Solving One and Two-dimensional Exponential Polynomial Systems

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Abstract

Three new symbolic-numeric algorithm for solving one and two special real analytic function systems are proposed. In this paper we only discuss real exponential polynomials but our approaches could be applied to other real analytic functions such as for instance logarithm or sinus polynomials.

We determine an interval which contains all real roots of a real exponential polynomial of one real variable and find the number of its roots. We also find the number of real solutions of systems of two rational bivariate exponential polynomial equations in generalized open rectangles.

Such tasks have already been studied by D. Richardson and K. Roach. Our novelty is to deal with non bounded sets.

1 Introduction

In this paper we study solving and counting the number of solutions of exponential polynomial equations in one and two variables.
Following the line initiated by Richardson we analyse three questions:

(a) finding an interval containing all real roots of $p(x, e^x)$,

(b) finding the number of real roots of a real exponential polynomial $p(x, e^x)$ in a non bounded interval,

(c) finding the number of real solutions (isolated points and branches of curves) of systems of two rational bivariate exponential polynomial equations of the form

$$\begin{array}{rcl} p(x, y, e^y) & = & 0, \\ A(x, e^x, y) & = & 0 \end{array}$$

in generalized open rectangles $I \times J \subset \mathbb{R}^2$ where at least one of the two intervals is bounded.

We propose algorithms for each of these tasks.

Our first algorithm determines an interval which contains all roots of a real exponential polynomial $p(x, e^x)$. Having this interval we can easily determine the last root. Richardson [3], [4] also deals with solving the last root problem using a Thom sequence. Our new approach only needs to find the last root of an exponential polynomial of degree one with respect to e^x whereas the Thom-Richardson's approach has to handle with many polynomials of higher degrees.

Our second algorithm is a variant of Richardson's approach. Using Sturm's theory Richardson reduces (b) to solving one algebraic polynomial. Our approach reduces (b) to solving two algebraic polynomials but they have significantly lower degrees.

Our third algorithm solves the counting problem (c). Recent papers such as [4] and [5] already study this task, but always in bounded open rectangles, whereas we treat unbounded cases as well.

Let us introduce some notations. Let $p(x, y)$ and $q(x, y)$ be bivariate polynomials. We will denote

- p_y (respectively p_x) the derivative of p with respect to y (respectively to x)
- $deg_y(p)$ (respectively $deg_x(p)$) the degree of p with respect to y (respectively to x)
- $lc_y(p)$ (respectively $lc_x(p)$) the leading coefficient of p with respect to y (respectively to x)
- $Res_y(p, q)$ (respectively $Res_x(p, q)$) the resultant of p and q with respect to y (respectively to x)

2 One-dimensional exponential polynomial equations

In this section we consider equations of the type

$$p(x, g(x)) = 0 \tag{1}$$

where p is a bivariate real polynomial and g a transcendental function such as for instance exp, ln, sin, cos, or tan.
Computer algebra systems like Maple or Matlab provide the numerical equation solvers **fsolve** and **fzero**, respectively. However, these solvers neither always return all solutions of (1) nor they always find a solution, even if one exists.
Our aim is to determine the number of real solutions of (1). From the Richardson's work we know that cylindrical algebraic decomposition and local Sturm sequence can be used in order to find the number of real roots of polynomials of x and $g(x)$ with rational coefficients.

Subsection 2.1 recalls Richardson's approach. In subsection 2.2 the last root problem and bounding the last root will be discussed. Finally in subsection 2.3 an extension for polynomials in x and $g(x)$ with real instead of rational coefficients is proposed.

2.1 Richardson's approach

Separation of the solutions

In [2], Richardson proves that the solutions of (1) are separated by the roots of $p_y(x, g(x))$ and those of some further univariate algebraic polynomials.
In the special case $g(x) = e^x$, Richardson shows the following result.

Theorem 1 *Assume $p(x, y)$ to be a bivariate integer polynomial. Then all the real roots of the exponential polynomial $p(x, e^x)$ are separated by the roots of the exponential polynomial $p_y(x, e^x)$ and the two algebraic polynomial $Res_y(p, p_y)$ and $Res_y(p, y\,p_y + p_x)$.*

The recursive algorithm directly suggested by this theorem is not very efficient in practice. But we can borrow from the proof of this theorem the following useful lemma to be used in subsection 2.3 later on.

Lemma 1 *Let $]a, b[$ be an open interval defined by the cylindrical decomposition of $p(x, y)$, let $y = \alpha_1(x)$, $y = \alpha_2(x)$, $\ldots$, $y = \alpha_l(x)$ be the implicit functions defined by $p(x, y) = 0$ on this segment. Then the real roots of $Res_y(p, y\,p_y + p_x)$ separate the roots of $e^x = \alpha_i(x)$ for all $i \in \{1, 2, \ldots, l\}$ in $]a, b[$.*

Sturm-Richardson's method

Richardson's second approach uses Sturm's theory for real analytic functions.
Like in the polynomial case, we can define local Sturm sequences in the following way.

Definition 1 *A finite sequence $(f_0, f_1, f_2, \ldots, f_n)$ of real analytic functions defined on $]a, b[$ is called a local Sturm sequences on $]a, b[$ if the following three conditions are verified:*

1. *the last function f_n does not vanish in $]a, b[$,*
2. *for all $k = 1, 2, \ldots, n \Leftrightarrow 1$,*
$$\forall x \in]a, b[\ \ (f_k(x) = 0 \ \Rightarrow \ f_{k+1}(x)\,f_{k-1}(x) < 0),$$
3. *$f_0'(\alpha)\,f_1(\alpha) > 0$ for every root $\alpha \in]a, b[$ of f_0.*

With this definition we have the following theorem.

Theorem 2 *Let f_0 be a real analytic function.*
If $S = (f_0, f_1, f_2, \ldots, f_n)$ is a local Sturm sequence on $]a, b[$ then the number of roots of f_0 in $]a, b[$ is given by the difference
$$Var(S, a^+) \Leftrightarrow Var(S, b^-)$$
of the number of sign changes of the sequence S right to a and left to b.

In [3] and [4], Richardson builds a local Sturm sequence for polynomials in x and $g(x)$ with the property that $g'(x)$ can be expressed as a polynomial in x and $g(x)$. Like as in the univariate case, the main tool of this construction is the Euclidean algorithm. Here, Euclidean division is applied to bivariate polynomials with respect to the second variable.
Let us rewrite Richardson's theorem with $g(x) = e^x$.

Definition 2 *A finite sequence $(p_0, p_1, p_2, \ldots, p_n)$ of real bivariate polynomials is called an opposite remainder sequence of p_0 and p_1 with respect to y if*
$$p_i(x, y) = \Leftrightarrow A_{i-1}(x)p_{i-2}(x, y) \Leftrightarrow Q_{i-1}(x, y)\,p_{i-1}(x, y)$$
with

(i) *$deg_y(p_i) < deg_y(p_{i-1})$ and $deg_y(p_n) = 0$,*

(ii) *$Q_i(x, y)$ is a quotient polynomial,*

(iii) *$A_i(x)$ is a positive multiplier (i.e., $\forall x \in \mathbb{R}$, $A_i(x) \geq 0$) such that $p_i(x, y) \in \mathbb{R}[x, y]$.*

Definition 3 *Let f be a real analytic function, f^* is a false derivative of f if f^* is continuous and if $f^*(x)$ and $f'(x)$ have the same sign (i.e. are either both less than zero, both greater than zero or both equal to zero) whenever $f(x) = 0$.*

Theorem 3 ([3], [4]) *Suppose that $p_0(x, e^x)$ is a square free rational polynomial in x and e^x not divisible by e^x. Assume $p_1(x, e^x)$ to be a true or a false derivative of p_0 (cf. [3]). Consider p_0 as a bivariate algebraic polynomial $p_0(x, y)$, and let $(p_0, p_1, p_2, \ldots, p_n)$ be an opposite remainder sequence of p_0 and p_1 with respect to y. Then $S = (p_0(x, e^x), p_1(x, e^x), p_2(x, e^x), \ldots, p_n(x, e^x))$ is a local Sturm sequence in $]a, b[$ if and only if for all $x \in]a, b[$, $p_n(x) \neq 0$.*

2.2 Last root problem

One can easily determine the sign of a polynomial in x and e^x at $\Leftrightarrow\infty$ and $+\infty$, and therefore we can apply the obvious extension of Theorem 3 for the situation $a = \Leftrightarrow\infty$ or $b = +\infty$. To solve the last root problem, Richardson [3] [4] proposes to use a Thom sequence. We use here another strategy consisting of two the following steps:

1. find an including interval $]\Leftrightarrow B, B[$ containing all roots of $p(x, e^x)$;
2. take B as the starting point of an exclusion method [1] to find the last root of $p(x, e^x)$ in some way, adapted to exponential polynomials.

The new approach here only needs the determination of the last root of one single exponential polynomial that is even more linear in e^x whereas the approach based on Thom sequences requires to find the last roots of a sequence of exponential polynomials. The length of this sequence is bounded by the product of $deg_x(p)$ and $deg_y(p)$.

Finding a bound B

We are going to describe an algorithm that finds an inclusion interval based on Cauchy's bound for univariate algebraic polynomials. If $A(x) = \sum_{i=0}^{k} a_i x^i$ is an algebraic polynomial of root radius ρ, $\rho = \max_{z/A(z)=0} |z|$, then Cauchy's bound $C(A)$ reads as
$$\rho \leq C(A) = 1 + \max_{i<k} \frac{|\,a_i\,|}{|\,a_k\,|}.$$

Let $p(x, e^x) = \sum_{i=0}^{n} A_i(x)e^{ix}$ with $A_i \in \mathbb{R}[x]$. For a certain index m we have $\lim_{x \to +\infty} (|A_m(x)| \Leftrightarrow |A_i(x)|) \geq 0$ for

$i = 1,\ 2,\ \ldots,\ n$.
In order to find a bound B as above the algorithm first finds such an m and then a bound B_1 such that

$$|A_m(x)| \geq |A_i(x)| \text{ for all } i \text{ and } x > B_1.$$

In the sequel, if A is an algebraic polynomial, A^+ will denote A multiplied by the sign of its leading coefficient. In fact we determine B_1 such that it verifies

$$\forall i \in \{1,\ 2,\ \cdots,\ n\}$$

$$\forall x \in]B_1, +\infty[:\ A_m^+(x) \geq A_i^+(x) \geq 0.$$

Then we choose

$$\boxed{B_1 = \max_{0 \leq i \leq n} (C(A_i^+), C(A_m^+ \Leftrightarrow A_i^+)).}$$

As an extension of Cauchy's theorem we can remark that the graph of $y(x) = 1 + \max\limits_{0 \leq i \leq n-1} \dfrac{A_i^+(x)}{A_n^+(x)}$ lies above all implicit functions of $p(x, y) = 0$ in $]B_1, +\infty[$.
Let us denote B_2 a real bigger than all the absolute values of the real roots of

$$Q(x, e^x) = A_n^+(x)\, e^x \Leftrightarrow (A_m^+(x) + A_n^+(x)).$$

Let us try to find such a B_2. From Subsection 2.1 we know that within $]B_1, +\infty[$ the roots of $Q(x, e^x)$ are separated by those of $R(x) = Res_y(Q, yQ_y + Q_x)$. Thus, let us define $h \in \mathbb{R}^+$ such that $Q(C(R) + h, e^{C(R)+h}) > 0$ (for $Q(C(R), e^{C(R)}) > 0$ we choose $h = 0$). We put

$$\boxed{B_2 = C(R) + h}$$

$$\boxed{B = max(B_1, B_2)}$$

Since e^x is increasing and $1 + \max\limits_{i=0 \cdots n-1} \dfrac{A_i^+(x)}{A_n^+(x)} < e^x$ when x tends to infinity B is right of all roots of $p(x, e^x)$. The algorithm below describes the essential steps to be performed.

Algorithm B

Input: $p(x, e^x) = \sum\limits_{i=0}^{n} A_i(x) e^{ix}$ and $h \in \mathbb{R}^+$
Output: bound B

- put $A_i^+ = \begin{cases} A_i & \text{if } lc(A_i) > 0 \\ \Leftrightarrow A_i & \text{if not} \end{cases}$
- find m such that
  $\forall i \in \{1,\ 2,\ \cdots,\ n\},\ \ \lim\limits_{x \to +\infty} (A_m^+ \Leftrightarrow A_i^+)(x) \geq 0$
- set $B_1 := \max\limits_{0 \leq i \leq n} (C(A_i^+), C(A_m^+ \Leftrightarrow A_i^+))$
- set $B_2 := C(A_n'^+ A_m^+ + A_n^+ A_m^+ + (A_n^+)^2 \Leftrightarrow A_n^+ A_m'^+)$
- compute $Q(x, e^x) = A_n^+(x)\, e^x \Leftrightarrow (A_m^+(x) + A_n^+(x))$;
  while $Q(B_2, e^{B_2}) < 0$ do $B_2 = B_2 + h$
- set $B := max(B_1, B_2)$

Remark. From an heuristic point of view one can take the Newton correction $\Leftrightarrow \dfrac{Q(B_2, e^{B_2})}{Q'(B_2, e^{B_2})}$ as the width h, and in practice B_2 is usually found after one step.

2.3 An approach for exponential polynomials with real coefficients

Let us see more precisely the construction of the local Sturm sequence for a square free rational polynomial $p_0(x, e^x)$ which does not have a factor of e^x. According to condition (iii) in Definition 2, polynomials $A_i(x)$ must be created such that $p_{i+1}(x, y) \in \mathbb{Q}[x, y]$ without modifying its sign, thus $A_i(x)$ must be positive for every $x \in]a, b[$.
A possible way of constructing these is to take

- $A_i(x) = lc_y(p_i)^{deg_y(p_{i-1}) - deg_y(p_i) + 1}$ if $deg_y(p_{i-1}) \Leftrightarrow deg_y(p_i)$ is odd, and
- $A_i(x) = lc_y(p_i)^{deg_y(p_{i-1}) - deg_y(p_i) + 2}$ if it is even.

This construction builds a sequence such that the degrees $deg_x(p_i)$ are big but we should keep in mind that we have to solve $p_n(x) = 0$.
If $p_0(x, e^x)$ has integer or rational coefficients we can use the gcd or the resultant to simplify $A_i(x)$ and $p_i(x, y)$. So in this case the two degrees $deg_x(p_n)$ and $deg_x(Res_y(p_0, p_1))$ are both of the same order.
In case of real coefficients, Theorem 3 goes through as well and the above construction of the A_i can be used, but such algebraic simplifications are difficult to realise in practice due to the presence of round off errors. For this case we propose another approach.

Theorem 4 *Let $p_0(x, e^x)$ be a square free real polynomial in x and e^x having not a factor of e^x, and let $p_1(x, y) = p_{0y}(x, y)$.*
Assume furthermore $(p_0,\ p_1,\ p_2,\ \ldots,\ p_n)$ to be the opposite remainder sequence of p_0 and p_1 with respect to y and let $S = (p_0(x, e^x),\ p_1(x, e^x),\ p_2(x, e^x),\ \ldots,\ p_n(x, e^x))$. Let X denote the set of roots of the product

$$Res_y(p_0, y\, p_1 + p_{0x}) \cdot lc_y(p_0) \cdot p_n.$$

If $]a, b[\cap X = \emptyset$, then the number of roots of $p_0(x, e^x)$ is given by

$$|\ Var(S, a^+) \Leftrightarrow Var(S, b^-)\ |\ .$$

Remark. 1) Note that p_{0y} is not a false derivative of p_0 and S is not a local Sturm sequence for $p_0(x, e^x)$ on $]a, b[$.
2) As $deg_y(p_{0y}) < deg_y(p_0')$ the opposite remainder sequence of p_0 and p_{0y} is simpler and shorter than the opposite remainder sequence of p_0 and any false or true derivative of p_0.
Proof of Theorem 4
For clarity let us denote for clarity for $x^* \in \mathbb{R}$.

$$S(x^*) = (p_0(x^*, y),\ p_1(x^*, y),\ p_2(x^*, y),\ \ldots,\ p_n(x^*, y)).$$

This proof is divided in two main parts. First we will prove that the number of roots of $p(x, e^x)$ in $]a, b[$ is the absolute value

$$|\ Var(S(a^+), e^{a^+}) \Leftrightarrow Var(S(a^+), +\infty)$$

$$\Leftrightarrow Var(S(b^-), e^{b^-}) + Var(S(b^-), +\infty)\ |\ .$$

Secondly, we will show that

$$Var(S(a^+), +\infty) = Var(S(b^-), +\infty).$$

1. Sturm sequence for fixed x
Let us denote $y = \alpha_1(x),\ y = \alpha_2(x),\ \ldots,\ y = \alpha_l(x)$ the implicit functions of $p(x, y) = 0$. According to Lemma 1

the real roots of $Res_y(p, y\,p_y + p_x)$ separate the roots of $e^x = \alpha_i(x)$ for all $i \in \{1,\ 2,\ \ldots,\ l\}$.
Moreover, for all fixed x^* in $]a, b[$ the polynomials $p(x^*, y)$ have the same number of real roots. Especially, $p(a^+, y)$ and $p(b^-, y)$ have the same number of real roots.
To know the number of roots of $p(x, e^x)$ we just have to find the number of roots of $p(a^+, y)$ in $]e^{a^+}, +\infty[$ and that of $p(b^-, y)$ in $]e^{b^-}, +\infty[$ using the classical Sturm's theorem for polynomials. For example, the number of roots of $p(a^+, y)$ in $]e^{a^+}, +\infty[$ is given by

$$Var(S(a^+), e^{a^+}) \Leftrightarrow Var(S(a^+), +\infty).$$

The subtraction of the two numbers will give us the number of solutions of (1).

2. Proof of $Var(S(a^+), +\infty) = Var(S(b^-), +\infty)$
We distinguish the following two cases:
case (a) $lc_y(p_i(x)) \neq 0$ for all i and for all $x \in]a, b[$. Then a real B_y exists such that $y = B_y$ is over all the implicitly defined functions of $p_i(x, y) = 0$ (for all $i \in \{0,\ 1,\ \cdots, n\}$) in the interval $]a, b[$. The polynomial $p_i(x, y)$ has a constant sign for all i and $y \in]B_y, +\infty[$, and $Var(S(a^+), y) = Var(S(b^-), y)$. So, $Var(S(a^+), +\infty) = Var(S(b^-), +\infty)$.

Case (b) $lc_y(p_i)(x^*) = 0$ for an integer i and for $x^* \in]a, b[$
Assume that i is the smallest integer such that $lc_y(p_i)(x^*) = 0$.

- $x^* \in X$ if $i = 0$, $i = n$ or $lc_y(p_{i+1})(x^*) = 0$,
- In the other case, $lc_y(p_{i-1})(x^*) \neq 0$ and $lc_y(p_{i+1})(x^*) \neq 0$. A real B_y exists such that $y = B_y$ is over all the implicit functions of $p_{i-1}(x, y) = 0$ and $p_{i+1}(x, y) = 0$ in an interval $]u, v[\subset]a, b[$, thus $p_{i-1}(x, y)$ and $p_{i+1}(x, y)$ has a constant sign in $]u, v[\times]B_y, +\infty[$. Moreover $p_{i-1}(x, y)\,p_{i+1}(x, y) < 0$ for all fixed x in $]a, b[$ thus $Var((p_{i-1}(u^+, .), p_i(u^+, .), p_{i+1}(u^+, .)), y)$ $= Var((p_{i-1}(v^-, .), p_i(v^-, .), p_{i+1}(v^-, .)), y)$ for $y \in]B_y, +\infty[$. Therefore, $Var((p_{i-1}(u^+, .), p_i(u^+, .), p_{i+1}(u^+, .)), +\infty)$ $= Var((p_{i-1}(v^-, .), p_i(v^-, .), p_{i+1}(v^-, .)), +\infty)$.
The extension of the argument to the whole sequence S and to the whole interval $]a, b[$ gives

$$Var(S(a^+), +\infty) \ = \ Var(S(b^-), +\infty).$$

$\square$

Example. The approaches of Theorems 3 and 4 have been implemented in Maple. Let us consider the following example
$P(x, e^x) =$
$(3.0x^2 \Leftrightarrow 7.0)e^{4x} \Leftrightarrow (x^5 \Leftrightarrow 3.0)e^{3x} \Leftrightarrow x e^{2x} \Leftrightarrow (x^3 + 1.0)e^x \Leftrightarrow x + 2.0$
In the first column we give the degrees of the polynomial of the opposite remainder sequence. In the column deg X we write the degree of the polynomial equation which has to be solved in order to find the elements of X. The cardinality of X is written in the column Nb(X).

i	$deg_y(p_i)$	$deg_x(p_i)$	deg X	Nb(X)	roots
0	4	5	91	13	-0.19260350
1	4	5	2		1.79741761
2	3	7			
3	2	17			
4	1	41			
5	0	91			

Approach of Theorem 3

i	$deg_y(p_i)$	$deg_x(p_i)$	deg X	Nb(X)	roots
0	4	5	48	11	-0.19260350
1	3	5	27		1.79741761
2	2	10	2		
3	1	20			
4	0	48			

Approach of Theorem 4

Complexity

More generally, let us consider a real exponential polynomial $p(x, e^x)$. Let $\mathcal{R}$ be the algorithm of the Richardson's approach (Theorem 3) and let $\mathcal{A}$ be the new algorithm (Theorem 4).
$\mathcal{R}$ creates a univariate polynomial of degree at most

$$d_1 = \frac{deg_x(p)}{2\sqrt{2}}\left((1+\sqrt{2})^{deg_y(p)+1} \Leftrightarrow (1 \Leftrightarrow \sqrt{2})^{deg_y(p)+1}\right).$$

However, $\mathcal{A}$ creates two polynomials of degree at most

$$d_2 = \frac{deg_x(p)}{2\sqrt{2}}\left((1+\sqrt{2})^{deg_y(p)} \Leftrightarrow (1 \Leftrightarrow \sqrt{2})^{deg_y(p)}\right)$$

and

$$d_3 = 2 deg_x(p) deg_y(p).$$

A simple analysis shows for the ratio $\dfrac{d_1}{d_2}$

- $2 \le \dfrac{d_1}{d_2} \le 2.5$
- $\displaystyle\lim_{deg_y(p) \to +\infty} \frac{d_1}{d_2} = 1 + \sqrt{2}$

It is easy to find an exponential polynomial such that these bounds are reached. For an exponential polynomial p such that $deg_x(p) = 5$ and $deg_y(p) = 4$ we have $d_1 = 145$, $d_2 = 60$ and $d_3 = 40$. For instance, let us consider
$p(x, e^x) = 4.0x^5\,e^{4x} \Leftrightarrow (7.0\,x^5 + x^4 + 18.0\,x^2 \Leftrightarrow 12.0)\,e^{3x} + (5.0\,x^5 \Leftrightarrow x + 4.0)\,e^{2x} \Leftrightarrow (3.0\,x^5 + 7.0\,x^3 \Leftrightarrow 5.0\,x^2 + 7.0)\,e^x + 2.0\,x^5 \Leftrightarrow x^4 + 8.0\,x^3 + 12.0.$
This table shows us that these bounds are reached.

i	$deg_y(p_i)$	$deg_x(p_i)$	deg X	Nb(X)	roots
0	4	5	145	14	-1.11724114
1	4	5	5		1.11481979
2	3	10			0.93028937
3	2	25			
4	1	60			
5	0	145			

Approach of Theorem 3

i	$deg_y(p_i)$	$deg_x(p_i)$	deg X	Nb(X)	roots
0	4	5	60	15	-1.11724114
1	3	5	40		1.11481979
2	2	10	5		0.93028937
3	1	25			
4	0	60			

Approach of Theorem 4

To fix the idea, here is a table which contains the maximum degree of algebraic equation to be solved to create X for an exponential polynomial p by the two approach.

$deg_y(p)$	4		5		7	
$deg_x(p)$	$\mathcal{R}$	$\mathcal{A}$	$\mathcal{R}$	$\mathcal{A}$	$\mathcal{R}$	$\mathcal{A}$
5	145	40 60	350	50 145	2040	70 845
10	290	80 120	700	100 290	4080	140 1690
15	435	120 180	1050	150 435	6120	210 2535
20	580	160 240	1400	200 580	8160	280 3380
30	870	240 360	2100	300 870	12240	420 5070
50	1450	400 600	3500	500 1450	20400	700 8450

3 Two-dimensional exponential polynomials systems

Let $p(x, y, z)$ and $A(x, t, y)$ be two rational trivariate polynomials and let $D =]x_a, x_b[\times]y_a, y_b[$ be a generalized rectangle. We want to find the number of solutions (isolated points or curves) of

$$\left\{ \begin{array}{rcl} p(x, y, e^y) & = & 0 \\ A(x, e^x, y) & = & 0 \end{array} \right. \qquad (2)$$

in D.

For bounded D Richardson [4] proposes an approach using cylindrical decomposition of $A(x, e^x, y)$ and Sturm's method applied to $p(x, y, e^y)$ on a implicit function of $A(x, e^x, y)$ to solve (2). We are going to find the number of solutions in D by using cylindrical decomposition only. This new approach applies generalized rectangles as well.

The main idea is to create subrectangles not containing critical points of the cylindrical decomposition for $p(x, y, e^y)$ and $A(x, e^x, y)$ in which one implicit function of $p(x, y, e^y) = 0$ does not cross more than once the same implicitly defined function of $A(x, e^x, y) = 0$.
In those rectangles, we can easily find the number of solutions of (2).

Now we assume that $]y_a, y_b[$ is bounded (otherwise we change the roles of x and y).
Assume that p and A are both square free and e^y occurs in p.

Cylindrical decomposition of the (x, y)-plane of $A(x, e^x, y)$

We know that we can find a cylindrical decomposition in D of the (x, y)-plane which is sign invariant for $A(x, e^x, y) = 0$. This decomposition creates a finite number of implicit functions of the form $y = \phi(x)$ and critical points which are obtained by solving the three equations $Res_y(A, A_y) = 0$, $A(x, e^x, y_a) = 0$ and $A(x, e^x, y_b) = 0$. We store this points in a set X.

Cylindrical decomposition of the (y, x)-plane of $p(x, y, e^y)$

In the same way, a cylindrical decomposition of the (y, x)-plane exists for $p(x, y, e^y)$ and creates a finite number of implicit functions of the form $x = \psi(y)$. The critical points of $p(x, y, e^y)$ are obtained by solving $Res_x(p, p_x) = 0$. Let us store these points into a set Y. We want to set up a cylindrical decomposition of the (x, y)-plane for $p(x, e^x, y) = 0$.

Strictly monotonic implicit function of the (y, x)-plane of $p(x, y, e^y) = 0$

We have to find intervals on the y-axis in which all the implicit functions are strictly monotonic so that inverse functions exist in such intervals.
For each function $x = \psi(y)$ we must determine where $\psi'(y)$ vanishes.
From $p(\psi(y), y, e^y) = 0$ we obtain by differentiation

$$\psi'(y) p_x(\psi(y), y, e^y) + p_y(\psi(y), y, e^y) = 0$$

In intervals not containing critical points this is equivalent to $\psi'(y) = \Leftrightarrow \dfrac{p_y(\psi(y), y, e^y)}{p_x(\psi(y), y, e^y)}$, and thus we have

$$\begin{array}{rl} & \exists y \in]y_1, y_2[\backslash Y \;:\; \psi'(y) = 0 \\ \Leftrightarrow & \exists y \in]y_1, y_2[\backslash Y \;:\; p_y(\psi(y), y, e^y) = 0 \\ \Leftrightarrow & \exists y \in]y_1, y_2[\backslash Y \;:\; \left\{ \begin{array}{rcl} p_y(x, y, e^y) & = & 0 \\ p(x, y, e^y) & = & 0 \end{array} \right. \end{array} .$$

This implies $\exists y \in]y_1, y_2[\backslash Y \;:\; Res_x(p, p_y) = 0$.

Let us add the roots of $Res_x(p, p_y)$ to Y. Now the restrictions of all the functions ψ in any interval contained in $]y_1, y_2[\backslash Y$ are strictly monotonic.

Cylindrical decomposition of the (x,y)-plane of $p(x, y, e^y)$

For all $y^* \in Y$ the solutions of $p(x, y^*, e^{y^*}) = 0$ are critical points of $p(x, y, e^y) = 0$ on the x-axis so we add them to X. We also have to add to X the solutions of $p(x, y_a, e^{y_a})$ and $p(x, y_b, e^{y_b})$. At this level of the construction, X contains all the critical points of the cylindrical decomposition of the (x, y)-plane of p and A. One implicit function $x = \psi(y)$ could generate several functions $y = \alpha_j(x)$).
Let us consider an interval I, $I \subset (]x_a, x_b[\backslash X)$. For every curve α of $p(x, y, e^y) = 0$ and for every curve ϕ of $A(x, e^x, y) = 0$ contained in $I \times]y_a, y_b[$, let us define $G(x) = \phi(x) \Leftrightarrow \alpha(x)$.

Separation of the roots of $G(x)$

We would like to separate the roots of G. We have to solve $G'(x) = 0$. Since

$$
\begin{aligned}
& G'(x) = 0 \\
\Leftrightarrow\quad & \phi'(x) \Leftrightarrow \alpha'(x) = 0 \\
\Leftrightarrow\quad & \Leftrightarrow \frac{A_x(x, e^x, \phi(x))}{A_y(x, e^x, \phi(x))} + \frac{p_x(x, \alpha(x), e^{\alpha(x)})}{p_y(x, \alpha(x), e^{\alpha(x)})} = 0 \\
\Leftrightarrow\quad & \Leftrightarrow A_x(x, e^x, \phi(x))\, p_y(x, \alpha(x), e^{\alpha(x)}) \\
& + p_x(x, \alpha(x), e^{\alpha(x)})\, A_y(x, e^x, \phi(x)) = 0
\end{aligned}
$$

This last equation is equivalent to the following system

$$
\begin{cases}
\Leftrightarrow A_x(x, e^x, y)\, p_y(x, y, e^y) + p_x(x, y, e^y)\, A_y(x, e^x, y) = 0 \\
p(x, y, e^y) = 0 \\
A(x, e^x, y) = 0
\end{cases}
$$

we obtain

$$
\begin{cases}
R(x, e^x, y) := Res_z(\Leftrightarrow A_x\, p_y + p_x\, A_y, p) = 0 \\
A(x, e^x, y) = 0
\end{cases}
$$

and finally $Res_y(R, A) = 0$.

Case of finite interval in the x-axis

- If $R(x, e^x, y)(mod A)$ is identically zero then p and $\Leftrightarrow A_x\, p_y + p_x\, A_y$ must have a common factor, according to Richardson. Either we can find a factor of p and then we do the factorization and we go back to the beginning, or p divides $\Leftrightarrow A_x\, p_y + p_x\, A_y$, using functions in $\mathbb{Q}[x][y])[z]$. In this very special case, we can determine if there is a part of a branch of $p(x, e^x, y) = 0$ which is identical to one of $A(x, e^x, y) = 0$.

- If $Res_y(R, A)(mod A)$ is not identically zero, we add the solutions of $Res_y(R, A) = 0$ to X. Between two consecutive points x_i and x_{i+1} of X, a branch of $A = 0$ does not cross a branch of $p = 0$ more than once. Thus to know the number of roots of the system (2) in $]x_i, x_{i+1}[$ we have just to solve the four equations $A(x_i, e^{x_i}, y) = 0$, $A(x_{i+1}, e^{x_{i+1}}, y) = 0$, $p(x_i, y, e^y) = 0$ and $p(x_{i+1}, y, e^y) = 0$ if this has not been done yet.

Case of infinite interval in the x-axis

Suppose that $x_b = +\infty$ and x_{max} is the maximal element of $X \cup \{C(Res_y(A, A_x))\}$. We want to know the number of solutions of system (2) in the generalized rectangle $]x_{max}, +\infty[\times]y_1, y_2[$. All the implicit functions contained in this rectangle are monotonic and have finite limits for $x \to +\infty$.
We can determine the finite limits of curves of $p(x, y, e^y) = 0$ because those limits are among the solutions of the equation $lc_x(p) = 0$.
Let α (respectively ϕ) be an implicit function of $p(x, y, e^y) = 0$ (respectively $A(x, e^x, y) = 0$) and let us denote $y^* = \lim_{x \to +\infty} \alpha(x)$. $G(x) = \phi(x) \Leftrightarrow \alpha(x)$ is strictly monotonic. Thus we arrive at two possibility:

- ϕ crosses the line $y = y^*$. Then ϕ crosses exactly once α.

- ϕ does not cross the line $y = y^*$. Then
  - either α and ϕ are not on the same side of the line $y = y^*$ and α never crosses ϕ on $]x_{max}, +\infty[\times]y_1, y_2[$,
  - or α and ϕ are in the same side. Let us suppose that α and ϕ lie in $]x_{max}, +\infty[\times]y^*, y_2[$. There are two cases
    - $\alpha(x_{max}) < \phi(x_{max})$ then α never crosses ϕ,
    - $\alpha(x_{max}) > \phi(x_{max})$ then if $\lim_{x \to +\infty} \neq y^*$ α crosses ϕ, if not α never crosses ϕ.

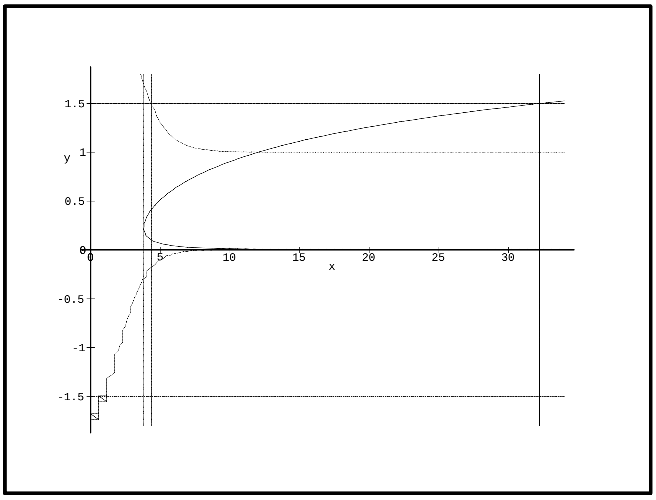
    The values of $\alpha(x_{max})$ and $\phi(x_{max})$ are among the solutions of $A(x_{max}, e^{x_{max}}, y)$ and $p(x_{max}, y, e^y)$.

Concluding remarks

The determination of five resultants and the resolution of several exponential polynomials and several polynomials allow to determine the number of solutions of the system (2) in a generalized rectangle.
Example. We want to know the number of solution of

$$
\begin{cases}
p(x, y, e^y) & = & 8\, x\, y\, e^{2y} + 2\, y\, e^y \Leftrightarrow 5\, y\, x^2 + 5 \\
A(x, e^x, y) & = & (e^x \Leftrightarrow x^2 + 4)\, y^2 \Leftrightarrow (e^x + x)\, y \Leftrightarrow 18
\end{cases}
$$

in the generalized rectangle $]3, +\infty[\times]\Leftrightarrow 1.5, 1, 5[$.



By solving $Res_y(A, A_y) = 0$, $A(x, e^x, \Leftrightarrow 1.5) = 0$ and $A(x, e^x, 1.5) = 0$ we obtain the set of critical points of A in $]3, +\infty[$ which is $X = \{x_1 = 4.34977794\}$
The solutions of $Res_x(p, p_x) = 0$ and $Res_x(p, p_y) = 0$ in $]\Leftrightarrow 1.5, 1.5[$ are $y_1 = 0$ and $y_2 = 0.22566237$, respectively.
The solution of $lc_x(p) = 0$ is $y = 0$ thus a branch of $p(x, y, e^y) = 0$ has a finite limit which is 0.
The equations $p(x, \Leftrightarrow 1.5, e^{-1.5}) = 0$ and $p(x, 0, e^0) = 0$ have both no solution in $]3, +\infty[$ and the solutions of $p(x, y_2, e^{y_2}) = 0$ and $p(x, 1.5, e^{1.5}) = 0$ are $x_2 = 3.80797556$ and $x_3 = 32.21320485$, respectively.
There is no root of $Res_y(Res_z(\Leftrightarrow A_x\, p_y + p_x\, A_y, p), A)$ in $]3, +\infty[$, thus at this step the cylindrical decomposition of the (x, y)-plane of $p(x, y, e^y)$ and $A(x, e^x, y)$ is determined and the set of critical points is

$$X = \{3,\ x_1,\ x_2,\ x_3\}$$

Over $]3, x_2[$
there is just one branch of a curve of A going through the points $(\Leftrightarrow 3, \Leftrightarrow .56850854)$ and $(x_2, \Leftrightarrow 0.30329348)$.

Over $]x_2, x_1[$

there is one branch of a curve of A going through the points $(x_2, \Leftrightarrow 0.30329348)$ and $(x_1, \Leftrightarrow 0.1918750747433021)$. The curve of p has exactly two branches the closures of which meet each other in the point $(x_2, 0.22566237)$ and go through the points $(x_1, 0.10026772)$ and $(x_1, 0.41291535)$, respectively. Thus the system has no solution in $]x_2, x_1[\times] \Leftrightarrow 1.5, 1.5[$.

Over $]x_1, x_3[$

there are two branches of a curve of A going through the points $(x_1, \Leftrightarrow 0.1918750747433021)$ and $(x_3, \Leftrightarrow 0.18418548\, 10^{-12})$, and $(x_1, 1.5)$ and $(x_3, 1.00000000)$. There are also two branches of a curve of p going through the points $(x_1, 0.10026772)$ and $(x_3, 0.1014564742)$, and $(x_1, 0.41291535)$ and $(x_3, 1.5)$, respectively. Thus the system has one solution in $]x_1, x_3[\times] \Leftrightarrow 1.5, 1.5[$.

Over $]x_3, +\infty[$

We verify that $C(Res_y(A, A_x)) = 27.06 < x_3$. There are two branches of curve of A but none of them cross the limit of the branch p which is $y = 0$. (We just verify that $A(x, e^x, 0)$ has no roots in $]x_3, +\infty[$, in our example $A(x, e^x, 0) = \Leftrightarrow 18)$. Between 27.06 and $+\infty$ the branches of curves of A are monotonic, the first one is increasing and remainds bellow $y = 0$ and the second one remains over the branch of p thus the system has no solution in $]x_3, +\infty[\times] \Leftrightarrow 1.5, 1.5[$.

To conclude, the system

$$\left\{ \begin{array}{rcl} p(x, y, e^y) & = & 8\, x\, y\, e^{2y} + 2\, y\, e^y \Leftrightarrow 5\, y\, x^2 + 5 \\ A(x, e^x, y) & = & (e^x \Leftrightarrow x^2 + 4)\, y^- (e^x + x)\, y \Leftrightarrow 18 \end{array} \right.$$

has just one solution in $]3, +\infty[\times] \Leftrightarrow 1.5, 1, 5[$.

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