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Pushdown Tree Automata, Algebraic Tree Systems, and Algebraic Tree Series

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The results of I. Guessarian (1983, *Math. Systems Theory* **16**, 237–263) are generalized to formal tree series. The following mechanisms are of equal power:

(i) algebraic tree systems,
(ii) pushdown tree automata,
(iii) restricted pushdown tree automata.

1. INTRODUCTION AND PRELIMINARIES

Guessarian [11] introduced the notion of a (topdown) pushdown tree automaton and showed that these pushdown tree automata recognize exactly the class of context-free tree languages. Here a tree language is called context-free iff it is generated by a context-free tree grammar. Moreover, she showed that pushdown tree automata are equivalent to restricted pushdown tree automata, i.e., to pushdown automata whose pushdown store is linear.

In this paper, we generalize these results of Guessarian [11] to formal tree series. We define pushdown tree automata whose behavior is a formal tree series and show that the class of behaviors of these pushdown tree automata coincides with the class of algebraic tree series. Here a tree series is called algebraic iff it is the initial component of the least solution of an algebraic tree system with initial function variable. Moreover, we show the equivalence of pushdown tree automata and restricted pushdown tree automata also in the case of formal tree series.

Essentially, the constructions in the case of formal tree series are the same as the constructions of Guessarian [11], while the proofs that these constructions are valid are totally different. The formal tree series approach has the usual advantages over the tree language approach: the proofs are separated from the constructions and are more satisfactory from a mathematical point of view and the results are more general; e.g., the general result includes the equivalence of unambiguous pushdown tree automata and unambiguous algebraic tree systems.

In Section 2, tree automata and linear systems are introduced as a framework for the forthcoming considerations. In Section 3, the equivalence of pushdown tree automata and algebraic tree systems is shown. The equivalence of pushdown tree automata and restricted pushdown tree automata is shown in Section 4. In the last section, we apply the yield-mapping to algebraic tree series and get the macro power series as a generalization of the OI languages generated by macro systems of Fischer [8].

It is assumed that the reader is familiar with the basics of semiring theory (see Kuich and Salomaa [15] and Kuich [13, Sect. 2]). Throughout the paper, $\langle A, +, \cdot, 0, 1\rangle$ denotes a *commutative continuous* semiring. This means:

(o) the multiplication $\cdot$ is commutative;

(i) A is partially ordered by the relation $\sqsubseteq$: $a \sqsubseteq b$ iff there exists a c such that $a + c = b$;

(ii) $\langle A, +, \cdot, 0, 1\rangle$ is a complete semiring;

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(iii) $\sum_{i\in I} a_i = \sup(\sum_{i\in E} a_i \mid E \subseteq I, E$ finite$)$, $a_i \in A$, $i \in I$, for an arbitrary index set I, where sup denotes the least upper bound with respect to $\sqsubseteq$.

Throughout the paper, we denote $\langle A, +, \cdot, 0, 1\rangle$ briefly by A.

Furthermore, $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_k \cup \cdots$ will always denote a *ranked alphabet*, where Σ_k, $k \geq 0$, contains the symbols of rank k and X will denote an *alphabet of leaf symbols*. By $T_\Sigma(X)$ we denote the set of *trees* formed by $\Sigma \cup X$. This set $T_\Sigma(X)$ is the smallest set formed according to the following conventions:

(i) if $\omega \in \Sigma_0 \cup X$ then $\omega \in T_\Sigma(X)$,

(ii) if $\omega \in \Sigma_k$, $k \geq 1$, and $t_1, \ldots, t_k \in T_\Sigma(X)$ then $\omega(t_1, \ldots, t_k) \in T_\Sigma(X)$.

If $\Sigma_0 \neq \emptyset$ then X may be the empty set ($\emptyset$ denotes the empty set).

By $A\langle\langle T_\Sigma(X)\rangle\rangle$ we denote the set of *formal tree series* over $T_\Sigma(X)$, i.e., the set of mappings $s : T_\Sigma(X) \to A$ written in the form $\sum_{t\in T_\Sigma(X)} (s,t)t$, where the coefficient (s,t) is the value of s for the tree $t \in T_\Sigma(X)$. For a formal tree series $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$, we define the *support* of s, $\mathrm{supp}(s) = \{t \in T_\Sigma(X) \mid (s,t) \neq 0\}$. By $A\langle T_\Sigma(X)\rangle$ we denote the set of tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$ that have finite support. A power series with finite support is called *polynomial*. (For more definitions see Kuich [14].)

For $\omega \in \Sigma_k$, $k \geq 0$, we define the mapping $\overline{\omega} : (A\langle\langle T_\Sigma(X)\rangle\rangle)^k \to A\langle\langle T_\Sigma(X)\rangle\rangle$ by

$$\overline{\omega}(s_1, \ldots, s_k) = \sum_{t_1,\ldots,t_k \in T_\Sigma(X)} (s_1, t_1) \cdots (s_k, t_k)\omega(t_1, \ldots, t_k),$$

$s_1, \ldots, s_k \in A\langle\langle T_\Sigma(X)\rangle\rangle$.

EXAMPLE 1.[2] If Σ is a finite ranked alphabet and X is a finite alphabet of leaf symbols then $T_\Sigma(X)$ is generated by the context-free grammar $G = (\{S\}, \Sigma \cup X, P, S)$, where $P = \{S \to \omega(S, \ldots, S) \mid \omega \in \Sigma_k, k \geq 1\} \cup \{S \to \omega \mid \omega \in \Sigma_0 \cup X\}$.

Sometimes it is more suggestive to employ a pictorial representation: The tree $\omega \in \Sigma_0 \cup X$ represents the rooted plane tree with just a single node labeled by ω; the tree $\omega(t_1, \ldots, t_k)$, $\omega \in \Sigma_k$, $t_1, \ldots, t_k \in T_\Sigma(X)$, $k \geq 1$, represents the rooted plane tree where the root is labeled by ω and has sons $t_1, \ldots, t_k$ (in this order).

Formal tree series have the advantage that the coefficient of a tree in a series can be used to give information about some quantity connected with that tree.

(i) (See Example 2.1 of Berstel and Reutenauer [3]) Define the height $h(t)$ of a tree t in $T_\Sigma(X)$ as follows:

$$h(t) = \begin{cases} 0 & \text{if } t \in \Sigma_0 \cup X, \\ 1 + \max\{h(t_i) \mid 1 \leq i \leq k\} & \text{if } t = \omega(t_1, \ldots, t_k), k \geq 1. \end{cases}$$

Now height is a formal tree series in $N\langle\langle T_\Sigma(X)\rangle\rangle$ defined as

$$\text{height} = \sum_{t \in T_\Sigma(X)} h(t)t.$$

Here N is the semiring of nonnegative integers.

(ii) Consider formal tree series s in $R_+\langle\langle T_\Sigma(X)\rangle\rangle$ such that $0 \leq (s,t) \leq 1$ for all $t \in T_\Sigma(X)$. Then (s,t) can be interpreted as a probability associated with the tree t. Here R_+ is the semiring of nonnegative reals.

(iii) Consider formal tree series s in $N^\infty\langle\langle T_\Sigma(X)\rangle\rangle$, where $N^\infty = N \cup \{\infty\}$. Then the coefficient (s,t) of $t \in T_\Sigma(X)$ can be interpreted as the number (possibly ∞) of distinct generations of t by some mechanism. (See Theorem 3.19.)

More examples can be found in Berstel and Reutenauer [3].

[2] In the examples we often refrain from our convention that the basic semiring is continuous.

Formal tree series induce continuous mappings called *substitutions* as follows. Let Y denote a set of variables, where $Y \cap (\Sigma \cup X) = \emptyset$, and consider a mapping $h : Y \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$. This mapping can be extended to a mapping $h : T_\Sigma(X \cup Y) \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ by $h(x) = x$, $x \in X$, and

$$\begin{aligned} h(\omega(t_1, \dots, t_k)) &= \overline{\omega}(h(t_1), \dots, h(t_k)) \\ &= \sum_{t'_1, \dots, t'_k \in T_\Sigma(X \cup Y)} (h(t_1), t'_1) \cdots (h(t_k), t'_k) \omega(t'_1, \dots, t'_k), \end{aligned}$$

for $\omega \in \Sigma_k$ and $t_1, \dots, t_k \in T_\Sigma(X \cup Y)$, $k \geq 0$. One more extension of h yields a mapping $h : A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ by defining $h(s) = \sum_{t \in T_\Sigma(X \cup Y)} (s, t) h(t)$. This last extension of h is a complete semiring morphism from $A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ into $A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$. It is a continuous mapping (see Corollary 2.15 of Kuich [14]).

Now let $s \in A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$. Then, by definition, the formal tree series s induces a mapping $s : (A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle)^Y \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ as follows: given $h : Y \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$, the value of s with argument h is simply $h(s)$, where h is the extended mapping. If $Y = \{y_1, \dots, y_n\}$ is finite, we use the following notation: $h : Y \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$, where $h(y_i) = s_i$, $1 \leq i \leq n$, is denoted by $(s_i, 1 \leq i \leq n)$ or $(s_1, \dots, s_n)$ and the value of s with argument h is denoted by $s(s_i, 1 \leq i \leq n)$ or $s(s_1, \dots, s_n)$. Intuitively, this is simply the substitution of the formal tree series $s_i \in A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ into the variables y_i, $1 \leq i \leq n$, of $s \in A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$. The mapping $s : (A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle)^Y \to A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$, i.e., the substitution of formal tree series into the variables of Y, is a continuous mapping (see Theorem 2.18 of Kuich [14]). Observe that $s(s_1, \dots, s_n) = \sum_{t \in T_\Sigma(X \cup Y)} (s, t) t(s_1, \dots, s_n)$.

In certain situations, formulae are easier to read if we use the notation $s[s_i/y_i, 1 \leq i \leq n]$ for the substitution of the formal tree series s_i into the variables y_i, $1 \leq i \leq n$, of s instead of the notation $s(s_i, 1 \leq i \leq n)$. So we will sometimes use this notation $s[s_i/y_i, 1 \leq i \leq n]$.

In the same way, $s \in A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle$ also induces a mapping $s : (A\langle\langle T_\Sigma(X)\rangle\rangle)^Y \to A\langle\langle T_\Sigma(X)\rangle\rangle$.

Our tree automata will be defined by transition matrices. Let $Y_k = \{y_1, \dots, y_k\}$, $k \geq 1$, and Y be sets of variables. A matrix $M \in (A\langle\langle T_\Sigma(X \cup Y_k)\rangle\rangle)^{I' \times I^k}$, $k \geq 1$, I' and I arbitrary index sets, induces a mapping

$$M : (A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle)^{I \times 1} \times \cdots \times (A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle)^{I \times 1} \to (A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle)^{I' \times 1}$$

(there are k argument vectors), defined by the entries of the resulting vector as follows: For $P_1, \dots, P_k \in (A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle)^{I \times 1}$ we define, for all $i \in I'$,

$$\begin{aligned} M(P_1, \dots, P_k)_i &= \sum_{i_1, \dots, i_k \in I} M_{i,(i_1, \dots, i_k)}\big((P_1)_{i_1}, \dots, (P_k)_{i_k}\big) \\ &= \sum_{i_1, \dots, i_k \in I} \sum_{t \in T_\Sigma(X \cup Y_k)} \big(M_{i,(i_1, \dots, i_k)}, t\big) t\big((P_1)_{i_1}, \dots, (P_k)_{i_k}\big). \end{aligned}$$

A matrix $M \in (A\langle\langle T_\Sigma(X \cup Y)\rangle\rangle)^{I' \times I^k}$, $k \geq 1$, is called *row finite* iff for each $i \in I'$ there are only finitely many $(i_1, \dots, i_k) \in I^k$ such that $M_{i,(i_1, \dots, i_k)} \neq 0$.

In the following, we need a notation for nodes in trees and for subtrees of trees. We take the notations used by Guessarian [11]. For $t \in T_\Sigma(X)$, D_t is the *tree domain* of t. A *node* in t is an element of D_t; $t(o) \in \Sigma \cup X$ is called the *label* of the node $o \in D_t$; o is also called an *occurence* of $t(o)$ in t. Observe that $t(\varepsilon)$ is the label of the root of t. The *subtree* of t at occurrence o (i.e., with root o) is denoted by $t \mid o$. Additional notations will be introduced whenever needed.

Throughout the paper, I (resp. Q) will denote an arbitrary (resp. a finite) index set.

2. TREE AUTOMATA AND LINEAR SYSTEMS

In this section we define tree automata and linear systems. These notions are a framework for the consideration of pushdown tree automata in the next section. The definitions are slightly adapted from Kuich [14]. The main result of this section is that (polynomial) tree automata and (polynomial) linear systems are equivalent mechanisms.

Our tree automata are a generalization of the nondeterministic root-to-frontier tree recognizers. (See Gécseg and Steinby [9, 10] and Kuich [14].) A *tree automaton* (with input alphabet Σ and leaf alphabet X)

$$A = (I, M, S, P)$$

is given by

(i) a nonempty set I of *states*,

(ii) a sequence $M = (M_k \mid k \geq 1)$ of *transition matrices* $M_k \in (A\langle\langle T_\Sigma(X \cup Y_k)\rangle\rangle)^{I \times I^k}$, $k \geq 1$,

(iii) $S \in (A\langle\langle T_\Sigma(X \cup Y_1)\rangle\rangle)^{1\times I}$, called the *initial state vector*,

(iv) $P \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$, called the *final state vector*.

Here and in the rest of this paper, $Y_k = \{y_1, \ldots, y_k\}$, $k \geq 1$, denotes an alphabet of variables and $Y_0 = \emptyset$. Our notation is similar to that used in Kuich [13–15]: I denotes a set of states that may be infinite (if the set of states or a part of it is finite, we use the traditional letter Q), and M denotes a sequence of transition matrices.

The *approximation sequence* $(\sigma^j \mid j \in N)$, $\sigma^j \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$, $j \geq 0$, *associated with* A is defined as follows:

$$\sigma^0 = 0, \quad \sigma^{j+1} = \sum_{k\geq 1} M_k(\sigma^j, \ldots, \sigma^j) + P, \qquad j \geq 0.$$

The *behavior* $\|A\| \in A\langle\langle T_\Sigma(X)\rangle\rangle$ of the tree automaton A is defined by

$$\|A\| = \sum_{i\in I} S_i(\sigma_i) = S(\sigma),$$

where $\sigma \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$ is the least upper bound of the approximation sequence associated with A. By Theorem 3.5 of Kuich [14], this least upper bound and, hence, the behavior of A exist.

A tree automaton $A = (I, M, S, P)$ is called *finite* iff I is finite. A tree automaton $A = (I, (M_k \mid k \geq 1), S, P)$ is called *simple* iff the entries of the transition matrices M_k, $k \geq 1$, of the initial state vector S and of the final state vector P have the following specific form:

(i) the entries of M_k, $k \geq 2$, are of the form $\sum_{f\in\Sigma_k} a_f f(y_1, \ldots, y_k)$, $a_f \in A$;

(ii) the entries of M_1 are of the form $\sum_{f\in\Sigma_1} a_f f(y_1) + ay_1$, $a_f, a \in A$;

(iii) the entries of P are of the form $\sum_{\omega\in\Sigma_0\cup X} a_\omega \omega$, $a_\omega \in A$;

(iv) the entries of S are of the form dy_1, $d \in A$.

Observe that the term ay_1 in (ii) corresponds to ε-moves in ordinary automata.

Consider the case that A is the semiring N^∞, i.e., we consider tree series in $N^\infty\langle\langle T_\Sigma(X)\rangle\rangle$. A simple tree automaton is called 1-*simple* iff all the coefficients a_f in (i) and (ii), a in (ii), a_ω in (iii), and d in (iv) belong to $\{0, 1\}$. For 1-simple tree automata, computations can be defined. Let $A = (I, (M_k \mid k \geq 1), S, P)$ be a 1-simple tree automaton with input alphabet Σ and leaf alphabet X, t a tree in $T_\Sigma(X)$, and D_t its tree domain. An *assignment* φ *for* t is a mapping $\varphi : D_t \to I^+$; i.e., to each node of t a finite nonempty sequence of states of A is assigned.

An assignment φ for t, where $\varphi(o) = (i(o, 1), \ldots, i(o, n_o))$, $o \in D_t$, $n_o \geq 1$, is called *computation for t* iff

$$w(\varphi) = \big(S_{i(\varepsilon,1)}, y_1\big) \prod_{o\in D_t} \prod_{k\geq 1} \prod_{t(o)\in\Sigma_k} \big((M_k)_{i(o,n_o),(i(o1,1),\ldots,i(ok,1))}, t(o)(y_1, \ldots, y_k)\big)$$
$$\times \prod_{1\leq j\leq n_o-1} \big((M_1)_{i(o,j),i(o,j+1)}, y_1\big) \prod_{t(o)\in\Sigma_0\cup X} \big(P_{i(o,n_o)}, t(o)\big) = 1.$$

The coefficient $(\|A\|, t)$ of the behavior of A is then the number (possibly ∞) of distinct computations for t (see Seidl [18, Proposition 3.1]).

Intuitively, given a computaton φ for t as above, the 1-simple tree automaton operates as follows on the input tree t. It starts at the root ε of t in state $i(\varepsilon, 1)$. (Since $S_{i(\varepsilon,1)} = y_1 \neq 0$, the state $i(\varepsilon, 1)$ is called an *initial state*.) Suppose now that A operates on some node $o \in D_t$ in the state $i(o, j)$, $1 \leq j \leq n_o$. If $j = n_o$ and $t(o) \in \Sigma_k$, $k \geq 1$, then $((M_k)_{i(o,n_o),(i(o1,1),\ldots,i(ok,1))}, t(o)(y_1, \ldots, y_k)) = 1$, A makes a parsing step and operates in the next steps at the sons of o (i.e., $o1, \ldots, ok$), in states $i(o1, 1), \ldots, i(ok, 1)$, respectively. If $j = n_o$ and $t(o) \in \Sigma_0 \cup X$, then $(P_{i(o,n_o)}, t(o)) = 1$ and A terminates the parse on this leaf and may continue elsewhere. If $1 \leq j \leq n_o - 1$, then $((M_1)_{i(o,j),i(o,j+1)}, y_1) = 1$ and A changes its state to $i(o, j + 1)$ without parsing the tree in this step.

THEOREM 2.1. *Consider a 1-simple tree automaton A and let $d(t)$, $t \in T_\Sigma(X)$, be the number (possibly ∞) of distinct computations of A for t. Then*

$$\|A\| = \sum_{t \in T_\Sigma(X)} d(t)t \in N^\infty\langle\langle T_\Sigma(X)\rangle\rangle.$$

We now turn to linear systems.

Let Z be an alphabet of variables. A *linear system* (with variables in $Z = \{z_i \mid i \in I\}$) is a system of formal equations $z_i = p_i$, $i \in I$, I an arbitrary index set, where each p_i is in $A\langle\langle T_\Sigma(X \cup Z_i)\rangle\rangle$. Here Z_i is, for each $i \in I$, a finite subset of Z. The linear system can be written in matrix notation as $z = p(z)$. Here z and p denote vectors whose ith component is z_i and p_i, $i \in I$, respectively. A *solution* to the linear system $z = p(z)$ is given by $\sigma \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$ such that $\sigma = p(\sigma)$. A solution σ of $z = p(z)$ is called *least* solution iff $\sigma \sqsubseteq \tau$ for all solutions τ of $z = p(z)$.

The *approximation sequence* $(\sigma^j \mid j \in N)$, $\sigma^j \in (A\langle\langle T_\Sigma(X)\rangle\rangle)^{I\times 1}$, $j \geq 0$, *associated with the linear system* $z = p(z)$ is defined as follows:

$$\sigma^0 = 0, \qquad \sigma^{j+1} = p(\sigma^j), \quad j \geq 0.$$

By a slight generalization of Theorem 3.1 of Kuich [14], the least upper bound $\sigma = \sup(\sigma^j \mid j \in N)$ of this approximation sequence exists and is the least solution of the linear system $z = p(z)$.

Our linear systems are a generalization of the systems of linear equations of Berstel and Reutenauer [3]. A linear system $z_i = p_i$, $i \in I$, is called *proper* iff $(p_i, z_j) = 0$ for all $i, j \in I$. Corollary 3.3 of Kuich [14] and an adaption of the proof of Proposition 6.1 of Berstel and Reutenauer [3] yield the next theorem.

THEOREM 2.2. *For each linear system there exists a proper one with the same least solution. A proper linear system has a unique solution.*

Clearly, this unique solution is at the same time the least solution.

We now show that tree automata and linear systems are mechanisms of equal power. For a given tree automaton $A = (I, M, S, P)$ as defined above we construct the linear system with variables in $Z = \{z_i \mid i \in I\}$

$$z_i = \sum_{k\geq 1} \sum_{i_1,\ldots,i_k \in I} (M_k)_{i,(i_1,\ldots,i_k)}\big(z_{i_1}, \ldots, z_{i_k}\big) + P_i, \qquad i \in I.$$

Here we have substituted the variables $z_{i_1}, \ldots, z_{i_k}$ for the variables $y_1, \ldots, y_k$ in $(M_k)_{i,(i_1,\ldots,i_k)}$ $(y_1, \ldots, y_k)$. In matrix notation, this linear system can be written as

$$z = \sum_{k\geq 1} M_k(z, \ldots, z) + P.$$

Here z is an $I \times 1$-vector whose ith component is the variable z_i, $i \in I$.

A proof that is analogous to the proof of Theorem 3.7 of Kuich [14] shows that the approximation sequences associated with this linear system and to the tree automaton A coincide. Consider now the linear system with variables in $\{z_0\} \cup Z$

$$z_0 = \sum_{i\in I} S_i(z_i), \qquad z = \sum_{k\geq 1} M_k(z, \ldots, z) + P.$$

Then the z_0-component of its least solution is equal to $\|A\|$.

Conversely, consider a linear system $z = p(z)$ as defined above. Let $Z_i = \{z_{i_1}, \ldots, z_{i_k}\}$, $i \in I$, and $p_i = p_i(z_{i_1}, \ldots, z_{i_k})$. Now construct the tree automaton $A = (Z, M, S, P)$, where, for all $i \in I$,

$$(M_k)_{z_i,(z_{i_1},\ldots,z_{i_k})}(y_1, \ldots, y_k) = p_i(y_1, \ldots, y_k), \qquad \text{if } k \geq 1,$$
$$P_{z_i} = p_i, \qquad \text{if } k = 0.$$

Moreover, choose a $z_{i_0} \in Z$ and let $S_{z_{i_0}}(y_1) = y_1$, $S_{z_i}(y_1) = 0$ for $z_i \neq z_{i_0}$.

It is easy to show that $\|A\|$ is equal to the z_{i_0}-component of the least solution of $z = p(z)$.

Theorem 2.3 summarizes the above considerations.

THEOREM 2.3. *A power series $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$ is a component of the least solution of a linear system iff s is the behavior of a tree automaton.*

We now consider polynomial tree automata and polynomial linear systems and show that they are mechanisms of equal power.

A tree automaton $A = (I, M, S, P)$ is called polynomial iff the following conditions are satisfied:

(i) $M = (M_k \mid 1 \leq k \leq \bar{k})$ is a finite sequence of transition matrices M_k whose entries are polynomials in $A\langle T_\Sigma(X \cup Y_k)\rangle$, $1 \leq k \leq \bar{k}$. (Technically speaking, this means that all transition matrices $M_{\bar{k}+j}$, $j \geq 1$, are equal to the zero matrix.) Moreover, the matrices M_k, $1 \leq k \leq \bar{k}$, are row finite.

(ii) The entries of the initial state vector S are of the form $S_i = d_i y_1$, $d_i \in A$, $i \in I$. Moreover, S is row finite.

(iii) The entries of the final state vector P are polynomials in $A\langle T_\Sigma(X)\rangle$.

A linear system (with variables in Z) $z_i = p_i$, $i \in I$, is called polynomial iff the following conditions are satisfied:

(i) Each p_i is a polynomial in $A\langle T_\Sigma(X \cup Z_i)\rangle$, $i \in I$.

(ii) For all $i \in I$, the cardinality of Z_i is smaller or equal to some constant $\bar{k}$.

The same constructions that proved Theorem 2.3 also prove the next theorem.

THEOREM 2.4. *A power series $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$ is a component of the least solution of a polynomial linear system iff s is the behavior of a polynomial tree automaton.*

EXAMPLE 2 (See Berstel and Reutenauer [3, Examples 6.2 and 4.2]). Our basic semiring is Z, the semiring of integers. Let $\Sigma = \Sigma_1 \cup \Sigma_2$, $\Sigma_1 = \{\ominus\}$, $\Sigma_2 = \{\oplus, \otimes\}$. We will evaluate arithmetic expressions with operators $\ominus$, $\oplus$, $\otimes$, and operands in the leaf alphabet X.

Define an interpretation *eval* of the elements of X, i.e., $\mathrm{eval}: X \to Z$. Extend it to a mapping $\mathrm{eval}: T_\Sigma(X) \to Z$ by $\mathrm{eval}(\ominus(t)) = -\mathrm{eval}(t)$, $\mathrm{eval}(\oplus(t_1, t_2)) = \mathrm{eval}(t_1) + \mathrm{eval}(t_2)$, $\mathrm{eval}(\otimes(t_1, t_2)) = \mathrm{eval}(t_1) \cdot \mathrm{eval}(t_2)$ for $t, t_1, t_2 \in T_\Sigma(X)$. Then $\mathrm{eval} = \sum_{t \in T_\Sigma(X)} \mathrm{eval}(t)t$ is a formal tree series in $Z\langle\langle T_\Sigma(X)\rangle\rangle$.

Consider the proper linear system

$$z_1 = \oplus(z_1, z_2) + \oplus(z_2, z_1) + \otimes(z_1, z_1) + (-1) \ominus (z_1) + \sum_{x \in X} \mathrm{eval}(x)x,$$

$$z_2 = \oplus(z_2, z_2) + \otimes(z_2, z_2) + \ominus(z_2) + \sum_{x \in X} x.$$

Let (σ_1, σ_2) be its unique solution. Then we claim that $\sigma_1 = \mathrm{eval}$, $\sigma_2 = \mathrm{char}$, where $\mathrm{char} = \sum_{t \in T_\Sigma(X)} t$. The claim is proven by substituting (eval, char) into the equations of the linear system:

$$\oplus(\mathrm{eval}, \mathrm{char}) + \oplus(\mathrm{char}, \mathrm{eval}) + \oplus(\mathrm{eval}, \mathrm{eval}) - \ominus(\mathrm{eval}) + \sum_{x \in \Sigma} \mathrm{eval}(x)x$$
$$= \sum_{t_1, t_2 \in T_\Sigma(X)} \mathrm{eval}(t_1) \oplus (t_1, t_2) + \sum_{t_1, t_2 \in T_\Sigma(X)} \mathrm{eval}(t_2) \oplus (t_1, t_2)$$
$$+ \sum_{t_1, t_2 \in T_\Sigma(X)} \mathrm{eval}(t_1)\mathrm{eval}(t_2) \otimes (t_1, t_2) + \sum_{t \in T_\Sigma(X)} -\mathrm{eval}(t) \ominus t + \sum_{x \in \Sigma} \mathrm{eval}(x)x$$

$$= \sum_{t_1,t_2\in T_\Sigma(X)} \mathrm{eval}(\oplus(t_1,t_2)) \oplus (t_1,t_2) + \sum_{t_1,t_2\in T_\Sigma(X)} \mathrm{eval}(\otimes(t_1,t_2)) \otimes (t_1,t_2)$$

$$+ \sum_{t\in T_\Sigma(X)} \mathrm{eval}(\ominus(t)) \ominus t + \sum_{x\in\Sigma} \mathrm{eval}(x)x$$

$$= \sum_{t\in T_\Sigma(X)} \mathrm{eval}(t)t = \mathrm{eval},$$

$$\oplus(\mathrm{char},\mathrm{char}) + \otimes(\mathrm{char},\mathrm{char}) + \ominus(\mathrm{char}) + \sum_{x\in\Sigma} x$$

$$= \sum_{t_1,t_2\in T_\Sigma(X)} \oplus(t_1,t_2) + \sum_{t_1,t_2\in T_\Sigma(X)} \otimes(t_1,t_2) + \sum_{t\in T_\Sigma(X)} \ominus(t) + \sum_{x\in\Sigma} x = \mathrm{char}.$$

Consider now the finite tree automaton $A = (Q, M, S, P)$, where $Q = \{z_1, z_2\}$, $M = (M_1, M_2)$, $S_{z_1} = y_1$, $S_{z_2} = 0$, $P_{z_1} = \sum_{x\in\Sigma} \mathrm{eval}(x)x$, $P_{z_2} = \sum_{x\in\Sigma} x$, and the nonnull entries of M_1 and M_2 are given by

$$\begin{aligned} (M_1)_{z_1,z_1} &= (-1)\ominus(y_1), & (M_1)_{z_2,z_2} &= \ominus(y_1),\\ (M_2)_{z_1,(z_1,z_1)} &= \otimes(y_1,y_2), & (M_2)_{z_1,(z_1,z_2)} &= \oplus(y_1,y_2),\\ (M_2)_{z_1,(z_2,z_1)} &= \oplus(y_1,y_2), & (M_2)_{z_2,(z_2,z_2)} &= \oplus(y_1,y_2) + \otimes(y_1,y_2). \end{aligned}$$

By Theorem 4.3 of Kuich [14], we obtain $\|A\| = \sigma_1 = \mathrm{eval}$.

Let $X = \{a, b, c\}$. We now perform computations of A on the tree $t = \oplus(\ominus(a), \otimes(b, c))$ similar to that defined for 1-simple tree automata. There are two computations φ_1, φ_2 for t such that $\omega(\varphi_1) \neq 0$, $\omega(\varphi_2) \neq 0$:

$$\varphi_1(\varepsilon) = z_1,\quad \varphi_1(1) = z_1,\quad \varphi_1(2) = z_2,\quad \varphi_1(11) = z_1,\quad \varphi_1(21) = z_2,\quad \varphi_1(22) = z_2;$$
$$\varphi_2(\varepsilon) = z_1,\quad \varphi_2(1) = z_2,\quad \varphi_2(2) = z_1,\quad \varphi_2(11) = z_2,\quad \varphi_2(21) = z_1,\quad \varphi_2(22) = z_1.$$

We now obtain

$$\begin{aligned} (\|A\|, \oplus(\ominus(a),\otimes(b,c))) &= \big(S_{z_1}, y_1\big)\big((M_2)_{z_1,(z_1,z_2)}, \oplus(y_1,y_2)\big)\big((M_1)_{z_1,z_1}, \ominus(y_1)\big)\\ &\quad\times \big(P_{z_1}, a\big)\big((M_2)_{z_2,(z_2,z_2)}, \otimes(y_1,y_2)\big)\big(P_{z_2}, b\big)\big(P_{z_2}, c\big)\\ &\quad+ \big(S_{z_1}, y_1\big)\big((M_2)_{z_1,(z_2,z_1)}, \oplus(y_1,y_2)\big)\big((M_1)_{z_2,z_2}, \ominus(y_1)\big)\\ &\quad\times \big(P_{z_2}, a\big)\big((M_2)_{z_1,(z_1,z_1)}, \otimes(y_1,y_2)\big)\big(P_{z_1}, b\big)\big(P_{z_1}, c\big)\\ &= -\mathrm{eval}(a) + \mathrm{eval}(b)\mathrm{eval}(c). \end{aligned}$$

3. PUSHDOWN TREE AUTOMATA AND ALGEBRAIC TREE SYSTEMS

In this section, a pushdown tree automaton is defined as a particular instance of a polynomial tree automaton as considered in Section 2. Such a pushdown tree automaton is a generalization of the pushdown tree automaton introduced by Guessarian [11]. Algebraic tree systems are then introduced as a generalization of the context-free tree grammars (see Rounds [17] and Gécseg and Steinby [10]). They are a particular instance of the second-order systems of Bozapalidis [5]. The main result of this section is that pushdown tree automata and algebraic tree systems are mechanisms of equal power. This generalizes Theorem 1 of Guessarian [11].

For the rest of the paper, Σ denotes a finite ranked alphabet and X denotes a finite leaf alphabet.

A pushdown tree automaton (with input alphabet Σ and leaf alphabet X)

$$P = (Q, \Gamma, Z, Y, M, S, p_0, P)$$

is given by

(i) a finite nonempty set Q of states;

(ii) a finite ranked alphabet $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_{\bar{m}}$ of pushdown symbols;

(iii) a finite alphabet $Z = \{z_1, \ldots, z_{\bar{m}}\}$ of pushdown variables; we use the notation $Z_m = \{z_1, \ldots, z_m\}$ for $1 \le m \le \bar{m}$ and $Z_0 = \emptyset$;

(iv) a finite alphabet $Y = \{y_1, \ldots, y_{\bar{k}}\}$ of variables; we use the notation $Y_k = \{y_1, \ldots, y_k\}$ for $1 \le k \le \bar{k}$;

(v) a finite sequence $M = (M_k \mid 1 \le k \le \bar{k})$ of pushdown tree transition matrices M_k of order k, $1 \le k \le \bar{k}$;

(vi) $S \in (A\langle T_\Sigma(X \cup Y_1)\rangle)^{1\times Q}$, called the initial state vector;

(vii) $p_0 \in \Gamma_0$, called the initial pushdown symbol;

(viii) a finite family $P = (P_{g(z_1,\ldots,z_m)} \mid g \in \Gamma_m, 0 \le m \le \bar{m})$ of final state vectors $P_{g(z_1,\ldots,z_m)} \in (A\langle T_\Sigma(X)\rangle)^{Q\times 1}$, $g \in \Gamma_m$, $0 \le m \le \bar{m}$.

Here a pushdown tree transition matrix of order k, $1 \le k \le \bar{k}$, is a matrix

$$M \in \left((A\langle T_\Sigma(X \cup Y_k)\rangle)^{Q\times Q^k}\right)^{T_\Gamma(Z)\times T_\Gamma(Z)^k}$$

which satisfies the following two conditions:

(i) for all $t, t_1, \ldots, t_k \in T_\Gamma(Z)$,

$$M_{t,(t_1,\ldots,t_k)} = \begin{cases} M_{g(z_1,\ldots,z_m),(v_1(z_1,\ldots,z_m),\ldots,v_k(z_1,\ldots,z_m))} & \text{if } g \in \Gamma_m, 1 \le m \le \bar{m}, t = g(u_1,\ldots,u_m), \\ \quad t_j = v_j(u_1,\ldots,u_m), 1 \le j \le k, \text{ for some } u_1,\ldots,u_m \in T_\Gamma(Z_m) \text{ and} \\ \quad v_1(z_1,\ldots,z_m),\ldots,v_k(z_1,\ldots,z_m) \in T_\Gamma(Z_m); \\ 0, \quad \text{otherwise.} \end{cases}$$

(ii) M is row finite, i.e., for each $g \in \Gamma_m$, $0 \le m \le \bar{m}$, there exists only finitely many blocks $M_{g(z_1,\ldots,z_m),(v_1,\ldots,v_k)}$, where $v_1, \ldots, v_k \in T_\Gamma(Z_m)$, that are unequal to zero;

Observe that if the root of t is labeled by $g \in \Gamma_m$, then $M_{t,(t_1,\ldots,t_k)} \ne 0$ implies $t, t_1, \ldots, t_k \in T_\Gamma(Z_m)$.

Intuitively, the definition of the pushdown tree transition matrix means that the action of the pushdown tree automaton with tree $t = g(u_1, \ldots, u_m)$ on its pushdown tape depends only on the label g of the root of t. Observe that a pushdown tree transition matrix of order k is defined by its finitely many nonnull blocks of the form $M_{g(z_1,\ldots,z_m),(v_1,\ldots,v_k)}$, $g \in \Gamma_m$.

Now let $Z_Q = \{(z_i)_q \mid 1 \le i \le \bar{m}, q \in Q\}$ be an alphabet of variables and denote $Z_Q^m = \{(z_i)_q \mid 1 \le i \le m, q \in Q\}$, $1 \le m \le \bar{m}$, $Z_Q^0 = \emptyset$. Define $F \in ((A\langle T_\Sigma(X \cup Z_Q)\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ by its entries as follows:

(i) $(F_t)_q = (P_{g(z_1,\ldots,z_m)})_q$ if $t = g(u_1, \ldots, u_m)$, $g \in \Gamma_m$, $0 \le m \le \bar{m}$, $u_1, \ldots, u_m \in T_\Gamma(Z_m)$, $q \in Q$;

(ii) $(F_{z_i})_q = (z_i)_q$, $1 \le i \le \bar{m}$, $q \in Q$;

(iii) $(F_t)_q = 0$, otherwise.

Hence, F_{z_i}, $1 \le i \le \bar{m}$, is a column vector of dimension Q whose q-entry, $q \in Q$, is the variable $(z_i)_q$.

The approximation sequence $(\tau^j \mid j \in N)$, $\tau^j \in ((A\langle T_\Sigma(X \cup Z_Q)\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$, $j \ge 0$, associated with P is defined as follows:

$$\tau^0 = 0, \qquad \tau^{j+1} = \sum_{1\le k\le \bar{k}} M_k(\tau^j, \ldots, \tau^j) + F, \quad j \ge 0.$$

This means that, for all $t \in T_\Gamma(Z)$, the block vectors τ_t^j of τ^j are defined by

$$\tau_t^0 = 0, \qquad \tau_t^{j+1} = \sum_{1 \le k \le \bar{k}} \sum_{t_1,\ldots,t_k \in T_\Gamma(Z)} (M_k)_{t,(t_1,\ldots,t_k)}\big(\tau_{t_1}^j, \ldots, \tau_{t_k}^j\big) + F_t, \quad j \ge 0.$$

Moreover, for all $t \in T_\Gamma(Z)$, $q \in Q$,

$$\begin{aligned}\big(\tau_t^0\big)_q &= 0,\\ \big(\tau_t^{j+1}\big)_q &= \sum_{1 \le k \le \bar{k}} \sum_{t_1,\ldots,t_k \in T_\Gamma(Z)} \sum_{q_1,\ldots,q_k \in Q} \big((M_k)_{t,(t_1,\ldots,t_k)}\big)_{q,(q_1,\ldots,q_k)}\Big(\big(\tau_{t_1}^j\big)_{q_1}, \ldots, \big(\tau_{t_k}^j\big)_{q_k}\Big) + (F_t)_q, \quad j \ge 0.\end{aligned}$$

Hence, for all $g \in \Gamma_m$, $0 \le m \le \bar{m}$, and all $u_1, \ldots, u_m \in T_\Gamma(Z_m)$, we obtain, for all $j \ge 0$,

$$\tau_{g(u_1,\ldots,u_m)}^{j+1} = \sum_{1 \le k \le \bar{k}} \sum_{v_1,\ldots,v_k \in T_\Gamma(Z_m)} (M_k)_{g(z_1,\ldots,z_m),(v_1,\ldots,v_k)}\Big(\tau_{v_1(u_1,\ldots,u_m)}^j, \ldots, \tau_{v_k(u_1,\ldots,u_m)}^j\Big) + P_{g(z_1,\ldots,z_m)}$$

and

$$\tau_{z_i}^{j+1} = F_{z_i}, \qquad z_i \in Z.$$

Let $\tau \in ((A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ be the least upper bound of the approximation sequence associated with P. Then the behavior $\|P\|$ of the pushdown tree automaton P is defined by

$$\|P\| = S\big(\tau_{p_0}\big) = \sum_{q \in Q} S_q\Big(\big(\tau_{p_0}\big)_q\Big).$$

Observe that, by the forthcoming Lemma 3.1, $\|P\|$ is a tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$. Furthermore, observe that $(\tau_t)_q \in A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle$, $t \in T_\Gamma(Z)$, $q \in Q$, induces a mapping from $(A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{\bar{m}|Q|}$ into $A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle$.

We now construct a polynomial tree automaton A that is "isomorphic" to the pushdown tree automaton P. Let $\hat{M}_k \in (A\langle T_\Sigma(X \cup Y_k)\rangle)^{(T_\Gamma(Z)\times Q)\times(T_\Gamma(Z)\times Q)^k}$, $1 \le k \le \bar{k}$, and $\hat{F} \in (A\langle T_\Sigma(X \cup Z_Q)\rangle)^{(T_\Gamma(Z)\times Q)\times 1}$ be the isomorphic copies of M_k, $1 \le k \le \bar{k}$, and F, respectively. Observe that $\hat{M}_k$, $1 \le k \le \bar{k}$, is row finite. Furthermore define $\hat{S} \in (A\langle T_\Sigma(X \cup Y_1)\rangle)^{1\times(T_\Gamma(Z)\times Q)}$ by $\hat{S}_{(p_0,q)} = S_q$, $\hat{S}_{(t,q)} = 0$, $t \ne p_0$, $q \in Q$. Specify the polynomial tree automaton A with input alphabet Σ and leaf alphabet $X \cup Z_Q$ by

$$A = (T_\Gamma(Z) \times Q, \hat{M}, \hat{S}, \hat{F}).$$

Then it is clear that $\|A\| = \|P\|$; i.e., our pushdown tree automaton fits into the general definition of a polynomial tree automaton. But for technical reasons, we prefer to work with transition matrices M_k in $((A\langle T_\Sigma(X \cup Y_k)\rangle)^{Q\times Q^k})^{T_\Gamma(Z)\times T_\Gamma(Z)^k}$, $1 \le k \le \bar{k}$, and with the final state vector F in $((A\langle T_\Sigma(X \cup Z_Q)\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$.

Clearly, this means that all notions concerning tree automata (e.g., simple tree automata) are also notions for pushdown tree automata.

Consider now the polynomial linear system constructed from A as in the proof of Theorem 2.3 and transfer it isomorphically to a linear system that "belongs" to P; i.e.,

$$y = \sum_{1 \le k \le \bar{k}} M_k(y, \ldots, y) + F. \tag{$*$}$$

Here $y \in (\{(y_t)_q \mid t \in T_\Gamma(Z), q \in Q\}^{Q\times 1})^{T_\Gamma(Z)\times 1}$ is a vector of variables $(y_t)_q$, $t \in T_\Gamma(Z)$, $q \in Q$, such that $(y_t)_q$ is the t-q-entry of y.

The equations of the linear system $(*)$ are, in block notation, for $t \in T_\Gamma(Z)$,

$$y_t = \sum_{1 \le k \le \bar{k}} \sum_{t_1,\dots,t_k \in T_\Gamma(Z)} (M_k)_{t,(t_1,\dots,t_k)}\big(y_{t_1},\dots,y_{t_k}\big) + F_t,$$

where y_t is a $Q \times 1$-vector, whose q-entry is the variable $(y_t)_q$, $q \in Q$, and for $t \in T_\Gamma(Z)$, $q \in Q$,

$$(y_t)_q = \sum_{1 \le k \le \bar{k}} \sum_{t_1,\dots,t_k \in T_\Gamma(Z)} \sum_{q_1,\dots,q_k \in Q} \big((M_k)_{t,(t_1,\dots,t_k)}\big)_{q,(q_1,\dots,q_k)}\big((y_{t_1})_{q_1},\dots,(y_{t_k})_{q_k}\big) + (F_t)_q.$$

Hence, for all $g \in \Gamma_m$, $0 \le m \le \bar{m}$, and all $u_1,\dots,u_m \in T_\Gamma(Z_m)$, the equations in matrix notation are

$$y_{g(u_1,\dots,u_m)} = \sum_{1 \le k \le \bar{k}} \sum_{v_1,\dots,v_k \in T_\Gamma(Z_m)} (M_k)_{g(z_1,\dots,z_m),(v_1,\dots,v_k)}\big(y_{v_1(u_1,\dots,u_m)},\dots,y_{v_k(u_1,\dots,u_m)}\big) + P_{g(z_1,\dots,z_m)},$$

and, for $z_i \in Z$,

$$y_{z_i} = F_{z_i}.$$

Here $v_i(u_1,\dots,u_m)$, $1 \le i \le k$, denotes $v_i[u_j/z_j, 1 \le j \le m]$. The least solution of this polynomial linear system is the least upper bound of the approximation sequence associated with P.

An example will illustrate the notions connected with pushdown tree automata.

Example 3 (Guessarian [11, Example 3]). The pushdown tree automaton M of Example 3 of Guessarian [11] is specified by our concepts as follows: The input alphabet is $F = \{b, c_1, c_2\}$, $\text{rank}(b) = 2$, $\text{rank}(c_i) = 0$, $i = 1, 2$; X is the empty set. $P = (Q, \Pi, \{z\}, \{y_1, y_2\}, M, S, Z_0, P)$, where $Q = \{q_0, q_1, q_2\}$, $\Pi = \{G, C, Z_0\}$, $\text{rank}(G) = 1$, $\text{rank}(C) = \text{rank}(Z_0) = 0$, $M = (M_1, M_2)$, $P = (P_C, P_{Z_0}, P_{G(z)})$,

(0) $((M_1)_{Z_0,G(C)})_{q_0,q_0} = y_1$,

(1) $((M_1)_{G(z),G(G(z))})_{q_0,q_0} = y_1$,

(2) $((M_2)_{G(z),(z,z)})_{q_0,(q_1,q_2)} = b(y_1, y_2)$,

(3) $((M_2)_{G(z),(z,z)})_{q_i,(q_i,q_i)} = b(y_1, y_2)$, $i = 1, 2$,

(4) $(P_C)_{q_i} = c_i$, $i = 1, 2$.

All other entries of the Z_0, C, and $G(z)$ block row of M_1 and M_2 are zero; moreover, $(P_C)_{q_0} = 0$ and $P_{Z_0} = 0$, $P_{G(z)} = 0$; furthermore, $S_{q_0} = y_1$, $S_{q_1} = S_{q_2} = 0$.

The important entries of the vectors of the approximation sequence associated with P are defined as follows for all $u \in T_\Pi(\{z\})$ and $j \ge 0$:

$$\big(\tau^{j+1}_{Z_0}\big)_{q_0} = \big(\tau^{j}_{G(C)}\big)_{q_0}, \quad \big(\tau^{j+1}_{Z_0}\big)_{q_i} = 0, i = 1, 2;$$

$$\big(\tau^{j+1}_{C}\big)_{q_0} = 0, \quad \big(\tau^{j+1}_{C}\big)_{q_i} = c_i, i = 1, 2; \quad \big(\tau^{j+1}_{z}\big)_{q_i} = z_{q_i}, i = 0, 1, 2;$$

$$\big(\tau^{j+1}_{G(u)}\big)_{q_0} = \big(\tau^{j}_{G(G(u))}\big)_{q_0} + b\big(\big(\tau^{j}_{u}\big)_{q_1}, \big(\tau^{j}_{u}\big)_{q_2}\big),$$

$$\big(\tau^{j+1}_{G(u)}\big)_{q_i} = b\big(\big(\tau^{j}_{u}\big)_{q_i}, \big(\tau^{j}_{u}\big)_{q_i}\big), \quad i = 1, 2.$$

Let $G^k(C) \in T_\Pi(\emptyset)$ be defined by $G^0(C) = C$, $G^{k+1}(C) = G(G^k(C))$, $k \ge 0$, and consider the equations for $G^k(C)$, $k \ge 0$, $j \ge 0$, $i = 1, 2$:

$$\big(\tau^{j+1}_{G^0(C)}\big)_{q_0} = 0, \quad \big(\tau^{j+1}_{G^0(C)}\big)_{q_i} = c_i;$$

$$\big(\tau^{j+1}_{G^k(C)}\big)_{q_0} = \big(\tau^{j}_{G^{k+1}(C)}\big)_{q_0} + b\big(\big(\tau^{j}_{G^{k-1}(C)}\big)_{q_1}, \big(\tau_{G^{k-1}(C)}\big)_{q_2}\big),$$

$$\big(\tau_{G^k(C)}\big)_{q_i} = b\big(\big(\tau^{j}_{G^{k-1}(C)}\big)_{q_i}, \big(\tau_{G^{k-1}(C)}\big)_{q_i}\big).$$

Let $\tau = \sup(\tau^j \mid j \in N)$. Then, for $k \geq 1$, $i = 1, 2$,

$$\begin{aligned}
\left(\tau_{G^0(C)}\right)_{q_0} &= 0, \quad \left(\tau_{G^0(C)}\right)_{q_i} = c_i;\\
\left(\tau_{G^k(C)}\right)_{q_0} &= \left(\tau_{G^{k+1}(C)}\right)_{q_0} + b\left(\left(\tau_{G^{k-1}(C)}\right)_{q_1}, \left(\tau_{G^{k-1}(C)}\right)_{q_2}\right),\\
\left(\tau_{G^k(C)}\right)_{q_i} &= b\left(\left(\tau_{G^{k-1}(C)}\right)_{q_i}, \left(\tau_{G^{k-1}(C)}\right)_{q_i}\right).
\end{aligned}$$

Hence $(\tau_{G^k(C)} \mid k \geq 0)$ is the least solution of the polynomial linear system

$$\begin{aligned}
(z_0)_{q_0} &= 0, \quad (z_0)_{q_i} = c_i, \ i = 1, 2;\\
(z_k)_{q_0} &= (z_{k+1})_{q_0} + b\big((z_{k-1})_{q_1}, (z_{k-1})_{q_2}\big), \ k \geq 1,\\
(z_k)_{q_i} &= b\big((z_{k-1})_{q_i}, (z_{k-1})_{q_i}\big), \ i = 1, 2, \ k \geq 1.
\end{aligned}$$

By Theorem 2.2 (see also Theorem 3.2 of Kuich [14]), $(\tau_{G^k(C)} \mid k \geq 0)$ is also the least solution of the linear system

$$\begin{aligned}
(z_0)_{q_0} &= 0, \quad (z_0)_{q_i} = c_i, \ i = 1, 2;\\
(z_k)_{q_0} &= \sum_{j \geq k-1} b\big((z_j)_{q_1}, (z_j)_{q_2}\big), \ k \geq 1,\\
(z_k)_{q_i} &= b\big((z_{k-1})_{q_i}, (z_{k-1})_{q_i}\big), \ i = 1, 2, \ k \geq 1.
\end{aligned}$$

This linear system is proper and has the unique solution $(\tau_{G^k(C)} \mid k \geq 0)$. Observe that this linear system is not polynomial.

Define now the trees $t_i^j \in T_F(\emptyset)$, $i = 1, 2$, $j \geq 0$, by

$$t_i^0 = c_i, \quad t_i^{j+1} = b\big(t_i^j, t_i^j\big), \ i = 1, 2, \ j \geq 0.$$

Let

$$(s_k)_{q_0} = \sum_{j \geq k-1} b\big(t_1^j, t_2^j\big), \ (s_k)_{q_i} = t_i^k, \ k \geq 1, \ (s_0)_{q_0} = 0, \quad (s_0)_{q_i} = c_i, \ i = 1, 2.$$

Then $((s_k)_{q_i} \mid k \geq 0, i = 0, 1, 2)$ is a solution of this proper linear system and, hence, $(s_k)_{q_i} = (\tau_{G^k(C)})_{q_i}$, $k \geq 0$, $i = 0, 1, 2$. Since $\|P\| = (\tau_{Z_0})_{q_0} = (\tau_{G(C)})_{q_0}$, we infer that $\|P\| = (s_1)_{q_0} = \sum_{j \geq 0} b(t_1^j, t_2^j)$.

This example indicates also a method to prove in a mathematically rigorous manner that the behavior of a pushdown tree automaton equals a certain formal tree series.

We now will show a result for pushdown tree automata that is analogous to Theorem 6.2 of Kuich [13] for pushdown automata. Intuitively, it states that the computations of the pushdown tree automaton governed by a pushdown tape with contents $t(u_1, \ldots, u_m)$ (i.e., $\tau_{t(u_1,\ldots,u_m)}$), where $t(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$ and $u_i \in T_\Gamma(Z_m)$, $1 \leq i \leq m$, are the same as the computations governed by a pushdown tape with contents $t(z_1, \ldots, z_m)$ (i.e., $\tau_{t(z_1,\ldots,z_m)}$) applied to the computations governed by pushdown tapes with contents $u_1, \ldots, u_m$ (i.e., $\tau_{t(z_1,\ldots,z_m)}[\tau_{u_i}/F_{z_i}, 1 \leq i \leq m]$). Four lemmas are needed before that result.

LEMMA 3.1. *Let $(\tau^j \mid j \in N)$ be the approximation sequence associated with the polynomial linear system $(*)$. Then for all $t \in T_\Gamma(Z_m)$, $0 \leq m \leq \overline{m}$,*

(i) $\tau_t^j \in (A\langle T_\Sigma(X \cup Z_Q^m)\rangle)^{Q \times 1}$, $\quad j \geq 0$,

(ii) $\tau_t \in (A\langle\langle T_\Sigma(X \cup Z_Q^m)\rangle\rangle)^{Q \times 1}$, *where* $\tau = \sup(\tau^j \mid j \in N)$.

Proof. The lemma holds for $t = z_i$, $1 \le i \le m$. The proof of (i) is now by induction on j. The case $j = 0$ being clear, we proceed with $j \ge 0$. Let $t = g(u_1, \ldots, u_m)$, where $g \in \Gamma_m$, $u_1, \ldots, u_m \in T_\Gamma(Z_m)$. By the induction hypothesis, $\tau^j_{v_i(u_1,\ldots,u_m)} \in (A\langle T_\Sigma(X \cup Z^m_Q)\rangle)^{Q\times 1}$, $1 \le i \le k$, and, furthermore, $F_{g(z_1,\ldots,z_m)} \in (A\langle T_\Sigma(X)\rangle)^{Q\times 1}$. Hence, $\tau^{j+1}_{g(u_1,\ldots,u_m)} \in (A\langle T_\Sigma(X \cup Z^m_Q)\rangle)^{Q\times 1}$. Clearly, the second statement of Lemma 3.1 is implied by the first statement. ■

This means that, for $t \in T_\Gamma(Z_m)$, $0 \le m \le \bar{m}$, and $q \in Q$, $(\tau_t)_q \in A\langle\langle T_\Sigma(X \cup Z^m_Q)\rangle\rangle$ induces a mapping from $(A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{m|Q|}$ into $A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle$.

LEMMA 3.2. *Let M be a pushdown tree transition matrix of order k, $k \ge 1$. Then, for $t(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$, $t(z_1, \ldots, z_m) \notin Z_m$, $t_j(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$, $1 \le j \le k$, and $u_i \in T_\Gamma(Z_m)$, $1 \le i \le m$, $1 \le m \le \bar{m}$,*

$$M_{t(u_1,\ldots,u_m),(t_1(u_1,\ldots,u_m),\ldots,t_k(u_1,\ldots,u_m))} = M_{t(z_1,\ldots,z_m),(t_1(z_1,\ldots,z_m),\ldots,t_k(z_1,\ldots,z_m))}.$$

Proof. Assume m is chosen minimal and let

$$t(z_1, \ldots, z_m) = g(s_1(z_1, \ldots, z_m), \ldots, s_r(z_1, \ldots, z_m)),$$

where $g \in \Gamma_r$, $s_j(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$, $1 \le j \le r$, $m \le r \le \bar{m}$. By the definition of a pushdown tree transition matrix of order k, there exist $v_j(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$, $1 \le j \le k$, such that $t_j(z_1, \ldots, z_m) = v_j(s_1(z_1, \ldots, z_m), \ldots, s_m(z_1, \ldots, z_m))$ and

$$M_{t(u_1,\ldots,u_m),(t_1(u_1,\ldots,u_m),\ldots,t_k(u_1,\ldots,u_m))} = M_{g(z_1,\ldots,z_r),(v_1,\ldots,v_k)}.$$

Moreover, again by the definition of a pushdown tree transition matrix of order k, we have

$$M_{g(s_1,\ldots,s_r),(v_1(s_1,\ldots,s_m),\ldots,v_k(s_1,\ldots,s_m))} = M_{g(z_1,\ldots,z_r),(v_1,\ldots,v_k)},$$

and the lemma is proven. ■

LEMMA 3.3. *Let $(\tau^j \mid j \in N)$ be the approximation sequence associated with the polynomial linear system $(*)$ and let τ be its least upper bound. Then, for all $t(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$, $1 \le m \le \bar{m}$, $u_i \in T_\Gamma(Z_m)$, $1 \le i \le m$, and all $j \ge 0$,*

$$\tau^j_{t(z_1,\ldots,z_m)}\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big] \sqsubseteq \tau_{t(u_1,\ldots,u_m)}.$$

Proof. If $t = z_k \in Z_m$, then the inequality of our lemma is $\tau^j_{z_k}[\tau_{u_i}/F_{z_i}, 1 \le i \le m] = \tau_{u_k} \sqsubseteq \tau_{u_k}$. We now proceed by induction on j. The induction basis being clear, let $j \ge 0$ and consider $t \notin Z$. Then we obtain by Lemma 3.2

$$\begin{aligned}
&\tau^{j+1}_{t(z_1,\ldots,z_m)}\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big] \\
&= \left(\sum_{1\le k\le \bar{k}} \sum_{t_1,\ldots,t_k\in T_\Gamma(Z_m)} (M_k)_{t(z_1,\ldots,z_m),(t_1,\ldots,t_k)}\big(\tau^j_{t_1}, \ldots, \tau^j_{t_k}\big)\right)\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big] + P_{t(z_1,\ldots,z_m)} \\
&= \sum_{1\le k\le \bar{k}} \sum_{t_1,\ldots,t_k\in T_\Gamma(Z_m)} (M_k)_{t(z_1,\ldots,z_m),(t_1,\ldots,t_k)} \\
&\quad \big(\tau^j_{t_1}\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big], \ldots, \tau^j_{t_k}\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big]\big) + P_{t(z_1,\ldots,z_m)} \\
&\sqsubseteq \sum_{1\le k\le \bar{k}} \sum_{t_1,\ldots,t_k\in T_\Gamma(Z_m)} (M_k)_{t(z_1,\ldots,z_m),(t_1,\ldots,t_k)}\big(\tau_{t_1(u_1,\ldots,u_m)}, \ldots, \tau_{t_k(u_1,\ldots,u_m)}\big) + P_{t(z_1,\ldots,z_m)} \\
&= \tau_{t(u_1,\ldots,u_m)}.
\end{aligned}$$

■

LEMMA 3.4. *Let $(\tau^j \mid j \in N)$ be the approximation sequence associated with the polynomial linear system $(*)$ and let τ be its least upper bound. Then, for all $t(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$, $1 \le m \le \bar{m}$, $u_i \in T_\Gamma(Z_m)$, $1 \le i \le m$, and all $j \ge 0$,*

$$\tau^j_{t(u_1,\ldots,u_m)} \sqsubseteq \tau_{t(z_1,\ldots,z_m)}\big[\tau_{u_i}/F_{z_i}; 1 \le i \le m\big].$$

Proof. If $t = z_k \in Z_m$, then the inequality of our lemma is $\tau^j_{u_k} \sqsubseteq \tau_{z_k}[\tau_{u_i}/F_{z_i}, 1 \le i \le m] = \tau_{u_k}$. We now proceed by induction on j. The induction basis being clear, let $j \ge 0$ and consider $t \notin Z$. Then we obtain by Lemma 3.2

$$\begin{aligned}
\tau^{j+1}_{t(u_1,\ldots,u_m)} &= \sum_{1\le k\le \bar{k}} \sum_{t_1,\ldots,t_k \in T_\Gamma(Z_m)} (M_k)_{t(z_1,\ldots,z_m),(t_1,\ldots,t_k)}\big(\tau^j_{t_1(u_1,\ldots,u_m)}, \ldots, \tau^j_{t_k(u_1,\ldots,u_m)}\big) + P_{t(z_1,\ldots,z_m)}\\
&\sqsubseteq \sum_{1\le k\le \bar{k}} \sum_{t_1,\ldots,t_k \in T_\Gamma(Z_m)} (M_k)_{t(z_1,\ldots,z_m),(t_1,\ldots,t_k)}\\
&\quad \big(\tau_{t_1}\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big], \ldots, \tau_{t_k}\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big]\big) + P_{t(z_1,\ldots,z_m)}\\
&= \left(\sum_{1\le k\le \bar{k}} \sum_{t_1,\ldots,t_k \in T_\Gamma(Z_m)} (M_k)_{t(z_1,\ldots,z_m),(t_1,\ldots,t_k)}\big(\tau_{t_1}, \ldots, \tau_{t_k}\big)\right)\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big] + P_{t(z_1,\ldots,z_m)}\\
&= \tau_{t(z_1,\ldots,z_m)}\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big]. \qquad \blacksquare
\end{aligned}$$

Lemmas 3.3 and 3.4 imply a fundamental result that plays in the theory of pushdown tree automata the same role as Theorem 6.2 of Kuich [13] in the theory of pushdown automata. Intuitively, Theorem 3.4 means the following: The computations of a pushdown tree automaton governed by a pushdown tape with contents $t(u_1, \ldots, u_m)$ are the same as the computations governed by a pushdown tape with contents $t(z_1, \ldots, z_m)$ applied as a mapping to the computations governed by pushdown tapes with contents $u_1, \ldots, u_m$.

THEOREM 3.5. *Let τ be the least solution of the polynomial linear system $(*)$. Then, for all $t(z_1, \ldots, z_m) \in T_\Gamma(Z_m)$, $1 \le m \le \bar{m}$, and $u_i \in T_\Gamma(Z_m)$, $1 \le i \le m$,*

$$\tau_{t(u_1,\ldots,u_m)} = \tau_{t(z_1,\ldots,z_m)}\big[\tau_{u_i}/F_{z_i}, 1 \le i \le m\big].$$

We now introduce algebraic tree systems. Let $\Phi = \{G_1, \ldots, G_n\}$, $\Phi \cap \Sigma = \emptyset$, be a finite ranked alphabet of function variables, where G_i has rank r_i, $1 \le i \le n$, and $\bar{m} = \max\{r_i \mid 1 \le i \le n\}$. Let $Z = \{z_1, \ldots, z_{\bar{m}}\}$ be a finite alphabet of variables and denote $Z_m = \{z_1, \ldots, z_m\}$, $1 \le m \le \bar{m}$, $Z_0 = \emptyset$.

Let $D = A\langle\langle T_\Sigma(X \cup Z_{r_1})\rangle\rangle \times \cdots \times A\langle\langle T_\Sigma(X \cup Z_{r_n})\rangle\rangle$ and consider tree series $s_i \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$. Then each s_i induces a function $\bar{s_i} : D \to A\langle\langle T_\Sigma(X \cup Z_{r_i})\rangle\rangle$. For $(\tau_1, \ldots, \tau_n) \in D$, we inductively define $\bar{s_i}(\tau_1, \ldots, \tau_n)$ to be

(i) z_m if $s_i = z_m$, $1 \le m \le r_i$; x if $s_i = x$, $x \in X$;

(ii) $\bar{f}(\bar{t_1}(\tau_1, \ldots, \tau_n), \ldots, \bar{t_r}(\tau_1, \ldots, \tau_n))$ if $s_i = f(t_1, \ldots, t_r)$, $f \in \Sigma_r, t_1, \ldots, t_r \in T_{\Sigma\cup\Phi}(X \cup Z_{r_i})$;

(iii) $\tau_j(\bar{t_1}(\tau_1, \ldots, \tau_n), \ldots, \bar{t}_{r_j}(\tau_1, \ldots, \tau_n))$ if $s_i = G_j(t_1, \ldots, t_{r_j})$, $G_j \in \Phi$, $t_1, \ldots, t_{r_j} \in T_{\Sigma\cup\Phi}(X \cup Z_{r_i})$;

(iv) $a\bar{t}(\tau_1, \ldots, \tau_n)$ if $s_i = at$, $a \in A$, $t \in T_{\Sigma\cup\Phi}(X \cup Z_{r_i})$;

(v) $\sum_{j\in J} \bar{r_j}(\tau_1, \ldots, \tau_n)$ if $s_i = \sum_{j\in J} r_j$, $r_j \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $j \in J$, for an arbitrary index set J.

We now show that the mappings $\bar{s_i}$, $1 \le i \le n$, and $\bar{s} = \langle \bar{s_1}, \ldots, \bar{s_n}\rangle$ are continuous. Hence, by the fixpoint theorem (see, e.g., Wechler [19, Sect. 1.5]), $\bar{s}$ has a least fixpoint in D. (See also Lemmas 3.1 and 3.2 of Engelfriet and Schmidt [6]; Bloom and Ésik [4]; Ésik [7].)

LEMMA 3.6. *Let $D = A\langle\langle T_\Sigma(X \cup Z_{r_1})\rangle\rangle \times \cdots \times A\langle\langle T_\Sigma(X \cup Z_{r_n})\rangle\rangle$ and consider tree series $s_i \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$. Then the mapping $\bar{s} : D \to D$, where $\bar{s} = \langle \bar{s_1}, \ldots, \bar{s_n}\rangle$, is continuous.*

Proof. Let $t_i \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$, and $(\tau_1, \ldots, \tau_n) \in D$. Then $(\overline{t_1}(\tau_1, \ldots, \tau_n), \ldots, \overline{t_n}(\tau_1, \ldots, \tau_n))$ is defined, in terms of recursive schemes and interpretations, by Guessarian [12] in Definition 4.22.

Given $t \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$, she proves in Lemma 4.24 of [12] by induction on the height of t that $\overline{t}$ is a continuous mapping. This means that the mappings $\overline{t}$, $t \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$, defined by items (i), (ii), and (iii) above, are continuous. Clearly, the mappings $a\overline{t}$, $a \in A$, $t \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$, of item (iv) above, are again continuous.

According to Manes and Arbib [16, Theorem 12, Sect. 8.3], arbitrary sums of continuous mappings are again continuous mappings. Hence, for $s_i \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$, the mapping $\overline{s_i} = \sum_{t\in T_{\Sigma\cup\Phi}(X\cup Z_{r_i})}(s_i, t)\overline{t}$, $1 \le i \le n$, is a continuous mapping. This implies that $\overline{s} = \langle\overline{s_1}, \ldots, \overline{s_n}\rangle$, $s_i \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$, is a continuous mapping. ■

An algebraic tree system $S = (\Phi, Z, \Sigma, E)$ (with function variables in Φ, variables in Z, and terminal symbols in Σ) has a set E of formal equations

$$G_i(z_1, \ldots, z_{r_i}) = s_i(z_1, \ldots, z_{r_i}), \qquad 1 \le i \le n,$$

where each s_i is in $A\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle$. A solution to the algebraic tree system S is given by $(\tau_1, \ldots, \tau_n) \in D$ such that $\tau_i = \overline{s_i}(\tau_1, \ldots, \tau_n)$, $1 \le i \le n$, i.e., by any fixpoint $(\tau_1, \ldots, \tau_n)$ of $\overline{s} = \langle\overline{s_1}, \ldots, \overline{s_n}\rangle$. A solution $(\sigma_1, \ldots, \sigma_n)$ of the algebraic tree system S is called least solution iff $\sigma_i \sqsubseteq \tau_i$, $1 \le i \le n$, for all solutions $(\tau_1, \ldots, \tau_n)$ of S. Since the least solution of S is nothing else than the least fixpoint of $\overline{s} = \langle\overline{s_1}, \ldots, \overline{s_n}\rangle$, the least solution of the algebraic system S exists in D.

THEOREM 3.7. *Let $S = (\Phi, Z, \Sigma, \{G_i = s_i \mid 1 \le i \le n\})$ be an algebraic tree system, where $s_i \in A\langle T_{\Sigma\cup\Phi}(X \cup Z_{r_i})\rangle$. Then the least solution of this algebraic tree system S exists in D and equals*

$$\mathrm{fix}(\overline{s}) = \sup(\overline{s}^j(0) \mid j \in N),$$

where $\overline{s}^j$ is the jth iterate of the mapping $\overline{s} = \langle\overline{s_1}, \ldots, \overline{s_n}\rangle : D \to D$.

Proof. By Lemma 3.6 and the fixpoint theorem. ■

Theorem 3.7 indicates how we can compute an approximation to the least solution of an algebraic tree system. The approximation sequence $(\tau^j \mid j \in N)$, where each $\tau^j \in D$ associated with the algebraic tree system $S = (\Phi, Z, \Sigma, \{G_i = s_i \mid 1 \le i \le n\})$ is defined as follows:

$$\tau^0 = 0, \qquad \tau^{j+1} = \overline{s}(\tau^j), \qquad j \in N.$$

Clearly, the least solution $\mathrm{fix}(\overline{s})$ of S is equal to $\sup(\tau^j \mid j \in N)$. An algebraic tree system $S = (\Phi, Z, \Sigma, \{G_i = s_i \mid 0 \le i \le n\}, G_0)$ (with function variables in $\Phi = \{G_0, G_1, \ldots, G_n\}$, variables in Z, terminal symbols in Σ) with initial function variable G_0 is an algebraic tree system $(\Phi, Z, \Sigma, \{G_i = s_i \mid 0 \le i \le n\})$ such that G_0 has rank 0. Let $(\tau_0, \tau_1, \ldots, \tau_n)$ be the least solution of $(\Phi, Z, \Sigma, \{G_i = s_i \mid 0 \le i \le n\})$. Then τ_0 is called the initial component of the least solution. Observe that $\tau_0 \in A\langle\langle T_\Sigma(X)\rangle\rangle$ contains no variables of Z.

Our algebraic tree systems are second-order systems in the sense of Bozapalidis [5] and are a generalization of the context-free tree grammars. (See Rounds [17], and Engelfriet and Schmidt [6, especially Theorem 3.4].)

A tree series in $A\langle\langle T_\Sigma(X)\rangle\rangle$ is called algebraic iff it is the initial component of the least solution of an algebraic tree system with initial function variable. An algebraic tree system $S = (\Phi, Z, \Sigma, \{G_i = s_i \mid 1 \le i \le n\})$ is called proper iff, for each $1 \le i \le n$, $(s_i, G_j(z_1, \ldots, z_{r_j})) = 0$, $1 \le j \le n$, and $(s_i, z_j) = 0$, $1 \le j \le \overline{m}$.

Bozapalidis [5] has shown the following important result in a more general setup.

THEOREM 3.8 (Bozapalidis [5; Proposition 9, Theorem 22]). *A proper algebraic tree system has a unique solution. Each algebraic tree series is the initial component of the unique solution of a proper algebraic tree system with initial funcion variable.*

Given a pushdown tree automaton $P = (Q, \Gamma, Z, Y, M, S, p_0, P)$, we now construct an equivalent algebraic tree system $S = (\Phi, Z_Q, \Sigma, E, y_0)$ with initial function variable y_0. Here $\Phi = \{y_0\} \cup$

$\{(y_{g(z_1,\dots,z_m)})_q \mid g \in \Gamma_m, 0 \le m \le \bar{m}, q \in Q\}$. The function variable $(y_{g(z_1,\dots,z_m)})_q$, $g \in \Gamma_m$, $0 \le m \le \bar{m}$, $q \in Q$, has the rank $m|Q|$. By definition, the $Q \times 1$-vector $y_{g(z_1,\dots,z_m)}$, $g \in \Gamma_m$, $0 \le m \le \bar{m}$, is the column vector with q-component $(y_{g(z_1,\dots,z_m)})_q$, $q \in Q$.

For the specification of the formal equations in E we have to introduce, for $t \in T_\Gamma(Z_m)$, $1 \le m \le \bar{m}$, vectors $\hat{y}_t$ in $(T_\Phi(Z_Q^m))^{Q\times 1}$ as follows:

$$\hat{y}_{g(u_1,\dots,u_m)} = y_{g(z_1,\dots,z_m)}\big(\hat{y}_{u_1},\dots,\hat{y}_{u_m}\big),\ g \in \Gamma_m,\ u_1,\dots,u_m \in T_\Gamma(Z_m),\ 1 \le m \le \bar{m};$$
$$\hat{y}_g = y_g,\ g \in \Gamma_0;\quad \big(\hat{y}_{z_i}\big)_q = (z_i)_q,\ 1 \le i \le \bar{m},\ q \in Q.$$

Written componentwise, the first equation reads

$$\big(\hat{y}_{g(u_1,\dots,u_m)}\big)_q = \big(y_{g(z_1,\dots,z_m)}\big)_q\big(\big(\hat{y}_{u_i}\big)_{q'}, 1 \le i \le m, q' \in Q\big)$$

for $g \in \Gamma_m$, $u_1,\dots,u_m \in T_\Gamma(Z_m)$, $1 \le m \le \bar{m}$, $q \in Q$. Observe that

$$\big(\hat{y}_{g(z_1,\dots,z_m)}\big)_q = \big(y_{g(z_1,\dots,z_m)}\big)_q((z_i)_{q'}, 1 \le i \le m, q' \in Q)$$

for $g \in \Gamma_m$, $1 \le m \le \bar{m}$, $q \in Q$. Observe further that $y_{g(z_1,\dots,z_m)}(\hat{y}_{u_1},\dots,\hat{y}_{u_m})$ means $y_{g(z_1,\dots,z_m)}[\hat{y}_{u_i}/F_{z_i}, 1 \le i \le m]$ and $(y_{g(z_1,\dots,z_m)})_q((\hat{y}_i)_{q'}, 1 \le i \le m, q' \in Q)$ means $(y_{g(z_1,\dots,z_m)})_q[(\hat{y}_i)_{q'}/(z_i)_{q'}, 1 \le i \le m, q' \in Q]$. The formal equations in E are now given in matrix notation:

$$\begin{aligned} y_0 &= S\big(y_{p_0}\big),\\ y_{g(z_1,\dots,z_m)}((z_i)_{q'}, 1 \le i \le m, q' \in Q) &= \Big(\sum_{1\le k\le \bar{k}} M_k(\hat{y},\dots,\hat{y}) + F\Big)_{g(z_1,\dots,z_m)}\\ &= \sum_{1\le k\le \bar{k}}\ \sum_{t_1,\dots,t_k \in T_\Gamma(Z_m)} (M_k)_{g(z_1,\dots,z_m),(t_1,\dots,t_k)}\big(\hat{y}_{t_1},\dots,\hat{y}_{t_k}\big)\\ &\quad + P_{g(z_1,\dots,z_m)},\ g \in \Gamma_m,\ 0 \le m \le \bar{m}. \end{aligned}$$

We now explicitly give the formal equations, except the first one, with index $q \in Q$. Observe that indexing by $q \in Q$ is needed only in examples. In theoretical considerations, we save the indexing by states $q, q_1,\dots,q_n$, i.e., we use the form given above.

$$\begin{aligned} &\big(y_{g(z_1,\dots,z_m)}\big)_q((z_i)_{q'}, 1 \le i \le m, q' \in Q)\\ &= \sum_{1\le k\le \bar{k}}\ \sum_{t_1,\dots,t_k \in T_\Gamma(Z_m)}\ \sum_{q_1,\dots,q_k \in Q} \big((M_k)_{g(z_1,\dots,z_m),(t_1,\dots,t_k)}\big)_{q,(q_1,\dots,q_k)}\big(\big(\hat{y}_{t_1}\big)_{q_1},\dots,\big(\hat{y}_{t_k}\big)_{q_k}\big)\\ &\quad + \big(P_{g(z_1,\dots,z_m)}\big)_q,\ g \in \Gamma_m,\ 0 \le m \le \bar{m},\ q \in Q. \end{aligned}$$

We denote this system of formal equations by $(**)$.

The systems $(*)$ and $(**)$ play in the theory of pushdown tree automata similar roles as the linear system $y = My + F$ of Theorem 6.3 and the algebraic system $y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi} y_\pi$ of Theorem 6.4 of Kuich [13], respectively, in the theory of pushdown automata: From a solution of the algebraic system a solution of the linear system can be easily constructed.

LEMMA 3.9. *If $\tau \in ((A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ is the least solution of the polynomial linear system $(*)$ then, for all $t \in T_\Gamma(Z)$,*

$$\hat{y}_t\big[\tau_{g(z_1,\dots,z_m)}\big/y_{g(z_1,\dots,z_m)}, g \in \Gamma_m, 0 \le m \le \bar{m}\big] = \tau_t.$$

Proof. The proof is by induction on the structure of the trees t in $T_\Gamma(Z)$. The lemma is true for $\hat{y}_{z_i}$, $z_i \in Z$, and $\hat{y}_g$, $g \in \Gamma_0$.

Now let $t(z_1, \dots, z_m) = g(u_1, \dots, u_r) \in T_\Gamma(Z_m)$, $1 \le m \le \bar{m}$, where $g \in \Gamma_r$, $u_j \in T_\Gamma(Z_m)$, $1 \le j \le r$, $m \le r \le \bar{m}$. (Here m is chosen minimal.) By definition,

$$\hat{y}_t = y_{g(z_1,\dots,z_r)}(\hat{y}_{u_1}, \dots, \hat{y}_{u_r}) = (y_{g(z_1,\dots,z_r)})\big[\hat{y}_{u_i}/F_{z_i}, 1 \le i \le m\big].$$

By the induction hypothesis, we obtain, for $1 \le i \le r$,

$$\hat{y}_{u_i}\big[\tau_{g(z_1,\dots,z_m)}/y_{g(z_1,\dots,z_m)}, g \in \Gamma_m, 0 \le m \le \bar{m}\big] = \tau_{u_i}.$$

Hence, by Theorem 3.5,

$$\begin{aligned}&\hat{y}_t\big[\tau_{g(z_1,\dots,z_m)}/y_{g(z_1,\dots,z_m)}, g \in \Gamma_m, 0 \le m \le \bar{m}\big]\\ &\quad = (\tau_{g(z_1,\dots,z_r)})\big[\tau_{u_i}/F_{z_i}, 1 \le i \le r\big] = \tau_{g(u_1,\dots,u_r)} = \tau_t.\end{aligned}$$ ■

The next theorem in the theory of pushdown tree automata plays the same role as the first sentence in the proof of Theorem 6.4 of Kuich [13] for pushdown automata: It shows how a solution of the algebraic system $y_p = \sum_{\pi \in \Gamma^*} M_{p,\pi} y_\pi$ is easily constructed from the least solution of the linear system $y = My + F$.

THEOREM 3.10. *If $\tau \in ((A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ is the least solution of the polynomial linear system* $(*)$ *then $(\tau_{g(z_1,\dots,z_m)} \mid g \in \Gamma_m, 0 \le m \le \bar{m})$ is a solution of the algebraic tree system* $(**)$.

Proof. Since τ is the least solution of $(*)$, we obtain, for $g \in \Gamma_m$, $0 \le m \le \bar{m}$, by Lemma 3.9

$$\begin{aligned}&\Big(\sum_{1\le k\le \bar{k}} M_k(\tau, \dots, \tau) + F\Big)_{g(z_1,\dots,z_m)}\\ &\quad = \sum_{1\le k\le \bar{k}} \sum_{v_1,\dots,v_k \in T_\Gamma(Z_m)} (M_k)_{g(z_1,\dots,z_m),(v_1,\dots,v_k)}(\tau_{v_1}, \dots, \tau_{v_k}) + F_{g(z_1,\dots,z_m)} = \tau_{g(z_1,\dots,z_m)}.\end{aligned}$$ ■

It remains to show that the solution of Theorem 3.10 for $(**)$ is the least solution. This will be shown in Theorem 3.13. Two lemmas are needed before this result.

Consider, for $g \in \Gamma_m$, $0 \le m \le \bar{m}$, column vectors $\tau_{g(z_1,\dots,z_m)} \in (A\langle\langle T_\Sigma(X \cup Z_Q^m)\rangle\rangle)^{Q\times 1}$ and define by these vectors column vectors $\tilde{\tau}_t \in (A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{Q\times 1}$, $t \in T_\Gamma(Z)$, as follows:

$$\begin{aligned}&\tilde{\tau}_{z_i} = F_{z_i},\ 1 \le i \le \bar{m}; \quad \tilde{\tau}_g = \tau_g,\ g \in \Gamma_0;\\ &\tilde{\tau}_{g(u_1,\dots,u_m)} = \tau_{g(z_1,\dots,z_m)}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big],\ g \in \Gamma_m,\ 1 \le m \le \bar{m},\ u_j \in T_\Gamma(Z_m),\ 1 \le j \le m.\end{aligned}$$

Observe that, for all $g \in \Gamma_m$, $0 \le m \le \bar{m}$, $\tilde{\tau}_{g(z_1,\dots,z_m)} = \tau_{g(z_1,\dots,z_m)}$. Define now $\tilde{\tau} \in ((A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ to have $\tilde{\tau}_t$ as its t-block vector, $t \in T_\Gamma(Z)$.

LEMMA 3.11. *For $t(z_1, \dots, z_m) \in T_\Gamma(Z_m)$ and $u_i \in T_\Gamma(Z_m)$, $1 \le i \le m$, $1 \le m \le \bar{m}$,*

$$\tilde{\tau}_{t(u_1,\dots,u_m)} = \tilde{\tau}_{t(z_1,\dots,z_m)}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big].$$

Proof. The proof is by induction on the structure of the tree t in $T_\Gamma(Z_m)$. The lemma is valid for $\tilde{\tau}_{z_i}$, $z_i \in Z$, and $\tilde{\tau}_{g(z_1,\dots,z_m)}$, $g \in \Gamma_m$. Let now $t(z_1, \dots, z_m) = g(s_1, \dots, s_r)$, where $g \in \Gamma_r$, $s_j \in T_\Gamma(Z_m)$, $1 \le j \le r$, $m \le r \le \bar{m}$. (Here m is chosen minimal.) By definition,

$$\tilde{\tau}_{t(u_1,\dots,u_m)} = \tau_{g(z_1,\dots,z_r)}\big[\tilde{\tau}_{s_j(u_1,\dots,u_m)}/F_{z_j}, 1 \le j \le r\big].$$

By induction hypothesis, we obtain

$$\tilde{\tau}_{s_j(u_1,\dots,u_m)} = \tilde{\tau}_{s_j(z_1,\dots,z_m)}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big], \qquad 1 \le j \le r.$$

Substitution yields

$$\begin{aligned}
\tilde{\tau}_{t(u_1,\dots,u_m)} &= \tau_{g(z_1,\dots,z_r)}\big[\tilde{\tau}_{s_j(z_1,\dots,z_m)}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big]/F_{z_j}, 1 \le j \le r\big] \\
&= \big(\tau_{g(z_1,\dots,z_r)}\big[\tilde{\tau}_{s_j(z_1,\dots,z_m)}/F_{z_j}, 1 \le j \le r\big]\big)\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big] \\
&= \tilde{\tau}_{g(s_1,\dots,s_r)}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big] = \tilde{\tau}_{t(z_1,\dots,z_m)}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big]. \qquad \blacksquare
\end{aligned}$$

The next lemma in the theory of pushdown tree automata plays the same role as Theorem 6.3 of Kuich [13] in the theory of pushdown automata. In this lemma a solution of the polynomial linear system $(*)$ is easily constructed from a solution of the algebraic tree system $(**)$.

LEMMA 3.12. *Let* $(\tau_{g(z_1,\dots,z_m)} \mid g \in \Gamma_m, 0 \le m \le \bar{m})$, $\tau_{g(z_1,\dots,z_m)} \in (A\langle\langle T_\Sigma(X \cup Z_Q^m)\rangle\rangle)^{Q\times 1}$, $g \in \Gamma_m$, $0 \le m \le \bar{m}$, *be a solution of the algebraic tree system* $(**)$. *Then* $\tilde{\tau} \in ((A\langle\langle T_\Sigma(X \cup Z_Q)\rangle\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ *is a solution of the polynomial linear system* $(*)$.

Proof. We substitute $\tilde{\tau}$ in the right side of $(*)$ and obtain, for $g \in \Gamma_m$, $u_1, \dots, u_m \in T_\Gamma(Z)$, $0 \le m \le \bar{m}$,

$$\begin{aligned}
&\Big(\sum_{1\le k\le \bar{k}} M_k(\tilde{\tau}, \dots, \tilde{\tau}) + F\Big)_{g(u_1,\dots,u_m)} \\
&= \sum_{1\le k\le \bar{k}} \sum_{v_1,\dots,v_k \in T_\Gamma(Z_m)} (M_k)_{g(z_1,\dots,z_m),(v_1,\dots,v_k)}\big(\tilde{\tau}_{v_1(u_1,\dots,u_m)}, \dots, \tilde{\tau}_{v_k(u_1,\dots,u_m)}\big) + F_{g(z_1,\dots,z_m)} \\
&= \sum_{1\le k\le \bar{k}} \sum_{v_1,\dots,v_k \in T_\Gamma(Z_m)} (M_k)_{g(z_1,\dots,z_m),(v_1,\dots,v_k)}\big(\tilde{\tau}_{v_1}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big], \dots, \\
&\quad \tilde{\tau}_{v_k}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big]\big) + F_{g(z_1,\dots,z_m)} \\
&= \Big(\sum_{1\le k\le \bar{k}} \sum_{v_1,\dots,v_k \in T_\Gamma(Z_m)} (M_k)_{g(z_1,\dots,z_m),(v_1,\dots,v_k)}\big(\tilde{\tau}_{v_1}, \dots, \tilde{\tau}_{v_k}\big)\Big)\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big] + F_{g(z_1,\dots,z_m)} \\
&= \Big(\sum_{1\le k\le \bar{k}} M_k(\tilde{\tau}, \dots, \tilde{\tau}) + F\Big)_{g(z_1,\dots,z_m)}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big] \\
&= \tilde{\tau}_{g(z_1,\dots,z_m)}\big[\tilde{\tau}_{u_i}/F_{z_i}, 1 \le i \le m\big] = \tilde{\tau}_{g(u_1,\dots,u_m)}.
\end{aligned}$$

Here we have applied Lemma 3.11 in the second equation. ■

THEOREM 3.13. *If* τ *is the least solution of the polynomial linear system* $(*)$ *then* $(\tau_{g(z_1,\dots,z_m)} \mid g \in \Gamma_m, 0 \le m \le \bar{m})$ *is the least solution of the algebraic tree system* $(**)$.

Proof. By Theorem 3.10, $(\tau_{g(z_1,\dots,z_m)} \mid g \in \Gamma_m, 0 \le m \le \bar{m})$ is a solution of $(**)$. Assume now that $(\sigma_g \mid g \in \Gamma_m, 0 \le m \le \bar{m})$ is a solution of $(**)$, too. Then, by Lemma 3.12, $\tilde{\sigma}$ is a solution of $(*)$. Hence, $\tau \sqsubseteq \tilde{\sigma}$ and $\tau_{g(z_1,\dots,z_m)} \sqsubseteq \tilde{\sigma}_{g(z_1,\dots,z_m)} = \sigma_g$, for all $g \in \Gamma_m$, $0 \le m \le \bar{m}$. ■

COROLLARY 3.14. *The initial component of the least solution of the algebraic tree system S coincides with* $\|P\|$.

COROLLARY 3.15. *The behavior of a pushdown tree automaton is an algebraic tree series.*

EXAMPLE 3 (continued). For the pushdown tree automaton P we now construct step-by-step the algebraic tree system S with initial function variable. By Corollary 3.14 this system S has the property that $\|P\|$ is the initial component of its least solution.

We first consider the linear system $(*)$ written in the form

$$\hat{y} = M_1(\hat{y}) + M_2(\hat{y}, \hat{y}) + F$$

and write down explicitly the equations for $\hat{y}_{G(z)}$, $\hat{y}_{Z_0}$, and $\hat{y}_C$:

$$\begin{aligned}
(\hat{y}_{G(z)})_{q_0} &= \big(\hat{y}_{G(G(z))}\big)_{q_0} + b\big((\hat{y}_z)_{q_1}, (\hat{y}_z)_{q_2}\big), \\
(\hat{y}_{G(z)})_{q_i} &= b\big((\hat{y}_z)_{q_i}, (\hat{y}_z)_{q_i}\big), && i = 1, 2, \\
(\hat{y}_{Z_0})_{q_0} &= \big(\hat{y}_{G(C)}\big)_{q_0}, \quad \big(\hat{y}_{Z_0}\big)_{q_i} = 0, && i = 1, 2, \\
(\hat{y}_C)_{q_0} &= 0, \quad (\hat{y}_C)_{q_i} = c_i, && i = 1, 2.
\end{aligned}$$

Now we express the components of $\hat{y}$ by $y_{G(z)}$, y_{Z_0}, and y_C and obtain the algebraic system $(**)$:

$$\begin{aligned}
\big(y_{G(z)}\big)_{q_0}\big(z_{q_0}, z_{q_1}, z_{q_2}\big) &= \big(y_{G(z)}\big)_{q_0}\big(\big(y_{G(z)}\big)_{q_0}\big(z_{q_0}, z_{q_1}, z_{q_2}\big), \\
&\qquad \big(y_{G(z)}\big)_{q_1}\big(z_{q_0}, z_{q_1}, z_{q_2}\big), \big(y_{G(z)}\big)_{q_2}\big(z_{q_0}, z_{q_1}, z_{q_2}\big)\big) + b\big(z_{q_1}, z_{q_2}\big), \\
\big(y_{G(z)}\big)_{q_i}\big(z_{q_0}, z_{q_1}, z_{q_2}\big) &= b\big(z_{q_i}, z_{q_i}\big), \qquad i = 1, 2, \\
\big(y_{Z_0}\big)_{q_0} &= \big(y_{G(z)}\big)_{q_0}\big((y_C)_{q_0}, (y_C)_{q_1}, (y_C)_{q_2}\big), \\
\big(y_{Z_0}\big)_{q_i} &= 0, \qquad i = 1, 2, \\
(y_C)_{q_0} &= 0, \\
(y_C)_{q_i} &= c_i, \qquad i = 1, 2.
\end{aligned}$$

The algebraic tree system $S = (\Phi, Z, F, E, y_0)$ is now specified by $\Phi = \{(y_{G(z)})_{q_i}, (y_{Z_0})_{q_i}, (y_C)_{q_i} \mid i = 0, 1, 2\} \cup \{y_0\}$, where the ranks of $(y_{G(z)})_{q_i}$, $(y_{Z_0})_{q_i}$, $(y_C)_{q_i}$ are 3, 0, 0, respectively, for $i = 0, 1, 2$; $Z = \{z_{q_0}, z_{q_1}, z_{q_2}\}$; E is the set of equations specified above augmented by the additional equation $y_0 = (y_{Z_0})_{q_0}$.

Observe that the construction of P from S is essentially the same construction as given by Guessarian [11] in her proof of Theorem 1. ■

We now want to show the converse to Corollary 3.15. Consider an algebraic tree system $S = (\Phi, Z, \Sigma, E, G_0)$ with initial function variable G_0. Let $\Phi = \{G_0, G_1, \ldots, G_n\}$ and denote the rank of G_i by r_i, $0 \le i \le n$. Let $\overline{m} = \max\{r_i \mid 1 \le i \le m\}$ and $Z = \{z_1, \ldots, z_{\overline{m}}\}$. Assume that the equations of E are given in the form

$$G_i\big(z_1, \ldots, z_{r_i}\big) = \sum_{1 \le j \le n_i} a_{ij} t_i^j, \qquad 0 \le i \le n,$$

where $a_{ij} \in A$ and $t_i^j \in T_{\Sigma \cup \Phi}(X \cup Z_{r_i})$, $1 \le j \le n_i$, $0 \le i \le n$.

We now construct an equivalent pushdown tree automaton $P = (Q, \Gamma, Z, Y, M, S, p_0, P)$. The pushdown tree automaton P is specified as follows: $|Q| = 1$, $\Gamma = \Sigma \cup X \cup \Phi$, $Z = \{z_1, \ldots, z_{\overline{m}}\}$, $Y = \{y_1, \ldots, y_{\overline{k}}\}$, where $\overline{k}$ is the maximal rank of a symbol in Σ, $p_0 = G_0$, and the entries of M, S, and P are given by:

(1) if $t_i^j(\varepsilon) \in \Phi$, $1 \le j \le n_i$, $0 \le i \le n$, then

$$(M_1)_{G_i(z_1, \ldots, z_{r_i}), t_i^j} = a_{ij} y_1;$$

(2a) if $t_i^j(\varepsilon) \in \Sigma_k$, $1 \le k \le \overline{k}$, $1 \le j \le n_i$, $0 \le i \le n$, then

$$(M_k)_{G_i(z_1, \ldots, z_{r_i}), (t_i^j|1, \ldots, t_i^j|k)} = a_{ij} t_i^j(\varepsilon)(y_1, \ldots, y_k);$$

(2b) if $t_i^j(\varepsilon) \in \Sigma_0 \cup X$, $1 \le j \le n_i$, $0 \le i \le n$, then

$$P_{G_i(z_1,\dots,z_{r_i})} = a_{ij} t_i^j(\varepsilon);$$

(3a) for $f \in \Sigma_k$, $1 \le k \le \overline{k}$,

$$(M_k)_{f(z_1,\dots,z_k),(z_1,\dots,z_k)} = f(y_1,\dots,y_k);$$

(3b) for $f \in \Sigma_0 \cup X$

$$P_f = f;$$

(4) $S = y_1$.

For this pushdown tree automaton P we construct the algebraic tree system $(**)$:

$$\begin{aligned}
&y_{G_i(z_1,\dots,z_{r_i})}(z_1,\dots,z_{r_i})\\
&\quad= \sum_{1\le j\le n_i} \sum_{t_i^j(\varepsilon)\in\Phi} (M_1)_{G_i(z_1,\dots,z_{r_i}),t_i^j}(\hat{y}_{t_i^j}) + \sum_{1\le k\le\overline{k}} \sum_{1\le j\le n_i} \sum_{t_i^j(\varepsilon)\in\Sigma_k} (M_k)_{G_i(z_1,\dots,z_{r_i}),(t_i^j|1,\dots,t_i^j|k)}(\hat{y}_{t_i^j|1},\dots,\hat{y}_{t_i^j|k})\\
&\qquad+ \sum_{1\le j\le n_i} \sum_{t_i^j(\varepsilon)\in\Sigma_0\cup X} P_{G_i(z_1,\dots,z_{r_i})}\\
&\quad= \sum_{1\le j\le n_i} a_{ij} \left(\sum_{t_i^j(\varepsilon)\in\Phi} \hat{y}_{t_i^j} + \sum_{1\le k\le\overline{k}} \sum_{t_i^j(\varepsilon)\in\Sigma_k} t_i^j(\varepsilon)(\hat{y}_{t_i^j|1},\dots,\hat{y}_{t_i^j|k}) + \sum_{t_i^j(\varepsilon)\in\Sigma_0\cup X} t_i^j(\varepsilon) \right), \quad 0\le i\le n;\\
&\qquad y_{f(z_1,\dots,z_k)}(z_1,\dots,z_k) = f(z_1,\dots,z_k),\ f\in\Sigma_k,\ 1\le k\le\overline{k};\quad y_f = f,\ f\in\Sigma_0\cup X.
\end{aligned}$$

Let τ be the least solution of the polynomial linear system $(*)$ for P. Then we obtain, for $t_i^j(\varepsilon) \in \Sigma_k$, $1 \le k \le \overline{k}$, $1 \le j \le n_i$, $0 \le i \le n$,

$$t_i^j(\varepsilon)(\tau_{t_i^j|1},\dots,\tau_{t_i^j|k}) = \tau_{t_i^j}.$$

We now substitute τ into the right side of the first equations of the algebraic tree system $(**)$:

$$\begin{aligned}
&\sum_{1\le j\le n_i} a_{ij} \left(\sum_{t_i^j(\varepsilon)\in\Phi} \tau_{t_i^j} + \sum_{1\le k\le\overline{k}} \sum_{t_i^j(\varepsilon)\in\Sigma_k} t_i^j(\varepsilon)(\tau_{t_i^j|1},\dots,\tau_{t_i^j|k}) + \sum_{t_i^j(\varepsilon)\in\Sigma_0\cup X} t_i^j(\varepsilon) \right)\\
&\quad= \sum_{1\le j\le n_i} a_{ij} \left(\sum_{t_i^j(\varepsilon)\in\Phi} \tau_{t_i^j} + \sum_{1\le k\le\overline{k}} \sum_{t_i^j(\varepsilon)\in\Sigma_k} \tau_{t_i^j} + \sum_{t_i^j(\varepsilon)\in\Sigma_0\cup X} \tau_{t_i^j} \right) = \sum_{1\le j\le n_i} a_{ij}\tau_{t_i^j}.
\end{aligned}$$

By Theorem 3.13, we have $\tau_{G_i(z_1,\dots,z_{r_i})} = \sum_{1\le i\le n_i} a_{ij}\tau_{t_i^j}$, $0 \le i \le n$. Hence, $(\tau_{G_i(z_1,\dots,z_{r_i})} \mid 0 \le i \le n) \cup (\tau_{f(z_1,\dots,z_k)} \mid f \in \Sigma_k, 1 \le k \le \overline{k}) \cup (\tau_f \mid f \in \Sigma_0 \cup X)$ is the least solution of S, and τ_{G_0} is the initial component of the least solution of S. Moreover, $\|P\| = \tau_{G_0}$.

Hence, we have proved the following theorem.

THEOREM 3.16. *The behavior of the pushdown automaton P coincides with the initial component of the least solution of the algebraic system S.*

The next corollary states the main result of Section 3.

COROLLARY 3.17. *The following statements on a formal tree series s in $A\langle\langle T_\Sigma(X)\rangle\rangle$ are equivalent*:

(i) *s is an algebraic tree series*;

(ii) *s is the behavior of a pushdown tree automaton*;

(iii) *s is the behavior of a simple pushdown tree automaton with one state.*

EXAMPLE 4. Consider the algebraic tree system $S = (\Phi, Z, \Sigma, E, Z_0)$ specified by

(i) $\Phi = \{G_0, G_1, G_2, Z_0\}$, where the ranks of G_0, G_1, G_2 are 3 and the rank of Z_0 is 0;

(ii) $Z = \{z_0, z_1, z_2\}$;

(iii) $\Sigma = \{b, c_1, c_2\}$, where the rank of b is 2 and the ranks of c_1, c_2 are 0;

(iv) the formal equations of E are

$$\begin{aligned} G_0(z_0, z_1, z_2) &= G_0(G_0(z_0, z_1, z_2), G_1(z_0, z_1, z_2), G_2(z_0, z_1, z_2)) + b(z_1, z_2), \\ G_i(z_0, z_1, z_2) &= b(z_i, z_i), \quad i = 1, 2, \\ Z_0 &= G_0(0, c_1, c_2). \end{aligned}$$

Our algebraic tree system S is a simplified version of the algebraic tree system S of Example 3. For this algebraic tree system S we now construct an equivalent pushdown tree automaton $P = (Q, \Gamma, Z, Y, M, S, Z_0, P)$, where $|Q| = 1$, $\Gamma = \{b, c_1, c_2\} \cup \{G_0, G_1, G_2, Z_0\}$, $Z = \{z_0, z_1, z_2\}$, $Y = \{y_1, y_2\}$, and the entries of M, S, and P are specified by

(1) $(M_1)_{G_0(z_0,z_1,z_2),G_0(G_0(z_0,z_1,z_2),G_1(z_0,z_1,z_2),G_2(z_0,z_1,z_2))} = y_1$, $(M_1)_{Z_0,G_0(0,c_1,c_2)} = y_1$,

(2) $(M_2)_{G_0(z_0,z_1,z_2),(z_1,z_2)} = b(y_1, y_2)$, $(M_2)_{G_i(z_0,z_1,z_2),(z_i,z_i)} = b(y_1, y_2)$, $i = 1, 2$,

(3) $(M_2)_{b(z_0,z_1),(z_0,z_1)} = b(y_1, y_2)$, $P_{c_i} = c_i$, $i = 1, 2$.

(4) $S = y_1$. ■

If our basic semiring is N^∞, i.e., if we consider tree series in $N^\infty\langle\langle T_\Sigma(X)\rangle\rangle$, we can draw some stronger conclusions.

Let $G = (\Phi, Z, \Sigma, R)$ be a context-free tree grammar, where $\Phi = \{G_1, \ldots, G_n\}$ and R is the set of rules

$$G_i\bigl(z_1, \ldots, z_{r_i}\bigr) \to t_i^j, \qquad 1 \le j \le n_i, 1 \le i \le n.$$

Denote by $d_i(t)$, $1 \le i \le n$, the number (possibly ∞) of distinct leftmost derivations of $t \in T_\Sigma(X \cup Z_{r_i})$ with respect to G and starting from G_i. Let $S = (\Phi, Z, \Sigma, E)$ be an algebraic tree system, where E is the set of formal equations

$$G_i\bigl(z_1, \ldots, z_{r_i}\bigr) = \sum_{1 \le j \le n_i} t_i^j, \qquad 1 \le i \le n.$$

Then there exists the following theorem.

THEOREM 3.18 (Bozapalidis [5, Theorem 11 ii]). *Let $G = (\Phi, Z, \Sigma, R)$ and $S = (\Phi, Z, \Sigma, E)$ be the context-free tree grammar and the algebraic tree system, respectively, considered above. Let $d_i(t)$, $1 \le i \le n$, be the number (possibly ∞) of distinct leftmost derivations of $t \in T_\Sigma(X \cup Z_{r_i})$ with respect to G and starting from G_i. Then the least solution of S is given by*

$$\left(\sum_{t \in T_\Sigma(X \cup Z_{r_i})} d_i(t)t \;\middle|\; 1 \le i \le n \right).$$

Theorems 3.18 and 2.1 and Corollary 3.17 yield the last results of this section.

THEOREM 3.19. *Let $d\colon T_\Sigma(X) \to N^\infty$. Then the following statements are equivalent*:

(i) *There exists a context-free tree grammar with initial function variable and with terminal alphabet Σ and leaf alphabet X such that the number (possibly ∞) of distinct leftmost derivations of $t \in T_\Sigma(X)$ from the initial function variable is given by $d(t)$.*

(ii) *There exists a 1-simple pushdown tree automaton with input alphabet Σ and leaf alphabet X such that the number (possibly ∞) of distinct computations for $t \in T_\Sigma(X)$ is given by $d(t)$.*

A context-free tree grammar with initial function variable and with terminal alphabet Σ and leaf alphabet X is called unambiguous iff, for all $t \in T_\Sigma(X)$, the number of distinct leftmost derivations of t with respect to G is either 1 or 0. A 1-simple pushdown tree automaton with terminal alphabet Σ and leaf alphabet X is called unambiguous iff, for all $t \in T_\Sigma(X)$, the number of distinct computations for t is either 1 or 0.

COROLLARY 3.20. *Let $L \subseteq T_\Sigma(X)$ be a tree language. Then L is generated by an unambiguous context-free tree grammar iff $\sum_{t\in L} t$ is the behavior of an unambiguous 1-simple pushdown tree automaton.*

4. RESTRICTED PUSHDOWN TREE AUTOMATA

A pushdown tree automaton $P = (Q, \Gamma, Z, Y, M, S, p_0, P)$ is called restricted iff $\Gamma = \{p_0\} \cup \Gamma_1$; i.e., except for the initial pushdown symbol p_0 of rank 0, all other pushdown symbols have rank 1. In this section we show that each pushdown tree automaton is equivalent to a restricted pushdown tree automaton. This equivalence is shown via algebraic tree systems.

We start with an arbitrary pushdown tree automaton whose behavior is, by Corollary 3.17, an algebraic tree series, say $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$. This algebraic tree series s is the initial component of the least solution of an algebraic tree system S with initial function variable. From S we will construct a simple restricted pushdown tree automaton P. In a next step, we construct from P an algebraic tree system $\underline{S}$ which plays the role of the algebraic tree system $(**)$ of Section 3 for P. Eventually, we construct from $\underline{S}$ an algebraic tree system S_1 with initial function variable. We claim that $\|P\| = s$. The proof of this claim runs as follows: By Theorem 3.8, we assume without loss of generality that S is proper. By Lemmas 4.1–4.6 we show that the initial component of the least solution of S_1 equals s. Since S_1 is constructed from P via $\underline{S}$ in such a manner that, by Corollary 3.14, the initial component of the least solution of S_1 equals $\|P\|$, our claim is proven in Theorem 4.7.

Let $S = (\Phi, Z, \Sigma, E, G_0)$ be the algebraic tree system with initial function variable, where $\Phi = \{G_0, G_1, \ldots, G_n\}$, G_i has rank r_i, $0 \le i \le n$, $r_0 = 0$, and E is the set of equations

$$G_i\big(z_1, \ldots, z_{r_i}\big) = \sum_{1\le j\le n_i} a_{ij} t_i^j, \qquad 0 \le i \le n,$$

$a_{ij} \in A$, $t_i^j \in T_{\Sigma\cup\Phi}(X \cup Z_{r_i})$, $1 \le i \le n_i$, $n_i \ge 0$. Here $Z = \{z_1, \ldots, z_r\}$, $r = \max\{r_i \mid 0 \le i \le n\}$, and $Z_{r_i} = \{z_1, \ldots, z_{r_i}\}$, $1 \le i \le n$, $Z_{r_0} = Z_0 = \emptyset$. From this algebraic tree system S we construct a simple restricted pushdown tree automaton $P = (Q, \Gamma, \{z\}, Y, M, S, G_0, P)$ such that $\|P\|$ coincides with the initial component of the least solution of the algebraic tree system S. The construction of P from S is essentially the same as given by Guessarian [11] in her proof of Theorem 3.

The items of P are as follows:

(i) $Q = \{(i, j, o) \mid o$ is an occurrence in t_i^j, $1 \le j \le n_i$, $0 \le i \le n\}$;

(ii) $\Gamma = \{G_0\} \cup \{(i, j, o) \mid o$ is an occurrence in t_i^j, where $t_i^j(o) \in \Phi$, $1 \le j \le n_i$, $0 \le i \le n\}$; G_0 has rank 0; the other pushdown symbols have rank 1;

(iii) $Y = \{y_1, \ldots, y_{\bar{k}}\}$, where $\bar{k} \ge 1$ is the maximal rank of a symbol in Σ.

The pushdown tree transition matrices of $M = (M_k \mid 1 \le k \le \bar{k})$, the initial state vector S, and the final state vectors of $P = (P_{G_0}, P_{(i,j,o)(z)} \mid (i, j, o) \in \Gamma)$ are now defined.

(i) The nonnull entries of S, of blocks of the G_0-block row of M_k, $1 \le k \le \bar{k}$, and of P_{G_0} are:

(1) $S_{(0,j,\varepsilon)} = a_{0j} y_1$, $1 \le j \le n_0$;

(2a) if $t_0^j(o) \in \Sigma_k$, $1 \le k \le \bar{k}$, $(0, j, o) \in Q$, then $((M_k)_{G_0,(G_0,\ldots,G_0)})_{(0,j,o),((0,j,o1),\ldots,(0,j,ok))} = t_0^j(o)(y_1, \ldots, y_k)$;

(2b) if $t_0^j(o) \in \Sigma_0$, $(0, j, o) \in Q$, then $(P_{G_0})_{(0,j,o)} = t_0^j(o)$;

(3) if $t_0^j(o) = G_k \in \Phi$, $(0, j, o) \in \Gamma$, then, for all $1 \le j_1 \le n_k$, $((M_1)_{G_0,(0,j,o)G_0})_{(0,j,o),(k,j_1,\varepsilon)} = a_{kj_1} y_1$.

(ii) The nonnull entries of blocks of the $(i', j', o')(z)$-block row of M_k, $1 \le k \le \bar{k}$, and of $P_{(i',j',o')(z)}$, $(i', j', o') \in \Gamma$, are:

(2a) if $t_i^j(o) \in \Sigma_k$, $1 \le k \le \bar{k}$, $(i, j, o) \in Q$, then $((M_k)_{(i',j',o')(z),((i',j',o')(z),\ldots,(i',j',o')(z))})_{(i,j,o),((i,j,o1),\ldots,(i,j,ok))} = t_i^j(o)(y_1, \ldots, y_k)$;

(2b) if $t_i^j(o) \in \Sigma_0$, $(i, j, o) \in Q$, then $(P_{(i',j',o')(z)})_{(i,j,o)} = t_i^j(o)$;

(3) if $t_i^j(o) = G_k \in \Phi$, $(i, j, o) \in \Gamma$, then, for all $1 \le j_1 \le n_k$, $((M_1)_{(i',j',o')(z),(i,j,o)(i',j',o')(z)})_{(i,j,o),(k,j_1,\varepsilon)} = a_{kj_1} y_1$;

(4) if $t_i^j(o) = z_m \in Z_{r_i}$, $(i, j, o) \in Q$, then $((M_1)_{(i',j',o')(z),z})_{(i,j,o),(i',j',o'm)} = y_1$.

In the proof of her Theorem 3, Guessarian [11] gives the following intuitive description of these entries of M, S, and P:

(1) The entries of S initialize the computations of P.

(2a) Reading $t_i^j(o) \in \Sigma_k$, $k \ge 1$, in a state corresponding to the node o in the tree t_i^j, the pushdown tree automaton P has to go down in the input tree without changing the pushdown contents.

(2b) Reading $t_i^j(o) \in \Sigma_0$ in a state corresponding to the node o in the tree t_i^j, the pushdown tree automaton P accepts.

(3) If a variable function symbol $G_k \in \Phi$ occurs at node o in the tree t_i^j, and P is in the state (i, j, o) corresponding to the node o in the tree t_i^j, G_k is stored on the pushdown tape, thus remembering the recursive call which shall be done later; moreover, P is repositioned in a state which corresponds to beginning the derivation of G_k (i.e., at the roots of the right-hand side trees $t_k^{j_1}$ corresponding to G_k).

(4) If a variable z_m occurs, indicating that the current recursive call has been completed, P takes a pop move, i.e., P continues the remembered recursive call; that means P is repositioned in a state that represents the mth argument of the popped variable function symbol.

We now construct an algebraic tree system $\bar{S} = (\bar{\Phi}, \bar{Z}, \Sigma, \bar{E})$ which plays the role of the algebraic tree system $(**)$ of Section 3 for P. Here $\bar{\Phi} = \{(Y_{G_0})_{(0,j,o)} \mid (0, j, o) \in Q\} \cup \{(Y_{(i',j',o')(z)})_{(i,j,o)} \mid (i', j', o') \in \Gamma, (i, j, o) \in Q\}$ is the alphabet of function variables; the rank of $(Y_{G_0})_{(0,j,o)}$, $1 \le j \le n_0$, is 0; the ranks of the other function variables are $|Q|$. The alphabet of variables is $\bar{Z} = \{z_q \mid q \in Q\}$. The equations of $\bar{E}$ are (we state only those equations whose right-hand side is unequal to 0):

(i)(2) if $t_0^j(o) \in \Sigma_k$, $0 \le k \le \bar{k}$, $(0, j, o) \in Q$, then $(Y_{G_0})_{(0,j,o)} = t_0^j(o)((Y_{G_0})_{(0,j,o1)}, \ldots, (Y_{G_0})_{(0,j,ok)})$;

(3) if $t_0^j(o) = G_k \in \Phi$, $(0, j, o) \in \Gamma$, then $(Y_{G_0})_{(0,j,o)} = \sum_{1 \le j_1 \le n_k} a_{kj_1} (Y_{(0,j,o)})_{(k,j_1,\varepsilon)} [(Y_{G_0})_{(0,j,om)}/z_{(0,j,om)}, 1 \le m \le r_k]$;

(ii)(2) if $t_i^j(o) \in \Sigma_k$, $0 \le k \le \bar{k}$, $(i, j, o) \in Q$, $(i', j', o') \in \Gamma$, then $(Y_{(i',j',o')})_{(i,j,o)} = t_i^j(o)((Y_{(i',j',o')})_{(i,j,o1)}, \ldots, (Y_{(i',j',o')})_{(i,j,ok)})$;

(3) if $t_i^j(o) = G_k \in \Phi$, $(i, j, o) \in \Gamma$, $(i', j', o') \in \Gamma$, then $(Y_{(i',j',o')})_{(i,j,o)} = \sum_{1 \le j_1 \le n_k} a_{kj_1} (Y_{(i,j,o)})_{(k,j_1,\varepsilon)} [(Y_{(i',j',o')})_{(i,j,om)}/z_{(i,j,om)}, 1 \le m \le r_k]$;

(4) if $t_i^j(o) = z_m \in Z_{r_i}$, $(i, j, o) \in Q$, $(i', j', o') \in \Gamma$, then $(Y_{(i',j',o')})_{(i,j,o)} = z_{(i',j',o'm)}$.

To simplify notations, we have used the following abbreviation: the function variable $(Y_{(i',j',o')})_{(i,j,o)}$ stands for the tree with root labeled by $(Y_{(i',j',o')})_{(i,j,o)}$ having as sons nodes labeled by the variables z_q, $q \in Q$, in some fixed order. In the equations of (3) the substitution means that the function variable $(Y_{G_0})_{(0,j,om)}$ or the "tree" $(Y_{(i',j',o')})_{(i,j,om)}$ is substituted for the variable $z_{(0,j,om)}$ or $z_{(i,j,om)}$, $1 \le m \le r_k$, respectively, while all other variables of $\bar{Z}$ remain unchanged. This latter simplification is due to the next lemma.

LEMMA 4.1. *Let σ be the least solution of $\bar{S}$. Then $(\sigma_{G_0})_{(0,j,o)} \in A\langle\langle T_\Sigma(X)\rangle\rangle$, $1 \le j \le n_0$, and $(\sigma_{(i',j',o')})_{(i,j,o)} \in A\langle\langle T_\Sigma(X \cup \{z_{(i',j',o'm)} \mid 1 \le m \le r_i\})\rangle\rangle$ for all $(i', j', o') \in \Gamma$, $(i, j, o) \in Q$.*

Proof. The proof is by induction on the upper index of the vectors σ^k, $k \in N$, of the approximation sequence associated with the algebraic tree system $\overline{S}$. By equations (ii)(4) of $\overline{E}$ the statement of the lemma is true for $(\sigma^k_{(i',j',o')})_{(i,j,o)} = z_{(i',j',o'm)}$, $k \geq 1$, where $t_i^j(o) \in Z_{r_i}$, $(i, j, o) \in Q$, $(i', j', o') \in \Gamma$. Since all other components of σ^1 are equal to 0, the statement of the lemma is true for σ^1. By equations (i)(2), (ii)(2), (i)(3), and (ii)(3) the statement of the lemma remains true in the induction step. ■

Recall that each $t \in T_{\Sigma\cup\Phi}(X \cup Z_{r_i})$, $0 \leq i \leq n$, induces a mapping $\overline{t}$ from D into $A\langle\langle T_\Sigma(X \cup Z_{r_i})\rangle\rangle$, where $D = A\langle\langle T_\Sigma(X)\rangle\rangle \times A\langle\langle T_\Sigma(X \cup Z_{r_1})\rangle\rangle \times \cdots \times A\langle\langle T_\Sigma(X \cup Z_{r_n})\rangle\rangle$. We denote the mapping induced by $t_i^j \mid o$, $(i, j, o) \in Q$, by $\tau_i^j(o)$. Observe that $\tau_i = \sum_{1\leq j\leq n_i} a_{ij}\tau_i^j(\varepsilon)(\tau_0, \ldots, \tau_n)$, $0 \leq i \leq n$, if $(\tau_0, \ldots, \tau_n)$ is a solution of the algebraic tree system S.

LEMMA 4.2. *Assume that $(\tau_0, \ldots, \tau_n)$ is the least solution of the algebraic tree system S. Let $(\sigma_{G_0})_{(0,j,o)} = \tau_0^j(o)(\tau_0, \ldots, \tau_n)$, $1 \leq j \leq n_0$, and $(\sigma_{(i',j',o')})_{(i,j,o)} = \tau_i^j(o)(\tau_0, \ldots, \tau_n)[z_{(i',j',o'm)}/z_m, 1 \leq m \leq r_i]$, $(i', j', o') \in \Gamma$, $(i, j, o) \in Q$. Then $((\sigma_{G_0})_{(0,j,o)} \mid (0, j, o) \in Q) \cup ((\sigma_{(i',j',o')})_{(i,j,o)} \mid (i', j', o') \in \Gamma, (i, j, o) \in Q)$ is a solution of the algebraic tree system $\overline{S}$.*

Proof. We consider the equations of $\overline{E}$, i.e., equations (i)(2), (i)(3), (ii)(2), (ii)(3), and (ii)(4).

<u>(i)(2)</u>: if $t_0^j(o) \in \Sigma_k$, $0 \leq k \leq \overline{k}$, $(0, j, o) \in Q$, then

$$t_0^j(o)\big(\tau_0^j(o1)(\tau_0, \ldots, \tau_n), \ldots, \tau_0^j(ok)(\tau_0, \ldots, \tau_n)\big) = \overline{t}(\tau_0, \ldots, \tau_n) = \tau_0^j(o)(\tau_0, \ldots, \tau_n),$$

where $t = t_0^j(o)(t_0^j \mid o1, \ldots, t_0^j \mid ok) = t_0^j \mid o$.

<u>(i)(3)</u>: if $t_0^j(o) = G_k \in \Phi$, $1 \leq j \leq n_0$, then

$$\sum_{1\leq j_1\leq n_k} a_{kj_1}\big(\tau_k^{j_1}(\varepsilon)(\tau_0, \ldots, \tau_n)\big[z_{(0,j,om)}/z_m, 1 \leq m \leq r_k\big]\big)\big[\tau_0^j(om)(\tau_0, \ldots, \tau_n)/z_{(0,j,om)}, 1 \leq m \leq r_k\big]$$
$$= \tau_k\big[\tau_0^j(om)(\tau_0, \ldots, \tau_n)/z_m, 1 \leq m \leq r_k\big] = \overline{t}(\tau_0, \ldots, \tau_n) = \tau_0^j(o)(\tau_0, \ldots, \tau_n),$$

where $t = G_k(t_0^j \mid o1, \ldots, t_0^j \mid or_k) = t_0^j \mid o$.

<u>(ii)(2)</u>: if $t_i^j(o) \in \Sigma_k$, $0 \leq k \leq \overline{k}$, $(i, j, o) \in Q$, $(i', j', o') \in \Gamma$, then

$$t_i^j(o)\big(\tau_i^j(o1)(\tau_0, \ldots, \tau_n)\big[z_{(i',j',o'm)}/z_m, 1\leq m\leq r_i\big], \ldots, \tau_i^j(ok)(\tau_0, \ldots, \tau_n)\big[z_{(i',j',o'm)}/z_m, 1\leq m\leq r_i\big]\big)$$
$$= t_i^j(o)\big(\tau_i^j(o1)(\tau_0, \ldots, \tau_n), \ldots, \tau_i^j(ok)(\tau_0, \ldots, \tau_n)\big)\big[z_{(i',j',o'm)}/z_m, 1 \leq m \leq r_i\big]$$
$$= \overline{t}(\tau_0, \ldots, \tau_n)\big[z_{(i',j',o'm)}/z_m, 1 \leq m \leq r_i\big] = \tau_i^j(o)(\tau_0, \ldots, \tau_n)\big[z_{(i',j',o'm)}/z_m, 1 \leq m \leq r_i\big],$$

where $t = t_i^j(o)(t_i^j \mid o1, \ldots, t_i^j \mid ok) = t_i^j \mid o$.

<u>(ii)(3)</u>: if $t_i^j(o) = G_k \in \Phi$, $(i, j, o) \in Q$, $(i', j', o') \in \Gamma$, then

$$\sum_{1\leq j_1\leq n_k} a_{kj_1}\big(\tau_k^{j_1}(\varepsilon)(\tau_0, \ldots, \tau_n)\big[z_{(i,j,om)}/z_m, 1 \leq m \leq r_k\big]\big)$$
$$\big[\tau_i^j(om)(\tau_0, \ldots, \tau_n)\big[z_{(i',j',o'm')}/z_{m'}, 1 \leq m' \leq r_i\big]/z_{(i,j,om)}, 1 \leq m \leq r_k\big]$$
$$= \tau_k\big[\tau_i^j(om)(\tau_0, \ldots, \tau_n)\big[z_{(i',j',o'm')}/z_{m'}, 1 \leq m' \leq r_i\big]/z_m, 1 \leq m \leq r_k\big]$$
$$= \overline{t}(\tau_0, \ldots, \tau_n)\big[z_{(i',j',o'm')}/z_{m'}, 1 \leq m' \leq r_i\big],$$

where $t = G_k(t_i^j \mid o1, \ldots, t_i^j \mid or_k) = t_i^j \mid o$.

<u>(ii)(4)</u>: if $t_i^j(o) = z_m \in Z_{r_i}$, $(i, j, o) \in Q$, $(i', j', o') \in \Gamma$, then

$$\tau_i^j(o)(\tau_0, \ldots, \tau_n)\big[z_{(i',j',o'm)}/z_m, 1 \leq m \leq r_i\big] = z_{(i',j',o'm)}. \quad ■$$

We now construct an algebraic tree system $S' = (\Phi', \overline{Z}, \Sigma, E')$ that is equivalent to $\overline{S}$. Here $\Phi' = \overline{\Phi} - \{(Y_{(i',j',o')})_{(i,j,o)} \mid t_i^j(o) \in Z_{r_i}, (i,j,o) \in Q, (i',j',o') \in \Gamma\}$, and the set E' of equations is constructed as follows: for $(i,j,o) \in Q$, $(i',j',o') \in \Gamma$, substitute $z_{(i',j',o'm)}$ for $(Y_{(i',j',o')})_{(i,j,o)}$ in the equations (i)(3), (ii)(2), and (ii)(3) of $\overline{E}$ if $t_i^j(o) = z_m \in Z_{r_i}$, and denote the resulting equations by (i)(3'), (ii)(2'), and (ii)(3'), respectively. Then E' contains the equations of (i)(2), (i)(3'), (ii)(2'), and (ii)(3').

Lemma 4.3. *If $\overline{\tau}$ is a solution of the algebraic tree system $\overline{S}$, then*

$$\tau' = \left(\left(\overline{\tau}_{G_0}\right)_{(0,j,o)} \,\middle|\, (0,j,o) \in Q\right) \cup \left(\left(\overline{\tau}_{(i',j',o')}\right)_{(i,j,o)} \,\middle|\, \left(Y_{(i',j',o')}\right)_{(i,j,o)} \in \Phi'\right)$$

is a solution of the algebraic tree system S'.

Proof. If $t_i^j(o) = z_m \in Z_{r_i}$, $(i,j,o) \in Q$, then, for $(i',j',o') \in \Gamma$, $(\overline{\tau}_{(i',j',o')})_{(i,j,o)} = z_{(i',j',om)}$. Hence, the substitution of $\overline{\tau}$ into the equations of $\overline{E}$ has the same effect as the substitution of τ' into the equations of E'. ∎

Lemma 4.4. *Assume that S is a proper algebraic tree system. Then S' is a proper algebraic tree system, too.*

Proof. We inspect the equations of (ii)(3'). Since S is proper, $t_i^j(o) = G_k \in \Phi$, $(i,j,o) \in \Gamma$, implies that not all $t_i^j(om)$, $1 \le m \le r_k$, are in Z_{r_i}. Hence, trees of the form $(Y_{(i,j,o)})_{(k,j_1,\varepsilon)}$ $(z_q, q \in Q)$ do not appear in the right-hand side of equations of (ii)(3'). Furthermore, since S is proper, $t_k^{j_1}(\varepsilon) \notin Z$ for $(k, j_1, \varepsilon) \in Q$. Hence, trees of the form $z_q, q \in Q$, do not appear in the right-hand side of equations (ii)(3').

Analogous considerations on the equations of (i)(3') show that S' is proper. ∎

Lemma 4.5. *Assume that S is a proper algebraic tree system with unique solution $(\tau_0, \ldots, \tau_n)$. Then*

$$\begin{aligned}&\left(\tau_0^j(o)(\tau_0,\ldots,\tau_n) \,\middle|\, (Y_{G_0})_{(0,j,o)} \in \overline{\Phi}\right)\\ &\quad \cup \left(\tau_i^j(o)(\tau_0,\ldots,\tau_n)\left[z_{(i',j',o'm)}/z_m, 1 \le m \le r_i\right] \,\middle|\, \left(Y_{(i',j',o')}\right)_{(i,j,o)} \in \overline{\Phi}\right)\end{aligned}$$

is the unique solution of the algebraic tree system $\overline{S}$.

Proof. By Lemma 4.2, the vector of the theorem is a solution of $\overline{S}$. By Lemmas 4.3 and 4.4 and Theorem 3.8, the algebraic tree system $\overline{S}$ has a unique solution. ∎

Lemma 4.6. *Assume that S is a proper algebraic tree system and consider the algebraic tree system $S_1 = (\overline{\Phi} \cup \{y_0\}, \overline{Z}, \Sigma, \overline{E} \cup \{y_0 = \sum_{1 \le j \le n_0} a_{0j}(Y_{G_0})_{(0,j,\varepsilon)}\}, y_0)$ with initial variable y_0. Then the initial components of the least solutions of S_1 and S coincide.*

Proof. Let $(\tau_0, \ldots, \tau_n)$ be the least solution of S. The least solution of S_1 is completely determined by the least (and unique) solution of $\overline{S}$ given in Lemma 4.5. Substitution of this solution into $\sum_{1 \le j \le n_0} a_{0j}(Y_{G_0})_{(0,j,\varepsilon)}$ yields $\sum_{1 \le j \le n_0} a_{0j}\tau_0^j(\varepsilon)(\tau_0, \ldots, \tau_n) = \tau_0$. ∎

Theorem 4.7. *Assume that S is a proper algebraic tree system. Then the behavior $\|P\|$ of P coincides with the initial component of the least solution of S.*

Proof. By Corollary 3.14 and Lemma 4.6 ∎

The next corollary augments the list of equivalent statements of Corollary 3.17.

Corollary 4.8. *The following statements on a formal tree series s in $A\langle\langle T_\Sigma(X)\rangle\rangle$ are equivalent*

(i) *s is an algebraic tree series*;

(ii) *s is the behavior of a restricted pushdown tree automaton*;

(iii) *s is the behavior of a simple restricted pushdown tree automaton.*

Proof. By Theorem 4.7. ∎

We now turn to formal tree series in $N^{\infty}\langle\langle T_{\Sigma}(X)\rangle\rangle$.

THEOREM 4.9. *Let $d : T_{\Sigma}(X) \to N^{\infty}$. Then the following statement is equivalent to the statements of Theorem 3.19*:

(iii) *There exists a* 1*-simple restricted pushdown tree automaton with input alphabet Σ and leaf alphabet X such that the number* (*possibly* ∞) *of distinct computations for $t \in T_{\Sigma}(X)$ is given by $d(t)$.*

COROLLARY 4.10. *Let $L \subseteq T_{\Sigma}(X)$ be a tree language. Then L is generated by an unambiguous context-free tree grammar iff $\sum_{t\in L} t$ is the behavior of an unambiguous 1-simple restricted pushdown tree automaton.*

5. MACRO POWER SERIES

In this last section, macro power series are introduced as a generalization of the OI languages of Fischer [8] and the indexed languages of Aho [1]. Throughout this section, the alphabets Φ and Z are defined as in Section 3: $\Phi = \{G_1, \ldots, G_n\}$, $\Phi \cap \Sigma = \emptyset$, where G_i has rank r_i, $1 \le i \le n$, is a finite ranked alphabet of function variables; and $Z = \{z_1, \ldots, z_{\bar{m}}\}$, where $\bar{m} = \max\{r_i \mid 1 \le i \le n\}$ is a finite alphabet of variables. We denote $Z_m = \{z_1, \ldots, z_m\}$, $1 \le m \le \bar{m}$, and $Z_0 = \emptyset$. We assume again throughout this section that Σ and X are finite.

We define $T(\Phi, X)$ to be the set of words over $\Phi \cup X \cup \{(\} \cup \{)\} \cup \{,\}$ satisfying the following conditions:

(i) $X \cup \{\varepsilon\} \subset T(\Phi, X)$;

(ii) if $t_1, t_2 \in T(\Phi, X)$ then $t_1 t_2 \in T(\Phi, X)$;

(iii) if $G \in \Phi$, where G is of rank $r \ge 0$, and $t_1, \ldots, t_r \in T(\Phi, X)$ then $G(t_1, \ldots, t_r) \in T(\Phi, X)$.

The words of $T(\Phi, X)$ are called terms over Φ and X. By $A\langle\langle T(\Phi, X)\rangle\rangle$ (resp. $A\langle T(\Phi, X)\rangle$) we denote the set of power series whose supports are subsets (resp. finite subsets) of $T(\Phi, X)$.

Let $D' = A\langle\langle (X \cup Z_{r_1})^*\rangle\rangle \times \cdots \times A\langle\langle (X \cup Z_{r_n})^*\rangle\rangle$ and consider power series $s_i \in A\langle\langle T(\Phi, X \cup Z_{r_i})\rangle\rangle$, $1 \le i \le n$. Then each s_i induces a function $\bar{s}_i : D' \to A\langle\langle (X \cup Z_{r_i})^*\rangle\rangle$. For $(\tau_1, \ldots, \tau_n) \in D'$, we define inductively $\bar{s}_i(\tau_1, \ldots, \tau_n)$ to be

(i) z_m if $s_i = z_m$, $1 \le m \le r_i$; x if $s_i = x$, $x \in X$;

(ii) $\bar{t}_1(\tau_1, \ldots, \tau_n)\bar{t}_2(\tau_1, \ldots, \tau_n)$ if $s_i = t_1 t_2$, $t_1, t_2 \in T(\Phi, X \cup Z_{r_i})$;

(iii) $\tau_j(\bar{t}_1(\tau_1, \ldots, \tau_n), \ldots, \bar{t}_{r_j}(\tau_1, \ldots, \tau_n))$ if $s_i = G_j(t_1, \ldots, t_{r_j})$, $G_j \in \Phi$, $t_1, \ldots, t_{r_j} \in T(\Phi, X \cup Z_{r_i})$;

(iv) $a \cdot \bar{t}(\tau_1, \ldots, \tau_n)$ if $s_i = at$, $a \in A$, $t \in T(\Phi, X \cup Z_{r_i})$;

(v) $\sum_{j\in J} \bar{r}_j(\tau_1, \ldots, \tau_n)$ if $s_i = \sum_{j\in J} r_j$, $r_j \in A\langle\langle T(\Phi, X \cup Z_{r_i})\rangle\rangle$, $j \in J$, where J is an arbitrary index set.

Analogous to the proof of Lemma 3.6 it can be shown that the mapping $\bar{s} : D' \to D'$, where $\bar{s} = \langle \bar{s}_1, \ldots, \bar{s}_n\rangle$, is continuous.

A macro system $S = (\Phi, Z, X, E)$ (with function variables in Φ, variables in Z and terminal symbols in X) has a set E of formal equations

$$G_i\bigl(z_1, \ldots, z_{r_i}\bigr) = s_i\bigl(z_1, \ldots, z_{r_i}\bigr), \quad 1 \le i \le n,$$

where each s_i is in $A\langle T(\Phi, X \cup Z_{r_i})\rangle$.

A solution to the macro system S is given by $(\tau_1, \ldots, \tau_n) \in D'$ such that $\tau_i = \bar{s}_i(\tau_1, \ldots, \tau_n)$, $1 \le i \le n$, i.e., by any fixpoint $(\tau_1, \ldots, \tau_n)$ of $\bar{s} = \langle \bar{s}_1, \ldots, \bar{s}_n\rangle$. A solution $(\sigma_1, \ldots, \sigma_n)$ of the macro system S is called least solution iff $\sigma_i \sqsubseteq \tau_i$, $1 \le i \le n$, for all solutions $(\tau_1, \ldots, \tau_n)$ of S. Since the least solution of S is nothing else than the least fixpoint of $\bar{s} = \langle \bar{s}_1, \ldots, \bar{s}_n\rangle$, the least solution of the macro system S exists in D'.

THEOREM 5.1. *Let* $S = (\Phi, Z, \Sigma, \{G_i = s_i \mid 1 \le i \le n\})$ *be a macro system, where* $s_i \in A\langle T(\Phi, X \cup Z_{r_i})\rangle$. *Then the least solution of this macro system* S *exists in* D' *and equals*

$$\operatorname{fix}(\bar{s}) = \sup(\bar{s}^i(0) \mid i \in N),$$

where $\bar{s}^i$, *is the* i*th iterate of the mapping* $\bar{s} = \langle \bar{s}_1, \ldots, \bar{s}_n\rangle : D' \to D'$.

Theorem 5.1 indicates how we can compute an approximation to the least solution of a macro system. The approximation sequence $(\tau^j \mid j \in N)$, where each $\tau^j \in D'$, associated with the macro system $S = (\Phi, Z, \Sigma, \{G_i = s_i \mid 1 \le i \le n\})$ is defined as follows:

$$\tau^0 = 0, \qquad \tau^{j+1} = \bar{s}(\tau^j),\ j \in N.$$

Clearly, the least solution $\operatorname{fix}(\bar{s})$ of S is equal to $\sup(\tau^j \mid j \in N)$. A macro system $S = (\Phi \cup \{G_0\}, Z, \Sigma, \{G_i = s_i \mid 0 \le i \le n\}, G_0)$ (with function variables in $\Phi \cup \{G_0\}$, variables in Z, terminal symbols in Σ) with initial function variable G_0 is a macro system $(\Phi \cup \{G_0\}, Z, \Sigma, \{G_i = s_i \mid 0 \le i \le n\})$ such that G_0 has rank 0. Let $(\tau_0, \tau_1, \ldots, \tau_n)$ be the least solution of $(\Phi \cup \{G_0\}, Z, \Sigma, \{G_i = s_i \mid 0 \le i \le n\})$. Then τ_0 is called the initial component of the least solution. Observe that $\tau_0 \in A\langle\langle T_\Sigma(X)\rangle\rangle$ contains no variables of Z.

A power series r in $A\langle\langle X^*\rangle\rangle$ is called a macro power series iff r is the initial component of the least solution of a macro system with initial function variable.

Analogous to the proof of Theorem 3.4 of Engelfriet and Schmidt [6] it can be shown that, in the case of the Boolean semiring, $r \in B\langle\langle X^*\rangle\rangle$ is a macro power series iff $\operatorname{supp}(r) \in X^*$ is an OI language in the sense of Definition 3.10 of Fischer [8]. Moreover, by Theorem 5.3 of Fischer [8], $r \in B\langle\langle X^*\rangle\rangle$ is a macro power series iff $\operatorname{supp}(r) \in X^*$ is an indexed language (see Aho [1]).

We now define a mapping $\operatorname{yd} : A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z)\rangle\rangle \to A\langle\langle T(\Phi, X \cup Z)\rangle\rangle$. For $s \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z)\rangle\rangle$, $\operatorname{yd}(s)$ is called the yield of s; $\operatorname{yd}(s)$ is defined inductively to be

(i) z_m if $s = z_m \in Z$; x if $s = x$, $x \in X$;

(ii) $\operatorname{yd}(t_1)\ldots\operatorname{yd}(t_r)$ if $s = f(t_1, \ldots, t_r)$, $f \in \Sigma_r$, $t_1, \ldots, t_r \in T_{\Sigma\cup\Phi}(X \cup Z)$, $r \ge 0$ (observe that $\operatorname{yd}(f) = \varepsilon$ if $f \in \Sigma_0$);

(iii) $G_i(\operatorname{yd}(t_1), \ldots, \operatorname{yd}(t_{r_i}))$ if $s = G_i(t_1, \ldots, t_{r_i})$, $t_1, \ldots, t_{r_i} \in T_{\Sigma\cup\Phi}(X \cup Z)$, $1 \le i \le n$;

(iv) $\sum_{t\in T_{\Sigma\cup\Phi}(X\cup Z)}(s, t)\operatorname{yd}(t)$ if $s = \sum_{t\in T_{\Sigma\cup\Phi}(X\cup Z)}(s, t)t$.

Observe that $\operatorname{yd}(s) \in A\langle\langle (X \cup Z)^*\rangle\rangle$ if $s \in A\langle\langle T_\Sigma(X \cup Z)\rangle\rangle$. Hence, our mapping yd is an extension of the usual yield-mapping (see Gécseg and Steinby [10, Sect. 14]).

We will connect algebraic tree series and macro power series by the yield-mapping in Corollary 5.6. A series of lemmas and theorems is needed before that result.

LEMMA 5.2. *For* $1 \le m \le \bar{m}$, *let* $s \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_m)\rangle\rangle$ *and* $\tau_1, \ldots, \tau_m \in A\langle\langle T_{\Sigma\cup\Phi}(X \cup Z_m)\rangle\rangle$. *Then* $\operatorname{yd}(s(\tau_1, \ldots, \tau_m)) = \operatorname{yd}(s)(\operatorname{yd}(\tau_1), \ldots, \operatorname{yd}(\tau_m))$.

Proof. The proof is by induction on the structure of trees in $T_{\Sigma\cup\Phi}(X \cup Z_m)$.

(i) If $s = z_i$, $1 \le i \le m$, or $s = x$, $x \in X$, or $s = f$, $f \in \Sigma_0$, the lemma is clearly true.

(ii) If $s = f(t_1, \ldots, t_r)$, $f \in \Sigma_r$, $t_1, \ldots, t_r \in T_{\Sigma\cup\Phi}(X \cup Z_m)$, $r \ge 1$, then we obtain

$$\begin{aligned}
\operatorname{yd}(s(\tau_1, \ldots, \tau_m)) &= \operatorname{yd}(f(t_1(\tau_1, \ldots, \tau_m), \ldots, t_r(\tau_1, \ldots, \tau_m)))\\
&= \operatorname{yd}(t_1(\tau_1, \ldots, \tau_m))\ldots\operatorname{yd}(t_r(\tau_1, \ldots, \tau_m))\\
&= (\operatorname{yd}(t_1)\ldots\operatorname{yd}(t_r))(\operatorname{yd}(\tau_1), \ldots, \operatorname{yd}(\tau_m)) = \operatorname{yd}(s)(\operatorname{yd}(\tau_1), \ldots, \operatorname{yd}(\tau_m)).
\end{aligned}$$

(iii) If $s = G_j(t_1, \ldots, t_{r_j})$, $G_j \in \Phi$, $t_1, \ldots, t_{r_j} \in T_{\Sigma\cup\Phi}(X \cup Z_m)$, then we obtain

$$\begin{aligned}
\operatorname{yd}(s(\tau_1, \ldots, \tau_m)) &= G_j\big(\operatorname{yd}\big(t_1(\tau_1, \ldots, \tau_m), \ldots, \operatorname{yd}\big(t_{r_j}(\tau_1, \ldots, \tau_m)\big)\big)\big)\\
&= G_j\big(\operatorname{yd}(t_1)(\operatorname{yd}(\tau_1), \ldots, \operatorname{yd}(\tau_m)), \ldots, \operatorname{yd}\big(t_{r_j}\big)(\operatorname{yd}(\tau_1), \ldots, \operatorname{yd}(\tau_m))\big)\\
&= \operatorname{yd}\big(G_j\big(t_1, \ldots, t_{r_j}\big)\big)(\operatorname{yd}(\tau_1), \ldots, \operatorname{yd}(\tau_m)) = \operatorname{yd}(s)(\operatorname{yd}(\tau_1), \ldots, \operatorname{yd}(\tau_m)).
\end{aligned}$$

For $s = \sum_{t\in T_{\Sigma\cup\Phi}(X\cup Z_m)}(s,t)t$ we now obtain

$$\begin{aligned}\mathrm{yd}(s(\tau_1,\ldots,\tau_m)) &= \sum_{t\in T_{\Sigma\cup\Phi}(X\cup Z_m)}(s,t)\mathrm{yd}(t(\tau_1,\ldots,\tau_m))\\ &= \sum_{t\in T_{\Sigma\cup\Phi}(X\cup Z_m)}(s,t)\mathrm{yd}(t)(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_m))\\ &= \mathrm{yd}(s)(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_m)).\end{aligned}$$ ■

LEMMA 5.3. *For $1 \le i \le n$, let $s_i \in A\langle\langle T_{\Sigma\cup\Phi}(X\cup Z_{r_i})\rangle\rangle$ and $(\tau_1,\ldots,\tau_n) \in D = A\langle\langle T_\Sigma(X\cup Z_{r_1})\rangle\rangle \times \cdots \times A\langle\langle T_\Sigma(X\cup Z_{r_n})\rangle\rangle$. Then $\mathrm{yd}(\overline{s_i}(\tau_1,\ldots,\tau_n)) = \overline{s_i'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n))$, where $s_i' = \mathrm{yd}(s_i)$, $1\le i\le n$.*

Proof. The proof is by induction on the structure of trees in $T_{\Sigma\cup\Phi}(X\cup Z_{r_i})$, $1\le i\le n$.

(i) If $s_i = z_m$, $1\le m\le r_i$, or $s_i = x$, $x\in X$, or $s_i = f$, $f\in\Sigma_0$, the lemma is clearly true.

(ii) If $s_i = f(t_1,\ldots,t_r)$, $f\in\Sigma_r$, $t_1,\ldots,t_r\in T_{\Sigma\cup\Phi}(X\cup Z_{r_i})$, $r\ge 1$, then

$$\overline{s_i}(\tau_1,\ldots,\tau_n) = f(\overline{t_1}(\tau_1,\ldots,\tau_n),\ldots,\overline{t_r}(\tau_1,\ldots,\tau_n))$$

and

$$\begin{aligned}&\mathrm{yd}(\overline{s_i}(\tau_1,\ldots,\tau_n)) = \mathrm{yd}(\overline{t_1}(\tau_1,\ldots,\tau_n))\ldots\mathrm{yd}(\overline{t_r}(\tau_1,\ldots,\tau_n))\\ &\quad=\overline{t_1'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n))\ldots\overline{t_r'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n)) = \overline{s_i'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n)),\end{aligned}$$

where $t_j' = \mathrm{yd}(t_j)$, $1\le j\le r$, and $s_i' = \mathrm{yd}(s_i)$.

(iii) If $s_i = G_j(t_1,\ldots,t_{r_j})$, $G_j\in\Phi$, $t_1,\ldots,t_{r_j}\in T_{\Sigma\cup\Phi}(X\cup Z_{r_i})$, then

$$\overline{s_i}(\tau_1,\ldots,\tau_n) = \tau_j\big(\overline{t_1}(\tau_1,\ldots,\tau_n),\ldots,\overline{t_{r_j}}(\tau_1,\ldots,\tau_n)\big)$$

and we obtain, by Lemma 5.2,

$$\begin{aligned}&\mathrm{yd}(\overline{s_i}(\tau_1,\ldots,\tau_n)) = \mathrm{yd}(\tau_j)\big(\mathrm{yd}(\overline{t_1}(\tau_1,\ldots,\tau_n)),\ldots,\mathrm{yd}\big(\overline{t_{r_j}}(\tau_1,\ldots,\tau_n)\big)\big)\\ &\quad= \mathrm{yd}(\tau_j)\big(\overline{t_1'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n)),\ldots,\overline{t_{r_j}'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n))\big)\\ &\quad= \overline{s_i'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n)),\end{aligned}$$

where $t_k' = \mathrm{yd}(t_k)$, $1\le k\le r_j$, and $s_i' = \mathrm{yd}(s_i)$.

For $s_i = \sum_{t\in T_{\Sigma\cup\Phi}(X\cup Z_{r_i})}(s_i,t)t$ we now obtain

$$\begin{aligned}\mathrm{yd}(\overline{s_i}(\tau_1,\ldots,\tau_n)) &= \sum_{t\in T_{\Sigma\cup\Phi}(X\cup Z_{r_i})}(s_i,t)\mathrm{yd}(\overline{t}(\tau_1,\ldots,\tau_n))\\ &= \sum_{t\in T_{\Sigma\cup\Phi}(X\cup Z_{r_i})}(s_i,t)\overline{t'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n)) = \overline{s_i'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n)),\end{aligned}$$

where $t' = \mathrm{yd}(t)$, $t\in T_{\Sigma\cup\Phi}(X\cup Z_{r_i})$, and $s_i' = \mathrm{yd}(s_i)$. ■

Given an algebraic tree system $S = (\Phi, Z, X, \{G_i(z_1,\ldots,z_{r_i}) = s_i \mid 1\le i\le n\})$, we define the macro system $\mathrm{yd}(S)$ to be $\mathrm{yd}(S) = (\Phi, Z, X, \{G_i(z_1,\ldots,z_{r_i}) = \mathrm{yd}(s_i) \mid 1\le i\le n\})$.

THEOREM 5.4. *Let $(\tau_1,\ldots,\tau_n)$ be a solution of the algebraic tree system S. Then $(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n))$ is a solution of the macro system $\mathrm{yd}(S)$.*

Proof. Assume that the equations of S are given by $G_i(z_1,\ldots,z_{r_i}) = s_i$, $1\le i\le n$. Then $\tau_i = \overline{s_i}(\tau_1,\ldots,\tau_n)$, $1\le i\le n$. We now apply the yield-mapping to both sides of this equation and obtain, by Lemma 5.3, $\mathrm{yd}(\tau_i) = \mathrm{yd}(\overline{s_i}(\tau_1,\ldots,\tau_n)) = \overline{s_i'}(\mathrm{yd}(\tau_1),\ldots,\mathrm{yd}(\tau_n))$, where $s_i' = \mathrm{yd}(s_i)$, $1\le i\le n$. Hence,

$(\mathrm{yd}(\tau_1), \ldots, \mathrm{yd}(\tau_n))$ is a solution of the macro system $\mathrm{yd}(S)$ with equations $G_i(z_1, \ldots, z_{r_i}) = \mathrm{yd}(s_i)$, $1 \le i \le n$. ■

THEOREM 5.5. *If* $(\tau_1, \ldots, \tau_n)$ *is the least solution of the algebraic tree system* S *then* $(\mathrm{yd}(\tau_1), \ldots, \mathrm{yd}(\tau_n))$ *is the least solution of the macro system* $\mathrm{yd}(S)$.

Proof. Assume that the equations of S are given by $G_i(z_1, \ldots, z_{r_i}) = s_i$, $1 \le i \le n$. Consider the approximation sequences $(\tau^j \mid j \in N)$ and $(\sigma^j \mid j \in N)$ of S and $\mathrm{yd}(S)$ with least upper bounds τ and σ, respectively. We claim that $\sigma^j = \mathrm{yd}(\tau^j)$ for all $j \ge 0$ and prove it by inducion on j.

The induction basis being clear, we prove the induction step. Let $j \ge 0$. Then, for $1 \le i \le n$

$$\begin{aligned}\sigma_i^{j+1} &= \overline{s_i'}\big(\sigma_1^j, \ldots, \sigma_n^j\big) = \overline{s_i'}\big(\mathrm{yd}\big(\tau_1^j\big), \ldots, \mathrm{yd}\big(\tau_n^j\big)\big)\\ &= \mathrm{yd}\big(\overline{s_i}\big(\tau_1^j, \ldots, \tau_n^j\big)\big) = \mathrm{yd}\big(\tau_i^{j+1}\big),\end{aligned}$$

where $s_i' = \mathrm{yd}(s_i)$. Hence, $\sigma_i = \mathrm{yd}(\tau_i)$ for all $1 \le i \le n$ and the theorem is proven. ■

Observe that Theorem 5.5 could also be proved by application of Theorem 34 of Bozapalidis [5].

COROLLARY 5.6. *If* s *is an algebraic tree series then* $\mathrm{yd}(s)$ *is a macro power series.*

LEMMA 5.7. *Let* $\Sigma = \{\cdot, e\}$, *where* $\cdot$ *and* e *have rank* 2 *and* 0, *respectively. Then, for each term* $w \in T(\Phi, X \cup Z)$, *there exists a tree* $t \in T_{\Sigma \cup \Phi}(X \cup Z)$ *such that* $w = \mathrm{yd}(t)$.

Proof. Obvious. ■

THEOREM 5.8. *Let* $\Sigma = \{\cdot, e\}$, *where* $\cdot$ *and* e *have rank 2 and 0, respectively. Then a power series* $r \in A\langle\langle X^*\rangle\rangle$ *is a macro power series iff there exists an algebraic tree series* $s \in A\langle\langle T_\Sigma(X)\rangle\rangle$ *such that* $\mathrm{yd}(s) = r$.

EXAMPLE 5. Let $S = (\Phi, Z, \Sigma, E, Z_0)$ be the algebraic tree system of Example 4 and consider the algebraic tree system $S' = (\Phi, Z, \Sigma', E, Z_0)$, where $\Sigma' = \{b\}$ and $X = \{c_1, c_2\}$. Then the initial component of the least solution of S' is $\sum_{j\ge0} b(t_1^j, t_2^j)$, where t_1^j and t_2^j, $j \ge 0$, are defined in Example 3. The macro system $\mathrm{yd}(S') = (\Phi, Z, \Sigma', E', Z_0)$ is specified by the following formal equations of E':

$$\begin{aligned}G_0(z_0, z_1, z_2) &= G_0(G_0(z_0, z_1, z_2), G_1(z_0, z_1, z_2), G_2(z_0, z_1, z_2)) + z_1 z_2,\\ G_i(z_0, z_1, z_2) &= z_i z_i, \quad i = 1, 2,\\ Z_0 &= G_0(0, c_1, c_2).\end{aligned}$$

The initial component of the least solution of $\mathrm{yd}(S')$ is $\sum_{j\ge0} c_1^{2^j} c_2^{2^j}$. ■

We now introduce yield automata. A yield automaton (with input alphabet Σ and leaf alphabet X)

$$A = (I, M, S, P)$$

is given by

(i) a nonempty set I of states,

(ii) a sequence $M = (M_k \mid k \ge 1)$ of transition matrices $M_k \in (A\langle\langle(X \cup Y_k)^*\rangle\rangle)^{I \times I^k}$, $k \ge 1$,

(iii) $S \in (A\langle\langle(X \cup Y_1)^*\rangle\rangle)^{1\times I}$, called the initial state vector,

(iv) $P \in (A\langle\langle X^*\rangle\rangle)^{I\times 1}$, called the final state vector.

The approximation sequence $(\sigma^j \mid j \in N)$, $\sigma^j \in (A\langle\langle X^*\rangle\rangle)^{I\times1}$, $j \ge 0$, associated with A is defined as follows:

$$\sigma^0 = 0, \qquad \sigma^{j+1} = \sum_{k\ge1} M_k(\sigma^j, \ldots, \sigma^j) + P, \quad j \ge 0.$$

The behavior $\|A\| \in A\langle\langle X^*\rangle\rangle$ of the yield automaton A is defined by

$$\|A\| = \sum_{i\in I} S_i(\sigma_i) = S(\sigma),$$

where $\sigma \in (A\langle\langle X^*\rangle\rangle)^{I\times 1}$ is the least upper bound of the approximation sequence associated with A.

Let $A = (I, M, S, P)$ be a tree automaton. Then we define the yield automaton yd(A) to be $\mathrm{yd}(A) = (I, \mathrm{yd}(M), \mathrm{yd}(S), \mathrm{yd}(P))$.

THEOREM 5.9. *Let A be a tree automaton. Then* $\|\mathrm{yd}(A)\| = \mathrm{yd}(\|A\|)$.

Proof. Let $A = (I, M, S, P)$ and let $(\sigma^j \mid j \in N)$ and $(\tau^j \mid j \in N)$ be the approximation sequences associated with A and yd(A), respectively. Then we claim that $\tau^j = \mathrm{yd}(\sigma^j)$, $j \geq 0$, and show it by induction on j. The induction basis being clear, let $j \geq 0$. Then we obtain, for all $i \in I$,

$$\begin{aligned}
\tau_i^{j+1} &= \sum_{k\geq 1}\sum_{i_1,\ldots,i_k\in I} \mathrm{yd}\big((M_k)_{i,(i_1,\ldots,i_k)}\big)\big(\tau_{i_1}^j,\ldots,\tau_{i_k}^j\big)\big) + \mathrm{yd}(P_i)\\
&= \sum_{k\geq 1}\sum_{i_1,\ldots,i_k\in I} \mathrm{yd}\big((M_k)_{i,(i_1,\ldots,i_k)}\big)\big(\mathrm{yd}\big(\sigma_{i_1}^j\big),\ldots,\mathrm{yd}\big(\sigma_{i_k}^j\big)\big) + \mathrm{yd}(P_i)\\
&= \mathrm{yd}\left(\sum_{k\geq 1}\sum_{i_1,\ldots,i_k\in I} (M_k)_{i,(i_1,\ldots,i_k)}\big(\sigma_{i_1}^j,\ldots,\sigma_{i_k}^j\big) + P_i\right) = \mathrm{yd}\big(\sigma_i^{j+1}\big).
\end{aligned}$$

Let σ and τ be the least upper bounds of the approximation sequences $(\sigma^j \mid j \in N)$ and $(\tau^j \mid j \in N)$, respectively. Then $\tau = \mathrm{yd}(\sigma)$ and we obtain

$$\|\mathrm{yd}(A)\| = \sum_{i\in I}\mathrm{yd}(S_i)(\mathrm{yd}(\sigma_i)) = \mathrm{yd}\left(\sum_{i\in I} S_i(\sigma_i)\right) = \mathrm{yd}(\|A\|). \qquad \blacksquare$$

A pushdown yield automaton (with input alphabet Σ and leaf alphabet X)

$$P = (Q, \Gamma, Z, Y, M, S, p_0, P)$$

is given by

(i) a finite nonempty set Q of states;

(ii) a finite ranked alphabet $\Gamma = \Gamma_0 \cup \Gamma_1$ of pushdown symbols, where $\Gamma_0 = \{p_0\}$;

(iii) an alphabet $Z = \{z\}$; z is called pushdown variable;

(iv) a finite alphabet $Y = \{y_1, \ldots, y_{\overline{k}}\}$ of variables;

(v) a finite sequence $M = (M_k \mid 1 \leq k \leq \overline{k})$ of pushdown yield transition matrices M_k of order k, $1 \leq k \leq \overline{k}$;

(vi) $S \in (A\langle(X \cup Y_1)^*\rangle)^{1\times Q}$, called the initial state vector;

(vii) p_0 is called the initial pushdown symbol;

(viii) a finite family $P = (P_{g(z)} \mid g \in \Gamma_1) \cup P_{p_0}$ of final state vectors in $(A\langle X^*\rangle)^{Q\times 1}$.

Here a pushdown yield transition matrix of order k, $1 \leq k \leq \overline{k}$, is a row finite matrix

$$M \in \big((A\langle(X \cup Y_k)^*\rangle)^{Q\times Q^k}\big)^{T_\Gamma(Z)\times T_\Gamma(Z)^k}$$

which satisfies the following condition for all $t, t_1, \ldots, t_k \in T_\Gamma(Z)$:

$$M_{t,(t_1,\ldots,t_k)} = \begin{cases} M_{g(z),(v_1(z),\ldots,v_k(z))} & \text{if } g \in \Gamma_1, t = g(u), t_j = v_j(u), 1 \leq j \leq k, \text{ for some } u \in T_\Gamma(Z);\\ 0, & \text{otherwise}. \end{cases}$$

Now let $Z_Q = \{z_q \mid q \in Q\}$ be an alphabet of variables. Define $F \in ((A\langle(X \cup Z_Q)^*\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ by its entries as follows:

(i) $(F_t)_q = (P_{g(z)})_q$ if $t = g(u)$, $g \in \Gamma_1$, $u \in T_\Gamma(Z)$, $q \in Q$;

(ii) $(F_z)_q = z_q$, $q \in Q$;

(iii) $(F_t)_q = 0$, otherwise.

Hence, F_z is a column vector of dimension Q whose q-entry, $q \in Q$, is the variable z_q.

The approximation sequence $(\tau^j \mid j \in N)$, $\tau^j \in ((A\langle(X \cup Z_Q)^*\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$, $j \geq 0$, associated with P is defined as follows:

$$\tau^0 = 0, \qquad \tau^{j+1} = \sum_{1\leq k\leq \bar{k}} M_k(\tau^j, \ldots, \tau^j) + F, \quad j \geq 0.$$

Let $\tau \in ((A\langle\langle(X \cup Z_Q)^*\rangle\rangle)^{Q\times 1})^{T_\Gamma(Z)\times 1}$ be the least upper bound of the approximation sequence associated with P. Then the behavior $\|P\| \in A\langle\langle X^*\rangle\rangle$ of the pushdown yield automaton P is defined by

$$\|P\| = S(\tau_{p_0}) = \sum_{q\in Q} S_q\big((\tau_{p_0})_q\big).$$

Theorem 5.9 yields now a corollary for restricted pushdown tree automata.

COROLLARY 5.10. *Let P be a restricted pushdown tree automaton. Then*

$$\|yd(P)\| = yd(\|P\|).$$

Now Corollaries 3.17, 4.8, and 5.10 yield the main result of this section.

THEOREM 5.11. *The following statements on a formal power series $s \in A\langle\langle X^*\rangle\rangle$ are equivalent*:

(i) *s is a macro power series*;

(ii) *s is the yield of the behavior of a simple pushdown tree automaton*;

(iii) *s is the yield of the behavior of a simple restricted pushdown tree automaton;*

(iv) *s is the behavior of a pushdown yield automaton.*

Theorem 5.11 shows that in the case of the Boolean semiring, the nested stack automata of Aho [2] are equivalent to pushdown yield automata. (See Theorem 5.3 and Corollary 5.10 of Fischer [8], and Theorem 5.3 of Aho [2].)

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