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A formula for the number of labelled trees

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ABSTRACT

Let $L(n, r)$ be the number of labelled trees with n points and r end-points. In this paper it is shown that the number $L(n, r)$ can be obtained from the formula

$$L(n,r) = \binom{n}{n-r} \sum_{i=0}^{n-r-1} (-1)^i \binom{n-r}{i} (n-r-i)^{n-2}.$$

1. Introduction

A tree is a connected graph with no cycles. For example, when the number of points $n = 4$, there are two trees. See Fig. 1.

A labelled tree of n points is a tree whose points are labelled by $1, 2, 3, \ldots, n$. Two labelled trees are isomorphic if their graphs are isomorphic and the corresponding points of the two trees have the same labels. Formal definitions of labelled graphs or trees can be found, for example, in [1] or [2]. Fig. 2 shows all labelled trees with four points. An end-point of a graph is the point whose degree is 1.

According to Cayley, there are n^{n-2} labelled trees of n points. For example, when $n = 4$ there are 2 trees, but if the trees are labelled, then there are 4 possible ways to label the left tree, and 12 ways to label the right tree of Fig. 1. Then, in total, there are $16 = 4^{4-2}$ labelled trees when $n = 4$.

We define $L(n, r)$ as the number of labelled trees of n points with the number of end-points equal to r. For example, $L(4, 3) = 4$, $L(4, 2) = 12$. See Fig. 2.

There are several methods for obtaining the value n^{n-2}. One of the methods is to make a one to one correspondence between a labelled tree of n points and $(n - 2)$-tuple $(a_1, a_2, \ldots, a_{n-2})$, where $a_1, a_2, \ldots, a_{n-2}$ are integers such that $1 \le a_i \le n$. We briefly mention the procedure for constructing $(a_1, a_2, \ldots, a_{n-2})$ from a given labelled tree. First, consider all end-points and delete the end-point whose label, say d_1, has the smallest value among all end-points. Let a_1 be the label of the point that is adjacent to the deleted end-point whose label is d_1. Now, we have $n - 1$ points left. We repeat the procedure until we finally obtain $a_1, a_2, \ldots, a_{n-2}$. So we have a particular $(n-2)$-tuple $(a_1, a_2, \ldots, a_{n-2})$ that corresponds to the given labelled tree. Conversely, there is a procedure for constructing the labelled tree from a given $(n-2)$-tuple. See the details in [1], or [2]. There are n^{n-2} ways to write $(a_1, a_2, \ldots, a_{n-2})$ and so there are n^{n-2} labelled trees with n points.

2. The values of $L(n, r)$

To prove the main result, we need the following lemma.

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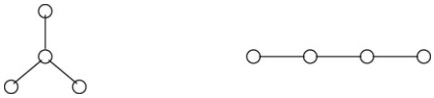

Fig. 1.

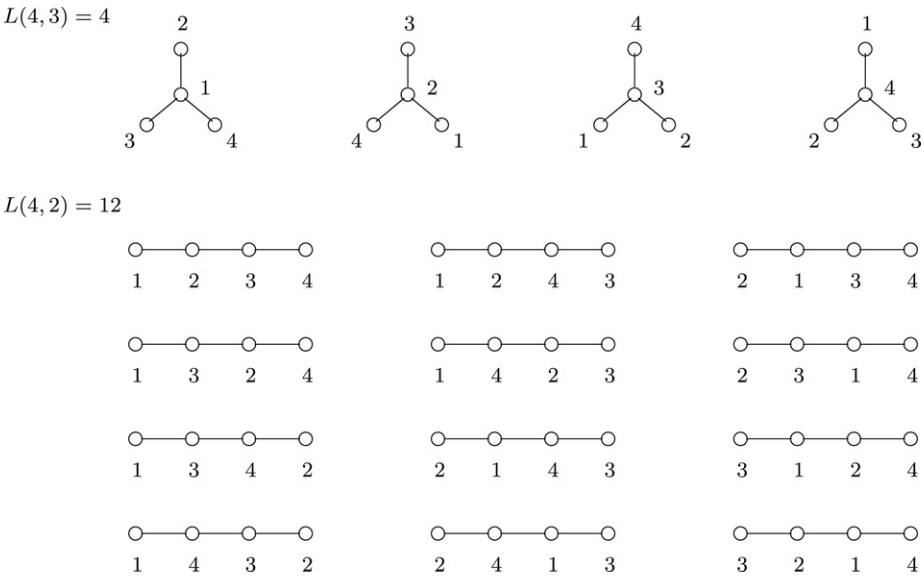

Fig. 2.

Lemma. *Let* $1, 2, 3, \ldots, n$ *be the labels of a labelled tree with* $n \geq 3$ *points that correspond to* $(a_1, a_2, \ldots, a_{n-2})$. *Then* $b_1, b_2, b_3, \ldots, b_k$ *are the labels of the end-points if and only if none of* $b_1, b_2, \ldots, b_k$ *appear in the* $(n-2)$*-tuple* $(a_1, a_2, \ldots, a_{n-2})$.

Proof. We note that, for $n \geq 3$, each of any two end-points can not be adjacent to the other. Next, we show that each of $a_1, a_2, a_3, \ldots, a_{n-2}$ in the $(n-2)$-tuple can not be the label of an end-point. For according to the process of constructing the $(n-2)$-tuple, a_1 is adjacent to a deleted end-point; therefore a_1 can not be the label of an end-point. Also, according to the process, a_2 is adjacent to a deleted point of degree 1; therefore a_2 can not be the label of an end-point. Similarly, each of $a_3, a_4, \ldots, a_{n-2}$ can not be the label of an end-point. Next, we show that any label, say b_j, that does not appear in the $(n-2)$-tuple must be a label of an end-point. Suppose there is a point whose label b_j does not appear in the $(n-2)$-tuple but is not an end-point. Then according to the process of deleting end-points one by one, at some steps, b_j must be adjacent to one of the deleting points and so b_j must be in the $(n-2)$-tuple, and this is a contradiction. Therefore, any b_j that does not appear in the $(n-2)$-tuple must be the label of an end-point. Hence, we have proved the lemma. □

Next, we shall find the value of $L(n, r)$.

Theorem. *Let* $L(n, r)$ *be the number of labelled trees with* $n \geq 3$ *points and* r *end-points. Then*

$$L(n, r) = \binom{n}{n-r} \sum_{i=0}^{n-r-1} (-1)^i \binom{n-r}{i} (n-r-i)^{n-2}.$$

Proof. Consider all $(n-2)$-tuples $(a_1, a_2, \ldots, a_{n-2})$ that consist of q different labels. Let r be the number of labels that are not in the $(n-2)$-tuple, and so $q + r = n$. According to the above lemma, the corresponding labelled trees have r end-points. Counting these labelled trees with r end-points is equivalent to counting the number of all $(n-2)$-tuples that consist of these q different labels. We shall use exponential generating function in counting all of these $(n-2)$-tuples. If needed, some readers may find the topics in exponential generating functions in most texts on combinatorics or discrete mathematics. For example, see [3,4]. Next, since each of the q labels must appear at least once in the $(n-2)$-tuple, the related generating function would be

$$\begin{aligned} g(x) &= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^q = (e^x - 1)^q \\ &= \binom{q}{0} e^{qx} - \binom{q}{1} e^{(q-1)x} + \binom{q}{2} e^{(q-2)x} - \cdots + (-1)^{q-1} \binom{q}{q-1} e^x + (-1)^q \binom{q}{q} \end{aligned}$$

$$
\begin{aligned}
&= \binom{q}{0}\left(1 + qx + \frac{(qx)^2}{2!} + \cdots + \frac{(qx)^{n-2}}{(n-2)!} + \cdots\right) \\
&\quad + (-1)\binom{q}{1}\left(1 + (q-1)x + \frac{((q-1)x)^2}{2!} + \cdots + \frac{((q-1)x)^{n-2}}{(n-2)!} + \cdots\right) \\
&\quad + (-1)^2\binom{q}{2}\left(1 + (q-2)x + \frac{((q-2)x)^2}{2!} + \cdots + \frac{((q-2)x)^{n-2}}{(n-2)!} + \cdots\right) \\
&\quad + \cdots + (-1)^{q-1}\binom{q}{q-1}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-2}}{(n-2)!} + \cdots\right) + (-1)^q.
\end{aligned}
$$

From the generating function above, the coefficient of $\frac{x^{n-2}}{(n-2)!}$ is $\sum_{i=0}^{q-1}(-1)^i\binom{q}{i}(q-i)^{n-2}$. This coefficient is the result of permutation of $a_1, a_2, \ldots, a_{n-2}$ in $(a_1, a_2, \ldots, a_{n-2})$ that consists of q labels. The coefficient obtained is equal to the number of labelled trees with r end-points and q particular non-end-points. This q labeling of non-end-points can be done in $\binom{n}{q}$ possible ways. Therefore, with fixed number of r, the total number of $(q-2)$-tuples that consist of q labels is $\binom{n}{q}\sum_{i=0}^{q-1}(-1)^i\binom{q}{i}(q-i)^{n-2}$. Hence, with $q = n - r$,

$$
L(n,r) = \binom{n}{n-r}\sum_{i=0}^{n-r-1}(-1)^i\binom{n-r}{i}(n-r-i)^{n-2}. \quad \square
$$

For example, when $n = 4$, we have results that agree with those in Fig. 2, i.e.,

$$
\begin{aligned}
L(4,2) &= \binom{4}{2}\sum_{i=0}^{1}(-1)^i\binom{2}{i}(2-i)^2 \\
&= \binom{4}{2}\left[(-1)^0\binom{2}{0}(2-0)^2 + (-1)^1\binom{2}{1}(2-1)^2\right] = 6[4-2] = 12 \\
L(4,3) &= \binom{4}{1}\sum_{i=0}^{0}(-1)^i\binom{1}{0}(1-i)^1 \\
&= \binom{4}{1}\left[(-1)^0\binom{1}{0}(1-0)^1\right] = 4.
\end{aligned}
$$

And, with $n = 5$, we have

$$
\begin{aligned}
L(5,2) &= \binom{5}{3}\sum_{i=0}^{2}(-1)^i\binom{3}{i}(3-i)^3 \\
&= \binom{5}{3}\left[(-1)^0\binom{3}{0}(3-0)^3 + (-1)^1\binom{3}{1}(3-1)^3 + (-1)^2\binom{3}{2}(3-2)^3\right] \\
&= 10[27-24+3] = 60 \\
L(5,3) &= \binom{5}{2}\sum_{i=0}^{1}(-1)^i\binom{2}{i}(2-i)^3 \\
&= \binom{5}{2}\left[(-1)^0\binom{2}{0}(2-0)^3 + (-1)^1\binom{2}{1}(2-1)^3\right] = 10[8-2] = 60
\end{aligned}
$$

$$
L(5,4) = \binom{5}{1}\sum_{i=0}^{0}(-1)^i\binom{1}{i}(1-i)^3 = \binom{5}{1}\left[(-1)^0\binom{1}{0}(1-0)^3\right] = 5.
$$

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