FO Model Checking on Map Graphs

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Abstract. For first-order logic model checking on monotone graph classes the borderline between tractable and intractable is well charted: it is tractable on all nowhere dense classes of graphs, and this is essentially the limit. In contrast to this, there are few results concerning the tractability of model checking on general, i.e. not necessarily monotone, graph classes.

We show that model checking for first-order logic on map graphs is fixed-parameter tractable, when parameterised by the size of the input formula. Map graphs are a geometrically defined class of graphs similar to planar graphs, but here each vertex of a graph is drawn homeomorphic to a closed disk in the plane in such a way that two vertices are adjacent if, and only if, the corresponding disks intersect. Map graphs may contain arbitrarily large cliques, and are not closed under edge removal.

Our algorithm works by efficiently transforming a given map graph into a nowhere dense graph in which the original graph is first-order interpretable. As a by-product of this technique we also obtain a model checking algorithm for FO on squares of trees.

1 Introduction

Starting with Courcelle's groundbreaking result [2] that model checking for monadic second-order logic (MSO) is fixed-parameter tractable on graphs of bounded tree width, efficient algorithms for model checking on restricted classes of structures have been thoroughly investigated. Since many well-known algorithmic problems on graphs (such as finding cliques, dominating sets, or vertex covers of a given size) can be rephrased as model checking problems, efficient algorithms for model checking immediately yield efficient algorithms for these problems as well. Therefore results showing the existence of such model checking algorithms are commonly referred to as *algorithmic meta theorems*.

For first-order logic (FO), model checking has been shown to be fixed-parameter tractable on a wide range of graph classes, cf. [4,6,9,14]. These results hinge on the fact that FO has very strong locality properties, and clever graph-theoretic tools for small-diameter graphs. In particular, the methods used in

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proving these results are well-behaved under edge-removal. A graph class which is closed under taking (not necessarily induced) subgraphs is called *monotone*, and for monotone graph classes, FO model checking is fixed-parameter tractable if, and only if, the graph class is nowhere dense [14] (modulo some minor technicalities).

Thus to overcome the barrier of sparse graphs, entirely different algorithmic techniques are necessary. Previous results for model checking on non-sparse graph classes are few. In particular, Courcelle's result has been generalised to graphs of bounded clique width [3], and there are results for FO model checking on partially ordered sets of bounded width [10] and on certain interval graphs, if these graphs are given as an interval representation [12]. Recently, Gajarský et al. obtained an efficient model checking algorithmic for FO on graphs that are FO-interpretable in graphs of bounded degree [11].

In this work we obtain a new algorithmic meta theorem for first-order logic:

Theorem 1. *The model checking problem for first-order logic on vertex coloured map graphs is fixed-parameter tractable, parameterised by the size of the input formula.*

Map graphs have been introduced by Chen et al. [1] as a generalisation of planar graphs. They are defined as graphs which can be drawn in the plane in a way such that to every vertex of the graph a region homeomorphic to a closed disk is drawn, and the regions corresponding to vertices u and v touch if, and only if, uv is an edge of the graph. Here, two regions are considered to touch already if they intersect (as point sets) in a single point. If instead one insists that regions intersect in a set containing a homeomorphic image of a line segment, one obtains the familiar notion of planar graphs.

Note that unlike planar graphs, map graphs may contain arbitrarily large cliques, and the class of map graphs is not closed under taking arbitrary subgraphs. The recognition problem for map graphs, i.e. deciding for a given an abstract graph $G = (V, E)$ whether it can be realised as a map graph, has been shown to be feasible in polynomial time by Thorup in the extended abstract [22]. However, Thorup's algorithm has a running time of roughly $O(|V|^{120})$, and no complete description of it has been published. Moreover, it does not produce a witness graph (which is a combinatorical description of a map drawing) if the input graph is found to be a map graph. Recently, Mnich et al. [19] have given a linear algorithm that decides whether a map graph has an outerplanar witness graph, and computes one if the answer is yes.

The graph input to our algorithm is given as an abstract graph (and not as, say, a geometric representation as a map), and we do not rely on Thorup's algorithm nor any results from [22]. Instead, we use Chen et al.'s classification of cliques in a map graph and show how to efficiently compute, given a map graph G, a graph R in which G is first-order interpretable and such that the class of all graphs arising in this way is nowhere dense. In fact, G is an induced subgraph of the *square* of R, i.e. the graph with the same vertex set as R in which two vertices are adjacent if, and only if, they have distance at most 2 in R. In Sect. 7

we show how know results on squares and square roots of graphs can be used to obtain further algorithmic meta theorems.

2 Preliminaries

2.1 Logic

We use standard definitions for first-order logic (FO), cf. [7,8,16]. In particular, $\bot$ and $\top$ denote false and true, respectively. We will only be dealing with finite, vertex coloured graphs as logical structures, i.e. finite structures with vocabularies of the form $\{E, P_1, \dots, P_k\}$, with a binary edge relation E and unary predicates $P_1, \dots, P_k$.

2.2 Graphs

We will be dealing with finite simple (i.e. loop-free and without multiple edges) undirected graphs, cf. [5,24] for an in-depth introduction. Thus a *graph* $G = (V, E)$ consists of some finite set V of *vertices* and a set $E \subseteq \binom{V}{2}$ of *edges*. A *clique* $C \subseteq V$ is a set of pairwise adjacent vertices, i.e. such that $uv \in E$ for all $u, v \in C$, $u \neq v$. The *neighbourhood* of a vertex $v \in V$ is defined as

$$N(v) := \{w \in V \mid vw \in E\}.$$

For a set $W \subseteq V$ of vertices we denote by $E[W] \subseteq E$ the set of edges that have both endpoints in W.

A *topological embedding* of a graph $H = (W, F)$ into a graph $G = (V, E)$, is an injective mapping $\iota : W \to V$ together with a set $\{p_{xy} \mid xy \in F\}$ of paths in G such that

- each path p_{xy} connects $\iota(x)$ to $\iota(y)$ and
- the paths p_{xy} share no internal vertices, and no $\iota(z)$ is an internal vertex of any of these paths.

If a topological embedding of H into G exists we say that H is a *topological minor* of G, written $H \preceq G$.

If all paths p_{xy} of a topological embedding have length at most r then the embedding is said to be *r-shallow*. The notion of an *r-shallow topological minor*, written $\preceq_r$, is defined accordingly. A class $\mathcal{C}$ of graphs is called *nowhere dense* if for every r there is an m with $K_m \not\preceq_r G$ for any $G \in \mathcal{C}$.

We relax these notions by allowing vertices of G to be used more than once but at most c times, for a constant c. Thus H is a topological minor *of complexity* $\le c$ of G (written $H \preceq^c G$) if there is a mapping $\iota : W \to V$ and paths p_{xy} connecting $\iota(x)$ to $\iota(y)$ for every $xy \in F$ such that no $v \in V$ is used more than c times as an internal vertex of some p_{xy} or as $\iota(x)$. Similarly for $H \preceq^c_r G$.

It is well known that $K_5 \not\preceq G$ for any planar graph G. While for every graph H and every $c \ge 2$ there is a planar graph G with $H \preceq^c G$, for every $c, r \in N$ there is some $m = m(c, r) \in \mathbb{N}$ such that $K_m \not\preceq^c_r G$ for any planar graph G (cf. [21, Sect. 4.8]).

2.3 Map Graphs

A graph $G = (V, E)$ is a *map graph* if there are sets $D_v \subseteq \mathbb{R}^2$, one for each $v \in V$, such that

- each D_v is homeomorphic to a closed disc (i.e. homeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$,
- D_v and D_w intersect only on their boundaries, for $v \neq w$, and
- $D_v \cap D_w \neq \emptyset$ if, and only if, $vw \in E$.

Chen et al. showed that G is a map graph if, and only if, there is a planar bipartite graph $H = (V \cup P, F)$ having the vertices of G as one side of its bipartition and such that $uv \in E$ iff $up, vp \in F$ for some $p \in P$; moreover we may assume that $|P| \leq 4|V|$ [1, Theorem 2.2, Lemma 2.3]. Such a graph H is called a *witness* for G. We call the elements of P the *points* of the witness, and refer the term vertex to elements of V.[1]

By [1, Theorem 3.1], every clique C in a map graph is of one (or more) of the following types (cf. Fig. 1):

pizza there is a $p \in P$ such that $pv \in F$ for all $v \in C$, or
pizza-with-crust there is a $v \in C$ and a $p \in P$ such that $pv \notin F$ but $pw \in F$ for all $w \in C \setminus \{v\}$, or
hamantasch there are $p, q, r \in P$ such that every $v \in C$ is adjacent to at least two of these points, or
rice ball $|C| \leq 4$ and any $p \in P$ is adjacent to at most two vertices in C.

Furthermore, the number of *maximal* cliques in a map graph with n vertices is bounded by $27n$ [1, Theorem 3.2].

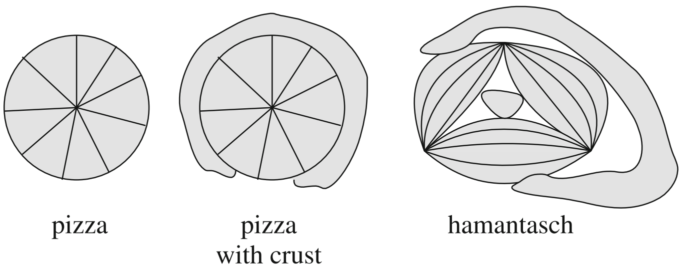

Fig. 1. The possible types of cliques in map graphs.

3 The Maximal Clique Graph

Let $G = (V, E)$ be a map graph and $H = (V \cup P, F)$ a planar witness graph for it. Let $C_1, \ldots, C_m \subseteq V$ be the maximal cliques in G. Then $m \leq 27 \cdot |V|$. We

[1] Elements of V are referred to as *nations* by Chen et al.

define the maximal clique graph $M = (V \cup W, F_M)$ as the bipartite graph with $W = \{w_C \mid C = C_1, \ldots, C_m\}$ and $v \in V$ and $w_C \in W$ adjacent if, and only if, $v \in C$.

Note that

– any witness graph H is, by definition, planar, but we do not know how to efficiently compute one from G,
– we can recover G from coloured versions of both H and M by first-order interpretations,
– we can compute M from G in polynomial time, because we can enumerate the maximal cliques of G in output-polynomial time [23] and there are only linearly many.

In Sect. 5 we define a graph similar to M which will indeed be nowhere dense. Before doing so we give a sequence of a map graphs G_n for which $K_n \preceq_2 M_n$, i.e. the class of maximal clique graphs of map graphs is *not* nowhere dense.

Let $G_n = (V_n, E_n)$ with

$$V_n := \underbrace{\{v_1, \ldots, v_n\}}_{=:V} \cup \underbrace{\{a_1, \ldots, a_m\}}_{=:A} \cup \underbrace{\{b_1, \ldots, b_m\}}_{=:B},$$

$$E_n := \binom{V}{2} \cup \binom{A}{2} \cup \binom{B}{2} \cup \{a_i b_i \mid 1 \leq i \leq m\} \cup \{v_i a_j, v_i b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\},$$

where $m := \binom{n}{2}$. This is a map graph, as witnessed by Fig. 2. The maximal cliques are the sets $V \cup A$, $V \cup B$, and

$$C_i := \{v_1, \ldots, v_n, a_i, b_i\}$$

for $i = 1, \ldots, m$, which can be seen as hamantasch or pizza-with-crust cliques. But then the maximal clique graph M_n contains a 2-subdivision of K_n as a subgraph, so $\{M_n \mid n \geq 1\}$ is not nowhere dense.

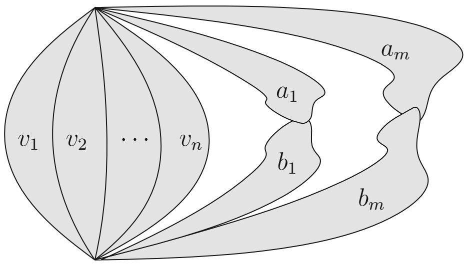

Fig. 2. The map graphs G_n whose maximal clique graphs contain large cliques as (even topological) 2-shallow minors.

We will deal with this by making sure that

- hamantasch cliques are of bounded size, and
- pizza-with-crust cliques with identical centre points are treated only once.

This will be the content of Sects. 4 and 5.

4 Neighbourhood Equivalence

We call two vertices $v, w \in V$ in a graph $G = (V, E)$ *neighbourhood equivalent*, written $v \sim w$, if

$$N(v) \setminus \{v, w\} = N(w) \setminus \{v, w\}.$$

This defines an equivalence relation on V; for transitivity note that $u \sim v$ and $v \sim w$ imply that $\{u, v, w\}$ is either a clique or an independent set. This equivalence relation has been studied before, e.g. in [11,15], where it is called twin relation. For the purpose of model checking we may prune a graph by removing neighbourhood equivalent vertices, as long as we keep track of their number, up to a given threshold.

Define the graph $G/{\sim}$ to be the graph with vertex set

$$V/{\sim} = \{[v] \mid v \in V\}$$

and vertices $[v] \neq [w]$ adjacent if, and only if, $vw \in E$. Note that this is independent of the particular choice of representatives, since $v \sim v'$ and $w \sim w'$ imply that $(vw \in E) \Leftrightarrow (v'w' \in E)$.

In $G/{\sim}$ no two vertices have identical neighbourhoods:

$$N([v]) = N([w]) \quad \Rightarrow \quad [v] = [w]$$

To see this, assume $[v] \neq [w]$. Wlog there is some $u \in V$ such that $uv \in E$ but $uw \notin E$. But then $[u][v] \in E/{\sim}$ but $[u][w] \notin E/{\sim}$, so $N([v]) \neq N([w])$.

We add information on the size of $[v]$ to the graph $G/{\sim}$ using unary predicates as follow: for $i \in \mathbb{N}$ we set

$$P_i(G) := \{[v] \mid |[v]| \geq i\} \subseteq V/{\sim},$$

and we let $G_{\sim,m}$ be the graph $G/{\sim}$ together with the unary predicates $P_1, \ldots, P_m$. Note that $G_{\sim,m}$ can be computed on input G and m in polynomial time, e.g. using colour refinement techniques. Our definition of $G_{\sim,m}$ is motivated by the following lemma, whose (straight-forward) proof we omit here:

Lemma 1. *For every $\varphi \in FO$ of quantifier rank m there is a $\psi \in FO$ of the same quantifier rank such that*

$$G \models \varphi \quad \Leftrightarrow \quad G_{\sim,m} \models \psi$$

for every graph G.

5 3-Connected Map Graphs

In the following we show how to perform FO model checking on 3-connected map graphs. Using Lemma 1 we may, and will, assume that no two vertices of the input graph have identical neighbourhoods. We start by showing that this together with 3-connectedness implies that hamantasch cliques have size at most 9.

Lemma 2. *Let $G = (V, E)$ be a 3-connected map graph, $H = (V \cup P, F)$ a witness graph for G and $p, q \in P$. Consider the vertices of G adjacent to both p and q:*

$$N_{p,q} := N(p) \cap N(q) = \{v \in V \mid pv, qv \in F\}.$$

Then all but (possibly) two of the vertices in $N_{p,q}$ have identical neighbourhoods in G: There are vertices $u, w \in V$ such that

$$N(v) = N(v')$$

for all $v, v' \in N_{p,q} \setminus \{u, w\}$.

Proof. We fix an arbitrary drawing of H. If $|N_{p,q}| \leq 2$ we are done. Otherwise we may order the vertices of $N_{p,q}$ as $\{v_1, v_2, \dots, v_\ell\}$ in such a way that for any $1 \leq i < j < k \leq \ell$, the vertex v_j is inside the region bounded by the cycle pv_iqv_kp in the drawing of H.

Now let $v \in V$ be a neighbour of v_j in G, for $1 < j < \ell$. Then there is a point $r \in P$ such that both v_j and v are adjacent to r in H. Then r must be inside the (closed) region bounded by $pv_{j-1}qv_{j+1}p$. If either $r = p$ or $r = q$, then v is adjacent to all vertices in $N_{p,q}$ and we are done.

Otherwise the point r must either be inside $pv_{j-1}qv_jp$ or inside $pv_jqv_{j+1}p$, but then either $\{v_{j-1}, v_j\}$ or $\{v_j, v_{j+1}\}$ disconnect v from v_{j+1} or v_{j-1}, contradicting the 3-connectedness of G. The lemma follows by choosing $u = v_1$ and $w = v_\ell$.

Since we assume all vertices of our graph to have unique neighbourhoods, it follows that

$$|N_{p,q}| \leq 3$$

for all $p, q \in P$. Thus if $C \subseteq V$ is a hamantasch-clique, then $|C| \leq 9$, because

$$C = N_{p,q} \cup N_{q,r} \cup N_{p,r}$$

for a suitable choice of points $p, q, r \in P$ in a witness $H = (V \cup P, F)$.

The Reduced Maximal Clique Graphs

Starting from a given 3-connected map graph G we now compute a graph which we call *reduced maximal clique graph* $M(G)$. Let $C_1, \dots, C_m \subseteq V$ be the maximal cliques of G, ordered in decreasing size:

$$|C_1| \geq |C_2| \geq \dots \geq |C_m|,$$

and cliques of the same size may appear in arbitrary order. By a result of Chen et al. [1, Theorem 3.2] we know that $m \leq 27 \cdot |V|$, and using an algorithm of Tsukiyama et al. [23] we may compute all of these in polynomial time.

We construct a bipartite graph $R = (V \cup U, A)$ such that for every $v, w \in V$ there is a $u \in U$ adjacent to both v and w if, and only if, $vw \in E$. In this case we say that the edge vw is *covered* by u. We process the cliques in descending size, keeping a set $S_i \subseteq E$ of edges which are already covered, a set $T_i \subseteq E$ of edges which will be covered by individual vertices, and sets U_i, A_i of vertices and edges in the graph that is created. Initially we have

$$S_0 := \emptyset, \quad T_0 := \emptyset, \quad U_0 := \emptyset, \text{ and } \quad A_0 := \emptyset$$

and do the following for $i = 1, \ldots, m$:

(R1) $E[C_i] \subseteq S_{i-1}$ then all edges of C_i are already covered and we ignore C_i (setting $S_i = S_{i-1}$, $T_i = T_{i-1}$ and so on),

(R2) otherwise, if there is a vertex $v \in C_i$ such that $E[C_i \setminus \{v\}] \subseteq S_{i-1}$ we set

$$\begin{aligned} S_i &:= S_{i-1}, & T_i &:= T_{i-1} \cup \{vw \mid w \in C_i, w \neq v\}, \\ U_i &:= U_{i-1}, \text{ and} & A_i &:= A_{i-1}, \end{aligned}$$

(R3) otherwise, if $|C_i| \leq 9$, we treat all edges in C_i as special edges:

$$\begin{aligned} S_i &:= S_{i-1} \\ T_i &:= T_{i-1} \cup E[C_i] \\ U_i &:= U_{i-1}, \text{ and} \\ A_i &:= A_{i-1} \end{aligned}$$

(R4) In all other cases we introduce a new vertex u_i connected to all vertices in C_i:

$$\begin{aligned} S_i &:= S_{i-1} \cup E[C_i] & T_i &:= T_{i-1} \\ U_i &:= U_{i-1} \cup \{u_i\}, \text{ and} & A_i &:= A_{i-1} \cup \{u_i v \mid v \in C_i\} \end{aligned}$$

At the end of this process we have a bipartite graph $(V \cup U_m, A_m)$, plus a set T_m of edges. For any edge $vw \in T_m \setminus S_m$ we add a new vertex u to U_m and connect it only to v and w. We call the resulting graph $R = R(G) := (V \cup U, A)$ the *reduced maximal clique graph* for G. Note that R is not uniquely determined by G but is also influenced by choices the algorithm makes at various stages.

By construction, the graph G is a half-square of R, i.e. for any $v, w \in V$ the edge vw is in E if, and only if, there is a $u \in U$ such that $uv, uw \in A$. Since R is not necessarily planar, it need not be a witness graph of G, but we will now show that the class of graphs arising in this way from map graphs is nowhere dense. Since we can easily recover G in a coloured version of R by a first-order interpretation, and since R can be constructed from G in polynomial time, we may use Grohe et al.'s model checking algorithm for first-order logic on nowhere dense classes of graphs [14] to obtain a model checking algorithm for first-order logic on 3-connected map graphs.

To show that the class

$$\{R(G) \mid G \text{ is a 3-connected map graph}\}$$

is indeed nowhere dense we choose an $r \in \mathbb{N}$ and assume that $K_{2m} \preceq_r R$. We will now show that in this case $K_m \preceq_r^c H'$ for some planar graph H', for an absolute constant c whose value will become apparent during the proof. Since this is not possible for large enough values of m, we conclude that for every r there is an m with $K_{2m} \not\preceq_r R$, and thus the class of reduced maximal clique graphs is nowhere dense.

We fix a witness graph $H = (V \cup P, F)$ of G with an arbitrary drawing. Suppose $K_{2m} \preceq_r R$. This means that there are vertices $x_1, \dots, x_{2m} \in V \cup U$ and pairwise internally vertex-disjoint paths p_{ij} connecting x_i and x_j, for $1 \le i < j \le 2m$. If we could map these vertices and paths *injectively* into H, we would obtain a topological K_{2m}-minor in H, contradicting the fact that H is planar if $2m \ge 5$. We can map the vertices in V to their respective counterparts in H. However,

(i) some maximal cliques (pizza-with-crust and hamantasch) do not correspond to single points in P, and
(ii) we may need to pass through points in P more than once.

We first deal with (i). This concerns vertices $u \in U$ that have been introduced to cover the edges of pizza-with-crust and hamantasch maximal cliques. Each $x_i \in V \cup U$ has degree $2m - 1$, which is > 9 if we choose m large enough. If $x_i = u \in U$ then u has been added by rule (R4) to R to cover the edges of a maximal clique C in G of size > 9. This clique must be either a pizza or pizza-with-crust, because all hamantasch cliques have size ≤ 9. Therefore there is a point $p \in P$ that is adjacent to all but at most one of the vertices in C. If there is a vertex $v \in V$ adjacent to x_i in R but not adjacent to p in H, we remove the x_j which is connected to x_i via the path containing v.

We do this for all the $2m$ vertices of the topological K_{2m} minor in R and, after relabelling the vertices, are left with a topological K_m-minor in R and a mapping of its vertices $x_1, \dots, x_m \in V \cup U$ to $y_1, \dots, y_m \in V \cup P$ such that:

- If $x_i \in V$, then $y_i = x_i$, and
- if $x_i \in U$, then $y_i \in P$, and all neighbours of x_i on paths of the K_m minor are also neighbours of y_i in H.

Furthermore, no $p \in P$ appears as y_i for more than one i: Obviously, any $p \in P$ can only be the centre vertex of at most one maximal clique of pizza type. It may be the centre vertex of more than one (in fact, an unbounded number of) maximal cliques of pizza-with-crust-type, but in this case it is also the centre vertex of a *larger* clique of pizza-type. It is precisely the purpose of rule (R2) in the construction of R to guarantee that only one of these maximal cliques results in a vertex in U.

It remains to map vertices on the paths connecting the x_i to vertices in H. Again we map vertices in V to their identical counterparts. For the remaining vertices, we do not need to preserve all adjacencies, but only their two neighbours on the path belonging to the topological minor. In the following, let xuy be a part of one of the paths connecting the x_i, with $x, y \in V$ and $u \in U$.

We make a case distinction, depending on how the vertex u was introduced to the graph R: If u was introduced using rule (R4) then there is a maximal clique C in G of size >9 containing both x and y. We make a case distinction on the type of C:

- If C is a maximal pizza-clique, then there is a $p \in P$ connected to exactly the elements of C, and we may map u to p.
- If C is a maximal pizza-with-crust-clique, there are two possibilities: If both x and y are connected to the centre point $p \in P$ of the pizza-with-crust, then we may map u to p. Using the same reasoning as above, we can ensure that no $p \in P$ is used more than once, because if it is the centre vertex of two or more pizzas-with-crust, then is also the centre vertex of an even larger pizza-clique, and by rule (R2) u could not have been introduced in this case. Finally, C may be a pizza-with-crust-clique, and x and y connected by a vertex $p \in P$ that is not the centre vertex of C. We can not bound the number of pairs x, y for which this happens, i.e. there may be arbitrarily many $x_1, \ldots, x_k, y_1, \ldots, y_k \in V$ such that
  - all x_i, y_i are adjacent to p,
  - each pair x_i, y_i belongs to some maximal clique C_i in G,
  - no $C_i \cup C_j$ is a clique for $1 \leq i < j \leq k$.

  However, in this case the paths $x_i p y_i$ do not cross but only touch at p, i.e. in the drawing of H, the pairs $x_i y_i$ are consecutive in the cyclic order of the neighbours $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ of p. Therefore we may split p into vertices $p_1, \ldots, p_k$, with each p_i adjacent to x_i and y_i, and still obtain a planar graph H'.

Otherwise, x and y are the endpoints of some edge $xy \in T_m$, and u was introduced to cover this edge. Then there is some $p \in P$ adjacent to both x and y in H. This p has degree at most 9, for otherwise the neighbours of p (plus possibly one other vertex) would form a maximal clique of size larger than 9, and there would be a vertex $u' \in U$ for this clique. Since p has degree ≤ 9 we may safely map u to p, because there can be at most $\binom{9}{2} = 36$ pairs of vertices that get routed through p in this way.

Thus after possibly splitting some vertices of H, we end up with a planar graph H' and an r-shallow topological embedding of K_m into H' of complexity at most 38, which gives the desired contradiction if m is large enough.

6 General Map Graphs

We briefly sketch how our algorithm can be adapted to map graphs that are not necessarily 3-connected. Recall that we needed 3-connectedness to bound the size of hamantasch cliques, which followed from Lemma 2.

Using Feferman and Vaught's composition theorem [18] we may treat connected components individually. Similarly, we may build a tree of 2-connected components (blocks) and process the blocks one by one. We are left with the case of 2-connected but not necessarily 3-connected graphs.

These can be tree-decomposed into parts which are cycles, parallel edges, or 3-connected map graphs, and such that these parts are glued together along edges (cf. [24], it is easy to see that the 3-connected parts in this decomposition are again map graphs). We could colour the edges of these component graphs with the FO[q]-types of the graphs attached to them, but this would result in 3-connected parts that are not necessarily map graphs. In fact, any graph can be encoded in a clique (which is a map graph) of the same size by colouring its edges with two colours.

Instead we introduce coloured vertices of degree 2 rather than colouring the edges. Essentially as in the proof of Lemma 2 we can then show that in any hamantasch clique, there can be only 9 different neighbourhood types if we neglect vertices of degree 2. Again using Feferman-Vaught, we can prune vertices from hamantasch cliques.

7 Squares of Trees

Algorithmic meta theorems for a logic L on a class $\mathcal{C}$ of structures immediately carry over to a structure $\mathcal{D}$ if

- every structure $A \in \mathcal{C}$ can be interpreted in a structure $A' \in \mathcal{D}$ using a L-interpretation that depends only on the classes $\mathcal{C}$ and $\mathcal{D}$, and
- the structure A' can be efficiently computed from A.

Courcelle's result for MSO model checking on graphs of bounded tree-width can be seen as an example of this, since for every graph G of tree-width k there is a tree T such that G is MSO-interpretable in T, using an interpretation that only depends on k, and T can be efficiently computed from G.

Our proof of Theorem 1 also uses this approach, with a very specific kind of FO-interpretation: The input graph G was interpretated as an induced subgraph of the square of the bipartite graph R computed in Sect. 5. Squares and square-roots of graphs have been studied in graph theory, cf. e.g. [13,20]. In particular, Lin and Skiena [17] showed that checking whether a given graph is a square of a tree, and computing such a tree, can be done in polynomial time. The key observation towards this algorithm is that if $G = (V, E)$ is a square of some tree $T = (V, F)$, then $v \in V$ is simplicial (i.e. $N(v)$ is a clique) if, and only if, v is a leaf of T.

Using Lin et al.'s result we immediately get:

Theorem 2. *Model checking for first-order logic on the class of (coloured) squares of trees is fixed-parameter tractable.*

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