Is zeroness of multivariate CDF decidable?

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Constructible differentially finite algebraic series in several variables

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Abstract

We extend the concept of CDF-series to the context of several variables, and show that the series solution of first-order differential equations $y' = x(t, y)$ and functional equation $y = x(t, y)$, with x CDF in two variables, are CDF-series. We also give many effective closure properties for CDF-series in several variables

1. CDF-series in one variable

We present in this paper, new properties and an extension to several variables of the concept of CDF-*series* introduced by Reutenauer and the first author in [4]. Throughout this text, the abbreviation CDF stands for "constructible differentially finite algebraic". A subset of CDF-series in one variables is discussed in [2] from the standpoint of their involvement in the complexity analysis of algorithms on increasing trees. CDF-series appear naturally in the study of enumeration problems in a manner similar to D-finite series which have been shown to have great importance in enumerative combinatorics by Gessel [5], Stanley [8] and Zeilberger [9]. On the other hand, Zeilberger has underlined how the "holonomic paradigm" (D-finite in several variables) allows for the automatic derivation of identities. In this light, Salvy and Zimmermann, and Plouffe and the first author have shown how one can obtain explicit or implicit forms for series out of a limited knowledge of their coefficients (see [7, 3]), supposing that the series in question lies in the class of D-finite series. The interest of considering the class of CDF-series for similar purposes is illustrated in [1] and [3]. However to fully exploit this approach it is essential to extend the concept of CDF-series to several variables.

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Recalling definitions of [4], we denote $\mathscr{C}(t)$ the class of CDF-series in one variable t defined as follows.

Definition 1. A formal power series $y = y(t) \in \mathbf{C}[\![t]\!]$ is said to be CDF (in one variable t) if there exists $k \geqslant 1$, and k series $y_1, \ldots, y_k \in \mathbf{C}[\![t]\!]$ with $y_1 = y$ satisfying

$$\begin{cases} y_1' = P_1(y_1, \ldots, y_k), \\ y_2' = P_2(y_1, \ldots, y_k), \\ \vdots \\ y_k' = P_k(y_1, \ldots, y_k), \end{cases} \tag{1}$$

with initial conditions $y_i(0) = y_{i0}$, for some polynomials $P_1, \ldots, P_k \in \mathbf{C}[y_1, \ldots, y_k]$.

We further say that this system is of *order k* and *degree m*, where m is the maximum total degree of the polynomials P_i.

Remark 2. In Definition 1, we may substitute polynomials $Q_j \in \mathbf{C}[t, y_1, \ldots, y_k]$ for the polynomials P_i, without changing the class $\mathscr{C}(t)$, since we can always add the equation $y_{k+1}' = 1$, with initial condition $y_{k+1,0} = 0$, to the system.

Another useful characterization for series in $\mathscr{C}(t)$ given in [4] is the following.

Lemma 3. *A series is $\mathscr{C}(t)$ if and only if it is contained in some finitely generated subalgebra of $\mathbf{C}[\![t]\!]$ which is closed for differentiation.*

what happens when going to $\mathbb{C}((t))$?

Using this lemma, one can show that the class $\mathscr{C}(t)$ enjoys nice effective closure properties. They are closed for addition, Cauchy-product, inversion (when defined), composition (when defined), compositional inversion (when defined) and integration. Furthermore, they contain the class of algebraic power series. To illustrate the effectiveness of these closure properties, let us suppose that $x_1 = x(t)$ and $y_1 = y(t)$ are in $\mathscr{C}(t)$, with systems

$$\begin{array}{ll} y_1' = P_1(y_1, \ldots, y_k), & x_1' = Q_1(x_1, \ldots, x_\ell), \\ \vdots & \vdots \\ y_k' = P_k(y_1, \ldots, y_k), & x_\ell' = Q_\ell(x_1, \ldots, x_\ell) \end{array} \tag{2}$$

for some polynomials P_i and Q_j. If $z_1 = z(t) = x(y(t))$, then adding the equations

$$\begin{aligned} z_1' &= Q_1(z_1, \ldots, z_\ell)P_1(y_1, \ldots, y_k), \\ z_2' &= Q_2(z_1, \ldots, z_\ell)P_1(y_1, \ldots, y_k), \\ &\vdots \\ z_k' &= Q_k(z_1, \ldots, z_\ell)P_1(y_1, \ldots, y_k) \end{aligned}$$

$z_i' = x_i'(y_1) \cdot y_1'$

to system (2) clearly gives a system of form (1) for z.

A more general closure property is the following.

a form of

Bekić's theorem

Proposition 4. *Let $x(t)$ be any series in $\mathscr{C}(t)$. Then the power series solution $y = y(t)$ of the differential equation $\boxed{y' = x(y)}$ with initial condition $y(0) = 0$, is a CDF-series.*

"infinite" differential equation

Proof. For $x = x_1$ let

$$x_i' = P_i(x_1, \ldots, x_k),$$

with $1 \leqslant i \leqslant k$, be a system of form (1) for x. We construct the following system for $y = y_1$:

$$y_1' = y_2,$$

$$\partial(x(y)) = (\partial x)(y) \cdot \partial y$$

$$y_2' = P_1(y_2, \ldots, y_{k+1})y_2,$$

$$\vdots$$

$$y_k' = P_k(y_2, \ldots, y_{k+1})y_2,$$

hence y verifies a system of form (1). $\square$

A similar proof gives the following proposition.

subclass of constructible algebraic

Proposition 5. *Let $x(t)$ be any series in $\mathscr{C}(t)$. Then the power series solution $y = y(t)$ of the functional equation $y = t \cdot x(y)$ is a CDF-series.*

$$\partial y = x(y) + t((\partial x)(y) \cdot \partial y) \Rightarrow \partial y = \frac{x(y)}{1 - t(\partial x)(y)}$$

also CDF

2. Comparison of $\mathscr{C}(t)$ with the class of D-finite series

Recall that a D-finite power series $y = y(t)$ is a series satisfying some linear differential equation

$$P_n(t)y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_0(t)y = 0,$$

where the p_i's are polynomials with coefficients in **C**. We denote $\mathscr{D}(t)$ the class of D-finite series. The properties of this class are discussed at length in a paper of Stanley [8].

As illustrated in [4], the classes $\mathscr{C}(t)$ and $\mathscr{D}(t)$ are incomparable. However, the following lemma shows that many D-finite series are in $\mathscr{C}(t)$.

the converse fails

Lemma 6. *Let y be a D-finite power series satisfying*

$$P_n(t)y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_0(t)y = 0,$$

with $P_n(t) \neq 0$. If $P_n(0) \neq 0$, then $y \in \mathscr{C}(t)$.

Invertible as a power series since of the form 1 + t*q(t)

Proof. The usual transformations of a linear differential equation into a system of equations gives

$$
\begin{aligned}
&y' = y_2,\\
&y_2' = y_3,\\
&\quad\vdots\\
&y_{n-1}' = y_n,\\
&P_n(t)y_n' = -P_{n-1}(t)y_n - P_{n-2}(t)y_{n-1} - \cdots - P_0(t)y.
\end{aligned}
$$

Since $P_n(0) \neq 0$, $1/P_n(t)$ is defined and in $\mathscr{C}(t)$. The last equation of (3) can thus be replaced by

$$y_n' = -P_{n-1}(t)Q(t)y_n - P_{n-2}(t)Q(t)y_{n-1} - \cdots - P_0(t)Q(t)y,$$

with further equations added for $Q(t) = 1/P_n(t)$.

3. CDF-series in several variables

By analogy with Zeilberger's theory of holonomic series (see [9]) extending the D-finite concept to several variables, we are led to introduce CDF-series in several variables.

Let us denote by $\boldsymbol{t}$ the vector of variables $(t_1, \ldots, t_n)$. In light of Lemma 3, we propose the following definition.

Definition 7. A formal power series $y = y_1 \in \mathbf{C}[\![\boldsymbol{t}]\!]$ is said to be CDF in the variables $t_1, \ldots, t_n$ if there exist k series $y_1, \ldots, y_k \in \mathbf{C}[\![\boldsymbol{t}]\!]$, and kn polynomials $P_{1,1}, \ldots, P_{k,n} \in \mathbf{C}[y_1, \ldots, y_k]$ satisfying

$$
\left\{
\begin{aligned}
&\frac{\partial}{\partial t_1} y_1 = P_{11}(y_1, \ldots, y_k),\\
&\frac{\partial}{\partial t_2} y_1 = P_{12}(y_1, \ldots, y_k),\\
&\qquad\vdots\\
&\frac{\partial}{\partial t_j} y_i = P_{ij}(y_1, \ldots, y_k),\\
&\qquad\vdots\\
&\frac{\partial}{\partial t_n} y_k = P_{kn}(y_1, \ldots, y_k),
\end{aligned}
\right.
$$

with initial conditions $y_i(0, \ldots, 0) = y_{(i,0)}$.

We say that k is the *order* and m the *degree* of system (4), if m is the maximal total degree of the polynomials P_{ij}. The class of all CDF-series in the variables $t_1, \ldots, t_n$ is denoted $\mathscr{C}(t) = \mathscr{C}(t_1, \ldots, t_n)$.

Second kind

Example 8. The exponential generating function S for Stirling numbers of ~~first kind~~ $S_{n,k}$

$$S(u,t) = \sum_{n,k \geqslant 0} S_{n,k} u^k \frac{t^n}{n!} = \exp(u(\exp(t) - 1))$$

is CDF since we have the system:

$$\begin{cases} \dfrac{\partial}{\partial u} S = YS - S, \\[2mm] \dfrac{\partial}{\partial t} S = YZS, \\[2mm] \dfrac{\partial}{\partial u} Y = 0, \quad \dfrac{\partial}{\partial u} Z = 1, \\[2mm] \dfrac{\partial}{\partial t} Y = Y, \quad \dfrac{\partial}{\partial t} Z = 0, \end{cases}$$

$\partial_u S = (e^t - 1) \cdot S$

$\partial_t S = u e^t \cdot S$

CDL if we regard e^t as a constant

$Z = u$

$Y = e^t$

(or via a change of variables $t := \log \tilde{t}$, $e^t \mapsto \tilde{t}$)

$\partial_{\tilde t}(S \circ_t \log \tilde t) = (\partial_t S \circ_t \log \tilde t) \cdot \partial_{\tilde t} \log \tilde t = u \tilde t \cdot \tilde S \cdot \frac{1}{\tilde t}$

with initial values $S(0,0) = 1$, $Y(0,0) = 1$, $Z(0,0) = 0$.

Example 9. The generating function for Laguerre polynomials

$$x(u,t) = \mathscr{L}^{(\alpha)}(u,t) = \sum_{n \geqslant 0} L_n^{(\alpha)}(u) t^n = 1/(1-t)^{\alpha+1} \exp(-ut/(1-t))$$

is CDF:

$$\frac{\partial}{\partial u} x = (\alpha + 1) yx - zy^2 x,$$

$$\frac{\partial}{\partial t} x = -wyx,$$

$$\frac{\partial}{\partial u} y = 0, \qquad \frac{\partial}{\partial u} z = 1, \qquad \frac{\partial}{\partial u} w = 0,$$

$$\frac{\partial}{\partial t} y = -y^2, \qquad \frac{\partial}{\partial t} z = 0, \qquad \frac{\partial}{\partial t} w = 1.$$

Proposition 10. *A series is CDF in the variables $t_1, \ldots, t_n$ if and only if it is contained in some finitely generated subalgebra of $\mathbf{C}[\![t_1, \ldots, t_n]\!]$ closed for all partial differentiations $\partial/\partial t_i$, $1 \leqslant i \leqslant n$.*

Proof. Straightforward translation of the condition. □

The following result follows as in the one-variable case.

reciprocal ? some sort of compositional inverse ?

Theorem 11. *The class $\mathscr{C}(t)$ is closed for addition, Cauchy-product, composition (when defined), inversion (when defined), partial derivation, evaluation (when defined), and contains the class of algebraic power series.*

(specialisation)

Proof. The several variable case differs from the one-variable case in only two instances: evaluation makes sense and they are not closed under integration. ! proof?

If for some $a \in \mathbf{C}$ the evaluation $z = x(t_1, \ldots, t_{h-1}, a, t_{h+1}, \ldots, t_n)$ is well defined, the system for z is obtained from the system for x by deletion of equations involving partial derivation with respect to t_h and replacing them with initial conditions of the form $z_j(0, \ldots, 0) = x_j(0, \ldots, 0, a, 0, \ldots, 0)$. □

Proposition 12. *Let $x \in \mathscr{C}(t_1, t_2)$. Then $y \in \mathbf{C}[\![t]\!]$ such that $\boxed{y' = x(t, y)}$ is CDF.*

Proof. Let $x = x_1$ and for $1 \leqslant j \leqslant k$ let $(\partial/\partial t_i)x_j = P_{j,i}(x_1, \ldots, x_k)$ for some polynomials $P_i \in \mathbf{C}[x_1, \ldots, x_k]$. We get a system of form (1):

$i = 1,2$

$$z_1' = z_2,$$

$$z_2' = P_{11}(z_2, \ldots, z_{k+1})z_2,$$

$$\vdots$$

$$z_{k+1}' = P_{k1}(z_2, \ldots, z_{k+1})z_2. \quad \square$$

$\bar{x} = (x_1, \ldots, x_d)$

$\bar{y} = (y_1, \ldots, y_k)$

$\Phi(\bar{x}, \bar{y}) \in \mathbb{Q}[\bar{x}, \bar{y}]^k$

Can be generalised : 1) $\Phi(0,0) = 0$

2) $J\Phi(0,0)$ invertible

$\Rightarrow$ unique solution $f(x) \in \mathbb{Q}[[x]]$ s.t. $f(0) = 0$, $\Phi(x, f(x)) = 0$

Hence y is $\mathscr{C}$.

The following result follows.

Corollary 13. *Let $x \in \mathscr{C}(t_1, t_2)$. Then $y \in \mathbf{C}[\![t]\!]$ such that $\boxed{y = t \cdot x(t, y)}$ is $\mathscr{C}$.*

The next characterization can sometimes be used to show that a given series is not $\mathscr{C}$. It is a generalization of a theorem of [4] to the case of several variables.

Theorem 14. *Let $y(t_1, \ldots, t_n) = \sum_{i \in \mathbf{N}^n} y_i t^i \in \mathscr{C}(t)$. Then the following conditions hold:*

(i) *The series y is analytic around 0. In fact, $|y_i| \leqslant c^{|i|}$ for some constant c.*

(ii) *If the y_i's lie in some subfield $\mathbf{K}$ of $\mathbf{C}$, then there exists a system of form (4) with all coefficients of the polynomials $P_{j,l}$ in $\mathbf{K}$.*

(iii) *If for all i: $y_i \in \mathbf{Q}$, then there exists $d \in \mathbf{N}$, $d \geqslant 1$ such that $d^{|i|} i! \, y_i \in \mathbf{Z}$.*

denominators : do not grow "too fast"

Proof. The proof is similar to that given in [4]. □

References

[1] F. Bergeron, M. Bousquet-Mélou and S. Dulucq, Standard paths in the composition poset, *Ann. Sci. Math. Québec*, submitted.

[2] F. Bergeron, P. Flajolet and B. Salvy, Varieties of increasing trees, Unité de Recherche INRIA-Rocquencourt, Rapports Techniques N. 1538, Programme 2, 1992.

[3] F. Bergeron and S. Plouffe, Computing the generating function of a series given its first few terms, *Experimental Math.* **1** (1992) 307–312.

[4] F. Bergeron and C. Reutenauer, Combinatoriqual resolution of systems of differential equations III: a special class of differentially algebraic series, *Eur. J. Combin.* **11** (1990) 501–512.

[5] I.M. Gessel, Symmetric functions and *P*-recursiveness, *J. Combin. Theory* **53** (1990) 257–285.

[6] I. Lipshitz, D-finite power series, *J. Algebra* **122** (1989) 353–373.

[7] B. Salvy and P. Zimmermann, GFUN: A Maple package for the manipulation of generating and holonomic functions in one variable, Unité de Recherche INRIA-ROCQUENCOURT, Rapports Techniques N. 143, Programme 2, 1992.

[8] R.P. Stanley, Differentiably finite power series, *Eur. J. Combin.* **1** (1980) 175–188.

[9] D. Zeilberger, A holonomic systems approach to special functions identities, *J. Comput. Appl. Math.* **32** (1990) 321–368.