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On the Complexity of H-Coloring

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Let H be a fixed graph, whose vertices are referred to as 'colors.' An H-coloring of a graph G is an assignment of 'colors' to the vertices of G such that adjacent vertices of G obtain adjacent 'colors.' (An H-coloring of G is just a homomorphism $G \rightarrow H$.) The following *H-coloring problem* has been the object of recent interest:

Instance: A graph G.
Question: Is it possible to H-color the graph G?

H-colorings generalize traditional graph colorings, and are of interest in the study of grammar interpretations. Several authors have studied the complexity of the H-coloring problem for various (families of) fixed graphs H. Since there is an easy H-colorability test when H is bipartite, and since all other examples of the H-colorability problem that were treated (complete graphs, odd cycles, complements of odd cycles, Kneser graphs, etc.) turned out to be NP-complete, the natural conjecture, formulated in several sources (including David Johnson's NP-completeness column), asserts that the H-coloring problem is NP-complete for *any* non-bipartite graph H. We give a proof of this conjecture. © 1990 Academic Press, Inc.

INTRODUCTION

Graph coloring problems arise in various contexts of both applied and theoretical natures [5, 12, 16, 17]. At the same time, k-colorability is one of the basic NP-complete problems. In fact, it is considered 'harder' than other typical NP-complete problems: It is believed that (unless $P = NP$) there does not exist a polynomial approximation algorithm guaranteed to color any graph with at most c times the minimum number of colors,

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for any constant c. This has only been proved for small constants c [4, 6]. Moreover, any *known* polynomial coloring algorithm uses $\Omega(n(\log\log n)^2/(\log n)^2)$ colors on some 3-colorable graph with n vertices [21]. This apparent difficulty of the graph coloring problem is not well understood, and it is reflected also in the more general H-coloring problem studied here. The complexity of the H-coloring problem was investigated by several authors [1, 2, 11, 15, 18, 19], but only special cases were settled; in particular, there is a simple H-colorability test when H is bipartite, and the problem is NP-complete when H is a complete graph, an odd cycle, or a member of a few other very restricted families [1, 11, 18, 19]. We prove that the H-coloring problem is NP-complete for *any* non-bipartite graph H. This was conjectured in [18]; cf. also [19] and [13]. Our proof is interesting not so much for the reductions we use, which are similar to those previously used, but rather for the intricate interplay of the various graphs, some quite complex, which must be employed in these reductions. These complications may help to explain why the problem had previously resisted solution.

Let G and H be graphs. A *homomorphism* $f: G \to H$ is a mapping f of $V(G)$ to $V(H)$ such that $f(g)$, $f(g')$ are adjacent vertices of H whenever g, g' are adjacent vertices of G. Since a homomorphism $c: G \to K_n$ is just an n-coloring of G, the term *H-coloring* of G has been employed to describe a homomorphism $G \to H$. Homomorphisms and H-colorings have been studied in various contexts [1–3, 7–10, 13–15, 17–20]; in particular, for their relation to grammars and interpretations, in [17]. Here we study the *H-coloring problem*, i.e., the decision problem "Is a given graph G H-colorable?" Clearly, each H-coloring problem is in the class NP. It is easy to see that if H is a bipartite graph then G is H-colorable if and only if G is 2-colorable. For some non-bipartite graphs H the H-coloring problem is NP-complete. Obviously, this is the case of K_n-coloring; moreover, C_{2k+1}-coloring is NP-complete according to [18, 19], where several other NP-completeness results of this type were obtained. (Also see [1, 2, 7, 11–15, 20].)

THEOREM 1. *If H is bipartite then the H-coloring problem is in* P. *If H is not bipartite then the H-coloring problem is* NP*-complete.*

THE REDUCTIONS

A. *The Indicator Construction*

Let I be a fixed graph, and let i and j be distinct vertices of I such that some automorphism of I maps i to j and j to i. The indicator construction (with respect to (I, i, j)) transforms a given graph H into the graph H^*

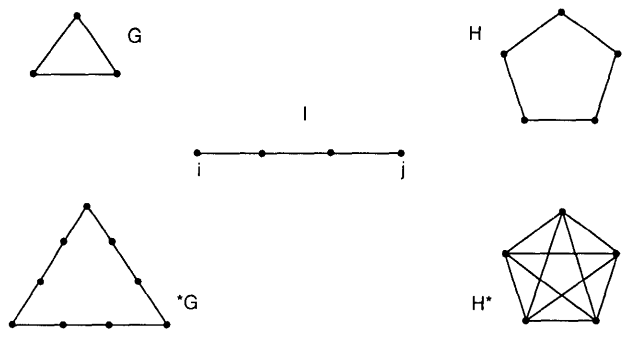


FIG. 1. Example indicator construction.

defined to have the same vertex set as H and to have as the edge set all pairs hh' for which there is a homomorphism of I to H taking i to h and j to h' (cf. Fig. 1). Because of our assumption on I, the edges of H^* will be undirected.

LEMMA 1. *If the H^*-coloring problem is* NP-*complete, then so is the H-coloring problem.*

(In applying Lemma 1 we need to be careful to ensure that H^* has no loops, i.e., that no homomorphism of I to H can map i and j to the same vertex. Otherwise the H^*-coloring problem will not be NP-complete: if H^* has a loop then any G admits an H^*-coloring—map all vertices of G to the vertex with a loop.)

Proof. Given a graph G, let $*G$ be the graph obtained from G by replacing each edge gg' by a disjoint copy of I, identifying i with g and j with g'. It is now easy to see from the definitions that there is a homomorphism $*G \to H$ if and only if there is a homomorphism $G \to H^*$.

Before introducing the next construction we need to review the following concepts [9, 10]: If H is a subgraph of H', then a *retraction* of H' to H is a homomorphism $r: H' \to H$ such that $r(h) = h$ for all vertices h of H. A graph is a *core* (or minimal graph [3]) if it does not admit a retraction to a proper subgraph; equivalently, H is a core if it does not admit a homomorphism to a proper subgraph. It is easy to see [10, 3] that every graph H' contains a unique (up to isomorphism) subgraph H which is a core and admits a retraction $r: H' \to H$; we call H the core of H'. Note that if H is a core of H', then there are homomorphisms $H \to H'$ (the inclusion) and $H' \to H$ (a retraction); thus G is H'-colorable if and only if it is H-colorable. This allows us to restrict our attention to cores H. (The core of a bipartite graph H is K_2; the core of a graph H with loops is one loop.

Now it follows that in both cases testing for H-colorability is easy. In particular, this proves the first half of Theorem 1.)

B. *The Sub-indicator Construction*

Let J be a fixed graph, with specified vertices j and $k_1, k_2, ..., k_t$. The sub-indicator construction (with respect to $J, j, k_1, k_2, ..., k_t$) transforms a given core H with t specified vertices $h_1, h_2, ..., h_t$, to its subgraph $H^\sim$ induced by the vertex set $V^\sim$ defined as follows: Let W be the graph obtained from the disjoint union of J and H by identifying each k_i with the corresponding h_i $(i = 1, 2, ..., t)$. A vertex v of H belongs to $V^\sim$ just if there is a retraction of W to H which maps the vertex j to v. (Cf. Fig. 2.)

LEMMA 2. *Let H be a core. If the $H^\sim$-coloring problem is* NP*-complete then so is the H-coloring problem.*

Proof. Given a graph G, we define ${}^\sim G$ as the graph obtained from the disjoint union of G, H, and $|V(G)|$ copies of J, by identifying, for every $i = 1, 2, ..., t$, the vertex k_i (in each copy of J) with the vertex h_i of H, and identifying each vertex g of G with the vertex j in the gth copy of J. If there is a homomorphism ${}^\sim G \to H$, then the copy of H which is a subgraph of ${}^\sim G$ must map *onto* H, because H is a core. It is then easy to see that there is a homomorphism $G \to H^\sim$. The converse, that the existence of a homomorphism $G \to H^\sim$ implies the existence of a homomorphism ${}^\sim G \to H$, is also easy to see.

C. *The Edge-sub-indicator Construction*

Let J be a fixed graph with a specified *edge jj'* and t specified vertices $k_1, k_2, ..., k_t$, such that some automorphism of J keeps each vertex k_i fixed while exchanging the vertices j and j'. The edge-sub-indicator construction

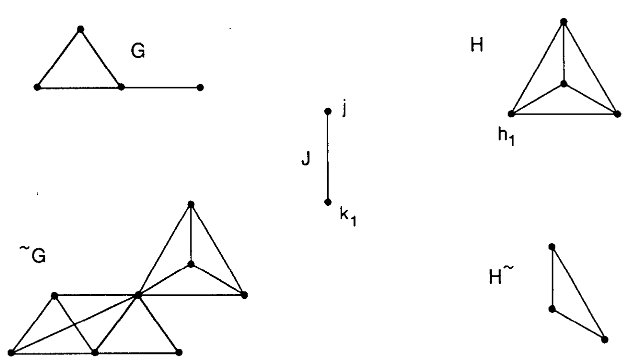


FIG. 2. Example sub-indicator construction.

transforms a given core H with t specified vertices $h_1, h_2, ..., h_t$ into its subgraph $H^\wedge$ determined by those edges hh' of H which are images of the edge jj' under retractions of W (defined as in B) to H (cf. Fig. 3). Note that because of our assumption on J, the edges of $H^\wedge$ are again undirected.

LEMMA 3. *Let H be a core. If the $H^\wedge$-coloring problem is* NP-*complete then so is the H-coloring problem.*

Proof. Given a graph G we let $^\wedge G$ denote the graph obtained from the disjoint union of G, H, and $|E(G)|$ copies of J by identifying, for each $i = 1, 2, ..., t$, the vertex k_i (in each copy of J) with the vertex h_i of H, and identifying each edge $e = gg'$ of G with the edge jj' in the eth copy of J. (Because of the symmetry of J, it does not matter whether g gets identified with j and g' with j', or the other way round.) As before, there is a homomorphism $^\wedge G \to H$ if and only if there is a homomorphism $G \to H^\wedge$.

Special cases of the first two constructions have been used in [18, 19]. The third construction is somewhat more cumbersome, but is crucial for our proof.

To prove the NP-completeness of the H-coloring problem for a particular non-bipartite H, we may appeal to an indicator construction and reduce the problem to proving the NP-completeness of the H^*-coloring problem; we shall always choose I, i, j in such a way that H^*, in addition to being undirected, has no loops, contains all the edges of H, and at least one more edge. Or we may appeal to a sub-indicator construction, and reduce the problem to proving the NP-completeness of the $H^\sim$-coloring problem; we shall always choose J, j, and the k_i's, so that $H^\sim$ is still non-bipartite, but has fewer vertices than H. (We cannot use the edge-sub-indicator construction by itself because it reduces the number of edges and thus counteracts the effect of the indicator construction. However, we shall

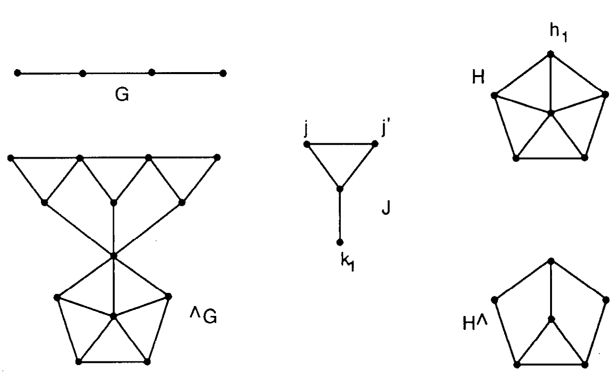


FIG. 3. Example edge-sub-indicator construction.

only be using it when it can be immediately followed by a sub-indicator construction.)

Hence, let H be a non-bipartite graph for which the H-coloring problem is *not* NP-complete and such that the H'-coloring problem *is* NP-complete for any non-bipartite H'

(1) with fewer vertices than H, or

(2) with the same number of vertices as H, but with more edges.

Clearly, if the Theorem does not hold then such an H must exist. Moreover, since each K_r-coloring problem *is* NP-complete (when $r \geqslant 3$), H has $n > 3$ vertices and $m < \binom{n}{2}$ edges. We shall proceed to derive a number of structural properties of the graph H, which will eventually imply that it cannot exist, thereby proving the Theorem. It follows from our earlier remarks that H is a core.

The Structure of Triangles

Our first goal is to prove that each edge of H belongs to a unique triangle. We do this in a sequence of steps:

(A1) *H contains a triangle.* Indeed, suppose that the shortest odd cycle C of H has k vertices, $k \geqslant 5$. Consider the indicator construction where the indicator I is a path of length three with endpoints i and j (as in Fig. 1). It transforms H into the graph H^* which is undirected (by the obvious symmetry of I), has no loops (because H has no triangles), contains all edges of H (it is easy to visualise how to "fold" I onto an arbitrary edge of H), and also contains some chords of C which were not present in H (because $k \geqslant 5$). According to our assumption (2) the H^*-coloring problem is NP-complete; by Lemma 1, the H-coloring problem is also NP-complete, contrary to assumption.

(A2) *H contains no K_4.* Otherwise we can use the sub-indicator $J = K_2$ with one endpoint j and the other k_1 (as in Fig. 2), and with h_1 being any vertex of H which belongs to a K_4. The transformed graph $H^{\sim}$ does not contain h_1, but does contain a triangle in its neighborhood. Thus $H^{\sim}$ is a non-bipartite graph with fewer vertices than H; this again contradicts our assumptions and Lemma 2.

(A3) *Each vertex of H belongs to a triangle.* Consider the sub-indicator construction with the disconnected sub-indicator J of Fig. 4 (the choice of

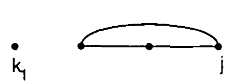


FIG. 4. A disconnected sub-indicator J.

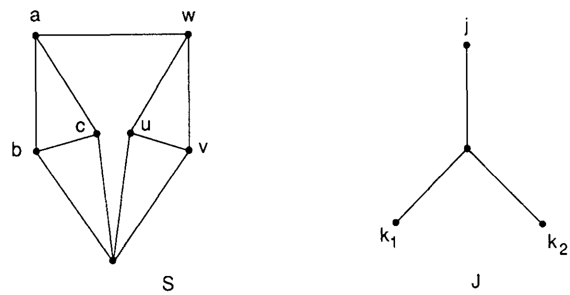


FIGURE 5

h_1 is irrelevant): $H^{\sim}$ consists of those vertices of H which belong to a triangle. By (A1) $H^{\sim}$ is non-empty and non-bipartite. If it were smaller than H we would obtain a contradiction with (1) and Lemma 2.

(A4) *Any two vertices of H have a common neighbor.* Use as sub-indicator a path of length two with endpoints j and k_1. If the vertices u and v of H have no common neighbour then setting $h_1 = u$ results in $H^{\sim}$ which does not contain v. Since u lies in a triangle by (A3), $H^{\sim}$ is non-bipartite and we obtain a contradiction with (1) and Lemma 2.

(A5) *There is no homomorphism* $S \to H$ (the graph S is given in Fig. 5). If such a homomorphism exists, let u', v', ..., be the images of the vertices $u, v, \ldots$, of S in H. The sub-indicator J of Fig. 5, with $h_1 = u'$ and $h_2 = v'$, yields a graph $H^{\sim}$ which contains the triangle $a'b'c'$ but does not contain the vertex w' (by (A2)). We obtain a contradiction as before. (We are grateful to Emo Welzl for this sub-indicator.)

(A6) *H contains no* K_4^-. The indicator construction with the indicator I of Fig. 6 transforms H into an H^* which is undirected (by the symmetry of I), has no loops (by A5), and contains all edges of H (by A4). (There is a 3-coloring of I with i and j being colored by different colors.) We will now show that if H contains a K_4^- then H^* has more edges than H, contrary to (2) and Lemma 1: Suppose K_4^- is a subgraph of H with u and v as above. Since H is a core, the neighborhoods of u and v cannot be

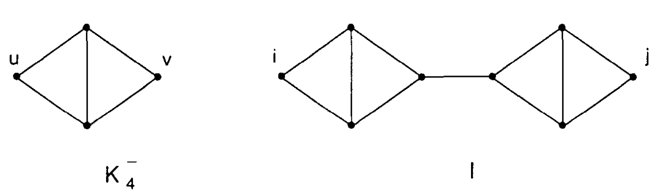


FIGURE 6

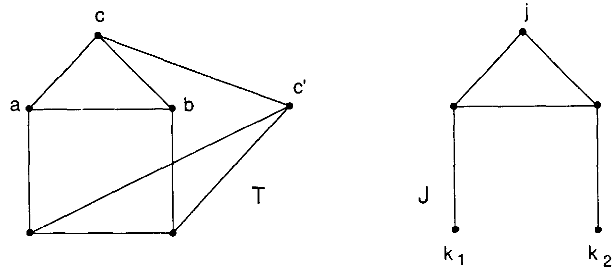


FIGURE 7

the same—thus some vertex w of H is adjacent to (say) v but not u. It is easy to construct a homomorphism $I \rightarrow H$ taking i to u and j to w (using (A3)). Thus uw is an edge of H^*, but not of H, contrary to (2) and Lemma 1.

(A7) *Each edge of H belongs to a unique triangle.* This now follows from (A4) and (A6).

In particular, *the graph spanned by the neighbors of any vertex of H is a union of disjoint edges.* Our next objective is to investigate the interconnections among the triangles of H.

(A8) *In H, any triangle abc and edge cc′ ($c' \neq a, b, c$) are contained in a subgraph T.* (The graph T is defined in Fig. 7.) The sub-indicator construction with the sub-indicator J of Fig. 7, applied to H with $h_1 = a$ and $h_2 = b$ results in an $H^{\sim}$ containing the triangle abc; thus $H^{\sim}$ is non-bipartite. If c' is a vertex of $H^{\sim}$, then it is the image of j under some retraction of W, and (using (A6) to see that all depicted vertices are distinct) we conclude that T is a subgraph of H. Otherwise $H^{\sim}$ has fewer vertices than H, contrary to (1) and Lemma 1.

Let U be the graph defined in Fig. 8.

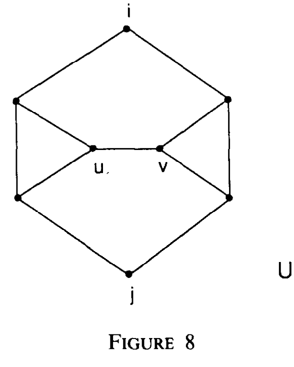
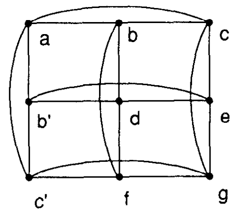


FIGURE 8



P (=$K_3 \times K_3$)

FIGURE 9

(A9) *For any homomorphism $U \to H$, the images of i and j are adjacent in H.* Consider the indicator construction with the indicator U of Fig. 8. The fact that H^* has no loops follows easily from (A5). Moreover, H^* contains all edges of H: Indeed, any edge of H belongs to a triangle, by (A7), and $I = U$ admits a homomorphism f onto a triangle, with $f(i) \neq f(j)$. If there were a homomorphism $U \to H$ with the images of i and j non-adjacent, then H^* would have strictly more edges than H, contrary to (2) and Lemma 1.

(A10) *In H, any two triangles abc, ab'c' are contained in a subgraph P* (from Fig. 9). We apply (A8) to $ab'c'$ and ab to obtain the triangle bdf with the additional edges $b'd$, $c'f$. Two applications of (A7) yield the two triangles $b'de$, $c'fg$. Finally three applications of (A9) imply the edges ce, cg, and eg. (Throughout, we appeal to (A6) to verify that all depicted vertices are distinct.)

THE STRUCTURE OF SQUARES

From now on we base our considerations on a fixed vertex r, chosen to be a vertex of maximum degree in H. By (A7), the neighborhood of r consists of $k \geq 2$ (say) disjoint edges a_1a_1', a_2a_2', ..., a_ka_k'. Let R denote the subgraph of H induced by the remaining vertices $V(H) - \{r, a_1, a_1', ..., a_k, a_k'\}$; according to (A4), each vertex x of R is adjacent to some a_i. By (A7), each edge uv of R belongs to a triangle uvw; if $w = a_i$, we label the edge uv by a_i. (Of course, the whole triangle uvw could belong to R, in which case none of the edges uv, uw, vw would be labelled.) If v in R is adjacent to some a_i, then the edge a_iv lies in a triangle a_ivw where w is also in R; hence v is incident with an edge labelled a_i. Note that (A6) implies that each edge obtains at most one label, and that two edges of the same label cannot intersect or have two of their endpoints adjacent. We shall state this as follows:

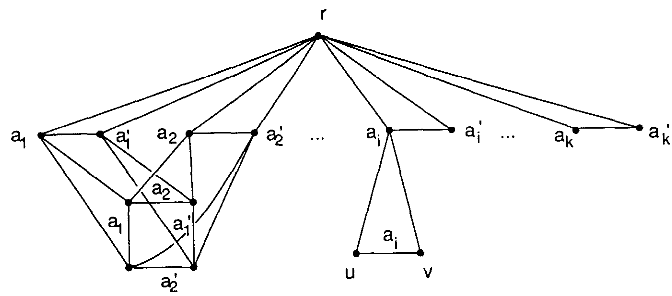


FIGURE 10

(B1) *In any path of length at most* 3, *no two edges have the same label.*

For the same reason, *no vertex can be incident with both an edge labelled* a_i *and an edge labelled* a'_i.

For any $i \neq j$ we can apply (A10) to the two triangles $ra_ia'_i$ and $ra_ja'_j$ to conclude that there is in R a four-cycle with edges consecutively labelled a_i, a_j, a'_i, a'_j (cf. Fig. 10).

Such a four-cycle will be called a *square*; there may, of course, be four-cycles in H (or even R) which are not "squares". There are at least $\binom{k}{2}$ squares, and they may intersect. Their structure is analyzed in this section, and it leads to a proof of Theorem 1.

(B2) *The squares are edge-disjoint.* If two squares intersect in an edge, it must be some squares $a_ia_ja'_ia'_j$ and $a_ia_ma'_ia'_m$ (because each edge has at most one label)—cf. Fig. 11. The indicator I from Fig. 11 admits an automorphism exchanging i and j, has a 3-coloring in which i and j obtain different colors, but has no 3-coloring in which i and j are given the same color. These facts imply that the graph H^* obtained from H by the indicator construction with this I is undirected, has no loops (any homomorphism $I \to H$ identifying i and j would have to map I to a triangle, i.e., be a 3-coloring, because of (A6)), and contains all edges of H

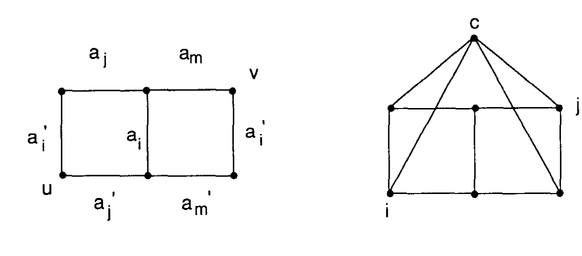


FIGURE 11

(because each of them lies in a triangle). Since there exists a homomorphism $I \to H$ taking c to a_i', i to u, and j to v, H^* also contains the edge uv, which did not belong to H (by (B1)). As always, this contradicts (2) and Lemma 1.

(B3) *H is $2k$-regular.* According to (A4) H is connected; hence it will suffice to prove that if r has the maximum degree $2k$, then all of its neighbors also have degree $2k$. The vertex a_i $(i = 1, 2, ..., k)$ lies in a triangle with the opposite edge labelled a_i; there are at least $k - 1$ such edges—one for each $j \neq i$, arising from the square $a_i a_j a_i' a_j'$—and by (B2) they are all distinct. Moreover, a_i also lies in the triangle $r a_i a_i'$; hence the degree of a_i is also $2k$.

It follows from the same proof that *each edge $a_i v$, for v in R, meets an edge labelled by a_i*, that *each labelled edge belongs to a square* (thus to a unique square), and that *H contains exactly one square labelled $a_i a_j a_i' a_j'$ for each $i \neq j$.*

(B4) *Each vertex of R belongs to a square.* This follows from the preceeding remarks: each vertex x of R must be adjacent to some a_i (or a_j') by (A4), and hence is incident with a labelled edge, and thus with a square.

(B5) *Each component of R^L is complete bipartite.* Consider the 23-point indicator I of Fig. 12. The symmetry condition with respect to i and j is obviously satisfied. Furthermore no homomorphism $f: I \to H$ can identify vertices i and j. Otherwise the images of i, u, and v form a triangle as illustrated in Fig. 13. Therefore $f(t_x) = f(v)$, $f(t_y) = f(i)$, and $f(t_z) = f(u)$ by (A7); this implies that $f(i)\, f(u)\, f(v)\, f(t_r)$ form a K_4, contrary to (A2). Since I admits a 3-coloring in which i and j obtain different colors, it follows that every edge of H belongs to H^*. If H^* has more edges than H, we have a contradiction with (2) and Lemma 1; hence we may assume that for any homomorphism $f: I \to H$, $f(i)$ and $f(j)$ are adjacent in H.

Consider any path of length three in R^L; say edges $b_i b_u$ labelled a_x (and contained in some square S_1), $b_u b_v$ labelled a_y (and contained in a square S_2), and $b_v b_j$ labelled a_z (and contained in a square S_3): It is easy

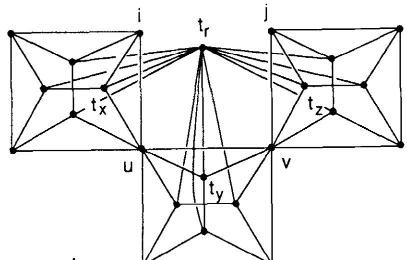


FIG. 12. The 23-point indicator I.

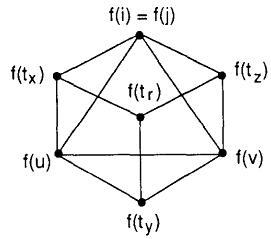


FIGURE 13

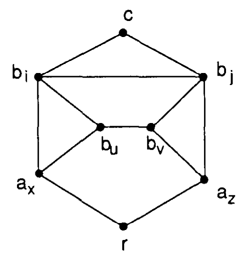


FIGURE 14

to construct a homomorphism $f: I \to H$ taking i to b_i, u to b_u, v to b_v, j to b_j, t_x to a_x, t_y to a_y, t_z to a_z, and t_r to r. (This can be done whether or not the squares S_1, S_2, and S_3 are disjoint.) Thus b_ib_j is an edge of H. We now show that b_ib_j is labelled, i.e., an edge of R^L. By (A7), the edge b_ib_j belongs to a unique triangle whose third vertex is some c. Then the graph U from Fig. 8 admits a homomorphism to H taking i to r, j to c, u to b_u and v to b_v (cf. Fig. 14). Consequently, c is adjacent to r in H, i.e., c is some a_q; thus b_ib_j is labelled by a_q. We have shown that in R^L every 3-path is in a 4-cycle. Since by (A7) there are no triangles in R^L, (B5) follows.

We conclude from (B1) and (B5) that *each label a_i (or a_i') occurs at most once in a component* of R^L. It also follows from (B5) and (A7) that *an unlabelled edge of R joins two vertices of different components* of R^L. Note that the *unique triangle containing an unlabelled edge of R has both other edges unlabelled* (and in R).

(B6) *Suppose xyz is a triangle in R with all three edges unlabelled, x is incident with an edge labelled a_i, z is incident with an edge labelled a_j, and yv is any edge of R such that v is incident with an edge labelled a_i'. Then v is also incident with an edge labelled a_j.* (Most often we shall be applying (B6) in situations where yv itself is labelled by a_i'.) The claim follows from (A9) applied to the homomorphism $U \to H$ taking u to x, v to a_i, i to v, and j to a_j (cf. Fig. 15).

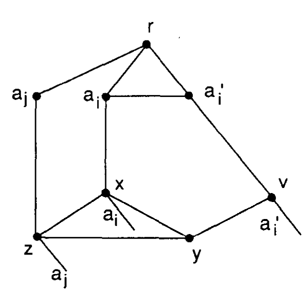


FIGURE 15

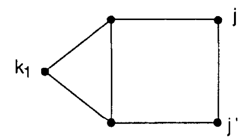


FIG. 16. The edge-sub-indicator J.

If uv is an unlabelled edge of R, and if u is incident with an edge labelled a_i while v is incident with an edge labelled a'_i, we *mark* the edge uv by the index i. Note that *each edge obtains at most one mark*: If the unlabelled edge uv of R obtained both marks i and j, then consider the third vertex w on the unique triangle uvw with unlabelled edges. According to (B4) it is incident with a labelled edge, say, of label a_m. If vv_1 and vv_2 are the edges with labels a_i (or a'_i) and a_j (or a'_j) at v, then, according to (B6), both v_1 and v_2 are incident with an edge labelled a_m, contradicting the fact that each component of R^L is bipartite and contains at most one edge of any label.

(B7) *Every unlabelled edge of R is marked by exactly one index i.* It only remains to verify that each unlabelled edge has at least one mark. Consider the edge-sub-indicator J of Fig. 16. We choose h_1 to be r. By the apparent symmetry of jj', the graph $H^\wedge$ is again undirected. It is easy to see that $H^\wedge$ contains all edges ra_i, ra'_i, all edges $a_ia'_i$, all edges a_iv with v in R, and (by the remarks following (B3)) precisely those edges of R for which there exists an index i such that one endpoint is incident with an edge labelled a_i and the other endpoint is incident with an edge labelled a'_i. Since we have already observed that each labelled edge belongs to a square, it follows that $H^\wedge$ contains all labelled edges; it also contains all marked (unlabelled) edges. Suppose there was in R an unlabelled edge uv without a mark; since every edge of H belongs to a unique triangle, there exists a vertex w in R with unlabelled edges uw, vw. We know that uv is not an edge of $H^\wedge$. At this point we cannot appeal to our assumptions because $H^\wedge$ has neither fewer vertices nor more edges than H. Therefore we let $K = H^\wedge$, $h_1 = w$, and consider the sub-indicator J which is a path of length two with endpoints k_1 and j. Note that u does not belong to $K^\sim$ because u and w have exactly one common neighbor, v, in H by (A7), and uv is not an edge of $K = H^\wedge$. Moreover, w is incident with a labelled edge, by (B4); say, some wx is labelled a_i. Then wxa_i is a triangle in $K^\sim$, so that $K^\sim$ is not bipartite. Since $K^\sim$ is a non-bipartite graph with fewer vertices than H, the $K^\sim$-coloring problem is NP-complete by assumption (1); hence the K-coloring problem (i.e., the $H^\wedge$-coloring problem) is NP-complete by Lemma 3, and the H-coloring problem is NP-complete by Lemma 2. This contradiction establishes that each unlabelled edge of R is marked by some index i.

We conclude that *in R there are labelled edges, forming the graph R^L which consists of complete bipartite components, and marked edges, forming edge-disjoint triangles each of which joins three different components of R^L.*

(B8) *R^L has a component K isomorphic to $K_{2,k}$.* Consider a vertex v of R, which has the maximum degree in R^L. If the degree is k, i.e., if v is incident with k labelled edges, then there is at v precisely one label from each pair $\{a_i, a_i'\}$. It now follows easily from (B5) and the first remark following it that the component of R^L containing v is isomorphic to $K_{2,k}$. On the other hand, if the degree of v in R^L is less than k, then v lies in a triangle vxy with marked edges. Suppose that the edge xy is marked by i. Then, according to (B1), v is not incident with any edges labelled a_i or a_i'. Let xw be labelled by a_i and yw' by a_i'. Thus (B6) implies that both w and w' have degree in R^L greater than v, contrary to our hypothesis.

We shall assume from now on that the special component $K = K_{2,k}$ has all edges with primed labels incident with the same vertex of degree k (cf. Fig. 17). Clearly, this involves no loss of generality, as we may rename the neighbours of r accordingly.

We shall call a vertex of R *positive* (respectively *negative*) if it is not incident with any edge labelled with a primed label (respectively unprimed label). A vertex which is neither positive nor negative shall be called *mixed.* Thus the special component K as shown in Fig. 17 has one positive vertex, one negative vertex, and k mixed vertices.

(B9) *If v is a mixed vertex of K, incident with edges labelled a_i and a_j', then v is adjacent to an endpoint of each edge labelled a_i' or a_j.* This is obvious for the edge labelled a_i' in K—it lies in the square containing v. Suppose that v is not adjacent to either endpoint of an edge u_1u_2 labelled a_i' which belongs to some component C of R^L. According to (A4), there exist vertices w_1 adjacent to v and u_1, and w_2 adjacent to v and u_2. Moreover,

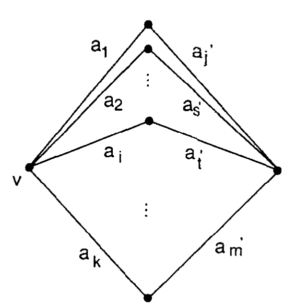


FIG. 17. The special component K.

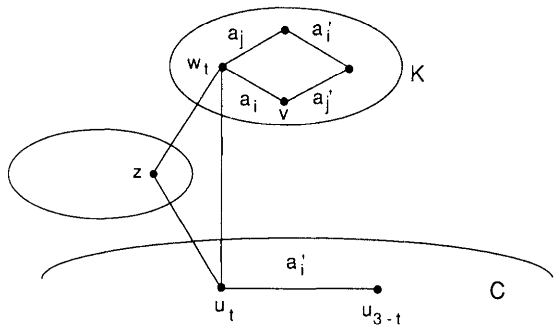


FIG. 18. $t = 1$ or 2.

$w_1 \neq w_2$, or else (A6) would imply that $w_1 = w_2 = a_i'$; this is impossible because v is not incident with an edge labelled a_i'.

If either w_t ($t = 1$ or 2) belongs to K, then it would be one of the two neighbors of v; it could not be incident with an edge labelled a_i' because of (B1). This leaves only one possible neighbor of v, say w_t ($t = 1, 2$) as shown in Fig. 18. Since $u_t w_t$ is not in a component of R^L, $u_t w_t$ is unlabelled, and it belongs to a unique triangle $u_t w_t z$ with unlabelled edges. By (B4), the vertex z, completing the triangle with unlabelled edges containing the edge $w_t u_t$, belongs to at least two labelled edges, and by (B1) their labels cannot be a_i. This yields a contradiction with (B6), at the vertex v.

However, neither w_t ($t = 1, 2$) can belong to a component of R^L different from C and K. Otherwise there is a triangle $w_t u_t z$ with three unlabelled edges of R. According to (B6), any label occuring at an edge incident to z also occurs at an edge incident to v. Hence z would have to be incident only with edges labelled a_i and a_j, and so lie in a unique square, labelled $a_i a_j a_i' a_j'$. Since there is only one square with these labels in H, this would mean that $z = v$, which contradicts our assumption that v is not adjacent to u_t.

Suppose w_t ($t = 1, 2$) belongs to C, and let the edge $w_t u_t$ be labelled a_s. There is a homomorphism $U \to H$ taking u to r, v to a_s, i to v, and j to u_{3-t} (consider the triangles $r a_i a_i'$ and $a_s w_t u_t$ as shown in Fig. 19). Hence by (A9) v is adjacent to u_{3-t}, contrary to our assumption.

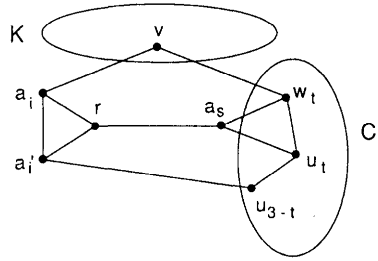


FIGURE 19

The last place for a vertex w_t $(t = 1, 2)$ is among the vertices adjacent to r; because v is only incident with edges labelled a_i and a_j, and because u_t cannot be incident with an edge labelled a_i (already being incident with an edge labelled a_i'), this would mean that $w_t = a_j$. Thus it is not possible for both w_1 and w_2 to be adjacent to r.

(B10) *Two mixed vertices cannot be adjacent.* We first prove that a mixed vertex of K is not adjacent to another mixed vertex: Obviously, a mixed vertex u of K is not adjacent to another mixed vertex of K. Thus we may restrict our attention to unlabelled edges: Let uv be an unlabelled edge, and let vw and uw also be unlabelled. Assume that u is incident with edges labelled a_i and a_j'; then one of uv, uw is marked by i and the other by j. Without loss of generality, let v be incident with an edge labelled a_j and w with an edge labelled a_i' (thus uv is marked by j and uw by i). Now (B6) implies that the label of any edge incident with v also occurs at the unique positive vertex of K, and the label of any edge incident with w also occurs at the unique negative vertex of K. Therefore v is a positive vertex, and w a negative vertex.

Now we prove that two arbitrary mixed vertices cannot be adjacent by a labelled edge. Otherwise, let a_i be the label of such an edge. Since K contains all labels, there is a mixed vertex v of K which is incident with an edge labelled a_i'. Then v must be adjacent to one of the endpoints of the edge labelled a_i by (B9), which contradicts the first paragraph of the present case. Next we consider two mixed vertices u and v adjacent by an unlabelled edge, belonging to a triangle uvw with unlabelled edges. Suppose that the edge uv is marked by i, namely that some edge ux is labelled a_i and some edge vy by a_i'. Then by what we have just observed x and y cannot be mixed, and hence x is positive and y negative. By (B6) any label of an edge incident with w also occurs at x and at y. This is only possible if x is not incident with any labelled edges, contrary to (B4).

Conclusion of the Proof. We now show that H is 3-colorable. This will show that the core of H is K_3 and hence H-coloring is NP-complete, contrary to our assumptions. Thus the Theorem will be proved. Color the vertex r as well as all mixed vertices by color 1; color all positive vertices of R as well as all primed neighbours of r by color 2; and color all negative vertices of R and all unprimed neighbours of r by color 3. We now show that this is a legal coloring. According to (B10), two vertices of color 1 cannot be adjacent. Moreover, two vertices of color 2 (respectively color 3) also cannot be adjacent. This is implied by the following remarks: Two positive (respectively negative) vertices of R cannot be adjacent. If they were adjacent by a labelled edge, then such an edge could not be part of a square, contradicting a remark made after (B3). They also could not be adjacent by an unlabelled edge, as it easily follows from (B10) and (B7) that any triangle of unlabelled edges joins a mixed vertex, a positive vertex, and a negative vertex.

Remark. The situation is less clear for directed graphs. Even a conjecture anticipating which H-coloring problems are polynomial and which are NP-complete does not suggest itself. Only a few results are known [2, 18]. There are some simple digraphs H (paths, cycles, transitive tournaments, etc.) for which polynomial H-coloring algorithms exist [2, 18]. Typically, they make use of results of the following type: There is a homomorphism $D \to H$ if and only if there is no homomorphism $H' \to D$ (for some fixed digraph H', depending on H) [2, 7, 14, 20]. (These results may be viewed as proving that H-colorability is in NP $\cap$ coNP, and are, in some sense, prototype results of this type; this line of study is pursued in [14, 20].) There are also a few classes of digraphs H with NP-complete H-colorability problems [18]. We note in passing that many NP-complete H-coloring problems may be produced by using the construction (of $*G$) from the proof of Lemma 1, with a suitable choice of the indicator I. Specifically, let (I, i, j) be a digraph indicator such that for graphs G and H, there is a homomorphism $G \to H$ if and only if there is a homomorphism of digraphs $*G \to *H$. Such indicators are called 'strongly rigid' in the terminology of [8]; they can be constructed to satisfy many addiditonal properties—assuring for example that $*H$ is an acyclic, or even balanced, digraph. (A digraph is acyclic if it has no directed cycles; it is balanced if it has the same number of forward and backward arcs on any cycle.) In any event, if $D = *H$ for such an I and a non-bipartite H, then the D-coloring problem is also NP-complete. Thus there are balanced (and hence also acyclic) digraphs H for which the H-coloring problem is NP-complete. Acyclic digraphs H with NP-complete H-coloring problems were also constructed by S. Burr and by W. Gutjahr and E. Welzl (personal communications).

Finally, it should be mentioned that any digraph H, such that the 'symmetric part' H_S of H (all pairs uv of vertices for which both uv and vu are arcs of H) is a non-bipartite graph, also results in an NP-complete H-coloring problem. This is an easy corollary of Theorem 1 and the observation that a *graph* G admits a homomorphism $G \to H_S$ if and only if it (viewed as a *digraph*) admits a homomorphism $G \to H$. It follows easily from this that *almost every digraph* H *gives an* NP-*complete* H-*coloring problem.*

Since the first version of this paper (Simon Fraser School of Computing Science Technical Report TR-86-4), there has been some progress on the problem of coloring by directed graphs. In particular, in a paper to appear in the *SIAM J. Discrete Math.*, J. Bang-Jensen, the first author, and G. MacGillivray prove that for semicomplete graphs H (and in particular for tournaments) the presence of two directed cycles makes the problem NP-complete, and otherwise it is polynomial.

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