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MODEL-THEORETIC METHODS IN THE STUDY OF ELEMENTARY LOGIC

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In order to study the structural and recursive properties of first-order theories, we introduce the notion of isomorphism of theories. Roughly speaking, two theories are isomorphic if there is a one-to-one recursive correspondence between sentences of one theory and sentences of the other which preserves implications valid in the theories. The main result of the paper is that there exists a finitely axiomatizable theory isomorphic to a theory constructed as a product of a fixed theory and an arbitrary axiomatizable theory. From this result, finitely axiomatizable theories can be constructed which reflect certain properties of arbitrary axiomatizable theories. One consequence is that there exists a sentence of elementary logic whose consequences form a recursively enumerable set of any prescribed degree of undecidability.

In Section 1 the basic notion of isomorphism is defined and some preliminary results are formulated (without proof). A simple theory H is given which is characterized up to isomorphism by the property of being decidable and being such that no finite extension is complete. The theory H also has the property that any axiomatizable theory is isomorphic to some axiomatizable extension of H. Section 1 also gives a definition of product of theories and states a necessary and sufficient condition for a theory to be isomorphic to the product of two other theories. Further basic results concerning isomorphism of theories are given in the abstract Hanf [62a]. In particular, a result stated there implies that the theories constructed in this paper could be formulated with one binary predicate symbol.

Section 2 gives a fairly detailed outline of the proof of the main result that given any axiomatizable theory T there exists a finitely axiomatizable theory F which is isomorphic to the product of T and the theory H mentioned above. The proof, which is model-theoretic in nature, makes use of a representation of a Turing machine in each model of F.

Section 3 states some consequences of the result and methods of

Section 2. Theorem 3.1, together with a well-known result of Friedberg and Muchnik, gives an example of a finitely axiomatizable theory of intermediate degree of undecidability. This solves a problem of Feferman [57]; the result was first stated in Hanf [62]. As pointed out by Rabin, it also gives an example of a decidable finitely axiomatizable theory whose decision method is not primitive recursive. The theory constructed in Theorem 3.1 has the additional property that no finite extension is essentially undecidable. This solves a problem of Tarski which appears as Problem 4 in Vaught [60]. This shows that the method of showing theories undecidable by proving them to be compatible with some finitely axiomatizable essentially undecidable theory does not apply to all finitely axiomatizable undecidable theories. Theorem 3.2 solves another problem of Tarski by showing that the above-mentioned method of showing undecidability also does not apply to all essentially hereditarily undecidable theories. Theorem 3.3 improves a result of Shoenfield by giving a finitely axiomatized example of an essentially undecidable theory of intermediate degree. Similarly Theorem 3.4, which was suggested by Cobham, improves a result of Ehrenfeucht by giving a finitely axiomatized example of an essentially undecidable theory which is recursively separable. Kreisel suggested that the author's method could be used to solve a problem stated by Feferman [60, footnote 23] as to whether there exists a decidable theory whose consistency cannot be proved in arithmetic. This turned out to be the case and the result is given in Theorem 3.5 and Corollary 3.6. Finally, in Section 3, one of a number of open problems concerning the existence of finitely axiomatizable theories is stated (many further problems can be stated in terms of the Lindenbaum–Tarski algebras of such theories). Two alternative conjectures are formulated in terms of the notion of isomorphism. The first conjecture implies that the variety of such theories is very broad and the second conjecture implies that the variety is very limited (at least for those theories that have countably many complete extensions). The verification of either conjecture would provide the solution to a number of open problems.

The author is indebted to a number of logicians for conversations at various stages of this research. In particular, the applications mentioned in the previous paragraph were suggested when only a general outline of a proof of Theorem 3.1 was known. The method of Theorem 2.1 was developed later in order to permit a unified presentation of the various results. The author is especially indebted to Professor Vaught,

whose questions and observations during the early stages of the research helped to pinpoint the key results and the principal obstacles to their proof. Finally, it should be mentioned that the author made essential use of the ideas of Minsky [61] in earlier versions of the proof of Theorem 3.1.

1. Isomorphism of theories. In this paper we consider theories formulated in first-order predicate logic with equality. The similarity type $\mu = \langle \mu_1, \mu_2, \ldots, \mu_n \rangle$ of a theory specifies the number n of predicate symbols and gives the number μ_i of places of the ith predicate symbol. We denote by L_μ the set of all sentences (formulas without free variables) formulated in similarity type μ. For any theory T, we denote by $K(T)$ the set of all relational structures of similarity type μ which satisfy all the theorems of T. Given two theories S and T of similarity type μ and ν, we say that S is isomorphic to T if there exist functions F and G such that

(i) F is a function mapping $K(S)$ one-to-one onto $K(T)$.

(ii) G is a recursive function mapping L_μ one-to-one onto L_ν.

(iii) For every sentence α of L_μ and relational structure $\mathfrak{A}$ of $K(S)$, α is true of $\mathfrak{A}$ just in case $G(\alpha)$ is true of $F(\mathfrak{A})$.

Note that if we drop the condition that G be recursive, the definition can be formulated for any language for which the notion of model and truth is defined. Dana Scott has pointed out that in the case of first-order logic the notion can be formulated in terms of sentences only. Two first-order theories are isomorphic if there exists a function G satisfying (ii) and such that

(iv) For all sentences α and β of L_μ, the sentence $\alpha \to \beta$ is a theorem of S just in case $G(\alpha) \to G(\beta)$ is a theorem of T.

Let H be the theory with a single binary relation symbol (thought of as a successor relation) and axioms expressing that every element has a unique successor and at most one predecessor and that there is exactly one element with no predecessor. Models of H all have a standard part isomorphic to the positive integers. In addition, they may have a number of non-standard parts each of which is isomorphic to all integers or is a loop consisting of a finite number of points. Every sentence is equivalent in H to a Boolean combination of sentences $\lambda(k, n)$ which state that there are at least k loops of n elements each. (This fact is quite elementary but can also be derived using the methods of Section 2.) Since the implications among the sentences $\lambda(k, n)$ are straightforward, H is decidable.

Theorem 1.1. *Suppose T is decidable and no finite extension of T is complete. Then T is isomorphic to H.*

Theorem 1.2. *Any axiomatizable theory T is isomorphic to some axiomatizable extension of H.*

Note: Theorem 1.2 remains true if "axiomatizable" is deleted in both of its occurrences.

Given two theories T_0 and T_1 of similarity types μ and ν, we construct the theory $T_0 \times T_1$ as follows: The similarity type of $T_0 \times T_1$ is $\langle 1 \rangle \frown \mu \frown \nu$; that is, $T_0 \times T_1$ has all the predicate symbols of T_0 and T_1 (assumed distinct) together with a new unary predicate P. P divides the universe of each model of $T_0 \times T_1$ into two parts A_0 and A_1. The axioms of $T_0 \times T_1$ insure that A_0 and A_1 are both non-empty, that no elements of A_0 satisfy predicates of T_1 or vice versa, and, finally, that all the theorems of T_0 hold when their quantifiers are restricted to A_0 and similarly for T_1 and A_1.

Lemma 1.3. *Suppose that S, T_0, and T_1 are theories and Γ_0 and Γ_1 are sets of sentences of S such that every sentence of S is equivalent in S to a Boolean combination of sentences of Γ_0 and Γ_1 and suppose that for $i = 0, 1$:*

(i) *For every $\alpha \in \Gamma_i$, there is a sentence of Γ_i equivalent in S to the negation of α.*

(ii) *There exists a one-to-one recursive function G_i mapping Γ_i onto the set of all sentences of T_i such that, for any $\alpha, \beta \in \Gamma_i$, $\alpha \to \beta$ is a theorem of S if and only if $G(\alpha) \to G(\beta)$ is a theorem of T_i.*

(iii) *If $\alpha \in \Gamma_i$ and $\beta \in \Gamma_{1-i}$ and $\alpha \to \beta$ is a theorem of S, then α is refutable or β is provable in S.*

Then S is isomorphic to $T_0 \times T_1$.

2. The main theorem.

Theorem 2.1. *If T is any axiomatizable theory, then there exists a finitely axiomatizable theory F such that $F \cong T \times H$.*

The proof of this theorem proceeds by first constructing a Turing machine M which "accepts" tapes which have written on them a description of a complete extension of T (Lemma 2.2). The theory F is then constructed so that each model of F incorporates a "diagram" of the machine M including an initial tape and subsequent states of the tape. The theory F requires that the machine accept the initial tape. Hence the initial tape represented in the model must have written on it a description of a complete extension of T. This permits a correspondence to be set up between an arbitrary sentence α of T and a sentence of F which says that α is a sentence in the complete extension represented by the initial tape of the machine. These sentences, although they completely determine the standard part of the model, do not completely describe the model. Additional sentences describing non-standard parts of the model are made to correspond to sentences of the theory H. Using Lemma 1.3, these two correspondences establish the required isomorphism.

The Turing machine we will construct has a single two-way infinite tape. The machine has a finite number m of internal states and each square of the tape can have one of n symbols, including $*$, 0, and 1, written on it. The machine is defined by a set M of quintuples $\langle i,j,p,q,t\rangle$ where $i, p<m$; $j, q<n$; and $t=0, 1$. For each state i and symbol j, there is at most one quintuple $\langle i, j, p, q, t\rangle$ in M. If the machine is in the ith state and reading the jth symbol, it will change to the pth state and will write the qth symbol in place of the symbol it read. If $t=0$, it will continue moving the tape in the same direction in which it was going and if $t=1$, the tape motion will be reversed. However, if there is no quintuple corresponding to i, j, the machine will halt.

Let g be a fixed recursive function mapping the set of all sentences formulated in the theory T one-to-one onto the positive integers. (Later in this section we will prescribe some further conditions on g in order to make a certain part of the proof work.) A tape for machine M will be said to *represent* an extension T' of T iff the following conditions hold:

(i) Exactly one $*$ appears on the tape.

(ii) Only 0's appear to the right of the $*$.

(iii) For each sentence σ, if $n=g(\sigma)$, then the nth square of the tape to the left of the $*$ has a 1 written on it if σ is a theorem of T' and has 0 written on it otherwise.

Lemma 2.2. *Given an axiomatizable theory T, there exists a Turing machine M such that*

(i) *If the initial tape does not represent a complete extension of T, then M halts after some finite number of steps.*

(ii) *If the initial tape represents a complete extension T' of T (and the machine is started so that it first reads the square just to the right of the* *), *then M never halts and every quintuple of M is used infinitely many times.*

Proof. The construction of machine M is simply a matter of combining some well-known techniques. Figure 2.1 shows the overall plan of the program for M. The program is based on the observation that, if T' is not a complete extension of T, then there exists a positive integer n such that either

(a) n is the Gödel number of a sentence σ such that neither σ nor $\neg\sigma$ is in T' or

(b) n is the Gödel number of a proof of the contradiction $(\forall x)(x \neq x)$ from premises taken from T' and the axioms of T.

The program looks successively at each possible n and halts if n satisfies (a) or (b). On the other hand, it is easily seen that if no n satisfies (a) or (b) then T' must be a complete extension of T and in this case the program continues in its loop forever.

Notice that each box of the flow chart in Figure 2.1 represents a simple recursive decision or operation. For example, there is a well-known recursive procedure for determining whether an integer n is the Gödel number of some sentence of a given similarity type (that of the theory T). The question "Is σ in T'?" is simply a matter of checking whether or not the $g(n)$th symbol to the left of the * is a 1. To illustrate how machine M might actually work, let us describe more fully the block which says "Is σ_j an axiom of T?". Upon arriving at this block the tape might have the following configuration:

$$\ldots \underbrace{0\,1\,1\,0\,1\,0}_{T'} * \underbrace{1\,1\,1\,1\,1\,1\,1\,1\,1\,1}_{n}\,0\,0\,\underbrace{1\,1\,1\,1\,1}_{\sigma_0}\,0\,\underbrace{1\,1\,1}_{\sigma_1}\,0\,0\,0 \ldots$$

To the left of the *, the tape has its original value representing a set T' of sentences. To the right is first a series of n 1's representing the number of times through the main loop. Let us suppose that n has been found to

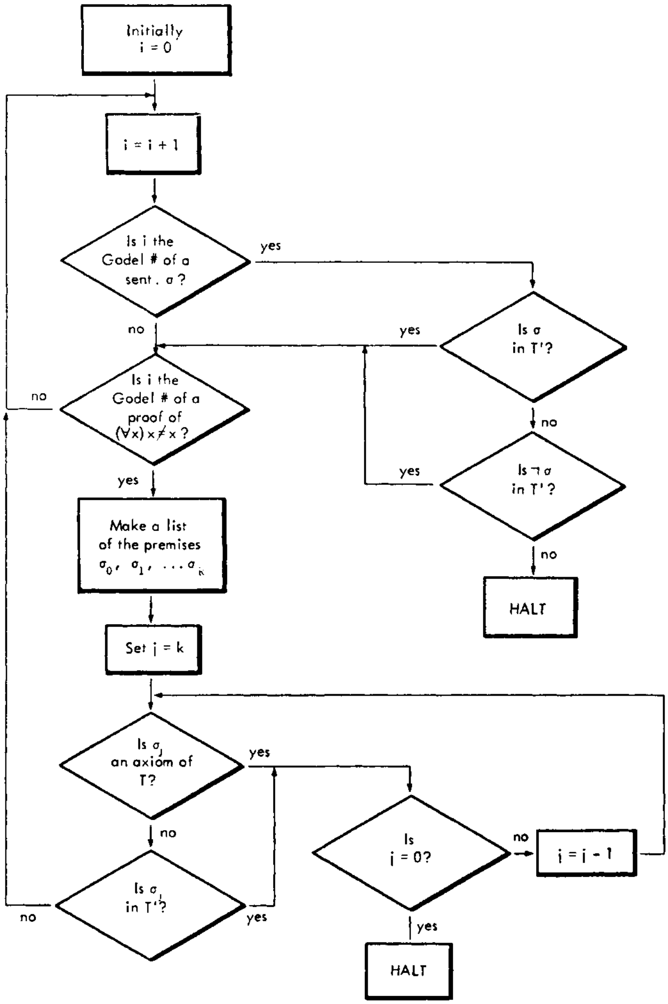

Fig. 2.1

be the Gödel number of a proof of the contradiction $(\forall x)(x \neq x)$ from premises σ_0, σ_1, and σ_2 and that σ_2 has already been found to be in T' and so has been removed from the list of premises. Thus σ_0 and σ_1 remain on the list of premises and we are ready to ask the question "Is σ_1 an axiom of T?". To answer the question, we simply compute the value of the characteristic function of the set of axioms of T. This is done by a sub-program which uses for its working space the portion of the tape to the right of the value σ_1. Thus the computation is not affected at all by other information on the tape. This sub-program is designed so that if anything but 0's were written in its working area, it would halt. If the value of the characteristic function is zero, the program goes on to determine whether σ_1 is written on the tape to the left of the *. If, on the other hand, the value is one, premise σ_1 is erased and the program goes on to examine σ_0.

It should now be clear how the details of the construction of machine M could be carried out. Notice that this can be done in such a way that each state-symbol combination that is used at all will be used infinitely many times. In the case of the computation of the characteristic function described above, this follows from the fact that each argument is used infinitely many times. It should also be pointed out that the machine M (and consequently the theory F) can be obtained recursively in terms of some standard method of presenting the axioms of T. This might require that the sub-program computing the characteristic function of the set of axioms be written interpretively so that the state-symbol combinations used in the sub-program can be effectively determined.

The theory F which will be constructed has two binary predicates R and S, m unary predicates $P_0, P_1, \ldots P_{m-1}$ corresponding to the m states of the machine M, and n unary predicates $Q_0, Q_1, \ldots Q_{n-1}$ corresponding to the n symbols that can be written on the tape. Figure 2.2 shows a part of a model of F. The dots, small circles, and small

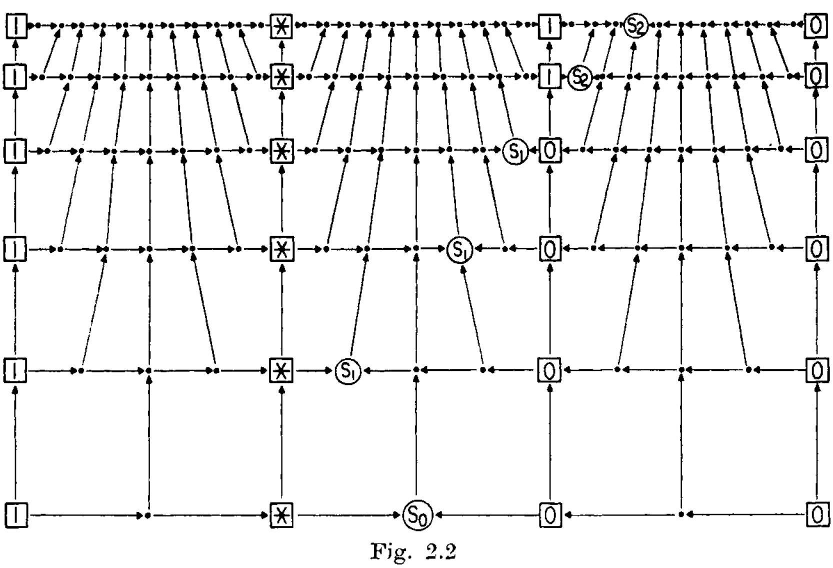

Fig. 2.2

squares represent elements of the model; the elements represented by circles are assumed to satisfy the unary predicate corresponding to the state indicated inside the circle and similarly, the elements represented by squares satisfy the predicate corresponding to the symbol written in the square; the elements represented by dots do not satisfy any unary predicate. The horizontal arrows indicate which elements are in the relation R and the vertical arrows indicate which elements are in the relation S. From the axioms which will be described later, it follows that any model of F has a set of elements which are grouped into a rectangular array by the relations R and S. The elements strung together by R form the rows of the array. Each row extends to infinity in both directions. There is an initial row in which every other element satisfies one of the predicates Q_j. Exactly one of the elements in this row satisfies the initial state predicate P_0. Each element in this initial row is related by S to a unique element of the second row and the same holds for each row and its succeeding row. As in figure 2.2, each "square" element (i.e., each element satisfying $Q_0, \ldots, Q_{n-1}$) in a row is succeeded in the next row by a square element. Furthermore, in each row, there are, in addition to the elements S related to the elements of the preceding row, new elements introduced on each side of each square element. Thus the number of filler elements between each pair of square elements increases from row to row; for example, in the third row, every sixth element is square. If we neglect the filler elements and look only at the square elements in a given row, we see that they form an image of a tape for machine M since each square element has one of the n symbols written on it (i.e., satisfies one of $Q_0, \ldots Q_{n-1}$). Also, in each row, there is exactly one element which satisfies one of the state predicates $P_0, \ldots P_{m-1}$. (This state element is also unique in that it is the R-successor of two elements; all the other elements in the row have one R-predecessor and one R-successor.) Thus each row determines not only a tape for machine M but also a state of M and a position of the reading head on the tape (determined by the position of the state element).

The fundamental property of the theory F is that in any model, the successive rows of the array accurately reflect the operations machine M would go through if started with the tape represented by the first row of the array. Actually, the 2nd, 5th, 14th, 41st, etc., rows of the array represent the state of the machine M and its tape after the first, second, third, fourth, etc. steps of its operation. (The distance between significant rows of the array increases as we proceed through successive rows because the state element must pass over more and more intervening filler elements. Because new filler elements are added in each row, the state element must move over two filler elements in order to make any progress at all.) Furthermore, because the axioms do not allow the machine to halt, we can conclude from Lemma 2.1 that the initial row of the array corresponds to a tape which represents a complete extension of the theory T. Thus at this point it is clear that for each theorem α of T, there will be a corresponding theorem of F which states that a certain element in the initial row of the array (of any model) satisfies Q_1. From this correspondence it is clear that F will have a degree of undecidability at least as great as that of T. Notice, however, that if the filler elements or some equivalent device had not been introduced, there would be no hope of proving that the degree of undecidability of F is at most that of T. For it could happen that even though T is decidable, the decision procedure given us for its axiom set could involve as intermediate steps the examination of proofs within, say, some system of arithmetic. In this case, it might be possible to formulate sentences of F which are provable just in case the corresponding sentences of arithmetic are provable.

We now formulate a notion of distance within a model and state a lemma which allows us to show elementary equivalence for various models of F. In Figure 2.2, the distance between any pair of elements is simply the smallest number of arrows that must be followed (either in the forward or reverse direction) to get from one element to the other. More precisely, given a model with binary relations R and S, we define the (open) sphere $Sp(n, a)$ of radius n about a point a recursively as follows:

$$\begin{aligned} Sp(1, a) &= a \\ Sp(n+1, a) &= \text{the set of points } x \text{ such that, for some } y \in Sp(n,a), \\ &\quad\ \text{either } xRy,\ yRx,\ xSy,\ ySx, \text{ or } x=y. \end{aligned}$$

The n-type τ of an element a of a model $\mathfrak{A}$ we define to be the isomorphism type of the sphere of radius n about a; i.e., the isomorphism type of $\mathfrak{A} \restriction Sp(n, a)$.

Lemma 2.3. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are models of the same similarity type (having two binary and arbitrarily many unary relations) and that each sphere of finite radius in $\mathfrak{A}$ or $\mathfrak{B}$ contains finitely many points. Then*

$\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent provided that, for each integer n and n-type τ either

(i) *Both $\mathfrak{A}$ and $\mathfrak{B}$ have infinitely many elements of n-type τ or*

(ii) *$\mathfrak{A}$ and $\mathfrak{B}$ have the same finite number of elements of n-type τ.*

Proof. We make use of the method of Ehrenfeucht [61, Theorem 7]. Given a length k of sequence, the "strategy" for picking elements is to insure that after partial sequences $a_0, \ldots, a_{j-1}$ and $b_0, \ldots, b_{j-1}$ have been picked, the following condition holds:

$$\text{(I)} \qquad \mathfrak{A} \upharpoonright \bigcup_{i<j} Sp(3^{k-j}, a_i) \cong \mathfrak{B} \upharpoonright \bigcup_{i<j} Sp(3^{k-j}, b_i).$$

To show elementary equivalence, one needs to show only that if (I) holds, and a_j is any element of $\mathfrak{A}$ then there exists an element b_j of $\mathfrak{B}$ such that (I) holds with j replaced by $j+1$. There are two cases: If

$$a_j \in \bigcup_{i<j} Sp(2 \cdot 3^{k-j-1}, a_i),$$

pick b_j to be the corresponding element (given by the isomorphism in (I)) of $\mathfrak{B}$. If not, pick b_j to be any element of $\mathfrak{B}$ which is outside

$$\bigcup_{i<j} Sp(2 \cdot 3^{k-j-1}, b_i)$$

and has the same 3^{k-j-1}-type as a_j.

The axioms of F are all expressed as "local" properties of the models. In fact they describe the 3-types of elements which can appear in the models. First there is an axiom which states that each open sphere of radius 3 contains less than 16 elements. This insures that there are only a finite number of 3-types to consider in further axioms. Then for any 3-type which cannot appear in the kind of two-dimensional array which we have described, there is an axiom stating that there is no element of that 3-type. This insures, for example, that in the array, whenever the state element lies next to a square element, the state element and square element in the next row of the array have the proper unary predicates to satisfy some quintuple in the definition of M. Finally, there is an axiom which states that there is exactly one square element which has no S predecessor and satisfies the unary predicate corresponding to the symbol *.

Using this last axiom as a starting point, we make use of the fact that each element has two neighbors in the horizontal direction and a successor in the vertical direction (elements not having this property were excluded by the earlier axioms) to show that each model of F has a two-dimensional array of the kind described earlier. This we call the standard part of the model. From the properties of machine M, we know that the initial row of this standard part corresponds to an initial tape which represents a complete extension of T. Furthermore, this initial row uniquely determines the entire two-dimensional array of the standard part of the model. The non-standard parts of a model may consist of any of the following: (i) a two-dimensional infinite array of filler elements, with or without a stripe of square elements running through it and with or without a series of state elements, (ii) a two-dimensional infinite array with an initial edge consisting of alternate square elements and filler elements (each square element has 0 or 1 written on it), (iii) a field of elements having an initial cyclical edge consisting of alternate square elements and filler elements with each successive row having the same number of square elements but more and more filler elements, (iv) fields of fillers which are cyclic in one direction but infinite in the other direction (those that are cyclic in the vertical direction may have a single stripe of square elements), and (v) fields of pure fillers which are cyclic in both directions.

Let Γ_0 be the set of sentences α_n where α_n states that the nth square element to the left of the * in the initial row of the model has a 1 written on it. Let Γ_1 be the set of all Boolean combinations of a set of sentences which describe the various types of cycles (as in (iii), (iv) and (v) above) that can occur in the non-standard parts of models. For example, corresponding to (iii) there are sentences $\beta(k, n, s)$ which state that there are at least k fields of elements having an initial edge with a cycle of n square elements (the unary predicates satisfied by these square elements are given by s which is an n-termed sequence of 0's and 1's). Using Lemma 2.3, we show that any two models of F are either elementarily equivalent or are distinguished by sentences of $\Gamma_0 \cup \Gamma_1$. For example, to show that a model having a non-standard part of the kind described in (ii) is elementarily equivalent to a model having no such part, we simply verify that each n-type of element occurring in the non-standard part also occurs infinitely many times in the standard part of any model. To make this true, we should have chosen the function g introduced above so that it forces every possible finite sequence of 0's and 1's to appear infinitely many times on the initial tape. This can be done by choosing a sequence of theorems of T (for example, the theorems $(\forall x)x=x$, $(\forall xy)(x=x \wedge y=y)$, ...) and a sequence of sentences refutable in T and mapping them in an appropriate way.

Now it is a simple matter to verify the hypotheses of Lemma 1.3 (taking F, T and H for S, T_0, and T_1). The fact that every sentence of F is equivalent to a Boolean combination of sentences of Γ_0 and Γ_1 follows from the fact that any two distinguishable models of F can be distinguished by sentences of $\Gamma_0 \cup \Gamma_1$. Condition (i) is obvious. The function G_0 of (ii) is obtained directly from the function g and the function G_1 is obtained by the method used to prove Theorem 1.1 (there is a straightforward decision method for sentences of Γ_1). Finally, condition (iii) follows from the fact that we can construct models of F having any combination of standard part and non-standard parts.

3. Conclusions and problems.

Theorem 3.1. *Given any recursively enumerable set E, there exists a finitely axiomatizable theory F such that:*

(i) *The set of theorems of F has the same truth table degree as E.*

(ii) *No finite extension of F is essentially undecidable.*

Proof. Apply Theorem 2.1 to the example of Feferman [57].

Theorem 3.2. *There exists an essentially hereditarily undecidable axiomatizable theory which is not compatible with any finitely axiomatizable essentially undecidable theory.*

Proof. Let E_1 and E_2 be disjoint recursively enumerable sets which are recursively inseparable. Let T be the extension of theory H obtained by adding axioms $\lambda(1, n)$ for every $n \in E_1$. Let F be a finitely axiomatizable theory isomorphic to $T \times H$. Finally, for each $n \in E_2$, add as an additional axiom of F the sentence which corresponds in the isomorphism to the sentence $\neg\lambda(1, n)$ of T. The resulting infinitely axiomatized theory has all the desired properties.

Theorem 3.3. *There exists an essentially undecidable finitely axiomatizable theory of any prescribed recursively enumerable degree of undecidability.*

Proof. Apply Theorem 2.1 to the example given by Shoenfield [58].

Theorem 3.4. *There exists a finitely axiomatizable essentially undecidable theory whose set of theorems is recursively separable from the set of sentences which are negations of theorems.*

Proof. Apply Theorem 2.1 to the example given by Ehrenfeucht [61a].

Theorem 3.5. *Given a formula Con(T) expressing the consistency of an arbitrary consistent axiomatizable theory T (in the terminology of Feferman [60], Con(T) can be the sentence Con_α for any* RE-*numeration α of the axioms of T), there exists a finitely axiomatizable decidable theory F such that the sentence*

$$\mathrm{Con}(T) \leftrightarrow \mathrm{Con}(F)$$

is provable in arithmetic.

Proof. Let M be a Turing machine which starts with a blank tape and examines all possible proofs to check that no contradiction can be derived from the axioms of T. M halts if a contradiction is found. The theory F is then constructed from M as in Section 2, and shown to be decidable by the methods given there. The proof that Con(T) is equivalent to Con(F) is elementary in view of the direct connections between a contradiction in T, the halting of M, and the resulting inconsistency of F. Kreisel has stated that it should even be possible to carry out the proof in primitive recursive (quantifier-free) arithmetic.

Corollary 3.6. *There exists a decidable consistent theory which can not be shown in Peano's arithmetic P to be consistent.*

Proof. In Theorem 3.5, take T to be P. If we had $\vdash_P \mathrm{Con}(F)$, we would have $\vdash_P \mathrm{Con}(P)$ contrary to Gödel's second underivability theorem.

Problem. *Does there exist a finitely axiomatizable undecidable theory with countably many complete extensions?*

Conjecture I. *Every axiomatizable theory is isomorphic to a finitely axiomatizable theory.*

Conjecture II. *Every finitely axiomatizable theory with countably many complete extensions is isomorphic to a finitely axiomatizable theory formulated with a finite number of unary predicates.*

Theorem 3.7. *Conjecture I implies a positive answer and Conjecture II implies a negative answer to the problem given above.*