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The lattice of subsemilattices of a semilattice

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This note makes two observations about lattices of subsemilattices. First, we establish relationship between direct decompositions of such lattices and ordinal sum decompositions of semilattices. Then we give a characterization of the subsemilattice-lattices.

Let us recall some terminology. L will always stand for a semilattice, whose operation will be denoted by $\circ$. The ordering on L is given by letting $l_1 \leq l_2$ iff $l_1 \circ l_2 = l_2$, i.e. L is always a join-semilattice. Subsemilattices of L, ordered by inclusion, form a subsemilattice-lattice denoted by $Sub\, L$. In $Sub\, L$ the meet operation is intersection, and the join operation is defined as follows: $L_1 \vee L_2 = L_1 \cup L_2 \cup \{l_1 \circ l_2 \mid l_1 \in L_1, l_2 \in L_2\}$. An element a of an arbitrary lattice $\mathscr{L}$ is called *neutral* if $m(a, x, y) = M(a, x, y)$ for all $x, y \in \mathscr{L}$, where $m(a, x, y) = (a \wedge x) \vee (a \wedge y) \vee (x \wedge y)$ and $M(a, x, y) = (a \vee x) \wedge (a \vee y) \wedge (x \vee y)$. Notice that $m(a, x, y) \leq M(a, x, y)$ holds in any lattice.

LEMMA 1. *Let L be a semilattice and L_0 its subsemilattice. Then L_0 is a neutral element of $Sub\, L$ iff $L - L_0$ is a subsemilattice of L and every element of L_0 is comparable with every element of $L - L_0$.*

Proof. Let L_0 be a subsemilattice of L such that $L - L_0$ is a subsemilattice of L as well and every element of L_0 is comparable with every element of $L - L_0$. We must prove that, for any $L_1, L_2 \in Sub\, L$, $M(L_0, L_1, L_2) \subseteq m(L_0, L_1, L_2)$. Let $x \in M(L_0, L_1, L_2)$. Since $L_0 \vee L_i = L_0 \cup L_i$, $i = 1, 2$, there are 12 cases, but only one of them is nontrivial: $x \in L_0$ and $x = l_1 \circ l_2$, where $l_1 \in L_1$, $l_2 \in L_2$. If l_1 and l_2 are comparable, then either $x \in L_1$ or $x \in L_2$; hence $x \in m(L_0, L_1, L_2)$. If l_1 and l_2 are not comparable, then $l_1, l_2 \in L_0$ and $x \in (L_0 \wedge L_1) \vee (L_0 \wedge L_2) \subseteq m(L_0, L_1, L_2)$. Conversely, if $L_0 \in Sub\, L$ and $L - L_0$ is not a subsemilattice of L, then there exist

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$l_1, l_2 \notin L_0$ such that $l_1 \circ l_2 \in L_0$. But then $m(L_0, \{l_1\}, \{l_2\}) \neq M(L_0, \{l_1\}, \{l_2\})$. If $L - L_0$ is a subsemilattice of L and there exist incomparable $l_1 \in L_0$, $l_2 \notin L_0$ and $l = l_1 \circ l_2$, then $m(L_0, L - L_0, \{l_2\}) \neq M(L_0, L - L_0, \{l_2\})$ if $l \notin L_0$ and $m(L_0, \{l_1\}, \{l_2\}) \neq M(L_0, \{l_1\}, \{l_2\})$ if $l \in L_0$. Hence, L_0 is not neutral.

LEMMA 2. *Sub* $L \simeq \mathscr{L}_1 \times \mathscr{L}_2$ *iff there exists a neutral element* L_0 *of Sub* L *and that* $\mathscr{L}_1 \simeq$ *Sub* L_0 *and* $\mathscr{L}_2 \simeq$ *Sub* $L - L_0$.

Proof. By Theorem 1 of [2, p. 152], the direct decompositions of *Sub* L into two factors are of form *Sub* $L \simeq (L_0] \times [L_0)$, where L_0 is neutral. By Lemma 1, φ : *Sub* $L - L_0 \to [L_0)$ defined by $\varphi(L') = L' \cup L_0$ is a lattice isomorphism if L_0 is neutral. The lemma follows now from the fact that *Sub* $L_0 \simeq (L_0]$. □

COROLLARY 1. *An arbitrary semilattice L can not be represented as an ordinal sum of its proper subsemilattices iff Sub L is directly indecomposable.* □

COROLLARY 2. *If L is finite, then Sub L is directly indecomposible iff it is subdirectly irreducible.*

Proof. One direction is obvious. To prove that a directly indecomposable *Sub* L is subdirectly irreducible, assume that $|L| \geq 2$, since *Sub* L for a one-element L is a two-element chain and, therefore, subdirectly irreducible. Let $\mathbf{1}$ be the greatest element of L. We will show that $\Theta(\varnothing, \{\mathbf{1}\})$ is a unique atom of the congruence lattice of *Sub* L. Since one-element subsemilattices are exactly the atoms of *Sub* L, it is enough to show that $\Theta(\varnothing, \{\mathbf{1}\}) \leq \Theta(\varnothing, \{l\})$ for each $l \in L$, $l \neq \mathbf{1}$ or, equivalently, that $\{\mathbf{1}\}/\varnothing \approx_\omega \{l\}/\varnothing$. Notice that if $l_1 \circ l_2 = l$ in L, then $\{l\}/\varnothing \sim_\omega \{l_1, l_2, l\}/\{l_2\} \sim_\omega \{l_1\}/\varnothing$ in *Sub* L.

Since *Sub* L is directly indecomposable, by Corollary 1 for any element $l \in L$, $l \neq \mathbf{1}$, there exists $l' \in L$ incomparable with l, i.e. $l \circ l' > l$. Since L is finite, for any $l \neq \mathbf{1}$ there is a finite sequence $l_0, l_1, \ldots, l_{2n}$, where $l_0 = l$, $l_{2n} = \mathbf{1}$, l_{2i} and l_{2i+1} are incomparable and $l_{2i+2} = l_{2i} \circ l_{2i+1}$, $i = 0, \ldots, n-1$. The existence of such a sequence and the observation made above immediately imply $\{\mathbf{1}\}/\varnothing \approx_\omega \{l\}/\varnothing$. □

Notice that any neutral element of *Sub* L is complemented. Neutral complemented elements of any lattice $\mathscr{L}$ form a Boolean sublattice of $\mathscr{L}$ denoted by *Cen* $(\mathscr{L})$ [2]. It follows from Lemma 1 that intersection of an arbitrary family of neutral elements of *Sub* L is neutral. Hence, *Cen* (*Sub* L) is a complete lattice. Moreover, intersection of all neutral elements containing $l \in L$ is an atom of *Cen* (*Sub* L). Therefore, *Cen* (*Sub* L) is an atomic Boolean lattice whose atoms are exactly ordinally indecomposable subsemilattices of L. From this we conclude

THEOREM 1. *Let L be an arbitrary semilattice. Then Sub L can be represented as a direct product of directly indecomposable lattices, Sub $L \simeq \prod_{i \in I}$ Sub L_i, where $L = \bigoplus_{i \in I} L_i$ is a representation of L as an ordinal sum of ordinally indecomposable subsemilattices.* □

In the finite case the structure of *Cen* (*Sub L*) allows us to list all the direct decompositions of *Sub L*. If $L = \bigoplus_{i \in I} L_i$, where each L_i is ordinally indecomposable and *Sub* $L \simeq \mathscr{L}_1 \times \cdots \mathscr{L}_m$, then there exist disjoint sets $I_1, \ldots, I_m \subseteq I$ such that $I_1 \cup \cdots \cup I_m = I$, $L_j' = \bigoplus_{i \in I_j} L_i$ and $\mathscr{L}_j \simeq$ *Sub* L_j' for $j = 1, \ldots, m$.

We conclude the paper by characterizing the subsemilattice-lattices. An atomistic lattice $\mathscr{L}$ is called *biatomic* [1] if for any two non-zero $x, y \in \mathscr{L}$ and an atom $z \leq x \vee y$ there exist atoms $x' \leq x, y' \leq y$ such that $z \leq x' \vee y'$. We say that a biatomic lattice $\mathscr{L}$ satisfies property (S_n) if for any ideal V generated by n atoms $a_1, \ldots, a_n \in \mathscr{L}$ there exists a finite semilattice L_V such that $V \simeq$ *Sub* L_V, and the natural embedding of ideals $V \to W$ induces the embedding of semilattices $L_V \to L_W$.

THEOREM 2. *A lattice $\mathscr{L}$ is isomorphic to Sub L for some semilattice L iff it is algebraic, biatomic and satisfies (S_3).*

Proof. The 'only if' part is obvious. To prove the 'if' part, denote the set of atoms of $\mathscr{L}$ by $A(\mathscr{L})$ and the set of atoms under $x \in \mathscr{L}$ by $A(x)$. Notice that (S_3) implies that $(*)$ for every $X \subseteq A(\mathscr{L})$ with $|X| \leq 3$ there exists a semilattice operation $\circ_X$ on $A(\bigvee X)$ such that $(\bigvee X] \simeq$ *Sub* $\langle A(\bigvee X), \circ_X \rangle$ and $\circ_Y = \circ_X |_{A(\bigvee Y)}$ for every $Y \subseteq A(\bigvee X)$ with $|Y| \leq 3$.

Define a binary operation $\circ$ on $A(\mathscr{L})$ by $a_1 \circ a_2 = a_1 \circ_{\{a_1, a_2\}} a_2$. Clearly, $\circ$ is idempotent and commutative. That $\circ$ is associative follows from $(*)$. Thus, $\circ$ is a semilattice operation on $A(\mathscr{L})$. Define $\varphi : \mathscr{L} \to$ *Sub* $\langle A(\mathscr{L}), \circ \rangle$ by $\varphi(y) = \{x \in A(\mathscr{L}) \mid x \leq y\}$. That φ is well-defined follows from $(*)$. The remaining properties of $\mathscr{L}$ guarantee that φ is an isomorphism. Thus, $\mathscr{L} \simeq$ *Sub* $\langle A(\mathscr{L}), \circ \rangle$. □

Remark. Neutral elements of a lattice *Sub L* were characterized in Lemma 1. One can easily check that a weaker condition characterizes distributive and standard elements. In fact, L_0 is a distributive element of *Sub L* iff it is standard iff for all $l_1 \in L_0$, $l_2 \notin L_0$ either $l_1 \leq l_2$ or $l_1 \circ l_2 \in L_0$.

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REFERENCES

[1] Bennett, M. K., *Biatomic lattices*, Algebra Universalis *24* (1987), 60–73.
[2] Grätzer, G. *General Lattice Theory*, Springer Verlag, Berlin, 1978.

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