THE COVERING PROBLEM

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ABSTRACT. An important endeavor in computer science is to precisely understand the expressive power of descriptive formalisms (such as fragments of first-order logic) over discrete structures (such as finite words, trees or graphs). Of course, the term "understanding" is not a precise mathematical notion. Therefore, carrying out this investigation requires a concrete objective to capture this understanding. In the literature, the standard choice for this objective is the *membership problem*, whose aim is to find a procedure deciding whether an input language can be defined in the formalism under investigation. This approach was cemented as the "right" one by the seminal work of Schützenberger, McNaughton and Papert on "first-order logic over finite words", and has been in use since then.

Unfortunately, membership questions are hard: for several fundamental formalisms, researchers have failed in this endeavor despite decades of investigation. In view of recent results on one of the most famous open questions, namely membership for all levels in the quantifier alternation hierarchy of first-order logic, an explanation may be that membership is too restrictive as a setting. Indeed, these new results were obtained by considering *more general* problems than membership, taking advantage of the increased flexibility of the enriched mathematical setting. Investigating such new problems opened a promising research avenue, which permitted to solve membership for natural fragments of first-order logic. However, many of these problems are *ad hoc*: for each fragment, the solution relies on a specific one. A unique new problem replacing membership as the right one is still missing.

The main contribution of this paper is a suitable candidate to play this role: the covering problem. We motivate this problem with several arguments. First, it admits an elementary set theoretic formulation, similar to membership. Second, we are able to reexplain or generalize all known results with this problem. Third, we develop a mathematical framework as well as a methodology tailored to the investigation of this problem. At last, for each class admitting a decidable membership, we are able to instantiate our methodology to solve this more general problem. In particular, this yields *constructive* solutions to membership. We illustrate our approach with algorithms solving covering (hence also membership and generalizations thereof, such as the problem called separation) for two classical fragments of first-order logic: level 1 in the quantifier alternation hierarchy of first-order logic, which consists of the so called *piecewise testable* languages, and the well-known 2-variable fragment of first-order logic.

1998 ACM Subject Classification: F.4.3.

Key words and phrases: Regular Languages, First-Order Logic, Membership Problem, Separation Problem, Covering Problem, Decidable Characterization.

Supported by the DeLTA project (ANR-16-CE40-0007).

LOGICAL METHODS
IN COMPUTER SCIENCE

DOI:10.2168/LMCS-???

1. Introduction

One of the most successful applications of the notion of regularity in computer science is the investigation of logics on discrete structures, such as words or trees. The story began in the 60s when Büchi [4], Elgot [10] and Trakhtenbrot [38] proved that the *regular languages* of finite words are exactly those that can be defined in monadic second order logic (MSO). This was later pushed to infinite words [5], to finite or infinite trees [34, 31], to labeled countable linear orders [6] and even to graphs of bounded tree width [2]. Such connections not only testify to robustness of the notion of regularity. Indeed, in the context of finite words, the connection was further exploited to investigate the expressive power of important *fragments* of MSO, by relying on a decision problem associated to any such fragment: the membership problem. This problem just asks whether the fragment under investigation forms a recursive class, *i.e.*, its statement is as follows: given a regular language as input, decide whether it is definable by a sentence of the fragment.

Obtaining membership algorithms is difficult. An oft-told and still open example is to decide the most natural fragment of MSO, namely first-order logic (FO), on finite binary trees. On finite words (a much simpler structure than binary trees), Schützenberger, McNaughton and Papert [32, 12] settled this question in the 70s. Their result was highly influential: it was often revisited [40, 9, 7, 20], and it paved the way to a series of results of the same nature. A famous example is Simon's Theorem [33], which yields an algorithm for the first level of the quantifier alternation hierarchy of FO. Other prominent examples are fragments of FO where the linear order on positions is replaced by the successor relation [3, 11, 41, 35] or consider the two-variable fragment of first-order logic [36]. The relevance of this approach is nowadays validated by a wealth of results.

The reason for this success is twofold. First, these results cemented membership as the "right" question: a solution conveys a deep intuition on the investigated logic. In particular, most results include a *generic method for building a canonical sentence* witnessing membership of an input language if it is expressible in the logic. Second, Schützenberger's solution established a suitable framework and a methodology for solving membership problems. This methodology is based on a canonical, finite and computable algebraic abstraction of a regular language: the *syntactic monoid*. The core of the approach is to translate the semantic question (*is the language definable in the fragment?*) into a purely syntactical, easy question to be tested on the syntactic monoid (*does the syntactic monoid satisfies some equation?*).

Unfortunately, this methodology seems to have reached its limits for the hardest questions. An emblematic example is the *quantifier alternation hierarchy of first-order logic*, which classifies sentences according to the number of alternations between $\exists$ and $\forall$ quantifiers in their prenex normal form. A sentence is Σ_i if its prenex normal form has $(i-1)$ alternations and starts with a block of existential quantifiers. A sentence is $\mathcal{B}\Sigma_i$ if it is a Boolean combination of Σ_i sentences. Obtaining membership algorithms for all levels in this hierarchy is a major open question, which has received a lot of attention (see [39, 37, 15, 16, 17, 18, 28, 19] for details and a complete bibliography). However, progress on this question has been slow: until recently, only the lowest levels have been solved, namely Σ_1 [1, 14], $\mathcal{B}\Sigma_1$ [33] and Σ_2 [1, 14].

It took years to solve higher levels. Recently, membership algorithms were obtained for the levels Σ_3 [25, 30], $\mathcal{B}\Sigma_2$ [25, 30] and Σ_4 [21, 22]. This was achieved by introducing new ingredients into Schützenberger's methodology: problems that are *more general than membership*. For each of these results, the strategy is the same:

- First, a well-chosen *more general* problem is solved for a *lower* level in the hierarchy.

– Then, this knowledge is *turned* into a membership algorithm for the level under investigation.

Let us illustrate what we mean by "more general problem" by presenting the simplest of them: the $\mathcal{C}$-*separation problem* (where $\mathcal{C}$ is a class of regular languages). This problem takes *two* languages as input—rather than just one for membership—and asks whether there exists a third one which:

– belongs to $\mathcal{C}$,
– contains the first language, and
– is disjoint from the second.

It is easy to see why this problem generalizes membership: a language belongs to a class $\mathcal{C}$ if and only if it is $\mathcal{C}$-separable from its complement. Being more general, such problems are also more difficult than membership. However, this generality makes them also more rewarding in the insight they provide on the investigated logic. This motivated a series of papers on the separation problem [26, 8, 23, 24, 27], which culminated in the three results above [25, 30, 21, 22]. However, while this avenue of research is very promising, it presently suffers **three major flaws**:

(1) The problems considered up until now form a jungle: each particular result relies actually not on separation itself, but rather on a specific *ad hoc* generalization of this problem. As an illustration, the results of [26, 25, 30, 21, 22] rely on *three* different such problems.
(2) Among the problems that were investigated, separation is the only one that admits a simple and generic set-theoretic definition (which is why it is favored as an example). On the other hand, for all other problems, the definition requires to introduce additional concepts, such as semigroups and Ehrenfeucht-Fraïssé games.
(3) In contrast to membership solutions, the solutions that have been obtained for these more general problems are *non-constructive*. For example, most of the solutions for separation do not include a generic method for building a separator language, when it exists, because the algorithms are designed around the idea of witnessing that the two inputs are *not* separable.

Contributions. Our objective in this paper is to address each of these three issues. Our first contribution is the definition of a *single general* problem, the "*covering problem*", which

a) encompasses *all variations* of the separation problem introduced so far to solve membership,
b) enjoys a simple, language-theoretic formulation (just as membership and separation).

This already addresses the first two issues. The second contribution is a *framework* and a *methodology* for solving this new problem, which were lacking for the separation problem. Finally, we illustrate this methodology with the presentation of algorithms solving the covering problem for several important fragments of first-order logic. Naturally, these algorithms are based on the general methodology developed as the second contribution, and they yield constructive solutions for the separation problem as a byproduct, which addresses the third issue.

Let us review these contributions in more details. As explained, the first one is to define a new problem, which we call *covering*, satisfying Items a) and b) above, in order to gain afterwards a methodology for solving separation in a constructive way.

First step: Extending separation to inputs that are sets. We start with a simple observation: we already have in hand *two orthogonal* generalizations of membership. The first is separation, that we aim at extending even further. The second is the straightforward but powerful generalization introduced by Schützenberger, with precisely a similar motivation as ours: setting up a methodology for solving membership. In order to define covering, a natural move is to combine both generalizations.

Before proceeding, let us recall Schützenberger's key idea: for testing whether a language L belongs to a fragment, one should not consider L *alone*. Instead, one should test whether *all* languages recognized by its syntactic monoid belong to the fragment. This seemingly more demanding problem is in fact equivalent to membership when the fragment enjoys some mild properties (as this set of languages contains L, and either all or none of its languages belong to the fragment). The motivation and the payoff for considering such *input sets* is that they have a nice algebraic structure, which can be leveraged to develop inductive arguments in order to successfully design membership algorithms. While simple, this idea is the core of most classical membership algorithms.

The definition of the *covering problem* builds on this idea: it generalizes separation to an input that is a *set* of languages rather than just a pair. Thus, covering is a (strict) generalization of separation to an arbitrary number of input languages.

Second step: Separation as an approximation problem. Carrying out this idea is not immediate, however: there is a discrepancy between the generalization for membership and that for "separation". Indeed, extending "membership" to a set of languages is obvious: simply solving membership for all languages of the set is enough for developing inductive arguments. A similar naive generalization for separation would be, given a finite set of languages, to test whether each pair of languages from the set is separable by a language in the fragment. Unfortunately, this turns out to be inadequate, because answering separation for all pairs of languages from a set is too weak to provide enough information.

A solution is to think of separation as an (over-)approximation problem. Given two input languages L_1 and L_2, it asks for a "good approximation" of L_1 (definable in the fragment under investigation) while L_2 serves as a quality measure—good approximations are those which do not intersect it. This point of view is amenable to generalization: in the *covering problem*, our inputs are pairs $(L_1, \mathbf{L}_2)$ where L_1 is a single language and $\mathbf{L}_2$ is a finite *set* of languages. The objective is still to approximate L_1 while $\mathbf{L}_2$ specifies what are the good approximations. Specifically, "covering" asks for a finite set of languages $\mathbf{K}$ (all belonging to the fragment under investigation) such that,

(1) The union of all languages in $\mathbf{K}$ includes L_1: $\mathbf{K}$ is a cover of L_1 (hence the name "covering").
(2) No language in $\mathbf{K}$ intersects all languages in $\mathbf{L}_2$.

In particular, the original separation problem is the special case of "covering" when the input set $\mathbf{L}_2$ is a singleton.

Third step: Abstracting the quality measure. In the covering problem, the input is made of two objects playing different roles: we have a language L_1 that needs to be covered and a set of languages $\mathbf{L}_2$ that serves as a quality measure specifying suitable covers. It is cleaner to separate[1] these roles. For this reason, we define *rating maps*, whose purpose is exclusively to evaluate the quality of a cover. This has two advantages: first, this makes it easier to pinpoint the hypotheses that we need on the set of languages and on the rating map. Second, it simplifies the notation.

Our algorithms apply to inputs and rating maps satisfying some mild assumptions. We will show that one can always effectively reduce any input to such a special one.

Benefits of the covering problem. The main advantage of the covering problem is that it comes with a generic framework and a generic methodology designed for solving it. This framework is our second contribution. It generalizes the original framework of Schützenberger for membership in a natural way and lifts all its benefits to a more general setting. In particular, we recover *constructiveness*: a solution to the covering problem associated to a particular fragment yields a generic way for building an actual optimal cover of the input set. Furthermore, its definition is modular: the covering problem is designed so that it can easily be generalized to accommodate future needs.

[1] No pun intended.

Finally, the relevance of our new framework is supported by the fact that we are able to obtain covering algorithms for the fragments that were already known to enjoy a decidable separation problem. In contrast to the previous algorithms, these more general ones are presented within a single unified framework. This is our third contribution. We present actual covering algorithms for five particular logics: first-order logic (FO), two-variable FO (FO^2) and three logics within the quantifier alternation hierarchy of FO (Σ_1, $\mathcal{B}\Sigma_1$ and Σ_2). We also illustrate our proof techniques for two of these cases, Σ_1 and FO^2. As explained, the payoff is that we obtain *effective* solutions to the covering problem. Hence, we obtain an effective method for building separators in the weaker separation problem.

Organization. We define the covering problem in Section 3. We then devote Sections 4 to 7 to the presentation of our general framework designed for tackling the covering problem when considering classes that are *lattices of regular languages*. We then illustrate this framework with a simple example in Section 8: the fragment Σ_1 in the quantifier alternation hierarchy of first-order logic. Finally, the last two sections are devoted to a simplification of our framework which may be used in the restricted case of classes that are Boolean algebras. We explain this simplification in Section 9. Finally, we present a detailed example for such a class in Section 10 (two-variable first-order logic: FO^2).

2. Preliminary definitions

In this section, we present the standard terminology that we shall need to formulate our results. Specifically, we define of classes of languages and their properties. Moreover, we introduce the standard membership and separation problems (which we shall generalize with the covering problem in the next section). Finally, we define stratifications for a class of languages. This is a mathematical tool which is often useful in proofs.

2.1. Finite words and classes of languages.

Throughout the whole paper, we fix a finite alphabet A and work with words over A. We denote by A^* the set of all finite words over A. We let ε be the empty word, and A^+ be the set $A^* \setminus \{\varepsilon\}$ of all nonempty words over A.

Given a word $w \in A^*$, we denote by $\mathbf{alph}(w)$ the set of letters appearing in w, that is, the smallest set $B \subseteq A$ such that $w \in B^*$. We say that $\mathbf{alph}(w)$ is the *alphabet* of w. Finally, for $B \subseteq A$, we write $B^{\circledast}$ for the set of words whose alphabet is exactly B, that is,

$$B^{\circledast} = \{w \in A^* \mid \mathbf{alph}(w) = B\}.$$

Observe that $B^{\circledast} \subseteq B^*$, and that $B^{\circledast} \subsetneq B^*$ when $B \neq \emptyset$.

A *language (over A)* is a subset of A^*. Furthermore, a *class of languages* $\mathcal{C}$ is simply a set of languages over A.

Remark 2.1. *When it is important to consider several alphabets, a class of languages is usually defined as a function that maps a finite alphabet A to a set of languages $\mathcal{C}(A)$ over A. However, we adopt a simpler terminology in this paper, since we do not need to deal with several alphabets.*

All classes that we consider in the paper satisfy robust properties. We present them now. We say that a class $\mathcal{C}$ of languages is a *lattice* if it contains $\emptyset$ and A^* and is closed under union and intersection. A *Boolean algebra* is a lattice that is additionally closed under complement. Finally, given a language $L \subseteq A^*$ and a word $u \in A^*$, the left quotient $u^{-1}L$ of L by u is the language

$$u^{-1}L \stackrel{\text{def}}{=} \{w \in A^* \mid uw \in L\}.$$

The right quotient Lu^{-1} of L by u is defined symmetrically. A class $\mathcal{C}$ is *quotienting* when it is closed under taking (left and right) quotients by words of A^*. In the paper, all classes that we consider are at least lattices.

Example 2.2. *Let* AT *be the class of languages consisting of all Boolean combinations of languages B^*, for some sub-alphabet $B \subseteq A$. Here, "*AT*" stands for "alphabet testable": a language is in* AT *when membership of a word in this language depends only on the set of letters occurring in the word. It is straightforward to verify that* AT *is a* **finite** *quotienting Boolean algebra, which will serve as an important example in the paper.*

Furthermore, we are only interested in regular languages, *i.e.*, the classes that we consider in the paper contain regular languages only. These are the languages that can be equivalently defined by nondeterministic finite automata, finite monoids or monadic second-order logic. In the paper, we shall use the automata and monoid definitions which we briefly recall below.

Automata. A nondeterministic finite automaton (NFA) is a tuple $\mathcal{A} = (Q, I, F, \delta)$, where Q is a finite set of states, I (resp. F) is the set of initial (resp. final) states, and $\delta \subseteq Q \times A \times Q$ is a set of transitions. For such an NFA and two states $q, r \in Q$, we shall write $L_{q,r} \stackrel{\text{def}}{=} \{w \in A^* \mid q \xrightarrow{w} r\}$ for the language of words labeling a run from state q to state r. It is well-know that a language is regular L when it is recognized by some NFA $\mathcal{A}$, *i.e.*, L is the union of all languages $L_{q,r}$ with $q \in I$ and $r \in F$.

Semigroups and monoids. A *semigroup* is a set S endowed with a binary associative operation $(s,t) \mapsto s \cdot t$. We will also write st instead of $s \cdot t$. An idempotent of a semigroup S is an element $e \in S$ such that $ee = e$. It is folklore that for any *finite* semigroup S, there exists a natural number $\omega(S)$ (denoted by ω when S is understood from the context) such that for any $s \in S$, the element s^ω is idempotent.

A monoid is a semigroup having a neutral element 1_S, *i.e.*, such that $1_S \cdot s = s \cdot 1_S = s$ for every element s of the monoid. In particular, A^+ is a semigroup (the binary operation is the concatenation of words) and A^* is a monoid, with ε as the neutral element. An *ordered monoid* is a monoid M together with an order relation $\leq$ on M which is compatible with the multiplication of M, that is, such that $s \leq s'$ and $t \leq t'$ imply $ss' \leq tt'$.

A *morphism* between two monoids M, M' is a map $\alpha : M \to M'$ such that $\alpha(1_M) = 1_{M'}$ and for all $s, t \in M$, we have $\alpha(st) = \alpha(s)\alpha(t)$. It is well-known that a language L is regular if and only if there exists a morphism from A^* into a finite monoid such that membership of the word in L is determined by its image under this morphism.

2.2. The membership and separation problems.

As announced above, in the paper, we only work with classes of regular languages. Usually, such a class $\mathcal{C}$ is associated to a syntax: the languages in $\mathcal{C}$ are those which can be described by (at least) one representation in this syntax (see the example of first-order logic below). When we have such a class of languages in hand, the most basic question is whether, for a regular language given as input, one can test membership of the language in the class $\mathcal{C}$. In other words, we want to determine whether there exists an algorithm deciding whether this input language admits a description in the given syntax. The corresponding decision problem is called $\mathcal{C}$-*membership* (or membership for $\mathcal{C}$).

Definition 2.3 (*Membership problem* for $\mathcal{C}$)**.**

Input: *A regular language L.*
Question: *Does L belong to $\mathcal{C}$?*

Recent solutions to the membership problem actually consider a more general problem, the $\mathcal{C}$-*separation problem* (or separation problem for $\mathcal{C}$). It is the following decision problem:

Definition 2.4 (*Separation problem* for $\mathcal{C}$)**.**
Input: *Two regular languages L_1, L_2.*
Question: *Does there exist a language K from $\mathcal{C}$ such that $L_1 \subseteq K$ and $K \cap L_2 = \emptyset$?*

We say that a language K such that $L_1 \subseteq K$ and $K \cap L_2 = \emptyset$ is a *separator* of (L_1, L_2). Observe that since regular languages are closed under complement, there is a straightforward reduction from membership to separation. Indeed, an input language L belongs to $\mathcal{C}$ when it can be $\mathcal{C}$-separated from its complement.

As we explained in the introduction, we shall not work directly with these two problems in the paper. Instead, we consider the more general *covering problem* (we define it in the next section).

2.3. ***First-order logic* and quantifier alternation.**

Most examples of classes that we shall consider in the paper are taken from logic. Here, we briefly recall the definition of first-order logic over words and its quantifier alternation hierarchy.

One may view a finite word as a logical structure composed of a linearly ordered sequence of positions labeled over A. In first-order logic (FO), one may use the following predicates:

(1) For each letter $a \in A$, a unary predicate P_a which selects positions labeled with an "a",
(2) A binary predicate "$<$" for the (strict) linear order between the positions.

A language L is said to be *first-order definable* when there exists an FO sentence φ such that $L = \{w \mid w \models \varphi\}$. One also denotes by FO the class of all first-order definable languages. It is folklore that FO is a quotienting Boolean algebra.

We shall also consider the quantifier alternation hierarchy of FO. It is natural to classify first-order sentences by counting their number of quantifier alternations. Let $n \in \mathbb{N}$. We say that a FO sentence is Σ_n (resp. Π_n) when its prenex normal form has either,

– *exactly* $n-1$ quantifier alternations (*i.e.*, exactly n blocks of quantifiers) starting with an $\exists$ (resp. $\forall$), or
– *strictly less* than $n-1$ quantifier alternations (*i.e.*, strictly less than n blocks of quantifiers).

For example, a formula whose prenex normal form is

$$\forall x_1 \exists x_2 \forall x_3 \forall x_4\ \varphi(x_1, x_2, x_3, x_4) \quad \text{(with } \varphi \text{ quantifier-free)}$$

is Π_3. In general, the negation of a Σ_n sentence is not a Σ_n sentence (it is a Π_n sentence). Hence it is relevant to define $\mathcal{B}\Sigma_n$ sentences as the Boolean combinations of Σ_n sentences. This gives an infinite hierarchy of classes of languages as presented in Figure 1.

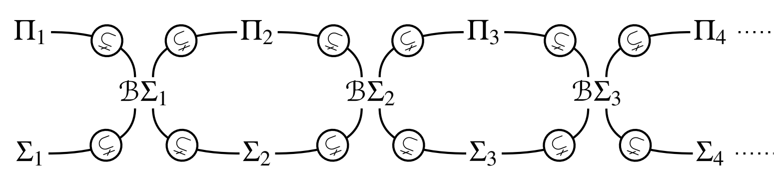


Figure 1: Quantifier Alternation Hierarchy

It is folklore that all classes Σ_n and Π_n in the hierarchy are quotienting lattices (but not Boolean algebras) and that all classes $\mathcal{B}\Sigma_n$ are quotienting Boolean algebras.

2.4. **Finite lattices and canonical preorders.** We finish with the presentation of an important mathematical tool for analyzing classes of languages: *stratifications*. The design principle behind this notion is that *finite* classes (*i.e.*, those containing finitely many distinct languages) are much simpler than others. Let us first explain why.

To any lattice $\mathcal{C}$, we associate a *canonical preorder* $\leqslant_{\mathcal{C}}$ over A^* as follows. For $w, w' \in A^*$, we write:

$$w \leqslant_{\mathcal{C}} w' \quad \text{if and only if} \quad \text{for all } L \in \mathcal{C}, \quad w \in L \ \Rightarrow\ w' \in L.$$

It is immediate from the definition that $\leqslant_{\mathcal{C}}$ is transitive and reflexive, making it a preorder.

Example 2.5. *Consider the finite class* AT *of Example 2.2 (*i.e.*, Boolean combinations of languages B^*, for some $B \subseteq A$). Since* AT *is a Boolean algebra, $\leqslant_{\text{AT}}$ is not only a preorder, but also an equivalence relation, which we denote by $\sim_{\text{AT}}$. One may verify that $w \sim_{\text{AT}} w'$ if and only if w and w' have the same alphabet (*i.e.*, $\mathbf{alph}(w) = \mathbf{alph}(w')$), or equivalently if w and w' belong to the same language $B^{\circledast}$. In other words, the $\sim_{\text{AT}}$-class of any word w is $(\mathbf{alph}(w))^{\circledast}$.*

We will use canonical preorders for *finite* lattices. In this particular case, the relation $\leqslant_{\mathcal{C}}$ has several nice properties. Before presenting them, we need a notation. For a class $\mathcal{C}$ of languages, we define the upper closure $\uparrow_{\mathcal{C}} L$ of a language $L \subseteq A^*$ as the set of words that are above some word of L:

$$\uparrow_{\mathcal{C}} L = \{w \in A^* \mid \exists v \in L,\ v \leqslant_{\mathcal{C}} w\}.$$

We write $\uparrow_{\mathcal{C}} w$ instead of $\uparrow_{\mathcal{C}} \{w\}$. A language $L \subseteq A^*$ is an *upper set* for $\leqslant_{\mathcal{C}}$ when $L = \uparrow_{\mathcal{C}} L$.

Lemma 2.6. *Let $\mathcal{C}$ be a finite lattice. Then, for any $L \subseteq A^*$, we have $L \in \mathcal{C}$ if and only if L is an upper set for $\leqslant_{\mathcal{C}}$. In particular, $\leqslant_{\mathcal{C}}$ has finitely many upper sets.*

Proof. Assume first that $L \in \mathcal{C}$. Then, for all $v \in L$ and all w such that $v \leqslant_{\mathcal{C}} w$, we have $w \in L$ by definition of $\leqslant_{\mathcal{C}}$. Hence, L is an upper set. Conversely, assume that L is an upper set. By definition of $\leqslant_{\mathcal{C}}$, the language $\uparrow_{\mathcal{C}} v$ is the intersection of all $L \in \mathcal{C}$ such that $v \in L$. Therefore, $\uparrow_{\mathcal{C}} v \in \mathcal{C}$ since $\mathcal{C}$ is a finite lattice (and is therefore closed under finite intersection). Finally, since L is an upper set, we have,

$$L = \bigcup_{v \in L} \uparrow_{\mathcal{C}} v.$$

Hence, since $\mathcal{C}$ is closed under union and is finite, L belongs to $\mathcal{C}$. □

While Lemma 2.6 states an equivalence, we mainly use the left to right implication (or rather its contrapositive). One may apply it to show that a language does **not** belong to $\mathcal{C}$. Indeed, by the lemma, proving that $L \notin \mathcal{C}$ is the same as proving that L is not an upper set for $\leqslant_{\mathcal{C}}$. In other words, one needs to exhibit $v, w \in A^*$ such that $v \leqslant_{\mathcal{C}} w$, $v \in L$ and $w \notin L$.

Example 2.7. *Assume that $A = \{a, b\}$ and consider the class* AT *of Example 2.5. The language $L = A^*aA^*bA^*$ does not belong to* AT*. Indeed, $ab \sim_{\text{AT}} ba$, $ab \in L$ and $ba \notin L$.*

As noted in Example 2.5, $\leqslant_{\text{AT}}$ is an equivalence relation. This property holds actually whenever $\mathcal{C}$ is a Boolean algebra: in this case, $\leqslant_{\mathcal{C}}$ is an equivalence relation, which we shall denote by $\sim_{\mathcal{C}}$. In this case, we say that $\sim_{\mathcal{C}}$ is the *canonical equivalence* associated to $\mathcal{C}$.

Lemma 2.8. *Let $\mathcal{C}$ be a finite Boolean algebra. Then the canonical preorder $\leqslant_{\mathcal{C}}$ is an equivalence relation $\sim_{\mathcal{C}}$, which admits the following direct definition:*

$$w \sim_{\mathcal{C}} w' \quad \textit{if and only if} \quad \textit{for all } L \in \mathcal{C}, \quad w \in L \ \Leftrightarrow\ w' \in L.$$

Thus, for any $L \subseteq A^*$, we have $L \in \mathcal{C}$ if and only if L is a union of $\sim_{\mathcal{C}}$-classes. In particular, $\sim_{\mathcal{C}}$ has finite index.

Proof. It is clear that when $w \sim_{\mathcal{C}} w'$, we have $w \leqslant_{\mathcal{C}} w'$ as well. We prove the reverse implication. Let $w, w' \in A^*$ be such that $w \leqslant_{\mathcal{C}} w'$. Let $L \in \mathcal{C}$ and observe that by closure under complement, we know that $A^* \setminus L \in \mathcal{C}$. Therefore, by definition of $w \leqslant_{\mathcal{C}} w'$,

$$\begin{array}{lcl} w \in L & \Rightarrow & w' \in L, \\ w \in A^* \setminus L & \Rightarrow & w' \in A^* \setminus L. \end{array}$$

One may now combine the first implication with the contrapositive of the second, which yields $w \in L \Leftrightarrow w' \in L$. We conclude that $w \sim_{\mathcal{C}} w'$: $\leqslant_{\mathcal{C}}$ and $\sim_{\mathcal{C}}$ are the same relation. Finally, $\sim_{\mathcal{C}}$ has finite index since it has finitely many upper sets by Lemma 2.6. □

Another important and useful property is that when $\mathcal{C}$ is quotienting, the canonical preorder $\leqslant_{\mathcal{C}}$ is compatible with word concatenation.

Lemma 2.9. *A finite lattice $\mathcal{C}$ is quotienting if and only if its associated canonical preorder $\leqslant_{\mathcal{C}}$ is compatible with word concatenation. That is, for any words u, v, u', v',*

$$u \leqslant_{\mathcal{C}} u' \quad \text{and} \quad v \leqslant_{\mathcal{C}} v' \quad \Rightarrow \quad uv \leqslant_{\mathcal{C}} u'v'.$$

Proof. First assume that $\mathcal{C}$ is quotienting and let u, u', v, v' be four words such that $u \leqslant_{\mathcal{C}} u'$ and $v \leqslant_{\mathcal{C}} v'$. We have to prove that $uv \leqslant_{\mathcal{C}} u'v'$. Let $L \in \mathcal{C}$ and assume that $uv \in L$. We use closure under left quotients to prove that $uv' \in L$ and then closure under right quotients to prove that $u'v' \in L$ which terminates the proof of this direction. Since $uv \in L$, we have $v \in u^{-1}L$. By closure under left quotients, we have $u^{-1}L \in \mathcal{C}$, hence, since $v \leqslant_{\mathcal{C}} v'$, we obtain that $v' \in u^{-1}L$ and therefore that $uv' \in L$. It now follows that $u \in L(v')^{-1}$. Using closure under right quotients, we obtain that $L(v')^{-1} \in \mathcal{C}$. Therefore, since $u \leqslant_{\mathcal{C}} u'$, we conclude that $u' \in L(v')^{-1}$ which means that $u'v' \in L$, as desired.

Conversely, assume that $\leqslant_{\mathcal{C}}$ is compatible with concatenation. Let $L \in \mathcal{C}$ and $w \in A^*$, we prove that $w^{-1}L \in \mathcal{C}$ (the proof for right quotients is symmetrical). By Lemma 2.6, we have to prove that $w^{-1}L$ is an upper set. Let $u \in w^{-1}L$ and $u' \in A^*$ be such that $u \leqslant_{\mathcal{C}} u'$. Since $\leqslant_{\mathcal{C}}$ is a compatible with concatenation, we have $wu \leqslant_{\mathcal{C}} wu'$. Hence, since L is an upper set (it belongs to $\mathcal{C}$) and $wu \in L$, we have $wu' \in L$. We conclude that $u' \in w^{-1}L$, which terminates the proof. □

Stratifications. While the above notions are useful, the downside is that they only apply to *finite* lattices. However, it is possible to lift their benefits to infinite classes with the notion of stratification. Consider an arbitrary *infinite* quotienting lattice $\mathcal{C}$. A *stratification* of $\mathcal{C}$ is an infinite sequence $\mathcal{C}_0, \dots, \mathcal{C}_k, \dots$ of *finite* lattices, called *strata* such that,

$$\text{For all } k, \quad \mathcal{C}_k \subseteq \mathcal{C}_{k+1} \quad \text{and} \quad \mathcal{C} = \bigcup_{k \in \mathbb{N}} \mathcal{C}_k.$$

Once we have a stratification of $\mathcal{C}$ in hand, we may associate a canonical preorder $\leqslant_k$ to each stratum $\mathcal{C}_k$. Proving that a language L does not belong to $\mathcal{C}$ now amounts to proving that it does not belong any stratum $\mathcal{C}_k$: for all $k \in \mathbb{N}$, one needs to exhibit $u, v \in A^*$ such that $u \leqslant_k v$, $u \in L$ and $v \notin L$.

3. The covering problem

In this section, we define the covering problem and establish the connection with separation. Furthermore, we outline the steps that we shall later take when devising a general approach for this problem in the next sections.

3.1. **Preliminary definitions.** Unlike the membership problem and as the separation problem, the covering problem takes **two** different objects as input. The first is a single language $L \subseteq A^*$. The second is a *finite multiset of languages* $\mathbf{L} = \{L_1, \dots, L_n\}$. Note that we speak of multisets here for the sake of allowing several copies of the same language in $\mathbf{L}$.

Remark 3.1. *Using multisets of languages here is quite natural. In practice, a "set of languages" is given by a set of recognizers (typically,* NFAs *or monoid morphisms). Since two distinct recognizers may define the same language, our input is indeed a multiset of languages. Another important point is that considering multisets is harmless. If* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *are distinct multisets for the same underlying set of languages, then the covering problems for instances* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *will be equivalent.*

Consider some class $\mathcal{C}$. Given an input language L and an input finite multiset of languages $\mathbf{L}$, the $\mathcal{C}$-covering problem asks whether there exists a $\mathcal{C}$*-cover of* L *which is separating for* $\mathbf{L}$. Let us first define what these notions mean.

Covers. Consider some language $L \subseteq A^*$. A *cover* of L is just a **finite** set of languages $\mathbf{K}$ such that,

$$L \subseteq \bigcup_{K \in \mathbf{K}} K.$$

We shall often look for covers of the universal language A^*. Indeed, this special case suffices when the investigated class $\mathcal{C}$ is a Boolean algebra (we discuss this point in Section 9). Such a cover will be called a *universal cover*.

Separating covers. Consider a finite multiset of languages $\mathbf{L}$ and a set $\mathbf{K}$ of languages. We say that $\mathbf{K}$ is *separating* for $\mathbf{L}$ when the following property holds:

$$\text{For all } K \in \mathbf{K}, \text{ there exists } L' \in \mathbf{L} \text{ such that } K \cap L' = \emptyset.$$

In other words, $\mathbf{K}$ is separating for $\mathbf{L}$ when no $K \in \mathbf{K}$ intersects each of the languages in $\mathbf{L}$. Note that while this definition makes sense for any set of languages $\mathbf{K}$, we are mainly interested in the case when $\mathbf{K}$ is a cover of some other language L. We illustrate this definition in Figure 2.

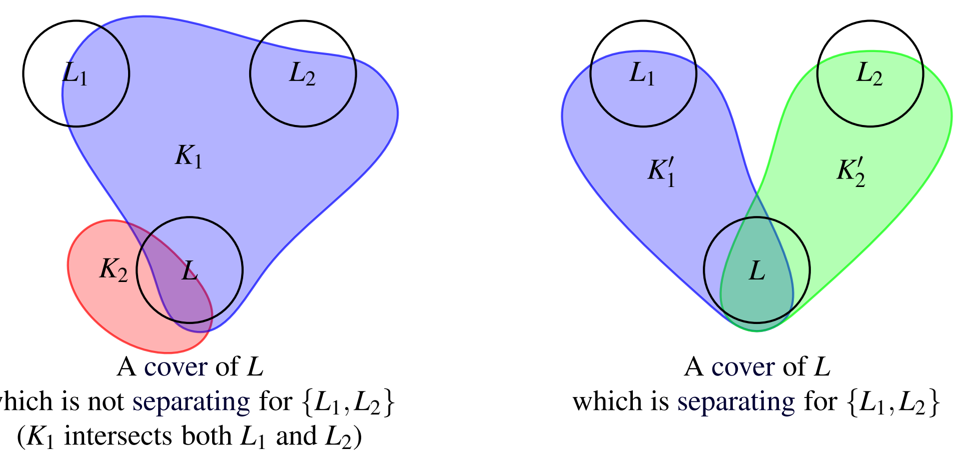


Figure 2: Two covers of L. The right one is separating for $\{L_1, L_2\}$ and the left one is not

A simple observation is that for any language L and any multiset of languages $\mathbf{L}$, there exists a cover of L which is separating for $\mathbf{L}$ if and only if the intersection between L and all languages in $\mathbf{L}$ is empty.

Lemma 3.2. *Let L be a language and $\mathbf{L}$ be a finite multiset of languages. There exists a cover of L which is separating for $\mathbf{L}$ if and only the following condition is satisfied:*

$$L \cap \bigcap_{L' \in \mathbf{L}} L' = \emptyset.$$

Proof. Assume first that there exists a cover $\mathbf{K}$ of L which is separating for $\mathbf{L}$. Since $\mathbf{K}$ is a cover of L, we have $L \subseteq \bigcup_{K \in \mathbf{K}} K$, and therefore,

$$L \cap \bigcap_{L' \in \mathbf{L}} L' \subseteq \bigcup_{K \in \mathbf{K}} \left(K \cap \bigcap_{L' \in \mathbf{L}} L' \right).$$

Moreover, since $\mathbf{K}$ is separating, for any $K \in \mathbf{K}$, there exists $L' \in \mathbf{L}$ such that $K \cap L' = \emptyset$. Therefore, it follows that $\bigcup_{K \in \mathbf{K}} (K \cap \bigcap_{L' \in \mathbf{L}} L') = \emptyset$ and we conclude that $L \cap \bigcap_{L' \in \mathbf{L}} L' = \emptyset$.

Conversely, assume that $L \cap \bigcap_{L' \in \mathbf{L}} L' = \emptyset$. Consider the following equivalence defined between words in L: $u, v \in L$ are equivalent when $u \in L' \Leftrightarrow v \in L'$ for all $L' \in \mathbf{L}$. We let $\mathbf{K}$ be the partition of L induced by this equivalence. Clearly, $\mathbf{K}$ is a cover of L. Moreover, one may verify that it is separating for $\mathbf{L}$ since we have $L \cap \bigcap_{L' \in \mathbf{L}} L' = \emptyset$. ☐

Remark 3.3. *For multisets $\mathbf{L}$ whose size is larger than 2, finding a cover of L which separating for $\mathbf{L}$ is less demanding than finding separators for all pairs of languages (L, L') where $L' \in \mathbf{L}$. For example, consider the alphabet $A = \{a, b, c\}$ and let $L = a^+ + b^+$, $L_1 = b^+ + c^+$ and $L_2 = c^+ + a^+$. It is impossible to separate the pairs (L, L_1) and (L, L_2) (as they pairwise intersect). However, $\{a^*, b^*\}$ is a cover of L which is separating for $\{L_1, L_2\}$.*

Naturally, as for separation, the covering problem restricts the set of allowed covers with a predefined class $\mathcal{C}$: we look for separating covers which are made of languages belonging to $\mathcal{C}$.

3.2. **The problem.** We may now state the covering problem for regular languages. As for separation and membership, it depends on a class $\mathcal{C}$ of languages that restricts the set of possible covers. Given a language L, a $\mathcal{C}$-*cover* of L is a cover $\mathbf{K}$ of L such that all languages $K \in \mathbf{K}$ belong to $\mathcal{C}$. Finally, if $\mathbf{L}$ is a finite multiset of languages, we say that the pair $(L, \mathbf{L})$ is $\mathcal{C}$-*coverable* when there exists a $\mathcal{C}$-cover of L which is separating for $\mathbf{L}$ (when $\mathbf{L}$ is clear from the context, we will simply say that such a cover is separating). The covering problem is a follows.

Definition 3.4 (*Covering problem* for $\mathcal{C}$)**.**

Input:	*A regular language L_1 and a finite multiset of regular languages $\mathbf{L}_2$.*
Question:	*Is $(L_1, \mathbf{L}_2)$ $\mathcal{C}$-coverable?*

There are two stages when solving the covering problem for a given class $\mathcal{C}$.

(1) *Stage One*: find an algorithm which *decides* the covering problem for $\mathcal{C}$ (such an algorithm is called a *covering algorithm* for $\mathcal{C}$).
(2) *Stage Two*: find an algorithm that actually *computes* representations for languages in a separating $\mathcal{C}$-cover when it exists (*i.e.*, when the answer to the question of Stage 1 is "yes").

Let us formally connect $\mathcal{C}$-covering with $\mathcal{C}$-separation: it is more general (provided that the class $\mathcal{C}$ is closed under union). Specifically, separation is the special case of covering when the multiset $\mathbf{L}_2$ is a singleton. While simple, this connection is important: many separation algorithms in the literature are actually based on covering.

Theorem 3.5. *Let $\mathcal{C}$ be a class closed under union and let L_1, L_2 be two languages. The following properties are equivalent:*
(1) L_1 is $\mathcal{C}$-separable from L_2.
(2) $(L_1, \{L_2\})$ is $\mathcal{C}$-coverable.

Proof. Assume first that L_1 is $\mathcal{C}$-separable from L_2. Then there exists $K \in \mathcal{C}$ such that $L_1 \subseteq K$ and $L_2 \cap K = \emptyset$. It follows that $\{K\}$ is a separating $\mathcal{C}$-cover of $(L_1, \{L_2\})$.

Conversely, assume that $(L_1, \{L_2\})$ is $\mathcal{C}$-coverable and let $\mathbf{K}$ be a separating $\mathcal{C}$-cover. We let $K = \bigcup_{K' \in \mathbf{K}} K'$. Clearly, $K \in \mathcal{C}$ by closure under union. Since $\mathbf{K}$ is a cover of L_1, we have $L_1 \subseteq K$ and since $\mathbf{K}$ is separating for $\{L_2\}$, we have $K' \cap L_2 = \emptyset$ for all $K' \in \mathbf{K}$. Thus, $K \cap L_2 = \emptyset$ which means that $K \in \mathcal{C}$ separates L_1 from L_2. □

Now that covering is defined, we need to explain the benefits of considering this problem rather than just separation. We do so by presenting a general framework whose purpose is to obtain covering algorithms for actual classes of languages. This approach is designed with both stages of the problem in mind: finding a decision algorithm and constructing separating covers when they exist. Consider some lattice $\mathcal{C}$. Our approach to $\mathcal{C}$-covering is obtained by combining two independent key ideas that we describe now.

(1) A first key ingredient is that when solving $\mathcal{C}$-covering for some input pair $(L_1, \mathbf{L}_2)$, one never works with $(L_1, \mathbf{L}_2)$ directly. Instead, one builds two new multisets of languages $\mathbf{H}_1$ and $\mathbf{H}_2$ (from L_1 and $\mathbf{L}_2$ respectively) and solves $\mathcal{C}$-covering for *all* input pairs in $\mathbf{H}_1 \times 2^{\mathbf{H}_2}$. The answer for the original pair $(L_1, \mathbf{L}_2)$ is then extracted from this information. While replacing a single input with several may seem counterintuitive, the key idea is that $\mathbf{H}_1$ and $\mathbf{H}_2$ have special properties which are crucially exploited by our algorithms. Note that this is where we need the fact that our inputs are regular: this hypothesis is required in order to carry out the construction of $\mathbf{H}_1$ and $\mathbf{H}_2$.
(2) Our second key ingredient is independent from the first one. When trying to solve $\mathcal{C}$-covering for some input pair $(L, \mathbf{L})$, we view the multiset $\mathbf{L}$ as a *quality measure* that one may use to evaluate $\mathcal{C}$-covers of L. In other words, we are browsing $\mathcal{C}$-covers of L in search for one which is good enough with respect to $\mathbf{L}$. This point of view allows us to reformulate $\mathcal{C}$-covering as a computational problem. One wants to build an object that always exists (regardless of whether the pair $(L, \mathbf{L})$ is indeed $\mathcal{C}$-coverable): a $\mathcal{C}$-cover of L which is *optimal for* $\mathbf{L}$. The main property of this object is that it is a $\mathcal{C}$-cover of L which is separating for any subset $\mathbf{H}$ of $\mathbf{L}$ such that $(L, \mathbf{H})$ is $\mathcal{C}$-coverable. Therefore, having it in hand is enough to solve $\mathcal{C}$-covering for all subsets of $\mathbf{L}$ (including $\mathbf{L}$ itself).

   In particular, this approach makes it possible generalize the $\mathcal{C}$-covering problem as an instance of a generic computational problem. This problem is parametrized by a new object that we name "rating map". Rating maps are algebraic objects that one may use to measure the quality of an arbitrary $\mathcal{C}$-cover, and that abstract the multiset $\mathbf{L}$ (which was also used in the $\mathcal{C}$-covering problem as a quality measure of a cover). The general problem asks to build a $\mathcal{C}$-cover of some language L which is as good as possible with respect to a given rating map ρ: an *optimal* $\mathcal{C}$*-cover of* L *for* ρ. The approach described above for $\mathcal{C}$-covering with input $(L, \mathbf{L})$ is just the instance of this abstract problem for a particular rating map that one may build from $\mathbf{L}$.

In Section 4, we shall detail Item (1), that is, the properties that we need on our input sets and how they are enforced. Item (2), *i.e.*, the notions of quality measure and of rating maps will be presented in Section 5. In Section 6, we shall further investigate the interplay between this second ingredient and the first one. Finally, in Section 7, we will summarize the notions that we have introduced and combine them to outline our general methodology for tackling $\mathcal{C}$-covering.

Remark 3.6. *While most results of the results involved in our framework make sense for any class $\mathcal{C}$ that is a lattice, using the full framework requires a* quotienting lattice of regular languages. *This hypothesis is required for using a lemma which is a crucial part of our approach (we shall point this out when we present the lemma).*

4. Multiplicative multisets

In this section, we present the first key ingredient involved in our general approach for the covering problem: multiplicative multisets of languages. A crucial idea when trying to obtain a covering algorithm for some class of languages $\mathcal{C}$ is that one always looks at several inputs simultaneously. Consider an input pair $(L_1, \mathbf{L}_2)$: L_1 is a language and $\mathbf{L}_2$ a finite multiset of languages. Rather than directly deciding whether $(L_1, \mathbf{L}_2)$ is $\mathcal{C}$-coverable, one considers **two** multisets of languages $\mathbf{H}_1$ and $\mathbf{H}_2$ (built from L_1 and $\mathbf{L}_2$ respectively) and solves $\mathcal{C}$-covering for *all* pairs in $\mathbf{H}_1 \times 2^{\mathbf{H}_2}$. This may seem counterintuitive: we replace one input with several. Yet, the key idea is that $\mathbf{H}_1$ and $\mathbf{H}_2$ enjoy special properties (such as being multiplicative) which are crucially exploited by our algorithms.

The section is organized as follows. First, we define multiplicative multisets and discuss their properties. Then, we show that we may restrict ourselves to multiplicative inputs without loss of generality. The argument is based on a new notion called *extension*. Finally, we introduce a second special kind of multisets: $\mathcal{D}$-compatible multisets (where $\mathcal{D}$ is any finite quotienting Boolean algebra), and we show how to use extension again to also enforce this property for our inputs.

4.1. Multiplicative multisets.

A multiplicative multiset of languages contains only regular languages and has a specific algebraic structure, which is connected to the concatenation operation of languages. While not all finite multisets of regular languages are multiplicative, we shall prove later that we may restrict the covering problem to such inputs without loss of generality. The two typical examples of multiplicative multisets are the following:

- For any morphism $\alpha : A^* \to M$, the multiset $\{\alpha^{-1}(s) \mid s \in M\}$ is multiplicative.
- For any NFA with set of states Q, the multiset $\{L_{q,r} \mid (q,r) \in Q^2\}$ is multiplicative.

The actual definition of multiplicative multisets of languages is designed to encompass these two examples in a single definition, which uses a unified notation. We say that a finite multiset of languages $\mathbf{L}$ is *multiplicative* when $\bigcup_{L \in \mathbf{L}} L = A^*$ and there is a semigroup multiplication "$\odot$" over $\mathbf{L}$ (we use the notation "$\odot$" to avoid confusion with language concatenation) that satisfies the following properties:

(1) For all $L_1, L_2 \in \mathbf{L}$, we have $L_1 L_2 \subseteq L_1 \odot L_2$.
(2) For all $L \in \mathbf{L}$, all words $w \in L$ and all factorizations $w = u_1 u_2$, there exist $L_1, L_2 \in \mathbf{L}$ such that $u_1 \in L_1$, $u_2 \in L_2$ and $L = L_1 \odot L_2$.
(3) Any language $E \in \mathbf{L}$ containing ε is idempotent: $\varepsilon \in E$ implies $E \odot E = E$.

When working with a multiplicative multiset, we will implicitly assume that we have a such a multiplication "$\odot$" for this multiset.

Let us emphasize the fact that while the multiplication "$\odot$" is connected with language concatenation, both operations are distinct and should not be confused. Therefore, we always use the symbol "$\odot$" explicitly: given two elements L_1, L_2 of a multiplicative multiset $\mathbf{L}$, we always write

L_1L_2 to denote the concatenation of L_1 with L_2 and $L_1 \odot L_2$ to denote the multiplication of L_1 with L_2 from the multiplicative multiset $\mathbf{L}$ containing L_1 and L_2.

Remark 4.1. *It is important here that* $\mathbf{L}$ *is a* **multiset** *of languages and not just a set: the algebraic structure is given to the multiset and not to the underlying set of languages. In particular, it may happen that* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *are different multisets whose underlying set is the same and that* $\mathbf{L}_1$ *is multiplicative while* $\mathbf{L}_2$ *is not (see Example 4.3).*

Let us present a few examples of actual multiplicative multisets.

Example 4.2. *If* L_0 *denotes the language of words of even length and* L_1 *denotes the language of words of odd length, the multiset* $\mathbf{L} = \{L_0, L_1\}$ *is multiplicative for the following multiplication:*

$$L_0 \odot L_0 = L_0 \quad L_1 \odot L_1 = L_0 \quad L_0 \odot L_1 = L_1 \quad L_1 \odot L_0 = L_1$$

Example 4.3. *Let* $L_0 = \{\varepsilon\}$, $L_1 = \{a\}$, $L_2 = \{c\}$, $L_3 = \{b\}$, $L_4 = \{b\}$, $L_5 = \{ab\}$, $L_6 = \{ab, c\}$ *and* $L_7 = A^*$ *(there are two copies of the language* $\{b\}$*). The multiset* $\mathbf{L} = \{L_0, L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$ *is multiplicative. Consider the following multiplication (products that are not presented are undefined):*

$$L_1 \odot L_3 = L_5 \quad L_1 \odot L_4 = L_6 \quad \text{and} \quad \text{for all } i,\ L_0 \odot L_i = L_i \odot L_0 = L_i$$

All other multiplications are equal to L_7*. One can verify that this multiplication satisfies all required properties. However, note that we do need the two copies of* $\{b\}$ *in the multiset (the multiset* $\mathbf{L}' = \{L_0, L_1, L_2, L_4, L_5, L_6, L_7\}$ *is not multiplicative).*

An important remark about multiplicative multisets is that they may only contain *regular languages*. We state and prove this fact in the next lemma.

Lemma 4.4. *Let* $\mathbf{L}$ *be a multiplicative multiset of languages. Then all* $L \in \mathbf{L}$ *are regular and one may build an NFA with* $|\mathbf{L}|$ *states which recognizes* L*.*

Proof. Let $L \in \mathbf{L}$, we construct an NFA $\mathcal{A} = (Q, I, F, \delta)$ recognizing L. We let $Q = \mathbf{L}$, $F = \{L\}$ and $I = \{H \in \mathbf{L} \mid \varepsilon \in H\}$. Note that I is not empty since $\bigcup_{L \in \mathbf{L}} L = A^*$ by definition. Finally, we let,

$$\delta = \{(H_1, a, H_2) \in Q \times A \times Q \mid \text{there exists } H' \in \mathbf{L} \text{ such that } a \in H' \text{ and } H_1 \odot H' = H_2\}.$$

By definition, $w = a_1 \cdots a_n \in A^*$ is accepted by $\mathcal{A}$ if and only if there exists $H_0, \cdots, H_n$ such that $\varepsilon \in H_0$, for all $i \geq 1$, $a_i \in H_i$ and $H_0 \odot H_1 \odot \cdots \odot H_n = L$. By Item (1) in the definition of a multiplicative multiset, the latter condition implies that $w \in L$ and by Item (2), the condition $w \in L$ implies it. Hence, we conclude that $w \in L$ if and only if it is accepted by $\mathcal{A}$, which terminates the proof. $\square$

Note that in view of Lemma 4.4, multiplicative multisets are finitely representable. A multiplicative multiset $\mathbf{L}$ is represented by NFAs for all $L \in \mathbf{L}$ (which exist since these languages are regular) and a multiplication table for "$\odot$".

Unfortunately, the converse of Lemma 4.4 is not true: there are finite multisets of regular languages whose union is A^* and that are not multiplicative. This is illustrated by the next example, which presents a multiset failing Condition (2).

Example 4.5. *For* $A = \{a, b\}$*, the multiset* $\mathbf{L} = \{\{ab\}, A^*\}$*, which only contains two regular languages, is not multiplicative. Indeed, if* $\mathbf{L}$ *were multiplicative, since* $a \cdot b \in \{ab\}$ *and since the only language in* $\mathbf{L}$ *containing a (resp. b) is* A^**, we would have* $\{ab\} = A^* \odot A^* \supseteq A^*$*, a contradiction.*

This observation is a problem: while we intend to restrict our covering algorithms to multiplicative inputs, our goal is to obtain a solution working for *all* regular inputs. However, we are able to deal with this issue using a new notion called *extension*.

4.2. **Extension.** Consider two finite multisets of languages $\mathbf{H}$ and $\mathbf{L}$. We say that $\mathbf{H}$ *extends* $\mathbf{L}$ if there exists a map $ex : \mathbf{L} \to 2^{\mathbf{H}}$ such that for any $L \in \mathbf{L}$, we have,

$$L = \bigcup_{H \in ex(L)} H.$$

When we work with two multisets of languages $\mathbf{H}$ and $\mathbf{L}$ such that the former extends the latter, we shall always assume implicitly that we have the map ex in hand. Additionally, we shall say that a subset $\mathbf{H}'$ of $\mathbf{H}$ is $\mathbf{L}$-*exhaustive* when it contains a language in $ex(L)$ for all $L \in \mathbf{L}$.

Observe that by definition, extension is a preorder relation on the set of finite multisets of languages. Note however that it is not an order, as for instance $\mathbf{L} = \{L_1, L_2\}$ and $\mathbf{H} = \{L_1, L_2, L_1 \cup L_2\}$ mutually extend one another. Furthermore, containment is stronger than extension: if $\mathbf{L}$ and $\mathbf{H}$ are two multisets of languages satisfying $\mathbf{H} \supseteq \mathbf{L}$, then $\mathbf{H}$ extends $\mathbf{L}$. However, containment is not the only example of extension, as shown by the preceding example and as illustrated by the following one.

Example 4.6. *Consider five arbitrary languages L_1, L_2, L_3, L_4 and L_5. Then the multiset $\mathbf{H} = \{L_1, L_2, L_3, L_4, L_5\}$ extends the multiset $\mathbf{L} = \{L_1 \cup L_2, L_2, L_3 \cup L_5\}$. Indeed, it suffices to define $ex : \mathbf{L} \to 2^{\mathbf{H}}$ as follows: $ex(L_1 \cup L_2) = \{L_1, L_2\}$, $ex(L_2) = \{L_2\}$ and $ex(L_3 \cup L_5) = \{L_3, L_5\}$.*

We denote by $\uplus$ the disjoint union of *multisets* of languages. For example, $\{L_1\} \uplus \{L_1, L_2\} = \{L_1, L_1, L_2\}$. The following immediate fact states that extension is compatible with disjoint union.

Fact 4.7. Let $\mathbf{L}_1, \mathbf{L}_2, \mathbf{H}_1$ and $\mathbf{H}_2$ be finite multisets of languages. Assume that $\mathbf{H}_1$ extends $\mathbf{L}_1$ and $\mathbf{H}_2$ extends $\mathbf{L}_2$, then $\mathbf{H}_1 \uplus \mathbf{H}_2$ extends $\mathbf{L}_1 \uplus \mathbf{L}_2$.

We may now state the main result about extension which connects it to the covering problem.

Lemma 4.8. *Let $\mathcal{C}$ be a lattice and consider a pair $(L_1, \mathbf{L}_2)$ where L_1 is a language and $\mathbf{L}_2$ a finite multiset of languages. Moreover, let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two multisets extending $\{L_1\}$ and $\mathbf{L}_2$, respectively. The two following properties are equivalent:*

(1) $(L_1, \mathbf{L}_2)$ is $\mathcal{C}$-coverable.

(2) For any $H_1 \in ex(L_1)$ and any $\mathbf{L}_2$-exhaustive $\mathbf{H}_2' \subseteq \mathbf{H}_2$, the pair $(H_1, \mathbf{H}_2')$ is $\mathcal{C}$-coverable.

Proof. Assume first that $(L_1, \mathbf{L}_2)$ is $\mathcal{C}$-coverable. Given $H_1 \in ex(L_1)$ and $\mathbf{H}_2' \subseteq \mathbf{H}_2$ which is $\mathbf{L}_2$-exhaustive, we have to show that $(H_1, \mathbf{H}_2')$ is $\mathcal{C}$-coverable. By hypothesis, we have a $\mathcal{C}$-cover $\mathbf{K}$ of L_1 which is separating for $\mathbf{L}_2$. Since $H_1 \subseteq L_1$, $\mathbf{K}$ is also a $\mathcal{C}$-cover of H_1. We show that $\mathbf{K}$ is separating for $\mathbf{H}_2'$ which concludes the proof of this direction. Let $K \in \mathbf{K}$, we have to find $H_2 \in \mathbf{H}_2'$ such that $K \cap H_2 = \emptyset$. Since $\mathbf{K}$ is separating for $\mathbf{L}_2$, there exists $L_2 \in \mathbf{L}_2$ such that $K \cap L_2 = \emptyset$. Moreover, since $\mathbf{H}_2'$ is $\mathbf{L}_2$-exhaustive, it contains $H_2 \in ex(L_2)$, and in particular, $H_2 \subseteq L_2$. Thus, $K \cap H_2 = \emptyset$.

Conversely, assume that for any $H_1 \in ex(L_1)$ and any $\mathbf{L}_2$-exhaustive $\mathbf{H}_2' \subseteq \mathbf{H}_2$, the pair $(H_1, \mathbf{H}_2')$ is $\mathcal{C}$-coverable. We have to show that $(L_1, \mathbf{L}_2)$ is $\mathcal{C}$-coverable. For any $H \in ex(L_1)$, we will construct a $\mathcal{C}$-cover $\mathbf{K}_H$ of H which is separating for $\mathbf{L}_2$. Since $L_1 = \bigcup_{H \in ex(L_1)} H$, it will follow that $\mathbf{K} = \bigcup_{H \in ex(L_1)} \mathbf{K}_H$ is a $\mathcal{C}$-cover of L_1 which is separating for $\mathbf{L}_2$, concluding the proof.

Let $H \in ex(L_1)$, we build $\mathbf{K}_H$. Let $\mathbf{G}_1, \dots, \mathbf{G}_n \subseteq \mathbf{H}_2$ be the list of all $\mathbf{L}_2$-exhaustive subsets of $\mathbf{H}_2$. By hypothesis, that for all $i \leq n$ there exists a $\mathcal{C}$-cover $\mathbf{R}_i$ of H which is separating for $\mathbf{G}_i$. Now let

$$\mathbf{K}_H = \{R_1 \cap \cdots \cap R_n \mid R_1 \in \mathbf{R}_1, \dots, R_n \in \mathbf{R}_n\}.$$

Clearly, $\mathbf{K}_H$ is a $\mathcal{C}$-cover of H since $\mathbf{R}_1, \dots, \mathbf{R}_n$ are all $\mathcal{C}$-covers of H. Let us show that it is separating for $\mathbf{L}_2$. We proceed by contradiction. Assume that $\mathbf{K}_H$ is **not** separating for $\mathbf{L}_2$. By definition it follows that there exists $K \in \mathbf{K}_H$ which intersects all languages $L_2 \in \mathbf{L}_2$. Moreover, since $\mathbf{H}_2$ extends $\mathbf{L}_2$, any $L_2 \in \mathbf{L}_2$ satisfies $L_2 = \bigcup_{H_2 \in ex(L_2)} H_2$. It follows that for all $L_2 \in \mathbf{L}_2$, K intersects some $H_2 \in ex(L_2)$. Consider the set $\mathbf{G}$ made of all these languages $H_2 \in ex(L_2)$ intersected by K for $L_2 \in \mathbf{L}_2$. By definition, $\mathbf{G}$ is $\mathbf{L}_2$-exhaustive. Hence, we have $i \leq n$ such that $\mathbf{G} = \mathbf{G}_i$. Finally, by definition of $\mathbf{K}_H$, $K \in \mathbf{K}_H$ is of the form $K = R_1 \cap \cdots \cap R_n$ for $R_1 \in \mathbf{R}_1, \dots, R_n \in \mathbf{R}_n$. In particular, $K \subseteq R_i$ which means that R_i intersects all $G \in \mathbf{G}_i$ since this is the case for K. This is a contradiction since $\mathbf{R}_i$ is separating for $\mathbf{G}_i$ by definition. $\square$

As announced, we shall use Lemma 4.8 to restrict our covering algorithms to classes of inputs that are strictly smaller than the one of all regular inputs (typically to multiplicative inputs). The only point that one needs to ensure is that for any finite multiset of regular languages $\mathbf{L}$, one can construct another multiset $\mathbf{H}$ extending $\mathbf{L}$ and belonging to the class of inputs that we choose. In that case, we obtain a reduction from arbitrary regular inputs to this chosen class.

More precisely, given an input pair $(L_1, \mathbf{L}_2)$, one first constructs $\mathbf{H}_1$ and $\mathbf{H}_2$ belonging to the desired class of inputs and extending $\{L_1\}$ and $\mathbf{L}_2$ respectively. Lemma 4.8 then states that deciding whether $(L_1, \mathbf{L}_2)$ is $\mathcal{C}$-coverable reduces to computing all $\mathcal{C}$-coverable pairs in $\mathbf{H}_1 \times 2^{\mathbf{H}_2}$. Let us point out that the proof of the lemma is constructive. When $(L_1, \mathbf{L}_2)$ is indeed $\mathcal{C}$-coverable, we are able to reconstruct a separating $\mathcal{C}$-cover for it provided that we have separating $\mathcal{C}$-covers for the pairs in $\mathbf{H}_1 \times 2^{\mathbf{H}_2}$. Altogether, we get reductions for both stages of the covering problem: deciding it and computing separating covers when they exist.

Finally, observe that when one is only interested in separation, Lemma 4.8 can be replaced by a simpler statement (which is an immediate corollary of the lemma since separation is a special case of covering by Theorem 3.5).

Corollary 4.9. *Let $\mathcal{C}$ be a lattice and consider two languages L_1 and L_2. Moreover, let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two multisets which extend $\{L_1\}$ and $\{L_2\}$ respectively. The two following properties are equivalent:*

(1) L_1 is $\mathcal{C}$-separable from L_2.
(2) For any $H_1 \in ex(L_1)$ and $H_2 \in ex(L_2)$, H_1 is $\mathcal{C}$-separable from H_2.

We now use extension to prove that we may restrict ourselves to multiplicative multisets without loss of generality. We show that given any multiset of regular languages, one may build a multiplicative multiset that extends it. This is stated in the next lemma, where for a set X, we denote by $|X|$ its number of elements, while for an automaton $\mathcal{A}$, we denote by $|\mathcal{A}|$ its number of states.

Proposition 4.10. *Let $\mathbf{L} = \{L_1, \dots, L_n\}$ be a finite multiset of regular languages. Then, one may construct a multiplicative multiset of languages $\mathbf{H}$ extending $\mathbf{L}$ in polynomial time with respect to the size of input recognizers for $L_1, \dots, L_n$. More precisely,*

- *If $L_1, \dots, L_n$ are given by n NFAs $\mathcal{A}_1, \dots, \mathcal{A}_n$, then $|\mathbf{H}| = |\mathcal{A}_1|^2 + \cdots + |\mathcal{A}_n|^2 + 1$.*
- *If $L_1, \dots, L_n$ are given by n monoids $M_1, \dots, M_n$, then $|\mathbf{H}| = |M_1| + \cdots + |M_n| + 1$.*

In view of Lemma 4.8, the main consequence of Proposition 4.10 is that when working on the covering problem for regular inputs, we will always be able to assume without loss of generality that we are given as input two *multiplicative* multisets $\mathbf{H}_1$ and $\mathbf{H}_2$, and that we need to compute all coverable pairs in $\mathbf{H}_1 \times 2^{\mathbf{H}_2}$. An important observation is that the construction is polynomial: the size of the multiplicative multiset that we construct is polynomial with respect recognizers for languages in the input multiset.

We now prove Proposition 4.10 by describing the construction. We begin by associating a canonical multiplicative multiset to any monoid morphism or NFA. This multiset is designed so that it extends the multiset of languages that are recognized by this morphism or NFA.

Canonical multiplicative multiset associated to a monoid morphism. Let $\alpha : A^* \to M$ be a morphism into a finite monoid M. We define the *canonical multiplicative multiset associated to* α as the following multiset:

$$\mathbf{H}_\alpha = \{\alpha^{-1}(s) \mid s \in \alpha(A^*)\}.$$

Remark 4.11. *Note that here, all languages* $\alpha^{-1}(s)$ *are different one from each other.*

That $\mathbf{H}_\alpha$ is multiplicative is straightforward: it suffices to define $\alpha^{-1}(s) \odot \alpha^{-1}(t) = \alpha^{-1}(st)$. One may verify that this multiplication indeed yields a multiplicative multiset. In fact, $\mathbf{L}$ is a monoid (isomorphic to $\alpha(A^*)$). The following fact is also immediate.

Fact 4.12. For any multiset of languages $\mathbf{L}$ that is made of languages recognized by α, the multiplicative multiset $\mathbf{H}_\alpha$ extends $\mathbf{L}$.

Canonical multiplicative multiset associated to an NFA. Let $\mathcal{A} = (Q, I, F, \delta)$ be an NFA. We define the *canonical multiplicative multiset associated to* $\mathcal{A}$ as the following multiset of languages:

$$\mathbf{H}_\mathcal{A} = \{L_{q,r} \mid (q,r) \in Q^2 \text{ and } L_{q,r} \neq \emptyset\} \uplus \{A^*\}.$$

Remark 4.13. *We need several copies of languages here, as it may happen that* $L_{q,r}$ *and* $L_{q',r'}$ *are the same language while* $(q,r) \neq (q',r')$. *In this case, it will be important to have two occurrences of this language in the multiset.*

Let us explain why $\mathbf{H}_\mathcal{A}$ is multiplicative. We begin by defining the multiplication "$\odot$". The language A^* is a zero: given any $L \in \mathbf{H}_\mathcal{A}$, we let $A^* \odot L = L \odot A^* = A^*$. Otherwise, consider $q,r,s,t \in Q$. If $r \neq s$, then $L_{q,r} \odot L_{s,t} = A^*$. Otherwise, $r = s$ and we define $L_{q,r} \odot L_{r,t} = L_{q,t}$.

Observe that this multiplication is well-defined because we are working with a multiset and not with the underlying set (it could be ambiguous otherwise). One can verify that this multiplication makes $\mathbf{H}_\mathcal{A}$ a multiplicative multiset. Moreover, the following fact is immediate.

Fact 4.14. Let L be the language recognized by $\mathcal{A}$. Then $\mathbf{H}_\mathcal{A}$ extends $\{L\}$.

Proof of Proposition 4.10. We may now finish the construction in Proposition 4.10. The proof is based on Fact 4.7 and the following one.

Fact 4.15. Let $\mathbf{H}_1, \dots, \mathbf{H}_n$ be multiplicative multisets. Then, the disjoint union $\mathbf{H}_1 \uplus \cdots \uplus \mathbf{H}_n \uplus \{A^*\}$ is multiplicative as well.

Before proving this fact, we finish the proof of Proposition 4.10. Let $\mathbf{L} = \{L_1, \dots, L_n\}$ be a finite multiset of regular languages. If $L_1, \dots, L_n$ are given by n NFAs $\mathcal{A}_1, \dots, \mathcal{A}_n$, we let $\mathbf{H} = \mathbf{H}_{\mathcal{A}_1} \uplus \cdots \uplus \mathbf{H}_{\mathcal{A}_n} \uplus \{A^*\}$. It is immediate from Facts 4.14 and 4.7 that $\mathbf{H}$ extends $\mathbf{L}$ and from Fact 4.15 that $\mathbf{H}$ is multiplicative. Moreover, it can be verified from the construction that $\mathbf{H}$ satisfies the properties described in the proposition.

If $L_1, \dots, L_n$ are given by n monoids $M_1, \dots, M_n$, let $\alpha_1, \dots, \alpha_n$ be the associated morphisms. We let $\mathbf{H} = \mathbf{H}_{\alpha_1} \uplus \cdots \uplus \mathbf{H}_{\alpha_n} \uplus \{A^*\}$. It is immediate from Facts 4.12 and 4.7 that $\mathbf{H}$ extends $\mathbf{L}$ and from Fact 4.15 that $\mathbf{H}$ is multiplicative. Moreover, it can be verified from the construction that $\mathbf{H}$ satisfies the properties described in the proposition.

It remains to prove Fact 4.15. Let $\mathbf{H}_1, \dots, \mathbf{H}_n$ be multiplicative multisets of languages. We have to prove that $\mathbf{H} = \mathbf{H}_1 \uplus \cdots \uplus \mathbf{H}_n \uplus \{A^*\}$ is multiplicative as well. For $1 \leq i \leq n$, we let $\odot_i$ be the multiplication on $\mathbf{H}_i$, making it a multiplicative multiset. We define a multiplication $\odot$ making $\mathbf{H}$ a multiplicative multiset as follows. For all $1 \leq i \leq n$ and all $S, T \in \mathbf{H}_i$, we define $S \odot T = S \odot_i T$. All other products are equal to A^*. One may verify that this yields a multiplication making $\mathbf{H}$ multiplicative.

4.3. **$\mathcal{D}$-compatible multisets.** We finish the section with another restriction on input multisets that is often encountered in covering algorithm: $\mathcal{D}$-compatibility. Here $\mathcal{D}$ is an arbitrary *finite* quotienting Boolean algebra that we fix for the definition. Intuitively, this restriction is used when considering covering for a class that is built from this finite quotienting Boolean algebra $\mathcal{D}$.

Remark 4.16. *This second restriction is far less important than the restriction of being multiplicative. The restriction of being multiplicative is essential to our methodology and appears in all algorithms. On the other hand, $\mathcal{D}$-compatibility is specific to some classes and could actually be avoided. However, using it makes the presentation simpler and more elegant.*

We explained in Section 2 that given any finite quotienting Boolean algebra $\mathcal{D}$, one may associate a canonical equivalence $\sim_{\mathcal{D}}$ over A^*: two words are equivalent when they belong to the same languages in $\mathcal{D}$. More precisely, given $w, w' \in A^*$,

$$w \sim_{\mathcal{D}} w' \quad \text{if and only if} \quad w \in L \Leftrightarrow w' \in L.$$

In particular, given a word $w \in A^*$, we denote by $[w]_{\mathcal{C}}$ its equivalence class with respect to $\sim_{\mathcal{D}}$. Moreover, since $\mathcal{D}$ is finite, $\sim_{\mathcal{D}}$ has finite index and the languages in $\mathcal{D}$ are exactly the unions of equivalence classes of $\sim_{\mathcal{D}}$. This has an important consequence: there exists a finest partition of A^* into languages of $\mathcal{D}$: the partition in $\sim_{\mathcal{D}}$-classes.

Remark 4.17. *Observe that in the special case when $\mathcal{D} = \mathrm{AT}$, there is a correspondence between $[w]_{\mathrm{AT}}$ and the alphabet $\mathbf{alph}(w)$ of w. Indeed, for each alphabet A, there is a bijection between the equivalence classes of $\sim_{\mathrm{AT}}$ and the sub-alphabets of A (the classes are the languages $B^{\circledast}$ for $B \subseteq A$).*

Another important point is that since $\mathcal{D}$ is closed under quotients, we know that $\sim_{\mathcal{D}}$ is a congruence for word concatenation (whose index is finite as explained above). Thus, the quotient set $A^*/{\sim_{\mathcal{D}}}$ is a finite monoid. We write "$\bullet$" its multiplication. Moreover the map $w \mapsto [w]_{\mathcal{C}}$ is a morphism from A^* to $A^*/{\sim_{\mathcal{D}}}$ (given $u, v \in A^*$, $[u]_{\mathcal{C}} \bullet [v]_{\mathcal{C}} = [uv]_{\mathcal{C}}$) which recognizes all languages in $\mathcal{D}(A)$ (they are the unions of equivalence classes).

Remark 4.18. *It is simple to verify that $A^*/{\sim_{\mathcal{D}}}$ is a multiplicative multiset whose multiplication is "$\bullet$" (it is the one associated to the morphism $w \mapsto [w]_{\mathcal{C}}$).*

$\mathcal{D}$-compatible multisets. We are now ready to define $\mathcal{D}$-compatibility. Consider a finite quotienting Boolean algebra $\mathcal{D}$ and let $\mathbf{H}$ be any finite multiset of languages. We say that $\mathbf{H}$ *is $\mathcal{D}$-compatible* when all $H \in \mathbf{H}$ satisfy the two following conditions:

(1) H is non-empty.
(2) H is included in some equivalence class of $\sim_{\mathcal{D}}$ (*i.e.*, for all $u, v \in L$, $[u]_{\mathcal{C}} = [v]_{\mathcal{C}}$).

In particular, observe that if $\mathbf{H}$ is $\mathcal{D}$-compatible, then given any $H \in \mathbf{H}$, $[H]_{\mathcal{C}}$ is well-defined as the unique equivalence class containing H (*i.e.*, $[H]_{\mathcal{C}} = [w]_{\mathcal{C}}$ for all $w \in H$). In particular, note that for any $H \in \mathbf{H}$, $[H]_{\mathcal{C}}$ is a language (an equivalence class of $\sim_{\mathcal{D}}$) such that $H \subseteq [H]_{\mathcal{C}}$.

This finishes the presentation of $\mathcal{D}$-compatibility. It remains to prove that we may assume without loss of generality that our inputs are both multiplicative and $\mathcal{D}$-compatible. We reuse extension: given any input multiset of regular languages $\mathbf{L}$, we explain how to build a second multiset $\mathbf{H}$ which *extends* $\mathbf{L}$ and is both multiplicative and $\mathcal{D}$-compatible. Combined with Lemma 4.8, this proves that we may restrict our inputs to those that are both multiplicative and $\mathcal{D}$-compatible.

Note that since we already know that we can extend any multiset of regular languages by a multiplicative multiset (this is Proposition 4.10), it suffices to present a construction which builds

$\mathcal{D}$-compatible multisets and preserves the property of being multiplicative. This is what we do now. Let $\mathbf{L} = \{L_1, \dots, L_n\}$ be a finite multiset of regular languages. We explain how to associate a new $\mathcal{D}$-compatible multiset $\mathbf{H}$ to $\mathbf{L}$. Given any $L \in \mathbf{L}$ and any class K in the (finite) quotient $A^*/{\sim_\mathcal{D}}$, we define $H_{L,K}$ as the following language,

$$H_{L,K} = L \cap K$$

We then define $\mathbf{H}$ as the multiset of all languages $H_{L,K}$ that are non-empty for $L \in \mathbf{L}$ and $K \in A^*/{\sim_\mathcal{D}}$. Let us prove that $\mathbf{H}$ satisfies the desired properties.

Proposition 4.19. *The multiset* $\mathbf{H}$ *is* $\mathcal{D}$*-compatible and extends* $\mathbf{L}$*. Moreover, if* $\mathbf{L}$ *is multiplicative, then* $\mathbf{H}$ *is multiplicative as well (and a multiplicative multiplication for* $\mathbf{H}$ *can be built from* $\mathbf{L}$*).*

Proof. It is immediate from the definition that $\mathbf{H}$ extends $\mathbf{L}$ and is $\mathcal{D}$-compatible. We concentrate on proving that if $\mathbf{L}$ is multiplicative, then $\mathbf{H}$ is multiplicative as well. Recall that the quotient set $A^*/{\sim_\mathcal{D}}$ is a monoid for the multiplication "$\bullet$". Let L_{H_1,K_1} and L_{H_2,K_2} in $\mathbf{H}$. We define the following multiplication:

$$H_{L_1,K_1} \odot H_{L_2,K_2} = H_{L_1 \odot L_2, K_1 \bullet K_2}$$

One may verify that this multiplication satisfies the axioms of multiplicative multisets. $\square$

Finally, let us point out that the construction described above may be costly depending on the number of equivalence classes in $A^*/{\sim_\mathcal{D}}$. We summarize the cost of building a multiplicative $\mathcal{D}$-compatible multiset from an arbitrary input multiset of regular languages in Figure 3.

	Size of a multiplicative and $\mathcal{D}$-compatible multiset extending $\mathbf{L}$
$\mathbf{L}$ is given by n NFAs $\mathcal{A}_1, \dots, \mathcal{A}_n$	$(\lvert\mathcal{A}_1\rvert^2 + \cdots + \lvert\mathcal{A}_n\rvert^2 + 1) \times \lvert A^*/{\sim_\mathcal{D}}\rvert$
$\mathbf{L}$ is given by n monoids $M_1, \dots, M_n$	$(\lvert M_1\rvert + \cdots + \lvert M_n\rvert + 1) \times \lvert A^*/{\sim_\mathcal{D}}\rvert$

Figure 3: Size of a multiplicative and $\mathcal{D}$-compatible extension for an input multiset of regular languages $\mathbf{L} = \{L_1, \dots, L_n\}$ over some alphabet A

5. Rating maps and optimal covers

This section details the second key ingredient involved in our approach to $\mathcal{C}$-covering. Given an input pair $(L, \mathbf{L})$, we view the finite multiset $\mathbf{L}$ as a quality measure for evaluating $\mathcal{C}$-covers of L. Our objective is to build such a $\mathcal{C}$-cover of L which is "optimal for this measure". The main point here is that this object always exists (regardless of whether $(L, \mathbf{L})$ is $\mathcal{C}$-coverable) and it is separating for any subset of $\mathbf{L}$ which is $\mathcal{C}$-coverable.

We shall actually work within a more general framework and consider a generic computational problem. It asks to build a $\mathcal{C}$-cover of some input language L that is optimal with respect to a parameter that we name a *rating map*. We shall then prove that for any finite multiset of languages $\mathbf{L}$, one may define a special rating map $\rho_\mathbf{L}$ such that the approach outlined above for $\mathcal{C}$-covering with input $\mathbf{L}$ corresponds to building a $\mathcal{C}$-cover which is optimal for this rating map.

Remark 5.1. *Considering this more general framework has two main benefits. First, working with abstract rating maps rather than the specific ones $\rho_{\mathbf{L}}$ associated to multisets $\mathbf{L}$ of languages simplifies the notation. Moreover, it yields more elegant presentations for covering algorithms.*

Remark 5.2. *This section is independent from the previous one. In other words, our two key ingredients are independent. We shall investigate the interplay between the two in Section 6: if we start from a multiset of languages $\mathbf{L}$ which is either multiplicative or $\mathcal{C}$-compatible how does this property translate on the associated rating map $\rho_{\mathbf{L}}$? In this section however, our multisets of languages are arbitrary.*

We first define rating maps. Then, we explain how to use them for measuring the quality of an arbitrary cover. Given a rating map ρ and a some finite set of language $\mathbf{K}$, we define the *ρ-imprint of* $\mathbf{K}$ which corresponds to this measure. We then use this new notion to define what an optimal $\mathcal{C}$-cover of a language L is for a given rating map ρ. Finally, we connect these definitions with our original goal: solving $\mathcal{C}$-covering.

5.1. **Rating maps.** In order to define a rating map, we first need a *rating set*. A *rating set* is simply a finite commutative and idempotent monoid $(R,+,0_R)$. Recall that being idempotent means that for all $r \in R$, we have $r+r=r$. The binary operation $+$ is called *addition*[2]. Given a rating set R, we define the relation "$\leq$" over R as follows:

$$\text{For all } r,s \in R, \quad r \leq s \text{ when } r+s=s.$$

Fact 5.3. The relation $\leq$ is a partial order, which makes $(R,+,0,\leq)$ an ordered monoid.

Proof. It can be verified that "$\leq$" is indeed a partial order (note that it is reflexive because R is idempotent). Let us check that it is compatible with addition. Let $r_1 \leq r_2$ and $s_1 \leq s_2$, we have to prove that $r_1+s_1 \leq r_2+s_2$. By definition of "$\leq$", $r_1 \leq s_1$ means that $r_2 = r_1+r_2$. Similarly, $s_2 = s_1+s_2$. Therefore, $r_2+s_2 = r_1+r_2+s_1+s_2 = (r_2+s_2)+(r_1+s_1)$ since addition is commutative. This exactly means that $r_1+s_1 \leq r_2+s_2$. □

Once we have a rating set R, a *rating map* for R is a monoid morphism $\rho : (2^{A^*},\cup,\emptyset) \to (R,+,0_R)$, *i.e.*, a map from 2^{A^*} to R satisfying the following properties:

(1) $\rho(\emptyset) = 0_R$.
(2) For all $K_1,K_2 \subseteq A^*$, we have $\rho(K_1 \cup K_2) = \rho(K_1)+\rho(K_2)$.

Fact 5.4. Any rating map $\rho : 2^{A^*} \to R$ is increasing:

$$\text{For all } K_1,K_2 \subseteq A^* \text{ such that } K_1 \subseteq K_2, \text{ we have } \rho(K_1) \leq \rho(K_2).$$

Proof. If $K_1 \subseteq K_2$, then $K_2 = K_1 \cup (K_2 \setminus K_1)$, whence $\rho(K_2) = \rho(K_1)+\rho(K_2 \setminus K_1)$. Since R is idempotent, we get $\rho(K_1)+\rho(K_2) = \rho(K_1)+\rho(K_1)+\rho(K_2 \setminus K_1) = \rho(K_1)+\rho(K_2 \setminus K_1) = \rho(K_2)$, which means that $\rho(K_1) \leq \rho(K_2)$. □

[2] It is often the case to denote by '$+$' a monoid operation when it is commutative. This choice is additionally motivated by the connection with "language union" in the definition of rating maps. Finally, we shall later consider a special class of rating sets, equipped with another binary operation, which we will denote multiplicatively.

Note that for the sake of improved readability, when applying a rating map ρ to a singleton set $K = \{w\}$, we shall write $\rho(w)$ for $\rho(\{w\})$.

Recall that given a rating map ρ, our goal is to define when an $\mathcal{C}$-cover is optimal for ρ, and to obtain algorithms for specific classes $\mathcal{C}$ computing such optimal $\mathcal{C}$-covers with respect to ρ. In order to carry out this computation, we will need some additional properties on ρ. The first one is called "niceness".

We say a rating map ρ is *nice* when it satisfies the following property:

$$\text{For any language } K \subseteq A^*, \quad \rho(K) = \sum_{w \in K} \rho(w). \tag{5.1}$$

Remark 5.5. *Observe that the sum in* (5.1) *is defined even when K is infinite. This is because R is finite, commutative and idempotent, hence the sum boils down to a finite one.*

Remark 5.6. *Not all rating maps are nice. Consider the rating set $R = \{0, 1, 2\}$ whose addition is defined by $i + j = max(i, j)$ for $i, j \in R$. We define $\rho : 2^{A^*} \to R$ by $\rho(\emptyset) = 0$ and for any nonempty $K \subseteq A^*$, $\rho(K) = 1$ if K is finite and $\rho(K) = 2$ if K is infinite. One may verify that ρ is a rating map which is not nice: for any infinite language K, we have $\rho(K) = 2$ while $\sum_{w \in K} \rho(w) = 1$.*

Remark 5.7. *It should be noticed that it is not clear how to finitely represent a rating map. However,* (5.1) *shows that any nice rating map is fully determined by the images of singleton languages $\{w\}$. While this does not yield a finite representation, in our algorithms, we shall work with rating maps having stronger properties that make them finitely representable (see Section 6).*

Canonical rating map associated to a finite multiset. While the above definition is abstract, we are mainly interested in a particular example of rating map which connects the framework presented here to the covering problem. Given a finite multiset of languages $\mathbf{L}$, observe that $2^{\mathbf{L}}$ is an ordered commutative idempotent monoid with union "$\cup$" as the addition.

Remark 5.8. *Since addition is union, the order is inclusion. Indeed, $\mathbf{H}_1 \subseteq \mathbf{H}_2$ if and only if $\mathbf{H}_1 \cup \mathbf{H}_2 = \mathbf{H}_2$.*

We use $2^{\mathbf{L}}$ as the rating set of a specific **nice** rating map $\rho_{\mathbf{L}} : 2^{A^*} \to 2^{\mathbf{L}}$ which we associate to $\mathbf{L}$. We define this rating map as follows:

$$\begin{array}{rccl} \rho_{\mathbf{L}} : & 2^{A^*} & \to & 2^{\mathbf{L}} \\ & K & \mapsto & \{L \in \mathbf{L} \mid L \cap K \neq \emptyset\}. \end{array}$$

Fact 5.9. For any finite multiset of languages $\mathbf{L}$, the mapping $\rho_{\mathbf{L}}$ is a nice rating map.

Proof. Let us first verify that $\rho_{\mathbf{L}}$ is a rating map. Clearly, $\rho_{\mathbf{L}}(\emptyset) = \{L \in \mathbf{L} \mid L \cap \emptyset \neq \emptyset\} = \emptyset$. Moreover given $K_1, K_2 \subseteq A^*$,

$$\begin{array}{rcl} \rho_{\mathbf{L}}(K_1 \cup K_2) & = & \{L \in \mathbf{L} \mid L \cap (K_1 \cup K_2) \neq \emptyset\} \\ & = & \{L \in \mathbf{L} \mid (L \cap K_1) \cup (L \cap K_2) \neq \emptyset\} \\ & = & \{L \in \mathbf{L} \mid (L \cap K_1) \neq \emptyset\} \cup \{L \in \mathbf{L} \mid (L \cap K_2) \neq \emptyset\} \\ & = & \rho_{\mathbf{L}}(K_1) \cup \rho_{\mathbf{L}}(K_2) \end{array}$$

It remains to show that $\rho_{\mathbf{L}}$ is nice. Consider $K \subseteq A^*$. We prove that $\rho_{\mathbf{L}}(K) = \sum_{w \in K} \rho_{\mathbf{L}}(w)$. Clearly, we have $\sum_{w \in K} \rho_{\mathbf{L}}(w) \subseteq \rho_{\mathbf{L}}(K)$ by Fact 5.4 since $\rho_{\mathbf{L}}$ is a rating map. Thus, it suffices to prove the converse inclusion. By definition, we have $\rho_{\mathbf{L}}(K) = \{L \in \mathbf{L} \mid L \cap K \neq \emptyset\}$. Thus, for any $L \in \rho_{\mathbf{L}}(K)$, there exists a word $w_L \in K \cap L$. It is now immediate that,

$$\rho_{\mathbf{L}}(K) \subseteq \sum_{L \in \mathbf{L}} \rho_{\mathbf{L}}(w_L) \subseteq \sum_{w \in K} \rho_{\mathbf{L}}(w)$$

This concludes the proof. $\square$

5.2. **Imprints.** Now that we have rating maps, we turn to imprints. Consider a rating map $\rho : 2^{A^*} \to R$. Given any finite set of languages $\mathbf{K}$, we define the ρ-imprint of $\mathbf{K}$. Intuitively, when $\mathbf{K}$ is a cover of some language L, this object measures the "quality" of $\mathbf{K}$.

Remark 5.10. *We are mainly interested in the case when* $\mathbf{K}$ *is a cover. However, the definition of* ρ*-imprint makes senses regardless of this hypothesis. In fact, it is often convenient in proofs to use it when* $\mathbf{K}$ *is not necessarily a cover.*

Intuitively, we want to define the ρ-imprint of $\mathbf{K}$ as the set $\rho(\mathbf{K}) \subseteq R$ of all images $\rho(K)$ for $K \in \mathbf{K}$. However, it will be convenient to use a slightly different definition which is equivalent for our objective and simplifies the notation. Observe that since R is an ordered set, we may apply a downset operation to subsets of R. For any $E \subseteq R$, we write:

$$\downarrow E = \{r \mid \exists r' \in E \text{ such that } r \leq r'\}.$$

The ρ*-imprint of* $\mathbf{K}$, denoted by $\mathcal{I}[\rho](\mathbf{K})$, is the set,

$$\begin{aligned} \mathcal{I}[\rho](\mathbf{K}) &= \downarrow\{\rho(K) \mid K \in \mathbf{K}\} \subseteq R \\ &= \{r \in R \mid \text{there exists } K \in \mathbf{K} \text{ such that } r \leq \rho(K)\}. \end{aligned}$$

Before we illustrate this notion with the rating maps $\rho_{\mathbf{L}}$ associated to finite multisets of languages, let us make a few observations about ρ-imprints. First observe that since any ρ-imprint is a subset of the evaluation set R of ρ, there are finitely many possible ρ-imprints, even though there are infinitely many sets of languages $\mathbf{K}$. Another simple observation is that all imprints are closed under downset.

Fact 5.11. Let $\rho : 2^{A^*} \to R$ be a rating map. For any finite set of languages $\mathbf{K}$, the ρ-imprint of $\mathbf{K}$ is closed under downset:

$$\downarrow\mathcal{I}[\rho](\mathbf{K}) = \mathcal{I}[\rho](\mathbf{K}).$$

In other words, for any $r \in \mathcal{I}[\rho](\mathbf{K})$ and any $r' \leq r$, we have $r' \in \mathcal{I}[\rho](\mathbf{K})$.

Finally, observe that when $\mathbf{K}$ is the cover of some language L, the ρ-imprint of $\mathbf{K}$ always contain some trivial elements within the evaluation set R. We define the *trivial* ρ*-imprint on* L as follows:

$$\mathcal{I}_{triv}[L,\rho] = \downarrow\{\rho(w) \mid w \in L\} \subseteq R.$$

When $\mathbf{K}$ is a cover of L, we know that for any $w \in L$, there exists $K \in \mathbf{K}$ such that $w \in K$. Thus, $\rho(w) \leq \rho(K)$ by Fact 5.4 and it is immediate by closure under downset that all $r \leq \rho(w)$ belong to $\mathcal{I}[\rho](\mathbf{K})$. Thus, we deduce the following fact.

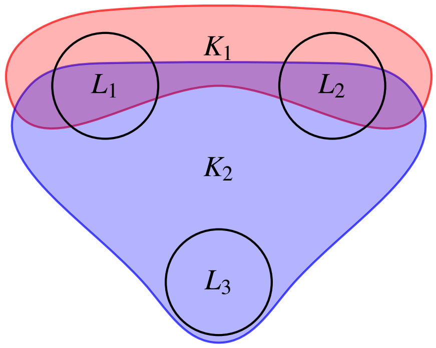
Fact 5.12. Let $\rho : 2^{A^*} \to R$ be a rating map. For any language L and any cover $\mathbf{K}$ of L, we have $\mathcal{I}_{triv}[L,\rho] \subseteq \mathcal{I}[\rho](\mathbf{K})$.

The special case of the rating maps $\rho_{\mathbf{L}}$. We now illustrate imprints with the special case of rating maps $\rho_{\mathbf{L}}$ associated to finite multisets of languages $\mathbf{L}$. In particular, we present a property which is specific to these rating maps and that we use to connect these definitions to the covering problem.

Consider a finite multiset of languages $\mathbf{L}$ and the associated rating map $\rho_{\mathbf{L}} : 2^{A^*} \to 2^{\mathbf{L}}$. For the sake of simplifying the notation, given a finite set of languages $\mathbf{K}$, we shall simply say $\mathbf{L}$*-imprint* for $\rho_{\mathbf{L}}$*-imprint* and write $\mathcal{I}[\mathbf{L}](\mathbf{K})$ for $\mathcal{I}[\rho_{\mathbf{L}}](\mathbf{K})$. Note that if we unravel the definitions for this specific case, we get,

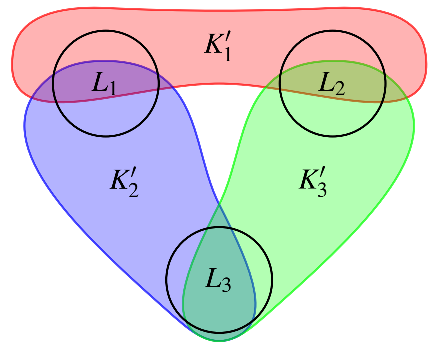
$$\mathcal{I}[\mathbf{L}](\mathbf{K}) = \{\mathbf{H} \subseteq \mathbf{L} \mid \text{there exists } K \in \mathbf{K} \text{ such that } H \cap K \neq \emptyset \text{ for all } H \in \mathbf{H}\} \subseteq 2^{\mathbf{L}}.$$

We illustrate this special case in Figure 4 below.



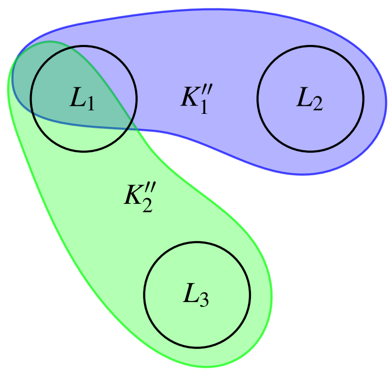
Cover $\mathbf{K} = \{K_1, K_2\}$

$$\mathcal{I}[\mathbf{L}](\mathbf{K}) = \left\{ \begin{array}{l} \{L_1,L_2,L_3\},\{L_1,L_2\}, \\ \{L_1,L_3\},\{L_2,L_3\}, \\ \{L_1\},\{L_2\},\{L_3\},\emptyset \end{array} \right\}$$



Cover $\mathbf{K}' = \{K'_1, K'_2, K'_3\}$

$$\mathcal{I}[\mathbf{L}](\mathbf{K}') = \left\{ \begin{array}{l} \{L_1,L_2\},\{L_1,L_3\}, \\ \{L_2,L_3\},\{L_1\},\{L_2\}, \\ \{L_3\},\emptyset \end{array} \right\}$$



Cover $\mathbf{K}'' = \{K''_1, K''_2\}$

$$\mathcal{I}[\mathbf{L}](\mathbf{K}'') = \left\{ \begin{array}{l} \{L_1,L_2\},\{L_1,L_3\}, \\ \{L_1\},\{L_2\},\{L_3\},\emptyset \end{array} \right\}$$

Figure 4: Examples of covers of $\mathbf{L} = \{L_1, L_2, L_3\}$ and their $\mathbf{L}$-imprints

Note that in this special case, given some arbitrary language L, we have a simpler definition of the trivial $\mathbf{L}$-imprint on L which we denote by $\mathcal{I}_{triv}[L,\mathbf{L}]$ (again for the sake of simplifying the notation).

Fact 5.13. Let L be and language and $\mathbf{L}$ a finite multiset of languages. The trivial $\mathbf{L}$-imprint on L is equal to the following set:

$$\mathcal{I}_{triv}[L,\mathbf{L}] = \left\{ \mathbf{H} \subseteq \mathbf{L} \mid L \cap \bigcap_{H \in \mathbf{H}} H \neq \emptyset \right\}.$$

Proof. By definition, $\mathbf{H} \in \mathcal{I}_{triv}[L,\mathbf{L}]$ if and only if $\mathbf{H} \subseteq \rho_{\mathbf{L}}(w)$ for some $w \in L$. By definition of $\rho_{\mathbf{L}}$, this is equivalent to saying that there exists $w \in L$ such that $w \in H$ for all $H \in \mathbf{H}$, *i.e.*, that $L \cap \bigcap_{H \in \mathbf{H}} H \neq \emptyset$. $\square$

We now connect $\mathbf{L}$-imprints to the covering problem. Given a finite set of languages $\mathbf{K}$, $\mathcal{I}[\mathbf{L}](\mathbf{K})$ records what are the subsets of $\mathbf{L}$ for which $\mathbf{K}$ is separating (or rather those for which it is **not** separating).

Lemma 5.14. *Let $\mathbf{L}$ be a finite multiset of languages and $\mathbf{K}$ a finite set of languages. Consider a subset $\mathbf{H}$ of $\mathbf{L}$. The following properties are equivalent:*

(1) $\mathbf{K}$ *is a separating for* $\mathbf{H}$.
(2) $\mathbf{H} \notin \mathcal{I}[\mathbf{L}](\mathbf{K})$.

Proof. By definition, $\mathbf{H} \notin \mathcal{I}[\mathbf{L}](\mathbf{K})$ if and only if for all $K \in \mathbf{K}$, there exists $H \in \mathbf{H}$ such that $K \cap H = \emptyset$. This is exactly the definition of $\mathbf{K}$ being separating for $\mathbf{H}$. ☐

5.3. **Optimality.** Given a rating map $\rho : 2^{A^*} \to R$ and a language L, the main intuition is that the "best" covers $\mathbf{K}$ of L are those with the smallest possible ρ-imprint $\mathcal{I}[\rho](\mathbf{K}) \subseteq R$ (with respect to inclusion). This is validated by the rating maps $\rho_{\mathbf{L}}$ associated to a finite multiset of languages $\mathbf{L}$. Indeed, in view of Lemma 5.14, given a cover $\mathbf{K}$, the smaller its $\mathbf{L}$-imprint is, the better it is at being separating for subsets of $\mathbf{L}$. For example, observe that in Figure 4, $\mathbf{K}''$ is better than $\mathbf{K}'$ which is itself better than $\mathbf{K}$. Of course, if we do not put any constraint on the covers that we may use, these notions are not very useful. There is always a trivial cover of L that is better than any other. Indeed, partitioning the words in $w \in A^*$ according to the value of $\rho_{\mathbf{L}}(w) \subseteq \mathbf{L}$ yields a cover whose imprint is $\mathcal{I}_{triv}[L, \mathbf{L}]$ and it is impossible to do better by Fact 5.12.

However, when we restrict ourselves to $\mathcal{C}$-*covers* for a fixed lattice $\mathcal{C}$, this "trivial best cover" is not necessarily a $\mathcal{C}$-cover, since there might be languages that do not belong to $\mathcal{C}$. Hence, it now makes sense to define a notion of "best" $\mathcal{C}$-cover.

Definition. We may now define optimality formally. Consider an arbitrary rating map $\rho : 2^{A^*} \to R$ and a lattice $\mathcal{C}$. Given a language L, an *optimal* $\mathcal{C}$-cover of L for ρ is a $\mathcal{C}$-cover $\mathbf{K}$ of L which satisfies the following property:

$$\mathcal{I}[\rho](\mathbf{K}) \subseteq \mathcal{I}[\rho](\mathbf{K}') \quad \text{for any } \mathcal{C}\text{-cover } \mathbf{K}' \text{ of } L.$$

As before, when working with the rating map $\rho_{\mathbf{L}}$ associated to a finite multiset of languages $\mathbf{L}$, we simply speak of optimal $\mathcal{C}$-covers of L for $\mathbf{L}$.

Finally, as we already announced in Section 3, an important special case is when L is the universal language A^*. In this case, we shall speak of *optimal universal* $\mathcal{C}$-*cover for* ρ.

Example 5.15. *We use the alphabet $A = \{a,b,c\}$. Consider the class* AT *of alphabet testable languages (*i.e.*, the Boolean combinations of languages of the form A^*aA^* for some $a \in A$). Let $\mathbf{L} = \{(ab)^+, b(ab)^+, c(ac)^+\}$. Consider*

$$\mathbf{K} = \{A^*bA^*,\ A^* \setminus (A^*bA^*)\}.$$

One may verify that $\mathbf{K}$ is an optimal universal AT*-cover for $\mathbf{L}$. Its $\mathbf{L}$-imprint is*

$$\{\{(ab)^+, b(ab)^+\}, \{(ab)^+\}, \{b(ab)^+\}, \{c(ac)^+\}, \emptyset\}.$$

Note that it contains $\{(ab)^+, b(ab)^+\}$. Therefore, by the necessary condition stated above, $(ab)^+$ and $b(ab)^+$ cannot be separated by alphabet testable languages. This can be verified directly, and used to prove that the cover $\mathbf{K}$ is indeed optimal. Note also that there are other optimal universal AT*-covers for $\mathbf{L}$. For instance*

$$\{A^*cA^*,\ A^* \setminus (A^*cA^*)\}$$

is another universal cover with the same $\mathbf{L}$-imprint as $\mathbf{K}$.

Note that in general, there can be infinitely many optimal $\mathcal{C}$-covers for a given rating map ρ. We now prove that there always exists at least one. In order to prove this, we need our hypothesis that $\mathcal{C}$ is a lattice (more precisely, we need $\mathcal{C}$ to be closed under finite intersection).

Lemma 5.16. *Let $\mathcal{C}$ be a lattice. Then, for any language L and any rating map $\rho : 2^{A^*} \to R$, there exists an optimal $\mathcal{C}$-cover of L for ρ.*

Proof. We already know that there exists a $\mathcal{C}$-cover of L since $\{A^*\}$ is such a cover. We prove that for any two $\mathcal{C}$-covers $\mathbf{K}'$ and $\mathbf{K}''$ of L, there exists a third $\mathcal{C}$-cover $\mathbf{K}$ of L such that $\mathcal{I}[\rho](\mathbf{K}) \subseteq \mathcal{I}[\rho](\mathbf{K}')$ and $\mathcal{I}[\rho](\mathbf{K}) \subseteq \mathcal{I}[\rho](\mathbf{K}'')$. Since there are only finitely possible ρ-imprints (they are all subsets of the finite rating set R), the lemma will follow.

We define $\mathbf{K} = \{K' \cap K'' \mid K' \in \mathbf{K}' \text{ and } K'' \in \mathbf{K}''\}$. Since $\mathbf{K}'$ and $\mathbf{K}''$ are $\mathcal{C}$-covers of L, the set $\mathbf{K}$ is also a cover of L. Moreover, it is a $\mathcal{C}$-cover since $\mathcal{C}$ is closed under intersection. Finally, it is immediate from Fact 5.4 that $\mathcal{I}[\rho](\mathbf{K}) \subseteq \mathcal{I}[\rho](\mathbf{K}')$ and $\mathcal{I}[\rho](\mathbf{K}) \subseteq \mathcal{I}[\rho](\mathbf{K}'')$. $\square$

An important remark is that the proof of Lemma 5.16 is non-constructive. Given a rating map $\rho : 2^{A^*} \to R$, computing an actual optimal $\mathcal{C}$-cover of some language L for ρ is a difficult problem in general. As seen in Theorem 5.21 below, when we work with the rating map $\rho_{\mathbf{L}}$ associated to some finite multiset of languages $\mathbf{L}$, this solves $\mathcal{C}$-covering for any pair $(L, \mathbf{H})$ where $\mathbf{H}$ is a subset of $\mathbf{L}$: whenever a $\mathcal{C}$-cover of L which is separating for $\mathbf{H} \subseteq \mathbf{L}$ exists, any optimal $\mathcal{C}$-cover of L is one. Before we present this theorem, let us make a key observation about optimal $\mathcal{C}$-covers.

Optimal imprint. By definition, given a lattice $\mathcal{C}$, a language L and a rating map $\rho : 2^{A^*} \to R$, all optimal $\mathcal{C}$-covers of L for ρ have the same ρ-imprint. Hence, this unique ρ-imprint is a *canonical* object for $\mathcal{C}$, L and ρ. We say that it is the *$\mathcal{C}$-optimal ρ-imprint on L* and we denote it by $\mathcal{I}_{\mathcal{C}}[L,\rho]$:

$$\mathcal{I}_{\mathcal{C}}[L,\rho] = \mathcal{I}[\rho](\mathbf{K}) \quad \text{for any optimal } \mathcal{C}\text{-cover } \mathbf{K} \text{ of } L \text{ for } \rho.$$

As usual, for the rating maps $\rho_{\mathbf{L}}$ associated to finite multisets of languages $\mathbf{L}$, we speak directly of $\mathcal{C}$-optimal $\mathbf{L}$-imprint on L and write $\mathcal{I}_{\mathcal{C}}[L,\mathbf{L}]$ for $\mathcal{I}_{\mathcal{C}}[L,\rho_{\mathbf{L}}]$.

Let us complete this definition with a few useful results about optimal imprints. We start with two simple facts that one may use to compare optimal imprints for different classes and languages.

Fact 5.17. Let $\rho : 2^{A^*} \to R$ be a rating map and consider two lattices $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{C} \subseteq \mathcal{D}$. Then, for any language L, we have $\mathcal{I}_{\mathcal{D}}[L,\rho] \subseteq \mathcal{I}_{\mathcal{C}}[L,\rho]$.

Proof. Consider an optimal $\mathcal{C}$-cover $\mathbf{K}$ of L for ρ. By definition, we have $\mathcal{I}_{\mathcal{C}}[L,\rho] = \mathcal{I}[\rho](\mathbf{K})$. Moreover, since $\mathcal{C} \subseteq \mathcal{D}$, $\mathbf{K}$ is also a $\mathcal{D}$-cover of L and we have $\mathcal{I}_{\mathcal{D}}[L,\rho] \subseteq \mathcal{I}[\rho](\mathbf{K})$. Altogether, this yields $\mathcal{I}_{\mathcal{D}}[L,\rho] \subseteq \mathcal{I}_{\mathcal{C}}[L,\rho]$ as desired. $\square$

Fact 5.18. Let $\rho : 2^{A^*} \to R$ be a rating map and consider two languages H, L such that $H \subseteq L$. Then, for any lattice $\mathcal{C}$, we have $\mathcal{I}_{\mathcal{C}}[H,\rho] \subseteq \mathcal{I}_{\mathcal{C}}[L,\rho]$.

Proof. Consider an optimal $\mathcal{C}$-cover $\mathbf{K}$ of L for ρ. By definition, we have $\mathcal{I}_{\mathcal{C}}[L,\rho] = \mathcal{I}[\rho](\mathbf{K})$. Moreover, since $H \subseteq L$, $\mathbf{K}$ is also a $\mathcal{C}$-cover of H and we have $\mathcal{I}_{\mathcal{C}}[H,\rho] \subseteq \mathcal{I}[\rho](\mathbf{K})$. Altogether, this yields $\mathcal{I}_{\mathcal{C}}[H,\rho] \subseteq \mathcal{I}_{\mathcal{C}}[L,\rho]$ as desired. $\square$

We finish the presentation with alternate definitions of optimal imprints which will be convenient. The first one is specific to finite classes. If $\leqslant$ is a preorder defined on A^*, w a word and K a language, we write $w \leqslant K$ to denote the fact that $w \leqslant u$ *for all* $u \in K$.

Lemma 5.19. *Let $\mathcal{C}$ be a* **finite** *lattice. Let L be a language and let $\rho : 2^{A^*} \to R$ a rating map. Given any $r \in R$, the following are equivalent:*
(1) $r \in \mathcal{I}_{\mathcal{C}}[L,\rho]$.
(2) There exist $w \in L$ and $K \subseteq A^*$ such that $w \leqslant_{\mathcal{C}} K$ and $r \leq \rho(K)$.

Proof. Assume first that $r \in \mathcal{I}_{\mathcal{C}}[L,\rho]$. For all $w \in L$, we let K_w be the upper set of w for $\leqslant_{\mathcal{C}}$: $K_w = \uparrow_{\mathcal{C}} w = \{u \in A^* \mid w \leqslant_{\mathcal{C}} u\} \in \mathcal{C}$. Since $\mathcal{C}$ is finite, there are finitely many languages K_w. Thus, $\mathbf{K} = \{K_w \mid w \in L\}$ is a $\mathcal{C}$-cover of L. Therefore, $\mathcal{I}_{\mathcal{C}}[L,\rho] \subseteq \mathcal{I}[\rho](\mathbf{K})$ by definition, and $r \in \mathcal{I}[\rho](\mathbf{K})$. It follows that there exists $K \in \mathbf{K}$ such that $r \leq \rho(K)$. By definition of $\mathbf{K}$, there exists $w \in L$ such that $K = K_w$, which means that $w \leqslant_{\mathcal{C}} K$, as desired.

Conversely assume that there exist $w \in L$ and $K \subseteq A^*$ such that $w \leqslant_{\mathcal{C}} K$ and $r \leq \rho(K)$. Consider some optimal $\mathcal{C}$-cover $\mathbf{H}$ of L for ρ. We have to show that $r \in \mathcal{I}[\rho](\mathbf{H})$. Since $w \in L$, there exists some $H \in \mathbf{H}$ such that $w \in H$. Moreover, since $H \in \mathcal{C}$, it must be an upper set for $\leqslant_{\mathcal{C}}$ by Lemma 2.6. Therefore, since $w \leqslant_{\mathcal{C}} K$, we obtain that $K \subseteq H$. It follows that $r \leq \rho(K) \leq \rho(H)$ which means that $r \in \mathcal{I}[\rho](\mathbf{H}) = \mathcal{I}_{\mathcal{C}}[L,\rho]$ as desired. ☐

Our second alternate definition of optimal imprints applies to any lattice admitting a stratification. It is actually a corollary of Lemma 5.19.

Corollary 5.20. *Let $\mathcal{C}$ be a lattice and let $(\mathcal{C}_k)_k$ be a stratification of $\mathcal{C}$ into finite lattices. For $k \in \mathbb{N}$, we write $\leqslant_k$ the canonical preorder of the stratum $\mathcal{C}_k$. Let L be a language and $\rho : 2^{A^*} \to R$ be a rating map. Given $r \in R$, the following are equivalent:*
(1) $r \in \mathcal{I}_{\mathcal{C}}[L,\rho]$.
(2) For all $k \in \mathbb{N}$, there exist $w \in L$ and $K \subseteq A^*$ such that $w \leqslant_k K$ and $r \leq \rho(K)$.

Proof. Assume first that $r \in \mathcal{I}_{\mathcal{C}}[L,\rho]$. By Fact 5.17, it follows that for all $k \in \mathbb{N}$, we have $r \in \mathcal{I}_{\mathcal{C}_k}[L,\rho]$. Combined with Lemma 5.19, this yields that for all $k \in \mathbb{N}$, there exist $w \in L$ and $K \subseteq A^*$ such that $w \leqslant_k K$ and $r \leq \rho(K)$. Conversely, assume that for all $k \in \mathbb{N}$, there exist $w \in L$ and $K \subseteq A^*$ such that $w \leqslant_k K$ and $r \leq \rho(K)$. By Lemma 5.19, it is immediate that $r \in \mathcal{I}_{\mathcal{C}_k}[L,\rho]$ for all $k \in \mathbb{N}$. Moreover, one may verify that there exists $k \in \mathbb{N}$ such that $\mathcal{I}_{\mathcal{C}}[L,\rho] = \mathcal{I}_{\mathcal{C}_k}[L,\rho]$ (this is because an optimal $\mathcal{C}$-cover of L for ρ is necessarily a $\mathcal{C}_k$-cover for some k). Thus, we get $r \in \mathcal{I}_{\mathcal{C}}[L,\rho]$ as desired. ☐

5.4. **Connection with the covering problem.** We are now ready to connect optimal $\mathcal{C}$-covers to the $\mathcal{C}$-covering problem. That is, we connect the notion of being an *optimal* $\mathcal{C}$-cover to that of being a *separating* $\mathcal{C}$-cover. We do so with the following theorem.

Theorem 5.21. *Let $\mathcal{C}$ be a lattice. Consider a language L and $\mathbf{L}$ a finite multiset of languages. Given any subset $\mathbf{H} \subseteq \mathbf{L}$, the following properties are equivalent:*
(1) $(L,\mathbf{H})$ is $\mathcal{C}$-coverable.
(2) $\mathbf{H} \notin \mathcal{I}_{\mathcal{C}}[L,\mathbf{L}]$.
(3) Any optimal $\mathcal{C}$-cover of L for $\mathbf{L}$ is separating for $\mathbf{H}$.

Proof. We prove that $3) \Rightarrow 1) \Rightarrow 2) \Rightarrow 3)$. Let us first assume that 3) holds, *i.e.*, that any optimal $\mathcal{C}$-cover of L for $\mathbf{L}$, is separating for $\mathbf{H}$. Since there exists at least one optimal $\mathcal{C}$-cover of L for $\mathbf{L}$ (see Lemma 5.16), $(L,\mathbf{H})$ is $\mathcal{C}$-coverable, *i.e.*, 1) holds.

We now prove that 1) $\Rightarrow$ 2). Assume that 1) holds, *i.e.*, that $(L,\mathbf{H})$ is $\mathcal{C}$-coverable. This means that there exists a $\mathcal{C}$-cover $\mathbf{K}$ of L which is separating for $\mathbf{H}$. It follows from Lemma 5.14 that $\mathbf{H} \notin \mathcal{I}[\mathbf{L}](\mathbf{K})$. Finally, since $\mathbf{K}$ is a cover of L, we have $\mathcal{I}_{\mathcal{C}}[L,\mathbf{L}] \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K})$ by definition and we conclude that $\mathbf{H} \notin \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$. We get that 2) holds.

It remains to prove that 2) $\Rightarrow$ 3). Assume that $\mathbf{H} \notin \mathcal{I}_{\mathcal{C}}[L,\mathbf{L}]$ and let $\mathbf{K}$ be an optimal $\mathcal{C}$-cover of L for $\mathbf{L}$. Since $\mathbf{K}$ is optimal, we know from our hypothesis that $\mathbf{H} \notin \mathcal{I}[\mathbf{L}](\mathbf{K})$. Hence, it follows from Lemma 5.14 that $\mathbf{K}$ is separating for $\mathbf{H}$. $\square$

In view of Theorem 5.21, both objectives in the $\mathcal{C}$-covering problem can now be reformulated with our new terminology. In order to decide $\mathcal{C}$-covering for a particular input $(L,\mathbf{L})$, it suffices to compute $\mathcal{I}_{\mathcal{C}}[L,\mathbf{L}]$, the $\mathcal{C}$-optimal $\mathbf{L}$-imprint on L (this is the second item in the theorem). Similarly, if a $\mathcal{C}$-cover of L which is separating for $\mathbf{L}$ exists, it suffices to compute an optimal $\mathcal{C}$-cover of L for $\mathbf{L}$ to obtain one. There are two main motivations for using this new formulation:

(1) The optimal $\mathbf{L}$-imprint on L, $\mathcal{I}_{\mathcal{C}}[L,\mathbf{L}]$, is a canonical object for $\mathcal{C}$, L and $\mathbf{L}$ which always exists regardless of whether the answer to $\mathcal{C}$-covering is "yes" or "no".
(2) We shall restrict ourselves to special input sets $\mathbf{L}$ (called "multiplicative"), to gain and exploit special algebraic properties on the rating map $\rho_{\mathbf{L}}$, which extend to the optimal $\mathbf{L}$-imprint on L, $\mathcal{I}_{\mathcal{C}}[L,\mathbf{L}]$. These properties will be crucial in our algorithms (see Sections 4 and 6 for details).

6. Multiplicative rating maps

We have now defined the two key ingredients involved in our general approach for the covering problem. This general approach is presented in Section 7. Here, we describe the interplay between these two key ingredients. More precisely, we investigate the properties of the rating maps $\rho_{\mathbf{L}}$, associated to a multiset $\mathbf{L}$ which is either multiplicative or $\mathcal{D}$-compatible for some finite quotienting Boolean algebra $\mathcal{D}$.

We first define a notion of multiplicative rating map and show that for any multiplicative multiset $\mathbf{L}$, the associated rating map $\rho_{\mathbf{L}}$ is multiplicative. We then do the same for $\mathcal{D}$-compatibility.

6.1. Multiplicative rating maps.

We first explain how the property of being multiplicative for a multiset translates on rating maps. We say that a rating map $\rho : 2^{A^*} \to R$ is *multiplicative* when its rating set R has more structure: it needs to be an *idempotent hemiring*. Moreover, ρ has to satisfy an additional property connecting this structure to language concatenation, namely, it has to be a morphism of hemirings. Let us first define hemirings.

A *hemiring* is a tuple $(R,+,\cdot)$ where R is a set and "+" and "$\cdot$" are two binary operations called addition and multiplication, such that the following axioms are satisfied:

- $(R,+)$ is a commutative monoid whose neutral element is denoted by 0_R.
- $(R,\cdot)$ is a semigroup.
- Multiplication distributes over addition, *i.e.*, for all $r,s,t \in R$ we have:

$$\begin{array}{rcl} r\cdot(s+t) & = & (r\cdot s)+(r\cdot t), \\ (r+s)\cdot t & = & (r\cdot t)+(s\cdot t). \end{array}$$

- The neutral element "0_R" of $(R,+)$ is a zero for $(R,\cdot)$, *i.e.*, for any $r \in R$:

$$0_R \cdot r = r \cdot 0_R = 0_R.$$

Note that we do not require a neutral element for the multiplication. A *semiring* is a hemiring $(R,+,\cdot)$ such that $(R,\cdot)$ is a monoid, *i.e.*, has a neutral element 1_R.

Remark 6.1. *Semirings generalize the more standard notion of rings. A ring $(R,+,\cdot)$ is a semiring for which $(R,+)$ is a group.*

While the notion of semiring is more common in mathematics than the one of hemiring, explicitly demanding a neutral element for R would make some constructions tedious. For this reason, we will work in a universe which is a hemiring only. Note however that the image of 2^{A^*} under the specific rating maps that we shall use in the sequel will actually be semirings. This will follow from the additional properties of these rating maps.

For the sake of simplifying the notation, we shall write rr' instead of $r \cdot r'$. We say that a hemiring R is *idempotent* when $r+r=r$ for any $r \in R$, *i.e.*, when the additive monoid $(R,+)$ is idempotent (on the other hand, note that there is no additional constraint on the multiplicative monoid $(R,\cdot)$). We know from Fact 5.3 that in an idempotent hemiring, the partial order "$\leq$", defined by $r \leq s$ when $r+s=s$, is compatible with addition. Actually, it is also compatible with multiplication.

Fact 6.2. Let R be an idempotent hemiring and let "$\leq$" be its induced ordering relation. For all $r_1,r_2,s_1,s_2 \in R$ such that $r_1 \leq r_2$ and $s_1 \leq s_2$, we have $r_1s_1 \leq r_2s_2$.

Proof. Assume that $r_1 \leq r_2$ and $s_1 \leq s_2$. Then by definition of "$\leq$", we have $r_2 = r_1 + r_2$ and $s_2 = s_1 + s_2$. Therefore, $r_2s_2 = (r_1+r_2)(s_1+s_2) = r_1s_1 + r_2s_1 + r_1s_2 + r_2s_2$, whence $r_1s_1 + r_2s_2 = r_2s_2$ since addition is idempotent. This shows that $r_1s_1 \leq r_2s_2$. ☐

Example 6.3. *A simple example of idempotent hemiring is the set 2^{A^*} of all languages over A. Indeed, it suffices to choose union as the addition (with the empty language as neutral element) and concatenation as the multiplication. More generally, any class of languages which is closed under union and concatenation and contains the empty language is a hemiring. The ordering is then simply language inclusion. Finally, note that $(2^{A^*},\cup,\cdot)$ is actually a semiring (with the singleton language $\{\varepsilon\}$ as the neutral element for concatenation).*

Now that we have idempotent hemirings, we may define multiplicative rating maps. Let $\rho : 2^{A^*} \to R$ be a rating map. Recall that this means that ρ satisfies the following properties:

(1) $\rho(\emptyset) = 0_R$.
(2) For all $K_1,K_2 \subseteq A^*$, we have $\rho(K_1 \cup K_2) = \rho(K_1) + \rho(K_2)$.

Assume that the rating set R is equipped with a second binary operation "$\cdot$" such that $(R,+,\cdot)$ is an idempotent hemiring. We say that ρ is *multiplicative* when $\rho : (2^{A^*},\cdot) \to (R,\cdot)$ is a semigroup morphism, *i.e.*, if ρ satisfies the following additional property:

(3) For any two languages $K_1,K_2 \subseteq A^*$, we have $\rho(K_1K_2) = \rho(K_1) \cdot \rho(K_2)$.

The special case of nice multiplicative rating maps. An important observation is that **nice** multiplicative rating maps have a finite representation. Indeed, as already noted in Remark 5.7, any nice rating map ρ is fully defined by the images of singleton languages $\{w\}$. For multiplicative rating maps, in turn, we have $\rho(w_1w_2) = \rho(w_1)\rho(w_2)$ (recall that we write $\rho(w)$ instead of $\rho(\{w\})$). Therefore, a nice multiplicative rating map is fully determined by the image of the singleton languages of the form $\{a\}$ for $a \in A$, and of $\{\varepsilon\}$.

This is important: since we have a finite representation of nice multiplicative rating maps, we shall be able consider algorithms which take nice multiplicative rating maps as input. Moreover, a crucial point is that given a nice multiplicative rating map $\rho : 2^{A^*} \to R$ and a regular language $K \subseteq A^*$ as input, we are able to evaluate $\rho(K)$. Indeed, by the following lemma, this amounts to checking whether regular languages have non-empty intersection.

Lemma 6.4. *Let $\rho : 2^{A^*} \to R$ is a nice multiplicative rating map and $K \subseteq A^*$ a regular language. Then $\rho(K)$ is the sum of all $r \in R$ such that $K \cap \{w \in A^* \mid \rho(w) = r\} \neq \emptyset$.*

Proof. Since ρ is nice, $\rho(K) = \sum_{w \in K} \rho(w)$. Therefore, the lemma is immediate. □

Connection with multiplicative multisets. Let us now verify that these definitions are indeed connected to multiplicative multisets. We show that when $\mathbf{L}$ is a multiplicative multiset, the associated rating map $\rho_{\mathbf{L}}$ is multiplicative.

Lemma 6.5. *Let $\mathbf{L}$ be a multiplicative multiset of languages. Then $\rho_{\mathbf{L}}$ is a multiplicative rating map.*

Proof. Recall that the rating set of $\rho_{\mathbf{L}}$ is the powerset $2^{\mathbf{L}}$. We first explain why this set is a hemiring when $\mathbf{L}$ is multiplicative. We know that $\mathbf{L}$ is is equipped with a multiplication "$\odot$". Clearly, this multiplication can be lifted to the powerset $2^{\mathbf{L}}$, making it a semigroup. Given $\mathbf{S}, \mathbf{T} \in 2^{\mathbf{L}}$, we define:

$$\mathbf{S} \odot \mathbf{T} = \{S \odot T \mid S \in \mathbf{S} \text{ and } T \in \mathbf{T}\}.$$

Note that $\emptyset \odot \mathbf{S} = \mathbf{S} \odot \emptyset = \emptyset$ for all $\mathbf{S} \in 2^{\mathbf{L}}$. One may also verify that the multiplication "$\odot$" over $2^{\mathbf{L}}$ distributes over union:

$$\begin{array}{rcl} (\mathbf{S}_1 \cup \mathbf{S}_2) \odot \mathbf{S}_3 & = & (\mathbf{S}_1 \odot \mathbf{S}_3) \cup (\mathbf{S}_2 \odot \mathbf{S}_3) \\ \mathbf{S}_1 \odot (\mathbf{S}_2 \cup \mathbf{S}_3) & = & (\mathbf{S}_1 \odot \mathbf{S}_2) \cup (\mathbf{S}_1 \odot \mathbf{S}_3) \end{array}$$

In particular, we obtain that $(2^{\mathbf{L}}, \cup, \odot)$ is an idempotent hemiring since we already know that $(2^{\mathbf{L}}, \cup, \emptyset)$ is an idempotent commutative monoid.

We now show that the rating map $\rho_{\mathbf{L}} : 2^{A^*} \to 2^{\mathbf{L}}$ is multiplicative for this structure of $2^{\mathbf{L}}$. We already know from Fact 5.9 that $\rho_{\mathbf{L}}$ is a rating map. It remains to show that it is multiplicative. For any two languages $K_1, K_2 \subseteq A^*$, we show that:

$$\rho_{\mathbf{L}}(K_1) \odot \rho_{\mathbf{L}}(K_2) \quad = \quad \rho_{\mathbf{L}}(K_1 K_2).$$

Let $L \in \rho_{\mathbf{L}}(K_1) \odot \rho_{\mathbf{L}}(K_2)$. By definition of "$\odot$", there exists $L_1 \in \rho_{\mathbf{L}}(K_1)$ and $L_2 \in \rho_{\mathbf{L}}(K_2)$ such that $L = L_1 \odot L_2$. By Item (1) in the definition of multiplicative multisets, this implies that $L_1 L_2 \subseteq L$. Moreover, since K_i intersects L_i for $i = 1, 2$ by definition of $\rho_{\mathbf{L}}$, we obtain that $K_1 K_2$ intersects $L_1 L_2$ and therefore L. Finally, $L \in \rho_{\mathbf{L}}(K_1 K_2)$.

Conversely, let $L \in \rho_{\mathbf{L}}(K_1 K_2)$. By definition, $K_1 K_2$ intersects L. Hence there exists $w_1 \in K_1$ and $w_2 \in K_2$ such that $w_1 w_2 \in L$. By Item (2) in the definition of multiplicative multisets, we obtain $L_1, L_2 \in \mathbf{L}$ such that $w_1 \in L_1$ and $w_2 \in L_2$ and $L = L_1 \odot L_2$. It is immediate that $L_1 \in \rho_{\mathbf{L}}(K_1)$ and $L_2 \in \rho_{\mathbf{L}}(K_2)$. Therefore, $L \in \rho_{\mathbf{L}}(K_1) \odot \rho_{\mathbf{L}}(K_2)$. □

Multiplicative rating maps and optimal imprints. We finish the presentation with a crucial property of multiplicative rating maps. It turns out that for classes of languages $\mathcal{C}$ which are quotienting lattices of regular languages, the structure of the multiplicative semigroup $(R, \cdot)$ is transferred to $\mathcal{C}$-*optimal* ρ-*imprints*. This result is why our framework is meant to be used for classes that are quotienting lattices of regular languages: it does not hold for arbitrary lattices.

Lemma 6.6. *Let $\mathcal{C}$ be a quotienting lattice of regular languages and let $\rho : 2^{A^*} \to R$ be a multiplicative rating map. Consider two languages L_1, L_2. Then, for any $r_1 \in \mathcal{I}_{\mathcal{C}}[L_1, \rho]$ and any $r_2 \in \mathcal{I}_{\mathcal{C}}[L_2, \rho]$, we have $r_1 r_2 \in \mathcal{I}_{\mathcal{C}}[L_1 L_2, \rho]$.*

Before, we prove Lemma 6.6, let us explain why it is important. Let $\mathcal{C}$ be a quotienting lattice and $\rho : 2^{A^*} \to R$ be a multiplicative rating map. By definition, we have in particular the following hypothesis on ρ:

$$\text{For any } K_1, K_2 \in \mathcal{C}, \quad \rho(K_1) \cdot \rho(K_2) = \rho(K_1 K_2). \tag{6.1}$$

A natural method for building an optimal $\mathcal{C}$-cover $\mathbf{K}$ of some language L for ρ is to start from $\mathbf{K} = \emptyset$ and to add new languages K in $\mathcal{C}$ to $\mathbf{K}$ until $\mathbf{K}$ covers L. By definition of ρ-imprints, for $\mathbf{K}$ to be optimal, we need all languages K that we add in $\mathbf{K}$ to satisfy $\rho(K) \in \mathcal{I}_{\mathcal{C}}[L,\rho]$. It follows from Lemma 6.6 and (6.1) that when $\mathcal{C}$ is a quotienting lattice, we may use concatenation to build new languages K. Assume that we have two languages L_1, L_2 such that $L_1L_2 \subseteq L$. If we have already built K_1 and K_2 in $\mathcal{C}$ such that $\rho(K_1) \in \mathcal{I}_{\mathcal{C}}[L_1,\rho]$ and $\rho(K_2) \in \mathcal{I}_{\mathcal{C}}[L_2,\rho]$, then we may add K_1K_2 to our $\mathcal{C}$-cover of L, provided this language belongs to $\mathcal{C}$, since by Lemmas 6.6 and (6.1), $\rho(K_1K_2) = \rho(K_1) \cdot \rho(K_2) \in \mathcal{I}_{\mathcal{C}}[L_1L_2,\rho] \subseteq \mathcal{I}_{\mathcal{C}}[L,\rho]$.

This is central for classes of languages defined through logic (such as first-order logic). Indeed, concatenation of languages is a fundamental process for building new languages in such classes. This means that for such classes, if K_1 and K_2 belong to $\mathcal{C}$, then so does K_1K_2. We finish with the proof of Lemma 6.6.

Proof of Lemma 6.6. Let $r_1 \in \mathcal{I}_{\mathcal{C}}[L_1,\rho]$ and any $r_2 \in \mathcal{I}_{\mathcal{C}}[L_2,\rho]$, our objective is to prove that $r_1r_2 \in \mathcal{I}_{\mathcal{C}}[L_1L_2,\rho]$. By definition, it suffices to prove that for any $\mathcal{C}$-cover $\mathbf{K}$ of L_1L_2, we have $r_1r_2 \in \mathcal{I}[\rho](\mathbf{K})$. Let $\mathbf{K}$ be a $\mathcal{C}$-cover of L_1L_2. Our objective is to find some language $K \in \mathbf{K}$ such that $r_1r_2 \leq \rho(K)$. We shall need the following claim which is based on the Myhill-Nerode theorem.

Claim. *There exists a language $H \in \mathcal{C}$ which satisfies the following two properties:*
(1) For any $u \in L_1$, there exists $K \in \mathbf{K}$ such that $H \subseteq u^{-1}K$.
(2) $r_2 \leq \rho(H)$

Proof. For any $u \in L_1$, consider the set $\mathbf{Q}_u = \{u^{-1}K \mid K \in \mathbf{K}\}$. Clearly, $\mathbf{Q}_u$ is a $\mathcal{C}$-cover of L_2 since $\mathbf{K}$ is a cover of L_1L_2 and $\mathcal{C}$ is closed under quotients. Moreover, we know by hypothesis on $\mathcal{C}$ that all languages in $\mathbf{K}$ are regular. Therefore, it follows from the Myhill-Nerode theorem that they have finitely many left quotients. Thus, while there may be infinitely many $u \in L_1$, there are only finitely many distinct sets $\mathbf{Q}_u$. It follows that we may use finitely many intersections to build a $\mathcal{C}$-cover $\mathbf{Q}$ of L_2 such that for any $Q \in \mathbf{Q}$ and any $u \in L_1$, there exists $K \in \mathbf{K}$ satisfying $Q \subseteq u^{-1}K$. This means that all $Q \in \mathbf{Q}$ satisfy the first item in the claim, we now pick one which satisfies the second one as well.

Since $r_2 \in \mathcal{I}_{\mathcal{C}}[L_2,\rho]$, and $\mathbf{Q}$ is a $\mathcal{C}$-cover of L_2, we have $r_2 \in \mathcal{I}[\rho](\mathbf{Q})$. Thus, we get $H \in \mathbf{Q}$ such that $r_2 \leq \rho(H)$ by definition. This concludes the proof of the claim. ☐

We may now finish the proof of Lemma 6.6. Let $H \in \mathcal{C}$ be as defined in the claim and consider the following set:

$$\mathbf{G} = \left\{ \bigcap_{v \in H} Kv^{-1} \mid K \in \mathbf{K} \right\}.$$

Observe that all languages in $\mathbf{G}$ belong to $\mathcal{C}$. Indeed, by hypothesis on $\mathcal{C}$, any $K \in \mathbf{K}$ is regular. Thus, it has finitely many right quotients by the Myhill-Nerode theorem and the language $\bigcap_{v \in H} Kv^{-1}$ is the intersection of finitely many quotients of languages in $\mathcal{C}$. By closure under intersection and quotients, it follows that $\bigcap_{v \in H} Kv^{-1} \in \mathcal{C}$. Moreover, $\mathbf{G}$ is a $\mathcal{C}$-cover of L_1. Indeed, given $u \in L_1$, we have $K \in \mathbf{K}$ such that $H \subseteq u^{-1}K$ by the first item in the claim. Hence, given any $v \in H$, we have $u \in Kv^{-1}$ and we obtain that $u \in \bigcap_{v \in H} Kv^{-1}$, which is an element of $\mathbf{G}$.

Therefore, since $r_1 \in \mathcal{I}_{\mathcal{C}}[L_1,\rho]$ by hypothesis, we have $r_1 \in \mathcal{I}[\rho](\mathbf{G})$ and we obtain $G \in \mathbf{G}$ such that $r_1 \leq \rho(G)$. Hence, since $r_2 \leq \rho(H)$ by the second item in the claim, we have $r_1r_2 \leq \rho(G) \cdot \rho(H)$. Moreover, since ρ is a multiplicative rating map, it satisfies $\rho(G) \cdot \rho(H) = \rho(GH)$ by definition, which yields,

$$r_1r_2 \leq \rho(GH).$$

Finally, by definition, $G = \bigcap_{v \in H} Kv^{-1}$ for some $K \in \mathbf{K}$. We show that $GH \subseteq K$. Since ρ is increasing by Fact 5.4, this will yield $r_1 r_2 \leq \rho(GH) \leq \rho(K)$ which concludes the proof of the lemma. Given $w \in GH$, we have $w = uv$ with $u \in G$ and $v \in H$. Moreover, since $v \in H$, we have $u \in Kv^{-1}$ by definition of H. This exactly says that $w = uv \in K$. ☐

6.2. $\mathcal{D}$**-compatible rating maps.** Finally, let us explain how the notion of $\mathcal{D}$-compatible multisets translates on rating maps. Recall that here, $\mathcal{D}$ is an arbitrary finite quotienting Boolean algebra.

Consider a rating map $\rho : 2^{A^*} \to R$. We say that ρ is $\mathcal{D}$*-compatible* when it satisfies the two following properties.

(1) For any language $K \subseteq A^*$ such $K \neq \emptyset$, we have $\rho(K) \neq 0_R$.
(2) For any $\sim_{\mathcal{D}}$-class $H \subseteq A^*$ and any arbitrary language $K \subseteq A^*$, if there exists $r \in R \setminus \{0_R\}$ such that $r \leq \rho(H)$ and $r \leq \rho(K)$, then we have $K \cap H \neq \emptyset$.

This concludes the definition. Let us show that the rating maps $\rho_{\mathbf{L}}$ associated to a $\mathcal{D}$-compatible multiset $\mathbf{L}$ are $\mathcal{D}$-compatible themselves.

Lemma 6.7. *Let* $\mathbf{L}$ *be a finite* $\mathcal{D}$*-compatible multiset of languages. Then, the rating map* $\rho_{\mathbf{L}}$ *is* $\mathcal{D}$*-compatible.*

Proof. We start with the first property. Let $K \subseteq A^*$ such that $K \neq \emptyset$. We have to show that $\rho_{\mathbf{L}}(K) = \neq \emptyset$. By definition, this amounts to finding $L \in \mathbf{L}$ such that $K \cap L \neq \emptyset$. This is immediate since we have $\bigcup_{L \in \mathbf{L}} L = A^*$ by definition of multiplicative multisets.

We turn to the second property. Let $H \subseteq A^*$ be a $\sim_{\mathcal{D}}$-class and $K \subseteq A^*$ an arbitrary language. Moreover, assume that there exists $\mathbf{H} \in 2^{\mathbf{L}} \setminus \{\emptyset\}$ such that $\mathbf{H} \subseteq \rho_{\mathbf{L}}(K)$ and $\mathbf{H} \subseteq \rho_{\mathbf{L}}(H)$. We have to show that $K \cap H \neq \emptyset$. Let $L \in \mathbf{H}$. By hypothesis, we have $L \in \rho_{\mathbf{L}}(H)$ which means that $L \cap H \neq \emptyset$. Since $\mathbf{L}$ is alphabet compatible, it follows that $L \subseteq H$. Finally, since $L \in \rho_{\mathbf{L}}(K)$, we obtain $L \cap K \neq \emptyset$ which yields $K \cap H \neq \emptyset$ as desired. ☐

7. General approach

In this section, we combine all notions that we have introduced so far. Then, we use them to outline a general methodology for handling the covering problem.

We begin by using the results presented in the three previous sections to reformulate covering as a computational problem: *computing optimal pointed imprints*. Then, we present our general methodology for approaching this new problem.

7.1. **Reformulating the covering problem.** For the presentation, we fix some quotienting lattice $\mathcal{C}$. Recall that the $\mathcal{C}$-covering problem is as follows:

Input: A regular language L_1 and a finite multiset of regular languages $\mathbf{L}_2$.
Question: Is $(L_1, \mathbf{L}_2)$ $\mathcal{C}$-coverable?

We use the results of the previous sections to reformulate this question as a more convenient one. We proceed in two steps.

First step: restriction to multiplicative sets. We showed in Section 4 that one may assume without loss of generality that our inputs satisfy more robust properties. Indeed, we know from Proposition 4.10 that we are able to build multiplicative multisets $\mathbf{H}_1$ and $\mathbf{H}_2$ extending $\{L_1\}$ and $\mathbf{L}_2$ respectively (additionally, by Proposition 4.19, we are also able to enforce that $\mathbf{H}_1$ and $\mathbf{H}_2$ are $\mathcal{D}$-compatible for some finite quotienting Boolean algebra $\mathcal{D}$ of our choice). Finally, we know from

Lemma 4.8 that having the pairs in $\mathbf{H}_1 \times 2^{\mathbf{H}_2}$ which are $\mathcal{C}$-coverable in hand is enough to decide whether the original pair $(L_1, \mathbf{L}_2)$ is $\mathcal{C}$-coverable.

Altogether, this means that we may reformulate $\mathcal{C}$-covering as follows. It suffices to obtain an algorithm for the following computational problem:

Input: Two multisets of languages $\mathbf{L}_1, \mathbf{L}_2$ which are multiplicative
(and possibly $\mathcal{D}$-compatible for some finite quotienting Boolean algebra $\mathcal{D}$).
Output: Compute all $\mathcal{C}$-coverable pairs in $\mathbf{L}_1 \times 2^{\mathbf{L}_2}$.

Second step: abstraction with rating maps. We now use the results of Sections 5 and 6 to abstract the above problem with multiplicative rating maps. By Theorem 5.21, the $\mathcal{C}$-coverable pairs in $\mathbf{L}_1 \times 2^{\mathbf{L}_2}$ are encoded in the $\mathcal{C}$-optimal $\mathbf{L}_2$-imprints for $L_1 \in \mathbf{L}_1$, *i.e.*, in the set $\mathcal{I}_{\mathcal{C}}[L_1, \mathbf{L}_2]$. Therefore, we may abstract the above problem as follows:

Input: A multiplicative multiset $\mathbf{L}$ and a nice multiplicative rating map $\rho : 2^{A^*} \to R$
(both possibly $\mathcal{D}$-compatible for some finite quotienting Boolean algebra $\mathcal{D}$).
Output: Compute all $\mathcal{C}$-optimal ρ-imprints $\mathcal{I}_{\mathcal{C}}[L, \rho]$ for $L \in \mathbf{L}$.

Remark 7.1. *Note that we restrict our inputs to* **nice** *multiplicative rating maps since these are the only ones that we are able to finitely represent in general. Moreover, we may assume that ρ is multiplicative and $\mathcal{D}$-compatible by Lemmas 6.5 and 6.7 since we use it to replace a multiplicative and $\mathcal{D}$-compatible multiset from our first step.*

For the sake of simplifying the presentation, we introduce a last notation which encodes all sets $\mathcal{I}_{\mathcal{C}}[L, \rho]$ for $L \in \mathbf{L}$ in a single object. Given any finite multiset $\mathbf{L}$ and any multiplicative rating map $\rho : 2^{A^*} \to R$, we define:

$$\mathcal{P}_{\mathcal{C}}[\mathbf{L}, \rho] = \{(L, r) \in \mathbf{L} \times R \mid r \in \mathcal{I}_{\mathcal{C}}[L, \rho]\} \subseteq \mathbf{L} \times R.$$

We say that $\mathcal{P}_{\mathcal{C}}[\mathbf{L}, \rho]$ is the $\mathcal{C}$-*optimal* $\mathbf{L}$-*pointed* ρ-*imprint*. Altogether we obtain the following proposition, which presents the new computational problem that we shall now consider.

Proposition 7.2. *Consider a lattice $\mathcal{C}$. Assume that there exists an algorithm which computes $\mathcal{P}_{\mathcal{C}}[\mathbf{L}, \rho]$ from a multiplicative multiset $\mathbf{L}$ and a nice multiplicative rating map $\rho : 2^{A^*} \to R$. Then, the $\mathcal{C}$-covering problem is decidable.*

Moreover, the result still holds when the algorithm computing $\mathcal{P}_{\mathcal{C}}[\mathbf{L}, \rho]$ is restricted to $\mathcal{D}$-compatible inputs for some finite quotienting Boolean algebra $\mathcal{D}$.

Remark 7.3. *Proposition 7.2 only deals with the first stage of the covering problem: getting an algorithm that decides it. However, our framework is also designed for handling the second step: computing separating covers when they exist. It is simple to verify from our results that this second step may be reformulated as the following problem:*

Input: *A multiplicative multiset $\mathbf{L}$ and a nice multiplicative rating map $\rho : 2^{A^*} \to R$
(both possibly $\mathcal{D}$-compatible for some finite quotienting Boolean algebra $\mathcal{D}$).*
Output: *Compute optimal $\mathcal{C}$-covers for ρ of all $L \in \mathbf{L}$.*

In view Proposition 7.2, we may now focus on the problem of computing optimal pointed imprints. We explain how to tackle this new problem now. Note that while Proposition 7.2 holds for any lattice, using the methodology that we present now requires at least a quotienting lattice of regular languages.

7.2. **General methodology.** A key design principle behind our framework is that our algorithms for computing optimal pointed imprints are formulated as *elegant characterization theorems*. Given a quotienting lattice $\mathcal{C}$, a multiplicative multiset $\mathbf{L}$ and a nice multiplicative rating map $\rho : 2^{A^*} \to R$, we characterize the $\mathcal{C}$-optimal $\mathbf{L}$-pointed ρ-imprint $\mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$ as the smallest subset of $\mathbf{L} \times R$ which

(1) includes trivial elements, and
(2) is closed under a list of operations.

We speak of a *characterization of* $\mathcal{C}$*-optimal pointed imprints*. Of course, some operations under which $\mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$ is closed are specific to $\mathcal{C}$, but importantly, some are *generic to all quotienting lattices*. We shall present some examples in the paper, and we will detail two of them in Sections 8 and 10.

The key idea is then that such a result yields a least fixpoint procedure for computing the desired set $\mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$, thus solving the covering problem by Proposition 7.2. Indeed, one starts from the set of trivial elements and saturates it with the operations in the list (of course, this depends on our ability to implement these operations, but this is straightforward in practice).

Remark 7.4. *We have to restrict ourselves to nice multiplicative rating maps for the computation, as we are not able to finitely represent arbitrary multiplicative rating maps in general. However, it turns out that the characterizations themselves often hold for all multiplicative rating maps. This will be the case for most of the examples that we present in the paper.*

As we just noticed, an important point is that most of the properties involved in our characterizations are generic to all quotienting lattices and independent from $\mathcal{C}$. Let us now describe these generic properties. First, this is the case for the trivial elements. Recall that given any language L and any multiplicative rating map $\rho : 2^{A^*} \to R$, we defined $\mathcal{I}_{triv}[L,\rho] \subseteq R$ as the set

$$\mathcal{I}_{triv}[L,\rho] = \{r \in R \mid \exists w \in L \text{ such that } r \leq \rho(w)\}.$$

We extend this notation to make it match our new terminology, "pointed imprints". Given a finite multiset $\mathbf{L}$ and a rating map $\rho : 2^{A^*} \to R$, we define,

$$\mathcal{P}_{triv}[\mathbf{L},\rho] = \{(L,r) \in \mathbf{L} \times R \mid r \in \mathcal{I}_{triv}[L,\rho]\} \subseteq \mathbf{L} \times R.$$

Let us point out that it is simple to compute $\mathcal{P}_{triv}[\mathbf{L},\rho]$ from $\mathbf{L}$ and ρ.

Lemma 7.5. *Consider a nice multiplicative rating map $\rho : 2^{A^*} \to R$. Then for any regular language L, one may compute $\mathcal{I}_{triv}[L,\rho]$. Moreover, for any multiplicative multiset $\mathbf{L}$, one may compute $\mathcal{P}_{triv}[\mathbf{L},\rho]$.*

Proof. Consider a regular language L. By definition,

$$\mathcal{I}_{triv}[L,\rho] = \{r \in R \mid \exists w \in L \text{ such that } r \leq \rho(w)\}.$$

Thus, it suffices to show that we may compute the set $T = \{\rho(w) \mid w \in L\}$. Let $\alpha : A^* \to R$ be the morphism (for multiplication) obtained by restricting ρ to A^*: for any word $w \in A^*$, $\alpha(w) = \rho(w)$. Clearly,

$$T = \{r \in R \mid \alpha^{-1}(r) \cap L \neq \emptyset\}.$$

Thus, we are able to compute T since this amounts to checking whether some given regular languages have nonempty intersection. It is then immediate that we may compute $\mathcal{P}_{triv}[\mathbf{L},\rho]$ for some multiplicative multiset $\mathbf{L}$ (since all $L \in \mathbf{L}$ are regular by Lemma 4.4). $\square$

Remark 7.6. *Observe that computing $\mathcal{P}_{triv}[\mathbf{L},\rho]$ does not even require the multiplicative rating map ρ to be nice. Indeed, the computation only requires the restriction of ρ to A^*, which is a morphism into a finite semigroup and therefore finitely representable, regardless of whether ρ is nice.*

We may now present the generic properties satisfied by all $\mathcal{C}$-optimal pointed imprints (provided that $\mathcal{C}$ is a quotienting lattice of regular languages). We do so in the following lemma.

Lemma 7.7. *Consider a multiplicative multiset $\mathbf{L}$, a multiplicative rating map $\rho : 2^{A^*} \to R$ and some quotienting lattice of regular languages $\mathcal{C}$. Then, the $\mathcal{C}$-optimal $\mathbf{L}$-pointed ρ-imprint $\mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho] \subseteq \mathbf{L} \times R$ contains $\mathcal{P}_{triv}[\mathbf{L},\rho]$ and satisfies the following closure properties:*

(1) **Downset***: For any $(L,r) \in \mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$ and any $r' \leq r$, we have $(L,r') \in \mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$.*
(2) **Multiplication***: For any $(H,s),(L,t) \in \mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$, we have $(H \odot L, st) \in \mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$.*

Proof. That $\mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$ contains $\mathcal{P}_{triv}[\mathbf{L},\rho]$ is immediate from Fact 5.12. That it is closed under downset follows from Fact 5.11. Finally closure under multiplication is a consequence of Lemma 6.6. Indeed, assume that $(H,s),(L,t) \in \mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$. This means that $s \in \mathcal{I}_{\mathcal{C}}[H,\rho]$ and $t \in \mathcal{I}_{\mathcal{C}}[L,\rho]$. Thus, we get from Lemma 6.6 that $st \in \mathcal{I}_{\mathcal{C}}[HL,\rho]$. Moreover, $HL \subseteq H \odot L$ by Item (1) in the definition of multiplicative multisets. Thus, we conclude from Fact 5.18 that $st \in \mathcal{I}_{\mathcal{C}}[H \odot L,\rho]$ which exactly says that $(H \odot L, st) \in \mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$. ☐

Finally, we present an alternate definition of $\mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$ which is based on stratifications (it is adapted from Corollary 5.20). Typically, it is used for proving that the $\mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$ satisfies the closure properties stated in our characterization theorems.

Lemma 7.8. *Let $\mathcal{C}$ be a lattice and let $(\mathcal{C}_k)_k$ be a stratification of $\mathcal{C}$ into finite Boolean algebras. For $k \in \mathbb{N}$, we write $\leqslant_k$ the canonical preorder of the stratum $\mathcal{C}_k$. Let $\mathbf{L}$ be a finite multiset of languages and $\rho : 2^{A^*} \to R$ be a rating map. Given $L \in \mathbf{L}$ and $r \in R$, the following are equivalent:*

(1) $(L,r) \in \mathcal{P}_{\mathcal{C}}[\mathbf{L},\rho]$.
(2) For all $k \in \mathbb{N}$, there exist $w \in L$ and $K \subseteq A^*$ such that $w \leqslant_k K$ and $r \leq \rho(K)$.

7.3. **Examples of characterization for quotienting lattices.** We finish with two examples of characterizations to illustrate what one can do within our framework. Both of them come from the quantifier alternation hierarchy of first-order logic over words: they are the levels Σ_1 and Σ_2. We start with a characterization for Σ_1 (which we shall detail and prove in the next section).

Example 7.9 (Characterization of Σ_1-optimal pointed imprints)**.** *This characterization holds for any multiplicative multiset $\mathbf{L}$ and any multiplicative rating map $\rho : 2^{A^*} \to R$. We shall prove that $\mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$ is the smallest subset of $\mathbf{L} \times R$ which contains $\mathcal{P}_{triv}[\mathbf{L},\rho]$ and satisfies the following closure properties:*

(1) **Downset***: For any $(L,r) \in \mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$ and any $r' \leq r$, we have $(L,r') \in \mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$.*
(2) **Multiplication***: For any $(H,s),(L,t) \in \mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$, we have $(H \odot L, st) \in \mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$.*
(3) **Σ_1-closure***: For any $L \in \mathbf{L}$ such that $\varepsilon \in L$, we have, $(L,\rho(A^*)) \in \mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$.*

Our second example is the level Σ_2. Note that we shall not prove this result in the paper. It is essentially adapted from the original separation algorithm of [25] and a proof for the formulation that we use below can be found in [22].

Example 7.10 (Characterization of Σ_2-optimal pointed imprints)**.** *This characterization is restricted to AT-compatible multisets $\mathbf{L}$ (see Example 2.2 for details on AT). Recall that this means that for any $L \in \mathbf{L}$, $[L]_{\mathrm{AT}}$ is well-defined as the unique $\sim_{\mathrm{AT}}$ class containing L.*

Given an AT-compatible multiplicative multiset $\mathbf{L}$ and a multiplicative rating map $\rho : 2^{A^*} \to R$, $\mathcal{P}_{\Sigma_2}[\mathbf{L},\rho]$ is the smallest subset of $\mathbf{L} \times R$ which contains, $\mathcal{P}_{triv}[\mathbf{L},\rho]$ and satisfies the following closure properties:

(1) **Downset***: For any $(L,r) \in \mathcal{P}_{\Sigma_2}[\mathbf{L},\rho]$ and any $r' \leq r$, we have $(L,r') \in \mathcal{P}_{\Sigma_2}[\mathbf{L},\rho]$.*
(2) **Multiplication***: For any $(H,s),(L,t) \in \mathcal{P}_{\Sigma_2}[\mathbf{L},\rho]$, we have $(H \odot L, st) \in \mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$.*
(3) Σ_2**-closure***: For any idempotent $(E,e) \in \mathcal{P}_{\Sigma_2}[\mathbf{L},\rho]$, we have $(E, e \cdot \rho([E]_{\mathrm{AT}}) \cdot e) \in \mathcal{P}_{\Sigma_2}[\mathbf{L},\rho]$.*

Remark 7.11. *Observe that both of the two above examples present classes which are lattices but not Boolean algebras. This is on purpose: when one works with Boolean algebras, our framework can be simplified. We shall explain how in Section 9 and present additional examples for Boolean algebras there.*

8. Example: the logic Σ_1

In this section, we illustrate our framework with a simple example taken from logic. Specifically, we prove a covering algorithm for the level Σ_1 within the quantifier alternation hierarchy of first-order logic. As expected, this algorithm is based on a characterization of Σ_1-optimal pointed imprints.

Remark 8.1. *Since this section is about illustrating our framework, we need a simple example and Σ_1 is an ideal candidate for this. However, it is known that covering is decidable for all levels in the alternation hierarchy of FO up to Σ_3 and all the algorithms may be formulated within our framework.*

We shall actually work with the an alternate definition of the class corresponding to Σ_1. It is language theoretic in nature and was shown by Perrin and Pin [13].

Lemma 8.2. *Consider a language $L \subseteq A^*$. Then L can be defined by a Σ_1 sentence if and only if it is a finite union of languages having the form $A^*a_1A^*a_2A^* \cdots A^*a_nA^*$ with $a_1,\dots,a_n \in A$.*

Remark 8.3. *The result of Perrin and Pin* [13] *is actually much stronger than Lemma 8.2. It establishes an exact correspondence between the quantifier alternation hierarchy and a language theoretic hierarchy called the Straubing-Thérien hierarchy. What we did above is stating this correspondence for the lowest level.*

We may now turn to our characterization of Σ_1-optimal imprints. We first present it and then turn to its proof.

8.1. Characterization of Σ_1-optimal imprints.

We start by describing the property characterizing Σ_1-optimal pointed imprints. Consider a multiplicative multiset $\mathbf{L}$ and a multiplicative rating map $\rho : 2^{A^*} \to R$.

Remark 8.4. *There is no additional constraint on $\mathbf{L}$ and ρ. We do not ask them to be $\mathcal{D}$-compatible for some finite quotienting Boolean algebra $\mathcal{D}$ (we use this for our second detailed example which is presented in Section 10). Moreover, ρ need not be nice for the characterization to hold.*

We say that a subset $S \subseteq \mathbf{L} \times R$ is Σ_1-*saturated* for ρ if it contains $\mathcal{P}_{triv}[\mathbf{L},\rho]$ and is closed under the following operations:

(1) *Downset*: For any $(L,r) \in S$ and any $r' \leq r$, we have $(L,r') \in S$.
(2) *Multiplication*: For any $(H,s),(L,t) \in S$, we have $(H \odot L, st) \in S$.

(3) Σ_1-*closure*: For any $L \in \mathbf{L}$ such that $\varepsilon \in L$, we have,

$$(L, \rho(A^*)) \in S$$

We are now ready to state the main theorem of this chapter: given any multiplicative multiset $\mathbf{L}$ and any multiplicative rating map $\rho : 2^{A^*} \to R$, the Σ_1-optimal $\mathbf{L}$-pointed ρ-imprint $\mathcal{P}_{\Sigma_1}[\mathbf{L}, \rho]$ is the smallest Σ_1-saturated subset of $\mathbf{L} \times R$ (with respect to inclusion).

Theorem 8.5 (Characterization of Σ_1-optimal imprints)**.** *Let $\mathbf{L}$ be a multiplicative multiset of languages and let $\rho : 2^{A^*} \to R$ a multiplicative rating map. Then, $\mathcal{P}_{\Sigma_1}[\mathbf{L}, \rho]$ is the smallest Σ_1-saturated subset of $\mathbf{L} \times R$.*

Observe that Theorem 8.5 yields a least fixpoint procedure for computing $\mathcal{P}_{\Sigma_1}[\mathbf{L}, \rho]$ from any multiplicative multiset $\mathbf{L}$ and *nice* multiplicative rating map ρ given as input. Indeed, we are able to compute $\mathcal{P}_{triv}[\mathbf{L}, \rho]$ (see Lemma 7.5) and all operations in the definition of Σ_1-saturated subsets are clearly implementable (for Σ_1-closure, one may check whether $\varepsilon \in L$ for a given $L \in \mathbf{L}$ since it is regular and $\rho(A^*)$ may be computed). Altogether, we get the desired corollary: Σ_1-covering is decidable.

Corollary 8.6. *The Σ_1-covering problem is decidable.*

We now turn to the proof of Theorem 8.5. Let $\mathbf{L}$ be a multiplicative multiset of languages and $\rho : 2^{A^*} \to R$ be a multiplicative rating map. We start by proving that the set $\mathcal{P}_{\Sigma_1}[\mathbf{L}, \rho]$ is Σ_1-saturated (from an algorithmic point of view this corresponds to soundness of the least fixpoint procedure: it only computes elements of $\mathcal{P}_{\Sigma_1}[\mathbf{L}, \rho]$).

8.2. **Soundness.** We show that $\mathcal{P}_{\Sigma_1}[\mathbf{L}, \rho] \subseteq \mathbf{L} \times R$ is Σ_1-saturated. Since Σ_1 is a quotienting lattice, we already know from Lemma 7.7 that $\mathcal{P}_{\Sigma_1}[\mathbf{L}, \rho]$ contains $\mathcal{P}_{triv}[\mathbf{L}, \rho]$ and is closed under downset and multiplication. Therefore, we may concentrate on Σ_1-closure.

Remark 8.7. *Typically, the soundness proofs for characterizations of optimal imprints involve stratifications and Lemma 7.8 (we shall use this approach for our second detailed example). However, Σ_1 being a very simple class, we manage directly in this case.*

Consider $L \in \mathbf{L}$ such that $\varepsilon \in L$, we need to show that $(L, \rho(A^*)) \in \mathcal{P}_{\Sigma_1}[\mathbf{L}, \rho]$. Let $\mathbf{K}$ be a ρ-optimal Σ_1-cover $\mathbf{K}$ of L. By definition, of $\mathcal{P}_{\Sigma_1}[\mathbf{L}, \rho]$, it suffices to prove that $\rho(A^*) \in \mathcal{I}[\rho](\mathbf{K})$. This follows from the next fact.

Fact 8.8. The only language $K \in \Sigma_1$ which contains ε is A^*.

Proof. By Lemma 8.2, we know that any language in Σ_1 is a finite union of languages having the form $A^*a_1A^*a_2 \cdots A^*a_nA^*$ with $n \in \mathbb{N}$ and $a_1, \dots, a_n \in A$. The only such language which contains ε is A^*, which terminates the proof. □

We may now finish the proof. Since $\varepsilon \in L$ and $\mathbf{K}$ is a cover of L, there exists $K \in \mathbf{K}$ which contains ε. Moreover, since $K \in \Sigma_1$, it follows from Fact 8.8 that $K = A^*$. Thus, $A^* \in \mathbf{K}$ which means that $\rho(A^*) \in \mathcal{I}[\rho](\mathbf{K})$, as desired.

8.3. **Completeness.** We now know that the set $\mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$ is Σ_1-saturated. Thus, it remains to prove that it is the smallest such set. Consider an arbitrary Σ_1-saturated subset $S \subseteq \mathbf{L} \times R$. We show that $\mathcal{P}_{\Sigma_1}[\mathbf{L},\rho] \subseteq S$. Note that from an algorithmic point of view, this direction of the proof corresponds to completeness of the least fixpoint procedure: we show that it computes *all* elements of $\mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$.

We proceed as follows. For each $L \in \mathbf{L}$, we build an Σ_1-cover $\mathbf{K}_L$ of L such that for any $r \in \mathcal{I}[\rho](\mathbf{K}_L)$, we have $(L,r) \in S$. By definition of $\mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$, it will then follow that,

$$\mathcal{P}_{\Sigma_1}[\mathbf{L},\rho] \subseteq \{(L,r) \in \mathbf{L} \times R \mid r \in \mathcal{I}[\rho](\mathbf{K}_L)\} \subseteq S.$$

Remark 8.9. *Since we already showed that $\mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$ itself is Σ_1-saturated, the special case when $S = \mathcal{P}_{\Sigma_1}[\mathbf{L},\rho]$ yields a Σ_1-cover $\mathbf{K}_L$ of all $L \in \mathbf{L}$ which satisfy*

$$\mathcal{I}[\rho](\mathbf{K}_L) = \mathcal{I}_{\Sigma_1}[L,\rho].$$

In other words, we are able to build optimal Σ_1-covers.

We may now start the construction. We first associate a language $K_w \in \Sigma_1$ to any word $w \in A^*$ and then use these languages to construct our covers $\mathbf{K}_L$. Let $w = a_1 \cdots a_n \in A^*$ be some word, we define,

$$K_w = A^*a_1A^*a_2A^* \cdots A^*a_nA^*.$$

Clearly, all languages K_w belong to Σ_1 by Lemma 8.2. We may now define our Σ_1-covers $\mathbf{K}_L$. Let $L \in \mathbf{L}$. We define,

$$\mathbf{K}_L = \{K_w \mid w \in L \text{ and } |w| \leq |\mathbf{L}|\}.$$

Clearly, all sets $\mathbf{K}_L$ are finite. It now remains to prove that they satisfy the desired properties: for all $L \in \mathbf{L}$, $\mathbf{K}_L$ is a Σ_1-cover of L such that for any $r \in \mathcal{I}[\rho](\mathbf{K}_L)$, we have $(L,r) \in S$

We begin by proving that $\mathbf{K}_L$ is a Σ_1-cover of L for all $L \in \mathbf{L}$. Since we already know that all languages K_w belong to Σ_1, it suffices to prove that $\mathbf{K}_L$ is a cover of L. Let $w \in L$ we need to find $K \in \mathbf{K}_L$ such that $w \in K$. We do so in the following lemma.

Lemma 8.10. *For any $w \in L$, there exists $K \in \mathbf{K}_L$ such that $w \in K$.*

Proof. We proceed by induction on $|w|$. When $|w| \leq |\mathbf{L}|$, we have $K_w \in \mathbf{K}_L$ and it is immediate by definition that $w \in K_w$. We now assume that $|w| > |\mathbf{L}|$. We prove that there exist $w_1, w_2 \in A^*$ and $v \in A^+$ such that $w = w_1vw_2$ and $w_1w_2 \in L$. Since $|w_1w_2| < w$, this yields $K \in \mathbf{K}_L$ such that $w_1w_2 \in K$. Moreover, since K is of the form $A^*a_1A^*a_2A^* \cdots A^*a_nA^*$ by definition, it will follow that $w = w_1vw_2$ belongs to K as well, finishing the proof.

Let $w = b_1 \cdots b_{|w|}$. Since $w \in L$, and $\mathbf{L}$ is a multiplicative multiset, Item (2) in the definition of multiplicative multisets yields $L_1, \dots, L_{|w|} \in \mathbf{L}$ such that $L = L_1 \odot \cdots \odot L_{|w|}$ and $b_i \in L_i$ for all i. Since $|w| > |\mathbf{L}|$, it follows by the pigeon-hole principle that there exist i, j with $i < j$ such that $L_1 \odot \cdots \odot L_i = L_1 \odot \cdots \odot L_j$. We let $w_1 = b_1 \cdots b_i$, $v = b_{i+1} \cdots b_j$ and $w_2 = b_{j+1} \cdots b_{|w|}$. By definition, we have $w = w_1vw_2$ and v is non-empty since $i < j$. Moreover,

$$\begin{array}{lcl} w_1 & \in & L_1 \odot \cdots \odot L_i = L_1 \odot \cdots \odot L_j, \\ w_2 & \in & L_{j+1} \odot \cdots \odot L_{|w|}. \end{array}$$

Thus, $w_1w_2 \in L_1 \odot \cdots \odot L_{|w|} = L$ which concludes the proof. ☐

It remains to prove that for any $L \in \mathbf{L}$ and $r \in \mathcal{I}[\rho](\mathbf{K}_L)$, we have $(L,r) \in S$. Recall that S is any Σ_1-saturated set. Since $r \in \mathcal{I}[\rho](\mathbf{K}_L)$, we have $K \in \mathbf{K}_L$ such that $r \leq \rho(K)$. Thus, since S is Σ_1-saturated and therefore closed under downset, it suffices to show that we have $(L,\rho(K)) \in S$.

By definition of $\mathbf{K}_L$, there exists $w \in L$ such that $K = K_w$. Therefore, $K = A^*a_1A^* \cdots A^*a_nA^*$ with $w = a_1 \cdots a_n$. Observe that we may view w as the concatenation $w = \varepsilon a_1 \varepsilon a_2 \varepsilon \cdots \varepsilon a_n \varepsilon$. Hence, since $w \in L$, Item (2) in the definition of multiplicative multisets yields $L_1,\dots,L_n,H_1,\dots,H_{n+1} \in \mathbf{L}$ such that,

(1) $H_1 \odot L_1 \odot H_2 \odot \cdots \odot H_n \odot L_n \odot H_{n+1} = L$.
(2) For all $i \leq n$, $a_i \in L_i$.
(3) For all $i \leq n+1$, $\varepsilon \in H_i$.

Since $a_i \in L_i$ for all $i \leq n$, we have $(L_i,\rho(a_i)) \in \mathcal{P}_{triv}[\mathbf{L},\rho]$. Hence, since S is Σ_1-saturated, we obtain $(L_i,\rho(a_i)) \in S$ for all $i \leq n$. Moreover, Σ_1-closure yields that $(H_i,\rho(A^*)) \in S$ for all $i \leq n+1$ since we have $\varepsilon \in H_i$. Thus, since $H_1 \odot L_1 \odot H_2 \odot \cdots \odot H_n \odot L_n \odot H_{n+1} = L$, it now follows from closure of S under multiplication that:

$$(L,\rho(A^*)\cdot\rho(a_1)\cdots\rho(A^*)\cdot\rho(a_n)\cdot\rho(A^*)) \in S$$

Finally, since ρ is a multiplicative rating map, we have,

$$\rho(K) = \rho(A^*a_1\cdots A^*a_nA^*) = \rho(A^*)\cdot\rho(a_1)\cdots\rho(A^*)\cdot\rho(a_n)\cdot\rho(A^*)$$

Thus, we conclude that $(L,\rho(K)) \in S$, which terminates the proof.

9. The paradise of Boolean algebras

In this section, we show that our framework is much simpler to work with when considering classes that are Boolean algebras. The key idea is that in this case, we may replace covering with a simpler problem called *universal covering*. Let us first define it.

Given a class $\mathcal{C}$, universal $\mathcal{C}$-covering is the restriction of $\mathcal{C}$-covering to input pairs $(L,\mathbf{L})$ where L is the universal language, *i.e.*, $L = A^*$. Therefore, there is only one object in the input: the finite multiset of languages $\mathbf{L}$. Let us state the problem properly.

Definition 9.1 (*Universal covering problem* for $\mathcal{C}$)**.**
Input: *A finite multiset of regular languages* $\mathbf{L}$.
Question: *Is* $(A^*,\mathbf{L})$ $\mathcal{C}$*-coverable?*

Therefore, the problem asks whether there exists a *universal cover* (*i.e.*, a cover of A^*) which is separating for the input multiset $\mathbf{L}$. For the sake of simplifying the presentation, when we consider universal covering, we shall often omit A^* and say that $\mathbf{L}$ is $\mathcal{C}$*-coverable* to indicate that $(A^*,\mathbf{L})$ is $\mathcal{C}$-coverable.

It turns out that when the investigated class $\mathcal{C}$ is a Boolean algebra, the general covering problem reduces to the special case of universal covering. We prove this in the following proposition.

Proposition 9.2. *Let* L_1 *be a language and* $\mathbf{L}_2$ *be a finite multiset of languages. Given any Boolean algebra* $\mathcal{C}$*, the two following properties are equivalent:*
(1) $(L_1,\mathbf{L}_2)$ *is* $\mathcal{C}$*-coverable.*
(2) $\{L_1\} \cup \mathbf{L}_2$ *is* $\mathcal{C}$*-coverable.*

Proof. We first prove the implication $1 \Rightarrow 2$. Assume that $(L_1, \mathbf{L}_2)$ is $\mathcal{C}$-coverable and let $\mathbf{K}$ be a $\mathcal{C}$-cover of L_1 which is separating for $\mathbf{L}_2$. Our goal is to find a universal $\mathcal{C}$-cover, which is separating for $\{L_1\} \cup \mathbf{L}_2$. Let K' be the following language:

$$K' = A^* \setminus \left(\bigcup_{K \in \mathbf{K}} K \right).$$

Note that since $\mathcal{C}$ is a Boolean algebra, we have $K' \in \mathcal{C}$. Let $\mathbf{K}' = \mathbf{K} \cup \{K'\}$. Clearly, $\mathbf{K}'$ is a universal $\mathcal{C}$-cover. We show that it is separating for $\{L_1\} \cup \mathbf{L}_2$ which concludes the proof for this direction. Given $K \in \mathbf{K}'$, either $K \in \mathbf{K}$, or $K = K'$. In the first case, we get $L_2 \in \mathbf{L}_2$ such that $K \cap L_2 = \emptyset$ since $\mathbf{K}$ is separating for $\mathbf{L}_2$. Otherwise, when $K = K'$, we have $K' \cap L_1 = \emptyset$ by definition of K', since $\mathbf{K}$ is a cover of L_1. This terminates the proof for this direction.

Conversely, assume that $\{L_1\} \cup \mathbf{L}_2$ is $\mathcal{C}$-coverable and let $\mathbf{K}$ be a universal $\mathcal{C}$-cover which is separating for $\{L_1\} \cup \mathbf{L}_2$. We define:

$$\mathbf{K}' = \{K \in \mathbf{K} \mid K \cap L_1 \neq \emptyset\}.$$

We claim that $\mathbf{K}'$ is a $\mathcal{C}$-cover of L_1 which is separating for $\mathbf{L}_2$. Indeed, we know that $L_1 \subseteq \bigcup_{K \in \mathbf{K}'} K$ since $\mathbf{K}$ is a cover of A^* and $\mathbf{K}'$ contains all languages in $\mathbf{K}$ that intersect L_1. Moreover, since $\mathbf{K}$ is separating for $\{L_1\} \cup \mathbf{L}_2$, we know that for any $K \in \mathbf{K}$, either $K \cap L_1 = \emptyset$ or $L \cap L_2 = \emptyset$ for some $L_2 \in \mathbf{L}_2$. For the languages $K \in \mathbf{K}'$, we know by definition that $K \cap L_1 \neq \emptyset$. Thus, there exists $L_2 \in \mathbf{L}_2$ such that $K \cap L_2 = \emptyset$. This concludes the proof. □

In view of Proposition 9.2, in order to solve the $\mathcal{C}$-covering problem when $\mathcal{C}$ is a Boolean algebra, it suffices to solve the simpler *universal* $\mathcal{C}$-covering problem. In the rest of this section, we shall revisit our framework and see how it simplifies in this case.

9.1. **Reformulating the general approach for Boolean algebras.** Recall that we showed in Section 7 that the general covering problem for a lattice $\mathcal{C}$ can be reformulated as the following computational problem:

Input: A multiplicative multiset $\mathbf{L}$ and a nice multiplicative rating map $\rho : 2^{A^*} \to R$
Output: Compute the $\mathcal{C}$-optimal $\mathbf{L}$-pointed ρ-imprint $\mathcal{P}_{\mathcal{C}}[\mathbf{L}, \rho]$.

In the special case of universal covering, the key idea is that it suffices to consider the case when the multiset $\mathbf{L}$ is a singleton $\mathbf{L} = \{A^*\}$ (which is clearly multiplicative for the only possible multiplication: $A^* \odot A^* = A^*$). Because of this, we are able to simplify our notation significantly .

However, let us start at the beginning and prove this claim by retracing the steps that we followed when reformulating covering as the above problem. We fix an arbitrary Boolean algebra $\mathcal{C}$.

First step: restriction to multiplicative sets. When considering universal $\mathcal{C}$-covering, the objective is to decide for some input multiset $\mathbf{L}$ whether the pair $(A^*, \mathbf{L})$ is $\mathcal{C}$-coverable (which we abbreviate as "$\mathbf{L}$ is $\mathcal{C}$-coverable").

By Proposition 4.10, we are able to build multiplicative multisets $\mathbf{H}_1$ and $\mathbf{H}_2$ extending $\{A^*\}$ and $\mathbf{L}$ respectively (additionally, by Proposition 4.19, we are also able to enforce that $\mathbf{H}_1$ and $\mathbf{H}_2$ are $\mathcal{D}$-compatible for some finite quotienting Boolean algebra $\mathcal{D}$ of our choice). Moreover, we showed in Lemma 4.8 that having the pairs in $\mathbf{H}_1 \times 2^{\mathbf{H}_2}$ which are $\mathcal{C}$-coverable in hand suffices to decide whether $(A^*, \mathbf{L})$ is $\mathcal{C}$-coverable. However, in this case, we do **not** have to compute $\mathbf{H}_1$: since $\{A^*\}$ is already a multiplicative multiset (which extends itself), we may simply use $\mathbf{H}_1 = \{A^*\}$.

Altogether, this means that we may reformulate universal $\mathcal{C}$-covering as follows. It suffices to obtain an algorithm for the following computational problem:

Input: A multiplicative multiset of languages $\mathbf{L}$.
(possibly $\mathcal{D}$-compatible for some finite quotienting Boolean algebra $\mathcal{D}$).
Output: Compute all $\mathcal{C}$-coverable pairs in $\{A^*\} \times 2^{\mathbf{L}}$.

Second step: abstraction with rating maps. By Theorem 5.21, the $\mathcal{C}$-coverable pairs in $\{A^*\} \times 2^{\mathbf{L}}$ are encoded in the $\mathcal{C}$-optimal $\mathbf{L}$-imprint on A^*, *i.e.*, in the set $\mathcal{I}_{\mathcal{C}}[A^*, \mathbf{L}]$. Therefore, we may abstract the above problem as follows:

Input: A nice multiplicative rating map $\rho : 2^{A^*} \to R$
(possibly $\mathcal{D}$-compatible for some finite quotienting Boolean algebra $\mathcal{D}$).
Output: Compute the $\mathcal{C}$-optimal ρ-imprint on A^*, $\mathcal{I}_{\mathcal{C}}[A^*, \rho]$.

For the sake of simplifying the presentation, we shall speak of the $\mathcal{C}$-*optimal universal ρ-imprint* for the set $\mathcal{I}_{\mathcal{C}}[A^*, \rho]$ and omit A^* in the notation. We define,

$$\mathcal{I}_{\mathcal{C}}[\rho] = \mathcal{I}_{\mathcal{C}}[A^*, \rho] \subseteq R.$$

Altogether we obtain the following proposition, which presents the computational problem that we shall consider when working with a Boolean algebra.

Proposition 9.3. *Consider a Boolean algebra $\mathcal{C}$. Assume that there exists an algorithm which computes $\mathcal{I}_{\mathcal{C}}[\rho]$ from an input nice multiplicative rating map $\rho : 2^{A^*} \to R$. Then, the $\mathcal{C}$-covering problem is decidable.*

Moreover, the result still holds when the algorithm computing $\mathcal{I}_{\mathcal{C}}[\rho]$ is restricted to $\mathcal{D}$-compatible inputs for some finite quotienting Boolean algebra $\mathcal{D}$.

In view Proposition 7.2, we may now focus on the problem of computing optimal universal imprints $\mathcal{I}_{\mathcal{C}}[\rho]$. Let us point out that these new objects are simpler than the optimal *pointed* imprints $\mathcal{P}_{\mathcal{C}}[\mathbf{L}, \rho]$ involved in our original general approach: they depend on one parameter rather than two (we removed the multiplicative multiset $\mathbf{L}$).

Remark 9.4. *Of course, using this simplification only makes sense for classes that are Boolean algebras. When considering a lattice which not closed under complement, we have to use optimal pointed imprints.*

We now adapt our methodology to this simplified problem. Note that as before, while Proposition 9.3 holds for any Boolean algebra, using our methodology requires at least a quotienting Boolean algebra of regular languages.

9.2. **Methodology for Boolean algebras.** In view of Proposition 9.3, given a quotienting Boolean algebra $\mathcal{C}$, our goal is to compute the $\mathcal{C}$-optimal universal ρ-imprint $\mathcal{I}_{\mathcal{C}}[\rho]$ from a multiplicative rating map $\rho : 2^{A^*} \to R$. Recall that for quotienting lattices, our methodology for computing optimal pointed imprints was based on characterization theorems. We use a similar approach. The key idea is now to characterize the $\mathcal{I}_{\mathcal{C}}[\rho] \subseteq R$ as the smallest subset of the rating set R including trivial elements and satisfying specific properties. In practice, such a characterization yields a least fixpoint procedure for computing $\mathcal{I}_{\mathcal{C}}[\rho]$ from ρ. We present some examples at the end of the section and detail a specific one in Section 10.

Of course, as before, some operations under which $\mathcal{I}_{\mathcal{C}}[\rho]$ is closed are specific to $\mathcal{C}$, but importantly, some are *generic to all quotienting Boolean algebras*. We present these generic properties now. Recall that given any language L and any multiplicative rating map $\rho : 2^{A^*} \to R$, we defined $\mathcal{I}_{triv}[L, \rho] \subseteq R$ as the set

$$\mathcal{I}_{triv}[L, \rho] = \{r \in R \mid \exists w \in L \text{ such that } r \leq \rho(w)\}.$$

Here, we only care about the special case when $L = A^*$, thus we simplify our notations by writing $\mathcal{I}_{triv}[\rho]$ for $\mathcal{I}_{triv}[A^*,\rho]$. Recall that we showed in Lemma 7.5 that $\mathcal{I}_{triv}[\rho]$ may be computed from ρ when ρ is nice. We may now present the generic properties satisfied by all $\mathcal{C}$-optimal universal imprints (provided that $\mathcal{C}$ is a quotienting Boolean algebra of regular languages). We do so in the following lemma.

Lemma 9.5. *Consider a multiplicative rating map $\rho : 2^{A^*} \to R$ and some quotienting Boolean algebra of regular languages $\mathcal{C}$. Then, the universal $\mathcal{C}$-optimal ρ-imprint $\mathcal{I}_{\mathcal{C}}[\rho] \subseteq R$ contains $\mathcal{I}_{triv}[\rho]$ and satisfies the following closure properties:*

(1) **Downset***: For any $r \in \mathcal{I}_{\mathcal{C}}[\rho]$ and any $r' \leq r$, we have $r' \in \mathcal{I}_{\mathcal{C}}[\rho]$.*
(2) **Multiplication***: For any $s,t \in \mathcal{I}_{\mathcal{C}}[\rho]$, we have $st \in \mathcal{I}_{\mathcal{C}}[\rho]$.*

Proof. That $\mathcal{I}_{\mathcal{C}}[\rho]$ contains $\mathcal{I}_{triv}[\rho]$ is immediate from Fact 5.12. That it is closed under downset follows from Fact 5.11. Finally closure under multiplication is a consequence of Lemma 6.6. Indeed, recall that $\mathcal{I}_{\mathcal{C}}[\rho] = \mathcal{I}_{\mathcal{C}}[A^*,\rho]$. Thus, since $A^*A^* = A^*$, it is immediate from Lemma 6.6 that when $s,t \in \mathcal{I}_{\mathcal{C}}[\rho]$, we have $st \in \mathcal{I}_{\mathcal{C}}[\rho]$. □

Finally, we present an alternate definition of $\mathcal{I}_{\mathcal{C}}[\rho]$ which is based on stratifications (it is adapted from Corollary 5.20). Typically, it is used for proving that the set $\mathcal{I}_{\mathcal{C}}[\rho]$ satisfies the closure properties stated in our characterization theorems.

Lemma 9.6. *Let $\mathcal{C}$ be a Boolean algebra and let $(\mathcal{C}_k)_k$ be a stratification of $\mathcal{C}$ into finite Boolean algebras. For $k \in \mathbb{N}$, we write $\sim_k$ the canonical equivalence of the stratum $\mathcal{C}_k$. Let $\rho : 2^{A^*} \to R$ be a rating map. Given $r \in R$, the following are equivalent:*

(1) $r \in \mathcal{I}_{\mathcal{C}}[\rho]$.
(2) For all $k \in \mathbb{N}$, there exists $K \subseteq A^*$ included in a $\sim_k$-class and such that $r \leq \rho(K)$.

9.3. **Examples of characterization for quotienting Boolean algebras.** We finish with three examples of characterizations for quotienting Boolean algebras. The three of them are fragments of FO. The first fragment is first-order logic itself and the characterization that we present is directly adapted from the separation algorithm of [26, 29]. However, there is no published proof of the statement that we present now. We leave this for further work.

Example 9.7 (Characterization of FO-optimal imprints)**.** *This characterization holds for any multiplicative multiset* **L** *and any* **nice** *multiplicative rating map $\rho : 2^{A^*} \to R$ (note the "nice" requirement). One may show that $\mathcal{I}_{\mathrm{FO}}[\rho]$ is the smallest subset of R which contains $\mathcal{I}_{triv}[\rho]$ and satisfies the following closure properties:*

(1) **Downset***: For any $r \in \mathcal{I}_{\mathrm{FO}}[\rho]$ and any $r' \leq r$, we have $r' \in \mathcal{I}_{\mathrm{FO}}[\rho]$.*
(2) **Multiplication***: For any $s,t \in \mathcal{I}_{\mathrm{FO}}[\rho]$, we have $st \in \mathcal{I}_{\mathrm{FO}}[\rho]$.*
(3) **FO-closure***: For any $s \in \mathcal{I}_{\mathrm{FO}}[\rho]$, we have $s^{\omega} + s^{\omega+1} \in \mathcal{I}_{\mathrm{FO}}[\rho]$.*

Our second example is the level 1 in the quantifier alternation hierarchy of first-order logic: $\mathcal{B}\Sigma_1$. The characterization is loosely inspired from the separation algorithm of [23]. Again, there is no published proof of the statement that we present now. We leave this for further work.

Example 9.8 (Characterization of $\mathcal{B}\Sigma_1$-optimal imprints)**.** *This characterization holds for any multiplicative rating map $\rho : 2^{A^*} \to R$. Recall that for any sub-alphabet $B \subseteq A$, $B^{\circledast}$ denotes the language $\{w \in A^* \mid \mathbf{alph}(w) = B\}$. One may show that $\mathcal{I}_{\mathcal{B}\Sigma_1}[\rho]$ is the smallest subset of R which contains $\mathcal{I}_{triv}[\rho]$ and satisfies the following closure properties:*

(1) **Downset**: *For any $r \in \mathfrak{I}_{\mathcal{B}\Sigma_1}[\rho]$ and any $r' \leq r$, we have $r' \in \mathfrak{I}_{\mathcal{B}\Sigma_1}[\rho]$.*
(2) **Multiplication**: *For any $s,t \in \mathfrak{I}_{\mathcal{B}\Sigma_1}[\rho]$, we have $st \in \mathfrak{I}_{\mathcal{B}\Sigma_1}[\rho]$.*
(3) $\mathcal{B}\Sigma_1$**-closure**: *For any $B \subseteq A$, we have $(\rho(B^{\circledast}))^{\omega} \in \mathfrak{I}_{\mathcal{B}\Sigma_1}[\rho]$.*

Our last example is the two-variable restriction of first-order logic: FO^2. We shall define FO^2 properly in the next section. The characterization that we present now is our second detailed example in the paper: we present a proof in the next section. Note that the statement is restricted to AT-compatible rating maps (recall that AT is the class of alphabet testable languages).

Example 9.9 (Characterization of FO^2-optimal imprints)**.** *This characterization holds for any* AT-compatible *multiplicative rating map $\rho : 2^{A^*} \to R$ (recall that* AT *is the class of alphabet testable languages). We shall prove that $\mathfrak{I}_{\mathrm{FO}^2}[\rho]$ is the smallest subset of R which contains $\mathfrak{I}_{triv}[\rho]$ and satisfies the following closure properties:*
(1) **Downset**: *For any $r \in \mathfrak{I}_{\mathcal{B}\Sigma_1}[\rho]$ and any $r' \leq r$, we have $r' \in \mathfrak{I}_{\mathcal{B}\Sigma_1}[\rho]$.*
(2) **Multiplication**: *For any $s,t \in \mathfrak{I}_{\mathcal{B}\Sigma_1}[\rho]$, we have $st \in \mathfrak{I}_{\mathcal{B}\Sigma_1}[\rho]$.*
(3) $\mathbf{FO}^2$**-closure**: *For any $B \subseteq A$ and any idempotents $e, f \in S$ such that $e, f \leq \rho(B^{\circledast})$, we have $e \cdot \rho(B^*) \cdot f \in S$.*

10. Example for Boolean algebras: two-variable first-order logic

In this final section, we illustrate our simplified framework for Boolean algebras by presenting a detailed proof for Example 9.9. In other words, we consider the class FO^2 corresponding to two-variable first-order logic. We first briefly recall the definition of FO^2 and then present the characterization.

10.1. Definition.

The two-variable fragment of first-order logic (denoted FO^2) is obtained by restricting the number of allowed variables within a single sentence to at most *two*. That is, we define FO^2 sentences as the first-order sentences which contain at most two distinct variables. Let us point out that while one is restricted to two variables only, it is possible to *reuse them*. For example, consider the following sentence,

$$\exists x \exists y \quad a(x) \quad \wedge \quad a(y) \quad \wedge \quad (x < y) \quad \wedge \quad \exists \underline{x}\ (a(\underline{x}) \wedge y < \underline{x})$$

This sentence defines the language $A^*aA^*aA^*aA^*$. Observe that in order to quantify the third position whose label is "a", we reused the variable x: this is allowed in FO^2. It is is folklore that the class of languages defined by FO^2 is a quotienting Boolean algebra.

We shall need the following classical stratification of FO^2 when proving our characterization. It is standard to classify first-order sentences (and therefore those in FO^2) using the notion of quantifier rank. Consider a sentence φ. We define the *rank* of φ as the longest sequence of nested quantifiers within its parsing tree. It is well-known and simple to verify that for any $k \in \mathbb{N}$, there are only finitely many non-equivalent FO^2 sentences of rank k. Therefore, we obtain a stratification of FO^2. For any $k \in \mathbb{N}$, the corresponding stratum is the class of languages that can be defined by an FO^2 sentence of rank at most k. It is immediate that all strata are Boolean algebras since Boolean connectives do not affect the rank of a sentence.

We now consider the canonical equivalences associated to the strata and state standard properties. Given $k \in \mathbb{N}$, the canonical equivalence $\cong_k$ associated to stratum k is defined as follows. For any two words $w, w' \in A^*$, we have $w \cong_k w'$ if and only if,

$$\text{For any } \mathrm{FO}^2 \text{ sentence } \varphi \text{ of rank at most } k, \quad w \models \varphi \iff w' \models \varphi$$

We now present two classical properties of the equivalences $\cong_k$, which we shall need when proving our characterizations. Since they are folklore, the proof is left to the reader (they are proved using Ehrenfeucht-Fraïssé arguments). The first property states that $\cong_k$ is a congruence for word concatenation (which means that each stratum is quotienting by Lemma 2.9).

Lemma 10.1. *For all $k \in \mathbb{N}$, $\cong_k$ is a congruence for concatenation. Given $w_1, w_2, w'_1, w'_2 \in A^*$ we have, $w_1 \cong_k w'_1$ and $w_2 \cong_k w'_2 \Rightarrow w_1 w_2 \cong_k w'_1 w'_2$.*

The second lemma states the characteristic property of two-variable first-order logic. In fact, our characterization of FO^2-optimal imprints is based on this property.

Lemma 10.2. *Let $B \subseteq A$ and $u, v \in A^*$ such that $\mathbf{alph}(u) = \mathbf{alph}(v) = B$. Finally, let $k, \ell \in \mathbb{N}$ such that $\ell \geq k$. Then for any $w, w' \in B^*$, $u^\ell w v^\ell \cong_k u^\ell w' v^\ell$.*

10.2. **Characterization of optimal imprints.** Now that we have defined FO^2, we may present our characterization of FO^2-optimal imprints. That is, given a multiplicative rating map ρ, we describe the set $\mathcal{I}_{FO^2}[\rho] = \mathcal{I}_{FO^2}[A^*, \rho]$ (as explained in Proposition 9.3, this is sufficient with respect to covering since FO^2 is a Boolean algebra).

An important point is that our characterization only applies to multiplicative rating maps which are AT-compatible. As we explained in the previous section, this is no restrictive with respect to FO^2-covering (see Proposition 9.3). Recall that AT is the finite quotienting Boolean algebra made of all Boolean combinations of languages A^*aA^* for $a \in A$. In particular, the classes of the associated canonical equivalence $\sim_{AT}$ are the languages $B^{\circledast} = \{w \in A^* \mid \mathbf{alph}(w) = B\}$ for $B \subseteq A$.

Remark 10.3. *As for Σ_1 in Section 8, our characterization does not require ρ to be nice: it holds for any AT-compatible multiplicative rating map. Of course, we only obtain an algorithm for computing $\mathcal{I}_{FO^2}[\rho]$ when ρ is nice. In fact, it does not even make sense to speak of an algorithm for all multiplicative rating maps since we are only able to finitely represent the nice ones.*

Consider an AT-compatible multiplicative rating map $\rho : 2^{A^*} \to R$ and a subset S of R. We say that S is FO^2-saturated (for ρ) if it contains $\mathcal{I}_{triv}[\rho]$ and is closed under the following operations:

(1) *Downset*: For any $r \in S$ and any $r' \leq r$, we have $r' \in S$.
(2) *Multiplication*: For any $s, t \in S$, we have, $st \in S$.
(3) FO^2-*closure*: For any $B \subseteq A$ and any idempotents $e, f \in S$ such that $e, f \leq \rho(B^{\circledast})$, we have,

$$e \cdot \rho(B^*) \cdot f \in S$$

We may now state the main theorem of this section: given any AT-compatible multiplicative rating map $\rho : 2^{A^*} \to R$, the FO^2-optimal ρ-imprint $\mathcal{I}_{FO^2}[\rho]$ is the smallest FO^2-saturated subset of R (with respect to inclusion).

Theorem 10.4 (Characterization of FO^2-optimal imprints)**.** *Let $\rho : 2^{A^*} \to R$ be an AT-compatible multiplicative rating map. Then, $\mathcal{I}_{FO^2}[\rho]$ is the smallest FO^2-saturated subset of R.*

Clearly, when we have a *nice* AT-compatible multiplicative rating map $\rho : 2^{A^*} \to R$ in hand, we may compute the smallest FO^2-saturated subset of R with a least fixpoint algorithm (it is immediate that FO^2-closure may be implemented). Thus, Theorem 10.4 yields a procedure for computing $\mathcal{I}_{FO^2}[\rho]$ from an input niceAT-compatible multiplicative rating map.

By Proposition 9.3, this yields an algorithm for FO^2-covering. Thus, we get the following corollary.

Corollary 10.5. *The* FO^2*-covering problem is decidable.*

This conclude the presentation of our main theorem for this section. We may now turn to its proof. Consider an AT-compatible multiplicative rating map $\rho : 2^{A^*} \to R$. Our objective is to show that $\mathcal{I}_{\mathrm{FO}^2}[\rho]$ is the smallest FO^2-saturated subset of R. We proceed in two steps:

(1) First, we show that $\mathcal{I}_{\mathrm{FO}^2}[\rho]$ is FO^2-saturated. This corresponds to soundness of the least fixpoint algorithm: all elements it computes belong to $\mathcal{I}_{\mathrm{FO}^2}[\rho]$.
(2) Then, we show that $\mathcal{I}_{\mathrm{FO}^2}[\rho]$ is smaller than all FO^2-saturated sets. This corresponds to completeness of the least fixpoint algorithm: it computes all elements of $\mathcal{I}_{\mathrm{FO}^2}[\rho]$. As we explained this direction is of particular interest: it yields a generic procedure for building optimal FO^2-covers.

10.3. **Soundness.** We prove the soundness part of Theorem 10.4. Our objective is to show that $\mathcal{I}_{\mathrm{FO}^2}[\rho]$ is FO^2-saturated. Since FO^2 is a quotienting Boolean algebra, we already know from Lemma 9.5 that $\mathcal{I}_{\mathrm{FO}^2}[\rho]$ contains $\mathcal{I}_{triv}[\rho]$ and is closed under downset and multiplication. Thus, we may focus on FO^2-*closure*. Given $B \subseteq A$ and any idempotents $e, f \in \mathcal{I}_{\mathrm{FO}^2}[\rho]$ such that $e, f \leq \rho(B^{\circledast})$, we have,

$$e \cdot \rho(B^*) \cdot f \in \mathcal{I}_{\mathrm{FO}^2}[\rho].$$

First observe that we may assume without loss of generality that $e \neq 0_R$ and $f \neq 0_R$. Indeed, otherwise, $e = 0_R$ or $f = 0_R$ and it is immediate that $e \cdot \rho(B^*) \cdot f = 0_R$ and $0_R \in \mathcal{I}_{\mathrm{FO}^2}[\rho]$ (more precisely, $0_R = \rho(\emptyset) \in \mathcal{I}_{triv}[\rho]$ and we already established that $\mathcal{I}_{triv}[\rho] \subseteq \mathcal{I}_{\mathrm{FO}^2}[\rho]$). Hence, we assume from now on that $e, f \neq 0_R$.

The proof relies on the alternate definition of optimal universal imprints which is based on stratifications (*i.e.*, on Lemma 9.6). We use the stratification of FO^2 introduced at the beginning of the section (recall that the canonical equivalences associated to the strata are denoted by $\cong_k$). In view of Lemma 9.6, it suffices to show that for any $k \in \mathbb{N}$, there exists some language $K \subseteq A^*$ included in a $\cong_k$-class such that $e \cdot \rho(B^*) \cdot f \leq \rho(K)$. Therefore, we fix $k \in \mathbb{N}$ and exhibit the appropriate K.

Since $e, f \in \mathcal{I}_{\mathrm{FO}^2}[\rho]$ by hypothesis, the converse direction of Corollary 5.20 yields $w_e, w_f \in A^*$ and $K_e, K_f \subseteq A^*$ which are both included in some $\cong_k$-class (possibly not the same one) and such that $e \leq \rho(K_e)$ and $f \leq \rho(K_f)$. We define K as follows:

$$K = (K_e)^k B^* (K_f)^k$$

We have to show that K is included in a $\cong_k$-class and that $e \cdot \rho(B^*) \cdot f \leq \rho(K)$. We start with the latter property which is simpler to establish. By definition of K, and since ρ is a multiplicative rating map (and thus a morphism for multiplication) we get,

$$(\rho(K_e))^k \cdot \rho(B^*) \cdot (\rho(K_f))^k = \rho(K).$$

Thus, since $e \leq \rho(K_e)$ and $f \leq \rho(K_f)$ by definition and e, f are idempotents, we get as desired that $e \cdot \rho(B^*) \cdot f \leq \rho(K)$.

It remains to show that K is included in a $\cong_k$-class. This is where we use the fact that ρ is AT-compatible: this property yields the following fact.

Fact 10.6. There exists $u \in K_e$ and $v \in K_f$ such that $\mathbf{alph}(u) = \mathbf{alph}(v) = B$.

Proof. We prove the existence of u, the argument is symmetrical for v. By hypothesis on e, we know that $e \neq 0_R$ and $e \leq \rho(B^{\circledast})$. Moreover, we have $e \leq \rho(K_e)$ by definition of K_e. Thus since $B^{\circledast}$ is a $\sim_{\mathrm{AT}}$-class, it is immediate by AT-compatibility of ρ that $K_e \cap B^{\circledast} \neq \emptyset$. Any word u in this intersection satisfies the conditions of the lemma. □

Let $u \in K_e$ and $v \in K_f$ be as described in the fact. We show that all word in K are $\cong_k$-equivalent to the word $u^k v^k$, which implies as desired that K is included in a $\cong_k$-class.

Recall that by definition, K_e, K_f are included in $\cong_k$-classes. Thus, all words in K_e (resp. K_f) are equivalent to $u \in K_e$ (resp $v \in K_f$). Moreover, we have $K = (K_e)^k B^* (K_f)^k$ by definition. Since $\cong_k$ is compatible with concatenation (this is Lemma 10.1), we get that for any word $w \in K$, there exists $x \in B^*$ such that,

$$w \cong_k u^k x v^k$$

Moreover, since $\mathbf{alph}(u) = \mathbf{alph}(v) = B$ and $x \in B^*$, we obtain from Lemma 10.2 that $u^k x v^k \cong_k u^k v^k$. Therefore, transitivity yields that all $w \in K$ are $\cong_k$-equivalent to $u^k v^k$, which concludes the proof.

10.4. **Completeness.** We turn to the difficult inclusion in Theorem 10.4 which corresponds to completeness of the least fixpoint procedure. Recall that an AT-compatible multiplicative rating map $\rho : 2^{A^*} \to R$ is fixed. We show that for any FO^2-saturated subset $S \subseteq R$, we have $\mathcal{I}_{\text{FO}^2}[\rho] \subseteq S$.

Our proof is a generic construction which builds a FO^2-cover $\mathbf{K}$ of A^* such that $\mathcal{I}[\rho](\mathbf{K}) \subseteq S$. Since $\mathcal{I}_{\text{FO}^2}[\rho] \subseteq \mathcal{I}[\rho](\mathbf{K})$ for any FO^2-cover $\mathbf{K}$ of A^*, this proves the desired result.

Remark 10.7. *Since we already showed that $\mathcal{I}_{\text{FO}^2}[\rho]$ itself is FO^2-saturated, the special case when $S = \mathcal{I}_{\text{FO}^2}[\rho]$ yields a universal FO^2-cover $\mathbf{K}$ which satisfies,*

$$\mathcal{I}[\rho](\mathbf{K}) \subseteq \mathcal{I}_{\text{FO}^2}[\rho].$$

In other words, we are able to build an optimal universal FO^2-cover.

The construction of $\mathbf{K}$ is achieved by induction on two parameters. The most important one is some sub-alphabet $B \subseteq A^*$: we reduce the construction of $\mathbf{K}$ to that of FO^2-covers of B^* for increasingly small $B \subseteq A$. We state this induction in Proposition 10.8 below.

Proposition 10.8. *Consider a FO^2-saturated set $S \subseteq R$. For any $B \subseteq A$ and any $t_\ell, t_r \in S$, there exists a FO^2-cover $\mathbf{K}$ of B^* such that $K \subseteq B^*$ for all $K \in \mathbf{K}$ and,*

$$t_\ell \cdot \mathcal{I}[\rho](\mathbf{K}) \cdot t_r \subseteq S$$

Before we prove Proposition 10.8, we use it to finish the proof of Theorem 10.4. Consider an arbitrary FO^2-saturated set $S \subseteq R$. We apply Proposition 10.8 in the case when $B = A$ and t_ℓ, t_r are both equal to $\rho(\varepsilon)$ (which belongs to $\mathcal{I}_{triv}[\rho]$ and therefore to S since S is FO^2-saturated). We obtain a universal FO^2-cover $\mathbf{K}$ such that $\rho(\varepsilon) \cdot \mathcal{I}[\rho](\mathbf{K}) \cdot \rho(\varepsilon) \subseteq S$. We show that this implies $\mathcal{I}[\rho](\mathbf{K}) \subseteq S$ which concludes the proof.

Let $r \in \mathcal{I}[\rho](\mathbf{K})$. By definition, we have $K \in \mathbf{K}$ such that $r \leq \rho(K)$. Since ρ is a multiplicative rating map, we have $\rho(K) = \rho(\varepsilon) \cdot \rho(K) \cdot \rho(\varepsilon)$. Thus, we obtain $\rho(K) \in \rho(\varepsilon) \cdot \mathcal{I}[\rho](\mathbf{K}) \cdot \rho(\varepsilon) \subseteq S$. Finally, since S is FO^2-saturated and $r \leq \rho(K)$, we conclude that $r \in S$.

It now remains to prove Proposition 10.8. We let $B \subseteq A$ and $t_\ell, t_r \in S$ be as in the statement of the proposition. Our objective is to construct a FO^2-cover $\mathbf{K}$ of B^* such that $K \subseteq B^*$ for all $K \in \mathbf{K}$ and $t_\ell \cdot \mathcal{I}[\rho](\mathbf{K}) \cdot t_r \subseteq S$.

The proof is by induction on three parameters that we define now. The first and most important one is the size of B. The other two also depend on t_ℓ and t_r respectively. Given $s_1, s_2 \in S$, we say that,

- s_2 is **left** *B-reachable* from s_1 when there exists $x \in S$ such that $x \leq \rho(B^*)$ and $s_2 \leq x \cdot s_1$.
- s_2 is **right** *B-reachable* from s_1 when there exists $x \in S$ such that $x \leq \rho(B^*)$ and $s_2 \leq s_1 \cdot x$.

One may verify that left and right B-reachability are both preorder relations on the set (this is because the multiplication of R is compatible with the order). For each $s \in S$, we define the *left B-index of s* (resp. the *right B-index of s*) as the number of elements that are left B-reachable (resp. right B-reachable) from s. We proceed by induction on the following parameters listed by order of importance:
(1) $|B|$
(2) The right B-index of t_ℓ.
(3) The left B-index of t_r.

We consider three cases depending on properties of B, t_ℓ and t_r that we define below. First observe that since S contains $\mathcal{I}_{triv}[\rho]$ by definition of FO^2-saturated sets, we know that for all $b \in B$, $\rho(b) \in S$. Moreover, S is closed under multiplication.

We may now present the three cases of our construction. They depend on the two following properties of t_ℓ and t_r respectively.

– We say that t_ℓ is *right saturated* (by B) if for all $b \in B$, there exists $y \in S$ such that $y \leq \rho(B^*)$ and t_ℓ is right B-reachable from $t_\ell \cdot y \cdot \rho(b)$.
– We say that t_r is *left saturated* (by B) if for all $b \in B$, there exists $y \in S$ such that $y \leq \rho(B^*)$ and t_r is left B-reachable from $\rho(b) \cdot y \cdot t_r$.

The base case happens when t_ℓ and t_r are respectively right and left saturated by B. When t_ℓ is not right saturated, we use induction on $|B|$ and the right B-index of t_ℓ. Finally, when t_r is not left saturated, we use induction on $|B|$ and the left B-index of t_r. Let us start with the base case.

Base Case: t_ℓ is right saturated by B and t_r is left saturated by B. In that case, we simply choose $\mathbf{K} = \{B^*\}$ (which is clearly a FO^2-cover of B^*). Therefore, we have to use our hypothesis to show that this choice satisfies the conditions in Proposition 10.8. We have to prove that,

$$t_\ell \cdot \mathcal{I}[\rho](\mathbf{K}) \cdot t_r \subseteq S$$

By definition of $\mathbf{K}$, $\mathcal{I}[\rho](\mathbf{K}) = \{r \in R \mid r \leq \rho(B^*)\}$. Therefore, since S is closed under downset (by definition of FO^2-saturated sets), it suffices to prove that,

$$t_\ell \cdot \rho(B^*) \cdot t_r \in S$$

This is what we do now using the following lemma.

Lemma 10.9. *There exist idempotents $e, f \in S$ such that $e, f \leq \rho(B^{\circledast})$, $t_\ell \leq t_\ell e$ and $t_r \leq f t_r$.*

Before we prove the lemma, we use it to conclude this case. Let e, f be as defined in the lemma. This yields,

$$t_\ell \cdot \rho(B^*) \cdot t_r \leq t_\ell \cdot e \cdot \rho(B^*) \cdot f \cdot t_r$$

Moreover, since $e, f \in S$ are idempotents such that $e, f \leq \rho(B^{\circledast})$, we know from FO^2-closure in the definition of FO^2-saturated sets that,

$$e \cdot \rho(B^*) \cdot f \in S$$

One may now use closure under multiplication and downset in the definition of FO^2-saturated sets to obtain that $t_\ell \cdot \rho(B^*) \cdot t_r \in S$ as desired. This concludes the proof of the base case. It remains to show Lemma 10.9.

Proof of Lemma 10.9. We only prove the existence of e using the fact that t_ℓ is right saturated (the existence of f is proved symmetrically using the fact that t_r is left saturated). By definition of right saturation, we know that for each $b \in B$, there exists $x_b, y_b \in S$ such that $x_b, y_b \leq \rho(B^*)$ and,

$$t_\ell \leq t_\ell x_b \cdot \rho(b) \cdot y_b$$

Let $B = \{b_1, \dots, b_n\}$, we define e as the following element,

$$e = (x_{b_1} \cdot \rho(b_1) \cdot y_{b_1} x_{b_2} \cdot \rho(b_2) \cdot y_{b_2} \cdots x_{b_n} \cdot \rho(b_n) \cdot y_{b_n})^\omega$$

By definition, e is idempotent and we have $t_\ell \leq t_\ell e$. Moreover, we have $e \in S$ since it is a multiplication of elements in S and FO^2-saturated sets are closed under multiplication. It remains to prove that $e \leq \rho(B^{\circledast})$.

By hypothesis, for any $b \in B$, we have $x_b, y_b \leq \rho(B^*)$. Thus, since ρ is a multiplicative rating map, we have,

$$e \leq \rho((B^* b_1 B^* b_2 \cdots B^* b_n B^*)^\omega)$$

Clearly, we have $(B^* b_1 B^* b_2 \cdots B^* b_n B^*)^\omega \subseteq B^{\circledast}$ since $b_1, \dots, b_n$ account for all letters in B. Thus, the above yields $e \leq \rho(B^{\circledast})$ since rating maps are increasing. $\square$

We are now finished with the base case. It remains to treat the case when either t_ℓ is not right saturated or t_r is not left saturated. These two cases are symmetrical. We consider the one when t_ℓ is not right saturated. The second case is left to the reader.

Induction Case: t_ℓ is not right saturated. By definition, since t_ℓ is not right saturated, this means that there exists some $b \in B$ such that for all $q \in S$ satisfying $q \leq \rho(B^*)$, t_ℓ is **not** right B-reachable from $t_\ell q \cdot \rho(b)$. We now assume that this letter b is fixed and use it to construct our FO^2-cover $\mathbf{K}$ of B^*.

Let $C = B \setminus \{b\}$. Using induction on the size of the alphabet (recall that this parameter is more important than the right B-index of t_ℓ and the left B-index of t_r), we obtain a FO^2-cover $\mathbf{H}$ of C^* such that $H \subseteq C^*$ for all $H \in \mathbf{H}$ and,

$$\rho(\varepsilon) \cdot \mathfrak{I}[\rho](\mathbf{H}) \cdot \rho(\varepsilon) \subseteq S$$

For all $H \in \mathbf{H}$, we define $t_H = t_\ell \cdot \rho(H) \cdot \rho(b)$. By definition, we have the following fact.

Fact 10.10. For all $H \in \mathbf{H}$, we have $t_H \in S$.

Proof. We show that $\rho(H) \in S$. Since S is FO^2-saturated, it will then follow from closure under multiplication that $t_H = t_\ell \cdot \rho(H) \cdot \rho(b) \in S$. We have $\rho(\varepsilon) \cdot \mathfrak{I}[\rho](\mathbf{H}) \cdot \rho(\varepsilon) \subseteq S$. Thus, $\rho(\varepsilon) \cdot \rho(H) \cdot \rho(\varepsilon) \in S$. Finally, since ρ is a multiplicative rating map, we have $\rho(H) = \rho(\varepsilon) \cdot \rho(H) \cdot \rho(\varepsilon)$ which concludes the proof. $\square$

Moreover, for any $H \in \mathbf{H}$, we have $H \subseteq C^* \subseteq B^*$, we have $\rho(H) \leq \rho(B^*)$ by Item 2 in the definition of rating maps. Therefore, by choice of the letter b, we know that t_ℓ is not right reachable from t_H. In particular, it follows that the right B-index of t_H is strictly smaller than the right B-index of t_ℓ. Hence, for any $H \in \mathbf{H}$, we may use induction to construct a FO^2-cover $\mathbf{K}_H$ of B^* such that,

$$t_H \cdot \mathfrak{I}[\rho](\mathbf{K}_H) \cdot t_r \subseteq S \tag{10.1}$$

We are now ready to construct our FO^2-cover $\mathbf{K}$ of B^* which satisfies the properties described in Proposition 10.8. We define,

$$\mathbf{K} = \mathbf{H} \cup \bigcup_{H \in \mathbf{H}} \{HbK \mid K \in \mathbf{K}_H\}$$

Clearly all $K \in \mathbf{K}$ satisfy $K \subseteq B^*$ (they are concatenations of languages satisfying this property). Therefore, it remains to prove that $\mathbf{K}$ is an FO^2-cover of B^* and that $t_\ell \cdot \mathcal{I}[\rho](\mathbf{K}) \cdot t_r \subseteq S$. We begin by proving that all languages $K \in \mathbf{K}$ are FO^2-definable. This is immediate by construction of $\mathbf{H}$ if $K \in \mathbf{H}$. Otherwise, $K = HbK'$ with $H \in \mathbf{H}$ and $K' \in \mathbf{K}_H$. Observe that by definition, $H \subseteq C^*$, $b \in B \setminus C$. Thus, a word w is in $K = HbK'$ if and only if it satisfies the following three properties:

(1) w contains the letter "b".
(2) The prefix of w obtained by keeping all positions that are strictly smaller than the leftmost one carrying a "b" belongs to H.
(3) The suffix of w obtained by keeping all positions that are strictly larger that the leftmost one carrying an "b" belongs to K'.

Hence, it suffices to show that these three properties can be expressed in FO^2. This is clear for the first property (it suffices to use the sentence "$\exists x\ P_b(x)$"). For the second and third properties, observe that one can select the position carrying the leftmost "b" using the following FO^2 formula:

$$\varphi(x) \ := \ P_b(x) \wedge \neg\exists y\ (P_b(y) \wedge y < x)$$

Therefore a sentence for the second property above can be obtained from a sentence Ψ_H that defines H (which exists by hypothesis on H) by restricting all quantifications to positions x that satisfy the formula $\exists y\, x < y \wedge \varphi(y)$. Similarly, a sentence for the third property can be obtained from a sentence $\Psi_{K'}$ that defines K' (which exists by hypothesis on K') by restricting all quantifications to positions x that satisfy the formula $\exists y\ y < x \wedge \varphi(y)$.

We now prove that $\mathbf{K}$ is a cover of B^*. Let $w \in B^*$, we need to find $K \in \mathbf{K}$ such that $w \in K$. If $w \in C^*$, it belongs to some $H \in \mathbf{H} \subseteq \mathbf{K}$ since $\mathbf{H}$ was constructed as a cover of C^*. Otherwise, $b \in \mathbf{alph}(w)$ and w can be decomposed as $w = ubv$ with $u \in C^*$ and $v \in B^*$ (the highlighted b is leftmost one in w). Since $\mathbf{H}$ is a cover of C^*, there exists $H \in \mathbf{H}$ such that $u \in H$. Moreover, since $\mathbf{K}_H$ is a cover of B^*, we get $K \in \mathbf{K}_H$ such that $v \in K$. It follows that $w \in HbK$ which belongs to $\mathbf{K}$ by definition.

Finally, we have to prove that $t_\ell \cdot \mathcal{I}[\rho](\mathbf{K}) \cdot t_r \subseteq S$. By definition of $\mathcal{I}[\rho](\mathbf{K})$ and closure under downset, it suffices to prove that for all $K \in \mathbf{K}$, $t_\ell \cdot \rho(K) \cdot t_r \in S$. We have two cases depending on K. If $K \in \mathbf{H}$, then, by definition of $\mathbf{H}$, we have,

$$\rho(K) = \rho(\varepsilon) \cdot \rho(K) \cdot \rho(\varepsilon) \in \rho(\varepsilon) \cdot \mathcal{I}[\rho](\mathbf{H}) \cdot \rho(\varepsilon) \subseteq S.$$

Moreover, since $t_\ell, t_r \in S$ it follows from closure under multiplication that $t_\ell \cdot \rho(K) \cdot t_r \in S$. Assume now that $K = HbK'$ for $H \in \mathbf{H}$ and $K' \in \mathbf{K}_H$. Since ρ is a multiplicative rating map, we have,

$$\rho(K) = \rho(H) \cdot \rho(b) \cdot \rho(K').$$

Recall that $t_H = t_\ell \cdot \rho(H) \cdot \rho(b)$. Therefore, $t_\ell \cdot \rho(K) \cdot t_r = t_H \cdot \rho(K') \cdot t_r$. Finally, since $K' \in \mathbf{K}_H$, we have $\rho(K') \in \mathcal{I}[\rho](\mathbf{K}_H)$. It now follows from (10.1) that $t_\ell \cdot \rho(K) \cdot t_r \in S$. This concludes the proof of Proposition 10.8.

11. Conclusion

In this paper, we introduced the covering problem, which we designed as a replacement of the two problems that are commonly used to investigate classes of languages: membership and separation. With covering, we get the best of both worlds: like for separation, the problem is flexible enough, so that we are able to obtain covering algorithms for classes that seem to be beyond reach when dealing with membership alone. Like for membership, we have a solid methodology and we recover what was missing in the case of separation: constructiveness. Moreover, the framework is adapted not only to Boolean algebras, but also to lattices.

As an application, we have presented covering algorithms for five fragments of first-order logic: FO itself, its two-variable fragment FO^2, as well as levels $\frac{1}{2}$, 1 and $\frac{3}{2}$ in the Straubing-Thérien hierarchy (denoted Σ_1, $\mathcal{B}\Sigma_1$ and Σ_2, respectively), and we have proved these algorithms for both Σ_1 and FO^2.

There are many natural questions in this line of research. One is to push further the investigation of this problem to other classes of languages, in particular for those of the quantifier alternation hierarchy. It would also be interesting to get precise complexity bounds for this problem, starting from automata as inputs, or from monoids. Finally, whether such problems are meaningful for other structures than words remains to be explored.

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