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A CONSTRUCTIVE PROOF OF A PERMUTATION-BASED GENERALIZATION OF SPERNER'S LEMMA

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In a recent paper, Gale has given an interesting generalization of the KKM lemma in combinatorial topology. We present a similar generalization of Sperner's well-known lemma and give a constructive proof. The argument uses the familiar idea of following simplicial paths in a triangulation. To demonstrate that the algorithm must work, orientation considerations are necessary. Gale's generalized KKM lemma is derived from the main result. A permutation-based generalization of Brouwer's fixed point theorem is also given.

Key words: Sperner's lemma, simplicial pivoting, fixed point algorithms.

1. Introduction

In a recent paper, Gale [6] has given an interesting generalization of the Knaster-Kuratowski-Mazurkiewicz (or the KKM) Lemma in combinatorial topology. It is possible to give a similar generalization of the Sperner's lemma. In this paper we present such a generalization and give a constructive proof. The argument is based on the familiar idea of following simplicial paths in a triangulation. We refer to the book by Todd [11] for a survey on path-following methods. Several combinatorial lemmas of related interest can be found in the papers by Freund [4, 5].

An important feature of the constructive method given in the present paper is that orientation considerations are necessary to demonstrate that the algorithm must terminate in finitely many steps. We refer to [7, 12] for a background on orientation in simplicial pivoting in an abstract setting.

The paper is organized as follows. In Section 2 we introduce the necessary terminology and prove the main result. A constructive method is illustrated in Section 3 by means of an example. We then derive Gale's generalized KKM lemma from the main result. Finally, a permutation-based generalization of Brouwer's fixed point theorem is given.

2. The main result

The $(n-1)$-dimensional unit simplex S^{n-1} is defined by

$$S^{n-1}=\left\{x\in R^n\colon x\geqslant 0,\ \sum_{i=1}^{n} x_i=1\right\}.$$

For $i=1,2,\dots,n$; we define the ith boundary S_i^{n-1} of S^{n-1} by

$$S_i^{n-1}=\{x\in S^{n-1}\colon x_i=0\}.$$

Orientation. Let σ be an $(n-1)$-simplex in S^{n-1} with vertices $v_1,\dots,v_n$. The order $v_1,\dots,v_n$ is said to determine the positive (negative) orientation of σ if

$$\det(v_1,\dots,v_n)$$

is positive (negative).

Let T be a triangulation of S^{n-1} which will be kept fixed throughout this section. A proper labelling ϕ is, by definition, a labelling of the vertices of T with integers from $1,2,\dots,n$ such that no vertex in S_i^{n-1} has label i, for all $i=1,2,\dots,n$. Suppose for each $i=1,2,\dots,n$; we have a labelling ϕ^i of the vertices of T by integers from $1,2,\dots,n$.

Definition. A *pair* (σ,f) consists of an $(n-1)$-simplex σ of T and a one-to-one function f which maps the vertices of σ onto $\{1,2,\dots,n\}$. The pair (σ,f) will be called *completely labelled* (c.l.) if

$$\{\phi^{f(v)}(v)\colon v \text{ a vertex of } \sigma\}=\{1,2,\dots,n\}.$$

Let (σ,f) be a c.l. pair and suppose $v_1,\dots,v_n$ are the vertices of σ. Then (σ,f) will be called *positive* if $v_1,\dots,v_n$ determines the positive (negative) orientation of σ and

$$\phi^{f(v_1)}(v_1),\dots,\phi^{f(v_n)}(v_n) \tag{1}$$

is an even (odd) permutation of $1,2,\dots,n$. Similarly (σ,f) will be called *negative* if $v_1,\dots,v_n$ determines the positive (negative) orientation of σ and the permutation (1) is odd (even).

Induced Orientation. Let σ be an $(n-1)$-simplex of T and let τ be an $(n-2)$-simplex which is a face of σ. Let $v_1,\dots,v_{n-1}$ be the vertices of τ and let $v_1,\dots,v_n$ be the vertices of σ. The ordering $v_1,\dots,v_{n-1}$ is said to determine the positive (negative) induced orientation of τ from σ if $v_1,\dots,v_n$ determines the positive (negative) orientation of σ.

The following properties of induced orientation will be needed.

(i) Let σ be an $(n-1)$-simplex of T and let τ^1 and τ^2 be distinct $(n-2)$-simplices, both of which are faces of σ. Let $v_1,\dots,v_{n-1}$ be the vertices of τ^1 and let $v_1,\dots,v_{n-2}$, v_n be the vertices of τ^2. If $v_1,\dots,v_{n-1}$ determines the positive (negative) induced orientation of τ^1 from σ, then $v_1,\dots,v_{n-2}$, v_n determines the negative (positive) induced orientation of τ^2 from σ.

(ii) Let σ^1, σ^2 be $(n-1)$-simplices of T and let τ be an $(n-2)$-simplex which is a face of both σ^1 and σ^2. Let $v_1,\dots,v_{n-1}$, u be the vertices of σ^1 and let $v_1,\dots,v_{n-1}$, v be the vertices of σ^2. If $v_1,\dots,v_{n-1}$ determines the positive (negative) induced orientation of τ from σ^1, then it determines the negative (positive) induced orientation of τ from σ^2.

Property (i) follows from the fact that the determinant changes sign if two columns are interchanged. To verify (ii), note that u and v must be on the opposite sides of the hyperplane spanned by $v_1, \ldots, v_{n-1}$ and the origin and hence $\det(v_1, \ldots, v_{n-1}, v)$ and $\det(v_1, \ldots, v_{n-1}, u)$ have opposite signs.

We can now state the main result.

Theorem 1. *Let T be a triangulation of S^{n-1} and let ϕ^i, $i=1,2,\ldots,n$ be n proper labellings of the vertices of T. Then the number of positive c.l. pairs exceeds the number of negative c.l. pairs by $n!$. In particular, there are at least $n!$ c.l. pairs.*

We illustrate the theorem by an example. Consider the triangulation of S^2 shown in Figure 1. The (proper) labels ϕ^i, $i=1,2,3$, are indicated as a column vector near each vertex. It can be verified that there are precisely six positive c.l. pairs given by (σ,f^1), (σ,f^2), (τ,f^3), (τ,f^4), (τ,f^5) and (τ,f^6) where σ is the simplex DBC and τ is the simplex ADC. The functions f^i, $i=1,2,\ldots,6$ are conveniently represented by the permutations 123, 132, 123, 132, 231 and 321 respectively. For example, $f^2:\{D,B,C\}\to\{1,2,3\}$ is defined as $f^2(D)=1$, $f^2(B)=3$, $f^2(C)=2$.

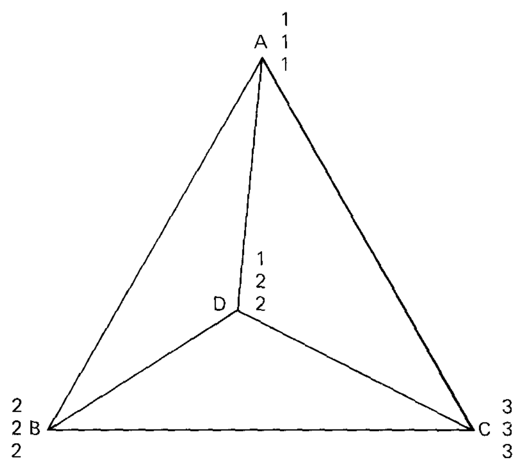

Fig. 1

To prove Theorem 1, we will make use of a combinatorial lemma of Ky Fan. We now quote the lemma from [3, Section 4].

Consider a house which naturally has rooms and doors. Some of the doors are *internal*, i.e., doors connecting two rooms, while the remaining doors are *external*, i.e., connecting a room with the outside of the house. We make the following assumptions:

Each room has 0, 1 or 2 doors. (2)

For every internal door, one side is painted white and the other side black. (3)

For every external door, both sides are
painted with the same colour, white or black. (4)

For any room with two doors, the sides of the two
doors, facing the interior of the room are painted
with different colours, one white and one black. (5)

A room will be considered *good* if it has exactly one door. If the side of the unique door facing the interior of a good room is white (black), we say that it is a good room with a white (black) door. Under this set up we have the following.

Lemma 2 [3, p. 596]. *The number of good rooms with a white door minus the number of good rooms with a black door is equal to the number of white external doors minus the number of black external doors.*

Proof of Theorem 1. We will assume that the triangulation T does not subdivide the boundary simplices S_i^{n-1} of S^{n-1} for any i. This assumption can safely be made, for if not, the simplex can be imbedded in a larger simplex by methods such as in Scarf [10, p. 192].

We now identify the rooms and the doors. Any pair (σ, f) will be called a room. We will say that (τ, g) is a door if conditions (a), (b) are satisfied:

(a) τ is an $(n-1)$-simplex of T and g is a one-to-one function which maps the vertices of τ into $\{1, 2, \ldots, n\}$.

(b) $\{\phi^{g(v)}(v)$: v a vertex of $\tau\} = \{1, 2, \ldots, n-1\}$.

If (σ, f) is a room and (τ, g) a door, we will say that (τ, g) is a door of (σ, f) if τ is a face of σ and if $f(v) = g(v)$ for all vertices v of τ. (For example, in Figure 1, (τ, g) is a door of (σ, f) where σ is the simplex DBC, τ is the simplex DB and the functions f, g are given by $f(D) = 1$, $f(B) = 3$, $f(C) = 2$, $g(B) = 3$, $g(C) = 2$.)

Now we turn to the choice of colours. Suppose (τ, g) is a door of (σ, f). Let $v_1, \ldots, v_{n-1}$ be the vertices of τ. Then (τ, g) will be painted white on the side facing the room (σ, f) if $v_1, \ldots, v_{n-1}$ determines the positive (negative) induced orientation of τ from σ and if the permutation

$$\phi^{g(v_1)}(v_1), \ldots, \phi^{g(v_{n-1})}(v_{n-1})$$

is an even (odd) permutation of $1, 2, \ldots, n-1$. Otherwise it will be painted black on the side facing the room (σ, f).

A door (τ, g) will be said to be external if τ is a boundary $(n-2)$-simplex of S^{n-1}. An external door will be painted with the same colour on both sides.

Note that a room (σ, f) is good if and only if it is a c.l. pair. Also, a good room with a white (black) door is precisely a positive (negative) c.l. pair. Condition (2) is easily verified. Conditions (3) and (5) follow from properties (ii), (i) respectively. Since T does not subdivide the boundary simplices of S^{n-1}, (τ, g) is an external door if and only if τ is S_n^{n-1}. Furthermore there are exactly $n\,!$ external doors and they are all painted white. It follows from Lemma 2 that the number of positive c.l. pairs exceeds the number of negative c.l. pairs by $n!$ and the proof is complete.

3. A constructive method

The proof of Lemma 2 given in [3] is by a pairing process. In particular, the process shows how to locate a good room by initiating a path at an external door. We will not reproduce the description of the pairing process here but we will illustrate how it leads to an algorithm to locate a c.l. pair in our case by an example.

Consider the triangulation of S^2 given in Figure 2. Suppose the (proper) labellings ϕ^i, $i=1,2,3$, are as follows:

	Vertices							
labels	A	B	C	D	E	F	G	H
ϕ^1	1	2	3	2	1	2	1	2
ϕ^2	1	2	3	1	1	1	2	2
ϕ^3	1	2	3	1	3	2	2	1

To initiate the algorithm we consider the simplex ABD and write down the labels of its vertices in the form of a matrix as follows.

A	B	D
1	2	2
1	2	1
1	2	1

Now choose any permutation of 1, 2, 3; say the identity permutation. The corresponding labels constitute the main diagonal of the matrix above. The labels are not all distinct, since the label 1 repeats. We must then remove vertex A and

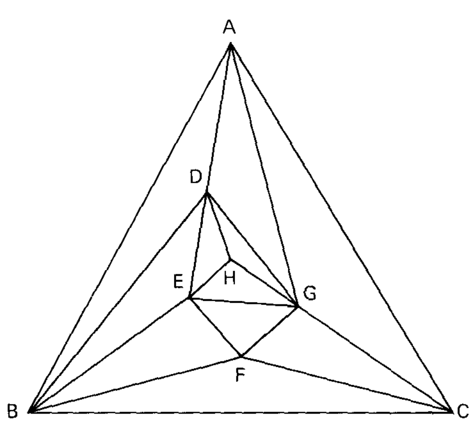

Fig. 2

introduce the vertex E so that we now move to the simplex EBD. Whenever we delete a vertex and introduce a new vertex, the new vertex will be written in the place of the old vertex without changing the positions of the other vertices. This device allows us to consider the same permutation (in this example, the identity permutation) at each stage of the algorithm.

The label-matrix now is

E	B	D
1	2	2
1	2	1
3	2	1

Again, the main diagonal does not carry distinct labels and we must remove vertex D, replacing it by F. It may be verified that by continuing this process, the following path of simplices is followed and we arrive at a c.l. pair as indicated: ABD, EBD, EBF, EGF, EGH, DGH, DEH, DEB, FEB, FEG, HEG, HED, HGD, AGD, AGC, with the final label-matrix:

A	G	C
1	1	3
1	2	3
1	2	3

The reason that the algorithm described by means of the example above must terminate by locating a c.l. pair is as follows. The fact that the algorithm does not get into a loop can be justified by an argument which is now well-known in this area and is commonly known as "the Lemke–Howson argument" or "the ghost story argument", see [8] or [2]. It hinges on the fact that any intermediate state in the algorithm has precisely two neighboring states.

Thus there are only two possibilities. The path initiated at an external door might terminate at a good room, i.e., a c.l. pair or it might lead to another external door. But in the later case, the external door must be of the opposite color (see [3, p. 597]). However, in our situation the external doors are all white. Thus the second possibility is ruled out and the algorithm must locate a c.l. pair. As seen from this argument, orientation considerations are necessary to demonstrate that the algorithm works, unlike in the case of the algorithms due to Cohen [1] for the classical Sperner's lemma, or the one due to Scarf [9].

Note that a different choice of the permutation at the initial stage would have led to a different c.l. pair and hence the algorithm can be used to find $n!$ different c.l. pairs.

It should also be remarked that the path generated by the algorithm can pass through the same simplex more than once (for instance, the simplex EBD in the example), although a different set of positions will appear on the main diagonal of the label-matrix at each visit to the same simplex. This is in contrast with the usual simplicial pivoting algorithms which never visit the same simplex again.

We now derive Gale's generalized KKM lemma from Theorem 1. The proof is similar to that of the classical KKM lemma using Sperner's lemma (see, e.g., [11, p. 10]) and is given here for completeness.

A family of closed sets $\mathscr{C}_1, \ldots, \mathscr{C}_n$ is called a KKM covering if for all nonempty proper subsets α of $\{1, 2, \ldots, n\}$,

$$\bigcap_{j\in\alpha} S_j^{n-1} \subset \bigcup_{j\notin\alpha} \mathscr{C}_j$$

and

$$S^{n-1} \subset \bigcup_{j=1}^{n} \mathscr{C}_j.$$

Theorem 3 [6, p. 63]. *For $i, j = 1, 2, \ldots, n$ let $\mathscr{C}_j^i$ be closed sets such that for each i, $\mathscr{C}_1^i, \ldots, \mathscr{C}_n^i$ is a KKM covering. Then there exists a permutation π of $1, 2, \ldots, n$ and a point $z \in S^{n-1}$ such that*

$$z \in \bigcap_{i=1}^{n} \mathscr{C}^i_{\pi(i)}.$$

Proof. Let T^k, $k = 1, 2, \ldots$, be a sequence of triangulations with mesh $T^k \to 0$. For each $i = 1, 2, \ldots, n$ define a labelling ϕ^i of the vertices of T^k, $k = 1, 2, \ldots$, as follows:

$$\phi^i(v) = \min\{j\colon v \in C_j^i,\ v \notin S_j^{n-1}\}$$

Since $\mathscr{C}_1^i, \ldots, \mathscr{C}_n^i$ is a KKM covering, ϕ^i is a proper labelling, $i = 1, 2, \ldots, n$. It follows by Theorem 1 that there exists an $(n-1)$-simplex σ^k of T^k with vertices $v_1^k, \ldots, v_n^k$ and a permutation π^k of $1, 2, \ldots, n$ such that

$$v_i^k \in \mathscr{C}^i_{\pi^k(i)}, \quad i = 1, 2, \ldots, n.$$

Since mesh $T^k \to 0$, there exists $z \in S^{n-1}$ such that for each i, a subsequence of v_i^k converges to z. Using a subsequence, if necessary, we may assume without loss of generality that π^k is the same for all k and we denote it by π. Since $\mathscr{C}_j^i$ are closed sets, it follows that

$$z \in \mathscr{C}^i_{\pi(i)}, \quad i = 1, 2, \ldots, n,$$

and the proof is complete.

We conclude by giving a permutation-based generalization of Brouwer's fixed point theorem.

Theorem 4. *Let $f^i : S^{n-1} \to S^{n-1}$ be continuous functions for $i = 1, 2, \ldots, n$. Then there exists $z \in S^{n-1}$ and a permutation π of $1, 2, \ldots, n$ such that*

$$z_{\pi(i)} \geqslant f^i_{\pi(i)}(z), \quad i = 1, 2, \ldots, n.$$

Proof. Define

$$\mathscr{C}_j^i = \{x \in S^{n-1}: x_j \geq f_j^i(x)\}, \quad i, j = 1, 2, \ldots, n.$$

Then for each i, $\mathscr{C}_j^i$, $j = 1, 2, \ldots, n$, is a KKM covering and the result follows from Theorem 3.

Brouwer's fixed point theorem follows from Theorem 4 by choosing all functions f^i to be the same, just as Sperner's classical lemma follows from Theorem 1 by choosing all labels to be identical.

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