ON FIXED-POINT CLONES

(extended abstract)

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Summary Expressions involving least and greatest fixed point operators are interpreted in the power-set algebra of (possibly infinite) trees and also in some abstract models. Initiality of the tree algebra is established via a "Mezei-and Wright-like" result on interpretation of fixed point terms. Then a reduction of these terms to Rabin automata on infinite trees is shown which yields some decidability results. A connection is exhibited between the hierarchy of alternating least and greatest fixed points and the hierarchy of Rabin pair indices of automata.

Introduction. Starting with a family of monotonic operations on a complete lattice, say f,g,..., one can consider the full clone generated by application of the least (μ) and greatest (ν) fixed point operators, beside the usual functional composition. The operations obtained in this way can be described by what we call μ-terms, the typical examples of which are $\mu x.f(x)$, $\nu y.g(x,\mu z.g(y,z))$, $\mu x.\nu y.g(y,x)$,... . Expressions of that kind appear in various types of μ-calculi (e.g. de Bakker and de Roever 1973, Emerson & Clarke 1980, Kozen 1983) where their capability to express computational phenomena including terminating, looping, repeating, fairness etc. is demonstrated. They have been also considered in some other contexts as semantics of nondeterministic recursive programs (Arnold&Nivat 1977), fair-merge problem (Park 1980) or characterization of context-free ∞-languages (Niwinski 1984). Note that greatest fixed points usually reflect infinite behaviour of a computation device and that there are some phenomena which require both extremal fixed point operators (Park 1980, Clarke&Emerson 1980).
The equational calculus of μ-terms considered in our paper is similar to the one developed by Wand 1972 and, as the latter, amenable to algebraic techniques. It is however more powerful because of incorporating greatest fixed point operator which was not present in Wand's system. The models in consideration, which we refer to as "good" μ-algebras are required to have a completely ordered and atomic carrier and completely additive basic operations. A good example of a good μ-algebra over a given signature is provided by the power-set algebra of (possibly infinite) trees over the signature. As a matter of fact, this turns out to be an initial algebra over the class of good μ-algebras, if, as morphisms, one takes the mappings preserving all μ-terms. Here a crucial rôle is played by the

concept of interpretation of a tree t with variables $x_1,\ldots,x_n$ (occurring on leaves) as an n-ary operation $t^{\underline{D}}$ on an algebra $\underline{D}$. A tree language L is then interpreted by $\bigcup \{ t^{\underline{D}} : t \in L \}$. It turns out that, for each μ-term $t(x_1,\ldots,x_n)$, in order to compute the value of t in $\underline{D}$, one may first compute the value of t in the power-set tree algebra and then interpret the resulting tree language in $\underline{D}$ in the above-mentioned sense. (This can be viewed as a "Mezei-and Wright-like" result in terms of Engelfriet&Schmidt 1978.)

Passing on to the power-set tree algebra, we find an elementary reduction of μ -terms to what we call ∞-tree grammars which are essentially equivalent to Rabin automata on infinite trees (Rabin 1972). That means, for any μ-term a grammar can be constructed generating the language defined by this μ-term. By results of Rabin 1969, this yields decidability of μ-terms equivalence problem. Now, a connection arises between the number of essential alternations in a μ-term and and the index of the corresponding grammar (i.e. the number of pairs in the acceptance condition), the latter being linearly bounded by the former.Again we exhibit a family of (μ-term definable) tree languages requiring arbitrary large indexes which implies that the hierarchy of alternating least and greatest fixed points can be, in general, infinite. Tree languages definable by Rabin's "special automata" (Rabin 1970, also called Bühi automata by Vardi and Wolper 1984) occupy exactly one level of this hierarchy. Thus, for a special case of monadic signature, because of equivalence of Rabin and Bühi automata on infinite words (MacNaughton 1966), the hierarchy collapses (this fact has been previously observed by Park and Tiuryn).

Because of space considerations, only main steps of proofs are indicated in this abstract. Full proofs will appear in a future paper. The proof of the result on hierarchy of indies of Rabin automata is given in Niwinski 1986.

I FIXED POINT TERMS AND THEIR REPRESENTATION BY INFINITE TREES

1. μ-terms and their semantics. By celebrated Knaster-Tarski fixed point theorem, a monotonic mapping $f : L \to L$ of a complete lattice $(L,\leq)$ has the least fixed point

$$\mu x.f(x) = \bigcap \{ u \in L : f(u) \leq u \}$$

and its dual, the greatest one

$$\nu x.f(x) = \bigcup \{ u \in L : u \leq f(u) \}$$

Throughout the paper, if an expression $t(y,x_1,\ldots,x_n)$ describes a mapping $L^{n+1} \to L$ then $\mu y.t(y,x_1,\ldots,x_n)$ denotes the mapping $L^n \to L$ which sends a tuple $(u_1,\ldots,u_n)$ onto the least fixed point of the equation $v = t(v,u_1,\ldots,u_n)$, similarly with $\nu y.t(y,x_1,\ldots,x_n)$ and greatest fixed point. This makes sense of expressions like $\mu x.\nu y.g(x,y)$, $f(\mu x.g(x,y))$ etc.Beside using this notation informally, we shall consider the concept of formal μ -terms defined as follows.

Let F be a finite signature (each f in F given with some arity) and V a set of (individual) variables. The set of μ-terms over F and V is the least set of expressions containing V and closed under application of function symbols from F and under least and under greatest fixed point operators (viz. $\mu x.t$, $\nu x.t$ are μ-terms whenever t is and $x \in V$).

The models in which μ-terms are to be interpreted, called μ-algebras, are of the form

$$\underline{D} = (D, \leq, f^{\underline{D}} : f \in F \ )$$

where $(D,\leq)$ is a complete lattice and all the operations $f^{\underline{D}}$ are monotonic in all variables. The interpretation of μ-terms in $\underline{D}$ follows the usual interpretation of terms and the meaning of μ and ν indicated above, we skip a formal definition. If $t(x_1,\ldots,x_n)$ is a μ-term with free (i.e. not bound by μ or ν) variables from among $x_1,\ldots,x_n$ and $d_1,\ldots,d_n \in D$ then $t^{\underline{D}}[x_1 \leftarrow d_1,\ldots,x_n \leftarrow d_n]$ denotes the value of the interpretation of t, $t^{\underline{D}}$, under the valuation of x_i by d_i, for $i=1,\dots,n$.

To illustrate the above-introduced concepts, let us note some simple properties. We have in any $\underline{D}$, $\mu x.\mu y.t(x,y)^{\underline{D}} = \mu x.t(x,x)^{\underline{D}}$, the similar for ν. For a system of equations, say

$$u = t_1^{\underline{D}}[x \leftarrow u,\ y \leftarrow v]$$
$$v = t_2^{\underline{D}}[x \leftarrow u,\ y \leftarrow v]\ ,$$

the first component of the least solution can be described by $\mu x.t_1[\mu y.t_2/y]^{\underline{D}}$, the similar for ν (μ-terms are capable to describe mutual fixed points). What is maybe less known, we have, in any $\underline{D}$, $\mu x.\nu y.t(x,y)^{\underline{D}} \leqslant \nu y.\mu x.t(x,y)^{\underline{D}}$ (Niwinski 1985).

Vector notation. Throughout the paper, whenever no confusion may arise, we abbreviate sequences like $x_1,\dots,x_n$, $d_1,\dots,d_m$, etc. by $\bar{x},\bar{d}$, ... and this is also applied in the notation for substitution, valuation etc., e.g. we write $t[\bar{t}/\bar{x}]$, $L[\bar{x}\leftarrow\bar{d}]$,..., instead of $t[t_1/x_1,\dots,t_n/x_n]$, $L[x_1\leftarrow d_1,\dots,x_k\leftarrow d_k]$,... .Again, we often identify a singleton $\{x\}$ with x.

2. Hierarchy of alternations. It is natural to consider the number of alternations of μ and ν as a certain "complexity measure". Considering the ideas of Park and Tiuryn (c.f. Park 1981), we introduce the following classification of μ-terms (over a given signature) :

$\Sigma^{\mu}_0 = \Pi^{\mu}_0$ is the set of all ordinary terms (without μ, ν).

Σ^{μ}_{n+1} is the least set containing $\Sigma^{\mu}_n \cup \Pi^{\mu}_n$ and closed under following rules: (i) If $t(x_1,\dots,x_m), t_1,\dots,t_m \in \Sigma_{n+1}$, t_i free for x in t, then $t[t_1/x_1,\dots,t_m/x_m] \in \Sigma_{n+1}$; (ii) If $t \in \Sigma_{n+1}$ then $\mu x.t \in \Sigma_{n+1}$.

Π_{n+1} is defined similarly with ν in place of μ .

Given a μ-algebra, we can ask whether the hierarchy of operations and elements of the algebra induced by the above classification of μ-terms does collapse or not. It is a simple observation that for a finite $\underline{D}$ the hierarchy reduces to $\Sigma^{\mu}_2 \cap \Pi^{\mu}_2$ (since, over $\underline{D}$, any t reduces to an algebraic combination of $\mu x.x$ and $\nu x.x$). Another example of short (Π^{μ}_2) hierarchy is provided by the power-set algebra of finite/infinite words with union and prefixing by single letters as basic operations (Park 1981, the similar holds also in presence of the product operation). Below we shall see an example of an infinite hierarchy.

3. Power-set algebra of trees. For a set X, X^* is the set of all finite words over X including the empty word λ . Given a set Σ , a Σ-valued tree is a mapping t : Dom $t \dashrightarrow \Sigma$ where $\emptyset \neq$ Dom $t \subseteq \omega^*$ is closed under initial segments (in the paper, ω denotes the set of natural numbers). Given t and $x \in$ Dom t, the subtree t_x of t is a tree with Dom $t_x = \{w: xw \in \text{Dom } t\}$ defined by $t_x(w) = t(xw)$. As usual, the elements of Dom t are called nodes and a node wn ($n \in \omega$) is a child of the node w. The childless nodes are called leaves and the only node without a parent (viz. λ) is the root. A path is an infinite sequence of nodes $w_0, w_1,\dots$ where each w_{n+1} is a child of w_n. An occurrence of $s \in \Sigma$ in t is any w such that $t(w) = s$, let occ(s,t) denote the set of all occurrences of s in t.

Now let F be a finite signature and V a finite set of variables (considered with arity 0). Extending, in the well-known way, the concept of a term, we define a syntactic tree over (F,V) as a $F \cup V$-valued tree t such that, for $w \in$ Dom t, if t(w) has arity n then the only children of w are $w1,\dots,wn$ (so, if n = 0, w is a leaf).

In the sequel, letters $x,y,z,x_i,\dots$ are reserved for variables. We write $t(x_1,\dots,x_n)$ to indicate that the variables occurring in a tree t are from among $x_1,\dots,x_n$, similarly for a set of trees $L(x_1,\dots,x_n)$. Given $t(x_1,\dots,x_n)$ and the sets of trees $L_1,\dots,L_n$, the substitution $t[L_1/x_1,\dots,L_n/x_n]$ is the set of trees obtained from t by simultaneous replacing each occurrence of x_i in t by some tree from L_i, in such a way that distinct occurrences of x_i may be replaced by distinct trees.

Proviso. We now fix a (finite) signature F. Given a set of variables V, T(V) denotes the set of all syntactic trees over (F,V). The binary function symbol $\cup$ is not in F. From now on unless otherwise stated, a μ-algebra is always of the type $F\cup\{\cup\}$ and $\cup$ is interpreted as the lattice union.

3.1.Definition. Given a finite set of variables V, $\underline{A}(V)$ is a μ-algebra

$$\underline{A}(V) = (A(V), \subseteq , \cup , f^{\underline{A}(V)} : f \in F) ,$$

where A(V) is the power-set of T(V), $\subseteq$ is the subset ordering, $\cup$ is the set union and, for $f\in F$, say n-ary,

$$f^{\underline{A}(V)}(L_1,\dots,L_n) =_{df} f(x_1,\dots,x_n)\left[L_1/x_1,\dots,L_n/x_n\right] .$$

If $V = \{x_1,\dots,x_n\}$ then $\underline{A}(V)$ will be written $\underline{A}(x_1,\dots,x_n)$ or simply $\underline{A}(\bar{x})$, and $\underline{A}(\emptyset)$ will be abbreviated by $\underline{A}$.

4.Interpreting trees like terms. We start with the following concept.

4.1.Definition.Let $t(x_1,\dots,x_n)$ be a tree in $T(\bar{x})$. Let $\underline{D}$ be a μ-algebra and let $d \in D$, $K_1,\dots,K_n \subseteq D$. An expansion of d by t under valuation of the variables $x_1,\dots,x_n$ by elements of $K_1,\dots,K_n$ respectively, is a D-valued tree, say t': Dom t' --> D with Dom t' = Dom t, satisfying the following conditions :

(i) $t'(\lambda) = d$,

(ii) if $t(w) = f \in F$, say n-ary, then $t'(w) \leqslant f^{\underline{D}}(t'(w1),\dots,t'(wn))$ (in particular, if f is a constant then $t'(w)\leqslant f^{\underline{D}}$),

(iii) if $t(w) = x_i$ then $t'(w) \leqslant k$, for some $k \in K_i$.

Let $\exp(t)\left[x_1\leftarrow K_1,\dots,x_n\leftarrow K_n\right]$ denote the set of all the d's in D for which an expansion as above exists.

The interpretation of a tree $t(x_1,\dots,x_n)$ in $\underline{D}$ under valuation of the variables $x_1,\dots,x_n$ by $d_1,\dots,d_n$ D respectively, is an element of D defined by

$$t^{\underline{D}}\left[x_1\leftarrow d_1,\dots,x_n\leftarrow d_n\right] =_{df}\bigcup \exp(t)\left[x_1\leftarrow \{d_1\} ,\dots,x_n\leftarrow \{d_n\}\right]$$

The interpretation of a set of trees, say $L(x_1,\dots,x_n)$ is defined by

$$L^{\underline{D}}\left[x_1\leftarrow d_1,\dots,x_n\leftarrow d_n\right] =_{df}\bigcup \left\{ t^{\underline{D}}\left[x_1\leftarrow d_1,\dots,x_n\leftarrow d_n\right] : t \in L \right\} .$$

4.2.Examples. (a) If $t \in T(\bar{x})$ is a finite tree (a term) then $t^{\underline{D}}\left[\bar{x}\leftarrow\bar{d}\right]$ in the above sense coincides with the usual interpretation of a term.

(b) For any $t(x_1,\dots,x_n) \in T(\bar{x})$,

$$t^{\underline{A}(\bar{x})}\left[x_1\leftarrow x_1,\dots,x_n\leftarrow x_n \right] = t$$

(note that in writting $x_i\leftarrow x_i$, x_i is considered first as a variable and then as an element $\{x_i\} \in A(\bar{x})$); more generally, for $t(\bar{x})$ and $L_1,\dots,L_n \in A(\bar{y})$ ($\bar{x}$ and $\bar{y}$ possibly overlaping),

$$t^{\underline{A}(\bar{y})}\left[\bar{x}\leftarrow \bar{L} \right] = t\left[\bar{L} / \bar{x}\right] .$$

From now on we shall be interested mainly in μ- algebras $\underline{D}$ possessing some special "good" properties. Let $0^{\underline{D}}$, $1^{\underline{D}}$ denote the least and greatest elements od D respectively. Recall that $a \in D$ is an atom if $a \neq 0^{\underline{D}}$ and there are no elements between $0^{\underline{D}}$ and a. $\underline{D}$ is atomic if every element is a union of atoms.

4.3.Definition. A μ-algebra $\underline{D}$ is good provided

(i) $(D,\leqslant)$ is atomic,

(ii) the basic operations $f^{\underline{D}}$ are completely additive,

(iii) $(D,\leqslant)$ satisfies the infinite distributivity law :

$$d\cap \bigcup C =_D\bigcup \left\{ d\cap c : c \in C\right\} ,$$

(iv) $T(\emptyset)^{\underline{D}} = 1^{\underline{D}}$.

Remark. Any complete Boolean algebra satisfies (iii). The technical condition (iv) may be imposed upon an algebra artificially by setting the interpretation of some constant, say c, $c^{\underline{D}} = 1^{\underline{D}}$.

Example. $\underline{A}$ is a good μ-algebra whereas $\underline{A}(\bar{x})$ is not (for non-empty $\bar{x}$) because of violating the condition (iv).

The following technical lemma can be derived from the definitions.
4.4. Lemma. Let $L \in A(x_1,\dots,x_n,y_1,\dots,y_m)$, $K_i \in A(\bar{y})$, $i = 1,\dots,n$. Let $\underline{D}$ be a good μ-algebra and $d_1,\dots,d_m \in D$. Then

$$L^{\underline{D}}\left[x_1 \leftarrow K_1^{\underline{D}}[\bar{y} \leftarrow \bar{d}],\dots,x_n \leftarrow K_n^{\underline{D}}[\bar{y} \leftarrow \bar{d}], \bar{y} \leftarrow \bar{d}\right] = L\left[K_1/x_1,\dots,K_n/x_n\right]^{\underline{D}}[\bar{y} \leftarrow \bar{d}] .$$

5. Parsing. This section introduces a technique to be used in proofs of our main results. We study fixed points of mappings of the form $L(\bar{x})^{\underline{D}}$ for a tree language L. The intuitive idea of what we call 'parsing' is the following. Consider $t \in T(\bar{x})$ and $L \in A(y,\bar{x})$. We "extract" from t an "initial segment belonging to L. The rest of t forms a forest and from each t' in this forest we again extract an initial segment in L, etc.

5.1. Definition. Let $t \in T(\bar{x})$, $L \in A(y,\bar{x})$. A parsing of t by L is a Dom $t \times L$ -valued tree, say p, such that for each w in Dom p, say $p(w) = (u,t')$, the following conditions are satisfied:
(i) if $w = \lambda$ then $u = \lambda$,
(ii) $t_u \in t'[T(\bar{x})/y]$ (t' is an "initial segment" of t_u),
(iii) for each $v \in occ(y,t')$, there is a child of w in Dom p valued by $(uv,t")$ for some t" and there are no other children of w.

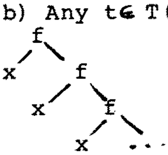
We say that t can be parsed by L if there exists a parsing of t by L . Given $L \in A(y,\bar{x})$, we set

$$L^y =_{df} \{ t \in T(\bar{x}) \ :\ t \text{ can be parsed by } L \} .$$

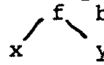
Recall that a tree is well-founded if it has no infinite paths. For $L \in A(y,\bar{x})$, we set

$$L^y_{<\omega} =_{df} \{ t \in T(\bar{x}) \ :\ \text{there exists a parsing of } t \text{ by } L \text{ which is well-founded} \} .$$

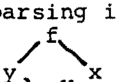
5.2. Examples. (a) Any $t(\bar{x})$ can be parsed by $\{y\}$. The parsing may be visualized as (λ,y)---(λ,y)---(λ,y)--- ... . (b) Any $t \in T(\bar{x})$ can be parsed by t (the parsing is just (λ,t)). (c)



may be parsed by



but not by



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(d) For $L = \{x, f(y,y)\}$, L^y ($L^y_{<\omega}$) consists of all (resp. all finite) trees over (f,x).
The following fact is a basic connection between fixed points and syntactic "manipulations" on trees.

5.3. Proposition. Let $L \in A(y,x_1,\dots,x_n)$. Let $\underline{D}$ be a good μ-algebra and $d_1,\dots,d_n \in D$. Then

(a) $\nu u.\ L^{\underline{D}}[y \leftarrow u, \bar{x} \leftarrow \bar{d}] = L^{y\,\underline{D}}[\bar{x} \leftarrow \bar{d}]$,

(b) $\mu u.\ L^{\underline{D}}[y \leftarrow u, \bar{x} \leftarrow \bar{d}] = L^{y\,\underline{D}}_{<\omega}[\bar{x} \leftarrow \bar{d}]$

Sketch of proof. (for the case b only) Verification that the right-hand side is indeed a fixed point is easy. It remains to show that $L^{\underline{D}}[y \leftarrow c, \bar{x} \leftarrow \bar{d}] \leq c$ implies $L^{y\,\underline{D}}_{<\omega}[\bar{x} \leftarrow \bar{d}] \leq c$. For consider $t \in L^y_{<\omega}$ and let p will be a well-founded parsing of t by L. We show that, for each node w of p, say $p(w) = (u,t')$, $t_u^{\underline{D}}[\bar{x} \leftarrow \bar{d}] \leq c$ (in particular $t_\lambda^{\underline{D}}[\bar{x} \leftarrow \bar{d}] \leq c$). If w is a leaf then t' must be free of occurrences of y which yields $t'^{\underline{D}}[\bar{x} \leftarrow \bar{d}] = t'[y \leftarrow c, \bar{x} \leftarrow \bar{d}] \leq L^{\underline{D}}[y \leftarrow c, \bar{x} \leftarrow \bar{d}] \leq c$ and, on the other hand $t' = t$. Now suppose w is not a leaf but all its children do satisfy the claim. (As before, $p(w) = (u,t')$) Note that t_u results from replacing in t' each occurrence of y, say v, by t_{uv} where, by hypothesis on children of w, $t_{uv}^{\underline{D}}[\bar{x} \leftarrow \bar{d}] \leq c$. Let $M = \{ t_{uv} : v \text{ is an occurrence of } y \text{ in } t' \}$. Therefore we have $t_u^{\underline{D}}[\bar{x} \leftarrow \bar{d}] \leq t'[M/y]^{\underline{D}}[\bar{x} \leftarrow \bar{d}] = t'^{\underline{D}}[y \leftarrow M^{\underline{D}}[\bar{x} \leftarrow \bar{d}], \bar{x} \leftarrow \bar{d}]$ (Lemma 4.4) $\leq t'^{\underline{D}}[y \leftarrow c, \bar{x} \leftarrow \bar{d}] \leq L^{\underline{D}}[y \leftarrow c, \bar{x} \leftarrow \bar{d}] \leq c$, as required. As p is well-founded, the above inductive argument completes the proof. □

6.Initiality of the tree algebra. We are ready to state our algebraic results. The following can be viewed as a "Mezei-and Wright-like" result on semantics of μ-terms.

6.1.Theorem. Let $t(x_1,\dots,x_n)$ be a μ-term. Let $\underline{D}$ be a good μ-algebra and $d_1,\dots,d_n \in D$. Then

$$t(\bar{x})^{\underline{D}}[\bar{x}\leftarrow\bar{d}] = (t^{\underline{A}(\bar{x})}[\bar{x}\leftarrow\bar{x}])^{\underline{D}}[\bar{x}\leftarrow\bar{d}] .$$

The proof is a bit involved technically though not very difficult. It applies the technique of parsing and proceeds by induction of a μ-term t. From the above one easily derives an initiality result.

6.2.Theorem. For any good μ-algebra $\underline{D}$, the mapping $\underline{A} \dashrightarrow \underline{D}$ given by $L \mapsto L^{\underline{D}}$ preserves all μ-terms, viz. for any μ-term $t(x_1,\dots,x_n)$ and $L_1,\dots,L_n \in \underline{A}$,

$$(t(\bar{x})^{\underline{A}}[x_1\leftarrow L_1,\dots,x_n\leftarrow L_n])^{\underline{D}} = t^{\underline{D}}[x_1\leftarrow L_1^{\underline{D}},\dots,x_n\leftarrow L_n^{\underline{D}}] .$$

In other words, $\underline{A}$ is initial over the class of good μ-algebras when as morphisms one takes the mappings preserving μ-terms.

II μ-TERMS vs. RABIN AUTOMATA ON INFINITE TREES

7. ∞-Tree grammars. We shall consider Rabin pair automata adopted to arbitrary signature. It is convenient, however, to use notions like grammar, derivation, variable,... rather than automaton, run, state... .

7.1.Definition. A regular) ∞-tree grammar over a signature S is a tuple

$$G = (S^G, V^G, V_O{}^G, R^G, C^G) ,$$

where $S^G = S$ is the set of terminal (ranked) symbols, V^G is the set of non-terminals also called variables (of arity 0), $V_O{}^G$ is the set of start variables, R^G is the set of rules and C^G is an accepting condition. Each rule is of the form $X_i \to f(X_1,\dots,X_k)$ or $X_i \to X_j$ (here the X_i's are variables and $f \in S$, it is understood that if arity of f is 0 then $X_i \to f$ may be a rule). The acceptance condition C^G is a list of pairs $(U_1,L_1),\dots,(U_n,L_n)$ where $U_i,L_i \subseteq V^G$, $i = 1,\dots,n$. Note that C^G may be empty. The number n is referred to as the index of the grammar.

The relation $\underset{G}{\to}$ associated to the grammar G (over finite syntactic trees over $(S,V^G)^G$)is defined as usual, $\overset{*}{\underset{G}{\to}}$ stands for reflexive-transitive closure of $\underset{G}{\to}$. Now consider an infinite derivation sequence $V_O{}^G \ni X \underset{G}{\to} t_1 \underset{G}{\to} t_2 \underset{G}{\to} \dots$. An infinite tree t is produced by this sequence if the following conditions are satisfied :

(i) $\mathrm{Dom}\ t = \bigcup_n \mathrm{Dom}\ t_n$,

(ii) for each w^n Dom t, $t(w) = t_m(w)$ for some m (and hence $= t_k(w)$ for all $k \geqslant m$),

(iii) Let, for an infinite path of t, say $P : w_0, w_1, \dots$, inf(P) be the set of variables x such that for infinitely many w_k, $t_m(w_k) = x$, for some m. For each infinite path P of t, there is a pair (U_i,L_i) in C^G such that $U_i \cap \mathrm{Inf}(P) \neq \emptyset$ and $\mathrm{Inf}(P) \cap L_i = \emptyset$.

(NB. Clearly, the rules of the form $X_i \to X_j$ could be eliminated which would simplify the above definition. We keep them however to make the constructions easier.)

The set of all (finite and infinite) trees produced by the grammar G will be denoted L(G) .

8.Reduction. Recall that unless otherwise stated we work with a fixed signature F (c.f. Section 3). Given a μ-term $t(x_1,\dots,x_n)$, we say that a grammar G_t represents t if $t^{\underline{A}(\bar{x})}[\bar{x}\leftarrow\bar{x}] = L(G_t)$. (Here G_t has as the terminal alphabet $F \cup \{x_1,\dots,x_n\}$. The reduction result will follow from a series of lemmas the proofs of which base on the technique of parsing (Proposition 5.3).

8.1. Lemma. Suppose the μ-terms $t(z_1,\dots,z_k), t_1(x_1,\dots,x_m),\dots,t_k(x_1,\dots,x_m)$ are represented by the grammars $G_t, G_{t_1},\dots,G_{t_k}$ respectively, each grammar of the index n. Then the μ-term $t[t_1/z_1,\dots,t_k/z_k]$ can be also represented by a grammar with the index n.

8.2. Lemma. Suppose a μ-term $t(y,x_1,\dots,x_m)$ is represented by a grammar G_t where $C^Gt = (U_1,L_1),\dots,(U_n,L_n)$. Then the following grammar G represents the μ-term $\mu y.t(y,x_1,\dots,x_m)$:

$S^G = F \cup \{x_1,\dots,x_m\}$,

$V^G = V^Gt \cup \{y\}$,

$V_0^G = V_0^Gt$,

$R^G = R^Gt \cup \{y \rightarrow z \ : \ z \in V_0^Gt\}$,

$C^G = (U_1,L_1\cup\{y\}),\dots,(U_n,L_n\cup\{y\})$ (if $C^Gt = \emptyset$ then $C^G = \emptyset$, too) .

8.3. Lemma. Suppose a μ-term $t(y,x_1,\dots,x_m)$ is represented by a tree grammar G_t with $C^Gt = (U_1,L_1),\dots,(U_n,L_n),(U_{n+1}^m,\emptyset)$. Then the following grammar G represents the μ-term $\nu y.t(y,x_1,\dots,x_m)$:

$S^G = F \cup \{x_1,\dots,x_m\}$,

$V^G = V^Gt \cup \{y, z_1\}$, where z_1 is a new variable,

$V_0^G = V_0^Gt$,

$R^G = R^Gt \cup \{y \rightarrow z \ : \ z \in V_0^Gt\} \cup$ CASE , where

Case $= \{y \rightarrow z_1\} \cup \{z_1 \rightarrow f(z_1,\dots,z_1) \ : \ f \in F\}$, if $y \in L(G_t)$

$= \emptyset$ otherwise ,

$C^G = (U_1,L_1),\dots,(U_n,L_n),(U_{n+1}\cup\{y,z_1\}, \emptyset)$.

Then, by induction on t, we can prove the following.

8.4. Theorem. For any μ-term t, one can construct a grammar G_t such that $t^{\underline{A}} = L(G_t)^{\underline{A}}$. Moreover, for t in $\Sigma^\mu_m \cup \Pi^\mu_m$ (c.f. Section 2), G_t may be chosen with index $\lfloor m/2 \rfloor + 1$.

9. Decidability result. Let us call a μ-term without free variables closed. By Theorem 6.2, we have, for closed μ-terms t_1,t_2, $t_1^{\underline{D}} = t_2^{\underline{D}}$ in any good algebra iff $t_1^{\underline{A}} = t_2^{\underline{A}}$. By Theorem 8.4 and the decidability results of Rabin 1969, we can conclude the following: It is decidable whether two closed μ-terms have the same interpretation over all good μ-algebras. It is elementarily decidable whether a μ-term defines a minimal element in each good μ-algebra.

10. Büchi automata and the monadic case. Let us call a grammar special if its acceptance condition consists of a single pair $(U,\emptyset)$. The grammars of that kind are essentially equivalent to 'special automata' of Rabin 1970 also called 'Büchi automata' by Vardi and Wolper 1984. It follows from Lemmas 8.1-3 that the tree languages definable by μ-terms from Π^μ_2 can be produced by special grammars. However, in this case, it is not very difficult to obtain a complete characterization.

10.1. Proposition. A tree language is definable by a μ-term in Π^μ_2 iff it can be produced by a special grammar.

A similar result was independently obtained by Takahashi 1985. For the case of monadic signature this implies collapsing of the fixed point hierarchy. Indeed, it follows from MacNaughton 1966, that, for automata on infinite words, the Büchi and Rabin acceptance conditions are equivalent in the defining power. So, we can state the following result (which was previously obtained by Park and Tiuryn) :

10.2.Proposition. For a signature F consisting of monadic symbols only, a tree language $L \in A$ is μ-term definable iff it is definable by a μ-term in Π^{μ}_{2}.

11.Infinity of the fixed point hierarchy. We are now to exhibit a family of tree languages M_n such that M_n can be defined only on the n+1th level of the fixed point hierarchy. Let, for $n \in \omega$, S_n be a signature consisting of the binary function symbols $a_0, a_1, \ldots, a_n$. Let $\underline{A}_n$ be the algebra $\underline{A}$ corresponding to the signature S_n (c.f. Section 3). We set :

$$M_0 = (\nu x_0.a_0(x_0,x_0))^{\underline{A}_0} \quad ,$$

$$M_1 = (\mu x_1.\nu x_0.a_0(x_0,x_0) \quad a_1(x_1,x_1))^{\underline{A}_1} \ ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$M_m = (\eta x_m. \ \ldots \ \nu x_2.\mu x_1.\nu x_0.a_0(x_0,x_0) \cup a_1(x_1,x_1) \cup \ \ldots \cup a_m(x_m,x_m))^{\underline{A}_m}$$

where $\eta = \mu$ or ν according as m is odd or even respectively.

Note that M_0 consists of a single tree. M_1 is the set of trees such that on each path, a_1 occurs only finitely often. M_2 consists of the trees where, on each path, a_1 occurs only finitely often unless it is"accompanied" by a_2, etc. We shall show that any grammar producing M_m must have an index at least $\lfloor m/2 \rfloor + 1$ which, togehter with Theorem 8.4 implies that M_m cannot be defined at the m th level in the fixed point hierarchy of $\underline{A}_m$. (It is not very difficult to "encode" the sets M_m over a fixed signature, for example S_1, with preservation of the above property hence we can have a single algebra with infinite fixed-point hierarchy.)

Some technical notions are needed. Let $\perp$ be a constant symbol which, unless otherwise stated, does not occur in the signatures in question. A partial tree over a signature S is any tree over $S \cup \{\perp\}$. Given a grammar G over S, $G_\perp$ is the grammar over $S \cup \{\perp\}$ which differs from G only in that it has additional rules $x \to \perp$ for each $x \in V^G$ $(=V^{G_\perp})$. We say that a partial tree t' occurs in a (partial) tree t if there is a subtree t" of t such that, for all $w \in$ Dom t', if $t'(w) \neq \perp$ then $t'(w) = t''(w)$. For a set of trees M, $M_\perp$ denotes the set of all partial trees occurring in the trees from M. A branch is a partial tree t such that any node $w \in$ Dom t is succeded by a leaf, say $wv \in$ Dom t .The following fact is a crucial argument of the proof.

11.1.Lemma. For all $m \in \omega$:

(a) Let m = 2n. Suppose that a grammar G of index n satisfies $L(G_\perp) \subsetneq (M_{2n})_\perp$. Then there exists a branch $B \in (M_{2n}) - L(G_\perp)$.

(b) Let m = 2n + 1. Suppose that a grammar G satisfies $L(G_\perp) \subsetneq (M_{2n+1})_\perp$ and $C^G = (U_1,L_1),\ldots,(U_n,L_n),(U_{n+1},\emptyset)$. Then there exists a branch $B \in (M_{2n+1})_\perp - L(G_\perp)$.

Sketch of proof. By induction on m. Let m = 0. If index of G is 0 then $L(G_\perp)$ may consist of finite trees only, so we can take as B any infinite branch in $(M_0)_\perp$. Suppose the claim holds for $0,\ldots,m-1$. We have to consider two cases.

(i) m = 2n + 1. Suppose $L(G_\perp) \subsetneq (M_{2n+1})_\perp$, $C^G = (U_1,L_1),\ldots,(U_n,L_n)$, $(U_{n+1},\emptyset)$. We can assume that all variables of G are reachable from the start variables. Let G' be a grammar obtained from G by restriction to the terminal alphabet S_{2n} and setting $V_0^{G'} = V^G$, $C^{G'} = (U_1,L_1),\ldots,(U_n,L_n)$. Clearly $L(G'_\perp) \subsetneq (M_{2n+1})_\perp$ but, as a_{2n+1} does not occur there, it is easy to check that also $L(G'_\perp) \subsetneq (M_{2n})_\perp$. Thus, by induction hypothesis, we have a branch $B \in (M_{2n})_\perp - L(G'_\perp)$. We now construct a branch B' as in the following figure (k is the cardinality of U_{n+1}) :

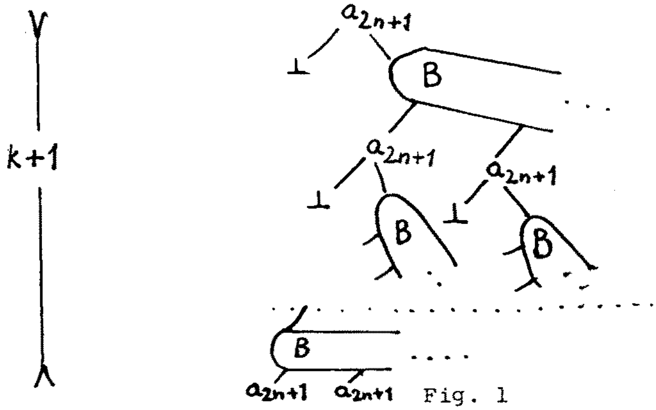

Fig. 1

It is not difficult to check that $B' \in (M_{2n+1})_\perp$. To show $B' \notin L(G_\perp)$ suppose the contrary. Consider a derivation in $G_\perp$ producing B'. Note that a fragment of this derivation producing an "occurence" of B in B' may not be itself a "succesful" derivation in $G'_\perp$. Hence, a variable from U_{n+1} must occur somewhere in tis fragment. Since this holds for any occurrence of B in B', we can derive that some variable from U_{n+1}, say x, occurs twice along a path, more precisely, there exists a fragment of our derivation $\ldots \xrightarrow[G]{} x \xrightarrow[G]{} \ldots \xrightarrow[G]{} t \xrightarrow[G]{}$ where x occurs in t and its occurrence is proceeded by an occurrence of a_{2n+1}. But from this we can deduce that $G_\perp$ produces a tree possessing an infinite path on which a_{2n+1} occurs infinitely often which contradicts the assumption $L(G_\perp) \subseteq (M_{2n+1})_\perp$ (c.f. definition of M_{2n+1} and Proposition 5.3).

(ii) $m = 2n$ $(n > 0)$. Suppose $L(G_\perp) \subseteq (M_{2n})_\perp$, $C^G = (U_1,L_1),\ldots,(U_n,L_n)$. Again we assume that the variables of G are reachable from V_0^G. Now, for $i = 1,\ldots,n$, let G_i be a grammar obtained from G by restriction to the terminal alphabet S_{2n-1}, "removing" all the variables of L_i (both from rules and the acceptance condition) and setting $V_0^{G_i} = V^{G_i}$. It is not difficult to show that $\bigcup_i L(G_{i\perp}) \subseteq (M_{2n-1})_\perp$ and that this set can be produced by a grammar, say G', with $C^{G'} = (U'_1,L'_1),\ldots,(U'_n,L'_n), L'_n = \emptyset$. Hence, by induction hypothesis, we have a branch $B \in (M_{2n-1})_\perp - \bigcup_i L(G_{i\perp})$. We now construct a branch B' as in the following figure:

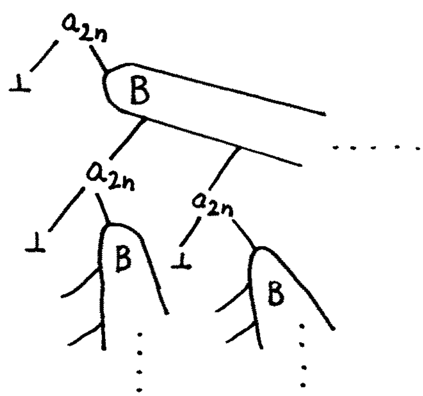

Fig. 2

Again, it is not difficult to check $B' \in (M_{2n})_\perp$. To prove $B \notin L(G_\perp)$

suppose the contrary. Consider a derivation of B' in G_L. Since no fragment of this derivation may be itself a derivation of B in $G_{i\mu}$, for i = 1,...,n, we can conclude that all the L_i's are nonempty (otherwise we obtain a contradiction) and then find an infinite path of B' on which some variables from all the sets L_i occur infinitely often which contradicts the definition of derivation.

From the lemma, we infer the following.

11.2.Theorem. For $m \in \omega$, the set M_m requires the index at least $\lfloor m/2 \rfloor +1$ in the producing grammar and hence cannot be defined on the m-th level of the fixed point hierarchy.

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