How Good Is a Strategy in a Game with Nature?

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We consider games with two antagonistic players—Éloïse (modelling a program) and Abélard (modelling a Byzantine environment)—and a third, unpredictable and uncontrollable player, which we call Nature. Motivated by the fact that the usual probabilistic semantics very quickly leads to undecidability when considering either infinite game graphs or imperfect-information, we propose two alternative semantics that lead to decidability where the probabilistic one fails: one based on counting and one based on topology.

CCS Concepts: • **Theory of computation** → **Automata over infinite objects**; **Tree languages**; *Verification by model checking*; • **Computing methodologies** → **Stochastic games**;

Additional Key Words and Phrases: Qualitative study of games, cardinality constraints, large sets of branches, tree automata

1 INTRODUCTION

An important problem in computer science is the specification and the verification of systems allowing *non-deterministic* behaviours. A non-deterministic behaviour can appear in several distinct contexts:

  (i) controllable behaviours (typically arising when the *program* is not fully specified, permitting to later restrict it);
 (ii) uncontrollable possibly Byzantine behaviours (typically arising from interactions of the program with its *environment*, e.g., a user);
(iii) uncontrollable unpredictable behaviours (usually arising from *nature,* often modelled by randomisation).

Here, we make an explicit distinction between the environment and nature: While we cannot assume that a user will not be malicious, the situation with nature is different, as we can accept a negligible set of bad behaviours, which implicitly means that they are very unlikely to appear. On top of this, one may also want to allow *imperfect-information* (typically arising when the protagonists—the program, the environment, and nature—share some public variables but also

have their own private variables) and/or *infinite* state systems (e.g., arising when modelling recursive procedures).

As the above-mentioned features are omnipresent in systems nowadays, their specification has already received a lot of attention, and there are several robust abstract mathematical models for them. There is also work on the specification side (i.e., on how to express a desirable behaviour of the system) and on the decidability of the fundamental question of whether a given specification is met by a given system. Unfortunately, whenever one combines any kind of non-determinism (interaction of the program with both nature and an uncontrollable environment) with one of the two others (either imperfect-information or an infinite number of states) it directly leads to undecidability. Moreover, if one is to recover decidability while still considering infinite arenas strong, unnatural restrictions are needed.

Two-player stochastic games on graphs are a natural way to model such systems. In a nutshell, a stochastic game is defined thanks to a directed graph whose vertices have been partitioned among two antagonistic players—Éloïse (modelling the program) and Abélard (modelling the Byzantine environment)—and a third, unpredictable and uncontrollable player, which we call Nature. The play starts with a token on a fixed initial vertex v_0 of the graph that is later moved by the players (the player owning the vertex where the token is chooses a neighbour to which the token is moved to, and so on forever) leading to an infinite path in the game graph. We are interested in *zero-sum* games, i.e., we consider a winning condition Ω consisting of a subset of plays, and we say that a play is winning for Éloïse if it belongs to Ω and otherwise it is winning for Abélard. A game $\mathbb{G}$ is such a graph together with a winning condition.

In the previous model, Nature usually comes with a probabilistic semantics (as in the seminal work of Condon [15]), i.e., any vertex controlled by Nature is associated with a probability distribution over its neighbours and this probability distribution is used to pick the next move when the token is on the corresponding vertex. The central concept is one of a strategy that maps any prefix of a play to the next vertex to move the token to. Once a strategy φ_E for Éloïse and a strategy φ_A for Abélard have been fixed, the set of all possible plays in the game where the players respect their strategies can be equipped with a probability measure $\mu_{v_0}^{\varphi_E, \varphi_A}$, and one can therefore define the **value** of the game as (φ_E and φ_A range over Éloïse and Abélard strategies, respectively):

$$\mathrm{Val}(\mathbb{G}) = \sup_{\varphi_E} \inf_{\varphi_A} \left\{ \mu_{v_0}^{\varphi_E, \varphi_A}(\Omega) \right\}.$$

Then, the following questions are of special interest:

(1) "*Decide whether the value of the game is larger than some given threshold η*" and its qualitative weakening "*Decide if the value is equal to* 1."

(2) "*When exists, compute an optimal strategy for Éloïse,*" where an optimal strategy φ_E for Éloïse is one such that $\mathrm{Val}(\mathbb{G}) = \inf_{\varphi_A} \{ \mu_{v_0}^{\varphi_E, \varphi_A}(\Omega) \}$ (note that such a strategy may not exist even if the graph is finite).

If the game is played on a finite graph and the winning condition is ω-regular, then all those questions can be answered and algorithms are known and their complexities, depending on the winning condition, range from P to PSPACE (see, e.g., Reference [13] for an overview).

Unfortunately, the landscape drastically changes as soon as one either considers infinite game graphs and/or imperfect-information (i.e., instead of knowing the exact state of the system, each player only knows that it belongs to some equivalence class). In particular, we have the following undecidability (somehow minimal) results:

- If the game graph is a pushdown graph, then even if Abélard is not part of the game, the qualitative analysis of reachability games is undecidable [18].

- If Éloïse has imperfect-information, then, even if the graph is finite and Abélard is not part of the game, almost-sure winning is undecidable for co-Büchi games [4].

In this work, we propose two alternative semantics that lead to decidable problems where the previous probabilistic approach fails. The main idea is to evaluate (for fixed strategies of Éloïse and Abélard) how "small" the set of resulting losing plays for Éloïse is.

Our first setting is based on *counting*. To evaluate how good a situation is for Éloïse (i.e., using some strategy φ_E against a strategy φ_A of Abélard), we simply count how many losing plays there are: the fewer, the better. Of special interest are those strategies for which, against any strategy of Abélard, the number of losing plays is at most countable. The idea of counting can be traced back to the work in References [6, 7] on automata with cardinality constraints. There is also work on the logical side with decidable results but that do not lead to efficient algorithms [5].

Our second setting is based on *topology*. To evaluate how good a situation is for Éloïse (i.e., using some strategy φ_E against a strategy φ_A of Abélard), we use a topological notion of "bigness"/ "smallness" given by the concept of large/meager set. The idea of using topology was considered previously in the context of *finite* Markov chains [36] and *finite* Markov decision processes [3].

The approach we follow in this article is to provide reductions to games that do not involve Nature. More precisely, a typical result will be to provide a transformation of a game involving Nature into a new game that no longer involves Nature and that is such that the algorithmic question considered on the original game reduces to another question on the new (two-player) game. In particular, when the latter is decidable it implies decidability of the original problem. To be as general as possible, we try to impose as few restrictions as possible on the underlying graph of the game (typically, we allow infinite graphs) and on the winning condition (many results are obtained for Borel conditions): this permits to obtain decidability results for a wide range of games.

The article starts with definitions of basic objects in Section 2, while Section 3 introduces the different settings we consider in this article. In Section 4, we focus on perfect-information games and we provide reductions for both the cardinality setting and the topological setting. Algorithmic consequences as well as consequences for automata on infinite trees are then discussed in Section 5. We then turn to the imperfect-information setting in Section 6 and discuss consequences of our results in Section 7. We summarize our results and propose some perspectives in Section 8.

2 PRELIMINARIES

We now introduce basic concepts that will be used throughout the article.

2.1 Sets

Let X be a set we denote by $Card(X)$ its cardinal. In this work, we will only need to consider[1] finite cardinals, $\aleph_0$ (the cardinality of the natural numbers) or $2^{\aleph_0}$ (the cardinality of the real numbers). A set is **countable** if its cardinal is smaller or equal to $\aleph_0$ (equivalently, the set is either finite or in bijection with the natural numbers).

If $S_1, \ldots, S_k$ are sets, then, we denote by $S = S_1 \uplus \cdots \uplus S_k$ the fact that they form a partition of S, i.e., $S = S_1 \cup \cdots \cup S_k$ and $S_i \cap S_j = \varnothing$ for every $i \neq j$.

Let S be a set, $\sim$ be an equivalence relation on S, and s be some element in S. Then, we denote by $[s]_{/\sim} = \{s' \mid s \sim s'\}$ the equivalence class of s for relation $\sim$ and by $S_{/\sim}$ the set of equivalence classes of $\sim$ on elements of S.

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[1] This is a consequence of the structure of the sets we consider (see Proposition 3.6) and has nothing to do with the continuum hypothesis.
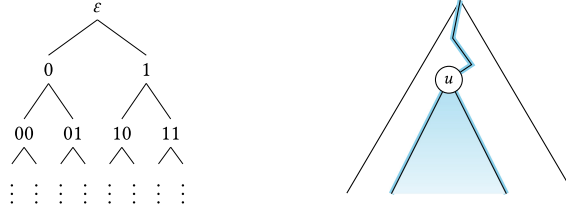
Fig. 1. On the left, first levels of the complete binary tree; on the right, the cone Cone(u).

2.2 Words

Let A be a (possibly infinite) set seen here as an **alphabet**. We denote by A^* the set of finite words over the alphabet A and by A^ω the set of infinite words over the alphabet A. If u is a word, then, we denote by $|u| \in \mathbb{N} \cup \{\omega\}$ its length. We denote by ε the empty word and we let $A^+ = A^* \backslash \{\varepsilon\}$.

If $u \in A^*$ and $v \in A^* \cup A^\omega$, then, we denote by $u \cdot v$ (or simply uv) the (possibly infinite) word obtained by concatenating u and v. A word $u \in A^*$ is a **prefix** of a word $w \in A^* \cup A^\omega$ if there exists some $v \in A^* \cup A^\omega$ such that $w = u \cdot v$, and we denote this situation by $u \sqsubseteq w$; moreover, if $u \neq w$, we say that u is a *strict* prefix (denoted by $u \sqsubset w$). A set $S \subseteq A^*$ is **prefix-closed** if for all $u \in S$ and $v \sqsubseteq u$ one has $v \in S$.

Let $(u_i)_{i \geq 0}$ be a sequence of finite words in A^* such that for all $i \geq 0$ one has $u_i \sqsubseteq u_{i+1}$ and for infinitely many $i \geq 0$ one has $u_i \sqsubset u_{i+1}$. We define its **limit** $u_\infty \in A^\omega$ as the unique infinite word such that for all $i \geq 0$, $u_i \sqsubset u_\infty$. Equivalently, $u_\infty = a_1 a_2 a_3 \cdots \in A^\omega$ where for all $k \geq 1$, a_k is the kth letter of any u_i such that $|u_i| \geq k$.

2.3 Trees

In this article, we consider various notions of trees that we introduce now (see Figure 1 for some illustrations). Let D be a (countable) set of directions; a **D-tree** (or simply a **tree** when D is clear) is a prefix-closed subset of D^*. A D-tree is *complete* if it equals D^*; it is *binary* if $Card(D) = 2$ (and in general one identifies D with $\{0, 1\}$).

For a given tree T, we refer to any element $u \in T$ as a **node**; if $T = \{0, 1\}^*$ is the complete binary tree, we refer to $u \cdot 0$ (respectively, $u \cdot 1$) as the left (respectively, right) son of u. The node ε is called the *root*.

In the sequel, we implicitly assume that the trees we consider do not contain leaves, i.e., for every node $u \in T$ there is some direction $d \in D$ such that $ud \in T$.

An (infinite) **branch** in a D-tree T is an infinite word $\pi \in D^\omega$ such that there is an increasing (for the prefix ordering) sequence of nodes $(u_i)_{i \geq 0}$ whose limit is π. A node u belongs to a branch π whenever $u \sqsubset \pi$. Branches in the complete D-tree exactly coincide with D^ω. For a node $u \in T$, the **cone** Cone$_T(u)$ is defined as the set of branches of T passing through u (i.e., Cone$_T(u) = \{\pi \mid \pi$ branch of T and $u \sqsubset \pi\}$).

Let A be a (countable) alphabet; an **A-labelled tree** t is a total function $t : Dom \to A$ where Dom is a tree. For a node $u \in Dom$, we call $t(u)$ the *label* of u; and for a branch $\pi = \pi_0 \pi_1 \cdots$ of the tree Dom, we call $t(\pi_0) t(\pi_0 \pi_1) t(\pi_0 \pi_1 \pi_2) \cdots \in A^\omega$ the label of π. For a node $u \in Dom$, we let $t[u]$ be the **subtree** rooted at u, i.e., $t[u] : Dom' \to A$ with $Dom' = \{v \mid u \cdot v \in Dom\}$ and $t[u](v) = t(uv)$. Finally, we call an **A-labelled D-tree** an A-labelled tree whose domain is a D-tree.

2.4 Graphs

A (directed) **graph** G is a pair (V, E) where V is a countable set of **vertices** and $E \subseteq V \times V$ is a set of **edges**. For a vertex v, we denote by $E(v)$ the set of its successors $\{v' \mid (v, v') \in E\}$ and in the

rest of the article (hence, this is implicit from now on), we only consider graphs that have no dead end, i.e., such that $E(v) \neq \emptyset$ for all v.

3 PERFECT-INFORMATION GAMES WITH NATURE: MAIN DEFINITIONS AND CONCEPTS

3.1 Definitions

In this article, we are interested in games involving two antagonistic players—Éloïse and Abélard—together with a third *uncontrollable and unpredictable* player called Nature. An *arena* is a tuple $\mathcal{G} = (G, V_E, V_A, V_N)$ where $G = (V, E)$ is a graph and $V = V_E \uplus V_A \uplus V_N$ is a partition of the vertices among the three players. We say that a vertex v is owned by Éloïse (respectively, by Abélard, respectively, by Nature) if $v \in V_E$ (respectively, $v \in V_A$, respectively, $v \in V_N$).

Éloïse, Abélard, and Nature play in $\mathcal{G}$ by moving a pebble along edges. A *play* from an initial vertex v_0 proceeds as follows: The player owning v_0 moves the pebble to a vertex $v_1 \in E(v_0)$. Then, the player owning v_1 chooses a successor $v_2 \in E(v_1)$ and so on forever. As we assumed that there is no dead end, a play is an infinite word $v_0 v_1 v_2 \cdots \in V^\omega$ such that for all $i \geq 0$, one has $v_{i+1} \in E(v_i)$. A *partial play* is a prefix of a play, i.e., it is a finite word $v_0 v_1 \cdots v_\ell \in V^*$ such that for all $0 \leq i < \ell$, one has $v_{i+1} \in E(v_i)$.

Example 3.1. Consider the arena depicted in Figure 2, where we adopt the following convention: a vertex owned by Éloïse (respectively, Abélard, respectively, Nature) is depicted by a circle (respectively, a square, respectively, a diamond). The underlying graph is $G = (V, E)$ where $V = \{E_i, A_i, N_i \mid i \geq 0\}$ and

$$E = \{(A_i, A_{i+1}), (A_i, N_i), (N_{i+1}, N_i), (N_i, E_i), (N_i, N_i), (E_{i+1}, E_i), (E_i, E_i) \mid i \geq 0\}.$$

Note that this graph is an example of a pushdown graph [30] (i.e., it can be presented as the transition graph of a pushdown automaton).

The partition of the vertices among the players is given by $V_E = \{E_i \mid i \geq 0\}$, $V_A = \{A_i \mid i \geq 0\}$ and $V_N = \{N_i \mid i \geq 0\}$.

The following sequence λ is an example of a play in that arena:

$$\lambda = A_0 A_1 A_2 N_2 N_2 N_2 N_1 E_1 E_1 E_1 E_0 E_0 E_0 \cdots .$$

A *strategy* for Éloïse is a function $\varphi_E : V^* V_E \to V$ assigning, to every partial play ending in some vertex $v \in V_E$, a vertex $v' \in E(v)$. Strategies for Abélard are defined likewise, i.e., as functions $\varphi_A : V^* V_A \to V$. A strategy φ is *positional* if for any two partial plays π and π' ending in the same vertex, we have $\varphi(\pi) = \varphi(\pi')$. When it is clear from the context that a strategy of Éloïse (respectively, Abélard) is positional, we will define it as a function from V_E (respectively, V_A) to V. A strategy for Éloïse φ_E is a *finite-memory* strategy if it can be implemented by a finite-memory machine that sequentially reads the vertices visited during the play; formally, we require that there is a finite set M, an element $m_0 \in M$, a function $\text{Up} : M \times V \to M$ and a function $\text{Move} : M \times V \to V$ such that $\varphi_E(v_0 \cdots v_i) = \text{Move}(m_i, v_i)$ for every partial play $v_0 \cdots v_i$ ending in a vertex $v_i \in V_E$ where we inductively define m_i by letting $m_{k+1} = \text{Up}(m_k, v_{k+1})$ for every $k \geq 0$. Finite-memory strategies for Abélard are defined likewise. The size of the memory used by such a strategy is defined as the size of M. Note that positional strategies correspond to the special case of finite-memory strategy where $M = \{m_0\}$ is a singleton.

In a given play $\lambda = v_0 v_1 \ldots$, we say that Éloïse (respectively, Abélard) *respects a strategy* φ if whenever $v_i \in V_E$ (respectively, $v_i \in V_A$) one has $v_{i+1} = \varphi(v_0 \cdots v_i)$.

With an initial vertex v_0 and a pair of strategies (φ_E, φ_A), we associate the set $\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A}$ of possible plays where each player respects his strategy, i.e., $\lambda \in \text{Outcomes}_{v_0}^{\varphi_E, \varphi_A}$ if and only if λ is
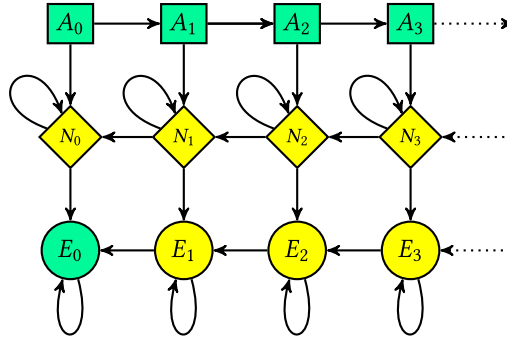
Fig. 2. Example of an infinite arena.

a play starting from v_0 where Éloïse respects φ_E and Abélard respects φ_A. In the classical setting where Nature is not involved (i.e., $V_N = \varnothing$), when the strategies of Éloïse and Abélard are fixed there is only one possible play, i.e., Outcomes$_{v_0}^{\varphi_E, \varphi_A}$ is a singleton. The presence of Nature induces a branching structure: Indeed, Outcomes$_{v_0}^{\varphi_E, \varphi_A}$ is the set of branches of the V-tree $T_{v_0}^{\varphi_E, \varphi_A}$ consisting of those partial plays where each player respects his strategy.[2]

Example 3.2. Consider again the arena from Example 3.1 (depicted in Figure 2) and define the following positional strategies φ_E and φ_A for Éloïse and Abélard:

- For every $i \geqslant 0$, $\varphi_E(E_i) = E_{i-1}$ if $i > 0$ and $\varphi_E(E_0) = E_0$.
- For every $i \geqslant 0$, $\varphi_A(A_i) = A_{i+1}$ if $i < 2$ and $\varphi_A(A_i) = N_i$ if $i \geqslant 2$.

Then the set Outcomes$_{A_0}^{\varphi_E, \varphi_A}$ consists of the plays described by the following ω-regular expression:

$$A_0 A_1 A_2 N_2 (N_2^\omega + (N_2^+ (E_2 E_1 E_0^\omega + N_1^\omega + N_1^+ (E_1 E_0^\omega + (N_0^\omega + N_0^+ E_0^\omega)))))),$$

i.e., a play in Outcomes$_{A_0}^{\varphi_E, \varphi_A}$ starts by moving the token to N_2 and can either get trapped forever in some N_i with $i \geqslant 2$ or eventually reach some E_i with $i \geqslant 2$ from where it goes to E_0 and stays there forever.

A **winning condition** is a subset $\Omega \subseteq V^\omega$ and a **game** is a tuple $\mathbb{G} = (G, \Omega, v_0)$ consisting of an arena, a winning condition, and an initial vertex v_0. In this article, we only consider winning conditions that are **Borel** sets, i.e., that belong to the σ-algebra defined from the basic open sets of the form KV^ω with $K \subseteq V^*$.

A well-known popular example of Borel winning conditions are the **parity** conditions. Let Col : $V \to C$ be a colouring function assigning to every vertex a colour in a *finite* set $C \subset \mathbb{N}$. Then one defines Ω_{Col} to be the set of all plays where the smallest infinitely often repeated colour is even, i.e.,

$$\Omega_{\mathrm{Col}} = \{v_0 v_1 v_2 \cdots \in V^\omega \mid \liminf(\mathrm{Col}(v_i))_i \text{ is even}\}.$$

Büchi (respectively, **co-Büchi**) conditions are those parity conditions where $C = \{0, 1\}$ (respectively, $C = \{1, 2\}$); it requires for a play to be winning to go infinitely (respectively, only finitely) often through vertices coloured by 0 (respectively, 1) and in general it is defined by a set of final (respectively, forbidden) vertices: those of colour 0 (respectively, 1).

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[2]We make here a slight abuse: Indeed, as we take as the root the trivial partial play v_0 and not ε, it breaks the definition of a tree as being a prefix closed set.
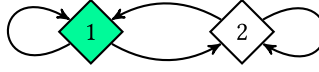
Fig. 3. A non-determined Büchi game where Nature plays alone.

A more general class of winning conditions is so-called ω-regular conditions. Such a condition $\Omega_{\tau,L}$ is defined thanks to a mapping $\tau : V \to A$ where A is a *finite* alphabet, and an ω-regular language L over the alphabet A (see, e.g., Reference [33] for definitions of ω-regular languages). Then one simply lets:

$$\Omega_{\tau,L} = \{v_0 v_1 v_2 \cdots \in V^\omega \mid \tau(v_0)\tau(v_1)\tau(v_2)\cdots \in L\}.$$

A play λ from v_0 is **won** by Éloïse if and only if $\lambda \in \Omega$; otherwise, λ is won by Abélard.

Example 3.3. Consider again the arena from Examples 3.1 and 3.2 depicted in Figure 2. Consider the Büchi condition defined by letting the final vertices (depicted in green in the picture) be the set $\{A_i \mid i \geqslant 0\} \cup \{E_0\}$. Then the play $\lambda = A_0 A_1 A_2 N_2 N_2 N_2 N_1 E_1 E_1 E_1 E_0 E_0 E_0 \cdots$ is won by Éloïse.

A strategy φ_E is (surely) **winning** for Éloïse in $\mathbb{G}$ if for any strategy φ_A of Abélard one has Outcomes$_{v_0}^{\varphi_E, \varphi_A} \subseteq \Omega$, i.e., she wins regardless of the choices of Abélard and Nature. Symmetrically, a strategy φ_A is (surely) **winning** for Abélard in $\mathbb{G}$ if for any strategy φ_E of Éloïse one has Outcomes$_{v_0}^{\varphi_E, \varphi_A} \cap \Omega = \varnothing$.

As the winning condition is Borel, it is a well-known result—Martin's determinacy Theorem [28]—that whenever $V_N = \varnothing$ the game is *determined*, i.e., either Éloïse or Abélard has a winning strategy. Due to Nature, it is easily seen that in many situations neither Éloïse nor Abélard has a winning strategy. For instance, consider the Büchi game depicted in Figure 3 where all vertices belong to Nature and where the final vertex is 1. The strategies for Éloïse and Abélard are both the trivial function with empty domain $\varphi_\varnothing$ and the set Outcomes$_1^{\varphi_\varnothing, \varphi_\varnothing} = 1 \cdot \{1,2\}^\omega$ contains plays that are winning for Éloïse (e.g., 1^ω) as well as plays that are winning for Abélard (e.g., 12^ω).

Another example of this situation is given by the Büchi game from Example 3.3 where it is easily observed that neither Éloïse nor Abélard has a winning strategy.

One way of solving this situation, i.e., to still evaluate how good a strategy/game is for Éloïse, is to equip Nature with a probabilistic semantics, leading to the well-known concept of *stochastic games*, which we briefly recall in the next section—the main focus of the present article being to propose alternative semantics (the cardinality one and the topological one) that lead to decidable problems where the previous probabilistic approach fails.

3.2 The Probabilistic Setting

We now briefly recall the concept of **stochastic games** [15, 34] (see also Reference [13] for an overview of the field and formal details on the objects below), which consists of equipping the games with Nature with a probabilistic semantics. In a nutshell, any vertex in V_N comes with a probability distribution over its neighbours and then, for a fixed tuple $(v_0, \varphi_E, \varphi_A)$, these probabilities are used to define a σ-algebra (taking as cones the sets of plays sharing a common finite prefix) and a probability measure $\mu_{v_0}^{\varphi_E, \varphi_A}$ on Outcomes$_{v_0}^{\varphi_E, \varphi_A}$. In particular, this permits to associate with any pair (φ_E, φ_A) a real in $[0,1]$ defined as the probability of the (mesurable) subset Outcomes$_{v_0}^{\varphi_E, \varphi_A} \cap \Omega$ in the previous space. Of special interest is the **value** of a given strategy φ_E of Éloïse, which estimates how good φ_E is for her:

$$\text{Val}_{\mathbb{G}}(\varphi_E) = \inf\left\{\mu_{v_0}^{\varphi_E, \varphi_A}(\Omega)\big|\varphi_A \text{ Abélard strategy}\right\}.$$

Finally, the value of the game is defined by taking the supremum of the values of Éloïse's strategies:

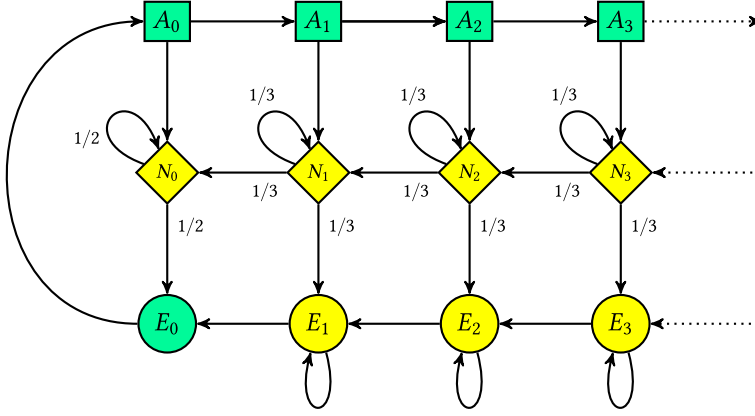$$\text{Val}(\mathbb{G}) = \sup\{\text{Val}_{\mathbb{G}}(\varphi_E) \mid \varphi_E \text{ Éloïse strategy}\}.$$

Fig. 4. Example of a stochastic game.

A strategy φ_E is optimal when $\mathrm{Val}_{\mathbb{G}}(\varphi_E) = \mathrm{Val}(\mathbb{G})$ and it is almost surely winning when $\mathrm{Val}_{\mathbb{G}}(\varphi_E) = 1$.

A deep result due to Martin [29] establishes the following determinacy result (φ_E and φ_A range over strategies of Éloïse and Abélard, respectively):

$$\sup_{\varphi_E} \inf_{\varphi_A} \left\{ \mu_{v_0}^{\varphi_E, \varphi_A}(\Omega) \right\} = 1 - \inf_{\varphi_A} \sup_{\varphi_E} \left\{ \mu_{v_0}^{\varphi_E, \varphi_A}(\Omega) \right\}.$$

Example 3.4. Consider the arena depicted in Figure 4. It is essentially a variant of the game from Examples 3.1–3.3 where we replaced the self-loop on E_0 by an edge from E_0 to A_0 and where we associate, with every node N_i owned by Nature, the following probability distribution d_i on its neighbours:

- $d_0(E_0) = d_0(N_0) = 1/2$;
- if $i > 0$, $d_i(E_i) = d_i(N_i) = d_i(N_{i-1}) = 1/3$.

Then it is easily seen that the positional strategy φ_E for Éloïse defined by letting $\varphi_E(E_i) = E_{i-1}$ if $i > 0$ and $\varphi_E(E_0) = A_0$ is almost surely winning. Indeed, for a fixed strategy φ_A the plays λ in $\mathrm{Outcomes}_{A_0}^{\varphi_E, \varphi_A}$ that are losing for Éloïse are included in the countable union (over all integer i and all integer k) of the finite sets of plays that get trapped in N_i forever after entering in it after having previously visited exactly k-times vertex E_0: hence, this set has measure 0.

3.3 The Cardinality Setting

We now propose a change of perspective based on counting: To evaluate how good a situation is for Éloïse (i.e., using some strategy φ_E against a strategy φ_A of Abélard), we simply count how many losing plays there are; the fewer there are, the better the situation is.

As a preliminary illustration, we revisit Example 3.4.

Example 3.5. Consider again the stochastic game from Example 3.4 (depicted in Figure 4) together with the positional strategy φ_E of Éloïse. Now forget about the stochastic view of Nature and think of it as being simply non-deterministic. Let φ_A be an arbitrary strategy of Abélard: as remarked in Example 3.4, the set of losing plays in $\mathrm{Outcomes}_{A_0}^{\varphi_E, \varphi_A}$ is countable. However, the set $\mathrm{Outcomes}_{A_0}^{\varphi_E, \varphi_A}$ is easily seen to be uncountable. Therefore, one can consider that the set of losing plays for Éloïse when using the strategy φ_E is somehow negligible with respect to the set of all plays.
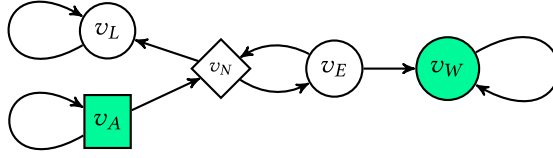
Fig. 5. Arena of Remark 3.9.

We first note the following proposition [1] that characterises the cardinals of the Borel subsets of an arbitrary set $\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A}$.

PROPOSITION 3.6. *For any arena, any initial vertex, any pair of strategies* (φ_E, φ_A), *and any Borel subset* $S \subseteq \text{Outcomes}_{v_0}^{\varphi_E, \varphi_A}$, *one has* $\text{Card}(S) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

PROOF. As $\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A}$ is the set of branches of a tree whose set of directions is countable (see, for instance, Theorem 3.11 in Reference [24]), it is a Polish space with the standard basis $\{\text{Cone}_{T_{v_0}^{\varphi_E, \varphi_A}}(v) \mid v \in T_{v_0}^{\varphi_E, \varphi_A}\}$. By Reference [24, Theorem 13.6], any Borel subset S is either countable or has cardinality $2^{\aleph_0}$. □

We define the cardinality leaking of an Éloïse's strategy as a measure of its quality.

Definition 3.7 (Cardinality Leaking of a Strategy). Let $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ be a game and let φ_E be a strategy of Éloïse. The **cardinality leaking** of φ_E is the cardinal $\text{CardLeak}(\varphi_E)$ defined by

$$\text{CardLeak}(\varphi_E) = \sup\{\text{Card}(\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega) \mid \varphi_A \text{ strategy of Abélard}\}.$$

Proposition 3.6 implies that $\text{CardLeak}(\varphi_E) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

The goal of Éloïse is to minimise the number of losing plays, hence, leading to the following concept:

Definition 3.8 (Leaking Value of a Game). Let $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ be a game. The **leaking value** of $\mathbb{G}$ is the cardinal $\text{LeakVal}(\mathbb{G})$ defined by

$$\text{LeakVal}(\mathbb{G}) = \inf\{\text{CardLeak}(\varphi_E) \mid \varphi_E \text{ strategy of Éloïse}\}.$$

Thanks to Proposition 3.6, it follows that $\text{LeakVal}(\mathbb{G}) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

In the reminder of this article, we consider that a strategy is good from the cardinality point of view if its cardinality leaking is countable. From a modelisation point of view, we agree that this notion can be questionable. In particular, it only makes sense if for all strategy φ_E and φ_A of Éloïse and Abélard, respectively, the set of outcomes is uncountable. A sufficient condition to ensure this last property is that all vertices of Nature have at least two successors and that every play visits infinitely many vertices of Nature. A stronger requirement that we will also consider is to look for strategy with a fixed finite cardinality leaking.

Remark 3.9. One can wonder whether the sup in the definition of $\text{CardLeak}(\varphi_E)$ can be replaced by a max, i.e., whether, against any fixed strategy of Éloïse, Abélard has always an "optimal" counter strategy.

Actually, this is not possible as exemplified by the Büchi game depicted in Figure 5 where coloured vertices (v_A and v_W) are the final ones—with v_A as initial vertex.

Consider the strategy φ_E of Éloïse consisting in a partial play ending in v_E (in v_L and v_W Éloïse has a single choice) to go to v_N if there are less occurrences of v_E than of v_A in the partial play and to go to v_W otherwise. Clearly, for any strategy φ_A of Abélard, $\text{Card}(\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega)$ is
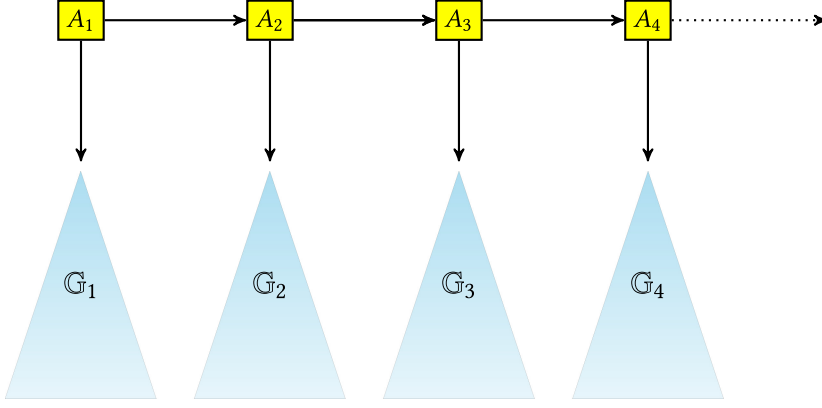
Fig. 6. Example of a reachability game played between Abélard and Nature. For all $n \geqslant 0$, the game $\mathbb{G}_n$ is only played by Nature and is such that exactly n plays are losing for Éloïse.

finite. However, for any $k \geqslant 0$, Abélard can ensure that there are k plays lost by Éloïse by looping $(k-1)$ times on the vertex v_A before going to v_N.

As cardinals are well ordered, Éloïse always has an "optimal" strategy for the leaking value criterion (i.e., we can replace the inf by a min in Definition 3.8).

PROPOSITION 3.10. *Let $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ be a game. There is a strategy φ_E of Éloïse such that $LeakVal(\mathbb{G}) = CardLeak(\varphi_E)$.*

Remark 3.11. It is natural to wonder if, for a strategy φ_E of Éloïse such that $CardLeak(\varphi_E) = LeakVal(\mathbb{G})$, there exists a strategy φ_A of Abélard that reaches $LeakVal(\mathbb{G})$, i.e., such that $Card(\text{Outcomes}_{v_0}^{\varphi_E,\varphi_A}\backslash\Omega) = LeakVal(\mathbb{G})$. By considering, for instance, the reachability game $\mathbb{G}$ depicted in Figure 6, we will see that such a strategy for Abélard may not necessarily exist. This game is played between Abélard and Nature and, hence, the empty strategy φ_E for Éloïse is optimal, i.e., $CardLeak(\varphi_E) = LeakVal(\mathbb{G})$. In this game, a strategy for Abélard is entirely characterised by the first index n (if it exists) such that Abélard moves from A_n to the game $\mathbb{G}_n$ (that is such that exactly n plays are losing for Éloïse in it). Hence, for all $n \geqslant 1$, we denote by φ_A^n the strategy of Abélard consisting in moving from A_i to A_{i+1} for all $i < n$ and going to $\mathbb{G}_n$ on A_n and by φ_A^∞ the strategy in which Abélard always moves from A_i to A_{i+1}. As there are exactly n losing plays for Éloïse in $\mathbb{G}_n$, we have $Card(\text{Outcomes}_{v_0}^{\varphi_E,\varphi_A^n}\backslash\Omega) = n$ and $Card(\text{Outcomes}_{v_0}^{\varphi_E,\varphi_A^\infty}\backslash\Omega) = 1$. It follows that the $LeakVal(\mathbb{G}) = \aleph_0$, which cannot be reached by any strategy of Abélard.

3.4 The Topological Setting

A notion of topological "bigness" and "smallness" is given by *large* and *meager* sets, respectively (see References [21, 36] for a survey of the notion). From the modelisation point of view, the intuition is that meager sets (the complements of large sets) are somehow negligible. In Reference [36], the authors give weight to this idea by showing that, for regular trees (i.e., those trees obtained by unfolding finite graphs), the set of branches satisfying an ω-regular condition is large if and only if it has probability 1 (in the sense of Section 3.2). However, they also show that, in general, even for the Büchi condition and when the tree is the unfolding of a pushdown graph, this is no longer true (see Reference [36, p. 27]).

Let t be a D-tree for some set D of directions. Then its set of branches can be seen as a topological space by taking as basic open sets the set of cones. A set of branches $B \subseteq D^\omega$ is **nowhere dense** if

for all node $u \in t$, there exists another node $v \in t$ such that $u \sqsubseteq v$ and such that v does not belong to any branch in B. A set of branches is ***meager*** if it is the countable union of nowhere dense sets. Finally, it is ***large*** if it is the complement of a meager set.

A natural topological criterion to consider that a strategy φ_E for Éloïse is good against a strategy φ_A of Abélard is that the set of plays lost by Éloïse is meager in the tree $T_{v_0}^{\varphi_E, \varphi_A}$.

Definition 3.12 (Topologically-good Strategies). Let $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ be a game and let φ_E be a strategy of Éloïse. We say that φ_E is ***topologically-good*** if and only if for any strategy φ_A of Abélard the set $\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \setminus \Omega$ of losing plays for Éloïse is meager in the tree $T_{v_0}^{\varphi_E, \varphi_A}$; or, equivalently, the set $\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \cap \Omega$ of plays won by Éloïse is large.

Example 3.13. Consider again the stochastic game from Example 3.4 (depicted in Figure 4) together with the positional strategy φ_E of Éloïse (also presented in Example 3.4). Now forget about the stochastic view of Nature and think of it as being simply non-deterministic. We claim that the strategy φ_E is topologically-good.

Indeed, let φ_A be an arbitrary strategy of Abélard: As already remarked in Example 3.4, the set of plays in $\text{Outcomes}_{A_0}^{\varphi_E, \varphi_A} \setminus \Omega$ is included in the countable union over all $i, k \geqslant 0$ of the plays $\Lambda_{i,k}$ where a play belongs to $\Lambda_{i,k}$ if it gets trapped forever in node N_i from round k, i.e., $\Lambda_{i,k} = V^{k-1} N_i^{\omega}$. Now, we remark that $\text{Outcomes}_{A_0}^{\varphi_E, \varphi_A} \cap \Lambda_{i,k}$ is nowhere dense in the tree $T_{v_0}^{\varphi_E, \varphi_A}$, as any partial play λ can be extended to another partial play λ' so any extension of the latter as an infinite play is not trapped forever in N_i from round k (it can be trapped forever, but later), i.e., for any node u_λ in $T_{v_0}^{\varphi_E, \varphi_A}$ there exists another node $u_{\lambda'}$ such that $u_{\lambda'} \sqsubseteq u_\lambda$ and $u_{\lambda'}$ does not belong to any branch in $\Lambda_{i,k}$. Hence, it means that the set $\text{Outcomes}_{A_0}^{\varphi_E, \varphi_A} \setminus \Omega$ is a countable union of nowhere dense sets, equivalently it is meager. Therefore, we conclude that φ_E is topologically-good.

We now recall a useful notion, Banach-Mazur games, to reason on meager sets. Banach-Mazur theorem gives a game characterisation of large and meager sets of branches (see, for instance, References [21, 24, 31]). The ***Banach-Mazur game*** on a tree t is a two-player game where Abélard and Éloïse choose alternatively a node in the tree, forming a branch: Abélard chooses first a node and then Éloïse chooses a descendant of the previous node and Abélard chooses a descendant of the previous node and so on forever. In this game, it is *always Abélard that starts* a play.

Formally, a *play* is an infinite sequence $u_1, u_2, \ldots$ of words in D^+ such that for all i one has $u_1 u_2 \cdots u_i \in t$, and the branch associated with this play is $u_1 u_2 \cdots$. A *strategy* for Éloïse is a mapping $\varphi : (D^+)^+ \to D^+$ that takes as input a finite sequence of words and outputs a word. A play $u_1, u_2, \ldots$ *respects* φ if for all $i \geqslant 1$, $u_{2i} = \varphi(u_1, \ldots, u_{2i-1})$. We define $Outcomes(\varphi)$ as the set of plays that respect φ and $\mathcal{B}(\varphi)$ as the set of branches associated with the plays in $Outcomes(\varphi)$.

The Banach-Mazur theorem[3] (see, e.g., Reference [21, Theorem 4]) states that a set of branches B is large if and only if there exists a strategy φ for Éloïse such that $\mathcal{B}(\varphi) \subseteq B$. Hence, if one thinks of B as a winning condition for Éloïse (i.e., she wins a play if and only if it belongs to B), it means that those sets B for which she has a winning strategy are exactly the large ones.

Furthermore, a folk result (see, e.g., Reference [21, Theorem 9]) about Banach-Mazur games states that when B is Borel,[4] one can look only at "simple" strategies, defined as follows: A ***decomposition-invariant strategy*** is a mapping $f : t \to D^+$ and we associate with f the strategy φ_f defined by $\varphi_f(u_1, \ldots, u_k) = f(u_1 \cdots u_k)$. Finally, we define $Outcomes(f) = Outcomes(\varphi_f)$

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[3]In Reference [21] the players of the Banach-Mazur game are called 0 and 1 and Player 0 corresponds to Abélard while player 1 corresponds to Éloïse. Hence, when using a statement from Reference [21] for our setting, one has to keep this in mind as well as the fact that one must replace the winning condition by its complement (hence, replacing "meager" by "large").

[4]This statement holds as soon as the Banach-Mazur games are determined and, hence, in particular for Borel sets.

and $\mathcal{B}(f) = \mathcal{B}(\varphi_f)$. The folk result states that for any Borel set of branches B, there exists a strategy φ such that $Outcomes(\varphi) \subseteq B$ if and only if there exists a decomposition-invariant strategy f such that $\mathcal{B}(f) \subseteq B$.

Example 3.14. Consider the game in Example 3.13 (depicted in Figure 4), fix again the same strategy φ_E for Éloïse and define as a strategy for Abélard a strategy where when the token is in some vertex in V_A Abélard moves it (by successive moves) to A_k, where k denotes the number of visits to vertex A_0 from the beginning of the play and from A_k moves it down to N_k.

As φ_E is topologically-winning, it means that the set $B = \text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \setminus \Omega$ is large in $T_{v_0}^{\varphi_E, \varphi_A}$. We illustrate the concept of Banach-Mazur game by defining a winning decomposition-invariant strategy f for Éloïse in the Banach-Mazur game on $T_{v_0}^{\varphi_E, \varphi_A}$. For that, it suffices to let $f(\lambda) = \lambda'$ where λ' is some arbitrary partial play extending λ by a path ending in node E_0 (such a path always exists). Then, it is straightforward to verify that $\mathcal{B}(f) \subseteq B$.

3.5   The Tree-language Setting

We now propose a last setting, which in some cases permits to capture the three previously defined ones. This setting only makes sense when the arena $\mathcal{G} = (G, V_E, V_A, V_N)$ comes with a mapping $\text{Col} : V \to C$, where $V = V_E \uplus V_A \uplus V_N$ denotes the set of vertices in the arena and C is a *finite* set. Fix a subset $\mathcal{L}$ of C-labeled V-trees.

For a given initial vertex v_0 and a pair (φ_E, φ_A) of strategies for Éloïse and Abélard, we can map the V-tree $T_{v_0}^{\varphi_E, \varphi_A}$ to a C-labelled V-tree where each node $v_0 v_1 \cdots v_k \in T_{v_0}^{\varphi_E, \varphi_A}$ is labelled by $\text{Col}(v_k)$. In the sequel, we overload notation $T_{v_0}^{\varphi_E, \varphi_A}$ to designate this tree.

Now, we say that a strategy φ_E is **$\mathcal{L}$-good** if and only if for every strategy φ_A of Abélard the C-labelled tree $T_{v_0}^{\varphi_E, \varphi_A}$ belongs to $\mathcal{L}$.

Let K be a subset of C^ω and let Ω_K be the winning condition defined by letting

$$\Omega_K = \{v_0 v_1 v_2 \cdots \mid \text{Col}(v_1)\text{Col}(v_2)\text{Col}(v_3) \in K\}.$$

Then the following trivially holds:

LEMMA 3.15. *Let C be a finite set and let $K \subseteq C^\omega$. Let $\mathcal{G} = (G, V_E, V_A, V_N)$ be an arena with vertices V and let $\text{Col} : V \to C$. Let $v_0 \in V$ be some initial vertex and let $\mathbb{G}$ be the game $\mathbb{G} = (\mathcal{G}, v_0, \Omega_K)$. Then the following holds:*

- (1) *Éloïse almost surely wins $\mathbb{G}$ if and only if she has an $\mathcal{L}_{\text{Stoc}}$-good strategy when playing in arena $\mathcal{G}$ starting from v_0 where $\mathcal{L}_{\text{Stoc}}$ is the set of C-labelled V-trees such that almost all branches[5] are labelled by a sequence in K.*
- (2) *The leaking value of $\mathbb{G}$ is countable (respectively, smaller than some fixed k) if and only if Éloïse has an $\mathcal{L}_{\text{Card}}$-good strategy when playing in arena $\mathcal{G}$ starting from v_0, where $\mathcal{L}_{\text{Card}}$ is the set of C-labelled V-trees such that all branches but countably many (respectively, but k) are labelled by a sequence in K.*
- (3) *Éloïse has a topologically-good strategy in $\mathbb{G}$ if and only she has an $\mathcal{L}_{\text{Topo}}$-good strategy when playing in arena $\mathcal{G}$ starting from v_0, where $\mathcal{L}_{\text{Topo}}$ is the set of C-labelled V-trees such that the subset of branches labelled by a sequence in K is large.*

Let C be a finite set. Then a C-labelled tree can be seen as a relational structure (see Reference [27] for basic concepts on relational structures) whose universe is the set of nodes of the tree

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[5]More formally, the set of branches that is labelled by a sequence in K has measure 1 for the Lebesgue measure obtained from the Carathéodory extension theorem when defining the measure of a cone, i.e., a set of branches sharing a common finite prefix $v_0 \cdots v_k$ as the product $\prod_{0 \leqslant i < k \mid v_i \in V_N} d_{v_i}(v_{i+1})$, where d_{v_i} denotes the probability distribution over the neighbours of a vertex $v_i \in V_N$.

and whose relations consist of a unary predicate for every element c in C (which holds in every node labelled by c) and a binary predicate for the parent/son relation.

Let $G = (V, E)$ be a graph, C be a finite set, $\text{Col} : V \rightarrow C$ be a mapping, $\mathcal{G} = (G, V_{\text{E}}, V_{\text{A}}, V_{\text{N}})$ be an arena, and v_0 be an initial vertex. We define the **unfolding** of $\mathcal{G}$ from v_0 as the C-labelled V-tree whose set of nodes is the set

$$\{v_1 \cdots v_k \mid v_0 v_1 \cdots v_k \text{ is a partial play in arena } \mathcal{G}\}$$

and where the root ε is labelled by $\text{Col}(v_0)$ and any other node $v_1 \cdots v_k$ is labelled by $\text{Col}(v_k)$. We see it as a relational structure as explained above with three extra unary predicate (on for each player) p_{E}, p_{A}, and p_{N} such that p_{E} (respectively, p_{A}, respectively, p_{N}) holds in a node u if and only if the last vertex of $v_0 \cdot u$ belongs to V_{E} (respectively, V_{A}, respectively, V_{N}).

One can wonder whether the existence of $\mathcal{L}$-good strategies can be decided for special classes of languages $\mathcal{L}$. The most natural one are those definable in monadic second order logic (MSO). As this is the only place in this article where we make use of logic, we refer the reader to Reference [35] for formal definitions and classical results regarding MSO logic. The following result was remarked by Paweł Parys [32] and it permits to derive decidability for several classes of arenas (see Corollary 3.18 below).

THEOREM 3.16 ([32]). *Let $G = (V, E)$ be a graph, let C be a finite set, let $\text{Col} : V \rightarrow C$ be a mapping, let $\mathcal{G} = (G, V_{\text{E}}, V_{\text{A}}, V_{\text{N}})$ be an arena, and let v_0 be an initial vertex. Let $\mathcal{L}$ be an MSO-definable set of C-labelled V-trees. Then there exists an MSO formula $\Phi_{\mathcal{L}-\text{good}}$ such that $\Phi_{\mathcal{L}-\text{good}}$ holds on the unfolding of $\mathcal{G}$ from v_0 if and only if Éloïse has an $\mathcal{L}$-good strategy from v_0.*

PROOF. Let $\Phi_{\mathcal{L}}$ be an MSO formula defining $\mathcal{L}$ (i.e., it holds in a tree if and only if the tree belongs to $\mathcal{L}$). Call $T_{\mathcal{G}}$ the unfolding of the arena $\mathcal{G}$ from v_0. The formula $\Phi_{\mathcal{L}-\text{good}}$ existentially quantifies a strategy for Éloïse, then universally quantifies a strategy of Abélard and then relativises the formula $\Phi_{\mathcal{L}}$ to the subtree induced by the respective strategies.

More formally, to quantify over a strategy—say, for Éloïse—one quantifies a set of nodes X such that for every node u owned by Éloïse (i.e., such that $p_{\text{E}}(u)$ holds) exactly one son of u belongs to X (it corresponds to the image by the strategy of the partial play $v_0 \cdot u$ associated with the node u): call $\text{Valid}_{\text{E}}(X)$ an MSO-formula checking that a set of nodes X satisfies the previous requirement. Symmetrically one defines a formula $\text{Valid}_{\text{A}}(Y)$ to check that a set Y correctly encodes a strategy of Abélard.

Next, we define $\text{Reach}(X, Y, Z)$ as a formula that holds if and only if Z is the set of vertices reachable from the root by following the strategies encoded by X and Y. The formula $\text{Reach}(X, Y, Z)$ simply states that Z is the smallest set that contains the root, and such that for every node u in Z if it satisfies p_{E} (respectively, p_{A}), then there is exactly one son of u in Z and it also belongs to X (respectively, Y), and if it satisfies p_{N}, then all its successors belong to Z. Hence, Z consists of all nodes in $T_{v_0}^{\varphi_X, \varphi_Y}$ (where φ_X and φ_Y are the strategies associated with X and Y, respectively).

Now, let $\Phi_{\mathcal{L}}^{\text{rel}}(Z)$ be the formula obtained from $\Phi_{\mathcal{L}}$ by relativising to Z, i.e., by guarding every quantification to nodes in Z, i.e., every $\exists x \, \Psi(x)$ in $\Phi_{\mathcal{L}}$ is replaced by $\exists x \, (x \in Z \wedge \Psi(x))$ and every $\exists X \, \Psi(X)$ in $\Phi_{\mathcal{L}}$ is replaced by $\exists X \, (X \subseteq Z \wedge \Psi(X))$.

Then, the formula $\Phi_{\mathcal{L}-\text{good}}$ is simply defined as

$$\Phi_{\mathcal{L}-\text{good}} = \exists X [\text{Valid}_{\text{E}}(X) \wedge \forall Y \forall Z \, (\text{Valid}_{\text{A}}(Y) \wedge \text{Reach}(X, Y, Z)) \Rightarrow \Phi_{\mathcal{L}}^{\text{rel}}(Z)]. \qquad \square$$

Thanks to Courcelle-Walukiewicz theorem [16] stating that every MSO-definable property on the unfolding of a structure is an MSO-definable property on the structure itself, we can lift Theorem 3.16, where we see an arena as a relational structure with predicates for the edge

relations, the image of the Col function, three predicates reflecting whether a vertex belongs to V_E, V_A, or V_N, and a last predicate to distinguish the initial vertex v_0.

COROLLARY 3.17. *Let $G = (V, E)$ be a graph, let C be a finite set, let $\mathrm{Col} : V \to C$ be a mapping, let $\mathcal{G} = (G, V_E, V_A, V_N)$ be an arena, and let v_0 be an initial vertex. Let $\mathcal{L}$ be an MSO-definable set of C-labelled V-trees. Then there exists an MSO formula $\Phi_{\mathcal{L}-\mathrm{good}}$ such that $\Phi_{\mathcal{L}-\mathrm{good}}$ holds on $\mathcal{G}$ if and only if Éloïse has an $\mathcal{L}$-good strategy from v_0.*

One can now wonder in which cases Corollary 3.17 can be combined with Lemma 3.15. A natural candidate for the criterion $K \subseteq C^\omega$ on branches is the parity condition[6] (in fact, it is the only reasonable option for $\mathcal{L}$ sets from Lemma 3.15 to be MSO-definable). We also restrict here to arenas with finite out-degree.

First note that it is known from Reference [10, Theorem 21] that the language of $\{a, b\}$-binary trees such that almost every branch contains a node label by a is not MSO-definable.[7] Hence, it follows that one can design a reachability game on a finite graph such that the associated language $\mathcal{L}_{\mathrm{Stoc}}$ by Lemma 3.15 point (1) is not MSO-definable and, therefore, for that game Corollary 3.17 is useless to decide existence of an almost-surely winning strategy. For the cardinality and the topological settings, the situation is much better.

COROLLARY 3.18. *Let $\mathbb{G} = (\mathcal{G}, v_0, \Omega)$ be a parity game played on an arena of finite out-degree. Then, for each of the following three problems, one can construct a formula Φ_Ω so the problem reduces to decide whether formula Φ_Ω holds on $\mathcal{G}$ (respectively, on the unfolding of $\mathcal{G}$).*

(1) Decide whether the leaking value of $\mathbb{G}$ is countable.
(2) Decide whether the leaking value of $\mathbb{G}$ is smaller than some fixed k.
(3) Decide whether Éloïse has a topologically-good strategy in $\mathbb{G}$.

PROOF. This is obtained by combining Lemma 3.15 and Theorem 3.16 (or Corollary 3.17, if one wants the statement on the arena and not on the unfolding) together with the fact that the languages $\mathcal{L}_{\mathrm{Card}}$ and $\mathcal{L}_{\mathrm{Topo}}$ from Lemma 3.15 are ω-regular (equivalently, MSO-definable) when considering parity conditions (see Reference [12] for a proof of this result). □

Remark 3.19. One directly obtains from Corollary 3.18 decidability over various classes of arenas that enjoy MSO-decidability (or whose unfolding does): finite arenas, pushdown arenas [37], or even CPDA arenas [22]. We will later discuss these results (and how to obtain them differently) in Section 5.3.

In the case where the graph is finite, one can also safely restrict the set of strategies for Éloïse and Abélard to finite-memory strategies. More precisely,

COROLLARY 3.20. *Let $G = (V, E)$ be a finite graph, let C be a finite set, let $\mathrm{Col} : V \to C$ be a mapping, let $\mathcal{G} = (G, V_E, V_A, V_N)$ be an arena, and let v_0 be an initial vertex. Let $\mathcal{L}$ be an MSO-definable set of C-labelled V-trees. Then the following are equivalent:*

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[6]We could consider more general ω-regular conditions, but they reduce to a parity condition via product of a game with a finite graph.
[7]We refer the reader to Reference [10, Theorem 21] for a complete proof of this statement, but here are the key arguments: Call $\mathcal{L}_a$ the language of $\{a, b\}$-binary trees such that almost every branch contains a node label by a. One first argues that for any regular tree t, if there is no cone in t whose branches only contain the letter b, then this tree t belongs to $\mathcal{L}_a$. Then, one lets $\mathcal{L}$ be the set of trees that does not belong to $\mathcal{L}_a$ but does not have a cone whose branches only contain b. Hence, $\mathcal{L}$ does not contain a regular tree and, assuming by contradiction that $\mathcal{L}_a$ is MSO-definable, basic properties of MSO-definable languages imply that $\mathcal{L}$ is empty. Finally, one builds a (non-regular) tree t_0 in $\mathcal{L}$, which leads to a contradiction with the MSO-definability of $\mathcal{L}_a$: the rough idea to construct t_0 is to pick, in every cone, a node that gets labelled by a and to choose that node deep enough to ensure that the set of branches containing a node labelled by a has a measure strictly smaller than 1.

- *Éloïse has an $\mathcal{L}$-good strategy from v_0.*
- *Éloïse has an $\mathcal{L}$-good finite-memory strategy from v_0.*
- *Éloïse has a finite memory strategy φ_E such that for every finite-memory strategy φ_A of Abélard, $T_{v_0}^{\varphi_E,\varphi_A}$ belongs to $\mathcal{L}$.*

PROOF. From the proof of Theorem 3.16, we get that an Éloïse's $\mathcal{L}$-good strategy can be defined in MSO, and as G is finite, it can be implemented by a finite transducer, hence, be finite-memory. Now, it remains to prove that one can without loss of generality restrict Abélard's strategies to be finite-memory. For that, we use the same argument. Fix a *finite-memory* strategy φ_E of Éloïse: the set of Abélard's strategies φ_A such that the set of losing plays for Éloïse is MSO definable in a synchronised product of the arena together with a transducer implementing strategy φ_E, i.e., it is MSO definable on a fixed finite graph. Hence, if this set is non-empty, it contains a finite-memory strategy. □

Note that the results from the previous two corollaries are somehow not very satisfactory. Indeed, they only apply to ω-regular winning conditions and, moreover, the computational complexity may be very costly (due to the fact that one works with MSO logic whose decidability is tower-exponential in the number of quantifier alternations, which in our case is quite high). For those reasons, we consider alternative approaches in the next section. This permits to significantly reduce the complexity for decidable instances (e.g., in the setting of pushdown arenas) and also to tackle, for the cardinality setting, winning conditions not captured by MSO logic, i.e., beyond ω-regular ones.

4 PERFECT-INFORMATION GAMES WITH NATURE: DECISION PROBLEMS

We first consider the following two problems regarding the leaking value of a game: (1) Is the leaking value at most $\aleph_0$? (2) For some given $k \in \mathbb{N}$, is the leaking value smaller or equal to k?

For both questions, we make no assumption on the game itself; namely, we do not restrict the class of arenas, neither the winning conditions. As we are working in such a general setting, we do not focus on decidability but rather on finding reductions to questions on games *without* Nature. More precisely, for both problems, we provide a transformation of the arena and of the winning condition such that the problem reduces to the existence of a winning strategy for Éloïse in a game without Nature played on the new arena and equipped with the new winning condition. This occupies Section 4.1 and Section 4.2.

Finally, in Section 4.3, we follow the same approach but for the existence of topologically-good strategies. However, we need to restrict our attention to games where Abélard is not playing.

Of course, for special classes of arenas and of winning conditions, those reductions imply decidability and various important consequences that we discuss in detail in Section 5.

4.1 A Game to Decide If the Leaking Value Is at Most $\aleph_0$

Our goal in this section is to design a technique to decide if the leaking value is at most $\aleph_0$ in a given two-player game with Nature for an arbitrary Borel winning condition.

Fix a graph $G = (V, E)$, an arena $\mathcal{G} = (G, V_E, V_A, V_N)$, and a game $\mathbb{G} = (\mathcal{G}, v_0, \Omega)$ where Ω is a *Borel* winning condition. We design a two-player perfect-information game *without Nature* $\widehat{\mathbb{G}} = (\widehat{\mathcal{G}}, v_0, \widehat{\Omega})$ such that Éloïse wins $\widehat{\mathbb{G}}$ if and only if $LeakVal(\mathbb{G}) \leqslant \aleph_0$.

Intuitively in the game $\widehat{\mathbb{G}}$, every vertex v of Nature is replaced by a gadget (See Figure 7) in which Éloïse announces a successor w of v (i.e., some $w \in E(v)$) that she wants to *avoid* and then Abélard chooses a successor of v. If he picks w, then we say that he *disobeys* Éloïse; otherwise, he *obeys* her. In vertices of Éloïse and Abélard, the game $\widehat{\mathbb{G}}$ works the same as the game $\mathbb{G}$. The
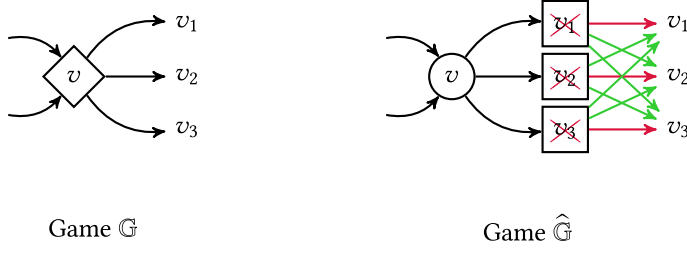
Fig. 7. Example of the gadget used to define $\widehat{\mathbb{G}}$, where ⊠ is a shorthand for (v, v_i), the node where Éloïse indicates she would prefer avoiding v_i from v.

winning condition $\widehat{\Omega}$ for Éloïse is either that the play (without the gadget nodes) belongs to Ω or that Abélard does not obey Éloïse infinitely often (i.e., after some point, Abélard always disobeys Éloïse). Remark that this is in particular the case if, after some point, no vertex corresponding to a vertex of Nature is encountered.

Formally, one defines $\widehat{G} = (\widehat{V}, \widehat{E})$ where $\widehat{V} = \widehat{V}_E \cup \widehat{V}_A$, $\widehat{V}_E = V_E \cup V_N$, $\widehat{V}_A = V_A \cup \{(v, w) \mid v \in V_N \text{ and } w \in E(v)\}$ and

$$\widehat{E} = E \setminus (V_N \times V) \;\cup\; \{(v, (v, w)) \mid v \in V_N \text{ and } w \in E(v)\}$$
$$\cup \; \{((v, w), w') \mid v \in V_N \text{ and } w, w' \in E(v)\}.$$

For ease of presentation, we view a partial play $\hat{\pi}$ in $\widehat{\mathbb{G}}$ as a partial play π in $\mathbb{G}$ together with a mapping associating to every prefix of π ending in V_N (with the possible exception of π itself) the successor that Éloïse wishes to avoid.

Formally, for a partial play $\hat{\pi}$ in $\widehat{\mathbb{G}}$, we denote by $[\![\hat{\pi}]\!]$ the partial play of $\mathbb{G}$ obtained by removing all occurrences of vertices in $V_N \times V$ from $\hat{\pi}$. A partial play $\hat{\pi}$ in $\widehat{\mathbb{G}}$ is entirely characterised by the pair (π, ξ), where π is the partial play $[\![\hat{\pi}]\!]$ and ξ is the mapping such for all $\pi' \sqsubseteq \pi$, $\xi(\pi') = w$ if and only if there exists $\hat{\pi}' \sqsubseteq \hat{\pi}$ with $[\![\hat{\pi}']\!] = \pi'$ and $\hat{\pi}'$ ends in a vertex of the form (v, w) for some $v \in V_N$. In the following, we do not distinguish between a pair (π, ξ) satisfying these conditions and the unique corresponding partial play. We adopt the same point of view for (infinite) plays.

Finally, the winning condition $\widehat{\Omega}$ is defined by

$$\widehat{\Omega} = \{(\lambda, \xi) \mid \lambda \in \Omega\} \cup \{(\lambda, \xi) \mid \exists^{<\infty} \pi v \sqsubset \lambda, \pi \in \mathrm{Dom}(\xi) \text{ and } v \neq \xi(\pi)\},$$

i.e., $\widehat{\Omega}$ contains those plays that project to a winning play in $\mathbb{G}$ as well as those plays where Abélard does not obey Éloïse infinitely often.

Remark 4.1. As Ω is assumed to be a Borel subset of plays in $\mathbb{G}$, $\widehat{\Omega}$ is a Borel subset of the set of plays in $\widehat{\mathbb{G}}$. Indeed, the second part of the condition (which does not involve Ω) is Borel. As the first part is the inverse image of Ω under the *continuous* mapping $\hat{\lambda} \mapsto [\![\hat{\lambda}]\!]$, it is also Borel. Using Borel determinacy [28] the game $\widehat{\mathbb{G}}$ is determined, i.e., either Éloïse or Abélard has a winning strategy in $\widehat{\mathbb{G}}$. Furthermore, remark that if Ω is ω-regular, then so is $\widehat{\Omega}$.

The following theorem relates the games $\mathbb{G}$ and $\widehat{\mathbb{G}}$:

THEOREM 4.2. *Let $\mathbb{G}$ be a game. The leaking value in $\mathbb{G}$ is at most $\aleph_0$ if and only if Éloïse has a winning strategy in $\widehat{\mathbb{G}}$.*

More precisely, from a winning strategy (respectively, positional winning strategy, respectively, finite-memory winning strategy) $\widehat{\varphi}_E$ of Éloïse in $\widehat{\mathbb{G}}$, we can define a strategy (respectively,

positional winning strategy, respectively, finite-memory winning strategy) φ_E for Éloïse in $\mathbb{G}$ such that $CardLeak(\varphi_E) \leqslant \aleph_0$.

Moreover, from a winning strategy (respectively, a positional winning strategy, respectively, finite-memory winning strategy) $\widehat{\varphi}_A$ for Abélard, we can define a strategy (respectively, a positional strategy, respectively, finite-memory winning strategy) φ_A for Abélard in $\mathbb{G}$ such that for any strategy φ_E of Éloïse $Card(\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega) = 2^{\aleph_0}$.

PROOF. First assume that Éloïse has a winning strategy $\widehat{\varphi}_E$ in $\widehat{\mathbb{G}}$. We define a strategy φ_E for her in $\mathbb{G}$ as follows: For any partial play π in $\mathbb{G}$ ending in V_E, if there exists a partial play of the form (π, ξ) in $\widehat{\mathbb{G}}$ in which Éloïse respects $\widehat{\varphi}_E$, then this play is unique and we let $\varphi_E(\pi) = \widehat{\varphi}_E((\pi, \xi))$. Otherwise, $\varphi_E(\pi)$ is undefined.

A straightforward induction shows that for each partial play π ending in V_E where Éloïse respects φ_E, the strategy φ_E is defined. Furthermore, remark that if $\widehat{\varphi}_E$ is positional (respectively, uses finite-memory), φ_E is also positional (respectively, also uses finite-memory).

Let us now prove that $CardLeak(\varphi_E) \leqslant \aleph_0$. For this, fix a strategy φ_A of Abélard in $\mathbb{G}$ and consider a play λ in $\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega$ that is losing for Éloïse. As Éloïse respects φ_E in λ, there exists by construction of φ_E, a unique play of the form (λ, ξ_λ) in $\widehat{\mathbb{G}}$ where Éloïse respects $\widehat{\varphi}_E$. As $\widehat{\varphi}_E$ is winning in $\widehat{\mathbb{G}}$, the corresponding play (λ, ξ_λ) is won by Éloïse and this can only be because Abélard obeys Éloïse only finitely often (indeed, recall that λ is losing for her in $\mathbb{G}$). Let π_λ be the longest prefix of λ of the form πv with $\pi \in \text{Dom}(\xi_\lambda)$ and $v \neq \xi_\lambda(\pi)$ (i.e., π_λ is the last time where Abélard obeys Éloïse).

We claim that $\lambda \in \text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega$ is uniquely characterised by π_λ. In particular, it will imply that $\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega$ is countable, as it can be injectively mapped into the countable set V^*.

Let $\lambda_1 \neq \lambda_2 \in \text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega$ and let (λ_1, ξ_1) and (λ_2, ξ_2) be the corresponding plays in $\widehat{\mathbb{G}}$. We will show that $\pi_{\lambda_1} \neq \pi_{\lambda_2}$. Consider the greatest common prefix π of λ_1 and λ_2. In particular, there exists $v_1 \neq v_2 \in V$ such that $\pi v_1 \sqsubset \lambda_1$ and $\pi v_2 \sqsubset \lambda_2$. As λ_1 and λ_2 respects the same strategies for Éloïse and Abélard, π must end in V_N. Moreover, for all prefixes of π (including π), ξ_{λ_1} and ξ_{λ_2} coincide. Let $w = \xi_{\lambda_1}(\pi) = \xi_{\lambda_2}(\pi)$ be the vertex Éloïse wants to avoid in π. Assume without loss of generality that $w \neq v_1$; i.e., Abélard obeys Éloïse at π in (λ_1, ξ_1). In particular, $\pi v_1 \sqsubseteq \pi_{\lambda_1}$: therefore, $\pi_{\lambda_1} \nsubseteq \pi_{\lambda_2}$ and thus $\pi_{\lambda_1} \neq \pi_{\lambda_2}$.

Conversely, assume that Éloïse has no winning strategy in $\widehat{\mathbb{G}}$. By Remark 4.1, Abélard has a winning strategy $\widehat{\varphi}_A$ in $\widehat{\mathbb{G}}$.

Using $\widehat{\varphi}_A$, we define a strategy φ_A of Abélard in $\mathbb{G}$ that is only partially defined. It can be turned into a full strategy by picking an arbitrary move for Abélard for all partial plays where it is not defined. This transformation can only increase the set of losing plays for Éloïse and, hence, we can work with φ_A as is.

The strategy φ_A uses as a memory a partial play in $\widehat{\mathbb{G}}$, i.e., with any partial play π in $\mathbb{G}$ where Abélard respects φ_A, we associate a partial play $\tau(\pi) = (\pi, \xi)$ in $\widehat{\mathbb{G}}$ where Abélard respects $\widehat{\varphi}_A$. The definition of both $\widehat{\varphi}_A$ and τ are done by induction.

Initially when $\pi = v_0$, one lets $\tau(\pi) = (v_0, \xi)$, where ξ is defined nowhere. Now, assume that the current partial play is π and that it ends in some vertex v and assume that $\tau(\pi) = (\pi, \xi)$.

- If $v \in V_A$, then $\varphi_A(\pi) = \widehat{\varphi}_A((\pi, \xi)) = v'$ and $\tau(\pi \cdot v') = (\pi \cdot v', \xi)$.
- If $v \in V_E$ and Éloïse moves to some v', then $\tau(\pi \cdot v') = (\pi \cdot v', \xi)$.
- If $v \in V_N$ and Nature moves to some v', then $\tau(\pi \cdot v')$ is defined only if there exists at least one $w \in E(v)$ such that $\widehat{\varphi}_A(\pi, \xi[\pi \mapsto w]) = v'$, where we denote by $\xi[\pi \mapsto w]$ the extension of ξ where π is mapped to w. In this case, if $\widehat{\varphi}_A(\pi, \xi[\pi \mapsto v']) = v'$, then, we take

$\tau(\pi \cdot v') = (\pi \cdot v', \xi[\pi \mapsto v'])$. Otherwise, we pick $w \in E(v)$ such that $\widehat{\varphi}_A(\pi, \xi[\pi \mapsto w]) = v'$ and set $\tau(\pi \cdot v') = (\pi \cdot v', \xi[\pi \mapsto w])$.

In the last case, remark that $\tau(\pi \cdot v')$ is always defined for at least one $v' \in E(v)$. Indeed, consider any node $w \in E(v)$ and set $v' = \widehat{\varphi}_A(\pi, \xi[\pi \mapsto w])$: then for this v' $\tau(\pi \cdot v')$ is defined. Furthermore, if it is defined for exactly one $v' \in E(v)$, then it is equal some to $(\pi \cdot v', \xi)$ with $\xi(\pi) = v'$: indeed, it means that $\widehat{\varphi}_A(\pi, \xi[\pi \mapsto w]) = v'$ for every w, and in particular for $w = v'$ and therefore $\tau(\pi \cdot v') = (\pi \cdot v', \xi[\pi \mapsto v'])$. This means in particular that Abélard disobeys Éloïse.

Finally, remark that if $\widehat{\varphi}_A$ is positional (respectively, uses finite-memory), then φ_A is also positional (respectively, also uses finite-memory).

Let φ_E be a strategy for Éloïse in $\mathbb{G}$. To prove that $CardLeak(\varphi_E) = 2^{\aleph_0}$, we will establish the following stronger result: $Card(\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \setminus \Omega) = 2^{\aleph_0}$.

First remark[8] that $\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \cap \Omega = \varnothing$. Indeed, consider a play $\lambda \in \text{Outcomes}_{v_0}^{\varphi_E, \varphi_A}$. By construction of φ_A, there exists a play of the form (λ, ξ) in $\widehat{\mathbb{G}}$ where Abélard respects $\widehat{\varphi}_A$: in particular, it implies that $\lambda \notin \Omega$.

It remains to show that $Card(\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A}) \geqslant 2^{\aleph_0}$. Consider the tree $T_{v_0}^{\varphi_E, \varphi_A}$ of all partial plays respecting both φ_E and φ_A. To show that $T_{v_0}^{\varphi_E, \varphi_A}$ has $2^{\aleph_0}$ branches, it is enough to show that every infinite branch in $T_{v_0}^{\varphi_E, \varphi_A}$ goes through infinitely many nodes with at least two successors.

Let λ be a branch in $T_{v_0}^{\varphi_E, \varphi_A}$ and let $\tau(\lambda) = (\lambda, \xi)$ be the corresponding play in $\widehat{\mathbb{G}}$. As $\tau(\lambda)$ is won by Abélard, he obeys Éloïse infinitely often during this play. Hence, there exists $\pi_1 v_1 \sqsubset \pi_2 v_2 \sqsubset \cdots \sqsubset \lambda$ such that for all $i \geqslant 1$, π_i ends in V_N and $\xi(\pi_i) \neq v_i$. As remarked previously for all $i \geqslant 0$, π_i has at least two successors in $T_{v_0}^{\varphi_E, \varphi_A}$ (as, otherwise, it would imply that Abélard disobeys Éloïse at π_i in $\tau(\lambda)$). □

Remark 4.3. One should think of the last part of the statement of Theorem 4.2 as a determinacy result in the spirit of Borel determinacy [28]. Indeed, it states that if Éloïse does not have a strategy that is good against every strategy of Abélard, then he has one that is bad (for her) against any of her strategies.

4.2   A Game to Decide If the Leaking Value Is Smaller Than Some k

Our goal in this section is to design a technique to decide if the leaking value is smaller than some fixed k in a given two-player game with Nature for an arbitrary Borel winning condition.

Fix a graph $G = (V, E)$, an arena $\mathcal{G} = (G, V_E, V_A, V_N)$, and a game $\mathbb{G} = (\mathcal{G}, v_0, \Omega)$, where Ω is a *Borel* winning condition. Fix a bound $k \geqslant 0$. We design a two-player perfect-information game *without Nature* $\check{\mathbb{G}}_k = (\check{\mathcal{G}}_k, s, \check{\Omega}_k)$ such that Éloïse wins $\check{\mathbb{G}}_k$ if and only if $LeakVal(\mathbb{G}) \leqslant k$.

Intuitively, the main vertices in the game $\check{\mathbb{G}}_k$ are pairs formed by a vertex from V together with an integer i such that $0 \leqslant i \leqslant k$, which indicates the maximum number of plays Éloïse is claiming that she may lose. A vertex (v_0, i) is controlled by the same player that controls v in $\mathbb{G}$. For technical reasons, we also add an initial vertex s that is controlled by Éloïse.

Informally, a play in $\check{\mathbb{G}}_k$ proceeds as follows: In the initial move, Éloïse goes from s to a vertex (v, i) for some $0 \leqslant i \leqslant k$. Then, we have the following situation, depending on the current vertex:

- If the play is in some vertex (v, i) with $v \in V_E$, then Éloïse can move to any (w, i) with $w \in E(v)$.
- If the play is in some vertex (v, i) with $v \in V_A$, then Abélard can move to any $(w, i, ?)$ that is a vertex controlled by Éloïse and from which she can decrease the integer value by going to any vertex (w, j) with $0 \leqslant j \leqslant i$.

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[8]This is no longer true for the full version of φ_A.

- If the play is in some vertex (v, i) with $v \in V_N$, then Éloïse can move to a vertex μ that stands for a function (we overload μ here) with domain $E(v)$ and that takes its value in $\{0, \ldots, i\}$ and is such that $\sum_{w \in E(v)} \mu(w) = i$, i.e., she indicates for every possible successor of v a new integer whose values sum to i. Then Abélard can choose any $w \in E(v)$ and the play goes to $(w, \mu(w))$.

Formally one defines $\breve{G}_k = (\breve{V}, \breve{E})$ where $\breve{V} = \breve{V}_E \cup \breve{V}_A$,

$$\breve{V}_E = \{s\} \cup \{(v, i) \mid v \in V_E \cup V_N, \ 0 \leqslant i \leqslant k\} \cup \{(v, i, ?) \mid v \in V_A, \ 0 \leqslant i \leqslant k\},$$

$$\breve{V}_A = \{(v, i) \mid v \in V_A, \ 0 \leqslant i \leqslant k\}$$
$$\cup \{\mu \mid \exists v \in V_N \text{ s.t. } \mu : E(v) \to \{0, \ldots, k\} \text{ and } \sum_{w \in E(v)} \mu(w) \leqslant k\}$$

and

$$\breve{E} = \{(s, (v_0, i)) \mid 0 \leqslant i \leqslant k\}$$
$$\cup \{((v, i), (w, i) \mid v \in V_E \text{ and } w \in E(v)\} \cup \{((v, i), (w, i, ?) \mid v \in V_A \text{ and } w \in E(v)\}$$
$$\cup \{((w, i, ?), (w, j) \mid i \geqslant j\}$$
$$\cup \{((v, i), \mu) \mid v \in V_N \text{ and } \mu : E(v) \to \{0, \ldots, k\} \text{ s.t. } \sum_{w \in E(v)} \mu(w) = i\}$$
$$\cup \{(\mu, (w, \mu(w))) \mid w \text{ is in the domain of } \mu\}.$$

Finally, we let $\breve{\mathcal{G}}_k = (\breve{G}_k, \breve{V}_E, \breve{V}_A)$.

Let $\breve{\lambda}$ be a play in $\breve{\mathcal{G}}_k$ and let us define $\rho(\breve{\lambda}) \in V^\omega$ to be the play obtained from $\breve{\lambda}$ by keeping only the vertices in $V \times \{0, \ldots, k\}$ and then projecting them on the V component. The play $\breve{\lambda}$ is winning for Éloïse if one of the following holds:

- $\rho(\breve{\lambda}) \in \Omega$; or
- no vertex of the form $(v, 0)$ is visited in $\breve{\lambda}$.

Call $\breve{\Omega}_k$ the corresponding winning condition, i.e., $\breve{\Omega}_k = \{\breve{\lambda} \mid \rho(\breve{\lambda}) \in \Omega\} \cup (\breve{V}^\omega \backslash \breve{V}^*(V \times \{0\})\breve{V}^\omega)$ and let $\breve{\mathbb{G}}_k = (\breve{\mathcal{G}}_k, s, \breve{\Omega}_k)$.

The following theorem relates both games $\mathbb{G}$ and $\breve{\mathbb{G}}_k$:

THEOREM 4.4. *Let $\mathbb{G}$ be a game and let $k \geqslant 0$ be an integer. The leaking value in $\mathbb{G}$ is smaller or equal to k if and only if Éloïse has a winning strategy in $\breve{\mathbb{G}}_k$.*

More precisely, from a winning strategy (respectively, finite-memory winning strategy) $\breve{\varphi}_E$ of Éloïse in $\breve{\mathbb{G}}_k$, we can define a strategy (respectively, finite-memory winning strategy) φ_E for Éloïse in $\mathbb{G}$ such that $CardLeak(\varphi_E) \leqslant k$.

PROOF. Assume first that $LeakVal(\mathbb{G}) \leqslant k$ and let φ_E be a strategy of Éloïse that witnesses it (note that as the leaking value is finite such an optimal strategy for Éloïse necessarily exists).

With any partial play λ in $\mathbb{G}$ where Éloïse respects φ_E, we associate an integer $\tau(\lambda) \leqslant k$ by letting

$$\tau(\lambda) = \max\{Card(\text{Outcomes}_\lambda^{\varphi_E, \varphi_A} \backslash \Omega) \mid \varphi_A \text{ Abélard's strategy}\},$$

where $\text{Outcomes}_\lambda^{\varphi_E, \varphi_A} = \text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \cap \lambda V^\omega$ denotes the (possibly empty) set of infinite plays that starts by λ and where Éloïse (respectively, Abélard) respects φ_E (respectively, φ_A). Remark that $\tau(v_0) = LeakVal(\mathbb{G})$. Moreover, when λ increases (with respect to the prefix ordering) τ is easily seen to be decreasing and, therefore, it implies that $\tau(\lambda)$ is always smaller or equal to k.

We now define a strategy $\breve{\varphi}_E$ for Éloïse in $\breve{\mathbb{G}}_k$ as follows, where we let $\lambda = \rho(\breve{\lambda})$:

- $\breve{\varphi}_E(s) = (v_0, LeakVal(\mathbb{G}))$;
- $\breve{\varphi}_E(\breve{\lambda}) = (\varphi(\lambda), i)$ if $\breve{\lambda}$ ends in some vertex (v, i) with $v \in V_E$;
- $\breve{\varphi}_E(\breve{\lambda}) = (w, \tau(\lambda \cdot w))$ if $\breve{\lambda}$ ends in some vertex $(w, i, ?)$;
- $\breve{\varphi}_E(\breve{\lambda}) = \mu$ if $\breve{\lambda}$ ends in some vertex (v, i) with $v \in V_N$ where $\mu : E(v) \to \{0, \ldots, k\}$ is defined by letting $\mu(w) = \tau(\lambda \cdot w)$.

It easily follows from the definition of $\breve{\varphi}_E$ that, for any partial play $\breve{\lambda}$ starting from s and where Éloïse respects $\breve{\varphi}_E$, one has the following:

(i) The play $\rho(\breve{\lambda})$ is a play in $\mathbb{G}$ that starts in v_0 and where Éloïse respects φ_E.
(ii) If $\breve{\lambda}$ ends in some vertex (v, i), then $i = \tau(\rho(\breve{\lambda}))$.

Note that the two above properties imply the following:

(iii) If some play $\breve{\lambda}$ where Éloïse respects $\breve{\varphi}_E$ eventually visits a vertex of the form $(v, 0)$, then $\rho(\breve{\lambda}) \in \Omega$.

We now establish that the strategy $\breve{\varphi}_E$ is winning for Éloïse in $\breve{\mathbb{G}}_k$. For that, assume toward a contradiction that there is a losing play $\breve{\lambda}$ for Éloïse where she respects $\breve{\varphi}_E$. By definition of $\breve{\Omega}_k$, it means that $\breve{\lambda}$ contains a vertex of the form $(v, 0)$ and that $\rho(\breve{\lambda}) \notin \Omega$, which contradicts property (iii) above.

We now turn to the converse implication. Hence, we assume that Éloïse has a winning strategy $\breve{\varphi}_E$ in $\breve{\mathbb{G}}_k$. Thanks to $\breve{\varphi}_E$, we define a strategy φ_E for Éloïse in $\mathbb{G}$. This strategy will associate with any partial play λ a partial play $\breve{\lambda}$ in $\breve{\mathbb{G}}_k$ where Éloïse respects $\breve{\varphi}_E$ and such that $\rho(\breve{\lambda}) = \lambda$. Initially, when $\lambda = v_0$, we let $\breve{\lambda} = s \cdot \breve{\varphi}(s)$. Now consider a partial play λ where Éloïse respects φ. Then, we do the following:

- If λ ends in some vertex $v \in V_E$ and if $\breve{\varphi}(\breve{\lambda}) = (w, i)$, then, we let $\varphi(\lambda) = w$ and we associate with $\lambda \cdot w$ the partial play $\breve{\lambda} \cdot (w, i)$.
- If λ ends in some vertex $v \in V_A$ (equivalently $\breve{\lambda}$ ends in some (v, i) with $v \in V_A$) and if Abélard moves to some vertex w, then, we associate with $\lambda \cdot w$ the partial play $\breve{\lambda} \cdot (w, i, ?) \cdot (w, j)$ where $(w, j) = \breve{\varphi}(\breve{\lambda} \cdot (w, i, ?))$.
- If λ ends in some vertex $v \in V_N$ and if Nature moves to some vertex w, then, we associate with $\lambda \cdot w$ the partial play $\breve{\lambda} \cdot \mu \cdot (w, \mu(w))$ where $\mu = \breve{\varphi}(\breve{\lambda})$.

Note that one easily verifies that $\breve{\lambda}$ is a partial play in $\breve{\mathbb{G}}_k$ where Éloïse respects $\breve{\varphi}_E$ and such that $\rho(\breve{\lambda}) = \lambda$.

In the remaining, we will prove that $CardLeak(\varphi_E) \leqslant k$, which implies that $LeakVal(\mathbb{G}) \leqslant k$. The proof goes by contradiction, assuming that Abélard has a strategy φ_A in $\mathbb{G}$ such that $Card(\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \setminus \Omega) > k$. Using φ_A, we define a strategy for Abélard in $\breve{\mathbb{G}}_k$ as follows, where we let $\lambda = \rho(\breve{\lambda})$:

- $\breve{\varphi}_A(\breve{\lambda}) = (w, i, ?)$ if $\breve{\lambda}$ ends in some vertex (v, i) with $v \in V_A$ and $\varphi_A(\lambda) = w$;
- $\breve{\varphi}_A(\breve{\lambda}) = (w, \mu(w))$ if $\breve{\lambda}$ ends in some vertex μ and w is such that $\mu(w) < Card(\text{Outcomes}_{\lambda}^{\varphi_E, \varphi_A} \setminus \Omega)$; in case several such w exist, $\breve{\varphi}_A$ chooses one minimising $\mu(w)$. Moreover, in case $\mu(w) = 0$, the strategy $\breve{\varphi}_A$ simply exhibits from that point a losing play extending the current one, which exists as $Card(\text{Outcomes}_{\lambda}^{\varphi_E, \varphi_A} \setminus \Omega) \geqslant 1$.

Remark that in the above definition (second item), the existence of such a w is verified by induction and is ensured by the initial assumption that $Card(\text{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega) > k$ together with the definition of $\check{\varphi}_A$ when the play ends in a μ-vertex. More precisely, in a partial play $\check{\lambda}$ ending in a vertex (v, i), we always have that $i < Card(\text{Outcomes}_{\check{\lambda}}^{\varphi_E, \varphi_A} \backslash \Omega)$.

We will show that the play $\check{\lambda}$ obtained when Éloïse respects $\check{\varphi}_E$ and Abélard respects $\check{\varphi}_A$ is won by Abélard, hence leading a contradiction. First note that if $\check{\lambda}$ eventually visits a vertex of the form $(v, 0)$, then by the definition of $\check{\varphi}_A$ one has $\lambda = \rho(\check{\lambda}) \notin \Omega$. Hence, it suffices to prove that $\check{\lambda}$ eventually visits a vertex of the form $(v, 0)$. Assume this is not the case. Hence, after some point all (main) vertices in $\check{\lambda}$ are of the form (v, i) with the same integer i: call $\check{\lambda}'$ the prefix of $\check{\lambda}$ ending in the first such vertex and consider the set $\text{Outcomes}_{\rho(\check{\lambda}')}^{\varphi_E, \varphi_A}$. This set contains at least $i + 1$ plays not in Ω. But as after $\check{\lambda}'$ the integer stays equal to i, it implies that $Card(\text{Outcomes}_{\rho(\check{\lambda}')}^{\varphi_E, \varphi_A} \backslash \Omega) \leqslant 1$ (the only possibly losing play being λ, as all other possible moves of Nature lead to a situation where no more play is losing for Éloïse as, otherwise, φ_A would indicate to mimic it). But, as $i \geqslant 1$ and $Card(\text{Outcomes}_{\rho(\check{\lambda}')}^{\varphi_E, \varphi_A} \backslash \Omega) \geqslant i$, it leads to a contradiction. Hence, $\check{\lambda}$ eventually visits a vertex of the form $(v, 0)$ and, therefore, as already noted, it implies that $\check{\lambda} \notin \check{\Omega}_k$. Hence, this contradicts the fact that $\check{\varphi}_E$ is losing and concludes the proof of the converse implication.

The fact that if $\check{\varphi}_E$ has finite-memory, then so does φ_E, is by definition. □

4.3 A Game to Decide the Existence of a Topologically-good Strategy

Our goal in this section is to design a technique to decide whether Éloïse has a topologically-good strategy in a perfect-information game with Nature. Unfortunately, we do not know how to obtain results in the general case, and, therefore, we focus on games where Abélard is not playing (i.e., one-player game with Nature). However, remember that, as already mentioned in Remark 3.19, if the underlying arena has an MSO decidable theory and if the winning condition is ω-regular, Corollary 3.18 implies decidability of the existence of a topologically-good strategy in the setting where both Éloïse and Abélard play.

4.3.1 Large Sets of Branches and Dense Set of Nodes. We start by recalling simple results from Reference [12] that provide a useful characterisation of large sets of branches in a tree. For this, fix a D-tree t for some set of directions D. Call a set of nodes $W \subseteq t$ **dense** if $\forall u \in t$, $\exists v \in W$ such that $u \sqsubseteq v$. Given a dense set of nodes W, the set $\mathcal{B}(W)$ of branches supported by W is defined as the set of branches π that have infinitely many prefixes in W. Formally,

$$\mathcal{B}(W) = \{\alpha \in D^\omega \mid \exists (u_i)_{i \geqslant 0} \in W^\omega \text{such that } \alpha \text{ is the limit of } (u_i)_{i \geqslant 0}\}.$$

Using the existence of decomposition-invariant winning strategies in Banach-Mazur games, the following lemma from Reference [12, Lemma 5] characterises large sets of branches (we repeat the proof for sake of completeness).

LEMMA 4.5. *Let t be a D-tree for some D and B be a Borel set of branches in t. Then, B is large if and only if there exists a dense set of nodes $W \subseteq t$ such that $\mathcal{B}(W) \subseteq B$.*

PROOF. Assume that B is large and let f be a decomposition-invariant strategy for Éloïse in the associated Banach-Mazur game (recall that we assumed B to be Borel). Consider the set:

$$W = \{vf(v) \mid v \in \{0, 1\}^*\}.$$

The set W is dense (as for all $v \in \{0, 1\}^*$, $v \sqsubset vf(v) \in W$). We claim that $\mathcal{B}(W)$ is included in B. Let π be a branch in $\mathcal{B}(W)$. As π has infinitely many prefixes in W, there exists a sequence of words $u_1, u_2, \ldots$ such that $u_1f(u_1) \sqsubset u_2f(u_2) \sqsubset \cdots \sqsubset \pi$. As the lengths of the u_i are strictly

increasing, there exists a sub-sequence $(v_i)_{i \geqslant 1}$ of $(u_i)_{i \geqslant 1}$ such that for all $i \geqslant 1$, $v_i f(v_i) \sqsubset v_{i+1}$. Now, consider the play in the Banach-Mazur game where Abélard first moves to v_1 and then Éloïse responds by going to $v_1 f(v_1)$. Then, Abélard moves to v_2 (which is possible, as $v_1 f(v_1) \sqsubset v_2$) and Éloïse moves to $v_2 f(v_2)$, and so on. In this play, Éloïse respects the strategy f and therefore wins. Hence, the branch π associated with this play belongs to B.

Conversely, let W be a dense set of nodes such that $\mathcal{B}(W) \subseteq B$. To show that B is large, we define a decomposition-invariant strategy f for Éloïse in the associated Banach-Mazur game. For all nodes u, we pick v of W such that u is a strict prefix of v (since W is dense, there must always exist such a v). Let $v = uu'$ and fix $f(u) = u'$. A play where Éloïse respects f goes through infinitely many nodes in W (as f always points to an element in W). Hence, the branch associated with the play belongs to $\mathcal{B}(W) \subseteq B$, which shows that f is winning for Éloïse.   □

To describe a dense set of nodes, we mark a path to this set in the tree as follows: Let t be a tree. A ***direction mapping*** is a mapping $d : t \to D$, and given a set of nodes W, we say that d *points to W* if for every node u there exists $d_1, \ldots, d_k \in D$ such that $ud_1 \cdots d_k \in W$ and for all $1 \leqslant j \leqslant k$, $d_j = d(ud_1 \cdots d_{j-1})$. We have the following result from Reference [12, Lemma 6] (we repeat the proof for the sake of completeness).

LEMMA 4.6. *A set of nodes W is dense if and only if there exists a direction mapping that points to W.*

PROOF. Assume that W is dense. We define $d(v)$ by induction on v as follows: Let v such that $d(v)$ is not yet defined, we pick a node $vd_1 \cdots d_k \in U$ (there must exist one, since W is dense), and for all $j \leqslant k$, we define

$$d(vi_1 \cdots d_{j-1}) = d_j.$$

The mapping is defined on every node and satisfies the requirement by definition. The other implication is straightforward (for all nodes v, there exists $vd_1 \cdots d_k \in W$).   □

4.3.2   Simulation Game. Fix a graph $G = (V, E)$, an arena $\mathcal{G} = (G, V_E, V_A, V_N)$ where we have $V_A = \varnothing$ (i.e., Abélard is not part of the game), and a game $\mathbb{G} = (\mathcal{G}, v_0, \Omega)$, where we assume that Ω is *Borel*. We assume that $v_0 \in V_E$ and that the game is turn-based, meaning that along a play the pebble alternatively visits V_E and V_N, which formally means that $E \subseteq V_E \times V_N \cup V_N \times V_E$. This restriction is not essential but highly simplifies the presentation.

We design a two-player perfect-information game without Nature such that Éloïse wins in $\widetilde{\mathbb{G}} = (\widetilde{\mathcal{G}}, v_0, \widetilde{\Omega})$ if and only if she has a topologically-good strategy in $\mathbb{G}$.

The arena $\widetilde{\mathcal{G}}$ of the game $\widetilde{\mathbb{G}}$ is quite similar to $\mathcal{G}$ and the main intuition is that Éloïse mimics a play against Nature in $\mathbb{G}$ and additionally describes a dense set of nodes W (thanks to a direction mapping and an explicit annotation of nodes in W) in the tree of possible outcomes. Abélard simulates the moves of Nature and he tries either to prove that W is not dense or that there is a losing play in $\mathcal{B}(W)$. Formally one defines $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ where $\widetilde{V} = \widetilde{V_E} \cup \widetilde{V_A}$, $\widetilde{V_E} = V_E$, $\widetilde{V_A} = V_N \times V_E \times \{\top, \bot\}$ and

$$\widetilde{E} = \{(v, (v', w, b)) \mid v' \in E(v), \ w \in E(v') \text{ and } b \in \{\top, \bot\}\} \cup \{((v, w, b), w') \mid w' \in E(v)\}.$$

Intuitively in a partial play λ, by choosing an edge from v to (v', w, b), Éloïse indicates that the direction mapping in $\lambda \cdot v'$ is to go to w; moreover, if $b = \top$, then she indicates that $\lambda \cdot v'$ is in the dense set W (remark that, due to the turn-based nature of the game, one can safely assume that the elements in W are always partial plays ending in a vertex in V_N). A play is winning for Éloïse if either it satisfies the winning condition while visiting infinitely many nodes marked as belonging

to the dense set or if at some point no more positions in W are reached while Abélard infinitely often selects a direction that is not the one given by the direction mapping (i.e., he does not let Éloïse have a chance to get to a position in W). Formally, the winning condition $\widetilde{\Omega}$ is defined by

$$\widetilde{\Omega} = \{v_0(v'_0, w_0, b_0)v_1(v'_1, w_1, b_1)v_2 \cdots \mid v_0 v'_0 v_1 v'_1 v_2 v'_2 \cdots \in \Omega \text{ and } \exists^\infty j \text{ s.t. } b_j = \top\}$$
$$\cup \{v_0(v'_0, w_0, b_0)v_1(v'_1, w_1, b_1)v_2 \cdots \mid \exists^{<\infty} j \text{ s.t. } b_j = \top \text{ and } \exists^\infty j \text{ s.t. } v_{j+1} \neq w_j)\}.$$

The following result connects the games $\mathbb{G}$ and $\widetilde{\mathbb{G}}$:

THEOREM 4.7. *Éloïse has a topologically-good strategy in $\mathbb{G}$ if and only if she has a winning strategy in $\widetilde{\mathbb{G}}$.*

More precisely, from a winning strategy (respectively, positional strategy, respectively, finite-memory strategy) $\widetilde{\varphi}_E$ of Éloïse in $\widetilde{\mathbb{G}}$, we can define a topologically-good strategy (respectively, positional strategy, respectively, finite-memory strategy) φ_E for Éloïse in $\mathbb{G}$.

PROOF. Assume that Éloïse has a topologically-good strategy in $\mathbb{G}$. Call φ this strategy and let t_φ be the set of all partial plays starting from v_0 where Éloïse respects φ. By definition, t_φ is a tree and its branches are those plays in $\mathbb{G}$ where Éloïse respects φ. As φ is topologically-good, the set of branches in t_φ that belongs to Ω is large and, therefore, thanks to Lemma 4.5, it contains a dense set of nodes W that, using Lemma 4.6, can be described by a direction mapping d.

Define a strategy $\widetilde{\varphi}$ in $\widetilde{\mathbb{G}}$ for Éloïse by letting $\widetilde{\varphi}(v_0(v'_0, w_0, b_0)v_1(v'_1, w_1, b_1) \ldots v_k) = (v'_k, w_k, b_k)$ where $v'_k = \varphi(v_0 v'_0 v_1 v'_1 \ldots v_k)$, $w_k = d(v_0 v_0 v_1 v'_1 \ldots v_k v'_k)$, and $b_k = \top$ if $v_0 v'_0 v_1 v'_1 \ldots v_k v'_k \in W$ and $b_k = \bot$ otherwise.

Now consider a play $\widetilde{\lambda} = v_0(v'_0, w_0, b_0)v_1(v'_1, w_1, b_1) \ldots$ in $\widetilde{\mathbb{G}}$ where Éloïse respects $\widetilde{\varphi}$: if it goes infinitely often through vertices in $V_N \times V_E \times \{\top\}$, then $v_0 v'_0 v_1 v'_1 \ldots$ is an infinite branch in t_φ that goes through infinitely many nodes in W, hence, belongs to Ω and so $\widetilde{\lambda} \in \widetilde{\Omega}$; otherwise, thanks to the direction mapping and the definition of $\widetilde{\varphi}$, it follows that if eventually Abélard always chooses to go from (v', w, b) to w, then one eventually reaches a vertex in $V_N \times V_E \times \{\top\}$ and therefore $\widetilde{\lambda} \in \widetilde{\Omega}$.

Conversely, assume that Éloïse has a winning strategy $\widetilde{\varphi}$ in $\widetilde{\mathbb{G}}$. We define a strategy φ for Éloïse in $\mathbb{G}$ as follows: The strategy φ is defined so a partial play λ in $\mathbb{G}$ (where she respects φ) is associated a partial play $\widetilde{\lambda}$ in $\widetilde{\mathbb{G}}$ (where she respects $\widetilde{\varphi}$). Initially $\lambda = \widetilde{\lambda} = v_0$. Let $\lambda = v_0 v'_0 v_1 v'_1 \cdots v_k$ be a partial play where she respects φ and let $\widetilde{\lambda} = v_0(v'_0, w_0, b_0)v_1(v'_1, w_1, b_1) \cdots v_k$; then call $\widetilde{\varphi}(\widetilde{\lambda}) = (v'_k, w'_k, b_k)$; define $\varphi(\lambda) = v'_k$ and $\widetilde{\lambda v'_k} = \widetilde{\lambda}(v'_k, w'_k, b_k)$. Now let t_φ be the set of all partial plays starting from v_0 where Éloïse respects φ. Define the set of nodes W in t_φ as those partial plays that end in V_N and such that $\widetilde{\lambda}$ ends in a vertex in $V_N \times V_E \times \{\top\}$ and define a direction mapping d in t_φ by letting, for any λ ending in V_N, $d(\lambda) = w$, where w is such that $\widetilde{\lambda}$ ends in a vertex in $V_N \times \{w\} \times \{\bot, \top\}$ (in other nodes there is a single son, so there is only one way to define d). As $\widetilde{\varphi}$ is winning, one easily deduces that d is a direction mapping that points to W and that $\mathcal{B}(W) \subseteq \Omega$. Therefore, the subset of branches of t_φ that satisfies Ω is large, meaning that φ is topologically-good. □

5 PERFECT-INFORMATION GAMES WITH NATURE: SOME CONSEQUENCES

We now discuss several consequences of our results in the perfect-information setting considered so far.

5.1   The Special Case of Parity Games Played on Finite Arenas

In the following, we argue that concepts of almost-surely winning strategies and of topologically-good strategies coincide in the special case where the arenas are *finite* and where one considers an ω-regular winning condition.

First recall that in the setting of stochastic games played on finite graphs and equipped with an ω-regular winning condition, it is well known (see, e.g., Reference [13]) that finite-memory strategies suffice for both players. Formally, Éloïse has an almost-surely strategy if and only if she has a finite-memory strategy φ_E such that for every finite-memory strategy φ_A of Abélard one has $\mu_{v_0}^{\varphi_E,\varphi_A}(\Omega) = 1$.

A similar property actually holds for the topological setting.

LEMMA 5.1. *Let* $\mathbb{G} = (\mathcal{G}, v_0, \Omega)$ *be a game with an ω-regular winning condition played on a finite arena. Then Éloïse has a topologically-good strategy if and only if she has a finite-memory strategy* φ_E *such that for every finite-memory strategy* φ_A *of Abélard the set* $\mathrm{Outcomes}_{v_0}^{\varphi_E,\varphi_A}\backslash\Omega$ *of losing plays for Éloïse is meager in the tree* $T_{v_0}^{\varphi_E,\varphi_A}$.

PROOF. In the case where Abélard is not part of the game, it is a direct consequence of Theorem 4.7 together with the fact that two-player games on finite graphs with an ω-regular enjoy finite-memory strategies. In the general setting, it is a consequence of Corollary 3.17. Indeed, topologically-good strategies are MSO definable and, therefore, when the graph is finite, they can be chosen to be regular (i.e., implemented by a finite transducer), hence, be finite-memory. Now it remains to prove that one can without loss of generality restrict Abélard's strategies to be finite-memory. For that, we use the same argument. Fix a *finite-memory* strategy φ_E of Éloïse: the set of Abélard's strategies φ_A such that the set $\mathrm{Outcomes}_{v_0}^{\varphi_E,\varphi_A}\backslash\Omega$ of losing plays for Éloïse is large in the tree $T_{v_0}^{\varphi_E,\varphi_A}$ is MSO definable in a synchronised product of the arena together with a transducer implementing strategy φ_E, i.e., it is MSO definable on a fixed finite graph. Hence, if this set is non-empty, it contains a finite-memory strategy. □

The following relates the stochastic and the topological settings in the special case where the arenas are *finite* and where one considers an ω-regular winning condition. Note that, here, we make no assumption on the probability distribution put on the transitions.

THEOREM 5.2. *Let* $\mathbb{G} = (\mathcal{G}, v_0, \Omega)$ *be a game with an ω-regular winning condition played on a finite arena. Then Éloïse almost-surely wins if and only she wins in the topological sense.*

PROOF. First recall that, as pointed out in Section 3.4, topological and probabilistic largeness coincide for ω-regular properties of regular trees. Moreover, as established above, one can safely (in both settings) restrict to finite-memory strategies (for both Éloïse and Abélard).

Assume that Éloïse has a finite-memory almost-surely winning strategy φ_E. We claim that it is topologically-good. Indeed, consider a finite-memory strategy φ_A of Abélard. As the arena is finite and as both strategies have finite-memory, the tree $T_{v_0}^{\varphi_E,\varphi_A}$ is regular and the set $\mathrm{Outcomes}_{v_0}^{\varphi_E,\varphi_A}\backslash\Omega$ of losing plays has measure 0, hence, is meager.

Conversely, assume that Éloïse has a finite-memory topologically-good strategy φ_E and let us prove that it is almost-surely winning. Indeed, consider a finite-memory strategy φ_A of Abélard. As the arena is finite and as both strategies have finite-memory, the tree $T_{v_0}^{\varphi_E,\varphi_A}$ is regular and the set $\mathrm{Outcomes}_{v_0}^{\varphi_E,\varphi_A}\backslash\Omega$ of losing plays is meager, hence, has measure 0. □

We believe that Theorem 5.2 is an important result, because it essentially means that in most of the situations (namely, when restricting to both ω-regular winning conditions and finite arenas) for which one can decide the existence of almost-surely winning strategies, the concept is the same as being topologically-good. Moreover, as one can decide existence of topologically-good

strategies for largest classes of games (as explained below in Section 5.3) it strengthens our belief that topologically-good strategies are a very valuable notion.

We now explain why Theorem 5.2 generalises previous work from References [36] and [3] (also, see Reference [9] for related questions). In Reference [36]—rephrased in our setting—Varacca and Völzer considered (among many other things) games where Nature plays alone and whose winning condition is ω-regular, and, in particular, they showed that if the arena is finite, then the set of outcomes (i.e., the set of all plays as Nature plays alone) is large if and only if it has probability 1, i.e., topological and probabilistic largeness coincide for ω-regular properties of finite Markov chains.

A natural question, addressed by Asarin et al. in Reference [3], is whether this is still true for Markov decision processes (i.e., a game with Eloise and Nature in the probabilistic setting). For this, they introduced a notion of three player games[9] (EBM-games) where Éloïse plays against Abélard, who is split into two sub-players—Banach who is good and Mazur who is evil. Banach starts playing for Abélard and after some time he decides to let Mazur play for a while and then Mazur let him play again and so on. Éloïse *does not observe* who—Banach or Mazur—is acting for Abélard. Say that Éloïse wins the game if she has a strategy such that Banach also has a strategy such that whatever Mazur does, the winning condition is satisfied. The main result of Reference [3] is that for an EBM-game on a finite arena with an ω-regular objective, Éloïse has a winning strategy iff she has an almost-surely winning strategy in the Éloïse-Nature game obtained by seeing the "Banach/Mazur" player as the single stochastic player Nature (for arbitrary probability distributions).

This result is a corollary of Theorem 5.2, as it is easily seen that in the Éloïse-Nature game obtained by merging the "Banach/Mazur" players as the single player Nature, Éloïse has a topologically-good strategy if and only if Éloïse wins the EBM-game. Indeed, she has a topologically-good strategy if and only if she has a strategy so in the induced Banach-Mazur game she has a strategy that wins against any strategy of Abélard; hence, it suffices to see Éloïse in the Banach-Mazur game as Banach and Abélard as Mazur.

Remark that our approach differs from Reference [3] by the fact that we reason by reduction instead of providing an ad hoc algorithm; moreover, topologically-good strategies make sense also for two-player games with Nature, while EBM-games do not extend naturally to capture a second antagonistic player.

5.2 Variant of Tree Automata

We now discuss consequences of our results in the cardinality setting for classes of automata on infinite trees.

A ***parity tree automaton*** $\mathcal{A}$ is a tuple $\langle A, Q, q_{ini}, \Delta, \text{Col} \rangle$ where A is a finite input alphabet, Q is a finite set of states, $q_{ini} \in Q$ is the initial state, $\Delta \subseteq Q \times A \times Q \times Q$ is a transition relation, and $\text{Col} : Q \to C$ is a colouring function.

Given an A-labelled complete binary tree t, a ***run*** of $\mathcal{A}$ over t is a Q-labelled complete binary tree ρ such that

 (i) the root is labelled by the initial state, i.e., $\rho(\varepsilon) = q_{ini}$;
 (ii) for every node $u \in \{0, 1\}^*$, $(\rho(u), t(u), \rho(u \cdot 0), \rho(u \cdot 1)) \in \Delta$.

A branch $\pi = \pi_0 \pi_1 \pi_2 \cdots$ is ***accepting*** in the run ρ if its labels satisfy the parity condition, i.e., $\liminf(\text{Col}(\rho(\pi_1 \cdots \pi_i)))_{i \geqslant 0}$ is even; otherwise, it is rejecting.

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[9]We change here the name of the players to stick to the presentation of this article and use EBM-game instead of the original name, ABM-game.

Classically, one declares that a tree t is accepted by $\mathcal{A}$ if there exists a run of $\mathcal{A}$ on t such that *all* branches in it are accepting. One denotes by $L(\mathcal{A})$ the set of accepted trees and such a language is called ***regular***.

Several relaxations of this criterion have been investigated in References [7, 10, 12].

- ***Automata with cardinality constraints***. Among others, one can consider the language $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$ of those trees for which there is a run with at least uncountably many accepting branches [7], and the language $L_{\leqslant\mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ of those trees for which there is a run with at most countably many rejecting branches [12].
- ***Automata with topological bigness constraints***: a tree belongs to $L_{\mathrm{Large}}^{\mathrm{Acc}}(\mathcal{A})$ if and only if there is a run whose set of accepting branches is large [12].
- ***Qualitative tree automata***: a tree belongs to $L_{=1}^{\mathrm{Acc}}(\mathcal{A})$ if and only if there is a run whose set of accepting branches has measure 1 [10].

Our results imply the following theorem [7, 12], where by *effectively regular*, we mean that the language is regular and that one can effectively construct an accepting automaton (in the statement below, starting from $\mathcal{A}$).

THEOREM 5.3. *For any parity tree automaton $\mathcal{A}$, $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$, $L_{\leqslant\mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ are effectively regular.*

PROOF. Start with the case $L_{\leqslant\mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$. One can think of the acceptance of a tree t as a game $\mathbb{G}$ where Éloïse labels the input by transitions and Nature chooses which branch to follow: $t \in L_{\leqslant\mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ iff the leaking value of this game is at most $\aleph_0$. Consider game $\widehat{\mathbb{G}}$ as in Theorem 4.2. This game (up to some small changes) is essentially the following: The play starts at the root of the tree; in a node u, Éloïse chooses a valid transition of the automaton and indicates a direction she wants to avoid and then Abélard chooses the next son; the winning condition is that either the parity condition is satisfied or finitely often Abélard obeys Éloïse. It is then easy to see this latter game as the "usual" acceptance game for some tree automaton with an ω-regular acceptance condition.

Now consider the case $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$. One can think of the acceptance of a tree t as a game $\mathbb{G}$ where Éloïse does nothing, Abélard labels the input by transitions, and Nature chooses which branch to follow; the winning condition is the complement of the parity condition: $t \in L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$ iff the leaking value of this game is $2^{\aleph_0}$. Again, one can consider game $\widehat{\mathbb{G}}$ as in Theorem 4.2 in which we know that Abélard has a winning strategy. Then switch the names of the players, complement the winning condition, and obtain an acceptance game for $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$, where in a node u Éloïse chooses a valid transition of the automaton, then Abélard indicates a direction he wants to avoid and then Éloïse chooses the next son; the winning condition is that the parity condition is satisfied and infinitely often Éloïse obeys Abélard. Then one can easily prove that this game is equivalent to the following game: In a node u, Éloïse chooses a valid transition of the automaton and may indicate a direction to follow, then Abélard chooses the next son (and if Éloïse indicated a direction to follow, he must respect it); the winning condition is that the parity condition is satisfied and infinitely often Éloïse does not indicate a direction. This latter game can easily be seen as the "usual" acceptance game for some tree automaton with an ω-regular acceptance condition. □

Remark 5.4. One can wonder whether a similar statement can be obtained for the languages $L_{\mathrm{Large}}^{\mathrm{Acc}}(\mathcal{A})$. In Reference [12] such languages are indeed shown to be effectively regular. One could use Theorem 4.7 to derive an alternative proof, but we omit it here, as the construction is far less

elegant than for automata with cardinality constraints (and, therefore, the gain compared with the direct approach in Reference [12] is unclear).

5.3 Games Played on Infinite Arenas

We claim that, in many contexts where the probabilistic approach leads to undecidability, the two approaches (cardinality and topological) that we proposed permit to obtain *positive* results for the main problem usually addressed: decide if Éloïse has a "good" strategy, and if so, compute it.

As this is not the core topic of the present article, we only briefly mention some of these contexts and, for each of them, point out the undecidability result in the probabilistic setting and the decidability result in the two-player game (without Nature) setting that combined with our main results (Theorem 4.2 / Theorem 4.7) leads to decidability in the cardinality/topological setting.

- Games played on pushdown graphs. These are games played on infinite graphs that can be presented as the transition graph of a pushdown automaton, as the one we considered in Example 3.1. They are of special interest, because, in particular, they permit to capture programs with recursion, and they are also the very first class of two-player games on infinite graphs that were shown to be decidable [37]. But when moving to the probabilistic setting, and already for Éloïse-Nature reachability games, they were shown (except under a quite strong restriction) to lead to undecidability [18]. In contrast, Éloïse-Abélard-Nature (respectively, Éloïse-Nature) parity games are decidable in the cardinality (respectively, topological) setting as a consequence of the decidability for the Éloïse-Abélard setting from Reference [37].

- Games played on higher-order and collapsible pushdown graphs. Handling higher-order recursion, a programming paradigm that has been widely adopted in the past decade, as all mainstream languages have added support for higher-order procedures,[10] is a crucial question in program verification (see, e.g., Reference [26] for a survey on that topic). One possible approach consists in finding an automata model capturing the behaviours of such programs (as pushdown automata do for order-1 recursion), and collapsible pushdown automata (as well as higher-order pushdown automata for a restricted class of program) form such a class [11, 22, 23]. As Éloïse-Abélard parity games played on transition graphs of collapsible pushdown automata are decidable [22], it turns out that Éloïse-Abélard-Nature (respectively, Éloïse-Nature) parity games played on collapsible pushdown graphs are decidable in the cardinality (respectively, topological) setting one. We believe this is an interesting starting point to study decidability of verification problems for programs with both higher-order recursion and uncontrollable and unpredictable behaviours. As an example, think of a jQuery program relying on a call to an external web service to complete a task: higher-order comes from using a call-back function to treat the answer of the web service while unpredictability comes from the fact that the web service may time out.

- A popular non-regular winning condition in pushdown game is the boundedness/unboundedness condition that imposes a restriction on how the stack height evolves during play. For stochastic games with Nature only (i.e., probabilistic pushdown automata) there are some positive results [17] but they break (because of Reference [18]) whenever Éloïse comes in. In the cardinality (respectively, topological) setting, we have decidability in the general case of Éloïse-Abélard-Nature (respectively, Éloïse-Nature), thanks to Theorem 4.2 (respectively, Theorem 4.7) combined with the results in References [8, 20].

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[10]For example, they were the major novelty in Java 8, they are central to Scala, and they are also at the core of JQuery, the most popular JavaScript library widely used in client-side web programming.

6　IMPERFECT-INFORMATION GAMES WITH NATURE

We now move to a richer setting where Éloïse has imperfect-information. The vertices of the game
are partitioned by an equivalence relation, and Éloïse does not observe exactly the current vertex
but only its equivalence class. In full generality, Abélard should also have imperfect-information,
but we are not able to handle this general case and, therefore, we assume here that he is perfectly
informed. Of course, as Éloïse has imperfect-information, we have to slightly change the definition
of the game (she now plays actions) and to restrict the strategies she can use. We also change how
Nature interacts with the players, but one can easily check that this setting captures the one we
gave in the perfect-information case.

One could wonder why we did not directly treat the imperfect-information case. There are two
main reasons for that. First, in the imperfect-information setting, we only have results for the
parity condition and not for any Borel condition. Second, the proof of Theorem 6.7 crucially uses
the results obtained in the perfect-information setting.

6.1　Definitions

An ***imperfect-information arena*** is a tuple $\mathcal{G} = (V_E, V_A, \Gamma, \Delta_E, \Delta_A, \sim)$ where V_E is a countable set
of Éloïse's vertices, V_A is a countable set of Abélard's vertices (we let $V = V_E \uplus V_A$), Γ is a possibly
uncountable set of Éloïse's actions, $\Delta_E : V_E \times \Gamma \to 2^V$ is Éloïse's transition function, $\Delta_A : V_A \to 2^V$
is Abélard's transition function, and $\sim$ is an equivalence relation on V. We additionally require
that the image by Δ_E (respectively, Δ_A) is never the empty set. We also require that there is no two
vertices $v_1 \in V_E$ and $v_2 \in V_A$ such that $v_1 \sim v_2$ (i.e., the $\sim$ relation always distinguishes between
vertices owned by different players).

As in the perfect-information setting, a play involves two antagonistic players—Éloïse and
Abélard—together with an unpredictable and uncontrollable player called Nature. It starts in some
initial vertex v_0 and when in some vertex v the following happens:

- if $v \in V_E$, Éloïse chooses an action γ and then Nature chooses the next vertex among those
  $v' \in \Delta_E(v, \gamma)$;
- if $v \in V_A$, Abélard chooses the next vertex $v' \in \Delta_A(v)$.

Then, the play goes on from v' and so on forever.

Hence, a ***play*** can be seen as an element in $(V_E \cdot \Gamma \cup V_A)^\omega$ compatible with Δ_E and Δ_A. More
formally, it is a sequence $\lambda = x_0 x_1 x_2 \cdots$ such that for all $i \geqslant 0$ if $x_i x_{i+1} \in V_E \cdot \Gamma$, then one has
$x_{i+2} \in \Delta_E(x_i, x_{i+1})$; and if $x_i \in V_A$, then one has $x_{i+1} \in \Delta_A(x_i)$. A ***partial play*** is a prefix of a play
that belongs to $(V_E \cdot \Gamma \cup V_A)^*$.

Two $\sim$-equivalent vertices are supposed to be indistinguishable by Éloïse and we extend $\sim$ as an
equivalence relation on V^*: $v_0 \ldots v_h \sim v'_0 \ldots v'_k$ if and only if $h = k$ and $v_i \sim v'_i$ for all $0 \leqslant i \leqslant k$;
we denote by $[\lambda]_{/_\sim}$ the equivalence class of $\lambda \in V^*$. An ***observation-based strategy*** for Éloïse
is a map $\varphi : (V^* V_E)_{/_\sim} \to \Gamma$. We say that Éloïse ***respects*** φ during a play $\lambda = v_0 \gamma_0 v_1 \gamma_1 v_2 \gamma_2 \cdots$
(where γ_i is the empty word when $v_i \in V_A$ and an action in Γ when $v_i \in V_E$) if and only if $\gamma_{i+1} =
\varphi([v_0 \cdots v_i]_{/_\sim})$ for all $i \geqslant 0$ such that $v_i \in V_E$.

Remark 6.1. One may expect a strategy for Éloïse to also depend on the actions she has played so
far, i.e., to be a map $\varphi_E : ((V_E \cdot \Gamma \cup V_A)^* \cdot V_E)_{/_\sim} \to \Gamma$ where $\sim$ is extended on Γ by letting $\gamma \sim x$ iff
$\gamma = x$ when $\gamma \in \Gamma$. But such a strategy can be mimicked by a strategy (in our sense) $\varphi'_E : V^* \to \Gamma$ by
letting $\varphi'_E([v_0 \cdots v_k]_{/_\sim}) = \varphi_E([v_0 \gamma_0 \cdots \gamma_{k-1} v_k]_{/_\sim})$ with $\gamma_i = \varphi_E([v_0 \gamma_0 \cdots \gamma_{i-1} v_i]_{/_\sim})$ when $v_i \in V_E$
and $\gamma_i = \varepsilon$ otherwise. Note that requiring to be observation-based does not interfere with the
previous trick.

A ***strategy*** for Abélard is a map $\varphi : (V_{\mathrm{E}} \cdot \Gamma \cup V_{\mathrm{A}})^* \cdot V_{\mathrm{A}} \to V$. We say that Abélard ***respects*** φ in the play $\lambda = v_0 \gamma_0 v_1 \gamma_1 v_2 \gamma_2 \cdots$ (again, γ_i is the empty word when $v_i \in V_{\mathrm{A}}$ and an action in Γ when $v_i \in V_{\mathrm{E}}$) if and only if $v_{i+1} = \varphi(v_0 \gamma_0 v_1 \cdots v_i)$ for all $i \geqslant 0$ such that $v_i \in V_{\mathrm{A}}$.

With an initial vertex v_0, a strategy φ_{E} of Éloïse, and a strategy φ_{A} of Abélard, we associate the set $\mathrm{Outcomes}_{v_0}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}}$ of all possible plays starting from v_0 and where Éloïse (respectively, Abélard) respects φ_{E} (respectively, φ_{A}).

In this part, we only have positive results for parity winning conditions, hence, we focus on this setting (but generalising the various notions to any Borel winning condition is straightforward). A parity winning condition is defined thanks to a colouring function $\mathrm{Col} : V \to C$ with a *finite* set of colours $C \subset \mathbb{N}$. We require that colouring function stays constant on the equivalence classes of the relation $\sim$ (i.e., $\mathrm{Col}(v) = \mathrm{Col}(v')$ for all $v \sim v'$).

Again, a play $\lambda = v_0 \gamma_0 v_1 \gamma_1 v_2 \gamma_2 \cdots$ (where $\gamma_i = \varepsilon$ when $v_i \in V_{\mathrm{A}}$) satisfies the parity condition if $\liminf(\mathrm{Col}(v_i))_{i \geqslant 0}$ is even; we denote by Ω_{Col} the set of plays satisfying the parity condition defined by the colouring function Col.

An ***imperfect-information parity game with Nature*** is a tuple $\mathbb{G} = (\mathcal{G}, \mathrm{Col}, v_0)$ consisting of an imperfect-information arena $\mathcal{G}$, a colouring function Col, and an initial vertex v_0.

Remark 6.2. A more symmetric notion of imperfect-information game would let Abélard also play actions (i.e., $\Delta_{\mathrm{A}} : V_{\mathrm{A}} \times \Gamma \mapsto 2^V$) while Nature would choose the successor as it does for Éloïse. Consider such a game $\mathcal{G} = (V_{\mathrm{E}}, V_{\mathrm{A}}, \Gamma, \Delta_{\mathrm{E}}, \Delta_{\mathrm{A}}, \sim)$ where Abélard plays actions. We can simulate it by a game $\mathcal{G} = (V_{\mathrm{E}}', V_{\mathrm{A}}, \Gamma, \Delta_{\mathrm{E}}', \Delta_{\mathrm{A}}', \sim')$ in our setting. For every vertex v of Abélard and every action $\gamma \in \Gamma$, we introduce a new vertex (v, γ) for Éloïse (i.e., $V_{\mathrm{E}}' = V_{\mathrm{E}} \cup V_{\mathrm{A}} \times \Gamma$). Furthermore, we set $\Delta_{\mathrm{A}}'(v) = \{(v, \gamma) \mid \gamma \in \Gamma\}$ and for every vertex of Éloïse of the form (v, γ), we take $\Delta_{\mathrm{E}}'((v, \gamma), \gamma') = \Delta_{\mathrm{A}}(v, \gamma)$ for all action $\gamma' \in \Gamma$. For the original vertices $v \in V_{\mathrm{E}}$ and for $\gamma \in \Gamma$, we take $\Delta_{\mathrm{E}}'(v, \gamma) = \Delta_{\mathrm{E}}(v, \gamma)$. Finally, the equivalence relation $\sim'$ coincides with $\sim$ and equates all new vertices. It is then easy to check that both games are equivalent.

To evaluate how good an Éloïse's strategy is, we can take exactly the same definitions and notations as we did in the perfect-information setting (this is why we do not repeat them here). Hence, we have the notions of ***cardinality leaking*** of a strategy (thanks to Definition 3.7), ***leaking value*** of a game (thanks to Definition 3.8), and ***topologically-good*** strategy (thanks to Definition 3.12).

Remark 6.3. For the same reason as in the prefect-information setting, we have that for any strategy φ_{E} one has $CardLeak(\varphi) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ and as a consequence that $LeakVal(\mathbb{G}) \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

Example 6.4. Consider the Büchi game where Éloïse and Abélard choose simultaneously and independently a bit in $\{0, 1\}$: If the bits are the same, then the game goes to a special vertex coloured by 0; otherwise, it goes to a special vertex coloured by 1, and then, in both cases, another round starts and so on forever. Hence, Éloïse wins if she infinitely often guesses correctly the choice of Abélard. To simulate the concurrent aspect of the choices of the player, we will use imperfect information: Abélard chooses first but Éloïse does not observe his choice, and she chooses second. Moreover, for her choice, Éloïse has a third option, which is to let Nature choose for her. Technically, once Abélard made his choice, Nature is also making a choice (hidden to Éloïse) and then Éloïse has three options: choose bit 0, choose bit 1, or pick the bit chosen by Nature.

Formally (see Figure 8 for an illustration) one defines the imperfect-information arena $\mathcal{G} = (V_{\mathrm{E}}, V_{\mathrm{A}}, \Gamma, \Delta_{\mathrm{E}}, \Delta_{\mathrm{A}}, \sim)$ where $V_{\mathrm{E}} = \{l, w, v_0, v_1, v_{0,0}, v_{0,1}, v_{1,0}, v_{1,0}\}$, $V_{\mathrm{A}} = \{v\}$, $\Gamma = \{\natural, 0, 1, N\}$, Δ_{A}, and Δ_{E} (we omit meaningless actions but could add a dummy state to handle them) are given by
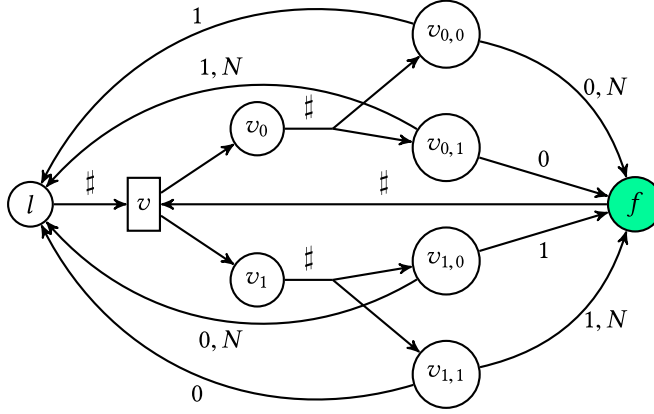
Fig. 8. Arena of Example 6.4: $\sim$-equivalent vertices are depicted in the same column, actions are written on edges, and f is the only vertex coloured by 0.

- $\Delta_A(v) = \{v_0, v_1\}$: i.e., Abélard encodes the choice of his bit by going either to v_0 or v_1;
- $\Delta_E(v_0, \sharp) = \{v_{0,0}, v_{0,1}\}$ and $\Delta_E(v_1, \sharp) = \{v_{1,0}, v_{1,1}\}$: this corresponds to the step where Nature is choosing its bit;
- $\Delta_E(v_{0,0}, 0) = \Delta_E(v_{0,0}, N) = \Delta_E(v_{0,1}, 0) = \Delta_E(v_{1,0}, 1) = \Delta_E(v_{1,1}, 1) = \Delta_E(v_{1,1}, N) = \{f\}$: this corresponds to either Éloïse choosing the same bit as Abélard or mimicking luckily the choice of Nature;
- $\Delta_E(v_{0,0}, 1) = \Delta_E(v_{0,1}, 1) = \Delta_E(v_{0,1}, 1) = \Delta_E(v_{1,0}, 0) = \Delta_E(v_{1,0}, N) = \Delta_E(v_{1,1}, 0) = \{l\}$: this corresponds to either Éloïse choosing a bit different from Abélard or mimicking unluckily the choice of Nature;
- $\Delta_E(f, \sharp) = \Delta_E(l, \sharp) = \{v\}$: this corresponds to start a new round.

And, $v_0 \sim v_1$ and $v_{0,0} \sim v_{0,1} \sim v_{1,0} \sim v_{1,1}$. The colouring function Col equals 1 everywhere except on f where it equals 0. Finally, we let $\mathbb{G} = (\mathcal{G}, \mathrm{Col}, v)$.

Consider an Éloïse's strategy φ_E that finitely often plays action N. Then, it is easily seen that $CardLeak(\varphi) = 2^{\aleph_0}$. Indeed, consider the strategy φ_A of Abélard that chooses the bit opposite to that prescribed by φ_E (and any bit when φ_E plays action N): then there are no winning plays for Éloïse in $\mathrm{Outcomes}_v^{\varphi_E, \varphi_A}$, hence, $CardLeak(\varphi) = 2^{\aleph_0}$.

Now, consider the strategy ψ_E of Éloïse that always plays action N from vertices in $\{v_{0,0}, v_{0,1}, v_{1,0}, v_{1,1}\}$. Then, one has $CardLeak(\varphi) = \aleph_0$. Indeed, consider any strategy ψ_A of Abélard. Then, for every $k \geqslant 0$, there are only finitely many plays in $\mathrm{Outcomes}_v^{\psi_E, \psi_A}$ that never visit f after the nth round (namely, the ones where Nature only makes incorrect choices after the kth round). As the set of losing plays is the countable union of the previous plays when k ranges over $\mathbb{N}$, there are only countably many losing plays in $\mathrm{Outcomes}_v^{\varphi_E, \varphi_A}$, hence, $CardLeak(\varphi) = \aleph_0$. As the set of losing plays is clearly a countable union of nowhere dense sets, it follows that ψ_E is also topologically-good.

6.2 Imperfect-information Two-player Games

We now introduce another version of games with imperfect-information where there are only two antagonist players—Éloïse and Abélard. The only difference with the previous model with Nature is that now the non-determinism induced by a choice of an action of Éloïse is resolved by Abélard. This concept was first considered in Reference [14] for finite arenas.

Let $\mathcal{G} = (V_E, V_A, \Gamma, \Delta_E, \Delta_A, \sim)$ be an imperfect-information arena. Then, a play involves two players Éloïse and Abélard: It starts in some initial vertex v_0 and when in some vertex v, the following happens:

- if $v \in V_E$, Éloïse chooses an action γ and then Abélard chooses the next vertex among those $v' \in \Delta_E(v, \gamma)$;
- if $v \in V_A$, Abélard chooses the next vertex $v' \in \Delta_A(v)$.

Then, the play goes on from v' and so on forever. Again, a **play** is an element in $(V_E \cdot \Gamma \cup V_A)^{\omega}$ and a **partial play** is a finite prefix of a play in $(V_E \cdot \Gamma \cup V_A)^*$.

Observation-based strategies for Éloïse are defined as for imperfect-information games with Nature. We shall later consider winning conditions that are slightly more general than parity conditions, hence, we allow any Borel subset Ω of $(V_E \cdot \Gamma \cup V_A)^{\omega}$.

An **imperfect-information two-player game** is a tuple $\mathbb{G} = (\mathcal{G}, \Omega, v_0)$ consisting of an arena of imperfect-information, a winning condition Ω, and an initial vertex v_0. A strategy φ_E of Éloïse is **winning** in $\mathbb{G}$ if any play starting from v_0 where Éloïse respects φ_E belongs to Ω.

Remark 6.5. Note that even for reachability conditions—i.e., when $\Omega = V^* F V^{\omega}$ for some non-empty $F \subseteq V$—and finite arena, imperfect-information two-player games are not determined, i.e., it can happen that none of the two players has a winning strategy. See Reference [14, Example 2.3].

6.3 Deciding Whether the Leaking Value Is at Most $\aleph_0$

Our goal in this section is to design a technique to decide whether Éloïse has a strategy with a cardinality leaking of at most $\aleph_0$ in an imperfect-information parity game with Nature.

For the rest of this section, we fix an imperfect-information parity game with Nature $\mathbb{G} = (\mathcal{G}, \mathrm{Col}, v_0)$, where $\mathcal{G} = (V_E, V_A, \Gamma, \Delta_E, \Delta_A, \sim)$ and we aim at deciding whether $LeakVal(\mathbb{G}) \leqslant \aleph_0$.

The approach has the same flavour as the one for the perfect-information case: We define an imperfect-information game without Nature, where Abélard is now in charge of simulating choices of Nature while Éloïse will indicate together with her action, a successor that she wants to avoid; moreover, Abélard will be forced (thanks to the winning condition) to respect her choices infinitely often.

To express the choice of Nature she wants to avoid while preserving the fact that she is imperfectly informed about the actual vertex, Éloïse will provide with her action $\gamma \in \Gamma$, a map $\theta : V \to V$ such that for all $v \in V$ one has $\theta(v) \in \Delta_E(v, \gamma)$; we denote by Θ_{γ} the set of such maps (for a given $\gamma \in \Gamma$). Intuitively, the meaning of Éloïse playing (γ, θ) is that she plays action γ and would prefer, for each vertex v, if the play turns to be in v, that the next vertex is not $\theta(v)$.

Remark 6.6. The map θ may be partial: What is important is that, if at some point the play can be in v, then $\theta(v)$ should be defined. In particular, if there are two bounds, one on the size of the $\sim$-equivalence classes of V and one on the out-degree of the vertices in G, then Θ_{γ} can be chosen to be finite (up to coding). This will be the case for pushdown games when discussing consequences in Section 7.

We define a *two-player* imperfect-information arena $\widehat{\mathcal{G}} = (V_E, V_A, \widehat{\Gamma}, \widehat{\Delta}_E, \Delta_A, \sim)$, where $\widehat{\Gamma} = \bigcup_{\gamma \in \Gamma} \{\gamma\} \times \Theta_{\gamma}$ and $\widehat{\Delta}_E(v, (\gamma, \theta)) = \Delta_E(v, \gamma)$. A play in $\widehat{\mathcal{G}}$ is of the form $v_0 v_0 v_1 v_1 v_2 \cdots$ where for all $i \geqslant 0$, v_i is equal to some (γ_i, θ_i) if $v_i \in V_E$ and is empty otherwise. For some $i \geqslant 0$, if $v_i \in V_E$ and $v_{i+1} \neq \theta_i(v_i)$, then we say that Abélard *obeys* Éloïse at this point.

We let $\widehat{\Omega}$ consist of those plays $v_0 v_0 v_1 v_2 \cdots$ such that either $v_0 v_1 v_2 \cdots \in \Omega_{\mathrm{Col}}$ or there are only finitely many i such that $v_i \in V_E$ and $v_{i+1} \neq \theta_i(v_i)$, i.e., either the play satisfies the

parity condition or eventually Abélard never obeys Éloïse. Finally, we denote by $\widehat{\mathbb{G}}$ the two-player imperfect-information game $(\widehat{\mathcal{G}}, \widehat{\Omega}, v_0)$. The next result relates $\mathbb{G}$ and $\widehat{\mathbb{G}}$.

THEOREM 6.7. *The leaking value of $\mathbb{G}$ is at most $\aleph_0$ if and only if Éloïse has a winning strategy in $\widehat{\mathbb{G}}$.*

More precisely, from a winning strategy (respectively, positional winning strategy, respectively, finite-memory winning strategy) $\widehat{\varphi}_E$ of Éloïse in $\widehat{\mathbb{G}}$, we can define a strategy (respectively, positional winning strategy, respectively, finite-memory winning strategy) φ_E for Éloïse in $\mathbb{G}$ such that $CardLeak(\varphi_E) \leqslant \aleph_0$.

PROOF. Let λ be a partial play in $\mathbb{G}$ (respectively, $\widehat{\mathbb{G}}$), we denote by $[\![\lambda]\!]$ the sequence of vertices in V^* obtained by removing the actions from λ.

First assume that Éloïse has a winning strategy $\widehat{\varphi}_E$ in $\widehat{\mathbb{G}}$. This direction is very similar to the perfect-information case. We define a strategy φ_E for Éloïse in $\mathbb{G}$ by letting $\varphi_E(\lambda) = \gamma$ whenever $(\gamma, \theta) = \widehat{\varphi}_E(\lambda)$. In particular, note that φ_E uses the same memory as $\widehat{\varphi}_E$. For any partial play λ in $\mathbb{G}$ in which Éloïse respects φ_E, there exists a unique play, denoted $\widehat{\lambda}$, in which Éloïse respects $\widehat{\varphi}_E$ and such that $[\![\lambda]\!] = [\![\widehat{\lambda}]\!]$. By taking the limit, we extend this notation from partial plays to plays.

Let us now prove that $CardLeak(\varphi_E) \leqslant \aleph_0$. For this, fix a strategy φ_A of Abélard in $\mathbb{G}$ and consider a play λ in $\mathrm{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega$.

As $\widehat{\varphi}_E$ is winning in $\widehat{\mathbb{G}}$, $\widehat{\lambda}$ (in which Éloïse respects $\widehat{\varphi}_E$) is won by Éloïse. As $\widehat{\lambda}$ does not satisfy the parity condition (because λ does not), Éloïse wins, because Abélard obeys her only finitely often. Let π_λ be the longest prefix π of λ such that $\widehat{\pi}$ is of the form $\pi' v(\gamma, \theta) v'$ with $v' \neq \theta(v)$ (i.e., it is the last time where Abélard obeys Éloïse). By convention, if Abélard never obeys Éloïse in λ, we let $\pi_\lambda = \varepsilon$.

We claim that $\lambda \in \mathrm{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega$ is uniquely characterised by π_λ. In particular, $\mathrm{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega$ is countable, as it can be injectively mapped into the countable set of partial plays in $\mathbb{G}$.

Let $\lambda_1 \neq \lambda_2 \in \mathrm{Outcomes}_{v_0}^{\varphi_E, \varphi_A} \backslash \Omega$. We will show that $\pi_{\lambda_1} \neq \pi_{\lambda_2}$. Consider the greatest common prefix π of λ_1 and λ_2. As λ_1 and λ_2 respect the same strategies for Éloïse and Abélard, π must end by some $v\gamma$ with v of Éloïse. In particular, there exists $v_1 \neq v_2 \in \Delta_E(v, \gamma)$ such that $\pi v_1 \sqsubset \lambda_1$ and $\pi v_2 \sqsubset \lambda_2$. The partial play $\widehat{\pi}$ ends in $v(\gamma, \theta)$ for some $\theta \in \Theta_\gamma$. Assume w.l.o.g. that $\theta(v) \neq v_1$; i.e., Abélard obeys Éloïse at $\widehat{\pi v_1}$ in λ_1. In particular, $\pi v_1 \sqsubseteq \pi_{\lambda_1}$; therefore, $\pi_{\lambda_1} \nsqsubseteq \pi_{\lambda_2}$ and thus $\pi_{\lambda_1} \neq \pi_{\lambda_2}$.

For the converse implication, as the game $\widehat{\mathbb{G}}$ may not be determined, we cannot proceed as in the perfect-information case.[11] Hence, assume that the leaking value of $\mathbb{G}$ is at most $\aleph_0$ and let φ_E be a strategy of Éloïse such that $LeakVal(\varphi_E) \leqslant \aleph_0$ (thanks to Remark 6.3, it exists).

To define a winning strategy in $\widehat{\mathbb{G}}$ for Éloïse, we consider a *perfect-information* parity game with Nature that we denote $\mathbb{P}$. In this game each vertex belongs either to Abélard or Nature.

To define $\mathbb{P}$, let $S \subseteq V^*$ be the set of all $[\![\pi]\!]$ for π a partial play respecting φ_E and let $\equiv$ be the equivalence relation on S defined for all $[\![\pi]\!], [\![\pi']\!] \in S$ by $[\![\pi]\!] \equiv [\![\pi']\!]$ if π and π' end in the same vertex and $\pi \sim \pi'$. In the rest of the proof, we will use letter $\eta, \eta', \ldots$ to denote elements in S. In particular, if $\eta = v_0 \cdots v_k$ and $\eta' = v_0' \cdots v_{k'}'$, we have $\eta \equiv \eta'$ if and only if $k = k'$, $v_k = v_{k'}'$ and $v_i \sim v_i'$ for all $0 \leqslant i < k$.

The vertices $V_\mathbb{P}$ are the equivalence classes $\equiv$. A vertex $[\eta]_{/\equiv} \in V_\mathbb{P}$ belongs to Abélard if η ends in a vertex of Abélard and it belongs to Nature otherwise. There is an edge from $[\eta]_{/\equiv}$ to $[\eta']_{/\equiv}$ if

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[11]Recall that in the perfect-information case, for the converse implication of the proof of Theorem 4.2, we were—thanks to determinacy—considering a winning strategy for Abélard in $\widehat{\mathbb{G}}$ and built out of it a winning strategy for him in $\mathbb{G}$.

η' extends η by one vertex. The initial vertex is $[v_0]_{/\equiv}$. Last, the parity condition is given by the mapping associating to $[\eta]_{/\equiv} \in V_{\mathbb{P}}$ the colour $\mathrm{Col}(v)$ of the the last vertex v of η.

A partial play ξ in $\mathbb{P}$ is of the form $\xi = [\eta_0]_{/\equiv}[\eta_1]_{/\equiv} \cdots [\eta_n]_{/\equiv}$, where $\eta_0 = v_0$ and for all $i < n$, η_{i+1} extends η_i by one vertex. With such a ξ, we naturally associate the play $\tau(\xi)$ in $\mathbb{G}$ defined by $v_0 \nu_0 v_1 \nu_1 \cdots v_n$, where for all $i \geqslant 0$, v_i is defined as the last vertex in η_i and ν_i is equal to $\varphi_{\mathrm{E}}([\eta_i]_{/\sim})$ if v_i belongs to Éloïse and ν_i is empty otherwise. It is easy to show that for all $i \leqslant n$, $v_0 v_1 \cdots v_i \equiv \eta_i$. Hence, as φ_{E} is observation-based, $\tau(\xi)$ respects φ_{E}. In fact, the continuous mapping τ establishes a one-to-one correspondance between the partial plays in $\mathbb{P}$ and the partial plays in $\mathbb{G}$ where Éloïse respects φ_{E}. By continuity, this mapping extends to plays.

The leaking value of $\mathbb{P}$ is at most $\aleph_0$: Indeed, any strategy $\varphi_{\mathrm{A}}^{\mathbb{P}}$ for Abélard in $\mathbb{P}$ can be lifted to a strategy φ_{A} in $\mathbb{G}$ such that $\{\tau(\xi) \mid \xi$ a play in $\mathbb{P}$, which respects $\varphi_{\mathrm{A}}^{\mathbb{P}}\}$ is equal to $\mathrm{Outcomes}_{v_0, \mathbb{G}}^{\varphi_{\mathrm{E}}, \varphi_{\mathrm{A}}}$, and, therefore, (as $LeakVal(\varphi_{\mathrm{E}}) \leqslant \aleph_0$) the set of losing plays for Éloïse in $\mathbb{P}$ when Abélard uses strategy $\varphi_{\mathrm{A}}^{\mathbb{P}}$ has cardinality at most $\aleph_0$.

Therefore, one can use Theorem 4.2 for the (perfect-information) game $\mathbb{P}$ and gets that Éloïse has a winning strategy in the game $\widehat{\mathbb{P}}$ (defined as in Theorem 4.2). As the winning condition of $\widehat{\mathbb{P}}$ is a disjunction of two parity conditions, the winning condition of $\widehat{\mathbb{P}}$ is a so-called Rabin condition.[12] Therefore, Éloïse has a *positional* winning strategy $\varphi_{\mathrm{E}}^{\widehat{\mathbb{P}}}$ in $\widehat{\mathbb{P}}$ [25]. For $\eta \in S$ ending with a vertex v of Éloïse, $\varphi_{\mathrm{E}}^{\widehat{\mathbb{P}}}$ associates to $[\eta]_{/\equiv}$ a pair $([\eta]_{/\equiv}, [\eta v']_{/\equiv})$ with $v' \in \Delta_{\mathrm{E}}(v, \varphi_E([\eta]_{/\sim}))$. This strategy is equivalently described by the mapping φ_B associating to $[\eta]_{/\equiv}$ the vertex v' in $\Delta_{\mathrm{E}}(v, \varphi_E([\eta]_{/\sim}))$.

The key property of this strategy is that any play λ in $\mathbb{G}$, which respects φ_{E} and such that λ has infinitely many prefixes of the form $\pi v \nu v'$ with $v \in V_{\mathrm{E}}$ and $v' \neq \varphi_B([\![\pi v]\!]_{/\equiv})$, satisfies the parity condition. Indeed, toward a contradiction, assume that λ does not satisfy the parity condition. Let $\lambda' = \tau^{-1}(\lambda)$ be the corresponding play in $\mathbb{P}$ and let (λ', φ_B) be the corresponding play in $\widehat{\mathbb{P}}$. None of these plays satisfies the parity condition. However, as (λ', φ_B) respects the positional winning strategy for Éloïse described by φ_B, it is won by Éloïse. This implies that Abélard only obeys Éloïse finitely often, which brings the contradiction.

To define a strategy for Éloïse in $\widehat{\mathbb{G}}$, we will mimic φ_{E} to choose the Γ-component (call γ the action) and use φ_B to choose the Θ_γ-component.

For this, we let $\widehat{\varphi}_{\mathrm{E}}([\pi]_{/\sim}) = (\gamma, \theta)$, where $\gamma = \varphi_{\mathrm{E}}([\pi]_{/\sim})$ and θ is defined as follows: Let $v \in V$: if there exists $\pi' \sim \pi$ ending with v, then, we take $\theta(v) = \varphi_B([\![\pi']\!]_{/\equiv})$; otherwise, we define $\theta(v) = w$ for some arbitrary $w \in \Delta(v, \gamma)$ (the value actually does not matter).

Now consider a play $\widehat{\lambda} = v_0 \nu_0 v_1 \nu_1 v_2 \cdots$ in $\widehat{\mathbb{G}}$ where Éloïse respects $\widehat{\varphi}_{\mathrm{E}}$, denote $\nu_i = (\gamma_i, \theta_i)$ when $\nu_i \neq \varepsilon$ (i.e., when $v_i \in V_{\mathrm{E}}$) and define $\gamma_i = \varepsilon$ when $\nu_i = \varepsilon$. By contradiction, assume that $\widehat{\lambda}$ is losing for Éloïse. Consider the play $\lambda = v_0 \gamma_0 v_1 \gamma_1 v_2 \cdots$: It is a play in $\mathbb{G}$ where Éloïse respects φ_{E} and as $\widehat{\lambda} \notin \widehat{\Omega}$ one also has $\lambda \notin \Omega_{\mathrm{Col}}$. But as $\widehat{\lambda}$ is losing for Éloïse, it means that for infinitely many i, one has $v_{i+1} \neq \theta_i(v_i)$, which implies that for infinitely many i, one has $v_{i+1} \neq \varphi_B([v_0 \cdots v_i]_{/\equiv})$. Therefore, as remarked previously, it implies that $\lambda \in \Omega_{\mathrm{Col}}$, hence, leading to a contradiction. □

Note that, due to page limit constraints, contrary to what we did in Section 4.2 for the perfect-information setting, we do not tackle here the problem of deciding whether the leaking value is smaller than some given k. However, we hope that it is clear that the approach we just developed, for the problem of deciding whether the leaking value is at most $\aleph_0$, to shift from perfect to imperfect-information, also leads to treat the other question as well.

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[12]We refer the reader not familiar with Rabin conditions to Reference [35] for a formal definition. Let us also stress that the Rabin condition is in fact on the sequence of edges taken during the play and not on the sequence of vertices visited. By a slight modification of $\widehat{\mathbb{P}}$, it can be transformed into a Rabin condition on the sequence of vertices visited.

6.4 Deciding the Existence of a Topologically-good Strategy

Our goal in this section is to design a technique to decide whether Éloïse has a topologically-good strategy in an imperfect-information parity game with Nature. We only have results in the case of games where Abélard is not playing (i.e., one-player game with Nature), hence, we implicitly assume this from now on.

We start by giving a useful characterisation of large sets of branches in a tree when the set of branches is defined by a parity condition. For this, fix a D-tree t for some set of directions D. Assume that we have a colouring function $\mathrm{Col} : t \to C$ for a finite set C of colours.

Call a ***local-strategy*** for Éloïse a pair (φ_f, φ_n) of two maps from t into $D \times \{\top, \bot\}$. For all node $u \in t$, we let $d_f(u)$ (respectively, $d_n(u)$) be the unique element such that $\varphi_f(u) \in \{d_f(u)\} \times \{\top, \bot\}$ (respectively, $\varphi_n(u) \in \{d_n(u)\} \times \{\top, \bot\}$).

A local-strategy is *valid* if the following holds:

(1) For every $u \in t$ both $u \cdot d_f(u)$ and $u \cdot d_n(u)$ are nodes in t; i.e., φ_f and φ_u indicate an existing son.

(2) For every $u \in t$ there is a node $v = u d_1 \cdots d_\ell$ such that $\varphi_f(v) \in D \times \{\top\}$ and $d_i = d_f(u d_1 \cdots d_{i-1})$ for all $i < \ell$; i.e., following φ_f leads to a node where the second component is $\top$.

(3) For every $u \in t$ there is a node $v = u d_1 \cdots d_\ell$ such that $\varphi_n(v) \in D \times \{\top\}$ and $d_i = d_n(u d_1 \cdots d_{i-1})$ for all $i < \ell$; i.e., following φ_n leads to a node where the second component is $\top$.

Take a valid local-strategy (φ_f, φ_n). A (φ_f, φ_n)-compatible branch is any branch in t that can be obtained as follows: one selects any node u_0 in t and then one lets v_0 be the shortest node satisfying property (2) above (w.r.t. node u_0), then one selects any node u_1 such that $v_0 \sqsubset u_1$ and one lets v_1 be the shortest node satisfying property (3) above (w.r.t. node u_1), then one selects any node u_2 such that $v_1 \sqsubset u_2$ and one lets v_2 be the shortest node satisfying property (3) above (w.r.t. node u_2), and so on forever (i.e., we use property (2) only in the first round and then we use property (3) forever).

We have the following lemma (whose proof follows the one of Reference [21, Proposition 13]).

LEMMA 6.8. *The set of branches satisfying the parity condition in t is large if and only if there is a valid local-strategy (φ_f, φ_n) such that any (φ_f, φ_n)-compatible branch satisfies the parity condition. Moreover, one can choose (φ_f, φ_n) such that $\varphi_f(u_1) = \varphi_f(u_2)$ and $\varphi_n(u_1) = \varphi_n(u_2)$ whenever $t[u_1] = t[u_2]$.*

PROOF. We rely on the characterisation of large sets by means of Banach-Mazur games.

Obviously, if there is a valid local-strategy (φ_f, φ_n) such that any (φ_f, φ_n)-compatible branch satisfies the parity condition, then it leads a winning strategy for Éloïse in the Banach-Mazur game. Indeed, for her first move, Éloïse goes down in the tree using φ_f until she ends up in a node whose father's second component by φ_f was $\top$ and in the next rounds she does similarly but using φ_n. The resulting play is a (φ_f, φ_n)-compatible branch, hence, it satisfies the parity condition.

We now prove the other implication, i.e., we assume that the set of branches satisfying the parity condition in t is large or equivalently that Éloïse wins the Banach-Mazur game. The beginning of the proof is very similar to the one that Banach-Mazur games with Muller winning condition admit positional strategies [21, Proposition 13]. Let u be a node in t, then one denotes by $C(u) = \{\mathrm{Col}(v) \mid u \sqsubseteq v\}$ the set of colours of nodes reachable from u in t. Obviously one has $C(w) \subseteq C(u)$ for all $u \sqsubseteq w$. In case one has $C(w) = C(u)$ for all $u \sqsubseteq w$, we say that u is a *stable* node (and so does its descendants). As the set of colours is finite, for all node u there is a stable node v such that $u \sqsubseteq v$.

We claim that for all stable node u, $\min C(u)$ is even. Indeed, assume that there is some stable u such that $\min C(u) = m$ is odd: then, a winning strategy (leading a contradiction) of Abélard in the Banach-Mazur game would consist to go to u in its first move and then whenever he has to play to go to a node with colour m (which he can always do by stability).

Now, we define a valid local-strategy (φ_f, φ_n) as follows: First, fix a total ordering on D. For every $u \in t$, call u_s the unique minimal (for the length lexicographic ordering) stable node such that $u \sqsubset u_s$: define $\varphi_f(u) = (d, x)$, where $u_s = u \cdot d \cdot w$ with $d \in D$ and $x = \top$ if $w = \varepsilon$ and $x = \bot$ otherwise. For every $u \in t$ that is stable, call u' the unique minimal (for the length lexicographic ordering) node with colour $\min C(u)$ and such that $u \sqsubset u'$: define $\varphi_n(u) = (d, x)$, where $u' = u \cdot d \cdot w$ with $d \in D$ and $x = \top$ if $w = \varepsilon$ and $x = \bot$ otherwise. For every $u \in t$ that is not stable, define $\varphi_n(u) = (d, \bot)$, where d is the minimal direction such that $ud \in t$ (the value of φ_n does not matter, but we want it to be the same in all isomorphic subtrees so we have to define it in a systematic way). From the definition, one directly gets that $\varphi_f(u_1) = \varphi_f(u_2)$ and $\varphi_n(u_1) = \varphi_n(u_2)$ whenever $t[u_1] = t[u_2]$.

The fact that (φ_f, φ_n) is valid is by definition and the fact that any (φ_f, φ_n)-compatible branch satisfies the parity condition is a direct consequence of the fact that for all stable node u one has $\min C(u)$ even. □

Recall that we assume that Abélard is not part of the game. Hence, we omit him in notations when considering the original game (i.e., we do not write V_A nor Δ_A).

For the rest of this section, we fix an imperfect-information one-player parity game with Nature $\mathbb{G} = (\mathcal{G}, \mathrm{Col}, v_0)$ where $\mathcal{G} = (V, \Gamma, \Delta, \sim)$ and we aim at deciding whether Éloïse has a topologically-good strategy.

The main idea is to define an imperfect-information game without Nature but with Abélard. In this game, Éloïse simulates a play in $\mathbb{G}$ and also describes a local-strategy for a Banach-Mazur game played on the outcomes; Abélard is in charge of simulating the Banach-Mazur game: Sometimes he chooses the directions and sometimes he plays what the local-strategy of Éloïse is indicating. Moreover, Éloïse does not observe who is currently playing in the Banach-Mazur game. The winning condition checks the parity condition as well as correctness of the simulation of the Banach-Mazur game (in particular, that no player plays eventually forever).

To describe the local-strategy, Éloïse will provide with any action $\gamma \in \Gamma$ a partial map $\theta : V \to (V \times \{\top, \bot\}) \times (V \times \{\top, \bot\})$ such that for all $v \in V$ one has $\theta(v) \in \Delta(v, \gamma) \times \{\top, \bot\} \times \Delta(v, \gamma) \times \{\top, \bot\}$; we denote by Θ_γ the set of such maps (for a given $\gamma \in \Gamma$).

We define a two-player imperfect-information arena (all vertices belong to Éloïse, so we omit vertices and the transition relation of Abélard) $\widetilde{\mathcal{G}} = (\widetilde{V}, \widetilde{\Gamma}, \widetilde{\Delta}, \approx)$, where $\widetilde{V} = V \times \{E, A\} \times \{f, n\}$ (the second component is used to remember who plays in the simulation of the Banach-Mazur game; the third component is f if the first move of Éloïse in the Banach-Mazur game has not yet been fully played), $(v, X, x) \approx (v', Y, y)$ if and only if $v \sim v'$ (Éloïse does not observe the second and third components), $\widetilde{\Gamma} = \bigcup_{\gamma \in \Gamma} \{\gamma\} \times \Theta_\gamma$ and $\widetilde{\Delta}((v, X, x), (\gamma, \theta))$ is as follows:

- If $X = A$, then it equals $\Delta(v, \gamma) \times \{E, A\} \times \{x\}$: Abélard can choose any successor and can decide to finish/continue his move in the Banach-Mazur component.
- If $X = E$, then it is the singleton consisting of node (v_x, Y, y) defined by letting[13] $\theta(v) = (v_f, y_f, v_n, y_n)$ and letting $Y = A$ and $y = n$ if $y_x = \top$ (we switch the player in the Banach-Mazur game) and $Y = E$ and $y = x$ if $y_x = \bot$ (she keeps playing).

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[13]In case $\theta(v)$ is undefined, Éloïse loses the play. We assume this never happens, but it can easily be captured in the winning condition by adding an extra vertex.

We let $\widetilde{\Omega}$ consist of those plays $(v_0, X_0, x_0)(v_1, X_1, x_1)(v_2, X_2, x_2) \cdots$ such that either (i) $v_0 v_1 v_2 \cdots$ satisfies the winning condition and one has $X_j = A$ for infinitely many j (i.e., Éloïse does not eventually play forever in the Banach-Mazur game) or (ii) there is some $N \geqslant 0$ such that one has $X_j = A$ for all $j \geqslant N$ (i.e., Abélard eventually plays forever in the Banach-Mazur game). In particular, $\widetilde{\Omega}$ is a (positive) Boolean combination of Ω and a parity condition.

Finally, we denote by $\widetilde{\mathbb{G}}$ the imperfect-information game $(\widetilde{\mathcal{G}}, \widetilde{\Omega}, (v_0, A, f))$. The following relates the games $\mathbb{G}$ and $\widetilde{\mathbb{G}}$:

THEOREM 6.9. *Éloïse has a topologically-good strategy in $\mathbb{G}$ if and only if she has a winning strategy in $\widetilde{\mathbb{G}}$.*

More precisely, from a winning strategy (respectively, positional strategy, respectively, finite-memory strategy) $\widetilde{\varphi}_E$ of Éloïse in $\widetilde{\mathbb{G}}$, we can define a topologically-good strategy (respectively, positional strategy, respectively, finite-memory strategy) φ_E for Éloïse in $\mathbb{G}$.

PROOF. Strategies $\widetilde{\varphi}$ for Éloïse in $\widetilde{\mathbb{G}}$ are in bijections with pairs made of a strategy φ in $\mathbb{G}$ together with a local-strategy (φ_f, φ_n) in the tree of the outcomes of φ in $\mathbb{G}$. Now, if $\widetilde{\varphi}$ is winning in $\widetilde{\mathbb{G}}$, then we have, thanks to the second part of $\widetilde{\Omega}$, that the local-strategy (φ_f, φ_n) is valid; and, thanks to the first part of $\widetilde{\Omega}$, that any compatible play is winning for Éloïse in the Banach-Mazur game. Hence, it implies that φ is topologically-good (the set of winning plays in $T_{v_0}^{\varphi}$ is large). Obviously, φ does not require more memory than $\widetilde{\varphi}$.

Conversely, if Éloïse has a topologically-good strategy φ in $\mathbb{G}$, then we can associate with φ a local-strategy (φ_f, φ_n) as in Lemma 6.8 (applied to $T_{v_0}^{\varphi}$). Using φ, φ_f, and φ_n, we define a winning strategy $\widetilde{\varphi}$ for Éloïse in $\widetilde{\mathbb{G}}$ as follows: We let $\widetilde{\varphi}([(v_0, X_0, x_0)(v_1, X_1, x_1) \cdots (v_k, X_k, x_k)]_{/\approx}) = (\gamma, \theta)$ where $\gamma = \varphi([v_0 v_1 \cdots v_k]_{/\sim})$ and θ is defined as follows: Let $v \in V$: if there is no $v_0' \cdots v_k' \in T_{v_0}^{\varphi} \cap [v_0 \cdots v_k]_{/\sim}$ with $v_k = v$, then we let $\theta(v)$ undefined; otherwise, choose such a $v_0' \cdots v_k'$ (the representative actually does not matter, thanks to the fact that (φ_f, φ_n) is the same in isomorphic subtrees) and define $\theta(v) = (\varphi_f(v_0' \cdots v_k'), \varphi_n(v_0' \cdots v_k'))$.

Now consider a play $\widetilde{\lambda} = (v_0, X_0, x_0)(v_1, X_1, x_1)(v_2, X_2, x_2) \cdots$ in $\widetilde{\mathbb{G}}$ where Éloïse respects $\widetilde{\varphi}$. If there are infinitely many i such that $X_i = E$, then there are infinitely many j such that $X_j = A$ (this is because (φ_f, φ_n) is valid). Moreover, if there are infinitely many i such that $X_i = E$, then the play $v_0 v_1 v_2 \cdots$ is a branch in $T_{v_0}^{\varphi}$ that is (φ_f, φ_n)-compatible and, therefore, it satisfies the parity condition by Lemma 6.8 and definition of (φ_f, φ_n). Therefore, the strategy $\widetilde{\varphi}$ is winning for Éloïse in $\widetilde{\mathbb{G}}$ and it concludes the proof.   □

7 IMPERFECT-INFORMATION GAMES WITH NATURE: SOME CONSEQUENCES

7.1 Imperfect-information Parity Games on Finite Graphs

For imperfect-information, in the case of finite arena, as soon as one considers co-Büchi conditions, almost-sure winning is undecidable even for Éloïse-Nature game where Éloïse is totally blind (all vertices are equivalent) [4]. Therefore, both the cardinality setting and the topological one are interesting alternatives to retrieve decidability: Indeed, thanks to Theorem 6.7 (respectively, Theorem 6.9) combined with the results in Reference [14], we get decidability for finite arena for any parity condition.

7.2 Imperfect-information Parity Games on Infinite Finite Graphs

There is very little work in the probabilistic setting about games with imperfect-information played on *infinite* arenas. The notable exception is the case of concurrent reachability games played

on single-exit recursive state machines[14] for which impressive results were obtained in Reference [19]. In the non-stochastic setting, it is easy to derive decidability results for *parity* game played on pushdown graphs when Éloïse perfectly observes the stack content but not the exact control state and Abélard is perfectly informed (see e.g., Reference [2]); this result can easily be extended for more general classes of graphs as collapsible pushdown graphs as defined in Reference [22]. Hence, thanks to Theorems 6.7 and 6.9, we obtain decidability results for games with Nature played on those classes of infinite arenas. Note that in the cardinality setting, even if we require that Abélard has perfect-information, our model captures concurrent games.

7.3 Probabilistic Automata

A temptation would be to consider cardinality/topological variants of probabilistic automata on infinite words [4], as such a machine can be thought of as an Éloïse-Nature game where Éloïse is totally blind: e.g., declare that an ω-word is accepted by an automaton if all but a countable number of runs on it are accepting (respectively, the set of accepting runs is large). However, a simple consequence of our results is that the languages defined in this way are always ω-regular.

8 CONCLUSION AND PERSPECTIVES

In this article, we provided several reductions that can later be used to obtain decidability results depending on the properties of both the arena and the winning condition. More precisely, we give transformations that associate with any game with Nature $\mathbb{G} = (G, \Omega, v_0)$ a game *without* Nature $\mathbb{G}' = (G', \Omega', v_0')$ on which the question (Is the leaking value is countable? Is the leaking value smaller than some threshold k? Is there a topologically-good strategy?) on the original game $\mathbb{G}$ is restated as whether Eve has a winning strategy in $\mathbb{G}'$. In all cases, the new arena G' is obtained from G by adding some gadgets, and the new winning condition Ω' is a Boolean combination of Ω with an ω-regular condition. Moreover, for some cases, we need extra hypotheses that we recall in the table below.

	$LeakVal(\mathbb{G}) \leqslant \aleph_0$? $LeakVal(\mathbb{G}) \leqslant k$?	Topologically-good?
Perfect-information		
	No extra hypothesis on $\mathbb{G}$	Eve + Nature only
	No extra hypothesis on Ω	No extra hypothesis on Ω
Imperfect-information		
	Adam perfect	Eve + Nature only
	Ω: parity	Ω: parity

Regarding perspectives, the most natural question is whether we can drop the restriction on Abélard not being part of the game for questions regarding the topological setting.

Another exciting problem is whether one can decide if the leaking value of a game is finite (without knowing *a priori* the bound). We believe that this problem should be decidable for parity games on finite graph but using different techniques than the one developed in this article.

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[14]Concurrency is a special instance of imperfect-information where Abélard is perfectly informed: He chooses an action that is stored on the state and cannot be observed by Éloïse, who next chooses an action that together with the one by Abélard leads to the next state (chosen by Nature). Recursive state machines are equivalent with pushdown automata; however, the single exit case quite strongly restricts the model.

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