

# Differential 2-rigs

= semiring

what about  
skew polynomial?

Fosco Loregian

folore@ttu.ee

Tallinn University of Technology  
Estonia

Todd Trimble

topological.musings@gmail.com  
Western Connecticut State University  
Danbury, CT, U.S.A.

## Abstract

We propose the notion of *differential 2-rig*, a category  $\mathcal{R}$  equipped with coproducts and a monoidal structure distributing over them, also equipped with an endofunctor  $\partial : \mathcal{R} \rightarrow \mathcal{R}$  that satisfies a categorified analogue of the Leibniz rule. This is intended as a tool to unify various applications of such categories to computer science, algebraic topology, and enumerative combinatorics. The theory of differential 2-rigs has a geometric flavour (for example, we prove that for every  $\otimes$ -monoid  $M \in \mathcal{R}$ , the derivative  $\partial M$  is a  $M$ -module), but boils down to a specialization of the theory of tensorial strengths on endofunctors, tightly connected to applicative endofunctors in functional programming, and captures the construction of Brzozowski derivatives in formal language theory. This builds a surprising connection between apparently disconnected fields. We provide an explicit construction to build free 2-rigs on a signature, and we prove various initiality results: for example, a certain category of colored species is the free differential 2-rig on a single generator.

**CCS Concepts:** • Theory of computation  $\rightarrow$  Logic.

**Keywords:** bimonoidal category, differential operator, monoidal strength, 2-monads, combinatorial species, analytic functors, Brzozowski derivative, free monoidal category

## ACM Reference Format:

Fosco Loregian and Todd Trimble. 2022. Differential 2-rigs. In *Proceedings of Logic in Computer Science (LICS22)*. ACM, New York, NY, USA, 16 pages. <https://doi.org/10.1145/nnnnnnn.nnnnnnn>

## 1 Introduction

The present paper formalizes the following idea: study a pair  $(\mathcal{R}, \partial)$ , where  $\mathcal{R}$  is a ‘categorified ring’ and  $\partial : \mathcal{R} \rightarrow \mathcal{R}$  a ‘linear endofunctor’ satisfying the ‘Leibniz rule’. *derivation*

Far from being a mere exercise in style, studying such structures unravels an interesting and expressive theory that

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than ACM must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from [permissions@acm.org](mailto:permissions@acm.org).

LICS22, August 2–5, Haifa, Israel

© 2022 Association for Computing Machinery.

ACM ISBN 978-x-xxxx-xxxx-x/YY/MM...\$15.00

<https://doi.org/10.1145/nnnnnnn.nnnnnnn>

has plenty of applications in mathematics and is tightly connected to notions of ‘derivation-like’ operations employed in computer science.

Adapting terminology from classical ring theory, such a pair  $(\mathcal{R}, \partial)$  could be termed a *differential 2-rig*, and  $\partial$  a *derivation* on  $\mathcal{R}$ ; the study of such structures could thus be viewed as a categorified version of *differential algebra*. Differential algebra is aimed at providing a completely algebraic framework in which to study derivatives à la Leibniz, and it is nowadays an important part of modern commutative algebra [8, III.10], finding application in differential, and algebraic, geometry [46].

Intuitively, a *2-rig* is a category  $\mathcal{R}$  equipped with two structures, one additive and one multiplicative, such that the latter ‘distributes’ over the former.

At its most basic level, this is the requirement that, for objects  $A, B \in \mathcal{R}$ , the endofunctors  $A \otimes -$  and  $- \otimes B$  distribute over coproducts:

$$A \otimes (B + C) \cong A \otimes B + A \otimes C \text{ and } (B + C) \otimes A \cong B \otimes A + C \otimes A,$$

but our main definition will be fairly more general, treating with other shapes of colimits apart from this basic one.

However, an important difference with classical ring theory is that the request that  $\mathcal{R}$  admits ‘additive inverses’ is an extremely restrictive one. This motivates our choice of terminology: a *rig*  $(R, +, \cdot)$  –also called a *semiring*– is a ring without negatives, i.e. an algebraic structure that satisfies all the axioms of a ring, but where  $(R, +)$  is just a commutative monoid.

**Literature on 2-rigs.** A motivating example of ‘categorified calculus on a 2-rig’ is Joyal’s theory of species and analytic functors [5, 26, 28] providing a categorical foundation for enumerative combinatorics and finding concrete applications as a model of PCF [22].

The category of combinatorial species (functors  $\Sigma^{\text{op}} \rightarrow \text{Set}$  from the category  $\Sigma$  of finite sets and bijections) is a prominent example of a 2-rig which supports a viable notion of derivative functor, obeying the usual rules of the calculus, and it will always be our motivating example and test-bench for definitions. So, our informal intuition is that a derivation on a 2-rig is an operator applied to an object that resembles a formal power series.<sup>1</sup>

<sup>1</sup>The equivalence between analytic functors, regarded as ‘categorified formal power series’, and species is a long-established result first proved in [28]; see also [1].

→ by Joyal

This situates our work on a different ground than another important piece of literature dealing with notions of derivation on a category, namely the theory of *differential categories* of Blute, Cockett et al. [7]. Differential categories were developed to provide a categorical doctrine for *differential linear logic*; as a rule of thumb, the fundamental difference between the two approaches lies in where the categorified derivative operation acts. In differential categories, every morphism has a derivative assigned via a so-called differential combinator; instead, we focus on deriving objects functorially and naturally. For the sake of completeness, we shall mention yet another approach to ‘categorical differentiation’ recently developed in [53] with applications to ZX calculus in mind; again, there seems to be no relation with our theory of differential 2-rigs, since derivations on their category  $\text{Mat-S}$  are not Leibniz on objects.

Elsewhere, terms like ‘2-rig’ or ‘rig categories’ have been appropriated by different authors to mean different things. For example, [2] defines a 2-rig to be a cocomplete symmetric monoidal category in which the monoidal product distributes over all colimits, and in [3], ‘2-rig’ has meant a Vect-enriched symmetric monoidal category with biproducts and idempotent splittings (where the distributivity is automatic). On the other hand, the term ‘rig category’ or ‘distributive category’ [36] has been used to mean a category with two monoidal products, one called ‘multiplicative’, which distributes over the other, called ‘additive’. It is easy to imagine variations on these themes: 2-rigs which are only finitely cocomplete, or that are assumed only to have *finite* coproducts (which we consider to be a baseline assumption). Alternatively, on the multiplicative side, one might want infinite products and a complete distributivity law over infinite coproducts. This type of ‘2-rig’ would be germane to the study of polynomial functors in the sense of [18, 31, 49–51], which have provided a unifying setting for studying numerous structures in applied category theory. Or, one could envisage other enhancements of the multiplicative monoidal structure: braided, pivotal, balanced, traced, with or without internal homs that are adjoint to the tensor, and so on.

Given the multiplicity of possible definitions of 2-rig, we believe it makes sense not to fix a single notion of 2-rig, but to be flexible and contemplate a whole spectrum of possible theories, or ‘doctrines’ of  $\mathbb{D}$ -rigs parametrized by  $\mathbb{D}$ , a 2-monad on  $\text{Cat}$  (locally small categories) whose algebras will possess colimits of a certain shape.

The first goal of this paper is thus to provide a generalized framework in which each of these instances can be studied on the same foot; our main definition for a ‘doctrine of 2-rigs’, Definition 4, is geared in this direction.

**Our main contributions.** Besides unifying most notions of 2-rig under a common framework, in this paper we

are also interested in seeing how different  $\mathbb{D}$ -rigs  $\mathcal{R}$ , for different doctrines  $\mathbb{D}$ , interact with an accompanying notion of derivation  $\partial : \mathcal{R} \rightarrow \mathcal{R}$  (roughly, functors which obey a Leibniz rule,  $\partial(A \otimes B) \cong \partial A \otimes B + A \otimes \partial B$ , see Definition 8). Unexpectedly, depending on the doctrine, derivations may be either virtually nonexistent (cf. Proposition 2 and Lemma 1), as is the case when the multiplicative structure is cartesian, or may exist in great plenitude, typically when the multiplicative structure enjoys a more ‘linear’ character (in the sense of Girard’s linear logic [20]). Within a doctrine  $\mathbb{D}$  where derivations are prevalent, they may also be used to give a notion of ‘dimension of a  $\mathbb{D}$ -rig’ (cf. Remark 4).

In such cases, one generally expects derivations to be potent and unifying tools. We show that this is the case, once again guided by the theory of species as motivating example, and the usage of derivatives in the hands of the ‘Montreal school of categorical combinatorics’ [6, 33, 34, 37, 38] (see also the more recent [42, 48]), where differential equations written in the category of species, as well as their solutions, are fruitfully interpreted combinatorially. We prove that categories of species are ‘necessary objects’ in a general theory of 2-rigs, because they arise as free objects for specific doctrines of 2-rigs and acquire a canonical choice of a differential structure. Moreover, in Theorem 2 we prove that the category of species on a countable set, equipped with a ‘shifting’ derivation operation, is the free differential 2-rig on one generator.

Derivations can also be used to shed light on the theory of operads; for example, recent results by Obradovich [43] show that ordinary (permutative) operads are certain types of monoids for a skew monoidal structure  $F' \otimes G$  defined using the derivative, and that cyclic operads [19] also admit an efficient description in terms of derivatives of species. Indeed, the category  $\text{Spc}$  of Joyal’s species and its multivariate versions are motivating examples, as their differential structure is virtually ideal. For example, not only does the derivative endofunctor  $\partial : \text{Spc} \rightarrow \text{Spc}$  yield categorified versions of all the good properties of ordinary derivatives, but it is also both a left and a right adjoint (cf. [47]).

As far as we know, and in spite the large effort to understand the properties of a *specific* instance of differential 2-rig (see [6, 33, 34, 37, 38] for the theory of ODEs in the category of species, and a variety of works by M. Fiore [13–15] that explored in detail the meaning of *bijective proofs* in terms of datatype structures), a systematic study of general properties of differential 2-rigs (a ‘synthetic 2-rig theory’, so to speak) has never been attempted.

Thus, one first aim of the present paper is to give all the various notions of 2-rig and 2-derivation their proper due, while balancing generality and applicability, and unifying diverse approaches. In developing the rudiments of this framework, we aim to clarify what is specific to the category  $\text{Spc}$  of species and what instead follows from a general theory

of 2-rigs, and concentrate on the latter to generalize the former (to other doctrines, other flavours of monoidality, other flavours of species –colored [41] or linear [37]).

As a showcase example, in subsection 5.2 we present a ‘chain rule’ that categorifies the well-known calculus theorem  $(f \circ g)'(x) = f'(g(x))g'(x)$  and that holds good across a broad spectrum of doctrines of 2-rigs, thus generalizing the chain rule true for species and proved by Joyal in his early works.

## Structure of the paper

In section 2 we introduce the main object of discussion of our paper: a 2-rig for a ‘doctrine’  $\mathbb{D}$ , i.e. a specified class of colimits, is a category having all colimits specified by  $\mathbb{D}$ , and a monoidal structure  $\otimes$  that distributes over said colimits; we show how this formalism is capable of encompassing most of the various notions of 2-rig scattered in the literature; in section 3 we outline the fundamental definition in order to arrive at a definition of derivation on a  $\mathbb{D}$ -rig  $\mathcal{R}$ , a pair of *tensorial strengths* interacting well with each other; to the best of our knowledge, [the characterization of tensorial strengths as lax natural transformations in \(11\) and \(13\) has not been accounted elsewhere](#). Section 4 is the heart of the paper: a differential 2-rig is defined in Definition 8; after this we concentrate on examples: species, in Example 4, and the Brzozowski derivative in Construction 1, and [we prove that for every  \$\otimes\$ -monoid  \$M\$ , its derivative  \$\partial M\$  is a  \$M\$ -module](#). In section 5 we provide the construction of free  $\mathbb{D}$ -rigs and prove that free 2-rigs acquire differential structures (Proposition 8) as well as various initiality results: for example, the category of ( $S$ -colored) species is the free cocomplete 2-rig on a single (on  $|S|$ ) generator; in section 6 we draw the conclusions of the paper and sketch ideas for future development: the opportunity to gain a geometric view on applicatives, through derivations on a 2-rig seems to be a promising prospect, as well as the application of our general theory to a synthetic approach to combinatorial differential equations.

## 2 Doctrines of 2-rigs

Before defining a notion of 2-rig doctrine, we present a few examples that play a guiding role and show a need for such a general notion.

**Example 1** (A list of motivating examples).

- *The category of presheaves  $[\mathcal{M}^{\text{op}}, \text{Set}]$  over a monoidal category  $\mathcal{M}$ , equipped with the Day convolution product induced from the monoidal product of  $\mathcal{M}$  (cf. [10]). Note that the convolution product tends to inherit other structures of the monoidal product  $\otimes$ ; e.g., the Day convolution  $(F, G) \mapsto F * G$  is symmetric (or braided, or cartesian monoidal) if  $\otimes$  is so. Meanwhile, the presheaf category is a free cocompletion of  $\mathcal{M}$ , and by design, the Day convolution preserves colimits in each of its separate arguments. Finally, one may consider presheaves*

*valued, and enriched in, some other category besides Set, such as the category of modules over a commutative ring, or the category of sup-lattices, or a thin category of ‘truth values’.*

- *The category of finite-dimensional vector bundles over a space or finitely generated projective modules over a commutative ring. These categories admit coproducts and tensor products, but not general colimits. Nor would one necessarily want to impose general colimits because of phenomena like ‘Eilenberg swindles’ [45]. These examples of ‘2-rigs’ are typically enriched in vector spaces or the like, and typically the only colimits envisaged are absolute colimits: those that are preserved by every (enriched) functor.*
- *Between these two extremes, one sometimes considers ‘2-rigs’ which have colimits over diagrams that are bounded in size. For example, one may consider the category of finite  $G$ -sets for some group  $G$  that may be infinite, or of continuous finite  $G$ -sets for some topological group  $G$ : these arise in Grothendieck’s approach to Galois theory. In this case, we are dealing only with finite colimits. Or, in other situations, as in the theory of locally  $\kappa$ -presentable categories, the compact objects will admit colimits over possibly infinitary diagrams but still bounded in size.*
- *It is automatic that for any monoidal closed category  $\mathcal{R}$ , tensoring on one side or the other by an object will preserve any colimits that happen to exist in the category. So long as those colimits include finite coproducts, we thus obtain a type of 2-rig.*

Guided by such examples, the following definitions are meant to describe a spectrum of notions of 2-rigs that have arisen in practice.

**Definition 1.** An [additive doctrine](#) is a 2-category whose objects are categories that admit all colimits of diagrams belonging to a prescribed class, including finite discrete diagrams whose colimits are finite coproducts (which we denote by  $+$ , and  $0$  for the empty coproduct). Morphisms are functors that preserve colimits of that class, and 2-cells are natural transformations between such functors.<sup>2</sup>

In each case, we may instead work with a stricter notion of additive doctrine where objects are categories with *chosen* colimits: these are strict algebras of a strict 2-monad, which is often technically convenient. Strict algebra morphisms preserve those chosen colimits strictly, which is not what one wants, but pseudomorphisms preserve colimits in the usual sense [35].

So, an additive doctrine is determined by a (strict or pseudo) 2-monad  $A$  on  $\text{Cat}$ , of which we consider the category

<sup>2</sup>A more general notion of additive doctrine is obtained by considering enriched analogues as well; in this paper, we mostly focus on the unenriched (i.e., Set-enriched) case.

of algebras. In short, the notion of an additive doctrine takes care of the additive monoid part of a 2-rig; as for the multiplicative part, we can similarly state the following definition.

**Definition 2.** A multiplicative doctrine is a 2-category that is monadic (in the 2-categorical sense) over the 2-category  $\mathbf{MCat}_s$  of monoidal categories, strong monoidal functors, and monoidal transformations, such that the composition of monadic functors,

$$U_M = (\mathcal{M} \rightarrow \mathbf{MCat}_s \rightarrow \mathbf{Cat}), \quad (1)$$

is also 2-monadic.

Intuitively, a multiplicative doctrine consists of a category of monoidal categories, possibly equipped with additional structure, that arises as the category of algebras for a monad on  $\mathbf{Cat}$ .

For example, the 2-category of symmetric monoidal categories is a multiplicative doctrine. Symmetric monoidal categories are strict algebras of a strict 2-monad, and pseudomorphisms coincide with strong symmetric monoidal functors.

One might also want to replace  $\mathbf{MCat}_s$  with the 2-category  $\mathbf{MCat}_l$  (having lax monoidal functors as 1-cells) or  $\mathbf{MCat}_c$  (colax functors), but we do not explore such a generalization here.<sup>3</sup>

In short, a multiplicative doctrine is given by a 2-monad  $\mathbf{M}$  on  $\mathbf{Cat}$  modelled over the 2-monad whose algebras are monoidal categories, of which we consider the 2-category of algebras.

Finally, we need a notion of what it means for a multiplicative doctrine to *distribute* over an additive doctrine. Intuitively, this is taken care by a *distributive law* in the sense of [4] between the two doctrines.

Let  $\mathbf{A}$  be the 2-monad for any additive doctrine in the sense above, and let  $\mathbf{P}$  be the 2-monad for the additive doctrine of all small-cocomplete categories, whose underlying functor  $P$  takes a locally small category  $C$  to the category consisting of presheaves  $C^{\text{op}} \rightarrow \mathbf{Set}$  that are small colimits of representable functors. We have an inclusion of 2-monads  $j : \mathbf{A} \rightarrow \mathbf{P}$ . Temporarily, let  $\mathbf{M}$  denote the 2-monad whose algebras are monoidal categories, with underlying functor  $M$ .

Now, the Day convolution monoidal structure provides for each monoidal category  $C$  a monoidal structure on the free small-cocompletion on its underlying category,  $PUC$ , and this construction also works as free cocompletion for monoidal categories [23].

In other words,  $PUC$  carries a canonical  $\mathbf{M}$ -algebra structure

$$MPUC \rightarrow PUC \quad (2)$$

pseudonatural in  $C$ , thus leading to an action

$$MPU \Rightarrow PU \quad (3)$$

and such an action is equivalent to a canonical distributive law between monads

$$\delta : \mathbf{M}\mathbf{P} \Rightarrow \mathbf{P}\mathbf{M}. \quad (4)$$

The only thing required to set up this distributive law is that Day-convolving on either side,  $A * -$  or  $- * A$ , preserves all small colimits. This remains true [52] for any restricted class of colimits coming from an additive doctrine given by a monad  $\mathbf{A}$  on  $\mathbf{Cat}$ ; thus, we obtain by restriction a distributive law

$$\delta' : \mathbf{M}\mathbf{A} \Rightarrow \mathbf{A}\mathbf{M} \quad (5)$$

or what is essentially the same, a canonical lifting  $\tilde{\mathbf{A}}$  as follows:

$$\begin{array}{ccc} \mathbf{MCat} & \xrightarrow{\tilde{\mathbf{A}}} & \mathbf{MCat} \\ U \downarrow & & \downarrow U \\ \mathbf{Cat} & \xrightarrow{\mathbf{A}} & \mathbf{Cat} \end{array} \quad (6)$$

**Definition 3.** A *distributivity of a multiplicative doctrine  $\mathbf{M}$  over an additive doctrine  $\mathcal{A} = \mathbf{A}\text{-Alg}$*  is a choice of lift  $\tilde{\mathbf{A}}$  in the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\tilde{\mathbf{A}}} & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathbf{MCat} & \xrightarrow{\mathbf{A}} & \mathbf{MCat} \end{array} \quad (7)$$

Thanks to [56, Definition 33, Remark 34] such distributivities are essentially unique. This is particularly the case when the unit of the monad for the multiplicative doctrine over  $\mathbf{MCat}$  is essentially surjective on objects (eso). In this case, the distributivity is uniquely given by the Day convolution structure at the underlying monoidal category level.

Now let  $\mathbf{M}$  denote the monad for the monadic functor  $U_M \rightarrow \mathbf{Cat}$  in (1). As above, a distributivity  $\tilde{\mathbf{A}}$  amounts to an  $\mathbf{M}$ -action  $\mathbf{M}\mathbf{A}U_M \Rightarrow \mathbf{A}U_M$ , which corresponds to a 2-distributive law  $\delta : \mathbf{M}\mathbf{A} \Rightarrow \mathbf{A}\mathbf{M}$  between 2-monads.

**Definition 4.** A doctrine of 2-rigs consists of an additive doctrine  $\mathbf{A}$ , a multiplicative doctrine  $\mathbf{M}$ , and a distributivity of  $\mathbf{M}$  over  $\mathbf{A}$ .

Using the distributive law, one obtains a structure of 2-monad  $\mathbf{AM}$  for the composition of functors. If one uses strict versions of the 2-monads  $\mathbf{A}$  and  $\mathbf{M}$ , one obtains a strict 2-monad  $\mathbb{D} = \mathbf{AM}$ , and a strict notion of  $\mathbb{D}$ -rig, with pseudomorphisms as an appropriate notion of *morphism* of  $\mathbb{D}$ -rigs.

<sup>3</sup>Also, it is well-known that the composition of monadic functors can fail to be monadic; to correct this shortcoming, various flatness conditions such as ‘preserving codescent objects’ may be imposed on a (2-)monadic functor  $G : \mathcal{M} \rightarrow \mathbf{MCat}$  to guarantee that the composition  $U_M = UG : \mathcal{M} \rightarrow \mathbf{Cat}$  is also monadic, but this issue is somewhat technical and it will not be pursued here.



**Remark 1.** As we have said, the intuitive idea of a ‘categorified rig’ (or ‘semiring’) can be formalized in various ways: the original notion of ‘distributive category’ given by Laplaza in [36] is more general because it only asks for the presence of two monoidal structures (the ‘additive’ one is not necessarily cocartesian). In such a setting, the complexity of the diagrams required to ensure coherence is daunting (cf. [12, 2.1.1]); our choice, where ‘coherence conditions’ follow automatically from universal properties, avoids this problem by design.

**Remark 2.** A minor warning about our definition: whereas ordinary rigs form discrete distributive categories, ordinary rigs do not give discrete 2-rigs in our sense, since the only discrete category admitting finite coproducts is the singleton. Nor is a 2-rig with a single object an exciting notion: a monoidal category with a single object is a commutative monoid; by the Eckmann-Hilton argument, the two operations of 2-rig collapse in one single commutative monoid structure, for which multiplication  $a \cdot - : M \rightarrow M$  is a monoid homomorphism.

We can build on the definition of a doctrine of 2-rigs and turn our attention to some specific examples of interest, where we assume something more on the additive or multiplicative doctrine in study (symmetry, or the presence of more shapes of colimit).

**Definition 5** (Terminological conventions for doctrines of 2-rigs).

- A doctrine is symmetric if the underlying multiplicative doctrine is the doctrine of symmetric monoidal categories. Similarly, we may speak of when a doctrine is braided, cartesian, semicartesian (meaning: the monoidal unit is the terminal object, see [17]), ..., referring to its multiplicative structure.
- A doctrine is cocomplete if the class of all small colimits gives the underlying additive doctrine. More generally, when a size bound is needed, we call a doctrine  $\kappa$ -cocomplete, for a regular cardinal  $\kappa$ , if its underlying additive doctrine is given by the class of colimits over all diagrams of size  $\kappa$  or less. (Following standard practice, we call finitary the  $\omega$ -cocomplete case.) A doctrine is  $\kappa$ -additive if its additive doctrine consists of categories that have coproducts over index sets of size  $\kappa$  or less.
- When the multiplicative doctrine is that of monoidal categories, then for an additive doctrine  $\mathcal{A}$  we may refer to the 2-rigs as monoidally  $\mathcal{A}$ -cocomplete categories.
- A doctrine is absolute if its additive doctrine consists of categories with finite coproducts in which all idempotent maps split.
- By default, ‘the’ doctrine of 2-rigs refers to the minimal notion of 2-rigs, where the multiplicative doctrine is just the doctrine of monoidal categories, and the additive doctrine is the  $\omega$ -additive doctrine.

- A closed 2-rig is a category  $\mathcal{R}$  as in Example 1 such that each  $A \otimes -$  and  $- \otimes B$  have right adjoints; in this case, of course, they preserve all colimits that exist in  $\mathcal{R}$ .

**Notation 1.** Combinations are possible: it is clear what a symmetric closed 2-rig is, or what a symmetric monoidally  $\mathcal{A}$ -cocomplete 2-rig is. With a small abuse of language, when we refer to a 2-rig as symmetric, cocomplete, ..., we are declaring that in the current context, we intend to consider it as an object of a 2-rig doctrine thus designated. When necessary, we tend to call just ‘2-rig’ an object of the minimal 2-rig doctrine.

**Example 2.** The following are examples of 2-rigs:

RA1) Any monoidal category  $(\mathcal{V}, \otimes, I)$  with the property that  $\otimes$  preserves  $\kappa$ -ary coproducts is a monoidally  $\kappa$ -additive category. This includes the category of sets, any cartesian closed category with finite coproducts, the category of modules over a ring  $R$ , or more generally, the category  $\text{Mod}_R^{\mathcal{V}}$  of modules over a monoid  $R$  in a suitable monoidal base  $\mathcal{V}$ .

RA2) In the same notation, the category  $[\mathcal{A}, \mathcal{V}]$  of  $\mathcal{V}$ -enriched presheaves over a (symmetric) monoidal  $\mathcal{V}$ -category  $(\mathcal{A}, \oplus, j)$ , endowed with the Day convolution monoidal structure

$$F * G := \int^{U, V \in \mathcal{A}} FU \times GV \times \mathcal{A}(U \oplus V, -) \quad (8)$$

is a (symmetric) closed 2-rig.

RA3) An example of a non-symmetric 2-rig is the category  $[\mathcal{A}, \mathcal{A}]_{\sqcup} \subseteq [\mathcal{A}, \mathcal{A}]$  of endofunctors  $F : \mathcal{A} \rightarrow \mathcal{A}$  that commute with finite coproducts.

### 3 Modules and strengths

Just as ordinary rings and rigs act on modules, so 2-rigs or  $\mathbb{D}$ -rigs (for a 2-rig doctrine  $\mathbb{D} = (\mathcal{A}, \mathcal{M}, \delta)$ ) act on 2-modules, sometimes called *actegories* (cf. [25]). For the same additive doctrine  $\mathcal{A}$ , if  $C$  is an  $\mathcal{A}$ -algebra, then we may form the endohom  $[C, C]$  of  $\mathcal{A}$ -algebra maps or  $\mathcal{A}$ -cocontinuous functors, and this endohom forms a monoidally  $\mathcal{A}$ -cocomplete category. If in addition  $C$  is a  $\mathbb{D}$ -rig, then it has an underlying monoidally  $\mathcal{A}$ -cocomplete category.

Keeping this in mind, we give the following definition to capture an action of  $\mathcal{R}$  on a category  $C$  as a suitable rig endomorphism.

**Definition 6.** A (left)  $\mathcal{R}$ -module structure (or actegory structure) on  $C$  is a monoidally  $\mathcal{A}$ -cocontinuous map

$$\mathcal{R} \rightarrow [C, C]. \quad (9)$$

**Example 3.** A simple example is  $\mathcal{R}$  acting on itself, so the map above takes an object  $R$  to the functor  $R \otimes - : \mathcal{R} \rightarrow \mathcal{R}$ . This is called the left Cayley action. For objects  $R$  of  $\mathcal{R}$  and  $C$  of  $C$ , we sometimes use  $R \odot C$  to denote values of left module actions. In some tautological cases, for example, the left Cayley action, we use ordinary tensor product notation  $R \otimes R'$ .

**Remark 3.** As a monoidal category,  $\mathcal{R}$  may also be construed as a one-object bicategory  $B\mathcal{R}$ , and an  $\mathcal{R}$ -module may be construed as a pseudofunctor of bicategories

$$B\mathcal{R} \rightarrow \mathbf{A}\text{-Alg} \quad (10)$$

that is locally  $\mathbf{A}$ -cocontinuous.

In this notation, we can provide a sensible notion for a morphism of modules.

**Definition 7.** Given  $\mathcal{R}$ -modules  $C, \mathcal{D} : B\mathcal{R} \rightarrow \mathbf{A}\text{-Alg}$ , a morphism from  $C$  to  $\mathcal{D}$  is a lax natural transformation  $C \rightarrow \mathcal{D}$ .

It is worth unpacking this very terse definition. Here a lax natural transformation takes the unique object of  $B\mathcal{R}$  to a 1-cell  $F : C \rightarrow \mathcal{D}$ , in other words an  $\mathbf{A}$ -continuous functor of this form. It takes 1-cells of  $B\mathcal{R}$ , i.e. objects  $R$  of  $\mathcal{R}$ , to 2-cells which take the form of families in  $\mathcal{D}$ ,

$$R \otimes FC \rightarrow F(R \otimes C), \quad (11)$$

that are natural in  $C$ . This 2-cell constraint is often called a *strength* on  $F$ ; we call it a *left strength*. The lax naturality axioms provide the usual axioms for a tensorial strength as defined e.g. in [30].

One can define right module structures by reversing the 1-cells of  $B\mathcal{R}$ , i.e., reversing the order of tensoring,

$$(B\mathcal{R})^{\text{op}} \rightarrow \mathbf{A}\text{-Alg}. \quad (12)$$

For example, we have a right Cayley action that takes an object  $R$  to  $- \otimes R$ . Then, a 2-cell constraint for a lax natural transformation between right module structures is called a *right strength*. It involves natural families, sometimes written as

$$FC \otimes R \rightarrow F(C \otimes R). \quad (13)$$

Similarly, one can define bimodules as homomorphisms

$$(B\mathcal{R})^{\text{op}} \times B\mathcal{R} \rightarrow [C, C] \quad (14)$$

(for example, there is an evident Cayley bimodule with  $\mathcal{R}$  acting on itself on both the left and right), and consider bistrorengths.

Here is one type of example that recurs frequently for us. Suppose given a  $\mathbb{D}$ -rig map  $\varphi : \mathcal{R} \rightarrow \mathcal{S}$ . This induces a homomorphism  $\varphi^{\text{op}} \times \varphi : B\mathcal{R}^{\text{op}} \times B\mathcal{R} \rightarrow B\mathcal{S}^{\text{op}} \times B\mathcal{S}$ , which composes with the Cayley bimodule of  $\mathcal{S}$ . Letting  $\alpha_{\mathcal{R}}$ ,  $\alpha_{\mathcal{S}}$  denote the Cayley bimodules, the data of a morphism from  $\alpha_{\mathcal{R}}$  to  $\alpha_{\mathcal{S}}(\varphi^{\text{op}} \times \varphi)$  entails an  $\mathbf{A}$ -cocontinuous functor  $G : \mathcal{R} \rightarrow \mathcal{S}$  with a ( $\varphi$ -augmented) left and right strength

$$\varphi(R) \otimes GR' \rightarrow G(R \otimes R'), \quad GR' \otimes \varphi(R) \rightarrow G(R' \otimes R). \quad (15)$$

## 4 Differential 2-rigs: basic theory

We now turn to the main definition of the present paper, that of a *derivation* on a 2-rig; in simple terms, if a 2-rig categorifies the notion of  $\text{ri}(\text{n})\text{g}$   $R$ , a derivation on a 2-rig categorifies the notion of derivation on  $R$ , widely used in commutative algebra and finding applications to the Galois theory of differential equations (see [46, 54]).

**Definition 8** (Derivation on a 2-rig). Let  $\mathbb{D}$  be a 2-rig doctrine, let  $\mathcal{R}$  be a  $\mathbb{D}$ -rig, and let  $\mathcal{M}$  be a  $\mathcal{R}$ -bimodule. An  $\mathcal{M}$ -valued derivation of  $\mathcal{R}$  is a bimodule morphism  $\partial$  from the Cayley bimodule of  $\mathcal{R}$  (cf. Example 3) to  $\mathcal{M}$ , such that the canonical natural maps

$$\begin{aligned} D1) \text{ I} : \partial C \otimes C' + C \otimes \partial C' &\rightarrow \partial(C \otimes C') \text{ (the 'leibnizator'),} \\ D2) \text{ i} : 0 &\rightarrow \partial(I) \end{aligned}$$

are isomorphisms.

Here the first arrow is defined by pairing the module right strength  $\partial C \otimes C' \rightarrow \partial(C \otimes C')$  with the module left strength  $C \otimes \partial C' \rightarrow \partial(C \otimes C')$ .

A differential  $\mathbb{D}$ -rig is a  $\mathbb{D}$ -rig  $\mathcal{R}$  equipped with a derivation from the Cayley bimodule of  $\mathcal{R}$  to itself.

**Example 4.** A paradigmatic example of a differential 2-rig is given by the category of Joyal species with its standard derivative functor, sending  $F : \Sigma^{\text{op}} \rightarrow \text{Set} : n \mapsto Fn$  to  $F' : n \mapsto F(n+1)$ , where  $n \in \Sigma$  is an  $n$ -element set.

In this example, the doctrine is that of symmetric monoidally cocomplete categories, and the category of species is the free symmetric monoidally cocomplete category on one object. This is the free cocompletion of the free symmetric monoidal category  $\Sigma$  on one object, equivalent to the category of presheaves  $\Sigma^{\text{op}} \rightarrow \text{Set}$  on the category of finite sets and bijections, equipped with the Day convolution product induced from the monoidal product on  $\Sigma$ , the groupoidal core of the cocartesian monoidal category of finite sets.

Besides the Leibniz rule, whose validity can be proved via elementary methods, the differential structure in the category of species satisfies two additional properties reminiscent of formal power series theory.<sup>4</sup> If  $S, T, U, V$  are objects of  $\Sigma$ , we have the following

**Proposition 1** (Generalised Leibniz rule for species). Let  $\partial$  be the standard derivation on species. We can think of the  $n$ -th derivative  $\partial^n F$  as a derivative 'with respect to a  $n$ -element set  $U$ ', since in case  $|U| = n$  one has

$$\partial^n F[A] = F[A + n] \cong F[A + U]. \quad (16)$$

Define  $F^{(U)}$  by the formula  $F^{(U)}[A] = F[A + U]$ . Now, let  $F, G : \Sigma \rightarrow \text{Set}$  be two combinatorial species; we have

$$(F * G)^{(U)}[C] \cong \sum_{S+T=U} (F^{(S)} * G^{(T)})[C]. \quad (17)$$

<sup>4</sup>Given the elementary nature of their proof, we believe both these results pertain to 'folklore' in circles of combinatorialists, but we could not find an appropriate reference for them.

*Proof.* In Appendix B, page 16.  $\square$

**Theorem 1** (A Taylor-Maclaurin formula for species). *Every species  $F : \Sigma \rightarrow \text{Set}$  has a ‘Taylor-Maclaurin’ expansion*

$$F(X + A) \cong \sum_{n=0}^{\infty} \frac{\partial^n F(A)}{n!} X^n \quad (18)$$

where  $X, A$  are sets and the quotient by  $n!$  is the quotient by the action of the symmetric group  $S_n$  over  $\partial^n F$ ; in coend notation,

$$\int^{n \in \mathbb{P}} \partial^n F(A) \times X^n \cong \int^n F(A + n) \times X^n \cong F(X + A) \quad (19)$$

(the coend takes care of the quotient by the action of the symmetric group).

*Proof.* In Appendix B, page 16.  $\square$

An interesting example of a derivation on a  $\mathbb{D}$ -rig having concrete applications to formal language theory is Brzozowski’s derivative on subsets of a free monoid (or on a set of regular expressions).

Here the additive doctrine of  $\mathbb{D}$  is the 2-category of sup-lattices (i.e., cocomplete 2-enriched categories), and the multiplicative doctrine is that of monoidal categories. Note that this is the only specific example of derivation on categories enriched over a base that is not  $\text{Set}$ .

**Construction 1** (Brzozowski’s derivative). *Let  $A$  be a non-empty set, and let  $R = 2^{A^*}$  be the semiring of formal languages on  $A$ . This is the collection of subsets of the free monoid on  $A$ , ordered by inclusion. It forms a  $\mathbb{D}$ -rig where*

- *suprema are given by unions of subsets;*
- *the monoidal product is given by ‘pointwise concatenation’,  $UV = \{uv \mid u \in U, v \in V\}$ . This is a case of an enriched Day convolution, making  $R$  a free  $\mathbb{D}$ -rig.*

Define a  $\mathbb{D}$ -rig morphism  $\varphi : R \rightarrow R$  by sending  $U \subseteq A^*$  to  $U \cap \{()\}$ , where  $()$  is the empty word in  $A^*$ . Make  $R$  into an  $R$ -bimodule  $R_\varphi$  by the rule  $U \cdot V \cdot W = \varphi(U) V W$ .

Then every word  $w$  in  $A^*$  defines a function  $\partial_w : R \rightarrow R$  by  $\partial_w U := w \rtimes U = \{x \in A^* \mid wx \in U\}$  (this is right adjoint to  $\{w\} \cdot -$ ). It is easy to verify that  $\partial_w : R \rightarrow R$  defines a derivation of  $R$  valued in  $R_\varphi$ .

In simple terms, the  $\partial_w$  searches for words in  $U$  having initial segment  $w$ , and the Leibniz rule

$$w \rtimes (UV) = (w \rtimes U)V + \varphi(U)(w \rtimes V) \quad (20)$$

holds in the sense that when this operation is performed on the set  $U \cdot V$ , it results in  $\partial_w U \cdot V$  (the words in  $U$  with initial segment  $w$ ), plus all the words in  $V$  with initial segment  $w$ , if  $U$  contains the empty word (in this case,  $V \subseteq U \cdot V$ , and thus we have to take into account  $w \rtimes V$ ).

There is a notion of morphism of differential 2-rig, and a notion of morphism of derivations: together, these define the category 2-Rig of differential 2-rigs, and the category

$\text{Der}(\mathcal{R}, \partial)$  of derivations on a given 2-rig. We will not investigate 2-categorical properties of 2-Rig, but the notion of morphism of derivation is necessary to turn Corollary 1 into an equivalence of categories, instead of just a correspondence on objects.

**Definition 9** (Morphism of differential 2-rigs). *Given differential 2-rigs  $(\mathcal{R}, \partial) \rightarrow (\mathcal{S}, \partial')$ , morphisms of differential 2-rigs are morphisms of 2-rigs  $F : \mathcal{R} \rightarrow \mathcal{S}$  such that  $\partial' \circ F = F \circ \partial$ .*

**Definition 10** (Morphism of derivations). *Let  $\mathcal{R}$  be a 2-rig, and  $\partial, \partial' : \mathcal{R} \rightarrow \mathcal{R}$  two derivations in the sense of Definition 8. A morphism of derivations  $\alpha : \partial \Rightarrow \partial'$  is a natural transformation of functors that is ‘compatible with the leibnizators’ of  $\partial, \partial'$ , i.e. such that the equality of 2-cells*

$$I' \circ (\alpha \otimes 1 + 1 \otimes \alpha) = (\alpha * \otimes) \circ I$$

holds if  $I$  (resp.,  $I'$ ) is the leibnizator of  $\partial$  (resp.,  $\partial'$ ).

Now we observe how some notions bearing on 2-rigs, particularly property-like notions for the multiplicative monoidal product, make sense independent of which doctrine of 2-rigs is considered. For example, a 2-rig (relative to any 2-rig doctrine) is *cartesian* if its multiplicative monoidal product is cartesian, and is *closed* if tensoring with an object on either side has a right adjoint.

A derivation  $\partial : \mathcal{R} \rightarrow \mathcal{R}$  is *trivial* if it is constantly 0.

**Proposition 2.** *A derivation  $\partial : \mathcal{R} \rightarrow \mathcal{R}$  on a cartesian 2-rig must be trivial.*

*Proof.* If  $X$  is an object of  $\mathcal{R}$ , then we have a map  $\partial(!) : \partial(X) \rightarrow \partial(1) = 0$ . But for any object  $A$  that admits a map  $f : A \rightarrow 0$ , we must have  $A \cong 0$ , because the composite

$$A \xrightarrow{(f, 1_A)} 0 \times A \xrightarrow{\pi_2} A \quad (21)$$

is  $1_A$ , and  $0 \times A \cong 0$  by distributivity.  $\square$

The description of derivations on a 2-rig in terms of tensorial strengths leads to another important ‘no-go theorem’ for derivations on a 2-rig:

**Lemma 1.** *Suppose that  $\mathcal{R}$  is a closed 2-rig and that the functor*

$$\mathcal{R}(I, -) : \mathcal{R} \rightarrow \text{Set} \quad (22)$$

*is faithful. Then any functor  $\partial : \mathcal{R} \rightarrow \mathcal{R}$  can carry at most one left strength and one right strength.*

*Proof.* Let  $[A, -]$  be the right adjoint of  $A \otimes - : \mathcal{R} \rightarrow \mathcal{R}$ . Then left strengths on  $T$  are in natural bijection with enrichment structures on  $T$ ,

$$[A, B] \rightarrow [TA, TB], \quad (23)$$

and by application of the faithful functor  $\mathcal{R}(I, -) : \mathcal{R} \rightarrow \text{Set}$ , such enrichment structures map one-to-one (not onto necessarily) to  $\text{Set}$ -enrichment structures  $\mathcal{R}(A, B) \rightarrow \mathcal{R}(TA, TB)$ . However, there is only one of these.  $\square$

**Remark 4.** Lemma 1 rules out many well-behaved categories from the list of nontrivial differential cocomplete 2-rigs: for example,  $\mathcal{R} = \text{Set}$  or  $\mathcal{R} = \text{Mod}_R$  satisfy the assumption that  $\mathcal{R}(I, -)$  is faithful. The idea is that every object in these categories is a colimit of copies of the monoidal unit  $I$ , i.e., a coequalizer of maps between coproducts of copies of  $I$ . If derivations preserve colimits and take  $I$  to 0, then, of course, every object maps to 0.

Intuitively:  $I$  is a ‘constant’, and sums/coproducts of constants (however many coproducts the additive doctrine allows) are also constants. Thus, these examples behave like ‘categories of constants’ and hence are ‘0-dimensional’ as seen through the lens of derivations.

The connection between derivations on 2-rigs and tensorial strengths deserves to be spelt out more explicitly: to this end, we provide a general procedure to turn every endofunctor  $F : \mathcal{R} \rightarrow \mathcal{R}$  on a 2-rig into a derivation.

#### 4.1 The universal construction of tensorial strengths

In this subsection, we shall show that every endofunctor  $F : \mathcal{A} \rightarrow \mathcal{A}$  admits a ‘best approximating’ lax derivation  $\Theta F$ , obtained as the free algebra for a comonad on  $[\mathcal{A}, \mathcal{A}]$ . This, in turn, follows from the fact that there is a comonad  $\Theta^L$  (resp.,  $\Theta^R$ ) on the category  $[\mathcal{A}, \mathcal{A}]$  of endofunctors of a monoidal category  $\mathcal{A}$ , that equips an endofunctor  $F$  with a cofree left (resp., right) tensorial strength.

**Proposition 3.** *Let  $\mathcal{A}$  be a complete and left (resp., right) symmetric monoidal closed category; then, there exists a comonad*

$$\Theta : [\mathcal{A}, \mathcal{A}] \rightarrow [\mathcal{A}, \mathcal{A}] \quad (24)$$

*on the category of endofunctors of  $\mathcal{A}$ , whose coalgebras are exactly the endofunctors equipped with a right (resp., left) tensorial strength.*

*Proof.* In Appendix B, page 15.  $\square$

This result entails that given an endofunctor  $F : \mathcal{R} \rightarrow \mathcal{R}$  on a symmetric 2-rig,  $F$  is best-approximated by a lax derivation  $\Theta F$ , obtained by endowing the functor  $F$  it with the cofree strength

$$\Theta F A \otimes B \xrightarrow{t_{AB}^L} \Theta F(A \otimes B) \xleftarrow{t_{AB}^R} A \otimes \Theta F B \quad (25)$$

using the universal property of coproducts, now one gets a map

$$F A \otimes B + A \otimes F B \xrightarrow{\begin{bmatrix} t_{AB}^L \\ t_{AB}^R \end{bmatrix}} F(A \otimes B) \quad (26)$$

The result remains true when the 2-rig  $\mathcal{R}$  is not symmetric, but a little more care is needed; in that case, one must define  $\Theta_R$  (resp.,  $\Theta_L$ ) exploiting a right (resp., left) closed structure on  $\mathcal{A}$ .

The relevance of this result lies in the fact that one can then formally invert the map in (26) and endow  $\mathcal{R}$  with a derivation canonically obtained from the pair  $(\mathcal{R}, F)$ .

We conclude the section turning our attention to the following result, that to the best of our knowledge is new, despite the simplicity of its proof: let  $M$  be an internal  $\otimes$ -monoid in a differential 2-rig  $(\mathcal{R}, \otimes, \partial)$ ; the derivative  $\partial M$  is an  $M$ -module.

**Proposition 4.** *Let  $\mathcal{R}$  be a 2-rig, and  $M$  a internal semigroup (resp., monoid) in  $\mathcal{R}$ , with multiplication  $m : M \otimes M \rightarrow M$  (and unit  $e : I \rightarrow M$ ); then the map  $\partial m : \partial M \otimes M + M \otimes \partial M \rightarrow \partial M$  amounts to a pair of actions  $i_R : \partial M \otimes M \rightarrow \partial M$  and  $i_L : M \otimes \partial M \rightarrow \partial M$  of  $M$  on its derivative object  $\partial M$ .*

*Proof.* In Appendix B, page 14.  $\square$

## 5 The construction of free 2-rigs

In this subsection, we turn our attention to constructions of derivations and differentials, restricting focus to symmetric 2-rig doctrines  $\mathbb{D}$ . Our main technique is to exploit the representability of derivations in the sense of Definition 11 and Proposition 5.

There are several reasons for restricting to symmetric 2-rigs  $\mathcal{R}$ . First, in ordinary algebra, the vast majority of applications of derivations is to commutative algebra; categorifying, it is then natural to consider symmetric monoidal structures. Moreover, tensoring functors  $R \otimes - : \mathcal{R} \rightarrow \mathcal{R}$  carry canonical (co)strengths, on account of the symmetry.

In the symmetric case, we can turn any left  $\mathcal{R}$ -module  $M$  into a right module or a bimodule by defining  $M \otimes R$  to be  $R \otimes M$ . We call bimodules arising this way *symmetric*. (Here it seems pointless to distinguish between  $\otimes$  and  $\odot$ , so we just write  $\otimes$  instead.)

**Definition 11** (Square-zero extensions). *Let  $\mathcal{R}$  be a  $\mathbb{D}$ -rig, and let  $M$  be a symmetric  $\mathcal{R}$ -bimodule. Define the square-zero extension  $\mathcal{R} \ltimes M$  of  $\mathcal{R}$  to be  $\mathcal{R} \times M$  as an  $\mathcal{A}$ -algebra, and equipped with a symmetric monoidal product defined by the formula*

$$(A, M) \boxtimes (B, N) := (A \otimes B, A \otimes N + M \otimes B), \quad (27)$$

*and with monoidal unit  $(I, 0)$ . The first projection  $\pi : \mathcal{R} \ltimes M \rightarrow \mathcal{R}$  makes this a  $\mathbb{D}$ -rig over  $\mathcal{R}$ .*

A straightforward computation allows determining the associators and unitors for the  $\boxtimes$  monoidal structure (one must use the compatibility between the left and right module structure on  $M$ ) and the left and right distributive maps.

An alternative presentation of the square zero extension, in the case where  $M$  is the Cayley bimodule of  $\mathcal{R}$  acting on itself, can be given as a ‘quotient’ 2-rig  $\mathcal{R}[Y]/(Y^2)$ : a categorification of an algebra of ‘dual numbers’, as explained in the following subsection. This 2-rig is denoted  $\mathcal{R}[\varepsilon]$ .

**Proposition 5.** *For a  $\mathbb{D}$ -rig  $S$  over  $\mathcal{R}$ , say  $\psi : S \rightarrow \mathcal{R}$ , there is a natural equivalence between maps  $\Phi : S \rightarrow \mathcal{R} \ltimes M$  in  $\mathbb{D}\text{-Rig}/\mathcal{R}$ , and  $\psi$ -augmented derivations  $\partial$  of  $S$  valued in  $M$ , where  $\partial = \pi_2 \Phi : S \rightarrow M$ .*



The proof is fairly routine since  $(\psi, \partial)$  being a (strong) symmetric monoidal functor means that we obtain isomorphisms

$$\partial(S) \otimes \psi(S') + \psi(S) \otimes \partial(S') \cong \partial(S \otimes S') \quad (28)$$

whose restrictions to the summands satisfy the strength coherence conditions, on account of the coherence conditions that obtain for a symmetric monoidal functor.

For example, we can use this proposition to reconstruct the standard derivative on Joyal species  $\text{Spc}$ , working over the doctrine  $\mathbb{D}$  of symmetric monoidally cocomplete categories. Consider  $\text{Spc}$  as a Cayley bimodule over itself, and form  $\text{Spc}[\varepsilon] = \text{Spc} \ltimes \text{Spc}$ .

As  $\text{Spc}$  is the free symmetric monoidally cocomplete category on one generator  $X$  (the representable functor  $\Sigma(-, 1)$ ), there is an equivalence of categories

$$\mathbb{D}\text{-Rig}(\text{Spc}, \text{Spc}[\varepsilon]) \simeq \text{Spc}[\varepsilon]. \quad (29)$$

This means any object  $(F, G)$  whatsoever of  $\text{Spc}[\varepsilon]$  induces a  $\mathbb{D}$ -rig map  $\Phi_{(F,G)} : \text{Spc} \rightarrow \text{Spc}[\varepsilon]$ , hence (by the Proposition) a  $\psi$ -augmented derivation for some  $\mathbb{D}$ -rig map  $\psi : \text{Spc} \rightarrow \text{Spc}$ .

Let us be more explicit. First we calculate  $\psi = \pi\Phi_{(F,G)} : \text{Spc} \rightarrow \text{Spc}$ . The pseudonaturality of the equivalence

$$\mathbb{D}\text{-Rig}(\text{Spc}, \mathcal{R}) \simeq \mathcal{R} \quad (30)$$

means  $\pi\Phi_{(F,G)}$  is the unique (essentially unique, i.e., unique up to unique isomorphism) symmetric monoidally cocontinuous functor  $\psi_F : \text{Spc} \rightarrow \text{Spc}$  that carries  $X$  to  $F$ . Proceeding in stages, the functor  $F : 1 \rightarrow \text{Spc}$  extends essentially uniquely to a symmetric monoidal functor  $\tilde{F} : \Sigma \rightarrow \text{Spc}$ , taking  $n$  to the  $n$ -fold Day convolution  $F^{\otimes n}$ . Then this extends essentially uniquely to a *cocontinuous* symmetric monoidal functor  $\text{Spc} = [\Sigma^{\text{op}}, \text{Set}] \rightarrow \text{Spc}$ , according to the formula

$$W = \int^{n:\Sigma} W(n) \cdot \Sigma(-, n) \mapsto \int^{n:\Sigma} W(n) \cdot F^{\otimes n}. \quad (31)$$

The last coend is an instance of the *substitution* product of species, denoted  $W \circ F$ . Whatever it is, the point is that  $\psi = (-) \circ F$ , where the right side is the essentially unique  $\mathbb{D}$ -rig map  $\text{Spc} \rightarrow \text{Spc}$  that extends  $F : 1 \rightarrow \text{Spc}$ . In particular, if  $F$  is the generator  $X$ , then  $\psi_X$  is the identity on  $\text{Spc}$ .

Now a derivation  $\partial : \text{Spc} \rightarrow \text{Spc}$  augmented by the identity is just an ordinary derivation, i.e., satisfies  $\partial(A \otimes B) \cong \partial(A) \otimes B + A \otimes \partial(B)$ . The composite

$$1 \xrightarrow{X} \text{Spc} \xrightarrow{\langle \text{id}, \partial \rangle} \text{Spc}[\varepsilon] \xrightarrow{\pi_2} \text{Spc} \quad (32)$$

is the component  $G$  of  $(F, G)$ , whereas the composition of the last two arrows is  $\partial$ . In other words,  $G = \partial(X)$ . If we want  $\partial$  to match the standard derivative of species, then we must have  $G = X' = I$ , the unit of Day convolution.

Therefore, under the natural equivalence of the proposition, the standard derivative of species corresponds to the  $\mathbb{D}$ -rig map  $\text{Spc} \rightarrow \text{Spc}[\varepsilon]$  that takes the generator  $X$  to  $(X, I)$ .

If we take  $X$  to some other element  $(X, G)$  instead, then the corresponding derivation  $\partial$  is defined by  $\partial(F) = F' \otimes G$ , because this is after all a derivation, and because  $\partial(X) \cong X' \otimes G \cong G$  is correct. Note then that every differential structure, i.e., every derivation on  $\text{Spc}$  augmented over the identity, is obtained by tensoring the standard derivative by some object.

**Remark 5.** In the analogy between species  $F, G$  and formal power series  $f, g$ , the substitution product corresponds to functional substitution  $(f \circ g)(x) = f(g(x))$ . The derivative of a substitution can be computed via the chain rule, known since Joyal [27]:

$$(F \circ G)' = (F' \circ G) \otimes G'. \quad (33)$$

We will provide a proof for the chain rule, valid in any  $\mathbb{D}$ -rig, in Appendix B, page 16.

### 5.1 Presentations of $\mathbb{D}$ -rigs

We begin with a discussion of free  $\mathbb{D}$ -rigs and then give a few sample constructions of other  $\mathbb{D}$ -rigs. To provide a presentation of an ordinary rig is tantamount to providing a coequalizer of two maps between free rigs since rigs form a category 2-monadic over  $\text{Set}$ .

Something like this is needed to construct some 2-rigs of interest (for example, the ‘dual numbers’ or *Kähler differentials* construction on a 2-rig generally involves a co-inverter of maps between free  $\mathbb{D}$ -rigs). The fact that under mild assumptions on  $\mathbb{D}$ —for example, if its multiplicative monad  $\mathbf{M}$  is finitary—the 2-category  $\mathbb{D}\text{-Rig}$  has bicolimits, ensures that similar such constructions exist and can provide presentations of 2-rigs as suitable 2-dimensional colimits [29] of diagrams of free 2-rigs.

Recall that a doctrine of 2-rigs  $\mathbb{D}$  consists of an additive doctrine given by a class of small colimits, a multiplicative doctrine that is 2-monadic over the doctrine of monoidal categories, and a distributivity.

If  $\mathbf{A}$  denotes the monad on  $\text{Cat}$  for the additive doctrine, then for a category  $C$ , the  $\mathbf{A}$ -cocompletion  $\mathbf{A}(C)$  is equivalent to the full (replete) subcategory of the small presheaf category  $P(C)$  obtained by taking the closure of the representable functors under the class of colimits on  $\mathbf{A}$ .

Using the distributivity, the monad for  $\mathbb{D}$  is the composite  $\mathbf{AM}$ . Hence, for every doctrine  $\mathbb{D}$ , the free  $\mathbb{D}$ -rig  $\mathbb{D}[C]$  on a category  $C$  is always formed according to the following procedure:

- Take the free multiplicative structure generated by  $C$ , i.e.,  $\mathbf{M}(C)$ .
- Take the free  $\mathbf{A}$ -cocompletion of that.

We have already seen an example of this in the case of Joyal species (in the doctrine  $\mathbb{D}$  of symmetric monoidally cocomplete categories): it is the free cocompletion  $[\Sigma^{\text{op}}, \text{Set}]$  of the free symmetric monoidal category  $\Sigma$  on a single generator. Likewise, we may define multivariate species, say for example species in two variables, as the category  $[\Sigma(2)^{\text{op}}, \text{Set}]$

equipped with Day convolution, where incidentally  $\Sigma(2)$  is equivalent to  $\Sigma \times \Sigma$ .

**Example 5.** *As a somewhat artificial but illustrative example, let us consider the free braided monoidally  $\aleph_1$ -additive category on one generator (so cocomplete for countable coproducts) and specify how its monoidal product works. If  $\mathbf{B}[1]$  is the free braided monoidal category on one generator (the braid group), with monoidal product  $\mu : \mathbf{B}[1] \times \mathbf{B}[1] \rightarrow \mathbf{B}[1]$ , and  $A$  is the countable coproduct cocompletion, then the monoidal product on the free  $\mathbb{D}$ -rig is a composite*

$$\mathbf{AB}[1] \times \mathbf{AB}[1] \rightarrow \mathbf{A}(\mathbf{B}[1] \times \mathbf{B}[1]) \xrightarrow{A\mu} \mathbf{AB}[1] \quad (34)$$

where only the first map needs explanation. Let  $X, Y$  be categories. An object in  $\mathbf{A}X$  is a countable product of representables  $\sum_i X(-, A_i)$ , and we define  $\mathbf{A}X \times \mathbf{A}Y \rightarrow \mathbf{A}(X \times Y)$  to be the evident map

$$\left\langle \sum_i X(-, A_i), \sum_j Y(-, B_j) \right\rangle \mapsto \sum_{i,j} X(-, A_i) \times Y(-, B_j).$$

For the remainder of this section, we return to symmetric 2-rigs (relative to some additive doctrine  $\mathbf{A}$ ), and proceed to categorify some commutative algebra. The 2-category of  $\mathbf{A}$ -algebras, being a 2-category of algebras for a KZ-monad, carries a monoidal product  $\odot$  (see [16]) characterized by the fact that for  $\mathbf{A}$ -algebras  $\mathcal{A}, \mathcal{B}, \mathcal{C}$ , functors  $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$  that are  $\mathbf{A}$ -cocontinuous in the separate  $\mathcal{A}$ -,  $\mathcal{B}$ -arguments are equivalent to  $\mathbf{A}$ -cocontinuous functors  $\mathcal{A} \odot \mathcal{B} \rightarrow \mathcal{C}$ .

Using the universal property one can show that if  $\mathcal{R}, \mathcal{S}$  are  $\mathbb{D}$ -rigs, meaning here symmetric monoidally  $\mathbf{A}$ -cocomplete categories, then  $\mathcal{R} \odot \mathcal{S}$  naturally acquires a  $\mathbb{D}$ -rig structure and is the coproduct of  $\mathcal{R}$  and  $\mathcal{S}$  in  $\mathbb{D}\text{-Rig}$ . In particular, let  $\mathcal{S} = \mathbb{D}[Y]$  be the free  $\mathbb{D}$ -rig on a single generator  $Y$ . We write  $\mathcal{R} \odot \mathbb{D}[Y]$  as  $\mathcal{R}[Y]$ ; this plays a role analogous to a polynomial rig  $C[Y]$  with coefficients in a rig  $C$ , and the construction is analogous to the ‘extension of scalars’ from the initial rig  $\mathbb{N}[Y]$  to the rig  $C[Y]$  obtained as a coproduct in the category of rigs.

**Remark 6.** *The formation of  $\mathcal{R}[Y]$  does not require working with symmetric 2-rigs: just as one can form a polynomial algebra  $R[x]$  over a noncommutative rig  $R$ , so one can form a ‘polynomial’ 2-rig  $\mathcal{R}[Y]$  over a monoidal 2-rig  $\mathcal{R}$ , by taking a tensor product  $\mathcal{R} \odot \mathbb{D}[Y]$ . But this tensor product will generally not be a coproduct in  $\mathbb{D}\text{-Rig}$ , if we work outside the symmetric context.*

Next, we sketch the construction of a  $\mathbb{D}$ -rig of Kähler differentials on  $\mathcal{R}$ . With a slight abuse of language, let  $0 : \mathbb{D}[Y] \rightarrow \mathbb{D}[Y]$  denote the essentially unique  $\mathbb{D}$ -rig morphism that takes  $Y$  to 0, and similarly let  $Y^2 : \mathbb{D}[Y] \rightarrow \mathbb{D}[Y]$  denote the morphism that takes  $Y$  to  $Y^{\otimes 2}$ . The unique map  $0 \rightarrow Y^2$  in  $\mathbb{D}[Y]$  transports across the equivalence

$$\mathbb{D}\text{-Rig}(\mathbb{D}[Y], \mathbb{D}[Y]) \simeq \mathbb{D}[Y] \quad (35)$$

to a symmetric monoidal natural transformation  $0 \Rightarrow Y^2$  between  $\mathbb{D}$ -rig maps  $0, Y^2 : \mathbb{D}[Y] \rightarrow \mathbb{D}[Y]$ . Then, tensoring this transformation with  $\mathcal{R}$ , we obtain a 2-cell in  $\mathbb{D}\text{-Rig}$ , as shown in the diagram below:

$$\mathcal{R}[Y] \begin{array}{c} \xrightarrow{0} \\ \Downarrow \\ \xrightarrow{Y^2} \end{array} \mathcal{R}[Y] \xrightarrow{q} \mathcal{R}[Y]/(Y^2) \quad (36)$$

The ‘quotient’ construction  $q : \mathcal{R}[Y] \rightarrow \mathcal{R}[Y]/(Y^2)$  we are after is a coinverter of this 2-cell in the 2-category  $\mathbb{D}\text{-Rig}$ .

It can be proved that the diagram (36) satisfies the universal property of a coinverter ([29, dual of (4.6)]).

Observe that for some 2-rig doctrines  $\mathbb{D}$ , this coinverter may be somewhat degenerate. For example, in the doctrine of cartesian 2-rigs (for any additive doctrine  $\mathbf{A}$ ), the condition that an arrow  $0 \rightarrow C^2$  is invertible in  $\mathcal{R}$  forces  $C \cong 0$  (because  $C$  is a retract of  $C^2$ ), and in this case the coinverter will just be the 2-rig map  $\mathcal{R}[Y] \rightarrow \mathcal{R}$  taking  $Y$  to 0 (cf. the fact that there are no nontrivial differentials on a cartesian 2-rig).

**Proposition 6.** *For a doctrine  $\mathbb{D}$  of symmetric 2-rigs, there is an equivalence  $\mathcal{R}[Y]/(Y^2) \simeq \mathcal{R} \ltimes \mathcal{R}$ .*

*Proof.* In Appendix B, page 15.  $\square$

In combination with Proposition 5, this means that  $\mathcal{R}[\varepsilon] = \mathcal{R}[Y]/(Y^2)$ , equipped with the evident  $\mathbb{D}$ -rig map  $\mathcal{R}[\varepsilon] \rightarrow \mathcal{R}$  taking  $Y$  to 0, represents augmented derivations.

Given its importance, it is worth providing an intuitive idea for the meaning of this construction.

In order to build  $\mathcal{R}[\varepsilon]$  as a universal object, we are particularly interested in studying 2-rig morphisms  $F : \mathcal{R}[Y] \rightarrow \mathcal{R}[Y]$  with the property of ‘killing a prescribed power of  $Y$ ’ in the free 2-rig  $\mathcal{R}[Y]$ ; classically the ring of Kähler differentials is obtained as the ring of ‘dual numbers’  $k[X]/(X^2)$  whose elements are classes  $a + bX$  of linear polynomials and  $X$  is interpreted as an infinitesimally small quantity such that  $X^2 \sim 0$ .

Let us start by observing that there is an obvious map

$$Y^2 \otimes - : P \mapsto Y^2 \otimes P; \quad (37)$$

in Proposition 9 we observed how every object  $P \in \mathcal{R}[Y]$  can be uniquely written as a ‘polynomial in  $Y$ ’; so, it makes sense to consider polynomials ‘divisible by  $Y^2$ ’,  $P(Y) = Y^2 \otimes R(Y)$  for some  $R \in \mathcal{R}[Y]$ .

This construction defines an ‘ideal’ in the sense that there is a projection whose ‘kernel’  $\ker q$  coincides with the subcategory  $\text{im}(Y^2 \otimes -)$  spanned by  $\{Y^2 \otimes R \mid R \in \mathcal{R}[Y]\}$  and obviously<sup>5</sup>

- $0 \in \ker q$ ;

<sup>5</sup>This notion of ideal is modeled on the notion of ideal in a semiring, [21, Ch. 6 and 8]; in order to get an interesting notion of quotient in absence of additive inverses, one needs to consider generic congruences on a rig, marking a quite clear difference with the ring-theoretic notion.

- $F, G \in \ker q \Rightarrow F + G \in \ker q$ ;
- $F \in \ker q$  and  $P$  any polynomial  $\Rightarrow F \otimes P \in \ker q$ .

Now, it can be proved that the coinverter  $\mathcal{R}[Y]/(Y^2)$  identifies with the full subcategory of  $\mathcal{R}[Y]$  spanned by objects

$$\mathcal{R}[Y]_{<2} := \{A + B \otimes Y \mid A, B \in \mathcal{R}\} \quad (38)$$

we can copy the classical proof of representability of derivations by the ring of dual numbers: derivations  $\partial : A \rightarrow A$  correspond naturally to algebra maps  $A \rightarrow A[Y]/(Y^2)$ , in an isomorphism  $\text{Der}(A, A) \cong \text{Alg}_A(A, A[Y]/(Y^2))$ .

It is easy to see that the subcategory in Equation (38) is a 2-rig when the operations are defined mimicking the case of rigs. What remains to show is that it can be identified with the coinverter in Equation (36); this is the aim of the following

**Proposition 7.** *There exists a coinverter diagram*

$$\mathcal{R}[Y] \begin{array}{c} \xrightarrow{0} \\ \Downarrow u \\ \xrightarrow{Y^2 \otimes -} \end{array} \mathcal{R}[Y] \xrightarrow{q} \mathcal{R}[Y]_{<2} \quad (39)$$

of 2-rigs and 2-rig morphisms.

*Proof.* In Appendix B, page 16.  $\square$

**Remark 7.** *Clearly, the above argument works for every ‘ideal’ of  $\mathcal{R}[Y]$  (at least, when the ideal is ‘principal’, i.e. we consider localization at a single morphism  $0 \rightarrow P$  for  $P \in \mathcal{R}[Y]$ ): the possibility to study ‘localization at an ideal’ builds a strong analogy with commutative algebra. This seems to be a point worthy of further investigation but out of the focus of the present discussion.*

We refrain from developing this more general theory, as for our purposes Proposition 7 is sufficient:

**Corollary 1.** *There is an equivalence of categories*

$$\text{Der}(\mathcal{R}, \mathcal{R}) \cong 2\text{-Rig}(\mathcal{R}, \mathcal{R}[Y]/(Y^2)) \quad (40)$$

or in other words, the category of derivations  $\mathcal{R} \rightarrow \mathcal{R}$  as in Definition 10 correspond to 2-rig morphisms  $\mathcal{R} \rightarrow \mathcal{R}[Y]/(Y^2)$ . More generally, there is an equivalence between derivations  $\mathcal{R} \rightarrow \mathcal{M}$  values in a  $\mathcal{R}$ -module  $\mathcal{M}$ , and algebra morphisms between  $\mathcal{R}$  and the square-zero extension of Definition 11.

The construction of free  $\mathbb{D}$ -rigs and Corollary 1 allow to provide examples of differentials on categories of multivariate (or ‘colored’, cf. [41]) species.

**Definition 12** (Partial derivative). *Let  $\mathbb{D}[S]$  be the free  $\mathbb{D}$ -rig on a set or discrete category of generators  $S$ . For  $s \in S$ , define the partial derivative*

$$\frac{\partial}{\partial s} : \mathbb{D}[S] \rightarrow \mathbb{D}[S] \quad (41)$$

to be the derivation that corresponds to the  $\mathbb{D}$ -rig map  $\mathbb{D}[S] \rightarrow \mathbb{D}[S][\varepsilon]$  that takes  $s$  to  $(s, I)$  and  $t \in S, t \neq s$ , to  $(t, 0)$ .<sup>6</sup>

Every differential on  $\mathbb{D}[S]$  is similarly formed from the  $\mathbb{D}[S]$ -rig maps  $\mathbb{D}[S] \rightarrow \mathbb{D}[S][\varepsilon]$  taking each  $s$  to  $(s, a_s)$  for some choice of ‘coefficients’  $a_s \in \mathbb{D}[S]$ . In the case where the additive doctrine admits arbitrary coproducts, this differential may be denoted

$$\partial = \sum_{s \in S} a_s \frac{\partial}{\partial s}. \quad (42)$$

Here is one more example of a differential 2-rig, bearing witness that differential structures on a symmetric 2-rig tend to be plentiful. The idea goes as follows: let  $\mathbb{D}[X, Y]$  be the free  $\mathbb{D}$ -rig over two generators; given any two polynomials  $p(X, Y), q(X, Y)$  we can build the ‘quotient 2-rig’ killing off the ‘ideal’ generated by  $\{p, q\}$  as a suitable 2-colimit.

**Example 6.** *Another example of a symmetric 2-rig presentation is suggested by the notation  $\mathcal{H} := \mathbb{D}[X, Y]/(Y^2 + 1 \cong X^2)$  where we categorify an algebra of polynomials on a hyperbola. Here we have two morphisms  $\mathbb{D}[T] \rightarrow \mathbb{D}[X, Y]$  to the free  $\mathbb{D}$ -rig on two generators, one taking  $T$  to  $Y^2 + 1$ , the other taking  $T$  to  $X^2$ ; to form  $\mathbb{D}[X, Y]/(Y^2 + 1 \cong X^2)$ , construct a co-iso-inserter ([9, 29]) between these two  $\mathbb{D}$ -rig maps.*

The differential  $\partial : \mathcal{H} \rightarrow \mathcal{H}$  is defined by  $\partial(X) = Y, \partial(Y) = X$ , and taking the co-iso-inserter  $\varphi : Y^2 + 1 \rightarrow X^2$  to the composite

$$\partial(Y^2 + 1) \cong \partial(Y^2) + \partial(1) \cong \partial(Y^2) \cong \partial Y \otimes Y + Y \otimes \partial Y \quad (43)$$

$$\cong X \otimes Y + Y \otimes X \xrightarrow{\sigma + \sigma} Y \otimes X + X \otimes Y = \partial X \otimes X + X \otimes \partial X \cong \partial(X^2) \quad (44)$$

where  $\sigma$  denotes an instance of the symmetry isomorphism.

**Proposition 8** (Free 2-rigs are differential). *The free 2-rigs  $\Sigma[Y], \Sigma[[Y]], \mathbb{M}[Y], \mathbb{M}[[Y]]$  all admit at least one nontrivial derivation, which is uniquely determined by the request that the ‘generator’  $Y$  goes to the monoidal (Day convolution) unit.*

From the universal property of  $\mathcal{R}[Y]$ , we deduce that it is the category generated under coproducts by formal expressions  $A_n \otimes Y^n$  where  $n \geq 0$  is an integer and  $A_n \in \mathcal{R}$ .

**Proposition 9.** *Every object in the differential 2-rig  $\mathcal{R}[Y]$  admits a unique representation as a formal sum like  $\sum_{i=0}^d A_i \otimes Y^i$ .*

*Proof.* In Appendix B, page 15.  $\square$

A particularly interesting example of a free 2-rig construction as differential 2-rig is where  $S$  is a countable set whose elements we denote  $\{Y, Y^{(1)}, Y^{(2)}, \dots, Y^{(n)}, \dots\}$ , that we interpret as the stock of all subsequent derivatives of a unique

<sup>6</sup>One can prove that the ‘Schwarz-Clairaut’s theorem’ of commutativity of composition of derivatives with respect different ‘indeterminates’. We refrain to provide such a proof in detail, as it is completely straightforward.

indeterminate  $Y$ . In other words, we construct a differential  $\partial : \mathbb{D}[S] \rightarrow \mathbb{D}[S]$  via the  $\mathbb{D}$ -rig map

$$\mathbb{D}[S] \rightarrow \mathbb{D}[S][\varepsilon] \quad (45)$$

that takes  $Y^{(i)}$  to  $(Y^{(i)}, Y^{(i+1)})$ , in effect defining  $\partial(Y^{(i)}) = Y^{(i+1)}$ . This construction has a parallel in differential algebra, see e.g. [54, Ch. 1]. Hence we obtain, by ‘scalar extension’ (tensoring with  $\mathcal{R}$ )

**Example 7** (The 2-rig of differential polynomials). *We can define the 2-rig of differential polynomials (with coefficients in a 2-rig  $\mathcal{R}$ ) using an infinite set of ‘indeterminates’  $\mathcal{Y} := \{Y = Y^{(0)}, Y^{(1)}, Y^{(2)}, \dots, Y^{(n)}, \dots\}$  as above, and defining the 2-rig  $\mathcal{R}[Y^\partial]$  as the free 2-rig of polynomials over  $\mathcal{Y}$ . This is a differential 2-rig where the differential  $\partial$  takes every ‘constant’  $C \odot I \in \mathcal{R} \odot \mathbb{D}[\mathcal{Y}]$  to 0, and  $\partial(Y^{(i)})$  to  $Y^{(i+1)}$ .*

The 2-rig  $\mathbb{D}[Y^\partial]$  defined above enjoys the following universal property: given a differential  $\mathbb{D}$ -rig  $\mathcal{S}$  and an element  $A \in \mathcal{S}$ , there exists a unique morphism of differential 2-rigs  $\bar{X} : \Sigma[Y^\partial] \rightarrow \mathcal{S}$  with the property that  $Y \mapsto A$ . In other words,

**Theorem 2.** *The free  $\mathbb{D}$ -rig of polynomials  $\Sigma[Y^\partial]$  of Example 7 is the free differential 2-rig on a single generator  $\{Y\}$ .*

**Remark 8.** *A slightly different way to put this theorem is: the monad  $\mathbb{E}$  on  $\text{Cat}$ , whose algebras are categories  $\mathcal{R}$  equipped with an endofunctor  $D : \mathcal{R} \rightarrow \mathcal{R}$ , distributes over the 2-rig monad  $\mathbf{AM}$  according to the Leibniz rule. (Intuitively, treat  $D$  as a differential operator so that  $D$  applied to a polynomial operator can be rewritten as a polynomial operator applied to  $D$ .) If  $\mathbb{D}$  is a symmetric 2-rig doctrine, then the free differential  $\mathbb{D}$ -rig on a set of generators  $S$  is*

$$\bigodot_{s \in S} \mathbb{D}[Y_s^\partial] = \mathbb{D}[\{Y_s^{(i)}\}_{s \in S, i \in \mathbb{N}}]. \quad (46)$$

## 5.2 The chain rule

In classical mathematical analysis, the *chain rule* asserts that the derivative of a composition of differentiable functions  $f, g$ , the derivative of the composition  $f \circ g$  at a point  $x$  is equal to  $f'(g(x))g'(x)$ . A completely analogous formula holds if, instead of differentiable functions, we take formal power series  $f, g$  and  $f \circ g$  is functional substitution of  $g$  into  $f$ .

If following [28] there is something to gain representing a combinatorial species as a categorified formal power series, a completely analogous formula shall hold; it follows from an easy computation that this is the case when the substitution  $F \circ G$  is interpreted as the *substitution product* of (48) (cf. for example [5, §1.4]).

In the present subsection, we provide a general form of a chain rule valid for a symmetric 2-rig doctrine. (With a little more care, this can be extended to handle the non-symmetric case.)

Let  $\mathbb{D}$  be a symmetric 2-rig doctrine. From the universal property of  $\mathbb{D}[1]$  it follows that there is an equivalence

$$\mathbb{D}\text{-Rig}(\mathbb{D}[1], \mathbb{D}[1]) \simeq \mathbb{D}[1] \quad (47)$$

where the right side should be read as the underlying category of  $\mathbb{D}[1]$ . To each object  $G$  of  $\mathbb{D}[1]$ , there is a corresponding  $\mathbb{D}$ -rig map denoted  $- \circ G : \mathbb{D}[1] \rightarrow \mathbb{D}[1]$ . Indeed, endofunctor composition on the left side  $\mathbb{D}\text{-Rig}(\mathbb{D}[1], \mathbb{D}[1])$  transports to a monoidal structure on  $\mathbb{D}[1]$  which, by abuse of notation, we denote as

$$\circ : \mathbb{D}[1] \times \mathbb{D}[1] \rightarrow \mathbb{D}[1]; \quad (48)$$

variously called the *substitution* monoidal product or *plethystic* monoidal product [41]. The unit for the substitution product is the generator  $X : 1 \rightarrow \mathbb{D}[1]$ .

The standard derivative  $\partial : \mathbb{D}[1] \rightarrow \mathbb{D}[1]$  is defined by  $\partial(X) = I$ , i.e., is given by the unique  $\mathbb{D}$ -rig map  $\mathbb{D}[1] \rightarrow \mathbb{D}[1][\varepsilon]$  that takes  $X$  to  $(X, I)$ .

**Theorem 3.** *Given species  $F, G$ , there is a canonical isomorphism*

$$(F \circ G)' = (F' \circ G) \otimes G'. \quad (49)$$

*Proof.* In Appendix B, page 16.  $\square$

## 6 Conclusions and future work

We introduced the notion of differential 2-rig as a unifying structure for many diverse instances of a category equipped with a derivation, an endofunctor that satisfies the Leibniz property. The guiding example of the category of species allows us to develop a ‘synthetic’ framework to treat on similar grounds the many applications that derivative endofunctors play in categorical algebra, combinatorics, and computer science. Such a unification is obtained using 2-dimensional monad theory, and allows for a slick construction of free 2-rigs on a ‘signature’, that happen to carry a canonical choice of differential structure. The initial differential 2-rig, a category of ‘abstract differential polynomials’ can be obtained in this way. The fact that the Leibniz property for an endofunctor on a 2-rig boils down to the invertibility of a map canonically obtained from a pair of tensorial strength structure hints at a connection between differential structures and *applicative* structures, widely used in functional programming [40, 44]. Given the ‘geometric’ flavour of differential 2-rig theory, this hints at a surprising connection between apparently disconnected fields. We claim this connection is worth unravelling and we plan to make this part of future investigation.

Another enticing future direction of investigation involves *differential equations*: for example, one can define a ‘differential polynomial endofunctor’ (DPE) in a similar fashion to the one polynomial functors are defined inductively (cf. [24, §2.2]), by declaring that all polynomial expressions  $\sum_{i=0}^n A_i \otimes \partial^i$  obtained from a differential  $\partial : \mathcal{R} \rightarrow \mathcal{R}$  on a 2-rig form the category  $\text{DPE}(\mathcal{R}, \partial)$ . The ‘solution’ for a



DPE  $F \in \text{DPE}(\mathcal{R}, \partial)$  now consists of its terminal coalgebra  $\sigma : E \rightarrow FE$ , i.e. an universal object satisfying  $E \cong FE$ . The theory of differential equations in the category of species has a long and well-established history: it was mostly developed by Leroux and Viennot [6, 38, 39, 55] Labelle [32] and other authors built on that [11, 42]. The general theory of combinatorial differential equations studied in these papers might fruitfully be framed into a more general theory of DPEs and their solutions.

A related field of investigation involves what we would like to call ‘special’ 2-rigs: some properties of the differential structure on the 2-rig of species can be derived ‘synthetically’ from the observation (outlined in [47]) that the derivative functor on species  $\partial$  sits in the middle of a triple of adjoints

$$L \dashv \partial \dashv R; \quad (50)$$

A differential 2-rig  $(\mathcal{R}, \partial)$  is then *special* if such a string of adjoints endofunctors of  $\mathcal{R}$  exists: to what extent does such an additional structure on a differential 2-rig allow for a ‘synthetic’ study of its differential properties?

## References

- [1] J. Adámek and J. Velebil. 2008. Analytic functors and weak pullbacks. *Theory Appl. Categ.* 21 (2008), No. 11, 191–209.
- [2] J.C. Baez and J. Dolan. 1998. Higher-Dimensional Algebra III. *n*-Categories and the Algebra of Opetopes. *Advances in Mathematics* 135, 2 (1998), 145–206. <https://doi.org/10.1006/aima.1997.1695>
- [3] J.C. Baez, J. Moeller, and T. Trimble. 2021. Schur Functors and Categorified Plethysm.
- [4] J. Beck. 1969. Distributive laws. In *Seminar on triples and categorical homology theory*. Springer, Springer, Germany, 119–140.
- [5] F. Bergeron, G. Labelle, and P. Leroux. 1998. *Combinatorial species and tree-like structures*. Vol. 67. Cambridge University Press, Cambridge, UK.
- [6] F. Bergeron and C. Reutenauer. 1990. Combinatorial Resolution of Systems of Differential Equations III: a Special Class of Differentially Algebraic Series. *European Journal of Combinatorics* 11 (1990), 501–512. [https://doi.org/10.1016/S0195-6698\(13\)80035-2](https://doi.org/10.1016/S0195-6698(13)80035-2)
- [7] R.F. Blute, J.R.B. Cockett, and Robert A.G. Seely. 2009. Cartesian differential categories. *Theory and Applications of Categories* 22, 23 (2009), 622–672.
- [8] N. Bourbaki. 2007. *Algèbres tensorielles, algèbres extérieures, algèbres symétriques*. Springer Berlin Heidelberg, Berlin, Heidelberg, 379–596.
- [9] J. Bourke. 2010. *Codescent objects in 2-dimensional universal algebra*. Ph.D. Dissertation. School of Mathematics and Statistics, University of Sydney.
- [10] B.J. Day. 1970. On Closed Categories of Functors. In *Reports of the Midwest Category Seminar, IV*. Springer, Berlin, Berlin, 1–38.
- [11] H. Décoste, G. Labelle, and P. Leroux. 1982. Une approche combinatoire pour l’itération de Newton-Raphson. *Advances in Applied Mathematics* 3, 4 (1982), 407–416. [https://doi.org/10.1016/S0196-8858\(82\)80013-4](https://doi.org/10.1016/S0196-8858(82)80013-4)
- [12] J. Elgueta. 2021. The groupoid of finite sets is biinitial in the 2-category of rig categories. *Journal of Pure and Applied Algebra* 225, 11 (2021), 106738. <https://doi.org/10.1016/j.jpaa.2021.106738>
- [13] M. Fiore. 2005. Mathematical Models of Computational and Combinatorial Structures. In *Foundations of Software Science and Computational Structures*, Vladimiro Sassone (Ed.). Springer Berlin Heidelberg, Berlin, Heidelberg, 25–46.
- [14] M. Fiore. 2012. Discrete Generalised Polynomial Functors. In *Automata, Languages, and Programming*. Springer Berlin Heidelberg, Germany, 214–226. [https://doi.org/10.1007/978-3-642-31585-5\\_22](https://doi.org/10.1007/978-3-642-31585-5_22)
- [15] M. Fiore. 2015. An Axiomatics and a Combinatorial Model of Creation/Annihilation Operators. arXiv:arXiv:1506.06402
- [16] Ignacio López Franco. 2011. Pseudo-commutativity of KZ 2-monads. *Advances in Mathematics* 228, 5 (2011), 2557–2605.
- [17] Tobias Fritz and Paolo Perrone. 2018. Bimonoidal Structure of Probability Monads. *Electronic Notes in Theoretical Computer Science* 341 (2018), 121–149. <https://doi.org/10.1016/j.entcs.2018.11.007>
- [18] N. Gambino and J. Kock. 2013. Polynomial functors and polynomial monads. In *Mathematical proceedings of the cambridge philosophical society*, Vol. 154. Cambridge University Press, Cambridge, UK, 153–192.
- [19] E. Getzler and M. Kapranov. 1995. Cyclic operads and cyclic homology. , 167–201 pages.
- [20] J.-Y. Girard. 1987. Linear logic. *Theoretical Computer Science* 50, 1 (1987), 1–101. [https://doi.org/10.1016/0304-3975\(87\)90045-4](https://doi.org/10.1016/0304-3975(87)90045-4)
- [21] J.S. Golan. 1999. *Semirings and their Applications* (1 ed.). Springer Netherlands, Netherlands.
- [22] R. Hasegawa. 2002. Two applications of analytic functors. *Theoretical Computer Science* 272, 1 (2002), 113–175. [https://doi.org/10.1016/S0304-3975\(00\)00349-2](https://doi.org/10.1016/S0304-3975(00)00349-2) Theories of Types and Proofs 1997.
- [23] G.B. Im and G.M. Kelly. 1986. A universal property of the convolution monoidal structure. *Journal of Pure and Applied Algebra* 43, 1 (1986), 75–88. [https://doi.org/10.1016/0022-4049\(86\)90005-8](https://doi.org/10.1016/0022-4049(86)90005-8)
- [24] B. Jacobs. 2017. *Introduction to Coalgebra*. Vol. 59. Cambridge University Press, Cambridge, UK.
- [25] G. Janelidze and G.M. Kelly. 2001. A note on actions of a monoidal category. *Theory Appl. Categ* 9, 61-91 (2001), 02.
- [26] A. Joyal. 1981. Une théorie combinatoire des séries formelles. *Adv. Math. (NY)* 42, 1 (1981), 1–82.
- [27] A. Joyal. 1981. Une théorie combinatoire des séries formelles. *Adv. in Math.* 42, 1 (1981), 1–82. [https://doi.org/10.1016/0001-8708\(81\)90052-9](https://doi.org/10.1016/0001-8708(81)90052-9)
- [28] A. Joyal. 1986. Foncteurs analytiques et especes de structures. In *Combinatoire énumérative*. Lecture Notes in Math., Vol. 1234. Springer, Berlin, Germany, 126–159. <https://doi.org/10.1007/BFb0072514>
- [29] G.M. Kelly. 1989. Elementary observations on 2-categorical limits. *Bulletin of the Australian Mathematical Society* 39 (1989), 301–317.
- [30] A. Kock. 1972. Strong functors and monoidal monads. *Archiv der Mathematik* 23, 1 (1972), 113–120.
- [31] J. Kock. 2017. Polynomial functors and combinatorial Dyson–Schwinger equations. *J. Math. Phys.* 58, 4 (2017), 041703.
- [32] G. Labelle. 1985. Une combinatoire sous-jacente au théoreme des fonctions implicites. *Journal of Combinatorial Theory, Series A* 40, 2 (1985), 377–393.
- [33] G. Labelle. 1986. On combinatorial differential equations. *J. Math. Anal. Appl.* 113, 2 (1986), 344–381. [https://doi.org/10.1016/0022-247X\(86\)90310-0](https://doi.org/10.1016/0022-247X(86)90310-0)
- [34] G. Labelle. 1990. Combinatorial Directional Derivatives and Taylor Expansions. *Discrete Math.* 79, 3 (feb 1990), 279–297. [https://doi.org/10.1016/0012-365X\(90\)90336-G](https://doi.org/10.1016/0012-365X(90)90336-G)
- [35] S. Lack. 2010. A 2-categories companion. In *Towards higher categories*. IMA Vol. Math. Appl., Vol. 152. Springer, New York, 105–191. [https://doi.org/10.1007/978-1-4419-1524-5\\_4](https://doi.org/10.1007/978-1-4419-1524-5_4)
- [36] M.L. Laplaza. 1972. Coherence for distributivity. In *Coherence in categories*. Springer, Germany, 29–65.
- [37] P. Leroux and G. X. Viennot. 1986. Combinatorial resolution of systems of differential equations, I. Ordinary differential equations. In *Combinatoire énumérative*, G. Labelle and P. Leroux (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 210–245.

- [38] P. Leroux and G. X. Viennot. 1988. Combinatorial resolution of systems of differential equations. IV. separation of variables. *Discrete Mathematics* 72, 1 (1988), 237–250. [https://doi.org/10.1016/0012-365X\(88\)90213-0](https://doi.org/10.1016/0012-365X(88)90213-0)
- [39] P. Leroux and G. X. Viennot. 1988. Combinatorial resolution of systems of differential equations. IV. separation of variables. *Discrete Mathematics* 72, 1 (1988), 237–250. [https://doi.org/10.1016/0012-365X\(88\)90213-0](https://doi.org/10.1016/0012-365X(88)90213-0)
- [40] C. McBride and R. Paterson. 2008. Applicative programming with effects. *J. Funct. Program.* 18 (2008), 1–13.
- [41] M. Méndez and O. Nava. 1993. Colored species, c-monoids, and plethysm, I. *Journal of combinatorial theory, Series A* 64, 1 (1993), 102–129.
- [42] M. Menni. 2008. Combinatorial functional and differential equations applied to differential posets. *Discrete Mathematics* 308, 10 (2008), 1864–1888. <https://doi.org/10.1016/j.disc.2007.04.035>
- [43] J. Obradović. 2017. *Cyclic operads: syntactic, algebraic and categorified aspects*. Ph.D. Dissertation. École doctorale Sciences mathématiques de Paris centre. <http://www.theses.fr/2017USPCC191> 2017USPCC191.
- [44] R. Paterson. 2012. Constructing applicative functors. In *International Conference on Mathematics of Program Construction*. Springer, Germany, 300–323.
- [45] V. Poénaru. 2007. What is... an infinite swindle. *Notices of the AMS* 54, 5 (2007), 619–622.
- [46] J.-F. Pommaret. 1994. Differential Galois Theory. In *Partial Differential Equations and Group Theory*. Springer, Germany, 259–318.
- [47] D. S. Rajan. 1993. The adjoints to the derivative functor on species. *Journal of Combinatorial Theory, Series A* 62, 1 (1993), 93–106. [https://doi.org/10.1016/0097-3165\(93\)90073-H](https://doi.org/10.1016/0097-3165(93)90073-H)
- [48] D. S. Rajan. 1993. The equations  $D^k Y = X^n$  in combinatorial species. *Discrete Mathematics* 118, 1 (1993), 197–206. [https://doi.org/10.1016/0012-365X\(93\)90061-W](https://doi.org/10.1016/0012-365X(93)90061-W)
- [49] D. Spivak. 2020. Poly: An abundant categorical setting for mode-dependent dynamics. , 11 pages.
- [50] D. Spivak and D.J. Myers. 2020. Dirichlet Functors are Contravariant Polynomial Functors. , 11 pages.
- [51] D. Spivak and D.J. Myers. 2020. Dirichlet Polynomials form a Topos. , 11 pages.
- [52] Kornél Szlachányi. 2005. Monoidal morita equivalence. *Contemp. Math.* 391 (2005), 353–370.
- [53] A. Toumi, R. Yeung, and G. de Felice. 2021. Diagrammatic Differentiation for Quantum Machine Learning. *EPTCS* 343 (2021), 132–144. <https://doi.org/10.4204/EPTCS.343.7> arXiv:arXiv:2103.07960
- [54] M. Van der Put and M.F. Singer. 2012. *Galois theory of linear differential equations*. Vol. 328. Springer Science & Business Media, Germany.
- [55] G. Viennot. 1980. Une interprétation combinatoire des coefficients des développements en série entière des fonctions elliptiques de Jacobi. *Journal of Combinatorial Theory, Series A* 29, 2 (1980), 121–133. [https://doi.org/10.1016/0097-3165\(80\)90001-1](https://doi.org/10.1016/0097-3165(80)90001-1)
- [56] Charles Walker. 2019. Distributive laws via admissibility. *Applied Categorical Structures* 27, 6 (2019), 567–617.

## 7 Appendix A: Coherence conditions for strengths

**Definition 13** (Morphism of  $\mathcal{R}$ -modules). *Given a monoidal 2-category, let  $\mathcal{R}$  be a pseudomonoid, and let  $\mathcal{M}, \mathcal{N}$  be two  $\mathcal{R}$ -bimodules. We denote the left and right unit constraints by  $j$  and  $k$ , and left and right associativity constraints by  $\alpha$  and  $\beta$ . A (lax) morphism of  $\mathcal{R}$ -bimodules  $F : \mathcal{M} \rightarrow \mathcal{N}$  is a 1-cell*

$\mathcal{M} \rightarrow \mathcal{N}$ , together with, 2-naturally in objects  $\mathcal{A}$ , maps

$$\begin{array}{ccc} C \otimes FM & \xrightarrow{\xi^L} & F(C \otimes M) \\ FM \otimes C' & \xrightarrow{\xi^R} & F(M \otimes C') \end{array} \quad (51)$$

for every  $C, C' : \mathcal{A} \rightarrow \mathcal{R}$  and  $M : \mathcal{A} \rightarrow \mathcal{M}$ .

These maps must satisfy the following coherence conditions (we give only the ones pertaining to the left constraints  $\lambda, \alpha$ ):

- naturality in both components; the diagrams

$$\begin{array}{ccc} F(C \otimes M) & \xleftarrow{\xi^L} & C \otimes FM \\ F(f \otimes u) \downarrow & & \downarrow f \otimes Fu \\ F(C' \otimes M') & \xleftarrow{\xi^L} & C' \otimes FM' \end{array} \quad (52)$$

are commutative, for every pair of morphisms  $f : C \rightarrow C'$  and  $u : M \rightarrow M'$ .

- compatibility with the monoidality of the action maps, in the form of compatibility with the isomorphisms  $C \otimes (C' \otimes M) \cong (C \otimes C') \otimes M$  witnessing the strong monoidality of the action functor and  $I \otimes M \cong M$ : the diagram

$$\begin{array}{ccc} & F(I \otimes M) & \\ \xi^L \swarrow & & \searrow Fj \\ I \otimes FM & \xrightarrow{j} & FM \\ F((C \otimes C') \otimes M) & \xrightarrow{F\alpha} & F(C \otimes (C' \otimes M)) \\ \xi^L \uparrow & & \uparrow \xi^L \\ (C \otimes C') \otimes FM & & C \otimes F(C' \otimes M) \\ \alpha \downarrow & & \uparrow \xi^L \\ C \otimes (C' \otimes FM) & \xlongequal{\quad} & C \otimes (C' \otimes FM) \end{array}$$

are commutative, for  $C, C' \in \mathcal{R}, M \in \mathcal{M}$ .

## 8 Appendix B: Proofs

*Proof of Proposition 4.* Let  $m : M \otimes M \rightarrow M$  be the multiplication of  $M$ ; the map  $\partial m$  is of the following form

$$\partial M \otimes M + M \otimes \partial M \xrightarrow{\partial m} \partial M \quad (53)$$

and by the universal property of coproducts, it can be written as the map  $\begin{bmatrix} i_R \\ i_L \end{bmatrix}$ , where

$$i_R : \partial M \otimes M \rightarrow \partial M \quad i_L : M \otimes \partial M \rightarrow \partial M. \quad (54)$$

Evidently, these maps are our candidate right and left actions of  $M$  over  $\partial M$ .

Now, the fact that  $m$  is associative is witnessed by the commutative square

$$\begin{array}{ccc} M \otimes M \otimes M & \xrightarrow{M \otimes m} & M \otimes M \\ m \otimes M \downarrow & & \downarrow m \\ M \otimes M & \xrightarrow{m} & M \end{array} \quad (55)$$

If we derive it, applying  $\partial$  to each map, we get the commutative square

$$\begin{array}{ccc} \partial M \otimes M \otimes M + M \otimes \partial M \otimes M + M \otimes M \otimes \partial M & & \\ \partial m \otimes M + m \otimes \partial M \swarrow & \partial M \otimes m + M \otimes \partial m \searrow & \\ \partial M \otimes M + M \otimes \partial M & & \partial M \otimes M + M \otimes \partial M \\ \downarrow \begin{bmatrix} i_R \\ i_L \end{bmatrix} & & \downarrow \begin{bmatrix} i_R \\ i_L \end{bmatrix} \\ \partial M & & \partial M \end{array}$$

which, thanks to the Leibniz action of  $\partial$  on morphisms, can be seen as the object- and morphism-wise sum of two diagrams, obtained taking the red and blue maps, respectively:

$$\begin{array}{ccc} \partial M \otimes M & \xrightarrow{\partial M \otimes m} & \partial M \otimes M \\ \partial m \otimes M \downarrow & & \downarrow i_R \\ \partial M & \xrightarrow{i_R} & \partial M \end{array} \quad \begin{array}{ccc} M \otimes \partial M & \xrightarrow{M \otimes m} & M \otimes \partial M \\ m \otimes \partial M \downarrow & & \downarrow i_L \\ M & \xrightarrow{i_L} & M \end{array} \quad (56)$$

these two diagrams witness exactly that  $i_R$  is a right action, and  $i_L$  is a left  $M$ -action on  $\partial M$ .  $\square$

*Proof of Proposition 9.* Inspecting the universal property: first of all, there is an obvious cospan of 2-rig morphisms  $\mathcal{R} \rightarrow \mathcal{R}[Y] \leftarrow \Sigma[Y]$  sending  $C$  to  $C \otimes Y^0$  and  $[n]$  to  $Y^n$ ; and given a diagram

$$\begin{array}{ccc} \mathcal{R} & & \\ \downarrow & \searrow G & \\ \Sigma[Y] \longrightarrow \mathcal{R}[Y] & & \mathcal{B} \\ & \searrow F & \\ & & \mathcal{B} \end{array} \quad (57)$$

we can define a unique dotted functor  $\begin{bmatrix} F \\ G \end{bmatrix} : \mathcal{R}[Y] \rightarrow \mathcal{B}$  as

$$\sum_{i=0}^d A_i \otimes Y^i \mapsto \sum_{i=0}^d G A_i \otimes (FY)^{\otimes n}, \quad (58)$$

since a 2-rig morphism  $F : \Sigma[Y] \rightarrow \mathcal{B}$  is completely determined by the image of  $Y = y(1)$ .  $\square$

*Proof of Proposition 3.* Let's examine the left closed case: this means that every  $A \otimes -$  has a right adjoint. The right closed case is analogous, mutatis mutandis.

The condition of having a right tensorial strength amounts to the presence of maps  $t_{AB} : A \otimes DB \rightarrow D(A \otimes B)$  satisfying suitable conditions.

The maps  $t_{AB}$  now transpose to

$$\hat{t}_{AB} : DB \longrightarrow [A, D(A \otimes B)] \quad (59)$$

and the  $\hat{t}_{AB}$ 's are natural in  $B$ , and a wedge in  $A$ : this means that there is a unique map

$$\hat{t}_B : DB \longrightarrow \int_A [A, D(A \otimes B)]; \quad (60)$$

we now claim that

- M1) the correspondence  $\lambda B. \int_A [A, D(A \otimes B)]$  is an endofunctor of  $\mathcal{A}$ ;
- M2) the correspondence  $\Theta : D \mapsto \lambda B. \int_A [A, D(A \otimes B)]$  is an endofunctor of  $[\mathcal{A}, \mathcal{A}]$ ; moreover, it is a comonad;
- M3) a  $\Theta$ -coalgebra is exactly an endofunctor equipped with a right tensorial strength, whose components are obtained from the coalgebra map by reverse-engineering the construction of  $\Theta$ .

The last part of the third claim is obvious; what remains of the third claim is an exercise on diagram chasing. Functoriality is evident from the canonical way in which we built  $\Theta$ , and  $DB \rightarrow \int_A [A, D(A \otimes B)]$  attach to the components of a natural transformation  $D \Rightarrow \Theta(D)$ .

It remains to show that  $\Theta$  is a comonad:

- the counit is obtained from the terminal wedge of  $\Theta(D)$ , taking the component on the monoidal unit (say,  $I$ ):

$$\int_A [A, D(A \otimes B)] \xrightarrow{\epsilon_B = \pi_I} [I, D(I \otimes B)] \cong DB \quad (61)$$

- the comultiplication is obtained from the following computation:

$$\begin{aligned} \Theta\Theta(D)(A) &= \int_B [B, \Theta(D)(A \otimes B)] \\ &\cong \int_B [B, \int_C [C, D(A \otimes B \otimes C)]] \\ &\cong \int_B \int_C [B, [C, D(A \otimes B \otimes C)]] \\ &\cong \int_B \int_C [B \otimes C, D(A \otimes B \otimes C)] \end{aligned}$$

It is evident, now, that the projections  $\pi_{B \otimes C}$  of the terminal wedge of  $\Theta(D)$  assemble into a morphism  $\sigma : \Theta \Rightarrow \Theta\Theta$  of the right type; moreover, this choice of  $\epsilon$  and  $\sigma$  is the unique that satisfies the counit equations of a comonad; showing that  $\sigma : \Theta \Rightarrow \Theta^2$  is coassociative is a matter of diagram chasing.  $\square$

*Proof of Proposition 6.* Let  $\mathbb{D}[Y] \rightarrow \mathcal{R} \ltimes \mathcal{R}$  be the essentially unique  $\mathbb{D}$ -rig map that takes  $Y$  to  $(0, I)$ , and let  $\mathcal{R} \rightarrow \mathcal{R} \ltimes \mathcal{R}$  be the map taking  $C$  to  $(C, 0)$ . By pairing these maps, we get a map  $\varepsilon : \mathcal{R}[Y] \rightarrow \mathcal{R} \ltimes \mathcal{R}$  out of the coproduct  $\mathcal{R}[Y] = \mathcal{R} \odot \mathbb{D}[Y]$ . It is clear that  $\varepsilon$  coinverts the 2-cell  $0 \Rightarrow Y^2$ . Given a  $\mathbb{D}$ -rig map  $F : \mathcal{R}[Y] \rightarrow \mathcal{S}$  that coinverts this 2-cell, define a map  $\bar{F} : \mathcal{R} \ltimes \mathcal{R} \rightarrow \mathcal{S}$  that takes  $(R, 0)$  to  $F(R)$ , and  $(0, I)$  to  $F(Y)$ . One may check that  $\bar{F}$  is a  $\mathbb{D}$ -rig map.  $\square$

*Proof of Theorem 3.* Let  $\partial$  denote the standard derivative, and denote the  $\mathbb{D}$ -rig map  $- \circ G$  by  $\varphi$ . Then the left side corresponds to the value at an object  $F$  of the composite  $\mathbb{D}$ -rig map

$$\mathbb{D}[1] \xrightarrow{\varphi} \mathbb{D}[1] \xrightarrow{\langle 1, \partial \rangle} \mathbb{D}[1][\varepsilon], \quad (62)$$

taking  $F$  to  $(F \circ G, (F \circ G)')$  and taking  $X$  to  $(G, G')$ . On the other hand,  $\varphi \circ \partial : \mathbb{D}[1] \rightarrow \mathbb{D}[1]$  is a  $\varphi$ -augmented derivation, and so is  $(\varphi \partial) \otimes G'$ . By Proposition 5, it corresponds to the  $\mathbb{D}$ -rig map  $\mathbb{D}[1] \rightarrow \mathbb{D}[1][\varepsilon]$  taking  $F$  to

$$(\varphi(F), (\varphi \partial(F)) \otimes G') = (F \circ G, (F' \circ G) \otimes G'). \quad (63)$$

This map is uniquely determined by where it sends the generator  $X$ , but this value on  $X$  is the same as before,

$$(X \circ G, (X' \circ G) \otimes G') = (G, G'). \quad (64)$$

This means the  $\mathbb{D}$ -rig maps

$$F \mapsto (F \circ G, (F \circ G)'), \quad F \mapsto (F \circ G, (F' \circ G) \otimes G') \quad (65)$$

coincide, and this completes the proof.  $\square$

### 8.1 Generalized Leibniz rule and Taylor formula

*Proof of Proposition 1.* Expand  $(F * G)^{(U)}[C] = (F * G)[C + U]$  using the fact that

$$(F * G)[C + U] = \sum_{A+B=C+U} FA \times GB. \quad (66)$$

For each indexing pair  $(A, B)$ , put  $A' = A \cap C$ ,  $B' = B \cap C$ ,  $S = A \cap U$ ,  $T = B \cap U$ . Then  $A = A' + S$  and  $B = B' + T$  and  $S + T = U$ . It follows that

$$\begin{aligned} (F * G)[C + U] &\cong \sum_{A+B=C+U} FA \times GB \\ &\cong \sum_{S+T=U} \sum_{A'+B'=C} F[A' + S] \times G[B' + T] \\ &\cong \sum_{S+T=U} \sum_{A'+B'=C} F^{(S)}[A'] \times G^{(T)}[B'] \\ &\cong \sum_{S+T=U} (F^{(S)} * G^{(T)})[C] \end{aligned}$$

This concludes the proof.  $\square$

*Proof of Theorem 1.* Let's first observe that we have the analytic functor formula

$$F(X) = \int^n F[n] \times X^n \quad (67)$$

which mimics the Maclaurin series expansion; this is obtained from the fact that  $F(-) \cong \text{Lan}_J F$ , and the integral on the right-hand side is exactly that Kan extension.

Now given an  $n$ -element set  $U$ , let's interpret  $\partial^n F(A) = \partial^{(U)} F(A) = F(U + A)$  as a species in the variable  $n$  but as analytic in the set-variable  $A$ . We have then the formula

$$\partial^n F(A) = \int^m F[m + n] \times A^m. \quad (68)$$

And thus we can categorify  $\sum_{n=0}^{\infty} \frac{\partial^n f(a)}{n!} x^n$  as the double co-end

$$\begin{aligned} &\int^{nm} F[m + n] \times A^m \times X^n \\ &\cong \int^{nmj} F[j] \times \Sigma(j, m + n) \times A^m \times X^n \\ &\cong \int^j F[j] \times \left( \int^{m,n} \Sigma(j, m + n) \times A^m \times X^n \right). \end{aligned}$$

Now, we have an isomorphism

$$\int^{mn} \Sigma(j, m + n) \times A^m \times X^n \cong (A + X)^j \quad (69)$$

which ultimately comes out of the fact that  $\text{Set}$  is an extensive category: there exists an equivalence of categories  $\text{Set}/A \times \text{Set}/X \cong \text{Set}/(A + X)$ . We conclude that

$$\int^j F[j] \times (A + X)^j \quad (70)$$

is the value  $F(A + X)$  of the analytic functor  $F(-)$ .  $\square$

*Proof of Proposition 7.* The universal property of the coinverter amounts to the following:

- c1) for each morphism of 2-rigs  $p : C[Y] \rightarrow X$  such that  $0 \rightarrow p(Y^2 \otimes R(Y))$  is invertible in  $X$ , there exists a unique (up to isomorphism)  $\bar{p} : C[Y]_{<2} \rightarrow X$  such that  $q \circ \bar{p} = p$ ;
- c2) for each natural transformation  $\alpha : p \Rightarrow p'$  of 2-rig morphisms with the property that the horizontal composition  $\alpha \boxtimes u$  is an isomorphism, there exists a unique  $\bar{\alpha} : \bar{q} \Rightarrow \bar{q}'$  such that  $q * \bar{\alpha} = \alpha$ .

Both properties descend from the fact that  $p$ , being a 2-rig morphism, preserves coproducts; if  $p(A + BY + RY^2) \cong p(A + BY) + p(RY^2)$ , and the initial arrow  $0 \rightarrow p(RY^2)$  is an isomorphism, the vertical right arrow in the commutative diagram

$$\begin{array}{ccc} p(A + BY) + 0 & \longrightarrow & p(A + BY) \\ \downarrow & & \downarrow \\ p(A + BY) + p(RY^2) & \longrightarrow & p(A + BY + RY^2) \end{array} \quad (71)$$

is an isomorphism; thus,  $p$  is uniquely determined by its action on  $C[Y]_{<2}$ , and  $\bar{p}(A + BY)$  can be defined just as  $p(A + BY)$ . For what concerns 2-cells  $\alpha : p \Rightarrow p'$ , a similar diagram

$$\begin{array}{ccc} p(A + BY + RY^2) & \longrightarrow & p'(A + BY + RY^2) \\ \downarrow \wr & & \downarrow \wr \\ p(A + BY) & \xrightarrow{\alpha_{A+BY}} & p'(A + BY) \\ \parallel & & \parallel \\ p(A + BY) & \xrightarrow{\bar{\alpha}_{A+BY}} & p'(A + BY) \end{array} \quad (72)$$

is commutative, so  $\alpha$  is uniquely determined by its components at objects  $A + BY$  of  $C[Y]_{<2}$ .  $\square$