COMPUTABLE FUNCTIONALS OF FINITE TYPES

Yu. L. Ershov

UDC 517.11:518.5

This paper is a continuation of [1]. Here we describe and study the concept of an f -space, which is then used to determine a class $\mathbf{C}$ of partial (continuous) functionals of all finite types over an arbitrary complete f -space (in particular, over the set of natural numbers N). This class of functionals has a high degree of universality which makes it possible to compare various classes of functionals with each other (to determine the "natural" action of one class of functionals on another).

The class of functionals $\mathbf{C}$ is the greatest class on which the computable functionals of the class $\{F_\sigma \mid \sigma \in T\}$, defined in [1], naturally act. The class $\{F_\sigma \mid \sigma \in T\}$ is compared with the Kleene–Kreisel functionals [2, 3].

The theory of computable partial continuous functionals constructed here is a very natural extension of the familiar earlier theory of computable functionals of type [(010)10], although not so much from the point of view of the analysis of the concept of computability, but rather from the mathematical functional point of view. It is also important to note that this theory realizes the idea of constructing computable functionals of higher types starting from just the class of generally recursive functions [4].

It appears that this theory has some conceptual proximity with Scott's program for constructing a general mathematical theory of computation [5]. Unfortunately, we have only comparatively recently become acquainted with this paper and a later paper [6] on the same theme, which has prevented explicit expression of the point of contact and the distance between these theories. It is only worth noting that it is the constructive (computational) aspect of Scott's theory which is least effectively realized in the subsequent publications. In his last article these connections are established more explicitly.

It should also be noted that some of the results in Sec. 7 on the solvability of the problem P have been obtained (in other terms) independently by Chernov who also studied certain topological and constructive properties of the space $\{P_\sigma \mid \sigma \in T\}$ (cf. the end of [1]), which are not studied in this paper.

1. Definition and Fundamental Properties of the f -Space

Let X be a topologically separable (T_0-) space. On the elements of X we define a partial order $\leqslant$ as follows: $x \leqslant y \rightleftharpoons x \in \bar{y}$ ($\bar{y}$ is the closure of the set $\{y\}$ containing a single point). In other words, $x \leqslant y \Longleftrightarrow$ for any open set V , if $x \in V$, then $y \in V$.

COROLLARY 1. If V is an open set, $x \leqslant y$ and $x \in V$, then $y \in V$.

Let us verify that the relation $x \leqslant y$ is a partial order.

1) $x \leqslant x$ – obviously;

2) $x \leqslant y \mathbin{\&} y \leqslant x \Rightarrow x = y$.

This property follows because X is separable. Indeed, if $x \neq y$, there is an open set V such that $x \in V$, while $y \notin V$ (or $y \in V$ and $x \notin V$); but then $x \nleqslant y$ $(y \nleqslant x)$.

3) $x \leqslant y \mathbin{\&} y \leqslant z \Rightarrow x \leqslant z$.

Translated from Algebra i Logika, Vol. 11, No. 4, pp. 367-437, July-August, 1972. Original article submitted August 28, 1972.

Indeed, if V is open and $x \preccurlyeq y$, then $x \preccurlyeq y \Longrightarrow y \in V$, while $y \preccurlyeq z \Longrightarrow z \in V$.

The open nonempty set V is said to be an f-set if there is an element $\sigma_V \in V$ such that $V = \{x \mid \sigma_V \preccurlyeq x\}$. We note that the element σ_V is defined uniquely (with respect to).

COROLLARY 2. If V is an f-set and $\{V_i \mid i \in I\}$ is a system of open sets, we have $V \subseteq \bigcup_{i \in I} V_i \Longleftrightarrow \exists i \in I \ (V \subseteq V_i)$.

Indeed, if $V \subseteq \bigcup_{i \in I} V_i$, there is an $i \in I$ such that $\sigma_V \in V_i$; but then, by Corollary 1, $V \subseteq V_i$.

The topological space X is said to be an f-space if the following conditions hold:

1. For any f-sets V_0 and V_1, if $V_0 \cap V_1 \neq \emptyset$, then $V_0 \cap V_1$ is an f-set.
2. The family of all f-sets, together with the empty set, forms a basis for the topology of X.

LEMMA 1. The T_1-space X is an f-space if and only if X is a discrete topological space.

Proof. For T_1-spaces the order $\preccurlyeq$ is trivial; i.e., it coincides with the relation of equality. Hence, f-sets (if they exist) contain only one point. If X is an f-space, every element x lies in some f-set. Consequently, for any $x \in X$, the set $\{x\}$ is open, i.e., the topology in X is discrete. Conversely, it is obvious that the discrete topological space X is an f-space. The lemma is proved.

The topological space X is said to be an f_0-space if X is an f-space and the space X itself is an f-set; i.e., there is an element $\sigma\,(\sigma_X)$ such that $\sigma \preccurlyeq x$ for any $x \in X$.

Let X be an arbitrary topological space, $\mathcal{B}$ a (fixed) basis for the topology of X, and Y be an f_0-space. Consider the family $C(X, Y)$ of all continuous mappings from X into Y. For any nonempty $V \in \mathcal{B}$ and any f-set W of Y, $\langle V, W \rangle$ denotes the set $\{\varphi \mid \varphi \in C(X,Y) \,\&\, \varphi(V) \subseteq W\}$. Let $V_i \in \mathcal{B}$, W_i be f-sets of Y, $i < k$, $U = \bigcap_{i<k} \langle V_i, W_i \rangle$.

LEMMA 2. The set U is nonempty if and only if for any $I \subseteq \{0, 1, \ldots, k-1\}$ if $\bigcap_{i \in I} V_i \neq \emptyset$, then $\bigcap_{i \in I} W_i \neq \emptyset$.

Proof. It is obvious that the condition is necessary since if $\varphi \in U$ $x \in \bigcap_{i \in I} V_i$, then $\varphi(x) \in \bigcap_{i \in I} W_i$. We prove sufficiency.

Let $A \rightleftharpoons \{I \mid I \subseteq \{0, 1, \ldots, k-1\}, \bigcap_{i \in I} V_i \neq \emptyset\}$, $V_I \rightleftharpoons \bigcap_{i \in I} V_i$, $V_I' \rightleftharpoons V_I \setminus \bigcup_{I \subset I'} V_{I'}$, $W_I \rightleftharpoons \bigcap_{i \in I} W_i$, $I, I' \subseteq \{0, 1, \ldots, k-1\}$. We define the mapping φ_U from X into Y as follows:

$$\varphi_U(x) = \begin{cases} \sigma_{W_I}, & \text{if } x \in V_I',\ I \in A; \\ \sigma_Y, & \text{if } x \notin \bigcup_{i<k} V_i. \end{cases}$$

We now verify that $\varphi_U \in C(X, Y)$ and $\varphi_U \in U$. To prove that φ_U is continuous, we show that $\varphi_U^{-1}(W)$ is open in W for any f-set X. Let $A_W \rightleftharpoons \{I \mid I \in A,\ W_I \subseteq W\} = \{I \mid I \in A,\ \sigma_{W_I} \in W\}$, then $\varphi_U^{-1}(W) = \bigcup V_I$. Indeed, if $I \in A_W$, $I' \supset I$ and $I' \in A$, then $I' \in A_W$ and so $\bigcup_{I \in A_W} V_I' = \bigcup_{I \in A_W} V_I$; but $V_I' = \varphi_U^{-1}(\sigma_{W_I})$ and

$$\varphi_U^{-1}(W) = \bigcup_{\sigma_{W_I} \in W} \varphi_U^{-1}(\sigma_{W_I}).$$

The condition $\sigma_{W_I} \in W$ is equivalent to $W_I \subseteq W$. Hence the equation $\varphi_U^{-1}(W) = \bigcup_{I \in A_W} V_I$ is proved. The set $\bigcup_{I \in A_W} V_I$ is open, and we have proved that φ_U is continuous. Let us show that $\varphi_U \in U$. It is sufficient to show that $\varphi_U \in \langle V_i, U_i \rangle$, $i < k$. If $x \in V_i$, then $x \in V_I'$ for some $I \in A$, $i \in I$; then $\varphi_U(x) = \sigma_{W_I} \in W_I \subseteq W_i$. Thus, $\varphi_U \in U$. The lemma is proved.

In the set $C(X,Y)$ we introduce a partial order $\leqslant$ thus:

$$\varphi_0 \leqslant \varphi_1 \Leftrightarrow \forall x \in X \quad (\varphi_0(x) \leqslant \varphi_1(x)).$$

Here the notation $\varphi_0(x) \leqslant \varphi_1(x)$ uses the partial order which is defined on the f_0-set Y. It is easy to verify that the relation defined above is a partial order.

THEOREM 1. The system of sets of the form

$$U = \bigcap_{i \in k} \langle V_i, W_i \rangle, \quad V_i \in \mathcal{B},$$

where W_i is an f-set of Y, $i<k$, $k \in N$, forms a basis for a topology on $C(X,Y)$. This topology is such that $C(X,Y)$ is an f_0-space, its f-sets are just the nonempty sets U of the form indicated above, and the partial order defined on $C(X,Y)$ by this topology coincides with the partial order defined above.

Proof. Let us show that every element φ belongs to some set of the form U. Since Y is an f-set in (because Y is an f_0-space), we have $\langle V, Y \rangle = C(X,Y)$ for any $V \in \mathcal{B}$. We now show separability. Let $\varphi_0 \neq \varphi_1 \in C(X,Y)$ and let $x \in X$ be such that $\varphi_0(x) \neq \varphi_1(x)$. Since Y is separable, there is an f-set W such that $\varphi_0(x) \in W$ and $\varphi_1(x) \notin W$ (or $\varphi_0(x) \notin W$ and $\varphi_1(x) \in W$). Now let us consider $\varphi_0^{-1}(W)$. Since φ_0 is continuous there can be found a $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq \varphi_0^{-1}(W)$. This inclusion implies that $\varphi_0 \in \langle V, W \rangle$. Since $x \in V$ and $\varphi_1(x) \notin W$, we have $\varphi_1 \notin \langle V, W \rangle$. Thus, the system of sets of the form U forms a basis for a separable topology on $C(X,Y)$.

We now prove that the order $\leqslant$, defined on $C(X,Y)$ above, coincides with the order defined by the topology on $C(X,Y)$. Indeed, let $\varphi_0 \leqslant \varphi_1$ and $\varphi_0 \in \langle V, W \rangle$, where $V \in \mathcal{B}$, and W is an f-set of Y. Then for $x \in V$, we have $\varphi_0(x) \in W$ and $\varphi_0(x) \leqslant \varphi_1(x)$, and consequently, $\varphi_1(x) \in W$ and $\varphi_1 \in \langle V, W \rangle$. Thus it follows that for any set $U = \bigcap_{i<k} \langle V_i, W_i \rangle$, if $\varphi_0 \leqslant \varphi_1$ and $\varphi_0 \in U$, we have $\varphi_1 \in U$, and consequently, also for any open set of the space $C(X,Y)$. Now let $\varphi_0 \nleqslant \varphi_1$ and let $x \in X$ be such that $\varphi_0(x) \nleqslant \varphi_1(x)$. Then, as at the beginning of the proof, we find $V \in \mathcal{B}$ and W, an f-set of Y, such that $x \in V$, $\varphi_0(x) \in W$, $\varphi_1(x) \notin W$ and $\varphi_0 \in \langle V, W \rangle$, $\varphi_1 \notin \langle V, W \rangle$. Thus, it also follows that the order $\leqslant$ coincides with the order defined by the topology on $C(X,Y)$.

Let $V_i \in \mathcal{B}$, let W_i be an f-set of Y, $i<k$, and let

$$U = \bigcap_{i<k} \langle V_i, W_i \rangle \neq \emptyset.$$

We can show that U is an f-set in $C(X,Y)$. To do this we consider the function $\varphi_U \in U$ constructed in Lemma 2. It easily follows from the definition of this function that if $\varphi \in U$, then $\varphi_U \leqslant \varphi$. With what has been proved above, we have $U = \{\varphi \mid \varphi_U \leqslant \varphi\}$. Consequently, U is an f-set.

Let U be an f-set in $C(X,Y)$. Then U is an open set and so $U = \bigcup_{i \in I} U_i$, where the U_i have the form of the finite product of sets of the form $\langle V, W \rangle$, and $V \in \mathcal{B}$, W being an f-set of Y. By Corollary 2, we have $U = U_i$ for some $i \in I$. The theorem is proved.

Note. The definition of the topology of $C(X,Y)$ depends on the choice of the basis $\mathcal{B}$. It appears that not only the definition, but also the topology itself depends on the choice of $\mathcal{B}$. In what follows we shall discuss only f-spaces which have a preferred basis (consisting of f-sets), which will be used in future without specific indication in the definition of the topology of $C(X,Y)$.

On the set $C(X,Y)$ there is always defined a topology of point convergence, defined by a prebasis of sets of the form $\langle x, U \rangle \Leftrightarrow \{f \mid f(x) \in U\}$, where $x \in X$, U open in Y.

LEMMA 3. The topology on $C(X,Y)$, defined in Theorem 1, is stronger than the topology of point convergence. If X is an f-space, $\mathcal{B}$ a basis, consisting of all the f-sets, these topologies coincide.

Proof. We can show that $\langle x, U\rangle$ is open in the topology defined in Theorem 1. Let $f \in \langle x, U\rangle$; then $f(x) \in U$, and there is a basis neighborhood (-set) $V \subseteq U$, such that $f(x) \in V$. Let $W \in B$ be such that $W \in f^{-1}(V)$ and $x \in W$; then $f \in \langle W, V\rangle$ and $\langle W, V\rangle \subseteq \langle x, U\rangle$. Thus, $\langle x, U\rangle$ is open. To prove the second assertion we note that for the f-sets V, U of X and Y respectively, we have $\langle V, U\rangle = \langle \sigma_V, U\rangle$. The lemma is proved.

The following assertion shows that $f(f_0)$-spaces are closed with respect to finite direct products.

PROPOSITION 1. Let X and Y be $f(f_0)$-spaces; then the set $X \times Y$, equipped with a topology of products, is also an $f(f_0)$-space. The subset $Z \neq \emptyset \subseteq X \times Y$ is an f-set in $X \times Y$ if and only if Z has the form $Z = V \times W$, where V is an f-set in X and W is an f-set in Y.

Proof. Sets of the form $V \times W$, where V is an f-set in X and W is an f-set in Y (and empty) form a basis for the (product) topology in the set $X \times Y$. It is easily verified that the partial order, defined by the topology in $X \times Y$, is the product of the partial orders of X and Y, respectively, i.e., $(x_0, y_0) \leq (x_1, y_1) \Leftrightarrow x_0 \leq x_1 \,\&\, y_0 \leq y_1$ for $x_0, x_1 \in X$, $y_0, y_1 \in Y$. It immediately follows from this observation that sets of the form $V \times W$, where V is an f-set of X, W an f-set of Y, are f-sets of $X \times Y$. And since they (with the empty set) form a basis for the topology, there are no other f-sets. The proposition is proved.

Indeed, a more general proposition is thus valid.

PROPOSITION 1'. Let X_i, $i \in I$, be an arbitrary family of $f(f_0)$ spaces and let almost all of them be f_0-spaces; then the set $\prod_{i\in I} X_i$ equipped with a product topology is an $f(f_0)$-space. The nonempty set $Z \subseteq \prod_{i\in I} X_i$ is an f-set in $\prod_{i\in I} X_i$ if and only if it has the form $Z = \prod_{i\in I} Y_i$, where Y_i is an f-set in X_i, $i \in I$, and $Y_i = X_i$ for almost all $i \in I$ (with the exception of a finite number).

The proof is similar to the proof of Proposition 1 and so we omit it.

The propositions which follow below on more specific features of f-spaces are typical of discrete spaces.

PROPOSITION 2. Let X and Y be f-spaces, let $X \times Y$ be equipped with a product topology, Z an arbitrary topological space. The mapping $f: X \times Y \to Z$ is continuous if and only if it is continuous with respect to each of its arguments.

Proof. In one direction the proposition is obvious. Let us show that it is valid in the other direction. Let $f: X \times Y \to Z$ be such that for each $x \in X$ the mapping $\lambda y f(x, y): Y \to Z$ is continuous and for any $y \in Y$ the mapping $\lambda x f(x, y): X \to Z$ is continuous. Let W be an arbitrary open set in Z; let $(x_0, y_0) \in f^{-1}(W)$. Since f is continuous in y, there is a basis f-set V in Y such that $y_0 \in V$ and $\{x_0\} \times V \subseteq f^{-1}(W)$. In particular $f(x, \sigma_V) \in W$, $(x_0, \sigma_V) \in f^{-1}(W)$. Using the continuity of f in x, we find a basis neighborhood U such that $x_0 \in U$ and $U \times \{\sigma_V\} \subseteq f^{-1}(W)$; but then $U \times V \subseteq f^{-1}(W)$, and $(x_0, y_0) \in U \times V$. The set $f^{-1}(W)$ is open and, consequently, the function f is continuous. The proposition is proved.

The following proposition shows that the topology on $C(X, Y)$ is natural.

PROPOSITION 3. Let X be an f-space and Y an f_0-space; then the mapping $\sigma: X \times C(X, Y) \to Y$ defined thus: $\sigma(x, \varphi) \rightleftharpoons \varphi(x)$ is continuous, i.e., $\sigma \in C(X \times C(X, Y), Y)$.

Proof. Let W be an f-set in Y and $(x, \varphi) \in \sigma^{-1}(W)$, i.e., $\sigma(x, \varphi) = \varphi(x) \in W$. Let us consider $\varphi^{-1}(W)$ and find an f-set V in X such that $x \in V$ and $V \subseteq \varphi^{-1}(W)$. (Such a V can be found since φ is continuous.) From the construction we see that $\varphi \in \langle V, W\rangle$ and $(x, \varphi) \in V \times \langle V, W\rangle$. We can show that $\sigma(V \times \langle V, W\rangle) \subseteq W$. Let $x' \in V$, $\varphi' \in \langle V, W\rangle$; then $\varphi'(x') = \sigma(x', \varphi') \in W$. The proposition is proved.

Note. The proposition remains valid for any topological space X if the topology in $C(X,Y)$ is specified as in Theorem 1 with respect to some basis $\mathcal{B}$ for the topology in X.

We prove another property of the topology of the space of functions.

PROPOSITION 4. Let X be an f-space; Y and Z, f_0 spaces; then the mapping of the composition of functions

$$\varkappa : C(X,Y) \times C(Y,Z) \to C(X,Z)$$

is continuous.

Proof. Let V be an f-set in X, W an f-set in Z, and $(g_0, g_1) \in \varkappa^{-1}(\langle V, W\rangle)$, i.e., $g_1^{-1}(W)$ $[\varkappa(g_0,g_1)](V) = g_1(g_0(V)) \subseteq W$ is an open set in Y and $g_0^{-1}(g_1^{-1}(W))$ is an open set in X, and $V \subseteq g_0^{-1}(g_1^{-1}(W))$. Consider the element $g_0(\sigma_V) \in g_1^{-1}(W)$; since the set $g_1^{-1}(W)$ is open, there is a basis neighborhood $U \subseteq Y$ such that $g_0(\sigma_V) \in U \subseteq g_1^{-1}(W)$. Then $g_0 \in \langle V, U\rangle$, $g_1 \in \langle U, W\rangle$, $(g_0, g_1) \in \langle V, U\rangle \times \langle U, W\rangle$. We can verify that $\langle V, U\rangle \times \langle U, W\rangle \in \varkappa^{-1}(W)$. If $g_0' \in \langle V, U\rangle$, $g_0' \in \langle U, W\rangle$, then $[\varkappa(g_0', g_1')](V) = g_1'(g_0'(V)) \subseteq g_1'(U) \subseteq W$; thus, $(g_0', g_1') \in \varkappa^{-1}(\langle V, W\rangle)$, $\langle V, U\rangle \times \langle U, W\rangle \subseteq \varkappa^{-1}(\langle V, W\rangle)$. Consequently, $\varkappa^{-1}(\langle V, W\rangle)$ is open and $\varkappa$ is continuous. The proposition is proved.

Now we prove a fundamental property of f-spaces which we shall need.

THEOREM 2. Let X, Y be f-spaces, Z an f_0-space; then the spaces $C(X \times Y, Z)$ and $C(X, C(Y,Z))$ are naturally homeomorphic.

Proof. Let $\varphi \in C(X \times Y, Z)$; then for all $x \in X$, let φ_x denote a mapping of Y into Z, defined thus: $\varphi_x \rightleftharpoons \lambda y\, \varphi(x,y)$. It follows from the continuity of φ that φ_x is continuous for all $x \in X$, i.e., $\varphi_x \in C(Y,Z)$. Thus, the function $\varphi \in C(X \times Y, Z)$ corresponds to the mapping $\lambda\varphi : X \to C(Y,Z)$ $(\lambda\varphi(x) \rightleftharpoons \varphi_x)$. We now prove that $\lambda\varphi \in C(X, C(Y,Z))$, i.e., that $\lambda\varphi$ is a continuous mapping from X into $C(Y,Z)$. Let W be an f-set of the space Y and V an f-set of Z; then $\langle W, V\rangle$ is an f-set of the space $C(Y,Z)$. We prove that $[\lambda\varphi]^{-1}(\langle W, V\rangle)$ is open. This is sufficient to prove the continuity of $\lambda\varphi$, since the sets of the form $\langle W, V\rangle$ form a prebasis for the topology in $C(Y,Z)$. Let $x \in [\lambda\varphi]^{-1}$ $(\langle W, V\rangle)$; then $\varphi_x \in \langle W, V\rangle$. Consider the element $\sigma_W \in W$. Then $\varphi_x(\sigma_W) \in V$, $\varphi(x, \sigma_W) \in V$. Since φ is continuous, there is an f-set U of the space X and an f-set W' of the space Y such that $(x, \sigma_W) \in U \times W'$ and $\varphi(U \times W') \subseteq V$. But since $\sigma_W \in W'$, we have $W \subseteq W'$ and $\varphi(U \times W) \subseteq V$. We now verify that $[\lambda\varphi](U) \subseteq \langle W, V\rangle$. Let $x' \in U$, $y' \in W$; then $\{[\lambda\varphi](x')\}(y') = \varphi_{x'}(y') = \varphi(x', y') \in V$, and consequently $[\lambda\varphi](x') = \varphi_{x'} \in \langle W, V\rangle$ for $x' \in U$. Noting that $x \in U$, we complete the proof that $\lambda\varphi$ is continuous. Thus, we have defined the mapping $\lambda : C(X \times Y, Z) \to C(X, C(Y,Z))$ $(\lambda(\varphi) \rightleftharpoons \lambda\varphi)$. We can verify that this mapping is continuous. Let U, W, and V be f-sets of the spaces X, Y, and Z respectively; then $\langle U, \langle W, V\rangle\rangle$ is an f-set of the space $C(X, C(Y,Z))$, and to prove that λ is continuous it is sufficient to prove that $\lambda^{-1}(\langle U, \langle W, V\rangle\rangle)$ is open in $C(X \times Y, Z)$ (the sufficiency follows from the fact that sets of the form $\langle U, \langle W, V\rangle\rangle$ form a prebasis for the topology in $C(X, C(Y,Z))$). We can show that $\lambda^{-1}(\langle U, \langle W, V\rangle\rangle) = \langle U \times W, V\rangle$. Let $\varphi \in \langle U \times W, V\rangle$; then, obviously, $\varphi_x \in \langle W, V\rangle$ for $x \in U$ and so $[\lambda\varphi](U) \subseteq \langle W, V\rangle$ and $\lambda\varphi \in \langle U, \langle W, V\rangle\rangle$. Conversely, let $\lambda\varphi \in \langle U, \langle W, V\rangle\rangle$ and let $(x,y) \in U \times W$; then $\varphi(x,y) = \varphi_x(y) = [\lambda\varphi(x)](y) \in V$, i.e., $\varphi(U \times W) \subseteq V$ and $\varphi \in \langle U \times W, V\rangle$. We have proved that λ is continuous.

Let us now construct the inverse mapping $\bar{\lambda} : C(X, C(Y,Z)) \to C(X \times Y, Z)$. Let $\varphi \in C(X, C(Y,Z))$; then we put $[\bar{\lambda}\varphi](x,y) = [\varphi(x)](y)$. Now $\bar{\lambda}\varphi$ is a mapping of $X \times Y$ into Z. We can show that this mapping is continuous. Let U be an f-set in Z; consider $[\bar{\lambda}\varphi]^{-1}(U)$. Let $(x,y) \in [\bar{\lambda}\varphi]^{-1}(U)$. Then

$[\bar{\lambda}\varphi](x,y)=[\varphi(x)](y)\in U$. Since $\varphi(x)\in C(Y,Z)$ is a continuous mapping of Y into Z, there is an f-set W in Y such that $y\in W$ and $W\subseteq[\varphi(x)]^{-1}(U)$. This inclusion implies that $\varphi(x)\in\langle W,U\rangle$. Since φ is continuous, there is an f-set V of X such that $x\in V$ and $V\subseteq\varphi^{-1}(\langle W,U\rangle)$. Consider the f-set $V\times W$ of the space $X\times Y$; $(x,y)\in V\times W$; if $(x',y')\in V\times W$, then $\bar{\lambda}\varphi(x',y')=[\varphi(x')](y')$, and since $x'\in V$, $\varphi\in\langle V,\langle W,U\rangle\rangle$, then $\varphi(x')\in\langle W,U\rangle$, and $y'\in W$ implies that $\bar{\lambda}\varphi(x',y')=[\varphi(x')](y')\in U$. Thus, $(x,y)\in V\times W$ and $V\times W\subseteq[\bar{\lambda}\varphi]^{-1}(U)$. Simple verification shows that λ and $\bar{\lambda}$ are mutually inverse mappings. The equation $\lambda^{-1}(\langle U,\langle W,V\rangle\rangle=\langle U\times W,V\rangle$ proved above implies that $\langle V,\langle W,U\rangle\rangle=\lambda(\langle V\times W,U\rangle)=\bar{\lambda}^{-1}(\langle V\times W,U\rangle)$. Thus, it follows that the mapping $\bar{\lambda}:C(X,C(Y,Z))\to C(X\times Y,Z)$ is continuous and that these spaces are homomorphic. The theorem is proved.

We now extend the concept of an f-space. The topological space X is said to be an $f^*(f_0^*)$-space if it is the retract of an $f(f_0)$-space. More exactly, if there is an $f(f_0)$-space Y and continuous mappings $i:X\to Y$ and $z:Y\to X$ such that the composition $z\circ i:X\to X$ is the identity mapping.

LEMMA 4. If the topological spaces X and X' are retracts of the topological spaces Y and Y', respectively, and the sets $C(X,X')$ and $C(Y,Y')$ are equipped with point convergence topologies, then $C(X,X')$ is the retract of $C(Y,Y')$.

Proof. Let $z:Y\to X$, $i:X\to Y$ be the mapping which achieves the retraction of Y into X and $\rho:Y'\to X'$, $\iota:X'\to Y'$ the retraction of Y' into X'. Then we define the mappings $R:C(Y,Y')\to C(X,X')$ and $J:C(X,X')\to C(Y,Y')$, thus: $R(f)=\rho\circ f\circ i$ for $f\in C(Y,Y')$; $J(g)=\iota\circ g\circ z$ for $g\in C(X,X')$. It is easy to verify that $R\circ J$ maps $C(X,X')$ identically into itself. We can verify that R and J are continuous mappings. Let V' be an open set in X'. Then $V=\rho^{-1}(V')$ is an open set in Y'. For $f\in C(Y,Y')$ and $x'\in X$ the following conditions are equivalent:

$$R(f)\in\langle x',V'\rangle\Longleftrightarrow(Rf)(x')\in V'\Longleftrightarrow\rho f i(x')\in V'\Longleftrightarrow$$
$$\Longleftrightarrow fi(x')\in V\Longleftrightarrow f\in\langle i(x'),V\rangle.$$

Thus, it follows that R is continuous. Similarly, if V is an open set in Y' and $V'=\iota^{-1}(V)$ for any $y\in Y$ and $f'\in C(X,X')$ the following conditions are equivalent:

$$J(f')\in\langle y,V\rangle\Longleftrightarrow f'\in\langle z(y),V'\rangle.$$

Thus, J is continuous. The lemma is proved.

We formulate corollaries of Lemma 4, Lemma 3, and Theorem 2, and some obvious properties of $f^*(f_0^*)$-spaces in the form of a theorem.

THEOREM 3. If X, Y are f^*-spaces and Z is an f_0^*-space, then

1) $X\times Y$ with product topology, is an f^*-space;

2) $C(Y,Z)$, with point convergence topology, is an f_0^*-space;

3) $C(X\times Y,Z)$ is homeomorphic with $C(X,C(Y,Z))$.

We now indicate a sufficient condition for the retract of an f-space to be an f-space itself. The monotonic (not necessarily continuous) mapping $\rho:X\to X$ of the f-space X into itself is said to be the closure of X if $\rho\rho(x)=\rho(x)$ and $x\leq\rho(x)$ for any $x\in X$.

PROPOSITION 4.* If X is an $f(f_0)$-space, ρ the closure of X, then the set $\rho(X)$ with its induced topology, is an $f(f_0)$-space.

* The proof of this proposition and also the formulation of the concluding propositions of this section use concepts which are defined later (in Sec. 2) and so at first reading they can be omitted.

Proof. We prove the following relations, from which the proposition at once follows:

a) $\check{x} \cap \rho(X) = \rho(x) \cap \rho(X)$ for any $x \in X_0$;

b) for any $x, y \in X_0$ if x and y are compatible, then

$$\check{\rho}(x) \cap \check{\rho}(y) \cap \rho(X) = \check{\rho}(x \cup y) \cap \rho(X).$$

Let us prove a). Since $x \leq \rho(x)$, we have $\check{x} = \{y \mid x \leq y\} \supseteq \check{\rho}(x) = \{y \mid \rho(x) \leq y\}$. Consequently, $\check{x} \cap \rho(X) \supseteq \check{\rho}(x) \cap \rho(X)$. Let $\rho(y) \in \check{x}$, i.e., $\rho(y) \geq x$; then $\rho\rho(y) \geq \rho(x)$, $\rho(y) \geq \rho(x)$, $\rho(y) \in \check{\rho}(x)$; consequently, $\check{x} \cap \rho(X) \subseteq \check{\rho}(x) \cap \rho(X)$. We have proved a). Relation b) is derived from the following: $\check{x} \cap \check{y} = x \check{\cup} y$, $\check{x} \cap \check{y} \cap \rho(X) = \check{\rho}(x) \cap \check{\rho}(y) \cap \rho(X) = (x \check{\cup} y) \cap \rho(X) = \rho(\check{x \cup y}) \cap \rho(X)$. The proposition is proved.

COROLLARY. Under the conditions of the proposition $\rho(X_0)$ is a basis subspace of $\rho(X)$.

If the closure ρ is also continuous, we say that $\rho(X)$ is the closed retract of X .

With obvious provisos, we formulate, without proof, the simple

PROPOSITION 5. If X is the closed retract of Y , X' the closed retract of Y' , then $C(X,X')$ is the closed retract of $C(Y, Y')$.

2. Further Properties of an f-Space

Let X be an f-space; let X_0 denote the subset of X consisting of all elements of the form σ_V , where V is an f-set. We call the elements themselves f-elements. We shall consider X_0 as a topological space with topology induced by the topology of X . We shall prove that

1. The space X_0 is an f-space.

Indeed, let V_0 denote the intersection $V \cap X_0$ for the f-set V of X . We can show that V_0 is an f-set in X_0 . To do this it is sufficient to verify that the order on X_0 , defined by the topology of X_0 , coincides with the restriction of the order, defined by the topology of X . Let us denote the first order by $\leq'$, and the second simply by $\leq$. Let $x \leq' y$, and let V be an f-set in X and $x \in V$; then $x \in V_0 = V \cap X_0$; since V_0 is open, $x \in V_0$ and $x \leq' y$, we have $y \in V_0$ and $y \in V (\supseteq V_0)$. Consequently, $x \leq' y \Rightarrow x \leq y$. Let $x \leq y$ and assume that $x \not\leq' y$. Then there is a V_0 such that $x \in V_0$, but $y \notin V_0$; but then $x \in V$ and $y \notin V_0$ (where $V_0 = V \cap X_0$, V is an f-set). Since $(V' \cap V'')_0 = V_0' \cap V_0''$, f-sets in X_0 of the form V_0 are closed with respect to intersection. Since any open set in X is the union of f-sets, any open set in X_0 is the union of f-sets of the form V , where V is an f-set in X . Thus, it follows that X_0 is an f-space and that all the f-sets of X_0 have the form V_0 , where $V_0 = V \cap X_0$, and V is an f-set of X .

2. X_0 is dense in X .

If W is an open set in X , there can be found an f-set and $V \subseteq W$ and $\sigma_V \in V \subseteq W$, $\sigma_V \in X_0 \cap V$.

We say that the space X_0 is a basis subspace of X . Two $f^{\bullet}$-spaces X and Y are said to be basically equivalent if their basis subspaces are homomorphic. We denote basic equivalence as follows: $X \sim Y$.

THEOREM 1. Let X^0, Y^0 be f-spaces, X', Y' , f_0-spaces; then

$$X^0 \sim Y^0, \quad X' \sim Y' \Rightarrow C(X^0, X') \sim C(Y^0, Y').$$

Proof. First we prove a lemma which we shall also need later. It indicates the condition for the inclusion of f-sets of the space $C(X, Y)$.

LEMMA. Let X be an f-space, Y an f_0-space. If the V_i are f-sets of X, the W_i f-sets of Y, $i \leq k$, then

1) if $U = \bigcap_{i<k} \langle V_i, W_i \rangle \neq \emptyset$, then

$$U \subseteq \langle V_k, W_k \rangle \Longleftrightarrow \exists i < k \ (\langle V_i, W_i \rangle \subseteq \langle V_k, W_k \rangle).$$

2) $\langle V_0, W_0 \rangle \subseteq \langle V_1, W_1 \rangle$ if and only if $W_1 = Y$ or $V_0 \supseteq V_1$ and $W_0 \subseteq W_1$.

Proof (of the lemma). 1) Let $U \neq \emptyset$ and $U \subseteq \langle V_k, W_k \rangle$; then $\sigma_U \ (= \varphi_U) \geq \sigma_{\langle V_k, W_k \rangle} \ (= \varphi_{\langle V_k, W_k \rangle})$. Consider $\varphi_U^{-1}(W_k)$, $\varphi_U^{-1}(W_k) = \bigcup \{V_i \mid W_i \subseteq W_k, \ i < k\}$. Since $\varphi_U \geq \varphi_{\langle V_k, W_k \rangle}$, i.e., $\varphi_U \in \langle V_k, W_k \rangle$, then $V_k \subseteq \bigcup_{\substack{i<k \\ W_i \subseteq W_k}}$. By Corollary 2 of Sec. 1, there is an $i<k$ such that $V_k \subseteq V_i$ and $W_i \subseteq W_k$. Thus, $\langle V_i, W_i \rangle \subseteq \langle V_k, W_k \rangle$.

2) If $W_1 = Y$, then $\langle V_1, W_1 \rangle = C(X, Y)$, and the inclusion $\langle V_0, W_0 \rangle \subseteq \langle V_1, W_1 \rangle = C(X, Y)$ is obvious. Let $W_1 \neq Y$, $\varphi_{\langle V_0, W_0 \rangle} \ (= \sigma_{\langle V_0, W_0 \rangle}) \in \langle V_1, W_1 \rangle$. If $x \in V_1$, then $\varphi_{\langle V_0, W_0 \rangle}(x) \in W_1$, and since $W_1 \neq Y$, we have $\varphi_{\langle V_0, W_0 \rangle}(x) \neq \sigma_Y$ from which $x \in V_0$ and $\varphi_{\langle V_0, W_0 \rangle}(x) = \sigma_{W_0} \in W_1$, consequently, $V_1 \subseteq V_0$ and $W_0 \subseteq W_1$. The converse assertion ($V_0 \supseteq V_1$, $W_0 \subseteq W_1 \Rightarrow \langle V_0, W_0 \rangle \subseteq \langle V_1, W_1 \rangle$) is obvious from the definition of the sets $\langle V, W \rangle$. The lemma is proved.

We return now to the proof of the theorem. The lemma just proved asserts that the inclusion relation (and, consequently, equality) between f-sets of $C(X, Y)$ can be expressed in terms of the inclusion relation of f-sets of the spaces X and Y, and so there is a formal correspondence Φ between the f-of the spaces $C(X^0, X^1)$ and $C(Y^0, Y^1)$ (which is defined as follows: if $\varphi_0 : X_0^0 \to Y_0^0$ and $\varphi_1 : X_0' \to Y_0'$ are homomorphisms of the basis subspaces of X^0, Y^0 and X', Y' respectively, then Φ associates the f-set $\bigcap_{i<k} \langle V_i, W_i \rangle$ of the space $C(X^0, X'))$ (the V_i are f-sets of X^0, the W_i are f-sets of X') with the f-set $\bigcap_{i<k} \langle V_i', W_i' \rangle$, where

$$V_i' \rightleftharpoons \{y \mid \varphi_0(\sigma_{V_i}) \leq y\}, \quad W_i' \rightleftharpoons \{y \mid \varphi_1(\sigma_{W_i}) \leq y\}, \quad i < k\ )$$

is a one-to-one mapping preserving the inclusion relation between f-sets. This mapping also specifies the desired homomorphism of the basis subspace of $C(X^0, X')$ onto the basis subspace of $C(Y^0, Y')$. The theorem is proved.

The theory of f-spaces can also be described in a purely algebraic language. Let X be an f-space, X_0 its basis subspace. Consider the following triplet $(X, X_0, \leq)$, where $\leq$ is a partial order on X defined by the topology of X. We verify the following properties of this triplet:

1) $X_0 \subseteq X$;

2) the relation $\leq$ is a partial order on X;

3) if $x_0, x_0' \in X_0$ and there is an $x \in X$ such that $x_0 \leq x$ and $x_0' \leq x$, then there is an element X_0 in $\bar{x}_0$ which is the least upper bound of the elements x_0 and x_0' in X;

4) for any element $x \in X$ there is an element $x_0 \in X_0$ such that $x_0 \leq x$;

5) for any elements $x, y \in X$, if $x \nleq y$, there is an element $x_0 \in X_0$, such that $x_0 \leq x$ and $x_0 \nleq y$.

Properties 1) and 2) are trivial. Let us verify 3). The condition for the existence of $x \in X$ such that $x_0 \leq x$ and $x_0' \leq x$ implies simply that the f-sets $\check{x}_0 \rightleftharpoons \{y \mid x_0 \leq y\}$ and $\check{x}_0' \rightleftharpoons \{y \mid x_0' \leq y\}$ have nonempty intersection ($x \in \check{x}_0 \cap \check{x}_0'$); but then $\check{x}_0 \cap \check{x}_0'$ itself is an f-set; i.e., it has the form $\check{\bar{x}}_0$, where $\bar{x}_0 \in X_0$. Obviously, $\bar{x}_0$ is just the required element. Property 4) follows from the fact that the f-sets form a basis for the topology of X, i.e., $X = \bigcup_{x_0 \in X_0} \check{x}_0$. Now let us verify 5). Let $x \not\leq y$; then there is an open set V such that $x \in V$ and $y \notin V$. Since we can take V to be a basis open set, $x_0 \leq x$ and $x_0 \not\leq y$ for x_0 such that $V = \check{x}_0$.

Note. Property 3) shows that the restriction of $\leq$ on X_0 makes X_0 a poset [1]; if 1)-3) hold, we shall say that X_0 is a subposet of the partially ordered set X.

THEOREM 2. The triplet $(X, X_0, \leq)$ is defined by a topological f-space if and only if conditions 1)-5) hold.

Proof. The necessity for conditions 1)-5) has already been verified. We can show the sufficiency of these conditions. For any element $x_0 \in X_0$ we define the set $\check{x}_0$ as follows: $\{y \mid x_0 \leq y\}$. Condition 4) shows that $X = \bigcup_{x_0 \in X_0} \check{x}_0$. Let us verify that the family of sets of the form $\check{x}_0, x_0 \in X$, with the empty set, is closed with respect to finite intersections. Let $x_0, y_0 \in X_0$ and $\check{x}_0 \cap \check{y}_0 \neq \emptyset$; then for $x \in \check{x}_0 \cap \check{y}_0$ we have $x_0 \leq x$ and $y_0 \leq x$. By 3) there is an element $z_0 \in X_0$ such that z_0 is the exact upper bound of the elements x_0 and y_0 (in X); but then $\check{x}_0 \cap \check{y}_0 = \check{z}_0$. Consequently, the family of sets $\{\emptyset, \check{x}_0 \mid x_0 \in X_0\}$ can be taken as the basis of a topology in X_0. We show that this topology is separable. Let $x \neq y$; then either $x \not\leq y$ or $y \not\leq x$. Suppose $x \not\leq y$; by property 5) there is an element $x_0 \in X_0$, such that $x_0 \leq x$ and $x_0 \not\leq y$. Then $x \in \check{x}_0$ and $y \notin \check{x}_0$. It remains to verify that all sets of the form $\check{x}_0$, $x_0 \in X_0$ are f-sets in this topology. To do this it is sufficient only to note that the order defined on X by the topology introduced above coincides with $\leq$, and this directly follows from the definition of the topology and the above discussion. The theorem is proved.

We now describe in algebraic terms the continuous mappings of f-spaces. Let X and Y be f-spaces with basis subspaces X_0 and Y_0, respectively. We denote the orders defined by the topologies of X and Y by $\leq$ and $\preccurlyeq$.

THEOREM 3. For an arbitrary mapping $g : X \to Y$ the following conditions are equivalent:

1) g is a continuous mapping;

2) g is a monotonic mapping (with respect to $\leq$ and $\preccurlyeq$) and for any $y_0 \in Y_0$ and any $x \in g^{-1}(\check{y}_0)$ there is an $x_0 \in X_0$ such that $x_0 \leq x$ and $x_0 \in g^{-1}(\check{y}_0)$.

The theorem follows almost directly from the properties of the topology of f-spaces.

COROLLARY. For any basis subspace X_0 and f-space Y the mapping $g : X_0 \to Y$ is continuous if and only if g is monotonic.

Using the algebraic description we have obtained for f-spaces, we can describe the set of all basically equivalent f-spaces with fixed basis subspace X_0.

Let $(X_0, \leq)$ be a poset; we may say that every nonempty subset $J \subseteq X_0$, satisfying the following conditions is an ideal.

1) $x \leq y$, $y \in J \Longrightarrow x \in J$;

2) $x_0, x_1 \in J \Longrightarrow$ there is an upper bound for x_0 and x_1 and $x_0 \cup x_1 \in J$;

In the second condition $x_0 \cup x_1$ denotes the exact upper bound of the elements x_0 and x_1. In what follows the elements x_0 and x_1 with an exact upper bound will be called compatible (incompatible otherwise). The set of all ideals in $(X_0, \leq)$ is denoted by $J(X_0, \leq)$. An example of an ideal is every principal ideal, an ideal of the form $J_x = \{y \mid y \leq x\}$ for $x \in X_0$. The set of all principal ideals is denoted by $J_0(X_0, \leq)$. On the set $J(X_0, \leq)$ an inclusion relation $\subset$ between ideals defines a partial order. It is easy to verify that the correspondence $j : x \rightsquigarrow J_x$ defines an isomorphism of $(X_0, \leq)$ and $(J_0(X_0, \leq), \subseteq)$.

THEOREM 4. Every triplet $(X, X_0, \leq)$ satisfying the conditions 1)-5) is isomorphic with the triplet $(J^X, J_0(X_0, \leq), \subseteq)$, where $J_0(X_0, \leq) \subseteq J^X \subseteq J(X_0, \leq)$; there is a unique isomorphism of $(X, X_0, \leq)$

into $(\mathcal{J}(X_0,\preccurlyeq), \mathcal{J}_0(X_0,\preccurlyeq), \subseteq)$ continuing the isomorphism j. Conversely, for every $\mathcal{J}'$ such that $\mathcal{J}_0(X_0,\preccurlyeq) \subseteq \mathcal{J}' \subseteq \mathcal{J}(X_0,\preccurlyeq)$ and $(X_0,\preccurlyeq)$ is a poset, the triplet $(\mathcal{J}', \mathcal{J}_0(X_0,\preccurlyeq), \subseteq)$ satisfies the conditions 1)-5).

Proof. Let $(X, X_0, \preccurlyeq)$ satisfy conditions 1)-5), let $x \in X$, and let $\mathcal{J}_x$ denote the set $\{y \mid y \in X_0$ and $y \preccurlyeq x\}$. We can verify that $\mathcal{J}_x$ is an ideal in $(X_0, \preccurlyeq)$. Indeed, that condition 1) of the definition of the ideal holds follows directly from the definition of the set $\mathcal{J}_x$. Further, if $x_0, x_1 \in \mathcal{J}_x$, then $x_0 \preccurlyeq x$ and $x_1 \preccurlyeq x$, and it follows from 3) that x_0 and x_1 are compatible (in X) and that $x_0 \cup x_1$ is the exact upper bound of x_0 and x_1 in X, and consequently that $x_0 \cup x_1 \preccurlyeq x$, i.e., $x_0 \cup x_1 \in \mathcal{J}_x$.

Let us now consider the mapping $j_X : x \rightsquigarrow \mathcal{J}_x$ of the set X into the set $\mathcal{J}(X_0,\preccurlyeq)$. We note that j_X continues the mapping (isomorphism) $j : X_0 \to \mathcal{J}_0(X_0,\preccurlyeq)$. We can verify that j_X is an isomorphism of the partially ordered set $(X,\preccurlyeq)$ into the partially ordered set $(\mathcal{J}(X_0,\preccurlyeq),\subseteq)$. Let $x, y \in X$ and $x \not\preccurlyeq y$; then, by 5), there is an $x_0 \in X_0$ such that $x_0 \preccurlyeq x$ and $x_0 \not\preccurlyeq y$. Then $x_0 \in \mathcal{J}_x$, $x_0 \notin \mathcal{J}_y$, and $\mathcal{J}_x \nsubseteq \mathcal{J}_y$. Hence, $x \not\preccurlyeq y \Longrightarrow \mathcal{J}_x \nsubseteq \mathcal{J}_y$. Let $x \preccurlyeq y$, $x_0 \in \mathcal{J}_x$; then $x_0 \preccurlyeq x$ and $x_0 \preccurlyeq y$, i.e., $x_0 \in \mathcal{J}_y$, and so $\mathcal{J}_x \subseteq \mathcal{J}_y$. Thus, $x \preccurlyeq y \Longleftrightarrow \mathcal{J}_x \subseteq \mathcal{J}_y$. Thus, it also follows that j_X is an isomorphism into j_X (taking different values) follows from the fact that $\preccurlyeq$ is a partial order, i.e., if $x \neq y$, then $x \not\preccurlyeq y$ or $y \not\preccurlyeq x$. Putting $\mathcal{J}^X \rightleftharpoons \{\mathcal{J}_x \mid x \in X\}$, we obtain the first assertion of the theorem. We now prove that j_X is unique (among isomorphisms containing j). Let j' be an arbitrary isomorphism of $(X,\preccurlyeq)$ into $(\mathcal{J}(X_0,\preccurlyeq), \subseteq)$ continuing j. Since for $x_0 \in X_0$, $x \in X$ from $x_0 \preccurlyeq x$ it follows that $j'(x_0) \subseteq j'(x)$, $j'(x_0) = \mathcal{J}_{x_0}$, $\mathcal{J}_{x_0} \subseteq j'(x)$, we have $\bigcup_{x_0 \in \mathcal{J}_x} \mathcal{J}_{x_0} \subseteq j'(x)$. We note now that $\bigcup_{x_0 \in \mathcal{J}_x} \mathcal{J}_{x_0} = \mathcal{J}_x$. This easily follows from the definition of $\mathcal{J}_x$. Thus, $\mathcal{J}_x \subseteq j'(x)$ for some $x \in X$. Assume that $j'(x) \neq j_X(x) = \mathcal{J}_x$ for some $x \in X$. Let $x_0 \in j'(x) \setminus \mathcal{J}_x$. Since $x_0 \notin \mathcal{J}_x$, we have $x_0 \not\preccurlyeq x$; but $j'(x_0) = \mathcal{J}_{x_0} \subseteq j'(x)$ and so j' is not an isomorphism. This contraction proves that j_X is unique.

We now prove the last assertion of the theorem. Let $\mathcal{J}'$ be such that $\mathcal{J}_0(X_0,\preccurlyeq) \subseteq \mathcal{J}' \subseteq \mathcal{J}(X_0,\preccurlyeq)$. We can show that the triplet $(\mathcal{J}', \mathcal{J}_0(X_0,\preccurlyeq), \subseteq)$ satisfies conditions 1)-5). Conditions 1) and 2) are trivial. Let us verify 3). Let $\mathcal{J}_{x_0}, \mathcal{J}_{x_1} \subseteq \mathcal{J}$, $x_0, x_1 \in X_0$, $\mathcal{J} \in \mathcal{J}' \subseteq \mathcal{J}(X_0,\preccurlyeq)$. The inclusions $\mathcal{J}_{x_0}, \mathcal{J}_{x_1} \subseteq \mathcal{J}$ show that $x_0, x_1 \in \mathcal{J}$; then, by condition 2) of the definition of the ideal x_0 and x_1 are compatible and $x_0 \cup x_1 \in \mathcal{J}$. Then $\mathcal{J}_{x_0 \cup x_1} \subseteq \mathcal{J}$. Thus, it follows that $\mathcal{J}_{x_0 \cup x_1}$ is the exact upper bound of the elements $\mathcal{J}_{x_0}$ and $\mathcal{J}_{x_1}$ (in $\mathcal{J}'$) and since $x_0 \cup x_1 \in X_0$, we have $\mathcal{J}_{x_0 \cup x_1} \in \mathcal{J}_0(X_0,\preccurlyeq)$. We have verified condition 3). Let us verify 5). Let $\mathcal{J}_0, \mathcal{J}_1 \in \mathcal{J}'$, $\mathcal{J}_0 \nsubseteq \mathcal{J}_1$; then there is an element $x_0 \in X_0$ such that $x_0 \in \mathcal{J}_0$ and $x_0 \notin \mathcal{J}_1$. Thus, $\mathcal{J}_{x_0} \subseteq \mathcal{J}_0$ and $\mathcal{J}_{x_0} \nsubseteq \mathcal{J}_1$. Condition 5) holds. Let us verify 4). Let $\mathcal{J} \in \mathcal{J}'$. Since $\mathcal{J}$ is an ideal in $(X_0,\preccurlyeq)$, then $\mathcal{J} \neq \emptyset$. And $x_0 \in \mathcal{J}$, for $\mathcal{J}_{x_0} \subseteq \mathcal{J}$. The theorem is proved.

If we use the notation of Theorem 4, from the discussion in the proof of that theorem we can formulate the

COROLLARY 1. Let X and X' be f-spaces with the same basis subspace X_0; then there is a homomorphism from X into X' which is an identity on X_0 if and only if $\mathcal{J}^X \subseteq \mathcal{J}^{X'}$ and this homomorphism (if it exists) is unique.

COROLLARY 2. For any element $x \in X$ we have

$$x = \sup\{x_0 \mid x_0 \in X_0,\ x_0 \preccurlyeq x\}.$$

The above considerations show that all the concepts relating to the theory of f-spaces can also be formulated quite simply in algebraic language. The possibility of this dual description is very convenient, since in particular cases experience in working with continuity and continuous functions is very useful, while in other cases, conversely, the algebraic description makes it possible to solve the problem quickly. The theorem just proved and its corollaries make it possible to give a complete description of all the basically equivalent f-spaces with fixed basis space X_0. From the point of view of mutual (homomorphic) imbedding, these spaces form a complete Boolean algebra, the least element being the space X_0 itself, while the greatest is the space $\mathcal{J}(X_0, \leqslant)$ of all ideals of the poset $(X_0, \leqslant)$. In all the f-spaces we have discussed a significant role is played by the basis subspace. Since the fundamental aim of using f-spaces in the following sections of the paper is to define functionals and computable functionals, it is worth indicating that f-spaces (more precisely, elements X_0 defining f-spaces) play the role of finite sets of one of the principal attributes of every extension (generalization) of the general theory of recursive functions. (This has repeatedly been emphasized by Kreisel [7].)

We note a further important property of the basis subspace X_0.

PROPOSITION 1. Let the triplet $(X, X_0, \leqslant)$ satisfy the conditions 1)-5); then the order $\leqslant$ in X is uniquely defined by the set of pairs $\{<x_0, x> \mid x_0 \in X_0,\ x_0 \leqslant x\}$.

Proof. If $x_1, x_2 \in X$, then $x_1 \leqslant x_2 \Longleftrightarrow \mathcal{J}_{x_1} \ (=\{x_0 \mid x_0 \in X_0,\ x_0 \leqslant x_1\}) \subseteq \mathcal{J}_{x_2} (=\{x_0 \mid x_0 \in X_0,\ x_0 \leqslant x_2\})$. The proposition is proved.

The importance of this proposition is that the order in X, and so also the topology on X, is completely defined by the order of X_0 and the possibility of comparing elements of X_0 and X. In view of the analogy, noted above, between elements of X_0 and finite sets, the specification of the topology (order) on X is defined by the reciprocal relation between "finite sets" and "variables." In all important cases this relation is recursively enumerable; i.e., it is potentially solvable (when it is true). This is an additional justification for introducing the definition of the concept of computable functionals of finite types, given in Sec. 8.

To conclude this section we indicate a canonical method of mapping f_0-spaces of f-spaces.

PROPOSITION 2. If $(X, X_0, \leqslant)$ is a triplet corresponding to the topology of an f-space on X, we can continue the order $\leqslant$ onto the set $X \cup \{\mathcal{O}\}$ $(\mathcal{O} \notin X)$. Thus, $\mathcal{O} \leqslant x$ for all $x \in X$, we obtain the triplet $(X \cup \{\mathcal{O}\},\ X_0 \cup \{\mathcal{O}\}, \leqslant)$ satisfying the conditions 1)-5). The corresponding topology on $X \cup \{\mathcal{O}\}$ specifies the structure of an f_0-space on that set and the imbedding of X in $X \cup \{\mathcal{O}\}$ is a homomorphism of X into $X \cup \{\mathcal{O}\}$.

The proof consists in a direct routine verification of all the assertions and so we omit it. If $F_0(X)$ denotes the space $X \cup \{\mathcal{O}\}$ defined in Proposition 2, it is easy to continue F_0 to a functor from the category of f-spaces into the category of f_0-spaces, putting $F_0(\varphi)(x) = \varphi(x)$, if $x \in X$ and $F_0(\varphi)(\mathcal{O}) = \mathcal{O}$ for $\varphi \in C(X, Y)$.

We note that the elements $C(X, F_0(Y))$ can be considered as partially continuous mappings from X into Y which have an open domain of definition [if a set of this domain is denoted by $C_p(X, Y)$, then $C(X, F_0(Y))$ and $C_p(X, Y)$ are naturally equivalent as bifunctors].

3. Complete f-spaces

In Theorem 4 of Sec. 2 we described all f-spaces X with given basis subspace X_0. Among these spaces there is a "greatest." Such spaces are the subject of this section.

THEOREM 1. Let X be an f-space, X_0 its basis subspace, $\leqslant$ an order in X defined by the topology. Then the following conditions are equivalent:

1) The triplet $(X, X_0, \leqslant)$ is naturally isomorphic with the triplet

$$(\mathcal{J}(X_0, \leqslant),\ \mathcal{J}_0(X_0, \leqslant), \subseteq).$$

2) Any nonempty directed subset $S \subseteq X_0$, not having a greatest element has the exact upper bound s in X and $s \in X \setminus X_0$; and if S_0, S_1 are directed subsets of X_0 and $\sup S_0 = \sup S_1$, then S_0 and S_1 are cofinal subsets of $S_0 \cup S_1$.

3) For any f -space X' with basis subspace X_0 the identity mapping of X_0 into X can be continuous (homomorphic) imbedding of X' in X .

Proof. 1) $\Longrightarrow$ 2). The natural isomorphism which is mentioned in 1) is the mapping $j_X : X \to \mathcal{J}(X_0, \leq)$, defined as follows: $j_X(x) = \{x_0 \mid x_0 \in X_0, x_0 \leq x\}$ for $x \in X$. Condition 1) implies that j_X maps X onto $\mathcal{J}(X_0, \leq)$. Let S be any nonempty directed subset of X_0 ; then the set $\mathcal{J}(S) \rightleftharpoons \{x_0 \mid$ and there is an $x_0' \in S$ such that $x_0 \leq x_0'$, $x_0 \in X_0\}$ is an ideal in $(X_0, \leq)$. But if S does not have a greatest element, then $\mathcal{J}(S)$ is not a principal ideal, i.e., $\mathcal{J}(S) \in \mathcal{J}(X_0, \leq) \setminus \mathcal{J}_0(X_0, \leq)$. Further, obviously, $\mathcal{J}(S)$ is the exact upper bound for the set $j_X(S)$; consequently, if $s \in X$ is an element such that $j_X(s) = \mathcal{J}(S)$, s is the exact upper bound for S and $s \in X \setminus X_0$. The equation $sup\, S_0 = sup\, S_1$ implies that $\mathcal{J}(S_0) = \mathcal{J}(S_1)$ but $S_0 \cup S_1 \subseteq \mathcal{J}(S_0) = \mathcal{J}(S_1)$ and S_i is cofinal with $\mathcal{J}(S_i)$, $i = 0, 1$.

2) $\Longrightarrow$ 3). We note first some properties of f -spaces satisfying the condition 2), more precisely, triplets $(X, X_0, \leq)$.

a) Any directed set $S \subseteq X$ has an exact upper bound $sup\, S$ and if $S_0 \rightleftharpoons \{x_0 \mid$ and there is an $x \in S$, $x_0 \leq x, x_0 \in X_0\}$, then $sup\, S = sup\, S_0$.

Indeed, S_0 is a directed subset of X_0 , and so $sup\, S_0$ exists. We can show that $x \leq sup\, S_0$ for $x \in S$. Assume that this is not so; then there is an $x_0 \in X_0$ such that $x_0 \leq x$ and $x_0 \not\leq sup\, S_0$, but $x_0 \in S_0$, which is impossible. Thus, $sup\, S_0$ is an upper bound for S ; that it is the least such follows from the fact that any upper bound for S is also an upper bound for S_0 .

b) If S is a directed set in X , $x_0 \in X_0$, then

$$x_0 \leq sup\, S \Longleftrightarrow \exists x \in S\, (x_0 \leq x).$$

Indeed, $\Longleftarrow$ is obvious. Let S_0 , as above, be the set $\{y_0 \mid \exists x \in S,\ y_0 \leq x,\ y_0 \in X_0\}$; then $sup S_0 = sup S$. Now put $S_1 \rightleftharpoons \{y_0 \mid y_0 \in X_0,\ y \leq sup\, S\}$; then, obviously, $sup\, S = sup\, S_0 = sup\, S_1$ further, S_0 and S_1 are cofinal subsets of $S_0 \cup S_1\ (= S_1)$ and so for $x_0 \in S_1$ in S_0 there can be found an element x_1 such that $x_0 \leq x_1$, but since $x_1 \in S_0$, there can be found an element $x_2 \in S$ such that $x_1 \leq x_2$. Then $x_0 \leq x_2$, $x_2 \in S$. The assertion is proved.

We now define the mapping μ from X' into X as follows:

$$\mu(x') \rightleftharpoons sup\{x_0 \mid x_0 \in X_0,\ x_0 \leq x'\} \quad \text{for } x' \in X'.$$

Clearly, μ extends the identity mapping of X_0 into X . Let us verify that μ is continuous. Let $V = \check{x}_0$ be a basis neighborhood in X , $x' \in \mu^{-1}(V)$; then $x_0 \leq \mu(x') = sup\{x_0' \mid x_0' \in X_0,\ x_0' \leq x'\}$ and, by property b), there is an $x_0' \in X_0$ such that $x_0 \leq x_0' \leq x'$. Then $x' \in \check{x}_0' = \{x'' \mid x'' \in X',\ x_0' \leq x''\} \subseteq \mu^{-1}(V)$ but $\check{x}_0'$ is open in X' and so $\mu^{-1}(V)$ is open; thus, μ is continuous. By Corollary 2 to Theorem 4 of Sec. 2, we have $x' = sup\{x_0 \mid x_0 \in X_0,\ x_0 \leq x'\}$ in X' (where the sup is taken over X'); then $\mu(x') = \mu(x'')$ implies that $x' = x''$ so that μ is not the identity mapping.

3) $\Longrightarrow$ 1). This directly follows from Corollary 1 to Theorem 4 of Sec. 2. The theorem is proved.

The f -space X is said to be complete if it satisfies the conditions of Theorem 1.

COROLLARY 1. If X is a complete space, for any nonempty directed subset $S \subseteq X$ there is an exact upper bound $sup S$.

COROLLARY 2. If X is a complete f_0 -space, for any nonempty set $Y \subseteq X$ there is an exact lower bound $(inf\ Y)$.

Proof. Let $Z \rightleftharpoons \{x \mid x \in X \text{ and } x \preccurlyeq y\}$ for any $y \in Y$. We note that the least element of X belongs to Z. Hence Z is not empty. In addition, Z is a directed set. By Corollary 1, $sup\, Z$ exists, and it is obviously the exact lower bound for X $(sup\, Z = inf\, X)$.

Example. Every space of discrete topology is a complete f-space.

PROPOSITION 1. If Y is an f-space with basis subspace Y_0, X a complete f-space, the mapping $g: Y \to X$ is continuous if and only if g is monotonic and $g(y) = sup\{g(y_0) \mid y_0 \in Y_0, y_0 \preccurlyeq y\}$ for $y \in Y$.

Proof. Let g be continuous; then g is monotonic and so $g(y) \geqslant sup\{g(y_0) \mid y_0 \in Y_0, y_0 \preccurlyeq y\}$ for $y \in Y$. Assume that $g(y) \not\preccurlyeq sup\{g(y_0) \mid y_0 \in Y, y_0 \preccurlyeq y\}$; then there is an $x_0 \in X_0$ such that $x_0 \preccurlyeq g(y)$ and $x_0 \not\preccurlyeq g(y_0)$ for all $y \in Y_0$, $y_0 \preccurlyeq y$. Therefore, $y \in g^{-1}(\check{x}_0)$, while $y_0 \notin g^{-1}(\check{x}_0)$ for all $y_0 \in Y_0$, $y_0 \preccurlyeq y$. But, since g is continuous, $g^{-1}(\check{x}_0)$ is open; then there is a $y_0 \in Y_0$ such that $\check{y}_0 \subseteq g^{-1}(\check{x}_0)$ and $y \in \check{y}_0$; but this implies that $y_0 \preccurlyeq y$ and $y_0 \in g^{-1}(\check{x}_0)$. This is a contradiction. Conversely, suppose g is monotonic and satisfies the condition $g(y) = sup\{g(y_0) \mid y_0 \in Y_0, y_0 \preccurlyeq y\}$. We can show that g is continuous. Let $x_0 \in X_0$, $y \in g^{-1}(\check{x}_0)$, i.e., $g(y) \geqslant x_0$, $x_0 \preccurlyeq sup\{g(y_0) \mid y_0 \in Y_0, y_0 \preccurlyeq y\}$. By property b), proved in Theorem 1, $x_0 \preccurlyeq g(y_0)$ for some $y_0 \in Y_0$, $y_0 \preccurlyeq y$. Then $y_0 \in g^{-1}(\check{x}_0)$, $\check{y}_0 \subseteq g^{-1}(\check{x}_0)$, and $y \in \check{y}_0$. Consequently, g is continuous. The proposition is proved.

COROLLARY 1. If Y is an f-space with basis subspace Y_0, X a complete f-space, then any monotonic mapping $g_0: Y_0 \to X$ can be continued (uniquely) to a continuous mapping g from Y into X.

Putting $g(y) \rightleftharpoons sup\{g_0(y_0) \mid y_0 \in Y_0, y_0 \preccurlyeq y\}$, we obtain the required mapping.

Note. Corollary 1 shows that the space $C(Y, X)$ depends only on the basis subspace for a complete f-space X, i.e., $Y \sim X' \Rightarrow C(Y, X) \approx C(Y', X)$.

COROLLARY 2. If X and Y are complete f-spaces, the mapping $g: X \to Y$ is continuous if and only if the following holds for any nonempty directed subset $S \subseteq X$:

$$g(sup\, S) = sup\, g(S).$$

PROPOSITION 2. If X is an f-space and Y a complete f_0-space, $C(X, Y)$ is a complete f_0-space.

Proof. In view of the note following Corollary 1 to Proposition 1, it is sufficient to consider the case when X coincides with the basis subspace X_0. Then $C(X_0, Y)$ consists of all monotonic mappings from X_0 into Y. The order defined by the topology on $C(X_0, Y)$ coincides with the order $f \preccurlyeq g \rightleftharpoons \forall x \in X_0\, (f(x) \preccurlyeq g(x))$, as was noted in the proof of Theorem 1 of Sec. 1. The f-elements of the space $C(X_0, Y)$ are functions defined by the finite sets $H = \{\langle x_0, y_0 \rangle, \ldots, \langle x_k, y_k \rangle\}$, such that $x_i \in X_0$, $y_i \in X_0$, $i = 0, 1, \ldots, k$; if $x_i \neq x_j$ for $i \neq j$ $x_i \preccurlyeq x_j$, then $y_i \preccurlyeq y_j$; if x_i and x_j are compatible in X then there is an $l \preccurlyeq k$ such that $x_l = x_i \cup x_j$. We note that for any element $x \in X$ if there is an x_i, $i \preccurlyeq k$, such that $x_i \preccurlyeq x$, there is also a largest element of the form x_i with this property, and we denote it by $x_{i(x)}$. The function g_H is defined by the set H as follows:

$$g_H(x) = \begin{cases} \sigma_Y, & \text{if there is no } x_i \preccurlyeq x, \\ y_{i(x)} - & \text{otherwise.} \end{cases}$$

We shall verify that $C(X_0, Y)$ satisfies condition 2) of Theorem 1. First, we note that any directed family $F \subseteq C(X_0, Y)$ has an exact upper bound in $C(X_0, Y)$. Indeed, putting $g_F(x) \rightleftharpoons sup\{g(x) \mid g \in F\}$, we obtain a monotonic function from X_0 to Y which is, obviously, the exact upper bound for F. Let S_0 and S_1 be directed families of f-elements in $C(X_0, Y)$ and $sup\, S_0 = sup\, S_1$. We can show that for any element g_H of S_1 there can be found an element $g_{H'}$ of S_0 such that $g_H \preccurlyeq g_{H'}$. Let H be as above;

since $g_H \leq sup\, S_0$ $(= sup\, S_1)$, then $g_H(x_i) = y_i \leq sup\{g_{H'}(x_i) | g_{H'} \in S_0\}$ for x_i. By property b) there is an H'_i such that $y_i \leq g_{H'_i}(x_i)$ and $g_{H'_i} \in S_0$. Since S_0 is a directed family, there can be found a $g_{H'} \in S_0$ such that $g_{H'} \geq g_{H'_i}$ for all $i \leq k$, but then, obviously, $g_{H'} \geq g_H$.

Similarly it can be proved that S_1 is cofinal in $S_0 \cup S_1$. It remains to prove that if the directed set S_0 of f-elements does not have a greatest element, then $sup\, S_0$ is not an f-element. Assume the contrary: $sup\, S_0 \notin S_0$ since S_0 does not have a greatest element. Let $S_1 \rightleftharpoons \{g \,|\, g$ be an f-element, $g \leq sup\, S_0\}$; then $sup\, S_0 = sup\, S_1$, $sup\, S_0 \in S_1$. By the property just proved, we can find in S_0 an element g such that $sup\, S_0 \leq g$, but $g \leq sup\, S_0$. This is a contradiction. This proposition is proved.

It appears that if the triplet $(X, X_0, \leq)$ corresponds to a complete f-space, the set X_0 is defined in terms of the pair $(X, \leq)$. We introduce the following definition: A partially ordered set $(X, \leq)$ is called a complete poset if it is a poset and any nonempty directed set S of the set X has an exact upper bound in X.

<u>PROPOSITION 3.</u> The partially ordered set $(X, \leq)$ corresponds to a complete topology of an f-space on X if and only if: $(X, \leq)$ is a complete poset and the set $X_0 \rightleftharpoons \{x | x \in X$, and if S is a directed set, then $x \leq sup\, S \Longleftrightarrow \exists s \in S\ (x \leq s)\}$ is dense in X, i.e., $x = sup\{x_0 | x_0 \in X_0, x_0 \leq x\}$ for any $x \in X$.

<u>Proof.</u> Necessity was proved by the proof of Theorem 1. We shall prove sufficiency. Let us verify that X_0 is a subposet of X, i.e., if $x_0, x_1 \in X_0$, and there is an $x \in X$, such that $x_0 \leq x$ and $x_1 \leq x$, there is an exact upper bound $x_0 \cup x_1$ for these elements in X and the element $x_0 \cup x_1$ in X_0. That the element $x_0 \cup x_1$ exists if x_0 and x_1 are compatible follows from the fact that X is a poset. Let S be a directed set, $x_0, x_1 \in X_0$, x_0, x_1, compatible and $x_0 \cup x_1 \leq sup\, S$, then $x_i \leq sup\, S$, $i = 0, 1$, and there are elements $x, x' \in S$ such that $x_0 \leq x$, $x_1 \leq x'$; but since S is directed, there is an $\bar{x} \in S$, such that $x \leq \bar{x}$ and $x' \leq \bar{x}$; but then $x_0 \leq \bar{x}$ and $x_1 \leq \bar{x}$ and so $x_0 \cup x_1 \leq \bar{x}$. Thus, we have proved that $x_0 \cup x_1 \in X_0$. Consequently, conditions 1)-3), characterizing the triplet $(X, X_0, \leq)$, have been verified. Condition 4) follows from the condition $x = sup\{x_0 | x_0 \in X_0, x_0 \leq x\}$. From the same condition 5) also follows. Indeed, if $x \not\leq y$, y cannot be the upper bound of the set $\{x_0 | x_0 \in X_0, x_0 \leq x\}$ and so there is an $x_0 \in X_0$, $x_0 \leq x$ and $x_0 \not\leq y$. Thus, on X we can define the topology of an f-space so that the basis subspace is X_0, and the order $\leq$ is defined by the topology. Let us show that X, equipped with this topology, becomes a complete f-space. Since $(X, \leq)$ is a complete poset, for all nonempty directed sets S there is an exact upper bound $sup\, S$. For the directed set S let S' denote the set $\{x_0 | x_0 \in X_0, \exists x \in S\,(x_0 \leq x)\}$. We prove the following auxiliary proposition:

If S_0, S_1 are two nonempty directed sets,

$$sup\, S_0 = sup\, S'_0 \text{ and } (sup\, S_0 = sup\, S_1 \Longleftrightarrow S'_0 = S'_1).$$

$$sup\, S_0 = sup\{x | x \in S_0\} = sup\{sup\{x_0 | x_0 \in X_0,\ x_0 \leq x\} \,|\, x \in S_0\} = sup\{x_0 | x_0 \in X_0, \exists x \in S_0\ (x_0 \leq x)\} = sup\, S'_0.$$

If $sup\, S_0 = sup\, S_1$ and $x_0 \in S'_0$, then $x_0 \leq sup\, S_0 = sup\, S_1$, and so $x_0 \leq x$ for some $x \in S_1$. This implies that $x_0 \in S'_1$, i.e., $S'_0 \subseteq S'_1$. Similarly, $S'_1 \subseteq S'_0$ and $S'_0 = S'_1$. The proposition is proved.

It follows from the proposition that if $S \subseteq X_0$ is directed and does not have a greatest element, then $sup\, S \notin X_0$ since otherwise S'_0 would have a greatest element and S_0 would be cofinal in S'_0. Similarly, if $S_0, S_1 \subseteq X_0$ are directed and $sup\, S_0 = sup\, S_1$, then S_0 and S_1 are cofinal in $S'_0 = S'_1\ (\supseteq S_0 \cup S_1)$. Thus, we have established that condition 2) of Theorem 1 holds; i.e., X is a complete f-space. The proposition is proved.

<u>Note.</u> If X is an f-space and $(X, \leq)$ is a complete poset, it is still not implied that X is complete. Thus, if X_0 is a complete poset which has nonprincipal ideals, then, having defined the topology on X_0 by the basis $\{\check{x}_0 | x_0 \in X_0\}$ we obtain an f-space coinciding with its basis which, obviously, is not complete.

We now give a category-theoretic description of some of the results obtained above. Let $\mathcal{P}$ be a category of posets whose morphisms are all monotonic mappings and let $\mathcal{F}_C$ be the category of complete f-spaces. We consider two functors: $C:\mathcal{P}\to\mathcal{F}_C$ and $\Lambda:\mathcal{F}_C\to\mathcal{P}$ defined as follows:

a) If $(X_0,\leq)$ is a poset, then $C(X_0,\leq)$ is a complete f-space corresponding to the triplet $(\mathcal{J}(X_0,\leq),\mathcal{J}_0(X_0,\leq),\subseteq)$; if $\varphi:(X_0,\leq)\to(Y_0,\leq)$ is a monotonic mapping, then $C(\varphi)$ is a continuous mapping from $C(X_0,\leq)$ into $C(Y_0,\leq)$ defined in accordance with Corollary 1 of Proposition 1;

b) if X is a complete f-space, then $\Lambda(X)=(X,\leq)$, where $\leq$ is the order defined by the topology; if $\varphi:X\to Y$ is continuous, then $\Lambda(\varphi)=\varphi$.

We now note that for $(X_0,\leq)\in\mathcal{P}$ and $Y\in\mathcal{F}_C$ we have the natural isomorphism of sets

$$Mor_{\mathcal{P}}((X_0,\leq),\Lambda(Y)) \text{ and } Mor_{\mathcal{F}_C}(C(X_0,\leq),Y)$$

because, by Corollary 1 of Proposition 1 any monotonic mapping of X_0 into $\Lambda(Y)$ can be continued (uniquely) to a continuous mapping of $C(X_0,\leq)$ into Y and, conversely, the restriction of the continuous mapping of $C(X_0,\leq)$ into Y on X_0 is a monotonic mapping of X_0 into $\Lambda(Y)$.

The above-mentioned facts can be formulated as follows:

PROPOSITION 4. The functors C and Λ form a conjugate pair of functors, C being conjugate on the left of the functor Λ (written $C\dashv\Lambda$).

We can indicate another pair of conjugate functors. If $\mathcal{F}$ is the category of f-spaces, the basis subspace of the complete f-space X corresponding to each f-space X^* can be extended by additional definitions to a functor $C^*:\mathcal{F}\to\mathcal{F}_C$ $(C^*(X)\rightleftharpoons X^*)$ (in accordance with the note following the proof of Proposition 1). Then, for the imbedding functor $I:\mathcal{F}_C\to\mathcal{F}$ there is a natural equivalence (homomorphism) (for any $X\in\mathcal{F}$ and $Y\in\mathcal{F}_C$); $C(X,I(Y))=C(X,Y)\approx C(C^*(X),Y)$.

PROPOSITION 5. The functors C^* and I form a pair of conjugate functors $(C^*\dashv I)$.

Let us now indicate one of the most important properties of complete f_0-spaces.

THEOREM 2. Let X be a complete f_0-space, and $g:X\to X$ a continuous mapping from X into itself; then g has a least fixed point; i.e., there is an element μg of the space X such that $g(\mu g)=\mu g$ and for any $x\in X$, if $g(x)=x$, then $\mu g\leq x$. The mapping $FP:C(X,X)\to X$, defined as follows: $FP(g)\rightleftharpoons\mu g$, is continuous, i.e., $FP\in C(C(X,X),X)$.

Proof. Let σ be the least element of X; consider the sequence $g^0(\sigma)\rightleftharpoons\sigma$, $g^1(\sigma)\rightleftharpoons g(\sigma),\dots,$ $g^{n+1}(\sigma)\rightleftharpoons g(g^n(\sigma)),\dots$; it is an increasing sequence S, and so $\sup S$ exists. Since $g(\sup S)=\sup g(S)=\sup S$, $\sup S$ is a fixed point of g. That every fixed point x is an upper bound for S follows from $g^n(\sigma)\leq x$, which is proved by induction on n: $\sigma\leq x$; if $g^n(\sigma)\leq x$ then $g^{n+1}(\sigma)\leq g(x)=x$. Thus, $\mu g\rightleftharpoons\sup\{\sigma,g(\sigma)\ \dots,g^n(\sigma),\dots\}$ is the least fixed point of g.

We now prove that the mapping $FP:C(X,X)\to X$ is continuous. Let $x_0\in X_0$. Consider $g\in FP^{-1}(\check{x}_0)$. This implies that $x_0\leq\mu g=\sup\{\sigma,g(\sigma),\dots,g^n(\sigma),\dots\}$. But then $x_0\leq g^n(\sigma)$ for some n. We can prove by induction on n for any $n>0$ the set $\{\bar g\mid x_0\leq\bar g^n(\sigma)\ ,\bar g\in C(X,X)\}$ is open in $C(X,X)$. For $n=1$ the set $\{\bar g\mid x_0\leq\bar g(\sigma)\}$ is a basically open set $\langle X,\check{x}_0\rangle$; assume the proposition holds for n, i.e., $\{\bar g\mid x_0\leq\bar g^n(\sigma),\ \bar g\in C(X,X)\}$ is open in $C(X,X)$. Consider the set $\{\bar g\mid x_0\leq\bar g^{n+1}(\sigma),\bar g\in C(X,X)\}$. We note that $\{\bar g\mid x_0\leq\bar g^n(\sigma)\}\subseteq\{\bar g\mid x_0\leq\bar g^{n+1}(\sigma)\}$, since $\bar g^n(\sigma)\leq\bar g^{n+1}(\sigma)$. Let $x_0\not\leq\bar g^n(\sigma)$, but $x_0\leq\bar g^{n+1}(\sigma)=\bar g(\bar g^n(\sigma))$. Consider $\bar g^{-1}(\check{x}_0)$; then $\bar g^n(\sigma)\in\bar g^{-1}(\check{x}_0)$, and there is an $x_1\in X_0$ such that $x_1\leq\bar g^n(\sigma)$ and

$\check{x}_1 \subseteq \bar{g}^{-1}(\check{x}_0)$. Then $\bar{g} \in \{g' \mid x_1 \leq g'^n(\sigma),\ g' \in C(X,X)\} \cap \langle \check{x}_1, \check{x}_0 \rangle$. The last set is open, by the induction hypothesis. Suppose now that g' is such that $g'^n(\sigma) \geq x_1$ and $g' \in \langle \check{x}_1, \check{x}_0 \rangle$; then $g'^{n+1}(\sigma) = g'(g'^n(\sigma)) \geq g'(x_1) \geq x_0$. Consequently, $\{g' \mid x_1 \leq g'^n(\sigma)\} \cap \langle \check{x}_1, \check{x}_0 \rangle \subseteq \{\bar{g} \mid x_0 \leq \bar{g}^{n+1}(\sigma)\}$, and so this set is open. But then $FP^{-1}(\check{x}_0) = \bigcup \{g \mid x_0 \leq g^n(\sigma)\}$ is open and FP is a continuous mapping from $C(X,X)$ into $\check{X}$. The theorem is proved.

Note. The minimal fixed point μg of every monotonic mapping g of a complete poset (with zero) into itself is also defined by the familiar relation

$$\mu g = \inf \{x \mid g(x) \leq x\} .$$

The following proposition shows that completeness is preserved when we pass from an f -space to an f_0 -space. We recall that at the end of Sec. 2 we defined the functor F_0 from the category of f -spaces into the category of f_0 -spaces.

PROPOSITION 6. If X is a complete f -space, $F_0(X)$ is a complete f_0 -space.

The proposition follows directly from the definition of the functor F_0 and Proposition 3.

4. Partial Functionals; λ -Models

In this section we define the concept of the class (model) of partial functionals of finite types over an arbitrary nonempty set S .

We introduce the following notation. For the sets S_0 and S_1 let $M(S_0, S_1)$ denote the set of all mappings from S_0 into S_1 and let $M_p(S_0, S_1)$ denote the set of all partial mappings from S_0 into S_1 . $M(S_0,S_1) \subseteq M_p(S_0,S_1)$. We note that there is a natural equivalence φ (a one-to-one correspondence) of of sets

$$M_p(S_0 \times S_1, S) \qquad (M(S_0 \times S_1, S))$$

and

$$M(S_0, M_p(S_1, S)) \qquad (M(S_0, M(S_1, S))) ,$$

which is defined as follows:

$$\text{if } f \in M_p(S_0 \times S_1, S) \text{ , the } [\varphi f](s_0) \simeq \lambda s_1 f(s_0, s_1),\ s_0 \in S_0 .$$

Here $\lambda s_1 f(s_0, s_1)$ denotes the (partial) mapping from S_1 into S which establishes a correspondence between the element $s_1 \in S_1$ and the element $f(s_0, s_1)$ of S (if $f(s_0,s_1)$ is defined; otherwise, $\lambda s_1 f(s_0, s_1)$ is not defined at the point s_1).

We now define the concept of the type of a functional. We denote the set of all types by T .

1) $0 \in T$, i.e., 0 is a type;

2) if $\sigma_0, \ldots, \sigma_{n-1}, \sigma_n \in T$, then $(\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n) \in T$.

Let $\bar{T}$ denote finite sequences of types: if $\sigma_0, \ldots, \sigma_{n-1} \in T$, then $(\sigma_0, \ldots, \sigma_{n-1}) \in \bar{T}$. If $\sigma \neq 0 \in T$, then let $\bar{\sigma}$ denote the sequence $(\sigma_0, \ldots, \sigma_{n-1})$, where $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)$.

The class $\mathcal{F} = \{F_\sigma \mid \sigma \in T\}$ of sets, indexed by types, is called a class of partial functionals over a set S if the following conditions hold:

1) $F_0 = S$;

2) if $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)$, and $\sigma_n \neq 0$, then

$$F_{\bar{\sigma}} \simeq \prod_{i<n} F_{\sigma_i} \ , \quad F_\sigma \subseteq M(F_{\bar{\sigma}}, F_{\sigma_n}) ,$$

i.e., F_σ consists of certain mappings from $F_{\bar\sigma}$ into F_{σ_n};

3) if $\sigma \neq 0$ and $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | 0)$, then

$$F_\sigma \subseteq M_p(F_{\bar\sigma}, F_0) = M_p(F_{\bar\sigma}, S).$$

We now define on the set T of all types an equivalent relation $\sim$ as the least equivalent relation satisfying the conditions:

1) if $\sigma = (\sigma_0, \ldots, \sigma_{k-1}, \sigma_k, \ldots, \sigma_{n-1} | \sigma_n)$, $\sigma' = (\sigma_0, \ldots, \sigma_{k-1} | (\sigma_k, \ldots, \sigma_{n-1} | \sigma_n))$, then $\sigma \sim \sigma'$;

2) if $\sigma_i \sim \sigma_i'$, $i < n$, then $(\sigma_0, \ldots, \sigma_{n-1} | \sigma_n) \sim (\sigma_0', \ldots, \sigma_{n-1}' | \sigma_n')$.

The class $\mathcal{F}$ of partial functionals over S is said to be a λ-model if, for each pair of equivalent types σ and σ', there is a mapping $S_{\sigma,\sigma'} : F_\sigma \longrightarrow F_{\sigma'}$ satisfying the following conditions:

1) $S_{\sigma,\sigma}$ is the identity mapping of F_σ into itself;

2) if $\sigma \sim \sigma'$, $\sigma' \sim \sigma''$, then $S_{\sigma',\sigma''} \circ S_{\sigma,\sigma'} = S_{\sigma,\sigma''}$;

3) if $\sigma = (\sigma_0, \ldots, \sigma_{k-1}, \sigma_k, \ldots, \sigma_{n-1} | \sigma_n)$, $\sigma' = (\sigma_0, \ldots, \sigma_{k-1} | (\sigma_k, \ldots, \sigma_{n-1} | \sigma_n))$, then $S_{\sigma,\sigma'}(f) = [\lambda f_k \ldots \lambda f_{n-1}] f$;

4) if $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | \sigma_n)$, $\sigma' = (\sigma_0', \ldots, \sigma_{n-1}' | \sigma_n')$ and $\sigma_i \sim \sigma_i'$, $i < n$, then

$$[S_{\sigma,\sigma'}(f)](\bar f) = S_{\sigma_n,\sigma_n'}(f(S_{\bar\sigma',\bar\sigma} \bar f)) \qquad \text{for } f \in F_\sigma, \bar f \in F_{\bar\sigma'}.$$

Note 1. Let us clarify condition 3). Since

$$F_\sigma \subseteq M(\prod_{i<k} F_{\sigma_i} \times \prod_{k \le i < n} F_{\sigma_i}, F_{\sigma_n}),$$

the mapping $f \rightsquigarrow [\lambda f_k \ldots \lambda f_{n-1}] f$ takes F_σ into

$$M(\prod_{i<k} F_{\sigma_i}, M(\prod_{k \le i < n} F_{\sigma_i}, F_{\sigma_n})),$$

so that it follows from 3) that the image of F_σ under this mapping lies in

$$F_{\sigma'} \subseteq M(\prod_{i<k} F_{\sigma_i}, M(\prod_{k \le i < n} F_{\sigma_i}, F_{\sigma_n})).$$

2. The notation $S_{\bar\sigma',\bar\sigma} \bar f$ is an abbreviation for

$$(S_{\sigma_0',\sigma_0}(f_0), \ldots, S_{\sigma_{n-1}',\sigma_{n-1}}(f_{n-1})).$$

3. It follows from conditions 1) and 2) that $S_{\sigma,\sigma'}$ is an equivalence between F_σ and $F_{\sigma'}$.

4. The system of mappings $S_{\sigma,\sigma'}$, if it exists for $\mathcal{F}$, is unique. This follows from 3) and 4) and the definition of the equivalence $\sim$.

We now define the concept of a special type (ST).

1) 0 is a special type $(0 \in ST)$;

2) if $\sigma_0, \ldots, \sigma_{n-1} \in ST$, then $(\sigma_0, \ldots, \sigma_{n-1} | 0) \in ST$.

We see from the definition that $ST \subseteq T$. Let s be a mapping from T into T, defined by induction:

$s(0) \leftrightharpoons 0$; if $s(\sigma_i)$, $i < n$, is defined and $s(\sigma_n) = (\tau_0, \ldots, \tau_{k-1} | 0)$, then

$$s((\sigma_0, \ldots, \sigma_{n-1} | \sigma_n)) \leftrightharpoons (s(\sigma_0), \ldots, s(\sigma_{n-1}), \tau_0, \ldots, \tau_{k-1} | 0).$$

We note that s is a mapping of T into ST and $\sigma \in ST$ for $s(\sigma) = \sigma$.

PROPOSITION 1. Let $\sigma_0, \sigma_1 \in T$; then $\sigma_0 \sim s(\sigma_0)$ and $\sigma_0 \sim \sigma_1 \iff s(\sigma_0) = s(\sigma_1)$.

Proof. First we verify by induction that $\sigma \sim s(\sigma)$. Now $s(0) = 0$ and as $0 \sim s(0)$. Assume that $\sigma_i \sim s(\sigma_i)$, $i < n$; let $s(\sigma_n) = (\tau_0, \ldots, \tau_{k-1} | 0)$; then

$$\sigma = (\sigma_0, \ldots, \sigma_{n-1} | \sigma_n) \sim (s(\sigma_0), \ldots, s(\sigma_{n-1}) | s(\sigma_n)) = (s(\sigma_0), \ldots, s(\sigma_{n-1}) |$$
$$| (\tau_0, \ldots, \tau_{k-1} | 0)) \sim (s(\sigma_0), \ldots, s(\sigma_{n-1}), \tau_0, \ldots, \tau_{k-1} | 0) = s(\sigma).$$

Thus, it follows that $s(\sigma_0) = s(\sigma_1) \Longrightarrow \sigma_0 \sim \sigma_1$. To prove the converse implication we have to show that $\sigma_0 \sim \sigma_1 \Rightarrow \sigma_0 = s(\sigma_1)$ for $\sigma_0 \in ST$. We shall prove this by induction on σ_0. If $\sigma_0 = 0$, this is obvious. Let $\sigma_0 = (\sigma_0', \ldots, \sigma_{n-1}' | 0)$, and for $\sigma_0', \ldots, \sigma_{n-1}'$ suppose the proposition holds. Since $\sigma_0 \sim \sigma_1$, there is a sequence of types $\tau_1, \ldots, \tau_s$ such that, putting $\tau_0 \doteq \sigma_0$, $\tau_{s+1} \doteq \sigma_1$; for any $l < s$ the pair τ_l, τ_{l+1} satisfies condition 1) or 2) of the definition of the equivalence $\sim$. We prove the proposition by induction on s. If $s = 0$, the pair σ_0, σ_1 satisfies condition 1) or condition 2). Suppose 1) holds; then $\sigma_1 = (\sigma_0', \ldots, \sigma_{k-1}' | (\sigma_k', \ldots, \sigma_{n-1}' | 0))$ for some $k < n$, and $s(\sigma_1) = (\sigma_0', \ldots, \sigma_{k-1}', \sigma_k', \ldots, \sigma_{n-1}' | 0) = \sigma_0$, since $\sigma_i' \in ST$. Suppose 2) holds; then $\sigma_1 = (\sigma_0'', \ldots, \sigma_{n-1}'' | 0)$ and $\sigma_i' \sim \sigma_i''$. $s(\sigma_1) = (s(\sigma_0''), \ldots, s(\sigma_{n-1}'') | 0)$, but, by the induction hypothesis $s(\sigma_i'') = \sigma_i'$, since $\sigma_i' \sim \sigma_i''$. Hence, $s(\sigma_1) = \sigma_0$. Suppose the position holds for s. We prove it for $s+1$. Consider the pair τ_{s+1}, σ_1. By the induction hypothesis $s(\tau_{s+1}) = \sigma_0$. Either 1) or 2) holds for the pair τ_{s+1}, σ_1. Consider condition 1). Then we at once see that $s(\tau_{s+1}) = s(\sigma_1)$, but $\sigma_0 = s(\tau_{s+1})$. Suppose 2) holds; then $\tau_{s+1} = (\tau_0', \ldots, \tau_{t-1}' | \tau_t')$, $\sigma_1 = (\sigma_0'', \ldots, \sigma_{t-1}'' | \sigma_t'')$ and $\tau_i' \sim \sigma_i''$, $i < t$. Since $s(\tau_{s+1}) = \sigma_0$, we have $s(\tau_i') = \sigma_i'$ for $i < t$ and $s(\tau_t') = (\sigma_t', \ldots, \sigma_{n-1}' | 0)$. Since $\tau_i' \sim \sigma_i''$, then $\sigma_i'' \sim \sigma_i'$ for $i < t$ and $s(\sigma_1) = (\sigma_0', \ldots, \sigma_{t-1}', \ldots | 0)$. To prove that $s(\sigma_1) = \sigma_0$ it is sufficient to show that $s(\tau_t') = s(\sigma_t'')$, which is proved by induction on n since $t > 0$. The proposition is proved.

PROPOSITION 2. Every λ -model $F = \{F_\sigma \mid \sigma \in T\}$ is uniquely (as a λ -model) defined by the family $F_S = \{F_\sigma \mid \sigma \in ST\}$.

This follows easily from Note 4.

We say that F_S is a special part of F. The converse (in a certain sense) of Proposition 2 holds.

The special class F_S of partial functionals over S is the family $\{F_\sigma \mid \sigma \in ST\}$, such that

1) $F_0 = S$;

2) $F_{(\sigma_0, \ldots, \sigma_{n-1} | 0)} \subseteq M_p(F_{\bar{\sigma}}, S)$;

3) for any $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | 0) \in ST$, $n > 1$ if $f \in F_\sigma$, $f_0 \in F_{\sigma_0}$, then $\lambda f_1 \ldots \lambda f_{n-1} f(f_0, f_1, \ldots, f_{n-1}) \in F_{(\sigma_1, \ldots, \sigma_{n-1} | 0)}$.

PROPOSITION 3. Every special class F_S of partial functionals over S defines (uniquely) a λ - model F such that F_S is a special part of F.

Proof. We define the sets F_σ for $\sigma \in T$.

We have already defined F_σ for special types σ. Let F_{σ_i}, $i < n$, already have been defined, together with the equivalences $S_{\sigma_i, s(\sigma_i)}$ from F_{σ_i} onto $F_{s(\sigma_i)}$. Let $s(\sigma_n) = (\tau_0, \ldots, \tau_{k-1} | 0)$, $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | \sigma_n)$, $s(\sigma) = (s(\sigma_0), \ldots, s(\sigma_{n-1}), \tau_0, \ldots, \tau_{k-1} | 0)$. We define the mapping $S_{\sigma, s(\sigma)}$ from $M(F_{\bar{\sigma}}, F_{\sigma_n})$ ($M_p(F_{\bar{\sigma}}, F_{\sigma_n})$ if $\sigma_n = 0$) into

$$M_p\left(\prod_{i<n} F_{s(\sigma_i)} \times \prod_{i<k} F_{\tau_i}, S\right)$$

as follows: for

$$f \in M(F_{\bar\sigma}, F_{\sigma_n}), \quad \bar f \in \prod_{i<n} F_{S(\sigma_i)}, \quad \bar g \in \prod_{i<k} F_{\tau_i},$$
$$[S_{\sigma, S(\sigma)}(f)](\bar f, \bar g) \leftrightharpoons [S_{\sigma_n, S(\sigma_n)}(f(S_{S(\bar\sigma), \bar\sigma}(\bar f)))](\bar g).$$

It is easily verified (using the fact that $S_{\sigma_i, S(\sigma_i)}$ and $S_{S(\sigma_i), \sigma_i}$ are equivalences) that $S_{\sigma, S(\sigma)}$ is an equivalence of the sets $M(F_{\bar\sigma}, F_{\sigma_n})$ and $M_p(\prod_{i<n} F_{S(\sigma_i)} \times \prod_{i<k} F_{\tau_i}, S)$. Put $F_\sigma \leftrightharpoons S^{-1}_{\sigma, S(\sigma)}(F_{S(\sigma)})$; the restriction of $S_{\sigma, S(\sigma)}$ on F_σ will be denoted by the same letter; $S_{S(\sigma), \sigma}$ is the mapping inverse to $S_{\sigma, S(\sigma)}$. The construction is complete. The routine verification that the class of partial functions over S just constructed forms a λ -model is left to the reader. The proposition is proved.

We now define an example of a λ -model $\mathcal{C} = \{C_\sigma \mid \sigma \in T\}$ of partial functionals over S which is fundamental for the sequel as follows:

All the sets C_σ are f -spaces.

Assuming S is a discrete space, we put $C_0 = S$. Let $C_{\sigma_0}, \ldots, C_{\sigma_{n-1}}, C_{\sigma_n}$ already have been defined and let C_{σ_n} be an f_0 -space for $\sigma \neq 0$; then we put

$$C_{(\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)} \leftrightharpoons C(\prod_{i<n} C_{\sigma_i}, C_{\sigma_n}) \quad , \text{ if } \sigma_n \neq 0 \text{ and}$$
$$C_{(\sigma_0, \ldots, \sigma_{n-1} \mid 0)} \leftrightharpoons C_p(\prod_{i<n} C_{\sigma_i}, S) \approx C(\prod_{i<n} C_{\sigma_i}, F_0(S)),$$

if $\sigma_n = 0$.

In the first case $(\sigma_n \neq 0)$ the topology on $C_{(\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)}$ is defined in the standard manner, since $\prod_{i<n} C_{\sigma_i}$ is an f -space, while C_{σ_n} is an f_0 -space (we note that $C_{(\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)}$ is an f_0 -space.)

In the second case the topology is defined as the topology carried over from $C(\prod_{i<n} C_{\sigma_i}, F_0(S))$, and in this case $C_{(\sigma_0, \ldots, \sigma_{n-1} \mid 0)}$ is an f_0 -space.

Theorem 2 of Sec. 1 shows that the class $\mathcal{C}$ of partial functionals over S is a λ -model.

A more general method of constructing a λ -model over S consists in specifying on S an arbitrary topology $\mathcal{S}$, which makes S an f -space. Then we define the class $\mathcal{C}(\mathcal{S}) = \{C(\mathcal{S})_\sigma \mid \sigma \in T\}$ of partial functionals in the same way as above.

$$C(\mathcal{S})_0 = S \qquad \text{(with topology } \mathcal{S} \text{)}.$$

If $C(\mathcal{S})_{\sigma_0}, \ldots, C(\mathcal{S})_{\sigma_{n-1}}, C(\mathcal{S})_{\sigma_n}$ have already been defined (with the topologies of f-spaces) and $C(\mathcal{S})_\sigma$ is an f_0 -space for $\sigma \neq 0$, we put

$$C(\mathcal{S})_{(\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)} \leftrightharpoons C(\prod_{i<n} C(\mathcal{S})_{\sigma_i}, C(\mathcal{S})_{\sigma_n}), \quad \text{if } \sigma_n \neq 0;$$
$$C(\mathcal{S})_{(\sigma_0, \ldots, \sigma_{n-1} \mid 0)} \leftrightharpoons C_p(\prod_{i<n} C(\mathcal{S})_{\sigma_i}, S), \quad \text{if} \quad \sigma_n = 0,$$

where the corresponding spaces of continuous functions have canonical topology.

Note. Different f -topologies $\mathcal{S}$ on S correspond to different λ -models $\mathcal{C}(\mathcal{S})$.

We note a number of properties of λ -models over S of the form $\mathcal{C}(\mathcal{S})$ where $\mathcal{S}$ is the topology of the f -space over S . The elements of the set $C(\mathcal{S})_{(\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)}$ are, in a natural sense, mappings (partial if $\sigma_n = 0$) from $\prod_{i<n} C(\mathcal{S})_{\sigma_i}$ into $C(\mathcal{S})_{\sigma_n}$:

1). The class $\mathcal{C}(\mathcal{S})$ is closed with respect to all permissible compositions of functions; moreover, the composition operations themselves are elements of the class $\mathcal{C}(\mathcal{S})$.

2. The operations of permutation and "splicing" of arguments of the same type do not go outside the limits of $\mathcal{C}(\mathcal{S})$; moreover, they belong to $\mathcal{C}(\mathcal{S})$.

3. If $\sigma \sim \sigma'$, the mapping $S_{\sigma,\sigma'}: C(\mathcal{S})_\sigma \to C(\mathcal{S})_{\sigma'}$ is a homomorphism of these spaces and $S_{\sigma,\sigma'} \in C(\mathcal{S})_{(\sigma|\sigma')}$.

The formulation of the properties 1 and 2 is not exact, although the meanings of these assertions are completely understood. One of the possible refinements would be the introduction of analogs of the operations $\zeta, \tau, \Delta, \nabla, *$ [8] for various types.

If S is equipped with the topology of f-spaces and $\mathcal{F}=\{F_\sigma \mid \sigma \in T\}$ is a λ-model over S, we say that the λ-model $\mathcal{F}$ is densely compatible with the topology $\mathcal{S}$, if we can introduce a topology of f-spaces such that all the following conditions are satisfied for all sets F_σ, $\sigma \in T$:

1. The topology on $F_0 (=S)$ coincides with the topology $\mathcal{S}$.

2. If $\sigma \neq 0$, then F_σ is an f_0-space.

3. If $\sigma = (\bar{\sigma} \mid \sigma')$, $\sigma' \neq 0$, then $F_\sigma \subseteq C(F_{\bar{\sigma}}, F_{\sigma'})$, and F_σ is a subspace in the canonical topology of the f_0-space $C(F_{\bar{\sigma}}, F_{\sigma'})$ containing a basis subspace.

3'. If $\sigma = (\bar{\sigma} \mid 0)$, then $F_\sigma \subseteq C_p(F_{\bar{\sigma}}, S)(\approx C(F_{\bar{\sigma}}, F_0(S))$ and F_σ is a subspace in the canonical topology of the f_0-space $C(F_{\bar{\sigma}}, F_0(S))$, containing a basis subspace.

We now introduce a concept which makes it possible to compare different classes of partial functionals over S. If $\mathcal{F}=\{F_\sigma \mid \sigma \in T\}$ and $\mathcal{G}=\{G_\sigma \mid \sigma \in T\}$ are two classes of partial functionals over S, then we say that the family of mappings $M=\{\mu_\sigma \mid \sigma \in T\}$ from $\mathcal{F}$ into $\mathcal{G}$, such that the following conditions hold, is a morphism $M : \mathcal{F} \to \mathcal{G}$.

1. $\mu_\sigma : F_\sigma \to G_\sigma$ for all $\sigma \in T$.

2. $\mu_0 = id_S$

3. If $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)$, $f \in F_\sigma$, $f_i \in F_{\sigma_i}$, $i < n$, then

$$\mu_{\sigma_n}(f(f_0, \ldots, f_{n-1})) = [\mu_\sigma f](\mu_{\sigma_0} f_0, \ldots, \mu_{\sigma_{n-1}} f_{n-1})$$

$$\text{(briefly, } \mu_{\sigma_n}[f(\bar{f})] = [\mu_\sigma f](\mu_{\bar{\sigma}} \bar{f})).$$

LEMMA. If $M=\{\mu_\sigma \mid \sigma \in T\}$ is a morphism from $\mathcal{F}$ into $\mathcal{G}$, all the mappings μ_σ, $\sigma \in T$, are one-to-one.

Proof. The proof is by induction on the type σ. For $\sigma=0$, the result is obvious from condition 2. Suppose the proposition is true for σ_n. Let $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)$, f and $f' \in F_\sigma$ and $f \neq f'$. Then

$$i<n,\ f(f_0, \ldots, f_{n-1}) \neq f'(f_0, \ldots, f_{n-1}).$$

for some $f_i \in F_{\sigma_i}$. Since $[\mu_\sigma f](\mu_{\sigma_0} f_0, \ldots, \mu_{\sigma_{n-1}} f_{n-1}) = \mu_{\sigma_n}(f(f_0, \ldots, f_{n-1})) \neq \mu_{\sigma_n}(f'(f_0, \ldots, f_{n-1})) = [\mu_\sigma f'](\mu_{\sigma_0} f_0, \ldots, \mu_{\sigma_{n-1}} f_{n-1})$, then $\mu_\sigma f \neq \mu_\sigma f'$. The lemma is proved.

The lemma shows that the morphism is essentially an imbedding of $\mathcal{F}$ in $\mathcal{G}$.

If there is at least one morphism from $\mathcal{F}$ into $\mathcal{G}$, we denote it thus: $\mathcal{F} \preccurlyeq \mathcal{G}$. Obviously, by definition, $\preccurlyeq$ is reflexive and transitive.

We now prove a fundamental proposition about topological models of the form $\mathcal{C}(\mathcal{S})$.

THEOREM. If $\mathcal{S}$ is the topology of a complete f-space on S, then for any class of partial functions $\mathcal{F}$ over S, densely compatible with the topology $\mathcal{S}$, we have $\mathcal{F} \preccurlyeq \mathcal{C}(\mathcal{S})$.

Proof. We shall construct the morphism $M=\{\mu_\sigma \mid \sigma \in T\}$ from $\mathcal{F}$ into $\mathcal{C}(\mathcal{S})$ by induction. Put $\mu_0 \rightleftharpoons id_S$.

Let $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)$, $\sigma_n \neq 0$, and suppose the mapping μ_{σ_i} from F_{σ_i} into $C(\mathcal{S})_{\sigma_i}$, $i < n$, has already been constructed, satisfying the following conditions: μ_{σ_i} is a homomorphic imbedding of F_{σ_i}

in $C(\mathcal{S})_{\sigma_i}$ and $\mu_{\sigma_i}(F_i)$ contains a basis subspace of the space $C(\mathcal{S})_{\sigma_i}$. In particular, the spaces F_{σ_i} and $C(\mathcal{S})_{\sigma_i}$ are basically equivalent ($F_{\sigma_i} \sim C(\mathcal{S})_{\sigma_i}$), and μ_{σ_i} is a homomorphism of the bases of F_{σ_i} and $C(\mathcal{S})_{\sigma_i}$. It follows from this that

$$C(\prod_{i<n} F_{\sigma_i}, F_{\sigma_n}) \sim C(\prod_{i<n} C(\mathcal{S})_{\sigma_i}, C(\mathcal{S})_{\sigma_n}) = C_\sigma(\mathcal{S}).$$

The system of homomorphisms μ_{σ_i}, $i \leq n$, defines the homomorphism μ'_σ of the basis subspace of the space $C(\prod_{i<n} F_{\sigma_i}, F_{\sigma_n})$ onto the basis subspace of the space $C_\sigma(\mathcal{S})$. By Theorem 1 of Sec. 3, or by Corollary 1 to Proposition 1 of Sec. 3, this homomorphism μ'_σ can be continued uniquely to the homomorphism μ''_σ from $C(\prod_{i<n} F_{\sigma_i}, F_{\sigma_n})$ into $C_\sigma(\mathcal{S})$. The restriction μ_σ of the mapping μ''_σ on $F_\sigma \subseteq C(\prod_{i<n} F_{\sigma_i}, F_{\sigma_n})$ is just the required mapping. The case $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | 0)$ can be discussed similarly. Thus the family M of mappings μ_σ, $\sigma \in T$, has been constructed. It remains to verify that M is a morphism from F into $\mathcal{C}(\mathcal{S})$. Let $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | \sigma_n)$, $\sigma_n \neq 0$, $f \in F_\sigma$, $f_i \in F_{\sigma_i}$, $i<n$. Consider

$$[\mu_\sigma f](\mu_{\sigma_0} f_0, \ldots, \mu_{\sigma_{n-1}} f_{n-1}).$$

If all the f, f_i belong to the basis subspace, the equation

$$[\mu_\sigma f](\mu_{\sigma_0} f_0, \ldots, \mu_{\sigma_{n-1}} f_{n-1}) = \mu_{\sigma_n}[f(f_0, \ldots, f_{n-1})]$$

follows directly from the definition of $\mu_\sigma (\supseteq \mu'_\sigma)$. For arbitrary f, f_i, $i<n$, this follows from the continuity of $\mu_\sigma, \mu_{\sigma_i}$, $i \leq n$ (cf. Proposition 1 of Sec. 3) and the validity of the equation for basis elements: $[\mu_\sigma f](\mu_{\sigma_0} f_0, \ldots, \mu_{\sigma_{n-1}} f_{n-1}) = [sup\{\mu_\sigma f' \mid f'$ is a basis element, $f' \leq f\}]$ $(sup\{\mu_{\sigma_0} f'_0 \mid f'_0$ is a basis element, $f'_0 \leq f_0\}, \ldots) = sup\{[\mu_\sigma f'](\mu_{\sigma_0} f'_0, \ldots) \mid f', f'_0, \ldots$ are basic, $f' \leq f, f'_0 \leq f_0, \ldots\} =$ $sup\{\mu_{\sigma_n}(f'(f'_0, \ldots)) \mid f', f'_0, \ldots$ are basic, $f' \leq f$, $f'_0 \leq f_0, \ldots\} = \mu_{\sigma_n}[f(f_0, \ldots)]$. The theorem is proved.

<u>Note.</u> The morphism which we constructed in the theorem is said to be canonical and is denoted by $K(K_F)$ $(K: F \to \mathcal{C}(\mathcal{S}))$.

The theorem just proved indicates a certain universality of the λ-model $\mathcal{C}(\mathcal{S})$. In the next section this property of universality will be extended suitably to certain classes of functionals defined everywhere.

5. Fertile Classes of Partial Functionals over a Complete f-Space

Let S be a complete f-space, $\mathcal{C}$ the λ-model of all partial functionals over S (reference to the topology $\mathcal{S}$ of S is omitted since it is fixed). In the following considerations we consider only λ-models of functionals over S, and so a number of definitions will refer only to special class of functionals.

Let $\mathcal{G}$ be a λ-model of functionals over S. We shall say that $\mathcal{G}$ is compatible with the topology of S if for all sets G_σ, $\sigma \in ST$, from $\mathcal{G}_S$ – the special part of the λ-model $\mathcal{G}$ – we can specify the structure of the topological space and define the basis of this topology B_σ so that the following conditions hold:

1. The topology on $G_0 (= S)$ coincides with the topology of S; the basis B_0 consists of all the f-sets of S.

2. If $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | 0) \in ST$, G_σ is a subspace of $C_p(\prod_{i<n} G_{\sigma_i}, S)$, where $C_p(\prod_{i<n} G_{\sigma_i}, S)$ is the set of all partially continuous mappings (with open domain of definition) from $\prod_{i<n} G_{\sigma_i}$ (with product topology) into S, and the topology on this set is defined by the prebasis of a set of the form

$$\langle \prod_{i<n} V_i, V \rangle \rightleftharpoons \{f \mid f \in C_p(\prod_{i<n} G_{\sigma_i}, S), \prod_{i<n} V_i \subseteq f^{-1}(V)\},$$

where $V_i \in B_{\sigma_i}$, $i<n$, $V \in B_0$. The basis B_σ is defined as the family of sets obtained by the restriction on G_σ of finite intersections of sets of the form

$$< \prod_{i<n} V_i , V > , \; V_i \in B_{\sigma_i} , i<n , V \in B_0 .$$

Let G be a λ -model of functionals over S, compatible with the topology of S. Let $G^{\vee}_\sigma$ $(\sigma \in ST)$ denote the set (topological space) $C_p(\prod_{i<n} G_{\sigma_i}, S)$. We shall say that G_S is a fertile class of functionals over S (or, that G is a fertile λ -model) if G_σ is dense in G^*_σ for any $\sigma \in ST$.

We now characterize fertile λ -models over S. We first define the concept of a formal neighborhood. We establish a correspondence between every f -element x of S and the symbol c_x. A formal neighborhood of type 0 is a symbol of the form c_x. The set of all formal neighborhoods of type 0 is denoted by B_0^*. If we have defined formal neighborhoods of type σ_i (sets $B^*_{\sigma_i}$), $i<n$, a formal neighborhood of type $\sigma = (\sigma_0, \dots, \sigma_{n-1} \mid 0)$ is an expression of the form

$$\bigcap_{i<k} < V_0^i \times \dots \times V_{n-1}^i , V^i >,$$

where $V_j^i \in B^*_{\sigma_j}$, $i<k$, $j<n$ and $V^i \in B_0^*, i<k$. We note that for any λ -model G, compatible with the topology of S and for any $\sigma \in ST$, there is a naturally defined mapping $\pi_\sigma^G : B_\sigma^* \to B_\sigma$ of the set of all formal neighborhoods of type σ into the set of all basis neighborhoods in G_σ $(\pi_0^G(c_x) \Leftarrow \check{x})$.

<u>PROPOSITION 1.</u> If G is a λ -model of functionals over S, compatible with the topology on S, G is fertile if and only if one of the following two equivalent conditions holds:

1. For any $\sigma \in ST$ and any formal neighborhood $V \in B^*_\sigma$ we have the equivalence

$$\pi_\sigma^C (V) \neq \emptyset \Longleftrightarrow \pi_\sigma^G (V) \neq \emptyset .$$

2. For any $\sigma \in ST$ and any $V_i^j \in B_{\sigma_i}$, $i<n$, $V^j \in B_0$, $j<k$, we have the equivalence

$$\bigcap_{j<k} < \prod_{i<n} V_i^j , V^j > \cap G_\sigma \neq \emptyset \Longleftrightarrow \forall I \subseteq \{0, \dots, k-1\} \Big[\bigcap_{j \in I} (\prod_{i<n} V_i^j) \neq \emptyset \Longrightarrow \bigcap_{j \in I} V^j \neq \emptyset \Big] .$$

<u>Proof.</u> We note that the topological space G^*_σ is an f -space (an f_0 -space if $\sigma \neq 0$) by Theorem 1 of Sec. 1. Then, by Lemma 2 of Sec. 1, the condition on the left of equivalence 2 is necessary and sufficient for

$$\bigcap_{j<k} < \prod_{i<n} V_i^j , V^j >$$

to be nonempty in G^*_σ, and so this equivalence is necessary and sufficient for G_σ to be in G^*_σ. Thus, we have shown that G is fertile if and only if condition (equivalence) 2 holds. That conditions 1 and 2 are equivalent is simply proved by induction on the type using Lemma 2 of Sec. 1. The proposition is proved.

<u>Note.</u> Every λ -model G over S, densely compatible with the topology of S, is obviously a fertile λ -model.

Let F and G be two λ -models over S; an S -morphism from F into G is the family of mappings $M_S = \{\mu_\sigma \mid \sigma \in ST\}$ for which

1) μ_σ is a mapping of F_σ into G_σ, $\sigma \in ST$;

2) $\mu_0 = id_S$;

3) for any $\sigma=(\sigma_0,\dots,\sigma_{n-1}|0)\in ST$, $f\in F_\sigma$, $f_i\in F_{\sigma_i}$, $i<n$, we have the (condition) equation

$$f(f_0,\dots,f_{n-1})=[\mu_\sigma(f)](\mu_{\sigma_0}(f_0),\dots,\mu_{\sigma_{n-1}}(f_{n-1})).$$

The following theorem is an extension of the theorem of the previous section. It indicates further properties of the universality of the space $\mathcal{C}$.

THEOREM 1. If $\mathcal{G}$ is a fertile λ -model over S , there is an S -morphism M from $\mathcal{G}$ into $\mathcal{C}$.

Proof. Let S_0 denote the basis subspace of S . We now define the sequence of mappings $Q=\{q_\sigma\mid\sigma\in ST\}$ (where $q_\sigma:G_\sigma\to P(C_\sigma)$ (= the set of all subsets of C_σ) , $\sigma\in ST$) inductively:

1. $q_0(s)\leftrightharpoons\{s\}$ for all $s\in G_0\ (=S)$;

2. for $g\in G_\sigma$, $\sigma=(\sigma_0,\dots,\sigma_{n-1}|0)\in ST$ we have

$$q_\sigma(g)\leftrightharpoons\{h\mid h\in C_\sigma\quad \forall g_i\in G_{\sigma_i},\ i<n\quad \forall h_i\in q_{\sigma_i}(g_i),\ i<n\,(h(\bar h)\preceq g(\bar g))\}.$$

Here $h(\bar h)\preceq g(\bar g)$ is an abbreviation for the relation: $h(h_0,\dots,h_{n-1})$ is not defined or $h(h_0,\dots,h_{n-1})$ and $g(g_0,\dots,g_{n-1})$ are defined and $h(h_0,\dots,h_{n-1})\preceq g(g_0,\dots,g_{n-1})$ in S . We note that the relation we have defined can be interpreted in the usual way in the space $F_0(S)$. Such a stipulation on the use of the relation $\preceq$ will also be used below in the proof without special mention.

By induction on the construction we shall prove the following properties of the family of mappings $Q=\{q_\sigma\mid\sigma\in ST\}$.

1. If $\sigma\in ST$, $g\in G_\sigma$, $h,h'\in q_\sigma(g)$, then h and h' are compatible (in C_σ as an ordered set) and $h\cup h'\in q_\sigma(g)$.

2. If $\sigma\in ST$, $g\in G_\sigma$, $V\in B_\sigma^*$, and $g\in\pi_\sigma^{\mathcal{C}}(V)$, we have

$$h\in q_\sigma(g)\cap\pi_\sigma^{\mathcal{C}}(V).$$

3. If $\sigma\in ST$, $g\in G_\sigma$, $V\in B_\sigma^*$ and $h\in q_\sigma(g)\cap\pi_\sigma^{\mathcal{C}}(V)$, then $g\in\pi_\sigma^{\mathcal{G}}(V)$.

For $\sigma=0$ properties 1-3 are obvious. Let $\sigma=(\sigma_0,\dots,\sigma_{n-1}|0)\in ST$ and for σ_i, $i<n$; suppose these properties hold. Before proving property 1 we indicate the following criterion for the compatibility of two functions of C_σ :

Let $\sigma=(\sigma_0,\dots,\sigma_{n-1}|0)\in ST$, h, and $h'\in C_\sigma=C_p(\prod_{i<n}C_{\sigma_i},S)$; then h and h' are compatible in C_σ if and only if for any $h_i\in C_{\sigma_i}$, $i<n$, provided $h(\bar h)(=h(h_0,\dots,h_{n-1}))$ and $h'(\bar h)$ are defined and the elements $h(\bar h)$ and $h'(\bar h)$ of S are compatible in S .

Since

$$C_p(\prod_{i<n}C_{\sigma_i},S)\approx C(\prod_{i<n}C_{\sigma_i},F_0(S)),$$

the assertion obviously holds. We note only that when h and h' are compatible, $h\cup h'$ is defined as follows:

$$[h\cup h'](\bar h)=\begin{cases} h(\bar h)\cup h'(\bar h), & \text{if } h(\bar h) \text{ and } h'(\bar h) \text{ are defined;}\\ h(\bar h), & \text{if } h(\bar h) \text{ is defined and } h'(\bar h) \text{ is not defined.}\\ h'(\bar h), & \text{if } h(\bar h) \text{ is not defined and } h'(\bar h) \text{ is defined;}\\ \text{is not defined if } h(\bar h) \text{ and } h'(\bar h) \text{ are not defined.} \end{cases}$$

We return to the proof of property 1. Assume that h and h' are compatible for some $g \in G_\sigma$, $h, h' \in q_\sigma(g)$. Then there are $h_i \in C_{\sigma_i}$, $i<n$ such that $h(\bar{h})$ and $h'(\bar{h})$ are defined and not compatible. Put $s \rightleftharpoons h(\bar{h})$, $s' \rightleftharpoons h'(\bar{h})$. Then $s = sup\{s_0 \mid s_0 \in S_0, s_0 \preccurlyeq s\}$, $s' = sup\{s_0' \mid s_0' \in S_0, s_0' \preccurlyeq s'\}$. If, for any $s_0, s_0' \in S_0$ such that $s_0 \preccurlyeq s$, $s_0' \preccurlyeq s'$, s_0 and s_0' were compatible, the set $S' \rightleftharpoons \{s_0 \cup s_0' \mid s_0, s_0' \in S_0, s_0 \preccurlyeq s, s_0' \preccurlyeq s'\}$ would be a directed set and, obviously, $sup\, S' \geq s, s'$, which is impossible since s and s' are not compatible. Consequently, there are $s_0, s_0' \in S_0$ such that $s_0 \preccurlyeq s$, $s_0' \preccurlyeq s'$ and s_0 and s_0' are not compatible; i.e., $\check{s}_0 \cap \check{s}_0' = \emptyset$. Since h and h', are continuous, we can find formal neighborhoods $V_i \in B^*_{\sigma_i}$, $i<n$, such that $h_i \in \pi^{\mathbf{C}}_{\sigma_i}(V_i)$ and

$$h \in \langle \prod_{i<n} \pi^{\mathbf{C}}_{\sigma_i}(V_i), \check{s}_0 \rangle,$$

while

$$h' \in \langle \prod_{i<n} \pi^{\mathbf{C}}_{\sigma_i}(V_i), \check{s}_0' \rangle.$$

Since the $\pi^{\mathbf{C}}_{\sigma_i}(V_i)$ are nonempty (they contain the h_i) and since $\mathbf{G}$ is fertile, it follows that the $\pi^{\mathbf{G}}_{\sigma_i}(V_i)$ are nonempty. Let $g_i \in \pi^{\mathbf{G}}_{\sigma_i}(V_i)$, $i<n$. By the induction hypothesis we have

$$h_i' \in q_{\sigma_i}(g_i) \cap \pi^{\mathbf{C}}_{\sigma_i}(V_i), \quad i<n.$$

But then $g(\bar{g}) \geq h(\bar{h}') \in \check{s}_0$, $g(\bar{g}) \geq h'(\bar{h}') \in \check{s}_0'$. Consequently, $g(\bar{g}) \in \check{s}_0 \cap \check{s}_0'$. This contradicts the condition $\check{s}_0 \cap \check{s}_0' = \emptyset$. We have proved that h and h' are compatible. That $h \cup h' \in q_\sigma(g)$ easily follows from the description of $h \cup h'$ given above and the definition of $q_\sigma(g)$. Property 1 is proved.

We turn to the proof of property 2. Let $g \in G_\sigma$. By property 1 the set $q_\sigma(g)$ is directed; let $\mu_\sigma(g) \rightleftharpoons sup\, q_\sigma(g)$. It is easily verified that $\mu_\sigma(g) \in q_\sigma(g)$ (for this we have only to note that the sup of the set of functions is computed pointwise, i.e., $(sup\, F)(\bar{h}) = \{sup\, h(\bar{h}) \mid h \in F\}$). We can show that if $V \in B^*_\sigma$ and $g \in \pi^{\mathbf{G}}_\sigma(V)$, then $\mu_\sigma(g) \in \pi^{\mathbf{C}}_\sigma(V)$. It is sufficient to prove this for V of the form $\langle V_0 \times \ldots \times V_{n-1}, C_{s_0} \rangle$, where $V_i \in B^*_{\sigma_i}$, $i<n$. If $g \in \pi^{\mathbf{G}}_\sigma(V)$, then

$$g \in \langle \pi^{\mathbf{G}}_{\sigma_0}(V_0) \times \ldots \times \pi^{\mathbf{G}}_{\sigma_{n-1}}(V_{n-1}), \check{s}_0 \rangle.$$

Put $V_i' \rightleftharpoons \pi^{\mathbf{G}}_{\sigma_i}(V_i)$, $V_i'' \rightleftharpoons \pi^{\mathbf{C}}_{\sigma_i}(V_i)$. Consider the function h', defined as follows:

$$h'(\bar{h}) \rightleftharpoons \begin{cases} s_0, & \text{if } \bar{h} \in V_0'' \times \ldots \times V_{n-1}''; \\ \text{is not defined otherwise.} \end{cases}$$

We note that h' is the least function in $\pi^{\mathbf{C}}_\sigma(\langle V_0 \times \ldots \times V_{n-1}, C_{s_0} \rangle) = \langle \pi^{\mathbf{C}}_{\sigma_0}(V_0) \times \ldots \times \pi^{\mathbf{C}}_{\sigma_{n-1}}(V_{n-1}), \pi^{\mathbf{C}}_0(C_{s_0}) \rangle = \langle V_0'' \times \ldots \times V_{n-1}'', \check{s}_0 \rangle$. We can show that $h' \in q_\sigma(g)$. Let $g_i \in G_{\sigma_i}$, $h_i \in q_{\sigma_i}(g_i)$, $i<n$, be arbitrary. If $\bar{h} = (h_0, \ldots, h_{n-1}) \notin V_0'' \times \ldots \times V_{n-1}''$, then $h'(\bar{h})$ is not defined and, of course, $h'(\bar{h}) \preccurlyeq g(\bar{g})$. If $\bar{h} \in V_0'' \times \ldots \times V_{n-1}''$, i.e., $h_i \in \pi^{\mathbf{C}}_{\sigma_i}(V_i)$, $i<n$, by property 3 we have $g_i \in \pi^{\mathbf{G}}_{\sigma_i}(V_i)$, $i<n$, and so $g(\bar{g}) = g(g_0, \ldots, g_{n-1}) \in \check{s}_0$ and $g(\bar{g}) \geq s_0 = h'(\bar{h})$. Since $\mu_\sigma(g) = sup\, q_\sigma(g)$ and $h' \in q_\sigma(g)$, we have $\mu_\sigma(g) \geq h'$ and $\mu_\sigma(g) \in \pi^{\mathbf{C}}_\sigma(V) = \langle V_0'' \times \ldots \times V_{n-1}'', \check{s}_0 \rangle$. Property 2 is proved.

We now prove property 3. Let $g \in G_\sigma$, $V \in B^*_\sigma$ and $h \in q_\sigma(g) \cap \pi^{\mathbf{C}}_\sigma(V)$. As above it is sufficient to consider the case when V has the form $\langle V_0 \times \ldots \times V_{n-1}, C_{s_0} \rangle$, $V_i \in B^*_{\sigma_i}$, $s_0 \in S_0$. Let V_i', V_i'', $i<n$,

be as above. Let the $g_i \in V_i'$, $i<n$, be arbitrary. By property 2, there can be found $h_i \in \varphi_{\sigma_i}(g_i) \cap V_i''$, $i<n$. Then $h(\bar{h}) \in \check{s}_0$, i.e., $h(\bar{h}) \geq s_0$. But $g(\bar{g}) \geq h(\bar{h}) \geq s_0$, consequently, $g \in \langle V_0' \times \ldots \times V_{n-1}', \check{s}_0 \rangle = \pi_\sigma^{\mathfrak{G}}(V)$. Property 3 is proved.

Let us now define $\mu_\sigma : G_\sigma \longrightarrow C_\sigma'$ as above in the proof of property 2. We can show that $M_S \rightleftharpoons \{\mu_\sigma \mid \sigma \in ST\}$ is a morphism from $\mathfrak{G}_S$ into $\mathfrak{C}_S$; i.e., for any $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | 0) \in ST$ and any $g \in G_\sigma$, $g_i \in G_{\sigma_i}$, $i<n$, we have

$$g(g_0, \ldots, g_{n-1}) = [\mu_\sigma(g)](\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1})).$$

Since $\mu_\sigma(g) \in \varphi_\sigma(g)$, $\mu_{\sigma_i}(g_i) \in \varphi_{\sigma_i}(g_i)$, $i<n$, we have

$$g(g_0, \ldots, g_{n-1}) \geq [\mu_\sigma(g)](\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1})).$$

Let us prove the converse inequality. If $g(g_0, \ldots, g_{n-1})$ is not defined, there is nothing to prove. If $g(g_0, \ldots, g_{n-1})$ is defined and equal to s, suppose $s_0 \in S_0$ and $s_0 \leq s$. Then there are $V_i \in B_{\sigma_i}^*$, $i<n$, such that $g_i \in \pi_{\sigma_i}^{\mathfrak{G}}(V_i)$ and $g \in \pi_\sigma^{\mathfrak{G}}(\langle \prod_{i<n} V_i, C_{s_0} \rangle)$. By property 2, there can be found $h_i \in \varphi_{\sigma_i}(g_i) \cap \pi_{\sigma_i}^{\mathfrak{C}}(V_i)$, $i<n$, and

$$h \in \varphi_\sigma(g) \cap \pi_{\sigma_i}^{\mathfrak{C}}(\langle \prod_{i<n} V_i, C_{s_0} \rangle).$$

But then $h(h_0, \ldots, h_{n-1}) \geq s_0$. Since $\mu_\sigma(g) \geq h$, $\mu_{\sigma_i}(g_i) \geq h_i$, $i<n$, we have $[\mu_\sigma(g)](\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1})) \geq s_0$. Consequently, $[\mu_\sigma(g)](\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1})) \geq \sup\{s_0 \mid s_0 \in S_0, s_0 \leq g(g_0, \ldots, g_{n-1})\} = g(g_0, \ldots, g_{n-1})$.

The theorem is proved.

It appears that all the mappings μ_σ $\sigma \in ST$ defined in the proof of Theorem 1 are continuous. At once we have proved the more precise assertion:

THEOREM 2. The family of mappings $M_S = \{\mu_\sigma \mid \sigma \in ST\}$, constructed in the proof of Theorem 1, is such that, for any $\sigma \in ST$, there is a continuous mapping $\mu_\sigma^* : G_\sigma^* \rightarrow C_\sigma$ and mappings $\rho_\sigma : C_\sigma \rightarrow G_\sigma^*$ such that $\mu_\sigma^* | G_\sigma = \mu_\sigma$, $\rho_\sigma \cdot \mu_\sigma^* = id_{G_\sigma^*}$ and $\mu_\sigma^*(\rho_\sigma(h)) \geq h$ for any $h \in C_\sigma$.

Proof. If $\sigma = 0$, then $G_\sigma = G_\sigma^* = S$ and all the mappings are identity mappings. Suppose $\sigma = (\sigma_0, \ldots, \sigma_{n-1}\ 0) \in ST$. The mapping $\mu_\sigma^* : G_\sigma^* \longrightarrow C_\sigma$ is defined as follows: for $g \in G_\sigma^* = C_p(\prod_{i<n} G_{\sigma_i}, S)$, $\mu_\sigma^*(g) \rightleftharpoons \sup\{h \mid h \in C_\sigma$ and for any $g_i \in G_{\sigma_i}$, $i<n$, $h_i \in \varphi_{\sigma_i}(g_i)$, $i<n$, $h(h_0, \ldots, h_{n-1}) \leq g(g_0, \ldots, g_{n-1})\}$. We see from the definition that $\mu_\sigma^* | G_\sigma = \mu_\sigma$. Let us prove that μ_σ^* is continuous. Let $V = \langle V_0 \times \cdots \times V_{n-1}, C_{s_0} \rangle \in B_\sigma^*$ and $\mu_\sigma^*(g) \in \pi_\sigma^{\mathfrak{C}}(V)$. We can show that $g \in \pi_\sigma^{\mathfrak{G}^*}(V)$ and if $g' \in \pi_\sigma^{\mathfrak{G}^*}(V)$, then $\mu_\sigma^*(g') \in \pi_\sigma^{\mathfrak{C}}(V)$. Indeed, if $g_i \in \pi_{\sigma_i}^{\mathfrak{G}}(V_i)$, $i<n$, as shown in the proof of property 2 of Theorem 1, $\mu_{\sigma_i}(g_i) \in \pi_{\sigma_i}^{\mathfrak{C}}(V)$, and hence $g(g_0, \ldots, g_{n-1}) \geq \mu_\sigma^*(g)(\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1})) \geq s_0$. Let

$$g' \in \pi_\sigma^{\mathfrak{G}^*}(V) - \langle \pi_{\sigma_0}^{\mathfrak{G}}(V_0) \times \ldots \times \pi_{\sigma_{n-1}}^{\mathfrak{G}}(V_{n-1}), \check{s}_0 \rangle,$$

Then, as in the proof of Theorem 1, it can be verified that, for h', the least function of the neighborhood $\langle \pi_{\sigma_0}^{\mathfrak{C}}(V_0) \times \ldots \times \pi_{\sigma_{n-1}}^{\mathfrak{C}}, \check{s}_0 \rangle$, we have $h' \leq \mu_\sigma^*(g')$, from which it at once follows that $\mu_\sigma^*(g') \in \pi_\sigma^{\mathfrak{C}}(V)$. We have proved that μ_σ^* is continuous.

We define the mapping $\rho_\sigma : C_\sigma \rightarrow G^*_\sigma$ as follows: for $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid 0)$, $h \in C_\sigma$, $g_i \in G_{\sigma_i}$, $i < n$, we have $[\rho_\sigma(h)]\,(g_0, \ldots, g_{n-1}) \leftrightharpoons h(\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1}))$.

Clearly, $\rho_\sigma(h) \in M_p(\prod_{i<n} G_{\sigma_i}, S)$. We can show that $\rho_\sigma(h) \in C_p(\prod_{i<n} G_{\sigma_i}, S) = G^*_\sigma$. Let $\rho_\sigma(h)(g_0, \ldots, g_{n-1}) \in \check{s}_0$, then $h(\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1})) \in \check{s}_0$ and, since h is continuous, there are neighborhoods $V_i \in \mathfrak{G}^*_{\sigma_i}$ such that $\mu_{\sigma_i}(g_i) \in \pi^{\mathbb{C}}_{\sigma_i}(V_i)$ and $h \in \langle \pi^{\mathbb{C}}_{\sigma_0}(V_0) \times \ldots \times \pi^{\mathbb{C}}_{\sigma_{n-1}}(V_{n-1}), \check{s}_0 \rangle$. Now if

$$h' \in \langle \pi^{\mathbb{C}}_{\sigma_0}(V_0) \times \ldots \times \pi^{\mathbb{C}}_{\sigma_{n-1}}(V_{n-1}), \check{s}_0 \rangle,$$

and $g'_i \in \pi^{\mathbb{G}}_{\sigma_i}(V_i)$, we have $\mu_{\sigma_i}(g'_i) \in \pi^{\mathbb{C}}_{\sigma_i}(V_i)$ and $[\rho_\sigma(h')](g'_0, \ldots, g'_{n-1}) = h'(\mu_{\sigma_0}(g'_0), \ldots, \mu_{\sigma_{n-1}}(g'_{n-1})) \in \check{s}_0$. Thus, we have proved that $\rho_\sigma(h)$ is continuous.

We now prove that $\rho_\sigma \circ \mu^*_\sigma = id_{G^*_\sigma}$. Let $g \in G^*_\sigma$; then, as in the proof of Theorem 1, it can be shown that

$$\mu^*_\sigma(g)(\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1})) = g(g_0, \ldots, g_{n-1})$$

for any $g_i \in G_{\sigma_i}, i < n$. But this implies that $\rho_\sigma \mu^*_\sigma(g) = g$, since $\rho_\sigma[\mu^*_\sigma(g)]\,(g_0, \ldots, g_{n-1}) = \mu^*_\sigma(g)(\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1})) = g(g_0, \ldots, g_{n-1})$ for all $g_i \in G_{\sigma_i}$, $i < n$.

It remains to prove that $\mu^*_\sigma(\rho_\sigma(h)) \geq h$. For $\sigma = 0$ it is obvious. Suppose the relation holds for σ_i, $i < n$, and let $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid 0)$ and $h \in C_\sigma$; then, for any $g_i \in G_{\sigma_i}$, $i < n$, and $h_i \in \varphi_{\sigma_i}(g_i)$, $i < n$, we can show that

$$h(h_0, \ldots, h_{n-1}) \leq \rho_\sigma h(g_0, \ldots, g_{n-1}).$$

Indeed, $\rho_\sigma h(g_0, \ldots, g_{n-1}) = h(\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1}))$, but $h_i \in \varphi_{\sigma_i}(g_i) \Rightarrow h_i \leq \mu_{\sigma_i}(g_i)$; hence, $(h(h_0, \ldots, h_{n-1}) \leq h(\mu_{\sigma_0}(g_0), \ldots, \mu_{\sigma_{n-1}}(g_{n-1}))$, and so $h \leq \mu^*_\sigma \rho_\sigma h$. The theorem is proved.

Using the terminology of Sec. 1, we can say that under the conditions of Theorem 1, $\mu^*_\sigma \rho_\sigma$ is the closure of C_σ $(\sigma \in ST)$.

Note 1. We can try to extend the S-morphism M_S, constructed in Theorem 1, to a morphism M from $\mathbb{G}$ into $\mathbb{C}$ by putting $\mu_\sigma \leftrightharpoons S_{S(\sigma),\sigma} \circ \mu_{S(\sigma)} \circ S_{\sigma, S(\sigma)}$ for $\sigma \in T$. In general the family of mappings $M = \{\mu_\sigma \mid \sigma \in T\}$ thus defined is not a morphism from $\mathbb{G}$ into $\mathbb{C}$.

Note 2. The mappings ρ_σ are not in general continuous, although we see from the proof that the mapping

$$\rho'_\sigma : C_\sigma \times G_\sigma \rightarrow S,$$

defined as follows: $\rho'_\sigma(f, \bar{g}) \leftrightharpoons [\rho_\sigma(f)]\,(\bar{g})$, is continuous.

We now define a class $\mathcal{D}$ of functionals which are continuous and defined everywhere over S (the topology on $\mathcal{D}_\sigma$, $\sigma \in ST$ is introduced in such a way that $\mathcal{D}$ is compatible with the topology of S) as follows:

1) $\mathcal{D}_0 \leftrightharpoons S$.

2) If $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid 0)$, then $\mathcal{D}_\sigma \leftrightharpoons C(\prod_{i<n} \mathcal{D}_{\sigma_i}, S)$.

Thus, we have defined the special part $\mathcal{D}_S$; the whole class $\mathcal{D}$ is defined as a λ-model with standard part $\mathcal{D}_S$. We indicate some conditions on S which ensure that λ-model $\mathcal{D}$ is fertile.

PROPOSITION 2. The class $\mathcal{D}$ of functionals which are continuous and defined everywhere over $\mathcal{S}$ is fertile if one of the following conditions holds:

1) $\mathcal{S}$ is a discrete topological space;

2) $\mathcal{S}$ is an f_0 -space.

Proof. It is easier to verify that condition 2 of Proposition 1 holds. Let $\mathcal{S}$ be a discrete topological space; then we prove that any basis neighborhood in $\mathcal{D}_\sigma$ is openly closed for any $\sigma \in ST$. Indeed, for $\sigma = 0$ this is obvious. Let $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid 0)$ and $V = \langle V_0 \times \ldots \times V_{n-1}, \check{s} \rangle$, where $V_i \in B_{\sigma_i}$, $s \in \mathcal{S} (= \mathcal{S}_0)$, then

$$\mathcal{D}_\sigma \setminus V = \bigcup \left\{ \langle V_0' \times \ldots \times V_{n-1}', \check{s}' \rangle \mid V_i' \in B_{\sigma_i}, \left[\prod_{i<n} V_i\right] \cap \left[\prod_{i<n} V_i'\right] \neq \emptyset, s' \neq s \right\}.$$

Indeed, the inclusion $\supseteq$ is obvious. Let $f \in \mathcal{D}_\sigma \setminus V$; then there is an $\bar{f} \in \prod_{i<n} V_i$ such that $f(\bar{f}) = s' \neq s$; since f is continuous there can be found $V_i' \in B_{\sigma_i}$, $i < n$, such that $\bar{f} \in \prod_{i<n} V_i'$ and $f \in \langle \prod_{i<n} V_i', \check{s}' \rangle$. This proves the converse assertion. Thus, all the basis neighborhoods of $\mathcal{D}_\sigma$ are openly closed. The right side of the equivalence in the condition of Proposition 1 can be reduced, in the case of a discrete space, to the following condition: if $V_i^j \in B_{\sigma_i}$, $i < n$, $s_j \in \mathcal{S}$, $j < k$, then

$$\left[\prod_{i<n} V_i^j\right] \cap \left[\prod_{i<n} V_i^{j'}\right] \neq \emptyset \Rightarrow s_j = s_{j'}$$

for $j \neq j' < k$. Then the function $f : \prod_{i<n} \mathcal{D}_{\sigma_i} \longrightarrow \mathcal{S}$, defined as follows:

$$f(\bar{f}) \leftrightharpoons \begin{cases} s_j & , \text{ if } \bar{f} \in \prod_{i<n} V_i^j, \\ s_0 & , \text{ if } \bar{f} \notin \bigcup_{j<k} \left[\prod_{i<n} V_i^j\right], \end{cases}$$

is correctly defined, is continuous (since the neighborhoods $\prod_{i<n} V_i^j$ are openly closed), and obviously belongs to the neighborhood $\bigcap_{j<k} \langle \prod_{i<n} V_i^j, \check{s}_j \rangle$. Thus, condition 2 of Proposition 1 holds and, consequently, $\mathcal{D}$ is a fertile λ -model when $\mathcal{S}$ is a discrete space. That condition 2 of Proposition 1 is valid when $\mathcal{S}$ is an f_0-space follows at once from the obvious observation that every partial continuous mapping with open domain of definition from any topological space X into an f_0 -space can be continued to a continuous mapping, defined everywhere. The proposition is proved.

We now make some observations on the "structure" $\mathcal{D}^*$. If $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid 0) \in ST$, then $\mathcal{D}_\sigma^* = C_p(\prod_{i<n} \mathcal{D}_{\sigma_i}, \mathcal{S})$ (by definition). If $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n)$ is an arbitrary type and $\sigma_n \neq 0$, we have $\mathcal{D}_\sigma^* \subseteq M(\prod_{i<n} \mathcal{D}_{\sigma_i}, \mathcal{D}_{\sigma_n}^*)$.

LEMMA. If $\sigma = (\sigma_0, \ldots, \sigma_{n-1} \mid \sigma_n) \in T$, $\sigma_n \neq 0$, then $f \in \mathcal{D}_\sigma^*$, for $f \in \mathcal{D}_\sigma$, if and only if $\bar{f} \in \prod_{i<n} \mathcal{D}_{\sigma_i}$, for $f(\bar{f}) \in \breve{\mathcal{D}}_{\sigma_n}$, i.e., $\mathcal{D}_\sigma = \mathcal{D}_\sigma^* \cap M(\prod_{i<n} \mathcal{D}_{\sigma_i}, \mathcal{D}_{\sigma_n})$.

Proof. If $S(\sigma_n) = (\sigma_0', \ldots, \sigma_{k-1}' \mid 0)$, we have $\sigma \sim \sigma' = (\sigma_0, \ldots, \sigma_{n-1}, \sigma_0', \ldots, \sigma_{k-1}' \mid 0)$. We note that $\mathcal{D}_\tau = C(\prod_{i<k} \mathcal{D}_{\tau_i}, \mathcal{S})$ and $\mathcal{D}_\tau^* = C_p(\prod_{i<k} \mathcal{D}_{\tau_i}, \mathcal{S})$ for $\tau = (\tau_0, \ldots, \tau_{k-1} \mid 0)$ (as in the case of the special type τ). Suppose now that $f \in \mathcal{D}_\sigma$; then $f' \leftrightharpoons S_{\sigma,\sigma'}(f) \in \mathcal{D}_{\sigma'}$, and hence $\bar{f} \in \prod_{i<n} \mathcal{D}_{\sigma_i}$

$$\lambda f_0', \ldots, \lambda f_{k-1}' \; f'(\bar{f}, f_0', \ldots, f_{k-1}'),$$

for any $\in \mathcal{D}_{S(\sigma_n)}$), is a function defined everywhere, i.e., $f(\bar{f}) = S_{S(\sigma_n),\sigma_n}(\lambda f_0', \ldots, \lambda f_{k-1}' \; f'(\bar{f}, \bar{f}')) \in \mathcal{D}_{\sigma_n}$. The converse is proved in exactly the same way (all the implications used above can be reversed, i.e., they are equivalences). The lemma is proved.

6. Indexed Sets with Approximations

As distinct from the previous sections, where the considerations were purely topological, beginning with this section we shall consider indexed sets. To understand what follows it is necessary to be familiar with [1]. In the last section of [1] the concept of the approximation of an indexed set was defined. It is more convenient to make a certain modification (essentially an extension) of this concept, for which we retain the former name.

An approximation of the indexed set $\gamma = (S, \nu)$ is a subobject (γ_0, μ) of this object for which the following conditions hold:

1) The predicate $R(x,y) \rightleftharpoons \{\langle x,y\rangle \mid \mu\nu_0 x \leq_\nu \nu y\}$ is recursively enumerable.

2) If S', S'' are two ν-fully enumerable subsets of S and $S' \not\subseteq S''$, there is an $s_0 \in S_0$, such that $\mu(s_0) \in S' \setminus S''$.

We note some properties of the concept we have introduced.

LEMMA 1. If (γ_0, μ) is an approximation of the separable indexed set γ, then γ_0 is a positively indexed set.

Proof. Since μ is a monomorphism,

$$\nu_0(x) = \nu_0(y) \Longleftrightarrow \mu\nu_0(x) = \mu\nu_0(y) \Longleftrightarrow (\mu\nu_0 x \leq_\nu \mu\nu_0 y \;\&\; \mu\nu_0 y \leq_\nu \mu\nu_0 x).$$

This predicate is recursively enumerable (if h is a generally recursive function such that $\mu\nu_0 = \nu h$, then $\mu\nu_0 x \leq_\nu \mu\nu_0 y \Longleftrightarrow R(x, h(y))$). The lemma is proved.

Note. If F_{0T} is the separability functor, defined in [1] from the fact that (γ_0, μ) is an approximation of γ it easily follows that $(F_{0T}(\gamma_0), F_{0T}(\mu))$ is an approximation of $F_{0T}(\gamma)$. Consequently, Lemma 1 can be reformulated as follows:

LEMMA 1'. If (γ_0, μ) is an approximation of γ, $F_{0T}(\gamma_0)$ is a positively indexed set.

In what follows we shall assume that all the enumerated sets we discuss are separable.

PROPOSITION 1. If (γ_0, μ) is an approximation of γ, the following assertions are obvious:

1) For any ν-fully enumerable set S' we have $S' = \{s \mid \exists s_0 \in S_0 \ (\mu s_0 \in S' \;\&\; \mu s_0 \leq_\nu s)\}$.

1'). If S' is ν-fully enumerable, $s \in S'$, there is an $s_0 \in S_0$ such that $\mu s_0 \in S'$ and $\mu s_0 \leq_\nu s$.

2) If $s, s' \in S$ and $s \not\leq_\nu s'$, there is an $s_0 \in S_0$, such that $\mu s_0 \leq_\nu s$ and $\mu s_0 \not\leq_\nu s'$.

3) If $\bar{s} \in S$, $s_0, s_1 \in S_0$, $\mu s_0 \leq_\nu \bar{s}$ and $\mu s_1 \leq \bar{s}$, there is an $s_2 \in S_0$, such that $\mu s_0 \leq_\nu \bar{s}$, $\mu s_0 \leq_\nu \mu s_2$ and $\mu s_1 \leq_\nu \mu s_2$.

Proof. 1) Let S' be ν-fully enumerable. Put $S'' \rightleftharpoons \{s \mid \exists s_0 \in S_0 \ (\mu s_0 \in S' \;\&\; \mu s_0 \leq s)\}$. It easily follows from the definition of an approximation that S'' is a ν-fully enumerable subset of S. Further, obviously $S'' \subseteq S'$. Assume that $S' \not\subseteq S''$. Then, by condition 2), there is an $s_0 \in S_0$, such that $\mu s_0 \in S' \setminus S''$. Since $\mu s_0 \in S'$, we have $\mu s_0 \in S''$, by the definition of the latter. This contradiction proves the assertion.

1') It is easy to see that 1) and 1') are equivalent.

2) If $s \not\leq_\nu s'$, there is a ν-fully enumerable set S' such that $s \in S'$ and $s' \notin S'$. By 1') there can be found an $s_0 \in S_0$ such that $\mu s_0 \in S'$ and $\mu s_0 \leq_\nu s$. Then $\mu s_0 \in S'$ implies that $\mu s_0 \not\leq_\nu s'$. Thus, $\mu s_0 \leq_\nu s$ and $\mu s_0 \not\leq_\nu s$. The assertion is proved.

3) Let $\bar{s} \in S$, $s_0, s_1 \in S_0$ be such that $\mu s_0 \leq_\nu \bar{s}$ and $\mu s_1 \leq_\nu \bar{s}$. Consider the sets $S' \rightleftharpoons \{s \mid \mu s_0 \leq_\nu s\}$, $S'' \rightleftharpoons \{s \mid \mu s_0 \leq_\nu s\}$. These sets are ν-fully enumerable, and so $\bar{S} \rightleftharpoons S' \cap S''$ is also ν-fully enumer-

able and $\bar{s} \in \bar{S}$. By 1') there can be found an $s_2 \in S_0$ such that $\mu s_2 \in \bar{S}$ and $\mu s_2 \le_\nu \bar{s}$. But $\mu s_2 \in \bar{S} = S' \cap S''$. Hence, $\mu s_0 \le \mu s_2$ and $\mu s_1 \le \mu s_2$. This completes the proof of 3) and of the proposition.

The most important corollary of this proposition is formulated separately.

PROPOSITION 2. Every indexed set has not more than one approximation. More precisely, if (γ_0, μ_0) and (γ_1, μ_1) are two approximations for γ, these subobjects are equal (equivalent).

Proof. We can show that there is a morphism $\mu: \gamma_0 \to \gamma$ such that the diagram

$$\begin{tikzcd} & \gamma & \\ \gamma_0 \arrow[ur, "\mu_0"] \arrow[rr, "\mu"] & & \gamma_1 \arrow[ul, "\mu_1"'] \end{tikzcd}$$

is commutative.

Let $s_0 \in S_0$. Consider the set $S' \rightleftharpoons \{s \mid s \in S, \mu_0 s_0 \le_\nu s\}$. This set is ν-fully enumerable. Hence, there is an $s_1 \in S_1$ such that $\mu_1 s_1 \in S'$ and $\mu_1 s_1 \le_\nu \mu_0 s_0$. Then, obviously, $\mu_0 s_0 = \mu_1 s_1$. Put $\mu s_0 \rightleftharpoons s_1$. This specifies a mapping μ from S_0 into S_1. This is indeed a mapping since μ_1 is a monomorphism, and so there is not more than one $s_1 \in S_1$ such that $\mu_0 s_0 = \mu_1 s_1$. Clearly, for any $n \in N$, we can effectively find $m \in N$ such that $\mu_0 \nu_0(n) \le_\nu \mu_1 \nu_1(m)$ and $\mu_1 \nu_1(m) \le_\nu \mu_0 \nu_0(n)$ (we note that the last two relations are recursively enumerable). This implies that μ is a morphism from γ_0 into γ_1. By the symmetry of the situation, there is an inverse mapping $\check{\mu}: S_1 \to S_0$ which is also a morphism. The proposition is proved.

Note. It is time to establish the connection between the concepts discussed here and the concepts in the preceding sections of the paper. If γ is a (separable) indexed set, we can introduce a topology on the basic set S defined by the basis consisting of all the ν-fully enumerable subsets. This is a separable topology (the separability of the topology is equivalent to the separability of the indexed set γ). If (γ_0, μ) is an approximation for γ, elements of the form μs_0 and only such are f-elements of the topological space S. This explains the uniqueness of the approximation.

The following proposition describes all the fully enumerable subsets of an indexed set with an approximation.

PROPOSITION 3. Let (γ_0, μ) be an approximation for the indexed set γ. The nonempty subset $S' \subseteq S$ is ν-fully enumerable if and only if there is a subobject (γ_1, μ_1) of the enumerated set γ_0 such that

$$s \in S' \Longleftrightarrow \exists s_1 \; (\mu \mu_1 s_1 \le_\nu s).$$

This is Proposition 14 of [1].

If (γ_0, μ) is an approximation for γ, the relation π, defined on the elements of S_0 as follows: $\pi(s_0, s_1) \Longleftrightarrow \mu s_0 \le_\nu \mu s_1$, is, obviously, a partial order relation on S_0. Of this order (we shall frequently denote it simply by $\le$) we shall say that it is induced by the order $\le_\nu$ on S_0.

LEMMA 2. The induced order is recursively enumerable. More precisely, if (γ_0, μ) is an approximation for γ, the predicate

$$\Pi(x, y) \rightleftharpoons \pi(\nu_0 x, \nu_0 y) (\Longleftrightarrow \nu_0 x \le \nu_0 y) \Longleftrightarrow \mu \nu_0 x \le_\nu \mu \nu_0 y$$

is recursively enumerable.

Proof. The proof follows directly from the definition of an approximation.

Note. The "imprecise" formulation of Lemma 2 becomes precise if we accept the general definition: an n-place predicate $R \subseteq S^n$ defined on the basis set of the indexed set $\gamma = (S, \nu)$ is said to be recursive (recursively enumerable) if the set n-ок $\{\langle x_0, \ldots, x_{n-1} \rangle \mid \langle \nu x_0, \ldots, \nu x_{n-1} \rangle \in R\}$ is recursive (recursively enumerable).

PROPOSITION 4. If (γ,μ_0) is an approximation for γ_0 and (γ,μ_1) an approximation for γ_1, and the induced orders coincide, there is not more than one morphism $\mu:\gamma_0\longrightarrow\gamma_1$ such that the diagram

$$\begin{tikzcd} \gamma_0 \arrow[rr, "\mu"] & & \gamma_1 \\ & \gamma \arrow[ul, "\mu_0"] \arrow[ur, "\mu_1"'] & \end{tikzcd}$$

is commutative. If there is such a morphism, it is a monomorphism.

Proof. Let $\mu:\gamma_0\longrightarrow\gamma_1$ be a morphism such that $\mu_0\mu=\mu_1$. Then

$$\{s \mid s\in\mathcal{S} \text{ and } \mu_0 s\preccurlyeq_{\nu_0} s_0\}=\{s \mid s\in\mathcal{S} \text{ and } \mu_1 s\preccurlyeq_{\nu_1}\mu s_0\}$$

for any $s_0\in\mathcal{S}_0$.

Indeed, $\mu_0 s\preccurlyeq_{\nu_0} s_0$ implies that $\mu\mu_0 s\preccurlyeq_{\nu_1}\mu s_0$, since the morphism is monotonic. But $\mu\mu_0=\mu_1$, hence, the inclusion $\subseteq$ is proved. Conversely, let $s_0\in\mathcal{S}_0$ and $s\in\mathcal{S}$ be such that $\mu_1 s\preccurlyeq_{\nu_1}\mu s_0$. Let $\mathcal{S}_1'\rightleftharpoons\{s_1 \mid s_1\in\mathcal{S}_1 \text{ and } \mu_1 s\preccurlyeq_{\nu_1} s_1\}$. $\mathcal{S}_1'$ is ν_1-fully enumerable; then $\mathcal{S}_0'\rightleftharpoons\mu^{-1}(\mathcal{S}_1')$ is ν_0-fully enumerable and $s_0\in\mathcal{S}_0'$. By 1') of Proposition 1, there can be found an $\bar{s}\in\mathcal{S}$, such that $\mu_0\bar{s}\preccurlyeq_{\nu_0} s_0$ and $\mu_0\bar{s}\in\mathcal{S}_0'$. But $\mu_0\bar{s}\in\mathcal{S}_0'$ implies that $\mu\mu_0\bar{s}\in\mathcal{S}_1'$, i.e., $\mu_1 s\preccurlyeq_{\nu_1}\mu\mu_0\bar{s}=\mu_1\bar{s}$. But since $\mu_1 s\preccurlyeq_{\nu_1}\mu_1\bar{s}\Longleftrightarrow\mu_0 s\preccurlyeq_{\nu_0}\mu_0\bar{s}$ we have $\mu_0 s\preccurlyeq_{\nu_0}\mu_0\bar{s}\preccurlyeq_{\nu_0} s_0$. The inclusion $\supseteq$, and with it, the equality, is proved. Since, by 1') of Proposition 1, the element $s_0\in\mathcal{S}_0$ $(s_1\in\mathcal{S}_1)$ is completely defined by the set $\{s \mid s\in\mathcal{S},\ \mu_0 s\preccurlyeq_{\nu_0} s_0\}$ $(\{s \mid s\in\mathcal{S},\ \mu_1 s\preccurlyeq_{\nu_1} s_1\})$, the two assertions of the proposition thus follow.

PROPOSITION 5. Let (γ_0,μ_0) be an approximation for γ, (γ',μ) the principal subobject of γ such that $\mu(\mathcal{S}')\supseteq\mu_0(\mathcal{S}_0)$. Then the (unique) morphism $\mu_0':\gamma_0\longrightarrow\gamma'$ is such that $\mu_0=\mu\mu_0'$ makes a pair (γ_0,μ_0') with the approximation for γ' and this approximation induces on $\mathcal{S}_0$ the same order.

Proof. The existence of a morphism $\mu_0':\gamma_0\longrightarrow\gamma'$, such that $\mu_0=\mu\mu_0'$ follows from Proposition 1 on p. 50 of [9]. The uniqueness of this morphism is obvious. Let f be a generally recursive function such that $\mu\nu'=\nu f$. Then

$$\mu_0'\nu_0 x\preccurlyeq_{\nu'}\nu' y\Longleftrightarrow\mu\mu_0'\nu_0 x\preccurlyeq_{\nu}\mu\nu' y\Longleftrightarrow\mu\nu_0 x\preccurlyeq_{\nu}\nu f(y).$$

This is recursively enumerable. Condition 1) of the definition of an approximation holds. Let $\bar{\mathcal{S}}'$ and $\bar{\mathcal{S}}''$ be two ν'-fully enumerable subsets of $\mathcal{S}'$; then $\mu_0'^{-1}(\bar{\mathcal{S}}')$ and $\mu_0'^{-1}(\bar{\mathcal{S}}'')$ are ν_0-fully enumerable subsets of $\mathcal{S}_0$. Put $\mathcal{S}^0\rightleftharpoons\{s \mid s\in\mathcal{S}$; there is an $s_0\in\mu_0'^{-1}(\bar{\mathcal{S}}'),\mu_0 s_0\preccurlyeq_{\nu} s\}$ and $\mathcal{S}^1\rightleftharpoons\{s \mid s\in\mathcal{S}$; there is an $s_0\in\mu_0'^{-1}(\bar{\mathcal{S}}'')$, $\mu_0 s_0\preccurlyeq_{\nu} s\}$. $\mathcal{S}^0$ and $\mathcal{S}^1$ are ν-fully enumerable and, as is easily verified, $\mu(\bar{\mathcal{S}}')=\mu(\mathcal{S}')\cap\mathcal{S}^0$, $\mu(\bar{\mathcal{S}}'')=\mu(\mathcal{S}')\cap\mathcal{S}^1$. Consequently, if $\bar{\mathcal{S}}'\not\subseteq\bar{\mathcal{S}}''$, then $\mathcal{S}^0\not\subseteq\mathcal{S}^1$ and there is an $s_0\in\mathcal{S}_0$ such that $\mu_0 s_0\in\mathcal{S}^0\setminus\mathcal{S}^1$. We now note that $\mu_0' s_0\in\bar{\mathcal{S}}'\setminus\bar{\mathcal{S}}''$. From this follows condition 2) of the definition of an approximation. The last assertion of the proposition is obvious. The proposition is proved.

The indexed set γ_0 is said to be complete over the approximation (γ,μ_0), if, for any indexed set γ_1 and approximation (γ,μ_1) such that both induced orders on $\mathcal{S}$ coincide, there is a morphism $\mu:\gamma_1\longrightarrow\gamma_0$ such that the diagram

$$\begin{tikzcd} \gamma_1 \arrow[rr, "\mu"] & & \gamma_0 \\ & \gamma \arrow[ul, "\mu_1"] \arrow[ur, "\mu_0"'] & \end{tikzcd}$$

is commutative.

PROPOSITION 6. If $\langle\gamma,\preccurlyeq\rangle$ is a pair consisting of a positively enumerated set γ and a recursively enumerable partial order $\preccurlyeq$ on $\mathcal{S}$, there is an indexed set γ (and it is unique to within equivalence over γ_0) and a morphism $\mu_0 : \gamma \longrightarrow \gamma_0$ such that (γ, μ_0) is an approximation for γ_0, the order induced on S by the order $\preccurlyeq_{\nu_0}$ coincides with $\preccurlyeq$., and γ_0 is complete over the approximation (γ, μ_0).

Proof. The recursively enumerable set $R \subseteq N$ is said to be compatible if it is nonempty and the following conditions hold:

1) if $x \in R$ and $x \sim_\nu y$, then $y \in R$;

2) if $x \in R$ and $\nu y \preccurlyeq \nu x$ (i.e., $\Pi(y, x)$), then $y \in R$;

3) if $x, y \in R$, there is a $z \in R$ such that $\nu x \preccurlyeq \nu z$ and $\nu y \preccurlyeq \nu z$.

We prove an auxiliary assertion.

LEMMA 3. The class of all compatible sets is a wn-subset of the indexed set $\Pi = (Pn, \pi)$.

Proof. We indicate an effective method of constructing a compatible set R^* from any nonempty nonrecursively enumerable set R.

Let $R_0 \subseteq R_1 \subseteq \ldots$ be a strictly computable sequence of finite sets such that $R = \bigcup_{i \in N} R_i$. Let Π_i be the predicate computed after i steps of an effective computation of the predicate Π (everywhere we assume that $\langle x, x \rangle \in \Pi_i$ and that Π_i is transitive).

Let x_0 be the least element of R_0 such that if $\Pi_0(x_0, y)$, then $\Pi_0(y, x_0)$ for any $y \in R_0$ (then $\nu x_0 = \nu y$). Let $x_0, \ldots, x_{n-1}$ already have been defined. We assume x_n is the least element of R_n such that $\Pi_n(x_{n-1}, x_n)$ and if $\Pi_n(x_n, y)$, then $\Pi_n(y, x_n)$ for any $y \in R_n$. The sequence $x_0, x_1, \ldots, x_n, \ldots$ has been effectively constructed. We note that if $m \preccurlyeq n$, then $\nu x_m \preccurlyeq \nu x_n$. And if R is a compatible set, for any $x \in R$ there can be found an $n \in N$ such that $\Pi_n(x, x_n)$. Then we put $R^* \rightleftharpoons \bigcup_{n \in N} \{ y \mid \Pi(y, x_n) \}$. From the definition we see at once that R^* is a compatible set and that if R is compatible, $R^* = R$. It follows from the above considerations that there is a partially recursive function g such that the domain of definition δg consists of all Post numbers of nonempty sets for any $n \in \rho g$, π_n is a compatible set, and if π_k is a compatible set, then $\pi_{g(k)} = \pi_k$. This implies that the class of all compatible sets forms a wn-subset of Π. The lemma is proved.

Let S_0 be the class of all compatible sets, and ν_0 the principal computable numeration of this family (it exists, since S_0 is a wn-subset of Π). We now define the mapping $\mu_0 : S \longrightarrow S_0$ as follows: if $s \in S$, then

$$\mu_0(s) \rightleftharpoons \{ y \mid \nu y \preccurlyeq s \}.$$

It is easy to verify that $s \in S$ is a compatible set for all $\mu_0(s)$ and that the mapping μ_0 is a morphism, even a monomorphism from γ into γ_0. Thus, (γ, μ_0) is a subobject of γ. We can show that it is an approximation. We note that $\mu_0 \nu x \preccurlyeq_{\nu_0} \nu_0 y \Longleftrightarrow x \in \nu_0(y)$ since the relation $\preccurlyeq_{\nu_0}$ coincides with the inclusion relation. Consequently, the relation $\mu_0 \nu x \preccurlyeq_{\nu_0} \nu_0 y$ is recursively enumerable, and condition 1) of the definition of an approximation holds. We now verify condition 2). Let S_0', S_0'' be two ν_0-fully enumerable subsets of S_0 and let $S_0' \nsubseteq S_0''$. It easily follows from the general properties of principal indexed families of recursively enumerable sets that any fully enumerable family is closed with respect to extensions. Let $R \in S_0' \setminus S_0''$; we can find x such that $x \in R$ and $\mu_0 \nu x \in R_0'$. Indeed, we can construct a numeration $\bar{\nu}$ of a family of compatible sets $\bar{S}$ such that $R \in \bar{S}$ and if $R' \in \bar{S} \setminus \{R\}$, then $R' = \mu_0 \nu x$ for $x \in R$ such that $\bar{\nu}^{-1}(R)$ is the complement to a recursively enumerable set of the nonrecursive set P. The construction is as follows: let $P_0 \subseteq P_1 \subseteq \ldots$ $(R_0 \subseteq R_1 \subseteq \ldots)$ be a strongly computable sequence of finite sets such that

$$P = \bigcup_{i \in N} P_i \qquad (R = \bigcup_{i \in N} R_i);$$

we put $\bar{\nu}(\kappa) \leftrightharpoons \{y \mid \nu y \leqslant \nu x_n\} = \mu_0 \nu x_n$, if $\kappa \in P_n \setminus P_{n-1}$ and x_n is the first number in order of computation of the set R such that we know that $\Pi(x, x_n)$ for all $x \in R_n$, and we put $\bar{\nu}(\kappa) \leftrightharpoons R$ if $x \notin P$. Then $\bar{\gamma} = (\bar{S}, \bar{\nu})$ is a subobject of γ_0, and S_0' must contain elements of $\bar{S}$ distinct from R. Thus, we have found an x such that $\mu_0 \nu x \in S_0'$. If $\mu_0 \nu x \in S_0''$, then $\mu_0 \nu x \subseteq R$ would imply that $R \in S_0''$, which is not so. The second condition for an approximation has been verified.

Let (γ, μ_1) be an approximation for γ_1 which induces on S the order $\leqslant$. For any $s \in S_1$, we put $\mu(s) \leftrightharpoons \{x \mid \mu_1 \nu x \leqslant_\nu s\}$. It is easy to verify (using assertion 3 of Proposition 1) that $\mu(s)$ is a compatible set, it being effectively constructed in accordance with the number of the element s. This shows that μ is a morphism from γ_1 into γ_0 which, obviously, is a morphism over γ. The proposition is proved.

It was shown in [10] and in [9] that the concept of a separable indexed set, when the set is finite is an abstract characteristic of finite families of recursively enumerable sets with computable numerations. The proposition proved above shows that the concept of a set which is complete over an approximation is also a sufficient condition for its representability [9, Sec. 9] as a subobject of Π. The theorem which follows indicates the exact result on the representability of a set with an approximation by subobjects of Π.

THEOREM. If the indexed set γ has an approximation, it is isomorphic with a subobject of Π. Such a γ is isomorphic with a $w\pi$-subobject of Π if and only if γ is complete over the approximation.

Proof. Let (γ_0, μ) be an approximation for γ. The mapping $\bar{\mu}: S \to P_n$ is defined as follows: for $s \in S$

$$\bar{\mu}(s) \leftrightharpoons \{x \mid \mu \nu_0 x \leqslant_\nu s\}.$$

It follows from Proposition 1 that $\bar{\mu}$ is a one-to-one mapping and it easily follows from condition 1) of the definition of an approximation that the numeration $\bar{\mu}\nu$ of the set $\bar{\mu}(S)$ is computable (since $x \in \bar{\mu}\nu y \Longleftrightarrow \mu\nu_0 x \leqslant_\nu \nu y = R(x, y)$). Thus, the first assertion of the theorem has been proved ($(\gamma = (S, \nu) \approx (\bar{\mu}(S), \bar{\mu}\nu))$). The sufficiency of the second assertion was proved in Proposition 6. Let us now prove necessity.

Let $S_1 \subseteq P_n$ and let ν_1 be a principal computable numeration of S_1, $\bar{\mu}: \gamma \to \gamma_1 = (S_1, \nu_1)$ an isomorphism. Then $\mu_0 = \bar{\mu}\mu : \gamma_0 \to \gamma_1$ is a morphism from γ_0 into γ_1. Since ν_1 is a principal computable enumeration, $R_0 \leqslant_{\nu_1} R_1 \Longleftrightarrow R_0 \subseteq R_1$ for $R_0, R_1 \in S_1$. Since $\bar{\mu}$ is an isomorphism, it follows that $s_0 \leqslant s_0' \Longleftrightarrow \mu_0 s_0 \subseteq \mu_0 s_0'$ for $s_0, s' \in S_0$. Let γ' and $\mu': \gamma_0 \to \gamma'$ be such that (γ_0, μ') is an approximation for γ' inducing on γ_0 the order $\leqslant$. For $x \in N$ we can effectively find a sequence $x_0, x_1, \ldots$ such that

1) $\mu'\nu_0 x_\kappa \leqslant_{\nu'} \nu' x$ for all $\kappa \in N$; 2) $\nu_0 x_\kappa \leqslant \nu_0 x_{\kappa+1}$ for all $\kappa \in N$; 3) for any $n \in N$, if $\mu'\nu_0 n \leqslant_{\nu'} \nu' x$, there is a κ such that $\nu_0 n \leqslant \nu_0 x_\kappa$.

Then the sequence of sets $R_0 \leftrightharpoons \mu_0 \nu_0 x_0, R_1 \leftrightharpoons \mu_0 \nu_0 x_1, \ldots, R_\kappa \leftrightharpoons \mu_0 \nu_0 x_\kappa$ is such that $R_i \in S_1$, $R_0 \subseteq R_1 \subseteq \ldots \subseteq R_\kappa \subseteq R_{\kappa+1} \subseteq \ldots$; this sequence is computable. We put

$$P_x \leftrightharpoons \bigcup_{\kappa \in N} R_\kappa \quad , \quad x \in N;$$

P_x is a recursively enumerable set which is effectively constructed with respect to x. We note, further, that P_x depends only on $\nu' x$, i.e., $\nu' x = \nu' y \Rightarrow P_x = P_y$. Moreover, it follows from 3) and Proposition 1 that $P_x = P_y \Rightarrow \nu' x = \nu' y$. Thus, γ' is isomorphic with the indexed set $(\mathcal{P}, \nu^*)$, where $\mathcal{P} \leftrightharpoons \{P_0, P_1, \ldots\}$ and $\nu^* \kappa \leftrightharpoons P_\kappa$ for $\kappa \in N$. Now ν^* is a computable enumeration.

Since ν_1 is a principal computable enumeration of the family S_1, and $\{R_i, i \in N\}$ is an effective increasing sequence of elements of S_1, by Lachlan's theorem [11], the union of this sequence belongs to S_1.

Thus, $P \subseteq S_1$ and $(\mathcal{P}, \nu^*)$ is a subobject of γ_1. Consider the diagram

$$\begin{tikzcd} & \gamma_0 \arrow[r, phantom, "\approx"] & \gamma_1 \\ \gamma \arrow[ur, "\mu"] \arrow[dr, "\mu'"'] & & \\ & \gamma' \arrow[uu, dashed, "\mu_0'"] \arrow[r, phantom, "\approx"] & (\mathcal{P}, \nu^*) \arrow[uu, "i"'] \end{tikzcd}$$

where the lower isomorphism is established by the mapping $\nu' x \rightsquigarrow P_x$ and i is the inclusion of $\mathcal{P}$ in S_1. It can be verified without particular difficulty that this diagram is commutative. Thus, it at once follows that there is a morphism $\mu_0' : \gamma' \to \gamma_0$ such that $\mu_0' \mu' = \mu$. This shows that γ is complete over the approximation. The theorem is proved.

COROLLARY. The indexed set γ has an approximation isomorphic with a principal subobject of Π if and only if γ is complete over the approximation.

7. Conditions for the Solvability of the Problem P

In this section we show that the fundamental discussions in Secs. 5 and 6 of [1] about the solvability of the problem P can be argued if we replace sn-subobjects of Π by indexed sets, complete over an approximation (the latter are the analog of the wider concept of wsn-subobjects which is obtained if, in the definition of an sn-subobject, we replace the condition that the function g is generally recursive by the condition that it is partially recursive.) Most of the assertions which follow are not proved since the proofs are obtained by almost word-for-word repetition of the corresponding proofs in [1].

Let (γ_0, μ_0) be an approximation for γ, (γ_1, μ_1) an approximation for γ' and let γ' be complete over the approximation. We denote the induced order on $S_0\,(S_1)$ by $\preccurlyeq_0\,(\preccurlyeq_1)$. Under these conditions we have:

PROPOSITION 1. The mapping $\mu : S \to S'$ is a morphism from γ into γ' if and only if the following conditions hold:

1. μ is monotonic; i.e., $s_0 \preccurlyeq_\gamma s_1 \Rightarrow \mu s_0 \preccurlyeq_{\gamma'} \preccurlyeq_{\gamma'} \mu s_1$ for $s_0 s_1 \in S$.

2. For any $s \in S$, μs is an exact upper bound (with respect to the order $\preccurlyeq_{\gamma'}$ on S') of the set $\{\mu\mu_0 s_0 \mid \mu_0 s_0 \preccurlyeq_\gamma s\}$.

3. The set of pairs $\Delta_\mu \leftrightharpoons \{\langle x, y\rangle \mid \mu_1 \nu_1 x \preccurlyeq_{\gamma'} \mu\mu_0 y\}$ is recursively enumerable.

Proof. The proof is similar to the proof of Proposition 9 of [1].

PROPOSITION 2. The mapping $\nu : N \to Mor(\gamma, \gamma')$ is computable if and only if the sequence of recursive enumerable sets $\{\Delta_{\nu(i)} \mid i \in N\}$ is computable.

This corresponds to Proposition 10 of [1].

PROPOSITION 3. The recursively enumerable set of pairs Δ has the form Δ_μ for some morphism from γ into γ' if and only if the following conditions hold:

1. For any $x \in N$ there is a $y \in N$ such that $\langle x, y\rangle \in \Delta$.

2. If $\langle x, y\rangle \in \Delta$, $\nu_0 x \preccurlyeq_0 \nu_0 x'$, $\nu_1 y' \preccurlyeq_1 \nu_1 y$, then $\langle x', y'\rangle \in \Delta$.

3. If $\langle x, y_0\rangle, \langle x, y_1\rangle \in \Delta$, there is a z such that $\langle x, z\rangle \in \Delta$ and $\nu_1 y_0 \preccurlyeq_1 \nu_1 z$, $\nu_1 y_1 \preccurlyeq_1 \nu_1 z$.

This corresponds to Lemma 10 of [1].

THEOREM 1. The problem P for the pair (γ, γ') can be solved if and only if the family of all recursively enumerable sets Δ, satisfying conditions 1-3 of Proposition 3, has a principal computable numeration.

This corresponds to Theorem 6 of [1] and follows directly from Propositions 2 and 3.

COROLLARY. If $P(\gamma, \gamma')$, is the $\mathbf{\Delta}$-family mentioned in Theorem 1, with principal computable numeration, $\mathfrak{Mor}\,(\gamma, \gamma') \approx \mathbf{\Delta}$.

This is a corollary of Proposition 2 and Theorem 1.

We now formulate a number of definitions.

We shall say that the indexed set γ has the properties:

C_0, if γ has an approximation;

C_1, if γ has an approximation (γ_0, μ) such that the general compatibility predicate for the order $\leq$ on S_0 induced by $\leq_\gamma$ is recursive;

C_2, if γ has an approximation (γ_0, μ) such that $(\gamma_0, \leq)$ is a constructive poset.

Note. We do not require here that γ_0 is solvable, only that the compatibility predicate (with respect to $\leq$) is recursive and that the partial operation U^* is partially recursive [the latter implies that there is a two-place partially recursive function g such that if $\nu_0 x$ and $\nu_0 y$ are compatible, then $g(x,y)$ is defined and $\nu_0 g(x,y) = \nu_0 x U^* \nu_0 y$]. These notes also concern the definition of a constructive YCA (cf. the following property).

C_3, if γ has an approximation (γ_0, μ) such that $(\gamma_0, \leq)$ is a constructive YC.

Instead of saying that the "indexed set γ has the property C_i " we shall say briefly that " γ is a C_i -indexed set" and sometimes more briefly that " $\gamma \in C_i$."

If γ is a C_i -indexed set, we shall say that γ is a C_{i0} -indexed set if the approximation for γ has a least element, $i \leq 3$.

If γ ; is a $C_i(C_{i0})$ —indexed set, we shall say that γ is a $C_i^*(C_{i0}^*)$ – indexed set if γ is complete over the approximation, $i \leq 3$.

We note some simple relations between the properties we have introduced.

1) If $i \geq j$, $C_i \Longrightarrow C_j$, $C_{i0} \Longrightarrow C_{j0}$, $C_{i0} \Longrightarrow C_i$; $C_i^* \Longrightarrow C_i$, $C_{i0}^* \Longrightarrow C_{i0}$, $i \leq 3$;

2) If γ is a $C_i(C_i^*)$ -indexed set, γ' a $C_j(C_j^*)$ -indexed set, $\kappa = min\{i,j\}$, then $\gamma \oplus \gamma'$ and $\gamma \times \gamma'$ are $C_\kappa(C_\kappa^*)$ indexed sets.

3) If γ is a $C_{i0}(C_{i0}^*)$ -indexed set, γ' a $C_{j0}(C_{j0}^*)$ -indexed set, $\kappa = min\{i,j\}$, then $\gamma \times \gamma'$ and is a $C_{\kappa 0}(C_{\kappa 0}^*)$ -indexed set.

4) If γ is a $C_i(C_i^*)$ -indexed set, then $F_\pi(\gamma)$ is a $C_{i0}(C_{i0}^*)$ -indexed set.

PROPOSITION 4. If $\gamma \in C_i\ (C_{i0}, C_i^*, C_{i0}^*)$, $i \leq 3$, and γ' is a closed retract of γ , then $\gamma' \in C_i\ (C_{i0}, C_i^*, C_{i0}^*)$.

Proof. Let (γ_0, μ) be an approximation for γ , $\bar{\mu}: \gamma \to \gamma'$ and $\iota: \gamma' \longrightarrow \gamma$ morphism achieving retraction; i.e., $\bar{\mu}\iota = id_S$, closedness implies that for any $s \in S$ we have $\iota\bar{\mu}s_\gamma \geq s$. Consider the equivalence relation $\sim$ over S_0 defined as follows: $s_0 \sim s_1 \Longleftrightarrow \bar{\mu}\mu s_0 = \bar{\mu}\mu s_1$. Let $S_0' = S_0/_\sim$ and $\gamma_0' = \gamma_0/_\sim = \gamma_0/_{\sim\bar{\mu}\mu}$ (cf. [9], p. 61) and let the mapping $\mu': S_0' \longrightarrow S'$ be defined as follows: $\mu'([s_0]_\sim) \leftrightharpoons \bar{\mu}\mu s_0$ is a morphism from γ_0' into γ' . We can show that (γ_0', μ') is an approximation for γ' . Let g be a single-place generally recursive function such that $\iota\nu' = \nu g$. Then for any $x, y \in N$ we have

$$\mu'\nu_0'(x) \leq_{\nu'} \nu'(y) \Longleftrightarrow \bar{\mu}\mu\nu_0(x) \leq_{\nu'} \nu'(y) \Longleftrightarrow$$
$$\Longleftrightarrow \mu\nu_0(x) \leq_\nu (\nu'(y)) = \nu g(y).$$

The last equivalence follows from the fact that the retract is closed; if $s_0 \in S_0$, $s' \in S$, then $\bar{\mu}\mu s_0 \leq_{\nu'} s' \Longleftrightarrow \iota\bar{\mu}\mu s_0 \leq_\nu \iota s' \Rightarrow \mu s_0 \leq_\nu \iota s'$; conversely, $\mu s_0 \leq_\nu \iota s \Rightarrow \bar{\mu}\mu s_0 \leq_{\nu'} \bar{\mu}\iota s' = s'$. Hence, $\bar{\mu}\mu s_0 \leq_{\nu'} \leq_{\nu'} s' \Longleftrightarrow \mu s_0 \leq_{\nu'} \iota s'$. The predicate $\mu\nu_0(x) \leq_\nu \nu g(y)$ is recursively enumerable ((γ_0, μ) is an approximation for γ). Hence, (γ_0', μ') is an approximation for γ' . We note also that if the $[s_0]_\sim, \ldots, [s_{n-1}]_\sim$ are compatible in $\langle S_0', \leq' \rangle$, then $s_0, \ldots, s_{n-1}$ are compatible in $\langle S_0, \leq \rangle$. Indeed, for $s_0 \in S_0$ and $s' \in S'$ it was shown above that $\mu'[s_0]_\sim \leq_{\nu'} s' \Longleftrightarrow \mu s_0 \leq_\nu \iota s'$. Hence the compatibility of $[s_0]_\sim, \ldots, [s_{n-1}]_\sim$ implies that there is an $s' \in S'$, such that $\mu'[s_i]_\sim \leq_{\nu'} s'$, $i < n$, but then, $\mu s_i \leq_\nu \iota s'$, $i < n$, and, by Proposition 1 of Sec. 6, the $s_0,\dots,s_{n-1}$ are compatible in $\langle S_0,\leqslant\rangle$. The converse is obvious; hence, the $[s_0]_\sim,\dots,[s_{n-1}]_\sim$ are compatible in $\langle S_0',\leqslant'\rangle \Longleftrightarrow s_0,\dots,s_{n-1}$ compatible in $\langle S_0,\leqslant\rangle$. It is also easily verified that if $\langle S_0,\leqslant\rangle$ is a poset, $\langle S_0',\leqslant'\rangle$ is also a poset and $[s_0]_\sim \cup^* [s_1]_\sim = [s_0 \cup^* s_1]_\sim$. The assertions of the proposition follow from these observations for $C_i\ (C_{i0})$, $i \leqslant 3$. From the theorem of the previous section and the fact that the retract of a wn-subobject of Π is isomorphic with a wn-subobject of Π, the assertions of the proposition follow for $C_i^*\ (C_{i0}^*)$, $i \leqslant 3$. The proposition is proved.

We now formulate the sufficient conditions for the solvability of the problem P and the properties of the solution in the form of a theorem.

THEOREM 2. Let γ and γ' be indexed sets. The problem P can be solved for the pair (γ,γ') if at least one of the following conditions holds:

1. $\gamma \in C_0$, but $\gamma' \in C_{30}^*$
2. $\gamma \in C_1$, but $\gamma' \in C_{20}^*$
3. $\gamma \in C_2$, but $\gamma' \in C_{00}^*$

If one of these conditions holds, then, when

condition 1 holds, we have $\mathfrak{Mor}(\gamma,\gamma') \in C_{30}^*$;

condition 2 holds, we have $\mathfrak{Mor}(\gamma,\gamma') \in C_{20}^*$;

condition 3 holds, we have $\mathfrak{Mor}(\gamma,\gamma') \in C_{00}^*$;

if also $\gamma' \in C_{10}^*$, then $\mathfrak{Mor}(\gamma,\gamma') \in C_{10}^*$.

Proof. The proof of the theorem is similar to the proof of Theorem 7 of [1]. However, the formulation is stronger than Theorems 7 and 9 of [1]; hence, as an illustration we give the proof of the fundamental theorem for the case when condition 2 holds. This case is only important for the remaining part of this paper.

Let (γ_0,μ_0) be an approximation for γ, $\leqslant_0$ the order on S_0 induced by the order $\leqslant_\nu$; let (γ_1,μ_1) be an approximation for γ', $\leqslant_1$ the order on S_1 induced by the order $\leqslant_{\nu'}$. The condition $\gamma \in C_1$ implies that for any finite set of natural numbers $\{x_0,\dots,x_{n-1}\}$ we can effectively know whether or not there is an $x \in N$, such that $\nu_0(x_i) \leqslant_0 \nu_0(x)$ for all $i < n$. The condition $\gamma' \in C_{20}^*$ implies that 1) there is an $x \in N$ such that $\nu_1 x \leqslant_1 \nu_1 y$ for all $y \in N$; without loss of generality for the sequel we can assume that $\nu_1 0 \leqslant_1 \nu_1 y$ for all $y \in N$; 2) for any pair $x_0, x_1 \in N$, we can effectively know whether the elements $\nu_1 x_0$ and $\nu_1 x_1$ are compatible with respect to the order $\leqslant_1$, i.e., whether there is an x_2 such that $\nu_1 x_0 \leqslant_1 \nu_1 x_2$, $\nu_1 x_1 \leqslant_1 \nu_1 x_2$; 3) there is a two-place partially recursive function g such that if $\nu_1 x_0$ and $\nu_1 x_1$ are compatible, $g(x_0,x_1)$ is defined and $\nu_1 g(x_0,x_1)$ is the exact upper bound for the elements $\nu_1 x_0$ and $\nu_1 x_1$ in $\langle S_1,\leqslant_1\rangle$ 4) γ' is complete over (γ_1,μ_1).

The finite set of pairs $A = \{\langle x_i, y_i\rangle \mid i < n\}$ is said to be permissible if the following conditions hold: that the set $\{\nu_0(x_i) \mid i \in I\}$, $I \subseteq \{0,1,\dots,n-1\}$

(*) is compatible in $\langle S_0,\leqslant_0\rangle$ implies that the set $\{\nu_1(y_i) \mid i \in I\}$ is compatible in $\langle S_1,\leqslant_1\rangle$.

Let F be the family of all finite permissible sets. It easily follows from the condition on $\langle \gamma_0,\leqslant_0\rangle$ and $\langle \gamma_1,\leqslant_1\rangle$ that F is a strongly recursive family since the property of being a permissible set can be verified effectively. Let ν^* be a strong numeration of F, i.e., such that $\nu^* \leqslant \gamma$ (here γ is a standard numeration of finite sets [9]).

The infinite set R is said to be permissible if any finite subset of it is permissible. The family $\mathcal{D}$ of all recursively enumerable permissible sets is obviously an sn-subset of Π; let δ be the corresponding numeration of $\mathcal{D}$ $(\pi_x \in \mathcal{D} \Longrightarrow \pi_{\delta(x)} = \pi_x \ \&\ \forall y\,(\pi_{\delta(y)} \subseteq \nu i_y))$. The imbedding i of the family F in $\mathcal{D}$ is obviously a morphism from $\mathbb{F} \rightleftharpoons (F,\nu^*)$ into $\vartheta \rightleftharpoons (\mathcal{D},\delta)$. Moreover, it is easily verified that $(\mathbb{F}, i)$ is an approximation for ϑ and that ϑ is complete over the approximation. We note that $\vartheta \in C_{20}^*$. Indeed, the induced order $\leqslant$ on F coincides with the inclusion relation for sets of F. If $M_0, M_1 \in F$, then M_0 and M_1 are compatible if and only if $M_0 \cup M_1 \in F$. If $M_0 \cup M_1 \in F$, this is the exact upper bound for M_0 and M_1. Since F is strongly recursive, $\langle \mathbb{F}, \subseteq \rangle$ is a constructive poset, $\phi \in F$ is the least element. Thus, $\vartheta \in C_{20}^*$. For any $R \in \mathcal{D}$ we now define the set $\Delta(R)$ as follows: $\Delta(R)$ is the least of the sets of pairs Δ which satisfy the following conditions:

1) $R \subseteq \Delta$;

2) $\langle x, y \rangle \in \Delta$ & $\nu_0 x \preccurlyeq_0 \nu_0 u$ & $\nu_1 v \preccurlyeq_1 \nu_1 y \Longrightarrow \langle u, v \rangle \in \Delta$;

3) $\langle x, 0 \rangle \in \Delta$ for all $x \in N$ (it should be noted that $\nu_1 0$ is the least element in $\langle S_1, \preccurlyeq_1 \rangle$);

4) $\langle x, y_0 \rangle, \langle x, y_1 \rangle \in \Delta$ & $\{\nu_1 y_0, \nu_1 y_1$ are compatible in $\langle S_1, \preccurlyeq_1 \rangle\}$ & $z = g(y_0, y_1) \Longrightarrow \langle x, z \rangle \in \Delta$.

It is easy to see from the definition that $\Delta(R)$ is recursively enumerable, and its Post number can be effectively determined from the Post number of R (or the δ number of R).

As in the proof of Theorem 7 of [1], it can be verified that $\Delta(R)$ is a permissible set and that the following conditions hold: $\Delta(\Delta(R)) = \Delta(R)$ and

1) For any $x \in N$ we have $\langle x, 0 \rangle \in \Delta(R)$ [this follows from 3)].

2) If $\langle x, y \rangle \in \Delta(R)$, $\nu_0 x \preccurlyeq_0 \nu_0 x'$, $\nu_1 y' \preccurlyeq_1 \nu_1 y$, then $\langle x', y' \rangle \in \Delta(R)$ [this follows from 2)].

3) If $\langle x, y_0 \rangle, \langle x, y_1 \rangle \in \Delta$, there is a z such that $\langle x, z \rangle \in \Delta(R)$ and $\nu_1 y_0 \preccurlyeq_1 \nu_1 z$, $\nu_1 y_1 \preccurlyeq_1 \nu_1 z$ [this follows from a) and the permissibility of $\Delta(R)$].

Conversely, if $R \in \mathcal{D}$ and conditions 1-3 hold for R (with R in place of $\Delta(R)$), then $\Delta(R) = R$. From this it follows that $\Delta(\mathcal{D})$ is an z-subset of ϑ, $\Delta(\vartheta)$ (the corresponding indexed set) is a retract, and obviously ϑ is closed. By Theorem 1 the problem P can be solved for the pair (γ, γ') by Proposition 4, $\Delta(\vartheta) \in C_{20}^*$ and by the Corollary to Theorem 1, $\mathfrak{mor}(\gamma, \gamma') \in C_{20}^*$. Theorem 1 is proved (for case 2).

Note 1. If instead of C_i (C_{i0}), $i \leq 3$, we introduce the more restricted class C_i^+ (C_{i0}^+), and require of the approximation that the indexed set γ_0 be solvable, we can occasionally sharpen Theorem 2, for example:

If $\gamma \in C_2^+$, $\gamma' \in C_{20}^{+*}$, then $P(\gamma, \gamma')$ and $\mathfrak{mor}(\gamma, \gamma') \in C_{20}^{+*}$.

To prove this refinement we have to consider the family F_0 of finite exact permissible sets of pairs of the form $A = \{\langle x_i, y_i \rangle \mid i < n\}$, where A is permissible and the following conditions hold:

a) If $\langle x, y \rangle, \langle x, y' \rangle \in A$, then $y = y'$;

b) If $\langle x_i, y_i \rangle, \langle x_j, y_j \rangle \in A$ and x_i, x_j are compatible, then there is a z such that $\langle g_0(x_i, x_j), z \rangle \in A$;

c) If $\nu_0 x_i \preccurlyeq_0 \nu_0 x_j$, then $\nu_1 y_i \preccurlyeq_1 \nu_1 y_j$.

Assume that the numerations ν_0 and ν_1 are unique; we can assert that $((F_0, \nu_0^*), \Delta)$ is an approximation for $\Delta(\vartheta)$.

Note 2. If we number all the positively indexed sets equipped with recursively enumerable partial orders in a reasonable way, Theorem 2 can be formulated in a more exact manner: when one of the conditions of the theorem holds, from the numbers of the approximations for γ and γ' we can effectively find the number of the approximation $\mathfrak{mor}(\gamma, \gamma')$ (similarly for the more exact structure-constructive poset for the properties C^+, etc.).

As in [1], we can extend the results of Theorem 2 by introducing new classes of indexed sets as follows: the indexed set γ has the property $\mathcal{D}_\beta^\alpha$ $(\gamma \in \mathcal{D}_\beta^\alpha)$, if γ is the retract of an indexed set with the property C_β^α; here $\alpha = \phi, *$; $\beta = 0, 1, 2, 3, 00, 10, 20, 30$.

Then a corollary of Theorem 2 and the results in [1] is

THEOREM 3. Let γ and γ' be indexed sets. The problem P can be solved for the pair (γ, γ') if at least one of the following conditions holds:

1. $\mathcal{Y} \in \mathcal{D}_0$, but $\mathcal{Y}' \in \mathcal{D}^*_{30}$.

2. $\mathcal{Y} \in \mathcal{D}_1$, but $\mathcal{Y}' \in \mathcal{D}^*_{20}$.

3. $\mathcal{Y} \in \mathcal{D}_2$, but $\mathcal{Y}' \in \mathcal{D}^*_{00}$.

If one of these conditions holds, then, when

condition 1 holds, we have $\mathfrak{Mor}(\mathcal{Y}, \mathcal{Y}') \in \mathcal{D}^*_{30}$;

condition 2 holds, we have $\mathfrak{Mor}(\mathcal{Y}, \mathcal{Y}') \in \mathcal{D}^*_{20}$;

condition 3 holds, we have $\mathfrak{Mor}(\mathcal{Y}, \mathcal{Y}') \in C^*_{00}$;

if also $\mathcal{Y}' \in \mathcal{D}^*_{10}$, then $\mathfrak{Mor}(\mathcal{Y}, \mathcal{Y}') \in \mathcal{D}^*_{10}$.

In concluding this section we note a further condition on the solvability of the problem P .

PROPOSITION 5. Let $(\mathcal{Y}, \mu_0)$ be an approximation for $\mathcal{Y}_0$, $(\mathcal{Y}, \mu_1)$ an approximation for $\mathcal{Y}_1$, $\mathcal{Y}_1$ complete over the approximation; these approximations induce on S an order, and $\mu : \mathcal{Y}_0 \longrightarrow \mathcal{Y}_1$ is a morphism such that $\mu\mu_0 = \mu_1$ and $(\mathcal{Y}_0, \mu)$ is a principal subobject of $\mathcal{Y}_1$. If $\mathcal{Y}' \in C_0$ and $P(\mathcal{Y}', \mathcal{Y}_1)$, the problem P can be solved for the pair $(\mathcal{Y}', \mathcal{Y}_0)$ if and only if the image of the set $Mor(\mathcal{Y}', \mathcal{Y}_0)$ in $Mor(\mathcal{Y}', \mathcal{Y}_1)$ (under the mapping $Mor(id_{S'}, \mu)$) is a principal subobject in $\mathfrak{Mor}(\mathcal{Y}', \mathcal{Y}_1)$. If $P(\mathcal{Y}', \mathcal{Y}_0)$, then $(\mathfrak{Mor}(\mathcal{Y}', \mathcal{Y}_0), Mor(id_{S'}, \mu))$ is a principal subobject of $\mathfrak{Mor}(\mathcal{Y}', \mathcal{Y}_1)$.

8. κf -Spaces and Computable Functionals

In one of the notes in Sec. 6 there was reference to the connection between separable sets and a certain T_0 -topology. If S is a fundamental set of separably indexed sets, then in discussing the topology on S below we shall always have in mind the topology defined by the basis of all completely enumerable subsets.

Note. If $\mathcal{Y}$ and $\mathcal{Y}'$ are indexed sets and $\mu : \mathcal{Y} \longrightarrow \mathcal{Y}'$ is a morphism, μ is a continuous mapping of S into S' . This follows from the fact that the inverse image of a fully enumerable set is always completely enumerable (when μ is a morphism). Thus, $Mor(\mathcal{Y}, \mathcal{Y}') \subseteq C(S, S')$.

DEFINITION. The topological space S is said to be a $\kappa f (\kappa f_0)$ -space if there is a numeration $\nu : N \longrightarrow S$ of the set S such that $\mathcal{Y} \leftrightharpoons (S, \nu) \in C_2$ (C_{20}) and the topology defined by the numeration ν coincides with the original topology.

COROLLARY. A $\kappa f(\kappa f_0)$ -space is an $f(f_0)$ -space.

This is a corollary of Proposition 1 of Sec. 6, the notes in Sec. 6, and the definition.

In what follows by a κf -space we shall understand not simply a topological space S , satisfying the definition, but an S with a numeration ν , which specifies on S the topology of a κf -space. Thus, κf - spaces are indexed sets of class C_2 .

The $\kappa f(\kappa f_0)$ -space $\mathcal{Y}$ is said to be complete if it is complete over an approximation. Using the notation of the previous section, we see that $\mathcal{Y}$ is a complete $\kappa f(\kappa f_0)$-space if and only if $\mathcal{Y} \in C^*_2(C^*_{20})$.

Many of the facts which were proved for $f(f_0)$-spaces have corresponding analogs for $\kappa f(\kappa f_0)$ - spaces. We enumerate some of them.

THEOREM 1. Let $\mathcal{Y}$ be a κf -space, $\mathcal{Y}'$ a complete κf_0 -space. Then $P(\mathcal{Y}, \mathcal{Y}')$, and $\mathfrak{Mor}(\mathcal{Y}, \mathcal{Y}')$ is a complete κf_0 -space.

This is a simple corollary of Theorem 2 of Sec. 7.

THEOREM 2. If $\mathcal{Y}_0$, $\mathcal{Y}_1$ are κf (κf_0 , complete κf , complete κf_0)-spaces, $\mathcal{Y}_2$ a complete κf_0 - space, then

1) $\mathcal{Y}_0 \times \mathcal{Y}_1$ is a κf (κf_0 , complete κf , complete κf_0)-space;

2) $\mathfrak{Mor}(\mathcal{Y}_0 \times \mathcal{Y}_1, \mathcal{Y}_2) \approx \mathfrak{Mor}(\mathcal{Y}_0, \mathfrak{Mor}(\mathcal{Y}_1, \mathcal{Y}_2))$;

3) if $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are basically equivalent (i.e., equivalent over a common approximation), then $\mathfrak{Mor}(\mathcal{Y}_0, \mathcal{Y}_2) \approx \mathfrak{Mor}(\mathcal{Y}_1, \mathcal{Y}_2)$;

4) if $\mathcal{Y}_0$ is a complete κf -space, then $F_\pi(\mathcal{Y}_0)$ is a complete κf_0 -space.

This theorem easily follows from the considerations of the preceding section and the results of [1].

Let $\mathcal{Y}$ be a complete κf -space. We define the λ -model $\mathcal{C}^\kappa$ of (partial) computable functionals over $\mathcal{Y}$ as follows:

1) If $\sigma = 0$, then $C_0^\kappa \leftrightharpoons \mathcal{Y}$:

2) If $\sigma = (\sigma_0, \dots, \sigma_{n-1} \mid 0)$, then

$$C_\sigma^\kappa \leftrightharpoons \mathfrak{Mor}_p\big(\prod_{i<n} C_{\sigma_i}^\kappa, \mathcal{Y}\big) \big(\approx \mathfrak{Mor}\big(\prod_{i<n} C_{\sigma_i}^\kappa, F_\pi(\mathcal{Y})\big)\big);$$

3) If $\sigma = (\sigma_0, \dots, \sigma_{n-1} \mid \sigma_n),\ \sigma_n \neq 0$, then $C_\sigma^\kappa \leftrightharpoons \mathfrak{Mor}\big(\prod_{i<n} C_{\sigma_i}^\kappa, C_{\sigma_n}^\kappa\big)$.

THEOREM 3. If $\mathcal{Y}$ is a complete κf -space and the corresponding topology on S makes S a complete f -space, $\mathcal{C}^\kappa$ is a λ -model of functionals over S , densely consistent with the topology on S . Consequently, there is a morphism $K : \mathcal{C}^\kappa \to \mathcal{C} (= \mathcal{C}(S))$.

This theorem is a corollary of the considerations in the proof of Theorem 2 of Sec. 7. The construction of the approximation in $\mathfrak{Mor}(\mathcal{Y}, \mathcal{Y}')$ shows that $Mor(\mathcal{Y}, \mathcal{Y}')$ and $C(S, S')$ are basically equivalent, where S and S' are equipped with the topologies defined by the numerations ν and ν' and $Mor(\mathcal{Y}, \mathcal{Y}')$ is considered with the topology defined by the numeration in $\mathfrak{Mor}(\mathcal{Y}, \mathcal{Y}')$.

Note. The very definition of the model $\mathcal{C}^\kappa$ depends not only on the topology on S , but on the existence of the numeration ν .

A particular case to which Theorem 3 can be applied is the indexed set $\mathbb{N} = (N, id)$. The corresponding λ -model $\mathcal{C}^\kappa$ is called the class of all (partial) computable functionals of finite types. We note that $\mathcal{C}^\kappa$ coincides with the class $\{\mathbb{F}_\sigma \mid \sigma \in T\}$ of indexed sets defined at the end of [1]. It is fully justified to consider the class $\mathcal{C}^\kappa$ as the most natural generalization (more precisely, extension) of the class of of partially recursive functions (partial functionals of type $(0|0)$). We enumerate some of the most useful attributes of the class $\mathcal{C}^\kappa$: all functionals of a fixed type are equipped with a Gōbel (Kleene) numeration; both Kleene theorems on recursion hold; all the functionals are continuous and monotonic; the class is closed with respect to the operations of primitive recursion, bar recursion, etc.

We introduce the question: What in the higher types corresponds to the generalization of generally recursive functions?

There are at least three quite natural approaches to the definition of generally recursive functionals:

1. The inductive definition (for $\sigma \in ST,\ \sigma = (\sigma_0, \dots, \sigma_{n-1} \mid 0$) of the class $\tilde{C}_\sigma^\kappa \subseteq C_\sigma^\kappa$ of partial computable functions, everywhere defined on $\prod_{i<n} \iota$

This leads precisely to the class of effective operations [3].

The two other definitions use the following diagram of morphisms and mappings:

$$\begin{tikzcd}
\mathcal{C}^\kappa \arrow[dr, "K"'] & \mathcal{D}^* \arrow[d, squiggly, shift left, "R"] \arrow[d, squiggly, shift right, "M^*"'] & \supseteq \ \mathcal{D} \arrow[dl, "M"] \\
& \mathcal{C} &
\end{tikzcd}$$

Here $\mathcal{D}^* \leftrightharpoons \{\mathcal{D}_\sigma^* \mid \sigma \in T\}$, M , M^* , and R are families of mappings constructed in Theorems 1 and 2 of Sec. 5 and extended formally to all $\sigma \in T$: $\mu_\sigma^* \leftrightharpoons S_{s(\sigma),\sigma} \circ \mu_{s(\sigma)} \circ S_{\sigma, s(\sigma)} \cdots$.

2. The class $\mathcal{D}^\kappa$ is defined as follows: $\mathcal{D}^\kappa \leftrightharpoons M^{-1}(K(\mathcal{C}^\kappa))$.

3. The class $\bar{\mathscr{D}}^{\kappa}$ is defined as follows: $\bar{\mathscr{D}}^{\kappa} \rightleftharpoons R(K(\mathbb{C}^{\kappa})) \cap \mathscr{D}$.

We note that $\mathscr{D}^{\kappa} \subseteq \bar{\mathscr{D}}^{\kappa}$.

The question whether the class $\mathscr{D}^{\kappa}$ forms a λ -model of functionals over $\mathcal{N}$ remains open. More precisely, it has not been proved that $\mathscr{D}^{\kappa}$ is closed with respect to composition. This property would easily be proved under the condition that M is a morphism from $\mathscr{D}$ into $\mathbb{C}$ (and not only an $\mathcal{S}$ -morphism).

Before proving that $\bar{\mathscr{D}}^{\kappa}$ is closed with respect to composition, we give another description of the classes $\mathscr{D}^{\kappa}$ and $\bar{\mathscr{D}}^{\kappa}$ in terms of functions on neighborhoods. Using the note 1 to Theorem 2 of Sec. 7, we can say that an approximation of every indexed set C_{σ}^{κ}, $\sigma \in T$ is an extended indexed set. Using the lemma of Sec. 2, we can also easily show that the order (induced by the order $\preccurlyeq_{V_\sigma}$) on the approximation is recursive. Further, each partial mapping $\varphi \in C_{p}(\prod_{i<n} C_{\sigma_i}, N)$ is defined uniquely by its restriction on the basis subspace.

In Sec. 5 we defined formal neighborhoods B_{σ}^{*} for all $\sigma \in T$; in a natural manner we can construct their numeration which defines a numeration of the basis neighborhoods of real spaces. We note that these numerations are equivalent, in the case of $\mathbb{C}^{\kappa}$ to the numerations of approximations, constructed in Theorem 2 of Sec. 7. (We recall that the elements of the approximations are identified with the f -elements of $\mathbb{C}^{\kappa}$, and so with $\mathbb{C}$, and also with the basis neighborhoods.) The numeration generated by the numerations of the formal neighborhoods will be said to be formal, and the corresponding numbers of the neighborhoods will also be said to be formal.

PROPOSITION 1. Let $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | 0) \in ST$; then the functional $\varphi \in C_{\sigma}$ belongs to $K_{\sigma}(C_{\sigma}^{\kappa})$ if and only if there is a (uniquely defined) single-place partial recursive function g such that if $\bar{f}$ is an f -element in $\prod_{i<n} C_{\sigma_i}$ and x is any of its formal numbers, then

a) $\varphi(\bar{f})$ is defined $\Longleftrightarrow g(x)$ is defined;

b) $\varphi(\bar{f})$ is defined $\Longrightarrow \varphi(\bar{f}) = g(x)$.

This proposition essentially reformulates the assertions proved above.

COROLLARY 1. The class $\bar{\mathscr{D}}^{\kappa}$ coincides with the class of recursively enumerable Kleene-Kreisel functionals [2, 3].

Indeed, using the observation that every basis neighborhood in $\mathscr{D}_{\sigma}$ (if $\sigma \neq 0$) is the union of all basis neighborhoods less than it, for every functional $\varphi \in C_{\sigma}^{\kappa}$ there is a functional $\varphi' \in C_{\sigma}^{\kappa}$, such that $\rho_{\sigma} K_{\sigma}(\varphi) = \rho_{\sigma} K_{\sigma}(\varphi')$, and the partially recursive function g' corresponding to φ' has a recursive domain of definition.

COROLLARY 2. The class $\bar{\mathscr{D}}^{\kappa}$ is closed with respect to composition.

Indeed, this was noted in Kleene's paper ([2], Sec. 1.7).

Let $f \in \mathscr{D}_{\sigma}$, $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | 0) \in ST$; we define the partial function g^{*} of N into N by the graph Γ as follows: $\langle x, y \rangle \in \Gamma \Longleftrightarrow$ if $\bar{v} \in B_{\bar{\sigma}}^{*}$ is a formal neighborhood of formal number x, and $f \in \langle \pi_{\sigma}^{\mathscr{D}}(\bar{v}), y \rangle$.

COROLLARY 3. $f \in \mathscr{D}_{\sigma}^{\kappa} \Longleftrightarrow g^{*}$ is partially recursive.

COROLLARY 4. If $\sigma = ((0|0)|0)$, then $\mathscr{D}_{\sigma}^{\kappa} \neq \bar{\mathscr{D}}_{\sigma}^{\kappa}$ (although, as already noted, $\mathscr{D}_{\sigma}^{\kappa} \subseteq \bar{\mathscr{D}}_{\sigma}^{\kappa}$).

To consider the classes $\mathscr{D}^{\kappa}$ and $\bar{\mathscr{D}}^{\kappa}$ as classes of functionals "into themselves," and not as classes of functionals over $\mathscr{D}$, we have to prove their extensionality; i.e., we have to prove that if $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | 0)$, $\varphi, \varphi' \in \mathscr{D}_{\sigma}^{\kappa}$ and for any $\bar{f} \in \mathscr{D}_{\bar{\sigma}}^{\kappa}$, $\varphi(\bar{f}) = \varphi'(\bar{f})$, then $\varphi = \varphi'$ (in $\mathscr{D}_{\sigma}$).

This property follows from the following proposition.

PROPOSITION 2. If φ is an f-element of $\mathcal{D}_\sigma$, $\sigma = (\sigma_0, \ldots, \sigma_{n-1} | 0)$, there is an $f \in \mathcal{D}_\sigma^\kappa$ such that $f \supseteq \mu_\sigma(\varphi)$.

This follows from Proposition 2 of Sec. 5 since the functional f defined there in the proof obviously belongs to $\mathcal{D}_\sigma^\kappa$.

COROLLARY. The class $\bar{\mathcal{D}}^\kappa$ can be considered as an ordinary class (λ -model) of functionals over N.

The class $\mathcal{D}^\kappa$ is obviously the most constructive, because, using the numbers of these functionals in C^κ, we can effectively compute the values of the composition (substitution). Nevertheless, it is difficult to say which of these three classes is the correct generalization of generally recursive functions (perhaps there is not a unique correct generalization).

LITERATURE CITED

1. Yu. L. Ershov, "Computable numerations of morphisms," Algebra i Logika, 10, No. 3, 247-308 (1971).
2. S. C. Kleene, "Countable functionals," in: Constructivity in Mathematics, Amsterdam (1959), pp. 81-100.
3. G. Kreisel, "Interpretation of analysis by means of constructive functionals of finite types," in: Constructivity in Mathematics, Amsterdam (1959), pp. 101-128.
4. M. Davis, "Computable functionals of arbitrary finite types," in: Constructivity in Mathematics, Amsterdam (1959), pp. 281-289.
5. D. Scott, "Outline of mathematical theory of computation," Proc. 4th Annual Princeton Conference on Information Science and Systems (1970), pp. 169-176.
6. D. Scott, "Continuous lattices," in: Topology, Algebraic Geometry, and Logic, Lecture Notes in Math. N 274 (1972), pp. 97-136.
7. G. Kreisel, "Some reasons for generalizing recursion theory," in: Logic Colloquium 69, Amsterdam (1971), pp. 139-198.
8. A. I. Mal'tsev, "Iterative algebras and Post varieties," Algebra i Logika, 5, No. 2, 59-68 (1968).
9. Yu. L. Ershov, The Theory of Numerations [in Russian], Vol. 1, Novosibirsk (1969).
10. Yu. L. Ershov, "Computable numerations," Algebra i Logika, 7, No. 5, 71-99 (1968).
11. A. H. Lachlan, "On the indexing of classes of recursively enumerable sets," J. Symb. Logic, 31, 10-22 (1966).