

# Learning-Based Mean-Payoff Optimization in an Unknown MDP under Omega-Regular Constraints

Jan Křetínský

Technische Universität München, Munich, Germany

Guillermo A. Pérez, Jean-François Raskin

Université libre de Bruxelles, Brussels, Belgium

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## Abstract

We formalize the problem of maximizing the mean-payoff value with high probability while satisfying a parity objective in a Markov decision process (MDP) with unknown probabilistic transition function and unknown reward function. Assuming the support of the unknown transition function and a lower bound on the minimal transition probability are known in advance, we show that in MDPs consisting of a single end component, two combinations of guarantees on the parity and mean-payoff objectives can be achieved depending on how much memory one is willing to use. (i) For all  $\varepsilon$  and  $\gamma$  we can construct an online-learning finite-memory strategy that almost-surely satisfies the parity objective and which achieves an  $\varepsilon$ -optimal mean payoff with probability at least  $1 - \gamma$ . (ii) Alternatively, for all  $\varepsilon$  and  $\gamma$  there exists an online-learning infinite-memory strategy that satisfies the parity objective surely and which achieves an  $\varepsilon$ -optimal mean payoff with probability at least  $1 - \gamma$ . We extend the above results to MDPs consisting of more than one end component in a natural way. Finally, we show that the aforementioned guarantees are tight, i.e. there are MDPs for which stronger combinations of the guarantees cannot be ensured.

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## 1 Introduction

**Reactive synthesis and online reinforcement learning.** Reactive systems are systems that maintain a continuous interaction with the environment in which they operate. When designing such systems, we usually face two partially conflicting objectives. First, to ensure a safe execution, we want some basic and critical properties to be enforced by the system no matter how the environment behaves. Second, we want the reactive system to be as efficient as possible given the actual observed behaviour of the environment in which the system is executed. As an illustration, let us consider a robot that needs to explore an unknown environment as fast and as efficiently as possible while avoiding any collision with other objects, human or robots in the environment. While operating the robot at low speed makes it easier to avoid collisions, it will impair our ability to explore the environment quickly even if this environment is free of other moving objects.

There has been, in the past, a large research effort to define mathematical models and algorithms in order to address the two objectives above, but in isolation only. To design safe control strategies, two-player zero-sum games with omega-regular objectives have been proposed [31, 4], while online-reinforcement-learning (RL, for short) algorithms for partially-specified Markov decision processes (MDPs) have been proposed (see e.g. [35, 23, 28, 30]) to learn strategies that reach optimal or near-optimal performance in the actual environment in which the system is executed. In this paper, we want to answer the following question: *How efficient can online-learning techniques be if one imposes that only correct executions, i.e. executions that satisfy a specified omega-regular objective (defined by a parity objective), are*

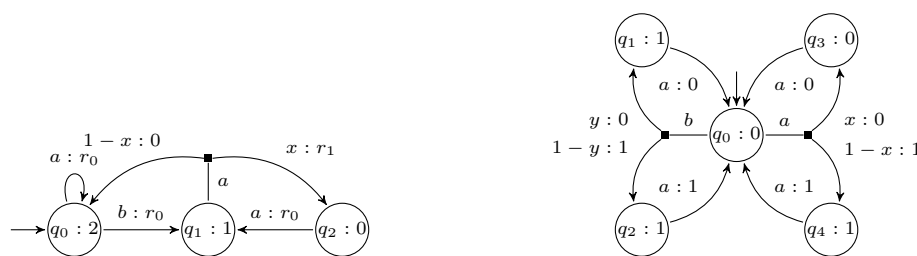
*explored during execution?* So, we want to understand how to combine (reactive) synthesis and RL in order to construct reactive systems that are safe, yet, at the same time, can adapt their behaviour according to the actual environment in which they execute.

**Contributions.** In order to answer in a precise way the question above, we consider a mathematical model which generalizes partially-specified MDPs with the mean payoff function and two-player zero-sum games with a parity objective. Assuming the support of the unknown transition function and a lower bound  $\pi_{\min}$  on the minimal transition probability are known in advance, we show that, in MDPs consisting of a single end component (EC), two combinations of guarantees on the parity and mean-payoff objectives can be achieved. (i) For all  $\varepsilon$  and  $\gamma$ , we show how to construct an online-learning finite-memory strategy which almost-surely satisfies the parity objective and which achieves an  $\varepsilon$ -optimal mean payoff with probability at least  $1 - \gamma$ , for all instantiations of the partially unknown MDP (Proposition 20). (ii) Alternatively, for all  $\varepsilon$  and  $\gamma$ , we show how to construct an online-learning infinite-memory strategy which satisfies the parity objective surely and which achieves an  $\varepsilon$ -optimal mean payoff with probability at least  $1 - \gamma$ , for all instantiations of the partially unknown MDP (Proposition 14). We extend the above results to MDPs consisting of more than one EC in a natural way (Theorem 21 and Theorem 16). We also study special cases that allow for improved optimality results as in the case of good ECs (Proposition 11 and Proposition 17). Finally, we show that there are partially-specified MDPs for which stronger combinations of the guarantees cannot be ensured.

**Example.** We illustrate in this example how to synthesize a finite-memory learning strategy that *almost-surely* wins the parity objective and ensures with *high probability* outcomes that are *near optimal* for the mean-payoff.<sup>1</sup> Consider the MDP on the right-hand side of Fig. 1 for which we know the support of the transition function but not the probabilities  $x$  and  $y$  (for simplicity the rewards are assumed to be known). First, note that while there is no surely winning strategy for the parity objective in this MDP, playing action  $a$  forever in  $q_0$  guarantees to visit state  $q_3$  infinitely many times with probability one, i.e. this is a strategy that almost-surely wins the parity objective. Clearly, if  $x > y$  then it is better to play  $b$  for optimizing the mean-payoff, otherwise, it is better to play  $a$ . As  $x$  and  $y$  are unknown, we need to learn estimates  $\hat{x}$  and  $\hat{y}$  for those values to take a decision. This can be done by playing  $a$  and  $b$  a number of times from  $q_0$  and by observing how many times we get up and how many times we get down. If  $\hat{x} > \hat{y}$ , we may choose to play  $b$  forever in order to optimize our mean payoff. Then we face two difficulties. First, after the learning episode, we may instead observe  $\hat{x} < \hat{y}$  while  $x \geq y$ . This is because we may have been unlucky and observed statistics that differ from the real distribution. Second, playing  $b$  always is not an option if we want to satisfy the parity objective with probability 1 (almost surely). In this paper, we give algorithms to overcome the two problems and compute a finite-memory strategy that satisfies the parity objective with probability 1 and is close to the optimal mean-payoff value with high probability. The finite-memory learning strategy produced by our algorithm works as follows in this example. First, it chooses  $n \in \mathbb{N}$  large enough so that trying  $a$  and  $b$  from  $q_0$  as many as  $n$  times allows to learn  $\hat{x}$  and  $\hat{y}$  such that  $|\hat{x} - x| \leq \varepsilon$  and  $|\hat{y} - y| \leq \varepsilon$  with probability at least  $1 - \gamma$ . Then, if  $\hat{x} > \hat{y}$  the strategy plays  $b$  for  $K$  steps and then  $a$  for 5 steps.  $K$  is chosen large enough so that the mean payoff of any outcome will be  $\varepsilon$ -close to the best obtainable mean payoff with probability at least  $1 - \gamma$ . Furthermore, as

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<sup>1</sup> We refer the reader interested in an example regarding *sure winning* for the parity objective to App. A.



■ **Figure 1** Two automata, representing unknown MDPs, are depicted in the figure. Actions label edges from states (circles) to distributions (squares); a probability-reward pair, edges from distributions to states; an action-reward pair, “Dirac” transitions; a name-priority pair, states.

$a$  is played infinitely many times, the upper-right state will be visited infinitely many times with probability 1. Hence, the strategy is also almost-surely satisfying the parity objective. Additionally, we also show in the paper that if we allow for learning all along the execution of the strategy then we can get, on this example, the exact optimal value and satisfy the parity objective almost surely. However, to do so, we need infinite memory.

**Related works.** In [11, 17, 8, 16], we initiated the study of a mathematical model that combines MDPs and two-player zero sum games. With this new model, we provide formal grounds to synthesize strategies that guarantee *both* some minimal performance against any adversary *and* a higher expected performance against a given expected behaviour of the environment, thus essentially combining the two traditional standpoints from games and MDPs. Following this approach, in [1], Almagor et al. study MDPs equipped with a mean-payoff and parity objective. They study the problem of synthesizing a strategy that ensures an expected mean-payoff value that is as large as possible while satisfying a parity objective surely. In [15], Chatterjee and Doyen study how to enforce almost surely a parity objective together with threshold constraint on the expected mean-payoff. See also [10], where mean-payoff MDPs with energy constraints are studied. In all those works, the transition probability and the reward function are *known* in advance. In contrast, we consider the more complex setting in which the reward function is *discovered on the fly during execution time* and the transition probabilities need to be *learned*.

In [20, 36, 22, 2], RL is combined with safety guarantees. In those works, there is a MDP with a set of unsafe states that must be avoided at all cost. This MDP is then restricted to states and actions that are safe and cannot lead to unsafe states. Thereafter, classical RL is exercised. The problem that is considered there is thus very similar to the problem that we study here with the difference that they only consider *safety constraints*. For safety constraints, the reactive synthesis phase and the RL can be entirely decoupled with a two-phase algorithm. A simple two-phase approach cannot be applied to the more general setting of parity objectives. In our more challenging setting, we need to intertwine the learning with the satisfaction of the parity objective in a non trivial way. It is easy to show that reducing parity to safety, as in [7], could lead to learning strategies that are arbitrary far from the optimal value that our learning strategies achieve. In [37], Topcu and Wen study how to learn in a MDP with a discounted-sum (and not mean-payoff) function and liveness constraints expressed as deterministic Büchi automata that must be enforced *almost surely*. Contrary to our setting, they do not consider general omega-regular specifications expressed as parity objectives nor *sure* satisfaction.

In [9], we apply reinforcement learning to MDP where even the topology is unknown, only

$\pi_{\min}$  and, for convenience, the size of the state space is given. It optimizes the probability to satisfy an omega-regular property; however, no mean payoff is involved. Our usage of  $\pi_{\min}$  follows [9, 18] which argue it is necessary for fully statistical analysis of unbounded-horizon properties, but also that this assumption is realistic in many scenarios.

**Structure of the paper.** In Sect. 2, we introduce the necessary preliminaries. In Sect. 3, we study online finite and infinite-memory learning strategies for mean-payoff objectives without omega-regular constraints. In Sect. 4, we study strategies for mean-payoff objectives under a parity constraint that must be enforced surely. In Sect. 5, we study strategies for mean-payoff objectives under a parity constraint that must be enforced almost surely.

## 2 Preliminaries

Let  $S$  be a finite set. We denote by  $\mathcal{D}(S)$  the set of all (*rational*) *probabilistic distributions* on  $S$ , i.e. the set of all functions  $f : S \rightarrow \mathbb{Q}_{\geq 0}$  such that  $\sum_{s \in S} f(s) = 1$ . For sets  $A$  and  $B$  and functions  $g : A \rightarrow \mathcal{D}(S)$  and  $h : A \times B \rightarrow \mathcal{D}(S)$ , we write  $g(s|a)$  and  $h(s|a, b)$  instead of  $g(a)(s)$  and  $h(a, b)(s)$  respectively. The *support* of a distribution  $f \in \mathcal{D}(S)$  is the set  $\text{supp}(f) \stackrel{\text{def}}{=} \{s \in S \mid f(s) > 0\}$ . The support of a function  $g : A \rightarrow \mathcal{D}(S)$  is the relation  $R \subseteq A \times S$  such that  $(a, s) \in R \stackrel{\text{def}}{\iff} g(s|a) > 0$ .

### 2.1 Markov chains

► **Definition 1** (Markov chains). A *Markov chain*  $\mathcal{C}$  (MC, for short) is a tuple  $(Q, \delta, p, r)$  where  $Q$  is a (potentially countably infinite) set of states,  $\delta$  is a (probabilistic) transition function  $\delta : Q \rightarrow \mathcal{D}(Q)$ ,  $p : Q \rightarrow \mathbb{N}$  is a priority function, and  $r : \text{supp}(\delta) \rightarrow [0, 1] \cap \mathbb{Q}$  is an (instantaneous) reward function.

A *run* of an MC is an infinite sequence of states  $q_0 q_1 \dots \in Q^\omega$  such that  $\delta(q_{i+1}|q_i) > 0$  for all  $0 \leq i$ . We denote by  $\text{Runs}^{q_0}(\mathcal{C})$  the set of all runs of  $\mathcal{C}$  that start with the state  $q_0$ .

Consider an initial state  $q_0$ . The *probability* of every measurable event  $\mathcal{A} \subseteq \text{Runs}^{q_0}(\mathcal{C})$  is uniquely defined [34, 27]. We denote by  $\mathbb{P}_{\mathcal{C}}^{q_0}[\mathcal{A}]$  the probability of  $\mathcal{A}$ ; for a measurable function  $f : \text{Runs}^{q_0}(\mathcal{C}) \rightarrow \mathbb{R}$ , we write  $\mathbb{E}_{\mathcal{C}}^{q_0}[f]$  for the *expected value* of the function  $f$  under the probability measure  $\mathbb{P}_{\mathcal{C}}^{q_0}[\cdot]$ .

**Parity and mean payoff.** Consider a run  $\varrho = q_0 q_1 \dots$  of  $\mathcal{C}$ . We say  $\varrho$  *satisfies the parity objective*, written  $\varrho \models \text{PARITY}$ , if the minimal priority of states along the run is even. That is to say  $\varrho \models \text{PARITY} \stackrel{\text{def}}{\iff} \liminf \{p(q_i) \mid i \in \mathbb{N}\}$  is even. In a slight abuse of notation, we sometimes write  $\text{PARITY}$  to refer to the set of all runs of a Markov chain which satisfy the parity objective  $\{\varrho \in \text{Runs}^{q_0}(\mathcal{C}) \mid \varrho \models \text{PARITY}\}$ . The latter set of runs is clearly measurable.

The *mean-payoff function*  $\underline{\text{MP}}$  is defined for all runs  $\varrho = q_0 q_1 \dots$  of  $\mathcal{C}$  as follows  $\underline{\text{MP}}(\varrho) \stackrel{\text{def}}{=} \liminf_{j \in \mathbb{N}_{>0}} \frac{1}{j} \sum_{i=0}^{j-1} r(q_i, q_{i+1})$ . This function is readily seen to be Borel definable [13], thus also measurable.

### 2.2 Markov decision processes

► **Definition 2** (Markov decision processes). A (*finite discrete-time*) *Markov decision process*  $\mathcal{M}$  (MDP, for short) is a tuple  $(Q, A, \alpha, \delta, p, r)$  where  $Q$  is a finite set of states,  $A$  a finite set of actions,  $\alpha : Q \rightarrow \mathcal{P}(A)$  a function that assigns to  $q$  its set of available actions,  $\delta : Q \times A \rightarrow \mathcal{D}(Q)$  a (partial probabilistic) transition function with  $\delta(q, a)$  defined for all

$q \in Q$  and all  $a \in \alpha(q)$ ,  $p : Q \rightarrow \mathbb{N}$  a priority function, and  $r : \text{supp}(\delta) \rightarrow [0, 1] \cap \mathbb{Q}$  a reward function. We make the assumption that  $\alpha(q) \neq \emptyset$  for all  $q \in Q$ .

A *history*  $h$  in an MDP is a finite state-action sequence that ends in a state and respects  $\alpha$  and  $\delta$ , i.e. if  $h = q_0 a_0 \dots a_{k-1} q_k$  then  $a_i \in \alpha(q_i)$  and  $\delta(q_{i+1}|q_i, a_i) > 0$  for all  $0 \leq i < k$ . We write  $\text{last}(h)$  to denote the state  $q_k$ . For two histories  $h, h'$ , we write  $h < h'$  if  $h$  is a *proper prefix* of  $h'$ .

► **Definition 3 (Strategies).** A *strategy*  $\sigma$  in an MDP  $\mathcal{M} = (Q, A, \alpha, \delta, p, r)$  is a function  $\sigma : (Q \cdot A)^* Q \rightarrow \mathcal{D}(A)$  such that  $\sigma(a|h) > 0 \implies a \in \alpha(\text{last}(h))$ .

We write that a strategy  $\sigma$  is *memoryless* if  $\sigma(h) = \sigma(h')$  whenever  $\text{last}(h) = \text{last}(h')$ ; *deterministic* if for all histories  $h$  the distribution  $\sigma(h)$  is Dirac.

Throughout this work we will speak of *steps*, *episodes*, and *following strategies*. We write that  $\sigma$  *follows*  $\tau$  (from the history  $h = q_0 a_0 \dots q_k$ ) *during*  $n$  *steps* if for all  $h' = q'_0 a'_0 \dots q'_\ell$ , such that  $h < h'$  and  $\ell \leq k + n$ , we have that  $\sigma(h') = \tau(h')$ . An episode is simply a finite sequence of steps, i.e. a finite infix of the history, during which one or more strategies may have been sequentially followed.

A *stochastic Moore machine*  $\mathcal{T}$  is a tuple  $(M, m_0, f_u, f_o)$  where  $M$  is a (potentially countably infinite) set of memory elements,  $m_0 \in M$  is the initial memory element,  $f_u : M \times Q \rightarrow M$  is an update function, and  $f_o : M \times Q \rightarrow \mathcal{D}(A)$  is an output function. The machine  $\mathcal{T}$  is said to implement a strategy  $\sigma$  if for all histories  $h = q_0 \dots a_{k-1} q_k$  we have  $\sigma(h) = f_o(m_k, q_k)$ , where  $m_k$  is inductively defined as  $m_i = f_u(m_{i-1}, q_{i-1})$  for all  $i \geq 1$ . It is easy to see that any strategy can be implemented by such a machine. A strategy  $\sigma$  is said to have *finite memory* if there exists a stochastic Moore machine that implements it and such that its set  $M$  of memory elements is finite.

A (possibly infinite) state-action sequence  $h = q_0 a_0 q_1 a_1 \dots$  is *consistent with strategy*  $\sigma$  if  $\sigma(a_i|q_0 a_0 \dots a_{i-1} q_i) > 0$  for all  $i \geq 0$ .

**From MDPs to MCs.** The MDP  $\mathcal{M}$  and a strategy  $\sigma$  implemented by the stochastic Moore machine  $(M, m_0, f_u, f_o)$  induce the MC  $\mathcal{M}^\sigma = (Q', \delta', p', r')$  where  $Q' = (Q \times M \times A) \cup (Q \times M)$ ,  $\delta'(\langle q', m', a' \rangle | s) = f_o(m, q) \cdot \delta(q'|q, a')$  for any  $s \in \{\langle q, m, a \rangle, \langle q, m \rangle\}$ ,  $p'(\langle q, m, a \rangle) = p'(\langle q, m \rangle) = p(q)$ , and  $r'(s, \langle q', m', a' \rangle) = r(q, a, q')$  for any  $s \in \{\langle q, m, a \rangle, \langle q, m \rangle\}$ . To avoid clutter, we write  $\mathbb{P}_{\mathcal{M}^\sigma}^{q_0}[\cdot]$  instead of  $\mathbb{P}_{\mathcal{M}^\sigma}^{q_0, m_0}[\cdot]$ .

A strategy  $\sigma$  is said to be *unichain* if  $\mathcal{M}^\sigma$  has a single recurrent class (i.e. bottom strongly-connected component).

**End components.** Consider a pair  $(S, \beta)$  where  $S \subseteq Q$  and  $\beta : S \rightarrow \mathcal{P}(A)$  gives a subset of actions allowed per state (i.e.  $\beta(q) \subseteq \alpha(q)$  for all  $q \in S$ ). Let  $\mathcal{G}_{(S, \beta)}$  be the directed graph  $(S, E)$  where  $E$  is the set of all pairs  $(q, q') \in S \times S$  such that  $\delta(q'|q, a) > 0$  for some  $a \in \beta(q)$ . We say  $(S, \beta)$  is an *end component* (EC) if the following hold: if  $a \in \beta(s)$ , for  $(s, a) \in S \times A$ , then  $\text{supp}(\delta(s, a)) \subseteq S$ ; and the graph  $\mathcal{G}_{(S, \beta)}$  is strongly connected. Furthermore, we say the EC  $(S, \beta)$  is *good (for the parity objective)* (a GEC, for short) if the minimal priority of a state from  $S$  is even; *weakly good* if it contains a GEC.

For ECs  $(S, \beta)$  and  $(S', \beta')$ , let us denote by  $(S, \beta) \subseteq (S', \beta')$  the fact that  $S \subseteq S'$  and  $\beta(s) \subseteq \beta'(s)$  for all  $s \in S$ . We denote by  $\text{MEC}_{\mathcal{M}}$  the set of all maximal ECs (MECs) in  $\mathcal{M}$  with respect to  $\subseteq$ . It is easy to see that for all  $(S, \cdot), (S', \cdot) \in \text{MEC}_{\mathcal{M}}$  we have that  $S \cap S' = \emptyset$ , i.e. every state belongs to at most one MEC.

**Model learning and robust strategies.** In this work we will “approximate” the stochastic dynamics of an unknown EC in an MDP. Below, we formalize what we mean by approximation.

► **Definition 4** (Approximating distributions). Let  $\mathcal{M} = (Q, A, \alpha, \delta, p, r)$  be an MDP,  $(S, \beta)$  an EC, and  $\varepsilon \in (0, 1)$ . We say  $\delta'$  is  $\varepsilon$ -close to  $\delta$  in  $(S, \beta)$ , denoted  $\delta' \sim_{(S, \beta)}^\varepsilon \delta$ , if  $|\delta'(q'|q, a) - \delta(q'|q, a)| \leq \varepsilon$  for all  $q, q' \in S$  and all  $a \in \beta(q)$ . If the inequality holds for all  $q, q' \in Q$  and all  $a \in \alpha(q)$ , then we write  $\delta' \sim^\varepsilon \delta$ .

A strategy  $\sigma$  is said to be (*uniformly*) *expectation-optimal* if for all  $q_0 \in Q$  we have  $\mathbb{E}_{\mathcal{M}^\sigma}^{q_0} [\underline{\mathbf{MP}}] = \sup_\tau \mathbb{E}_{\mathcal{M}^\tau}^{q_0} [\underline{\mathbf{MP}}]$ . The following result captures the idea that some expectation-optimal strategies for MDPs whose transition function have the same support are “robust”. That is, when used to play in another MDP with the same support and close transition functions, they achieve near-optimal expectation.

► **Lemma 5** (Adapted from [14, Theorem 5]). Consider values  $\varepsilon, \eta_\varepsilon \in (0, 1)$  such that  $\eta_\varepsilon \leq \frac{\pi_{\min}}{2} \left( \left(1 + \frac{\varepsilon}{2}\right)^{\frac{1}{2|Q|}} - 1 \right)$ , and a transition function  $\delta'$  such that  $\text{supp}(\delta) = \text{supp}(\delta')$  and  $\delta \sim^{\eta_\varepsilon} \delta'$ . For all memoryless deterministic expectation-optimal strategies  $\sigma$  in  $(Q, A, \alpha, \delta', p, r)$ , for all  $q_0 \in Q$ , it holds that  $|\mathbb{E}_{\mathcal{M}^\sigma}^{q_0} [\underline{\mathbf{MP}}] - \sup_\tau \mathbb{E}_{\mathcal{M}^\tau}^{q_0} [\underline{\mathbf{MP}}]| \leq \varepsilon$ .

We say a strategy  $\sigma$  such as the one in the result above is  $\varepsilon$ -robust-optimal (with respect to the expected mean payoff).

### 2.3 Automata as proto-MDPs

We study MDPs with unknown transition and reward functions. It is therefore convenient to abstract those values and work with *automata*.

► **Definition 6** (Automata). A (*finite-state parity*) *automaton*  $\mathcal{A}$  is a tuple  $(Q, A, T, p)$  where  $Q$  is a finite set of states,  $A$  is a finite alphabet of actions,  $T \subseteq Q \times A \times Q$  is a transition relation, and  $p : Q \rightarrow \mathbb{N}$  is a priority function. We make the assumption that for all  $q \in Q$  we have  $(q, a, q') \in T$  for some  $(a, q') \in A \times Q$ .

A transition function  $\delta : Q \times A \rightarrow \mathcal{D}(Q)$  is then said to be *compatible* with  $\mathcal{A}$  if  $\forall (q, a) \in Q \times A : \text{supp}(\delta(q, a)) = \{q' \mid T(q, a, q')\}$ . For a transition function  $\delta$  compatible with  $\mathcal{A}$  and a reward function  $r : T \rightarrow [0, 1] \cap \mathbb{Q}$ , we denote by  $\mathcal{A}_{\delta, r}$  the MDP  $(Q, A, \alpha_T, \delta, p, r)$  where  $a \in \alpha_T(q) \stackrel{\text{def}}{\iff} \exists (q, a, q') \in T$ . It is easy to see that the sets of ECs of MDPs  $(Q, A, \alpha_T, \delta, p, r)$  and  $(Q, A, \alpha_T, \delta', p, r')$  coincide for all  $\delta'$  compatible with  $\mathcal{A}$  and all reward functions  $r'$ . Hence, we will sometimes speak of the ECs of an automaton.

**Transition-probability lower bound.** Let  $\pi_{\min} \in [0, 1] \cap \mathbb{Q}$  be a *transition-probability lower bound*. We say that  $\delta$  is *compatible* with  $\pi_{\min}$  if for all  $(q, a, q') \in Q \times A \times Q$  we have that: either  $\delta(q'|q, a) \geq \pi_{\min}$  or  $\delta(q'|q, a) = 0$ .

## 3 Learning for MP: the Unconstrained Case

In this section, we focus on the design of optimal learning strategies for the mean-payoff function in the unconstrained single-end-component case. That is, we have an unknown strongly connected MDP with no parity objective.

We consider, in turn, learning strategies that use finite and infinite memory. Whereas classical RL algorithms focus on achieving an optimal expected value (see, e.g., [35]; cf. [6]),



we prove here that a stronger result is achievable: one can ensure—using finite memory only—outcomes that are close to the best expected value with high probability. Further, with infinite memory the optimal outcomes can be ensured with probability 1. In both cases, we argue that our results are tight.

For the rest of this section, let us fix an automaton  $\mathcal{A} = (Q, A, T, p)$  such that  $(Q, \alpha_T)$  is an EC, and some  $\pi_{\min} \in (0, 1]$ .

**Yardstick.** Let  $\delta$  be a transition function compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and  $r$  be a reward function. The optimal mean-payoff value that is achievable in the unique EC  $(Q, \alpha_T)$  is defined as  $\mathbf{Val}(Q, \alpha_T) \stackrel{\text{def}}{=} \sup_{\sigma} \mathbb{E}_{\mathcal{A}_{\delta, r}}^{q_0} [\mathbf{MP}]$  for any  $q_0 \in Q$ . Indeed, it is well known that the value on the right-hand side of the definition is the same for all states in the same EC.

Note that this value can always be obtained by a memoryless deterministic [21] and unichain [11] expectation-optimal strategy when  $\delta$  and  $r$  are known. We will use this value as a yardstick for measuring the performance of the learning strategies we describe below.

**Model learning.** Our strategies learn approximate models of  $\delta$  and  $r$  to be able to compute near-optimal strategies. To obtain those models, we use an approach based on ideas from probably approximately correct (PAC) learning. Namely, we will execute a random exploration of the MDP for some number of steps and obtain an empirical estimation of its stochastic dynamics, see e.g. [33]. We say that a memoryless strategy  $\lambda$  is a (*uniform random*) *exploration strategy* for a function  $\beta : Q \rightarrow \mathcal{P}(A)$  if  $\lambda(a|q) = 1/|\beta(q)|$  for all  $q \in Q$ . Each time the random exploration enters a state  $q$  and chooses an action  $a$ , we say that it performs an experiment on  $(q, a)$ , and if the state reached is  $q'$  then we say that the result of the experiment is  $q'$ . Furthermore, the value  $r(q, a, q')$  is then known to us. To learn an approximation  $\delta'$  of the transition function  $\delta$ , and to learn  $r$ , the learning strategy remembers statistics about such experiments. If the random exploration strategy is executed long enough then it collects sufficiently many experiment results to accurately approximate the transition function  $\delta$  and the exact reward function  $r$  with high probability.

The next lemma gives us a bound on the number of  $|Q|$ -step episodes for which we need to exercise such a strategy to obtain the desired approximation with at least some given probability. It can be proved via a simple application of Hoeffding's inequality.

► **Lemma 7.** *For all ECs  $(S, \beta)$  and all  $\varepsilon, \gamma \in (0, 1)$  one can compute  $n \in \mathbb{N}$  (exponential in  $|Q|$  and polynomial in  $|A|$ ,  $\pi_{\min}^{-1}$ ,  $\ln(\gamma^{-1})$ , and  $\varepsilon^{-1}$ ) such that following an exploration strategy for  $\beta$  during  $n$  (potentially non-consecutive) episodes of  $|Q|$ -steps suffices to collect enough information to be able to compute a transition function  $\delta'$  such that  $\mathbb{P} \left[ \delta' \sim_{(S, \beta)}^{\varepsilon} \delta \right] \geq 1 - \gamma$ .*

### 3.1 Finite memory

We now present a family of finite memory strategies  $\sigma_{\text{fin}}$  that force, given any  $\varepsilon, \gamma \in (0, 1)$ , outcomes with a mean payoff that is  $\varepsilon$ -close to the optimal expected value with probability higher than  $1 - \gamma$ . The strategy  $\sigma_{\text{fin}}$  is defined as follows.

1. First,  $\sigma_{\text{fin}}$  follows the model learning strategy above for  $L$  steps, according to Lemma 7, in order to obtain an approximation  $\delta'$  of  $\delta$  such that  $\delta' \sim^{\eta} \delta$  with probability at least  $1 - \gamma$ . A reward function  $r'$  is also constructed from the observed rewards.
2. Then,  $\sigma_{\text{fin}}$  follows a memoryless deterministic expectation-optimal strategy  $\tau$  for  $\mathcal{A}_{\delta', r'}$ . The following result tells us that if the learning phase is sufficiently long, then we can obtain, with  $\sigma_{\text{fin}}$ , a near-optimal mean payoff with high probability.

► **Proposition 8.** *For all  $\varepsilon, \gamma \in (0, 1)$ , one can compute  $L \in \mathbb{N}$  such that for the resulting finite memory strategy  $\sigma_{\text{fin}}$ , for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\text{min}}$ , and for all reward functions  $r$ , we have  $\mathbb{P}_{\mathcal{A}_{\delta, r}^{\sigma_{\text{fin}}}}^{q_0} [\varrho : \underline{\mathbf{MP}}(\varrho) \geq \mathbf{Val}(Q, \alpha_T) - \varepsilon] \geq 1 - \gamma$ .*

**Proof.** We will make use of Lemma 5. For that purpose, let  $\eta = \min\{\varepsilon, \eta_\varepsilon\}$  where  $\eta_\varepsilon$  is as in the statement of the lemma. Next, we set  $L = |Q|n$  where  $n$  is as dictated by Lemma 7 using  $\eta$  and  $\gamma$ . By Lemma 7, with probability at least  $1 - \gamma$  our approximation  $\delta'$  is such that  $\delta' \sim^\eta \delta$ . It follows that, since  $\eta \leq \varepsilon$ , we have  $\text{supp}(\delta) = \text{supp}(\delta')$  and we now have learned  $r$ , again with probability  $1 - \gamma$ . Finally, since  $\eta \leq \eta_\varepsilon$ , Lemma 5 implies the desired result. ◀

► **Remark (Finite-memory implementability).** Note that  $\sigma_{\text{fin}}$ , as we described it previously, is not immediately seen to be a computable finite stochastic Moore machine. However, for all possible runs of length  $L$ , we can compute  $\delta'$ —an approximation of  $\delta$ —and  $r'$ . Using that information, the required finite-memory expectation-optimal strategy  $\tau$  can be computed. We encode these (finitely many) strategies into the machine implementing  $\sigma_{\text{fin}}$  so that it only has to choose which one to follow forever after the (finite) learning phase has ended. Hence, one can indeed construct a finite-memory strategy realizing the described strategy.

**Optimality.** The following tells us that we cannot do better with finite memory strategies.

► **Proposition 9.** *Let  $\mathcal{A}$  be the single-EC automaton on the right-hand side of Fig. 1 and  $\pi_{\text{min}} \in (0, 1]$ . For all  $\varepsilon, \gamma \in (0, 1)$ , the following two statements hold.*

- *For all finite memory strategies  $\sigma$ , there exist  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\text{min}}$ , and a reward function  $r$ , such that  $\mathbb{P}_{\mathcal{A}_{\delta, r}^{\sigma}}^{q_0} [\varrho : \underline{\mathbf{MP}}(\varrho) \geq \mathbf{Val}(Q, \alpha_T) - \varepsilon] < 1$ .*
- *For all finite memory strategies  $\sigma$ , there exist  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\text{min}}$ , and a reward function  $r$  such that  $\mathbb{P}_{\mathcal{A}_{\delta, r}^{\sigma}}^{q_0} [\varrho : \underline{\mathbf{MP}}(\varrho) < \mathbf{Val}(Q, \alpha_T)] \geq \gamma$ .*

**Proof sketch.** With a finite-memory strategy we cannot satisfy a stronger guarantee than being  $\varepsilon$ -optimal with probability at least  $1 - \gamma$  in this example. Indeed, as we can only use finite memory, we can only learn imprecise models of  $\delta$  and  $r$ . That is, we will always have a non-zero probability to have approximated  $x$  or  $y$  arbitrarily far from their actual values. It should then be clear that neither optimality with high probability nor almost-sure  $\varepsilon$ -optimality can be achieved. ◀

### 3.2 Infinite memory

While we have shown that probably approximately optimal is the best that can be obtained with finite memory learning strategies, we now establish that with infinite memory, one can guarantee almost sure optimality.

To this end, we define a strategy  $\sigma_\infty$  which operates in episodes consisting of two phases: learning and optimization. In episode  $i \in \mathbb{N}$ , the strategy does the following.

1. It first follows an exploration strategy  $\lambda$  for  $\alpha_T$  during  $L_i$  steps, there exist models  $\delta_i$  and  $r_i$  based on the experiments obtained throughout the  $\sum_{j=0}^i L_j$  steps during which  $\lambda$  has been followed so far.
2. Then,  $\sigma_\infty$  follows a unichain memoryless deterministic expectation-optimal strategy  $\sigma_{\text{MP}}^{\delta_i}$  for  $\mathcal{A}_{\delta_i, r_i}$  during  $O_i$  steps.

One can then argue that  $\sigma_\infty$  can be instantiated so that in every episode the finite average obtained so far gets ever close to  $\mathbf{Val}(Q, \alpha_T)$  with ever higher probability. This is achieved by choosing the  $L_i$  as an increasing sequence so that the approximations  $\delta_i$  get ever better



with ever higher probability. Then, the  $O_i$  are chosen so as to compensate for the past history, for the time before the induced Markov chain reaches its limit distribution, and for the future number of steps that will be spent learning in the next episode. The latter then allows us to use the Borel-Cantelli lemma to show that in the unknown EC we can obtain its value almost surely. The proof of this result is technically challenging and given in full details in the appendix.

► **Proposition 10.** *One can compute a sequence  $(L_i, O_i)_{i \in \mathbb{N}}$  such that  $L_i \geq |Q|$  for all  $i \in \mathbb{N}$ ; additionally the resulting strategy  $\sigma_\infty$  is such that for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and for all reward functions  $r$ , we have  $\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}^{\sigma_\infty} [\varrho : \underline{\mathbf{MP}}(\varrho) \geq \mathbf{Val}(Q, \alpha_T)] = 1$ .*

**Optimality.** Note that  $\sigma_\infty$  is optimal since it obtains with probability 1 the best value that can be obtained when the MDP is fully known, i.e. when  $\delta$  and  $r$  are known in advance.

## 4 Learning for MP under a Sure Parity Constraint

We show here how to design learning strategies that obtain near-optimal mean-payoff values while ensuring that all runs satisfy a given parity objective with certainty.

First, we note that all such learning strategies must avoid entering states  $q$  from which there is no strategy to enforce the parity objective with certainty. Hence, we make the hypothesis that all such states have been removed from the automaton  $\mathcal{A}$ , and so we assume that for all  $q_0 \in Q$  there exists a strategy  $\sigma_{\text{par}}$  such that for all functions  $\delta$  compatible with  $\mathcal{A}$ , for all reward functions  $r$ , and for all  $\varrho \in \text{Runs}^{q_0}(\mathcal{A}_{\delta,r}^{\sigma_{\text{par}}})$ , we have  $\varrho \models \text{PARITY}$ . It is worth noting that, in fact, there exists a memoryless deterministic strategy such that the condition holds for all  $q_0 \in S$  [4, 3]. Notice the swapping of the quantifiers over the initial states and the strategy, this is why we say it is *uniformly winning (for the parity objective)*. The set of states to be removed, along with a uniformly winning strategy, can be computed quasi-polynomial time [12]. We say that an automaton with no states from which there is no winning strategy is *surely good*.

We study the design of learning strategies for mean-payoff optimization under *sure* parity constraints for increasingly complex cases.

### 4.1 The case of a single good EC

Consider a surely-good automaton  $\mathcal{A} = (Q, A, T, p)$  such that  $(Q, \alpha_T)$  is a GEC, i.e. the minimal color of a state in the EC is even, and some  $\pi_{\min} \in (0, 1]$ .

**Yardstick.** For this case, we use as yardstick the optimal expected mean-payoff value:  $\mathbf{Val}(Q, \alpha_T) = \sup_{\sigma} \mathbb{E}_{\mathcal{A}_{\delta,r}^{q_0}}^{\sigma} [\underline{\mathbf{MP}}]$ .

**Learning strategy.** We show here that it is possible to obtain an optimal mean-payoff with high probability. Note that our solution extends a result given by Almagor et al. [1] for *known* MDPs. The main idea behind our solution is to use the strategy  $\sigma_\infty$  from Proposition 10 in a controlled way: we verify that during all successive learning and optimization episodes, the minimal parity value that is visited is even. If during some episode, this is not the case, then we resort to a strategy  $\sigma_{\text{par}}$  that enforces the parity objective with certainty. Such  $\sigma_{\text{par}}$  is guaranteed to exist as  $\mathcal{A}$  is surely good.

► **Proposition 11.** *For all  $\gamma \in (0, 1)$ , there exists a strategy  $\sigma$  such that for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and for all reward functions  $r$ , we have  $\varrho \models \text{PARITY}$  for all  $\varrho \in \text{Runs}^{q_0}(\mathcal{A}_{\delta,r}^\sigma)$  and  $\mathbb{P}_{\mathcal{A}_{\delta,r}^\sigma}^{q_0}[\varrho : \underline{\text{MP}}(\varrho) \geq \text{Val}(Q, \alpha_T)] \geq 1 - \gamma$ .*

**Proof sketch.** We modify  $\sigma_\infty$  so as to “give up” on optimizing the mean payoff if the minimal even priority has not been seen during a long sequence of episodes. This will guarantee that the measure of runs which give up on the mean-payoff optimization is at most  $\gamma$ .

First, recall that we can instantiate  $\sigma_\infty$  so that  $L_i \geq |Q|$  for all  $i \in \mathbb{N}$ . Hence, with some probability  $\zeta > 0$ , during every learning phase, we visit a state with even minimal priority. We can then find a sequence  $n_1, n_2, \dots \in \mathbb{N}^\omega$  of natural numbers such that  $\prod_{j=i}^\infty (1 - \zeta^{n_j}) \geq 1 - \gamma$ , for some  $i \in \mathbb{N}$ . Given this sequence, we apply the following monitoring. If for  $\ell \in \mathbb{N}$  we write  $N_\ell \stackrel{\text{def}}{=} \sum_{k=1}^{\ell-1} n_k$ , then at the end of the  $\ell$ -th episode we verify that during some learning phase from  $L_{N_\ell}, L_{N_\ell+1}, \dots, L_{N_\ell+n_\ell}$  we have visited a state with minimal even priority, otherwise we switch to a parity-winning strategy forever. ◀

**Optimality.** The following proposition tells us that the guarantees from Proposition 11 are indeed optimal w.r.t. our chosen yardstick.

► **Proposition 12.** *Let  $\mathcal{A}$  be the single-GEC automaton on the left-hand side of Fig. 1 and  $\pi_{\min} \in (0, 1]$ . For all parity-winning strategies  $\sigma$ , there exist  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and a reward function  $r$ , such that  $\mathbb{P}_{\mathcal{A}_{\delta,r}^\sigma}^{q_0}[\varrho : \underline{\text{MP}}(\varrho) \geq \text{Val}(Q, \alpha_T)] < 1$ .*

**Proof sketch.** Consider a reward function such that  $r_0 = 0$  and  $r_1 = 1$  and an arbitrary  $\delta$ . It is easy to see that  $\text{Val}(Q, \alpha_T) = 1$ . However, any strategy that ensures the parity objective is satisfied surely must be such that, with probability  $\gamma > 0$ , it switches to follow a strategy  $q_2 \mapsto (a \mapsto 1)$  forever. Hence, with probability at least  $\gamma$  its mean-payoff is sub-optimal. ◀

## 4.2 The case of a single EC

We now turn to the case where the surely-good automaton  $\mathcal{A} = (Q, A, T, p)$  consists of a unique, not necessarily good, EC  $(Q, \alpha_T)$ . Let us also fix some  $\pi_{\min} \in (0, 1]$ .

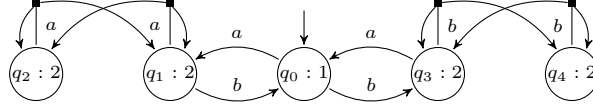
An important observation regarding single-end-component MDPs that are surely good is that they contain at least one GEC as stated in the following lemma.

► **Lemma 13.** *For all surely-good automata  $\mathcal{A} = (Q, A, T, p)$  such that  $(Q, \alpha_T)$  is an EC there exists  $(S, \beta) \subseteq (Q, \alpha_T)$  such that  $(S, \beta)$  is a GEC in  $\mathcal{A}_{\delta,r}$  for all  $\delta$  compatible with  $\mathcal{A}$  and all reward functions  $r$ , i.e.  $(Q, \alpha_T)$  is weakly good.*

**Yardstick.** Let  $\delta$  and  $r$  be fixed in the single EC, our yardstick for this case is defined as follows:  $\text{sVal}(Q, \alpha_T) \stackrel{\text{def}}{=} \max_{q \in Q} \sup \left\{ \mathbb{E}_{\mathcal{A}_{\delta,r}^\sigma}^q[\underline{\text{MP}}] \mid \sigma \text{ is a parity-winning strategy} \right\}$ . That is  $\text{sVal}(Q, \alpha_T)$  is the best MP expectation value that can be obtained from a state  $q \in Q$  with a parity-winning strategy. It is remarkable to note that we take the maximal value over all states in  $Q$ . As noted by Almagor et al. [1], this value is not always achievable even when  $\delta$  and  $r$  are a priori known, but it can be approached arbitrarily close.

**Learning strategy.** The following proposition tells us that we can obtain a value close to  $\text{sVal}(Q, \alpha_T)$  with arbitrarily high probability while satisfying the parity objective surely.

► **Proposition 14.** *For all  $\varepsilon, \gamma \in (0, 1)$  there exists a strategy  $\sigma$  such that for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and for all reward functions  $r$ , we have  $\varrho \models \text{PARITY}$  for all  $\varrho \in \text{Runs}^{q_0}(\mathcal{A}_{\delta,r}^\sigma)$  and  $\mathbb{P}_{\mathcal{A}_{\delta,r}^\sigma}^{q_0}[\varrho : \underline{\text{MP}}(\varrho) \geq \text{sVal}(Q, \alpha_T) - \varepsilon] \geq 1 - \gamma$ .*



■ **Figure 2** An automaton for which it is impossible to learn to obtain near-optimal mean-payoff almost surely or optimal mean-payoff with high probability, while satisfying the parity objective. For clarity, probability and reward placeholders have been omitted.

**Proof sketch.** We define a strategy  $\sigma$  as follows. Let  $\eta = \min\{\pi_{\min}, \eta_{\varepsilon/2}\}$  for  $\eta_{\varepsilon/2}$  as defined for Lemma 5. The strategy  $\sigma$  plays as follows.

1. It first computes  $\delta'$  such that  $\delta' \sim^\eta \delta$  with probability at least  $1 - \gamma/4$  and a reward function  $r'$  by following an exploration strategy  $\lambda$  for  $\alpha_T$  during  $J_0$  steps (see Lemma 7).
2. It then selects a contained maximal good EC (MGEC) with maximal value (see Lemma 13) and tries to reach it with probability at least  $1 - \gamma/4$  by following  $\lambda$  during  $J_1$  steps.
3. Finally, if the component is reached, it follows a strategy  $\tau$  as in Proposition 11 with  $\gamma/4$  from then onward.

If the learning “fails” or if the component is not reached, the strategy reverts to following a winning strategy forever. (A failed learning phase is one in which the approximated distribution function does not have  $T$  as its support.) ◀

**Optimality.** The following states that we cannot improve on the result of Proposition 14.

► **Proposition 15.** *Let  $\mathcal{A}$  be the single-EC automaton in Fig. 2 and  $\pi_{\min} \in (0, 1]$ . For all  $\varepsilon, \gamma \in (0, 1)$ , the two following statements hold.*

- *For all strategies  $\sigma$ , there exist  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and a reward function  $r$ , such that  $\mathbb{P}_{\mathcal{A}_{\delta,r}}^{q_0} [\varrho : \underline{\mathbf{MP}}(\varrho) \geq \mathbf{sVal}(Q, \alpha_T) - \varepsilon] < 1$ .*
- *For all strategies  $\sigma$ , there exist  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and a reward function  $r$ , such that  $\mathbb{P}_{\mathcal{A}_{\delta,r}}^{q_0} [\varrho : \underline{\mathbf{MP}} \geq \mathbf{sVal}(Q, \alpha_T)] < 1 - \gamma$ .*

**Proof sketch.** Observe that the MEC is not a good EC. However, it does contain the GECs with states  $\{q_1, q_2\}$  and  $\{q_3, q_4\}$  respectively. Now, since those two GECs are separated by  $q_0$ , whose priority is 1, any winning strategy must at some point stop playing to  $q_0$  and commit to a single GEC. Thus, the learning of the global EC can only last for a finite number of steps. It is then straightforward to argue that near-optimality with high-probability is the best achievable guarantee. ◀

### 4.3 General surely-good automata

In this section, we generalize our approach from single-EC automata to general automata. We will argue that, under a sure parity constraint, we can achieve a near-optimal meanpayoff with high probability in any MEC in which we end up with non-zero probability. That is, given that we stay in that EC with non-zero probability.

► **Theorem 16.** *Consider a surely-good automaton  $\mathcal{A} = (Q, A, T, p)$  and some  $\pi_{\min} \in (0, 1]$ . For all  $\varepsilon, \gamma \in (0, 1)$  there exists a strategy  $\sigma$  such that for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and all reward functions  $r$ , we have*

- $\varrho \models \text{PARITY}$  for all  $\varrho \in \text{Runs}^{q_0}(\mathcal{A}_{\delta,r}^\sigma)$  and
- $\mathbb{P}_{\mathcal{A}_{\delta,r}}^{q_0} [\varrho : \underline{\mathbf{MP}}(\varrho) \geq \mathbf{sVal}(S, \beta) - \varepsilon \mid \text{Inf} \subseteq S] \geq 1 - \gamma$  for all  $(S, \beta) \in \text{MEC}_{\mathcal{A}_{\delta,r}}$  such that  $(S, \beta)$  is weakly good and  $\mathbb{P}_{\mathcal{A}_{\delta,r}}^{q_0} [\text{Inf} \subseteq S] > 0$ .

**Proof sketch.** The strategy  $\sigma$  we construct follows a parity-winning strategy  $\sigma_{\text{par}}$  until a state contained in a weakly good MEC, that has not been visited before, is entered. In this case, the strategy follows  $\tau$  (the strategy from Proposition 14). Observe that when  $\tau$  switches to  $\sigma_{\text{par}}$  (a parity-winning strategy) it may exit the end component. If this happens, then the component is marked as visited and  $\sigma_{\text{par}}$  is followed until a new—not previously visited—maximal good end component is entered. In that case, we switch to  $\tau$  once more. Crucially, the new strategy  $\sigma$  ignores MECs that are revisited  $\blacktriangleleft$

► **Remark (On the choice of MECs to reach).** The strategy constructed for the proof of Theorem 16 has to deal with leaving a MEC due to the fallbacks to the parity-winning strategy  $\sigma_{\text{par}}$ . However, surprisingly, instead of actually following  $\sigma_{\text{par}}$ , upon entering a new MEC it has to restart the process of achieving a satisfactory mean-payoff. Indeed, otherwise the overall mass of sub-optimal runs from various MECs (each smaller than  $\gamma$ ) could get concentrated in a single MEC, thus violating the advertised guarantees.

The strategy could be simplified as follows. First, we follow any strategy to reach a bottom MEC (BMEC)—that is, a MEC from which no other MEC is reachable. By definition, the winning strategy can be played here and the MEC cannot be escaped. Therefore, in the BMEC we run the strategy as described, and after the fallback we indeed simply follow  $\sigma_{\text{par}}$ . If we did not reach a BMEC after a long time, we could switch to the fallback, too. While this strategy is certainly simpler, our general strategy has the following advantage. Intuitively, we can force the strategy to stay in any current good MEC, even if it is not bottom, and thus maybe achieve a more satisfactory mean-payoff. Further, whenever we want, we can force the strategy to leave the current MEC and go to a lower one. For instance, if the current estimate of the mean payoff is lower than what we hope for, we can try our luck in a lower MEC. We further comment on the choice of unknown MECs in the conclusions.

## 5 Learning for MP under an Almost-Sure Parity Constraint

In this section, we turn our attention to learning strategies that must ensure a parity objective not with certainty (as in previous section) but *almost surely*, i.e. with probability 1. As winning almost surely is less stringent, we can hope both for a stricter yardstick (i.e. better target values) and also better ways of achieving such high values. We show here that this is indeed the case. Additionally, we argue that several important learning results can now be obtained with finite-memory strategies.

As previously, we make the hypothesis that we have removed from  $\mathcal{A}$  all states from which the parity objective cannot be forced with probability 1 (no such state can ever be entered). Note that to compute the set of states to remove, we do not need the knowledge of  $\delta$  but only the support as given by  $\mathcal{A}$ . States to remove can be computed in polynomial time using graph-based algorithms (see, e.g., [5]). An automaton  $\mathcal{A}$  which contains only almost-surely winning states for the parity objective is called *almost-surely good*.

We have, as in the previous section, that for all automata  $\mathcal{A}$  there exists a memoryless deterministic strategy such that for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$ , for all  $r$ , the measure of the subset of  $\varrho \in \text{Runs}^{q_0}(\mathcal{A}_{\mu,r}^\sigma)$  such that  $\varrho \models \text{PARITY}$  is equal to 1 (see e.g. [5]). Such a strategy is said to be *uniformly almost-sure winning (for the parity objective)*. In the sequel, we denote such a strategy  $\sigma_{\text{par}}^{\text{as}}$ .

We now study the design of learning strategies for mean-payoff optimization under *almost-sure* parity constraints for increasingly complex cases.

## 5.1 The case of a good end component

Consider an automaton  $\mathcal{A} = (Q, A, T, p)$  such that  $(Q, \alpha_T)$  is a GEC, and some  $\pi_{\min} \in (0, 1]$ .

**Yardstick.** For this case, we use as a yardstick the optimal expected mean-payoff value:  $\text{Val}(Q, \alpha_T) = \sup_{\sigma} \mathbb{E}_{\mathcal{A}_{\delta, r}^{q_0}} [\underline{\text{MP}}]$  for any  $q_0 \in Q$ .

**Learning strategies.** We start by noting that  $\sigma_{\infty}$  from Section 3 also ensures that the parity objective is satisfied almost surely when exercised in a GEC.

► **Proposition 17.** *One can compute a sequence  $(L_i, O_i)_{i \in \mathbb{N}}$  such that for the resulting strategy  $\sigma_{\infty}$  we have that for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and for all reward functions  $r$ , we have  $\mathbb{P}_{\mathcal{A}_{\delta, r}^{\sigma_{\infty}}}^{q_0} [\text{PARITY}] = 1$  and  $\mathbb{P}_{\mathcal{A}_{\delta, r}^{\sigma_{\infty}}}^{q_0} [\varrho : \underline{\text{MP}}(\varrho) \geq \text{Val}(Q, \alpha_T)] = 1$ .*

**Proof.** By Proposition 10, one can choose parameter sequences such that  $L_i \geq |Q|$  for all  $i \in \mathbb{N}$  and so that we obtain the second part of the claim. Then, since in every episode we have a non-zero probability of visiting a minimal even priority state, we obtain the first part of the claim as a simple consequence of the second Borel-Cantelli lemma. ◀

We now turn to learning using finite memory only. Consider parameters  $\varepsilon, \gamma \in (0, 1)$ . Let  $\eta = \min\{\pi_{\min}, \eta_{\varepsilon/4}\}$  for  $\eta_{\varepsilon/4}$  as defined for Lemma 5. The strategy  $\tau_{\text{fin}}$  that we construct does the following.

1. First, it computes  $\delta'$  such that  $\delta' \sim^{\eta} \delta$  with probability at least  $1 - \gamma$  and a reward function  $r'$  by following an exploration strategy  $\lambda$  for  $\alpha_T$  during  $J$  steps (see Lemma 7).
  2. Then, it computes a unichain deterministic expectation-optimal strategy  $\sigma_{\text{MP}}^{\delta', r'}$  for  $\mathcal{A}_{\delta', r'}$  and repeats the following forever: follow  $\sigma_{\text{MP}}^{\delta', r'}$  for  $O$  steps, then follow  $\lambda$  for  $|Q|$  steps.
- Using the fact that, in a finite MC with a single BSCC, almost all runs obtain the expected mean payoff and the assumption that the EC is good, one can then prove the following result.

► **Proposition 18.** *For all  $\varepsilon, \gamma \in (0, 1)$  one can compute  $L, O \in \mathbb{N}$  such that for the resulting strategy  $\tau_{\text{fin}}$ , for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and for all reward functions  $r$ , we have  $\mathbb{P}_{\mathcal{A}_{\delta, r}^{\tau_{\text{fin}}}}^{q_0} [\text{PARITY}] = 1$  and  $\mathbb{P}_{\mathcal{A}_{\delta, r}^{\tau_{\text{fin}}}}^{q_0} [\varrho : \underline{\text{MP}}(\varrho) \geq \text{Val}(Q, \alpha_T) - \varepsilon] \geq 1 - \gamma$ .*

**Optimality.** Obviously, the result of Proposition 17 is optimal as we obtain the best possible value with probability one. We claim that the result of Proposition 18 is also optimal as we have seen that when we use finite learning, we cannot do better than  $\varepsilon$ -optimality with high probability, this can be proved on the example of Fig. 2 with a similar argument to the one that has been developed for the proof of Proposition 15.

## 5.2 The case of a single end component

Consider an almost-surely-good automaton  $\mathcal{A} = (Q, A, T, p)$  such that  $(Q, \alpha_T)$  is an EC and some  $\pi_{\min} \in (0, 1]$ . The EC is not necessarily good but as the automaton is almost-surely-good, then we have the analogue of Lemma 13 in this context.

► **Lemma 19.** *For all almost-surely-good automata  $\mathcal{A} = (Q, A, T, p)$  such that  $(Q, \alpha_T)$  is an EC there exists  $(S, \beta) \subseteq (Q, \alpha_T)$  such that  $(S, \beta)$  is a GEC in  $\mathcal{A}_{\delta, r}$  for all  $\delta$  compatible with  $\mathcal{A}$  and all reward functions  $r$ , i.e.  $(Q, \alpha_T)$  is weakly good.*

**Yardstick.** As a yardstick for this case, we use the following value:  $\mathbf{asVal}(Q, \alpha_T) \stackrel{\text{def}}{=} \max\{\mathbf{Val}(S, \beta) \mid (S, \beta) \subseteq (Q, \alpha_T) \text{ and } (S, \beta) \text{ is a GEC}\}$ . That is,  $\mathbf{asVal}(Q, \alpha_T)$  is the best expected mean-payoff value that can be obtained in a GEC included in the EC. Such a good EC exists by Lemma 19.

**Learning strategy.** We will now prove an analogue of Proposition 14. For any given  $\varepsilon, \gamma \in (0, 1)$  we define the strategy  $\sigma$  as follows.

1. First, it follows an exploration strategy  $\lambda$  for  $\alpha_T$  during sufficiently many steps (say  $K$ ) to compute an approximation  $\delta'$  of  $\delta$  such that  $\delta' \sim^{\eta\varepsilon/4} \delta$  with probability at least  $1 - \gamma/2$ ; and a reward function  $r'$  (see Lemma 7).
2. Next, it selects a GEC  $(S, \beta)$  with maximal value  $\pm \frac{\varepsilon}{4}$  (see Lemma 19) and computes for it a strategy  $\tau$  as in Proposition 18 with  $\varepsilon_1/2$  and  $\gamma/2$ .
3. Finally,  $\sigma$  switches to  $\lambda$  and follows it until  $(S, \beta)$  is reached and follows  $\tau$  from then onward.

It is straightforward to prove the following about the constructed strategy.

► **Proposition 20.** *For all  $\varepsilon, \gamma \in (0, 1)$  one can construct a finite-memory strategy  $\sigma$  such that for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and for all reward functions  $r$ , we have  $\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}[\text{PARITY}] = 1$  and  $\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}[\varrho : \mathbf{MP}(\varrho) \geq \mathbf{asVal}(Q, \alpha_T) - \varepsilon] \geq 1 - \gamma$ .*

See the remark in Sect. 4.3 for a comment on the finite-memory implementability of  $\sigma$ .

**Optimality.** Using the same example and reasoning as in Proposition 15, we can show that this result is optimal and cannot be improved. Also note that using infinite memory would not help as shown with the example of Fig. 2, where the learning needs to be finite and enforcing the almost sure parity does not require infinite memory.

### 5.3 General almost-surely-good automata

As a final contribution, we now generalize our approach to general almost-surely-good automata.

► **Theorem 21.** *Consider an almost-surely-good automaton  $\mathcal{A} = (Q, A, T, p)$  and some  $\pi_{\min} \in (0, 1]$ . For all  $\varepsilon, \gamma \in (0, 1)$  one can compute a finite-memory strategy  $\sigma$  such that for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and all reward functions  $r$ , we have*

- $\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}[\text{PARITY}] = 1$  and
- $\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}[\varrho : \mathbf{MP}(\varrho) \geq \mathbf{asVal}(S, \beta) - \varepsilon \mid \text{Inf} \subseteq S] \geq 1 - \gamma$  for all  $(S, \beta) \in \text{MEC}_{\mathcal{A}_{\delta,r}}$  such that  $(S, \beta)$  is weakly good and  $\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}[\text{Inf} \subseteq S] > 0$ .

**Proof sketch.** The argument to prove the above result is simple:  $\sigma$  follows a strategy  $\sigma_{\text{par}}^{\text{as}}$  that ensures satisfying the parity objective almost surely. Then, if the run reaches a state contained in a weakly good MEC,  $\sigma$  switches to  $\tau$  as described in Proposition 20. ◀

See the remark in Sect. 3.1 for a word on how to modify  $\sigma$  to favour some unknown MECs.

## 6 Conclusion

As future work, we would like to study different configurations resulting from relaxations of the assumptions we make in this work (i.e. full support,  $\pi_{\min}$ , and bounded reward). Further, we would like to obtain model-free learning algorithms ensuring the same guarantees we



give here. Finally, we have left open the choice of strategy driving the visits to MECs in Theorems 16 and 21 (as long as it satisfies the parity objective). Indeed, the question of computing an “optimal” such strategy in view of the unknown components of the MDP can be addressed in different ways. One such way would be to model the problem as a Canadian traveler problem [26].

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**A Another example**

Consider the MDP on the left-land side of Fig. 1 for which we know the support of the transition function but we do not know the value of the rewards  $r_1$ ,  $r_2$ , and the transition probability  $x$ . Note that playing  $a$  forever surely wins the parity objective from everywhere but it may not be optimal for mean-payoff. To play optimally, we need to learn about the values  $r_1$ ,  $r_2$ , and  $x$ . Assume that we play for  $n$  steps  $a$  and  $b$  uniformly at random when in state  $q_0$ . This will allow us to hopefully reach a number of times  $q_1$  and  $q_2$ , and so to learn  $r_0$  and  $r_1$ , and compute an estimation  $\hat{x}$  of  $x$ . If  $\hat{x} \cdot r_1 > r_0$ , we may want to conclude that the optimal strategy is to always play  $b$  in state  $q_0$ . But we face here two major difficulties. First, after the learning episode of  $n$  steps, we can observe  $\hat{x} \cdot r_1 > r_0$  while  $x \cdot r_1 \leq r_0$ , this is because we may have been unlucky and observed statistics that differ for the real distribution. Second, playing  $b$  always is not an option if we want to surely win parity. In this paper, we give algorithms to overcome the two problems. First, we use PAC like results to determine the right number of learning step  $n$  needed to reduce probability that  $x \cdot r_1 \leq r_0$  while observing  $\hat{x} \cdot r_1 > r_0$ . Second, we show how to combine the optimal mean-payoff strategy for the learned model  $\hat{x}$  with a strategy that enforces the parity objective surely in order to obtain with high probability a strategy that is close to the optimal for the mean-payoff and enforce the parity objective surely. In our example, the strategy constructed by our algorithm will do the following: given  $\varepsilon, \gamma \in (0, 1)$ , it chooses  $n \in \mathbb{N}$  large enough, learn  $\hat{x}$  such that  $|\hat{x} - x| \leq \varepsilon$  with probability more than  $1 - \gamma$ , then if  $\hat{x} \cdot r_1 \leq r_0$ , play  $a$  forever. Otherwise, determine a sequence of natural numbers  $n_1, n_2, \dots \in \mathbb{N}^\omega$  such that  $\prod_{i=0}^{\infty} (1 - (\hat{x} - \varepsilon)^{n_i}) \geq 1 - \gamma'$ , and play  $b$  for episodes of  $n_i$  steps. If during one of this episodes, the state  $q_2$  is not visited (i.e. the parity objective is endangered as the minimal color seen during the episode is odd), then switch to playing  $a$  forever. Our results establish that this strategy is winning the parity objective surely and its mean-payoff is  $\varepsilon$ -close to the optimal value with probability larger than  $(1 - \gamma) \cdot (1 - \gamma')$ . We also show that this is the best guarantee that any learning procedure can give us.

**B A word on Lemma 5**

The result we cite, [14, Theorem 5], comes from a work of Chatterjee that focuses on stochastic parity games with the same support. (In their nomenclature, they are structurally equivalent.) There, they derive robustness bounds for MDPs with the discounted-sum function and use them to obtain robustness bounds for MDPs with a parity objective.

We are, however, extending those results to MDPs with the mean-payoff function (cf. [19]) making use of an observation by Solan [29]: robustness bounds for discounted-sum MDPs extend directly to mean-payoff MDPs if they do not depend on the discount factor. As can be observed in the cited result, this is indeed the case. Furthermore, we need not adapt the bound in our context since we are assuming that the reward function in our MDPs with mean-payoff function assigns transitions rewards between 0 and 1.

**C Proof of Lemma 7**

We recall Hoeffding's concentration bound for the binomial distribution.

► **Proposition 22** (Hoeffding's inequalities). *Let  $X_1, \dots, X_n$  be independent random variables with domain bounded by the interval  $[0, 1]$  and let  $M \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n X_i$ . For all  $0 < \varepsilon < 1$  the following hold.*



- $\mathbb{P}[\mathbb{E}[M] - M \geq \varepsilon] \leq \exp(-2n\varepsilon^2)$
- $\mathbb{P}[M - \mathbb{E}[M] \geq \varepsilon] \leq \exp(-2n\varepsilon^2)$
- $\mathbb{P}[|M - \mathbb{E}[M]| \geq \varepsilon] \leq 2\exp(-2n\varepsilon^2)$

We learn an approximation  $\delta'$  of the transition function  $\delta$  by following a strategy in the MDP and remembering the number of “experiments” we have conducted for each pair  $(q, a) \in Q \times A$ . We then stop sampling when a sufficient number of experiments has been carried out. To obtain an approximation that matches a desired confidence interval, we need a bound on how many experiments have to be carried out.

► **Lemma 23.** *Consider an MDP  $\mathcal{M} = (Q, A, \alpha, \delta, p, r)$ , an end component  $(S, \beta)$  in  $\mathcal{M}$ , and  $\varepsilon, \gamma \in (0, 1)$ . If we let  $X_{q,a}^{q'}$  denote a Bernoulli random variable with success probability  $\delta(q'|q, a)$ , then*

$$k \geq \frac{\ln(2|Q|^2|A|) - \ln(\gamma)}{2\varepsilon^2} \quad (1)$$

*samples of each  $X_{q,a}^{q'}$ , for all  $q, q' \in Q$  and all  $a \in \beta(q)$ , suffice to be able to compute a transition function  $\delta'$  such that*

$$\mathbb{P}[\delta' \sim_{(S, \beta)}^\varepsilon \delta] \geq 1 - \gamma.$$

**Proof.** Let us denote by  $d_{q,a}^{q'}$  the empirical mean of the  $k$  samples of  $X_{q,a}^{q'}$ . It follows from Hoeffding’s two-sided inequality and from our choice of  $k$  that

$$\mathbb{P}\left[\left|d_{q,a}^{q'} - \delta(q'|q, a)\right| \geq \varepsilon\right] \leq \frac{\gamma}{|Q|^2|A|}$$

for all  $q, q' \in Q$  and all  $a \in \beta(q)$ . Hence, the probability that for some  $q, q' \in Q$  and some  $a \in \beta(q)$  we get  $\left|d_{q,a}^{q'} - \delta(q'|q, a)\right| \geq \varepsilon$  is at most  $\gamma$ . ◀

We can now prove the result.

**Proof of Lemma 7.** Let us start by recalling a tail bound for binomial distributions that follows from Hoeffding’s inequalities. Let  $X_1, \dots, X_m$  be independent Bernoulli random variables with success probability  $\mu$ . We want an upper bound on the probability that the random variable  $Y \stackrel{\text{def}}{=} \sum_{i=1}^m X_i$  (with binomial distribution) is less than some desired threshold  $k \in \mathbb{N}$ . If  $k \leq m\mu$  then

$$\begin{aligned} & \mathbb{P}[Y \leq k - 1] \\ &= \mathbb{P}[-Y \geq -k + 1] \\ &= \mathbb{P}[m\mu - Y \geq m\mu - k + 1] \\ &= \mathbb{P}\left[\mu - \frac{1}{m} \sum_{i=1}^m X_i \geq \frac{m\mu - k + 1}{m}\right] \end{aligned}$$

and  $0 < (m\mu - k + 1)/m < 1$ . From one of Hoeffding’s one-sided inequalities we then obtain that

$$\mathbb{P}[Y \leq k - 1] \leq \exp\left(-2 \frac{(m\mu - k + 1)^2}{m}\right). \quad (2)$$

Let us consider an arbitrary state  $q' \in e$ . Observe that, from any state  $q_0 \in e$ , the measure of runs  $q_0 a_0 \dots q_{|Q|}$  that start from  $q_0$  and contain the infix  $q'a$ , i.e.  $q_i a_i = q'a$  for

some  $0 \leq i \leq |Q|$ , while following a uniform random exploration strategy  $\lambda$  for  $\beta$  during  $|Q|$  steps is at least

$$\mu \stackrel{\text{def}}{=} \left( \frac{\pi_{\min}}{|A|} \right)^{|Q|}.$$

At this point we would like to fix the value of  $k$  (used in our discussion above) by using Lemma 23 with  $\varepsilon$  and  $\gamma/2$ . Therefore, we will henceforth have

$$k \stackrel{\text{def}}{=} \left\lceil \frac{\ln(4|Q|^2|A|) - \ln(\gamma)}{2\varepsilon^2} \right\rceil.$$

Consider two states  $q, q' \in Q$  and an action  $a \in A$ . We want to ensure that  $X_{q,a}^{q'}$ , as defined in Lemma 23, is sampled at least  $k$  times with high probability. For this purpose, we let  $W_{q,a}$  be a Bernoulli random variable with success probability  $\mu$ . Intuitively,  $W_{q,a} = 1$  indicates that we have reached  $q'$  and from it played  $a$  while following  $\lambda$  during  $|Q|$  steps. We will use the bound given in Equation (2) to obtain a lower bound on the number  $n$  of times the strategy  $\lambda$  has to be followed for  $|Q|$  steps. That is, since  $\exp(-2(n\mu - k + 1)^2/n)$  is eventually decreasing, we can compute  $n$  large enough so that  $n \geq k/\mu$  and

$$\mathbb{P} \left[ \sum_{i=1}^n W_{q,a} \leq k - 1 \right] \leq \frac{\gamma}{2|Q||A|}.$$

Observe that  $n$  will be polynomial in  $\mu$  and  $1/\mu$  (thus exponential in  $|Q|$  and polynomial in  $|A|$  and  $1/\pi_{\min}$ ), and also in  $k$  (and thus polynomial in  $1/\varepsilon$  and  $\ln(1/\gamma)$ ) since it suffices to take  $n$  larger than the maximum among  $k/\mu$  and the second root of

$$(n\mu - k + 1)^2 - \frac{n}{2} (\ln(2|Q||A|) - \ln(\gamma)) \geq 0.$$

Hence, the probability that, after following  $\lambda$  for  $n$  episodes of  $|Q|$  steps in  $(S, \beta)$ , some  $X_{q,a}^{q'}$  has not yet been sampled sufficiently many times is at most  $\gamma/2$ .

From the above discussion it follows that after following  $\lambda$  for  $n$  (not necessarily consecutive) episodes of  $|Q|$  steps each, we have

$$\mathbb{P} \left[ \delta' \not\sim_{(S,\beta)}^\varepsilon \delta \mid \exists W_{q,a} : \sum_{i=1}^n W_{q,a} \leq k - 1 \right] \cdot \mathbb{P} \left[ \exists W_{q,a} : \sum_{i=1}^n W_{q,a} \leq k - 1 \right] \leq \frac{\gamma}{2}$$

where  $\delta'$  is as constructed in Lemma 23 (i.e. from the empirical mean of the samples of the  $X_{q,a}^{q'}$ ). To conclude, we observe that that we also have

$$\mathbb{P} \left[ \delta' \not\sim_{(S,\beta)}^\varepsilon \delta \mid \forall W_{q,a} : \sum_{i=1}^n W_{q,a} \geq k \right] \cdot \mathbb{P} \left[ \forall W_{q,a} : \sum_{i=1}^n W_{q,a} \geq k \right] \leq \frac{\gamma}{2}$$

by Lemma 23. The fact that

$$\mathbb{P} \left[ \delta' \not\sim_{(S,\beta)}^\varepsilon \delta \right] \leq \gamma$$

then follows from the law of total probability.  $\blacktriangleleft$

## D

 On the convergence of the finite averages

Let us fix an MDP  $\mathcal{M} = (Q, A, \alpha, \delta, p, r)$  for this section.

The following result is repeatedly used throughout the paper.

► **Lemma 24.** *If  $(Q, \alpha)$  is an EC then for all  $q_0 \in Q$ , for all unichain deterministic memoryless strategies  $\mu$ , we have*

- (i)  $\mathbb{P}_{\mathcal{M}^\mu}^{q_0} [\varrho : \underline{\mathbf{MP}}(\varrho) \geq \mathbb{E}_{\mathcal{M}^\mu}^{q_0} [\underline{\mathbf{MP}}]] = 1$ ; and
- (ii) *for all  $\varepsilon \in (0, 1)$ , one can compute  $M(\varepsilon) \in \mathbb{N}$  (dependent only on  $\pi_{\min}$ ,  $|Q|$ , and  $|A|$ ) such that  $\mathbb{P}_{\mathcal{M}^\mu}^{q_0} [\varrho : \forall k \geq M(\varepsilon), \mathbf{MP}(\varrho(..k)) \geq \mathbb{E}_{\mathcal{M}^\mu}^{q_0} [\underline{\mathbf{MP}}] - \varepsilon] \geq 1 - \varepsilon$ .*

Before we prove the above lemma we recall a result by Tracol which is slightly weaker.

► **Proposition 25** ([32, Proposition 2]). *If  $(Q, \alpha)$  is an EC then for all  $q_0 \in Q$ , for all unichain deterministic memoryless strategies  $\mu$ , for all  $\varepsilon \in (0, 1)$ , one can compute  $K_0 \in \mathbb{N}$  and  $c_1, c_2 > 0$  (dependent only on  $\pi_{\min}$ ,  $|Q|$ , and  $|A|$ ) such that  $\forall k \geq K_0$  we have*

$$\mathbb{P}_{\mathcal{M}^\mu}^{q_0} [\varrho : \mathbf{MP}(\varrho(..k)) \geq \mathbb{E}_{\mathcal{M}^\mu}^{q_0} [\mathbf{MP}_k] - \varepsilon] \geq 1 - c_1 \cdot \exp(-k \cdot c_2 \cdot \varepsilon^2)$$

where  $\mathbf{MP}_k$  is the function such that  $\varrho \mapsto \mathbf{MP}(\varrho(..k))$ .

► **Remark.** We observe that Tracol's result depends on a bound for the mixing time of the induced Markov chain. From results in [24] it follows that one can compute such a bound even in unknown chains.

We will need one final ingredient before proving the advertised lemma: a strengthening of Tracol's result in which the comparison inside the probability operator is with the expected mean payoff and not the expectation of the finite average.

► **Proposition 26.** *If  $(Q, \alpha)$  is an EC then for all  $q_0 \in Q$ , for all unichain deterministic memoryless strategies  $\mu$ , for all  $\varepsilon \in (0, 1)$ , one can compute  $K_0 \in \mathbb{N}$  and  $c_1, c_2 > 0$  (dependent only on  $\pi_{\min}$ ,  $|Q|$ , and  $|A|$ ) such that  $\forall k \geq K_0$  we have*

$$\mathbb{P}_{\mathcal{M}^\mu}^{q_0} [\varrho : \mathbf{MP}(\varrho(..k)) \geq \mathbb{E}_{\mathcal{M}^\mu}^{q_0} [\underline{\mathbf{MP}}] - \varepsilon] \geq 1 - c_1 \cdot \exp(-k \cdot c_2 \cdot \varepsilon^2). \quad (3)$$

**Proof.** Our first observation is that, by definition of limit, if we were to replace Equation (3) by

$$\mathbb{P}_{\mathcal{M}^\mu}^{q_0} \left[ \varrho : \forall k \geq M(\varepsilon), \mathbf{MP}(\varrho(..k)) \geq \lim_{\ell \in \mathbb{N}_{>0}} \mathbb{E}_{\mathcal{M}^\mu}^{q_0} [\mathbf{MP}_\ell] - \varepsilon \right] \geq 1 - c_1 \cdot \exp(-k \cdot c_2 \cdot \varepsilon^2)$$

the claim would then be implied by Proposition 25. Hence, it suffices to prove that

$$\mathbb{E}_{\mathcal{M}^\mu}^{q_0} [\underline{\mathbf{MP}}] = \lim_{\ell \in \mathbb{N}_{>0}} \mathbb{E}_{\mathcal{M}^\mu}^{q_0} [\mathbf{MP}_\ell]. \quad (4)$$

Recall that the reward function is bounded, i.e. all rewards are in  $[0, 1]$ . Then, from the ergodic theorem for bounded irreducible unichain reward Markov chains [25, Theorem 1.10.2] we get that

$$\mathbb{P}_{\mathcal{M}^\mu}^{q_0} \left[ \varrho : \text{the limit } \lim_{\ell \in \mathbb{N}_{>0}} \mathbf{MP}(\varrho(..\ell)) \text{ exists} \right] = 1.$$

(Technically, the ergodic theorem applies only to strongly-connected MCs. However, it clearly extends to unichain MCs for the mean-payoff function since it is prefix-independent and almost all runs reach the unique BSCC with probability 1.) Finally, we can now apply Lebesgue's dominated convergence theorem and conclude that Equation 4 does indeed hold. ◀

We will now prove the lemma in two parts.

**Proof of Lemma 24.**

**Item ii**

Remark that for all  $\varepsilon$ , there exists  $K_1 \geq K_0$  such that

$$1 - c_1 \cdot \exp(-k \cdot c_2 \cdot \varepsilon^2) \leq 1 - 2^{-k}.$$

Recall that (from [8, Proof of Lemma 12]) we know that  $\lim_{i \rightarrow \infty} \prod_{j=i}^{\infty} (1 - 2^{-j}) = 1$ . The latter means that

$$\prod_{j=K_2}^{\infty} (1 - 2^{-j}) \geq 1 - \varepsilon$$

for some  $K_2 \geq K_1$ . Let us set  $M(\varepsilon)$  to be the minimal such  $K_2$ .

Let us denote by  $E_\ell$  the event

$$\bigcap_{k=K_2}^{\ell} \{\varrho : \mathbf{MP}(\varrho(..k)) \geq \mathbb{E}_{\mathcal{M}^\sigma}^{q_0} [\underline{\mathbf{MP}}] - \varepsilon\}.$$

It follows from Proposition 26 that the probability measure of  $E_\ell$  is at least  $\prod_{k=K_2}^{\ell} (1 - 2^{-k})$ . Furthermore, we have that  $E_j \subseteq E_i$  for all  $i \leq j$ . Hence, we get (see [5, Page 756]) that

$$\mathbb{P}_{\mathcal{M}^\mu}^{q_0} \left[ \bigcap_{\ell \geq K_2} E_\ell \right] \geq \prod_{j=K_2}^{\infty} (1 - 2^{-j}) \geq 1 - \varepsilon$$

which concludes the proof.

**Item i**

We will now make use of item ii to prove item i. Consider a sequence  $(\varepsilon_i)_{i \in \mathbb{N}}$  such that  $\varepsilon_i = 2^{-i}$ . It should be clear that, if we write  $E_i$  for the event

$$\{\varrho \mid \exists K_0 \in \mathbb{N}, \forall k \geq K_0, \mathbf{MP}(\varrho(..k)) \geq \mathbb{E}_{\mathcal{M}^\sigma}^{q_0} [\underline{\mathbf{MP}}] - \varepsilon_i\}$$

we have that  $E_k \subseteq E_j$  for all  $j \leq k$ . Furthermore, it follows from item ii that  $\mathbb{P}_{\mathcal{M}^\mu}^{q_0} [E_i] \geq 1 - 2^{-i}$  for all  $i \geq 0$ . Hence, we can once more use the limit of the probabilities of the  $E_i$  and conclude that

$$\mathbb{P}_{\mathcal{M}^\mu}^{q_0} \left[ \bigcap_{i \in \mathbb{N}} E_i \right] = \lim_{i \in \mathbb{N}} 1 - \varepsilon_i = 1$$

which proves the claim since the event measured above corresponds to the set of runs whose mean payoff is at least the expected mean payoff  $\blacktriangleleft$

## E Proof of Proposition 10

Let  $S_i$  denote the sum of all steps of all episodes  $j < i$ , i.e.  $S_i \stackrel{\text{def}}{=} \sum_{j=0}^{i-1} L_j + O_j$ .

In the following lemma, we state the guarantees that  $\sigma_\infty$  enforces when the sequence  $(L_i, O_i)_{i \in \mathbb{N}}$  of parameters is chosen appropriately. We need to introduce some notation. Let  $\varrho = q_0 a_0 \dots$  be a run and  $k, \ell \in \mathbb{N}$  such that  $k \leq \ell$ . We denote by  $\mathbf{MP}(\varrho(k..\ell))$  the (finite) average of the  $(k, \ell)$ -infix of  $\varrho$ , i.e.  $\mathbf{MP}(\varrho(k..\ell)) \stackrel{\text{def}}{=} \frac{1}{\ell-k} \sum_{i=k}^{\ell-1} w(q_i, a_i, q_{i+1})$ . We write  $\varrho(..\ell)$  instead of  $\varrho(0..\ell)$ .

► **Lemma 27.** *For all sequences  $(\varepsilon_i)_{i \in \mathbb{N}}$  such that  $0 < \varepsilon_k < \varepsilon_j$  for all  $j < k$ , one can compute  $(L_i, O_i)_{i \in \mathbb{N}}$  such that  $L_i \geq |Q|$  for all  $i \in \mathbb{N}$ ; additionally, for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$ , and for all reward functions  $r$ , we have*

$$\forall i \geq 1, \mathbb{P}_{\mathcal{A}_{\delta, r}^{q_0}}^{\sigma_\infty} [\varrho : \forall k \in (S_i, S_{i+1}], \mathbf{MP}(\varrho(..k)) \geq \mathbf{Val}(Q, \alpha_T) - \varepsilon_i] \geq 1 - \varepsilon_i.$$

**Proof.** Let  $(L_i)_{i \in \mathbb{N}}$  be such that with probability at least  $1 - \varepsilon_{i+1}/4$  we have that

- $\delta_i \sim^\eta \delta$  for  $\eta$  smaller than  $\pi_{\min}$  (see Lemma 7) and
- smaller than  $\eta_\varepsilon$  so that  $\sigma_{\mathbf{MP}}^{\delta_i}$  is  $(\varepsilon_{i+2}/4)$ -robust-optimal with respect to the expected mean payoff (see Lemma 5)

for all  $i \in \mathbb{N}$ . Observe that to approximate  $\delta$  we need to follow  $\lambda$  for episodes of  $|Q|$  steps. The  $L_i$  can thus be assumed to be multiples of  $|Q|$ .

For the optimization part of each episode, we set  $O_i = M(\varepsilon_{i+2}/4) + \max\{0, P_i, F_i\}$  where  $P_i$  and  $F_i$  are inductively defined as follows

$$P_i \stackrel{\text{def}}{=} \left\lceil \left( S_i + L_i + M\left(\frac{\varepsilon_{i+2}}{4}\right) \right) \left( \frac{2(R_i - \varepsilon_{i+2})}{\varepsilon_{i+2}} \right) \right\rceil,$$

with  $R_i = \max\{r_i(t) \mid t \in \text{supp}(\delta_i)\}$ , and

$$F_i \stackrel{\text{def}}{=} \left\lceil \left( S_i + L_i + M\left(\frac{\varepsilon_{i+2}}{4}\right) + P_i + L_{i+1} + M\left(\frac{\varepsilon_{i+3}}{4}\right) \right) \left( \frac{\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}}{\varepsilon_{i+1} - \varepsilon_{i+2}} \right) \right\rceil$$

for all  $i \in \mathbb{N}$ . It is easy to see that, since the  $\varepsilon_i$  are decreasing, the  $P_i$  and  $F_i$  are eventually positive. Additionally, the  $P_i$  depend only on the length of the history after  $L_i$ ; the  $F_i$  has the same dependencies plus  $P_i$  and  $M(\varepsilon_{i+2}/4)$ . The existence of such sequences of integers is therefore guaranteed.

Consider an arbitrary  $i \geq 1$ . It follows from our choice of  $L_{i-1}$  that  $\delta_{i-1} \sim^\eta \delta$ , for  $\eta < \pi_{\min}$ , with probability at least  $1 - \varepsilon_i/4$ . Hence, with the same probability, we also have that  $\text{supp}(\delta_{i-1}) = \text{supp}(\delta)$  thus also that  $r_{i-1}$  coincides with  $r$  (since we have seen all positive-probability transitions and witnessed their rewards). Also from our choice of  $\eta$ , and with the same probability, we have that  $\sigma_{\mathbf{MP}}^{\delta_{i-1}}$  is  $(\varepsilon_{i+1}/4)$ -robust-optimal. If we write  $M_i = L_i + M(\varepsilon_{i+2}/4)$  then from the above arguments and Lemmas 5 and 24 item ii we get that

$$\begin{aligned} & \mathbb{P}_{\mathcal{A}_{\delta, r}^{q_0}}^{\sigma_\infty} \left[ \varrho : \forall k \in (M_{i-1}, S_i], \mathbf{MP}(\varrho(M_{i-1}..k)) \geq \mathbf{Val}(Q, \alpha_T) - \frac{\varepsilon_{i+1}}{2} \right] \\ & \geq (1 - \varepsilon_i/4)(1 - \varepsilon_{i+1}/4) \\ & \geq 1 - \frac{\varepsilon_i + \varepsilon_{i+1}}{4} \\ & \geq 1 - \frac{\varepsilon_i}{2}, \text{ since } \varepsilon_{i+1} < \varepsilon_i. \end{aligned} \tag{5}$$

Moreover, from our choice of  $P_i$  we have that if  $r_i$  coincides with  $r$  then

$$\begin{aligned}
 P_{i-1} &\geq (S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4))(R_{i-1} - \varepsilon_{i+1})(2/\varepsilon_{i+1}) \\
 \implies P_{i-1} &\geq (S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4))(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1})(2/\varepsilon_{i+1}), \\
 &\text{since } 0 \leq \mathbf{Val}(Q, \alpha_T) < R_{i-1} \\
 \iff P_{i-1}(\varepsilon_{i+1}/2) &\geq (S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4))(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}) \\
 \iff P_{i-1}(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}/2) - P_{i-1}(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}) \\
 &\geq (S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4))(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}) \\
 \iff P_{i-1}(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}/2) \\
 &\geq (S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4) + P_{i-1})(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}) \\
 \iff \frac{P_{i-1}(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}/2)}{S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4) + P_{i-1}} &\geq \mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}.
 \end{aligned}$$

Hence, we get that if we write  $N_i = M_i + P_i$  then

$$\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}^{\infty} [\varrho : \forall k \in (N_{i-1}, S_i], \mathbf{MP}(\varrho(..k)) \geq \mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}] \geq 1 - \frac{\varepsilon_i}{2}. \quad (6)$$

Note that Equation (6) holds for all  $i \geq 1$ . It follows that the desired result holds for  $k \in (N_i, S_{i+1}]$  since  $\varepsilon_{i+1}/2 < \varepsilon_{i+1} < \varepsilon_i$ . Therefore, to conclude the proof, all that remains is to argue that  $F_{i-1}$  is large enough so that the claim also holds for all  $k \in (S_i, N_i]$  (with the desired probability).

Observe that from our choice of  $F_i$  we have that

$$\begin{aligned}
 F_{i-1} &\geq \frac{(S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4) + P_{i-1} + L_i + M(\varepsilon_{i+2}/4))(\mathbf{Val}(Q, \alpha_T) - \varepsilon_i)}{\varepsilon_i - \varepsilon_{i+1}} \\
 \iff F_{i-1}(\varepsilon_i - \varepsilon_{i+1}) \\
 &\geq (S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4) + P_{i-1} + L_i + M(\varepsilon_{i+2}/4))(\mathbf{Val}(Q, \alpha_T) - \varepsilon_i) \\
 \iff F_{i-1}(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}) - F_{i-1}(\mathbf{Val}(Q, \alpha_T) - \varepsilon_i) \\
 &\geq (S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4) + P_{i-1} + L_i + M(\varepsilon_{i+2}/4))(\mathbf{Val}(Q, \alpha_T) - \varepsilon_i) \\
 \iff F_{i-1}(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1}) \\
 &\geq (S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4) + P_{i-1} + F_{i-1} + L_i + M(\varepsilon_{i+2}/4))(\mathbf{Val}(Q, \alpha_T) - \varepsilon_i) \\
 \iff \frac{F_{i-1}(\mathbf{Val}(Q, \alpha_T) - \varepsilon_{i+1})}{S_{i-1} + L_{i-1} + M(\varepsilon_{i+1}/4) + P_{i-1} + F_{i-1} + L_i + M(\varepsilon_{i+2}/4)} &\geq (\mathbf{Val}(Q, \alpha_T) - \varepsilon_i).
 \end{aligned}$$

The above inequality implies that

$$\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}^{\infty} [\varrho : \forall k \in (S_i, M_i], \mathbf{MP}(\varrho(..k)) \geq \mathbf{Val}(Q, \alpha_T) - \varepsilon_i] \geq 1 - \frac{\varepsilon_i}{2} \quad (7)$$

since all rewards are assumed to be non-negative. Furthermore, Equations (5) and (7) allow us to conclude that

$$\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}^{\infty} [\varrho : \forall k \in (S_i, S_{i+1}], \mathbf{MP}(\varrho(..k)) \geq \mathbf{Val}(Q, \alpha_T) - \varepsilon_i] \geq (1 - \varepsilon_i/2)(1 - \varepsilon_{i+1}/2).$$

The proof is thus complete since  $(1 - \frac{\varepsilon_i}{2})(1 - \frac{\varepsilon_{i+1}}{2}) \geq (1 - \varepsilon_i)$  because  $\varepsilon_{i+1} < \varepsilon_i$ .  $\blacktriangleleft$

We are now ready to prove the proposition making use of the above lemma.



**Proof of Proposition 10.** Let  $\varepsilon_i$  be  $2^{-i}$  for all  $i \in \mathbb{N}$ . Clearly, we have that  $0 < \varepsilon_k < \varepsilon_j < 1$  for all  $j < k$ .

It follows from Lemma 27 that we can compute  $(L_i)_{i \in \mathbb{N}}$  and  $(O_i)_{i \in \mathbb{N}}$  such that

$$\forall i \geq 1, \mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}} [\varrho : \forall k \in (S_i, S_{i+1}], \underline{\mathbf{MP}}(\varrho(..k)) \geq \mathbf{Val}(Q, \alpha_T) - \varepsilon_i] \geq 1 - \varepsilon_i.$$

Henceforth, we will refer to the event in the above equation as  $E_i$  and to its complement as  $\overline{E_i}$ . Observe that  $D \stackrel{\text{def}}{=} \bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} E_j$  consists only of runs whose mean payoff is at least  $\mathbf{Val}(Q, \alpha_T)$ . Hence, to conclude, it suffices to show that the complement  $\overline{D}$  of  $D$  has probability 0. Since  $\sum_{i \in \mathbb{N}} \mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}} [\overline{E_i}] < \infty$ , then the Borel-Cantelli lemma gives us that

$$\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}} \left[ \bigcap_{i \in \mathbb{N}} \bigcup_{j \geq i} \overline{E_j} \right] = 0.$$

◀

## F Proof of Proposition 11

Let us define a strategy  $\sigma$  which follows  $\sigma_\infty$  while keeping a counter  $k$  initially set to  $K_0$  (whose value will depend on  $\gamma$ ). Intuitively,  $k$  keeps track of how many times we have tried to reach a state with minimal even priority since the last time it was reset. At the end of episode  $i$ , the counter  $k$  is incremented by 1 if

$$\left(1 - (\pi_{\min}/|A|)^{|Q|}\right)^\ell \leq 2^{-k} \quad (8)$$

where  $\ell \stackrel{\text{def}}{=} |\{0 \leq j \leq i \mid L_j \geq |Q|\}|$ . Additionally, if no episode between  $i$  and the last time  $k$  was incremented contains a visit to a state with the minimal even priority,  $\sigma$  switches to follow  $\sigma_{\text{par}}$  forever. Observe that if  $L_i \geq |Q|$  for infinitely many  $i$ , then the expression in the left part of Inequality (8) decreases monotonically. (Hence, for runs which never switch to  $\sigma_{\text{par}}$ ,  $k$  will be increased infinitely often.)

**Proof of Proposition 11.** Let the  $L_i$  and  $O_i$  be chosen as in Proposition 10. Further, let  $S_i \stackrel{\text{def}}{=} \sum_{j=0}^{i-1} L_j + O_i$  as in Section 3. Additionally let the sequence  $(J_i)_{i \in \mathbb{N}}$  be such that  $J_0 = 0$  and  $J_i = S_{\ell+1}$ , where  $\ell$  is the minimal natural number such that,  $\left(1 - \left(\frac{\pi_{\min}}{|A|}\right)^{|Q|}\right)^\ell \leq 2^{-K_0+i-1}$  for all  $i \geq 1$ .

It is not hard to see that for all  $K_0 \in \mathbb{N}$  and all  $m \geq K_0$ , for all priorities  $x$  such that  $p(q) = x$  for some  $q \in Q$ , we have that

$$\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}} [q_0 q_1 \cdots : \forall i \in [K_0, m], \exists j \in [J_i, J_{i+1}], p(q_j) = x] \geq \prod_{k=K_0}^m (1 - 2^{-k}). \quad (9)$$

Indeed, this follows from the fact that every  $L_i$  is at least  $|Q|$  steps long and that  $(Q, \alpha_T)$  is an end component. Let  $E_m$  denote the event in the above equation. Observe that for  $x$  the minimal even priority in the end component, the probability that  $\sigma$  does not switch to  $\sigma_{\text{par}}$  is at least

$$\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}} \left[ \bigcap_{m \geq 0} E_m \right].$$

It then follows from the fact that  $\lim_{i \rightarrow \infty} \prod_{k=i}^{\infty} (1 - 2^{-k}) = 1$  [8, Proof of Lemma 12] that we can choose  $K_0$  large enough so that  $\prod_{k=K_0}^{\infty} (1 - 2^{-k}) = 1 - \gamma$ . Hence, since  $E_j \subseteq E_i$  for all  $i \leq j$ , Equation (9) gives us that the probability that  $\sigma$  does not switch to  $\sigma_{\text{par}}$  is at least

$$\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}} \left[ \bigcap_{m \geq 0} E_m \right] \geq \prod_{k=K_0}^{\infty} (1 - 2^{-k}) = 1 - \gamma.$$

This already implies  $\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}} [\varrho : \underline{\mathbf{MP}}(\varrho) \geq \mathbf{Val}(Q, \alpha_T)] = 1 - \gamma$  by Proposition 10 since  $\sigma$  follows  $\sigma_{\infty}$  if it does not switch to  $\sigma_{\text{par}}$ .

To conclude, we argue that all runs consistent with  $\sigma$  satisfy the parity objective. Indeed, if along a run,  $\sigma$  starts following  $\sigma_{\text{par}}$ , then the run satisfies the parity objective from then onward. Thus, it the whole run satisfies the objective by prefix-independence. If along a run,  $\sigma$  does not switch to  $\sigma_{\text{par}}$ , then the run can be cut into finite segments of increasing length which contain at least one visit to a state with the minimal even priority. (If this were not the case, there would have been a switch to  $\sigma_{\text{par}}$ .) Hence, that priority is seen infinitely often and the parity objective is satisfied. ◀

## G Proof of Lemma 13

**Proof of Lemma 13.** Since  $\mathcal{A}$  is surely good, from all  $q \in Q$  there is a parity-winning strategy. For simplicity, let  $\sigma_{\text{par}}$  be a uniform memoryless deterministic winning strategy implementable by a stochastic Moore machine with a single memory element  $m_0$ .

Consider an arbitrary  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\min}$  and an arbitrary reward function  $r$ . Clearly, in  $\mathcal{A}_{\delta,r}^{\sigma_{\text{par}}}$  there cannot be any cycles  $\chi$  such that the minimal priority of a state in  $\chi$  is odd—otherwise this would contradict the fact that  $\sigma_{\text{par}}$  is uniformly winning.

Let  $C \subseteq Q \times \{m_0\}$  be a strongly-connected component in  $\mathcal{A}_{\delta,r}^{\sigma_{\text{par}}}$ . From the above arguments we have that the minimal priority in  $C$  must be even. It should then be clear that  $(S, \beta)$ , where  $S = \{q \mid (q, m_0) \in C\}$  and  $\beta(q) = \text{supp}(\sigma_{\text{par}}(q))$  for all  $q \in S$ , is a good end component in  $\mathcal{A}_{\delta,r}^{\sigma_{\text{par}}}$ . Since  $\delta$  and  $r$  were arbitrary, the result follows. ◀

## H Proof of Proposition 14

**Proof of Proposition 14.** For the sure satisfaction of the parity objective we just observe that every run is eventually consistent with a strategy  $\tau$  obtained from Proposition 11 or with a winning strategy. Hence, by prefix-independence of the parity objective, the claim holds.

The approximated transition function  $\delta'$  is such that  $\delta' \sim^{\eta} \delta$  with probability at least  $1 - \gamma/4$ . Hence, with the same probability, since  $\eta \leq \pi_{\min}$ , we have that  $\text{supp}(\delta') = \text{supp}(\delta)$  and that  $r' = r$ . By Lemma 5, and because of our choice of  $\eta$ , the value of our chosen MGEC in  $\mathcal{A}_{\delta',r'}$  is “off” by at most  $\varepsilon$ . That is, if the chosen MGEC is  $(S, \beta)$  and  $(S', \beta')$  is a MGEC with maximal value then

$$|\mathbf{Val}(S, \beta) - \mathbf{Val}(S', \beta')| \leq \varepsilon.$$

Now, let us consider the strategy  $\tau$  obtained from Proposition 11 with  $\gamma/4$ . Recall that it guarantees that

$$\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}} [\varrho : \underline{\mathbf{MP}}(\varrho) \geq \mathbf{Val}(S, \beta)] \geq 1 - \gamma/4.$$

From the above arguments, and the fact that we follow  $\lambda$  until the probability that we have reached  $(S, \beta)$  is at least  $1 - \gamma/4$ , it follows that

$$\mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}}[\varrho : \underline{\mathbf{MP}}(\varrho) \geq V - \varepsilon] \geq (1 - \gamma/4)^3 \geq 1 - \gamma$$

where  $V = \max\{\mathbf{Val}(S, \beta) \mid (S, \beta) \subseteq (Q, \alpha_T) \text{ is an MGEC}\}$ . Thus, to conclude, it suffices to argue that  $\mathbf{sVal}(Q, \alpha_T) \leq V$ .

This fact had already been observed in [1] but we sketch a proof of it here for completeness. First, let us point out that the maximum over states in the definition of  $\mathbf{sVal}(\cdot)$  is not needed because the value is the same for all states (as a consequence of it being an end component). Second, recall that the mean-payoff function is prefix-independent, and that almost all runs consistent with a strategy in an MDP are eventually trapped in an end component [5, Theorem 10.120]. It then follows then  $\mathbf{sVal}(Q, \alpha_T)$  is bounded by a convex combination of the elements from  $\{\mathbf{Val}(S, \beta) \mid (S, \beta) \subseteq (Q, \alpha_T) \text{ is an MGEC}\}$ . There is no need to consider other ECs since  $\mathbf{sVal}(Q, \alpha_T)$  is a supremum over all winning strategies only (hence, if runs consistent with them were trapped in bad end components, it would not be winning). The desired result then follows from properties of convex combinations. ◀

## I Proof of Theorem 16

**Proof of Theorem 16.** We now argue that  $\sigma$  satisfies the parity objective surely. If, along a run, eventually  $\sigma_{\text{par}}$  is followed forever then by choice of  $\sigma_{\text{par}}$  and by prefix-independence of the parity objective, we have that the run satisfies it. Otherwise, the run eventually stays forever in an MGEC, and it is following  $\sigma_{\infty}$ . Then, by Proposition 11 the run satisfies the parity objective from then onward. Hence, the run satisfies the parity objective (again by prefix-independence).

Let us now focus on the mean payoff. Consider an arbitrary MGEC component  $(S, \beta)$  with non-zero probability of being reached under  $\sigma$ . Since  $\tau$  (from Proposition 14) is followed the first time  $(S, \beta)$  is entered, by Proposition 11, the definition of conditional probability, and prefix-independence of the mean payoff we get the desired result. ◀

## J Proof of Proposition 18

Let us first focus on a simplified version of the strategy  $\tau_{\text{fin}}$ . Namely, let us suppose that we have access to  $\sigma_{\text{MP}}^{\delta'}$  a  $(\varepsilon/4)$ -robust-optimal strategy. The new strategy  $\varphi$  then plays in episodes consisting of  $|Q|$  steps during which  $\lambda$  is followed, then  $O$  steps during which  $\sigma_{\text{MP}}^{\delta'}$  is followed.

It should be clear that we can obtain, in a similar way to how it is done for Lemma 27 an  $O$  large enough so that we have obtain  $(\varepsilon/2)$ -average-optimal episodes. That is, if we forget about all the past and focus only on the steps contained in the current episode. Indeed, the  $O$  steps only need to account for the  $|Q|$  sub-optimal steps carried out previously in the same episode. We thus obtain the following result.

► **Lemma 28.** *Given  $\sigma_{\text{MP}}^{\delta'}$  a  $(\varepsilon/4)$ -robust-optimal strategy, one can compute  $O \in \mathbb{N}$  such that for all  $q_0 \in Q$ , for all  $\delta$  compatible with  $\mathcal{A}$  and  $\pi_{\text{min}}$ , and for all reward functions  $r$ , we have*

$$\forall i \in \mathbb{N}, \mathbb{P}_{\mathcal{A}_{\delta,r}^{q_0}} \left[ \varrho : \forall k \in (S_i, S_{i+1}], \mathbf{MP}(\varrho(S_i..k)) \geq \mathbf{Val}(Q, \alpha_T) - \frac{\varepsilon}{2} \right] \geq 1 - \frac{\varepsilon}{2}.$$

We now proceed with the proof of the full claim.

**Proof of Proposition 18.** First, let us observe that  $\tau_{\text{fin}}$  indeed ensures almost-sure satisfaction of the parity objective because it follows  $\lambda$  for  $|Q|$  steps infinitely often. Thus, with non-zero probability we visit a state with minimal even priority in every episode. The second Borel-Cantelli lemma then gives us that the parity objective is satisfied almost surely.

Let us assume that we have access to  $\sigma_{\text{MP}}^{\delta'}$  as for Lemma 28. Now, using Lemma 28 and the Bellman optimality equations for the limit of expected averages [27, 5] we obtain that

$$\lim_{\ell \geq 1} \mathbb{E}_{\mathcal{A}_{\delta,r}^{\varphi}}^{q_0} [\text{MP}_{\ell}] \geq \text{Val}(Q, \alpha_T) - \varepsilon.$$

Observe now that  $\varphi$  is a unichain strategy. Hence, the equality we have established in Equation 4 holds and we get that

$$\mathbb{E}_{\mathcal{A}_{\delta,r}^{\varphi}}^{q_0} [\underline{\text{MP}}] \geq \text{Val}(Q, \alpha_T) - \varepsilon.$$

Furthermore, by Lemma 24 item i the expectation is achieved with probability 1.

Now, all that remains is to show that can obtain  $\sigma_{\text{MP}}^{\delta'}$  a  $(\varepsilon/4)$ -robust-optimal strategy with probability  $1 - \gamma$ . However, this is a direct consequence of Lemma 7, so the proof is complete.  $\blacktriangleleft$

## K Proof of Lemma 19

**Proof of Lemma 19.** The proof of the claim goes almost identical to the argument used for the proof of Lemma 13. Let  $\sigma_{\text{par}}^{\text{as}}$  be a uniform memoryless deterministic almost-sure winning strategy implementable by a stochastic Moore machine with a single memory element  $m_0$ . Consider arbitrary  $\delta$  and  $r$ . In  $\mathcal{A}_{\delta,r}^{\sigma_{\text{par}}^{\text{as}}}$  there cannot be any bottom strongly-connected components with a state whose minimal priority is odd.

Let  $C \subseteq Q \times \{m_0\}$  be any bottom strongly-connected component in  $\mathcal{A}_{\delta,r}^{\sigma_{\text{par}}^{\text{as}}}$ . From the above arguments we have that the minimal priority in  $C$  must be even. It should then be clear that  $(S, \beta)$ , where  $S = \{q \mid (q, m_0) \in C\}$  and  $\beta(q) = \text{supp}(\sigma_{\text{par}}(q))$  for all  $q \in S$ , is a good end component in  $\mathcal{A}_{\delta,r}^{\sigma_{\text{par}}}$ . Since  $\delta$  and  $r$  were arbitrary, the result follows.  $\blacktriangleleft$

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