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Connected choice and the Brouwer fixed point theorem

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We study the computational content of the Brouwer Fixed Point Theorem in the Weihrauch lattice. Connected choice is the operation that finds a point in a non-empty connected closed set given by negative information. One of our main results is that for any fixed dimension the Brouwer Fixed Point Theorem of that dimension is computably equivalent to connected choice of the Euclidean unit cube of the same dimension. Another main result is that connected choice is complete for dimension greater than or equal to two in the sense that it is computably equivalent to Weak Kőnig's Lemma. While we can present two independent proofs for dimension three and upward that are either based on a simple geometric construction or a combinatorial argument, the proof for dimension two is based on a more involved inverse limit construction. The connected choice operation in dimension one is known to be equivalent to the Intermediate Value Theorem; we prove that this problem is not idempotent in contrast to the case of dimension two and upward. We also prove that Lipschitz continuity with Lipschitz constants strictly larger than one does not simplify finding fixed points. Finally, we prove that finding a

connectedness component of a closed subset of the Euclidean unit cube of any dimension greater than or equal to one is equivalent to Weak Kőnig's Lemma. In order to describe these results, we introduce a representation of closed subsets of the unit cube by trees of rational complexes.

Keywords: Computable analysis; Weihrauch lattice; reverse mathematics; choice principles; connected sets; fixed point theorems.

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1. Introduction

In this paper, we continue with the programme to classify the computational content of mathematical theorems in the Weihrauch lattice (see [26, 9, 8, 40, 41, 11, 29]). This lattice is induced by Weihrauch reducibility, which is a reducibility for multi-valued partial functions $f :\subseteq X \rightrightarrows Y$ on represented spaces X, Y. Intuitively, $f \leq_{\mathrm{W}} g$ reflects the fact that f can be realized with a single application of g as an oracle. Hence, if two multi-valued functions are equivalent in the sense that they are mutually reducible to each other, then they are equivalent as computational resources, as far as computability is concerned.

Many theorems in mathematics are actually of the logical form

$$(\forall \, x \in X)(\exists \, y \in Y) \, P(x, y)$$

and such theorems can straightforwardly be represented by a multi-valued function $f : X \rightrightarrows Y$ with $f(x) := \{y \in Y : P(x, y)\}$ (sometimes partial f are needed, where the domain captures additional requirements that the input x has to satisfy). In some sense, the multi-valued operation f directly reflects the computational task of the theorem to find some suitable y for any x. Hence, in a very natural way the classification of a theorem can be achieved via a classification of the corresponding multi-valued function that represents the theorem.

Theorems that have been compared and classified in this sense include Weak Kőnig's Lemma WKL, the Hahn–Banach Theorem [26], the Baire Category Theorem [14], Banach's Inverse Mapping Theorem, the Open Mapping Theorem, the Uniform Boundedness Theorem, the Intermediate Value Theorem [8], the Bolzano–Weierstraß Theorem [11], Nash Equilibria [41], the Radon–Nikodym Theorem [29] and the Vitali Covering Theorem [10]. In this paper, we classify the Brouwer Fixed Point Theorem.

Theorem 1.1 (Brouwer Fixed Point Theorem 1911). *Every continuous function $f : [0,1]^n \rightarrow [0,1]^n$ has a fixed point $x \in [0,1]^n$, i.e. a point such that $f(x) = x$.*

This theorem was first proved by Hadamard in 1910 and later by Brouwer [19], after whom it is named the Brouwer Fixed Point Theorem. Brouwer is known as one of the founders of intuitionism, which is one of the well-studied varieties of constructive mathematics and ironically, the theorem that he is best known for does

not admit any constructive proof.[a] This fact has been confirmed in many different ways, most relevant for us is the counterexample in Russian constructive analysis by Orevkov [39], which was transferred into computable analysis by Baigger [1]. Baigger's counterexample shows that from dimension two upward (i.e. $n \geq 2$) there are computable functions $f : [0,1]^n \to [0,1]^n$ without computable fixed point x. Baigger's proof actually proceeds by encoding a Kleene tree (implicitly via a pair of computably inseparable sets) into a suitable computable function f and hence it can be seen as a reduction of Weak Kőnig's Lemma to the Brouwer Fixed Point Theorem.[b] The essential geometrical content of this construction is that the map

$$A \mapsto (A \times [0,1]) \cup ([0,1] \times A)$$

maps *arbitrary* non-empty closed sets $A \subseteq [0,1]$ to *connected* non-empty closed subsets of $[0,1]^2$ such that any pair in the resulting set has at least one component that is in A.

Constructions similar to those used for the above counterexamples have been utilized in order to prove that the Brouwer Fixed Point Theorem is equivalent to Weak Kőnig's Lemma in reverse mathematics [46, 45, 34] and to analyze computability properties of fixable sets [37], but a careful analysis of these reductions reveals that none of them can be straightforwardly transferred into a *uniform* reduction in the sense that we are seeking here. The problem is that there is no uniform way to select a component x_i of a pair (x_1, x_2) such that $x_i \in A$, given that at least one of the components has this property. The results cited above essentially characterize the complexity of the fixed points themselves, whereas we want to characterize the complexity of finding a fixed point, given the function. This requires full uniformity.

In the Weihrauch lattice, the Brouwer Fixed Point Theorem of dimension n is represented by the multi-valued function $\mathsf{BFT}_n : \mathcal{C}([0,1]^n, [0,1]^n) \rightrightarrows [0,1]^n$ that maps any continuous function $f : [0,1]^n \to [0,1]^n$ to the set of its fixed points $\mathsf{BFT}_n(f) \subseteq [0,1]^n$. The question now is where BFT_n is located in the Weihrauch lattice? It easily follows from a meta theorem presented in [8] that the Brouwer Fixed Point Theorem BFT_n is reducible to Weak Kőnig's Lemma WKL for any dimension n, i.e. $\mathsf{BFT}_n \leq_W \mathsf{WKL}$. However, for which dimensions n do we also obtain the inverse reduction? Clearly not for $n = 0$, since BFT_0 is computable, and clearly not for $n = 1$, since BFT_1 is equivalent to the Intermediate Value Theorem IVT and hence not equivalent to WKL, as proved in [8].[c]

In order to approach this question for a general dimension n, we introduce a choice principle CC_n that we call *connected choice* and which is just the closed choice operation restricted to connected subsets. That is, in the sense discussed

[a]However, as noticed already by Brouwer himself, the theorem admits an approximative constructive version [20].
[b]See [42] for a discussion of these counterexamples.
[c]We mention that in Bishop style constructive reverse mathematics the Intermediate Value Theorem is equivalent to Weak Kőnig's Lemma [31], as parallelization is freely available in this framework.

above CC_n is the multi-valued function that represents the following mathematical statement: every non-empty connected closed set $A \subseteq [0,1]^n$ has a point $x \in [0,1]^n$. Since closed sets are represented by negative information (i.e. by an enumeration of open balls that exhaust the complement), the computational task of CC_n consists in finding a point in a closed set $A \subseteq [0,1]^n$ that is promised to be non-empty and connected and that is given by negative information.

One of our main results, proved in Sec. 4, is that the Brouwer Fixed Point Theorem is equivalent to connected choice for each fixed dimension n, i.e.

$$\mathsf{BFT}_n \equiv_{\mathrm{W}} \mathsf{CC}_n.$$

This result allows us to study the Brouwer Fixed Point Theorem in terms of the operation CC_n that is easier to handle since it involves neither function spaces nor fixed points. This is also another instance of the observation that several important theorems are equivalent to certain choice principles (see [8]) and many important classes of computable functions can be calibrated in terms of choice (see [6]). For instance, closed choice on Cantor space $\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$ and on the unit cube $\mathsf{C}_{[0,1]^n}$ are both easily seen to be equivalent to Weak Kőnig's Lemma WKL, i.e. $\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]^n}$ for any $n \geq 1$. Studying the Brouwer Fixed Point Theorem in the form of CC_n now amounts to comparing $\mathsf{C}_{[0,1]^n}$ with its restriction CC_n.

Our second main result, proved in Secs. 6 and 7, is that from dimension two upward connected choice is equivalent to Weak Kőnig's Lemma, i.e. $\mathsf{CC}_n \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]}$ for $n \geq 2$. In Sec. 6, we present a proof for dimension $n \geq 3$ that is based on the geometrical construction

$$A \mapsto (A \times [0,1] \times \{0\}) \cup (A \times A \times [0,1]) \cup ([0,1] \times A \times \{1\})$$

that maps an *arbitrary* non-empty closed set $A \subseteq [0,1]$ to a *pathwise connected* non-empty closed subset of $[0,1]^3$ that has the property that from any of its points we can compute a point of the original set A in a uniform sense. This construction seems to require at least dimension three in a crucial sense. The same is true for an alternative combinatorial proof of the same result that we provide. The proof for dimension 2 is presented in Sec. 7 and is based on a more involved inverse limit construction and hence on an entirely different idea. It only yields a connected (not necessarily pathwise connected) set in general. We are left with the open question whether pathwise connected choice of dimension two is equivalent to connected choice of dimension two.

In Sec. 5, we show that Lipschitz continuity with a Lipschitz constant $L > 1$ does not simplify finding fixed points. Using results of Neumann [38], we obtain a trichotomy of the problem of finding fixed points for Lipschitz continuous functions that depends on whether the Lipschitz constant L satisfies $L < 1$, $L = 1$ or $L > 1$.

In order to prove some of our results, we use a representation of closed sets by trees of so-called *rational complexes*, which we introduce in Sec. 3. It can be seen as a generalization of the well-known representation of co-c.e. closed subsets of Cantor

space $\{0,1\}^{\mathbb{N}}$ by binary trees. As a side result we prove that finding a connectedness component of a closed set for any fixed dimension from one upward is equivalent to Weak Kőnig's Lemma. This yields conclusions along the line of earlier studies of connected components in [36].

Finally, we provide a so-called Displacement Principle in Sec. 8 that helps us in Sec. 9 to show that CC_1 is neither idempotent nor a cylinder.

In the following section, we start with a short summary of relevant definitions and results regarding the Weihrauch lattice.

2. The Weihrauch Lattice

In this section, we briefly recall some basic results and definitions regarding the Weihrauch lattice. The original definition of Weihrauch reducibility is due to Weihrauch and has been studied for many years (see [48–50, 27, 3, 4]). Only recently it has been noticed that a certain variant of this reducibility yields a lattice that is very suitable for the classification of mathematical theorems (see [26, 40, 41, 9, 8, 6, 11]). The basic reference for all notions from computable analysis is [51], and a survey on Weihrauch complexity can be found in [12]. The Weihrauch lattice is a lattice of multi-valued functions on represented spaces. A *representation* δ of a set X is just a surjective partial map $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. In this situation we call (X, δ) a *represented space*. In general we use the symbol "$\subseteq$" in order to indicate that a function is potentially partial. Using represented spaces we can define the concept of a realizer. For $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Y \rightrightarrows Z$ we write $g \circ f$ or gf for the *composition* defined by $(g \circ f)(x) := \{z \in Z : (\exists y \in f(x))\ z \in g(y)\}$ and $\mathrm{dom}(g \circ f) := \{x \in \mathrm{dom}(f) : f(x) \subseteq \mathrm{dom}(g)\}$.

Definition 2.1 (Realizer). Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces. A function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called a *realizer* of f, in symbols $F \vdash f$, if $\delta_Y F(p) \in f \delta_X(p)$ for all $p \in \mathrm{dom}(f \delta_X)$.

Realizers allow us to transfer the notions of computability and continuity and other notions available for Baire space to any represented space; a function between represented spaces will be called *computable*, if it has a computable realizer, etc. Now we can define Weihrauch reducibility. By $\langle\ ,\ \rangle : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ we denote the standard pairing function, defined by $\langle p, q \rangle(2n) := p(n)$ and $\langle p, q \rangle(2n+1) := q(n)$ for all $p, q \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

Definition 2.2 (Weihrauch reducibility). Let f, g be multi-valued functions on represented spaces. Then f is said to be *Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable functions $K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $K \langle \mathrm{id}, GH \rangle \vdash f$ for all $G \vdash g$. Moreover, f is said to be *strongly Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{sW}} g$, if there are computable functions K, H such that $KGH \vdash f$ for all $G \vdash g$.

The difference between ordinary and strong Weihrauch reducibility is that the "output modifier" K has direct access to the original input in case of ordinary Weihrauch reducibility, but not in case of strong Weihrauch reducibility. We note that the relations $\leq_{\mathrm{W}}$, $\leq_{\mathrm{sW}}$ and $\vdash$ implicitly refer to the underlying representations, which we will only mention explicitly if necessary. It is known that these relations only depend on the underlying equivalence classes of representations, but not on the specific representatives (see [9, Lemma 2.11]). The relations $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ are reflexive and transitive, thus they induce corresponding partial orders on the sets of their equivalence classes (which we refer to as *Weihrauch degrees* or *strong Weihrauch degrees*, respectively). These partial orders will be denoted by $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ as well. In this way one obtains a distributive bounded lattice for $\leq_{\mathrm{W}}$ which we call the *Weihrauch lattice* (for details see [40] and [9]). We use $\equiv_{\mathrm{W}}$ and $\equiv_{\mathrm{sW}}$ to denote the respective equivalences regarding $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$, and by $<_{\mathrm{W}}$ and $<_{\mathrm{sW}}$ we denote strict reducibility.

The Weihrauch lattice is equipped with a number of useful algebraic operations that we summarize in the next definition. We use $X \times Y$ to denote the ordinary set-theoretic *product*, $X \sqcup Y := (\{0\} \times X) \cup (\{1\} \times Y)$ in order to denote *disjoint sums* or *coproducts*, by $\bigsqcup_{i=0}^{\infty} X_i := \bigcup_{i=0}^{\infty}(\{i\} \times X_i)$ we denote the *infinite coproduct*. By X^i we denote the *i*-fold product of a set X with itself, where $X^0 = \{()\}$ is some canonical computable singleton. By $X^* := \bigsqcup_{i=0}^{\infty} X^i$ we denote the set of all *finite sequences over X* and by $X^{\mathbb{N}}$ the set of all *infinite sequences over X*. All these constructions have parallel canonical constructions on representations and the corresponding representations are denoted by $[\delta_X, \delta_Y]$ for the product of (X, δ_X) and (Y, δ_Y), $\delta_X \sqcup \delta_Y$ for the coproduct and δ_X^* for the representation of X^* and $\delta_X^{\mathbb{N}}$ for the representation of $X^{\mathbb{N}}$ (see [9, 40, 6] for details). We will always assume that these canonical representations are used, if not mentioned otherwise.

Definition 2.3 (Algebraic operations). Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions on represented spaces. Then we define the following operations:

(1) $f \times g :\subseteq X \times Z \rightrightarrows Y \times W, (f \times g)(x,z) := f(x) \times g(z)$ (product)

(2) $f \sqcap g :\subseteq X \times Z \rightrightarrows Y \sqcup W, (f \sqcap g)(x,z) := f(x) \sqcup g(z)$ (sum)

(3) $f \sqcup g :\subseteq X \sqcup Z \rightrightarrows Y \sqcup W$, with $(f \sqcup g)(0,x) := \{0\} \times f(x)$
and $(f \sqcup g)(1,z) := \{1\} \times g(z)$ (coproduct)

(4) $f^* :\subseteq X^* \rightrightarrows Y^*, f(i,x) := \{i\} \times f^i(x)$ (finite parallelization)

(5) $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, f(x_n) := \mathsf{X}_{i=0}^{\infty} f(x_i)$ (parallelization)

In this definition and in general we denote by $f^i :\subseteq X^i \rightrightarrows Y^i$ the *i*-th fold product of the multi-valued map f with itself. For f^0 we assume that $X^0 := \{()\}$ is a canonical singleton for each set X and hence f^0 is just the constant operation on that set. It is known that $f \sqcap g$ is the *infimum* of f and g with respect to strong as well as ordinary Weihrauch reducibility (see [9], where this operation was denoted by $f \oplus g$). Correspondingly, $f \sqcup g$ is known to be the *supremum* of f and g with

respect to $\leq_{\mathrm{W}}$ (see [40]). The two operations $f \mapsto \widehat{f}$ and $f \mapsto f^*$ are known to be *closure operators* in the corresponding lattices, which means $f \leq_{\mathrm{W}} \widehat{f}$ and $\widehat{f} \equiv_{\mathrm{W}} \widehat{\widehat{f}}$, and $f \leq_{\mathrm{W}} g$ implies $\widehat{f} \leq_{\mathrm{W}} \widehat{g}$ and analogously for finite parallelization (see [9, 40]).

We use some terminology related to these algebraic operations. We say that f is a *cylinder* if $f \equiv_{\mathrm{sW}} \mathrm{id} \times f$ where $\mathrm{id} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ always denotes the identity on Baire space, if not mentioned otherwise. Cylinders f have the property that $g \leq_{\mathrm{W}} f$ is equivalent to $g \leq_{\mathrm{sW}} f$ (see [9]). We say that f is *idempotent* if $f \equiv_{\mathrm{W}} f \times f$. We say that a multi-valued function on represented spaces is *pointed*, if it has a computable point in its domain. For pointed f and g we obtain $f \sqcup g \leq_{\mathrm{sW}} f \times g$. If $f \sqcup g$ is idempotent, then we also obtain the inverse reduction. The finite parallelization f^* can also be considered as *idempotent closure* as $f \equiv_{\mathrm{W}} f^*$ holds if and only if f is idempotent and pointed. We call f *parallelizable* if $f \equiv_{\mathrm{W}} \widehat{f}$ and it is easy to see that $\widehat{f}$ is always idempotent. The properties of pointedness and idempotency are both preserved under equivalence and hence they can be considered as properties of the respective degrees.

A particularly useful multi-valued function in the Weihrauch lattice is closed choice (see [26, 9, 8, 6]) and it is known that many notions of computability can be calibrated using the right version of choice. We will focus on closed choice for computable metric spaces, which are separable metric spaces such that the distance function is computable on the given dense subset. We assume that computable metric spaces are represented via their Cauchy representation (see [51] for details).

By $\mathcal{A}_-(X)$ we denote the set of closed subsets of a metric space X, where the index "$-$" indicates that we work with negative information. These are given by the representation $\psi_- : \mathbb{N}^{\mathbb{N}} \to \mathcal{A}_-(X)$, defined by $\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)}$, where B_n is some standard enumeration of the open balls of X with center in the dense subset and rational radius. The computable points in $\mathcal{A}_-(X)$ are called *co-c.e. closed sets*. We now define closed choice for the case of computable metric spaces.

Definition 2.4 (Closed choice). Let X be a computable metric space. Then the *closed choice* operation of this space X is defined by $\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$ with $\mathrm{dom}(\mathsf{C}_X) := \{A \in \mathcal{A}_-(X) : A \neq \emptyset\}$.

Intuitively, C_X takes as input a non-empty closed set in negative description (i.e. given by ψ_-) and it produces an arbitrary point of this set as output. Hence, $A \mapsto A$ means that the multi-valued map C_X maps the input $A \in \mathcal{A}_-(X)$ to the set $A \subseteq X$ as a set of possible outputs. We mention a couple of properties of closed choice for specific spaces.

The omniscience principle LLPO has turned out to be very useful and it is closely related to closed choice. We recall the definition.

Definition 2.5 (Omniscience principle). We define $\mathsf{LLPO} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ by

$$j \in \mathsf{LLPO}(p) \Leftrightarrow (\forall n \in \mathbb{N}) p(2n+j) = 0$$

for all $j \in \{0, 1\}$, where $\mathrm{dom}(\mathsf{LLPO}) := \{p \in \mathbb{N}^{\mathbb{N}} : p(k) \neq 0 \text{ for at most one } k\}$.

It is easy to see that $C_{\{0,1\}} \equiv_{\mathrm{sW}} \mathsf{LLPO}$. We mention that closed choice can also be used to characterize the computational content of many theorems. By WKL we denote the straightforward formalization of Weak Kőnig's Lemma. Since we will not use WKL in any formal sense here, we refer the reader to [26, 9] for precise definitions.

Fact 2.6 (Weak Kőnig's Lemma). $\mathsf{WKL} \equiv_{\mathrm{sW}} C_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{sW}} C_{[0,1]^n} \equiv_{\mathrm{sW}} C_{[0,1]^{\mathbb{N}}} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$ for all $n \geq 1$.

3. Closed Sets and Trees of Rational Complexes

In this section, we want to describe a representation of closed sets $A \subseteq [0,1]^n$ that is useful for the study of connectedness. It is well known that closed subsets of Cantor space can be characterized exactly as sets of infinite paths of trees (see for instance [22]). We describe a similar representation of closed subsets of the unit cube $[0,1]^n$ of the Euclidean space. While in the case of Cantor space clopen balls are associated to each node of the tree, we now associate finite complexes of rational balls to each node. While infinite paths lead to points of the closed set in case of Cantor space, they now lead to connectedness components (which can be seen as a generalization, since the connectedness components in Cantor space are singletons).

This representation of closed subsets $A \subseteq [0,1]^n$ of the unit cube will enable us to analyze the relation between connected choice and the Brouwer Fixed Point Theorem in the next section. In this section, we will use this representation in order to prove the result that finding a connectedness component of a closed set A is computationally exactly as difficult as Weak Kőnig's Lemma.

We first fix some topological terminology that we are going to use. We work with the *maximum norm* $\| \ \|$ on $\mathbb{R}^n$, defined by $\|(x_1, \ldots, x_n)\| := \max\{|x_i| : i = 1, \ldots, n\}$. By $d(x, A) := \inf_{a \in A} \|x - a\|$ we denote the *distance* of $x \in \mathbb{R}^n$ to $A \subseteq \mathbb{R}^n$. By $d_A : \mathbb{R}^n \to \mathbb{R}$ we denote the corresponding *distance function* given by $d_A(x) := d(x, A)$. By $B(x, r) := \{y \in \mathbb{R}^n : \|x - y\| < r\}$ we denote the *open ball* with center x and radius r and by $B[x, r] := \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$ the corresponding *closed ball*. Since we are using the maximum norm, all these balls are open or closed cubes, respectively (if the radius is positive). By ∂A we denote the topological *boundary*, by $\overline{A}$ the *closure* and by A° the *interior* of a set A. If the underlying space X is clear from the context, then $A^{\mathrm{c}} := X \setminus A$ denotes the *complement* of A.

We are now prepared to define rational complexes, which are just finite sets of rational closed balls whose union is connected and that pairwise intersect at most on their boundary.

Definition 3.1 (Rational complex). We call a set $R := \{B[c_1, r_1], \ldots, B[c_k, r_k]\}$ of finitely many closed balls $B[c_i, r_i]$ with rational center $c_i \in \mathbb{Q}^n$ and positive rational radius $r_i \in \mathbb{Q}$ an $(n\text{-}dimensional)$ *rational complex* if $\bigcup R$ is connected and $B_1, B_2 \in R$ with $B_1 \neq B_2$ implies $B_1^\circ \cap B_2^\circ = \emptyset$. We say that a rational complex

is *non-empty*, if $\bigcup R \neq \emptyset$. By $\mathrm{C}\mathbb{Q}^n$ we denote the set of n-dimensional rational complexes.

Each rational complex R can be represented by a list of the corresponding rational numbers $c_1, r_1, \ldots, c_k, r_k$ and we implicitly assume in the following that this representation is used for the set of rational complexes $\mathrm{C}\mathbb{Q}^n$.

In order to organize the rational complexes that are used to approximate sets it is suitable to use trees. We recall that a *tree* is a set $T \subseteq \mathbb{N}^*$ that is closed under prefix, i.e. $u \sqsubseteq v$ and $v \in T$ implies $u \in T$. A function $b : \mathbb{N} \to \mathbb{N}$ is called a *bound* of a tree T if $w \in T$ implies $w(i) \leq b(i)$ for all $i = 0, \ldots, |w| - 1$, where $|w|$ denotes the *length* of the word w. A tree is called *finitely branching*, if it has a bound. A tree of rational complexes is understood to be a finitely branching tree T (together with a bound) such that to each node of the tree a rational complex is associated, with the property that these complexes are compactly included in each other if we proceed along paths of the tree and they are disjoint on any particular level of the tree. We write $A \Subset B$ for two sets $A, B \subseteq \mathbb{R}^n$ if the closure $\overline{A}$ of A is included in the interior B° of B and we say that A is *compactly included* in B in this case.

Definition 3.2 (Tree of rational complexes). We call (T, f) a *tree of rational complexes* if $T \subseteq \mathbb{N}^*$ is a finitely branching tree and $f : T \to \mathrm{C}\mathbb{Q}^n$ is a function such that for all $u, v \in T$ with $u \neq v$

(1) $u \sqsubseteq v \Rightarrow \bigcup f(v) \Subset \bigcup f(u)$,
(2) $|u| = |v| \Rightarrow \bigcup f(u) \cap \bigcup f(v) = \emptyset$.

In the following we assume that finitely branching trees T are represented as a pair (χ_T, b), where $\chi_T : \mathbb{N}^* \to \{0, 1\}$ is the characteristic function of T and $b : \mathbb{N} \to \mathbb{N}$ is a bound of T. Correspondingly, trees (T, f) of rational complexes are then represented in a canonical way by (χ_T, b, f). We now define which set $A \subseteq [0, 1]^n$ is represented by such a tree (T, f) of rational complexes.

Definition 3.3 (Closed sets and trees of rational complexes). We say that a closed set $A \subseteq \mathbb{R}^n$ is *represented* by a tree (T, f) of n-dimensional rational complexes if one obtains $A = \bigcap_{i=0}^{\infty} \bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$.

It is clear that in this way any tree (T, f) of rational complexes actually represents a compact set A. This is because $\bigcup f(w)$ is compact for each $w \in T$ and since T is finitely branching, the set $T \cap \mathbb{N}^i$ is finite for each i, hence $\bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$ is compact and hence A is compact too. Vice versa, every compact set $A \subseteq \mathbb{R}^n$ can be represented by a tree of n-dimensional rational complexes. For $[0, 1]^n$ we prove the uniform result that even the map $(T, f) \mapsto A$ is computable and has a computable multi-valued right inverse. We assume that trees of rational complexes are represented as specified above and closed sets A are represented as points in $\mathcal{A}_-([0, 1]^n)$. We recall that a *connectedness component* of a set A is a connected subset of A that is not included in any larger connected subset of A. Any connectedness component

of a subset A is closed in A. If $A = \emptyset$, then the only connectedness component is the empty set, otherwise connectedness components are always non-empty.

Proposition 3.4 (Closed sets and complexes). *Let $n \geq 1$. The map $(T, f) \mapsto A$ that maps every tree of n-dimensional rational complexes (T, f) to the closed set $A \subseteq [0,1]^n$ represented by it, is computable and has a multi-valued computable right inverse. An analogous result holds restricted to infinite trees of non-empty rational complexes and non-empty closed A.*

Proof. It is clear that, given (T, f) and a bound b of T we can actually compute $A \in \mathcal{A}_-([0,1]^n)$. First, we can explicitly determine all finitely many $w \in T \cap \mathbb{N}^i$ using the bound b and compute $C_i := \bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w) \in \mathcal{A}_-([0,1]^n)$ for each i. Since intersection of sequences of closed sets is computable on $\mathcal{A}_-([0,1]^n)$, we can also compute $A := \bigcap_{i=0}^{\infty} C_i$.

We note that if (T, f) is an infinite tree of non-empty rational complexes then the C_i form a decreasing chain of non-empty compact sets and hence $A = \bigcap_{i=0}^{\infty} C_i$ is non-empty too by Cantor's Intersection Theorem.

For the other direction, let us assume that $A \subseteq [0,1]^n$ is given as the complement of a union of rational open balls $B(c_i, r_i)$. We use the larger cube $Q := [-1, 2]^n$ and we assume that $A = Q \cap (\bigcup_{i=0}^{\infty} B(c_i, r_i))^c$ with $B(c_i, r_i) \cap Q \neq \emptyset$ for all i. Now we show how we can compute a tree (T, f) of rational complexes together with a bound b that represents A. We proceed inductively over the length $i = |w| = 0, 1, 2, \ldots$ of words in the tree T.

We start with the empty node $\varepsilon \in T$ and we assign $f(\varepsilon) = \{Q\}$ to it. Let us now assume that $T \cap \mathbb{N}^i$ has been completely determined, $f(w)$ has been fixed for all $w \in T \cap \mathbb{N}^i$ and $b(j)$ has been determined for all $j < i$. We now determine $T \cap \mathbb{N}^{i+1}$, $f(w)$ for words $w \in T \cap \mathbb{N}^{i+1}$, and $b(i)$. The following is applied to each $w \in T \cap \mathbb{N}^i$:

(1) First, we copy each rational complex $f(w)$ into $f(w0)$.
(2) Then the points $B := \{x : d(x, \partial \bigcup f(w)) < 2^{-i-1}\}$ which are close to the boundary are removed from $\bigcup f(w0)$. That is $f(w0)$ is refined such that the resulting union is the original one minus B and all new balls in $f(w0)$ intersect at most on their boundaries. This guarantees $\bigcup f(w0) \Subset \bigcup f(w)$ (but it might destroy the property that $\bigcup f(w0)$ is connected).
(3) In the next step $U := \bigcup_{j=0}^{i} B(c_j, r_j - 2^{-i})$ is removed from $\bigcup f(w0)$. This means that $f(w0)$ is refined such that the union is the original union minus U and all new balls in $f(w0)$ intersect at most on their boundaries. This guarantees that the tree of rational complexes will eventually represent A (we subtract 2^{-i} from the radius here in order to ensure that there is enough space for removing the boundary stripe B in the next step (2) of the induction without removing anything of A).
(4) Now $\bigcup f(w0)$ is not necessarily connected, but it has only finitely many connectedness components $C_0, \ldots, C_k$ that can all be explicitly determined as rational complexes. We copy these rational complexes into $f(w0), \ldots, f(wk)$ such that

$\bigcup f(wj) = C_j$ for $j = 0, \ldots, k$ afterward. Then all the $\bigcup f(wj)$ are pairwise disjoint and $\bigcup f(wj) \Subset \bigcup f(w)$ for all $j = 0, \ldots, k$. Should the only connectedness component C_0 be the empty set, then we stop the tree T at this point and add no words wj to it.

After this procedure has been completed for all finitely many $w \in T \cap \mathbb{N}^i$, we choose $b(i)$ as the maximal number k (of connectedness components) that occurred for any of these words w. It is clear that then $v(i) \leq b(i)$ for all $v \in T \cap \mathbb{N}^{i+1}$. Moreover, we also have $\bigcup f(wj) \cap \bigcup f(vl) = \emptyset$ for all $w, v \in \mathbb{N}^{i+1}$ with $v \neq w$ since $\bigcup f(w) \cap \bigcup f(v) = \emptyset$ and $\bigcup f(wj) \Subset \bigcup f(w)$ and $\bigcup f(vl) \Subset \bigcup f(v)$.

Altogether, (T, f) as constructed here is a tree of rational complexes with bound b. We still need to prove that the set $A_{(T,f)}$ represented by (T, f) is actually A. Let us denote by $A_i := \bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$ the closed set represented by the union of all the complexes of height i. In particular $A_{(T,f)} = \bigcap_{i \in \mathbb{N}} A_i$. If $x \in Q \backslash A$, then there are some i, j such that $x \in B(c_j, r_j - 2^{-i})$ and hence x is removed from all the complexes of height i of the tree in step (3) above. Hence $x \notin A_i$, which implies $A_{(T,f)} \subseteq A$. Let, on the other hand, $x \in A$. Then clearly $x \in A_0 = Q$ and has distance from ∂A_0 at least 1. By induction one can show that for each i the distance $d(x, \partial A_i)$ is at least 2^{-i} and hence x cannot be removed in step (2) (and also not in step (3) since only points outside A are removed there). This induction shows that $x \in A_i$ for all i and hence $x \in A_{(T,f)}$. Altogether we have proved $A = A_{(T,f)}$.

We note that if A is a non-empty set, then there is always at least one non-empty connectedness component C_0 in step (4) of the algorithm and the computed tree is automatically an infinite tree of non-empty rational complexes. If A is the empty set, then the computed tree is a finite tree of non-empty rational complexes. $\qquad\square$

We note that this proof in particular shows that we can restrict the investigation in general to trees of non-empty rational complexes (even if we want to include the empty closed set). The previous result has a lot of interesting applications. For instance, if A is represented by (T, f), then the sets $A_i := \bigcup_{w \in T \cap \mathbb{N}^i} \bigcup f(w)$ of height i used in the previous proof are of very special form. They are finite unions of connected sets that are themselves finite unions of rational closed balls. It is easy to see that for a co-c.e. closed A the resulting sequence $(A_i)_{i \in \mathbb{N}}$ is automatically a computable sequence of *bi-computable* sets A_i, which means that the sequences $(d_{A_i})_{i \in \mathbb{N}}$ and $(d_{A_i^c})_{i \in \mathbb{N}}$ of characteristic functions are computable (see [28] for more information on bi-computable sets). This is because the maps $R \mapsto d_{\bigcup R}$ and $R \mapsto d_{(\bigcup R)^c}$ of type $C\mathbb{Q}^n \to \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ are easily seen to be computable. This leads to the following corollary.

Corollary 3.5. *For every non-empty co-c.e. closed set $A \subseteq [0, 1]^n$ there is a computable sequence $(A_i)_{i \in \mathbb{N}}$ of bi-computable compact sets $A_i \subseteq [-1, 2]^n$ that is compactly decreasing, i.e. $A_{i+1} \Subset A_i$ for all $i \in \mathbb{N}$ and such that $A = \bigcap_{i \in \mathbb{N}} A_i$.*

The representation of closed sets $A \subseteq [0,1]^n$ by trees of rational complexes also has the advantage that connectedness components of A can easily be expressed in terms of the tree structure. This is made precise by the following lemma. By $[T] := \{p \in \mathbb{N}^{\mathbb{N}} : (\forall i) \ p|_i \in T\}$ we denote the set of *infinite paths* of T, which is also called the *body* of T. Here $p|_i = p(0) \cdots p(i-1) \in \mathbb{N}^*$ denotes the *prefix* of p of length i for each $i \in \mathbb{N}$. According to the following lemma there is bijection between $[T]$ and the set of connectedness components of a non-empty closed set $A \subseteq [0,1]^n$.

Lemma 3.6 (Connectedness components). *Let (T,f) be an infinite tree of n-dimensional non-empty rational complexes and let $A \subseteq [0,1]^n$ be the non-empty closed set represented by (T,f). Then the sets $C_p := \bigcap_{i=0}^{\infty} \bigcup f(p|_i)$ for $p \in [T]$ are exactly all connectedness components of A (without repetitions).*

Proof. Let $A \subseteq [0,1]^n$ be represented by (T,f). First, it is clear that every set $C_p = \bigcap_{i=0}^{\infty} \bigcup f(p|_i)$ is included in A for $p \in [T]$. We claim that also $\bigcup_{p \in [T]} C_p = A$. If $x \in A$, then for every i there is a unique $w_i \in T \cap \mathbb{N}^i$ such that $x \in \bigcup f(w_i)$. Since $w \sqsubseteq w_i$ and $w \neq w_i$ imply $\bigcup f(w_i) \subseteq \bigcup f(w)$, it follows that $T_x := \{w_i : i \in \mathbb{N}\}$ constitutes an infinite finitely branching tree and by Weak Kőnig's Lemma this tree has an infinite path p such that $x \in C_p$. Now we claim that $\bigcup_{p \in [T]} C_p$ is even a pairwise disjoint union. Let $x \in C_p \cap C_q$ for $p, q \in [T]$ with $p \neq q$. Then there is an $i \in \mathbb{N}$ such that $p|_i \neq q|_i$ and we have $x \in \bigcup f(p|_i) \cap \bigcup f(q|_i)$. This contradicts the fact that (T,f) is a tree of rational complexes. Hence, the union $\bigcup_{p \in [T]} C_p$ is a disjoint union. By definition of a tree of rational complexes, $\bigcup f(p|_i)$ is connected and compact for every $i \in \mathbb{N}$. It follows that C_p is connected, since the intersection of a sequence of continua is a continuum (i.e. connected and compact, see for instance [25, Corollary 6.1.19]). Altogether, this proves the claim. □

As another interesting result we can deduce from Proposition 3.4 a classification of the operation that determines a connectedness component. We first define this operation. For brevity, we denote by $\mathcal{A}_n$ the subspace of non-empty closed subsets of $\mathcal{A}_-([0,1]^n)$.

Definition 3.7 (Connectedness components). By $\mathsf{Con}_n : \mathcal{A}_n \rightrightarrows \mathcal{A}_n$ we denote the map with $\mathsf{Con}_n(A) := \{C : C \text{ is a connectedness component of } A\}$ for every $n \geq 1$.

We note that the Weihrauch degree of Weak Kőnig's Lemma has been defined and studied in [26, 9, 8, 6, 11]. Here we prove that the problem Con_n of finding a connectedness component of a closed set has the same strong Weihrauch degree as Weak Kőnig's Lemma for every dimension $n \geq 1$.

Theorem 3.8 (Connectedness components). $\mathsf{Con}_n \equiv_{\mathrm{sW}} \mathsf{WKL}$ *for $n \geq 1$.*

Proof. Given a set $A \subseteq [0,1]^n$, we can compute a tree (T,f) of rational complexes that represents A (together with a bound b of T). With the help of Weak Kőnig's

Lemma WKL we can find an infinite path $p \in [T]$ of T (since the bound b is available). Then $C = \bigcap_{i=0}^{\infty} \bigcup f(p|_i)$ is a connectedness component of A by Lemma 3.6 and given T, f, p we can actually compute $C \in \mathcal{A}_n$. This proves $\mathsf{Con}_n \leq_{\mathrm{W}} \mathsf{WKL}$ and since WKL is a cylinder (see [9]) this even implies $\mathsf{Con}_n \leq_{\mathrm{sW}} \mathsf{WKL}$.

For the other direction, $\mathsf{WKL} \leq_{\mathrm{sW}} \mathsf{Con}_1$ we use a standard computable embedding $\iota : \{0,1\}^{\mathbb{N}} \to [0,1]$ of Cantor space into the unit interval with a computable right inverse. Given a tree T with infinite paths we can compute the set $A = [T] \in \mathcal{A}(\{0,1\}^{\mathbb{N}})$ of infinite paths and hence we can also compute $\iota(A) \in \mathcal{A}_1$ (see [7]). Using Con_1 we obtain a connectedness component $C \in \mathcal{A}_1$ of $\iota(A)$. Since $\iota(\{0,1\}^{\mathbb{N}})$ is totally disconnected, any connectedness component C of $\iota(A)$ is actually a singleton and hence we can compute x with $C = \{x\}$ (since $[0,1]$ is compact). Hence $p = \iota^{-1}(x)$ is an infinite path in T. This proves $\mathsf{WKL} \leq_{\mathrm{sW}} \mathsf{Con}_1$ and the higher dimensional case can be treated analogously (using the canonical embedding of $[0,1]$ into $[0,1]^n$). □

In [36], Le Roux and Ziegler studied computability properties of closed sets and their connectedness components. For instance, they prove that any co-c.e. closed set with only finitely many connectedness components has only co-c.e. closed connectedness components and any co-c.e. closed set without co-c.e. closed connectedness components has continuum cardinality many connectedness components. This can easily be deduced from the previous theorem as well as many other properties of connectedness components. For instance, it is well known that there exists a computable tree with countably many infinite paths and a unique non-isolated infinite path that is not even limit computable (see [21, Theorem 2.18]). This implies the following result, which resolves the Open Question 4.10 in [36].

Corollary 3.9. *There exists a non-empty co-c.e. closed set $A \subseteq [0,1]$ with only countably many connectedness components one of which is not co-c.e. closed (and it is not even the set of accumulation points of a computable sequence).*

We mention that a closed set is the set of accumulation points of a computable sequence if and only if it has a limit computable name (i.e. if it is co-c.e. closed in the halting problem, see [36, 11]). Another consequence of Lemma 3.6 using the Low Basis Theorem (see [47]) is that every co-c.e. closed set has a low connectedness component in the sense that it is low in the space $\mathcal{A}_-([0,1]^n)$. We describe this result in the special case of the representation of closed sets considered here.

Corollary 3.10. *Let $A \subseteq [0,1]^n$ be co-c.e. closed. Then there is a computable sequence $(A_i)_{i \in \mathbb{N}}$ of bi-computable closed sets $A_i \subseteq [0,1]^n$ and a low $p \in \mathbb{N}^{\mathbb{N}}$ such that $\bigcap_{i=0}^{\infty} A_{p(i)}$ is a connectedness component of A (which is then, in particular, the set of accumulation points of a computable sequence).*

We close this section by mentioning that one can use the representation of closed sets by trees of rational complexes in order to prove that the function $(A, x) \mapsto C$ that maps any non-empty closed set A together with a point $x \in A$ to the

connectedness component C of A that contains x is computable. The point x guides the path in the tree of rational complexes that one has to take. This result was already proved in [36]. We formulate a non-uniform corollary here.

Corollary 3.11. *Every connectedness component of a co-c.e. closed set $A \subseteq [0,1]^n$ that contains a computable point $x \in [0,1]^n$ is itself co-c.e. closed.*

We note that in the one-dimensional case an inverse holds true: every non-empty connected co-c.e. closed set $A \subseteq [0,1]$ contains a computable point. However, the analogue statement is no longer true from dimension two upward (see Corollary 6.6). Further interesting results on connected co-c.e. closed sets can be found in [33].

4. Brouwer's Fixed Point Theorem and Connected Choice

In this section, we want to prove that the Brouwer Fixed Point Theorem is computably equivalent to connected choice for any fixed dimension. We first define these two operations. By $\mathcal{C}(X,Y)$ we denote the *set of continuous functions $f : X \to Y$* and for short we write $\mathcal{C}_n := \mathcal{C}([0,1]^n, [0,1]^n)$.

Definition 4.1 (Brouwer Fixed Point Theorem). By $\mathsf{BFT}_n : \mathcal{C}_n \rightrightarrows [0,1]^n$ we denote the operation defined by $\mathsf{BFT}_n(f) := \{x \in [0,1]^n : f(x) = x\}$ for $n \in \mathbb{N}$.

We note that BFT_n is well defined, i.e. $\mathsf{BFT}_n(f)$ is non-empty for all f, since by the Brouwer Fixed Point Theorem every $f \in \mathcal{C}_n$ admits a fixed point x, i.e. with $f(x) = x$. We can also consider the infinite-dimensional version of the Brouwer Fixed Point Theorem on the Hilbert cube $[0,1]^{\mathbb{N}}$, which is represented by the map $\mathsf{BFT}_\infty : \mathcal{C}([0,1]^{\mathbb{N}}, [0,1]^{\mathbb{N}}) \rightrightarrows [0,1]^{\mathbb{N}}$ with $\mathsf{BFT}_\infty(f) := \{x \in [0,1]^{\mathbb{N}} : f(x) = x\}$. This can also be seen as a special case of the Schauder Fixed Point Theorem and hence BFT_∞ is well defined too. We now define connected choice.

Definition 4.2 (Connected choice). By $\mathsf{CC}_n :\subseteq \mathcal{A}_n \rightrightarrows [0,1]^n$ we denote the operation defined by $\mathsf{CC}_n(A) := A$ for all non-empty connected closed $A \subseteq [0,1]^n$ and $n \in \mathbb{N}$. We call CC_n *connected choice (of dimension n)*.

Hence, connected choice is just the restriction of closed choice to connected sets. We also use the following notation for the set of fixed points of a function $f \in \mathcal{C}_n$.

Definition 4.3 (Set of fixed points). By $\mathsf{Fix}_n : \mathcal{C}_n \to \mathcal{A}_n$ we denote the function with $\mathsf{Fix}_n(f) := \{x \in [0,1]^n : f(x) = x\}$.

It is easy to see that Fix_n is computable, since $\mathsf{Fix}_n(f) := (f - \mathrm{id}|_{[0,1]^n})^{-1}\{0\}$ and it is well known that closed sets in $\mathcal{A}_n$ can also be represented as zero sets of continuous functions (see [18, 17]). We note that the Brouwer Fixed Point Theorem can be decomposed to $\mathsf{BFT}_n \supseteq \mathsf{CC}_n \circ \mathsf{Con}_n \circ \mathsf{Fix}_n$.

The main result of this section is that the Brouwer Fixed Point Theorem and connected choice are (strongly) equivalent for any fixed dimension n (see Theorem 4.9). An important tool for both directions of the proof is the representation of closed sets by trees of rational complexes. We start with the direction

$\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{BFT}_n$ that can be seen as a uniformization of an earlier construction of Baigger [1] that is in turn built on results of Orevkov [39].

We first formulate a stronger conclusion that we can derive from Proposition 3.4 in case of connected sets. In order to express these stronger conclusions we first recall the notion of effective pathwise connectedness as it was introduced in [5]. Essentially, a set is called *effectively pathwise connected*, if for every two points in the set we can compute a path that connects these two points entirely within this set.[d] We need a uniform such notion for sequences.

Definition 4.4 (Effectively pathwise connected). Let $(A_i)_{i \in \mathbb{N}}$ be a sequence of non-empty closed sets $A_i \subseteq \mathbb{R}^n$. Then $(A_i)_{i \in \mathbb{N}}$ is called *pathwise connected*, if there is a function $U :\subseteq \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathcal{C}([0,1], \mathbb{R}^n)$, such that for every $p \in U(i, x, y)$ with $x, y \in A_i$ we obtain $p(0) = x$, $p(1) = y$ and $\mathrm{range}(p) \subseteq A_i$. Such a U is called a *witness of pathwise connectedness*. If there is a computable such witness U, then $(A_i)_{i \in \mathbb{N}}$ is called *effectively pathwise connected*.

If a (name of a realizer of a) witness U of pathwise connectedness of $(A_i)_{i \in \mathbb{N}}$ can be computed from A, then we say that $(A_i)_{i \in \mathbb{N}}$ is *pathwise connected uniformly in A*. We note that any rational complex $R \subseteq \mathrm{C}\mathbb{Q}^n$ is connected and also automatically pathwise connected, due to the simple structure of such complexes. It is easy to see that there is a computable map that maps any rational complex $R \in \mathrm{C}\mathbb{Q}^n$ to a witness of pathwise connectedness of $\bigcup R$. By $d(A, B) := \inf_{a \in A, b \in B} \|a - b\|$ we denote the *minimal distance* between sets $A, B \subseteq \mathbb{R}^n$. We note that $d(A, B^c) > 0$ is equivalent to $A \Subset B$ for non-empty compact $A, B \subseteq \mathbb{R}^n$.

Proposition 4.5 (Connected sets). *Given a non-empty connected closed set* $A \subseteq [0,1]^n$ *we can compute sequences of distance functions* $(d_{A_i})_{i \in \mathbb{N}}$ *and* $(d_{A_i^c})_{i \in \mathbb{N}}$ *for non-empty closed sets* $A_i \subseteq [-1,2]^n$ *such that*:

(1) $A = \bigcap_{i=0}^{\infty} A_i$,
(2) $d(A_{i+1}, A_i^c) > 0$ *for all* $i \in \mathbb{N}$,
(3) $(A_i)_{i \in \mathbb{N}}$ *is pathwise connected uniformly in A*.

Proof. Given a non-empty connected closed $A \subseteq [0,1]^n$ we can compute an infinite tree of non-empty rational complexes (T, f) that represents A by Proposition 3.4. Since A is connected, A is its only connectedness component and by Lemma 3.6 there is exactly one infinite path $p \in [T]$. If we can find this path, then $A_i := \bigcup f(p|_i)$ is a sequence of closed sets $A_i \subseteq [-1,2]^n$ with $A_{i+1} \Subset A_i$ for all i, which implies $d(A_{i+1}, A_i^c) > 0$ and $A = \bigcap_{i=0}^{\infty} A_i$. Since $f(p|_i)$ is a rational complex, it is straightforward how to determine d_{A_i} and $d_{A_i^c}$, given this complex and since $\bigcup f(p|_i)$ is connected it is also automatically pathwise connected and a witness U for pathwise connectedness can be easily computed.

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[d]We mention that the Warsaw circle is an example of a set that is pathwise connected but not effectively so, not even with respect to some oracle.

It remains to show how we can compute the unique infinite path p in T. For each fixed i there are only finitely many words $w_0, \ldots, w_k \in T \cap \mathbb{N}^i$ and due to connectedness of A and $\bigcup f(w_j)$ and the fact that all the $\bigcup f(w_j)$ are pairwise disjoint, it follows that there is exactly one such w_a with $A \subseteq \bigcup f(w_a)$. Due to compactness of the $\bigcup f(w_j)$ all the other w_j with $j \neq a$ will eventually be covered by negative information given as input for A and if this happens it can be recognized. Hence, one just needs to wait until all the $\bigcup f(w_j)$ except one are covered by negative information in order to identify w_a. Then $w_a \sqsubseteq p$ and by a repetition of this procedure for each i one can compute p. $\qquad\square$

Now we use Proposition 4.5 to prove that every non-empty connected closed set $A \subseteq [0,1]^n$ can be effectively translated into a continuous function $f \in \mathcal{C}_n$ that has all its fixed points in A. The idea is to compute a compactly decreasing sequence $(A_i)_{i \in \mathbb{N}}$ of closed sets according to the previous proposition together with points $x_i \in A_i$ and paths p_i in A_i that connect x_{i+1} with x_i. In some sense we then use these paths like Ariadne's thread in order to construct a function f that is a modified identity with all fixed points shifted toward A along the given paths. By $\|f\| := \sup_{x \in [0,1]^n} \|f(x)\|$ we denote the supremum norm for continuous functions $f : [0,1]^n \to [0,1]^n$.

Lemma 4.6. $\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{BFT}_n$ *for all* $n \geq 1$.

Proof. Given a non-empty closed and connected set $A \subseteq [0,1]^n$, we will compute a function $f \in \mathcal{C}_n$ such that all fixed points of f are included in A. First, we compute the sequences $(d_{A_i})_{i \in \mathbb{N}}$ and $(d_{A_i^c})_{i \in \mathbb{N}}$ according to Proposition 4.5.

Without loss of generality, we can assume that $A \subseteq [2^{-3}, 1 - 2^{-3}]^n$ and all $A_i \subseteq [2^{-4}, 1 - 2^{-4}]^n =: Q$. This can always be achieved using a suitable computable homeomorphism $T : [-1, 2]^n \to [2^{-4}, 1 - 2^{-4}]^n$ that is applied to all input data and afterward the fixed point x that is found is transferred back by $T^{-1}(x)$.

Since we can compute the sequences of distance functions $(d_{A_i})_{i \in \mathbb{N}}$ we can also find a sequence of points $(x_i)_{i \in \mathbb{N}}$ with $x_i \in A_i$ for all $i \in \mathbb{N}$. Since $(A_i)_{i \in \mathbb{N}}$ is pathwise connected uniformly in A, we can also compute a sequence $(p_i)_{i \in \mathbb{N}}$ of continuous functions $p_i : [0,1] \to [0,1]^n$ such that $p_i(0) = x_{i+1}$, $p_i(1) = x_i$ and $\mathrm{range}(p_i) \subseteq A_i$. We can also uniformly compute a sequence $(D_i)_{i \in \mathbb{N}}$ of functions $D_i : [0,1]^n \to [0,1]$ defined by

$$D_i(x) := \frac{d(x, A_{i+1})}{d(x, A_{i+1}) + d(x, A_i^c)}$$

for all $x \in [0,1]^n$ and $i \in \mathbb{N}$. Since $d(A_{i+1}, A_i^c) > 0$ for all $i \in \mathbb{N}$, it follows that the denominator is always non-zero and hence the functions D_i are well defined. We obtain $D_i(x) = 0 \Leftrightarrow x \in A_{i+1}$ and $D_i(x) = 1 \Leftrightarrow x \in \overline{A_i^c}$.

We now compute a continuous function $f : [0,1]^n \to [0,1]^n$ with $\mathsf{BFT}_n(f) \subseteq A$. The function f will be defined as $f := \mathrm{id} + 2^{-4} \sum_{i=0}^{\infty} g_i$ using further continuous functions g_i. As an abbreviation we write $G_i := \sum_{j=0}^{i} g_j$ for the partial sums. We

also use the abbreviations $C_n := \sum_{i=n}^{\infty} 2^{-3i-1}$ and we note that $C_n \leq 2^{-3n}$ for all $n \in \mathbb{N}$. We start with

$$g_0(x) := \begin{cases} 2^{-1} \dfrac{x_2 - x}{\|x_2 - x\|} d(x, A_1) & \text{if } x \notin A_2, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in [0,1]^n$. In the next step we define inductively

$$g_{i+1}(x) := \begin{cases} 2^{-3i-4} \dfrac{G_i(x)}{\|G_i(x)\|} & \text{if } x \notin A_{i+1}, \\ 2^{-3i-4} \dfrac{p_{i+2}(D_{i+1}(x)) - x}{\|p_{i+2}(D_{i+1}(x)) - x\|} D_{i+1}(x) & \text{if } x \in A_{i+1} \backslash A_{i+2}, \\ 0 & \text{if } x \in A_{i+2} \end{cases}$$

for all $x \in [0,1]^n$ and $i \in \mathbb{N}$.

We first prove that all g_i and $\sum_{i=0}^{\infty} g_i(x)$ are well defined and

$$x \in A = \bigcap_{i=0}^{\infty} A_i \Leftrightarrow \sum_{i=0}^{\infty} g_i(x) = 0 \Leftrightarrow f(x) = x. \tag{1}$$

The second equivalence follows immediately from the definition of f (once we know that the g_i and $\sum_{i=0}^{\infty} g_i$ are well defined). If $x \in \bigcap_{i=0}^{\infty} A_i$, then it follows immediately that $g_i(x) = 0$ for all i and hence $\sum_{i=0}^{\infty} g_i(x) = 0$. If $x \notin \bigcap_{i=0}^{\infty} A_i$, then there is a minimal $m \in \mathbb{N}$ with $x \notin A_m$, since $(A_i)_{i \in \mathbb{N}}$ is decreasing. If $m \in \{0, 1\}$, then $x \notin A_1$ and hence $x \notin A_2$. Since $x_2 \in A_2$ it follows that $\|x_2 - x\| \neq 0$. We also obtain $d(x, A_1) > 0$ and thus $g_0(x) \neq 0$. This implies

$$\sum_{i=0}^{\infty} g_i(x) = g_0(x) + \sum_{i=1}^{\infty} 2^{-3i-1} \frac{g_0(x)}{\|g_0(x)\|} = \frac{x_2 - x}{\|x_2 - x\|}(2^{-1}d(x, A_1) + C_1) \neq 0. \tag{2}$$

If $m > 1$, then $x \in A_{m-1} \backslash A_m$ and it follows that $g_i(x) = 0$ for $i \leq m - 2$. Since $\text{range}(p_m) \subseteq A_m$ and $x \notin A_m$, it follows that $\|p_m(D_{m-1}(x)) - x\| \neq 0$. We also have $D_{m-1}(x) \neq 0$ and hence $g_{m-1}(x) \neq 0$. This implies

$$\sum_{i=0}^{\infty} g_i(x) = g_{m-1}(x) + \sum_{i=m}^{\infty} g_i(x)$$

$$= \frac{p_m(D_{m-1}(x)) - x}{\|p_m(D_{m-1}(x)) - x\|}(2^{-3m+2}D_{m-1}(x) + C_m) \neq 0.$$

These two cases together prove the first equivalence in (1) together with the fact that all g_i and $\sum_{i=0}^{\infty} g_i(x)$ are well defined. We can also conclude from Eq. (1) that A is exactly the set of fixed point of f.

Next we want to show that by $f := \text{id} + 2^{-4} \sum_{i=0}^{\infty} g_i$ actually a continuous function of type $f : [0,1]^n \to [0,1]^n$ is defined. We show that $f([0,1]^n) \subseteq [0,1]^n$. If $x \in [0,1]^n \backslash A_0$, then Eq. (2) implies $f(x) = x + 2^{-4} \frac{x_2 - x}{\|x_2 - x\|}(2^{-1}d(x, A_1) + C_1)$, which means that f moves x toward $x_2 \in A_0 \subseteq Q$ and, in particular, $f(x) \in [0,1]^n$.

If $x \in A_0 \subseteq Q$, then $f(x) = x + 2^{-4} \sum_{i=0}^{\infty} g_i(x) \in [0,1]^n$ since $\|g_i\| = \sup_{x \in [0,1]^n} \|g_i(x)\| \leq 2^{-3i-1}$ and hence $\|2^{-4} \sum_{i=0}^{\infty} g_i(x)\| \leq 2^{-4} C_0 \leq 2^{-4}$. Now we prove that f is also continuous. First we show that each function g_i is continuous. We start with g_0. If x approaches ∂A_2 from the outside, then eventually $d(x, A_1) = 0$ and hence $g_0(x) = 0$. This means that g_0 continuous. We now continue with g_{i+1}. If $x \in \partial A_{i+1} = \partial \overline{A_{i+1}^{\mathrm{c}}}$, then $D_{i+1}(x) = 1$ and hence $p_{i+2}(D_{i+1}(x)) = x_{i+2}$ and we obtain

$$g_{i+1}(x) = 2^{-3i-4} \frac{p_{i+2}(D_{i+1}(x)) - x}{\|p_{i+2}(D_{i+1}(x)) - x\|} D_{i+1}(x) = 2^{-3i-4} \frac{x_{i+2} - x}{\|x_{i+2} - x\|}.$$

If, on the other hand, x approaches ∂A_{i+1} from the outside of A_{i+1}, then $D_i(x) \to 0$ and x is eventually in A_i and hence $g_j(x) = 0$ for $j \leq i - 1$ and $G_i = g_i$. In case $i > 0$ we use $D_i(x) \to 0$ in order to conclude

$$g_{i+1}(x) = 2^{-3i-4} \frac{G_i(x)}{\|G_i(x)\|} = 2^{-3i-4} \frac{p_{i+1}(D_i(x)) - x}{\|p_{i+1}(D_i(x)) - x\|} \to 2^{-3i-4} \frac{x_{i+2} - x}{\|x_{i+2} - x\|}.$$

In case of $i = 0$ we obtain

$$g_1(x) = 2^{-4} \frac{G_0(x)}{\|G_0(x)\|} = 2^{-4} \frac{x_2 - x}{\|x_2 - x\|}.$$

Finally, if x approaches ∂A_{i+2} from the outside, then $D_{i+1}(x) \to 0$ and x is eventually in A_{i+1}. Hence

$$g_{i+1}(x) = 2^{-3i-4} \frac{p_{i+2}(D_{i+1}(x)) - x}{\|p_{i+2}(D_{i+1}(x)) - x\|} D_{i+1}(x) \to 0.$$

Altogether, this proves that the case distinction in the definition of g_i is continuous and it is also computable since

(1) $x \notin A_{i+1} \Leftrightarrow D_i(x) > 0$,
(2) $x \in A_{i+1} \backslash A_{i+2} \Leftrightarrow D_i(x) = 0$ and $D_{i+1}(x) > 0$,
(3) $x \in A_{i+2} \Leftrightarrow D_{i+1}(x) = 0$.

Hence all the functions g_i and f are continuous and can be uniformly computed in the input A. We also obtain $\mathsf{BFT}_n(f) = A$ by Eq. (1), which proves $\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{BFT}_n$. $\qquad\square$

We note that the proof shows more than necessary. We only need that $\mathsf{BFT}_n(f) \subseteq A$ and we even obtain equality.

For the other direction $\mathsf{BFT}_n \leq_{\mathrm{sW}} \mathsf{CC}_n$ of the reduction we uniformize ideas presented by Miller [37, Sec. 2.3]. He proved the following result in terms of simplicial complexes. We note that rational complexes can be effectively converted into corresponding simplicial complexes.

Proposition 4.7 (Topological index [37]). *There is a computable topological index function* $\mathrm{ind} :\subseteq \mathcal{C}_n \times \mathrm{C}\mathbb{Q}^n \to \mathbb{Z}$ *such that for all* $f \in \mathcal{C}_n$ *and* $S, S_1, S_2 \in \mathrm{C}\mathbb{Q}^n$

such that f has no fixed points on $\partial \bigcup S_1$ and $\partial \bigcup S_2$ the following holds:

(1) $\mathrm{ind}(f, S)$ *is defined if and only if* $f(x) \neq x$ *for all* $x \in \partial \bigcup S$.
(2) $\mathrm{ind}(f, S) \neq 0$ *implies that* $f(x) = x$ *for some* $x \in \bigcup S$.
(3) $\mathrm{ind}(f, \{[0, 1]^n\}) \neq 0$.
(4) *If* $\{x \in \bigcup S_1 : f(x) = x\} = \{x \in \bigcup S_2 : f(x) = x\}$, *then one obtains* $\mathrm{ind}(f, S_1) = \mathrm{ind}(f, S_2)$.
(5) *If* $\bigcup S_1$ *and* $\bigcup S_2$ *are disjoint, then* $\mathrm{ind}(f, S_1 \cup S_2) = \mathrm{ind}(f, S_1) + \mathrm{ind}(f, S_2)$.

The proof of this result uses simplicial homology theory and, more specifically, the local topological degree. The effectivization follows the lines of classically known results in algebraic topology. Computability aspects of homology have also been studied in a discrete setting by Kaczynski *et al.* [32] and in the context of computable analysis by Collins [23, 24]. We essentially use Miller's ideas to reduce the Brouwer Fixed Point Theorem uniformly to connected choice. First we prove that the map $\mathsf{Con}_n \circ \mathsf{Fix}_n$ is computable (which might be surprising in light of Theorem 3.8).

Proposition 4.8. $\mathsf{Con}_n \circ \mathsf{Fix}_n : \mathcal{C}_n \rightrightarrows \mathcal{A}_n$ *is computable for all* $n \in \mathbb{N}$.

Proof. Given a continuous function $f \in \mathcal{C}_n$ we can easily compute the set of fixed points $A := \{x \in [0, 1]^n : f(x) = x\} \in \mathcal{A}_n$. Using Proposition 3.4 we can compute a tree (T, f) of rational complexes that represents A. Using Proposition 4.7 we can now identify an infinite path p in T and hence by Lemma 3.6 a connectedness component C of A.

We start with the empty node ε in T. Given a node $w \in T$, we construct an extension $wi \in T$ that is part of an infinite path as follows. Let us assume that $S_0 = f(w0), \ldots, S_k = f(wk)$ are the rational complexes that we need to consider. Due to the definition of a tree of rational complexes we know that $\bigcup S_i \cap \bigcup S_j = \emptyset$ for $i \neq j$. Since $A \Subset \bigcup_{j=0}^{k} \bigcup S_j$, it is clear that f cannot have any fixed point on any of the boundaries $\partial \cup S_j$, and hence we can compute the indexes $\mathrm{ind}(f, S_0), \ldots, \mathrm{ind}(f, S_k)$ by Proposition 4.7(1). One of them, say $\mathrm{ind}(f, S_i)$, must be different from 0, as one can see inductively using Proposition 4.7(3)–(5). By Proposition 4.7(2), this means that f has a fixed point in S_i, which means that $A \cap \bigcup S_i \neq \emptyset$. We use this wi as an extension of w and we proceed inductively in the same manner.

Altogether, this algorithm produces an infinite path p of T and hence we can compute the connected component $C := \{\bigcap_{i=0}^{\infty} \bigcup f(p|_i) : p \in [T]\} \in \mathcal{A}_n$ of A by Lemma 3.6. This shows that $\mathsf{Con}_n \circ \mathsf{Fix}_n$ is computable. $\qquad\square$

Since $\mathsf{BFT}_n(f) \supseteq \mathsf{CC}_n \circ \mathsf{Con}_n \circ \mathsf{Fix}_n(f)$ we can directly conclude $\mathsf{BFT}_n \leq_{\mathrm{sW}} \mathsf{CC}_n$ for all n. Together with Lemma 4.6 we obtain the following theorem.

Theorem 4.9 (Brouwer Fixed Point Theorem). $\mathsf{BFT}_n \equiv_{\mathrm{sW}} \mathsf{CC}_n$ *for all* $n \in \mathbb{N}$.

It is easy to see that in general the Brouwer Fixed Point Theorem and connected choice are not independent of the dimension. In case of $n = 0$ the space $[0, 1]^n$ is the

one-point space $\{0\}$ and hence $\mathsf{BFT}_0 \equiv_{\mathrm{sW}} \mathsf{CC}_0$ are both computable. In case of $n = 1$ connected choice was already studied in [8] and it was proved that it is equivalent to the Intermediate Value Theorem IVT (see [8, Definition 6.1 and Theorem 6.2]).

Corollary 4.10 (Intermediate Value Theorem). $\mathsf{IVT} \equiv_{\mathrm{sW}} \mathsf{BFT}_1 \equiv_{\mathrm{sW}} \mathsf{CC}_1$.

It is also easy to see that the Brouwer Fixed Point Theorem BFT_2 in dimension two is more complicated than in dimension one. For instance, it is known that the Intermediate Value Theorem IVT always offers a computable function value for a computable input, whereas this is not the case for the Brouwer Fixed Point Theorem BFT_2 by Baigger's counterexample [1]. We continue to discuss this topic in Sec. 6.

Here we point out that Proposition 4.8 implies that the fixed point set $\mathsf{Fix}_n(f)$ of every computable function $f : [0,1]^n \to [0,1]^n$ has a co-c.e. closed connectedness component. Miller observed that also any co-c.e. closed superset of such a set is the fixed point set of some computable function and the following result is a uniform version of this observation. We denote by $(f, g) :\subseteq X \rightrightarrows Y \times Z$ the *juxtaposition* of two functions $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq X \rightrightarrows Z$, defined by $(f, g)(x) = (f(x), g(x))$.

Theorem 4.11 (Fixability). $(\mathsf{Fix}_n, \mathsf{Con}_n \circ \mathsf{Fix}_n)$ *is computable and has a multivalued computable right inverse for all* $n \in \mathbb{N}$.

Proof. It follows directly from Proposition 4.8 and the fact that Fix_n is computable that $(\mathsf{Fix}_n, \mathsf{Con}_n \circ \mathsf{Fix}_n)$ is computable for all $n \in \mathbb{N}$. We now describe how a right inverse $R :\subseteq \mathcal{A}_n \times \mathcal{A}_n \rightrightarrows \mathcal{C}_n$ can be computed. First, given (A, C) such that $A \in \mathcal{A}_n$ and C is a connectedness component of A, we can find some $f \in \mathcal{C}_n$ such that $\mathsf{Fix}_n(f) = C$ following the algorithm that is specified in the proof of Lemma 4.6. We can also find a continuous $g : [0,1]^n \to [0,1]$ such that $g^{-1}\{0\} = A$ (see [18]). Then we can also compute a continuous h with

$$h(x) := (1 - g(x))x + f(x)g(x)$$

and since this is a convex combination of id and f, it follows that h is actually a continuous function $h : [0,1]^n \to [0,1]^n$. Finally,

$$h(x) = x \Leftrightarrow (f(x) - x)g(x) = 0 \Leftrightarrow x \in C \cup A = A.$$

That is, $\mathsf{Fix}_n(h) = A$. Hence the function R with $(A, C) \mapsto h$ is a suitable computable right inverse of $(\mathsf{Fix}_n, \mathsf{Con}_n \circ \mathsf{Fix}_n)$. $\qquad\square$

Roughly speaking, a closed set $A \in \mathcal{A}_n$ together with one of its connectedness components is as good as a continuous function $f \in \mathcal{C}_n$ with A as set of fixed points. As a non-uniform corollary we obtain immediately Miller's original result [37, Theorem 2.6.1].

Corollary 4.12 (Fixable sets [37]). *A set* $A \subseteq [0,1]^n$ *is the set of fixed points of a computable function* $f : [0,1]^n \to [0,1]^n$ *if and only if it is non-empty and co-c.e. closed and contains a co-c.e. closed connectedness component.*

We can also derive other interesting results from Theorem 4.11. For instance we can derive an upper bound on how complex a continuous function needs to be that has an arbitrary given non-empty co-c.e. closed set as fixed point set.

Corollary 4.13. *Let $A \subseteq [0,1]^n$ be a non-empty co-c.e. closed set. Then there is a continuous function $f : [0,1]^n \to [0,1]^n$ that is low as a point in $\mathcal{C}_n$ and has A as fixed point set.*

This result follows from an application of the Uniform Low Basis Theorem [6, Theorem 8.3] since Fix_n has a right inverse that is reducible to WKL by Theorems 4.11 and 3.8.[e]

5. Lipschitz Continuity

In this section, we want to discuss the question whether Lipschitz continuity of a function $f : [0,1]^n \to [0,1]^n$ simplifies finding fixed points in any way, compared to a function f that is just continuous.[f] We recall that a function $f : [0,1]^n \to [0,1]^n$ is called *Lipschitz continuous* with constant $L \geq 0$, if

$$\|f(x) - f(y)\| \leq L \cdot \|x - y\|$$

holds for all $x, y \in [0,1]^n$. As before, $\| \ \|$ denotes the maximum norm on Euclidean space. We are going to prove in this section that a Lipschitz constant $L > 1$ as an extra constraint does not simplify finding fixed points. We first need a refined version of Proposition 4.5.

Proposition 5.1. *Given a non-empty connected closed set $A \subseteq [0,1]^n$ we can compute sequences of distance functions $(d_{A_i})_{i \in \mathbb{N}}$ and $(d_{A_i^c})_{i \in \mathbb{N}}$ for non-empty closed sets $A_i \subseteq [-1,2]^n$, a sequence $(x_i)_{i \in \mathbb{N}}$ of points in $[-1,2]^n$ and a sequence $(p_i)_{i \in \mathbb{N}}$ of paths $p_i : [0,1] \to [-1,2]^n$ such that for all $i \in \mathbb{N}$:*

(1) $\bigcap_{i=0}^{\infty} A_i = A$,
(2) $d(A_{i+1}, A_i^c) \geq 2^{-i-1}$,
(3) $x_i \in A_i$,
(4) $\mathrm{range}(p_i) \subseteq A_i$ and $p_i(0) = x_i$, $p_i(1) = x_{i+1}$,
(5) $d(\mathrm{range}(p_i), A_i^c) \geq 2^{-i}$,
(6) $p_i : [0,1] \to [-1,2]^n$ is Lipschitz continuous with constant $L = 1$.

Proof. We start as in the proof of Proposition 4.5 with a tree (T, f) of rational complexes that represent A and from which we compute sequences of distance functions $(d_{A_i})_{i \in \mathbb{N}}$ and $(d_{A_i^c})_{i \in \mathbb{N}}$ satisfying condition (1). Then we compute a sequence

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[e]We mention that a function f that is low as a point in $\mathcal{C}_n$ is not necessarily low as a function in the sense that $f \leq_{\mathrm{sW}} \mathsf{L}$ (where $\mathsf{L} = \mathsf{J}^{-1} \circ \lim$ is the composition of the inverse of the Turing jump J and the limit operation), but one only obtains $f \leq_{\mathrm{W}} \mathsf{L}$ here (see [6, 11] for a discussion of low functions).

[f]We would like to thank Ulrich Kohlenbach for raising this question.

of points $(x_i)_{i\in\mathbb{N}}$ and paths $(p_i)_{i\in\mathbb{N}}$ linking them satisfying conditions (3) and (4) as in the proof of Lemma 4.6.

The construction of the p_i allows us to choose a constant-speed parameterization, i.e. a p_i that is Lipschitz continuous with constant $L_i \in \mathbb{N}$, and moreover we can compute a sequence $(L_i)_{i\in\mathbb{N}}$ of corresponding constants. Now for any $(i,j) \in \mathbb{N}^2$ with $j < L_i$, define $A_{i,j} := A_i$, $x_{i,j} := p_i(L_i^{-1} \cdot j)$ and $p_{i,j}(t) := p_i(L_i^{-1} \cdot (j+t))$ for $t \in [0,1]$. The purpose of these refinements is to obtain $p_{i,j}$ that are Lipschitz continuous for constant $L = 1$. Now we can determine new sequences $(A_i')_{i\in\mathbb{N}}$, $(x_i')_{i\in\mathbb{N}}$, $(p_i')_{i\in\mathbb{N}}$ by a lexicographic ordering of the double sequences $(A_{i,j})_{i\in\mathbb{N},j<L_i}$, $(x_{i,j})_{i\in\mathbb{N},j<L_i}$ and $(p_{i,j})_{i\in\mathbb{N},j<L_i}$, respectively. These are clearly computable from the original ones. Moreover, the conditions (1), (3) and (4) remain unaffected, while condition (6) is now satisfied, too for the sequence $(A_i')_{i\in\mathbb{N}}$.

In order to satisfy condition (2), the construction employed in the proof of Proposition 3.4 to obtain a tree (collapsing to a single path here) of rational complexes as a name for a closed set is reused. First, we determine an enumeration of rational balls $B(c_i, r_i)$ such that $A = Q \cap (\bigcup_{i=0}^{\infty} B(c_i, r_i))^c$ with $Q := [-1,2]^n$ with the additional property that $B(c_i, r_i) \subseteq (A_i')^c$. Now we construct a new tree (T', f') of rational complexes that represents A following the algorithm in the proof of Proposition 3.4 with this particular enumeration of balls $B(c_i, r_i)$ and we use again the method in the proof of Proposition 4.5 to obtain sequences of distance functions $(d_{A_i''})_{i\in\mathbb{N}}$ and $(d_{(A_i'')^c})_{i\in\mathbb{N}}$ for non-empty closed sets $A_i'' \subseteq [-1,2]^n$. The additional property $B(c_i, r_i) \subseteq (A_i')^c$ guarantees that the new sets are supersets of the original ones, i.e. $A_i' \subseteq A_i''$ for all $i \in \mathbb{N}$. The extra margin of 2^{-i} provided by step (3) of the construction in the proof of Proposition 3.4 even guarantees that $d(A_i', (A_i'')^c) \geq 2^{-i}$ and hence, in particular, $d(\mathrm{range}(p_i), (A_i'')^c) \geq 2^{-i}$. An inspection of step (2) of that construction reveals that we also obtain $d(A_{i+1}'', (A_i'')^c) \geq 2^{-i-1}$ for all $i \in \mathbb{N}$. Hence, the sequence $(A_i'')_{i\in\mathbb{N}}$ still satisfies the corresponding conditions (1), (3), (4) and (6) and it additionally also satisfies conditions (2) and (5). $\blacksquare$

With some extra calculations we can now prove a refined version of Lemma 4.6 for arbitrary Lipschitz constant $L > 1$.

Theorem 5.2 (Lipschitz continuity). *Given a non-empty connected closed set $A \subseteq [0,1]^n$ and a real number $L > 1$ we can compute a continuous function $f : [0,1]^n \to [0,1]^n$ that is Lipschitz continuous with constant L and such that A is the set of fixed points of f.*

Proof. Let $0 < \varepsilon < 1$. Suppose we can find a continuous function f that is Lipschitz continuous with constant $L > 0$ and has A as set of fixed points. Then we can compute the function $\mathrm{id} + \frac{\varepsilon}{1+L}(f - \mathrm{id})$ that has the same fixed points as f and is Lipschitz continuous with constant $1 + \varepsilon$. Hence, it is sufficient to prove that we can compute a function f that is Lipschitz continuous with constant $L = 6$. We prove that if the construction of the proof of Lemma 4.6 is carried out with the conditions provided by Proposition 5.1, then the resulting function f is such a function.

First, we provide a simplified expression for $f(x)$. We use the abbreviation $P_x := \frac{p_{n+1}D_n(x)-x}{\|p_{n+1}D_n(x)-x\|}$. If $x \in A$, then $f(x) = x$. If $x \in A_n \backslash A_{n+1}$ with $n > 0$, then

$$f(x) = x + 2^{-4}g_n(x) + 2^{-4} \sum_{i=n+1}^{\infty} \left(2^{-3i-1} \frac{\sum_{j=n}^{i-1} g_j(x)}{\|\sum_{j=n}^{i-1} g_j(x)\|} \right)$$

$$= x + 2^{-4}g_n(x) + 2^{-4} \sum_{i=n+1}^{\infty} \left(2^{-3i-1} \frac{g_n(x)}{\|g_n(x)\|} \right)$$

$$= x + 2^{-4}g_n(x) + 2^{-4}C_{n+1} \frac{g_n(x)}{\|g_n(x)\|}$$

$$= x + 2^{-4}(2^{-3n-1}D_n(x) + C_{n+1})P_x.$$

By continuity, this expression remains true for $x \in \overline{A_n \backslash A_{n+1}}$.

Now we want to estimate a Lipschitz constant for f and we distinguish a number of cases.

Case 1. $x \in A_n \backslash A_{n+1}$ with $n > 0$, $y \in A$. In this situation, we obtain $f(y) = y$ and $\|x - y\| \geq d(A_{n+2}, A_{n+1}^c) \geq 2^{-n-2}$. We recall that $C_{n+1} \leq 2^{-3n-3}$ and we estimate:

$$\|f(x) - f(y)\| = \|x + 2^{-4}(2^{-3n-1}D_n(x) + C_{n+1})P_x - y\|$$

$$\leq \|x - y\| + 2^{-3n-5}\|(D_n(x) + 1)P_x\|$$

$$\leq \|x - y\| + 2^{-3n-4}$$

$$\leq 2\|x - y\|.$$

Case 2. $x, y \in \overline{A_n \backslash A_{n+1}}$ with $n > 0$. We use $d(A_n^c, A_{n+1}) \geq 2^{-n-1}$ and we will need the following bound:

$$\|D_n(x) - D_n(y)\|$$

$$= \left\| \frac{d(x, A_{n+1})}{d(x, A_{n+1}) + d(x, A_n^c)} - \frac{d(y, A_{n+1})}{d(y, A_{n+1}) + d(y, A_n^c)} \right\|$$

$$= \left\| \frac{(d(x, A_{n+1}) - d(y, A_{n+1}))d(y, A_n^c) + (d(y, A_n^c) - d(x, A_n^c))d(y, A_{n+1})}{(d(x, A_{n+1}) + d(x, A_n^c))(d(y, A_{n+1}) + d(y, A_n^c))} \right\|$$

$$\leq \frac{\|(d(x, A_{n+1}) - d(y, A_{n+1}))d(y, A_n^c)\| + \|(d(y, A_n^c) - d(x, A_n^c))d(y, A_{n+1})\|}{\|d(A_n^c, A_{n+1})\| \cdot \|(d(y, A_{n+1}) + d(y, A_n^c))\|}$$

$$\leq 2^{n+1} \cdot \frac{d(x, y)d(y, A_n^c) + d(x, y)d(y, A_{n+1})}{d(y, A_{n+1}) + d(y, A_n^c)}$$

$$\leq 2^{n+1}\|x - y\|.$$

Now we also use the abbreviation $N_x := \|p_{n+1}D_n(x) - x\|$. Using the fact that p_{n+1} is Lipschitz continuous with constant 1, we obtain

$$\|N_x - N_y\| \leq \|p_{n+1}D_n(x) - p_{n+1}D_n(y)\| + \|x - y\|$$

$$\leq \|D_n(x) - D_n(y)\| + \|x - y\|$$

$$\leq 2^{n+2}\|x - y\|.$$

We note that $2^{-n-1} \leq N_x \leq 2$ since $d(\mathrm{range}(p_{n+1}), A_{n+1}^{\mathrm{c}}) \geq 2^{-n-1}$. For the same reason also $\|p_{n+1}D_n(y) - x\| \leq 2$. Hence

$$\|P_x - P_y\|$$

$$= N_x^{-1}N_y^{-1}\|p_{n+1}D_n(x)N_y - xN_y - p_{n+1}D_n(y)N_x + N_xy\|$$

$$\leq 2^{2n+2}\|N_y(p_{n+1}D_n(x) - p_{n+1}D_n(y)) + (N_y - N_x)(p_{n+1}D_n(y) - x)$$

$$\quad - xN_x + N_xy\|$$

$$\leq 2^{2n+2}(N_x\|x - y\| + N_y\|p_{n+1}D_n(x) - p_{n+1}D_n(y)\|$$

$$\quad + \|p_{n+1}D_n(y) - x\| \cdot \|N_x - N_y\|)$$

$$\leq 2^{2n+3}(\|x - y\| + \|D_n(x) - D_n(y)\| + \|N_x - N_y\|)$$

$$\leq 2^{2n+3}(1 + 2^{n+1} + 2^{n+2})\|x - y\|$$

$$\leq 2^{3n+6}\|x - y\|.$$

Since $2^{3n+1}C_{n+1} + D_n(y) \leq 2$ and using the previous estimations, we finally obtain

$$\|f(x) - f(y)\|$$

$$\leq \|x - y\| + 2^{-3n-5}\|(D_n(x) + 2^{3n+1}C_{n+1})P_x - (D_n(y) + 2^{3n+1}C_{n+1})P_y\|$$

$$= \|x - y\| + 2^{-3n-5}\|(D_n(x) - D_n(y))P_x + (2^{3n+1}C_{n+1} + D_n(y))(P_x - P_y)\|$$

$$\leq \|x - y\| + 2^{-3n-5}\|D_n(x) - D_n(y)\| + 2^{-3n-4}\|P_x - P_y\|$$

$$\leq (1 + 2^{-2n-4} + 4)\|x - y\|$$

$$\leq 6\|x - y\|.$$

Case 3. $x, y \in [0,1]^n$ not satisfying the conditions from our first or second case. Without loss of generality we can assume $[0,1]^n \subseteq A_1$. The straight line from x to y either intersects $\overline{A}$, or is composed of a finite number of line segments each fully included in some $\overline{A_i \backslash A_{i+1}}$ with $i > 0$. In the former case, pick some z from the intersection of the line and A. We obtain with the help of the estimations above

$$\|f(x) - f(y)\| \leq \|f(x) - f(z)\| + \|f(z) - f(y)\|$$

$$\leq 2\|x - z\| + 2\|z - y\| = 2\|x - y\|.$$

In the latter case, we obtain $\|f(x) - f(y)\| \leq 6\|x - y\|$ with a similar argument. In this case we use finitely many points z_j where the line segments touch. $\qquad\qquad$ □

If we denote by $\mathsf{BFT}_{n,L}$ the problem BFT_n restricted to functions that are Lipschitz continuous with constant L, then we can formulate our main result on Lipschitz continuous functions as follows (using Theorems 4.9 and 5.2).

Corollary 5.3. $\mathsf{BFT}_{n,L} \equiv_{\mathrm{sW}} \mathsf{CC}_n$ *for all* $n \in \mathbb{N}$ *and* $L > 1$.

We mention that the problem $\mathsf{BFT}_{n,L}$ is obviously computable for $L < 1$, since fixed points of contractions are uniquely determined by the Banach Fixed Point Theorem. The boundary case $L = 1$ has been studied by Neumann [38, Theorem 5.8] in the context of more general versions of the Browder-Göhde-Kirk Fixed Point Theorem. In this case $\mathsf{BFT}_{n,1} \equiv_{\mathrm{W}} \mathsf{XC}_n$, where XC_n denotes convex choice for the space $[0,1]^n$, i.e. $\mathsf{C}_{[0,1]^n}$ restricted to convex sets. Convex choice was further studied by Le Roux and Pauly [35, Corollary 3.31] and they proved among other results that one actually obtains a strictly increasing chain of problems with increasing dimension, i.e.

$$\mathsf{CC}_1 \equiv_{\mathrm{W}} \mathsf{XC}_1 <_{\mathrm{W}} \mathsf{XC}_2 <_{\mathrm{W}} \mathsf{XC}_3 <_{\mathrm{W}} \cdots <_{\mathrm{W}} \mathsf{C}_{[0,1]}.$$

Hence, in general one has a trichotomy for the complexity of $\mathsf{BFT}_{n,L}$ in the cases $L < 1, L = 1$ and $L > 1$. In the one-dimensional case, one is left with a dichotomy since it follows from Neumann's result that

$$\mathsf{BFT}_{1,1} \equiv_{\mathrm{W}} \mathsf{XC}_1 \equiv_{\mathrm{W}} \mathsf{CC}_1 \equiv_{\mathrm{W}} \mathsf{BFT}_1.$$

6. Aspects of Dimension

In this section, we want to discuss aspects of dimension of connected choice and the Brouwer Fixed Point Theorem. Our main result is that connected choice is computably universal or complete from dimension three upward in the sense that it is strongly equivalent to Weak Kőnig's Lemma, which is one of the degrees of major importance. In order to prove this result, we use the following geometric construction.

Proposition 6.1 (Twisted cube). *The function*

$$T :\subseteq \mathcal{A}_-[0,1] \to \mathcal{A}_3, \quad A \mapsto (A \times [0,1] \times \{0\}) \cup (A \times A \times [0,1]) \cup ([0,1] \times A \times \{1\})$$

is computable and maps non-empty closed sets $A \subseteq [0,1]$ *to non-empty pathwise connected closed sets* $T(A) \subseteq [0,1]^3$.

Here tuples $(x_1, x_2, x_3) \in T(A)$ have the property that at least one of the first two components provide a solution $x_i \in A$, and the third component lets us pick one that surely does. If x_3 is close to 1, then surely $x_2 \in A$ and if x_3 is close to 0, then surely $x_1 \in A$. If x_3 is neither close to 0 nor 1, then both $x_1, x_2 \in A$.

Hence, there is a computable function H such that $\mathsf{C}_{[0,1]} = H \circ \mathsf{CC}_3 \circ T$, which proves $\mathsf{C}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{CC}_3$. Together with Theorem 4.9 and Fact 2.6 we obtain the following conclusion.

Theorem 6.2 (Completeness of three dimensions). *For $n \geq 3$ we obtain*

$$\mathsf{CC}_n \equiv_{\mathrm{sW}} \mathsf{BFT}_n \equiv_{\mathrm{sW}} \mathsf{BFT}_\infty \equiv_{\mathrm{sW}} \mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}.$$

Proof. We note that the reduction $\mathsf{CC}_n \leq_{\mathrm{sW}} \mathsf{C}_{[0,1]^n}$ holds for all $n \in \mathbb{N}$, since connected choice is a just a restriction of closed choice and the equivalences

$$\mathsf{C}_{[0,1]^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]^n} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{WKL}$$

are known for all $n \geq 1$ by Fact 2.6. The equivalence $\mathsf{CC}_n \equiv_{\mathrm{sW}} \mathsf{BFT}_n$ has been proved in Theorem 4.9 for all $n \in \mathbb{N}$. We mention that $\mathsf{BFT}_n \leq_{\mathrm{sW}} \mathsf{BFT}_\infty$ can be proved as follows. The function

$$K : \mathcal{C}([0,1]^n, [0,1]^n) \to \mathcal{C}([0,1]^{\mathbb{N}}, [0,1]^{\mathbb{N}}), \quad f \mapsto ((x_i) \mapsto (f(x_1, \ldots, x_n), 0, 0, 0, \ldots))$$

is computable and together with the projection on the first n-coordinates this yields the reduction $\mathsf{BFT}_n \leq_{\mathrm{sW}} \mathsf{BFT}_\infty$. Since

$$\mathcal{C}([0,1]^{\mathbb{N}}, [0,1]^{\mathbb{N}}) \to \mathcal{A}_-([0,1]^{\mathbb{N}}), \quad f \mapsto (f - \mathrm{id}_{[0,1]^{\mathbb{N}}})^{-1}\{0\}$$

is computable too, it follows that $\mathsf{BFT}_\infty \leq_{\mathrm{sW}} \mathsf{C}_{[0,1]^{\mathbb{N}}}$ holds. Finally, $\mathsf{C}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{CC}_n$ follows for $n \geq 3$ from Proposition 6.1. $\qquad\square$

In particular, we get the Baigger counterexample for dimension $n \geq 3$ as a consequence of Theorem 6.2. A superficial reading of the results of Orevkov [39] and Baigger [1] can lead to the wrong conclusion that they actually provide a reduction of Weak Kőnig's Lemma to the Brouwer Fixed Point Theorem BFT_n of any dimension $n \geq 2$. However, this is only correct in a non-uniform way and the corresponding uniform result will be settled in Sec. 7 with different methods and does not follow from the known constructions. The Orevkov-Baigger result is built on the following fact.

Proposition 6.3 (Mixed cube). *The function*

$$M :\subseteq \mathcal{A}_-[0,1] \to \mathcal{A}_2, \quad A \mapsto (A \times [0,1]) \cup ([0,1] \times A)$$

is computable and maps non-empty closed sets $A \subseteq [0,1]$ to non-empty pathwise connected closed sets $M(A) \subseteq [0,1]^2$.

It follows straightforwardly from the definition that the pairs $(x, y) \in M(A)$ are such that one out of two components x, y is actually in A. In order to express the uniform content of this fact, we introduce the concept of a fraction.

Definition 6.4 (Fractions). Let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function and $0 < n \leq m \in \mathbb{N}$. We define the *fraction* $\frac{n}{m}f :\subseteq X \rightrightarrows Y^m$ by

$$\frac{n}{m}f(x) := \{(y_1, \ldots, y_m) \in \text{range}(f)^m : |\{i : y_i \in f(x)\}| \geq n\}$$

for all $x \in \text{dom}(\frac{n}{m}f) := \text{dom}(f)$.

The idea of a fraction $\frac{n}{m}f$ is that it provides m potential answers for f, at least $n \leq m$ of which have to be correct. The uniform content of the Orevkov–Baigger construction is then summarized in the following result.

Proposition 6.5 (Connected choice in dimension two). $\frac{1}{2}\mathsf{C}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{CC}_2 \leq_{\mathrm{sW}} \mathsf{C}_{[0,1]}$.

Proof. With Proposition 6.3 we obtain $\frac{1}{2}\mathsf{C}_{[0,1]} = \mathsf{CC}_2 \circ M$ and hence $\frac{1}{2}\mathsf{C}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{CC}_2$. The other reduction follows from $\mathsf{CC}_2 \leq_{\mathrm{sW}} \mathsf{C}_{[0,1]^2} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$. $\square$

That is, given a closed set $A \subseteq [0,1]$ we can utilize connected choice CC_2 of dimension 2 in order to find a pair of points (x, y) one of which is in A. This result directly implies the counterexample of Baigger [1] because the fact that there are non-empty co-c.e. closed sets $A \subseteq [0,1]$ without computable point immediately implies that $\frac{1}{2}\mathsf{C}_{[0,1]}$ is not non-uniformly computable (i.e. there are computable inputs without computable outputs) and hence CC_2 is also not non-uniformly computable.

Corollary 6.6 ([39, 1]). *There exists a computable function $f : [0,1]^2 \to [0,1]^2$ that has no computable fixed point $x \in [0,1]^2$. There exists a non-empty connected co-c.e. closed subset $A \subseteq [0,1]^2$ without computable point.*

We mention that Proposition 6.5 does not directly imply $\mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{CC}_2$, since $\frac{1}{2}\mathsf{C}_{[0,1]} <_{\mathrm{W}} \mathsf{CC}_2$. In fact, we can prove an even stronger result which shows that $\frac{1}{2}\mathsf{C}_{[0,1]}$ computes almost nothing, not even choice for the two point space.[g] This means that Proposition 6.5 has very little uniform content.

Proposition 6.7. $\mathsf{C}_{\{0,1\}} \not\leq_{\mathrm{W}} \frac{1}{2}\mathsf{C}_{[0,1]}$.

Proof. We use $\mathsf{C}_{\{0,1\}} \equiv_{\mathrm{sW}} \mathsf{LLPO}$ and by ψ_- we denote the representation of $\mathcal{A}_1$. We recall that $\mathsf{LLPO} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ is defined such that for $j \in \{0,1\}$ and $p \in \{0,1\}^{\mathbb{N}}$ it holds that

$$j \in \mathsf{LLPO}(p) \Leftrightarrow (\forall i)p(2i + j) = 0,$$

where $\text{dom}(\mathsf{LLPO})$ contains all sequences p such that $p(k) \neq 0$ for at most one k.

Let us now assume that $\mathsf{LLPO} \leq_{\mathrm{W}} \frac{1}{2}\mathsf{C}_{[0,1]}$ holds. Then there are continuous H, K such that $H\langle \text{id}, FK \rangle$ realizes LLPO whenever F realizes $\frac{1}{2}\mathsf{C}_{[0,1]}$. We consider the

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[g]Such problems have been called *indiscriminative* in [13].

inputs $p_{ji} := 0^{2i+j+1}10^{\mathbb{N}}$ and $p_\infty := 0^{\mathbb{N}}$ for LLPO. We obtain $\mathsf{LLPO}(p_{ji}) = \{j\}$ for $j \in \{0,1\}$ and $\mathsf{LLPO}(p_\infty) = \{0,1\}$. Now we let $K_{ji} := \psi_-(K(p_{ji}))$ and $K_\infty := \psi_-(K(p_\infty))$. These sets are all non-empty compact subsets of $[0,1]$, hence there are $x_{ji} \in K_{ji}$ with names q_{ji} (with respect to the signed-digit representation of $[0,1]$). Due to compactness for each $j \in \{0,1\}$ there is some convergent subsequence (q_{ji_k}) of q_{ji} and we let $q_j := \lim_{k\to\infty} q_{ji_k}$ and $x_j := \lim_{k\to\infty} x_{ji_k}$.

Now we claim that $x_j \in K_\infty$ for both $j \in \{0,1\}$ and by symmetry it suffices to prove this for $j = 0$. Let us assume that $x_0 \notin K_\infty$. Then by continuity of K there exist some open neighborhood U of x_0 and some $k \in \mathbb{N}$ such that $U \cap \psi_-(K(r)) = \emptyset$ for all $r \in \mathrm{dom}(\psi_- K)$ with $0^k \sqsubseteq r$. Almost all p_{0i} satisfy this condition, which implies $U \cap K_{0i} = \emptyset$ for almost all i. This contradicts the construction of x_0. Hence $x_0 \in K_\infty$ follows and analogously $x_1 \in K_\infty$.

Hence there is some realizer F_∞ of $\frac{1}{2}\mathsf{C}_{[0,1]}$ with $F_\infty K(p_\infty) = \langle q_0, q_1 \rangle$. Without loss of generality we can assume $H\langle p_\infty, \langle q_0, q_1 \rangle\rangle = 0$. There are also realizers F_k of $\frac{1}{2}\mathsf{C}_{[0,1]}$ with $F_k K(p_{1i_k}) = \langle q_{0i_k}, q_{1i_k} \rangle$, since the second component contains a correct answer. Hence $H\langle p_{1i_k}, \langle q_{0i_k}, q_{1i_k} \rangle\rangle = 1$ has to hold. Continuity of H now implies $H\langle p_\infty, \langle q_0, q_1 \rangle\rangle = 1$, which is a contradiction. □

In the following result we summarize the known relations for connected choice in dependency of the dimension.

Proposition 6.8. *We obtain* $\mathsf{CC}_0 <_\mathrm{W} \mathsf{CC}_1 <_\mathrm{W} \mathsf{CC}_2 \leq_\mathrm{W} \mathsf{CC}_n \equiv_\mathrm{W} \mathsf{C}_{[0,1]}$ *for all* $n \geq 3$.

Proof. It is clear that $\mathsf{CC}_n \leq_\mathrm{sW} \mathsf{CC}_{n+1}$ holds for all $n \in \mathbb{N}$, since the computable map $A \mapsto A \times [0,1]$ maps connected closed sets of dimension n to such sets of dimension $n + 1$. The reduction $\mathsf{CC}_0 <_\mathrm{W} \mathsf{CC}_1$ is strict, since CC_0 is computable and CC_1 is not. The reduction $\mathsf{CC}_1 <_\mathrm{W} \mathsf{CC}_2$ is strict, since CC_1 is non-uniformly computable (since any non-empty connected co-c.e. closed set $A \subseteq [0,1]$ is either a singleton and hence computable or it has non-empty interior and contains even a rational point) and CC_2 is not non-uniformly computable by Corollary 6.6. □

In Sec. 7, we are going to prove that also $\mathsf{CC}_2 \equiv_\mathrm{sW} \mathsf{C}_{[0,1]}$ holds.

We close this section with a second proof of Theorem 6.2 that uses a combinatorial argument as a replacement for the geometric construction provided in Proposition 6.1. It also indicates special properties of dimension two, which are not shared by higher dimensions. First, one can extend Proposition 6.3 straightforwardly to higher dimensions (by choosing $A \mapsto (A \times A \times [0,1]) \cup (A \times [0,1] \times A) \cup ([0,1] \times A \times A)$ in dimension three and so forth) and that leads to the following generalization of Proposition 6.5.

Proposition 6.9. $\frac{n-1}{n}\mathsf{C}_{[0,1]} \leq_\mathrm{sW} \mathsf{CC}_n \leq_\mathrm{sW} \mathsf{C}_{[0,1]}$ *for all* $n \geq 2$.

On the other hand, one can use a majority voting strategy to obtain the following result.

Proposition 6.10 (Majority vote). $\frac{k}{n}\mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$ *if* $2k > n \geq k > 0$.

Proof. It is clear that $\frac{k}{n}\mathsf{C}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$ holds. Hence, we only need to prove $\widehat{\mathsf{LLPO}} \leq_{\mathrm{sW}} \frac{k}{n}\mathsf{C}_{[0,1]}$. In the first step we show $\widehat{\mathsf{LLPO}} \leq_{\mathrm{sW}} \frac{k}{n}\widehat{\mathsf{LLPO}}$. Given some answer $(p_1, \ldots, p_n) \in \frac{k}{n}\widehat{\mathsf{LLPO}}(q)$, a solution $p \in \widehat{\mathsf{LLPO}}(q)$ can be obtained by bitwise majority voting: for any $i \in \mathbb{N}$, we let $p(i) := 1$ if and only if $|\{j : p_j(i) = 1\}| \geq k$ and $p(i) := 0$ otherwise. This guarantees majority since $2k > n$. To complete the proof it suffices to show $\frac{k}{n}\widehat{\mathsf{LLPO}} \leq_{\mathrm{sW}} \frac{k}{n}\mathsf{C}_{[0,1]}$. We know that $\widehat{\mathsf{LLPO}} \leq_{\mathrm{sW}} \mathsf{C}_{[0,1]}$ and hence there are computable H, K such that $HFK \vdash \widehat{\mathsf{LLPO}}$ whenever $F \vdash \mathsf{C}_{[0,1]}$ holds. Without loss of generality, we can assume that we use a total representation for $[0,1]$ and hence H has to be total since $\mathsf{C}_{[0,1]}$ is surjective. This implies that $H^n FK \vdash \frac{k}{n}\widehat{\mathsf{LLPO}}$ whenever $F \vdash \frac{k}{n}\mathsf{C}_{[0,1]}$, which completes the proof. $\qquad\square$

We note that $\frac{n-1}{n}$ satisfies $2(n-1) > n$ if and only if $n \geq 3$. This does constitute a second proof of Theorem 6.2. Moreover, Proposition 6.7 shows that the claim of Proposition 6.10 does not hold for $n = 2$ and $k = 1$. This illustrates from a combinatorial perspective why dimension two is special.

7. The Two-Dimensional Case

The goal of this section is to prove that connected choice CC_n is equivalent to $\mathsf{C}_{[0,1]}$ in the two-dimensional case $n = 2$. The construction required for the proof of $\mathsf{C}_{[0,1]} \leq_{\mathrm{sW}} \mathsf{CC}_2$ is much more involved than in the three-dimensional case and it is essentially based on an inverse limit construction.

Theorem 7.1 (Two-dimensional case). $\mathsf{CC}_2 \equiv_{\mathrm{sW}} \mathsf{BFT}_2 \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$.

Proof. By Theorem 4.9 and Fact 2.6 it is sufficient to show $\widehat{\mathsf{LLPO}} \leq_{\mathrm{sW}} \mathsf{CC}_2$. In order to make the proof more understandable, we structure it into several parts.

Preparation of the input. In order to organize the input information, we replace $\widehat{\mathsf{LLPO}}$ by an equivalent problem LLPO_∞ that we now define. In the following we denote pairs $(n, b) \in \mathbb{N} \times \{0, 1\}$ for simplicity by n_b. We say that a word $w \in (\mathbb{N} \times \{0, 1\})^*$ is *repetition-free*, if no number appears twice in the first component, i.e. if $w = n_{0b_0} n_{1b_1} \ldots n_{kb_k}$, then $n_i \neq n_j$ for all $i, j \leq k$ with $i \neq j$. We introduce the following sets of repetition-free words.

Definition 7.2 (Repetition-free words). For all $n \in \mathbb{N}$ we define the sets

(1) $W_n := \{w \in (\{0, \ldots, n-1\} \times \{0, 1\})^* : w \text{ repetition-free}\}$,
(2) $W_* := \bigcup_{n \in \mathbb{N}} W_n$,

(3) $W_{\mathbb{N}} := \{p \in (\mathbb{N} \times \{0,1\})^{\mathbb{N}} : (\forall k)\, p|_k \in W_*\}$,

(4) $W_{\infty} := W_* \cup W_{\mathbb{N}}$.

For instance, $W_2 = W_1 \cup \{1_0, 1_1, 0_0 1_0, 0_0 1_1, 0_1 1_0, 0_1 1_1, 1_0 0_0, 1_0 0_1, 1_1 0_0, 1_1 0_1\}$, where $W_1 = W_0 \cup \{0_0, 0_1\}$ and $W_0 = \{\varepsilon\}$.

We note that we use a representation δ_{W_∞} to represent W_∞ that enumerates the content of words. More precisely, we consider $p = \langle n_0, b_0 \rangle \langle n_1, b_1 \rangle \langle n_2, b_2 \rangle \ldots$ with $n_i \in \mathbb{N}, b_i \in \{0,1\}$ as a name of a sequence $n_{0_{b_0}} n_{1_{b_1}} n_{2_{b_2}} \ldots$ in which we remove all the $n_{i_{b_i}}$ with $n_i = 0$ or with an n_i that occurred already earlier in the sequence and then we replace all the remaining $n_{i_{b_i}}$ by $(n_i - 1)_{b_i}$. The resulting object is a finite or infinite sequence $q \in W_\infty$ and we set $\delta_{W_\infty}(p) = q$.

Now we can define the problem LLPO_∞.

Definition 7.3. $\mathsf{LLPO}_\infty : W_\infty \rightrightarrows \{0,1\}^{\mathbb{N}}$ is defined by

$$\mathsf{LLPO}_\infty(p) := \{q \in \{0,1\}^{\mathbb{N}} : (\forall n, b)(q(n) = b \Rightarrow n_b \notin \mathrm{range}(p))\}$$

for all $p \in W_\infty$.

Claim 7.4. $\widehat{\mathsf{LLPO}} \equiv_{\mathrm{sW}} \mathsf{LLPO}_\infty$.

Proof. "$\widehat{\mathsf{LLPO}} \leq_{\mathrm{sW}} \mathsf{LLPO}_\infty$": Given an input $p = \langle p_0, p_1, \ldots \rangle$ for $\widehat{\mathsf{LLPO}}$ we generate a repetition-free sequence $q \in W_\infty$ as follows. As soon as we learn from p_n that $b \notin \mathsf{LLPO}(p_n)$, then we write n_b into the output. Hence, n_b occurs in the output sequence if and only if the nth copy of LLPO does not allow the result b. Hence, the resulting sequence $q \in W_\infty$ satisfies $\mathsf{LLPO}_\infty(q) = \widehat{\mathsf{LLPO}}(p)$. "$\mathsf{LLPO}_\infty \leq_{\mathrm{sW}} \widehat{\mathsf{LLPO}}$": Vice versa, given a sequence $q \in W_\infty$, we can generate a suitable input $p = \langle p_0, p_1, \ldots \rangle$ to LLPO as follows. We start all p_i with zeros and as soon as we read some n_b in q we modify p_n so that it contains a 1 in some still available position, i.e. we set $p_n(2k + b) = 1$ for large enough k. All other positions of p_n will be filled with 0. In this way we obtain a sequence p with $\mathsf{LLPO}_\infty(q) = \widehat{\mathsf{LLPO}}(p)$. $\qquad \square$

Now our goal is now to prove $\mathsf{LLPO}_\infty \leq_{\mathrm{sW}} \mathsf{CC}_2$. In fact, for convenience we replace $[0,1]^2$ by $B_0 := [0,1] \times [0,3]$ and we show $\mathsf{LLPO}_\infty \leq_{\mathrm{sW}} \mathsf{CC}_{B_0}$, where CC_{B_0} is a connected choice for the space B_0. It is clear that $\mathsf{CC}_{B_0} \equiv_{\mathrm{sW}} \mathsf{CC}_2$.

Overview of the Proof. Given a repetition-free sequence $p \in W_\infty$, i.e. an input to LLPO_∞, we will compute a connected non-empty set

$$A(p) := \{x \in B_0 : (\forall n \in \mathbb{N})\, f_{n-1}^{-1} \circ \cdots \circ f_0^{-1}(x) \in E_n(s_n(p))\} \subseteq \mathbb{R}^2$$

that is defined by an inverse limit construction. That means that the functions $f_n : B_{n+1} \hookrightarrow B_n$ are computable embeddings of certain rectangles $B_n \subseteq \mathbb{R}^2$ (called *blocks*) into each other and $E_n(s_n(p)) \subseteq B_n$ are certain subsets that consist of a union of finitely many squares (called *tiles*) within B_n. These sets E_n will be constructed such that they reflect the information encoded in a certain portion

$s_n(p) \in W_{n+1}$ of p and this encoding will be organized such that any point $y \in A(p)$ will allow us to compute some possible value of $\mathsf{LLPO}_\infty(p)$. We first describe the construction of the discrete structure of these blocks B_n and certain subsets $S_n \subseteq B_n$ (called *snakes*) that are completely independent of the input p. In a second step we describe how the sets E_n are constructed as subsets of the snakes S_n in dependence of the input p. Then we define the computable embeddings f_n such that they preserve the information encoded in the sets E_n in a particular way. In the next step we show that the sets $A(p)$ can be computed from p and that they allow us to recover the information $\mathsf{LLPO}_\infty(p)$ from any $y \in A(p)$. Finally, we show that the sets $A(p)$ are non-empty and connected.

The discrete structure of blocks and snakes within them. We will now describe a discrete structure within $\mathbb{R}^2$ that will be used to represent information from repetition-free words. This structure consists of certain *blocks* $B_n := [0, w_n] \times [0, h_n]$ of a suitable *width* w_n and a suitable *height* h_n. We call subsets of the form $[i, i+1] \times [j, j+1] \subseteq \mathbb{R}^2$ *tiles*. Within the blocks B_n we identify subsets $G_n, M_n, R_n \subseteq B_n$ that are unions of tiles. The sets G_n will be displayed in green and they will be used to encode certain bits of value 1. The sets R_n will be displayed in red and they will be used to encode certain bits of value 0. The sets M_n will be displayed in gray and they are middle sets that are used to separate bits. The union $S_n := G_n \cup M_n \cup R_n$ will constitute a connected chain of tiles, called *snake*. The construction proceeds inductively using previous parts of the same color that are shifted using certain *corner points* c_n. For a point $c \in \mathbb{R}^2$ and a set $A \subseteq \mathbb{R}^2$, we use the notation $c + A := \{c + x : x \in A\}$.

Definition 7.5 (Blocks). We define numbers $w_n, h_n \in \mathbb{N}$, points $c_n \in \mathbb{R}^2$ and sets $B_n, G_n, M_n, R_n, S_n \subseteq \mathbb{R}^2$ for all $n \in \mathbb{N}$ as follows:

(1) $w_n := 2n(w_{n-1} + 1) - 1$ for $n > 0$, $w_0 := 1$ (width)

(2) $h_n := 2n(h_{n-1} - 1) + 5$ for $n > 0$, $h_0 := 3$ (height)

(3) $c_n := (w_{n-1} + 1, h_{n-1} - 1)$ for $n > 0$ (corner)

(4) $B_n := [0, w_n] \times [0, h_n]$ (block)

(5) $G_n := [0, w_n] \times [h_n - 1, h_n] \cup \bigcup_{k=0}^{2n-1} (kc_n + G_{n-1})$ (green)

(6) $R_n := [0, w_n] \times [h_n - 3, h_n - 2] \cup \bigcup_{k=0}^{2n-1} (kc_n + R_{n-1})$ (red)

(7) $M_n := ([0, 1] \times [h_n - 2, h_n - 1]) \cup ([w_n - 1, w_n] \times [h_n - 4, h_n - 3])$
       $\cup \bigcup_{k=0}^{2n-1} (kc_n + M_{n-1}) \cup \bigcup_{k=1}^{2n-1} (kc_n + ([-1, 0] \times [0, 1]))$ (middle)

(8) $S_n := G_n \cup R_n \cup M_n$ (snake)

The construction is illustrated in Figs. 1 and 2. The sets G_n, M_n, and R_n are green, gray, and red, respectively. The following observations are immediate.

Claim 7.6. $G_n, M_n, R_n, S_n \subseteq \mathbb{R}^2$ *are closed and they satisfy for all* $n \in \mathbb{N}$:

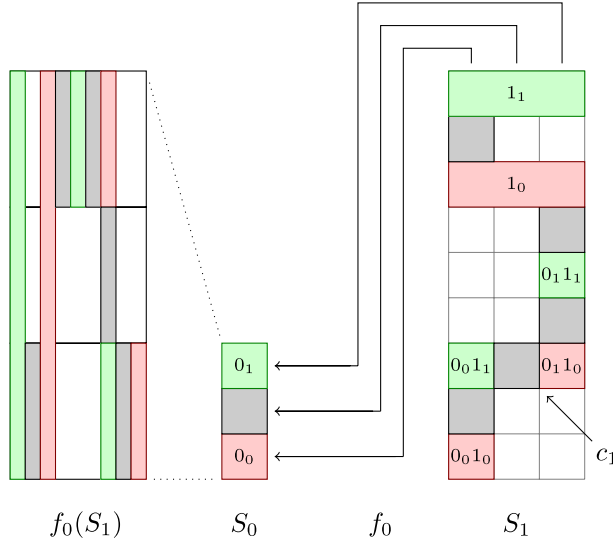(1) $S_n = G_n \cup M_n \cup R_n \subseteq B_n$ *and* $G_n \cap R_n = \emptyset$,

Fig. 1.   The embedding $f_0 : B_1 \hookrightarrow S_0$.

(2) S_n is a chain of tiles, i.e. all tiles in S_n but $[0,1] \times [0,1]$ and $[w_n - 1, w_n] \times [h_n - 1, h_n]$ are edge-connected to exactly two other tiles in S_n,

(3) $kc_n + S_{n-1} \subseteq S_n$ for all $0 \leq k \leq 2n$.

Coding of repetition-free words in the discrete structure. In this part of the proof we describe how the discrete structure with the snakes S_n can be used to encode repetition-free words into sets $E_n \subseteq S_n$ that depend on these words. For this purpose we first define a map that removes the number k from a repetition-free word and decrements all entries that are greater than k by one.

Definition 7.7. We define $r : \mathbb{N} \times W_* \to W_*$ as follows for $u, v \in W_*, k \in \mathbb{N}$:

(1) $r(k, u) := u$ if $u \in W_k$,
(2) $r(k, uk_b v) := r(k, uv)$,
(3) $r(k, u(n+1)_b v) := r(k, u)n_b r(k, v)$ for $n + 1 > k$.

We will use this map r in order to define a map $E_n : W_{n+1} \to \mathcal{A}_-(S_n)$ that shows how we encode repetition-free words $w \in W_{n+1}$ as closed subsets of S_n.

Definition 7.8. For all $n \in \mathbb{N}$ we define a map $E_n : W_{n+1} \to \mathcal{A}_-(S_n)$ inductively as follows for $w \in W_{n+1}$:

(1) $E_n(\varepsilon) := S_n$,
(2) $E_n(n_0 w) := [0, w_n] \times [h_n - 3, h_n - 2]$,
(3) $E_n(n_1 w) := [0, w_n] \times [h_n - 1, h_n]$,
(4) $E_n(k_b w) := (2k + b)c_n + E_{n-1}(r(k, w))$ for $k < n$.
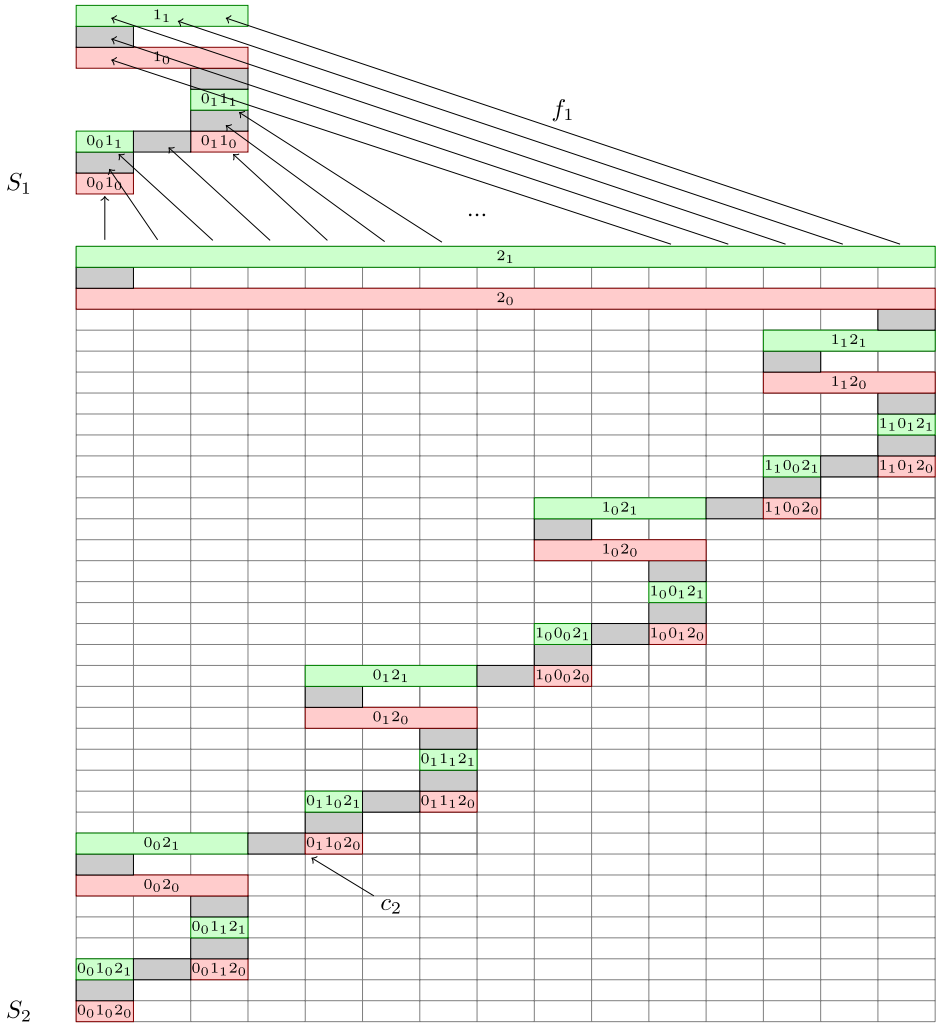
Fig. 2. The embedding $f_1 : B_2 \hookrightarrow S_1$.

The sets $E_0(w), E_1(w)$ and $E_2(w)$ are illustrated in Figs. 1 and 2 by the given words w. By comparing the recursive definition of E_n with the definitions of G_n and R_n, we see that if $n_0 \in \text{range}(w)$, then $E_n(w) \subseteq G_n$ and if $n_1 \in \text{range}(w)$, then $E_n(w) \subseteq R_n$. Together with the fact that $G_n \cap R_n = \emptyset$ this will enable us to recover the bits b with $n_b \in \text{range}(w)$ from $E_n(w)$.

Claim 7.9. *For all* $n, k \in \mathbb{N}$, $b \in \{0, 1\}$ *and* $w \in W_{n+1}$ *we have*

(1) $E_n(w) \subseteq S_n$,
(2) $E_n(wk_b) \subseteq E_n(w)$,
(3) $n_0 \in \text{range}(w) \Rightarrow E_n(w) \subseteq R_n$, *and* $n_1 \in \text{range}(w) \Rightarrow E_n(w) \subseteq G_n$.

Proof. We prove all claims by induction.

(1) By induction on w, which holds trivially for $w = \varepsilon$. For the inductive case: $E_n(n_b w) \subseteq S_n$ by definition, and for $k < n$ we have

$$E_n(k_b w) = (2k + b)c_n + E_{n-1}(r(k, w)) \subseteq (2k + b)c_n + S_{n-1}$$

by induction hypothesis, so $E_n(k_b w) \subseteq S_n$ by Claim 7.6.

(2) By induction on w. Base case,

$$E_n(k_b) = (2k + b)c_n + E_{n-1}(\varepsilon) = (2k + b)c_n + S_{n-1} \subseteq S_n = E_n(\varepsilon)$$

by Claim 7.6. Inductive case, $w = i_d u$. First subcase, $i = n$: $E_n(n_d u k_b) = E_n(n_d u)$. Second subcase, $i < n$, so

$$E_n(i_d u k_b) = (2i + d)c_n + E_{n-1}(r(i, u k_b)) = (2i + d)c_n + E_{n-1}(r(i, u)k_b')$$

with $k' \in \{k, k - 1\}$. By induction hypothesis $E_{n-1}(r(i, u)k_b') \subseteq E_{n-1}(r(i, u))$ so $E_n(i_d u k_b) \subseteq (2i + d)c_n + E_{n-1}(r(i, u)) = E_n(i_d u)$.

(3) We prove only the first statement, and by (2) it suffices to prove that $E_n(u n_0) \subseteq R_n$ for all $u \in W_n$. By induction on u. In the base case, $E_n(\varepsilon n_0) \subseteq R_n$ holds by definition. In the inductive case we obtain

$$E_n(k_b u n_0) = (2k + b)c_n + E_{n-1}(r(k, u)(n - 1)_0) \subseteq (2k + b)c_n + R_{n-1}$$

by induction hypothesis. Since $(2k + b)c_n + R_{n-1} \subseteq R_n$ by definition, it follows that $E_n(k_b u n_0) \subseteq R_n$. $\qquad\square$

Finally, we mention that the recursive definition of E_n together with Claim 7.9(2) implies the following.

Claim 7.10. *$E_n : W_{n+1} \to \mathcal{A}_-(B_n)$ is computable for all $n \in \mathbb{N}$.*

Computable embedding of blocks into snakes. As stated above, the S_i on different levels are connected via computable embeddings $f_n : B_{n+1} \hookrightarrow S_n$. The precise form of the f_n is irrelevant for our purposes, we merely demand that they map every *stripe* $C_{n+1}^k := [k, k + 1] \times [0, h_{n+1}]$ for $k \leq w_n - 1$ in B_{n+1} to a specific tile in S_n. Clearly, adjacent stripes have to be mapped into edge-adjacent tiles for a continuous embedding to exist, and this requirement is sufficient. We will state our specific requirements inductively.

Claim 7.11. *There exists a computable sequence $(f_n)_n$ of computable embeddings $f_n : B_{n+1} \hookrightarrow S_n$ such that for $n > 0$, $i < w_n$ and $k \leq 2n - 1$*

(1) $f_0(C_1^0) \subseteq [0, 1] \times [0, 1]$ *(base cases)*

(2) $f_0(C_1^1) \subseteq [0, 1] \times [1, 2]$

(3) $f_0(C_1^2) \subseteq [0, 1] \times [2, 3]$

(4) $f_n(C_{n+1}^{k(w_n+1)+i}) \subseteq kc_n + f_{n-1}(C_n^i)$ *(snake)*

(5) $f_n(C_{n+1}^{k(w_n+1)-1}) \subseteq kc_n + ([-1, 0] \times [0, 1])$ *if* $k \geq 1$ *(padding stripe)*

(6) $f_n(C_{n+1}^{w_{n+1}-2w_n-2}) \subseteq [w_n - 1, w_n] \times [h_n - 4, h_n - 3]$ (*padding stripe*)

(7) $f_n(C_{n+1}^{w_{n+1}-2w_n-1+i}) \subseteq [w_n - i - 1, w_n - i] \times [h_n - 3, h_n - 2]$ (*snake*)

(8) $f_n(C_{n+1}^{w_{n+1}-w_n-1}) \subseteq [0, 1] \times [h_n - 2, h_n - 1]$ (*padding stripe*)

(9) $f_n(C_{n+1}^{w_{n+1}-w_n+i}) \subseteq [i, i + 1] \times [h_n - 1, h_n]$ (*snake*)

Proof. Clearly, adjacent stripes have to be mapped into edge-adjacent tiles for a continuous embedding $f_n : B_{n+1} \hookrightarrow S_n$ to exist, and this requirement is sufficient. According to the definition above this requirement is satisfied and since each snake S_n consists of finitely many tiles, there is also a computable embedding f_n that satisfies these requirements. Since the inductive definition of the sequence $(S_n)_n$ is computable in a uniform way depending on n, it follows that there is also a computable sequence $(f_n)_n$ of embeddings that satisfies the given requirements. $\square$

The embeddings $f_0 : B_1 \hookrightarrow S_0$ and $f_1 : B_2 \hookrightarrow S_1$ are illustrated in Figs. 1 and 2. The case of f_1 is already a prototype for the general situation of embeddings $f_n : B_{n+1} \hookrightarrow S_n$ with $n \geq 1$. Essentially, consecutive stripes of B_{n+1} are mapped by f_n into consecutive tiles in S_n one by one.

The definition of the embeddings f_n matches the definition of the sets E_m in such a way that they are preserved in a particular way. In order to express this result precisely, we first define a function $s_n : W_\infty \to W_{n+1}$ that removes all irrelevant information from the input sequence or word p, i.e. it removes all entries with a first component $k > n$.

Definition 7.12. For all $n \in \mathbb{N}$ we define a function $s_n : W_\infty \to W_{n+1}$ by:

(1) $s_n(p) := \varepsilon$ if $k > n$ for all $k_b \in \text{range}(p)$,

(2) $s_n(k_b u) := k_b s_n(u)$ if $k \leq n$ and

(3) $s_n(k_b u) := s_n(u)$ if $k > n$.

Now we can derive the following from the respective definitions.

Claim 7.13. $E_{m+1}(w) \subseteq f_m^{-1}(E_m(s_m(w)))$ *for all* $w \in W_{m+2}$ *and* $m \in \mathbb{N}$.

The reduction function. Now we can define the actual reduction function that we are going to use for the reduction $\mathsf{LLPO}_\infty \leq_{\mathrm{sW}} \mathsf{CC}_{B_0}$.

Definition 7.14. We define a function

$$A : W_\infty \to \mathcal{A}_-(B_0), \quad p \mapsto \bigcap_{n=0}^{\infty} (f_0 \circ \cdots \circ f_{n-1})(E_n(s_n(p))).$$

Claim 7.15. $A : W_\infty \to \mathcal{A}_-(B_0)$ *is computable and given a point* $x \in A(p)$, *we can computably reconstruct some* $q \in \mathsf{LLPO}_\infty(p)$.

Proof. Since $(s_n)_n$ and $(E_n)_n$ are computable sequences, and since $E_n(s_n(p)) \subseteq B_n$ and $(B_n)_n$ is a computable sequence of compact sets, it follows that also

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$((f_0 \circ \cdots \circ f_{n-1})(E_n(s_n(p))))_n$ is a computable sequence of compact sets whose intersection is compact and can be computed as a closed set. Given a point $x \in A(p)$ we can reconstruct $\mathsf{LLPO}_\infty(p)(n)$ by computing $f_{n-1}^{-1} \circ \cdots \circ f_0^{-1}(x) \in E_n(s_n(p))$. By Claim 7.9 we have that $n_0 \in \mathrm{range}(p)$ implies $E_n(s_n(p)) \subseteq R_n$ and $n_1 \in \mathrm{range}(p)$ implies $E_n(s_n(p)) \subseteq G_n$. Since R_n and G_n consist of finitely many tiles and are clearly disjoint, we can find one possible value for $\mathsf{LLPO}_\infty(p)(n) = b$ such that $n_b \notin \mathrm{range}(p)$. $\qquad \square$

What remains to be proved in order to conclude that A yields the reduction $\mathsf{LLPO}_\infty \leq_{\mathrm{sW}} \mathsf{CC}_{B_0}$ is to show that $A(p)$ is always non-empty and connected.

Claim 7.16. *If the sets*

$$A_m(w) := \bigcap_{n=0}^{m} (f_n \circ \cdots \circ f_{m-1})^{-1}(E_n(s_n(w))) \subseteq B_m$$

are non-empty and connected for all $w \in W_*$, then so is $A(p)$ for all $p \in W_\infty$.

Proof. If $A_m(w)$ is non-empty and connected for all $w \in W_*$, then so is

$$K_m := f_0 \circ \cdots \circ f_{m-1}(A_m(s_m(p))) = \bigcap_{n=0}^{m} f_0 \circ \cdots \circ f_{n-1}(E_n(s_n(p))) \subseteq B_0$$

for all $p \in W_\infty$. Since the K_m are also compact, $A(p) = \bigcap_{m=0}^{\infty} K_m$ is a decreasing chain of non-empty continua and hence itself a non-empty continuum by [25, Corollary 6.1.19]. $\qquad \square$

Note that any $A_m(w)$ is a union of tiles in S_m. We can be more specific, though, and this will be useful in the proof.

Connectedness of the sets $A(p)$. By mutual induction we define the notions of a segment in S_n and a slice in B_n. This will help us to prove that the sets $A(p)$ are connected.

Definition 7.17 (Segments and slices). Let the notions of a *segment* in S_n and of a *slice* in B_n be defined by mutual induction for all $n \in \mathbb{N}$:

(1) B_0 is a slice in B_0.
(2) If G is a segment in S_n, then $f_n^{-1}(G)$ is a slice in B_{n+1}.
(3) S_n is a segment in S_n.
(4) If L is a slice in B_n, then $L \cap ([0, w_n] \times [h_n - 3, h_n - 2])$ is a segment in S_n.
(5) If L is a slice in B_n, then $L \cap ([0, w_n] \times [h_n - 1, h_n])$ is a segment in S_n.
(6) If G is a segment in S_{n-1} and $k \leq 2n - 1$, then $kc_n + G$ is a segment in S_n.

Claim 7.18. *We obtain the following for all $n \in \mathbb{N}$:*

(1) B_n *is a slice in* B_n.
(2) *Every slice in* B_n *is of the form* $[a, b] \times [0, h_n]$ *with* $a < b$.

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(3) *Every segment in S_n is non-empty and connected.*

(4) $E_n(w)$ *is a segment in S_n for every $w \in W_{n+1}$.*

Proof. (1) follows from Definition 7.17(1)–(3).

(2) and (3) By mutual induction on the definition of segments and slices.

(4) That $E_n(\varepsilon) = S_n$ is a segment follows from Definition 7.17(3). That $E_n(n_b w)$ is a segment follows from Definition 7.17(4), (5) with the help of (1). The case $E_n(k_b w)$ with $k < n$ follows from Definition 7.17(6) by induction. $\qquad\square$

Next we show that the sets A_m and E_m are essentially identical.

Claim 7.19. $A_m = E_m \circ s_m$ *for all $m \in \mathbb{N}$.*

Proof. For $m = 0$ the claim follows from the definition. An inspection of the definition of $A_m(w)$ from Claim 7.16 shows that for $m > 0$

$$A_m(w) = f_{m-1}^{-1}(A_{m-1}(w)) \cap E_m(s_m(w)) \tag{3}$$

for all $w \in W_*$. We prove by induction on $m \in \mathbb{N}$ that $A_m(w) = E_m(s_m(w))$ for all $w \in W_*$. For $m = 0$ this holds by definition of A_0. Given the induction claim for m, we obtain by Claim 7.13

$$E_{m+1}(w) \subseteq f_m^{-1}(E_m(s_m(w))) \subseteq f_m^{-1}(A_m(w))$$

for all $w \in W_{m+2}$. This in turn implies $A_{m+1}(w) = E_{m+1}(s_{m+1}(w))$ for all $w \in W_*$ using Eq. (3). $\qquad\square$

This easily implies the final claim.

Claim 7.20. $A(p)$ *is non-empty and connected for all $p \in W_\infty$.*

Proof. By Claim 7.16 it is sufficient to show that all the sets $A_m(w)$ for $w \in W_*$ are connected and non-empty. However, this is the case according to Claims 7.19 and 7.18(3),(4). $\qquad\square$

Together with Claim 7.15 this completes the proof of $\mathsf{LLPO}_\infty \leq_{\mathrm{sW}} \mathsf{CC}_{B_0}$ and hence it completes the proof of Theorem 7.1.

Even though Theorem 7.1 completes our characterization of the Brouwer Fixed Point Theorem in dimension 2, it raises some further questions. The construction provided by Proposition 6.1 has the property that the values of T are even pathwise connected sets. Let us denote by PWCC_n the restriction of CC_n to pathwise connected sets. Then a part of Theorem 6.2 can be strengthened to the following result.

Corollary 7.21 (Pathwise connected choice). $\mathsf{PWCC}_n \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]}$ *for all $n \geq 3$.*

However, we are left with the following open question.

Question 7.22. Is $\mathsf{PWCC}_2 \equiv_W \mathsf{C}_{[0,1]}$?

At least the construction in the proof of Theorem 7.1 does not answer this question since it yields a connected set $A(p)$ that is not pathwise connected.

We can draw some further interesting conclusions from the construction of the sets $A(p)$ that is related to the work of Iljazović [30], who studied computability properties of chainable decomposable continua.[h] We recall that a continuum $A \subseteq [0,1]^n$ is called *decomposable* if it is the union of two of its proper subcontinua. And A is called *chainable*, if for every $\varepsilon > 0$ there exists an ε-chain $C_0, \ldots, C_m$ that covers A. For $C_0, \ldots, C_m \subseteq [0,1]^n$ to be an ε-*chain* means that the $C_0, \ldots, C_m$ are non-empty open sets with $\mathrm{diam}(C_i) < \varepsilon$ and such that $C_i \cap C_j \neq \emptyset$ holds if and only if $|i - j| \leq 1$. The following is a consequence of [30, Theorem 44].

Proposition 7.23 ([30]). *Every co-c.e. chainable and decomposable continuum $A \subseteq [0,1]^n$ contains a dense subset of computable points.*

The construction of the sets $A(p)$ in the proof of Theorem 7.1 guarantees that there is a computable point p such that $A(p)$ does not contain any computable point. It is also easy to see that the sets $A(p)$ are chainable. As a conclusion we obtain the following.

Corollary 7.24. *There is a non-empty co-c.e. chainable continuum $A \subseteq [0,1]^2$ that does not contain any computable point.*

As a consequence of this result and Proposition 7.23 it follows that the corresponding set A is necessarily indecomposable.

8. The Displacement Principle

In this section, we want to prove a displacement principle that provides some information on the power of binary choice $\mathsf{C}_{\{0,1\}}$ on the left-hand side of a reduction. We will apply this principle in Sec. 9 to prove that CC_1 is not idempotent. In order to prove our result we first need to study the convergence relation of $\mathcal{A}_-(X)$ induced by ψ_-. This convergence relation can be characterized in terms of *closed upper limits* as defined by Hausdorff. For a sequence $(A_i)_{i \in \mathbb{N}}$ of closed subsets of a topological space X the *closed upper limit* of $(A_i)_{i \in \mathbb{N}}$ is defined by

$$\mathrm{Ls}(A_i) := \bigcap_{k=0}^{\infty} \overline{\bigcup_{i=k}^{\infty} A_i}.$$

The common notation Ls is derived from the fact that this is also called the *topological limit superior* of $(A_i)_{i \in \mathbb{N}}$. The set $\mathrm{Ls}(A_i)$ is always closed and possibly empty.

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[h]Thanks to an anonymous referee for pointing out this connection.

If X is compact and all the A_i are non-empty, then $\mathrm{Ls}(A_i)$ is also compact and non-empty by Cantor's Intersection Theorem. We mention the following known characterization of the topological limit superior by Choquet (see, for instance, [2, Proposition 5.2.2]).

Fact 8.1 (Choquet). Let X be a Hausdorff space and let $\mathcal{N}_x$ denote the set of open neighborhoods of $x \in X$. For each sequence $(A_i)_{i \in \mathbb{N}}$ of closed sets $A_i \subseteq X$ one has

$$\mathrm{Ls}(A_i) = \{x \in X : (\forall U \in \mathcal{N}_x)(\forall k)(\exists i \geq k)\, U \cap A_i \neq \emptyset\}.$$

It is well known that the topological limit superior (and the related topological limit inferior) are used to define Kuratowski–Painlevé convergence, which is closely related to convergence with respect to the Fell topology (see [2, Chap. 5]). Here we characterize the convergence relation of $\mathcal{A}_-(X)$ in terms of the topological limit superior. For a sequence $(A_i)_{i \in \mathbb{N}}$ and a set A in $\mathcal{A}_-(X)$ we write $A_i \to A$ if there are p_i and p such that $\psi_-(p_i) = A_i$, $\psi_-(p) = A$ and $p_i \to p$. We note that this convergence relation on $\mathcal{A}_-(X)$ is not unique in general, i.e. one sequence $(A_i)_{i \in \mathbb{N}}$ can have many different limits. The following result gives an exact characterization.

Lemma 8.2 (Closed upper limit). *Let X be a computable metric space and let $A_i, A \in \mathcal{A}_-(X)$ for all $i \in \mathbb{N}$. Then $A_i \to A$ if and only if $\mathrm{Ls}(A_i) \subseteq A$.*

Proof. Let p_i and p be such that $\psi_-(p_i) = A_i$, $\psi_-(p) = A$.

We now assume $p_i \to p$. Let $x \notin A$. Then there is some basic open neighborhood B_m of x that is eventually listed in position j of p. Since the p_i converge to p, there is a $k \in \mathbb{N}$ such that B_m is also listed in position j of p_i for all $i \geq k$. According to Fact 8.1 this means that $x \notin \mathrm{Ls}(A_i)$. Hence, we have proved $\mathrm{Ls}(A_i) \subseteq A$.

Let us now assume that $\mathrm{Ls}(A_i) \subseteq A$. It suffices to find q_i with $\psi_-(q_i) = A_i$ and $q_i \to p$. We choose $q_i := p|_{m_i} p_i$, where $p|_{m_i}$ is the prefix of p of suitable length m_i. It is clear that $q_i \to p$ follows if the m_i are increasing without bound, so we need to prove that we can choose such m_i with $\psi_-(q_i) = \psi_-(p_i)$. We note that for each n the set $U = B_{p(n)}$ does not intersect A, i.e. $U \cap A = \emptyset$ and hence there is some k such that for all $i \geq k$ we have $U \cap A_i = \emptyset$ by Fact 8.1. That means that we can add the ball $B_{p(n)}$ to the negative information of A_i without changing A_i. This guarantees the existence of a suitable unbounded increasing sequence m_i. $\qquad\square$

This result implies that the convergence relation on $\mathcal{A}_-(X)$ induced by ψ_- is the convergence relation of the upper Fell topology and hence ψ_- is admissible with respect to this topology (which was already known, see [44]). We introduce some further terminology. If $\mathcal{S} \subseteq \mathcal{A}_-(X)$, then we denote by

$$\overline{\mathcal{S}} := \{A \in \mathcal{A}_-(X) : (\exists (A_i)_{i \in \mathbb{N}} \in \mathcal{S}^{\mathbb{N}})\, \mathrm{Ls}(A_i) \subseteq A\}$$

the *sequential closure* of $\mathcal{S}$ in $\mathcal{A}_-(X)$ and by

$$2\mathcal{S} := \{A \in \mathcal{S} : (\exists A_1, A_2 \in \mathcal{S})(A_1 \cap A_2 = \emptyset \text{ and } A_1 \cup A_2 \subseteq A)\}$$

the set of those sets in $\mathcal{S}$ that have two disjoint subsets in $\mathcal{S}$. By $\mathsf{C}_X|_{\mathcal{S}}$ we denote the restriction of C_X to $\mathcal{S}$.

Theorem 8.3 (Displacement Principle). *Let f be a multi-valued function on represented spaces, let X be a computable metric space and let $\mathcal{S} \subseteq \mathcal{A}_-(X)$. Then*

$$f \times \mathsf{C}_{\{0,1\}} \leq_{\mathrm{W}} \mathsf{C}_X|_{\mathcal{S}} \Rightarrow f \leq_{\mathrm{W}} \mathsf{C}_X|_{\mathcal{S} \cap 2\overline{\mathcal{S}}}.$$

An analogous statement holds with $\leq_{\mathrm{W}}$ replaced by $\leq_{\mathrm{sW}}$ in both instances.

Proof. We use the computable metric space (X, δ_X) and represented spaces (Y, δ_Y) and (Z, δ_Z). We assume that f is of type $f :\subseteq Y \rightrightarrows Z$ and we use $\mathsf{C}_{\{0,1\}} \equiv_{\mathrm{sW}} \mathsf{LLPO}$ (see [6]). Let $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be computable functions that witness the reduction $f \times \mathsf{LLPO} \leq_{\mathrm{W}} \mathsf{C}_X|_{\mathcal{S}}$, i.e. $H\langle\mathrm{id}, GK\rangle \vdash f \times \mathsf{LLPO}$ whenever $G \vdash \mathsf{C}_X|_{\mathcal{S}}$.

We recall that $\mathsf{LLPO} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ is defined such that for $j \in \{0,1\}$ and $p \in \{0,1\}^{\mathbb{N}}$ it holds that $j \in \mathsf{LLPO}(p) \Leftrightarrow (\forall i)\, p(2i+j) = 0$, where $\mathrm{dom}(\mathsf{LLPO})$ contains all sequences p such that $p(k) \neq 0$ for at most one k. We consider the inputs $p_{j,i} := 0^{2i+j+1}10^{\mathbb{N}}$ and $p_\infty := 0^{\mathbb{N}}$ for LLPO. We obtain $\mathsf{LLPO}(p_{j,i}) = \{j\}$ for $j \in \{0,1\}$ and $\mathsf{LLPO}(p_\infty) = \{0,1\}$. For every $p \in \mathrm{dom}(f\delta_Y)$, $i \in \mathbb{N}$ and $j \in \{0,1\}$ we now define

$$A_{j,i}^p := \psi_- K\langle p, p_{j,i}\rangle \quad \text{and} \quad A_\infty^p := \psi_- K\langle p, p_\infty\rangle.$$

Since $p_{j,i} \to p_\infty$ for $i \to \infty$, continuity of K implies $\mathrm{Ls}(A_{j,i}^p) \subseteq A_\infty^p$ for $j \in \{0,1\}$ by Lemma 8.2. Now we consider the corresponding subsets of $\mathrm{dom}(H)$:

$$B_j^p := \bigcup_{i=0}^{\infty} \langle\{\langle p, p_{j,i}\rangle\} \times \delta_X^{-1}(A_{j,i}^p)\rangle, \quad B_\infty^p := \langle\{\langle p, p_\infty\rangle\} \times \delta_X^{-1}(A_\infty^p)\rangle.$$

By π_j we denote the projection on the jth component of a tuple in Baire space. Then $h := \delta_{\mathbb{N}}\pi_2 H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is a computable function such that $h|_{B_j^p}$ is constant with value j for $j \in \{0,1\}$. We claim that due to continuity of h this implies $\mathrm{Ls}(A_{0,i}^p) \cap \mathrm{Ls}(A_{1,i}^p) = \emptyset$. Let us assume that q is such that $\delta_X(q) \in \mathrm{Ls}(A_{0,i}^p) \cap \mathrm{Ls}(A_{1,i}^p)$. Then $\delta_X(q) \in A_\infty^p$ and hence $r := \langle\langle p, p_\infty\rangle, q\rangle \in B_\infty^p \subseteq \mathrm{dom}(h)$. Let now U be a neighborhood of r and let $j \in \{0,1\}$. By Fact 8.1 the point $\delta_X(q)$ is a cluster point of the sequence $(A_{j,i}^p)_{i\in\mathbb{N}}$ and hence there is a sequence $(q_i)i \in \mathbb{N}$ with $\delta_X(q_i) \in A_{j,i}^p$ for all i with a subsequence that converges to q. Hence, for some sufficiently large i we obtain $\langle\langle p, p_{j,i}\rangle, q_i\rangle \in B_j^p \cap U$, which means $B_j^p \cap U \neq \emptyset$ for $j \in \{0,1\}$. Hence $h|_U$ has to take both values 0 and 1 on any neighborhood U of r, which contradicts continuity of h. This proves the claim $\mathrm{Ls}(A_{0,i}^p) \cap \mathrm{Ls}(A_{1,i}^p) = \emptyset$.

Altogether, we have proved $A_\infty^p \in 2\overline{\mathcal{S}}$ and $A_\infty^p \in \mathcal{S}$ is clear. We now define computable functions $H', K' :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$H'\langle p, q\rangle := \pi_1 H\langle\langle p, p_\infty\rangle, q\rangle \quad \text{and} \quad K'(p) = \pi_1 K\langle p, p_\infty\rangle.$$

Then $H'\langle\mathrm{id}, GK'\rangle \vdash f$ whenever $G \vdash \mathsf{C}_X|_{\mathcal{S} \cap 2\overline{\mathcal{S}}}$, i.e. $f \leq_{\mathrm{W}} \mathsf{C}_X|_{\mathcal{S} \cap 2\overline{\mathcal{S}}}$. If H does not depend on the first component, then $H' = \pi_1 H$ also does not depend on the

first component. Hence the claim also holds for strong reducibility $\leq_{\mathrm{sW}}$ in place of $\leq_{\mathrm{W}}$. □

If $\mathcal{S}$ only contains non-empty closed sets $A \subseteq X$ and X is compact, then $\overline{\mathcal{S}}$ also contains only non-empty sets and $2\overline{\mathcal{S}}$ contains only sets that have at least two points. Hence we obtain the following corollary, where $\mathcal{U}(X) := \{\{x\} : x \in X\}$ denotes the set of singleton subsets of X.

Corollary 8.4. *Let f be a multi-valued function on represented spaces, let X be a compact computable metric space and let $\mathcal{S} \subseteq \mathcal{A}_-(X)\backslash\{\emptyset\}$. Then*

$$f \times \mathsf{C}_{\{0,1\}} \leq_{\mathrm{W}} \mathsf{C}_X|_{\mathcal{S}} \Rightarrow f \leq_{\mathrm{W}} \mathsf{C}_X|_{\mathcal{S}\backslash\mathcal{U}(X)}.$$

An analogous statement holds with $\leq_{\mathrm{W}}$ replaced by $\leq_{\mathrm{sW}}$ in both instances.

9. Idempotency of Connected Choice in Dimension One

The goal of this section is to prove that connected choice CC_1 of dimension one is not idempotent, i.e. $\mathsf{CC}_1 \not\equiv_{\mathrm{W}} \mathsf{CC}_1 \times \mathsf{CC}_1$. For this purpose we use CC_1^-, which is just the restriction of CC_1 to such connected sets that are not singletons. In [8], it was proved that $\mathsf{CC}_1^- \leq_{\mathrm{W}} \mathsf{C}_\mathbb{N}$, which follows since one can just guess a rational number in a non-degenerate interval and with finitely many mind changes one can find a correct number. Using the Displacement Principle we can prove the following result.

Proposition 9.1. $\mathsf{CC}_1 <_{\mathrm{W}} \mathsf{CC}_1 \times \mathsf{C}_{\{0,1\}}$.

Proof. It is clear that $\mathsf{CC}_1 \leq_{\mathrm{W}} \mathsf{CC}_1 \times \mathsf{C}_{\{0,1\}}$. Let us assume that also $\mathsf{CC}_1 \times \mathsf{C}_{\{0,1\}} \leq_{\mathrm{W}} \mathsf{CC}_1$. Then by Corollary 8.4 we obtain $\mathsf{CC}_1 \leq_{\mathrm{W}} \mathsf{CC}_1^-$. Since $\mathsf{CC}_1^- \leq_{\mathrm{W}} \mathsf{C}_\mathbb{N}$ by [8, Proposition 3.8]), we obtain $\mathsf{CC}_1 \leq_{\mathrm{W}} \mathsf{C}_\mathbb{N}$, which is a contradiction to [8, Lemma 4.9]. □

While this result shows that binary choice $\mathsf{C}_{\{0,1\}}$ enhances the power of connected choice CC_1 if multiplied with it, products of binary choice with itself are not that powerful, as the next result shows.

Proposition 9.2. $\mathsf{C}_{\{0,1\}}^* \leq_{\mathrm{sW}} \mathsf{CC}_1^-$.

Proof. Given a pair $\langle n, p \rangle$ as input to $\mathsf{C}_{\{0,1\}}^*$ we need to construct a non-degenerate connected closed set $A \subseteq [0,1]$ any point of which allows us to find a point in $\mathsf{C}_{\{0,1\}}^n$ for input p. The input p describes a product $A_1 \times \cdots \times A_n$ of non-empty sets $A_i \subseteq \{0,1\}$ by an enumeration of the complement.

In order to construct A we use an auxiliary tree of rational complexes with branching degree $2n$ in which each complex exists exactly of one rational interval $[a,b]$ with $a < b$. More precisely, we start with the root $[\frac{1}{2n+2}, \frac{1}{2n+1}]$ and on each

successor node of the tree we use $2n$ canonical pairwise disjoint subintervals of the previous interval, sorted in the natural order.

We now describe how we use this tree to construct A. Given p we start to produce the root interval $\left[\frac{1}{2n+2}, \frac{1}{2n+1}\right]$ as long as no negative information on any of the sets $A_1, \ldots, A_n$ is available. If A_k is the first of these sets that is determined by p, then we proceed with child node number $2k-1$ or $2k$ depending on whether $A_k = \{0\}$ or $A_k = \{1\}$. We then produce a description of the interval associated with this child node until further information on one of the sets $A_{k+1}, \ldots, A_n$ becomes available, in which case we proceed inductively as described above.

Altogether, this procedure produces an interval I that is somewhere between the root level (in case that all the sets A_i remain undetermined) and level n below the root level of the tree (in case that all the sets A_i are eventually determined). Given a point $x \in I$, we can find one of the (at most two) intervals J on level n that are closest to x and included in I. Given J, we can reconstruct all decisions in the above algorithm and in this way we can produce a point $(x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n$. $\square$

We mention that one can use the level (as introduced by Hertling [27]) to prove that $\mathsf{C}^*_{\{0,1\}} <_{\mathrm{sW}} \mathsf{CC}_1^-$. One can show that CC_1^- has no level, whereas the level of $\mathsf{C}^*_{\{0,1\}}$ is ω_0. Since the level is preserved downward by reducibility, it follows that the reduction must be strict.

We arrive at the main result of this section.

Theorem 9.3 (Non-idempotency). $\mathsf{CC}_1 <_{\mathrm{W}} \mathsf{CC}_1 \times \mathsf{CC}_1 <_{\mathrm{W}} \mathsf{CC}_2$.

Proof. First, it is clear that $\mathsf{CC}_1 <_{\mathrm{W}} \mathsf{CC}_1 \times \mathsf{C}_{\{0,1\}} \leq_{\mathrm{W}} \mathsf{CC}_1 \times \mathsf{CC}_1$ holds by Propositions 9.1 and 9.2. Second, it is also clear that $\mathsf{CC}_1 \times \mathsf{CC}_1 \leq_{\mathrm{W}} \mathsf{CC}_2$, since the product map $(A, B) \mapsto A \times B$ is computable on closed sets and the product of two connected sets is connected. Finally, $\mathsf{CC}_1 \times \mathsf{CC}_1$ is non-uniformly computable, whereas CC_2 is not by Proposition 6.5 and hence $\mathsf{CC}_1 \times \mathsf{CC}_1 <_{\mathrm{W}} \mathsf{CC}_2$. $\square$

In particular, CC_1 is not idempotent and the same reasoning that was used in the proof shows that $\mathsf{CC}_2 \not\leq_{\mathrm{W}} \mathsf{CC}_1^*$ holds for the idempotent closure CC_1^*. That means that not even an arbitrary finite number of copies of CC_1 in parallel is powerful enough to compute connected choice in dimension two. With Corollary 4.10 we obtain the following conclusion of Theorem 9.3.

Corollary 9.4. *The Brouwer Fixed Point Theorem* BFT_1 *of dimension one and the Intermediate Value Theorem* IVT *are both not idempotent.*

This means that two realizations of the Intermediate Value Theorem in parallel are more powerful than just one.

A problem related to idempotency is whether CC_n is a cylinder. Again it is clear that CC_n is a cylinder for $n \geq 2$, which follows from Theorems 6.2 and 7.1 and the

fact that $C_{[0,1]}$ is a cylinder. We can use the techniques of this section to prove that CC_1 is not a cylinder.

Theorem 9.5 (Cylinder). CC_1 *is not a cylinder.*

Proof. Let us assume that $\mathrm{id} \times CC_1 \leq_{sW} CC_1$. Then $\mathrm{id} \times C_{\{0,1\}} \leq_{sW} CC_1$ follows by Proposition 9.2 and hence $\mathrm{id} \leq_{sW} CC_1^-$ by Corollary 8.4. Since CC_1^- has a realizer that always selects a rational number, we obtain $CC_1^- \leq_{sW} CC_1^-|^{\mathbb{Q} \cap [0,1]}$, where $CC_1^-|^{\mathbb{Q} \cap [0,1]}$ denotes the restriction of CC_1^- to $\mathbb{Q} \cap [0,1]$ in the image. By [11, Proposition 13.2] this implies that $\mathrm{range}(\mathrm{id})$ is countable, which is a contradiction! □

10. Conclusions

We have systematically studied the uniform computational content of the Brouwer Fixed Point Theorem for any fixed dimension and we have obtained a systematic classification for all dimensions. A problem that we have left open is the status of pathwise connected choice of dimension two. Besides solving this open problem, one can proceed into several different directions. For one, one could study generalizations of the Brouwer Fixed Point Theorem, such as the Schauder Fixed Point Theorem or the Kakutani Fixed Point Theorem. On the other hand, one could study results that are based on the Brouwer Fixed Point Theorem, such as equilibrium existence theorems in computable economics (see for instance [43]). Nash equilibria existence theorems for bimatrix games have been studied in [41], and they can be seen to be strictly simpler than the general Brouwer Fixed Point Theorem (in fact they can be considered as linear version of it).

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