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H-fields and their Liouville extensions

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Dedicated to the memory of Maxwell Rosenlicht (*1924–1999*).

Abstract. We introduce H-fields as ordered differential fields of a certain kind. Hardy fields extending $\mathbb{R}$, as well as the field of logarithmic-exponential series over $\mathbb{R}$ are H-fields. We study Liouville extensions in the category of H-fields, as a step towards a model theory of H-fields. The main result is that an H-field has at most two Liouville closures.

Contents

Introduction

There are two algebraically flavoured theories about the asymptotic behaviour at infinity of real valued functions on halflines $(a, +\infty)$, with $a \in \mathbb{R}$. One is the theory of Hardy fields (see Bourbaki [5], Rosenlicht [17], [18]).

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The other approach studies the field of “transseries” (Écalle [7], Van der Hoeven [8]), also called the field of “logarithmic-exponential series” (“LE-series” for short), see [6].

Hardy fields are ordered differential fields of germs at $+\infty$ of real valued C^1-functions on halflines $(a, +\infty)$. Their elements are *analytic* objects, amenable to analytic methods, see for example Boshernitzan [3]. In contrast, LE-series are *formal* objects, and typically occur as “expansions” of elements of Hardy fields. These series can be manipulated formally and combinatorially.

In order to find out how these two views of “orders of infinity” are related, we introduce the purely algebraic notion of “H-field”. Before defining H-fields we state here a result which involves only Hardy fields and the field $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ of real LE-series. It will be obtained in Sect. 6 of this paper.

Theorem *Let* $e\colon K \to \mathbb{R}((x^{-1}))^{\mathrm{LE}}$ *be an ordered differential field embedding of the Hardy field* $K \supseteq \mathbb{R}$ *into the field of real LE-series such that* $e(r) = r$ *for* $r \in \mathbb{R}$. *Then* e *extends to an ordered differential field embedding from the Liouville closure* $\mathrm{Li}(K)$ *of* K *into the field of real LE-series.*

The Liouville closure $\mathrm{Li}(K)$ of K is the smallest real closed Hardy field extension of K that is closed under exponentiation and integration; see [3] or [17] for existence of $\mathrm{Li}(K)$. (Thus if $f \in \mathrm{Li}(K)$, then $\exp(f) \in \mathrm{Li}(K)$, and $f = g'$ for some $g \in \mathrm{Li}(K)$.)

Viewing e as a formal expansion operator and its inverse as a summation operator, the theorem says that such operators automatically extend to the Liouville closures of their domains of definition. We remark that integration can create new kinds of divergence in series expansions.

Conventions used throughout this paper

For a field K we put $K^\times := K \setminus \{0\}$, the multiplicative group of K. “Differential field” will mean “ordinary differential field of characteristic 0”, and the derivative of an element a of a differential field will be written as a'. If a differential field is denoted by the capital K, then we shall write C for its constant field, while if a differential field is denoted by another symbol, say L, then its constant field will be written as C_L. An “ordering” on a field is a linear order on the field compatible with the field operations, in the sense of “ordered field”. An *ordered differential field* is just a differential field with an ordering on the field, and no relation between derivation and ordering is assumed. For a linearly ordered set S, $a \in S$ and $A \subseteq S$, put

$$S^{<a} := \{x \in S : x < a\}, \quad S^{<A} := \{x \in S : x < A\}$$

and define $S^{>a}$ and $S^{>A}$ similarly.

Definition *An* H**-field** *is an ordered differential field* K *such that*

(H1) $a \in K, a > C \Rightarrow a' > 0$,
(H2) $\mathcal{O} = C + \mathfrak{m}$, *where* $\mathcal{O} := \{a \in K : |a| \le |c| \text{ for some } c \in C\}$, *and* $\mathfrak{m}$ *is the maximal ideal of the convex subring* $\mathcal{O}$ *of* K.

Thus Hardy fields extending $\mathbb{R}$ are H-fields, and so are ordered differential subfields of $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ that contain $\mathbb{R}$ (see [6], Prop. 4.3). The real closure of an H-field (equipped with the unique derivation that extends the given derivation) is again an H-field, see Corollary 3.10. An H-field K is said to be **Liouville closed** if K is real closed and for each $a \in K$ there exist $y, z \in K$ such that $y' = a$ and $z \neq 0$ with $z'/z = a$. In particular, maximal Hardy fields are Liouville closed H-fields in this sense, as is $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ (see [17], Cor. 1, [6], Cor. 5.7).

A **Liouville extension** of a differential field K is a differential field extension L of K such that C_L is algebraic over C and for each $a \in L$ there are $t_1, \dots, t_n \in L$ with $a \in K(t_1, \dots, t_n)$ and for each $i = 1, \dots, n$ one of the following holds:

(1) t_i is algebraic over $K(t_1, \dots, t_{i-1})$,
(2) $t_i' \in K(t_1, \dots, t_{i-1})$,
(3) $t_i \neq 0$ and $t_i'/t_i \in K(t_1, \dots, t_{i-1})$.

A **Liouville closure** of an H-field K is a Liouville closed H-field extension L of K such that L is a Liouville extension of K. (For example, if K is a Hardy field extending $\mathbb{R}$, then $\mathrm{Li}(K)$ as defined above is indeed a Liouville closure of K in this sense.)

We can now state the main result of this paper, proved in Sect. 6.

Theorem *Let* K *be an* H*-field. Then one of the following occurs:*

(I) K *has exactly one Liouville closure up to isomorphism over* K,
(II) K *has exactly two Liouville closures up to isomorphism over* K.

Remarks. Let K be an H-field, $v\colon K^\times \to \Gamma = v(K^\times)$ the (Krull) valuation on K whose valuation ring is the convex hull of C in K. Then

(1) $v(a'/a) < v(b')$ for all $a, b \in K^\times$ with $0 < v(a), v(b)$. There is at most one $\gamma \in \Gamma$ such that $v(a'/a) < \gamma < v(b')$ for all such a, b.
(2) If $\{v(a'/a) : a \in K^\times, 0 < v(a)\}$ has a largest element, then no γ as in (1) exists and Case (I) of the theorem occurs. ((I) can also occur in other ways.)
(3) If there is γ as in (1), then Case (II) of the theorem occurs, and in one Liouville closure L_1 of K, all $s \in K$ with $v(s) = \gamma$ have the form b' with $v(b) < 0$, while in another Liouville closure L_2 of K all $s \in K$ with $v(s) = \gamma$ have the form b' with $v(b) > 0$.

(4) The situation of (3) is *the* fork in the road towards a Liouville closure: (II) is equivalent to the existence of a Liouville H-field extension of K for which the hypothesis of 3 is satisfied. See Sect. 6 for details.

The "fork in the road" of remark (3) manifests itself already in the value group. Indeed, after a preliminary first section, we study in Sect. 2 the "asymptotic couples" (enriched value groups) that are associated to H-fields. We establish there several extension lemmas for these couples, as tools in obtaining corresponding extension results about H-fields in sections 3–5.

The present paper is also meant as a further step (after [1] and [6]) towards a model theory of the differential field of LE-series. A key problem is whether the theory of H-fields has something like a model completion with $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ as model. The "fork in the road" phenomenon is an obstruction, but one that seems managable.

We freely use elementary differential algebra. The "logarithmic-derivative" identity $\frac{(ab)'}{ab} = \frac{a'}{a} + \frac{b'}{b}$ (for non-zero a, b in a differential field) is in the background of several computations. We shall also use results on valued fields, in particular concerning henselization, and pseudo-Cauchy sequences. (See e.g. [10] or [11].)

Further conventions and notations

A valued field is just a field equipped with a valuation ring of the field, and the corresponding (Krull) valuation on the field is generally indicated by v, even if more than one valued field is in play. In particular, the valuation of an element a of a valued field will always be indicated by $v(a)$. If a valued field is indicated by the capital K, then its valuation ring will always be written as $\mathcal{O}$, the maximal ideal of $\mathcal{O}$ as $\mathfrak{m}$, the value group $v(K^\times)$ as Γ. Also $\Gamma^* := \Gamma \setminus \{0\}$, and $\Gamma_\infty := \Gamma \cup \{\infty\}$, with the usual convention on extending the addition and ordering on Γ to Γ_∞. If a valued field is indicated by the capital L, then the corresponding objects are denoted by the same symbols with a subscript L attached, for example $\mathcal{O}_L$ is the valuation ring of L. The residue fields of valued fields K and L are denoted by $\mathrm{res}(K) := \mathcal{O}/\mathfrak{m}$ and $\mathrm{res}(L) := \mathcal{O}_L/\mathfrak{m}_L$. Thus a valued field extension $K \subseteq L$ gives rise to a field extension $\mathrm{res}(K) \subseteq \mathrm{res}(L)$ and to an ordered group extension $\Gamma \subseteq \Gamma_L$. A *valued differential field* is just a differential field equipped with a valuation ring of the field, and no relation between derivation and valuation is assumed, in contrast to our use of the term "differential-valued field", which is a valued differential field where the derivation and valuation are linked in a certain way, see Sect. 1 below.

The terms "embedding" and "extension" are used as in model theory. For example, let K and L be valued differential fields. Then an *embedding*

$h\colon K \to L$ is a differential field embedding from K into L such that for all $a \in K$ we have: $a \in \mathcal{O} \Leftrightarrow h(a) \in \mathcal{O}_L$. We say that L *extends* K (notation: $K \subseteq L$) if the underlying set of K is a subset of the underlying set of L and the inclusion map $K \hookrightarrow L$ is an embedding (tacitly: of valued differential fields).

Throughout we let m and n range over $\mathbb{N} := \{0, 1, 2, \dots\}$.

1 H-fields and differential-valued fields

In this section we obtain some simple properties of H-fields, and relate this notion to earlier work by Robinson [12] and Rosenlicht [15].

Lemma 1.1 *Let K be an H-field, and $a, b \in \mathfrak{m}$, $b \neq 0$. Then $a'b/b' \in \mathfrak{m}$.*

Proof. Replacing b by $-b$ if necessary we may assume that $b > 0$. Then $1/b > C$, so $-b'/b^2 > 0$, that is, $b' < 0$. Let $c > 0$ in C. Then $c + a > \mathfrak{m}$ and $c - a > \mathfrak{m}$, so $(c+a)/b > C$ and $(c-a)/b > C$. Taking derivatives in the last two relations gives $a'b - (c+a)b' > 0$ and $-a'b - (c-a)b' > 0$. Dividing by $b' < 0$ gives $-(c-a) < a'b/b' < c + a$. This holds for all positive $c \in C$, so $a'b/b' \in \mathfrak{m}$. $\square$

We consider from now on an H-field as an ordered valued differential field by taking as the valuation ring the convex hull of its constant field. The lemma says that the valuation of an H-field is a differential valuation in the sense of Rosenlicht [15]. Some arguments and constructions with H-fields become more transparent in the setting of differential-valued fields, or even pre-differential-valued fields, and we now turn to these objects.

Differential-valued fields

A **differential valuation** on a differential field K is by definition a valuation on K with valuation ring $\mathcal{O}$ such that

(DV1) $\mathcal{O} = C + \mathfrak{m}$;
(DV2) if $a, b \in \mathfrak{m}$ and $b \neq 0$, then $a'b/b' \in \mathfrak{m}$.

By (DV1) the constant field C maps isomorphically onto the residue field $\mathcal{O}/\mathfrak{m}$ under the residue class map $\mathcal{O} \to \mathcal{O}/\mathfrak{m}$. In terms of the valuation v we can reformulate (DV2) as follows

(DV2) $\big(v(a) > 0, v(b) > 0 \text{ and } b \neq 0\big) \Rightarrow v(a') > v(b'/b)$.

Note that *in the presence of* (DV1) we can replace (DV2) by

(DV2′) $\big(v(a) \geq 0, v(b) > 0 \text{ and } b \neq 0\big) \Rightarrow v(a') > v(b'/b)$.

We define a **differential-valued field** to be a valued differential field K such that $\mathcal{O}$ is the valuation ring of a differential valuation on K.

Suppose K is a valued differential subfield of a differential-valued field. It may happen that K is not differential-valued, because of the possible failure of clause (DV1) in the definition of differential valuation. But K is clearly a pre-differential-valued field in the following sense.

Definition 1.2 *A* **pre-differential-valued field** *is a valued differential field that satisfies* (DV2′) *above.*

In a pre-differential-valued field K the valuation is trivial on the constant field: otherwise there would be a non-zero constant b of positive valuation, contradicting (DV2′) above for $a = 0$. (Thus the map $\varepsilon \mapsto \varepsilon' \colon \mathfrak{m} \to K$ is one-to-one.) Note also that any differential field with the trivial valuation is a pre-differential-valued field. We shall prove in §4 that every pre-differential-valued field is a valued differential subfield of some differential-valued field.

Theorem 1 from [15] gives in effect several equivalent conditions for a valued differential field to be a pre-differential-valued field (without using this terminology), and its corollaries 1 and 2 state some consequences that we shall use below.

Pre-H-fields

If K is a differential subfield of an H-field, then K with the induced ordering and valuation is clearly a pre-H-field in the following sense.

Definition 1.3 *A* **pre-H-field** *is an ordered valued differential field K such that*

(PH1) *K is a pre-differential-valued field,*
(PH2) *the valuation ring is convex with respect to the ordering,*
(PH3) $f \in K, f > \mathcal{O} \Rightarrow f' > 0$.

Examples. Every H-field is a pre-H-field. Every Hardy field (not necessarily extending $\mathbb{R}$) is a pre-H-field. Any ordered differential field with the trivial valuation is a pre-H-field.

Let K be a pre-H-field. Then the function $\varepsilon \mapsto \varepsilon' \colon \mathfrak{m} \to K$ is strictly decreasing. This follows from the additivity of this map, noting that $\varepsilon' < 0$ for $0 < \varepsilon \in \mathfrak{m}$ because $f = \varepsilon^{-1} > \mathcal{O}$. Thus the derivation is also strictly decreasing on each coset $a + \mathfrak{m}$ in K. However, the behaviour of the derivation "in the large" is better described in terms of the valuation as follows:

Lemma 1.4 *Let K be a pre-H-field and $a, b \in K^\times$ with $v(a) < v(b)$. Then* $\frac{a'}{a} > \frac{b'}{b}$.

Proof. Write $a = bc$. Then $v(c) < 0$, so $c'/c > 0$, thus $a'/a = b'/b + c'/c > b'/b$. $\square$

Invariance under change of derivation

If K is a pre-differential-valued field and $a \in K^{\times}$, then K remains a pre-differential-valued field if we replace its derivation ∂ by the derivation $a\partial$, keeping the same valuation. The constant field of K is invariant under this change. Similarly, if K is a pre-H-field, and $a \in K^{>0}$, then K remains a pre-H-field if we replace its derivation ∂ by the derivation $a\partial$, keeping the same ordering and valuation.

This way of changing the derivation is not directly used in the present paper, but we expect it to become an effective tool in later developments. Such a change of derivation would correspond to a "change of independent variable" if the elements of K were (germs of) functions.

Robinson's work

Abraham Robinson [12] defined a *regular ordered differential field* to be an ordered differential field K with an element $x > C$ satisfying $x' = 1$ and such that if $y \in K$ with $y > C$, then $y' > 0$. He shows that some basic Hardy field asymptotics can be done in this abstract setting. However, he also constructs a regular ordered differential field K containing an $l > 0$ such that $l' = 1/x$, but $l < c$ for some constant $c \in C$, see [12], pp. 331–332. We note here that this example cannot be embedded as ordered valued differential field in any H-field, with valuation ring the convex hull of C.

Robinson shows this pathology of "$l = \log x$" disappears if the constant field is *archimedean*. (See Remark 1.5 below.) However, being archimedean notoriously fails to be a first-order property (in the logical sense). Thus he may have intended his example as indicating an obstacle to a possible model theory of Hardy fields. In any case, this pathology vanishes in the first-order setting of pre-H-fields. Indeed, we conjecture that the asymptotic rules valid in all Hardy fields will hold also for all pre-H-fields K with a distinguished element $x > \mathcal{O}$, $x' = 1$. ("Asymptotic rule" means "universal property in the natural language of ordered valued differential fields".)

The following remark shows in particular that regular ordered differential fields with archimedean constant field are pre-H-fields for the valuation whose valuation ring is the convex hull of the constant field.

Remark 1.5 Let K be an ordered differential field satisfying axiom (H1) for H-fields. Give K the valuation with $\mathcal{O} =$ convex hull of C. Identify C in the natural way with a subfield of the residue field $\operatorname{res}(K)$, which carries a natural ordering.

Suppose C is dense in $\operatorname{res}(K)$. (This assumption is satisfied if C is archimedean.) *Then K is a pre-H-field.* To see this, let $a, b \in K$ with $v(a) \geq 0$ and $b \neq 0$, $v(b) > 0$; we have to show $v(a') > v(b'/b)$. Passing from a to $-a$ if necessary, we may assume $a' \leq 0$; similarly, we may assume $b > 0$, so $a'b/b' \geq 0$. Let $0 < c \in C$; choose $d \in C$ such that $\overline{a} - c < d < \overline{a}$ in $\operatorname{res}(K)$, where $\overline{a}$ is the residue class of a. Then we have $(a-d)/b > C$, and taking derivatives and then dividing by $b' < 0$, we get $a'b/b' - (a-d) < 0$, hence $0 \leq a'b/b' < a - d < c$. Since this holds for all $c \in C^{>0}$, we get $a'b/b' \in \mathfrak{m}$ as required.

2 Asymptotic couples

Let K be a pre-differential-valued field. By Corollary 1 of [15], $v(a')$ is uniquely determined by $v(a)$ for $a \in K^{\times}$ with $v(a) \neq 0$. Thus v induces a map $\psi\colon \Gamma^* \to \Gamma$ given by

$$\psi\big(v(a)\big) = v(a') - v(a) = v(a'/a) \quad \text{ for } a \in K^{\times} \text{ with } v(a) \neq 0.$$

We also put $\psi(0) := \infty$. Following Rosenlicht [16] we call (Γ, ψ) the **asymptotic couple** of K. It encodes key features of the valued differential field K in a similar way as the value group does for a valued field.

Lemma 2.1 *Let $\alpha, \beta \in \Gamma$. Then*

(1) $\psi(\alpha + \beta) \geq \min\{\psi(\alpha), \psi(\beta)\}$*, that is, $\psi\colon \Gamma \to \Gamma_{\infty}$ is a valuation on the ordered abelian group Γ,*
(2) $\psi(r\alpha) = \psi(\alpha)$ *for $r \in \mathbb{Z} \setminus \{0\}$,*
(3) $\psi(\alpha) < \psi(\beta) + |\beta|$ *for $\alpha \neq 0$.*

Proof. This is Theorem 4 of [15] extended from differential-valued fields to pre-differential-valued fields. The proof there goes through. $\square$

With id denoting the identity function on Γ, we further note that

$$\begin{aligned} \psi(\Gamma^*) = \psi\big(\Gamma^{>0}\big) &= \big\{v(a'/a) : 0 \neq a \in \mathfrak{m}\big\}, \\ (\operatorname{id} + \psi)(\Gamma^*) &= \big\{v(a') : a \in K^{\times}, v(a) \neq 0\big\}, \\ (\operatorname{id} + \psi)\big(\Gamma^{>0}\big) &= \big\{v(a') : 0 \neq a \in \mathfrak{m}\big\}. \end{aligned}$$

In the case that K is a pre-H-field, the map ψ is decreasing on $\Gamma^{>0}$:

Lemma 2.2 *Let K be a pre-H-field and $\gamma, \delta \in \Gamma$. Then*

$$0 < \gamma < \delta \quad \Longrightarrow \quad \psi(\gamma) \geq \psi(\delta). \tag{2.1}$$

Proof. Suppose $0 < \gamma < \delta$. Choose $a, b > \mathcal{O}$ such that $v(a) = -\delta$ and $v(b) = -\gamma$, so $v(a) < v(b) < 0$. By Lemma 1.4 we have $a'/a > b'/b > 0$, so $v(a'/a) \leq v(b'/b)$, that is, $\psi(\gamma) \geq \psi(\delta)$. $\square$

Independently of their connection to pre-differential-valued fields we define (as in [16]) an **asymptotic couple** to be a pair (Γ, ψ) where Γ is an ordered abelian group and $\psi\colon \Gamma^* = \Gamma \setminus \{0\} \to \Gamma$ is a function such that Lemma 2.1 holds, where ψ is extended to all of Γ by setting $\psi(0) := \infty$. We will refer to (1), (2), (3) of Lemma 2.1 as axioms (1), (2), (3), respectively, for asymptotic couples.

The rest of this section concerns only asymptotic couples in this abstract sense. The main fact we prove about them is Theorem 2.6 below. Readers may prefer to skip the rest of this section and return when it gets used later in the paper.

Notation. Let Γ be an ordered abelian group. We define an equivalence relation $\sim$ on Γ by

$$\alpha \sim \beta \quad :\Longleftrightarrow \quad |\alpha| \leq m|\beta| \text{ and } |\beta| \leq n|\alpha| \text{ for some } m, n > 0.$$

The equivalence class of an element $\alpha \in \Gamma$ is written as $[\alpha]$, and is called its **archimedean class**. (This notation is used with a slightly different meaning in [1].) We let $[\Gamma]$ denote the set of archimedean classes of Γ, and $[\Gamma^*] := [\Gamma] \setminus \big\{[0]\big\}$. We linearly order $[\Gamma]$ by setting

$$[\alpha] < [\beta] \quad :\Longleftrightarrow \quad n|\alpha| < |\beta| \text{ for all } n > 0.$$

An embedding $i\colon \Gamma \to \Gamma'$ of ordered abelian groups induces an embedding $[\Gamma] \to [\Gamma']$ of linearly ordered sets. In case $\Gamma \subseteq \Gamma'$ and i is the inclusion map we regard $[\Gamma]$ as an ordered subset of $[\Gamma']$ via this induced embedding.

Basic properties of asymptotic couples

In the following, let (Γ, ψ) be an asymptotic couple. We set $\Psi := \psi(\Gamma^*)$.

Remark. For $\alpha \in \Gamma$ we define $\psi + \alpha\colon \Gamma^* \to \Gamma$ by

$$(\psi + \alpha)(x) := \psi(x) + \alpha \qquad \text{for } x \in \Gamma^*.$$

Then $(\Gamma, \psi + \alpha)$ is also an asymptotic couple, with $(\psi + \alpha)(\Gamma^*) = \Psi + \alpha$. Replacing the derivation ∂ of a pre-differential-valued field K by $a\partial$, $a \in K^\times$, has the effect that the asymptotic couple (Γ, ψ) of K gets replaced by $(\Gamma, \psi + \alpha)$, with $\alpha := v(a)$.

Proposition 2.3 (Basic properties of ψ; cf. [16], [1].) *Let* $\alpha, \beta \in \Gamma$.

(1) *If* $\alpha, \beta < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, *then* $\psi(\alpha - \beta) > \min\{\alpha, \beta\}$. *In particular, if* $\alpha, \beta \neq 0$, *then* $\psi\big(\psi(\alpha) - \psi(\beta)\big) > \min\big\{\psi(\alpha), \psi(\beta)\big\}$.
(2) *If* $\alpha, \beta \neq 0$, *then* $n\big(\psi(\beta) - \psi(\alpha)\big) < |\alpha|$.
(3) *If* $\alpha, \beta \neq 0$ *and* $\alpha \neq \beta$, *then* $\big[\psi(\alpha) - \psi(\beta)\big] < [\alpha - \beta]$.

(4) *The map* $x \mapsto x + \psi(x) \colon \Gamma^* \to \Gamma$ *is strictly increasing.*
(5) $(\mathrm{id} + \psi)\big(\Gamma^{<0}\big) = (-\mathrm{id} + \psi)\big(\Gamma^{>0}\big) \subseteq \big\{\gamma \in \Gamma : \gamma < \delta$ *for some* $\delta \in \Psi\big\}$.

Proof. For (1), let $\alpha < \beta < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. Then $\psi(\beta - \alpha) + (\beta - \alpha) > \beta$, hence $\psi(\beta - \alpha) > \alpha$, as required.— Property (2) is Theorem 5 in [15].— For (3), let $\alpha, \beta \neq 0$ with $\gamma := \alpha - \beta \neq 0$. We have to show that then $n\big|\psi(\alpha) - \psi(\beta)\big| < |\gamma|$ for all n. If $\psi(\gamma) > \psi(\beta)$, then by axiom (1), $\psi(\alpha) = \psi(\beta)$. Suppose $\psi(\gamma) \leq \psi(\beta)$. Then by axiom (1) again we have $\psi(\gamma) \leq \psi(\alpha)$, hence by (2):

$$n\psi(\gamma) \leq n\psi(\beta) < n\psi(\gamma) + |\gamma|, \qquad n\psi(\gamma) \leq n\psi(\alpha) < n\psi(\gamma) + |\gamma|.$$

Thus $n\big|\psi(\alpha) - \psi(\beta)\big| < |\gamma|$ in all cases.— Property (4) follows easily from (3).— The equality in (5) follows from ψ being an even function, and the inclusion is clear. $\square$

We shall consider Γ as a subgroup of the divisible abelian group $\mathbb{Q}\Gamma = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ via the embedding $\gamma \mapsto 1 \otimes \gamma$. We also equip $\mathbb{Q}\Gamma$ with the unique linear order that makes it into an ordered abelian group containing Γ as ordered subgroup. Using part (2) of the previous proposition, ψ extends uniquely to a map $(\mathbb{Q}\Gamma)^* \to \mathbb{Q}\Gamma$, also denoted by ψ, such that $(\mathbb{Q}\Gamma, \psi)$ is an asymptotic couple. Note that $\psi\big((\mathbb{Q}\Gamma)^*\big) = \Psi$ and $[\mathbb{Q}\Gamma] = [\Gamma]$. If $\dim_{\mathbb{Q}} \mathbb{Q}\Gamma$ is finite, then $\Psi = \psi(\Gamma^*)$ is a finite set.

The following proposition generalizes part of Proposition 3.1 in [1]; the proof given there goes through.

Proposition 2.4 *There is at most one element* $\gamma \in \Gamma$ *such that*

$$\Psi < \gamma < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big). \tag{2.2}$$

If Ψ *has a largest element, then there is no* $\gamma \in \Gamma$ *satisfying* (2.2). $\square$

In thinking about asymptotic couples, we found the picture in Sect. 1 of [1] very helpful, and we advise the reader to have a look. It gives an impression of how the maps ψ and $\mathrm{id} + \psi$ on Γ^* behave, especially in the case that ψ is decreasing on $\Gamma^{>0}$.

Lemma 2.5 *The following conditions on an element* $\alpha \in \Gamma$ *are equivalent:*

(1) $\alpha \in (\mathrm{id} + \psi)(\Gamma^*)$.
(2) $\psi\big(\alpha - \psi(\beta)\big) = \psi(\beta)$ *for some* $\beta \in \Gamma^*$.
(3) $\psi\big(\alpha - \psi(\beta)\big) \leq \psi(\beta)$ *for some* $\beta \in \Gamma^*$.

Proof. For (1) $\Rightarrow$ (2), assume $\alpha = \beta + \psi(\beta)$ for some $\beta \in \Gamma^*$. Then $\alpha - \psi(\beta) = \beta \neq 0$, so $\psi\big(\alpha - \psi(\beta)\big) = \psi(\beta)$. Conversely, if $\beta \in \Gamma^*$ and $\psi\big(\alpha - \psi(\beta)\big) = \psi(\beta)$, then

$$\alpha = \big(\alpha - \psi(\beta)\big) + \psi(\beta) = \big(\alpha - \psi(\beta)\big) + \psi\big(\alpha - \psi(\beta)\big),$$

showing that $\alpha \in (\mathrm{id}+\psi)(\Gamma^*)$. This proves (2) $\Rightarrow$ (1). The implication (2) $\Rightarrow$ (3) being trivial, it remains to show (3) $\Rightarrow$ (1). So assume we have $\beta \in \Gamma^*$ with $\psi\big(\alpha-\psi(\beta)\big) \leq \psi(\beta)$, but $\alpha \notin (\mathrm{id}+\psi)(\Gamma^*)$. By (1) $\Leftrightarrow$ (2), $\psi\big(\alpha-\psi(\beta)\big) < \psi(\beta)$. By passing from ψ to $\psi-\alpha$, if necessary, we may assume $\alpha=0$. Hence $\psi^2(\beta)=\psi\big(-\psi(\beta)\big)<\psi(\beta)$, so by Proposition 2.3, (1),

$$\psi\big(\psi(\beta)-\psi^2(\beta)\big) > \min\big\{\psi(\beta),\psi^2(\beta)\big\} = \psi^2(\beta). \tag{2.3}$$

Also $\psi^2(\beta)\neq\psi^3(\beta)$ by (1) $\Leftrightarrow$ (2). Now (2.3) implies

$$\min\big\{\psi^2(\beta),\psi^3(\beta)\big\} = \psi\big(\psi(\beta)-\psi^2(\beta)\big) > \psi^2(\beta),$$

a contradiction. $\square$

Remark. Since $\psi(\Gamma^*)=\psi\big((\mathbb{Q}\Gamma)^*\big)$, this lemma implies that for $\alpha\in\Gamma$ we have:

$$\alpha\in(\mathrm{id}+\psi)(\Gamma^*) \quad\Longleftrightarrow\quad \alpha\in(\mathrm{id}+\psi)(\mathbb{Q}\Gamma^*).$$

The following result was proved in [14], but the first part only under the assumption that Ψ is well-ordered.

Theorem 2.6 *The set $\Gamma\setminus(\mathrm{id}+\psi)(\Gamma^*)$ has at most one element. If $\max\Psi$ exists, then $\Gamma\setminus(\mathrm{id}+\psi)(\Gamma^*)=\{\max\Psi\}$.*

Proof. Suppose $\alpha\neq\beta$ in Γ and $\alpha,\beta\notin(\mathrm{id}+\psi)(\Gamma^*)$. Then for all $\gamma\in\Gamma^*$ we have $\psi\big(\alpha-\psi(\gamma)\big)>\psi(\gamma)$ and $\psi\big(\beta-\psi(\gamma)\big)>\psi(\gamma)$. Applying this to $\gamma:=\alpha-\beta=\big(\alpha-\psi(\gamma)\big)-\big(\beta-\psi(\gamma)\big)$, we get

$$\psi(\alpha-\beta)\geq\min\big\{\psi\big(\alpha-\psi(\gamma)\big),\psi\big(\beta-\psi(\gamma)\big)\big\}>\psi(\gamma),$$

a contradiction.— Now suppose $\max\Psi$ exists. If $\max\Psi=\alpha+\psi(\alpha)$ for some $\alpha\in\Gamma^*$, then $\alpha<0$ by axiom (3) for asymptotic couples, hence $\psi(\alpha)=\max\Psi-\alpha>\max\Psi\geq\psi(\alpha)$, a contradiction. $\square$

Lemma 2.7 *If $\alpha\in\Gamma$, $\alpha\neq0$, $\alpha\notin(\mathrm{id}+\psi)(\Gamma^*)$, then $\psi^2(\alpha)=\psi(\alpha)$.*

Proof. Note that $\psi(\alpha)\neq0$, since $\alpha\neq\alpha+\psi(\alpha)$. By 2.5, (1) $\Leftrightarrow$ (3), we have $\psi\big(\alpha-\psi(\alpha)\big)>\psi(\alpha)$, so $\psi^2(\alpha)=\psi\big(\psi(\alpha)-\alpha+\alpha\big)=\psi(\alpha)$. $\square$

Remarks. Assume that $(\mathrm{id}+\psi)(\Gamma^*)=\Gamma\setminus\{\beta_0\}$, $\beta_0\in\Gamma$. Then we can express the unique solution $x\in\Gamma^*$ to any equation $x+\psi(x)=\alpha$ with $\alpha\in\Gamma\setminus\{\beta_0\}$ in terms of α and β_0, namely $x=\alpha-\psi(\beta_0-\alpha)$. To see this, note that $\gamma:=\beta_0-\alpha\notin\big(\mathrm{id}+(\psi-\alpha)\big)(\Gamma^*)$, $\gamma\neq0$, hence $(\psi-\alpha)(\gamma)\neq0$ and

$$\psi(\gamma)-\alpha=(\psi-\alpha)(\gamma)=(\psi-\alpha)^2(\gamma)=\psi\big(\alpha-\psi(\gamma)\big)-\alpha$$

by Lemma 2.7. This says $-x=\psi(x)-\alpha$, so $x+\psi(x)=\alpha$.

Now suppose (Γ,ψ) is the asymptotic couple of the differential-valued field K, and $a \in K^\times$. As in [18], call $b \in K$ an **asymptotic integral of** a if b' is close to a in the sense that $v(a-b') > v(a)$. It is easy to see (cf. [14], Sect. 3) that a has an asymptotic integral in K if and only if $v(a) \in (\mathrm{id}+\psi)(\Gamma^*)$. Reinterpreting the previous remark in this setting, we get: If $b_0 \in K^\times$ has no asymptotic integral, then for all $a \in K^\times$ with $v(a) \neq v(b_0)$, the element $b := a/\big((b_0/a)'/(b_0/a)\big) \in K$ satisfies $v(b') = v(a)$, and hence there exists a non-zero constant c such that cb is an asymptotic integral of a.

In [1], we indicate an asymptotic couple (Γ,ψ) such that $\max\Psi$ exists, and one that contains an element β with $\Psi < \beta < (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$. In fact, the asymptotic couples considered in that paper satisfy (2.1) in Lemma 2.2, and for such an asymptotic couple (Γ,ψ) and $\beta \in \Gamma$ we have:

$$\beta \notin (\mathrm{id}+\psi)(\Gamma^*) \quad\Longleftrightarrow\quad \beta = \max\Psi \text{ or } \Psi < \beta < (\mathrm{id}+\psi)\big(\Gamma^{>0}\big).$$

(See Lemma 3.1 in [1], which generalizes to the asymptotic couples satisfying (2.1) in Lemma 2.2 above.)

However, "$(\mathrm{id}+\psi)(\Gamma^*) \neq \Gamma$" can also occur in other ways. This came as a surprise to us, and explains some case distinctions to be made in later sections. In Example 2.8 below we construct an asymptotic couple (Γ,ψ) with an element $\beta \in \Gamma \setminus (\mathrm{id}+\psi)(\Gamma^*)$, but $\beta > (\mathrm{id}+\psi)(\alpha)$ for some $\alpha \in \Gamma^{>0}$. Example 2.9 describes an asymptotic couple (Γ,ψ) with an element $\beta \in \Gamma \setminus (\mathrm{id}+\psi)(\Gamma^*)$ and $\beta < \psi(\alpha)$ for some $\alpha \in \Gamma^*$. In both examples we arrange $\beta = 0$. (Results from [16], §2, imply that there exists a differential-valued field whose asymptotic couple is isomorphic to that of Example 2.8; similarly for Example 2.9.)

Example 2.8 Let Γ be the free abelian group on generators

$$\alpha = \psi^0(\alpha), \quad \psi^1(\alpha), \quad \psi^2(\alpha), \quad \dots \tag{2.4}$$

Setting $\varepsilon_0 := \alpha + \psi^1(\alpha)$ and $\varepsilon_n := \psi^n(\alpha) - \psi^{n+1}(\alpha)$ for $n > 0$, we see that

$$\Gamma = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}\psi^n(\alpha) = \mathbb{Z}\alpha \oplus \bigoplus_{n\in\mathbb{N}} \mathbb{Z}\varepsilon_n,$$

with $\psi^n(\alpha) = -\alpha + \varepsilon_0 - \varepsilon_1 - \cdots - \varepsilon_{n-1}$ for $n > 0$. We make Γ into an ordered group by imposing $\alpha > 0$, $\varepsilon_n < 0$ and

$$[0] < \cdots < [\varepsilon_{n+1}] < [\varepsilon_n] < \cdots < [\varepsilon_1] < [\varepsilon_0] < [\alpha].$$

Now define a map $\psi\colon \Gamma^* \to \Gamma$ by

$$\psi\left(\sum r_n \psi^n(\alpha)\right) := \psi^{m+1}(\alpha), \quad \text{where } m = \min\{n \in \mathbb{N} : r_n \neq 0\}.$$

(Note that this definition implies $\psi\bigl(\psi^m(\alpha)\bigr) = \psi^{m+1}(\alpha)$, in accordance with our notation for the generators (2.4) of Γ.) We have $\psi^n(\alpha) < \psi^{n+1}(\alpha)$ for all $n > 0$.

We verify that (Γ, ψ) is an asymptotic couple. Axioms (1) and (2) are straightforward to check. For (3), let $\gamma, \delta \in \Gamma^*$, $\delta > 0$,

$$\gamma = \sum r_n \psi^n(\alpha), \quad \delta = \sum s_n \psi^n(\alpha), \quad \text{with } r_n, s_n \in \mathbb{Z}.$$

We have to show $\psi(\gamma) < \delta + \psi(\delta)$. Set

$$m := \min\{n \in \mathbb{N} : r_n \neq 0\}, \quad p := \min\{n \in \mathbb{N} : s_n \neq 0\}.$$

If $m \leq p$, then

$$\psi(\gamma) = \psi^{m+1}(\alpha) \leq \psi^{p+1}(\alpha) = \psi(\delta) < \delta + \psi(\delta).$$

So assume $m > p$. Then

$$0 < \psi^{m+1}(\alpha) - \psi^{p+1}(\alpha) = -\varepsilon_{p+1} - \varepsilon_{p+2} - \cdots - \varepsilon_m.$$

Let $q := \max\{n \in \mathbb{N} : s_n \neq 0\}$, $q \geq p$. If $p > 0$, then

$$\delta = \mu\bigl(-\alpha + \varepsilon_0 - \varepsilon_1 - \cdots - \varepsilon_{p-1}\bigr) - \sum_{k=0}^{q-p-1} \left(\sum_{n=p+k+1}^{q} s_n \right) \varepsilon_{p+k},$$

where $\mu := \sum_{n=p}^{q} s_n$. Hence (since $s_p \neq 0$) either $[\delta] = [\alpha]$ (if $\mu \neq 0$), or $[\delta] = [\varepsilon_p]$ (if $\mu = 0$). If $p = 0$, we have similarly

$$\delta = \mu'\alpha + (s_0 - \mu')\varepsilon_0 - \sum_{k=1}^{q-1} \left(\sum_{n=k+1}^{q} s_n \right) \varepsilon_k,$$

with $\mu' := s_0 - \sum_{n=1}^{q} s_n$. Thus if $\mu' \neq 0$, we have $[\delta] = [\alpha]$, and if $\mu' = 0$, we have $[\delta] = [\varepsilon_0]$. In any case, $[\delta] > [\varepsilon_{p+1}]$, and it follows that $\psi^{m+1}(\alpha) - \psi^{p+1}(\alpha) < \delta$, as required.

Also note that $\psi(\gamma) \neq \gamma$ for all $\gamma \in \Gamma^*$. So (Γ, ψ) is an asymptotic couple with $0 > \alpha + \psi(\alpha)$, $\alpha \in \Gamma^{>0}$, but $0 \notin (\mathrm{id} + \psi)(\Gamma^*)$.

Example 2.9 We slightly modify the previous example to obtain an asymptotic couple (Γ, ψ) and an element $\alpha \in \Gamma^{>0}$ with $\psi(\alpha) > 0$, where $0 \notin (\mathrm{id} + \psi)(\Gamma^*)$. Take $\Gamma = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}\psi^n(\alpha)$ as before, but now let $\varepsilon_n := \psi^n(\alpha) - \psi^{n+1}(\alpha)$ for all $n \in \mathbb{N}$. Again

$$\Gamma = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}\psi^n(\alpha) = \mathbb{Z}\alpha \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{Z}\varepsilon_n,$$

with $\psi^n(\alpha) = \alpha - \varepsilon_0 - \varepsilon_1 - \cdots - \varepsilon_{n-1}$ for each $n \in \mathbb{N}$. We make Γ into an ordered group by setting $\alpha > 0$ and $\varepsilon_n < 0$ for all $n \in \mathbb{N}$, as well as

$$[0] < \cdots < [\varepsilon_{n+1}] < [\varepsilon_n] < \cdots < [\varepsilon_1] < [\varepsilon_0] < [\alpha].$$

Define $\psi\colon \Gamma^* \to \Gamma$ as above. Note that $0 < \alpha < \psi(\alpha)$ and $\psi(\gamma) \neq \gamma$ for all $\gamma \in \Gamma^*$. That (Γ, ψ) is an asymptotic couple follows much as in the previous example.

Embedding results

Let (Γ_1, ψ_1) and (Γ_2, ψ_2) be asymptotic couples. An **embedding**

$$h\colon (\Gamma_1, \psi_1) \to (\Gamma_2, \psi_2)$$

is an embedding $h\colon \Gamma_1 \to \Gamma_2$ of ordered abelian groups such that

$$\psi_2\big(h(\gamma)\big) = h\big(\psi_1(\gamma)\big) \text{ for } \gamma \in \Gamma_1^*.$$

If $\Gamma_1 \subseteq \Gamma_2$ and the inclusion $\Gamma_1 \hookrightarrow \Gamma_2$ is an embedding $(\Gamma_1, \psi_1) \to (\Gamma_2, \psi_2)$, then we call (Γ_2, ψ_2) an **extension** of (Γ_1, ψ_1).

In the next four lemmas we fix an asymptotic couple (Γ, ψ), and show that if $\beta \in \Gamma \setminus (\mathrm{id} + \psi)(\Gamma^*)$, then (Γ, ψ) can be embedded into an asymptotic couple (Γ_1, ψ_1) with $\beta \in (\mathrm{id} + \psi_1)(\Gamma_1^*)$.

Lemma 2.10 *Let $\beta \in \Gamma$ and $\Psi < \beta < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. Then there is an asymptotic couple $\big(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha\big)$ extending (Γ, ψ), with $\alpha > 0$, such that:*

(1) $\alpha + \psi^\alpha(\alpha) = \beta$.
(2) *Given any embedding $i\colon (\Gamma, \psi) \to (\Gamma', \psi')$ of asymptotic couples and any element $\alpha' \in \Gamma'$ with $\alpha' > 0$ and $\alpha' + \psi'(\alpha') = i(\beta)$, there is a unique extension of i to an embedding $j\colon \big(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha\big) \to (\Gamma', \psi')$ with $j(\alpha) = \alpha'$.*

Proof. Let $\Gamma^\alpha := \Gamma \oplus \mathbb{Z}\alpha$ be an ordered group extension of Γ such that $0 < n\alpha < \Gamma^{>0}$ for all $n > 0$. We extend ψ to a function $\psi^\alpha\colon \big(\Gamma^\alpha\big)^* \to \Gamma^\alpha$ by

$$\psi^\alpha(\gamma + r\alpha) := \begin{cases} \psi(\gamma), & \text{if } \gamma \neq 0, \\ \beta - \alpha, & \text{otherwise,} \end{cases}$$

for $\gamma \in \Gamma$, $r \in \mathbb{Z}$, with $\gamma + r\alpha \neq 0$. Note that $\alpha + \psi^\alpha(\alpha) = \beta$.

It is tedious but routine to check that axiom (1) holds for $(\Gamma^\alpha, \psi^\alpha)$. Axiom (2) is trivially satisfied.— For axiom (3), we take $\gamma + r\alpha, \delta + s\alpha \in \Gamma^*$ ($\gamma, \delta \in \Gamma$, $r, s \in \mathbb{Z}$) with $\delta + s\alpha > 0$ (hence $\delta > 0$ if $\delta \neq 0$); we have to show $\psi^\alpha(\gamma + r\alpha) < \psi^\alpha(\delta + s\alpha) + (\delta + s\alpha)$. We can assume $\psi^\alpha(\gamma + r\alpha) >$

$\psi^\alpha(\delta+s\alpha)$ (otherwise $\psi^\alpha(\gamma+r\alpha) \le \psi^\alpha(\delta+s\alpha) < \psi^\alpha(\delta+s\alpha)+(\delta+s\alpha)$). We claim that $\delta > 0$. If not, then $\delta = 0$, so $\gamma \ne 0$ and

$$\psi^\alpha(\gamma + r\alpha) > \psi^\alpha(\delta + s\alpha) = \beta - \alpha,$$

hence $\psi^\alpha(\gamma+r\alpha) = \psi(\gamma)$, and thus $\beta-\psi(\gamma) < \alpha$. This implies $\beta-\psi(\gamma) < 0$, contradicting $\beta > \Psi$. So $\delta > 0$ as claimed. If moreover $\gamma = 0$, then $[0] < [\alpha] < [\Gamma^*]$ and $\beta < (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$ imply $\beta - (s+1)\alpha < \delta + \psi(\delta)$, i.e.

$$\psi^\alpha(\gamma + r\alpha) - \psi^\alpha(\delta + s\alpha) = \beta - \alpha - \psi(\delta) < \delta + s\alpha,$$

as required. Similarly, if $\gamma \ne 0$, we get

$$\psi^\alpha(\gamma + r\alpha) = \psi(\gamma) < \psi(\delta) + (\delta + s\alpha),$$

by axiom (3) for (Γ, ψ), and $[0] < [\alpha] < [\Gamma^*]$.

Let now $i\colon (\Gamma, \psi) \to (\Gamma', \psi')$ be an embedding of asymptotic couples, and $\alpha' \in \Gamma'$ with $\alpha' > 0$ and $i(\beta) = \alpha' + \psi'(\alpha')$.

Then $0 < n\alpha' < i\big(\Gamma^{>0}\big)$ for all $n > 0$: For $n = 1$ this is because $\mathrm{id}+\psi'$ is strictly increasing on $(\Gamma')^{>0}$. Assume there is a counterexample. Take $n > 0$ minimal such that $0 < i(\gamma) \le n\alpha'$ for some $\gamma \in \Gamma^{>0}$. So $n > 1$. Then

$$i\big(\gamma + \psi(\gamma)\big) \le n\alpha' + \psi'(n\alpha') = (n-1)\alpha' + i(\beta),$$

so $0 < i\big(\gamma + \psi(\gamma) - \beta\big) \le (n-1)\alpha'$, contradicting the minimality of n.

It follows that i extends to an embedding $\Gamma^\alpha \to \Gamma$ of ordered groups which sends α to α'. It is easy to check that this is in fact an embedding $\big(\Gamma^\alpha, \psi^\alpha\big) \to (\Gamma', \psi')$. □

In a similar way one shows:

Lemma 2.11 *Let $\beta \in \Gamma$ and $\Psi < \beta < (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$. Then there is an asymptotic couple $\big(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha\big)$ extending (Γ, ψ), with $\alpha < 0$, such that:*

(1) $\alpha + \psi^\alpha(\alpha) = \beta$.
(2) *Given any embedding $i\colon (\Gamma, \psi) \to (\Gamma', \psi')$ of asymptotic couples and any element $\alpha' \in \Gamma'$ with $\alpha' < 0$ and $\alpha' + \psi'(\alpha') = i(\beta)$, there is a unique extension of i to an embedding $j\colon \big(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha\big) \to (\Gamma', \psi')$ with $j(\alpha) = \alpha'$.* □

Lemma 2.12 *Let $\beta \in \Gamma \setminus (\mathrm{id}+\psi)(\Gamma^*)$ and suppose $\beta \le \psi(\gamma)$ for some $\gamma \in \Gamma^*$. Then (Γ, ψ) extends to an asymptotic couple $(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha)$ such that:*

(1) $\alpha + \psi^\alpha(\alpha) = \beta$.
(2) *Given any embedding $i\colon (\Gamma, \psi) \to (\Gamma', \psi')$ of asymptotic couples and any element $\alpha' \in \Gamma'$ with $\alpha'+\psi'(\alpha') = i(\beta)$, there is a unique extension of i to an embedding $j\colon (\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha) \to (\Gamma', \psi')$ with $j(\alpha) = \alpha'$.*

Proof. Let $C := \{\gamma \in (\mathbb{Q}\Gamma)^* : \gamma + \psi(\gamma) < \beta\}$, so $C \subseteq (\mathbb{Q}\Gamma)^{<0}$ is closed downward. Let $\Gamma^\alpha := \Gamma \oplus \mathbb{Z}\alpha$ be an ordered abelian group extension of Γ such that in $\mathbb{Q}\Gamma^\alpha$ the element α realizes the cut C in $\mathbb{Q}\Gamma$, i.e. $C < \alpha < (\mathbb{Q}\Gamma) \setminus C$. In particular $\alpha < 0$. Note that for $\gamma \in \Gamma$, $r \in \mathbb{Z}$, we have $\gamma + r\alpha > 0$ if and only if

(1′) $\gamma = 0$ and $r < 0$, or
(2′) $\gamma \neq 0$ and $\gamma + r\big(\beta - \psi(\gamma)\big) > 0$.

(This is clear if $\gamma = 0$ or $r = 0$. If $\gamma \neq 0$, $r \neq 0$, say $r > 0$, then $\gamma + r\alpha > 0$ if and only if $-\gamma/r < \alpha$, which is equivalent, by the definition of C, to $-\gamma/r + \psi(-\gamma/r) < \beta$, that is, to $\gamma + r\big(\beta - \psi(\gamma)\big) > 0$. If $r < 0$, one argues in a similar way, using also the remark after Lemma 2.5.) We extend ψ to a map $\psi^\alpha \colon \big(\Gamma^\alpha\big)^* \to \Gamma^\alpha$ by setting

$$\psi^\alpha(\gamma + r\alpha) := \begin{cases} \psi(\gamma), & \text{if } \gamma \neq 0, \\ \beta - \alpha, & \text{otherwise,} \end{cases}$$

for $\gamma \in \Gamma$, $r \in \mathbb{Z}$, with $\gamma + r\alpha \neq 0$. Observe that $\psi^\alpha(\gamma + r\alpha) = \min\{\psi(\gamma), \beta - \alpha\}$: This is clear if $\beta = \psi(\gamma)$ or $\gamma = 0$; if $\gamma \neq 0$ and $\beta \neq \psi(\gamma)$, then $\psi(\gamma) < \psi\big(\psi(\gamma) - \beta\big)$ by Lemma 2.5, hence $\psi(\gamma) - \beta + \big(\beta - \psi\big(\psi(\gamma) - \beta\big)\big) < 0$, giving $\psi(\gamma) < \beta - \alpha$ by the remarks above.

We claim that $(\Gamma^\alpha, \psi^\alpha)$ is an asymptotic couple. Verifying axioms (1) and (2) presents no problem. For axiom (3), let $\gamma, \delta \in \Gamma$, $r, s \in \mathbb{Z}$, with $\gamma + r\alpha \neq 0$, $\delta + s\alpha > 0$; we have to show

$$\psi^\alpha(\gamma + r\alpha) < \psi^\alpha(\delta + s\alpha) + (\delta + s\alpha).$$

We may assume $\psi^\alpha(\gamma + r\alpha) > \psi^\alpha(\delta + s\alpha)$. We claim that $\delta \neq 0$. Otherwise, just like in the proof of Lemma 2.10, it follows that $\gamma \neq 0$ and $\beta - \psi(\gamma) < \alpha$. Hence $\beta \neq \psi(\gamma)$ and $\big(\beta - \psi(\gamma)\big) + \psi\big(\beta - \psi(\gamma)\big) < \beta$, since α realizes the cut C. So $\psi(\gamma) > \psi\big(\beta - \psi(\gamma)\big)$, contradicting $\beta \notin (\mathrm{id} + \psi)(\Gamma^*)$ by Lemma 2.5. So $\delta \neq 0$ as claimed. Since $\delta + s\alpha > 0$ this gives

$$s\big(\psi(\delta) - \beta\big) < \delta \tag{2.5}$$

by (2′) above.

First assume $\gamma = 0$. By Lemma 2.5 we have $\psi\big(\beta - \psi(\delta)\big) > \psi(\delta)$, and thus $\psi\big(\beta - \psi(\delta) - \delta\big) = \psi(\delta)$; from (2.5) we obtain

$$\big(\beta - \psi(\delta) - \delta\big) + (s+1)\psi\big(\beta - \psi(\delta) - \delta\big) < (s+1)\beta.$$

Therefore (using the definition of C) $\beta - \psi(\delta) - \delta < (s+1)\alpha$, hence

$$\psi^\alpha(\gamma + r\alpha) - \psi^\alpha(\delta + s\alpha) = \beta - \alpha - \psi(\delta) < \delta + s\alpha,$$

as required.

Next assume $\gamma \neq 0$. Then $\psi^\alpha(\gamma + r\alpha) - \psi^\alpha(\delta + s\alpha) = \psi(\gamma) - \psi(\delta)$, so it is enough to show that $\psi(\gamma) - \psi(\delta) - \delta < s\alpha$. Now, by axiom (3), $\psi(\gamma) \neq \psi(\delta) + \delta$, since otherwise $\delta < 0$, hence $\psi(\gamma) = \psi(\delta) + \delta < \psi(\delta)$, contradicting the assumption $\psi^\alpha(\gamma + r\alpha) > \psi^\alpha(\delta + s\alpha)$. So we are reduced to showing that

$$\big(\psi(\gamma) - \psi(\delta) - \delta\big) + s\psi\big(\psi(\gamma) - \psi(\delta) - \delta\big) < s\beta,$$

that is, since $\psi\big(\psi(\gamma) - \psi(\delta)\big) > \min\big\{\psi(\gamma), \psi(\delta)\big\} = \psi(\delta)$ by Proposition 2.3 (1), to

$$\psi(\gamma) < \psi(\delta) + \delta + s\big(\beta - \psi(\delta)\big).$$

To prove this, we may assume, by passing from ψ to $\psi - \beta$, if necessary, that $\beta = 0$; hence $\psi(\eta) < \psi^2(\eta)$ for all $\eta \in \Gamma^*$, by Lemma 2.5. Hence $\psi(\delta + \psi(\delta)) = \psi(\delta)$, and thus $\psi\big(s\psi(\delta) - \delta - \psi(\delta)\big) = \psi(\delta)$, in particular $s\psi(\delta) - \delta - \psi(\delta) \neq 0$. To get a contradiction, assume

$$-\psi(\gamma) \leq s\psi(\delta) - \delta - \psi(\delta).$$

If $\psi(\gamma) \neq 0$, we use that $\mathrm{id} + \psi$ is increasing on Γ^* to obtain (by (2.5))

$$\begin{aligned} -\psi(\gamma) + \psi^2(\gamma) &\leq \big(s\psi(\delta) - \delta - \psi(\delta)\big) + \psi\big(s\psi(\delta) - \delta - \psi(\delta)\big) \\ &= s\psi(\delta) - \delta < 0, \end{aligned}$$

contradicting $\psi(\gamma) < \psi^2(\gamma)$. But if $\psi(\gamma) = 0$, then $0 < s\psi(\delta) - \delta - \psi(\delta)$, hence by axiom (3)

$$0 = \psi(\gamma) < \big(s\psi(\delta) - \delta - \psi(\delta)\big) + \psi\big(s\psi(\delta) - \delta - \psi(\delta)\big) = s\psi(\delta) - \delta,$$

contradicting $s\psi(\delta) < \delta$.

To prove part (2) of the statement, we may assume that $(\Gamma', \psi') \supseteq (\Gamma, \psi)$, and $\alpha' \in (\Gamma')^*$ is such that $\alpha' + \psi'(\alpha') = \beta$; we then have to show that there is an embedding of $(\Gamma^\alpha, \psi^\alpha)$ into (Γ', ψ') which is the identity on Γ and sends α to α'. Since $\mathrm{id} + \psi$ is strictly increasing on Γ^*, α and α' realize the same cut (namely C) in $\mathbb{Q}\Gamma$. Hence there is an embedding $\Gamma^\alpha = \Gamma \oplus \mathbb{Z}\alpha \to \Gamma'$ of ordered groups that is the identity on Γ and sends α to α'. Clearly $\psi'(\gamma + r\alpha') = \min\big\{\psi(\gamma), \beta - \alpha'\big\}$ for $\gamma \in \Gamma$, $r \in \mathbb{Z}$, with $\gamma + r\alpha' \neq 0$. Hence the map $\Gamma^\alpha \to \Gamma'$ is in fact an embedding $(\Gamma^\alpha, \psi^\alpha) \to (\Gamma', \psi')$. □

Lemma 2.13 *Let $\beta \in \Gamma \setminus (\mathrm{id} + \psi)(\Gamma^*)$, and suppose $\gamma + \psi(\gamma) < \beta$ for some $\gamma \in \Gamma^{>0}$. Then (Γ, ψ) extends to an asymptotic couple $(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha)$ such that:*

(1) $\alpha + \psi^\alpha(\alpha) = \beta$.
(2) *Given any embedding $i\colon (\Gamma, \psi) \to (\Gamma', \psi')$ of asymptotic couples and any element $\alpha' \in \Gamma'$ with $\alpha' + \psi'(\alpha') = i(\beta)$, there is a unique extension of i to an embedding $j\colon (\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha) \to (\Gamma', \psi')$ with $j(\alpha) = \alpha'$.*

Proof. We proceed as in the proof of the last lemma, except that the cut C is now defined as $C := \{0\} \cup \big\{\gamma \in (\mathbb{Q}\Gamma)^* : \gamma + \psi(\gamma) < \beta\big\}$. Then $\alpha > 0$, and for $\gamma \in \Gamma$, $r \in \mathbb{Z}$ we have $\gamma + r\alpha > 0$ if and only if $\gamma = 0$ and $r > 0$, or $\gamma \neq 0$ and $\gamma + r\big(\beta - \psi(\gamma)\big) > 0$. $\square$

Remark. In Lemmas 2.10–2.13, $\Psi^\alpha := \psi^\alpha\big((\Gamma^\alpha)^*\big) = \Psi \cup \{\beta - \alpha\}$ has maximum $\beta - \alpha$, so

$$(\mathrm{id} + \psi^\alpha)\big((\Gamma^\alpha)^*\big) = \Gamma^\alpha \setminus \{\beta - \alpha\}.$$

Hence $(\mathrm{id} + \psi^\alpha)\big((\Gamma^\alpha)^{>0}\big)$ is closed upward and there is no $\gamma \in \Gamma^\alpha$ with $\Psi^\alpha < \gamma < (\mathrm{id} + \psi^\alpha)\big((\Gamma^\alpha)^{>0}\big)$.

In the next two lemmas, needed in Sect. 5, we let (Γ, ψ) be an **asymptotic couple of H-type,** that is, an asymptotic couple with the property that ψ is decreasing on $\Gamma^{>0}$: $0 < \alpha \leq \beta \Rightarrow \psi(\alpha) \geq \psi(\beta)$. (By Lemma 2.2 asymptotic couples of pre-H-fields have this property.) Note that then ψ is constant on archimedean classes of Γ, i.e. for $\alpha, \beta \in \Gamma^*$ with $[\alpha] = [\beta]$ we have $\psi(\alpha) = \psi(\beta)$. Also, given any $\gamma \in \Gamma^{>0}$ and setting $\delta := \psi(\gamma) - \gamma$, we get an H_0-couple $(\mathbb{Q}\Gamma, \psi - \delta)$ with distinguished positive element γ, as defined in Sect. 6 of [1].

The first lemma is an analog of Lemma 3.2 in [1], with a similar proof:

Lemma 2.14 *Let $i\colon \Gamma \to \Gamma'$ an embedding of ordered abelian groups such that the induced map $[\Gamma] \to [\Gamma']$ is bijective. There is a unique function $\psi'\colon (\Gamma')^* \to \Gamma'$ such that (Γ', ψ') is an asymptotic couple of H-type and $i\big(\psi(\gamma)\big) = \psi'\big(i(\gamma)\big)$ for all $\gamma \in \Gamma^*$.* $\square$

We now generalize Lemma 3.6 in [1]. Given a cut C in a linearly ordered set $(S, <)$ (i.e. C is a downward closed subset of S) we say that an element a of a linearly ordered set extending $(S, <)$ **realizes the cut** C if $C < a < S \setminus C$.

Lemma 2.15 *Let C be a cut in $[\Gamma^*]$ and $\beta \in \Gamma$ such that $\beta < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, $\psi(\gamma) \leq \beta$ for all $\gamma \in \Gamma^*$ with $[\gamma] \notin C$, and $\beta \leq \psi(\delta)$ for all $\delta \in \Gamma^*$ with $[\delta] \in C$. Then there exists an asymptotic couple $(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha)$ of H-type extending (Γ, ψ), with $\alpha > 0$, such that:*

(1) *$[\alpha] \notin [\Gamma^*]$ realizes the cut C in $[\Gamma^*]$, $\psi^\alpha(\alpha) = \beta$.*
(2) *Given any embedding i of (Γ, ψ) into an asymptotic couple (Γ', ψ') of H-type and any element $\alpha' \in (\Gamma')^{>0}$ such that $[\alpha'] \notin \big[i(\Gamma^*)\big]$ realizes the cut $\big\{\big[i(\delta)\big] : [\delta] \in C\big\}$ in $\big[i(\Gamma^*)\big]$ and $\psi'(\alpha') = i(\beta)$, there is a unique extension of i to an embedding $j\colon (\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha) \to (\Gamma', \psi')$ with $j(\alpha) = \alpha'$.*

Proof. Embed the ordered abelian group Γ into an ordered abelian group $\Gamma^\alpha := \Gamma \oplus \mathbb{Z}\alpha$ with $\alpha > 0$ such that $[\alpha] \notin [\Gamma]$ realizes the cut C in $[\Gamma^*]$.

(See proof of Lemma 3.6 in [1].) Note that then $[\Gamma^\alpha] = [\Gamma] \cup \big\{[\alpha]\big\}$. We extend $\psi\colon \Gamma^* \to \Gamma$ to a map $\psi^\alpha\colon (\Gamma^\alpha)^* \to \Gamma$ by setting

$$\psi^\alpha(\gamma + r\alpha) := \min\big\{\psi(\gamma), \beta\big\} \quad \text{for } \gamma \in \Gamma,\ r \in \mathbb{Z},\ r \neq 0.$$

(So $\psi^\alpha\big((\Gamma^\alpha)^*\big) = \Psi \cup \{\beta\}$.) A tedious but routine checking of cases shows that ψ^α is decreasing on $(\Gamma^\alpha)^{>0}$, and that axioms (1) and (2) for asymptotic couples hold for $(\Gamma^\alpha, \psi^\alpha)$. Axiom (3) can be verified in a similar way as the corresponding part in the proof of Lemma 3.6 in [1], in the case $\boldsymbol{k} = \mathbb{Q}$. This proves part (1) of the lemma. Part (2) is routine. $\square$

Remark. In the context of Lemma 2.15, assume that $(\mathrm{id} + \psi)(\Gamma^*) = \Gamma$, that is, Ψ has no largest element, and there is no $\gamma \in \Gamma$ with $\Psi < \gamma < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. Then also $\Psi^\alpha := \psi^\alpha\big((\Gamma^\alpha)^*\big) = \Psi \cup \{\beta\}$ has no largest element, and there is no $\gamma \in \Gamma^\alpha$ such that $\Psi^\alpha < \gamma < (\mathrm{id} + \psi^\alpha)\big((\Gamma^\alpha)^{>0}\big)$. To see this, suppose $\gamma \in \Gamma^\alpha$ and $\Psi^\alpha < \gamma < (\mathrm{id} + \psi^\alpha)\big((\Gamma^\alpha)^{>0}\big)$. Then $\gamma \notin \Gamma$, hence $\gamma \notin \mathbb{Q}\Gamma \subseteq \mathbb{Q}\Gamma^\alpha$. Since Ψ has no maximum, $[\Gamma^*]$ has no minimum. Therefore $\Gamma^{>0}$ is coinitial in $\big(\mathbb{Q}\Gamma\big)^{>0}$, and thus $\Psi = \psi\big((\mathbb{Q}\Gamma)^*\big) < \gamma < (\mathrm{id} + \psi)\big(\mathbb{Q}\Gamma\big)^{>0}$. Lemma 4.5 in [1] then implies $[\Gamma^\alpha] = [\mathbb{Q}\Gamma \oplus \mathbb{Q}\gamma] = [\Gamma]$, contradicting $[\alpha] \notin [\Gamma]$.

3 Algebraic extensions of H-fields

In this section we show that an algebraic extension field of a pre-differential-valued field K is again a pre-differential-valued field. (Here the algebraic extension is equipped with the unique derivation that extends the derivation of K, and with *any* valuation that extends the one of K.) Similarly, we show that an ordered valued algebraic extension field L of a pre-H-field K such that $\mathcal{O}_L$ is the convex hull in L of $\mathcal{O}$ is again a pre-H-field.

In all lemmas in this section we make the following standing assumption:

> *K is a pre-differential-valued field, and $K \subseteq L$ is a valued differential field extension.* (Thus $\Gamma \subseteq \Gamma_L$ and $\operatorname{res}(K) \subseteq \operatorname{res}(L)$.)

Under various extra hypotheses (specified in the lemmas) we shall then derive that L is also a pre-differential-valued field. In remarks following these lemmas we consider the case that K is in addition a pre-H-field.

Lemmas 3.1, 3.3 and 3.7 are modifications of results in [15].

Lemma 3.1 *Assume $\Gamma = \Gamma_L$ and suppose $U \supseteq K$ is a K-linear subspace of L such that $L = \big\{u_1/u_2 : u_1, u_2 \in U, u_2 \neq 0\big\}$ and*

$$\big(u \in U, v(u) \geq 0, b \in K^\times, v(b) > 0\big) \quad \Longrightarrow \quad v(u') > v(b'/b). \tag{3.1}$$

Then L is a pre-differential-valued field.

Proof. Let $f, g \in L$, $v(f) \geq 0$, $g \neq 0$, $v(g) > 0$. We have to show that $v(f') > v(g'/g)$. Write $f = a/u$ with $a, u \in U$, $u \neq 0$. Dividing a and u by an element of K of valuation $v(u)$ we reduce to the case that $v(f) = v(a) \geq 0$ and $v(u) = 0$. From $f' = \frac{1}{u}a' - \frac{f}{u}u'$ and (3.1) we obtain $v(f') > v(b'/b)$ for all $b \in K^\times$ with $v(b) > 0$. Now write $g = bh$ with $b \in K$ and $v(g) = v(b) > 0$, $v(h) = 0$. Then $g' = b'h + bh'$. But $v(b'h) = v(b') < v(bh')$ by the above result with h instead of f. Thus $v(g') = v(b')$. Hence $v(f') > v(b'/b) = v(g'/g)$ as desired. □

Remarks.

(1) The lemma above remains valid if we add the assumption $\mathrm{res}(K) = \mathrm{res}(L)$ and replace in (3.1) the condition $v(u) \geq 0$ by $v(u) > 0$. To see this, note that with the added assumption each $u \in U$ with $v(u) \geq 0$ is of the form $u = a + u_1$ with $a \in \mathcal{O}$ and $u_1 \in U$ with $v(u_1) > 0$.

(2) Suppose K is a pre-H-field and L is equipped with an ordering extending that of K in which $\mathcal{O}_L$ is convex. Then the lemma above holds with the conclusion that L is a pre-H-field. To see this, let $f \in L$ with $f > \mathcal{O}_L$, and take positive $g \in K$ with $v(g) = v(f) < 0$. Then $\frac{f}{g} - \frac{f'}{g'} \in \mathfrak{m}_L$, since L is a pre-differential-valued field, and $f/g > \mathfrak{m}_L$, so $f'/g' > \mathfrak{m}_L$. Since $g' > 0$ this implies $f' > 0$.

The following is an easy consequence of the above, and deserves special mention, since we shall use it several times.

Corollary 3.2 *Let K be a pre-H-field, and L a pre-differential-valued field extension of K such that as valued field L is an immediate extension of K. Then the field L has a unique ordering extending that of K in which $\mathcal{O}_L$ is convex. With this ordering L is a pre-H-field and $\mathcal{O}_L$ is the convex hull of $\mathcal{O}$ in L.*

Proof. The existence and uniqueness of the ordering of L with the stated properties is well-known (see for example [10], III, Sect. 2, Satz 11). For the reader's convenience we briefly indicate the argument. Write each $f \in L^\times$ as $f = g(1 + \varepsilon)$ with $g \in K^\times$ and $\varepsilon \in L$ such that $v(f) = v(g)$ and $v(\varepsilon) > 0$. Then in any ordering of L with the stated properties we must have $f > 0$ in L if and only if $g > 0$ in K. This equivalence also shows how to define the desired ordering of L. The last statement of the lemma now follows from remark (2) after Lemma 3.1. □

Lemma 3.3 *Assume* $\mathrm{res}(K) = \mathrm{res}(L)$, *and suppose $T \supseteq K^\times$ is a multiplicative subgroup of $L^\times$ such that $L = K(T)$ (as fields), each element of $K[T] \setminus \{0\}$ is of the form $t_1 + \cdots + t_k$ with $k \geq 1$, $t_1, \ldots, t_k \in T$ and $v(t_1) < v(t_i)$ for $2 \leq i \leq k$, and such that*

$$\big(a, b \in T, v(a) \geq 0, v(b) > 0\big) \quad \Longrightarrow \quad v(a') > v(b'/b). \tag{3.2}$$

Then L is a pre-differential-valued field.

Proof. Let $f, g \in L^\times$, $v(f) \geq 0$, $v(g) > 0$. We have to show that $v(f') > v(g'/g)$. By the assumptions on T we can write

$$g = b \cdot \frac{\sum_{i=1}^m a_i}{\sum_{j=1}^n b_j}$$

where $m, n \geq 1$, $b \in T$, $a_i, b_j \in T$ for all i, j, $a_1 = b_1 = 1$, $v(a_i) > 0$ for $2 \leq i \leq m$, and $v(b_j) > 0$ for $2 \leq j \leq n$. So $v(g) = v(b) > 0$, and we claim that $v(g') = v(b')$. For this, note that

$$\frac{g'}{g} = \frac{b'}{b}\left(1 + \frac{\sum_{i>1} a_i' b/b'}{1 + \sum_{i>1} a_i} - \frac{\sum_{j>1} b_j' b/b'}{1 + \sum_{j>1} b_j}\right).$$

By (3.2) we have $v(a_i' b/b'), v(b_j' b/b') > 0$ for $i, j > 1$. Thus $v(g'/g) = v(b'/b)$ by the equation above, that is, $v(g') = v(b')$ as claimed.

First assume $v(f) = 0$, and write $f = a + h$ with $a \in K$, $v(a) = 0$ and $h \in L$, $v(h) > 0$. (This is possible since $\operatorname{res}(K) = \operatorname{res}(L)$.) Then $f' = a' + h'$. By the claim above, applied to h instead of g, we find $t \in T \cup \{0\}$ with $v(h) = v(t)$ and $v(h') = v(t')$. Hence $v(f') \geq \min\{v(a'), v(t')\} > v(b'/b) = v(g'/g)$, as desired. If $v(f) > 0$, the argument goes through by setting $a := 0$, $h := f$. $\square$

Remarks.

(1) The proof shows that the case $v(a) = 0$ of (3.2) is only needed when $a \in K$. Thus if K is a differential-valued field, we can replace (3.2) by

$$\big(a, b \in T, v(a) > 0, v(b) > 0\big) \quad \Longrightarrow \quad v(a') > v(b'/b).$$

(2) Suppose that K is a pre-H-field, and that L is equipped with an ordering extending that of K in which $\mathcal{O}_L$ is convex. Add to the assumptions of the lemma that for all $t \in T$ with $v(t) < 0$ we have $\operatorname{sgn}(t) = \operatorname{sgn}(t')$. With this extra assumption the lemma above yields the conclusion that L is a pre-H-field. To see this, let $f \in L$ with $f > \mathcal{O}_L$, and take $t \in T$ with $v(t) = v(f) < 0$. Then $\frac{f}{t} - \frac{f'}{t'} \in \mathfrak{m}_L$. If $t > 0$, then $f/t > \mathfrak{m}_L$, so $f'/t' > \mathfrak{m}_L$, and thus $f' > 0$, using $t' > 0$. If $t < 0$ the argument is similar, using $t' < 0$.

The following easy fact will be used repeatedly in the next sections, in conjunction with the previous lemma.

Lemma 3.4 *Let $K \subseteq L$ be an extension of pre-differential-valued fields such that $\operatorname{res}(K) = \operatorname{res}(L)$. If the valuation of K is a differential valuation, so is the valuation of L, and $C_L = C$.* $\square$

Lemma 3.5 *Suppose L (as valued field extension of K) is a henselization of K. Then L is a pre-differential-valued field.*

Proof. If the valuation is trivial ($K = \mathcal{O}$), then $L = K$ and there is nothing to prove. So we shall assume the valuation is not trivial. We now replace the assumption that L is a henselization of K by the assumption that $L = K(x)$ for some $x \in L^\times$ that is algebraic over K, with $v(x) > 0$, and whose minimum polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ over K has coefficients in $\mathcal{O}$, with $a_1 \notin \mathfrak{m}$ and $a_0 \in \mathfrak{m}$. We are allowed to make this reduction because the henselization of K is a directed union of subextensions of K that have this form.

Claim 1. $v(x) = v(a_0)$ and $v(x') = v(a_0')$.

That $v(x) = v(a_0)$ follows from $f(x) = 0$ and $v(a_i x^i) > v(a_1 x) = v(x)$ for $2 \le i \le n$ (with $a_n := 1$). Next, $f'(x) = a_1 + \sum_{i=2}^n i a_i x^{i-1}$, so $v(a_1) = 0$ gives $v\big(f'(x)\big) = 0$. Now $0 = f(x)' = f'(x)x' + a_0' + \sum_{i=1}^{n-1} a_i' x^i$, hence

$$x' = -\frac{a_0' + \sum_{i=1}^{n-1} a_i' x^i}{f'(x)}. \tag{3.3}$$

For $1 \le i < n$ we have $v(a_i' x^i) \ge v(a_i') + v(a_0) > v(a_0')$, where the last inequality uses that K is a pre-differential-valued field. Thus (3.3) implies $v(x') = v(a_0')$, which finishes the proof of Claim 1.

Claim 2. For each $\alpha \in \mathcal{O}[x]$ there exists $a \in \mathcal{O}$ such that $v(\alpha') \ge v(a')$.

This property certainly holds for $\alpha \in \mathcal{O}$, as well as for $\alpha = x$ by Claim 1, and is clearly inherited under taking sums and products. Thus Claim 2 follows.

Let $\mathfrak{n} := (\mathfrak{m}, x)\mathcal{O}[x]$, a maximal ideal of $\mathcal{O}[x]$. We note that $\mathcal{O}_L = S^{-1}\mathcal{O}[x]$ with $S = 1 + \mathfrak{n}$.

Claim 3. For each $\alpha \in \mathcal{O}_L$ there exists $a \in \mathcal{O}$ such that $v(\alpha') \ge v(a')$.

This follows from Claim 2 by writing $\alpha \in \mathcal{O}_L$ as $\alpha = \beta/(1+\varepsilon)$ with $\beta \in \mathcal{O}[x]$ and $\varepsilon \in \mathfrak{n}$, and differentiating this quotient, using $v(\varepsilon) > 0$.

We finish the proof by noting that by Claim 3 the hypothesis of Lemma 3.1 holds for $U := L$. □

Remark. Suppose in addition to the hypothesis of the last lemma that K is also equipped with an ordering making it a pre-H-field. Then by Corollary 3.2 there is a unique ordering of L extending that of K in which $\mathcal{O}_L$ is convex; with this ordering L is a pre-H-field.

Lemma 3.6 *Suppose that the field extension $L|K$ is of finite degree, with $[L : K] = \big[\operatorname{res}(L) : \operatorname{res}(K)\big] = n > 1$. Then L is a pre-differential-valued field.*

Proof. Take $x \in \mathcal{O}_L$ such that $\operatorname{res}(L) = \operatorname{res}(K)[\overline{x}]$ where $\overline{x}$ is the residue class of x in $\operatorname{res}(L)$. Let $f(X) = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0 \in \mathcal{O}[X]$ be the minimum polynomial of x over K, so its reduction $\overline{f}(X)$ is the minimum polynomial of $\overline{x}$ over $\operatorname{res}(K)$. Hence $\overline{f'}(\overline{x}) \neq 0$, that is, $v\big(f'(x)\big) = 0$. In combination with

$$x' = -\frac{\sum_{i=0}^{n-1} c_i' x^i}{f'(x)}$$

this implies that $v(x') \geq v(c_i')$ for some $i \in \{0, \ldots, n-1\}$.

We now write an arbitrary element $u \in \mathcal{O}_L$ as $u = u_0 + u_1x + \cdots + u_{n-1}x^{n-1}$ with all $u_j \in \mathcal{O}$. Then

$$u' = u_0' + u_1'x + u_1x' + \cdots + u_{n-1}'x^{n-1} + (n-1)u_{n-1}x^{n-2}x'.$$

Considering the terms in this sum and using what we just proved about x', we see that $v(u') \geq v(a')$ for some $a \in \mathcal{O}$. We also note that necessarily $\Gamma = \Gamma_L$. Thus the hypotheses of Lemma 3.1 hold for $U = L$. $\square$

Remark. Suppose that in the situation of the lemma, K is a pre-H-field and L is equipped with an ordering extending that of K in which $\mathcal{O}_L$ is a convex subring of L. Then L is a pre-H-field as well. (This follows from remark (2) after Lemma 3.1.)

Next we extend a lemma in [15].

Lemma 3.7 *Let p be a prime number, and let $L = K\big(u^{1/p}\big)$ where $u \in K^\times$ with $v(u) \notin p \cdot v(K^\times)$. Then L is a pre-differential-valued field.*

Proof. Let $u^{i/p} := \big(u^{1/p}\big)^i$ for $i \in \mathbb{Z}$, and put $T := \bigcup_{i=0}^{p-1} K^\times u^{i/p}$. Thus T is a multiplicative subgroup of $L^\times$. By general valuation theory, $\operatorname{res}(K) = \operatorname{res}(L)$, and each element of L has the form $t_1 + \cdots + t_k$ with $t_1, \ldots, t_k \in T$, $v(t_1) < \cdots < v(t_k)$. The remaining hypothesis (3.2) of Lemma 3.3 now follows as in the proof of a similar lemma on p. 316 of [15]. $\square$

Remark. Suppose that in the situation of the lemma K is a pre-H-field and L is given an ordering extending that of K such that $\mathcal{O}_L$ is convex in L. Then L is a pre-H-field. This follows from part (2) of the remark after Lemma 3.3: Let $t = au^{i/p} \in T$ ($a \in K^\times, 0 \leq i < p$) with $v(t) < 0$. Then $v(t^p) < 0$ and $t^p \in K^\times$, so $p(t'/t) = (t^p)'/t^p > 0$ by Lemma 1.4, hence $\operatorname{sgn} t = \operatorname{sgn} t'$.

Corollary 3.8 *Suppose K is a pre-differential-valued field and the valued differential field extension $L|K$ is algebraic. Then L is a pre-differential-valued field.*

Proof. The property of being a pre-differential-valued field is inherited by valued differential subfields, so we may as well assume by Lemma 3.5 that K is henselian, and that L is an algebraic closure of K. We then reach L in two steps. In the first step we pass from K to its maximal unramified extension K^{unr} inside L. The valuation on K^{unr} remains a pre-differential valuation by Lemma 3.6. In the second step we obtain L as a purely ramified extension of K^{unr}, and now Lemma 3.7 can be applied. □

Remark 3.9 If K is a differential-valued field, then its algebraic closure $\widetilde{K}$, with any valuation extending the valuation of K, is also a differential-valued field. (Since $C_{\widetilde{K}}$ = algebraic closure of C, and $\operatorname{res}(\widetilde{K})$ = algebraic closure of $\operatorname{res}(K)$.)

Theorem 6 of [15] asserts more generally that if L is any valued differential field extension of a differential-valued field K and $L|K$ is algebraic, then L is also differential-valued. However, this Theorem 6 is incorrect. (The problem is with the first sentence of its proof. The argument *following* that sentence is correct, but only treats the case that L is an algebraic closure of K. The referee informed us that Rosenlicht later became aware of this error.) Below is a counterexample.

Counterexample. Let K be the Hardy field $\mathbb{Q}(x)$, with $x > \mathbb{Q}$, $x' = 1$. Then $C = \mathbb{Q}$ maps onto the residue field, so K is a differential-valued field. Let $L = K(y)$ be the Hardy field with $y = (2 + x^{-1})^{1/2}$. Since $L = \mathbb{Q}(y)$, it follows that $\mathbb{Q}$ is algebraically closed in L, so $\mathbb{Q}$ is also the constant field of L. But the residue class of y is $\sqrt{2}$. Thus L is algebraic over K, but is not a differential-valued field, only a pre-differential-valued field.

The L in this example is a Hardy field, and thus also a pre-H-field, while the Hardy field K is even an H-field. Hence the example also shows that an ordered valued differential extension field of an H-field may very well be algebraic over that H-field, and still fail to be an H-field itself. But we do have the following positive result.

Corollary 3.10 *Suppose K is a pre-H-field and L is an ordered valued differential field extension of K such that $L|K$ is algebraic and $\mathcal{O}_L$ is the convex hull in L of $\mathcal{O}$. Then L is a pre-H-field.*

Proof. Since ordered valued differential subfields of pre-H-fields are again pre-H-fields, we may assume, first, that L is real closed, and second, by Lemma 3.5 and Corollary 3.2, that K is henselian. Now proceed as in the proof of Corollary 3.8, also using the remarks after Lemma 3.6 and Lemma 3.7. □

In particular, the real closure of a pre-H-field K is again a pre-H-field, where the valuation on the real closure has as valuation ring the convex hull of $\mathcal{O}$, and the derivation extends the derivation of K. If moreover K is an H-field, then its real closure is again an H-field. (Since the constant field of the real closure of K is the real closure of the constant field of K, and the same holds for the residue fields.)

4 Embedding pre-H-fields into H-fields

Here we show that each pre-differential-valued field extends to a differential-valued field, and that each pre-H-field extends to an H-field. We actually construct such an extension that is minimal in a certain sense, and is determined up to isomorphism by this minimality property, see Corollary 4.6 below.

Let K be a pre-differential-valued field, with associated asymptotic couple (Γ, ψ), and suppose its valuation is *not* a differential valuation. Hence for a certain $r \in \mathcal{O}$ we have $r' \neq \varepsilon'$ for all $\varepsilon \in \mathfrak{m}$. Put $s := r'$. Thus for K to extend to a differential-valued field, there must exist an element y in some valued differential field extension of K such that $y' = s$ and $v(y) > 0$, and such that $K(y)$ with its induced valuation and derivation is a pre-differential-valued field.

Conversely, in order to construct a differential-valued field extension of K, we consider the field extension $L = K(y)$ of K, with y transcendental over K, and we extend the derivation and valuation of K to a derivation and valuation on $K(y)$ such that $y' = s$ and $v(y) > 0$. The key fact to be established is that (under mild assumptions on K) this can be done in a unique way such that $K(y)$ remains a pre-differential-valued field. The analysis splits into two cases:

Special case. $v(s) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$.

Other case. $v(s) \geq \gamma + \psi(\gamma)$ for some $\gamma \in \Gamma^{>0}$.

The next lemma covers the “special case” (and more). The “other case” is harder: by Lemma 4.2 we reduce to the situation that $(\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$ is closed upward, and then we construct in Proposition 4.3 a pseudo-Cauchy sequence and let y be a pseudo-limit.

Lemma 4.1 *Let K be a pre-differential-valued field and $\Psi < v(s) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, $s \in K$. Let $L = K(y)$ be a field extension of K with y transcendental over K. Equip L with the unique derivation extending that of K such that $y' = s$. Then there exists exactly one valuation on L extending the one on K such that L is a pre-differential-valued field with $v(y) > 0$.*

Proof. Let L be given a valuation making it into a pre-differential-valued field with $v(y) > 0$, and with asymptotic couple (Γ_L, ψ_L). Then $v(y) + \psi_L\big(v(y)\big) = v(s)$, so $\Gamma^{<0} < nv(y) < 0$ for all $n > 0$, cf. proof of

Lemma 2.10. This condition determines a valuation on L extending the valuation on K.

Conversely, equip L with the unique valuation extending that of K such that $0 < nv(y) < \Gamma^{>0}$ for all $n > 0$. We show that then L is a pre-differential-valued field. Note that $\Gamma_L := v(L^\times) = \Gamma \oplus \mathbb{Z}v(y)$. We extend ψ to a map $\psi_L \colon (\Gamma_L)^* \to \Gamma_L$ such that (Γ_L, ψ_L) is an asymptotic couple with $v(y) + \psi_L\bigl(v(y)\bigr) = v(s)$, as in Lemma 2.10, with $\beta := v(s)$.

We shall verify the conditions of Lemma 3.3 for the multiplicative subgroup $T := \bigcup_{j\in\mathbb{Z}} K^\times y^j$ of $L^\times$. From general valuation theory we know that $\operatorname{res}(K) = \operatorname{res}(L)$, and that every element of $K[T] \setminus \{0\}$ is of the form $t_1 + \cdots + t_k$ with $k \geq 1$, $t_1, \dots, t_k \in T$, and $v(t_1) < \cdots < v(t_k)$.

Let $f, g \in T$ with $v(f) \geq 0$ and $v(g) > 0$. We have to show that then $v(f') > v(g'/g)$. First assume $v(f) > 0$. Write $f = ay^p$ and $g = by^q$ with $a, b \in K^\times$ and $p, q \in \mathbb{Z}$. Then

$$f' = a'y^p + pay^{p-1}y' = ay^p\bigl((a'/a) + p(y'/y)\bigr)$$

and $v(f) = v(a) + p\alpha$, so we get $v\bigl((a'/a) + p(y'/y)\bigr) = \psi_L\bigl(v(f)\bigr)$ and hence $v(f') = v(f) + \psi_L\bigl(v(f)\bigr)$. Similarly, $v(g'/g) = \psi_L\bigl(v(g)\bigr)$. So

$$v(f') = v(f) + \psi_L\bigl(v(f)\bigr) > \psi_L\bigl(v(g)\bigr) = v(g'/g),$$

by axiom (3) for asymptotic couples, applied to (Γ_L, ψ_L). If $v(f) = 0$, then $f \in K^\times$, so $v(f') \geq v(s) > \Psi_L$. Hence in particular, $v(f') > v(g'/g)$. $\square$

Remarks.

(1) In this lemma we did not assume the existence of $r \in \mathcal{O}$ such that $s = r'$, although this is the relevant case for constructing a differential-valued field extension of K as explained in the beginning of this section. The extra generality is needed later when we construct Liouville closures.

(2) With the hypotheses of the lemma, suppose also that K is differential-valued. Then there is a unique valuation on L that extends the valuation on K and makes L a pre-differential-valued field with $v(y) < 0$. The proof is similar to that of Lemma 4.1, first deriving that such a valuation satisfies $\Gamma^{<0} < nv(y) < 0$ for all $n > 0$, then using Lemma 2.11 instead of 2.10, and using the stronger assumption on K to handle the case $v(f) = 0$.

(3) With the assumptions on K and s of the lemma, let E be a differential-valued field extension of K with an element $y \in E$ such that $y' = s$ and $v(y) \geq 0$. Subtracting a constant from y if necessary, we can achieve $v(y) > 0$. Then the beginning of the proof of the lemma shows that $0 < nv(y) < \Gamma^{>0}$ for all $n > 0$. Hence y is *transcendental* over K, and thus the pre-differential-valued subfield $L = K(y)$ of E is exactly as described in the lemma.

(4) With the assumptions on K and s of the lemma, suppose that in addition K is differential-valued, and let E be a differential-valued field extension of K, $y \in E$ such that $y' = s$ and $v(y) < 0$. As in the last remark it follows that y is transcendental over K, hence $L = K(y)$ is exactly the pre-differential-valued field as described in Remark (2).

(5) In the setting of the lemma, assume moreover that K is a differential-valued field. Then also L is a differential-valued field with $C_L = C$, for the valuation on L of the lemma, as well as for the valuation on L described in Remark (3). (Use Lemma 3.4.)

(6) In the setting of the lemma, let L have the valuation described there. Let in addition K be equipped with an ordering making it a pre-H-field. We claim:

    There is a unique ordering on L making it a pre-H-field extension of K. If K is an H-field, so is L with that unique ordering, and $C_L = C$.

    The last part of the claim follows from the first part and remark (5). To see that the first part of the claim holds, we distinguish two cases:

    (a) $s < 0$. Then $\mathfrak{m} < y < K^{>\mathfrak{m}}$ for any ordering of L as in the claim. Conversely, this inequality uniquely determines an ordering making L an ordered extension field of K. One easily verifies using remark (2) after Lemma 3.3 with $T := \bigcup_{j\in\mathbb{Z}} K^{\times}y^j$ and Lemma 1.4 that L with this ordering is indeed a pre-H-field.

    (b) $s > 0$. The same argument as in (a) shows that $K^{<\mathfrak{m}} < y < \mathfrak{m}$ (so $y < 0$) uniquely determines the ordering of L satisfying our claim.

(7) Suppose that in the context of the lemma, K is equipped with an ordering making it an H-field. As in (6), one shows that for the valuation on L such that $v(y) < 0$, there exists a unique ordering on L making it a pre-H-field extension of K. (If $s < 0$, then $K^{<\mathcal{O}} < y < \mathcal{O}$, and if $s > 0$, then $\mathcal{O} < y < K^{>\mathcal{O}}$.) In fact, with that valuation and ordering, L is an H-field with $C_L = C$.

Along similar lines, but using Lemmas 2.12 and 2.13 instead of Lemma 2.10, one shows:

Lemma 4.2 *Let K be a pre-differential-valued field, $s \in K$ with $v(s) \notin (\mathrm{id} + \psi)(\Gamma^*)$, and suppose there exists $\gamma \in \Gamma^{>0}$ such that $v(s) \leq \psi(\gamma)$ or $v(s) > \gamma + \psi(\gamma)$. Let $L = K(y)$ with y transcendental over K. Equip L with the unique derivation extending the derivation on K such that $y' = s$. Then there is a unique valuation on L extending the valuation on K such that L is a pre-differential-valued field and $v(y) \neq 0$.* □

Remarks.

(1) With the assumptions on K and s as in the lemma, let E be a differential-valued field extension of K with an element $y \in E$ such that $y' = s$.

Subtracting a constant from y if necessary, we may assume $v(y) \neq 0$. Then y is transcendental over K, and the pre-differential-valued subfield $K(y)$ of E is exactly as described in the lemma.

(2) If in this lemma K is even a differential-valued field, then L is also a differential-valued field with $C = C_L$, by Lemma 3.4.

(3) If in this lemma K is equipped with an ordering making it a pre-H-field, then $v(s) = \max \Psi$ (hence $\Gamma^{<0} < nv(y) < 0$ for all $n > 0$), and there is a unique ordering on $L = K(y)$ making L into a pre-H-field extension of K. This is established as in remark (7) after Lemma 4.1.

Proposition 4.3 *Let K be a pre-differential-valued field, henselian as valued field, with $(\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$ closed upward, and let $s \in K$ be such that $v(s) \in (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$ but $s \neq \varepsilon'$ for all $\varepsilon \in \mathfrak{m}$. Let $L = K(y)$ be a field extension of K with y transcendental over K, and let L be equipped with the unique derivation extending the derivation of K such that $y' = s$.*

Then there is a unique valuation of L that makes L a pre-differential-valued field extension of K with $v(y) \neq 0$. Moreover, this valuation makes L an immediate extension of K with $v(y) > 0$.

Proof. Put $S := \big\{v(s - \varepsilon') : \varepsilon \in \mathfrak{m}\big\}$.

Claim 1. The set S has no largest element.

To see this, note first that $v(s) \in S$. Let $\gamma \in S$ with $\gamma \geq v(s)$, and write $\gamma = v(s - \varepsilon')$ with $\varepsilon \in \mathfrak{m}$. Since $\{v(b') : b \in \mathfrak{m}\}$ is closed upward by assumption, there exists $b \in \mathfrak{m}$ with $v(b') = \gamma$. Thus for some $u \in K$ with $v(u) = 0$ we have $v(s - \varepsilon' - ub') > \gamma$. Now $v(u'b) > v(b') = \gamma$, so $v\big(s - \varepsilon' - (ub)'\big) > \gamma$. This proves Claim 1.

Let $\kappa = \mathrm{cofinality}(S)$, so κ is an infinite cardinal. Let $(\varepsilon_\lambda)_{\lambda<\kappa}$ be a sequence in $\mathfrak{m}$ such that $\big(v(s - \varepsilon'_\lambda)\big)$ is a strictly increasing sequence in S, and cofinal in S. Then $v(s - \varepsilon'_\lambda) = v\big((\varepsilon_\lambda - \varepsilon_\mu)'\big)$ for $\lambda < \mu < \kappa$, and hence $v\big((\varepsilon_\lambda - \varepsilon_\mu)'\big) < v\big((\varepsilon_\mu - \varepsilon_\nu)'\big)$ for $\lambda < \mu < \nu < \kappa$. Hence $v(\varepsilon_\lambda - \varepsilon_\mu) < v(\varepsilon_\mu - \varepsilon_\nu)$ for $\lambda < \mu < \nu < \kappa$. Thus (ε_λ) is a pseudo-Cauchy sequence.

Claim 2. The pseudo-Cauchy sequence (ε_λ) has no pseudo-limit in K.

To see this, suppose $\varepsilon \in K$ is a pseudo-limit of this sequence. We have $v(\varepsilon - \varepsilon_\lambda) = v(\varepsilon_\mu - \varepsilon_\lambda) > 0$ for $\lambda < \mu < \kappa$, in particular $\varepsilon \in \mathfrak{m}$. Also, $v(\varepsilon' - \varepsilon'_\lambda) = v(\varepsilon'_\mu - \varepsilon'_\lambda)$ for $\lambda < \mu < \kappa$. Hence $v(s - \varepsilon') \geq v(s - \varepsilon'_\lambda)$ for all $\lambda < \kappa$, contradicting Claim 1. This proves Claim 2.

Assume for a moment that L is given a valuation making it a pre-differential-valued field extension of K with $v(y) \neq 0$. Then $v(y) > 0$, since $v(y) < 0$ would give $v(y') < v(\varepsilon')$ for all $\varepsilon \in \mathfrak{m}$. Also, since $y' = s$, the sequence $\big(v(y' - \varepsilon'_\lambda)\big)$ is strictly increasing, and thus $\big(v(y - \varepsilon_\lambda)\big)$ is strictly increasing. So y is a pseudo-limit of (ε_λ).

Since K is henselian, it follows from Claim 2 that $K(y)$ has a unique valuation that extends the valuation of K in which y is a pseudo-limit of (ε_λ). (See [10], III, Sect. 3, Sätze 6 and 7, Lemma 11.) In the following $K(y)$ is equipped with this valuation. (We just saw that no other choice is possible.) Then $K(y)|K$ is an immediate extension, $v(y) > 0$, and $v(y-\varepsilon_\lambda) = v(\varepsilon_\mu - \varepsilon_\lambda)$ for $\lambda < \mu < \kappa$. Note also that for any polynomial $f(Y) \in K[Y]$ we have $v\big(f(y)\big) = v\big(f(\varepsilon_\lambda)\big)$ eventually. (Here and below "eventually" means "for all sufficiently large $\lambda < \kappa$".) It remains to establish the following.

Claim 3. L is a pre-differential-valued field.

It will suffice to verify the hypotheses of Lemma 3.1 for $U = K[y]$. In view of the remark following the proof of that lemma we consider a polynomial $f(Y) \in K[Y]$ such that $v\big(f(y)\big) > 0$ and an element $b \in K^\times$ with $v(b) > 0$, and have to show that $v\big(f(y)'\big) > v(b'/b)$. We may assume $f(Y) \notin K$. Write $f(Y) = c_nY^n + \cdots + c_0$ ($c_i \in K$), and put $f^*(Y) := c_n'Y^n + \cdots + c_0'$, so that

$$f(y)' = f^*(y) + f'(y)y' = f^*(y) + f'(y)s,$$

so we are reduced to showing that $v\big(f^*(y) + f'(y)s\big) > v(b'/b)$.

Now $v\big(f^*(y) + f'(y)s\big) = v\big(f^*(\varepsilon_\lambda) + f'(\varepsilon_\lambda)s\big)$ eventually, so it is enough to show that $v\big(f^*(\varepsilon_\lambda) + f'(\varepsilon_\lambda)s\big) > v(b'/b)$ eventually. We have

$$\begin{aligned} f(\varepsilon_\lambda)' &= f^*(\varepsilon_\lambda) + f'(\varepsilon_\lambda)\varepsilon_\lambda' \\ &= f^*(\varepsilon_\lambda) + f'(\varepsilon_\lambda)s + f'(\varepsilon_\lambda)(\varepsilon_\lambda' - s). \end{aligned} \tag{4.1}$$

Now $v\big(f(\varepsilon_\lambda)\big) = v\big(f(y)\big) > 0$ eventually, so $v\big(f(\varepsilon_\lambda)'\big)$ is eventually constant and $v\big(f(\varepsilon_\lambda)'\big) > v(b'/b)$ eventually. Also, $v\big(f^*(\varepsilon_\lambda) + f'(\varepsilon_\lambda)s\big) = v\big(f^*(y) + f'(y)s\big)$ eventually, while

$$v\big(f'(\varepsilon_\lambda)(\varepsilon_\lambda' - s)\big) = v\big(f'(y)\big) + v(\varepsilon_\lambda' - s)$$

is eventually strictly increasing. It cannot happen that

$$v\big(f'(\varepsilon_\lambda)(\varepsilon_\lambda' - s)\big) < v\big(f^*(y) + f'(y)s\big)$$

eventually, since then, by (4.1), $v\big(f(\varepsilon_\lambda)'\big) = v\big(f'(\varepsilon_\lambda)(\varepsilon_\lambda' - s)\big)$ would both be eventually constant and eventually strictly increasing. Hence

$$v\big(f'(\varepsilon_\lambda)(\varepsilon_\lambda' - s)\big) > v\big(f^*(y) + f'(y)s\big)$$

eventually, and thus $v\big(f^*(\varepsilon_\lambda) + f'(\varepsilon_\lambda)s\big) = v\big(f(\varepsilon_\lambda)'\big) > v(b'/b)$ eventually, as desired. □

Remarks.

(1) Assume K and s are as in the proposition. Let E be any differential-valued field extension of K, and $y \in E$ with $y' = s$. After subtracting a constant from y if necessary, we may assume $v(y) > 0$. Then y is *transcendental* over K. (This is because the sequence (ε_λ) in the proof of the proposition has y as pseudo-limit, so we can invoke [10], III, §3, Lemma 11.) Hence the pre-differential-valued subfield $K(y)$ of L is exactly as described in the proposition.

(2) If in this proposition K is even a differential-valued field, then L is too, with $C_L = C$. (Use Lemma 3.4.)

(3) If in this proposition K is equipped with an ordering making it a pre-H-field (H-field), then there is a unique ordering on L extending that on K for which L is a pre-H-field (H-field with $C_L = C$, respectively). This follows from Corollary 3.2 and Remark (3).

Theorem 4.4 *Let K be a pre-differential-valued field. Then K has a differential-valued field extension $\widehat{K}$ such that any embedding of K into any differential-valued field L can be extended uniquely to an embedding from $\widehat{K}$ into L.*

Proof. For the purpose of this proof only, a pre-differential-valued field L is said to be *nice* if there is no $r \in \mathcal{O}_L \setminus C_L$ with $v(r') < (\mathrm{id} + \psi_L)\big(\Gamma_L^{>0}\big)$, and $(\mathrm{id} + \psi_L)\big(\Gamma_L^{>0}\big)$ is closed upward in Γ_L. We first introduce a nice pre-differential-valued field extension K_0 of K as follows:

(1) If there exists $r \in \mathcal{O} \setminus C$ such that $v(r') < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, then we put $s := r'$ for such an r, and take $K_0 := K(y)$ with $v(y) > 0$, as in Lemma 4.1.

(2) If there exists $s \in K^\times$ with $v(s) \notin (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, but $v(s) > \gamma + \psi(\gamma)$ for some $\gamma \in \Gamma^{>0}$, then for such an s we put $K_0 := K(y)$ as in Lemma 4.2.

(3) Otherwise, let $K_0 := K$.

Note that $\mathrm{res}(K_0) = \mathrm{res}(K)$, and that K_0 is nice by the remark after Lemma 2.13. Note also that being nice is preserved under immediate extensions of pre-differential-valued fields. Thus, starting with K_0 and iterating and alternating applications of Lemma 3.5 and Proposition 4.3 we extend K_0 to a nice henselian differential-valued field K' such that $\mathrm{res}(K') = \mathrm{res}(K)$ and any embedding of K into a nice henselian differential-valued field L can be extended to an embedding of K' into L.

Let D be the constant field of K'; so D maps isomorphically onto $\mathrm{res}(K') = \mathrm{res}(K)$ under the residue map $\mathcal{O}_{K'} \to \mathrm{res}(K')$. Put $\widehat{K} := K(D)$, so $\widehat{K}$ is a differential-valued subfield of K'. We now show that $\widehat{K}$ has the desired universal property.

Let $i\colon K \to L$ be any embedding of K into a differential-valued field L. Extend i to an embedding $j : K' \to L'$ of valued differential fields. Then $j(D)$ is the unique subfield of $C_{L'}$ that maps isomorphically onto $\operatorname{res}\bigl(i(K)\bigr)$ under the residue map $\mathcal{O}_L \to \operatorname{res}(L)$. Now $C_{L'} = C_L$ gives $j(\widehat{K}) \subseteq L$. Thus $j|\widehat{K}\colon \widehat{K} \to L$ is the required embedding, which this argument also shows to be uniquely determined by i. $\square$

Remark. The universal property in this theorem determines $\widehat{K}$ up to *unique* isomorphism as a valued differential field extension of K. Note also that the proof shows that $\operatorname{res}(\widehat{K}) = \operatorname{res}(K)$. However, it can happen that $\Gamma_{\widehat{K}} \neq \Gamma$. In any case, we can determine $\bigl(\Gamma_{\widehat{K}}, \psi_{\widehat{K}}\bigr)$ explicitly in terms of K and (Γ, ψ):

Corollary 4.5 *Let K be a pre-differential-valued field.*

(1) *Let $r \in \mathcal{O} \setminus C$ and $v(r') \notin (\mathrm{id} + \psi)\bigl(\Gamma^{>0}\bigr)$. Take the unique $y \in \widehat{K}$ such that $y' = r'$ and $v(y) > 0$. Then $\bigl(\Gamma_{\widehat{K}}, \psi_{\widehat{K}}\bigr) = (\Gamma^{\alpha}, \psi^{\alpha})$ with $\alpha := v(y)$ and $\beta := v(r')$, where $(\Gamma^{\alpha}, \psi^{\alpha})$ is as in Lemma* 2.10 *if $v(r') < (\mathrm{id} + \psi)\bigl(\Gamma^{>0}\bigr)$, and as in Lemma* 2.13 *otherwise.*
(2) *If no r as in the previous case exists, then $\Gamma_{\widehat{K}} = \Gamma$.*

Proof. Suppose r is as in case (1). Then we put $K_0 := K(y)$ for the y defined in that case. The proof of Theorem 4.4 shows that the K' introduced there is an immediate extension of K_0. Hence $\widehat{K}$ is an immediate extension of K_0 as well, and the desired result follows from Lemmas 2.10 and 2.13.

Suppose we are in case (2). This splits into two subcases. The first subcase is that we are in case (3) of the definition of K_0 in the proof of Theorem 4.4. We then use that $\widehat{K}$ is an immediate extension of $K_0 = K$. In the second subcase we have an s as in case (2) of the definition of K_0 in the proof of Theorem 4.4 such that moreover s has no antiderivative in K. Using the notations from that proof, and with $y \in K'$ such that $y' = s$ and $v(y) > 0$ we then have $C_{K(y)} = C$. The desired result $\Gamma_{\widehat{K}} = \Gamma$ easily follows from the claim that $v(y^n) \notin \Gamma_{\widehat{K}}$ for all $n > 0$. Suppose this claim is false. Choose $n > 0$ minimal with $v(y^n) \in \Gamma_{\widehat{K}}$, say $y^n = ua$ with $a \in \widehat{K}$, $u \in K'$, $v(u) = 0$. Then

$$ny'y^{n-1} = u'a + ua',$$

hence $v(y^{n-1}) = -v(s) + v(u'a + ua') = v(a'/s) \in \Gamma_{\widehat{K}}$. Thus $n = 1$ by minimality of n, that is, $v(y) \in \Gamma_{\widehat{K}}$. Then $v(s) = v(y') \in (\mathrm{id} + \psi_{\widehat{K}})\bigl(\Gamma_{\widehat{K}}^{>0}\bigr)$, so there exists $\varepsilon \in \mathfrak{m}_{\widehat{K}}$ with $\varepsilon' = s = y'$; hence $y = \varepsilon \in \widehat{K} = K(D)$. But from general differential algebra (see e.g. [13], p. 292) we know that $K(y)$ and D are linearly disjoint over $C_{K(y)} = C$, a contradiction. $\square$

Corollary 4.6 *Let K be a pre-H-field. Then K has an H-field extension $\widehat{K}$ such that any embedding of K into any H-field L can be extended uniquely to an embedding from $\widehat{K}$ into L.*

Proof. Let $\widehat{K}$ be as in Theorem 4.4. Then $\widehat{K}$ carries a unique ordering extending the ordering of K in which the valuation ring is convex, using the proof of the corollary above, Corollary 3.2, and remarks after 4.1, 4.2, and 4.3. This ordering makes $\widehat{K}$ an H-field extension of K with the desired universal property. $\square$

5 Simple transcendental Liouville extensions

In building the Liouville closures of an H-field K we have to consider three more types of extensions of the form $K(y)$ with y transcendental over K, and either $y' \in K$, or $y'/y \in K$. This is done in the three lemmas below. The first two also make sense for differential-valued fields.

Lemma 5.1 *Let K be a henselian differential-valued field. Let $s \in K$ be such that $S := \{v(s-a') : a \in K\} < (\mathrm{id}+\psi)(\Gamma^{>0})$, and S has no largest element. Let $L = K(y)$ be a field extension of K with y transcendental over K, and let L be equipped with the unique derivation extending the derivation of K such that $y' = s$. Then there is a unique valuation of L that makes L a pre-differential-valued field extension of K. With this valuation L is a differential-valued field, and an immediate extension of K with $v(y) < 0$.*

Proof. Let $\kappa = \mathrm{cofinality}(S)$, so κ is an infinite cardinal. Let $(a_\lambda)_{\lambda<\kappa}$ be a sequence in K such that the sequence $\big(v(s-a'_\lambda)\big)$ is strictly increasing and cofinal in S, with $v(s-a'_\lambda) > v(s)$ for all λ. Then $v(s-a'_\lambda) = v\big((a_\lambda - a_\mu)'\big)$ for $\lambda < \mu < \kappa$, and hence $v\big((a_\lambda - a_\mu)'\big) < v\big((a_\mu - a_\nu)'\big)$ for $\lambda < \mu < \nu < \kappa$. Note that $v(a_\lambda - a_\mu) < 0$ for $\lambda < \mu < \kappa$, since otherwise $v(s-a'_\lambda) = v\big((a_\lambda - a_\mu)'\big) > \Psi$, so $v(s-a'_\lambda) \in (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$, contradicting the hypothesis. Hence $v(a_\lambda - a_\mu) < v(a_\mu - a_\nu)$ for $\lambda < \mu < \nu < \kappa$, by Proposition 2.3, (4). Thus (a_λ) is a pseudo-Cauchy sequence. From $v(s-a'_\lambda) > v(s)$, we obtain $v(s) = v(a'_\lambda) < (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$, so $v(a_\lambda) < 0$ for all λ.

Claim. The pseudo-Cauchy sequence (a_λ) has no pseudo-limit in K.

To see this, suppose $a \in K$ is a pseudo-limit of this sequence. Then we have $v(a - a_\lambda) = v(a_\mu - a_\lambda) < 0$, so $v(a' - a'_\lambda) = v(a'_\mu - a'_\lambda)$ for $\lambda < \mu < \kappa$. Hence $v(s-a') \geq v(s-a'_\lambda)$ for all $\lambda < \kappa$, contradicting the assumption that S has no largest element. This proves the claim.

Assume for a moment that L is given a valuation making it a pre-differential-valued field extension of K. Then $v(y) < 0$, since $v(y) \geq 0$ would give $v(s) = v(y') > \Psi$. Also, since $y' = s$, the sequence $\big(v(y' - a'_\lambda)\big)$ is strictly increasing, and thus $\big(v(y - a_\lambda)\big)$ is strictly increasing. So y is a pseudo-limit of (a_λ).

Since K is henselian, it follows from the claim above that $K(y)$ has a unique valuation that extends the valuation of K in which y is a pseudo-limit of (a_λ). (See proof of Proposition 4.3.) In the following $K(y)$ is equipped with this valuation. (We just saw that no other choice is possible.) Then $K(y)|K$ is an immediate extension, $v(y) < 0$, and $v(y - a_\lambda) = v(a_\mu - a_\lambda)$ for $\lambda < \mu < \kappa$. Note also that for any polynomial $f(Y) \in K[Y]$ we have $v\big(f(y)\big) = v\big(f(a_\lambda)\big)$ eventually. In view of Lemma 3.5 it remains to show that L is a pre-differential-valued field. This last property is obtained as in the proof of Proposition 4.3 from an analogue of Claim 3 there, with the sequence (a_λ) instead of (ε_λ). $\square$

Remarks.

(1) If K, s and L are as in the lemma and K carries in addition an ordering making it an H-field, then there is a unique ordering on L making it a pre-H-field extension of K, and with this ordering L is an H-field. (By Corollary 3.2.)

(2) Let K be a henselian H-field, $s \in K$, and $S := \big\{v(s-a') : a \in K\big\} \subseteq \Gamma_\infty$. Then we are in exactly one of the following three cases:
   (a) S has a maximum β. If $\beta = \infty$, then s has an antiderivative in K. Suppose $\beta \in \Gamma$. Then $\beta \notin (\mathrm{id}+\psi)(\Gamma^*)$. To see this, write $\beta = v(s - a')$ with $a \in K$. Assuming $\beta \in (\mathrm{id}+\psi)(\Gamma^*)$, take $b \in K^\times$ with $v(b) \neq 0$ such that $v(b') = \beta$. Then for some $u \in K$ with $v(u) = 0$ we have $v\big(s-a'-ub'\big) > \beta$, hence $v\big(s-(a+ub)'\big) > \beta$, a contradiction.
   (b) $S < (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$ and S has no maximum: the situation of Lemma 5.1.
   (c) $S \cap (\mathrm{id}+\psi)\big(\Gamma^{>0}\big) \neq \emptyset$ and S has no maximum. Take $a \in K$ with $v(s-a') \in (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$. Then Proposition 4.3 applies with $s-a'$ in place of s.

(3) A result in [9] says that if K is a *maximally valued* differential-valued field with $(\mathrm{id}+\psi)(\Gamma^*) = \Gamma$, then every element of K has an integral in K. We recover this fact here in a very different way: Suppose K is a differential-valued field with $(\mathrm{id}+\psi)(\Gamma^*) = \Gamma$. Then by Lemmas 3.5, 5.1 and Proposition 4.3, there exists an immediate extension $L|K$ of differential-valued fields such that every element a of L has an integral in L. (The condition $(\mathrm{id}+\psi)(\Gamma^*) = \Gamma$ means that every element of K has an asymptotic integral in K, see remarks after Lemma 2.7.)

Lemma 5.2 *Let K be a henselian differential-valued field, with the set $(\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$ closed upward, and $s \in K$ with $v(s) \in (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$ and $s \neq a'/a$ for all $a \in K^\times$. Let $L = K(y)$ be a field extension of K with y transcendental over K, and equip L with the unique derivation extending the derivation of K such that $\frac{y'}{1+y} = s$. Then there is a unique* *valuation of L that makes L a pre-differential-valued field extension of K with $v(y) \neq 0$. With this valuation L is a differential-valued field, and an immediate extension of K with $v(y) > 0$.*

Proof. Put $S := \left\{v\left(s - \frac{\varepsilon'}{1+\varepsilon}\right) : \varepsilon \in \mathfrak{m}\right\}$.

Claim 1. The set S has no largest element.

To see this, note first that $v(s) \in S$. Let $\gamma \in S$ with $\gamma \geq v(s)$, and write $\gamma = v\left(s - \frac{\varepsilon'}{1+\varepsilon}\right)$ with $\varepsilon \in \mathfrak{m}$. Since $\left\{v(b') : b \in \mathfrak{m}\right\}$ is closed upward, there exists $b \in \mathfrak{m}$ with $v(b') = \gamma$. Thus for some $u \in K$ with $v(u) = 0$ we have $s - \frac{\varepsilon'}{1+\varepsilon} = ub'$. Now $v(u'b) > v(b') = \gamma$, so with $\delta \in \mathfrak{m}$ such that $(1+\varepsilon)(1+ub) = 1+\delta$ we have

$$s - \frac{\delta'}{1+\delta} = s - \frac{\varepsilon'}{1+\varepsilon} - \frac{(ub)'}{1+ub} = ub' - \frac{(ub)'}{1+ub} = \frac{u^2bb' - u'b}{1+ub},$$

hence $v\left(s - \frac{\delta'}{1+\delta}\right) > \gamma$. This proves Claim 1.

Let $\kappa = \operatorname{cofinality}(S)$, so κ is an infinite cardinal. Let $(\varepsilon_\lambda)_{\lambda<\kappa}$ be a sequence in $\mathfrak{m}$ such that $\left(v\left(s - \frac{\varepsilon'_\lambda}{1+\varepsilon_\lambda}\right)\right)$ is a strictly increasing sequence in S, and cofinal in S. Then $v\left(s - \frac{\varepsilon'_\lambda}{1+\varepsilon_\lambda}\right) = v\left(\frac{\varepsilon'_\lambda}{1+\varepsilon_\lambda} - \frac{\varepsilon'_\mu}{1+\varepsilon_\mu}\right)$ for $\lambda < \mu < \kappa$. Note that for $\lambda < \mu < \kappa$ we have

$$\frac{\varepsilon'_\lambda}{1+\varepsilon_\lambda} - \frac{\varepsilon'_\mu}{1+\varepsilon_\mu} = \frac{(1+\varepsilon_\mu)(\varepsilon_\lambda - \varepsilon_\mu)' - \varepsilon'_\mu(\varepsilon_\lambda - \varepsilon_\mu)}{(1+\varepsilon_\lambda)(1+\varepsilon_\mu)},$$

and since $v\big((\varepsilon_\lambda - \varepsilon_\mu)'\big) < v\big(\varepsilon'_\mu(\varepsilon_\lambda - \varepsilon_\mu)\big)$, this gives

$$v\left(\frac{\varepsilon'_\lambda}{1+\varepsilon_\lambda} - \frac{\varepsilon'_\mu}{1+\varepsilon_\mu}\right) = v\big((\varepsilon_\lambda - \varepsilon_\mu)'\big).$$

It follows that $v\big((\varepsilon_\lambda - \varepsilon_\mu)'\big) < v\big((\varepsilon_\mu - \varepsilon_\nu)'\big)$ for $\lambda < \mu < \nu < \kappa$. Hence $v(\varepsilon_\lambda - \varepsilon_\mu) < v(\varepsilon_\mu - \varepsilon_\nu)$ for $\lambda < \mu < \nu < \kappa$. Thus (ε_λ) is a pseudo-Cauchy sequence.

Claim 2. The pseudo-Cauchy sequence (ε_λ) has no pseudo-limit in K.

To see this, suppose $\varepsilon \in K$ is a pseudo-limit of this sequence. Then we have $v(\varepsilon - \varepsilon_\lambda) = v(\varepsilon_\mu - \varepsilon_\lambda) > 0$ for $\lambda < \mu < \kappa$, in particular $\varepsilon \in \mathfrak{m}$. Hence $v(\varepsilon' - \varepsilon'_\lambda) = v(\varepsilon'_\mu - \varepsilon'_\lambda) > 0$ for $\lambda < \mu < \kappa$. By the same computations as those preceding Claim 2 this gives $v\left(\frac{\varepsilon'}{1+\varepsilon} - \frac{\varepsilon'_\lambda}{1+\varepsilon_\lambda}\right) = v\left(\frac{\varepsilon'_\mu}{1+\varepsilon_\mu} - \frac{\varepsilon'_\lambda}{1+\varepsilon_\lambda}\right)$ for $\lambda < \mu < \kappa$. Hence $v\left(s - \frac{\varepsilon'}{1+\varepsilon}\right) \geq v\left(s - \frac{\varepsilon'_\lambda}{1+\varepsilon_\lambda}\right)$ for all $\lambda < \kappa$, contradicting Claim 1. This proves Claim 2.

Assume for a moment that L is given a valuation making it a pre-differential-valued field extension of K with $v(y) \neq 0$. Then $v(y) > 0$, since if $v(y) < 0$, we get $\psi\big(v(y)\big) = v(s) \in (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, which is impossible. Since $\frac{y'}{1+y} = s$, the sequence $\left(v\left(\frac{y'}{1+y} - \frac{\varepsilon_\lambda'}{1+\varepsilon_\lambda}\right)\right)$ is strictly increasing. Again by estimates as above this implies that $\big(v(y - \varepsilon_\lambda)\big)$ is strictly increasing. So y is a pseudo-limit of (ε_λ).

Since K is henselian, it follows from Claim 2 that $K(y)$ has a unique valuation that extends the valuation of K in which y is a pseudo-limit of (ε_λ) (see proof of Proposition 4.3). In the following $K(y)$ is equipped with this valuation. (We just saw that no other choice is possible.) Then $K(y)|K$ is an immediate extension, $v(y) > 0$, and $v(y - \varepsilon_\lambda) = v(\varepsilon_\mu - \varepsilon_\lambda)$ for $\lambda < \mu < \kappa$. Note also that for any polynomial $f(Y) \in K[Y]$ we have $v\big(f(y)\big) = v\big(f(\varepsilon_\lambda)\big)$ eventually. By Lemma 3.5 it remains to establish:

Claim 3. L is a pre-differential-valued field.

We omit the proof of this claim because it is almost identical to that of Claim 3 in the proof of Proposition 4.3, except that $y' = (1 + y)s$ instead of $y' = s$. $\square$

Remark. If K, s and L are as in the lemma and K carries in addition an ordering making it an H-field, then there is a unique ordering on L making it a pre-H-field extension of K and with this ordering L is an H-field. (By Corollary 3.2.)

Lemma 5.3 *Let K be a real closed H-field and $s \in K^{<0}$ such that for each $a \in K^\times$, there exists $\gamma \in \Gamma^*$ with $v(s - a'/a) \leq \psi(\gamma)$. Let $L = K(y)$ be a field extension of K with y transcendental over K, and let L be equipped with the unique derivation extending the derivation of K such that $y'/y = s$. Then there is a unique pair consisting of a valuation of L and an ordering on L that makes L a pre-H-field extension of K with $y > 0$. With this valuation and ordering L is an H-field with $v(y) \notin \Gamma$ and $v(y) > 0$.*

Proof. Suppose $L = K(y)$ is equipped with a valuation and an ordering making L a pre-H-field extension of K with $y > 0$.

Claim 1. $v(y) \notin \Gamma$. Otherwise we can write $y = au$ with $a \in K^\times$ and $u \in L$ with $v(u) = 0$. Then $s - (a'/a) = u'/u$, hence $v\big(s - (a'/a)\big) > \Psi$, contradicting the assumption on s.

Claim 2. $v(y) > 0$. This is because $v(y) < 0$ would imply $s = y'/y > 0$.

Claim 3. $v(y) < v(b) \iff s > b'/b$, for all $b \in K^\times$ with $v(b) > 0$. This follows from Lemma 1.4.

Claim 3 shows how $v(y)$ determines a cut in Γ. Thus in constructing a valuation of L and ordering of L with the desired properties, the three claims above leave no choice: we equip L with the unique valuation extending the valuation of K such that $0 < v(y) \notin \Gamma$ realizes the cut in Γ described in the Claim 3 above, and with the unique ordering extending the ordering of K in which the valuation ring of L is convex, and with $y > 0$. It remains to show that with this valuation and ordering L is an H-field.

Put $\eta := v(y)$, so $\Gamma_L = \Gamma \oplus \mathbb{Z}\eta$. Note that if $a \in K^\times$ and $j \in \mathbb{Z}$, then $0 < v(a) + j\eta$ if and only if either

(1) $v(a) = 0$ and $j > 0$, or
(2) $v(a) \neq 0$ and $a'/a + js < 0$.

This is clear if $v(a) = 0$, or $v(a) \neq 0$ and $j = 0$, by Claim 2. Assume $v(a) \neq 0$ and $j < 0$. Let $d \in K$ be a solution to the equation $d^j = 1/|a|$. We have $0 < v(a) + j\eta$ if and only if $\eta < v(d)$, which is equivalent to $s > (-1/j) \cdot (a'/a)$, by definition of the cut in Γ realized by $v(y)$, that is, to $a'/a + js < 0$. If $v(a) \neq 0$, $j > 0$, one argues similarly.

We observe that for any $a \in K^\times$ and $j \in \mathbb{Z}$ with $v(a) + j\eta > 0$, we have $v(a'/a + js) \leq \psi(\gamma)$ for some $\gamma \in \Gamma^*$. To see this, we may assume $j \neq 0$. Let $d \in K$ be a solution to $d^j = 1/|a|$; then $v(a'/a + js) = v(s - d'/d) \leq \psi(\gamma)$ for some $\gamma \in \Gamma^*$. Also note that if $b \in K^\times$ is another element such that $v(a) = v(b)$, we have $v(a'/a + js) = v(b'/b + js)$: Write $a = ub$ with $u \in K^\times$, $v(u) = 0$; then $v(u'/u) = v(u') > \Psi$, so $v(u'/u) > v(b'/b + js)$, and $a'/a + js = (b'/b + js) + u'/u$, implying

$$v(a'/a + js) = \min\bigl\{v(b'/b + js), v(u'/u)\bigr\} = v(b'/b + js),$$

as required. Now extend $\psi\colon \Gamma^* \to \Gamma$ to a map $\psi_L\colon \Gamma_L^* \to \Gamma$ by

$$\psi_L\bigl(v(a) + j\eta\bigr) := v(a'/a + js) \quad \text{for } a \in K^\times, j \in \mathbb{Z} \text{ with } v(a) + j\eta \neq 0.$$

By the previous remarks, ψ_L is well-defined. Also note that the criterion given above for $v(a) + j\eta > 0$ implies that ψ_L is decreasing on $\Gamma_L^{>0}$. (Hence ψ_L is constant on archimedean classes of Γ_L, since obviously $\psi_L(r\gamma) = \psi_L(\gamma)$ for all $\gamma \in \Gamma_L^*$, $r \in \mathbb{Z} \setminus \{0\}$.) We claim:

Claim 4. (Γ_L, ψ_L) is an asymptotic couple.

First assume $[\Gamma] = [\Gamma_L]$. Then if $a \in K^\times$, $j \in \mathbb{Z}$ with $v(a) + j\eta \neq 0$, we choose $b \in K^\times$ with $\bigl[v(b)\bigr] = \bigl[v(a) + j\eta\bigr]$. Hence $\psi_L\bigl(v(a) + j\eta\bigr) = \psi_L\bigl(v(b)\bigr) = v(b'/b) = \psi\bigl(v(b)\bigr)$. So it follows from Lemma 2.14 that (Γ_L, ψ_L) is an asymptotic couple. Now suppose $[\Gamma] \neq [\Gamma_L]$, so $[\Gamma_L] = [\Gamma] \cup \bigl\{[\gamma_0]\bigr\}$ for some $\gamma_0 \in \Gamma_L^*$ with $[\gamma_0] \notin [\Gamma]$. (Apply Lemma 5.3 in [1] to $\boldsymbol{k} = \mathbb{Q}$, $V_0 = \Gamma$, $V = \mathbb{Q}\Gamma_L$.) Then

$$\psi_L\bigl(v(a) + j\gamma_0\bigr) = \min\bigl\{v(a'/a), \psi_L(\gamma_0)\bigr\} \quad \text{for all } a \in K^\times, j \in \mathbb{Z}, j \neq 0.$$

Lemma 2.15 now implies that $\bigl(\Gamma_0, \psi_L|\Gamma_0^*\bigr)$ is an asymptotic couple, where $\Gamma_0 := \Gamma \oplus \mathbb{Z}\gamma_0 \subseteq \Gamma_L$. Since $\mathbb{Q}\Gamma_0 = \mathbb{Q}\Gamma_L$, we see that (Γ_L, ψ_L) is an asymptotic couple as well.

By Claim 1 and general valuation theory we have $\operatorname{res}(K) = \operatorname{res}(L)$. Using this fact and Claim 4 above we can prove very quickly:

Claim 5. L is a differential-valued field.

To see this, put $T := \bigcup_{j\in\mathbb{Z}} K^\times y^j$. We consider elements $f = ay^p$ and $g = by^q$ of T with $a, b \in K^\times$ and $v(f) > 0$ and $v(g) > 0$. By Lemmas 3.4 and 3.5 it suffices to derive $v(f') > v(g'/g)$. Clearly $v(f) = v(a) + p\eta$, and $v(f') = v(a) + p\eta + v(a'/a + ps) = v(f) + \psi_L\big(v(f)\big)$. Similarly, $v(g'/g) = \psi_L\big(v(g)\big)$. Now $\psi_L\big(v(g)\big) < v(f) + \psi_L\big(v(f)\big)$, since (Γ_L, ψ_L) is an asymptotic couple, by Claim 4.

To complete the proof of the lemma, it remains to show:

Claim 6. L is an H-field.

By Remark (2) after Lemma 3.3, we must show $t'/t > 0$ for all $t \in T$ with $v(t) < 0$. Write $t = ay^j$ with $a \in K^\times$, $j \in \mathbb{Z}$; we may assume $j \neq 0$. Let $d \in K$ as before be a solution to the equation $d^j = 1/|a|$. First assume $j < 0$. Then $v(a) + j\eta = v(t) < 0$ implies $v(d) > v(y)$, so $s < d'/d = (-1/j) \cdot (a'/a)$, by the definition of the cut in Γ realized by $v(y)$. Thus $t'/t = a'/a + js > 0$, as required. The case $j > 0$ is similar. $\square$

Remarks. Let K, s and L be as in Lemma 5.3.

(1) The proof of the lemma shows that $C_L = C$, and that Ψ is cofinal in Ψ_L. In particular, if Ψ has a maximum, then Ψ_L has the same maximum.
(2) If $\Psi_L < \gamma < (\mathrm{id} + \psi)\big(\Gamma_L^{>0}\big)$ for some $\gamma \in \Gamma_L \setminus \Gamma$, then $[\Gamma] = [\Gamma_L]$, by the remark after Lemma 2.15.

6 Liouville closures

In this section we prove the results on Liouville closures stated in the introduction. After some generalities on Liouville closed H-fields we define "Liouville towers". It is natural to construct Liouville closures by building such towers. But in showing that, up to isomorphism, there can be *at most two* Liouville closures of a given H-field we face a difficulty:

(1) Lemma 4.1 and remark (2) following it present a choice: either make s a derivative of an infinitesimal, or make it a derivative of an infinitely large element. This leads to non-isomorphic Liouville closures.
(2) Applying Lemma 5.3 might create a new element in the value group that produces the situation of Lemma 4.1.

An example (after Lemma 6.5 below) shows that (2) can really occur. Fortunately, it can be arranged to occur *at most once* in building Liouville closures via towers. This is what we accomplish in the proofs of the main

Theorems 6.9, 6.10 and 6.11. We also give an application to Hardy fields and LE-series, see Theorem 6.7.

Generalities on Liouville closed H-fields

We note that a real closed H-field K is Liouville closed if and only if each equation $y' + ay = b$ with $a, b \in K$ has a solution in K. It follows easily that a Liouville closed H-field remains Liouville closed when we replace its derivation ∂ by a multiple $a\partial$ with $a \in K^{>0}$. If K is a Liouville closed H-field, then Ψ is downward closed in the value group Γ, and Γ is the disjoint union of Ψ and $(\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. There are two kinds of Liouville closed H-fields K: those with $\Psi < 0$, and those with $0 \in \Psi$. One kind can be changed into the other kind by replacing its derivation ∂ by a suitable multiple $a\partial$ with $a \in K^{>0}$.

Lemma 6.1 *Let L be a Liouville extension of the differential field K. Then* $\mathrm{card}(L) = \mathrm{card}(K)$.

Proof. Define a chain of differential subfields $K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ of L:

$$K_{n+1} = \begin{cases} \text{algebraic closure of } K_n \text{ in } L & \text{for } n \equiv 0 \pmod 3 \\ K_n\big(\{a \in L : a' \in K_n\}\big) & \text{for } n \equiv 1 \pmod 3 \\ K_n\big(\{a \in L^\times : a'/a \in K_n\}\big) & \text{for } n \equiv 2 \pmod 3 \end{cases}$$

Clearly $\mathrm{card}(K_n) = \mathrm{card}(K)$ for all n (by induction), and $L = \bigcup_n K_n$, so $\mathrm{card}(L) = \mathrm{card}(K)$. $\square$

Lemma 6.2 *Let K be a differential-valued field* (*respectively, an H-field*)*, and let $(K_i)_{i\in I}$ be a family of differential-valued subfields of K* (*respectively, of H-subfields of K*)*. Then $\bigcap_i K_i$ is a differential-valued subfield of K* (*respectively, an H-subfield of K*)*.*

Proof. With $\mathcal{O}_i$ the valuation ring of K_i, the valuation ring of $\bigcap_i K_i$ is $\bigcap_i \mathcal{O}_i$. Now, given any $a \in \bigcap_i \mathcal{O}_i$, there is unique $c \in C$ and unique $c_i \in C_{K_i}$ for each i such that $v(a - c) > 0$ and $v(a - c_i) > 0$ for each i. Hence all c_i are equal to c, and thus $c \in \bigcap_i K_i$. We have now checked that the valuation of $\bigcap_i K_i$ satisfies condition (1) in Sect. 1 for differential valuations. Condition (2) is obviously also satisfied. $\square$

Lemma 6.3 *Let K be a Liouville closed H-field. Then:*

(1) *K has no proper Liouville extension with the same constants as K.*
(2) *Let $(K_i)_{i\in I}$ be a family of Liouville closed H-subfields of K, all with the same constants as K. Then $\bigcap_i K_i$ is a Liouville closed H-subfield of K.*

Proof. Suppose L is a proper Liouville extension of the differential field K with the same constants as K. Up to K-isomorphism the only proper algebraic extension field of K is $K(i)$ with $i^2 = -1$, and as a differential field extension of K it contains the constant $i \notin C$. Hence L must contain a solution $y \notin K$ to an equation $y' = a$ with $a \in K$, or a solution $z \notin K$ with $z \neq 0$ to an equation $z'/z = b$ with $b \in K$. But given y as above, take $y_0 \in K$ with $y_0' = a$, and then $y - y_0 \in C_L \setminus C$, contradiction. Similarly, given z as above, take $z_0 \in K^\times$ with $z_0'/z_0 = b$, and note that then $z/z_0 \in C_L \setminus C$, contradiction. This proves the first statement.

For the second statement we first remark that $\bigcap_i K_i$ is an H-subfield of K by Lemma 6.2. Now note that a similar argument as in (1) shows that for $a, b \in \bigcap_i K_i$ the equation $y' = a$ has the same solutions in all K_i, and so does the equation $z'/z = b$ (subject to $z \neq 0$). □

Let $K \subseteq L$ be an extension of differential fields. Then the subfield of L generated by any collection of intermediate Liouville extension fields is also a differential subfield of L and a Liouville extension of K. Hence there exists a biggest Liouville extension of K contained in L. If $L \subseteq M$ is a further differential field extension and $M|K$ is a Liouville extension, so is $M|L$.

Liouville towers

Let K be an H-field. A **Liouville tower on** K is a strictly increasing chain $(K_\lambda)_{\lambda \leq \mu}$ of H-fields, indexed by the ordinals less than or equal to some ordinal μ, such that

(1) $K_0 = K$,
(2) if λ is a limit ordinal, $0 < \lambda \leq \mu$, then $K_\lambda = \bigcup_{\iota < \lambda} K_\iota$,
(3) for $\lambda < \lambda + 1 \leq \mu$, *either*
  (a) $K_{\lambda+1}$ is a real closure of K_λ,
  or K_λ is already real closed, $K_{\lambda+1} = K_\lambda(y_\lambda)$ with $y_\lambda \notin K_\lambda$ (hence y_λ is transcendental over K_λ), and one of the following holds, where $(\Gamma_\lambda, \psi_\lambda)$ denotes the asymptotic couple of K_λ and $\Psi_\lambda := \psi_\lambda(\Gamma_\lambda^*)$:
  (b) $y_\lambda' = s_\lambda \in K_\lambda$ with $v(y_\lambda) > 0$ and $\Psi_\lambda < v(s_\lambda) < (\mathrm{id} + \psi_\lambda)\big(\Gamma_\lambda^{>0}\big)$.
  (c) $y_\lambda' = s_\lambda \in K_\lambda$ with $v(y_\lambda) < 0$ and $\Psi_\lambda < v(s_\lambda) < (\mathrm{id} + \psi_\lambda)\big(\Gamma_\lambda^{>0}\big)$.
  (d) $y_\lambda' = s_\lambda \in K_\lambda$ with $v(s_\lambda) = \max \Psi_\lambda$.
  (e) $y_\lambda' = s_\lambda \in K_\lambda$ with $v(s_\lambda) \in (\mathrm{id} + \psi_\lambda)\big(\Gamma_\lambda^{>0}\big)$, and $s_\lambda \neq \varepsilon'$ for all $\varepsilon \in K_\lambda$ with $v(\varepsilon) > 0$.
  (f) $y_\lambda' = s_\lambda \in K_\lambda$ such that $S_\lambda := \big\{v(s_\lambda - a') : a \in K_\lambda\big\} < (\mathrm{id} + \psi_\lambda)\big(\Gamma_\lambda^{>0}\big)$, and S_λ has no largest element.
  (g) $\frac{y_\lambda'}{1+y_\lambda} = s_\lambda \in K_\lambda$ with $v(y_\lambda) \neq 0$, $v(s_\lambda) \in (\mathrm{id} + \psi_\lambda)\big(\Gamma_\lambda^{>0}\big)$, and $s_\lambda \neq a'/a$ for all $a \in K_\lambda^\times$.

(h) $\frac{y'_\lambda}{y_\lambda} = s_\lambda \in K_\lambda^{<0}$ with $y_\lambda > 0$, and for each $a \in K_\lambda^\times$ there is $\gamma \in \Gamma_\lambda^*$ such that $v(s_\lambda - a'/a) \leq \psi_\lambda(\gamma)$.

The H-field K_μ is called the **top** of the tower $(K_\lambda)_{\lambda \leq \mu}$. Note that clause (a) corresponds to the last part of Sect. 3, (b) to Lemma 4.1, (c) to remark (2) following Lemma 4.1, (d) to Lemma 4.2, (e) to Proposition 4.3, and (f), (g) and (h) to Lemmas 5.1, 5.2 and 5.3, respectively.

Remark. Let a tower as above be given. Then:

(1) K_μ is a Liouville extension of K.
(2) The constant field C_μ of K_μ is a real closure of C if $\mu > 0$.
(3) $\operatorname{card}(K_\mu) = \operatorname{card}(K)$, hence $\mu < \operatorname{card}(K)^+$.

For (1) and (2) use results of sections 3–5 to show by induction on $\lambda \leq \mu$ that K_λ is a Liouville extension of K, and that the constant field of K_λ is the real closure of C for $\lambda > 0$. Item (3) follows from (1) by Lemma 6.1.

Because of remark (3) there is for each H-field K a *maximal* Liouville tower $(K_\lambda)_{\lambda \leq \mu}$ on K, "maximal" meaning that it cannot be extended to a Liouville tower $(K_\lambda)_{\lambda \leq \mu+1}$ on K. It follows easily that then K_μ is Liouville closed, and hence a Liouville closure of K. Thus each H-field has a Liouville closure. We now turn to the question to what extent such a Liouville closure is unique.

Lemma 6.4 *Let $K \subseteq L$ be an extension of H-fields such that L is Liouville closed. Then there is a unique H-field K' such that $K \subseteq K' \subseteq L$ and K' is a Liouville closure of K.*

Proof. Let K' be the top of some maximal Liouville tower on K consisting of H-subfields of L. It is easy to see that K' is Liouville closed, and hence a Liouville closure of K. This proves "existence". "Uniqueness": Let K' be any H-field such that $K \subseteq K' \subseteq L$ and K' is a Liouville closure of K. By Lemma 6.3, (1) K' has no proper Liouville extension inside L. Thus K' is necessarily the largest Liouville extension of K contained in L. □

The K' in this lemma is called the **Liouville closure of K inside L.**

For the purpose of this section, a **gap** in an H-field K is a $\gamma \in \Gamma$ such that $\Psi < \gamma < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. Note that for an H-field K we have:

(1) If Ψ has a largest element, then K has no gap.
(2) If K is Liouville closed, then K has no gap.
(3) A gap in K remains a gap in its real closure, by the remark after Lemma 2.5.

Lemma 6.5 *Let K be an H-field, $(K_\lambda)_{\lambda \leq \mu}$ a Liouville tower on K such that no K_λ with $\lambda < \mu$ has a gap. Then every embedding of K into a*

Liouville closed H-field L can be extended to an embedding of K_μ into L. If K_μ is also Liouville closed, then K_μ is the unique Liouville closure of K, up to isomorphism over K.

This follows from the uniqueness properties in the results of sections 3–5, together with Lemma 6.4.

Example

Let $\mathbb{L}^{\mathbb{Q}}$ be the multiplicative subgroup of $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ generated by the rational powers l_n^a ($a \in \mathbb{Q}$) of the iterated logarithms l_n of x, that is, $l_0 = x$, $l_{n+1} = \log(l_n)$. Thus $\mathbb{L}^{\mathbb{Q}}$ is the set of products $l_0^{a_0} l_1^{a_1} \cdots l_n^{a_n}$ with $a_0, \ldots, a_n \in \mathbb{Q}$, and if also $b_0, \ldots, b_n \in \mathbb{Q}$, then

$$l_0^{a_0} l_1^{a_1} \cdots l_n^{a_n} < l_0^{b_0} l_1^{b_1} \cdots l_n^{b_n} \iff (a_0, \ldots, a_n) < (b_0, \ldots, b_n) \text{ lexicographically.}$$

We consider the formal series field $\mathbb{R}((\mathbb{L}^{\mathbb{Q}}))$ as an ordered field as usual (see [6], (1.1)), and equip it with the derivation that is trivial on $\mathbb{R}$, sends l_n to $1/l_0 l_1 \cdots l_{n-1}$ (in particular x to 1), and that commutes with infinite summation in $\mathbb{R}((\mathbb{L}^{\mathbb{Q}}))$. We refer to [2] for more details, and for proofs of some facts used in this *Example*, such as the fact that this derivation makes $\mathbb{R}((\mathbb{L}^{\mathbb{Q}}))$ into a real closed H-field extension of $\mathbb{R}(l_n : n \in \mathbb{N})$. Let E be the real closure of $\mathbb{R}(l_n : n \in \mathbb{N})$ inside $\mathbb{R}((\mathbb{L}^{\mathbb{Q}}))$, so $\mathbb{R}((\mathbb{L}^{\mathbb{Q}}))|E$ is an immediate extension of valued fields. As in [7], p. 289, (7.9), we put

$$\Lambda := l_1 + l_2 + l_3 + \cdots = \sum_{n=1}^{\infty} l_n \in \mathbb{R}((\mathbb{L}^{\mathbb{Q}})),$$

so

$$\Lambda' = \frac{1}{l_0} + \frac{1}{l_0 l_1} + \frac{1}{l_0 l_1 l_2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{l_0 l_1 \cdots l_n}.$$

Let $s = -\Lambda'$ and K be the real closure inside $\mathbb{R}((\mathbb{L}^{\mathbb{Q}}))$ of the H-subfield $E(s, s', s'', \ldots)$ of $\mathbb{R}((\mathbb{L}^{\mathbb{Q}}))$ generated by s over E. One can show ([2]) that then K has no gap, and that the hypothesis of Lemma 5.3 holds for K and s. The conclusion of that lemma then gives us an H-field extension $L = K(y)$ of K such that y is transcendental over K, $y > 0$ and $y'/y = s$. Then $v(y)$ is a gap in L, see [2].

Thus (h) is special in that it can *create* a gap, while none of the extensions of type (b)–(g) can produce a gap that wasn't already there. This explains the perhaps curious arrangement of the proofs of the main theorems below.

Application to Hardy fields and LE-series

In the realm of LE-series gaps don't occur, as the next lemma states in more detail. For positive infinite f in $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ we put $\mathrm{L}^0(f) := f$, and $\mathrm{L}^{n+1}(f) := \log\bigl(\mathrm{L}^n(f)\bigr)$. Then the sequence $\bigl\{\mathrm{L}^n(f)\bigr\}$ is coinitial in the set of positive infinite elements of $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$, see [6].

Lemma 6.6 *The H-field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ is Liouville closed, and thus has no gap. More generally, if K is any H-subfield of $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ not contained in $\mathbb{R}$, then K has no gap, and any two Liouville closures of K are isomorphic over K.*

Proof. See [6] for the fact that $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ is Liouville closed. Let K be an H-subfield of $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ not contained in $\mathbb{R}$. Then the derivation on K is non-trivial, and hence its valuation is non-trivial. Take some positive infinite $f \in K$. If $v\bigl(\mathrm{L}^n(f)\bigr) \in \Gamma$ for all n, then the remark preceding the lemma implies that $\Gamma^{<0}$ is cofinal in $\Gamma_{\mathrm{LE}}^{<0}$, and hence K has no gap (as $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ has none). Suppose that $v\bigl(\mathrm{L}^n(f)\bigr) \notin \Gamma$ for some n, and take n minimal with this property. Then $n > 0$ since $f \in K$, and $v\bigl(\mathrm{L}^n(f)'\bigr) = \psi\bigl(v\bigl(\mathrm{L}^{n-1}(f)\bigr)\bigr) \in \Psi \setminus (\mathrm{id} + \psi)(\Gamma^*)$, so $v\bigl(\mathrm{L}^n(f)'\bigr) = \max \Psi$, and thus K has no gap. Next we build a Liouville tower on K whose top is the Liouville closure of K inside $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$. Since none of the H-fields in this tower can have a gap, Lemma 6.5 implies that each Liouville closure of K is isomorphic over K to the Liouville closure of K inside $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$. $\square$

This lemma implies the following theorem stated in the introduction:

Theorem 6.7 *Let $K \supseteq \mathbb{R}$ be a Hardy field and $e\colon K \to \mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ an embedding of ordered differential fields with $e|\mathbb{R} = \mathrm{id}_{\mathbb{R}}$. Then e can be extended to an embedding $\mathrm{Li}(K) \to \mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ of ordered differential fields.*

To see how this follows, note first that if $K = \mathbb{R}$, then by extending e to $\mathbb{R}(x)$ we reduce to the case $K \neq \mathbb{R}$. We can then apply the lemma in view of the fact that an ordered differential field embedding between H-fields also respects the valuation, and is thus an H-field embedding.

Constructing Liouville closures

Let $\alpha \subseteq \bigl\{$(a), (b), (c), (d), (e), (f), (g), (h)$\bigr\}$ with (a) $\in \alpha$, and let K be an H-field. Then the definition of "α-tower on K" is identical to that of "Liouville tower on K", except that in clause (3) of that definition only the items from α occur.

Lemma 6.8 *Let K be an H-field such that Ψ has a largest element. There exists a Liouville tower on K with top L such that:*

(1) *No H-field in the tower has a gap.*
(2) *Ψ_L has a maximum.*
(3) *For each $a \in K$ there exist $y, z \in L$ with $y' = a$ and $z \neq 0$, $z'/z = a$.*

Proof. Let $\alpha := \{\text{(a), (e), (f), (g), (h)}\}$. Take a maximal α-tower $(K_\lambda)_{\lambda \le \mu}$ on K. By remarks following Proposition 4.3 and Lemmas 5.1–5.3, induction on λ shows that each Ψ_λ has maximum $\max \Psi$. In particular, no K_λ has a gap. By maximality with respect to (a), (g) and (h), there is for each $s \in K$ some $y \in K_\mu^\times$ with $y'/y = s$. (Use Lemmas 5.2 and 5.3.) Take $s \in K_\mu$ with $v(s) = \max \Psi$. By Lemma 4.2, there is an H-field extension $L := K_\mu(y)$ of K_μ such that y is transcendental over K_μ and $y' = s$. Then Ψ_L again has a maximum, namely $\psi_L\big(v(y)\big) > \max \Psi$. In particular L has no gap. It only remains to show that each element of K has an anti-derivative in L. Suppose $t \in K$ has no anti-derivative in K_μ. Then the maximality property of the tower with respect to (a), (e) and (f) implies that $\max \Psi = v(t - a')$ for some $a \in K_\mu$. (See Lemma 5.1 and remark (2) following it.) Hence $t - a' = cs + d$ for some $c \in C_\mu$ and $d \in K_\mu$ with $v(d) > \max \Psi$. So $d = e'$ for an $e \in K_\mu$, and thus $t = (a + cy + e)'$ in L. $\square$

Let us write K' for the real closure of an H-field extension L of K as in this lemma. Then $\Psi_{K'} = \Psi_L$, so $\Psi_{K'}$ also has a largest element. Thus we can iterate this operation, and form $K'' := (K')'$, $K''' := (K'')'$, and so on. Taking the union of the increasing sequence of H-fields built in this way, and applying Lemma 6.5, we obtain:

Theorem 6.9 *Let K be an H-field K such that Ψ has a largest element. Then K has a Liouville closure L such that any embedding of K into a Liouville closed H-field M extends to an embedding of L into M. Any two Liouville closures of K are isomorphic over K.* $\square$

Theorem 6.10 *Let K be an H-field with a gap $\gamma \in \Gamma$. Then K has Liouville closures L_1 and L_2, such that any embedding of K into a Liouville closed H-field M extends to an embedding of L_1 or of L_2 into M, depending on whether the image of γ in Γ_M lies in $(\mathrm{id} + \psi_M)\big(\Gamma_M^{<0}\big)$ or in $(\mathrm{id} + \psi_M)\big(\Gamma_M^{>0}\big)$. Each Liouville closure of K is K-isomorphic to L_1 or to L_2, but L_1 and L_2 are not K-isomorphic.*

Proof. Take $s \in K$ such that $v(s) = \gamma$. Let $K_1 := K(y_1)$ and $K_2 := K(y_2)$ be H-field extensions of K with y_i transcendental over K and $y_i' = s$, for $i = 1, 2$, such that $v(y_1) < 0$ and $v(y_2) > 0$. (Such K_i exist by Lemma 4.1 and remark (2) following it.) Then Ψ_{K_1} and Ψ_{K_2} both have a largest element. Let L_1 and L_2 be Liouville closures of K_1 and K_2 respectively. Let an embedding of K into a Liouville closed H-field M be given. If the image of γ in Γ_M lies in $(\mathrm{id} + \psi_M)\big(\Gamma_M^{<0}\big)$, then we can extend that embedding to an embedding of K_1 into M, and hence by the previous theorem, to an embedding of L_1 into M. If the image of γ in Γ_M lies in $(\mathrm{id} + \psi_M)\big(\Gamma_M^{>0}\big)$, then we can similarly extend that embedding to an embedding of L_2 into M. It is now routine to show that L_1 and L_2 as defined here have all the properties claimed in the theorem. $\square$

Remark. Let K be an ordered field, and equip K with the trivial derivation and trivial valuation. Then K is an H-field with $\Gamma = \{0\}$, and has gap $0 = v(1)$. The two Liouville closures L_1 and L_2 of K in the theorem satisfy $\Psi_{L_1} < 0$ and $0 \in \Psi_{L_2}$. Replacing the derivation ∂ of L_1 by a suitable multiple $a\partial$, $0 < a \in L_1$, we obtain a K-isomorphic copy of L_2.

A more interesting H-field with a gap is the H-field L constructed in the "Example" after Lemma 6.5, with gap $v(y)$.

The two theorems above concern two special cases, and we now turn to the general situation. Let K be an H-field. Take a maximal Liouville tower $(K_\lambda)_{\lambda \le \mu}$ on K. Then K_μ is a Liouville closure of K. We have two cases:

(I) No K_λ in the tower has a gap. See Lemma 6.5.

(II) Some K_λ in the tower has a gap. Take λ minimal with this property. Let L_1 and L_2 be the two Liouville closures of K_λ as in the last theorem. Given any embedding of K into a Liouville closed H-field M, we can first extend it to an embedding of K_λ into M, and then, by the last theorem, to an embedding of L_1 or of L_2 into M. It follows that L_1 and L_2 are Liouville closures of K (not isomorphic over K), and that any Liouville closure of K is K-isomorphic to L_1 or to L_2.

Summarizing the above, we have the following more precise version of the Main Theorem stated in the introduction.

Theorem 6.11 *Let K be an H-field. Then K has at least one and at most two Liouville closures, up to isomorphism over K. Any embedding of K into a Liouville closed H-field M extends to an embedding of some Liouville closure of K into M. Moreover, the following are equivalent:*

(1) *K has two Liouville closures, not isomorphic over K.*

(2) *There exists a Liouville H-field extension $L \supseteq K$ with a gap.*

(3) *There exists a Liouville H-field extension $L \supseteq K$ with a gap such that L embeds over K into any Liouville closed H-field extension of K.* $\square$

Remark. Let K be an H-field such that Ψ has no largest element, and let L be a Liouville closure of K. Then the theorem implies that L is up to K-isomorphism the only Liouville closure of K if and only if $\Gamma^{>0}$ is coinitial in $\Gamma_L^{>0}$.

7 Concluding remarks

In this paper we have shown that several results on Hardy fields extend to the abstract setting of H-fields, with rather different proofs. More significant is that the material around "Liouville closures" *demands* the H-field setting: Hardy fields and the field $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ of LE-series obscure the "fork in the road" phenomenon. Our real motive for introducing H-fields is the possibility of a model theory for the *differential* field $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$, analogous to the subject of real closed fields and semialgebraic sets as a model theory for the field $\mathbb{R}$.

In a follow-up paper [2] we treat various other issues on H-fields:

(1) Constant field extensions of H-fields.
(2) Adjoining powers to H-fields.
(3) Equipping Liouville closed H-fields with an exponential function.
(4) Algebraic-topological properties of differential polynomials over H-fields.
(5) Generalized series constructions producing H-fields with given constant field and asymptotic couple.

A critical question is how gaps can arise in H-fields. The example of the last section illustrates one possibility. Can the trouble caused by gaps be localized in Liouville extensions, and bypassed in other kinds of differentially algebraic extensions? More specifically:

> Can a differentially algebraic H-field extension of a *Liouville closed* H-field K have a gap?

The answer is "no" for $K = \mathbb{R}((x^{-1}))^{\mathrm{LE}}$, to be proved in [2]. Perhaps unfortunately, it is "yes" in some other cases, as will be shown elsewhere.

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