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Counting Labelled Three-Connected and Homeomorphically Irreducible Two-Connected Graphs

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Labelled three-connected graphs and labelled two-connected graphs with no vertices of degree 2 are counted using methods similar to those used by Riddell to count labelled two-connected graphs.

STATEMENT OF RESULTS

A *graph* will be assumed to be finite and unoriented, with no loops or multiple edges; if multiple edges are to be allowed, the term *multigraph* will be used. A graph or multigraph will be called *k-connected* if at least k vertices and their incident edges must be removed to disconnect it (a complete graph is considered to be k-connected for any k). A *block* (respectively, *multiblock*) is a 2-connected graph (respectively, multigraph) with at least 2 vertices, and a *brick* is a 3-connected graph with at least 4 vertices. A *labelling* of a graph or multigraph with n vertices is a 1–1 correspondence from the set $\{1, 2,..., n\}$ onto the set of its vertices.

Let $A(x, y)$ be the mixed exponential generating function $\sum_{n,m} A_{n,m} x^n y^m/n!$, where $A_{n,m}$ is the number of labelled graphs with n vertices and m edges, and let $C(x, y)$ and $B(x, y)$ be analogous generating functions which count labelled connected graphs and labelled blocks, respectively. The following formulae, due to Riddell [10], appear in one-variable form in [6, pp. 3–11]:

$$A(x, y) = \sum_{n=1}^{\infty} x^n (1+y)^{\binom{n}{2}}/n!; \tag{1}$$

$$C(x, y) = \log(1 + A(x, y)); \tag{2}$$

$$\partial B(z, y)/\partial z = \log(z/x), \tag{3}$$

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where

$$z = x\,\partial C(x, y)/\partial x. \tag{4}$$

In this article the following formulae are derived. Let $H(x, y)$ and $F(x, y)$ count labelled blocks with no vertices of degree less than 3 and labelled bricks, respectively. Then

$$H(x, R) = B(x, y) - (x^2/2)\int_0^y \exp(S(x, t))\,dt, \tag{5}$$

where

$$R(x, y) = (1 + y)\exp(S(x, y)) - 1 \tag{6}$$

and

$$S(x, y) = xR(x, y)[R(x, y) - S(x, y)]; \tag{7}$$

$$(2/x^2)\,\partial F(x, D)/\partial D = \log(K(x, y)) - P(x, y), \tag{8}$$

where

$$K(x, y) = (2/x^2)\,\partial B(x, y)/\partial y, \tag{9}$$

$$D(x, y) = (1 + y)\,K(x, y) - 1, \tag{10}$$

and

$$P(x, y) = xD(x, y)[D(x, y) - P(x, y)]. \tag{11}$$

1. Counting Labelled Three-Connected Graphs

To prove (8)–(11) we use B. A. Trakhtenbrot's canonical network decomposition theorem [14] expressed below as Proposition 1.1. A *network* N is a multigraph with two distinguished vertices, called its *poles* and labelled 0 and ∞, such that the multigraph N^* obtained from N by adding an edge between the poles of N is 2-connected. A vertex of N which is not a pole is called an *internal vertex*. A *chain* is a network consisting of 2 or more edges connected in series with the poles at its terminal vertices. A *bond* is a network consisting of 2 or more edges connected in parallel. A *pseudo-brick* is a network N such that N^* is a brick. If M is a multigraph or a network, then EM denotes its edge-set.

Let M be a multiblock or a network with $m \geqslant 2$ edges and let $X = \{N_e, e \in EM\}$ be a set of networks, disjoint from each other and from M, each having at least one edge. Let $G = M(X)$ be the multiblock or network obtained from M by choosing an orientation (u, v) of each edge $e = \{u, v\}$ in EM and replacing e by N_e, identifying the pole 0 of N_e with u and the pole ∞ with v. Then $G = M(X)$ is called a *superposition* with *core* M and *components* $N_e \in X$. A *decomposition* of a multiblock or a network G is a representation of G as a superposition: $G = M(X)$. A network N is called, respectively, an *h-network*, a *p-network* or an *s-network* if its admits a decomposition whose core is, respectively, a pseudo-brick, a bond or a chain. A drawing of each of these types of decomposition is given in Fig. 2 of [18]. A p-network (respectively, an s-network) is called a *series union* (*parallel union*) of its components.

Trakhtenbrot's canonical network decomposition theorem can be stated as follows.

PROPOSITION 1.1. *Any network with at least* 2 *edges belongs to exactly one of the* 3 *classes*: *h-networks*, *p-networks*, *s-networks*. *An h-network has a unique decomposition and a p-network* (*respectively*, *an s-network*) *can be uniquely decomposed into components which are not themselves p-networks* (*s-networks*), *where uniqueness is up to orientation of the edges of the core*, *and also up to their order if the core is a bond.*

Proof. Proofs of this theorem in Russian can be found in [14, pp. 240–244; 17, pp. 178–184; and 7, pp. 31–44], and the reader can easily construct one using a similar argument for maps given in [16, pp. 260–263]. ∎

Using Proposition 1.1, we can now prove (8)–(11) by applying to labelled networks the techniques used in [6, p. 10] for treating labelled graphs with only one distinguished vertex. In a labelled network, the poles do not receive labels other than 0 or ∞; only the n internal vertices receive labels from $\{1, 2,..., n\}$. For the rest of this section, a network or a graph will be assumed to be labelled and without parallel edges, and each edge $\{u, v\}$ of the core of a superposition is assumed to be given the orientation (u, v), where $u < v$.

Since $B(x, y)$ counts blocks, and since a network with non-adjacent poles can be obtained by distinguishing, orienting and then deleting any edge of an arbitrary block, all such networks are counted by $K(x, y)$ of (9), where the exponent of x is the number of internal vertices. Then $D(x, y)$ of (10) counts all the networks with at least one edge.

Now let $P(x, y)$ count the s-networks, so that $E = D(x, y) - P(x, y)$ counts all the networks which are not s-networks. Their series unions are distinct and exhaust all the s-networks, by Proposition 1.1 for s-networks decomposition. But series unions are ordered k-tuples, $k \geqslant 2$; so $P(x, y) = xE^2(1 - xE)^{-1}$, and substituting for E yields (11).

Let U count non-p-networks with at least 2 edges. These have non-adjacent poles, and together with the zero-edge network, are all the non-p-networks with non-adjacent poles. Their parallel unions, which also have non-adjacent poles, are distinct and exhaust all the p-networks with non-adjacent poles, by Proposition 1.1 for p-network decomposition. But parallel unions are unordered k-tuples, $k \geqslant 2$; so $U = \log(K(x, y))$.

By the first assertion of Proposition 1.1, the right side of (8) counts the h-networks. But $(2/x^2)\, \partial F(x, y)/\partial y$ counts the pseudo-bricks; so by Proposition 1.1 for h-network decomposition, the left side of (8) also counts h-networks. This completes the proof of formulae (8)–(11).

We note the following generalization, which requires no further proof.

PROPOSITION 1.2. *Let X be a set of bricks, X' be the set of pseudo-bricks N such that $N^* \in X$, X'' be the set of networks obtained by requiring the cores of h-networks to be taken from X', and Y be the set of blocks N^* such that $N \in X''$. Then* (8)–(11) *are valid if $F(x, y)$ counts X and $B(x, y)$ counts Y.* ∎

Trakhtenbrot's theorem was part of a study made together with V. A. Kuznetzov [9] of networks, called "strongly-connected networks," and pseudo-bricks which, together with the networks with 1 and 2 edges, are called "indecomposable networks," and the two classes of Boolean functions they code. Drawings of all the indecomposible networks with at most 10 edges appear at the end of [9]. It turns out [18] that repeated network decomposition is essentially equivalent to the unique decomposition of multiblocks into bricks, bonds and polygons, where the uniqueness condition is not the maximality of the components as in [15, Chap. 11], but the non-adjacency of two components if both are bonds or if both are polygons. The sufficiency of this condition was conjectured in [12] and recently proved in [2] and [3]. We have used this "decomposition into 3-connected components" and a modification of the methods of [11] to count unlabelled bricks [18]. Here we note that the set Y of Proposition 1.2 is the set of blocks whose 3-connected components include only bricks taken from X.

2. Counting Labelled Homeomorphically Irreducible 2-Connected Graphs

To prove (5)–(7) we use the classical series–parallel decomposition of a multiblock, expressed below as Proposition 2.1. A drawing of this type of decomposition is given in Fig. 3 of [18]. A *series-parallel network* (SPN) can be defined inductively as either the 1-edge network or else the series union or parallel union of SPN's. A block or multiblock $G = N^*$ which can be obtained from some series–parallel network N by adding an edge between the poles of N is called a *series–parallel graph* (SPG) or *series–parallel multigraph* (SPM), respectively. An *H-block* is a block without vertices of degree <3.

PROPOSITION 2.1. *Let G be a multiblock with at least* 2 *edges.*

(a) *If G is an* SPM, *then for* any *edge* $e=\{u,v\}$ *of G, deleting and orienting e yields an* SPN *with poles and u and v.*

(b) *If G is not an* SPM, *then G has a unique decomposition whose core is an H-block and whose components are* SPN's.

Proof. Part (a) follows from the well-known characterization of an SPN as a network with no "Wheatstone bridge"—that is, a network N is an SPN iff N^* has no homeomorph of K_4, the complete graph on 4 vertices. Clearly this is a property of the multiblock $G=N^*$ and not of the particular edge one deletes to make N.

To prove part (b), we define a *homeomorphic reduction* on a multiblock G to consist of either replacing a vertex of degree 2—with distinct neighbors—and its incident edges by an edge joining these neighbours, or of deleting one edge from a set of parallel edges. Successive homeomorphic reductions will eventually reduce G to some homeomorphically irreducible block G_0, which is either a single edge or an H-block. If G_0 is a single edge, then G must be an SPM, since the existence in G of a homeomorph of K_4 precludes reducibility to a single edge [5]. If G_0 is a H-block, then by an argument similar to the one in [5] it follows that any sequence of reductions will reduce G to G_0: the crucial point is that 2 reductions commute unless G is a triangle, which is an SPG. Reversing these reductions turns each edge of G_0 into an SPN, yielding the required unique decomposition of G. ∎

Now let $R(x, y)$ count the SPN's assumed to be labelled and without parallel edges, and let $S(x, y)$ count those which are s-networks. Clearly SPN's are characterized as networks in which no h-networks appear at any level of decomposition or, equivalently, SPM's are just multiblocks with no bricks among their 3-connected components. By Proposition 1.2 with $X=\phi$ and part (a) of Proposition 2.1, the SPG's can be counted from (8)–(11) after first setting the left side of (8) to 0. Thus (6) and (7) follow from (8), (10) and (11), and by integrating (9) and setting the lower limit of integration to 0 to exclude the zero-edge, 2-vertex graph it follows that the last term in (5) counts the SPG's. Since $H(x, y)$ counts H-blocks, the left side of (5) counts those blocks which are not SPG's, by part (b) of Proposition 2.1. This completes the proof of formulae (5)–(7).

Labelled graphs with no vertices of degree 2 were counted in [8] along with those that are connected. So labelled graphs with at least one vertex of degree 2 are counted by connectivity, since such a graph cannot be 3-connected.

We have also counted unlabelled H-blocks [18].

3. Numerical Solution of the Equations

For the remainder of this article, if the names of the arguments of a function are omitted, they are assumed to be x and y, and partial derivatives are expressed by subscripting, so that B_{xx} means $\partial^2 B/\partial x^2$.

TABLE I

The number of Labelled 3-Connected (F) and Homeomorphically Irreducible 2-Connected (H) n-Vertex m-Edge Graphs for $n \leqslant 10$

n	m	H	F	n	m	H	F
4	6	1	1	8	12	19320	16800
				8	13	515760	442680
5	8	15	15	8	14	2821500	2485920
5	9	10	10	8	15	7207396	6629056
5	10	1	1	8	16	11163523	10684723
				8	17	11924808	11716068
6	9	70	70	8	18	9459226	9409806
6	10	537	492	8	19	5831560	5824980
6	11	735	690	8	20	2872737	2872317
6	12	395	395	8	21	1147676	1147576
6	13	105	105	8	22	373156	373156
6	14	15	15	8	23	98112	98112
6	15	1	1	8	24	20475	20475
				8	25	3276	3276
7	11	5670	5040	8	26	378	378
7	12	32375	28595	8	27	28	28
7	13	63945	58905	8	28	1	1
7	14	66090	63990				
7	15	42602	42392				
7	16	18732	18732				
7	17	5880	5880				
7	18	1330	1330				
7	19	210	210				
7	20	21	21				
7	21	1	1				

Table continued

TABLE I (*continued*)

n	m	H	F	n	m	H	F
9	14	3787560	3197880	10	15	11052720	9238320
9	15	59121720	50828400	10	16	681515100	577432800
9	16	333188100	296711100	10	17	8579598300	7488142200
9	17	1040804100	962902080	10	18	51121236600	46189596600
9	18	2158303224	2061518844	10	19	188523083700	175880023200
9	19	3277818432	3200708952	10	20	491009360625	469919266740
9	20	3872947050	3830943438	10	21	975949118145	951063537600
9	21	3704885712	3688441200	10	22	1556478133290	1534460236200
9	22	2948201280	2943415800	10	23	2061536771430	2046277331640
9	23	1987998768	1986963048	10	24	2324010011625	2315459369700
9	24	1149824529	1149664509	10	25	2270132385381	2266183117296
9	25	574550928	574535052	10	26	1946802611250	1945288222920
9	26	248787882	248787126	10	27	1479734628330	1479253936440
9	27	93290260	93290260	10	28	1003586008995	1003461253560
9	28	30163059	30163059	10	29	610052393295	610026517620
9	29	8340552	8340552	10	30	333216921144	333212790864
9	30	1947540	1947540	10	31	163688109840	163687633560
9	31	376992	376992	10	32	72270520875	72270485595
9	32	58905	58905	10	33	28618931775	28618930515
9	33	7140	7140	10	34	10128741210	10128741210
9	34	630	630	10	35	3187559826	3187559828
9	35	36	36	10	36	885933085	885933085
9	36	1	1	10	37	215540145	215540145
				10	38	45379260	45379260
				10	39	8145060	8145060
				10	40	1221759	1221759
				10	41	148995	148995
				10	42	14190	14190
				10	43	990	990
				10	44	45	45
				10	45	1	1

Logarithms and exponentials were computed by a two-variable version of [6, p. 9, formula 1.2.8]—a similar generalization appears in [4, p. 406]. Equations (3), (5) and (8) were solved using a two-variable version of the method described in [6, p. 11, formula 1.3.10], modified by subtracting the appropriate multiples of all the coefficients in the kth power of z/x, R/y and

TABLE II

The Number of Labelled Homeomorphically Irreducible 2-Connected Graphs with $n \leqslant 20$ Vertices

n	
4	1
5	26
6	1858
7	236856
8	53458832
9	21494404400
10	15580475076986
11	20666605559464968
12	50987322515860980236
13	23774756491323236720265 6
14	21257083955793721009155 53094
15	368861871325528386062521 37372776
16	1253964424003393931277014555990072272
17	8409662829146640773436066915556694718 6944
18	1118032145815837410654709542950829449877375 6658
19	29561960650275207201209121086401216157362077036 55176
20	155790616837543983819737575065259505080916866572960692 7348

TABLE III

The Number of Labelled 3-Connected Graphs with $n \leqslant 20$ Vertices

n	
4	1
5	26
6	1768
7	225096
8	51725352
9	21132802554
10	15463799747936
11	20604021770403328
12	50928019401158515328
13	23764442394892899419750 4
14	21253732969001664521998 61760
15	368841339031946270145317 23872256
16	1253940482615318472758477553881715712
17	8409609209963148495102012115056605964 4928
18	1118029820109289475580283563674984587056641 2288
19	29561940980974602381353631386683920626403549276 73344
20	155790584232837889549164420452597373661003024526969900 2368

D/y, respectively, from the right side of (3), (5) and (8), respectively, before replacing those coefficients in the memory by those of the $k+1$st power. The theoretical estimates of the time required to solve these equations up to vertices by these methods are $O(n^4)$ operations for (6) and (7), $O(n^6)$ for (2), (11) and (5), $O(n^7)$ for (3) and $O(n^8)$ for (8), where an "operation" is a multiple-integer-precision multiplication or addition. Using FORTRAN multiple-integer-precision routines we have computed the numbers $H_{n,m}$ and $F_{n,m}$ for $n \leqslant 17$ and all relevant m in roughly one hour of computing time on the BESM-6 computer at Moscow State University. Table I contains the $H_{n,m}$ and $F_{n,m}$ for $n \leqslant 10$.

We have also developed a method of counting labelled bricks and labelled homeomorphically irreducible blocks by number of vertices alone up to n vertices in $O(n^3)$ operations. The basic idea is first to integrate (3), (8) and the integrand of (5) analytically (with lower limit zero), then to solve for $H(x, 1)$ by finding $B(x, y)$ and the integral of (5) as power series in x subject to the condition that $R=1$, and finally to solve for $F(x, 1)$ by finding $B(x, y)$ and $\log((1+y)/2)$ as power series in x subject to the condition that $D=1$. We have counted labelled homeomorphically irreducible blocks with up to 34 vertices in 10 minutes of computer time and labelled bricks with up to 37 vertices in 20 minutes. Tables II and III contain the numbers of labelled homeomorphically irreducible blocks and labelled bricks, respectively, with from 4 to 20 vertices.

4. Comparison with Wormald's Enumeration of Labelled 3-Connected Graphs

After the first draft of this paper had been submitted for publication, we learned of two independent enumerations of labelled bricks [1, 19]. We demonstrate the equivalence of Eqs. (8)–(11) with Eq. (1) of [19], using a method suggested by the referee with the original aim of improving upon Eqs. (8)–(11). It was proved in [13] by calculus and in [20] combinatorially that $B(x, y)$ satisfies the partial differential equation

$$x^2(1+B_{xx}(1-xB_{xx})^{-1})=2(1+y)\,B_y. \qquad (12)$$

Calculations similar to those in [13] yield the following second-order second-degree PDE for $F(x, D)$:

$$(1+D)\,F_D=(x^2/2)\,F_{xx}-(x^4D^4/4)/(1+xD)^2+(x^4/4)(W_x^2/W_D) \qquad (13)$$

where

$$W(x, D)=\log(1+D)-xD^2/(1+xD)-(2/x^2)\,F_D(x, D). \qquad (14)$$

The basic idea is to express y and the derivatives of B in terms of x, D and F and its derivatives and substitute into (12). Define $W(x, D)$ as in (14); then from (8), (10), (11) and (14) we have

$$y = -1 + \exp(W(x, D)) \tag{15}$$

and from (9),

$$B_y(x, y) = (x^2/2)(D + 1)\exp(-W(x, D)). \tag{16}$$

Now $B_D = B_y y_D$; computing B_D from (15) and (16) and integrating (with lower limit zero) yields

$$B(x, y(x, D)) = (x^2/2)[(D + 1)W - T] \tag{17}$$

where

$$T(x, D) = \int_0^D W(x, t)\, dt. \tag{18}$$

Differentiating the left side of (17) with respect to x yields $B_x + B_y y_x$; so from (15), (16) and (17) we obtain

$$B_x(x, y(x, D)) = x[(D + 1)W - T] - (x^2/2)\, T_x. \tag{19}$$

Differentiating the left side of (19) with respect to x and D yields $B_{xx} + B_{xy} y_D$ and $B_{xy} y_D$, respectively; so if we let $G(x, D)$ be the right side of (19), we obtain

$$B_{xx} = G_x - G_D\, y_x / y_D. \tag{20}$$

Another expression for B_{xx} is obtained by substituting from (9) and (10) into (12):

$$B_{xx} = D(1 + xD)^{-1}. \tag{21}$$

Equating (20) and (21) and substituting for G we have

$$D(1 + xD)^{-1} - (D + 1)W + T + 2xT_x + (x^2/1)\, T_{xx} = (x^2/2)(W_x^2/W_D). \tag{22}$$

Finally, evaluating the integral in (18) to find T and then substituting for T and W in the left side of (22) *but not in the right side* and simplifying yields (13). And changing D to y and W to T in (13) and (14) yields Eq. (1) of [19].

Equations (13) and (14), and hence Eq. (1) of [19], are solvable in $O(n^6)$, an improvement over the $O(n^8)$ required for (8), but not over the $O(n^3)$ needed to count labelled bricks by number of vertices alone.

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REFERENCES

1. N. M. BOGOLYUBOV, A. N. VASILYEV, AND Y. M. PISMAK, Enumeration of labelled 3-connected graphs, *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov iomi*, in press. English transl. *J. Soviet Math.*, in press.
2. W. H. CUNNINGHAM, "A Combinatorial Decomposition Theory," Ph. D. Thesis, University of Waterloo, 1973.
3. W. H. CUNNINGHAM AND J. EDMONDS, A combinatorial decomposition theory, *Canad. J. Math.* **32** (1980), 734–765.
4. E. N. GILBERT, Enumeration of labelled graphs, *Canad. J. Math.* **8** (1956), 405–411.
5. F. HARARY, J, KRARUP, AND A. SCHWENK, Graphs suppressible to an edge, *Canad. Math. Bull.* **15** (1971), 201–204.
6. F. HARARY AND E. M. PALMER, "Graphical Enumeration," Academic Press, New York/London, 1973.
7. S. V. JABLONSKI, "The theory of graphs and networks," Institut Prikladnoy Matematiki, Akad. Nauk SSSR, Moscow, 1972. [Russian]
8. D. M. JACKSON AND J. W. REILLY, The enumeration of homeomorphically irreducible labelled graphs, *J. Combin. Theory Ser. B* **19** (1975), 272–286.
9. V. A. KUZNETZOV, On non-repeating contact schemes and non-repeating superpositions of functions of the algebra of logic, *Trudi Mat. Inst. Akad. Nauk SSSR* **51** (1958), 186–225. [Russian]
10. R. J. RIDDELL, Contribution to the theory of condensation, Dissertation, University of Michigan, Ann Arbor, 1951.
11. R. W. ROBINSON, Enumeration of non-separable graphs, *J. Combin. Theory* **9** (1970), 327–356.
12. R. TARJAN AND J. HOPCROFT, Finding the triconnected components of a graph, *SIAM J. Comput.* **2** (1973), 135-158.
13. H. N. V. TEMPERLEY, On the enumeration of Mayer cluster integrals, *Proc. Phys. Soc.* **72** (1958), 1141–1144.
14. B. A. TRAKHTENBROT, Towards a theory of non-repeating contact schemes, *Trudi Mat. Inst. Akad. Nauk SSSR* **51** (1958), 226–269. [Russian]
15. W. T. TUTTE, "Connectivity in Graphs," University of Toronto Press, Toronto, 1966.
16. W. T. TUTTE, A census of planar maps, *Canad. J. Math.* **15** (1963), 249–271.
17. F. Y. VETUKHNOVSKI, "Graphs and Networks," Diskretnaya matematika i matematicheskie voprosi kibernetiki 1, Ed: pp. 149–206, Nauka, Moscow, 1974. [Russian]
18. T. R. S. WALSH, Counting unlabelled three-connected and homeomorphically irreducible two-connected graphs, *J. Combin. Theory Ser. B* **32** (1982), 12–32.
19. N. WORMALD, Enumeration of labelled graphs. I. 3-connected graphs, *J. London Math. Soc. (2)* **19** (1979), 7–12.
20. N. WORMALD AND E. M. WRIGHT, The exponential generating function of labelled blocks, *Discrete Math.* **25** (1979), 93–96.