A Factorization Algorithm for Linear Ordinary Differential Equations

Fritz Schwarz
GMD, Institut F1
Postfach 1240
5205 St. Augustin
West Germany

Abstract. The reducibility and factorization of linear homogeneous differential equations are of great theoretical and practical importance in mathematics. Although it has been known for a long time that factorization is in principle a decision procedure, its use in an automatic differential equation solver requires a more detailed analysis of the various steps involved. Especially important are certain auxiliary equations, the so-called *associated equations*. An upper bound for the degree of its coefficients is derived. Another important ingredient is the computation of optimal estimates for the size of polynomial and rational solutions of certain differential equations with rational coefficients. Applying these results, the design of the factorization algorithm *LODEF* and its implementation in the Scratchpad II Computer Algebra System is described.

1 Reducibility of Differential Equations

The concept of *reducibility of a differential equation* came into existence in the second half of the last century when many efforts were made to obtain a theory for solving differential equations by analogy to that for algebraic equations which had been created by Lagrange and Galois. According to Frobenius, a linear homogeneous differential equation $P(y)$ with coefficients from a given field is said to be *reducible* if there exists another linear equation $Q(y)$ of lower order with coefficients of the same type which has its solutions in common with $P(y)$. If an equation is not reducible it is called *irreducible*. A reducible equation may be decomposed according to

$$P(y) = R(Q(y)), \quad \textit{or} \quad P = RQ \tag{1}$$

for short where R is a differential operator which is obtained from P and Q by a procedure similar to Euclid's algorithm for determining the greatest common divisor of two polynomials. If n and m are the orders of P and Q respectively with $n > m$, the order of R is $n - m$. If there exists no equation of order less than m over the same coefficient domain which has its solution in common with Q, the latter is called an *irreducible factor* of P. $R(y) = 0$ may be reducible as well and a corresponding decomposition may be obtained in the same way as for $P(y)$. This process is continued until the decomposition

$$P = Q_\lambda Q_{\lambda-1} \cdots Q_2 Q_1$$

into irreducible components is obtained. It is not unique. The arbitrariness involved is described by the following fundamental theorem due to Landau [1]. In any two decompositions of the differential equation $P(y) = 0$ into irreducible components, the number of factors and its orders are the same up to permutations. There is a one-to-one correspondence between pairs of factors from either decomposition such that both are of the same kind. This restriction does not exclude the possibility that there are equations which allow infinitely many decompositions into irreducible components.

If a differential equation is irreducible, its differential Galois group is transitive. From a theoretical point of view this is an important constraint and has its obvious counterpart in the Galois theory of algebraic equations. If an equation is reducible, its decomposition into irreducible components is usually an important step towards finding its solutions. Assume that a decomposition (1) has been obtained and that $\{y_1, \ldots, y_m\}$ is a fundamental system for $Q(y) = 0$. Obviously it also solves $P(y) = 0$. To obtain the remaining elements of a fundamental system for this latter equation, a fundamental system for the $(n-m)$th order equation $R(z)=0$, denoted by $\{z_{m+1},\ldots,z_n\}$, must first be found. Then the $n-m$ nonhomogeneous equations $Q(y)=z_k$ for $k=m+1,\ldots,n$ have to be solved. Due to the fact that a fundamental system for the corresponding homogeneous equation is already known, this may be achieved by quadratures. The corresponding solutions $\{y_{m+1},\ldots y_n\}$ obviously satisfy also $P(y_k)=0$ and represent the remaining elements of the fundamental system aimed for. The factorization of P has reduced the solution of an nth order equation to solving two homogeneous equations of mth and $(n-m)$th order respectively and quadratures. An example for a decomposition is

$$x^3y''+x(2x-1)y'+y=(x\frac{d}{dx}-1)[x^2y'+(x-1)y]=0.$$

Applying the notation introduced above the homogeneous equations corresponding to the two factors have the solutions

$$z_2=x \ \ and \ \ y_1=\frac{1}{x}e^{-\frac{1}{x}}$$

respectively. The second element of a fundamental system for the complete equation follows from

$$x^2y'+(x-1)y=z_2=x$$

and leads to

$$y_2=\frac{1}{x}e^{-\frac{1}{x}}\int e^{\frac{1}{x}}dx.$$

Another example is

$$y''+(x-1)y'-xy=(\frac{d}{dx}+x)(y'-y)=0$$

with

$$z_2=e^{-\frac{1}{2}x^2} \ \ and \ \ y_1=e^x.$$

From

$$y'-y=z_2=e^{-\frac{1}{2}x^2}$$

there follows

$$y_2=e^x\int e^{-\frac{x^2}{2}-x}dx.$$

An example with a factorization depending on a parameter a is [1]

$$y''-\frac{2}{x}y'+\frac{2}{x^2}y=[\frac{d}{dx}-\frac{1}{x(1+ax)}][y'-\frac{1+2ax}{x(1+ax)}y].$$

Due to its importance both for theoretical and practical purposes there arises the question how to obtain the factorization into irreducible components. First of all the factorization must be performable in a finite number of steps. This has been known to be true since the last century. However, the fact that a process is finite does not necessarily mean that it can be utilized it for practical purposes such as solving equations. As is well known from related problems, there may be such a large number of calculations involved that these problems are intractable even for large-scale computers. What is needed is an estimate for the cost that is involved in the average and in the worst case for obtaining a decomposition into irreducible components for those parameters of the input that are relevant in applications.

It is the purpose of this article to provide this analysis for the factorization of linear homogeneous differential equations with rational function coefficients, to describe the correponding algorithm and how it is implemented in the Scratchpad II computer algebra system. This is a prerequisite for using it as part of an automatic differential equation solver. As it will turn out later, certain auxiliary equations, the so-called *associated equations* will play a key role in the factorization algorithm. The way in which they are generated means that they may grow to considerable size. This topic is discussed in detail in Section 2. Another important prerequisite are estimates for the size of polynomial and rational solutions of certain differential equations. These are considered in Section 3. Applying these results, in Section 4 the factorization algorithm *LODEF* and its implementation in the Scratchpad II Computer Algebra System will be described.

In the remaining part of this Introduction the notation which is used throughout the rest of the paper will be fixed. A general reference for all subjects which are raised in this article is the three-volume work by Schlesinger [2] and the literature quoted there. A more recent reference which includes a large collection of solved examples is the book by Kamke [3].

Throughout this article a linear homogeneous differential equation for the unknown function y depending on x is written in the form

$$p_0y^{(n)}+p_1y^{(n-1)}+p_2y^{(n-2)}+\ldots+p_{n-1}y'+p_ny=0. \quad (2)$$

The coefficients p_i are polynomials in x over the integers of degree d_i, i.e.

$$p_i=\sum_{j=0}^{d_i}p_{i,j}x^j$$

with $p_{i,j}$ integer. Furthermore the m_i are defined by

$$m_i=d_i-(n-i) \ \ for \ \ i=0,\ldots,n.$$

For some purposes it is more appropriate to rewrite (2) in the form

$$y^{(n)}+q_1y^{(n-1)}+q_2y^{(n-2)}+\ldots+q_{n-1}y'+q_ny=0 \quad (3)$$

where

$$q_i \equiv \frac{p_i}{p_0} = \sum_{\sigma=1}^{s} \sum_{j=d_i^\sigma}^{-1} q_{i,j}^\sigma (x - x_\sigma)^j + \sum_{j=0}^{d_i} q_{i,j} x^j$$

with $d_0^\sigma = 0$, $d_i^\sigma \leq -1$, $q_{i,j}^\sigma$, $q_{i,j}$ rational for $\sigma = 1, \ldots, s$, $1 \leq i, j \leq n$. This equation shows that the q_i are assumed to be represented as partial fractions, i.e. its complete factorization must be known. The position of the poles are x_σ for $\sigma = 1, \ldots, s$. Finally the m_i^σ are defined by

$$m_i^\sigma = d_i^\sigma - (n - i) \quad \textit{for} \quad i = 0, \ldots, n \quad \textit{and} \quad \sigma = 0 \ldots s.$$

The scaling symmetry of (2) and (3) which corresponds to the infinitesimal generator $y\frac{\partial}{\partial y}$ may be utilized to lower its order by one. To this end the invariant $u(x)$ is introduced as new dependent variable by

$$y' = uy.$$

Defining the functional $\phi_n(u)$ through

$$y^{(n)} = \phi_n(u) y$$

with $\phi_1(u) = u$ the ϕ's obey the recursion relation

$$\phi_{n+1}(u) = \frac{d\phi_n(u)}{dx} + u\phi_n(u)$$

for $n \geq 1$ from which it follows easily

$$\begin{aligned} y'' &= (u^2 + u')y, \\ y^{(3)} &= (u^3 + 3uu' + u'')y, \\ &\vdots \\ y^{(n)} &= [u^n + \ldots + nu^{(n-2)}u + u^{(n-1)}]y. \end{aligned}$$

Substituting the expressions for the derivatives of y into (3) leads to the equation

$$\phi_n + q_1\phi_{n-1} + q_2\phi_{n-2} + \ldots + q_{n-1}\phi_1 + q_n = 0. \quad (4)$$

It is a *nonlinear* differential equation of order $n - 1$ for the unknown function u which may be considered as a *generalized Riccati equation*. The coefficients q_i however enter linearly. A similar equation follows from (2). For $n = 2$ and $n = 3$ the equations

$$u' + u^2 + q_1 u + q_2 = 0,$$

$$u'' + (3u + q_1)u' + u^3 + q_1 u^2 + q_2 u + q_3 = 0$$

are obtained.

A linear independent set of n solutions $\{y_1, \ldots, y_n\}$ of (2) or (3) is called a *fundamental system*. For such a system the *Wronskian*

$$D^n(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots y_n \\ y_1' & y_2' & \ldots y_n' \\ \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots y_n^{(n-1)} \end{vmatrix} \quad (5)$$

is different from zero. More general, $D_k^n(y_1, \ldots, y_n)$ for $k = 0, 1, \ldots, n$ is defined as

$$\frac{\partial}{\partial y^{(n-k)}} \begin{vmatrix} y & y_1 & y_2 & \ldots y_n \\ y' & y_1' & y_2' & \ldots y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \ldots y_n^{(n)} \end{vmatrix} \quad (6)$$

There are the obvious relations

$$D_0^n = D^n \quad \textit{and} \quad \frac{dD^n}{dx} = D_1^n. \quad (7)$$

Expanding the determinant (11) with respect to the first column and comparing the result with (3), the representation

$$q_k = (-1)^k \frac{D_k^n(y_1, \ldots, y_n)}{D^n(y_1, \ldots, y_n)} \quad (8)$$

of the coefficients q_k is obtained. For $k = 1$ it follows with (12)

$$q_1 = -\frac{D^{n'}}{D^n} \quad \textit{or} \quad D^{n'} + q_1 D^n = 0 \quad (9)$$

which is known as *Liouville's relation*. It may be considered as a first order differential equation for D^n in terms of q_1. The associated equations which have been mentioned above are generalizations of Liouville's relation. Its structure will be investigated in detail in the subsequent Section 2.

2 Associated Equations

Liouville's equation may be generalized for determinants which are formed out of subsets $\{y_1, \ldots, y_m\}$ with $m < n$ of a fundamental system of (2) or (3) in the following way. Consider the matrix

$$\begin{pmatrix} y_1 & y_2 & \ldots y_m \\ y_1' & y_2' & \ldots y_m' \\ \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots y_m^{(n-1)} \end{pmatrix} \quad (10)$$

There are $\binom{n}{m}$ square $m \times m$ matrices that may be formed out of its rows for any fixed value of m. Their determinants may be considered as new functions. This set of functions is closed under differentiation if the original differential equation is used to substitute derivatives of order higher than $n-1$. By suitable differentiations and elimination, for each of these functions an $\binom{n}{m}$th order linear differential equation may be obtained. These equations are called *associated equations* for the original one. As is well known, elimination often suffers from the tremendous growth of intermediate expressions or of the result. To make use of these associated equations it is therefore of utmost importance to have an estimate for the size of its coefficients.

Let the determinants which are formed out of the $m \times m$ submatrices of (15) be denoted by z_ν for $\nu = 1, \ldots, \binom{n}{m}$. The key for understanding the structure of the associated equations is to decompose the z's into several subsets with a well-defined coefficient size. To this end the full range for the index ν is subdivided into $n-m+1$ subintervals $I^{(k)}$ which are defined by

$$I^{(k)} = \{\nu | \binom{k}{m} + 1 \le \nu \le \binom{k+1}{m}, m-1 \le k \le n-1\}. \tag{11}$$

An index ν belongs to $I^{(k)}$ if in the corresponding z_ν the highest derivatives of the y's which are involved are exactly of order k. The subinterval $I^{(k)}$ contains $\binom{k}{m-1}$ elements. Combined they form the complete range because

$$\sum_{k=m-1}^{n-1} \binom{k}{m-1} = \binom{n}{m}.$$

The enumeration of the z's is chosen such that the subintervals with index $k = m-1$, $k = m, \ldots$ are traversed in increasing order. This implies $z_1 = D^m(y_1, \ldots y_m)$.

Each z'_ν may be expressed as a linear homogeneous function of the z's by using the original differential equation (2) for substituting derivatives of order higher than $n-1$. Due to the enumeration which has been chosen for the z's, these relations may be written in the form

$$z'_\nu = \sum_{\mu \in I^{(k)} \cup I^{(k+1)}} a_{\nu\mu} z_\mu, \tag{12}$$

$a_{\nu\mu}$ integer for $\nu \in I^{(k)}$, $m-1 \le k \le n-2$ and

$$p_0 z'_\nu = \sum_{\mu=1}^{\binom{n}{m}} a_{\nu\mu} z_\mu, \tag{13}$$

$a_{\nu\mu}$ linear homogeneous in $p_0, \ldots, p_n$ for $\nu \in I^{(n-1)}$. If for a fixed value of ν any of the equations (17) or (18) is differentiated $\binom{n}{m} - 1$ times and all derivatives z'_μ for $\mu \ne \nu$ which appear at the right hand side are substituted by using (17) or (18), a set of $\binom{n}{m}$ equations expressing the $z_\nu^{(\lambda)}$ in terms of the z_μ is obtained. Its structure may be described as follows.

$$z_\nu^{(\lambda)} = \sum_{\mu \in I^{(k)} \cup I^{(k+1)} \cup \ldots \cup I^{(k+\lambda)}} b_{\nu\mu} z_\mu, \tag{14}$$

$b_{\nu\mu}$ integer for $\nu \in I^{(k)}$, $m-1 \le k \le n-2$, $1 \le \lambda \le n-k-1$ and

$$p_0^{\lambda-n+k+1} z_\nu^{(\lambda)} = \sum_{\mu=1}^{\binom{n}{m}} b_{\nu\mu} z_\mu, \tag{15}$$

$b_{\nu\mu}$ homogeneous of degree $\lambda - n + k + 1$ in $p_0, \ldots p_n$ and its derivatives for $m-1 \le k \le n-1$, $n-k \le \lambda$.

For each z_ν a differential equation of order $\binom{n}{m}$ may be obtained from the system (19), (20) by the following procedure. If $k \le n-2$, there are $n-k-1$ equations (19) which may be applied to eliminate as many variables z_μ, $\mu \ne \nu$ by expressing them as linear combinations with integer coefficients of the remaining ones. Substituting these relations into (20) transforms the latter into a quadratic system without changing the degree of homogeneity of the coefficients. If $k = n-1$ there are no equations of the type (19) and system (20) is quadratic from the beginning. The variables of this system are rearranged such that z_ν corresponds to the last column of the coefficient matrix. It is transformed into row echelon form by applying Bareiss' version of the Gauss elimination scheme. Upon completion the last equation contains only a single term proportional to z_ν at the right hand side and a polynomial in the derivatives of it at the left hand side, i.e. it is the associated equation for z_ν. The equation for z_1 is simply called the $(n-m)$th associated equation.

To obtain the desired bound for the coefficients, a general property of the Bareiss elimination scheme will be needed which is formulated as follows.

Lemma 1 *Let $\{a_{ij}\}$ be a non-singular square $L \times L$ matrix with the property that its elements in the i-th row are homogeneous of degree i in some variables for all j. Applying the Bareiss two-step elimination scheme to transform it into row echelon form, upon completion the diagonal element in the Lth lower right corner has degree $\frac{1}{2}L(L+1)$.*

Proof. Define $\{a_{ij}\} = \{a_{ij}^{(0)}\}$ and let the elements after the k-th step be $a_{ij}^{(k)}$ for $1 \le k \le L-1$ with degree $d_i^{(k)}$ for all j. The two-step Bareiss scheme is defined by

$$a_{ij}^{(k)} = \frac{a_{kk}^{(k-1)} a_{ij}^{(k-1)} - a_{kj}^{(k-1)} a_{ik}^{(k-1)}}{a_{k-1,k-1}^{(k-2)}}$$

with $a_{0,0}^{(-1)} = 1$. The division is always exact. From this it follows for the degrees

$$d_i^{(k)} = d_k^{(k-1)} + d_i^{(k-1)} - d_{k-1}^{(k-2)}.$$

This recurrence relation has the solution

$$d_i^{(k)} = i + \frac{1}{2}k(k+1)$$

with $d_i^{(0)} = i$. Upon completion after the $(L-1)$th step it has the value

$$d_L^{(L-1)} = \frac{1}{2}L(L+1).$$

This completes the proof of the Lemma.

The quadratic system which has been transformed into triangular form above to obtain the associated equation has exactly the structure which is required by this Lemma. Therefore it may be immediately applied to obtain the following result.

Theorem 1 *The coefficients of the associated equation for z_ν are homogeneous in the variables $p_0, \ldots, p_n$. If $\nu \in I^{(k)}$ which is defined by (16), an upper bound $B_{\nu(k)}$ for its degree is*

$$B_{\nu(k)} = \frac{1}{2}[\tbinom{n}{m} - n + k + 1][\tbinom{n}{m} - n + k + 2].$$

This is a worst-case estimate.

It is instructive to evaluate the bound of Theorem 1 for some specific cases. For $n = 3$ and $n = 4$ it is given in the tables

$\mathbf{n=3}$	$k=1$	$k=2$
$m=2$	3	6

and

$\mathbf{n=4}$	$k=1$	$k=2$	$k=3$
$m=2$	10	15	21
$m=3$		6	10

Furthermore the exact result is derived for $n = 3$ and $m = 2$. In this case $k = 1$ or 2, $\binom{n}{m} = 3$ and the intervals are $I^{(1)} = \{1\}$, $I^{(2)} = \{2, 3\}$. The differential equation is

$$p_0 y''' + p_1 y'' + p_2 y' + p_3 y = 0. \qquad (16)$$

There are three second-order determinants

$$z_1 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \; z_2 = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}, \; z_3 = \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}$$

from which the expressions

$$z_1' = z_2$$
$$p_0 z_2' = -p_2 z_1 - p_1 z_2 + p_0 z_3$$
$$p_0 z_3' = p_3 z_1 - p_1 z_3$$

for the first order derivatives follow. Proceeding as described above, the third order equations for z_1 and z_3 are

$$p_0^2 z_1''' + 2p_0 p_1 z_1'' + (p_1^2 + p_0 p_2 - p_0' p_1 + p_0 p_1') z_1'$$
$$+ (p_1 p_2 - p_0 p_3 - p_0' p_2 + p_0 p_2') z_1 = 0 \qquad (17)$$

$$p_0^2 p_3^2 z_2''' + 2p_0 p_3 (p_1 p_3 + p_0' p_3 - p_0 p_3') z_2''$$
$$+ (p_1^2 p_3^2 + p_0 p_2 p_3^2 + p_0' p_1 p_3^2 + 2p_1' p_0 p_3^2 - 3p_3' p_0 p_1 p_3$$
$$+ 2p_3'^2 p_0^2 - 2p_0' p_3' p_0 p_3 + p_0'' p_0 p_3^2 - p_3'' p_0^2 p_3) z_2'$$
$$+ (p_1 p_2 p_3^2 - p_0 p_3^3 + p_1' p_1 p_3^2 - p_3' p_1^2 p_3 - 2p_1' p_3' p_0 p_3$$
$$+ 2p_3'^2 p_0 p_1 - p_3'' p_0 p_1 p_3 + p_0 p_1'' p_3^2) z_2 = 0 \qquad (18)$$

From the table for $n = 3$ it follows that the bound for z_1 is 3 whereas the coefficients in (22) are only of degree 2. Similarly the bound for z_2 is 6 whereas the coefficients in (23) is only of degree 4. In both cases an overall monomial factor has been dropped. It is not clear in the moment whether this is true in general.

3 Bounds for Polynomial and Rational Solutions

The algorithm *LODEF* requires in several places the solutions of a certain structure for equations of the type (2) or (3). This may be accomplished in two steps. At first the size of the respective solution is limited by deriving bounds for suitable parameters. Secondly within that finite parameter space all possible solutions are obtained by explicit construction. The three cases to be considered are polynomial solutions, rational solutions and solutions with a rational logarithmic derivative. They are parametrized by the number and the order of their poles. The answer is given in terms of Bound 1, 2 or 3 respectively. The notation is the same as in the introduction.

Bound 1 *(Polynomial solutions). Let $(i_0, i_1 \ldots i_n)$ be a permutation of $(0, 1, \ldots, n)$ such that*

$$m_{i_0} = m_{i_1} = \ldots = m_{i_l} > m_{i_{l+1}} \geq \ldots \geq m_{i_n}$$

for $0 \leq l \leq n$ and

$$P(N) \equiv \sum_{k=0}^{l} p_{i_k, d_{i_k}} \frac{N!}{(N - n + i_k)!}. \qquad (19)$$

An upper bound for the degree of a polynomial solution of (2) is

$$N_{max} = max(n, integer \ \ solutions \ \ of \ \ P(N) = 0).$$

The cost for obtaining this bound is not higher than that for factorizing a univariate polynomial of degree n over the integers. Without additional assumptions on the coefficients p_i this bound cannot be improved.

Proof. Assume that (2) has a solution y of the form

$$y = a_0 + a_1 x + \ldots + a_N x^N. \qquad (20)$$

Substituting for y and equating the coefficients of the various powers of x to zero leads to a system of linear equations for the expansion coefficients a_i. For $N \geq n$ the equation for the leading coefficient a_N is

$$a_N P(N) = 0$$

which implies $a_N = 0$ except if $P(N) = 0$. For $N < n$ the equations for the leading coefficient in general do not have the simple structure (3) due to possible cancellations between several terms. This bound is saturated for

$$x^2 y'' - 6y = 0$$

with the fundamental system

$$y_1 = x^3, \qquad y_2 = x^{-2}.$$

Here $d_0 = 2$, $d_1 = d_2 = 0$, $m_0 = m_2 = 0$, $m_1 = -1$ and (19) reads

$$N(N-1) - 6 = (N-3)(N+2) = 0.$$

The solution $N = 3$ corresponds to y_1. This completes the proof.

Special cases of this result have been known for a long time. Mambriani [4] considers two special cases of (2). The first case is characterized by $d_i \leq n - i - 1$ for $i = 0 \ldots n-2$ and $p_1 = a_1 x + b_1$, $p_0 = a_0 x + b_0$, i.e. $d_{n-1} = d_n = 1$. It follows that $m_i \leq -1$ for $i = 0 \ldots n-2$, $m_{n-1} = 0$, $m_n = 1$. As a consequence of (19) it follows immediately that $a_0 = 0$ is necessary for a polynomial solution to exist. For the modified problem, $m_n = 0$ and (19) implies now $a_1 N + b_0 = 0$ for the degree N of a polynomial solution. Due to the special structure of the equation under consideration Mambriani is able to give the polynomial solution explicitly. In the second case of Mambriani $d_i = n - i$ for $i = 0, \ldots, n$, i.e. $m_i = 0$ for all i. The condition for a polynomial solution (19) may be simplified by $l = n$, $i_k = k$.

Perron [5] considers equations with $d_0 = n-1$, $d_i \leq n - i$ for $i = 1, \ldots, n$ and obtains for this special case a condition similar to (19).

Bound 2 *(Rational solutions). For a fixed value $\sigma \in \{1, 2, \ldots, s\}$ let $(i_0^\sigma, i_1^\sigma \ldots i_n^\sigma)$ be a permutation of $(0, 1, \ldots, n)$ such that*

$$m^\sigma_{i^\sigma_0} = m^\sigma_{i^\sigma_1} = \ldots = m^\sigma_{i^\sigma_l} < m^\sigma_{i^\sigma_{l+1}} \leq \ldots \leq m^\sigma_{i^\sigma_n}$$

for $0 \leq l \leq n$ and

$$Q^\sigma(M^\sigma) \equiv \sum_{k=0}^{l} q^\sigma_{i^\sigma_k, d_{i^\sigma_k}} \frac{(-1)^{n-i^\sigma_k} M^\sigma!}{(M^\sigma - n + i^\sigma_k)!}. \qquad (21)$$

A lower bound for the degree of the leading term at $x = x_\sigma$ for a solution of (3) is

$M^\sigma_{min} =$

$$min(negative \ integer \ solutions \ of \ Q^\sigma(M) = 0)$$

A bound for the degree of the polynomial part is obtained by multiplying (3) with the lcm of the denominators of the coefficients q_i and applying Bound 1 to the resulting equation of the form (2).

Proof. Assume that (3) has a solution y of the form

$$y = \sum_{\sigma=1}^{s} \sum_{k=M^\sigma}^{-1} b^\sigma_k (x - x_\sigma)^k + \sum_{k=0}^{N} a_k x^k. \qquad (22)$$

Substituting it at the left hand side and expanding it around $x = x_\sigma$ leads to a system of linear equations for the coefficients b^σ_k. For $k \leq -1$ the equation for the leading term $b^\sigma_{M^\sigma}$ is

$$b^\sigma_{M^\sigma} Q^\sigma(M^\sigma) = 0.$$

This implies $b^\sigma_{M^\sigma} = 0$ except if $Q^\sigma(M^\sigma) = 0$ for each finite pole. For $x \to \infty$ the first term in (22) does not contribute and the bound for the polynomial part is determined by Bound 1.

Bound 3 *(Solutions with rational logarithmic derivative). If there exists a solution y of (3) with a rational logarithmic derivative y'/y, a bound for the degree of its poles may be obtained by the following procedure. For each σ determine the smallest negative integer M^σ_{min} in the range*

$$\lfloor min_{1 \leq i,j \leq n,\, i \neq j} \frac{d^\sigma_i - d^\sigma_j}{i - j} \rfloor \leq M^\sigma \leq -1 \qquad (23)$$

such that at least two of the expressions $(n-i)M^\sigma + d^\sigma_i$ have the same value. The integer $\mid M^\sigma_{min} \mid$ is an upper bound for the order of a pole at $x = x_\sigma$. An upper bound for the degree of the polynomial part is obtained by determining the largest inter N_{max} in the range

$$0 \leq N \leq \lceil max_{1 \leq i,j \leq n,\, i \neq j} \frac{d^i - d^j}{i - j} \rceil$$

for which at least two of the expressions $(n-i)N + d^i$ have the same value. In addition there may occur an unspecified number of first-order poles of y'/y with an integer residuum R in the range $0 \le R \le n-1$ corresponding to the the polynomial part of y.

Proof. Transform (3) into the corresponding Riccati equation (8). Assume that u has a pole of order $\mid M^\sigma \mid$ at x_σ, i.e. around x_σ the expansion

$$u = b^\sigma_{M^\sigma}(x - x_\sigma)^{M^\sigma} + O[(x - x_\sigma)^{M^\sigma+1}]$$

with $M^\sigma \le -1$ is valid. Then $\phi_n(u)$ has a pole of order $\mid nM^\sigma \mid$ there. This is true for $n = 1$ because by definition $\phi_1(u) = u$. Assume that it is true for $n > 1$, i.e.

$$\phi_n(u) = b^\sigma_{nM^\sigma}(x - x_\sigma)^{nM^\sigma} + o[(x - x_\sigma)^{nM^\sigma+1}].$$

Then the proposition follows from

$$\phi_{n+1}(u) = b^\sigma_{(n+1)M^\sigma}(x-x_\sigma)^{(n+1)M^\sigma} + o[(x-x_\sigma)^{(n+1)M^\sigma+1}]$$

which is obtained by substituting the expression for ϕ_n in the recursion (6). In the expansion of (8) around the pole $x = x_\sigma$ the ith term contains the leading power $(n-i)M^\sigma + d^\sigma_i$ for $i \in \{1, 2, \ldots, n\}$. To satisfy it in the vicinity of x_σ a cancellation of the leading powers must occur. This is only possible if at least two of them are identical to each other. The minimum at the left hand side of (33) arises from minimizing M^σ in

$$(n-i)M^\sigma + d^\sigma_i = (n-j)M^\sigma + d^\sigma_j$$

which is obviously the condition for two terms having the same leading singularity at x_σ.

If the polynomial part of u has degree N, i.e. if

$$u = a_N x^N + a_{N-1}x^{N-1} + \ldots$$

then it follows by a simlar reasoning as above

$$\phi_n(u) = a_{nN}x^{nN} + a_{nN-1}x^{nN-1} + \ldots .$$

Consequently the ith term in (8) contains the leading power $(n-i)N + d^i$ for $i = 1 \ldots n$. A cancellation is only possible if there are at least two terms with the same leading degree in the range (34). The maximum at the right hand side follows from maximizing N in

$$(n-i)N + d_i = (n-j)N + d_j.$$

The second part follows immediately from

$$\phi_n\left(\frac{R}{x - x_0}\right) = \frac{R(R-1)\ldots(R-n+1)}{(x - x_0)^n}.$$

with R integer and x_0 arbitrary. This completes the proof.

4 The Factorization Algorithm. Implementation in Scratchpad II

All ingredients for designing the factorization algorithm are now available. Let

$$y^{(m)} + r_1 y^{(m-1)} + r_2 y^{(m-2)} + \ldots + r_{m-1}y' + r_m y = 0 \tag{24}$$

with $m < n$ be a possible factor of (3) with rational coefficients r_i and a fundamental system $\{y_1, \ldots, y_m\}$. Then all members of this fundamental system also satisfy the nth order equation (3). The coefficients r_k may be expressed in terms of the determinants $D^m(y_1, \ldots, y_m)$ and $D^m_k(y_1, \ldots, y_m)$ according to (13). On the other hand, these determinants satisfy certain equations which are associated with (3). This suggests the following algorithm for obtaining an equation (38) which is satisfied by a subset of a fundamental system of (3). Choose a fixed m in the interval $1 \le m \le n-1$ beginning with $m = 1$. Construct the associated equations for z_ν, $\nu \in I^{\nu(k)}$ for $k = m-1$ and $k = m$ in the notation of Section 2. The other associated equations are not required for the factorization. For a rational coefficient r_1 to exist the equation for z_1 must have a solution with a rational logarithmic derivative due to Liouville's relation. If this is true, the equations for the remaining coefficients r_k, $k = 2, \ldots, m$ are obtained from the associated equations for the determinants $D^m_k(y_1, \ldots, y_m)$ by substituting

$$D^m_k = (-1)^k r_k D^m.$$

If each of these equations has a rational solution, a candidate for a factor of (3) has been found. Finally it has to be verified that a true factor has been found. These steps are formalized in terms of the algorithm *LODEF* given in Figure 1. It takes a linear homogeneous equation with rational coefficients as input and returns the right irreducible factor of lowest order. Applying it iteratively a complete decomposition into irreducible factors of the input is obtained.

It is instructive to see how the algorithm *LODEF* operates in terms of a detailed example. Consider the equation

$$y''' + \frac{x-1}{x}y'' + \frac{x^2-2}{x}y' + \frac{2}{x^2}y =$$

$$\left(\frac{d}{dx} - \frac{1}{x}\right)\left(y'' + y' + \frac{x^2-1}{x}y\right) = 0. \tag{25}$$

This does not have a right factor of first order because there is no solution with rational logarithmic derivative.

Algorithm LODEF. Given a linear ordinary differential equation

$$y^{(n)} + q_1 y^{(n-1)} + q_2 y^{(n-2)} + \ldots + q_{n-1} y' + q_n y = 0 \qquad (*)$$

with rational coefficients $q_k \in \mathbf{Z}(x)$, it returns the right factor

$$y^{(m)} + r_1 y^{(m-1)} + r_2 y^{(m-2)} + \ldots + r_{m-1} y' + r_m y = 0$$

of lowest order $m < n$ with $r_k \in \mathbf{Z}(x)$. If no genuine factor exists the input equation is returned unchanged.

$m := 0$

S1 $m := m + 1$. If $m = n$ return input equation.

S2 Determine associated equations.

S3 Determine solution of equation for r_1 found in S2 with rational logarithmic derivative and determine r_1 from it. If none exists goto S1.

k:=1

S4 $k := k + 1$. If $k > m$ goto S7.

S5 Determine equation for r_k.

S6 Find rational solution of equation determined in S5 and determine r_k from it. If none exists goto S1 else goto S4.

S7 From the coefficients r_k construct a factor of $(*)$ and return.

Figure 1: The algorithm *LODEF* or **L**inear **O**rdinary **D**ifferential **E**quation **F**actorizer.

For a second order right factor to exist the associated equation for z_1

$$z_1''' + 2\frac{x-1}{x} z_1'' + \frac{x^3 + x^2 - 4x + 2}{x^2} z_1' + \frac{x^3 - 2x + 2}{x^2} z_1 = 0 \qquad (26)$$

must have a solution with a rational logarithmic derivative. Applying the construction described in Bound 3, the values $M^0_{min} = -4$ and $N_{max} = 5$ are obtained as bounds for the order of the single pole at $x_0 = 0$ and at infinity respectively. Substituting an ansatz with the appropriate number of terms into the Riccati equation corresponding to (40) leads to system of algebraic equations for the coefficients with the unique solution $y'/y = u = -1$. It turns out that there are no additional first order poles with integer residues in this case. Therefore $r_1 = 1$. To obtain the equation for r_2, the substitution $z_2 = ve^{-x}$ has to be performed in the associated equation

$$z_2''' + 2\frac{x+1}{x} z_2'' + \frac{x^3 + x^2 + 2x - 1}{x^2} z_2' + \frac{x^4 - x^3 - x + 1}{x^3} z_2 = 0 \qquad (27)$$

with the result

$$v''' - \frac{x-2}{x} v'' + \frac{x^3 - x - 1}{x^2} v' - \frac{x^3 - 1}{x^3} v = 0.$$

For the single pole $x = x_0$ one gets $d^0_1 = -1$, $d^0_2 = -1$, $d^0_3 = -3$ and $q^0_{1,-1} = 2$, $q^0_{2,-2} = -1$, $q^0_{3,-3} = 1$. From this it follows $m^0_1 = m^0_2 = m^0_3 = -3$ and Bound 2 yields

$$-M(M+1)(M+2) + 2M(M+1) + M + 1 = M^2(M+1).$$

The zero $M = -1$ is the lower bound for the degree of the pole. Similarly for the polynomial part it follows that $d^0 = d^1 = d^3 = 0$, $d^2 = 1$, i.e. $m^0 = 0$, $m^1 = 1$ and $m^2 = m^3 = 3$. With $p_{2,4} = 1$ and $p_{3,3} = -1$ Bound 1 yields the equation $N - 1 = 0$, i.e. $N = 1$ bounds the order of the polynomial part. With an appropriate ansatz the rational solution

$$v = \frac{x^2 - 1}{x}$$

as a candidate for the coefficient r_2 in a possible factor is obtained. Substituting these values for r_1 and r_2 into the ansatz (38) it is easily verified that a true factor of the third order equation (39) has been found. From this decompositon a fundamental system of (39) in terms of the two Bessel functions $J_1(x)$ and $Y_1(x)$ is easily obtained.

To perform automatically the extensive calculations usually involved in a factorization, the algorithm *LODEF* and a complete factorization package based on it have been implemented in the Scratchpad II computer algebra system [6]. It consists essentially of four parts, the two domains `LODE` and `ODE` abbreviating

```
LinearOrdinaryDifferentialEquation(U:GcdDomain,V:Symbol)

OrdinaryDifferentialEquation(U:GcdDomain,V:Symbol)
```

defining the respective datatypes and providing the basic operations for these objects, and the packages `LODEP` and `LODEF`

```
LinearOrdinaryDifferentialEquationPackage(V:Symbol)

LinearOrdinaryDifferentialEquationFactorizer(V:Symbol)
```

in which general operations for differential equations and the proper factorization are realized. The availability of such an implementation is a prerequisite for experimenting with the factorization and thereby gaining experience from it. The details of this implementation will be reported elsewhere.

5 Concluding Remarks

It has been the goal of this article to describe an algorithm for factoring linear differential equations that may be used in an automatic differential equation solver. For such an algorithm to work safely, all expressions that have to be handled must be kept under control. The bounds presented in this article serve this purpose. Typically the order of equations which occur in applications is limited by $n \sim 4$. The algorithm *LODEF* has proved extremely useful for finding closed form solutions in these cases as the application of our Scratchpad package has shown. The collection of equations by Kamke [3] has been used as a proving ground. It is hard to imagine the existence of any other method that is to such a degree purely algorithmic and universally applicable at the same time. It seems fair to say that a factorization package should be included as part of any automatic differential equation solver. A generalization to linear equations with more general coefficients, nonlinear or even partial differential equations would be extremely interesting. Somewhat complementary to this application oriented approach, in a recent article Grigor'ev [7] has analyzed the factorization with the emphasis on asymptotic bounds.

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