From determinacy to Nash equilibrium

Stéphane Le Roux, TU Darmstadt

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Abstract

Game theory is usually considered applied mathematics, but a few game-theoretic results, such as Borel determinacy, were developed by mathematicians for mathematics in a broad sense. These results usually state determinacy, *i.e.* the existence of a winning strategy in games that involve two players and two outcomes saying who wins. In a multi-outcome setting, the notion of winning strategy is irrelevant yet faithfully replaced with the notion of (pure) Nash equilibrium. This article shows that every determinacy result over a game structure, *e.g.* a tree, is transferable into existence of multi-outcome (pure) Nash equilibrium over the same game structure. The equilibrium-transfer theorem requires cardinal and order-theoretic conditions on the strategies and the preferences respectively, whereas counter-examples show that every requirement is important, including the players being two only. As examples of application, this article generalises (by invoking!) Borel determinacy, positional determinacy of parity games, and finite-memory determinacy of Müller games.

Keywords: Borel Determinacy, parity games, Müller games, transfer from determinacy to multi-outcome Nash equilibrium

1 Introduction

Game theory is the theory of competitive interactions between decision makers having different interests. Its primary purpose is to further understand such real-world interactions through mathematical modelling. Apart from some earlier related works, the field of game theory is usually said to be born in the first part of the 20th century, especially thanks to von Neumann [11], but also Borel [1] and some others. Since then, it has been applied to many areas such as economics, political science, evolutionary biology, *etc.* Conversely, specific problems in these areas have been raising new questions and have thus been helpful in developing game theory.

Game theory has also provided a useful point of view for abstract areas such as descriptive set theory and theoretical informatics. In the games that are usually involved therein, two players play alternately and infinitely often, and only then one player eventually wins, whereas the opponent loses. For instance, Martin's Theorem on Borel determinacy [7],[8] (generalising [3] and [13]) relates Borel sets to the existence of winning strategies in such games built on infinite trees; similarly, some program-verification results, such as [5] or [4] (generalising [2] and [9]), relate specific logical properties to the existence/computation of simple winning strategies in such games built on graphs. In such frameworks, existence of a winning strategy (of some sort) is called determinacy, and the games that enjoy it are said to be determined.

Game theory also studies games with many players and many possible outcomes, where it is no longer a mere matter of winning instead of losing, but each agent has (complex) preferences over the possible outcomes instead. There, the notion of winning strategy is irrelevant yet faithfully replaced with the notion of (pure, not mixed) Nash equilibrium, introduced in [10]. This multi-outcome framework should not only constitute a more accurate modelling tool, but also trigger phenomena (involving Nash equilibrium) that cannot occur in the two-outcome framework. On the contrary, this article essentially shows that determinacy (over a two-player game structure instantiated with two outcomes) is equivalent to existence of Nash equilibrium (over the same game structure instantiated with many outcomes). It actually shows more than this under some weak conditions that are detailed in the remainder of the introduction. This equilibrium-transfer theorem will be proved for games in normal form, *i.e.* a class of games where many other classes of games can be faithfully embedded as far as Nash equilibrium is concerned.

Unlike traditional games in normal form, the definition below involves abstract outcomes (instead of mere real-valued payoff functions) and preferences that are arbitrary (instead of transitive, reflexive, total, *etc.*). It is important since there is no reason why games, *e.g.*, with real-valued payoff functions should account for all possible games, as shown in Section 5.

Definition 1 (Games in normal form) *Such a game is an object* $\langle A, \otimes_{a\in A} S_a, O, v, (\prec_a)_{a\in A}\rangle$ *complying with the following:*

- A *is a non-empty set (of players),*
- $\otimes_{a\in A} S_a$ *is a non-empty Cartesian product (whose elements are the strategy profiles and where* S_a *represents the strategies available to player* a*),*
- O *is a non-empty set (of possible outcomes),*
- $v : \otimes_{a\in A} S_a \to O$ *(uses the outcomes to value the strategy profiles),*
- *Each* $\prec_a$ *is a binary relation over* O *(modelling the preference of player* a*).*

The traditional notion of Nash equilibrium is rephrased below in the abstract setting with a subtle semantic change (but remains the same in extension): each binary relation $\prec_a$, which I call preference, is the complement of the inverse of what is traditionally called preference.

Definition 2 (Nash equilibrium) *Let* $g = \langle A, \otimes_{a\in A} S_a, O, v, (\prec_a)_{a\in A}\rangle$ *be a game in normal form. A strategy profile* s *in* $S := \otimes_{a\in A} S_a$ *is a Nash equilibrium if it makes every agent* a *stable, i.e.* $v(s) \not\prec_a v(s')$ *for all* $s' \in S$ *that differ from* s *at most at the* a*-component.*

$$NE(s) \quad := \quad \forall a \in A, \forall s' \in S, \quad \neg(v(s) \prec_a v(s') \,\wedge\, \forall b \in A - \{a\},\, s_b = s'_b)$$

Proposition 3 below is a very simple version of the equilibrium-transfer theorem, but the very basic idea is already there. It uses only an abstraction of zero-sum games, *i.e.* two-player games where preferences are inverses of each other. It is named after von Neumann's Minimax Theorem (see, *e.g.*, [11]) to hint at the similarities, although it is neither a generalisation nor a special case of it. (For instance, the Minimax Theorem involves infinitely many mixed strategies whereas Proposition 3 involves finitely many pure strategies only.) Note that $v(s_1, S_2)$ below refers to $\{v(s_1, s_2) \mid s_2 \in S_2\}$.

Proposition 3 (Minimax transfer) *Let* $\langle \{1, 2\}, S_1 \times S_2, O, v, <_1, <_2\rangle$ *be a two-player game in normal form and let us assume the following:*

1. $<_2 = <_1^{-1}$ *is a strict linear order over the finite domain* O*.*
2. $\forall P \subseteq O, \quad (\forall x \in P \forall y \in O, x <_1 y \Rightarrow y \in P)$
   $\Rightarrow \qquad (\exists s_1 \in S_1,\, v(s_1, S_2) \subseteq P) \vee (\exists s_2 \in S_2,\, v(S_1, s_2) \subseteq O \backslash P)$
   Informally, for every $<_1$*-terminal interval of outcomes, either player 1 can enforce it or player 2 can exclude it.*

Then the game $\langle\{1,2\}, S_1 \times S_2, O, v, <_1, <_2\rangle$ *has a Nash equilibrium and all Nash equilibria yield the same outcome.*

Proof Let P be the smallest $<_1$-terminal (or $<_1$-upper) interval that player 1 can enforce *via* some strategy s_1, and let m be the $<_1$-minimum (and the $<_2$-maximum!) of P. Since player 1 cannot enforce $P-\{m\}$, player 2 can enforce $(O\backslash P)\cup\{m\}$ *via* some strategy s_2. Therefore (s_1, s_2) is a Nash equilibrium yielding outcome m, and any strategy profile that does not yield m may be improved upon by some player i *via* s_i. $\square$

The equilibrium-transfer theorem is now stated in its most useful form, when considering a multi-outcome two-player game and a predicate on its strategies: If arbitrarily mapping the outcomes to $\{win, lose\}$ (or equivalently to $\{(1,0),(0,1)\}$) yields a game with a winning strategy that satisfies the predicate, the original game has a Nash equilibrium whose two components satisfy the predicate, provided that either the heights of the preferences are finite (*i.e.* there exists a finite bound to the length of all chains of the preferences), or the inverse relations of the preferences are well-founded and if one player has uncountably many strategies, the opponent has finitely many strategies.

This result may be interpreted from different points of view. Game theorist: to prove existence of Nash equilibrium for a large class of games, I may just prove existence of Nash equilibrium (or equivalently of winning strategy) for the subclass involving two outcomes only. Logician: my determinacy result, which was meant for logic only, can be extended to a wider Nash-equilibrium result that may interest game theorists, provided that I define a multi-outcome version of my games. Lazy mathematician: I might generalise existing determinacy results for free, without knowing how they are proved or even what they mean! Computer scientist: since I consider only computable strategies, which are countably many, I can extend my determinacy result to infinite-height preferences; moreover, if there is always a winning strategy that is easy to compute, there is always a Nash equilibrium that is easy to compute; even uniformly when outcomes a finitely many, as discussed later.

All the conditions of application of the equilibrium-transfer theorem are useful: First, when the inverse of some preference is not well-founded, it is easy to build a game without Nash equilibrium. Second, this article defines a game such that the strategies of one player only are countably many, one preference only has finite height and the inverse of the other preference is still well-founded, but there is no Nash equilibrium although arbitrarily rewriting the outcomes with win and lose yields a determined game. Third, there is no obvious three-player version of the equilibrium-transfer theorem: indeed, for every natural n, there exists a finite three-player game with preferences of heights $n+1$ and without Nash equilibrium, although arbitrarily replacing the preferences with preferences of height at most n yields a game with Nash equilibrium. (Of course, it does not mean that the conditions of application of the theorem cannot be effectively weakened for specific classes of games though.)

Note that, in the area of graph games for program verification, [12] has already investigated extensions of determinacy in various directions, namely for subgame perfect equilibrium (a stronger notion of Nash equilibrium), for n-player games instead of two-player games, or for payoff functions in $\{0,1\}^n$ instead of $\{(0,1),(1,0)\}$. For instance, Theorem 4.19. in [12] states that any initialised two-player parity game has a positional subgame perfect equilibrium and Theorem 4.20. states that any initialised finite multiplayer parity game has a finite-state subgame perfect equilibrium.

Section 2 presents the equilibrium-transfer theorem and a basic algorithmic remark; Section 3 invokes the equilibrium-transfer theorem to generalise Martin's theorem on Borel determinacy, positional determinacy of parity games (with infinitely many priorities), and finite-memory determinacy of Müller games; Section 4 gives counterexamples to reasonable candidates to generalise the theorem; and Section 5 concludes and shows in passing that linearly ordered preferences do not account for partially ordered preferences.

2 The equilibrium-transfer theorem

This section proves the theorem by transfinite induction on the preferences. The three main ingredients of the proof are: an equilibrium-reflecting reduction that shrinks games in terms of preferences, a property on functions from $\mathbb{N}^2$ to $\mathbb{N}$ that enables a diagonal argument when shrinking games is not possible, and a finite-case version of the theorem, which itself relies on lifting binary relations to the power set of their domains. This lift, defined below, is the basic idea of the equilibrium transfer: especially, it overcomes the difficulty that the proof of the minimax transfer does not scale up for preferences that are not inverses of each other.

Definition 4 *A binary relation $\prec$ on a set S may be lifted to the power set of S as below.*

$$\forall A, B \subseteq S, \quad A \prec^{\mathcal{P}} B := \exists a \in A \backslash B, \forall b \in B \backslash A,\, a \prec b$$

Lemma 5 *Let $\prec$ be a binary relation on a set S. If $\prec$ is a strict linear order, $\prec^{\mathcal{P}}$ is a strict partial order.*

Proof A strict partial order is a transitive and irreflexive binary relation. A strict linear order is a strict partial order such that any two distinct elements are comparable. Assume that $\prec$ is as strict linear order. Since $\prec^{\mathcal{P}}$ is irreflexive by definition, it suffices to show that $\prec^{\mathcal{P}}$ is transitive. Assume that $A \prec^{\mathcal{P}} B$ and $B \prec^{\mathcal{P}} C$ with respective witnesses $a \in A\backslash B$ and $b \in B\backslash C$. First note that $a \neq b$ since $a \notin B$ and $b \in B$. Now let us case-split to show that $A \prec^{\mathcal{P}} C$.

- Assume that $a \prec b$, so $\neg(b \prec a)$ by transitivity and irreflexivity assumptions, so $a \notin C\backslash B$ since b is a witness for $B \prec^{\mathcal{P}} C$. Together with $a \notin B$ it yields $a \notin C$, so $a \in A\backslash C$. Now let x be in $C\backslash A$. If $x \in B$, then $x \in B\backslash A$, and $a \prec x$ since a is a witness for $A \prec^{\mathcal{P}} B$. If $x \notin B$, then $x \in C\backslash B$, and $b \prec x$ since b is a witness for $B \prec^{\mathcal{P}} C$, so $a \prec x$ by transitivity. Therefore $A \prec^{\mathcal{P}} C$ is witnessed by a.
- Assume that $b \prec a$, so $\neg(a \prec b)$ by transitivity and irreflexivity assumptions, so $b \notin B\backslash A$ since a is a witness for $A \prec^{\mathcal{P}} B$. Together with $b \in B$ it yields $b \in A$, so $b \in A\backslash C$. Now let x be in $C\backslash A$. If $x \notin B$, then $x \in C\backslash B$, and $b \prec x$ since b is a witness for $B \prec^{\mathcal{P}} C$. If $x \in B$, then $x \in B\backslash A$, and $a \prec x$ since a is a witness for $A \prec^{\mathcal{P}} B$, so $b \prec x$ by transitivity. Therefore $A \prec^{\mathcal{P}} C$ is witnessed by b.

□

The lemma below states a bit more than a mere finitely-many-outcome version of the forthcoming theorem; it sounds a bit less natural too, due to Condition 3 (which is obviously fulfilled when the outcomes are finitely many), but it is very useful in the proof of the theorem. Note that Condition 1, where $v(s_1, S_2)$ refers to $\{v(s_1, s_2) \mid s_2 \in S_2\}$, amounts to determinacy of any two-outcome version of the original game where the winning strategies must satisfy some additional predicate. Also note that Condition 1 defines the notions of enforcement and exclusion.

Lemma 6 (Finitary equilibrium transfer) *Let $\langle \{1,2\}, S_1 \times S_2, O, v, \prec_1, \prec_2 \rangle$ be a two-player game in normal form, let $R_1 \subseteq S_1$ and $R_2 \subseteq S_2$, and let us assume the following:*

1. $\forall P \subseteq O, (\exists s_1 \in R_1,\, v(s_1, S_2) \subseteq P) \vee (\exists s_2 \in R_2,\, v(S_1, s_2) \subseteq O\backslash P)$
   That is, for every subset of outcomes, either player 1 can enforce it via some strategy in R_1 or player 2 can exclude it via some strategy in R_2.
2. *Both preferences $\prec_1$ and $\prec_2$ are acyclic.*
3. $\exists s_1 \in S_1,\, |\{o \in O \mid \exists s_2 \in S_2,\, v(s_1, s_2) \preceq_1 o\}| < \infty \quad \vee$
   $\exists s_2 \in S_2,\, |\{o \in O \mid \exists s_1 \in S_1,\, v(s_1, s_2) \preceq_2 o\}| < \infty$
   That is, player i can enforce a subset of outcomes whose $\prec_i$-upper cone is finite.

Then the game $\langle\{1,2\}, S_1 \times S_2, O, v, \prec_1, \prec_2\rangle$ has a Nash equilibrium in $R_1 \times R_2$.

Proof First note that if one player i can enforce a subset of outcome (*via* some strategy in S_i), he/she can enforce it *via* some strategy in R_i, since the opponent cannot exclude it and by assumption (2). Assume that one player, *e.g.*, player 1 can enforce C a finite $\prec_1$-upper cone. Since $\prec_1$ is acyclic, so is its restriction $\prec_1|_C$ to C; let $<$ be a strict linear extension of $\prec_1|_C$, so $<^{\mathcal{P}}$ is a strict partial order by lemma 5. (Actually a strict linear order but this fact is not needed here.) Since C is finite, so is $\mathcal{P}(C)$, so let M be a $<^{\mathcal{P}}$-maximal (actually the $<^{\mathcal{P}}$-greatest) subset of C that player 1 can enforce and let $s_1 \in R_1$ be a strategy enforcing M.

Since M is finite and non-empty and since $\prec_2$ is acyclic, let m be $\prec_2|_M$-maximal and let $X := \{x \in M \mid x < m\} \cup \{x \in C \mid m < x\}$. Since $M <^{\mathcal{P}} X$ by definition 4 and since $X \subseteq C$ by definition of M and X, player 1 cannot enforce X by definition of M, so player 2 can enforce $O \backslash X$ by assumption. Let $s_2 \in R_2$ be a strategy enforcing $O\backslash X$, so that $v(s_1, s_2) \in M \cap (O\backslash X) = \{m\}$.

Player 2 is stable since m is $\prec_2$-maximal in M enforced by s_1. Let $o \in O$ such that $m \prec_1 o$, so $o \in C$ be definition of C, so $o \in X$ by definition of X. Therefore m is $\prec_1$-maximal among $O\backslash X$ enforced by 2. Therefore the strategy profile $(s_1, s_2) \in R_1 \times R_2$ is a Nash equilibrium. $\square$

There is a straightforward algorithmic consequence of the proof of Lemma 6. Namely, finding a suitable Nash equilibrium in a two-player game that involves n outcomes requires at most n (resp. 2) calls to the function w_a (resp. w_s) expecting a two-outcome two-player game and returning the winning player (resp. a suitable winning strategy). Indeed, let us first call w_a to check whether player 1 can exclude his/her least-preferred outcome. If yes (resp. no), from now on let us only consider subsets of outcomes excluding (resp. including) the least-preferred outcome of player 1. Then let us call w_a and check whether player 1 can exclude his/her second-least-preferred outcome, and so on. This procedure calls w_a at most n times to determine the $<_1^{\mathcal{P}}$-greatest subset of outcomes that player 1 can enforce. Then let us call w_s once to determine a strategy of player 1 that enforces this set, and once to determine a strategy of player 2 that excludes the $<_1^{\mathcal{P}}$-successor of the $<_1^{\mathcal{P}}$-greatest subset.

When considering infinitely many outcomes, there may not exist a maximal subset that a given agent can enforce. Nonetheless, the lemma above and the following two lemmas will be combined several times to prove the theorem by transfinite induction on (the order types of the inverses of) the preferences.

Lemma 7 below relies on the remark that if an agent can exclude a lower/downward interval of least-preferred outcomes, no Nash equilibrium will ever yield such outcomes. So, the excludable least-preferred outcomes may just be merged into one single worst outcome of the agent, and become the best outcome of the opponent: indeed, this reduction does not create any Nash equilibrium but yields in many cases a smaller game in terms of outcomes and especially of preferences, thus enabling a step in the transfinite induction. Lemma 7 is named after the well-known elimination of dominated strategies (see, *e.g.*, [6]), which simplifies a game through its set of strategies only, in order to suggest that both procedures may complement each other nicely. (Although not in this article.)

Lemma 7 (Elimination of dominated outcomes) *Let $g = \langle\{1,2\}, S_1 \times S_2, O, v, <_1, <_2\rangle$ be a two-player game in normal form with strict linear preferences. Let $e \in S_1$ and $o \in O$ and assume that $o <_1 v(e, s_2)$ for all $s_2 \in S_2$. Let $g' := \langle\{1,2\}, S_1 \times S_2, O', v', <'_1, <'_2\rangle$, where*

- $O' := \{x \in O \mid o \leq_1 x\}$
- *$v'(s) := v(s)$ if $v(s) \in O'$ and $v'(s) := o$ otherwise.*
- *$<'_1$ is the restrictions of $<_1$ to O'.*
- $x <'_2 y := x \neq o \wedge (x <_2 y \vee y = o)$.

Then every Nash equilibrium of g' is also Nash equilibrium of g.

Moreover, if the inverse relations of $<_1$ and $<_2$ are well-orders, the order types of $(<'_1)^{-1}$ and $(<'_2)^{-1}$ are not greater than those of $(<_1)^{-1}$ and $(<_2)^{-1}$ respectively. Furthermore, if o is not the $<_1$-least element of O, the order type of $(<'_1)^{-1}$ is less than that of $(<_1)^{-1}$, and if o is the $<_2$-least element of infinite O, the order type of $(<'_2)^{-1}$ is less than that of $(<_2)^{-1}$.

Proof Let s be a Nash equilibrium of g'. Since $o <_1 v(e, s_2)$ by assumption about e, the outcome $v(e, s_2)$ is in O' by definition of O', so $o <'_1 v'(e, s_2)$ by definitions of v' and $<'_1$. Since $v'(e, s_2) \leq'_1 v'(s)$ by definition of NE and since $<'_1$ is also strict linear, we have $o <'_1 v'(s)$, so $v'(s) \in O' - \{o\}$ and $v(s) = v'(s)$ by definitions of O' and v'.

Now let us prove by contradiction that both players are stable w.r.t. s and g. If $v(s) <_1 v(x, s_2)$ for some $x \in S_1$, then $v(x, s_2) \in O'$ since $v(s) \in O'$ and by definition of O', so $v'(s) <'_1 v'(x, s_2)$ by definitions of v' and $<'_1$, which contradicts s being an NE of g'. If $v(s) <_2 v(s_1, y)$ for some $y \in S_2$, then $v'(s) = o$ since $v'(s_1, y) \leq'_2 v'(s)$ (by definition of NE) and by definition of $<'_2$, which is also a contradiction. $\square$

In some cases of the transfinite-inductive proof of the theorem, Lemma 7 above cannot be applied. Then, one may invoke the lemma below, *i.e.* the contraposition of its bottom-top implication. For instance, when no agent is able to exclude their least-preferred outcomes by enforcing (finite) sets, *i.e* sets A and B in Lemma 8, one can built a set C that contradicts Condition 1 of Lemma 6.

Lemma 8 *Let $f : \mathbb{N}^2 \to \mathbb{N}$. The following two propositions are equivalent.*

1. *There exists a subset of the naturals that intersects each $f(n, \mathbb{N})$ and whose complement intersects each $f(\mathbb{N}, n)$, where $f(n, \mathbb{N}) := \{f(n, k) \mid k \in \mathbb{N}\}$.*

2. *There exist A and B disjoint subsets of the naturals such that either $f(n, \mathbb{N})$ and A overlap or $f(n, \mathbb{N})\backslash(A \cup B)$ is infinite, and likewise, either $f(\mathbb{N}, n)$ and B overlap or $f(\mathbb{N}, n)\backslash(A \cup B)$ is infinite.*

The statement is formalised below.

$$\begin{array}{c} \forall f : \mathbb{N}^2 \to \mathbb{N}, \\ \exists C \subseteq \mathbb{N}, \forall n \in \mathbb{N}, \quad f(n, \mathbb{N}) \cap C \neq \emptyset \quad \wedge \quad f(\mathbb{N}, n) \cap \mathbb{N}\backslash C \neq \emptyset \\ \Updownarrow \\ \exists A, B \subseteq \mathbb{N}, A \cap B = \emptyset \quad \wedge \quad \forall n \in \mathbb{N}, \quad (f(n, \mathbb{N}) \cap A \neq \emptyset \quad \vee \quad |f(n, \mathbb{N})\backslash(A \cup B)| = \aleph_0) \quad \wedge \\ \qquad\qquad\qquad\qquad (f(\mathbb{N}, n) \cap B \neq \emptyset \quad \vee \quad |f(\mathbb{N}, n)\backslash(A \cup B)| = \aleph_0) \end{array}$$

Proof For the top-bottom implication $A := C$ and $B := \mathbb{N}\backslash C$ witness the claim. To prove the bottom-top implication let us define two sequences of subsets of $\mathbb{N}$ as follows, by mutual induction.

$$\begin{array}{rcl} X_0 & := & A \\ Y_0 & := & B \\ X_{n+1} & := & X_n \cup \{min(f(n, \mathbb{N})\backslash(X_n \cup Y_n))\} \text{ if } f(n, \mathbb{N}) \cap A = \emptyset, \text{ otherwise } X_{n+1} := X_n \\ Y_{n+1} & := & Y_n \cup \{min(f(\mathbb{N}, n)\backslash(X_{n+1} \cup Y_n))\} \text{ if } f(\mathbb{N}, n) \cap B = \emptyset, \text{ otherwise } Y_{n+1} := Y_n \end{array}$$

The inductive steps above are well-defined by the assumed disjunctions and since the $X_n\backslash A$ and $Y_n\backslash B$ are finite by construction. It is provable by induction on n that X_n and Y_n are disjoint for all n, and so are $X := \bigcup_{n=0}^{\infty} X_n$ and $Y := \bigcup_{n=0}^{\infty} Y_n$. Now note that that $C := X$ witnesses the claim since X_{n+1} (resp. Y_{n+1}) intersects $f(n, \mathbb{N})$ (resp. $f(\mathbb{N}, n)$) by construction. $\square$

The proof of Theorem 9 starts with a case splitting on Condition 2. The first case is exactly Lemma 6; the second case is reduced to Lemma 6; and the third case, which involves countably many strategies only, is proved by transfinite induction on the preferences and performs nested case splits. These case splits are solved either by invoking Lemma 7 and the induction hypothesis, or Lemma 8 to contradicts Condition 1, or by exhibiting a Nash equilibrium in the remaining cases. (Note that Theorem 9 proves slightly more than what Section 1 has promised, since it refers to finite upper cones instead of finitely many strategies.)

Theorem 9 (Equilibrium transfer) *Let $\langle\{1,2\}, S_1 \times S_2, O, v, \prec_1, \prec_2\rangle$ be a two-player game in normal form, let $R_1 \subseteq S_1$ and $R_2 \subseteq S_2$, and let us assume the following:*

1. $\forall P \subseteq O,\ (\exists s_1 \in R_1,\ v(s_1, S_2) \subseteq P) \vee (\exists s_2 \in R_2,\ v(S_1, s_2) \subseteq O \backslash P)$
   That is, for every subset of outcomes, either player 1 can enforce it via some strategy in R_1 or player 2 can exclude it via some strategy in R_2.
2. *One of the following assertions holds:*
   - *The preferences are acyclic and one player i can enforce a finite $\prec_i$-upper cone.*
   - *The preferences have finite height.*
   - *S_1 and S_2 are countable and the inverses of the preferences are well-founded.*

Then the game $\langle\{1,2\}, S_1 \times S_2, O, v, \prec_1, \prec_2\rangle$ has a Nash equilibrium in $R_1 \times R_2$.

Proof The first case is proved by Lemma 6. For the second case, let $n \in \mathbb{N}$ bound the height of both $\prec_1$ and $\prec_2$, and let $\rho_1 : O \to \{0, \dots, n-1\}$ and $\rho_2 : O \to \{0, \dots, n-1\}$ be corresponding rank functions, that is, $x \prec_i y$ implies $\rho_i(x) < \rho_i(y)$. Consider the game $\langle\{1,2\}, S_1 \times S_2, \{0, \dots, n-1\}^2, (\rho_1 \circ v, \rho_2 \circ v), \prec'_1, \prec'_2\rangle$ where $(i,j) \prec'_1 (k,l)$ iff $i < k$ and $(i,j) \prec'_2 (k,l)$ iff $j < l$. By Lemma 6 the derived game has a Nash equilibrium, which happens to be a Nash equilibrium for the original game too, by property of ρ_i.

It suffices to prove the statement for well-ordered preferences. Also note that if one player i can enforce a subset of outcome (*via* some strategy in S_i), he/she can enforce it *via* some strategy in R_i, since the opponent cannot exclude it and by assumption (3). Let us define a well-founded binary relation over pairs of ordinals, as follows: $(\alpha, \beta) \prec (\gamma, \delta) := (\alpha < \gamma \wedge \beta \leq \delta) \vee (\alpha \leq \gamma \wedge \beta < \delta)$, where $<$ is the usual well-order over ordinals, and let us proceed by induction (w.r.t $\prec$) on the pairs of order types that correspond to the inverses of the preferences.

If one order type is finite, it suffices to invoke Lemma 6, so from now on both order types are infinite.

If one player can exclude two or more of his/her least-preferred outcomes, Lemma 7 and the induction hypothesis prove the claim, so from now on no player can exclude such a downward interval. Especially, at least one order type is not a limit ordinal.

Let us yet consider the case when one order type, say for player 2, is a limit ordinal. So player 2 cannot enforce any finite set, otherwise excluding an infinite $\prec_2$-downward interval; so player 1 can enforce some finite set, by assumption (3) and (contraposition of) Lemma 8 instantiated with empty A and B. Let $x_0 \prec_1 \cdots \prec_1 x_n$ be the $n+1$ least preferred outcomes of player 1. Since player 2 cannot enforce $\{x_0, \dots, x_n\}$, player 1 can exclude it, so $n = 0$ and the order type of $(\prec_1)^{-1}$ is a limit ordinal plus one. Therefore every finite set that player 1 can enforce contains x_0. Instantiating (contraposition of) Lemma 8 with $A := \{x_0\}$ and $B := \emptyset$ contradicts assumption (3), so from now on both order types are not limit ordinals. Let x and y be the $\prec_1$-least and $\prec_2$-least outcomes respectively.

If players 1 and 2 cannot exclude x and y respectively, by assumption they can enforce y and x *via* some strategies s_1 and s_2 respectively, so $x = y$ and (s_1, s_2) is a Nash equilibrium. From now on one player can exclude his/her least preferred outcome, e.g. player 1 can exclude x.

If $x = y$, Lemma 7 and the induction hypothesis prove the claim, so from now on $x \neq y$.

If player 2 cannot exclude y, player 1 can enforce y *via* some strategy s_1 (but cannot exclude any proper $\prec_1$-downward interval), so y is the second $\prec_1$-least outcome. Therefore player 2 can enforce $\{x, y\}$ *via* some strategy s_2, and (s_1, s_2) is a Nash equilibrium. From now on player 2 can exclude y.

If y (resp. x) is not the preferred outcome of player 1 (resp. 2), Lemma 7 and the induction hypothesis prove the claim. From now on y (resp. x) is the preferred outcome of player 1 (resp. 2).

If both order types are limit ordinals plus one, every finite set enforceable by players 1 (resp. 2) contains x (resp. y). Instantiating (contraposition of) Lemma 8 with $A := \{x\}$ and $B := \{y\}$ contradicts assumption (3), so from now on one player, e.g., player 1 has a second least-preferred outcome z, and $z \neq y$.

Player 2 and 1 can enforce $\{x, z\}$ and the interval $[y, z]$ *via* some strategies s_2 and s_1, respectively, so (s_1, s_2) is a Nash equilibrium. $\square$

3 Applications of the equilibrium-transfer theorem

Section 3.1 generalises Martin's Theorem on Borel determinacy, from descriptive set theory, and Section 3.2 generalises two determinacy results from theoretical informatics, namely positional determinacy of parity games and finite-memory determinacy of Müller games.

3.1 Generalisation of Borel Determinacy

An infinite two-player alternate game consists of two players that play alternately and infinitely many times. In addition, the same non-empty set of choices C is available at each stage, so the first player picks an element in C, then the second player picks an element in C, then the first player picks an element in C again, and so on. The underlying structure of such a game is a leafless and uniform rooted tree. Moreover, each infinite sequence of choices is mapped to some outcome and both players have preferences over the outcomes. In a Gale-Stewart game [3] there are two outcomes only, each being preferred by one player.

Definition 10 (Infinite two-player alternate games and strategies) *An infinite 2-player alternate game is an object $\langle C, O, v, \prec_1, \prec_2 \rangle$ complying with the following:*

- *C is a non-empty set (of choices).*
- *O is a non-empty set (of possible outcomes of the game).*
- *$v : C^\omega \to O$ (uses outcomes to value the infinite sequences of choice).*
- *$\prec_1$ and $\prec_2$ are binary relations over O (modelling the preference of player 1 and 2 respectively).*

A function of type $C^{2*} \to C$ (resp. $C^{2*+1} \to C$) is a strategy for player 1 (resp. 2).

A strategy of a player tells what he/she would play at each node of the game. Since the tree has a uniform structure, it is convenient to represent a strategy of the first (resp. second) player by a function of type $C^{2*} \to C$ (resp. $C^{2*+1} \to C$), where C^{2*} represents the finite sequences on C of even (resp. odd) length. When both players have chosen their individual strategies, their combination induces a unique play, *i.e.* a unique infinite sequence of choices.

Definition 11 (Induced play) *Given a game $g = \langle C, W\rangle$, and $s_1 : C^{2*} \to C$, and $s_2 : C^{2*+1} \to C$, let us define $p(s_1, s_2)$ through its prefixes, inductively as below, where $p_{<n}$ is the prefix of p of length n and the symbol $\cdot$ represents concatenation.*

- $p(s_1, s_2)_{<2n+1} := p(s_1, s_2)_{<2n} \cdot s_1(p(s_1, s_2)_{<2n})$
- $p(s_1, s_2)_{<2n+2} := p(s_1, s_2)_{<2n+1} \cdot s_2(p(s_1, s_2)_{<2n+1})$

An infinite two-player alternate game $\langle C, O, v, \prec_1, \prec_2\rangle$ may be translated into a game in normal form $\langle\{1, 2\}, (C^{2*} \to C) \times (C^{2*+1} \to C), O, v \circ p, \prec_1, \prec_2\rangle$, which provides the infinite two-player alternate game framework with a natural notion of Nash equilibrium.

Before stating Borel determinacy below, let us recall that a subset of a topological space X is called Borel if it belongs to the smallest collection of subsets of X which contains all the open sets and is closed under complementation and countable unions.

Theorem 12 (Martin [7], [8]) *Let C be a non-empty set and $v : C^\omega \to \{win, lose\}$ be such that $v^{-1}(win)$ is a Borel set of C^ω for the product topology. The game below is determined. $\langle\{1, 2\}, (C^{2*} \to C) \times (C^{2*+1} \to C), \{win, lose\}, v \circ p, \{(lose, win)\}, \{(win, lose)\}\rangle$.*

The generalisation of Martin's theorem below is a straightforward corollary of both Martin's Theorem itself and the equilibrium transfer theorem.

Corollary 13 *Let $\langle C, O, v, \prec_1, \prec_2\rangle$ be an infinite 2-player alternate game and assume the following three conditions.*

- *O is countable.*
- *$\prec_1$ and $\prec_2$ have finite height.*
- *For all $o \in O$, the pre-image $v^{-1}(o)$ is Borel.*

Then the game $\langle C, O, v, \prec_1, \prec_2\rangle$ has a Nash equilibrium.

Proof Thanks to the above-mentioned embedding (of alternate games into games in normal form) and the finite height assumption, it suffices to check Condition 1 from Theorem 9. Let $P \subseteq O$ and assume that for all s_1 in $C^{2*} \to C$, there exists s_2 in $C^{2*+1} \to C$ such that $v(s_1, s_2) \notin P$. So player 1 has no winning strategy for the game $\langle\{1, 2\}, (C^{2*} \to C) \times (C^{2*+1} \to C), \{win, lose\}, v \circ p, \{(lose, win)\}, \{(win, lose)\}\rangle$ where $v^{-1}(win) := P$. Since $v(P)$ is a countable union of Borel sets, by assumption, Martin's Theorem implies that player 2 has a winning strategy. So there exists s_2 in $C^{2*+1} \to C$ such that $v(s_1, s_2) \notin P$ for all s_1 in $C^{2*} \to C$, which proves Condition 1 from Theorem 9 . $\square$

3.2 Generalisations on parity games and Müller games

These infinite two-agent games are played on graphs. Unfolding these graphs yields infinite trees, so Borel determinacy may imply determinacy for these games. However, Borel determinacy does not say whether there exist simple winning strategies, or more generally winning strategies satisfying some predicate. So, the results that are generalised in Section 3.2 are not mere corollaries of Borel determinacy. The definitions below rephrase, *e.g.*, [4].

Definition 14 (Arena and strategy) *An arena is an object $\langle V, V', E, C, \gamma\rangle$ complying with the following:*

- *V is a non-empty set of vertexes.*

- *$V' \subseteq V$ are the vertexes owned by player 1.*
- *$E \subseteq V \times V$ are the edges of a sink-free graph, i.e. $\forall x \in V, xE \neq \emptyset$.*
- *C is a non-empty set of colours.*
- *$\gamma : V \to C$ assigns a colour to each of the vertexes.*

A strategy of player 1 (resp. 2) is a function of (dependent) type $V^* \to \forall v \in V', vE$ (resp. $V^* \to \forall v \in V \backslash V', vE$). A strategy profile is a function of (dependent) type $V^* \to \forall v \in V, vE$. The combination (s_1, s_2) of a strategy s_1 for players 1 and s_2 for player 2 amounts to a strategy profile, which, when starting from a given vertex, induces a unique infinite sequence of colours $\Gamma(s_1, s_2) \in C^{\mathbb{N}}$.

Definition 15 (multi-outcome priority/Müller games) *A multi-outcome priority (resp. Müller) game is an object $\langle \mathcal{G}, O, e, \prec_1, \prec_2 \rangle$ complying with the following:*

- *$\mathcal{G}$ is an arena where $C = \mathbb{N}$ (resp. finite arena) as defined above.*
- *O is a non empty set of outcomes.*
- *$e : \mathbb{N} \cup \{\perp\} \to O$ (resp. $e : \mathcal{P}(C) \to O$)*
- *$\prec_1$ and $\prec_2$ are binary relations over O, the preferences.*

For every infinite sequence of colours Γ, let $cl(\Gamma)$ be its cluster set, i.e. the set of the colours occurring infinitely often in Γ. The outcome induced by a sequence of colours $\Gamma \in C^{\mathbb{N}}$, i.e. by a strategy profile and a starting vertex, is:

- *For Müller, $e \circ cl(\Gamma)$.*
- *For priority, $e \circ min \circ cl(\Gamma)$ if $cl(\Gamma) \neq \emptyset$, otherwise $e(\perp)$.*

Note that setting $O := \{win, lose\}$ and $lose \prec_1 win$ and $win \prec_2 lose$ (plus $e(2n) := e(\perp) := win$ and $e(2n+1) := lose$ for a multi-outcome priority game) in the definition above yields a parity (resp. Müller) game, "up to isomorphism".

It was proved in [5] that Müller games are determined through finite-memory strategies and in [4] that parity games with priorities in $\mathbb{N}$ are positionally determined. Since these are determinacy results, let us extend them to multi-outcome settings below. (Note that one need not know what positional or finite-memory means.)

Corollary 16 *Every multi-outcome Müller game (initiated with a starting vertex) with acyclic preferences has a finite-memory Nash equilibrium.*

Proof Let $\langle V, V', E, C, \gamma, O, e, \prec_1, \prec_2 \rangle$ be a multi-outcome Müller game with acyclic preferences, and let $v_0 \in V$ be the starting vertex. Since it is naturally embedded into a game in normal form (as far as NE are concerned), it suffices to invoke Lemma 6 to prove the claim, where Conditions 2 and 3 are fulfilled by assumption and finiteness of the game respectively. Let us prove below that Condition 1 is also fulfilled.

Let $P \subseteq O$ and assume that $e \circ cl \circ \Gamma(s_1, S_2) \nsubseteq P$ for all s_1 in R_1, the finite-memory strategies of player 1. Let $\langle V, V', E, C, \gamma, \{win, lose\}, e', \{(lose, win)\}, \{(win, lose)\} \rangle$ be a derived Müller game (starting with v_0) where $e'(F) := win$ iff $e(F) \in P$. Let us rephrase the above assumption: $e' \circ cl \circ \Gamma(s_1, S_2) \nsubseteq \{win\}$ for all s_1 in R_1, that is, player 1 has no winning strategy in the derived game, so by [5] player 2 has a winning strategy in R_2, the finite-memory strategies of player 2. So let s_2 be in R_2 such that $e' \circ cl \circ \Gamma(s_1, s_2) = lose$ for all s_1 in R_1, therefore $e \circ cl \circ \Gamma(S_1, s_2) \subseteq O \backslash P$. $\square$

Corollary 17 *Every multi-outcome priority game where preferences have finite height has a positional Nash equilibrium.*

Proof Let $\langle V, V', E, \mathbb{N}, \gamma, O, e, \prec_1, \prec_2\rangle$ be a multi-outcome priority game where the preferences have finite height, and let $v_0 \in V$ be the starting vertex. Since it is naturally embedded into a game in normal form (as far as NE are concerned), and thanks to the finite height assumption, it suffices to check Condition 1 from Theorem 9 to prove the claim.

Let $P \subseteq O$ and assume that $e(\perp) \in P$. (If not, swap players 1 and 2 and subsets P and $O\backslash P$ in the remainder of the proof.) Let us further assume that for all s_1 in R_1, the positional strategies of player 1, there exists s_2 in S_2, the strategies of player 2, such that $cl \circ \Gamma(s_1, s_2) \neq \emptyset$ and $e \circ min \circ cl \circ \Gamma(s_1, s_2) \notin P$. Let $\langle V, V', E, C, \gamma', \{win, lose\}, e, \{(lose, win)\}, \{(win, lose)\}\rangle$ be a derived priority game (starting with v_0) where $\gamma'(v) := 2\gamma(v)$ if $e \circ \gamma(v) \in P$ and $\gamma'(v) := 2\gamma(v)+1$ if $e \circ \gamma(v) \notin P$, which yields a new function Γ' from strategy profiles to infinite sequences of naturals. Let us rephrase the above assumption: for all s_1 in R_1, there exists s_2 in S_2 such that $cl \circ \Gamma'(s_1, s_2) \neq \emptyset$ and $e \circ min \circ cl \circ \Gamma'(s_1, s_2) = lose$, that is, player 1 has no winning strategy in the derived game, so by [4] player 2 has a winning strategy in R_2. So let s_2 be in R_2 such that $e \circ min \circ cl \circ \Gamma'(s_1, s_2) = lose$ for all s_1 in R_1, therefore $e \circ min \circ cl \circ \Gamma(S_1, s_2) \subseteq O\backslash P$. $\square$

Unfortunately I could not invoke in Corollary 17 the countable-well-founded version of Theorem 9 since I do not know whether the winning strategy from [4] can always be chosen in a given countable set, *e.g.* whether a computable winning strategy always exists.

4 Limitations of transfer possibilities

Proposition 18 below shows that Condition 2 of Theorem 9 is difficult to weaken in general, then Proposition 20 rules out a reasonable three-player version of equilibrium transfer.

In the game below, one preference has finite height (but not both preferences, otherwise equilibrium transfer would hold), the other preference has well-founded inverse, one player has countably many strategies (but not both players, otherwise equilibrium transfer would hold), but equilibrium transfer does not hold.

Proposition 18 *Let I (resp. C) be the infinite (resp. cofinite) subsets of the naturals. Consider the game in normal form $\langle \{1,2\}, (C \cup \{\alpha, \beta\}) \times (I \cup \{\alpha, \beta\}), \mathbb{N} \cup \{a, b\}, v, \prec_1, \prec_2\rangle$ where $0 \prec_1 a$ and $b \prec_1 a$ and $n+1 \prec_1 n$ for all $n \in \mathbb{N}$, and $a \prec_2 b$ and $n \prec_2 b$ for all $n \in \mathbb{N}$, and where v is as below and min refers to the usual order over $\mathbb{N}$.*

$$
\begin{array}{llll}
v: & (C \cup \{\alpha,\beta\}) \times (I \cup \{\alpha,\beta\}) & \to & \mathbb{N} \cup \{a,b\} \\
 & (X,Y) & \mapsto & a \quad\quad\quad\quad\quad\quad\quad\quad\quad \textit{if } X = Y \in \{\alpha,\beta\} \\
 & & & b \quad\quad\quad\quad\quad\quad\quad\quad\quad \textit{if } \{X,Y\} = \{\alpha,\beta\} \\
 & & & min(X) \quad\quad\quad\quad\quad\quad\;\; \textit{if } (X,Y) \in C \times \{\alpha,\beta\} \\
 & & & min(Y) \quad\quad\quad\quad\quad\quad\;\; \textit{if } (X,Y) \in \{\alpha,\beta\} \times I \\
 & & & min(X \cap Y - \{min\, X \cap Y\}) \quad \textit{if } X, Y \notin \{\alpha,\beta\}
\end{array}
$$

The game has no Nash equilibrium although the following formula holds:
$\forall P \subseteq \mathbb{N} \cup \{a,b\}, (\exists s_1 \in S_1,\, v(s_1, S_2) \subseteq P) \vee (\exists s_2 \in S_2,\, v(S_1, s_2) \subseteq O\backslash P)$
Also, the result still holds when swapping I and C.

Proof First, let $P \subseteq \mathbb{N} \cup \{a,b\}$. If $P \cap \mathbb{N}$ is cofinite, player 1 can enforce it (by playing it) since $v(X,Y)$ is in X for all $X \in C$ and $Y \in I \cup \{\alpha,\beta\}$, by definition of v; so a fortiori player 1 can enforce P. If $P \cap \mathbb{N}$ is not cofinite, $\mathbb{N}\backslash P$ is infinite, so player 2 can enforce it, and a fortiori $(\mathbb{N} \cup \{a,b\})\backslash P$.

Second, let us show below that the game has no Nash equilibrium.

- $v(\alpha,\alpha) <_2 v(\alpha,\beta)$ and $v(\beta,\beta) <_2 v(\beta,\alpha)$.

- $v(\alpha, \beta) <_1 v(\beta, \beta)$ and $v(\beta, \alpha) <_1 v(\alpha, \alpha)$.
- If $X \in C$ and $Y \in \{\alpha, \beta\}$ then $v(X, Y) <_1 v(Y, Y) = a$.
- If $X \in \{\alpha, \beta\}$ and $Y \in I$ then $v(X, Y) <_2 v(X, X') = b$, where $\{X, X'\} = \{\alpha, \beta\}$.
- If $X, Y \notin \{\alpha, \beta\}$ then $v(X, Y) <_1 v(\alpha, Y)$ since $min(X \cap Y - \{minX \cap Y\}) < min(Y)$.

$\square$

The equilibrium-transfer theorem considers two-player games only, which raises the issue of the existence of a three-player version of the theorem. The determinacy condition of the equilibrium-transfer theorem, *i.e.* Condition 1 of Theorem 9, may be rephrased using maps from the set of outcomes of the original game into the set of outcomes $\{(1,0),(0,1)\}$. In the case of three players, a natural extension of this condition would require existence of Nash equilibrium when arbitrarily mapping the set of outcomes of the original game into, *e.g.*, the set $\{(1,0,0),(0,1,0),(0,0,1)\}$. A stronger condition, *i.e.* more likely to lead to a three-player version of the equilibrium-transfer theorem, would map the outcomes into $\{0,1\}^3$. Even stronger conditions would map the outcomes into $\{0,1,2,\dots n\}^3$ for some $n \geq 2$. Nonetheless, even the last and strongest condition fails to guarantee equilibrium transfer. The remark below proves the simpler case $\{0,1\}^3$ and Proposition 20 afterwards generalises it to $\{0,1,2,\dots n\}^3$.

Remark 19 *Let a, b and c be three players, let l and r be two available strategies, let x, y and z be three possible outcomes, define three transitive preferences through $z <_a y <_a x$ and $x <_b y <_b z$ and $<_c := <_b$, and define an outcome function as follows: $v(l,l,l) = v(l,r,l) = v(r,l,l) := y$ and $v(r,r,l) = v(l,l,r) = v(l,r,r) := z$ and $v(r,l,r) = v(r,r,r) = x$. See the graphical representation below, where one of players a, b, and c choosing l yields top row, left columns, and left array respectively.*

y	y
z	y

z	z
x	x

- *The game $\langle\{a,b,c\},\{l,r\}^3,\{x,y,z\},v,(<_d)_{d\in\{1,2,3\}}\rangle$ has no Nash equilibrium.*
- *Let $\{0,1\}^3$ be an alternative set of outcomes and for $i,j,k,n \in \{0,1\}$ let $(0,i,j) \prec_a (1,k,n)$ and $(i,0,j) \prec_b (k,1,n)$ and $(i,j,0) \prec_c (k,n,1)$. Then for all $f : \{x,y,z\} \to \{0,1\}^3$ the game $\langle\{a,b,c\},\{l,r\}^3,\{0,1\}^3,f\circ v,(\prec_d)_{d\in\{a,b,c\}}\rangle$ has a Nash equilibrium.*

Proof The original game has no Nash equilibrium, by construction. Let $f : \{x,y,z\} \to \{0,1\}^3$ and from now on let us consider the modified game only, assume that it has no Nash equilibrium, and draw a contradiction. Both players a and b are stable *w.r.t.* the strategy profile (l,l,l) since $v(l,l,l) = v(l,r,l) = v(r,l,l) = y$ by construction, so $f(y) = f\circ v(l,l,l) \prec_c f\circ v(l,l,r) = f(z)$. So player c is stable *w.r.t.* the strategy profile (l,l,r), and so is player b since $v(l,l,r) = v(l,r,r) = z$, therefore $f(z) = f\circ v(l,l,r) \prec_a f\circ v(r,l,r) = f(x)$. So player a is stable *w.r.t.* the strategy profile (r,r,r), and so is player b since $v(r,l,r) = v(r,r,r) = x$, so $f(x) = f\circ v(r,r,r) \prec_c f\circ v(r,r,l) = f(y)$. Therefore $f(x) \prec_c f(y) \prec_c f(z)$, contradiction since $\prec_c$ has no chain of length 2. $\square$

Proposition 20 *For every natural $2 \leq n$ there exists a finite three-player game in normal form that complies with the following:*

- *The preferences are linear orders.*
- *The game has no Nash equilibrium.*
- *Replacing the preferences with preferences that have no chain of length n yields a game with a Nash equilibrium.*

Proof Let $2 \leq n$ be a natural, let $<_b$ and $<_c$ be the restrictions of the usual order over the naturals to $\{0,\dots,n\}$, and let $<_a := <_b^{-1}$, that is, $n <_a n-1 <_a \dots <_a 1 <_a 0$. Let us define the outcome function $v : \{1,\dots,n\}^3 \to \{0,\dots,n\}$ below.

- For $1 \leq i \leq n$ let $v(n,i,n) := 0$.
- For $1 \leq i < n$ let $v(i,i,n) := i+1$.
- Otherwise let $v(\cdot,\cdot,n)$ return n.
- For $1 \leq i < n$ let $v(n,1,i) := n$.
- For $1 < i < n$ and $1 \leq j \leq n$ let $v(i,j,i) := v(j,i,i) := i$.
- Otherwise let v return 1.

The game $\langle \{a,b,c\}, \{1,2,3,4\}^3, \{0,1,2,3,4\}, v, (<_d)_{d\in\{a,b,c\}}\rangle$ is represented below, where player a chooses the row, b the column, and c the array.

1	1	1	1
1	1	1	1
1	1	1	1
4	1	1	1

1	2	1	1
2	2	2	2
1	2	1	1
4	2	1	1

1	1	3	1
1	1	3	1
3	3	3	3
4	1	3	1

2	4	4	4
4	3	4	4
4	4	4	4
0	0	0	0

Let us show that the game $\langle \{a,b,c\}, \{1,\dots,n\}^3, \{0,\dots,n\}, v, (<_d)_{d\in\{a,b,c\}}\rangle$ witnesses the claim. First, the preferences are linear orders indeed. Second, let us show that there is no Nash equilibrium by case-splitting below.

- If $i,k \neq n$, then $v(i,j,k) <_c v(i,j,n)$.
- If $i \neq n$, then $v(i,j,n) <_a 0 = v(n,j,n)$.
- $v(n,j,n) = 0 <_c v(n,j,1)$.
- $v(n,1,1) = n \prec_a 1 = v(1,1,1)$ and if $j \neq 1$, then $v(n,j,1) = 1 \prec_b n = v(n,1,1)$.
- If $j \neq 1$ and $1 < k < n$, then $v(n,j,k) <_b n = v(n,1,k)$.
- If $1 < k < n$, then $v(n,1,k) = n <_a 1 = v(1,1,k)$.

Third by contraposition, let $\prec_a$, $\prec_b$ and $\prec_c$ be arbitrary acyclic preferences, assume that the game $\langle \{a,b,c\}, \{1,\dots,n\}^3, \{0,\dots,n\}, v, (\prec_d)_{d\in\{1,2,3\}}\rangle$ has no Nash equilibrium, and let us prove that $\prec_c = <_c$, thus contradicting the assumption on the chains. If $i \neq n$, by construction $v(i,i,i) = i = v(j,i,i) = v(i,j,i)$ and $v(i,i,j) \in \{1,i,i+1\}$. By assumption there is no Nash equilibrium and the preferences are acyclic, so $i \prec_c 1$ or $i \prec_c i+1$ if $1 \leq i < n$, so $1 \prec_c 2$, and it is provable by induction that $i \prec_c i+1$ for all $1 \leq i < n$. Now it suffices to prove $0 \prec_c 1$ to conclude. By assumption the strategy profile $(n-1,n,n)$ is not a Nash equilibrium, so $v(n-1,n,n) = n \prec_a 0 = v(n,n,n)$ since $v(i,n,n) = v(n-1,j,n) = n$ for $i \neq n$, since $v(n-1,n,k) \in \{1,n-1,n\}$, and by assumption of acyclic preferences. Now, the profile (n,n,n) is not a Nash equilibrium either. Since $v(n,j,n) = 0$, since $v(i,n,n) = n$ if $i \neq n$, and since $n \prec_a 0$, the players a and b are sable. Since $v(n,n,k) \in \{0,1\}$, we must have $v(n,n,n) = 0 \prec_c 1$. $\square$

In addition to strengthening the determinacy condition to obtain a three-player version of the equilibrium-transfer theorem, one may also require a zero-sum condition on the preferences, since transfer of equilibrium is easier to prove for zero-sum games. However, defining $z(n) := (-2n, n, n)$ and using the outcome function $z \circ v$ instead of v for the game in the proof of Proposition 20 above yields a zero-sum game that is also a counterexample to the weakest three-player version of equilibrium transfer.

5 Conclusion

This article has shown that every determinacy result over a given game structure is transferable into existence of multi-outcome Nash equilibrium over the same game structure in three cases: when one player can enforce a finite upper cone of outcomes, or when both preferences have finite height, or when strategies are countable and preferences have well-founded inverses. As examples of application of the equilibrium-transfer theorem, Borel determinacy, finite-memory determinacy of Müller games, and positional determinacy of parity games have been generalised. In each case, the application involves the design of a multi-outcome version of the determined games, and some reasonable bookkeeping is required to check Condition 1 of Theorem 9 since the games at hand are not in normal form. The three examples are meant to give an idea of the various possible applications of the theorem: the first example comes from descriptive set theory, the others from theoretical informatics; the first example does not use any additional predicate, the second example discriminates finite-memory strategies, and the third example discriminates positional strategies; unfortunately all examples involve finite-height preferences only.

Unexpectedly, the finite-height condition of Theorem 9 leads to an interesting phenomenon about preferences: linear orders do not account for partial orders. Indeed the remark below considers two finite-height preferences, therefore fulfilling Condition 2 of Theorem 9, yet for all possible linear extensions of these preferences, equilibrium transfer does not hold!

Remark 21 *Let $\prec_1$ and $\prec_2$ be two binary relations over $\mathbb{N}$ that are defined by $2n \prec_1 2n+1$ and $\prec_2 := \prec_1^{-1}$. For all $<_1$ and $<_2$ linear extensions of $\prec_1$ and $\prec_2$ respectively, there exists a game satisfying Condition 1 of Theorem 9, but without Nash equilibrium.*

Proof If the inverse of $<_i$ is not a well-order, the game $\langle \{i\}, \mathbb{N}, \mathbb{N}, id, <_i \rangle$ has no Nash equilibrium although determinacy holds, so let us assume that the inverses of $<_1$ and $<_2$ are well-orders. Since the sequence $(2n)_{n\in\mathbb{N}}$ has no $<_1$-increasing subsequence, it has a $<_1$-decreasing subsequence $(2\phi(n))_{n\in\mathbb{N}}$ (as a consequence of Ramsey Theorem). Since the sequence $(2\phi(n))_{n\in\mathbb{N}}$ has has no $<_2$-increasing subsequence, it a $<_2$-decreasing subsequence $(2\phi\circ\psi(n))_{n\in\mathbb{N}}$. By setting $a := 2\phi\circ\psi(0)+1$ and $b := 2\phi \circ \psi(0)$ and $x_n = 2\phi \circ \psi(n+1)$, we obtain a preference structure that includes the one from Proposition 18, so the game built therein witnesses also for the remark at hand. $\square$

Nonetheless, it is often very convenient to consider linearly ordered preferences only, when actually done without loss of generality. The remark above just exemplifies that one ought to be very cautious because a loss of generality may occur.

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References

[1] Emile Borel. La théorie du jeu et les equations intégrales à noyau symétrique. *Comptes Rendus Hebdomadaires des séances de l'académie des Sciences*, 173:1304–1308, 1921.

[2] A. Emerson and C. Jutla. Tree automata, mu-calculus and determinacy. In *32nd IEEE Symp. on Foundations of Computer Science*, page 368377, 1991.

[3] D. Gale and F. M. Stewart. Infinite games with perfect information. *Annals of Math. Studies*, 28:245–266, 1953.

[4] Erich Grädel and Igor Walukiewicz. Positional determinacy of games with infinitely many priorities. *Logical Methods in Computer Science*, 2006.

[5] Y. Gurevich and L. Harrington. Trees, automata, and games. In *STOC82*, page 6065. ACM Press, 1982.

[6] Robert Duncan Luce and Howard Raiffa. *Games and Decisions*. John Wiley and Sons, New York, 1957.

[7] Donald A. Martin. Borel determinacy. *Annals of Mathematics*, 102:363371, 1975.

[8] Donald A. Martin. A purely inductive proof of borel determinacy. In *Recursion theory (Ithaca, N.Y., 1982), Proceedings of Symposia in Pure Mathematics*, volume 42, page 303308. American Mathematical Society, 2006.

[9] A. Mostowski. Games with forbidden positions. Research report 78, University of Gdansk, 1991.

[10] John Nash. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36:48–49, 1950.

[11] John von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior*. Princeton Univ. Press, Princeton, 1944.

[12] Michael Ummels. Rational behaviour and strategy construction in infinite multiplayer games, 2005. Master's thesis supervised by Erich Grädel.

[13] Philip Wolfe. The strict determinateness of certain infinite games. *Pacific Journal of Mathematics*, 5:841847, 1955.