

A Primer on Strategic Games

Krzysztof R. Apt

(so *not* Krzystof and *definitely not* Krystof)

CWI, Amsterdam, the Netherlands,

University of Amsterdam

Overview

- Best response,
- Nash equilibrium,
- Weak/strict dominance,
- Iterated elimination of strategies,
- Mixed strategies,
- Variations on the definition,
- Pre-Bayesian games,
- Mechanism design: implementation in dominant strategies.

Strategic Games: Definition

Strategic game for $n \geq 2$ players:

- (possibly infinite) set S_i of strategies,
- payoff function $p_i : S_1 \times \dots \times S_n \rightarrow \mathbb{R}$,

for each player i .

Basic assumptions:

- players choose their strategies simultaneously,
- each player is rational: his objective is to maximize his payoff,
- players have common knowledge of the game and of each others' rationality.

Three Examples

Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

The Battle of the Sexes

	F	B
F	2, 1	0, 0
B	0, 0	1, 2

Matching Pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Three Main Concepts

● **Notation:** $s_i, s'_i \in S_i$, $s, s', (s_i, s_{-i}) \in S_1 \times \dots \times S_n$.

● s_i is a **best response** to s_{-i} if

$$\forall s'_i \in S_i \quad p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

● s is a **Nash equilibrium** if $\forall i$ s_i is a best response to s_{-i} :

$$\forall i \in \{1, \dots, n\} \quad \forall s'_i \in S_i \quad p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

Intuition: In a Nash equilibrium no player can gain by *unilaterally* switching to another strategy.

● s is **Pareto efficient** if for no s'

$$\begin{aligned} \forall i \in \{1, \dots, n\} \quad p_i(s') &\geq p_i(s), \\ \exists i \in \{1, \dots, n\} \quad p_i(s') &> p_i(s). \end{aligned}$$

Nash Equilibrium

Prisoner's Dilemma: 1 Nash equilibrium

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

The Battle of the Sexes: 2 Nash equilibria

	F	B
F	2, 1	0, 0
B	0, 0	1, 2

Matching Pennies: no Nash equilibrium

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Dominance

- s'_i is strictly dominated by s_i if

$$\forall s_{-i} \in S_{-i} \ p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}),$$

- s'_i is weakly dominated by s_i if

$$\begin{aligned} \forall s_{-i} \in S_{-i} \ p_i(s_i, s_{-i}) &\geq p_i(s'_i, s_{-i}), \\ \exists s_{-i} \in S_{-i} \ p_i(s_i, s_{-i}) &> p_i(s'_i, s_{-i}). \end{aligned}$$

Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Why a **dilemma**?

- (D, D) is the unique **Nash equilibrium**,
- For each player C is **strictly dominated** by D ,
- (C, C) is a **Pareto efficient** outcome in which each player has a $>$ payoff than in (D, D) .

Prisoner's Dilemma for n Players

Assume $k_i(n - 1) > l_i > 0$ for all i .

$$p_i(s) := \begin{cases} k_i |s_{-i}(C)| + l_i & \text{if } s_i = D \\ k_i |s_{-i}(C)| & \text{if } s_i = C. \end{cases}$$

- For $n = 2$, $k_i = 2$ and $l_i = 1$ we get the original Prisoner's Dilemma game.
- $p_i(C^n) = k_i(n - 1) > l_i = p_i(D^n)$,
so for all players C^n yields a $>$ payoff than D^n .
- For all players strategy C is strictly dominated by D :
 $p_i(D, s_{-i}) - p_i(C, s_{-i}) = l_i > 0$.

Quiz

	H	T	E
H	1, -1	-1, 1	-1, -1
T	-1, 1	1, -1	-1, -1
E	-1, -1	-1, -1	-1, -1

- What are the Nash equilibria of this game?

Answer

	H	T	E
H	1, -1	-1, 1	-1, -1
T	-1, 1	1, -1	-1, -1
E	-1, -1	-1, -1	-1, -1

- (E, E) is the only Nash equilibrium.
- It is a Nash equilibrium in weakly dominated strategies.

IESDS: Example 1

	L	M	R
T	3, 0	2, 1	1, 0
C	2, 1	1, 1	1, 0
B	0, 1	0, 1	0, 0

- B is strictly dominated by T ,
- R is strictly dominated by M .

By eliminating them we get:

	L	M
T	3, 0	2, 1
C	2, 1	1, 1

IESDS, Example 1ctd

	L	M
T	3, 0	2, 1
C	2, 1	1, 1

Now C is strictly dominated by T , so we get:

	L	M
T	3, 0	2, 1

Now L is strictly dominated by M , so we get:

	M
T	2, 1

We solved the game by IESDS.

IESDS

Theorem

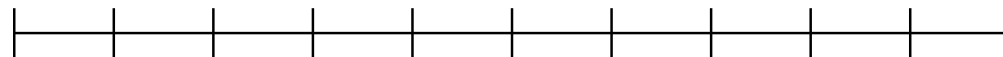
- If G' is an outcome of IESDS starting from a **finite** G , then s is a Nash equilibrium of G' iff it is a Nash equilibrium of G .
- If G is **finite** and is solved by IESDS, then the resulting joint strategy is a **unique** Nash equilibrium of G .
- (**Gilboa, Kalai, Zemel, '90**) Outcome of IESDS is unique (**order independence**).

IESDS: Example

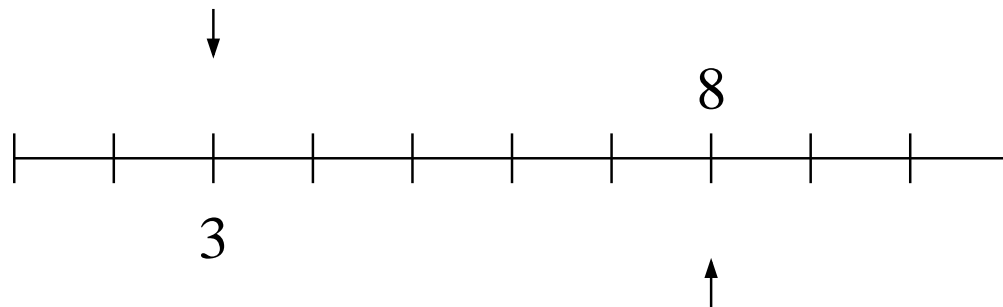
Location game (Hotelling '29)

- 2 companies decide **simultaneously** their **location**,
- customers choose the closest vendor.

Example: Two bakeries, one (discrete) street.



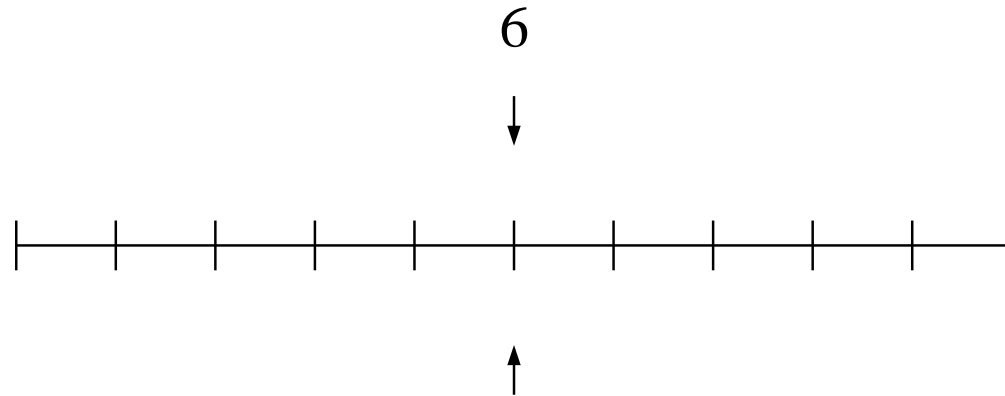
For instance:



Then $\text{baker}_1(3, 8) = 5$, $\text{baker}_2(3, 8) = 6$.

Where do I put my bakery?

Answer



Then:

$$\text{baker}_1(6, 6) = 5.5,$$

$$\text{baker}_2(6, 6) = 5.5.$$

- $(6, 6)$ is the outcome of IESDS.
- Hence $(6, 6)$ is a unique **Nash equilibrium**.

IEWDS

Theorem

- If G' is an outcome of IEWDS starting from a **finite** G and s is a Nash equilibrium of G' , then s is a Nash equilibrium of G .
- If G is **finite** and is solved by IEWDS, then the resulting joint strategy is **a** Nash equilibrium of G .
- Outcome of IEWDS does not need to be unique (**no order independence**).

IEWDS: Example 1

Beauty-contest game (Moulin, '86)

- each set of strategies = $\{1, \dots, 100\}$,
- payoff to each player:
1 is split equally between the players whose submitted number is closest to $\frac{2}{3}$ of the average.

Example

submissions: 29, 32, 29; average: 30,
payoffs: $\frac{1}{2}, 0, \frac{1}{2}$.

- This game is solved by IEWDS.
- Hence it has a Nash equilibrium, namely $(1, \dots, 1)$.

IEWDS: Example 2

The following game has two Nash equilibria:

	X	Y	Z
A	2, 1	0, 1	1, 0
B	0, 1	2, 1	1, 0
C	1, 1	1, 0	0, 0
D	1, 0	0, 1	0, 0

- D is weakly dominated by A ,
- Z is weakly dominated by X .

By eliminating them we get:

	X	Y
A	2, 1	0, 1
B	0, 1	2, 1
C	1, 1	1, 0

Example 2, ctd

	X	Y
A	2, 1	0, 1
B	0, 1	2, 1
C	1, 1	1, 0

Next, we get

	X
A	2, 1
B	0, 1
C	1, 1

and finally

	X
A	2, 1

IEWDS: Example 3

	L	R
T	1, 1	1, 1
B	1, 1	0, 0

can be reduced both to

	L	R
T	1, 1	1, 1

and to

	L
T	1, 1
B	1, 1

Infinite Games

Consider the game with

- $S_i := \mathbb{N}$,
- $p_i(s) := s_i$.

Here

- every strategy is strictly dominated,
- in one step we can eliminate
 - all strategies,
 - all $\neq 0$ strategies,
 - one strategy per player.

Infinite Games (2)

Conclusions For infinite games

- IESDS is **not** order independent,
- definition of order independence has to be modified.

IENBR: Example 1

	X	Y
A	2, 1	0, 0
B	0, 1	2, 0
C	1, 1	1, 2

- No strategy strictly or weakly dominates another one.
- C is **never a best response**.

Eliminating it we get

	X	Y
A	2, 1	0, 0
B	0, 1	2, 0

from which in two steps we get

	X
A	2, 1

IENBR

Theorem

- If G' is an outcome of IENBR starting from a **finite** G , then s is a Nash equilibrium of G' iff it is a Nash equilibrium of G .
- If G is **finite** and is solved by IENBR, then the resulting joint strategy is a **unique** Nash equilibrium of G .
- (**Apt, '05**) Outcome of IENBR is unique (**order independence**).

IENBR: Example 2

Location game on the open real interval $(0, 100)$.

$$p_i(s_i, s_{3-i}) := \begin{cases} s_i + \frac{s_{3-i} - s_i}{2} & \text{if } s_i < s_{3-i} \\ 100 - s_i + \frac{s_i - s_{3-i}}{2} & \text{if } s_i > s_{3-i} \\ 50 & \text{if } s_i = s_{3-i} \end{cases}$$

- No strategy strictly or weakly dominates another one.
- Only 50 is a best response to some strategy (namely 50).
- So this game is solved by IENBR, in one step.

Mixed Extension of a Finite Game

- Probability distribution over a finite non-empty set A :

$$\pi : A \rightarrow [0, 1]$$

such that $\sum_{a \in A} \pi(a) = 1$.

- Notation: ΔA .

Fix a finite strategic game $G := (S_1, \dots, S_n, p_1, \dots, p_n)$.

- Mixed strategy of player i in G : $m_i \in \Delta S_i$.
- Joint mixed strategy: $m = (m_1, \dots, m_n)$.

Mixed Extension of a Finite Game (2)

- Mixed extension of G :

$$(\Delta S_1, \dots, \Delta S_n, p_1, \dots, p_n),$$

where

$$m(s) := m_1(s_1) \cdot \dots \cdot m_n(s_n)$$

and

$$p_i(m) := \sum_{s \in S} m(s) \cdot p_i(s).$$

- **Theorem** (Nash '50) Every mixed extension of a finite strategic game has a Nash equilibrium.

2 Examples

Matching Pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

- $(\frac{1}{2} \cdot H + \frac{1}{2} \cdot T, \frac{1}{2} \cdot H + \frac{1}{2} \cdot T)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: 0.

The Battle of the Sexes

	F	B
F	2, 1	0, 0
B	0, 0	1, 2

- $(\frac{2}{3} \cdot F + \frac{1}{3} \cdot B, \frac{1}{3} \cdot F + \frac{1}{3} \cdot B)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: $\frac{2}{3}$.

Variations on the Definition

- **Strategic games with qualitative preferences**
(Osborne, Rubinstein '94)
 $(S_1, \dots, S_n, \succeq_1, \dots, \succeq_n)$, where each \succeq_i is a **preference relation** on $S_1 \times \dots \times S_n$.
- **Strategic games with parametrized preferences**
(Apt, Rossi, Venable '08)
Each player i has a set of strategies S_i and a preference relation $\succeq(s_{-i})$ on S_i **parametrized** by $s_{-i} \in S_{-i}$.
- **Conversion/preference games**
(Le Roux, Lescanne, Vestergaard '08)
The game consists of a set S of **situations** and for each player i a **preference relation** \succeq_i on S and a **conversion relation** \rightarrow_i on S .

Graphical Games

(Kearns, Littman, Singh '01)

- Each player i has a set of **neighbours** $neigh(i)$.
- Payoff for player i is a function

$$p_i : \times_{j \in neigh(i) \cup \{i\}} S_j \rightarrow \mathbb{R}.$$

Dominance by a Mixed Strategy

Example

	X	Y	Z
A	2, —	0, —	1, —
B	0, —	2, —	1, —
C	1, —	1, —	0, —
D	1, —	0, —	0, —

- D is weakly dominated by A ,
- C is weakly dominated by $\frac{1}{2} \cdot A + \frac{1}{2} \cdot B$,
- D is strictly dominated by $\frac{1}{2} \cdot A + \frac{1}{2} \cdot C$.

Iterated Elimination of Strategies

Consider weak dominance by a mixed strategy.

	X	Y	Z
A	2, 1	0, 1	1, 0
B	0, 1	2, 1	1, 0
C	1, 1	1, 0	0, 0
D	1, 0	0, 1	0, 0

- D is weakly dominated by A ,
- Z is weakly dominated by X ,
- C is weakly dominated by $\frac{1}{2} \cdot A + \frac{1}{2} \cdot B$.

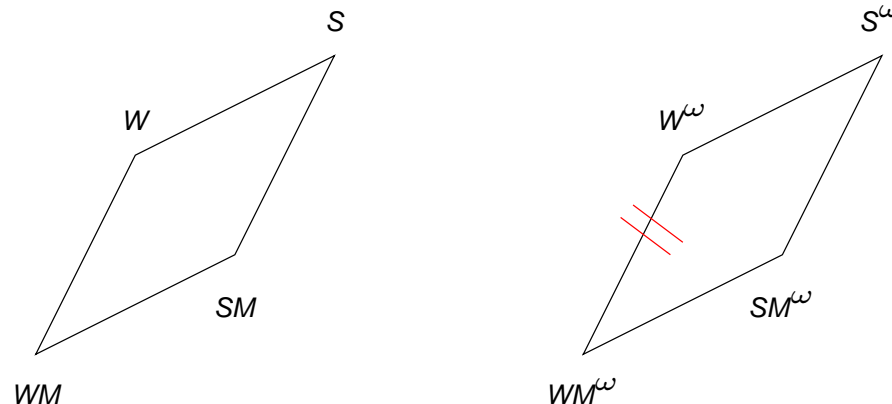
By eliminating them we get the final outcome:

	X	Y
A	2, 1	0, 1
B	0, 1	2, 1

Relative Strength of Strategy Elimination

- Weak dominance by a **pure strategy** is less powerful than weak dominance by a **mixed strategy**, but
- iterated elimination using weak dominance by a **pure strategy** (W^ω) can be more powerful than iterated elimination using weak dominance by a **mixed strategy** (MW^ω).

In general (Apt '07):



Best responses to Mixed Strategies

- s_i is a **best response** to m_{-i} if

$$\forall s'_i \in S_i \quad p_i(s_i, m_{-i}) \geq p_i(s'_i, m_{-i}).$$

- $\text{support}(m_i) := \{a \in S_i \mid m_i(a) > 0\}$.

- **Theorem** (Pearce '84) In a 2-player finite game

- s_i is **strictly dominated** by a mixed strategy iff it is not a **best response** to a mixed strategy.
- s_i is **weakly dominated** by a mixed strategy iff it is not a **best response** to a mixed strategy with **full** support.

IESDMS

Theorem

- If G' is an outcome of IESDMS starting from G , then m is a Nash equilibrium of G' iff it is a Nash equilibrium of G .
- If G is solved by IESDMS, then the resulting joint strategy is a **unique** Nash equilibrium of G .
- (**Osborne, Rubinstein, '94**) Outcome of IESDMS is unique (**order independence**).

IESDMS: Example

Beauty-contest game

- each set of strategies = $\{1, \dots, 100\}$,
- payoff to each player:
1 is split equally between the players whose submitted number is closest to $\frac{2}{3}$ of the average.
- This game is solved by IESDMS, in 99 steps.
- Hence it has a **unique** Nash equilibrium, $(1, \dots, 1)$.

IEWDMS

Theorem

- If G' is an outcome of IEWDMS starting from G and m is a Nash equilibrium of G' , then m is a Nash equilibrium of G .
- If G is solved by IEWDMS, then the resulting joint strategy is a Nash equilibrium of G .
- Outcome of IEWDS does not need to be unique (no order independence).
- Every mixed extension of a finite strategic game has a Nash equilibrium in which no pure strategy is weakly dominated by a mixed strategy.

Rationalizable Strategies

- Introduced in [Bernheim '84](#) and [Pearce '84](#).
- Strategies in the outcome of IENBRM.
- Subtleties in the definition ...

Theorem

- ([Bernheim '84](#)) If G' is an outcome of IENBRM starting from G , then m is a Nash equilibrium of G' iff it is a Nash equilibrium of G .
- If G is solved by IESDMS, then the resulting joint strategy is a [unique](#) Nash equilibrium of G .
- ([Apt '05](#)) Outcome of IENBRM is unique ([order independence](#)).

Pre-Bayesian Games

(Hyafil, Boutilier '04, Ashlagi, Monderer, Tennenholtz '06,)

- In a strategic game after each player selected his strategy each player knows all the payoffs (**complete information**).
- In a **pre-Bayesian game** after each player selected his strategy each player knows only **his** payoff (**incomplete information**).
- This is achieved by introducing (private) **types**.

Pre-Bayesian Games: Definition

Pre-Bayesian game for $n \geq 2$ players:

- (possibly infinite) set A_i of actions,
- (possibly infinite) set Θ_i of (private) types,
- payoff function $p_i : A_1 \times \dots \times A_n \times \Theta_i \rightarrow \mathbb{R}$,

for each player i .

Basic assumptions:

- Nature moves first and provides each player i with a θ_i ,
- players do not know the types received by other players,
- players choose their actions simultaneously,
- each player is rational (wants to maximize his payoff),
- players have common knowledge of the game and of each others' rationality.

Nash Equilibrium

- A **strategy** for player i :

$$s_i(\cdot) \in A_i^{\Theta_i}.$$

- Joint strategy $s(\cdot)$ is a **Nash equilibrium** if each $s_i(\cdot)$ is a best response to $s_{-i}(\cdot)$:

$$\forall \theta \in \Theta \quad \forall i \in \{1, \dots, n\} \quad \forall s'_i(\cdot) \in A_i^{\Theta_i} \\ p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq p_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$$

- **Note:** For each $\theta \in \Theta$ we have **one** strategic game. $s(\cdot)$ is a Nash equilibrium if for each $\theta \in \Theta$ the joint action $(s_1(\theta_1), \dots, s_n(\theta_n))$ is a Nash equilibrium in the θ -game.

Example

• $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,

• $A_1 = A_2 = \{F, B\}$.

		<i>L</i>	
		<i>F</i>	<i>B</i>
<i>U</i>	<i>F</i>	2, 1	0, 0
	<i>B</i>	0, 0	1, 2
		<i>F</i>	<i>B</i>
<i>D</i>	<i>F</i>	0, 0	1, 1
	<i>B</i>	2, 2	0, 0

		<i>R</i>	
		<i>F</i>	<i>B</i>
<i>F</i>	<i>F</i>	0, 0	1, 1
	<i>B</i>	2, 2	0, 0
		<i>F</i>	<i>B</i>
<i>B</i>	<i>F</i>	2, 1	0, 0
	<i>B</i>	0, 0	1, 2

• **Strategies**

$s_1(U) = F$, $s_1(D) = B$,

$s_2(L) = F$, $s_2(R) = B$

form a Nash equilibrium.

But ...

Nash equilibrium does not need to exist in mixed extensions of finite pre-Bayesian games.

Example: Mixed extension of the following game.

• $\Theta_1 = \{U, B\}, \Theta_2 = \{L, R\},$

• $A_1 = A_2 = \{C, D\}.$

		<i>L</i>	
		<i>C</i>	<i>D</i>
<i>U</i>	<i>C</i>	2, 2	0, 3
	<i>D</i>	3, 0	1, 1
<i>B</i>	<i>C</i>	2, 2	0, 3
	<i>D</i>	3, 0	1, 1

		<i>R</i>	
		<i>C</i>	<i>D</i>
<i>C</i>	<i>C</i>	2, 2	0, 3
	<i>D</i>	3, 0	1, 1
<i>D</i>	<i>C</i>	0, 3	2, 2
	<i>D</i>	1, 1	3, 0

Intelligent Design

Economist.com

Search Economist.com

Log in: e-mail Password

☐ Remember me

12 weeks for €26 [SUBSCRIBE NOW >>](#)

Finance & Economics

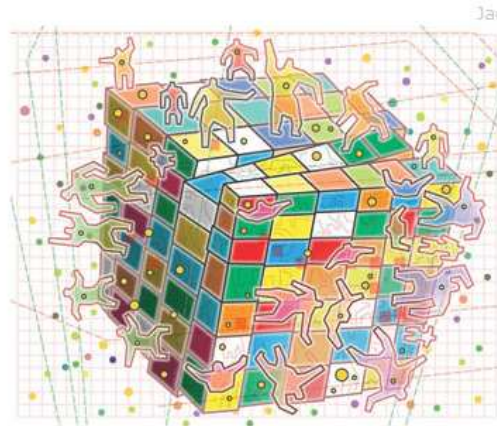
Economics focus

Intelligent design

Oct 18th 2007

From *The Economist* print edition

A theory of an intelligently guided invisible hand wins the Nobel prize



"WHAT on earth is mechanism design?" was the typical reaction to this year's Nobel prize in economics, announced on October 15th. In this era of "Freakonomics", in which everyone is discovering their inner economist, economics has become unexpectedly sexy. So what possessed the Nobel committee to honour a subject that sounds so thoroughly dismal? Why didn't they follow the lead of the peace-prize judges, who know not to let technicalities about being true to the meaning of the award get in the way of good headlines?

In fact, despite its dreary name, mechanism design is a hugely important area of economics, and underpins much of what dismal scientists do today. It goes to the heart of one of the biggest challenges in economics: how to arrange our economic interactions so that, when everyone behaves in a self-interested manner, the result is something we all like. The word "mechanism" refers to the institutions and the rules of the game that govern our economic activities, which can range from a Ministry of Planning in a command economy to the internal organisation of a company to trading in a market.

Intelligent Design

A theory of an intelligently guided invisible hand wins the Nobel prize

WHAT on earth is **mechanism design**? was the typical reaction to this year's Nobel prize in economics, announced on October 15th.

[...]

In fact, despite its dreary name, mechanism design is a hugely important area of economics, and underpins much of what dismal scientists do today. It goes to the heart of one of the biggest challenges in economics: how to arrange our economic interactions so that, **when everyone behaves in a self-interested manner, the result is something we all like.**

(The Economist, Oct. 18th, 2007)

Mechanism Design

Decision problem for n players:

- set D of decisions,
- for each player i a set Θ_i of (private) types Θ_i
- and a utility function

$$v_i : D \times \Theta_i \rightarrow \mathcal{R}.$$

- **Intuition:** $v_i(d, \theta_i)$ represents the benefit to player i of type θ_i from decision $d \in D$.
- When the individual types are $\theta_1, \dots, \theta_n$
 $\sum_{i=1}^n v_i(d, \theta_i)$ represents the social welfare from the decision $d \in D$.

Decision Rules

- **Decision rule** is a function

$$f : \Theta_1 \times \dots \times \Theta_n \rightarrow D.$$

- Decision rule f is **efficient** if

$$\sum_{i=1}^n v_i(f(\theta), \theta_i) \geq \sum_{i=1}^n v_i(d, \theta_i)$$

for all $\theta \in \Theta$ and $d \in D$.

- **Intuition:** f is efficient if it always yields a best decision for the society.

Set up

- each player i receives/has a **type** θ_i ,
- each player i submits to the **central authority** a type θ'_i ,
- the central authority computes **decision**

$$d := f(\theta'_1, \dots, \theta'_n),$$

and communicates it to each player i .

Example: Public Project Problem

- c : cost of the public project (e.g., a bridge),
- $D = \{0, 1\}$,
- each θ_i is \mathbb{R}_+ ,
- $v_i(d, \theta_i) := d(\theta_i - \frac{c}{n})$,
- $f(\theta) := \begin{cases} 1 & \text{if } \sum_{i=1}^n \theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$
- **Note:** f is efficient.

Manipulations

An optimal strategy for individual i :

- if $\theta_i \geq \frac{c}{n}$ submit $\theta'_i = c$.
- if $\theta_i < \frac{c}{n}$ submit $\theta'_i = 0$.

Example $c = 30$

player	type
A	6
B	7
C	25

Revised Set-up: Direct Mechanisms

- each player i receives/has a type θ_i ,
- each player i submits to the central authority a type θ'_i ,
- the central authority computes decision

$$d := f(\theta'_1, \dots, \theta'_n),$$

and taxes

$$(t_1, \dots, t_n) := g(\theta'_1, \dots, \theta'_n) \in \mathbb{R}^n,$$

and communicates to each player i both d and t_i .

- final utility function for player i :

$$u_i(d, \theta_i) = v_i(d, \theta_i) + t_i.$$

Groves Mechanisms

• $t_i(\theta') := \sum_{j \neq i} v_j(f(\theta'), \theta'_j) + h_i(\theta'_{-i})$, where

$h_i : \Theta_{-i} \rightarrow \mathbb{R}$ is an arbitrary function.

• **Intuition:**

$$\sum_{j \neq i} v_j(f(\theta'), \theta'_j)$$

is the **social welfare** with i excluded from decision $f(\theta')$.

• **Note:**

$$u_i((f, t)(\theta), \theta_i) = \sum_{j=1}^n v_j(f(\theta), \theta_j) + h_i(\theta_{-i}).$$

Groves Mechanisms, ctd

- Direct mechanism (f, t) is **incentive compatible** if for all $\theta \in \Theta$, $i \in \{1, \dots, n\}$ and $\theta'_i \in \Theta_i$

$$u_i((f, t)(\theta_i, \theta_{-i}), \theta_i) \geq u_i((f, t)(\theta'_i, \theta_{-i}), \theta_i).$$

- Theorem (**Groves '73**)
Suppose f is efficient. Then each Groves mechanism is **incentive compatible**.
- Groves mechanism with tax function $t := (t_1, \dots, t_n)$ is **feasible** if $\sum_{i=1}^n t_i(\theta) \leq 0$ for all θ .

Special Case: Pivotal Mechanism

- $h_i(\theta_{-i}) := -\max_{d \in D} \sum_{j \neq i} v_j(d, \theta'_j).$

- Then

$$t_i(\theta') := \sum_{j \neq i} v_j(f(\theta'), \theta'_j) - \max_{d \in D} \sum_{j \neq i} v_j(d, \theta'_j) \leq 0.$$

- **Conclusion** Pivotal mechanism is feasible.

Example

Suppose $c = 30$ and $n = 3$.

player	type	tax	u_i
A	6	0	-4
B	7	0	-3
C	25	-7	8

player	type	tax	u_i
A	4	-5	-5
B	3	-6	-6
C	22	0	0

Relation to pre-Bayesian Games

- Strategy $s_i(\cdot)$ is **dominant** if for all $a \in A$ and $\theta_i \in \Theta_i$

$$\forall a \in A \ p_i(s_i(\theta_i), a_{-i}, \theta_i) \geq p_i(a_i, a_{-i}, \theta_i).$$

- A pre-Bayesian game is of a **revelation-type** if $A_i = \Theta_i$ for all $i \in \{1, \dots, n\}$.
- So in a revelation-type pre-Bayesian game the strategies of player i are the functions on Θ_i .
- A strategy for player i is called **truth-telling** if it is the identity function $\pi_i(\cdot)$.

Relation to pre-Bayesian Games, ctd

- Mechanism design (as discussed here) can be viewed as an instance of the revelation-type pre-Bayesian games.
- With each direct mechanism (f, t) we can associate a revelation-type pre-Bayesian game:
 - Each Θ_i as in the mechanism,
 - Each $A_i = \Theta_i$,
 - $p_i(\theta'_i, \theta_{-i}, \theta_i) := u_i((f, t)(\theta'_i, \theta_{-i}), \theta_i)$.
- **Note** Direct mechanism (f, t) is incentive compatible iff in the associated pre-Bayesian game for each player truth-telling is a dominant strategy.
- **Conclusion** In the pre-Bayesian game associated with a Groves mechanism, $(\pi_1(\cdot), \dots, \pi_i(\cdot))$ is a dominant strategy Nash equilibrium.