ON THE SEQUENCE OF CONSECUTIVE MATRIX POWERS OF BOOLEAN MATRICES IN THE MAX-PLUS ALGEBRA

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In this paper we consider sequences of consecutive powers of boolean matrices in the max-plus algebra, which is one of the frameworks that can be used to model certain classes of discrete event systems. The ultimate behavior of a sequence of consecutive max-plus-algebraic powers of a boolean matrix is cyclic. First we derive upper bounds for the length of the cycles as a function of the size of the matrix. Then we study the transient part of the sequence of consecutive powers of a max-plus-algebraic boolean matrix, and we derive upper bounds for the length of this transient part. These results can then be used in the max-plus-algebraic system theory for discrete event systems.

1 Introduction

1.1 Overview

We consider the sequence of the consecutive powers of a "boolean" matrix in the max-plus algebra, which has maximization and addition as basic operations. Our main motivation for studying this topic lies in the system theory for discrete event systems (DESs). Typical examples of DESs are flexible manufacturing systems, telecommunication networks, parallel processing systems and traffic control systems. The class of the DESs essentially consists of man-made systems that contain a finite number of resources (e.g., machines, communications channels or processors) that are shared by several users (e.g., product types, information packets or jobs).

There are many modeling and analysis techniques for DESs, such as queuing theory, (extended) state machines, max-plus algebra, formal languages, automata, temporal logic, generalized semi-Markov processes, Petri nets, perturbation analysis, computer simulation and so on. In general, models that describe the behavior of a DES are nonlinear in conventional algebra. However, there is a class of DESs – the max-linear DESs – that can be described by a model that is "linear" in the max-plus algebra.[1,2,3] The model of a max-linear DES can be characterized by a triple of matrices (A, B, C), which are called the system matrices of the model.

The sequence of consecutive powers of a max-plus-algebraic boolean matrix reaches a "cyclic" behavior after a finite number of terms. This ultimate behavior has already been studied extensively by several authors.[1,4,5] We study the *transient* behavior of the sequence of the consecutive max-plus-algebraic powers of a max-plus-algebraic *boolean* matrix. We give upper bounds for the length of the transient part of the sequence as a function of structural parameters of the matrix.

We also study the cyclicity of a matrix. The cyclicity of the system matrix A of the model of a max-linear DES determines the length of the cycles of the ultimate cyclic behavior of the DES. We give an upper bound for the cyclicity of a matrix as a function of the size of the matrix. This corresponds to an upper bound for the length of the cycles of the ultimate cyclic behavior of a max-linear DES as a function of the minimal system order.

One of the open problems in the max-plus-algebraic system theory for DESs is the minimal realization problem,[1] which consists in determining the system matrices of the model of a max-linear DES starting from its "impulse response" such that the dimensions of the system matrices are as small as possible. In order to tackle the general minimal realization problem it is useful to first study a simplified version of this problem: the boolean minimal realization problem, in which only models with boolean system matrices are considered. The results of this paper on the length of the transient part of the sequence of consecutive matrix powers of a max-plus-algebraic boolean matrix can be used to obtain some results for the boolean minimal realization problem in the max-plus algebra:[6] they can be used to obtain a lower bound for the minimal system order (i.e., the smallest possible size of the system matrix A) and to prove that the boolean minimal realization problem in the max-plus algebra is decidable (and can be solved in a time that is bounded from above by a function that is exponential in the minimal system order).

1.2 Notation and definitions

If A is an m by n matrix and if $\alpha \subseteq \{1, 2, \ldots, m\}$, $\beta \subseteq \{1, 2, \ldots, n\}$ then $A_{\alpha\beta}$ is the submatrix of A obtained by removing all rows that are not indexed by α and all columns that are not indexed by β. The m by n zero matrix is denoted by $O_{m\times n}$.

If S is a set, then the number of elements of S is denoted by $\#S$. If γ is a set of positive integers then the least common multiple of the elements of γ is denoted by lcm γ, and the greatest common divisor of the elements of γ is denoted by gcd γ.

2 Max-plus algebra and graph theory

2.1 Max-plus algebra

The basic operations of the max-plus algebra[1,3] are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$x \oplus y = \max(x, y)$$
$$x \otimes y = x + y$$

with $x, y \in \mathbb{R} \cup \{-\infty\}$. Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The structure $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the max-plus algebra.

The operations $\oplus$ and $\otimes$ are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then we have

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$$

for all i, j. If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then we have

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$$

for all i, j. Note that these definitions resemble the definitions of the sum and the product of matrices in linear algebra but with $+$ replaced by $\oplus$ and $\times$ replaced by $\otimes$. This analogy is one of the reasons why we call $\oplus$ the max-plus-algebraic addition and $\otimes$ the max-plus-algebraic multiplication.

The matrix E_n is the max-plus-algebraic identity matrix: we have $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The matrix $\mathcal{E}_{m \times n}$ is the max-plus-algebraic zero matrix: $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j.

The max-plus-algebraic matrix power of the matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is defined as follows:

$$A^{\otimes^0} = E_n$$
$$A^{\otimes^k} = A \otimes A^{\otimes^{k-1}} \qquad \text{for } k = 1, 2, \ldots$$

If we permute the rows or the columns of the max-plus-algebraic identity matrix, we obtain a max-plus-algebraic permutation matrix. If $P \in \mathbb{R}_\varepsilon^{n \times n}$ is a max-plus-algebraic permutation matrix, then we have $P \otimes P^T = P^T \otimes P = E_n$. A matrix $R \in \mathbb{R}_\varepsilon^{m \times n}$ is a max-plus-algebraic upper triangular matrix if $r_{ij} = \varepsilon$ for all i, j with $i > j$.

Define $\mathbb{B} = \{0, \varepsilon\}$. A matrix with entries in $\mathbb{B}$ is called a max-plus-algebraic *boolean* matrix.

2.2 Graph theory

We assume that the reader is familiar with basic concepts from graph theory such as directed graph, subgraph, path and circuit.

A directed graph $\mathcal{G}$ with set of vertices $\mathcal{V}$ is called strongly connected if for any two different vertices v_i, $v_j \in \mathcal{V}$ there exists a path from v_i to v_j. A maximal strongly connected subgraph (m.s.c.s.) $\mathcal{G}_{\text{sub}}$ of a directed graph $\mathcal{G}$ is a strongly connected subgraph that is maximal, i.e., if we add an extra vertex (and some extra arcs) of $\mathcal{G}$ to $\mathcal{G}_{\text{sub}}$ then $\mathcal{G}_{\text{sub}}$ is no longer strongly connected.

The cyclicity of an m.s.c.s. is the greatest common divisor of the lengths of all the circuits of the given m.s.c.s. If an m.s.c.s. or a graph contains no circuits then its cyclicity is equal to 0 by definition. The cyclicity $c(\mathcal{G})$ of a graph $\mathcal{G}$ is the least common multiple of the nonzero cyclicities of its m.s.c.s.'s.

If we have a directed graph $\mathcal{G}$ with set of vertices $\mathcal{V} = \{1, 2, \ldots, n\}$ and if we associate a real number a_{ij} with each arc $(j, i) \in \mathcal{A}$, then we say that $\mathcal{G}$ is a weighted directed graph. We call a_{ij} the weight of the arc (j, i). Note that the first subscript of a_{ij} corresponds to the final (and not the initial) vertex of the arc (j, i).

Definition 2.1 (Precedence graph) *Consider a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$. The precedence graph of A, denoted by $\mathcal{G}(A)$, is a weighted directed graph with vertices $1, 2, \ldots, n$ and an arc (j, i) with weight a_{ij} for each $a_{ij} \neq \varepsilon$.*

Let $\mathcal{G}$ be a weighted directed graph with set of vertices $\mathcal{V} = \{1, 2, \ldots, n\}$. The weight of a path $i_1 \to i_2 \to \cdots \to i_l$ is defined as the sum of the weights of the arcs that compose the path: $a_{i_2 i_1} + a_{i_3 i_2} + \cdots + a_{i_l i_{l-1}}$. The average weight of a circuit is defined as the weight of the circuit divided by the length of the circuit. A circuit of $\mathcal{G}(A)$ is called critical if it has maximum average weight. The critical graph $\mathcal{G}^c(A)$ consists of those vertices and arcs of $\mathcal{G}(A)$ that belong to some critical circuit of $\mathcal{G}(A)$.

Consider a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$. The cyclicity of a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is denoted by $c(A)$ and is equal to the cyclicity of the critical graph of the precedence graph of A. So $c(A) = c(\mathcal{G}^c(A))$. Note that if $A \in \mathbb{B}^{n \times n}$ then every circuit in $\mathcal{G}(A)$ is critical, which implies that $c(A) = c(\mathcal{G}^c(A)) = c(\mathcal{G}(A))$.

2.3 Max-plus algebra and graph theory

Now we give a graph-theoretic interpretation of the max-plus-algebraic matrix power. Let $A \in \mathbb{R}_\varepsilon^{n \times n}$. If $k \in \mathbb{N}_0$ then we have

$$(A^{\otimes^k})_{ij} = \max_{i_1, i_2, \ldots, i_{k-1}} \left(a_{i i_1} + a_{i_1 i_2} + \cdots + a_{i_{k-1} j} \right)$$

for all i, j. Hence, $(A^{\otimes^k})_{ij}$ is the maximal weight of all paths of $\mathcal{G}(A)$ of length k that have j as their initial vertex and i as their final vertex — where the maximal weight is equal to ε by definition if there does not exist a path of length k from j to i.

Definition 2.2 (Irreducibility) *A matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is called irreducible if its precedence graph is strongly connected.*

If we reformulate this in the max-plus algebra then a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ is irreducible if

$$(A \oplus A^{\otimes^2} \oplus \cdots \oplus A^{\otimes^{n-1}})_{ij} \neq \varepsilon \quad \text{for all } i, j \text{ with } i \neq j,$$

since this condition means that for two arbitrary vertices i and j of $\mathcal{G}(A)$ with $i \neq j$ there exists at least one path (of length 1, 2, ... or $n-1$) from j to i. Note that it is not necessary to consider paths with a length that is larger than $n-1$. By definition any 1 by 1 matrix is irreducible. Moreover, the only max-plus-algebraic zero matrix that is irreducible is the 1 by 1 max-plus-algebraic zero matrix $[\varepsilon]$.

Definition 2.3 (Max-plus-algebraic eigenvalue) *Let $A \in \mathbb{R}_\varepsilon^{n\times n}$. If there exist a number $\lambda \in \mathbb{R}_\varepsilon$ and a vector $v \in \mathbb{R}_\varepsilon^n$ with $v \neq \mathcal{E}_{n\times 1}$ such that $A \otimes v = \lambda \otimes v$ then we say that λ is a max-plus-algebraic eigenvalue of A and that v is a corresponding max-plus-algebraic eigenvector.*

It can be shown that every square matrix with entries in $\mathbb{R}_\varepsilon$ has at least one max-plus-algebraic eigenvalue. [1] However, in contrast to linear algebra, the number of max-plus-algebraic eigenvalues of an n by n matrix is in general less than n. If a matrix is irreducible, it has is only one max-plus-algebraic eigenvalue. [2]

The only irreducible matrix that has a max-plus-algebraic eigenvalue that is equal to ε is the 1 by 1 max-plus-algebraic zero matrix $[\varepsilon]$.

The max-plus-algebraic eigenvalue has the following graph-theoretic interpretation. Consider $A \in \mathbb{R}_\varepsilon^{n\times n}$. If $\lambda_{\max}$ is the maximal average weight over all elementary circuits of $\mathcal{G}(A)$, then $\lambda_{\max}$ is a max-plus-algebraic eigenvalue of A. For formulas and algorithms to determine max-plus-algebraic eigenvalues and eigenvectors the interested reader is referred to the book of Baccelli *et al.* [1] and the references given therein.

For irreducible matrices we have: [1,2,5]

Theorem 2.4 *If $A \in \mathbb{R}_\varepsilon^{n\times n}$ is irreducible, then*

$$\exists k_0 \in \mathbb{N} \text{ such that } \forall k \geqslant k_0 : A^{\otimes^{k+c}} = \lambda^{\otimes^c} \otimes A^{\otimes^k}$$

where λ is the (unique) max-plus-algebraic eigenvalue of A and c is the cyclicity of A.

In the next section we shall discuss upper bounds for the integer k_0 that appears in Theorem 2.4 if A is a max-plus-algebraic boolean matrix. We also extend this theorem to max-plus-algebraic boolean matrices that are not irreducible.

Let $A \in \mathbb{R}_\varepsilon^{n\times n}$. In general we have $c(A) \leqslant \exp\left(\frac{n}{e}\right)$, and $c(A) \leqslant n$ if A is irreducible. [7]

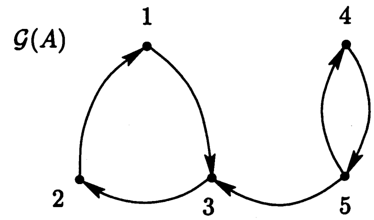

Figure 1: The precedence graph of the matrix A of Example 2.5. All the arcs have weight 0.

Example 2.5 Consider the matrix

$$A = \begin{bmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \end{bmatrix} .$$

The precedence graph of A is represented in Figure 1. Since there does not exist a path from node 1 to node 4, $\mathcal{G}(A)$ is not strongly connected. Hence, A is not irreducible. The graph $\mathcal{G}(A)$ has two m.s.c.s.'s: $\mathcal{G}_1$ with vertices 1, 2 and 3 and $\mathcal{G}_2$ with vertices 4 and 5. We have $c(\mathcal{G}_1) = 3$ and $c(\mathcal{G}_2) = 2$. Hence, $c(\mathcal{G}) = 6$. Note that $c(\mathcal{G}) \leqslant \exp\left(\frac{5}{e}\right) \approx 6.29$. □

3 Consecutive max-plus-algebraic matrix powers of a max-plus-algebraic boolean matrix

In this section we study the length of the transient part of the sequence $\{A^{\otimes^k}\}_{k=1}^{\infty}$ where A is a max-plus-algebraic boolean matrix. First we consider irreducible matrices and then we present the theorem that covers the general case. Unless explicitly mentioned, the proofs of theorems given in this section can be found in a paper of De Schutter *et al.* [7] In that paper we also give some examples of matrices for which the sequence of the consecutive max-plus-algebraic matrix powers exhibits the longest possible transient behavior.

For matrices with cyclicity 0 we have:

Theorem 3.1 *Let $A \in \mathbb{B}^{n\times n}$. If $c(A) = 0$ then we have $A^{\otimes^k} = \mathcal{E}_{n\times n}$ for all $k \geqslant n$.*

From now on we only consider matrices with a cyclicity that is larger than or equal 1. For irreducible matrices with a positive cyclicity we have:

Theorem 3.2 *Let $A \in \mathbb{B}^{n\times n}$ be irreducible and let $c = c(A) > 0$. If we define*

$$k_{n,c} = \begin{cases} (n-1)^2 + 1 & \text{if } c = 1 \\ \max(n-1, & \\ \qquad \dfrac{n^2-1}{2} + \dfrac{n^2}{c} - 3n + 2c) & \text{if } c > 1, \end{cases} \tag{1}$$

then we have $A^{\otimes^{k+c}} = A^{\otimes^{k}}$ *and* $A^{\otimes^{k}} \oplus A^{\otimes^{k+2}} \oplus \ldots \oplus A^{\otimes^{k+c-1}} = O_{n\times n}$ *for all* $k \geqslant k_{n,c}$.

Before we consider the general case, we first need the following lemma:[1]

Lemma 3.3 *If* $A \in \mathbb{R}_\varepsilon^{n\times n}$ *then there exists a max-plus-algebraic permutation matrix* $P \in \mathbb{R}_\varepsilon^{n\times n}$ *such that the matrix* $\hat{A} = P \otimes A \otimes P^T$ *is a max-plus-algebraic block upper triangular matrix of the form*

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \ldots & \hat{A}_{1l} \\ \varepsilon & \hat{A}_{22} & \ldots & \hat{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \ldots & \hat{A}_{ll} \end{bmatrix} \qquad (2)$$

with $l \geqslant 1$ *and where the matrices* $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ *are square and irreducible. The matrices* $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ *are uniquely determined to within simultaneous permutation of their rows and columns, but their ordering in (2) is not necessarily unique.*

This lemma is also the max-plus-algebraic equivalent of a result of Harary.[8] A proof of the uniqueness assertion can be found in the book of Brualdi and Ryser[9] (Theorem 3.2.4).[a]

The form in (2) is called the max-plus-algebraic Frobenius normal form of the matrix A. Note that if A is irreducible then there is only one block in (2) and then A is a max-plus-algebraic Frobenius normal form of itself.

Let $A \in \mathbb{B}^{n\times n}$. If $\hat{A} = P \otimes A \otimes P^T$ is the max-plus-algebraic Frobenius normal form of A, then we have $A = P^T \otimes \hat{A} \otimes P$. Hence,

$$A^{\otimes^{k}} = \left(P^T \otimes \hat{A} \otimes P\right)^{\otimes^{k}} = P^T \otimes \hat{A}^{\otimes^{k}} \otimes P$$

for all $k \in \mathbb{N}$. Therefore, we may consider without loss of generality the sequence $\left\{\hat{A}^{\otimes^{k}}\right\}_{k=1}^{\infty}$ instead of the sequence $\left\{A^{\otimes^{k}}\right\}_{k=1}^{\infty}$. Furthermore, since the transformation from A to $\hat{A}$ corresponds to a simultaneous reordering of the rows and columns of A (or to a reordering of the vertices of $\mathcal{G}(A)$), we have $c(A) = c(\hat{A})$.

Theorem 3.4 *Let* $\hat{A} \in \mathbb{B}^{n\times n}$ *be a matrix of the form (2) where the matrices* $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ *are irreducible. Define sets* $\alpha_1, \alpha_2, \ldots, \alpha_l$ *such that* $\hat{A}_{\alpha_i\alpha_j} = \hat{A}_{ij}$ *for all* i, j *with* $i \leqslant j$. *Let* $n_i = \#\alpha_i$ *and* $c_{ii} = c_i = c(\hat{A}_{ii})$ *for all* i.

Define:

$$\lambda_i = \begin{cases} \varepsilon & \text{if } \hat{A}_{ii} = [\varepsilon] \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots, l$. *Define*

$$S_{ij} = \big\{ \{i_0, i_1, \ldots, i_s\} \subseteq \{1, 2, \ldots, l\} \;\big|\; i = i_0 < i_1 < \ldots < i_s = j \text{ and } \hat{A}_{i_r i_{r+1}} \neq \mathcal{E} \text{ for } r = 0, 1, \ldots, s-1 \big\}$$

for all i, j *with* $i < j$.

Let $\lambda_{ii} = \lambda_i$ *and* $k_{ii} = k_i = k_{n_i,c_i}$ *for* $i = 1, 2, \ldots, n$ *where* k_{n_i,c_i} *is defined as in (1) with* $k_{n_i,0} = 0$ *by definition.*

For each i, j *with* $i < j$ *we define for each* $\gamma \in S_{ij}$:

$$\delta_\gamma = \{t \in \gamma \,|\, \lambda_t \neq \varepsilon\}$$

$$c_\gamma = \begin{cases} \gcd\{c_t \,|\, t \in \delta_\gamma\} & \text{if } \delta_\gamma \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Define

$$\Gamma_{ij} = \big\{\, t \,\big|\, \exists \gamma \in S_{ij} \text{ such that } t \in \gamma \big\}$$

$$\Delta_{ij} = \big\{\, t \,\big|\, t \in \Gamma_{ij} \text{ and } \lambda_t \neq \varepsilon \big\}$$

$$\lambda_{ij} = \begin{cases} 0 & \text{if } \Delta_{ij} \neq \emptyset \\ \varepsilon & \text{otherwise} \end{cases}$$

$$c_{ij} = \begin{cases} \operatorname{lcm}\{c_\gamma | \gamma \in S_{ij}\} & \text{if } \lambda_{ij} \neq \varepsilon \text{ and } c_\gamma \neq 1 \text{ for each } \gamma \in S_{ij} \text{ with } \delta_\gamma \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

$$r_{ij} = \begin{cases} \arg\max\{n_t \,|\, t \in \Delta_{ij}\} & \text{if } \lambda_{ij} \neq \varepsilon \\ 1 & \text{otherwise} \end{cases}$$

$$k_{ij} = \begin{cases} \displaystyle\sum_{t\in\Delta_{ij}} k_{n_t,c_t} + \#\Gamma_{ij} - 1 + \max\left(0, \max_{\substack{\gamma\in S_{ij} \\ \delta_\gamma\neq\emptyset}} \left\{ \frac{n_{r_{ij}}^2}{c_\gamma} - 3n_{r_{ij}} + 2c_\gamma \right\}\right) & \text{if } \lambda_{ij} \neq \varepsilon \\ \#\Gamma_{ij} & \text{if } \Gamma_{ij} \neq \emptyset \text{ and } \lambda_{ij} = \varepsilon \\ 1 & \text{if } \Gamma_{ij} = \emptyset \end{cases}$$

for all i, j *with* $i < j$.

Then we have for all i, j *with* $i \leqslant j$:

$$\left(\hat{A}^{\otimes^{k+c_{ij}}}\right)_{\alpha_i\alpha_j} = \left(\hat{A}^{\otimes^{k}}\right)_{\alpha_i\alpha_j} \qquad \text{for all } k \geqslant k_{ij}$$

and

$$\left(\hat{A}^{\otimes^{k}} \oplus \hat{A}^{\otimes^{k+1}} \oplus \ldots \oplus \hat{A}^{\otimes^{k+c_{ij}-1}}\right)_{\alpha_i\alpha_j} =$$

[a] Although this theorem is stated for $(0,1)$-matrices, there is a one-to-one correspondence between a max-plus-algebraic boolean matrix and a $(0,1)$-matrix if we let 0 and ε correspond with 1 and 0 respectively.

$$\begin{cases} \mathcal{O}_{n_i \times n_j} & \textit{if } \lambda_{ij} \neq \varepsilon \\ \mathcal{E}_{n_i \times n_j} & \textit{if } \lambda_{ij} = \varepsilon \end{cases} \qquad \textit{for all } k \geqslant k_{ij} \ .$$

For all i, j with $i > j$ we have

$$\left(\hat{A}^{\otimes^k}\right)_{\alpha_i \alpha_j} = \mathcal{E}_{n_i \times n_j} \qquad \textit{for all } k \in \mathbb{N} \ .$$

Remark 3.5 Let us now comment on the various variables and sets that appear in the formulation of Theorem 3.4.
It is easy to verify that λ_i is the max-plus-algebraic eigenvalue of the matrix $\hat{A}_{ii}$ for each i.
Let C_i be the m.s.c.s. of $\mathcal{G}(\hat{A})$ that corresponds to $\hat{A}_{ii}$ for $i = 1, 2 \ldots, l$. So α_i is the set of vertices of C_i.
If $\{i_0 = i, i_1, \ldots, i_{s-1}, i_s = j\} \in S_{ij}$ then there exists a path from a vertex in C_{i_r} to a vertex in $C_{i_{r-1}}$ for each $r = 1, 2, \ldots, s$. Since each m.s.c.s. C_i of $\mathcal{G}(\hat{A})$ either is strongly connected or consists of only one vertex, this implies that there exists a path from a vertex in C_j to a vertex in C_i that passes through $C_{i_{s-1}}, C_{i_{s-2}}, \ldots, C_{i_1}$.
If $S_{ij} = \emptyset$ then there does not exist any path from a vertex in C_j to a vertex in C_i.
The set Γ_{ij} is the set of indices of the m.s.c.s.'s of $\mathcal{G}(\hat{A})$ through which some path from a vertex of C_j to a vertex of C_i passes. ◇

Now we give an example in which the various sets and indices that appear in the formulation of Theorem 3.4 are illustrated.

Example 3.6 Consider the matrix

$$A = \left[\begin{array}{c|cc|ccc|c} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right] .$$

This matrix is in max-plus-algebraic Frobenius normal form and its block structure is indicated by the horizontal and vertical lines. The precedence graph of A is represented in Figure 2. We have

$$A^{\otimes^2} = \left[\begin{array}{c|cc|ccc|c} \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & 0 & \varepsilon \\ \hline \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right], \quad A^{\otimes^3} = \left[\begin{array}{c|cc|ccc|c} \varepsilon & 0 & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right],$$

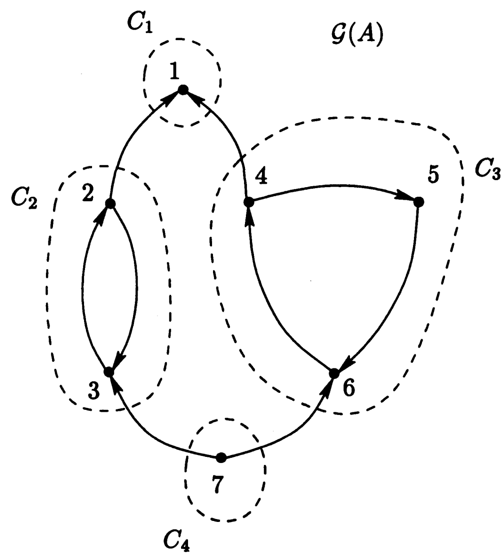

Figure 2: The precedence graph $\mathcal{G}(A)$ of the matrix A of Example 3.6. All the arcs have weight 0. C_1, C_2, C_3 and C_4 are the m.s.c.s.'s of $\mathcal{G}(A)$.

$$A^{\otimes^4} = \left[\begin{array}{c|cc|ccc|c} \varepsilon & \varepsilon & 0 & 0 & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right], \quad A^{\otimes^5} = \left[\begin{array}{c|cc|ccc|c} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & 0 & 0 \\ \hline \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right],$$

$$A^{\otimes^6} = \left[\begin{array}{c|cc|ccc|c} \varepsilon & \varepsilon & 0 & \varepsilon & 0 & \varepsilon & 0 \\ \hline \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right], \quad A^{\otimes^7} = \left[\begin{array}{c|cc|ccc|c} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right],$$

$$A^{\otimes^8} = \left[\begin{array}{c|cc|ccc|c} \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & 0 & \varepsilon \\ \hline \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right], \quad A^{\otimes^9} = \left[\begin{array}{c|cc|ccc|c} \varepsilon & 0 & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right],$$

$$A^{\otimes^{10}} = \left[\begin{array}{c|cc|ccc|c} \varepsilon & \varepsilon & 0 & 0 & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & 0 \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array}\right], \ldots$$

So $A^{\otimes^{k+6}} = A^{\otimes^{k}}$ for all $k \geqslant 2$.
We have $\alpha_1 = \{1\}$, $\alpha_2 = \{2,3\}$, $\alpha_3 = \{4,5,6\}$ and $\alpha_4 = \{7\}$. Furthermore, $\lambda_1 = \lambda_4 = \varepsilon$ and $\lambda_2 = \lambda_3 = 0$. Let us now look at the sequence $\left\{ \left(A^{\otimes^{k}}\right)_{\alpha_1\alpha_4} \right\}_{k=1}^{\infty}$. We have $S_{14} = \{\gamma_1, \gamma_2\}$ with $\gamma_1 = \{1,2,4\}$ and $\gamma_2 = \{1,3,4\}$. So $\delta_{\gamma_1} = \{2\}$, $c_{\gamma_1} = 2$, $\delta_{\gamma_2} = \{3\}$ and $c_{\gamma_2} = 3$. We have $\Gamma_{14} = \{1,2,3,4\}$, $\Delta_{14} = \{2,3\}$, $\lambda_{14} = 0$, $c_{14} = \operatorname{lcm}(2,3) = 6$ and $r_{14} = 3$. Hence,

$$
\begin{aligned}
k_{14} &= k_{2,2} + k_{3,3} + \#\Gamma_{14} - 1 + \\
&\qquad \max\left(0, \frac{n_3^2}{c_{\gamma_1}} - 3n_3 + 2c_{\gamma_1}, \frac{n_3^2}{c_{\gamma_2}} - 3n_3 + 2c_{\gamma_2}\right) \\
&= \frac{3}{2} + 4 + 4 - 1 + \max\left(0, \frac{9}{2} - 9 + 4, \frac{9}{3} - 9 + 6\right) \\
&= \frac{17}{2} .
\end{aligned}
$$

Note that we indeed have $\left(A^{\otimes^{k+6}}\right)_{\alpha_1\alpha_4} = \left(A^{\otimes^{k}}\right)_{\alpha_1\alpha_4}$ for all $k \geqslant 9$. □

As a consequence of the theorems given above we have: [6]

Theorem 3.7 *Let $A \in \mathbb{B}^{n \times n}$ with $c(A) = 1$. Then we have $A^{\otimes^{k+1}} = A^{\otimes^{k}}$ for all $k \geqslant (n-1)^2 + 1$.*

Theorem 3.8 *Let $A \in \mathbb{B}^{n \times n}$ and let c be the cyclicity of A. We have $A^{\otimes^{k+c}} = A^{\otimes^{k}}$ for all $k \geqslant 2n^2 - 3n + 2$.*

Remark 3.9 The results for the length of the transient behavior of the sequence of consecutive matrix powers that have been obtained in this paper do not only hold for matrices in the boolean max-plus algebra: they hold for matrices in any boolean algebra, i.e., a structure of the form $(\{\mathtt{o}, \mathtt{1}\}, \boxplus, \boxtimes)$ such that the operations $\boxplus$ and $\boxtimes$ applied on $\mathtt{o}$ and $\mathtt{1}$ yield the results of Table 1 and where $\boxplus$ and $\boxtimes$ are associative and $\boxtimes$ is distributive with respect to $\boxplus$. Other examples of boolean algebras are: $(\{\text{false}, \text{true}\}, \text{or}, \text{and})$, $(\{0,1\}, \max, \cdot)$, $(\{\emptyset, \mathbb{N}\}, \cup, \cap)$ and $(\{-\infty, \infty\}, \max, \min)$. ◇

4 Conclusions

In this paper we have considered sequences of consecutive max-plus-algebraic matrix powers of a max-plus-algebraic boolean matrix, and we have given some upper bounds for the length of the transient part of the sequence of consecutive matrix powers of a max-plus-algebraic boolean matrix. These results can be used in the max-plus-algebraic system theory for DESs. [6]

Topics for future research are the derivation of tighter upper bounds for the cyclicity of a matrix, the derivation of tighter upper bounds for the length of the transient part of the sequence of consecutive max-plus-algebraic matrix powers of a max-plus-algebraic boolean matrix (possibly for special cases), and extension of our results to general max-plus-algebraic matrices.

$\boxplus$	o	1
o	o	1
1	1	1

$\boxtimes$	o	1
o	o	o
1	o	1

Table 1: The operations $\boxplus$ and $\boxtimes$ for a boolean algebra $(\{\mathtt{o}, \mathtt{1}\}, \boxplus, \boxtimes)$.

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