Left-linear bounded term rewriting systems are inverse recognizability preserving

Irène Durand and Marc Sylvestre

LaBRI, Université Bordeaux 1
351 Cours de la libération, 33405 Talence cedex, France

Abstract

Bounded rewriting for linear term rewriting systems has been defined in (I. Durand, G. Sénizergues, M. Sylvestre. Termination of linear bounded term rewriting systems. Proceedings of the 21st International Conference on Rewriting Techniques and Applications) as a restriction of the usual notion of rewriting. We extend here this notion to the whole class of left-linear term rewriting systems, and we show that bounded rewriting effectively inverse-preserves recognizability. The *bounded class* (BO) is, by definition, the set of left-linear systems for which every derivation can be replaced by a bottom-up derivation. The class BO contains (strictly) several classes of systems which were already known to be inverse recognizability preserving: the left-linear growing systems, and the inverse Right-Linear Finite-Path Overlapping systems.

1998 ACM Subject Classification Primary: F.4.2, Secondary: F.3.2, F.4.1

Keywords and phrases Term rewriting, preservation of recognizability, rewriting strategies

Digital Object Identifier 10.4230/LIPIcs.xxx.yyy.p

Category Regular Research Paper

1 Introduction

General framework. A term rewriting system (TRS) $\mathcal{R}$ is said to be recognizability preserving (respectively inverse recognizability preserving) if for every recognizable set of terms T the set $[T](\to^*_{\mathcal{R}}) = \{s \mid \exists t \in T, t \to^*_{\mathcal{R}} s\}$ (resp. $(\to^*_{\mathcal{R}})[T] = \{s \mid \exists t \in T, s \to^*_{\mathcal{R}} t\}$) is recognizable. Many efforts have been made for finding subclasses of $TRSs$ which preserve (or inverse-preserve) recognizability. Each identification of a more general class of TRSs preserving recognizability yields almost directly a new decidable call-by-need class [2], decidability results for confluence, accessibility, joinability. Many such classes have been defined by imposing syntactical restrictions on the rewrite rules (e.g. growing TRSs [10, 7] and Finite-Path Overlapping systems [14, 15]). Another way is to use a *strategy*, i.e. some restrictions on the derivations rather than on the rules, to ensure preservation of recognizability. Various such stategies where studied in [5, 11, 13, 4, 3]. In this paper, we extend the bounded rewriting for linear TRS to left-linear $TRSs$ that may have non-right linear rules and we prove that this strategy is inverse recognizability preserving.

From linear $TRSs$ to left linear $TRSs$. Bounded rewriting for linear $TRSs$ is essentially a new version of bottom-up rewriting [4] which is easier to define and has better properties. The reader may refer to [3] for more details on bounded rewriting for linear $TRSs$. Intuitively, for a linear TRS $\mathcal{R}$, a derivation is k-bounded ($\mathsf{lbo}(k)$) if when a rule is applied, the parts of the substitution located at a depth greater than k are not used further in the derivation, i.e. do not match a left-handside of a rule applied further. A linear TRS $\mathcal{R}$ is $\mathsf{bo}(k)$ if for any derivation $s \to^*_{\mathcal{R}} t$ there exists a $\mathsf{bo}(k)$ derivation $s \ {}_k\!\!\to^*_{\mathcal{R}} t$. The

submitted to 22nd International Conference on Rewriting Techniques and Applications.
Editor: M. Schmidt-Schauß Editor; pp. 1–15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

class of linear $\mathsf{lbo}(k)$ $TRSs$ is denoted by $\mathsf{LBO}(k)$. One goal of this paper is to propose an extension of bounded rewriting to left-linear $TRSs$. This extension cannot be the trivial one: even if nothing in the definition of $\mathsf{LBO}(k)$ TRSs requires the linear condition, keeping this definition unchanged would handle to define a class of systems containing only linear $TRSs$ (see example 4.14). To solve this problem, we introduce a binary symbol E and a set $\mathcal{E}$ of three rules to manipulate this symbol: the introduction rule $x \to E(x, x)$, and two selections rules $E(x, y) \to x$ and $E(x, y) \to y$. Let $\mathcal{R}$ be a left-linear TRS over a signature $\mathcal{F}$. Roughly speaking, a derivation in $\mathcal{R} \cup \mathcal{E}$ is k-bounded if when a rule is applied, the parts of the substitution located at a depth greater than k do not match a left-handside of a rule of $\mathcal{R}$ applied further. A derivation in $s \to^*_{\mathcal{R}} t$ is k-bounded convertible ($\mathsf{boc}(k)$) if there exists a $\mathsf{bo}(k)$ derivation $s\ {}_k{\to}_{\mathcal{R}\cup\mathcal{E}}\ t$. Note that this definition does not constrain the application of the rules of $\mathcal{E}$. A TRS is $\mathsf{bo}(k)$ if every derivation is $\mathsf{boc}(k)$. Bounded rewriting is formally defined in section 4. The class of $\mathsf{bo}(k)$ $TRSs$ is denoted by $\mathsf{BO}(k)$. Let us see how we use the symbol E. Suppose that $\mathsf{f}(\mathsf{a}) \to_{\mathsf{f}(x)\to\mathsf{g}(x,x)} \mathsf{g}(\mathsf{a}, \mathsf{a}) \to_{\mathsf{a}\to\mathsf{b}} \mathsf{g}(\mathsf{a}, \mathsf{b})$. The symbol E is used to apply the rule $\mathsf{a} \to \mathsf{b}$ before the rule $\mathsf{f}(x) \to \mathsf{g}(x, x)$. First, we use E to create an envelop which contains a and b: $\mathsf{f}(\mathsf{a}) \to_{x\to E(x,x)} \mathsf{f}(E(\mathsf{a}, \mathsf{a})) \to_{\mathsf{a}\to\mathsf{b}} \mathsf{f}(E(\mathsf{a}, \mathsf{b}))$. Then we can apply the rule $\mathsf{f}(x) \to \mathsf{g}(x, x)$, and use the selections rules to obtain $\mathsf{g}(\mathsf{a}, \mathsf{b})$: $\mathsf{f}(E(\mathsf{a}, \mathsf{b})) \to \mathsf{g}(E(\mathsf{a}, \mathsf{b}), E(\mathsf{a}, \mathsf{b})) \to_{E(x,y)\to x} \mathsf{g}(\mathsf{a}, E(\mathsf{a}, \mathsf{b})) \to_{E(x,y)\to y} \mathsf{g}(\mathsf{a}, \mathsf{b})$. The introduction of the symbol E can be viewed as a counterpart of the construction of the powerset automaton in the extension of Jacquemard's saturation method [7] by Nagaya and Toyama [10] (this saturation method is used to prove that left-linear growing $TRSs$ are inverse recognizability preserving).

Inverse recognizability preserving. In section 5, we prove that bounded rewriting for left-linear $TRSs$ is inverse recognizability preserving. This result is obtained by simulating $\mathsf{bo}(k)$-derivations using a ground tree transducer. The idea of simulating $\mathsf{bo}(k)$-derivations is similar to the idea developed in [4] where bottom-up(k) derivations are simulated using a ground TRS. This simulation yields directly to the inverse preservation result since GTTs are inverse recognizability preserving.

Strongly bounded systems. In section 6, we introduce a subclass of $\mathsf{BO}(k)$ called the strongly bounded class ($\mathsf{SBO}(k)$). The membership problem for $\mathsf{SBO}(k)$ is decidable whereas the membership problem for $\mathsf{BO}(0)$ is undecidable. The class of strongly bounded TRSs contains inverse Right-Linear Finite-Path Overlapping TRSs and left-linear growing TRSs strongly bounded. Note that a full version of this paper is available at ADRESSE.

2 Preliminaries

Given a set E, we denote by $\mathcal{P}(E)$ its powerset i.e. the set of all its subsets. Its cardinality is denoted by $\mathsf{Card}(E)$. A finite *word* over an alphabet A is a map $u : [0, \ell - 1] \to A$, for some $\ell \in \mathbb{N}$. The integer ℓ is the *length* of the word u and is denoted by $|u|$. The set of words over A is denoted by A^* and endowed with the usual *concatenation* operation $u, v \in A^* \mapsto u \cdot v \in A^*$. The *empty* word is denoted by ε. Assume that the set A is ordered. We denote by $\preceq_{\mathsf{Lex}_A}$ the lexicographic order on the set of words A^*. We may omit Lex_A when it is clear from the context. We assume the reader familiar with terms and automaton (see e.g. [1] or [16] for an introduction). We call *signature* a set $\mathcal{F}$ of symbols with arity $ar : \mathcal{F} \to \mathbb{N}$. The subset of symbols with arity $m \in \mathbb{N}$ is denoted by $\mathcal{F}_m$. As usual, a finite set $P \subseteq \mathbb{N}^*$ is called a *tree-domain* (or, *domain*, for short) iff for every $u \in \mathbb{N}^*, i \in \mathbb{N}$

$(u \cdot i \in P \Rightarrow u \in P)$ & $(u \cdot (i+1) \in P \Rightarrow u \cdot i \in P)$. We call $P' \subseteq P$ a *subdomain* of P iff, P' is a domain and, for every $u \in P, i \in \mathbb{N}$ $(u \cdot i \in P'$ & $u \cdot (i+1) \in P) \Rightarrow u \cdot (i+1) \in P'$.

A (first-order) *term* on a signature $\mathcal{F}$ is a partial map $t : \mathbb{N}^* \to \mathcal{F}$ whose domain is a non-empty tree-domain and which respects the arity assignment. We denote by $\mathcal{T}(\mathcal{F}, \mathcal{V})$ the set of first-order terms over the signature $\mathcal{F} \cup \mathcal{V}$, where $\mathcal{F}$ is a signature and $\mathcal{V}$ is a denumerable set of variables of arity 0.

The domain of t is also called its set of *positions* and denoted by $\mathcal{P}\mathsf{os}(t)$. The set of variables of t is denoted by $\mathcal{V}\mathsf{ar}(t)$. A variable x is said to occur at depth n in t if there exists a position $u \in \mathcal{P}\mathsf{os}(t)$ such that $t(u) = x$ and $|u| = n$. The root symbol of t is denoted by $\mathsf{root}(t)$. Given a set of symbols and variables $A \subseteq \mathcal{F} \cup \mathcal{V}$ and a term t, the set of positions $u \in \mathcal{P}\mathsf{os}(t)$ such that $t(u) \in A$ is denoted by $\mathcal{P}\mathsf{os}_A(t)$ and the set of position $u \in \mathcal{P}\mathsf{os}(t)$ such that $t(u) \notin A$ is denoted by $\mathcal{P}\mathsf{os}_{\setminus A}(t)$. Let X be either A or $\setminus A$ and $u \in \mathcal{P}\mathsf{os}_X(t)$. We denote by $\mathcal{P}\mathsf{os}_X{}^{\preceq u}(t)$ (respectively $\mathcal{P}\mathsf{os}_X{}^{\prec u}(t)$) the set of positions $v \in \mathcal{P}\mathsf{os}_X(t)$ such that $v \preceq u$ (resp. $v \prec u$) and by $\mathcal{P}\mathsf{os}_X{}^{\succeq u}(t)$ (respectively $\mathcal{P}\mathsf{os}_X{}^{\succ u}(t)$) the set of positions $v \in \mathcal{P}\mathsf{os}_X(t)$ such that $v \succeq u$ (resp. $v \succ u$). When $A = \{f\}$ for some $f \in \mathcal{F} \cup \mathcal{V}$ we may denote $\mathcal{P}\mathsf{os}_f(t)$ (respectively $\mathcal{P}\mathsf{os}_{\setminus f}(t)$) instead of $\mathcal{P}\mathsf{os}_{\{f\}}(t)$ (resp. $\mathcal{P}\mathsf{os}_{\setminus\{f\}}(t)$). A *substitution* σ is a mapping from $\mathcal{V}$ into $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The substitution σ is naturally extended to a morphism $\sigma : \mathcal{T}(\mathcal{F}) \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, where $\sigma(f(t_1, .., t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$, for each $f \in \mathcal{F}_n$, $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. We identify a substitution and its extension to $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Substitutions will often be used in postfix notation: $t\sigma$ is the result of applying σ to the term t. The *depth* of a term t is defined by $dpt(t) := \sup\{\mathsf{Card}(\mathcal{P}\mathsf{os}^{\preceq u}(t)) \mid u \in \mathcal{P}\mathsf{os}_{\setminus\mathcal{V}}(t)\}$. This definition is extended to substitutions $dpt(\sigma) := \max\{dpt(x\sigma) \mid x \in \mathcal{V}\}$. For a term t and a symbol $f \in \mathcal{F}$, we define $dpt_{\setminus f}(t)$ by: $dpt_{\setminus f}(t) := \sup\{\mathsf{Card}(\mathcal{P}\mathsf{os}_{\setminus f}{}^{\preceq u}(t)) \mid u \in \mathcal{P}\mathsf{os}_{\setminus\{f\}\cup\mathcal{V}}(t)\}$. This definition is extended to substitutions $dpt_{\setminus f}(\sigma) := \max\{dpt_{\setminus f}(x\sigma) \mid x \in \mathcal{V}\}$. The set of *leaves* of t is the set $\mathcal{P}\mathsf{os}_{\mathcal{V}\cup\mathcal{F}_0}(t)$ and is also denoted by $\mathcal{L}\mathsf{v}(t)$. For a variable $x \in \mathcal{V}\mathsf{ar}(t)$, the set of positions $\mathcal{P}\mathsf{os}_x(t)$ is also denoted by $\mathcal{P}\mathsf{os}(t, x)$. Given a term t and $u \in \mathcal{P}\mathsf{os}(t)$ the *subterm of t at u* is denoted by t/u and defined by $\mathcal{P}\mathsf{os}(t/u) = \{w \mid uw \in \mathcal{P}\mathsf{os}(t)\}$ and $\forall w \in \mathcal{P}\mathsf{os}(t/u)$, $t/u(w) = t(uw)$.

A term that does not contain twice the same variable is called *linear*. Given a linear term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $x \in \mathcal{V}\mathsf{ar}(t)$, we denote by $\mathsf{pos}(t, x)$ the position of x in t. A term containing no variable is called *ground*. The set of ground terms is abbreviated to $\mathcal{T}(\mathcal{F})$ or $\mathcal{T}$ whenever $\mathcal{F}$ is understood.

Let S be a set. We denote by $\mathsf{Seq}(S) = \{(s_0, \ldots, s_n) | n \in \mathbb{N}, s_i \in S\}$ the set of *sequences* built with elements in S. The *length* of a sequence $\overrightarrow{s}$ is the number of elements in this sequence and is denoted by $|\overrightarrow{s}|$. For every $s \in S$, we identify the sequence (s) and s. A sequence which does not contains twice the same element is *linear*. We define the concatenation operation on sequences $s = (s_1, \ldots, s_n), t = (t_1, \ldots, t_m)$ by $s \cdot t := (s_1, \ldots, s_n, t_1, \ldots, t_m)$. Let $\overrightarrow{s} = (s_1, \ldots, s_n)$ be a sequence of terms. We use the notation $\overrightarrow{s}\sigma$ where σ is a substitution, for the sequence $(s_1\sigma, \ldots, s_n\sigma)$. For m sequences of terms $\overrightarrow{s_1}, \ldots, \overrightarrow{s_m}$ such that $|s_1| + |s_2| + \ldots + |s_m| = n$, we denote by $f(\overrightarrow{s_1}, \ldots, \overrightarrow{s_m})$ the term $f(\overrightarrow{s_1} \cdot \overrightarrow{s_2} \cdot \ldots \cdot \overrightarrow{s_m})$.

Let $\overrightarrow{z} = (z_1, \ldots, z_n)$ be a sequence of terms. We define the sequence $\overrightarrow{\mathsf{lin}}(\overrightarrow{z})$ by induction on the length of $\overrightarrow{z}$

- if $|\overrightarrow{z}| \leq 1$, then $\overrightarrow{\mathsf{lin}}(\overrightarrow{z}) = \overrightarrow{z}$,
- if $|\overrightarrow{z}| > 1$, then $\overrightarrow{\mathsf{lin}}(\overrightarrow{z}) := \overrightarrow{\mathsf{lin}}(z_1, \ldots, z_{n-1}) \cdot (z_n)$, if $\forall j < n, z_j \neq z_n$ and $\overrightarrow{\mathsf{lin}}(\overrightarrow{z}) := \overrightarrow{\mathsf{lin}}(z_1, \ldots, z_{n-1})$ if $\exists j < n, z_j = z_n$

So, $\overrightarrow{\mathsf{lin}}(\overrightarrow{z})$ is obtained from $\overrightarrow{z}$ by erasing all terms that are present more than once, keeping the left-most copy.

Among all variables in $\mathcal{V}$ there is a special one $\square$. A term containing $n > 0$ occurrences of $\Box$ is called a *context*. A context is usually denoted as $C[]_{u_1,\dots,u_n}$ where the u_i are the positions of the $\Box$ given in lexicographic order. We denote by $C[t_1,\dots,t_n]_{u_1,\dots,u_n}$ the term obtained from $C[]_{u_1,\dots,u_n}$ by replacing, for every $i \in \{1,\dots,n\}$, the symbol $\Box$ at position u_i by the term t_i. Let t be a term, and $\{u_1,\dots,u_n\} \subset \mathcal{P}\mathsf{os}(t)$ be a set of incomparable positions given in lexicographic order. We denote by $t[]_{u_1,\dots,u_m}$ the context obtained from t by replacing each subterm t/u_i at a position u_i by a leaf labeled by $\Box$.

A *rewrite rule* over the signature $\mathcal{F}$ is a pair $l \to r$ of terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$ which satisfy $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(l)$. We call l (resp. r) the *left-handside* (resp. *right-handside*) of the rule (*lhs* and *rhs* for short). A rule is *linear* if both its left and right-handsides are linear. A rule is *left-linear* if its left-handside is linear.

A *term rewriting system* (TRS for short) is a pair $(\mathcal{R},\mathcal{F})$ where $\mathcal{F}$ is a signature and $\mathcal{R}$ a finite set of rewrite rules over the signature $\mathcal{F}$. When $\mathcal{F}$ is clear from the context or contains exactly the symbols of $\mathcal{R}$, we may omit $\mathcal{F}$ and write simply $\mathcal{R}$. We denote by $\mathsf{LHS}(\mathcal{R})$ the set of lhs of $\mathcal{R}$, and by $\mathsf{RHS}(\mathcal{R})$ the set of rhs of $\mathcal{R}$.

Given a TRS $(\mathcal{R},\mathcal{F})$, and two ground terms s,t, we say that there exists an *$\mathcal{R}$-rewriting step* between s and t in $\mathcal{R}$ and write $s \to_{\mathcal{R},l\to r,\sigma,v} t$ if there exist a position $v \in \mathcal{P}\mathsf{os}(s)$, a rule $l \to r \in \mathcal{R}$, and a substitution σ such that $s = s[l\sigma]_v$ and $t = s[r\sigma]_v$. We may omit $\mathcal{R}, l \to r, \sigma$, or v when they are clear from the context. The term $l\sigma$ is called a *redex* of s, and $r\sigma$ is called the *contractum* of $r\sigma$. We denote by $\to^+$ the transitive closure of $\to$, by $\to^{0,1}$ its reflexive closure, and by $\to^*$ its reflexive and transitive closure. We say that there exists a derivation from s to t in $\mathcal{R}$ when $s \to_{\mathcal{R}}^* t$. The *length* of a derivation is the number of step in this derivation. A n-step derivation from s to t is denoted by $s \to^n t$. Let $d = s_0 \to_{\mathcal{R}} s_1 \to \dots \to_{\mathcal{R}} s_n$ be a derivation and let $i,j \in \mathbb{N}$, $i \leq j$. We denote by $d_{i,j}$ the derivation $s_i \to_{\mathcal{R}} s_{i+1} \to \dots \to_{\mathcal{R}} s_j$. More generally, the notation defined in [8] will be used in proofs.

A TRS is *linear* (resp. *left-linear*) if each of its rules is linear (resp. left-linear). A TRS $\mathcal{R}$ is *growing* [7] if every variable of a right-handside is at depth at most 1 in the corresponding left-handside.

We shall consider finite bottom-up term (tree) automata only [1] (which we abbreviate to *f.t.a.*). An automaton $\mathcal{A}$ is given by a 4-tuple $(\mathcal{F},\mathcal{Q},\mathcal{Q}_f,\Gamma)$ where $\mathcal{F}$ is a signature, $\mathcal{Q}$ is a set of symbols of arity 0, called the set of states and such that $\mathcal{Q} \cap \mathcal{F}_0 = \emptyset$, $\mathcal{Q}_f \subseteq \mathcal{Q}$ is the set of final states, Γ is the set of transitions. Every element of Γ has either the form $q \to r$ for some $q,r \in \mathcal{Q}$, or $f(q_1,\dots,q_m) \to q$ for some $m \geq 0, f \in \mathcal{F}_m, q_1,\dots,q_m \in \mathcal{Q}$. Note that we can have rules of the form $c \to q$ with $c \in \mathcal{F}_0$, and $q \in \mathcal{Q}$. We may consider *f.t.a.* over an infinite signature. The set of rules Γ can be viewed as a TRS over the signature $\mathcal{F} \cup \mathcal{Q}$. We then denote by $\to_{\mathcal{A}}$ the one-step rewriting relation generated by Γ. Given an automaton $\mathcal{A}$, the set of terms accepted by $\mathcal{A}$ is defined by: $\mathcal{L}(\mathcal{A}) := \{t \in \mathcal{T}(\mathcal{F}) \mid \exists q \in \mathcal{Q}_f, t \to_{\mathcal{A}}^* q\}$. A set of terms T is *recognizable* if there exists a term automaton $\mathcal{A}$ such that $T = \mathcal{L}(\mathcal{A})$. The automaton $\mathcal{A}$ is called *deterministic* if there is no rule of the form $q \to r$ for some $q,r \in \mathcal{Q}$ and for every $t, \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}), u, u' \in \mathcal{Q}$, $(t \to u \in \Gamma \;\&\; t \to u' \in \Gamma) \Rightarrow (u = u')$. The automaton $\mathcal{A}$ is called *complete* if for every $m \geq 0$, $f \in \mathcal{F}_m$ and m-tuple of states $(q_1,\dots,q_m) \in \mathcal{Q}^m$, there exists $q \in \mathcal{Q}$ such that $f(q_1,\dots,q_m) \to q \in \Gamma$.

Ground tree transducers have been introduced in [9]. A *ground tree transducer* (GTT) is a pair $V = (\mathcal{A}_1,\mathcal{A}_2)$ of f.t.a. automata over a signature $\mathcal{F}$. Let $\mathcal{A}_1 = (\mathcal{F},\mathcal{Q}_1,\mathcal{Q}_{1f},\Gamma_1)$, $\mathcal{A}_2 = (\mathcal{F},\mathcal{Q}_2,\mathcal{Q}_{2f},\Gamma_2)$. The relation recognized by V is the set $\mathcal{L}(V) = \{(t,t') \mid t,t' \in \mathcal{T}(\mathcal{F}), \exists s \in \mathcal{T}(\mathcal{F} \cup (\mathcal{Q}_1 \cap \mathcal{Q}_2)),\ t \to_{\mathcal{A}_1}^* s, t' \to_{\mathcal{A}_2}^* s\}$. A set $T \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ is said to be *recognizable* by a GTT if there exists a GTT V such that $T = \mathcal{L}(V)$. The reflexive and transitive closure of the relation $\mathcal{L}(V)$ is recognizable by a GTT (see e.g. [1]). A *ground recognizable*

TRS (GRS) $(\mathcal{F}, \mathcal{G})$ is a (possibly infinite) TRS of the form $\mathcal{G} = \{l \to r \mid i \in I, l \in R_i, r \in K_i\}$, where I is a finite set, R_i and K_i for all $i \in I$ are recognizable sets of terms over $\mathcal{F}$. One can easily check that the relation $\to_{\mathcal{G}}^*$ is recognizable by a GTT.

Given a TRS $\mathcal{R}$ and a set of terms T, we define $(\to_{\mathcal{R}}^*)[T] = \{s \in \mathcal{T}(\mathcal{F}) \mid s \to_{\mathcal{R}}^* t$ for some $t \in T\}$ and $[T](\to_{\mathcal{R}}^*) = \{s \in \mathcal{T}(\mathcal{F}) \mid t \to_{\mathcal{R}}^* s$ for some $t \in T\}$. A TRS $\mathcal{R}$ is *recognizability preserving* if $[T](\to_{\mathcal{R}}^*)$ is recognizable for every recognizable T. A TRS $\mathcal{R}$ is *inverse recognizability preserving* if $(\to_{\mathcal{R}}^*)[T]$ is recognizable for every recognizable T or equivalently if $\mathcal{R}^{-1}$ is recognizability preserving.

We shall illustrate many of our definition with the following left-linear TRS $(\mathcal{F}_1, \mathcal{R}_1)$ and the following complete deterministic automaton $\mathcal{A}_1$.

▶ **Example 2.1.** $\mathcal{F}_1 = \{\mathsf{a}, \mathsf{b}, \mathsf{f}(), \mathsf{h}(), \mathsf{g}(,), \mathsf{i}(,)\}$ is a signature, $\{x, y\}$ is a set of variables, $\mathcal{R}_1 = \{\mathsf{a} \to \mathsf{b}, \mathsf{f}(x) \to \mathsf{g}(x, x), \mathsf{h}(\mathsf{b}) \to \mathsf{b}, \mathsf{g}(\mathsf{h}(x), y) \to \mathsf{i}(x, y)\}$ is a set of rules, $\mathcal{A}_1 = (\mathcal{F}, \mathcal{Q}_{\mathcal{A}_1}, \{\mathsf{q}_f\}, \Gamma_{\mathcal{A}_1})$ with $\mathcal{Q}_{\mathcal{A}_1} = \{\mathsf{q}_f, \mathsf{q}_\mathsf{a}, \mathsf{q}_\mathsf{b}\, \mathsf{q}_\perp\}$, $\Gamma_{\mathcal{A}_1} = \{\mathsf{a} \to \mathsf{q}_\mathsf{a}, \mathsf{b} \to \mathsf{q}_\mathsf{b}, \mathsf{h}(\mathsf{q}_\mathsf{a}) \to \mathsf{q}_\mathsf{a}, \mathsf{h}(\mathsf{q}_\mathsf{b}) \to \mathsf{q}_\mathsf{b}, \mathsf{h}(\mathsf{q}_\perp) \to \mathsf{q}_\perp, \mathsf{i}(\mathsf{q}_\mathsf{a}, \mathsf{q}_\mathsf{b}) \to \mathsf{q}_f\} \cup \{\mathsf{f}(\mathsf{q}) \to \mathsf{q}_\perp \mid \mathsf{q} \in \mathcal{Q}_{\mathcal{A}_1}\} \cup \{\mathsf{g}(\mathsf{q}, \mathsf{q}') \to \mathsf{q}_\perp \mid \mathsf{q}, \mathsf{q}' \in \mathcal{Q}_{\mathcal{A}_1}\}$ $\cup \{\mathsf{i}(\mathsf{q}, \mathsf{q}') \to \mathsf{q}_\perp \mid \mathsf{q}, \mathsf{q}' \in \mathcal{Q}_{\mathcal{A}_1}, (\mathsf{q}, \mathsf{q}') \neq (\mathsf{q}_\mathsf{a}, \mathsf{q}_\mathsf{b})\}$. We have $\mathcal{L}(\mathcal{A}_1) = \{\mathsf{i}(t_1, t_2) \mid t_1 \in \{\mathsf{a}, \mathsf{h}(\mathsf{a}), \ldots, \mathsf{h}(\mathsf{h}(\ldots(\mathsf{a}))), \ldots\}, t_2 \in \{\mathsf{b}, \mathsf{h}(\mathsf{b}), \ldots, \mathsf{h}(\mathsf{h}(\ldots(\mathsf{b}))), \ldots\}\}$

3 The TRS $\mathcal{E}$

From now on, until the end of this paper, we denote by $\mathcal{F}$ a finite signature, and by $\mathcal{R}$ a left-linear rewrite TRS over $\mathcal{F}$. Let $E \notin \mathcal{F}$ be a fresh symbol of arity 2.

▶ **Definition 3.1.** Let $x, y \in \mathcal{V}$. The left-linear term rewrite TRS $\mathcal{E}$ is the system over $\mathcal{F} \cup \{E\}$ with the rules

$$x \to E(x, x) \quad (1) \qquad E(x, y) \to x \text{ and } E(x, y) \to y \quad (2)$$

Rule (1) is the *introduction rule*. Rules (2) are the *selection rules*. Note that the TRS $\mathcal{R} \cup \mathcal{E}$ is left-linear. This TRS will be used to define bounded rewriting (section 4) and has the following property.

▶ **Proposition 3.2.** *Let $s, t \in \mathcal{T}(\mathcal{F})$. We have $s \to_{\mathcal{R}}^* t$ iff $s \to_{\mathcal{R} \cup \mathcal{E}}^* t$.*

4 Bounded rewriting

4.1 Marked Terms

We define the signature of marked symbols: $\mathcal{F}^{\mathbb{N}} = \{f^i \mid i \in \mathbb{N}, f \in \mathcal{F}\}$. The operation $\mathsf{m}()$ returns the mark of a marked symbol: for $f \in \mathcal{F}, i \in \mathbb{N}, \mathsf{m}(f^i) = i$. We extend this operation to the symbol E: $\mathsf{m}(E) = 0$, and to variables: $\forall x \in \mathcal{V}, \mathsf{m}(x) = 0$. We denote by $\mathcal{F}^{\leq k}$ the signature $\mathcal{F}^{\leq k} = \{f^i \mid i \in \{0, \ldots, k\}, f \in \mathcal{F}\}$ and by $\mathcal{F}^{\geq k}$ the signature $\mathcal{F}^{\geq k} = \{f^i \mid i \geq k, f \in \mathcal{F}\}$. Marked terms are elements of $\mathcal{T}_M := \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\}, \mathcal{V})$. The operation m extends to marked terms: if $t \in \mathcal{V}$, $\mathsf{m}(t) = 0$, otherwise, $\mathsf{m}(t) = \mathsf{m}(\mathsf{root}(t))$. We use $\mathsf{mmax}(t)$ to denote the maximal mark on t. We denote by t^i the term obtained by setting all the marks in t at a position $u \in \mathcal{P}\mathsf{os}_{\setminus E}(t)$ to i. We extend this notation to sets of terms ($S^i = \{s^i | s \in S\}$), and to substitutions ($\sigma^i : x \to (x\sigma)^i$). For every $f \in \mathcal{F}$, we identify f^0 and f ; it follows that $\mathcal{T}(\mathcal{F}) \subset \mathcal{T}(\mathcal{F}^{\mathbb{N}})$, and $\mathcal{T}(\mathcal{F} \cup \{E\}) \subset \mathcal{T}_M$. We denote by $\bar{t}$ (or $\widehat{t}$) a marked term such that $\bar{t}^0 = t$. The same rule will apply with substitutions and contexts. For a set of terms $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$, we denote by $T^{\mathbb{N}}$ the set of terms $\{\bar{t} \in \mathcal{T}_M \mid t \in T\}$.

▶ **Example 4.1.** $\mathsf{m}(\mathsf{f}^3(E(\mathsf{a}^4, \mathsf{b}^1))) = 3, \mathsf{m}(x) = 0, \mathsf{m}(E(\mathsf{a}^1, \mathsf{b}^2)) = 0, \mathsf{mmax}(\mathsf{f}^3(E(\mathsf{a}^4, \mathsf{b}^1))) = 4, \mathsf{mmax}(E(\mathsf{a}^1, \mathsf{b}^2)) = 2$, and if $\bar{t} = \mathsf{g}^3(\mathsf{a}^0, E(x, \mathsf{b}^2))$, then $\bar{t}^1 = \mathsf{g}^1(\mathsf{a}^1, E(x, \mathsf{b}^1))$.

From now on and until the end of section 5, let us fix, a language $T \subseteq \mathcal{T}(\mathcal{F})$ recognized by a complete deterministic automaton, $\mathcal{A} = (\mathcal{F}, \mathcal{Q}_{\mathcal{A}}, \mathcal{Q}_{f,\mathcal{A}}, \Gamma_{\mathcal{A}})$. We start giving some technical definitions and lemmas.

The automaton $\mathcal{A}_{\mathcal{P}}$

▶ **Definition 4.2.** We denote by $\overline{\mathcal{A}}$ the (infinite) automaton $\overline{\mathcal{A}} := (\mathcal{F}^{\mathbb{N}}, \mathcal{Q}_{\mathcal{A}}, \mathcal{Q}_{f,A}, \Gamma_{\overline{\mathcal{A}}})$, with: $\Gamma_{\overline{\mathcal{A}}} = \{f^i(q_1, \ldots, q_n) \to q \mid i \in \mathbb{N}, (f(q_1, \ldots, q_n) \to q) \in \Gamma_{\mathcal{A}}\}$.

Note that $\overline{\mathcal{A}}$ is deterministic and complete over $\mathcal{F}^{\mathbb{N}}$, and contains all the rules $c^i \to q$ for $i \in \mathbb{N}$, $c \in \mathcal{F}_0$, $(c \to q) \in \Gamma_{\mathcal{A}}$.

▶ **Lemma 4.3.** *Let $\bar{t}, \widehat{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}), q \in \mathcal{Q}_{\mathcal{A}}, m > 0$. If $\bar{t} \to_{\overline{\mathcal{A}}}^{m} q$ then $\widehat{t} \to_{\overline{\mathcal{A}}}^{m} q$.*

▶ **Definition 4.4.** We define the (infinite) automaton $\mathcal{A}_{\mathcal{P}} := \{\mathcal{F}^{\mathbb{N}} \cup \{E\}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{f,\mathcal{P}}, \Gamma_{\mathcal{P}}\}$ built from $\overline{\mathcal{A}}$, where $\mathcal{Q}_{\mathcal{P}} = \mathcal{P}(\mathcal{Q}_{\mathcal{A}})$, $\mathcal{Q}_{f,\mathcal{P}} = \{\{q\} \mid q \in \mathcal{Q}_{f,\mathcal{A}}\}$, $\Gamma_{\mathcal{P}} = \{E(S_1, S_2) \to S_1 \cup S_2 \mid S_1, S_2 \in \mathcal{Q}_{\mathcal{P}}\} \cup \{f^i(S_1, \ldots, S_n) \to S_{f^i(S_1,\ldots,S_n)} \mid i \in \mathbb{N}, f \in \mathcal{F}_n, S_1, \ldots, S_n \in \mathcal{Q}_{\mathcal{P}}\}$ with $S_{f^i(S_1,\ldots,S_n)} = \{q \in \mathcal{Q}_{\mathcal{A}} \mid \forall j \in \{1, \ldots, n\}, \exists s_j \in S_j \text{ s.t. } f^i(s_1, \ldots, s_n) \to q \in \Gamma_{\overline{\mathcal{A}}}\}$

Note that $\mathcal{A}_{\mathcal{P}}$ contains all the rules $c^i \to \{q\}$ for $c \in \mathcal{F}_0$, $i \in \mathbb{N}$, $c \to q \in \Gamma_{\mathcal{A}}$, and that $\mathcal{A}_{\mathcal{P}}$ is deterministic and complete over $\mathcal{F}^{\mathbb{N}} \cup \{E\}$. The language recognized by $\mathcal{A}_{\mathcal{P}}$ is $\mathcal{L}(\mathcal{A}_{\mathcal{P}}) = T$. Every term $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}})$ has a unique normal form reduced to a state and denoted by $\mathsf{nf}_{\mathcal{A}_{\mathcal{P}}}(\bar{t})$. Since $\mathcal{A}_{\mathcal{P}}$ erases marks $\mathsf{nf}_{\mathcal{A}_{\mathcal{P}}}(\bar{t}) = \mathsf{nf}_{\mathcal{A}_{\mathcal{P}}}(t)$. We extend the operation m to $\mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}, \mathcal{V})$ by setting $\mathsf{m}(S) = 0$, for all $S \in \mathcal{Q}_{\mathcal{P}}$.

▶ **Example 4.5.** Let us consider the automaton $\mathcal{A}_1$ from example 2.1. The following rules belong to the set of rules of $\mathcal{A}_{1\mathcal{P}}$: $E(\{\mathsf{q_a}\}, \{\mathsf{q_b}\}) \to \{\mathsf{q_a}, \mathsf{q_b}\}, \mathsf{a}^3 \to \{\mathsf{q_a}\}, \mathsf{h}^2(\{\mathsf{q_a}, \mathsf{q}_\perp\}) \to \{\mathsf{q_a}, \mathsf{q}_\perp\}, \mathsf{i}^1(\{\mathsf{q_a}, \mathsf{q_b}\}, \{\mathsf{q_b}\}) \to \{\mathsf{q}_\perp, \mathsf{q}_f\}$, $E(\{\mathsf{q_a}, \mathsf{q}_f\}, \{\mathsf{q_b}, \mathsf{q_a}\}) \to \{\mathsf{q_a}, \mathsf{q_b}, \mathsf{q}_f\}$, $\mathsf{g}^4(\{\mathsf{q_a}\}, \{\mathsf{q_b}\}) \to \{\mathsf{q}_\perp\}$.

▶ **Definition 4.6.** We denote by $\mathcal{A}_{\mathcal{P}}^+$ the automaton $(\mathcal{F}^{\mathbb{N}} \cup \{E\}, \mathcal{Q}_{\mathcal{P}}, \mathcal{Q}_{f,\mathcal{P}}, \Gamma_{\mathcal{P}}^+)$, where $\Gamma_{\mathcal{P}}^+ = \Gamma_{\mathcal{P}} \cup \{S \to S' \mid S \in \mathcal{Q}_{\mathcal{P}}, S' \subset S\}$.

The language recognized by the automaton $\mathcal{A}_{\mathcal{P}}^+$ is $\mathcal{L}(\mathcal{A}_{\mathcal{P}}^+) = (\to_{\{E(x,y)\to x, E(x,y)\to y\}}^*)[T]$.

▶ **Definition 4.7.** For an automaton $\mathcal{B} = (\mathcal{F}^{\mathbb{N}} \cup \{E\}, \mathcal{Q}_{\mathcal{B}}, \mathcal{Q}_{\mathcal{B},f}, \Gamma_{\mathcal{B}})$ and for every $n \in \mathbb{N}$, we denote by $\mathcal{B}^{\leq n}$ (respectively $\mathcal{B}^{\geq n}$) the automaton $\mathcal{B}^{\leq n} := (\mathcal{F}^{\leq n} \cup \{E\}, \mathcal{Q}_{\mathcal{B}}, \mathcal{Q}_{\mathcal{B},f}, \Gamma_{\mathcal{B}}{}^{\leq n})$ (resp. $\mathcal{B}^{\geq n} := (\mathcal{F}^{\geq n} \cup \{E\}, \mathcal{Q}_{\mathcal{B}}, \mathcal{Q}_{\mathcal{B},f}, \Gamma_{\mathcal{B}}{}^{\geq n})$) where $\Gamma_{\mathcal{B}}{}^{\leq n} := \{l \to r \in \Gamma_{\mathcal{B}} \mid l, r \in \mathcal{T}(\mathcal{F}^{\leq n} \cup \{E\} \cup \mathcal{Q}_{\mathcal{B}})\}$ (resp. $\Gamma_{\mathcal{B}}{}^{\geq n} := \{l \to r \in \Gamma_{\mathcal{B}} \mid l, r \in \mathcal{T}(\mathcal{F}^{\geq n} \cup \{E\} \cup \mathcal{Q}_{\mathcal{B}})\}$).

Note that for all $n \in \mathbb{N}$, $\mathcal{A}_{\mathcal{P}}^+{}^{\leq n}$ and $\mathcal{A}_{\mathcal{P}}{}^{\leq n}$ are finite automata. The automaton $\mathcal{A}_{\mathcal{P}}^+{}^{\geq k+1}$ will be used to define the top of a marked term, i.e. the top part of the term that could be used in a k-bounded derivation (see definition 5.3). The automaton $\mathcal{A}_{\mathcal{P}}^{+\leq k}$ will be a part of the GRS $\mathcal{G}$ used to simulate k-bounded derivations (see definition 5.15).

▶ **Definition 4.8.** For all linear terms $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}, \mathcal{V})$, for all $n \in \mathbb{N}$, we define $\bar{t} \odot n$ as the unique marked term such that $(\bar{t} \odot n)^0 = t$, and, $\forall u \in \mathcal{P}\mathsf{os}_{\mathcal{F}^{\mathbb{N}}}(t)$, $\mathsf{m}(\bar{t} \odot n/u) = \max(\mathsf{m}(\bar{t}/u), \mathsf{Card}(\mathcal{P}\mathsf{os}_{\setminus E}{}^{\prec u}(t)) + n)$.

▶ **Example 4.9.** Let $\overline{t_1} = E(\mathsf{f}^0(E(\mathsf{a}^0, E(\{\mathsf{q_a}, \mathsf{q_b}\}, \mathsf{h}^0(\mathsf{b}^0)))), \mathsf{a}^0)$, $\overline{t_2} = \mathsf{g}^0(E(x, \mathsf{f}^3(\mathsf{a}^1)), E(\mathsf{f}^2(\mathsf{b}^0), y))$, $\overline{t_3} = \mathsf{g}^1(\mathsf{g}^2(\{\mathsf{q_a}\}, E(\mathsf{a}^3, \mathsf{b}^1)), \mathsf{a}^0)$. We have $\overline{t_1} \odot 0 = E(\mathsf{f}^0(E(\mathsf{a}^1, E(\{\mathsf{q_a}, \mathsf{q_b}\}, \mathsf{h}^1(\mathsf{b}^2)))), \mathsf{a}^1)$, $\overline{t_1} \odot 2 = E(\mathsf{f}^2(E(\mathsf{a}^3, E(\{\mathsf{q_a}, \mathsf{q_b}\}, \mathsf{h}^3(\mathsf{b}^4)))), \mathsf{a}^2)$, $\overline{t_2} \odot 0 = \mathsf{g}^0(E(x, \mathsf{f}^3(\mathsf{a}^2)), E(\mathsf{f}^2(\mathsf{b}^2), y))$, $\overline{t_2} \odot 1 = \mathsf{g}^1(E(x, \mathsf{f}^3(\mathsf{a}^3)), E(\mathsf{f}^2(\mathsf{b}^3), y))$, $\overline{t_3} \odot 0 = \mathsf{g}^1(\mathsf{g}^2(\{\mathsf{q_a}\}, E(\mathsf{a}^3, \mathsf{b}^2)), \mathsf{a}^1)$, $\overline{t_3} \odot 3 = \mathsf{g}^3(\mathsf{g}^4(\{\mathsf{q_a}\}, E(\mathsf{a}^5, \mathsf{b}^5)), \mathsf{a}^4)$.

We extend this notation to sets of marked terms ($S \odot n := \{s \odot n \mid s \in S\}$), and to marked substitution ($\sigma \odot n : x \mapsto x\sigma \odot n$).

4.2 Marked rewriting

From now on and until the end of this paper, let us fix an integer $k > 0$. We introduce here the rewrite relation $\circ\!\!\rightarrow$ between marked terms.

▶ **Definition 4.10** (Marked rewriting step). A ground marked term $\overline{s} \in \mathcal{T}_M$ rewrites to a ground marked term $\overline{t} \in \mathcal{T}_M$ in $\mathcal{R} \cup \mathcal{E}$ if there exist a rule $l \to r \in \mathcal{R} \cup \mathcal{E}$, a position $v \in \mathcal{P}\mathsf{os}(s)$, a marked term $\overline{l}$, and a marked substitution $\overline{\sigma}$ such that : $\overline{s} = \overline{s}[\overline{l\sigma}]_v, \overline{t} = \overline{s}[r(\overline{\sigma} \odot j)]_v$, where: $j = 0$, if $l \to r \in \mathcal{E}$, and $j = 1$, if $l \to r \in \mathcal{R}$. We then just write $\overline{s} \circ\!\!\rightarrow_{\mathcal{R}\cup\mathcal{E}, l\to r, \sigma, v} \overline{t}$.

We may omit $\mathcal{R} \cup \mathcal{E}$, $l \to r$, σ, or v when they are clear from the context. We use two different marking ($j = 0$ or $j = 1$) depending on the rule applied only to properly extend the notion of weakly bottom-up for linear TRS (defined in [4]) to left-linear TRS (see section 6). This notion is helpful to prove that several already known classes of TRS belong to the class of bounded TRS. Let us give some properties of marked derivations.

Associated marked derivation

Every derivation

$$d : s_0 = s_0[l_0\sigma_0]_{v_0} \to s_0[r_0\sigma_0]_{v_0} = s_1 \to \ldots \to s_{n-1}[r_{n-1}\sigma_{n-1}]_{v_{n-1}} = s_n. \tag{3}$$

is mapped to a marked derivation $\overline{d}$ called the *marked derivation associated* to d

$$\overline{d} : \overline{s_0} = \overline{s_0}[\overline{l_0\sigma_0}]_{v_0} \circ\!\!\rightarrow \overline{s_0}[r_0(\overline{\sigma_0} \odot i_0)]_{v_0} = \overline{s_1} \circ\!\!\rightarrow \ldots \circ\!\!\rightarrow \overline{s_{n-1}}[r_{n-1}(\overline{\sigma_{n-1}} \odot i_{n-1})]_{v_{n-1}} = \overline{s_n} \tag{4}$$

where $\overline{s_0} = s_0$. Note that this map is unique since the position v_j, the rule (l_j, r_j), and $\overline{s_j}$ completely determine $\overline{s}_{j+1}$. From now on, each time we deal with a derivation $s \to^* t$ between two terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, we may implicitly decompose it as (3) where n is the length of the derivation, $s = s_0$ and $t = s_n$.

4.3 Bounded derivations and bounded systems

▶ **Definition 4.11** (Bounded derivations). A marked rewriting step $\overline{s} \circ\!\!\rightarrow_{\mathcal{R}\cup\mathcal{E}, l\to r, v} \overline{t}$ is *k-bounded* ($\mathsf{bo}(k)$) if $l \to r \in \mathcal{E}$ or if $l \to r \in \mathcal{R}$ and the following assertion holds:

$$(l \notin \mathcal{V} \Rightarrow \mathsf{mmax}(\overline{l}) \le k) \text{ and } (l \in \mathcal{V} \Rightarrow \sup(\{\mathsf{m}(\overline{s}/u) \mid u \prec v\}) \le k)$$

A marked derivation in $\mathcal{R} \cup \mathcal{E}$ is $\mathsf{bo}(k)$ if all its rewriting steps are $\mathsf{bo}(k)$. A derivation in $\mathcal{R} \cup \mathcal{E}$ is $\mathsf{bo}(k)$ if the associated marked derivation is $\mathsf{bo}(k)$.
A derivation $s \to^*_{\mathcal{R}} t, s, t \in \mathcal{T}(\mathcal{F})$ is *k-bounded convertible* ($\mathsf{boc}(k)$) if there exists a $\mathsf{bo}(k)$-derivation $s \to^*_{\mathcal{R}\cup\mathcal{E}} t$ in $\mathcal{R} \cup \mathcal{E}$. The left-linear TRS $\mathcal{R}$ is k-bounded if every derivation in $\mathcal{R}$ is $\mathsf{boc}(k)$. We denote by $\mathsf{BO}(k)$ the class of k-bounded TRS and by BO the class $\bigcup_{k\in\mathbb{N}} \mathsf{BO}(k)$.

▶ **Example 4.12.** Let $\mathcal{R}_1$ be the TRS of example 2.1. The following derivation $d : \mathsf{f}(\mathsf{h}(\mathsf{a})) \rightarrow_{\mathsf{f}(x)\rightarrow\mathsf{g}(x,x)} \mathsf{g}(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{a})) \rightarrow_{\mathsf{a}\rightarrow\mathsf{b}} \mathsf{g}(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{b})) \rightarrow_{\mathsf{g}(\mathsf{h}(x),y)\rightarrow\mathsf{i}(x,y)} \mathsf{i}(\mathsf{a},\mathsf{h}(\mathsf{b})) \rightarrow_{\mathsf{h}(\mathsf{b})\rightarrow\mathsf{b}} \mathsf{i}(\mathsf{a},\mathsf{b})$ is $\mathsf{bo}(2)$ since in the associated marked derivation $\overline{d} : \mathsf{f}^0(\mathsf{h}^0(\mathsf{a}^0)) \circ\!\!\rightarrow \mathsf{g}^0(\mathsf{h}^1(\mathsf{a}^2),\mathsf{h}^1(\mathsf{a}^2)) \circ\!\!\rightarrow \mathsf{g}^0(\mathsf{h}^1(\mathsf{a}^2),\mathsf{h}^1(\mathsf{b}^0)) \circ\!\!\rightarrow \mathsf{i}^0(\mathsf{a}^2,\mathsf{h}^1(\mathsf{b}^2)) \circ\!\!\rightarrow \mathsf{i}^0(\mathsf{a}^2,\mathsf{b}^0)$ the maximal mark that appears on a lhs is 2. The derivation d is $\mathsf{boc}(1)$ since there is a derivation $d' : \mathsf{f}(\mathsf{h}(\mathsf{a})) \rightarrow_{x\rightarrow E(x,x)} \mathsf{f}(\mathsf{h}(E(\mathsf{a},\mathsf{a})))$ $\rightarrow_{\mathsf{a}\rightarrow\mathsf{b}} \mathsf{f}(\mathsf{h}(E(\mathsf{a},\mathsf{b}))) \rightarrow_{x\rightarrow E(x,x)} \mathsf{f}(E(\mathsf{h}(E(\mathsf{a},\mathsf{b})),\mathsf{h}(E(\mathsf{a},\mathsf{b})))) \rightarrow_{E(x,y)\rightarrow x} \mathsf{f}(E(\mathsf{h}(\mathsf{a}),\mathsf{h}(E(\mathsf{a},\mathsf{b}))))$ $\rightarrow_{E(x,y)\rightarrow y} \mathsf{f}(E(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{b}))) \rightarrow_{\mathsf{h}(\mathsf{b})\rightarrow\mathsf{b}} \mathsf{f}(E(\mathsf{h}(\mathsf{a}),\mathsf{b})) \rightarrow_{\mathsf{f}(x)\rightarrow\mathsf{g}(x,x)} \mathsf{g}(E(\mathsf{h}(\mathsf{a}),\mathsf{b}),E(\mathsf{h}(\mathsf{a}),\mathsf{b}))$ $\rightarrow_{E(x,y)\rightarrow x} \mathsf{g}(\mathsf{h}(\mathsf{a}),E(\mathsf{h}(\mathsf{a}),\mathsf{b})) \rightarrow_{E(x,y)\rightarrow y} \mathsf{g}(\mathsf{h}(\mathsf{a}),\mathsf{b}) \rightarrow_{\mathsf{g}(\mathsf{h}(x),y)\rightarrow\mathsf{i}(x,y)} \mathsf{i}(\mathsf{a},\mathsf{b})$ which is $\mathsf{bo}(1)$ since in the associated marked derivation $\overline{d'} : \mathsf{f}^0(\mathsf{h}^0(\mathsf{a}^0)) \rightarrow_{x\rightarrow E(x,x)} \mathsf{f}^0(\mathsf{h}^0(E(\mathsf{a}^0,\mathsf{a}^0))) \rightarrow_{\mathsf{a}\rightarrow\mathsf{b}}$ $\mathsf{f}^0(\mathsf{h}^0(E(\mathsf{a}^0,\mathsf{b}^0))) \rightarrow_{x\rightarrow E(x,x)} \mathsf{f}^0(E(\mathsf{h}^0(E(\mathsf{a}^1,\mathsf{b}^1)),\mathsf{h}^0(E(\mathsf{a}^1,\mathsf{b}^1)))) \rightarrow_{E(x,y)\rightarrow x}$ $\mathsf{f}^0(E(\mathsf{h}^0(\mathsf{a}^1),\mathsf{h}(E(\mathsf{a}^1,\mathsf{b}^1)))) \rightarrow_{E(x,y)\rightarrow y} \mathsf{f}^0(E(\mathsf{h}^0(\mathsf{a}^1),\mathsf{h}^0(\mathsf{b}^1))) \rightarrow_{\mathsf{h}(\mathsf{b})\rightarrow\mathsf{b}} \mathsf{f}^0(E(\mathsf{h}^0(\mathsf{a}^1),\mathsf{b}^0))$ $\rightarrow_{\mathsf{f}(x)\rightarrow\mathsf{g}(x,x)} \mathsf{g}^0(E(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1),E(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1)) \rightarrow_{E(x,y)\rightarrow x} \mathsf{g}^0(\mathsf{h}^1(\mathsf{a}^2),E(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1))$ $\rightarrow_{E(x,y)\rightarrow y} \mathsf{g}^0(\mathsf{h}^1(\mathsf{a}^2),\mathsf{b}^1) \rightarrow_{\mathsf{g}(\mathsf{h}(x),y)\rightarrow\mathsf{i}(x,y)} \mathsf{i}^0(\mathsf{a}^2,\mathsf{b}^1)$ the maximal mark that appears on a lhs is 1.

Notation: Let $\overline{s},\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\})$.

- $\overline{s} \;{}_{k}\!\circ\!\!\rightarrow^*_{\mathcal{R}\cup\mathcal{E}} \overline{t}$ means that there exists a marked $\mathsf{bo}(k)$-derivation from $\overline{s}$ to $\overline{t}$.
- $s \;{}_{k}\!\!\rightarrow^*_{\mathcal{R}} t$ means that there exists a $\mathsf{boc}(k)$ derivation $s \rightarrow^*_{\mathcal{R}} t$.

Let us recall the notion of linear bo rewriting defined in [3] and which will be denoted here lbo to avoid confusion.

▶ **Definition 4.13.** Let $\mathcal{R}$ be a linear *TRS*. A marked rewriting step $\overline{s} \circ\!\!\rightarrow_{\mathcal{R},l\rightarrow r,v} \overline{t}$ is *linear k-bounded* ($\mathsf{lbo}(k)$) if the following assertion holds

$$(l \notin \mathcal{V} \Rightarrow \mathsf{mmax}(\overline{l}) \leq k), \text{ and } (l \in \mathcal{V} \Rightarrow \sup(\{\mathsf{m}(\overline{s}/u) \mid u \prec v\}) \leq k) \tag{5}$$

A marked derivation is $\mathsf{lbo}(k)$ if all its rewriting steps are $\mathsf{bo}(k)$. A derivation in $\mathcal{R}$ is $\mathsf{lbo}(k)$ if the associated marked derivation is $\mathsf{bo}(k)$. The *TRS* $\mathcal{R}$ is linear k-bounded if every derivation $s \rightarrow_{\mathcal{R}} t$ can be replaced by a $\mathsf{lbo}(k)$ derivation from s to t. We denote by $\mathsf{LBO}(k)$ the class of linear k-bounded TRS and by LBO the class $\bigcup_{k\in\mathbb{N}} \mathsf{LBO}(k)$.

By definition, $\mathsf{LBO}(k) \subseteq \mathsf{BO}(k)$. Moreover, one can easily check that for every linear *TRS* $\mathcal{R}$, $\mathcal{R} \in \mathsf{LBO}(k)$ iff $\mathcal{R} \in \mathsf{BO}(k)$. Since the $\mathsf{LBO}(0)$ membership problem is undecidable, the $\mathsf{BO}(0)$ membership problem is undecidable too. Note that in the definition of an $\mathsf{lbo}(k)$ derivation, nothing requires the linear condition. But if we consider $\mathsf{lbo}(k)$-derivations for left-linear TRSs, then the class $\mathsf{LBO}(k)$ does not contains left-linear *TRSs* with non right-linear rules. This is illustrated in the following example.

▶ **Example 4.14.** Let $\mathcal{R}_2 = \{\mathsf{f}(x) \rightarrow \mathsf{g}(x,x), \mathsf{a} \rightarrow \mathsf{b}\}$ and let $k \in \mathbb{N}$. There is a $\mathsf{bo}(0)$-derivation $\mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a}))\ldots)) \rightarrow_{\mathcal{E}} \mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{a}))\ldots))) \rightarrow_{\mathsf{a}\rightarrow\mathsf{b}} \mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{b}))\ldots))) \rightarrow$ $\mathsf{g}(\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{b}))\ldots)),\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{b}))\ldots))) \rightarrow_{E(x,y)\rightarrow x} \mathsf{g}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a})\ldots)),\mathsf{f}(\ldots(\mathsf{f}(E(\mathsf{a},\mathsf{b}))\ldots)))$ $\rightarrow_{E(x,y)\rightarrow y} \mathsf{g}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a}))\ldots),\mathsf{f}(\ldots(\mathsf{f}(\mathsf{b}))\ldots))$ but there is no $\mathsf{lbo}(k)$ derivation from $\mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a}))\ldots))$ to $\mathsf{g}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a}))\ldots),\mathsf{f}(\ldots(\mathsf{f}(\mathsf{b}))\ldots))$. The *TRS* $\mathcal{R}_2$ is $\mathsf{bo}(0)$ but $\mathcal{R}_2$ does not belong to LBO.

4.4 Well-marked derivation

▶ **Definition 4.15** (well-marked). A term $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}, \mathcal{V})$ is *well-marked* if these two assertions holds

1. for all $w \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(s)$, for all $v \preceq w$, $\mathsf{m}(\overline{s}/v) \leq k$,
2. for all $w \in \mathcal{L}\mathsf{v}(s)\backslash\mathcal{P}\mathsf{os}_{\mathcal{V}}(s)$, one of these two assertions holds

a. $\mathsf{m}(\overline{s}/v) \leq k$ for all $v \preceq w$,
b. there exists $u \in \mathcal{P}\mathsf{os}_{\mathcal{F}^{\mathbb{N}}}{}^{\preceq w}(\overline{s})$ such that:
- $\mathsf{m}(\overline{s}/v) \leq k$ for all $v \prec u$,
- $\mathsf{m}(\overline{s}/v) = k + 1 + \mathsf{Card}(\mathcal{P}\mathsf{os}_{\setminus E}{}^{\prec v}(s)) - \mathsf{Card}(\mathcal{P}\mathsf{os}_{\setminus E}{}^{\prec u}(s))$, for all $v \in \mathcal{P}\mathsf{os}_{\mathcal{F}^{\mathbb{N}}}{}^{\preceq w}(s)$ such that $v \succeq u$.

A marked derivation is *well-marked* if every term in the derivation is well-marked

So, a term is well-marked if for every $w \in \mathcal{L}\mathsf{v}(t)$, the sequence of marks on the symbols of $\mathcal{F}$ that appear on the branch containing w has the form: $m_0, m_1, \ldots, m_n, k+1, k+2, \ldots, k+l$ with $m_i \leq k$ in case $2b$. is satisfied and $m_0, m_1, \ldots, m_n$ with $m_i \leq k$ in case 1. or $2a$. is satisfied. Note that an unmarked term is well-marked, and that condition $2a$. is equivalent to $\mathsf{m}(\overline{s}/v) \leq k$ for all $v \prec u$, $\mathsf{m}(\overline{s}/u) = k+1$ and $\mathsf{m}(\overline{s}/v) - \mathsf{m}(\overline{s}/u) = \mathsf{Card}(\mathcal{P}\mathsf{os}_{\setminus E}{}^{\prec v}(s)) - \mathsf{Card}(\mathcal{P}\mathsf{os}_{\setminus E}{}^{\prec u}(s))$ for all $v \in \mathcal{P}\mathsf{os}_{\mathcal{F}^{\mathbb{N}}}{}^{\preceq w}(s)$ such that $v \succeq u$.

▶ **Example 4.16.** Let $k = 3$ and let $\mathcal{R}_1$ and $\mathcal{A}_1$ be the TRS and the automaton from example 2.1. The terms $\mathsf{f}^1(E(\mathsf{f}^2(\mathsf{a}^2), x)), \mathsf{f}^0(E(\mathsf{f}^3(\mathsf{a}^4), x))$, $\mathsf{f}^0(\mathsf{f}^3(E(\mathsf{f}^4(\mathsf{a}^5), \mathsf{f}^3(\mathsf{a}^3)))), \mathsf{f}^2(E(\mathsf{f}^0(\mathsf{a}^4), x)), \mathsf{f}^4(\mathsf{f}^5(E(\mathsf{f}^6(\{\mathsf{q}_\mathsf{a}, \mathsf{q}_\perp\}), \mathsf{f}^6(\mathsf{b}^7))))$ are well-marked. The term $\mathsf{f}^4(E(\mathsf{f}^5(\mathsf{a}^6), x))$ is not since condition 1. is not satisfied. The term $\overline{t_1} = \mathsf{f}^2(\mathsf{f}^3(E(\mathsf{f}^4(\mathsf{a}^4), \mathsf{f}^3(\mathsf{a}^4))))$, is not well-marked since condition $2a$. and condition $2b$. are not satisfied ($\mathsf{m}(\overline{t_1}/000) = 4$ and $\mathsf{m}(\overline{t_1}/0000) = 4$).
The term $\overline{t_2} = \mathsf{f}^2(\mathsf{f}^3(E(\mathsf{f}^4(\mathsf{a}^6), \mathsf{f}^3(\{\mathsf{q}_\mathsf{b}, \mathsf{q}_f\}))))$ is not well-marked since condition $2a$. and condition $2b$. are not satisfied ($\mathsf{m}(\overline{t_2}/000) = 4$ and $\mathsf{m}(\overline{t_2}/0000) = 6$).

▶ **Lemma 4.17.** *A* $\mathsf{bo}(k)$*-derivation is well-marked iff it is starting on a well-marked term.*

5 Main result

The main theorem of this section (and of the paper) is the following.

▶ **Theorem 5.1.** *Let $\mathcal{R}$ be some (finite) left-linear rewriting* TRS *over a signature $\mathcal{F}$. Let T be some recognizable subset of $\mathcal{T}(\mathcal{F})$ and let $k > 0$. Then, the set $({}_k\!\!\to^*_{\mathcal{R}})[T]$ is recognizable too.*

To obtain this result, we simulate $\mathsf{bo}(k)$-derivations using a GTT. The construction of the proof can be divided into three steps:

- First, we define the top part $\mathsf{Top}(\overline{t})$ of a well-marked term $\overline{t}$ which is the only part of $\overline{t}$ that can be rewritten using a rule of $\mathcal{R}$ in a $\mathsf{bo}(k)$-derivation.
- Then we define a GRS $\mathcal{G}$ which has the following properties:
  - If $\overline{s} \to^*_{\mathcal{G}} \overline{t}$, then there exists $\overline{t'}$ such that $\overline{s}\ {}_k\!\!\circ\!\!\to^*_{\mathcal{R}} \overline{t'} \to^*_{\mathcal{A}_{\mathcal{P}}{}^+} \overline{t}$ (lifting rewriting in $\mathcal{G}$ to $\mathcal{R}$).
  - If $\overline{s}\ {}_k\!\!\circ\!\!\to^*_{\mathcal{R}} \overline{t}$ then $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{G}} \mathsf{Top}(\overline{t})$ (projecting rewriting in $\mathcal{R}$ to $\mathcal{G}$).
- From these two properties of $\mathcal{G}$ and using some technical lemmas, we obtain the simulation lemma 5.23. The relation $\to^*_{\mathcal{G}}$ is recognizable by a GTT, and since GTTs inverse preserve recognizability, we obtain theorem 5.1.

Top of a marked term

By definition of a $\mathsf{bo}(k)$-derivation, a symbol in a term $\overline{t}$ can match a *lhs* of a rule of $\mathcal{R}$ only if the mark that appears on this symbol is smaller or equal to k. This leads us to define the top part of a well-marked term $\overline{t}$ which is (intuitively) obtained by replacing all the useless subterms $\overline{t}/u$ by their normal form $\mathsf{nf}_{\mathcal{A}_{\mathcal{P}}}(\overline{t}/u)$.

▶ **Definition 5.2.** Let $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}, \mathcal{V})$ be well-marked. We define $\mathsf{Topd}(\bar{t})$ the top domain of $\bar{t}$ as: $u \in \mathsf{Topd}(\bar{t})$ iff $u \in \mathcal{P}\mathsf{os}(t)$ and $\forall v \prec u, \mathsf{m}(\bar{t}/v) \leq k$.

▶ **Definition 5.3.** Let $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}, \mathcal{V})$ be well-marked, and let A be a subdomain of $\mathcal{P}\mathsf{os}(t)$ such that $\mathsf{m}(\bar{t}/u) \leq k$ iff $u \in A$. We denote by $\mathsf{Red}_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}}(\bar{t}, A)$ the unique term such that

- $\mathcal{P}\mathsf{os}(\mathsf{Red}_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}}(\bar{t}, A)) = A$,
- $\bar{t} \to^*_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}} \mathsf{Red}_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}}(\bar{t}, A)$,
- for all $\overline{t'}$ such that $\mathcal{P}\mathsf{os}(\overline{t'}) = A$ and $\bar{t} \to^*_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}} \overline{t'}$, we have $\overline{t'} \to^*_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}} \mathsf{Red}_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}}(\bar{t}, A)$.

▶ **Definition 5.4.** Let $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\} \cup \mathcal{Q}_{\mathcal{P}}, \mathcal{V})$ be well-marked. We denote by $\mathsf{Top}(\bar{t})$ the term $\mathsf{Red}_{\mathcal{A}_{\mathcal{P}}^{\geq k+1}}(\bar{t}, \mathsf{Topd}(\bar{t}))$.

▶ **Example 5.5.** Let $k = 3$ and let $\mathcal{R}_1$ and $\mathcal{A}_1$ be the TRS and the automaton from example 2.1. Let $\overline{t_0} = \mathsf{f}^0(E(\{\mathsf{q_a}\}, \mathsf{g}^0(\mathsf{a}^0, \mathsf{b}^0)))$, $\overline{t_1} = \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^0(\mathsf{a}^3, \mathsf{b}^4)))$, $\overline{t_2} = \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^3(\mathsf{a}^4, \mathsf{b}^4)))$, $\overline{t_3} = \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^4(\mathsf{a}^5, \mathsf{b}^5)))$, $\overline{t_4} = \mathsf{f}^4(E(\{\mathsf{q_a}\}, \mathsf{g}^5(\mathsf{a}^6, \mathsf{b}^6)))$. Note that these terms are well-marked. We have $\mathsf{Topd}(\overline{t_0}) = \mathsf{Topd}(\overline{t_1}) = \mathsf{Topd}(\overline{t_2}) = \mathcal{P}\mathsf{os}(t_0)$, $\mathsf{Topd}(\overline{t_3}) = \{\epsilon, 0, 00, 01\}$, $\mathsf{Topd}(\overline{t_4}) = \{\epsilon\}$ and $\overline{t_0} \to^0_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{Top}(\overline{t_0}) = \overline{t_0}$, $\overline{t_1} \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{Top}(\overline{t_1}) = \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^0(\mathsf{a}^3, \{\mathsf{q_b}\})))$, $\overline{t_2} \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^3(\mathsf{a}^4, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{Top}(\overline{t_2}) = \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^3(\{\mathsf{q_a}\}, \{\mathsf{q_b}\})))$, $\overline{t_3} \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^4(\mathsf{a}^5, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^2(E(\{\mathsf{q_a}\}, \mathsf{g}^4(\{\mathsf{q_a}\}, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{Top}(\overline{t_3}) = \mathsf{f}^2(E(\{\mathsf{q_a}\}, \{\mathsf{q}_\perp\}))$, $\overline{t_4} \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^4(E(\{\mathsf{q_a}\}, \mathsf{g}^5(\mathsf{a}^6, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^4(E(\{\mathsf{q_a}\}, \mathsf{g}^5(\{\mathsf{q_a}\}, \{\mathsf{q_b}\}))) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^4(E(\{\mathsf{q_a}\}, \{\mathsf{q}_\perp\})) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{f}^4(\{\mathsf{q_a}, \mathsf{q}_\perp\}) \to_{\mathcal{A}_{1\mathcal{P}}^{\geq 4}} \mathsf{Top}(\overline{t_4}) = \{\mathsf{q}_\perp\}$

5.1 Definition of the GRS $\mathcal{G}$ used for the simulation

The set $\mathcal{B}_{\leq n}$, the operation $\odot_e$, and some other definitions

Let $\{e_i \mid i \geq 1\}$ be a new set of symbols such that $\mathsf{arity}(e_i) = i$, and $\mathcal{F} \cap \{e_i \mid i \geq 1\} = \emptyset$. Let $\mathcal{T}_e = \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{e_i \mid i \geq 1\})$. Let $\vec{t} = (\overline{t_1}, \ldots, \overline{t_n})$ be a sequence of terms in $\mathcal{T}_e$. We denote by $e(\vec{t})$ the term $e_n(\overline{t_1}, \ldots, \overline{t_n})$. The sequence $\vec{t}$ is *linear* if the $\overline{t_j}$ are pairwise distinct. Let $\bar{t} \in \mathcal{T}_e$. We denote by $\mathcal{P}\mathsf{os}_e(t)$ the set of positions $\bigcup_{i \in \mathbb{N}} \mathcal{P}\mathsf{os}_{e_i}(t)$, and by $\mathcal{P}\mathsf{os}_{\backslash e}(t)$ the set $\mathcal{P}\mathsf{os}(t) \backslash \mathcal{P}\mathsf{os}_e(t)$. The term $\bar{t}$ is in *e-normal form* if $\forall j \in \mathbb{N}$, $\forall u \in \mathcal{P}\mathsf{os}_{e_j}(t)$, the sequence $(t/u \cdot 0, \ldots, t/u \cdot j-1)$ is linear and such that $\forall i \in \{0, \ldots, j-1\}$, $u \cdot i \in \mathcal{P}\mathsf{os}_{\backslash e}(t)$. Let $n \in \mathbb{N}$. We denote by $\mathcal{B}_{\leq n}$ the set of terms $\bar{t} \in \mathcal{T}_e$ in e-normal form and such that $dpt_{\backslash e}(\bar{t}) \leq n$. Since $\mathcal{B}_{\leq n}$ is finite, there exists an integer N_n such that $\mathcal{B}_{\leq n} \subseteq \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{e_i | 1 \leq i \leq N_n\})$.

▶ **Definition 5.6.** Let $\bar{s}, \bar{t} \in \mathcal{B}_{\leq k+2}$. We define $l_e(\bar{s}, \bar{t})$ by:

- $l_e(\bar{s}, \bar{t}) = e(\overrightarrow{\mathsf{lin}}(\bar{s}, \bar{t}))$ if $\epsilon \in \mathcal{P}\mathsf{os}_{\backslash e}(s) \cap \mathcal{P}\mathsf{os}_{\backslash e}(t)$,
- $l_e(\bar{s}, \bar{t}) = e(\overrightarrow{\mathsf{lin}}(\bar{s}, \bar{t}/0, \ldots, \bar{t}/n-1))$ if $\epsilon \in \mathcal{P}\mathsf{os}_{\backslash e}(s)$ and $\epsilon \in \mathcal{P}\mathsf{os}_{e_n}(t)$ for some $n \in \mathbb{N}$,
- $l_e(\bar{s}, \bar{t}) = e(\overrightarrow{\mathsf{lin}}(\bar{s}/0, \ldots, \bar{s}/n-1, t))$ if $\epsilon \in \mathcal{P}\mathsf{os}_{\backslash e}(t)$ and $\epsilon \in \mathcal{P}\mathsf{os}_{e_n}(s)$ for some $n \in \mathbb{N}$,
- $l_e(\bar{s}, \bar{t}) = e(\overrightarrow{\mathsf{lin}}(\bar{s}/0, \ldots, \bar{s}/n-1, \bar{t}/0, \ldots, \bar{t}/m-1))$ if $\epsilon \in \mathcal{P}\mathsf{os}_{e_n}(s)$ and $\epsilon \in \mathcal{P}\mathsf{os}_{e_m}(t)$ for some $n, m \in \mathbb{N}$.

Note that $l_e(\bar{s}, \bar{t}) \in \mathcal{B}_{\leq k+2}$.

▶ **Definition 5.7.** Let $\bar{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{e_i \mid i \geq 1\})$, $n \in \mathbb{N}$. We denote by $\bar{s} \odot_e n$ the unique term such that:

- $(\bar{s} \odot_e n)^0 = s$,
- $\forall u \in \mathcal{P}\mathsf{os}_{\backslash \{e\} \cup \mathcal{Q}_{\mathcal{P}}}(s), \mathsf{m}((\bar{s} \odot_e n)/u) = \max(\mathsf{m}(\bar{s}/u), \mathsf{Card}(\mathcal{P}\mathsf{os}_{\backslash e}{}^{\prec u}(s)) + n)$.

We extend this definition to substitutions $(\bar{\sigma} \odot_e n : x \mapsto x\bar{\sigma} \odot_e n)$.

Note that if $\bar{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}})$, $\bar{s} \odot_e n = \bar{s} \odot n$.

▶ **Definition 5.8.** Let $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{e_i \mid i \geq 1\})$, and let $n \in \mathbb{N}$. We define $\mathsf{cpr}(\overline{s})$ the compressed form of $\overline{s}$ by:

- $\mathsf{cpr}(\overline{s}) = \overline{s}$ if $s \in \mathcal{F}_0{}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}}$,
- $\mathsf{cpr}(\overline{s}) = \mathsf{root}(\overline{s})(\mathsf{cpr}(\overline{s}/0), \ldots, \mathsf{cpr}(\overline{s}/m-1))$ if $\mathsf{root}(s) \in \mathcal{F}_m$ for some $m \in \mathbb{N}$,
- $\mathsf{cpr}(\overline{s}) = e(\overrightarrow{\mathsf{lin}}(\mathsf{cpr}(\overline{s}/0), \ldots, \mathsf{cpr}(\overline{s}/m-1)))$ if $\mathsf{root}(s) = e_m$ for some $m \in \mathbb{N}$.

We extend this definition to substitutions ($\mathsf{cpr}(\overline{\sigma}) : x \mapsto \mathsf{cpr}(x\overline{\sigma})$).

Note that if $\overline{s}$ is in e-normal form, $\overline{s} = \mathsf{cpr}(\overline{s})$. Note also that if $\overline{s}$ is such that $\forall j \in \mathbb{N}$, $\forall u \in \mathcal{P}\mathsf{os}_{e_j}(s)$, $\forall i \in \{0, \ldots, j-1\}$, $u \cdot i \in \mathcal{P}\mathsf{os}_{\setminus e}(s)$, then $\mathsf{cpr}(\overline{s})$ is in e-normal form.

▶ **Definition 5.9.** Let $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}, \mathcal{V})$, $\mathcal{V}\mathsf{ar}(t) = \{x_1, \ldots, x_n\}$, and let $\{x_{i,j} \mid i, j \in \mathbb{N}\}$ be a set of variables. For all $1 \leq i \leq n$, let $j_i = \mathsf{Card}(\mathcal{P}\mathsf{os}(t, x_i))$, and let $\mathcal{P}\mathsf{os}(t, x_i) = \{v_{1,1}, \ldots, v_{1,j_i}\}$ where the $v_{p,q}$ are given in lexicographic order. We denote by $\mathsf{lin}(\overline{t})$ the term $\overline{t}[x_{1,1}, \ldots, x_{1,j_1}, \ldots, x_{n,1}, \ldots, x_{n,j_n}]_{v_{1,1},\ldots,v_{1,j_1},\ldots v_{n,1},\ldots v_{n,j_n}}$.

From now on, each time we use the notation $\mathsf{lin}(\overline{t})$, we implicitly suppose that $\mathcal{V}\mathsf{ar}(t) = \{x_1, \ldots, x_n\}$, and that the variables in $\mathcal{V}\mathsf{ar}(\mathsf{lin}(t))$ are denoted $x_{i,j}$ as in definition 5.9.

Overview of the simulation

Let us give an overview of the proof of the projecting and lifting lemmas used to simulate $\mathsf{bo}(k)$-derivations by a GTT (lemmas 5.19 and 5.16).

- First, we associate to each term $\overline{t} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ such that $dpt_{\setminus E}(\overline{t}) \leq k+2$ an e-normal form in $\mathcal{B}_{k+2}$. The computation of e-normal forms is assigned to the automaton Ψ which replaces each stack of E in $\overline{t}$ by the symbol e and then "compresses" the resulting term. For example, the e-normal form of the term $E(E(E(a^0, b^1), f^1(E(a^0, b^1))), a^0)$ is $e(a^0, b^1, f^1(e(a^0, b^1)))$.
- Then, for every rule $l \to r \in \mathcal{R} \cup \mathcal{E}$, every term $\overline{l} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ and every substitution $\overline{\tau} : \mathcal{V} \to \mathcal{B}_{\leq k+2}$ such that $\overline{\tau} \odot_e a : \mathcal{V} \to \mathcal{B}_{\leq k+2}$ (where $a = 1$ if $l \to r \in \mathcal{R}$, and $a = 0$ otherwise), we define a GRS $\mathcal{G}_{\overline{l},\overline{r},\tau} = (L, R)$, where L is the recognizable set containing all the terms $\overline{l\sigma}$ that have $\overline{l\tau}$ for e-normal form and R is the recognizable set containing all the terms $\mathsf{lin}(r)(\overline{\sigma} \odot a)$ such that for all $x_{i,j} \in \mathcal{V}\mathsf{ar}(\mathsf{lin}(r))$, the e-normal form of $(x_{i,j}\overline{\sigma} \odot a)$ is $\mathsf{cpr}(x_i\overline{\tau} \odot a)$.

The GRS $\mathcal{G}$ over $\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\}$ is hence defined as the union of all the GRS $\mathcal{G}_{\overline{l},r,\overline{\tau}}$ and $\mathcal{A}_{\mathcal{P}}^{+\leq k}$.
Now, let us see how the simulation works. The simulation is based on the projecting lemma 5.19 and the lifting lemma 5.16. Let us start with the projecting lemma. Let us suppose that we have a rewriting step $\overline{s} = \overline{s}[\overline{l\sigma}]_v \;{}_k{\circ\!\!\longrightarrow}_{\mathcal{R} \cup \mathcal{E}}\; \overline{t} = \overline{s}[\overline{r\sigma} \odot a]$ and that $v \in \mathsf{Topd}(\overline{s}) \backslash \mathcal{L}\mathsf{v}(\mathsf{Top}(\overline{s}))$ (the other case $v \notin \mathsf{Topd}(\overline{s}) \backslash \mathcal{L}\mathsf{v}(\mathsf{Top}(\overline{s}))$ is not treated here but can be found in the projecting lemma proof available in the full version of this paper). The projecting lemma 5.19 claims that $\mathsf{Top}(\overline{s}) = \mathsf{Top}(\overline{s}[])[\overline{l}\mathsf{Top}(\overline{\sigma})]_v \to^*_{\mathcal{G}} \mathsf{Top}(\overline{t}) = \mathsf{Top}(\overline{s}[])[r\mathsf{Top}(\overline{\sigma} \odot a)]$. We obtain this derivation in two steps:

- First, we cut the useless part of $\mathsf{Top}(\overline{s})$ using $\mathcal{A}_{\mathcal{P}}^{\leq k}$, i.e. the parts of $x\overline{\sigma}$ that are marked by an integer greater than k in $x\overline{\sigma} \odot a$. Let us denote by $\overline{\sigma'}$ the substitution obtained after this step (i.e. the unique substitution such that $\overline{\sigma'} \odot a = \mathsf{Top}(\overline{\sigma} \odot a)$).
- After that, we use the GRS $\mathcal{G}(\overline{l}, r, \overline{\tau})$ to simulate the rewriting step, where $\overline{\tau}$ is the e-normal form associated to $x\overline{\sigma'}$, and we obtain the required derivation $\mathsf{Top}(\overline{s}) = \overline{s}[\overline{l\sigma}]_v \to_{\mathcal{A}_{\mathcal{P}}^{\leq k}} \mathsf{Top}(\overline{s})[\overline{l\sigma'}]_v \to_{\mathcal{G}(\overline{l},r,\overline{\tau})} \mathsf{Top}(\overline{t}) = \mathsf{Top}(\overline{s})[r\mathsf{Top}(\overline{\sigma} \odot a)]_v$.

Now, let us see how the lifting lemma works. Let $\overline{s} = \overline{s}[l\overline{\sigma}] \to_{\mathcal{G}(\overline{l},r,\overline{\tau})} \overline{t} = \overline{s}[\mathsf{lin}(r)\overline{\sigma'}]$. We want to prove that there exists a term $\overline{s'}$ and a derivation $\overline{s} \;{}_k{\circ\!\!\longrightarrow}_{\mathcal{R} \cup \mathcal{E}}\; \overline{s'} \to^*_{\mathcal{A}_{\mathcal{P}}} \overline{t}$. First, we apply the rule $l \to r$. We obtain a derivation $\overline{s} = \overline{s}[\overline{l}\overline{\sigma}]_v \ {}_k\!\circ\!\!\to_{\mathcal{R}} \overline{s}[r(\overline{\sigma} \odot a)]$. We then use the e-normal form proposition 5.12, and some other technical lemmas to obtain a term $\overline{s'}$ and a derivation $\overline{s} \ {}_k\!\circ\!\!\to_{\mathcal{R}} \overline{s}[r(\overline{\sigma} \odot a)] \ {}_k\!\circ\!\!\to^*_{\mathcal{E}} \overline{s'} \to_{\mathcal{A_P}} \overline{t}$.

The automaton Ψ

▶ **Definition 5.10.** We denote by Ψ the automaton: $\Psi = (\mathcal{F}^{\leq k} \cup \mathcal{Q_P} \cup \{E\}, \mathcal{Q}_\Psi, \emptyset, \Gamma_\Psi)$, where $\mathcal{Q}_\Psi = \{\langle \overline{t} \rangle \mid \overline{t} \in \mathcal{B}_{\leq k+2}\}$, and: $\Gamma_\Psi = \{E(\langle \overline{t_1} \rangle, \langle \overline{t_2} \rangle) \to \langle l_e(\overline{t_1}, \overline{t_2}) \rangle \mid \overline{t_1}, \overline{t_2} \in \mathcal{B}_{\leq k+2}\}$ $\cup \{f^j(\langle \overline{t_1} \rangle, \ldots, \langle \overline{t_n} \rangle) \to \langle f^j(\overline{t_1}, \ldots, \overline{t_n}) \rangle \mid n \in \mathbb{N}\backslash 0, j \in \{0, \ldots, k\}, \ f \in \mathcal{F}_n, \overline{t_1}, \ldots, \overline{t_n} \in \mathcal{B}_{\leq k+1}\} \cup \{\overline{c} \to \langle \overline{c} \rangle \mid \overline{c} \in \mathcal{F}_0^{\leq k} \cup \mathcal{Q_P}\}$.

The automaton Ψ associates to each term $\overline{s} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q_P} \cup \{E\})$ such that $dpt_{\backslash E}(s) \leq k+2$ a unique state $\langle \overline{t} \rangle$ such that $\overline{t} \in \mathcal{B}_{\leq k+2}$. The term $\overline{t}$ is in e-normal form and is said to be the e-normal form associated to $\overline{s}$.

▶ **Lemma 5.11.** *Let $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q_P} \cup \{E\})$, $i \leq k+2$. We have $\exists \overline{t} \in \mathcal{B}_{\leq i}, \overline{s} \to^*_\Psi \langle \overline{t} \rangle$ iff $(dpt_{\backslash E}(\overline{s}) \leq i \wedge \overline{s} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q_P} \cup \{E\}))$.*

The previous lemma is a straightforward consequence of the definition of Ψ. Moreover, the following proposition holds.

▶ **Proposition 5.12** (e-normal form proposition)**.** *Let $\overline{s}, \overline{t} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q_P} \cup \{E\}), \overline{u} \in \mathcal{B}_{\leq k+2}$, $n \in \mathbb{N}$. If $\overline{s} \odot n \to^*_\Psi \langle \overline{u} \rangle$ and $\overline{t} \odot n \to^*_\Psi \langle \overline{u} \rangle$, then $\overline{s} \odot n \circ\!\!\to^*_{\mathcal{E}} \overline{t} \odot n$.*

The GRS $\mathcal{G}$

For every linear term $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q_P} \cup \{E\}, \mathcal{V})$, and every substitution $\overline{\sigma} : \mathcal{V} \to \mathcal{B}_{\leq k+2}$, the set $\{\overline{t}\,\overline{\sigma'} \mid \overline{\sigma'} : \mathcal{V} \to \mathcal{T}(\mathcal{F}^k \cup \mathcal{Q_P} \cup \{E\}), \forall x \in \mathcal{V}, x\overline{\sigma'} \to^*_\Psi \langle x\overline{\sigma} \rangle\}$ is recognizable.

▶ **Definition 5.13.** We denote by Λ_a the set of substitutions $\overline{\tau} : \mathcal{V} \to \mathcal{B}_{\leq k+2}$ such that $(\overline{\tau} \odot_e a) : \mathcal{V} \to \mathcal{B}_{\leq k+2}$.

▶ **Definition 5.14.** Let $l \to r \in \mathcal{R} \cup \mathcal{E}$, $\overline{l} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q_P} \cup \{E\}, \mathcal{V})$. Let $a = 0$ if $l \to r \in \mathcal{E}$, and let $a = 1$ if $l \to r \in \mathcal{R}$. Let $\tau \in \Lambda_a$, and let $L = \{\overline{l}\overline{\sigma} \mid \overline{\sigma} : \mathcal{V} \to \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q_P} \cup \{E\}), \forall i \in \{1, \ldots, n\}, x_i\overline{\sigma} \to^*_\Psi \langle x_i\overline{\tau} \rangle\}$, $R = \{\mathsf{lin}(r)(\overline{\sigma} \odot a) \mid \overline{\sigma} : \mathcal{V} \to \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q_P} \cup \{E\}), \forall i \in \{1, \ldots, m\}, \forall j \in \{1, \ldots, \mathsf{Card}(\mathcal{P}\mathsf{os}(r, x_i))\}, x_{i,j}\overline{\sigma} \odot a \to^*_\Psi \langle \mathsf{cpr}(x_i\overline{\tau} \odot_e a) \rangle\}$, be two recognizable sets. We denote by $\mathcal{G}(\overline{l}, r, \overline{\tau})$ the GRS $\{l \to r \mid l \in L, r \in R\}$ over $\mathcal{F}^{\leq k} \cup \mathcal{Q_P} \cup \{E\}$.

Note that since in the definition of R $x_{i,j}\overline{\sigma'} \odot a \to^*_\Psi \langle \mathsf{cpr}(x_i\overline{\tau}) \odot a \rangle$, by lemma 5.11, $x_{i,j}\overline{\sigma'} \odot a \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q_P} \cup \{E\})$.

▶ **Definition 5.15.** We denote by $\mathcal{G}$ the GRS over $\mathcal{F}^{\leq k} \cup \{E\} \cup \mathcal{Q_P}$

$$\Big(\bigcup_{l \to r \in \mathcal{R}, \overline{l} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \{E\}, \mathcal{V}), \overline{\tau} \in \Lambda_1} \mathcal{G}_{\overline{l}, r, \overline{\tau}} \Big) \cup \Big(\bigcup_{l \to r \in \mathcal{E}, \overline{\tau} \in \Lambda_0} \mathcal{G}_{\overline{l}, r, \overline{\tau}} \Big) \cup \Gamma_{\mathcal{P}(\mathcal{A})^{+ \leq k}}$$

The transitive and reflexive closure of a GRS is recognizable by a GTT. The GTT recognizing $\to^*_{\mathcal{G}}$ will be used to simulate $\mathsf{bo}(k)$-derivations in $\mathcal{R} \cup \mathcal{E}$ (see lemma 5.23).

5.2 Simulation of $\mathsf{bo}(k)$-derivations

Lifting lemma

The lifting lemma simulates a derivation $\overline{s} \to^*_{\mathcal{A}^+_{\mathcal{P}}} \overline{s'} \to_{\mathcal{G}} \overline{t}$ by a $\mathsf{bo}(k)$-derivation in $\mathcal{R} \cup \mathcal{E}$ followed by a derivation in $\mathcal{A_P}^+$. The proof can be found in the full version of this article.

▶ **Lemma 5.16** (lifting lemma). *Let $l \to r \in \mathcal{R} \cup \mathcal{E}$, $\bar{l} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$, let $a = 0$ if $l \to r \in \mathcal{E}$ and $a = 1$ if $l \to r \in \mathcal{R}$, and let $\overline{\sigma} \in \Lambda_a$ be a substitution. Let $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \{E\})$, $\overline{s'}, \overline{t} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ be such that $\overline{s} \to^*_{\mathcal{A}^+_{\mathcal{P}}} \overline{s'} \to_{\mathcal{G}(\bar{l},r,\overline{\sigma})} \overline{t}$. There exists $\overline{t'} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \cup\{E\})$ such that $\overline{s}\ {}_k{\circ\!\!\to}^+_{\mathcal{R}\cup\mathcal{E}} \overline{t'} \to^*_{\mathcal{A}^+_{\mathcal{P}}} \overline{t}$.*

▶ **Example 5.17.** Let $\mathcal{R}_1$ and $\mathcal{A}_1$ be the TRS and the automaton of example 2.1 and let $k = 1$. Let $\overline{s} = \mathsf{f}^0(E(\mathsf{h}^0(\mathsf{a}^1), E(\mathsf{a}^0, \mathsf{a}^1))) \to_{\mathcal{A}_{\mathcal{P}}{}^+} \overline{s'} = \mathsf{f}^0(E(\mathsf{f}^0(\{\mathsf{q}_\mathsf{a}\}), E(\mathsf{a}^0, \mathsf{a}^1)))$. Let $\overline{\tau}$ be such that $x\overline{\tau} = e(\mathsf{h}(\{\mathsf{q}_\mathsf{a}\}), \mathsf{a}^0, \mathsf{a}^1)$. One can check that $\overline{s'} \to_{\mathcal{G}(\mathsf{f}^0(x),\mathsf{g}^0(x,x),\overline{\tau})} \overline{t} = \mathsf{g}^0(E(E(\mathsf{h}^1(\{\mathsf{q}_\mathsf{a}\}), \mathsf{h}^1(\{\mathsf{q}_\mathsf{a}\})), \mathsf{a}^1), E(\mathsf{h}^1(\{\mathsf{q}_\mathsf{a}\}), E(\mathsf{a}^1, \mathsf{a}^1)))$. First, we apply the rule $\mathsf{f}(x) \to \mathsf{g}(x,x)$ to obtain a $\mathsf{bo}(1)$-derivation $\overline{s'}\ {}_1{\circ\!\!\to}_{\mathcal{R}} \overline{t'} = \mathsf{g}^0(E(\mathsf{h}^1(\mathsf{a}^2), E(\mathsf{a}^1, \mathsf{a}^1)), E(\mathsf{h}^1(\mathsf{a}^2), E(\mathsf{a}^1, \mathsf{a}^1)))$. Then, we apply several steps in $\mathcal{E}$ followed by several steps in $\mathcal{A}_{\mathcal{P}}{}^+$ to obtain a derivation $\overline{t'}\ {}_1{\circ\!\!\to}^*_{\mathcal{E}} \mathsf{g}^0(E(E(\mathsf{h}^1(\mathsf{a}^2), \mathsf{h}^1(\mathsf{a}^2)), \mathsf{a}^1), E(\mathsf{h}^1(\mathsf{a}^2), E(\mathsf{a}^1, \mathsf{a}^1))) \to^*_{\mathcal{A}_{\mathcal{P}}{}^+} \overline{t}$. Hence, $\overline{s}\ {}_1{\circ\!\!\to}_{\mathcal{R}\cup\mathcal{E}} \overline{t'} \to^*_{\mathcal{A}_{\mathcal{P}}{}^+} \overline{t}$.

▶ **Corollary 5.18** (lifting n-steps). *Let $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$, $\overline{s'}, \overline{t} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ be such that $\overline{s} \to^*_{\mathcal{G}} \overline{t}$. There exists $\overline{t'} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$ such that $\overline{s}\ {}_k{\circ\!\!\to}^*_{\mathcal{R}\cup\mathcal{E}} \overline{t'} \to^*_{\mathcal{A}^+_{\mathcal{P}}} \overline{t}$.*

Projecting lemma

The projecting lemma simulates one $\mathsf{bo}(k)$-step $\overline{s}\ {}_k{\circ\!\!\to}_{\mathcal{R}\cup\mathcal{E},v} \overline{t}$ by a derivation in $\mathcal{G}$ from $\mathsf{Top}(\overline{s})$ to $\mathsf{Top}(\overline{t})$. The proof is given in the full version of this paper.

▶ **Lemma 5.19** (Projecting one step). *Let $\overline{s}$, $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$, $v \in \mathcal{P}\mathsf{os}(s)$ be such that $\overline{s}$ is well-marked and $\overline{s}\ {}_k{\circ\!\!\to}_{\mathcal{R}\cup\mathcal{E},v} \overline{t}$.*

- *If $\forall u \prec v$, $\mathsf{m}(\overline{s}/u) \leq k$ then there exist a term $\overline{s'} \in \mathcal{T}(\mathcal{F}^{\leq k} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$, a substitution $\overline{\sigma'} : \mathcal{V} \to \mathcal{B}_{\leq k+2}$ such that $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}_{\mathcal{P}}{}^{\leq k}} \overline{s'} \to_{\mathcal{G}(\bar{l},r,\overline{\sigma'})} \mathsf{Top}(\overline{t})$,*
- *otherwise, $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}^+_{\mathcal{P}}{}^{\leq k}} \mathsf{Top}(\overline{t})$.*

▶ **Example 5.20.** Let us consider the TRS $\mathcal{R}_1$, and the automaton $\mathcal{A}_1$ of example 2.1, and $k = 1$. We have the following derivation between these two well-marked terms $\overline{s} = \mathsf{f}^1(E(\mathsf{a}^0, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^0(\mathsf{h}^0(\mathsf{a}^1)))))) \to_{\mathsf{f}(x)\to\mathsf{g}(x,x),\overline{\sigma}} \overline{t} =$
$\mathsf{g}(E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\mathsf{h}^2(\mathsf{a}^3))))), E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\mathsf{h}^2(\mathsf{a}^3))))))$, where
$x\overline{\sigma} = E(\mathsf{a}^0, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^0(\mathsf{h}^0((\mathsf{a}^1))))))$. We have $\mathsf{Top}(\overline{s}) = \overline{s}$ and
$\mathsf{Top}(\overline{t}) = \mathsf{g}(E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\{\mathsf{q}_\perp\})))), E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\{\mathsf{q}_\perp\})))))$. First, we cut the "useless" part of $\overline{s}$ using $\mathcal{A}_{1\mathcal{P}}{}^{\leq 1}$ i.e. the part of $x\overline{\sigma}$ that is marked by an integer greater that 1 in $x\overline{\sigma} \odot 1 = E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\mathsf{h}^2(\mathsf{a}^3)))))$. We obtain the following derivation $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}^{\leq k}_{1\mathcal{P}}} \mathsf{f}^1(E(\mathsf{a}^0, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^0(\{\mathsf{q}_\perp\})))))$. We are now ready to apply the step of the GRS that simulates the rule $l \to r$. Let $x\overline{\sigma'} = E(\mathsf{a}^0, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\{\mathsf{q}_\perp\}))))$. The e-normal form associated to $x\overline{\sigma'}$ is $x\overline{\tau} = e(\mathsf{a}^0, \mathsf{a}^1, \mathsf{b}^1, \mathsf{h}^1((\{\mathsf{q}_\perp\})))$. Moreover, the e-normal form associated to $x\overline{\sigma'} \odot 1 = E(\mathsf{a}^1, E(\mathsf{a}^1, E(\mathsf{b}^1, \mathsf{h}^1(\{\mathsf{q}_\perp\}))))$ is $\mathsf{cpr}(x\overline{\tau} \odot_e 1) = e(\mathsf{a}^1, \mathsf{b}^1, \mathsf{h}^1(\{\mathsf{q}_\perp\}))$. Hence, we obtain the derivation $\mathsf{Top}(\overline{s}) \to_{\mathcal{A}^{\leq 1}_{1\mathcal{P}}} \mathsf{f}^1(x\overline{\sigma'}) \to_{\mathcal{G}(\bar{l},r,\overline{\tau})} \mathsf{g}(x\overline{\sigma'} \odot 1, x\overline{\sigma'} \odot 1) = \mathsf{Top}(\overline{t})$.

▶ **Corollary 5.21** (Projecting n -steps). *Let $\overline{s}$, $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}} \cup \mathcal{Q}_{\mathcal{P}} \cup \{E\})$, $v \in \mathcal{P}\mathsf{os}(s)$ be such that $\overline{s}$ is well-marked and $\overline{s}\ {}_k{\circ\!\!\to}^+_{\mathcal{R}\cup\mathcal{E},v} \overline{t}$. We have $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{G}} \mathsf{Top}(\overline{t})$.*

▶ **Lemma 5.22.** *Let $s \in \mathcal{T}(\mathcal{F})$, $q \in \mathcal{Q}_{\mathcal{A}}$. We have $\exists \overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$, $s\ {}_k{\circ\!\!\to}^*_{\mathcal{R}\cup\mathcal{E}} \overline{t} \to^*_{\mathcal{A}} q$ iff $s \to^*_{\mathcal{G}} \{q\}$.*

Inverse preservation of the recognizability

▶ **Lemma 5.23** (simulation lemma). *We have* $(\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F}) = (\,{}_k\!\!\to^*_{\mathcal{R}})[T]$.

Proof. Let $s \in (\,{}_k\!\!\to^*_{\mathcal{R}})[T]$. By definition, there exist $t \in \mathcal{T}(\mathcal{F})$ and $q \in \mathcal{Q}_{f,\mathcal{A}}$ such that $s \,{}_k\!\!\to^*_{\mathcal{R}}\, t \to^*_{\mathcal{A}} q$. By definition of a $\mathsf{boc}(k)$ derivation, there exists a marked term $\overline{t}$ such that $s \,{}_k\!\circ\!\!\to^*_{\mathcal{R}\cup\mathcal{E}}\, \overline{t}$. By lemma 4.3, since $t \to^*_{\mathcal{A}} q$, we have $\overline{t} \to^*_{\overline{\mathcal{A}}} q$. By lemma 5.22, $s \to^*_{\mathcal{G}} \{q\}$, and since $\{q\} \in \mathcal{Q}_{f,\mathcal{P}}$, we have $s \in (\to_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}]$. Hence, $(\,{}_k\!\!\to^*_{\mathcal{R}})[T] \subseteq (\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F})$.
Let $s \in (\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F})$. There exists $q \in \mathcal{Q}_{f,\mathcal{A}}$ such that $s \to^*_{\mathcal{G}} \{q\}$. By lemma 5.22, there exists $\overline{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ such that $s \,{}_k\!\circ\!\!\to^*_{\mathcal{R}\cup\mathcal{E}}\, \overline{t} \to^*_{\overline{\mathcal{A}}} q$. By lemma 3.2, $s \to^*_{\mathcal{R}} t$, and since there exists a $\mathsf{bo}(k)$ marked derivation from s to $\overline{t}$, $s \,{}_k\!\!\to^*_{\mathcal{R}}\, t$. By lemma 4.3, since $\overline{t} \to^*_{\overline{\mathcal{A}}} q$, we have $t \to^*_{\mathcal{A}} q$. So, $t \in T$, and $s \in (\,{}_k\!\!\to^*_{\mathcal{R}})[T]$. ◀

We are now ready to prove theorem 5.1.

▶ **Theorem 5.1.** *Let $\mathcal{R}$ be some (finite) left-linear* TRS *over a signature $\mathcal{F}$. Let T be some recognizable subset of $\mathcal{T}(\mathcal{F})$ and let $k > 0$. Then, the set $(\,{}_k\!\!\to^*_{\mathcal{R}})[T]$ is recognizable too.*

By lemma 5.23, $(\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F}) = (\,{}_k\!\!\to^*_{\mathcal{R}})[T]$. The relation $\to^*_{\mathcal{G}}$ is recognizable by a GTT, and since GTTs are inverse recognizability preserving (see e.g. [1]), $(\to^*_{\mathcal{G}})[\mathcal{Q}_{f,\mathcal{P}}] \cap \mathcal{T}(\mathcal{F})$ is recognizable, and thus $(\,{}_k\!\!\to^*_{\mathcal{R}})[T]$ is recognizable.

▶ **Corollary 5.24.** *Every* $\mathsf{BO}(k)$ TRS *inverse-preserve recognizability.*

6 Strongly bounded TRSs

We introduce here strongly bounded TRSs. The reader may refer to the full version of the article for more details.

▶ **Definition 6.1.** A marked step $\overline{s} \circ\!\!\to_{\mathcal{R}\cup\mathcal{E}, l \circ\!\to r, v} \overline{t}$ is *weakly bottom-up* (wbu for short) if $l \to r \in \mathcal{E}$ or if $l \to r \in \mathcal{R}$ and the following assertion holds:

$$(l \notin \mathcal{V} \Rightarrow \mathsf{m}(\overline{l}) = 0) \text{ and } (l \in \mathcal{V} \Rightarrow \sup(\{\mathsf{m}(\overline{s}/u) \mid u \prec v) = 0).$$

A marked derivation is wbu if all its rewriting steps are wbu. A derivation $s \to_{\mathcal{R}\cup\mathcal{E}} t$ is wbu if the associated marked derivation is wbu. A derivation $s \to^*_{\mathcal{R}} t$ is *weakly bottom-up convertible* (wbuc for short) if there exists a wbu derivation $s \to^*_{\mathcal{R}\cup\mathcal{E}} t$. Let $k \in \mathbb{N}$. A TRS is strongly k-bounded ($\mathsf{SBO}(k)$ for short) if every wbu derivation starting on a term $\overline{s} \in \mathcal{T}(\mathcal{F})$ is $\mathsf{bo}(k)$. We denote by $\mathsf{SBO}(k)$ the class of $\mathsf{SBO}(k)$ TRSs. Finally, the class of strongly bounded TRS SBO is defined by: $\mathsf{SBO} = \bigcup_{k\in\mathbb{N}} \mathsf{SBO}(k)$.

Note that every marked derivation in $\mathcal{E}$ is wbu. Roughly speaking, a wbu derivation is a derivation in which the rules of $\mathcal{R}$ are applied going from the bottom to the top. Moreover, every derivation is wbuc and $\mathsf{SBO}(k) \subset \mathsf{BO}(k)$. The class SBO contains Inverse Right-Linear Finite-Path Overlapping *TRSs* [14], and left-linear growing TRSs [10]. Moreover, the membership problem for $\mathsf{SBO}(k)$ is decidable, whereas the membership problem for $\mathsf{BO}(0)$ is undecidable.

7 Perspectives

Here are some natural perspectives of development for this work.

- The method developed here might be used also for testing some termination properties and might leads to a proof of the decidability of the termination of left-linear growing TRSs as conjectured in [10].
- A dual notion of *top-down* rewriting should be defined (at least for linear TRSs). The class would presumably extend the class of Layered Transducing TRSs defined in [12].
- The TRSs considered in [6] and the TRSs considered here might be treated in a unifed manner for the linear case and if so, might be extended to the left-linear case.

Some work in this directions has been undertaken by the authors.

References

1 H. Comon, M. Dauchet, R. Gilleron, C. Löding, F. Jacquemard, D. Lugiez, S. Tison, and M. Tommasi. Tree automata techniques and applications. Available on: `http://www.grappa.univ-lille3.fr/tata`, 2007. release October, 12th 2007.

2 I. Durand and A. Middeldorp. Decidable call-by-need computations in term rewriting. *Inf. Comput.*, 196(2):95–126, 2005.

3 I. Durand, G. Sénizergues, and M. Sylvestre. Termination of linear bounded term rewriting systems. In Christopher Lynch, editor, *Proceedings of the 21st International Conference on Rewriting Techniques and Applications*, pages 341–356. LIPIcs, July 2010.

4 Irène Durand and Géraud Sénizergues. Bottom-up rewriting for words and terms. *CoRR*, abs/0903.2554, 2009.

5 Z. Fülöp, E. Jurvanen, M. Steinby, and S. Vágvölgyi. On one-pass term rewriting. In *MFCS*, pages 248–256, 1998.

6 A. Geser, D. Hofbauer, and J. Waldmann. Match-bounded string rewriting systems. *Journal Applicable Algebra in Engineering, Communication and Computing*, 15(3-4):149–171, November, 2004.

7 F. Jacquemard. Decidable approximations of term rewriting systems. In *Proceedings of the 7th International Conference on Rewriting Techniques and Applications*, volume 1103, pages 362–376, 1996.

8 J.W. Klop. Term rewriting systems. In *Handbook of Logic in Computer Science, Vol. 2*, pages 1–116. Oxford University Press, 1992.

9 P. Lescanne, T. Heuillard, M. Dauchet, and S. Tison. Decidability of the confluence of ground term rewriting systems. Research Report RR-0675, INRIA, 1987.

10 T. Nagaya and Y. Toyama. Decidability for left-linear growing term rewriting systems. In *RTA '99: Proceedings of the 10th International Conference on Rewriting Techniques and Applications*, pages 256–270, London, UK, 1999. Springer-Verlag.

11 P. Réty and J. Vuotto. Tree automata for rewrite strategies. *J. Symb. Comput.*, 40(1):749–794, 2005.

12 H. Seki, T. Takai, Y. Fujinaka, and Y. Kaji. Layered transducing term rewriting system and its recognizability preserving property. In *Proceedings of the 13th International Conference on Rewriting Techniques and Applications*, volume 2378. Springer Verlag, 2002.

13 F. Seynhaeve, S. Tison, and M. Tommasi. Homomorphisms and concurrent term rewriting. In *FCT*, pages 475–487, 1999.

14 T. Takai, Y. Kaji, and H. Seki. Right-linear finite path overlapping term rewriting systems effectively preserve recognizability. In *RTA*, pages 246–260, 2000.

15 T. Takai, Y. Kaji, and H. Seki. Termination property of inverse finite path overlapping term rewriting system is decidable. *IEICE transactions on information and systems*, 85(3):487–496, 2002-03-01.

16 Terese. *Term Rewriting Systems by Terese*, volume 55. Cambridge University Press, 2003.