A Fixed Point Theorem on Lexicographic Lattice Structures

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Abstract

We introduce the notion of a *lexicographic lattice structure*, namely a lattice whose elements can be viewed as *stratified entities* and whose ordering relation compares elements in a lexicographic manner with respect to their strata. These lattices arise naturally in many non-monotonic formalisms, such as normal logic programs, higher-order logic programs with negation, and boolean grammars. We consider functions over such lattices that may overall be non-monotonic, but retain a restricted form of monotonicity inside each stratum. We demonstrate that such functions always have a least fixed point which is also their least pre-fixed point. Moreover, we prove that the sets of pre-fixed and post-fixed points of such functions, are complete lattices. For the special case of a trivial lexicographic lattice structure whose elements essentially consist of a unique stratum, our theorem gives as a special case the well-known Knaster-Tarski fixed point theorem. Moreover, our work considerably simplifies and extends recent results on non-monotonic fixed point theory, providing in this way a useful and convenient tool in the semantic investigation of non-monotonic formalisms.

CCS Concepts: • **Theory of computation** → *Logic*; *Denotational semantics.*

Keywords: lexicographic ordering, lattices, fixpoint

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1 Introduction

The use of fixed point theorems has a long tradition in theoretical computer science. In particular, fixed points of monotonic functions defined over partially ordered sets, have been the cornerstone tool for obtaining the denotational semantics of programming languages that support recursion [1, 15, 19, 21]. A well-known such result is the Knaster-Tarski fixed point theorem [20], which applies to monotonic functions defined over complete lattices. The semantics of classical (ie., negation-less) logic programs [15, 22], heavily relies on this theorem: the set of Herbrand interpretations of a logic program is a complete lattice and the semantics of the program is the least fixed point of a monotonic function, called the *immediate consequence operator* of the program. However, things change radically when we extend logic programs with negation in rule bodies. The immediate consequence operator becomes non-monotonic and in many cases it does not even have a fixed point. In such cases, the Knaster-Tarski theorem is no longer applicable. The pioneering papers on the foundations of logic programs with negation, such as [13, 17], used clever but somewhat ad-hoc techniques to define the fixed point semantics of the programs. This state of affairs made apparent the need to develop a solid *fixed point theory for non-monotonic functions* on which such developments could be based. The purpose of the present paper is to contribute towards this research direction.

The starting point for the development of a fixed point theory for non-monotonic functions, was the work of Melvin Fitting [12], who used the abstract framework of lattices and operators on lattices in order to characterize all major semantic approaches for logic programming. Building on Fitting's framework, Denecker, Marek and Truszczyński developed *approximation fixed point theory* [7, 8] which uses bilattices in order to compute the fixed points of functions that exhibit non-monotonicity. In a different direction, Ésik and Rondogiannis introduced an axiomatic framework [10, 11] that defines lattices on which a class of functions that are generally non-monotonic, can be shown to have least fixed points. The work in the present paper follows the research direction initiated in [10, 11]. However, the theoretical framework that we develop is much simpler than the one proposed in [10, 11] and the proofs of all theorems are much more direct. In the rest of this section, we give an intuitive outline of our approach. A detailed comparison of our results with those of [7, 8, 10–12] is given in Section 7.

We start by introducing the notion of a *lexicographic lattice structure*, which is a complete lattice $(L, \sqsubseteq)$ together with a sequence of preordering relations. The elements of L, when viewed through the preordering relations, can be regarded as *entities that consist of multiple strata*. The ordering relation $\sqsubseteq$ compares the elements of L in a lexicographic manner with respect to their strata (starting the comparison from the lower strata and proceeding towards the higher ones). It turns out that such lattice structures arise naturally in many non-monotonic formalisms, such as normal logic programs [15], higher-order logic programs with negation [4], boolean grammars [14, 16], and so on. The reason that these lattice structures appear to be appropriate for the study of non-monotonic systems, is because many of these systems are also structured in a stratified manner. For example, it is well known that logic programs with negation admit a natural stratification, called *dynamic stratification* in [17]. Moreover, it is also known that the well-founded model of any such program, is the most "preferred" one with respect to a lexicographic ordering [18].

We consider functions of the form $f : L \rightarrow L$ that are, in general, non-monotonic with respect to $\sqsubseteq$ but retain a restricted form of monotonicity with respect to the preordering relations that stratify the lattice L. We call such functions *stratified monotonic*. We demonstrate that a stratified monotonic function always has a least fixed point which is also its least pre-fixed point. Moreover, we prove that the sets of pre-fixed and post-fixed points of such functions, are complete lattices. For the special case of a trivial lexicographic lattice structure whose elements essentially consist of a unique stratum, our theorem gives as a special case the well-known Knaster-Tarski fixed point theorem. Therefore, the main contributions of the present work can be outlined as follows:

- We propose the notion of a *lexicographic lattice structure* as a tool for modeling different non-monotonic formalisms. The characterization of a lexicographic lattice structure relies on three simple properties, giving in this way a more compact approach than the one introduced in [11]. We then derive a novel fixed point theorem for stratified monotonic functions over such structures. The proof of the proposed theorem is much more direct than the central theorem of [11]. Our theorem has been verified through the Coq proof assistant.
- We demonstrate that the sets of pre-fixed and post-fixed points of stratified monotonic functions are complete lattices. This result together with our main theorem, show a striking analogy with the Knaster-Tarski fixed point theorem. Actually, we demonstrate that the Knaster-Tarski theorem follows as a simple special case of our developments. We present in detail two applications of the proposed theory in systems that involve non-monotonicity: the former is in the semantics of logic programs with negation and the latter is in constructing the fixed points of non-monotonic functions over transfinite sequences.

The rest of the paper is organized as follows. Section 2 introduces lexicographic lattice structures. Section 3 demonstrates several mathematical properties of these structures. Section 4 establishes the main fixed point theorem of the paper. Section 5 establishes properties of the pre-fixed and post-fixed points of functions over lexicographic lattice structures. Section 6 presents two applications of the theoretical results obtained in the paper. Finally, Section 7 discusses related and future work.

2 Lexicographic Lattice Structures

Consider a complete lattice $(L, \sqsubseteq)$, whose least element will be denoted by $\perp$ and the lub and glb operations by $\bigsqcup$ and $\bigsqcap$ respectively. In order to define the notion of a *lexicographic lattice structure*, we assume that $\sqsubseteq$ can be "constructed" using a sequence of preorderings $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$, where $\kappa > 0$ is an ordinal. Actually, as we are going to see, *every* complete lattice $(L, \sqsubseteq)$ can be "constructed" in a trivial way using such preorderings; using this trivial construction, we will be able to obtain as a special case of our theorem the well-known Knaster-Tarski fixed point theorem. Of course, we will be mostly interested in the case where $\sqsubseteq$ is "constructed" in a non-trivial way from the preorderings; in this case our fixed point theorem will be applicable to a much broader class of functions, namely functions that are potentially non-monotonic.

Before giving any formal definitions, we present the intuition behind the above notions, using a well-known example. Let us take L to be the set of ω-words (ie., $\kappa = \omega$ in our example) over a finite alphabet Σ. We assume that the elements of Σ are alphabetically ordered. Let us take the $\sqsubseteq$ relation to be the lexicographic comparison of ω-words. One can easily verify that the set of ω-words under the lexicographic ordering, is a complete lattice. Consider now for each $\alpha < \omega$, the preordering $\sqsubseteq_\alpha$ to be the relation that compares two ω-words up to their α-th elements: given two ω-words x and y, we write $x \sqsubset_\alpha y$ iff x and y are identical at all positions less than α and the sequence x contains at position α a character of Σ that is alphabetically "smaller" than the corresponding character of y at the same position. Notice now that the lexicographic ordering $\sqsubseteq$ can be constructed using the relations $\sqsubseteq_\alpha$: given two ω-words x and y, it holds $x \sqsubset y$ iff there exists some α such that $x \sqsubset_\alpha y$. In the following, we formalize the above ideas. We start with some simple notational conventions:

Definition 2.1. Let $(L, \sqsubseteq)$ be a complete lattice. Let $\kappa > 0$ be an ordinal and let $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ be a sequence of preorderings over L. For all $x, y \in L$, we write $x \sqsubset y$ if $x \sqsubseteq y$ and $x \neq y$. For every $\alpha < \kappa$ and $x, y \in L$, we write $x =_\alpha y$ iff $x \sqsubseteq_\alpha y$ and $y \sqsubseteq_\alpha x$. We write $x \sqsubset_\alpha y$ iff $x \sqsubseteq_\alpha y$ but $x =_\alpha y$ does not hold. Given $X \subseteq L$ we will write $X \sqsubseteq_\alpha y$ (respectively, $X =_\alpha y$) iff for every $x \in X, x \sqsubseteq_\alpha y$ (respectively, $x =_\alpha y$). For all $\alpha \leq \kappa$, we define $(x]_\alpha = \{y \in L : \forall \beta < \alpha \; x =_\beta y\}$; moreover, for all $\alpha < \kappa$ we define $[x]_\alpha = \{y \in L : x =_\alpha y\}$.

In our setting, we will insist that the partial order $\sqsubseteq$ and the preorderings $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ are closely related in the sense that the latter relations determine the former one:

Definition 2.2. Let $(L, \sqsubseteq)$ be a complete lattice. Let $\kappa > 0$ be an ordinal and let $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ be a set of preorderings over L. We will say that the relation $\sqsubseteq$ is *determined* by the preorderings $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ if for all $x, y \in L$, $x \sqsubset y$ iff $x \sqsubset_\alpha y$ for some $\alpha < \kappa$.

We can now define lexicographic lattice structures:

Definition 2.3. Let $(L, \sqsubseteq)$ be a complete lattice. Let $\kappa > 0$ be an ordinal, let $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ be a set of preorderings over L, and assume that $\sqsubseteq$ is determined by these preorderings. The triple $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa}\rangle$ will be called a *lexicographic lattice structure* if the following three properties hold:

Property 1. For every $\alpha < \kappa$ and for all $x, y \in L$, if $x \sqsubseteq_\alpha y$, then $x =_\beta y$, for all $\beta < \alpha$.

Property 2. For all $x, y \in L$, if $x =_\alpha y$ for all $\alpha < \kappa$, then $x = y$.

Property 3. For every $x \in L$ and for every ordinal $\alpha < \kappa$, $[x]_\alpha$ has a $\sqsubseteq$-least and a $\sqsubseteq$-greatest element.

Let $\alpha < \kappa$ be an ordinal and $X \subseteq (x]_\alpha$ for some $x \in L$. We denote the $\sqsubseteq$-least element of $[\bigsqcup X]_\alpha$ as $\bigsqcup_\alpha X$ and the $\sqsubseteq$-greatest element of $[\bigsqcap X]_\alpha$ as $\bigsqcap_\alpha X$. Obviously, $\bigsqcup X =_\alpha \bigsqcup_\alpha X$ and $\bigsqcap X =_\alpha \bigsqcap_\alpha X$.

The intuition behind the above definition can be outlined as follows. First of all, one can think of the elements of L as entities consisting of κ levels. More generally, Property 1 states that each successive preordering relation provides a more accurate comparison of the elements of L. Property 2 states that if two elements of L are indistinguishable with respect to all our preordering relations, then the two elements must coincide. Finally, Property 3 states that if we consider the set of elements of L that have the same "prefix" until their stratum α, then this set has a least and a greatest element. Notice that in Property 3 we require the existence of *both* a least *and* a greatest element. Actually, it can be shown (we omit the details) that both assumptions are needed in order to derive the fixed point theorem of the paper.

One can verify that there exist several natural application domains in which lexicographic lattice structures can be used. One of the most natural ones, is the set of ω-words discussed earlier in this section. We can now state this in a more formal way. Given a finite alphabet Σ whose elements are ordered by a relation $<$, consider the triple $\langle \Sigma^\omega, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\omega}\rangle$, where, for all $x, y \in \Sigma^\omega$ and for all $\alpha < \omega$:

$$x \sqsubset_\alpha y \text{ iff } [\forall \beta < \alpha (x(\beta) = y(\beta)) \wedge (x(\alpha) < y(\alpha))]$$

and

$$x \sqsubset y \text{ iff } \exists \alpha < \omega [\forall \beta < \alpha (x(\beta) = y(\beta)) \wedge (x(\alpha) < y(\alpha))]$$

It is not hard to check that the requirements of Definition 2.3 are all satisfied. In particular, Property 3 holds because for every ω-sequence x and every $\alpha < \omega$, the $\sqsubseteq$-least (respectively, $\sqsubseteq$-greatest) element of $[x]_\alpha$ is the sequence that is identical to x at all indices $\beta \leq \alpha$ and at all indices that are greater than α it has a constant value that coincides with the alphabetically least (respectively, greatest) element of Σ.

More generally, the application domains in which lexicographic lattice structures appear to be applicable, are sets that have a natural stratification and are accompanied by a natural lexicographic ordering. Indicatively, we mention the set of infinite-valued interpretations of logic programs with negation [18], the set of interpretations of higher-order logic programs with negation [4], the set of interpretations of boolean grammars [14, 16], the set of transfinite sequences over complete lattices [5], and so on. Two of these applications will be presented in detail in Section 6.

Remark 1. For *every* complete lattice $(L, \sqsubseteq)$ we can create a trivial lexicographic lattice structure: simply take $\kappa = 1$ and $\sqsubseteq_0$ to be equal to $\sqsubseteq$. Obviously, $\sqsubseteq_0$ determines $\sqsubseteq$ (because they coincide). Moreover, the triple $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<1}\rangle = \langle L, \sqsubseteq, \{\sqsubseteq\}\rangle$ satisfies the three properties of Definition 2.3.

Remark 2. In our proofs we can safely assume that the ordinal κ in the definition of lexicographic lattice structures, is always a *limit* one. The formal justification for this remark is identical to the one given in [11][page 23]. Intuitively, given a lexicographic lattice structure $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa}\rangle$, where κ is a successor ordinal, we can create a structure $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\lambda}\rangle$, where λ is the least limit ordinal that is greater than κ, and the relations $\sqsubseteq_\beta$, for $\kappa \leq \beta < \lambda$, are all equal to the identity relation over L. It can be easily seen that the new structure is a lexicographic lattice one (ie., it satisfies the properties of Definition 2.3) and it can be used interchangeably with the initial structure for the purposes of the paper.

Remark 3. Let $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa}\rangle$ be a lexicographic lattice structure. For all $x, y \in L$ and $\alpha < \kappa$, let us define $x \sqsupseteq y$ iff $y \sqsubseteq x$ and $x \sqsupseteq_\alpha y$ iff $y \sqsubseteq_\alpha x$. The *dual of a proposition* is obtained by replacing each occurrence of $\sqsubseteq$ by $\sqsupseteq$, each occurrence of $\sqsubseteq_\alpha$ by $\sqsupseteq_\alpha$, each occurrence of $\bigsqcup$ by $\bigsqcap$, each occurrence of $\bigsqcup_\alpha$ by $\bigsqcap_\alpha$, and finally, for any relation $\circ$ each occurrence of $\circ$-greatest by $\circ$-least and each occurrence of $\circ$-least by $\circ$-greatest. It is clear that all of the properties in Definition 2.3 imply the dual of themselves. For each proposition P that can be proved using these properties, the dual of P can be established by the dual of the proof of P. Therefore, our theory is closed under dual operation.

In the rest of the paper, we will assume that we have a fixed lexicographic lattice structure $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa}\rangle$. This will allow us to use κ, L, $\sqsubseteq$, and the preorderings $\sqsubseteq_\alpha$ freely in lemmas, definitions, and so on (avoiding in this way to repeat statements such as "Let $\kappa > 0$ be an ordinal, let L be a complete lattice, ...", and so on).

3 Some Consequences of the Properties

In this section we establish some properties of lexicographic lattice structures that are implied by the definitions of the previous section. These properties, apart from providing a better understanding of lexicographic lattice structures, will all be used in the proofs of the more demanding results of the subsequent sections.

Lemma 3.1. *Let $\alpha < \kappa$ and let $x, y, z \in L$ such that $x \sqsubset_\alpha y$ and $y =_\alpha z$. Then, $x \sqsubset_\alpha z$.*

Proof. Since $x \sqsubset_\alpha y$, we also have $x \sqsubseteq_\alpha y$. By the transitivity of $\sqsubseteq_\alpha$, $x \sqsubseteq_\alpha z$. Suppose, for the sake of contradiction, that $z \sqsubseteq_\alpha x$. By the transitivity of $\sqsubseteq_\alpha$, $y \sqsubseteq_\alpha x$ (contradiction). Thus, $x \sqsubset_\alpha z$. □

Lemma 3.2. *For each $x \in L$, for every ordinal $\alpha < \kappa$ and for any non-empty $X \subseteq (x]_\alpha$, $\bigsqcup X \in (x]_\alpha$.*

Proof. Let x' be an element in X. We are going to prove that $\bigsqcup X =_\beta x'$ for all $\beta < \alpha$. We have that $x' \sqsubseteq \bigsqcup X$. If $x' = \bigsqcup X$ or $x' \sqsubset_\beta \bigsqcup X$ for some $\beta \geq \alpha$ we are done. Assume, for the sake of contradiction, that $x' \sqsubset_\beta \bigsqcup X$ for some $\beta < \alpha$. Let y be the $\sqsubseteq$-greatest element of $[x']_\beta$. Clearly $X \subseteq [x']_\beta$, so that y is a $\sqsubseteq$-upper bound of X. By Property 3, $x' =_\beta y$. By Lemma 3.1, $y \sqsubset_\beta \bigsqcup X$, so that $y \sqsubset \bigsqcup X$, contradicting the definition of $\bigsqcup X$. □

Lemma 3.3. *Let $\alpha < \kappa$ and suppose $x, y \in L$ such that $x =_\beta y$ for all $\beta < \alpha$. If $x \sqsubseteq y$ then $x \sqsubseteq_\alpha y$.*

Proof. By the definition of $\sqsubseteq$, we have $x = y$ or $x \sqsubset_\beta y$ for some $\beta < \kappa$. If $x = y$, by Property 2 we get $x =_\alpha y$ which implies that $x \sqsubseteq_\alpha y$. If $x \sqsubset_\beta y$ for some β, then it must be the case that $\beta \geq \alpha$, because $x =_\beta y$ for all $\beta < \alpha$. If $\beta = \alpha$, then $x \sqsubset_\alpha y$, so $x \sqsubseteq_\alpha y$ clearly holds. If $\beta > \alpha$, by Property 1 $x =_\alpha y$, so that $x \sqsubseteq_\alpha y$ again. □

Lemma 3.4. *For each $x \in L$, for every ordinal $\alpha < \kappa$ and for any non-empty $X \subseteq (x]_\alpha$, if $X \sqsubseteq_\alpha x$ then $\bigsqcup X \sqsubseteq_\alpha x$.*

Proof. Let x be an arbitrary element of L and suppose non-empty $X \subseteq (x]_\alpha$ such that $X \sqsubseteq_\alpha x$. Let $X_1 = X \cap [x]_\alpha$, $X_2 = X \setminus X_1$ and y be the $\sqsubseteq$-greatest element of $[x]_\alpha$. Clearly $X_2 \sqsubset_\alpha x$. We have that $X_1 \sqsubseteq y$ and $X_2 \sqsubset x \sqsubseteq y$, so that y is a $\sqsubseteq$-upper bound of X. Thus $\bigsqcup X \sqsubseteq y$. By Property 3, $y =_\alpha x$ and by Lemma 3.2, $\bigsqcup X \in (x]_\alpha$. By Lemma 3.3 $\bigsqcup X \sqsubseteq_\alpha y$ and by the transitivity of $\sqsubseteq_\alpha$, $\bigsqcup X \sqsubseteq_\alpha x$. □

Lemma 3.5. *Let $x \in L$ and $\alpha < \kappa$ be an ordinal. Suppose that $X \subseteq (x]_\alpha$ and $z \in (x]_\alpha$ with $X \sqsubseteq z$. Then $\bigsqcup_\alpha X \sqsubseteq_\alpha z$.*

Proof. Since $X \sqsubseteq z$, for each $x \in X$ either $x = z$ or there is some ordinal β with $x \sqsubset_\beta z$. Since $z \in (x]_\alpha$, we must have $\beta \geq \alpha$. In either case, $x \sqsubseteq_\alpha z$, so that $X \sqsubseteq_\alpha z$. By Lemma 3.4, $\bigsqcup X \sqsubseteq_\alpha z$. Thus, by the definition of $\bigsqcup_\alpha X$, we have that $\bigsqcup_\alpha X \sqsubseteq_\alpha z$. □

Lemma 3.6. *For each $x \in L$, for every ordinal $\alpha < \kappa$ and for any non-empty $X \subseteq (x]_\alpha$, $X \sqsubseteq_\alpha \bigsqcup_\alpha X$.*

Proof. By definition $X \sqsubseteq \bigsqcup X$. By Lemma 3.2, $\bigsqcup X \in (x]_\alpha$, thus by Lemma 3.3, $X \sqsubseteq_\alpha \bigsqcup X$. Also $\bigsqcup X =_\alpha \bigsqcup_\alpha X$, so by the transitivity of $\sqsubseteq_\alpha$, $X \sqsubseteq_\alpha \bigsqcup_\alpha X$. □

Lemma 3.7. *For each $x \in L$, for every ordinal $\alpha < \kappa$, for any non-empty $X \subseteq (x]_\alpha$ and for every $z \in (x]_\alpha$, if $X \sqsubseteq_\alpha z$ then $\bigsqcup_\alpha X \sqsubseteq_\alpha z$.*

Proof. Suppose $z \in (x]_\alpha$ such that $X \sqsubseteq_\alpha z$. By Lemma 3.4 we have $\bigsqcup X \sqsubseteq_\alpha z$. Also $\bigsqcup_\alpha X =_\alpha \bigsqcup X$. By transitivity of $\sqsubseteq_\alpha$ we have $\bigsqcup_\alpha X \sqsubseteq_\alpha z$. □

Lemma 3.8. *For each $x \in L$, for every ordinal $\alpha < \kappa$, for any non-empty $X \subseteq (x]_\alpha$ and for every $z \in (x]_\alpha$, if $X \sqsubseteq_\alpha z$ then $\bigsqcup_\alpha X \sqsubseteq z$.*

Proof. By Lemma 3.7, $\bigsqcup_\alpha X \sqsubseteq_\alpha z$. So we have $\bigsqcup_\alpha X \sqsubset_\alpha z$ or $\bigsqcup_\alpha X =_\alpha z$. In the first case $\bigsqcup_\alpha X \sqsubset z$, so that $\bigsqcup_\alpha X \sqsubseteq z$. In the second case $z \in [\bigsqcup X]_\alpha$ and $\bigsqcup_\alpha X$ is the $\sqsubseteq$-least element of $[\bigsqcup X]_\alpha$, so that $\bigsqcup_\alpha X \sqsubseteq z$ again. □

Lemma 3.9. *For each $x \in L$, for every ordinal $\alpha < \kappa$ and for any non-empty $X \subseteq (x]_\alpha$, if for some $y \in L$, $y \sqsubseteq_\alpha X$ then $y \sqsubseteq_\alpha \bigsqcup_\alpha X$.*

Proof. Let $z \in X$. We have $y \sqsubseteq_\alpha z$ and by Lemma 3.6, $z \sqsubseteq_\alpha \bigsqcup_\alpha X$. Thus, by transitivity, $y \sqsubseteq_\alpha \bigsqcup_\alpha X$. □

Lemma 3.10. *Let $x, y \in L$. If $x \sqsubseteq y$ then $x \sqsubseteq_0 y$.*

Proof. It follows from Lemma 3.3 for the case $\alpha = 0$. □

Lemma 3.11. *Suppose $x \in L$. Then $\perp \sqsubseteq_0 x$.*

Proof. It follows from Lemma 3.10 for the case $x = \perp$. □

Lemma 3.12. *For each $x \in L$, for every ordinal $\alpha < \kappa$ and for any non-empty $X \subseteq (x]_\alpha$, $\bigsqcup_\alpha X$ is the $\sqsubseteq$-least element and a $\sqsubseteq_{\alpha+1}$-least element of $[\bigsqcup_\alpha X]_\alpha$.*

Proof. Suppose $z \in [\bigsqcup_\alpha X]_\alpha$. Note that since $\bigsqcup_\alpha X =_\alpha \bigsqcup X$, $[\bigsqcup_\alpha X]_\alpha = [\bigsqcup X]_\alpha$. Thus $z \in [\bigsqcup X]_\alpha$. By definition, $\bigsqcup_\alpha X \sqsubseteq z$. Since by Property 1, $z =_\beta \bigsqcup_\alpha X$ for any $\beta < \alpha + 1$, by Lemma 3.3 we have $\bigsqcup_\alpha X \sqsubseteq_{\alpha+1} z$. □

Lemma 3.13. *Let $\alpha \leq \kappa$ be an ordinal and $(x_\beta)_{\beta<\alpha}$ be a sequence of elements of L such that each x_β is the $\sqsubseteq$-least element of $[x_\beta]_\beta$ and $x_\beta =_\beta x_\gamma$ whenever $\beta < \gamma < \alpha$. If $x = \bigsqcup\{x_\beta : \beta < \alpha\}$, then $x_\beta =_\beta x$ for all $\beta < \alpha$ and x is the $\sqsubseteq$-least element of $(x]_\alpha$; moreover, if $\alpha < \kappa$, then x is also the $\sqsubseteq$-least element of $[x]_\alpha$.*

Proof. When $\alpha = 1$, the result follows easily. Assume $\alpha \geq 2$. Since $x_\beta =_\beta x_\gamma$ whenever $\beta < \gamma < \alpha$, we have $x_\gamma \in [x_\beta]_\beta$. Moreover, since each x_β is the $\sqsubseteq$-least element of $[x_\beta]_\beta$, $x_\beta \sqsubseteq x_\gamma$ whenever $\beta < \gamma < \alpha$.

We now show that for all $\beta < \alpha$, $x_\beta =_\beta x$. Note that since $x_\beta \sqsubseteq x_\gamma$ whenever $\beta < \gamma < \alpha$ we have $\bigsqcup\{x_\beta : \beta < \alpha\} = \bigsqcup\{x_\gamma : \beta < \gamma < \alpha\}$. Also, $\{x_\gamma : \beta < \gamma < \alpha\} \subseteq (x_\beta]_{\beta+1}$ because for any $\delta < \beta + 1$ and for each $y \in \{x_\gamma : \beta < \gamma < \alpha\}$, $y =_\delta x_\delta =_\delta x_\beta$. Thus, by Lemma 3.2, $x \in (x_\beta]_{\beta+1}$, so that $x_\beta =_\beta x$.

Now suppose $y \in (x]_\alpha$. We have $y =_\beta x =_\beta x_\beta$ for all $\beta < \alpha$, hence $x_\beta \sqsubseteq y$ for all $\beta < \alpha$ and $x = \bigsqcup\{x_\beta : \beta < \alpha\} \sqsubseteq y$. To show that x is also the $\sqsubseteq$-least element of $[x]_\alpha$, simply observe that $[x]_\alpha \subseteq (x]_\alpha$. ◻

4 The Fixed Point Theorem

In this section we develop a novel fixed point theorem for functions $f : L \to L$, which are not necessarily monotonic with respect to $\sqsubseteq$ (and therefore the traditional theorems of fixed point theory do not apply to them). Instead, we will require that the functions we consider exhibit a restricted form of monotonicity, which we term *stratified monotonicity*.

Definition 4.1. Let $\alpha < \kappa$ be an ordinal. A function $f : L \to L$ is called α-monotonic if for all $x, y \in L$, if $x \sqsubseteq_\alpha y$ then $f(x) \sqsubseteq_\alpha f(y)$. Moreover, f is called *stratified monotonic* if it is α-monotonic for all $\alpha < \kappa$.

The main theorem we establish in this section (Theorem 4.4) states that if a function is stratified monotonic, then it has a least pre-fixed point which is also its least fixed point. The construction of this pre-fixed point is performed in stages. We start with the bottom element $\perp$ of L and we take successive approximations, namely $\perp, f(\perp), f(f(\perp)), \ldots$. We demonstrate that after a certain point these approximations converge with respect to the relation $=_0$, giving us an element x_0 of L such that $f(x_0) =_0 x_0$. We now start with x_0 and take successive approximations $x_0, f(x_0), f(f(x_0)), \ldots$. These approximations eventually converge with respect to the relation $=_1$, giving us an element $x_1 \in L$ such that $f(x_1) =_1 x_1$. We continue with the next level, and so on. We demonstrate that the element $\bigsqcup\{x_\alpha : \alpha < \kappa\}$ of L is the desired pre-fixed point.

Before we obtain the main theorem of the paper, we need a technical lemma which essentially proves that at each level α we can obtain a fixed point with respect to $=_\alpha$. Our main theorem then uses this lemma for each level of the construction.

Lemma 4.2. *Suppose that $f : L \to L$ is α-monotonic, where $\alpha < \kappa$. If $x \in L$ and $x \sqsubseteq_\alpha f(x)$, then there is some $y \in L$ such that:*

- $x \sqsubseteq_\alpha y =_\alpha f(y)$
- *For all $z \in L$, if $x \sqsubseteq_\alpha z$ and $f(z) \sqsubseteq_\alpha z$, then $y \sqsubseteq_\alpha z$*
- *y is the $\sqsubseteq$-least element and a $\sqsubseteq_{\alpha+1}$-least element of $[y]_\alpha$.*

Proof. Let us define $x_0 = x$, $x_\gamma = f(x_\delta)$ if $\gamma = \delta + 1$ and $x_\gamma = \bigsqcup_\alpha\{x_\delta : \delta < \gamma\}$ if $\gamma > 0$ is a limit ordinal. To demonstrate that the definition for the case of limit ordinals makes sense, we prove by induction on γ that $x \sqsubseteq_\alpha x_\gamma$ for all γ. It is clear when $\gamma = 0$. Suppose that $\gamma > 0$ and our claim holds for all ordinals less than γ. If $\gamma = \delta + 1$, then $x \sqsubseteq_\alpha x_\delta$ by the induction hypothesis, thus $x \sqsubseteq_\alpha f(x) \sqsubseteq_\alpha f(x_\delta) = x_\gamma$ by the assumption that f is α-monotonic. If γ is a limit ordinal, then $x_\gamma = \bigsqcup_\alpha\{x_\delta : \delta < \gamma\}$, and since $x \sqsubseteq_\alpha x_\delta$ for all $\delta < \gamma$, also $x \sqsubseteq_\alpha x_\gamma$ by Lemma 3.9. We establish three claims that will lead us to the proof of the lemma:

Claim 1. For all γ, $x_\gamma \sqsubseteq_\alpha f(x_\gamma)$.

Proof of Claim 1. We prove this claim by induction on γ. When $\gamma = 0$, it holds by assumption. Suppose now that $\gamma = \delta + 1$. By the induction hypothesis, we have that $x_\delta \sqsubseteq_\alpha f(x_\delta) = x_\gamma$, and since f is α-monotonic, we have $x_\gamma = f(x_\delta) \sqsubseteq_\alpha f(x_\gamma)$. Finally, suppose that $\gamma > 0$ is a limit ordinal, then $x_\gamma = \bigsqcup_\alpha\{x_\delta : \delta < \gamma\}$. By Lemma 3.6, we have $x_\delta \sqsubseteq_\alpha x_\gamma$ for all $\delta < \gamma$. By the α-monotonicity of f and the induction hypothesis, we have $x_\delta \sqsubseteq_\alpha f(x_\delta) \sqsubseteq_\alpha f(x_\gamma)$ for all $\delta < \gamma$. By Lemma 3.7, it follows $x_\gamma \sqsubseteq_\alpha f(x_\gamma)$. ◇

Claim 2. For all ordinals $\beta < \gamma$, $x_\beta \sqsubseteq_\alpha x_\gamma$.

Proof of Claim 2. Again, we prove this claim by induction on γ. When $\gamma = 0$, our claim is trivial. Suppose now that $\gamma > 0$. If $\gamma = \delta + 1$, then $\beta \leq \delta$ and $x_\beta \sqsubseteq_\alpha x_\delta \sqsubseteq_\alpha f(x_\delta) = x_\gamma$ by the induction hypothesis and Claim 1. If $\gamma > 0$ is a limit ordinal, then $x_\beta \sqsubseteq_\alpha x_\gamma$ by Lemma 3.6. ◇

Claim 3. Suppose that $x \sqsubseteq_\alpha z$ and $f(z) \sqsubseteq_\alpha z$. Then $x_\gamma \sqsubseteq_\alpha z$ for all γ.

Proof of Claim 3. We proceed by induction on γ. Since $x_0 = x$, our claim is clear for $\gamma = 0$. Suppose that $\gamma > 0$ and our claim holds for all ordinals less than γ. If $\gamma = \delta + 1$, then since $x_\delta \sqsubseteq_\alpha z$ by the induction hypothesis, we have $x_\gamma = f(x_\delta) \sqsubseteq_\alpha f(z) \sqsubseteq_\alpha z$ by assumption and since f is α-monotonic. Thus, $x_\gamma \sqsubseteq_\alpha z$. If γ is a limit ordinal, then $x_\gamma = \bigsqcup_\alpha\{x_\delta : \delta < \gamma\}$. By the induction hypothesis we have $x_\delta \sqsubseteq_\alpha z$ for all $\delta < \gamma$, and therefore, by Lemma 3.7 we get $x_\gamma \sqsubseteq_\alpha z$. ◇

Now, there is an ordinal λ_0 such that $x_\gamma =_\alpha x_\delta$ for all $\gamma, \delta \geq \lambda_0$ (since otherwise the cardinality of the set $\{x_\gamma : \gamma \text{ is an ordinal}\}$ would exceed the cardinality of L). Let λ denote the least limit ordinal with $\lambda \geq \lambda_0$. Let $y = x_\lambda$. By the definition of y, we have that $f(y) =_\alpha y$ and by Claim 2 $x \sqsubseteq_\alpha y$ (since $x = x_0$). Suppose that $z \in L$ with $x \sqsubseteq_\alpha z$ and $f(z) \sqsubseteq_\alpha z$. Then, by Claim 3 above, $x_\gamma \sqsubseteq_\alpha z$ for all γ, thus $y \sqsubseteq_\alpha z$. Finally, since $y = \bigsqcup_\alpha\{x_\gamma : \gamma < \lambda\}$, by Lemma 3.12 we have that y is the $\sqsubseteq$-least element and $\sqsubseteq_{\alpha+1}$-least element of $[y]_\alpha$. ◻

Below for any $x \in L$ and ordinal $\alpha < \kappa$ with $x \sqsubseteq_\alpha f(x)$, we will denote the element y constructed above by $f_\alpha(x)$. We have shown above that when $x \sqsubseteq_\alpha f(x)$, then $f_\alpha(x)$ satisfies the three properties of Lemma 4.2.

Remark 4. According to Lemma 4.2 $f_\alpha(x) =_\alpha f(f_\alpha(x))$ and $f_\alpha(x)$ is a $\sqsubseteq_{\alpha+1}$-least element of $[f_\alpha(x)]_\alpha$. Thus $f_\alpha(x) \sqsubseteq_{\alpha+1} f(f_\alpha(x))$.

Theorem 4.3. *Suppose that $f : L \to L$ is stratified monotonic. Then f has a $\sqsubseteq$-least pre-fixed point which is also its $\sqsubseteq$-least fixed point.*

Proof. Let us define for each ordinal $\alpha < \kappa$, $x_\alpha = f_\alpha(y_\alpha)$, where $y_0 = \perp$, $y_\alpha = x_\beta$ when $\alpha = \beta + 1$ and $y_\alpha = \bigsqcup\{x_\beta : \beta < \alpha\}$ when α is a limit ordinal. We will prove that $\bigsqcup\{x_\alpha : \alpha < \kappa\}$ is the desired least pre-fixed point and least fixed point of f.

We start by proving simultaneously by transfinite induction on α the following auxiliary statements:

- $y_\alpha \sqsubseteq_\alpha f(y_\alpha)$ for all $\alpha < \kappa$.
- $x_\gamma =_\gamma x_\alpha$ whenever $\gamma < \alpha < \kappa$.
- x_α is the $\sqsubseteq$-least element of $[x_\alpha]_\alpha$ for all $\alpha < \kappa$.
- $x_\alpha =_\alpha f(x_\alpha)$ for all $\alpha < \kappa$.

When $\alpha = 0$ the first claim holds by Lemma 3.11. The second one is trivial. We apply Lemma 4.2 for the case $\alpha = 0$ and $x = y_0$, thus $x_0 = f_0(y_0)$ is the $\sqsubseteq$-least element of $[x_0]_0$ and $x_0 =_0 f(x_0)$.

Suppose now that $\alpha = \beta + 1$. By the induction hypothesis, we have that $y_\beta \sqsubseteq_\beta f(y_\beta)$. We apply Lemma 4.2 for the case $\alpha = \beta$ and $x = y_\beta$ and by Remark 4 we have $x_\beta \sqsubseteq_{\beta+1} f(x_\beta)$, so that $y_\alpha \sqsubseteq_\alpha f(y_\alpha)$. By applying Lemma 4.2 for the case $\alpha = \alpha$ and $x = y_\alpha$ we have $x_\beta = y_\alpha \sqsubseteq_\alpha f_\alpha(y_\alpha) = x_\alpha$, so that by Property 1, $x_\beta =_\gamma x_\alpha$ for all $\gamma < \alpha$. If $\gamma < \alpha$, then $\gamma \leq \beta$, so by the induction hypothesis $x_\gamma =_\gamma x_\beta$ for all $\gamma \leq \beta$. It follows $x_\gamma =_\gamma x_\alpha$ whenever $\gamma < \alpha$. Finally, by Lemma 4.2, x_α is the $\sqsubseteq$-least element of $[x_\alpha]_\alpha$ and $x_\alpha =_\alpha f(x_\alpha)$.

Suppose now that $\alpha > 0$ is a limit ordinal and our claims hold for all ordinals less than α. By Lemma 3.13 we have that $x_\beta =_\beta y_\alpha$ for all $\beta < \alpha$ and that y_α is the $\sqsubseteq$-least element of $(y_\alpha]_\alpha$. Since f is β-monotonic, $f(x_\beta) =_\beta f(y_\alpha)$ for all $\beta < \alpha$ and by the induction hypothesis $x_\beta =_\beta f(x_\beta)$ for all $\beta < \alpha$, so that $f(y_\alpha) \in (y_\alpha]_\alpha$. Since y_α is the $\sqsubseteq$-least element of $(y_\alpha]_\alpha$, we get by Lemma 3.3 that $y_\alpha \sqsubseteq_\alpha f(y_\alpha)$. By applying Lemma 4.2 for the case $\alpha = \alpha$ and $x = y_\alpha$ we have $y_\alpha \sqsubseteq_\alpha f_\alpha(y_\alpha) = x_\alpha$, so that by Property 1, $y_\alpha =_\beta x_\alpha$ for all $\beta < \alpha$, hence $x_\beta =_\beta x_\alpha$ whenever $\beta < \alpha$. Also, again by Lemma 4.2, x_α is the $\sqsubseteq$-least element of $[x_\alpha]_\alpha$ and $x_\alpha =_\alpha f(x_\alpha)$.

Let us define $x_\infty = \bigsqcup\{x_\alpha : \alpha < \kappa\}$. By Lemma 3.13, $x_\infty =_\alpha x_\alpha$ for all $\alpha < \kappa$. Now since f is α-monotonic, also $f(x_\infty) =_\alpha f(x_\alpha) =_\alpha x_\alpha$ for all $\alpha < \kappa$. Thus, by Property 2, $f(x_\infty) = x_\infty$. It remains to show that x_∞ is the least pre-fixed point of f with respect to $\sqsubseteq$.

Suppose that $f(z) \sqsubseteq z$. We want to prove by induction that for all $\alpha < \kappa$, either $x_\gamma \sqsubset_\gamma z$ for some $\gamma < \alpha$, or $x_\gamma \sqsubseteq_\gamma z$ for all $\gamma \leq \alpha$. It then follows that $x_\infty \sqsubseteq z$.

When $\alpha = 0$ then by Lemma 3.10 and Lemma 3.11 it holds that $f(z) \sqsubseteq_0 z$ and $\perp \sqsubseteq_0 z$; thus, by Lemma 4.2, $x_0 \sqsubseteq_0 z$.

Suppose now that $\alpha > 0$. If $x_\gamma \sqsubset_\gamma z$ for some $\gamma < \alpha$, then we are done. So suppose that this is not the case, ie., $x_\gamma =_\gamma z$ for all $\gamma < \alpha$.

Suppose that $\alpha = \beta + 1$. Then $x_\beta =_\beta z$ and thus $z \in [x_\beta]_\beta$. Since x_β is $\sqsubseteq$-least in $[x_\beta]_\beta$, we have $x_\beta \sqsubseteq z$ and thus, by Lemma 3.3, $x_\beta \sqsubseteq_\alpha z$. We conclude that $x_\beta \sqsubseteq_\alpha f(x_\beta) \sqsubseteq_\alpha f(z)$, ie., $x_\beta \sqsubseteq_\alpha f(z)$. If $f(z) \sqsubset_\gamma z$ for some $\gamma < \alpha$, then by $x_\beta \sqsubseteq_\alpha f(z) \sqsubset_\gamma z$ we have $x_\beta \sqsubset_\gamma z$, contradicting $x_\beta =_\beta z$. Thus, since $f(z) \sqsubseteq z$ and $f(z) =_\gamma z$ for all $\gamma < \alpha$, we must have $f(z) \sqsubseteq_\alpha z$. Since $x_\alpha = f_\alpha(x_\beta)$ and $x_\beta \sqsubseteq_\alpha z$, we conclude by Lemma 4.2 that $x_\alpha \sqsubseteq_\alpha z$.

Suppose that $\alpha > 0$ is a limit ordinal. Then, as shown above, $y_\alpha =_\gamma x_\gamma$ for all $\gamma < \alpha$. Since also $x_\gamma =_\gamma z$ for all $\gamma < \alpha$, we have $z \in (y_\alpha]_\alpha$. But y_α is the $\sqsubseteq$-least and a $\sqsubseteq_\alpha$-least element of $(y_\alpha]_\alpha$, so that $y_\alpha \sqsubseteq_\alpha z$ and thus $y_\alpha \sqsubseteq_\alpha f(y_\alpha) \sqsubseteq_\alpha f(z)$; therefore $y_\alpha \sqsubseteq_\alpha f(z)$. Suppose that $f(z) \sqsubset_\gamma z$ for some $\gamma < \alpha$. Then $y_\alpha \sqsubseteq_\alpha f(z) \sqsubset_\gamma z$ and thus $y_\alpha \sqsubset_\gamma z$, contradicting $y_\alpha \sqsubseteq_\alpha z$. Thus, $f(z) =_\gamma z$ for all $\gamma < \alpha$. Since $f(z) \sqsubseteq z$ and $f(z) =_\gamma z$ for all $\gamma < \alpha$, we have $f(z) \sqsubseteq_\alpha z$. Since $y_\alpha \sqsubseteq_\alpha z$ and $f(z) \sqsubseteq_\alpha z$, by Lemma 4.2 we have that $x_\alpha \sqsubseteq_\alpha z$. □

Using duality, we get the following:

Theorem 4.4. *Suppose that $f : L \to L$ is stratified monotonic. Then f has a $\sqsubseteq$-least pre-fixed point which is also its $\sqsubseteq$-least fixed point. Moreover, f has a $\sqsubseteq$-greatest post-fixed point which is also its $\sqsubseteq$-greatest fixed point.*

The above theorem has as a special case the well-known Knaster-Tarski fixed point theorem when restricted to the case of monotonic functions.

Corollary 4.5 (Knaster-Tarski Fixed Point Theorem)**.** *Suppose that $f : L \to L$ is monotonic with respect to $\sqsubseteq$. Then f has a $\sqsubseteq$-least pre-fixed point which is also its $\sqsubseteq$-least fixed point. Moreover, f has a $\sqsubseteq$-greatest post-fixed point which is also its $\sqsubseteq$-greatest fixed point.*

Proof. By Remark 1, for every complete lattice $(L, \sqsubseteq)$ we can create the trivial lexicographic lattice structure $\langle L, \sqsubseteq, \{\sqsubseteq\}\rangle$ by taking $\kappa = 1$ and $\sqsubseteq_0$ to be equal to $\sqsubseteq$. Then, f is 0-monotonic because it is monotonic with respect to $\sqsubseteq$, and therefore it is stratified monotonic (because it is α-monotonic for all $\alpha < \kappa = 1$). The result follows directly by Theorem 4.4. □

The main theorem of this section (Theorem 4.4) has been formally verified using the Coq proof assistant.[1]

[1] The code can be retrieved from https://github.com/acharal/lexicographic-lattice-structures

5 Pre-fixed points and Post-fixed points

In this section we demonstrate that the sets of pre-fixed and post-fixed points of a stratified monotonic function $f : L \to L$, are both complete lattices. Central in our proofs is the following lemma which demonstrates that we can obtain the least upper bound of a subset of L as the least upper bound of a sequence of successive approximations.

Definition 5.1. Let $X \subseteq L$. For each $\alpha < \kappa$, we define $x_\alpha \in L$ and $X_\alpha, Y_\alpha \subseteq X$. Let $Y_0 = X$. For each $\alpha < \kappa$, we define x_α, X_α and Y_α as follows:

$$\begin{aligned} Y_\alpha &= \bigcap_{\beta<\alpha} X_\beta,\ \alpha > 0 \\ x_\alpha &= \begin{cases} \bigsqcup\{x_\beta : \beta < \alpha\} & \text{if } Y_\alpha = \emptyset \\ \bigsqcup_\alpha Y_\alpha & \text{if } Y_\alpha \neq \emptyset \end{cases} \\ X_\alpha &= \{x \in Y_\alpha : x =_\alpha x_\alpha\} \end{aligned}$$

The sequence $\{x_\alpha\}_{\alpha<\kappa}$ is called the *sequence of approximations defined by* X.

Lemma 5.2. *Let $X \subseteq L$ and let $\{x_\alpha\}_{\alpha<\kappa}$ be the sequence of approximations defined by X. Then, $\bigsqcup X = \bigsqcup\{x_\alpha : \alpha < \kappa\}$.*

Proof. If $X = \emptyset$ then $x_\alpha = \perp$ for all $\alpha < \kappa$ and the lemma obviously holds. Assume therefore that $X \neq \emptyset$. From Definition 5.1, we observe that if $Y_\alpha = \emptyset$ then $X_\alpha = \emptyset$. Moreover, it is easy to see that:

$$Y_\alpha = \bigcap_{\beta<\alpha} X_\beta = \{x \in X : \forall\beta < \alpha\ x =_\beta x_\beta\}$$

and:

$$X_\alpha = \{x \in Y_\alpha : x =_\alpha x_\alpha\} = \{x \in X : \forall\beta \leq \alpha\ x =_\beta x_\beta\}$$

It is clear from the above that $X_\beta \supseteq X_\alpha$ whenever $\beta < \alpha$.

We show that for all $\beta < \alpha < \kappa$, $x_\beta =_\beta x_\alpha$, and x_α is the $\sqsubseteq$-least element of $[x_\alpha]_\alpha$. We argue by induction on α. When $\alpha = 0$ our claim follows by Lemma 3.12. Suppose now that $\alpha > 0$ and the claim is true for all ordinals less than α. Let $\beta < \alpha$. We distinguish two cases:

Case 1: $Y_\alpha \neq \emptyset$. Let y be a fixed element of Y_α. Since $y \in X_\beta$, we have that $x_\beta =_\beta y$. Also, since $x_\alpha = \bigsqcup_\alpha Y_\alpha$, by Lemma 3.6 we get $y \sqsubseteq_\alpha x_\alpha$ which implies that $y =_\beta x_\alpha$. Thus, $x_\beta =_\beta x_\alpha$ by transitivity. Also, by Lemma 3.12, x_α is the $\sqsubseteq$-least element of $[x_\alpha]_\alpha$.

Case 2: $Y_\alpha = \emptyset$. Then, $x_\alpha = \bigsqcup\{x_\beta : \beta < \alpha\}$. Using Lemma 3.13 we get $x_\alpha =_\beta x_\beta$ for all $\beta < \alpha$ and x_α is the $\sqsubseteq$-least element of $[x_\alpha]_\alpha$.

Let $x_\infty = \bigsqcup\{x_\beta : \beta < \kappa\}$. Note that by Lemma 3.13, for all $\alpha < \kappa$, $x_\infty =_\alpha x_\alpha$. Let $X_\infty = \bigcap_{\alpha<\kappa} X_\alpha$. Our aim is to prove that $x_\infty = \bigsqcup X$. Moreover, we prove that either $X_\infty = \emptyset$ or $X_\infty = \{x_\infty\}$, ie., $X_\infty \subseteq \{x_\infty\}$.

Proof of $X_\infty \subseteq \{x_\infty\}$. Suppose that $y \in X_\infty$. Then for all α, $x_\infty =_\alpha x_\alpha =_\alpha y$ which implies that $x_\infty = y$. ⋄

Proof of $X \sqsubseteq x_\infty$. Let $y \in X$. There are two cases, either $y \in X_\infty$ or $y \notin X_\infty$. If $y \in X_\infty$ then $y = x_\infty$ and clearly $y \sqsubseteq x_\infty$. If $y \notin X_\infty$, then there is a least ordinal α less than κ such that $y \notin X_\alpha$. We have $y \in Y_\alpha$ and thus $y \sqsubseteq_\alpha x_\alpha$, since $x_\alpha = \bigsqcup_\alpha Y_\alpha$. But $y \neq_\alpha x_\alpha$ since $y \notin X_\alpha$. Thus $y \sqsubset_\alpha x_\alpha$ and therefore $y \sqsubseteq x_\infty$. ⋄

To complete the proof, it remains to show that x_∞ is the *least* among the upper bounds of X. Therefore suppose that for some z, $X \sqsubseteq z$. We need to prove that $x_\infty \sqsubseteq z$. This is clear when $X_\infty \neq \emptyset$, since in this case $x_\infty \in X$. Suppose now that $X_\infty = \emptyset$. We prove by induction that for all $\alpha < \kappa$ either $x_\alpha \sqsubseteq_\alpha z$, or there is some $\beta < \alpha$ with $x_\beta \sqsubset_\beta z$.

When $\alpha = 0$, $x_0 = \bigsqcup_0 X$. Since $X \sqsubseteq z$, by Lemma 3.10 we have $X \sqsubseteq_0 z$. Thus, $x_0 \sqsubseteq_0 z$. Suppose that $\alpha > 0$ and the claim is true for all ordinals less than α. If there is some $\beta < \alpha$ with $x_\beta \sqsubset_\beta z$ then we are done. Otherwise $x_\beta =_\beta z$ for all $\beta < \alpha$. We distinguish two cases. If $Y_\alpha \neq \emptyset$, then $x_\alpha = \bigsqcup_\alpha Y_\alpha$, where $Y_\alpha = \bigcap_{\beta<\alpha} X_\beta = \{y \in X : \forall\beta < \alpha\ x_\beta =_\beta y\}$. Since $x_\beta =_\beta z$ for all $\beta < \alpha$ and $Y_\alpha \sqsubseteq z$, it follows by Lemma 3.5 that $x_\alpha \sqsubseteq_\alpha z$. If, on the other hand, $Y_\alpha = \emptyset$, then $x_\alpha = \bigsqcup\{x_\beta : \beta < \alpha\}$ and by Lemma 3.13 x_α is the $\sqsubseteq$-least element of $(x_\alpha]_\alpha$. Also, since $x_\beta =_\beta z$ for all $\beta < \alpha$, it is $z \in (x_\alpha]_\alpha$. Therefore, $x_\alpha \sqsubseteq z$ which implies that $x_\alpha \sqsubseteq_\alpha z$.

To complete the proof of the fact that $x_\infty \sqsubseteq z$, first note that if for all $\alpha < \kappa$ it is $x_\alpha =_\alpha z$, then, since it also holds $x_\infty =_\alpha x_\alpha$ for all $\alpha < \kappa$, we have $x_\infty =_\alpha z$ for all $\alpha < \kappa$, which implies that $x_\infty = z$ and therefore $x_\infty \sqsubseteq z$. Consider, on the other hand, the case that there exists $\alpha < \kappa$ such that $x_\alpha \sqsubset_\alpha z$. But then, since $x_\alpha =_\alpha x_\infty$, we get $x_\infty \sqsubset_\alpha z$ which implies that $x_\infty \sqsubseteq z$. □

Lemma 5.3. *Suppose that $f : L \to L$ is stratified monotonic and let X be a set of post-fixed points of f with respect to $\sqsubseteq$. Let $x = \bigsqcup X$ and let $\{x_\alpha\}_{\alpha<\kappa}$ be the sequence of approximations defined by X. Then for all $\alpha < \kappa$, either there exists $\beta < \alpha$ such that $x_\beta \sqsubset_\beta f(x_\beta)$, or for all $\beta \leq \alpha$, $x_\beta =_\beta f(x_\beta)$.*

Proof. Our claim is clear when X is empty since in that case $x_\alpha = \perp$ for all $\alpha < \kappa$ and the lemma obviously holds. So below we assume that X is nonempty. Recall now that for all $\alpha < \kappa$, x_α is the $\sqsubseteq$-least element of $[x_\alpha]_\alpha$.

Suppose that the claim does not hold and let δ denote the least ordinal for which it fails. Then $x_\beta =_\beta f(x_\beta)$ for all $\beta < \delta$ and $x_\delta \not\sqsubseteq_\delta f(x_\delta)$. By the properties of the sequence $\{x_\alpha\}_{\alpha<\kappa}$, we have that $x_\beta =_\beta x_\delta$ and thus also $f(x_\beta) =_\beta f(x_\delta)$ for all $\beta < \delta$. We conclude that $x_\beta =_\beta x_\delta =_\beta f(x_\delta)$ for all $\beta < \delta$. We distinguish the following two cases:

Case 1: $Y_\delta = \emptyset$. Then $x_\delta = \bigsqcup\{x_\beta : \beta < \delta\}$, and by Lemma 3.13 x_δ is the $\sqsubseteq$-least element of $(x_\delta]_\delta$. Since $x_\beta =_\beta x_\delta =_\beta f(x_\delta)$ for all $\beta < \delta$, $f(x_\delta) \in (x_\delta]_\delta$. It follows that $x_\delta \sqsubseteq f(x_\delta)$ which implies that $x_\delta \sqsubseteq_\delta f(x_\delta)$, a contradiction.

Case 2: $Y_\delta \neq \emptyset$. Then $x_\delta = \bigsqcup_\delta Y_\delta$. Recall that $Y_\delta = \{z \in X : \forall\beta < \delta\ z =_\beta x_\beta\}$. But we know that $x_\beta =_\beta f(x_\beta)$ for all $\beta < \delta$, and by our assumption on f, also $f(x_\beta) =_\beta f(z)$

for all $z \in Y_\delta$ and $\beta < \delta$. So we conclude that $z =_\beta f(z)$ holds for all $z \in Y_\delta$ and $\beta < \delta$. Since X is a set of post-fixed points of f, we have $z \sqsubseteq f(z)$ for all $z \in Y_\delta$. Therefore, $z \sqsubseteq_\delta f(z)$ for all $z \in Y_\delta$. By the definition of x_δ, for all $z \in Y_\delta$, $z \sqsubseteq_\delta x_\delta$ and thus $z \sqsubseteq_\delta f(z) \sqsubseteq_\delta f(x_\delta)$. Thus, $f(x_\delta)$ is a $\sqsubseteq_\delta$-upper bound of Y_δ, and by the definition of x_δ, we have $x_\delta \sqsubseteq_\delta f(x_\delta)$, a contradiction again.

This completes the proof of the lemma. □

We now easily get the following corollary:

Corollary 5.4. *Suppose that $f : L \to L$ is stratified monotonic and let $X \subseteq L$ be a set of post-fixed points of f with respect to $\sqsubseteq$. Then $\bigsqcup X$ is a post-fixed point of f with respect to $\sqsubseteq$.*

Proof. Let $x = \bigsqcup X$. If $x =_\alpha f(x)$ for all $\alpha < \kappa$, then $x = f(x)$. Otherwise, by Lemma 5.3, there is some $\alpha < \kappa$ with $x =_\alpha x_\alpha \sqsubset_\alpha f(x_\alpha) =_\alpha f(x)$ and then $x \sqsubset_\alpha f(x)$ and consequently $x \sqsubseteq f(x)$. □

The above results lead to the following theorem:

Theorem 5.5. *Suppose that $f : L \to L$ is stratified monotonic. Then, the set of post-fixed points of f form a complete lattice.*

Proof. It is well known (see [6][Definition 2.4, page 34 and Theorem 2.31, page 47]) that a partially ordered set $(P, \sqsubseteq)$ is a complete lattice if P has a bottom element and every nonempty subset $X \subseteq P$ has a least upper bound in P. Notice now that $\perp$ is a post-fixed point of f. The theorem follows from this observation together with Corollary 5.4. □

By duality, we also get the following theorem:

Theorem 5.6. *Suppose that $f : L \to L$ is stratified monotonic. Then, the set of pre-fixed points of f form a complete lattice.*

6 Two Applications of the Fixed Point Theorem

In this section we present two applications of the fixed point theorem developed in this paper. The first is an application to the semantics of logic programs with negation and the second an application regarding sequences of (possibly) transfinite length.

6.1 The Infinite-Valued Semantics

In [18] it is demonstrated that for every logic program P with negation, there exists a special model which can be taken as its intended semantics. It is also shown in [18] that this model is the least fixed point of the *immediate consequence operator* T_P of the program P. The proof of this result is given in [18] in a lengthy and somewhat ad-hoc way. In the following, we demonstrate that this result is a simple consequence of the theory developed in the present paper. To avoid an extensive presentation of the material in [18], we will introduce only the basic notions that are needed for this result to be established. The interested reader should consult [18] for additional details.

The basic notion that needs to be introduced, is that of an *infinite-valued interpretation*. Such interpretations are used in [18] to give meaning to logic programs with negation. Intuitively, an infinite-valued interpretation is a generalization of classical interpretations of logic programs [15] to an infinite-valued logic which contains one truth value F_α and one T_α for each countable ordinal α, together with a neutral truth value 0. Intuitively, F_0 and T_0 are the classical *False* and *True* values and 0 is the *undefined* value. The intuition behind the new values is that they express different levels of truth and falsity. Let V be this set of truth values, ie.,

$$V = \{F_\alpha : \alpha < \Omega\} \cup \{T_\alpha : \alpha < \Omega\} \cup \{0\}$$

where Ω is the first uncountable ordinal. We will need the following definition from [18]:

Definition 6.1. The *order* of a truth value is defined as follows: $order(T_\alpha) = \alpha$, $order(F_\alpha) = \alpha$ and $order(0) = \Omega$.

Let Z be a non-empty set of variables. Intuitively, the variables in Z are used to construct propositional logic programs in [18]. For the purposes of this subsection, it suffices to know that Z is just a non-empty set. Then:

Definition 6.2. An *infinite-valued interpretation* (or simply, *interpretation*) I is a function from the set Z to the set of truth values V.

Following [18], we define various relations on interpretations:

Definition 6.3. Let $I \in V^Z$ be an interpretation and let $v \in V$. Then $I \parallel v = \{z \in Z \mid I(z) = v\}$.

Definition 6.4. Let $I, J \in V^Z$ be interpretations and $\alpha < \Omega$. We write $I =_\alpha J$, if for all $\beta \leq \alpha$, $I \parallel T_\beta = J \parallel T_\beta$ and $I \parallel F_\beta = J \parallel F_\beta$.

Definition 6.5. Let $I, J \in V^Z$ be interpretations and $\alpha < \Omega$. We write $I \sqsubset_\alpha J$, if for all $\beta < \alpha$, $I =_\beta J$ and either $I \parallel T_\alpha \subset J \parallel T_\alpha$ and $I \parallel F_\alpha \supseteq J \parallel F_\alpha$, or $I \parallel T_\alpha \subseteq J \parallel T_\alpha$ and $I \parallel F_\alpha \supset J \parallel F_\alpha$. We write $I \sqsubseteq_\alpha J$ if $I =_\alpha J$ or $I \sqsubset_\alpha J$. We write $I \sqsubset J$ if there exists $\alpha < \Omega$ such that $I \sqsubset_\alpha J$. We write $I \sqsubseteq J$ if $I \sqsubset J$ or $I = J$.

We can now show that the set of infinite-valued interpretations together with the above relations, forms a lexicographic lattice structure.

Lemma 6.6. *Let L be the set of infinite-valued interpretations. Let the relations $\sqsubseteq$ and $\{\sqsubseteq_\alpha\}_{\alpha<\Omega}$ be as in Definition 6.5. Then, the triple $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\Omega}\rangle$ is a lexicographic lattice structure.*

Proof. The set L is a complete lattice (see [11]). Property 1 holds directly due to Definition 6.5. To verify that Property 2 holds, let I, J be infinite-valued interpretations and assume that for all $\alpha < \Omega$ it is $I =_\alpha J$. For any $\alpha < \Omega$ it follows by the definition of the $=_\alpha$ relation (Definition 6.4) that given an arbitrary $z \in Z$, $I(z) = T_\alpha$ iff $J(z) = T_\alpha$ and $I(z) = F_\alpha$ iff $J(z) = F_\alpha$; this implies that $I(z) = 0$ iff $J(z) = 0$. Therefore, for all $z \in Z$, $I(z) = J(z)$, ie., $I = J$. To verify Property 3, consider an arbitrary interpretation I and let $\alpha < \Omega$ be an ordinal. The set $[I]_\alpha$ has a $\sqsubseteq$-least element J defined as:

$$J(z) = \begin{cases} I(z) & \text{if } order(I(z)) \leq \alpha \\ F_{\alpha+1} & \text{otherwise} \end{cases}$$

and a $\sqsubseteq$-greatest element K defined as follows:

$$K(z) = \begin{cases} I(z) & \text{if } order(I(z)) \leq \alpha \\ T_{\alpha+1} & \text{otherwise} \end{cases}$$

It is straightforward to verify using Definition 6.5 that J and K are indeed the $\sqsubseteq$-least and $\sqsubseteq$-greatest elements of $[I]_\alpha$. ◻

In [18] an operator $T_P : L \to L$ is defined for every logic program P, where L is the set of infinite-valued interpretations. It is demonstrated that for every $\alpha < \Omega$, T_P is α-monotonic. Moreover, it is demonstrated through a lengthy reasoning, that T_P has a least fixed point (see Sections 6 and 7 in [18]), which is taken as the intended meaning of the program. This result can now be obtained in a much easier way as a direct consequence of the theory developed in this paper: since T_P is α-monotonic for all $\alpha < \Omega$, and since L is a lexicographic lattice structure, it follows from Theorem 4.4 that T_P has a least fixed point.

6.2 Transfinite Sequences over Complete Lattices

In this subsection we consider (possibly transfinite) sequences over complete lattices. This is actually a generalization of ω-words discussed in Section 2. Sets of transfinite sequences over complete lattices have been studied by Henry Crapo in [5], where they are referred as *lexicographic lattices*. In this subsection we demonstrate that these sets induce, in a natural way, lexicographic lattice structures.

In the rest of this subsection we assume that κ is a fixed ordinal and $(Q, \leq)$ is a complete lattice. The set Q^κ of the functions from κ to Q can be viewed as sequences of length κ over Q. These sequences have an intuitive lexicographic order: suppose $f, g \in Q^\kappa$ such that $f \neq g$ and let α be the least ordinal such that $f(\alpha) \neq g(\alpha)$. Then $f \sqsubset g$ if $f(\alpha) < g(\alpha)$ and $g \sqsubset f$ if $g(\alpha) < f(\alpha)$.

Definition 6.7. Let $f, g \in Q^\kappa$. We define $f \sqsubseteq_\alpha g$ if $f(\beta) = g(\beta)$ for all $\beta < \alpha$ and $f(\alpha) \leq g(\alpha)$. We write $f =_\alpha g$ if $f \sqsubseteq_\alpha g$ and $g \sqsubseteq_\alpha f$. We write $f \sqsubset_\alpha g$ if $f \sqsubseteq_\alpha g$ but $f =_\alpha g$ does not hold. We write $f \sqsubset g$ if $f \sqsubset_\alpha g$ for some $\alpha < \kappa$. We write $f \sqsubseteq g$ if $f \sqsubset g$ or $f = g$.

The following lemmas hold:

Lemma 6.8. *For each $\alpha < \kappa$, $\sqsubseteq_\alpha$ is a preorder.*

Proof. Let $\alpha < \kappa$ be an ordinal and $f, g, h \in Q^\kappa$. We have that $f(\beta) = f(\beta)$ for all $\beta < \alpha$ and $f(\alpha) \leq f(\alpha)$ by the reflexivity of $\leq$, so that $f \sqsubseteq_\alpha f$. Suppose now that $f \sqsubseteq_\alpha g \sqsubseteq_\alpha h$. Then $f(\beta) = g(\beta) = h(\beta)$ for all $\beta < \alpha$ and $f(\beta) \leq g(\beta) \leq h(\beta)$. by the transitivity of $\leq$ and the equality, so that $f \sqsubseteq_\alpha h$. Thus $\sqsubseteq_\alpha$ is reflexive and transitive. ◻

Lemma 6.9. *(See [5][Proposition 2]) $\sqsubseteq$ is a partial order.*

Lemma 6.10. *(See [5][Theorem 1]) $(Q^\kappa, \sqsubseteq)$ is a complete lattice.*

Lemma 6.11. *$\sqsubseteq$ is determined by the sequence $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$.*

Proof. Let $f, g \in Q^\kappa$. By definition, $f \sqsubset g$ iff $f \sqsubset_\alpha g$ for some $\alpha < \kappa$. ◻

Definition 6.12. Let $\alpha > \kappa$ be an ordinal and $f \in Q^\kappa$. We define $f|_\alpha, f|^\alpha \in Q^\kappa$ such that:

$$f|_\alpha(\beta) = \begin{cases} f(\beta), & \beta \leq \alpha \\ \bot, & \text{otherwise} \end{cases}$$

$$f|^\alpha(\beta) = \begin{cases} f(\beta), & \beta \leq \alpha \\ \top, & \text{otherwise} \end{cases}$$

where $\bot$ is the least element and $\top$ is the greatest element of Q.

Lemma 6.13. *The triple $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa} \rangle$ is a lexicographic lattice structure.*

Proof. Property 1 holds by definition. For Property 2, if for some $f, g \in Q^\kappa$, $f =_\alpha g$ for all $\alpha < \kappa$ then $f(\alpha) = g(\alpha)$ for all $\alpha < \kappa$, so that $f = g$. Finally, for Property 3, for any $f \in Q^\kappa$ and $\alpha < \kappa$, $f|_\alpha$ is the $\sqsubseteq$-least element and $f|^\alpha$ is the $\sqsubseteq$-greatest element of $[x]_\alpha$. ◻

We now derive a simple lemma for a class of functions over sequences that always have a least fixed point. A function $T : Q^\kappa \to Q^\kappa$ will be called *past-dependent* if, intuitively speaking, for every $f \in Q^\kappa$ and for every ordinal $\alpha < \kappa$, the value of the sequence $T(f)$ at index α depends only on the values that the sequence f has at ordinal indices that are strictly less than α. More formally:

Definition 6.14. The function $T : Q^\kappa \to Q^\kappa$ will be called *past-dependent* if the following condition holds: for each $\alpha < \kappa$ and for all $f, g \in Q^\kappa$, if $f(\beta) = g(\beta)$ for all $\beta < \alpha$ then $T(f)(\beta) = T(g)(\beta)$ for all $\beta \leq \alpha$.

We have the following simple lemma:

Lemma 6.15. *Every past-dependent function is stratified monotonic.*

Proof. Let $T : Q^\kappa \to Q^\kappa$ be a past-dependent function. We show that T is α-monotonic for each $\alpha < \kappa$. Suppose $\alpha < \kappa$ and let $f, g \in Q^\kappa$ such that $f \sqsubseteq_\alpha g$. By definition, $f(\beta) = g(\beta)$ for all $\beta < \alpha$. By hypothesis, $T(f)(\beta) = T(g)(\beta)$ for all $\beta \leq \alpha$, so that $T(f) \sqsubseteq_\alpha T(g)$. ◻

Corollary 6.16. *Every past-dependent function has a least fixed point and a greatest fixed point.*

Proof. Immediate by Theorem 4.4. □

Example 6.17. Let $\kappa = \omega$ and $Q = \{a, b\}$. $\{a, b\}^\omega$ are the infinite strings over alphabet $\{a, b\}$. Let $\leq$ be the usual ordering on $\{a, b\}$. Obviously, $(\{a, b\}, \leq)$ is a complete lattice. Using Lemma 6.13, $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\omega}\rangle$ is a lexicographic lattice structure and $\sqsubseteq$ is the usual lexicographic ordering over strings.

Let's define $T : \{a, b\}^\omega \to \{a, b\}^\omega$ such that for all $f \in \{a, b\}^\omega$ and $n \in \omega$:

$$T(f)(n) = \begin{cases} b, & n = 0 \\ f(n-1), & n \text{ is odd} \\ f(n-2), & n \text{ is even and greater than } 0 \end{cases}$$

Although T is not monotonic under the lexicographic ordering, it follows by Lemma 6.15 that T is stratified monotonic. By Corollary 6.16, T has a least fixed point and a greatest fixed point. □

7 Related and Future Work

Fitting's work [12] can be regarded as the pioneering approach for developing an abstract fixed point theory for non-monotonic functions. Although groundbreaking, Fitting's approach is centered around logic programming. More specifically, Fitting demonstrated that four different semantics of logic programs can be studied by considering two operators on a bilattice of 4-valued interpretations. Based on Fitting's work, Denecker, Marek and Truszczyński developed *approximation fixed point theory* [7, 8]. The key idea behind their approach can be intuitively described as follows. Assume we have a complete lattice $(L, \sqsubseteq)$ and we are given a (possibly non-monotonic) function $f : L \to L$. It is possible that one can not study the fixed point behaviour of f directly (for example, f may not have fixed points due to its non-monotonicity). Instead, we can study the fixed point behaviour of operators that *approximate* f and are defined on the bilattice L^2. Elements of L^2 can be thought of as approximating the elements of L from below and above (an element $x \in L$ is approximated by a pair $(y, z) \in L^2$ if $y \sqsubseteq x \sqsubseteq z$). Therefore, we can consider *approximating operators* of f which are functions of the form $A_f : L^2 \to L^2$. Notice that A_f must be closely related to f and it must also possess a form of monotonicity known as *Fitting monotonicity*. Then, the fixed points of A_f can give insight on the operator f itself. As it is demonstrated in [7, 8] several semantic approaches to logic programming can be captured using the above approach.

The work of Ésik and Rondogiannis, introduced in [11], proceeds in a different direction. The starting point is a complete lattice $(L, \leq)$, a sequence of preorderings $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ on L, and a partial order $\sqsubseteq$ on L. Four axioms are then introduced which restrict the "behaviour" of the above partial orders. The main theorem of [11] states that if L is a partial order that obeys the given axioms, then every function $f : L \to L$ that is monotonic with respect to $\sqsubseteq_\alpha$, for all $\alpha < \kappa$, has a least pre-fixed point with respect to $\sqsubseteq$ which is also the least fixed-point of f. The results of [11] were subsequently extended in [10] to obtain various theorems regarding the pre-fixed and post-fixed points of f. The main difference between the works in [7, 8] and those in [10, 11], is that the former theory applies on the approximators of a given function while the latter applies directly on the function itself. In other words, the approach of [7, 8] copes with non-monotonicity by considering approximating operators of the function f and exploiting the Fitting-monotonicity of these operators in order to construct fixed points, while the approach of [10, 11] exploits restricted forms of monotonicity that the function f possesses in order to find its least fixed point. A deeper understanding of the relative powers of these two approaches is currently lacking. A step towards this direction has been performed in [3, 9] where it was shown that the semantics of logic programs with negation obtained through the approach of [11], satisfies all identities of *iteration theories* [2], which is not the case when we consider the semantics obtained through the approach of [7, 8]. However, a deeper comparison of the two approaches is required. This is definitely a very interesting topic for further investigation, which however is outside the scope of the present paper.

Our present work follows the research direction initiated in [10, 11]. However, the theoretical framework that we develop is much simpler than the one proposed in [10, 11]. The novel features of our development with respect to that of [10, 11] can be outlined as follows. The work in [10, 11] is based on two distinct complete lattices, namely $(L, \leq)$ and $(L, \sqsubseteq)$, the former having a "pointwise" flavour while the latter a "lexicographic" one. Both partial orders must exist in order for the results of [10, 11] to be applicable. Two of the axioms proposed in [11] are actually introduced in order to characterize the relationship and the behaviour of these two partial orders. This makes the approach of [10, 11] relatively complicated and the corresponding proofs rather involved and lengthy. In contrast to the above, our approach uses a unique complete lattice $(L, \sqsubseteq)$, our axioms are fewer (three instead of four) and simpler, and the proofs of all our results are significantly simpler than those of [10, 11]. Last but not least, our main result appears as a natural extension of the classical Knaster-Tarski fixed point theorem: it intuitively states that given a complete lattice $(L, \sqsubseteq)$ and a function $f : L \to L$ that possesses a restricted form of monotonicity, there always exists a least (and a greatest) fixed point of f. This main theorem of our paper (Theorem 4.4) has been verified through the Coq proof assistant.

The main unresolved open question of our work is to verify whether the set of fixed points of a stratified monotonic function, forms a complete lattice. We believe that this is indeed the case, but we have been unable to establish it until now. Proving this result would make the analogy with the Knaster-Tarski fixed point theorem even firmer. It seems however, that the technique we have used to establish Theorem 5.5 can not be easily adapted in order to establish this conjecture. We are currently investigating alternative proof approaches.

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