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NOTES AND COMMENTS

RANDOM SERIAL DICTATORSHIP AND THE CORE FROM RANDOM ENDOWMENTS IN HOUSE ALLOCATION PROBLEMS

BY ATİLA ABDULKADİROĞLU AND TAYFUN SÖNMEZ[1]

1. INTRODUCTION

WHEN A COLLEGE GRADUATE decides to pursue a higher degree at a particular institution, one of the first challenges she faces is finding an apartment. Most institutions have on-campus housing available that is often subsidized and hence more appealing than its alternatives. Usually there are several types of on-campus housing and the attractiveness of each type changes from person to person. Therefore housing offices need to find "mechanisms" to allocate available housing among the applicants who might have various preferences. In this paper we deal with this class of problems to which we refer as *house allocation problems*.[2] Formally, there are n agents who collectively own n indivisible objects, say houses, and each agent has preferences over objects.[3] An allocation is a *matching* of houses to agents and a *matching mechanism* is a systematic procedure to select a matching for each problem. A widely studied class of matching mechanisms is the class of *simple serial dictatorships*: For a given ordering of agents, the agent who is ordered first is assigned her top choice, the agent ordered second is assigned her top choice among the remaining houses, and so on. These matching mechanisms are not considered very desirable as they discriminate between the agents. However this difficulty can be handled by randomly determining an ordering and using the induced simple serial dictatorship. We refer to this mechanism as the *random serial dictatorship*. Of course this mechanism selects lotteries over matchings instead of matchings and we refer to such mechanisms as *lottery mechanisms*.

Our first contribution is the introduction of a (seemingly) alternative lottery mechanism. For this purpose we need to introduce a related class of problems, namely the

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[2]See Hylland and Zeckhauser (1979) and Zhou (1990).

[3]In most real life applications houses are scarce. For instance, it is not possible to accommodate all graduate students with on-campus housing. Therefore it is important to deal with the model where there are more agents than houses. Our results extend to this case in a direct way given that agents prefer having any house to having nothing at all. In most applications agents can forgo their assignments and hence this assumption will be satisfied.

housing markets (Shapley and Scarf (1974)).[4] This class differs from house allocation problems in only one point: In house allocation problems agents collectively own a set of houses whereas in the housing markets each agent owns a particular house. Roth and Postlewaite (1977) show that whenever preferences are strict, there is a unique matching in the *core* of each housing market, and moreover this matching coincides with the unique *competitive allocation*. The following mechanism is a counterpart to the *core* in the context of *house allocation problems*: For each house allocation problem randomly choose an endowment matching with uniform distribution, and select the core (or equivalently the competitive allocation) of the induced housing market. We refer to this mechanism as the *core from random endowments*. Our main result is that the *core from random endowments* is equivalent to the *random serial dictatorship*. That is, for all house allocation problems both mechanisms select precisely the same lottery over matchings and thus they are two different formulations of the same lottery mechanism.

Our analysis has two by-products that may be of independent interest. First, we develop some tools associated with *Gale's top trading cycles algorithm* (the algorithm that determines the unique matching in the core of a housing market), that improve our understanding of it and are essential for the proof of our main result. We believe that these tools might be useful in other applications. Second, we obtain a characterization of Pareto efficient matching mechanisms: A matching mechanism is Pareto efficient if and only if it is a *serial dictatorship* (Satterthwaite and Sonnenschein (1981)).

2. BASIC MODELS

2.1. *House Allocation Problems*

Consider the following class of problems: there is a group of n agents who collectively own n indivisible objects, say houses. Each agent has use for one and only one house and each agent has preferences over the set of houses. Formally, a *house allocation problem* is a triple (A, H, P). The first component $A = \{a_1, a_2, \dots, a_n\}$ is a finite set of agents. The second component $H = \{h_1, h_2, \dots, h_n\}$ is a finite set of houses. Note that $|A| = |H| = n$. The last component $P = (P_{a_1}, P_{a_2}, \dots, P_{a_n})$ is a list of preference relations, each preference relation being a strict preference on the set of houses H. Let R_a denote the "at least as good as" relation associated with the preference relation P_a for all $a \in A$. That is, for all h, $h' \in H$ we have hR_ah' if and only if hP_ah' or $h = h'$. Throughout the paper we fix A, H, P and suppress them whenever we can so as to simplify the notation.

The *choice* $X_a(H')$ of an agent $a \in A$ from a set of houses $H' \subseteq H$ is the best house among H'. That is,

$$X_a(H') = h' \quad \Leftrightarrow \quad h' \in H' \text{ and } h'P_ah \text{ for all } h \in H' \setminus \{h'\}.$$

A (*house*) *matching* μ is a bijection from A to H. For all $a \in A$, we refer to $\mu(a)$ as the *assignment* of agent a under μ. Let $\mathcal{M}$ be the set of all matchings. Note that there are $n!$ matchings, that is $|\mathcal{M}| = n!$.

Agent a's preference relation P_a initially defined over H, is extended to the set of matchings in the following natural way: agent a prefers the matching μ to the matching μ' if and only if he prefers his assignment under μ to his assignment under μ'. We slightly abuse the notation and also use P_a to denote this extension.

[4]See Moulin (1995) for an extensive analysis of housing markets.

A matching $\mu \in \mathcal{M}$ is *Pareto efficient* if there is no matching $\nu \in \mathcal{M}$ such that $\nu(a) R_a \mu(a)$ for all $a \in A$ and $\nu(a) P_a \mu(a)$ for some $a \in A$. Let $\mathcal{E}$ be the set of Pareto efficient matchings.

A *(housing) lottery* m is a probability distribution over matchings, $m = (m_1, m_2, \ldots, m_{n!})$, with $\sum_k m_k = 1$ and $m_k \geq 0$ for all k. We denote the lottery that assigns probability 1 to matching μ by m^{μ}. Let $\Delta\mathcal{M}$ be the set of all lotteries.

REMARK 1: In this paper we do not impose any structure on the preferences over the set of lotteries. What we gain by this generality is that our results are valid under any such structure. What we lose is that we cannot study *ex-ante Pareto efficiency* of the mechanisms that we introduce in Section 2.3. However an impossibility result due to Zhou (1990) makes the limitations of imposing this requirement very clear. We elaborate on this point in Section 2.3.

2.2. *Housing Markets*

Next we define a class of problems, namely *housing markets* (Shapley and Scarf (1974)), which differ from *house allocation problems* in only one respect. There is a group of n agents, each of whom owns one house and has preferences over the set of houses. An allocation is a permutation of the houses among the agents. Formally, a *housing market* is a four-tuple (A, H, P, μ). The first component $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set of agents. The second component $H = \{h_1, h_2, \ldots, h_n\}$ is a finite set of houses. The third component $P = (P_{a_1}, P_{a_2}, \ldots, P_{a_n})$ is a list of preference relations, where the preference relation P_a of each agent a is a strict preference on the set of houses H. Finally μ is a matching, which is interpreted as an *(initial) endowment*. Thus, the only difference between the class of house allocation problems and the class of housing markets is that in the former agents collectively own a set of houses, whereas in the latter each agent owns a particular house. In this model too, we extend preferences to the set of matchings in a natural way. We also consider the case where A, H, P are fixed. Hence, it suffices to specify an endowment matching to define a housing market.

The following concept is central to this paper:

A matching η is in the *core* (*defined by weak domination*) of the housing market μ if there is no coalition $T \subseteq A$ and matching ν such that:

(i) $\nu(a) \in \{h \in H\colon h = \mu(a') \text{ for some } a' \in T\}$ for all $a \in T$,

(ii) $\nu(a) R_a \eta(a)$ for all $a \in T$,

(iii) $\nu(a) P_a \eta(a)$ for some $a \in T$.

Roth and Postlewaite (1977) show that there is a unique matching in the core of each housing market. We denote the unique matching in the core of the housing market μ by $\mathcal{C}(\mu)$. There is a well-known algorithm, namely *Gale's top trading cycles algorithm*, to determine this matching. This algorithm is essential to this paper and we analyze it in Section 3.

2.3. *Matching and Lottery Mechanisms*

A *matching mechanism* is a systematic procedure to select a matching for each house allocation problem. Similarly a *lottery mechanism* is a systematic procedure to select a lottery for each house allocation problem. Next we introduce two classes of matching mechanisms and two lottery mechanisms. We need more notation for that.

Let f: $\{1,2,\ldots,n\} \to A$ be a bijection and $\mathscr{F}$ be the class of all such bijections. Note that $|\mathscr{F}| = n!$. We refer to each of these bijections as an *ordering* of the agents. That is, for any $f \in \mathscr{F}$, agent $f(1)$ is first, agent $f(2)$ is second, and so on.

Given any ordering $f \in \mathscr{F}$ of the agents, define the *simple serial dictatorship induced by* f, φ^f as

$$\begin{aligned}
\varphi^f(f(1)) &= X_{f(1)}(H),\\
\varphi^f(f(2)) &= X_{f(2)}(H \setminus \{\varphi^f(f(1))\}),\\
&\vdots \qquad\qquad \vdots\\
\varphi^f(f(i)) &= X_{f(i)}\left(H \setminus \bigcup_{j=1}^{i-1} \{\varphi^f(f(j))\}\right),\\
&\vdots \qquad\qquad \vdots\\
\varphi^f(f(n)) &= X_{f(n)}\left(H \setminus \bigcup_{j=1}^{n-1} \{\varphi^f(f(j))\}\right).
\end{aligned}$$

That is, the agent who is first gets his top choice, the agent who is second gets his top choice among those remaining, and so on. We have $n!$ simple serial dictatorships, each of which is induced by a different ordering of the agents. Of course some of these mechanisms may select the same matching. Let $\mathscr{F}^{\eta}$ denote the set of orderings for which the induced simple serial dictatorship selects $\eta \in \mathscr{M}$. Formally, $\mathscr{F}^{\eta} = \{f \in \mathscr{F}\colon \varphi^f = \eta\}$.

Any simple serial dictatorship φ^f does not treat agents symmetrically. Agent $f(1)$ always gets his top choice whereas agent $f(n)$ gets whatever is remaining after everyone else has chosen. For that reason, simple serial dictatorships are not considered very desirable. On the other hand, the following lottery mechanism and its variants are commonly used in real life applications.[5]

Define the *random serial dictatorship*, ψ^{rsd}, as

$$\psi^{rsd} = \sum_{f \in \mathscr{F}} \frac{1}{n!} m^{\varphi^f}.$$

That is, each simple serial dictatorship is selected with equal probability, or equivalently an ordering is randomly chosen with uniform distribution and the induced simple serial dictatorship is used.

Next we define another class of matching mechanisms. Given any matching $\mu \in \mathscr{M}$ define the *core from assigned endowments* μ, φ^{μ} as $\varphi^{\mu} = \mathscr{C}(\mu)$. That is, φ^{μ} selects the unique matching in the core for the housing market defined by the endowment matching μ. We have $n!$ different matchings and hence $n!$ such matching mechanisms. Note that different endowment matchings might lead to the same matching η in the core. Let $\mathscr{M}^{\eta}$ be the set of endowment matchings for which η is the resulting matching in the core. Formally, $\mathscr{M}^{\eta} = \{\mu \in \mathscr{M}\colon \varphi^{\mu} = \eta\}$.

Obviously these mechanisms, which are central to housing markets, are somewhat artificial when they are used for house allocation problems. We do not try to justify them. We instead use them to define the following lottery mechanism. The *core from random*

[5] Some examples include graduate housing at Stanford University, University of Michigan, University of Rochester, undergraduate housing at Carnegie Mellon University, University of Michigan, and allocation of clinical positions at Northwestern College of Chiropractic in Minnesota.

endowments, ψ^{cre} is defined as

$$\psi^{cre} = \sum_{\mu \in \mathcal{M}} \frac{1}{n!} m^{\varphi^{\mu}}.$$

That is, each φ^{μ} is chosen with equal probability, or equivalently endowments are randomly chosen with uniform distribution and the core of the associated housing market is selected. The only difference between the housing markets and the house allocation problems is that in the former each agent owns a particular house whereas in the latter the grand coalition owns all the houses. We interpret this as each agent having a right to a uniform distribution over the set of houses and consider the *core from random endowments* to be a counterpart to the core in the context of house allocation problems.

The *core from random endowments* has many appealing properties. It is *ex-post Pareto efficient*, *anonymous*, and *strategy-proof*.[6] Although we do not impose any structure on preferences over lotteries, we know that this mechanism cannot be *ex-ante Pareto efficient* since Zhou (1990) shows that there is no lottery mechanism that is *ex-ante Pareto efficient*, *anonymous*, and *strategy-proof*.[7]

3. GALE'S TOP TRADING CYCLES ALGORITHM

Gale's top trading cycles algorithm can be used to find the unique matching in the core of a housing market. It can be described as follows:

Step 1: At step 1, each agent points to the agent who owns his most preferred house. Since the number of agents is finite, there is at least one cycle (a cycle is an ordered list of agents $\{a'_1, a'_2, \ldots, a'_m\}$ where a'_1 points to a'_2, a'_2 points to $a'_3, \ldots, a'_m$ points to a'_1). In each cycle the corresponding trades are performed and all agents belonging to a cycle are removed together with their assignments. (Note that all of them are assigned their most preferred houses.) If there are remaining agents we go to the next step.

$\vdots \qquad\qquad\qquad\qquad \vdots$

Step t: At step t, each remaining agent points to the agent who owns his most preferred house among those remaining in the market. In each cycle the corresponding trades are performed and all agents belonging to a cycle are removed together with their assignments. If there are remaining agents we go to the next step.

By the finiteness of n, at least one cycle forms at each step so that this algorithm terminates in at most n steps. As we have mentioned, the algorithm is central to this paper and here we further analyze it. We need more notation for that.

Let $\mu \in \mathcal{M}$. We partition the set of agents according to the step in which they belong to a cycle and trade. Let $A(\mu) = \{A^1(\mu), A^2(\mu), \ldots, A^{k_\mu}(\mu)\}$ be this partition, to which we refer as the *cycle structure* for μ.

[6] Here *strategy-proofness* holds regardless of whether the agents report their preferences before or after the lottery stage of the mechanism. See Roth (1982).

[7] On the other hand, if one gives up *strategy-proofness*, then there are lottery mechanisms that satisfy the other two properties. See Hylland and Zeckhauser (1979) and Zhou (1990) for examples of such mechanisms.

Let $H^0 = \varnothing$ and for all $t \in \{1, 2, \ldots, k_\mu\}$ let

$$H^t(\mu) = \{h \in H \colon \mu(a) = h \text{ for some } a \in A^t\}.$$

That is, $H^t(\mu)$ is the set of houses that are owned by agents in $A^t(\mu)$ under the endowment matching μ.

Let $\eta = \mathscr{C}(\mu)$. Then by Gale's top trading cycles algorithm we have

$$\forall t \in \{1, 2, \ldots, k_\mu\}, \qquad \forall a \in A^t(\mu), \qquad X_a\left(H \setminus \bigcup_{s=0}^{t-1} H^s(\mu)\right) = \eta(a).$$

Consider any agent in $A^t(\mu)$ at Step $(t-1)$. This agent will take part in a cycle only in the next step. Therefore his favorite house among those left at step $(t-1)$ is either in $H^{t-1}(\mu)$ or in $H^t(\mu)$. Note that there should be at least one agent in $A^t(\mu)$ whose favorite house among those left at step $(t-1)$ is in $H^{t-1}(\mu)$; otherwise agents in $A^t(\mu)$ would form one or several cycles and trade at step $(t-1)$. Therefore we have

$$\forall a \in A^1(\mu), \qquad X_a(H) \in H^1(\mu),$$

$$\forall a \in A^t(\mu), \qquad X_a\left(H \setminus \bigcup_{s=0}^{t-2} H^s(\mu)\right) \in H^{t-1}(\mu) \cup H^t(\mu) \quad (t = 2, \ldots, k_\mu).$$

Based on this observation, for all $t \in \{2, \ldots, k_\mu\}$ we partition the set $A^t(\mu)$ into the sets of *satisfied* agents $S^t(\mu)$ and *unsatisfied* agents $U^t(\mu)$ where

$$S^t(\mu) = \left\{a \in A^t(\mu) \colon X_a\left(H \setminus \bigcup_{s=0}^{t-2} H^s(\mu)\right) \in H^t(\mu)\right\},$$

$$U^t(\mu) = \left\{a \in A^t(\mu) \colon X_a\left(H \setminus \bigcup_{s=0}^{t-2} H^s(\mu)\right) \in H^{t-1}(\mu)\right\}.$$

Note that we have

$$\forall t \in \{2, \ldots, k_\mu\}, \qquad U^t(\mu) \neq \varnothing.$$

At step $(t-1)$ agents in $S^t(\mu)$ point to an agent in $A^t(\mu)$ whereas agents in $U^t(\mu)$ point to an agent in $A^{t-1}(\mu)$. The agents in the latter group only in the next step point to an agent in $A^t(\mu)$ and this allows agents in $A^t(\mu)$ to form one or several cycles. At step $(t-1)$ agents in $A^t(\mu)$ form one or several *chains* each of which are headed by an agent in $U^t(\mu)$ who is possibly followed by agents in $S^t(\mu)$. Formally the *chain structure* of $A^t(\mu)$ is a partition $\{C_1^t(\mu), C_2^t(\mu), \ldots, C_{r_t}^t(\mu)\}$ where each *chain* $C_i^t(\mu) = \{a_{i1}^t(\mu), a_{i2}^t(\mu), \ldots, a_{in_i}^t(\mu)\}$ is such that

$$a_{i1}^t(\mu) \in U^t(\mu),$$

$$a_{ij}^t(\mu) \in S^t(\mu) \quad \text{for all } j \in \{2, \ldots, n_i\},$$

$$X_{a_{i1}^t(\mu)}\left(H \setminus \bigcup_{s=0}^{t-2} H^s(\mu)\right) \in H^{t-1}(\mu),$$

$$X_{a_{i(j+1)}^t(\mu)}\left(H \setminus \bigcup_{s=0}^{t-2} H^s(\mu)\right) = \mu\left(a_{ij}^t(\mu)\right) \quad \text{for all } j \in \{1, 2, \ldots, n_i - 1\}.$$

We refer to agent $a_{i1}^t(\mu)$ as the *head* and agent $a_{in_i}^t(\mu)$ as the *tail* of the chain $C_i^t(\mu)$. Let $T^t(\mu)$ denote the set of tails in $A^t(\mu)$. That is, $T^t(\mu)=\bigcup_{i=1}^{r_t}\{a_{in_i}^t(\mu)\}$. Note that at step t (the agents in $A^{t-1}(\mu)$ have already left together with the set of houses $H^{t-1}(\mu)$ and only then), each agent in $U^t(\mu)$ points to one of these tails (and each of them points to a different one), which in turn converts these chains into one or several cycles. Note that, unsatisfied agents pointing to the tail of a chain does not in general create a cycle; it only does here because these agents are leaving the market at step t.

4. RESULTS

4.1. *Simple Serial Dictatorships and Cores from Assigned Endowments*

Our first result establishes a strong link between the simple serial dictatorships and the cores from assigned endowments. It shows that members of both classes select Pareto efficient matchings with the same "frequency," in the sense that for each Pareto efficient matching the numbers of mechanisms from each class that select it are the same.

THEOREM 1: *For any house allocation problem, the number of simple serial dictatorships selecting a Pareto efficient matching η is the same as the number of cores from assigned endowments selecting η. That is, for all $\eta\in\mathscr{E}$, we have $|\mathscr{M}^\eta|=|\mathscr{F}^\eta|$.*

The proof of this result involves constructing a mapping $f\colon \mathscr{M}^\eta\to\mathscr{F}^\eta$ and showing that this mapping is one-to-one and onto. Construction of this mapping heavily relies on the tools that we introduced in Section 3 concerning Gale's top trading cycles algorithm: Given the cycle structure $A(\mu)$ of μ, the mapping $f(\mu)$ orders agents in $A^1(\mu)$ before agents in $A^2(\mu)$, agents in $A^2(\mu)$ before agents in $A^3(\mu)$, and so on. Agents in $A^1(\mu)$ are ordered based on the index of their endowments. Chains in $A^t(\mu)$, $t>1$, are ordered based on the index of the endowments of the tails, and agents within each chain are ordered based on their order in the chain, starting with the head. It turns out that any two matchings that lead to the same ordering under this mapping should have the same cycle structure, which in turn implies the equivalence of these two matchings.

What about matchings that are not Pareto efficient? Can they be selected by members of either classes? As the following lemma states, the answer is negative. Simple serial dictatorships and cores from assigned endowments always select Pareto efficient matchings and conversely, for each Pareto efficient matching there is at least one mechanism from each class that selects it. We need the following additional notation to present this result. Define

$$\varphi^{\mathscr{F}}=\{\eta\in\mathscr{M}\colon \varphi^f=\eta \text{ for some } f\in\mathscr{F}\},$$

$$\varphi^{\mathscr{M}}=\{\eta\in\mathscr{M}\colon \varphi^\mu=\eta \text{ for some } \mu\in\mathscr{M}\}.$$

LEMMA 1: $\varphi^{\mathscr{M}}=\varphi^{\mathscr{F}}=\mathscr{E}$.

REMARK 2: Satterthwaite and Sonnenschein (1981) extend the class of simple serial dictatorships by allowing the orderings to be preference dependent. They refer to members of this class as *serial dictatorships*. An immediate corollary of Lemma 1 is that a matching mechanism is Pareto efficient if and only if it is a serial dictatorship. Moreover a matching mechanism is Pareto efficient if and only if it is the core, where the endowment matching is obtained via an arbitrary matching mechanism. Therefore, we obtain a dual characterization of Pareto efficient matching mechanisms.

4.2. *Random Serial Dictatorship and the Core from Random Endowments*

Now we are ready to establish a strong relation between the two lottery mechanisms we study.

THEOREM 2: *The random serial dictatorship is precisely the same lottery mechanism as the core from random endowments. That is,* $\psi^{rsd} = \psi^{cre}$.

PROOF: We have $n!$ *simple serial dictatorships* and $n!$ *cores from assigned endowments*. By Lemma 1 the members of both classes select Pareto efficient matchings and by Theorem 1 the number of *simple serial dictatorships* selecting a particular Pareto efficient matching η is the same as the number of *cores from assigned endowments* selecting η. Therefore *random serial dictatorship* which randomly selects a *simple serial dictatorship* with uniform distribution leads to the same lottery as the *core from random endowments* which randomly selects a *core from assigned endowment* with uniform distribution. *Q.E.D.*

As we have already mentioned, the *random serial dictatorship* is widely used in real life applications of house allocation problems, mostly due to its computational ease.[8] We believe this result provides more justification to its use.

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APPENDIX

PROOF OF LEMMA 1: We will prove the lemma via three claims.

CLAIM 1: $\varphi^{\mathcal{M}} \subseteq \varphi^{\mathcal{F}}$.

[8] In many real life applications there are several types of housing and agents rank types of houses rather than houses themselves. We can fit such applications to our model by assuming that agents are indifferent between houses of the same type (or two rooms in the same dormitory) and they have strict preferences over types of houses. The *random serial dictatorship* naturally extends to this more general model. Extending the *core from random endowments* is not trivial since the core of a housing market may be empty for this case. One natural extension is as follows: Order the houses of the same type in an exogenous way; break the indifference in the preferences based on this ordering; randomly choose the endowments using the uniform distribution; and select the core of the induced housing market (with constructed strict preferences). Our equivalence result extends to this more general model in a direct way.

Proof of Claim 1: Let $\eta \in \varphi^{\mathscr{M}}$. Then there exists $\mu \in \mathscr{M}$ with $\varphi^{\mu} = \eta$. Let $A(\mu) = \{A^1(\mu), A^2(\mu), \dots, A^{k_\mu}(\mu)\}$ be the cycle structure of μ. By Gale's top trading cycles algorithm we have

$$\forall t \in \{1,2,\dots,k_\mu\}, \quad \forall a \in A^t(\mu), \quad X_a\left(H \setminus \bigcup_{s=0}^{t-1} H^s(\mu)\right) = \eta(a).$$

Let f: $\{1,2,\dots,n\} \to A$ be the ordering such that

$$\forall t, \bar{t} \in \{1,2,\dots,k_\mu\}, \quad \forall a \in A^t(\mu), \quad \forall \bar{a} \in A^{\bar{t}}(\mu), \quad t < \bar{t} \Rightarrow f^{-1}(a) < f^{-1}(\bar{a}).$$

That is, f orders agents in $A^1(\mu)$ before agents in $A^2(\mu)$; agents in $A^2(\mu)$ before agents in $A^3(\mu)$ and so on. We will show by induction on i that for all $i \in \{1,2,\dots,n\}$ we have $\varphi^f(f(i)) = \eta(i)$.

By Gale's top trading cycles algorithm and the construction of f, we have $\varphi^f(f(1)) = X_{f(1)}(H) = \eta(f(1))$. Next suppose $\varphi^f(f(j)) = \eta(f(j))$ for all $j \in \{1,2,\dots,i-1\}$ where $2 \le i \le n$. Let $f(i) \in A^t(\mu)$. We have the following:

1. $X_{f(i)}(H \setminus \bigcup_{s=0}^{t-1} H^s(\mu)) = \eta(f(i))$ by Gale's top trading cycles algorithm;
2. $\bigcup_{s=0}^{t-1} H^s(\mu) \subseteq \bigcup_{j=1}^{i-1} \eta(f(j))$ by the construction of f, and hence $H \setminus \bigcup_{j=1}^{i-1} \eta(f(j)) \subseteq H \setminus \bigcup_{s=0}^{t-1} H^s(\mu)$;
3. $\eta(f(i)) \in H \setminus \bigcup_{j=1}^{i-1} \eta(f(j))$;

which imply $X_{f(i)}(H \setminus \bigcup_{j=1}^{i-1} \eta(f(j))) = \eta(f(i))$ and therefore

$$\varphi^f(f(i)) = X_{f(i)}\left(H \setminus \bigcup_{j=1}^{i-1} \varphi^f(f(j))\right) = X_{f(i)}\left(H \setminus \bigcup_{j=1}^{i-1} \eta(f(j))\right) = \eta(f(i)).$$

Hence by induction we have $\varphi^f = \eta$, completing the proof of Claim 1.

Claim 2: $\varphi^{\mathscr{F}} \subseteq \mathscr{E}$.

Proof of Claim 2: Let $\eta \in \varphi^{\mathscr{F}}$. Then there exists $f \in \mathscr{F}$ such that $\varphi^f = \eta$. Let $\nu \in \mathscr{M}$ be such that $\nu(a) R_a \eta(a)$ for all $a \in A$. We will show that for all $i \in \{1,2,\dots,n\}$ we have $\nu(i) = \eta(i)$, by induction on i which in turn will prove that η cannot be Pareto dominated.

Consider agent $f(1)$. We have $\varphi^f(f(1)) = \eta(f(1)) = X_{f(1)}(H)$ and therefore $\eta(f(1)) R_{f(1)} \nu(f(1))$. This, together with the relation $\nu(f(1)) R_{f(1)} \eta(f(1))$ and preferences being strict, imply that $\nu(f(1)) = \eta(f(1))$.

Next suppose $\nu(f(j)) = \eta(f(j))$ for all $j \in \{1,2,\dots,i-1\}$ where $2 \le i \le n$. We want to show that $\nu(f(i)) = \eta(f(i))$. We have

$$\begin{aligned}\varphi^f(f(i)) = \eta(f(i)) &= X_{f(i)}\left(H \setminus \bigcup_{s=1}^{i-1} \varphi^f(f(s))\right)\\ &= X_{f(i)}\left(H \setminus \bigcup_{s=1}^{i-1} \eta(f(s))\right) = X_{f(i)}\left(H \setminus \bigcup_{s=1}^{i-1} \nu(f(s))\right)\end{aligned}$$

as well as $\nu(f(i)) \in H \setminus \bigcup_{s=1}^{i-1} \nu(f(s))$ and therefore $\eta(f(i)) R_{f(i)} \nu(f(i))$. This, together with the relation $\nu(f(i)) R_{f(i)} \eta(f(i))$ and preferences being strict imply that $\nu(f(i)) = \eta(f(i))$. Hence by induction we have $\nu = \eta$ showing $\eta \in \mathscr{E}$. This completes the proof of Claim 2.

Claim 3: $\mathscr{E} \subseteq \varphi^{\mathscr{M}}$.

Proof of Claim 3: Let $\eta \in \mathscr{E}$. Consider the matching mechanism φ^{η}. We have $\varphi^{\eta}(a) R_a \eta(a)$ for all $a \in A$ since $\varphi^{\eta} = \mathscr{E}(\eta)$. Therefore the relation $\eta \in \mathscr{E}$ together with the preferences being strict imply $\varphi^{\eta}=\eta$ which in turn implies $\eta \in \varphi^{\mathscr{M}}$, completing the proof of Claim 3. Claims 1–3 complete the proof of the lemma. *Q.E.D.*

PROOF OF THEOREM 1: Let $\eta \in \mathscr{E}$. We will prove the theorem by constructing a mapping from $\mathscr{M}^{\eta}$ into $\mathscr{F}^{\eta}$ and showing that it is *one-to-one* and *onto*.

For any $\mu \in \mathscr{M}^{\eta}$, the ordering $f(\mu)$ is obtained as follows:

1. Find the cycle structure $A(\mu)=\{A^1(\mu), A^2(\mu), \ldots, A^{k_\mu}(\mu)\}$ for μ.
2. For all $t \in \{2,\ldots,k_\mu\}$ partition $A^t(\mu)$ into its chains.
3. Order the agents in $A^1(\mu)$ based on the index of their endowments, starting with the agent whose house has the smallest index. (Recall that the endowment of agent $a \in A^1(\mu)$ is $\mu(a)$.)
4. Order the agents in $A^t(\mu)$ $(t=2,\ldots,k_\mu)$ as follows:

(a) Order the agents in the same chain subsequently based on their order in the chain, starting with the head.

(b) Order the chains based on the index of the endowments of the tails of the chains (starting with the chain whose tail has the house with the smallest index).

5. Order the agents in $A^t(\mu)$ before the agents in $A^{t+1}(\mu)$ $(t=1,2,\ldots,k_\mu-1)$. That is, agents in $A^1(\mu)$ are ordered before agents in $A^2(\mu)$; agents in $A^2(\mu)$ are ordered before agents in $A^3(\mu)$ and so on.

This construction uniquely determines an ordering. We want to show that f is a *one-to-one* and *onto* mapping from $\mathscr{M}^{\eta}$ into $\mathscr{F}^{\eta}$. We will establish this in three steps.

STEP 1: *The range of the mapping f is contained in $\mathscr{F}^{\eta}$. That is, $\varphi^{f(\mu)}=\eta$ for all $\mu \in \mathscr{M}^{\eta}$.*

PROOF OF STEP 1: Let $\mu \in \mathscr{M}^{\eta}$. We have $\varphi^{\mu}=\eta$. Therefore, by Gale's top trading cycles algorithm

$$\forall t \in \{1,2,\ldots,k_\mu\}, \qquad \forall a \in A^t(\mu), \qquad X_a\left(H \Big\backslash \bigcup_{s=0}^{t-1} H^s(\mu)\right)=\eta(a).$$

Moreover by construction $f(\mu)$ orders agents in $A^1(\mu)$ before the agents in $A^2(\mu)$, agents in $A^2(\mu)$ before the agents in $A^3(\mu)$, and so on. Hence the simple serial dictatorship associated with $f(\mu)$, namely $\varphi^{f(\mu)}$, assigns each agent $a \in A$ the house $\eta(a)$. This completes the proof of Step 1. (See the proof of Claim 1 in Lemma 1 for a rigorous proof of this last statement.)

STEP 2: *$f(\mu)$ is a one-to-one mapping. That is,*

$$\forall \mu, \bar{\mu} \in \mathscr{M}^{\eta}, \qquad f(\bar{\mu})=f(\mu) \Rightarrow \bar{\mu}=\mu.$$

PROOF OF STEP 2: We will prove Step 2 via two claims.

CLAIM 1: $\forall \mu,\ \bar{\mu} \in \mathscr{M}^{\eta},\ \ f(\bar{\mu})=f(\mu) \Rightarrow A(\bar{\mu})=A(\mu)$.

PROOF OF CLAIM 1: Let μ, $\overline{\mu} \in \mathscr{M}^{\eta}$. Without loss of generality assume $f(\overline{\mu}) = f(\mu) = f$ orders the agents as $a_1, a_2, \dots, a_n$. Let

$$A(\mu) = \Big\{ \underbrace{\{a_1, \dots, a_{m_1}\}}_{A^1}, \underbrace{\{a_{m_1+1}, \dots, a_{m_2}\}}_{A^2}, \dots$$
$$\dots, \underbrace{\{a_{m_{t-1}+1}, \dots, a_{m_t}\}}_{A^t}, \dots, \underbrace{\{a_{m_{k-1}+1}, \dots, a_n\}}_{A^k} \Big\},$$
$$A(\overline{\mu}) = \Big\{ \underbrace{\{a_1, \dots, a_{\overline{m}_1}\}}_{\overline{A}^1}, \underbrace{\{a_{\overline{m}_1+1}, \dots, a_{\overline{m}_2}\}}_{\overline{A}^2}, \dots$$
$$\dots, \underbrace{\{a_{\overline{m}_{t-1}+1}, \dots, a_{\overline{m}_t}\}}_{\overline{A}^t}, \dots, \underbrace{\{a_{\overline{m}_{\overline{k}-1}+1}, \dots, a_n\}}_{\overline{A}^{\overline{k}}} \Big\}.$$

We want to show that $\overline{k} = k$ and $\overline{A}^t = A^t$ for all $t \in \{1, 2, \dots, k\}$. We proceed by induction.

Suppose $\overline{A}^1 \neq A^1$. Without loss of generality suppose that $\overline{m}_1 < m_1$. Then we have $a_{\overline{m}_1+1} \in A^1$. Moreover as agent $a_{\overline{m}_1+1}$ is ordered first in $\overline{A}^2$, she is also ordered first among the agents in her chain and hence by the construction of $f(\overline{\mu})$ we must have $a_{\overline{m}_1+1} \in U^2(\overline{\mu})$. Therefore $X_{a_{\overline{m}_1+1}}(H) \in H^1(\overline{\mu})$. In addition the relations $a_{\overline{m}_1+1} \in \overline{A}^2$ and $\overline{\mu} \in \mathscr{M}^{\eta}$ imply $\eta(a_{\overline{m}_1+1}) \in H^2(\overline{\mu})$ and hence $X_{a_{\overline{m}_1+1}}(H) \neq \eta(a_{\overline{m}_1+1})$. But we also have $a_{\overline{m}_1+1} \in A^1$ and $\mu \in \mathscr{M}^{\eta}$, contradicting this relation. Therefore $\overline{A}^1 = A^1$.

Next suppose $\overline{A}^r = A^r$ for all $r \in \{1, 2, \dots, t-1\}$ where $2 \leq t \leq \min\{k, \overline{k}\}$. We have $\overline{m}_{t-1} = m_{t-1}$ by the induction hypothesis. We want to show that this implies $\overline{A}^t = A^t$. Suppose that is not the case. Without loss of generality suppose that $\overline{m}_t < m_t$. Then we have $a_{\overline{m}_t+1} \in A^t$. Moreover as agent $a_{\overline{m}_t+1}$ is ordered first among those in $\overline{A}^{t+1}$, he is also ordered first among those agents in his chain and hence by the construction of $f(\overline{\mu})$ we must have $a_{\overline{m}_t+1} \in U^{t+1}(\overline{\mu})$. Therefore $X_{a_{\overline{m}_t+1}}(H \setminus \bigcup_{s=1}^{t-1} H^s(\overline{\mu})) \in H^t(\overline{\mu})$. In addition the relations $a_{\overline{m}_t+1} \in \overline{A}^{t+1}$ and $\overline{\mu} \in \mathscr{M}^{\eta}$ imply $\eta(a_{\overline{m}_t+1}) \in H^{t+1}(\overline{\mu})$ and hence $X_{a_{\overline{m}_t+1}}(H \setminus \bigcup_{s=1}^{t-1} H^s(\overline{\mu})) \neq \eta(a_{\overline{m}_t+1})$. However the induction hypothesis together with μ, $\overline{\mu} \in \mathscr{M}^{\eta}$ ensures that $H^s(\overline{\mu}) = H^s(\mu)$ for all $s \in \{1, 2, \dots, t-1\}$ and therefore $X_{a_{\overline{m}_t+1}}(H \setminus \bigcup_{s=1}^{t-1} H^s(\mu)) \neq \eta(a_{\overline{m}_t+1})$, contradicting $a_{\overline{m}_t+1} \in A^t$. Therefore $\overline{A}^t = A^t$. This also proves that $\overline{k} = k$ and hence $A(\overline{\mu}) = A(\mu)$ by induction, completing the proof of Claim 1.

CLAIM 2: *Suppose μ, $\overline{\mu} \in \mathscr{M}^{\eta}$ is such that $A(\overline{\mu}) = A(\mu)$. Then*

$$f(\overline{\mu}) = f(\mu) \Rightarrow \overline{\mu} = \mu.$$

PROOF OF CLAIM 2: Let μ, $\overline{\mu} \in \mathscr{M}^{\eta}$ be such that $A(\overline{\mu}) = A(\mu) = \{A^1, A^2, \dots, A^k\}$. Suppose $f(\overline{\mu}) = f(\mu) = f$. Then we have $H^t(\overline{\mu}) = H^t(\mu)$ for all $t \in \{1, 2, \dots, k\}$. We will show

$$\forall t \in \{1, 2, \dots, k\}, \quad \forall a \in A^t, \qquad \overline{\mu}(a) = \mu(a)$$

by induction on t.

Consider agents in A^1. We have $H^1(\overline{\mu}) = H^1(\mu)$. By construction f orders agents in $A^1(\mu)$ based on the index of their endowments. Therefore the relation $f(\overline{\mu}) = f(\mu)$ implies that $\overline{\mu}(a) = \mu(a)$ for all $a \in A^1$.

Next suppose

$$\forall r \in \{1, 2, \dots, t-1\}, \qquad \forall a \in A^r, \qquad \overline{\mu}(a) = \mu(a).$$

We want to show that $\overline{\mu}(a) = \mu(a)$ for all $a \in A^t$. We have $H^r(\overline{\mu}) = H^r(\mu)$ for all $r \in \{1, \dots, t-1\}$. Therefore

$$U^t(\overline{\mu}) = \{a \in A^t \colon X_a(H \setminus \textstyle\bigcup_{s=0}^{t-2} H^s(\overline{\mu})) \in H^{t-1}(\overline{\mu})\}$$
$$= \{a \in A^t \colon X_a(H \setminus \textstyle\bigcup_{s=0}^{t-2} H^s(\mu)) \in H^{t-1}(\mu)\} = U^t(\mu),$$

and also

$$S^t(\bar{\mu}) = A^t \setminus U^t(\bar{\mu}) = A^t \setminus U^t(\mu) = S^t(\mu).$$

These relations together with the relation $f(\bar{\mu}) = f(\mu)$ and the construction of f (item 4) imply that we have the same chain structure for $\bar{\mu}$ and μ. (Recall that f orders agents in a chain subsequently based on their order in the chain, starting with the head of the chain who is the only member of the chain that is an element of U^t. Therefore for a given ordering f, the set of agents in U^t uniquely determines the chain structure for A^t.) Let this common chain structure be $\{C_1^t, C_2^t, \dots, C_m^t\}$. Here for all $i \in \{1, 2, \dots, m\}$, we have $C_i^t = \{a_{i1}^t, \dots, a_{in_i}^t\}$ with $a_{i1}^t \in U^t$, and $a_{ij}^t \in S^t$ for all $j \in \{2, \dots, n_i\}$. Moreover for all $i \in \{1, \dots, m\}$, and all $j \in \{1, \dots, n_i - 1\}$,

$$\mu(a_{ij}^t) = X_{a_{i(j+1)}^t}\left(H \setminus \bigcup_{s=0}^{t-2} H^s(\mu)\right) = X_{a_{i(j+1)}^t}\left(H \setminus \bigcup_{s=0}^{t-2} H^s(\bar{\mu})\right) = \bar{\mu}(a_{ij}^t)$$

by the definition of a chain and the induction hypothesis. But since the chain structure is the same for matchings $\bar{\mu}$ and μ, the set of tails is also the same for both matchings. That is, $T^t(\bar{\mu}) = T^t(\mu) = T^t$ and therefore the last relation is equivalent to $\bar{\mu}(a) = \mu(a)$ for all $a \in A^t \setminus T^t$. Finally by the construction of f (item 4) tails of chains are ordered based on their endowments and moreover

$$\begin{aligned} &\{h \in H\colon \bar{\mu}(a) = h \text{ for some } a \in T^t\} \\ &\quad = H^t \setminus \{h \in H\colon \bar{\mu}(a) = h \text{ for some } a \in A^t \setminus T^t\} \\ &\quad = H^t \setminus \{h \in H\colon \mu(a) = h \text{ for some } a \in A^t \setminus T^t\} \\ &\quad = \{h \in H\colon \mu(a) = h \text{ for some } a \in T^t\}. \end{aligned}$$

That is, the set of agents $T^t \subseteq A^t$ collectively own the same set of houses under endowments μ and $\bar{\mu}$ and therefore the relation $f(\bar{\mu}) = f(\mu)$ implies $\bar{\mu}(a) = \mu(a)$ for all $a \in T^t$ and hence $\bar{\mu}(a) = \mu(a)$ for all $a \in A^t$. Therefore by induction we have $\bar{\mu} = \mu$, completing the proof of Claim 2.

Claim 1 together with Claim 2 prove that f is a *one-to-one* mapping from $\mathscr{M}^\eta$ into $\mathscr{F}^\eta$, completing the proof of Step 2.

STEP 3: *$f(\mu)$ is an onto mapping.*

PROOF OF STEP 3: By Step 1 and Step 2 we have $|\mathscr{F}^\eta| \geq |\mathscr{M}^\eta|$ for all $\eta \in \mathscr{E}$; therefore

$$\sum_{\eta \in \mathscr{E}} |\mathscr{F}^\eta| \geq \sum_{\eta \in \mathscr{E}} |\mathscr{M}^\eta|$$

and hence by Lemma 1

$$\sum_{\eta \in \varphi^{\mathscr{F}}} |\mathscr{F}^\eta| \geq \sum_{\eta \in \varphi^{\mathscr{M}}} |\mathscr{M}^\eta|.$$

But the left-hand side of the inequality is equal to the number of orderings and the right-hand side of it is equal to the number of matchings both of which are equal to $n!$. Hence we should have $|\mathscr{M}^\eta| = |\mathscr{F}^\eta|$ for all $\eta \in \mathscr{E}$, completing the proofs of both Step 3 as well as the theorem. *Q.E.D.*

REFERENCES

HYLLAND, A., AND R. ZECKHAUSER (1979): "The Efficient Allocation of Individuals to Positions," *Journal of Political Economy*, 87, 293–314.

MOULIN, H. (1995): *Cooperative Microeconomics. A Game Theoretic Introduction*. Princeton: Princeton University Press.

ROTH, A. E. (1982): "Incentive Compatibility in a Market with Indivisibilities," *Economics Letters*, 9, 127–132.

ROTH, A. E., AND A. POSTLEWAITE (1977): "Weak versus Strong Domination in a Market with Indivisible Goods," *Journal of Mathematical Economics*, 4, 131–137.

SATTERTHWAITE, M. A., AND H. SONNENSCHEIN (1981): "Strategy-Proof Allocation Mechanisms at Differentiable Points," *Review of Economic Studies*, 48, 587–597.

SHAPLEY, L., AND H. SCARF (1974): "On Cores and Indivisibility," *Journal of Mathematical Economics*, 1, 23–28.

ZHOU, L. (1990): "On a Conjecture by Gale about One-Sided Matching Problems," *Journal of Economic Theory*, 52, 123–135.