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GRAPH CONGRUENCES AND WREATH PRODUCTS*

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Results on graph congruences have recently been used to decide membership in varieties defined by wreath products. The general technique for doing so is explained. In particular an efficient algorithm is obtained for deciding when a semigroup divides a wreath product of a commutative monoid with locally trivial semigroup.

Des résultats concernant les congruences sur les graphes ont récemment été utilisés pour décider du problème d'appartenance à des variétés définies par le produit en couronne. On explique ici la technique générale qui permet de le faire. En particulier, on obtient un bon algorithme pour décider si un semigroupe divise un produit en couronne d'un monoide commutatif avec un semigroupe localement trivial.

0. Introduction

A natural problem arising in automaton theory is to determine when a given machine can be simulated by a serial connection of components with certain properties. This problem can be dealt with in purely algebraic terms, replacing machines by semigroups, serial connection by wreath product and simulation by division. Recent work on such problems have used a graph-theoretic interpretation of the wreath product ([2, VIII.6], [3], [6]): the idea is to associate with the wreath product of two semigroups S and T a graph G_T and a graph congruence β_S on G_T; properties of β_S are then interpreted back in the algebraic context.

In this paper we will formalize this point of view and give several examples of questions that can be advantageously interpreted in this framework. In particular we establish an algorithm to decide when a semigroup can be decomposed as a wreath product of a commutative monoid and a locally trivial semigroup.

Let S, T be semigroups. We say that T divides S and write $T \prec S$ if T is a morphic image of a subsemigroup of S. An S-variety $\mathbf{V}$ is a collection of finite semigroups closed under division and finite direct product. A similar definition is made for M-variety with monoids being considered instead. We refer the reader to [2] for terms not explicitely defined here.

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The wreath product of S and T is defined as

$$S \circ T = S^{T^{\cdot}} \times T,$$

where $T^{\cdot}$ is the smallest monoid containing T and $S^{T^{\cdot}}$ stands for the set of all functions $f: T^{\cdot} \to S$; the product of $S \circ T$ is defined by

$$(f, u)(g, v) = (h, uv)$$

where, for any $t \in T^{\cdot}$, $th = (tf)((tu)g)$. The wreath product of two S-varieties $\mathbf{V}$ and $\mathbf{W}$ is defined as the S-variety

$$\mathbf{V} * \mathbf{W} = \{U: U < S \circ T, S \in \mathbf{V}, T \in \mathbf{W}\}.$$

One can adapt this definition to the case where $\mathbf{V}$ or $\mathbf{W}$ is a M-variety. We consider the following question: given a semigroup U and varieties $\mathbf{V}$ and $\mathbf{W}$, how can one decide if $U \in \mathbf{V} * \mathbf{W}$ or not?

The notion of congruence will play a central role in our approach. For any finite set A denote by A^+ (A^*) the free semigroup (monoid) generated by A: we say that a semigroup S is A-generated iff there exists a congruence γ on A^+ such that S is isomorphic to A^+/γ. An S-variety $\mathbf{V}$ is locally finite iff for any A there are finitely many A-generated semigroups in $\mathbf{V}$; equivalently there exists for each A a congruence γ_A such that an A-generated semigroup S is in $\mathbf{V}$ iff S is a morphic image of A^+/γ_A. The same concepts can be defined in terms of monoids, A^* and M-varieties instead. If $\{\mathbf{V}_i\}_{i \in I}$ is a collection of S-varieties (M-varieties) we define $\bigvee_i \mathbf{V}_i$ to be the smallest S-variety (M-variety) containing $\bigcup_i \mathbf{V}_i$. Unless otherwise specified any congruence we discuss has finite index and every non-free semigroup or monoid is finite.

Example 0.1. For any $x \in A^*$, $a \in A$, $k \geq 0$, we denote by $|x|$ the length of x, by $|x|_a$ the number of factorizations of x of the form $x = x_0 a x_1$, by $x\alpha_k$ the prefix of length $\min\{|x|, k\}$ of x, and by $x\omega_k$ the suffix of length $\min\{|x|, k\}$ of x. On A^+ we define the congruence α_k by $x \, \alpha_k \, y$ iff $x\alpha_k = y\alpha_k$, and the congruence ω_k by $x \, \omega_k \, y$ iff $x\omega_k = y\omega_k$. Considering the S-varieties

$$\mathbf{D}_k = \{S: \text{for all } e = e^2,\ Se = e \text{ and } S^k = S^{k+1}\}$$

and

$$\mathbf{Ll}_k = \{S: \text{for all } e = e^2,\ eSe = e \text{ and } S^k = S^{k+1}\}$$

we have that an A-generated semigroup S is in $\mathbf{D}_k$ ($\mathbf{Ll}_k$) iff S is a morphic image of A^+/ω_k ($A^+/\alpha_k \cap \omega_k$). Thus $\mathbf{D}_k$ and $\mathbf{Ll}_k$ are locally finite, but $\mathbf{D} = \bigvee_k \mathbf{D}_k$ and $\mathbf{Ll} = \bigvee_k \mathbf{Ll}_k$ are not. Next for any $t \geq 0$, $q \geq 1$, we define on A^* the congruence $\gamma_{t,q}$ by $x \, \gamma_{t,q} \, y$ iff for all $a \in A$, $|x|_a = |y|_a$ or $|x|_a, |y|_a \geq t$ and $|x|_a \equiv |y|_a \pmod q$. Considering the M-variety

$$\mathbf{Com}_{t,q} = \{M: \text{for all } m, n,\ mn = nm,\ m^t = m^{t+q}\}$$

we have that an A-generated monoid M is in $\mathbf{Com}_{t,q}$ iff M is a morphic image of $A^*/\gamma_{t,q}$. Thus $\mathbf{Com}_{t,q}$ is finitely generated although $\mathbf{Com} = \bigvee_{t,q} \mathbf{Com}_{t,q}$ is not.

1. Graph congruences

A graph $G=(V,A,\alpha,\omega)$ consists in a set V of vertices, a set A of edges and two mappings $\alpha,\omega: A\to V$ which to each edge a assigns the start vertex $a\alpha$ and the end vertex $a\omega$ of that edge. Two edges a,b are consecutive iff $a\omega=b\alpha$. A path of length $n, n>0$, is a sequence of n consecutive edges; if $x=a_1\cdots a_n$ is a path with $a_1,\ldots,a_n\in A$, we let $x\alpha=a_1\alpha$ and $x\omega=a_n\omega$. For each vertex v we allow an empty path 1_v of length 0 for which $1_v\alpha=1_v\omega=v$. Two paths x,y are coterminal, written $x\sim y$, iff $x\alpha=y\alpha$, $x\omega=y\omega$. A loop is a path x such that $x\alpha=x\omega$. We define

$$x\gamma=\{a: a\in A, x=x_0ax_1\}$$

and

$$xv=\{v: v\in V \text{ and } x=x_0x_1 \text{ with } x_0\omega=v\}.$$

An equivalence β on the set P of all paths in G is a graph congruence iff $x\,\beta\, y$ implies $x\sim y$, and $x_1\,\beta\, y_1$, $x_2\,\beta\, y_2$, $x_1\omega=x_2\alpha$ imply $x_1x_2\,\beta\, y_1y_2$. A congruence γ on A^* induce a graph congruence $\bar\gamma$ on G by defining for any two paths x,y, $x\,\bar\gamma\, y$ iff $x\sim y$ and $x\,\gamma\, y$, with the interpretation that $1_v\,\bar\gamma\, x$ if $1\,\gamma\, x$; on the other hand, for a graph congruence β there might exist more than one congruence γ on A^* such that $\bar\gamma=\beta$. One way of inducing a monoid M_β from a graph congruence β is to let $M_\beta=P/\beta\cup\{1,0\}$ with the product defined as $[x]_\beta[y]_\beta=[xy]_\beta$ if $x\omega=y\alpha$ and $[x]_\beta[y]_\beta=0$ otherwise.

For any M-variety $\mathbf{V}$ the graph congruence β is said to be a $\mathbf{V}$-congruence iff there exists γ on A^* such that $A^*/\gamma\in\mathbf{V}$ and $\bar\gamma\subseteq\beta$. In particular if $M_\beta\in\mathbf{V}$, then β is a $\mathbf{V}$-congruence, but this condition is obviously not necessary; see Example 1.1. We make the following observations. If V contains only one vertex, then $P=A^*$ and β is a graph congruence on G iff β is a congruence on A^*. For any subgraph $G'=(V',E',\alpha',\omega')$ of G the restriction $\beta_{G'}$ of β to G' is a $\mathbf{V}$-congruence whenever β is a $\mathbf{V}$-congruence. For any vertex v the set $M_v=\{[x]_\beta: x\alpha=x\omega=v\}$ forms a monoid; we say that β is a locally $\mathbf{V}$-congruence iff $M_v\in\mathbf{V}$ for all vertices v. Clearly any $\mathbf{V}$-congruence is also a locally $\mathbf{V}$-congruence. The status of the converse depends on the specific $\mathbf{V}$ considered; see Examples 1.3 and 1.4 below.

Example 1.1. Let a be an edge such that $a\alpha$ and $a\omega$ belong to distinct strongly connected components of G. Define $x\,\beta\, y$ iff $x\sim y$ and $|x|_a\geq 1$ iff $|y|_a\geq 1$; note that $|x|_a$ can only be 0 or 1. We show that β is a $\mathbf{V}$-congruence for any non-trivial $\mathbf{V}$. Let $M\in\mathbf{V}$ be any non-trivial monoid and let $\phi: A^*\to M$ be any morphism, such that $b\phi\neq 1$ iff $b=a$; the induced congruence ϕ on A^* is such that $A^*/\phi\in\mathbf{V}$ and $\bar\phi\subseteq\beta$, thus proving the claim. Note that in particular β is a $\mathbf{G}$-congruence for any non-trivial variety $\mathbf{G}$ of groups although $M_\beta\notin\mathbf{G}$.

We now show that, for non-trivial $\mathbf{V}$, testing if β is a $\mathbf{V}$-congruence can be done by considering only the restrictions of β to the various strongly connected components of the graph G. Suppose G contains r strongly connected components: for

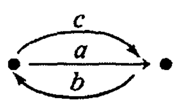
$i=1,\dots,r$ let $G_i=(V_i, A_i, \alpha_i, \omega_i)$ be the subgraphs of G that they determine and let β_i be the restriction of β to G_i.

Theorem 1.2. *β is a* **V**-*congruence iff β_i is a* **V**-*congruence for $i=1,\dots,r$.*

Proof. If β is a **V**-congruence, then the restriction of β to any subgraph of G is a **V**-congruence. Suppose now that β_i is a **V**-congruence for $i=1,\dots,r$; there then exists ϕ_i: $A_i^* \to M_i$, $M_i \in \mathbf{V}$, such that $x\bar{\phi}_i y$ implies $x\beta_i y$: we extend this morphism to ϕ_i: $A^* \to M_i$ by letting $a\phi=1$ if $a \notin A_i$. Let also $A_0=A-\bigcup_i A_i$, i.e. A_0 consists of those edges a for which $a\alpha$ and $a\omega$ belong to distinct strongly connected components of G. If $A_0=\{a_1,\dots,a_s\}$, choose for each $j=1,\dots,s$ a non-trivial $M_{r+j} \in \mathbf{V}$ and a morphism $\phi_{r+j}: A^* \to M_{r+j}$ such that $b\phi_{r+j} \neq 1$ iff $b=a_j$. Let $\phi: A^* \to M_1 \times \cdots \times M_{r+s}$ be defined by $a\phi=(a\phi_1,\dots,a\phi_{r+s})$. If $x \sim y$ and $x\phi=y\phi$, then $x=x_0b_1x_1\cdots b_nx_n$, $y=y_0b_1y_1\cdots b_ny_n$ where $b_1,\dots,b_n \in A_0$, and for $j=0,\dots,n$, $x_j, y_j \in A_{i_j}^*$, $x_j \sim y_j$ and $x_j\phi_{i_j}=y_j\phi_{i_j}$. Hence $x_j \beta_{i_j} y_j$ and since β_{i_j} is the restriction of β, also $x_j \beta y_j$. This proves that $x\beta y$ and β is thus a **V**-congruence. □

Example 1.3. Let **H** be a non-trivial M-variety consisting only of groups and suppose that β is a locally **H**-congruence. We show that β is a **H**-congruence. By Theorem 1.2 we can assume that G is strongly connected. Select a vertex v and for each vertex v_i choose paths x_i, y_i such that $x_i\alpha=y_i\omega=v$, $x_i\omega=y_i\alpha=v_i$ and $y_ix_i \, \beta \, 1_{v_i}$. Define $\phi: A^* \to M_v$ by $a\phi=[x_iay_j]_\beta$ if $a\alpha=v_i$, $a\omega=v_j$. If $x\alpha=v_i$, $x\omega=v_j$, then $x\phi=[x_ixy_j]_\beta$. Now if $x \sim y$ and $x\phi=y\phi$, we get $x_ixy_j \, \beta \, x_iyy_j$. Multiplying on the left by y_i and on the right by x_j we deduce that $x\beta y$, so that β is a **V**-congruence.

Example 1.4. Let G be the following graph:



Define $x\beta y$ iff $x \sim y$ and ($x=y$ or $|x|, |y|>3$). Let x, y be loops about some vertex v. If $x=1_v$ or $y=1_v$, then $xy=yx$; otherwise $|xy|>3$ so that $xy\beta yx$. Thus β is a locally **Com**-congruence. On the other hand β cannot be a **Com**-congruence since $A^*/\gamma \in \mathbf{Com}$ would imply $abc\,\bar{\gamma}\,cba$, so that $\bar{\gamma}$ would not refine β.

The next theorem is basically a result of Straubing [7]. Let B_n denote the monoid $\{(i,j): 1 \le i, j \le n\} \cup \{0,1\}$ with multiplication defined as $(i,j)(k,p)=(i,p)$ if $j=k$ and $(i,j)(k,p)=0$ otherwise. Note that

$$B_n < B_2 \times \cdots \times B_2 \quad (\lceil \log_2 n \rceil \text{ times}).$$

Theorem 1.5. *If $B_2 \in$* **V**, *then β is a* **V**-*congruence iff $M_\beta \in$* **V**.

Proof. Consider the morphism $\phi: A^* \to M_\beta$ induced by $a\phi=[a]_\beta$. It is clear that $A^*/\phi < M_\beta$ and that $\bar{\phi} \subseteq \beta$. Thus if $M_\beta \in \mathbf{V}$, β is a **V**-congruence. Conversely if β is a $\mathbf{V}$-congruence there exists a congruence γ on A^* such that $A^*/\gamma \in \mathbf{V}$ and $\bar{\gamma} \subseteq \beta$. Let the vertices of G be $v_1, \ldots, v_n$. Consider the submonoid M of $A^*/\gamma \times B_n$ generated by the set $\{([x]_\gamma, (i,j))$; there exists a path y such that $x\gamma y$, $y\alpha = v_i$, $y\omega = v_j\}$. Define $\phi : M \to M_\beta$ by $([x]_\gamma, (i,j))\phi = [y]_\beta$: it is easily verified that ϕ is a morphism. Hence $M_\beta < A^*/\gamma \times B_n$ and $M_\beta \in \mathbf{V}$. □

Example 1.6. One important source of examples of graphs consists of what are known as semigroup automata. Let T be an A-generated semigroup: for clarity, if $x \in A^*$ we write $\mathbf{x}$ for the value in T of the string x. The graph G_T is defined as $(T^\cdot, T^\cdot \times A, \alpha, \omega)$ where $(v,a)\alpha = v$ and $(v,a)\omega = v\mathbf{a}$. For $x = a_1 \cdots a_n \in A^*$, $v \in T^\cdot$, we write x_v to denote the path 1_v if $n = 0$, or the path $(v, a_1)(v\mathbf{a}_1, a_2) \cdots (v\mathbf{a}_1 \cdots \mathbf{a}_{n-1}, a_n)$ if $n > 0$. This establishes a bijection between $\{x_v : x \in A^*, v \in T^\cdot\}$ and the set P_T of paths in G_T.

2. Graph congruences and wreath products

Let T be an A-generated semigroup, β a graph congruence on G_T. This induces a congruence on A^+ by requesting that $x\beta^*Ty$ iff $x_v \beta y_v$ for all $v \in T^\cdot$.

Lemma 2.1. *If β is a $\mathbf{V}$-congruence, then $A^+/\beta^*T < M \circ T$ for some $M \in \mathbf{V}$.*

Proof. Let δ be a congruence on $(T^\cdot \times A)^*$ such that $M = (T^\cdot \times A)^*/\delta \in \mathbf{V}$ and $\bar{\delta} \subseteq \beta$. Define $\phi : A^+ \to M \circ T$ by $a\phi = (f_a, \mathbf{a})$ where $vf_a = [a_v]\delta$. One verifies that $x\phi = (f_x, \mathbf{x})$ where $vf_x = [x_v]_\delta$. Thus $x\phi = y\phi$ implies $\mathbf{x} = \mathbf{y}$ which in turn implies that $x_v \sim y_v$ for any $v \in T^\cdot$. Also $x\phi = y\phi$ implies $x_v \,\delta\, y_v$: since $\bar{\delta} \subset \beta$, we get $x_v \beta y_v$ for all $v \in T^\cdot$, hence $x\beta^*Ty$. This shows that $A^+/\beta^*T < M \circ T$. □

We can now relate graph congruences and wreath products. We will consider the case when $\mathbf{V}$ is an M-variety and $\mathbf{W}$ is an S-variety, but other combinations can be similarly handled.

Theorem 2.2. *Let $S = A^+/\delta$ be an A-generated semigroup. Then $S \in \mathbf{V} * \mathbf{W}$ iff $\delta \supseteq \beta^*T$ where $T \in \mathbf{W}$ is an A-generated semigroup and β is a $\mathbf{V}$-congruence on G_T.*

Proof. If $\delta \supseteq \beta^*T$, then $S < A^+/\beta^*T < M \circ T$ for some $M \in \mathbf{V}$ by Lemma 2.1. Hence $S \in \mathbf{V} * \mathbf{W}$. For the converse let δ be a congruence on A^+ such that $A^+/\delta < M \circ T'$ with $M \in \mathbf{V}$ and $T' \in \mathbf{W}$. There exists $\phi : A^+ \to M \circ T'$ such that $x\phi = y\phi$ implies $x\,\delta\, y$. Let $\pi : M \circ T' \to T'$ be the projection. Let $T = A^+\phi\pi$: thus $T < T'$ and T is an A-generated semigroup in $\mathbf{W}$. Define an equivalence on P_T by $x_v \beta y_v$ iff $x\phi = y\phi$; β is clearly a $\mathbf{V}$-congruence. We now have $x\beta^*Ty$ iff $x\phi = y\phi$ which implies $x\,\delta\, y$; thus $\delta \supseteq \beta^*T$ as was to be shown. □

Example 2.3. Let $S = A^+/\delta$ and $T = A^+/\gamma$ be A-generated semigroups. Define δ' on

G_T by $x_v \delta' y_v$ iff $x_v \sim y_v$ and $x \delta y$; δ' is a graph congruence and $\delta'^*T = \delta \cap \gamma \subseteq \delta$. Thus, if δ' is a **V**-congruence we get $S < M \circ T$ for some $M \in \mathbf{V}$, but this condition is not necessary. Note that the monoid $M_{\delta'}$ is equal to $\Phi^{\cdot}$ where Φ is the derived semigroup of the morphism $\phi: A^+/\delta \cap \gamma \to A^+/\gamma$, as introduced in [8].

Example 2.4. Let S and T be as in the previous example. Define $\hat{\delta}$ on G_T by $x_v \hat{\delta} y_v$ iff $x_v \sim y_v$ and $wx \delta wy$ for any $w \in A^*$ such that $\bar{w} = v$; $\hat{\delta}$ is a graph congruence and $\hat{\delta}^*T = \delta \cap \gamma \subseteq \delta$. Note also that $\delta' \subseteq \hat{\delta}$. In the language of [5], $\hat{\delta}$ is a geometric parametrization. It is clear that if $\hat{\delta}$ is a **V**-congruence, then $S < M \circ T$ for some $M \in \mathbf{V}$. Conversely suppose that $\delta \supseteq \beta^*T$ for some **V**-congruence β; define on G_T: $x_v \beta_1 y_v$ iff $x_u \beta y_u$ for all $u \in T^{\cdot} v$. It is easily verified that β_1 is a **V**-congruence. Moreover $x_v \beta_1 y_v$ implies that $(wx)_u \beta (wy)_u$ for all $u \in T^{\cdot}$ and all w such that $\bar{w} = v$. Thus $wx \delta wy$ for all such w and so $\beta_1 \subseteq \hat{\delta}$. This proves that $\hat{\delta}$ is a **V**-congruence.

Example 2.5. We now consider the question of deciding membership in $\mathbf{V} * \mathbf{W}$. Suppose that **W** is a locally finite S-variety and let γ be the congruence on A^+ such that an A-generated semigroup T belongs to **W** iff T is a morphic image of A^+/γ. Then, for any congruence δ on A^+, $\delta \supseteq \beta^*T$ for some A-generated $T \in \mathbf{W}$ and **V**-congruence β iff $\delta \supseteq \beta_1 * A^+/\gamma$ for some **V**-congruence β_1. By Example 2.4 this is true iff the congruence $\hat{\delta}$ on $G = G_{A^+/\gamma}$ is a **V**-congruence. We can thus decide the membership problem in the following cases.

(i) If **V** is locally finite; let γ_1 be the congruence on $(A^+/\gamma^{\cdot} \times A)^*$ generating **V** for this alphabet. Then $S = A^+/\delta \in \mathbf{V} * \mathbf{W}$ iff $\bar{\gamma}_1 \subseteq \hat{\delta}$.

(ii) If **V** is a variety of groups with a decidable membership problem. If **V** is the trivial variety, then $\mathbf{V} * \mathbf{W} = \mathbf{W}$; membership is then decidable since **W** is locally finite. If **V** is non-trivial, then by Example 1.3, $\hat{\delta}$ is a **V**-congruence iff the monoid $M_v \in \mathbf{V}$ for each vertex v of G, which is decidable by assumption on **V**.

(iii) If **V** is a variety containing the monoid B_2 and with a decidable membership problem. Then $\hat{\delta}$ is a **V**-congruence iff $M_{\hat{\delta}} \in \mathbf{V}$ by Theorem 1.5. Hence the question is decidable.

Example 2.6. The previous example stresses the importance of **W** being locally finite. For example, consider the S-variety $\mathbf{D} = \{S\colon Se = e \text{ for all } e = e^2\}$ introduced in Example 0.1. **D** is not locally finite but $T \in \mathbf{D}$ iff there exists $k \geq 0$ such that $T \in \mathbf{D}_k$, and $\mathbf{D}_k$ is locally finite. Thus when trying to decide membership of $S \in \mathbf{V} * \mathbf{D}$, it is useful to be able to bound the value of k that need to be considered in the right-hand factor. This question has been solved in [7] in the following way. Given $S = A^+/\delta$ one considers the graph $G_E = (V_E, A_E, \alpha, \omega)$ where

$$V_E = \{e\colon e \in S, e = e^2\}, \qquad A_E = \{(e, esf, f)\colon e, s, f \in S, e = e^2, f = f^2\},$$

$$(e, esf, f)\alpha = e, \qquad (e, esf, f)\omega = f,$$

and the graph congruence δ_E defined by $p \delta_E q$ iff $p \sim q$ and $\bar{p} = \bar{q}$ where $\bar{p} = e$ if

$p=1_e$ and

$$p=e_0s_1e_1\cdots e_{n-1}s_ne_n \text{ if } p=(e_0,e_0s_1e_1,e_1)\cdots(e_{n-1},e_{n-1}s_ne_n,e_n).$$

It is then possible to show that if there exists $k\geq 0$ such that $\hat{\delta}$ is a $\mathbf{V}$-congruence on G_T where $T=A^+/\omega_k$, then δ_E is a $\mathbf{V}$-congruence as well; conversely if δ_E is a $\mathbf{V}$-congruence, then $\hat{\delta}$ is also a $\mathbf{V}$-congruence on G_T for $T=A^+/\omega_k$ where $k=|S|$. Thus suppose that $S\in\mathbf{V}*\mathbf{D}$; there exists $T=A^+/\omega_k$ such that $\hat{\delta}$ is a $\mathbf{V}$-congruence on G_T. This implies that δ_E is a $\mathbf{V}$-congruence on G_E which in turn implies that $\hat{\delta}$ is a $\mathbf{V}$-congruence on G_T with $T=A^+/\omega_k$, $k=|S|$. Thus $S\in\mathbf{V}*\mathbf{D}_{|S|}$ by Theorem 2.2.

Example 2.7. Let $k\geq 0$ and $T_1=A^+/\alpha_k\cap\omega_k$, $T_2=A^+/\omega_k$, $G_1=G_{T_1}$, $G_2=G_{T_2}$, β_1 a congruence on G_1. Let $A^k=\{z_1,\dots,z_r\}$ and define $\phi: A_2^*\to A_1^*\times\cdots\times A_1^*$ (r times) by $(v,a)\phi=(1,\dots,1)$ if $v=[w]_{\psi_k}$ with $|w|<k$, and $(v,a)\phi=((v_1,a),\dots,(v_r,a))$ if $v=[w]_{\omega_k}$ with $|w|=k$, where $v_i=[z_iw]_{\alpha_k\cap\omega_k}$. Define a congruence β_2 on G_2 by $x_v\,\beta_2\,y_v$ iff $x_v\sim y_v$, $x_v\phi\pi_i\,\beta_1\,y_v\phi\pi_i$, where π_i is the projection on the ith coordinate, and $|x_v|_b=|y_v|_b$ for any edge in $B_2=\{b: b\in A_2, b\alpha, b\omega$ lie in different strongly connected components$\}$. It is verified that if β_1 is a $\mathbf{V}$-congruence for some non-trivial $\mathbf{V}$, then β_2 is also a $\mathbf{V}$-congruence. Every path in G_2 has the form pq with $p\in B_2^*$, $q\in(A_2-B_2)^*$: also $(pq)\phi=q\phi$. Suppose $x_v\,\beta_2\,y_v$ for all $v\in V_2$. Then $x_v=pq$, $y_v=pq'$ and $q\phi\pi_i\,\beta_1\,q'\phi\pi_i$ for $i=1,\dots,r$. Let now $u\in V_1$: if $u=[z_i W]_{\alpha_k\cap\omega_k}$ with $|w|=k$, then $x_u\,\beta_1\,y_u$ since $x_u=(x_{[w]_{\omega_k}})\phi\pi_i$; if $u=[w]_{\alpha_k\cap\omega_k}$ with $|w|<k$, then

$$x_u=(u,a_1)(u_1,a_2)\cdots(u_{n-1},a_n)\cdots(u_{m-1},a_m)$$

where $(u_i,a_{i+1})\in A_1-B_1$ iff $i\geq n$. Letting $v=[w]_{\omega_k}\in V_2$ and supposing $x_v\,\beta_2\,y_v$ we deduce that $y_u\in P_1$ must have the form

$$(u,a_1)(u_1,a_2)\cdots(u_{n-1},a_n)(u_n',a_n')\cdots(u_{s-1}',a_s').$$

Moreover $(u_n,a_{n+1})\cdots(u_{m-1},a_m)=q\phi\pi_i$ if $wa_1\cdots a_n=z_i$ and similarly $(u_m',a_{m+1})\cdots(u_{s-1}',a_s')=q'\phi\pi_i$. Thus for any $u\in V_1$, $x_u\,\beta_1\,y_u$. We have shown that $\beta_1*T_1\subseteq\beta_2*T_2$ and this proves that $\mathbf{V}*\mathbf{LI}_k\subseteq\mathbf{V}*\mathbf{D}_k$ for any non-trivial $\mathbf{V}$. Since the other inclusion is trivial, equality holds. This result was first noticed in [7].

We end this section by some remarks about S-varieties of the form $\mathbf{LV}=\{S: eSe\in\mathbf{V}$ for all $e=e^2\in S\}$. It is clear that $S\in\mathbf{LV}$ iff the graph congruence δ_E on G_E has the property that $M_v\in\mathbf{V}$ for any vertex v, i.e. δ_E is a locally $\mathbf{V}$-congruence. Since this property holds when δ_E is a $\mathbf{V}$-congruence, we deduce that $\mathbf{V}*\mathbf{D}\subseteq\mathbf{LV}$. We can characterize in graph-theoretic terms the case when equality will hold.

Theorem 2.8. $\mathbf{V}*\mathbf{D}=\mathbf{LV}$ *iff every locally* $\mathbf{V}$*-congruence is a* $\mathbf{V}$*-congruence.*

Proof. The sufficiency of the condition follows from the remarks above. Conversely, if $\mathbf{V}$ is a variety of groups, then the result follows by Example 1.3. Else $\mathbf{V}$ contains the monoid $\{1,0\}$ and thus whenever $S\in\mathbf{V}$, $S\cup\{0\}\in\mathbf{V}$ as well. Now let β be a local **V**-congruence on a graph G. One easily checks that $S_\beta = M_\beta - \{1\} \in \mathbf{LV}$. If $S_\beta \in \mathbf{V} * \mathbf{D}$, then the congruence δ_E on the graph of idempotents G_E of S_β would be a **V**-congruence, i.e. there would exist a congruence γ on A_E^* such that $A_E^*/\gamma \in \mathbf{V}$ and $\bar{\gamma} \subseteq \delta_E$. Consider $\phi: A^* \to A_E^*$ defined by $a\phi = ([1_{\alpha a}]_\beta, [a]_\beta, [1_{a\omega}]_\beta)$. Then $A^*/\phi\gamma \in \mathbf{V}$ and clearly $\overline{\phi\gamma} \subseteq \beta$. $\square$

It follows from Example 1.3 that $\mathbf{H} * \mathbf{D} = \mathbf{LH}$ for any non-trivial variety of groups, and from Example 1.4 that $\mathbf{Com} * \mathbf{D} \neq \mathbf{LCom}$. In the next section we proceed to give an effective characterization of the S-variety $\mathbf{Com} * \mathbf{D}$.

3. An effective characterization of Com * D

The semigroup $S = A^+/\delta$ is in $\mathbf{Com} * \mathbf{D}$ iff it is in $\mathbf{Com} * \mathbf{D}_{|S|}$. Thus it suffices to be able to decide if the congruence δ on G_T with $T = A^+/\omega_{|S|}$ is a **Com**-congruence. This question is resolved by the

Theorem 3.1. *β is a* **Com**-*congruence iff $xyz \,\beta\, zyx$ for any paths x, y, z such that $x \sim z$.*

Since also $S \in \mathbf{Com} * \mathbf{D}$ iff the graph congruence δ_E on the graph of idempotents G_E of S is a **Com**-congruence, we deduce that $S \in \mathbf{Com} * \mathbf{D}$ iff $e_1se_2te_1ue_2 = e_1ue_2te_1se_2$ for every $s, t, u, e_1 = e_1^2$, $e_2 = e_2^2 \in S$. This yields a reasonable algorithm for deciding membership in $\mathbf{Com} * \mathbf{D}$.

We will need some more definitions. Let γ_∞ be the congruence of infinite index on A^* defined by $x \gamma_\infty y$ iff $|x|_a = |y|_a$ for all $a \in A$. For any graph G, let θ denote the graph conguence generated by the relation $xyz \,\theta\, zyx$ for $x \sim z$; θ also has infinite index. Finally for any $t \geq 0$, $q \geq 1$, $\theta_{t,q}$ is the congruence on G generated by $xyz \,\theta\, zyx$ for $x \sim z$, and $x^t \theta x^{t+q}$ for all loops x. It is trivial that $\theta \subseteq \bar{\gamma}_\infty$ and $\theta_{t,q} \subseteq \bar{\gamma}_{t,q}$. Lemma 3.2 below shows that $\theta = \bar{\gamma}_\infty$. Since β is a **Com**-congruence iff $\beta \supseteq \bar{\gamma}_{t,q}$ for some $t \geq 0$, $q \geq 1$, Theorem 3.1 is equivalent to the existence, for any t, q, of $t' \geq 0$, $q' \geq 1$ such that $\bar{\gamma}_{t',q'} \subseteq \theta_{t,q}$.

Lemma 3.2. $\bar{\gamma}_\infty \subseteq \theta$.

Proof. Let $x \bar{\gamma}_\infty y$. If $|x| \leq 1$, then $x = y$ and $x \,\theta\, y$. Otherwise $x = ax_0$, $y = by_0$: if $a = b$ then $x_0 \bar{\gamma}_\infty y_0$ and by induction hypothesis $x_0 \,\theta\, y_0$, so that $x \,\theta\, y$. If $a \neq b$, we must have $x = ax_1bx_2$, $y = by_1ay_2$, where $x_1 = a_1 \cdots a_n$. If $n = 0$, then a is a loop so that $y \,\theta\, aby_1y_2$. Let then $n > 0$. Suppose first that $y_1 = w_1a_1z_1$; then $a \sim bw$, and $y \,\theta\, aa_1a_1bw_1y_2$. Suppose next that for some i between 2 and n we have $y_1 = w_1a_iz_1$ and $y_2 = w_2a_{i-1}bw_1z_2$; then $bw_1 \sim aw_2a_{i-1}$ so that $y \,\theta\, aw_2a_{i-1}a_iz_1bw_1z_2$. Otherwise it must be that $y_2 = wa_nz$; in this case $by_1 \sim awa_n$ and $y \,\theta\, awa_nby_1z$. We have shown that y can always be transformed to a path of the form az and the lemma follows. $\square$

Lemma 3.3. *For any $G=(V, A, \alpha, \omega)$, $t \geq 0$, there exists $n=n(G,t)$ such that $|x|_a > n$ implies $x\,\theta\, w(ay)^t z$.*

Proof. Let $n=|V|+t(2^{|A|}-1)+1$ and suppose that $x=x_0ax_1 \cdots ax_{n+1}$. By commuting loops if necessary we can assume that $(x_0ax_1 \cdots ax_{|V|})v=xv$. Suppose that, for some $j>|V|$, $ax_j=w_0w_1w_2$ for some loop w_1 with $1 \neq w_1 \neq ax_j$; since $x_0ax_1 \cdots ax_{|V|}=u_0u_1$ with $u_0\omega=w_1\alpha$, we have

$$u_1ax_{|V|+1} \cdots w_0 \sim w_1 \quad \text{and} \quad x\,\theta\, u_0w_1u_1ax_{|V|+1} \cdots w_0w_2 \cdots ax_{n+1}.$$

Continuing in this way we can transform x to $x_0ax_1 \cdots ax_{n+1}$ where for each $j>|V|$, $|ax_j|_b \leq 1$ for any edge b. By our choice of n there must be t indices $j_1, \dots, j_t$ such that $ax_{j_1}\, \bar{\gamma}_\infty \cdots \bar{\gamma}_\infty\, ax_{j_t}$. By Lemma 2.1 they are all congruent to the same ay. Using commutativity of loops again we get that $x\,\theta\, w(ay)^t z$. □

Lemma 3.4. *Let $x, y \in P$ be such that for all $a \in A$, $|x|_a > |y|_a$. Then $x\,\theta\, wyz$ for some $w, z \in P$.*

Proof. If $|y|=0$, the result is trivial. Let $y=y_0a$; by induction hypothesis we know that $x\,\theta\, wy_0z$. Since $\theta \subseteq \bar{\gamma}_\infty$ we have $|w|_a+|z|_a>1$. If $w=w_0aw_1aw_2$, then $aw_1 \sim 1_{a\alpha}$ and $x\,\theta\, w_0aw_2y_0az_1z_0az_2$. If $z=z_0az_1az_2$, then $az_1 \sim 1_{a\alpha}$ and $x\,\theta\, wy_0az_1z_0az_2$. If $w=w_0aw_1$ and $z=z_0az_1$, then $z_0 \sim 1_{a\alpha}$ and $x\,\theta\, w_0z_0aw_1y_0az_1$. The result follows. □

Lemma 3.5. *Let $x \in P$ and y be a loop about x such that $|y|_a=1$ and $|x|_a>t \geq 1$ for any $a \in y\gamma$. Then there exists z such that $x\,\theta\, zy^t$.*

Proof. If $t=1$, Lemma 3.4 says that $x\,\theta\, wyz$ for some $w, z \in P$. Since y is a loop and $x \sim wyz$, we get $x\,\theta\, wzy$. If $t>1$, using Lemma 3.4 again yields that $x\,\theta\, wyz$ so that $x\,\theta\, wzy$. Now $|wz|_a>t-1$, so by induction $wz\,\theta\, uy^{t-1}$. Hence $x\,\theta\, uy^t$. □

Lemma 3.6. *Let y be a loop such that $|y|_a \equiv 0 \pmod q$ for all a; then $y\,\theta\, z^q$ for some loop z.*

Proof. For any $a \in A$, let k_a be such that $|y|_a=k_aq$. Construct a graph $G_0=(V_0, A_0, \alpha, \omega)$ where $V_0=V$, $A_0=\{a_1, \dots, a_{k_a}: a \in y\gamma\}$, $a_i\alpha=a\alpha$, $a_i\omega=a\omega$. Consider the path y_0 obtained from y by replacing the first q occurences of a by a_1, the next q occurences of a by a_2 and so on. Then $|y_0|_{a_i}=q$ for all $a_i \in A_0$. Since y is a loop it must be that $\sum_{a\alpha=v} k_a=\sum_{a\omega=v} k_a$ for any $v \in V$; this implies that for any vertex v of V_0, the number of edges entering v is equal to the number of edges leaving v. Such graphs admit eulerian paths, i.e. there exists a loop z_0 such that $|z_0|_{a_i}=1$ for all $a_i \in A_0$ [1]. Then $y_0\, \gamma_\infty\, z_0^q$ and by Lemma 3.2, $y_0\,\theta\, z_0^q$. Reverting back any occurence of a_i in z_0 to an a, we get a loop z in G such that $y\,\theta\, z^q$. □

Lemma 3.7. *Let x be a path and y be a loop about $x\omega$ such that $|x|_a>t \geq 1$ for any $a \in y\gamma$. Then $x\,\theta_{t,q}\, xy^q$.*

Proof. If $|y|=0$ there is nothing to prove. If $|y|_a=1$ for all $a\in y\gamma$, then by Lemma 3.5, $x\,\theta\,zy^t$, so that $x\,\theta_{t,q}\,zy^t$ as well. Since $y^t\,\theta_{t,q}\,y^{t+q}$, we deduce that $x\,\theta_{t,q}\,xy^q$. If $|y|_a>1$ for some $a\in y\gamma$ it must be that $y=y_0y_1y_2$ for some loop y_1 such that $0<|y_1|<|y|$. We get by induction on y that $x\,\theta_{t,q}\,x(y_0y_2)^q$. Since $|xy_0|_a>t$ for any $a\in y_1\gamma$, we also have $xy_0\,\theta_{t,q}\,xy_0y_1^q$ and thus $x\,\theta_{t,q}\,xy_0y_1^qy_2(y_0y_2)^{q-1}$. The fact that $y^q\,\bar{\gamma}_\infty\,y_0y_1^qy_2(y_0y_2)^{q-1}$ implies that $x\,\theta_{t,q}\,xy^q$ by using Lemma 3.2. □

Lemma 3.8. *For any graph $G, r\geq 0$, $q\geq 1$, there exists $m=m(G,r)$ such that $x\,\bar{\gamma}_{m,q}\,y$ implies that there exists $w\in P$ such that $|w|_a\geq|x|_a$ for all $a\in E$, $x\,\bar{\gamma}_{r,q}\,w$ and $y\,\theta_{r,q}\,w$.*

Proof. Let $m(G,r)=n(G,r)+1$, where n is the function defined in lemma 3.3, and suppose $x\,\bar{\gamma}_{m,q}\,y$. For each a such that $|x|_a>|y|_a$, it must be that $|y|_a>n(G,r)$. Consider any such edge a; by Lemma 3.3, $y\,\theta\,u_0(au)^ru_1$. Hence $y\,\theta_{r,q}\,u_0(au)^{r+kq}u_1$ for all $k\geq 0$; choose k such that $r+kq\geq|x|_a$. We thus obtained a word w_1 such that $y\,\theta_{r,q}\,w_1$ and $|w_1|_b\geq|y|_b$ for all $b\in E$; in particular $|w_1|_b>n(G,r)$ for any b such that $|x|_b>|w_1|_b$. Reapplying Lemma 3.3 as often as necessary we construct a word w such that $|w|_a\geq|x|_a$ for all $s\in A$ and $y\,\theta_{r,q}\,w$; thus also $y\,\bar{\gamma}_{r,q}\,w$ and, since $m\geq r$, $x\,\bar{\gamma}_{r,q}\,w$. □

Lemma 3.9. *For any G, $t\geq 1$, $q\geq 1$, there exists $r=r(G,t)$ such that $x\,\bar{\gamma}_{r,q}\,y$ and $|y|_a\geq|x|_a$ for all $a\in A$ imply $y\,\theta_{t,q}\,xz$ for some loop z.*

Proof. Let $r_i=t+i$. Let p be the longest common prefix of x and y and write $x=px_0$, $y=py_0$. We claim that if $|y_0\gamma|\leq i$, $x\,\bar{\gamma}_{r_i,q}\,y$ and $|x|_a\leq|y|_a$, then the lemma holds. If $i=0$, then $x=y$ and we take $z=1$. Let $i>0$. If $x_0=1$, then $x\,\bar{\gamma}_{r_i,q}\,xy_0$: hence for any $a\in y_0\gamma$ we have that $|y_0|_a\equiv 0 \pmod q$ and $|x|_a>t\geq 1$. By Lemma 3.6, $y_0\,\theta\,z^q$ and by Lemma 3.7, $x\,\theta_{t,q}\,xz^q$. This shows that $x\,\theta_{t,q}\,xy_0$. Otherwise $x=pax_1$, $y=pby_1$ and since $|x|_a\leq|y|_a$, $y_1=wau$. First suppose that $bw=w_0w_1$ and $u=u_0u_1$ with $w_0\omega=u_0\omega$. Then $w_0\sim au_0$ and $y\,\theta\,pau_0w_1w_0u_1$, so that $y\,\bar{\gamma}_{r_i,q}\,pau_0w_1w_0u_1$ as well. The length of the common prefix can thus be increased until we get to a point where $x=pax_1$, $y=pbwau$ and $(bw)v\cap uv=\emptyset$. Observe that $x_1\gamma\cap(bw)\gamma=\emptyset$ as well: indeed if $x_1=x_2cx_3$ with $c\in(bw)\gamma$ and $x_2\gamma\cap(bw)=\emptyset$, then it must be that $x_2\gamma\subseteq u\gamma$, hence $x_2\omega=c\alpha\in(bw)v\cap uv$, a contradiction. We can thus assume that $(bw)\gamma\cap(au)\gamma=\emptyset$ and $x_1\gamma\subseteq u\gamma$. This implies that $|bw|_c\equiv 0 \pmod q$ and that $|p|_c>t+i-1$. By Lemma 3.6, $bw\,\theta\,s^q$ and by Lemma 3.7, $p\,\theta_{t+i-1,q}\,ps^q$: thus $y\,\theta_{t+i-1,q}\,pau$ and $y\,\bar{\gamma}_{t+i-1,q}\,pau$ also. Moreover $|pau|_c\geq|x|_c$ for all $c\in A$ and $|u_y|\leq i-1$. Using the induction hypothesis on i, the claim is proved. Hence the lemma holds with $r=t+|A|$. □

We are now in position to prove Theorem 3.1.

Proof of Theorem 1. If β is a **Com**-congruence, then there exists $t\geq 0$, $q\geq 1$ such that $\bar{\gamma}_{t,q}\subseteq\beta$: hence $xyz\,\beta\,zyx$ for any paths x, y, z such that $x\sim z$. Conversely suppose that β satisfies the given equation: since β has finite index it also satisfies $x^t \beta x^{t+q}$ for all loops x, for some $t \geq 0$, $q \geq 1$. Thus $\theta_{t,q} \subseteq \beta$. If $t=0$ β is a locally $\mathbf{Com}_{0,q}$-congruence hence also a $\mathbf{Com}_{0,q}$-congruence by Example 1.3. If $t \geq 1$, we claim that $\bar{\gamma}_{p,q} \subseteq \theta_{t,q}$ for $p = m(G, r(G, t+1))$. By Lemma 3.8 we can find w such that $x\, \bar{\gamma}_{r,q}\, w$, $w\, \theta_{t,q}\, y$ and $|w|_a \geq |x|_a$ for all $a \in A$. By Lemma 3.9, $w\, \theta_{t+1,q}\, xz$ for some loop z. Since $x\, \bar{\gamma}_{r,q}\, w$ we also have $x\, \bar{\gamma}_{t+1,q}\, w$ and $w\, \theta_{t+1,q}\, xz$ implies $w\, \bar{\gamma}_{t+1,q}\, xz$. Also $x\, \bar{\gamma}_{t+1,q}\, xz$ so that $|z|_a \equiv 0 \pmod q$ and $|x|_a > t$ for all $a \in A$. By Lemma 3.6, $z\, \theta\, u^q$ for some loop u and by Lemma 3.7, $x\, \theta_{t,q}\, xu^q$. Hence $x\, \theta_{t,q}\, xz\, \theta_{t,q}\, w\, \theta_{t,q}\, y$. $\square$

By Example 1.3 and Theorem 2.8, it follows that

$$\mathbf{Com}_{0,q} * \mathbf{Ll} = \mathbf{LCom}_{0,q} \quad \text{for any } q \geq 1$$

and that

$$\mathbf{G}_{\mathrm{com}} * \mathbf{Ll} = \mathbf{LG}_{\mathrm{com}}$$

where $\mathbf{G}_{\mathrm{com}}$ denotes the variety of abelian groups. Also let $S \in \mathbf{LCom}_{1,q}$; then the graph congruence δ_E on the graph G_E of idempotents of S_1 satisfies $xy\, \delta_E\, yx$ and $x\, \delta_E\, x^{q+1}$ for any loops x, y. Thus whenever $x \sim z$, we get

$$xyz\, \delta_E\, (xy)^{q+1} z = x(yx)^q yz\, \delta_E\, xyz(yx)^q\, \delta_E\, zyx(yx)^q\, \delta_E\, zyx$$

and δ_E is also a $\mathbf{Com}$-congruence. This says that $\mathbf{LCom}_{1,q} \subseteq \mathbf{Com} * \mathbf{Ll}$: it can in fact be shown [4] that $\mathbf{LCom}_{1,q} = \mathbf{Com}_{1,q} * \mathbf{Ll}$. Example 1.4 shows that $\mathbf{LCom}_{t,q} \not\subseteq \mathbf{Com} * \mathbf{Ll}$ when $t \geq 2$.

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