Vector Addition Tree Automata & Multiplicative Exponential Linear Logic

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Overview

- Introduction
- An example
- The automata side
- The linear logic side
- Bridging the two sides

Introduction

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$$
\begin{tikzcd}
\text{!-Horn fragment} \arrow[r, dash] \arrow[d, dash] & \begin{array}{c}\text{Vector Addition}\\ \text{System with States}\\ \simeq \text{Petri Nets}\end{array} \arrow[r, dash] \arrow[d, dash] & \text{Finite State Automata} \arrow[d, dash] \\
\text{IMELL} \arrow[r, dash] & \begin{array}{c}\text{Vector Addition}\\ \text{Tree Automata}\end{array} \arrow[r, dash] & \text{Tree Automata}
\end{tikzcd}
$$

An example

Find a proof in IMELL of

$$!((A \multimap B \multimap S) \multimap S), !(S \multimap S \multimap S), !(A \multimap S), !(B \multimap S) \vdash S$$

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is equivalent to find a closed linear λ-term of type S on the signature:

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Let $\mathbf{x} = (n_1, n_2) \in \mathbb{N}^2$, $\mathcal{T}_S[\mathbf{x}]$ stands for the set of terms of type S with n_1 free variables of type A and n_2 free variables of type B.

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$$\begin{array}{rcll}
\mathcal{T}_S[\mathbf{x}] & ::= & f\ (\lambda_ : A\ \lambda_ : B\ \mathcal{T}_S[\mathbf{x} + (1,1)]) & \\
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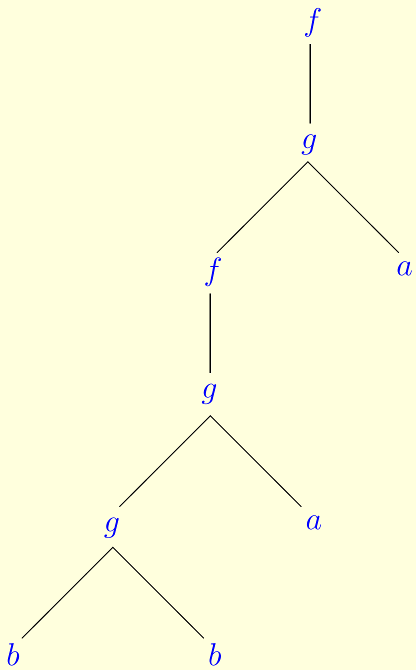
Putting it as a (tree automata like) rewriting system:

$$\begin{array}{rcl}
a & \to & q[(1,0)] \\
b & \to & q[(0,1)] \\
f(q[\mathbf{x}]) & \to & q[\mathbf{x} - (1,1)] \\
g(q[\mathbf{x}_1], q[\mathbf{x}_2]) & \to & q[\mathbf{x}_1 + \mathbf{x}_2]
\end{array}$$

An example



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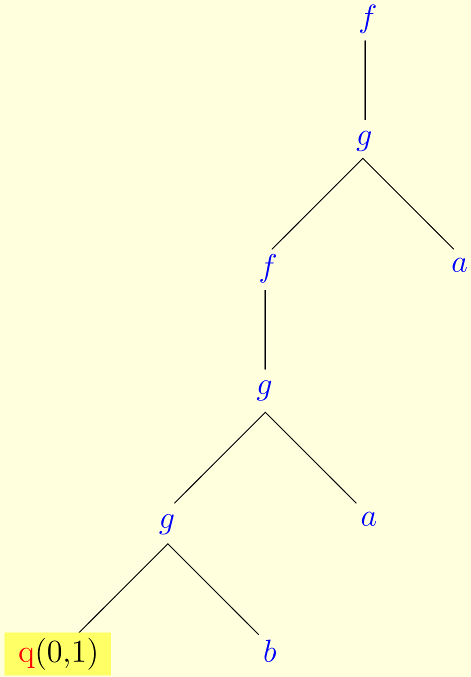

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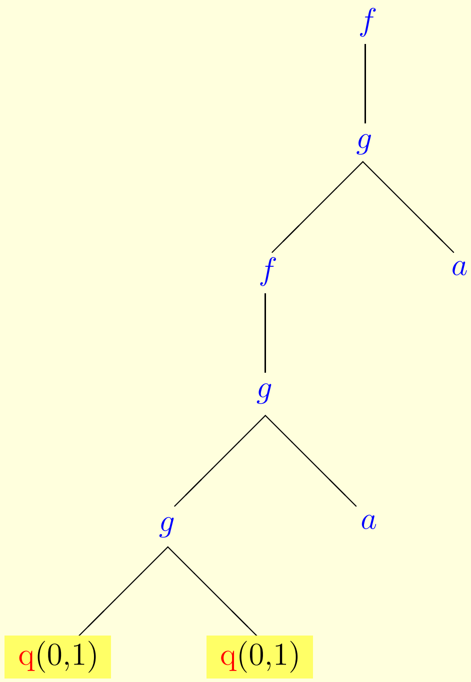

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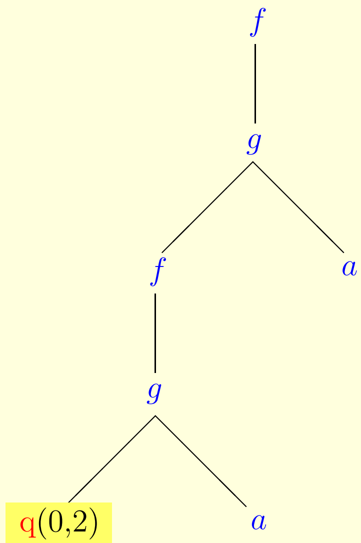

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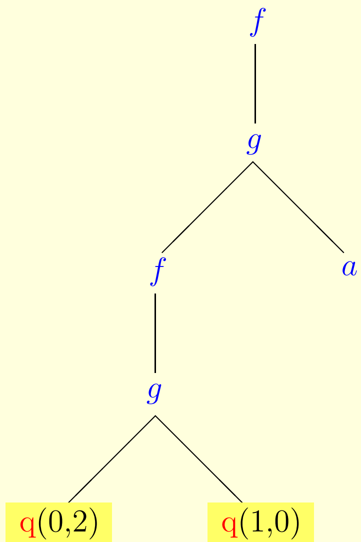

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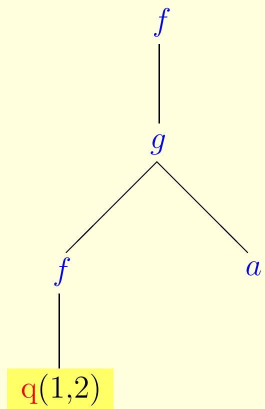

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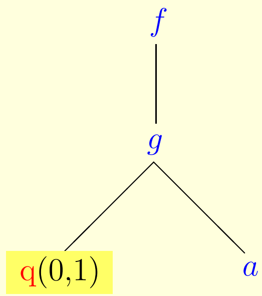

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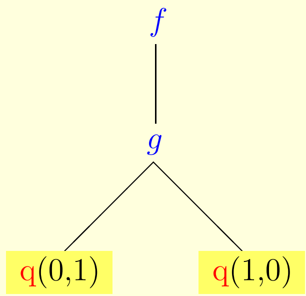

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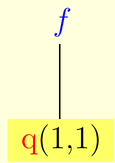

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Vector Addition Tree Automata

A k-VATA is a quadruple $\langle \mathcal{F}, Q, C_f, \Delta \rangle$ where:

- $\mathcal{F}$ is a ranked alphabet;
- Q is a finite set of states;
- C_f is a finite set of accepting configurations (i.e. elements of $Q \times \mathbb{N}^k$);
- Δ is a finite set of transition rules of the form:

$$f(q_0[\mathbf{x}_0], \ldots, q_{n-1}[\mathbf{x}_{n-1}]) \rightarrow q\left[\sum_{i \in n} (\mathbf{x}_i - \mathbf{c}_i) + \mathbf{c}\right]$$

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where:

$f \in \mathcal{F}$ is a functional symbol of arity n;

$q_0 \ldots q_{n-1} \in Q$;

$\mathbf{c}, \mathbf{c}_0, \ldots \mathbf{c}_{n-1} \in \mathbb{N}^k$ are given vectors, proper to the transition rule;

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The rewriting relation is induced by the transition rules with the constraint that $\mathbf{x}_i - \mathbf{c}_i \in \mathbb{N}^k$ (this corresponds to the positivity condition in VASS).

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Proposition 1 *For any k-VATA $\mathcal{A}$ there exists a k-VATA $\mathcal{A}'$ in normal form such that $\mathcal{L}_{\mathcal{A}} = \emptyset$ iff $\mathcal{L}_{\mathcal{A}'} = \emptyset$.*

Linear Logic

IMELL

$$\mathcal{F} \quad ::= \quad \mathbf{1} \mid \mathcal{A} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \multimap \mathcal{F} \mid\ !\mathcal{F}$$

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From IMELL to IMELL$_0$

Proposition 2 *IMELL is decidable iff IMELL$_0$ is decidable.*

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Let $\Gamma \vdash A$ an IMELL sequent.

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Add Σ, the following set of formulas to the antecedent of the sequent:

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$!(p_F \multimap \mathbf{1})$,

$!(p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap p_F)$ whenever $p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap F^*$ is provable.

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$!(p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap p_F)$ whenever $p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap F^*$ is provable.

- If IMELL$_0$ is decidable then the construction is effective.

From IMELL to IMELL$_0$

Proposition 2 *IMELL is decidable iff IMELL$_0$ is decidable.*

Proof

Let $\Gamma \vdash A$ an IMELL sequent.

Define $\Gamma^* \vdash A^*$ to be the sequent obtained by replacing each exponential subformula $!F$ by a fresh atomic proposition p_F.

Add Σ, the following set of formulas to the antecedent of the sequent:

$!(p_F \multimap F^*)$,

$!(p_F \multimap p_F \otimes p_F)$,

$!(p_F \multimap \mathbf{1})$,

$!(p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap p_F)$ whenever $p_{F_1} \multimap \cdots \multimap p_{F_n} \multimap F^*$ is provable.

- If IMELL$_0$ is decidable then the construction is effective.
- $\Gamma \vdash A$ is provable if and only if $\Sigma, \Gamma^* \vdash A^*$ is provable. $\square$

From $\mathbf{IMELL}_0$ to $\mathbf{IMELL}_0^{\multimap}$

Proposition 3 *$IMELL_0$ is decidable iff $IMELL_0^{\multimap}$ is decidable.*

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Proof Let b a fresh atomic proposition, we define two translation on multiplicative formulas:

$A^+ = A^- \multimap b$, for any formula A

$\mathbf{1}^- = b$

$a^- = a \multimap b$

$(A \otimes B)^- = A^+ \multimap (B^+ \multimap b)$

$(A \multimap B)^- = (A^+ \multimap B^+) \multimap b$

and extend it to IMELL_0 with $(!A)^+ = !(A^+)$.

From IMELL_0 to $\mathrm{IMELL}_0^{\multimap}$

Proposition 3 *IMELL_0 is decidable iff $\mathrm{IMELL}_0^{\multimap}$ is decidable.*

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$(A \multimap B)^- = (A^+ \multimap B^+) \multimap b$

and extend it to IMELL_0 with $(!A)^+ = !(A^+)$.

- $\Gamma \vdash A$ an IMELL_0 sequent is provable iff $\Gamma^+ \vdash A^+$ is provable. $\square$

From $\mathbf{IMELL}_0^{\multimap}$ to s-$\mathbf{IMELL}_0^{\multimap}$

Proposition 4 *$IMELL_0^{\multimap}$ is decidable iff s-$IMELL_0^{\multimap}$ is decidable.*

From IMELL$_0^{\multimap}$ to s-IMELL$_0^{\multimap}$

Proposition 4 *IMELL$_0^{\multimap}$ is decidable iff s-IMELL$_0^{\multimap}$ is decidable.*

Proof

Lemma 1 *Let $\Gamma \vdash A$ an IMELL sequent.*

$$\underbrace{\ldots F \ldots F \ldots}_{\Gamma} \vdash \underbrace{\ldots F \ldots}_{A} \iff !(p \multimap F), !(F \multimap p), \underbrace{\ldots p \ldots p \ldots}_{\Gamma} \vdash \underbrace{\ldots p \ldots}_{A}$$

From IMELL$_0^{\multimap}$ to s-IMELL$_0^{\multimap}$

Proposition 4 *IMELL$_0^{\multimap}$ is decidable iff s-IMELL$_0^{\multimap}$ is decidable.*

Proof

Lemma 1 *Let $\Gamma \vdash A$ an IMELL sequent.*

$$\underbrace{\ldots F \ldots F \ldots}_{\Gamma} \vdash \underbrace{\ldots F \ldots}_{A} \iff\ !(p \multimap F), !(F \multimap p), \underbrace{\ldots p \ldots p \ldots}_{\Gamma} \vdash \underbrace{\ldots p \ldots}_{A}$$

$$!\Sigma, \Gamma \vdash A \text{ is a provable IMELL}_0^{\multimap} \text{ sequent}$$
$$\Updownarrow$$
$$!\Sigma', \underbrace{\Gamma'}_{atomic} \vdash a \text{ is a provable IMELL}_0^{\multimap} \text{ sequent}$$
$$\Updownarrow$$
$$!\underbrace{\Sigma''}_{\leq 2 `\multimap`}, \underbrace{\Gamma'}_{atomic} \vdash a \text{ is a provable IMELL}_0^{\multimap} \text{ sequent}$$

$\square$

From VATA to IMELL

Let $\mathcal{A} = \langle \mathcal{F}, Q, \{(q_f, \mathbf{0})\}, \Delta \rangle$ a k-VATA in normal form.

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We define the set of atomic types $A = Q \cup \{a_0, \ldots, a_{k-1}\}$ and Σ by:

$$\begin{array}{rcl}
\Delta & & \Sigma \\
f \rightarrow q[\mathbf{e}_i] & \rightsquigarrow & a_i \multimap q \\
f(q_0[\mathbf{x}_0]) \rightarrow q[\mathbf{x}_0 - \mathbf{e}_i] & \rightsquigarrow & (a_i \multimap q_0) \multimap q \\
f(q_0[\mathbf{x}_0], q_1[\mathbf{x}_1]) \rightarrow q[\mathbf{x}_0 + \mathbf{x}_1] & \rightsquigarrow & q_0 \multimap q_1 \multimap q
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\end{array}$$

Proposition 5 $\mathcal{L}(\mathcal{A}) \neq \varnothing$ *iff* $!\Sigma \vdash q_f$.

From IMELL to VATA

Let $!\Sigma, \Gamma \vdash a_0$ an s-IMELL$_0^{\multimap}$ sequent and $\{a_0, \ldots, a_{k-1}\}$ an enumeration of the atomic formulas of the sequent.

From IMELL to VATA

Let $!\Sigma, \Gamma \vdash a_0$ an s-IMELL$_0^{\multimap}$ sequent and $\{a_0, \ldots, a_{k-1}\}$ an enumeration of the atomic formulas of the sequent. We define the set of state $Q = \{q_0, \ldots, q_{k-1}\}$, the only final configuration $(q_0, |\Gamma|)$ and the set of transitions:

$$
\begin{array}{rcl}
\Sigma & & \Delta \\
 & \rightsquigarrow & c_i \longrightarrow q_i[\mathrm{e}_i] \\
a_j \multimap a_l & \rightsquigarrow & f(q_j[\mathbf{x}]) \longrightarrow q_l[\mathbf{x}] \\
(a_j \multimap a_l) \multimap a_m & \rightsquigarrow & g(q_l[\mathbf{x}]) \longrightarrow q_m[\mathbf{x} - \mathrm{e}_j] \\
a_j \multimap a_l \multimap a_m & \rightsquigarrow & h(q_j[\mathbf{x}_0], q_l[\mathbf{x}_1]) \longrightarrow q_m[\mathbf{x}_0 + \mathbf{x}_1]
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a_j \multimap a_l \multimap a_m & \rightsquigarrow & h(q_j[\mathbf{x}_0], q_l[\mathbf{x}_1]) \longrightarrow q_m[\mathbf{x}_0 + \mathbf{x}_1]
\end{array}
$$

Proposition 6 $!\Sigma, \Gamma \vdash a_0$ *is provable in s-IMELL$_0^{\multimap}$ iff* $\mathcal{L}(\mathcal{A}) \neq \varnothing$.

Conclusion and future work

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- Future work:

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