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Transitivity of Permutation Groups on Unordered Sets

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1. Introduction

A permutation group on a set X is said to be *t-homogeneous* if it acts transitively on the set $\binom{X}{t}$ of t-element subsets of X. Livingstone and Wagner [3, Theorem 2] showed that a finite t-homogeneous group of degree at least $2t$ is $(t-1)$-transitive, and is t-transitive if $t \geqq 5$. Bercov and Hobby [1] observed that the theorem fails for infinite groups: the group of order-preserving permutations of the real numbers is t-homogeneous for every non-negative integer t, but is not even doubly transitive.

A preliminary result of Livingstone and Wagner [3, Theorem 1] asserts that a finite group of degree $n \geqq 2t$ has at least as many orbits on t-sets as on $(t-1)$-sets. As a corollary, if G is t-homogeneous, it is also $(t-1)$-homogeneous. The proof involved character theory of the symmetric group; Kantor [2] gave a proof in similar spirit using incidence matrices. Wielandt [5] gave a short number-theoretic proof of the corollary, which improved the bound ($n \geqq 2t$) slightly and also remained valid in the infinite case. The infinite version of the corollary was also proved by Bercov and Hobby [1] using Ramsey's theorem. In Section 2 I shall prove the full theorem for finite and infinite groups using a method similar to Kantor's. This not only shows that a t-homogeneous group is $(t-1)$-homogeneous but also gives a relation between the setwise stabilisers of t-sets and $(t-1)$-sets in such a group. Section 3 gives some results "in the opposite direction" in $(t+1)$-homogeneous groups, one of which uses a weak form of Ramsey's theorem.

The known examples of groups which are t-homogeneous for all t but not t-transitive for all t are

(i) groups of order-preserving permutations of certain linear orders (e.g. $\mathbb{R}$), not 2-transitive;

(ii) groups of order-preserving or reversing permutations of certain linear orders (e.g. $\mathbb{R}$), 2- but not 3-transitive;

(iii) groups preserving certain circular orders (e.g. S^1), 2- but not 3-transitive;

(iv) groups preserving or reversing certain circular orders (e.g. S^1), 3- but not 4-transitive.

The main theorem of this paper (Theorem 6.1) asserts that there are no more. Further, it seems likely that a t-homogeneous but not $(t-1)$-transitive group must be of one of these types. In support of this, McDermott showed that a 3-homogeneous but not 2-transitive group is an automorphism group of a linear order, and in Section 5 I show that a 4-homogeneous but not 3-transitive group is of one of the first three types. A tool in the proofs is the axiomatisation of various relations that occur, which is given in Section 4.

I am grateful to J.P.J. McDermott for drawing my attention to this problem.

2. Livingstone and Wagner's Theorem 1

If an incidence matrix with r rows and c columns has rank r, then any group of automorphisms has at least as many orbits on columns as on rows. The same result, suitably formulated, holds in the infinite case also. For any set X, let $\mathbb{Q}^X$ be the vector space (over $\mathbb{Q}$) of functions from X to $\mathbb{Q}$, with pointwise operations.

Proposition 2.1. *Let X and Y be sets and $\rho \subseteq X \times Y$ a relation between them. Suppose that*

(i) *for all $y \in Y$, the set $\rho^*(y) = \{x \in X \mid (x, y) \in \rho\}$ is finite;*

(ii) *the linear transformation $\theta: \mathbb{Q}^X \to \mathbb{Q}^Y$ defined by*

$$(f\theta)(y) = \sum_{x \in \rho^*(y)} f(x)$$

has kernel $\{0\}$.

Let G be a group of automorphisms of ρ, with m orbits in X and n orbits in Y. Then $m \leqq n$.

Proof. If n is infinite, then hypotheses (i) and (ii) imply that G has n orbits in ρ and that every $x \in X$ is the first component of some member of ρ; so G has at most n orbits in X. Thus we may suppose n is finite.

There is a natural action of G on $\mathbb{Q}^Y$; let W be the set of fixed elements. A function $f \in \mathbb{Q}^Y$ belongs to W if and only if it is constant on each G-orbit in Y; so W is a $\mathbb{Q}$-vector space of dimension n. Similarly, let V be the set of fixed elements of G in $\mathbb{Q}^X$. Since θ intertwines the actions of G, we have $V\theta \subseteq W$, and $\dim V = \dim V\theta \leqq n$ since $\ker \theta = \{0\}$. Thus m is finite and $m \leqq n$.

Theorem 2.2. *Let G be a permutation group on X; let s and t be integers with $0 \leqq s \leqq t$, $s + t \leqq |X|$. Then G has at least as many orbits in $\binom{X}{t}$ as it has in $\binom{X}{s}$.*

Proof. Let ρ be the relation of inclusion between $\binom{X}{s}$ and $\binom{X}{t}$; we show it satisfies the conditions of Proposition 2.1 Condition (i) is clear, and we must verify (ii).

Suppose first that $|X|=s+t$. Then we can identify any t-set with the unique s-set disjoint from it, and ρ becomes the relation of disjointness. For the remainder of the proof, S denotes an arbitrary s-subset of X. Suppose $f\in\ker\theta$; that is,

$$\sum_{S\cap S_0=\emptyset} f(S)=0 \qquad \text{for all } S_0\in\binom{X}{s}.$$

For $0\leqq j\leqq s$, $J\in\binom{X}{j}$, let $F(J)$ be the sum of $f(S)$ over all S disjoint from J. Then Since

$$\binom{t-j}{s-j}F(J)=\sum_{S\cap J=\emptyset}\ \sum_{\substack{J\subset S_0\\ S\cap S_0=\emptyset}} f(S)=\sum_{S_0\supset J}\ \sum_{S\cap S_0=\emptyset} f(S)=0;$$

so $F(J)=0$. Now let $I\in\binom{X}{i}$, $J\in\binom{X}{j}$, where $i+j\leqq s$, and suppose $I\cap J=\emptyset$; let $F(I,J)$ be the sum of $f(S)$ over all S containing I and disjoint from J. Since $F(\emptyset,J)=0$ and

$$F(I,J)=F(I-\{x\},J)-F(I-\{x\},J\cup\{x\})$$

for $x\in I$, it follows by induction on i that $F(I,J)=0$. In particular, $f(S)=F(S,\emptyset)=0$ for all $S\in\binom{X}{s}$. Thus $f=0$.

Now suppose $|X|\geqq s+t$ (we allow X to be infinite). Given any $S\in\binom{X}{s}$, choose $Y\in\binom{X}{s+t}$ with $S\subseteq Y$. For any $T\in\binom{Y}{t}$, we have

$$\sum_{S'\subset T} f(S')=0.$$

By the preceding paragraph, $f(S')=0$ for all $S'\in\binom{Y}{s}$; in particular, $f(S)=0$. Since S was arbitrary, $f=0$.

Proposition 2.3. *Let G be a permutation group on X; let s and t be integers with $0\leqq s\leqq t$, $s+t\leqq|X|$, and suppose G is t-homogeneous. Then*

(i) *G is s-homogeneous;*

(ii) *if $s\geqq 1$ and $S\in\binom{X}{s}$, $T\in\binom{X}{t}$, then the number of G_S-orbits in S is not greater than the number of G_T-orbits in T. In particular, if G_T is transitive on T, then G_S is transitive on S; and if $s\geqq 2$ and G_S acts trivially on S, then G_T acts trivially on T.*

(Here G_S denotes the set-wise stabiliser of S in G, etc.)

Proof. (i) is immediate from Theorem 2.2.

Since G is transitive and s-homogeneous, there is a one-to-one correspondence between the set of G_S-orbits in S $\left(S\in\binom{X}{s}\right)$ and the set of G_x-orbits in $\binom{X-\{x\}}{s-1}$ $(x\in X)$; each corresponds to the set of G-orbits on pairs (x,S) with $x\in S\in\binom{X}{s}$. Similar remarks hold with t and T replacing s and S. Now the first statement of (ii) is immediate on applying Theorem 2.2 to the permutation group G_x on $X-\{x\}$, and the second is a special case. For the last statement, it is sufficient to prove it when $t=s+1$; the general result follows by induction on $t-s$. Now if G_T is non-trivial on T, it cannot fix a point of T (or there would be a permutation acting non-trivially on a $(t-1)$-set), so it has at most $\frac{1}{2}t$ orbits. Then $t-1\leqq\frac{1}{2}t$, whence $t=2$, $s=1$, contrary to assumption.

3. The Ramsey Argument and Related Results

The following results are in the opposite direction to Proposition 2.3 (ii).

Proposition 3.1. *Suppose G is $(t+1)$-homogeneous on the infinite set X. Let $S\in\binom{X}{t-1}$, $T\in\binom{X}{t}$. Suppose that, for some integer d with $1\leqq d\leqq\frac{1}{2}(t-1)$, and for any d-subset D of S, there is an element $g\in G_S$ such that $D\cap Dg=\varnothing$. Then G_T has an orbit of size greater than d in T.*

Proof. Suppose the theorem is false. Then, in particular, G_T is not transitive on T. As before, its orbits correspond naturally to the G_x-orbits in $\binom{X-\{x\}}{t-1}$. By Ramsey's theorem, there is a t-subset of $X-\{x\}$, all of whose $(t-1)$-subsets belong to the same orbit. That is, there is a $(t+1)$-set U containing a point x such that, for all t-subsets T of U containing x, x lies in the same G_T-orbit. (We identify G_T-orbits in T for different T in the obvious way.) Since G is $(t+1)$-homogeneous, any $(t+1)$-subset U of X contains such a point x. Now let $T=\{x_1,\dots,x_t\}$ be a t-set, with the numbering chosen as follows: $\{x_1,\dots,x_k\}$ is the relevant G_T-orbit in T (so $k\leqq d$), and there exists an element $g\in G$ fixing $\{x_1,\dots,x_{t-1}\}$ setwise and satisfying $x_i g=x_{k+i}$ for $1\leqq i\leqq k$. Clearly g cannot fix x_t; let $x_{t+1}=x_t g$. Let $\rho=(x_1,\dots,x_t)^G$, so $(x_{k+1},\dots,x_{2k},\dots,x_{t+1})\in\rho$. Now, given any $(t+1)$-set U, there is a point $x\in U$ such that, whenever the components of a member of ρ all lie in U and include x, then x lies in one of the first k places. (We shall refer to this as the "Ramsey argument".) However, it is clear that the set $U=\{x_1,\dots,x_{t+1}\}$ contains no such point.

Proposition 3.2. *Let G be t-homogeneous on the infinite set X. Then the stabiliser of $t-1$ points fixes no further point.*

Proof. Suppose $H=G_{x_1\dots x_{t-1}}$ fixes x_t. There are two possibilites:

(i) H is properly contained in the stabiliser of some $t-1$ of the points $x_1,\dots,x_t$. Since G is $(t-1)$-homogeneous, all $(t-1)$-point stabilisers are conjugate, and we have $H<H^g$ for some g, where H fixes a point not fixed by H^g. Then $H<H^g<H^{g^2}<\cdots$, and for each non-negative integer i, H^{g^i} fixes a point not fixed by $H^{g^{i+1}}$. So H fixes infinitely many points.

(ii) The stabiliser of any $t-1$ of the points $x_1,\dots,x_t$ is equal to H. Since G is t-homogeneous, the stabiliser of any $t-1$ points fixes any further point, that is, $H=1$, and again H fixes infinitely many points.

But the number of H-orbits in $X-\{x_1,\dots,x_{t-1}\}$ is not greater than the number of G-orbits on ordered t-tuples of distinct points, which is at most $t!$ since G is t-homogeneous.

Suppose a group G is t-homogeneous. For $i \leqq t$ let $G(I)$ be the pointwise stabiliser of an i-set I, and $K_i \cong G_I/G(I)$ the group of permutations induced on I by its setwise stabiliser G_I, with $k_i=|K_i|$. For $i \leqq t-1$ let m_i be the number of orbits of $G(I)$ on $X-I$. Note that G is $(i+1)$-transitive if and only if $m_i=1$.

Proposition 3.3. *Suppose G is t-homogeneous on the infinite set X. Then*

(i) $m_i=(i+1)\,k_i/k_{i+1}$ *for* $i \leqq t-1$;

(ii) $m_i \leqq m_{i+1}$ *for* $i \leqq t-2$;

(iii) $k_i k_{i+2} \leqq \left(\dfrac{i+2}{i+1}\right) k_{i+1}{}^2$ *for* $i \leqq t-2$.

Proof. (i) The number of G-orbits on ordered i-tuples of distinct points is $i!/k_i$; thus the number of orbits on ordered $(i+1)$-tuples is given by the expressions $i!\,m_i/k_i=(i+1)!/k_{i+1}$. (ii) is immediate from Proposition 3.2, and (iii) from (i) and (ii).

A corollary of this is a converse of the last part of Proposition 2.3.

Corollary 3.4. *Let s and t be positive integers with $s<t$, and suppose G is a $(t+1)$-homogeneous group on the infinite set X, $S\in\binom{X}{s}$, $T\in\binom{X}{t}$, and $G_T{}^T=1$. Then $G_S{}^S=1$.*

Proof. We have $k_t=1$, and $t>1$; so, by Proposition 3.3 (iii), $k_{t-1}=1$. The result follows after $t-s$ steps. Note that such a group is not 2-transitive; so, by McDermott's theorem, it is an automorphism group of a linear order.

Lemma 3.5. *Suppose G is t-homogeneous on the infinite set X, $T\in\binom{X}{t}$, and $G_T{}^T$ is 2-homogeneous, with $t>2$. Then G is $(t-1)$-transitive. In particular, a t-homogeneous but not $(t-1)$-transitive group can be transitively extended at most once.*

Proof. G is transitive on pairs (S,T) with $S\in\binom{X}{t-2}$, $T\in\binom{X}{t}$, $S\subset T$; so G_S is 2-homogeneous (and hence primitive) on $X-S$. Then the normal subgroup $G(S)$ is transitive on $X-S$, and so $m_{t-2}=1$.

Remark. In examples (i)–(iv) in the Introduction, K_t is the identity, a cyclic group of order 2 with at most one fixed point (for $t>1$), a cyclic group of order t, and a dihedral group of order $2t$ (for $t>2$) respectively.

4. Characterisations of Certain Relations

A *linear order* on a set X is a binary relation λ on X which is irreflexive ($(x,x)\notin\lambda$), skew-symmetric ($(x,y)\in\lambda$ or $(y,x)\in\lambda$ but not both, for $x\neq y$), and transitive ($(x,y)\in\lambda$ and $(y,z)\in\lambda$ imply $(x,z)\in\lambda$). Since none of these axioms uses more than three points, it follows that a binary relation on a set of size at least 3 is a linear order if and only if its restriction to every 3-subset is a linear order. Indeed, we could look on this statement as an axiomatic definition of a linear order, if we understand that a linear order on a 3-set $\{x, y, z\}$ consists of (x, y), (y, z) and (x, z) (or the result of applying some permutation of $\{x, y, z\}$ to these). I shall give similar axiomatisations of other relations which occur in the next two sections.

Given a set X with a linear order λ, the *betweenness* relation β corresponding to λ is the ternary relation consisting of those triples (x, y, z) of distinct points for which either $(x, y), (y, z) \in \lambda$ or $(z, y), (y, x) \in \lambda$. (We can read this as "$y$ is between x and z".) There are just two linear orders which induce a given betweenness relation β, namely λ and its converse, so a permutation preserving β must preserve or reverse λ. The typical betweenness relation on a 4-set $\{w, x, y, z\}$ consists of the triples (w, x, y), (w, x, z), (w, y, z), (x, y, z) and their "reverses" ((y, x, w) etc.).

Theorem 4.1. *A ternary relation β on a set of size at least 4 is a betweenness relation if and only if its restriction to any 4-set is a betweenness relation.*

Proof. The necessity of the condition is clear; so assume that it holds for the relation β. First we show that the restriction of β to any 5-subset is a betweenness relation. Let $Y = \{a, b, c, d, e\}$ be a 5-set, where we may assume $(a, b, c), (b, c, d) \in \beta$. If $(e, a, b) \in \beta$ or $(a, e, b) \in \beta$, then the restriction of β to Y is completely determined. (For example, if $(e, a, b) \in \beta$, then $(a, b, c) \in \beta$ so $(e, a, c), (e, b, c) \in \beta$, etc.) So assume $(a, b, e) \in \beta$. Then $(e, b, c) \in \beta$ is impossible, and $(b, e, c) \in \beta$ determines the restriction of β to Y, so we can assume $(b, c, e) \in \beta$. Continuing similarly we find all possibilities. In each case the restriction of β to Y is a betweenness relation.

Now construct a graph Γ as follows. The vertices of Γ are ordered pairs of distinct elements of X; we join any two of (x, y), (x, z), and (y, z) with an edge whenever $(x, y, z) \in \beta$. Then, for any four points w, x, y, z, (w, x) is adjacent to either (w, y) or (y, w), and to either (x, y) or (y, x), and is joined by a path of length 2 to either (y, z) or (z, y). We claim that (x, y) and (y, x) lie in different connected components. (If not, then they are joined by a path of length at most 5; the coordinates of its vertices involve at most 5 points of X. But the restriction of β to this 5-set is known, and it is easily checked that no such path exists.) Thus Γ has two connected components, each of which is an irreflexive and skew-symmetric binary relation. From the definition of Γ, the restriction of one of these relations (say λ) to any 3-set is a linear order, so λ is a linear order; and β is the betweenness relation induced by λ.

A *circular order* γ can be defined as follows. Let X' be a set linearly ordered by λ, and ∞ a point not contained in X', $X = X' \cup \{\infty\}$. Then γ contains all triples (∞, x, y) $((x, y) \in \lambda)$ and (x, y, z) $((x, y), (y, z) \in \lambda)$, together with cyclic permutations of these. Thus the typical circular order on the 4-set $\{w, x, y, z\}$ consists of (w, x, y), (x, y, z), (y, z, w) and (z, w, x), together with their cyclic permutations.

Theorem 4.2. *For a ternary relation γ on a set X of at least four points, the following are equivalent:*

(i) *γ is a circular order;*

(ii) *γ is closed under cyclic permutations of components, and for all $x \in X$, $\lambda_x = \{(y, z) \mid (x, y, z) \in \gamma\}$ is a linear order on $X - \{x\}$;*

(iii) *the restriction of γ to any 4-subset of X is a circular order.*

Proof. (i) implies (ii): The first part is clear from the definition. For the second, note that λ_x is clearly irreflexive and skew-symmetric, so we must show that $(x, a, b), (x, b, c) \in \gamma$ imply $(x, a, c) \in \gamma$. This is clear if one of x, a, b, c is ∞; so suppose not. There are three possibilities. Firstly, $(x, a), (a, b), (x, b) \in \lambda$ (where λ is the linear order on $X - \{\infty\}$ defining γ). Then either $(b, c), (x, c) \in \lambda$ or $(c, x), (c, b) \in \lambda$. In the first case, $(a, c) \in \lambda$, so $(x, a, c) \in \gamma$. In the second, $(c, x, a) \in \gamma$. The other two possibilities, viz. $(a, b), (b, x), (a, x) \in \lambda$, or $(b, x), (x, a), (b, a) \in \lambda$, are dealt with similarly.

(ii) implies (iii): trivial.

(iii) implies (i): Suppose (iii) holds. Choose a point $\infty \in X$ and let $\lambda = \lambda_\infty$. Then the restriction of λ to any 3-set is a linear order, so λ is a linear order on $X - \{\infty\}$. We must show that $(x, y, z) \in \gamma$ if and only if two of $(x, y), (y, z), (z, x)$ belong to λ; this can be seen by considering the 4-set $\{\infty, x, y, z\}$. Finally, it is clear that γ is closed under cyclic permutations.

The *separation* relation induced by a circular order γ is the quaternary relation

$$\sigma = \{(w, x, y, z) \mid (w, x, y), (x, y, z), (y, z, w) \in \gamma \ \text{ or } \ (y, x, w), (z, y, x), (w, z, y) \in \gamma\}.$$

We can read (w, x, y, z) as "w and y separate x and z".

Theorem 4.3. *A quaternary relation σ on a set X of size at least 5 is a separation relation if and only if its restriction to any 5-set is a separation relation.*

Proof. Necessity is clear; we prove sufficiency. By hypothesis and Theorem 4.1, for any point $w \in X$, the relation

$$\beta_w = \{(x, y, z) \mid (w, x, y, z) \in \sigma\}$$

is a betweenness relation induced by two mutually converse linear orders on $X - \{w\}$, of which a typical one is λ_w. We say the linear orders λ_x and λ_y are *associated* if either

(i) they are equal, or

(ii) for some z, either $(y, z) \in \lambda_x$, $(z, x) \in \lambda_y$, or $(z, y) \in \lambda_x$, $(x, z) \in \lambda_y$ (with $x \neq y$).

To check that λ_x is not associated with both λ_y and its converse (by means of two different points z, z') we need only consider the 4-set $\{x, y, z, z'\}$. Association is clearly reflexive and symmetric; its transitivity follows by considering a set of 3 points. Thus it is an equivalence relation with just two equivalence classes, such that one of λ_x and its converse belongs to each class. If C is one class, the relation $\gamma = \{(x, y, z) \mid (y, z) \in \lambda_x \text{ for } \lambda_x \in C\}$ is a circular order inducing the given separation relation.

5. 4-Homogeneous but Not 3-Transitive Groups

As an introduction, I sketch McDermott's proof that a 3-homogeneous but not 2-transitive group G of degree greater than 4 is an automorphism group of a linear order. Since G is 2-homogeneous, it has two orbits on ordered pairs of distinct elements; let λ be one of these (so λ is irreflexive and skew-symmetric). Given $x \in X$, we may assume (replacing λ by its converse if necessary) that there exist distinct points $y, z \in X$ such that $(x, y), (x, z) \in \lambda$. Whether or not (y, z) belongs to λ, the restriction of λ to $\{x, y, z\}$ is a linear order. Since G is 3-homogeneous, the restriction of λ to any 3-subset is a linear order, so λ is a linear order preserved by G.

Theorem 5.1. *Suppose G is 4-homogeneous but not 3-transitive on the infinite set X. Then either*

(i) *there is a linear order on X preserved or reversed by all members of G; or*

(ii) *there is a circular order on X preserved by G.*

Proof. By Proposition 2.3 (i) and McDermott's theorem, we may assume G is 2-transitive. By Propositon 3.1, the set-wise stabiliser of a 3-set acts non-trivially on it, and so as a group of order 2 or 3. We call the two cases A and B respectively.

Case A. Let $\{a, b, c\}$ be a 3-set, and assume that G contains a permutation $(a\,c)(b)\ldots$. Let β be the G-orbit containing (a, b, c) and (c, b, a). Since G is 2-transitive, it contains an element g interchanging a and b. This element cannot fix c; let $d = c\,g$. Then $(b, a, d), (d, a, b) \in \beta$.

Consider the restriction of β to $\{a, b, c, d\}$. The Ramsey argument shows that, either one of a, b, c, d occurs in the second place in every member of β containing it, or one occurs in the first or third place in every member of β containing it. Clearly the second alternative holds, and the letter involved is c or d; we may suppose it is c. By Proposition 3.2, $\{y \mid (x, y, z) \in \beta\}$ has size at least 2, for any given x and z; so some pair of letters occurs in the first and third places in at least two members of β. This pair must be $\{c, d\}$ or $\{c, a\}$. In the first case, β contains (c, a, d), (c, b, d), and their reverses. Then the restriction of β to $\{a, b, c, d\}$ is a betweenness relation; by Theorem 4.1, β is a betweenness relation, and conclusion (i) of the theorem holds. In the second case, $(a, d, c), (c, d, a) \in \beta$, and the fourth pair is either (d, b, c) and (c, b, d) (case A1) or (b, d, c) and (c, d, b) (case A2).

In case A1, the existence of a permutation $(a)(b)(c)(d\,e\ldots)\ldots \in G$ (Proposition 3.2) shows that $(b, a, e), (a, e, c), (e, b, c) \in \beta$. Since d and e now occur symmetrically, we may assume that β contains (e, d, b) (rather than (d, e, b)); consideration of $\{a, b, d, e\}$ and $\{b, c, d, e\}$ show that β also contains (a, e, d) and (d, e, c). Now the restriction of β to $\{a, c, d, e\}$ is wrong.

In case A2, there is a permutation $h = (b\,c)(d)\ldots \in G$. It cannot fix a, since $(a, c, b) \notin \beta$; say $a\,h = e$. Then β contains (e, c, b), (c, e, d) and (e, d, b). There are now four possibilities for the restriction of β to $\{a, b, c, e\}$: it may contain (b, a, e) and (c, a, e), (b, e, a) and (c, e, a), (b, a, e) and (a, c, e), or (e, b, a) and (c, e, a). In each case, attempts to extend β to the whole of $\{a, b, c, d, e\}$ lead to contradictions.

Case B. Let γ be an orbit of G on ordered triples of distinct elements. Then γ is closed under cyclic permutations. If γ is not a circular order, then Theorem 4.2 implies that there are points a, b, c, d such that $(b, c), (c, d), (d, b) \in \lambda_a$; that is, γ contains (a, b, c), (a, c, d), (a, d, b), and cyclic permutations of these. Non-existence now follows from a general result:

Lemma 5.2. *There does not exist a $(t+1)$- homogeneous group G on an infinite set X in which K_t is the alternating group A_t, and the restriction to a $(t+1)$-set U of a G-orbit ρ on ordered t-tuples contains all images under A_t (fixing u) of some t-tuple containing a point $u \in U$.*

Proof. Suppose such a group exists. If $(x_1, \ldots, x_t) \in \rho$, there is an element $g \in G$ fixing $x_1, \ldots, x_{t-1}$, and mapping x_t to a new point x_{t+1} (Proposition 3.2); so $(x_1, \ldots, x_{t-1}, x_{t+1}) \in \rho$. These two t-tuples are related by an odd permutation of the set $\{x_1, \ldots, x_{t+1}\}$. So we can describe $\rho | U$ as consisting of all images under A_t of some ordered t-tuple not containing u, together with the result of substituting u for any other letter in any of these. If we know three members of $\rho | U$ such that the first is related to each of the others by an odd permutation, then the letter not occuring in the first is u, and $\rho | U$ is determined. We shall write $(u | u_1, \ldots, u_t)$ to mean $(u_1, \ldots, u_t) \in \rho$ and u is the distinguished point of $U = \{u, u_1, \ldots, u_t\}$.

Since G is $(t+1)$-homogeneous, by Proposition 3.2 there are points $a_1, \ldots, a_{t+2}$ with $(a_1 | a_2, \ldots, a_t, a_{t+1})$ and $(a_1 | a_2, \ldots, a_t, a_{t+2})$. Because of the symmetry between a_{t+1} and a_{t+2}, we may assume $(a_{t+1} | a_2, \ldots, a_{t-1}, a_{t+2})$ rather than $(a_{t+2} | a_2, \ldots, a_t, a_1, a_{t+1})$. Then ρ contains $(a_2, \ldots, a_t, a_{t+1})$, $(a_2, \ldots, a_t, a_{t+2})$, and $(a_2, \ldots, a_{t-1}, a_{t+1}, a_{t+2})$; so $(a_{t+1} | a_2, \ldots, a_t, a_{t+2})$ holds, and $(a_{t+1}, a_3, \ldots, a_t, a_{t+2}) \in \rho$. Also ρ contains $(a_1, a_3, \ldots, a_t, a_{t+1})$, $(a_1, a_3, \ldots, a_t, a_{t+2})$, and $(a_{t+1}, a_3, \ldots, a_{t-1}, a_1, a_{t+2})$; so $(a_{t+2} | a_1, a_3, \ldots, a_t, a_{t+1})$ holds, and $(a_{t+2}, a_3, \ldots, a_t, a_{t+1}) \in \rho$. But these two members of $\rho | \{a_3, \ldots, a_{t+2}\}$ differ by an odd permutation.

Remarks. 1. If the full automorphism group of a linear order is 2-homogeneous, then it is t-homogeneous for all t. So by McDermott's theorem, a group which is 3-homogeneous but not 2-transitive has a supergroup which is t-homogeneous for all t but not 2-transitive. Theorem 5.1 and similar considerations imply that a 4-homogeneous but not 3-transitive group has a supergroup t-homogeneous for all t but not 3-transitive. Note also that, if G is 4-homogeneous on X and $|G_T{}^T| = 2$ for $T \in \binom{X}{3}$, then G has a subgroup of index 2.

2. If a linear order admits a 2-homogeneous group, then it is dense (every open interval is non-empty). It was shown by Cantor that a countable dense linear order is isomorphic to $\mathbb{Q}$. So a 3-homogeneous but not 2-transitive group of countable degree is a subgroup of the group of order-preserving permutations of $\mathbb{Q}$. (There are many such subgroups: functions with bounded support, piecewise-linear functions, etc.) Similarly a 4-homogeneous but not 3-transitive group of countable degree either preserves or reverses the order on $\mathbb{Q}$, or preserves the circular order on the complex roots of unity.

6. The Main Theorem

Theorem 6.1. *Suppose G is a permutation group on the infinite set X which is t-homogeneous for all positive integers t; suppose that G is r-transitive but not $(r+1)$-transitive, for some positive integer r. Then $r \leqq 3$, and there is a linear or circular order on X preserved or reversed by all elements of G.*

Proof. Assume that the hypotheses of the theorem hold. By Theorem 5.1, we may assume $r \geqq 3$. First we require a few lemmas concerning the numbers m_i defined before Proposition 3.3.

Lemma 6.2. *If $m_i = m_{i+1}$, then m_i divides $\frac{1}{2}(i+1)(i+2)$.*

Proof. Suppose $I \in \binom{X}{i}$. Then $G(I)$ has m_i orbits on $X - I$, none of them a singleton; if y is any point outside I, then $G(I)_y$ has m_i orbits on $X - (I \cup \{y\})$. So $G(I)$ is doubly transitive on each of its orbits. Thus, of the m_i^2 $G(I)$-orbits on ordered pairs of points outside I, m_i consist of pairs interchanged by elements of $G(I)$. This means that a proportion $1/m_i$ of the G-orbits on ordered $(i+2)$-tuples have the property that the elements in the last two places can be interchanged by a transposition. Since this proportion is the same for all pairs of places, K_{i+2} contains $\binom{i+2}{2} \Big/ m_i$ transpositions.

Corollary 6.3. *$m_i \leqq i+2$ for all i.*

Proof. We cannot have $m_i = m_{i+1} = i+3$. So if $m_i \geqq i+3$ then $m_j \geqq j+3$ for all $j \geqq i$. Then $k_j = j\,k_{j-1}/m_{j-1} \leqq \left(\frac{j}{j+2}\right) k_{j-1}$, so

$$k_j \leqq \frac{(i+2)(i+1)}{(j+2)(j+1)} k_i$$

for all $j \geqq i$, a contradiction when $(j+2)(j+1) > (i+2)!$

Lemma 6.4. *If $(m_i, i+1) = 1$ then K_{i+1} is transitive.*

Proof. Let K_{i+1} have an orbit of length d. Then K_i contains a subgroup of K_{i+1} of index d, so k_{i+1}/d divides $k_i = k_{i+1}\, m_i/(i+1)$. It follows that $i+1$ divides $d\,m_i$; so if $(m_i, i+1) = 1$ then $d = i+1$.

Now we list the quintuples (r, m, n, K, L), where r, m and n are integers with $r \geqq 3$, $2 \leqq m \leqq r+2$, $m \leqq n \leqq r+3$, K is a subgroup of index m in S_{r+1}, and L a subgroup of index mn in S_{r+2} in which the stabiliser of any point is a subgroup of K. This is obtained by using the lower bounds for the indices of subgroups of S_k: [4 Theorem 14.2] for primitive subgroups, and obvious bounds for imprimitive or intransitive subgroups.

(i) $m = r+1$, $n = r+2$, $K = S_r$ (fixing a point), $L = S_r$ (fixing two points);

(ii) $m = 2$, $n = r+2$, $K = A_{r+1}$, $L = A_{r+1}$ (fixing a point);

(iii) $r = 3$, $m = 2$, $n = 6$, $K = A_4$, $L = D_{10}$;

(iv) $r = 3$, $m = 3$, $n = 4$, $K = D_8$, $L = D_{10}$;

(v) $r = 3$, $m = 3$, $n = 5$, $K = L = D_8$;

(vi) $r = 3$, $m = 4$, $n = 5$, $K = S_3$, $L = Z_3 \times Z_2$;

(vii) $r = 3$, $m = 4$, $n = 6$, $K = S_3$, $L = Z_5$;

(viii) $r = 4$, $m = 2$, $n = 6$, $K = A_5$, $L = PSL(2, 5)$;

(ix) $r = 4$, $m = 5$, $n = 6$, $K = S_4$, L has index 2 in S_2 wr S_3;

(x) $r=4$, $m=n=6$, $K=L=\mathrm{Hol}(Z_5)$;

(xi) $r=5$, $m=6$, $n=7$, $K=L=PGL(2,5)$.

We must examine these to see whether we can have $m=m_r$, $n=m_{r+1}$, $K=K_{r+1}$, $L=K_{r+2}$ in a group satisfying the hypotheses of the theorem. We can immediately eliminate cases (vii) and (ix), in which K is intransitive and L is transitive (Proposition 2.3); case (viii), in which L is 2-transitive (Lemma 3.5); case (x), by Lemma 6.2; and case (ii), in which the hypotheses of Lemma 5.2 are satisfied. Case (iii) is dealt with by the following result:

Lemma 6.5. *Suppose G is $(t+1)$-homogeneous, and K_t is contained in the alternating group A_t. Then the subgroup $K_{t+1}{}^+$ of even permutations in K_{t+1} is intransitive.*

Proof. Let $\rho=(x_1,\ldots,x_t)^G$ and suppose the conclusion is false. Then, for $U\in\binom{X}{t+1}$, $\rho|U$ consists of images of some t-tuple under even permutations only. But ρ contains $(x_1,\ldots,x_t)$ and $(x_1,\ldots,x_{t-1},x_{t+1})$, with $x_{t+1}\neq x_t$ (Proposition 3.2).

If K_{i+1} is intransitive and $m_i=i+1$, Corollary 6.3 and Lemma 6.4 imply that $m_j=j+1$, and so $k_j=k_i$, for all $j\geqq i$. In case (xi), we have $m_7=8$ and $k_8=120$, so K_8 fixes two points and acts as $PGL(2,5)$ on the remaining points. Now the following lemma deals with cases (i) and (xi):

Lemma 6.6. *There does not exist a $2t$-homogeneous group (with $t\geqq 3$) having the following properties:*

(i) K_t is 2-transitive, and any transitive representation of it with degree t is 2-transitive;

(ii) K_{t+1} and K_{t+2} fix one and two points respectively;

(iii) $|K_i|=|K_t|$ for $t\leqq i\leqq 2t$.

Proof. Choose $T\in\binom{X}{t}$. Since $m_t=t+1$, $G(T)$ has orbits $Y_1,\ldots,Y_{t+1}$ on $X-T$, none of them a singleton. G_T has two orbits on $X-T$ and four on $\binom{X-T}{2}$, since K_{t+1} has two orbits on points and K_{t+2} has four orbits on pairs. If $Y_1\cup\cdots\cup Y_s$ and $Y_{s+1}\cup\cdots\cup Y_{t+1}$ were G_T-orbits in $X-T$ with $1<s<t$, then G_T would have at least five orbits in $\binom{X-T}{2}$, viz. pairs belonging to the same Y_i, to different Y_i in the same G_T-orbit, and to different G_T-orbits. So we may assume G_T fixes Y_{t+1} and acts transitively (hence 2-transitively) on $\{Y_1,\ldots,Y_t\}$.

For $y\in Y_i$, $G(T)_y$ has $t+2$ orbits in $X-(T\cup\{y\})$. It cannot split Y_j into two orbits for $j\neq i$. For if this occured with $i=t+1$, then every orbit Y_k with $k\leqq t$ would split. (Choose $g\in G_T$ mapping Y_j to Y_k; g normalises $G(T)$.) A similar argument (using the double transitivity of G_T on $\{Y_1,\ldots,Y_t\}$) applies if $i\leqq t$. So $G(T)$ has rank 3 on each Y_i.

Now K_{t+2} has seven orbits on ordered pairs; so G_T has seven orbits on ordered pairs of points outside T. Thus three of its four orbits on unordered pairs must be asymmetric. Clearly the orbit $\{\{y,y'\}\mid y\in Y_i,\ y'\in Y_j,\ i,j\leqq t,\ i\neq j\}$ is symmetric. So $G(T)$ is 2-homogeneous on Y_i for all i, and G_T is 2-homogeneous on Y_{t+1}.

For $y, y' \in Y_i$, $G(T)_{yy'}$ has $t+3$ orbits in $X-(T \cup \{y, y'\})$, of which at least three lie in $Y_i - \{y, y'\}$. It follows that $G(T)$ is 3-homogeneous on Y_i. By McDermott's theorem, it preserves a linear order on Y_i. An easy induction shows that $G(T)$ is k-homogeneous on Y_i for all $k \leqq t$.

Let $<$ be the linear order on Y_{t+1} (which is preserved by G_T). Choose $y_1, \ldots, y_t \in Y_{t+1}$ with $y_1 < y_2 < \cdots < y_t$, and any $g \in G_T$. Then $y_1 g < \cdots < y_t g$, so there is an element $h \in G(T)$ such that $y_1 gh = y_1, \ldots, y_t gh = y_t$. Thus $(G_T)_{y_1 \ldots y_t}$ induces the whole of K_t on T.

On the other hand, if a group preserves a linear order on each of its orbits, then the setwise stabiliser of any finite set fixes it pointwise. Now $(G_T)(T') \leqq G(T')$, where $T' = \{y_1, \ldots, y_t\}$, so $(G_T)(T') = G(T')_T$ acts trivially on T, a contradiction.

In case (vi), write $(w \mid x\,y\,z)$ to mean that $G_{\{w, x, y, z\}}$ fixes w. Let $T = \{a, b, c, d, e\}$ be a 5-set with $g = (a\,b\,c)(d\,e) \cdots \in G_T$. Then we have $(d \mid a\,b\,c)$ and $(e \mid a\,b\,c)$, and without loss of generality $(a \mid b\,d\,e)$. Then the Ramsey argument gives $(c \mid b\,d\,e)$, but applying g gives $(b \mid c\,d\,e)$.

Case (v) falls to an argument resembling Lemma 5.2. Let $\rho = (a, b, c, d)^G$, where we assume ρ is invariant under cyclic shifts and reversals. Since $K_5 = D_8$, we can suppose ρ contains $(x, a, b, c), (x, b, c, d), (x, c, d, a), (x, d, a, b)$ and (a, b, c, d); we shall write $(x \mid a\,b\,c\,d)$. By Proposition 3.2, there exists e with $(x \mid a\,b\,c\,e)$; we may assume $(d \mid a\,b\,c\,e)$. From these we deduce $(d \mid x\,b\,c\,e)$ and $(e \mid x\,c\,d\,a)$, whence $(x, d, c, e), (x, c, d, e) \in \rho$, a contradiction.

Finally, consider case (iv). Let $\sigma = (a, b, c, d)^G$ be invariant under cyclic shifts and reversals, and suppose $(a, b, c, e) \in \sigma$. Now we can write down the eight permutations of $\{a, b, c, d, e\}$ which map (a, b, c, d) to the elements of $\sigma \mid \{a, b, c, e\}$. Among these must be an element of order 5 in $G_U{}^U$, $U = \{a, b, c, d, e\}$. The only permutations of order 5 turn out to be $(a\,e\,d\,c\,b)$ and $(a\,b\,c\,e\,d)$. For each of these, $\sigma \mid U$ is determined as the set of images of $\sigma \mid \{a, b, c, d\}$, and in either case we find that $\sigma \mid U$ is a separation relation. By Theorem 4.3, σ is a separation relation. (This argument shows that a 5-homogeneous but not 4-transitive group, in which the stabiliser of a 5-set acts transitively on it, preserves a separation relation.)

Remarks. 1. The proof of Theorem 6.1 shows in fact that there is a function f such that a group which is $f(r)$-homogeneous and r- but not $(r+1)$-transitive satisfies the conclusions of the theorem. (It is enough that $m_i \leqq i+2$ for $i \leqq 2(r+1)$; this will hold if $f(r)(f(r)+1) > (2r+4)!$ However, this is almost certainly far from the best possible result, which should be $f(r) = r+2$.)

2. There are many examples of infinite permutation groups which are t-transitive for every integer t. Among them are

(i) the group of all permutations of X whose support has cardinality less than some cardinal number a;

(ii) the group of all permutations of X with finite support which act evenly on their supports;

(iii) the homeomorphism group of a manifold such as $\mathbb{R}^n$, S^n, $\mathbb{R}P^n$ $(n > 1)$.

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