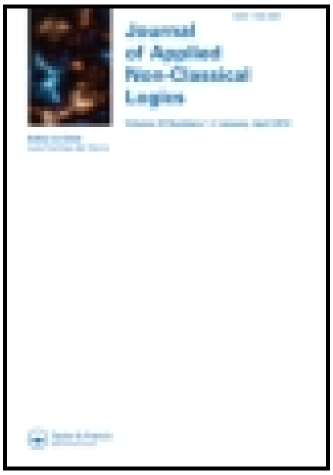

Journal of Applied Non-Classical Logics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tncl20

μ-programs, uniform interpolation and bisimulation quantifiers for modal logics ★

Giovanna D'Agostino [a] , Giacomo Lenzi [b] & Tim French [c]

[a] University of Udine, Department of Mathematics and Computer Science , Viale delle Scienze 206, 33100 , Udine , Italy

[b] University of Pisa, Department of Mathematics , Via Buonarroti 2, 56127 , Pisa , Italy

[c] The University of Western Australia, School of Computer Science and Software Engineering , 35 Stirling Highway, 6009 , Perth , Western Australia

Published online: 13 Apr 2012.

μ-programs, uniform interpolation and bisimulation quantifiers for modal logics *

Giovanna D'Agostino* — **Giacomo Lenzi**** — **Tim French*****

* *University of Udine,*
Department of Mathematics and Computer Science
Viale delle Scienze 206, 33100 Udine (Italy)

dagostin@dimi.uniud.it

** *University of Pisa,*
Department of Mathematics
Via Buonarroti 2, 56127 Pisa (Italy)

lenzi@mail.dm.unipi.it

*** *The University of Western Australia,*
School of Computer Science and Software Engineering
35 Stirling Highway, 6009 Perth, (Western Australia)

tim@csse.uwa.edu.au

ABSTRACT. *We consider the relation between the uniform interpolation property and the elimination of non-standard quantifiers (the bisimulation quantifiers) in the context of the μ-calculus. In particular, we isolate classes of frames where the correspondence between these two properties is nicely smooth.*

KEYWORDS: *modal logics, uniform interpolations, bisimulation quantifiers*

1. Introduction

The property of uniform interpolation for a logic states that, given a formula ϕ and a sublanguage L' of the language $L(\phi)$ of ϕ, we can find an L'-formula θ having the same logical consequences as ϕ, if we restrict to consequences that don't use symbols in $L(\phi) \setminus L(\theta)$. In the context of extended modal logics, such a property

$\star$. The first author has been partially supported by grant INTAS No. 04-77-7080 and by the McTafi Project.

is known to hold e.g. for the basic modal logic K, for Gödel Löb's Logic, and for the μ-calculus, while it does not hold e.g. for $K4, S4$. Since uniform interpolation implies the modularization of the logic, it is important to know if a logic enjoys it, and if it does not, if it is possible to extend the logic in a natural way in order to achieve uniform interpolation. In this paper we prove that for natural classes of modal logic, the idempotent transductions classes, the extension with least fixed points of definable monotone operators provides for such an extension. If we further restrict to classes of transitive frames, we can also prove that this extension is minimal.

Our approach to uniform interpolation is via bisimulation quantifiers: if we interpret a formula $\widetilde{\exists} P\phi$ as true in a model M whenever we can find a bisimilar (up to P) model N where ϕ is true, then, under some conditions, the elimination of such quantifiers is sufficient to ensure the uniform interpolation of the logic. We take advantage of the fact that in the whole class of frames the elimination of bisimulation quantifiers holds both for modal logic and for the μ-calculus, and we consider special classes of frames (the idempotent transduction classes) where the problem of elimination of quantifiers can be reduced to the elimination in the whole class. This provides a proof of the elimination of bisimulation quantifiers in the μ-calculus of an idempotent transduction class.

We also consider the problem of the robustness of uniform interpolants. If a formula ϕ of a logic L has a uniform interpolant θ in L, then it is possible that θ will no longer be a uniform interpolant for a logic L' based on a richer language: in this paper we provide an example of classes of frames where a modal formula ϕ has a modal uniform interpolant θ which is not a uniform interpolant in the μ-calculus over the same class of frames. We prove however that this is not the case in modal logic over an idempotent transduction transitive class: if a modal uniform interpolant exists at all, then it is a *robust* uniform interpolant in the sense that it works not only for modal logic but also for the μ-calculus.

This paper is organized as follows. In Section 2 we briefly recall some definitions and state some results which are needed in the rest of the paper. In Section 3 we define the idempotent transduction classes of frames, and prove that the μ-calculus over such classes is closed under bisimulation quantifiers and enjoys uniform interpolation. In Section 3 we restrict to transitive frames, prove the minimality of the μ-calculus with respect to extensions of modal logic having uniform interpolation, and the robustness of modal uniform interpolants. Finally, in Section 4 we prove that the property of robustness is not always granted for uniform interpolants.

2. Preliminaries and notation

To understand the paper the reader is supposed to be familiar with the basic definition of Modal Logic and bisimulations. To fix notation, we briefly introduce the μ-calculus.

The set of formulas of the modal μ calculus is defined over a non empty set of atomic actions Λ and a countably infinite set of propositions $Prop$ as

$$\alpha := P \mid \alpha \vee \alpha \mid \neg\alpha \mid \langle a\rangle\alpha \mid \mu P\alpha \ ,$$

where P ranges over propositions and the fixed point operator μ only applies to formulas α where P appears positively (i.e. is under an even number of negations).
A μ-formula ϕ is a modal formula if it does not contain fixed points.
Bound and free occurrences of propositions in a formula ϕ are defined as usual (with P bound in $\mu P\phi$). The language $L(\phi)$ of a μ-formula ϕ is the set of propositions occurring free in ϕ.

We interpret μ formulas over Kripke models, i.e. tuples

$$M = (D^M, r^M, R_a^M, \ldots, P_1^M, \ldots)$$

where D^M is the domain, r^M is an element of the domain, $R_a^M \subseteq D^M \times D^M$ for each $a \in \Lambda$, and $P_i^M \subseteq D^M$, for all i.

Given a Kripke model M, a μ-formula is interpreted in M as a subset $[\![\phi]\!]_M$ of D^M, defined as follows:

$$\begin{array}{lcl} [\![P]\!]_M & := & P^M \\ [\![\neg\phi]\!]_M & := & D^M \setminus [\![\phi]\!]_M \\ [\![\phi \vee \psi]\!]_M & := & [\![\phi]\!]_M \cup [\![\psi]\!]_M \\ [\![\langle a\rangle\phi]\!]_M & := & \{s \in D^M \mid [\![\phi]\!]_M \cap \{t : sR^M t\} \neq \emptyset\} \\ [\![\mu P\phi]\!]_M & := & \bigcap\{S \subseteq D^M \mid [\![\phi]\!]_{M[P:=S]} \subseteq S\} \end{array}$$

where $M[P := S]$ is equal to M except that the proposition P is evaluated as S. Notice that $[\![\mu P\phi]\!]_M$ is the least fixpoint of the monotone operator $S \mapsto [\![\phi]\!]_{M[P:=S]}$.

In the following, we denote $s \in [\![\phi]\!]_M$ by $(M, s) \models \phi$. $M \models \phi$ is used to denote $(M, r^M) \models \phi$. $\Gamma \models \phi$ denotes logical consequence: if $M \models \Gamma$ then $M \models \phi$ for every model M.

If C is a class of Kripke frames (i.e., first order structures for the language $\{r\} \cup \{R_a : a \in \Lambda\}$) we say that a Kripke model M is a C-*model* (or $M \in C$) if the frame underlying M is in C; $\Gamma \models_C \phi$ stands for logical consequence restricted to C-models: $\Gamma \models_C \phi$ means that, for all $M \in C$, if $M \models \Gamma$ then $M \models \phi$. A formula ϕ is said to be *valid* in C if $M \models \phi$, for all $M \in C$. The set of μ-formulas (modal formulas) which are valid in C is denoted by $\mu(C)$ (ML(C), respectively).

2.1. *Uniform interpolants*

If C is a class of frames, ϕ is a μ-formula and $P \in Prop$, then we say that a formula θ in the language $L(\phi) \setminus \{P\}$ is a C-*uniform interpolant* for ϕ w.r.t. P if, for all μ-formulas ψ not containing P it holds:

$$\models_C \phi \to \psi \quad \Leftrightarrow \quad \models_C \theta \to \psi.$$

If we restrict the above condition to *modal* formulas ψ we obtain the corresponding notion of a *modal uniform interpolant.*

Given a class C of frames we say that the μ-calculus (or modal logic) enjoys the *uniform interpolation* property over C if for every (modal) formula ϕ and $P \in Prop$ there exists a (modal) C-uniform interpolant θ.

Uniform interpolation is known to hold e.g. for modal logic and the μ-calculus over the class of all frames, and for modal logic over transitive and conversely well founded frames (where the μ-calculus is expressively equal to modal logic). On the other hand, uniform interpolation fails in e.g. $K4, S4$.

2.2. *Bisimulation quantifiers*

The property of uniform interpolation is related to the possibility of eliminating bisimulation quantifiers, that we now briefly introduce. The language of Bisimulation Quantifiers extends the one of modal μ-calculus by allowing the existential propositional quantification $\widetilde{\exists} P\phi$. The μ-semantics is extended by:

$$M, s \models_C \widetilde{\exists} P\phi \Leftrightarrow \exists N \in C, N \sim_{\neq P} M, \text{ and } N \models_C \phi,$$

where $N \sim_{\neq P} M$ stands for: there is a bisimulation between N and M which respects all propositional variables except P.

To give an example of the semantics of bisimulation quantifiers, consider the class C of all frames and the formula $\phi := \widetilde{\exists} P(\Diamond P \wedge \Diamond\neg P)$. Then

$$M \models_C \phi \text{ iff the root has at least one successor.}$$

In fact, in a model, there exists a P such that $\Diamond P \wedge \Diamond\neg P$ holds if and only if the root has at least two successors (just put one successor of the root in P and one out of P); and a model M is bisimilar to one where the root has two successors if and only if, in M, the root has at least one successor.

Notice that, if we change the class C, then the semantics of a formula in a model may change: e.g. if C' is the class of functional frames (where every world has exactly one successor) then the formula $\widetilde{\exists} P(\Diamond P \wedge \Diamond\neg P)$ is unsatisfiable in C'.

The μ-calculus is said to be *closed under bisimulation quantifiers* over a class C of frames if for every μ-formula ϕ and proposition P there exists a μ-formula θ having the same semantics as $\widetilde{\exists} P\phi$ over C. The same definition applies to modal logic over C.

One can easily show that a formula θ of the μ-calculus which is semantically equivalent to $\widetilde{\exists} P\phi$ is a uniform interpolant for ϕ w.r.t. P. It follows: if the μ-calculus is closed under bisimulation quantifiers over C, then it enjoys the uniform interpolation property, and the same holds for modal logic over C.

We say that the uniform interpolant θ of ϕ w.r.t. P *behaves semantically as a bisimulation quantifier over* C if θ is semantically equivalent to $\widetilde{\exists} P\phi$ over C. In general, uniform interpolants w.r.t. P do not always have the same semantics as the closure under the bisimulation quantifier. However, we shall use the following result:

THEOREM 1 ([DAG 06]). — *Consider a class C of finite transitive frames. Then a uniform interpolant of an ML-formula ψ behaves as a bisimulation quantifiers over C.*

3. Transduction invariant classes

In this Section we introduce the notion of a transduction invariant class of frames, and prove that the μ-calculus is always closed under bisimulation quantifiers over these classes. In order to give the definition of these classes we first need the notion of μ-programs over a class of atomic programs Λ, and of the μ-formulas extended with the μ-programs (extended μ-formulas for short). As usual, the definition is given by a double recursion:

DEFINITION 2. — *Let Λ be a set of atomic programs. The μ-programs π over Λ are:*

$$\pi := a|\pi;\pi|\pi \cup \pi|\pi^*|\alpha?;$$

where $a \in \Lambda$ and

$$\alpha := P \mid \neg\phi \mid \alpha \vee \alpha \mid \langle\pi\rangle\phi \mid \mu P\alpha,$$

provided P is under an even number of negations in α.

A μ-program is *closed* if it contains no free propositions. The set of closed μ-programs over Λ is denoted by $Progr(\Lambda)$. If $a \in \Lambda$, then a^*, $a;a^*$, $a \cup (\top)?$, $(\mu P[a]P)?;a;a^*$ are examples of closed μ-programs over Λ.

The semantics of μ-programs and μ-formulas is defined in the usual way. E.g., given a Kripke models M, and a μ-program π, π^M is a binary relation on M defined inductively in such a way that

$$a^M = R_a^M, \quad (\alpha?)^M = \{(w,w) \in M^2 : M \models \alpha\},$$

$$(\pi^*)^M \text{ is the reflexive and transitive closure of } \pi^M.$$

Since $\langle\pi^*\rangle\phi$ is equivalent to $\mu P(\phi \vee \langle\pi\rangle P)$ and similar equivalences are true for the other program constructs, there is an obvious translation between extended μ-formulas and ordinary μ-formulas, but the explicit use of programs allow the definition of substitutions of programs in a formula.

DEFINITION 3. — *A transduction is defined as a function Π from Λ to $Progr(\Lambda)$.*

A transduction is extended in a unique way from atomic programs to programs and to extended μ-formulas by imposing that Π commutes with the operators over programs and formulas: if π is a program $\Pi(\pi)$ is a program and if α is a μ-formula, $\Pi(\alpha)$ is an extended μ-formula; e.g. if $\Pi(a) = (\mu Q[a]Q)?; a; a^*$ and $\Pi(b) = a^*$, then

$$\Pi(a \cup b) = ((\mu Q[a]Q)?; a; a^*) \cup a^*, \text{ and } \Pi(\mu P[a]P) = \mu P[(\mu Q[a]Q)?; a; a^*)]P.$$

We can also define the result of applying a transduction to a Λ-frame F: $\Pi(F)$ is the frame which has the same domain as F while the interpretation of an atomic relation R_a is given by:

$$(R_a)^{\Pi(F)} = R^F_{\Pi(a)};$$

(notice that $\Pi(a)$ does not contain free propositions, since $\Pi(a) \in Progr(\Lambda)$). If M is a model based on a frame F we define $\Pi(M)$ as the model over $\Pi(F)$ which has the same interpretation of the propositions as M.

We denote by C_Π the result of applying the transduction Π to the class $Fr(\Lambda)$ of all Λ-frames.

$$C_\Pi = \{\Pi(F) : F \in Fr(\Lambda)\}.$$

We have the following adjunction:

LEMMA 4. — *For all models M, extended μ-formulas ϕ and μ-transductions Π it holds*

$$M \models \Pi(\phi) \quad \Leftrightarrow \quad \Pi(M) \models \phi.$$

PROOF. — This is shown by induction over ϕ. Particularly, we will show that if for models M and for all proper subformulas ψ of ϕ we have

$$M \models \Pi(\psi) \Leftrightarrow \Pi(M) \models \psi,$$

then for all models M,

$$M \models \Pi(\phi) \Leftrightarrow \Pi(M) \models \phi.$$

The base case of this induction, where ψ is a propositional atom, is trivially true and the inductive cases for propositional operators are also easy to show. So supppose that $M \models \Pi(\psi) \Leftrightarrow \Pi(M) \models \psi$ and $\phi = \mu P\psi$. From the definition of Π we note that $\Pi(\mu P\psi) = \mu P\Pi(\psi)$. Since $\| \mu P\psi \|_M = \bigcap\{S \subseteq D^M \mid \| \psi \|_{M[P:=S]} \subseteq S\}$ we have

$$\begin{aligned} \| \Pi(\mu P\psi) \|_M &= \| \mu P\Pi(\psi) \|_M \\ &= \bigcap\{S \subseteq D^M \mid \| \Pi(\psi) \|_{M[P:=S]} \subseteq S\} \\ &= \bigcap\{S \subseteq D^M \mid \| \psi \|_{\Pi(M)[P:=S]} \subseteq S\} \\ &= \| \mu P\psi \|_{\Pi(M)} \end{aligned}$$

as required.

Finally, if $\phi = \langle a \rangle \psi$, then by the definition of Π we have $\Pi(\phi) = \langle \Pi(a) \rangle \Pi(\psi)$. Then

$$\begin{aligned} \parallel \Pi(\langle a \rangle \psi) \parallel_M &= \parallel \langle \Pi(a) \rangle \Pi(\psi) \parallel_M \\ &= \{s \in D^M \mid \parallel \Pi(\psi) \parallel_M \cap \{t \;:\; sR^M_{\Pi(a)} t\} \neq \emptyset\} \\ &= \{s \in D^M \mid \parallel \psi \parallel_{\Pi(M)} \cap \{t \;:\; sR^{\Pi(M)}_a t\} \neq \emptyset\} \\ &= \parallel \langle a \rangle \psi \parallel_{\Pi(M)}, \end{aligned}$$

and the lemma follows. ■

We now declare a transduction Π to be *idempotent* if applying it two times to a frame is the same as applying it only once:

DEFINITION 5 ([FRE 06]). — *A transduction* Π *is* idempotent *if for all* Λ *frames* F *we have* $\Pi(\Pi(F)) = \Pi(F)$.

Examples of idempotent transduction over $\Lambda = \{a\}$ are:

– $\Pi(a) = a \cup (\top)?$,

– $\Pi(a) = a^*$,

– $\Pi(a) = a^*; a$.

– $\Pi(a) = (\mu P[a]P)?; a; a^*$;

the corresponding classes C_Π are, respectively:

– the class of reflexive frames,

– the class of reflexive and transitive frames,

– the class of transitive frames,

– the class of conversely well founded transitive frames.

From Lemma 4 we easily obtain:

LEMMA 6. — *If* Π *is an idempotent transduction then* ϕ *and* $\Pi(\phi)$ *are equivalent over* C_Π.

From Marco Hollenberg's characterization of the *bisimulation safe fragment of monadic second-order logic* [HOL 98] we obtain:

LEMMA 7. — *If* Π *is a transduction, then*

$$M \sim N \Rightarrow \Pi(M) \sim \Pi(N).$$

PROOF. — Suppose that $B \subseteq D^M \times D^N$ is a bisimulation from M to N. We will show that B is also a bisimulation from $\Pi(M)$ to $\Pi(N)$. By induction on the complexity of a program $\pi \in Progr(\Lambda)$ we can prove that, for all $(s, s') \in B$,

Zig for all $t \in D^M$ where $(s,t) \in R^M_\pi$, there is some $t' \in D^N$ where $(s',t') \in R^N_\pi$ and $(t,t') \in B$, and

Zag for all $t' \in D^N$ where $(s', t') \in R^N_\pi$, there is some $t \in D^M$ where $(s, t) \in R^M_\pi$ and $(t, t') \in B$;

(notice that the base case for the atomic programs $\pi = a \in \Lambda$ is given by the Zig-Zag clauses in the definition of bisimulation). Applying this inductive argument for all programs $\Pi(a)$ where $a \in \Lambda$ we have

1) $(r^{\Pi(M)}, r^{\Pi(N)}) \in B$ (since $r^{\Pi(M)} = r^M$ and $r^{\Pi(N)} = r^N$),

2) for all $(s, s') \in B$, for all $P \in Prop$, $s \in P^{\Pi(M)} \Leftrightarrow s' \in P^{\Pi(N)}$, (since $P^{\Pi(M)} = P^M$ and $P^{\Pi(N)} = P^N$),

3) for all $(s, s') \in B$, for all $t \in D^{\Pi(M)} = D^M$ where $(s, t) \in R_a^{\Pi(M)} = R^M_{\Pi(a)}$, there is some $t' \in D^N = D^{\Pi(N)}$ where $(s', t') \in R^N_{\Pi(a)} = R_a^{\Pi(N)}$ and $(t, t') \in B$, and

4) for all $(s, s') \in B$, for all $t' \in D^N = D^{\Pi(N)}$ where $(s', t') \in R_a^{\Pi(N)} = R^N_{\Pi(a)}$, there is some $t \in D^M = D^{\Pi(M)}$ where $(s, t) \in R^M_{\Pi(a)} = R_a^{\Pi(M)}$ and $(t, t') \in B$.

Therefore B is a bisimulation from $\Pi(M)$ to $\Pi(N)$. ■

THEOREM 8. — *If Π is an idempotent transduction then the μ-calculus is closed under bisimulation quantifiers on the class C_Π.*

PROOF. — Let ϕ be a μ-formula, and consider the μ-formula which is equivalent to the extended μ-formula $\Pi(\phi)$. Since the μ-calculus is closed under bisimulation quantifiers over the class $Fr(\Lambda)$ of all Λ-frames [DAG 00], there exists a μ-formula ψ which is semantically equivalent to $\widetilde{\exists} P \Pi(\phi)$, that is: for all Λ-models M it holds:

$$M \models \psi \quad \Leftrightarrow \quad \exists N \in Fr(\Lambda) \qquad M \sim_{\neq P} N \models \Pi(\phi).$$

We want to prove that, if $M \in C_\Pi$, then

$$M \models \psi \quad \Leftrightarrow \quad \exists N \in C_\Pi \qquad M \sim_{\neq P} N \models \phi.$$

($\Rightarrow$) Let $M \in C_\Pi$ be such that $M \models \psi$, and let N be a $Fr(\Lambda)$-model such that $M \sim_{\neq P} N \models \Pi(\phi)$. Then, since $M \in C_\Pi$ and Π is idempotent we have $M \sim_{\neq P} \Pi(N)$, and by Lemma 4 we have $\Pi(N) \models \phi$, $\Pi(N) \in C_\Pi$.

($\Leftarrow$) If $\exists N \in C_\Pi$ with $M \sim_{\neq P} N \models \phi$, then, since ϕ and $\Pi(\phi)$ are equivalent on C_Π we have $N \models \Pi(\phi)$ and $M \models \psi$. ■

COROLLARY 9. — *If Π is an idempotent transduction then the μ-calculus enjoys uniform interpolation over the class C_Π.*

COROLLARY 10. — *Suppose Π is an idempotent transduction such that, for any modal formula ϕ, the extended μ-formula $\Pi(\phi)$ is equivalent to a modal formula over the class $Fr(\Lambda)$ of all frames. Then $ML(C_\Pi)$ is closed under bisimulation quantifiers (and hence it enjoys uniform interpolation).*

PROOF. — If $\Pi(\phi)$ is equivalent to a modal formula over $Fr(\Lambda)$, then (since modal logic is closed under bisimulation quantifiers over $Fr(\Lambda)$) there exists a *modal* formula ψ which has the same semantics as $\widetilde{\exists} P \Pi(\phi)$ over $Fr(\Lambda)$. From the proof of Theorem 8 we see that ψ is equivalent to $\widetilde{\exists} P \phi$ over the class C_Π. ■

The above Corollary gives, in particular, a quick proof of uniform interpolation for the modal logic T of reflexive frames, because if $\Pi(a) = a \cup (\top)?$ then:
Π is idempotent;
the class of reflexive frames is equal to C_Π;
if ϕ is modal then $\Pi(\phi)$ is modal (e.g, $\Pi([a]P) = P \wedge [a]P$).

Uniform interpolants and bisimulation quantifiers on transitive transduction invariant classes

LEMMA 11. — *Suppose C only contains transitive frames and any C-satisfiable μ-formula has a finite C-model. Then a uniform interpolant of a modal formula ϕ w.r.t $ML(C)$, if it exists, is also a uniform interpolant for ϕ w.r.t. $\mu(C)$.*

PROOF. — Consider a modal formula ϕ, a proposition P and a modal uniform interpolant θ for ϕ w.r.t. P. Since $\mu(C)$ has the finite model property, we also have $ML(C) = ML(Cfin)$, where $Cfin$ is the class of all finite frames in C. By Proposition 1 we know that θ behaves as the bisimulation quantifier formula $\widetilde{\exists} P \phi$ over $Cfin$. We use this to prove that θ is a uniform interpolant of ϕ w.r.t. $\mu(C)$. Let ψ be a μ-formula such that $\models_C \phi \to \psi$, where ψ does not contain the variable P, and suppose by contradiction that $\not\models_C \theta \to \psi$: by the f.m.p. of $\mu(C)$ there exists a model M in $Cfin$ such that $M \models \theta \wedge \neg\psi$. But θ behaves as $\widetilde{\exists} P \phi$ over $Cfin$, hence there exists N in $Cfin$ such that $N \models \phi$ and $N \sim_{\neq P} M$. From $\models_C \phi \to \psi$ it follows that $N \models \psi$; since $N \sim_{\neq P} M$ and P does not appear in ψ, we also have and $N \models \neg\psi$, a contradiction. ■

From Lemma 4 it easily follows that the μ-calculus over transducton invariant classes has the finite model property; hence, from Lemma 11 we obtain:

COROLLARY 12. — *Modal uniform interpolants of modal formulae over an idempotent transitive transduction class C_Π, when they exists, are also uniform interpolants w.r.t. $\mu(C_\Pi)$.*

In Section 4 we shall see that this kind of *robustness* of modal interpolants is not always granted if the class C is not an idempotent class.

The following corollary extends in a uniform way a result obtained by Visser about uniform interpolant of $S4$-formulae.

COROLLARY 13. — *Modal uniform interpolants of modal formulae over idempotent transitive transduction classes behave as bisimulation quantifiers.*

PROOF. — Let C_Π be an idempotent transduction class, and let θ be a modal uniform interpolant for ϕ w.r.t. P in $ML(C_\Pi)$. By the above corollary θ is a uniform interpolant w.r.t. ϕ in $\mu(C)$; by Theorem 8 we know that there exists a μ-formula θ' which is semantically equivalent to $\widetilde{\exists}P\phi$ over C_Π. But θ and θ', being uniform interpolants for the same formula, must be equivalent, hence θ behaves as a bisimulation quantifier. ■

Finally, we want to prove that, in a sense, there are no extensions of $ML(C_\Pi)$ which are strictly weaker than $\mu(C_\Pi)$ and have uniform interpolation.

More precisely, given a class C of frames, we consider a set L of μ-formulas which contains all modal logic formulas and is closed under substitutions:
if $\phi(P_1, \ldots, P_n), \psi_1, \ldots, \psi_n \in L$ then $\phi(\psi_1, \ldots, \psi_n)$ is (equivalent over C_Π to) a formula in L.
Given such an L, we let

$$L(C) = \{\phi \in L : \ \phi \text{ is } C\text{-valid}\}.$$

$L(C)$ is called a μ-extension of $ML(C)$.

THEOREM 14. — *Let C_Π be an idempotent transduction class of transitive frames. Then $\mu(C_\Pi)$ is minimal w.r.t. expressive power among the μ-extensions of $ML(C_\Pi)$ having uniform interpolation.*

PROOF. — To prove the Theorem we first notice that the proof of Theorem 1 given in [DAG 06] also holds for the μ-extensions $L(C)$: if θ is the $L(C)$ uniform interpolant of an L-formula ϕ w.r.t. P over a class C of finite transitive frames, then θ behaves semantically as the formula $\widetilde{\exists}P\phi$ over C.

Using this, we prove that if $L(C_\Pi)$ has uniform interpolation then for any μ-formula ϕ there exists an L formula which is equivalent to ϕ over C_Π.

To show this, we first prove that L is closed under bisimulation quantifiers. Consider an L-formula ψ and its uniform interpolant θ w.r.t. P: we shall prove that θ behaves as $\widetilde{\exists}P\psi$ over C_Π. Since $\mu(C_\Pi)$ has the finite model property, and any L-formula is a μ-formula, we know that $L(C_\Pi) = L(C_{\Pi FIN})$, where $C_{\Pi FIN}$ is the class of finite frames in C_Π. By Theorem 1, θ behaves as $\widetilde{\exists}P\psi$ over $C_{\Pi FIN}$. This in turn implies that θ is a uniform interpolant for the formula ψ in $\mu(C_\Pi)$, and by Theorem 8, we know that θ behaves as $\widetilde{\exists}P\psi$ over all C_Π.

Finally, we notice that any extension of $ML(C_\Pi)$ which is closed under bisimulation quantifiers can express all μ-formulae, because over transitive frames we have

$$\nu P\phi \equiv \widetilde{\exists}P(P \wedge (P \to \phi) \wedge \Box(P \to \phi)).$$

■

From [VIS 96, GHI 02, GHI 95] we know that neither $K4$ nor $S4$ have uniform interpolation. It follows:

COROLLARY 15. — *If C is the class of transitive (transitive and reflexive) frames, any μ-extension $L(C)$ of modal logic having uniform interpolation coincides with $\mu(C)$.*

4. Robust uniform interpolants

In this section we consider again Corollary 12 and prove that the generalization of this Corollary to an arbitrary class of (transitive) frames does not hold.

DEFINITION 16. — *Let C be a class of frames, and ϕ a modal formula. A uniform modal interpolant θ of ϕ w.r.t. P is called a* robust *uniform interpolant if θ is also a uniform interpolant of ϕ w.r.t. the logic $\mu(C)$.*

Notice that if θ is semantically equivalent to $\widetilde{\exists} P\phi$ over C, then θ is a robust uniform interpolant for ϕ w.r.t. P. However, we can prove that this semantical behaviour is not a necessary condition for a uniform interpolant to be robust. We consider the logic $GL.3 = ML(C)$ where C is the class of frames over $\Lambda = \{a\}$ where R_a is a strict, linear ordering without strictly ascending infinite R_a-chains. Let ϕ be the modal formula

$$\phi = (T \Rightarrow S) \wedge (S \Rightarrow \Box(\neg T)) \wedge (T \vee \Diamond(T)),$$

where $A \Rightarrow B$ stands for $(A \to B) \wedge \Box(A \to B)$. Then the modal formula $\theta = S \vee \Diamond S$ is a uniform robust interpolant for the formula ϕ w.r.t. to T. First, fixed points are definable in $GL3$ (because they are already definable in GL), hence it is enough to show that θ is a uniform interpolant of ϕ in $GL3$. This can be seen as follows:

1) $\models_C \phi \to \theta$;

2) if $\models_C \phi \to \psi$, and T does not appear in ψ, then $\models_C \theta \to \psi$: otherwise, by the finite model property of $GL3$, there would be a finite strict linear order M where $\theta \wedge \neg\psi$ holds. Consider the model M' which is like M except that $T^{M'} = \{w\}$, where w is a node in M which satisfies S and such that for all v in M, if $vR_a w$ then $v \not\models S$. Then $M' \models \phi$, hence $M' \models \psi$, and since ψ does not contain T, we obtain $M \models \psi$, a contradiction.

On the other hand, θ does not behave semantically as $\widetilde{\exists} T\phi$ over C: consider the model M where the domain is $\{\omega\} \cup \omega$, the root is ω, the accessibility relation is $>$ between ordinals, and all points in ω satisfies S, but ω does not. Then $S \vee \Diamond(S)$ is true in M, but $M \not\models_C \widetilde{\exists} T\phi$.

There are uniform interpolants which are not robust. In the following, we provide an example of such a formula. Let $\Lambda = \{a\}$, and, for any $n \geq 1$, let F_n be the transitive frame having domain equal to $\{1, \dots, n\}$ and $R_a = \{(k, m) : k < m\}$. We consider the frame F obtained by taking the disjoint union of all F_n for $n \in \omega$ to which we add a new root 0, which is related to all other points.

We have:

LEMMA 17. — *A modal formula ψ which is satisfiable in F is also satisfiable in a transitive frame with an ascending R_a-chain.*

Given the lemma, we can consider the class C of transitive frames containing F together with all transitive frames having an ascending R_a-chain.

Then the formula $\phi = P \wedge (P \to \Diamond P) \wedge \Box(P \to \Diamond P)$ has $\top$ as uniform interpolant w.r.t. P in $ML(C)$, but $\top$ is not a uniform interpolant w.r.t. the μ-calculus, because

$$\models_C \phi \to \nu P \Diamond P \text{ and } \not\models_C \top \to \nu P \Diamond P.$$

Hence, the modal uniform interpolant $\top$ is not robust.

We are left to prove Lemma 17. Given a finite set of propositional variables $\mathcal{Q}$ and $n \in \omega$, let $\theta_1, \ldots, \theta_s$ be the finite set of formulas in $\mathcal{Q}$ characterizing models modulo n-bisimulations in the language $\mathcal{Q}$, that is, for all models M there exists a unique i such that

$$N \models \theta_i \Leftrightarrow M \text{ is } n\text{-bisimular to } N \text{ in the language } \mathcal{Q}.$$

The formulas $\theta_1, \ldots, \theta_s$ are called $\mathcal{Q}, n$-characters.

LEMMA 18. — *Fix $n \in \omega$ and $\mathcal{Q}$. Suppose M is a model based on a frame $F_h = (\{1, \ldots, h\}, <)$ in which a $\mathcal{Q}, n$-character θ holds in the root and in at least $n - 1$ other nodes. Then the model M is n-bisimilar w.r.t. $\mathcal{Q}$ to a model M' based on a non conversely well founded transitive frame.*

PROOF. — Consider a countable number of copies M_1, M_2, ... of M, and let r_1, r_2, ... be the respective roots. The model M' is obtained from the disjoint union of the copies $M_1, M_2, \ldots$ by adding to the accessibility relation all pairs (r_i, r_j) where $i < j$, and closing for transitivity.

We claim that M and M' are n-bisimilar w.r.t. $\mathcal{Q}$. To construct an n-bisimulation between M and M', we consider the first n nodes in M which satisfy the character θ: we denote them by $i_0, i_1, \ldots, i_{n-1}$, following the order in M (hence, i_0 is the root in M). We then define the n-bisimulation $(Z_h)_{h=0}^n$ as follows: for $0 \leq h \leq n$, let Z_h be the union of the set

$$\{(u, u') \in M \times M' : u, u' \text{ are } h\text{-bisimilar w.r.t. } \mathcal{Q}\}$$

with

$$\{(i_0, r_j) : j \in \omega\} \cup \{(i_1, r_j) : j \in \omega\} \cup \ldots \cup \{(i_{n-h}, r_j) : j \in \omega\}$$

in other words: if r is a root of one of the M_i's, then a pair (i_j, r) is in Z_h if and only if after i_j there are still at least $h - 1$ nodes of the form i_k with $k > j$. E.g. the pair (i_0, r_j) is in Z_h, for all $0 \leq h \leq n$ and all j, while the pair (i_1, r_j) is in Z_h for all $0 \leq h \leq n - 1$, and so on.

Then one can check that $(Z_h)_{h=0}^n$ is an n-bisimulation between M and M'. ■

PROOF (OF LEMMA 17). — Let N be a model which is based on the frame F such that $N \models \psi$, let n be the modal depth of ψ, and let $\mathcal{Q}$ be the set of ψ variables. Since N has branches of arbitrary finite height, there exists a $\mathcal{Q}, n$-character θ and a branch in N such that θ holds in at least n nodes of the branch different from the root. Let w be the first node in the branch after the root that makes θ true. Let M be the N-submodel which is rooted in w, and let M' be as in Lemma 18. Consider the model

N' which is like N, except that we substitute the submodel M starting from w with the model M' and we close for transitivity. Then N' is n-bisimular to N w.r.t. the language $\mathcal{Q}$, and hence satisfies ψ. ■

5. Conclusions

In this paper we considered special classes of frames, defined by means of idempotent transductions, in which the behaviour of bisimulation quantifiers over the modal logic and the μ-logic of the class is similar to the one over the class of all frames. In particular, we showed that the μ-calculus is always closed under bisimulation quantifiers over idempotent transductions classes, and that modal uniform interpolants behaves as bisimulation quantifiers if we restirct to transitive classes of frames. We also noticed that uniform interpolants behaving as bisimulation quantifiers enjoy a *robustness* property which is not shared by all uniform interpolants. So, for example, although logics as $K4, S4$ do not contain all their uniform interpolants, when a specific formula of these logics has a uniform interpolant, this is very well behaved. It then makes sense to look for a useful characterization of formulae in $K4, S4$ having all uniform interpolants, and, more general, to do the same in idempothent transuctions logics. As a first basic question one could ask: is the set of formulae having all uniform interpolants a decidable set?

6. References

[DAG 00] D'AGOSTINO G., HOLLENBERG M., "Logical questions concerning the μ-calculus", *Journal of Symbolic Logic*, vol. 65, 2000, p. 310–332.

[DAG 06] D'AGOSTINO G., LENZI G., "A Note on Bisimulation Quantifiers and Fixed Points over Transitive Frames", Technical Report num. 1626, March 2006, Department of Mathematics, University of Pisa.

[FRE 06] FRENCH T., "Bisimulation Quantifiers for Modal Logics", PhD Thesis, University of Western Australia, 2006.

[GHI 95] GHILARDI S., ZAWADOWSKI. M., "Undefinability of Propositional Quantifiers in the Modal System $S4$", *Studia Logica*, vol. 55, 1995, p. 259–271.

[GHI 02] GHILARDI S., ZAWADOWSKI M., *Sheaves, Games and Model Completions (a categorical approach to non classical propositional logics)*, Trends in Logic Series, Kluwer, 2002.

[HOL 98] HOLLENBERG M., "Logic and Bisimulation", PhD Thesis, University of Utrecht, Vol XXIV, Zeno Institute of Philosophy, 1998.

[VIS 96] VISSER A., "Uniform interpolation and layered bisimulation", *Goedel '96*, vol. 6 of *Lecture Notes in Logic*, Springer, 1996, p. 139–164.