Probabilistic computability and choice

Vasco Brattka [a,b,∗,1], Guido Gherardi [a], Rupert Hölzl [c,2]

[a] Faculty of Computer Science, Universität der Bundeswehr München, Germany
[b] Department of Mathematics & Applied Mathematics, University of Cape Town, South Africa
[c] Department of Mathematics, Faculty of Science, National University of Singapore, Singapore

A R T I C L E   I N F O

A B S T R A C T

We study the computational power of randomized computations on infinite objects, such as real numbers. In particular, we introduce the concept of a Las Vegas computable multi-valued function, which is a function that can be computed on a probabilistic Turing machine that receives a random binary sequence as auxiliary input. The machine can take advantage of this random sequence, but it always has to produce a correct result or to stop the computation after finite time if the random advice is not successful. With positive probability the random advice has to be successful. We characterize the class of Las Vegas computable functions in the Weihrauch lattice with the help of probabilistic choice principles and Weak Weak Kőnig's Lemma. Among other things we prove an Independent Choice Theorem that implies that Las Vegas computable functions are closed under composition. In a case study we show that Nash equilibria are Las Vegas computable, while zeros of continuous functions with sign changes cannot be computed on Las Vegas machines. However, we show that the latter problem admits randomized algorithms with weaker failure recognition mechanisms. The last mentioned results can be interpreted such that the Intermediate Value Theorem is reducible to the jump of Weak Weak Kőnig's Lemma, but not to Weak Weak Kőnig's Lemma itself. These examples also demonstrate that Las Vegas computable functions form a proper superclass of the class of computable functions and a proper subclass of the class of non-deterministically computable functions. We also study the impact of specific lower bounds on the success probabilities, which leads to a strict hierarchy of classes. In particular, the classical technique of probability amplification fails for computations on infinite objects. We also investigate the dependency on the underlying probability space. Besides Cantor space, we study the natural numbers, the Euclidean space and Baire space.

∗ Corresponding author.
  E-mail addresses: Vasco.Brattka@cca-net.de (V. Brattka), Guido.Gherardi@unibw.de (G. Gherardi), r@hoelzl.fr (R. Hölzl).

1. Introduction

What is the computational power of a sequence of coin flips for computations on real numbers? While the power of randomized algorithms has been studied in the discrete setting for a long time (for a survey see for instance the text book by Motwani and Raghavan [33]), very little is known for computations on real numbers.[3]

In the discrete setting of decision problems randomization has no impact on what can be computed in principle, which follows from the Theorem of Sacks and its predecessors (see the discussion of Theorem 14.10), but it might have an impact on the computational complexity (whether it does or not is still not known for polynomial time complexity).

As we will see, randomization actually increases the computational power in general for computations of multi-valued functions in the infinite setting; and the question is to which extent it does so.

The purpose of this study is to analyze this question from the following perspective. Given a problem (a partial multi-valued function) $f :\subseteq X \rightrightarrows Y$:

- Imagine that a Turing machine upon input of $x \in X$ receives a second auxiliary input $r \in R$ and is supposed to produce a result $y \in f(x)$ with the help of this additional "random advice" r.
- We will require that such a computation is successful for every fixed x with a certain probability, i.e., the set S_x of successful advices r for input x has to have a certain measure.
- Additional conditions can be imposed on how the machine has to recognize the possible failure of an advice during the course of the computation.

Hence, this scenario can be seen as a way to formalize randomized algorithms over infinite objects x, y, where the computation is performed using some additional "random" input r. Now we can distinguish several ways in which this scenario can be refined:

1. **Probability space.** The probability space R can be chosen in different ways:
   (a) $R = 2^{\mathbb{N}}$ reflects the situation where the computation depends on a sequence of "coin tosses" (i.e., a sequence of zeros and ones),
   (b) $R = \mathbb{N}$ reflects the situation where the computation depends on a randomly chosen natural number,
   (c) $R = \mathbb{N} \times 2^{\mathbb{N}}$ reflects the situation where the computation depends on a randomly chosen natural number and a randomly chosen sequence of zeros and ones (as we will see, one can more or less equivalently work with $R = \mathbb{R}$ and choose a real number $x \in \mathbb{R}$),
   (d) $R = \mathbb{N}^{\mathbb{N}}$ reflects the situation where the computation depends on a randomly chosen sequence of natural numbers.
   In each case R is equipped with some natural canonical (probability) measure μ. We also allow measures which are not probability measures.
2. **Success probability.** Different success probabilities can be imposed:
   (a) $\mu(S_x) > 0$ for all admissible inputs x reflects positive success probability, which is the weakest meaningful requirement in this regard,
   (b) $\mu(S_x) \in I$ reflects more generally a success probability in some fixed interval $I \subseteq \mathbb{R}$ for all admissible inputs x.
3. **Failure recognition.** Finally, we can impose different conditions on how the machine has to recognize the failure of an advice r:
   (a) Las Vegas algorithms[4] require that the machine always produces a correct result and otherwise recognizes at some finite stage that the advice is unsuccessful and stops the computation in this case.
   (b) Higher order probabilistic algorithms are defined by weaker failure recognition mechanisms of the machine.

The most important and most natural scenario for us is the one with a sequence of coin tosses $R = 2^{\mathbb{N}}$, with positive success probability in $I = (0, 1]$ and a Las Vegas failure recognition mechanism. In this setting we will simply speak of *Las Vegas computability*.

We briefly summarize some major results that we are going to provide. After the introduction of some preliminaries in Section 2 we formally introduce the concept of Las Vegas computability in Section 3 and we characterize it with the help of a probabilistic choice operation $\mathsf{P}_I\mathsf{C}_X$. Intuitively, the problem $\mathsf{P}_I\mathsf{C}_X$ is the problem of finding a point in a given closed set $A \subseteq X$ of measure $\mu(A) \in I$ (we assume that X is a topological space equipped with a suitable measure μ and $I \subseteq \mathbb{R}$ is an interval). In case of $I = (0, \infty]$ we briefly write PC_X instead of $\mathsf{P}_I\mathsf{C}_X$ and in this form this problem was already introduced and studied under the name *positive choice* by the first author and Arno Pauly in [14]. Using the concept of Weihrauch reducibility $\leq_{\mathrm{W}}$ we show that

$$f \text{ is Las Vegas computable} \iff f \leq_{\mathrm{W}} \mathsf{PC}_{2^{\mathbb{N}}}.$$

We assume that Cantor space $2^{\mathbb{N}}$ is equipped with the usual uniform measure. Intuitively, the above characterization means that a function f is Las Vegas computable if and only if it can be computed using the resource of probabilistic choice $PC_{2^{\mathbb{N}}}$ exactly once during the course of the computation. We also prove that Weihrauch reducibility $\leq_W$ cannot be replaced by strong Weihrauch reducibility $\leq_{sW}$ here (the strong reducibility is different in that the only information that is available after usage of the oracle is the answer of the oracle; in particular the original input is not available afterwards).

In Section 4 we prove a general Independent Choice Theorem 4.3 with the help of the Theorem of Fubini (that generalizes a corresponding result of the first author, de Brecht and Pauly [9] on non-deterministic computations). This theorem implies, in particular, that Las Vegas computable functions are closed under composition, i.e.,

f and g Las Vegas computable $\implies g \circ f$ Las Vegas computable.

Closure under composition is in a certain sense a minimal requirement for a reasonable class of functions from a "practical programming" perspective.

The Independent Choice Theorem 4.3 also shows that a similar result cannot just be obtained for randomized computations with positive probabilities, but more generally whenever the intervals I of probabilities are closed under multiplication. In Section 5 we show that over the probability space $R = 2^{\mathbb{N}}$ we can only get three classes of functions in this way: the computable functions, the Las Vegas computable functions and the non-deterministically computable functions.[5]

In Section 6 we study the setting of the probability space $R = \mathbb{N} \times 2^{\mathbb{N}}$ and we call the functions below $PC_{\mathbb{N} \times 2^{\mathbb{N}}}$ *Las Vegas computable with finitely many mind changes*. We prove that these functions are exactly those functions that one can obtain if one composes a Las Vegas computable function f with a function g that is computable with finitely many mind changes in either order. In particular, the class of functions that are Las Vegas computable with finitely many mind changes is closed under composition as well. This result is interesting, since functions that are computable with finitely many mind changes are of independent interest and have been used for instance in learning theory (see for instance [17]).

In Section 7 we use the Smith-Volterra Cantor set construction in order to show that the probability space $R = 2^{\mathbb{N}}$ can almost (up to some arbitrarily small measure) be replaced by the unit interval $R = [0, 1]$, equipped with the Lebesgue measure. In fact, the corresponding classes are even exactly equivalent if arbitrary positive probabilities are allowed and, in particular, we obtain

f is Las Vegas computable $\iff f \leq_W PC_{[0,1]}$.

We also prove that $R = \mathbb{N} \times 2^{\mathbb{N}}$ and $R = \mathbb{R}$ lead exactly to the same classes of probabilistically computable functions, no matter what kind of intervals are imposed on the probabilities. In particular,

f is Las Vegas computable with finitely many mind changes $\iff f \leq_W PC_{\mathbb{R}}$.

In Section 8 we collect some definitions and observations regarding Weak Weak Kőnig's Lemma WWKL, in particular we have $PC_{2^{\mathbb{N}}} \equiv_W WWKL$ and thus

f is Las Vegas computable $\iff f \leq_W WWKL$.

In Section 9 we briefly discuss jumps WWKL$'$ and discrete jumps WWKL$^{\Delta}$ of WWKL and we show that

f is Las Vegas computable with finitely many mind changes $\iff f \leq_W WWKL^{\Delta}$.

The functions $f \leq_W WWKL'$ form an even larger class and they can be seen as probabilistically computable with weaker failure recognition mechanisms. For all these functions failure of the random advice can, in particular, be recognized "in the limit".

In Section 10 we study the problem ε-WWKL that was introduced by Dorais et al. [19] and that is Weihrauch equivalent to $P_{(\varepsilon,1]}C_{2^{\mathbb{N}}}$, the probabilistic choice problem for closed sets $A \subseteq 2^{\mathbb{N}}$ with $\mu(A) > \varepsilon$. We prove that the lower bounds that are imposed on the probability lead to a strict hierarchy of problems, i.e.,

ε-WWKL $\leq_W \delta$-WWKL $\iff \varepsilon \geq \delta$.

A similar result has independently been proved by Dorais et al. [19, Proposition 4.7].

The aforementioned result can be interpreted such that probability amplification fails for Las Vegas computable functions. Intuitively, this is because we are dealing with infinite computations and even if we perform two randomized computations in parallel we need to start producing some definite output possibly before we might know that one of the computations fails. Using a version of the Lebesgue Density Lemma LDL we prove in Section 11 that probability amplification works for Las Vegas computable functions with finitely many mind changes, i.e.,

$PC_{\mathbb{N} \times 2^{\mathbb{N}}} \equiv_W PC_{\mathbb{R}} \equiv_W P_{(\varepsilon,\infty]}C_{\mathbb{R}}$ for every $\varepsilon > 0$.

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[5] A multi-valued function is *non-deterministically computable* if it can be computed using an infinite sequence of coin tosses with failure recognition, but without any further restrictions on the success probability; however, we still require that there has to be at least one successful guess.

This also shows that there is a trade-off between the different aspects of probabilistic computations that we consider: the underlying probability space, the imposed probability and the failure recognition mechanism. If we want to achieve a certain guaranteed probability, then this might be feasible for the price of changing the underlying probability space (for instance from $PC_{2^\mathbb{N}}$ to $PC_{\mathbb{N} \times 2^\mathbb{N}}$) or for the price of allowing a weaker failure recognition mechanism (for instance from $PC_{2^\mathbb{N}}$ to $PC'_{2^\mathbb{N}}$).

As a side result we obtain that the Lebesgue Density Lemma LDL itself is equivalent to choice on natural numbers, i.e., $LDL \equiv_W C_\mathbb{N}$. In Section 12 we briefly discuss an algebraic operation $f + g$ that mimics parallel computations as they occur in the usual probability amplification method.

In Section 13 we prove that single-valued functions $f : X \to Y$ on computable metric spaces X, Y, which are below $\frac{1}{2}$-WWKL$^{(n)}$ (the n-fold jump of $\frac{1}{2}$-WWKL) are always computable, i.e., for all $n \in \mathbb{N}$ we have

$$f \leq_W \frac{1}{2}\text{-WWKL}^{(n)} \implies f \text{ computable.}$$

The underlying idea of a "majority-vote" is the same that has been used for the classical proof of the Theorem of Sacks 14.10, which we discuss in Section 14.

More generally, we discuss *probabilistic functions* f in Section 14, which have been called functions that are computable with *random advice* in an earlier study of the first author and Arno Pauly [14]. Intuitively, a probabilistic f is an f that can be computed with random advice irrespectively of any kind of uniformity or failure recognition method in this regard. In [14] it was already proved that WKL is not probabilistic. We show that all functions below $PC_R^{(n)}$ for R among $\mathbb{N}, 2^\mathbb{N}, \mathbb{N} \times 2^\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ are probabilistic. Hence, a proof that some function is not probabilistic shows that it cannot be computed with any of the mentioned resources. In particular, we obtain

$$\text{WKL} \not\leq_W \text{WWKL}^{(n)}$$

for all $n \in \mathbb{N}$. We prove another result that generalizes the Theorem of Sacks 14.10 in a certain sense, namely for suitable single-valued $f : X \to Y$ we have that

$$f \text{ probabilistic} \implies f \text{ maps computable inputs } x \text{ to computable outputs } f(x).$$

In the remaining sections of the paper we present case studies in which we investigate certain computational problems with regards to the question of whether they admit a Las Vegas algorithm. In Section 15 we start with the problem of finding a zero $x \in [0, 1]$ of a continuous function $f : [0, 1] \to \mathbb{R}$ that changes its sign, i.e., $f(0) \cdot f(1) < 0$. This problem is also known as the Intermediate Value Theorem IVT. Using the finite extension method we prove that there is no Las Vegas algorithm for this problem (not even with finitely many mind changes), however it is probabilistic and even admits a uniform probabilistic algorithm with a weaker failure recognition mechanism in the sense that IVT $\leq_W$ WWKL$'$. Altogether, we obtain

$$\text{IVT} \not\leq_W \text{WWKL and IVT} \leq_W \text{WWKL}'.$$

The second case study concerns the computation of Nash equilibria NASH. This problem was studied by Arno Pauly [38,40] who proved that it is equivalent to the idempotent closure RDIV* of robust division RDIV, i.e., NASH $\equiv_W$ RDIV*. Intuitively, robust divisions can be used to solve linear equations (and inequalities) in a compact domain and by using this operation repeatedly, one can determine Nash equilibria. In Section 16 we first prove that there is a Las Vegas algorithm for robust division and since WWKL is idempotent (which means WWKL$^* \leq_W$ WWKL) we can conclude in Section 17 that

$$\text{NASH} \leq_W \text{WWKL.}$$

This implies that there is a Las Vegas algorithm to compute Nash equilibria. As a side result we prove that robust division (and hence Nash equilibria NASH) cannot be computed with any fixed positive success probability and that Nash equilibria cannot be reduced to the Intermediate Value Theorem, i.e.,

$$\text{NASH} \not\leq_W \text{IVT.}$$

Hence, the two problems IVT and NASH are incomparable by the above results. The exact relation between most of the problems studied in this paper is shown in the diagram in Fig. 1 in the concluding Section 18.

Altogether, our case study proves that there are problems of practical interest that are Las Vegas computable, but not computable (such as Nash equilibria) and that there are other problems of practical interest which are probabilistically and non-deterministically computable, but not Las Vegas computable, such as the problem of finding zeros of continuous functions with sign changes. The latter problem also illustrates that there are problems of practical interest, which admit randomized algorithms with weaker failure recognition methods (here the failure is recognizable only in the limit), but not Las Vegas algorithms. This proves that the distinctions that we have made are meaningful and can be illustrated with problems of practical importance.

2. Preliminaries

In this section we give a brief introduction into the Weihrauch lattice and we provide some basic notions from probability theory.

Pairing functions We are going to use some standard pairing functions in the following that we briefly summarize. As usual, we denote by $\langle n, k \rangle := \frac{1}{2}(n + k + 1)(n + k) + k$ the Cantor pair of two natural numbers $n, k \in \mathbb{N}$ and by $\langle p, q \rangle(n) := p(k)$ if $n = 2k$ and $\langle p, q \rangle(n) = q(k)$, if $n = 2k + 1$, the pairing of two sequences $p, q \in \mathbb{N}^{\mathbb{N}}$. By $\langle k, p \rangle(n) := kp$ we denote the obvious pairing of a number $k \in \mathbb{N}$ with a sequence $p \in \mathbb{N}^{\mathbb{N}}$. We also define a pairing function $\langle p_0, p_1 \rangle := \langle \langle p_0(0), p_1(0) \rangle, \langle \overline{p_0}, \overline{p_1} \rangle \rangle$, for $p_0, p_1 \in \mathbb{N} \times 2^{\mathbb{N}}$, where $\overline{p_i}(n) = p_i(n + 1)$. Finally, we use the pairing function $\langle p_0, p_1, p_2, \ldots \rangle \langle i, j \rangle := p_i(j)$ for sequences $(p_i)_i$ in $\mathbb{N}^{\mathbb{N}}$.

The Weihrauch lattice The original definition of Weihrauch reducibility is due to Klaus Weihrauch and has been studied for many years (see [44–46,25,7,8]). More recently it has been noticed that a certain variant of this reducibility yields a lattice that is very suitable for the classification of the computational content of mathematical theorems (see [23,38,39,11,10,9,12]). The basic reference for all notions from computable analysis is Weihrauch's textbook [47]. The Weihrauch lattice is a lattice of multi-valued functions on represented spaces.

A *representation* δ of a set X is just a surjective partial map $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. In this situation we call (X, δ) a *represented space*. By $\mathbb{N} := \{0, 1, 2, \ldots\}$ we denote the set of *natural numbers*. In general we use the symbol "$\subseteq$" in order to indicate that a function is potentially partial. We work with partial multi-valued functions $f :\subseteq X \rightrightarrows Y$ where $f(x) \subseteq Y$ denotes the set of possible values upon input $x \in \text{dom}(f)$. If f is single-valued, then for the sake of simplicity we identify $f(x)$ with the single element y in it. We denote the *composition* of two (multi-valued) functions $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Y \rightrightarrows Z$ either by $g \circ f$ or by gf. It is defined by

$$g \circ f(x) := \{z \in Z : (\exists y \in Y)(z \in g(y) \text{ and } y \in f(x))\},$$

where $\text{dom}(g \circ f) := \{x \in X : f(x) \subseteq \text{dom}(g)\}$. Using represented spaces we can define the concept of a realizer.

Definition 2.1 (*Realizer*). Let $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ be a multi-valued function on represented spaces. A function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called a *realizer* of f, in symbols $F \vdash f$, if $\delta_Y F(p) \in f \delta_X(p)$ for all $p \in \text{dom}(f \delta_X)$.

Realizers allow us to transfer the notions of computability and continuity and other notions available for Baire space to any represented space; a function between represented spaces will be called *computable*, if it has a computable realizer, etc. Now we can define Weihrauch reducibility.

Definition 2.2 (*Weihrauch reducibility*). Let f, g be multi-valued functions on represented spaces. Then f is said to be *Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable functions $K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $H\langle \text{id}, GK \rangle \vdash f$ for all $G \vdash g$. Moreover, f is said to be *strongly Weihrauch reducible* to g, in symbols $f \leq_{\mathrm{sW}} g$, if an analogous condition holds, but with the property $HGK \vdash f$ in place of $H\langle \text{id}, GK \rangle \vdash f$.

We can always tacitly assume that K, H are defined on the minimal necessary domains that consist of those names that are actually involved. More precisely, let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ be multi-valued functions on represented spaces (X, δ_X), (Y, δ_Y), (W, δ_W) and (Z, δ_Z), and let $f \leq_{\mathrm{W}} g$ hold according to the above definition; then we say that H and K have *minimal domains* if

1. $\text{dom}(K) = \{p \in \mathbb{N}^{\mathbb{N}} : \delta_X(p) \in \text{dom}(f)\}$ and
2. $\text{dom}(H) = \{\langle p, q \rangle : p \in \text{dom}(K) \text{ and } \delta_Z(q) \in g(\delta_W(K(p)))\}$.

We use an analogous terminology in case of $f \leq_{\mathrm{sW}} g$.

The difference between ordinary and strong Weihrauch reducibility is that the "output modificator" H has direct access to the original input in case of ordinary Weihrauch reducibility, but not in case of strong Weihrauch reducibility. There are algebraic and other reasons to consider ordinary Weihrauch reducibility as the more natural variant. For instance, one can characterize the reduction $f \leq_{\mathrm{W}} g$ as follows: $f \leq_{\mathrm{W}} g$ holds if and only if a Turing machine can compute f in such a way that it evaluates the "oracle" g exactly on one (usually infinite) input during the course of its computation (see [41, Theorem 7.2]). We will use the strong variant $\leq_{\mathrm{sW}}$ of Weihrauch reducibility mostly for technical purposes, for instance it is better suited to study jumps (see below).

We note that the relations $\leq_{\mathrm{W}}$, $\leq_{\mathrm{sW}}$ and $\vdash$ implicitly refer to the underlying representations, which we will only mention explicitly if necessary. It is known that these relations only depend on the underlying equivalence classes of representations, but not on the specific representatives (see Lemma 2.11 in [11]). The relations $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ are reflexive and transitive, thus they induce corresponding partial orders on the sets of their equivalence classes (which we refer to as *Weihrauch degrees* or *strong Weihrauch degrees*, respectively). These partial orders will be denoted by $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ as well. The induced

lattice and semi-lattice, respectively, are distributive (for details see [39] and [11]). We use $\equiv_W$ and $\equiv_{sW}$ to denote the respective equivalences regarding $\leq_W$ and $\leq_{sW}$, by $<_W$ and $<_{sW}$ we denote strict reducibility and by $|_W, |_{sW}$ we denote incomparability in the respective sense.

The algebraic structure The partially ordered structures induced by the two variants of Weihrauch reducibility are equipped with a number of useful algebraic operations that we summarize in the next definition. We use $X \times Y$ to denote the ordinary set-theoretic *product*, $X \sqcup Y := (\{0\} \times X) \cup (\{1\} \times Y)$ in order to denote *disjoint sums* or *coproducts*, by $\bigsqcup_{i=0}^{\infty} X_i := \bigcup_{i=0}^{\infty}(\{i\} \times X_i)$ we denote the *infinite coproduct*. By X^i we denote the *i*-fold product of a set X with itself, where $X^0 = \{()\}$ is some canonical singleton (i.e., we identify $()$ with the empty word ε). By $X^* := \bigsqcup_{i=0}^{\infty} X^i$ we denote the set of all *finite sequences over X* and by $X^{\mathbb{N}}$ the set of all *infinite sequences over X*. All these constructions have parallel canonical constructions on representations and the corresponding representations are denoted by $[\delta_X, \delta_Y]$ for the product of (X, δ_X) and (Y, δ_Y), by δ_X^n for the *n*-fold product of (X, δ_X) with itself, where $n \in \mathbb{N}$ and δ_X^0 is a representation of the one-point set $\{()\} = \{\varepsilon\}$. By $\delta_X \sqcup \delta_Y$ we denote the representation of the coproduct, by δ_X^* the representation of X^* and by $\delta_X^{\mathbb{N}}$ the representation of $X^{\mathbb{N}}$. For instance, $(\delta_X \sqcup \delta_Y)$ can be defined by $(\delta_X \sqcup \delta_Y)\langle n, p \rangle := (0, \delta_X(p))$ if $n = 0$ and $(\delta_X \sqcup \delta_Y)\langle n, p \rangle := (1, \delta_Y(p))$, otherwise. Likewise, $\delta_X^*\langle n, p \rangle := (n, \delta_X^n(p))$. See [47] or [11,39,9] for details of the definitions of the other representations. We will always assume that these canonical representations are used, if not mentioned otherwise.

Definition 2.3 (*Algebraic operations*). Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions. Then we define the following operations:

1. $f \times g :\subseteq X \times Z \rightrightarrows Y \times W, (f \times g)(x, z) := f(x) \times g(z)$ (product)
2. $f \sqcap g :\subseteq X \times Z \rightrightarrows Y \sqcup W, (f \sqcap g)(x, z) := f(x) \sqcup g(z)$ (sum)
3. $f \sqcup g :\subseteq X \sqcup Z \rightrightarrows Y \sqcup W$, with $(f \sqcup g)(0, x) := \{0\} \times f(x)$ and $(f \sqcup g)(1, z) := \{1\} \times g(z)$ (coproduct)
4. $f^* :\subseteq X^* \rightrightarrows Y^*, f^*(i, x) := \{i\} \times f^i(x)$ (finite parallelization)
5. $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, \widehat{f}(x_n) := \underset{i \in \mathbb{N}}{\times} f(x_i)$ (parallelization)

In this definition and in general we denote by $f^i :\subseteq X^i \rightrightarrows Y^i$ the *i*-th fold product of the multi-valued map f with itself (f^0 is the constant function on the canonical singleton). It is known that $f \sqcap g$ is the *infimum* of f and g with respect to strong as well as ordinary Weihrauch reducibility (see [11], where this operation was denoted by $f \oplus g$). Correspondingly, $f \sqcup g$ is known to be the *supremum* of f and g with respect to ordinary Weihrauch reducibility $\leq_W$ (see [39]). This turns the partially ordered structure of Weihrauch degrees (induced by $\leq_W$) into a lattice, which we call the *Weihrauch lattice*. The two operations $f \mapsto \widehat{f}$ and $f \mapsto f^*$ are known to be closure operators in this lattice (see [11,39]).

There is some useful terminology related to these algebraic operations. We say that f is a *a cylinder* if $f \equiv_{sW} \text{id} \times f$ where $\text{id} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ always denotes the identity on Baire space, if not mentioned otherwise. Cylinders f have the property that $g \leq_W f$ is equivalent to $g \leq_{sW} f$ (see [11]). We say that f is *idempotent* if $f \equiv_W f \times f$ and *strongly idempotent*, if $f \equiv_{sW} f \times f$. We say that a multi-valued function on represented spaces is *pointed*, if it has a computable point in its domain. For pointed f and g we obtain $f \sqcup g \leq_{sW} f \times g$. The properties of pointedness and idempotency are both preserved under equivalence and hence they can be considered as properties of the respective degrees. For a pointed f the finite parallelization f^* can also be considered as *idempotent closure* since one can easily show that idempotency is equivalent to $f \equiv_W f^*$ in this case. We call f *parallelizable* if $f \equiv_W \widehat{f}$ and it is easy to see that $\widehat{f}$ is always idempotent. Analogously, we call f *strongly parallelizable* if $f \equiv_{sW} \widehat{f}$.

More generally, we define *countable coproducts* $\bigsqcup_{i \in \mathbb{N}} f_i :\subseteq \bigsqcup_{i \in \mathbb{N}} X_i \rightrightarrows \bigsqcup_{i \in \mathbb{N}} Y_i$ for a sequence (f_i) of multi-valued functions $f_i :\subseteq X_i \rightrightarrows Y_i$ on represented spaces and then it denotes the operation given by $(\bigsqcup_{i \in \mathbb{N}} f_i)(i, u) := \{i\} \times f_i(u)$. Using this notation we obtain $f^* = \bigsqcup_{i \in \mathbb{N}} f^i$. In [9] a multi-valued function on represented spaces has been called *join-irreducible* if $f \equiv_W \bigsqcup_{n \in \mathbb{N}} f_n$ implies that there is some n such that $f \equiv_W f_n$. Analogously, we can define *strong join-irreducibility* using strong Weihrauch reducibility in both instances. We can also define a *countable sum* $\bigsqcap_{i \in \mathbb{N}} f_i :\subseteq \times_{i \in \mathbb{N}} X_i \rightrightarrows \bigsqcup_{i \in \mathbb{N}} Y_i$, defined by $(\bigsqcap_{i \in \mathbb{N}} f_i)(x_i)_i := \bigsqcup_{i \in \mathbb{N}} f_i(x_i)$.

One should note however, that $\bigsqcap$ and $\bigsqcup$ do not provide infima and suprema of sequences. By a result of Higuchi and Pauly [27, Proposition 3.15] the Weihrauch lattice has no non-trivial suprema (i.e., a sequence (s_n) has a supremum s if and only if s is already the supremum of a finite prefix of the sequence (s_n)) and likewise by [27, Corollary 3.18] the pointed Weihrauch degrees do not have non-trivial infima. In particular, the Weihrauch lattice is not complete.[6]

Compositional products While the Weihrauch lattice is not completed, some suprema and some infima exist in general. The following result was proved by the first author and Pauly in [15] and ensures the existence of an important maximum.

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[6] We note, however, that for the continuous variant of Weihrauch reducibility the objects $\bigsqcup_{n \in \mathbb{N}} f_n$ and $\bigsqcap_{n \in \mathbb{N}} f_n$ are suprema and infima of the sequence $(f_n)_n$, respectively, and the corresponding continuous version of the Weihrauch lattice is actually countably complete (see [27]).

Proposition 2.4 *(Compositional products). Let f, g be multi-valued functions on represented spaces. Then the following Weihrauch degree exists:*

$$f * g := \max\{f_0 \circ g_0 : f_0 \leq_W f \text{ and } g_0 \leq_W g\} \qquad \qquad \text{(compositional product)}$$

Here $f * g$ is defined over all $f_0 \leq_W f$ and $g_0 \leq_W g$ which can actually be composed (i.e., the target space of g_0 and the source space of f_0 have to coincide). In this way $f * g$ characterizes the most complicated Weihrauch degree that can be obtained by first performing a computation with the help of g and then another one with the help of f. Since $f * g$ is a maximum in the Weihrauch lattice, we can consider $f * g$ as some fixed representative of the corresponding degree. It is easy to see that $f \times g \leq_W f * g$ holds. We can also define the *strong compositional product* by

$$f *_s g := \sup\{f_0 \circ g_0 : f_0 \leq_{sW} f \text{ and } g_0 \leq_{sW} g\}$$

(but we neither claim that it exists in general nor that it is a maximum). The compositional products were originally introduced in [12].

Jumps In [12] the first two authors and Marcone introduced *jumps* or *derivatives* f' of multi-valued functions f on represented spaces. We recall that the *jump* $f' :\subseteq (X, \delta'_X) \rightrightarrows (Y, \delta_Y)$ of a multi-valued function $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ on represented spaces is obtained by replacing the input representation δ_X by its jump $\delta'_X := \delta_X \circ \lim$. This leads to $f' \equiv_{sW} f *_s \lim$ (see [12, Corollary 5.16]). By $f^{(n)}$ we denote the n-fold jump. Here

$$\lim :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, \langle p_0, p_1, p_2, \ldots \rangle \mapsto \lim_{n \to \infty} p_n$$

is the limit operation on Baire space $\mathbb{N}^\mathbb{N}$ with respect to the product topology on $\mathbb{N}^\mathbb{N}$. Hence, a δ'_X-name p of a point $x \in X$ is a sequence that converges to a δ_X-name of x. This means that a δ'_X-name typically contains significantly less accessible information on x than a δ_X-name. Hence, f' is typically harder to compute than f, since less input information is available for f'.

The jump operation $f \mapsto f'$ plays a similar role in the Weihrauch lattice as the Turing jump operation does in the Turing semi-lattice. In a certain sense f' is a version of f on the "next higher" level of complexity (which can be made precise using the Borel hierarchy [12]). It was proved in [12] that the jump operation $f \mapsto f'$ is monotone with respect to strong Weihrauch reducibility $\leq_{sW}$, but not with respect to ordinary Weihrauch reducibility $\leq_W$. This is one reason why it is beneficial to extend the study of the Weihrauch lattice to strong Weihrauch reducibility.

Closed choice A particularly useful multi-valued function in the Weihrauch lattice is closed choice (see [23,11,10,9]) and it is known that many notions of computability can be calibrated using the right version of choice. We recall that a subset $U \subseteq X$ of a represented space X is open with respect to the final topology of its representation,[7] if and only if its characteristic function

$$\chi_U : X \to \mathbb{S}, x \mapsto \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

is continuous, where $\mathbb{S} = \{0, 1\}$ is Sierpiński space (equipped with the topology $\{\emptyset, \{1\}, \mathbb{S}\}$). Analogously, U is *c.e. open* if χ_U is computable, where $\mathbb{S}$ is equipped with its Standard representation $\delta_\mathbb{S}$ defined by $\delta_\mathbb{S}(p) := 0$ if $p(n) = 0$ for all n and $\delta_\mathbb{S}(p) := 1$, otherwise. Closed and *co-c.e. closed* sets $A \subseteq X$ are sets whose complement $U := X \setminus U$ is open and c.e. open, respectively (see [9,16] for more details). For subsets $A \subseteq \mathbb{N}$ of natural numbers this leads to the usual notion of c.e. and co-c.e. sets. The co-c.e. closed subsets $A \subseteq 2^\mathbb{N}$ of Cantor space are exactly the usual Π^0_1-classes.

In general, if (X, δ_X) is a represented space then we always assume that this space is endowed with the final topology of its representation. We denote by $\mathcal{A}_-(X)$ the set of closed subsets of X represented with respect to negative information. More precisely, we can define a representation ψ_- of $\mathcal{A}_-(X)$ by

$$\psi_-(p) = A :\Longleftrightarrow [\delta_X \to \delta_\mathbb{S}](p) = \chi_{X \setminus A},$$

where $[\delta_X \to \delta_\mathbb{S}]$ is the canonical function space representation in the category of represented spaces (see [47]). This means that a ψ_--name p of a closed set $A \subseteq X$ is a name for the characteristic function $\chi_{X \setminus A}$ of its complement.

We are mostly interested in closed choice for computable metric spaces X, which are separable metric spaces such that the distance function is computable on the given dense subset. We assume that computable metric spaces are represented via their Cauchy representations (see [47] for details). In this special case a computably equivalent definition of ψ_- can be obtained by

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[7] If δ_X is the representation of X, then $\{U \subseteq X : \delta_X^{-1}(U) \text{ open in } \mathrm{dom}(\delta_X)\}$ is called the *final topology of δ_X*.

$$\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)},$$

where B_n is some standard enumeration of the open balls of X with center in the dense subset and rational radius. Here a ψ_--name p of a closed set $A \subseteq X$ is a list of sufficiently many open rational balls whose union exhausts exactly the complement of A. We are now prepared to define closed choice.

Definition 2.5 *(Closed choice).* Let X be a represented space. The *closed choice* problem of the space X is defined by

$$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$$

with $\mathrm{dom}(\mathsf{C}_X) := \{A \in \mathcal{A}_-(X) : A \neq \emptyset\}$.

Intuitively, C_X takes as input a non-empty closed set in negative description (i.e., given by ψ_-) and it produces an arbitrary point of this set as output. Hence, $A \mapsto A$ means that the multi-valued map C_X maps the input $A \in \mathcal{A}_-(X)$ to the set $A \subseteq X$ as a set of possible outputs. We mention some classes of functions that can be characterized by closed choice. The following results have mostly been proved in [9]:

Proposition 2.6. *Let f be a multi-valued function on represented spaces. Then:*

1. $f \leq_{\mathrm{W}} \mathsf{C}_1 \iff f$ *is computable,*
2. $f \leq_{\mathrm{W}} \mathsf{C}_\mathbb{N} \iff f$ *is computable with finitely many mind changes,*
3. $f \leq_{\mathrm{W}} \mathsf{C}_{2^\mathbb{N}} \iff f$ *is non-deterministically computable,*
4. $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \iff f$ *is effectively Borel measurable.*

In the latter case (4) we have to assume that $f : X \to Y$ is single-valued and defined on computable complete metric spaces X, Y.

Here and in general we identify each natural number $n \in \mathbb{N}$ with the corresponding finite subset $n = \{0, 1, \ldots, n-1\}$. The problem C_0, i.e., closed choice for the empty set $0 = \emptyset$, is the bottom element of the Weihrauch lattice. Also C_2 plays a significant role, since it is equivalent to LLPO, the so-called *lesser limited principle of omniscience* as it is known from constructive mathematics. In [12] we have characterized the jumps $\mathsf{C}'_X \equiv_{\mathrm{W}} \mathsf{CL}_X$ for computable metric spaces X using the cluster point problem CL_X of X. We also use the *limited principle of omniscience* $\mathrm{LPO} : \mathbb{N}^\mathbb{N} \to \{0, 1\}$, which is simply the characteristic function of the constant zero sequence $\widehat{0} \in \mathbb{N}^\mathbb{N}$.

Some measure theory We now introduce some notation from measure theory. We consider measures as non-negative functions into $[0, \infty]$. On the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ we can use the *geometric probability measure* μ, induced by $\mu(\{n\}) = 2^{-n-1}$ for all $n \in \mathbb{N}$. This leads to a product measure $\mu_{\mathbb{N}^\mathbb{N}}$ on Baire space $\mathbb{N}^\mathbb{N}$ with $\mu_{\mathbb{N}^\mathbb{N}}(w\mathbb{N}^\mathbb{N}) = \prod_{i=0}^{|w|-1} 2^{-w(i)-1}$ for all words $w \in \mathbb{N}^*$. Similarly, we use the uniform measure μ_2 on $2 = \{0, 1\}$ with $\mu_2(\{i\}) = \frac{1}{2}$ for $i \in \{0, 1\}$ and the induced product measure $\mu_{2^\mathbb{N}}$ on Cantor space $2^\mathbb{N}$ with $\mu_{2^\mathbb{N}}(w2^\mathbb{N}) = 2^{-|w|}$ for all words $w \in 2^* = \{0, 1\}^*$. Often, we rather use the *counting measure* $\mu_\mathbb{N}$ on $\mathbb{N}$ that is induced by $\mu_\mathbb{N}(\{n\}) = 1$ for all $n \in \mathbb{N}$. If not mentioned otherwise, we assume that $\mathbb{N}$ is endowed with the counting measure. On $\mathbb{R}$ and $[0, 1]$ we use the Lebesgue measure λ. We recall that a measure μ on X is called *finite* if $\mu(X) < \infty$ and σ-*finite* if there exists a sequence $(X_n)_n$ of measurable sets $X_n \subseteq X$ with $X = \bigcup_{n=0}^{\infty} X_n$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. All the measures mentioned here are σ-finite, in fact all except the counting measure on $\mathbb{N}$ and the Lebesgue measure on $\mathbb{R}$ are even probability measures. We only use Borel measures, i.e., measures for which exactly all sets in the Borel σ-algebra generated by the underlying topology are measurable sets. This is because we want to assume that all closed sets are measurable. If not mentioned otherwise, we always assume in the following that $\mathbb{N}$ is endowed with the counting measure $\mu_\mathbb{N}$ and $2^\mathbb{N}, \mathbb{N}^\mathbb{N}$ are endowed with the product measures $\mu_{2^\mathbb{N}}, \mu_{\mathbb{N}^\mathbb{N}}$ as standard measures, respectively. Likewise, $\mathbb{R}$ and $[0, 1]$ are always endowed with the Lebesgue measure λ. If we just write μ for a measure on one of these spaces, then this refers to the corresponding standard measure.

Given two σ-finite measures μ_X on X and μ_Y on Y, we obtain a unique *product measure* $\mu_X \otimes \mu_Y$ on $X \times Y$ with respect to the corresponding product σ-algebra and this measure is σ-finite too (see [3, Theorem 23.3]). The product measure $\mu_X \otimes \mu_Y$ satisfies

$$(\mu_X \otimes \mu_Y)(A \times B) = \mu_X(A) \cdot \mu_Y(B)$$

for all corresponding measurable sets $A \subseteq X$ and $B \subseteq Y$. In the following we assume that we always use this product measure on product spaces. For instance, $\mathbb{N} \times 2^\mathbb{N}$ is endowed with the measure $\mu_\mathbb{N} \otimes \mu_{2^\mathbb{N}}$ and so forth.

A basic computability theoretic property of all measures used here is that they are upper semi-computable on closed sets.

Lemma 2.7 *(Semi-computability of measures).* *The measures $\mu_{2^\mathbb{N}} : \mathcal{A}_-(2^\mathbb{N}) \to \mathbb{R}$ and $\lambda : \mathcal{A}_-([0, 1]) \to \mathbb{R}$ are upper-semi computable.*

We will also occasionally use the fact that two measures μ_1, μ_2 on $2^{\mathbb{N}}$ coincide on closed sets if they coincide on all open balls $w2^{\mathbb{N}}$.

Lemma 2.8 *(Identity of measures). Let μ_1, μ_2 be two Borel measures on a subspace $R \subseteq \mathbb{N}^{\mathbb{N}}$ of Baire space, at least one of which is finite. If*

$$\mu_1(w\mathbb{N}^{\mathbb{N}} \cap R) = \mu_2(w\mathbb{N}^{\mathbb{N}} \cap R)$$

for all $w \in \mathbb{N}^*$, then $\mu_1 = \mu_2$.

Proof. Firstly, if one of the involved measures is finite, then the other is finite too, since $\mu_1(R) = \mu_2(R)$. It follows by σ-additivity that $\mu_1(U) = \mu_2(U)$ for every open $U \subseteq R$, since every such open set U can be written as a disjoint union of balls $w\mathbb{N}^{\mathbb{N}} \cap R$. Finally, all finite Borel measures on Polish spaces are outer regular (see [3, Lemma 26.2]) and hence μ_1, μ_2 even coincide completely under the given conditions. □

By an *interval I* we mean any interval of real numbers with open or closed end points and including ∞ as a possible right end point. An interval $[a, \infty]$ can be used to accommodate a measure that is infinite. We note that $[\infty, \infty]$ is not considered as an interval here, but singletons $[a, a] = \{a\}$ for $a \in \mathbb{R}$ are allowed. If we want to exclude the case of the closed right end point ∞, then we speak about an interval $I \subseteq \mathbb{R}$. If we want to exclude the case of the open right end point ∞, then we speak about intervals I without the open endpoint ∞. We will need the following statement on the monotonicity of Lebesgue integrals. We adopt the usual convention in measure theory (see [3]) that

$$0 \cdot \infty = 0 \text{ and } x \cdot \infty = \infty \text{ for } x > 0.$$

We also assume that these products commute and we can set similar conventions for negative numbers (that we are not going to use).

Lemma 2.9 *(Monotonicity). Let X be a topological space with a σ-finite Borel measure μ and let I be a non-empty interval that does not have ∞ as an open endpoint. Let $A \subseteq X$ be measurable and let $f : X \to \mathbb{R}$ be a non-negative measurable function such that $f(x) \in I$ for all $x \in A$. Then*

$$\int_A f \, \mathrm{d}\mu \in \mu(A) \cdot I.$$

Proof. If $\mu(A) = 0$, then $\mu(A) \cdot I = \{0\}$ since I is non-empty and $\int_A f \, \mathrm{d}\mu = 0$. Let now $\mu(A) > 0$ and $a, b \in \mathbb{R}$. Then we obtain

1. $a \leq f(x)$ for all $x \in A \Longrightarrow \mu(A)a \leq \int_A f \, \mathrm{d}\mu$,
2. $f(x) \leq b$ for all $x \in A \Longrightarrow \int_A f \, \mathrm{d}\mu \leq \mu(A)b$,
3. $a < f(x)$ for all $x \in A \Longrightarrow \mu(A)a < \int_A f \, \mathrm{d}\mu$,
4. $f(x) < b$ for all $x \in A \Longrightarrow \int_A f \, \mathrm{d}\mu < \mu(A)b$.

Here the first two properties (1) and (2) are just consequences of the monotonicity of the Lebesgue integral (see [3, Theorem 12.4]), whereas the other two strong monotonicity properties (3) and (4) follow from the first two properties together with the following observation: for any constant $c \in \mathbb{R}$ we have that $\int_A f \, \mathrm{d}\mu - \mu(A)c = \int_A f - c \, \mathrm{d}\mu = 0$ implies $f = c$ almost everywhere by [3, Theorem 13.2] and hence $f(x) = c$ for some $x \in A$ since $\mu(A) > 0$. Altogether, suitable combinations of the above statements prove the claim for all bounded intervals I. It clearly also holds if I is of the form $I = (a, \infty]$ or $I = [a, \infty]$. □

We mention that the above result cannot be extended to the cases (a, ∞) or $[a, \infty)$, since for instance $\int_{(0,1]} \frac{1}{x} \, \mathrm{d}x = \infty$, even though $\frac{1}{x} < \infty$ for all $x \in (0, 1]$.

3. Las Vegas computability and probabilistic choice

In this section we would like to formalize the notion of a Las Vegas computable multi-valued function $f :\subseteq X \rightrightarrows Y$ as it was intuitively described in the introduction and we will show that this notion can be characterized by a suitably defined probabilistic choice operation. Since we do not want to formalize probabilistic Turing machines on infinite sequences in a technical way here, we will just use the notion of an ordinary computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ in order to introduce our concept of randomized computations. Essentially, we will use two such functions F_1, F_2, which play the following roles:

1. F_2 is a *failure recognition function* that takes a name p of the input $x \in X$ and a "random advice" $r \in R$ and indicates whether r is successful on input p. Here $\delta_{\mathbb{S}} F_2 \langle p, r \rangle = 0$ indicates success, i.e., the set

$$S_p := \{ r \in R : \delta_{\mathbb{S}} F_2 \langle p, r \rangle = 0 \}$$

   (which is closed in R) is the *set of successful advices* on input p.
2. F_1 is the *computation function*, i.e., it also takes a name p of the input and the "random advice" r and it actually computes the correct result for $f :\subseteq (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ in the sense that

$$\delta_Y F_1 \langle p, r \rangle \in f \delta_X(p)$$

   for all successful advices $r \in S_p$.

By the nature of Sierpiński space $(\mathbb{S}, \delta_{\mathbb{S}})$ the failure event $\delta_{\mathbb{S}} F_2 \langle p, r \rangle = 1$ is the one that can be recognized in finite time, while success $\delta_{\mathbb{S}} F_2 \langle p, r \rangle = 0$ just means absence of failure in the long run. If no failure occurs, then $F_1 \langle p, r \rangle$ will be a correct result on input p with advice r in the long run. Having these interpretations in mind we are now prepared to give the formal definition, which is actually a refined version of non-deterministic computability as defined in [9].

Definition 3.1 *(Las Vegas computability).* Let (X, δ_X) and (Y, δ_Y) be represented spaces, let $R \subseteq \mathbb{N}^{\mathbb{N}}$, let μ_R be a Borel measure on R and let I be some interval. A multi-valued function $f :\subseteq X \rightrightarrows Y$ is said to be *Las Vegas computable over R with measure in I*, if there exist two computable functions $F_1, F_2 :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\langle \mathrm{dom}(f \delta_X) \times R \rangle \subseteq \mathrm{dom}(F_2)$ and for each $p \in \mathrm{dom}(f \delta_X)$ the following hold:

1. $S_p := \{ r \in R : \delta_{\mathbb{S}} F_2 \langle p, r \rangle = 0 \}$ is non-empty and $\mu_R(S_p) \in I$,
2. $\delta_Y F_1 \langle p, r \rangle \in f \delta_X(p)$ for all $r \in S_p$.

If f is Las Vegas computable over $R = 2^{\mathbb{N}}$ with measure $\mu_{2^{\mathbb{N}}}$ and values in $I = (0, 1]$, then we say for short that f is *Las Vegas computable*. If the same holds over $R = \mathbb{N} \times 2^{\mathbb{N}}$ with $\mu_{\mathbb{N}} \otimes \mu_{2^{\mathbb{N}}}$ and $I = (0, \infty]$, then we say for short that f is *Las Vegas computable with finitely many mind changes*. The latter terminology will become clearer in Section 6 and in particular by Corollary 6.4.

We mention that Las Vegas computability over $2^{\mathbb{N}}$ with probabilities in $I = [0, 1]$ is the same as non-deterministic computability as originally introduced by Martin Ziegler [50,49] and further studied in [9]. The above definition is just an adaption of [9, Definition 7.1]. In [9, Theorem 7.2] non-deterministic computations over R were characterized with the help of the closed choice principle C_R and here we transfer this result to the probabilistic setting.

We introduce corresponding probabilistic choice principles by generalizing the corresponding definition in [14]. By $\mathsf{P}_I \mathsf{C}_X$ we denote the closed choice operation restricted to closed subsets of X whose measure is in the interval I.

Definition 3.2 *(Probabilistic choice).* Let X be a represented space together with a Borel measure μ_X on X and let I be an interval. By

$$\mathsf{P}_I \mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$$

we denote the choice operation restricted to $\mathrm{dom}(\mathsf{P}_I \mathsf{C}_X) := \{ A : \mu_X(A) \in I \}$. We call $\mathsf{P}_I \mathsf{C}_X$ *probabilistic choice* of X with respect to I.

We usually abbreviate the interval I, for instance, by writing "> 0" instead of $(0, \infty]$ and we use analogous abbreviations for other intervals. We also write $\mathsf{PC}_X := \mathsf{P}_{>0} \mathsf{C}_X = \mathsf{P}_{(0, \infty]} \mathsf{C}_X$, which was already studied under the name *positive choice* in [14]. We also obtain ordinary closed choice as special case: $\mathsf{C}_X = \mathsf{P}_{\geq 0} \mathsf{C}_X = \mathsf{P}_{[0, \infty]} \mathsf{C}_X$. Theorem 7.2 from [9] can now directly be transferred to our setting.

Theorem 3.3 *(Las Vegas computability).* *Let X and Y be represented spaces, let $R \subseteq \mathbb{N}^{\mathbb{N}}$ be endowed with a Borel measure, let I be an interval and let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function. Then the following are equivalent to each other:*

1. $f \leq_{\mathrm{W}} \mathsf{P}_I \mathsf{C}_R$,
2. *f is Las Vegas computable over R with measure in I.*

The proof is literally the same as the proof of Theorem 7.2 in [9], with the extra observation that the success sets S_p in case of Las Vegas computability have to satisfy analogous measure requirements as the sets in the domain of $\mathsf{P}_I \mathsf{C}_R$. As a special case of Theorem 3.3 we obtain the following corollary.

Corollary 3.4 *(Las Vegas computability). Let f be a multi-valued function on represented spaces. Then*

1. $f \leq_W PC_{2^\mathbb{N}}$ *if and only if f is Las Vegas computable,*
2. $f \leq_W PC_{\mathbb{N} \times 2^\mathbb{N}}$ *if and only if f is Las Vegas computable with finitely many mind changes.*

Theorem 3.3 raises the question of whether there is a difference between ordinary and strong Weihrauch reducibility to $P_I C_R$, in other words, whether $P_I C_R$ is a cylinder. The following result shows that this is typically not the case. We first introduce a notation.

Definition 3.5 *(Cardinality). Let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function. Then we denote by $\#f$ the supremum of the cardinalities $|M|$ of sets $M \subseteq \mathrm{dom}(f)$ such that $\{f(x) : x \in M\}$ contains only pairwise disjoint sets. We call $\#f$ the cardinality of f.*

Obviously, the cardinality $\#f$ is always bounded from above by the cardinality of $\mathrm{dom}(f)$. It is clear that strong reductions preserve cardinalities in the following sense.

Proposition 3.6 *(Cardinality). $f \leq_{sW} g \implies \#f \leq \#g$.*

It is folklore that for a σ-finite measure there cannot be an uncountable number of pairwise disjoint sets of positive measure. For completeness we include the proof.

Proposition 3.7. *Let X be a space that is equipped with a σ-finite measure. Then there can be at most countably many pairwise disjoint measurable sets $A \subseteq X$ of positive measure.*

Proof. Since X is σ-finite, there is a sequence (X_i) of measurable sets $X_i \subseteq X$ with $X = \bigcup_{i=0}^{\infty} X_i$ and $\mu(X_i) < \infty$ for all $i \in \mathbb{N}$. Let $\mathcal{F}$ be a family of pairwise disjoint sets $A \subseteq X$ of positive measure and let

$$\mathcal{F}_{n,k} := \{A \in \mathcal{F} : \mu(A \cap X_n) > 2^{-k}\}$$

for all $n, k \in \mathbb{N}$. Then $\mathcal{F} = \bigcup_{n,k \in \mathbb{N}} \mathcal{F}_{n,k}$. There cannot be more than $\mu(X_n) \cdot 2^k$ many pairwise disjoint sets in $\mathcal{F}_{n,k}$. Hence any set $\mathcal{F}_{n,k}$ is finite and hence $\mathcal{F}$ is a countable union of finite sets and countable itself. $\square$

Hence, we obtain the following.

Proposition 3.8 *(Cardinality of probabilistic choice). If X is a represented space that is equipped with a σ-finite measure and I is an interval with $0 \notin I$, then $\#P_I C_X \leq |\mathbb{N}|$.*

Since $\#\mathrm{id}_{\mathbb{N}^\mathbb{N}} = |\mathbb{N}^\mathbb{N}|$, we get $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \not\leq_{sW} P_I C_X$ in this situation. This yields the following corollary.

Corollary 3.9 *(Probabilistic choice is not a cylinder). If X is a represented space that is equipped with a σ-finite measure and I is an interval with $0 \notin I$, then $P_I C_X$ is not a cylinder.*

This means that typically (namely under the conditions given in Corollary 3.9) we cannot replace Weihrauch reducibility $\leq_W$ by strong Weihrauch reducibility $\leq_{sW}$ in Theorem 3.3.

4. Products of probabilistic choice

We now want to compare different probabilistic choice operations with each other and in particular we want to consider products of probabilistic choice. As a first obvious observation we note that probabilistic choice is always monotone in the interval of probabilities (or measure values).

Proposition 4.1 *(Monotonicity). Let I, J be intervals and let X be some represented space endowed with some Borel measure. Then*

$$I \subseteq J \implies P_I C_X \leq_{sW} P_J C_X.$$

We mention another obvious result on products. For two intervals $I, J \subseteq \mathbb{R}$ we denote by $I \cdot J := \{x \cdot y : x \in I, y \in J\}$ the *arithmetical product* of the two sets. It is not too difficult to see that products of non-negative intervals are always intervals (the case $[0, a] \cdot [\infty, \infty] = \{0, \infty\}$ cannot occur since $[\infty, \infty]$ is not considered as an interval here.) We say that an interval I is *closed under product*, if $I \cdot I \subseteq I$.

Proposition 4.2 *(Products). Let X and Y be represented spaces, both endowed with σ-finite Borel measures. Let I, J be intervals. Then we obtain*

$$\mathsf{P}_I \mathsf{C}_X \times \mathsf{P}_J \mathsf{C}_Y \leq_{sW} \mathsf{P}_{I \cdot J} \mathsf{C}_{X \times Y}.$$

The proof is straightforward, noting that the map

$$\times : \mathcal{A}_-(X) \times \mathcal{A}_-(Y) \to \mathcal{A}_-(X \times Y), (A, B) \mapsto A \times B$$

is computable and satisfies the property that the measure of the output is the product of the measures of the inputs.

In [9] an Independent Choice Theorem 7.3 was proved that allows to conclude that non-deterministically computable functions are closed under composition. With an additional invocation of the Theorem of Fubini we can transfer this theorem and its proof to the probabilistic setting. This theorem can also be seen as a generalization of Proposition 4.2 for ordinary Weihrauch reducibility in the case of $R, S \subseteq \mathbb{N}^{\mathbb{N}}$, since $f \times g \leq_{W} f * g$.

Theorem 4.3 *(Independent Choice). Let $R, S \subseteq \mathbb{N}^{\mathbb{N}}$ both be endowed with σ-finite Borel measures and let I, J be intervals, such that ∞ is not an open endpoint of I. Then*

$$\mathsf{P}_I \mathsf{C}_R * \mathsf{P}_J \mathsf{C}_S \leq_{W} \mathsf{P}_{I \cdot J} \mathsf{C}_{R \times S}.$$

Proof. If one of the intervals I, J is empty, then $I \cdot J$ is empty and the claim holds. Hence, we can assume that I, J are both non-empty. We consider represented spaces (X, δ_X), (Y, δ_Y) and (Z, δ_Z). Let now $f :\subseteq Y \rightrightarrows Z$ and $g :\subseteq X \rightrightarrows Y$ be Las Vegas computable over R, S, respectively with measures in I, J, respectively. Let μ_R, μ_S denote the σ-finite measures of R, S, respectively. Due to Theorem 3.3 it suffices to show that $f \circ g$ is Las Vegas computable over $R \times S$ with measure in $I \cdot J$. Intuitively, we can choose an advice $(r, s) \in R \times S$ and use advice r for f and advice s for g. More precisely, let f and g be Las Vegas computable using computable functions F_1, F_2 and G_1, G_2 according to Definition 3.1, respectively. We define H_1 and H_2 that witness Las Vegas computability of $f \circ g$ over $R \times S$ with measure in $I \cdot J$. We can define a computable H_1 by

$$H_1 \langle p, \langle r, s \rangle \rangle := F_1 \langle G_1 \langle p, s \rangle, r \rangle$$

and there exists a computable H_2 such that

$$\delta_{\mathbb{S}} H_2 \langle p, \langle r, s \rangle \rangle = \begin{cases} 1 & \text{if } \delta_{\mathbb{S}} G_2 \langle p, s \rangle = 1 \\ \delta_{\mathbb{S}} F_2 \langle G_1 \langle p, s \rangle, r \rangle & \text{otherwise} \end{cases}$$

for all $p \in \operatorname{dom}(fg\delta_X)$ and all $(r, s) \in R \times S$. Such a computable H_2 exists, since $\delta_{\mathbb{S}} G_2 \langle p, s \rangle = 0$ implies that $\delta_Y G_1 \langle p, s \rangle \in g(\delta_X(p)) \subseteq \operatorname{dom}(f)$. Now we verify that H_1 and H_2 satisfy conditions (1) and (2) of Definition 3.1 for $f \circ g$. To this end, let $p \in \operatorname{dom}(fg\delta_X)$.

Let $(r, s) \in R \times S$ be such that $\delta_{\mathbb{S}} H_2 \langle p, \langle r, s \rangle \rangle = 0$. Then $\delta_{\mathbb{S}} G_2 \langle p, s \rangle = 0$ and $\delta_{\mathbb{S}} F_2 \langle G_1 \langle p, s \rangle, r \rangle = 0$. Hence by conditions (2) for g and f we obtain $\delta_Y G_1 \langle p, s \rangle \in g\delta_X(p)$ and hence $\delta_Z F_1 \langle G_1 \langle p, s \rangle, r \rangle \in fg\delta_X(p)$, which proves condition (2) for $f \circ g$.

It remains to prove that condition (1) holds for $f \circ g$. For this purpose we consider for our fixed p the following sets (which are closed in S, R, and $R \times S$, respectively):

- $S_p := \{ s \in S : \delta_{\mathbb{S}} G_2 \langle p, s \rangle = 0 \}$,
- $R_{p,s} := \{ r \in R : \delta_{\mathbb{S}} F_2 \langle G_1 \langle p, s \rangle, r \rangle = 0 \}$ for all $s \in S_p$,
- $T_p := \{ (r, s) \in R \times S : \delta_{\mathbb{S}} H_2 \langle p, \langle r, s \rangle \rangle = 0 \}$.

Intuitively, the set S_p is the set of successful advices for the Las Vegas computation of g on input p, $R_{p,s}$ is the set of successful advices of the Las Vegas computation of f on input $G_2 \langle p, s \rangle$, provided $s \in S_p$ and T_p is the set of all successful advices (r, s) for the Las Vegas computation of $f \circ g$ on input p. By condition (2) for f and g we know that $\mu_S(S_p) \in J$ and $\mu_R(R_{p,s}) \in I$ for all $s \in S_p$. By definition of H_2 we obtain

$$T_p = \{ (r, s) \in R \times S : s \in S_p \text{ and } r \in R_{p,s} \}.$$

By the Theorem of Fubini for measurable sets (see [3, Theorem 23.3]) and Lemma 2.9 this yields

$$(\mu_R \otimes \mu_S)(T_p) = \int_{S_p} \mu_R(R_{p,s}) \, d\mu_S \in I \cdot \mu_S(S_p) \subseteq I \cdot J,$$

where the integrand is understood to be the function $s \mapsto \mu_R(R_{p,s})$. This shows that condition (1) also holds for $f \circ g$. $\square$

In order to complete our results on products we need another notion. A function $f : X \to Y$ on spaces X, Y that are equipped with measures μ_X, μ_Y, respectively, is called *measure preserving* if $\mu_X(f^{-1}(A)) = \mu_Y(A)$ for all closed $A \subseteq Y$.

A function $f : X \to Y$ is called a *computable isomorphism* if it is bijective and f as well as f^{-1} are computable. We will exploit the fact that the usual Cantor pairing functions (as introduced in Section 2) are computable and measure preserving.

Lemma 4.4 *(Pairing functions). The following functions are computable isomorphisms and measure preserving. We assume that $\mathbb{N}$ is endowed with the counting measure.*

1. $2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}, (p, q) \mapsto \langle p, q \rangle$,
2. $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, (p, q) \mapsto \langle p, q \rangle$,
3. $\mathbb{N} \times \mathbb{N} \to \mathbb{N}, (n, k) \mapsto \langle n, k \rangle$,
4. $(\mathbb{N} \times 2^{\mathbb{N}}) \times (\mathbb{N} \times 2^{\mathbb{N}}) \to \mathbb{N} \times 2^{\mathbb{N}}, (p, q) \mapsto \langle p, q \rangle$.

In the special case of spaces R that are equipped with a pairing mechanism that is a computable isomorphism and measure preserving (as for the spaces given in Lemma 4.4) and I is closed under product, we can exploit the fact that we obtain

$$\mathsf{P}_I \mathsf{C}_{R \times R} \equiv_{\mathrm{sW}} \mathsf{P}_I \mathsf{C}_R.$$

This yields the following corollary.

Corollary 4.5 *(Products and pairing). If I is an interval that is closed under product and does not contain ∞ as an open endpoint and $R \subseteq \mathbb{N}^{\mathbb{N}}$ is equipped with a σ-finite measure and a corresponding pairing function that is a computable isomorphism as well as measure preserving, then we obtain*

$$\mathsf{P}_I \mathsf{C}_{R \times R} \equiv_{\mathrm{sW}} \mathsf{P}_I \mathsf{C}_R \equiv_{\mathrm{sW}} \mathsf{P}_I \mathsf{C}_R \times \mathsf{P}_I \mathsf{C}_R \equiv_{\mathrm{W}} \mathsf{P}_I \mathsf{C}_R * \mathsf{P}_I \mathsf{C}_R.$$

In particular, $\mathsf{P}_I \mathsf{C}_R$ is strongly idempotent and closed under composition.

We note that by Lemma 4.4 the assumption on the pairing function applies to all the spaces R among $\mathbb{N}, 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}, \mathbb{N} \times 2^{\mathbb{N}}$ with the respective canonical measures. In particular, we get the following corollary.

Corollary 4.6 *(Closure under composition). The classes of multi-valued functions that are Las Vegas computable and Las Vegas computable with finitely many mind changes, respectively, are both closed under composition.*

5. Intervals that are closed under product

In Corollary 4.5 we have seen that intervals that are closed under product lead to very natural notions of probabilistic computability, since the corresponding classes of functions are closed under composition. For the case of coin tosses, i.e., for the space $2^{\mathbb{N}}$, we will see that we only obtain three distinct classes in this way: $\mathsf{C}_1, \mathsf{C}_{2^{\mathbb{N}}}$ and $\mathsf{PC}_{2^{\mathbb{N}}}$. These classes correspond exactly to the following classes of multi-valued functions: computable, non-deterministically computable and Las Vegas computable ones, respectively.

We start by considering the types of intervals that are closed under product. In case that we are using a probability measure, we only have to consider intervals $I \subseteq [0, 1]$ and we can easily see which of those are closed under product.

Lemma 5.1 *(Intervals closed under product). An interval $I \subseteq [0, 1]$ is closed under product if and only if one of the following cases holds:*

1. $I = \{1\}$,
2. $0 \in I$,
3. $I = (0, b)$ or $I = (0, b]$ for some $b \in (0, 1]$.

These three cases lead exactly to the three choice principles $\mathsf{C}_1, \mathsf{C}_{2^{\mathbb{N}}}$ and $\mathsf{PC}_{2^{\mathbb{N}}}$, respectively.

Proposition 5.2 *(Choice for intervals that are closed under product). Let $I \subseteq [0, 1]$ be an interval that is closed under product. We obtain:*

$$\mathsf{P}_I \mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \begin{cases} \mathsf{C}_1 & \text{if } I = \{1\} \\ \mathsf{C}_{2^{\mathbb{N}}} & \text{if } 0 \in I \\ \mathsf{PC}_{2^{\mathbb{N}}} & \text{if } I = (0, b) \text{ or } I = (0, b] \text{ for some } b \in (0, 1] \end{cases}.$$

Proof. If a closed set $A \subseteq 2^{\mathbb{N}}$ has full measure 1, then it must be identical to the whole space $2^{\mathbb{N}}$ and hence it contains the constant zero sequence that can be computed. This proves that $\mathsf{P}_{=1}\mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{C}_1$.

An arbitrary non-empty closed set $A \subseteq 2^{\mathbb{N}}$ can be paired with the constant zero sequence 0^{ω} to $B = \langle 0^{\omega}, A \rangle$ and this set has measure 0. Hence, the computable map $A \mapsto B$ yields the reduction $\mathsf{C}_{2^{\mathbb{N}}} \leq_{\mathrm{sW}} \mathsf{P}_I \mathsf{C}_{2^{\mathbb{N}}}$ if $0 \in I$. The other direction is obvious. This proves $\mathsf{P}_I \mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{P}_{\geq 0} \mathsf{C}_{2^{\mathbb{N}}} = \mathsf{C}_{2^{\mathbb{N}}}$ if $0 \in I$.

Now let $I = (0, b)$ and $J = (0, c)$ with $b, c \in (0, 1]$. We claim

$$\mathsf{P}_{(0,b)}\mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{P}_{(0,c)}\mathsf{C}_{2^{\mathbb{N}}}.$$

If $b \leq c$, then the direction $\leq_{\mathrm{sW}}$ is obvious. For the other direction, we need to map a given closed set $A \subseteq 2^{\mathbb{N}}$ with positive measure $\mu(A) < c$ in a computable way to a closed set $B \subseteq 2^{\mathbb{N}}$ with positive measure $\mu(B) < b \leq c$ such that we can recover a point of the original set A from any point in B. For this purpose we use the map $A \mapsto 0^n A$ that adds a suitable prefix 0^n of length n to any point in A, where n depends on $\frac{c}{b}$ and guarantees that $0^n A$ has a small enough measure. This yields the desired reduction. The proof for intervals of type $I = (0, b]$ is analogous. $\quad\square$

We mention that the second case also includes $\mathsf{P}_{=0}\mathsf{C}_{2^{\mathbb{N}}}$, i.e., choice for non-empty closed zero sets. While the results in this section show that *upper* bounds on the probability do not really lead to meaningful distinctions, we will see in Section 10 that *lower* bounds can lead to such distinctions.

6. Las Vegas computability with finitely many mind changes

We recall that we want to call f Las Vegas computable with finitely many mind changes if $f \leq_{\mathrm{W}} \mathsf{PC}_{\mathbb{N} \times 2^{\mathbb{N}}}$. The purpose of this section is to get some further insights into the $\mathsf{PC}_{\mathbb{N} \times 2^{\mathbb{N}}}$. We start with some comments on $\mathsf{PC}_{\mathbb{N}}$.

Lemma 6.1 *(Probabilistic choice on natural numbers). We obtain*

$$\mathsf{PC}_{\mathbb{N}} = \mathsf{P}_{[0,\infty]}\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{P}_{[0,\infty)}\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}},$$

if $\mathbb{N}$ is equipped with the counting measure or the geometric measure.

Proof. Let $\mathbb{N}$ be endowed with the counting measure. Then $\mathsf{PC}_{\mathbb{N}} = \mathsf{P}_{[0,\infty]}\mathsf{C}_{\mathbb{N}} = \mathsf{C}_{\mathbb{N}}$ and $\mathsf{P}_{[0,\infty)}\mathsf{C}_{\mathbb{N}}$ is closed choice restricted to finite subsets $A \subseteq \mathbb{N}$. Hence we get

$$\mathsf{UC}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{P}_{[0,\infty)}\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{P}_{[0,\infty]}\mathsf{C}_{\mathbb{N}} = \mathsf{C}_{\mathbb{N}},$$

where $\mathsf{UC}_{\mathbb{N}}$ denotes choice for singletons. In [12, Proposition 3.8] we have proved $\mathsf{UC}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$ and hence the equivalence follows. If $\mathbb{N}$ is endowed with the geometric measure (which is finite), then $\mathsf{PC}_{\mathbb{N}} = \mathsf{P}_{[0,\infty]}\mathsf{C}_{\mathbb{N}} = \mathsf{P}_{[0,\infty)}\mathsf{C}_{\mathbb{N}} = \mathsf{C}_{\mathbb{N}}$. $\quad\square$

It follows from the Independent Choice Theorem 4.3 that

$$\mathsf{C}_{\mathbb{N}} \times \mathsf{PC}_{2^{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * \mathsf{PC}_{2^{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{PC}_{\mathbb{N} \times 2^{\mathbb{N}}}.$$

Here we assume that $\mathbb{N}$ is endowed with the counting measure. We now want to prove that we can also get the inverse reductions in the above situation.

Lemma 6.2. $\mathsf{PC}_{\mathbb{N} \times 2^{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{PC}_{2^{\mathbb{N}}} * \mathsf{C}_{\mathbb{N}}.$

Proof. Given a closed set $A \subseteq \mathbb{N} \times 2^{\mathbb{N}}$ of positive measure we can compute the sequence (A_n) of sections

$$A_n := \{p \in 2^{\mathbb{N}} : (n, p) \in A\}.$$

Since the measure $\mu : \mathcal{A}_-(2^{\mathbb{N}}) \to \mathbb{R}$ is upper semi-computable by Lemma 2.7, we obtain that

$$B := \{\langle n, k \rangle \in \mathbb{N} : \mu(A_n) \geq 2^{-k}\}$$

is co-c.e. in the original set A. Since A has positive measure, it follows that B is non-empty and hence we can use $\mathsf{C}_{\mathbb{N}}$ to find a point $\langle n, k \rangle \in B$. Given $\langle n, k \rangle \in B$ and the original input A, we can use $\mathsf{PC}_{2^{\mathbb{N}}}$ to find a point $p \in A_n$. Then $(n, p) \in A$. This proves the claim. $\quad\square$

Next we prove that $\mathsf{PC}_{2^{\mathbb{N}}}$ commutes with $\mathsf{C}_{\mathbb{N}}$. Essentially, we exploit for this proof that $\mathsf{PC}_{2^{\mathbb{N}}}$ is a fractal (fractals are defined after Theorem 15.4).[8]

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[8] Arno Pauly pointed out that Lemma 6.3 holds more generally for suitably defined uniform fractals f in place of $\mathsf{PC}_{2^{\mathbb{N}}}$ too.

Lemma 6.3. $\mathsf{PC}_{2^{\mathbb{N}}} * \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \times \mathsf{PC}_{2^{\mathbb{N}}}$.

Proof. It suffices to prove that $f \leq_{\mathrm{W}} \mathsf{PC}_{2^{\mathbb{N}}} * \mathsf{C}_{\mathbb{N}}$ implies $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \times \mathsf{PC}_{2^{\mathbb{N}}}$ for all $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$. Let then $f \leq_{\mathrm{W}} \mathsf{PC}_{2^{\mathbb{N}}} * \mathsf{C}_{\mathbb{N}}$. Then upon input of $p \in \mathrm{dom}(f)$ there is a computation of a machine M with finitely many mind changes that produces finitely many partial outputs $v_0, \dots, v_k \in \mathbb{N}^*$ before it finally produces an infinite output $q \in \mathbb{N}^{\mathbb{N}}$ that is a name for a set $A \subseteq 2^{\mathbb{N}}$ of positive measure. A name r for a point in A together with q and p finally allow to compute a point in $f(p)$. The basic idea is to replace the computation of M with finitely many mind changes by an ordinary computation that produces $v_0 v_1 \dots v_k q$ instead. The problem is that the latter sequence might not be a name of a set $A \subseteq 2^{\mathbb{N}}$ of positive measure. However, this can be rectified.

Firstly, there is a computable function $g : \mathbb{N}^* \to \mathbb{N}^*$ such that $g(v) = v$ if v is a valid prefix of a name of a set of positive measure and such that $g(v)$ is always a valid prefix of a name of a set of positive measure (this can be achieved by replacing certain portions of negative information by dummy information). In this sense g "cleans-up" the output. A set of positive measure described by only finitely many open balls in its complement also automatically has a non-empty interior. Moreover, we can assume that g is monotone, i.e., that $v \sqsubseteq w$ implies $g(v) \sqsubseteq g(w)$. We assume that (w_n) is a bijective effective standard enumeration of 2^*. Then there is a computable function $h :\subseteq \mathbb{N}^* \times \mathbb{N} \to \mathbb{N}$ such that $h(v, i) = j$, where j is minimal with the property that $w_j 2^{\mathbb{N}}$ is left uncovered by the negative information v and $w_i \sqsubseteq w_j$, which is possible whenever an open subset of $w_i \mathbb{N}^{\mathbb{N}}$ is left uncovered by v. Hence, h finds an "unspoiled region" where the computation can continue. Finally, there is a computable function $s : \mathbb{N}^* \times \mathbb{N} \to \mathbb{N}^*$ that has the property that if v describes a closed set $A \subseteq 2^{\mathbb{N}}$, then $s(v, i)$ describes the set $w_i A$. Hence s "shifts the output" to a possibly unspoiled region. We can also assume that s is monotone in the first component, i.e., $w \sqsubseteq v$ implies $s(w, i) \sqsubseteq s(v, i)$.

We now describe an algorithm that transfers the original computation of machine M with finitely many mind changes into a regular computation of an infinite output together with a sequence (n_i) of natural numbers. We start with n_0 such that w_{n_0} is the empty word. Whenever the output of M is extended to w, then we convert it to the "cleaned-up" output $g(w)$ and we write the number n_0 repeatedly to the output stream of numbers. When a first mind change happens, then the output $u_0 := g(v_0)$ has been produced so far. In case of this event we compute $n_1 := h(u_0, n_0)$ (i.e., an unspoiled region, which exists since u_0 is cleaned-up). We continue to read the pieces of information w produced by M after the first mind change and we proceed writing $s(g(w), n_1)$ to the output (i.e., a cleaned-up version of the information that follows shifted to the unspoiled region) and we write n_1 repeatedly to the stream of natural numbers until possibly the second mind change happens, in which case we continue inductively. In general, in between the i-th and the $(i+1)$-st mind change we have produced the output $u_i := s(g(v_i), n_i)$. When the $(i+1)$-st mind change happens we compute $n_{i+1} := h(u_i, n_i)$ and we start writing $s(g(w), n_{i+1})$ on the output and the number of n_{i+1} into the stream of numbers from now on (for the partial output w of M that follows). Eventually (when $i = k+1$) no further mind change happens and we continue with the last step forever, writing $s(g(w), n_{k+1})$ on the output tape and the number n_{k+1} into the stream of numbers.

In this phase after the last mind change w will consist of longer and longer prefixes of q and hence it will be already clean (i.e., $g(w) = w$ at this stage). Altogether, we end up writing an output that constitutes a name of the set $w_{n_{k+1}} A$ and a sequence of numbers $n_0, n_1, \dots, n_k, n_{k+1}$ with possible repetitions of each n_i and infinitely many repetitions of the last n_{k+1}. With the help of $\mathsf{C}_{\mathbb{N}}$ we can compute the value n_{k+1} from this list. Given a point $r' \in w_{n_{k+1}} A$ and n_{k+1} we can easily recover a point $r \in A$, which together with p, q allows to find some point in $f(p)$. Altogether, this proves $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \times \mathsf{PC}_{2^{\mathbb{N}}}$. $\quad\square$

Now Lemmas 6.2 and 6.3 together with Theorem 4.3 yield the following characterization.

Corollary 6.4. $\mathsf{PC}_{2^{\mathbb{N}}} \times \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \times \mathsf{PC}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{PC}_{2^{\mathbb{N}}} * \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * \mathsf{PC}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{PC}_{\mathbb{N} \times 2^{\mathbb{N}}}$.

We obtain the following corollary that expresses this result in different terms using Proposition 2.6 and Theorem 3.3.

Corollary 6.5 (*Computability with finitely many mind changes*). *Let f be a multi-valued function on represented spaces. Then the following are equivalent:*

1. *f is Las Vegas computable with finitely many mind changes,*
2. *$f = g \circ h$ with some g that is Las Vegas computable and some h that is computable with finitely many mind changes,*
3. *$f = g \circ h$ with some g that is computable with finitely many mind changes and some h that is Las Vegas computable.*

This result actually justifies to call the f below $\mathsf{PC}_{\mathbb{N} \times 2^{\mathbb{N}}}$ Las Vegas computable with finitely many mind changes.

Since the uniform measure on Cantor space is finite, it does not matter whether we define $\mathsf{PC}_{2^{\mathbb{N}}}$ using the interval $(0, \infty]$ or $(0, \infty)$. Likewise, it does not matter for $\mathbb{N}$ by Lemma 6.1. The proof of Corollary 6.4 also goes through in both cases. Hence we obtain the following corollary, which says that also $\mathsf{PC}_{\mathbb{N} \times 2^{\mathbb{N}}}$ can be defined using $(0, \infty]$ or $(0, \infty)$.

Corollary 6.6. $\mathsf{PC}_{\mathbb{N} \times 2^{\mathbb{N}}} = \mathsf{P}_{(0, \infty]} \mathsf{C}_{\mathbb{N} \times 2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{P}_{(0, \infty)} \mathsf{C}_{\mathbb{N} \times 2^{\mathbb{N}}}$.

We mention that also the equivalence class of $PC_{\mathbb{N} \times 2^{\mathbb{N}}}$ does not depend on whether $\mathbb{N}$ is equipped with the counting measure or the geometric measure. This is because the domains of $P_{(0,\infty]}C_{\mathbb{N} \times 2^{\mathbb{N}}}$ are identical in both cases and hence the multi-valued functions are identical.

Lemma 6.7. $PC_{\mathbb{N} \times 2^{\mathbb{N}}}$ *does not depend on whether $\mathbb{N}$ is equipped with the counting measure or the geometric measure.*

7. Changes of the probability space

In this section we compare probabilistic choice for $[0, 1]$ and $2^{\mathbb{N}}$ as well as probabilistic choice for $\mathbb{R}$ and $\mathbb{N} \times 2^{\mathbb{N}}$.

We recall that a *computable embedding* $f : X \to Y$ is a computable injective function, such that the partial inverse $f^{-1} :\subseteq Y \to X$ is computable too. If there is such a computable embedding f such that range(f) is co-c.e. closed, then $C_X \leq_{sW} C_Y$ follows (see Corollary 4.3 in [9]). Likewise, if there is a computable surjection $s :\subseteq X \to Y$ with a co-c.e. closed domain dom(s), then also $C_Y \leq_{sW} C_X$ follows (see Proposition 3.7 in [9]). We will implicitly use these ideas in the following and the proofs work for probabilistic choice too with some assumptions on measure preservation.

We note that the binary representation

$$\rho_2 : 2^{\mathbb{N}} \to [0, 1], p \mapsto \sum_{i=0}^{\infty} \frac{p(i)}{2^{i+1}}$$

is surjective, computable and measure-preserving. This yields immediately the reduction $P_I C_{[0,1]} \leq_{sW} P_I C_{2^{\mathbb{N}}}$. For the other direction we use the usual Smith–Volterra–Cantor set construction.

Lemma 7.1 *(Smith–Volterra–Cantor set).* *For every computable $\varepsilon \in [0, 1)$ there exists a computable embedding $f_\varepsilon : 2^{\mathbb{N}} \to [0, 1]$ such that*

$$\lambda(f_\varepsilon(A)) = \varepsilon \cdot \mu(A)$$

for every closed $A \subseteq 2^{\mathbb{N}}$. Moreover, range$(f_\varepsilon)$ *is co-c.e. closed in this situation.*

Proof. We consider the classical Smith–Volterra–Cantor set construction (see the ε-Cantor set in [1, Lemma 18.9]). Given a computable $\varepsilon \in [0, 1)$ we choose $\delta := 1 - \varepsilon$ and given the unit interval $[0, 1]$ we inductively construct a sequence (I_w) of closed intervals $I_w \subseteq \mathbb{R}$ indexed by binary words $w \in \{0, 1\}^*$ as follows. For the empty word e and words $w \in \{0, 1\}^{n-1}$, $n \geq 1$ and symbols $c \in \{0, 1\}$ we define

- $I_e := [0, 1]$,
- $I_{wc} := \begin{cases} [a, a + \frac{b-a}{2} - \frac{\delta}{2^{2n}}] & \text{if } c = 0 \\ [a + \frac{b-a}{2} + \frac{\delta}{2^{2n}}, b] & \text{if } c = 1 \end{cases}$ where $[a, b] := I_w$.

In other words, I_{wc} is constructed from I_w with $|w| = n - 1$ by removing a middle piece of length $\frac{\delta}{2^{2n-1}}$ and I_{w0} is the left half of the result, while I_{w1} is the right half. The set

$$C_\varepsilon := \bigcap_{n=0}^{\infty} \bigcup_{w \in \{0,1\}^n} I_w$$

is called the Smith–Volterra–Cantor set and due to the construction we obtain

$$\lambda(C_\varepsilon) = 1 - \sum_{n=1}^{\infty} 2^{n-1} \frac{\delta}{2^{2n-1}} = 1 - \delta = \varepsilon.$$

Now we define a computable function $f_\varepsilon : 2^{\mathbb{N}} \to [0, 1]$ by

$$\{f_\varepsilon(p)\} := \bigcap_{w \sqsubseteq p} I_w,$$

where the function value is meant to be the unique real in the given set (the value is unique by Cantor's Intersection Theorem as diam$(I_w) \leq 2^{-|w|}$). It is easy to see that f_ε is computable due to the inductive nature of the construction and because ε and hence δ are computable. Moreover, range$(f_\varepsilon) = f_\varepsilon(2^{\mathbb{N}}) = C_\varepsilon$ and due to the symmetry of the construction we obtain $\lambda(f_\varepsilon(w2^{\mathbb{N}})) = 2^{-|w|}\varepsilon = \varepsilon \cdot \mu(w2^{\mathbb{N}})$. Due to Lemma 2.8, this implies $\lambda(f_\varepsilon(A)) = \varepsilon \cdot \mu(A)$ for any closed $A \subseteq 2^{\mathbb{N}}$. Since $2^{\mathbb{N}}$ is computably compact, it follows that $f_\varepsilon(2^{\mathbb{N}}) = C_\varepsilon$ is computably compact too by [48, Theorem 3.3] and, in particular, co-c.e. closed. This also implies that the partial inverse f_ε^{-1} is computable (see for instance [9, Corollary 6.7]). □

Altogether, we obtain the following result.

Proposition 7.2 *(Cantor space and the unit interval).* Let I be an interval and let $\varepsilon \in [0, 1)$ be computable. Then we obtain

$$\mathsf{P}_I \mathsf{C}_{[0,1]} \leq_{sW} \mathsf{P}_I \mathsf{C}_{2^{\mathbb{N}}} \leq_{sW} \mathsf{P}_{\varepsilon I} \mathsf{C}_{[0,1]}.$$

For the interval $I = (0, \infty]$ we can just choose $\varepsilon = \frac{1}{2}$ and we obtain the following result that was already proved in [14, Corollary 19].

Corollary 7.3. $\mathsf{PC}_{[0,1]} \equiv_{sW} \mathsf{PC}_{2^{\mathbb{N}}}$.

In case of the spaces $\mathbb{N} \times 2^{\mathbb{N}}$ and $\mathbb{R}$ we can obtain a stronger result.

Proposition 7.4 *(Real numbers).* Let I be an interval. Then we obtain

$$\mathsf{P}_I \mathsf{C}_{\mathbb{R}} \equiv_{sW} \mathsf{P}_I \mathsf{C}_{\mathbb{N} \times 2^{\mathbb{N}}}.$$

Here $\mathbb{N}$ is equipped with the counting measure.

Proof. For the proof of $\mathsf{P}_I \mathsf{C}_{\mathbb{R}} \leq_{sW} \mathsf{P}_I \mathsf{C}_{\mathbb{N} \times 2^{\mathbb{N}}}$ we use the function

$$f : \mathbb{N} \times 2^{\mathbb{N}} \to \mathbb{R}, (n, p) \mapsto \begin{cases} \frac{1}{2}n + \rho_2(p) & \text{if } n \text{ is even} \\ -\frac{1}{2}(n+1) + \rho_2(p) & \text{if } n \text{ is odd} \end{cases},$$

which is defined with the help of the binary representation ρ_2. This function f is computable, surjective and measure-preserving (since the Lebesgue measure λ is translation-invariant) and hence we obtain the desired reduction.

For the proof of $\mathsf{P}_I \mathsf{C}_{\mathbb{N} \times 2^{\mathbb{N}}} \leq_{sW} \mathsf{P}_I \mathsf{C}_{\mathbb{R}}$ we use the function $f_{\frac{1}{2}}$ from the Smith–Volterra–Cantor set construction and we define

$$g : \mathbb{N} \times 2^{\mathbb{N}} \to \mathbb{R}, (n, p) \mapsto 3n + 2 \cdot f_{\frac{1}{2}}(p).$$

The function g is injective and even a computable embedding and it satisfies

$$\lambda(g(A)) = 2\lambda(f_{\frac{1}{2}}(A)) = \mu(A)$$

for any closed set $A \subseteq \mathbb{N} \times 2^{\mathbb{N}}$, where μ is the measure on $\mathbb{N} \times 2^{\mathbb{N}}$. Finally, $\text{range}(g) = 3\mathbb{N} + \text{range}(2 \cdot f_{\frac{1}{2}})$ is clearly a co-c.e. closed set. This is because $2^{\mathbb{N}}$ is computably compact and hence $\text{range}(f_{\frac{1}{2}})$ is computably compact too and $\text{range}(g)$ is the union of clearly separated copies of $\text{range}(2 \cdot f_{\frac{1}{2}})$ and hence it is co-c.e. closed. Altogether, this proves $\mathsf{P}_I \mathsf{C}_{\mathbb{N} \times 2^{\mathbb{N}}} \leq_{sW} \mathsf{P}_I \mathsf{C}_{\mathbb{R}}$. □

We give an example that shows that this stronger type of result cannot be achieved for the unit interval $[0, 1]$ and Cantor space $2^{\mathbb{N}}$, hence Proposition 7.2 is in a certain sense optimal. The reason for this difference are the different connectedness properties of $[0, 1]$ and $2^{\mathbb{N}}$. While the first one does not have two disjoint closed subsets of measure $\frac{1}{2}$, the second one does.

Proposition 7.5. $\mathsf{P}_{\geq \frac{1}{2}} \mathsf{C}_{[0,1]} <_W \mathsf{P}_{\geq \frac{1}{2}} \mathsf{C}_{2^{\mathbb{N}}}$.

Proof. The positive part of the reduction follows from Proposition 7.2. In order to prove the strictness, we claim that $\mathsf{C}_2 \leq_{sW} \mathsf{P}_{\geq \frac{1}{2}} \mathsf{C}_{2^{\mathbb{N}}}$ and $\mathsf{C}_2 \not\leq_W \mathsf{P}_{\geq \frac{1}{2}} \mathsf{C}_{[0,1]}$.

The first part is easy to see as $2^{\mathbb{N}}$ can be subdivided into $A_i := i2^{\mathbb{N}}$ for $i \in \{0, 1\}$ and $\mu(A_0) = \mu(A_1) = \frac{1}{2}$. Now any subset $C \subseteq \{0, 1\}$ is computably mapped to $A_C = \bigcup_{i \in C} A_i$ and given some $q \in A_C$ one can directly recover an i with $q \in A_i$. This yields a computable reduction $\mathsf{C}_2 \leq_{sW} \mathsf{P}_{\geq \frac{1}{2}} \mathsf{C}_{2^{\mathbb{N}}}$.

In order to prove the negative claim, we assume for a contradiction that we have $\mathsf{C}_2 \leq_W \mathsf{P}_{\geq \frac{1}{2}} \mathsf{C}_{[0,1]}$. Then there are computable H, K such that $H\langle \text{id}, FK \rangle$ is a realizer of C_2 whenever F is a realizer of $\mathsf{P}_{\geq \frac{1}{2}} \mathsf{C}_{[0,1]}$. We recall that we can assume that H is defined on the minimal required domain and that we use the signed-digit representation of $[0, 1]$. Let p be a name of $\{0, 1\}$. Then $K(p)$ is a name of a closed set $A \subseteq [0, 1]$ with $\lambda(A) \geq \frac{1}{2}$. Let $A_i \subseteq A$ be the set of all points that have only names q such that $H\langle p, q \rangle$ is mapped to (a name of) $i \in \{0, 1\}$. We write $H\langle p, q \rangle = i$ in this situation. Let $A_2 \subseteq A$ be a set of all those points that have names q with $H\langle p, q \rangle = 0$ as well as names q with $H\langle p, q \rangle = 1$. Then $A = A_0 \cup A_1 \cup A_2$ is a disjoint union, where one of the A_i's might be empty. Let N be the set of names of the points in A, which is compact

for the signed-digit representation. Since H is uniformly continuous[9] on the compact set $\{p\} \times N$, there is a finite prefix $w \sqsubseteq p$ such that $H\langle w\mathbb{N}^{\mathbb{N}}, q\rangle$ is a singleton for every name q of a point in A. Since K is continuous, we can assume that w is long enough such that $K(w\mathbb{N}^{\mathbb{N}})$ only contains names of sets A' with $\lambda(A' \setminus A) \leq \frac{(1-\lambda(A))}{3}$. There also exists an $i \in \{0, 1\}$ such that $\lambda(A_i) \leq \frac{1}{2}\lambda(A)$. Let now p' be a name of $C := \{i\}$ that extends w, i.e., $w \sqsubseteq p'$. Then $K(p')$ is a name of a set A' with $\lambda(A') \geq \frac{1}{2}$. We claim that $A' \cap A \subseteq A_i$, for otherwise there is an $x \in A' \cap A$ with a name q such that $H\langle p', q\rangle = 1 - i \notin C$ and there is a realizer F of $\mathsf{P}_{\geq \frac{1}{2}}\mathsf{C}_{[0,1]}$ such that $FK(p') = q$ in contradiction to the assumption and the choice of C. Hence, if $\lambda(A) < 1$, then we obtain

$$\lambda(A') \leq \lambda(A_i) + \lambda(A' \setminus A) \leq \frac{1}{2}\lambda(A) + \frac{1 - \lambda(A)}{3} < \frac{1}{2}$$

in contradiction to the assumption. If, on the other hand, $\lambda(A) = 1$, then $A = [0, 1]$ and by continuity of H we obtain $d(A_0, A_1) := \inf\{|a - b| : a \in A_0, b \in A_1\} > 0$. This implies $\lambda(A_2) > 0$ and hence we can even assume $\lambda(A_i) < \frac{1}{2}$ for some $i \in \{0, 1\}$. Similarly to above, this yields the contradiction $\lambda(A') \leq \lambda(A_i) + \lambda(A' \setminus A) < \frac{1}{2}$. $\square$

This proves that Proposition 7.2 cannot be extended to the case $\varepsilon = 1$.

8. Weak Weak Kőnig's Lemma

It is easy to see that $\mathsf{PC}_{2^{\mathbb{N}}}$ is essentially equivalent to Weak Weak Kőnig's Lemma as it is known from reverse mathematics [43]. By Tr we denote the set of binary trees $T \subseteq 2^*$, represented via their characteristic functions. Then Weak Kőnig's Lemma $\mathsf{WKL} :\subseteq \mathrm{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$ is the problem to map a binary tree T to set $[T]$ of its infinite paths. The domain $\mathrm{dom}(\mathsf{WKL})$ consists of all infinite binary trees. It has been proved in [11, Corollary 83 and Theorem 8.5] that $\mathsf{C}_{2^{\mathbb{N}}} \equiv_{sW} \mathsf{WKL}$ (see also [23,9,12]). The proof is essentially based on the fact that the map

$$[\,] : \mathrm{Tr} \rightarrow \mathcal{A}_{-}(2^{\mathbb{N}}), T \mapsto [T]$$

that maps infinite binary trees to closed subsets of Cantor space (with respect to negative information) is computable and has a multi-valued computable inverse. Analogously to Weak Kőnig's Lemma, we can define the computational counterpart of Weak Weak Kőnig's Lemma, as it is used in reverse mathematics [43].

Definition 8.1 *(Weak Weak Kőnig's Lemma). Weak Weak Kőnig's Lemma* is the problem

$$\mathsf{WWKL} :\subseteq \mathrm{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$$

restricted to the set $\mathrm{dom}(\mathsf{WWKL}) = \{T : \mu([T]) > 0\}$ of those trees T, whose set of infinite paths $[T]$ has positive measure.

Weak Weak Kőnig's Lemma was already studied in the Weihrauch lattice in [14] and the following result was noticed. This observation can be proved using the computable map $T \mapsto [T]$ above which also preserves the respective measure conditions.

Proposition 8.2 *(Weak Weak Kőnig's Lemma).* $\mathsf{WWKL} \equiv_{sW} \mathsf{PC}_{2^{\mathbb{N}}} \equiv_{sW} \mathsf{PC}_{[0,1]}$.

We mention that by Corollary 4.5 we obtain the following.

Corollary 8.3. $\mathsf{WWKL} * \mathsf{WWKL} \equiv_{W} \mathsf{WWKL}$.

Theorem 3.3 yields the following characterization.

Corollary 8.4 *(Las Vegas computability). The following are equivalent to each other:*

1. $f \leq_{W} \mathsf{WWKL}$,
2. f *is Las Vegas computable.*

It is important to note that WWKL can be separated from WKL. This was proved in [14, Theorem 20] and independently in [19, Proposition 4.2].

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[9] Strictly speaking, we use the following property stronger than uniform continuity on K: if $f : X \rightarrow Y$ is a continuous function on metric spaces (X, d_X) and (Y, d_Y) and $K \subseteq X$ is compact, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$ and all $y \in K$ we obtain that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$. This can be proved analogously to the fact that $f|_K$ is uniformly continuous. When we refer to "uniform continuity" of a function $f : X \rightarrow Y$ on K, then we mean this property.

Corollary 8.5 *(Weak and Weak Weak Kőnig's Lemma).* WWKL $<_W$ WKL.

We will prove a more general result in Section 14, see Corollary 14.8. Corollary 8.5 can also be rephrased using classes of functions as follows.

Corollary 8.6. *Every Las Vegas computable multi-valued function is non-deterministically computable, but there are non-deterministically computable multi-valued functions that are not Las Vegas computable.*

To emphasize multi-valuedness is important here, since all single-valued non-deterministically computable functions (on computable metric spaces) are automatically computable, as proved in [11, Corollary 8.8].

Dorais et al. [19] also introduced a quantitative version ε-WWKL of Weak Weak Kőnig's Lemma that requires a probability above some threshold ε. We define this version here and we will study it starting from Section 10.

Definition 8.7 *(Quantitative Weak Weak Kőnig's Lemma).* For $\varepsilon \in \mathbb{R}$ we denote by ε-WWKL the restriction of Weak Weak Kőnig's Lemma WWKL to the set $\mathrm{dom}(\varepsilon\text{-WWKL}) = \{T : \mu([T]) > \varepsilon\}$.

From the aforementioned results it is clear that we obtain the following corollary.

Corollary 8.8. ε-WWKL $\equiv_{sW} \mathsf{P}_{>\varepsilon}\mathsf{C}_{2^{\mathbb{N}}}$ *for all $\varepsilon \in \mathbb{R}$.*

9. Jumps of Weak Weak Kőnig's Lemma

In this section we want to mention some observations on the jump of Weak Weak Kőnig's Lemma. We will also see that there is a certain trade-off between the complexity of the underlying spaces X in $\mathsf{PC}_X^{(n)}$ and the number of jumps n. We start with mentioning that with iterated jumps of WWKL one actually climbs up the Borel hierarchy. We recall that a multi-valued map $f :\subseteq X \rightrightarrows Y$ on Polish spaces X and Y is called $\mathbf{\Sigma}_n^0$-*measurable* if preimages

$$f^{-1}(U) := \{x \in X : f(x) \cap U \neq \emptyset\}$$

of open sets $U \subseteq Y$ are $\mathbf{\Sigma}_n^0$-sets in the Borel hierarchy relatively to $\mathrm{dom}(f)$ (see [8] for more details).

Proposition 9.1. WWKL$^{(n)}$ *is $\mathbf{\Sigma}_{n+2}^0$-measurable, but not $\mathbf{\Sigma}_{n+1}^0$-measurable with respect to the Borel hierarchy for all $n \in \mathbb{N}$.*

Proof. Since WWKL$^{(n)} \leq_{sW}$ WKL$^{(n)} \leq_{sW} \lim^{(n)}$ it is clear that WWKL$^{(n)}$ is $\mathbf{\Sigma}_{n+2}^0$-measurable. This is because $\lim^{(n)}$ is $\mathbf{\Sigma}_{n+2}^0$-measurable and hence so is WWKL$^{(n)}$ by [8, Proposition 7.5]. On the other hand, $\mathsf{C}_2^{(n)} \leq_{sW}$ WWKL$^{(n)}$ (as one can easily show) and $\mathsf{C}_2^{(n)}$ is not $\mathbf{\Sigma}_{n+1}^0$-measurable. Hence WWKL$^{(n)}$ is not $\mathbf{\Sigma}_{n+1}^0$-measurable by [8, Proposition 7.5]. We sketch the proof that $\mathsf{C}_2^{(n)}$ is not $\mathbf{\Sigma}_{n+1}^0$-measurable, which can be proved by induction. It is clear that C_2 is not continuous, which proves the case $n = 0$. For $n = 1$ we note that $\mathsf{C}_2' \equiv_{sW} \mathsf{CL}_2$ by [12, Theorem 9.4] where CL_2 denotes the cluster point problem of $\{0, 1\}$, also called the infinite pigeonhole principle. Since $\mathsf{CL}_2^{-1}\{0\}$ is the set of binary sequences that contain infinitely many zeros, which is known to be $\mathbf{\Pi}_2^0$-complete (see, for instance, [30, Exercise II.23.1]), it is not a $\mathbf{\Sigma}_2^0$-set and hence $\mathsf{C}_2' \equiv_{sW} \mathsf{CL}_2$ is not $\mathbf{\Sigma}_2^0$-measurable. We note that for a convergent sequence (p_i) in $\{0, 1\}^{\mathbb{N}}$ we obtain

$$\lim_{i \to \infty} p_i(k) = 0 \iff (\exists j)(\forall i \geq j)\, p_i(k) = 0 \iff (\forall j)(\exists i \geq j)\, p_i(k) \neq 1.$$

This implies that each further application of a limit adds exactly one quantifier (the above equivalence allows to choose whether it is an existential or a universal one). From this it follows by induction that the preimage $(\mathsf{CL}_2^{(n)})^{-1}(\{0\})$ is $\mathbf{\Pi}_{n+2}^0$-complete and hence not a $\mathbf{\Sigma}_{n+2}^0$ set (using suitable complete sets of higher levels of the Borel hierarchy) and hence $\mathsf{C}_2^{(n+1)} \equiv_{sW} \mathsf{CL}_2^{(n)}$ is not $\mathbf{\Sigma}_{n+2}^0$-measurable. □

In particular, we obtain the following by [8, Proposition 7.5].

Corollary 9.2. WWKL$^{(n)} <_W$ WWKL$^{(n+1)}$ *for all $n \in \mathbb{N}$.*

We also introduce discrete jumps of Weihrauch degrees. In [9] the discrete limit map $\lim_{\Delta} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, \ldots \rangle \mapsto \lim_{i \to \infty} p_i$ was studied, which is the limit map with respect to the discrete topology on $\mathbb{N}^{\mathbb{N}}$, i.e., $\lim_{\Delta}$ is the restriction of $\lim$ to eventually constant sequences. This leads to the *discrete jump* $\delta_X^{\Delta} := \delta_X \circ \lim_{\Delta}$ of a representation and analogously to the *discrete jump* $f^{\Delta} :\subseteq (X, \delta_X^{\Delta}) \rightrightarrows (Y, \delta_Y)$ of a multi-valued function. It is easy to see that we obtain $f^{\Delta} \equiv_{sW} f *_s \lim_{\Delta}$. In [12, Fact 3.7] it was proved that $\lim_{\Delta} \equiv_{sW} (\mathrm{id} \times \mathsf{C}_{\mathbb{N}})$, i.e., $\lim_{\Delta}$ is strongly equivalent to the cylindrification of $\mathsf{C}_{\mathbb{N}}$. We can use this concept to express the following result.

Theorem 9.3 *(Discrete jump).* $\mathsf{WWKL}^\Delta \equiv_{\mathrm{sW}} \mathsf{PC}_{2^\mathbb{N}}^\Delta \equiv_{\mathrm{sW}} \mathsf{PC}_\mathbb{R} \equiv_{\mathrm{sW}} \mathsf{PC}_{\mathbb{N} \times 2^\mathbb{N}}$.

Proof. It follows from Proposition 7.4 that $\mathsf{PC}_\mathbb{R} \equiv_{\mathrm{sW}} \mathsf{PC}_{\mathbb{N} \times 2^\mathbb{N}}$ and from Proposition 8.2 that $\mathsf{PC}_{2^\mathbb{N}}^\Delta \equiv_{\mathrm{sW}} \mathsf{WWKL}^\Delta$.

A close inspection of the proof of Lemma 6.2 shows that the number n that occurs in the pair $\langle n, k \rangle$ of that proof can be encoded in the set A_n, by replacing it by $C_n := 01^n 0 A_n$. Then an application of $\mathsf{PC}_{2^\mathbb{N}}$ to C_n yields a point p from which a point in the original set A can be reconstructed without any direct access to the original input. This proves

$$\mathsf{PC}_{\mathbb{N} \times 2^\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{PC}_{2^\mathbb{N}} *_{\mathrm{s}} (\mathrm{id} \times \mathsf{C}_\mathbb{N}) \equiv_{\mathrm{sW}} \mathsf{PC}_{2^\mathbb{N}} *_{\mathrm{s}} \lim_\Delta \equiv_{\mathrm{sW}} \mathsf{PC}_{2^\mathbb{N}}^\Delta.$$

On the other hand, exactly the same proof as the proof of Lemma 6.3 shows

$$\mathsf{PC}_{2^\mathbb{N}} *_{\mathrm{s}} \lim_\Delta \leq_{\mathrm{sW}} \mathsf{C}_\mathbb{N} \times \mathsf{PC}_{2^\mathbb{N}}.$$

Finally, Proposition 4.2 implies $\mathsf{C}_\mathbb{N} \times \mathsf{PC}_{2^\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{PC}_{\mathbb{N} \times 2^\mathbb{N}}$, which completes the proof. □

We mention that by Corollary 4.5 we obtain the following.

Corollary 9.4. $\mathsf{WWKL}^\Delta * \mathsf{WWKL}^\Delta \equiv_{\mathrm{W}} \mathsf{WWKL}^\Delta$.

If we combine Theorem 9.3 with Corollary 6.4 and Proposition 8.2, then we obtain the following corollary.

Corollary 9.5. $\mathsf{WWKL}^\Delta \equiv_{\mathrm{W}} \mathsf{WWKL} * \mathsf{C}_\mathbb{N}$.

This yields the following characterization of Las Vegas computability with finitely many mind changes.

Corollary 9.6 *(Las Vegas computability with finitely many mind changes). The following are equivalent to each other:*

1. *$f \leq_{\mathrm{W}} \mathsf{WWKL}^\Delta$,*
2. *f is Las Vegas computable with finitely many mind changes.*

Theorem 9.3 together with $\lim_\Delta \leq_{\mathrm{sW}} \lim$ yields the positive content of the following corollary. The fact that the reduction is strict follows from $\mathsf{PC}_{\mathbb{N} \times 2^\mathbb{N}} \leq_{\mathrm{W}} \lim$, which holds by [9, Theorem 8.7] (an intuitive explanation of this fact is that there is a limit computation that always selects the left most path in a given infinite tree $T \subseteq \mathbb{N} \times \{0, 1\}^*$), which means that $\mathsf{PC}_{\mathbb{N} \times 2^\mathbb{N}}$ is $\boldsymbol{\Sigma}_2^0$-measurable (by [8, Proposition 7.5]), whereas WWKL' is not $\boldsymbol{\Sigma}_2^0$-measurable by Proposition 9.1.

Corollary 9.7. $\mathsf{PC}_{\mathbb{N} \times 2^\mathbb{N}} <_{\mathrm{sW}} \mathsf{PC}_{2^\mathbb{N}}' \equiv_{\mathrm{sW}} \mathsf{WWKL}'$.

In [12] the cluster point problem of a space was studied and it was proved that the cluster point problem is the jump of the closed choice problem of the same space. This result straightforwardly generalizes to the probabilistic setting. We will call the problem to find a cluster point of a sequence that has a set of cluster points of a certain measure the *probabilistic cluster point problem*.

Definition 9.8 *(Probabilistic cluster point problem).* Let X be a computable metric space that is equipped with a Borel measure μ and let I be some interval. Then we call

$$\mathsf{P}_I \mathsf{CL}_X :\subseteq X^\mathbb{N} \rightrightarrows X, (x_n) \mapsto \{x \in X : x \text{ is a cluster point of } (x_n)\}$$

the *probabilistic cluster point problem* with measure in I, where $\mathrm{dom}(\mathsf{P}_I \mathsf{CL}_X)$ is the set of all sequences (x_n) in X, with a non-empty set of cluster points C that satisfies $\mu(C) \in I$.

We use similar abbreviations as for $\mathsf{P}_I \mathsf{C}_X$, for instance $\mathsf{PCL}_X := \mathsf{P}_{>0} \mathsf{CL}_X := \mathsf{P}_{(0,\infty]} \mathsf{CL}_X$ etc. Now we obtain the following general result.

Theorem 9.9 *(Probabilistic cluster point problem). Let X be a computable metric space that is equipped with a Borel measure and let I be an interval. Then*

$$(\mathsf{P}_I \mathsf{C}_X)' \equiv_{\mathrm{sW}} \mathsf{P}_I \mathsf{CL}_X.$$

Proof. We just adapt results that have been provided in [12]. If we denote by $\mathsf{L}_X :\subseteq X^{\mathbb{N}} \rightrightarrows \mathcal{A}_-(X)$ the surjective map that maps each sequence (x_n) to the set $\mathsf{L}_X(x_n)$ of its cluster points, then we obtain

$$\mathsf{P}_I \mathsf{CL}_X = \mathsf{P}_I \mathsf{C}_X \circ \mathsf{L}_X.$$

By [12, Proposition 9.2] we have $\mathsf{L}_X \leq_{sW} \lim$ and together with [12, Theorem 5.14] we obtain $\mathsf{P}_I \mathsf{CL}_X \leq_{sW} (\mathsf{P}_I \mathsf{C}_X)'$. The other direction $(\mathsf{P}_I \mathsf{C}_X)' \leq_{sW} \mathsf{P}_I \mathsf{CL}_X$ follows by [12, Corollary 9.5], which states that the jump $(\mathsf{L}_X^{-1})'$ of the multi-valued inverse of L_X is computable. $\square$

Together with Proposition 8.2 and the fact that jumps are monotone with respect to $\leq_{sW}$ we obtain the following corollary.

Corollary 9.10. $\mathsf{WWKL}' \equiv_{sW} \mathsf{PCL}_{2^{\mathbb{N}}} \equiv_{sW} \mathsf{PCL}_{[0,1]}$.

10. Changes of the probability values

In this section we discuss the dependency of probabilistic choice on different lower bounds on the probability. Intuitively, it should make choice easier if the measure of the set that one chooses from increases. Indeed, in some cases this is strictly so. For instance, one easily obtains

$$\frac{1}{2}\text{-WWKL} <_{sW} \frac{1}{3}\text{-WWKL} <_{sW} \frac{1}{4}\text{-WWKL} <_{sW} \ldots$$

While the reductions $\leq_{sW}$ are obvious, the strictness of these reductions follows since $\mathsf{C}_n \leq_{sW} \frac{1}{n+1}\text{-WWKL}$ and $\mathsf{C}_n \not\leq_{sW} \frac{1}{n}\text{-WWKL}$. The latter is true for mere counting reasons, which show that there cannot be n disjoint subsets of $2^{\mathbb{N}}$ all with measure $> \frac{1}{n}$. In fact, $\#\frac{1}{n}\text{-WWKL} = n - 1$ and $\#\mathsf{C}_n = n$ and Proposition 3.6 yields the result. As a consequence one obtains $\varepsilon\text{-WWKL} <_{sW} \mathsf{WWKL}$ for all $\varepsilon > 0$, which was also proved by Dorais et al. in [19, Proposition 4.7].

Here we generalize the above observation in several respects: for one we prove that the strictness of the reduction even occurs for ordinary Weihrauch reducibility $\leq_W$, secondly we separate the probabilistic choice principles for arbitrary probabilities (not just for fractions of the form $\frac{1}{n}$) and lastly we prove the result for both spaces, the unit interval $[0,1]$ and Cantor space $2^{\mathbb{N}}$. Instead of C_n as above we use a slightly more general variant of choice. For non-negative integers a, b we denote by $\mathsf{C}_{a,b}$ the closed choice operation for the set $b = \{0, 1, \ldots, b-1\}$, restricted to subsets $C \subseteq b$ with cardinality $|C| \geq a$. In other words,

$$\mathsf{C}_{a,b} := \mathsf{P}_{\geq a} \mathsf{C}_b.$$

Equivalent problems have been studied under the name $\mathsf{LLPO}_{b,b-a}$ by Mylatz [34]. He also classified the exact relation of the problems $\mathsf{LLPO}_{n,m}$ and $\mathsf{LLPO}_{k,l}$ to each other in terms of number theoretic properties of n, m, k and l [34, Satz 15]. Other finite choice principles with restricted cardinality have already been studied by Pauly and Le Roux in [31]. We point out that the negative part of the following proof is very similar to the proof of Proposition 7.5 (and to a lesser extent to the proof of Theorem 16.6).

Theorem 10.1 *(Probability dependency). Let* $\varepsilon, \delta \in [0, 1]$ *and* $X = 2^{\mathbb{N}}$ *or* $X = [0, 1]$. *Then*

$$\mathsf{P}_{>\varepsilon} \mathsf{C}_X \leq_W \mathsf{P}_{>\delta} \mathsf{C}_X \iff \mathsf{P}_{>\varepsilon} \mathsf{C}_X \leq_{sW} \mathsf{P}_{>\delta} \mathsf{C}_X \iff \varepsilon \geq \delta.$$

Proof. "$\impliedby$" these reductions are clear and follow from Proposition 4.1.

"$\implies$" Let $\varepsilon < \delta$. Then there are positive integers $a < b$ with $\varepsilon < \frac{a}{b} \leq \delta$. By Proposition 7.2 it suffices to show $\mathsf{C}_{a,b} \leq_W \mathsf{P}_{>\varepsilon} \mathsf{C}_{[0,1]}$ and $\mathsf{C}_{a,b} \not\leq_W \mathsf{P}_{>\delta} \mathsf{C}_{2^{\mathbb{N}}}$, since this implies $\mathsf{P}_{>\varepsilon} \mathsf{C}_X \not\leq_W \mathsf{P}_{>\delta} \mathsf{C}_X$ for both $X = [0, 1]$ and $X = 2^{\mathbb{N}}$.

In order to prove the first statement, we select b consecutive disjoint closed intervals $I_0, \ldots, I_{b-1} \subseteq [0, 1]$ with rational endpoints of equal length $l = \lambda(I_i)$ for all $i \in B = \{0, 1, \ldots, b-1\}$ with $\frac{\varepsilon}{a} < l < \frac{1}{b}$. The function that maps a subset $C \subseteq B$ to $A_C := \bigcup_{i \in C} I_i$ is computable and if C is of cardinality $|C| \geq a$, then $\lambda(A_C) \geq al > \varepsilon$. Since given a point $x \in A_C$ one can easily recover the unique number $i \in B$ of the interval I_i with $x \in I_i$, one obtains the reduction $\mathsf{C}_{a,b} \leq_{sW} \mathsf{P}_{>\varepsilon} \mathsf{C}_{[0,1]}$.

Let us now assume for a contradiction that $\mathsf{C}_{a,b} \leq_W \mathsf{P}_{>\delta} \mathsf{C}_{2^{\mathbb{N}}}$. Then there are computable H, K such that $H\langle \mathrm{id}, FK \rangle \vdash \mathsf{C}_{a,b}$, whenever $F \vdash \mathsf{P}_{>\delta} \mathsf{C}_{2^{\mathbb{N}}}$ holds. Let p be a name of the entire set $B = \{0, 1, \ldots, b-1\}$. Then $K(p)$ is a name of a closed set $A \subseteq 2^{\mathbb{N}}$ with measure $\mu(A) > \delta$. Now $H\langle p, q \rangle$ is a (name of a) point $i \in B$ for every $q \in A$. For simplicity we write $H\langle p, q \rangle = i$ in this situation. Let

$$A_i := \{q \in A : H\langle p, q \rangle = i\}$$

for all $i \in B$. By definition the sets $A_0, \ldots, A_{b-1}$ are pairwise disjoint. By a version of the Pigeonhole Principle there must be a set $C \subseteq B$ of cardinality $|C| = a$ such that $A_C := \bigcup_{i \in C} A_i$ has measure $\mu(A_C) \leq \frac{a}{b} \mu(A)$.

Since H is uniformly continuous on the compact set $\{p\} \times A$, it follows that there is some finite prefix $w \sqsubseteq p$ such that $H\langle w\mathbb{N}^\mathbb{N}, q\rangle$ is (a name of) a singleton for every $q \in A$. Moreover, since K is continuous we can assume that w is long enough such that $K(w\mathbb{N}^\mathbb{N})$ only contains names of sets $A' \subseteq 2^\mathbb{N}$ with $\mu(A' \setminus A) \le \frac{a}{b}(1 - \mu(A))$. Now there is a name p' of the set C with $w \sqsubseteq p'$ and hence $K(p')$ is a name of a set A' as above. It is clear that $A' \cap A \subseteq A_C$: for if $q \in A' \cap A$, then there is a realizer F of $\mathsf{P}_{>\delta}\mathsf{C}_{2^\mathbb{N}}$ such that $FK(p') = q$ and hence $H\langle p, q\rangle = H\langle p', q\rangle = H\langle p', FK(p')\rangle \in C$, which implies $q \in A_C$. Finally, the measure of A' satisfies

$$\mu(A') \le \mu(A_C) + \mu(A' \setminus A) \le \frac{a}{b} \le \delta$$

in contradiction to the requirement $\mu(A') > \delta$. $\quad\square$

In particular, we obtain the following result.

Corollary 10.2. $\varepsilon\text{-WWKL} \le_\mathrm{W} \delta\text{-WWKL} \iff \varepsilon \ge \delta$ *for* $\varepsilon, \delta \in [0, 1]$.

This result was independently proved by Dorais et al. [19, Proposition 4.7]. Since the proof of Theorem 10.1 includes the case $\frac{a}{b} = \delta$, we obtain the following.

Corollary 10.3. $\mathsf{C}_n \not\le_\mathrm{W} \frac{1}{n}\text{-WWKL}$ *for all* $n \ge 1$.

Besides ε-WWKL we can also consider a $*$-version of this principle that we define next. Essentially, the definition is $*\text{-WWKL} := \bigsqcup_{n\in\mathbb{N}} 2^{-n}\text{-WWKL}$, which can be understood as a uniform version of the logical statement "$(\forall n)\ 2^{-n}$-WWKL." We phrase the principle slightly more precisely.

Definition 10.4. We define $*\text{-WWKL} :\subseteq \mathbb{N} \times \mathrm{Tr} \rightrightarrows 2^\mathbb{N}$ by

$$*\text{-WWKL}(n, T) := 2^{-n}\text{-WWKL}(T),$$

where $\mathrm{dom}(*\text{-WWKL}) := \{(n, T) \in \mathbb{N} \times \mathrm{Tr} : \mu([T]) > 2^{-n}\}$.

That is the input to $*$-WWKL is a pair (n, T), where n is a natural number and T is a tree such that the set of infinite paths $[T]$ of T satisfies $\mu([T]) > 2^{-n}$. The output is an infinite path $p \in [T]$. It follows from Proposition 4.2 that $*$-WWKL is idempotent. Similarly, we can define a lower counterpart $(1 - *)\text{-WWKL} := \bigsqcap_{n\in\mathbb{N}}(1 - 2^{-n})\text{-WWKL}$, which can be understood as corresponding to the logical statement "$(\exists n)\ (1 - 2^{-n})$-WWKL." Also in this case we repeat the definition for clarity.

Definition 10.5. We define $(1 - *)\text{-WWKL} :\subseteq \mathrm{Tr}^\mathbb{N} \rightrightarrows 2^\mathbb{N}$ by

$$(1 - *)\text{-WWKL}(T_n)_n := \bigsqcup_{n\in\mathbb{N}}(1 - 2^{-n})\text{-WWKL}(T_n),$$

where $\mathrm{dom}((1 - *)\text{-WWKL}) := \{(T_n)_n \in \mathrm{Tr}^\mathbb{N} : (\forall n \in \mathbb{N})\ \mu([T_n]) > 1 - 2^{-n}\}$.

Thus, the input to $(1 - *)$-WWKL is a sequence $(T_n)_n$ of trees with $\mu([T_n]) > 1 - 2^{-n}$ and the output is an infinite path $p \in [T_n]$ of one of these trees T_n together with the information n to which tree the path belongs. It is clear that Corollary 10.2 implies the following.

Corollary 10.6. $(1 - *)\text{-WWKL} <_\mathrm{W} \varepsilon\text{-WWKL} <_\mathrm{W} *\text{-WWKL} \le_\mathrm{sW} \text{WWKL}$ *for every* $\varepsilon \in (0, 1)$.

We want to show that the latter reduction is strict too. We use $\mathsf{PCC}_{[0,1]}$, which is closed choice on $[0, 1]$ restricted to connected sets of positive measure. In other words, this is choice restricted to proper intervals, which was already considered in [10] under the name C_I^- and in [13] under the name CC_1^-. In [10, Proposition 3.8] it was proved that $\mathsf{PCC}_{[0,1]} \le_\mathrm{W} \mathsf{C}_\mathbb{N}$ holds.

We first prove that $\mathsf{PCC}_{[0,1]}$ is join-irreducible. We mention that due to distributivity of the Weihrauch lattice g is *join-irreducible* (in the sense introduced in Section 2) if and only if $g \le_\mathrm{W} \bigsqcup_{i=0}^\infty f_i$ implies that there exists $i \in \mathbb{N}$ with $g \le_\mathrm{W} f_i$. We prove a slightly more general result that we apply to other problems than $\mathsf{PCC}_{[0,1]}$ at a later stage.

Lemma 10.7. *Every restriction* $\mathsf{C}_{[0,1]}|_\mathcal{C}$ *of closed choice to a set* $\mathcal{C}$ *of closed subsets* $A \subseteq [0, 1]$ *with* $[0, 1] \in \mathcal{C}$ *is join-irreducible.*

Proof. Let $(f_i)_i$ be a sequence of multi-valued functions and let $f := \bigsqcup_{i=0}^\infty f_i$. Let us assume that $\mathsf{C}_{[0,1]}|_\mathcal{C} \le_\mathrm{W} f$ holds. Then there are computable H, K such that $H\langle \mathrm{id}, FK\rangle$ is a realizer of $\mathsf{C}_{[0,1]}|_\mathcal{C}$ for every realizer F of f. Now let p be a name of

$[0, 1]$. Then $K(p)$ is a name of a pair (n, x), such that x is an input to f_n. Since K is continuous, there is a finite prefix $w \sqsubseteq p$ such that $K(w\mathbb{N}^\mathbb{N})$ only contains names of pairs (n, x) with the same fixed n. Now there is a computable function L that transforms any name p' of a closed set $A \subseteq [0, 1]$ into a name $q = L(p')$ of the same set that starts with w, i.e., such that $w \sqsubseteq q$. This is because w contains no negative information that overlaps with $[0, 1]$. Hence, the functions H, KL witness the reduction $\mathsf{C}_{[0,1]}|_{\mathcal{C}} \leq_W f_n$. $\square$

We get the following immediate corollary.

Corollary 10.8. $\mathsf{PCC}_{[0,1]}$ *is join-irreducible.*

Now we are prepared to prove the following.

Proposition 10.9. $\mathsf{PCC}_{[0,1]} \nleq_W *\text{-WWKL}$.

Proof. Let us assume for a contradiction that $\mathsf{PCC}_{[0,1]} \leq_W *\text{-WWKL}$ holds. Since $\mathsf{PCC}_{[0,1]}$ is join-irreducible we can conclude by Corollary 10.8, there is some n such that $\mathsf{PCC}_{[0,1]} \leq_W 2^{-n}\text{-WWKL}$. Now we obtain

$$\mathsf{C}_{2^n} \leq_W \mathsf{K}_\mathbb{N} \equiv_{sW} \mathsf{C}_2^* \leq_W \mathsf{PCC}_{[0,1]} \leq_W 2^{-n}\text{-WWKL}$$

in contradiction to Corollary 10.3. Here $\mathsf{K}_\mathbb{N}$ denotes choice for finite subsets of $\mathbb{N}$ that are given together with an upper bound of the set and $\mathsf{K}_\mathbb{N} \equiv_{sW} \mathsf{C}_2^*$ has been proved in [12, Proposition 10.9]. The reduction $\mathsf{C}_2^* \leq_W \mathsf{PCC}_{[0,1]}$ has been proved in [13, Proposition 7.2]. $\square$

We note that this proof also yields another proof of the strictness of $\mathsf{C}_2^* <_W \mathsf{PCC}_{[0,1]}$ since $\mathsf{C}_2^* \leq *\text{-WWKL}$ (compare the remark after [13, Proposition 7.2]). Since we also obtain $\mathsf{PCC}_{[0,1]} \leq_W \mathsf{PC}_{[0,1]} \equiv_W \mathsf{WWKL}$, we arrive at the following corollary of Proposition 10.9.

Corollary 10.10. $*\text{-WWKL} <_W \mathsf{WWKL}$.

11. The Lebesgue Density Lemma

In the following we will need a simple version of the Lebesgue Density Theorem, which we will call the Lebesgue Density Lemma, and for that purpose we will classify its Weihrauch degree. The classical Lebesgue Density Theorem (see [4, 5.8(ii)]), which is a special case of the Lebesgue Differentiation Theorem for measurable sets, says that for every measurable set $A \subseteq \mathbb{R}^n$

$$\lim_{\varepsilon \to 0} \frac{\lambda(A \cap B(x, \varepsilon))}{\lambda(B(x, \varepsilon))} = 1$$

for almost all $x \in A$ (where $B(x, \varepsilon)$ denotes the ball around x with radius ε). We will use a special case of this theorem in Cantor space that is in for-all-exists form (see [20, Theorem 1.2.3] for a direct proof of this special case).

Lemma 11.1 *(Lebesgue Density Lemma). For every closed $A \subseteq 2^\mathbb{N}$ with $\mu(A) > 0$ and every $k \in \mathbb{N}$ there exists a word $w \in \{0, 1\}^*$ such that*

$$\frac{\mu(A \cap w2^\mathbb{N})}{2^{-|w|}} \geq 1 - \frac{1}{2^k}.$$

We consider the following multi-valued function as a representative of the Lebesgue Density Lemma in the Weihrauch lattice:

$$\mathsf{LDL} :\subseteq \mathcal{A}_-(2^\mathbb{N}) \times \mathbb{N} \rightrightarrows \{0, 1\}^*, (A, k) \mapsto \left\{ w \in \{0, 1\}^* : \frac{\mu(A \cap w2^\mathbb{N})}{2^{-|w|}} \geq 1 - \frac{1}{2^k} \right\}$$

with $\mathrm{dom}(\mathsf{LDL}) = \{(A, k) : \mu(A) > 0\}$. It is easy to see that the Lebesgue Density Lemma is equivalent to $\mathsf{C}_\mathbb{N}$.

Theorem 11.2 *(Lebesgue Density Lemma). $\mathsf{LDL} \equiv_{sW} \mathsf{C}_\mathbb{N}$.*

Proof. We first prove $\mathsf{LDL} \leq_{sW} \mathsf{C}_\mathbb{N}$. By (w_i) we denote some effective standard enumeration of $\{0, 1\}^*$. Given a closed set $A \subseteq 2^\mathbb{N}$ with $\mu(A) > 0$ and $k \in \mathbb{N}$ the Lebesgue Density Lemma 11.1 guarantees that

$$B := \left\{ i \in \mathbb{N} : \frac{\mu(A \cap w_i 2^{\mathbb{N}})}{2^{-|w_i|}} \geq 1 - \frac{1}{2^k} \right\}$$

is non-empty and since the measure $\mu : \mathcal{A}_-(2^{\mathbb{N}}) \to \mathbb{R}$ is upper semi-computable by Lemma 2.7 it follows that B is co-c.e. closed in A. Hence, $\mathsf{C}_{\mathbb{N}}$ can determine some point $i \in B$ which yields the desired result w_i.

Now we prove $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{LDL}$. We recall that by $\mathsf{UC}_{\mathbb{N}}$ we denote choice for singletons $\{n\}$. In fact, since $\mathsf{UC}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$ by [12, Proposition 3.8] it suffices to prove $\mathsf{UC}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{LDL}$. Hence, given a singleton $\{n\} \subseteq \mathbb{N}$ by an enumeration of its complement, we need to find the number n. Given $\{n\}$ by an enumeration of its complement, we can compute (negative information on) the closed set

$$A_n := 0^n 1 \{0, 1\}^{\mathbb{N}} \cup \{0^\omega\} \subseteq \{0, 1\}^{\mathbb{N}}$$

and we obtain $\mu(A_n) > 0$. Then LDL will produce upon input of $(A_n, 2)$ a word $w \in \{0, 1\}^*$ such that $\mu(A_n \cap w 2^{\mathbb{N}}) > 2^{-|w|-1}$. In order to ensure this condition, the word w has to have prefix $0^n 1$, which yields the number n.   □

By Corollary 9.5 we have $\mathsf{WWKL}^\Delta \equiv_{\mathrm{W}} \mathsf{WWKL} * \mathsf{C}_{\mathbb{N}}$. We can now factorize WWKL^Δ also using $\varepsilon\text{-}\mathsf{WWKL}$ for arbitrarily large $\varepsilon < 1$. As a preparation we prove the following lemma.

Lemma 11.3. $\mathsf{WWKL} \leq_{\mathrm{W}} \varepsilon\text{-}\mathsf{WWKL} * \mathsf{C}_{\mathbb{N}}$ *for all* $\varepsilon \in [0, 1)$.

Proof. Let $\varepsilon \in [0, 1)$ and let $k \in \mathbb{N}$ be such that $1 - \frac{1}{2^k} > \varepsilon$. By Theorem 11.2 it suffices to prove $\mathsf{WWKL} \leq_{\mathrm{W}} \varepsilon\text{-}\mathsf{WWKL} * \mathsf{LDL}$. Given a binary tree T with a set A of infinite paths of positive measure, we apply LDL to (A, k) in order to obtain a w such that $\frac{\mu(A \cap w 2^{\mathbb{N}})}{2^{-|w|}} > 1 - \frac{1}{2^k} > \varepsilon$. Hence, the subtree T_w of T that starts in node w (i.e., $u \in T_w \iff wu \in T$) has a set A_w of infinite paths of measure $\mu(A_w) > \varepsilon$ and hence $\varepsilon\text{-}\mathsf{WWKL}$ yields an infinite path p_w in A_w. Given this path p_w and w we can compute $p = w p_w \in A$.   □

Now we can prove the following main result of this section.

Theorem 11.4. $\mathsf{WWKL}^\Delta \equiv_{\mathrm{W}} \varepsilon\text{-}\mathsf{WWKL} * \mathsf{C}_{\mathbb{N}}$ *for all* $\varepsilon \in [0, 1)$.

Proof. Let $\varepsilon \in [0, 1)$. Since $\varepsilon\text{-}\mathsf{WWKL} \leq_{\mathrm{sW}} \mathsf{WWKL}$, we obtain by Corollary 9.5

$$\varepsilon\text{-}\mathsf{WWKL} * \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{WWKL} * \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{WWKL}^\Delta.$$

On the other hand, Corollary 9.5 and Lemma 11.3 yield

$$\mathsf{WWKL}^\Delta \leq_{\mathrm{W}} \mathsf{WWKL} * \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \varepsilon\text{-}\mathsf{WWKL} * \mathsf{C}_{\mathbb{N}} * \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \varepsilon\text{-}\mathsf{WWKL} * \mathsf{C}_{\mathbb{N}}.$$

The last reduction follows since $\mathsf{C}_{\mathbb{N}}$ is closed under composition, which in turn follows from the obvious closure under composition of the class of functions computable with finitely many mind changes [9, Corollary 7.6].   □

This result can also be interpreted such that on real numbers probabilistic choice does not depend on the value of the probability.

Theorem 11.5. $\mathsf{PC}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{P}_{>\varepsilon} \mathsf{C}_{\mathbb{R}}$ *for all* $\varepsilon \geq 0$.

Proof. Let us suppose that the claim is true for $\varepsilon = 2$. Then we obtain for all $\varepsilon \in [0, 1]$ by Proposition 4.1

$$\mathsf{PC}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{P}_{>2} \mathsf{C}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{P}_{>\varepsilon} \mathsf{C}_{\mathbb{R}} \leq_{\mathrm{W}} \mathsf{PC}_{\mathbb{R}}.$$

This means that it follows that the claim also holds for $\varepsilon \in [0, 1]$. What remains is to prove the claim for $\varepsilon > 1$ (which includes the case $\varepsilon = 2$). Hence, let $\varepsilon > 1$. Then there exists $\delta \in (0, 1)$ and $n \in \mathbb{N}$ with $n \geq 1$ such that $\delta \cdot n > \varepsilon$. We note that $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{P}_{=n} \mathsf{C}_{\mathbb{N}}$ (which can be proved analogously to $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{UC}_{\mathbb{N}} = \mathsf{P}_{=1} \mathsf{C}_{\mathbb{N}}$). With Theorems 9.3, 11.4, 4.3, Propositions 4.1 and 7.4 and Corollary 8.8 we obtain

$$\mathsf{PC}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{WWKL}^\Delta \equiv_{\mathrm{W}} \delta\text{-}\mathsf{WWKL} * \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{P}_{>\delta} \mathsf{C}_{2^{\mathbb{N}}} * \mathsf{P}_{=n} \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{P}_{>\delta \cdot n} \mathsf{C}_{2^{\mathbb{N} \times \mathbb{N}}} \leq_{\mathrm{W}} \mathsf{P}_{>\varepsilon} \mathsf{C}_{\mathbb{R}}.$$

The inverse reduction is clear by Proposition 4.1.   □

This result is in sharp contrast to Theorem 10.1. While probabilistic choice on Cantor space sensitively depends on lower bounds on the probability, probabilistic choice on the Euclidean space does not. Intuitively, this is because Cantor space is compact, whereas Euclidean space offers "enough space" to enlarge the measure of sets.

12. Probability amplification

Theorem 10.1 shows that the technique of probability amplification, which is well-known from the theory of randomized algorithms [33] fails for Las Vegas computability over infinite objects. The reason is that we are dealing with infinite computations and if we run two instances of a probabilistic algorithm with different guesses in parallel, then we need to decide at some finite time which output we are going to choose. This is simply not possible in general. The positive content of probability amplification can however be captured in an algebraic way. In order to express it precisely, we introduce the parallel sum of two degrees.

Definition 12.1 *(Parallel sums).* Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ be multi-valued functions. Then we define the *parallel sum* $f + g : X \times W \rightrightarrows Y \times Z$ by

$$(f + g)(x, w) := (f(x) \times \text{range}(g)) \cup (\text{range}(f) \times g(w))$$

for all $(x, w) \in \text{dom}(f + g) := \text{dom}(f) \times \text{dom}(g)$.

The parallel sum $f + g$ captures an operation that takes inputs x, w for both f and g and it produces a pair (y, z) such that $y \in f(x)$ or $z \in g(w)$, i.e., only one of the two results is guaranteed to be correct.[10] We note that for f, g with computable points in the range one obtains $f + g \leq_W f \sqcap g$. The concept of a parallel sum is closely related to the concept of a fraction as introduced in [13]. We can now express probability amplification with sums as follows.

Proposition 12.2 *(Probability amplification). Let X and Y be represented spaces with σ-finite Borel probability measures μ_X and μ_Y, respectively and let $a, b \in [0, 1]$ and $c := 1 - (1 - a)(1 - b)$. Then*

$$\mathsf{P}_{>a}\mathsf{C}_X + \mathsf{P}_{>b}\mathsf{C}_Y \leq_{\mathrm{sW}} \mathsf{P}_{>c}\mathsf{C}_{X \times Y}.$$

Proof. Given closed sets $A \subseteq X$ and $B \subseteq Y$ with $\mu_X(A) > a$ and $\mu_Y(B) > b$ we can compute $C := (A \times Y) \cup (X \times B)$ and we obtain $(\mu_X \otimes \mu_Y)(C) > c$. This yields the reduction. □

In case that X is a space that has a measure preserving pairing mechanism, we can replace X, Y and $X \times Y$ by X in this result using Corollary 4.5. In particular, we obtain the following.

Corollary 12.3. *Let $a, b \in [0, 1]$ and $c := 1 - (1 - a)(1 - b)$. Then*

$$a\text{-WWKL} + b\text{-WWKL} \leq_{\mathrm{sW}} c\text{-WWKL}.$$

We also note that in the situation of Proposition 12.2 with $X = Y$ one single mind change allows us to identify the successful one among the two parallel computations, i.e., $\mathsf{P}_{>a}\mathsf{C}_X \sqcap \mathsf{P}_{>b}\mathsf{C}_X \equiv_W \mathsf{P}_{>\max(a,b)}\mathsf{C}_X \leq_W \mathsf{C}_2 * (\mathsf{P}_{>a}\mathsf{C}_X + \mathsf{P}_{>b}\mathsf{C}_X)$.

13. Majority vote

In this section we will prove that any suitable single-valued function f below any jump $(\frac{1}{2}\text{-WWKL})^{(n)}$ is computable. The idea is that a simple majority vote after an exhaustive search will yield the result if more than half of the random advices do the job. We will consider single-valued functions $f : X \to Y$ to computable metric spaces Y. The majority vote technique works for these spaces since the consistency of approximations can be recognized. We will make this statement more precise. If (X, δ) is a represented space, then we denote by $\mathcal{F}(X)$ the set of finite subsets of X that is represented in the canonical way by $\delta_{\mathcal{F}(X)}$, which is defined by

$$\delta_{\mathcal{F}(X)}\langle n, p_0, \ldots, p_n\rangle := \{\delta(p_0), \ldots, \delta(p_n)\}.$$

Moreover, we recall that every represented space (X, δ) induces a dual represented space $(\mathcal{O}(X), \delta^\circ)$, where $\mathcal{O}(X)$ is the topology of X (i.e., the final topology of δ) and $\delta^\circ(p) := X \setminus \psi_-(p)$ is the representation of open subsets via their characteristic functions to Sierpiński space. In the following lemma we consider $\mathbb{N}^{\mathbb{N}}$ as a represented space via the identity $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}}$ as representation.

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[10] We warn the reader that the parallel sum is not monotone for (strong) Weihrauch reducibility and hence it cannot be considered as an operation on the Weihrauch lattice.

Lemma 13.1 *(Cauchy representation). Let X be a computable metric space. The (suitably defined) Cauchy representation δ of X satisfies*

1. $\Delta : \mathcal{O}(\mathbb{N}^{\mathbb{N}}) \to \mathcal{O}(X), U \mapsto \delta(U)$ *is computable,*
2. $C := \{W \in \mathcal{F}(\mathbb{N}^*) : \bigcap_{w \in W} \delta(w\mathbb{N}^{\mathbb{N}}) \neq \emptyset\}$ *is c.e.*

We say that a set $W \in \mathcal{F}(\mathbb{N}^*)$ is consistent, *if $W \in C$.*

Proof. Let (X, d, α) be a computable metric space (see [47] for definitions). Then we can define a version of the Cauchy representation δ of X by $\delta(p) := \lim_{n \to \infty} \alpha(p(n))$ with

$$\mathrm{dom}(\delta) := \{p \in \mathbb{N}^{\mathbb{N}} : (\exists x \in X)(\forall i \in \mathbb{N}) \, d(x, \alpha p(i)) < 2^{-i}\}.$$

This representation δ is computably equivalent to other standard versions of the Cauchy representation of X and we obtain

$$\delta(w\mathbb{N}^{\mathbb{N}}) = \bigcap_{i=0}^{|w|-1} B(\alpha w(i), 2^{-i})$$

for every $w \in \mathbb{N}^*$. Secondly, for $U = \bigcup_{w \in W} w\mathbb{N}^{\mathbb{N}}$ with $W \subseteq \mathbb{N}^*$ we obtain $\delta(U) = \bigcup_{w \in W} \delta(w\mathbb{N}^{\mathbb{N}})$. Since finite intersections and countable unions of open sets are computable, we can conclude that Δ is computable. Since

$$\{\mathcal{U} \in \mathcal{F}(\mathcal{O}(X)) : \bigcap \mathcal{U} \neq \emptyset\}$$

is c.e. open and Δ is computable, it follows that C is c.e. $\square$

Now we can prove our main result on majority votes. We point out that the multi-valued function g in the following result need not be computable.

Theorem 13.2 *(Majority vote). Let X be a represented space, let Y be a computable metric space and let $f : X \to Y$ be a single-valued function. If*

$$f \leq_{\mathrm{W}} \frac{1}{2}\text{-WWKL} \circ g$$

for some g, then f is computable.

Proof. Let (Y, d, α) be a computable metric space. Without loss of generality, we can assume that δ is the Cauchy representation of Y according to Lemma 13.1. Let $f \leq_{\mathrm{W}} \frac{1}{2}\text{-WWKL} \circ g$ and let $g :\subseteq Z \rightrightarrows \mathrm{Tr}$. Then there are computable H, K, such that $H\langle \mathrm{id}, GK\rangle$ is a realizer of f whenever G is a realizer of $\frac{1}{2}\text{-WWKL} \circ g$. Given a name p of a point $x \in \mathrm{dom}(f)$, $K(p)$ is a name of a point $z \in \mathrm{dom}(g)$ such that every tree $T \in g(z)$ satisfies $\mu([T]) > \frac{1}{2}$ and every infinite path $q \in [T]$ yields a name $H\langle p, q\rangle$ of $f(x)$, i.e., $\delta H\langle p, q\rangle = f(x)$. We need to prove that there is a computable realizer F of f. Upon input of p one can use $H\langle p, q\rangle$ for varying q in order to obtain such a realizer by majority vote. For every p that is a name of a point $x \in \mathrm{dom}(f)$ as above we can compute a monotone function $h_p : \{0, 1\}^* \to \mathbb{N}^*$ that approximates $q \mapsto H\langle p, q\rangle$. Since every $A = [T]$ for binary trees T is a compact set, the function H is uniformly continuous on $\{p\} \times A$. That means that for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that for every $q \in A$ it holds that $|h_p(q|_n)| \geq k + 2$. On the other hand, $\mu(A) > \frac{1}{2}$ and hence there is a finite set $W \in \mathcal{F}(\{0, 1\}^*)$ that consists of more than half of the words in $\{0, 1\}^n$ and such that $\{h_p(w) : w \in W\}$ is consistent and $|h(w)| \geq k + 2$ for all $w \in W$. Since consistency is a c.e. property by Lemma 13.1, we can find for our given k a suitable n and a corresponding set W by exhaustive search. As soon as we have found it, we compute an approximation $\alpha(i) \in \bigcap_{w \in W} \delta(h_p(w)\mathbb{N}^{\mathbb{N}})$ of $f(x)$, which is possible since δ is computably open by Lemma 13.1. We claim that $d(\alpha(i), f(x)) < 2^{-k}$. We note that if W consists of half of the words in $\mathbb{N}^n$, then due to the measure condition at least one of the $h_p(w)$ for $w \in W$ has to be a prefix of a correct δ-name q of $f(x)$. Hence, for this w we have $f(x) \in \delta(h_p(w)\mathbb{N}^{\mathbb{N}})$ and hence we obtain for $a := \alpha(h_p(w)(k + 1))$

$$d(\alpha(i), f(x)) \leq d(\alpha(i), a) + d(a, f(x)) \leq 2^{-k-1} + 2^{-k-1} = 2^{-k}.$$

If we proceed with the above algorithm for $k = 0, 1, 2, \dots$ and each fixed given input p, then we obtain a computable realizer F of f with respect to the representation δ on the output side. $\square$

In general we obtain $h^{(n)}$ from $h :\subseteq X \rightrightarrows Y$ by replacing the representation δ of X by its n-fold jump $\delta^{(n)}$ on the input side. Since $h^{(n)} \equiv_{\mathrm{sW}} h \circ \delta^{(n)}$, we obtain the following corollary of Theorem 13.2.

Corollary 13.3 *(Majority vote). Let X be a represented space, let Y be a computable metric space and let $f : X \to Y$ be a single-valued function. If*

$$f \leq_W \frac{1}{2}\text{-WWKL}^{(n)}$$

for some $n \in \mathbb{N}$, then f is computable.

Theorem 13.2 and Corollary 13.3 automatically also hold true for all $\varepsilon > \frac{1}{2}$ instead of $\frac{1}{2}$. It is easy to see that LPO $\leq_W \varepsilon$-WWKL$'$ for every $\varepsilon < \frac{1}{2}$ and LPO is single-valued (with the computable metric space $\{0, 1\}$ on the output side). Hence Theorem 13.2 and Corollary 13.3 cannot be generalized to the case of $\varepsilon < \frac{1}{2}$.

Corollary 13.3 analogously holds for $\frac{1}{2}$-WWKL$^\Delta$ instead of $\frac{1}{2}$-WWKL$'$. Since LPO $\leq_W C_\mathbb{N} \leq_W$ WWKL$^\Delta$ by Theorem 11.4, we obtain the following.

Corollary 13.4. $\frac{1}{2}$-WWKL$^\Delta <_W$ WWKL$^\Delta$.

This shows that Theorem 11.4 can in general not be improved to the statement that WWKL$^\Delta$ is equivalent to $(\varepsilon$-WWKL$)^\Delta$. Since LPO $\leq_W$ WWKL$^\Delta$ is single-valued, we also get the following corollary.

Corollary 13.5. WWKL$^\Delta \not\leq_W \frac{1}{2}$-WWKL$^{(n)}$ *for all $n \in \mathbb{N}$.*

14. Probabilistic degrees

In this section we would like to capture the non-uniform content of probabilistic computability. Intuitively, we want to call a Weihrauch degree *probabilistic*, if it can be computed with some random advice, irrespectively of any failure recognition mechanisms that Las Vegas machines come equipped with. A suitable notion of random advice has already been introduced and studied by the first author and Arno Pauly in [14]. We repeat the definition for our setting.

Definition 14.1 *(Probabilistic degrees).* Let (X, δ_X), (Y, δ_Y) be represented spaces. A multi-valued function $f :\subseteq X \rightrightarrows Y$ is called *probabilistic*, if there exists a computable function $F :\subseteq \mathbb{N}^\mathbb{N} \times 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\mu(\{r \in 2^\mathbb{N} : \delta_Y F(p, r) \in f \delta_X(p)\}) > 0$ for all $p \in \mathrm{dom}(f \delta_X)$. A Weihrauch degree is called *probabilistic*, if it has a probabilistic member.

We emphasize that the condition in this definition implies that the sets $A_p := \{r \in 2^\mathbb{N} : \delta_Y F(p, r) \in f \delta_X(p)\}$ have to be measurable, but they are not required to be closed and they do not need to depend on p in any uniform way (in contrast to the sets S_p in Definition 3.1). It follows from results below that Cantor space $2^\mathbb{N}$ could be equivalently replaced by Baire space $\mathbb{N}^\mathbb{N}$ in the above definition. The following characterization of probabilistic degrees follows from [14, Theorem 11]. We also give a direct proof.

Proposition 14.2 *(Probabilistic degrees). A multi-valued function f on represented spaces is probabilistic, if and only if $f \leq_W g$ for some $g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows 2^\mathbb{N}$ such that $\mu_{2^\mathbb{N}}(g(p)) > 0$ for all $p \in \mathrm{dom}(g)$.*

Proof. Let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function on represented spaces (X, δ_X) and (Y, δ_Y) and let $g :\subseteq \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}$ be as stated above. Let $f \leq_W g$ be witnessed by computable H, K. We define $A_p := gK(p)$ for all $p \in D := \mathrm{dom}(f \delta_X)$. Without loss of generality, we can assume that H has minimal domain, i.e., $\mathrm{dom}(H) = \{\langle p, r \rangle : p \in D, r \in A_p\}$. We define $F :\subseteq \mathbb{N}^\mathbb{N} \times 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $F(p, r) := H\langle p, r \rangle$. Then F is computable and by assumption we have $\mu_{2^\mathbb{N}}(A_p) > 0$ and $\delta_Y F(p, r) = \delta_Y H\langle p, r \rangle \in f \delta_X(p)$ for all $p \in D$ and $r \in A_p$. Hence, f is probabilistic. If, on the other hand, f is probabilistic, then there is a computable $F :\subseteq \mathbb{N}^\mathbb{N} \times 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $A_p := \{r \in 2^\mathbb{N} : \delta_Y F(p, r) \in f \delta_X(p)\}$ satisfies $\mu_{2^\mathbb{N}}(A_p) > 0$ for all $p \in D := \mathrm{dom}(f \delta_X)$. If we define $g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows 2^\mathbb{N}$ by $g(p) := A_p$ for all $p \in D$, then we obtain $f \leq_W g$. $\square$

It is clear that it follows from this proposition that probabilistic degrees are closed downwards with respect to Weihrauch reduction.

Proposition 14.3. *If $f \leq_W g$ and g is probabilistic, then f is probabilistic.*

The case of Baire space can be reduced to Cantor space in the non-uniform setting due to the following lemma.

Lemma 14.4 *(Embedding of Baire into Cantor space). The following map is a computable embedding with a measure preserving inverse:*

$$\iota : \mathbb{N}^\mathbb{N} \to 2^\mathbb{N}, p \mapsto 1^{p(0)} 01^{p(1)} 01^{p(2)} \ldots.$$

Proof. It is clear that ι and its partial inverse are computable. For $w \in \mathbb{N}^*$ we obtain

$$\mu_{2^{\mathbb{N}}}(\iota(w\mathbb{N}^{\mathbb{N}})) = 2^{-\sum_{i=0}^{|w|-1}(w(i)+1)} = \prod_{i=0}^{|w|-1} 2^{-w(i)-1} = \mu_{\mathbb{N}^{\mathbb{N}}}(w\mathbb{N}^{\mathbb{N}}).$$

By Lemma 2.8 this proves that $\mu_{2^{\mathbb{N}}}(\iota(A)) = \mu_{\mathbb{N}^{\mathbb{N}}}(A)$ holds for all measurable $A \subseteq \mathbb{N}^{\mathbb{N}}$.　□

We point out that this embedding does not have a closed range and it does not preserve closedness. Hence it cannot be used in a uniform setting. The following lemma provides a fully uniform reduction in the opposite direction for sets. We use the *signum function* $\mathrm{sgn} : \mathbb{N} \to \mathbb{N}$, defined by $\mathrm{sgn}(0) := 0$ and $\mathrm{sgn}(n+1) = 1$ for all $n \in \mathbb{N}$ and its extension $\mathrm{sgn} : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$, defined by $\mathrm{sgn}(p)(n) := \mathrm{sgn}(p(n))$ for all $p \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

Lemma 14.5 *(Signum). The map*

$$J : 2^{(2^{\mathbb{N}})} \to 2^{(\mathbb{N}^{\mathbb{N}})}, A \mapsto \mathrm{sgn}^{-1}(A)$$

has the property that $\mu_{\mathbb{N}^{\mathbb{N}}}(J(A)) = \mu_{2^{\mathbb{N}}}(A)$ for all measurable $A \subseteq 2^{\mathbb{N}}$ and its restriction $J : \mathcal{A}_-(2^{\mathbb{N}}) \to \mathcal{A}_-(\mathbb{N}^{\mathbb{N}})$ to closed sets is computable.

Proof. Since $\mathrm{sgn} : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ is computable, it follows that J maps measurable sets to measurable sets and closed sets to closed sets and that the restriction of J to closed sets is computable. The claim on the measure can be proved by induction. We claim that $\mu_{\mathbb{N}^{\mathbb{N}}}(J(w2^{\mathbb{N}})) = \mu_{2^{\mathbb{N}}}(w\mathbb{N}^{\mathbb{N}})$ holds for all $w \in \{0,1\}^*$. This is clear for the empty word w. Suppose it holds for a given word w. Then we obtain

$$\mu_{\mathbb{N}^{\mathbb{N}}}(J(1w2^{\mathbb{N}})) = \sum_{i=1}^{\infty} 2^{-i-1} \mu_{\mathbb{N}^{\mathbb{N}}}(J(w2^{\mathbb{N}})) = \frac{1}{2}\mu_{2^{\mathbb{N}}}(w2^{\mathbb{N}}) = \mu_{2^{\mathbb{N}}}(1w2^{\mathbb{N}})$$

and likewise $\mu_{\mathbb{N}^{\mathbb{N}}}(J(0w2^{\mathbb{N}})) = \mu_{2^{\mathbb{N}}}(0w2^{\mathbb{N}})$. Hence the claim follows by structural induction. This implies that $\mu_{\mathbb{N}^{\mathbb{N}}}(J(A)) = \mu_{2^{\mathbb{N}}}(A)$ for all measurable $A \subseteq 2^{\mathbb{N}}$ by Lemma 2.8.　□

Lemma 14.4 and 14.5 show that we could equivalently use functions $g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ in Proposition 14.2. The following result, which follows from Proposition 14.2 shows that being probabilistic is a necessary criterion for being probabilistically computable in any sense that we consider here.

Theorem 14.6 *(Las Vegas computability and probabilistic degrees). If there is a g with $f \leq_{\mathrm{W}} \mathrm{PC}_{\mathbb{N}^{\mathbb{N}}} \circ g$, then f is probabilistic.*

By Lemma 14.5 and due to the fact that the signum function $\mathrm{sgn} : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ is computable, we obtain $\mathrm{PC}_{2^{\mathbb{N}}} \leq_{\mathrm{sW}} \mathrm{PC}_{\mathbb{N}^{\mathbb{N}}}$. By a slight variant of this result we also obtain $\mathrm{PC}_{\mathbb{N} \times 2^{\mathbb{N}}} \leq_{\mathrm{sW}} \mathrm{PC}_{\mathbb{N}^{\mathbb{N}}}$ and hence

$$\mathrm{PC}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathrm{PC}_{\mathbb{N} \times 2^{\mathbb{N}}} \leq_{\mathrm{sW}} \mathrm{PC}_{\mathbb{N}^{\mathbb{N}}}.$$

Altogether, we get the following corollary.

Corollary 14.7 *(Probabilistic degrees). If $f \leq_{\mathrm{W}} \mathrm{PC}_R^{(n)}$ for some $n \in \mathbb{N}$ and R is among $\mathbb{N}, 2^{\mathbb{N}}, \mathbb{N} \times 2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$, then f is probabilistic.*

The core observation in this context is that Weak Kőnig's lemma is not probabilistic [14, Theorem 20].

Proposition 14.8. WKL *is not probabilistic.*

The proof is essentially based on an earlier result of Jockusch and Soare (see [29, Theorem 5.3]) which shows that the set of those points in Cantor space from which one can compute a separating set for two given disjoint c.e. sets that are computably inseparable has measure 0. Since the separation problem is equivalent to Weak Kőnig's Lemma by [23, Theorem 6.7], one obtains Proposition 14.8.

As a consequence it follows with Proposition 14.3 that everything above WKL is also not probabilistic. This applies, in particular, to the limit map lim. Corollary 14.7 and Proposition 14.8 yield the following.

Corollary 14.9. WKL $\not\leq_{\mathrm{W}} \mathrm{PC}_{\mathbb{N}^{\mathbb{N}}}^{(n)}$ *and, in particular,* WKL $\not\leq_{\mathrm{W}}$ WWKL$^{(n)}$ *for every $n \in \mathbb{N}$.*

An important classical result that yields further insights into non-uniform randomized computations is the Theorem of Sacks (see [42] and for recent presentations see [36, Theorem 5.1.12], [20, Corollary 8.12.2]).

Theorem 14.10 *(Sacks 1963 [42]). Let $A \subseteq 2^{\mathbb{N}}$ be a set such that $\mu(A) > 0$ and let $q \in \mathbb{N}^{\mathbb{N}}$ be such that $q \leq_T r$ for every $r \in A$. Then q is computable.*

In a certain sense Theorem 13.2 captures the uniform content of the Theorem of Sacks. An early predecessor of the Theorem of Sacks is the Theorem of de Leeuw, Moore, Shannon and Shapiro [18] that makes a corresponding statement with respect to c.e. sets (see also [20, Theorem 8.12.1]).

Theorem 14.11 *(de Leeuw, Moore, Shannon and Shapiro 1956 [18]). Let $A \subseteq 2^{\mathbb{N}}$ be a set such that $\mu(A) > 0$ and let $B \subseteq \mathbb{N}$ be such that B is c.e. in r for every $r \in A$. Then B is c.e.*

The Theorem of de Leeuw, Moore, Shannon and Shapiro also holds true in a relativized version that states that if $\mu(A) > 0$ and B is c.e. in $p \oplus r$ for every $r \in A$, then B is c.e. in p. Here we use this relativized version in order to prove the following result that shows that certain single-valued probabilistic functions map computable inputs to computable outputs.[11]

In fact, using the notion of representation reducibility introduced by Joseph Miller [32], one can even express a stronger result. We recall that for represented spaces (X, δ_X) and (Y, δ_Y) and $x \in X$ and $y \in Y$ we say that y is *representation reducible* to x, if there exists a partial computable function $g :\subseteq X \to Y$ such that $g(x) = y$. In symbols this is denoted by $y \leq_r x$.[12] We recall that a T_0-space X with countable basis $(U_i)_i$ has a standard representation δ_X given by $\delta_X(p) = x :\iff \{n \in \mathbb{N} : n + 1 \in \text{range}(p)\} = \{n \in \mathbb{N} : x \in U_n\}$ (see [47]). In other words, p is a name of x if it encodes a list of all basic properties U_n of x.

Theorem 14.12 *(Single-valued probabilistic degrees). Let X be a represented space and let Y be a T_0-space with countable base and standard representation. If a single-valued function $f : X \to Y$ is probabilistic, then f maps computable inputs to computable outputs. In fact, even $f(x) \leq_r x$ holds for all $x \in X$ in this case.*

Proof. Let δ_X be the representation of X and let δ_Y be a standard representation of Y with respect to a countable base $(U_i)_i$. Let $f : X \to Y$ be single-valued and probabilistic. Then there is a computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that the sets $A_p := \{r \in 2^{\mathbb{N}} : \delta_Y F(p, r) = f \delta_X(p)\}$ satisfy $\mu(A_p) > 0$ for all $p \in D := \text{dom}(f \delta_X)$. We fix some δ_X-name p of some $x \in X$. Then $F(p, r)$ is a δ_Y-name of $f(x)$, i.e., $F(p, r)$ is an enumeration of $B := \{n \in \mathbb{N} : f(x) \in U_n\}$. Hence, B is c.e. in $p \oplus r$ for every $r \in A_p$. By the relativized version of the Theorem of de Leeuw, Moore, Shannon and Shapiro 14.11 we obtain that B is c.e. in p. This means that $\delta_Y^{-1}(f(x))$ is Muchnik reducible to $\delta_X^{-1}(x)$ and hence $f(x) \leq_r x$ by [32, Corollary 4.3]. In particular, f maps computable inputs to computable outputs. □

15. Zeros of continuous functions with sign changes

In this section we would like to prove that there is no Las Vegas algorithm for computing zeros of continuous functions with changing signs (not even one with additional finitely many mind changes). There is, however, a probabilistic algorithm of second order for finding such zeros.

By IVT we denote the Intermediate Value Theorem, i.e., the problem: given a continuous function $f : [0, 1] \to \mathbb{R}$ that changes its sign (i.e., $f(0) \cdot f(1) < 0$), find an $x \in [0, 1]$ with $f(x) = 0$. In [10, Theorem 6.2] it was proved that $\text{IVT} \equiv_{sW} \text{CC}_{[0,1]}$, where $\text{CC}_{[0,1]}$ denotes choice for closed sets $A \subseteq [0, 1]$ restricted to connected sets (i.e., closed intervals). We now prove that $\text{CC}_{[0,1]}$ cannot be reduced to probabilistic choice $\text{PC}_{[0,1]}$. For that purpose we use a finite extension construction to obtain a name of a connected closed set that is mapped to a set of measure zero (given a potential reduction).

Proposition 15.1. $\text{CC}_{[0,1]} \not\leq_W \text{PC}_{[0,1]}$.

Proof. By Proposition 8.2 it suffices to show $\text{CC}_{[0,1]} \not\leq_W \text{WWKL}$. Let us assume for a contradiction that $\text{CC}_{[0,1]} \leq_W \text{WWKL}$. Then there are computable H, K such that $H\langle \text{id}, FK \rangle$ is a realizer of $\text{CC}_{[0,1]}$ for every realizer F of WWKL. Without loss of generality, we can assume as usual that H, K have minimal domains. Let now p_0 be a name of the unit interval $I_0 := [0, 1]$. Then $K(p_0)$ is a name of a tree T_0 such that the set of infinite paths $A_0 = [T_0]$ satisfies $\mu(A_0) > 0$. Since H is uniformly continuous on the compact set $\{p_0\} \times A_0$, it follows that there is a finite prefix $w_0 \sqsubseteq p_0$ such that $H\langle w_0 \mathbb{N}^{\mathbb{N}}, q \rangle$ produces its output in $[0, 1]$ with precision smaller than $\frac{1}{3}$ (i.e., any two outputs x, x' named in that set satisfy $|x - x'| < \frac{1}{3}$), uniformly for all names q of points in A_0. Since K is continuous, we can also assume that w_0 is long enough such that $K(w_0 \mathbb{N}^{\mathbb{N}})$ only contains names of trees with closed sets $B \subseteq 2^{\mathbb{N}}$ of infinite paths with $\mu(B \setminus A_0) \leq \frac{1}{2}(1 - \mu(A_0))$. Now there are

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[11] We thank Mathieu Hoyrup for pointing out that the Theorem of de Leeuw, Moore, Shannon and Shapiro 14.11 can be used to generalize our initial version of Theorem 14.12, which was originally only formulated for certain metric spaces Y. At the same time its use simplified the proof.

[12] This can also be rephrased such that $\delta_Y^{-1}(y)$ is Medvedev reducible to $\delta_X^{-1}(x)$. Miller proved that for computable metric spaces X and Y this is equivalent to $\delta_Y^{-1}(y)$ being Muchnik reducible to $\delta_X^{-1}(x)$, see [32, Corollary 4.3].

names r_0 and r_1 of the intervals $J_0 := [0, \frac{1}{3}]$ and $J_1 := [\frac{2}{3}, 1]$, respectively such that $w_0 \sqsubseteq r_0$ and $w_0 \sqsubseteq r_1$. Hence $K(r_0)$ and $K(r_1)$ are names of infinite trees S_0 and S_1 with sets of infinite paths $B_0 = [S_0]$ and $B_1 = [S_1]$, respectively. We claim that $A_0 \cap B_0 \cap B_1 = \emptyset$. For if $q \in A_0 \cap B_0 \cap B_1$, then $H\langle r_0, q \rangle$ and $H\langle r_1, q \rangle$ have to be names of points $x_0, x_1 \in [0, 1]$, respectively with $|x_0 - x_1| < \frac{1}{3}$ according to the choice of w_0. On the other hand, there is a realizer F of WWKL with $F(r_0) = F(r_1) = q$ and thus $x_0 \in J_0 = [0, \frac{1}{3}]$ and $x_1 \in J_1 := [\frac{2}{3}, 1]$, which is impossible. Hence $A_0 \cap B_0 \cap B_1 = \emptyset$ and there is an $i \in \{0, 1\}$ with $\mu(B_i \cap A_0) \leq \frac{1}{2}\mu(A_0)$ and we obtain

$$\mu(B_i) = \mu(B_i \cap A_0) + \mu(B_i \setminus A_0) \leq \frac{1}{2}\mu(A_0) + \frac{1}{2}(1 - \mu(A_0)) = \frac{1}{2}.$$

We now choose $I_1 := J_i$, $p_1 := r_i$ and $A_1 := B_i$ and we continue the construction inductively. If at stage $n > 0$ the interval I_n with $\operatorname{diam}(I_n) = \frac{1}{3^n}$, the sequence p_n and a set A_n with $\mu(A_n) \leq \frac{1}{2^n}$ have been determined, then we continue as follows. There exists a prefix $w_n \sqsubseteq p_n$ such that H on w_n guarantees precision $\frac{1}{3^{n+1}}$ and K guarantees measure $\mu(B \setminus A_n) \leq \frac{1}{2}(\frac{1}{2^n} - \mu(A_n))$ in an analogous way as above. We select the left and right third J_0 and J_1 of I_n together with corresponding names r_0 and r_1, respectively, such that $w_n \sqsubseteq r_0$ and $w_n \sqsubseteq r_1$. By B_0, B_1 we denote the sets named by $K(r_0), K(r_1)$, respectively. As above, $A_n \cap B_0 \cap B_1 = \emptyset$ and there exists $i \in \{0, 1\}$ such that $\mu(B_i \cap A_n) \leq \frac{1}{2}\mu(A_n)$. Analogously to above, we obtain $\mu(B_i) \leq \frac{1}{2}\mu(A_n) + \frac{1}{2}(\frac{1}{2^n} - \mu(A_n)) = \frac{1}{2^{n+1}}$. For the next stage we choose $I_{n+1} := J_i$, $p_{n+1} := r_i$ and $A_{n+1} := B_i$.

Altogether, this construction yields a strictly increasing sequence w_n of prefixes of names p_n of closed intervals I_n with $\operatorname{diam}(I_n) = \frac{1}{3^n}$ that are mapped to names $K(p_n)$ of closed sets A_n with $\mu(A_n) \leq \frac{1}{2^n}$. These names p_n converge to a name p of a singleton interval $\{x\} \subseteq [0, 1]$ and by continuity of K this name p is mapped to a name $K(p)$ of a tree T with a set $A = [T]$ of infinite paths such that $\mu(A) = 0$. This is a contradiction to the assumption. $\quad\square$

It is easy to see that the inverse reduction is also not possible. This is because $\mathsf{CC}_{[0,1]}$ has some computable outputs for any computable input and $\mathsf{PC}_{[0,1]}$ does not (for instance a universal Martin-Löf test yields examples of co-c.e. closed subsets $A \subseteq [0, 1]$ of positive measure without computable points).

Proposition 15.2. $\mathsf{PC}_{[0,1]} \not\leq_{\mathrm{W}} \mathsf{CC}_{[0,1]}$.

This can also be proved in a topological way that does not refer to computable inputs and outputs and such a proof will follow from Proposition 17.4. Altogether we obtain that connected and probabilistic choice on $[0, 1]$ are incomparable.

Corollary 15.3 (*Connected and probabilistic choice*). $\mathsf{CC}_{[0,1]} |_{\mathrm{W}} \mathsf{PC}_{[0,1]}$.

This means, in particular, that there is no Las Vegas algorithm that computes zeros of continuous functions with changing sign. Next we want to show that it does not help to have finitely many mind changes additionally to a Las Vegas algorithm or, in other words, there is also no Las Vegas algorithm over the probability space $\mathbb{N} \times 2^{\mathbb{N}}$. This follows from Corollary 15.3 using the following choice elimination principle that was proved in [31] by Le Roux and Pauly and is based on the Baire category technique introduced in [10].

Theorem 15.4 (*Discrete choice elimination*). $f \leq_{\mathrm{W}} g * \mathsf{C}_{\mathbb{N}}$ implies $f \leq_{\mathrm{W}} g$ for total fractals f.

We recall that a *fractal* f is a multi-valued function on represented spaces such that there exists a $g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ with non-empty domain and such that $g|_A \equiv_{\mathrm{W}} f$ for every clopen $A \subseteq \mathbb{N}^{\mathbb{N}}$ for which $A \cap \operatorname{dom}(g)$ is non-empty. Moreover, f is called a *total fractal* if g can be chosen to be total. It is implicit in the proof of [10, Proposition 4.9] that $\mathsf{CC}_{[0,1]}$ is a total fractal and we include the proof here for completeness.

Lemma 15.5. $\mathsf{CC}_{[0,1]}$ *is a total fractal.*

Proof. In [10, Proposition 3.6] it was proved that $\mathsf{CC}_{[0,1]} \equiv_{\mathrm{W}} \mathsf{B}_I$, where

$$\mathsf{B}_I :\subseteq \mathbb{R}_< \times \mathbb{R}_> \rightrightarrows \mathbb{R}, \ (a, b) \mapsto [a, b]$$

is the boundedness principle that maps a left real number a and a right real number b with $a \leq b$ to the closed interval $[a, b]$, i.e., $\operatorname{dom}(\mathsf{B}_I) = \{(a, b) \in \mathbb{R}_< \times \mathbb{R}_> : a \leq b\}$. It is easy to see that $\operatorname{dom}(\mathsf{B}_I)$ is co-c.e. closed. We assume that $\mathbb{R}_<$ and $\mathbb{R}_>$ are represented by $\rho_<$ and $\rho_>$, which are the representations of reals by strictly increasing and decreasing sequences of rational numbers, respectively. We represent $\mathbb{R}$ by the usual Cauchy representation ρ. Let $G :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be the function that maps every pair $\langle p, q \rangle \in \mathbb{N}^{\mathbb{N}}$ to all names of r with $\rho_<(p) \leq \rho(r) \leq \rho_>(q)$. By definition this G is partial, but since the domain is co-c.e. closed, one can easily extend it to an equivalent total $F :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$: as soon as a prefix $\langle p|_{n+1}, q|_{n+1} \rangle$ of the input is inconsistent, since the represented sequences of rational numbers are either not increasing in case of p or not decreasing in case of q or since the rational number given by $p(n)$ is not smaller than the one given by $q(n)$, F just

maps $\langle p, q \rangle$ to the value $G\langle p', q' \rangle$ where p' and q' are some canonical consistent extensions of $p|_n$ and $q|_n$, respectively. This guarantees that $F \equiv_W G \equiv_W B_I \equiv_W CC_{[0,1]}$. Moreover, it is easy to see that $F|_A \equiv_W F$ for every non-empty clopen $A \subseteq \mathbb{N}^{\mathbb{N}}$. It suffices to consider A of the form $A = w\mathbb{N}^{\mathbb{N}}$ with $w \in \mathbb{N}^*$. Given a prefix w of a $[\rho_<, \rho_>]$-name of an interval $[a, b]$, the interval is only described by w up to some rational numbers c, d with $c < a \leq b < d$. Now any other given interval $[a', b']$ can be mapped by a computable affine transformation $T : \mathbb{R} \to \mathbb{R}$ to an interval $[T(a'), T(b')]$ with $c < T(a') \leq T(b') < d$ and given a point $x \in [T(a'), T(b')]$ one can easily recover a point $T^{-1}(x) \in [a', b']$. This proves $F_{w\mathbb{N}^{\mathbb{N}}} \equiv_W F$. Altogether, this shows that $CC_{[0,1]}$ is a total fractal. $\square$

Using Corollary 15.3 we obtain the following result.

Theorem 15.6. $CC_{[0,1]} \,|_W\, PC_{\mathbb{R}}$.

Proof. Since $PC_{[0,1]} \leq_W PC_{\mathbb{R}}$ we obtain $PC_{\mathbb{R}} \not\leq_W CC_{[0,1]}$ by Proposition 15.2. Now we prove $CC_{[0,1]} \not\leq_W PC_{\mathbb{R}}$. For one we have $PC_{\mathbb{R}} \equiv_W PC_{2^{\mathbb{N}}} * C_{\mathbb{N}} \equiv_W PC_{[0,1]} * C_{\mathbb{N}}$ due to Corollaries 6.4, 7.3 and Proposition 7.4. If we assume for a contradiction $CC_{[0,1]} \leq_W PC_{\mathbb{R}}$, then $CC_{[0,1]} \leq_W PC_{[0,1]} * C_{\mathbb{N}}$ follows and hence $CC_{[0,1]} \leq_W PC_{[0,1]}$ follows with Theorem 15.4 on discrete choice elimination since $CC_{[0,1]}$ is a total fractal by Lemma 15.5. This contradicts Proposition 15.1. $\square$

On the other hand, it is easy to see that there is a probabilistic algorithm of second order that can compute zeros of functions with sign change. We first give an intuitive description of this algorithm before we formulate the result using choice:

1. A continuous function $f : [0, 1] \to \mathbb{R}$ with $f(0) \cdot f(1) < 0$ is given as input.
2. Guess a binary sequence or, equivalently, a bit $b \in \{0, 1\}$ and a point $x \in [0, 1]$.
3. Interpret the guess $b = 1$ such that the zero set $f^{-1}\{0\}$ contains no open intervals and use the trisection method to compute a zero $z \in [0, 1]$ with $f(z) = 0$ in this case (disregarding x).
4. Interpret the guess $b = 0$ such that the zero set $f^{-1}\{0\}$ does contain an open interval and check whether $f(x) = 0$ in this case. Stop after finite time if this test fails and output x otherwise.

This is not a Las Vegas algorithm, since the failure of the algorithm in case of $b = 0$ cannot be recognized computably. However, the algorithm succeeds with a positive probability in any case since x is disregarded in case $b = 0$ and there is a set of successful guesses of positive measure in case $b = 1$. Additionally, even in case $b = 0$ it is not too difficult to recognize failure, even though it is not computable. We prove that this algorithm is probabilistic of second order and for simplicity we express this using choice again.

Proposition 15.7. $CC_{[0,1]} \leq_W PC'_{[0,1]}$.

Proof. By Proposition 8.2 it suffices to prove $CC_{[0,1]} \leq_W WWKL'$. Given a name p of a closed interval $I \subseteq [0, 1]$ we compute a sequence $(T_n)_n$ of trees that converges to a tree T with a set $A := [T] \subseteq \{0, 1\}^{\mathbb{N}}$ of infinite paths with $\mu(A) > 0$. Firstly, we can compute a tree $T_I \subseteq 0\{0, 1\}^*$ such that the binary representation ρ_2 maps $[T_I]$ to $\frac{1}{2}I$. By m_n we denote the measure of the approximation of I that is determined by $p|_n$, the prefix of p of length n. Without loss of generality, we can assume that we can compute m_n as a positive rational number. Now we can compute a sequence $(T_n)_n$ of trees

$$T_n := T_I \cup 1\{0, 1\}^k$$

where k is maximal with $m_n < 2^{-k}$. Then the sequence $(T_n)_n$ clearly converges to a tree T with

$$[T] = \begin{cases} [T_I] & \text{if } I \text{ is not a singleton} \\ [T_I] \cup 1\{0, 1\}^{\mathbb{N}} & \text{otherwise} \end{cases}$$

Let us denote by K the computable map that maps p to a sequence of names of the trees $(T_n)_n$. Given an infinite path $q \in [T]$ and the original name p we can reconstruct a point $x \in I$ as follows: if the path q starts with $q(0) = 0$, then we compute $x = 2 \cdot \rho_2(q)$ and if the path q starts with $q(0) = 1$, then we know that I must be a singleton and we use p to compute x with $I = \{x\}$. This describes a computable function H such that $H\langle id, FK \rangle$ is a realizer of $CC_{[0,1]}$ whenever F is a realizer of $WWKL'$. $\square$

One can also ask whether Proposition 15.7 can be strengthened to the statement $CC_{[0,1]} \leq_{sW} PC'_{[0,1]}$. However, for mere cardinality reasons this is not possible. Since there cannot be an uncountable number of pairwise disjoint sets $A \subseteq [0, 1]$ of positive measure it follows that $\#PC^{(n)}_{[0,1]} = |\mathbb{N}|$ (see Proposition 3.8), while obviously $\#CC_{[0,1]} = |\mathbb{R}|$. We obtain the following consequence of Proposition 3.6.

Corollary 15.8. $CC_{[0,1]} \not\leq_{sW} PC^{(n)}_{[0,1]}$ *for all* $n \in \mathbb{N}$.

Another interpretation of our results is that Weak Weak Kőnig's Lemma does not compute the Intermediate Value Theorem (nor the other way around), but the jump of Weak Weak Kőnig's Lemma does.

Corollary 15.9 *(Intermediate Value Theorem and Weak Weak Kőnig's Lemma).* IVT $|_W$ WWKL *and* IVT $\leq_W$ WWKL$'$.

We note that the results of this section also yield another proof of the fact that WKL $\not\leq_W$ WWKL (see Corollary 14.9), since IVT $\leq_W$ WKL.

16. Robust division

In this section we would like to prove that there is a Las Vegas algorithm for robust division

$$\text{RDIV} : [0, 1] \times [0, 1] \rightrightarrows [0, 1], (x, y) \mapsto \begin{cases} \{\frac{x}{\max(x,y)}\} & \text{if } y \neq 0 \\ [0, 1] & \text{otherwise} \end{cases}$$

Robust division can be used for solving linear equations and inequalities in compact domains and it has been defined and studied by Arno Pauly [38,40]. For instance it is easy to see that robust division can be used to find solutions of linear equations $ax = b$ for $a, b \in \mathbb{R}$ in a compact domain. Robust division is related to all-or-unique choice that we define next.

Definition 16.1 *(All-or-unique choice).* Let X be a represented space. Then by AUC$_X$ we denote the *all-or-unique choice* operation of X, which is C$_X$ restricted to

$$\text{dom}(\text{AUC}_X) := \{X\} \cup \{\{x\} : x \in X\},$$

i.e., the entire set or singletons.

One readily verifies the following result (see [40, Proposition 5.2.1.3]).

Proposition 16.2 *(Robust division).* RDIV $\equiv_{sW}$ AUC$_{[0,1]}$.

Now we will further study the relation of all-or-unique choice AUC$_{[0,1]}$ to probabilistic choice.

Theorem 16.3 *(All-or-unique and probabilistic choice).* AUC$_{[0,1]} <_W$ PCC$_{[0,1]}$.

Proof. Given a name p of a set $A \subseteq [0, 1]$, which is either the whole interval or a singleton, we compute the name q of a proper closed interval $I \subseteq [0, 1]$ as follows: as long as p does not contain any negative information (i.e., p is still compatible with a name of the full interval), we just copy p to q. If the first negative information in p appears at position n, then we continue to read p until we know the singleton $A = \{x\}$ given by it up to precision 2^{-n}. At that point we have rational numbers a, b with $x \in [a, b]$ and $b - a \leq 2^{-n}$ and we just extend the output q to a name of the interval $I = [a, b]$. This describes a computable function K that maps p to q. The output produced by K is a name for an interval of the form $I = [a, b]$, where $I = [0, 1]$ if and only if p is a name of $[0, 1]$. Given a name of a point $y \in I$ and the original input p, we can recover a point $x \in A$ as follows: we read p and produce an approximation of y up to precision 2^{-n} as long as $p|_n$ does not contain any negative information. In the moment where we find some negative information in p, we stop using y and we just compute an output x with $A = \{x\}$ by inspection of p. Note that this is always possible, since the approximation of y that we have produced so far can always be extended to x. This describes a computable function H such that $H\langle \text{id}, FK \rangle$ is a realizer of AUC$_{[0,1]}$ whenever F is a realizer of PCC$_{[0,1]}$.

It is clear that PCC$_{[0,1]} \not\leq_W$ AUC$_{[0,1]}$: while AUC$_{[0,1]} \leq_W$ LPO can be computed with one mind change, C$_2^* \leq_W$ PCC$_{[0,1]}$ (which holds by [13, Proposition 7.2]) implies that PCC$_{[0,1]}$ cannot be computed with any finite number of mind changes. □

In other words, we have proved that robust division can be reduced to Weak Weak Kőnig's Lemma.

Corollary 16.4. RDIV $<_W$ WWKL.

This means that there is a Las Vegas algorithm for robust division. Now one can ask whether there is a Las Vegas algorithm for robust division with a fixed positive success probability. We will show that this is not the case and we start with an observation that follows from Lemma 10.7.

Corollary 16.5. AUC$_{[0,1]}$ *is join-irreducible.*

Now we can conclude that no fixed positive success probability is sufficient for robust division.

Theorem 16.6. $\mathsf{AUC}_{[0,1]} \not\leq_W *\text{-WWKL}$.

Proof. Let us assume for a contradiction that $\mathsf{AUC}_{[0,1]} \leq_W *\text{-WWKL}$. Since $\mathsf{AUC}_{[0,1]}$ is join-irreducible by Corollary 16.5, we obtain that there exists an $n \in \mathbb{N}$ with $\mathsf{AUC}_{[0,1]} \leq_W 2^{-n}\text{-WWKL}$. Let H, K be computable functions such that $H\langle \mathrm{id}, FK\rangle$ is a realizer of $\mathsf{AUC}_{[0,1]}$ for every realizer F of 2^{-n}-WWKL. Let p be a name of $[0,1]$, which is mapped to a name $K(p)$ of a tree T with a set $A = [T]$ of infinite paths such that $\mu(A) > 2^{-n}$. Now we consider $2^n + 1$ distinct points $x_0, x_1, \ldots, x_{2^n} \in [0,1]$ and let $m \in \mathbb{N}$ be such that $2^{-m} < \min\{|x_i - x_j| : i, j \in \{0, \ldots, 2^n\}, i \neq j\}$. Since H is uniformly continuous on the compact set $\{p\} \times A$, there is some prefix $w \sqsubseteq p$ such that all names in $H\langle w\mathbb{N}^{\mathbb{N}}, q\rangle$ determine their results with precision better than 2^{-m-1} for all $q \in A$. Since K is continuous, we can assume that w is long enough such that $K(w\mathbb{N}^{\mathbb{N}})$ contains only names of trees S with sets $B = [S]$ of infinite paths such that $\mu(B \setminus A) \leq 2^{-n}(1 - \mu(A))$. Now we consider extensions $p_0, p_1, \ldots, p_{2^n}$ of w which are names of the singletons $\{x_0\}, \{x_1\}, \ldots, \{x_{2^n}\}$, respectively. Then $K(p_0), \ldots, K(p_{2^n})$ are names of trees $T_0, \ldots, T_{2^n}$ with sets of infinite paths $A_0, \ldots, A_{2^n}$, respectively. Since $\mu(A_i) > 2^{-n}$ for all $i = 0, \ldots, 2^n$, it is clear that there are distinct $k, j \in \{0, \ldots, 2^n\}$ such that $A_j \cap A_k \cap A \neq \emptyset$, since otherwise we obtain for some $i = 0, \ldots, 2^n$

$$\mu(A_i) = \mu(A_i \cap A) + \mu(A_i \setminus A) \leq \frac{1}{2^n + 1}\mu(A) + 2^{-n}(1 - \mu(A)) < 2^{-n}$$

in contradiction to the assumption. Let now $q \in A_j \cap A_k \cap A$. Then $H\langle p_j, q\rangle$ is a name of x_j and $H\langle p_k, q\rangle$ is a name of x_k and hence $|x_j - x_k| < 2^{-m}$ according to the choice of w. This is in contradiction to the definition of m. □

Together with Theorem 16.3 this gives us an alternative proof of Proposition 10.9. Theorem 16.6 also implies $\mathsf{AUC}_{[0,1]} \not\leq_W \mathsf{C}_2^*$, which was proved in a different way by Arno Pauly in [40, Theorem 5.2.1.4].

Similarly as before for Corollary 15.8 we can ask whether Theorem 16.3 can be strengthened to the statement $\mathsf{AUC}_{[0,1]} \leq_{sW} \mathsf{PCC}_{[0,1]}$. However, again for mere cardinality reasons this is not possible. As above we note that $\#\mathsf{PC}_{[0,1]}^{(n)} = |\mathbb{N}|$, while obviously $\#\mathsf{AUC}_{[0,1]} = |\mathbb{R}|$. We obtain the following consequence of Proposition 3.6.

Corollary 16.7. $\mathsf{AUC}_{[0,1]} \not\leq_{sW} \mathsf{PC}_{[0,1]}^{(n)}$ for all $n \in \mathbb{N}$.

17. Nash equilibria

In this section we would like to prove (based on results of Arno Pauly) that there is a Las Vegas algorithm to compute Nash equilibria. We recall from [38,37] that a pair $A, B \in \mathbb{R}^{m \times n}$ of $m \times n$-matrices is called a *bi-matrix game*. Any vector $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ with $s_i \geq 0$ for all $i = 1, \ldots, m$ and $\sum_{j=1}^{m} s_j = 1$ is called a *mixed strategy*. By S^m we denote the set of these mixed strategies of dimension m. Then a *Nash equilibrium* is a pair $(x, y) \in S^n \times S^m$ of strategies such that

1. $x^{\mathrm{T}} A y \geq w^{\mathrm{T}} A y$ for all $w \in S^n$ and
2. $x^{\mathrm{T}} B y \geq x^{\mathrm{T}} B z$ for all $z \in S^m$.

John F. Nash [35] proved that for any bi-matrix game there exists a Nash equilibrium. By $\mathsf{NASH}_{n,m} : \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ we denote the corresponding problem

$$\mathsf{NASH}_{n,m}(A, B) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : (x, y) \text{ is a Nash equilibrium for } (A, B)\}$$

of finding a Nash-equilibrium for an $m \times n$ bi-matrix game and by $\mathsf{NASH} := \bigsqcup_{n,m \in \mathbb{N}} \mathsf{NASH}_{n,m}$ we denote the coproduct of all such games for finite $m, n \in \mathbb{N}$. By [38, Theorem 28] it follows that NASH is strongly idempotent, i.e., $\mathsf{NASH} \times \mathsf{NASH} \leq_{sW} \mathsf{NASH}$. Like in [38, Theorem 24] we will use a variant of the well-known *matching pennies game* (see [37]), which has a unique Nash equilibrium, in order to prove the following result.

Lemma 17.1. NASH *is a cylinder.*

Proof. Since NASH is strongly idempotent and $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \equiv_{sW} \mathrm{id}_{[0,1]}$, it suffices to prove $\mathrm{id}_{[0,1]} \leq_{sW} \mathsf{NASH}$. Given some input $a \in [0, 1]$, we can compute the bi-matrix game (A, B) given by

$$A := \begin{pmatrix} 1 & -1 \\ -1 & a \end{pmatrix}, \ B := \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We claim that the unique Nash equilibrium (x, y) of the game (A, B) is given by $x = (x_1, x_2) := (\frac{1}{2}, \frac{1}{2})$ and $y := (y_1, y_2)$ with $y_1 := \frac{1+a}{3+a}$ and $y_2 := 1 - y_1$. This yields the desired reduction, since a can be recovered from the unique output $(x, y) = \mathsf{NASH}_{2,2}(A, B)$ by $a = \frac{2y_1}{1-y_1} - 1$. It remains to prove the claim, which amounts to check that the above pair $(x, y) \in S^2 \times S^2$ is the unique pair that satisfies

1. $(y_1 - y_2)x_1 + (ay_2 - y_1)x_2 \geq (y_1 - y_2)w_1 + (ay_2 - y_1)w_2$ and
2. $(x_2 - x_1)y_1 + (x_1 - x_2)y_2 \geq (x_2 - x_1)z_1 + (x_1 - x_2)z_2$

for all $w = (w_1, w_2), z = (z_1, z_2) \in S^2$. If we consider the case $y_1, y_2 \notin \{0, 1\}$, then the only way to satisfy the second constraint (2) is by balancing both addends, i.e., $x_2 - x_1 = x_1 - x_2$, which yields $x_1 = x_2 = \frac{1}{2}$. This is because an unbalanced pair $x_2 - x_1 \neq x_1 - x_2$ would always allow to increase the weight y_1 or y_2 of the larger component of the pair, which is possible if both weights y_1 and y_2 are smaller than 1. The corresponding modified pair of weights z_1, z_2 would then violate (2). Likewise, if we consider the case $x_1, x_2 \notin \{0, 1\}$ balancing the addends in the first constraint (1) yields $y_1 - y_2 = ay_2 - y_1$, which implies $y_1 = \frac{1+a}{3+a}$ after the substitution $y_2 = 1 - y_1$. Now we still need to consider the case where we allow $x_1, x_2, y_1, y_2 \in \{0, 1\}$. For instance, if $y_2 = 1$, then $y_1 = 0$ and the second constraint (2) can only be satisfied if $x_1 - x_2 \geq x_2 - x_1$, which means if $x_1 \geq x_2$. In this case the first constraint can only be satisfied if $-1 = y_1 - y_2 \geq ay_2 - y_1 = a$, which is impossible for $a \in [0, 1]$. Likewise, the other cases with values in $\{0, 1\}$ can be ruled out. What remains is the above unique Nash equilibrium (x, y). □

Arno Pauly proved that the problem NASH is Weihrauch equivalent to the idempotent closure $\mathsf{AUC}^*_{[0,1]}$ of all-or-unique choice on the unit interval (see [38]).

Theorem 17.2 *(Nash equilibria, Pauly 2010 [38]).* $\mathsf{NASH} \equiv_{sW} \mathsf{AUC}^*_{[0,1]}$.

The proof of $\mathsf{NASH} \equiv_W \mathsf{RDIV}^*$ can be found in [38, Corollary 40]. Moreover, the equivalence $\mathsf{RDIV}^* \equiv_{sW} \mathsf{AUC}^*_{[0,1]}$ holds by Proposition 16.2. It is easy to see that $\mathsf{AUC}^*_{[0,1]}$ is a cylinder and hence we obtain by Lemma 17.1 that even strong Weihrauch equivalence holds as stated in the previous result. Since WWKL is idempotent, we obtain the following immediate conclusion of Theorem 16.3.

Corollary 17.3. $\mathsf{NASH} \leq_W \mathsf{WWKL}$.

This means that there is a Las Vegas algorithm for computing Nash equilibria. In terms of RDIV this algorithm is quite involved and can be found in [38], while we have shown how RDIV can be computed in Las Vegas style (see the proof of Theorem 16.3 and Corollary 16.4). An obvious question is whether Corollary 17.3 can be improved to the reduction $\mathsf{NASH} \leq_W \mathsf{PCC}_{[0,1]}$, which would mean that there is a Las Vegas algorithm, whose random guesses can always be organized in a connected interval. We will see that this is not the case and we will even obtain $\mathsf{NASH} \nleq_W \mathsf{CC}_{[0,1]}$. We start with the following result, which improves [13, Proposition 7.1].

Proposition 17.4. $\mathsf{C}_2 \times \mathsf{AUC}_{[0,1]} \nleq_W \mathsf{CC}_{[0,1]}$.

Proof. Let us assume for a contradiction that $\mathsf{C}_2 \times \mathsf{AUC}_{[0,1]} \leq_W \mathsf{CC}_{[0,1]}$. Then there are computable H, K such that $H\langle \mathrm{id}, FK \rangle$ is a realizer of $\mathsf{C}_2 \times \mathsf{AUC}_{[0,1]}$ whenever F is a realizer of $\mathsf{CC}_{[0,1]}$. Without loss of generality, we can assume that we represent $[0, 1]$ with the following signed-digit representation:

$$\rho :\subseteq \{-1, 0, 1\}^{\mathbb{N}} \to [0, 1], p \mapsto \sum_{n=0}^{\infty} p(n)2^{-n}.$$

Let now p be a name of $\{0, 1\} \times [0, 1]$. Then $K(p)$ is the name of an interval I and H is uniformly continuous on the compact set $\{p\} \times N$, where N is the set of all names of points in I (this set is compact, since we are using a signed-digit representation).

(a) Hence there is a number $n \in \mathbb{N}$ such that all points in $H\langle p|_n\mathbb{N}^{\mathbb{N}}, q|_n\mathbb{N}^{\mathbb{N}}\rangle$ with $q \in N$ have a fixed first discrete component in $\{0, 1\}$. Since we use the signed-digit representation ρ for $[0, 1]$, each $q|_n = q(0)...q(n-1)$ determines a point $x \in I$ up to precision 2^{-n+1}, more precisely, if $x, y \in I$ are points with $|x - y| \leq 2^{-n+1}$, then x, y have names r, s, respectively, with $r|_n = s|_n$.

(b) There is also a number $k \geq n$ such that all second components of the points $H\langle p|_k\mathbb{N}^{\mathbb{N}}, q|_k\mathbb{N}^{\mathbb{N}}\rangle$ with $q \in N$ determine values in $[0, 1]$, which are identical up to precision 2^{-n-2}; more precisely, if $r, s \in \pi_2 H\langle p|_k\mathbb{N}^{\mathbb{N}}, q|_k\mathbb{N}^{\mathbb{N}}\rangle$ for some $q \in N$, then r, s are names of points $x, y \in [0, 1]$ with $|x - y| < 2^{-n-2}$.

We now choose 2^{n+2} equi-distant points $x_1, \ldots, x_{2^{n+2}} \in [0, 1]$ including the endpoints $0, 1$. In particular, $|x_i - x_j| > 2^{-n-2}$ for all i, j with $i \neq j$. Let now q_a, q_b be names of the two endpoints of I and let $a, b \in [0, 1]$ be the value of the second components of $H\langle p, q_a \rangle$ and $H\langle p, q_b \rangle$, respectively. We fix some $x \in [0, 1]$. Let $A_x = \{0, 1\} \times \{x\}$. Then due to continuity of K there is a name p_i of A_x for every $i \in \mathbb{N}$ such that $p|_{k+i}$ is a prefix of p_i and $K(p_i)$ is a name of an interval I_i such that $\sup_{y \in I_i} \mathrm{dist}_I(y) < 2^{-i}$ (where $\mathrm{dist}_I(y) := \inf\{|z - y| : z \in I\}$). We note that $p_i \to p$ for $i \to \infty$. Let us assume that $I_i \nsubseteq I^\circ$ for all i (where I° denotes the interior of I). Then there is a sequence of points (y_i) with $y_i \in I_i$ and a corresponding sequence

of names (q_i) such that $q_i \to q_a$ or $q_i \to q_b$ for $i \to \infty$. Without loss of generality, we assume $q_i \to q_a$. Due to continuity of H we obtain

$$H\langle p, q_a \rangle = H\left\langle \lim_{i \to \infty} p_i, \lim_{i \to \infty} q_i \right\rangle = \lim_{i \to \infty} H\langle p_i, q_i \rangle.$$

Since the second component of $H\langle p, q_a \rangle$ is a name for a and the second components of all the $H\langle p_i, q_i \rangle$ are names for x, we obtain $x = a$. Hence, in the general case we have $x \in \{a, b\}$. In other words, if $x \notin \{a, b\}$, then A_x has a name p_x with prefix $p|_k$ such that $K(p_x)$ is a name of an interval I_x with $I_x \subseteq I^\circ$. Among the 2^{n+2} points x_i there are at least 2^{n+1}, which are different from a, b. Let us assume, without loss of generality, that the points $x_1, \ldots, x_{2^{n+1}}$ are all different from a, b. Now we claim that

1. $I_{x_i} \cap I_{x_j} = \emptyset$ for different $i, j \in \{1, \ldots, 2^{n+1}\}$,
2. $\lambda(I_{x_i}) > 2^{-n}$ for all $i \in \{1, \ldots, 2^{n+1}\}$.

Together this is clearly a contradiction since

$$\sum_{i=1}^{2^{n+1}} \lambda(I_{x_i}) > 2^{n+1} 2^{-n} > 1 = \lambda([0, 1]).$$

We first prove (1). Let us assume that $z \in I_{x_i} \cap I_{x_j}$ for $i \neq j$ and q is a name of z. It follows that the second components of $H\langle p_{x_i}, q \rangle$ and $H\langle p_{x_j}, q \rangle$ are names of x_i and x_j, respectively, and since p_{x_i} and p_{x_j} have the prefix $p|_k$ in common, we obtain $|x_i - x_j| < 2^{-n-2}$ in contradiction to the choice of these points x_i, x_j. We now prove (2). Let $i \in \{1, \ldots, 2^{n+1}\}$. Due to continuity of K we can choose some $m \geq k$ such that all intervals J named in $K(p_{x_i}|_m \mathbb{N}^\mathbb{N})$ satisfy $\sup_{y \in J} \text{dist}_{I_{x_i}}(y) < 2^{-n-1}$ and $J \subseteq I$ (the latter is possible since $I_{x_i} \subseteq I^\circ$). Now the two sets $\{0\} \times \{x_i\}$ and $\{1\} \times \{x_i\}$ have names p_0, p_1 that share the common prefix $p_{x_i}|_m$. Let $J_0 \subseteq I$ and $J_1 \subseteq I$ be the intervals named by $K(p_0)$ and $K(p_1)$, respectively and let $x \in J_0$ and $y \in J_1$. If $|x - y| \leq 2^{-n+1}$, then there are names $q, r \in N$ of x, y, respectively, such that $q|_n = r|_n$ and hence $H\langle p_0, q \rangle$ and $H\langle p_1, r \rangle$ must name identical first components, which is a contradiction to the choice of p_0, p_1. Hence we obtain $\inf\{|x - y| : x \in J_0, y \in J_1\} \geq 2^{-n+1}$. But this implies

$$\lambda(I_{x_i}) = \sup\{|x - y| : x, y \in I_{x_i}\} > 2^{-n+1} - 2 \cdot 2^{-n-1} = 2^{-n}.$$

This completes the proof. $\square$

Since $C_2 \times AUC_{[0,1]} \leq_W AUC^*_{[0,1]}$ we obtain the following corollary.

Corollary 17.5. $AUC^*_{[0,1]} \not\leq_W CC_{[0,1]}$.

This means that a method to compute zeros of continuous functions with changing signs cannot help to compute Nash equilibria.

Corollary 17.6. $NASH \not\leq_W IVT$.

We mention that Proposition 17.4 together with $C_2 \times AUC_{[0,1]} \leq_W PC_{[0,1]}$ also yields another proof of Proposition 15.2. Since $C_2 \leq_W AUC_{[0,1]} \leq_W PCC_{[0,1]} \leq_W CC_{[0,1]}$ we also obtain the following corollary of Proposition 17.4.

Corollary 17.7. $AUC_{[0,1]}$, $PCC_{[0,1]}$ and $CC_{[0,1]}$ are not idempotent.

The first fact can also easily be deduced from the number of mind changes required and the latter fact was already proved in [13, Theorem 7.3] in a slightly different way. We mention that the situation also yields another instance of a difference between suprema and products, since we obtain

$$C_2 \sqcup AUC_{[0,1]} <_W C_2 \times AUC_{[0,1]}.$$

In particular, Corollary 17.5 implies that $AUC^*_{[0,1]} \not\leq_W PCC_{[0,1]}$. As a final result in this section we would like to clarify the inverse relation between $PCC_{[0,1]}$ and $AUC^*_{[0,1]}$. The separation can be achieved using the concept of a level (as introduced by Hertling [24,25]), which is preserved downwards by Weihrauch reducibility (i.e., if $f \leq_W g$, then the level of f is less or equal to the level of g). Since $AUC^*_{[0,1]} \leq_W LPO^*$, it follows that $AUC^*_{[0,1]}$ has at most the level of LPO^*, which is ω (the first transfinite ordinal), while $PCC_{[0,1]}$ has no level, since its entire domain consists of points of discontinuity. This implies $PCC_{[0,1]} \not\leq_W LPO^*$ and altogether we obtain the following result.
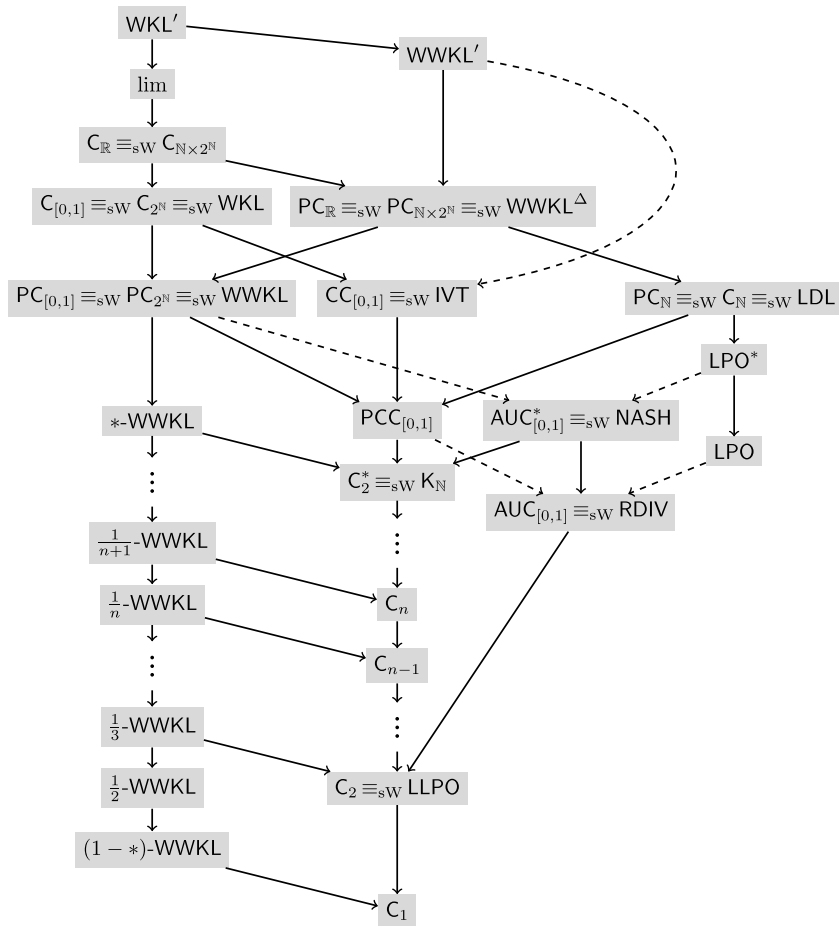
Fig. 1. (Probabilistic choice in the Weihrauch lattice). All solid lines indicate strong Weihrauch reductions $\leq_{sW}$ against the direction of the arrow, i.e., if $f \leq_{sW} g$, then the arrow points from g to f (which corresponds to the direction of logical implication). All dashed lines indicate that we only have ordinary Weihrauch reductions $f \leq_W g$ in those cases. The diagram is complete (with regards to ordinary Weihrauch reducibility) up to transitivity.

Corollary 17.8. $PCC_{[0,1]} \mid_W AUC^*_{[0,1]}$.

In particular, we obtain the following corollary.

Corollary 17.9. $IVT \mid_W NASH$.

18. Conclusions

In Fig. 1 we illustrate the fragment of the Weihrauch lattice that we have studied in this paper.

We emphasize that many separations presented in this paper hold for purely topological reasons. That is, we get analogous results if we replace (strong) Weihrauch reducibility by its topological counterpart that is defined analogously, but with continuous H, K instead of computable H, K. In particular, Propositions 7.5, 15.1, and 17.4, as well as Theorems 10.1 and 16.6, hold analogously for the topological variant of Weihrauch reducibility.

There are numerous structural questions that we have not addressed or answered in our study. We mention some examples:

1. Is $WWKL'$ closed under composition?
2. Or is $WWKL' * WWKL' \equiv_W WWKL''$?
3. Is $WWKL' \leq_W PC_{\mathbb{N}^{\mathbb{N}}}$?

The techniques used to prove $WKL' * WKL' \equiv_W WKL''$ in [12] cannot be directly transferred to the case of WWKL, since the proofs crucially exploit that WKL is a cylinder, which WWKL is not according to Corollary 3.9.

Of course, it would be very interesting to find out whether further concrete problems (besides determining zeros or computing Nash equilibria) admit Las Vegas algorithm or other types of randomized algorithms. In one forthcoming paper we will study the Vitali Covering Theorem and other results from measure theory from this perspective and in a second paper we will investigate the relation between Martin-Löf randomness and Weak Weak Kőnig's Lemma.

Acknowledgments

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