# Interpolation, Amalgamation and Combination (The Non-disjoint Signatures Case)

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**Abstract.** In this paper, we study the conditions under which existence of interpolants (for quantifier-free formulae) is modular, in the sense that it can be transferred from two first-order theories $T_1, T_2$ to their combination $T_1 \cup T_2$. We generalize to the non-disjoint signatures case the results from [3]. As a surprising application, we relate the Horn combinability criterion of this paper to superamalgamability conditions known from propositional logic and we use this fact to derive old and new results concerning fusions transfer of interpolation properties in modal logic.

## 1 Introduction

Craig's interpolation theorem [5] applies to first order formulae and states that whenever the formula $\phi \to \psi$ is valid, then it is possible to find a formula $\theta$ such that (i) $\phi \to \theta$ is valid; (ii) $\theta \to \psi$ is valid, and (iii) $\theta$ is defined over the common symbols of $\phi$ and $\psi$. Interpolation theory has a long tradition in non-classical logics (see for instance the seminal papers by L.L. Maksimova [12,13]) and has been recently introduced also in verification, after the work of McMillan (see, e.g., [15]). Intuitively, the interpolant $\theta$ can be seen as an over-approximation of $\phi$ with respect to $\psi$: thus, for example, in the abstraction-refinement phase of software model checking [10], interpolants are used to compute increasingly precise over-approximations of the set of reachable states.

Of particular importance for verification techniques are those algorithms capable of computing *quantifier-free* interpolants in presence of some background theory. This is so because several symbolic verification problems are formalized by representing sets of states and transitions as quantifier-free formulae. Unfortunately, Craig's interpolation theorem does not guarantee that it is always possible to compute quantifier-free interpolants when reasoning *modulo a first-order theory*: in fact, for certain first-order theories, it is known that quantifiers must occur in interpolants of quantifier-free formulae [11]. Even when quantifier-free interpolants exist for single theories, this might not be anymore the case when considering their combinations (see e.g. Example 3.5 below). Since verification techniques frequently require to reason in combinations of theories, methods to modularly combine available interpolation algorithms are indeed desirable.

The study of the modularity property of quantifier-free interpolation was first started in [27], where the disjoint signatures convex case was solved; in [3] - the


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journal version of [2] - the non-convex (still disjoint) case was also thoroughly investigated. The analysis in [3] is large-spectrum: combinability of quantifier-free interpolation is first semantically analyzed (where it is related to strong sub-amalgamability), then it is syntactically characterized and finally suitable combination algorithms are designed.

This paper intends to be a first contribution for an extension to the non-disjoint signatures case. Given the complexity of the problem, we shall limit to semantic investigations, leaving for future research the subsequent, algorithmically oriented aspects. However, we show that our semantic techniques can be quite effective in practice: in fact, we show how to use them in order to establish that some theories combining integers and common datatypes (lists, trees, etc.) indeed enjoy quantifier-free interpolation. In addition, we employ our results in order to get interesting information concerning the transfer of interpolation properties to the fusion of modal logics: in fact, not only we show how to obtain Wolter's interpolation fusion transfer theorem [26] for normal modal logics, but we also identify a modular interpolation property for the non-normal case.

In attacking combination problems for non-disjoint signatures, we follow the model-theoretic approach successfully employed in [6,9,17–20]; this approach relies on the notion of $T_0$-compatibility, in order to identify modular conditions for combinability. The reason why this approach works can roughly be explained as follows. In combining a model of a theory $T_1$ with a model of a theory $T_2$, one needs to produce a superstructure of both of them: in such a superstructure, additional constraints in the shared subsignature might turn out to be satisfied and $T_0$-compatibility is meant to keep satisfiability of constraints in superstructures under control inside $T_1$ and $T_2$. This is because $T_0$-compatibility refers to model-completeness and model-completeness is the appropriate technique [4] to talk about satisfiability of quantifier-free formulae in extended structures.

The paper is organized as follows: in Sect. 2, we introduce notations and basic ingredients from the literature; in Sect. 3 we obtain a first general result (Theorem 3.2) and show how to use it in examples taken from verification theories. In the final Sect. 4, we apply our results to modal logic (Corollary 4.3 and Theorem 4.7); the proofs of the results from this last section require some algebraic logic background, so they are moved to the (online available from authors' web page) manuscript [7] for space reasons.

## 2 Formal Preliminaries

We adopt the usual first-order syntactic notions of signature, term, atom, (ground) formula, sentence, and so on. Let $\Sigma$ be a first-order signature; we assume the binary equality predicate symbol '=' to be added to any signature (so, if $\Sigma = \emptyset$, then $\Sigma$ just contains equality). The signature obtained from $\Sigma$ by adding it a set $\underline{a}$ of new constants (i.e., 0-ary function symbols) is denoted by $\Sigma^{\underline{a}}$. A *positive clause* is a disjunction of atoms. A *constraint* is a conjunction of literals. A formula is *quantifier-free* (or open) iff it does not contain quantifiers. A $\Sigma$*-theory* $T$ is a set of sentences (called the axioms of $T$) in the signature $\Sigma$ and it is *universal* iff it has universal closures of open formulae as axioms.

We also assume the usual first-order notion of interpretation and truth of a formula, with the proviso that the equality predicate $=$ is always interpreted as the identity relation. We let $\bot$ denote a ground formula which is true in no structure. A formula $\varphi$ is *satisfiable* in $\mathcal{M}$ iff its *existential* closure is true in $\mathcal{M}$. A $\Sigma$-structure $\mathcal{M}$ is a *model* of a $\Sigma$-theory $T$ (in symbols $\mathcal{M} \models T$) iff all the sentences of $T$ are true in $\mathcal{M}$. If $\varphi$ is a formula, $T \models \varphi$ ('$\varphi$ *is a logical consequence of* $T$') means that the universal closure of $\varphi$ is true in all the models of $T$. $T$ is *consistent* iff it has a model, i.e., if $T \not\models \bot$. A sentence $\varphi$ is $T$-consistent iff $T \cup \{\varphi\}$ is consistent. A $\Sigma$-theory $T$ is *complete* iff for every $\Sigma$-sentence $\varphi$, either $\varphi$ or $\neg\varphi$ is a logical consequence of $T$. $T$ admits *quantifier elimination* iff for every formula $\varphi(\underline{x})$ there is a quantifier-free formula $\varphi'(\underline{x})$ such that $T \models \varphi(\underline{x}) \leftrightarrow \varphi'(\underline{x})$ (notations like $\varphi(\underline{x})$ mean that $\varphi$ has free variables only among the tuple $\underline{x}$).

If $\Sigma_0 \subseteq \Sigma$ is a subsignature of $\Sigma$ and if $\mathcal{M}$ is a $\Sigma$-structure, the $\Sigma_0$-*reduct* of $\mathcal{M}$ is the $\Sigma_0$-structure $\mathcal{M}_{|\Sigma_0}$ obtained from $\mathcal{M}$ by forgetting the interpretation of function and predicate symbols from $\Sigma \setminus \Sigma_0$. A $\Sigma$-*homomorphism* (or, simply, a homomorphism) between two $\Sigma$-structures $\mathcal{M}$ and $\mathcal{N}$ is any mapping $\mu : |\mathcal{M}| \longrightarrow |\mathcal{N}|$ among the support sets $|\mathcal{M}|$ of $\mathcal{M}$ and $|\mathcal{N}|$ of $\mathcal{N}$ satisfying the condition

$$\mathcal{M} \models \varphi \quad \Rightarrow \quad \mathcal{N} \models \varphi \tag{1}$$

for all $\Sigma^{|\mathcal{M}|}$-atoms $\varphi$ (here $\mathcal{M}$ is regarded as a $\Sigma^{|\mathcal{M}|}$-structure, by interpreting each additional constant $a \in |\mathcal{M}|$ into itself and $\mathcal{N}$ is regarded as a $\Sigma^{|\mathcal{M}|}$-structure by interpreting each additional constant $a \in |\mathcal{M}|$ into $\mu(a)$). In case condition (1) holds for all $\Sigma^{|\mathcal{M}|}$-literals, the homomorphism $\mu$ is said to be an *embedding* and if it holds for all first order formulae, the embedding $\mu$ is said to be *elementary*. If $\mu : \mathcal{M} \longrightarrow \mathcal{N}$ is an embedding which is just the identity inclusion $|\mathcal{M}| \subseteq |\mathcal{N}|$, we say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ or that $\mathcal{N}$ is an *extension* of $\mathcal{M}$. A $\Sigma$-structure $\mathcal{M}$ is said to be *generated by* a set $X$ included in its support $|\mathcal{M}|$ iff there are no proper substructures of $\mathcal{M}$ including $X$.

Given a signature $\Sigma$ and a $\Sigma$-structure $\mathcal{A}$, we indicate with $\Delta_\Sigma(\mathcal{A})$ the *diagram* of $\mathcal{A}$: this is the set of sentences obtained by first expanding $\Sigma$ with a fresh constant $\bar{a}$ for every element $a$ from $|\mathcal{A}|$ and then taking the set of ground $\Sigma^{|\mathcal{A}|}$-literals which are true in $\mathcal{A}$ (under the natural expanded interpretation mapping $\bar{a}$ to $a$).

Finally, we point out that all the above definitions can be extended in a natural way to many-sorted signatures (we shall use many-sorted theories in some examples).

### 2.1 Model Completion and $T_0$-compatibility

We recall a standard notion in Model Theory, namely the notion of a *model completion* of a first order theory [4] (we limit the definition to universal theories, because we shall use only this case):

**Definition 2.1.** *Let $T_0$ be a universal $\Sigma$-theory and let $T_0^\star \supseteq T_0$ be a further $\Sigma$-theory; we say that $T_0^\star$ is a model completion of $T_0$ iff:* (i) *every model of $T_0$*

*can be embedded into a model of $T_0^\star$;* (ii) *for every model $\mathcal{M}$ of $T_0$, we have that $T_0^\star \cup \Delta_\Sigma(\mathcal{M})$ is a complete theory in the signature $\Sigma^{|\mathcal{M}|}$.*

Being $T_0$ universal, condition (ii) is equivalent to the fact that $T_0^\star$ *has quantifier elimination*; we recall also that the model completion $T_0^\star$ of a theory $T_0$ is unique, if it exists (see [4] for these results and for examples).

We also recall the concept of $T_0$-compatibility [6,9], which is crucial for our combination technique.

**Definition 2.2.** *Let $T$ be a theory in the signature $\Sigma$ and let $T_0$ be a universal theory in a subsignature $\Sigma_0 \subseteq \Sigma$. We say that $T$ is $T_0$-*compatible *iff $T_0 \subseteq T$ and there is a $\Sigma_0$-theory $T_0^\star$ such that:*

(i) *$T_0 \subseteq T_0^\star$;*
(ii) *$T_0^\star$ is a model completion of $T_0$;*
(iii) *every model of $T$ can be embedded, as a $\Sigma$-structure, into a model of $T \cup T_0^\star$.*

Notice that if $T_0$ is the empty theory over the empty signature, then $T_0^*$ is the theory axiomatizing an infinite domain, and the requirement of $T_0$-compatibility is equivalent to the stably infinite requirement of the Nelson-Oppen schema [16, 24] (in the sense that $T$ is $T_0$-compatible iff it is stably infinite). We remind that a theory $T$ is stably infinite iff every $T$-satisfiable quantifier-free formula (from the signature of $T$) is satisfiable in an infinite model of $T$. By compactness, it is possible to show that $T$ is stably infinite iff every model of $T$ embeds into an infinite one.

We shall see many examples of $T_0$-compatible theories (for various $T_0$) during the paper, here we just underline that $T_0$-compatibility is a modular condition. The following result is proved in [6] (as Proposition 4.4):

**Proposition 2.3.** *Let $T_1$ be a $\Sigma_1$-theory and let $T_2$ be a $\Sigma_2$-theory; suppose they are both compatible with respect to a $\Sigma_0$-theory $T_0$ (where $\Sigma_0 := \Sigma_1 \cap \Sigma_2$). Then $T_1 \cup T_2$ is $T_0$-compatible too.*

## 2.2 Interpolation and Amalgamation

We say that a theory $T$ *has quantifier-free interpolation* iff the following hold, for every pair of quantifier free formulae $\varphi(\underline{x}, \underline{y}), \psi(\underline{y}, \underline{z})$: if $T \models \varphi(\underline{x}, \underline{y}) \to \psi(\underline{y}, \underline{z})$, then there exists a quantifier-free formula $\theta(\underline{y})$ such that $T \models \varphi(\underline{x}, \underline{y}) \to \theta(\underline{y})$ and $T \models \theta(\underline{y}) \to \psi(\underline{y}, \underline{z})$. We underline that the requirement that $\theta$ is quantifier-free is essential: in general such a $\theta(\underline{y})$ exists by the Craig interpolation theorem, but it is not quantifier-free even if $\varphi, \psi$ are such.[1] Quantifier-free interpolation property can be semantically characterized using the following notions, introduced in [1,3]:

[1] Notice that in the above definition free function and predicate symbols (not already present in the signature $\Sigma$ of $T$) are not allowed; allowing them (and requiring that only shared symbols occur in the interpolant $\theta$) produces a different stronger definition, which is nevertheless reducible to quantifier-free interpolation in the combination with the theory of equality with uninterpreted function symbols (see [3]).

**Definition 2.4.** *A theory $T$ has the* sub-amalgamation property *iff, for given models $\mathcal{M}_1$ and $\mathcal{M}_2$ of $T$ sharing a common substructure $\mathcal{A}$, there exists a further model $\mathcal{M}$ of $T$ endowed with embeddings $\mu_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$ and $\mu_2 : \mathcal{M}_2 \longrightarrow \mathcal{M}$ whose restrictions to the support of $\mathcal{A}$ coincide. The triple $(\mathcal{M}, \mu_1, \mu_2)$ (or, by abuse, $\mathcal{M}$ itself) is said to be a $T$-*sub-amalgama *of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}$.*

**Definition 2.5.** *A theory $T$ has the* strong sub-amalgamation property *if the $T$-sub-amalgama $(\mathcal{M}, \mu_1, \mu_2)$ of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}$ can be chosen so as to satisfy the following additional condition: if for some $m_1, m_2$ we have $\mu_1(m_1) = \mu_2(m_2)$, then there exists an element $a$ in $|\mathcal{A}|$ such that $m_1 = a = m_2$.*

If $T$ is universal, then every substructure of a model of $T$ is itself a model of $T$: in these cases, we shall drop the prefix sub- and directly speak of 'amalgamability', 'strong amalgamability' and '$T$-amalgama'. The following fact is proved in [3], as Theorem 3.3:

**Theorem 2.6.** *A theory $T$ has the sub-amalgamation property iff it admits quantifier-free interpolants.*

## 3 Conditions for Combination

The main result from [3] says that if $T_1, T_2$ have disjoint signatures, are both stably infinite and both enjoy the strong sub-amalgamation property, then the combined theory $T_1 \cup T_2$ also has the strong sub-amalgamation property[2] (and so it has quantifier-free interpolation).

In this paper, we try to extend the above results to the non-disjoint signatures case. The idea, already shown to be fruitful for combined satisfiability problems in [6], is to use $T_0$-compatibility as the proper generalization of stable infiniteness.

We shall first obtain a rather abstract sufficient condition for transfer of quantifier-free interpolation property to combined theories; nevertheless, we show that such sufficient condition generalizes the disjoint signatures result from [3] and is powerful enough to establish the quantifier-free interpolation property for some natural combined theories arising in verification. Then we move to the case in which the shared theory $T_0$ is Horn and obtain as a corollary a specialized result which is quite effective in modal logic applications.

### 3.1 Sub-amalgamation Schemata

Let $T_0, T$ be theories in their respective signatures $\Sigma_0, \Sigma$ such that $\Sigma_0 \subseteq \Sigma$, $T_0$ is universal and $T_0 \subseteq T$. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\Sigma$-models of $T$ with a common substructure $\mathcal{A}$, we call the triple $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ a *$T$-fork* (or, simply, a *fork*).

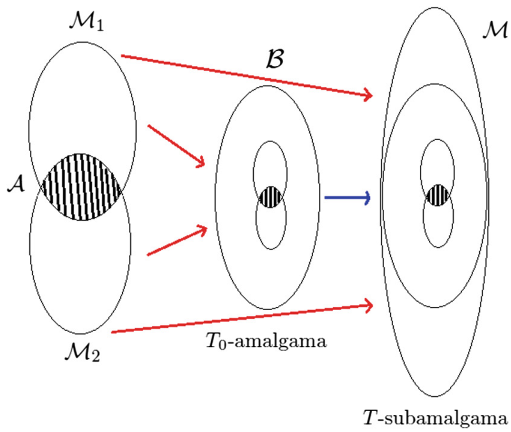
[2] It is possible to characterize syntactically strong sub-amalgamability in terms of a suitable 'equality interpolating' condition [3]. That sub-amalgamability needs to be strenghtened to strong sub-amalgamability in order to get positive combination results is demonstrated by converse facts also proved in [3].

The *sub-amalgamation schema* $\sigma^T_{T_0}$ (of $T$ over $T_0$) is the following function, associating sets of $T_0$-amalgama with $T$-forks:[3]

$$\sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})] := \left\{ \begin{array}{l} \text{the set of all } (\mathcal{B}, \nu_1, \nu_2) \text{ s.t.} \\ (i)\ (\mathcal{B}, \nu_1, \nu_2) \text{ is a } T_0\text{-amalgama of the } \Sigma_0\text{-reducts of} \\ \qquad \mathcal{M}_1 \text{ and } \mathcal{M}_2 \text{ over the } \Sigma_0\text{-reduct of } \mathcal{A}; \\ (ii)\ \mathcal{B} \text{ is generated, as } \Sigma_0\text{-structure, by the union of} \\ \qquad \text{the images of } \nu_1 \text{ and } \nu_2; \\ (iii)\ (\mathcal{B}, \nu_1, \nu_2) \text{ is embeddable in the } \Sigma_0\text{-reduct of a} \\ \qquad T\text{-sub-amalgama of the fork } (\mathcal{M}_1, \mathcal{M}_2, \mathcal{A}). \end{array} \right\}$$

Condition (iii) means that there is a $T$-sub-amalgama $(\mathcal{M}, \mu_1, \mu_2)$ such that $\mathcal{B}$ is a $\Sigma_0$-substructure of $\mathcal{M}$ and that $\mu_1, \mu_2$ coincide with $\nu_1, \nu_2$ on their domains.

Condition (ii) ensures that, disregarding isomorphic copies, $\sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$ is a set and not a proper class. Recall that $T_0$ is universal, so that substructures of models of $T_0$ are also models of $T_0$. This ensures that the following Proposition trivially holds:




**Proposition 3.1.** *$T$ is sub-amalgamable iff $\sigma^T_{T_0}$ is not empty (i.e. iff we have that $\sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})] \neq \emptyset$, for all forks $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$).*

One side of the inclusion of the following Theorem is also immediate; for the other one, $T_0$-compatibility is needed (we shall prove the theorem in Subsect. 3.2 below).

**Theorem 3.2.** *Let $T_1$ and $T_2$ be two theories in their respective signatures $\Sigma_1, \Sigma_2$; assume that they are both $T_0$-compatible, where $T_0$ is a universal theory*

[3] It is not difficult to realize (using well-known Löwenheim-Skolem theorems [4]) that one can get all the results in the paper by limiting this definition to forks among structures whose cardinality is bounded by the cardinality of set of the formulae in our signatures (signatures are finite or countable in all practical cases).

*in the signature* $\Sigma_0 := \Sigma_1 \cap \Sigma_2$. *The following hold for the amalgamation schema of* $T_1 \cup T_2$ *over* $T_0$:

$$\sigma_{T_0}^{T_1 \cup T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})] \;=\; \sigma_{T_0}^{T_1}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}] \cap \sigma_{T_0}^{T_2}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}]$$

*for every* $(T_1 \cup T_2)$*-fork* $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ *(here, with* $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_i}$ *we indicate the* $T_i$*-fork obtained by taking reducts to the signature* $\Sigma_i$*).*

Despite its abstract formulation, Theorem 3.2 is powerful enough to imply the main disjoint signatures result of [3] and also to work out interesting examples.

*Example 3.3. (The disjoint signature case).* Let $S_0, S_1, S_2$ be sets such that $S_0 \subseteq S_1, S_0 \subseteq S_2$; the amalgamated sum $S_1 +_{S_0} S_2$ of $S_1, S_2$ over $S_0$ is just the set-theoretic union $S_1 \cup S_2$ in which elements from $S_1 \setminus S_0$ are renamed away so as to be different from the elements of $S_2 \setminus S_0$. With this terminology, a theory $T$ is strongly sub-amalgamable iff its sub-amalgamation schema over the empty theory $T_0$ is such that $\sigma_{T_0}^{T}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$ always contains the amalgamated sum of the supports of $\mathcal{M}_1, \mathcal{M}_2$ over the support of $\mathcal{A}$. Thus, Theorem 3.2 says in particular that if $T_1, T_2$ are both stably infinite and strongly sub-amalgamable, then so is $T_1 \cup T_2$ (and the last is in particular quantifier-free interpolating).

*Example 3.4 (Lists and Trees).* Consider $T_0 := T_S$, the 'theory of increment' [20]; $T_S$ has the monosorted signature $\Sigma_S := \{0 : NUM,\ s : NUM \longrightarrow NUM\}$ and it is axiomatized by the following sentences:

$$\forall x \forall y\ s(x) = s(y) \to x = y \qquad \text{(injectivity)}$$
$$\forall x\ s^n(x) \neq x \ \ \text{for all } n \in \mathbb{N},\ n > 0$$

This theory is universal and it admits as a model-completion $T_S^*$ the theory obtained by adding the axiom $\forall x \exists y\ x = s(y)$. Hence, $T_S$ is amalgamable for general reasons [4] (but notice that it is not strongly amalgamable).

Now consider the theory $T_{LS}$ of 'lists endowed with length' [20]. This is a many-sorted theory; its signature $\Sigma_{LS}$ contains, besides $\Sigma_S$-symbols, the additional sorts $LISTS, ELEM_L$, the additional set of function symbols $\{nil : LISTS,\ car : LISTS \longrightarrow ELEM_L,\ cdr : LISTS \longrightarrow LISTS,\ cons : ELEM_L \times LISTS \longrightarrow LISTS,\ l : LISTS \longrightarrow NUM\}$ and a single unary relation symbol $atom : LISTS$. The axioms of $T_{LS}$ are the following:

| | |
|---|---|
| 1. $car(cons(x, y)) = x$ | 5. $\neg atom(x) \to cons(car(x), cdr(x)) = x$ |
| 2. $cdr(cons(x, y)) = y$ | 6. $\neg atom(cons(x, y))$ |
| 3. $l(nil) = 0$ | 7. $atom(nil)$ |
| 4. $l(cons(x, y)) = s(l(y))$ | |

This theory is $T_S$-compatible [20]; below, we show that *every* $T_S$-amalgama of the $T_S$-reducts of two models of $T_{LS}$ (sharing a common submodel) can be embedded in a $T_{LS}$-amalgama (since $T_{LS}$ is universal we can speak of amalgams instead of sub-amalgams).

Let a $T_{LS}$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ be given and let $\mathcal{B}$ be any amalgam of the $T_S$-reducts of $\mathcal{M}_1, \mathcal{M}_2$. We sketch the definition of a $T_{LS}$-amalgam $\mathcal{M}$ of the fork (based on $\mathcal{B}$). The support $NUM^{\mathcal{M}}$ is the support of $\mathcal{B}$ and $ELEM_L^{\mathcal{M}_1} \cup ELEM_L^{\mathcal{M}_2}$ is the support of $ELEM_L^{\mathcal{M}}$. It remains to define $LISTS^{\mathcal{M}}$;[4] we take $LISTS^{\mathcal{M}}$ to be the union of $LISTS^{\mathcal{M}_1}$, $LISTS^{\mathcal{M}_2}$ and of $LT$, where $LT$ is the set containing the pairs $(x, l)$, with $x \in LIST^{\mathcal{M}_{3-j}} \setminus LIST^{\mathcal{M}_j}$ and $l$ a finite list of elements from $ELEM_L^{\mathcal{M}_1} \cup ELEM_L^{\mathcal{M}_2}$ which begins with an element in $ELEM_L^{\mathcal{M}_j}$ $(j = 1, 2)$. In other words, an element in $LT$ has the form:

$$(x, (e_1, e_2, ..., e_n))$$

where (1) $j = 1, 2$; (2) $e_1$ is in $ELEM_L^{\mathcal{M}_j}$; (3) $x$ is in $LISTS^{\mathcal{M}_{3-j}}$; and (4) $e_i$ $(i > 1)$ is in $ELEM_L^{\mathcal{M}_1} \cup ELEM_L^{\mathcal{M}_2}$. $\Sigma_{LS}$-operations and relations can be defined in the obvious way so that axioms 1–7 above hold and so that the inclusions $\mathcal{M}_1 \subseteq \mathcal{M}$ and $\mathcal{M}_2 \subseteq \mathcal{M}$ are embeddings.

Let us now consider the theory $T_{BS}$ of binary trees endowed with size functions [20]. This is also a many-sorted theory: its signature $\Sigma_{BS}$ has the symbols of the signature $\Sigma_S$ of the theory of increment plus the set of function symbols $\{null : TREES,\ bin : ELEM_T \times TREES \times TREES \longrightarrow TREES,\ l_L : TREES \longrightarrow NUM,\ l_R : TREES \longrightarrow NUM\}$. The axioms of $T_{LS}$ are the following:

| | |
|---|---|
| 1. $l_L(null) = 0$ | 2. $l_R(null) = 0$ |
| 3. $l_L(bin(e, t_1, t_2)) = s(l_L(t_1))$ | 4. $l_R(bin(e, t_1, t_2)) = s(l_R(t_2))$ |

It can be showed that this theory is $T_S$-compatible [20]. By arguments similar to those we employed for $T_{LS}$, it is possible to show that *every* $T_S$-amalgama of the $T_S$-reducts of two models of $T_{BS}$ (sharing a common submodel) can be embedded in a $T_{BS}$-amalgama.

In conclusion, by (the multi-sorted version of) Theorem 3.2 we get that for every $(T_{LS} \cup T_{BS})$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, the amalgamation schema for this fork $\sigma_{T_S}^{T_{LS} \cup T_{BS}}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$, being equal to the intersection of $\sigma_{T_S}^{T_{LS}}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_{LS}}]$ and of $\sigma_{T_S}^{T_{BS}}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_{BS}}]$, contains all the amalgams of the $\Sigma_S$-reduced fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_S}$ and hence it is trivially not empty. This guarantees that $T_{LS} \cup T_{BS}$ has quantifier-free interpolation by Proposition 3.1.

*Example 3.5 (Where combined quantifier-free interpolation fails).* Let $T_0$ be the theory of linear orders (its signature $\Sigma_0$ has just a binary relation symbol $<$ and the axioms of $T_0$ say that $<$ is irreflexive, transitive and satisfies the trichotomy condition $x < y \vee x = y \vee y < x$). This is a universal theory and admits a model completion $T_0^*$, which is the theory of dense linear orders without endpoints [4]; it is easily seen also that $T_0$ is strongly sub-amalgamable. We consider the signature $\Sigma_1$ of linear orders endowed with an extra unary relation symbol $P$ and we let let $T_1$ be the theory obtained by adding to $T_0$ the following axiom:

$$\forall x \forall y\ (P(x) \wedge \neg P(y) \rightarrow x < y)$$

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[4] We can freely assume that $ELEM_L^{\mathcal{M}_1} \cap ELEM_L^{\mathcal{M}_2} = ELEM_L^{\mathcal{A}}$ and $LIST^{\mathcal{M}_1} \cap LIST^{\mathcal{M}_2} = LIST^{\mathcal{A}}$.

It is not difficult to see that $T_1$ is $T_0$-compatible and also strongly-sub-amalgamable. We shall be interested in the combination of $T_1$ with a partially renamed copy of itself: this is the $\Sigma_2 := \Sigma_0 \cup \{Q\}$-theory $T_2$ axiomatized by the axioms of $T_0$ and

$$\forall x \forall y \ (Q(x) \wedge \neg Q(y) \to x < y)$$

Quantifier-free interpolation fails in $T_1 \cup T_2$, because sub-amalgamability fails: to see this fact, just consider a fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ such that there exists an element $a \in |\mathcal{M}_1| \backslash |\mathcal{A}|$ which satisfies $P \wedge \neg Q$ and another element $b \in |\mathcal{M}_2| \backslash |\mathcal{A}|$ that satisfies $Q$ and $\neg P$. Notice that we have $\sigma^{T_1}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}] \cap \sigma^{T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}] = \emptyset$ although both $\sigma^{T_i}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_i}]$ are not empty (the sub-amalgamation schemata here 'do not match').

## 3.2 Proof of Theorem 3.2

This subsection is entirely devoted to the proof of Theorem 3.2. We begin by recalling some standard results from model theory and by introducing some preliminary lemmata. The following easy fact is proved in [3], as Lemma 3.7:

**Lemma 3.6.** *Let $\Sigma_1, \Sigma_2$ be two signatures and $\mathcal{A}$ be a $\Sigma_1 \cup \Sigma_2$-structure; then $\Delta_{\Sigma_1 \cup \Sigma_2}(\mathcal{A})$ is logically equivalent to $\Delta_{\Sigma_1}(\mathcal{A}) \cup \Delta_{\Sigma_2}(\mathcal{A})$.*

An easy but nevertheless important basic result, called *Robinson Diagram Lemma* [4], says that, given any $\Sigma$-structure $\mathcal{B}$, the embeddings $\mu : \mathcal{A} \longrightarrow \mathcal{B}$ are in bijective correspondence with expansions of $\mathcal{B}$ to $\Sigma^{|\mathcal{A}|}$-structures which are models of $\Delta_\Sigma(\mathcal{A})$. The expansions and the embeddings are related in the obvious way: $\bar{a}$ is interpreted as $\mu(a)$.

The following Lemma is proved using this property of diagrams:

**Lemma 3.7.** *Let $T_0, T$ be theories in their respective signatures $\Sigma_0, \Sigma$ such that $\Sigma_0 \subseteq \Sigma$ and $T_0 \subseteq T$; let $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ be a $T$-fork. For a $T_0$-amalgam $(\mathcal{B}, \nu_1, \nu_2)$ the following conditions are equivalent (we suppose that the support of $\mathcal{B}$ is disjoint from the supports of $\mathcal{M}_1, \mathcal{M}_2$):*

(i) $(\mathcal{B}, \nu_1, \nu_2) \in \sigma^{T}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$;
(ii) *the following theory* $(*)$ *is consistent*

$$\begin{aligned}
&T \cup \Delta_\Sigma(\mathcal{M}_1) \cup \Delta_\Sigma(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup \\
&\cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\
&\{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.
\end{aligned}$$

*Furthermore, in case $T$ is $T_0$-compatible, we can equivalently put $T \cup T_0^\star$ instead of $T$ in the theory* $(*)$ *mentioned in* (ii) *above.*

*Proof.* By the above mentioned property of diagrams, the consistency of $(*)$ means that there is a model $\mathcal{N} \models T$ and there are three embeddings

$$\mu_1 : \mathcal{M}_1 \longrightarrow \mathcal{N}, \quad \mu_2 : \mathcal{M}_2 \longrightarrow \mathcal{N}, \quad \nu : \mathcal{B} \longrightarrow \mathcal{N}$$

(the last one is a $\Sigma_0$-embedding, the first two are $\Sigma$-embeddings) such that $\nu \circ \nu_1 = \mu_1$ and $\nu \circ \nu_2 = \mu_2$. Since $\mu_1, \mu_2$ agree on the support of $\mathcal{A}$, the triple $(\mathcal{N}, \mu_1, \mu_2)$ is a $T$-sub-amalgam of the fork. To make $\mathcal{B}$ a substructure of $\mathcal{N}$, it is sufficient to make a renaming of the elements in the image of $\nu$ (so that $\nu$ becomes an inclusion). Thus consistency of $(*)$ means precisely that $(\mathcal{B}, \nu_1, \nu_2) \in \sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$.

Since, by $T_0$-compatibility, every model of $T$ can be embedded into a model of $T \cup T^*$, the consistency of $(*)$ is the same of the consistency of $T^\star \cup (*)$. $\dashv$

We need a further result from model theory to be found in textbooks like [4]; it can be seen as a combination result 'ante litteram':

**Lemma 3.8.** *[Joint Consistency] Let $\Theta_1, \Theta_2$ be two signatures and let $\Theta_0 := \Theta_1 \cap \Theta_2$; suppose that the $\Theta_1$-theory $U_1$ and the $\Theta_2$-theory $U_2$ are both consistent and that there is a $\Theta_0$-theory $U_0$ which is complete and included both in $U_1$ and in $U_2$. Then, $U_1 \cup U_2$ is also consistent.*

*Proof.* There are basically two proofs of this result, one by Craig's interpolation Theorem and another one by a double chain argument. The interested reader is referred to [4]. $\dashv$

We can now *prove Theorem* 3.2; the Theorem concerns theories $T_1, T_2$ (in their respective signatures $\Sigma_1, \Sigma_2$) which are both $T_0$-compatible with respect to a universal theory $T_0$ in the shared signature $\Sigma_0 := \Sigma_1 \cap \Sigma_2$.

Fix a $T_1 \cup T_2$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$. On one side, it is evident that if $(\mathcal{B}, \nu_1, \nu_2)$ belongs to $\sigma^{T_1 \cup T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$, then it also belongs to $\sigma^{T_1}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}] \cap \sigma^{T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}]$.

Vice versa, suppose that $(\mathcal{B}, \nu_1, \nu_2)$ belongs to $\sigma^{T_1}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_1}]$ and to $\sigma^{T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_2}]$; in order to show that it belongs to $\sigma^{T_1 \cup T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$, in view of Lemmas 3.6 and 3.7 (recall also Proposition 2.3), we need to show that the following theory (let us call it $U$) is consistent:

$$\begin{aligned} & T_1 \cup T_2 \cup T_0^\star \cup \Delta_{\Sigma_1}(\mathcal{M}_1) \cup \Delta_{\Sigma_1}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup \\ & \cup \Delta_{\Sigma_2}(\mathcal{M}_1) \cup \Delta_{\Sigma_2}(\mathcal{M}_2) \cup \\ & \cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\ & \{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}. \end{aligned}$$

The idea is to use Robinson Joint Consistency Lemma 3.8 and split $U$ as $U_1 \cup U_2$. Now $U$ is a theory in the signature $\Sigma_1 \cup \Sigma_2 \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$; we let (for $i = 1, 2$) $U_i$ be the following theory in the signature $\Sigma_i \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$:

$$\begin{aligned} & T_i \cup T_0^\star \cup \Delta_{\Sigma_i}(\mathcal{M}_1) \cup \Delta_{\Sigma_i}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup \\ & \cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\ & \{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}. \end{aligned}$$

Notice that $U_i$ is consistent by Lemma 3.7 because our assumption is that $(\mathcal{B}, \nu_1, \nu_2)$ belongs to $\sigma^{T_i}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_i}]$. We now only have to identify a complete theory $U_0$ included in $U_1 \cap U_2$. The shared signature of $U_1$ and $U_2$ is $\Sigma_0 \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$ and we take as $U_0$ the theory

$$\begin{aligned}
&T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{M}_1) \cup \Delta_{\Sigma_0}(\mathcal{M}_2) \cup \Delta_{\Sigma_0}(\mathcal{B}) \cup \\
&\cup \{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\
&\{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.
\end{aligned}$$

By the definition of a model-completion ($T_0^\star$ is a model-completion of $T_0$), we know that $T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{B})$ is a complete theory in the signature $\Sigma_0 \cup |\mathcal{B}|$. Now it is sufficient to observe that every $\Sigma_0 \cup |\mathcal{M}_1| \cup |\mathcal{M}_2| \cup |\mathcal{B}|$-sentence is equivalent, modulo $U_0 \supseteq T_0^\star \cup \Delta_{\Sigma_0}(\mathcal{B})$, to a $\Sigma_0 \cup |\mathcal{B}|$-sentence: this is clear because $U_0$ contains the sentences

$$\begin{aligned}
&\{\bar{a}_1 = \bar{b} \parallel b \in |\mathcal{B}|, a_1 \in |\mathcal{M}_1|, \nu_1(a_1) = b\} \cup \\
&\{\bar{a}_2 = \bar{b} \parallel b \in |\mathcal{B}|, a_2 \in |\mathcal{M}_2|, \nu_2(a_2) = b\}.
\end{aligned}$$

which can be used to eliminate the constants from $|\mathcal{M}_1| \cup |\mathcal{M}_2|$. ⊣

### 3.3 When the Shared Theory is Horn

Theorem 3.2 gives modular information to determine the combined sub-amalgamation schema, but it is not a modular result itself. In fact, a modular result should identify a condition $C$ on a single (standing alone) theory such that whenever $T_1, T_2$ satisfy $C$, then $T_1 \cup T_2$ is sub-amalgamable and also satisfies $C$. To get a modular sufficient condition, we need to specialize our framework. In doing that, we are still guided by what happens in the disjoint signatures case. Although we feel that suitable conditions could be identified without Horn hypotheses, we prefer to assume that the shared theory is universal Horn to get simpler statements of our results below.

Recall that a $\Sigma$-theory $T$ is *universal Horn* iff it can be axiomatized via Horn clauses (i.e. via formulae of the form $A_1 \wedge \cdots \wedge A_n \to B$, where the $A_i$ are atoms and $B$ is either an atom or $\bot$). In universal Horn theories, it is possible to show that if amalgamation holds, then there is always a minimal amalgama, as stated in the following fact (which is basically due to the universal property of pushouts, see [7] for a proof):

**Proposition 3.9.** *Let $T$ be a universal Horn theory having the amalgamation property; given a $T$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, there exists a $T$-amalgam $(\mathcal{M}, \mu_1, \mu_2)$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{A}$ such that for every other $T$-amalgam $(\mathcal{M}', \mu_1', \mu_2')$ there is a unique homomorphism $\nu : \mathcal{M} \longrightarrow \mathcal{M}'$ such that $\nu \circ \mu_i = \mu_i'$ ($i = 1, 2$).*

We call the amalgam mentioned in the above Proposition (which is unique up to isomorphism) the *minimal $T$-amalgam* of the $T$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$; the homomorphism $\nu$ (which needs not to be an embedding) is called the *comparison* homomorphism.

Let now $T$ be a $\Sigma$-theory and let $T_0 \subseteq T$ be a universal Horn $\Sigma_0$-theory having the amalgamation property (with $\Sigma_0 \subseteq \Sigma$). We say that $T$ is $T_0$-*strongly sub-amalgamable* if the sub-amalgamation schema $\sigma^T_{T_0}$ always contains the minimal $T_0$-amalgama (meaning that for every $T$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, we have that the minimal $T_0$-amalgama of $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ belongs to $\sigma^T_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$). Notice that, whenever $T_0$ is the empty theory in the empty signature, being $T_0$-strongly sub-amalgamable is the same as being strongly sub-amalgamable.

Theorem 3.2 immediately implies the following:

**Theorem 3.10.** *If $T_1, T_2$ are both $T_0$-compatible and $T_0$-strongly sub-amalgamable (over an amalgamable universal Horn theory $T_0$ in their common subsignature $\Sigma_0$), then so it is $T_1 \cup T_2$.*

*Proof.* Since $T_1$ and $T_2$ are $T_0$-strongly sub-amalgamable, their sub-amalgamation schemata $\sigma^{T_i}_{T_0}$ $(i = 1, 2)$ always contain minimal $T_0$-amalgamas. By Theorem 3.2 ($T_1$ and $T_2$ are also $T_0$-compatible), this implies that for every $T_1 \cup T_2$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$, the minimal amalgama $\mathcal{B}$ of $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})_{|\Sigma_0}$ belongs to the set $\sigma^{T_1 \cup T_2}_{T_0}[(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})]$. Using Proposition 2.3, we conclude that also $T_1 \cup T_2$ is $T_0$-compatible and $T_0$-strongly sub-amalgamable. $\dashv$

## 4 Applications to Modal Logic

Theorem 3.10 (obtained as a generalization of the analogous result from [3] for the disjoint signatures case) has surprising applications to modal logic. To get such applications, we need to reformulate it in the case of Boolean algebras with operators: the reformulation needs a further Theorem, showing that $T_0$-strong sub-amalgamability, in case $T_0$ is the theory of Boolean algebras, is nothing but the superamalgamability property known from algebraic logic. Let us recall the last property and state the Theorem we are still missing. For space reasons, all proofs in this section are deferred to [7].

In the following, we let $BA$ be the theory of Boolean algebras; a *BAO-equational theory*[5] is any theory $T$ whose signature extends the signature of Boolean algebras and whose axioms are all equations and include the Boolean algebra axioms. In [7] we shall recall in detail how BAO-equational theories are related to modal propositional logics via Lindenbaum constructions. The *fusion* of two BAO-equational theories $T_1$ and $T_2$ is just their combination $T_1 \cup T_2$ (when speaking of the fusion of $T_1$ and $T_2$, we assume that $T_1$ and $T_2$ share only the Boolean algebras operations and no other symbol).

The following Proposition is proved in [6] (proof is reported in [7]):

**Proposition 4.1.** *Every BAO-equational theory is BA-compatible.*

We say that a BAO-equational theory $T$ has the *superamalgamation* property iff for every $T$-fork $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{A})$ there exists a $T$-amalgam $(\mathcal{M}, \mu_1, \mu_2)$ such that

[5] BAO stands for 'Boolean algebras with operators'.

for every $a_1 \in |\mathcal{M}_1|, a_2 \in |\mathcal{M}_2|$ such that $\mu_1(a_1) \leq \mu_2(a_2)$ there exists $a_0 \in |\mathcal{A}|$ such that $a_1 \leq a_0$ holds in $\mathcal{M}_1$ and $a_0 \leq a_2$ holds in $\mathcal{M}_2$.[6]

We can now state our second main result (see [7] for the proof):

**Theorem 4.2.** *A BAO-equational theory $T$ has the superamalgamation property iff it is BA-strongly amalgamable.*

As an immediate consequence, from Theorem 3.10, we get:

**Corollary 4.3.** *If two BAO-equational theories $T_1$ and $T_2$ both have the superamalgamability property, so does their fusion.*

### 4.1 Superamalgamability and Interpolation in Propositional Logic

Corollary 4.3 immediately implies Wolter's result [26] on fusion transfer of Craig interpolation property for normal modal logics and says something new for non-normal modal logics too. To see all this, we only need to recall some background from propositional logic. For simplicity, we deal only with unary modalities (and, consequently, we shall consider only BAO-theories whose non-Boolean symbols are unary function symbols), however we point out that the extension to $n$-ary modalities is straightforward.

A *modal signature* $\Sigma_M$ is a set of unary operation symbols; from $\Sigma_M$, propositional *modal formulae* are built using countably many propositional variables, the operation symbols in $\Sigma_M$, the Boolean connectives $\cap, \cup, \sim$ and the constants 1 for truth and 0 for falsity. We use the letters $x, x_1, \ldots, y, y_1, \ldots$ to denote propositional variables and the letters $t, t_1, \ldots, u, u_1, \ldots$ to denote propositional formulae; $t \Rightarrow u$ and $t \Leftrightarrow u$ are abbreviations for $(\sim t) \cup u$ and for $(t \Rightarrow u) \cap (u \Rightarrow t)$, respectively. We use notations like $t(\underline{x})$ (resp. $\Gamma(\underline{x})$) to say that the modal formula $t$ (the set of modal formulae $\Gamma$) is built up from a set of propositional variables included in the tuple $\underline{x}$.

The following definition is taken from [21], pp. 8–9:

**Definition 4.4.** *A* classical modal logic *$L$ based on a modal signature $\Sigma_M$ is a set of modal formulae that*

(i) *contains all classical propositional tautologies;*
(ii) *is closed under uniform substitution of propositional variables by propositional formulae;*
(iii) *is closed under the modus ponens rule ('from $t$ and $t \Rightarrow u$ infer $u$');*
(iv) *is closed under the replacement rules, which are specified as follows. We have one such rule for each $o \in \Sigma_M$, namely:*

$$\frac{t \Leftrightarrow u}{o(t) \Leftrightarrow o(u)}$$

[6] We recall that in every Boolean algebra (more generally, in every semilattice) $x \leq y$ is defined as $x \cap y = x$, where $\cap$ is the meet operation.

*A classical modal logic $L$ is said to be* normal *iff for every modal operator $o \in \Sigma_M$, $L$ contains the modal formulae $o(1)$ and $o(y \Rightarrow z) \Rightarrow (o(y) \Rightarrow o(z))$.*

Since classical modal logics (based on a given modal signature) are closed under intersections, it makes sense to speak of the least classical modal logic $[S]$ containing a certain set of propositional formulae $S$. If $L = [S]$, we say that $S$ is a set of *axiom schemata* for $L$.

If $L_1$ is a classical modal logic over the modal signature $\Sigma_M^1$ and $L_2$ is a classical modal logic over the modal signature $\Sigma_M^2$ and $\Sigma_M^1 \cap \Sigma_M^2 = \emptyset$, the *fusion* $L_1 \oplus L_2$ is the modal logic $[L_1 \cup L_2]$ over the modal signature $\Sigma_M^1 \cup \Sigma_M^2$.

Given a modal logic $L$, a set of modal formulae $\Gamma$ and a modal formula $t$, the *global consequence relation* $\Gamma \vdash_L t$ holds iff there is a finite list of modal formulae $t_0, \dots, t_n$ such that: (i) $t_n$ is $t$; (ii) each $t_i$ is either a member of $L$ or a member of $\Gamma$ or is obtained from previous members of the list by applying one of the two inference rules from Definition 4.4 (i.e. modus ponens and replacement).

Global consequence relation should be contrasted with *local consequence relation*, to be indicated with $\vdash_L \Gamma \Rightarrow t$: this holds iff there are $g_1, \dots, g_n \in \Gamma$ such that $\bigcap_{i=1}^n g_i \Rightarrow t$ belongs to $L$. If $\Gamma$ consists of a single modal formula $g$, below we write $g \vdash_L t$ and $\vdash_L g \Rightarrow t$ instead of $\{g\} \vdash_L t$ and of $\vdash_L \{g\} \Rightarrow t$.

In case $L$ is normal, one can reduce the global consequence relation to the local one: in fact, it is not difficult to see by induction that the following fact ('deduction theorem') holds:

$$\Gamma \vdash_L t \qquad \text{iff} \qquad \vdash_L o\Gamma \Rightarrow t$$

where $o\Gamma$ is some finite set of modal formulae (depending on $t$) obtained from $\Gamma$ by prefixing a string of modal operators (i.e. elements of $o\Gamma$ are modal formulae of the kind $o_1(o_2 \cdots o_n(g) \cdots)$, for $g \in \Gamma$ and $n \geq 0$, $o_1, \dots, o_n \in \Sigma_M$).

Due to the presence of local and global consequence relations, we can formulate two different versions of the Craig's interpolation theorem:

**Definition 4.5.** *Let $L$ be a classical modal logic in a modal signature $\Sigma_M$.*

(i) *We say that $L$ enjoys the* local *interpolation property iff whenever we have $\vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$ for two modal formulae $t_1, t_2$, then there is a modal formula $u(\underline{x})$ such that $\vdash_L t_1 \Rightarrow u$ and $\vdash_L u \Rightarrow t_2$.*
(ii) *We say that $L$ enjoys the* global *interpolation property iff whenever we have $t_1(\underline{x}, \underline{y}) \vdash_L t_2(\underline{x}, \underline{z})$ for two modal formulae $t_1, t_2$, then there is a modal formula $u(\underline{x})$ such that $t_1 \vdash_L u$ and $u \vdash_L t_2$.*

For *normal* modal logics, in view of the above deduction theorem, it is easy to see that the local interpolation property implies the global one (but it is not equivalent to it, see [13]). In the non-normal case, there is no deduction theorem available, so that in order to have an interpolation property encompassing both the local and the global versions, it seems that a different notion needs to be introduced. This is what we are doing now.

Given a modal logic $L$ and two sets of modal formulae $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z})$, let us call an *$\underline{x}$-residue chain* a tuple of modal formulae $C(\underline{x}) = g_1(\underline{x}), \dots, g_k(\underline{x})$

such that we have $\Gamma_1 \cup \{g_1, \ldots, g_{2i}\} \vdash_L g_{2i+1}$ and $\Gamma_2 \cup \{g_1, \ldots, g_{2j-1}\} \vdash_L g_{2j}$, for all $i$ such that $0 \leq 2i < n$ and for all $j$ such that $0 < 2j \leq n$.

**Definition 4.6.** *Let $L$ be a classical modal logic in a modal signature $\Sigma_M$.*

(iii) *We say that $L$ enjoys the* comprehensive *interpolation property iff whenever we have $\Gamma_1(\underline{x}, \underline{y}), \Gamma_2(\underline{x}, \underline{z}) \vdash_L t_1(\underline{x}, \underline{y}) \Rightarrow t_2(\underline{x}, \underline{z})$ for two modal formulae $t_1, t_2$ and for two finite sets of modal formulae $\Gamma_1, \Gamma_2$, there are an $\underline{x}$-residue chain $C(\underline{x})$ and a modal formula $u(\underline{x})$ such that we have $\Gamma_1, C \vdash_L t_1 \Rightarrow u$ and $\Gamma_2, C \vdash_L u \Rightarrow t_2$.*

Notice that the comprehensive interpolation property implies both the local and the global interpolation properties; moreover, in the normal case, via deduction theorem, it is easily seen that the comprehensive interpolation property is equivalent to the local interpolation property. Our final result, giving an extension of Wolter's result [26] to non-normal case, is the following:

**Theorem 4.7.** *If the modal logics $L_1$ and $L_2$ both have the comprehensive interpolation property, so does their fusion $L_1 \oplus L_2$.*

The proof of the above Theorem is reported in [7] for space reasons; in fact, it requires some background, but only routine work. The idea is the following. One first recall that classical modal logics are in bijective correspondence with BAO-equational theories. Under this correspondence, in the normal case, global interpolation property coincides with quantifier-free interpolation (alias amalgamation property) and local interpolation property coincides with superamalgamability [13] (see [8] for a proof operating in a general context). Using similar techniques as in the above mentioned papers, in the non-normal general case, we show that *the comprehensive interpolation property coincides with superamalgamability*. Now it is sufficient to apply Corollary 4.3.

## 5 Conclusions and Future Work

In this paper we considered the problem of transferring the quantifier-free interpolation property from two theories to their union, in the case where the two theories share symbols other than pure equality.

We are not aware of previous papers attacking this problem. One should however mention a series of papers (e.g. [22,23,25]) analyzing the problem of transferring, in a hierarchical way, interpolation properties to theory extensions. This problem is related to ours, but it is different because there interpolation is assumed to hold for a basic theory $T_0$ and conditions on super-theories $T \supseteq T_0$ are analyzed in order to be able to extend interpolation to them. In our case, we are given interpolation properties for component theories $T_1, T_2$ and we are asked for modular conditions in order to transfer the property to $T_1 \cup T_2$.

To this aim, we obtained a sufficient condition (Theorem 3.2) in terms of sub-amalgamation schemata; we used such result to get a modular condition in case the shared theory is universal Horn (Theorem 3.10). For equational theories extending the theory of Boolean algebras, this modular condition turns out to be equivalent to the superamalgamability condition known from algebraic logic [14]. Thus, our results immediately imply the fusion transfer of local interpolation property [26] for classical normal modal logics. In the general non-normal case, the modularity of superamalgamability can be translated into a fusion transfer result for a new kind of interpolation property (which we called 'comprehensive interpolation property').

Still, many problems need to be faced by future research. Our combinability conditions should be characterizable from a syntactic point of view and, from such syntactic characterizations, we expect to be able to design concrete combined interpolation algorithms. Concerning modal logic, besides the old question about modularity of local interpolation property in the non-normal case, new questions arise concerning the status of the new comprehensive interpolation property: is it really stronger than other forms of interpolation property (e.g. than the local one)? Are there different ways of specifying it? Is it modular also for modal logics on a non-classical basis?

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