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TWO-WAY REPRESENTATIONS AND WEIGHTED AUTOMATA

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Abstract. We study the series realized by weighted two-way automata, that are strictly more powerful than weighted one-way automata. To this end, we consider the Hadamard product and the Hadamard iteration of formal power series. We introduce two-way representations and show that the series they realize are the solutions of fixed-point equations. In rationally additive semirings, we prove that two-way automata are equivalent to two-way representations, and, for some specific classes of two-way automata, rotating and sweeping automata, we give a characterization of the series that can be realized.

Mathematics Subject Classification. 68Q45, 68Q70.

1. Introduction

Two-way finite automata were introduced at the very beginning of the theory of automata. It was then proved [17, 21] that they are not more powerful than one-way automata. Many papers have studied the succintness of two-way automata with respect to one-way automata (*cf.* for instance [16]). In this paper, we study *weighted* two-way automata. This model is strictly more powerful than weighted one-way automata: they have been introduced in [1] where two-way $\mathbb{Z}$-automata that are equivalent to one-way $\mathbb{Z}$-automata have been characterized, and in the framework of probabilistic automata [2].

In this paper, some classes of series realized by two-way automata are characterized. To this end, we describe first different classes of formal power series closed under rational operations, or other operations like the Hadamard product or the mirror. Rational series naturally appear as the behaviour of (one-way) weighted automata. The combination of the runs of two different automata on the same inputs naturally leads to consider Hadamard product; likewise, the ability to use the same automaton over the same input several times is reflected by the Hadamard iteration. When considering two-way machines, it is also normal consider the mirror operation to express that the input can be read from right to left.

Two-way weighted automata are then defined; they are straighforward extensions from two-way automata and weighted one-way automata. Like one-way automata, these automata are finite-memory *memoryless* machines: at each step of the computation the automaton has access to no other information that the current state and the letter read on the input. The value of the weight which is computed along the run does not influence the actions of the automaton. This makes this model more restrictive (and less powerful) than models involving pebbles or stacks (*cf.* for instance [12]), but it is consistent with the study of Hadamard series and is suitable for an algebraic study.

Keywords and phrases. Two-way automata, weighted automata, formal power series.

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We introduce then two-way representations which are algebraic models that are extensions of linear representations. As proved in the last part, they are, under some assumptions on the semiring of weights, equivalent to two-way automata. The set of series realizable by such representations, called *two-way recognizable series*, is closed under sum, Hadamard product, Hadamard iteration, mirror, and left quotient.

We then show that two-way recognizable series are solutions of some fixed-point systems. The resolution of such a system amounts to compute an explicit expression for the series realized by a two-way representation. This allows to prove that rotating representations exactly realize the series which are in the closure of the rational series under sum, Hadamard product and Hadamard iteration; likewise, sweeping representations exactly realize the series which are in the closure of the rational series under the same operations plus the mirror.

Finally, we show that, in the case of *rationally additive semirings*, each weighted two-way automaton corresponds to a two-way representation and realizes the same series. We also used weighted two-way automata to prove that the set of two-way recognizable series are not closed under Cauchy product and Kleene star.

2. Formal power series

If A is a finite alphabet of symbols, A^* is the set of words over A; this set is naturally endowed with the concatenation as multiplication; this operation is associative and admits the *empty word* ε (the word with no letter) as neutral element. For every word $w = w_1 \dots w_n$, $|w| = n$ denotes the length of w; the mirror of w is the word $\overline{w} = w_n \dots w_1$.

A semiring is an algebra $(\mathbb{K}, +, .)$, where the addition $+$ is associative and commutative, the product . is associative and distributes over $+$; $\mathbb{K}$ contains two distinct elements 0 and 1, 0 is neutral for the addition and is an annihilator for the product, 1 is neutral for the product. The semiring $\mathbb{K}$ is *commutative* if its product is. Moreover, we assume that every semiring is endowed with a *partial* unary operation $*$ (star) such that, for every x, y in $\mathbb{K}$,

(P) if $(x.y)^*$ exists, so does $(y.x)^*$, and $(y.x)^* = 1 + y.(x.y)^*.x$;
(S) if x^*, $(y.x^*)^*$, and $(x+y)^*$ exist, $(x+y)^* = x^*.(y.x^*)^*$.

The domain of this operation always contains 0 and $0^* = 1$. A semiring with a complete star operation is a *Conway* semiring [9]. If $\mathbb{K}$ is a Conway semiring, for every positive integer n, the semiring of $\mathbb{K}$-matrices with size n is canonically endowed with a star operation. If $n = 1$, $M^* = [M_{1,1}^*]$, otherwise, for every decomposition of M, it holds:

$$M = \left[\begin{array}{c|c} X & Y \\ \hline Z & T \end{array}\right] \Longrightarrow M^* = \left[\begin{array}{c|c} (X + Y \cdot T^* \cdot Z)^* & (X + Y \cdot T^* \cdot Z)^* \cdot Y \cdot T^* \\ \hline (T + Z \cdot X^* \cdot Y)^* \cdot Z \cdot X^* & (T + Z \cdot X^* \cdot Y)^* \end{array}\right],$$

where X and T are square matrices. Then, the semiring $\mathbb{K}^{n \times n}$ is also Conway (*cf.* [5]).

The extension of Identity (P) to nonsquare matrices is straighforward:

Lemma 2.1. *Let $\mathbb{K}$ be a Conway semiring and let m and n be two nonnegative integers. If M is in $\mathbb{K}^{m \times n}$ and N is in $\mathbb{K}^{n \times m}$, then $(M \cdot N)^* = \mathsf{Id}_m + M \cdot (N \cdot M)^* \cdot N$.*

2.1. Operations on series

Let $\mathbb{K}$ be a semiring and A be an alphabet. A formal power series s in $\mathbb{K}\langle\langle A^* \rangle\rangle$ is a mapping from A^* to $\mathbb{K}$; for every word w, we denote $\langle s, w \rangle$ the image (or *coefficient*) of w in s and s is formally denoted as an infinite sum: $s = \sum_{w \in A^*} \langle s, w \rangle w$. The *support* of a formal power series s is the set of words w such that $\langle s, w \rangle$ is different from zero. A series with a finite support is called a polynomial.

The set $\mathbb{K}\langle\langle A^*\rangle\rangle$ is naturally endowed with a number of operations:

$$s+t=\sum_{w\in A^*}(\langle s,w\rangle+\langle t,w\rangle)w \text{ (Sum)}; \qquad \overline{s}=\sum_{w\in A^*}\langle s,\overline{w}\rangle w \text{ (Mirror)};$$

$$s\cdot t=\sum_{w\in A^*}\left(\sum_{u\cdot v=w}\langle s,u\rangle\cdot\langle t,v\rangle\right)w \text{ (Cauchy product)};$$

$$s\odot t=\sum_{w\in A^*}(\langle s,w\rangle\cdot\langle t,w\rangle)w \text{ (Hadamard product)}.$$

The algebra $(\mathbb{K}\langle\langle A^*\rangle\rangle,+,\cdot)$ is a semiring; the unit element for the Cauchy product is the constant series 1. The star operator of this semiring is the *Kleene star*. It is first defined on *proper* series:

$$\langle s,\varepsilon\rangle=0 \quad\Longrightarrow\quad s^*=\sum_{w\in A^*}\left(\sum_{k=0}^{\infty}\langle s^k,w\rangle\right)w=\sum_{w\in A^*}\left(\sum_{k=0}^{|w|}\langle s^k,w\rangle\right)w.$$

It is then extended to series s such that $\langle s,\varepsilon\rangle^*$ exists; to this end, we consider the proper part s_p of s: $s=\langle s,\varepsilon\rangle+s_p$, and set $s^*=\langle s,\varepsilon\rangle^*(s_p\cdot\langle s,\varepsilon\rangle^*)^*$.

If $\mathbb{K}$ is a Conway semiring, so is the semiring $(\mathbb{K}\langle\langle A^*\rangle\rangle,+,\cdot)$ (*cf.* [5]). The sum, the Cauchy product and the Kleene star are called *rational operations*.

The algebra $(\mathbb{K}\langle\langle A^*\rangle\rangle,+,\odot)$ is also a semiring; the unit element for the Hadamard product is A^* (actually the characteristic series of the language A^*). The star in this semiring is the *Hadamard iteration* which exists if and only if the star of every coefficient exists; for every word w, $\langle s^{\circledast},w\rangle=\langle s,w\rangle^*$. Since every operation is pointwise, if $\mathbb{K}$ is a Conway semiring, so is $(\mathbb{K}\langle\langle A^*\rangle\rangle,+,\odot)$. The sum, the Hadamard product and the Hadamard iteration are called *pointwise operations*.

Remark 2.2. The mirror operation commutes with the pointwise operations. Moreover, if $\mathbb{K}$ is commutative, the mirror operation anticommutes with the Cauchy product ($\overline{s\cdot t}=\overline{t}\cdot\overline{s}$) and commutes with the Kleene star.

2.2. A Hierarchy of series

In this paper, we consider the following hierarchy of series in $\mathbb{K}\langle\langle A^*\rangle\rangle$:

- $\mathbb{K}\langle A^*\rangle$ is the set of polynomials of $\mathbb{K}\langle\langle A^*\rangle\rangle$;
- $\mathbb{K}\mathsf{Rat}A^*$ is the closure of $\mathbb{K}\langle A^*\rangle$ under rational operations;
- $\mathbb{K}\mathsf{Had}A^*$ is the closure of $\mathbb{K}\mathsf{Rat}A^*$ under pointwise operations[2];
- $\mathbb{K}\mathsf{RH}A^*$ is the closure of $\mathbb{K}\langle A^*\rangle$ under both rational and pointwise operations.

In the commutative case, $\mathbb{K}\mathsf{Rat}A^*$ is closed under Hadamard product (*cf.* [19]), but not under Hadamard iteration. We also consider $\mathbb{K}\mathsf{MirRat}A^*$, $\mathbb{K}\mathsf{MirHad}A^*$, and $\mathbb{K}\mathsf{MirRH}A^*$, which are the closure by mirror of $\mathbb{K}\mathsf{Rat}A^*$, $\mathbb{K}\mathsf{Had}A^*$, and $\mathbb{K}\mathsf{RH}A^*$ respectively. If the semiring $\mathbb{K}$ is commutative or A has only one letter, the closure by mirror does not create larger families. It can be noticed that $\mathbb{K}\mathsf{MirHad}A^*$ is the closure of $\mathbb{K}\mathsf{MirRat}A^*$ under pointwise operations.

Example 2.3. Let $A=\{a,b\}$ and $\mathbb{K}$ be the semiring of rational languages over $B=\{x,y\}$. The polynomial $P=\{x\}\,a+\{y\}\,b$ is the series that maps a onto $\{x\}$ and b onto $\{y\}$; thus, for every word w, $\langle P^*,w\rangle$ is the singleton that contains the word $w_{A\mapsto B}$ obtained from w in replacing every a by x and every b by y.

For every word w, $\langle(P^*)\odot(P^*),w\rangle=(w\cdot w)_{A\mapsto B}$, thus $(P^*)\odot(P^*)$ is a series of $\mathbb{K}\mathsf{Had}A^*$ which is not rational. thus when K is not commutative rational series over words are not closed under Hadamard product
In [13], it is shown that the series $(a+b)^*b((\{x\}a)^*)^{\circledast}b(a+b)^*$ is in $\mathbb{K}\mathsf{RH}A^*$ but not in $\mathbb{K}\mathsf{Had}A^*$. Notice that in this last example, $\mathbb{K}=\mathsf{Rat}\{x\}^*$ is commutative.

[2] This family is slightly different from the family introduced in [8] of series which are finite sums of series of the form $s\odot t^{\circledast}$, where s and t are rational series; nevertheless, by results from [8], they coincide on commuttive idempotent semirings.

3. Weighted two-way automata

Weighted two-way automata are extensions of classical two-way automata. We consider, like for instance in [4], that, during the computation of two-way automata, the move of the input only depends on the state and not on the transition. It does not alter the expressivity of the model.

Definition 3.1. Let $\mathbb{K}$ be a semiring, A an alphabet and $\Box$ a special symbol, called endmarker, which does not belong to A. A two-way $\mathbb{K}$-automaton over A is a tuple $(Q_+, Q_-, A, \Box, \mathbb{K}, E, I, T)$, where:

- $Q = Q_+ \cup Q_-$ is a finite set of states;
- $E : Q \times A \times Q \cup Q_+ \times \Box \times Q_- \cup Q_- \times \Box \times Q_+ \longrightarrow \mathbb{K}$ is the transition function;
- $I : Q_+ \longrightarrow \mathbb{K}$ is the initial function, and $T : Q_+ \longrightarrow \mathbb{K}$ is the final function.

States in Q_+ are called *forward* states, states in Q_- are *backward* states. The set of transitions is the support of E, the set of initial states is the support of I and the set of final states is the support of T. For every transition $e = (p, a, q)$, $\lambda(e) = a$ is the label of e, $\sigma(e) = p$ is the source, and $\tau(e) = q$ is the target of e.

A path in such an automaton is a triple $\pi = (p, (e_i)_{i\in[1;k]}, q)$, where $(e_i)_{i\in[1;k]}$ is a sequence of transitions, and k is a nonnegative integer, such that, for every i in $[2; k]$, $\sigma(e_i) = \tau(e_{i-1})$. Moreover, if k is positive, $\sigma(e_1) = p$ and $\tau(e_k) = q$; if $k = 0$, the path is *empty* and $p = q$.

At each step of a computation, the head on the input is at a given position; depending on the letter at this position and on the state, the automaton performs a transition; it then reaches a new state, and, depending on the nature of the state, respectively forward or backward, the head respectively moves to the left or to the right before the next step. For each path $\pi = (p, (e_i)_{i\in[1;k]}, q)$, the move of the head on the input after each transition is given by the function $\delta_\pi : [0; k] \to \mathbb{Z}$ inductively defined as:

$$\delta_\pi(0) = 0, \qquad \forall i \in [1; k],\ \delta_\pi(i) = \begin{cases} \delta_\pi(i-1) + 1 & \text{if} \quad \tau(e_i) \in Q_+; \\ \delta_\pi(i-1) - 1 & \text{if} \quad \tau(e_i) \in Q_-. \end{cases}$$

The global move of the head after π is $\delta(\pi) = \delta_\pi(k)$.

A word $w = w_1 \dots w_n$ is *admissible* for a path $\pi = (p, (e_i)_{i\in[1;k]}, q)$ at position r in $[1; n]$ if, starting with the head at position r, the automaton can follow path π. Formally, for every i in $[0; k-1]$, $r + \delta_\pi(i)$ is in $[1; n]$, and $\lambda(e_{i+1}) = w_{r+\delta_\pi(i)}$.

A word is admissible for a path if it is admissible at some position. If a path admits some admissible words, there is a shortest one, which is the *label* of the path.

There are four particular types of non empty paths. Let $\pi = (p, (e_i)_{i\in[1;k]}, q)$ be a path with label $w = w_1 \dots w_n$;

- π is *forward* if p and q are in Q_+, w is admissible for π at position 1, and $\delta(\pi) = n$; thus π reads w starting from the left end of w, to the right end of w.
- π is *backward* if p and q are in Q_-, w is admissible for π at position n, and $\delta(\pi) = -n$; thus π reads w starting from the right end of w, to the left end of w.
- π is *backward-turn* if p is in Q_+, q is in Q_-, w is admissible for π at position 1, and $\delta(\pi) = -1$; thus π reads w starting from the left end of w and comes back to the left end.
- π is *forward-turn* if p is in Q_-, q is in Q_+, w is admissible for π at position n, and $\delta(\pi) = 1$; thus π reads w starting from the right end of w and comes back to the right end.

For every backward-turn (*resp.* forward-turn) path, we say that a word w is *strongly admissible* if the label of the path is a prefix (*resp.* a suffix) of w.

A *computation* on the word $w = w_1 \dots w_n$ of A^* is a path $\pi = (p, (e_i)_{i\in[1;k]}, q)$ where p is initial, q is final, $\Box w \Box$ is admissible for π, and the head starts on the first letter of w (which is the second position in the word $\Box w \Box$)

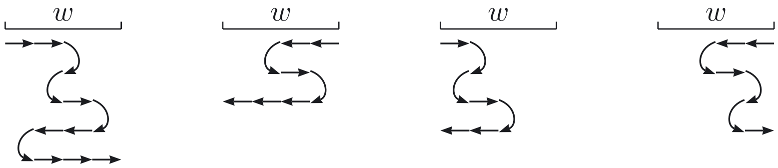

Figure 1. Geometrical representations of paths in a two-way automaton: first and second, a forward and a backward paths with label w, then a backward-turn and a forward-turn paths with w as strongly admissible word.

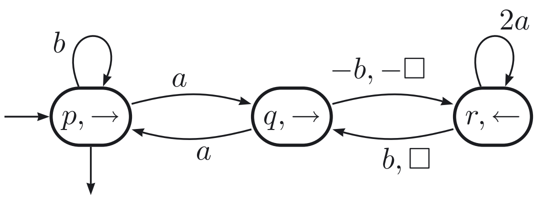

Figure 2. The two-way $\mathbb{Z}$-automaton $\mathcal{A}_1$; p and q are forward states, r is a backward state.

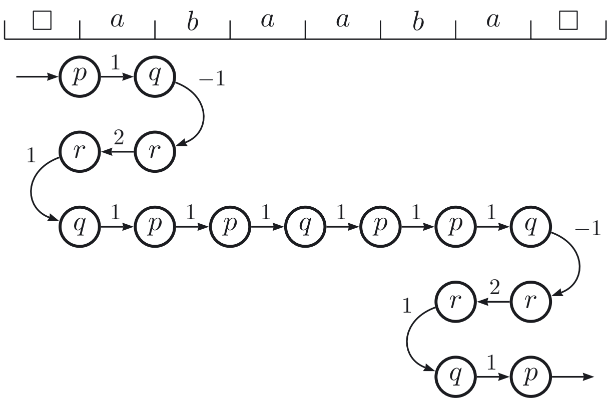

Figure 3. A computation of $\mathcal{A}_1$ on the word $abaaba$.

and ends on the right endmarker. Formally, the latest condition is equivalent to $\delta(\pi) = n$.
The weight of a computation $\pi = (p, (e_i)_{i\in[1;k]}, q)$ is the product

$$\mathsf{w}(\pi) = I(p) \cdot \left(\prod_{i=1}^{k} E(e_i)\right) \cdot T(q).$$

A word w is *accepted* by an automaton if there exists a computation on w in this automaton.

Notice that the weight of a computation is computed "one-way", like the *trace* of a two-way computation defined in [15].

Example 3.2. The two-way $\mathbb{Z}$-automaton of Figure 2 accepts every word. A word w with m blocks of a with odd lengths and k letters a in these blocks is accepted with a weight equal to $(-1)^m 2^k$. Figure 3 shows a computation of this automaton on the word $abaaba$.

Definition 3.3. A two-way $\mathbb{K}$-automaton $\mathcal{A}$ is *valid*, if, for every word w, the sum of the weights of the computations on w is defined. In this case, $\mathcal{A}$ realizes a series $|\mathcal{A}|$ where, for every word w, $\langle |\mathcal{A}|, w\rangle$ is the sum of the weights of the computations on w.

In this paper, we consider some classes of two-way automata.

Definition 3.4. Let $\mathcal{A} = (Q_+, Q_-, A, \square, \mathbb{K}, E, I, T)$ be a two-way $\mathbb{K}$-automaton.

- $\mathcal{A}$ is *one-way* if $Q_- = \emptyset$.
- $\mathcal{A}$ is *sweeping* if every half-turn is labeled by the endmarker: for every transition e, if $\sigma(e)$ and $\tau(e)$ are not both in Q_- or both in Q_+, then $\lambda(e) = \square$.
- $\mathcal{A}$ is *rotating* if it is sweeping and, for every p in Q_-, for every letter a, there is a unique transition with source p and label a; this transition is a loop on p with weight 1.
- $\mathcal{A}$ is deterministic if
  – it contains at most one initial state,
  – for every letter (or endmarker) a and every state p, there is at most one transition e with $\sigma(e) = p$ and $\lambda(e) = a$,
  – for every final state p, there is no transition e with $\sigma(e) = p$ and $\lambda(e) = \square$.
- $\mathcal{A}$ is *simple* if for every input, the number of computations is finite.

Obviously, a simple two-way weighted automaton is always valid, without any assumption on the semiring of weights. It is decidable whether a two-way $\mathbb{K}$-automaton is simple [1], and if $\mathbb{K}$ is commutative, every simple two-way $\mathbb{K}$-automaton realizes a rational series [1] (and, like in the unweighted case, it may be smaller than equivalent one-way $\mathbb{K}$-automata). Moreover, every unambiguous[3] one-way $\mathbb{K}$-automaton can be simulated by a deterministic two-way $\mathbb{K}$-automaton [6].

4. Two-way representations

The linear representation of a one-way $\mathbb{K}$-automaton maps every word w to a matrix $\mu(w)$ such that $\mu(w)_{p,q}$ is the sum of the weights of the path from state p to state q with label w. The matrix $\mu(w)$ is the product of the matrices corresponding to every letter of w. This gives an efficient way to compute the weight of a given word in such an automaton. We extend this notion in the two-way case.

4.1. A new product of matrices

We first introduce a new product on matrices depending on a decomposition of $\mathbb{K}$-matrices with size $m+n$:

$$M = \left[\begin{array}{c|c} \overrightarrow{M} \in \mathbb{K}^{m\times m} & \overset{\hookleftarrow}{M} \in \mathbb{K}^{m\times n} \\ \hline \overset{\hookrightarrow}{M} \in \mathbb{K}^{n\times m} & \overleftarrow{M} \in \mathbb{K}^{n\times n} \end{array}\right].$$

Intuitively, such a matrix represents the weight on some parts of computations in a two-way automaton: $\overrightarrow{M}_{p,q}$ represents forward paths from p to q, $\overleftarrow{M}_{p,s}$ backward paths from p to s, $\overset{\hookrightarrow}{M}_{r,q}$ forward-turn paths from r to q, and $\overset{\hookleftarrow}{M}_{r,s}$ backward-turn paths from r to s. If M and N are two matrices in $\mathbb{K}^{(m+n)\times(m+n)}$, we set:

$$M \mathbin{\text{ƨ}} N = \left[\begin{array}{c|c} \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N} & \overset{\hookleftarrow}{M} + \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overset{\hookleftarrow}{N} \cdot \overleftarrow{M} \\ \hline \overset{\hookrightarrow}{N} + \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N} & \overleftarrow{N} \cdot (\overset{\hookrightarrow}{M} \cdot \overset{\hookleftarrow}{N})^* \cdot \overleftarrow{M} \end{array}\right].$$

[3]With only one computation for each accepted input.

This product, inspired by the definitions made in [4] for two-way finite automata, requires that the star of matrices is defined; it depends on the pair (m,n), and not only on $m+n$; if needed, it can be explicitely precised as $M \mathbin{\text{ƨ}}_n^m N$.

To explain the mean of these formulae, we consider the following example. Assume that M (*resp.* N) represents computations on the word u (*resp.* v); $\overset{\hookleftarrow}{M}_{p,q}$ is for instance the sum of the weights of all backward-turn paths from p to q for which u is strongly admissible. $M \mathbin{\text{ƨ}} N$ must be defined in such a way that $\overset{\hookleftarrow}{M \mathbin{\text{ƨ}} N}_{p,q}$ is the sum of the weights of all backward-turn paths from p to q for which uv is strongly admissible. Let Π be this set of paths. The label of every path in Π is a prefix of uv; Π can be split into two parts: the paths with label prefix of u, and the paths with label longer than u. The sum of the paths of the first part is $\overset{\hookleftarrow}{M}_{p,q}$; every path π in the second part is the concatenation of paths $\pi_0, \ldots \pi_{2k}$ with $k>0$, where π_0 is a forward path with label u, for every i in $[0;k-1]$, π_{2i+1} is a backward-turn path for which v is strongly admissible, for every i in $[1;k-1]$, π_{2i} is a forward-turn path for which u is strongly admissible, and π_{2k} is a backward path with label u. The sum of the weights of all the paths of Π with the same decomposition is

$$\overrightarrow{M} \cdot \left(\prod_{i=0}^{k-1} \overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M} \right) \cdot \overset{\hookleftarrow}{N} \cdot \overleftarrow{M}.$$

Proposition 4.1. *Let $\mathbb{K}$ be a Conway semiring and let m and n be two nonnegative integers. The product $\mathbin{\text{ƨ}}$ on matrices in $\mathbb{K}^{m+n \times m+n}$ is associative and the unit is the usual identity matrix Id_{m+n}.*

Proof. To prove that $(M \mathbin{\text{ƨ}} N) \mathbin{\text{ƨ}} P = M \mathbin{\text{ƨ}} (N \mathbin{\text{ƨ}} P)$, we prove that the equality holds for the four blocks of these matrices.

$$\begin{aligned}
\overrightarrow{(M \mathbin{\text{ƨ}} N) \mathbin{\text{ƨ}} P} &= \overrightarrow{M \mathbin{\text{ƨ}} N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{M \mathbin{\text{ƨ}} N})^* \cdot \overrightarrow{P} \\
&= \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot (\overset{\hookleftarrow}{N} + \overleftarrow{N} \cdot (\overset{\hookrightarrow}{M} \cdot \overset{\hookleftarrow}{N})^* \cdot \overset{\hookrightarrow}{M} \cdot \overrightarrow{N})^* \cdot \overrightarrow{P} \\
&= \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N} \cdot ((\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot (\overset{\hookrightarrow}{M} \cdot \overset{\hookleftarrow}{N})^* \cdot \overset{\hookrightarrow}{M} \cdot \overrightarrow{N})^* \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overrightarrow{P} \\
&= \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot (\overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^*)^* \cdot \overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overrightarrow{P} \\
&= \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M} + \overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overrightarrow{P} \\
&= \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N \mathbin{\text{ƨ}} P} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N \mathbin{\text{ƨ}} P} = \overrightarrow{M \mathbin{\text{ƨ}} (N \mathbin{\text{ƨ}} P)} \cdot
\end{aligned}$$

$$\begin{aligned}
\overset{\hookleftarrow}{M \mathbin{\text{ƨ}} (N \mathbin{\text{ƨ}} P)} &= \overset{\hookleftarrow}{M} + \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N \mathbin{\text{ƨ}} P} \cdot \overset{\hookrightarrow}{M})^* \cdot \overset{\hookleftarrow}{N \mathbin{\text{ƨ}} P} \cdot \overleftarrow{M} \\
&= \overset{\hookleftarrow}{M} + \overrightarrow{M} \cdot [(\overset{\hookleftarrow}{N} + \overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N}) \cdot \overset{\hookrightarrow}{M}]^* \cdot (\overset{\hookleftarrow}{N} + \overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N}) \cdot \overleftarrow{M} \\
&= \overset{\hookleftarrow}{M} + \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot [\overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^*]^* \\
&\qquad \cdot (\overset{\hookleftarrow}{N} + \overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N}) \cdot \overleftarrow{M} \\
&= \overset{\hookleftarrow}{M} + \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot [\overrightarrow{N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^*]^* \cdot \overset{\hookleftarrow}{N} \cdot \overleftarrow{M} \\
&\quad + \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N} \cdot [(\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N}]^* \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overleftarrow{M}
\end{aligned}$$

$$\begin{aligned}
&= \overset{\hookleftarrow}{M} + \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overset{\hookleftarrow}{N} \cdot \overleftarrow{M} + \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N} \\
&\quad \cdot [(\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N}]^* \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overset{\hookleftarrow}{N} \cdot \overleftarrow{M} \\
&\quad + \overrightarrow{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N} \cdot [\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N} + \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N}]^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overleftarrow{M}
\end{aligned}$$

$$
\begin{aligned}
&= \overset{\hookleftarrow}{M \mathbin{\text{ƨ}} N} \\
&\quad + \overrightarrow{M \mathbin{\text{ƨ}} N} \cdot [\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N} + \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N}]^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overset{\hookleftarrow}{N} \cdot \overleftarrow{M} \\
&\quad + \overrightarrow{M \mathbin{\text{ƨ}} N} \cdot [\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N} + \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N}]^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overleftarrow{M} \\
&= \overset{\hookleftarrow}{M \mathbin{\text{ƨ}} N} + \overrightarrow{M \mathbin{\text{ƨ}} N} \cdot [\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{N} + \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot \overset{\hookrightarrow}{M} \cdot (\overset{\hookleftarrow}{N} \cdot \overset{\hookrightarrow}{M})^* \cdot \overrightarrow{N}]^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{N} \cdot (\overset{\hookrightarrow}{M} \cdot \overset{\hookleftarrow}{N})^* \cdot \overleftarrow{M} \\
&= \overset{\hookleftarrow}{M \mathbin{\text{ƨ}} N} + \overrightarrow{M \mathbin{\text{ƨ}} N} \cdot (\overset{\hookleftarrow}{P} \cdot \overset{\hookrightarrow}{M \mathbin{\text{ƨ}} N})^* \cdot \overset{\hookleftarrow}{P} \cdot \overleftarrow{M \mathbin{\text{ƨ}} N} = \overset{\hookleftarrow}{(M \mathbin{\text{ƨ}} N) \mathbin{\text{ƨ}} P}.
\end{aligned}
$$

Similar computations show that $\overleftarrow{M \mathbin{\text{ƨ}} (N \mathbin{\text{ƨ}} P)} = \overleftarrow{(M \mathbin{\text{ƨ}} N) \mathbin{\text{ƨ}} P}$ and $\overset{\hookrightarrow}{(M \mathbin{\text{ƨ}} N) \mathbin{\text{ƨ}} P} = \overset{\hookrightarrow}{M \mathbin{\text{ƨ}} (N \mathbin{\text{ƨ}} P)}$. □

4.2. Definition of two-way representations

Definition 4.2. Let $\mathbb{K}$ be a Conway semiring and let A be an alphabet. Let m and n be two nonnegative integers. A two-way representation over A^* with dimension $m+n$ is a tuple $\rho = (I, \mu, \Diamond, T)$, where I and T are vectors in $\mathbb{K}^m$, μ is a morphism from A^* into $(\mathbb{K}^{(m+n)\times(m+n)}, \mathbin{\text{ƨ}})$, and $\Diamond$ is a matrix in $\mathbb{K}^{(m+n)\times(m+n)}$ such that $\overrightarrow{\Diamond} = \mathsf{Id}_m$ and $\overleftarrow{\Diamond} = 0$.

The series realized by ρ is the series $|\rho|$ defined by

$$|\rho| = \sum_{w \in A^*} \left(I \cdot \overrightarrow{\Diamond \mathbin{\text{ƨ}} \mu(w) \mathbin{\text{ƨ}} \Diamond} \cdot T \right) w.$$

A series is *two-way* $\mathbb{K}$-*recognizable* if it can be realized by a two-way $\mathbb{K}$-representation.

For every word w, $\overrightarrow{\mu}$ is the application which maps w on $\overrightarrow{\mu(w)}$; the applications $\overset{\hookleftarrow}{\mu}$, $\overset{\hookrightarrow}{\mu}$ and $\overleftarrow{\mu}$ are defined likewise.

Example 4.3. We consider a two-way representation of a randow walk on the input. This example comes from [2]. Let $\rho = (I, \mu, \Diamond, T)$ be the two-way $(\mathbb{Q}_+ \cup \{\infty\})$-representation over $\{a\}^*$ with size $1+1$ defined by:

$$I = [1], \qquad T = [1], \qquad \Diamond = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right], \qquad \mu(a) = \left[\begin{array}{c|c} 1/2 & 1/2 \\ \hline 1/2 & 1/2 \end{array}\right]$$

Hence, the weight of a^n in $|\rho|$ is equal to $\overrightarrow{\mu}(a^n)$ for every n. By induction, for every n, $\overrightarrow{\mu}(a^n) = [1/(n+1)]$ and $\overset{\hookleftarrow}{\mu}(a^n) = [n/(n+1)]$. Therefore, $|\rho| = \sum_{k \geqslant 0} \dfrac{a^k}{k+1}$. This series is not in $\mathbb{Q}\mathsf{Rat}a^*$; by [18], it is even not in the closure by the Hadamard product. Nevertheless, it belongs to $\mathbb{Q}\mathsf{Had}a^*$:

$$|\rho| = \left(a^* - \left(\frac{a}{2}\right)^* \cdot \left(\frac{a}{2}\right)^* \right)^{\circledast} \odot \left(\frac{a}{2}\right)^*.$$

4.3. Effective computation

We describe here an algorithm to compute the coefficient of a word $w = w_1 \dots w_k$ in $|\rho|$, close to the one presented in [12] for the evaluation in pebble weighted automata. For every i in $[0; k]$, we set $X_i = I, \dots, \overrightarrow{\Diamond \mathbin{\text{ƨ}} \mu(w_1, \dots, w_i)}$ and $Y_i = \overset{\hookrightarrow}{\Diamond \mathbin{\text{ƨ}} \mu(w_1, \dots, w_i)}$. It holds

$$\begin{cases} X_0 = I, \\ Y_0 = \overset{\hookrightarrow}{\Diamond}, \end{cases} \qquad \forall i \in [1; k], \begin{cases} X_i = X_{i-1} \cdot (\overset{\hookleftarrow}{\mu}(w_i) \cdot Y_{i-1})^* \cdot \overrightarrow{\mu}(w_i), \\ Y_i = \overset{\hookrightarrow}{\mu}(w_i) + \overleftarrow{\mu}(w_i) \cdot Y_{i-1} \cdot (\overset{\hookleftarrow}{\mu}(w_i) \cdot Y_{i-1})^* \cdot \overrightarrow{\mu}(w_i). \end{cases}$$

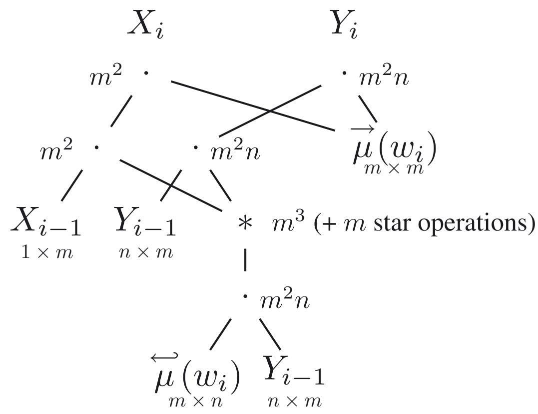

FIGURE 4. Evaluation scheme of X_i and Y_i.

The complexity of the computation of (X_k, Y_k) depends on the complexity of the operations in the semiring $\mathbb{K}$; usually, the addition is less expensive than the multiplication and the star; moreover, the star of a matrix of size n can be computed with $O(n^3)$ multiplications and $O(n)$ star operations(for instance with the McNaughton-Yamada algorithm [14]). Notice that we consider the naive algorithm for the multiplication of matrices: for every m, n, r, the multiplication of M in $\mathbb{K}^{m\times r}$ and N in $\mathbb{K}^{r\times n}$ can be performed with $O(mnr)$ multiplications. We evaluate the complexity of each step of the induction. Assume that ρ is a representation with size $m+n$; for every i, X_i is a vector of size m and Y_i is a matrix of size $n \times m$. Figure 4 shows the evaluation scheme for X_i and Y_i, with the cost of each operation. Moreover, if n is negligible w.r.t. m, $(\overleftrightarrow{\mu}(w_i) \cdot Y_{i-1})^* = \mathsf{Id} + \overleftrightarrow{\mu}(w_i) \cdot (Y_{i-1} \cdot \overleftrightarrow{\mu}(w_i))^* \cdot Y_{i-1}$ can be computed with $O(m^2 n)$ multiplications and $O(n)$ star operations. Finally, since $\langle|\rho|, w\rangle = X_k \cdot (\overleftrightarrow{\Diamond} \cdot Y_k)^* \cdot T$, we obtain:

Proposition 4.4. *Let ρ be a two-way representation of size $m+n$. The coefficient of a word of length k in $|\rho|$ is computed with $O(km^2n)$ multiplications and $O(k\min(m,n))$ star operations.*

4.4. Representation of a two-way automaton.

Definition 4.5. Let $\mathbb{K}$ be a Conway semiring and let $\mathcal{A} = (Q_+, Q_-, A, \Box, \mathbb{K}, E, I, T)$ be a two-way $\mathbb{K}$-automaton. The representation of $\mathcal{A}$, with dimension $Q_+ + Q_-$, is $\rho = (I, \mu, \Diamond, T)$ defined by:

$$\forall a \in A,\ \forall (p,q) \in Q \times Q,\ \mu(a)_{p,q} = E(p,a,q)$$
$$\forall (p,q) \in Q_+ \times Q_- \cup Q_- \times Q_+,\ \Diamond_{p,q} = E(p,\Box,q).$$

In the case where $\mathbb{K}$ is the Boolean semiring, the *monoid* $\mathcal{M}$ generated by $\{\mu(a) \mid a \in A^*\}$ (with ƨ as product) is the (finite) transition monoid of the two-way automaton, defined in [3]; μ is a morphism from A^* into $\mathcal{M}$, and the language accepted by the automaton is $\mu^{-1}(P)$, with $P = \{M \in \mathcal{M} \mid I \cdot \overrightarrow{\Diamond\, \text{ƨ}\, M\, \text{ƨ}\, \Diamond} \cdot T = 1\}$. It is a proof of the classical result [17, 21]: languages accepted by two-way finite automata are recognizable languages.

In general, the Conway properties do not tell how the sum of the weights of all computations on a given word can be computed in the two-way automaton. Hence, the computations made with the representation do not imply that the two-way automaton is valid. In Section 7, we describe a framework where the star and the infinite sums are related and where two-way representations and two-way automata are equivalent models.

The following classes of two-way $\mathbb{K}$-representations correspond to classes defined for two-way $\mathbb{K}$-automata above.

Definition 4.6. Let $\rho = (I, \mu, \Diamond, T)$ be a two-way $\mathbb{K}$-representation of size $m + n$.

- ρ is *sweeping* if for every letter a, $\overset{\hookleftarrow}{\mu}(a) = 0$ and $\overset{\hookrightarrow}{\mu}(a) = 0$.
- ρ is *rotating* if it is sweeping and, for every letter a, $\overleftarrow{\mu}(a) = \mathsf{Id}$.
- If $n = 0$, ρ is *one-way*. In this case, $\Diamond$ can be ignored and the representation is a linear $\mathbb{K}$-representation, as defined in [11]. By the Kleene–Schützenberger Theorem [20], a series over A^* can be realized by a one-way $\mathbb{K}$-representation if and only if it is in $\mathbb{K}\mathsf{Rat}A^*$.

4.5. Closure properties

Like in the case of linear (one-way) representations, the set of series realized by two-way representations is closed by a number of operations.

Proposition 4.7. *The set of series realized by two-way (resp. sweeping, resp. rotating) $\mathbb{K}$-representations is closed under the pointwise operations.*

Proof. The following constructions realize the pointwise operations; they clearly preserve the sweeping and the rotating properties.

Let $\rho_i = (I_i, \mu_i, \Diamond_i, T_i)$ be a representation of size $m_i + n_i$, for i in $\{1, 2\}$.

The *sum* $|\rho_1| + |\rho_2|$ is realized by this representation with size $(m_1 + m_2) + (n_1 + n_2)$:

$$\left(\left[\begin{array}{c|c} I & J \end{array}\right], \left[\begin{array}{cc|cc} \overrightarrow{\mu_1} & 0 & \overset{\hookleftarrow}{\mu_1} & 0 \\ 0 & \overrightarrow{\mu_2} & 0 & \overset{\hookleftarrow}{\mu_2} \\ \hline \overset{\hookrightarrow}{\mu_1} & 0 & \overleftarrow{\mu_1} & 0 \\ 0 & \overset{\hookrightarrow}{\mu_2} & 0 & \overleftarrow{\mu_2} \end{array}\right], \left[\begin{array}{c|cc} \mathsf{Id} & \overset{\hookleftarrow}{\Diamond}_1 & 0 \\ & 0 & \overset{\hookleftarrow}{\Diamond}_2 \\ \hline \begin{array}{cc} \overset{\hookrightarrow}{\Diamond}_1 & 0 \\ 0 & \overset{\hookrightarrow}{\Diamond}_2 \end{array} & 0 & \end{array}\right], \left[\begin{array}{c} T_1 \\ \hline T_2 \end{array}\right] \right).$$

The *Hadamard product* $|\rho_1| \odot |\rho_2|$ is realized by this representation with size $(m_1 + m_2) + (n_1 + 1 + n_2)$:

$$\left(\left[\begin{array}{c|c} I_1 & 0 \end{array}\right], \left[\begin{array}{cc|ccc} \overrightarrow{\mu_1} & 0 & \overset{\hookleftarrow}{\mu_1} & 0 & 0 \\ 0 & \overrightarrow{\mu_2} & 0 & 0 & \overset{\hookleftarrow}{\mu_2} \\ \hline \overset{\hookrightarrow}{\mu_1} & 0 & \overleftarrow{\mu_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \overset{\hookrightarrow}{\mu_2} & 0 & 0 & \overleftarrow{\mu_2} \end{array}\right], \left[\begin{array}{c|ccc} \mathsf{Id} & \overset{\hookleftarrow}{\Diamond}_1 & T_1 & 0 \\ & 0 & 0 & \overset{\hookleftarrow}{\Diamond}_2 \\ \hline \begin{array}{cc} \overset{\hookrightarrow}{\Diamond}_1 & 0 \\ 0 & I_2 \\ 0 & \overset{\hookrightarrow}{\Diamond}_2 \end{array} & & 0 & \end{array}\right], \left[\begin{array}{c} 0 \\ \hline T_2 \end{array}\right] \right).$$

The *Hadamard iteration* $|\rho_1|^{\circledast}$ is realized by this representation with size $(1 + m_1) + (n_1 + 1)$:

$$\rho_{\circledast} = \left(\left[\begin{array}{c|c} 1 & I_1 \end{array}\right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \overrightarrow{\mu_1} & \overset{\hookleftarrow}{\mu_1} & 0 \\ \hline 0 & \overset{\hookrightarrow}{\mu_1} & \overleftarrow{\mu_1} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{c|cc} \mathsf{Id} & 0 & 0 \\ & \overset{\hookleftarrow}{\Diamond}_1 & T_1 \\ \hline \begin{array}{cc} 0 & \overset{\hookrightarrow}{\Diamond}_1 \\ 0 & I_1 \end{array} & 0 & \end{array}\right], \left[\begin{array}{c} 1 \\ \hline T_1 \end{array}\right] \right).$$

The correctness of the construction for the sum is straighforward. The proof of the correctness of the construction for the Hadamard product is similar to the proof for the Hadamard iteration that we detail here. Let

$\rho_{\circledast} = (I_{\circledast}, \mu_{\circledast}, \lozenge_{\circledast}, T_{\circledast})$. We prove by induction that, for every word w,

$$I_{\circledast} \cdot \overrightarrow{\lozenge_{\circledast} \text{ƨ}\, \mu_{\circledast}(w)} = \left[\begin{array}{c|c} 1 & I_1 \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \end{array}\right], \qquad \overleftrightarrow{\lozenge_{\circledast} \text{ƨ}\, \mu_{\circledast}(w)} = \left[\begin{array}{c|c} 0 & \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \\ \hline 0 & I \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \end{array}\right].$$

It is true if w if the empty word, and, if it is true for w, for every letter a:

$$\begin{aligned}
&(\overleftrightarrow{\mu_{\circledast}}(a) \cdot \overleftrightarrow{\lozenge_{\circledast} \text{ƨ}\, \mu_{\circledast}(w)})^* \cdot \overrightarrow{\mu_{\circledast}}(a) \\
&\quad = \left(\left[\begin{array}{c|c} 0 & 0 \\ \hline \overleftarrow{\mu}_1(a) & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \\ \hline 0 & I_1 \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \end{array}\right] \right)^* \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \overrightarrow{\mu}_1(a) \end{array}\right] \\
&\quad = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \overleftarrow{\mu}_1(a) \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \end{array}\right]^* \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \overrightarrow{\mu}_1(a) \end{array}\right] \\
&\quad = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & (\overleftarrow{\mu}_1(a) \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)})^* \cdot \overrightarrow{\mu}_1(a) \end{array}\right].
\end{aligned}$$

$$\begin{aligned}
I_{\circledast} . \overrightarrow{\lozenge_{\circledast} \text{ƨ}\, \mu_{\circledast}(wa)} &= \left[\begin{array}{c|c} 1 & I_1 . \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & (\overleftarrow{\mu}_1(a) \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)})^* \cdot \overrightarrow{\mu}_1(a) \end{array}\right] \\
&= \left[\begin{array}{c|c} 1 & I_1 \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \cdot (\overleftarrow{\mu}_1(a) \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)})^* \cdot \overrightarrow{\mu}_1(a) \end{array}\right] = \left[\begin{array}{c|c} 1 & I_1 \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(wa)} \end{array}\right].
\end{aligned}$$

$$\begin{aligned}
&\overleftrightarrow{\lozenge_{\circledast} \text{ƨ}\, \mu_{\circledast}(wa)} \\
&= \left[\begin{array}{c|c} 0 & \overleftrightarrow{\mu}_1(a) \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & \overleftarrow{\mu}_1(a) \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \\ \hline 0 & I_1 \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & (\overleftarrow{\mu}_1(a) \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)})^* \cdot \overrightarrow{\mu}_1(a) \end{array}\right] \\
&= \left[\begin{array}{c|c} 0 & \overleftrightarrow{\mu}_1(a) + \overleftarrow{\mu}_1(a) \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \cdot (\overleftarrow{\mu}_1(a) \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)})^* \cdot \overrightarrow{\mu}_1(a) \\ \hline 0 & I_1 \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \cdot (\overleftarrow{\mu}_1(a) \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)})^* \cdot \overrightarrow{\mu}_1(a) \end{array}\right] \\
&= \left[\begin{array}{c|c} 0 & \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(wa)} \\ \hline 0 & I \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(wa)} \end{array}\right].
\end{aligned}$$

We then compute $(\overleftrightarrow{\lozenge}_{\circledast} \cdot \overleftrightarrow{\lozenge_{\circledast} \text{ƨ}\, \mu_{\circledast}(w)})^*$:

$$\begin{aligned}
\left(\left[\begin{array}{c|c} 0 & 0 \\ \hline \overleftrightarrow{\lozenge}_1 & T_1 \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & \overleftrightarrow{\lozenge_1 \text{ƨ}_1\, \mu(w)} \\ \hline 0 & I_1 \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \end{array}\right] \right)^* &= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & (\overleftrightarrow{\lozenge}_1 \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} + T_1 \cdot I_1 \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)})^* \end{array}\right] \\
&= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & (\overleftrightarrow{\lozenge}_1 \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)})^* \cdot (T_1 \cdot I_1 \cdot \overrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)} \cdot (\overleftrightarrow{\lozenge}_1 \cdot \overleftrightarrow{\lozenge_1 \text{ƨ}\, \mu_1(w)})^*)^* \end{array}\right].
\end{aligned}$$

Thus, $(I_\circledast \cdot \overrightarrow{\Diamond_\circledast \,\text{ƨ}\, \mu_\circledast(w)}) \cdot (\overleftrightarrow{\Diamond}_\circledast \cdot \overset{\hookrightarrow}{\Diamond_\circledast \,\text{ƨ}\, \mu_\circledast(wa)})^* \cdot T_\circledast$ is equal to

$$
\begin{aligned}
&\left[\begin{array}{c|c} 1 & I_1 \cdot \overrightarrow{\Diamond_1 \,\text{ƨ}\, \mu_1(w)} \end{array}\right] \cdot \left(\left[\begin{array}{c|c} 0 & 0 \\ \hline \overleftrightarrow{\Diamond}_1 & T_1 \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & \overset{\hookrightarrow}{\Diamond_1 \,\text{ƨ}\, \mu_1(w)} \\ \hline 0 & I_1 \cdot \overrightarrow{\Diamond_1 \,\text{ƨ}\, \mu_1(w)} \end{array}\right] \right)^* \cdot \left[\begin{array}{c} 1 \\ \hline T_1 \end{array}\right] \\
=&1 + I_1 \cdot \overrightarrow{\Diamond_1 \,\text{ƨ}\, \mu_1(w)} \cdot (\overleftrightarrow{\Diamond}_1 \cdot \overset{\hookrightarrow}{\Diamond_1 \,\text{ƨ}\, \mu_1(w)})^* \cdot (T_1 \cdot I_1 \cdot \overrightarrow{\Diamond_1 \,\text{ƨ}\, \mu_1(w)} \cdot (\overleftrightarrow{\Diamond}_1 \cdot \overset{\hookrightarrow}{\Diamond_1 \,\text{ƨ}\, \mu_1(w)})^*)^* \cdot T_1 \\
=&1 + I_1 \cdot \overrightarrow{\Diamond_1 \,\text{ƨ}\, \mu_1(w)} \cdot (\overleftrightarrow{\Diamond}_1 \cdot \overset{\hookrightarrow}{\Diamond_1 \,\text{ƨ}\, \mu_1(w)})^* \cdot T_1 \cdot (I_1 \cdot \overrightarrow{\Diamond_1 \,\text{ƨ}\, \mu_1(w)} \cdot (\overleftrightarrow{\Diamond}_1 \cdot \overset{\hookrightarrow}{\Diamond_1 \,\text{ƨ}\, \mu_1(w)})^* \cdot T_1)^* \\
=&1 + \langle |\rho_1|, w\rangle \cdot \langle |\rho_1|, w\rangle^* = \langle |\rho_1|, w\rangle^* = \langle |\rho_1|^{\circledast}, w\rangle,
\end{aligned}
$$

which is the coefficient of w in $|\rho_\circledast|$. □

Proposition 4.8. *The set of series realized by two-way (resp. sweeping) $\mathbb{K}$-representations is closed under mirror.*

Let $\rho = (I, \mu, \Diamond, T)$ be a two-way representation of size $m + n$. The following two-way representation with size $(n + 2) + m$ realizes the mirror of $|\rho|$.

$$
\left(\left[\begin{array}{c|c|c} 1 & 0 & 0 \end{array}\right], \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & \overleftarrow{\mu} & 0 & \overset{\hookrightarrow}{\mu} \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & \overleftrightarrow{\mu} & 0 & \overrightarrow{\mu} \end{array}\right], \left[\begin{array}{c|c|c|c} \multicolumn{3}{c|}{\mathsf{Id}} & \begin{array}{c} I \\ \hline \overset{\hookrightarrow}{\Diamond} \\ \hline 0 \end{array} \\ \hline 0 & \overleftrightarrow{\Diamond} & T & 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ \hline 1 \\ \hline \end{array}\right] \right).
$$

This construction preserves the sweeping property.

From Propositions 4.7 and 4.8, it comes:

Proposition 4.9. *Let $\mathbb{K}$ be a Conway semiring and A an alphabet.*

Every series in $\mathbb{K}\mathsf{Had}A^*$ can be realized by a rotating $\mathbb{K}$-representation.

Every series in $\mathbb{K}\mathsf{MirHad}A^*$ can be realized by a sweeping $\mathbb{K}$-representation.

We shall see further that two-way recognizable series are not closed under Cauchy product and Kleene star. Nevertheless, they are closed under left (and right) quotients:

Proposition 4.10. *Let $\rho = (I, \mu, \Diamond, T)$ be a two-way $\mathbb{K}$-representation and let w be a word. Then, the series $w^{-1}|\rho| = \sum_{v\in A^*} \langle |\rho|, wv\rangle v$ is realized by the two-way $\mathbb{K}$-representation $(I \cdot \overrightarrow{\Diamond \,\text{ƨ}\, \mu(w)}, \mu, \Diamond', T)$, with $\overset{\hookrightarrow}{\Diamond'} = \overset{\hookrightarrow}{\Diamond \,\text{ƨ}\, \mu(w)}$, and $\overleftrightarrow{\Diamond'} = \overleftrightarrow{\Diamond}$.*

Proof. For every word v, $\overrightarrow{\mu(v) \,\text{ƨ}\, \Diamond'} = \overrightarrow{\mu}(v) \cdot (\overleftrightarrow{\Diamond'} \cdot \overset{\hookrightarrow}{\mu}(v))^* = \overrightarrow{\mu(v) \,\text{ƨ}\, \Diamond}$, and $\overleftrightarrow{\mu(v) \,\text{ƨ}\, \Diamond'} = \overleftrightarrow{\mu}(v) + \overrightarrow{\mu}(v) \cdot (\overleftrightarrow{\Diamond'} \cdot \overset{\hookrightarrow}{\mu}(v))^* \cdot \overleftrightarrow{\Diamond'} \cdot \overleftarrow{\mu}(v) = \overleftrightarrow{\mu(v) \,\text{ƨ}\, \Diamond}$, hence

$$
\begin{aligned}
I \cdot \overrightarrow{\Diamond \,\text{ƨ}\, \mu(w)} \cdot \overrightarrow{\Diamond' \,\text{ƨ}\, \mu(v) \,\text{ƨ}\, \Diamond'} \cdot T &= I \cdot \overrightarrow{\Diamond \,\text{ƨ}\, \mu(w)} \cdot (\overleftrightarrow{\mu(v) \,\text{ƨ}\, \Diamond'} \cdot \overset{\hookrightarrow}{\Diamond'})^* \cdot \overrightarrow{\mu(v) \,\text{ƨ}\, \Diamond'} \cdot T \\
&= I \cdot \overrightarrow{\Diamond \,\text{ƨ}\, \mu(w)} \cdot (\overleftrightarrow{\mu(v) \,\text{ƨ}\, \Diamond} \cdot \overset{\hookrightarrow}{\Diamond \,\text{ƨ}\, \mu(w)})^* \cdot \overrightarrow{\mu(v) \,\text{ƨ}\, \Diamond} \cdot T \\
&= I \cdot \overrightarrow{\Diamond \,\text{ƨ}\, \mu(w) \,\text{ƨ}\, \mu(v) \,\text{ƨ}\, \Diamond} \cdot T = \langle w^{-1}|\rho|, v\rangle \cdot
\end{aligned}
$$

□

5. Two-way-recognizable series as fixed points

In the case of one-way $\mathbb{K}$-automata, if M is the transition matrix of an automaton with final vector T, and X is the vector with entry in $\mathbb{K}\langle\langle A^*\rangle\rangle$ such that X_p is the series realized if p is the initial state (with weight 1), X fulfills the fixed-point equation $X = T + M \cdot X$. If each entry of M is a linear combination of letters, $S = M^* \cdot T$ is the unique solution of the equation, and thus the series realized by the automaton is $I \cdot M^* \cdot T$, where I is the initial vector. This is the foundation of algorithms that convert one-way automata to rational expressions.

We set a similar equation for two-way representations. We consider the matrix algebras respectively induced by the Cauchy and the Hadamard products of series. We use the same symbol to denote the product of matrices as the product of scalars. Notice that $M \odot N$ is *not* the Hadamard (entrywise) product of matrices. More precisely,

$$\begin{aligned}\forall M \in&(\mathbb{K}\langle\langle A^*\rangle\rangle)^{m\times k}, N \in (\mathbb{K}\langle\langle A^*\rangle\rangle)^{k\times n},\ \forall (i,j) \in [1;m] \times [1;n]\\ (M \cdot N)_{i,j} &= \sum_{h=1}^{k} M_{i,h} \cdot N_{h,j}, \quad \text{and } (M \odot N)_{i,j} = \sum_{h=1}^{k} M_{i,h} \odot N_{h,j}.\end{aligned}$$

Let A_n^* be the diagonal matrix with entries A^* (the dimension may be omitted); it is the unit for the matrix product involving the Hadamard product of series. We use these algebras to characterize series realized by a two-way $\mathbb{K}$-representation as fixed-points.

Theorem 5.1. *Let $\mathbb{K}$ be a Conway semiring. Let $\rho = (I, \mu, \Diamond, T)$ be a two-way $\mathbb{K}$-representation and let $M = \sum_{a\in A} \mu(a)a$. Then $|\rho| = I \cdot (X_0 \cdot \overleftrightarrow{\Diamond})^{\circledast} \odot Y_0$, where (X_0, Y_0) is the unique solution of the system*

$$\begin{cases} X = \overleftrightarrow{\Diamond} + \overleftrightarrow{M} \cdot \mathsf{A}^* + (\overrightarrow{M} \cdot X) \odot (\overleftrightarrow{M} \cdot X)^{\circledast} \odot (\overleftarrow{M} \cdot \mathsf{A}^*), \\ Y = T + \overrightarrow{M} \cdot Y + (\overrightarrow{M} \cdot X) \odot (\overleftrightarrow{M} \cdot X)^{\circledast} \odot (\overleftrightarrow{M} \cdot Y). \end{cases} \tag{5.1}$$

In equation (5.1), X comes from the inductive description of backward-turn paths, and Y from the inductive description of forward paths to final states. For instance, starting from a given state, a forward path (contributing to Y) is either the empty path (term T), or a forward transition followed by a forward path ($\overrightarrow{M} \cdot Y$), or a path where the head comes back (potentially many times) at the first position (last term). The move of the head to the right is reflected by the Cauchy product; the fact that when the head is at the same position, it must read the same letter is reflected by the Hadamard product.

Proof. Assume first that there exists a solution (X_0, Y_0). We prove that $(X_0, Y_0) = (\overleftrightarrow{S}, \overrightarrow{S} \cdot T)$, with $S = \sum_{w\in A^*} (\mu(w) ƨ \Diamond)w$. We first prove by induction that $X_0 = \overleftrightarrow{S}$. The matrix M is proper, hence $\langle X_0, \varepsilon\rangle = \overleftrightarrow{\Diamond} = \overleftrightarrow{\mu(\varepsilon) ƨ \Diamond}$. Let w be a word such that $\langle X_0, w\rangle = \overleftrightarrow{\mu(w) ƨ \Diamond}$. Then, for every letter a,

$$\begin{aligned}\langle X_0, aw\rangle &= \langle \overleftrightarrow{M} \cdot \mathsf{A}^*, aw\rangle + \langle \overrightarrow{M} \cdot X_0, aw\rangle \cdot \langle \overleftrightarrow{M} \cdot X_0, aw\rangle^* \cdot \langle \overleftarrow{M} \cdot \mathsf{A}^*, aw\rangle\\ &= \langle \overleftrightarrow{M}, a\rangle + \langle \overrightarrow{M}, a\rangle \cdot \langle X_0, w\rangle \cdot (\langle \overleftrightarrow{M}, a\rangle \cdot \langle X_0, w\rangle)^* \cdot \langle \overleftarrow{M}, a\rangle\\ &= \overleftrightarrow{\mu}(a) + \overrightarrow{\mu}(a) \cdot \overleftrightarrow{\mu(w) ƨ \Diamond} \cdot (\overleftrightarrow{\mu}(a) \cdot \overleftrightarrow{\mu(w) ƨ \Diamond})^* \cdot \overleftarrow{\mu}(a)\\ &= \overleftrightarrow{\mu(a) ƨ \mu(w) ƨ \Diamond} = \overleftrightarrow{\mu(aw) ƨ \Diamond} = \langle \overleftrightarrow{S}, aw\rangle.\end{aligned}$$

Likewise, we prove by induction that $Y_0 = \overrightarrow{S} \cdot T$. It is then easy to check that $(\overleftrightarrow{S}, \overrightarrow{S} \cdot T)$ is actually a solution to the system. Finally, for every word w,

$$\begin{aligned}\langle I \cdot (X_0 \cdot \overset{\hookrightarrow}{\Diamond})^{\circledast} \odot Y_0, w\rangle &= I \cdot (\overleftrightarrow{\mu(w)\,\text{ƨ}\,\Diamond} \cdot \overset{\hookrightarrow}{\Diamond})^{*} \cdot \overrightarrow{\mu(w)\,\text{ƨ}\,\Diamond} \cdot T \\ &= I \cdot \overrightarrow{\Diamond} \cdot (\overleftrightarrow{\mu(w)\,\text{ƨ}\,\Diamond} \cdot \overset{\hookrightarrow}{\Diamond})^{*} \cdot \overrightarrow{\mu(w)\,\text{ƨ}\,\Diamond} \cdot T = I \cdot \overrightarrow{\Diamond\,\text{ƨ}\,\mu(w)\,\text{ƨ}\,\Diamond} \cdot T.\end{aligned}$$ □

Theorem 5.1 is an implicit characterization of the series realized by a two-way representation. Unfortunately, equation (5.1) is not easy to solve (and does not always accept solutions in $\mathbb{K}\mathsf{RH}A^*$). In the next section, we solve it in the particular case of sweeping representations.

6. Rotating and sweeping representations

In this section, we characterize series realized by rotating and sweeping representations and we show that they can be denoted by explicit expressions involving rational, pointwise and mirror operators.

Proposition 6.1. *If $\rho = (I, \mu, \Diamond, T)$ is a rotating $\mathbb{K}$-representation,*

$$|\rho| = I \cdot \big(\overrightarrow{M}^{*} \cdot \overleftrightarrow{\Diamond} \cdot \overset{\hookrightarrow}{\Diamond}\big)^{\circledast} \odot \overrightarrow{M}^{*} \cdot T, \text{ with } M = \sum_{a \in A} \mu(a) a.$$

Proof. Since ρ is rotating, equation (5.1) reduces to $X = \overleftrightarrow{\Diamond} + \overrightarrow{M} \cdot X,\ Y = T + \overrightarrow{M} \cdot Y$. The solution of this system is $(X_0, Y_0) = (\overrightarrow{M}^{*} \cdot \overleftrightarrow{\Diamond}, \overrightarrow{M}^{*} \cdot T)$. □

Proposition 6.2. *If $\rho = (I, \mu, \Diamond, T)$ is a sweeping $\mathbb{K}$-representation,*

$$|\rho| = I \cdot \big((\overrightarrow{M}^{*} \cdot \overleftrightarrow{\Diamond}) \odot (\overline{\overleftarrow{M}^{*}} \cdot \overset{\hookrightarrow}{\Diamond})\big)^{\circledast} \odot \overrightarrow{M}^{*} \cdot T, \text{ with } M = \sum_{a \in A} \mu(a) a.$$

Proof. Since ρ is sweeping, equation (5.1) reduces to

$$X = \overleftrightarrow{\Diamond} + (\overrightarrow{M} \cdot X) \odot (\overleftarrow{M} \cdot \mathsf{A}^*), \qquad Y = T + \overrightarrow{M} \cdot Y.$$

It immediatly comes $Y = \overrightarrow{M}^{*} \cdot T$, and we prove by induction on the length of words that, for every word w, $\langle X, w\rangle = \langle (\overrightarrow{M}^{*} \cdot \overleftrightarrow{\Diamond}) \odot \overline{\overleftarrow{M}^{*}}, w\rangle$.

It is true when $w = \varepsilon$ ($\overrightarrow{M}$ and $\overleftarrow{M}$ are proper), and if it is true for a word w, then, for every letter a,

$$\begin{aligned}\langle X, aw\rangle &= \langle \overleftrightarrow{\Diamond} + (\overrightarrow{M} \cdot X) \odot (\overleftarrow{M} \cdot A^*), aw\rangle = \langle \overrightarrow{M}, a\rangle \cdot \langle X, w\rangle \cdot \langle \overleftarrow{M}, a\rangle \\ &= \langle \overrightarrow{M}, a\rangle \cdot \langle (\overrightarrow{M}^{*} \cdot \overleftrightarrow{\Diamond}) \odot \overline{\overleftarrow{M}^{*}}, w\rangle \cdot \langle \overleftarrow{M}, a\rangle \\ &= \langle \overrightarrow{M}, a\rangle \cdot \langle \overrightarrow{M}^{*}, w\rangle \cdot \overleftrightarrow{\Diamond}) \cdot \langle \overleftarrow{M}^{*}, \overline{w}\rangle \cdot \langle \overleftarrow{M}, a\rangle \\ &= \langle \overrightarrow{M}^{*}, aw\rangle \cdot \overleftrightarrow{\Diamond}) \cdot \langle \overleftarrow{M}^{*}, \overline{aw}\rangle = \langle (\overrightarrow{M}^{*} \cdot \overleftrightarrow{\Diamond}) \odot \overline{\overleftarrow{M}^{*}}, aw\rangle.\end{aligned}$$

Finally, $X = (\overrightarrow{M}^{*} \cdot \overleftrightarrow{\Diamond}) \odot \overline{\overleftarrow{M}^{*}}$. The proposition follows then by Theorem 5.1. □

Since the star of matrices can be effectively computed by the usual formulae, these propositions state that, for rotating and sweeping representations, an explicit expression (involving rational, pointwise and mirror operators) representing the realized series can be computed.

The following theorem results from Propositions 4.9, 6.1, and 6.2.

Theorem 6.3. *Let $\mathbb{K}$ be a Conway semiring. A series can be realized by a rotating $\mathbb{K}$-representation if and only if it is in $\mathbb{K}\mathsf{Had}A^*$. A series can be realized by a sweeping $\mathbb{K}$-representation if and only if it is in $\mathbb{K}\mathsf{MirHad}A^*$.*

7. Two-way automata over rationally additive semirings

7.1. Equivalence between Automata and representations

We consider now *rationally additive semirings* introduced in [10], and we prove in this framework the equivalence between two-way weighted automata and two-way representations.

Definition 7.1. [10] A semiring $\mathbb{K}$ is a rationally additive semiring if it is equipped with an operator $\sum$ defined on some countable families such that, for every countable set I and every family s in $\mathbb{K}^I$,

(1) if I is finite, $\sum_{i\in I} s_i$ exists and is the sum of elements of s;
(2) for every element x of $\mathbb{K}$, $\sum_{i\in\mathbb{N}} x^i$ exists and x^* is defined as $x^* = \sum_{i\in\mathbb{N}} x^i$;
(3) for every element x of $\mathbb{K}$, if $\sum s$ exists, so do $\sum_{i\in I} x\cdot s_i$ and $\sum_{i\in I} s_i \cdot x$, and

$$\sum_{i\in I} x\cdot s_i = x\cdot(\sum s) \qquad \text{and} \quad \sum_{i\in I} s_i\cdot x = (\sum s)\cdot x;$$

assume that I is the disjoint union of sets $(I_j)_{j\in J}$;
(4) if for every j in J, $r_j = \sum_{i\in I_j} s_i$ exists, and $\sum_{j\in J} r_j$ exists, then $\sum s$ exists and $\sum s = \sum_{j\in J} r_j$;
(5) if for every j in J, $r_j = \sum_{i\in I_j} s_i$ exists, and $\sum s$ exists, then $\sum_{j\in J} r_j$ exists and $\sum_{j\in J} r_j = \sum s$.

Example 7.2. Many positive semirings are rationally additive or can be completed to be rationally additive, for instance $\mathbb{N}\cup\{\infty\}$, $\mathbb{Q}_+\cup\{\infty\}$, $(\mathbb{N}\cup\{\infty\},\min,+)$, the regular languages, every complete lattice, or every positive finite semiring. Nevertheless, many current semirings are not rationally additive. For instance, no ring is a rationally additive semiring (there is no consistent definition of the star of -1). The study of two-way automata and represention in semirings that are not rationally additive requires further investigation. Specific cases have already been studied: $\mathbb{Z}$ in [1], or $(\mathbb{Z}\cup\{\infty\},\min,+)$ in [7].

Proposition 7.3 ([10], Prop. 4)**.** *A rationally additive semiring is a Conway semiring.*

This means that, in rationally additive semirings, the combinatorial approach of star (as sum of powers) and the axiomatic approach meet. As a consequence, we show that, in this framework, the combinatorial description of series by two-way automata is equivalent to the algebraic description by two-way representations.

Proposition 7.4. *Let $\mathbb{K}$ be a rationally additive semiring, and let $\rho = (I,\mu,\Diamond,T)$ be the representation of a two-way $\mathbb{K}$-automaton $\mathcal{A} = (Q_+,Q_-,A,\Box,\mathbb{K},E,I,T)$. Let $w = w_1\ldots w_n$ be a non empty word of A^*; let p,q in Q_+ and r,s be in Q_-. We consider the four following sets*:

- *$F(w)_{p,q}$ is the set of forward paths from p to q with label w,*
- *$B(w)_{r,s}$ is the set of backward paths from p to q with label w,*
- *$BT(w)_{p,s}$ is the set of backward-turn paths from p to q where w is strongly admissible,*
- *$FT(w)_{r,q}$ is the set of forward-turn paths from p to q where w is strongly admissible.*

Then, for each of these sets, the sum of the weights of the paths exists and

$$\sum_{\pi\in F(w)_{p,q}} \mathsf{w}(\pi) = \overrightarrow{\mu}(w)_{p,q}, \qquad \sum_{\pi\in BT(w)_{p,s}} \mathsf{w}(\pi) = \overleftrightarrow{\mu}(w)_{p,s},$$

$$\sum_{\pi\in FT(w)_{r,q}} \mathsf{w}(\pi) = \overleftrightarrow{\mu}(w)_{r,q}, \qquad \sum_{\pi\in B(w)_{r,s}} \mathsf{w}(\pi) = \overleftarrow{\mu}(w)_{r,s}.$$

Proof. The proof is by induction on the length of w. If $n = 1$, each set is either empty and there is no transition from p to q with label a in $\mathcal{A}$, or it is the singleton that contains the transition in $\mathcal{A}$ between the two states with label w. The equalities are therefore true by Definition 4.5.

We assume that the proposition is true for every w of length at most n. Let $w = ua$ where u is a word with length n and a is a letter. For every path $\pi = (p, (e_i)_{i\in[1;k]}, q)$ in $F(w)_{p,q}$, we set $R_\pi = \{i \in [1; k-1] \mid \delta_\pi(i) = n\}$. Let $F(w)^{(t)}_{p,q} = \{\pi \in F(w)_{p,q} \mid |R_\pi| = t\}$; obviously $F(w)_{p,q} = \bigcup_{t\in\mathbb{N}} F(w)^{(t)}_{p,q}$.

We consider now the (partial) operation of concatenation of paths: if $(p, \pi = (e_i)_{i\in[1;k]}, q)$ and $\zeta = (r, (f_i)_{i\in[1;\ell]}, s)$ are two paths with $q = r$, then $\pi \cdot \zeta$ is the path from p to s resulting from the concatention of the two sequences. This operation extends to sets of paths.

We can now decompose the paths in $F(w)^{(t)}_{p,q}$:

$$F(w)^{(t)}_{p,q} = \bigcup_{\substack{r_1,\ldots,r_t\in Q_+,\\ s_1,\ldots,s_{t-1}\in Q_-}} F(u)_{p,r_1} \cdot BT(a)_{r_1,s_1} \cdot FT(u)_{s_1,r_2} \ldots FT(u)_{s_{t-1},r_t} \cdot F(a)_{r_t,q}.$$

By induction hypothesis, the sum of the weights in each of these sets is defined, hence, by ([10], Prop. 3), the sum of the weights of paths in $F(w)^{(t)}_{p,q}$ is defined and

$$\begin{aligned}\sum_{\pi\in F(w)^{(t)}_{p,q}} \mathsf{w}(\pi) &= \sum_{\substack{r_1,\ldots,r_t\in Q_+,\\ s_1,\ldots,s_{t-1}\in Q_-}} \overrightarrow{\mu}(u)_{p,r_1} \cdot \overleftrightarrow{\mu}(a)_{r_1,s_1} \ldots \overleftrightarrow{\mu}(u)_{s_{t-1},r_t} \cdot \overrightarrow{\mu}(a)_{r_t,q} \\ &= \left(\overrightarrow{\mu}(u) \cdot \left(\overleftrightarrow{\mu}(a) \cdot \overleftrightarrow{\mu}(u)\right)^{t-1} \cdot \overrightarrow{\mu}(a)\right)_{p,q}. \end{aligned} \tag{7.1}$$

By ([10], Thm. 9), if $\mathbb{K}$ is a rationally additive semiring, so is the semiring of matrices $\mathbb{K}^{m\times m}$ for every m. Therefore, the sum over t of $\left(\overleftrightarrow{\mu}(a) \cdot \overleftrightarrow{\mu}(u)\right)^t$ is defined. Hence, by the fourth axiom of rationally additive semirings, the sum of the weights of paths in $F(w)$ exists and

$$\sum_{\pi\in F(w)_{p,q}} \mathsf{w}(\pi) = \left(\overrightarrow{\mu}(u) \cdot \left(\overleftrightarrow{\mu}(a) \cdot \overleftrightarrow{\mu}(u)\right)^* \overrightarrow{\mu}(a)\right)_{p,q} = \overrightarrow{\mu}(ua)_{p,q}.$$

We define now $BT(w)^{(t)}_{p,s}$ in a similar way, and we can decompose $BT(w)^{(t)}_{p,s}$:

$BT(w)^{(0)}_{p,s} = BT(u)_{p,s}$ and for every positive t,

$$BT(w)^{(t)}_{p,s} = \bigcup_{\substack{q_1,\ldots q_t\in Q_+,\\ r_1,\ldots r_t\in Q_-}} F(u)_{p,q_1} \cdot BT(a)_{q_1,r_1} \cdot FT(u)_{r_1,q_2} \ldots BT(a)_{q_t,r_t} \cdot B(u)_{r_t,s}.$$

By the same arguments, the sum of the weights of paths in $BT(w)^{(t)}_{p,q}$ exists and

$$\sum_{\pi\in BT(w)^{(t)}_{p,s}} \mathsf{w}(\pi) = \left(\overrightarrow{\mu}(u) \cdot \left(\overleftrightarrow{\mu}(a) \cdot \overleftrightarrow{\mu}(u)\right)^{t-1} \cdot \overleftrightarrow{\mu}(a) \cdot \overleftarrow{\mu}(u)\right)_{p,s}.$$

Finally, we get

$$\sum_{\pi\in BT(w)_{p,s}} \mathsf{w}(\pi) = \overleftrightarrow{\mu}(u)_{p,s} + \left(\overrightarrow{\mu}(u) \cdot \left(\overleftrightarrow{\mu}(a) \cdot \overleftrightarrow{\mu}(u)\right)^* \cdot \overleftrightarrow{\mu}(a) \cdot \overleftarrow{\mu}(u)\right)_{p,s} = \overleftrightarrow{\mu}(ua)_{p,s}.$$

The proof is similar for $FT(w)_{r,q}$ and $B(w)_{r,s}$. □

Theorem 7.5. *Let $\mathcal{A}$ be a weighted two-way automaton over a rationally additive semiring. Then, the automaton $\mathcal{A}$ is valid and the series realized by $\mathcal{A}$ is the same as the series realized by the representation of $\mathcal{A}$.*

Proof. The proof is similar to the proof of Proposition 7.4. Let $w = w_1 \dots w_n$ be a word in A^*; for every pair of states p, q in Q_+, let $S(w)_{p,q}$ be the set of paths $\pi = (e_i)_{i\in[1;k]}$ from p to q such that $\Box w$ is admissible for π at position 2 (the first position of w) and $\delta(\pi) = n$; like in the proof of Proposition 7.4, we show that the sum of the weights of $S(w)_{p,q}$ exists and is equal to $(\Diamond \wr \mu(w))_{p,q}$.

Likewise, if p is in Q_- and q in Q_+, we set $S(w)_{p,q}$ as the set of forward-turn paths π from p to q such that $\Box w$ is admissible for π at the last position, and $\delta(\pi) = 1$; the sum of the weights of these paths is equal to $(\Diamond \wr \mu(w))_{p,q}$.

Finally, for every pair of states p, q in Q_+, let $C(w)_{p,q}$ be the set of computations π from p to q; let $R_\pi = \{i \in [1; k-1] \mid \delta_\pi(i) = n\}$ and, for every t, let $C(w)^{(t)}_{p,q} = \{\pi \in C(w)_{p,q} \mid |R_\pi| = t\}$. Then, for every t,

$$C(w)^{(t)}_{p,q} = \bigcup_{\substack{r_1,\dots r_t \in Q_+,\\ s_1,\dots s_t \in Q_-}} S(w)_{p,r_1} \cdot f_{r_1,s_1} S(w)_{s_1,r_2} \dots f_{r_t,s_t} S(w)_{s_t,q},$$

where every f_{r_i,s_i} is a transition with label $\Box$. Hence, by the same arguments as in the proof of Proposition 7.4, the sum of the weights of the paths in $C(w)_{p,q}$ exists and

$$\sum_{\pi \in C(w)_{p,q}} \mathsf{w}(\pi) = (\overrightarrow{\Diamond \wr \mu(w)} \cdot (\overleftrightarrow{\Diamond} \cdot \overleftarrow{\Diamond \wr \mu(w)})^*)_{p,q} = (\overrightarrow{\Diamond \wr \mu(w) \wr \Diamond})_{p,q}.$$

The sum of the weights of the computations with label w that start in an initial state p and end in a final state q is therefore $I_p \cdot \overrightarrow{\Diamond \wr \mu(w) \wr \Diamond} \cdot T_q$.

Finally, the sum of the weights of the computations of $\mathcal{A}$ with label w exists and

$$\langle |\mathcal{A}|, w\rangle = I \cdot \overrightarrow{\Diamond \wr \mu(w) \wr \Diamond} \cdot T \cdot$$

$\Box$

7.2. Non closure properties

In this part, we first prove that two-way recognizable series are in general not closed under Cauchy product or the Kleene star.

Proposition 7.6. *The set of two-way recognizable series is not closed under Cauchy product or Kleene star.*

The proof is based on the following example.

Example 7.7. Let $\mathbb{K} = \mathcal{P}(\{x, y\}^*)$. We consider the series $r = \sum_{k=0}^{\infty} (x^k y^k) a^k$.

This series can be realized by a sweeping two-way $\mathbb{K}$-automaton. We consider now

$$r \cdot r = \sum_{k=0}^{\infty} \sum_{i=0}^{k} (x^i y^i x^{k-i} y^{k-i}) a^k.$$

Assume that $r \cdot r$ is realized by a two-way automaton $\mathcal{A}$ with n states. Let m be a positive integer; we consider the computations with label a^m; we first prove that for every output $w = x^i y^i x^{m-i} y^{m-i}$ there exists a computation $\pi = (p, (e_r)_{r\in[1;k]}, q)$ with this output such that the transition e_d that outputs the last y of the first block of y's satisfies $\min(\delta_\pi(d), m - \delta_\pi(d)) \leqslant 2n^2$: in this run, the last y of the first block of y's is produced when the head is "close" (independently from m and i) to the left or the right end of the input.

For $w = x^i y^i x^{m-i} y^{m-i}$ fixed, we consider a computation $\pi = (p, (e_r)_{r\in[1;k]}, q)$ with output w such that $\Delta = \min(\delta_\pi(d), m - \delta_\pi(d))$ is minimal, where d is the index of the transition which output the last y; among all candidates, we chose a computation with minimal length. If $\Delta \leqslant 2n^2$, our assumption holds. Otherwise, d is in $[2n^2 + 1; k - 2n^2]$. Every subpath of length n contains a circuit (with positive length), *i.e.* a subpath that starts and ends in the same state. Thus, the path $(e_r)_{r\in[d-n^2;d-1]}$ (*resp.* $(e_r)_{r\in[d+1;d+n^2]}$) contains n non-overlapping circuits $(c_j)_{j\in[1;n]}$ (*resp.* $(c'_j)_{j\in[1;n]}$) of length at most n. We denote $\delta(c_j) = \delta_{c_j}(l_j)$, where l_j is the length of c_j. $\delta(c_j)$ (*resp.* $\delta(c'_j)$) is not null, otherwise we could build a shorter computation by removing it. Notice also that $\delta(c_j)$ (*resp.* $\delta(c'_j)$) is in $[-n;n]$.

First case, if there exists j and j' such that $\delta(c_j)\cdot\delta(c'_{j'}) < 0$, the insertion of $|\delta(c'_{j'})|$ instances of circuit c_j immediatly after c_j and $|\delta(c_j)|$ instances of circuit $c'_{j'}$ immediatly after $c'_{j'}$ in π leads to a computation with label a^m where the position of the head on the input is shifted of $\delta(c_j)|\delta(c'_{j'})|$ for every transition occuring between circuits c_j and c'_j. Since $\Delta > 2n^2$, for every transition e_r, with r in $[d-n^2, d+n^2]$, it holds $\delta_\pi(r) > n^2$, hence the shift preserves the admissibility of the computation. Notice that the output of c_j and $c'_{j'}$ is necessarily the empty word, otherwise, we build a computation with an output which is not consistent with r. This insertion can be iterated while the transition that outputs the last y of the first block is more than $2n^2$ far away from one end of the input.

Second case, if all circuits correspond to a move in the same direction, we use the following combinatorial lemma: if $(a_j)_{j\in[1;n]}$ and $(b_j)_{j\in[1;n]}$ are two vectors of integers in $[1;n]$, there exist J and J' non empty subsets of $[1;n]$ such that $\sum_{j\in J} a_j = \sum_{j\in J'} b_j$. Hence, there exist J and J' such that the sum of $(\delta(c_j))_{j\in J}$ is equal to $(\delta(c'_j))_{j\in J'}$. Another computation can be built from π in removing circuits in $\{c_j \mid j\in J\}$ and inserting another copy of each circuit in $\{c'_j \mid j \in J'\}$ next to the first occurence. The position of the head on the input for the transition that outputs the last y of the first block is shifted of $-\sum_{j\in J} a_j$. If circuits in $\{c'_j \mid j\in J'\}$ are removed and circuits in $\{c_j \mid j \in J\}$ are inserted, we get another computation with a shift in the other direction. This proves that π is not minimal and leads to a contradiction.

We have stated that, for every i in $[1;m]$ there exists a computation π_i on input a^m such that the position of the head on the transition that ouputs the last y of the first block is at most $2n^2$ far away from one end of the input. Hence, if m is larger than $4n^3$ there exist at least two distinct i and i' such that the head is at the same position and the automaton is in the same state at this step of the computation. Hence, there is a computation that begins as π_i and ends as $\pi_{i'}$ and outputs $x^iy^ix^{m-i'}y^{m-i'}$, which is in contradiction with the fact that $\mathcal{A}$ realizes $r\cdot r$.

Likewise r^* can not be realized by a two-way automaton $\mathcal{A}$.

Proposition 7.8. *Let A and B be two alphabets, let φ be a length-preserving morphism from A^* into B^*, and let ρ be a two-way representation over A^*. Then*

$$\varphi(|\rho|) = \sum_{w\in A^*} \langle|\rho|, w\rangle\varphi(w)$$

is not necessarily two-way recognizable.

Proof. We use the same example; let r' be the following series over $A^* = \{a, b\}^*$ with coefficients in $\mathcal{P}(\{x, y\}^*)$:

$$r' = 1 + \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(x^{k+1}y^{k+1}x^hy^h)a^kba^h.$$

This series is two-way recognizable, but the morphism that maps both a and b onto a sends r' to $r\cdot r$ which is not recognizable. □

8. $\mathbb{Q}$ as counterexample

In the field $\mathbb{Q}$, the sum of powers of x exists if and only if x is in $]-1;1[$, and in this interval, $x^* = \frac{1}{1-x}$. Thus $\mathbb{Q}$ is not a Conway semiring. The two following examples show that, in general, the results of our paper does not provide any conversion

- from a valid weighted two-way automaton to a consistent two-way representation;

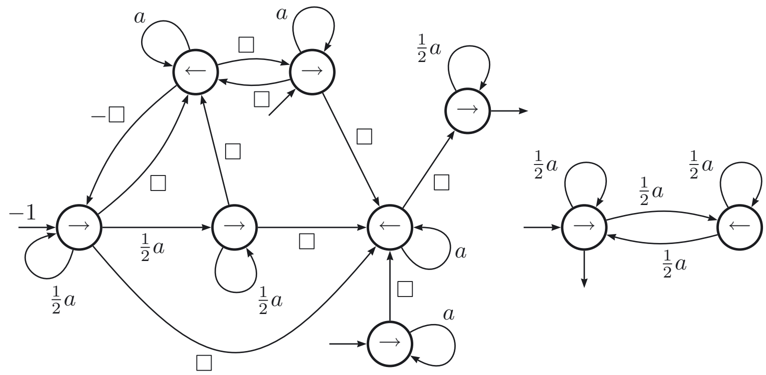

FIGURE 5. The sweeping $\mathbb{Q}$-automaton $\mathcal{A}_2$ and the two-way $\mathbb{Q}$-automaton $\mathcal{A}_3$.

- from an expression describing a series in $\mathbb{K}\mathsf{Had}A^*$ to a valid weighted two-way automaton.

Example 8.1. Let $\rho_1 = (I_1, \mu_1, \Diamond_1, T_1)$ be the representation of the deterministic automaton $\mathcal{A}_1$ of Figure 2. This representation is not consistent:

$$\overrightarrow{\mu_1(b) \wr \mu_1(b)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \right)^* \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}^* \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and the star of this matrix does not exist. Nevertheless, $\mathcal{A}_1$ is a deterministic automaton and is obviously valid.

Example 8.2. We consider the series $\displaystyle\sum_{k\geqslant 0} \frac{a^k}{k+1} = \left(a^* - \left(\frac{a}{2}\right)^* \cdot \left(\frac{a}{2}\right)^* \right)^{\circledast} \odot \left(\frac{a}{2}\right)^* .$

We can apply the constructions of Proposition 4.7 to build a two-way $\mathbb{Q}$-automaton from this expression. The result is the automaton $\mathcal{A}_2$ of Figure 5 (left). This automaton is not valid: for every word w, there are an infinite number of computations with label w and weight $\frac{1}{2^{|w|}}$. Nevertheless, there exists a two-way $\mathbb{Q}$-automaton corresponding to the representation that realizes this series; it is drawn on Figure 5 (right), and this one is valid.

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