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Residuated fuzzy logics with an involutive negation

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Abstract. Residuated fuzzy logic calculi are related to continuous t-norms, which are used as truth functions for conjunction, and their residua as truth functions for implication. In these logics, a negation is also definable from the implication and the truth constant $\overline{0}$, namely $\neg\varphi$ is $\varphi \to \overline{0}$. However, this negation behaves quite differently depending on the t-norm. For a nilpotent t-norm (a t-norm which is isomorphic to Łukasiewicz t-norm), it turns out that $\neg$ is an involutive negation. However, for t-norms without non-trivial zero divisors, $\neg$ is Gödel negation. In this paper we investigate the residuated fuzzy logics arising from continuous t-norms without non-trivial zero divisors and extended with an involutive negation.

1. Introduction

Residuated fuzzy (many-valued) logic calculi are related to continuous t-norms which are used as truth functions for the conjunction connective, and their residua as truth functions for the implication. Main examples are Łukasiewicz (Ł), Gödel (G) and product (Π) logics, related to Łukasiewicz t-norm ($x * y = \max(0, x + y - 1)$), Gödel t-norm ($x * y = \min(x, y)$) and product t-norm ($x * y = x \cdot y$) respectively. In the fifties Rose and Rosser [7] provided completeness results for Łukasiewicz logic and Dummet [2] for Gödel logic, and recently three of the authors [5] axiomatized product logic. More recently, Hájek [4] has proposed the axiomatic system BL corresponding to a generic continuous t-norm and having Ł, G and Π as extensions.

In all these logics, a negation is also definable from the implication and the truth constant $\overline{0}$, namely $\neg\varphi$ is $\varphi \to \overline{0}$. However, this negation behaves quite differently depending on the t-norm.

Nilpotent t-norms are continuous t-norms $*$ such that each element $x \in (0,1)$ is nilpotent, that is, there exists $n \in N$ such that $x * \overset{n}{\ldots} * x = 0$. It has been shown that nilpotent t-norms are exactly those which are isomorphic to Łukasiewicz t-norm. For nilpotent t-norms, it turns out that its residuum $\Rightarrow$ defines an involutive negation[1] $n : [0,1] \to [0,1]$ as

$$n(x) = (x \Rightarrow 0),$$

that is, n is a non-increasing involution in [0, 1]. In particular, for Łukasiewicz implication $(x \Rightarrow 0) = 1 - x$.

Among t-norms which are not nilpotent, we are interested in those which do not have *non-trivial zero divisors*, i.e., which verify:

$$\forall x, y \in [0,1], x * y = 0 \text{ iff } (x = 0 \text{ or } y = 0).$$

This condition characterizes those t-norms for which the negation definable from its residuum, i.e. $n(x) = (x \Rightarrow 0)$, is not any longer a strong negation but Gödel negation, that is:

$$(x \Rightarrow 0) = \begin{cases} 1, \text{ if } x = 0, \\ 0, \text{ otherwise.} \end{cases}$$

If we restrict ourselves to continuous t-norms, this is the case of the so-called *strict* t-norms (i.e. those which are isomorphic to product), the *minimum* t-norm, and those t-norms which are ordinal sums not having a t-norm isomorphic to Łukasiewicz t-norm in the first square around the point $(0,0)$ (see the Appendix for further details). Observe that Łukasiewicz t-norm has zero divisors, which it is not the case of product and minimum t-norms.

In this paper we investigate the many-valued residuated logics arising from continuous t-norms without non-trivial zero divisors and extended with an involutive negation. In the next section we provide the main results about the residuated fuzzy logics BL and BL_Δ needed for the paper. In Section 3 and 4, we first define SBL, the schematic extension of the basic logic BL accounting for those logics in which the negation $\neg$ defined above is Gödel negation, and afterwards we extend it with an involutive negation and present completeness results for the resulting logic $\mathrm{SBL}_\sim$. In Section 5 we show how the standard completeness theorems for Gödel and product logics generalize when both logics are extended with the involutive negation. In Section 6 predicate calculi for SBL, product and Gödel logics with involutive negation are studied. Finally, in Section 7 we extend product and Gödel logics with involutive negation by introducing a truth-constant for each rational of [0, 1] and we discuss Pavelka-style completeness results for both logics.

[1] Also called *strong* negation in the literature on fuzzy set connectives.

2. Background: the basic fuzzy logics BL and BL$_\Delta$

Here we summarize some important notions and facts from [4].

2.1. The basic fuzzy logic BL

The language of the basic logic BL is built in the usual way from a (countable) set of propositional variables, a conjunction $\&$, an implication $\rightarrow$ and the truth constant $\bar{0}$. Further connectives are defined as follows:

$$\begin{aligned}
\varphi \wedge \psi &\text{ is } \varphi \& (\varphi \rightarrow \psi),\\
\varphi \vee \psi &\text{ is } ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi),\\
\neg\varphi &\text{ is } \varphi \rightarrow \bar{0},\\
\varphi \equiv \psi &\text{ is } (\varphi \rightarrow \psi)\&(\psi \rightarrow \varphi).
\end{aligned}$$

The following formulas are the *axioms* of BL:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $(\varphi \& \psi) \rightarrow \varphi$
(A3) $(\varphi \& \psi) \rightarrow (\psi \& \varphi)$
(A4) $(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \& (\psi \rightarrow \varphi))$
(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$
(A5b) $((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A7) $\bar{0} \rightarrow \varphi$

The *deduction rule* of BL is modus ponens.

If one takes a continuous t-norm $*$ for the truth function of $\&$ and the corresponding residuum[2] $\Rightarrow$ for the truth function of $\rightarrow$ (and evaluating $\bar{0}$ by 0) then all the axioms of BL become 1-tautologies (have identically the truth value 1). And since modus ponens preserves 1-tautologies, all formulas provable in BL are 1-tautologies.

It has been shown [4] that the well-known Łukasiewicz logic is the extension of BL by the axiom

(Ł) $\neg\neg\varphi \rightarrow \varphi$,

and Gödel logic is the extension of BL by the axiom

(G) $\varphi \rightarrow (\varphi \& \varphi)$.

Finally, product logic is just the extension of BL by the following two axioms:

(Π1) $\neg\neg\chi \rightarrow (((\varphi \& \chi) \rightarrow (\psi \& \chi)) \rightarrow (\varphi \rightarrow \psi))$,
(Π2) $\varphi \wedge \neg\varphi \rightarrow \bar{0}$.

[2] The residuum $\Rightarrow$ is the binary function on [0, 1] defined as $(x \Rightarrow y) = \sup\{z \in [0,1] \mid x * z \leq y\}$.

2.2. BL-algebras and a completeness theorem

A BL-*algebra* is an algebra

$$\mathbf{L} = (L, \cap, \cup, *, \Rightarrow, 0, 1)$$

with four binary operations and two constants such that

(i) $(L, \cap, \cup, 0, 1)$ is a lattice with the greatest element 1 and the least element 0 (with respect to the lattice ordering $\leq$),
(ii) $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e. $*$ is commutative, associative and $1 * x = x$ for all x,
(iii) the following conditions hold for all x, y, z:
 (1) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$
 (2) $x \cap y = x * (x \Rightarrow y)$
 (3) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

Thus, in other words, a BL-algebra is a *residuated lattice* satisfying (2) and (3). The class of all BL-algebras is a variety. Moreover, each BL-algebra can be decomposed as a subdirect product of linearly ordered BL-algebras.

Defining $\neg x = (x \Rightarrow 0)$, it turns out that MV-*algebras* are BL-algebras satisfying $\neg\neg x = x$, G-*algebras* are BL-algebras satisfying $x * x = x$, and finally, *product algebras* are BL-algebras satisfying

$$x \cap \neg x = 0$$
$$(\neg\neg z \Rightarrow ((x * z \Rightarrow y * z) \Rightarrow (x \Rightarrow y))) = 1.$$

The logic BL is sound with respect to **L**-tautologies: if φ is provable in BL then φ is an **L**-tautology for each BL-algebra **L** (i.e. has the value $1_{\mathbf{L}}$ for each evaluation of variables by elements of **L** extended to all formulas using operations of **L** as truth functions).

Theorem 1. BL *is complete, i.e. for each formula φ the following three conditions are equivalent:*

(i) *φ is provable in* BL,
(ii) *for each* BL-*algebra* **L**, *φ is an* **L**- *tautology,*
(iii) *for each linearly ordered* BL-*algebra* **L**, *φ is an* **L**-*tautology.*

This theorem also holds if we replace BL by a *schematic extension*[3] $\mathcal{C}$ of BL, and BL-algebras by the corresponding $\mathcal{C}$-algebras (BL-algebras in which all axioms of $\mathcal{C}$ are tautologies).

Note that we also get *strong completeness* for provability in theories over BL. For completeness theorems of the three main many-valued logics (Łukasiewicz, Gödel and product) see [4].

[3] A calculus which results from BL by adding some axiom schemata.

2.3. The extended basic fuzzy logic BL$_\Delta$

Now we expand the language of BL by a new unary (projection) connective Δ whose truth function (denoted also by Δ) is defined as follows:

$$\Delta x = \begin{cases} 1, \text{ if } x = 1 \\ 0, \text{ otherwise} \end{cases}$$

The *axioms* of the extended basic logic BL$_\Delta$ (first formulated by Baaz in [1]) are those of BL plus:

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$
(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
(Δ3) $\Delta\varphi \to \varphi$
(Δ4) $\Delta\varphi \to \Delta\Delta\varphi$
(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

Deduction rules of BL$_\Delta$ are modus ponens and *necessitation*: from φ derive $\Delta\varphi$.

A Δ-algebra is a structure $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow, 0, 1, \Delta)$ which is a BL-algebra expanded by a unary operation Δ satisfying the following conditions:

$\Delta x \cup \neg\Delta x = 1$
$\Delta(x \cup y) \leq \Delta x \cup \Delta y$
$\Delta x \leq x$
$\Delta x \leq \Delta\Delta x$
$(\Delta x) * (\Delta(x \Rightarrow y)) \leq \Delta y$
$\Delta 1 = 1$

The notions of **L**-evaluation and **L**-tautology easily generalize to BL$_\Delta$ and Δ-algebras. The decomposition of any BL$_\Delta$ algebra as a subdirect product of linearly ordered ones also holds. Notice that in linearly ordered Δ-algebras we have that $\Delta 1 = 1$ and $\Delta a = 0$ for $a \neq 1$. Then the above completeness theorem for BL extends to BL$_\Delta$ as follows.

Theorem 2. BL$_\Delta$ *is complete, i.e. for each formula* φ *the following three conditions are equivalent:*

(i) φ *is provable in* BL$_\Delta$,
(ii) *for each* Δ*-algebra* **L**, φ *is an* **L**-*tautology,*
(iii) *for each linearly ordered* Δ*-algebra* **L**, φ *is an* **L**-*tautology.*

A *strong completeness* result for provability in theories over BL$_\Delta$ is also given in [4].

3. The basic strict fuzzy logic SBL

In this section we introduce the strict basic logic SBL, an extension of the basic logic BL for which the linearly ordered BL-algebras that satisfy

SBL axioms are those having Gödel negation. In the next section we shall introduce an involutive negation over SBL.

Definition 1. Axioms of the basic strict fuzzy logic SBL are those of BL plus the following axiom:
(STR) $(\varphi \& \psi \to \overline{0}) \to ((\varphi \to \overline{0}) \vee (\psi \to \overline{0}))$.

An equivalent expression of the axiom (STR) is:

$$\neg(\varphi \& \psi) \to (\neg\varphi \vee \neg\psi),$$

where $\neg\varphi$ is $\varphi \to \overline{0}$. Notice that (STR) is a theorem in both product and Gödel logics. Moreover, it can be shown that SBL proves $\varphi \wedge \neg\varphi \to \overline{0}$ (cf. [4] Sect. 4.1).

Definition 2. An SBL-algebra is a BL-algebra $(L, \cap, \cup, *, \Rightarrow, 0, 1)$ verifying this further condition:

$$((x * y) \Rightarrow 0) \ = \ (x \Rightarrow 0) \cup (y \Rightarrow 0).$$

Note that this condition is equivalent to the seemingly weaker condition

$$(((x * y) \Rightarrow 0) \Rightarrow ((x \Rightarrow 0) \cup (y \Rightarrow 0))) = 1$$

or, equivalently,

$$((x * y) \Rightarrow 0) \ \leq \ (x \Rightarrow 0) \cup (y \Rightarrow 0).$$

To show the converse inequality just observe that

$$(x \Rightarrow 0) \leq ((x * y) \Rightarrow 0) \text{ since } x * y \leq x,$$

and similarly $(y \Rightarrow 0) \leq ((x * y) \Rightarrow 0)$.

Examples of SBL-algebras are the algebras $([0,1], \max, \min, *, \Rightarrow, 0, 1)$, where $*$ is a t-norm without non-trivial zero divisors and $\Rightarrow$ its corresponding residuum, and the quotient algebra SBL/$\equiv$ of provably equivalent formulas.

In linearly ordered SBL-algebras, the above condition implies

$$x * y = 0 \text{ iff } (x = 0 \text{ or } y = 0). \tag{1}$$

Indeed, if $x*y = 0$ then $((x*y) \Rightarrow 0) = 1$, thus $(x \Rightarrow 0) \cup (y \Rightarrow 0) = 1$, which, due to linearity, gives $(x \Rightarrow 0) = 1$ or $(y \Rightarrow 0) = 1$, i.e. $x = 0$ or $y = 0$.

Moreover, this condition identifies linearly ordered SBL-algebras with linearly ordered BL-algebras which have Gödel negation.

Lemma 1. *A linearly ordered* BL-*algebra is an* SBL-*algebra iff it satisfies* (1)*, and iff the negation* $\neg x = (x \Rightarrow 0)$ *is Gödel negation.*

Proof. We have shown above that linearly ordered SBL-algebras satisfy (1). In a linearly ordered BL-algebra satisfying (1), $\neg x = (x \Rightarrow 0) = \sup\{z \mid x * z \leq 0\}$ which is 1 if $x = 0$ and is 0 otherwise due to (1). Finally if a linearly ordered BL-algebra has Gödel negation then we easily get the condition of SBL-algebra. Indeed, if $x*y = 0$ then both sides in the condition of SBL-algebras equal 1, and if $x * y > 0$ then both sides equal 0. □

Theorem 3 (Completeness). *The logic* SBL *is complete w.r.t. the class of linearly ordered* SBL-*algebras.*

This follows immediately from [4] 2.3.22, noticing that SBL is a schematic extension of BL.

4. Extending SBL by an involutive negation

Now we extend SBL with a unary connective $\sim$. The *semantics* of $\sim$ is an arbitrary strong negation function

$$n : [0,1] \to [0,1]$$

which is a decreasing involution, i.e. $n(n(x)) = x$ and $n(x) \leq n(y)$ whenever $x \geq y$. It turns out that with both negations, $\neg$ and $\sim$, the projection connective Δ is definable:

$$\Delta\varphi \text{ is } \neg{\sim}\varphi$$

Moreover, notice that having an involutive negation in the logic enriches, in a non-trivial way, the representational power of the logical language. For instance, a strong disjunction $\varphi\underline{\vee}\psi$ is definable now as ${\sim}({\sim}\varphi \,\&{\sim}\psi)$, with truth function the *t-conorm* $\oplus$ defined as $x \oplus y = n(n(x) * n(y))$, and a contrapositive implication $\varphi \hookrightarrow \psi$ is definable as ${\sim}\varphi\underline{\vee}\psi$, with truth function the *strong implication* function $\overset{c}{\Rightarrow}$ defined as $(x \overset{c}{\Rightarrow} y) = {\sim}x \oplus y$. Although these new connectives may be interesting for future development, we shall make no further use of them in the rest of the paper.

Definition 3. Axioms of $\mathrm{SBL}_{\sim}$ are those of SBL plus

$({\sim}1)$	$({\sim}{\sim}\varphi) \equiv \varphi$	*(Involution)*
$({\sim}2)$	$\neg\varphi \to {\sim}\varphi$	
$({\sim}3)$	$\Delta(\varphi \to \psi) \to \Delta({\sim}\psi \to {\sim}\varphi)$	*(Order Reversing)*
$(\Delta 1)$	$\Delta\varphi \vee \neg\Delta\varphi$	
$(\Delta 2)$	$\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$	
$(\Delta 5)$	$\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$	

where $\Delta\varphi$ is $\neg{\sim}\varphi$. Deduction rules of $\mathrm{SBL}_{\sim}$ are those of BL_{Δ}, that is, *modus ponens* and *necessitation* for Δ.

Lemma 2. $\mathrm{SBL}_\sim$ *proves*
$(\Delta 3)$ $\Delta\varphi \to \varphi$
$(\Delta 4)$ $\Delta\varphi \to \Delta\Delta\varphi$
and thus $\mathrm{SBL}_\sim$ *extends* BL_Δ. *Moreover, in* $\mathrm{SBL}_\sim$ *the following is a derived inference rule:*
(CP) *from* $\varphi \to \psi$ *derive* $\sim\psi \to \sim\varphi$.

Proof. $(\Delta 3)$ is an easy consequence of the definition of the connective Δ, (~ 2) and (~ 1). $(\Delta 4)$ comes from ($\sim$3) taking 1 for φ and φ for ψ. Finally, if $\mathrm{SBL}_\sim$ proves $\varphi \to \psi$, it also proves $\Delta(\varphi \to \psi)$ (necessitation), and thus it proves $\Delta(\sim\psi \to \sim\varphi)$ (axiom ($\sim$3)). By (Δ3), it also proves $\sim\psi \to \sim\varphi$. □

Lemma 3. $\mathrm{SBL}_\sim$ *proves the following De Morgan laws:*
$(DM1)$ $\sim(\varphi \wedge \psi) \equiv (\sim\varphi \vee \sim\psi)$
$(DM2)$ $\sim(\varphi \vee \psi) \equiv (\sim\varphi \wedge \sim\psi)$

Proof. We prove (DM1). Clearly, BL proves $\varphi \wedge \psi \to \varphi$, $\varphi \wedge \psi \to \psi$, and by the above derived inference rule, $\mathrm{SBL}_\sim$ proves $\sim\varphi \to \sim(\varphi \wedge \psi)$ and $\sim\psi \to \sim(\varphi \wedge \psi)$, and thus it proves also $(\sim\varphi) \vee (\sim\psi) \to \sim(\varphi \wedge \psi)$.

On the other direction, BL proves both $\sim\varphi \to (\sim\varphi \vee \sim\psi)$ and $\sim\psi \to (\sim\varphi \vee \sim\psi)$. Applying again the above rule and (~ 1), we have that $\mathrm{SBL}_\sim$ proves $\sim(\sim\varphi \vee \sim\psi) \to \varphi$ and $\sim(\sim\varphi \vee \sim\psi) \to \psi$, and thus it proves $\sim(\sim\varphi \vee \sim\psi) \to (\varphi \wedge \psi)$, and finally, by the inference rule again, it proves $\sim(\varphi \wedge \psi) \to (\sim\varphi \vee \sim\psi)$. This completes the proof. □

In SBL$_\sim$ the classical deduction theorem fails, but we have the same weaker formulation as in BL$_\Delta$ (see [4] 2.4.14).

Theorem 4 (Deduction theorem). *Let* T *be a theory over* SBL$_\sim$. *Then* $T \cup \{\varphi\} \vdash \psi$ *iff* $T \vdash \Delta\varphi \to \psi$.

Definition 4. An $\mathrm{SBL}_\sim$-algebra is a structure $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow, \sim, 0, 1)$ which is an SBL-algebra expanded with a unary operation $\sim$ satisfying the following conditions:
$(A_\sim 1)$ $\sim\sim x = x$
$(A_\sim 2)$ $\neg x \leq \sim x$
$(A_\sim 3)$ $\Delta(x \Rightarrow y) = \Delta(\sim y \Rightarrow \sim x)$
$(A_\sim 4)$ $\Delta x \cup \neg\Delta x = 1$
$(A_\sim 5)$ $\Delta(x \cup y) \leq \Delta x \cup \Delta y$
$(A_\sim 6)$ $\Delta x * (\Delta(x \Rightarrow y)) \leq \Delta y$
where $\neg x = (x \Rightarrow 0)$ and $\Delta x = (\sim x \Rightarrow 0)$.

Examples of $\mathrm{SBL}_\sim$-algebras are:

– The algebras $([0,1], \max, \min, *, \Rightarrow, n, 0, 1)$ of the unit interval of the real line with any strict t-norm $*$, its corresponding residuated implication $\Rightarrow$ and with any involutive negation function $n : [0,1] \to [0,1]$.

– The quotient algebra $\mathrm{SBL}_\sim/\!\equiv$ of provably equivalent formulas. Indeed, since $\mathrm{SBL}_\sim$ is an extension of BL_Δ, we only need to check that $\equiv$ is a congruence w.r.t. the involutive negation $\sim$. So, assume $\varphi \to \psi$ is provable. Then, using the necessitation rule, $\Delta(\varphi \to \psi)$ is also provable, and by axiom (~ 3) we get $\Delta(\sim\psi \to \sim\varphi)$, and finally, by $(\Delta 3)$ we prove $\sim\psi \to \sim\varphi$.

Lemma 4. *In any* $\mathrm{SBL}_\sim$*-algebra the following properties hold:*
(1) $\sim 0 = 1$
(2) $\sim 1 = 0$
(3) $\Delta 1 = 1$

Proof. (1) By $(A_\sim 2)$, $\sim 0 \geq \neg 0$ and, by definition, $\neg 0 = (0 \Rightarrow 0) = 1$.
(2) Using (1), $\sim 1 = \sim\sim 0$, and by $(A_\sim 1)$, $\sim\sim 0 = 0$.
(3) By definition, $\Delta 1 = (\sim 1 \Rightarrow 0)$, but from (2) $\sim 1 = 0$, and thus $\Delta 1 = (0 \Rightarrow 0) = 1$. □

Lemma 5. $\mathrm{SBL}_\sim$ *is sound with respect to the class of* $\mathrm{SBL}_\sim$*-algebras.*

Soundness of axioms is straightforward from the definition of $\mathrm{SBL}_\sim$-algebras and the soundness of the necessitation inference rule for Δ is a consequence of (3) of previous Lemma.

Lemma 6. *In any* $\mathrm{SBL}_\sim$*-algebra the following properties hold:*
(1) $(\sim x) \cap (\sim y) = \sim(x \cup y)$, $(\sim x) \cup (\sim y) = \sim(x \cap y)$
(2) *If* $x \leq y$, *then* $\sim y \leq \sim x$
(3) $\Delta(x \Rightarrow y) \leq (\Delta x \Rightarrow \Delta y)$
(4) *If* $x \leq y$, *then* $\Delta x \leq \Delta y$
(5) $\Delta x \leq x$
(6) $\Delta x * \Delta\neg x = 0$
(7) $\Delta x = \Delta\Delta x$
(8) $\Delta x = 1$ *iff x = 1*
(9) $\Delta x * \Delta y = \Delta(x * y)$
(10) $\Delta\sim x = \Delta\neg x = \neg x$

Proof. (1), (3), (5) and (7) are obvious consequences of the soundness of $\mathrm{SBL}_\sim$, in particular (1) follows from (DM1), (DM2), (3) follows from $(\Delta 5)$ and (5) and (7) follow from $\Delta 3$ and $\Delta 4$ respectively.
(2): If $x \leq y$ then $1 = (x \Rightarrow y) = \Delta(x \Rightarrow y) = \Delta(\sim y \Rightarrow \sim x) \leq (\sim y \Rightarrow \sim x)$, thus $\sim y \leq \sim x$.
(4): If $x \leq y$, applying (3) we obtain $1 = \Delta 1 = \Delta(x \Rightarrow y) \leq (\Delta x \Rightarrow \Delta y)$. Therefore $\Delta x \leq \Delta y$.
(6) is a direct consequence of ($A_\sim 6$) taking $y = 0$.
(8) follows from (3) of Lemma 4 and (5).

(9): It is proved in [4] 2.4.11 (4).
(10): From ($A_\sim 3$), taking $x = 1$ and $y = \sim x$, we obtain $\Delta(1 \Rightarrow \sim x) = \Delta(\sim\sim x \Rightarrow 0)$ and then $\Delta\sim x = \Delta\neg x$. The equality $\Delta\sim x = \neg x$ follows immediately from the definition of Δ. □

Lemma 7. (1) *In a linearly ordered* $\mathrm{SBL}_\sim$*-algebra,* $\Delta x = 0$ *for all* $x \neq 1$.
(2) *The class of* $\mathrm{SBL}_\sim$*-algebras is a variety.*

Proof. (1) Within a linearly ordered $\mathrm{SBL}_\sim$-algebra, by ($A_\sim 4$), we have that $\Delta x \cup \neg\Delta x = \max(\Delta x, \neg\Delta x) = 1$. Therefore either $\Delta x = 1$, which implies $x = 1$, or $(\Delta x \Rightarrow 0) = 1$, which implies $\Delta x = 0$.
(2) It is obvious from the definition of $\mathrm{SBL}_\sim$-algebras since all axioms can be written as equations. □

For proving the subdirect representation theorem for $\mathrm{SBL}_\sim$-algebras we will need some previous definitions and results.

Definition 5. *A subset F of an* $\mathrm{SBL}_\sim$*-algebra* $\mathbf{L}$ *is a filter if it satisfies:*

(F1) *If* $a, b \in F$*, then* $a * b \in F$
(F2) *If* $a \in F$ *and* $b \geq a$*, then* $b \in F$
(F3) *If* $(a \Rightarrow b) \in F$*, then* $(\sim b \Rightarrow \sim a) \in F$

F *is a prime filter if it is a filter and*

(F4) *For all* $a, b \in L$, $(a \Rightarrow b) \in F$ *or* $(b \Rightarrow a) \in F$

Remark 1. If $a \in F$ then $\Delta a \in F$. Indeed, by property (F3), $(1 \Rightarrow a) = a \in F$ implies $(\sim a \Rightarrow \sim 1) = (\sim a \Rightarrow 0) = \neg\sim a = \Delta a \in F$.

Lemma 8.

(1) *The relation* $a \equiv_F b$ *iff* $(a \Rightarrow b) \in F$ *and* $(b \Rightarrow a) \in F$*, is a congruence relation over an* $\mathrm{SBL}_\sim$*-algebra.*
(2) *The quotient of* $\mathbf{L}$ *by* $\equiv_F$ *is an* $\mathrm{SBL}_\sim$*-algebra.*
(3) *The quotient algebra is linearly ordered iff* F *is a prime filter.*
(4) *Linearly ordered* $\mathrm{SBL}_\sim$*-algebras* $\mathbf{L}$ *are simple, that is, the only filters of a linearly ordered* $\mathrm{SBL}_\sim$ *algebra* $\mathbf{L}$ *are* $\{1\}$ *and* L *itself.*

Proof. The proofs for (1), (2) and (3) are analogous to those for Δ-algebras. The proof of (4) reduces to showing that the only filters of a linearly ordered $\mathrm{SBL}_\sim$-algebra $\mathbf{L}$ are $\{1\}$ and the full algebra L itself. This is true because if a filter F has an element $a \neq 1$, then, by Remark 1 and Lemma 7, $\Delta a = 0 \in F$ and therefore $F = L$. □

Theorem 5. *Any* $\mathrm{SBL}_\sim$*-algebra is a subdirect product of linearly ordered* $\mathrm{SBL}_\sim$*-algebras.*

Notice that this theorem is actually a subdirect decomposition theorem because linearly ordered algebras are simple, and so subdirectly irreducible, which is not the case for other related algebras like BL, SBL, Π or MV algebras. The proof of this theorem is as usual and the only critical point is the proof of the following lemma.

Lemma 9. *Let* $\mathbf{L}$ *be an* $\mathrm{SBL}_\sim$*-algebra and* $a \in L$*. If* $a \neq 1$*, there is a prime filter* F *on* $\mathbf{L}$ *not containing* a*.*

Proof. The proof is very analogous to that for BL_Δ. The interesting point to remark is that the least filter containing another filter F and an element z is

$$F' = \{u \mid \exists v \in F, u \geq v * \Delta z\}.$$

It can be checked that F' is indeed a filter, in particular, we check that condition (F3) is satisfied. If $(x \Rightarrow y) \in F'$, it means that, for some $v \in F$, $(x \Rightarrow y) \geq v * \Delta z$, and then $(\sim y \Rightarrow \ \sim x) \geq \Delta(\sim y \Rightarrow \ \sim x) = \Delta(x \Rightarrow y) \geq \Delta(v * \Delta z) = \Delta v * \Delta\Delta z = \Delta v * \Delta z$, thus also $(\sim y \Rightarrow \ \sim x) \in F'$ since if $v \in F$, then $\Delta v \in F$ as well.

Then the sketch of the proof is as follows. Let F be a filter not containing a (there exists at least one, $F = \{1\}$). Let $x, y \in L$ such that neither $(x \Rightarrow y), (y \Rightarrow x) \notin F$. Using the above definition, we can build then F_1 and F_2 as the filters generated by F and $x \Rightarrow y$ and $y \Rightarrow x$ respectively. Then one can prove that at least one of these two filters does not contain a. In this way, a sequence of nested filters not containing a can be built. Finally, the prime filter which we are looking for is the union of that sequence of filters. □

Theorem 6. $\mathrm{SBL}_\sim$ *is complete w.r.t. the class of* $\mathrm{SBL}_\sim$*-algebras. In more details, for each formula* φ*, the following are equivalent:*

(i) $\mathrm{SBL}_\sim \vdash \varphi$,
(ii) φ *is an* $\mathbf{L}$*-tautology for each* $\mathrm{SBL}_\sim$*-algebra* $\mathbf{L}$*,*
(iii) φ *is an* $\mathbf{L}$*-tautology for each linearly ordered* $\mathrm{SBL}_\sim$*-algebra* $\mathbf{L}$*.*

Proof. The proof is fully analogous to the proof of Theorem 2.3.19 of [4]. In particular, the implication (i) $\Rightarrow$ (iii) is soundness and trivial to verify; (iii) $\Rightarrow$ (ii) follows from the subdirect product representation and (ii) $\Rightarrow$ (i) is proved by showing that the algebra of classes of mutually provably equivalent formulas is an $\mathrm{SBL}_\sim$-algebra whose largest element is the class of all $\mathrm{SBL}_\sim$-provable formulas. □

5. Standard completeness

In this section we turn our attention to the corresponding extensions of product and Gödel logics with an involutive negation. Of course both product and Gödel logics are extensions of SBL, and therefore their corresponding extensions will be extensions of SBL$_\sim$.

Definition 6. Let G$_\sim$ be Gödel logic G extended by a new negation $\sim$, by axioms $(\sim 1), (\sim 2), (\sim 3), (\Delta 1), (\Delta 2), (\Delta 5)$ and by the necessitation for $\Delta = \neg \sim$.

Similarly for $\Pi_\sim$ (product logic with an involutive negation). G$_\sim$-algebras and $\Pi_\sim$-algebras are defined in the obvious way.

Remark 2. Since both G$_\sim$ and $\Pi_\sim$ extend SBL$_\sim$, the corresponding completeness theorems are proved by the obvious modification of the completeness proof for SBL$_\sim$. But note that both G and Π satisfy corresponding standard completeness theorems, i.e. G$\vdash \varphi$ iff φ is a tautology over the standard G-algebra $[0,1]_{\mathrm{G}}$ (i.e. the real interval $[0,1]$ with Gödel truth functions) and similarly for Π and the standard product algebra. Does this generalize for G$_\sim$ and $\Pi_\sim$? We first show that the answer for G$_\sim$ is positive.

Definition 7. The standard G$_\sim$-algebra is the unit interval $[0,1]$ with Gödel truth functions extended by the involutive negation $\sim x = 1 - x$.

Theorem 7 (Standard completeness for $\mathrm{G}_\sim$). *For each* $\mathrm{G}_\sim$*-formula* φ, $\mathrm{G}_\sim$ *proves* φ *iff* φ *is a tautology over the standard* $\mathrm{G}_\sim$*-algebra.*

Proof. Let φ be a formula and let **L** be a linearly ordered G$_\sim$-algebra such that for an **L**-evaluation e, $e(\varphi) < 1_{\mathbf{L}}$. Let X be a finite subset of L containing $0_{\mathbf{L}}$, $1_{\mathbf{L}}$, the values $e(\psi)$ for all subformulas ψ of φ and containing with each a also its involutive negation $\sim a$. Assume X has $(k+1)$ elements $0_{\mathbf{L}} = a_0 < a_1 < \ldots < a_{k-1} < a_k = 1_{\mathbf{L}}$. Let $f(a_i) = \frac{i}{k}$ for $i = 0, \ldots, k$. Observe that f is a partial isomorphism of X onto $\{\frac{i}{k} \mid 0 \le i \le k\}$; indeed, it preserves minimum as well as truth functions of implication and of both negations. Hence defining $e'(p) = f(e(p))$ for each propositional variable p occurring in φ, we get $e'(\varphi) = f(e(\varphi)) < 1$. Thus φ is not a tautology over the standard G$_\sim$-algebra. $\square$

For $\Pi_\sim$ we shall get only a weaker result.

Definition 8. A semistandard $\Pi_\sim$-algebra has the form $([0,1], \max, \min, *, \Rightarrow, n, 0, 1)$, where $*$ is the product of real numbers restricted to $[0,1]$ and $\Rightarrow$ is its residuum (Goguen implication); n is an arbitrary decreasing involution on $[0,1]$ (i.e. $x \le y$ implies $n(x) \ge n(y)$ and $n(n(x)) = x$).

Lemma 10. *Let* $0 < a_0 < a_1 < \ldots < a_k < 1$ *be reals. Then there is a decreasing involution* n *on* $[0,1]$ *such that* $n(a_i) = a_{k-i}$ *for* $i = 0, \ldots, k$ *(obvious).*

Theorem 8 (Semistandard completeness for $\Pi_\sim$). *For each $\Pi_\sim$-formula φ, $\Pi_\sim$ proves φ iff φ is an $\mathbf{L}$-tautology for each semistandard $\Pi_\sim$-algebra $\mathbf{L}$.*

The proof of this theorem is by the obvious modification of the proof of standard completeness of Π.

Remark 3. A natural question to pose is whether one could get standard completeness also for $\Pi_\sim$, that is, whether any 1-tautology over the standard $\Pi_\sim$-algebra $([0,1], \max, \min, *, \Rightarrow, n_s, 0, 1)$, where $n_s(x) = 1 - x$, is also a 1-tautology over any semistandard $\Pi_\sim$-algebra. The answer turns out to be negative. Indeed, one can show that the formula $(\sim\varphi \mathbin{\&} \varphi) \to (\sim(\sim\varphi \mathbin{\&} \varphi))^3$, where ψ^3 means $\psi\&\psi\&\psi$, is a 1-tautology over the standard $\Pi_\sim$-algebra. However, it is not a 1-tautology for some semistandard algebras with a strong negation n different from n_s. In particular, for the simplest piecewise linear strong negation n having its fixed point (equilibrium) $x_0 = 0.8$, the antecedent gets the value 0.64, its $\sim$-negation 0.84 and the succedent $0.84^3 = 0.5927$ when φ is evaluated to the value x_0.

6. Predicate calculi

We are going to show how far the completeness theorems for fuzzy predicate logics presented in [4], Chapter V, generalize for the present situation.

First observe that the notions of a language, its interpretations and formulas generalize trivially. We recall that given an SBL$_\sim$-algebra $\mathbf{L}$, an $\mathbf{L}$-interpretation of a language consisting of some predicates $P \in$ *Pred* and constants $c \in$ *Const* is a structure

$$\mathbf{M} = (M, (r_P)_{P \in \mathit{Pred}}, (m_c)_{c \in \mathit{Const}})$$

where $M \neq \emptyset$, $r_P : M^{ar(P)} \to L$, and $m_c \in M$ (for each $P \in$ *Pred*, $c \in$ *Const*).

The value $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ of a formula (where $v(x) \in M$ for each variable x) is defined inductively: for φ being $P(x, \ldots, c, \ldots)$,

$$\|P(x, \ldots, c, \ldots)\|^{\mathbf{L}}_{\mathbf{M},v} = r_P(v(x), \ldots, m_c, \ldots),$$

the value commutes with connectives (including $\sim$), and

$$\|(\forall x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables, except } x\}$$

if this infimum exists, otherwise undefined, and similarly for $\exists x$ and $\sup$. $\mathbf{M}$ is $\mathbf{L}$-*safe* if all infs and sups needed for definition of the value of any formula exist in $\mathbf{L}$.

The axioms of Hájek basic predicate logic BL$\forall$ are (see [4]) the axioms of BL plus the following set of five axioms for quantifiers:

($\forall 1$) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)
($\exists 1$) $\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
($\forall 2$) $(\forall x)(\psi \to \varphi) \to (\psi \to (\forall x)\varphi)$ (x not free in ψ)
($\exists 2$) $(\forall x)(\psi \to \varphi) \to ((\exists x)\psi \to \varphi)$ (x not free in φ)
($\forall 3$) $(\forall x)(\psi \vee \varphi) \to ((\exists x)\psi \vee \varphi)$ (x not free in φ)

Rules of inference are modus ponens and generalization (from φ infer $(\forall x)\varphi$).

Now, we define the predicate calculus SBL$\forall_\sim$ by taking as axioms those of SBL$_\sim$ plus the above five axioms for quantifiers, and with modus ponens, generalization and necessitation. Obviously SBL$\forall_\sim$ extends BL$\forall$. However, it is worth noticing that in SBL$\forall_\sim$ one quantifier is definable from the other one and the involutive negation, for instance $(\exists x)\varphi$ is $\sim(\forall x)(\sim\varphi)$. Thus the above set of axioms for quantifiers could certainly be simplified.

Theorem 9 (Completeness). *Let T be a theory over* SBL$\forall_\sim$, *φ a formula. T proves φ over* SBL$\forall_\sim$ *iff $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = 1_{\mathbf{L}}$ for each* SBL$_\sim$*-algebra $\mathbf{L}$, each $\mathbf{L}$-safe $\mathbf{L}$-model of T and each v.*

Proof. Inspect the corresponding proof in [4] Chapter V and see that the proof for SBLL$_\sim$ is similar (using the present deduction theorem). □

Now let us turn to standard completeness. We recall that neither Łukasiewicz predicate logic Ł$\forall$ nor product predicate logic $\Pi\forall$ have a recursive axiom system which would be complete in the usual sense with respect to models over the corresponding standard algebra. But Gödel predicate logic G$\forall$ does: the axiom system consists of the above five axioms for quantifiers $(\forall 1), (\forall 2), (\forall 3), (\exists 1), (\exists 2)$, together with the axioms of the propositional calculus G. One can show that this system is simply complete for G$\forall$, not only with respect to all $\mathbf{L}$-models, but also just with respect to the standard G-algebra.

We shall show that this extends to G$\forall_\sim$. When inspecting the proof for G$\forall$ then we see that the following two lemmas, analogous to Lemmas 5.3.1. and 5.3.2. in [4], are sufficient to get the result.

Lemma 11. *Let $\mathbf{L}$ be a countable linearly ordered* G$_\sim$*-algebra. Then there is a countable densely linearly ordered* G$_\sim$*-algebra $\mathbf{L}'$ such that $\mathbf{L} \subseteq \mathbf{L}'$ and the identical embedding of $\mathbf{L}$ into $\mathbf{L}'$ preserves all infinite suprema and infima existing in $\mathbf{L}$. In addition we may assume that $\mathbf{L}'$ has an element h such that $\sim h = h$.*

Proof. Handle h first. Clearly, $\mathbf{L}$ has at most one element h such that $\sim h = h$. Let $P = \{x \in L \mid x > \sim x\}$ and $N = \{x \in L \mid x < \sim x\}$. If $\mathbf{L}$ has no h, just add a new element h with $\sim h = h$ and define $x < h$ for $x \in N$, $h < x$ for $x \in P$. It is evident that this makes $\mathbf{L}$ to a new G$_\sim$-algebra $\mathbf{L}^+$ and the embedding of $\mathbf{L}$ into $\mathbf{L}^+$ preserves all sups and infs (since if a set $X \subseteq N$ has a sup in $\mathbf{L}$ the sup must lie in N; similarly for P and inf). Thus assume $\mathbf{L}$ to have an h with $h = \sim h$.

Now apply the technique of the proof of [4] 5.3.1. of putting a copy of rationals from $(0, 1)$ into each "hole" (x, y) (a pair of elements of $\mathbf{L}$ such that y is the successor of x). Let the copy be $C_x = \{(x, r) \mid 0 < r < 1,$ r rational$\}$. Observe that (x, y) is a hole iff $(\sim y, \sim x)$ is a hole; thus in the new algebra $\mathbf{L}'$ containing a copy of $\mathbf{L}$ extend the operation $\sim$ induced by $\mathbf{L}$ to the new elements as follows: if (x, y) is a hole then

$$\sim(x, r) = (\sim y, 1 - r)$$

for $r \in (0, 1)$. This makes $\mathbf{L}'$ a $\mathrm{G}_\sim$-algebra. The rest is as in [4]. □

Lemma 12. *Let* $\mathbf{L}$ *be a countable densely linearly ordered* $\mathrm{G}_\sim$*-algebra with a fixed point* $h = \sim h$ *of* $\sim$*. Then there is an isomorphism* f *of* $\mathbf{L}$ *onto the* $\mathrm{G}_\sim$*-algebra* $Q \cap [0, 1]_{\mathrm{G}_\sim}$ *of rationals with min, max, Gödel implication and the involutive negation* $n_s(x) = 1 - x$*.*

Proof. First, define $f(1_{\mathbf{L}}) = 1$. $f(0_{\mathbf{L}}) = 0$ and $f(h) = \frac{1}{2}$. Then define the values of f for $h \leq x \leq 1_{\mathbf{L}}$ making f and order isomorphism of $[h, 1_{\mathbf{L}}]$ to $[\frac{1}{2}, 1]$ in the usual way. For $0_{\mathbf{L}} \leq x \leq h$ define $f(x) = 1 - f(\sim x)$. This makes f to an order isomorphism of $\mathbf{L}$ with $Q \cap [0, 1]_{\mathrm{G}_\sim}$ preserving all sups and infs and commuting with $\sim$. Thus f is just an isomorphism of the two $\mathrm{G}_\sim$-algebras in question and preserves sups and infs. □

Theorem 10 (Standard Completeness). *Let* T *be a theory over* $\mathrm{G}\forall_\sim$ *(where the language is at most countable).* $T \vdash \varphi$ *iff* $\|\varphi\|_{\mathbf{M}} = 1$ *for each* $[0, 1]_{\mathrm{G}_\sim}$*-model* $\mathbf{M}$ *of* T*.*

The proof is as in [4], Theorem 5.3.3.

7. Adding truth constants

Rational Pavelka Logic (RPL) (see [3]) is an extension of Łukasiewicz logic Ł by adding a truth constant $\overline{r}$ for each rational $r \in [0, 1]$ together with the following two book-keeping axioms for truth constants:

(RPL1) $\overline{r} \& \overline{s} \equiv \overline{r * s}$

(RPL2) $\overline{r} \to \overline{s} \equiv \overline{r \Rightarrow s}$

where $*$ and $\Rightarrow$ are Łukasiewicz t-norm and implication respectively. An evaluation e of propositional variables by reals from [0, 1] extends to an evaluation of all formulas as in Łukasiewicz logic over the standard MV-algebra provided that $e(\overline{r}) = r$ for each rational r.

The following is a (Pavelka-style) form of the strong completeness of RPL. Let T be a theory and define the *truth degree* of a formula φ in T as

$||\varphi||_T = \inf\{e(\varphi) \mid e$ is a model of $T\}$, and the *provability degree* of φ over T as $|\varphi|_T = \sup\{r \mid T \vdash \overline{r} \to \varphi\}$. Then the completeness of RPL says that the provability degree of φ in T is just equal to the truth degree of φ over T, that is, $||\varphi||_T = |\varphi|_T$.

Remark 4. The provability degree is a supremum, which is not necessarily attained as a maximum; for an infinite T, $|\varphi|_T = 1$ does not always imply $T \vdash \varphi$. (For finite T it does, see [6] and [4] 3.3.14.)

As it has been noticed elsewhere (e.g. [4]), a complete analogy to RPL for product and Gödel logics is impossible, due to the discontinuity of Goguen and Gödel implication truth functions. However, we show that it is possible to introduce truth-constants in product logic provided we also introduce one infinitary deduction rule to overcome the discontinuity problem of Goguen implication at the point $(0,0)$ of the unit square. However, the problem is not as simple for Gödel logic since Gödel implication is discontinuous in all points of the diagonal of $[0,1)\times[0,1)$. It must be noticed that Hájek presents in [4] a reformulation of Takeuti-Titani predicate logic [9], denoted TT∀, which includes rational truth-constants and contains Łukasiewicz, Gödel and product predicate logics as its sublogics. Nevertheless we think it is of interest to present next how propositional product logic (with involutive negation) can be endowed with rational constants resulting a very simple sublogic of TT∀ (when taking $n(x) = 1 - x$)). Finally we also discuss the case of Gödel logic.

7.1. Rational product logic

The language of rational product logic (RΠL) will be the same as the language of RPL, and we take as axioms of RΠL the axioms of product logic plus

(RΠL1) $\overline{r}\&\overline{s} \equiv \overline{r\cdot s}$

(RΠL2) $\overline{r} \to \overline{s} \equiv \overline{r \Rightarrow s}$

where $\cdot$ is usual product of reals and $\Rightarrow$ is Goguen implication function. Deductions rules are modus ponens and the following infinitary rule:

$$\text{from } \varphi \to \overline{r}, \text{ for each } r > 0, \text{ derive } \varphi \to \overline{0}.$$

A *theory* T over RΠL is just a set of formulas. The set $Cn_{\mathrm{R}\Pi\mathrm{L}}(T)$ of all provable formulas in T is the smallest T' containing T as a subset, containing all axioms of RΠL and closed under all deduction rules. For simplicity we shall denote $\varphi \in Cn_{\mathrm{R}\Pi\mathrm{L}}(T)$ by $T \vdash \varphi$. By definition, a theory T is *consistent* if $T \nvdash \overline{0}$. Further, a theory T is *complete* if $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$ for each pair φ, ψ. The notions of provability and truth degree of a formula φ in a theory T, denoted by $|\varphi|_T$ and $||\varphi||_T$ respectively, are the same as for RPL. Our purpose is to show completeness (Pavelka-style) for RΠL. The main steps are the following.

Lemma 13. *$T \vdash \overline{0}$ iff $T \vdash \overline{r}$ for some $r < 1$.*

Proof. If $T \vdash \overline{r}$ for some $r < 1$, then $T \vdash \overline{r^n}$ for each natural n, and thus $T \vdash \overline{r'}$ for any $r' < 1$. Then the infinitary rule does the job. □

Next step is just to check that the following known three results for rational Pavelka logic easily extend to rational product logic (cf. [4] 2.4.2, 3.3.7 and 3.3.8 (1) respectively).

Lemma 14.

1. *Each consistent theory T can be extended to a consistent and complete theory T'.*
2. *If T does not prove $(\overline{r} \to \varphi)$ then $T \cup \{\varphi \to \overline{r}\}$ is consistent.*
3. *If T is complete, then $\sup\{r \mid T \vdash \overline{r} \to \varphi\} = \inf\{r \mid T \vdash \varphi \to \overline{r}\}$.*

Lemma 15. *If T is complete, the provability degree commutes with connectives.*

Proof. We have only to check that $|\varphi \to \psi|_T$ = 1 when $|\varphi|_T = 0$, since the truth function of conjunction (product) is continuous and Goguen implication $\Rightarrow$ is also continuous for $x \neq 0$. (The interested reader may check that the corresponding proof for rational Pavelka logic in [4] 3.3.8 (2) also applies in these cases.)

Therefore, assume $|\varphi|_T = 0 = \inf\{r \mid T \vdash \varphi \to \overline{r}\}$. This means that $T \vdash \varphi \to \overline{r}$ for every $r > 0$, and using the infinitary rule, T proves $\varphi \to \overline{0}$. But $\overline{0} \to \psi$ is provable in product logic, and thus T also proves $\varphi \to \psi$, and thus $|\varphi \to \psi|_T$ = 1. □

Finally, one can also easily check that $|\varphi|_T \geq ||\varphi||_T$. (cf. [4] 3.3.9.). The other inequality is just due to the soundness of RΠL. Then we may state the following Pavelka-style completeness for rational product logic.

Theorem 11. *In* RΠL *we have $||\varphi||_T = |\varphi|_T$, for any theory T and any formula φ.*

This completeness result easily extends to product logic with an involutive negation since strong negations in [0, 1] are continuous functions. For a given strong negation function n in $[0, 1]$, axioms of the corresponding RΠL$_\sim$ (rational product logic with strong negation) are those of $\Pi_\sim$ plus the book-keeping axioms:

(RΠL1) $\overline{r}\&\overline{s} \equiv \overline{r \cdot s}$,

(RΠL2) $\overline{r} \to \overline{s} \equiv \overline{r \Rightarrow s}$,

(RΠL$_\sim$) $\sim\overline{r} \equiv \overline{n(r)}$

Deduction rules are those of RΠL, i.e. modus ponens and the infinitary rule.

Theorem 12. *In* $\mathrm{R}\Pi\mathrm{L}_{\sim}$ *we have* $||\varphi||_T = |\varphi|_T$*, for any theory* T *and any formula* φ.

But now, due to the strong negation and the infinitary deduction rule in $\mathrm{R}\Pi\mathrm{L}_{\sim}$, Pavelka-style completeness can be improved to completeness in the classical sense.

Corollary 1. $\mathrm{R}\Pi\mathrm{L}_{\sim}$ *is strongly complete, i.e., for any theory* T *and any formula* φ, $T \vdash \varphi$ *iff* φ *is true in all models of* T.

Proof. It suffices to show that if $|\varphi|_T = 1$ then $T \vdash \varphi$. So, suppose $T \vdash \overline{r} \to \varphi$ for all rationals $r < 1$. Applying the $\mathrm{SBL}_{\sim}$ inference rule of Lemma 2, T proves $\sim\varphi \to \overline{n(r)}$ for all $r < 1$, that is, T proves $\sim\varphi \to \overline{r}$ for all $r > 0$. Now, using the infinitary inference rule of $\mathrm{R}\Pi\mathrm{L}_{\sim}$, T proves $\sim\varphi \to \overline{0}$, and applying again the above mentioned inference rule, we get $T \vdash \sim\sim\varphi$, i.e. $T \vdash \varphi$. □

Remark 5. An inspection of the proof of (Pavelka-style) completeness of the rational Pavelka predicate calculus RPL$\forall$ (see [4] 5.4.10) shows that the above completeness extends to the case of the predicate calculi $\mathrm{R}\Pi\mathrm{L}\forall$ and $\mathrm{R}\Pi\mathrm{L}\forall_{\sim}$, defined as the obvious extensions of $\Pi\forall$ and $\Pi\forall_{\sim}$ respectively by truth constants and the corresponding book-keeping axioms. Clearly, now $\|\varphi\|_T = \inf\{\|\varphi\|_{\mathbf{M}} \mid \mathbf{M} \text{ model of } T\}$.

Theorem 13. *Both* $\mathrm{R}\Pi\mathrm{L}\forall$ *and* $\mathrm{R}\Pi\mathrm{L}\forall_{\sim}$ *satisfy* $|\varphi|_T = \|\varphi\|_T$ *for each theory* T *and formula* φ.

7.2. Rational Gödel logic with involutive negation

The language of rational Gödel logic with involutive negation ($\mathrm{RGL}_{\sim}$) will be the same as the language of $\mathrm{R}\Pi\mathrm{L}_{\sim}$. Then $\mathrm{RGL}_{\sim}$ is the extension of $\mathrm{G}_{\sim}$ with the following book-keeping axioms:

(RGL1) $\overline{r}\&\overline{s} \equiv \overline{\min(r,s)}$,

(RGL2) $\overline{r} \to \overline{s} \equiv \overline{r \Rightarrow s}$,

($\mathrm{RGL}_{\sim}$) $\sim\overline{r} \equiv \overline{1-r}$,

where $\Rightarrow$ is Gödel implication function.

As already mentioned, to get completeness, besides modus ponens and the necessitation for Δ, in this case we would need, for each real $\alpha \in [0,1)$ the following infinitary rule:

from $\varphi \to \overline{r}$ and $\overline{s} \to \psi$, for all rationals r, s such that $r > \alpha > s$, derive $\varphi \to \psi$.

Unfortunately this set of inference rules is not denumerable. The problem remains whether it is possible to overcome the discontinuity problems of $\Rightarrow$ with a denumerable set of axioms and rules. Nevertheless we can show the following completeness result (cf. Remark 4).

Theorem 14 (Completeness). RGL$_\sim$ *proves* φ *iff* $e(\varphi) = 1$ *for each evaluation* e.

Proof. (Sketch) If RGL$_\sim$ $\not\vdash \varphi$ then there is a countable linearly ordered G$_\sim$-algebra $\mathbf{M}$ with elements interpreting rational constants $r_1, \ldots, r_n$ occurring in φ and an $\mathbf{M}$-evaluation e of variables such that $e_\mathbf{M}(\varphi) < 1_\mathbf{M}$. Using the technique of Lemma 10, we may assume $\mathbf{M}$ to to be densely linearly ordered. Further assume $\frac{1}{2}$ be one of the r_i's. Then we may find an isomorphism of $\mathbf{M}$ onto rationals from $[0, 1]$, respecting $\sim$ and sending the $\mathbf{M}$-interpretation of $\overline{r_i}$ to r_i ($i = 1, \ldots, n$). □

As a direct corollary, taking into account that in Gödel logic the deduction theorem holds, we get the following completeness result for finite theories.

Corollary 2. *Let* T *be a finite theory over* RGL$_\sim$. *Then* T *proves* φ *iff* $e(\varphi) = 1$ *for each evaluation* e *which is a model of* T.

Also here the generalization for predicate calculus is easy — check [4] 5.3.3.

Theorem 15. *Let* T *be a finite theory over* RGL$\forall_\sim$, *let* φ *be a formula.* T *proves* φ *iff* φ *is true in each model of* T.

Remark 6. Without truth constants we have a strong completeness for arbitrary theories (over G$\forall$, G$\forall_\sim$); here only for finitely axiomatized theories. On the other hand, we have "classical" completeness (provable = true in all models), not just Pavelka-style completeness.

8. Conclusions

The logic SBL$_\sim$, together with its extensions $\Pi_\sim$ and G$_\sim$ and their corresponding Pavelka-like extensions RΠL$_\sim$ and RGL$_\sim$, proposed in this paper fill an existing gap between, on one side the basic logic SBL, the extension of the basic logic BL resulting from fixing the negation to Gödel negation, and on the other side the strong Takeuti-Titani fuzzy logic TT$\forall$, with three residuated pairs of connectives, and in particular with both Gödel and an involutive negations. However, it should be also noticed that Ł$_\Delta$, Łukasiewicz logic extended with the projection connective Δ, proposed by Hájek in [4], is also a fuzzy logic exhibiting both kinds of negation.

The following remains to be an open problem: is RΠL complete in the classical sense, i.e. does it prove all $[0, 1]_\Pi$-tautologies? For the corresponding predicate calculus the answer is negative, as it is for Łukasiewicz logic (see [4] for details).

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Appendix: Some basic notions about t-norms

This appendix only contains some necessary definitions and properties of t-norms which are used in the paper. For more extensive surveys about t-norms the interested reader is referred to the monograph [8].

Definition 9. A t-norm $*$ is a binary operation on the real unit interval $[0, 1]$ which is associative, commutative, non-decreasing, and fulfils the following boundary conditions: $1 * x = x$ and $0 * x = 0$, for all $x \in [0, 1]$.

Most well-known examples of t-norms are:
(i) *Łukasiewicz t-norm*: $x * y = \max(0, x + y - 1)$
(ii) *Product t-norm*: $x * y = x \cdot y$
(iii) *Gödel t-norm*: $x * y = \min(x, y)$
These examples are important since, as the theorem below shows, any continuous t-norm is either isomorphic to one of these, or it is a combination (ordinal sum) of them. First we introduce some more definitions:

1. An element $x \in [0, 1]$ is *idempotent* for a t-norm $*$ if $x * x = x$. $E(*)$ will denote the set of idempotent elements of $*$.
2. An element $x \in [0, 1]$ is *nilpotent* for a t-norm $*$ if there exists some natural n such that $x * \overset{n}{\ldots} * x = 0$.
3. A continuous t-norm is *Archimedean* if it has no idempotents except 0 and 1.
4. An archimedean t-norm is *strict* if it has no nilpotent elements except 0. Otherwise it is called *nilpotent*.

For each continuous t-norm $*$, $E(*)$ is a closed subset of $[0, 1]$. Let us denote by $E^c(*)$ its complement (a countable union of disjoint open intervals) and define the following set:

$$\mathcal{I}(E(*)) = \{[a, b] \mid a, b \in E(*), a \neq b, (a, b) \subseteq E^c(*)\}.$$

Then the following representation theorem, due to Ling, for continuous t-norms holds (see [4] for a proof).

Theorem 16. *If $*_I$ denotes the restriction of a continuous t-norm $*$ to $I \times I$, $I \in \mathcal{I}(E(*))$, then:*

1. *For each* $I \in \mathcal{I}(E(*))$, $*_I$ *is isomorphic either to the product t-norm (on* $[0,1]$*) or to Łukasiewicz t-norm (on* $[0,1]$*).*
2. *If* $x, y \in [0,1]$ *are such that there is no* $I \in \mathcal{I}(E(*))$ *with* $x, y \in I$*, then* $x * y = \min(x, y)$.

Basically, this theorem says that, for any continuous t-norm $*$, we can identify along the diagonal of $[0,1]^2$ a set of smaller adjacent squares (sharing one single point of the diagonal, which will be an idempotent) where inside these squares we have either Gödel, product or Łukasiewicz t-norm (up to isomorphisms), and outside these squares we have $x * y = \min(x, y)$. Such combinations are known as *ordinal sums*.

Finally the next proposition characterizes continuous t-norms which have non-trivial zero divisors.

Proposition 1. *A continuous t-norm* $*$ *has non-trivial zero divisors iff it is an ordinal sum such that there exists* $I_0 = [0, a] \in \mathcal{I}(E(*))$ *and* $*_{I_0}$ *is isomorphic to Łukasiewicz t-norm.*

Proof. Let $x > 0$ be an idempotent of $*$. According to Theorem 16, $x * y = \min(x, y)$ for all $y \in (0, 1]$. For each $z \in [x, 1]$ we obtain $z * y \geq x * y \geq \min(x, y) > 0$, hence z is not a zero divisor. Thus all non-trivial zero divisors must belong to an interval $I_0 = [0, a] \in \mathcal{I}(E(*))$ and the problem reduces to the discussion of an Archimedean t-norm isomorphic to $*_{I_0}$. □

As a consequence of Proposition 1, each continuous t-norm $*$ without non-trivial zero divisors must be of one of the following forms:

1. either $\inf(E(*) - \{0\}) = 0$ (0 is a cluster point of $E(*)$), or
2. there is an interval $I_0 = [0, a] \in \mathcal{I}(E(*))$ and $*_{I_0}$ is isomorphic to the product t-norm.

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