Vol. 33, 1979 437

Affine parts of monads

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HARALD LINDNER

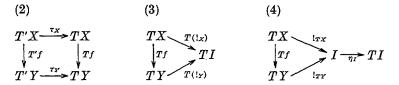
We introduce the notion of the affine part of a monad on a finitely complete category. This generalizes known constructions for algebraic theories. The inclusion-functor of the subcategory of affine monads is coadjoint. The affine part of a monad is characterized in terms of idempotent operations. The affine part of a (cartesian-) monoidal monad is monoidal, hence the corresponding Eilenberg-Moore situation is monoidal, too.

Let \mathscr{V} be a finitely complete category. We choose a terminal object I of \mathscr{V} . For any $X \in |\mathscr{V}|$ we denote by $!_X$ the unique morphism from X to I.

1. Definition and proposition. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad (triple, triad, standard construction) on \mathscr{V} . There is a monad $\mathbb{T}' = (T', \eta', \mu')$ on \mathscr{V} , called the affine part of \mathbb{T} , together with a monad morphism $\tau \colon \mathbb{T}' \to \mathbb{T}$, determined both uniquely up to isomorphism, such that for every $X \in |\mathscr{V}|$ the diagram (1) is an equalizer diagram.

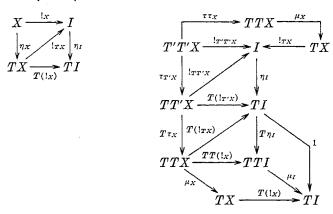
(1)
$$T'X \xrightarrow{\tau_X} TX \xrightarrow{T(!x)} TI$$

Proof. For every $X \in |\mathscr{V}|$ we choose an equalizer $\tau_X \colon T'X \to TX$ of the pair $(T(!_X), \eta_I(!_{TX}))$. There is a unique way of extending the assignment $X \mapsto T'X$ to a functor $T' \colon \mathscr{V} \to \mathscr{V}$, such that $\tau = \{\tau_X \mid X \in |\mathscr{V}|\}$ becomes a natural transformation, i.e. the diagram (2) commutes for every $f \colon X \to Y$ in \mathscr{V} . This follows from the commutative diagrams (3), (4):



If $\tau: T' \to T$ is going to be a morphism of monads $\mathbb{T}' \to \mathbb{T}$, the diagrams (5), (6) must commute for every $X \in |\mathscr{V}|$:

If η' and μ' exist, they are uniquely determined by (5), (6), since τ is a pointwise monomorphism. On the other hand, the following two commutative diagrams imply the existence of η' and μ' :



Since τ is a pointwise monomorphism, the naturality of η' and μ' follows from the naturality of η , μ , and τ . Finally, $\tau \colon \mathbb{T}' \to \mathbb{T}$ is a morphism of monads by the construction of η' and μ' (cf. (5), (6)).

The assignment $\mathbb{T} \mapsto \mathbb{T}'$ is natural: if $\vartheta \colon \mathbb{S} \to \mathbb{T}$ is a morphism of monads, there is a unique morphism of monads $\vartheta' \colon \mathbb{S}' \to \mathbb{T}'$ such that the diagram (7) commutes:

$$\begin{array}{ccc} & & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ &$$

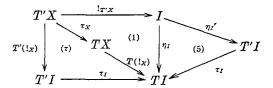
A monad isomorphic to a monad of the form T' with T any monad is called affine.

- **2.** Lemma. Let S be a monad on V. The following are equivalent:
- (i) S is an affine monad,
- (ii) the canonical morphism $\sigma: S' \to S$ is an isomorphism,
- (iii) the diagram (8) is commutative for all $X \in |\mathcal{V}|$.

(8)
$$SX \xrightarrow{!_{SX}} I$$

$$S(!_{X}) \xrightarrow{\eta_{I}} SI$$

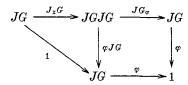
Proof. (iii) \Rightarrow (ii) \Rightarrow (i) are trivial. To prove (i) \Rightarrow (iii) we choose \mathbb{T} with $\mathbb{S} \cong \mathbb{T}'$ and consider the following commutative diagram:



In this way we obtain a functor G from the category $\mathscr{M}(\mathscr{V})$ of monads on \mathscr{V} to the category $\mathscr{A}(\mathscr{V})$ of affine monads on \mathscr{V} (these categories are in general illegitimate, i.e. they "live" in a higher universe).

3. Proposition. The functor $G: \mathcal{M}(\mathcal{V}) \to \mathcal{A}(\mathcal{V})$ is adjoint to the inclusion functor $J: \mathcal{A}(\mathcal{V}) \to \mathcal{M}(\mathcal{V})$.

Proof. The assignment $\mathbb{T}\mapsto (\tau\colon\mathbb{T}'\to\mathbb{T})$ defines a natural transformation $\varphi\colon JG\to 1_{\mathscr{M}(\mathscr{Y})}$. Since J is fully faithful and φJ is an isomorphism (by lemma 2), there is a unique $\chi\colon 1_{\mathscr{M}(\mathscr{Y})}\to GJ$ such that $(\varphi J)(J\chi)=1_J$. The second equation $(G\varphi)(\chi G)=1_G$ is a consequence of the following commutative diagram, because φ is a pointwise equalizer and J is faithful:



From now on we assume \mathscr{V} to be a cartesian closed category $(\mathscr{V}, \otimes, I, \alpha, \lambda, \varrho, \gamma)$, i.e. $\otimes : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ is the productfunctor, I is a terminal object, $\alpha, \lambda, \varrho, \gamma$ are compatible with the projections, etc. We denote the internal Hom-functor of \mathscr{V} by $\mathscr{V}(-, -)$. We proceed to show that our previous constructions can be lifted to the \mathscr{V} -enriched or symmetric monoidal closed level.

Let $\mathbb{T} = (T, \eta, \mu)$ be a \mathscr{V} -monad on \mathscr{V} . The \mathscr{V} -naturality of a natural transformation $\tau \colon T' \to T$ is expressed by the commutativity of the diagram (9):

(9)
$$\begin{array}{ccc}
\mathscr{V}(X,Y) & \xrightarrow{T_{X,Y}} & \mathscr{V}(TX,TY) \\
& & \downarrow^{T_{X,Y}} & \downarrow^{\mathscr{V}(\tau_{X},TY)} \\
\mathscr{V}(T'X,T'Y) & \xrightarrow{\mathscr{V}(T'X,\tau_{Y})} & \mathscr{V}(T'X,TY)
\end{array}$$

The \mathscr{V} -functor $\mathscr{V}(TX, -) \colon \mathscr{V} \to \mathscr{V}$ is adjoint, hence it preserves equalizers. Therefore, the (ordinary) functor $T' \colon \mathscr{V} \to \mathscr{V}$ can be equipped with the structure of a \mathscr{V} -functor making $\tau \colon T' \to T$ a \mathscr{V} -natural transformation, if and only if $\mathscr{V}(\tau_X, TY) \ T_{X,Y}$ equalizes $\mathscr{V}(T'X, T(!_Y))$ and $\mathscr{V}(T'X, \eta_I) \ \mathscr{V}(T'X, !_{TY})$ for all $X, Y \in |\mathscr{V}|$; but this is true, as the following two commutative diagrams prove:

The diagrams for the \mathscr{V} -functoriality of T' and the \mathscr{V} -naturality of η' , μ' follow from (5), (6) and the corresponding diagrams for T, η , μ since τ is a pointwise monomorphism. In this way we obtain a \mathscr{V} -monad \mathbb{T}' , together with a morphism of \mathscr{V} -monads $\tau \colon \mathbb{T}' \to \mathbb{T}$. The assignment $\mathbb{T} \mapsto (\tau \colon \mathbb{T}' \to \mathbb{T})$ belongs to a natural transformation $\widetilde{\varphi} \colon \widetilde{J}\widetilde{G} \to 1_{\widetilde{\mathscr{M}}(\mathscr{V})}$, the counit of an adjunction

(10)
$$\tilde{J} \longrightarrow \tilde{G}$$

where \tilde{J} denotes the inclusion of the category $\tilde{\mathscr{A}}(\mathscr{V})$ of affine \mathscr{V} -monads into the category $\tilde{\mathscr{M}}(\mathscr{V})$ of \mathscr{V} -monads on \mathscr{V} .

We shall now describe the affine part of a \mathscr{V} -monad in terms of (idempotent) operations. If $\mathbb{T}=(T,\eta,\mu)$ is a \mathscr{V} -monad on \mathscr{V} , the object $\mathscr{V}(Y,TX)$ (for $X,Y\in |\mathscr{V}|)$ of \mathscr{V} can be considered as the \mathscr{V} -object of Y-tuples of X-ary operations of \mathbb{T} . Let (A,a) be a \mathbb{T} -algebra, i.e. $a\colon TA\to A$ satisfies $a\eta_A=1_A$ and $a\mu_A=a(Ta)$. The applying of a Y-tuple of X-ary \mathbb{T} -operations to a X-tuple of elements of A, yielding a Y-tuple of elements of A, is an operation on the level of sets, which on the \mathscr{V} -level is described by an "applying"-morphism $W_{Y,X,a}$:

The composing of operations is described on the \mathscr{V} -level by the "composition"-morphism $\varkappa_{Z,Y,X}$ (of the Kleisli- \mathscr{V} -category of \mathbb{T}):

(12)
$$\begin{array}{ccc}
\mathscr{V}(Y,TX) \otimes \mathscr{V}(Z,TY) & \xrightarrow{\varkappa_{Z,Y,X}} \mathscr{V}(Z,TX) \\
\downarrow^{T_{Y,TX} \otimes 1} & \downarrow^{\mathscr{V}(Z,\mu_{X})} \\
\mathscr{V}(TY,TTX) \otimes \mathscr{V}(Z,TY) & \xrightarrow{\mu_{Z,TX,TTX}^{\mathscr{V}}} \mathscr{V}(Z,TTX)
\end{array}$$

The composing of operations is compatible with the applying of operations to

 \mathbb{T} -algebras, i.e. the diagram (13) commutes for every \mathbb{T} -algebra (A,a) and for all $X,Y,Z\in [\mathscr{V}]$:

In fact, the composition morphism $\varkappa_{Z,Y,X}$ is a particular applying morphism, namely W_{Z,Y,μ_X} . The commutativity of the diagram (13) follows by an evident diagram chase.

Now let \mathbb{T}' be the affine part of the monad \mathbb{T} . We will show that for all $Z, Y \in |\mathscr{Y}|$, $\mathscr{V}(Z, T'Y)$ is the subobject of all *idempotent* Z-tuples of Y-ary \mathbb{T} -operations (prop. 4 below). A Z-tuple of Y-ary operations $\omega: I \to \mathscr{V}(Z, TY)$ is called *idempotent* iff it equalizes the pair of morphisms (14) (a not necessarily commutative diagram), where $\lceil \delta_Y \rceil$ denotes the name of δ_Y (diagram (15)):

4. Proposition. Let $\mathbb{T}=(T,\eta,\mu)$ be a \mathscr{V} -monad on \mathscr{V} . For all $Z,Y\in |\mathscr{V}|$, $\mathscr{V}(Z,\tau_Y)\colon \mathscr{V}(Z,T'Y)\to \mathscr{V}(Z,TY)$ is an equalizer of the pair of morphisms (14), i.e. $\mathscr{V}(Z,T'Y)$ is the subobject of $\mathscr{V}(Z,TY)$ of idempotent operations.

Since $\mathcal{V}(Z, -)$: $\mathcal{V} \to \mathcal{V}$ preserves equalizers, this follows from (1) and the following two commutative diagrams (16), (17), the proof of which we leave to the reader:

(16)
$$\begin{array}{c}
\mathscr{V}(Z,TY) \xrightarrow{\mathscr{V}(Z,\mathbb{I}_{TY})} \mathscr{V}(Z,I) \\
\downarrow ! \qquad \qquad & \downarrow \mathscr{V}(Z,\eta_{I}) \\
I \xrightarrow{\Gamma \delta_{Z} \Gamma} & \mathscr{V}(Z,TI)
\end{array}$$

$$\mathscr{V}(Z,TY) \xrightarrow{\mathscr{V}(Z,T(\mathbb{I}_{Y}))} & \mathscr{V}(Z,TI) \\
\downarrow 1 \otimes \mathscr{V}(Z,TY) \xrightarrow{\Gamma \delta_{Y} \Gamma \otimes 1} \mathscr{V}(Y,TI) \otimes \mathscr{V}(Z,TY)$$
The inclusion of idempotent operations $\mathscr{V}(Z,\tau_{Y})$ is naturally $\mathscr{V}(Z,\tau_{Y})$.

The inclusion of idempotent operations $\mathscr{V}(Z, \tau_Y)$ is natural with respect to Z: if $f: X \to Z$ is a morphism, the diagram (18) commutes:

(18)
$$\begin{aligned}
\mathscr{V}(Z, T'Y) &\xrightarrow{\mathscr{V}(Z, \tau_Y)} \mathscr{V}(Z, TY) \\
&& &\downarrow \mathscr{V}(f, T'Y) & &\downarrow \mathscr{V}(f, TY) \\
&& && &\swarrow (X, T'Y) &\xrightarrow{\mathscr{V}(X, \tau_Y)} \mathscr{V}(X, TY)
\end{aligned}$$

If, in particular, X = I, then $\mathcal{V}(f, T'Y)$ can be interpreted as the fth projection, mapping an operation to its fth component. Hence every component of an idempotent operation is idempotent. In the case $\mathcal{V} = Ens$, (the category of sets) the converse of this statement also holds (cf. [2]).

5. Proposition. Let $\mathbb{T}=(T,\psi,\eta,\mu)$ be a monoidal monad on \mathscr{V} . There is a unique monoidal structure ψ' on the affine part of (T,η,μ) , such that $\mathbb{T}'=(T',\psi',\eta',\mu')$ is a monoidal monad and $\tau\colon\mathbb{T}'\to\mathbb{T}$ is a monoidal monad transformation. Furthermore, if \mathbb{T} is symmetric monoidal then so is \mathbb{T}' .

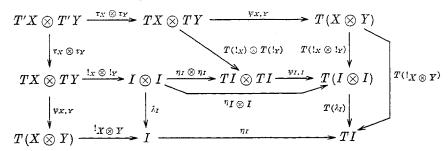
Proof. If τ is going to be a monoidal transformation, the diagram (19) must commute for all $X, Y \in |\mathscr{V}|$:

(19)
$$T'X \otimes T'Y \xrightarrow{\psi'x,Y} T'(X \otimes Y)$$

$$\tau_X \otimes \tau_Y \downarrow \qquad \qquad \downarrow^{\tau_X \otimes Y}$$

$$TX \otimes TY \xrightarrow{\psi_{X,Y}} T(X \otimes Y)$$

Since $\tau_{X \otimes Y}$ is an equalizer of $T(!_{X \otimes Y})$, $\eta_I(!_{T(X \otimes Y)})$, the uniqueness and existence of a morphism $\psi'_{X,Y} \colon T'X \otimes T'Y \to T'(X \otimes Y)$ making (19) commute is implied by the exterior of the following commutative diagram:



Since τ is a pointwise monomorphism, the required diagrams for ψ' follow from the corresponding diagrams for ψ . This completes the proof of the proposition 5.

As an application we mention that the Eilenberg-Moore situation of a monad inherits from it the pleasant properties of being (symmetric) monoidal or closed, respectively (cf. [1, 3, 4, 5, 7]).

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Eingegangen am 11. 5. 1979

Anschrift des Autors:

Harald Lindner Universität Düsseldorf Mathematisches Institut II Universitätsstraße 1 D-4000 Düsseldorf