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Enumerating a class of lattice paths

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Abstract

Let $\mathscr{D}_0(n)$ denote the set of lattice paths in the xy-plane that begin at (0,0), terminate at (n,n), never rise above the line $y=x$ and have step set $\mathscr{S}=\{(k,0):k\in\mathbb{N}^+\}\cup\{(0,k):k\in\mathbb{N}^+\}$. Let $\mathscr{E}_0(n)$ denote the set of lattice paths with step set $\mathscr{S}$ that begin at (0,0) and terminate at (n,n). Using primarily the symbolic method (R. Sedgewick, P. Flajolet, An Introduction to the Analysis of Algorithms, Addison-Wesley, Reading, MA, 1996) and the Lagrange inversion formula we study some enumerative problems associated with $\mathscr{D}_0(n)$ and $\mathscr{E}_0(n)$.

Keywords: Lattice path; Lagrange inversion formula; Narayana number

1. Introduction

For any positive integer n let $\mathscr{C}_0(n)$ denote the set of lattice paths in the xy-plane that:

(i) have step set $S^{(1)}=\{(1,0),(0,1)\}$,
(ii) begin at (0,0) and terminate at (n,n),
(iii) never rise above the line $y=x$.

It is of course well known that $|\mathscr{C}_0(n)|=[1/(n+1)]\binom{2n}{n}$. More generally if $n\geqslant k\geqslant 0$ then the ballot number $[(n-k+1)/(n+1)]\binom{n+k}{k}$ counts the lattice paths that begin at (0,0), terminate at (n,k), never rise above the line $y=x$ and have step set $S^{(1)}$. The present work originated in an attempt to enumerate the class $\mathscr{D}_0(n)$ of lattice paths that satisfy conditions (ii) and (iii) and have step set $\mathscr{S}=\{(k,0):k\in\mathbb{N}^+\}\cup\{(0,k):k\in\mathbb{N}^+\}$.

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Given a subset S of $\mathbb{N} \times \mathbb{N}$ we define a (*lattice*) *path with step set* S to be a finite sequence $\Gamma = s_1 s_2 \cdots s_k$ where $s_i \in S$ for all i. Suppose $s_i = (p_i, q_i)$ for each $i = 1, 2, \ldots, k$. If Γ begins at (p_0, q_0) then Γ passes sequentially through the lattice points (p_0, q_0), $(p_0 + p_1, q_0 + q_1)$, $(p_0 + p_1 + p_2, q_0 + q_1 + q_2)$, and so on, terminating at (p, q) where $p = \sum_{i=0}^{k} p_i$ and $q = \sum_{i=0}^{k} q_i$. By an *underdiagonal* path we mean a path in the xy-plane that begins at (0,0) and never rises above the line $y = x$.

For integers $n \geqslant r \geqslant 0$ let $d_{n,r}$ denote the number of underdiagonal paths that terminate at (n, r) and have step set $\mathscr{S} = \{(k, 0) : k \in \mathbb{N}^+\} \cup \{(0, k) : k \in \mathbb{N}^+\}$. For any integer $k \geqslant 0$ we denote by $\mathscr{D}_k$ the set of all underdiagonal paths that terminate on the line $x - y = k$ and have step set $\mathscr{S}$. We define $D_k(t) = \sum_{n \geqslant k} d_{n,n-k} t^n$.

Here is a summary of the paper's contents. In Section 2 we enumerate paths in $\mathscr{D}_0$. In Section 3, we present several recurrences for the generating functions $D_k(t)$. One of these recurrences implies that $d_{k+r,r}$ is divisible by 2^{k-1} for any integers $k > 0$ and $r \geqslant 0$. We discuss in Section 4 a second enumeration of paths in $\mathscr{D}_0$ in terms of Narayana numbers. Interrupting our study of $\mathscr{D}_0$ we present in Section 5 an example of a class of underdiagonal paths with unrestricted steps. In Section 6 we enumerate paths in $\mathscr{D}_0$ with respect to the number of steps. In Section 7 we briefly study a second class of lattice paths with step set $\mathscr{S}$.

2. Enumerating paths in $\mathscr{D}_0$

In this section, we enumerate paths in $\mathscr{D}_0$ by first finding a functional equation satisfied by D_0. To lend concreteness to the exposition let us calculate $d_{n,n}$ for a few small values of n. If we set $d_{0,0} = 1$ and initially set all other $d_{n,k}$ to zero, then using the relation $d_{n,k} = \sum_{j=0}^{k-1} d_{n,j} + \sum_{j=1}^{n-k} d_{n-j,k}$ we can generate recursively the following partial array d:

$d_{n,k}$	$k = 0$	1	2	3	4	5	6	7
$n = 0$	1							
1	1	1						
2	2	3	5					
3	4	8	17	29				
4	8	20	50	107	185			
5	16	48	136	336	721	1257		
6	32	112	352	968	2370	5091	8925	
7	64	256	880	2640	7116	17304	37185	65445

Our first goal is to enumerate the sets $\mathscr{D}_0(n)$; their cardinalities appear along the main diagonal of d. To facilitate our work we introduce another generating function: for each integer $n \geqslant 1$ let $\bar{\mathscr{D}}_0(n)$ denote the set of paths in $\mathscr{D}_0(n)$ that do not intersect the line $y = x$ except at(0,0) and (n, n). Paths in $\bar{\mathscr{D}}_0(n)$ are called *primitive*. We now define $\bar{D}_0(t) = \sum_{n \geqslant 1} \bar{d}_n t^n$ where $\bar{d}_n = |\bar{\mathscr{D}}_0(n)|$. Henceforth, we shall often abbreviate $d_{n,n}$

by d_n. Let $\mathscr{J}$ denote the set of nonempty paths in $\mathscr{D}_0$ and let J be the corresponding generating function. Clearly

$$D_0 = 1 + J. \tag{2.1}$$

We begin with a basic relation between the generating functions D_0 and $\bar{D}_0$.

Lemma 2.1. *The generating functions D_0 and $\bar{D}_0$ satisfy*

$$\bar{D}_0 = t + 4t(D_0 - 1).$$

Proof. We verify that $\sum_{n\geqslant 1} \bar{d}_n t^n = t + 4t \sum_{n\geqslant 1} d_n t^n$. This is equivalent to showing that $\bar{d}_n = 4d_{n-1}$ for all $n \geqslant 2$. To prove this we note that any path in $\mathscr{D}_0(n-1)$, say $\Gamma = s_1 s_2 s_3 \cdots s_k$, gives rise to four paths in $\bar{\mathscr{D}}_0(n)$. In order to construct these four paths, say Γ_1, Γ_2, Γ_3, Γ_4, let us assume that $s_1 = (p, 0)$ for some $p \geqslant 1$ and $s_k = (0, q)$ for some $q \geqslant 1$. Put $s_1^+ = (p+1, 0)$, $s_k^+ = (0, q+1)$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Define $\Gamma_1 = s_1^+ s_2 s_3 \cdots s_k^+$, $\Gamma_2 = s_1^+ s_2 s_3 \cdots s_k e_2$, $\Gamma_3 = e_1 s_1 s_2 s_3 \cdots s_k^+$ and $\Gamma_4 = e_1 s_1 s_2 s_3 \cdots s_k e_2$. Clearly these new paths Γ_i belong to $\bar{\mathscr{D}}_0(n)$.

In this construction every path in $\bar{\mathscr{D}}_0(n)$ arises exactly once from some path in $\mathscr{D}_0(n-1)$, as is easily seen by reversing the process. The result now follows. □

By applying (2.1) we can rewrite the statement of Lemma 2.1 in the form

$$\bar{D}_0 = t(1 + 4J). \tag{2.2}$$

We obtain a further relation between $\bar{D}_0$ and J by noting that every path in $\mathscr{J}$ decomposes uniquely into a path in $\bar{\mathscr{D}}_0$ followed by a path in $\mathscr{D}_0$ (the first return decomposition). This immediately implies

$$J = \bar{D}_0(1 + J). \tag{2.3}$$

Eliminating J from (2.1) and (2.3) gives

$$D_0(1 - \bar{D}_0) = 1. \tag{2.4}$$

Eliminating $\bar{D}_0$ from (2.2) and (2.3) gives

$$J = t(1 + 4J)(1 + J). \tag{2.5}$$

Eliminating J from (2.1) and (2.5) gives $D_0 - 1 = tD_0[1 + 4(D_0 - 1)]$.

Rearranging terms in this last equation yields

$$4tD_0^2 - (1 + 3t)D_0 + 1 = 0, \tag{2.6}$$

which implies that

$$d_n + 3d_{n-1} = 4\sum_{k=0}^{n-1} d_{n-k-1} d_k \tag{2.7}$$

for all $n \geqslant 1$.

The referee outlined an elegant derivation of (2.5) that subsumes Lemma 2.1; here are the details. Consider underdiagonal paths with step set $S^{(1)}=\{E,N\}$ where $E=(1,0)$ and $N=(0,1)$. We call the vertex between a pair of consecutive Es a *double-east point* and that between a pair of consecutive Ns a *double-north point*. These are the counterparts of the better-known doublerise and doublefall points [5] of classical Dyck paths. Assume the double-east and double-north points come in two colors, say red and blue. Let $\mathscr{P}_0$ denote the set of these two-colored underdiagonal paths.

It is easy to see that $\mathscr{P}_0$ and $\mathscr{D}_0$ are in bijective correspondence.

Furthermore, every nonempty path $\Gamma \in \mathscr{P}_0$ is the concatenation of a primitive path $E\Gamma_1 N$ with a path Γ_2. If Γ_1 is nonempty then $E\Gamma_1 N$ has one of the four forms $Er\Gamma^* rN$, $Er\Gamma^* bN$, $Eb\Gamma^* rN$ and $Eb\Gamma^* bN$, where r and b indicate the colors of the corresponding double points and $\Gamma^* \in \mathscr{P}_0$. It follows that if we let G denote the generating function of the paths in $\mathscr{P}_0$ then $G = 1 + t[4(G-1)+1]G$; this is equivalent to (2.5) since $J = G - 1$.

Let us now enumerate paths in $\mathscr{D}_0$.

Theorem 2.2. (a) *The generating function* $D_0(t)$ *is given by* $D_0(t) = 1/8t[1+3t-(9t^2-10t+1)^{1/2}]$. *Moreover, for all* $n \geqslant 1$, $d_n = \sum_{k=\lfloor n/2 \rfloor}^{n} [1/(k+1)]\binom{2k}{k}\binom{k+1}{n-k} \times (-9)^{n-k}5^{2k-n+1}/2^{n+3}$.

(b) *For all* $n \geqslant 1$, $d_n = \sum_{k=1}^{n} (1/n)\binom{n}{k}\binom{n}{k-1}4^{n-k}$.

(c) *For all* $n \geqslant 1$, $d_n = (1/4)\sum_{k=0}^{n} [1/(n+1)]\binom{n+1}{k}\binom{2n-k}{n-k}3^k$.

(d) *The generating function* $\bar{D}_0(t)$ *is given by* $\bar{D}_0(t) = (1/2)[1-3t-(9t^2-10t+1)^{1/2}]$. *Moreover, for all* $n \geqslant 2$, $\bar{d}_n = \sum_{k=\lfloor (n-1)/2 \rfloor}^{n-1} [1/(k+1)]\binom{2k}{k}\binom{k+1}{n-k-1}(-9)^{n-k-1}5^{2k-n+2}/2^n$.

(e) *For all* $n \geqslant 2$, $\bar{d}_n = \sum_{k=0}^{n-1} (1/n)\binom{n}{k}\binom{2n-k-2}{n-k-1}3^k$.

Proof. (a) Solving for D_0 in (2.6) gives $D_0(t) = (1/8t)[1+3t-(9t^2-10t+1)^{1/2}]$. Using the binomial theorem to expand $(9t^2-10t+1)^{1/2}$ we find

$$D_0(t) = 1 + t + \sum_{n \geqslant 2} \sum_{k=\lfloor n/2 \rfloor}^{n} \frac{1}{k+1}\binom{2k}{k}\binom{k+1}{n-k}\frac{(-9)^{n-k}10^{2k-n+1}}{4^{k+2}} t^n.$$

Thus for all $n \geqslant 2$,

$$d_n = \sum_{k=\lfloor n/2 \rfloor}^{n} \frac{1}{k+1}\binom{2k}{k}\binom{k+1}{n-k}\frac{(-9)^{n-k}5^{2k-n+1}}{2^{n+3}}.$$

Notice that this identity also holds for $n = 1$.

(b) Solving for t in (2.5) gives $t = J/(1+J)(1+4J) = H(J)$, say. By the Lagrange inversion formula [2,12,14] we find $[t^n]H^{-1}(t) = (1/n)\sum_{k=0}^{n-1}\binom{n}{n-1-k}\binom{n}{k}4^k$. Replacing the index k by $n-k$ yields

$$d_n = [t^n]H^{-1}(t) = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k-1}\binom{n}{n-k}4^{n-k} = \sum_{k=1}^{n}\frac{1}{n}\binom{n}{k-1}\binom{n}{k}4^{n-k}$$

for all $n \geqslant 1$.

(c) By part (b) we find

$$d_n = \sum_{j=1}^{n} \frac{1}{n}\binom{n}{j}\binom{n}{j-1}4^{n-j} = \frac{1}{4}\sum_{j=0}^{n-1}\frac{1}{n}\binom{n}{j+1}\binom{n}{j}4^{n-j}$$

$$= \frac{1}{4}\sum_{j=0}^{n-1}\frac{1}{n}\binom{n}{j+1}\binom{n}{n-j}\sum_{k=0}^{n-j}\binom{n-j}{k}3^k$$

$$= \frac{1}{4}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\frac{1}{n}\binom{n}{j+1}\binom{n}{k}\binom{n-k}{n-k-j}3^k$$

$$= \frac{1}{4}\sum_{k=0}^{n}\frac{1}{n}\binom{n}{k}3^k\sum_{j=0}^{n-k}\binom{n}{j+1}\binom{n-k}{n-k-j}.$$

By Vandermonde's convolution formula the inner sum is $\binom{2n-k}{n-k+1}=\binom{2n-k}{n-1}$. Thus $d_n = (1/4)\sum_{k=0}^{n}(1/n)\binom{n}{k}\binom{2n-k}{n-1}3^k$. The result now follows since $(1/n)\binom{n}{k}\binom{2n-k}{n-1} = [1/(n+1)]\binom{n+1}{k}\binom{2n-k}{n-k}$.

(d) By (2.4) and part (a) we find $\bar{D}_0(t) = \frac{1}{2}[1 - 3t - (9t^2 - 10t + 1)^{1/2}]$. In the proof of Lemma 2.1 we saw that $\bar{d}_n = 4d_{n-1}$ for all $n \geqslant 2$. Consequently part (a) implies that

$$\bar{d}_n = \sum_{k=\lfloor (n-1)/2\rfloor}^{n-1}\frac{1}{k+1}\binom{2k}{k}\binom{k+1}{n-k-1}\frac{(-9)^{n-k-1}5^{2k-n+2}}{2^n}$$

for all $n \geqslant 2$.

(e) This follows from part (c) together with the fact that $\bar{d}_n=4d_{n-1}$ for all $n \geqslant 2$. □

Other representations for d_n, for example,

$$d_n = \sum_{k=0}^{n-1}\frac{1}{n}\binom{n}{k}\binom{2n-k}{n+1}3^k$$

can easily be found as in the proof of part (c) of Theorem 2.2. In the intervening months since the initial submission of this manuscript Sulanke [16] and Woan [18] (see also [19]) independently published enumerations of $\mathscr{D}_0$. I thank the referee for bringing these works to my attention.

3. Recurrences involving $D_k(t)$

Let c be the infinite lower triangular array with (n,r)-entry $c_{n,r} = [(n-r+1)/(n+1)]\binom{n+r}{n}$; here $n \geqslant r \geqslant 0$. And for each integer $k \geqslant 0$ let $C_k(t) = \sum_{n\geqslant k} c_{n,n-k}t^n$ be the generating function for the kth diagonal of c. It is easy to see that $C_k = tC_0C_{k-1}$ for all $k \geqslant 1$. Iterating this gives $C_k = (tC_0)^kC_0$ for all $k \geqslant 1$. Since $tC_0^2 = C_0 - 1$ it follows that $C_k = (tC_0)^{k-1}(C_0 - 1)$ for all $k \geqslant 1$ and $C_k = C_{k-1} - tC_{k-2}$ for all $k \geqslant 2$. In this section, our goal is to find analogous recurrences for the generating functions $D_k(t)$.

Our first theorem relates the generating functions D_1 and D_0.

Theorem 3.1. *The generating functions D_1 and D_0 satisfy $D_1 = tD_0(2D_0 - 1)$.*

Proof. Let A be the subset of $\mathscr{D}_1$ consisting of those paths that intersect the main diagonal $y = x$ only at the origin, and let $B = \mathscr{D}_1 \backslash A$. We claim that any nonempty path $\Gamma_0 \in \mathscr{D}_0$ gives rise to two paths in A. To see this suppose $\Gamma_0 = s_1 s_2 s_3 \cdots s_k$, where, say $s_1 = (p, 0)$ for some $p \geqslant 1$. As in the proof of Lemma 2.1 we set $s_1^+ = (p + 1, 0)$ and $e_1 = (1, 0)$. Then $\Gamma_1 = e_1 \Gamma_0$ and $\Gamma_2 = s_1^+ s_2 s_3 \cdots s_k$ are the desired paths. It follows that if we let t mark path length in the horizontal direction then A has generating function $t + 2t(D_0 - 1)$.

Any path in B decomposes uniquely into a nonempty path $\Gamma_0 \in \mathscr{D}_0$ followed by a path $\Gamma \in A$. Therefore, B has generating function $(D_0 - 1)[t + 2t(D_0 - 1)]$. Consequently $\mathscr{D}_1$ has generating function $D_1 = t + 2t(D_0 - 1) + (D_0 - 1)[t + 2t(D_0 - 1)] = tD_0(2D_0 - 1)$. □

An argument similar to that of Theorem 3.1 proves our first recurrence for D_k.

Theorem 3.2. *Let D_k be the generating function for the kth diagonal of d. Then $D_k = 2tD_0 D_{k-1}$ for all $k \geqslant 2$.*

Proof. Let A be the subset of $\mathscr{D}_k$ consisting of those paths that intersect the main diagonal $y = x$ only at the origin, and let $B = \mathscr{D}_k \backslash A$. As in the proof of Theorem 3.1 any path $\Gamma \in \mathscr{D}_{k-1}$ gives rise to two paths in A. Thus A has generating function $2tD_{k-1}$. And any path in B decomposes uniquely into a nonempty path $\Gamma_0 \in \mathscr{D}_0$ followed by a path $\Gamma_1 \in A$. Thus, B has generating function $(D_0 - 1)2tD_{k-1}$. Therefore, $\mathscr{D}_k$ has generating function $D_k = 2tD_{k-1} + (D_0 - 1)2tD_{k-1} = 2tD_0 D_{k-1}$. □

Corollary 3.3. *For all $k \geqslant 2$, $D_k = 2^{k-1}(tD_0)^k(2D_0 - 1)$. In particular every entry on the kth diagonal of d is divisible by $2^{k-1} = d_{k,0}$.*

Proof. Iterating the result in Theorem 3.2 gives $D_k = (2tD_0)^{k-1} D_1$. From Theorem 3.1 we thus obtain $D_k = (2tD_0)^{k-1} tD_0(2D_0 - 1) = 2^{k-1}(tD_0)^k(2D_0 - 1)$. In particular every entry on the kth diagonal of d is divisible by $2^{k-1} = d_{k,0}$. □

From (2.6) and Theorem 3.1 it follows that

$$2D_1 = -1 + (1 + t)D_0. \tag{3.1}$$

We can in turn use (3.1) to derive an expression for D_2. We begin by applying Theorem 3.2 to write $D_2 = tD_0 \cdot 2D_1 = tD_0[-1 + (1 + t)D_0] = t(1 + t)D_0^2 - tD_0$. Therefore $2D_2 = (1 + t)2tD_0^2 - 2tD_0 = (1 + t)(tD_0 + D_1) - 2tD_0 = 2D_1 + (t - 1)(tD_0 + D_1)$, that is,

$$2D_2 = 2D_1 + (t - 1)(tD_0 + D_1). \tag{3.2}$$

Eq. (3.2) is a special case of the following result.

Theorem 3.4. *Let D_k be the generating function for the kth diagonal of d. Then $2D_k = 2D_{k-1} + (t-1)\sum_{i=0}^{k-1} t^{k-i-1}D_i$ for all $k \geqslant 2$.*

Proof. This follows from Theorem 3.1 and Corollary 3.3 by a straightforward but somewhat lengthy calculation. □

The functions D_k satisfy a second-order linear recurrence as given in the next theorem. Here again the proof, which is a straightforward calculation based on Theorem 3.1 and Corollary 3.3, is omitted.

Theorem 3.5. *Let D_k be the generating function for the kth diagonal of d. Then $2D_k = (1+3t)D_{k-1} - 2tD_{k-2}$ for all $k \geqslant 3$.*

4. Enumerating paths in $\mathscr{D}_0$ in terms of Narayana numbers

We begin this section with what seems a natural proof of the result in Theorem 2.2(b), that is, $d_n = \sum_{k=1}^{n}(1/n)\binom{n}{k}\binom{n}{k-1}4^{n-k}$. We proceed by first classifying each lattice path according to the number of its horizontal and vertical segments. This approach relies on Narayana numbers.

In connection with his work regarding partial orders on integer partitions Narayana [8] proved that for each $k = 1, 2, \ldots, n$ the number of underdiagonal paths that terminate at (n, n) and have a total of $2k$ horizontal and vertical segments is given by $N(n,k) = (1/n)\binom{n}{k}\binom{n}{k-1}$. Suppose Γ is such a path. Define the *shape* of Γ denoted by $\sigma(\Gamma)$, to be the underdiagonal path in $\mathscr{D}_0(n)$ whose steps are the $2k$ horizontal and vertical segments of Γ. We claim there are 4^{n-k} paths Γ' in $\mathscr{D}_0(n)$ satisfying $\sigma(\Gamma') = \sigma(\Gamma)$. To see this, let us assume that Γ has segments of lengths $j_1, j_2, \ldots, j_{2k}$. For each $i = 1, 2, \ldots, 2k$ there are $d_{j_i,0} = 2^{j_i-1}$ distinct ways to construct a segment of length j_i with steps from $\mathscr{S}$. It follows that there are $2^{j_1-1}2^{j_2-1}\cdots 2^{j_{2k}-1} = 4^{n-k}$ paths Γ' in $\mathscr{D}_0$ satisfying $\sigma(\Gamma') = \sigma(\Gamma)$. Thus, for any $n \geqslant 1$ the total number of paths in $\mathscr{D}_0(n)$ is $d_n = \sum_{k=1}^{n} N(n,k)4^{n-k} = \sum_{k=1}^{n}(1/n)\binom{n}{k}\binom{n}{k-1}4^{n-k}$.

Let $F(x,t) = \sum_{n\geqslant 1}\sum_{k=1}^{n} N(n,k)t^k x^n$ be the generating function of the Narayana numbers. Stanley [14, Exercise 6.36] shows that

$$xF^2 + (xt + x - 1)F + xt = 0 \tag{4.1}$$

and thus

$$F(x,t) = \frac{1}{2x}(1 - x - xt - [(1 - x - xt)^2 - 4x^2t]^{1/2}). \tag{4.2}$$

Replacing x by xt and t by t^{-1} in (4.2) gives $F(xt, t^{-1}) = (1/2xt)(1 - x - xt - [(1 - x - xt)^2 - 4x^2t]^{1/2}) = t^{-1}F(x,t)$. Defining $\mathscr{N}_n(t) = \sum_{k=1}^{n}(1/n)\binom{n}{k}\binom{n}{k-1}t^{n-k}$ we obtain $F(xt, t^{-1}) = \sum_{n\geqslant 1}\mathscr{N}_n(t)x^n$. For convenience we set $G(x,t) = F(xt, t^{-1})$, so that $G(x,t) = \sum_{n\geqslant 1}\mathscr{N}_n(t)x^n$. It follows from the preceding remarks that $d_n = \mathscr{N}_n(4)$ for all $n \geqslant 1$. Therefore, $1 + \sum_{n\geqslant 1}\mathscr{N}_n(4)x^n = 1 + F(4x, 4^{-1}) = (1/8x)[1 + 3x - (9x^2 - 10x + 1)^{1/2}]$ is the generating function for paths in $\mathscr{D}_0$, in agreement with Theorem 2.2(a).

The function $G(x,x)$ has an interesting interpretation: $1+G(x,x)=(1/2x^2)(1-x+x^2-[1-2x-x^2-2x^3+x^4]^{1/2})$ is the generating function for the number of *secondary structures* with n vertices; see [14, Exercise 6.43(a)] for definitions and references. From the expansion

$$1+G(x,x)=1+\sum_{n\geqslant 1}\mathcal{N}_n(x)x^n=1+\sum_{n\geqslant 1}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{n-k}\binom{n-k}{k+1}\binom{n-k}{k}x^n$$

we deduce that the number of secondary structures with n vertices is $\sum_{k=0}^{\lfloor n/2\rfloor}[1/(n-k)]\binom{n-k}{k+1}\binom{n-k}{k}$. Schmitt and Waterman [11] gave a combinatorial proof of a stronger result, namely, that $(1/k)\binom{n-k}{k+1}\binom{n-k-1}{k-1}=[1/(n-k)]\binom{n-k}{k+1}\binom{n-k}{k}$ is the number of secondary structures with n vertices that have exactly k *pairs*.

The sequences $(\mathcal{N}_n(1))_{n\geqslant 1}$ and $(\mathcal{N}_n(2))_{n\geqslant 1}$ are well known. Indeed $1+\sum_{n\geqslant 1}\mathcal{N}_n(1)x^n=1+F(x,1)=(1/2x)[1-(1-4x)^{1/2}]$ is the generating function for the Catalan numbers $c_n=[1/(n+1)]\binom{2n}{n}$. And $1+\sum_{n\geqslant 1}\mathcal{N}_n(2)x^n=1+F(2x,2^{-1})=(1/4x)[1+x-(x^2-6x+1)^{1/2}]$ is the generating function for the number of arbitrary bracketings (parenthesizations) of a string of length n. The enumeration of such bracketings is known as Schröder's second problem (see [14, Example 6.2.8]). It follows that $\mathcal{N}_n(2)=s_n$ is the nth little Schröder number for all $n\geqslant 1$. Stanley [14, Exercise 6.39] provides a wealth of combinatorial interpretations for s_n.

The polynomials $\mathcal{N}_n(t)=\sum_{k=1}^{n}(1/n)\binom{n}{k}\binom{n}{k-1}t^{n-k}$ have an additional representation that is worth noting. To derive it we first observe that (4.1) entails

$$xtG^2+x(1+t)G+x=G, \tag{4.3}$$

a functional equation which appears in [9]. Solving for x gives $x=G/(1+G)(1+tG)=H(G)$, say. Applying the Lagrange inversion formula we find

$$[x^n]H^{-1}(x)=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n-1}{2k}t^k(1+t)^{n-1-2k}.$$

This implies that

$$\mathcal{N}_n(t)=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}c_k\binom{n-1}{2k}t^k(1+t)^{n-2k-1}, \tag{4.4}$$

where $c_k=[1/(k+1)]\binom{2k}{k}$ is the kth Catalan number. Expanding the right side of (4.4) and then comparing coefficients yields $N(n,k)=\sum_{r=0}^{k-1}c_r\binom{n-1}{2r}\binom{n-2r-1}{k-r-1}$. A combinatorial proof of this result appears in Simion and Ullman [13]. Setting $t=-1$ in (4.4) gives

$$\sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}(-1)^{n-k}=\begin{cases}0 & \text{if } n=2r,\\ (-1)^r c_r & \text{if } n=2r+1,\end{cases}$$

an identity proved in [1] by using properties of a certain symmetric chain decomposition of the lattice of noncrossing partitions previously constructed in [13]. Setting $t=1$ in

(4.4) gives an identity found originally by Touchard [17]:

$$c_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} c_k \binom{n-1}{2k} 2^{n-2k-1}. \tag{4.5}$$

Eq. (4.5) has a well-known combinatorial interpretation, namely, that the Motzkin paths of length $n-1$ in which the level steps have weight 2 (i.e., can be assigned one of two colors) are counted by the Catalan number c_n. Setting $t=2$ in (4.4) gives another representation for little Schröder numbers: $s_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} c_k \binom{n-1}{2k} 2^k 3^{n-2k-1}$. Gouyou-Beauchamps and Vauquelin [7] applied the DSV method of Schützenberger to construct a bijective proof of this last identity. Setting $t=4$ in (4.4) yields another representation for the numbers $d_n : d_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} c_k \binom{n-1}{2k} 4^k 5^{n-2k-1}$. As pointed out by the referee this implies that Motzkin paths of length $n-1$ in which the level steps have weight 5 and the down steps (or up steps) have weight 4 are counted by the number d_n. It remains an open problem to find a bijective proof of this fact.

Several authors have studied the polynomials $\mathcal{N}_n(t)$ in one context or another. For instance:

(i) Rogers [9] showed that the polynomials $\mathcal{N}_n(t)$ satisfy the nonlinear recurrence

$$\mathcal{N}_n = (1+t)\mathcal{N}_{n-1} + t\sum_{k=1}^{n-2} \mathcal{N}_k \mathcal{N}_{n-k-1} \tag{4.6}$$

for all $n \geqslant 3$. In addition he showed that $\mathcal{N}_n(k)$ is the number of increasing bipartite graphs, with partite sets of cardinality n, each of whose edges is colored independently with one of k colors.

(ii) For positive integers k and n Rogers and Shapiro [10] showed that $k\mathcal{N}_n(k)$ is the number of ordered (i.e., rooted plane) trees with n edges in which each of the *eldest branches* has weight k.

(iii) Sulanke [14] showed that the polynomials $(\mathcal{N}_n(t))_{n\geqslant 1}$ satisfy the second-order linear recurrence $\mathcal{N}_1(t)=1$, $\mathcal{N}_2(t)=1+t$ and

$$\begin{aligned}(n+1)&\mathcal{N}_n(t)\\ &=(2n-1)(1+t)\mathcal{N}_{n-1}(t)-(n-2)(t-1)^2\mathcal{N}_{n-2}(t)\end{aligned} \tag{4.7}$$

for all $n \geqslant 3$.

(iv) Bonin et al. [1] studied a polynomial closely related to $\mathcal{N}_n(t)$. These authors let $\mathrm{Sch_L}(n)$ denote the set of underdiagonal lattice paths that terminate at (n,n) and have step set $S^d = \{(1,0),(1,1),(0,1)\}$. (It is well known that $|\mathrm{Sch_L}(n)|$ is the Schröder number r_n.) And for any path $P \in \mathrm{Sch_L}(n)$ they let $\mathrm{diag}(P)$ denote the number of occurrences in P of the step (1,1). They then defined a q-analog of the Schröder number r_n by the equation $d_n(q) = \sum_{P\in \mathrm{Sch_L}(n)} q^{\mathrm{diag}(P)}$. One of the properties of $d_n(q)$ they derived is that $d_n(q) = \sum_{k\geqslant 1}(1/n)\binom{n}{k}\binom{n}{k-1}(1+q)^k$. From this it follows that $d_n(q) = (1+q)\mathcal{N}_n(1+q)$.

5. Underdiagonal paths with unrestricted steps

Let us digress for a moment to consider a class of underdiagonal paths with unrestricted steps. Let $\mathscr{G}(x) = 1 + \sum_{n \geqslant 1} g_n x^n$ be the generating function for the number of underdiagonal paths that terminate at (n, n) and have step set $S^* = \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$. As an application of his theorem on multiplicative decompositions of Laurent series, Gessel [6, Example 4, p. 329] proved that $\mathscr{G}(x) = (1/8x)[1 + 2x - (4x^2 - 12x + 1)^{1/2}]$ and thus

$$g_n = \sum_{k=0}^{n} (-1)^k \frac{2^{2n-k}}{2n-k+1} \binom{2n-k+1}{n-k+1, n-k, k}.$$

We show that in fact $g_n = 2^n s_n$.

It is well known [14, Exercise 6.39] that there are s_n underdiagonal paths that terminate at (n, n) and have step set $S = \{(0,1)\} \cup \{(k,0) : k \in \mathbb{N}^+\}$. We claim that from any such path $\Gamma = s_1 s_2 \cdots s_k$ we can construct 2^n underdiagonal paths Γ' that terminate at (n, n) and have step set S^*. To verify this let us assume that $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$ are the vertical unit steps in Γ. For each $j = 1, 2, \ldots, n$ we have two choices: either replace the sequence of steps $\cdots s_{i_j-2} s_{i_j-1} s_{i_j} s_{i_j+1} \cdots$ by $\cdots s_{i_j-2}(s_{i_j-1} + s_{i_j}) s_{i_j+1} \cdots$ or leave it unaltered. In this way, we associate with Γ a total of 2^n underdiagonal paths that terminate at (n, n) and have step set S^*. Thus $g_n = 2^n s_n$.

Stanley [14, Exercise 6.39] shows that $4g_n = 2^{n+2} s_n$ is the number of simple graphs $G(V, E)$ on the vertex set $V = \{1, 2, \ldots, n+2\}$ with the property that if $a < b < c < d$ belong to V then it is not the case that both $\{a, c\}$ and $\{b, d\}$ belong to E. Such graphs are called *noncrossing*. We refer the reader to [4] for an interesting study of noncrossing combinatorial configurations.

Let $\bar{\mathscr{G}}(t) = \sum_{n \geqslant 0} \bar{g}_n t^n$ where $\bar{g}_n$ denotes the number of primitive underdiagonal paths that terminate at (n, n) and have step set S^*. In order to find the coefficients $\bar{g}_n$ we first note that the same argument that justifies (2.4) also implies $\mathscr{G}(1 - \bar{\mathscr{G}}) = 1$, or $\bar{\mathscr{G}} = (\mathscr{G} - 1)\mathscr{G}^{-1}$. Since $\mathscr{G} = (1/8x)[1 + 2x - (4x^2 - 12x + 1)^{1/2}]$ we obtain $\bar{\mathscr{G}} = \frac{1}{2}[1 - 2x - (4x^2 - 12x + 1)^{1/2}]$. As is well known the generating function of the Schröder numbers r_n is given by $R(x) = \sum_{n \geqslant 0} r_n x^n = (1/2x)[1 - x - (1 - 6x + x^2)^{1/2}]$. Since $R(2x) = (1/4x)[1 - 2x - (1 - 12x + 4x^2)^{1/2}]$ we find $\sum_{n \geqslant 1} 2^n r_{n-1} x^n = 2xR(2x) = \frac{1}{2}[1 - 2x - (4x^2 - 12x + 1)^{1/2}] = \bar{\mathscr{G}}(x) = \sum_{n \geqslant 1} \bar{g}_n x^n$. Equating coefficients gives $\bar{g}_n = 2^n r_{n-1}$ for all $n \geqslant 1$.

6. Enumerating paths in $\mathscr{D}_0$ with respect to number of steps

We resume our study of $\mathscr{D}_0$. For any path $\Gamma \in \mathscr{D}_0(n)$ let $\lambda(\Gamma)$ denote the number of steps in Γ. In this section, we propose to enumerate paths in $\mathscr{D}_0(n)$ with respect to the parameter λ. To this end we define the polynomial $\mathscr{P}_n(s) = \sum_{\Gamma \in \mathscr{D}_0(n)} s^{\lambda(\Gamma)}$. If for each $j = 2, 3, \ldots, 2n$ we let λ_j denote the number of paths $\Gamma \in \mathscr{D}_0(n)$ that have exactly j steps then $\mathscr{P}_n(s) = \sum_{j=2}^{2n} \lambda_j s^j$. Our goal is to compute explicitly the coefficients λ_j.

Let $G(t,s)=\sum_{n\geqslant 0}\mathscr{P}_n(s)t^n$ be the generating function of the polynomials $\mathscr{P}_n(s)$, where t marks size and s marks number of steps. In the same way that (2.6) was derived for the generating function $D_0(t)$ one can show that

$$G-1=tG[s^2+(1+s)^2(G-1)]. \tag{6.1}$$

The Lagrange inversion formula now gives $\mathscr{P}_n(s)=[t^n]G(t,s)=\sum_{k=1}^{n}(1/n)\binom{n}{k}\binom{n}{k-1}s^{2k}(1+s)^{2n-2k}$. Let us summarize these remarks.

Theorem 6.1. *For any positive integer n define* $\mathscr{P}_n(s)=\sum_{\Gamma\in\mathscr{D}_0(n)}s^{\lambda(\Gamma)}$ *where* $\lambda(\Gamma)$ *denotes the number of steps in* Γ. *Then* $\mathscr{P}_n(s)=\sum_{k=1}^{n}(1/n)\binom{n}{k}\binom{n}{k-1}s^{2k}(1+s)^{2n-2k}$. *Moreover, the number of paths in* $\mathscr{D}_0(n)$ *with j steps is given by* $\lambda_j=[s^j]\mathscr{P}_n(s)=\sum_{k=1}^{n}(1/n)\binom{n}{k}\binom{n}{k-1}\binom{2n-2k}{j-2k}$.

From the equation $\mathscr{P}_n(x)=\sum_{k=1}^{n}(1/n)\binom{n}{k}\binom{n}{k-1}x^{2k}(1+x)^{2n-2k}$ we immediately obtain $\mathscr{P}_n(x)=x^{2n}\mathscr{N}_n((1+x^{-1})^2)$. Simply for illustration we exhibit the first few polynomials $\mathscr{P}_n(x)$:

$$\begin{aligned}
&\mathscr{P}_1(x)=x^2,\\
&\mathscr{P}_2(x)=x^2+2x^3+2x^4,\\
&\mathscr{P}_3(x)=x^2+4x^3+9x^4+10x^5+5x^6,\\
&\mathscr{P}_4(x)=x^2+6x^3+21x^4+44x^5+57x^6+42x^7+14x^8,\\
&\mathscr{P}_5(x)=x^2+8x^3+38x^4+116x^5+240x^6+336x^7+308x^8+168x^9+42x^{10},\\
&\mathscr{P}_6(x)=x^2+10x^3+60x^4+240x^5+680x^6+1392x^7+2060x^8\\
&\qquad\qquad +2160x^9+1530x^{10}+660x^{11}+132x^{12}.
\end{aligned}$$

It follows from (6.1) that the polynomials $\mathscr{P}_n(x)$ satisfy the nonlinear recurrence $\mathscr{P}_1=x^2$, $\mathscr{P}_2=x^2[x^2+(1+x)^2]$ and $\mathscr{P}_n=[x^2+(1+x)^2]\mathscr{P}_{n-1}+(1+x)^2\sum_{k=1}^{n-2}\mathscr{P}_k\mathscr{P}_{n-k-1}$ for all $n\geqslant 3$. Using the fact that $\mathscr{P}_n(x)=x^{2n}\mathscr{N}_n((1+x^{-1})^2)$, $n\geqslant 1$, together with (4.7) one can also show that the polynomials $\mathscr{P}_n$ satisfy the second-order linear recurrence $\mathscr{P}_1=x^2$, $\mathscr{P}_2=x^2[x^2+(1+x)^2]$ and $(n+1)\mathscr{P}_n(x)=(2n-1)[(1+x)^2+x^2]\mathscr{P}_{n-1}(x)-(n-2)(1+x)^2[(1+x)^2-x^2]\mathscr{P}_{n-2}(x)$ for all $n\geqslant 3$.

A variant of the polynomial $\mathscr{P}_n(x)$ appeared in some recent work of Denise and Simion [3]. As part of a program to find the generating function $\mathscr{G}_n$ for Dyck paths (counted according to their length) whose number of *exterior pairs* is n, these authors defined combinatorially a sequence of polynomials $\mathscr{R}_n=\mathscr{R}_n(x)$ satisfying the recurrence $\mathscr{R}_1=1$ and $\mathscr{R}_n=x^2(1-x)^2\sum_{k=1}^{n-2}\mathscr{R}_k\mathscr{R}_{n-k-1}+(1-2x+2x^2)\mathscr{R}_{n-1}$ for all $n>1$. They further showed that $\mathscr{R}_n(x)=\sum_{k=0}^{n-1}(-1)^k c_{k+1}\binom{n-1}{k}x^k(1-x)^k$, where c_{k+1} is the $(k+1)$st Catalan number. Denise and Simion's interest in the polynomials $\mathscr{R}_n(x)$ arose from their discovery that

$$\mathscr{G}_0 = \frac{1-x}{1-2x} \quad \text{and} \quad \mathscr{G}_n = \frac{x^{n+2}(1-x)}{(1-2x)^{2n+1}}\,\mathscr{R}_n \quad \text{for } n \geqslant 1.$$

The polynomials $\mathscr{P}_n(x)$ and $\mathscr{R}_n(x)$ are related by the equation $\mathscr{P}_n(x) = t^2\mathscr{R}_n(-x)$, that is,

$$\sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} x^{2k}(1+x)^{2n-2k} = x^2 \sum_{k=0}^{n-1} c_{k+1}\binom{n-1}{k} x^k(1+x)^k. \quad (6.2)$$

This can be verified by a straightforward calculation. As an immediate corollary we deduce that the number of paths $\Gamma \in \mathscr{D}_0(n)$ with j steps is given by

$$\lambda_j = [t^j][t^2\mathscr{R}_n(-t)] = \sum_{r=0}^{\lfloor (j-2)/2 \rfloor} c_{j-r-1}\binom{n-1}{r}\binom{n-r-1}{j-2r-2}. \quad (6.3)$$

It is an open problem to find a combinatorial explanation for Eq. (6.2).

One can use (6.2) to generate numerical identities. For instance:

(i) setting $t=1$ yields another representation for the number $d_n = \mathscr{N}_n(4)$, namely, $d_n = \sum_{k=0}^{n-1} c_{k+1}\binom{n-1}{k}2^k$;
(ii) setting $t = -\frac{1}{2}$ yields $c_n = \sum_{k=0}^{n-1}(-1)^k c_{k+1}\binom{n-1}{k}4^{n-k-1}$;
(iii) setting $t = -\frac{1}{3}$ yields $d_n = \sum_{k=0}^{n-1}(-1)^k c_{k+1}\binom{n-1}{k}2^k g^{n-k-1}$.

7. Arbitrary paths with step set $\mathscr{S}$

For each integer $n \geqslant 1$ let $\mathscr{E}_0(n)$ denote the set of paths with step set $\mathscr{S} = \{(k,0): k \in \mathbb{N}^+\} \cup \{(0,k): k \in \mathbb{N}^+\}$ that begin at (0,0) and terminate at (n,n). Thus paths in $\mathscr{E}_0(n)$, in contrast to those in $\mathscr{D}_0(n)$, may rise above the line $y=x$. Set $e_n = |\mathscr{E}_0(n)|$ and $E_0(t) = 1 + \sum_{n\geqslant 1} e_n t^n$. The first few terms of the sequence $(e_n)_{n\geqslant 0}$ appear along the main diagonal of the following partial array e. We denote by $e_{n,k}$ the entry in the nth row and kth column of e. The array e can be generated by setting $e_{0,0}=1$, initially setting all other $e_{n,k}$ to zero and then recursively computing $e_{n,k} = \sum_{i=0}^{n-1} e_{i,k} + \sum_{j=0}^{k-1} e_{n,j}$.

$e_{n,k}$	$k=0$	1	2	3	4	5	6	7
$n=0$	1	1	2	4	8	16	32	64
1	1	2	5	12	28	64	144	320
2	2	5	14	37	94	232	560	1328
3	4	12	37	106	289	760	1944	4864
4	8	28	94	289	838	2329	6266	16428
5	16	64	232	760	2329	6802	19149	52356
6	32	144	560	1944	6266	19149	56190	159645
7	64	320	1328	4864	16428	52356	159645	470010

In this section, we study some enumerative aspects of e analogous to those of d given in preceding sections. We begin by finding E_0.

Theorem 7.1. (a) *The generating functions* E_0 *and* D_0 *satisfy* $E_0 = 1 + 2(D_0 - 1)E_0$.

(b) *The generating function* E_0 *satisfies the functional equation* $(9t-1)E_0^2 - (9t-1)E_0 + 2t = 0$.

(c) *The generating function* $E_0(t)$ *is given by* $E_0(t) = (9t - 1 - (9t^2 - 10t + 1)^{1/2})/2(9t-1)$.

Proof. (a) We note that every nonempty path in $\mathscr{E}_0$ either:

(i) begins with a horizontal step and thus decomposes uniquely into a nonempty path in $\mathscr{D}_0$ followed by a (possibly empty) path in $\mathscr{E}_0$, or
(ii) is the reflection in the line $y = x$ of a path of type (i).

The result is now immediate.

(b) Eliminate D_0 from (2.6) and the equation in part (a).

(c) This follows from part (b). $\square$

One can also obtain E_0 from the theory of algebraic generating functions. For if we let $F(x,t)$ denote the generating function of the numbers $e_{n,k}$ then

$$F(x,t) = \left(1 - \sum_{n\geqslant 1} x^n - \sum_{k\geqslant 1} t^k\right)^{-1} = \frac{(1-x)(1-t)}{1 - 2(x+t) + 3xt}.$$

Moreover, E_0 is algebraic since it is the *diagonal* of the rational function F. One can now extract E_0 from F by means of a well-known procedure based on Puiseux's theorem. For a discussion of this procedure we refer the interested reader to Stanley [14, Section 6.3].

Theorem 7.1(c) leads to a second-order linear recurrence for $(e_n)_{n\geqslant 0}$.

Theorem 7.2. *The sequence* $(e_n)_{n\geqslant 0}$ *satisfies the second-order linear recurrence* $ne_n = 2(5n-3)e_{n-1} - 9(n-2)e_{n-2}$ *for all* $n \geqslant 3$, *with initial conditions* $e_1 = 2$ *and* $e_2 = 14$.

Proof. From Theorem 7.1(c) we find $(9t^2 - 10t + 1)^{1/2} = (9t-1)(1-2E_0)$. Differentiating with respect to t gives

$$\frac{9t-5}{(9t^2-10t+1)^{1/2}} = -2(9t-1)E_0' + 9(1-2E_0).$$

Differentiating this latter equation now gives

$$-2[(9t-1)E_0'' + 18E_0'] = -\frac{16}{(9t^2-10t+1)^{3/2}}.$$

Therefore $(9t^2 - 10t + 1)[(9t - 1)E_0'' + 18E_0'] = 8/(9t^2 - 10t + 1)^{1/2} = -4(9t - 1)E_0'$. This can be rewritten in the form

$$(9t^2 - 10t + 1)E_0'' + 2(9t - 7)E_0' = 0. \tag{7.1}$$

Substituting the series $E_0 = \sum_{n\geqslant 0} e_n t^n$, $E_0 = \sum_{n\geqslant 1} n e_n t^{n-1}$ and $E_0'' = \sum_{n\geqslant 2} n(n-1)e_n t^{n-2}$ into (7.1) yields

$$\sum_{n\geqslant 2} (n+1)[(n+2)e_{n+2} - 2(5n+7)e_{n+1} + 9ne_n]t^n = 0,$$

which implies $(n+2)e_{n+2} = 2(5n+7)e_{n+1} - 9ne_n$ for all $n \geqslant 2$. Replacing n by $n-2$ gives $ne_n = 2(5n-3)e_{n-1} - 9(n-2)e_{n-2}$ for all $n \geqslant 4$. Notice that this recurrence also holds for $n = 3$. $\square$

It is easy to find an explicit expression for e_n. For if we set $K = E_0 - 1$ then Theorem 7.1(b) implies that $t = K(K+1)/(9K^2 + 9K + 2)$. Applying the Lagrange inversion formula now yields

Corollary 7.3. *For any $n \geqslant 1$ let $e_n = |\mathscr{E}_0(n)|$. Then $e_n = \sum_{k=0}^{n-1} (-1)^{n-k-1} 2^{n-k} 9^k \binom{n}{k} \binom{2n-2k-2}{n-k-1}$.*

A direct enumeration of the paths in $\mathscr{E}_0(n)$ leads to another representation for e_n.

Theorem 7.4. *For each $n \geqslant 1$ let e_n denote the number of lattice paths with step set $\mathscr{S} = \{(k,0) : k \in \mathbb{N}^+\} \cup \{(0,k) : k \in \mathbb{N}^+\}$ that begin at (0,0) and terminate at (n,n). Then $e_n = \sum_{k=1}^{n} 4^{n-k} \binom{n-1}{k-1} [\binom{n}{k} + \binom{n-1}{k-1}]$.*

Proof. Any path Γ in $\mathscr{E}_0(n)$ corresponds to an ordered pair (α, β) of compositions of n. The parts of α and β are simply the lengths of the horizontal and vertical segments of Γ. For any composition γ of n let $l(\gamma)$ denote the number of parts of γ. Note that if (α, β) is the composition pair associated with a path $\Gamma \in \mathscr{E}_0(n)$ then $|l(\alpha) - l(\beta)| \leqslant 1$. It is easy to count the number of pairs (α, β) of compositions of n satisfying $|l(\alpha) - l(\beta)| \leqslant 1$: since n has $\binom{n-1}{k-1}$ compositions into exactly k parts there are $\sum_{k=1}^{n} \binom{n-1}{k-1}^2$ pairs of compositions (α, β) for which $l(\alpha) = l(\beta)$. And there are $\sum_{k=1}^{n} \binom{n-1}{k-1}\binom{n-1}{k}$ pairs of compositions (α, β) for which $|l(\alpha) - l(\beta)| = 1$.

We now observe that if Γ has k horizontal segments and k vertical segments then there are 2^{2n-2k} paths Γ' in $\mathscr{E}_0(n)$ with $\sigma(\Gamma') = \sigma(\Gamma)$. And if Γ has $k+1$ horizontal and k vertical segments or k horizontal and $k+1$ vertical segments then there are $2^{2n-2k-1}$ paths Γ' in $\mathscr{E}_0(n)$ with $\sigma(\Gamma') = \sigma(\Gamma)$. Finally we note that there are two paths in $\mathscr{E}_0(n)$ associated with an ordered pair (α, β) of compositions of n. This is because we can take the segments of α to be horizontal and those of β to be vertical or vice versa. It follows therefore that

$$e_n = 2\sum_{k=1}^{n} 2^{2n-2k} \binom{n-1}{k-1}^2 + 2\sum_{k=1}^{n} 2^{2n-2k-1} \binom{n-1}{k-1}\binom{n-1}{k}$$

$$= \sum_{k=1}^{n} 2^{2n-2k} \binom{n-1}{k-1} \left[2\binom{n-1}{k-1} + \binom{n-1}{k} \right]$$

$$= \sum_{k=1}^{n} 4^{n-k} \binom{n-1}{k-1} \left[\binom{n}{k} + \binom{n-1}{k-1} \right]. \qquad \square$$

Our final result concerns the diagonals of e. For each $k \geqslant 1$ let $\mathscr{E}_k$ denote the set of lattice paths with step set $\mathscr{S} = \{(k,0): k \in \mathbb{N}^+\} \cup \{(0,k): k \in \mathbb{N}^+\}$ that begin at the origin and terminate on the kth diagonal $y = x - k$. Let $E_k(t) = \sum_{n \geqslant k} e_{n,n-k} t^n$ be the corresponding generating function. For convenience we rewrite the equation in Theorem 7.1(a) in the form

$$D_0 = \frac{3E_0 - 1}{2E_0}. \tag{7.2}$$

Theorem 7.5. *In terms of the notation introduced above*:

(a) $E_k = E_0 D_k$ *for all* $k \geqslant 1$,
(b) $E_k = 2tD_0E_{k-1}$ *for all* $k \geqslant 2$,
(c) $E_k = tE_{k-1}((3E_0 - 1)/E_0)$ *for all* $k \geqslant 2$,
(d) $E_k = 2^{k-1}E_0(tD_0)^k(2D_0 - 1)$ *for all* $k \geqslant 2$,
(e) $E_k = \frac{1}{2}(2E_0 - 1)t^k((3E_0 - 1)/E_0)^k$ *for all* $k \geqslant 2$,
(f) $2E_k = (1 + 3t)E_{k-1} - 2tE_{k-2}$ *for all* $k \geqslant 3$.

Proof. (a) Any path in $\mathscr{E}_k$ decomposes uniquely into a (possibly empty) path in $\mathscr{E}_0$ followed by a path in $\mathscr{D}_k$, whence the result.

(b) If $k \geqslant 2$ then by part (a) and Theorem 3.2 we find $E_k = E_0D_k = E_0 2tD_0D_{k-1} = 2tD_0E_{k-1}$.

(c) This follows from part (b) and (7.2).

(d) Part (a) and Corollary 3.3 imply that $E_k = E_0D_k = 2^{k-1}E_0(tD_0)^k(2D_0 - 1)$ for all $k \geqslant 2$. Consequently each entry on the kth diagonal of e is divisible by 2^{k-1}.

(e) This follows from part (d) and (7.2).

(f) If $k \geqslant 3$ then part (a) and Theorem 3.5 imply $2E_k = E_0 2D_k = E_0[(1+3t)D_{k-1} - 2tD_{k-2}] = (1 + 3t)E_{k-1} - 2tE_{k-2}$. $\square$

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