# $\exists\mathbb{R}$-Completeness for Decision Versions of Multi-Player (Symmetric) Nash Equilibria

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As a result of a series of important works [7–9, 15, 23], the complexity of two-player Nash equilibrium is by now well understood, even when equilibria with special properties are desired and when the game is symmetric. However, for multi-player games, when equilibria with special properties are desired, the only result known is due to Schaefer and Štefankovič [28]: that checking whether a three-player Nash Equilibrium (3-Nash) instance has an equilibrium in a ball of radius half in $l_\infty$-norm is $\exists\mathbb{R}$-complete, where $\exists\mathbb{R}$ is the class of decision problems that can be reduced in polynomial time to Existential Theory of the Reals.

Building on their work, we show that the following decision versions of 3-Nash are also $\exists\mathbb{R}$-complete: checking whether (i) there are two or more equilibria, (ii) there exists an equilibrium in which each player gets at least $h$ payoff, where $h$ is a rational number, (iii) a given set of strategies are played with non-zero probability, and (iv) all the played strategies belong to a given set.

Next, we give a reduction from 3-Nash to symmetric 3-Nash, hence resolving an open problem of Papadimitriou [25]. This yields $\exists\mathbb{R}$-completeness for symmetric 3-Nash for the last two problems stated above as well as completeness for the class $\text{FIXP}_a$, a variant of FIXP for strong approximation. All our results extend to $k$-Nash for any constant $k \geq 3$.

CCS Concepts: • **Theory of computation** → *Exact and approximate computation of equilibria*;

Additional Key Words and Phrases: Nash equilibrium, symmetric games, decision problems, existential theory of reals, FIXP

**ACM Reference format:**
Jugal Garg, Ruta Mehta, Vijay V. Vazirani, and Sadra Yazdanbod. 2018. $\exists\mathbb{R}$-Completeness for Decision Versions of Multi-Player (Symmetric) Nash Equilibria. *ACM Trans. Econ. Comput.* 6, 1, Article 1 (January 2018), 23 pages.
https://doi.org/10.1145/3175494

## 1 INTRODUCTION

Nash equilibrium (NE) is arguably the most important and well-studied solution concept within game theory, and understanding its complexity has led to an impressive theory, which was discovered largely over the past decade. We denote by $k$-Nash the problem of computing a NE in a

An extended abstract of this work appeared in the proceedings of the *42nd International Colloquium on Automata, Languages and Programming (ICALP)*. This work is supported by NSF Grants CCF-0914732 and CCF-1216019.

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https://doi.org/10.1145/3175494

$k$-player game for a constant $k$. For the case of 2-Nash, the seminal results of Daskalakis, Goldberg, and Papadimitriou [9] and Chen, Deng, and Teng [7] exactly characterized the complexity of this problem, namely it is PPAD-complete. This leads us to another basic question: of finding a $k$-Nash solution that satisfies special properties, e.g., has a payoff of at least $h$ for each player. For the case of two players, these questions were first studied by Gilboa and Zemel for non-symmetric games [15] and later by Conitzer and Sandholm for symmetric games [8]. Both articles considered 2-Nash under numerous special properties and showed them all to be NP-complete. More recently, Bilò and Mavronicolas [3] extended the results of Gilboa and Zemel to win-lose games, in which all payoffs are either 0 or 1. Thus, the complexity of the two-player case is very well understood.

Although the two-player case is the most classical and well-studied case, it is also important to study the complexity of the multi-player case, especially in the context of new applications arising on the Internet and other large networks where multiple players are locked in strategic situations. Indeed, there has been much activity on this front, e.g., see References [2, 17, 26], but the picture is not as clear as the two-player case. A fundamental difference between 2-Nash and $k$-Nash, for $k \geq 3$, is that whereas the former always admits an equilibrium that can be written using rational numbers [18], the latter require irrational numbers in general, as shown by Nash himself [22] (we will assume that all numbers in the given instance are rational). It is easy to see that in the latter case, equilibria are algebraic numbers. This difference makes the multi-player case much harder.

Daskalakis, Goldberg, and Papadimitriou [9] showed that for $k$-player games, $k \geq 3$, finding an $\epsilon$-approximate Nash equilibrium is PPAD-complete. The complexity of exact equilibrium was resolved by Etessami and Yannakakis [11], who showed this case to be complete for their class FIXP. How about the complexity of finding a $k$-Nash solution that satisfies special properties? Due to the inherent difficulty of dealing with irrational numbers, this problem remained open until 2011, when Schaefer and Štefankovič [28] formally defined class $\exists\mathbb{R}$, and showed that checking if a three-player game has a NE in which every strategy is played with probability at most 0.5 (**InBox**) is $\exists\mathbb{R}$-complete. $\exists\mathbb{R}$ is the class of "yes" instances of existentially quantified formulas with bases $\{+, -, *, \wedge, \vee, =, <, >\}$ on real numbers; we note that this class was informally known and used earlier than Reference [28], e.g., see Reference [6]. In Reference [10], Datta showed that an arbitrary semi-algebraic set can be encoded as *totally mixed* NE of a three-player game. However, the reduction is not polynomial time and therefore is not applicable to show $\exists\mathbb{R}$-completeness of the decision problems in 3-Nash. Recently, in Reference [19] Levy gave another construction to *precisely* capture any compact semi-algebraic set of mixed-strategies of a game as a projection of Nash equilibrium strategies of another game with additional binary players; however, no bound is provided on the number of additional players.

Our first set of results extends $\exists\mathbb{R}$-completeness to NE computation with a number of special properties in $\geq$ 3-player games: ($i$) checking if a game has more than one NE (**NonUnique**). NE where, ($ii$) each player gets at least $h$ payoff (**MaxPayoff**), ($iii$) a given set of strategies are played with positive probability (**Subset**), or ($iv$) all the played strategies belong to a given set (**Superset**).

Our second set of results deals with symmetric games. Symmetry arises naturally in numerous strategic situations and with the growth of the Internet, on which typically users are indistinguishable, such situations are only becoming more ubiquitous. In a *symmetric game* all players participate under identical circumstances, i.e., strategy sets and payoffs. Thus, the payoff of player $i$ depends only on the strategy, $s$, played by her and the multiset of strategies, $S$, played by the others, without reference to their identities. Furthermore, if any other player $j$ were to play $s$ and the remaining players $S$, the payoff to $j$ would be identical to that of $i$. A *symmetric Nash equilibrium* (SNE) is a NE in which all players play the same strategy. Nash [22], while providing game theory with its central solution concept, also defined the notion of a symmetric game and proved, in a separate theorem, that such games always admit a symmetric equilibrium.

Table 1. Dichotomy for NE

| | 2-Nash | $k$-Nash, $k \geq 3$ |
|---|---|---|
| Nature of solution | Rational [18] | Algebraic; irrational example [22] |
| Complexity | PPAD-complete [7, 9, 23] | FIXP-complete [11] |
| Practical algorithms | Lemke-Howson [18] | ? |
| Decision problems | NP-complete [8, 15] | ∃ℝ-complete: [28]<br>$\mathcal{CP}$ (Theorems 4.20, 4.26) |

Table 2. Dichotomy for Symmetric NE

| | Symmetric 2-Nash | Symmetric $k$-Nash, $k \geq 3$ |
|---|---|---|
| Nature of solution | Rational [18] | Algebraic; irrational example<br>$\mathcal{CP}$ together with [22] |
| Complexity | PPAD-complete [7, 9, 23] | FIXP$_a$-complete: $\mathcal{CP}$ (Theorem 7.4) |
| Practical algorithms | Lemke-Howson [18, 27] | ? |
| Decision problems | NP-complete [8] | ∃ℝ-complete: $\mathcal{CP}$ (Theorem 7.2) |

A simple reduction is known from 2-Nash to symmetric 2-Nash, and it shows that the latter is also PPAD-complete. The questions studied by Gilboa and Zemel [15] for two-player games were studied by Conitzer and Sandholm [8] for symmetric games and were shown to be NP-complete. On the other hand, no reduction is known from 3-Nash to symmetric 3-Nash. Indeed, after giving the reduction from 2-Nash to symmetric 2-Nash, Papadimitriou [25] states, "Amazingly, it is not clear how to generalize this proof for three player games!"

To obtain our results on symmetric $k$-player games, for $k \geq 3$, we first give a reduction from 3-Nash to symmetric 3-Nash, hence settling the open problem of Reference [25]. This also enables us to show that symmetric 3-Nash is complete for the class FIXP$_a$, Strong Approximation FIXP, which is a variant of FIXP that is restricted to working with rational numbers only. It also yields ∃ℝ-completeness for **Superset** and **Subset** in such games. Once the three-player case is settled, we prove analogous results for symmetric $k$-player games, for $k > 3$.

Reference [14] gave a dichotomy for NE, showing a qualitative difference between 2-Nash and $k$-Nash along three different criteria; see Table 1. The results of this article add a fourth criterion to this dichotomy, namely complexity of decision problems. Additionally, we get an analogous dichotomy for symm etric NE; see Table 2. Results of current article are indicated by $\mathcal{CP}$ in the tables.

We note that the results of the current article were first presented in Reference [13]. In that article, we had left the open problem of extending our ∃ℝ-completeness results to decision versions of other 3-Nash and symmetric 3-Nash problems. Since then, there has been much progress on this open problem. First, Bilò and Mavronicolas [4] showed that the three-player versions of all problems studied by Gilboa and Zemel [15] are ∃ℝ-complete; moreover, they do so via a unified reduction from **InBox**. Next, the same authors [5] showed ∃ℝ-completeness for several decision versions of symmetric 3-Nash, this time via reductions from **Subset**.

### 1.1 Technical Overview

We first give the main idea behind our reduction from 3-Nash to symmetric 3-Nash (Theorem 5.5). We will reduce the given game $(A, B, C)$, where each tensor is of size $m \times n \times p$, to a symmetric game, $D$, of size $l \times l \times l$, where $l = m + n + p$ (see Section 2.1 for the description of (symmetric) games). In this game, under each symmetric NE, the strategy of each player can be decomposed into three vectors, say $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, of dimension $m, n, p$, respectively. An essential condition for recovering a Nash equilibrium for the original game $(A, B, C)$ is that each of these three vectors be non-zero; this is also the most difficult part of the reduction.

To achieve this, we construct a $3 \times 3 \times 3$ symmetric game $G$ all of whose symmetric NE are of full support, even though it is only partially specified (see Equation (6)). We "blow up" $G$ to derive $D$, which is of size $l \times l \times l$, and the unspecified entries of $G$ create room where tensors $A, B, C$ are "inserted." Now, if $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a symmetric NE of $D$, then so is $(\sum_i x_i, \sum_j y_j, \sum_k z_k)$ of $G$. As a result, each vector $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \neq 0$. Next, we show that if these vectors are scaled to probability vectors, they form a NE for $(A, B, C)$. Additional arguments yield $\exists\mathbb{R}$-completeness for **Subset** and **Superset** for symmetric $k$-Nash (Theorems 5.6 and 5.7).

Next, we give the idea for showing that symmetric 3-Nash is complete for the class $\text{FIXP}_a$ (Theorem 6.4). Note that we are unable to show that symmetric 3-Nash is complete for the class FIXP itself, since we don't see how to express the solution to the given instance $(A, B, C)$ as a rational linear projection of the solution of the reduced symmetric game $D$, a requirement of FIXP reductions [11].

Under $\text{FIXP}_a$, given an instance $I$ and a rational $\epsilon > 0$, we need to compute a vector $\boldsymbol{x}$ that is within (additive) $\epsilon$ distance from some solution, i.e., $\exists \boldsymbol{x}^* \in Sol(I)$ such that $|\boldsymbol{x}^* - \boldsymbol{x}|_\infty \leq \epsilon$, in time polynomial in $size[I]$ and $\log(1/\epsilon)$. In the above reduction, obtaining a solution of $(A, B, C)$ involves, e.g., dividing $\boldsymbol{x}$ by $\sum_i x_i$. If the latter is very small, then this may give us a vector that is very far away from a solution of $(A, B, C)$, even though $x$ may be close to a solution of $D$.

We get around this problem by a small change in the above reduction, namely, we need to multiply the tensors $A, B, C$ by a small constant $\epsilon'$ before they are "inserted" at the appropriate places in $G$ to get symmetric game $D$. This ensures that vector $(\sum_i x_i, \sum_j y_j, \sum_k z_k)$ is approximately $(1/3, 1/3, 1/3)$. As a result, given a point close to a solution of $D$, we can get a point "close" to a solution of $(A, B, C)$.

Next, we describe how we show $\exists\mathbb{R}$-completeness for the four decision problems, mentioned in the previous section, for $k$-Nash. To show hardness in the case of three players, we reduce **InBox**, which is known to be $\exists\mathbb{R}$-complete for 3-Nash [28], to each of **MaxPayoff**, **Subset**, and **Superset**, and then from **MaxPayoff** to **NonUnique**. Hardness for the $k$-Nash, $k > 3$, follows, since 3-Nash reduces to $k$-Nash trivially by introducing dummy players. To show containment in $\exists\mathbb{R}$, we give a non-linear complementarity problem (NCP) formulation that exactly captures NE of a given game (Theorems 3.1 and 3.2).

Next, we briefly explain the reduction from **InBox** to **MaxPayoff** for the two-player case (see Section 4.1 for details); three-player case is an extension of it (Section 4.2). Let the given game be represented by two payoff matrices $(A, B)$ of size $m \times n$, one for each player. The **InBox** problem is to check if it has a NE in which all strategies are played with at most 0.5 probability. We reduce it to checking if another game $(C, D)$ has a NE in which every player gets payoff at least $h > 0$ (**MaxPayoff**). Without loss of generality (wlog), we can assume that $A, B > 0$.

We construct $m(n+1) \times n(m+1)$ matrices $C$ and $D$, where the top-left block is set to $A + h$ and $B + h$, respectively. This ensures that if each player gets payoff $h$ at a NE, then strategies from this block are played with non-zero probability, and normalizing them gives a NE of $(A, B)$. The latter follows, since NE set remains invariant under additive scaling of payoffs. To retrieve a NE in 0.5 ball, we ensure that if any of these strategies is (relatively) played with more than 0.5 probability then a sequence of deviations leads to both players playing only among their last $mn$ strategies where payoff is zero ($<h$).

In particular, suppose the second player plays $\boldsymbol{y}$ in the top-left block. The last $mn$ strategies of the row player are divided into $n$ blocks of size $m$, one for each $y_j$, $j \leq n$ such that if $y_j > 0.5$ then best response of the first player is to deviate to $j$th block. The payoff of the second player is set to −1 in these blocks, so then $y_j$ fetches −1 and second player is forced to deviate to her last $mn$ strategies where both get zero. Similarly for the first player.

**Organization:** In Section 2, we formally define the (symmetric) $k$-Nash problem, their decision problems, and discuss the complexity classes ∃ℝ and FIXP. Membership in ∃ℝ for decision problems in (symmetric) $k$-Nash is shown in Section 3. In Section 4, we show that decision problems in $k$-player games, for any constant $k \geq 3$, are ∃ℝ-complete. ∃ℝ-completeness of decision problems in *symmetric* 3-Nash is shown in Section 5. In Section 6, we show that computing an equilibrium in symmetric 3-Nash is $\text{FIXP}_a$-complete. Since symmetric 3-Nash does not trivially reduce to symmetric $k$-Nash, we extend the ∃ℝ and $\text{FIXP}_a$-completeness results for the latter in Section 7 for any constant $k \geq 3$.

## 2 PRELIMINARIES

In this section, we formally define the (symmetric) $k$-Nash problem, and their decision problems. Further, we discuss the complexity classes ∃ℝ and FIXP.

**Notations:** Vectors are represented in bold-face letters, and $i$th coordinate of vector $\boldsymbol{x}$ is denoted by $x_i$, and $\boldsymbol{x}^{-i}$ denotes the vector $\boldsymbol{x}$ with $i$th coordinate removed. $\mathbf{1}$ and $\mathbf{0}$ represent all ones and all zeros vector, respectively, of appropriate dimension. For integers $k < l$, $\boldsymbol{x}(k : l) = (x_k, x_{k+1}, \ldots, x_l)$. We use $[n]$ to denote set $\{1, \ldots, n\}$ and $[k : l]$ to denote $\{k, k+1, \ldots, l\}$. If $\boldsymbol{x}$ is of $m$ dimensional, then by $\sigma(\boldsymbol{x})$ we mean $\sum_{i=1}^{m} x_i$, and $\eta(\boldsymbol{x}) = \boldsymbol{x}/\sigma(\boldsymbol{x})$. Concatenation of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $(\boldsymbol{x}|\boldsymbol{y})$. Given a matrix $A$ and $h \in \mathbb{R}$, $A + h$ denotes the matrix $A$ with $h$ added to each of its entries. Further, $A(i, :)$ is its $i$th row and $A(:, j)$ is its $j$th column.

### 2.1 (Symmetric) $k$-Nash

For a given $k$-player game let $S_i, i \in [k]$ be the set of pure strategies of player $i$, and let $\boldsymbol{S} = \times_{i\in[k]} S_i$. The payoffs of player $i$ can be represented by a $k$-dimensional tensor $A_i$, such that $A_i(\boldsymbol{s})$ denotes the payoff she gets when $\boldsymbol{s} \in \boldsymbol{S}$ is played. Players may randomize among their strategies. Let $\Delta_i$ denote the set of mixed strategy profiles of player $i$, and let $\boldsymbol{\Delta} = \times_{i\in[k]} \Delta_i$. Expected payoff of player $i$ from $\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k) \in \boldsymbol{\Delta}$ is $\pi_i(\boldsymbol{x}) = \sum_{\boldsymbol{s}\in\boldsymbol{S}} (\Pi_{i\in[k]} \boldsymbol{x}^i_{s_i}) A_i(\boldsymbol{s})$.

*Definition 2.1.* (Nash Equilibrium (NE) [22]) $\boldsymbol{x} \in \boldsymbol{\Delta}$ is said to be a NE if no player gains by unilateral deviation. Formally, $\forall i,\ \forall \boldsymbol{x}'^i \in \Delta_i,\ \pi_i(\boldsymbol{x}) \geq \pi_i(\boldsymbol{x}'^i, \boldsymbol{x}^{-i})$.

Let $\pi_i(s, \boldsymbol{x}^{-i})$ denote the payoff $i$ receives when she plays $s \in S_i$ and others play as per $\boldsymbol{x}^{-i}$. It is easy to see that $\boldsymbol{x}$ is a NE iff [22]

$$\forall i \in [k],\ \ \forall s \in S_i,\ \ x^i_s > 0 \ \ \Rightarrow \ \ \pi_i(s, \boldsymbol{x}^{-i}) = \max_{t\in S_i} \pi_i(t, \boldsymbol{x}^{-i}). \tag{1}$$

**Symmetric $k$-Nash:** In a symmetric game the players are indistinguishable. Their strategy sets are identical ($S$) and payoffs are symmetric represented by one tensor $A$. For a player, the payoff she gets by playing $s' \in S$, when others are playing $\boldsymbol{s} \in S^{(k-1)}$, is $A(s', \boldsymbol{s})$. Further, who is playing what in $\boldsymbol{s}$ does not matter. Formally, $A$ satisfies $A(s', \boldsymbol{s}) = A(s', \boldsymbol{s}_\tau)$ for all permutations $\tau$ of $(1, \ldots, k-1)$, where $\boldsymbol{s}_\tau$ is the corresponding permuted vector.

A profile $\boldsymbol{x} \in \boldsymbol{\Delta}$ is called *symmetric* if $\boldsymbol{x}^i = \boldsymbol{x}^j,\ \forall i, j \in [k]$, thus one vector $\boldsymbol{x} \in \Delta$ is enough to denote a symmetric profile. At a symmetric strategy profile all the players get the same payoff, and we denote it by $\pi(\boldsymbol{x})$. The problem of computing a symmetric NE (SNE) of a symmetric game is called *symmetric k-Nash*.

Note that the description of a (symmetric) $k$-player game takes $O(km^k)$ space, where $m = \max_i |S_i|$, which is exponential in $m$ and $k$. To keep it polynomial, we consider $k$ as a constant.

Further, wlog $(A_1, \ldots, A_k) > 0$, because adding a constant to the tensors does not change the set of NE.

**2-Nash:** The payoff tensors in case of two-player game are matrices, say $(A, B)$, $A$ for player one and $B$ for player two. If the first player plays $i$ and second plays $j$, then their respective payoffs are $A_{ij}$ and $B_{ij}$. Game is said to be symmetric if $B = A^T$. A mixed strategy is $(\boldsymbol{x}, \boldsymbol{y}) \in \Delta_1 \times \Delta_2$, and respective payoffs at such a strategy are $\boldsymbol{x}^T A\boldsymbol{y}$ and $\boldsymbol{x}^T B\boldsymbol{y}$. 2-Nash is the problem of finding a Nash equilibrium of such a game, i.e., strategy $(\boldsymbol{x}, \boldsymbol{y})$ such that

$$\boldsymbol{x}^T A\boldsymbol{y} \geq \boldsymbol{x}'^T A\boldsymbol{y}, \ \ \forall \boldsymbol{x}' \in \Delta_1 \quad \text{and} \quad \boldsymbol{x}^T B\boldsymbol{y} \geq \boldsymbol{x}^T B\boldsymbol{y}', \ \ \forall \boldsymbol{y}' \in \Delta_2.$$

The NE characterization of Equation (1) reduces to:

$$\forall i \in S_1, \ x_i > 0 \Rightarrow (A\boldsymbol{y})_i = \max_{k \in S_1} (A\boldsymbol{y})_k; \ \forall j \in S_2, \ y_j > 0 \Rightarrow (\boldsymbol{x}^T B)_j = \max_{k \in S_2} (\boldsymbol{x}^T B)_k. \tag{2}$$

**3-Nash:** It is the $k$-Nash problem with $k = 3$ players. We will represent such a game by three three-dimensional tensors $(A, B, C)$; $A$ for player one, $B$ for player two, and $C$ for player three. If player one plays $i$, two plays $j$, and three plays $k$, then their respective payoffs are $A_{ijk}$, $B_{ijk}$, and $C_{ijk}$. If the game is symmetric, then we have $A_{ijk} = A_{ikj} = B_{jik} = B_{kij} = C_{jki} = C_{kji}$. A mixed strategy is denoted by $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \Delta_1 \times \Delta_2 \times \Delta_3$. Thus, NE characterization of Equation (1) reduces to

$$\begin{array}{lll} \forall i \in S_1, \ \ x_i > 0 & \Rightarrow & \sum_{j \in S_2, k \in S_3} A_{ijk} y_j z_k = \max_{l \in S_1} \sum_{j \in S_2, k \in S_3} A_{ljk} y_j z_k, \\ \forall j \in S_2, \ \ y_j > 0 & \Rightarrow & \sum_{i \in S_1, k \in S_3} B_{ijk} x_i z_k = \max_{l \in S_2} \sum_{i \in S_1, k \in S_3} B_{ilk} x_i z_k, \\ \forall k \in S_3, \ \ z_k > 0 & \Rightarrow & \sum_{i \in S_1, j \in S_2} C_{ijk} x_i y_j = \max_{l \in S_3} \sum_{i \in S_1, j \in S_2} C_{ijl} x_i y_j. \end{array} \tag{3}$$

**Decision Problems:** Computational complexity of numerous decision problems have been studied for 2-Nash and 3-Nash [8, 15]. In this article, we consider the following:

- **NonUnique**: Does there exist more than one NE?
- **MaxPayoff**: Given a rational number $h$, does there exist a NE where every player gets payoff at least $h$?
- **Subset**: Given sets $T_i \subset S_i$, $\forall i \in [k]$, does there exist a NE where every strategy in $T_i$ is played with positive probability by player $i$?
- **Superset**: Given sets $T_i \subset S_i$, $\forall i \in [k]$, does there exist a NE where all the strategies outside $T_i$ are played with zero probability by player $i$?
- **InBox**: Does there exists a NE where every strategy is played with probability less than or equal 0.5?

All but last problem have been shown to be NP-complete in case of 2-Nash [8, 15], and the last one is shown to be $\exists\mathbb{R}$-complete in case of 3-Nash [28]. In this article, we show $\exists\mathbb{R}$-completeness for the first four decision problems for $k$-Nash, and for third and fourth for symmetric $k$-Nash, where $k \geq 3$ in both cases.

## 2.2 Existential Theory of Reals (ETR)

An instance $I$ of the *existential theory of reals* (ETR) consists of a sentence of the form

$$(\exists x_1, \ldots, x_n) \ \ \phi(x_1, \ldots, x_n),$$

where $\phi$ is a quantifier-free $(\wedge, \vee, \neg)$-Boolean formula over the predicates (sentences) defined by signature $\{0, 1, -1, +, *, <, \leq, =\}$ over variables that take real values. The problem is to check if the sentence is true. The size of the problem is $n + size(\phi)$, where $n$ is the number of variables and $size(\phi)$ is the minimum number of signatures needed to represent $\phi$. Schaefer and Štefankovič

[28] defined complexity class ∃ℝ as the downward closure of these problems under polynomial-time many-one reductions (we refer the reader to Reference [28] for more details on ∃ℝ, and its relation with other classes like PSPACE). Note that the number of signatures required to represent any rational number is linear in its bit-length. Schaefer and Štefankovič showed that for 3-Nash, problem **InBox** is ∃ℝ-complete.

### 2.3 The Class FIXP and its Variant $\mathrm{FIXP}_a$

Etessami and Yannakakis [11] defined the class FIXP to capture complexity of the exact fixed point problems with algebraic solutions. A FIXP problem is to find a fixed-point of a function $F : D \rightarrow D$ over a convex, compact domain $D$, i.e., find $\boldsymbol{x} \in D$ s.t. $F(\boldsymbol{x}) = \boldsymbol{x}$. The function is given by an arithmetic circuit $C$ with $\{\min, \max, +, -, *, /\}$ gates, rational constants, and $n$ input/output; $size[C] = n + \#$ gates + bit-length(constants). Given $\boldsymbol{\lambda} \in D$ to $C$ as an input, all its gates are well defined.

Fixed-points of $F$ may be irrational. Therefore, to remain faithful to Turing machine computation, Etessami and Yannakakis [11] defined a discrete class $\mathrm{FIXP}_a$.

**(Strong) Approximation $\mathrm{FIXP}_a$:** Given circuit $C$ defining function $F$, and a rational $\epsilon > 0$, compute a vector $\boldsymbol{x}$ that is within (additive) $\epsilon$ distance in $l_\infty$ norm from $\boldsymbol{x}^*$ where $F(\boldsymbol{x}^*) = \boldsymbol{x}^*$ (a fixed-point), in time polynomial in $size[C]$ and $\log(1/\epsilon)$.

Theorem 2.2. *[11] Given a three-player game $(A_1, A_2, A_3)$, computing its NE is FIXP-complete. The corresponding (Strong) Approximation is complete for* $\mathrm{FIXP}_a$.

To handle irrational solutions in reductions to show FIXP-hardness, the reduction has to provide a linear function that maps solutions of the reduced instance to the solutions of the original instance. The constants used in the linear function have to be of size polynomial in the size of the given instance [11].

## 3 (SYMMETRIC) $K$-NASH: CONTAINMENT IN ∃ℝ

In this section, we show that the first four decision problems described in Section 2.1 are in ∃ℝ, for $k$-Nash as well as symmetric $k$-Nash. For a $k$-player game $(A_1, \ldots, A_k)$, NE characterization of Equation (1) can be reformulated as a set of polynomial inequalities as follows, where variable $x^i_s$ captures the probability with which player $i$ plays $s \in S_i$, and variable $\lambda_i$ captures her best payoff. Recall function $\pi_i(s, \boldsymbol{x}^{-i})$ from Section 2.1 representing payoff of player $i$ when she plays $s \in S_i$ and others play as per $\boldsymbol{x}^{-i}$:

$$\forall i \in [k], \;\; \forall s \in S_i, \;\; x^i_s \geq 0; \;\; \pi_i(s, \boldsymbol{x}^{-i}) \leq \lambda_i; \;\; x^i_s(\pi_i(s, \boldsymbol{x}^{-i}) - \lambda_i) = 0; \;\; \sum_{s \in S_i} x^i_s = 1. \tag{4}$$

It is easy to see that strategy profile $\boldsymbol{x} \in \boldsymbol{\Delta}$ satisfies Equation (1) if and only if it satisfies Equation (4).

Theorem 3.1. *Given a $k$-player game $(A_1, \ldots, A_k)$, for a constant $k$, the problems of* **NonUnique**, **MaxPayoff**, **Subset**, *and* **Superset** *are in* ∃ℝ.

Proof. To frame **NonUnique** as an ∃ℝ problem, take two copies of Equation (4) each with different sets of variables, say $\boldsymbol{x}$ and $\boldsymbol{y}$, and add $|\boldsymbol{x} - \boldsymbol{y}|^2 > 0$ to it. This system has a feasible solution $(\boldsymbol{x}, \boldsymbol{y})$ if and only if the game has two NE $\boldsymbol{x} \neq \boldsymbol{y}$. Thus, containment of **NonUnique** in ∃ℝ follows.

For **MaxPayoff**, add $\forall i \in [k], \; \pi_i(\boldsymbol{x}) \geq h$ to the system Equation (4). It has a feasible solution $\boldsymbol{x}$ if and only if $\boldsymbol{x}$ is a NE of the game where payoff received by every player is at least $h$, implying **MaxPayoff** is in ∃ℝ.

Similarly, to formulate **Subset**, add $\forall i \in [k],\ \forall s \in T_i,\ x_s^i > 0$ to (4). And for **Superset**, add $\forall i \in [1:k],\ \forall s \in S_i \setminus T_i,\ x_s^i = 0$ to Equation (4). □

Given a symmetric game $A$, the following system of polynomial inequalities (similar to Equation (4)) exactly captures its symmetric NE, where variable $x_s$ captures the probability of playing strategy $s \in S$ and $\lambda$ captures the payoff:

$$\forall s \in S, \quad x_s \geq 0; \quad \pi(s, \boldsymbol{x}) \leq \lambda; \quad x_s(\pi(s, \boldsymbol{x}) - \lambda) = 0 \ \text{ and } \ \sum_s x_s = 1.$$

The proof for the next theorem follows similar to that of Theorem 3.1.

Theorem 3.2. *Given a symmetric $k$-player game $A$, for a constant $k$, the problems of* **NonUnique***,* **MaxPayoff***,* **Subset***, and* **Superset** *for symmetric NE are in $\exists\mathbb{R}$.*

## 4 $K$-NASH: $\exists\mathbb{R}$-COMPLETENESS FOR DECISION PROBLEMS

In this section, we show that **MaxPayoff**, **Subset**, **Superset**, and **NonUnique** are $\exists\mathbb{R}$-complete in $k$-player games, for any constant $k \geq 3$. Containment in $\exists\mathbb{R}$ follows from Theorem 3.1 from Section 3. We show $\exists\mathbb{R}$-hardness for these four decision problems in the case of three-player games next, and since three-player games trivially reduce to $k$-player games, for $k > 3$, by adding $k-3$ dummy players with one strategy each, the result will follow for the latter as well. To show hardness for **MaxPayoff**, **Subset**, and **Superset**, we reduce from **InBox** (in Section 4.1), and for **NonUnique**, we reduce from **MaxPayoff** (in Section 4.3).

### 4.1 $\exists\mathbb{R}$-hardness: InBox to MaxPayoff, Subset, and Superset

To convey the main ideas, we first describe the reduction in two-player games and later generalize it to the three-player case in Section 4.2. We show the reduction from **InBox** to **MaxPayoff**, and from the intermediate lemmas, reduction to **Subset** and **Superset** will follow. Let the given two-player game be represented by $m \times n$-dimensional payoff matrices $(A, B) > 0$.

For $a \geq 0$, let $\mathcal{B}_a = [0,\ a]^{m+n}$ be a ball of radius $a$ at origin in $l_\infty$ norm. We will construct another game $(C, D)$, with $m(n+1) \times n(m+1)$-dimensional matrices, and show that it has a NE where each player gets at least $h > 0$ payoff (**MaxPayoff**) if and only if the game $(A, B)$ has a NE in $\mathcal{B}_{0.5}$ (**InBox**). First, we define a couple of notations required for the construction.

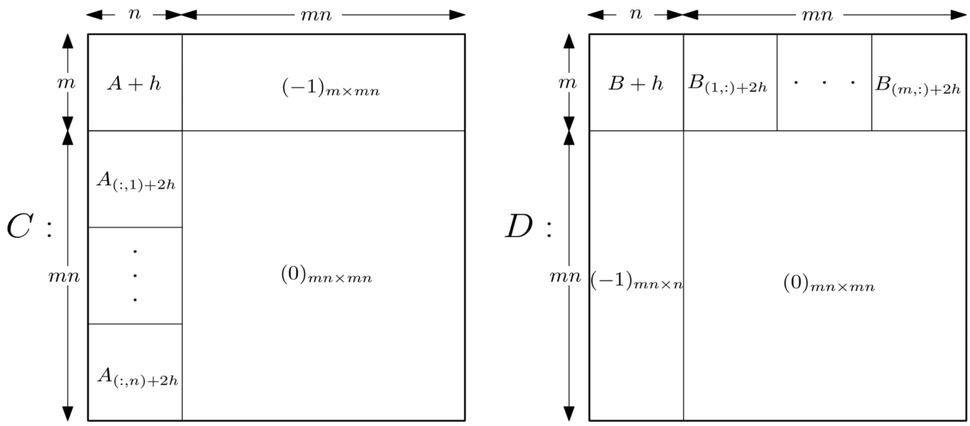
*Definition 4.1.* Let $i$ and $j$ be integers where $i \in [m]$ and $j \in [n]$, and $h$ be a real number. We define the following operators:
$A_{(i,:)+h}$ : matrix $A$ with $h$ added to the entries in its $i$th row, and
$A_{(:,j)+h}$ : matrix $A$ with $h$ added to the entries in its $j$th column.

*Definition 4.2.* Given a matrix $M$ of size $a \times b$ and integers $r$ and $s$ such that $a + r - 1 \leq m(n+1)$ and $b + s - 1 \leq n(m+1)$, define $[M]_{r,s}$ to be an $m(n+1) \times n(m+1)$-dimensional matrix where $M$ is copied starting at position $(r, s)$, and all other coordinates are set to zero.

Using the above notations, we construct matrices $C$ and $D$ as follows, where $h > 0$:

$$C = [A + h]_{1,1} + [(-1)_{m \times mn}]_{1,n+1} + \textstyle\sum_{j \in [n]} [A_{(:,j)+2h}]_{jm+1,1}, \text{ and}$$
$$D = [B + h]_{1,1} + [(-1)_{mn \times n}]_{m+1,1} + \textstyle\sum_{i \in [m]} [B_{(i,:)+2h}]_{1,in+1}.$$

The next lemma follows from the construction of $C, D$. Recall that $\sigma(\boldsymbol{x}) = \sum_i x_i$.




Lemma 4.3. *Given a strategy* $(\boldsymbol{x}', \boldsymbol{y}')$ *of game* $(C, D)$*, let* $\boldsymbol{x} = \boldsymbol{x}'(1 : m)$*,* $\boldsymbol{y} = \boldsymbol{y}'(1 : n)$*,* $\alpha = h * \sigma(\boldsymbol{y}) - \sigma(\boldsymbol{y}'(n+1 : (m+1)n))$*, and* $\beta = h * \sigma(\boldsymbol{x}) - \sigma(\boldsymbol{x}'(m+1 : (n+1)m))$*. Then,*

$$(C\boldsymbol{y}')_i \;=\begin{cases}\alpha + (A\boldsymbol{y})_i & \text{if } i \in [m]\\ 2hy_{\lfloor (i-1)/m \rfloor} + (A\boldsymbol{y})_r & \text{if } i \in [m+1, m(n+1)],\; r = ((i-1) \bmod m) + 1.\end{cases}$$

$$(\boldsymbol{x}'^T D)_j = \begin{cases}\beta + (\boldsymbol{x}^T B)_j & \text{if } j \in [n]\\ 2hx_{\lfloor (j-1)/n \rfloor} + (\boldsymbol{x}^T B)_r & \text{if } j \in [n+1, n(m+1)], r = ((j-1) \bmod n) + 1.\end{cases}$$

Before the formal reduction, here is a brief intuition. Note that in $(C, D)$, we have copied $(A + h, B + h)$ in the top-left $m \times n$ block, we call it *first block* now on. Since adding a constant does not change NE of a game, if strategies from only the first block are played with non-zero probability at a NE of $(C, D)$, then they give a NE of $(A, B)$ as well. Also, the payoff achieved at such a NE are at least $h$, a solution of **MaxPayoff**, using Lemma 4.3.

To guarantee a NE in $\mathcal{B}_{0.5}$ for game $(A, B)$ (solution of **InBox**), we make use of the blocks added after the first block in both the directions. In particular, in Lemma 4.3, if $\exists j \in [n],\ y_j > 0.5 * \sigma(\boldsymbol{y})$, then for the first player her first $m$ strategies are worse than those from block $[mj + 1 : mj + m]$, forcing her to play only from her last $mn$ strategies. This will force the second player to move away from the first block too (or else he gets negative payoff), and thereby leading to a NE where both play from the last $mn$ strategies and both get zero payof—also not a solution of **MaxPayoff**. We will use these observations crucially in the reduction.

We show that solutions of **InBox** in game $(A, B)$, i.e., $(\boldsymbol{x}, \boldsymbol{y})$ such that $\boldsymbol{x}, \boldsymbol{y} \le 0.5$, are retained as NE of $(C, D)$. The proof uses the fact that in $C$ and $D$, top-left block encodes $A$ and $B$, respectively.

Lemma 4.4. $(A, B)$ *has a NE* $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{B}_{0.5}$ *iff* $(\boldsymbol{x}', \boldsymbol{y}') = ((\boldsymbol{x}, 0_{mn}), (\boldsymbol{y}, 0_{mn}))$ *is a NE of* $(C, D)$.

Proof. To prove forward direction, it suffices to check if strategy profile $(\boldsymbol{x}', \boldsymbol{y}')$ satisfies Equation (2) for game $(C, D)$. We show the conditions for the first player, namely involving $C$, and proof for the second player follows similarly. As last $mn$ strategies in $\boldsymbol{y}'$ are not played at all, we have $\alpha = h \sum_{j \in [n]} y_i - \sum_{j \in [n+1, n(m+1)]} y'_j = h * 1 - 0 = h$. This together with Lemma 4.3 gives

$$i \in [m],\ (C\boldsymbol{y}')_i = h + (A\boldsymbol{y})_i \quad \Rightarrow \quad \max_{i \in [m]} (C\boldsymbol{y}')_i = h + \max_{i \in [m]} (A\boldsymbol{y})_i.$$

For $i \in [m+1, m(n+1)]$, let $r = ((i-1) \mod m) + 1$ and $k = \lfloor (i-1)/m \rfloor$. Then, using Lemma 4.3 and the fact that $y_k \le 0.5$, we have

$$(C\boldsymbol{y}')_i \le 2h(0.5) + (A\boldsymbol{y})_r = h + (A\boldsymbol{y})_r = (C\boldsymbol{y}')_r.$$

In other words strategies $[1:m]$ give at least as much payoff as the rest. Since $(\boldsymbol{x}, \boldsymbol{y})$ is a NE of game $(A, B)$, if $x'_i = x_i > 0$ then $(C\boldsymbol{y}')_i = h + (A\boldsymbol{y})_i = h + \max_{k\in[m]}(A\boldsymbol{y})_k = \max_{k\in[m(n+1)]}(C\boldsymbol{y}')_k$.

For the reverse direction, $\exists i \in [m]$ s.t. $x'_i > 0$ and hence $\forall j \in [n]$, $(C\boldsymbol{y}')_i \geq (C\boldsymbol{y}')_{mj+i} \Rightarrow 2hy_j \leq h \Rightarrow y_j \leq 0.5$. Similarly, $\boldsymbol{x} \leq 0.5$ follows. □

Lemma 4.4 maps a solution of **InBox** in game $(A, B)$ to a NE of $(C, D)$ where players play only among their first $m, n$ strategies, respectively. Clearly, at such a NE both the players in game $(C, D)$ get at least $h$ payoff, therefore it is also a solution of **MaxPayoff** in $(C, D)$. Next, we show a reverse mapping: a NE of $(C, D)$ where both players play some of first $m, n$ strategies, gives a NE of game $(A, B)$. Recall that for vector $\boldsymbol{x}$, $\eta(\boldsymbol{x}) = \boldsymbol{x}/\sigma(\boldsymbol{x})$.

Lemma 4.5. *If $(\boldsymbol{x}', \boldsymbol{y}')$ is a NE of game $(C, D)$ s.t. $\boldsymbol{x} = \boldsymbol{x}'[1:m]$ and $\boldsymbol{y} = \boldsymbol{y}'[1:n]$ are non-zero, then $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}))$ is a NE for game $(A, B)$, and $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y})) \in \mathcal{B}_{0.5}$.*

Proof. As $\sigma(\boldsymbol{x}), \sigma(\boldsymbol{y}) > 0$, to show $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}))$ is a NE of $(A, B)$ it suffices to show the following.

$$\begin{aligned} \forall i \in [m], x_i > 0 \ &\Rightarrow\ (A\boldsymbol{y})_i = max_{k\in[m]}(A\boldsymbol{y})_k, \text{ and} \\ \forall j \in [n], y_j > 0 \ &\Rightarrow\ (\boldsymbol{x}^T B)_j = max_{k\in[n]}(\boldsymbol{x}^T B)_k. \end{aligned}$$

We show that the first one holds and the proof for the second follows similarly. Let

$$\lambda = max_{k\in[m(n+1)]}(C\boldsymbol{y}')_k \text{ and } \lambda' = max_{k\in[m]}(C\boldsymbol{y}')_k = \alpha + \max_{k\in[m]}(A\boldsymbol{y})_k \text{ (Using Lemma 4.3).}$$

As $\exists i \in [m], x'_i > 0$, we have $\lambda' = \lambda$. Thus, we get

$$\forall i \in [m], x_i > 0 \Rightarrow (C\boldsymbol{y}')_i = \lambda \Rightarrow \alpha + (A\boldsymbol{y})_i = \alpha + \max_{k\in[m]}(A\boldsymbol{y})_k \Rightarrow (A\boldsymbol{y})_i = \max_{k\in[m]}(A\boldsymbol{y})_k.$$

For the second part, to the contrary suppose $\exists j \in [n]$, $(\eta(\boldsymbol{y}))_j = \frac{y_j}{\sigma(\boldsymbol{y})} > 0.5 \Rightarrow 2y_j > \sigma(\boldsymbol{y})$. Then, for some $i \in [m]$, we have $x'_i > 0$ and $(C\boldsymbol{y}')_i \leq h\sigma(\boldsymbol{y}) + (A\boldsymbol{y})_i < 2hy_j + (A\boldsymbol{y})_i = (C\boldsymbol{y}')_{jm+i} \leq \lambda$, a contradiction to $(\boldsymbol{x}', \boldsymbol{y}')$ being a NE of game $(C, D)$. □

Lemmas 4.4 and 4.5 imply that game $(A, B)$ has a NE in $\mathcal{B}_{0.5}$ if and only if game $(C, D)$ has a NE where both players play some of first $m, n$ strategies, respectively. If we show that to get payoff of at least $h$ in the latter game, players have to play some of first $m, n$ strategies, then clearly the reduction will follow.

Lemma 4.6. *Given a strategy profile $(\boldsymbol{x}', \boldsymbol{y}')$, if $\boldsymbol{x}'^T C\boldsymbol{y}' \geq h$ and $\boldsymbol{x}'^T D\boldsymbol{y}' \geq h$ then $\boldsymbol{x} = \boldsymbol{x}'(1:m)$ and $\boldsymbol{y} = \boldsymbol{y}'(1:n)$ are non-zero.*

Proof. If $\boldsymbol{y} = \boldsymbol{0}$, then $\forall i \in [m(n+1)]$ we have $(C\boldsymbol{y}')_i \leq 0$ using Lemma 4.3, and in turn $\boldsymbol{x}'^T C\boldsymbol{y}' \leq 0$. Similarly, if $\boldsymbol{x} = 0$, then $\forall j \in [n(m+1)]$ we have $(\boldsymbol{x}'^T D)_j \leq 0$, and then $\boldsymbol{x}'^T D\boldsymbol{y}' \leq 0$. Lemma follows using the fact that $h > 0$. □

The next theorem follows using Lemmas 4.4, 4.5, and 4.6.

Theorem 4.7. *Game $(A, B)$ has a NE in $\mathcal{B}_{0.5}$ if and only if game $(C, D)$ has a NE where every player gets payoff at least $h$.*

Next theorem shows reduction from **InBox** to **Superset** using Lemma 4.4.

Theorem 4.8. *Game $(A, B)$ has a NE in $\mathcal{B}_{0.5}$ if and only if game $(C, D)$ has a NE where all the strategies played with non-zero probability by the first and second player are from $T_1 = [1:m]$ and $T_2 = [1:n]$, respectively.*

Lemmas 4.4 and 4.5 imply that one of the first $m, n$ strategies are played with non-zero probability by respective players in game $(C, D)$ if and only if game $(A, B)$ has a NE in $\mathcal{B}_{0.5}$. Thus, next theorem gives a Turing reduction from **InBox** to **Subset**.

Theorem 4.9. *Game $(A, B)$ has a NE in $\mathcal{B}_{0.5}$ if and only if $\exists i \in [m], \exists j \in [n]$ such that for $T_1 = \{i\}$ and $T_2 = \{j\}$, game $(C, D)$ has a NE where all strategies of $T_1$ and $T_2$ are played with non-zero probability.*

Leveraging on the intuition presented for the reduction on two-player games, next we extend Theorems 4.7, 4.8, and 4.9 to three-player games to get the hardness results for the same.

### 4.2 3-Nash: InBox to MaxPayoff, Subset, and Superset

Like in the two player case, given a three-player game with $m \times n \times p$-dimensional payoff tensors $(A, B, C)$, we will create a game $(D, E, F)$ of size $m(n+1) \times n(p+1) \times p(m+1)$ and insert the original game in the first block with $h$ added. We start with the definitions, analogous to that of 4.1 and 4.2.

*Definition 4.10.* For $i \in [m]$, $j \in [n]$, $k \in [p]$, and a real number $h$, define
$A_{(i,:,:)+h}$ : Tensor $A$ with $h$ added to the entries $A_{ij'k'}$ $\forall j' \in [n], \forall k' \in [p]$,
$A_{(:,j,:)+h}$ : Tensor $A$ with $h$ added to the entries $A_{i'jk'}$ $\forall i' \in [m], \forall k' \in [p]$, and
$A_{(:,:,k)+h}$ : Tensor $A$ with $h$ added to the entries $A_{i'j'k}$ $\forall i' \in [m], \forall j' \in [n]$.

*Definition 4.11.* Given a tensor $T$ of size $a \times b \times c$ and integers $r, s, t$ s.t. $a + r - 1 \le m(n+1)$, $b + s - 1 \le n(p+1)$ and $c + t - 1 \le p(m+1)$, define $[T]_{r,s,t}$ to be an $m(n+1) \times n(p+1) \times p(m+1)$-dimensional tensor where $T$ is copied starting at position $(r, s, t)$, and all other coordinates are set to zero.

Construct game $(D, E, F)$ as follows, given $(A, B, C)$ and a scalar $h > 0$.

$$\begin{array}{l} D = [A+h]_{1,1,1} + [(-1)_{m,n(p+1),mp}]_{1,1,p+1} + \sum_{j\in[n]} [A_{(:,j,:)+2h}]_{jm+1,1,1}, \\ E = [B+h]_{1,1,1} + [(-1)_{mn,n,(m+1)p}]_{m+1,1,1} + \sum_{k\in[p]} [B_{(:,:,k)+2h}]_{1,kp+1,1}, \\ F = [C+h]_{1,1,1} + [(-1)_{m(n+1),np,p}]_{1,n+1,1} + \sum_{i\in[m]} [C_{(i,:,:)+2h}]_{1,1,in+1}. \end{array} \tag{5}$$

We will mimic the proof of 2-Nash to 3-Nash next, i.e., Lemmas 4.3, 4.4, and 4.5. In the proof of each of these lemmas, argument for the second player follows similar to that for the first player due to symmetry in the construction of the reduced game. Therefore, in what follows, we will focus on the first player again, and argument for the second and third player follows similarly.

Recall that $\pi_i(\boldsymbol{x})$ for $\boldsymbol{x} \in \Delta$ represents the payoff of player $i$ when played profile is $\boldsymbol{x}$. Since we will be dealing with two games in this section, to resolve ambiguity, we superscript it with the payoff tensor under consideration. To denote payoff from a pure-strategy $i$ with respect to tensor $A$, when other two are playing $\boldsymbol{y}, \boldsymbol{z}$, we use $\pi_1^A(i, \boldsymbol{y}, \boldsymbol{z})$, even if $\boldsymbol{y}, \boldsymbol{z}$ are not probability distributions.

Next lemma follows from the construction of game $(D, E, F)$ in Equation (5).

Lemma 4.12. *Let $\boldsymbol{y}'$ and $\boldsymbol{z}'$ be vectors of sizes $n(p+1)$ and $p(m+1)$, respectively. Let $\boldsymbol{y} = \boldsymbol{y}'[1:n]$, $\boldsymbol{z} = \boldsymbol{z}'[1:p]$ and $\alpha = h * \sigma(\boldsymbol{y})\sigma(\boldsymbol{z}) - \sum_{j\in[p+1,p(m+1)]} z'_j$. We have*

$$\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \begin{cases} \alpha + \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) & \text{if } i \in [m] \\ 2hy_{\lfloor (i-1)/m \rfloor} + \pi_1^A(r, \boldsymbol{y}, \boldsymbol{z}) & \text{if } i \in [m+1, m(n+1)], \\ & \quad \text{where } r = ((i-1) \bmod m) + 1 \end{cases}.$$

Let $\mathcal{B}_{0.5} = [0, 0.5]^{m+n+p}$. Using the payoff structure in game $(D, E, F)$, we show the next lemma.

Lemma 4.13. *Game $(A, B, C)$ has a NE $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathcal{B}_{0.5}$ iff $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') = ((\boldsymbol{x}, 0_{mn}), (\boldsymbol{y}, 0_{np}), (\boldsymbol{z}, 0_{mp}))$ is a NE of the game $(D, E, F)$.*

Proof. The proof is similar to that of Lemma 4.4. For the forward direction, we show the first condition of Equation (3) characterizing 3-Nash, and other two follow similarly. Note that again $\alpha = h$, and hence $\max_{i\in[m]} \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = h + \max_{i\in[m]} \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z})$ (Using Lemma 4.12).

Further, $\forall j \in [n]$ and $\forall i \in [m]$, we have $\pi_1^D(jm+i, \boldsymbol{y}', \boldsymbol{z}') = 2hy_j + \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) \le h + \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) = \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}')$. Thus, the first $m$ strategies are at least as good as last $[m+1, m(n+1)]$. We get $\forall i \in [m(n+1)]$, $x_i' > 0 \Rightarrow \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \max_{s \in [m(n+1)]} \pi_1^D(s, \boldsymbol{y}', \boldsymbol{z}')$. Argument for the second and third player follows similarly using the fact that $\boldsymbol{z} \le 0.5$ and $\boldsymbol{x} \le 0.5$, respectively.

For the reverse direction, $\exists i \in [m]$, $x_i' > 0$ and hence $\forall j \in [n]$, $\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') \ge \pi_1^D(mj + i, \boldsymbol{y}', \boldsymbol{z}') \Rightarrow 2hy_j \le h \Rightarrow y_j \le 0.5$. Similarly, $\boldsymbol{x} \le 0.5$ and $\boldsymbol{z} \le 0.5$ follows by arguing for third and second players, respectively. □

Next, we obtain a solution of **InBox** for game $(A, B, C)$ from a NE of $(D, E, F)$ where players play some strategies from the first $m$, $n$, and $p$ strategies, respectively, with non-zero probability.

Lemma 4.14. *If $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ is a NE of game $(D, E, F)$ such that the vectors $\boldsymbol{x} = \boldsymbol{x}'[1:m]$, $\boldsymbol{y} = \boldsymbol{y}'[1:n]$, and $\boldsymbol{z} = \boldsymbol{z}'[1:p]$ are non-zero, then $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z}))$ is a NE for game $(A, B, C)$, and $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z})) \in \mathcal{B}_{0.5}$.*

Proof. As $\sigma(\boldsymbol{x}), \sigma(\boldsymbol{y}), \sigma(\boldsymbol{z}) > 0$, profile $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z}))$ is well-defined. To show that it is a NE of game $(A, B, C)$, it suffices to show the following for the first player, and a similar argument follows for the other two players:

$$\forall i \in [m],\ x_i > 0 \quad \Rightarrow \quad \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) = \max_{l \in [m]} \pi_1^A(l, \boldsymbol{y}, \boldsymbol{z}).$$

Let $\lambda = \max_{k \in [m(n+1)]} \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}')$, and $\lambda' = \max_{k \in [m]} \pi_1^D(k, \boldsymbol{y}', \boldsymbol{z}') = \alpha + \max_{k \in [m]} \pi_1^A(k, \boldsymbol{y}, \boldsymbol{z})$ (Using Lemma 4.12). As $\exists i \in [m], x_i' > 0$, we have $\lambda' = \lambda$. Thus, we get

$$\begin{aligned} \forall i \in [m],\ \ x_i > 0 &\Rightarrow x_i' > 0 \\ &\Rightarrow \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \lambda \\ &\Rightarrow \alpha + \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) = \alpha + \max_{k \in [m]} \pi_1^A(k, \boldsymbol{y}', \boldsymbol{z}') \\ &\Rightarrow \pi_1^A(i, \boldsymbol{y}', \boldsymbol{z}') = \max_{k \in [m]} \pi_1^A(k, \boldsymbol{y}', \boldsymbol{z}'). \end{aligned}$$

For the second part, to the contrary suppose $\exists j \in [n]$, $(\eta(\boldsymbol{y}))_j = \frac{y_j}{\sigma(\boldsymbol{y})} > 0.5 \Rightarrow 2y_j > \sigma(\boldsymbol{y})$. Then, for some $i \in [m]$, we have $x_i' > 0$ and $\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') \le h\sigma(\boldsymbol{y}) + \pi_1^A(i, \boldsymbol{y}', \boldsymbol{z}') < 2hy_j + \pi_1^A(i, \boldsymbol{y}', \boldsymbol{z}') = \pi_1^D(jm+i, \boldsymbol{y}', \boldsymbol{z}') \le \lambda$, a contradiction to $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ being a NE of game $(D, E, F)$. □

Now, if we can relate the NE of $(D, E, F)$ where at least one of the first $m$, $n$, and $p$ strategies are played by the first, second, and third players, respectively, and the payoff received at the NE by all the players, then **InBox** to **MaxPayoff** reduction will follow.

Lemma 4.15. *Given a strategy profile $\boldsymbol{d} = (\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ of game $(D, E, F)$, if $\pi_i(\boldsymbol{d}) \ge h > 0$, $i = 1, 2, 3$, then $\boldsymbol{x} = \boldsymbol{x}'(1:m)$, $\boldsymbol{y} = \boldsymbol{y}'(1:n)$ and $\boldsymbol{z} = \boldsymbol{z}'(1:p)$ are non-zero.*

Proof. If $\boldsymbol{y} = \boldsymbol{0}$, then $\forall i \in [m(n+1)]$ we have $\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') \le 0$ using Lemma 4.12, and in turn $\pi_1(\boldsymbol{d}) \le 0$. Similarly, if $\boldsymbol{z} = 0$, then we get $\pi_2(\boldsymbol{d}) \le 0$, and if $\boldsymbol{x} = 0$ then $\pi_3(\boldsymbol{d}) \le 0$. Lemma follows using the fact that $h > 0$. □

The next theorem, for **InBox** to **MaxPayoff** reduction, follows using Lemmas 4.13, 4.14, and 4.15.

Theorem 4.16. *Game $(A, B, C)$ has a NE in $\mathcal{B}_{0.5}$ if and only if game $(D, E, F)$ has a NE where every player gets payoff at least $h$.*

The next theorem showing reduction from **InBox** to **Superset** follows using Lemma 4.13.

Theorem 4.17. *Game $(A, B, C)$ has a NE in $\mathcal{B}_{0.5}$ if and only if game $(D, E, F)$ has a NE where all the strategies played with non-zero probability by players are from $T_1 = [1:m]$, $T_2 = [1:n]$ and $T_3 = [1:p]$, respectively.*

Next theorem follows using Lemmas 4.13 and 4.14, and it gives a Turing machine reduction (many-to-one) from **InBox** to **Subset**.

THEOREM 4.18. *Game $(A, B, C)$ has a NE in $\mathcal{B}_{0.5}$ if and only if $\exists i \in [m], \exists j \in [n], \exists k \in [p]$ such that for $T_1 = \{i\}$, $T_2 = \{j\}$ and $T_3 = \{k\}$, game $(D, E, F)$ has a NE where all strategies of $T_1, T_2, T_3$ are played with non-zero probability.*

From Theorem 4.18, it follows that to solve **InBox** for game $(A, B, C)$, we will need to solve $(mnp)$ many instances of **Subset** in game $(D, E, F)$, we get many-to-one reduction from **InBox** to **Subset**. Theorems 4.16, 4.17, and 4.18 together with ∃ℝ-hardness of **InBox** in 3-Nash, and Theorem 3.1 gives the next result.

THEOREM 4.19. *The problems of* **MaxPayoff**, **Subset** *and* **Superset** *are ∃ℝ-complete in three-player games.*

A three-player game can be reduced to a $k$-player game for $k > 3$ trivially, without changing its set of NE, by adding $k - 3$ dummy players with one strategy each (and payoff tensor $A_i = [h]$ to get reduction for **MaxPayoff**). And therefore, the next theorem follows from Theorem 4.19.

THEOREM 4.20. *Given a $k$-player game $(A_1, \ldots, A_k)$, for a constant $k \geq 3$, the problems of* **MaxPayoff**, **Subset**, *and* **Superset** *are ∃ℝ-complete.*

In the next section, we show ∃ℝ-completeness for **NonUnique**, by reducing **MaxPayoff** to **NonUnique** in three-player games.

### 4.3 MaxPayoff to NonUnique

In this section, we reduce **MaxPayoff** to **NonUnique** in a three-player game. Let $(A, B, C)$ be a given game, and for a given rational number $h > 0$, we are asked to check if it has a NE where all three players get payoff at least $h$. We will reduce this problem to checking if game $(D, E, F)$ has more than one equilibrium. Tensors $A, B, C$ are of size $m \times n \times p$, where $m, n, p$ are number of strategies of player 1, 2, 3, respectively. Let $m' = m + 1, n' = n + 1, p' = p + 1$, and $D, E, F$ be of size $m' \times n' \times p'$, where

$$\begin{array}{ll} \forall i \in [m], j \in [n], k \in [p], & D_{ijk} = A_{ijk},\ E_{ijk} = B_{ijk},\ F_{ijk} = C_{ijk} \\ \forall j \in [n'], k \in [p'], & D_{m'jk} = h \\ \forall i \in [m'], k \in [p'], & E_{in'k} = h \\ \forall i \in [m'], j \in [n'], & F_{ijp'} = h. \end{array}$$

The rest of the entries in $D, E, F$ are set to zero. Basically, we added one extra strategy for each player and made sure that the player gets payoff $h$ when she plays this extra strategy regardless of what others play.

LEMMA 4.21. *Let $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ be a strategy profile for game $(D, E, F)$, and $\boldsymbol{x} = \boldsymbol{x}'(1 : m), \boldsymbol{y} = \boldsymbol{y}'(1 : n)$ and $\boldsymbol{z} = \boldsymbol{z}'(1 : p)$. Then,*

- $\pi_1^D(m', \boldsymbol{y}', \boldsymbol{z}') = h$, $\pi_2^E(\boldsymbol{x}', n', \boldsymbol{z}') = h$, *and* $\pi_3^F(\boldsymbol{x}', \boldsymbol{y}', p') = h$.
- $\forall i \in [1 : m]$, $\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z})$. $\forall j \in [1 : n]$, $\pi_2^E(\boldsymbol{x}', j, \boldsymbol{z}') = \pi_2^B(\boldsymbol{x}, j, \boldsymbol{z})$. $\forall k \in [1 : p]$, $\pi_3^F(\boldsymbol{x}', \boldsymbol{y}', k) = \pi_3^C(\boldsymbol{x}, \boldsymbol{y}, k)$.

PROOF. The first part follows by construction. For the second part, we show $\forall i \in [1 : m]$, $\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z})$. The rest can be proven similarly. Recall that $n' = n + 1$ and

$p' = p + 1$. We have

$$\begin{aligned}\pi_1^D(i, \boldsymbol{y}', z') &= \textstyle\sum_{j\in[n'],k\in[p']} D_{ijk} y'_j z'_k \\ &= \textstyle\sum_{j\in[n],k\in[p]} D_{ijk} y_j z_k + \sum_{k\in[p']} D_{in'k} y'_{n'} z'_k + \sum_{j\in[n']} D_{ijp'} y'_j z'_{p'} \\ &= \textstyle\sum_{j\in[n],k\in[p]} A_{ijk} y_j z_k \\ &= \pi_1^A(i, \boldsymbol{y}, z),\end{aligned}$$

where the third equality holds, because $D_{ijk} = A_{ijk},\ \forall j \in [n], \forall k \in [p]$, and $D_{ijk} = 0$ if either $j = n'$ or $k = p'$. □

Next, we show that game $(D, E, F)$ has a trivial pure NE where all players play their extra strategy.

Lemma 4.22. *Pure-strategy profile $(m', n', p')$ is a NE of game $(D, E, F)$.*

Proof. When players two and three are playing strategy $n'$ and $p'$, respectively, then $\forall i \in [m]$ payoff $D_{in'p'}$ of the first player is zero, while $D_{m'n'p'} = h > 0$. Therefore, playing $m'$ is the best response for her. Similarly, we can argue for players two and three. □

Except for the trivial NE established in Lemma 4.22 if game $(D, E, F)$ has another equilibrium, then we need to construct a solution of **MaxPayoff** in game $(A, B, C)$.

Lemma 4.23. *If $(\boldsymbol{x}', \boldsymbol{y}', z') \neq (m', n', p')$ is a NE of game $(D, E, F)$, then $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(z))$ is a NE of game $(A, B, C)$ with payoff at least $h$ to each player, where $\boldsymbol{x} = \boldsymbol{x}'(1 : m), \boldsymbol{y} = \boldsymbol{y}'(1 : n)$ and $z = z'(1 : p)$.*

Proof. First, we show that $\sigma(\boldsymbol{x}), \sigma(\boldsymbol{y}), \sigma(z) > 0$. To the contrary, suppose $z = \boldsymbol{0}$ and wlog $\boldsymbol{x} \neq \boldsymbol{0}$. Then, $z'_{p'} = 1$, and $\exists i \in [m],\ x'_i > 0$ with payoff $\pi_1^D(i, \boldsymbol{y}', z') = \pi_1^A(i, \boldsymbol{y}, z) = 0$ (Lemma 4.21), a contradiction, because player one will deviate to $m'$ that always fetches payoff $h > 0$. Similar contradiction can be derived if $\sigma(\boldsymbol{y}) = 0$ or $\sigma(\boldsymbol{x}) = 0$.

We will show that $\eta(\boldsymbol{x})$ is a best response of the first player when other two are playing $\eta(\boldsymbol{y})$ and $\eta(z)$, respectively, in $(A, B, C)$, and that her payoff is at least $h$. Argument for other players follow similarly. Let $\lambda = \max_{s\in[m]} \sum_{j\in[n],k\in[p]} A_{sjk} y_j z_k$. It suffices to show that $\forall i \in [m],\ x_i > 0 \Rightarrow \sum_{j\in[n],k\in[p]} A_{ijk} y_j z_k = \lambda$ and $\lambda \geq h$, because normalization will increase the payoff of all the pure-strategies, and that too by the same factor.

Let $\lambda' = \max_{i\in[m']} \pi_1^D(i, \boldsymbol{y}', z')$, then $\lambda = \lambda'$, because $\exists i \in [m], x_i > 0$ and payoff at $i$ is $\lambda$.

$$x_i > 0 \Rightarrow x'_i > 0 \Rightarrow \pi_1^D(i, \boldsymbol{y}', z') = \lambda' \Rightarrow \sum_{j\in[n],k\in[p]} A_{sjk} y_j z_k = \lambda.$$

Now, since each player gets payoff $h$ from their last strategy in game $(D, E, F)$ (Lemma 4.21), other strategies played with non-zero probabilities have to fetch payoff at least $h$ and hence $\lambda = \lambda' \geq h$ follows. □

We also need to establish that if game $(A, B, C)$ has a feasible solution for **MaxPayoff** then game $(D, E, F)$ has more than one equilibrium.

Lemma 4.24. *If $(\boldsymbol{x}, \boldsymbol{y}, z)$ is a Nash equilibrium of $(A, B, C)$ where every player gets payoff at least $h$, then $((\boldsymbol{x}|0), (\boldsymbol{y}|0), (z|0))$ is a NE of game $(D, E, F)$.*

Proof. Let $\boldsymbol{x}' = (\boldsymbol{x}|0), \boldsymbol{y}' = (\boldsymbol{y}|0)$ and $z' = (z|0)$. We will show that $\boldsymbol{x}'$ is a best response for player one against $\boldsymbol{y}', z'$ in $(D, E, F)$, and cases for other two players follow similarly. Let $\lambda = \max_{i\in[m]} \pi_1^A(i, \boldsymbol{y}, z)$ and $\lambda' = \max_{i\in[m']} \pi_1^D(i, \boldsymbol{y}', z')$. Since $\lambda \geq h$ and $\pi_1^D(m', \boldsymbol{y}', z') = h$ (Lemma 4.21), we get $\lambda = \lambda'$, and the lemma follows. □

Using Lemmas 4.22, 4.23, and 4.24, we get the next theorem.

Theorem 4.25. *Game $(A, B, C)$ has a NE where every player gets at least h payoff iff game $(D, E, F)$ has more than one equilibrium.*

As argued in Section 4.2, a three-player game can be trivially reduced to a $k$-player game, for $k > 3$, by adding $k - 3$ dummy players. Therefore, next theorem follows using Theorems 3.1, 4.19, and 4.25.

Theorem 4.26. *Given a $k$-player game $(A_1, \ldots, A_k)$, for a constant $k \geq 3$, the problem of* **NonUnique** *is $\exists\mathbb{R}$-complete.*

Since there is no reduction known from 3-Nash to symmetric 3-Nash, results of this section do not follow directly for symmetric 3-Nash, or symmetric $k$-Nash for that matter. In the next section, we show $\exists\mathbb{R}$-completeness results for symmetric Nash equilibria in three-player symmetric games.

## 5 SYMMETRIC 3-NASH: ∃ℝ-COMPLETENESS

Containment in $\exists\mathbb{R}$ for various decision versions of symmetric 3-Nash is shown in Section 3 (Theorem 3.2). In this section, we show $\exists\mathbb{R}$-hardness for **Subset** and **Superset** for symmetric 3-Nash, by giving a reduction from 3-Nash to symmetric 3-Nash.

Let the given game be $(A, B, C)$, where each tensor is of size $m \times n \times p$. Let $D$ denote the reduced symmetric game, which will be of size $l \times l \times l$, where $l = m + n + p$. Let $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ be a NE of $(A, B, C)$. We will show that there are positive numbers $\alpha, \beta, \gamma$ such that $(\boldsymbol{d}, \boldsymbol{d}, \boldsymbol{d})$ is a NE of the reduced game, where $\boldsymbol{d}$ is a $l$-dimensional vector $(\alpha\boldsymbol{x}|\beta\boldsymbol{y}|\gamma\boldsymbol{z})$. Furthermore, let $(\boldsymbol{d}, \boldsymbol{d}, \boldsymbol{d})$ be a NE of the reduced game, where $\boldsymbol{d}$ decomposes into vectors $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}'$ of dimension $m, n, p$, respectively. Scaling these vectors gives a NE $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of game $(A, B, C)$. This will yield mapping in both directions.

Essential to this reduction is the $3 \times 3 \times 3$ symmetric game $G(a, b, c)$ given below. We represent the payoff tensor of the first player by three $3 \times 3$ matrices, one for each of her pure strategy. Here $a, b, c$ are any non-negative reals:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & a & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

Lemma 5.1. *If $(\alpha, \beta, \gamma)$ is a symmetric NE of game G, then $\alpha, \beta, \gamma > 0$.*

Proof. We will first show that $G$ has no symmetric NE of support one or two. This involves a case analysis of which we present one representative case each. First, observe that $(\alpha, \beta, \gamma) = (1, 0, 0)$ cannot be a symmetric NE, since player 1 should play $(0, 0, 1)$ if the other two players play the given strategy. Next, consider the strategy $(\alpha, \beta, 0)$ with the first two components non-zero. Because the matrix corresponding to second strategy of the first player has all zeros in the upper left $2 \times 2$ sub-matrix, she will be strictly better off playing the third strategy instead of the second. Hence, any symmetric NE of $G$ must be of full support. □

From $G$, we derive a symmetric game $D$, which is of size $l \times l \times l$, by blowing up each of the three strategies of $G$ to $m, n, p$ number of strategies, respectively. Copy 0s and 1s to their respective blocks, and replace $a, b, c$ with tensors $A, B, C$, respectively, after appropriate rotation. For example, we have $G(1, 2, 3) = a$, which is replaced by $A$ as is, while $G(1, 3, 2) = a$ is replaced by $A$ after rotation so that first, second and third dimensions correspond to players one, three, and two, respectively. In general, $G(i_1, i_2, i_3)$ is replaced by an appropriate tensor after rotation such that first, second, and third dimensions correspond to players $i_1, i_2$ and $i_3$, respectively, where $i_1 \neq i_2 \neq i_3 \neq i_1$. The following is the formal description of $D$:

$$\begin{array}{c} D_{stu} = \\ 1 \leq s,t,u \leq l \end{array} \begin{cases} A_{s(t-m)(u-m-n)} & \text{if } s \leq m \;\&\; m < t \leq m+n \;\&\; m+n < u \leq l \\ A_{s(u-m)(t-m-n)} & \text{if } s \leq m \;\&\; m < u \leq m+n \;\&\; m+n < t \leq l \\ B_{t(s-m)(u-m-n)} & \text{if } t \leq m \;\&\; m < s \leq m+n \;\&\; m+n < u \leq l \\ B_{u(s-m)(t-m-n)} & \text{if } u \leq m \;\&\; m < s \leq m+n \;\&\; m+n < t \leq l \\ C_{t(u-m)(s-m-n)} & \text{if } t \leq m \;\&\; m < u \leq m+n \;\&\; m+n < s \leq l \\ C_{u(t-m)(s-m-n)} & \text{if } u \leq m \;\&\; m < t \leq m+n \;\&\; m+n < s \leq l \\ 1 & \text{if } s \leq m \;\&\; m < t = u \leq m+n, \\ 1 & \text{if } m < s \leq m+n \;\&\; m+n < t = u \leq l \\ 1 & \text{if } m+n < s \leq l \;\&\; t = u \leq m \\ 0 & \mathit{Otherwise}. \end{cases} \tag{7}$$

In the above game, suppose two players are playing mixed-strategy $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$, where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are of dimensions $m, n, p$, respectively. Then, from strategy $s$ the third player receives payoff:

$$\pi^D(s, \boldsymbol{d}) = \begin{cases} (\sigma(\boldsymbol{y}))^2 + 2\sum_{j\in[n],k\in[p]} A_{sjk} y_j z_k, & \text{if } s \leq m, \\ (\sigma(\boldsymbol{z}))^2 + 2\sum_{i\in[m],k\in[p]} B_{isk} x_i z_k & \text{if } m < s \leq m+n. \\ (\sigma(\boldsymbol{x}))^2 + 2\sum_{i\in[m],j\in[n]} C_{ijs} x_i y_j & \text{if } m+n < s \leq l \end{cases} \tag{8}$$

Wlog, we assume that $A, B, C \geq 0$, and hence $D \geq 0$. We consider $\frac{0}{0}$ as 0.

Lemma 5.2. *If $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$ is a SNE of game $D$, then $(\sigma(\boldsymbol{x}), \sigma(\boldsymbol{y}), \sigma(\boldsymbol{z}))$ is a NE of $G(a, b, c)$, where $a = \frac{\max_{s\leq m} \sum_{jk} A_{sjk} y_j z_k}{\sigma(\boldsymbol{y})\sigma(\boldsymbol{z})}$, $b = \frac{\max_{s\leq n} \sum_{i,k} B_{isk} x_i z_k}{\sigma(\boldsymbol{x})\sigma(\boldsymbol{z})}$, $c = \frac{\max_{s\leq p} \sum_{i,j} C_{ijs} x_i y_j}{\sigma(\boldsymbol{x})\sigma(\boldsymbol{y})}$.*

Proof. Let $\alpha = \sigma(\boldsymbol{x})$, $\beta = \sigma(\boldsymbol{y})$, and $\gamma = \sigma(\boldsymbol{z})$. Clearly, the payoffs from three strategies of $G$ are, respectively, $\beta^2 + 2a\beta\gamma$, $\gamma^2 + 2b\alpha\gamma$, and $\alpha^2 + 2c\alpha\beta$. Observe that these are also the best payoffs among strategies $[1:m]$, $[m+1:m+n]$, and $[m+n+1:l]$, respectively, in game $D$. Let the maximum among these three be $\lambda$. Then, we have

$$\alpha = \sigma(\boldsymbol{x}) > 0 \Rightarrow \exists i \leq m,\ x_i > 0 \Rightarrow \pi^D(i, \boldsymbol{d}) = \beta^2 + 2a\beta\gamma = \lambda.$$

Similarly, we can show that if $\beta > 0$ then payoff at the second strategy is $\lambda$, and if $\gamma > 0$ then the third gives $\lambda$. Hence, $(\alpha, \beta, \gamma)$ is a symmetric NE of game $G$. □

Lemmas 5.1 and 5.2 imply that at any SNE $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$, all three components $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ of the strategy profile are non-zero. Next, we show that normalizing each gives a NE of the original game $(A, B, C)$. Recall the notations $\sigma(\boldsymbol{x}) = \sum_i x_i$ and $\eta(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\sigma(\boldsymbol{x})}$.

Lemma 5.3. *If $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$ is a SNE of game $D$, then $\sigma(\boldsymbol{x}), \sigma(\boldsymbol{y}), \sigma(\boldsymbol{z}) > 0$, and $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z}))$ is a NE of game $(A, B, C)$.*

Proof. From Lemmas 5.1 and 5.2 it follows that vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are non-zero. Thus, the first part follows.

Let $\boldsymbol{x}' = \eta(\boldsymbol{x}), \boldsymbol{y}' = \eta(\boldsymbol{y})$, and $\boldsymbol{z}' = \eta(\boldsymbol{z})$; clearly these are well-defined due to the first part. We will show that $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ satisfies conditions Equation (3) characterizing NE of game $(A, B, C)$. We do this for the first condition, the rest two follow similarly. Let $\lambda$ denote the maximum payoff of a player in symmetric game $D$ when others are playing $\boldsymbol{d}$. For strategy $s \in S_1$ of the first player, we have

$$\begin{aligned} x'_s > 0 \Rightarrow{} & x_s > 0 \\ & \Rightarrow \pi^D(s, \boldsymbol{d}) = \lambda \ \text{(Using Equations (8) and (3))} \\ & \Rightarrow \pi^D(s, \boldsymbol{d}) \geq \pi^D(s', \boldsymbol{d}),\ \ \forall s' \leq m \\ & \Rightarrow \textstyle\sum_{j\in[n],k\in[p]} A_{sjk} y'_j z'_k \geq \sum_{j\in[n],k\in[p]} A_{s'jk} y'_j z'_k, \forall s' \leq m. \end{aligned}$$

□

The mapping from SNE of game $D$ to NE of game $(A, B, C)$ established in Lemma 5.3 implies that computing SNE in symmetric games is no easier than computing a NE in normal games. We extend this reduction to $k$-Nash in Section 7. Next, we show a mapping in reverse direction, i.e., from NE of $(A, B, C)$ to a SNE of $D$, to obtain $\exists\mathbb{R}$-hardness results for a number of decision problems in symmetric 3-Nash.

Lemma 5.4. *Let $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ be a NE of $(A, B, C)$, and let $(\alpha, \beta, \gamma)$ be a NE of game $G(a, b, c)$, where $a, b, c$ are set to payoffs of the first, second, and third players, respectively, at the NE of game $(A, B, C)$. Then, $\boldsymbol{d} = (\alpha\boldsymbol{x}|\beta\boldsymbol{y}|\gamma\boldsymbol{z})$ is a SNE of game $D$.*

Proof. Clearly, $a = \max_{i \in S_1} \sum_{j,k} A_{ijk} y_j z_k$, $b = \max_{j \in S_2} \sum_{i,k} B_{ijk} x_i z_k$ and $c = \max_{i,j} C_{ijk} x_i y_j$. Let $\boldsymbol{x}' = \alpha\boldsymbol{x}$, $\boldsymbol{y}' = \beta\boldsymbol{y}$ and $\boldsymbol{z}' = \gamma\boldsymbol{z}$, then clearly $\boldsymbol{d} = (\boldsymbol{x}'|\boldsymbol{y}'|\boldsymbol{z}')$ is a mixed-strategy, i.e., $\sigma(\boldsymbol{d}) = 1$. Since $\alpha, \beta, \gamma > 0$ (Lemma 5.1), we have $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}' \neq 0$. In symmetric game $D$, let $a' = \max_{s \le m} \pi^D(s, \boldsymbol{d}) = \beta^2 + 2a\beta\gamma$, $b' = \max_{m < s \le m+n} \pi^D(s, \boldsymbol{d}) = \gamma^2 + 2b\alpha\gamma$, and $c' = \max_{m+n < s \le l} \pi^D(s, \boldsymbol{d}) = \alpha^2 + 2c\alpha\beta$. Note that $a', b', c'$ are payoffs from the three strategies at $(\alpha, \beta, \gamma)$ in game $G$. Since $(\alpha, \beta, \gamma)$ is a NE of $G$, we have $a' = b' = c'$ (using Lemma 5.1).

As $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a NE of game $(A, B, C)$, we get

$$\forall i \in [m],\ x'_i > 0 \Rightarrow x_i > 0 \Rightarrow \sum_{j,k} A_{ijk} y_j z_k = a \Rightarrow \pi^D(i, \boldsymbol{d}) = a'.$$

Similarly, we get $\forall j \in [n],\ y'_j > 0 \Rightarrow \pi^D(m + j, \boldsymbol{d}) = b'$, and $\forall k \in [p],\ z'_k > 0 \Rightarrow \pi^D(m + n + k, \boldsymbol{d}) = c'$. Lemma follows using the fact that $a' = b' = c'$. □

The next theorem summaries the relation between NE of game $(A, B, C)$ and SNE of game $D$, and follows using Lemmas 5.3 and 5.4.

Theorem 5.5. *Profile $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$ is a SNE of game $D$ iff $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z}))$ is a NE of game $(A, B, C)$.*

We showed a number of $\exists\mathbb{R}$-completeness results for 3-Nash in Section 4. Since support of a NE remains intact in the reduction from 3-Nash to symmetric 3-Nash as shown in Theorem 5.5, next we show $\exists\mathbb{R}$-completeness of **Subset** and **Superset** problems for symmetric 3-Nash.

Theorem 5.6. *Given a symmetric game $D$ and a subset $T \subset S$, it is $\exists\mathbb{R}$-complete to check if there exists a SNE $\boldsymbol{x}$ s.t. $x_s > 0,\ \forall s \in T$ (**Subset**).*

Proof. Theorem 4.19 establishes that checking if game $(A, B, C)$ has a NE where strategies in $T_i \subset S_i,\ i = 1, 2, 3$ are played with non-zero probability is $\exists\mathbb{R}$-complete. Let $l = m + n + p$. Construct a symmetric game $D$ of size $l \times l \times l$ from $G$ of Equation (6) by blowing it up and replacing $a$, $b$, and $c$ with $A$, $B$, and $C$, respectively. Construct $D$ as given in Equation (7).

Let $T = T_1 \cup \{j + m \mid j \in T_2\} \cup \{k + m + n \mid k \in T_3\}$. Using Theorem 5.5 it follows that game $(A, B, C)$ has a NE where strategies of $T_i$ are played with positive probability if and only if game $D$ has a symmetric NE where strategies of $T$ are played with positive probability. Since size of $D$ is $O(size(A, B, C))$, $\exists\mathbb{R}$-hardness follows.

Containment in $\exists\mathbb{R}$ follows from Theorem 3.2. □

The next theorem follows similarly using Theorems 4.19 and 5.5.

Theorem 5.7. *Given a symmetric game $D$ and a subset $T \subset S$, it is $\exists\mathbb{R}$-complete to check if there exists a SNE $\boldsymbol{x}$ s.t. $x_s = 0,\ \forall s \in S \setminus T$ (**Superset**).*

## 6 SYMMETRIC 3-NASH: $\mathrm{FIXP}_a$-COMPLETENESS

Even though Theorem 5.5 reduces 3-Nash, which is known to be FIXP-complete [11], to symmetric 3-Nash, we do not get FIXP-hardness for the latter. This is because to obtain a solution, say $\boldsymbol{x}$, of former requires *division* among the coordinates of a solution, say $\boldsymbol{d}$, of the latter. While FIXP reduction requires that every $x_i$ is a linear function of some $d_j$, with rational coefficients [11] (to handle irrational solutions under Turing reduction). Since there always exists a strong approximate solution that constitutes only rational numbers, such a requirement is not needed for FIXP$_a$. Leveraging on this, we give a reduction for strong approximation in 3-Nash to strong approximation in symmetric 3-Nash in this section, to get FIXP$_a$-completeness for the latter.

First, for containment in FIXP$_a$, we show a more general result, namely for symmetric $k$-Nash, $k \geq 3$. And later we show the hardness for the three-player case. We show that symmetric $k$-Nash, for a constant $k$, is in FIXP, and consequently strong approximation is in FIXP$_a$. Let the given game be represented by tensor $A$ and let the set of pure strategies of players be $S$. At a symmetric NE all players play the same mixed-strategy. Consider a function $F : \Delta \rightarrow \Delta$ as follows, where $\boldsymbol{x}' = F(\boldsymbol{x})$ for an $\boldsymbol{x} \in \Delta$:

$$\forall s \in S, \quad x'_s = \frac{x_s + \max\{\pi^A(s, \boldsymbol{x}) - \pi^A(\boldsymbol{x}), 0\}}{1 + \sum_s \max\{\pi^A(s, \boldsymbol{x}) - \pi^A(\boldsymbol{x}), 0\}}. \tag{9}$$

Nash [22] proved that fixed-points of $F$ are exactly the symmetric NE of game $A$.

Theorem 6.1. *The problem of computing a symmetric NE in a symmetric $k$-player game, for a constant $k$, is in FIXP, and corresponding strong approximation is in* FIXP$_a$.

Proof. The operations used in defining $F$ are $+, -, *, /$, and max. Further, domain of $F$ is convex and compact, and function is well-defined over the domain. Thus, finding fixed-points of $F$ is in FIXP by definition. Since description of $F$ is $O(size(A))$, this together with Nash's result [22] imply that finding a symmetric NE of $A$ is also in FIXP. Further, for a given $\epsilon > 0$ if $\boldsymbol{x}$ is $\epsilon$-near to an actual fixed-point $\boldsymbol{x}^*$, i.e., $|\boldsymbol{x} - \boldsymbol{x}^*|_\infty < \epsilon$, then $\boldsymbol{x}$ is also a strong approximate symmetric NE of game $A$. Containment in FIXP$_a$ follows. □

For FIXP$_a$-hardness result, we need to compute a strategy profile $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ that is $\epsilon$-near to an actual equilibrium of $(A, B, C)$, given a symmetric profile $\boldsymbol{d}$ $\epsilon'$-near to a symmetric NE $\boldsymbol{d}^*$ of $D$, where distances are measured in $l_\infty$ norm.

In reduction of Theorem 5.5, obtaining solution of $(A, B, C)$ involves, e.g., dividing $\boldsymbol{x}$ by $\sigma(\boldsymbol{x})$. If the latter is very small, then this may give us a vector that is very far from a solution of $(A, B, C)$, even when $\boldsymbol{d}$ may be close to $\boldsymbol{d}^*$. To get around this, next we make sure that $\sigma(\boldsymbol{x})$ is big enough.

Wlog, we assume that all entries of $A, B, C \in [0, 0.1]$, as adding constants to $A, B, C$ or scaling them by positive constants does not change its set of NE. In that case, payoffs of a player at its NE is in $[0, 0.1]$. The $a, b, c$ of Lemma 5.2 are also in $[0, 0.1]$. Thus, if we can lower bound the NE strategy $(\alpha, \beta, \gamma)$ of game $G$ with such $a, b, c$, then we get a lower bound on $\sigma(\boldsymbol{x})$, $\sigma(\boldsymbol{y})$, and $\sigma(\boldsymbol{z})$ as desired.

Lemma 6.2. *If $(\alpha, \beta, \gamma)$ is a NE of game $G(a, b, c)$, where $a, b, c \in [0, 0.1]$, then $\frac{1}{4} \leq \alpha, \beta, \gamma \leq \frac{1}{2}$.*

Proof. Note that $\alpha, \beta, \gamma > 0$ because of Lemma 5.1. Therefore, each of the three strategies fetch the same payoff, i.e., $\beta^2 + 2a\beta\gamma = \gamma^2 + 2b\alpha\gamma = \alpha^2 + 2c\alpha\beta$. We show that none of $\alpha, \beta, \gamma < 1/4$, and the upper bound follows, because $\alpha + \beta + \gamma = 1$. There are two cases for each, and we show them for $\alpha$. For $\beta$ and $\gamma$ they follow similarly.

**Case I:** $\alpha < 1/4$, and $\beta, \gamma \geq 1/4$.
As $\beta + \gamma \geq 3/4$, wlog let $\beta \geq 3/8$. Then, we have $\beta^2 + 2a\beta\gamma \geq 9/64 + 3a/16$, and $\alpha^2 + 2c\alpha\beta \leq 1/16 + c/2$. The above equality gives $9/64 + 3a/16 \leq 1/16 + c/2 \Rightarrow 5/64 \leq c/2 - 3a/16 \Rightarrow c \geq 10/64 \geq 0.1$, a contradiction.

**Case II:** $\alpha, \gamma < 1/4$, and $\beta > 1/2$.
$\beta^2 + 2a\beta\gamma \geq 1/4$ and $\gamma^2 + 2c\alpha\gamma \leq 1 + 2c/16$. Thus, we have $4 \leq 1 + 2c \Rightarrow c \geq 3/2$, a contradiction. □

Next, we show that strong approximate symmetric NE of game $D$ maps to a strong approximate NE of $(A, B, C)$, under the mapping of Theorem 5.5.

Lemma 6.3. *Let $\boldsymbol{d}^* = (\boldsymbol{x}^*|\boldsymbol{y}^*|\boldsymbol{z}^*)$ be a symmetric Nash equilibrium of game D, and $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$ be such that $|\boldsymbol{d} - \boldsymbol{d}^*|_\infty \leq \epsilon$. Then, $|\frac{x_i}{\sigma(\boldsymbol{x})} - \frac{x_i^*}{\sigma(\boldsymbol{x}^*)}| \leq \epsilon'$, $\forall i$; $|\frac{y_j}{\sigma(\boldsymbol{y})} - \frac{y_j^*}{\sigma(\boldsymbol{y}^*)}| \leq \epsilon'$, $\forall j$; and $|\frac{z_k}{\sigma(\boldsymbol{z})} - \frac{z_k^*}{\sigma(\boldsymbol{z}^*)}| \leq \epsilon'$, $\forall k$, where $\epsilon = \frac{\epsilon'}{20l}$.*

Proof. Lemmas 5.2 and 6.2 give us $\frac{1}{4} \leq \sigma(\boldsymbol{x}^*) \leq \frac{1}{2}$. Using this, we obtain bounds on $\sigma(\boldsymbol{x})$:

$$\forall i \leq m,\ |x_i - x_i^*| \leq \epsilon \Rightarrow |\sigma(\boldsymbol{x}) - \sigma(\boldsymbol{x}^*)| \leq m\epsilon \Rightarrow \sigma(\boldsymbol{x}^*) - m\epsilon \leq \sigma(\boldsymbol{x}) \leq \sigma(\boldsymbol{x}^*) + m\epsilon.$$

Assuming $\epsilon < \frac{1}{20m}$, we get that $\frac{1}{5} \leq \sigma(\boldsymbol{x}) \leq \frac{2}{3}$. Next, consider the quantity we wish to bound:

$$\begin{aligned}\left|\frac{x_i}{\sigma(\boldsymbol{x})} - \frac{x_i^*}{\sigma(\boldsymbol{x}^*)}\right| &\leq 20|x_i \textstyle\sum_k x_k^* - x_i^* \sum_k x_k| \\ &\leq 20|x_i(m\epsilon + \textstyle\sum_k x_k) - (x_i - m\epsilon)\sum_k x_k| \\ &\leq 20(m\epsilon(x_i + \textstyle\sum_k x_k)) \\ &\leq 20(m+1)\epsilon \leq \epsilon'.\end{aligned}$$

Similar argument suffices to show $\forall j$, $|\frac{y_j}{\sigma(\boldsymbol{y})} - \frac{y_j^*}{\sigma(\boldsymbol{y}^*)}| \leq \epsilon'$, and $\forall k$, $|\frac{z_k}{\sigma(\boldsymbol{z})} - \frac{z_k^*}{\sigma(\boldsymbol{z}^*)}| \leq \epsilon'$. □

From Theorem 5.5, we know that a symmetric NE $\boldsymbol{d}^* = (\boldsymbol{x}^*|\boldsymbol{y}^*|\boldsymbol{z}^*)$ maps to a NE $(\boldsymbol{x'}^*, \boldsymbol{y'}^*, \boldsymbol{z'}^*) = (\eta(\boldsymbol{x}^*), \eta(\boldsymbol{y}^*), \eta(\boldsymbol{z}^*))$ of game $(A, B, C)$. Lemma 6.3 implies that finding a profile $(\boldsymbol{x'}, \boldsymbol{y'}, \boldsymbol{z'})$ that is $\epsilon'$ near to $(\boldsymbol{x'}^*, \boldsymbol{y'}^*, \boldsymbol{z'}^*)$, for any $\epsilon' < 1$ reduces to finding a symmetric profile $\boldsymbol{d}$ that is $\frac{\epsilon'}{20l}$ near to $\boldsymbol{d}^*$. Clearly, there is such a $\boldsymbol{d}$ with size $poly\{size(A, B, C), \log(\frac{1}{\epsilon'})\}$, and therefore it can be mapped to a solution of $(A, B, C)$ in polynomial time. Since such an approximation in 3-Nash is FIXP$_a$-hard [11], and symmetric 3-Nash is in FIXP (Theorem 6.1), the next theorem follows.

Theorem 6.4. *Symmetric* 3-*Nash is* FIXP$_a$-*complete.*

Since there is no trivial reduction from symmetric three-player game to symmetric $k$-player game, in the next section, we extend Theorems 5.6, 5.7, and 6.4 to symmetric $k$-Nash, to obtain all the results for the latter.

## 7 SYMMETRIC $K$-NASH: ∃ℝ AND FIXP$_A$ COMPLETENESS

Building on the construction of Section 5, in this section, we reduce $k$-Nash to symmetric $k$-Nash. Given a $k$-player game $\mathcal{A} = (A_1, \ldots, A_k)$, we construct a symmetric game $D$ where the set of strategies of each player is $S = \cup_i S_i$, such that NE of game $\mathcal{A}$ maps to symmetric NE of game $D$, and vice versa. Note that $D$ will be a $k$-dimensional tensor with $l = \sum_i m_i$ coordinates in each dimension, where $m_i = |S_i|$. First, we construct a symmetric game $G$ (similar to that of (6)), which has now $k$-players each with $k$ strategies. As players are identical in symmetric games, the payoff of a player from her pure-strategy depends on which strategies are played by how many players; it doesn't matter who played what. Therefore, the non-zero entries of $G$ may be represented as follows, where $a_1, \ldots, a_k$ are non-negative numbers.

$$\begin{aligned}&G(i, i+1, \ldots, i+1) = 1,\ \forall i < k; \quad G(k, 1, \ldots, 1) = 1; \\ &G(i, \{1, \ldots, i-1, i+1, \ldots, k\}) = a_i, \forall i \leq k, \forall \text{ permutations of } \{1, \ldots, i-1, i+1, \ldots, k\};\end{aligned}$$

Set the rest of entries of $G$ to zero.

Similar to Lemma 5.1, it follows that all symmetric NE of $G$ are of full support. Next, we can blow up $G$ to construct $D$. Note that $G$ is a $k$-dimensional tensor with length $k$ in each dimension, i.e., for any $(i_1, \ldots, i_k)$th entry of $G$ each $i_j \in [k]$. In every dimension, $j$th element will represent $j$th player of game $\mathcal{A}$ when mapped to $D$, and therefore in game $D$ it will be replaced by $m_j = |S_j|$ many elements. Thus, $D$ will be $k$-dimensional tensor with length $l = \sum_{j=1}^{k} m_j$ in each dimension.

If $G(i_1, \ldots, i_k)$ is zero or one, then we replace it by a $k$-dimensional tensor of all zeros or all ones, respectively, of dimension $m_{i_1} \times \cdots \times m_{i_k}$. Note that if $G(i_1, \ldots, i_k) = a_i$ for some $i \in [k]$, then set $\{i_1, \ldots, i_k\} = [k]$; i.e., every player is represented. We will replace this entry in $G$ by tensor $A_i$ from game $\mathcal{A}$ after appropriate rotation so that its $j$th dimension corresponds to $j$th player.

Like Lemma 5.2, we can show that if $\boldsymbol{d} = (\boldsymbol{x}^1 | \ldots | \boldsymbol{x}^k)$ is a symmetric NE of game $D$ then $(\sigma(\boldsymbol{x}^1), \ldots, \sigma(\boldsymbol{x}^k))$ is a symmetric NE of game $G$, thereby showing that each of these sums are strictly positive. Here, $a_i$ is set to the best payoff achieved among the strategies of $\boldsymbol{x}^i$ divided by $\Pi_{j \neq i} \sigma(\boldsymbol{x}^j)$. Further, $\boldsymbol{d}$ being a NE it ensures that if a coordinate $j$ of $\boldsymbol{x}^i$ is non-zero then payoff from $j$th strategy, among strategies corresponding to $\boldsymbol{x}^i$ is the best. This sets the stage to obtain NE of game $\mathcal{A}$ from $\boldsymbol{d}$, namely $(\eta(\boldsymbol{x}^1), \ldots, \eta(\boldsymbol{x}^k))$ (similar to Lemma 5.3).

For the reverse mapping, let $\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k)$ be a NE of game $\mathcal{A}$, and let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a symmetric NE of $G$, where $a_i$ is set to the payoff player $i$ receives at the given NE of $\mathcal{A}$. Then, it follows that $\boldsymbol{d} = (\alpha_1 \boldsymbol{x}^1 | \ldots | \alpha_k \boldsymbol{x}^k)$ is a symmetric NE of $D$. The brief reason is as follows: the best payoff from $i$th block of strategies is $a'_i = \alpha_{i+1}^{k-1} + (k-1)! a_i \Pi_{j \neq i} \alpha_i$, and $\boldsymbol{x}$ being a NE of $\mathcal{A}$, non-zero strategies of $\boldsymbol{x}^i$ fetch best payoff to player $i$, namely $a_i$. Hence, in $\boldsymbol{d}$ the strategies played with non-zero probability within block $i$ fetch payoff $a'_i$. Since $a_i$ is the maximum payoff of player $i$ from any of its pure strategies, $a'_i$ is also maximum among the payoffs from the strategies within the block. Furthermore, $a'_i$ is also the payoff from $i$th strategy in game $G$, and $\boldsymbol{\alpha}$ being a NE with full support, it ensures that all $a'_i$s are same. Thus, in $\boldsymbol{d}$ best payoffs are the same across blocks, and therefore it is a symmetric NE of game $D$.

The next theorem follows from the above discussion (of this section).

Theorem 7.1. *Profile $\boldsymbol{d} = (\boldsymbol{x}^1 | \ldots | \boldsymbol{x}^k)$ is a symmetric NE of game $D$ if and only if $(\eta(\boldsymbol{x}^1), \ldots, \eta(\boldsymbol{x}^k))$ is a NE of game $(A_1, \ldots, A_k)$.*

Using Theorem 7.1 together with Theorems 3.2 and 4.19, we get the following $\exists\mathbb{R}$-completeness results.

Theorem 7.2. *For symmetric $k$-Nash, problems* **Subset** *and* **Superset** *are $\exists\mathbb{R}$-complete, where $k \geq 3$ is a constant.*

A normal form $k$-player game can be reduced to $k+1$-player game trivially by adding a dummy player with one strategy and any payoff, and therefore FIXP$_a$-hardness of Theorem 2.2 extends to $k$-Nash for $k \geq 3$. However, such a reduction is not possible in the case of symmetric games, because the resulting game has to satisfy the symmetry conditions (see Section 2.1). Therefore, FIXP$_a$-hardness for symmetric 3-Nash does not extend to symmetric $k$-Nash for $k > 3$. We show this result using the fact that $k$-Nash is FIXP$_a$-hard together with Theorem 7.1.

As done in Section 6, we need to lower bound each $\sigma(\boldsymbol{x}^i)$ for a given symmetric NE $\boldsymbol{d} = (\boldsymbol{x}^1 | \ldots | \boldsymbol{x}^k)$ of game $D$. Similar to Lemma 6.2, we show the following. Define

$$\Gamma = \frac{1}{(k-1)!} \left( \left( \frac{k}{k-1} \right)^{(k-1)} - 1 \right).$$

LEMMA 7.3. *Let* $(\alpha_1, \ldots, \alpha_k)$ *be a NE of game* $G$ *with* $a_i \in [0,\ \Gamma],\ \forall i \in [k]$*, then* $\frac{1}{k+1} \leq \alpha_i \leq \frac{1}{k-1},\ \forall i \in [k]$.

PROOF. Suppose not, and wlog let $\alpha_1 < \frac{1}{k+1}$. Then, $\exists i \neq 1,\ \alpha_i > \frac{k}{k^2-1}$, let it be $i = 2$ (wlog). Then, the payoff of first player from strategy one is

$$\alpha_2^{(k-1)} + (k-1)! a_1 \prod_{i=2}^{k} \alpha_i \geq \left(\frac{k}{k^2-1}\right)^{(k-1)}.$$

Given that $d$ non-negative numbers sum up to one, then their product is maximized when each number is 1/d. Using this, from strategy $k$, player one gets

$$\alpha_1^{(k-1)} + (k-1)! a_k \prod_{i=1}^{(k-1)} \alpha_i < \frac{1}{(k+1)^{(k-1)}} + \frac{(k-1)!\ a_k}{(k+1)(k-1)^{(k-2)}}.$$

Since at equilibrium both the strategies are played with non-zero probability, we get

$$\left(\frac{k}{k^2-1}\right)^{(k-1)} < \frac{1}{(k+1)^{(k-1)}} + \frac{(k-1)!\ a_k}{(k+1)(k-1)^{(k-2)}} \Rightarrow \frac{1}{(k-1)!}\left(\left(\frac{k}{k-1}\right)^{(k-1)} - 1\right) < a_k,$$

which is a contradiction to $a_k \leq \Gamma$. □

We can wlog assume that $A_1, \ldots, A_k \in [0, \Gamma]$, since NE remains unchanged when all the payoffs are scaled additively, or multiplicatively by a positive constant. This will ensure that payoff of each player in $\mathcal{A}$ at any NE is in $[0, \Gamma]$.

Since we know that if $\boldsymbol{d} = (\boldsymbol{x}^1 | \ldots | \boldsymbol{x}^k)$ is a symmetric NE of game $D$ then $(\sigma(\boldsymbol{x}^1), \ldots, \sigma(\boldsymbol{x}^k))$ is a symmetric NE of game $G$, from Lemma 7.3, we get that each of $\sigma(\boldsymbol{x}^i)$ is lower bounded by $\frac{1}{k+1}$. Finally, using this lower bound, we can show that if $|\boldsymbol{d} - \boldsymbol{d}^*|_\infty < \epsilon$ where $\boldsymbol{d}^*$ is a symmetric NE, then $|x_s^i/\sigma(\boldsymbol{x}^i) - x_s^{*^i}/\sigma(\boldsymbol{x}^{*^i})| < \epsilon',\ \ \forall i \in [1:k], \forall s \in S_i$, where $\epsilon = \frac{\epsilon'}{2(k+1)^2(\sum_i |S_i|)}$ and $\epsilon < \frac{1}{5(k+1)(\max_{i \in [k]} |S_i|)}$ (similar to Lemma 6.3). In other words, the strategy profile $(\eta(\boldsymbol{x}^1), \ldots, \eta(\boldsymbol{x}^k))$ obtained from $\boldsymbol{d}$ is $\epsilon'$-near to NE $(\eta(\boldsymbol{x}^{*^1}), \ldots, \eta(\boldsymbol{x}^{*^k}))$ obtained from $\boldsymbol{d}^*$ for game $\mathcal{A}$. Thus, FIXP$_a$-hardness follows for symmetric $k$-Nash, and we get the next result using Theorem 6.1.

THEOREM 7.4. *For a constant* $k \geq 3$*, symmetric* $k$*-Nash is* FIXP$_a$*-complete.*

## 8 DISCUSSION

There is a reduction from symmetric 2-Nash to 2-Nash using the notion of imitation games [20]. Is there an analogous reduction from symmetric $k$-Nash to $k$-Nash, for $k \geq 3$? For the case of two-player games, Papadimitriou [24] asked the complexity of finding a non-symmetric equilibrium in a symmetric game. This was recently shown to be NP-complete [21]. What is the complexity of the analogous question for $k$-player games, for $k \geq 3$? For the case of two-player games, the question of counting the number of equilibria, even those satisfying special properties, is typically #P-complete. What is the complexity of analogous questions for $k$-player games, for $k \geq 3$? Are they PSPACE-complete? Another question is whether our reduction from 3-Nash to symmetric 3-Nash creates a one-to-one correspondence between solutions of the two problems. If so, then intractability of counting 3-Nash solutions will carry over to counting symmetric 3-Nash solutions.

For $k$-player games, $k \geq 3$, finding an $\epsilon$-approximate Nash equilibrium was shown to be in the class PPAD by Reference [9]. Equilibrium questions that are in this class have admitted complementary pivot algorithms that are practical, e.g., for 2-Nash [18] and for market equilibrium under separable, piecewise-linear concave utility functions [12]. Are there practical algorithms for finding an $\epsilon$-approximate Nash equilibrium in $k$-player games, $k \geq 3$?

We next come to other results on NE satisfying certain properties for two-player games. First, Reference [8] showed that finding a exact NE that approximately maximizes properties such as social welfare is NP-hard. Next, Reference [16] showed that finding an approximately Nash equilibrium that maximizes the social welfare is as hard as finding a planted clique in a random graph $G(n, 1/2)$, and Reference [1] showed the same hardness for the following three problems: finding an approximate NE with payoff more than $v$, finding two approximate Nash equilibria that are far apart and finding an approximate NE with a small support. Resolving the complexity of analogous problems for three-player games is open.

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Received December 2016; revised May 2017; accepted July 2017