The Complexity of Reachability in Parametric Markov Decision Processes[☆]

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Abstract

This article presents the complexity of reachability decision problems for parametric Markov decision processes (pMDPs), an extension to Markov decision processes (MDPs) where transitions probabilities are described by polynomials over a finite set of parameters. In particular, we study the complexity of finding values for these parameters such that the induced MDP satisfies some maximal or minimal reachability probability constraints. We discuss different variants depending on the comparison operator in the constraints and the domain of the parameter values. We improve all known lower bounds for this problem, and notably provide ETR-completeness results for distinct variants of this problem.

Keywords: Parametric Markov decision processes, Formal verification, Existential theory of the reals, Computational complexity, Parameter synthesis

1. Introduction

Markov decision processes (MDPs) are the model to reason about sequential processes under (stochastic) uncertainty and non-determinism. Markov chains (MCs) are MDPs without non-determinism. Often, probability distributions in these models are difficult to assess precisely during design time of a system. This shortcoming has led to interval MCs [33, 14, 52, 49] and interval MDPs (also known as bounded-parameter MDPs) [26, 55, 42], which allow for interval-labelled transitions. Analysis under interval Markov models is often too pessimistic: The actual probabilities on the transitions are considered to be non-deterministically and locally chosen. Intuitively, consider the probability of a coin-flip yielding heads in some uncertain environment. In interval models, the probability may vary with the local memory state of an agent acting in this environment. Such behaviour is unrealistic. Parametric MCs and

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MDPs [19, 41, 28, 22] (pMCs, pMDPs) overcome this limitation by adding dependencies (or couplings) between various transitions—they add global restrictions to the selection of the probability distributions. Intuitively, the probability of flipping heads can be arbitrary, but should be independent of an agent's local memory. Such couplings are similar to restrictions on schedulers in decentralised/partially observable MDPs, considered in e.g., [4, 25, 50].

Technically, pMDPs label their transitions with polynomials over a finite set of parameters. Fixing all parameter values in a pMDP yields an MDP. The synthesis problem considered in this article asks to find parameter values such that the induced MDPs satisfy reachability constraints. Such reachability constraints state that the probability — under some/all possible ways to resolve non-determinism in the MDP — to reach a target state is (strictly) above or below a threshold. A sample synthesis problem is thus: "Are there parameter values such that for all possible ways to resolve the non-determinism, the probability to reach a target state exceeds 1/2?" Variants of the synthesis problem are obtained by varying the reachability constraints, and the domain of the parameter values. Parameter synthesis is supported by the model checkers PRISM [40] and Storm [21], and dedicated tools PARAM [29] and PROPhESY [20]. The complexity of the decision problems corresponding to parameter synthesis is mostly open.

This article significantly extends complexity results for parameter synthesis in pMCs and pMDPs. Tables 1 and 2 on pages 13 and 18 give an overview of new results: Most prominently, we establish completeness for the *Existential Theory of the Reals* (ETR) of reachability problems for pMCs with non-strict comparison operators, and **NP**-hardness for pMCs with strict comparison operators. For pMDPs with universal non-determinism, it establishes ETR-completeness for any comparison operator. For existential non-determinism, the synthesis problems are mostly equivalent to their pMC counterparts. When considering pMDPs with a fixed number of variables, we establish **NP** upper bounds for parameter synthesis under existential or universal non-determinism. These results are partially based on properties of pMDPs scattered over earlier works (see below), and use a strong connection between polynomial inequalities and parameter synthesis.

Finally, pMDPs are interesting generalisations of other models: Most importantly, [36] shows that parameter synthesis in pMCs is equivalent to the synthesis of finite-state controllers (with a-priori fixed bounds) of partially observable MDPs (POMDPs) [46] under reachability constraints. Thus, as a side product we improve complexity bounds [53, 11] for (a-priori fixed) memory bounded strategies in POMDPs.

Related work

Various results in this article extend work by Chonev [15], who studied augmented interval Markov chains, a model that coincides with pMCs. Our work also builds upon results by Hutschenreiter *et al.* [31], in particular upon the result that pMCs with an a-priori fixed number of parameters can be checked in **P**. Furthermore, they study the complexity of PCTL model checking of pMCs. The complexity of finite-state controller synthesis in POMDPs has been studied in [53, 11]. Some of the proofs for ETR-completeness presented here reuse ideas from [48].

Methods (and implementations) to analyse pMCs by computing their characteristic solution function are considered in [19, 29, 20, 31, 23, 32, 22, 24]. Sampling-based approaches to find feasible (i.e., satisfying) instantiations considered by [28, 13], while [3, 18] utilise optimisation methods. Finally, [44] presents a method to prove the absence of solutions in pMDPs by iteratively considering simple stochastic games [17]. Some other works on Markov models with structurally equivalent yet parameterised dynamics include [10, 51, 12, 7]. Parameter synthesis with statistical guarantees has been explored in, e.g., [5]. Novel methods for parametric models under Boolean parameters (i.e., parameter values are restricted to zero or one) have recently been presented in [8, 9]. Further work on parameter synthesis in Markov models has been surveyed in [35].

Contributions

The main contribution of this paper is a concise and complete discussion of the complexity landscape for parameter synthesis in pMCs and pMDPs, as summarised in Tables 1 and 2. In particular, we consider a set of decision problems that ask whether there exists a parameter valuation of a particular type such that, if we substitute a pMDP (or pMC) with this valuation, the resulting MDP (or MC) satisfies a quantitative or qualitative reachability property.

The tables contain some known results (mentioned above) that are now part of a larger picture, but they also contain various new results. We consider the following theorems central contributions.

- Parameter synthesis in pMCs is ETR-complete for non-strict relations regarding quantitative reachability (Theorem 8). Conceptually, this means that parameter synthesis is as hard as answering whether a multivariate polynomial has a root. Interestingly, this result can be established using very simple pMCs.
- Parameter synthesis in pMDPs is **ETR**-complete for any relation regarding quantitative reachability (Theorem 10). This result is a straightforward adaption of deep results about the existential theory of the reals.
- The results above are independent of whether or not the parameter valuations are graph-preserving. Graph-preserving valuations simplify matters as they allow for stronger continuity assumptions, and are therefore standard in tool support for parameter synthesis. The results above show that they provide, from a complexity point of view, no benefit.
- Parameter synthesis for qualitative reachability is **NP**-hard in general (Theorem 1) but various special cases can be decided in polynomial time (Theorem 2). Results for pMCs and pMDPs coincide. To the best of our knowledge, the results cover all classes considered in the literature on parameter synthesis in pMDPs.
- For any fixed number of parameters, pMDP parameter synthesis is in $NP(Theorem\ 12)$. We would like to stress that this result is non-trivial, as parameter values may be real-valued.

The presented results extend some results in [54] by providing examples, full proofs, and novel results on qualitative variants of the reachability problem. The presentation is partially based on [34].

2. Preliminaries

We assume familiarity with basic graph, automata, and complexity theory. Below, we present our notation for the theory of the reals and Markov models.

2.1. Existential theory of the reals

The first-order theory of the reals is the set of all valid sentences in the first-order language $(\mathbb{R}, +, \cdot, 0, 1, <)$. The existential theory of the reals (written ETR, for short) restricts the language to (purely) existentially quantified sentences. The complexity of deciding membership, i.e. whether a sentence is (true) in the theory of the reals, is in **PSPACE** [6] and **NP**-hard. A careful analysis of its complexity is given in [45]. In particular, deciding membership for sentences with an a-priori fixed upper bound on the number of variables is in polynomial time. We write **ETR** to denote the complexity class [48] of problems with a polynomial-time many-one reduction to deciding membership in the existential theory of the reals.

2.2. Markov models

Markov models are stochastic state models that exhibit the Markov property: Given any current state, the probability distribution describing the next state is independent of previous states. In this work by Markov models we mean discrete-time Markov chains (MCs) [16, 38, 27] and Markov decision processes (MDPs) [30, 43, 37]. We mostly follow the notation from [2].

Markov decision processes and chains. A Markov decision process (MDP) is a tuple $M:=(S,\iota,Act,P)$ where S is a finite set of states, $\iota\in S$ is an initial state, Act is a finite set of actions, and $P\colon S\times Act\times S\nrightarrow [0,1]$ is a partial transition probability function such that for all $s\in S$, $\alpha\in Act$ we either have that $\sum_{s'\in S} P(s,\alpha,s')=1$ or $P(s,\alpha,s')=\bot$ (undefined) for all s'. Let $Act(s):=\{\alpha\in Act\mid \forall s': P(s,\alpha,s')\ne\bot\}$ denote the available actions in state s. Without loss of generality, we assume that $|Act(s)|\ge 1$ for all $s\in S$. Furthermore, we refer to a transition $P(s,\alpha,s)=1$ as a self-loop.

A (discrete-time) Markov chain (MC) is an MDP such that |Act(s)| = 1 for all states $s \in S$. We may denote an MC as tuple $D := (S, \iota, P)$ with S, ι as for MDPs and transition probability function $P: S \times S \to [0, 1]$.

Paths and sets thereof. We fix an MDP $M := (S, \iota, Act, P)$. A path is an (in)finite sequence $\pi := s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} \dots$, where $s_i \in S$, $\alpha_i \in Act(s_i)$, and $P(s_i, \alpha_i, s_{i+1}) \neq 0$ for all $i \in \mathbb{N}$. The set Π^M of paths in M is the union of finite paths Π^M_{fin} and infinite paths Π^M_{∞} . The notions of paths carry over to MCs (actions are omitted).

(actions are omitted). For finite $\pi = s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} \dots s_n$, we furthermore define the $length |\pi| := n+1$ of π and $last(\pi) := s_n$. For infinite paths, we set $|\pi| := \infty$. For any path $\pi = s_0 \xrightarrow{\alpha_0} s_1 \dots$ we set $\pi@i := s_i$. A path π visits a state s, if there is some $i \in \mathbb{N}$ such that $\pi@i = s$.

We mostly consider paths between or from fixed states. Let $S' \subseteq S$ be a subset of the states and $s \in S$ a state. The set $\Pi^M(S') := \{\pi \in \Pi^M \mid \mathsf{first}(\pi) \in S'\}$ contains all paths starting in some $s \in S'$. We simplify the notation of the

set $\Pi^M(\{s\})$ to $\Pi^M(s)$. Analogously, $\Pi^M_{\mathsf{fin}}(s)$, $\Pi^M_{\infty}(s)$, $\Pi^M_{\mathsf{fin}}(S')$, $\Pi^M_{\infty}(S')$ define the (in)finite paths starting in some $s' \in S'$, respectively. The set

$$\Pi^{M}(\lozenge S') := \{ \pi \in \Pi^{M}_{\mathrm{fin}} \mid \mathsf{last}(\pi) \in S' \land \forall i < |\pi| : \pi@i \notin S' \}$$

contains all finite paths that end in S' with no proper prefix visiting a state $s \in S'$. Again, we simplify notation of $\Pi^M(\lozenge\{s\})$ to $\Pi^M(\lozenge s)$. Similarly, for any horizon $h \in \mathbb{N}$, the set

$$\Pi^{M}(\lozenge^{\leq h}S') \coloneqq \{\pi \in \Pi^{M}_{\mathrm{fin}} \mid \mathsf{last}(\pi) \in S' \land |\pi| \leq h \land \forall i < |\pi| : \pi@i \notin S'\}$$

contains all paths of length at most h that end in S' with no proper prefix visiting a state $s \in S'$. For subsets $S', T \subseteq S$, the sets

$$\Pi^M(S', \lozenge T) := \Pi^M(S') \cap \Pi^M(\lozenge T)$$

denote the paths of states starting in S' and reaching T. We simplify notation for singleton sets as above.

Underlying graphs. MDPs may be considered as annotated graphs and this perspective helps in describing many operations on MDPs. For an MDP $M = (S, \iota, Act, P)$, the underlying digraph of M is G(M) := (S, E(M)) with

$$E(M) := \{(s, s') \mid \exists \alpha \in Act(s) : P(s, \alpha, s') \neq 0\}.$$

This definition allows to lift various definitions from graphs to MDPs.

MDP strategies and induced chains. To define a probability measure over paths, action choices have to be resolved. Actions are resolved using strategies. A strategy for an MDP $M=(S,\iota,Act,P)$ is a (measurable) function $\sigma\colon\Pi^M_{\mathsf{fin}}\to Distr(Act)$ such that $supp(\sigma(\pi))\subseteq \mathrm{Act}(\mathsf{last}(\pi))$ for all $\pi\in\Pi^M_{\mathsf{fin}}$.

- A strategy is memoryless if for all $\pi, \pi' \in \Pi^M_{\mathsf{fin}}$ we have $\mathsf{last}(\pi) = \mathsf{last}(\pi') \implies \sigma(\pi) = \sigma(\pi')$.
- A strategy is deterministic if for all $\pi \in \Pi^M_{\text{fin}}$ we have $|supp(\sigma(\pi))| = 1$.

The set of all strategies of M is \mathfrak{S}^M , the set of all memoryless strategies is \mathfrak{S}^M_m , and the set of all deterministic and memoryless strategies is Σ^M . Notice that $|\Sigma^M| \leq |Act|^S$, so in particular there are only finitely many strategies that are deterministic and memoryless. We may use the function signature $\sigma \colon \Pi^M_{\text{fin}} \to Act$ for deterministic strategies, and $\sigma \colon S \to Distr(Act)$ for memoryless strategies.

Let $\sigma \in \mathfrak{S}^M$ be a strategy. The *induced MC* of M and σ is given by $M[\sigma] := (\Pi^M_{\mathsf{fin}}, \iota, P[\sigma])$ where

$$P[\sigma](\pi,\pi') \coloneqq \begin{cases} P(\mathsf{last}(\pi),\alpha,s') \cdot \sigma(\pi)(\alpha) & \text{if } \pi' = \pi \xrightarrow{\alpha} s', \\ 0 & \text{otherwise.} \end{cases}$$

For memoryless strategies σ , the MC $M[\sigma]$ may be identified with the finite MC $M[\sigma]' := (S, \iota, P[\sigma]')$ where

$$P[\sigma](s,s') \coloneqq \sum_{\alpha \in \mathbb{A}\mathsf{ct}(s)} P(s,\alpha,s') \cdot \sigma(s)(\alpha).$$

Formally, $M[\sigma]$ is probabilistic bisimilar (cf. [2, Sect. 10.4.2]) to $M[\sigma]'$. In particular, all reachability probabilities are preserved. This equivalence justifies using $M[\sigma]'$ as redefinition of $M[\sigma]$ for memoryless strategies.

Reachability probabilities. A probability measure $Pr^D: \Pi_{\text{fin}}^D \to [0,1]$ for finite paths $\pi = s_0 s_1 \dots s_n$ is given by the product of transition probabilities, referred to as the mass of the path: $Pr^D(\pi) := \prod_{i=0}^{n-1} P(s_i, s_{i+1})$. The unique probability measure for infinite paths $Pr^D: \Pi_{\infty}^D \to [0,1]$ is defined by the usual cylinder set construction, see [2] for details.

We define the reachability probability $Pr_D(s \models \Diamond T)$ for reaching T from state s as follows:

$$Pr_D(s \models \Diamond T) \coloneqq \sum_{\pi \in \Pi^D(s, \Diamond T)} Pr^D(\pi).$$

The reachability probability $Pr_D(\lozenge T)$ for reaching T in the MC D is then defined as the reachability probability from the initial state.

Optimal strategies for reachability. A classical result we use in this work is the fact that deterministic and memoryless strategies suffice in order to optimize reachability probabilities in MDPs [43].

Proposition 1. For any given MDP M, it holds that

$$\sup_{\sigma \in \mathfrak{S}^M} Pr_{M[\sigma]}(\lozenge T) = \sup_{\sigma \in \Sigma^M} Pr_{M[\sigma]}(\lozenge T)$$

and thus since Σ^M is a finite set, the suprema may be replaced by maxima. An analogous statement holds for infima and minima.

Computing reachability values. A final result which we will repeatedly make use of is the well-known fact that minimal and maximal reachability probabilities are polynomial-time computable.

Proposition 2. For any given MDP M, the values $\max_{\sigma \in \Sigma^M} Pr_{M[\sigma]}(\lozenge T)$ and $\min_{\sigma \in \Sigma^M} Pr_{M[\sigma]}(\lozenge T)$ can be computed in polynomial time.

This follows from a straightforward encoding of the value into a linear program. We refer the reader to [2, 43] for a proof.

3. Parametric Markov decision processes

In this section we introduce parametric MDPs and parametric MCs and provide examples of what can be modelled by them.

Example 1. The Knuth-Yao algorithm [39] uses repeated coin flips to model a six-sided die. It uses a fair coin to obtain each possible outcome ('one', 'two', ..., 'six') with probability ¹/6. Figure 1a depicts an MC of a variant in which two unfair coins are flipped in an alternating fashion. Flipping the coins yields heads with probability ²/5 (gray states) or ₹/10 (white states), respectively. Accordingly, the probability of tails is ³/5 and ³/10, respectively. The event of throwing a 'two' corresponds to reaching the state □ in the MC. Assume now a specification requiring the probability to obtain 'two' to be larger than ³/20. Knuth-Yao's original algorithm satisfies this property as using a fair coin results in ¹/6 as probability to reach □. The biased model however, does not satisfy the property; in fact, □ is reached with probability ¹/10. We may now ask ourselves: how unfair may these coins be and still satisfy the property? Vice versa, we can ask: Assuming that the probability to throw heads is between ²/5 and ³/5, does the property hold for all admissible probabilities of throwing heads?

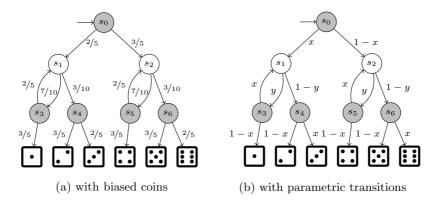
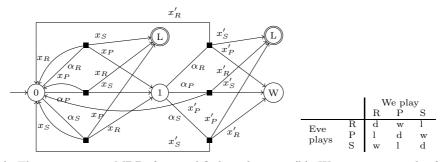


Figure 1: Two variants of the Knuth-Yao die



(a) The parametric MDP for modified rock-paper-(b) Winning a round of scissors $% \left(\frac{1}{2}\right) =0$

Figure 2: Playing (a slightly modified) rock-paper-scissors.

When analysing parameterised MCs, we consider reachability properties on some or all instantiations. In the example above, we asked whether all 'almost fair' coins satisfy reaching \square with at least some given probability.

Example 2. We consider a slightly modified variant of rock-paper-scissors, in which we play against Eve. Eve selects behind her back either Rock (R), Paper (P) or Scissors (S). Now, without knowing what she selected, we have to make a (randomised) decision. Then, we win the round as usual, in accordance with Table 2b. We are interested in minimising the probability that we ever loose a round. Luckily, we thus either (1) win by playing (draws) forever, or (2) Eve gives up if we win twice in a row. We model this protocol by the pMDP in Figure 2a. Initially, Eve selects either (R), (P) or (S). Then, we select with some probability x_P paper, x_R rock, and x_S scissors. This probability is independent of the selection of Eve. If we lose, we go to a target state L (duplicated to avoid clutter). Otherwise, if we win, we move to the next round, and if we draw, we restart the game. In the second round, we may choose a different distribution over the actions. The distribution is represented by x_P', x_R', x_S' . After two wins, we reach the sink state W.

We want to know how we should choose the actions to avoid reaching L. On a technical level, this questions corresponds to asking for the right values for x_R, x_P, x_S . In the worst-case, Eve has some transcendental powers and always

counters optimally. What is the best strategy for us, i.e., what is the minimal probability of reaching the target state L? And how may we randomise if we merely want to ensure that we reach L with a probability less than 90%? Clearly, we have to randomise, otherwise Eve will easily counter our moves.

We now formally define parametric MDPs and their instantiations.

3.1. Fundamentals

Parametric MDPs extend MDPs with a set X of parameters and an adapted transition relation: Probabilities are no longer expressed by values from [0,1], but by rational functions over X^1 .

Definition 1 (pMDP [28]). A parametric Markov Decision Process (pMDP) \mathcal{M} is a tuple $(S, \iota, Act, X, \mathcal{P})$ with a finite set S of states, an initial state $\iota \in S$, a finite set Act of actions, a finite set X of parameters, and a parametric probabilistic transition $\mathcal{P} \colon S \times Act \times S \to \mathbb{Q}(X)$.

We limit ourselves to polynomial pMDPs: A pMDP \mathcal{M} is polynomial if

$$\forall s, s' \in S, \alpha \in Act(s) : \mathcal{P}(s, \alpha, s') \in \mathbb{Q}[X].$$

The rationale is twofold. First, instantiating rational functions yields undefined values when the denominator becomes zero. That induces an additional case distinction which is undesirable for a concise presentation of the theory. Furthermore, we want to use transition probabilities in the ETR constraints. However, the ETR does not 'natively' support rational functions².

As pMDPs extend (finite) MDPs, the concepts of paths, graphs, and strategies as in Section 2 carry over naturally. Likewise, parametric MCs are obtained as an extension to MCs, and are a special case of pMDPs.

Definition 2 (pMC). A parametric Markov chain (pMC) \mathcal{D} is a pMDP $(S, \iota, Act, X, \mathcal{P})$ such that |Act(s)| = 1 for all states $s \in S$. We identify the parametric probabilistic transition of \mathcal{D} with a function $\mathcal{P} : S \times S \to \mathbb{Q}(X)$.

Example 3. Figure 1b depicts a parametric version of the biased Knuth-Yao die from Example 1. It has parameters $X = \{x, y\}$, where x is the probability of outcome heads in grey states and y the same for white states. The probability for tails is then 1-x and 1-y, respectively. Figure 2a depicts the pMDP for our rock-paper-scissors variant. The parameters are $X = \{x_R, x_P, x_S, x_R', x_P', x_S'\}$.

Parameters are thus variables that may be substituted by concrete values. Not all valuations are meaningful in the context of pMDPs. We are only interested in valuations for pMDPs that yield MDPs. Such *valuations* are called well-defined.

Definition 3 (Well-defined valuation). Let \mathcal{M} be a pMDP with parameters X. A valuation val: $X \to \mathbb{R}$ is well-defined for \mathcal{M} if:

¹For technical reasons, we exclude non-rational probabilities which are allowed in MDPs. The field of rational functions (polynomials) with coefficients in \mathbb{Q} is denoted $\mathbb{Q}(X)$ ($\mathbb{Q}[X]$, respectively).

²It would require, e.g., defining whether 0/0 > 0.

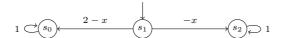


Figure 3: An unrealisable pMC

- probabilities are non-negative, i.e., $\mathcal{P}(s, \alpha, s')[\text{val}] \geq 0$ for all $s, s' \in S, \alpha \in \text{Act}(s)$.
- outgoing probabilities induce distributions, i.e., $\sum_{s' \in S} \mathcal{P}(s, \alpha, s')[\text{val}] = 1$ for all $s \in S$, $\alpha \in Act(s)$.

The set $Val_{\mathcal{M}}^{\mathrm{wd}}$ consists of all well-defined valuations for \mathcal{M} .

Well-defined valuations of X for \mathcal{M} are just called valuations for \mathcal{M} .

Example 4. The well-defined valuations for the pMDP in Figure 2a are

$$\begin{split} \operatorname{Val}^{\operatorname{wd}}_{\mathcal{M}} &= \{\operatorname{val} \in \operatorname{Val} \mid \ \operatorname{val}(x_P) + \operatorname{val}(x_R) + \operatorname{val}(x_S) = 1, \\ & \operatorname{val}(x_P), \operatorname{val}(x_R), \operatorname{val}(x_S) \geq 0, \\ & \operatorname{val}(x_P') + \operatorname{val}(x_R') + \operatorname{val}(x_S') = 1, \\ & \operatorname{val}(x_P'), \operatorname{val}(x_R'), \operatorname{val}(x_S') \geq 0\}. \end{split}$$

A valuation $\{x_R, x_P, x_S, x_R', x_P', x_S' \mapsto 2/3\}$ is not well-defined, as the sum of $val(x_R), val(x_P), val(x_S)$ exceeds one. Some pMDPs do not have any well-defined valuation, e.g., the pMC in Figure 3. It can be readily checked that no x satisfies 2-x+(-x)=1 and $-x\geq 0$.

Definition 4 (Realisable). A pMDP \mathcal{M} is realisable, if $Val_{\mathcal{M}}^{wd} \neq \emptyset$.

Let \mathcal{M} be a pMDP and $\mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ a valuation. The instantiation of \mathcal{M} with val is the MDP $\mathcal{M}[\mathsf{val}] \coloneqq (S, \iota, Act, P)$ with

$$P(s, \alpha, s') := \mathcal{P}(s, \alpha, s')[\mathsf{val}] \quad \text{for all } s, s' \in S, \alpha \in \mathbb{A}\mathsf{ct}(s).$$

Example 5. Reconsider the pMC in Figure 1b. Observe that the valuation val with $val(x) = \frac{2}{5}$ and $val(y) = \frac{7}{10}$ is well-defined and yields the MC in Figure 1a.

The notion of strategies carries over from MDPs to pMDPs and thus induced MCs carry over to induced pMCs.

It is helpful and natural to consider a pMDP \mathcal{M} as a generator for an, in general, uncountable set $\langle \mathcal{M} \rangle$ of instantiated MDPs.

Definition 5 (Generator). The generator of pMDP \mathcal{M} is the set

$$\langle \mathcal{M} \rangle \coloneqq \{ \mathcal{M}[\mathsf{val}] \mid \mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}} \}.$$

Let $R \subseteq \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ be a set of valuations for \mathcal{M} . We define the set

$$\langle \mathcal{M} \mid R \rangle \coloneqq \{ \mathcal{M}[\mathsf{val}] \mid \mathsf{val} \in R \}.$$

Thus, $\langle \mathcal{M} \rangle = \langle \mathcal{M} \mid \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}} \rangle$. For this set, and sets yet to be introduced, we often omit everything but the superscript and write, e.g., $\langle \mathcal{M} \mid \mathrm{wd} \rangle$.

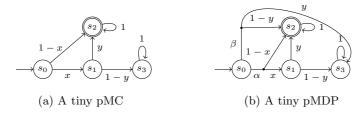


Figure 4: Two small, acyclic models

3.2. Solution functions of parametric models

Each well-defined parameter valuation yields an instantiation for which the measures are defined. Hence, we map valuations to reachability probabilities. *Solution functions* capture this mapping.

Definition 6 (Solution function). For a pMC \mathcal{D} and a state s, let the (probability) solution function $\mathsf{sol}_{s \to T}^{\mathcal{D}} \colon \mathsf{Val}_{\mathcal{D}}^{\mathrm{wd}} \to [0,1]$ be

$$\operatorname{sol}_{s \to T}^{\mathcal{D}}(\operatorname{val}) \coloneqq Pr_{\mathcal{D}[\operatorname{val}]}(s \models \lozenge T).$$

We omit the initial state whenever possible and omit \mathcal{M} and T whenever they are clear from the context.

Example 6. Consider the pMC \mathcal{D} in Figure 4a. There are two paths to the target state. The probability $f := x \cdot y + 1 - x$ to reach the target is the sum over the probabilities over these two paths. For any well-defined instantiation val, the probability to reach the target in $\mathcal{D}[\text{val}]$ is f[val]. Thus,

$$\mathsf{sol}_T^{\mathcal{D}} = x \cdot y + 1 - x.$$

3.3. Graph-preserving valuations

Recall that $\langle \mathcal{M} \rangle$ considers the instantiations that are induced by well-defined valuations. Below, we consider a restriction on the valuations. In the analysis of parameter-free MDPs, it is often essential to exploit the topology of the MDP, e.g., when computing zero-states. In $\langle \mathcal{M} \rangle$, not all MDPs have the same topology. The goal below is to consider a restriction on the valuations such that all MDPs in $\langle \mathcal{M} \rangle$ have the same topology as \mathcal{M} . The topology changes, if a transition is present in the pMDP but not in its instantiation.

Definition 7 (Vanishing transitions). Let \mathcal{M} be a pMDP with $\mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$. We call a transition (s, α, s) vanishing under val if

$$\mathcal{P}(s, \alpha, s') \neq 0$$
 and $\mathcal{P}(s, \alpha, s')[val] = 0$.

The set $Vanish_{\mathcal{M}}(val) \subseteq S \times Act \times S$ contains all vanishing transitions.

A valuation preserves the topology if no transitions vanish, formally:

Definition 8 (Graph-preserving valuations). Let \mathcal{M} be a pMDP. A valuation $\mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ is graph-preserving if $\mathsf{Vanish}_{\mathcal{M}}(\mathsf{val}) = \emptyset$. The set $\mathsf{Val}^{\mathrm{gp}}_{\mathcal{M}}$ contains all graph-preserving valuations.

The generator for this class is $\langle \mathcal{M} \mid \mathsf{Val}_{\mathcal{M}}^{\mathrm{gp}} \rangle$, also denoted $\langle \mathcal{M} \mid \mathrm{gp} \rangle$.

Example 7. Let us again consider Figure 1b. If we set val(x) := 0, then various transitions disappear, in particular the transitions from s_0 to s_1 , and from s_3 to s_1 . Thus, any valuation with val(x) = 0 is not graph-preserving.

There exist realisable pMCs without graph-preserving instantiations, e.g., any realisable pMC with states s, s_1, s_2 such that $\mathcal{P}(s, s_1) = x$, $\mathcal{P}(s, s_2) = x + 1$.

3.4. Other sets of valuations

Sets of valuations may have particular characteristics. For example, when all valuations are graph-preserving, it is a graph-preserving set of valuations. Slightly weaker, the set is *graph-consistent*, when all valuations induce the same topology (but not necessarily the topology of the corresponding pMDP).

Definition 9 (Graph-consistent sets of valuations). A graph-consistent set R of valuations is a subset of the well-defined valuations such that for all val, val' $\in R$:

$$Vanish_{\mathcal{M}}(val) = Vanish_{\mathcal{M}}(val').$$

It is maximally graph-consistent, if no true superset of R is graph-consistent.

Example 8. Let us again consider Figure 1b. As we have seen previously, valuations with val(x) are not graph-preserving. However, the set

$$\{\mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{D}} \mid \mathsf{val}(x) = 0 \land 0 < \mathsf{val}(y) < 1\}$$

 $is \ maximally \ graph-consistent.$

Note that inside graph-consistent sets of valuations the sets of vanishing transitions are invariant. Formally, we have from [29] the following property.

Proposition 3. Let \mathcal{M} be a pMDP with target states T and R a graph-consistent set of valuations. For all $M, M' \in \langle \mathcal{M} \mid R \rangle$ and $\sigma \in \Sigma^{\mathcal{M}}$:

$$Pr_{M}^{\sigma}(\lozenge T) = 0 \text{ implies } Pr_{M'}^{\sigma}(\lozenge T) = 0, \text{ and } Pr_{M}^{\sigma}(\lozenge T) = 1 \text{ implies } Pr_{M'}^{\sigma}(\lozenge T) = 1.$$

A proof of this claim follows directly from the graph-based algorithms for qualitative properties [2], that is, whether the maximal or minimal reachability probabilities are precisely 0 or 1. The same graph-based algorithms suggest that removing transitions does not increase the number of states from which the reachability probability is positive:

Lemma 1. Let \mathcal{M} be a pMDP with target states T. For all $\mathsf{val} \in \mathsf{Val}^\mathsf{gp}_{\mathcal{M}}$ and all $\sigma \in \Sigma^{\mathcal{M}}$ we have that:

$$Pr_{\mathcal{M}[\mathsf{val}]}^{\sigma}(\lozenge T) > 0$$
 iff there exists some $M \in \langle \mathcal{M} \rangle$ s.t. $Pr_{M}^{\sigma}(\lozenge T) > 0$.

Boolean valuations. A final class of valuation sets that we consider is the restriction to $\{0,1\}$. Formally, a valuation $\mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ is a Boolean valuation if $\mathsf{val}(x) \in \{0,1\}$ for all $x \in X$. We write \mathbb{B} for the set of all Boolean valuations.

3.5. Problem statement

The question we address in this article is whether some instantiation of \mathcal{M} is such that its maximal or minimal probability of eventually reaching T compares with $\lambda \in \{0, 1/2, 1\}$ in some desired way. In symbols, for $\mathcal{Q} \in \{\exists, \forall\}$, $R \subseteq \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$, and $\bowtie \in \{\leq, <, >, \geq\}$, we consider the decision problem

$$\exists \ \mathsf{val} \in R, \mathcal{Q} \ \sigma \in \Sigma^{\mathcal{M}}: \ Pr^{\sigma}_{\mathcal{M}[\mathsf{val}]}(\lozenge T) \bowtie \lambda.$$

Assumptions. When studying the complexity of reachability problems, we will mostly focus on $simple\ pMDPs$. A pMDP \mathcal{M} is simple if

- $\mathcal{P}(s, \alpha, s') \in \{x, 1 x \mid x \in X\} \cup \mathbb{Q}_{>0} \text{ for all } s, s' \in S \text{ and } \alpha \in Act; \text{ and }$
- $\sum_{s' \in S} \mathcal{P}(s, \alpha, s') \equiv 1$ for all $s \in S$ and $\alpha \in Act(s)$.

The well-defined and graph-preserving valuations for simple pMDPs are $[0,1]^X$ and $(0,1)^X$ respectively.

Note that such pMDPs are essentially a model of sequential parametric Bernoulli experiments. The reason we restrict our study to simple pMDPs is to avoid the complexity being governed by the subproblem of checking whether there is some well-defined valuation, which in general is an **ETR**-hard problem.

Proposition 4 (From [41]). Given a polynomial pMDP with at least two states, determining whether it is realisable is **ETR**-hard.

We give here a simple proof of the claim using a lemma that will be proved in the sequel.

Proof. Consider a pMC with two states and a single transition between them with probability $f \in \mathbb{Q}[X]$. The constraints for well-definedness collapse to f = 1, or equivalently f - 1 = 0. For multivariate polynomials of degree at least four, answering this question is **ETR**-hard — see Lemma 3.

Encoding of the input. Let \mathcal{M} be a simple pMDP with and a set T of target states. We analyse the decision problems according to whether the set X of parameters from \mathcal{M} has bounded size — with a-priori fixed bound — or arbitrary size. It remains to fix an encoding for polynomials with rational coefficients. Henceforth, we assume the exponents of such polynomials are given as binary-encoded integers and the (rational) coefficients as pairs of integers, also encoded in binary.

4. Qualitative Reachability Problems

Table 1 summarises the results for qualitative reachability in (simple) pMDPs and pMCs. Consider a pMDP \mathcal{M} and let $R \subseteq \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ and $\mathcal{Q} \in \{\exists, \forall\}$. For convenience, we give names to the questions asking whether there exists some pMDP $M \in \langle \mathcal{M} \mid R \rangle$ with the following properties.

- Positive reachability: $Q \sigma \in \Sigma^{\mathcal{M}} : Pr_{\mathcal{M}[\mathsf{val}]}^{\sigma}(\lozenge T) > 0$
- Unsure reachability: $Q \sigma \in \Sigma^{\mathcal{M}} : Pr_{\mathcal{M}[\mathsf{val}]}^{\sigma}(\lozenge T) < 1$
- Safety: $Q \sigma \in \Sigma^{\mathcal{M}}$: $Pr_{\mathcal{M}[\mathsf{val}]}^{\sigma}(\lozenge T) \leq 0$

	Fixed #	Arbitrary # parameters		
	parameters	graph-preserving	well-defined	Boolean
> 0	in P Thm 2	in P Thm 2	in P Thm 2	NP-complete
- 0	111 1 1 1 1 1 1 2	111 1 1 nm 2	111 1 1 nm 2	Thm 1, Prop 5
< 1	,,	"	NP-complete	"
\ 1			Thm 1, Prop 7	
≤ 0	,,	"	NP-complete	,,
0			Thm 1, Prop 6	
≥ 1	"	"	"	"

Table 1: The complexity landscape for qualitative reachability in simple pMCs and pMDPs. Observe that the decision problems for pMCs and pMDPs (for maximal and minimal reachability probability values) are different, but (with respect to the considered classes) the categorisation coincides. Unlisted combinations of comparison operators and thresholds yield trivial decision problems.

• Almost-sure reachability: $Q \sigma \in \Sigma^{\mathcal{M}} : Pr_{\mathcal{M}[\mathsf{val}]}^{\sigma}(\lozenge T) \geq 1.$

Note that the decision problem changes depending on R and, for pMDPs, on Q. Together, these problems are the *qualitative reachability problems*. Table 1 lists their computational complexity for a fixed number of parameters, and the complexity if the parametric model contains arbitrarily many parameters. In the latter case, we make a distinction based on whether the parameter valuations range over the well-defined, graph-preserving or Boolean valuations.

4.1. Upper bounds

Towards a general upper bound, recall that inside graph-consistent valuation sets the sets of vanishing transitions are invariant. The following is a corollary of Proposition 3 and the fact that (maximal and minimal) reachability values in MDPs are computable in polynomial time.

Theorem 1. The qualitative reachability problems for simple pMDPs are all decidable in NP.

Indeed, one can guess a graph-consistent set of valuations by, for instance, guessing an assignment of the parameters with values 0, 1/2, or 1, for all of them. In the instantiated MDP one can verify that the property holds in polynomial time.

There are three particular cases in which the problem is tractable: when considering graph-preserving valuations only, when the problem is positive reachability, and when the number of parameters is fixed.

Theorem 2. The following problems for simple pMDPs are decidable in polynomial time:

- all the qualitative reachability problems with respect to graph-preserving valuations;
- the positive reachability problems that include graph-preserving valuations; and
- $\bullet \ \ all \ the \ qualitative \ reachability \ problems \ for \ a \ fixed \ number \ of \ parameters.$

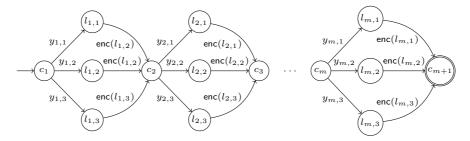


Figure 5: pMC construction for NP-hardness of positive reachability in pMCs.

The main idea behind the proof is the same as for the previous claim. Indeed, one can guess a graph-consistent set of valuations by choosing a 'dummy variable assignment' giving a value of 0, 1/2, or 1, for all of them. We observe that the set of all such valuations forms a finite partition of the set of well-defined valuations:

Lemma 2. Let \mathcal{M} be a pMDP with parameters X. The set $\mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ may be partitioned into at most $3^{|X|}$ maximal graph-consistent sets of valuations.

We can now argue that the theorem holds.

Proof of Theorem 2. If we only consider graph-preserving valuations then the structure remains fixed. Hence, the qualitative reachability problems are essentially equivalent to their parameter-free counterparts (obtained, e.g., by assigning $\frac{1}{2}$ to all parameters) and therefore in **P**.

For positive reachability, we observe that removing transitions is never beneficial and (non-empty) graph-preserving valuations are, in that sense, optimal for positive reachability — see Lemma 1. Hence, the positive reachability problems can be decided by considering any graph-preserving instantiation (e.g., assigning $\frac{1}{2}$ for all parameters). Therefore, for well-defined instantiations, positive reachability is in **P**.

When we have a fixed number of parameters, even when ranging over all well-defined instantiations, there are only constantly many different graph-consistent valuation sets — see Lemma 2. Consequently, the problem reduces to a constant number of problems in $\bf P$.

In the sequel we give **NP**-lower bounds for the remaining cases.

4.2. Lower bounds for Boolean valuations

This hardness result crucially depends on the absence of graph-preserving instantiations and is inspired by a construction in [31].

Proposition 5. The qualitative reachability problems with respect to Boolean valuations are NP-hard even for acyclic simple pMCs.

Proof. We show a reduction from 3SAT to prove positive reachability is **NP**-hard and comment on how to adapt the argument for the other problems. Let

$$\psi \coloneqq c_1 \wedge \dots \wedge c_m$$

be a given 3SAT-formula, i.e. the clauses c_i are of the form

$$c_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$$

where the $l_{i,j}$ are *literals* (variables or negated variables). Let $\boldsymbol{x} = \{x_1, \dots, x_k\}$ be the variables of ψ . The pMC for ψ is outlined in Figure 5. Formally, the pMC $\mathcal{D} := (S, \iota, X, \mathcal{P})$ is defined as follows: The 4m + 2 states

$$S := \{c_i \mid 1 \le i \le m+1\} \cup \{l_{i,j} \mid 1 \le i \le m, 1 \le j \le 3\} \cup \{\bot\} \text{ with } \iota := c_1,$$

3m + k parameters

$$X := \{\tilde{x} \mid x \in \mathbf{x}\} \cup \{y_{i,j} \mid 1 \le i \le m, 1 \le j \le 3\},\$$

and with transitions

$$\mathcal{P}(s,s') := \begin{cases} y_{i,j} & \text{if } s = c_i, s' = l_{i,j} \text{ for some } 1 \leq i \leq m, 1 \leq j \leq 3, \\ \operatorname{enc}(l_{i,j}) & \text{if } s = l_{i,j}, s' = c_{i+1} \text{ for some } 1 \leq i \leq m, 1 \leq j \leq 3, \\ 1 - \operatorname{enc}(l_{i,j}) & \text{if } s = l_{i,j}, s' = \bot \text{ for some } 1 \leq i \leq m, 1 \leq j \leq 3, \\ 0 & \text{otherwise,} \end{cases}$$

using

$$\operatorname{enc}(l_{i,j}) \coloneqq \begin{cases} \tilde{x} & \text{if } l_{i,j} = x, \\ 1 - \tilde{x} & \text{if } l_{i,j} = \overline{x}. \end{cases}$$

The target states are $T = \{c_{m+1}\}$. It should be clear that this construction can be realised in polynomial time.

We will argue that ψ is satisfiable if and only if there exists $D \in \langle \mathcal{D} \mid \mathbb{B} \rangle$ such that $Pr_D(\lozenge T) > 0$. Intuitively, the variables $l_{i,j}$ represent the witness literal for each satisfied clause, i.e., the literal that makes the clause true. The parameters \tilde{x} correspond to the x variables in the 3SAT-formula as follows: For a valuation val of variables x in ψ and a valuation val' of X such that val' $(\tilde{x}) = 1$ iff val(x) = true it holds that:

$$enc(l_{i,j})[val'] = 1 \iff val(l_{i,j}) = true.$$

Formally, first assume there exists a satisfying assignment val for ψ . Then, this assignment makes at least one literal $l_{i,*}$ in every clause c_i true. We consider val' with the corresponding $y_{i,*}$ assigned to 1 and \tilde{x} assigned 1 iff val(x) = true. Then, in the MC $\mathcal{D}[\text{val'}]$, there is a path from ι to T.

Now assume that there exists an MC $D \in \langle \mathcal{D} \mid \mathbb{B} \rangle$ with a path from ι to T. Observe that this path in D is the only path to the target. We construct a satisfying assignment val for ψ . This path goes through a set of $l_{i,*}$. These become the witness literals that make all the clauses true. The assignment to the variables \boldsymbol{x} are obtained from the occurrences of \tilde{x} along the path, or equivalently, by lookup from the witness literals given by the path.

For safety, almost-sure, and unsure reachability, we observe that the probability to reach c_{m+1} in \mathcal{D} is either zero or one for any Boolean valuation so the corresponding proofs are straightforward adaptions of the one given above. \square

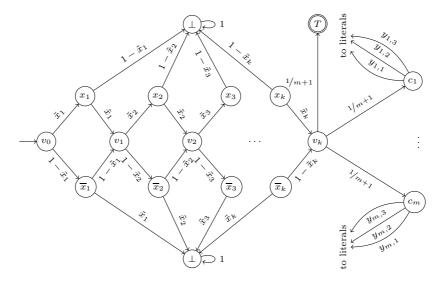


Figure 6: pMC construction for NP-hardness of almost-sure reachability in pMCs.

4.3. Lower bounds for well-defined valuations

We have argued that positive reachability is in **P**. We now show that all other qualitative problems are **NP**-complete. We begin with the almost-sure reachability and safety problems.

Proposition 6 (From [15]). The safety and almost-sure reachability problems are NP-hard even for simple pMCs.

We deliberately recall the proof from [15] rather than adapting the construction used to prove Proposition 5, as the former is a crucial step towards Proposition 12. The essential idea here is to enforce Boolean valuations.

Proof. We reduce from 3SAT once more. To that end, let $\psi = c_1 \wedge \cdots \wedge c_m$ be a given 3SAT-formula with clauses c_i of the form $c_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$ and variable set $\boldsymbol{x} = \{x_1, \dots, x_k\}$. The pMC for ψ is outlined in Figure 6, where state \bot is duplicated to avoid clutter. Formally, the pMC $\mathcal{D} = (S, \iota, X, \mathcal{P})$ is defined as follows:

$$S := \{ v_i \mid 0 \le i \le k \} \ \uplus \ \{ x_i, \overline{x_i} \mid 1 \le i \le k \} \ \uplus \ \{ c_i \mid 1 \le i \le m \} \ \uplus \ \{ T, \bot \}$$

are the 3k+m+3 states, $v_0=\iota$ is the initial state, T and \bot indicate target and sink respectively,

$$X := \{ \tilde{x} \mid x \in x \} \cup \{ y_{i,j} \mid 1 \le i \le m, 1 \le j \le 3 \}$$

are the k+3m parameters, for all $1 \leq i \leq m$ and $1 \leq j \leq k$ we define the

transition probabilities as

$$\begin{split} \mathcal{P}(v_{i-1},x_i) &\coloneqq \tilde{x}_i, & \mathcal{P}(v_{i-1},\overline{x_i}) \coloneqq 1 - \tilde{x}_i, \\ \mathcal{P}(x_i,v_i) &\coloneqq \tilde{x}_i, & \mathcal{P}(\overline{x_i},v_i) \coloneqq 1 - \tilde{x}_i, \\ \mathcal{P}(x_i,\bot) &\coloneqq 1 - \tilde{x}_i, & \mathcal{P}(\overline{x_i},\bot) \coloneqq \tilde{x}_i, \\ \mathcal{P}(v_k,c_i) &\coloneqq \frac{1}{m+1}, & \mathcal{P}(v_k,T) \coloneqq \frac{1}{m+1}, \\ \mathcal{P}(c_i,x_i) &\coloneqq y_{i,r} \text{ if } l_{i,r} = x_i \text{ (in } \psi), & \mathcal{P}(c_i,\overline{x_i}) \coloneqq y_{i,r} \text{ if } l_{i,r} = \overline{x}_i \text{ (in } \psi). \end{split}$$

We let $\mathcal{P}(s,t) = 0$ for each pair (s,t) of states not specified above.

Observe that under any well-defined valuation, there are exactly two bottom strongly connected components, namely \bot and T. As a consequence:

for all
$$D \in \langle \mathcal{D} \rangle$$
: $Pr_D(\Diamond T) + Pr_D(\Diamond \{\bot\}) = 1.$ (1)

We will argue that ψ is satisfiable if and only if there exists $D \in \langle \mathcal{D} \rangle$ such that $Pr_D(\Diamond T) \geq 1$. For convenience, we write 1 and 0 instead of true and false respectively.

First, assume ψ is satisfiable. Choose some satisfying assignment val for ψ . We construct $\mathsf{val}' \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{D}}$ in two steps. First, let $\mathsf{val}'(\tilde{x}_i) = \mathsf{val}(x_i) \in \{0,1\}$ for all $1 \leq i \leq k$. Thus, \bot is unreachable. Second, for each clause c_i , select one literal $l_{i,j}$ which makes c_i true, and set $\mathsf{val}'(y_{i,j}) = 1$. Set all other $y_{i,j}$ to 0. It follows from Equation (1) that $Pr_{\mathcal{D}[\mathsf{val}']}(\lozenge T) = 1$.

Now, assume there is a well-defined valuation val' such that $Pr_{\mathcal{D}[\mathsf{val'}]}(\lozenge T) \geq 1$. Then, using Equation (1), no path leads to \bot . For \tilde{x}_i , that means that necessarily $\mathsf{val'}(\tilde{x}_i) \in \{0,1\}$. Note that $\mathsf{val'}$ must be such that we can choose for each c_i a literal $l_{i,j}$ (a $y_{i,j}$ set to 1) which surely reaches v_k again. These $l_{i,j}$ are exactly the witness literals making every clause true. It follows that the assignment for \tilde{x} gives rise to a satisfying valuation val for ψ .

To conclude we observe that the construction can be easily adapted to show NP-hardness of the safety problem. $\hfill\Box$

To close this section we describe how to adapt the construction used to prove Proposition 5 in order to show the unsure reachability problem is also \mathbf{NP} -hard.

Proposition 7. The unsure reachability problems are NP-hard even for simple pMCs.

Proof sketch. We reuse the pMC from Proposition 5 depicted in Figure 5 and extend it with a transition from c_{m+1} to c_1 with probability 1. Further, we set \bot as the only target state.

Observe that if c_{m+1} is reached with probability 1 then no probability 'leaks' to the \perp -state. Hence the target is reached with probability 0. Otherwise, the target is the only bottom strongly connected component and the probability to reach that becomes 1. The result thus follows from an almost identical argument as the one given for Proposition 5.

5. Quantitative Reachability Problems

Table 2 summarises the results we will present in this section. We use the following notation for conciseness: For $Q \in \{\exists, \forall\}$ and $\bowtie \in \{\leq, <, >, \geq\}$, let

$$\mathcal{Q}_{\text{REACH}_{\text{wd}}}^{\bowtie} \stackrel{\text{def}}{\Longleftrightarrow} \exists \text{ val} \in \text{Val}_{\mathcal{M}}^{\text{wd}}, \mathcal{Q} \ \sigma \in \Sigma^{\mathcal{M}} : Pr_{\mathcal{M}[\text{val}]}^{\sigma}(\lozenge T) \bowtie \frac{1}{2}$$

	Fixed #		Arbitrary # parameters	
		parameters	well-defined	graph-preserving
pMC	$_{\mathrm{REACH}}\geq/\leq$	in \mathbf{P}	— ETR-complete Thm 8 —	
		Thm 11		
	REACH>	,,	$\mathbf{NP} ext{-}\mathrm{hard}$	$REACH_{wd}^{>}$ -complete
			Prop 12	Prop 9, Prop 10
	REACH<	,,	$\mathbf{NP} ext{-}\mathrm{hard}$	$REACH_{wd}^{>}$ -complete
			Prop 12	Prop 8
pMDP	$\exists_{\text{REACH}} \geq / \leq$	in \mathbf{NP}	— ETR-complete (trivial) —	
		Prop 15		
	$\exists \text{REACH}^{>}$	"	— REACH $^>_{ m wd}$ -complete Prop 13, Prop 14 —	
	$\exists \text{REACH}^{<}$,,	REACH < complete	REACH ^{>} _{wd} -hard
			Prop 13	(trivial)
	$\forall \mathrm{REACH}^{\bowtie}$	in \mathbf{NP}	— ETR-complete Thm 10 —	
		Thm 12		

Table 2: The complexity landscape for *quantitative* reachability in simple pMDPs. All problems are in **ETR**.

be the *quantitative reachability problems*. We write $\mathcal{Q}_{REACH_{gp}^{\bowtie}}$ whenever we consider graph-preserving instantiations. We write $\mathcal{Q}_{REACH_{gp}^{\bowtie}}$ to denote both the wd and gp variants. Furthermore, if \mathcal{M} is a pMC we omit the quantifier, e.g. $REACH_{gp}^{<}$.

Fixed threshold. Note that we have fixed a threshold of $^1/2$. This is without loss of generality as any given rational threshold λ may be reduced to $^1/2$: Simply prepend a transition with probability $^1/2$ to the initial state, one with probability $^1/2(1-\lambda)$ to the target state and a third one with probability $^1/2\lambda$ to a sink state. Then it can be readily checked that the reachability probability in the original model compares to λ in some desired way iff it compares to $^1/2$ in the modified model.

We first show that well-defined and graph-preserving sets of valuations are *semialgebraic*, i.e., they can be described by an ETR formula. Then we give a detailed account on how to encode the reachability problems for pMCs into the ETR. First, we consider reachability probabilities and the easier case of graph-preserving valuation subsets, then in general for well-defined valuation subsets. We then show the lifted encodings to pMDPs.

5.1. ETR encoding for pMCs

Below, we show that sets of all well-defined or graph-preserving valuations are indeed semialgebraic. The following set of constraints is a natural encoding of Definition 3.

Constraints 1 (Well-defined sets of valuations). The following constraints capture well-defined valuations for a polynomial pMDP \mathcal{M} :

$$0 \leq \mathcal{P}(s, \alpha, s') \leq 1 \text{ for all } s, s' \in S, \alpha \in \mathbb{A}\mathsf{ct}(s) \quad (with \ \mathcal{P}(s, \alpha, s') \neq 0),$$

$$\sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \text{ for all } s \in S, \alpha \in \mathbb{A}\mathsf{ct}(s).$$

We denote the corresponding formula for this constraint system with $\Phi_{wd}^{\mathcal{M}}$.

The constraints ensure that (1) all (non-zero) transitions are evaluated to a probability, and (2) transition probabilities describe distributions. It follows that the set of well-defined valuations of some \mathcal{M} is semialgebraic.

Example 9. Recall the pMC \mathcal{D} for the Knuth-Yao die from Figure 1b, with the well-defined valuations as in Example 4. We have:

$$\begin{split} \Phi_{\mathrm{wd}}^{\mathcal{D}} = & \quad x \geq 0 \, \wedge \, 1 - x \geq 0 \, \wedge \, x + 1 - x = 1 \\ & \quad \wedge \, y \geq 0 \, \wedge \, 1 - y \geq 0 \, \wedge \, y + 1 - y = 1. \end{split}$$

This formula simplifies to $0 \le x \le 1 \land 0 \le y \le 1$.

Now recall the pMDP \mathcal{M} for rock-paper-scissors from Figure 2a, with the well-defined valuations as in Example 4. We get:

$$\Phi_{\text{wd}}^{\mathcal{M}} = x_R \ge 0 \land x_P \ge 0 \land x_S \ge 0 \land x_R + x_P + x_S = 1$$
$$\land x_R' \ge 0 \land x_P' \ge 0 \land x_S' \ge 0 \land x_R' + x_P' + x_S' = 1.$$

This encoding is easily extended with strict inequalities to describe graph-preserving valuations, based on Definition 8.

We now move to the more interesting question of how to actually encode reachability. We start with pMCs, which we consider extensively as the ideas for pMDPs are mostly straightforward extensions.

5.1.1. Qualitative analysis

Before we treat quantitative problems, we start with the qualitative ones.

Definition 10. Let \mathcal{D} be a pMC. The zero-states for valuation $val \in Val_{\mathcal{D}}^{wd}$ are

$$S^{\mathsf{val},T}_{-0} \coloneqq \{ s \in S \mid \mathsf{sol}^{\mathcal{D}}_{s \to T}[\mathsf{val}] = 0 \}$$

containing the states that reach the target with probability zero in instantiation val and the one-states for valuation val $\in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{D}}$ is the set

$$S^{\operatorname{val},T}_{=1} := \{s \in S \mid \operatorname{sol}_{s \to T}^{\mathcal{D}}[\operatorname{val}] = 1\}$$

containing all states that reach the target almost surely.

These sets vary for different valuations. However, for any val, val' $\in Val_{\mathcal{D}}^{\mathrm{wd}}$

$$\mathsf{Vanish}_{\mathcal{D}}(\mathsf{val}) \subseteq \mathsf{Vanish}_{\mathcal{D}}(\mathsf{val}') \text{ implies } S^{\mathsf{val},T}_{=0} \subseteq S^{\mathsf{val}',T}_{=0},$$

and

$$\mathsf{Vanish}_{\mathcal{D}}(\mathsf{val}) = \mathsf{Vanish}_{\mathcal{D}}(\mathsf{val}') \text{ implies } S^{\mathsf{val},T}_{=0} = S^{\mathsf{val}',T}_{=0}.$$

Essentially, removing transitions may cut states from having a path to the target states, but never adds new paths.

Computing the sets. Proposition 3 justifies the notation $S_{=0}^{R,T}$ and $S_{=1}^{R,T}$ for graph-consistent R as being the (unique) sets $S_{=0}^{\mathsf{val},T}$, $S_{=1}^{\mathsf{val},T}$ for any $\mathsf{val} \in R$, respectively. Crucially, for any fixed graph-consistent valuation set, the sets $S_{=0}^{\mathsf{val},T}$ and $S_{=1}^{\mathsf{val},T}$ may be computed on the parameter-free $\mathcal{D}[\mathsf{val}]$. However, when regarding a non-graph-consistent valuation set, this does not necessarily suffice. The essential idea is to encode the graph-algorithm together with a ranking function.

5.1.2. Quantitative analysis

We move from the qualitative setting to a quantitative one. The principle is again to generalise the parameter-free case. We develop the encoding in two steps: First, we consider an encoding only valid for graph-preserving valuations. In particular, it requires the zero-states to be known a-priori. Later, we combine this encoding with the earlier qualitative encodings to compute the zero-states on the fly.

Graph-preserving case. We lift the classical equation system for parameter-free MCs to polynomial pMCs.

Constraints 2. Let \mathcal{D} be a polynomial pMC. We assume a graph-preserving valuation set $R \subseteq \mathsf{Val}_{\mathcal{D}}^{\mathsf{gp}}$. Consider real variables $\{p_s \mid s \in S\}$ and variables for the parameters X of \mathcal{D} :

$$p_s = 1$$
 for all $s \in T$,
 $p_s = 0$ for all $s \in S_{=0}^{R,T}$,
 $p_s = \sum_{s' \in S} \mathcal{P}(s, s') \cdot p_{s'}$ for all $s \in S \setminus (T \cup S_{=0}^{R,T})$.

We denote the corresponding formula with $\Phi_{gp}^{\mathcal{D}}$.

Note that the constraints do not actually depend on R, only the fact that R is graph-preserving matters. The constraints are essentially identical to those for parameter-free MCs. The key difference is that the transition probabilities are no longer constants. Therefore (in general³) the encodings are non-linear.

Recall that we have to restrict the parameter valuations accordingly and encode that the induced probability in the initial state compares $\bowtie 1/2$. We add these constraints and obtain:

Theorem 3. Let \mathcal{D} be a polynomial pMC with target states T and let $R \subseteq \mathsf{Val}_{\mathcal{D}}^{\mathsf{gp}}$ be a semialgebraic set given by ψ_R . We define

$$\psi \coloneqq \Phi_{\mathrm{gp}}^{\mathcal{D}} \wedge p_{\iota} \bowtie 1/2 \wedge \psi_{R}.$$

Then, for all $val \in Val$,

val
$$satisfies \ \psi \ iff \ val \in R \wedge Pr_{\mathcal{D}[val]}(\lozenge T) \bowtie 1/2.$$

Well-defined case. We extend the encoding to any well-defined valuation set. An essential assumption before was that the set of zero-states is fixed and may be precomputed. This assumption is no longer valid. We thus encode the computation of the zero-states using the encoding for positive reachability.

Constraints 3. Let \mathcal{D} be a polynomial pMC with states S. Consider Boolean variables $\{q_s \mid s \in S\}$, real variables $\{p_s, r_s \mid s \in S\}$, and variables for the

 $^{^3}$ The notable exceptions are systems where the parameters only occur in states where all successor states are sink- or target-states.

parameters:

$$p_{s} = 1 \qquad \qquad for \ all \ s \in T,$$

$$q_{s} \ is \ true \qquad \qquad for \ all \ s \in T,$$

$$q_{s} \leftrightarrow \bigvee_{s' \in S} (\mathcal{P}(s, s') > 0 \land (q_{s'} \land r_{s} > r_{s'})) \qquad for \ all \ s \in S \setminus T,$$

$$\neg q_{s} \rightarrow p_{s} = 0 \qquad \qquad for \ all \ s \in S \setminus T,$$

$$q_{s} \rightarrow p_{s} = \sum_{s' \in S} \mathcal{P}(s, s') \cdot p_{s'} \qquad for \ all \ s \in S \setminus T.$$

We denote the corresponding formula with $\Phi_{wd}^{\mathcal{D}}$.

The meaning of the variables is as before: The variables q_s determine whether we have to compute the non-zero probability to the target or whether this probability is zero. The r_s variables are auxiliary variables ranking the states. The specialised constraints for the graph-preserving case (Constraints 2) are obtained by setting all variables of non-zero states q_s to true. The following theorem is the analogue to Theorem 3.

Theorem 4. Let \mathcal{D} be a polynomial pMC with target states T and let $R \subseteq \mathsf{Val}_{\mathcal{D}}^{\mathrm{wd}}$ be a semialgebraic set given by ψ_R . We define

$$\psi \coloneqq \Phi_{\mathrm{wd}}^{\mathcal{D}} \wedge p_{\iota} \bowtie 1/2 \wedge \psi_{R}.$$

Then, for any $val \in Val$,

val
$$satisfies \ \psi \ iff \ val \in R \wedge Pr_{\mathcal{D}[val]}(\lozenge T) \bowtie 1/2.$$

5.1.3. Alternative encoding via solution functions

The encodings presented above contain $\mathcal{O}(|S| + |X|)$ many variables. As solving ETR is exponential in the number of variables, the large number of variables is a significant hurdle. In this section, we present encodings that prevent the dependency on the number of states, by incorporating the solution function.

We reconsider the encoding for pMCs under the assumption that we may precompute the zero-states. Reachability in an MC corresponds to a linear equation system $A \cdot \vec{p} = \vec{b}$ where \vec{p} is the solution vector, and A is a matrix and b is a vector (see, e.g., [2]). For pMCs, $A \cdot \vec{p} = \vec{b}$ may be viewed as a linear equation system over the field $\mathbb{Q}(X)$ of rational functions with rational coefficients. That is, the entries of A are no longer rational numbers, but rational functions instead [31, 19].

By basic linear algebra, we obtain that for all pMCs $\mathcal D$ with targets T, there exists $f\in \mathbb Q(X)$ such that

$$\operatorname{\mathsf{sol}}_T^{\mathcal{D}}[\operatorname{\mathsf{val}}] = f[\operatorname{\mathsf{val}}] \text{ for all } \operatorname{\mathsf{val}} \in \operatorname{\mathsf{Val}}_{\mathcal{D}}^{\operatorname{gp}}.$$

The rational function f is exactly the entry q_{ι} of the unique solution q for the system $A \cdot \vec{p} = \vec{b}$. Thus, solving linear equation systems (symbolically) is sufficient to find these solution functions. We observe that f is the restriction of $\mathsf{sol}_T^{\mathcal{D}}$ to $\mathsf{Val}_T^{\mathsf{gp}}$. We denote this restriction with $\mathsf{gpsol}_T^{\mathcal{D}}$.

We conclude this alternative ETR encoding by stating its main property.

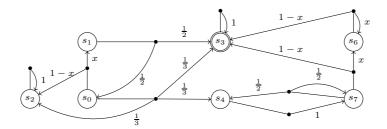


Figure 7: Example pMDP

Theorem 5. Let \mathcal{D} be a polynomial pMC with target states T. Let $_{gp}sol_{T}^{\mathcal{D}} = f/g$ for polynomials f and g be the solution function of \mathcal{D} on graph-preserving valuations and let $R \subseteq Val_{\mathcal{D}}^{gp}$ be a semialgebraic set given by ψ_{R} . We define

$$\psi := ((g > 0 \land f \bowtie 1/2 \cdot g) \lor (g < 0 \land 1/2 \cdot g \bowtie f)) \land \psi_R.$$

Then, for any $val \in Val$,

$$\text{val } \textit{satisfies } \psi \textit{ iff } \text{val} \in R \land Pr_{\mathcal{D}[\text{val}]}(\lozenge T) \bowtie {}^{1}\!/{2}.$$

5.2. ETR encoding for pMDPs

In this section, we generalise the encodings from pMCs to pMDPs. We distinguish between existential and universal nondeterminism. Together, this subsection establishes that for every pMDP the set of all valuations giving a positive answer to reachability problems is semialgebraic.

5.2.1. Qualitative analysis

Again, we first give some preliminary considerations regarding the qualitative case before moving to the quantitative setting.

Definition 11. Let \mathcal{M} be a pMDP. The exist-zero states for a valuation $\mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ is the set

$$S^{\mathrm{val},T}_{\exists=0} \coloneqq \{s \in S \mid \exists \sigma \in \Sigma^{\mathcal{M}} \ s.t. \ \operatorname{sol}_{s \to T}^{\mathcal{M}[\sigma]}[\mathrm{val}] = 0\}$$

containing the states that reach the target with probability zero in instantiation val, and the exist-one states is the set

$$S^{\mathrm{val},T}_{\exists=1} \coloneqq \{s \in S \mid \exists \sigma \in \Sigma^{\mathcal{M}} \ s.t. \ \operatorname{sol}_{s \to T}^{\mathcal{M}[\sigma]}[\mathrm{val}] = 1\}.$$

The sets $S_{\forall=0}^{\mathsf{val},T}$ and $S_{\forall=1}^{\mathsf{val},T}$ are defined analogously.

Example 10. Consider the pMDP in Figure 7, with $T = \{s_3\}$. First, assume $val := \{x \mapsto 1/2\}$. We have

$$S_{\exists=0}^{\text{val},T} = \{s_2, s_4, s_7\}, \text{ and } S_{\forall=0}^{\text{val},T} = \{s_2\}.$$

For val $:= \{x \mapsto 0\}$, we have

$$S_{\exists=0}^{\mathsf{val},T} = \{s_0, s_2, s_4, s_7\}, \ and \ S_{\forall=0}^{\mathsf{val},T} = \{s_2\},$$

and for val $:= \{x \mapsto 1\}$, we have

$$S_{\exists = 0}^{\mathsf{val}, T} = \{s_2, s_4, s_6, s_7\} = S_{\forall = 0}^{\mathsf{val}, T}$$

5.2.2. Quantitative analysis

For pMDPs, we omit the special case of graph-preservation. Instead, we consider existential and universal nondeterminism separately. Contrary to pMCs, but in line with parameter-free MDPs, we now also have to distinguish properties with lower bounds and properties with upper bounds.

Existential nondeterminism. Existential nondeterminism is conceptually simple, as we existentially quantify over both parameter values and strategies. In a game-theoretic sense, one player chooses both parameter values and strategies, and we may just generalise the pMC encoding and use the ETR (where the player selects the values for all variables). We may, however, avoid variables for the strategies by observing that the quantification over strategies is over a finite set, and that this choice may be represented by a (finite) disjunction. This disjunction ranges over exponentially many strategies. We avoid this explicit blowup by recalling that the nondeterminism is resolved locally. Instead of a disjunction over all strategies, we make disjunctions over the local action choices, similar to the encoding of the qualitative case. These insights yield a compact encoding, detailed below.

Constraints 4 (Upper-bounded reachability, existential nondeterminism).

$$\begin{aligned} p_s &= 1 & & \textit{for all } s \in T, \\ q_s & \textit{is true} & & \textit{for all } s \in T, \\ q_s &\leftrightarrow \bigwedge_{\alpha \in \mathbb{A} \text{ct}(s)} \bigvee_{s' \in S} \left(\mathcal{P}(s, \alpha, s') > 0 \to (q_{s'} \land r_s > r_{s'}) \right) & \textit{for all } s \in S \setminus T, \\ \neg q_s &\to p_s &= 0 & & \textit{for all } s \in S \setminus T, \\ q_s &\to \bigvee_{\alpha \in \mathbb{A} \text{ct}(s)} \left(p_s = \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'} \right) & & \textit{for all } s \in S \setminus T. \end{aligned}$$

We refer to the corresponding formula as $\Phi_{\mathrm{wd}}^{\leq,\exists}(\mathcal{M})$.

Under existential nondeterminism, we can freely choose the action at every state: the probability of reaching the target is the sum over the probabilities of reaching the target from the successors after taking this action. As we can choose the action, we thus have a disjunction over equalities for every state and these disjunctions are guarded by the flag that the probability is positive from this state, as for pMCs.

For upper bounds on the reachability probability, under existential nondeterminism, the strategy tries to minimise the probability. In particular, the strategies sets states to probability zero if there is any strategy to do so. As the interpretation of the q_s variables is positive reachability, i.e., q_s is **true** iff it is not a zero-state, we obtain a conjunction over all actions in the encoding.

Below, we give the encoding for lower bounds on the probability.

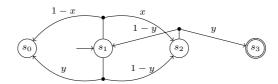


Figure 8: Small pMDP to illustrate encodings

Constraints 5 (Lower-bounded reachability, existential nondeterminism).

$$\begin{aligned} p_s &= 1 & & \text{for all } s \in T, \\ q_s & \text{is true} & & \text{for all } s \in T, \\ q_s &\leftrightarrow \bigvee_{\alpha \in \mathbb{A} \text{ct}(s)} \bigvee_{s' \in S} \left(\mathcal{P}(s,\alpha,s') > 0 \to (q_{s'} \land r_s > r_{s'}) \right) & \text{for all } s \in S \setminus T, \\ \neg q_s &\to p_s &= 0 & & \text{for all } s \in S \setminus T, \\ q_s &\to \bigvee_{\alpha \in \mathbb{A} \text{ct}(s)} \left(p_s = \sum_{s' \in S} \mathcal{P}(s,\alpha,s') \cdot p_{s'} \right) & & \text{for all } s \in S \setminus T. \end{aligned}$$

We refer to the corresponding formula as $\Phi_{\mathrm{wd}}^{\trianglerighteq,\exists}(\mathcal{M})$.

For lower bounds, (only) the computation of the zero states changes, as we now try to avoid setting a state to probability zero. Thus, we only set the reachability probability to zero if all actions lead to zero-states. Again, as the interpretation of q_s is that s is not a zero-state, we obtain a disjunction over the actions.

Theorem 6. Let \mathcal{M} be a pMDP with target states T. Let $R \subseteq \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ be a semialgebraic set given by ψ_R . For $\bowtie \in \{\leq, <, \geq, >\}$ we define

$$\psi \coloneqq \Phi_{\mathrm{wd}}^{\bowtie,\exists}(\mathcal{M}) \wedge p_{\iota} \bowtie 1/2 \wedge \psi_{R}.$$

Then, for any $val \in Val$,

$$\text{val } satisfies \ \psi \ \textit{iff} \ \text{val} \in R \land \exists \ \sigma \in \Sigma^{\mathcal{M}}: \ Pr^{\sigma}_{\mathcal{M}[\text{val}]}(\lozenge T) \bowtie {}^{1}\!/{2}.$$

Example 11. Consider the pMDP \mathcal{M} in Figure 8 and let $R \subseteq \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ be an arbitrary semialgebraic set. We first consider the encoding from Theorem 6 for an upper bound (i.e., $\bowtie = \leq$). For conciseness, we simplified several constraints.

$$\begin{aligned} p_1 &\leq 1/2 \wedge \psi_R \wedge p_0 = 0 \wedge \neg q_0 \wedge p_3 = 1 \wedge q_3, \\ q_1 &\leftrightarrow \Big(\big(x > 0 \wedge q_2 \wedge r_1 > r_2 \big) \wedge \big(1 - y > 0 \wedge q_2 \wedge r_1 > r_2 \big) \Big), \\ q_2 &\leftrightarrow \Big(\big(y > 0 \big) \vee \big(1 - y > 0 \wedge q_1 \wedge r_2 > r_1 \big) \Big), \\ q_1 &\to \Big(\big(p_1 = x \cdot p_2 \big) \vee \big(p_1 = (1 - y) \cdot p_2 \big) \Big) \wedge \neg q_1 \to p_1 = 0, \\ q_2 &\to \big(p_2 = (1 - y) \cdot p_1 + y \big) \wedge \neg q_1 \to p_2 = 0. \end{aligned}$$

Below, we give the encoding for $\bowtie = \ge$. Compared to the encoding above, only the first constraint and one further connective in the second line changed:

$$\begin{aligned} p_1 & \geq \sqrt[1]{2} \wedge \psi_R \wedge p_0 = 0 \wedge \neg q_0 \wedge p_3 = 1 \wedge q_3, \\ q_1 & \leftrightarrow \Big(\big(x > 0 \wedge q_2 \wedge r_1 > r_2 \big) \vee \big(1 - y > 0 \wedge q_2 \wedge r_1 > r_2 \big) \Big), \\ & \dots \end{aligned}$$

Universal nondeterminism. For the universal case, we existentially quantify over parameter values and universally over strategies. The insight is that the universal quantification is over a finite domain and may therefore be turned in a conjunction, analogously to the existential case above. However, when applying the conjunction locally at the states, we have to ensure that we do not expect all equalities to hold simultaneously. Instead, we adapt the encoding of the Bellman inequations from parameter-free MDPs. All further ideas are then straightforward analogues. Naturally we have to change the zero-states to the universal case.

Constraints 6 (Upper-bounded reachability, universal nondeterminism).

$$\begin{aligned} p_s &= 1 & & \textit{for all } s \in T, \\ q_s & \textit{is true} & & \textit{for all } s \in T, \\ q_s &\leftrightarrow \bigvee_{\alpha \in \mathbb{A} \mathsf{ct}(s)} \bigvee_{s' \in S} \left(\mathcal{P}(s, \alpha, s') > 0 \to (q_{s'} \land r_s > r_{s'}) \right) & \textit{for all } s \in S \setminus T, \\ \neg q_s &\to p_s &= 0 & & \textit{for all } s \in S \setminus T, \\ q_s &\to \bigwedge_{\alpha \in \mathbb{A} \mathsf{ct}(s)} \left(p_s \geq \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'} \right) & & \textit{for all } s \in S \setminus T. \end{aligned}$$

We refer to the corresponding formula as $\Phi_{\text{wd}}^{\leq,\forall}(\mathcal{M})$.

For lower-bounded reachability, it suffices to change the zero-state computation and the inequalities on the probabilities. We refer to the corresponding formula as $\Phi_{\mathrm{wd}}^{\trianglerighteq,\forall}(\mathcal{M})$. The accompanying encoding is then similar to the existential case

Theorem 7. Let \mathcal{M} be a pMDP with target states T. Let $R \subseteq \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ be a semialgebraic set given by ψ_R . For $\bowtie \in \{\leq, <, \geq, >\}$ we define

$$\psi \coloneqq \Phi_{\mathrm{wd}}^{\bowtie, \forall}(\mathcal{M}) \wedge p_{\iota} \bowtie 1/2 \wedge \psi_R$$

Then, for any $val \in Val$,

$$\text{val } \textit{satisfies } \psi \textit{ iff } \text{val} \in R \land \forall \ \sigma \in \Sigma^{\mathcal{M}}: \ Pr^{\sigma}_{\mathcal{M}[\text{val}]}(\lozenge T) \bowtie {}^{1}\!/2.$$

5.3. Lower bounds

Following the results from the previous sections, we have an **ETR** upper bound. For Boolean valuations, we even have an **NP** upper bound by guessing which parameters are assigned to one. Therefore, the lower bound for the qualitative and the upper bound for the quantitative case coincide.

We first reduce some entries in the table to each other. The remainder of this section then first considers hardness results for the case of arbitrary parameters, and then shows better upper bounds for the case where the number of parameters is fixed.

Considerations for the comparison relations. The number of combinations that we need to consider is significantly reduced by a couple of reductions that follow from structural properties of pMCs. Intuitively, the first property is based on the duality of target and bad states:

Proposition 8. For every $Q \in \{\exists, \forall\}$, there are polynomial-time many-one reductions

- \bullet among the problems $\mathcal{Q}\mathtt{REACH}^>_\mathtt{gp}$ and $\mathcal{Q}\mathtt{REACH}^<_\mathtt{gp}$ and
- among the problems $QREACH_{gp}^{\geq}$ and $QREACH_{gp}^{\leq}$.

Proof. We prove only the first item for $Q = \exists$. All other cases may be proven analogously. First, we deduce from [2, Thm. 10.122 and Thm. 10.127] that, in polynomial time, and without regarding the actual transition probabilities, we can compute from \mathcal{M} and a target set T, a target set T' such that for each $M \in \langle \mathcal{M} \mid \text{gp} \rangle$:

$$\max_{\sigma \in \Sigma^{\mathcal{M}}} Pr_{M}^{\sigma}(\lozenge T) = 1 - \min_{\sigma \in \Sigma^{\mathcal{M}}} Pr_{M}^{\sigma}(\lozenge T').$$

Please observe that the step above in general does not work without the restriction to graph-preserving instantiations. We combine this to obtain:

$$\begin{split} \exists \ \mathsf{val} \in \mathsf{Val}^{\mathrm{gp}}_{\mathcal{M}}, \ \exists \ \sigma \in \Sigma^{\mathcal{M}}: \ Pr^{\sigma}_{M[\mathsf{val}]}(\lozenge T) > \frac{1}{2} \\ \iff \exists \ \mathsf{val} \in \mathsf{Val}^{\mathrm{gp}}_{\mathcal{M}}: \ \max_{\sigma \in \Sigma^{\mathcal{M}}} Pr^{\sigma}_{M[\mathsf{val}]}(\lozenge T) > \frac{1}{2} \\ \iff \exists \ \mathsf{val} \in \mathsf{Val}^{\mathrm{gp}}_{\mathcal{M}}: \ \left(1 - \min_{\sigma \in \Sigma^{\mathcal{M}}} Pr^{\sigma}_{M[\mathsf{val}]}(\lozenge T')\right) > \frac{1}{2} \\ \iff \exists \ \mathsf{val} \in \mathsf{Val}^{\mathrm{gp}}_{\mathcal{M}}: \ \min_{\sigma \in \Sigma^{\mathcal{M}}} Pr^{\sigma}_{M[\mathsf{val}]}(\lozenge T') < \frac{1}{2} \\ \iff \exists \ \mathsf{val} \in \mathsf{Val}^{\mathrm{gp}}_{\mathcal{M}}, \exists \ \sigma \in \Sigma^{\mathcal{M}}: \ Pr^{\sigma}_{M[\mathsf{val}]}(\lozenge T') < \frac{1}{2}. \end{split}$$

For strict lower-bounded reachability, we can restrict our attention to graph-preserving parameter instantiations.

Proposition 9. REACH $_{\mathrm{wd}}^{>}$ is polynomially reducible to REACH $_{\mathrm{gp}}^{>}$.

This proposition is an immediate consequence of the semi-continuity of the solution function for simple pMCs [36, Thm. 5]. Conversely, we can also construct gadgets that avoid valuations which are not graph-preserving. Using the gadget in Figure 9 we can ensure that for any non graph-preserving instantiation, the probability to reach the target is 0, while the reachability probabilities for

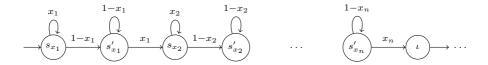


Figure 9: Gadget for the reduction from Prop. 10. ι is the initial state of the given pMC.

graph-preserving instantiations are not affected. Together with semi-continuity of the solution function, we deduce:

Proposition 10. REACH $_{\rm gp}^{>}$ is polynomially reducible to REACH $_{\rm wd}^{>}$, and similarly REACH $_{\rm gp}^{\geq}$ is polynomially reducible to REACH $_{\rm wd}^{\geq}$.

Proof. Let \mathcal{D} be a simple pMC. We extend \mathcal{D} with the gadget outlined in Figure 9. Formally, we construct a pMC \mathcal{D}' with states $S' := S \cup \{s_x, s_x' \mid x \in X\}$, initial state s_{x_1} and

$$\mathcal{P}'(s,s') := \begin{cases} \mathcal{P}(s,s) & \text{if } s,s' \in S, \\ x & \text{if } s = s' = s_x, \\ 1-x & \text{if } s = s' = s'_x, \\ 1-x & \text{if } s = s_x \text{ and } s' = s'_x, \\ x & \text{if } s = s'_x \text{ and } s' = \text{next}(s'_x), \\ 0 & \text{otherwise.} \end{cases}$$

where $\operatorname{next}(s_x')$ is s_{x+1} if $x=x_i$ for some i<|X|, and ι if i=|X|, where ι is the initial state of \mathcal{D} . The pMC \mathcal{D}' is only linearly larger than \mathcal{D} . Observe that the construction of the gadget may be adapted for non-simple pMCs (with different well-defined parameter valuations). By construction, we have for every $\operatorname{val} \in \operatorname{Val}_{\mathcal{D}}^{\operatorname{wd}}$ and $T \subseteq S$:

$$Pr_{\mathcal{D}'[\mathsf{val}]}(\lozenge T) = Pr_{\mathcal{D}'[\mathsf{val}]}(\lozenge \{\iota\}) \cdot Pr_{\mathcal{D}[\mathsf{val}]}(\lozenge T).$$

We observe the following:

$$\forall \ \mathsf{val} \in \mathsf{Val}^{\mathrm{gp}}_{\mathcal{D}} : Pr_{\mathcal{D}'[\mathsf{val}]}(\lozenge\{\iota\}) = 1 \ \text{ and thus } \ Pr_{\mathcal{D}'[\mathsf{val}]}(\lozenge T) = Pr_{\mathcal{D}[\mathsf{val}]}(\lozenge T)$$

and

$$\forall \mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}} \setminus \mathsf{Val}^{\mathrm{gp}}_{\mathcal{M}} : Pr_{\mathcal{D}'[\mathsf{val}]}(\Diamond\{\iota\}) = 0 \text{ and thus } Pr_{\mathcal{D}'[\mathsf{val}]}(\Diamond T) = 0.$$

Together, we deduce:

$$\exists \; \mathsf{val} \in \mathsf{Val}^{\mathrm{gp}}_{\mathcal{D}} : Pr_{\mathcal{D}[\mathsf{val}]}(\lozenge T) \trianglerighteq {}^{1\!\!}/_{2} \quad \Longleftrightarrow \quad \exists \; \mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{D}'} : Pr_{\mathcal{D}'[\mathsf{val}]}(\lozenge T) \trianglerighteq {}^{1\!\!}/_{2}.$$

We are not aware of any such reductions for upper bounds.

 $^{^4}$ Which is some adequate union of particular maximal end components in $\mathcal{M}.$

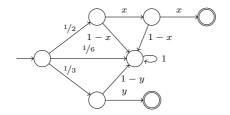


Figure 10: pMC for the polynomial $\frac{1}{2}x^2 + \frac{1}{3}y$

5.3.1. pMCs with arbitrarily many parameters

We first consider the upper right part of Table 2: reachability in pMCs with an unbounded number of parameters.

Non-strict inequalities. First, we establish the following theorem.

Theorem 8. The REACH $_*^{\leq}$ and REACH $_*^{\geq}$ problems are all **ETR**-complete even for acyclic pMCs.

For this result, we reduce from the following ETR-hard problem.

Definition 12. The modified-closed-bounded-4-feasibility (mb4FEAS-c) problem asks: Given a (non-negative) polynomial f of degree 4, does there exist some val: $X \to [0,1]$ such that $f[val] \le 0$? The modified-open-bounded-4-feasibility (mb4FEAS-o) problem is analogously defined with val ranging over (0,1).

This problem easily reduces to its \geq -variant by multiplying f with -1.

Lemma 3. The mb4FEAS-c and mb4FEAS-o problems are ETR-hard.

Proof sketch. Essentially, one reduces from the existence of common roots of quadratic polynomials lying in a unit ball, which is **ETR**-complete [47, Lemma 3.9]. The reduction to mb4FEAS follows the reduction⁵ between unconstrained variants (i.e., variants in which the position of the root is not constrained) of the same decision problem [48, Lemma 3.2]. □

Before presenting a proof of our **ETR**-hardness claim we recall the following result hinted at by Chonev [15]. More precisely, we consider the question: Given a polynomial f, does there exist a (simple, acyclic) pMC such that $\mathsf{sol}_T^{\mathcal{D}} = f$? We start with a positive example.

Example 12. The polynomial $f = \frac{1}{2}x^2 + \frac{1}{3}y$ corresponds to the solution function of the pMC in Figure 10.

The polynomial in the example is easy to translate. In particular, all coefficients are positive and they sum up to a value less than one. In the pMC, all transitions of the form 1-x (for any parameter x) go to the sink state immediately. To handle negative coefficients, we are going to make a more flexible use of the 1-x transitions. We first reformulate the polynomials.

 $^{^5}$ Essentially the polynomial f in mb4FEAS is constructed by taking the sum-of-squares of the quadratic polynomials, and further operations are adequately shifting the polynomial.

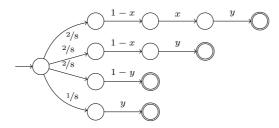


Figure 11: pMC with $_{\text{gp}}^{\mathbb{P}}\mathsf{sol}_{T}^{\mathcal{D}} = -2x^{2}y + y + 2/8$

Lemma 4. ([15, Remark 1]) Let $f \in \mathbb{Q}[x]$ be a polynomial. We can rewrite f as:

$$f = \sum_{i=1}^{m} a_i \cdot h_i + b \qquad \text{with } h_i = \prod_{1 \le j \le k} x_j^{e_{i,j}} \cdot (1 - x_j)^{e'_{i,j}}$$
 (2)

with $a_i \in \mathbb{Q}_{>0}$, $e_{i,j}, e'_{i,j} \in \mathbb{N}$, and $b \in \mathbb{Q}$.

Proof. Observe that a monomial $-x_1 \cdot \cdots \cdot x_d$ of degree $d \geq 0$ may be written as

$$-x_1 \cdot \dots \cdot x_d = -1 + \sum_{i=1}^d (1 - x_i) \cdot x_{i+1} \cdot \dots \cdot x_d, \tag{3}$$

which is proved by induction on d: For d = 0, both sides are -1 (an empty product equals 1). For $d \ge 0$, we multiply both sides of (3) by x_{d+1} to obtain

$$-x_1 \cdot \dots \cdot x_d \cdot x_{d+1} = -x_{d+1} + \sum_{i=1}^d (1 - x_i) \cdot x_{i+1} \cdot \dots \cdot x_d \cdot x_{d+1}$$

$$= (1 - x_{d+1}) - 1 + \sum_{i=1}^d (1 - x_i) \cdot x_{i+1} \cdot \dots \cdot x_d \cdot x_{d+1}$$

$$= -1 + \sum_{i=1}^{d+1} (1 - x_i) \cdot x_{i+1} \cdot \dots \cdot x_d \cdot x_{d+1}.$$

Hence applying (3) to every term of f we obtain Equation (2) where the a_i are positive rational coefficients, the h_i are nonempty products of terms from $\{x, (1-x) \mid x \in X\}$ and $b \in \mathbb{Q}$ is a constant term. We may assume that $b \leq 0$, otherwise $b = b \cdot x + b \cdot (1-x)$ for any $x \in X$ and we may "pull" b inside the sum.

We will show that this reformulation allows to translate and scale a polynomial f such that there exists a pMC \mathcal{D} with targets T and

$$\frac{f+A}{B} = \operatorname{sol}_T^{\mathcal{D}} \quad \text{ for some } A \in \mathbb{Q}_{\geq 0}, B \in \mathbb{Q}_{> 0}$$

Example 13. Consider the polynomial $-2x^2y + y$. We reformulate this to:

$$2 \cdot ((1-x)xy + (1-x)y + (1-y) - 1) + y$$

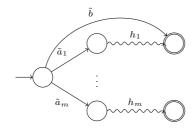


Figure 12: The essential construction of the pMC in Proposition 11: Any probability mass not drawn goes to a sink.

and then to

$$2 \cdot (1-x)xy + 2 \cdot (1-x)y + 2 \cdot (1-y) + y - 2.$$

After shifting upwards (with +2) and rescaling (with $\frac{1}{8}$), we can construct the pMC \mathcal{D} depicted in Figure 11.

Formally, we show the following slightly more general proposition.

Proposition 11. ([15]) Let f be a polynomial. For any A and B sufficiently large, there exists a pMC D with targets T such that

$$\frac{f+A}{B} = \operatorname{sol}_T^{\mathcal{D}}.$$

Moreover, if d is the total degree of f, t the number of terms in f and κ a bound on the (bit-)size of the coefficients and the thresholds μ , λ , then \mathcal{D} may be constructed in time $\mathcal{O}(\text{poly}(d,t,\kappa))$.

Proof. Recall that f may be written as

$$f = \sum_{i=1}^{m} a_i \cdot h_i + b$$
 with $a_i \in \mathbb{Q}_{\geq 0}$ and $b \in \mathbb{Q}_{< 0}$.

Let A > b. We reformulate:

$$f + A = \sum_{i=1}^{m} a_i \cdot h_i + b'$$
 with $a_i \in \mathbb{Q}_{\geq 0}$ and $b' = (b + A) \in \mathbb{Q}_{\geq 0}$.

Let $B > \sum_{i=1}^{m} a_i + b'$. We can write

$$\tilde{f} := \frac{f+A}{B} = \sum_{i=1}^{m} \tilde{a_i} \cdot h_i + \tilde{b}$$

with $\tilde{a}_i, \tilde{b} \in \mathbb{Q}_{\geq 0}$ and $\sum_{i=1}^m \tilde{a}_i + \tilde{b} < 1$. The modified polynomial \tilde{f} naturally corresponds to a simple acyclic pMC $\tilde{\mathcal{D}}$ with $\mathsf{sol}_T^{\mathcal{D}} = \tilde{f}$ as shown in Figure 12.

For the complexity of the construction, notice that m in the sum (2) is in $\mathcal{O}(td)$ where d and t are bounds on the total degree and the number of terms of f, respectively. The h_i are products of at most d terms. The a_i are the absolute values of the original coefficients of f and b is the sum of at most t of those coefficients. Hence a_i , b, A, B and the polynomial \tilde{f} may be computed in time $\mathcal{O}(poly(t,d,\kappa))$. The same then also holds for the pMC $\tilde{\mathcal{D}}$.

Proof of Theorem 8. The reduction from mb4FEAS-c to REACH[≤]_{wd} consists in constructing for a given polynomial a pMC using Propostion 11 with $\mu=0$ and $\lambda = \frac{1}{2}$. For REACH \leq , we reduce from the open variant and notice that as the construction in Propostion 11 preserves all satisfying instantiations val: $X \rightarrow$ [0,1] it, in particular, also preserves them on the graph-preserving parameter valuations. For \geq , we apply Proposition 11 on -f.

Observe that there are instances of the mb4FEAS problems which admit a unique solution, and this solution may be irrational. In contrast, if there exists a solution for a constraint f > 0, then there exist infinitely many (rational) solutions. To the best of our knowledge, the complexity of a variant of these problems with strict bounds is open. Therefore, we have no ETR-hardness proof for REACH with strict bounds. In general, conjunctions of strict inequalities are also ETR-complete [48]. We exploit this in the proof of Theorem 10.

Strict inequalities. We now move to the problems with strict inequalities.

Theorem 9. REACH $\stackrel{>}{_{\star}}$ and REACH $\stackrel{<}{_{\star}}$ are **NP**-hard.

Using Propositions 10 and 9, we may restrict our attention to well-defined parameter valuation sets. Recall Proposition 6: Almost-sure reachability is **NP**-hard. A more refined analysis of the 3SAT-reduction yields:

Proposition 12. REACH $_{\mathrm{wd}}^{>}$ and REACH $_{\mathrm{wd}}^{<}$ are **NP**-hard.

Proof sketch. Reconsider the construction in Figure 6. We first show the following claim to simplify our proof afterwards:

Auxiliary claim. If ψ is unsatisfiable, then for all $\mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{D}}$ there exists some clause c_{i^*} , such that $\mathcal{P}(l_{i^*,j},\perp)[\mathsf{val}] \geq \frac{1}{2}$ for all $j \in \{1,2,3\}$, or more formally

 ψ is unsatisfiable

$$\Longrightarrow \forall \operatorname{val} \in \operatorname{Val}^{\operatorname{wd}}_{\mathcal{D}}, \ \exists i^* \in \{1, \dots, m\}, \forall j \in \{1, 2, 3\}: \ \mathcal{P}(l_{i^*, j}, \bot)[\operatorname{val}] \ge \frac{1}{2}. \tag{4}$$

Proof of the auxiliary claim. Let ψ be satisfiable and assume towards contradiction that for some val and for every clause c_i there is a 'witness' literal $l_{i,j}$ with $\mathcal{P}(l_{i,j}, \perp)[\mathsf{val}] < \frac{1}{2}$. Together with the definition of \mathcal{P} , we conclude either 1. $l_{i,j}$ is a variable x, and $\mathsf{val}(\tilde{x}) > \frac{1}{2}$ or

- 2. $l_{i,j}$ is a negated variable \overline{x} , and $val(\tilde{x}) < \frac{1}{2}$.

We now construct a satisfying assignment for ψ : Consider an assignment val_{ψ} for ψ , with

$$\mathsf{val}_{\psi}(x) \coloneqq \begin{cases} \mathsf{true}, & \text{if } \mathsf{val}(\tilde{x}) > \frac{1}{2}, \\ \mathsf{false}, & \text{if } \mathsf{val}(\tilde{x}) < \frac{1}{2}, \\ \text{arbitrary}, & \text{if } \mathsf{val}(\tilde{x}) = \frac{1}{2}. \end{cases}$$

In both case 1 and 2 above, val_{ψ} satisfies clause c_i . Thus ψ is satisfiable, contradiction.

Proof for correctness of reduction. We only show:

$$\psi$$
 is unsatisfiable $\iff \forall \operatorname{val} \in \operatorname{Val}_{\mathcal{D}}^{\operatorname{wd}} : \operatorname{Pr}_{\mathcal{D}[\operatorname{val}]}(\lozenge T) \le \frac{2}{3},$ (5)

which is equivalent to:

$$\psi \text{ is satisfiable } \Longleftrightarrow \ \exists \ \mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{D}} : Pr_{\mathcal{D}[\mathsf{val}]}(\lozenge T) > \frac{2}{3}.$$

Again let ψ be unsatisfiable and fix a parameter valuation val and set $D := \mathcal{D}[\text{val}]$. We show $Pr_D(\lozenge T) \leq \frac{2}{3}$. Let i^* be like in the auxiliary claim (4). The idea here is that c_{i^*} is the (potentially only) unsatisfied clause. By construction of \mathcal{D} and the auxiliary claim,

$$Pr_D(l_{i^*,j} \models \lozenge^{>0} v_k) \le 1 - \mathcal{P}(l_{i^*,j},\bot)[\mathsf{val}] \le \frac{1}{2}$$

for all $j \in \{1, 2, 3\}$. Hence

$$Pr_D(c_{i^*} \models \lozenge^{>0} \ v_k) = \sum_{i=1}^3 \mathsf{val}(y_{i^*,j}) \cdot Pr_D(l_{i^*,j} \models \lozenge^{>0} \ v_k) \leq \frac{1}{2} \sum_{i=1}^3 \mathsf{val}(y_{i^*,j}) = \frac{1}{2}.$$

Consequently, for $Pr_D(v_k \models \lozenge^{>0} v_k)$ it holds that

$$Pr_{D}(v_{k} \models \lozenge^{>0} v_{k}) = \mathcal{P}(v_{k}, c_{i^{*}}) \cdot Pr_{D}(c_{i^{*}} \models \lozenge^{>0} v_{k})$$

$$+ \sum_{i \neq i^{*}} \mathcal{P}(v_{k}, c_{i}) \cdot Pr_{D}(c_{i} \models \lozenge^{>0} v_{k})$$

$$\leq \frac{1}{m+1} \cdot \frac{1}{2} + \frac{m-1}{m+1} = \frac{2m-1}{2(m+1)}.$$

Plugging this into the equation

$$Pr_D(v_k \models \lozenge^{>0} T) = \frac{1}{m+1} + Pr_D(v_k \models \lozenge^{>0} v_k) \cdot Pr_D(v_k \models \lozenge^{>0} T)$$

yields $Pr_D(v_k \models \lozenge^{>0} T) \leq \frac{2}{3}$. All paths from v_0 to T go through v_k , thus:

$$Pr_D(\lozenge T) = Pr_D(v_0 \models \lozenge^{>0} v_k) \cdot Pr_D(v_k \models \lozenge^{>0} T) \le Pr_D(v_k \models \lozenge^{>0} T) \le \frac{2}{3}.$$

The remainder of the proof is analogous to the proof of Proposition 6. The proof for threshold $^{1}/_{2}$ follows by applying the argument sketched on page 18 \Box

5.3.2. pMDPs with arbitrarily many parameters

We now move to the lower-right corner of Table 2, and consider pMDPs without bound on the number of parameters. For the results, we distinguish whether the quantifier over the strategies is existential or universal.

Existential nondeterminism. We remove the nondeterminism by reducing to pMCs with additional variables for the nondeterminism. (Recall such local resolution of the nondeterminism is valid because of Proposition 1.) This reduction however requires an arbitrary range for the parameters. More formally, we obtain:

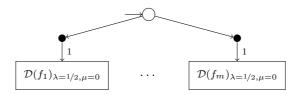


Figure 13: Construction for the proof of Theorem 10

Proposition 13. There are polynomial-time many-one reductions among $REACH_{wd}^{\bowtie}$ and $\exists REACH_{wd}^{\bowtie}$.

Minor adaptions to Proposition 1 and Proposition 10 yield:

Proposition 14. There are polynomial-time many-one reductions among the problems $\exists REACH_{gp}^{>}$ and $REACH_{wd}^{>}$.

Universal nondeterminism. We now consider universal nondeterminism. Contrary to pMCs, we obtain **ETR**-completeness for pMDPs and any comparison relation:

Theorem 10. $\forall REACH_*^{\bowtie}$ are all ETR-complete even for acyclic pMDPs.

Non-strict relations are already trivially **ETR**-hard via Theorem 8. For the strict relations, we reduce from the following problem.

Definition 13. The bounded-conjunction-of-inequalities (bcon4INEQ-c) problem asks: Given a family of polynomials f_1, \ldots, f_m of degree 4, does there exist some val: $X \to [0,1]$ such that $\bigwedge_{i=1}^m f_i[\text{val}] < 0$? The open variant (bcon4INEQ-o) may be defined analogously.

By a straightforward reduction from mb4FEAS (adapted from [48, Thm 4.1]) we obtain that:

Lemma 5. The bcon4INEQ-o/c problems are ETR-hard.

Proof of Theorem 10. We show the reduction from the bcon4INEQ problems to $\forall \text{REACH}_{\text{wd}}^{>}$. For given f_1, \ldots, f_m , we construct pMCs

$$\mathcal{D}(f_1)_{\lambda=1/2,\mu=0},\ldots,\mathcal{D}(f_m)_{\lambda=1/2,\mu=0}$$
 with target states T_i

by applying Proposition 11 to f_i (with $\lambda = \frac{1}{2}$ and $\mu = 0$). Then, we construct a pMDP as outlined in Figure 13. We take the disjoint union of the pMCs and adding a fresh initial state, with nondeterministic actions into each pMC. Formally, let $\mathcal{D}(f_i)_{\lambda=1/2,\mu=0} = (S_i, \iota^i, X, \mathcal{P}_i)$. We construct a pMDP $\mathcal{M} := (S, \iota, Act, X, \mathcal{P})$ with

$$S := \bigcup S_i \cup \{s_0\}, \iota := s_0, Act := \{\alpha_i \mid 1 \le i \le m\},\$$

and \mathcal{P} given by:

$$\mathcal{P}(s, \alpha, s') \coloneqq \begin{cases} \mathcal{P}_i(s, s') & \text{if } s, s' \in S_i, \alpha = \alpha_i \text{ for some } i, \\ 1 & \text{if } s = s_0, s' = \iota^i, \alpha = \alpha_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

We consider target states $T := \bigcup T_i$. The construction is in polynomial time. The pMDP \mathcal{M} has m strategies $\sigma_1, \ldots, \sigma_m$ with $\sigma_i := \{s_0 \mapsto \alpha_i\}$ (all other states have trivial nondeterminism).

By construction, there exists $val \in Val_{\mathcal{M}}^{wd}$ such that:

$$Pr_{M[\mathrm{val}]}^{\sigma_i}(\lozenge T) < \frac{1}{2} \quad \mathrm{iff} \quad f_i[\mathrm{val}] < 0.$$

Then,

$$\exists \; \mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}: \; \bigwedge_{i} Pr^{\sigma_{i}}_{\mathcal{M}[\mathsf{val}]}(\lozenge T) < \frac{1}{2} \quad \mathrm{iff} \quad \exists \; \mathsf{val} \in [0,1]^{X} \; \bigwedge_{i} f_{i}[\mathsf{val}] < 0,$$

or equivalently,

$$\exists \; \mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}, \; \forall \sigma \in \Sigma^{\mathcal{M}}: \; Pr^{\sigma_i}_{\mathcal{M}[\mathsf{val}]}(\lozenge T) < \frac{1}{2} \quad \mathrm{iff} \quad \exists \; \mathsf{val} \in [0,1]^X \; \bigwedge_i f_i[\mathsf{val}] < 0.$$

5.4. Upper bounds with a fixed number of parameters

While the **ETR**-completeness may be considered bad news, as it renders the problem intractable in general, there is also good news. In particular, for any fixed number of parameters, the (parametric) complexity is lower.

In our considerations, we focus on graph-preserving instantiations, as the analysis of pMDP \mathcal{M} and $\mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}$ corresponds to analysing constantly many pMDPs on $\mathsf{Val}^{\mathrm{gp}}_{\mathcal{M}'}$ — see Lemma 2.

Theorem 11 (From [31, 1]). For any fixed number K, given a pMC \mathcal{D} with at most K parameters, determining whether there is a $D \in \langle \mathcal{D} \rangle$ such that $Pr_D(\lozenge T) \bowtie \lambda$ is in P.

That is, in the fixed parameter case, REACH $^{\bowtie}$ is in **P**.

pMDPs. With the positive result for pMCs in place, we turn our attention to pMDPs. However, we can no longer simply eliminate all state variables in the ETR encoding. Observe that a reduction from pMDPs with existential nondeterminism to pMCs does not work: it requires the introduction of additional parameters (depending on the number of states). Indeed, the precise complexity for the problem remains open. Below, we establish $\bf NP$ -membership for all variants.

For pMDPs with existential nondeterminism, \mathbf{NP} -membership is straightforward.

Proposition 15. In the fixed parameter case, $\exists REACH_*^{\bowtie}$ is in NP.

Proof. Guess a memoryless strategy. The strategy can be stored using polynomially many bits⁶. Construct the induced pMC, and verify it in \mathbf{P} .

For pMDPs with universal nondeterminism, \mathbf{NP} -membership is more involved.

⁶contrary to guessing parameter values, as they are real numbers.

Theorem 12. In the fixed parameter case, $\forall REACH_*^{\bowtie}$ is in **NP**.

The essential trick for **NP**-membership for universal nondeterminism is guessing an optimal strategy and verifying the induced pMC together with checking that the strategy is indeed optimal. For the verification step, we will make use of the following ETR encoding based on the Bellman optimality equations for minimising strategies in parameter-free MDPs. For conciseness, we give it here only for graph-preserving valuations.

Constraints 7. Let \mathcal{M} be a pMDP and consider a set of valuations $R \subseteq \mathsf{Val}_{\mathcal{M}}^{\mathsf{gp}}$. Let $\sigma \in \Sigma^{\mathcal{M}}$ and let $h_s/g_s := \mathsf{sol}_{s \to T}^{\mathcal{M}[\sigma]}$ for any $s \in S$. The constraints over variables for all parameters are:

$$h_s[\mathrm{val}] \cdot \prod_{s'' \neq s} g_{s''}[\mathrm{val}] \ \leq \ \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot h_{s'}[\mathrm{val}] \cdot \prod_{s'' \neq s'} g_{s''}[\mathrm{val}]$$

for all $s \in S$, $\alpha \in Act$.

Proof of Theorem 12. We only give the proof for the \geq -relation, the other cases are analogous. Observe that

$$\begin{split} \exists \ \mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}}, \forall \sigma \in \Sigma^{\mathcal{M}} : Pr^{\sigma}_{\mathcal{M}[\mathsf{val}]}(\lozenge T) \geq 1/2 \\ \iff \exists \ \mathsf{val} \in \mathsf{Val}^{\mathrm{wd}}_{\mathcal{M}} : \min_{\sigma \in \Sigma^{\mathcal{M}}} Pr^{\sigma}_{\mathcal{M}[\mathsf{val}]}(\lozenge T) \geq 1/2, \end{split}$$

which means that it is sufficient and necessary for the answer to the problem to be positive that there be a *somewhere optimal strategy* which, for the valuation for which it is minimal, induces a reachability probability of at least 1/2. Hence, we may guess a somewhere minimal strategy and check its minimality using Constraints 7 with a conjunction that the initial state satisfies the threshold⁷. This conjunction only has parameters X, and can thus be checked in \mathbf{P} .

6. Conclusions

We have given a concise overview of known and new results regarding the complexity of parameter synthesis. In particular, the new results clarify that the general case of parameter synthesis is **ETR**-complete, as, e.g., asking whether (Boolean combination of) polynomials have a common root. These results motivate the usage of SMT solvers for ETR to practically solve parameter synthesis problems. In practice, however, such approaches still lack behind abstraction-refinement based approaches.

Some complexity bounds provided in this paper are not tight. The most interesting problem seems to be a lower bound for parameter synthesis in pMDPs with a single parameter and quantitative reachability. Another question is the precise complexity class of parameter synthesis in pMCs with arbitrarily many parameters and strict bounds on the reachability probability.

Finally, there seems to be a large zoo of practically relevant subclasses of pMDP synthesis problems whose complexity may still be explored.

⁷Technically, one has to find the zero states and make them sinks. Recall that zero states can be computed using graph-based algorithms for pMDPs and MDPs alike [2].

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References

- [1] Christel Baier, Christian Hensel, Lisa Hutschenreiter, Sebastian Junges, Joost-Pieter Katoen, and Joachim Klein. Parametric Markov chains: PCTL complexity and fraction-free Gaussian elimination. *Inf. Comp.*, 2020.
- [2] Christel Baier and Joost-Pieter Katoen. *Principles of Model Checking*. MIT Press, 2008.
- [3] Ezio Bartocci, Radu Grosu, Panagiotis Katsaros, C. R. Ramakrishnan, and Scott A. Smolka. Model repair for probabilistic systems. In *TACAS*, volume 6605 of *LNCS*, pages 326–340. Springer, 2011.
- [4] Daniel S. Bernstein, Robert Givan, Neil Immerman, and Shlomo Zilberstein. The complexity of decentralized control of Markov decision processes. *Math. Oper. Res.*, 27(4):819–840, 2002.
- [5] Luca Bortolussi and Simone Silvetti. Bayesian statistical parameter synthesis for linear temporal properties of stochastic models. In *TACAS*, volume 10806 of *LNCS*, pages 396–413. Springer, 2018.
- [6] John F. Canny. Some algebraic and geometric computations in PSPACE. In STOC, pages 460–467. ACM, 1988.
- [7] Milan Ceska, Frits Dannenberg, Nicola Paoletti, Marta Kwiatkowska, and Lubos Brim. Precise parameter synthesis for stochastic biochemical systems. *Acta Inf.*, 54(6):589–623, 2017.
- [8] Milan Ceska, Christian Hensel, Sebastian Junges, and Joost-Pieter Katoen. Counterexample-driven synthesis for probabilistic program sketches. In *FM*, volume 11800 of *LNCS*, pages 101–120. Springer, 2019.
- [9] Milan Ceska, Nils Jansen, Sebastian Junges, and Joost-Pieter Katoen. Shepherding hordes of markov chains. In *TACAS*, volume 11428 of *LNCS*, pages 172–190. Springer, 2019.
- [10] Krishnendu Chatterjee. Robustness of structurally equivalent concurrent parity games. In FOSSACS, volume 7213 of LNCS, pages 270–285. Springer, 2012.
- [11] Krishnendu Chatterjee, Martin Chmelik, and Jessica Davies. A symbolic SAT-based algorithm for almost-sure reachability with small strategies in POMDPs. In AAAI, pages 3225–3232. AAAI Press, 2016.
- [12] Taolue Chen, Yuan Feng, David S. Rosenblum, and Guoxin Su. Perturbation analysis in verification of discrete-time Markov chains. In *CONCUR*, volume 8704 of *LNCS*, pages 218–233. Springer, 2014.

- [13] Taolue Chen, Ernst Moritz Hahn, Tingting Han, Marta Z. Kwiatkowska, Hongyang Qu, and Lijun Zhang. Model repair for Markov decision processes. In TASE, pages 85–92. IEEE Computer Society, 2013.
- [14] Taolue Chen, Tingting Han, and Marta Z. Kwiatkowska. On the complexity of model checking interval-valued discrete time Markov chains. *Inf. Process. Lett.*, 113(7):210–216, 2013.
- [15] Ventsislav Chonev. Reachability in augmented interval Markov chains. In RP, volume 11674 of LNCS, pages 79–92. Springer, 2019.
- [16] Kai Lai Chung. Markov Chains. Springer, 1967.
- [17] Anne Condon. Computational models of games. ACM Distinguished Dissertations. MIT Press, 1989.
- [18] Murat Cubuktepe, Nils Jansen, Sebastian Junges, Joost-Pieter Katoen, and Ufuk Topcu. Synthesis in pMDPs: A tale of 1001 parameters. In ATVA, volume 11138 of LNCS, pages 160–176. Springer, 2018.
- [19] Conrado Daws. Symbolic and parametric model checking of discrete-time Markov chains. In *ICTAC*, volume 3407 of *LNCS*, pages 280–294. Springer, 2004.
- [20] Christian Dehnert, Sebastian Junges, Nils Jansen, Florian Corzilius, Matthias Volk, Harold Bruintjes, Joost-Pieter Katoen, and Erika Ábrahám. Prophesy: A probabilistic parameter synthesis tool. In CAV, volume 9206 of LNCS, pages 214–231. Springer, 2015.
- [21] Christian Dehnert, Sebastian Junges, Joost-Pieter Katoen, and Matthias Volk. A storm is coming: A modern probabilistic model checker. In *CAV* (2), volume 10427 of *LNCS*, pages 592–600. Springer, 2017.
- [22] Karina Valdivia Delgado, Scott Sanner, and Leliane Nunes de Barros. Efficient solutions to factored MDPs with imprecise transition probabilities. Artif. Intell., 175(9-10):1498–1527, 2011.
- [23] Antonio Filieri, Giordano Tamburrelli, and Carlo Ghezzi. Supporting self-adaptation via quantitative verification and sensitivity analysis at run time. *IEEE Trans. Software Eng.*, 42(1):75–99, 2016.
- [24] Paul Gainer, Ernst Moritz Hahn, and Sven Schewe. Accelerated model checking of parametric Markov chains. In ATVA, volume 11138 of LNCS, pages 300–316. Springer, 2018.
- [25] Sergio Giro, Pedro R. D'Argenio, and Luis María Ferrer Fioriti. Distributed probabilistic input/output automata: Expressiveness, (un)decidability and algorithms. *Theor. Comput. Sci.*, 538:84–102, 2014.
- [26] Robert Givan, Sonia Leach, and Thomas Dean. Bounded-parameter Markov decision processes. *Artif. Intell.*, 122(1-2):71–109, 2000.
- [27] Olle Häggström. Finite Markov Chains and Algorithmic Applications, volume 52 of London Mathematical Society Student Texts. Cambridge University Press, 2002.

- [28] Ernst Moritz Hahn, Tingting Han, and Lijun Zhang. Synthesis for PCTL in parametric Markov decision processes. In NASA Formal Methods, volume 6617 of LNCS, pages 146–161. Springer, 2011.
- [29] Ernst Moritz Hahn, Holger Hermanns, and Lijun Zhang. Probabilistic reachability for parametric Markov models. STTT, 13(1):3–19, 2010.
- [30] R.A. Howard. Dynamic probabilistic systems: Semi-Markov and decision processes. Number 2 in Series in Decision and Control. John Wiley & Sons, 1971.
- [31] Lisa Hutschenreiter, Christel Baier, and Joachim Klein. Parametric Markov chains: PCTL complexity and fraction-free Gaussian elimination. In *Gan-dALF*, volume 256 of *EPTCS*, pages 16–30, 2017.
- [32] Nils Jansen, Florian Corzilius, Matthias Volk, Ralf Wimmer, Erika Ábrahám, Joost-Pieter Katoen, and Bernd Becker. Accelerating parametric probabilistic verification. In QEST, volume 8657 of LNCS, pages 404–420. Springer, 2014.
- [33] Bengt Jonsson and Kim Guldstrand Larsen. Specification and refinement of probabilistic processes. In *LICS*, pages 266–277. IEEE Computer Society, 1991.
- [34] Sebastian Junges. Parameter Synthesis in Markov Models. PhD thesis, RWTH Aachen University, 2020.
- [35] Sebastian Junges, Erika Abraham, Christian Hensel, Nils Jansen, Joost-Pieter Katoen, Tim Quatmann, and Matthias Volk. Parameter synthesis for Markov models. *CoRR*, abs/1903.07993, 2019.
- [36] Sebastian Junges, Nils Jansen, Ralf Wimmer, Tim Quatmann, Leonore Winterer, Joost-Pieter Katoen, and Bernd Becker. Finite-state controllers of POMDPs using parameter synthesis. In *UAI*, pages 519–529. AUAI Press, 2018.
- [37] Lodewijk Kallenberg. Markov Decision Processes. Lecture Notes. University of Leiden, 2011.
- [38] John G Kemeny and J Laurie Snell. Markov Chains. Springer, 1976.
- [39] D. Knuth and A. Yao. The complexity of nonuniform random number generation, In: Algorithms and Complexity: New Directions and Recent Results. Academic Press, 1976.
- [40] Marta Z. Kwiatkowska, Gethin Norman, and David Parker. PRISM 4.0: Verification of probabilistic real-time systems. In CAV, volume 6806 of LNCS, pages 585–591. Springer, 2011.
- [41] Ruggero Lanotte, Andrea Maggiolo-Schettini, and Angelo Troina. Parametric probabilistic transition systems for system design and analysis. *Formal Asp. Comput.*, 19(1):93–109, 2007.

- [42] Alberto Puggelli, Wenchao Li, Alberto L. Sangiovanni-Vincentelli, and Sanjit A. Seshia. Polynomial-time verification of PCTL properties of MDPs with convex uncertainties. In CAV, volume 8044 of LNCS, pages 527–542. Springer, 2013.
- [43] Martin L. Puterman. Markov Decision Processes: Discrete Stochastic Dynamic Programming. John Wiley & Sons, 1994.
- [44] Tim Quatmann, Christian Dehnert, Nils Jansen, Sebastian Junges, and Joost-Pieter Katoen. Parameter synthesis for Markov models: Faster than ever. In ATVA, volume 9938 of LNCS, pages 50–67. Springer, 2016.
- [45] James Renegar. On the computational complexity and geometry of the first-order theory of the reals, part I: introduction. preliminaries. the geometry of semi-algebraic sets. the decision problem for the existential theory of the reals. J. Symb. Comput., 13(3):255–300, 1992.
- [46] Stuart J. Russell and Peter Norvig. Artificial Intelligence A Modern Approach (3. ed.). Pearson Education, 2010.
- [47] Marcus Schaefer. Realizability of graphs and linkages. In *Thirty Essays on Geometric Graph Theory*, pages 461–482. Springer New York, 2013.
- [48] Marcus Schaefer and Daniel Stefankovic. Fixed points, Nash equilibria, and the existential theory of the reals. *Theory Comput. Syst.*, 60(2):172–193, 2017.
- [49] Koushik Sen, Mahesh Viswanathan, and Gul Agha. Model-checking Markov chains in the presence of uncertainties. In TACAS, volume 3920 of LNCS, pages 394–410. Springer, 2006.
- [50] Sven Seuken and Shlomo Zilberstein. Formal models and algorithms for decentralized decision making under uncertainty. AAMAS, 17(2):190–250, 2008.
- [51] Eilon Solan. Continuity of the value of competitive Markov decision processes. *Journal of Theoretical Probability*, 16(4):831–845, 2003.
- [52] Jeremy Sproston. Qualitative reachability for open interval Markov chains. In *RP*, volume 11123 of *LNCS*, pages 146–160. Springer, 2018.
- [53] Nikos Vlassis, Michael L. Littman, and David Barber. On the computational complexity of stochastic controller optimization in POMDPs. TOCT, 4(4):12:1–12:8, 2012.
- [54] Tobias Winkler, Sebastian Junges, Guillermo A. Pérez, and Joost-Pieter Katoen. On the complexity of reachability in parametric Markov decision processes. In *CONCUR*, volume 140 of *LIPIcs*, pages 14:1–14:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.
- [55] Di Wu and Xenofon D. Koutsoukos. Reachability analysis of uncertain systems using bounded-parameter Markov decision processes. Artif. Intell., 172(8-9):945–954, 2008.