Model Transformations in Decidability Proofs for Monadic Theories

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Abstract. We survey two basic techniques for showing that the monadic second-order theory of a structure is decidable. In the first approach, one deals with finite fragments of the theory (given for example by the restriction to formulas of a certain quantifier rank) and – depending on the fragment – reduces the model under consideration to a simpler one. In the second approach, one applies a global transformation of models while preserving decidability of the theory. We suggest a combination of these two methods.

1 Introduction

Half a century ago, the first papers appeared on decidability of monadic second-order theories using concepts of automata theory. In 1958, Büchi, Elgot, and independently Trakhtenbrot announced the first results on the "logic-automata connection", showing that the weak monadic second-order theory of the successor structure $(\mathbb{N}, +1)$ of the natural numbers is decidable. Results on the unrestricted monadic second-order theory (short: MSO-theory) were then established by Büchi [4] (decidability of S1S, the monadic theory of $\mathcal{N} = (\mathbb{N}, +1)$) and by Rabin [13] (decidability of S2S, the monadic theory of the infinite binary tree $\mathcal{T}_2 = (\{0,1\}^*, \mathrm{Succ}_0, \mathrm{Succ}_1)$). All these results were shown by the transformation of formulas into finite automata (over infinite words, respectively over infinite trees).

The present note deals with the ongoing research in establishing larger and larger classes of infinite structures for which the MSO-theory is decidable (or in other words: the model-checking problem with respect to MSO-logic is decidable). We recall two methods that have been introduced for this purpose. The first is a transformation of the structure $\mathcal{S}$ under consideration into a simpler structure using a pumping argument. This method involves a finite equivalence that allows to compress certain parts of $\mathcal{S}$ to smaller ones. The finite equivalence takes into account only a finite fragment of the MSO-theory of $\mathcal{S}$; thus the transformation has to be done separately for any such fragment. The second method shows decidability of the entire MSO-theory of a structure $\mathcal{S}$ in one step, in which $\mathcal{S}$ is obtained by a transformation of another structure whose MSO-theory is known to be decidable. The purpose of this paper is to describe both methods and to suggest that a combination of them may be useful.

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As a prerequisite we discuss possibilities to define equivalences that determine "finite fragments" of a theory. These equivalences come in two forms: referring to automata with a certain number of states, and referring to formulas of a certain quantifier rank. We recall these equivalences and their use in the next section.

The paper is a discussion of methods rather than an exposition of results, and thus adheres to an informal style and assumes knowledge of the basics on monadic theories (as found e.g. in [11]).

The MSO-theory of a structure $\mathcal{S}$ is denoted $\mathrm{MTh}(\mathcal{S})$. In this paper we focus on the case of labelled transition systems, i.e. vertex- and edge-labelled graphs. We use the format $G = (V, (E_a)_{a\in A}, (P_b)_{b\in B})$ with finite label alphabets A, B, where E_a is the set of edges labelled a and $P_b \subseteq V$ the set of vertices labelled b.

2 Equivalences

A natural approach for showing decidability of the (say monadic) theory of a structure is to settle the problem for any given finite fragment of the theory, and for this to apply a composition of submodels. A standard option is to restrict to sentences up to some given quantifier rank. Another approach refers to automata with a given number of states (when formulas are known to be equivalent to automata). An extreme case is that one only considers a single automaton (corresponding to a single formula). In each of these cases one derives a corresponding equivalence between structures, and we call the equivalence classes "types". One now tries to compose a model from "simple" parts that has the same type as the original model, and at the same time to compose its type from the types of the components.

This approach has been most successful over (labelled) linear orderings, but it can also be applied with more technical work over more complex structures like trees and certain graphs. For the theory S1S the method involves a composition of an ω-word α (identified here with a labelled ω-ordering) from finite segment orderings (words). It turns out possible to represent α as an "infinite sum" of summand models such that the types of all finite summands are the same (except for the first summand). This allows to deduce the corresponding type of α from the two constituent types (the initial, respectively the repeated type). A composition of this simple form is guaranteed by Ramsey's Infinity Lemma [14]. The types of the segments define a finite coloring of the set of pairs (i, j) (with (i, j) we associate the type of the segment $\alpha[i, j)$). By Ramsey's Lemma there is an infinite "homogeneous" set $H = \{h_0 < h_1 < \ldots\}$: All segments $\alpha[h_i, h_j)$ with $i < j$, and in particular all segments $\alpha[h_i, h_{i+1})$ have the same type. In addition to Ramsey's Lemma one needs also a summation result for the types: First, the type of a concatenation (sum) of two words is determined by (and computable from) their types; so type equivalence is a congruence with respect to concatenation. Second, the type of an infinite sum of words of the same type is determined by (and computable from) this type.

This composition occurs in two versions in the literature. In the first version, one refers to a given Büchi automaton $\mathcal{A}$ and defines theory-fragments via the transition structure of $\mathcal{A}$: One declares two segments (i.e., finite words) u, v equivalent (written $u \sim_{\mathcal{A}} v$) if the following holds: $\mathcal{A}$ can proceed from state p to state q via u iff this is possible via v, and this is possible with a visit to a final state via u iff it is possible via v. It is easy to show that this equivalence relation is a congruence of finite index and that the $\sim_{\mathcal{A}}$-type of a finite word w determines whether $\mathcal{A}$ accepts the infinite word $www\ldots$.

In the world of logic, one cannot achieve the congruence property on the level of a single formula. One obtains it when passing to the level of formula sets classified by the measure of quantifier rank: Call u, v n-equivalent (short $u \equiv_n v$) if u and v satisfy the same sentences of quantifier rank $\leq n$. Then again we obtain an equivalence relation of finite index. The fact that $\equiv_n$ is a congruence is not as immediate as for $\sim_{\mathcal{A}}$ but can be established by the standard method of Ehrenfeucht-Fraïssé games. An analogous congruence can also be introduced in the domain of automata: Define $u \approx_n v$ if $u \sim_{\mathcal{A}} v$ holds for each automaton with $\leq n$ states. It is then clear that the sequences $\equiv_n$ and $\approx_n$ are compatible in the sense that they mutually refine each other and hence that their intersections coincide.

When monadic formulas can be transformed into automata, it is often convenient to work with the relations $\sim_{\mathcal{A}}$ or $\approx_n$. This connection to automata exists over words and trees. Over generalized domains, such as dense labelled orderings, it is hard and maybe unnatural to try to invent suitable "automata"; here the logical equivalence has the advantage to be applicable directly. This is a key aspect in the "composition method" as developed by Shelah [17].

3 Reduction to Periodic Structures

In a pioneering paper, Elgot and Rabin [10] studied structures $(\mathbb{N}, \mathrm{Succ}, P)$ with a designated set $P \subseteq \mathbb{N}$ and showed for certain P that $\mathrm{MTh}(\mathbb{N}, \mathrm{Succ}, P)$ is decidable. Note that there are examples of recursive predicates P such that $\mathrm{MTh}(\mathbb{N}, \mathrm{Succ}, P)$ is undecidable. (Consider a recursively enumerable nonrecursive set S with enumeration $s_0, s_1, \ldots$, and introduce P by its characteristic sequence $\chi_P := 10^{s_0}10^{s_1}1\ldots$. Then P is recursive, and we have $n \in S$ iff there is a number in P such that its $(n+1)$-st successor is the next P-number; thus S is reducible to $\mathrm{MTh}(\mathbb{N}, \mathrm{Succ}, P)$.) There are also predicates P where the decidability of $\mathrm{MTh}(\mathbb{N}, \mathrm{Succ}, P)$ is open. The most prominent example is the prime number predicate $\mathbb{P}$.

The examples P given in [10] such that $\mathrm{MTh}(\mathbb{N}, \mathrm{Succ}, P)$ is decidable are the predicate of the factorial numbers, the predicate of powers of k (for fixed k), and the predicate of k-th powers (for fixed k). Another predicate to which the method can be applied is the set $\{2 \uparrow n \mid n \geq 0\}$ of "hyperpowers of 2", inductively defined by $2 \uparrow 0 = 1$ and $2 \uparrow (n+1) = 2^{2 \uparrow n}$. Further examples are given in [6].

The starting point for the decidability proof is the transformation of an S1S-formula $\varphi(X)$ into a Büchi automaton $\mathcal{A}_\varphi$ that accepts a 0-1-sequence iff it is the characteristic sequence χ_P of a predicate P satisfying $\varphi(X)$. This allows to restate the decision problem for $\mathrm{MTh}(\mathbb{N}, \mathrm{Succ}, P)$ as follows:

$(*)$ *Decide for any Büchi automaton whether it accepts the fixed ω-word χ_P.*

As an example consider the predicate $P = \mathrm{Fac}$ of factorial numbers. Given a Büchi automaton $\mathcal{A}$ over the characteristic sequence χ_{Fac} of the factorial predicate, one can shorten (by a pumping argument) the segments of successive zeros between any two letters 1 in χ_{Fac} in such a way that (1) the distance between two successive letters 1 in the new sequence χ' is bounded, and (2) $\mathcal{A}$ accepts χ_{Fac} iff $\mathcal{A}$ accepts χ'. More precisely, one replaces each segment 10^m1 by the shortest segment $10^{m'}1$ such that $0^m \sim_{\mathcal{A}} 0^{m'}$. It turns out that regardless of the choice of $\mathcal{A}$ the resulting "compressed" sequence χ' is *ultimately periodic*. Since in this case the acceptance problem can be decided, the problem $(*)$ is decidable.

This compression is done for the equivalence $\sim_{\mathcal{A}}$ associated with a given automaton $\mathcal{A}$; similarly, one can also use the relation $\equiv_n$ or $\approx_n$ in place of $\sim_{\mathcal{A}}$ and thus capture all formulas of quantifier rank $\leq n$, respectively all automata with $\leq n$ states, in one step. In all three cases we deal with a "finite fragment" of the theory. The essence thus is the transformation of the given structure to a simpler one (namely, an ultimately periodic one) that is equivalent with respect to a finite fragment of the MSO-theory.

It is important to note that the transformation into a periodic model is *computable* in the parameter $\mathcal{A}$, respectively n. Even without insisting on this computability requirement, for each ω-word χ such a transformation into an ultimately periodic structure *exists* (given $\mathcal{A}$, respectively n) — again by applying Ramsey's Lemma.

Recently, it was shown that the Elgot-Rabin method can be "uniformized" in the following sense ([15]): One considers the logical equivalence $\equiv_n$ (or its automata theoretic analogue $\approx_n$) and observes that $\equiv_{n+1}$ is a refinement of $\equiv_n$. An iterative application of Ramsey's Lemma yields for any χ_P a "uniformly homogeneous" set $H_P = \{h_0 < h_1 < \ldots\}$ which supplies periodic decompositions for all values of n simultaneously: For each n, all segments $\chi_P[h_i, h_{i+1})$ with $i \geq n$ are $\equiv_n$-equivalent. As a consequence, the truth of a sentence of quantifier-depth n is determined by the $\equiv_n$-types of $\chi_P[0, h_n)$ and $\chi_P[h_n, h_{n+1})$; in fact we have $\chi_P \equiv_n \chi_P[0, h_n) + (\chi_P[h_n, h_{n+1}))^\omega$.

Again, this decomposition is possible for each χ_P. One can show that a uniformly homogeneous set H_P exists which is recursive in P'' (the second recursion theoretic jump of P). A *recursive* uniformly homogeneous set H_P exists iff $\mathrm{MTh}(\mathbb{N}, \mathrm{Succ}, P)$ is decidable. As an illustration consider a predicate P where the decidability of $\mathrm{MTh}(\mathbb{N}, \mathrm{Succ}, P)$ is unsettled, namely the prime number predicate $\mathbb{P}$. Let $H_{\mathbb{P}} = \{h_0 < h_1 < \ldots\}$ be a corresponding — currently unknown — uniformly homogeneous set. We may be interested in the truth of the sentence TPH (twin prime hypothesis) saying "there are infinitely many twin primes", i.e. pairs $(m, m+2)$ with $m, m+2 \in \mathbb{P}$. The truth of TPH is open. Since TPH can be written as a monadic sentence of quantifier-depth 5, it suffices to inspect the segment $\chi_{\mathbb{P}}[h_5, h_6)$ of $\chi_{\mathbb{P}}$ for an occurrence of twin primes, to check whether TPH holds. If TPH fails then the last twin prime pair would appear up to number h_5 and none in $\chi_{\mathbb{P}}[h_5, h_6)$.

A similar theory of model transformation can be developed for expansions of the binary tree by a unary predicate P, i.e. for models $(\{0,1\}^*, \mathrm{Succ}_0, \mathrm{Succ}_1, P)$. The desired "compression" of the structure is then a regular tree. The situation is much more complicated than it is over $\mathcal{N}$. First, the composition technique is technically more involved. Second, one does not know (as yet) a decomposition that corresponds to Ramsey's Lemma over ω-words. For recent work in this direction see [12].

The case of labelled tree structures is also interesting for its connection with Seese's conjecture of decidability of monadic theories. The conjecture can be stated as follows (see [2]): A structure has a decidable MSO-theory iff it can be MSO-interpreted in an expansion $\mathcal{T} = (\{0,1\}^*, \mathrm{Succ}_0, \mathrm{Succ}_1, \overline{P})$ of the binary tree by a finite tuple $\overline{P}$ of unary predicates such that $\mathrm{MTh}(\mathcal{T})$ is decidable.

4 Transformations Preserving Decidability

In his celebrated paper [13], Rabin starts with many applications of his main result, the decidability of $\mathrm{MTh}(\mathcal{T}_2)$, before entering the tedious proof. Starting from $\mathrm{MTh}(\mathcal{T}_2)$, several other theories are shown to be decidable by the method of interpretation. An MSO-interpretation of a relational structure $\mathcal{S}$ in $\mathcal{T}_2$ is an MSO-description of the universe and the relations of a copy of $\mathcal{S}$ in $\mathcal{T}_2$. Given such a description, the decidability of $\mathrm{MTh}(\mathcal{S})$ can be derived from the decidability of $\mathrm{MTh}(\mathcal{T}_2)$.

Another very powerful transformation that preserves the decidability of the MSO-theory is the "iteration" of a given structure in the form of a tree-like model. We use here a simple form of iteration which is appropriate for transition graphs as considered in this paper: the unfolding $U(G)$ of a graph G (from a definable vertex v_0). The structure $U(G)$ is a tree whose vertices are the finite paths $\pi = v_0 a_1 v_1 \dots a_m v_m$ in G where $(v_i, v_{i+1}) \in E_{a_{i+1}}$, and the pair $(\pi, (\pi a_{m+1} v_{m+1}))$ belongs then to the edge relation $E_{a_{m+1}}$ of $U(G)$. A fundamental result of Muchnik (announced in [16]) says that $\mathrm{MTh}(U(G))$ is decidable if $\mathrm{MTh}(G)$ is. Proofs are given in [7,9,18] and the expository paper [1].

Caucal observed in [5] that a large class of infinite graphs arises if MSO-interpretation and unfolding are applied in alternation. The *Caucal hierarchy* is a sequence $\mathcal{C}_0, \mathcal{C}_1, \dots$ of classes of graphs where

- $\mathcal{C}_0$ consists of the finite graphs,
- $\mathcal{C}_{n+1}$ consists of the graphs obtained from the graphs in $\mathcal{C}_n$ by an unfolding and a subsequent MSO-interpretation.

The original definition referred to a different transformation (inverse rational mappings rather than MSO-interpretations); for the equivalence between the two see [8]. The hierarchy is strict; a method to separate the levels is presented in [3].

Let $\mathcal{C} = \bigcup_i \mathcal{C}_i$. Each structure in $\mathcal{C}$ has a decidable MSO-theory. The class $\mathcal{C}$ contains an abundance of structures, and the extension of the higher levels is not well understood. Many interesting examples occur on the first three levels. $\mathcal{C}_1$ is the class of "prefix-recognizable graphs" (encompassing the transition graphs of pushdown automata). Moreover, the expansions of $\mathcal{N}$ by the predicate of the factorial numbers, by the powers of (some fixed) k, and by the k-th powers (k fixed), respectively, are all in $\mathcal{C}_3$. This provides a more uniform proof of decidability than by the Elgot-Rabin method: It is no more necessary to provide a structure decomposition for each finite theory fragment; rather the membership of the structure in $\mathcal{C}$ suffices as the decidability proof for the full MSO-theory.

Only very few structures are known that have a decidable MSO-theory but are located outside $\mathcal{C}$. An example noted in the literature (see [8,2]) is the structure $(\mathbb{N}, \mathrm{Succ}, P)$ where P is the set of 2-hyperpowers $2 \uparrow n$. We shall generate this structure in an extension of the Caucal hierarchy.

5 Limit Models

By an iterated application of interpretations and unfoldings one can generate finite trees t_i ($i = 0, 1, 2, \ldots$) where t_i has height $2 \uparrow i$ and $2 \uparrow (i+1)$ leaves. For $i = 0$ we take the binary tree consisting of the root and two sons. In order to construct t_{i+1}, consider t_i with $2 \uparrow i$ leaves. Along the frontier we introduce two identical edge relations S_1, S_2 (and as universe we take their common domain): For both S_1 and S_2 start from the rightmost leaf, proceed leaf by leaf towards the left to the leftmost leaf (which yields $2 \uparrow (i-1)$ edges), and continue with one more step to the parent of the leftmost leaf. Clearly S_1, S_2 are MSO-definable in t_i. The unfolding of this successor structure from its root, which is the rightmost leaf of t_i, gives the desired tree t_{i+1} of height $2 \uparrow (i+1)$ with $2 \uparrow (i+2)$ leaves.

Let $\prod_i t_i$ be the "limit model" of the t_i ($i \geq 0$) where the rightmost leaf of t_i coincides with the root of t_{i+1}. An interpretation in the limit model will generate a structure $(\mathbb{N}, \mathrm{Succ}, P)$ which does not belong to the Caucal hierarchy: As copy of $(\mathbb{N}, \mathrm{Succ})$ one uses the infinite sequence of leaves, and one declares as P-elements the "first leaves" of the t_i. The difference between successive P-elements is then $(2 \uparrow (i+1)) - 1$ (for $i = 0, 1, \ldots$). By a refinement of the construction one can also generate a copy of $(\mathbb{N}, \mathrm{Succ}, H)$ where H is the set of hyperpowers of 2. For this, we use a structure $\prod_i t'_i$ where each t'_i contributes only $2 \uparrow (i+1) - 2 \uparrow i$ rather than $2 \uparrow (i+1) - 1$ leaves. Technically we work with the t_i as above but expanded by a singleton predicate Q that marks the $((2 \uparrow i) + 1)$-st of its $2 \uparrow (i+1)$ leaves. To construct the t'_i inductively, one starts with t_1 and fixes Q_{t_1} as the set containing the second leaf. For the step from t'_i to t'_{i+1} we have to proceed (in the definition of Q) from a number of the form $2^k - k$ to $2^{2^k} - 2^k$; this is possible (using a little technical work) by observing that $2^{2^k} - 2^k = 2^k(2^{2^k - k} - 1)$.

The model $\prod_i t_i$ (and similarly $\prod_i t'_i$) is generated by an infinite sequence of interpretations and unfoldings. However, each of the interpretations is based on the same formulas defining the universe and the relations, and for each unfolding one uses the same formula for defining the root vertex. So we speak of an *interpretation-unfolding scheme* that generates $\prod_i t_i$ ($\prod_i t'_i$, respectively). In our example we referred to a single definable vertex for the "next unfolding"; in this case we speak of a *linear* interpretation-unfolding scheme.

The limit models $\prod_i t_i$ and $\prod_i t'_i$ have decidable MSO-theories. To show this, we have (presently) to resort to the "non-uniform" method of model reduction (see Section 3). This requires to invoke the equivalences $\equiv_n$ and the associated n-types. We observe that the n-type of t_{i+1} is computable from the n-type of t_i (similarly for t'_i and t'_{i+1}). Then — by finiteness of the set of n-types — the generated sequence of n-types is ultimately periodic, which allows to compute the n-type of the limit model.

We do not know whether by a linear interpretation-unfolding scheme (and an extra interpretation in the limit model) one can generate a structure $(\mathbb{N}, \mathrm{Succ}, P)$ whose monadic theory is undecidable. This is connected with the old problem to find (in any way) a non-artificial — i.e. number theoretically meaningful — recursive predicate P such that $\mathrm{MTh}(\mathbb{N}, \mathrm{Succ}, P)$ is undecidable.

It seems interesting to analyze also non-linear schemes. In this case, one would allow to expand a given model at several vertices, which leads to a tree-like construction. The formula that defines the set of vertices where the unfolding takes place is then satisfied by several vertices. We show that such a scheme can lead to a structure with an undecidable MSO-theory. Consider a Turing machine M (say with set Q of states and tape alphabet Σ) that accepts a non-recursive language. As initial model we use the tree $\mathcal{S}_0$ of all initial configurations $q_0 a_1 \dots a_n$ with initial state q_0 and input word $a_1 \dots a_n$ (coded by paths with successive edge labels $q_0, a_1, \dots, a_n, \$$ where \$ is an endmarker). This is an infinite tree with a decidable MSO-theory. By an interpretation-unfolding scheme we generate, level by level, a tree model $\mathcal{S}_M$ in which all M-computations can be traced as paths. A word w will be accepted by M iff a sentence φ_w is true in $\mathcal{S}_M$ that expresses the following: From the vertex after the initial path $q_0 w \$$ there is a path to a configuration with an accepting state of M.

We describe the interpretation-unfolding scheme. Consider a tree $\mathcal{S}_k$ that is generated at level k of the construction. We first treat the case $k > 0$ and later $k = 0$. Let r be the root of $\mathcal{S}_k$. The next unfolding will take place at any vertex v which is the source of an \$-labelled edge and such that between r and v there is no other \$-labelled edge. Vertex v is the end of a path from r labelled by an M-configuration, say $w_1 bqaw_2$. We define a structure $\mathcal{S}_v$ as follows: The universe is given by the path from r to v, the sequence of edge labels along the path gives the next M-configuration after $w_1 bqaw_2$, and there is a new \$-labelled edge from v back to r. So we obtain $\mathcal{S}_v$ from $\mathcal{S}_k$ by adding the \$-edge and by changing the bqa-labelled path segment to a new one according to the table of M. (In the case the length of the configuration increases by one, the original \$-labelled edge (v, v') gets a letter from Σ, and the \$-labelled back-edge to r starts at v'.) It is clear that for each pair $(q, a) \in Q \times \Sigma$ (which fixes an M-transition) the respective structure $\mathcal{S}_v$ can be defined; a disjunction over all (q, a) gives the desired interpretation. The unfolding of $\mathcal{S}_v$ at v will produce an infinite sequence of finite paths labelled with the new configuration. In the subsequent step, only the first such path will stay (by the definition of the new v and the new $\mathcal{S}_v$).

In the initial step (where $k = 0$) the initial configuration is simply copied; this ensures that the first copy stays unmodified by the construction.

Clearly one can express in the limit model $\mathcal{S}_M$ by an MSO-sentence φ_w that for input w there is an accepting computation of M. So $\mathrm{MTh}(\mathcal{S}_M)$ is undecidable.

Interpretation-unfolding schemes are thus a powerful tool to generate models (and in general too powerful to obtain only structures with a decidable monadic theory). By a linear interpretation scheme it was possible to synthesize a structure $(\mathbb{N}, \mathrm{Succ}, P)$ (namely, where P is the set of hyperpowers of 2) which was previously just given a priori. The decidability of its monadic theory was shown using the "non-uniform" method of model reduction. An open issue is to find easily verified conditions that ensure decidability of the MSO-theory of a limit model. Also one can study schemes that involve transfinite stages of construction.

6 Conclusion

We surveyed two techniques for proving that the MSO-theory of an infinite labelled graph is decidable: the "non-uniform" method of model reduction à la Elgot and Rabin and two "uniform" types of model transformation, namely MSO-interpretations and unfoldings. We proposed to study models that are generated by an infinite number of steps involving the latter two operations. It was illustrated that in this context a combination of the uniform and the non-uniform approach gives a further small step in building up more infinite graphs that have a decidable MSO-theory.

We have not touched the rich landscape of recent studies on other types of interpetations and of model composition. A good survey on the state-of-the-art is given in [2]. On the side of logics, one should note that for applications in infinite-state verification weaker logics than MSO-logic are of interest, for which the class of structures with a decidable model-checking problem can then be expanded.

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References

1. Berwanger, D., Blumensath, A.: The monadic theory of tree-like structures. In: [11], pp. 285–302
2. Blumensath, A., Colcombet, T., Löding, C.: Logical theories and compatible operations. In: Flum, J., et al. (eds.) Logic and Automata, pp. 73–106. Amsterdam Univ. Press, Amsterdam (2007)
3. Blumensath, A.: On the structure of graphs in the Caucal hierarchy. Theor. Comput. Sci. 400, 19–45 (2008)
4. Büchi, J.R.: On a decision method in restricted second-order arithmetic. In: Nagel, E., et al. (eds.) Logic, Methodology, and Philosophy of Science: Proceedings of the 1960 International Congress, pp. 1–11. Stanford Univ. Press (1962)

5. Caucal, D.: On infinite graphs having a decidable monadic theory. In: Diks, K., Rytter, W. (eds.) MFCS 2002. LNCS, vol. 2420, pp. 165–176. Springer, Heidelberg (2002)
6. Carton, O., Thomas, W.: The monadic theory of morphic infinite words and generalizations. In: Nielsen, M., Rovan, B. (eds.) MFCS 2000. LNCS, vol. 1893, pp. 275–284. Springer, Heidelberg (2000)
7. Courcelle, B.: The monadic second-order logic of graphs IX: machines and their behaviours. Theor. Comput. Sci. 151, 125–162 (1995)
8. Carayol, A., Wöhrle, S.: The Caucal hierarchy of infinite graphs in terms of logic and higher-order pushdown automata. In: Pandya, P.K., Radhakrishnan, J. (eds.) FSTTCS 2003. LNCS, vol. 2914, pp. 112–123. Springer, Heidelberg (2003)
9. Courcelle, B., Walukiewicz, I.: Monadic second-order logic, graph coverings and unfoldings of transition systems. Ann. Pure Appl. Logic 92, 35–62 (1998)
10. Elgot, C.C., Rabin, M.O.: Decidability and undecidability of extensions of second (first) order theory of (generalized) successor. J. Symb. Logic 31, 169–181 (1966)
11. Grädel, E., Thomas, W., Wilke, T. (eds.): Automata, Logics, and Infinite Games. LNCS, vol. 2500. Springer, Heidelberg (2002)
12. Montanari, A., Puppis, G.: A contraction method to decide MSO theories of deterministic trees. In: Proc. 22nd IEEE Symposium on Logic in Computer Science (LICS), pp. 141–150
13. Rabin, M.O.: Decidability of second-order theories and automata on infinite trees. Trans. Amer. Math. Soc. 141, 1–35 (1969)
14. Ramsey, F.P.: On a problem of formal logic. Proc. London Math. Soc. s2-30, 264–286 (1930)
15. Rabinovich, A., Thomas, W.: Decidable theories of the ordering of natural numbers with unary predicates. In: Ésik, Z. (ed.) CSL 2006. LNCS, vol. 4207, pp. 562–574. Springer, Heidelberg (2006)
16. Semenov, A.: Decidability of monadic theories. In: Chytil, M.P., Koubek, V. (eds.) MFCS 1984. LNCS, vol. 176, pp. 162–175. Springer, Heidelberg (1984)
17. Shelah, S.: The monadic theory of order. Ann. Math. 102, 379–419 (1975)
18. Walukiewicz, I.: Monadic second-order logic on tree-like structures. Theor. Comput. Sci. 275, 311–346 (2002)