = \mathbf{m}_e$ in N if and only if $(\mathbf{m}_{i-1}^- + \mathbf{m}_i^+) \in \mathbf{R}(t_i)$ for all $1 \le i \le n$ if and only if $(\mathbf{m}_0^- + \mathbf{m}_e^+) \in \mathbf{R}(t_1) \circ_P \mathbf{R}(t_2)...\circ_P \mathbf{R}(t_n)$ if and only if $(\mathbf{m}_0^- + \mathbf{m}_e^+) \in \mathbf{R}(T) \circ_P \mathbf{R}(T)...\circ_P \mathbf{R}(T) = *_P^n(\mathbf{R}(T))$ if and only if $(\mathbf{m}_0^- + \mathbf{m}_e^+) \in *_P(\mathbf{R}(T)) = \mathbf{R}(N)$ and only this multiset can "match" according to $\hat{P}$. $\square$

Example:
Consider the following Petri net N with
$\mathbf{R}(t_1) = \{p_2^- \mapsto 1, p_1^+ \mapsto 2\} + \mathbf{Id}_P$,
$\mathbf{R}(t_2) = \{p_1^- \mapsto 3, p_2^+ \mapsto 2\} + \mathbf{Id}_P$,
and thus, $\mathbf{R}(T) = \left\{ \begin{bmatrix} p_2^- , p_1^+ \\ 1 , 2 \end{bmatrix}, \begin{bmatrix} p_1^- , p_2^+ \\ 3 , 2 \end{bmatrix} \right\}$.
Let furthermore $\mathbf{m}_0 = \{p_1 \mapsto 1, p_2 \mapsto 1\}$
and $\mathbf{m}_e = \{p_1 \mapsto 1, p_2 \mapsto 2\}$.
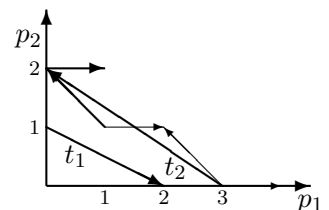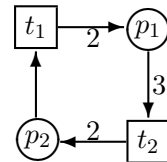

By concatenating, we get for example
$\mathbf{R}(t_1 t_2) = \mathbf{R}(t_1) \circ_P \mathbf{R}(t_2) = \begin{bmatrix} p_2^- , p_1^- , p_2^+ \\ 1 , 1 , 2 \end{bmatrix} + \mathbf{Id}_P$ and

$\mathbf{R}(t_1 t_1 t_2) = \begin{bmatrix} p_2^- , p_2^+ , p_1^+ \\ 2 , 2 , 1 \end{bmatrix} + \mathbf{Id}_P$.

In this way we get $(\mathbf{m}_0^- + \mathbf{m}_e^+) = \begin{bmatrix} p_1^- , p_2^- , p_1^+ , p_2^+ \\ 1 , 1 , 1 , 2 \end{bmatrix}$

$\in \mathbf{R}(t_1 t_2 t_1 t_1 t_2) = \begin{bmatrix} p_1^- , p_2^- , p_1^+ , p_2^+ \\ 1 , 1 , 1 , 2 \end{bmatrix} + \mathbf{Id}_P \subseteq \mathbf{R}(N)$.

By further concatenating and iterating, we get

$$\mathbf{R}((t_1 t_2)^*) = \begin{bmatrix} p_1^- , p_1^+ \\ 1 , 1 \end{bmatrix} + \begin{bmatrix} p_1^- , p_2^+ \\ 1 , 1 \end{bmatrix}^* + \mathbf{Id}_P,$$

$$\mathbf{R}(t_2 t_1) = \mathbf{R}(t_2) \circ_P \mathbf{R}(t_1) = \begin{bmatrix} p_1^- , p_2^+ , p_1^+ \\ 3 , 1 , 2 \end{bmatrix} + \mathbf{Id}_P,$$

$$\mathbf{R}((t_2 t_1)^*) = \begin{bmatrix} p_1^- , p_1^+ \\ 2 , 2 \end{bmatrix} + \begin{bmatrix} p_1^- , p_2^+ \\ 1 , 1 \end{bmatrix}^* + \mathbf{Id}_P,$$

$$\mathbf{R}(t_2 t_1 t_1) = \begin{bmatrix} p_1^- , p_1^+ \\ 3 , 4 \end{bmatrix} + \mathbf{Id}_P,$$

$$\mathbf{R}((t_2 t_1 t_1)^*) = \begin{bmatrix} p_1^- , p_1^+ \\ 3 , 3 \end{bmatrix} + \begin{bmatrix} p_1^+ \\ 1 \end{bmatrix}^* + \mathbf{Id}_P,$$

$$\mathbf{R}((t_1 t_1 t_2)^*) = \begin{bmatrix} p_2^- , p_2^+ \\ 2 , 2 \end{bmatrix} + \begin{bmatrix} p_1^+ \\ 1 \end{bmatrix}^* + \mathbf{Id}_P,$$

$$\mathbf{R}((t_1 t_2 t_1)^*) = \begin{bmatrix} p_2^- , p_2^+ , p_1^- , p_2^+ \\ 1 , 1 , 1 , 1 \end{bmatrix} + \begin{bmatrix} p_1^+ \\ 1 \end{bmatrix}^* + \mathbf{Id}_P,$$

...,

which finally yields $\mathbf{R}(N) = \mathbf{R}(T^*) = *_P \left(\left\{ \begin{bmatrix} p_2^- , p_1^+ \\ 1 , 2 \end{bmatrix} , \begin{bmatrix} p_1^- , p_2^+ \\ 3 , 2 \end{bmatrix} \right\} \right) =$

$$\left\{ \begin{bmatrix} p_2^- , p_1^+ \\ 1 , 2 \end{bmatrix} , \begin{bmatrix} p_1^- , p_2^+ \\ 3 , 2 \end{bmatrix} \right\}^* + \mathbf{Id}_P \cup$$

$$\left\{ \begin{bmatrix} p_2^- , p_2^+ \\ 1 , 1 \end{bmatrix} , \begin{bmatrix} p_1^- , p_1^+ \\ 2 , 2 \end{bmatrix} \right\} + \left\{ \begin{bmatrix} p_1^- , p_2^+ \\ 1 , 1 \end{bmatrix} , \begin{bmatrix} p_2^- , p_1^+ \\ 1 , 2 \end{bmatrix} , \begin{bmatrix} p_1^- , p_2^+ \\ 3 , 2 \end{bmatrix} \right\}^* + \mathbf{Id}_P \cup$$

$$\left\{ \begin{bmatrix} p_1^- , p_1^+ \\ 3 , 3 \end{bmatrix} , \begin{bmatrix} p_2^- , p_2^+ \\ 2 , 2 \end{bmatrix} , \begin{bmatrix} p_2^- , p_2^+ , p_1^- , p_2^+ \\ 1 , 1 , 1 , 1 \end{bmatrix} , \begin{bmatrix} p_1^- , p_2^+ \\ 3 , 2 \end{bmatrix} , \begin{bmatrix} p_2^- , p_1^+ , p_2^+ \\ 2 , 2 , 1 \end{bmatrix} , \begin{bmatrix} p_1^- , p_2^- , p_1^+ \\ 1 , 1 , 3 \end{bmatrix} , \right.$$

$$\left. \begin{bmatrix} p_2^- , p_1^+ \\ 2 , 4 \end{bmatrix} \right\} + \left\{ \begin{bmatrix} p_2^- , p_1^+ \\ 1 , 2 \end{bmatrix} , \begin{bmatrix} p_1^- , p_2^+ \\ 3 , 2 \end{bmatrix} , \begin{bmatrix} p_1^- , p_2^+ \\ 1 , 1 \end{bmatrix} , \begin{bmatrix} p_1^+ \\ 1 \end{bmatrix} , \begin{bmatrix} p_2^+ \\ 1 \end{bmatrix} \right\}^* + \mathbf{Id}_P$$

2.2 Petri nets with inhibitor arcs

An *inhibitor arc* from a place to a transition means that the transition can only fire if no token is on that place. We describe such a Petri net as the 6-tuple $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ with the places P, the transitions T, the weight function $W \in \mathbb{N}^{P \times T \cup T \times P}$, the inhibitor arcs $I \subseteq P \times T$ and, the start and end markings $\mathbf{m}_0, \mathbf{m}_e \in \mathbb{N}^P$. We will denote an inhibitor arc in the pictures by ●——— .

A transition $t \in T$ can fire from a marking $\mathbf{m} \in \mathbb{N}^P$ to a marking $\mathbf{m}' \in \mathbb{N}^P$, denoted by $\mathbf{m}[t\rangle\mathbf{m}'$ if

$$\mathbf{m} - W(., t) = \mathbf{m}' - W(t, .) \in \mathbb{N}^P \text{ and } \forall p \in P \ (p, t) \in I \rightarrow \mathbf{m}(p) = 0.$$

A firing sequence $w = t_1...t_n \in T^*$ can fire from $\mathbf{m}_0$ to $\mathbf{m}_n$, denoted by $\mathbf{m}_0[w\rangle\mathbf{m}_n$, if there exist intermediate markings $\mathbf{m}_1, ... \mathbf{m}_{n-1}$ with $\mathbf{m}_0[t_1\rangle\mathbf{m}_1[t_2\rangle...[t_n\rangle\mathbf{m}_n$.

The *reachability problem* for a Petri net $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ is to decide, whether there exists a $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$.

In the following two lemmata, we restrict the cases for which we have to regard the reachability problem. (See the [17] for proofs).

Lemma 2.2 *Each Petri net $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ can be changed in such a way that the condition $\forall p \in P, t \in T \ (p, t) \in I \rightarrow W(t, p) = 0$ holds without changing the inhibitor arcs or the reachability problem.*

Lemma 2.3 *Each Petri net $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ can be changed in a way such that the condition $\forall p \in P, t \in T \ (p,t) \in I \rightarrow \mathbf{m}_0(p) = \mathbf{m}_e(p) = 0$ holds by changing neither the inhibitor arcs, the condition in Lemma 2.2 nor the reachability problem.*

2.3   The reachability relation for Petri nets with one inhibitor arc

Let us consider a Petri-net $N = (P, T, W, \{(p_1, \hat{t})\}, \mathbf{m}_0, \mathbf{m}_e)$ having the property of lemmata 2.2 and 2.3. As in the case of no inhibitor arcs, we can describe by $\mathbf{R}(N') = *_P(\mathbf{R}(T \setminus \{\hat{t}\}))$ for the Petri net $N' = (P, T \setminus \{\hat{t}\}, W \mid_{\overline{P \times \{\hat{t}\} \cup \{\hat{t}\} \times P}})$ the reachability relation for firing sequences $w \in (T \setminus \{\hat{t}\})^*$. The restriction to those firing sequences starting and ending with markings without tokens on p_1 can be described by $\mathbf{R}(t_{p_1}(N')) := \{\mathbf{r} \in \mathbf{R}(N') \mid \mathbf{r}(p_1^-) = \mathbf{r}(p_1^+) = 0\}$. In $\mathbf{R}(N) := *_{P \setminus \{p_1\}}(\mathbf{R}(t_{p_1}(N')) \cup \mathbf{R}(\hat{t}))$ with $\mathbf{R}(\hat{t}) = \mathbf{c}_{\hat{t}} + \mathrm{Id}_{P \setminus \{p_1\}}$ we iterate these parts together with the alternative of using $\hat{t}$. Generalizing Lemma 2.1 we get the following:

Lemma 2.4 *Given a Petri-net $N = (P, T, W, \{(p_1, \hat{t})\}, \mathbf{m}_0, \mathbf{m}_e)$ with only one inhibitor arc $(p_1, \hat{t})$ having the property of lemmata 2.2 and 2.3, then there is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$ if and only if*

$$\mathbf{m}_0^+ \circ_{P \setminus \{p_1\}} \mathbf{R}(N) \circ_{P \setminus \{p_1\}} \mathbf{m}_e^- = (\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^- \setminus \{p_1^+, p_1^-\}} \mathbf{R}(N) = \{\emptyset\}.$$

(If there is no such $w \in T^*$, we have $(\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^- \setminus \{p_1^+, p_1^-\}} \mathbf{R}(N) = \emptyset$.)

Proof. Since $\mathbf{m}_0(p) = \mathbf{m}_e(p) = 0$ according to Lemma 2.3, a firing sequence $w \in T^*$ can be decomposed in minimal firing sequences $w_1...w_k = w$ having the property $\mathbf{m}_0[w_1\rangle\mathbf{m}_1[w_2\rangle...[w_k\rangle\mathbf{m}_k$ with $\mathbf{m}_k = \mathbf{m}_e$ such that $\mathbf{m}_i(p_1) = 0$ for all $i \leq k$.

Each w_i is either in $(T \setminus \{\hat{t}\})^*$ or equal to $\hat{t}$ by Lemma 2.2 and minimality of w_i. This holds since the occurrence of $\hat{t}$ in a w_i with $|w_i| > 1$ would mean that, at that time during the firing of w_i, there is no token on p_1, and thus, w_i would not be minimal.
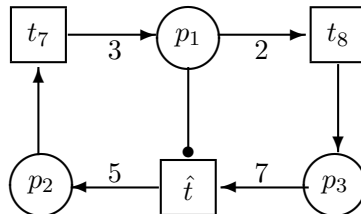
If $w_i \in (T \setminus \{\hat{t}\})^*$ then $\mathbf{m}_{i-1}^- + \mathbf{m}_i^+ \in \mathbf{R}(N')$. Then from $\mathbf{m}_{i-1}(p_1) = 0$ and $\mathbf{m}_i(p_1) = 0$ it follows that $\mathbf{m}_{i-1}^- + \mathbf{m}_i^+ \in \mathbf{R}(t_{p_1}(N'))$. Otherwise, if $w_i = \hat{t}$, we have $\mathbf{m}_{i-1}^- + \mathbf{m}_i^+$ in $\mathbf{R}(\hat{t})$.

Concatenating all with the operator $*_{P \setminus \{p_1\}}$ leads to $\mathbf{m}_0^- + \mathbf{m}_e^+$ is in $\mathbf{R}(N)$, which means $(\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^-} \mathbf{R}(N) = \{\emptyset\}$.

The other direction follows simply by composing firing sequences.   □

Example:
Consider the Petri net N

with the start marking $\{p_2 \mapsto 4, p_3 \mapsto 2\}$ and the end marking $\{p_2 \mapsto 4, p_3 \mapsto 3\}$. We have $\mathbf{R}(t_7) = \{p_2^- \mapsto 1, p_1^+ \mapsto 3\} + \mathbf{Id}_P$, $\mathbf{R}(t_8) = \{p_1^- \mapsto 2, p_3^+ \mapsto 1\} + \mathbf{Id}_P$ and $\mathbf{R}(\hat{t}) = \{p_3^- \mapsto 7, p_2^+ \mapsto 5\} + \mathbf{Id}_{P \setminus \{p_1\}}$. This yields

$$\mathbf{R}(N') = \mathbf{R}((P, \{t_7, t_8\}, W\mid_{\overline{P \times \{\hat{t}\} \cup \{\hat{t}\} \times P}})) = *_P \left(\left\{ \begin{bmatrix} p_2^- & p_1^+ \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} p_1^- & p_3^+ \\ 2 & 1 \end{bmatrix} \right\} \right) =$$

$$\left\{ \begin{bmatrix} p_2^- & p_1^+ \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} p_1^- & p_3^+ \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} p_2^- & p_1^+ & p_3^+ \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} p_2^- & p_1^- & p_3^+ \\ 1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} p_2^- & p_1^+ & p_3^+ \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^+ \\ 2 & 3 \end{bmatrix} \right\}^* + \mathbf{Id}_P$$

and $\mathbf{R}(t_{p_1}(N')) = \mathbf{R}(N') \circ_{\{(p_1^-, x), (p_1^+, y)\}} \{\emptyset\} = \left\{ \begin{bmatrix} p_2^- & p_3^+ \\ 2 & 3 \end{bmatrix} \right\}^* + \mathbf{Id}_{\{p_2, p_3\}}$.

We can cut the firing sequences in $(t_7 + t_8 + \hat{t})^* = ((t_7 + t_8)^* + \hat{t})^*$ into parts in $(t_7 + t_8)^*$ and $\hat{t}$ all starting and ending with no token on p_1. This yields $\mathbf{R}(N) := *_{\{p_2, p_3\}}(\mathbf{R}(t_{p_1}(N')) \cup \mathbf{R}(\hat{t}))$

$$\left\{ \begin{bmatrix} p_2^- & p_3^+ \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} p_3^- & p_2^+ \\ 7 & 5 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^- & p_2^+ \\ 2 & 4 & 5 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^- & p_2^+ \\ 4 & 1 & 5 \end{bmatrix}, \begin{bmatrix} p_3^- & p_2^+ & p_3^+ \\ 7 & 3 & 3 \end{bmatrix}, \begin{bmatrix} p_3^- & p_2^+ & p_3^+ \\ 7 & 1 & 6 \end{bmatrix}, ..., \right.$$
$$\left. \begin{bmatrix} p_2^- & p_3^- & p_2^+ \\ 4 & 2 & 8 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^- & p_2^+ & p_3^+ \\ 5 & 1 & 1 & 7 \end{bmatrix}, \begin{bmatrix} p_3^- & p_2^+ & p_3^+ \\ 6 & 4 & 3 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^- & p_2^+ & p_3^+ \\ 4 & 2 & 4 & 3 \end{bmatrix} \right\}^* + \mathbf{Id}_{\{p_2, p_3\}}.$$

3 Nested Petri Nets as normal form for expressions

From now on we use the variables t, T, N as expressions describing transitions, sets of transitions and (sub-)nets. For an expression e, we will always define a *carrier set* $C(e) \supseteq \{a \mid \exists \mathbf{f} \in \mathbf{R}(e), \mathbf{f}(a) > 0\}$. The function $\mathbf{R}$ was in the previous section giving the reachability relation $\mathbf{R}(e) \subseteq \mathbb{N}^{C(e)}$ for an e of the form t, N or T. Now, we use $\mathbf{R}$ as the evaluation function for an expression where the expression operators $*_P, \circ_Q, \cup$ and $+$, and the operator $\cap$ will always be defined on expressions such that they commute with $\mathbf{R}$.

Let the expression for an *elementary transition* have the form $t = L_t$, where L_t is an expression for the *linear set* $\mathbf{L}_t = \mathbf{R}(L_t) = \mathbf{c}_t + \Gamma_t^*$ described by a (constant) multiset $\mathbf{c}_t$ and a finite set of (period-) multisets Γ_t. For example, in Sections 2.1 and 2.3, we have $\Gamma_t = \{\{p^- \mapsto 1, p^+ \mapsto 1\} \mid p \in P\}$ leading to $\Gamma_t^* = \mathbf{Id}_P$. We have $C(t) := C(L_t) := P^- \cup P^+ \cup \{a \mid \exists \mathbf{f} \in \{\mathbf{c}_t\} \cup \Gamma_t, \mathbf{f}(a) > 0\}$.

Let the expression for *sets of transitions* be $T = t_1 \cup t_2 ... \cup t_l$ for expressions for transitions $t_i \in T$ for $i \leq l$, and the expression for a *sub-net* with places P_T and transitions T be $N = *_{P_T}(T)$. Let $C(N) := C(T) := \bigcup_{t \in T} C(t)$.

Let the expression for a *generalized transition* have the form $t = L_t \hat{\circ}_{A_t} K_t$, where L_t again expresses a linear set, and K_t is a set of sub-nets and interpreted as expression $K_t = \sum_{N_i \in K_t} N_i$ where the $C(N_i)$ are pairwise disjoint and $A_t := C(K_t) := \bigcup_{N_i \in K_t} C(N_i)$.

We define $C(t) := \{a \mid \exists \mathbf{f} \in \{\mathbf{c}_t\} \cup \Gamma_t, \mathbf{f}(a) > 0\} \setminus A_t$. This means that the behavior of t is mainly described by the linear set $\mathbf{c}_t + \Gamma_t^* \subseteq \mathbb{N}^{C(t) \cup A_t}$ but it is additionally controlled by the reachability in the sub-nets N_i.

For example, the reachability question for a complete net in Lemma 2.1 and Lemma 2.4 is formulated as the control by a (sub-)net in t with $\mathbf{R}(t) = (\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^-} \mathbf{R}(N)$ respectively $\mathbf{R}(t) = (\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^- \setminus \{p_1^+, p_1^-\}} \mathbf{R}(N)$ in a way

that already has this normal form for expressions with $c_t = (\mathbf{m}_0^- + \mathbf{m}_e^+)$ and $\Gamma_t = \emptyset$. However, the behavior on the outside is trivial ($\emptyset$ or $\{\emptyset\}$) because $C(t) = \emptyset$ in this case. The same holds for $(\mathbf{m}_0^- + \mathbf{m}_e^+)\hat{\odot}_{P_{T_{g-1}}^+ \cup P_{T_{g-1}}^-} *_{P_{T_{g-1}}}(T_{g-1})$ in Theorem 5.1.

The decision algorithm below will start with an expression of the form $T = \{t\}$ by keeping in mind that, according to Lemma 2.4, $\mathbf{R}(T) = \mathbf{R}(t) = \{\emptyset\}$ if there is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$. Otherwise $\mathbf{R}(T) = \mathbf{R}(t) = \emptyset$ if there is not.

An example for the general case is $\mathbf{R}(t_{p_1}(N')) := \{\mathbf{r} \in \mathbf{R}(N') \mid \mathbf{r}(p_1^-) = \mathbf{r}(p_1^+) = 0\}$ form Section 2.3. Here we need Lemma 3.1 to construct the expression $t_{\{p_1\}}(N')$ of the desired normal form with $\mathbf{R}(t_{p_1}(N')) = \mathbf{R}(t_{\{p_1\}}(N'))$. Again, t_h in Theorem 5.1 has the normal form for expressions.

Lemma 3.1 *Let N be an expression for a subnet. Then, we can construct an equivalent expression for a transition $t(N)$ with $\mathbf{R}(t(N)) = \mathbf{R}(N)$ and $t_{P'}(N)$ with $\mathbf{R}(t_{P'}(N)) = \{\mathbf{m} \in \mathbf{R}(N) \mid \forall p \in P' \; \mathbf{m}(p^-) = \mathbf{m}(p^+) = 0\}$.*

Proof. Define $t(N)$ by $\mathbf{c}_{t(N)} := \mathbf{c}_{t_{P'}(N)} := \emptyset$, $\Gamma_{t(N)} := \{\{q \mapsto 1, \; \hat{q} \mapsto 1\} \mid q \in C(N)\}$ and $K_{t(N)} := \{\hat{N}\}$ where $\hat{N}$ is the result of replacing all occurrences of some $q \in C(N)$ in N by $\hat{q}$. This means that we make the $C(\hat{N})$ disjoint to $C(t(N))$.

The restriction of places in P' to 0 is done by $\Gamma_{t_{P'}(N)} := \{\mathbf{m} \in \Gamma_{t(N)} \mid \forall p \in P' \; \mathbf{m}(p^-) = \mathbf{m}(p^+) = 0\}$. $\qquad\square$

Example (continued):
We identify $t_7 = \{\hat{p}_2^- \mapsto 1, \hat{p}_1^+ \mapsto 3\} + \mathbf{Id}_{\{\hat{p}_1,\hat{p}_2,\hat{p}_3\}}$, $t_8 = \{\hat{p}_1^- \mapsto 2, \hat{p}_3^+ \mapsto 1\} + \mathbf{Id}_{\{\hat{p}_1,\hat{p}_2,\hat{p}_3\}}$ and $\hat{t} = \{p_3^- \mapsto 7, p_2^+ \mapsto 5\} + \mathbf{Id}_{\{p_2,p_3\}}$. This yields the expressions $T_1 = t_7 \cup t_8$ and $N_1 = *_{\{\hat{p}_1,\hat{p}_2,\hat{p}_3\}}(T_1)$. On the next level, we get the generalized transition $t_2 = t_{p_1}(N')$

$$\left(\emptyset + \left\{\begin{bmatrix} \hat{p}_2^- \\ 1 \end{bmatrix}, \begin{bmatrix} \hat{p}_3^- \\ 1 \end{bmatrix}, \begin{bmatrix} \hat{p}_2^+ \\ 1 \end{bmatrix}, \begin{bmatrix} \hat{p}_3^+ \\ 1 \end{bmatrix}\right\}^*\right) \hat{\odot}_{\{\hat{p}_2^-,\hat{p}_3^-,\hat{p}_2^+,\hat{p}_3^+\}} N_1,$$
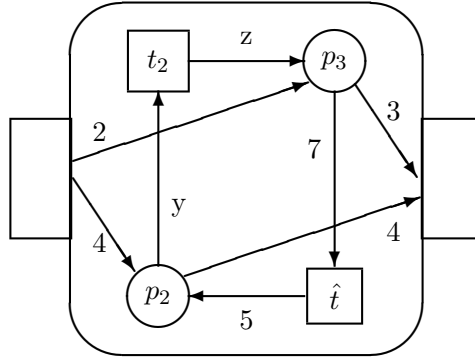
which we visualize as
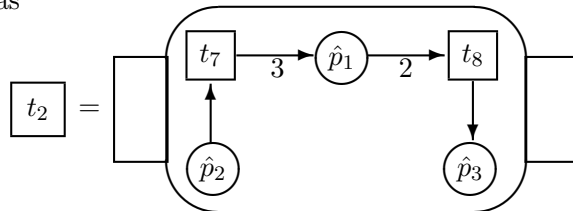
$T_2 = t_2 \cup \hat{t}$ and $N_2 = *_{\{p_2,p_3\}}(T_2)$. On the top level, we get

$$T_3 = t_3 = \begin{bmatrix} p_2^- & p_3^- & p_2^+ & p_3^+ \\ 4 & 2 & 4 & 3 \end{bmatrix} \hat{\odot}_{\{p_2^-,p_3^-,p_2^+,p_3^+\}} N_2,$$

which we visualize as follows:

3.1 The property $\mathcal{T}$

In order to decide the emptiness problem for expressions, we want to establish a normal form $\mathcal{T}$, which corresponds to the condition Θ in [12]:

Definition 3.2 An expression T has the property $\mathcal{T}$ if $\forall t \in T, \forall N_i = *_{P_{T_i}}(T_i) \in K_t$ the following conditions hold:

(i) In recursive manner, T_i has
   (a) the property $\mathcal{T}$, and
   (b) For all $t' \in T_i$ it holds $\forall \mathbf{g} \in \{\mathbf{c}_{t'}\} \cup \Gamma_{t'} \ \exists w_{\mathbf{g}} \in C(t') \ \mathbf{g}(w_{\mathbf{g}}) = 1$,
   $\forall \mathbf{g}' \in \bigcup_{t' \in T_i} \{\mathbf{c}_{t'}\} \cup \Gamma_{t'} \setminus \{\mathbf{g}\} \ \mathbf{g}'(w_{\mathbf{g}}) = 0$.

   This condition says that the number of times where $\mathbf{g}$ is used is exactly the number of occurrence of the *witness (place)* $w_{\mathbf{g}}$.

(ii) $\forall \mathbf{g} \in \{\mathbf{c}_t\} \cup \Gamma_t, \forall p \in P_{T_i} \ \mathbf{g}(p^-) - ind(\mathbf{g})(p^-) = \mathbf{g}(p^+) - ind(\mathbf{g})(p^+)$, where

$$ind(\mathbf{g}) := \sum_{t' \in T_i, \mathbf{g}' \in \{\mathbf{c}_{t'}\} \cup \Gamma_{t'}} \mathbf{g}(w_{\mathbf{g}'})\mathbf{g}'$$

describes the indirect effect of $\mathbf{g}$ using the property about the witness places in Condition 1 in the recursion for T_i. This property says that $\mathbf{g}(w_{\mathbf{g}'})$ is exactly the number of times that $\mathbf{g}'$ is used. Thus, $ind(\mathbf{g})$ contains a quantitative information about the firing sequences which are allowed by $\mathbf{g}$. The condition says that (disregarding the real control by the sub-net N_i) the quantitative information is consistent with the expected control.

(iii) $\forall w \in C(N_i) \setminus (P_{T_i}^+ \cup P_{T_i}^-) \ \sum_{g \in \Gamma_t} \mathbf{g}(w) > 0$. This condition says that each witness appears in a period and, thus, the use of each interior transition and period is unlimited.

(iv) There are multisets $\exists \mathbf{m}_+, \mathbf{m}_- \in \mathbf{R}(N_i)$ with $\forall p \in P_{T_i}$

$$\mathbf{m}_+|_{P_{T_i}^-} \in (\mathbf{c}_t + \Gamma_t^*)|_{P_{T_i}^-} \wedge ((\forall \mathbf{g} \in \Gamma_t \ \mathbf{g}(p^-) = 0) \to \mathbf{m}_+(p^+) > \mathbf{m}_+(p^-)) \wedge$$

$$\mathbf{m}_-|_{P_{T_i}^+} \in (\mathbf{c}_t + \Gamma_t^*)|_{P_{T_i}^+} \wedge ((\forall \mathbf{g} \in \Gamma_t \ \mathbf{g}(p^+) = 0) \to \mathbf{m}_-(p^-) > \mathbf{m}_-(p^+)).$$

This condition says that there is a firing sequence in the sub-net N_i quantita-

tively described by $\mathbf{m}_+$. This firing sequence starts with a marking available by $\mathbf{c}_t + \Gamma_t^*$ and increases all those places which cannot be increased by Γ_t.

(v) $\mathbf{c}_t|_{C(t)} \in \mathbf{R}(t)$. This condition says that transition t can fire without the use of one of its periods in Γ_t.

3.2   The size of an expression

To prove the termination of the algorithm in Section 4, we have to define an ordering on a size S which is Noetherian and decreasing in every step of the algorithm:

A list (tuple, respectively) is smaller than another if the first i elements are equal and the $i+1$'th element is smaller (or not existing). A multiset $\mathbf{m}$ is smaller than a multiset $\mathbf{m}'$ if there is an e with $\mathbf{m}(e) < \mathbf{m}'(e)$ and $\mathbf{m}(e') = \mathbf{m}'(e')$ for all $e' > e$. (Thus multisets may as well be interpreted as a descending ordered list using lexicographic order.)
The smallest size is $S(\emptyset)$. Accordingly, if $T = \emptyset$ then T has the property $\mathcal{T}$.

The size $S(T) = \sum_{t \in T} \{S(t) \mapsto 1\}$ is a multiset of all sizes $S(t)$ with $t \in T$. The size of t is $S(t) := (S(K_t), b_2, b_5 + |\Gamma_t|)$. Here, $b_i = 0$ if Condition $\mathcal{T}.i$ is fulfilled, and $b_i = 1$ otherwise. The size $S(K_t) = \sum_{N_i \in K_t} \{S(N_i) \mapsto 1\}$ of a set of nets is a multiset of the sizes $S(N_i)$ of the nets $N_i \in K_t$. The size of a net is

$$S(N_i) := (\mathbf{s}_m + \{|P_{T_i}| \mapsto 1\}, S(T_i), b_{1b}, |C(N_i)|)$$

with $\mathbf{s}_m := max\{\mathbf{s} \mid \exists \mathbf{g}, f, b_2, b'_{1b}, e, \mathbf{s}'\ \mathbf{s}'((\mathbf{s}, \mathbf{g}, b'_{1b}, f)) > 0, S(T_i)((\mathbf{s}', b_2, e)) > 0\}$. In other words, the first component is a multiset in $\mathbb{N}^{\mathbb{N}}$ which is obtained by taking the maximal of such multisets of all first components in the size of a subnet of one of the transitions in T_i (respectively $\emptyset$ if none exists) and adding the current number of places. The second component contains the recursion. The reason for this complicated construction comes from Section 4.4 where the recursion-depth increases but the size has to decrease. Furthermore, this causes $S(N_i)$ to be greater than the size of its occurring subnets. This is also necessary in parts where the
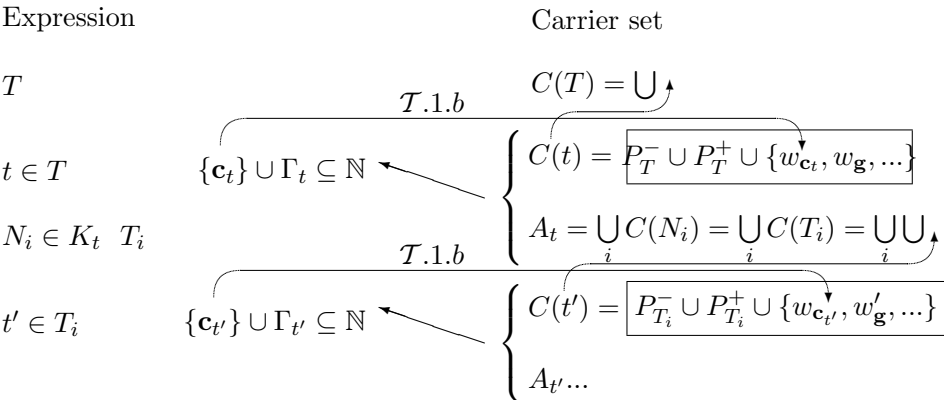

Fig. 1. An overview over the expressions and their carrier sets.

algorithm works recursively since it follows that $S(K_{t'}) < S(K_t)$ for all t' contained one or more levels deeper in K_t.

Example (continued):
$S(t_7) = S(t_8) = (\emptyset, 0, 3)$, $S(T_1) = \{(\emptyset, 0, 3) \mapsto 2\}$,
$S(N_1) = (\{3 \mapsto 1\}, \{(\emptyset, 0, 3) \mapsto 2\}, 1, 6)$, $S(t_2) = (\{S(N_1) \mapsto 1\}, 1, 4)$,
$S(T_2) = \{S(t'_2) \mapsto 1, (\emptyset, 0, 2) \mapsto 1\}$, $S(N_2) = (\{3 \mapsto 1, 2 \mapsto 1\}, S(T_2), 1, 4)$.

Lemma 3.3 *The ordering on S defined above is Noetherian*

Proof. As shown in [4], the set of descending ordered lists of elements of a Noetherian ordered set is again Noetherian. The first components of the quadruples $S(N)$ are descending lists of natural numbers and, thus, Noetherian.

Assume by contradiction that $\mathbf{x}$ is the smallest first component such that there is an infinite descending sequence of quadruples

$$S(N) = (\mathbf{x}, \mathbf{y}_1, b'_1, n_1), (\mathbf{x}, \mathbf{y}_2, b'_2, n_2), ...$$

In all quadruples appearing in all lists in all triples appearing in any $\mathbf{y}_i$, the first component must always be smaller than $\mathbf{x}$ and, therefore, their order must be Noetherian. Thus, the lists which are the first components of the triples are also ordered Noetherian. Since the other components are natural numbers, the triples and the $\mathbf{y}_i$'s are also ordered Noetherian. Since the first component $\mathbf{x}$ must remain constant, and the third and forth components are natural numbers, we get a contradiction; thus, $S(T)$ is Noetherian.   □

3.3 Additional operators working on expressions

The following lemma is used to restrict the semilinear part in a transition t as it will be needed to establish the property $\mathcal{T}.2$

Lemma 3.4 *Let $t = L_t \hat{\circ}_{A_t} K_t$ be an expressions for a transition and L be (an expression for) a semi linear set. Then, we can construct an expression $T' = t|_L$ (with $\mathbf{R}(T') = (\mathbf{R}(L_t) \cap \mathbf{R}(L)) \hat{\circ}_{A_t} \mathbf{R}(K_t)$) where the occurring sizes $S(t')$ with $t' \in T'$ can increase relatively to $S(t)$ only in the last position in the triple.*

Proof. Using Presburger arithmetic [8],[5], we can calculate for every $t \in T$ the semi-linear set

$$L_t \cap L =: \bigcup_{j=1}^{l} L_j$$

resulting in finitely many linear sets L_j, and define $T' := t|_L := \{L_j \hat{\circ}_{A_t} K_t \mid j \le l\}$.□

An Example is given in Subsection 4.2 where $L_t \cap L$ consists of only one linear set and thus T' consists of only one transition.

The following Lemma 3.5 and Lemma 3.1 allow us to bring every expression into the normal form as nested Petri nets:

Lemma 3.5 *Let T and T' be expressions for sets of transitions, and Q be a relation. Then, we can construct an expression $T'' := T \circ_Q T'$ (with $\mathbf{R}(T'') = \mathbf{R}(T) \circ_Q \mathbf{R}(T')$)*

where the occurring sizes $S(t)$ can increase only in the last position in the triple and sum up in the first position.

Proof. We may assume that $\bigcup\limits_{t\in T} A_t$, $\bigcup\limits_{t\in T'} A_t$ and $\pi_1(Q)\cup\pi_2(Q)$ are pairwise disjoint (otherwise replace elements by copies). We define

$$T'' := \left\{ L_j \hat{\circ}_{A_t\cup A_{t'}}(K_t + K_{t'}) \,\middle|\, t\in T, t'\in T', j\le r \,, L_t\circ_Q L_{t'} =: \bigcup_{j=1}^{r} L_j \right\}$$

using Presburger arithmetics (since $L_t\circ_Q L_{t'}$ is semilinear see Subsection 1.3). It holds $\mathbf{R}(T)\circ_Q\mathbf{R}(T') = \bigcup\limits_{t\in T, t'\in T'} \mathbf{R}(t)\circ_Q\mathbf{R}(t') =$

$$\bigcup_{t\in T, t'\in T'} ((\mathbf{L}_t\hat{\circ}_{A_t}(\mathbf{R}(N_1)+...))\circ_Q(\mathbf{L}_{t'}\hat{\circ}_{A_{t'}}(\mathbf{R}(N_1')+...))) =$$

$$\bigcup_{t\in T, t'\in T'} ((\mathbf{R}(N_1)+...)\hat{\circ}_{A_t^{-1}}\mathbf{L}_t\circ_Q\mathbf{L}_{t'}\hat{\circ}_{A_{t'}}(\mathbf{R}(N_1')+...)) =$$

$$\bigcup_{t\in T, t'\in T'} (\mathbf{L}_t\circ_Q\mathbf{L}_{t'})\hat{\circ}_{A_t\cup A_{t'}}(\mathbf{R}(N_1)+...+\mathbf{R}(N_1')+...) = \mathbf{R}(T'')$$

since A_t, $A_{t'}$ and $\pi_1(Q)\cup\pi_2(Q)$ are pairwise disjoint. (see Subsection 1.2.)   □

Examples are given in Subsections 4.3 and 4.4.

From this we can conclude some decidability result for the first order formulas with PLUS and the *monotone transitive closure mTC* defined as follows:

Given a formula $\phi(x_1,...,x_k,x_1',...,x_k')$, then $\mathrm{mTC}(\phi)$ denotes the smallest set $S\subset\mathbb{N}^{2k}$ containing all of the following:

- $(x_1,...,x_k,x_1,...,x_k)$ for $(x_1,...,x_k)\in\mathbb{N}^k$ (this stands for the identity),
- $(x_1,...,x_k,x_1',...,x_k')$ for $\phi(x_1,...,x_k,x_1',...,x_k')$,
- $(x_1,...,x_k,x_1'',...,x_k'')$ for $(x_1,...,x_k,x_1',...,x_k')$, $(x_1',...,x_k',x_1'',...,x_k'')\in S$, and
- $(x_1+x_1'',...,x_k+x_k'',x_1'+x_1'',...,x_k'+x_k'')$ for a $(x_1,...,x_k,x_1',...,x_k')\in S$ and $(x_1'',...,x_k'')\in\mathbb{N}^k$.

Corollary 3.6 *The emptiness and satisfiability is decidable for formulas with an FO+PLUS-formula inside and $\wedge,\vee,\exists$ and mTC operators outside.*

Proof. We can express linear sets by a t and, thus, semilinear sets by a T. Now, observe that the operators work on expressions of the form T as follows: We can express $\wedge$ corresponding to $\cap$ with $\circ_Q$ (see Section 1.2) and apply Lemma 3.5. For $\vee$ this follows simply from T being already a union. The existential quantifier is done by removing the element (thus, releasing the control from the outside) and the operator mTC is done by using Lemma 3.1. Then we construct T' according to Theorem 4.1. According to Condition 5 of property $\mathcal{T}$, $\mathbf{R}(T) = \mathbf{R}(T')$ is empty if and only if $T' = \emptyset$.   □

4 The main algorithm

The idea of the algorithm is to reduce T if one of the conditions is not fulfilled. For Condition 2, Presburger arithmetics is used to transfer the implicit quantitative restriction by the witness places to the explicit restriction of the transitions. Condition 3 ensures that all quantitative controls are unlimited. Condition 4 ensures that all places are unlimited. A covering graph construction deciding Condition 4 uses the algorithm recursively (like for Condition 1) for every step. Here, the current marking of a node is being included as a restriction to the semilinear set. Limited places are deleted at the cost of a larger structure. This larger structure, however, contains parts which are generated by restricting parts. This restriction might cause them to loose the property $\mathcal{T}$ reached by a previous recursive step. However, as we will see because of their smaller size, the property $\mathcal{T}$ can be established again and the whole algorithm will still terminate.

Theorem 4.1 *For every expression T, we can effectively construct a T' with $\mathbf{R}(T) = \mathbf{R}(T')$ such that T' has property $\mathcal{T}$.*

Proof. The expression T' in the Theorem is computed by the following algorithm where the details are explained in the subsections:

> **function** reacheq(T):
> **begin**
>   **repeat**
>   i:= 1
>   **while** i≤5 and $\forall t \in T, \forall N \in K_t$ Condition $\mathcal{T}$.i fulfilled
>   **do** i:=i+1 **od**
>   **if** i=6 **then return** T
>   **else** T:=T' for T' according to subsection 4.i **fi**
>   **until** i=6
> **end** reacheq

in each step $S(T)$ decreases ($S(reacheq(T)) < S(T)$ if $T \neq reacheq(T)$); due to Lemma 3.3 the algorithm terminates. □

The following table shows how the size $S(t)$ can change during the steps of Chapter 4:

		$S(t)$						
		$S(N_i)$						
			$S(t')$ for $t' \in T_i$					
	$\mathbf{s}_m + \{|P_{T_i}| \mapsto 1\}$	$S(K_{t'})$	b_2	$b_5 + |\Gamma_{t'}|$	b_{1b}	$|C(N_i)|$	b_2	$b_5 + |\Gamma_t|$
4.1	-	-	-	-	↓	↑	↑	-
4.2	-	-	-	-	-	-	↓	↑
4.3	-	-	-	-	-	↓	↑	↑
4.4	↓	↑	↑	↑	↑	↑	↑	↑
4.5	-	-	-	-	-	-	-	↓

Corollary 4.2 *The reachability problem for a Petri net with one inhibitor arc is decidable.*

Proof. According to Lemmata 2.4 and 3.1, we can construct an expression T where $\mathbf{R}(T) = \{\emptyset\}$ (and is not empty) if and only if there is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$. Then, we construct T' according to Theorem 4.1. According to Condition 5 of property $\mathcal{T}$, $\mathbf{R}(T) = \mathbf{R}(T')$ is empty if and only if $T' = \emptyset$.   □

Remark: Alternatively, instead of using Lemma 2.4, we can construct T for a given net $N = (P, T, W, \{(p_1, \hat{t})\}, \mathbf{m}_0, \mathbf{m}_e)$ directly by connecting three sub-nets with a linear set in such a way that the first sub-net expresses firing sequences in $(T \setminus \{\hat{t}\})^*$ starting with $\mathbf{m}_0$ before the first firing of $\hat{t}$, the second contains only one transition which has a sub-net itself expressing firing sequences $\in (\hat{t}(T \setminus \{\hat{t}\})^*)^*$ ending with markings without tokens on p_1 and the third sub-net expresses firing sequences in $(T \setminus \{\hat{t}\})^*$ ending with $\mathbf{m}_e$.

4.1   Condition 1 Recursion and introducing witnesses

Let Condition 1 be not fulfilled by T_i; let $T'_i := reacheq(T_i)$, which terminates by induction since $S(T_i) < S(T)$.

For all $t_j \in T'_i$ let G_j be the set of all $\mathbf{g} \in \{\mathbf{c}_{t_j}\} \cup \Gamma_{t_j}$ not having a witness. Add witnesses$\{w_{\mathbf{g}'} \mid \mathbf{g} \in G_j\}$ to $C(t'_j) := C(t_j) \cup \{w_{\mathbf{g}'} \mid \mathbf{g} \in G_j\}$ by replacing each $\mathbf{g}$ in G_j by $\mathbf{g}' := \mathbf{g} + \{w_{\mathbf{g}'} \mapsto 1\}$ in

$$T''_i := \left\{ t'_j \;\middle|\; \begin{array}{l} t_j \in T'_i, A_{t'_j} = A_{t_j}, K_{t'_j} = K_{t_j}, \\ \Gamma_{t'_j} = \Gamma_{t_j} \setminus G_j \cup \{\mathbf{g}' \mid \mathbf{g} \in G_j \setminus \{\mathbf{c}_t\} \text{ and } \mathbf{c}_{t'_j} := \mathbf{c}_{t_j} \text{ if } \mathbf{c}_{t_j} \notin G_j \end{array} \right\}.$$

Now, we set $t' := L_{t'}\hat{\circ}_{A_{t'}}K_{t'}$ with $K_{t'} = K_t \setminus \{N_i\} \cup \{N''_i\}$, $N''_i = *_{P_{T''_i}}(T''_i)$, $A_{t'} = A_t \cup C(N''_i)$ and $\Gamma_{t'} := \Gamma_t \cup \{\{w \mapsto 1\} \mid w \in C(N''_i) \setminus C(N_i)\}$, and let $T' := T \setminus \{t\} \cup \{t'\}$. Since $\mathbf{R}(T_i) = \mathbf{R}(T''_i)|_{C(T_i)}$, we have $\mathbf{R}(N_i) = \mathbf{R}(N''_i)|_{C(T_i)}$; thus, $\mathbf{R}(t) = \mathbf{R}(t')$; thus, $\mathbf{R}(T) = \mathbf{R}(T')$.

Since $S(t_j) = S(t'_j)$ for all $t_j \in T'_i$, the size $S(T''_i) = S(T'_i)$ remains unchanged. The only increase was $|C(N''_i)| \geq |C(N_i)|$ but but we have either $S(T''_i) = S(T'_i) < S(T_i)$, or in case $T'_i = T_i$, we have now $b_{1b} = 0$. From that follows that $S(N''_i) < S(N_i)$; thus, $S(t') < S(t)$ and $S(T') < S(T)$.

Example (continued):
Since the expression T_1 does not fulfill Condition 1, we add the two witnesses $w_{\mathbf{c}_{t'_7}}$ and $w_{\mathbf{c}_{t'_8}}$. For simplicity, we omit the witnesses for the periods for $\mathbf{I}d_P$ in the elementary transitions. So we replace t_7 and t_8 by $t'_7 = \{\hat{p}_2^- \mapsto 1, \hat{p}_1^+ \mapsto 3, w_{\mathbf{c}_{t'_7}} \mapsto 1\} + \mathbf{I}d_{\hat{p}_1, \hat{p}_2, \hat{p}_3}$ and $t'_8 = \{\hat{p}_1^- \mapsto 2, \hat{p}_3^+ \mapsto 1, w_{\mathbf{c}_{t'_8}} \mapsto 1\} + \mathbf{I}d_{\hat{p}_1, \hat{p}_2, \hat{p}_3}$. This yields the expressions $T''_1 = t'_7 \cup t'_8$ and $N''_1 = *_{\{\hat{p}_1, \hat{p}_2, \hat{p}_3\}}(T''_1)$. On the next level, we get $t'_2 = (\emptyset + \{\{p_2^- \mapsto 1, \hat{p}_2^- \mapsto 1\}, \{p_3^- \mapsto 1, \hat{p}_3^- \mapsto 1\}, \{p_2^+ \mapsto 1, \hat{p}_2^+ \mapsto 1\}, \{p_3^+ \mapsto 1, \hat{p}_3^+ \mapsto 1\}, \{w_{\mathbf{c}_{t'_7}} \mapsto 1\}, \{w_{\mathbf{c}_{t'_8}} \mapsto 1\}\}^*), \hat{\circ}_{\{\hat{p}_2^-, \hat{p}_3^-, \hat{p}_2^+, \hat{p}_3^+\}} N''_1$ for the generalized transition and $T'_2 = t'_2 \cup \hat{t}$.

The new sizes are now $S(t'_7) = S(t'_8) = (\emptyset, 0, 3) = S(t_7)$, $S(T''_1) = \{(\emptyset, 0, 3) \mapsto 2\} = S(T_1)$,

$S(N_1'') = (\{3 \mapsto 1\}, \{(\emptyset, 0, 3) \mapsto 2\}, 0, 8) < S(N_1),$
$S(t_2') = (\{S(N_1'') \mapsto 1\}, 1, 6) < S(t_2),$
$S(T_2') = \{S(t_2') \mapsto 1, (\emptyset, 0, 2) \mapsto 1\} < S(T_2).$

4.2 Condition 2 Quantitative consistency

Let Condition 2 be not fulfilled by T_i. The set $\mathbf{L} :=$

$$\left\{ \mathbf{g} \in \mathbb{N}^{C_\mathbf{L}} \,\middle|\, \forall p \in \bigcup_{N_i \in K_t} P_{T_i} \; \mathbf{g}(p^-) - ind(\mathbf{g})(p^-) = \mathbf{g}(p^+) - ind(\mathbf{g})(p^+) \right\}$$

on the carrier set $C_\mathbf{L} = C(t) \cup \bigcup_{N_i \in K_t} C(N_i)$ is a Presburger set. Since $\mathbf{R}(t) \subseteq$ $\mathbf{L}|_{C(t)}$ follows from the definition of $\mathbf{R}(t)$ and the function ind, we can set $T' :=$ $T \setminus \{t\} \cup t|_L$ using Lemma 3.4. In other words, we have cut something away which could not have been in $\mathbf{R}(T)$ anyway.

Since b_2 is now 0 for each $t_j \in t|_L$ and $S(K_{t_j})$ remains the same as $S(K_t)$, according to Lemma 3.4, it holds $S(T') < S(T)$.

Example (continued):
We see that t_2' does not fulfill Condition 2 when we look at the resulting equation

$$\mathbf{g}(\hat{p}^-) - \mathbf{g}(w_{\mathbf{c}_{t_7'}})\mathbf{c}_{t_7'}(\hat{p}^-) - \mathbf{g}(w_{\mathbf{c}_{t_8'}})\mathbf{c}_{t_8'}(\hat{p}^-) = \mathbf{g}(\hat{p}^+) - \mathbf{g}(w_{\mathbf{c}_{t_7'}})\mathbf{c}_{t_7'}(\hat{p}^+) - \mathbf{g}(w_{\mathbf{c}_{t_8'}})\mathbf{c}_{t_8'}(\hat{p}^+)$$

for all $p \in P$ characterizing L. This is equivalent to the following three equations: $2\mathbf{g}(w_{\mathbf{c}_{t_8'}}) = 3\mathbf{g}(w_{\mathbf{c}_{t_7'}})$, $\mathbf{g}(\hat{p}_2^-) - \mathbf{g}(w_{\mathbf{c}_{t_7'}}) = \mathbf{g}(\hat{p}_2^+)$, $\mathbf{g}(\hat{p}_3^-) = \mathbf{g}(\hat{p}_3^+) - \mathbf{g}(w_{\mathbf{c}_{t_8'}})$. Their solutions are described by the linear set $L_{t_2''} = L_{t_2'} \cap L =$

$$\emptyset + \left\{ \begin{bmatrix} p_2^- & \hat{p}_2^- & p_2^+ & \hat{p}_2^+ \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} p_3^- & \hat{p}_3^- & p_3^+ & \hat{p}_3^+ \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} w_{\mathbf{c}_{t_7'}} & w_{\mathbf{c}_{t_8'}} & p_2^- & \hat{p}_2^- & p_3^- & \hat{p}_3^- \\ 2 & 3 & 2 & 2 & 3 & 3 \end{bmatrix} \right\}^*$$

and yield $t_2'' = t_2'|_L = L_{t_2''} \hat{\circ}_{\{\hat{p}_2^-, \hat{p}_2^+, \hat{p}_3^-, \hat{p}_3^+\}} N_1''$ with $S(t_2'') = (\{S(N_1'') \mapsto 1\}, 0, 3) < S(t_2')$. Since $T_2'' = t_2'' \cup \hat{t}$ fulfills the remaining properties, we can continue one level higher.

Adding the witnesses leads to $L_{t_2'''} =$

$$\emptyset + \left\{ \begin{bmatrix} p_2^- & \hat{p}_2^- & p_2^+ & \hat{p}_2^+ & w_1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} p_3^- & \hat{p}_3^- & p_3^+ & \hat{p}_3^+ & w_2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} w_{\mathbf{c}_{t_7'}} & w_{\mathbf{c}_{t_8'}} & p_2^- & \hat{p}_2^- & p_3^- & \hat{p}_3^- & w_3 \\ 2 & 3 & 2 & 2 & 3 & 3 & 1 \end{bmatrix} \right\}^*$$

(we omit the witness for $\emptyset$.) with $S(t_2''') = S(t_2'') = (\{S(N_1'') \mapsto 1\}, 0, 3)$.

Defining $T_2''' = t_2''' \cup \hat{t}$ with $S(T_2''') = S(T_2'')$ and $N_2''' = *_{\{\hat{p}_1, \hat{p}_2, \hat{p}_3\}}(T_2''')$ with $S(N_2''') = (\{3 \mapsto 1, 2 \mapsto 1\}, S(T_2'''), 0, 8) < S(N_2'') = (\{3 \mapsto 1, 2 \mapsto 1\}, S(T_2'''), 1, 4)$ we get

$$t_3' = \left(\begin{bmatrix} p_2^- & p_3^- & p_2^+ & p_3^+ \\ 4 & 2 & 4 & 3 \end{bmatrix} + \left\{ \begin{bmatrix} w_1 \\ 1 \end{bmatrix}, \begin{bmatrix} w_2 \\ 1 \end{bmatrix}, \begin{bmatrix} w_3 \\ 1 \end{bmatrix}, \begin{bmatrix} w_{\mathbf{c}_{\hat{t}'}} \\ 1 \end{bmatrix} \right\}^* \right) \circ_{\{p_2^-, p_2^+, p_3^-, p_3^+\}} N_2'''.$$

Establishing Condition 2 leads to

$$t_3'' = \left(\begin{bmatrix} p_2^- & p_3^- & p_2^+ & p_3^+ & w_3 & w_{\mathbf{c}_{\hat{t}'}} \\ 4 & 2 & 4 & 3 & 5 & 2 \end{bmatrix} + \left\{ \begin{bmatrix} w_1 \\ 1 \end{bmatrix}, \begin{bmatrix} w_2 \\ 1 \end{bmatrix} \right\}^* \right) \circ_{\{p_2^-, p_2^+, p_3^-, p_3^+\}} N_2'''.$$

4.3   Condition 3 Elimination of witnesses

Let Condition 3 be not fulfilled by witness $w \in C(N_i) \setminus (P^+_{T_i} \cup P^-_{T_i})$. This means that we can replace N_i by some expression $\hat{T}$ with $\mathbf{R}(\hat{T}) = \mathbf{R}(N_i)\hat{\mathrm{o}}_{\{w\}}\mathbf{c}_t|_w$ since for all $\mathbf{m} \in \mathbf{L}_t$, we have $\mathbf{m}(w) = \mathbf{c}_t(w)$. Then, we can set

$$T' := T \setminus \{t\} \cup (L_t|_{\overline{\{w\}}} \hat{\mathrm{o}}_{A_t \setminus C(N_i)}(K_t \setminus \{N_i\}))\hat{\mathrm{o}}_{C(N_i) \setminus \{w\}}\hat{T}$$

which repaces the transition $t = L_t \mathrm{o}_{A_t} K_t$ by all those sets of transitions which result from using Lemma 3.5 (because $\hat{T}$ is not a net). This means that N_i is removed and the equivalent $\hat{T}$ is plugged in at the same range; thus, $\mathbf{R}(t) = \mathbf{R}(L_t) |_{\overline{\{w\}}}$ $\hat{\mathrm{o}}_{A_t \setminus \{w\}}(\mathbf{R}(K_t \setminus \{N_i\}) + \mathbf{R}(\hat{T}))$.

To create $\hat{T} = \bigcup_{\gamma} T_{\gamma}$, we consider every possible combination γ (including the order of the summands) of $\mathbf{c}_t(w) = \sum_{m=1}^{l_\gamma} \mathbf{g}_m(w)$ with $\mathbf{g}_m \in \mathbf{c}_{t_m} + \{\mathbf{g} \in \Gamma_{t_m} \mid \mathbf{g}(w) > 0\}^*$, $\mathbf{g}_m(w) > 0$ and $t_m \in T_i$ and build t'_m with $L_{t'_m} = \mathbf{g}_m|_{\overline{\{w\}}} + \{\mathbf{g} \in \Gamma_{t_m} \mid \mathbf{g}(w) = 0\}^*$ and $K_{t'_m} := K_{t_m}$. The expressions t'_m describe the parts in which w was used. In $N'_i = *_{P_{T'_i}}(T'_i)$ with $T'_i :=$

$$\{t''' \mid t'' \in T_i, \mathbf{c}_{t''} = \mathbf{c}_{t'''}, K_{t'''} = K_{t''}, \mathbf{c}_{t''}(w) = 0, \Gamma_{t'''} = \{\mathbf{g} \in \Gamma_{t''} \mid \mathbf{g}(w) = 0\}\},$$

we filter out everything which affects w; thus, $C(N'_i) = C(T'_i) = C(T_i) \setminus \{w\}$ and $\mathbf{R}(N'_i) = \{\mathbf{m} \in \mathbf{R}(N_i) \mid \mathbf{m}(w) = 0\}$. Then, using Lemma 3.1, we construct $t(N'_i)$ which has now the property $\mathbf{R}(t(N'_i)) = \{\mathbf{m} \in \mathbf{R}(N_i) \mid \mathbf{m}(w) = 0\}$. Now, we define
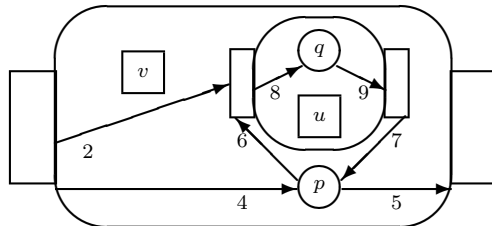
$$T_\gamma := t(N'_i) \mathrm{o}_{P_{T'_i}} t'_1 \mathrm{o}_{P_{T'_i}} t(N'_i) \mathrm{o}_{P_{T'_i}} t'_2 \mathrm{o}_{P_{t_i}} ... \mathrm{o}_{P_{T'_i}} t'_{l_\gamma} \mathrm{o}_{P_{T'_i}} t(N'_i)$$

again using Lemma 3.5.

It holds $S(K_{t'}) < S(K_t)$ for every new t' in T' because of $S(N'_i) < S(N_i)$. This in turn follows from $|C(T'_i)| = |C(T_i)| - 1$ and $S(N) < S(N_i)$ for all $N \in K_{t_m}$, and $m \leq l_\gamma$ for all γ.
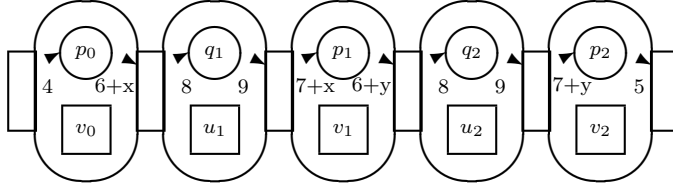
It holds $S(T') < S(T)$ since $S(t') < S(t)$ for every t'.

Example: Consider t with $\mathbf{c}_t = \left[\frac{w}{2}, \frac{p^-}{4}, \frac{p^+}{5} \right]$, $\forall \mathbf{g} \in \Gamma_t \, \mathbf{g}(w) = 0$, $K_t = \{*_{\{p\}}(v \cup t_j)\}$, and $\mathbf{c}_{t_j} = \left[\frac{w}{1}, \frac{p^-}{6}, \frac{p^+}{7}, \frac{q^-}{8}, \frac{q^+}{9} \right]$, $K_{t_j} = \{*_{\{q\}}(u)\}$.

Then t' is defined such that $\mathbf{c}_{t'} = \left[\frac{p^-_0}{4}, \frac{p^+_0}{6}, \frac{q^-_1}{8}, \frac{q^+_1}{9}, \frac{p^-_1}{7}, \frac{p^+_1}{6}, \frac{q^-_2}{8}, \frac{q^+_2}{9}, \frac{p^-_2}{7}, \frac{p^+_2}{5} \right]$, furthermore, $\left[\frac{p^-_1}{1}, \frac{p^+_0}{1} \right]$, $\left[\frac{p^-_2}{1}, \frac{p^+_1}{1} \right] \in \Gamma_{t'}$ and

$K_t = \{*_{\{p_0\}}(v_0), *_{\{q_1\}}(u_1), *_{\{p_1\}}(v_1), *_{\{q_2\}}(u_2), *_{\{p_2\}}(v_2)\}$, where p_i, q_i, v_i and u_i are replacements caused by disjointness condition in Lemma 3.5.

The variables x and y illustrate the effect of the periods in $\Gamma_{t'}$ which originate from the (omitted) periods of t_j.

4.4 Condition 4 Elimination of bounded places

Condition 4 is decidable by two *covering graph* constructions for every i working as follows: Every node in the covering graph $CG_{(i,+)}$ ($CG_{(i,-)}$, respectively) has a marking from $(\mathbb{N} \cup \{\omega\})^{P_{T_i}^-}$ ($(\mathbb{N} \cup \{\omega\})^{P_{T_i}^+}$, respectively). The root of the covering graph $CG_{(i,+)}$ has the marking $\mathbf{c}_t|_{P_{T_i}^-} + \omega^{\{p^- | \exists \mathbf{g} \in \Gamma \mathbf{g}(p^-) > 0\}}$.

For a node in $CG_{(i,+)}$ marked with $\mathbf{m}$, we construct T_i' with $\mathbf{R}(T_i') = \{\mathbf{g} \in \mathbf{R}(T_i) \mid \mathbf{g}|_{P_{T_i}^-} \leq \mathbf{m}\}$ using Lemma 3.4 as $T_i' := \{t'|_{\{\mathbf{g} \in \mathbf{L}_{t'}| \ \mathbf{g}|_{P_{T_i}^-} \leq \mathbf{m}\}} \mid t' \in T_i\}$. This restricts the allowed multisets to those which are possible starting with the limited marking $\mathbf{m}$. All $K_{t'}$ with $t' \in T_i'$ appear in the subnet N_i in t (unchanged by Lemma 3.4). For all $N' \in K_{t'}$, we have $S(N') < S(N_i)$ since the first component in $S(N_i)$ is $\{|P_{T_i}| \mapsto 1\}$ plus the maximum of everything one level deeper. Therefore, we have $S(K_{t'}) < S(K_t)$ for all $t' \in T_i'$ and, thus, $S(T_i') < S(T)$. This allows us to compute $T_i'' := reacheq(T_i')$ recursively.

For every $t'' \in T_i''$, (since we know from Condition $\mathcal{T}.5$ that $\mathbf{c}_{t''}$ alone can fire), we add a new node

$$\mathbf{m}' := \mathbf{m} - \mathbf{c}_{t''}|_{P_{T_i}^-} + \{p^- \mapsto (\mathbf{c}_{t''}(p^+) + \omega \sum_{\mathbf{g} \in \Gamma_{t''}} \mathbf{g}(p^+)) \mid p \in P_{T_i}\}$$

to the covering graph $CG_{(i,+)}$. According to Corollary 4.4, there is no limit for the number of appearances of the multi-sets in $\Gamma_{t''}$ in firing sequences. This allows us to label those places p^- with ω where $\mathbf{g}(p^+) > 0$ for a $\mathbf{g} \in \Gamma_{t''}$.

If $\mathbf{m}' > \mathbf{m}''$ for an $\mathbf{m}''$ on the path from the root to $\mathbf{m}$, then we set $\mathbf{m}' := \mathbf{m}' + \omega(\mathbf{m}' - \mathbf{m}'')$. This is because we can lift the marking of those places p^- with $(\mathbf{m}' - \mathbf{m}'')(p^-) > 0$ by repeating the firing sequence corresponding to the path from $\mathbf{m}''$ to $\mathbf{m}'$ arbitrarily many times.

If $\mathbf{m}' \leq \mathbf{m}'' \in Path(\mathbf{m}')$, then we need not calculate the successors of $\mathbf{m}'$ since we already had better chances at $\mathbf{m}''$.

According to [3], there are only finite sets of incomparable multi-sets over a finite set $P_{T_i}^-$. It, therefore, follows that every path must terminate.

If for all i a node marked with $\omega^{P_{T_i}^-}$ is in $CG_{(i,+)}$ and, analogously, a node marked with $\omega^{P_{T_i}^+}$ is in $CG_{(i,-)}$, then the Condition 4 is fulfilled. Otherwise, we can

calculate without loss of generality

$$k := \min_{\sigma \in \{+,-\}} \; \max_{path \subseteq CG_{(i,\sigma)}} \; \min_{p \in P_{T_i}} \; \max_{\mathbf{m} \in path} \mathbf{m}(p^{\sigma})$$

This means that in every path in $CG_{(i,+)}$ or $CG_{(i,-)}$, there is a place p such that on this path there are never more than k tokens on p^- or p^+ respectively.

Now, we can replace in $T' := T \setminus \{t\} \cup \bigcup\limits_{p \in P_{T_i}} U(p)$ the transition t by all those sets of transitions $U(p)$, described in the following sub section, which are generated by restricting t in such a way that, in the subnet N_i, there can never be more than k tokens on p.

In order to show that $S(T') < S(T)$ we have to show that each $S(t') < S(t)$ for every t' in every $U(p)$.

4.4.1 Elimination of places

As in the construction of a regular expression from a finite automaton having the states $0, ...k$, we define for all $l, j, h \leq k$ an expression $T_{j,h}^{l-1}$ describing corresponding firing sequences with the following property: They start with a marking $\mathbf{m}_0$ with $\mathbf{m}_0(p) = j$, end with a marking $\mathbf{m}_1$ with $\mathbf{m}_1(p) = h$, and meanwhile the number tokens on p is always less than l. This allows us to remove the place p since its information is no longer necessary. Therefore, we have $P_{T_{j,h}^{l-1}} = P'_{T_i} := P_{T_i} \setminus \{p\})$.

For an inductive definition, we start with the case of an immediate success where there is no 'meanwhile': This means

$$T_{j,h}^{-1} := T_i \hat{\circ}_{\{p^-,p^+\}} \{\{p^- \mapsto j, p^+ \mapsto h\}\}$$

is constructed using Lemma 3.5. (We can write $\{\{p^- \mapsto j, p^+ \mapsto h\}\}$ as $\{t_{j,h}\}$ with $\mathbf{c}_{t_{j,h}} = \{p^- \mapsto j, p^+ \mapsto h\}$ and $\Gamma_{t_{j,h}} = K_{t_{j,h}} = \emptyset$.) Recursively, we define

$$T_{l,l}^{l} := \{t(N_{l,l}^{l-1})\} := \{t(*_{P'_{T_i}}(T_{l,l}^{l-1}))\}$$

using Lemma 3.1. Then with Lemma 3.5, we construct

$$T_{l,h}^{l} := T_{l,l}^{l} \circ_{P'_{T_i}} T_{l,h}^{l-1} \text{ for } h \neq l,$$
$$T_{j,l}^{l} := T_{j,l}^{l-1} \circ_{P'_{T_i}} T_{l,l}^{l} \text{ for } j \neq l, \text{ and}$$
$$T_{j,h}^{l} := T_{j,l}^{l-1} \circ_{P'_{T_i}} T_{l,l}^{l} \circ_{P'_{T_i}} T_{l,h}^{l-1} \cup T_{j,h}^{l-1} \text{ for } h \neq l \wedge j \neq l.$$

Now we define

$$U(p) = (L_t |_{\overline{\{p^-,p^+\}}} \hat{\circ}_{A_t \setminus C(N_i)} (K_t \setminus \{N_i\})) \hat{\circ}_{C(N_i) \setminus \{p^-,p^+\}} T_{\mathbf{c}_t(p^-),\mathbf{c}_t(p^+)}^{k}$$

using Lemma 3.5. We have $S(N') < S(N_i)$ for every $N' \in K_{t'}$ with $t' \in T_{\mathbf{c}_t(p^-),\mathbf{c}_t(p^+)}^{k}$ because for the corresponding first components s' and s_i of the 4-tuples, we have $\mathbf{s}'(|P'_{T_i}|) = \mathbf{s}_i(|P'_{T_i}|) + k + 1$ but $\mathbf{s}'(|P_{T_i}|) = \mathbf{s}_i(|P_{T_i}|) - 1$ (It holds $|P'_{T_i}| = |P_{T_i}| - 1$). Thus, $S(t'') < S(t)$ for every $t'' \in U(p)$.

Example: Let $t = (\mathbf{c} + \Gamma^*)\hat{\circ}_{C(N_i)} *_{\{p\}\cup P}(N_i)$ with $\mathbf{c}(p^-) = 1$, $N_i = v \cup w \cup t_j$, $C(N_i) = \{p^-, p^+\} \cup P^- \cup P^+$ and $t_j = (\mathbf{c}_j + \Gamma_j^*)\hat{\circ}_{\{q^-,q^+\}\cup Q^-\cup Q^+} *_{\{q\}\cup Q}(u)$ with $\mathbf{c}_j(p^+) = 1$, $\mathbf{c}_j(q^-) = 8$ and $\mathbf{c}_j(q^+) = 9$ look like
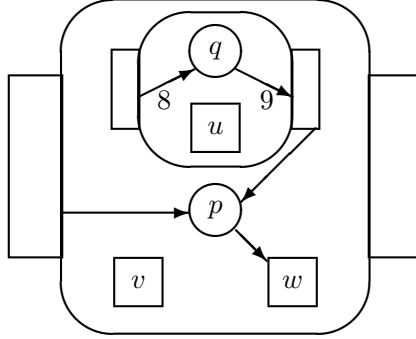
and $k = 1$. Furthermore, we assume no other occurrence of p in any other constant or period. This means that the firing sequences are restricted to the regular expression $((wv^*t_j) + v)^*wv^*$. This corresponds to $T_{0,0}^{-1}$ and $T_{1,1}^{-1}$ to consist only of a copy of v, $T_{1,0}^{-1}$ only of a copy of w and $T_{0,1}^{-1}$ only of a copy of t_j.

We get $T_{0,0}^0 = t(*_P(T_{0,0}^{-1}))$, $T_{1,1}^0 = T_{1,0}^{-1}\circ_P T_{0,0}^0\circ_P T_{0,1}^{-1} \cup T_{1,1}^{-1}$; in the end every new transition t' in $(\mathbf{c} + \Gamma^*)|_{\overline{\{p^-,p^+\}}} \hat{\circ}_{C(N_i)\backslash\{p^-,p^+\}}T_{1,0}^1$ with $T_{1,0}^1 = T_{1,1}^1\circ_P T_{1,0}^0 = t(*_P(T_{1,1}^0))\circ_P T_{1,0}^{-1}\circ_P T_{0,0}^0$ now looks like
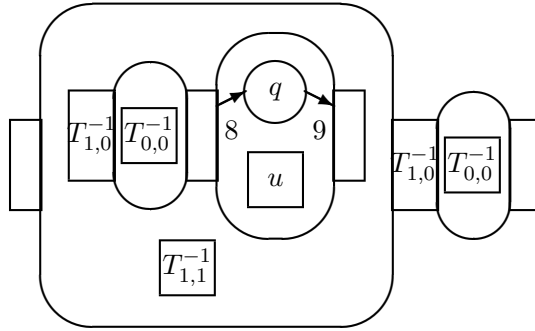
4.5   Condition 5 Making the constant firing

If Condition 5 is not fulfilled for t then, according to Corollary 4.4, for $\mathbf{f} = \sum_{\mathbf{g}\in\Gamma} \mathbf{g}$, there exists a (smallest) k such that $(c + k\mathbf{f})|_{C(t)}\in \mathbf{R}(t)$. So we decompose L_t such that $\mathbf{R}(L_t) = \mathbf{R}(L_t + k\mathbf{f}) \cup \bigcup_{\mathbf{g}\in\Gamma}\bigcup_{j\leq k} \mathbf{R}(\mathbf{c}_t + j\mathbf{g} + (\Gamma_t \backslash \{\mathbf{g}\})^*)$. Set

$$T' := T \backslash \{t\} \cup \{t' \mid K_{t'} = K_t, \Gamma'_t = \Gamma_t \wedge \mathbf{c}_{t'} = \mathbf{c}_t + k\mathbf{f})\}$$

$$\cup \{t' \mid \exists j \leq k, \mathbf{g} \in \Gamma\ \Gamma'_t = \Gamma_t \backslash \{\mathbf{g}\}) \wedge \mathbf{c}_{t'} = \mathbf{c}_t + j\mathbf{g})\}.$$

Since Conditions 1 and 2 are not affected, b_2 and $S(K_t)$ do not change. The size

$S(t')$ is smaller than $S(t)$ since b_5 is now zero respectively $|\Gamma \setminus \{\mathbf{g}\}| < |\Gamma|$; thus, it holds $S(T') < S(T)$.

Lemma 4.3 *If the conditions 1 - 4 hold for t, then it holds*

$$\forall \mathbf{f} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^* \forall \mathbf{e} \in (\Gamma_t \cup -\Gamma_t)^* \exists k \geq 2 \ \left\{ (\mathbf{c}_t + k\mathbf{f})|_{C(t)}, (\mathbf{c}_t + k\mathbf{f} + \mathbf{e})|_{C(t)} \right\} \subseteq \mathbf{R}(t)$$

The proof is in [17] where we show that we can build up firing sequences which compensate the 'odd' firing sequences resulting from condition 4, from the constant, and from the 'odd' indirect firing sequences in order to find a $\mathbf{c}_t$ fulfilling condition 5 according to the following corollary:

Corollary 4.4 *If the conditions 1 - 4 hold for t, then it holds*

$$\forall \mathbf{f} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^* \exists k \geq 2 \ (\mathbf{c}_t + k\mathbf{f})|_{C(t)} \in \mathbf{R}(t)$$

# 5   The reachability relation for Petri nets with inhibitor arcs

Now, we generalize Lemma 2.4 by using the operators $\cup, \circ_Q$ and $*_Q$ over finite sets of multisets in a nested way. This allows us to express the reachability problem in a Petri net for which there exists an ordering of the places such that a place has an inhibitor arc to all those transitions which have an inhibitor arc from a preceding place:

Theorem 5.1 *In a Petri-net $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ with*

$$\exists \mathbf{g} \in \mathbb{N}_+^P \ \forall p, p' \in P \ \mathbf{g}(p) \leq \mathbf{g}(p') \rightarrow (\forall t \in T \ (p', t) \in I \rightarrow (p, t) \in I),$$
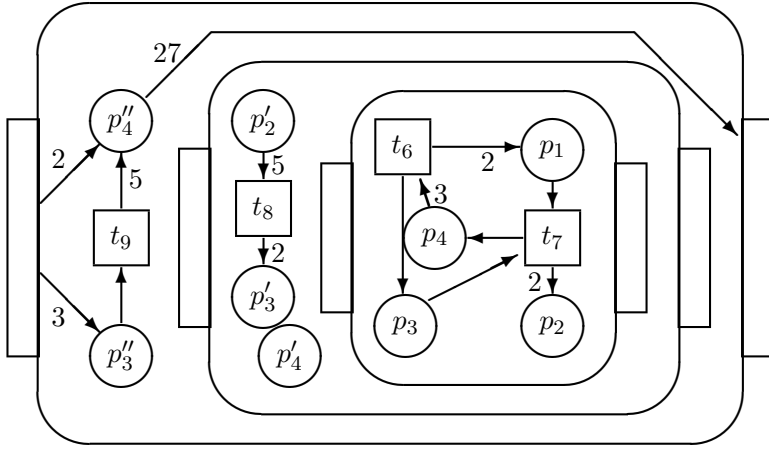
we can construct an expression T_g such that there is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$ if and only if $\mathbf{R}(T_g)$ is $(= \{\emptyset\}$ and$)$ not empty.

Example: For the Petri net
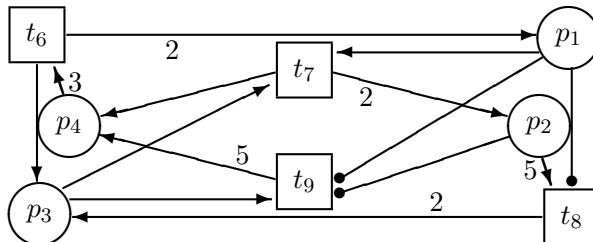
with the start marking $\{p_3 \mapsto 3, p_4 \mapsto 2\}$ and the end marking $\{p_4 \mapsto 27\}$ we can easily find the function $\mathbf{g}$ with $\mathbf{g}(p_1) = 1$, $\mathbf{g}(p_2) = 2$ and $\mathbf{g}(p_3) = \mathbf{g}(p_4) = 3$. The

construction gives us an expression which we visualise as

with the innermost level

$$T_1 = \left\{ \begin{bmatrix} p_4^- & p_1^+ & p_3^+ \\ 3 & , & 2 & , & 1 \end{bmatrix}, \begin{bmatrix} p_1^- & p_3^- & p_2^+ & p_4^+ \\ 1 & , & 1 & , & 2 & , & 1 \end{bmatrix} \right\} + \mathbf{Id}_P.$$

This enables the firing sequence $w = t_6 t_7 t_7$ from $\begin{bmatrix} p_3 & , & p_4 \\ 1 & & 3 \end{bmatrix}$ to $\begin{bmatrix} p_2 & , & p_4 \\ 4 & & 2 \end{bmatrix}$ on the inner-most level as $\begin{bmatrix} p_3^- & , & p_4^- & , & p_2^+ & , & p_4^+ \\ 1 & & 3 & & 4 & & 2 \end{bmatrix} \in \mathbf{R}(*_{P_{T_1}}(T_1)) = \mathbf{R}(t_2) \subseteq \mathbf{R}(T_2)$. Together with $\begin{bmatrix} p_2^- & , & p_3^+ \\ 5 & & 2 \end{bmatrix} \in \mathbf{R}(T_2)$ for t_8, we get the firing sequence $w' = (w)(w)t_8(w)t_8(w)t_8(w)t_8$ from $\begin{bmatrix} p_3 & , & p_4 \\ 2 & & 7 \end{bmatrix}$ to $\begin{bmatrix} p_3 & , & p_4 \\ 5 & & 2 \end{bmatrix}$ on the next level as $\begin{bmatrix} p_3^- & , & p_4^- & , & p_3^+ & , & p_4^+ \\ 2 & & 7 & & 5 & & 2 \end{bmatrix} \in \mathbf{R}(*_{P_{T_2}}(T_2)) = \mathbf{R}(t_3) \subseteq \mathbf{R}(T_3)$. Together with $\begin{bmatrix} p_3^- & , & p_4^+ \\ 1 & & 5 \end{bmatrix} \in \mathbf{R}(T_3)$ for t_9, this enables the firing se-quence $w'' = t_9(w')t_9^5$ from $\begin{bmatrix} p_3 & , & p_4 \\ 3 & & 2 \end{bmatrix}$ to $\begin{bmatrix} p_4 \\ 27 \end{bmatrix}$ on the following level as $\begin{bmatrix} p_3^- & , & p_4^- & , & p_4^+ \\ 3 & & 2 & & 27 \end{bmatrix} \in \mathbf{R}(*_{P_{T_3}}(T_3)) = \mathbf{R}(t_4) = \mathbf{R}(T_4)$.

The general construction is in [17].

With Theorem 4.1 we derive the following:

Corollary 5.2 *The reachability problem for a Petri net* $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ *with*

$$\exists \mathbf{g} \in \mathbb{N}_+^P \ \forall p, p' \in P \ \mathbf{g}(p) \leq \mathbf{g}(p') \rightarrow (\forall t \in T \ (p', t) \in I \rightarrow (p, t) \in I),$$

is decidable.

This is optimal since the non-existence of such an ordering would mean that there are two inhibitor arcs $(p, t), (p', t')$ without (p, t') or (p', t) being an inhibitor arc, which would already mean undecidability of the reachability problem for such instances.

6 Priority-Multicounter-Automata

We define a priority-multicounter-automaton by a restrictive zero-test according to an order of the counters in the following way: the first counter can be tested for zero at any time; the second counter can only be tested for zero simultaneously with the

first counter; any further counter can only be tested for zero simultaneously with all preceding counters. Formally, this reads as follows:

A *priority-multicounter-automaton* is a one-way automaton described by the 6-tuple

$$A = (k, Z, \Sigma, \delta, z_0, E)$$

with the set of states Z, the input alphabet Σ, the transition relation

$$\delta \subseteq (Z \times (\Sigma \cup \{\lambda\}) \times \{0 \ldots k\}) \times (Z \times \{-1, 0, 1\}^k),$$

initial state z_0, the accepting states $E \subseteq Z$, the set of configurations $C_A = Z \times \Sigma^* \times \mathbb{N}^k$, the initial configuration $\sigma_A(x) = \langle z_0, x, \underbrace{0, \ldots, 0}_{k} \rangle$ and configuration transition relation

$$\langle z, ax, n_1, \ldots, n_k \rangle \;\vdash_A\; \langle z', x, n_1 + i_1, \ldots, n_k + i_k \rangle$$

if and only if $z, z' \in Z, a \in \Sigma \cup \{\lambda\}, \langle (z, a, j), (z', i_1, \ldots i_k) \rangle \in \delta, \; \forall i \le j \; n_i = 0$.

The language recognized by an priority-multicounter-automaton A is $L(A) = \{w \mid \exists z_e \in E \; \exists n_1, \ldots, n_k \in \mathbb{N} \; \langle z_0, w, 0, \ldots, 0 \rangle \vdash_A^* \langle z_e, \lambda, n_1, \ldots, n_k \rangle\}$. A priority-multicounter-automaton can be changed in such a way that it has only one accepting state z_e and that all counters are empty while accepting. Thus, $L(A) = \{w \mid \langle z_0, w, 0, \ldots, 0 \rangle \vdash_A^* \langle z_e, \lambda, 0, \ldots, 0 \rangle\}$.

Using Theorem 5.1, we show that the emptiness problem of the accepted language is decidable for priority-multicounter-automata. The same holds for the halting problem by constructing an automaton which contains its input in the states.

Theorem 6.1 *The emptiness problem for priority-multicounter-automata is decidable.*

Proof. Given A we construct a Petri net (P, T, W, I, m_0, m_1) with the places $P := \{1 \ldots k\} \cup Z$, the transitions $T = \delta$, the weights W with
$W(z, ((z', a, j), (z'', V))) := 1$ if $z = z'$ else $:= 0$;
$W(((z', a, j), (z'', V)), z) := 1$ if $z = z''$ else $:= 0$;
$W(i, ((z', a, j), (z'', V))) := 1$ if $V(i) = -1$ else $:= 0$; and
$W(((z', a, j), (z'', V)), i) := 1$ if $V(i) = 1$ else $:= 0$;
the inhibitor arcs $I := \{(i, ((z', a, j), (z'', V))) \mid i \le j\}$, the start marking $\mathbf{m}_0 := \{z_0 \mapsto 1\}$, and the end marking $\mathbf{m}_1 := \{z_e \mapsto 1\}$ which is reachable from $\mathbf{m}_0$ if and only if $L(A) \ne \emptyset$. According to Corollary 4.2 with $\mathbf{g}(i) = i$ for $i \le k$ and $\mathbf{g}(z) = k + 1$ for $z \in Z$, this is decidable. $\qquad\square$

Consequences of this result for the decidability of the synchronizability and maximality of semi-trace languages are formulated in [16]. Further consequences are described in [6]: Every unary language is recursive if it is generated by a graph-controlled grammar with an arbitrary number of nonterminal symbols but only one of the nonterminal symbols being allowed to be used in the appearance checking mode. Theorem 6.1 implies the optimality the results in [6] proving the computational completeness of graph-controlled grammars, programmed grammars, and

matrix grammars with a certain nonterminal complexity.

The classes *k-PMC* of languages accepted by a priority-multicounter-automaton with $k > 0$ counters (and also their union) are incomparable to the class *LIN* of linear languages and it holds *(k-1)-PMC $\subsetneq$ k-PMC*. This is because

$$\{a^{n_1}ba^{n_2}...ba^{n_{k+1}}\$a^{n_{k+1}}b...a^{n_2}ba^{n_1} \mid \forall i \leq k+1 \; n_i \in \mathbb{N}\} \notin k-PMC.$$

This can be shown by constructing T fulfilling property $\mathcal{T}$ and, then, by using Lemma 4.3 to find two different words in the language where the automaton has the same configuration reading \$. With the same argument, this also holds for the classes *k-BLIND* and *k-PBLIND* in [7]. Furthermore, $\{(a^nb)^m \mid n, m \in \mathbb{N}\}$ cannot be accepted by a priority-multicounter-automaton (Theorem 3.2 in [16]).

7 Restricted Priority- Multipushdown- Automata

We define a priority-multipushdown-automaton by a different treatment of one of the two pushdown symbols according to an order of the pushdown stores in the following way: let the pushdown alphabet be $\{0, 1\}$. A 0 can be pushed to and popped from every pushdown store independently, but a 1 can only be pushed to or popped from a pushdown store if all pushdown stores with a lower order are empty. Furthermore, the restriction requires that if a 1 is popped from a pushdown store, then a 1 cannot be pushed anymore to this store until it is empty.

Theorem 7.1 *The emptiness problem for restricted priority-multipushdown-automata is decidable.*

This generalizes the result in [10] that $\underline{LIN}\%D_1'^*$ (the class of languages generated by linear grammar and deletion of semi Dyck words) is recursive. We conjecture that decidability still holds in the unrestricted case but, even in the special case of a pushdown automaton with additional weak counters (without zero-test), this is still an open problem.

Acknowledgement

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