Deciding Finiteness of Petri Nets up to Bisimulation

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Abstract. We study the following problems for strong and weak bisimulation equivalence: given a labelled Petri net and a finite transition system, are they equivalent?; given a labelled Petri net, is it equivalent to some (unspecified) finite transition system? We show that both problems are decidable for strong bisimulation and undecidable for weak bisimulation.

1 Introduction

The decidability of equivalence notions for infinite-state systems has been extensively studied in the last years. Among other results, it has been shown that trace equivalence is undecidable for Basic Process Algebra (BPA) and Basic Parallel Processes (BPP), while bisimulation equivalence is decidable in both cases [1, 2, 4].

For arbitrary labelled Petri nets (called just Petri nets in the rest of this introduction), all the equivalence notions commonly used in the literature are undecidable [8, 6]. Therefore, in order to obtain positive results some constraints have to be imposed on the nets accepted as problem instances.

The case in which one of the two Petri nets to be compared is bounded has been considered in [9, 11]. For interleaving equivalence notions like language, trace or bisimulation equivalence, a bounded Petri net is equivalent to a finite transition system. So, loosely speaking, a Petri net is compared against a regular behaviour. The authors of [9, 11] study the following two problems:

- The *equivalence* problem (EP): given a Petri net and a finite transition system, are they equivalent?
- The *finiteness* problem (FP): given a Petri net, is it equivalent to some (unspecified) finite transition system?

Notice that the versions of EP and FP for deterministic or general context-free grammars and language equivalence are classical results of language theory.

[1] This author acknowledges the support of DAAD; he is also partly supported by the Grant Agency of the Czech Republic, Grant No. 201/96/0195.

[2] Supported by the Project SAM of the Sonderforschungsbereich 342.

In [11], Valk and Vidal-Naquet showed that EP and FP are undecidable for language equivalence. In the years following the publication of their paper, other equivalences based on the notion of transition system were found to be more appropriate for the verification of concurrent systems; two of the most commonly used today are trace and bisimulation equivalence.[3] In [9], EP and FP were studied again for these two equivalences (as well as for the simulation preorders). EP was shown to be decidable in both cases, while FP was shown to be undecidable for trace equivalence. FP for bisimulation was left open.

We show in this paper that FP is decidable for bisimulation, and that both EP and FP are undecidable for weak bisimulation.

The paper is structured as follows. Sections 2 and 3 contain some preliminaries. Sections 4 and 5 are devoted to EP and FP for bisimulation, respectively. Sections 6 and 7 study the same problems for weak bisimulation. (Although the decidability of the equivalence problem was already proved in [9], we prove it again in Section 4; the new proof is arguably simpler, and all the lemmas are needed in Section 5 anyway.)

2 Preliminary definitions

Let Act be a finite set of *actions*, containing a distinguished element τ called the *invisible* or *silent* action. A (*rooted, labelled*) *transition system* over Act consists of a set of states S with a distinguished initial state and a relation $\xrightarrow{a} \subseteq S \times S$ for every action $a \in Act$.

A (*labelled, place/transition Petri*) *net* over Act is a tuple $N = (P, T, W, M_0, \ell)$ where

- P and T are finite and disjoint sets of *places* and *transitions*, respectively;
- $W: (P \times T) \cup (T \times P) \to \mathbb{N}$ is a *weight function*;
- $M_0: P \to \mathbb{N}$ is the *initial marking* of N; and
- $\ell: T \to Act$ is a *labelling*, which associates an action to each transition.

If $W(x, y) \in \{0, 1\}$ for every x, y then N is called *ordinary*.

We fix some total ordering $p_1, p_2, \ldots, p_n$ of the places of P. A marking of N, i.e., a mapping $M: P \to \mathbb{N}$ attaching $M(p)$ *tokens* to a place p, is also denoted by the vector $(M(p_1), \ldots, M(p_n))$.

A transition t is *enabled* at a marking M if $M(p) \geq W(p, t)$ for every place p. A transition t enabled at M may *fire* or *occur* yielding the marking M' given by $M'(p) = M(p) + W(t, p) - W(p, t)$ for every $p \in P$. This is denoted by $M \xrightarrow{t} M'$. For any $a \in Act$, $M \xrightarrow{a} M'$ denotes that $M \xrightarrow{t} M'$ for some transition t such that $\ell(t) = a$. $M \longrightarrow M'$ denotes that there exists some sequence $t_1 \ldots t_k$ of transitions such that $M \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \cdots \xrightarrow{t_k} M'$.

[3] The language of a Petri net is defined as the set of words corresponding to occurrence sequences which lead to some final marking, while its trace set contains the words corresponding to all occurrence sequences.

The *reachability set* of a marking M is $\mathcal{R}(M) = \{M' \mid M \longrightarrow M'\}$. The reachability set of N is $\mathcal{R}(N) = \mathcal{R}(M_0)$. The transition system of N has $\mathcal{R}(N)$ as set of states, the initial marking M_0 as initial state, and the relations $\xrightarrow{a}$ between markings as relations.

Let N_1 and N_2 be two nets. A *strong bisimulation* is a relation $\mathcal{B}$ between the markings of N_1 and N_2 such that for all $(M_1, M_2) \in \mathcal{B}$ and for all $a \in Act$:

- if $M_1 \xrightarrow{a} M_1'$ then $M_2 \xrightarrow{a} M_2'$ for some M_2' such that $(M_1', M_2') \in \mathcal{B}$; and
- if $M_2 \xrightarrow{a} M_2'$ then $M_1 \xrightarrow{a} M_1'$ for some M_1' such that $(M_1', M_2') \in \mathcal{B}$.

For every action $a \in Act$, define $\stackrel{a}{\Longrightarrow} = (\xrightarrow{\tau})^* \xrightarrow{a} (\xrightarrow{\tau})^*$. A relation $\mathcal{B}$ between the markings of N_1 and N_2 is a *weak bisimulation* if whenever $(M_1, M_2) \in \mathcal{B}$ then for all $a \in Act \setminus \{\tau\}$:

- if $M_1 \xrightarrow{a} M_1'$ then $M_2 \stackrel{a}{\Longrightarrow} M_2'$ for some M_2' such that $(M_1', M_2') \in \mathcal{B}$; and
- if $M_2 \xrightarrow{a} M_2'$ then $M_1 \stackrel{a}{\Longrightarrow} M_1'$ for some M_1' such that $(M_1', M_2') \in \mathcal{B}$.

and, moreover

- if $M_1 \xrightarrow{\tau} M_1'$ then $(M_1', M_2) \in \mathcal{B}$ or $M_2 \stackrel{\tau}{\Longrightarrow} M_2'$ for some M_2' such that $(M_1', M_2') \in \mathcal{B}$; and
- if $M_2 \xrightarrow{a} M_2'$ then $(M_1, M_2') \in \mathcal{B}$ or $M_1 \stackrel{\tau}{\Longrightarrow} M_1'$ for some M_1' such that $(M_1', M_2') \in \mathcal{B}$.

M_1 is *strongly bisimilar* to M_2, denoted by $M_1 \sim M_2$, if $(M_1, M_2) \in \mathcal{B}$ for some strong bisimulation $\mathcal{B}$. Similarly, M_1 is *weakly bisimilar* to M_2, denoted by $M_1 \approx M_2$, if $(M_1, M_2) \in \mathcal{B}$ for some weak bisimulation $\mathcal{B}$. Two nets are strongly (weakly) bisimilar if their initial markings are strongly (weakly) bisimilar.

Notice that in the case of strong bisimulation τ plays the same rôle as any other action.

We use a stratified characterisation of strong bisimulation. Given two nets N_1 and N_2, the relations $\sim_i$ between their sets of markings are inductively defined as follows. First, we stipulate that $M_1 \sim_0 M_2$ for all pairs (M_1, M_2). Then for every $n \in \mathbb{N}$ we let $M_1 \sim_{n+1} M_2$ whenever for every $a \in Act$:

- if $M_1 \xrightarrow{a} M_1'$ then $M_2 \xrightarrow{a} M_2'$ for some M_2' such that $M_1' \sim_n M_2'$; and
- if $M_2 \xrightarrow{a} M_2'$ then $M_1 \xrightarrow{a} M_1'$ for some M_1' such that $M_1' \sim_n M_2'$.

Clearly, $\sim_i$ is an equivalence relation, and $\sim_{i+1} \subseteq \sim_i$ for every $i \geq 0$. Again we apply these relations to nets by considering their initial marking.

For every action a and every marking M of a net N, the set of markings M' such that $M \xrightarrow{a} M'$ is finite, because N has finitely many transitions. It is well known that in this case we have $M_1 \sim M_2$ iff $M_1 \sim_n M_2$ for every $n \in \mathbb{N}$; moreover, the problem 'Is $N_1 \sim_n N_2$?' is decidable for every $n \in \mathbb{N}$.

We also make use of the following property:

Proposition 1. *Let $N = (P, T, W, M_0, \ell)$ be a net, and let $k = max\{W(p,t) \mid p \in P, t \in T\}$. Let M_1, M_2 be two markings of N satisfying the following property for some number n: for every place p, $M_1(p) \neq M_2(p)$ implies $M_1(p) \geq kn$ and $M_2(p) \geq kn$. Then $M_1 \sim_n M_2$.*

Proof: The occurrence of a transition removes at most k tokens from a place. Therefore, the transition systems of N with M_1 and M_2 as initial markings are isomorphic up to depth n, which implies $M_1 \sim_n M_2$. ■ 1

Notice that every finite transition system is the transition system of some net up to renaming of the states. Using this fact we can immediately extend the definitions above to the case in which N_1 and/or N_2 is replaced by a finite transition system. We use R to denote a finite transition system, r, r' to range over its states, and r_0 to denote its initial state.

To finish the section we formally define the problems we are going to study

> **Strong equivalence problem:** given a Petri net N and a finite transition system R, is $N \sim R$?

> **Strong finiteness problem:** given a Petri net N, is N finite with respect to strong bisimilarity (*b-finite*, for short), i.e. is there a finite transition system R such that $N \sim R$?

We also study the *weak equivalence problem* and the *weak finiteness problem*, which are defined analogously, substituting weak bisimulation ($\approx$) for strong bisimulation ($\sim$), and *weakly b-finite* for b-finite.

3 Auxiliary results

We recall a well-known lemma, which can be easily proved by induction on $|P|$:

Lemma 2. *Given an infinite sequence of mappings $M_1, M_2, M_3, \ldots$ of the type $P \rightarrow \mathbb{N}$, for a finite set P, there are indices $1 \leq i_1 < i_2 < i_3 < \ldots$ such that $M_{i_1} \leq M_{i_2} \leq M_{i_3} \ldots$ ($\leq$ defined componentwise).*

We also need a semidecidability result. Its proof is very similar to the proof of Theorem 6.5 in [5], and is omitted here. Let P be a finite set and let $\mathcal{M} \subseteq \mathbb{N}^P$. An element $p \in P$ is *unbounded in* $\mathcal{M}$ if for every $k \geq 0$ there exists $M_k \in \mathcal{M}$ such that $M_k(p) \geq k$. The set $\mathcal{M}$ is *linear* if there exist M_b (basis), and $M_1, \ldots, M_n$ (periods) in $\mathbb{N}^P$ such that

$$\mathcal{M} = \{M_b + \sum_{i=1}^{n} a_i M_i \mid a_1, \ldots, a_n \in \mathbb{N}\}.$$

Lemma 3. *Let N be a Petri net with initial marking M_0, and let $\mathcal{M}_1, \mathcal{M}_2$ be linear sets of markings of N. It is semidecidable if the set*

$$\mathcal{M}_{12} = \{M_1 \in \mathcal{M}_1 \mid \exists M_2 \in \mathcal{M}_2 \colon M_0 \longrightarrow M_1 \longrightarrow M_2\}$$

has the same unbounded places as $\mathcal{M}_1$.[4] ■ 3

4 Decidability of strong equivalence

The strong equivalence problem has already been shown to be decidable in [9]. The proof given there relies on some general results about semilinear sets and Presburger arithmetic. Here we look in greater detail at the $\sim_n$ equivalence classes, which enables us to give a more self contained proof. All the lemmas of this section are also used in the next one.

We consider ordinary nets, because Proposition 1 has a particularly simple form for them. The results can be easily generalised to arbitrary nets by carrying the constant k of Proposition 1 through the proofs.

We fix an ordinary Petri net $N = (P, T, W, M_0, \ell)$, and a finite transition system R having n states and initial state r_0.

We start with a simple observation. Given two states r and r' of R, we have $r \sim_n r'$ whenever $r \sim_{n-1} r'$. This is easily seen to be true by noting that the sequence of approximation equivalences $\sim_0$, $\sim_1$, ... on the states of R must stabilise within n steps (every equivalence relation on this set has at most n equivalence classes).

We examine the structure of the classes $\sim_n$ on the markings of N.

Definition 4. A marking L of N is *n-bounded* if $L \leq (n, n, \ldots, n)$. For every n-bounded marking L, we define $L^{\geq n}$ as the set of markings M such that if $L(p) < n$, then $M(p) = L(p)$, and if $L(p) = n$, then $M(p) \geq L(p)$. A marking M of N is *incompatible with R* if $M \not\sim_n r$ for every state r of R. The set of markings incompatible with R is denoted by $Inc(R)$. ■ 4

In the sequel we use the symbols L, L_1, L_2 for n-bounded markings. Note that $M_1 \sim_n M_2$ for any $M_1, M_2 \in L^{\geq n}$ (cf. Proposition 1), and that there are only finitely many n-bounded markings. Obviously, $Inc(R) = L_1^{\geq n} \cup \ldots \cup L_k^{\geq n}$ for some $L_1, \ldots, L_k$, where each L_i is n-bounded for $1 \leq i \leq k$.

Lemma 5. *$N \sim R$ iff the following two conditions hold:*

(1) *$N \sim_n R$; and*
(2) *$\mathcal{R}(N) \cap L^{\geq n} = \emptyset$ for every n-bounded marking $L \in Inc(R)$.*

Proof: ($\Rightarrow$): (1) follows immediately from the definitions. For (2), let L be an n-bounded marking such that $L \in Inc(R)$. Then $L^{\geq n} \subseteq Inc(R)$. Since $N \sim R$,

[4] Ernst Mayr has observed that this problem is probably even decidable.

every marking of $\mathcal{R}(N)$ is bisimilar to some state of R, and so $\mathcal{R}(N) \cap Inc(R) = \emptyset$. Therefore $\mathcal{R}(N) \cap L^{\geq n} = \emptyset$.

($\Leftarrow$): Define $\mathcal{B} = \{(M, r) \mid M \in \mathcal{R}(N) \text{ and } M \sim_n r\}$. We show that $\mathcal{B}$ is a strong bisimulation containing (M_0, r_0). $(M_0, r_0) \in \mathcal{B}$ follows from (1). Let $(M, r) \in \mathcal{B}$ and assume $M \xrightarrow{a} M'$ for some action a. Then there exists a state r' such that $r \xrightarrow{a} r'$ and $M' \sim_{n-1} r'$. We have $M' \in \mathcal{R}(N)$ and $M' \in L^{\geq n}$ for some n-bounded marking L. It follows from (2) that $L \notin Inc(R)$, and hence R contains a state r'' such that $M' \sim_n r''$. So we get $r' \sim_{n-1} r''$. By our observation at the beginning of the section, $r' \sim_n r''$, and hence $M' \sim_n r'$. This implies $(M', r') \in \mathcal{B}$, and so $\mathcal{B}$ is a strong bisimulation. ■ 5

Theorem 6. *It is decidable whether $N \sim R$ or not.*

Proof: It suffices to prove that (1) and (2) in Lemma 5 are decidable. (1) is clearly decidable. Since there exist finitely many n-bounded markings, the set of n-bounded markings $L \in Inc(R)$ is effectively constructible. The decidability of the reachability problem for submarkings [3] easily implies the decidability of the problem $\mathcal{R}(N) \cap L^{\geq n} = \emptyset$ for a given n-bounded marking L. So (2) is decidable as well. ■ 6

5 Decidability of strong finiteness

In this section we prove that the strong finiteness problem is decidable. Again, the proof is carried out for ordinary nets, but it can be easily generalised.

Let N be a Petri net, and let M be a marking of N. M is infinite with respect to strong bisimilarity (*b-infinite* for short) if there exist infinitely many markings $M_1, M_2, M_3, \ldots$ reachable from M such that $M_i \not\sim M_j$ for $i \neq j$. The net N is *b-infinite* if its initial marking is b-infinite. Since the strong equivalence problem is decidable, the strong finiteness problem is semidecidable. Therefore, it suffices to show that b-infiniteness is semidecidable.

We fix an ordinary Petri net $N = (P, T, W, M_0, \ell)$ for the rest of the section, and introduce a little notation. Let $M_1: P_1 \to \mathbb{N}$ and $M_2: P_2 \to \mathbb{N}$ be two mappings, where P_1, P_2 is a partition of P such that $P_2 \neq \emptyset$. (M_1, M_2) denotes the marking of N whose projection on P_1 is M_1, and whose projection on P_2 is M_2. We say 'a marking (M_1, M_2) of N' instead of 'a partition P_1, $P_2 \neq \emptyset$ of P and mappings $M_1: P_1 \to \mathbb{N}$, $M_2: P_2 \to \mathbb{N}$'. $(M_1, -)$ denotes the marking M_1 of the net obtained from N by removing all places that do not belong to P_1, together with their incident arcs. We make abuse of language, and speak of 'the marking $(M_1, -)$ of N'. We also say 'a marking $(M_1, -)$ of N' instead of 'a partition P_1, $P_2 \neq \emptyset$ of P and a mapping $M_1: P_1 \to \mathbb{N}$'.

An argument similar to that of Proposition 1 proves that $M' \geq (i, \ldots, i)$ for some number i implies $(M, M') \sim_i (M, -)$.

Lemma 7. *If $(M, M_1) \sim (M, M_2) \sim (M, M_3) \sim \ldots$ and $M_1 < M_2 < M_3 \ldots$ ($<$ defined componentwise), then $(M, M_1) \sim (M, -)$.*

Proof: For every $i \geq 0$ there surely exists an index j such that $M_j \geq (i, i, \ldots, i)$. Then $(M, -) \sim_i (M, M_j)$ holds and, since $(M, M_1) \sim (M, M_j)$, we also have $(M, -) \sim_i (M, M_1)$. Therefore, $(M, -) \sim_i (M, M_1)$ for every $i \geq 0$, and so $(M, -) \sim (M, M_1)$. ■ 7

Lemma 8. *N is b-infinite iff there exists a marking $(M, -)$ of N that satisfies either*

(1) *$(M, -)$ is b-infinite and there exists a chain $M_1 < M_2 < M_3 \ldots$ such that $(M, M_i) \in \mathcal{R}(N)$ for every $i \geq 1$, or*
(2) *$(M, -)$ is b-finite and there exists a chain $M_1 < M_2 < M_3 \ldots$ such that $(M, M_i) \in \mathcal{R}(N)$ and $(M, M_i) \not\sim (M, -)$ for every $i \geq 1$.*

Proof: ($\Rightarrow$): If N is b-infinite, then there exists an infinite set of pairwise non-bisimilar reachable markings. By Dickson's Lemma, we can extract from this set an infinite subsequence $(M, M_1), (M, M_2), (M, M_3), \ldots$ such that $M_1 < M_2 < M_3 < \ldots$. So either (1) or (2) holds, according to whether $(M, -)$ is b-finite or b-infinite.

($\Leftarrow$): Define $\mathcal{M} = \{(M, M_i) \mid i \geq 1\}$. Assume (1) holds. If $\mathcal{M}$ contains infinitely many pairwise non-bisimilar markings, then N is b-infinite, and we are done. Otherwise, $\mathcal{M}$ contains infinitely many pairwise bisimilar markings. By Lemma 7, all these markings are bisimilar to $(M, -)$ and hence b-infinite. So N is b-infinite. Assume now that (2) holds. If $\mathcal{M}$ contains infinitely many pairwise bisimilar markings, then by Lemma 7 all of them are bisimilar to $(M, -)$, contradicting the condition $(M, M_i) \not\sim (M, -)$ for every $i \geq 0$. So $\mathcal{M}$ contains infinitely many pairwise non-bisimilar markings, which implies that N is b-infinite.

■ 8

Theorem 9. *It is decidable whether N is b-finite or not.*

Proof: We proceed by induction on the number of places of N. If N has no places, then it is clearly b-finite. Assume now that N has some places. By Theorem 6, the b-finiteness of N is semidecidable. So it suffices to prove that the b-infiniteness of N is also semidecidable, or, equivalently, that conditions (1) and (2) of Lemma 8 are semidecidable.

For that, we enumerate all markings $(M, -)$ of N for all partitions P_1, P_2 such that $P_2 \neq \emptyset$. Given a marking $(M, -)$, we can decide by induction hypothesis if it is b-finite or b-infinite; moreover:

(1) The existence of a chain $M_1 < M_2 < M_3 \ldots$ such that $(M, M_i) \in \mathcal{R}(N)$ for every $i \geq 1$ is semidecidable.

Let $\mathcal{M}_1$ be the set of all markings of the form (M, M'); it is obviously a linear set where exactly the places of P_2 are unbounded. Now the desired semidecidability follows from Lemma 3 (putting e.g. $\mathcal{M}_2 = \mathcal{M}_1$).

(2) If $(M, -)$ is b-finite, then the existence of a chain $M_1 < M_2 < M_3 \ldots$ such that $(M, M_i) \in \mathcal{R}(N)$ and $(M, M_i) \not\sim (M, -)$ for every $i \geq 1$ is semidecidable.

Assume that $(M, -)$ is b-finite. Then by exhaustive search and Theorem 6 a finite transition system R can be found such that $(M, -) \sim R$. Let n denote the number of states of R.

Say a chain is *adequate* if it satisfies the conditions of (2). We prove that there exists an adequate chain iff there exists an n-bounded marking L of N satisfying the following two conditions

(a) $L \in Inc(R)$; and
(b) there exists a chain $M_1 < M_2 < M_3 \ldots$ and markings $M'_1, M'_2, M'_3, \ldots \in L^{\geq n}$ such that $M_0 \longrightarrow (M, M_i) \longrightarrow M'_i$ for every $i \geq 1$.

($\Rightarrow$): Let $M_1 < M_2 < M_3 \ldots$ be an adequate chain. There exists an index i_0 such that $M_i \geq (n, n, \ldots, n)$ for every $i \geq i_0$. For $i \geq i_0$ we have $(M, M_i) \not\sim (M, -)$ by assumption (and so $(M, M_i) \not\sim R$), but $(M, M_i) \sim_n (M, -)$ (and so $(M, M_i) \sim_n R$). By Lemma 5 there exists an n-bounded marking $L_i \in Inc(R)$ such that $\mathcal{R}((M, M_i)) \cap L_i^{\geq n} \neq \emptyset$.
By the pigeonhole principle there exists an n-bounded marking L and infinitely many indices $i_1 < i_2 < i_3 \ldots$ such that $L = L_{i_1} = L_{i_2} \ldots$. Clearly, L satisfies (a) and the subchain $M_{i_1} < M_{i_2} < M_{i_3} \ldots$ satisfies (b).

($\Leftarrow$): Let M_i be an arbitrary marking of the chain given by (b). We prove $(M, M_i) \not\sim (M, -)$, which shows that the chain is adequate. Since $M_0 \longrightarrow (M, M_i) \longrightarrow M'_i$ for some marking $M'_i \in L^{\geq n}$, we have $\mathcal{R}((M, M_i)) \cap L^{\geq n} \neq \emptyset$. By (a) and Lemma 5 we have $(M, M_i) \not\sim R$, which together with $(M, -) \sim R$ implies $(M, M_i) \not\sim (M, -)$.

It remains to prove the semidecidability of conditions (a) and (b) for a given n-bounded marking L. Condition (a) is clearly decidable. For condition (b), apply Lemma 3 with $\mathcal{M}_1$ as the set of markings of N of the form (M, M'), and $\mathcal{M}_2 = L^{\geq n}$. Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are linear sets. ■ 9

6 Undecidability of weak equivalence

We show that the weak equivalence problem is undecidable. In fact, we prove that neither of the problems $N \approx R$ and $N \not\approx R$ is semidecidable; as a corollary of this result, we also find a fixed transition system R_{fix} with 7 states such that $N \approx R_{fix}$ is undecidable. The proof is by reduction from the

Containment problem: given two Petri nets N_1, N_2 having the same number of places, and a bijection f from the places of N_1 onto the places of N_2, is $f(\mathcal{R}(N_1)) \subseteq \mathcal{R}(N_2)$? (where f is extended to markings and sets of markings in the obvious way.)

The undecidability of this problem was proved by Rabin by means of a reduction from Hilbert's 10th problem. A reduction from the halting problem for counter machines can be found in [8].

Let $N_1 = (P_1, T_1, W_1, M_{10}, \ell_1)$, $N_2 = (P_2, T_2, W_2, M_{20}, \ell_2)$ be two Petri nets, and let $f: P_1 \to P_2$ be a bijection. We construct another net N which is weakly bisimilar to the state r_1 of the finite transition system R shown in Figure 1 if $f(\mathcal{R}(N_1)) \not\subseteq \mathcal{R}(N_2)$ and weakly bisimilar to the state r_5 if $f(\mathcal{R}(N_1)) \subseteq \mathcal{R}(N_2)$.

(The state r_0 of R is used in the next section.) Without loss of generality, we assume that (a) the sets of places and transitions of N_1 and N_2 are disjoint, (b) $f(M_{10}) = M_{20}$, (c) $|\mathcal{R}(N_2)| \geq 2$, and (d) $(0,\ldots,0) \notin \mathcal{R}(N_1) \cup \mathcal{R}(N_2)$.

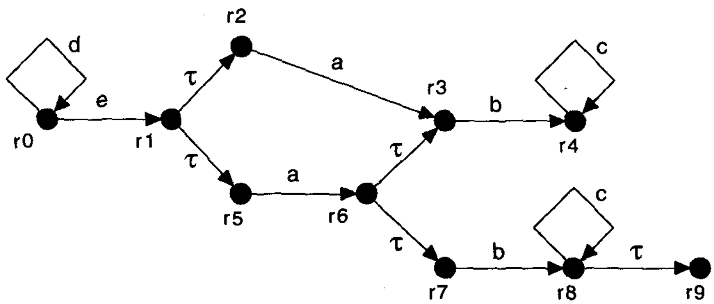

Fig. 1. The transition system R

Instead of giving a formal definition of N, which would be tedious, we describe it informally at a level of detail which suffices to follow our arguments. We need the following notion: a place p is a *run place* of a set T of transitions if $W(p,t) = 1 = W(t,p)$ for every $t \in T$. The transitions of T can only occur when p holds at least one token.

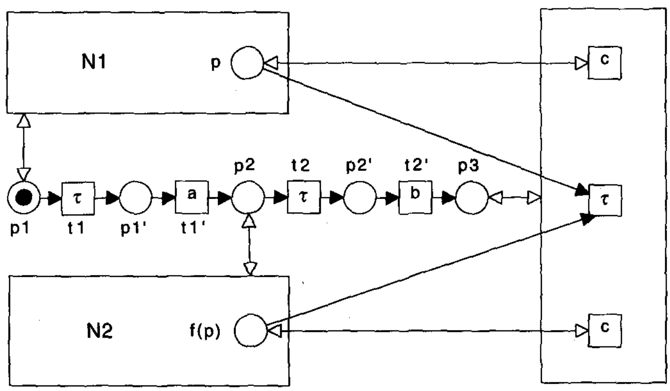

Fig. 2. Scheme net N

Figure 2 shows a schema of the net N. To construct N, we first take the union of N_1 and N_2 (the sets of places and transitions of N_1 and N_2 are disjoint by assumption (a)), labelling all transitions by τ. Then we add some further places and transitions: the place p_1 is a run-place of T_1 (graphically represented by a double pointed white arrow), and contains initially one token. This token can be moved by a new τ-transition t_1 to a place p_1', and then by an a-transition t_1' to p_2, which is a run-place of T_2. From p_2, the token can be moved by another τ-transition t_2 to p_2' and by a b-transition t_2' to p_3, which is a run-place of an additional set of transitions. This set contains:

- a τ-transition for every pair $(p, f(p))$ of places of N_1 and N_2. The transition has p and $f(p)$ as input places, and no output place – when it occurs, it simultaneaously decreases the marking of p and $f(p)$;
- a c-transition for each place p of N_1 and N_2; the transition has p as unique input and output place.

We denote a marking of N as a vector with three components: the first and the third components are the projections of the marking on N_1 and N_2, respectively, while the second indicates which place of the set $\{p_1, p_1', p_2, p_2', p_3\}$ currently holds a token. The initial marking of N is (M_{10}, p_1, M_{20}).

From the initial marking, N can execute some τ-moves corresponding to transitions of N_1. If at some moment t_1 occurs (this is also a τ-move), then a marking (M_1, p_1', M_{20}) is reached, and the marking M_1 becomes 'frozen'. Then, after an a-move, N can execute some τ-moves corresponding to transitions of N_2. Again, if at some moment t_2 occurs, then a marking (M_1, p_2', M_2) is reached, and the marking M_2 becomes 'frozen'.

The following Proposition is easy to prove making use of the asumptions (a) to (d):

Proposition 10. *If $f(\mathcal{R}(N_1)) \subseteq \mathcal{R}(N_2)$, then $N \approx r_5$. If $f(\mathcal{R}(N_1)) \not\subseteq \mathcal{R}(N_2)$, then $N \approx r_1$.* ∎ 10

We now have:

Theorem 11. *Neither the weak equivalence problem nor the weak non-equivalence problem are semidecidable.*

Proof: Follows from $r_1 \not\approx r_5$ and Proposition 10. ∎ 11

It follows from $r_1 \not\approx r_5$ that the problem $N \approx r_5$ is undecidable. The transition system R_{fix} announced at the beginning of the section is obtained by removing r_0, r_1, r_2 from R, together with their adjacent arcs.

7 Undecidability of the weak finiteness problem

We show the undecidability of the weak finiteness problem by means of a reduction from the

> **Halting problem for 2-counter machines and 0-input:** does a given 2-counter machine C halt when both counters are initialised to 0?

whose undecidability is well known. Let C be an arbitrary 2-counter machine. We construct another 2-counter machine C' which for input $(x, 0)$ computes as follows: first, it checks if $x = 2^k$ for some $k \geq 0$; if this is the case, then it sets the counters to 0 and simulates C, otherwise it halts. We have:

- if C halts for $(0,0)$, then C' halts for every input $(x,0)$, $x \geq 0$;
- if C does not halt for $(0,0)$ then C' halts for $(x,0)$ iff x is not a power of 2.

We now use a procedure described in [8]: it accepts the 2-counter machine C' and outputs two nets N_1, N_2 with distinguished places c_{11}, c_{12} and c_{21}, c_{22} (initially unmarked), together with a bijection f between the places of N_1 and N_2 satisfying $f(c_{11}) = c_{21}$ and $f(c_{12}) = c_{22}$. N_1 and N_2 satisfy the following property for every $x \geq 0$:

$$C' \text{ halts for input } (x,0) \quad \text{iff} \quad f(\mathcal{R}(N_1^{x,0})) \not\subseteq \mathcal{R}(N_2^{x,0})$$

where $N_i^{x,0}$ denotes the result of changing the initial marking of N_i by putting exactly x tokens on c_{i1}.

We apply the construction of the Section 6 to the nets N_1 and N_2, and obtain this way a net N. We modify this net in the following way . First, we remove the token from p_1. Second, we add some new places and transitions:

- a place p_0, initially marked with one token;
- a d-transition with p_0 as only input place, and p_0, c_{11} and c_{21} as output places (i.e., p_0 is a run-place for this transition);
- an e-transition, with p_0 as input place and p_1 as output place.

Let N' be the result of this modification. From its initial marking, N' can repeatedly execute the d-transition, by which it puts an arbitrary number of tokens x on the places c_{11} and c_{21}. Then, it may execute the e-transition. After that, the place p_1 carries a token, and N' behaves like the net we would obtain by applying the construction of the last section to $N_1^{x,0}$ and $N_2^{x,0}$.

Proposition 12. *N' is weakly b-finite iff the counter machine C halts for* $(0,0)$.

Proof: ($\Rightarrow$): If C does not halt for $(0,0)$ then C' halts for $(x,0)$ iff x is not a power of 2. Therefore $f(\mathcal{R}(N_1^{x,0})) \subseteq \mathcal{R}(N_2^{x,0})$ if $x = 2^k$ for some $k \geq 0$, and $f(\mathcal{R}(N_1^{x,0})) \not\subseteq \mathcal{R}(N_2^{x,0})$ otherwise. Let R be the transition system of Figure 1 with states $r_0, \ldots, r_9$. We have for any x and the unique marking M reached after executing the sequence d^x in N':

(1) if x is not a power of 2 then there is M' s.t. $M \stackrel{e}{\Longrightarrow} M'$ and $M' \approx r_1$;
(2) if x is a power of 2 then there is no such M'.

We prove that N' is weakly b-infinite by contradiction. Assume that N' is weakly bisimilar to some transition system R' with n states. Let r'_0 be the initial state of R', and let r' be a state such that $r'_0 \xrightarrow{u} r'$, where u is a sequence of actions whose projection on the set of visible actions is d^{2^n}. By the pumping lemma, $r'_0 \xrightarrow{vw^ix} r'$ for sequences v, w, x and for every $i \geq 0$, where $u = vwx$ and the projection of w on the set of visible actions is a nonempty sequence of d's. By (1) and (2) we have that there is r'' in R' s.t. $r' \stackrel{e}{\Longrightarrow} r''$ and $r'' \approx r_1$, and at the same time there is no such r'' – a contradiction. ($\Leftarrow$): If C halts for $(0,0)$, then C' halts for every input $(x,0)$, $x \geq 0$. Therefore after the occurrence of the e-transition we always have $f(\mathcal{R}(N_1^{x,0})) \not\subseteq \mathcal{R}(N_2^{x,0})$, independently of the value of x. It is then clear that $N' \approx r_0$. So N' is weakly b-finite. ■ 12

Theorem 13. *Neither the weak finiteness problem nor the weak infiniteness problems are semidecidable.*

Proof: By Proposition 12, C does not halt for $(0,0)$ iff N' is weakly b-infinite. So the weak infiniteness problem is not semidecidable. We can also change C' in the following way: if x is not a power of 2, then C' enters an infinite loop. In this case, C does not halt for $(0,0)$ iff the net N' is weakly b-finite. So the weak finiteness problem is not semidecidable either. ∎ 13

8 Conclusions

We have shown that the finiteness problem is decidable for Petri nets, while the weak equivalence and weak finiteness problems are undecidable. The finiteness problem for Basic Process Algebra has been recently studied in [10]. The results are similar to ours: undecidable for trace equivalence, but decidable for bisimulation equivalence. Finally, since BPPs are bisimilar to a particular class of Petri nets (called communication-free nets in [4]), the decidability of the finiteness problem for BPP and bisimulation follows as a corollary of our results.

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