FO Model Checking of Interval Graphs

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Motivation

Algorithmic Meta-Theorems

General statements on the existence of an efficient algorithmic solution for a *class of problems*.

Example (Courcelle's theorem)

All MSO_2-definable problems can be solved in linear (FPT) time on graphs of bounded treewidth.

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Definition (First-order (FO) logic on graphs)

- logical connectives, quantification over elements
- objects: vertices x, predicates: $edge(x, y)$

E.g. $\forall x, y.\, edge(x, y) \implies x \in C \vee y \in C$ "C is a vertex cover"

Story so far

- Any FO property can be tested in XP (i.e $n^{f(\phi)}$) time
- We ask for FPT (i.e. $f(\phi) \cdot n^{O(1)}$) algorithms

Existing results

- bounded degree graphs [Seese]
- locally bounded treewidth [Frick, Grohe]
- locally excluding minor [Dawar, Grohe, Kreutzer]
- locally bounded expansion [D., K.][Dvořák, Kráľ, Thomas]

These classes are all *sparse*.

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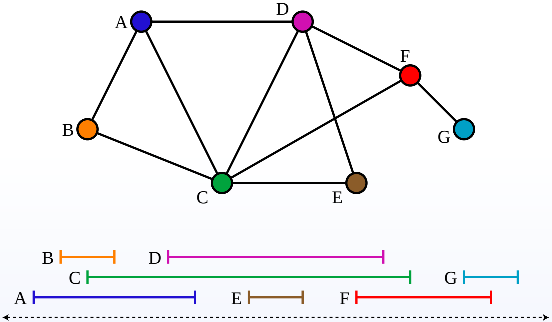
Can we obtain a similar results for some *dense* class?

Interval Graphs

Definition

Intersection graphs of *intervals on the real line*.



[image source:Wikipedia]

L-interval graphs

Interval lengths are taken from some fixed set L .

Unit interval graphs: $L = \{1\}$

Is it all lost?

Theorem

For any dense subset L of $[1, 1+\epsilon]$, all graphs can be FO-interpreted in the class of L-interval graphs.

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For any dense subset L of $[1, 1+\epsilon]$, all graphs can be FO-interpreted in the class of L-interval graphs.

- implies **W[2]-hardness of FO model checking**
- i.e. not in FPT (unless something "bad" happens)
- the interpretation is simple and polynomially bounded
- for MSO_1 a similar result holds even for unit interval graphs

Can we restrict L just a little bit?

Main result

Theorem

For any finite set $L \subseteq \mathbb{R}$ any FO-definable property can be tested in time $\mathcal{O}(n \log n)$ on L-interval graphs.

- Includes $\textsc{IndependentSet}$, $\textsc{DominatingSet}$, $\textsc{SubgraphIsomorphism}$ ...
- Nearly tight result (by the previous slide).
- Easy proof when L is *rational*.
- Much harder when it is not.

Using Locality

Simple case: unit interval graphs

1. *Gaifman's Theorem* tells us that to check any FO-sentence we need to look only at **neighbourhoods of bounded radius**
2. unit interval graphs have *locally bounded cliquewidth*
3. using a result of [Courcelle et al.] we obtain an FPT algorithm for FO model-checking

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Extension to any finite set L of rationals

- the key is to show that L-interval graphs of *bounded radius* have *bounded cliquewidth*

Proof ingredients

- representing interval graphs by *endpoint sequences*
- assume no two endpoints are the same
- look at endpoints w.r.t. *linear combinations* of lengths from L (coefficients depend on the quantifier rank d of ϕ)

Beyond rationals

- remove interval w if there are “too many” within an ϵ-distance of some point a
- give a winning strategy of the Duplicator in the d-round Ehrenfeucht-Fraïssé game for G and $G \setminus w$
- once we remove enough intervals, the graph has *bounded degree* and we apply the result of [Seese]

Conclusions

We showed that:

- There are *natural somewhere dense* classes for which FO model checking is decidable in FPT.
- Our result for L-interval graphs is almost tight.

Natural questions

- Some other dense classes?
- Broader meta-theorem?
- What about *nowhere dense* graph classes?

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Thank you for your attention