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Decidable call-by-need computations in term rewriting

Irène Durand[a], Aart Middeldorp[b,*]

[a]Laboratoire Bordelais de Recherche en Informatique, Université de Bordeaux I, 33405 Talence Cedex, France
[b]Institute of Computer Science, University of Innsbruck, 6020 Innsbruck, Austria

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Abstract

The theorem of Huet and Lévy stating that for orthogonal rewrite systems (i) every reducible term contains a needed redex and (ii) repeated contraction of needed redexes results in a normal form if the term under consideration has a normal form, forms the basis of all results on optimal normalizing strategies for orthogonal rewrite systems. However, needed redexes are not computable in general. In the paper we show how the use of approximations and elementary tree automata techniques allows one to obtain decidable conditions in a simple and elegant way. Surprisingly, by avoiding complicated concepts like index and sequentiality we are able to cover much larger classes of rewrite systems. We also study modularity aspects of the classes in our hierarchy. It turns out that none of the classes is preserved under signature extension. By imposing various conditions we recover the preservation under signature extension. By imposing some more conditions we are able to strengthen the signature extension results to modularity for disjoint and constructor-sharing combinations.
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* Corresponding author. Fax: +43 512 507 9887.
E-mail addresses: Irene.Durand@labri.fr (I. Durand), aart.middeldorp@uibk.ac.at (A. Middeldorp).
URL: informatik.uibk.ac.at/users/ami (A. Middeldorp).

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1. Introduction

The following theorem of Huet and Lévy [12] forms the basis of all results on optimal normalizing rewrite strategies for orthogonal term rewrite systems: Every reducible term contains a needed redex, i.e., a redex which is contracted in every rewrite sequence to normal form, and repeated contraction of needed redexes results in a normal form, if the term under consideration has a normal form. Unfortunately, needed redexes are not computable in general. Hence, in order to obtain a *computable* optimal rewrite strategy, we are left to find (1) decidable approximations of neededness and (2) decidable properties of rewrite systems which ensure that every reducible term has a needed redex identified by (1). Starting with the seminal work of Huet and Lévy [12] on strong sequentiality, these issues have been extensively investigated in the literature [2,13,14,17,21,25,29]. In all these works Huet and Lévy's notions of index, ω-reduction, and sequentiality figure prominently.

In this paper we present an approach to decidable call-by-need computations in which issues (1) and (2) above are addressed directly. Besides facilitating understanding this enables us to cover much larger classes of rewrite systems. For instance, a trivial consequence of our work is that every orthogonal right-ground rewrite system admits a computable call-by-need strategy whereas none of the sequentiality-based approaches cover all such systems. Our approach is based on the easy but fundamental observation that needed redexes are *uniform* but not *independent* of other redexes in the same term. Uniformity means that only the position of a redex in a term counts for determining neededness.

From [12,25,2] we extract the important concept of *approximation mapping*, which is used to parameterize our framework. An approximation mapping transforms a rewrite system into a simpler one such that every rewrite step in the former can be simulated in the latter. We identify *regularity preservingness* as the key property that an approximation mapping α must have in order to obtain a decidable class CBN_α consisting of all rewrite systems that have the property that at least one of the needed redexes in every reducible term can be computed by α. Consequently, every rewrite system in CBN_α admits a computable call-by-need strategy. Inspired by Comon [2], our decidability results heavily rely on tree automata techniques. However, by assigning a greater role to *ground tree transducers* we do not need to rely on weak second-order monadic logic.

Not much is known about the complexity of the problem of deciding membership in one of the classes that guarantees a computable call-by-need strategy to normal form. Comon [2] showed that strong sequentiality of a left-linear rewrite system can be decided in exponential time. Moreover, for left-linear rewrite systems satisfying the additional syntactic condition that whenever two proper subterms of left-hand sides are unifiable one of them matches the other, strong sequentiality can be decided in polynomial time. The class of forward-branching systems (Strandh [27]), a proper subclass of the class of orthogonal strongly sequential systems, coincides with the class of transitive systems (Toyama et al. [30]) and can be decided in quadratic time (Durand [8]). For classes higher in the hierarchy only double exponential upper bounds are known [10]. Consequently, it is of obvious importance to have results available that enable to split a rewrite system into smaller components such that membership in CBN_α of the components implies membership of the original system in CBN_α.

Such *modularity* results have been extensively studied for basic properties like confluence and termination, see [24] for a recent overview. The simplest kind of modularity results are concerned with enriching the signature. Most properties of rewrite systems are preserved under *signature extension*. Two notable exceptions are the normal form property and the unique normal form property (with respect to reduction), see Kennaway et al. [15]. Also some properties dealing with ground terms are not preserved under signature extension. Consider for instance the property that every ground term is innermost terminating, the rewrite system consisting of the two rewrite rules $\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{f}(x))$ and $\mathsf{f}(\mathsf{a}) \to \mathsf{a}$, and add a new constant b. It turns out that for no α, membership in CBN_α is preserved under signature extension. We present several sufficient conditions which guarantee the preservation under signature extension.

Since preservation under signature extension does not give rise to a very useful technique for splitting a system into smaller components, we also consider combinations of systems without common function symbols as well as constructor-sharing combinations.

The remainder of this paper is organized as follows. In the next section we recall the necessary background of term rewriting and tree automata. In Section 3 we give a brief introduction to call-by-need strategies. In Section 4 we present sufficient conditions for neededness in terms of approximations. Several approximations are defined in Section 5. In Section 6 we present our framework for decidable call-by-need computations to normal form. Section 7 contains a comparison with the sequentiality-based approach. In Section 8 we present our signature extension results and in Section 9 these results are extended to modularity. The proofs of most of the results in these two sections are given in Appendix A. We make some concluding remarks in Section 10.

Many of the results presented in this paper were first announced in [9,11].

2. Preliminaries

Familiarity with the basic notions of term rewriting (see, e.g. [1,16]) will be helpful in the sequel. A term rewrite system (TRS for short) $\mathcal{R}$ over a signature $\mathcal{F}$ consists of rewrite rules $l \to r$ between terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$ that satisfy $l \notin \mathcal{V}$ and $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(l)$. Here $\mathcal{V}$ is a countably infinite set of variables. If the second condition is not imposed we find it useful to speak of extended TRSs (eTRSs). Such systems arise naturally when we approximate TRSs, as explained in Section 5. When applying a rewrite rule $l \to r$ of an eTRS, variables in $\mathcal{V}\mathsf{ar}(r) \setminus \mathcal{V}\mathsf{ar}(l)$ may be instantiated by arbitrary terms.

A ground term does not contain variables. A linear term does not contain multiple occurrences of the same variable. A redex is an instance of the left-hand side of a rewrite rule. The set of all ground redexes of a TRS $\mathcal{R}$ is denoted by $\mathsf{REDEX}(\mathcal{R})$. A normal form is a term without redexes. The set of all ground normal forms of a TRS $\mathcal{R}$ is denoted by $\mathsf{NF}(\mathcal{R})$. The root symbol of a term t is denoted by $\mathsf{root}(t)$. A term is root-stable if it cannot be rewritten to a redex. An eTRS is left-linear (right-linear, linear) if the left-hand sides (right-hand sides, both left- and right-hand sides) of its rewrite rules are linear terms. An eTRS is right-ground (ground) if the right-hand sides (left- and right-hand sides) of its rewrite rules are ground terms. A left-linear TRS without critical pairs is orthogonal. Orthogonal TRSs have the property that every term has at most one normal form.

We write $s \mathrel{+\!\!\!\!\!\!\;\!\!\!\!\!\to} t$ if t can be obtained from s by contracting a, possibly zero, number of redexes at pairwise disjoint positions in s. In other words, $s = C[s_1,\ldots,s_n]$ and $t = C[t_1,\ldots,t_n]$ for some context C and terms $s_1,\ldots,s_n, t_1,\ldots,t_n$ with $s_i \to t_i$ for all $1 \leqslant i \leqslant n$. The relation $\mathrel{+\!\!\!\!\!\!\;\!\!\!\!\!\to}$ is called parallel rewriting.

A rewrite rule $l \to r$ is collapsing if r is a variable. A redex with respect to a collapsing rewrite rule is also called collapsing and so is an eTRS that contains a collapsing rewrite rule. A redex is innermost if it does not contain smaller redexes. A redex in a term is outermost if it is not a proper subterm of another redex in the same term.[1] Let $\mathcal{R}$ be a TRS over the signature $\mathcal{F}$. A function symbol in $\mathcal{F}$ is called defined if it is the root symbol of a left-hand side of a rewrite rule in $\mathcal{R}$. All other function symbols in $\mathcal{F}$ are called constructors. We use $\mathcal{F}_\mathcal{D}$ and $\mathcal{F}_\mathcal{C}$ to denote the set of defined symbols and the set of constructors. Terms in $\mathcal{T}(\mathcal{F}_\mathcal{C}, \mathcal{V})$ are called constructor terms.

In the remainder of this section we recall some basic definitions and results concerning tree automata. Much more information can be found in [3]. A (finite bottom-up) *tree automaton* is a quadruple $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Delta)$ consisting of a finite signature $\mathcal{F}$, a finite set Q of states, disjoint from $\mathcal{F}$, a subset $Q_f \subseteq Q$ of final states, and a set of transition rules Δ. Every transition rule is of the form $f(q_1, \ldots, q_n) \to q$ with $f \in \mathcal{F}$ and $q_1, \ldots, q_n, q \in Q$ or $q \to q'$ with $q, q' \in Q$. The latter rules are called ϵ-transitions. So a tree automaton $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Delta)$ is simply a finite ground TRS Δ over the signature $\mathcal{F} \cup Q$ whose rewrite rules have a special shape, together with a subset Q_f of Q. The induced rewrite relation on $\mathcal{T}(\mathcal{F} \cup Q)$ is denoted by $\to_\mathcal{A}$. A ground term $t \in \mathcal{T}(\mathcal{F})$ is accepted by $\mathcal{A}$ if $t \to_\mathcal{A}^+ q$ for some $q \in Q_f$. The set of all such terms is denoted by $L(\mathcal{A})$. A subset $L \subseteq \mathcal{T}(\mathcal{F})$ is called *regular* if there exists a tree automaton $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Delta)$ such that $L = L(\mathcal{A})$. It is well known that the set $\mathcal{T}(\mathcal{F})$ of all ground terms is regular. Other well-known properties are stated in the following two lemmata.

Lemma 1.

(1) *Regular languages are effectively closed under Boolean operations.*
(2) *Membership and emptiness are decidable for regular languages.*

Lemma 2. *If $\mathcal{R}$ is a finite left-linear TRS then* $\mathsf{REDEX}(\mathcal{R})$ *and* $\mathsf{NF}(\mathcal{R})$ *are regular.*

A *ground tree transducer* is a pair $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ of tree automata over the same signature $\mathcal{F}$. Let $s, t \in \mathcal{T}(\mathcal{F})$. We say that the pair (s, t) is accepted by $\mathcal{G}$ if $s \to_\mathcal{A}^* u$ and $t \to_\mathcal{B}^* u$ for some term $u \in \mathcal{T}(\mathcal{F} \cup Q)$ where Q is the set of common states of $\mathcal{A}$ and $\mathcal{B}$. The set of all such pairs is denoted by $L(\mathcal{G})$. Observe that $L(\mathcal{G})$ is a binary relation on $\mathcal{T}(\mathcal{F})$. A binary relation on ground terms is called *regular* if there exists a ground tree transducer that accepts it. Every regular relation R is parallel, i.e., $C[s_1, \ldots, s_n] \; R \; C[t_1, \ldots, t_n]$ whenever $s_1 \; R \; t_1$, …, $s_n \; R \; t_n$, for all contexts C and terms $s_1, \ldots, s_n, t_1, \ldots, t_n$. (The parallel rewrite relation $\mathrel{+\!\!\!\!\!\rightarrow}$ defined above is parallel. Actually, $\mathrel{+\!\!\!\!\!\rightarrow}$ is the smallest parallel relation that contains $\to$, i.e., the parallel closure of $\to$.) Ground tree transducers were introduced by Dauchet and Tison [6] in order to prove that confluence is a decidable property of ground TRSs. In this paper we make use of the following closure properties. They can be proved by adding appropriate ϵ-transitions. Part (2) originates from [4].

Lemma 3. *Let R be a regular relation on $\mathcal{T}(\mathcal{F})$.*

(1) *The transitive closure R^+ of R is regular.*
(2) *If $L \subseteq \mathcal{T}(\mathcal{F})$ is regular then $R[L]$[2] $= \{s \mid s \; R \; t$ for some $t \in L\}$ is regular.*

[1] Here the position of the redex is important. Depending on the context, by redex we either mean the subterm or its position.

[2] In the literature $R[L]$ often denotes the different set $\{t \mid s \; R \; t$ for some $s \in L\}$. We find our choice more convenient.

We would like to emphasize that there are other notions of regularity for binary relations in the literature. The one defined above suffices for our purposes. (In [5] regular relations are called *GTT-relations*.)

3. Call-by-need strategies

Given a TRS and a term, a rewrite strategy specifies which part(s) of the term to evaluate. If a TRS admits infinite computations, certain rewrite strategies may fail to reduce terms to their normal forms.

Example 4. Consider the TRS $\mathcal{R}$ consisting of the rewrite rules

$$\begin{array}{rlrl} 0 + y & \to y & \mathsf{fib} & \to \mathsf{f}(0, \mathsf{s}(0)) \\ \mathsf{s}(x) + y & \to \mathsf{s}(x + y) & \mathsf{f}(x, y) & \to x : \mathsf{f}(y, x + y) \\ \mathsf{nth}(0, y : z) & \to y & \mathsf{nth}(\mathsf{s}(x), y : z) & \to \mathsf{nth}(x, z) \end{array}$$

for computing Fibonacci numbers. The term $t = \mathsf{nth}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))), \mathsf{fib})$ admits the normal form $\mathsf{s}(\mathsf{s}(0))$: [3]

$$\begin{aligned} t &\to \mathsf{nth}(3, \mathsf{f}(0, 1)) \to \mathsf{nth}(3, 0 : \mathsf{f}(1, 0 + 1)) \to \mathsf{nth}(2, \mathsf{f}(1, 0 + 1)) \\ &\to \mathsf{nth}(2, \mathsf{f}(1, 1)) \to \mathsf{nth}(2, 1 : \mathsf{f}(1, 1 + 1)) \to \mathsf{nth}(1, \mathsf{f}(1, 1 + 1)) \\ &\to \mathsf{nth}(1, \mathsf{f}(1, \mathsf{s}(0 + 1))) \to \mathsf{nth}(1, \mathsf{f}(1, 2)) \to \mathsf{nth}(1, 1 : \mathsf{f}(2, 1 + 2)) \\ &\to \mathsf{nth}(0, \mathsf{f}(2, 1 + 2)) \to \mathsf{nth}(0, \mathsf{f}(2, \mathsf{s}(0 + 2))) \to \mathsf{nth}(0, \mathsf{f}(2, 3)) \\ &\to \mathsf{nth}(0, 2 : \mathsf{f}(3, 2 + 3)) \to 2 \end{aligned}$$

but an eager (innermost) strategy will produce an infinite rewrite sequence:

$$\begin{aligned} t &\to \mathsf{nth}(3, \mathsf{f}(0, 1)) \to \mathsf{nth}(3, 0 : \mathsf{f}(1, 0 + 1)) \to \mathsf{nth}(3, 0 : \mathsf{f}(1, 1)) \\ &\to \mathsf{nth}(3, 0 : (1 : \mathsf{f}(1, 1 + 1))) \to^2 \mathsf{nth}(3, 0 : (1 : \mathsf{f}(1, 2))) \\ &\to \mathsf{nth}(3, 0 : (1 : (1 : \mathsf{f}(2, 1 + 2)))) \to^2 \mathsf{nth}(3, 0 : (1 : (1 : \mathsf{f}(2, 3)))) \\ &\to \mathsf{nth}(3, 0 : (1 : (1 : (2 : \mathsf{f}(3, 2 + 3))))) \to^3 \mathsf{nth}(3, 0 : (1 : (1 : (2 : \mathsf{f}(3, 5))))) \\ &\to \cdots \end{aligned}$$

If a term t has a normal form then we can always compute a normal form of t by computing the reducts of t in a breadth-first manner until we encounter a normal form. However, this is a highly inefficient way to compute normal forms. In practice, normal forms are computed by adopting a suitable strategy for selecting the redexes which are to be contracted in each step. A strategy is called *normalizing* if it succeeds in computing normal forms for all terms that admit a normal form. For the class of *orthogonal* TRSs several normalization results are known (see, e.g., Klop [16]). For instance, O'Donnell [23] proved that the *parallel-outermost* strategy

[3] In the rewrite sequences we denote $\mathsf{s}^n(0)$ by n for $n = 1, 2, 3, 5$.

(which contracts in a single step all outermost redexes in parallel) is normalizing for all orthogonal TRSs. However, parallel-outermost is not an optimal[4] strategy as it may perform useless steps.

Example 5. Consider the TRS $\mathcal{R}$ consisting of the rewrite rules

$$\begin{array}{rclcrcl} \mathsf{0} + y & \to & y & \qquad & \mathsf{0} \times y & \to & \mathsf{0} \\ \mathsf{s}(x) + y & \to & \mathsf{s}(x + y) & & \mathsf{s}(x) \times y & \to & (x \times y) + y \end{array}$$

Faced with the term $t = (\mathsf{0} \times \mathsf{s}(\mathsf{0})) \times (\mathsf{0} + \mathsf{s}(\mathsf{0}))$, the parallel-outermost strategy computes its normal form $\mathsf{0}$ by contracting three redexes in two steps:

$$\underline{(\mathsf{0} \times \mathsf{s}(\mathsf{0}))} \times \underline{(\mathsf{0} + \mathsf{s}(\mathsf{0}))} \mathrel{+\!\!\!\!\!\!\to} \underline{\mathsf{0} \times \mathsf{s}(\mathsf{0})} \to \mathsf{0}$$

The normal form $\mathsf{0}$ can also be reached by contracting just two redexes:

$$\underline{(\mathsf{0} \times \mathsf{s}(\mathsf{0}))} \times (\mathsf{0} + \mathsf{s}(\mathsf{0})) \to \underline{\mathsf{0} \times (\mathsf{0} + \mathsf{s}(\mathsf{0}))} \to \mathsf{0}$$

So redex $\mathsf{0} + \mathsf{s}(\mathsf{0})$ in t is not needed to reach the normal form.

An optimal strategy selects only *needed* redexes. Formally, a redex Δ in a term t is needed if in every rewrite sequence from t to normal form a *descendant* of Δ is contracted. The latter concept is defined as follows. Let $A\colon s = s[l\sigma]_p \to s[r\sigma]_p = t$ be a rewrite step in an eTRS and let $q \in \mathcal{P}\mathsf{os}(s)$. The set $q\backslash A$ of descendants of q in t is defined as follows:

$$q\backslash A = \begin{cases} \{q\} & \text{if } q < p \text{ or } q \parallel p, \\ \{pp_3p_2 \mid r|_{p_3} = l|_{p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(l), \\ \varnothing & \text{otherwise.} \end{cases}$$

The notion of descendant extends naturally to rewrite sequences. Orthogonal (e)TRSs have the property that descendants of redex positions are again redex positions.

Example 6. In the displayed rewrite sequence $\mathsf{nth}(3, \mathsf{fib}) \to^* 2$ in Example 4 non-needed redexes are contracted. For instance, redex $1 + 2$ in the term $\mathsf{nth}(0, \mathsf{f}(2, 1 + 2))$ is non-needed:

$$\mathsf{nth}(0, \underline{\mathsf{f}(2, 1 + 2)}) \to \underline{\mathsf{nth}(0, 2 : \mathsf{f}(1 + 2, 2 + (1 + 2)))} \to 2$$

The following theorem of Huet and Lévy [12] forms the basis of all results on optimal normalizing reduction strategies for orthogonal TRSs.

[4] An optimal strategy uses the least number of redex contractions to normalize terms.

Theorem 7. *Let $\mathcal{R}$ be an orthogonal TRS.*

(1) *Every reducible term contains a needed redex.*
(2) *Repeated contraction of needed redexes results in a normal form, whenever the term under consideration has a normal form.*

So, for orthogonal TRSs, the strategy that always selects a needed redex for contraction is normalizing and optimal.[5] Unfortunately, needed redexes are not computable in general. Hence, in order to obtain a *computable* optimal strategy, we need to find (1) decidable approximations of neededness and (2) (decidable) classes of rewrite systems which ensure that every reducible term has a needed redex identified by (1).

In the sequentiality-based approach (see Section 7) issue (1) is addressed as follows. Basically, to determine whether an outermost redex Δ in a term $t = C[\Delta]$ is needed, Δ is replaced by a fresh symbol $\bullet$ and all other outermost redexes in t are replaced by Ω which represents an unknown term. It is then investigated whether $\bullet$ can disappear from the resulting Ω-term t' by using some computable notion of partial reduction. If this is not the case, then we may conclude that redex Δ in t is needed. Since neededness of redex Δ in t is solely determined by its position in t (cf. Lemma 9), replacing redex Δ in t by $\bullet$ incurs no loss of generality. However, by replacing all other outermost redexes by Ω, essential information may be lost for determining the neededness of Δ. This is illustrated in the following example, which shows that needed redexes are not independent of other redexes.

Example 8. Consider again the TRS of Example 5. An arbitrary redex Δ is needed in the term $(\mathsf{0} + \mathsf{s}(\mathsf{0})) \times \Delta$ but not in the term $(\mathsf{0} \times \mathsf{s}(\mathsf{0})) \times \Delta$:

$$\underline{(\mathsf{0} \times \mathsf{s}(\mathsf{0}))} \times \Delta \to \underline{\mathsf{0} \times \Delta} \to \mathsf{0}$$

In the next section we present a new approach to the problem of determining neededness of a given redex in a term which does not abstract from the other redexes in the term.

4. Decidable approximations of neededness

In the remaining part of the paper we are dealing with finite TRSs only. Moreover, we consider rewriting on ground terms only, except in Section 7 for reasons explained there. So we assume that the set of ground terms is non-empty. It is undecidable whether a redex in a term is needed with respect to a given (orthogonal) TRS. In this section we present decidable sufficient conditions for a redex to be needed.

We start with an easy lemma that provides an alternative definition of neededness, not depending on the notion of descendant. Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$. We assume the existence of a constant $\bullet$ not appearing in $\mathcal{F}$ and we view $\mathcal{R}$ as a TRS over the extended signature $\mathcal{F}_\bullet = \mathcal{F} \cup \{\bullet\}$. So $\mathsf{NF}(\mathcal{R})$ consists of all terms in $\mathcal{T}(\mathcal{F}_\bullet)$ that are in normal form. Let $\mathcal{R}_\bullet$ be the TRS $\mathcal{R} \cup \{\bullet \to \bullet\}$. Note that $\mathsf{NF}(\mathcal{R}_\bullet)$ coincides with $\mathsf{NF}(\mathcal{R}) \cap \mathcal{T}(\mathcal{F})$.

[5] We ignore here the problem of duplication of (needed) redexes, which can be solved if common subterms are shared.

Lemma 9. *Let $\mathcal{R}$ be an orthogonal TRS over a signature $\mathcal{F}$. Redex Δ in term $C[\Delta] \in \mathcal{T}(\mathcal{F})$ is needed if and only if there is no term $t \in \mathsf{NF}(\mathcal{R}_\bullet)$ such that $C[\bullet] \to_{\mathcal{R}}^* t$.*

Proof. Let $A: s \to^* t$ be a rewrite sequence and Δ a redex in s. We write $\Delta \perp A$ if no descendant of Δ is contracted in A. So a redex Δ in a term s is needed if and only if $A: s \to^* t$ with $\Delta \perp A$ implies that t is not a normal form.

For the "only if" direction we suppose there is a term $t \in \mathsf{NF}(\mathcal{R}_\bullet)$ such that $C[\bullet] \to_{\mathcal{R}}^* t$. Replacing every occurrence of $\bullet$ by Δ yields a sequence $A: C[\Delta] \to_{\mathcal{R}}^* t$ with $\Delta \perp A$. Hence Δ is not needed.

For the "if" direction we suppose that Δ is not needed. So there exists a rewrite sequence $A: C[\Delta] \to_{\mathcal{R}}^* t$ with $t \in \mathsf{NF}(\mathcal{R}_\bullet)$ and $\Delta \perp A$. Replacing every descendant of Δ in A by $\bullet$ yields a sequence $C[\bullet] \to_{\mathcal{R}}^* t$. (Here we use orthogonality. Note that because t is a normal form there are no descendants of Δ in t left.) $\square$

An immediate consequence of this lemma is the folklore result that only the position of a redex in a term is important for determining neededness. So if redex Δ in term $C[\Delta]$ is needed then so is redex Δ' in $C[\Delta']$.

Using the notation introduced in Section 2, the preceding lemma can be rephrased as follows: Redex Δ in $C[\Delta] \in \mathcal{T}(\mathcal{F})$ is needed if and only if $C[\bullet] \notin (\to_{\mathcal{R}}^*)[\mathsf{NF}(\mathcal{R}_\bullet)]$. Since membership for regular languages is decidable but neededness undecidable, it follows that $(\to_{\mathcal{R}}^*)[\mathsf{NF}(\mathcal{R}_\bullet)]$ is not regular in general. The key to decidability is to extend $\to_{\mathcal{R}}^*$ to $\to_{\mathcal{S}}^*$ for some suitable eTRS $\mathcal{S}$ such that $(\to_{\mathcal{S}}^*)[\mathsf{NF}(\mathcal{R})]$ becomes regular.

Definition 10. Let $\mathcal{R}$ and $\mathcal{S}$ be eTRSs over the same signature. We say that $\mathcal{S}$ *approximates* $\mathcal{R}$ if $\to_{\mathcal{R}} \subseteq \to_{\mathcal{S}}^*$ and $\mathsf{NF}(\mathcal{R}) = \mathsf{NF}(\mathcal{S})$.

Definition 11. An *approximation mapping* is a mapping α from eTRSs to eTRSs with the property that $\alpha(\mathcal{R})$ approximates $\mathcal{R}$ for all eTRSs $\mathcal{R}$. We write $\mathcal{R}_\alpha$ for $\alpha(\mathcal{R})$. We say that α is *regularity preserving* if $(\to_{\mathcal{R}_\alpha}^*)[L]$ is regular for all eTRSs $\mathcal{R}$ and regular L. We define a partial order $\leqslant$ on approximation mappings as follows: $\alpha \leqslant \beta$ if and only if $\mathcal{R}_\beta$ approximates $\mathcal{R}_\alpha$, for every eTRS $\mathcal{R}$. Note that the identity mapping is the minimum element of this partial order.

Needless to say, we are only interested in *computable* approximation mappings that are *effectively* regularity preserving. This means that there is an algorithm which, given a tree automaton for L, constructs a tree automaton for $(\to_{\mathcal{R}_\alpha}^*)[L]$. The regularity preserving approximation mappings that we introduce in the next section have this property.

Definition 12. Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$ and α an approximation mapping. We say that redex Δ in $C[\Delta] \in \mathcal{T}(\mathcal{F})$ is *α-needed* if $C[\bullet] \notin (\to_{\mathcal{R}_\alpha}^*)[\mathsf{NF}(\mathcal{R}_\bullet)]$. The set of all such terms $C[\bullet]$ is denoted by $\mathsf{NEED}(\mathcal{R}_\alpha)$.

In the following we abbreviate $\to_{\mathcal{R}_\alpha}$ to $\to_\alpha$ when the $\mathcal{R}$ can be inferred from the context.

Lemma 13. *Let $\mathcal{R}$ be an orthogonal TRS and α an approximation mapping. Every α-needed redex is needed.*

Proof. Let Δ be an α-needed redex in $C[\Delta]$. So $C[\bullet] \notin (\to_{\mathcal{R}_\alpha}^*)[\mathsf{NF}(\mathcal{R}_\bullet)]$. Since $\mathcal{R}_\alpha$ approximates $\mathcal{R}$, we have $\to_{\mathcal{R}} \subseteq \to_{\mathcal{R}_\alpha}^*$ by definition and thus also $\to_{\mathcal{R}}^* \subseteq \to_{\mathcal{R}_\alpha}^*$. Hence $C[\bullet] \notin (\to_{\mathcal{R}}^*)[\mathsf{NF}(\mathcal{R}_\bullet)]$. Because $\mathcal{R}$ is orthogonal, we obtain the neededness of Δ from Lemma 9. $\square$

Only in Lemma 13 do we require orthogonality. For decidability issues, left-linearity suffices. The following example shows that both left-linearity and non-overlappingness are required for Lemmata 9 and 13.

Example 14. First of all, consider the left-linear overlapping TRS consisting of the single rewrite rule

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{a}$$

and the term $\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a})))$. Since contracting either of the two redexes immediately gives a normal form, neither of the two redexes is needed. On the other hand, for any approximation mapping α, including the identity mapping, redex $\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a})))$ is α-needed since $\bullet$ is an $\mathcal{R}_\alpha$-normal form which does not belong to $\mathsf{NF}(\mathcal{R}_\bullet)$.

Next consider the non-left-linear non-overlapping TRS consisting of the three rewrite rules

$$\mathsf{f}(x, x) \to \mathsf{a} \quad \mathsf{b} \to \mathsf{c} \quad \mathsf{c} \to \mathsf{b}$$

and the term $\mathsf{f}(\mathsf{b}, \mathsf{c})$. Again, it is easy to see that neither of the two redexes is needed. Replacing either of them by $\bullet$ yields a term which, for two of the three approximation mappings α defined in the next section as well as for the identity mapping, does not $\mathcal{R}_\alpha$-rewrite to a normal form in $\mathsf{NF}(\mathcal{R}_\bullet)$.

Lemma 15. *Let $\mathcal{R}$ be a left-linear TRS and α an approximation mapping. If α is regularity preserving then $\mathsf{NEED}(\mathcal{R}_\alpha)$ is regular.*

Proof. We have

$$\mathsf{NEED}(\mathcal{R}_\alpha) = (\to^*_{\mathcal{R}_\alpha})[\mathsf{NF}(\mathcal{R}_\bullet)]^c \cap \mathsf{M}_\bullet{}^6$$

where $\mathsf{M}_\bullet$ is the subset of $\mathcal{T}(\mathcal{F}_\bullet)$ consisting of all terms that contain exactly one occurrence of $\bullet$. The regularity of $\mathsf{M}_\bullet$ is easily shown. Hence the regularity of $\mathsf{NEED}(\mathcal{R}_\alpha)$ is a consequence of Lemmata 1 and 2. □

Since membership for regular tree languages is decidable, we obtain the following result.

Corollary 16. Let $\mathcal{R}$ be a left-linear TRS and α a regularity preserving approximation mapping. It is decidable whether a redex in a term is α-needed.

Naturally, a better approximation can identify more needed redexes.

Lemma 17. *Let α and β be approximation mappings. If $\alpha \leqslant \beta$ then $\mathsf{NEED}(\mathcal{R}_\beta) \subseteq \mathsf{NEED}(\mathcal{R}_\alpha)$, for every TRS $\mathcal{R}$.*

5. Approximations

In this section we define three approximation mappings that are known to be regularity preserving. We give new proofs for two of these results. The approximations differ in the way they treat

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[6] Here $(\to^*_{\mathcal{R}_\alpha})[\mathsf{NF}(\mathcal{R}_\bullet)]^c$ denotes the complement of $(\to^*_{\mathcal{R}_\alpha})[\mathsf{NF}(\mathcal{R}_\bullet)]$ (with respect to $\mathcal{T}(\mathcal{F}_\bullet)$).

the right-hand sides of the rewrite rules of the original TRS. The left-hand sides are not affected, and hence the second requirement in the definition of approximation is trivially satisfied.[7]

Definition 18. Let $\mathcal{R}$ be a TRS. The *strong approximation* $\mathcal{R}_{\mathsf{s}}$ is obtained from $\mathcal{R}$ by replacing the right-hand side of every rewrite rule by a fresh variable.

Example 19. For the TRS $\mathcal{R}$ of Example 5, the eTRS $\mathcal{R}_{\mathsf{s}}$ consists of the following rules:

$$\begin{array}{rr} \mathsf{0} + y \to z & \mathsf{0} \times y \to z \\ \mathsf{s}(x) + y \to z & \mathsf{s}(x) \times y \to z \end{array}$$

The idea of approximating a TRS by ignoring the right-hand sides of its rewrite rules is due to Huet and Lévy [12]. A better approximation is obtained by preserving the non-variable parts of the right-hand sides of the rewrite rules.

Definition 20. Let $\mathcal{R}$ be a TRS. The *nv approximation* $\mathcal{R}_{\mathsf{nv}}$ is obtained from $\mathcal{R}$ by replacing all occurrences of variables in the right-hand side of every rewrite rule by distinct fresh variables.

Example 21. For the TRS $\mathcal{R}$ of Example 5, the eTRS $\mathcal{R}_{\mathsf{nv}}$ consists of the following rules:

$$\begin{array}{ll} \mathsf{0} + y \to y' & \mathsf{0} \times y \to \mathsf{0} \\ \mathsf{s}(x) + y \to \mathsf{s}(x' + y') & \mathsf{s}(x) \times y \to (x' \times y') + y'' \end{array}$$

The idea of approximating a TRS by ignoring the variables in the right-hand sides of the rewrite rules is due to Oyamaguchi [25]. Note that $\mathcal{R}_{\mathsf{nv}} = \mathcal{R}$ whenever $\mathcal{R}$ is right-ground. Hence for every orthogonal right-ground TRS $\mathcal{R}$, a redex is needed if and only if it is nv-needed.

Definition 22. An eTRS is called *growing* if for every rewrite rule $l \to r$ the variables in $\mathcal{V}\mathsf{ar}(l) \cap \mathcal{V}\mathsf{ar}(r)$ occur at depth 1 in l. Let $\mathcal{R}$ be a TRS. The *growing approximation* $\mathcal{R}_{\mathsf{g}}$ is defined as the growing eTRS that is obtained from $\mathcal{R}$ by renaming the variables in the right-hand sides that occur at a depth greater than 1 in the corresponding left-hand sides.

Example 23. For the TRS $\mathcal{R}$ of Example 5, the eTRS $\mathcal{R}_{\mathsf{g}}$ consists of the following rules:

$$\begin{array}{ll} \mathsf{0} + y \to y & \mathsf{0} \times y \to \mathsf{0} \\ \mathsf{s}(x) + y \to \mathsf{s}(x' + y) & \mathsf{s}(x) \times y \to (x' \times y) + y \end{array}$$

Note that the occurrences of y in the right-hand sides of the rules of $\mathcal{R}$ are not renamed since they occur at depth 1 in the corresponding left-hand sides.

Growing TRSs, introduced by Jacquemard [13], are a proper extension of the shallow TRSs considered by Comon [2]. The growing approximation defined above stems from Nagaya and Toyama [22]. It extends the growing approximation in [13] in that the right-linearity requirement is dropped.

[7] Since we deal exclusively with left-linear TRSs in this paper, there is no need to modify the left-hand sides. In [18] the definitions are adapted such that the resulting TRSs are left-linear. This is useful for automated termination analysis, but violates the second requirement in Definition 10. This requirement, however, plays no role in [18].

The mapping s that assigns to every eTRS $\mathcal{R}$ the eTRS $\mathcal{R}_{\mathsf{s}}$ is an approximation mapping. In the same fashion, Definitions 20 and 22 define approximation mappings nv and g. We clearly have $\mathsf{g} \leqslant \mathsf{nv} \leqslant \mathsf{s}$.

Example 24. Consider again the TRS $\mathcal{R}$ of Example 5. Let Δ_1 and Δ_2 be arbitrary redexes and consider the term

$$t = \underbrace{(\mathsf{0} + \mathsf{s}(\Delta_1))}_{\Delta_3} + \Delta_2$$

All three redexes are needed (since $\mathcal{R}$ is non-erasing). The following rewrite sequences show that Δ_1 and Δ_2 are not s-needed:

$$\begin{array}{l} (\mathsf{0} + \mathsf{s}(\bullet)) + \Delta_2 \rightarrow_{\mathsf{s}} \mathsf{0} + \Delta_2 \rightarrow_{\mathsf{s}} \mathsf{0} \\ (\mathsf{0} + \mathsf{s}(\Delta_1)) + \bullet \rightarrow_{\mathsf{s}} \mathsf{0} + \bullet \rightarrow_{\mathsf{s}} \mathsf{0} \end{array}$$

Redex Δ_3 is s-needed since all s-reducts of $\bullet + \Delta_2$ are of the form $\bullet + t'$. For the nv approximation the situation is the same. Redexes Δ_1 and Δ_2 are not nv-needed—the above s-rewrite sequences are also nv-rewrite sequences—but Δ_3 is. With respect to the growing approximation, Δ_1 is not g-needed:

$$(\mathsf{0} + \mathsf{s}(\bullet)) + \Delta_2 \rightarrow_{\mathsf{g}} \mathsf{s}(\bullet) + \Delta_2 \rightarrow_{\mathsf{g}} \mathsf{s}(\mathsf{0} + \Delta_2) \rightarrow_{\mathsf{g}} \mathsf{s}(\Delta_2) \rightarrow_{\mathsf{g}}^* t'$$

for some normal form t' (which depends on redex Δ_2). However, Δ_2 is g-needed. The reason is that we cannot get rid of $\bullet$ in the term $(\mathsf{0} + \mathsf{s}(\Delta_1)) + \bullet$ since the second argument of $+$ is never erased by the rules in $\mathcal{R}_{\mathsf{g}}$.

Theorem 25. *The approximation mappings* s, nv, *and* g *are regularity preserving.*

Nagaya and Toyama [22] proved the above result for the growing approximation; the tree automaton that recognizes $(\rightarrow_{\mathsf{g}}^*)[L]$ is defined as the limit of a finite saturation process. This saturation process is similar to the ones defined in Comon [2] and Jacquemard [13], but by working exclusively with deterministic tree automata, non-right-linear rewrite rules can be handled.

Below we give a very simple proof of Theorem 25 for the strong and nv approximations, using ground tree transducers.

Lemma 26. *Let $\mathcal{R}$ be a left-linear TRS. The relations $\rightarrow_{\mathsf{s}}^*$ and $\rightarrow_{\mathsf{nv}}^*$ are regular.*

Proof. According to Lemma 3(1) regular relations are closed under transitive closure. Since $\twoheadrightarrow^+ = \rightarrow^*$ it therefore suffices to show that $\twoheadrightarrow_{\mathsf{s}}$ and $\twoheadrightarrow_{\mathsf{nv}}$ are regular. First we show the regularity of $\twoheadrightarrow_{\mathsf{nv}}$. Let $\mathcal{R}_{\mathsf{nv}} = \{l_i \rightarrow r_i \mid 1 \leqslant i \leqslant n\}$. Define the ground tree transducer $\mathcal{G}_{\mathsf{nv}}$ as the pair of tree automata $\mathcal{A}$ and $\mathcal{B}$ that accept in state i all instances of l_i and r_i, respectively. Moreover, we may assume that the two tree automata share no other states. Hence $L(\mathcal{G}_{\mathsf{nv}}) = \twoheadrightarrow_{\mathsf{nv}}$. The regularity of $\twoheadrightarrow_{\mathsf{s}}$ is obtained by replacing $\mathcal{B}$ by the tree automaton $\mathcal{C}$ that accepts in state i all terms. $\square$

We illustrate the construction of $\mathcal{G}_{\mathsf{nv}}$ and $\mathcal{G}_{\mathsf{s}}$ in the proof of the above lemma on a small example.

Table 1
The tree automata $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in the proof of Lemma 26

$\mathsf{a} \to *$	$\mathsf{a} \to \langle * \rangle$	$\mathsf{a} \to \langle * \rangle$
$\mathsf{b} \to *$	$\mathsf{b} \to \langle * \rangle$	$\mathsf{b} \to \langle * \rangle$
$\mathsf{f}(*,*) \to *$	$\mathsf{f}(\langle * \rangle, \langle * \rangle) \to \langle * \rangle$	$\mathsf{f}(\langle * \rangle, \langle * \rangle) \to \langle * \rangle$
$\mathsf{g}(*) \to *$	$\mathsf{g}(\langle * \rangle) \to \langle * \rangle$	$\mathsf{g}(\langle * \rangle) \to \langle * \rangle$
$\mathsf{h}(*) \to *$	$\mathsf{h}(\langle * \rangle) \to \langle * \rangle$	$\mathsf{h}(\langle * \rangle) \to \langle * \rangle$
$\bullet \to *$	$\bullet \to \langle * \rangle$	$\bullet \to \langle * \rangle$
$\mathsf{a} \to [\mathsf{a}]$	$\mathsf{b} \to \langle \mathsf{b} \rangle$	
$\mathsf{b} \to [\mathsf{b}]$	$\mathsf{h}(\langle * \rangle) \to \langle \mathsf{h}(*) \rangle$	
$\mathsf{g}(*) \to [\mathsf{g}(*)]$	$\mathsf{h}(\langle \mathsf{h}(*) \rangle) \to \langle \mathsf{h}(\mathsf{h}(*)) \rangle$	
$\mathsf{f}(*, [\mathsf{b}]) \to [\mathsf{f}(*, \mathsf{b})]$		
$\mathsf{f}([\mathsf{g}(*)], [\mathsf{a}]) \to 1$	$\mathsf{f}(\langle \mathsf{h}(\mathsf{h}(*)) \rangle, \langle * \rangle) \to 1$	$\langle * \rangle \to 1$
$\mathsf{h}([\mathsf{a}]) \to 2$	$\mathsf{h}(\langle \mathsf{b} \rangle) \to 2$	$\langle * \rangle \to 2$
$\mathsf{h}([\mathsf{f}(*, \mathsf{b})]) \to 3$	$\langle * \rangle \to 3$	$\langle * \rangle \to 3$

Example 27. Table 1 shows the tree automata $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ used in the proof of the above lemma for the following TRS $\mathcal{R}$:

$$\begin{aligned} 1&: \mathsf{f}(\mathsf{g}(x), \mathsf{a}) \to \mathsf{f}(\mathsf{h}(\mathsf{h}(x)), x) \\ 2&: \quad\quad \mathsf{h}(\mathsf{a}) \to \mathsf{h}(\mathsf{b}) \\ 3&: \mathsf{h}(\mathsf{f}(x, \mathsf{b})) \to x \end{aligned}$$

Note that only states $1, 2$, and 3 are shared between $\mathcal{A}$ and $\mathcal{B}$ and between $\mathcal{A}$ and $\mathcal{C}$. Consider the tree automaton $\mathcal{A}$. Its states are $*$, $[\mathsf{a}]$, $[\mathsf{b}]$, $[\mathsf{g}(*)]$, and $[\mathsf{f}(*, \mathsf{b})]$. In state $*$ all ground terms are accepted. The purpose of the second group of transition rules is to recognize all ground instances of proper non-variable subterms of the left-hand sides of $\mathcal{R}$. So in state $[\mathsf{a}]$ only the term a is accepted, whereas in state $[\mathsf{f}(*, \mathsf{b})]$ all ground terms of the form $\mathsf{f}(t, \mathsf{b})$ are accepted. The third group of transition rules corresponds to the left-hand sides of $\mathcal{R}$.

The regularity preservingness of s and nv is an immediate consequence of Lemmata 26 and 3(2). (Since $\to_{\mathsf{g}}^{*}$ need not be a regular relation,[8] ground tree transducers are not useful for obtaining the regularity preservingness of g.)

It is easy to see that s-needed redexes in a term are always outermost. The same is true for nv-needed redexes in terms that have a normal form. However, g-needed redexes in normalizing terms need not be outermost. For instance, the TRS $\mathcal{R}$:

$$\mathsf{f}(x) \to \mathsf{g}(x) \quad \mathsf{a} \to \mathsf{b}$$

is growing and hence $\mathcal{R}_{\mathsf{g}} = \mathcal{R}$. Innermost redex a in the term $\mathsf{f}(\mathsf{a})$ is g-needed because there is no term $t \in \mathsf{NF}(\mathcal{R}_{\bullet})$ such that $\mathsf{f}(\bullet) \to_{\mathcal{R}}^{*} t$. Note that a is not nv-needed as $\mathsf{f}(\bullet) \to_{\mathsf{nv}} \mathsf{g}(\mathsf{b})$ with $\mathsf{g}(\mathsf{b}) \in \mathsf{NF}(\mathcal{R}_{\bullet})$.

[8] It is not difficult to show that $\to_{\mathsf{g}}^{*}$ is not regular for the TRS $\mathcal{R} = \{\mathsf{f}(x) \to x\}$ over the signature consisting of unary function symbols f and g, and a constant a.

Takai et al. [28] introduced the class of left-linear inverse finite path overlapping rewrite systems and showed that Theorem 25 is true for the corresponding approximation mapping. Growing rewrite systems constitute a proper subclass of the class of inverse finite path overlapping rewrite systems. Since the definition of this class is rather difficult, we do not consider the inverse finite path overlapping approximation here. We note, however, that our results easily extend. Another complicated regularity preserving approximation mapping can be extracted from the recent paper by Seki et al. [26].

6. Call-by-need computations to normal form

A TRS $\mathcal{R}$ admits decidable call-by-need computations to normal form if there exists an approximation mapping α such that α-needed redexes are computable and, moreover, every reducible term has an α-needed redex. In Section 4 we addressed the first issue. This section is devoted to the second issue. The following definition is readily understood.

Definition 28. Let α be an approximation mapping. The class of TRSs $\mathcal{R}$ such that every reducible term in $\mathcal{T}(\mathcal{F})$ has an α-needed redex is denoted by CBN_α. Here $\mathcal{F}$ denotes the signature of $\mathcal{R}$.

Lemma 29. *Let $\mathcal{R}$ be an orthogonal TRS.*

(1) *If $\mathcal{R}$ is right-ground then $\mathcal{R} \in \mathsf{CBN}_{\mathsf{nv}}$.*
(2) *If $\mathcal{R}$ is growing then $\mathcal{R} \in \mathsf{CBN}_{\mathsf{g}}$.*

Proof. According to Theorem 7(1) every reducible term contains a needed redex. If $\mathcal{R}$ is right-ground then $\mathcal{R} = \mathcal{R}_{\mathsf{nv}}$ and thus all needed redexes are nv-needed. Hence $\mathcal{R} \in \mathsf{CBN}_{\mathsf{nv}}$. If $\mathcal{R}$ is growing then $\mathcal{R} = \mathcal{R}_{\mathsf{g}}$ and thus all needed redexes are g-needed. Hence $\mathcal{R} \in \mathsf{CBN}_{\mathsf{g}}$. $\square$

The next lemma is an easy consequence of Lemma 17.

Lemma 30. *Let α and β be approximation mappings. If $\alpha \leqslant \beta$ then $\mathsf{CBN}_\beta \subseteq \mathsf{CBN}_\alpha$.*

Proof. Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$ that belongs to CBN_β. So every reducible term t in $\mathcal{T}(\mathcal{F})$ has a β-needed redex. So $t = C[\Delta]$ with Δ a β-needed redex. By definition $C[\bullet] \in \mathsf{NEED}(\mathcal{R}_\beta)$. Lemma 17 yields $C[\bullet] \in \mathsf{NEED}(\mathcal{R}_\alpha)$. Hence redex Δ is α-needed in t. It follows that $\mathcal{R}$ belongs to CBN_α. $\square$

Below we show that membership of a left-linear TRS in CBN_α is decidable for any regularity preserving approximation mapping α. The proof is a straightforward consequence of the following result.

Theorem 31. *Let $\mathcal{R}$ be a left-linear TRS and let α be a regularity preserving approximation mapping. The set of terms that have an α-needed redex is regular.*

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. Define the relation $\mathsf{mark}^\bullet_\mathcal{R}$ on $\mathcal{T}(\mathcal{F}_\bullet)$ as the parallel closure of $\{(\Delta, \bullet) \mid \Delta \in \mathcal{T}(\mathcal{F}) \text{ is a redex}\}$. The set of terms that have an α-needed redex coincides with

$$\mathsf{mark}^\bullet_\mathcal{R}[\mathsf{NEED}(\mathcal{R}_\alpha)] \cap \mathcal{T}(\mathcal{F})$$

If we can show that the relation $\mathsf{mark}^{\bullet}_{\mathcal{R}}$ is regular then the result follows from Lemmata 1, 3(2), and 15. Let $\mathcal{A}$ be a tree automaton with a unique final state ! that accepts $\mathsf{REDEX}(\mathcal{R}) \cap \mathcal{T}(\mathcal{F})$ and let $\bullet \to\ !$ be the single transition rule of the tree automaton $\mathcal{B}$. It is not difficult to see that the ground tree transducer $(\mathcal{A}, \mathcal{B})$ accepts $\mathsf{mark}^{\bullet}_{\mathcal{R}}$. □

Theorem 32. *Let $\mathcal{R}$ be a left-linear TRS and let α be a regularity preserving approximation mapping. It is decidable whether $\mathcal{R} \in \mathsf{CBN}_\alpha$.*

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$. The TRS $\mathcal{R}$ belongs to CBN_α if and only if the set

$$A = \mathsf{NF}(\mathcal{R})^c \setminus \{t \in \mathcal{T}(\mathcal{F}) \mid t \text{ has an } \alpha\text{-needed redex}\}$$

is empty. According to Lemmata 1, 2 and Theorem 31, A is regular. Hence the emptiness of A is decidable by Lemma 1. □

Because $\mathcal{R}_\alpha$-needed redexes need not be needed for a left-linear TRS $\mathcal{R}$ (Example 14), membership in CBN_α does not guarantee that $\mathcal{R}$ admits a computable call-by-need strategy; orthogonality is needed to draw that conclusion.

It should not come as a surprise that a better approximation covers a larger class of TRSs. This is expressed formally in the next lemma.

Lemma 33. *We have $\mathsf{CBN}_\mathsf{s} \subsetneq \mathsf{CBN}_\mathsf{nv} \subsetneq \mathsf{CBN}_\mathsf{g}$, even when these classes are restricted to orthogonal TRSs.*

Proof. From Lemma 30 we obtain $\mathsf{CBN}_\mathsf{s} \subseteq \mathsf{CBN}_\mathsf{nv} \subseteq \mathsf{CBN}_\mathsf{g}$. Consider the orthogonal TRSs

$$\begin{array}{llll} \mathcal{R}_1 : & \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{a} & \mathsf{f}(\mathsf{b},x,\mathsf{a}) \to \mathsf{b} & \mathsf{f}(x,\mathsf{a},\mathsf{b}) \to \mathsf{c} \\ \mathcal{R}_2 : & \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{a} & \mathsf{f}(\mathsf{b},x,\mathsf{a}) \to \mathsf{b} & \mathsf{f}(x,\mathsf{a},\mathsf{b}) \to x \end{array}$$

According to Lemma 29 $\mathcal{R}_1 \in \mathsf{CBN}_\mathsf{nv}$ and $\mathcal{R}_2 \in \mathsf{CBN}_\mathsf{g}$. So it remains to show that $\mathcal{R}_1 \notin \mathsf{CBN}_\mathsf{s}$ and $\mathcal{R}_2 \notin \mathsf{CBN}_\mathsf{nv}$. We have

$$\begin{array}{llll} (\mathcal{R}_1)_\mathsf{s} : & \mathsf{f}(\mathsf{a},\mathsf{b},x) \to y & \mathsf{f}(\mathsf{b},x,\mathsf{a}) \to y & \mathsf{f}(x,\mathsf{a},\mathsf{b}) \to y \\ (\mathcal{R}_2)_\mathsf{nv} : & \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{a} & \mathsf{f}(\mathsf{b},x,\mathsf{a}) \to \mathsf{b} & \mathsf{f}(x,\mathsf{a},\mathsf{b}) \to y \end{array}$$

Let Δ be the redex $\mathsf{f}(\mathsf{a},\mathsf{a},\mathsf{b})$. In $(\mathcal{R}_1)_\mathsf{s}$ and $(\mathcal{R}_2)_\mathsf{nv}$ we have $\Delta \to t$ for every term t. The following rewrite sequences in $(\mathcal{R}_1)_\mathsf{s}$ show that none of the redexes in $\mathsf{f}(\Delta,\Delta,\Delta)$ is s-needed:

$$\begin{array}{l} \mathsf{f}(\bullet,\Delta,\Delta) \to \mathsf{f}(\bullet,\mathsf{a},\Delta) \to \mathsf{f}(\bullet,\mathsf{a},\mathsf{b}) \to \mathsf{a} \\ \mathsf{f}(\Delta,\bullet,\Delta) \to \mathsf{f}(\mathsf{b},\bullet,\Delta) \to \mathsf{f}(\mathsf{b},\bullet,\mathsf{a}) \to \mathsf{a} \\ \mathsf{f}(\Delta,\Delta,\bullet) \to \mathsf{f}(\mathsf{a},\Delta,\bullet) \to \mathsf{f}(\mathsf{a},\mathsf{b},\bullet) \to \mathsf{a} \end{array}$$

Hence $\mathcal{R}_1 \notin \mathsf{CBN}_\mathsf{s}$. The following rewrite sequences in $(\mathcal{R}_2)_\mathsf{nv}$ show that none of the redexes in $\mathsf{f}(\Delta,\Delta,\Delta)$ is nv-needed:

$$\begin{array}{l} \mathsf{f}(\bullet,\Delta,\Delta) \to \mathsf{f}(\bullet,\mathsf{a},\Delta) \to \mathsf{f}(\bullet,\mathsf{a},\mathsf{b}) \to \mathsf{a} \\ \mathsf{f}(\Delta,\bullet,\Delta) \to \mathsf{f}(\mathsf{b},\bullet,\Delta) \to \mathsf{f}(\mathsf{b},\bullet,\mathsf{a}) \to \mathsf{b} \\ \mathsf{f}(\Delta,\Delta,\bullet) \to \mathsf{f}(\mathsf{a},\Delta,\bullet) \to \mathsf{f}(\mathsf{a},\mathsf{b},\bullet) \to \mathsf{a} \end{array}$$

Consequently, $\mathcal{R}_2 \notin \mathsf{CBN}_\mathsf{nv}$. □

7. Sequentiality

In this section we relate our classes CBN_α to the ones based on the sequentiality concept of Huet and Lévy. The following definitions originate from [12].

Definition 34. Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$. Let $\mathcal{F}_\Omega = \mathcal{F} \cup \{\Omega\}$ with Ω a fresh constant. The prefix order $\leqslant$ on $\mathcal{T}(\mathcal{F}_\Omega, \mathcal{V})$ is defined as follows: $s \leqslant t$ if t can be obtained from s by replacing some Ωs by terms in $\mathcal{T}(\mathcal{F}_\Omega, \mathcal{V})$. A term in $\mathcal{T}(\mathcal{F}_\Omega, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})$ that is in normal form with respect to $\mathcal{R}$ is called an Ω-*normal form*. Let P be a predicate on $\mathcal{T}(\mathcal{F}_\Omega, \mathcal{V})$.

- An Ω-position p in a term $t \in \mathcal{T}(\mathcal{F}_\Omega, \mathcal{V})$ is called an *index* with respect to P if $s|_p \neq \Omega$ for all terms $s \geqslant t$ such that $P(s)$ holds.
- The predicate P is called *sequential* if every Ω-normal form has an index.

Definition 35. Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$. The predicate nf is defined on $\mathcal{T}(\mathcal{F}_\Omega, \mathcal{V})$ as follows: nf(t) if and only if $t \to^*_{\mathcal{R}} u$ for some normal form $u \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. We say that $\mathcal{R}$ is *sequential* if nf is a sequential predicate.

The explanation for not restricting the above definitions to ground terms will be given after Example 41.

Huet and Lévy remarked that sequentiality is undecidable and that sequentiality indices are not computable in general. They identified a decidable subclass, the class of *strongly sequential* TRSs, in which every Ω-normal form admits at least one computable index. This subclass, as well as several later extensions, is defined below using the concept of approximation mapping.

Definition 36. Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$ and let α be an approximation mapping. The predicate nf_α is defined on $\mathcal{T}(\mathcal{F}_\Omega, \mathcal{V})$ as follows: $\mathrm{nf}_\alpha(t)$ if and only if $t \to^*_\alpha u$ for some normal form $u \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. We say that $\mathcal{R}$ is α-*sequential* if nf_α is a sequential predicate.

The class of s-sequential TRSs coincides with the class of strongly sequential TRSs of Huet and Lévy. The class of nv-sequential TRSs coincides with the class of *NVNF-sequential* TRSs of Nagaya et al. [21], which is an extension of the class of *NV-sequential* TRSs of Oyamaguchi [25]. The latter class is defined using the nv approximation mapping but with a different predicate $\mathrm{term}_{\mathsf{nv}}$: $\mathrm{term}_{\mathsf{nv}}(t)$ if and only if $t \to^*_{\mathsf{nv}} u$ for some *term* $u \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. The class of g-sequential TRSs properly contains all *growing sequential* TRSs of Jacquemard [13], cf. the paragraph following Example 23.

Below we compare the classes defined in Definition 36 with our CBN_α classes. The following lemma connects nf_α-indices with α-needed redexes.

Lemma 37. *Let $\mathcal{R}$ be a left-linear TRS over a signature $\mathcal{F}$ and let α be an approximation mapping. If a position p in a term $t \in \mathcal{T}(\mathcal{F}_\Omega)$ is an nf_α-index then redex Δ in the term $s[\Delta]_p$ is α-needed, for all terms $s \geqslant t$ and redexes Δ.*

Proof. Suppose Δ is not an α-needed redex in the term $s[\Delta]_p$. Then there exists a normal form $u \in \mathcal{T}(\mathcal{F})$ such that $s[\bullet]_p \to^*_\alpha u$. Since $\mathcal{R}_\alpha$ is left-linear and $\bullet$ does not appear in its rewrite rules, we obtain $s[\Omega]_p \to^*_\alpha u$ from $s[\bullet]_p \to^*_\alpha u$ by replacing all occurrences of $\bullet$ by Ω. It follows that $\mathrm{nf}_\alpha(s[\Omega]_p)$ holds. We have $s[\Omega]_p \geqslant t$ as $t|_p = \Omega$. Hence p is not an nf_α-index position. $\square$

Corollary 38. Let α be an approximation mapping. Every left-linear α-sequential TRS belongs to CBN_α.

Proof. Let $\mathcal{R}$ be a left-linear α-sequential TRS. We show that every reducible term s has an α-needed redex. Let t be the Ω-normal form obtained from s by replacing all outermost redexes by Ω. Because $\mathcal{R}$ is α-sequential, t has an nf_α-index, say at Ω-position p. We obviously have $s \geqslant t$. According to the previous lemma the redex at position p in s is α-needed. We conclude that $\mathcal{R} \in \mathsf{CBN}_\alpha$. $\square$

The reverse directions do not hold in general. For the strong approximation this is kind of surprising since redexes carry the same information as Ω because the former can reduce to any term.

Example 39. Consider the TRS $\mathcal{R}$

$$\mathsf{f}(x,\mathsf{g}(y),\mathsf{h}(z)) \to x \quad \mathsf{f}(\mathsf{h}(z),x,\mathsf{g}(y)) \to x \quad \mathsf{f}(\mathsf{g}(y),\mathsf{h}(z),x) \to x \quad \mathsf{a} \to \mathsf{a}$$

over the signature $\mathcal{F}$ consisting of all symbols appearing in the rewrite rules. As $\mathsf{NF}(\mathcal{R}) = \varnothing$, $\mathcal{R}$ trivially belongs to $\mathsf{CBN_S}$. However, $\mathcal{R}$ is not strongly sequential since the Ω-normal form $\mathsf{f}(\Omega,\Omega,\Omega)$ does not have an nf_S-index:

$$\mathsf{f}(\Omega,\mathsf{g}(\mathsf{a}),\mathsf{h}(\mathsf{a})) \to_\mathsf{S} x \quad \mathsf{f}(\mathsf{h}(\mathsf{a}),\Omega,\mathsf{g}(\mathsf{a})) \to_\mathsf{S} x \quad \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{h}(\mathsf{a}),\Omega) \to_\mathsf{S} x$$

The following lemma states that for orthogonal TRSs the discrepancy between strong sequentiality and $\mathsf{CBN_S}$ can only occur if there are no ground normal forms.

Lemma 40. *Let $\mathcal{R}$ be an orthogonal TRS over a signature $\mathcal{F}$ such that $\mathsf{NF}(\mathcal{R}) \neq \varnothing$. If $\mathcal{R} \in \mathsf{CBN_S}$ then $\mathcal{R}$ is strongly sequential.*

Proof. Suppose that $\mathcal{R}$ is not strongly sequential. So there exists an Ω-normal form $t \in \mathcal{T}(\mathcal{F}_\Omega,\mathcal{V})$ without nf_S-indices. Let $u \in \mathcal{T}(\mathcal{F})$ be the term obtained from t by replacing all occurrences of Ω by a ground redex. (Since the empty TRS is trivially strongly sequential, $\mathcal{R}$ contains at least one rule.) We claim that u has no S-needed redexes. Let P be the set of Ω-positions in t, which coincides with the set of redex positions in u because of orthogonality. Let $p \in P$. We show that the redex in u at position p is not S-needed. Since p is not an nf_S-index position in t, we have $\mathrm{nf}_\mathsf{S}(s)$ for some term $s \in \mathcal{T}(\mathcal{F}_\Omega,\mathcal{V})$ with $s \geqslant t$ and $s|_p = \Omega$. Without loss of generality we assume that p is the only Ω-position in s. There exists a rewrite sequence $A\colon s \to_\mathsf{S}^* s'$ with $s' \in \mathcal{T}(\mathcal{F},\mathcal{V})$ a normal form. Since there is no Ω in s', A must contain a rewrite step at a position $q < p$. Let $s_1 \to_\mathsf{S} s_2$ be the first such step. By simply replacing every occurrence of Ω by a variable, we may assume that the remainder $s_2 \to_\mathsf{S}^* s'$ of A does not contain any occurrences of Ω. We will now transform A into a sequence $B\colon u[\bullet]_p \to_\mathsf{S}^* u'$ with $u' \in \mathsf{NF}(\mathcal{R}_\bullet)$, which implies that redex $u|_p$ is not S-needed. By replacing every variable in A by some constant we obtain the sequence $\hat{A}\colon \hat{s} \to_\mathsf{S}^* \hat{s}_1 \to_\mathsf{S} \hat{s}_2 \to_\mathsf{S}^* \hat{s}'$, where $\hat{s}'$ need not be in normal form. Next we replace all occurrences of Ω in $\hat{s} \to_\mathsf{S}^* \hat{s}_1$ by $\bullet$, yielding $\hat{u} \to_\mathsf{S}^* \hat{u}_1$. Because redexes S-rewrite to all possible terms and $\hat{u}|_p = \bullet$, we clearly have $u[\bullet]_p \to_\mathsf{S}^* \hat{u}$. Note that $\hat{u}_1$ contains a single occurrence of $\bullet$, at position p, and a redex at position q. We obtain $\hat{u}_1 \to_\mathsf{S} \hat{s}_2$ by contracting this redex. Combining the various parts yields $u[\bullet]_p \to_\mathsf{S}^* \hat{s}'$. If we can S-rewrite $\hat{s}'$ to a ground normal form then we obtain the desired rewrite sequence B. It is easy to see that repeatedly replacing redexes by any ground normal form, whose existence is guaranteed by the assumption $\mathsf{NF}(\mathcal{R}) \neq \varnothing$, will terminate in a ground normal form. $\square$

The following example shows that Lemma 40 need not be true for left-linear TRSs.

Example 41. Consider the left-linear TRS $\mathcal{R}$

$$\begin{array}{lll} \mathsf{g}(\mathsf{f}(x,\mathsf{a})) \to \mathsf{a} & \mathsf{f}(\mathsf{g}(x),\mathsf{g}(y)) \to \mathsf{a} & \mathsf{f}(\mathsf{g}(x),\mathsf{f}(y,z)) \to \mathsf{a} \\ \mathsf{g}(\mathsf{f}(\mathsf{a},x)) \to \mathsf{a} & \mathsf{f}(\mathsf{f}(x,y),\mathsf{f}(z,u)) \to \mathsf{a} & \mathsf{f}(\mathsf{f}(x,y),\mathsf{g}(z)) \to \mathsf{a} \end{array}$$

The Ω-normal form $\mathsf{g}(\mathsf{f}(\Omega,\Omega))$ has no nf_{s}-indices:

$$\mathsf{g}(\mathsf{f}(\Omega,\mathsf{a})) \to_{\mathsf{s}} \mathsf{a} \quad \mathsf{g}(\mathsf{f}(\mathsf{a},\Omega)) \to_{\mathsf{s}} \mathsf{a}$$

and hence $\mathcal{R}$ is not strongly sequential. Membership in $\mathsf{CBN}_{\mathsf{s}}$ is not hard to prove.[9]

The reader may wonder why the definitions in this section are not restricted to ground terms. The reason is that the standard decision procedure for nf_{s}-indices requires the existence of variables. To see this, let us recall the details of this procedure [12,17].

A term $t \in \mathcal{T}(\mathcal{F}_\Omega,\mathcal{V})$ is redex-compatible if $t \leqslant u$ for some redex u. The relation $\to_\Omega$ is defined as follows: $C[t] \to_\Omega C[\Omega]$ for every context C and redex-compatible term $t \neq \Omega$. The relation $\to_\Omega$ is confluent and terminating, and hence every term t admits a unique normal form with respect to $\to_\Omega$, which is denoted by $\omega(t)$. Now, an Ω-position p in t is an nf_{s}-index if and only if $p \in \mathcal{P}\mathsf{os}(\omega(t[\bullet]_p))$. The proof of this equivalence (see [17, Lemma 4.8]) relies on the existence of variables.

Returning to Example 39, we have $\omega(\mathsf{f}(\bullet,\Omega,\Omega)) = \omega(\mathsf{f}(\Omega,\bullet,\Omega)) = \omega(\mathsf{f}(\Omega,\Omega,\bullet)) = \Omega$, confirming that the term $\mathsf{f}(\Omega,\Omega,\Omega)$ indeed lacks nf_{s}-indices. If we would restrict the above sequentiality definitions to ground terms, then all Ω-positions would become nf_{s}-indices; because of the rewrite rule $\mathsf{a} \to \mathsf{a}$ there are no ground normal forms without Ω and hence $\mathrm{nf}_{\mathsf{s}}(t)$ fails as soon as t contains an occurrence of Ω.[10]

After this digression we return to the comparison between CBN_α and α-sequentiality. It is easy to show that $\mathsf{CBN}_{\mathsf{nv}}$ properly includes the class of nv-sequential TRSs (and hence also the class of NV-sequential TRSs introduced by Oyamaguchi [25]).

Example 42. Consider the TRS $\mathcal{R}_1$ defined in the proof of Lemma 33. The following rewrite steps show that the Ω-normal form $\mathsf{f}(\Omega,\Omega,\Omega)$ does not have an index with respect to $\mathrm{nf}_{\mathsf{nv}}$:

$$\mathsf{f}(\Omega,\mathsf{a},\mathsf{b}) \to_{\mathsf{nv}} \mathsf{c} \quad \mathsf{f}(\mathsf{b},\Omega,\mathsf{a}) \to_{\mathsf{nv}} \mathsf{b} \quad \mathsf{f}(\mathsf{a},\mathsf{b},\Omega) \to_{\mathsf{nv}} \mathsf{a}$$

Since $\mathcal{R}_1 \in \mathsf{CBN}_{\mathsf{nv}}$, it follows that the class of nv-sequential TRSs is a proper subclass of $\mathsf{CBN}_{\mathsf{nv}}$.

It is interesting to note that the same example illustrates that Huet and Lévy's sequentiality concept does *not* capture the class of (orthogonal) TRSs that admit a (computable or otherwise) call-by-need strategy. Since $\mathcal{R}_1$ is right-ground, we have $\to_{\mathsf{nv}} = \to_{\mathcal{R}_1}$ and thus $\mathrm{nf}_{\mathsf{nv}} = \mathrm{nf}$. Hence $\mathcal{R}_1$ is not sequential. Because $\mathcal{R}_1$ is orthogonal and belongs to $\mathsf{CBN}_{\mathsf{nv}}$, it obviously admits a computable call-by-need strategy. Since $\mathcal{R}_1$ is not g-sequential but belongs to $\mathsf{CBN}_{\mathsf{g}}$, it is clear that $\mathsf{CBN}_{\mathsf{g}}$ properly includes the class of g-sequential TRSs (and thus the class of growing sequential TRSs).

[9] Membership can also be verified by the Autowrite tool; see the description preceding Definition 44.

[10] It follows that the suggestion made in the Footnote 2 in [2] to simulate variables by enriching the signature is mandatory rather than optional.

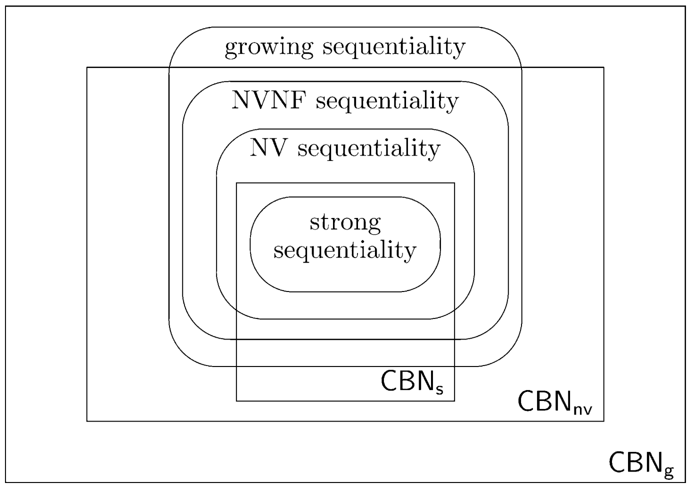

Fig. 1. Comparison.

Fig. 1 summarizes the findings of this section. Concerning the placement of CBN_s, the TRS $\mathcal{R}$ in Example 39 is not nv-sequential. To show that CBN_s contains TRSs that are not g-sequential, we need to slightly modify the example.

Example 43. Consider the TRS $\mathcal{R}$

$$\begin{array}{ll} \mathsf{f}(x,\mathsf{g}(y),\mathsf{h}(z)) \to \mathsf{i}(\mathsf{g}(x)) & \mathsf{i}(\mathsf{g}(x)) \to x \\ \mathsf{f}(\mathsf{h}(z),x,\mathsf{g}(y)) \to \mathsf{i}(\mathsf{g}(x)) & \qquad \mathsf{a} \to \mathsf{a} \\ \mathsf{f}(\mathsf{g}(y),\mathsf{h}(z),x) \to \mathsf{i}(\mathsf{g}(x)) & \end{array}$$

over the signature $\mathcal{F}$ consisting of all symbols appearing in the rewrite rules. We have $\mathcal{R} \in \mathsf{CBN}_\mathsf{s}$ because $\mathsf{NF}(\mathcal{R}) = \varnothing$. The TRS $\mathcal{R}$ is not g-sequential since the Ω-normal form $\mathsf{f}(\Omega,\Omega,\Omega)$ does not have an nf_g-index:

$$\begin{array}{l} \mathsf{f}(\Omega,\mathsf{g}(\mathsf{a}),\mathsf{h}(\mathsf{a})) \to_\mathsf{g} \mathsf{i}(\mathsf{g}(\Omega)) \to_\mathsf{g} x \\ \mathsf{f}(\mathsf{h}(\mathsf{a}),\Omega,\mathsf{g}(\mathsf{a})) \to_\mathsf{g} \mathsf{i}(\mathsf{g}(\Omega)) \to_\mathsf{g} x \\ \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{h}(\mathsf{a}),\Omega) \to_\mathsf{g} \mathsf{i}(\mathsf{g}(\Omega)) \to_\mathsf{g} x \end{array}$$

8. Signature extension

In this section we study the question whether membership in CBN_α is preserved after adding new function symbols. This entails that we need to be a bit more precise about the underlying signature in our notation. From now on we write $\mathsf{NF}(\mathcal{R},\mathcal{F})$ for the set of ground normal forms of an eTRS $\mathcal{R}$ over a signature $\mathcal{F}$. Furthermore, an α-needed redex with respect to a TRS $\mathcal{R}$ over the signature $\mathcal{F}$ will often be called $(\mathcal{R}_\alpha,\mathcal{F})$-needed in the sequel.

Many of the examples presented in this and the next section have been verified by Autowrite. This tool, described in Durand [7], checks membership in CBN_α for $\alpha \in \{\mathsf{s},\mathsf{nv},\mathsf{g}\}$ by using the direct (as opposed to the ground tree transducer constructions of Sections 5 and 6) tree automata constructions described in [10].

Definition 44. We say that a class $\mathcal{C}$ of TRSs is *preserved under signature extension* if $(\mathcal{R}, \mathcal{G}) \in \mathcal{C}$ for all $(\mathcal{R}, \mathcal{F}) \in \mathcal{C}$ and $\mathcal{F} \subseteq \mathcal{G}$.

Our first example shows that $\mathsf{CBN_s}$ is not preserved under signature extension.

Example 45. Consider the TRS $(\mathcal{R}, \mathcal{F})$ of Example 39. Let $\mathcal{G} = \mathcal{F} \cup \{\mathsf{b}\}$ with b a constant. We have $(\mathcal{R}, \mathcal{G}) \notin \mathsf{CBN_s}$ as the term $\mathsf{f(a, a, a)}$ has no $(\mathcal{R}_\mathsf{s}, \mathcal{G})$-needed redex:

$$\begin{array}{l}
\mathsf{f}(\bullet, \mathsf{a}, \mathsf{a}) \to_\mathsf{s} \mathsf{f}(\bullet, \mathsf{g(a)}, \mathsf{a}) \to_\mathsf{s} \mathsf{f}(\bullet, \mathsf{g(a)}, \mathsf{h(a)}) \to_\mathsf{s} \mathsf{b} \\
\mathsf{f}(\mathsf{a}, \bullet, \mathsf{a}) \to_\mathsf{s} \mathsf{f}(\mathsf{h(a)}, \bullet, \mathsf{a}) \to_\mathsf{s} \mathsf{f}(\mathsf{h(a)}, \bullet, \mathsf{g(a)}) \to_\mathsf{s} \mathsf{b} \\
\mathsf{f}(\mathsf{a}, \mathsf{a}, \bullet) \to_\mathsf{s} \mathsf{f}(\mathsf{g(a)}, \mathsf{a}, \bullet) \to_\mathsf{s} \mathsf{f}(\mathsf{g(a)}, \mathsf{h(a)}, \bullet) \to_\mathsf{s} \mathsf{b}
\end{array}$$

One may wonder whether there are any non-trivial counterexamples, where non-trivial means that the set of ground normal forms is non-empty. Surprisingly, the answer is yes, provided we consider an approximation mapping α that is at least as good as nv.

Example 46. Consider the TRS $\mathcal{R}$

$$\begin{array}{lll}
\mathsf{f}(x, \mathsf{a}, \mathsf{b}) \to \mathsf{g}(x) & \mathsf{f(a, a, a)} \to \mathsf{g(a)} & \mathsf{g(a)} \to \mathsf{g(a)} \\
\mathsf{f}(\mathsf{b}, x, \mathsf{a}) \to \mathsf{g}(x) & \mathsf{f(b, b, b)} \to \mathsf{g(a)} & \mathsf{g(b)} \to \mathsf{g(b)} \\
\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{g}(x) & \quad \mathsf{e}(x) \to x &
\end{array}$$

over the signature $\mathcal{F}$ consisting of all symbols appearing in the rewrite rules. First we show that $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN_{nv}}$. It is not difficult to show that the only $(\mathcal{R}_\mathsf{nv}, \mathcal{F})$-normalizable terms are a, b, and $\mathsf{e}(t)$ for every $t \in \mathcal{T}(\mathcal{F})$. Since a and b are normal forms, we only have to show that every $\mathsf{e}(t)$ contains an $(\mathcal{R}_\mathsf{nv}, \mathcal{F})$-needed redex, which is easy since $\mathsf{e}(t)$ itself is an $(\mathcal{R}_\mathsf{nv}, \mathcal{F})$-needed redex. Let $\mathcal{G} = \mathcal{F} \cup \{\mathsf{c}\}$ with c a constant. We have $(\mathcal{R}, \mathcal{G}) \notin \mathsf{CBN_{nv}}$ as the term $\mathsf{f(e(a), e(a), e(a))}$ has no $(\mathcal{R}_\mathsf{nv}, \mathcal{G})$-needed redex:

$$\begin{array}{l}
\mathsf{f}(\bullet, \mathsf{e(a)}, \mathsf{e(a)}) \to_\mathsf{nv} \mathsf{f}(\bullet, \mathsf{a}, \mathsf{e(a)}) \to_\mathsf{nv} \mathsf{f}(\bullet, \mathsf{a}, \mathsf{b}) \to_\mathsf{nv} \mathsf{g(c)} \\
\mathsf{f}(\mathsf{e(a)}, \bullet, \mathsf{e(a)}) \to_\mathsf{nv} \mathsf{f}(\mathsf{b}, \bullet, \mathsf{e(a)}) \to_\mathsf{nv} \mathsf{f}(\mathsf{b}, \bullet, \mathsf{a}) \to_\mathsf{nv} \mathsf{g(c)} \\
\mathsf{f}(\mathsf{e(a)}, \mathsf{e(a)}, \bullet) \to_\mathsf{nv} \mathsf{f}(\mathsf{a}, \mathsf{e(a)}, \bullet) \to_\mathsf{nv} \mathsf{f}(\mathsf{a}, \mathsf{b}, \bullet) \to_\mathsf{nv} \mathsf{g(c)}
\end{array}$$

For $\alpha = \mathsf{s}$ there is no non-trivial counterexample.

Theorem 47. *The subclass of* $\mathsf{CBN_s}$ *consisting of all orthogonal TRSs* $(\mathcal{R}, \mathcal{F})$ *such that* $\mathsf{NF}(\mathcal{R}, \mathcal{F}) \neq \varnothing$ *is preserved under signature extension.*

We refrain from giving the proof at this point since the statement easily follows from Theorem 52 below, whose proof is presented in detail in Appendix A. (See also the discussion following Corollary 60). We just show the necessity of the orthogonality condition.

Example 48. Consider the left-linear TRS $\mathcal{R}$

$$\begin{array}{ll}
\mathsf{f}(x, \mathsf{a}) \to \mathsf{a} & \mathsf{g}(\mathsf{f}(\mathsf{a}, x), y) \to \mathsf{a} \\
\mathsf{g}(x, \mathsf{a}) \to \mathsf{a} & \mathsf{g}(x, \mathsf{f}(y, z)) \to \mathsf{a} \\
 & \mathsf{g}(x, \mathsf{g}(y, z)) \to \mathsf{a}
\end{array}$$

over the signature $\mathcal{F} = \{\mathsf{a}, \mathsf{f}, \mathsf{g}\}$. Autowrite is able to verify that $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN_s}$. Let $\mathcal{G} = \mathcal{F} \cup \{\mathsf{c}\}$ with c a constant. The TRS $(\mathcal{R}, \mathcal{G})$ does not belong to $\mathsf{CBN_s}$ because the term $\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{a},\mathsf{a}),\mathsf{f}(\mathsf{a},\mathsf{a})),\mathsf{c})$ lacks $(\mathcal{R}_\mathsf{s}, \mathcal{G})$-needed redexes:

$$\mathsf{g}(\mathsf{f}(\bullet,\mathsf{f}(\mathsf{a},\mathsf{a})),\mathsf{c}) \to_\mathsf{s} \mathsf{g}(\mathsf{f}(\bullet,\mathsf{a}),\mathsf{c}) \to_\mathsf{s} \mathsf{g}(\mathsf{a},\mathsf{c})$$
$$\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{a},\mathsf{a}),\bullet),\mathsf{c}) \to_\mathsf{s} \mathsf{g}(\mathsf{f}(\mathsf{a},\bullet),\mathsf{c}) \to_\mathsf{s} \mathsf{a}$$

Note that here only the rewrite rules $\mathsf{f}(x,\mathsf{a}) \to \mathsf{a}$ and $\mathsf{g}(\mathsf{f}(\mathsf{a},x),y) \to \mathsf{a}$ are used. The remaining rules of $\mathcal{R}$ are needed to ensure that $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN_s}$.

Our second result states that for any approximation mapping α the subclass of CBN_α consisting of all left-linear TRSs $\mathcal{R}$ with the property defined below is preserved under signature extension.

Definition 49. We say that a TRS $\mathcal{R}$ has *external* normal forms if there exists a ground normal form which is not an instance of a proper non-variable subterm of a left-hand sides of a rewrite rule in $\mathcal{R}$.

Note that the TRS of Example 46 lacks external normal forms as both ground normal forms a and b appear in the left-hand sides of the rewrite rules. Further note that it is decidable whether a left-linear TRS has external normal forms by straightforward tree automata techniques. Finally note that the external normal form property is satisfied whenever there exists a constant not occurring in the left-hand sides of the rewrite rules.

Theorem 50. *Let α be an approximation mapping. The subclass of CBN_α consisting of all left-linear TRSs with external normal forms is preserved under signature extension.*

The proof is given in Appendix A. Note that for $\alpha = \mathsf{s}$ the above theorem is a special case of Theorem 47 since the existence of an external normal form implies the existence of a ground normal form.

Our final signature extension result is about TRSs without external normal form. Such TRSs are quite common.

Example 51. Consider the TRS $\mathcal{R}$ of Example 5 over the signature $\mathcal{F}$ consisting of all symbols appearing in the rewrite rules. Since every ground normal form is of the form $\mathsf{s}^n(\mathsf{0})$ for some $n \geqslant 0$, it follows that $\mathcal{R}$ lacks external normal forms.

We denote by $\mathsf{WN}(\mathcal{R}, \mathcal{F})$ the set of all ground terms in $\mathcal{T}(\mathcal{F})$ that rewrite in $\mathcal{R}$ to a normal form in $\mathsf{NF}(\mathcal{R}, \mathcal{F})$. If no confusion can arise, we just write $\mathsf{WN}(\mathcal{R})$. Let $\mathcal{F} \subseteq \mathcal{G}$. We denote by $\mathsf{WN}(\mathcal{R}, \mathcal{G}, \mathcal{F})$ the set of terms in $\mathcal{T}(\mathcal{F})$ that have a normal form with respect to $(\mathcal{R}, \mathcal{G})$.

The condition $\mathsf{WN}(\mathcal{R}_\alpha, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_\alpha, \mathcal{G}, \mathcal{F})$ in Theorem 52 expresses that the set of $\mathcal{R}_\alpha$-normalizable terms in $\mathcal{T}(\mathcal{F})$ is not enlarged by allowing terms in $\mathcal{T}(\mathcal{G})$ to be substituted for the variables in the rewrite rules. We stress that this condition is decidable for left-linear $\mathcal{R}$ and regularity preserving α by standard tree automata techniques.

Theorem 52. *Let $\mathcal{R}$ be an orthogonal TRS over a signature $\mathcal{F}, \alpha \in \{\mathsf{s}, \mathsf{nv}\}$, and $\mathcal{F} \subseteq \mathcal{G}$ such that $\mathsf{WN}(\mathcal{R}_\alpha, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_\alpha, \mathcal{G}, \mathcal{F})$. If $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN}_\alpha$ and $\mathcal{R}_\alpha$ is collapsing then $(\mathcal{R}, \mathcal{G}) \in \mathsf{CBN}_\alpha$.*

The necessity of the $\mathsf{WN}(\mathcal{R}_\alpha, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_\alpha, \mathcal{G}, \mathcal{F})$ condition for collapsing $\mathcal{R}_\alpha$ is a consequence of Example 46. The TRS $\mathcal{R}$ in that example is a collapsing orthogonal TRS with $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN_{nv}}$,

$(\mathcal{R}, \mathcal{G}) \notin \mathsf{CBN}_{\mathsf{nv}}$, and $\mathsf{WN}(\mathcal{R}_{\mathsf{nv}}, \mathcal{F}) \neq \mathsf{WN}(\mathcal{R}_{\mathsf{nv}}, \mathcal{G}, \mathcal{F})$ as witnessed by the term $\mathsf{f}(\mathsf{a}, \mathsf{a}, \mathsf{b})$. The following example shows the necessity of the collapsing condition.

Example 53. Consider TRS $\mathcal{R}$

$$
\begin{array}{rclrcl}
\mathsf{f}(x, \mathsf{a}, \mathsf{b}(y, z)) & \to & \mathsf{c}(\infty) & \mathsf{g}(x) & \to & \mathsf{b}(x, \infty) \\
\mathsf{f}(x, \mathsf{a}, \mathsf{c}(y)) & \to & \infty & \mathsf{h}(\mathsf{a}) & \to & \infty \\
\mathsf{f}(\mathsf{a}, \mathsf{a}, \mathsf{a}) & \to & \infty & \mathsf{h}(\mathsf{b}(\mathsf{a}, x)) & \to & \mathsf{a} \\
\mathsf{f}(\mathsf{a}, \mathsf{b}(x, y), z) & \to & \mathsf{a} & \mathsf{h}(\mathsf{b}(\mathsf{b}(x, y), z)) & \to & \mathsf{b}(\infty, \infty) \\
\mathsf{f}(\mathsf{a}, \mathsf{c}(x), y) & \to & \infty & \mathsf{h}(\mathsf{b}(\mathsf{c}(x), y)) & \to & \infty \\
\mathsf{f}(\mathsf{b}(x, y), z, \mathsf{a}) & \to & \mathsf{a} & \mathsf{h}(\mathsf{c}(x)) & \to & \infty \\
\mathsf{f}(\mathsf{b}(x, y), \mathsf{b}(z, u), \mathsf{b}(v, w)) & \to & \infty & \mathsf{i}(\mathsf{a}, \mathsf{a}) & \to & \infty \\
\mathsf{f}(\mathsf{b}(x, y), \mathsf{b}(z, u), \mathsf{c}(v)) & \to & \infty & \mathsf{i}(\mathsf{a}, \mathsf{b}(x, y)) & \to & \infty \\
\mathsf{f}(\mathsf{b}(x, y), \mathsf{c}(z), \mathsf{b}(u, v)) & \to & \infty & \mathsf{i}(\mathsf{a}, \mathsf{c}(x)) & \to & \infty \\
\mathsf{f}(\mathsf{b}(x, y), \mathsf{c}(z), \mathsf{c}(u)) & \to & \infty & \mathsf{i}(\mathsf{b}(x, y), z) & \to & \infty \\
\mathsf{f}(\mathsf{c}(x), \mathsf{a}, \mathsf{a}) & \to & \infty & \mathsf{i}(\mathsf{c}(x), y) & \to & \mathsf{a} \\
\mathsf{f}(\mathsf{c}(x), \mathsf{b}(y, z), \mathsf{a}) & \to & \infty & \infty & \to & \infty \\
\mathsf{f}(\mathsf{c}(x), \mathsf{b}(y, z), \mathsf{c}(u)) & \to & \infty & & & \\
\mathsf{f}(\mathsf{c}(x), \mathsf{b}(y, z), \mathsf{b}(u, v)) & \to & \infty & & & \\
\mathsf{f}(\mathsf{c}(x), \mathsf{c}(y), z) & \to & \infty & & &
\end{array}
$$

over the signature $\mathcal{F}$ consisting of all symbols appearing in the rewrite rules and let $\mathcal{G} = \mathcal{F} \cup \{\mathsf{d}\}$ with d a constant. One easily checks that the term $\mathsf{i}(\mathsf{f}(\Delta, \Delta, \Delta), \mathsf{d})$ with $\Delta = \mathsf{h}(\mathsf{g}(\mathsf{d}))$ lacks $(\mathcal{R}_{\mathsf{nv}}, \mathcal{G})$-needed redexes and hence $(\mathcal{R}, \mathcal{G}) \notin \mathsf{CBN}_{\mathsf{nv}}$. Autowrite is able to verify $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN}_{\mathsf{nv}}$ and $\mathsf{WN}(\mathcal{R}_{\mathsf{nv}}, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_{\mathsf{nv}}, \mathcal{G}, \mathcal{F})$.

The next example shows the necessity of the restriction to $\alpha \in \{\mathsf{s}, \mathsf{nv}\}$.

Example 54. Consider the orthogonal TRS $\mathcal{R}$

$$
\begin{array}{rclrcl}
\mathsf{f}(x, \mathsf{a}, \mathsf{b}(y), z) & \to & \mathsf{h}(z) & \mathsf{h}(\mathsf{a}) & \to & \infty \\
\mathsf{f}(\mathsf{b}(x), y, \mathsf{a}, z) & \to & \mathsf{h}(z) & \mathsf{h}(\mathsf{b}(x)) & \to & \infty \\
\mathsf{f}(\mathsf{a}, \mathsf{b}(x), y, z) & \to & \mathsf{h}(z) & \mathsf{i}(\mathsf{b}(x)) & \to & \mathsf{j}(\infty, x) \\
\mathsf{f}(\mathsf{a}, \mathsf{a}, \mathsf{a}, z) & \to & \infty & \mathsf{i}(\mathsf{a}) & \to & \infty \\
\mathsf{f}(\mathsf{b}(x), \mathsf{b}(y), \mathsf{b}(z), u) & \to & \infty & \mathsf{j}(x, \mathsf{a}) & \to & \mathsf{a} \\
\infty & \to & \infty & \mathsf{j}(x, \mathsf{b}(y)) & \to & \mathsf{b}(\mathsf{a})
\end{array}
$$

over the signature $\mathcal{F}$ consisting of all symbols appearing in the rewrite rules. Note that the growing approximation only modifies the rule $\mathsf{i}(\mathsf{b}(x)) \to \mathsf{j}(\infty, x)$ into $\mathsf{i}(\mathsf{b}(x)) \to \mathsf{j}(\infty, y)$. Let $\mathcal{G} = \mathcal{F} \cup \{\mathsf{c}\}$ with c a constant. As the term $\mathsf{f}(\mathsf{i}(\mathsf{b}(\mathsf{c})), \mathsf{i}(\mathsf{b}(\mathsf{c})), \mathsf{i}(\mathsf{b}(\mathsf{c}))), \mathsf{c})$ lacks $(\mathcal{R}_{\mathsf{g}}, \mathcal{G})$-needed redexes, $(\mathcal{R}, \mathcal{G}) \notin \mathsf{CBN}_{\mathsf{g}}$. Autowrite is able to verify $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN}_{\mathsf{g}}$ and $\mathsf{WN}(\mathcal{R}_{\mathsf{g}}, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_{\mathsf{g}}, \mathcal{G}, \mathcal{F})$. Note that $\mathcal{R}$ is not collapsing. This is not essential, since adding the single collapsing rule $\mathsf{k}(x) \to x$ to $\mathcal{R}$ does not affect any of the above properties.

We show that Theorem 47 is a special case of Theorem 52 by proving that for $\alpha = \mathsf{s}$ the condition $\mathsf{WN}(\mathcal{R}_{\alpha}, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_{\alpha}, \mathcal{G}, \mathcal{F})$ is a consequence of $\mathsf{NF}(\mathcal{R}, \mathcal{F}) \neq \varnothing$.

Lemma 55. *Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$. If* $\mathsf{NF}(\mathcal{R},\mathcal{F}) \neq \varnothing$ *then* $\mathsf{WN}(\mathcal{R}_{\mathsf{s}},\mathcal{F}) = \mathcal{T}(\mathcal{F})$.

Proof. If $\mathsf{NF}(\mathcal{R},\mathcal{F}) \neq \varnothing$ then there must be a constant $c \in \mathsf{NF}(\mathcal{R},\mathcal{F})$. Define the TRS $\mathcal{R}' = \{l \to c \mid l \to r \in \mathcal{R}\}$ over the signature $\mathcal{F}$. Clearly $\to_{\mathcal{R}'} \subseteq \to_{\mathsf{s}}$. The TRS $\mathcal{R}'$ is terminating since every rewrite step reduces the number of function symbols in $\mathcal{F} \setminus \{c\}$. Since $\mathcal{R}_{\mathsf{s}}$ and $\mathcal{R}'$ have the same normal forms, it follows that $\mathcal{R}_{\mathsf{s}}$ is weakly normalizing. $\square$

Proof of Theorem 47. Let $\mathcal{R}$ be an orthogonal TRS over a signature $\mathcal{F}$ such that $(\mathcal{R},\mathcal{F}) \in \mathsf{CBN}_{\mathsf{s}}$. Let $\mathcal{F} \subseteq \mathcal{G}$. We have to show that $(\mathcal{R},\mathcal{G}) \in \mathsf{CBN}_{\mathsf{s}}$. If $\mathcal{R} = \varnothing$, this is trivial. Otherwise $\mathcal{R}_{\mathsf{s}}$ is collapsing and the result follows from Theorem 52 provided that $\mathsf{WN}(\mathcal{R}_{\mathsf{s}},\mathcal{F}) = \mathsf{WN}(\mathcal{R}_{\mathsf{s}},\mathcal{G},\mathcal{F})$. From Lemma 55 we obtain $\mathsf{WN}(\mathcal{R}_{\mathsf{s}},\mathcal{F}) = \mathcal{T}(\mathcal{F})$ and $\mathsf{WN}(\mathcal{R}_{\mathsf{s}},\mathcal{G},\mathcal{F}) = \mathsf{WN}(\mathcal{R}_{\mathsf{s}},\mathcal{G}) \cap \mathcal{T}(\mathcal{F}) = \mathcal{T}(\mathcal{G}) \cap \mathcal{T}(\mathcal{F}) = \mathcal{T}(\mathcal{F})$. $\square$

We conclude this section by remarking that we have to use Theorem 52 only once. After adding a single new function symbol we obtain an external normal form and hence we can apply Theorem 50 for the remaining new function symbols.

9. Modularity

The results obtained in the previous section form the basis for the modularity results presented in this section. We first consider disjoint combinations.

Definition 56. We say that a class $\mathcal{C}$ of TRSs is *modular* (*for disjoint combinations*) if $(\mathcal{R} \cup \mathcal{R}', \mathcal{F} \cup \mathcal{F}') \in \mathcal{C}$ for all $(\mathcal{R},\mathcal{F}), (\mathcal{R}',\mathcal{F}') \in \mathcal{C}$ such that $\mathcal{F} \cap \mathcal{F}' = \varnothing$.

To simplify notation, in the remainder of this section we write $\mathcal{S}$ for $\mathcal{R} \cup \mathcal{R}'$ and $\mathcal{G}$ for $\mathcal{F} \cup \mathcal{F}'$.

The condition in Theorem 50 is insufficient for modularity as shown by the following example.

Example 57. Consider the TRS $\mathcal{R}$

$$\mathsf{f}(x,\mathsf{a},\mathsf{b}) \to \mathsf{a} \quad \mathsf{f}(\mathsf{b},x,\mathsf{a}) \to \mathsf{a} \quad \mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{a}$$

over the signature $\mathcal{F}$ consisting of all symbols appearing in the rewrite rules and the TRS $\mathcal{R}' = \{\mathsf{g}(x) \to x\}$ over the signature $\mathcal{F}'$ consisting of a constant c in addition to g. Both TRSs have external normal forms and belong to $\mathsf{CBN}_{\mathsf{nv}}$, as one easily shows. Their union does not belong to $\mathsf{CBN}_{\mathsf{nv}}$ as the term $\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}))$ has no $(\mathcal{S}_{\mathsf{nv}},\mathcal{G})$-needed redex:

$$\begin{array}{l}
\mathsf{f}(\bullet,\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a})) \to_{\mathsf{nv}} \mathsf{f}(\bullet,\mathsf{a},\mathsf{g}(\mathsf{a})) \to_{\mathsf{nv}} \mathsf{f}(\bullet,\mathsf{a},\mathsf{b}) \to_{\mathsf{nv}} \mathsf{a} \\
\mathsf{f}(\mathsf{g}(\mathsf{a}),\bullet,\mathsf{g}(\mathsf{a})) \to_{\mathsf{nv}} \mathsf{f}(\mathsf{b},\bullet,\mathsf{g}(\mathsf{a})) \to_{\mathsf{nv}} \mathsf{f}(\mathsf{b},\bullet,\mathsf{a}) \to_{\mathsf{nv}} \mathsf{a} \\
\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(\mathsf{a}),\bullet) \to_{\mathsf{nv}} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a}),\bullet) \to_{\mathsf{nv}} \mathsf{f}(\mathsf{a},\mathsf{b},\bullet) \to_{\mathsf{nv}} \mathsf{a}
\end{array}$$

If we forbid collapsing rules like $\mathsf{g}(x) \to x$, modularity holds. The following theorem is proved along the lines of the proof of Theorem 50; because there are no collapsing rules and the eTRSs are left-linear, aliens (see Appendix A) cannot influence the possibility to perform a rewrite step in the non-alien part of a term.

Theorem 58. *Let α be an arbitrary approximation mapping. The subclass of* CBN_{α} *consisting of all left-linear TRSs $\mathcal{R}$ with external normal forms such that $\mathcal{R}_{\alpha}$ is non-collapsing is modular.*

The following result is the modularity counterpart of Theorem 52. The proof is given in Appendix A.

Theorem 59. *Let* $(\mathcal{R},\mathcal{F})$ *and* $(\mathcal{R}',\mathcal{F}')$ *be disjoint orthogonal TRSs and* $\alpha \in \{\mathsf{s},\mathsf{nv}\}$ *such that both* $\mathsf{WN}(\mathcal{R}_\alpha,\mathcal{G},\mathcal{F}) = \mathsf{WN}(\mathcal{R}_\alpha,\mathcal{F})$ *and* $\mathsf{WN}(\mathcal{R}'_\alpha,\mathcal{G},\mathcal{F}') = \mathsf{WN}(\mathcal{R}'_\alpha,\mathcal{F}')$. *If* $(\mathcal{R},\mathcal{F}),(\mathcal{R}',\mathcal{F}') \in \mathsf{CBN}_\alpha$ *and both* $\mathcal{R}_\alpha$ *and* $\mathcal{R}'_\alpha$ *are collapsing then* $(\mathcal{S},\mathcal{G}) \in \mathsf{CBN}_\alpha$.

It is rather surprising that the presence of collapsing rules helps to achieve modularity; for most properties of TRSs collapsing rules are an obstacle for modularity (see, e.g., Middeldorp [20]).

The next result is the modularity counterpart of Theorem 47. It is an easy corollary of the preceding theorem.

Corollary 60. The subclass of CBN_s consisting of all orthogonal TRSs $(\mathcal{R},\mathcal{F})$ such that $\mathsf{NF}(\mathcal{R},\mathcal{F}) \neq \varnothing$ is modular.

Using Huet and Lévy's characterization of strong sequentiality by means of increasing indices, Klop and Middeldorp [17] showed that strong sequentiality is a modular property of orthogonal TRSs. Since membership in CBN_s coincides with strong sequentiality for orthogonal TRSs with ground normal forms (Lemma 40), this provides another proof of Corollary 60. Actually, in [17] it is remarked that it is sufficient that the left-hand sides of the two strongly sequential rewrite systems do not share function symbols. One easily verifies that for our modularity results it is sufficient that $\mathcal{R}_\alpha$ and $\mathcal{R}'_\alpha$ do not share function symbols. Actually, we can go a step further by considering so-called *constructor-sharing* combinations. In such combinations the participating systems may share constructors but not defined symbols.

Definition 61. Two TRSs $(\mathcal{R},\mathcal{F})$ and $(\mathcal{R}',\mathcal{F}')$ *share constructors* if $\mathcal{F}_\mathcal{D} \cap \mathcal{F}' = \mathcal{F}'_\mathcal{D} \cap \mathcal{F} = \varnothing$. We say that a class $\mathcal{C}$ of TRSs is *constructor-sharing modular* if $(\mathcal{R} \cup \mathcal{R}',\mathcal{F} \cup \mathcal{F}') \in \mathcal{C}$ for all TRSs $(\mathcal{R},\mathcal{F}),(\mathcal{R}',\mathcal{F}') \in \mathcal{C}$ that share constructors.

It can be shown that the results obtained in this section extend to constructor-sharing combinations, provided we strengthen the requirements in Theorems 58 and 59 by forbidding the presence of *constructor-lifting* rules. A rewrite rule $l \to r$ is called constructor-lifting if $\mathsf{root}(r)$ is a shared constructor. In Appendix A we give a detailed proof of the extension of Theorem 58. The proof of Theorem 59 is easily extended to constructor-sharing combinations and hence omitted.

Theorem 62. *Let* $(\mathcal{R},\mathcal{F})$ *and* $(\mathcal{R}',\mathcal{F}')$ *be left-linear constructor-sharing TRSs with external normal forms and without constructor-lifting rules and let* α *be an approximation mapping such that* $\mathcal{R}_\alpha$ *and* $\mathcal{R}'_\alpha$ *are non-collapsing. If* $(\mathcal{R},\mathcal{F}),(\mathcal{R}',\mathcal{F}') \in \mathsf{CBN}_\alpha$ *then* $(\mathcal{S},\mathcal{G}) \in \mathsf{CBN}_\alpha$.

The reason for excluding constructor-lifting rules in Theorem 62 is shown in the following example.

Example 63. Consider the TRS $\mathcal{R}$

$$\mathsf{f}(x,\mathsf{c}(\mathsf{a}),\mathsf{c}(\mathsf{b})) \to \mathsf{a} \quad \mathsf{f}(\mathsf{c}(\mathsf{b}),x,\mathsf{c}(\mathsf{a})) \to \mathsf{a} \quad \mathsf{f}(\mathsf{c}(\mathsf{a}),\mathsf{c}(\mathsf{b}),x) \to \mathsf{a}$$

over the signature $\mathcal{F}$ consisting of all symbols appearing in the rewrite rules and the TRS $\mathcal{R}' = \{\mathsf{g}(x) \to \mathsf{c}(x)\}$ over the signature $\mathcal{F}'$ consisting of a constant d in addition to g and c. Both TRSs have external normal forms, lack collapsing rules, and belong to $\mathsf{CBN}_{\mathsf{nv}}$. Their union does not belong to $\mathsf{CBN}_{\mathsf{nv}}$ as the term $\mathsf{f}(\mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{a}), \mathsf{g}(\mathsf{a}))$ has no $(\mathcal{S}_{\mathsf{nv}}, \mathcal{G})$-needed redex. Note that $\mathcal{R}$ and $\mathcal{R}'$ share the constructor c and hence $\mathsf{g}(x) \to \mathsf{c}(x)$ is constructor-lifting.

Theorem 64. *Let $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$ be orthogonal constructor-sharing TRSs without constructor-lifting rules and $\alpha \in \{\mathsf{s}, \mathsf{nv}\}$ such that $\mathsf{WN}(\mathcal{R}_\alpha, \mathcal{G}, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_\alpha, \mathcal{F})$ and $\mathsf{WN}(\mathcal{R}'_\alpha, \mathcal{G}, \mathcal{F}') = \mathsf{WN}(\mathcal{R}'_\alpha, \mathcal{F}')$. If $(\mathcal{R}, \mathcal{F}), (\mathcal{R}', \mathcal{F}') \in \mathsf{CBN}_\alpha$ and both $\mathcal{R}_\alpha$ and $\mathcal{R}'_\alpha$ are collapsing then $(\mathcal{S}, \mathcal{G}) \in \mathsf{CBN}_\alpha$.*

Again, it is essential that constructor-lifting rules are excluded.

Example 65. Consider the TRSs $\mathcal{R}$

$$\begin{array}{ll} \mathsf{f}(x, \mathsf{a}, \mathsf{b}) \to \mathsf{c}(\mathsf{g}(x)) & \mathsf{g}(x) \to \mathsf{g}(\mathsf{a}) \\ \mathsf{f}(\mathsf{b}, x, \mathsf{a}) \to \mathsf{c}(\mathsf{g}(x)) & \mathsf{h}(x) \to x \\ \mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{c}(\mathsf{g}(x)) & \end{array}$$

and $\mathcal{R}' = \{\mathsf{i}(\mathsf{a}) \to \mathsf{a}, \mathsf{i}(\mathsf{c}(x)) \to x\}$ over the signatures $\mathcal{F}$ and $\mathcal{F}'$ consisting of function symbols that appear in their respective rewrite rules. The two TRSs are obviously collapsing and share the constructors a and c. One easily verifies that both TRSs belong to $\mathsf{CBN}_{\mathsf{nv}}$ and that $\mathsf{WN}(\mathcal{R}_{\mathsf{nv}}, \mathcal{G}, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_{\mathsf{nv}}, \mathcal{F})$ and $\mathsf{WN}(\mathcal{R}_{\mathsf{nv}}, \mathcal{G}, \mathcal{F}') = \mathcal{T}(\mathcal{F}') = \mathsf{WN}(\mathcal{R}'_{\mathsf{nv}}, \mathcal{F}')$. However, the union of the two TRSs does not belong to $\mathsf{CBN}_{\mathsf{nv}}$ as the term $\mathsf{i}(\mathsf{f}(\mathsf{h}(\mathsf{a}), \mathsf{h}(\mathsf{a}), \mathsf{h}(\mathsf{a})))$ has no $(\mathcal{S}_{\mathsf{nv}}, \mathcal{G})$-needed redex.

For the strong approximation we need of course not exclude constructor-sharing rules. Moreover, the two conditions $\mathsf{WN}(\mathcal{R}_{\mathsf{s}}, \mathcal{G}, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_\alpha, \mathcal{F})$ and $\mathsf{WN}(\mathcal{R}'_{\mathsf{s}}, \mathcal{G}, \mathcal{F}') = \mathsf{WN}(\mathcal{R}'_\alpha, \mathcal{F}')$ are always satisfied (cf. the proof of Theorem 47). Hence we can state the final result of the paper.

Corollary 66. Let $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$ be orthogonal constructor-sharing TRSs with ground normal forms. If $(\mathcal{R}, \mathcal{F}), (\mathcal{R}', \mathcal{F}') \in \mathsf{CBN}_{\mathsf{s}}$ then $(\mathcal{S}, \mathcal{G}) \in \mathsf{CBN}_{\mathsf{s}}$.

10. Conclusion

In this paper we introduced a new framework for the study of call-by-need computations in term rewriting. Our framework is parameterized by the concept of approximation mapping and we showed that regularity preservingness is the key to decidability, which is obtained by applying simple tree automata techniques. We performed a detailed study of the modularity aspects of our framework and we showed that our framework provides a better approximation to neededness than the sequentiality notions originating from the seminal paper of Huet and Lévy [12].

What we did not address in this paper is the important issue of compiling call-by-need strategies. The knowledge that every reducible term has at least one computable needed redex is clearly insufficient to obtain an efficient call-by-need strategy. Testing the redexes in a reducible term one by one until a needed redex is encountered is unattractive. Moreover, after a needed redex is identified and contracted, the search for a needed redex in the obtained term has to start from scratch. Huet and Lévy showed that every strongly sequential orthogonal TRS admits a so-called *matching dag*, which implements an efficient call-by-need strategy. Since in our framework neededness of a redex may depend on other redexes in a term, it is highly unlikely that a similar data structure exists for the efficient compilation of call-by-need strategies for the TRSs in CBN_α for $\alpha \in \{\mathsf{nv}, \mathsf{g}\}$.

Another issue we did not address is call-by-need strategies to root-stable forms. In [19] it is shown that *root-neededness* is more fundamental than neededness when it comes to infinitary normalization. However, root-stability is undecidable and, unlike neededness, root-neededness of a redex is not determined by its position. This considerably complicates the quest for a computable call-by-need strategy to root-stable forms. The interested reader is referred to [9, Sections 6 and 7] for some preliminary results in this direction.

Appendix

A. Proofs for Sections 8 and 9

A reducible term without $(\mathcal{R}_\alpha, \mathcal{F})$-needed redexes is called $(\mathcal{R}_\alpha, \mathcal{F})$-*free*. A *minimal*-free term has the property that none of its proper subterms is free.

The proofs of our signature extension results follow the same strategy. We consider a TRS $\mathcal{R}$ over a signature $\mathcal{F}$ such that $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN}_\alpha$. Let $\mathcal{G}$ be an extension of $\mathcal{F}$. Assuming that $(\mathcal{R}, \mathcal{G}) \notin \mathsf{CBN}_\alpha$, we consider a minimal $(\mathcal{R}_\alpha, \mathcal{G})$-free term t in $\mathcal{T}(\mathcal{G})$. By replacing the maximal subterms of t that start with a function symbol in $\mathcal{G} \backslash \mathcal{F}$—such subterms will be called *aliens* or more precisely $\mathcal{G} \backslash \mathcal{F}$-aliens in the sequel—by a suitable term in $\mathcal{T}(\mathcal{F})$, we obtain an $(\mathcal{R}_\alpha, \mathcal{F})$-free term t' in $\mathcal{T}(\mathcal{F})$. Hence $(\mathcal{R}, \mathcal{F}) \notin \mathsf{CBN}_\alpha$, contradicting the assumption.

We start with a useful lemma which is used repeatedly in the sequel.

The subset of $\mathsf{WN}(\mathcal{R}, \mathcal{G}, \mathcal{F})$ consisting of those terms that admit a normalizing rewrite sequence in $(\mathcal{R}, \mathcal{G})$ containing a root rewrite step is denoted by $\mathsf{WNR}(\mathcal{R}, \mathcal{G}, \mathcal{F})$. If $\mathcal{F} = \mathcal{G}$ then we just write $\mathsf{WNR}(\mathcal{R}, \mathcal{F})$ or even $\mathsf{WNR}(\mathcal{R})$ if the signature is clear from the context. We also find it convenient to write $\mathsf{WN}_\bullet(\mathcal{R}, \mathcal{G}, \mathcal{F})$ for $\mathsf{WN}(\mathcal{R}_\bullet, \mathcal{G}_\bullet, \mathcal{F}_\bullet)$ and $\mathsf{WNR}_\bullet(\mathcal{R}, \mathcal{G}, \mathcal{F})$ for $\mathsf{WNR}(\mathcal{R}_\bullet, \mathcal{G}_\bullet, \mathcal{F}_\bullet)$.

Lemma 67. *Let $\mathcal{R}$ be a left-linear TRS and α an approximation mapping. Every minimal $\mathcal{R}_\alpha$-free term belongs to* $\mathsf{WNR}(\mathcal{R}_\alpha)$.

Proof. Let $\mathcal{F}$ be the signature of $\mathcal{R}$ and let $t \in \mathcal{T}(\mathcal{F})$ be a minimal-free term. For every redex position p in t we have $t[\bullet]_p \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha)$. Let p' be the minimum position above p at which a contraction takes place in any rewrite sequence from $t[\bullet]_p$ to a normal form in $\mathcal{T}(\mathcal{F})$ and define $P = \{p' \mid p \text{ is a redex position in } t\}$. Let p^* be a minimal position in P. We show that $p^* = \epsilon$. If $p^* > \epsilon$ then we consider the term $t|_{p^*}$. Let q be a redex position in $t|_{p^*}$. There exists a redex position p in t such that $p = p^*q$. We have $t|_{p^*}[\bullet]_q = (t[\bullet]_p)|_{p^*} \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha)$ by the definition of p^*. Since $t|_{p^*}$ has at least one redex, it follows that $t|_{p^*}$ is free. As $t|_{p^*}$ is a proper subterm of t we obtain a contradiction to the minimality of t. Hence $p^* = \epsilon$. So there exists a redex position p in t and a rewrite sequence $A\colon t[\bullet]_p \to^+_{\mathcal{R}_\alpha, \mathcal{F}_\bullet} u \in \mathsf{NF}(\mathcal{R}, \mathcal{F})$ that contains a root rewrite step. Because $\mathcal{R}_\alpha$ is left-linear and $\bullet$ does not occur in the rewrite rules of $\mathcal{R}_\alpha$, $\bullet$ cannot contribute to this sequence. It follows that if we replace in A every occurrence of $\bullet$ by $t|_p$ we obtain an $(\mathcal{R}_\alpha, \mathcal{F})$-rewrite sequence from t to u with a root rewrite step. $\square$

In particular, minimal-free terms are not root-stable.

Proof of Theorem 50. Let $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN}_\alpha$ and let $c \in \mathsf{NF}(\mathcal{R}, \mathcal{F})$ be an external normal form. Let $\mathcal{F} \subseteq \mathcal{G}$. We have to show that $(\mathcal{R}, \mathcal{G}) \in \mathsf{CBN}_\alpha$. Suppose to the contrary that $(\mathcal{R}, \mathcal{G}) \notin \mathsf{CBN}_\alpha$. According to Lemma 67 there exists a term $t \in \mathsf{WNR}(\mathcal{R}_\alpha, \mathcal{G})$ without $(\mathcal{R}_\alpha, \mathcal{G})$-needed redex. Let t' be the term in $\mathcal{T}(\mathcal{F})$ obtained from t by replacing every $\mathcal{G}\backslash\mathcal{F}$-alien by c. Because t is not root-stable, we have $t \to^*_{\mathcal{R}_\alpha, \mathcal{G}} l\sigma$ for some left-hand side l. Replacing in this sequence every $\mathcal{G} \setminus \mathcal{F}$-alien by c, yields a sequence $t' \to^*_{\mathcal{R}_\alpha, \mathcal{F}} l\sigma'$. So t' cannot be a normal form. Since $(\mathcal{R}, \mathcal{F}) \in \mathsf{CBN}_\alpha$, t' contains an $(\mathcal{R}_\alpha, \mathcal{F})$-needed redex Δ, say at position p. Because c is an external normal form, Δ is also a redex in t. Since t has no $(\mathcal{R}_\alpha, \mathcal{G})$-needed redexes, there exists a rewrite sequence $t[\bullet]_p \to^+_{\mathcal{R}_\alpha, \mathcal{G}_\bullet} u$ with $u \in \mathsf{NF}(\mathcal{R}_\bullet, \mathcal{G})$. If we replace in this rewrite sequence every $\mathcal{G}\backslash\mathcal{F}$-alien by c, we obtain a rewrite sequence $t'[\bullet]_p \to^+_{\mathcal{R}_\alpha, \mathcal{F}_\bullet} u'$. Because c does not unify with a proper non-variable subterm of a left-hand side of a rewrite rule, it follows that $u' \in \mathsf{NF}(\mathcal{R}_\bullet, \mathcal{F})$. Hence Δ is not an $(\mathcal{R}_\alpha, \mathcal{F})$-needed redex in t', yielding the desired contradiction. $\square$

Before we can prove Theorem 52, we need a few preliminary results.

Definition 68. Let $\mathcal{R}$ be a TRS. Two redexes Δ_1, Δ_2 are called *pattern equal*, denoted by $\Delta_1 \approx \Delta_2$, if they have the same redex pattern, i.e., they are redexes with respect to the same rewrite rule.

Lemma 69. *Let $\mathcal{R}$ be an orthogonal TRS, $\alpha \in \{\mathsf{s}, \mathsf{nv}\}$, and suppose that $\Delta \approx \Delta'$. If $C[\Delta] \in \mathsf{WN}(\mathcal{R}_\alpha)$ then $C[\Delta'] \in \mathsf{WN}(\mathcal{R}_\alpha)$.*

Proof. Let $C[\Delta] \to^* t$ be a normalizing rewrite sequence in $\mathcal{R}_\alpha$. If we replace every descendant of Δ by Δ' then we obtain a (possibly shorter) normalizing rewrite sequence $C[\Delta'] \to^* t$. The reason is that every descendant Δ'' of Δ satisfies $\Delta'' \approx \Delta$ due to orthogonality and hence if Δ'' is contracted to some term u then Δ rewrites to the same term because the variables in the right-hand sides of the rewrite rules in $\mathcal{R}_\alpha$ are fresh, due to the assumption $\alpha \in \{\mathsf{s}, \mathsf{nv}\}$. Moreover, as t is a normal form, there are no descendants of Δ left. Note that the resulting sequence can be shorter since rewrite steps below a descendant of Δ are not mimicked. $\square$

The above lemma does not hold for the growing approximation, as shown by the following example.

Example 70. Consider the TRS $\mathcal{R}$

$$\mathsf{f}(x) \to x \quad \mathsf{a} \to \mathsf{b} \quad \mathsf{c} \to \mathsf{c}$$

We have $\mathcal{R}_\mathsf{g} = \mathcal{R}$. Consider the redexes $\Delta = \mathsf{f}(\mathsf{a})$ and $\Delta' = \mathsf{f}(\mathsf{c})$. Clearly $\Delta \approx \Delta'$. Redex Δ admits the normal form b, but Δ' has no normal form.

Orthogonality is also necessary for Lemma 69.

Example 71. Consider the TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{b} \quad \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{a} \quad \mathsf{g}(x) \to \mathsf{a} \quad \mathsf{b} \to \mathsf{b}$$

We have $\mathcal{R}_\mathsf{nv} = \mathcal{R}$. Consider the context $C = \mathsf{f}(\Box)$ and the pattern equivalent redexes $\Delta = \mathsf{g}(\mathsf{a})$ and $\Delta' = \mathsf{g}(\mathsf{b})$. The term $C[\Delta]$ admits the normal form a, but $C[\Delta']$ has no normal form.

Lemma 72. *Let $\mathcal{R}$ be an orthogonal TRS over a signature $\mathcal{F}$, $\alpha \in \{\mathsf{s}, \mathsf{nv}\}$, and $\mathcal{F} \subseteq \mathcal{G}$. If $\mathsf{WN}(\mathcal{R}_\alpha, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_\alpha, \mathcal{G}, \mathcal{F})$ then $\mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{F}) = \mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{G}, \mathcal{F})$.*

Proof. The inclusion $\mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{F}) \subseteq \mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{G}, \mathcal{F})$ is obvious. For the reverse inclusion we reason as follows. Let $t \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{G}, \mathcal{F})$ and consider a rewrite sequence A in $(\mathcal{R}_\alpha, \mathcal{G}_\bullet)$ that normalizes t. We may write $t = C[t_1, \ldots, t_n]$ such that $t_1, \ldots, t_n$ are the maximal subterms of t that are rewritten in A at their root positions. Hence A can be rearranged into A':

$$t \to^*_{\mathcal{R}_\alpha, \mathcal{G}_\bullet} C[\Delta_1, \ldots, \Delta_n] \to^*_{\mathcal{R}_\alpha, \mathcal{G}_\bullet} C[u_1, \ldots, u_n]$$

for some redexes $\Delta_1, \ldots, \Delta_n$ and normal form $C[u_1, \ldots, u_n] \in \mathcal{T}(\mathcal{G})$. Since the context C cannot contain $\bullet$, all occurrences of $\bullet$ are in the substitution parts of the redexes $\Delta_1, \ldots, \Delta_n$. If we replace in $C[\Delta_1, \ldots, \Delta_n]$ every $\mathcal{G}_\bullet \backslash \mathcal{F}$-alien by some ground term $c \in \mathcal{T}(\mathcal{F})$, we obtain a term $t' = C[\Delta'_1, \ldots, \Delta'_n]$ with $\Delta'_i \in \mathcal{T}(\mathcal{F})$ and $\Delta_i \approx \Delta'_i$ for every i. Repeated application of Lemma 69 yields $t' \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{G})$. Because $\bullet$ cannot contribute to the creation of a normal form, we actually have $t' \in \mathsf{WN}(\mathcal{R}_\alpha, \mathcal{G})$ and thus $t' \in \mathsf{WN}(\mathcal{R}_\alpha, \mathcal{G}, \mathcal{F})$ as $t' \in \mathcal{T}(\mathcal{F})$. The assumption yields $t' \in \mathsf{WN}(\mathcal{R}_\alpha, \mathcal{F})$. Since $\mathsf{WN}(\mathcal{R}_\alpha, \mathcal{F}) \subseteq \mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{F})$ clearly holds, we obtain $t' \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{F})$. Now, if we replace in the first part of A' every $\mathcal{G} \backslash \mathcal{F}$-alien by c then we obtain a (possibly shorter) rewrite sequence $t \to^*_{\mathcal{R}_\alpha, \mathcal{F}_\bullet} C[\Delta''_1, \ldots, \Delta''_n] \in \mathcal{T}(\mathcal{F}_\bullet)$ with $\Delta_i \approx \Delta''_i$ and thus also $\Delta'_i \approx \Delta''_i$ for every i. Repeated application of Lemma 69 yields $C[\Delta''_1, \ldots, \Delta''_n] \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{F})$ and therefore $t \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha, \mathcal{F})$ as desired. □

We note that for $\alpha = \mathsf{s}$ the preceding lemma is a simple consequence of Lemma 55 below. The following example shows that the restriction to $\alpha \in \{\mathsf{s}, \mathsf{nv}\}$ is essential.

Example 73. Consider TRS $\mathcal{R}$

$$\begin{array}{rlrl}
\mathsf{f}(x, \mathsf{a}) & \to \mathsf{a} & \mathsf{h}(x, \mathsf{a}, \mathsf{a}) & \to \mathsf{i} \\
\mathsf{f}(\mathsf{a}, \mathsf{b}(x)) & \to \mathsf{i} & \mathsf{h}(x, \mathsf{a}, \mathsf{b}(y)) & \to \mathsf{i} \\
\mathsf{f}(\mathsf{b}(x), \mathsf{b}(y)) & \to \mathsf{i} & \mathsf{h}(x, \mathsf{b}(y), \mathsf{a}) & \to \mathsf{i} \\
\mathsf{g}(\mathsf{a}, \mathsf{a}) & \to \mathsf{i} & \mathsf{h}(x, \mathsf{b}(y), \mathsf{b}(z)) & \to \mathsf{b}(\mathsf{g}(y, \mathsf{f}(x, z))) \\
\mathsf{g}(\mathsf{b}(x), \mathsf{a}) & \to \mathsf{i} & \mathsf{i} & \to \mathsf{b}(\mathsf{i}) \\
\mathsf{g}(x, \mathsf{b}(y)) & \to \mathsf{a} & &
\end{array}$$

over the signature $\mathcal{F}$ consisting of all symbols appearing in the rewrite rules and let $\mathcal{G} = \mathcal{F} \cup \{\mathsf{c}\}$ with c a constant. The term $t = \mathsf{h}(\bullet, \mathsf{i}, \mathsf{i})$ belongs to $\mathsf{WN}_\bullet(\mathcal{R}_\mathsf{g}, \mathcal{G}, \mathcal{F})$:

$$t \to^+_{\mathcal{R}_\mathsf{g}, \mathcal{G}_\bullet} \mathsf{h}(\bullet, \mathsf{b}(\mathsf{i}), \mathsf{b}(\mathsf{i})) \to_{\mathcal{R}_\mathsf{g}, \mathcal{G}_\bullet} \mathsf{b}(\mathsf{g}(\mathsf{c}, \mathsf{f}(\bullet, \mathsf{a}))) \to_{\mathcal{R}_\mathsf{g}, \mathcal{G}_\bullet} \mathsf{b}(\mathsf{g}(\mathsf{c}, \mathsf{a}))$$

However, one easily verifies that there is no normal form $u \in \mathsf{NF}(\mathcal{R}_\mathsf{g}, \mathcal{F})$ such that $t \to^*_{\mathcal{R}_\mathsf{g}, \mathcal{F}_\bullet} u$. Hence $\mathsf{WN}_\bullet(\mathcal{R}_\mathsf{g}, \mathcal{F}) \neq \mathsf{WN}_\bullet(\mathcal{R}_\mathsf{g}, \mathcal{G}, \mathcal{F})$. Using the observations that (i) every term $t \in \mathcal{T}(\mathcal{F})$ rewrites to a or a term of the form $\mathsf{b}(u)$ and (ii) the only rewrite rule of $\mathcal{R}_\mathsf{g}$ where c can be introduced is $\mathsf{h}(x, \mathsf{b}(y), \mathsf{b}(z)) \to \mathsf{b}(\mathsf{g}(y', \mathsf{f}(x, z')))$ but every redex in $\mathcal{T}(\mathcal{F})$ of the form $\mathsf{h}(s, \mathsf{b}(t), \mathsf{b}(u))$ rewrites to $\mathsf{b}(\mathsf{a})$ without using c:

$$\begin{array}{rll}
\mathsf{h}(s, \mathsf{b}(t), \mathsf{b}(u)) & \to_{\mathcal{R}_\mathsf{g}, \mathcal{F}} \mathsf{b}(\mathsf{g}(\mathsf{a}, \mathsf{f}(s, \mathsf{b}(\mathsf{a})))) & \\
& \to^+_{\mathcal{R}_\mathsf{g}, \mathcal{F}} \mathsf{b}(\mathsf{g}(\mathsf{a}, \mathsf{i})) & \text{because } s \to^* \mathsf{a} \text{ or } s \to^* \mathsf{b}(s') \\
& \to_{\mathcal{R}_\mathsf{g}, \mathcal{F}} \mathsf{b}(\mathsf{g}(\mathsf{a}, \mathsf{b}(\mathsf{i}))) \to_{\mathcal{R}_\mathsf{g}, \mathcal{F}} \mathsf{b}(\mathsf{a}) &
\end{array}$$

it can be readily checked that $\mathsf{WN}(\mathcal{R}_\mathsf{g}, \mathcal{F}) = \mathsf{WN}(\mathcal{R}_\mathsf{g}, \mathcal{G}, \mathcal{F})$. (Autowrite is able to check this equality automatically.)

A redex is called *flat* if it does not contain smaller redexes.

Lemma 74. *Let* $(\mathcal{R},\mathcal{F})$ *and* $(\mathcal{S},\mathcal{G})$ *be orthogonal TRSs and* $\alpha \in \{\mathsf{s},\mathsf{nv}\}$ *such that* $(\mathcal{R},\mathcal{F}) \subseteq (\mathcal{S},\mathcal{G})$ *and* $\mathsf{WN}(\mathcal{S}_\alpha,\mathcal{G},\mathcal{F}) = \mathsf{WN}(\mathcal{R}_\alpha,\mathcal{F})$. *If* $t \in \mathsf{WNR}(\mathcal{S}_\alpha,\mathcal{G})$ *and* $\mathsf{root}(t) \in \mathcal{F}$ *then there exists a flat* $\mathcal{R}$*-redex* Θ *in* $\mathcal{T}(\mathcal{F})$. *Moreover, if* $\mathcal{R}_\alpha$ *is collapsing then we may assume that* Θ *is* $\mathcal{R}_\alpha$*-collapsing.*

Proof. From $t \in \mathsf{WNR}(\mathcal{S}_\alpha,\mathcal{G})$ we infer that $t \to^*_{\mathcal{S}_\alpha,\mathcal{G}} \Delta$ for some redex $\Delta \in \mathsf{WN}(\mathcal{S}_\alpha,\mathcal{G})$. By considering the first such redex it follows that Δ is a redex with respect to $(\mathcal{R}_\alpha,\mathcal{G})$. If we replace in Δ the subterms below the redex pattern by an arbitrary ground term in $\mathcal{T}(\mathcal{F})$ then we obtain a redex $\Delta' \in \mathcal{T}(\mathcal{F})$ with $\Delta \approx \Delta'$. Lemma 69 yields $\Delta' \in \mathsf{WN}(\mathcal{S}_\alpha,\mathcal{G})$ and thus $\Delta' \in \mathsf{WN}(\mathcal{S}_\alpha,\mathcal{G},\mathcal{F}) = \mathsf{WN}(\mathcal{R}_\alpha,\mathcal{F})$. Hence $\mathsf{NF}(\mathcal{R},\mathcal{F}) = \mathsf{NF}(\mathcal{R}_\alpha,\mathcal{F}) \neq \varnothing$. Therefore, using orthogonality, we obtain a flat redex $\Theta \in \mathcal{T}(\mathcal{F})$ by replacing the variables in the left-hand side of any rewrite rule in $\mathcal{R}$ by terms in $\mathsf{NF}(\mathcal{R},\mathcal{F})$. If $\mathcal{R}_\alpha$ is collapsing then we take any $\mathcal{R}_\alpha$-collapsing rewrite rule. $\square$

Proof of Theorem 52. If $(\mathcal{R},\mathcal{F})$ has external normal forms then the result follows from Theorem 50. So we assume that $(\mathcal{R},\mathcal{F})$ lacks external normal forms. We also assume that $\mathcal{R} \neq \varnothing$ for otherwise the result is trivial. Suppose to the contrary that $(\mathcal{R},\mathcal{G}) \notin \mathsf{CBN}_\alpha$. According to Lemma 67 there exists a term $t \in \mathsf{WNR}(\mathcal{R}_\alpha,\mathcal{G})$ without $(\mathcal{R}_\alpha,\mathcal{G})$-needed redex. Lemma 74 (with $\mathcal{S} = \mathcal{R}$) yields a flat redex $\Theta \in \mathcal{T}(\mathcal{F})$. Since $\mathcal{R}_\alpha$ is collapsing, we may assume that Θ is $\mathcal{R}_\alpha$-collapsing. Let t' be the term in $\mathcal{T}(\mathcal{F})$ obtained from t by replacing every $\mathcal{G}\backslash\mathcal{F}$-alien by Θ. Let P be the set of positions of those aliens. Since t' is reducible, it contains an $(\mathcal{R}_\alpha,\mathcal{F})$-needed redex, say at position q. We show that $t'[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha,\mathcal{G})$. We consider two cases.

(1) Suppose that $q \in P$. Since $t \in \mathsf{WNR}(\mathcal{R}_\alpha,\mathcal{G})$, $t \to^*_{\mathcal{R}_\alpha,\mathcal{G}} \Delta$ for some redex $\Delta \in \mathsf{WN}(\mathcal{R}_\alpha,\mathcal{G}) \subseteq \mathsf{WN}_\bullet(\mathcal{R}_\alpha,\mathcal{G})$. Since the root symbol of every alien belongs to $\mathcal{G}\backslash\mathcal{F}$, aliens cannot contribute to the creation of Δ and hence we may replace them by arbitrary terms in $\mathcal{T}(\mathcal{G}_\bullet)$ and still obtain a redex that is pattern equal to Δ. We replace in t the alien at position q by $\bullet$ and every alien at position $p \in P \setminus \{q\}$ by $t'|_p = \Theta$. This gives $t'[\bullet]_q \to^*_{\mathcal{R}_\alpha,\mathcal{G}_\bullet} \Delta'$ with $\Delta' \approx \Delta$. Lemma 69 yields $\Delta' \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha,\mathcal{G})$ and hence $t'[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha,\mathcal{G})$.
(2) Suppose that $q \notin P$. Since Θ is flat, it follows by orthogonality that q is also a redex position in t. Since t is an $(\mathcal{R}_\alpha,\mathcal{G})$-free term, $t[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha,\mathcal{G})$. Because Θ is a collapsing redex and $\alpha \in \{\mathsf{s},\mathsf{nv}\}$, we have $\Theta \to_{\mathcal{R}_\alpha,\mathcal{G}} t|_p$ for all $p \in P$. Hence $t'[\bullet]_q \to^*_{\mathcal{R}_\alpha,\mathcal{G}_\bullet} t[\bullet]_q$ and thus $t'[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha,\mathcal{G})$.

As $t' \in \mathcal{T}(\mathcal{F})$, we have $t'[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{R}_\alpha,\mathcal{G},\mathcal{F})$ and thus $t'[\bullet]_q \notin \mathsf{WN}_\bullet(\mathcal{R}_\alpha,\mathcal{F})$ by Lemma 72, contradicting the assumption that q is the position of an $(\mathcal{R}_\alpha,\mathcal{F})$-needed redex in t'. $\square$

For the proof of Theorem 59, the counterpart of Theorem 52, we need the following preliminary lemma. In the remainder of the appendix we have $\mathcal{S} = \mathcal{R} \cup \mathcal{R}'$ and $\mathcal{G} = \mathcal{F} \cup \mathcal{F}'$.

Lemma 75. *Let* $(\mathcal{R},\mathcal{F})$ *and* $(\mathcal{R}',\mathcal{F}')$ *be disjoint TRSs. If* $\alpha \in \{\mathsf{s},\mathsf{nv}\}$ *then* $\mathsf{WN}(\mathcal{S}_\alpha,\mathcal{G},\mathcal{F}) \subseteq \mathsf{WN}(\mathcal{R}_\alpha,\mathcal{G},\mathcal{F})$.

Proof. We consider here the more complicated case $\alpha = \mathsf{nv}$. Let $s \in \mathsf{WN}(\mathcal{S}_{\mathsf{nv}},\mathcal{G},\mathcal{F})$, so $s \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$ for some normal form $t \in \mathsf{NF}(\mathcal{S}_{\mathsf{nv}},\mathcal{G})$. By induction on the length n of $s \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$ we show that $s \to^*_{\mathcal{R}_{\mathsf{nv}},\mathcal{G}} t$. In order to make the induction work we prove this statement for all $s \in \mathcal{T}(\mathcal{G})$ such that in $s \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$ no redex inside an $\mathcal{G}\backslash\mathcal{F}$-alien of s is contracted. If $n = 0$ then the statement is trivial. If $n > 0$ then there exists a term $s' \in \mathcal{T}(\mathcal{G})$ such that $s \to_{\mathcal{R}_{\mathsf{nv}},\mathcal{G}} s' \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$. Note that the rewrite rule $l \to r$ applied in the step from s to s' must come from $\mathcal{R}_{\mathsf{nv}}$ because redexes inside $\mathcal{G}\backslash\mathcal{F}$-aliens of s are not contracted. We have $s = C[l\sigma]$ and $s' = C[r\sigma]$ for some context C and substitution σ. If $\sigma(x) \in \mathcal{T}(\mathcal{F})$ for all $x \in \mathcal{V}\mathsf{ar}(r)$ then we can apply the induction hypothesis to $s' \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$. This yields $s' \to^*_{\mathcal{R}_{\mathsf{nv}},\mathcal{G}} t$ and thus $s \to^*_{\mathcal{R}_{\mathsf{nv}},\mathcal{G}} t$ as desired. If $\sigma(x) \in \mathcal{T}(\mathcal{G})\backslash\mathcal{T}(\mathcal{F})$ for some $x \in \mathcal{V}\mathsf{ar}(r)$ then s' contains new $\mathcal{G}\backslash\mathcal{F}$-aliens. If no redexes are contracted in these aliens in the $(\mathcal{S}_{\mathsf{nv}}, \mathcal{G})$-rewrite sequence to t then we can again apply the induction hypothesis. Otherwise we have to modify $s' \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$ first. Let p be the position of a $\mathcal{G}\backslash\mathcal{F}$-alien in s' such that a redex in $s'|_p$ is contracted in $s' \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$. We distinguish two cases. If in $s' \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$ no step takes place at a position strictly above p, then we replace $s'|_p$ by $t|_p$. Otherwise, let $u \to_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} u'$ be the first step in $s' \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$ in which a redex is contracted at a position strictly above p. In this case we replace $s'|_p$ by $u|_p$. It is easy to see that we end up with a rewrite sequence $s'' \to^*_{\mathcal{S}_{\mathsf{nv}},\mathcal{G}} t$ whose length is less than $n - 1$ and with the property that redexes inside $\mathcal{G}\backslash\mathcal{F}$-aliens of s'' are not contracted. Hence we can apply the induction hypothesis, which yields $s'' \to^*_{\mathcal{R}_{\mathsf{nv}},\mathcal{G}} t$. Because r is a linear term, we may write $s'' = C[r\sigma']$ for some substitution σ'. Since we are dealing with the nv approximation, $s \to_{\mathcal{R}_{\mathsf{nv}},\mathcal{G}} s''$ and therefore $s \to^*_{\mathcal{R}_{\mathsf{nv}},\mathcal{G}} t$ as desired. □

Let us illustrate the construction in the above proof on a small example.

Example 76. Consider the TRSs $\mathcal{R}$

$$\begin{array}{rlrl} \mathsf{f}(x) \to \mathsf{g}(x,x) & & \mathsf{g}(\mathsf{a},\mathsf{a}) \to \mathsf{g}(\mathsf{a},\mathsf{a}) \\ \mathsf{g}(\mathsf{a},\mathsf{b}) \to \mathsf{c} & & \mathsf{g}(\mathsf{b},\mathsf{b}) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \end{array}$$

and $\mathcal{R}' = \{\mathsf{h}(x) \to x\}$ over the signatures $\mathcal{F}$ and $\mathcal{F}'$ consisting of function symbols that appear in their respective rewrite rules. The $(\mathcal{S}_{\mathsf{nv}}, \mathcal{G})$-rewrite sequence

$$\mathsf{f}(\mathsf{a}) \to_{\mathcal{R}_{\mathsf{nv}}} \mathsf{g}(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{a})) \to_{\mathcal{R}'_{\mathsf{nv}}} \mathsf{g}(\mathsf{a},\mathsf{h}(\mathsf{a})) \to_{\mathcal{R}'_{\mathsf{nv}}} \mathsf{g}(\mathsf{a},\mathsf{b}) \to_{\mathcal{R}_{\mathsf{nv}}} \mathsf{c}$$

is transformed into

$$\mathsf{f}(\mathsf{a}) \to_{\mathcal{R}_{\mathsf{nv}}} \mathsf{g}(\mathsf{a},\mathsf{b}) \to_{\mathcal{R}_{\mathsf{nv}}} \mathsf{c}$$

Note that simply replacing all $\mathcal{G}\backslash\mathcal{F}$-aliens by some constant in $\mathcal{F}$ does not work.

The reverse inclusion does not hold in general.

Example 77. Consider the TRSs $\mathcal{R} = \{\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}), \mathsf{g}(x) \to \mathsf{f}(x)\}$ and $\mathcal{R}' = \{\mathsf{b} \to \mathsf{b}\}$ over the signatures $\mathcal{F}$ and $\mathcal{F}'$ consisting of function symbols that appear in their respective rewrite rules. The term $\mathsf{f}(\mathsf{b})$ is a normal form with respect to $(\mathcal{R}_{\mathsf{nv}}, \mathcal{G})$ and hence $\mathsf{g}(\mathsf{a}) \in \mathsf{WN}(\mathcal{R}_{\mathsf{nv}}, \mathcal{G}, \mathcal{F})$. One easily verifies that $\mathsf{g}(\mathsf{a}) \notin \mathsf{WN}(\mathcal{S}_{\mathsf{nv}}, \mathcal{G}, \mathcal{F})$.

Proof of Theorem 59. We assume that both $\mathcal{R}$ and $\mathcal{R}'$ are non-empty, for otherwise the result follows from Theorem 52. Suppose to the contrary that $(\mathcal{S}, \mathcal{G}) \notin \mathsf{CBN}_\alpha$. According to Lemma 67 there exists a term $t \in \mathsf{WNR}(\mathcal{S}_\alpha, \mathcal{G})$ without $(\mathcal{S}_\alpha, \mathcal{G})$-needed redex. Assume without loss of generality that

$\mathsf{root}(t) \in \mathcal{F}'$. Lemma 74 yields a flat $\mathcal{R}'_\alpha$-collapsing redex $\Theta \in \mathcal{T}(\mathcal{F}')$. Let t' be the term in $\mathcal{T}(\mathcal{F}')$ obtained from t by replacing every $\mathcal{G}\backslash\mathcal{F}'$-alien by Θ. Let P be the set of positions of those aliens. Since t' is reducible, it contains an $(\mathcal{R}'_\alpha, \mathcal{F}')$-needed redex, say at position q. We show that $t'[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$. Because Θ is a collapsing redex, we have $\Theta \to_{\mathcal{R}_\alpha,\mathcal{G}} t|_p$ for all $p \in P$. Hence $t' \to^*_{\mathcal{R}_\alpha,\mathcal{G}_\bullet} t$ and thus, by orthogonality, $t'[\bullet]_q \to^*_{\mathcal{R}_\alpha,\mathcal{G}_\bullet} t[\bullet]_q$. Hence it suffices to show that $t[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$. We distinguish two cases.

(1) Suppose that $q \in P$. Since $t \in \mathsf{WNR}(\mathcal{S}_\alpha, \mathcal{G})$, $t \to^*_{\mathcal{S}_\alpha,\mathcal{G}} \Delta$ for some redex $\Delta \in \mathsf{WN}(\mathcal{S}_\alpha, \mathcal{G}) \subseteq \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$. We distinguish two further cases.
   (a) If $t|_q$ is a normal form then it cannot contribute to the creation of Δ and hence by replacing it by $\bullet$ we obtain $t[\bullet]_q \to^*_{\mathcal{S}_\alpha,\mathcal{G}} \Delta'$ with $\Delta \approx \Delta'$. Lemma 69 yields $\Delta' \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$ and thus $t[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$.
   (b) Suppose $t|_q$ is reducible. Because t is a minimal-free term, $t|_q$ contains an $(\mathcal{S}_\alpha, \mathcal{G})$-needed redex, say at position q'. So $t|_q[\bullet]_{q'} \notin \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$. In particular, $t|_q[\bullet]_{q'}$ does not $(\mathcal{S}_\alpha, \mathcal{G})$-rewrite to a collapsing redex, for otherwise it would rewrite to a normal form in one extra step. Hence the root symbol of every reduct of $t|_q[\bullet]_{q'}$ belongs to $\mathcal{F}$. Since qq' is not the position of an $(\mathcal{S}_\alpha, \mathcal{G})$-needed redex in t, $t[\bullet]_{qq'} \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$. Since any normalizing $(\mathcal{S}_\alpha, \mathcal{G})$-rewrite sequence must contain a rewrite step at a position above q, we may write $t[\bullet]_{qq'} \to^*_{\mathcal{S}_\alpha,\mathcal{G}} C[\Delta'] \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$ such that Δ' is the first redex above position q. Since $\mathsf{root}(\Delta') \in \mathcal{F}'$, the subterm $t|_q[\bullet]_{q'}$ of $t[\bullet]_{qq'}$ does not contribute to the creation of Δ' and hence $t[\bullet]_q \to^*_{\mathcal{S}_\alpha,\mathcal{G}} C[\Delta'']$ with $\Delta'' \approx \Delta'$. Lemma 69 yields $C[\Delta''] \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$ and thus $t[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$.

(2) Suppose that $q \notin P$. Since Θ is flat, q cannot be below a position in P. It follows by orthogonality that q is also a redex position in t. Since t is an $(\mathcal{S}_\alpha, \mathcal{G})$-free term, $t[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G})$.

As $t' \in \mathcal{T}(\mathcal{F}')$, we have $t'[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G}, \mathcal{F}')$. Since $\mathsf{WN}_\bullet(\mathcal{S}_\alpha, \mathcal{G}, \mathcal{F}') \subseteq \mathsf{WN}_\bullet(\mathcal{R}'_\alpha, \mathcal{G}, \mathcal{F}') = \mathsf{WN}_\bullet(\mathcal{R}'_\alpha, \mathcal{F}')$ by Lemmata 75 and 72, we obtain $t'[\bullet]_q \in \mathsf{WN}_\bullet(\mathcal{R}'_\alpha, \mathcal{F}')$, contradicting the assumption that q is the position of an $(\mathcal{R}'_\alpha, \mathcal{F}')$-needed redex in t'. □

Proof of Theorem 62. Let $\mathcal{C} = \mathcal{F}_\mathcal{C} \cap \mathcal{F}'_\mathcal{C}$ be the set of common constructors. Let $\mathcal{H} = \mathcal{F} \cup \mathcal{C}$ and $\mathcal{H}' = \mathcal{F}' \cup \mathcal{C}$. According to Theorem 50 the TRSs $(\mathcal{R}, \mathcal{H})$ and $(\mathcal{R}', \mathcal{H}')$ belong to CBN_α. Suppose to the contrary that $(\mathcal{S}, \mathcal{G}) \notin \mathsf{CBN}_\alpha$. (As before, $\mathcal{S} = \mathcal{R} \cup \mathcal{R}'$ and $\mathcal{G} = \mathcal{F} \cup \mathcal{F}'$.) According to Lemma 67 there exists a term $t \in \mathsf{WNR}(\mathcal{S}_\alpha, \mathcal{G})$ without $(\mathcal{S}_\alpha, \mathcal{G})$-needed redex. We assume without loss of generality that $\mathsf{root}(t) \in \mathcal{F}_\mathcal{D}$. Let c be an external normal form of $(\mathcal{R}, \mathcal{F})$. Let t' be the term obtained from t by replacing every $\mathcal{G} \setminus \mathcal{H}$-alien by c. Note that $t' \in \mathcal{T}(\mathcal{H})$. Because $\mathcal{R}_\alpha$ is left-linear and $\mathcal{R}'_\alpha$ lacks both collapsing and constructor-lifting rules, contractions in the $\mathcal{G} \setminus \mathcal{H}$-aliens of t cannot create a redex in the non-alien part of t. Since t is not root-stable, the latter exists and thus t' contains a redex as well. Because $(\mathcal{R}, \mathcal{H}) \in \mathsf{CBN}_\alpha$, t' must contain an $(\mathcal{R}_\alpha, \mathcal{H})$-needed redex Δ, say at position p. Because c is an external normal form, Δ is also a redex in t and hence there exists a rewrite sequence $t[\bullet]_p \to^+_{\mathcal{S}_\alpha,\mathcal{G}_\bullet} u$ with $u \in \mathsf{NF}(\mathcal{R}_\bullet, \mathcal{G})$. If we replace in this rewrite sequence every $\mathcal{G} \setminus \mathcal{H}$-alien by c, we obtain a rewrite sequence $t'[\bullet]_p \to^+_{\mathcal{R}_\alpha,\mathcal{H}_\bullet} u'$. Because c does not unify with a proper non-variable subterm of a left-hand side of a rewrite rule, it follows that $u' \in \mathsf{NF}(\mathcal{R}_\bullet, \mathcal{H})$. Hence Δ is not an $(\mathcal{R}_\alpha, \mathcal{H})$-needed redex in t', yielding the desired contradiction. □

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