

# Note on a Translation from First-Order Logic into the Calculus of Relations Preserving Validity and Finite Validity

**Yoshiki Nakamura**

*Department of Computer Science*

*Tokyo Institute of Technology*

*nakamura.yoshiki.ny@gmail.com*

**Abstract.** In this note, we give a linear-size translation from formulas of first-order logic into equations of the calculus of relations preserving validity and finite validity. Our translation also gives a linear-size conservative reduction from formulas of first-order logic into formulas of the three-variable fragment of first-order logic.

**Keywords:** first-order logic, relation algebra

## 1. Introduction

*The calculus of relations* (CoR, for short) [1] is an algebraic system with operations on binary relations. As binary relations appear everywhere in computer science, CoR and relation algebras can be applied to various areas, such as databases and program development and verification [2]. W.r.t. binary relations, CoR has the same expressive power as the three-variable fragment of [first-order predicate logic with equality](#) (FO3<sub>=</sub>) (where all predicate symbols are binary) [3], so CoR has strictly less expressive power than [first-order predicate logic with equality](#) (FO<sub>=</sub>). For example, CoR [equations](#) cannot characterize the class of structures s.t. “its cardinality is greater than or equal to 4”, whereas it can be characterized by the FO<sub>=</sub> [formula](#)  $\forall x_1, \forall x_2, \forall x_3, \exists y, (\neg y = x_1) \wedge (\neg y = x_2) \wedge (\neg y = x_3)$  where  $x_1, x_2, x_3, y$  are pairwise distinct variables.

Nevertheless, there is a recursive translation from FO<sub>=</sub> [formulas](#) into CoR [equations](#) (resp. FO3<sub>=</sub> [formulas](#)) equations preserving [validity](#) [3] (see also [4, 5]).

In this paper, we give another translation from FO= **formulas** into CoR **equations** preserving **validity**, slightly refined in that it satisfies both of the following:

1. Our translation preserves both **validity** and **finite validity** (so, it also gives a *conservative reduction* [6, Def. 2.1.35]).
2. Our translation is in linear-size (i.e., the output size is bounded by a linear function in the input size) and recursive.

The first refinement is useful, e.g., in finding counter-models (because if there exists a finite counter-model in the pre-translated **formula**, then there also exists a finite counter-model in the post-translated **formula**). Such a translation is already known (e.g., [6, Cor. 3.1.8 and Thm. 3.1.9]), but via encodings of Turing-machines and domino problems. Our translation presents a **conservative reduction** from FO= **formulas** to FO3= **formulas**, directly. Thanks to this, we also have the second refinement, which shows that the **validity** (resp. **finite validity**) problem of FO= **formulas** and that of CoR **equations** are equivalent under linear-size translations, as the converse direction immediately follows from the **standard translation** from CoR **equations** into FO3= [1] (Prop. 2.2).

Our translation is not so far from known encodings (e.g., [4, 5]) in that they and our translation use pairing (2-tupling) functions, but in our translation, we use *non-nested*  $k$ -tupling functions where  $k$  is an arbitrary natural number, instead of arbitrarily nested pairing functions. For constructions using arbitrarily nested pairing functions, we need infinitely many vertices even if the base universe is finite (as there is no surjective function from  $X$  to  $X^2$  when  $X$  is finite and  $\#X \geq 2$ ). Thanks to the modification above, our construction preserves both **validity** and **finite validity**. Additionally, to preserve the output size in linear to the input size, we apply a cumulative sum technique.

This paper is structured as follows. In Sect. 2, we give basic definitions of FO= and CoR. In Sect. 3, we give a translation from FO= **formulas** into CoR **equations** preserving **validity** and **finite validity**. In Sect. 4, we additionally give a Teseitin translation for CoR, which is useful for reducing the number of alternations of operations.

## 2. Preliminaries

We write  $\mathbb{N}$  for the set of all non-negative integers. For a set  $A$ , we write  $\#A$  for the cardinality of  $A$  and  $\wp(A)$  for the power set of  $A$ .

A **structure**  $\mathfrak{A}$  over a set  $A$  is a tuple  $\langle |\mathfrak{A}|, \{a^{\mathfrak{A}}\}_{a \in A} \rangle$ , where

- the universe  $|\mathfrak{A}|$  is a non-empty set of vertices,
- each  $a^{\mathfrak{A}} \subseteq |\mathfrak{A}|^2$  is a binary relation on  $|\mathfrak{A}|$ .

We say that a **structure**  $\mathfrak{A}$  is **finite** if  $|\mathfrak{A}|$  is finite. For **structures**  $\mathfrak{A}, \mathfrak{B}$  over a set  $A$ , we say that  $\mathfrak{A}$  and  $\mathfrak{B}$  are **isomorphic** if there is a bijective map  $f: |\mathfrak{A}| \rightarrow |\mathfrak{B}|$  such that for all  $x, y \in |\mathfrak{A}|$  and  $a \in A$ , we have  $\langle x, y \rangle \in a^{\mathfrak{A}} \iff \langle f(x), f(y) \rangle \in a^{\mathfrak{B}}$ .

## 2.1. First-order logic

Let  $\Sigma$  be a countably infinite set of binary predicate symbols and  $\mathbf{V}$  be a countably infinite set of *variables*. The set of *formulas* in *first-order predicate logic with equality* (FO<sub>=</sub>) is defined by:

$$\varphi, \psi, \rho ::= a(x, y) \mid x = y \mid \neg\psi \mid \psi \wedge \rho \mid \exists x, \psi \quad (a \in \Sigma \text{ and } x, y \in \mathbf{V})$$

We write  $V(\varphi)$  for the set of free and bound *variables* occurring in a *formula*  $\varphi$ . For  $k \geq 0$ , we write  $\text{FO}k_{=}$  for the set of all *formulas*  $\varphi$  s.t.  $V(\varphi) \leq k$ . A *sentence* is a *formula* not having any free *variable*. We use parenthesis in ambiguous situations and use the following notations:

$$\begin{aligned} \varphi \vee \psi &\triangleq \neg((\neg\varphi) \wedge (\neg\psi)) & \forall x, \psi &\triangleq \neg\exists x, \neg\psi \\ \text{t} &\triangleq \exists x, x = x & \text{f} &\triangleq \neg\exists x, x = x \end{aligned}$$

We write  $\bigwedge \Gamma$  for the *formula*  $\varphi_1 \wedge \dots \wedge \varphi_n$  where  $\Gamma = \{\varphi_1, \dots, \varphi_n\}$  is a finite set (and  $\varphi_1, \dots, \varphi_n$  are ordered by a total order). The *size*  $\|\varphi\| \in \mathbb{N}$  of a *formula*  $\varphi$  is defined by:

$$\begin{aligned} \|a(x, y)\| &\triangleq 1 + 2 & \|x = y\| &\triangleq 1 + 2 & \|\neg\psi\| &\triangleq 1 + \|\psi\| \\ \|\psi \wedge \rho\| &\triangleq 1 + \|\psi\| + \|\rho\| & \|\exists x, \psi\| &\triangleq 1 + 1 + \|\psi\| \end{aligned}$$

For a *structure*  $\mathfrak{A}$  over  $\Sigma$ , the *semantics*  $\llbracket \varphi \rrbracket^{\mathfrak{A}} \subseteq |\mathfrak{A}|^{\mathbf{V}}$  of a *formula*  $\varphi$  over  $\mathfrak{A}$  is defined as follows where  $|\mathfrak{A}|^{\mathbf{V}}$  denotes the set of functions from  $\mathbf{V}$  to  $|\mathfrak{A}|$ :

$$\begin{aligned} \llbracket a(x, y) \rrbracket^{\mathfrak{A}} &\triangleq \{f: \mathbf{V} \rightarrow |\mathfrak{A}| \mid \langle f(x), f(y) \rangle \in a^{\mathfrak{A}}\} \\ \llbracket x = y \rrbracket^{\mathfrak{A}} &\triangleq \{f: \mathbf{V} \rightarrow |\mathfrak{A}| \mid f(x) = f(y)\} \\ \llbracket \neg\psi \rrbracket^{\mathfrak{A}} &\triangleq |\mathfrak{A}|^{\mathbf{V}} \setminus \llbracket \psi \rrbracket^{\mathfrak{A}} \\ \llbracket \psi \wedge \rho \rrbracket^{\mathfrak{A}} &\triangleq \llbracket \psi \rrbracket^{\mathfrak{A}} \cap \llbracket \rho \rrbracket^{\mathfrak{A}} \\ \llbracket \exists x, \psi \rrbracket^{\mathfrak{A}} &\triangleq \{f: \mathbf{V} \rightarrow |\mathfrak{A}| \mid \text{for some } v \in |\mathfrak{A}|, f[v/x] \in \llbracket \psi \rrbracket^{\mathfrak{A}}\} \end{aligned}$$

Here,  $f[v/x]$  denotes the function  $f$  in which the value of  $f(x)$  has been replaced with  $v$ .

For a *formula*  $\varphi$  and a *structure*  $\mathfrak{A}$ , we say that  $\varphi$  is *true* on  $\mathfrak{A}$ , written  $\mathfrak{A} \models \varphi$ , if  $\llbracket \varphi \rrbracket^{\mathfrak{A}} = |\mathfrak{A}|^{\mathbf{V}}$ . We say that a *formula*  $\varphi$  is *valid* (resp. *finitely valid*) if  $\llbracket \varphi \rrbracket^{\mathfrak{A}} = |\mathfrak{A}|^{\mathbf{V}}$  holds for all *structures* (resp. all *finite structures*)  $\mathfrak{A}$ . We say that two *formulas*  $\varphi, \psi$  are *semantically equivalent* if the *formula*  $\varphi \leftrightarrow \psi$  is *valid*. Additionally, we say that a *formula*  $\varphi$  is *satisfiable* (resp. *finitely satisfiable*) if  $\llbracket \varphi \rrbracket^{\mathfrak{A}} \neq \emptyset$  holds for some *structure* (resp. *finite structure*)  $\mathfrak{A}$ .

**Remark 2.1.** Function and constant symbols can be encoded by predicate symbols with functionality axiom (see, e.g., [7, Sect. 19.4]) and each predicate symbol (of arbitrary arity) can be encoded by binary predicate symbols (see, e.g., [7, Lem. 21.2]). Thus by well-known facts, we can give a linear-size translation from formulas of first-order logic with predicate and function symbols of arbitrary arity into FO<sub>=</sub> *formulas* preserving *validity* and *finite validity*. Here, the *size* of a  $k$ -ary atomic formula  $\|a(x_1, \dots, x_k)\|$  should depend on  $k$  such as  $\|a(x_1, \dots, x_k)\| = 1 + k$ . Hence, it suffices to consider about the FO<sub>=</sub> above (equality = can also be eliminated, see, e.g., [7, Sect. 19.4], but we introduce it only for convenience).

## 2.2. The calculus of relations

Let  $\Sigma$  be a countably infinite set of *(term) variables*. The set of *terms* in *the calculus of relations* (CoR) is defined by:

$$t, s, u ::= a \mid \mathbf{I} \mid s^- \mid s \cap u \mid s \cdot u \mid s^\smile \quad (a \in \Sigma)$$

We write  $\bigcap \Gamma$  for the *term*  $t_1 \cap \dots \cap t_n$  where  $\Gamma = \{t_1, \dots, t_n\}$  is a finite set (and  $t_1, \dots, t_n$  are ordered by a total order). We use parenthesis in ambiguous situations and use the following notations:

$$\begin{array}{ll} t \cup s & \triangleq (t^- \cap s^-)^- & t \dagger s & \triangleq (t^- \cdot s^-)^- \\ \top & \triangleq \mathbf{I} \cup \mathbf{I}^- & \perp & \triangleq \top^- \end{array}$$

The *size*  $\|t\| \in \mathbb{N}$  of a *term*  $t$  is defined by:

$$\begin{array}{lll} \|a\| & \triangleq 1 & \|\mathbf{I}\| & \triangleq 1 & \|s^-\| & \triangleq 1 + \|s\| \\ \|s \cap u\| & \triangleq 1 + \|s\| + \|u\| & \|s \cdot u\| & \triangleq 1 + \|s\| + \|u\| & \|s^\smile\| & \triangleq 1 + \|s\| \end{array}$$

The *semantics*  $\llbracket t \rrbracket^{\mathfrak{A}} \subseteq |\mathfrak{A}|^2$  of a *term*  $t$  over a *structure*  $\mathfrak{A}$  over  $\Sigma$  is defined by:

$$\begin{array}{l} \llbracket a \rrbracket^{\mathfrak{A}} \triangleq \{\langle v, v' \rangle \in |\mathfrak{A}|^2 \mid \langle v, v' \rangle \in a^{\mathfrak{A}}\} \\ \llbracket \mathbf{I} \rrbracket^{\mathfrak{A}} \triangleq \{\langle v, v' \rangle \in |\mathfrak{A}|^2 \mid v = v'\} \\ \llbracket s^- \rrbracket^{\mathfrak{A}} \triangleq |\mathfrak{A}|^2 \setminus \llbracket s \rrbracket^{\mathfrak{A}} \\ \llbracket s \cap u \rrbracket^{\mathfrak{A}} \triangleq \llbracket s \rrbracket^{\mathfrak{A}} \cap \llbracket u \rrbracket^{\mathfrak{A}} \\ \llbracket s \cdot u \rrbracket^{\mathfrak{A}} \triangleq \{\langle v, v' \rangle \in |\mathfrak{A}|^2 \mid \text{for some } v'' \in |\mathfrak{A}|, \langle v, v'' \rangle \in \llbracket s \rrbracket^{\mathfrak{A}} \text{ and } \langle v'', v' \rangle \in \llbracket u \rrbracket^{\mathfrak{A}}\} \\ \llbracket s^\smile \rrbracket^{\mathfrak{A}} \triangleq \{\langle v, v' \rangle \in |\mathfrak{A}|^2 \mid \langle v', v \rangle \in \llbracket s \rrbracket^{\mathfrak{A}}\} \end{array}$$

We say that a CoR *term*  $t$  and an FO<sub>=</sub> *formula*  $\varphi$  with two distinct free *variables*  $x_1$  and  $x_2$  are *semantically equivalent w.r.t. binary relations* if  $\llbracket t \rrbracket^{\mathfrak{A}} = \{\langle f(x_1), f(x_2) \rangle \mid f \in \llbracket \varphi \rrbracket^{\mathfrak{A}}\}$  holds for all *structures*  $\mathfrak{A}$ . It is well-known that we can translate CoR *terms* into FO<sub>3=</sub> *formulas*.

**Proposition 2.2. (the *standard translation* theorem [1])**

Let  $x_1$  and  $x_2$  be distinct *variables*. There is a linear-size translation from CoR *terms* into FO<sub>3=</sub> *formulas* with two free *variables*  $x_1$  and  $x_2$  preserving the *semantic equivalence w.r.t. binary relations*.

**Proof:**

[Proof Sketch] Because we can express each operations in CoR by using FO<sub>3=</sub> *formulas* (see also [8, Fig. 1]).  $\square$

Moreover, the set of *quantifier-free formulas* in CoR is inductively defined as follows:

$$\varphi, \psi, \rho ::= t = s \mid \neg \psi \mid \psi \wedge \rho \quad (t, s \text{ are terms in CoR})$$

We say that  $t = s$  is an **equation**. An **inequation**  $t \leq s$  is an abbreviation of the **equation**  $t \cup s = s$ . We use the following notations:

$$\begin{aligned} \varphi \vee \psi &\triangleq \neg((\neg\varphi) \wedge (\neg\psi)) & \forall x, \psi &\triangleq \neg(\exists x, (\neg\psi)) \\ \varphi \rightarrow \psi &\triangleq (\neg\varphi) \vee \psi & \varphi \leftrightarrow \psi &\triangleq (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \end{aligned}$$

The **size**  $\|\varphi\| \in \mathbb{N}$  of a **quantifier-free formula**  $\varphi$  is defined by:

$$\|t = s\| \triangleq 1 + \|t\| + \|s\| \quad \|\neg\psi\| \triangleq 1 + \|\psi\| \quad \|\psi \wedge \rho\| \triangleq 1 + \|\psi\| + \|\rho\|$$

The **semantic relation**  $\mathfrak{A} \models \varphi$ , where  $\varphi$  is a **quantifier-free formula** and  $\mathfrak{A}$  is a **structure** over  $\Sigma$ , is defined by:

$$\begin{aligned} \mathfrak{A} \models t = s &\iff \llbracket t \rrbracket^{\mathfrak{A}} = \llbracket s \rrbracket^{\mathfrak{A}} \\ \mathfrak{A} \models \neg\psi &\iff \text{not } \mathfrak{A} \models \psi \\ \mathfrak{A} \models \psi \wedge \rho &\iff \mathfrak{A} \models \psi \text{ and } \mathfrak{A} \models \rho \end{aligned}$$

For a **quantifier-free formula**  $\varphi$  and a **structure**  $\mathfrak{A}$ , we say that  $\varphi$  is **true** on  $\mathfrak{A}$  if  $\mathfrak{A} \models \varphi$ . Similarly for FO<sub>=</sub> formulas, we say that a **quantifier-free formula**  $\varphi$  is **valid** (resp. **finitely valid**) if  $\mathfrak{A} \models \varphi$  holds for all **structures** (resp. all **finite structures**)  $\mathfrak{A}$ . We say that two **quantifier-free formulas**  $\varphi, \psi$  are **semantically equivalent** if the **quantifier-free formula**  $\varphi \leftrightarrow \psi$  is **valid**.

It is also well-known that we can translate CoR **quantifier-free formulas** into CoR **equations** preserving the **semantic equivalence**.

**Proposition 2.3. (Schröder-Tarski translation theorem [1])**

There is a linear-size translation from a given **quantifier-free formula**  $\varphi$  in CoR into a **term**  $t$  such that  $\varphi$  and  $(t = \top)$  are **semantically equivalent**.

**Proof:**

[The proof is from [1].] First, by using  $(s = u) \leftrightarrow ((s \cap u) \cup (s^- \cap u^-) = \top)$ , we translate a given **quantifier-free formula** into a **quantifier-free formula** s.t. each **equation** is of the form  $t = \top$ . Second, by using the following two **semantic equivalence**, we eliminate logical connectives:

$$\neg(s = \top) \iff \top \cdot s^- \cdot \top = \top \quad (s = \top) \wedge (u = \top) \iff s \cap u = \top$$

Then we have obtained the desired **equation** of the form  $t = \top$ . □

**Remark 2.4.** There is also a translation from FO3<sub>=</sub> **formulas** with two free **variables**  $x_1$  and  $x_2$  into CoR **terms** preserving the **semantic equivalence w.r.t. binary relations** (i.e., the converse direction of Prop. 2.2) [3, 9, 8, 10], but the best known translation is an exponential-size translation and it is open whether there is a subexponential-size translation [8, 10]. This paper's translation given in Sect. 3 only preserves **validity** and **finite validity** and does not preserve the **semantic equivalence w.r.t. binary relations**, but it is a linear-size translation.

### 3. A translation from first-order logic into CoR

We consider the following **structure** transformation.

**Definition 3.1. (*k*-tuple structure)**

Let  $\mathfrak{A}$  be a **structure** over  $\Sigma$ . For  $k \geq 1$ , the *k*-tuple structure of  $\mathfrak{A}$ , written  $\mathfrak{A}^{(k)}$ , is the **structure** over  $\Sigma^{(k)} \triangleq \Sigma \cup \{U\} \cup \{\pi_i, Q_i, E_{[1,i]}, E_{[i,k]} \mid 1 \leq i \leq k\}$  defined as follows:

$$\begin{aligned}
 |\mathfrak{A}^{(k)}| &= |\mathfrak{A}|^k \\
 a^{\mathfrak{A}^{(k)}} &= \{\langle \langle v, \dots, v \rangle, \langle w, \dots, w \rangle \rangle \mid \langle v, w \rangle \in a^{\mathfrak{A}} \text{ for } a \in \Sigma\} \\
 \pi_i^{\mathfrak{A}^{(k)}} &= \{\langle \langle v_1, \dots, v_k \rangle, \langle v_i, \dots, v_i \rangle \rangle \mid v_1, \dots, v_k \in |\mathfrak{A}|\} \\
 U^{\mathfrak{A}^{(k)}} &= \{\langle v, \dots, v \rangle \mid v \in |\mathfrak{A}|\} \\
 E_{[i,i']}^{\mathfrak{A}^{(k)}} &= \{\langle \langle v_1, \dots, v_k \rangle, \langle v'_1, \dots, v'_k \rangle \rangle \in |\mathfrak{A}|^k \times |\mathfrak{A}|^k \mid v_j = v'_j \text{ for } i \leq j \leq i'\} \\
 Q_i^{\mathfrak{A}^{(k)}} &= \{\langle \langle v_1, \dots, v_k \rangle, \langle v'_1, \dots, v'_k \rangle \rangle \in |\mathfrak{A}|^k \times |\mathfrak{A}|^k \mid v_j = v'_j \text{ for } 1 \leq j \leq k \text{ s.t. } j \neq i\}
 \end{aligned}$$

We write *k*-TUPLE for the class of all *k*-tuple structures. Fig. 1 gives a graphical example of *k*-tuple structures. This construction preserves the **finiteness**, so the following holds.

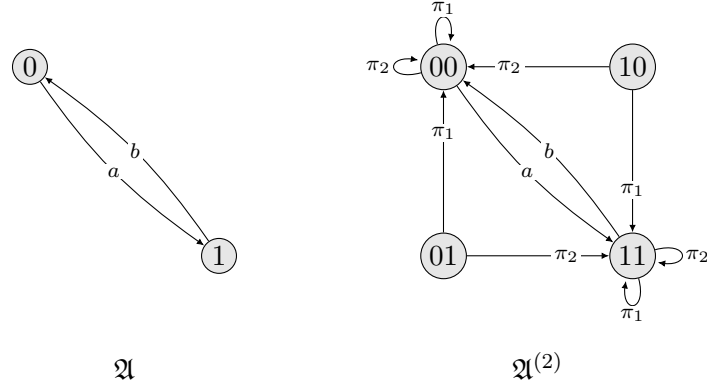


Figure 1. Example of *k*-tuple structures (when  $k = 2$ )

**Proposition 3.2.** For all **structure**  $\mathfrak{A}$  and  $k \geq 1$ ,  $\mathfrak{A}$  is **finite** if and only if  $\mathfrak{A}^{(k)}$  is **finite**.

Additionally note that in *k*-tuple structures,  $U, Q_i, E_{[i,i']}$  can be defined by using  $\pi_i$  as follows:

$$U = \bigcap_{1 \leq j \leq k} \pi_j \quad E_{[i,i']} = \bigcap_{i \leq j \leq i'} \pi_j \cdot \pi_j^\sim \quad Q_i = \bigcap_{1 \leq j \leq k; j \neq i} \pi_j \cdot \pi_j^\sim$$

Thus,  $U, Q_i, E_{[i,i']}$  does not change the expressive power, but they are introduced only for reducing the output size in linear. By using two cumulative sums  $\{E_{[1,i]}\}_{i=1}^k$  and  $\{E_{[i,k]}\}_{i=1}^k$ , we can succinctly express  $\{Q_i\}_{i=1}^k$  as  $Q_i = E_{[1,i-1]} \cap E_{[i+1,n]}$ .

We show that the class of (the isomorphism closure of)  $k$ -tuple structures can be characterized by using equations in CoR. Let  $\Gamma^{(k)}$  be the following finite set of equations where  $i$  ranges over  $1 \leq i \leq k$  and  $E_{[1,0]}, E_{[k+1,k]}$  denote  $\top$ :

$$\begin{aligned}
 U &= \bigcap_{1 \leq j \leq k} \pi_j & E_{[1,i]} &= E_{[1,i-1]} \cap (\pi_i \cdot \pi_i^\smile) \\
 E_{[i,k]} &= E_{[i+1,k]} \cap (\pi_i \cdot \pi_i^\smile) & Q_i &= E_{[1,i-1]} \cap E_{[i+1,k]} \\
 U &\leq \text{I} & & \text{(U is a subset of the identity relation)} \\
 \pi_i^\smile \cdot \pi_i &\leq \text{I} & & \text{(\pi_i is functional)} \\
 \text{I} &\leq \pi_i \cdot U \cdot \pi_i^\smile & & \text{(\pi_i is serial and its codomain is } \{w \mid \langle w, w \rangle \in U\}) \\
 \top \cdot U &\leq Q_i \cdot \pi_i & & \text{(Existence of } \langle u_1, \dots, u_{i-1}, u'_i, u_{i+1}, \dots, u_k \rangle \text{ from } \langle u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_k \rangle) \\
 \text{I} &= E_{[1,k]} & & \text{(If each } \pi_j\text{-edge has the same target, the vertices are the same)} \\
 \top \cdot U \cdot \top &= \top & & \text{(U is not empty)} \\
 a &\leq U \cdot \top \cdot U & & \text{(a is a subset of } \{w \mid \langle w, w \rangle \in U\} \times \{w \mid \langle w, w \rangle \in U\})
 \end{aligned}$$

The set  $\Gamma^{(k)}$  can characterize the class of the isomorphism closure of  $k$ -tuple structures as follows:

**Lemma 3.3.** Let  $\mathfrak{A}$  be a structure over  $\Sigma^{(k)}$ . Then

$$\mathfrak{A} \models \bigwedge \Gamma^{(k)} \iff \mathfrak{A} \in \text{I}(k\text{-TUPLE}).$$

Here,  $\text{I}(k\text{-TUPLE})$  denotes the isomorphism closure of  $k\text{-TUPLE}$ .

**Proof:**

$\Leftarrow$ : This direction can be shown by checking that each equation holds on  $k$ -tuple structures.  $\Rightarrow$ : Let  $U_0 = \{v \mid \langle v, v \rangle \in U^{\mathfrak{A}}\}$ . Combining  $\pi_i^\smile \cdot \pi_i \leq \text{I}$  and  $\text{I} \leq \pi_i \cdot U \cdot \pi_i^\smile$  yields that  $\pi_i^{\mathfrak{A}}$  is a function from  $|\mathfrak{A}|$  to  $U_0$ . Let  $f: |\mathfrak{A}| \rightarrow U_0^k$  be the function defined by  $f(v) = \langle \pi_1^{\mathfrak{A}}(v), \dots, \pi_k^{\mathfrak{A}}(v) \rangle$ . Then  $f$  is bijective as follows. Let  $v_0 \in |\mathfrak{A}|$  be any vertex. Let  $w_1, \dots, w_k$  be s.t.  $f(v_0) = \langle w_1, \dots, w_k \rangle$ . For any  $w'_1 \in U_0$ , by  $\pi_1 \cdot \top \cdot U \leq Q_1 \cdot \pi_1$  and  $\langle v_0, w_1 \rangle \in \pi_1^{\mathfrak{A}}$ ,  $\langle w_1, w'_1 \rangle \in \llbracket \top \rrbracket^{\mathfrak{A}}$ , and  $w'_1 \in U_0$ , there is  $v_1$  such that  $\langle v_0, v_1 \rangle \in Q_1^{\mathfrak{A}}$  and  $\langle v_1, w'_1 \rangle \in \pi_1^{\mathfrak{A}}$ . Then by  $Q_1 = E_{[2,k]}$ , we have  $f(v_1) = \langle w'_1, w_2, \dots, w_k \rangle$ . Similarly, for any  $w'_2 \in U_0$ , by  $\pi_2 \cdot \top \cdot U \leq Q_2 \cdot \pi_2$  and  $\langle v_1, w_2 \rangle \in \pi_2^{\mathfrak{A}}$ ,  $\langle w_2, w'_2 \rangle \in \llbracket \top \rrbracket^{\mathfrak{A}}$ , and  $w'_2 \in U_0$ , there is  $v_2$  such that  $\langle v_1, v_2 \rangle \in Q_2^{\mathfrak{A}}$  and  $\langle v_2, w'_2 \rangle \in \pi_2^{\mathfrak{A}}$ . Then by  $Q_2 = E_{[1,1]} \cap E_{[3,k]}$ , we have  $f(v_2) = \langle w'_1, w'_2, w_3, \dots, w_k \rangle$ . By applying this method iteratively, we have that for any  $w'_1, \dots, w'_k \in U_0$ , there is  $v$  s.t.  $f(v) = \langle w'_1, \dots, w'_k \rangle$ . Hence,  $f$  is surjective. Also, if  $f(v) = \langle w_1, \dots, w_k \rangle = f(v')$ , then by  $\text{I} = E_{[1,k]}$ , we have  $v = v'$ . Hence,  $f$  is injective. Therefore,  $f$  is bijective. We now define  $\mathfrak{B}$  as the structure over  $\Sigma$ , where

$$|\mathfrak{B}| = U_0 \qquad a^{\mathfrak{B}} = a^{\mathfrak{A}} \text{ for } a \in \Sigma.$$

Here,  $\mathfrak{B}$  is indeed an structure, because  $U_0$  is not empty by  $\top \cdot U \cdot \top = \top$  and  $a^{\mathfrak{B}} \subseteq U_0^2$  by  $a \leq U \cdot \top \cdot U$ . Then  $f$  is an isomorphism from  $\mathfrak{A}$  to  $\mathfrak{B}^{(k)}$ . Hence, this completes the proof.  $\square$

Using *k*-tuple structures, we can give the following translation.

**Definition 3.4.** Let  $k \geq 1$  and  $X = \{x_1, \dots, x_k\}$  where  $x_1, \dots, x_k$  are pairwise distinct variables. For each formula  $\varphi$  of  $V(\varphi) \subseteq X$ , the term  $T^{(k)}(\varphi)$  is inductively defined as follows:

$$\begin{aligned} T^{(k)}(a(x_i, x_j)) &\triangleq (\pi_i \cdot a \cdot \pi_j^\sim) \cap I \\ T^{(k)}(\neg\psi) &\triangleq (T^{(k)}(\psi))^- \\ T^{(k)}(\psi \wedge \rho) &\triangleq T^{(k)}(\psi) \cap T^{(k)}(\rho) \\ T^{(k)}(\exists x_i, \psi) &\triangleq (Q_i \cdot T^{(k)}(\psi) \cdot Q_i^\sim) \cap I. \end{aligned}$$

**Lemma 3.5.** Let  $k \geq 1$  and  $X = \{x_1, \dots, x_k\}$  where  $x_1, \dots, x_k$  are pairwise distinct variables. Let  $\mathfrak{A}$  be a structure. For all formulas  $\varphi$  of  $V(\varphi) \subseteq X$  and all  $v_1, \dots, v_k \in |\mathfrak{A}|$ , we have:

$$\llbracket T^{(k)}(\varphi) \rrbracket^{\mathfrak{A}^{(k)}} = \{ \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \mid \{x_1 \mapsto v_1, \dots, x_k \mapsto v_k\} \in \llbracket \varphi \rrbracket^{\mathfrak{A}} \upharpoonright X \}.$$

Here,  $\llbracket \varphi \rrbracket^{\mathfrak{A}} \upharpoonright X$  denotes the set  $\{f \upharpoonright X \mid f \in \llbracket \varphi \rrbracket^{\mathfrak{A}}\}$  where  $f \upharpoonright X$  is the restriction of  $f$  to  $X$ .

**Proof:**

By induction on the structure of  $\varphi$ .

Case  $\varphi = a(x_i, x_j)$ : Since  $T^{(k)}(\varphi) = (\pi_i \cdot a \cdot \pi_j^\sim) \cap I$ , we have:

$$\begin{aligned} &\llbracket (\pi_i \cdot a \cdot \pi_j^\sim) \cap I \rrbracket^{\mathfrak{A}^{(k)}} \\ &= \{ \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \mid \langle \langle v_i, \dots, v_i \rangle, \langle v_j, \dots, v_j \rangle \rangle \in a^{\mathfrak{A}^{(k)}} \} \\ &= \{ \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \mid \langle v_i, v_j \rangle \in a^{\mathfrak{A}} \} \quad (\text{Def. of } a^{\mathfrak{A}^{(k)}}) \\ &= \{ \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \mid \{x_1 \mapsto v_1, \dots, x_k \mapsto v_k\} \in \llbracket a(x_i, x_j) \rrbracket^{\mathfrak{A}} \upharpoonright X \} \end{aligned}$$

Case  $\varphi = \neg\psi$ : Since  $T^{(k)}(\varphi) = T^{(k)}(\psi)^-$ , we have:

$$\begin{aligned} &\llbracket T^{(k)}(\psi)^- \rrbracket \\ &= |\mathfrak{A}^{(k)}|^2 \setminus \llbracket T^{(k)}(\psi) \rrbracket \\ &= |\mathfrak{A}^{(k)}|^2 \setminus \{ \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \mid \{x_1 \mapsto v_1, \dots, x_k \mapsto v_k\} \in \llbracket \psi \rrbracket^{\mathfrak{A}} \upharpoonright X \} \quad (\text{IH}) \\ &= \{ \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \mid \{x_1 \mapsto v_1, \dots, x_k \mapsto v_k\} \notin \llbracket \psi \rrbracket^{\mathfrak{A}} \upharpoonright X \} \\ &= \{ \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \mid \{x_1 \mapsto v_1, \dots, x_k \mapsto v_k\} \in \llbracket \neg\psi \rrbracket^{\mathfrak{A}} \upharpoonright X \} \end{aligned}$$

Case  $\varphi = \psi \wedge \rho$ : Since  $T^{(k)}(\varphi) = T^{(k)}(\psi) \cap T^{(k)}(\rho)$ , we have:

$$\begin{aligned} &\llbracket T^{(k)}(\psi) \cap T^{(k)}(\rho) \rrbracket \\ &= \llbracket T^{(k)}(\psi) \rrbracket^{\mathfrak{A}^{(k)}} \cap \llbracket T^{(k)}(\rho) \rrbracket^{\mathfrak{A}^{(k)}} \\ &= \{ \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \mid \{x_1 \mapsto v_1, \dots, x_k \mapsto v_k\} \in (\llbracket \psi \rrbracket^{\mathfrak{A}} \upharpoonright X) \cap (\llbracket \rho \rrbracket^{\mathfrak{A}} \upharpoonright X) \} \quad (\text{IH}) \\ &= \{ \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \mid \{x_1 \mapsto v_1, \dots, x_k \mapsto v_k\} \in \llbracket \psi \wedge \rho \rrbracket^{\mathfrak{A}} \upharpoonright X \} \end{aligned}$$



Case  $\varphi = \exists x_i, \psi$ : Since  $T^{(k)}(\varphi) = (Q_i \cdot T^{(k)}(\psi) \cdot Q_i^\sim) \cap \mathbf{l}$ , we have:

$$\begin{aligned}
& \llbracket (Q_i \cdot T^{(k)}(\psi) \cdot Q_i^\sim) \cap \mathbf{l} \rrbracket^{\mathfrak{A}^{(k)}} \\
&= \{ \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \mid \langle \langle v'_1, \dots, v'_k \rangle, \langle v'_1, \dots, v'_k \rangle \rangle \in \llbracket T^{(k)}(\psi) \rrbracket^{\mathfrak{A}^{(k)}} \\
&\quad \text{for some } v'_1, \dots, v'_k \in |\mathfrak{A}| \text{ s.t. } v'_j = v_j \text{ for } 1 \leq j \leq k \text{ s.t. } j \neq i \} \\
&\quad \text{(note that } \llbracket T^{(k)}(\psi) \rrbracket^{\mathfrak{A}^{(k)}} \subseteq \llbracket \mathbf{l} \rrbracket^{\mathfrak{A}^{(k)}} \text{ by IH)} \\
&= \{ \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \mid \{x_1 \mapsto v'_1, \dots, x_k \mapsto v'_k\} \in \llbracket \psi \rrbracket^{\mathfrak{A}^{(k)}} \upharpoonright X \\
&\quad \text{for some } v'_1, \dots, v'_k \in |\mathfrak{A}| \text{ s.t. } v'_j = v_j \text{ for } 1 \leq j \leq k \text{ s.t. } j \neq i \} \quad \text{(IH)} \\
&= \{ \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \mid \{x_1 \mapsto v_1, \dots, x_k \mapsto v_k\} \in \llbracket \exists x_i, \psi \rrbracket^{\mathfrak{A}^{(k)}} \upharpoonright X \}
\end{aligned}$$

□

Combining the two above, we have obtained the following main lemma.

**Lemma 3.6.** Let  $k \geq 1$  and  $X = \{x_1, \dots, x_k\}$  where  $x_1, \dots, x_k$  are pairwise distinct **variables**. Let  $\varphi$  be an FO= **formula** of  $V(\varphi) \subseteq X$ . Then

$$(\bigwedge \Gamma^{(k)}) \rightarrow T^{(k)}(\varphi) \geq \mathbf{l} \text{ is [finitely] valid} \iff \varphi \text{ is [finitely] valid.}$$

**Proof:**

We have:

$$\begin{aligned}
& (\bigwedge \Gamma^{(k)}) \rightarrow T^{(k)}(\varphi) \geq \mathbf{l} \text{ is valid} \\
&\iff \langle v, v \rangle \in \llbracket T^{(k)}(\varphi) \rrbracket^{\mathfrak{A}} \text{ for all } \mathfrak{A} \text{ s.t. } \mathfrak{A} \models \bigwedge \Gamma^{(k)} \text{ and all } v \in |\mathfrak{A}| \\
&\iff \langle \langle v_1, \dots, v_k \rangle, \langle v_1, \dots, v_k \rangle \rangle \in \llbracket T^{(k)}(\varphi) \rrbracket^{\mathfrak{B}^{(k)}} \text{ for all } \mathfrak{B} \text{ and } v_1, \dots, v_k \in |\mathfrak{B}| \quad \text{(Lem. 3.3)} \\
&\iff \{x_1 \mapsto v_1, \dots, x_k \mapsto v_k\} \in \llbracket \varphi \rrbracket^{\mathfrak{B}} \upharpoonright X \text{ for all } \mathfrak{B} \text{ and } v_1, \dots, v_k \in |\mathfrak{B}| \quad \text{(Lem. 3.5)} \\
&\iff \varphi \text{ is valid.}
\end{aligned}$$

For **finite validity**, it is shown in the same way because  $\mathfrak{B}$  is **finite** iff  $\mathfrak{B}^{(k)}$  is **finite** (Prop. 3.2). □

**Theorem 3.7.** There is a linear-size translation from FO= **formulas** into CoR **equations** preserving **validity** and **finite validity**.

**Proof:**

By Lem. 3.6 with the **Schröder-Tarski translation** (Prop. 2.3). □

**Remark 3.8.** We do not know whether our translation works for the equational theory of (possibly non-representable) relation algebras. This is because our construction is not compatible with *quasi-projective relation algebras*—relation algebras having elements  $p$  and  $q$  s.t.  $p^\sim \cdot p \leq \mathbf{l}$ ,  $q^\sim \cdot q \leq \mathbf{l}$ , and  $p^\sim \cdot q = \top$ . (As quasi-projective relation algebras are representable [3], this class is useful to show that a given translation works also for the equational theory of relation algebras, see e.g., [5].)

### 3.1. Reducing to a more restricted syntax of CoR

Moreover, we can eliminate converse  $\smile$  and identity  $l$  by using translations given in [11].

**Proposition 3.9.** ([11, Lem. 7, 9])

There is a linear-size translation from CoR **equations** into CoR **equations** without  $\smile$  nor  $l$  preserving **validity** and **finite validity**.

**Proposition 3.10.** ([11, Lem. 7, 9, 11, 16])

There is a polynomial-size translation from CoR **equations** into CoR **equations** with one variable and without  $\smile$  nor  $l$  preserving **validity** and **finite validity**.

**Remark 3.11.** The translation in [11, Lem. 11] (for reducing the number of **variables** to one) is not a linear-size translation, as the output size is not bounded in linear (bounded in quadratic) to the input size.

By Thm. 3.7 with the propositions above, we also have the following:

**Corollary 3.12.** There is a linear-size translation from FO<sub>=</sub> **formulas** into CoR **equations** without  $\smile$  nor  $l$  preserving **validity** and **finite validity**.

**Corollary 3.13.** There is a polynomial-size translation from FO<sub>=</sub> **formulas** into CoR **equations** with one variable and without  $\smile$  nor  $l$  preserving **validity** and **finite validity**.

**Corollary 3.14.** There is a polynomial-size translation from FO<sub>=</sub> **formulas** into FO3 **formulas** (without equality) with one binary predicate symbol preserving **validity** and **finite validity**.

## 4. Tseitin translation for CoR

By a similar argument as the *Tseitin translation* [12], which is a translation from propositional formulas into conjunctive normal form preserving validity in proposition logic (see also the Plaisted-Greenbaum translation [13] for FO<sub>=</sub> and the translation from FO2<sub>=</sub> into the Scott class [14, 15]), we can translate into CoR **terms** with bounded alternation of operations.

For each **term**  $t$ , we introduce a fresh **variable**  $a_t$ . Then for a **term**  $t$ , we define the set of **equations**  $\Gamma_t$  as follows:

$$\begin{aligned} \Gamma_b &\triangleq \{a_b = b\} & \Gamma_{s-} &= \Gamma_s \cup \{a_{s-} = a_s^-\} & \Gamma_{s \cap u} &= \Gamma_s \cup \Gamma_u \cup \{a_{s \cap u} = a_s \cap a_u\} \\ \Gamma_l &\triangleq \{a_l = l\} & \Gamma_{s \smile} &= \Gamma_s \cup \{a_{s \smile} = a_s^{\smile}\} & \Gamma_{s \cdot u} &= \Gamma_s \cup \Gamma_u \cup \{a_{s \cdot u} = a_s \cdot a_u\} \end{aligned}$$

Then we have the following:

**Lemma 4.1.** For all CoR **terms**  $t$ , we have:

$$t = \top \text{ is [finitely] } \textbf{valid} \iff (\bigwedge \Gamma_t) \rightarrow a_t = \top \text{ is [finitely] } \textbf{valid}.$$

**Proof:**

For all **structures**  $\mathfrak{A}$  s.t.  $\mathfrak{A} \models \bigwedge \Gamma_t$ , we have  $\mathfrak{A} \models s = a_s$  for all subterms  $s$  of  $t$ , by straightforward induction on  $s$ . Thus it suffices to prove that  $t = \top$  is [finitely] **valid**  $\iff \Gamma_t \rightarrow t = \top$  is [finitely] **valid**.  $\implies$ : Trivial.  $\impliedby$ : Since  $a_s$  is not occurring in  $t$ , we can easily transform a **structure**  $\mathfrak{A}$  s.t.  $\mathfrak{A} \models t = \top$  into a **structure**  $\mathfrak{A}'$  s.t.  $\mathfrak{A}' \models \bigwedge \Gamma_t$  and  $\mathfrak{A}' \not\models t = \top$  by only modifying  $a_s^{\mathfrak{A}}$  appropriately. Hence this completes the proof.  $\square$

**Example 4.2.** The **equation**  $((b \cdot c)^- \cdot d)^- = \top$  is translated into the following **quantifier-free formula** preserving **validity** and **finite validity** (we omit **equations** for each **variables**  $b, c, d$ , as they are verbose):

$$\bigwedge \left\{ \begin{array}{ll} a_{b \cdot c} = a_b \cdot a_c, & a_{(b \cdot c)^-} = a_{b \cdot c}^-, \\ a_{(b \cdot c)^- \cdot d} = a_{(b \cdot c)^-} \cdot a_d, & a_{((b \cdot c)^- \cdot d)^-} = a_{(b \cdot c)^- \cdot d}^- \end{array} \right\} \rightarrow a_{((b \cdot c)^- \cdot d)^-} = \top$$

This is **semantically equivalent** to the following **equation**:

$$(\top \cdot \bigcup \left\{ \begin{array}{l} (a_{b \cdot c} \cap (a_b^- \dagger a_c^-)), (a_{b \cdot c}^- \cap (a_b \cdot a_c)), \\ (a_{(b \cdot c)^-} \cap a_{b \cdot c}), (a_{(b \cdot c)^-}^- \cap a_{b \cdot c}^-) \\ (a_{(b \cdot c)^- \cdot d} \cap (a_{(b \cdot c)^-}^- \dagger a_d^-)), (a_{(b \cdot c)^- \cdot d}^- \cap (a_{(b \cdot c)^-} \cdot a_d)), \\ (a_{((b \cdot c)^- \cdot d)^-} \cap a_{(b \cdot c)^- \cdot d}), (a_{((b \cdot c)^- \cdot d)^-}^- \cap a_{(b \cdot c)^- \cdot d}^-) \end{array} \right\} \cdot \top) \cup a_{((b \cdot c)^- \cdot d)^-} = \top$$

By using the translation above (and replacing complemented **variables**  $b^-$  with fresh **variables**  $c$  and introducing axiom  $b^- = c$ ), we can translate each CoR **equation** without  $\mid$  nor  $\_ \smile$  into an **equation** of the form  $(\top \cdot (\bigcup \Gamma) \cdot \top) \cup a = \top$ , where  $\Gamma$  is a finite set of **terms** of one of the following forms:

$$b \cap c \quad b^- \cap c^- \quad b \cap (c \dagger d) \quad b \cap (c \cdot d) \quad b \cap (c \cap d)$$

In this form, the number of alternations of operations, in particular the operations  $\cdot$  and  $\dagger$  (and similarly,  $\cdot$  and  $\_ \smile$ ), is reduced. Hence, we have obtained the following:

**Theorem 4.3.** There is a linear-size translation from CoR **equations** into **equations** of the form  $t = \top$ , where  $t$  is in the level  $\Sigma_2^{\text{CoR}}$  of the **dot-dagger alternation hierarchy** [10], preserving **validity** and **finite validity**.

**Proof:**

By Lem. 4.1 (with Prop. 2.3).  $\square$

**4.1. Linear-size conservative reduction to the Gödel class  $[\forall^3 \exists^*, (0, \omega), (0)]$** 

Additionally, we note that by using the argument above, we give a linear-size translation from  $\text{FO}_=$  **formulas** into  $[\forall^3 \exists^*, (0, \omega), (0)]$  **sentences** (i.e., **sentences** of the form  $\forall x, \forall y, \forall z, \exists w_1, \dots, \exists w_n, \varphi$  where  $n \geq 0$  and  $\varphi$  is quantifier-free, has only binary predicate symbols and does not have constant symbols, function symbols or non-binary predicate symbols, see e.g., [6] for the notation of the prefix-vocabulary class.  $\text{FO}_=$  in this paper corresponds to the class  $[\text{all}, (0, \omega), (0)]_=$  preserving **satisfiability** and **finite satisfiability**.

For example, let us recall the translated [equation](#) in Example 4.2. By the [standard translation](#) (Prop. 2.2), this [equation](#) is [semantically equivalent w.r.t. binary relations](#) to the following FO3 [sentence](#) (without equality):

$$\exists x, \exists y, \bigvee \left\{ \begin{array}{l} (a_{b.c}(x, y) \wedge (\forall w_1, \neg a_b(x, w_1) \vee \neg a_c(w_1, y))), \\ (\neg a_{b.c}(x, y) \wedge (\exists z, a_b(x, z) \wedge a_c(z, y))), \\ a_{(b.c)-}(x, y) \wedge a_{b.c}(x, y), \neg a_{(b.c)-}^-(x, y) \wedge \neg a_{b.c}^-(x, y), \\ ((a_{(b.c)-}.d(x, y) \wedge (\forall w_2, \neg a_{(b.c)-}(x, w_2) \vee \neg a_d(w_2, y))), \\ (\neg a_{(b.c)-}.d(x, y) \wedge (\exists z, a_{(b.c)-}(x, z) \wedge a_d(z, y))), \\ a_{((b.c)-.d)-}(x, y) \wedge a_{(b.c)-.d}(x, y), \neg a_{((b.c)-.d)-}(x, y) \wedge \neg a_{(b.c)-.d}(x, y) \end{array} \right\} \\ \vee (\forall w_3, \forall w_4, a_{((b.c)-.d)-}(w_3, w_4))$$

By taking the prenex normal form of the [sentence](#) above in the ordering of  $x, y, z, w_1, w_2, w_3, w_4$ , we can obtain an  $[\exists^3 \forall^*, (0, \omega), (0)]$  [sentence](#) (note that  $(\exists z, \psi \vee \rho) \leftrightarrow ((\exists z, \psi) \vee (\exists z, \rho))$ ). Thus, as a corollary of Thm. 4.3, we can translate CoR [equations](#) without  $\mathsf{l}$  into  $[\exists^3 \forall^*, (0, \omega), (0)]$  [sentences](#) preserving [validity](#) and [finite validity](#). Hence, we also have the following:

**Corollary 4.4.** There is a linear-size [conservative reduction](#) from  $\text{FO} =$  [formulas](#) into  $[\forall^3 \exists^*, (0, \omega), (0)]$  [sentences](#).

**Proof:**

By Cor. 3.13 with the translation above, there is a linear-size translation from  $\text{FO} =$  [formulas](#) into  $[\exists^3 \forall^*, (0, \omega), (0)]$  [sentences](#) preserving [validity](#) and [finite validity](#). Hence this completes the proof (by considering negated [formulas](#)).  $\square$

## References

- [1] Tarski A. On the Calculus of Relations. *The Journal of Symbolic Logic*, 1941. **6**(3):73–89. doi:10.2307/2268577.
- [2] Givant S. Introduction to Relation Algebras, volume 1. Springer International Publishing, 2017. ISBN 978-3-319-65234-4. doi:10.1007/978-3-319-65235-1.
- [3] Tarski A, Givant S. A Formalization of Set Theory without Variables, volume 41. American Mathematical Society, 1987. ISBN 978-0-8218-1041-5. doi:10.1090/coll/041.
- [4] Maddux RD. Finitary Algebraic Logic. *Mathematical Logic Quarterly*, 1989. **35**(4):321–332. doi:10.1002/malq.19890350405.
- [5] Andr  ka H, N  meti I. Reducing First-order Logic to Df3, Free Algebras. In: Cylindric-like Algebras and Algebraic Logic, Bolyai Society Mathematical Studies, pp. 15–35. Springer. ISBN 978-3-642-35025-2, 2013. doi:10.1007/978-3-642-35025-2\_2.
- [6] B  rger E, Gr  del E, Gurevich Y. The Classical Decision Problem. Springer, 1997. ISBN 3-540-42324-9. URL <https://link.springer.com/9783540423249>.

- [7] Boolos GS, Burgess JP, Jeffrey RC. Computability and Logic. Cambridge University Press, 5 edition, 2007. ISBN 978-0-521-87752-7. doi:10.1017/CBO9780511804076.
- [8] Nakamura Y. Expressive Power and Succinctness of the Positive Calculus of Relations. In: RAMiCS, volume 12062 of LNCS. Springer, 2020 pp. 204–220. doi:10.1007/978-3-030-43520-2\_13.
- [9] Givant S. The Calculus of Relations as a Foundation for Mathematics. *Journal of Automated Reasoning*, 2007. **37**(4):277–322. doi:10.1007/s10817-006-9062-x.
- [10] Nakamura Y. Expressive power and succinctness of the positive calculus of binary relations. *Journal of Logical and Algebraic Methods in Programming*, 2022. **127**:100760. doi:10.1016/j.jlamp.2022.100760.
- [11] Nakamura Y. The Undecidability of FO3 and the Calculus of Relations with Just One Binary Relation. In: ICLA, volume 11600 of LNCS. Springer, 2019 pp. 108–120. doi:10.1007/978-3-662-58771-3\_11.
- [12] Tseitin GS. On the Complexity of Derivation in Propositional Calculus. In: Automation of Reasoning: 2: Classical Papers on Computational Logic 1967–1970, Symbolic Computation, pp. 466–483. Springer. ISBN 978-3-642-81955-1, 1983. doi:10.1007/978-3-642-81955-1\_28.
- [13] Plaisted DA, Greenbaum S. A Structure-preserving Clause Form Translation. *Journal of Symbolic Computation*, 1986. **2**(3):293–304. doi:10.1016/S0747-7171(86)80028-1.
- [14] Scott D. A decision method for validity of sentences in two variables. *Journal of Symbolic Logic*, 1962. **27**(337).
- [15] Grädel E, Kolaitis PG, Vardi MY. On the Decision Problem for Two-Variable First-Order Logic. *The Bulletin of Symbolic Logic*, 1997. **3**(1):53–69. doi:10.2307/421196.

## A. A construction from FO to FO3 not via CoR

In the following, we present a more direct translation from FO= [formulas](#) into FO3= [formulas](#) not via CoR (this is almost immediately obtained from Sect. 3 with the [standard translation](#) from CoR [terms](#) into FO3= [formulas](#) (Prop. 2.2)).

Let  $\Gamma_{\text{FO3=}}^{(k)}$  be the following finite set of FO3= [formulas](#) where  $i$  ranges over  $1 \leq i \leq k$ ,

$E_{[1,0]}(x, y)$  and  $E_{[k+1,k]}(x, y)$  denote the **formula**  $t$ , and  $x, y, z$  are pairwise distinct **variables**:

$$\begin{aligned}
\forall x, \forall y, U(x, y) &\leftrightarrow \bigwedge_{1 \leq j \leq k} \pi_j(x, y) \\
\forall x, \forall y, (E_{[1,i]}(x, y) &\leftrightarrow (E_{[1,i-1]}(x, y) \wedge (\exists z, (\pi_i(x, z) \wedge \pi_i(y, z)))) \\
\forall x, \forall y, (E_{[i,k]}(x, y) &\leftrightarrow (E_{[i+1,k]}(x, y) \wedge (\exists z, (\pi_i(x, z) \wedge \pi_i(y, z)))) \\
\forall x, \forall y, Q_i(x, y) &\leftrightarrow (E_{[1,i-1]}(x, y) \wedge E_{[i+1,k]}(x, y)) \\
\forall x, \forall y, (U(x, y) &\rightarrow x = y) && (U \text{ is a subset of the identity relation}) \\
\forall x, \forall y, \forall z, ((\pi_i(x, y) \wedge \pi_i(x, z)) &\rightarrow y = z) && (\pi_i \text{ is functional}) \\
\forall x, \exists y, (\pi_i(x, y) \wedge U(y, y)) && (\pi_i \text{ is serial and its codomain is } \{w \mid \langle w, w \rangle \in U\}) \\
\forall x, \forall y, (U(y, y) \rightarrow (\exists z, Q_i(x, z) \wedge \pi_i(z, y))) && (\text{Existence of } \langle u_1, \dots, u_{i-1}, u'_i, u_{i+1}, \dots, u_k \rangle \text{ from } \langle u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_k \rangle) \\
\forall x, \forall y, (x = y) &\leftrightarrow E_{[1,k]}(x, y) && (\text{If each } \pi_j\text{-edge has the same target, the vertices are the same}) \\
\exists x, U(x, x) && (U \text{ is not empty}) \\
\forall x, \forall y, a(x, y) &\rightarrow (U(x, x) \wedge U(y, y)) && (a \text{ is a subset of } \{w \mid \langle w, w \rangle \in U\} \times \{w \mid \langle w, w \rangle \in U\})
\end{aligned}$$

**Lemma A.1.** Let  $\mathfrak{A}$  be a **structure** over  $\Sigma^{(k)}$ . Then

$$\mathfrak{A} \models \bigwedge \Gamma_{\text{FO3}=}^{(k)} \iff \mathfrak{A} \in \text{I}(k\text{-TUPLE}).$$

**Proof:**

We can check  $(\bigwedge \Gamma_{\text{FO3}=}^{(k)})$  and  $(\bigwedge \Gamma^{(k)})$  are **semantically equivalent**. Thus by Lem. 3.3.  $\square$

**Definition A.2.** Let  $k \geq 3$  and  $X = \{x_1, \dots, x_k\}$  where  $x_1, \dots, x_k$  are pairwise distinct **variables**. For each **formula**  $\varphi$  of  $V(\varphi) \subseteq X$  and  $z \in \{x_1, x_2, x_3\}$ , the FO3 **formula**  $T_z^{(k)}(\varphi)$  of  $V \subseteq \{x_1, x_2, x_3\}$  is inductively defined as follows, where  $z' = \min(\{x_1, x_2, x_3\} \setminus \{z\})$  and  $z'' = \min(\{x_1, x_2, x_3\} \setminus \{z, z'\})$  under the ordering  $x_1 < x_2 < x_3$ :

$$\begin{aligned}
T_z^{(k)}(a(x_i, x_j)) &\triangleq \exists z', \exists z'', \pi_i(z, z') \wedge a(z', z'') \wedge \pi_j(z, z'') \\
T_z^{(k)}(\neg \psi) &\triangleq \neg T_z^{(k)}(\psi) \\
T_z^{(k)}(\psi \wedge \rho) &\triangleq T_z^{(k)}(\psi) \wedge T_z^{(k)}(\rho) \\
T_z^{(k)}(\exists x_i, \psi) &\triangleq \exists z', Q_i(z, z') \wedge T_{z'}^{(k)}(\psi).
\end{aligned}$$

**Lemma A.3.** Let  $k \geq 3$  and  $X = \{x_1, \dots, x_k\}$  where  $x_1, \dots, x_k$  are pairwise distinct **variables**. Let  $\mathfrak{A}$  be a **structure**. For all **formulas**  $\varphi$  of  $V(\varphi) \subseteq X$ , all  $z \in \{x_1, x_2, x_3\}$ , and all  $u_1, \dots, u_k \in |\mathfrak{A}|$ , we have:

$$\{z \mapsto \langle u_1, \dots, u_k \rangle\} \in \llbracket T_z^{(k)}(\varphi) \rrbracket^{\mathfrak{A}^{(k)}} \upharpoonright \{z\} \iff \{x_1 \mapsto u_1, \dots, x_k \mapsto u_k\} \in \llbracket \varphi \rrbracket^{\mathfrak{A}} \upharpoonright X.$$

**Proof:**

By induction on the structure of  $\varphi$  (similarly for Lem. 3.5).  $\square$

Combining the two above, we have obtained the following main lemma.

**Lemma A.4.** Let  $k \geq 3$  and  $X = \{x_1, \dots, x_k\}$  where  $x_1, \dots, x_k$  are pairwise distinct variables. Let  $\varphi$  be a formula of  $V(\varphi) \subseteq X$  and  $z \in \{x_1, x_2, x_3\}$ . Then

$$(\bigwedge \Gamma_{\text{FO3}=}^{(k)} \rightarrow T_z^{(k)}(\varphi) \text{ is [finitely] valid}) \iff \varphi \text{ is [finitely] valid.}$$

**Proof:**

We have:

$$\begin{aligned} & (\bigwedge \Gamma_{\text{FO3}=}^{(k)} \rightarrow T_z^{(k)}(\varphi) \text{ is valid}) \\ \iff & \{z \mapsto v\} \in \llbracket T_z^{(k)}(\varphi) \rrbracket^{\mathfrak{A}} \upharpoonright \{z\} \text{ for all } \mathfrak{A} \text{ s.t. } \mathfrak{A} \models \bigwedge \Gamma_{\text{FO3}=}^{(k)} \text{ and all } v \in |\mathfrak{A}| \\ \iff & \{z \mapsto \langle u_1, \dots, u_k \rangle\} \in \llbracket T_z^{(k)}(\varphi) \rrbracket^{\mathfrak{B}^{(k)}} \upharpoonright \{z\} \text{ for all } \mathfrak{B} \text{ and } u_1, \dots, u_k \in |\mathfrak{B}| \quad (\text{Lem. A.1}) \\ \iff & \{x_1 \mapsto u_1, \dots, x_k \mapsto u_k\} \in \llbracket \varphi \rrbracket^{\mathfrak{B}} \upharpoonright X \text{ for all } \mathfrak{B} \text{ and } u_1, \dots, u_k \in |\mathfrak{B}| \quad (\text{Lem. A.3}) \\ \iff & \varphi \text{ is valid.} \end{aligned}$$

For finite validity, it is shown in the same way, because  $\mathfrak{B}$  is finite iff  $\mathfrak{B}^{(k)}$  is finite.  $\square$

Additionally, note that equality can be eliminated in  $\text{FO3}_=$ .

**Proposition A.5.** There is a linear-size conservative reduction from  $\text{FO3}_=$  formulas into  $\text{FO3}$  formulas without equality.

**Proof:**

[Proof Sketch] See, e.g., [7, Prop. 19.13] for  $\text{FO}_=$ . This can be proved by replacing each occurrence of equality  $=$  with a fresh binary predicate symbol  $E$  and then adding axioms of that  $E$  is an equivalence relation and of that each binary predicate  $a$  satisfies the congruence law w.r.t.  $E$ :

$$\forall x, \forall x', \forall y, \forall y', (E(x, x') \wedge E(y, y')) \rightarrow (a(x, y) \leftrightarrow a(x', y'))$$

While the formula above is not in  $\text{FO3}$ , the construction in [7, Prop. 19.13] still works for  $\text{FO3}$  by replacing the formula with the conjunction of the following two formulas:

$$\begin{aligned} & \forall x, \forall x', \forall y, E(x, x') \rightarrow (a(x, y) \leftrightarrow a(x', y)) \\ & \forall x, \forall y, \forall y', E(y, y') \rightarrow (a(x, y) \leftrightarrow a(x, y')) \end{aligned}$$

$\square$

**Theorem A.6.** There is a linear-size conservative reduction from  $\text{FO}_=$  formulas into  $\text{FO3}$  formulas (without equality).

**Proof:**

By Lem. A.4 with Prop. A.5.  $\square$