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Viscosity Solutions of Isaacs' Equations and Differential Games with Lipschitz Controls

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It is demonstrated that the upper and lower values of a two-person, zero-sum differential game solve the respective upper and lower Isaacs' equations in the viscosity sense (introduced by Crandall and Lions (*Trans. Amer. Math. Soc.* **277** (1983), 1–42). Since such solutions are unique, this yields a fairly simple proof that the game has value should the minimax condition hold. As a further application of viscosity techniques, a new and simpler proof that the upper and lower values can be approximated by the values of certain games with Lipschitz controls is given.

1. INTRODUCTION: VISCOSITY SOLUTIONS

Recent work of Crandall and Lions [5], expanded upon in Lions [14] and later reformulated and simplified in part by Crandall, Evans, and Lions [4], has introduced a new notion of generalized solutions for certain fully nonlinear, first-order partial differential equations. For the Hamilton–Jacobi equation in particular these new, so-called *viscosity solutions* exist and, more

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importantly, are unique under a wide variety of hypotheses. A major advance here beyond previous work (cf. Benton [3]) is the elimination of any requirement that the nonlinearities be convex.

Our aim in this paper is to demonstrate the utility of viscosity solution methods in studying various problems in two-person zero-sum differential game theory and to rederive with greatly simplified proofs some of the main results of Fleming [7], Friedman [8, 9], and Barron [1, 2]. (Notice, however, that our hypotheses are stronger in several instances.)

Some definitions and historical remarks are appropriate here. A *differential game* comprises an ordinary differential equation of the form

$$\text{(ODE)}\qquad \begin{cases} \dfrac{dx(s)}{ds} = f(s, x(s), y(s), z(s)), & t < s \leqslant T, \\ x(t) = x, \end{cases}$$

where $y(\cdot)$ and $z(\cdot)$ are the controls exercised by players y and z, and an associated payoff

$$\text{(P)}\qquad P(y(\cdot), z(\cdot)) \equiv g(x(T)) + \int_t^T h(s, x(s), y(s), z(s))\, ds.$$

(Full definitions are provided in Sect. 2.) The goal of y is to maximize P and the goal of z is to minimize P. The basic mathematical question is to discover, if possible, optimal strategies y and z should employ.

This last problem is difficult and need not in general have any very good solution. Nevertheless, Isaacs in [10] (cf. also [11]) initiated the study of (ODE), (P) by dynamic programming methods, already applied by R. Bellman to control (= one-person game) problems. The idea, loosely speaking, is first to define $V(t, x)$ as the payoff of the game above, provided y and z each play optimally, and next to discover a PDE that V formally satisfies. Then, conversely, should we find a smooth enough solution of this PDE, it must equal V; and a further analysis of the PDE leads to the synthesis of optimal controls for y and z. We will not address here this last, important aspect of the theory and refer the reader instead to [8].

There are profound difficulties in making rigorous the procedure outlined above. Fleming in [7] and after him Friedman [8, 9], Elliott and Kalton [6], and others have undertaken this task. Here we will follow Friedman and his definition of the *upper value* V^+ and the *lower value* V^- associated with the game: these correspond to the limits of certain discrete approximations, in the first of which y and in the second of which z has an advantage. (See Sect. 2 for complete definitions.) Friedman [8, 9] has shown that $V^\pm$ are Lipschitz, that V^+ solves Isaacs' equation (I^+) (defined in Sect. 2) a.e. and that V^- solves Isaacs' equation (I^-) a.e. These facts alone are not particularly useful as there are in general several distinct functions solving these equations a.e. This last difficulty pervades much of the existing game theory literature.

Our contribution here is to show in Theorem 3.1 that $V^{\pm}$ are viscosity solution of ($\mathrm{I}^{\pm}$), an important fact as such solutions are unique. A simple consequence is that $V^+ = V^-$, and so the game has value, should the *minimax* (or *Isaacs'*) *condition* hold; see Corollary 3.2. This last assertion was first proved by Fleming [7] using a rather complicated approximation via stochastic differential games; see also Friedman [9]. Souganidis [15] has independently obtained similar results,[1] and Lions [14] had earlier observed the connections between dynamic programming and viscosity solutions for control problems.

As a further demonstration of viscosity solution methods we present in Section 4 a greatly simplified proof of Barron's result [1] that $V^{\pm}$ can be approximated by the values of games with Lipschitz controls. An analogous assertion for nondegenerate stochastic differential games has been established by Jensen [12] (cf. also Jensen and Lions [13]).

We conclude by recording here the relevant definitions and properties of viscosity solutions.

Assume H: $[0, T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is continuous, and g: $\mathbb{R}^m \to \mathbb{R}$ is bounded, uniformly continuous. A bounded, uniformly continuous function u: $[0, T] \times \mathbb{R}^m \to \mathbb{R}$ is called a *viscosity solution* of the Hamilton–Jacobi equation

$$\text{(HJ)} \qquad \begin{cases} u_t + H(t, x, Du) = 0 & \text{in } (0, T) \times \mathbb{R}^m, \quad (1.1) \\ u(T, x) = g(x) & \text{in } \mathbb{R}^m, \quad (1.2) \end{cases}$$

provided (1.2) holds and for each $\phi \in C^1((0, T) \times \mathbb{R}^m)$,

(a) if $u - \phi$ attains a local maximum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^m$, then

$$\phi_t(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \geqslant 0 \tag{1.3}$$

and

(b) if $u - \phi$ attains a local minimum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^m$, then

$$\phi_t(t_0, x_0) + H(t_0, x_0, D\phi(t_0, x_0)) \leqslant 0. \tag{1.4}$$

See [4] for a proof that if u is a viscosity solution of (HJ) and if u is differentiable at some point (t_0, x_0), then

$$u_t(t_0, x_0) + H(t_0, x_0, Du(t_0, x_0)) = 0.$$

[1] Souganidis' proof is similar to and antedates ours by several months; the priority of discovery is his.

Assume next

$$\begin{aligned} &|H(t,x,p)-H(t,x,\hat{p})| \leqslant C\,|p-\hat{p}|, \\ &|H(t,x,p)-H(\hat{t},\hat{x},p)| \leqslant C(|x-\hat{x}|+|t-\hat{t}|)(1+|p|) \end{aligned} \tag{1.5}$$

for all x, $\hat{x}$, p, $\hat{p} \in \mathbb{R}^m$, $0 \leqslant t$, $\hat{t} \leqslant T$, and some constant C. Then there exists at most one viscosity solution of (HJ): this uniqueness assertion is the principal result of [5].

Remarks. We have described here the appropriate definition for the terminal value problem (1.1), (1.2); this is, as we shall see, the kind of PDE arising in game theory applications. A viscosity solution of the initial value problem (1.1),

$$u(x,0)=g(x) \qquad \text{in } \mathbb{R}^m, \tag{1.2$'$}$$

is defined by reversing the inequalities in (1.3), (1.4).

Note from [5] that uniqueness for viscosity solutions of (HJ) holds under various weaker assumptions than (1.5). Also, the simplified uniqueness proof given in [4] (for the case that H is independent of (t,x)) extends without much trouble to the present situation.[2]

2. The Upper and Lower Values

Fix $T > t \geqslant 0$, $x \in \mathbb{R}^m$, and consider then the ordinary differential equation

$$\text{(ODE)} \qquad \begin{cases} \dfrac{dx(s)}{ds} = f(s, x(s), y(s), z(s)), & t < s \leqslant T, \\ x(t) = x. \end{cases}$$

Here

$$y(\cdot) : [t,T] \to Y$$

and

$$z(\cdot) : [t,T] \to Z$$

are given measurable functions (called the *controls* employed by players y and z, respectively) and $Y \subset \mathbb{R}^k$, $Z \subset \mathbb{R}^l$ are given compact subsets, the *control sets*. In addition we will assume

$$f\colon [0,T] \times \mathbb{R}^m \times Y \times Z \to \mathbb{R}^m$$

[2] *Note added in proof.* A forthcoming paper by Evans and Souganidis (*Indiana Univ. Math. J.*, to appear) simplifies much of the following by adapting the approach of Elliott–Kalton [6] to differential games.

is uniformly continuous and satisfies

$$\begin{aligned} |f(t,x,y,z)| &\leqslant C_1, \\ |f(t,x,y,z)-f(t,\hat{x},y,z)| &\leqslant C_1\,|x-\hat{x}| \end{aligned} \tag{2.1}$$

for some constant C_1 and all $0 < t \leqslant T$, $x, \hat{x} \in \mathbb{R}^m$, $y \in Y$, $z \in Z$.

Then for each pair of controls $y(\cdot)$ and $z(\cdot)$ (ODE) has a unique, absolutely continuous solution $x(\cdot)$, called the *response* of the system to the controls $y(\cdot)$ and $z(\cdot)$.

Associated with these dynamics is the *payoff*

$$\text{(P)} \qquad P(y(\cdot), z(\cdot)) \equiv g(x(T)) + \int_t^T h(s, x(s), y(s), z(s))\, ds,$$

where $g\colon \mathbb{R}^m \to \mathbb{R}$ satisfies

$$\begin{aligned} |g(x)| &\leqslant C_2, \\ |g(x)-g(\hat{x})| &\leqslant C_2\,|x-\hat{x}|, \end{aligned} \tag{2.2}$$

and $h\colon [0, T] \times \mathbb{R}^m \times Y \times Z \to \mathbb{R}$ is uniformly continuous and satisfies

$$\begin{aligned} |h(t,x,y,z)| &\leqslant C_2, \\ |h(t,x,y,z)-h(t,\hat{x},y,z)| &\leqslant C_2\,|x-\hat{x}| \end{aligned} \tag{2.3}$$

for some constant C_2 and all $0 \leqslant t \leqslant T$, $x, \hat{x} \in \mathbb{R}^m$, $y \in Y$, $z \in Z$.

The goal of player y is to maximize P and the goal of player z is to minimize P. How should these players choose their controls?

As noted in Section 1, one major approach to this problem, initiated by Isaacs [10] and made rigorous by Fleming [7], Friedman [8, 9], and Elliott and Kalton [6], consists of studying the value functions $V^{\pm}$ and the PDE they satisfy.

We begin with a number of definitions taken from [8].

Choose first a positive integer n and then define

$$\delta = \delta_n \equiv \frac{T-t}{n}.$$

Divide $[t, T]$ into n subintervals

$$I_j \equiv \{t \mid t_{j-1} < t < t_j\} \qquad (j = 1, \ldots, n),$$

where

$$t_j \equiv t + j\delta \qquad (j = 0, \ldots, n).$$

Let Y_j (resp. Z_j) denote the set of all measurable mappings from I_j into Y (resp. Z); we will identify any two such mappings which agree a.e.

Next, denote by $\Gamma^{\delta,j}$ any map from $Z_1 \times Y_1 \times \cdots \times Z_{j-1} \times Y_{j-1} \times Z_j$ into Y_j. Any collection

$$\Gamma^{\delta} \equiv (\Gamma^{\delta,1},..., \Gamma^{\delta,n})$$

is then called an *upper δ-strategy for y*. Furthermore,

$$\Delta^{\delta} \equiv (\Delta^{\delta,1},..., \Delta^{\delta,n})$$

is an *upper δ-strategy for z*, where for each j, $\Delta^{\delta,j}$ is any map from $Y_1 \times Z_1 \times \cdots \times Y_{j-1} \times Z_{j-1} \times Y_j$ into Z_j.

Similarly, a *lower δ-strategy for y* is a collection

$$\Gamma_{\delta} \equiv (\Gamma_{\delta,1},..., \Gamma_{\delta,n}),$$

where $\Gamma_{\delta,1}$ is any function in Y_1 and $\Gamma_{\delta,j}$ $(j=2,.\dot{}.,n)$ is any mapping from $Y_1 \times Z_1 \times \cdots \times Y_{j-1} \times Z_{j-1}$ into Y_j. A lower *δ-strategy for z* is a collection

$$\Delta_{\delta} \equiv (\Delta_{\delta,1},..., \Delta_{\delta,n}),$$

where $\Delta_{\delta,1}$ is any function in Z_1 and $\Delta_{\sigma,j}$ $(j=2,.\dot{}.,n)$ is any mapping from $Z_1 \times Y_1 \times \cdots \times Z_{j-1} \times Y_{j-1}$ into Z_j.

Given a pair $(\Delta_{\delta}, \Gamma^{\delta})$ as above we define the controls $y^{\delta}(\cdot)$, $z_{\delta}(\cdot)$, with components y_j, z_j on I_j, this way:

$$z_j = \Delta_{\delta,1}, \qquad y_1 = \Gamma^{\delta,1}(z_1)$$

and

$$z_j = \Delta_{\delta,j}(z_1, y_1,..., z_{j-1}, y_{j-1}),$$
$$y_j = \Gamma^{\delta,j}(z_1, y_1,..., z_{j-1}, y_{j-1}, z_j)$$

for $j=2,.\dot{}.,n$. Call $(y^{\delta}(\cdot), z_{\delta}(\cdot))$ the *outcome* of $(\Delta_{\delta}, \Gamma^{\delta})$ and denote by $x^{\delta}(\cdot)$ the corresponding solution of (ODE). We also write

$$P[\Delta_{\delta}, \Gamma^{\delta}] = P(y^{\delta}(\cdot), z_{\delta}(\cdot))$$

to denote the associated payoff.

Given a pair $(\Gamma_{\delta}, \Delta^{\delta})$ we similarly define $x_{\delta}(\cdot)$, $y_{\delta}(\cdot)$, $z^{\delta}(\cdot)$, $P[\Gamma_{\delta}, \Delta^{\delta}]$.

Finally let us set

$$V^{\delta} \equiv \inf_{\Delta_{\delta}} \sup_{\Gamma^{\delta}} P[\Delta_{\delta}, \Gamma^{\delta}] = \sup_{\Gamma^{\delta}} \inf_{\Delta_{\delta}} P[\Delta_{\delta}, \Gamma^{\delta}].$$

This is the *upper δ-value*, and the second equality here is [8, Theorem 1.4.1]. Analogously,

$$V_{\delta} \equiv \inf_{\Delta^{\delta}} \sup_{\Gamma_{\delta}} P[\Gamma_{\delta}, \Delta^{\delta}] = \sup_{\Gamma_{\delta}} \inf_{\Delta^{\delta}} P[\Gamma_{\delta}, \Delta^{\delta}]$$

is the *lower δ-value.*

See [8, Theorems 1.4.3 and 1.4.4] for proofs that

$$V^{\delta_n} \geqslant V^{\delta_m} \geqslant V_{\delta_m} \geqslant V_{\delta_n}$$

for $\delta_n = (T-t)/n$, $\delta_m = (T-t)/m$, $m = kn$, $k \geqslant 1$. Hence

$$V^+ \equiv \lim_{n \to \infty} V^{\delta_n} \tag{2.4}$$

and

$$V^- \equiv \lim_{n \to \infty} V_{\delta_n} \tag{2.5}$$

exist; call V^+ the *upper value* of the differential game and V^- the *lower value*.

We will write $V^\delta = V^\delta(t, x)$, $V_\delta = V_\delta(t, x)$, $V^\pm = V^\pm(t, x)$ to display the dependence on (t, x) in (ODE) and (P).

According to [8, Theorem 2.6.4],

$$V^{\delta_n}(t, x) \to V^+(t, x),$$
$$V_{\delta_n}(t, x) \to V^-(t, x)$$

uniformly for (t, x) contained in compact subsets of $[0, T) \times \mathbb{R}^m$, $\delta_n = (T-t)/n$, $n = 1, 2, \ldots$.

3. Viscosity Solutions of Isaacs' Equations

It is known— and we will reprove this below—that $V^\pm$ are uniformly Lipschitz continuous, are thus differentiable almost everywhere, and satisfy a.e. *Isaacs' equations*

$$(\mathrm{I}^+) \qquad \begin{cases} V_t^+ + H^+(t, x, DV^+) = 0 & (0 \leqslant t \leqslant T,\ x \in \mathbb{R}^m) \\ V^+(T, x) = g(x) \end{cases}$$

and

$$(\mathrm{I}^-) \qquad \begin{cases} V_t^- + H^-(t, x, DV^-) = 0 & (0 \leqslant t \leqslant T,\ x \in \mathbb{R}^m) \\ V^-(T, x) = g(x), \end{cases}$$

where

$$H^+(t, x, p) \equiv \min_{z \in Z} \max_{y \in Y} \{f(t, x, y, z) \cdot p + h(t, x, y, z)\}$$

and

$$H^-(t, x, p) \equiv \max_{y \in Y} \min_{z \in Z} \{f(t, x, y, z) \cdot p + h(t, x, y, z)\}$$

are, respectively, the *upper* and *lower Hamiltonians*.

Our principal assertion is that $V^{\pm}$ in fact satisfy Isaacs' equations in the viscosity sense.

THEOREM 3.1. *V^+ is the viscosity solution of* (I^+) *and V^- is the viscosity solution of* (I^-).

COROLLARY 3.2. *If for all $0 \leqslant t \leqslant T$, $x, p \in \mathbb{R}^m$,*

$$H^+(t, x, p) = H^-(t, x, p), \qquad \text{(minimax condition)}$$

then

$$V^+ \equiv V^-.$$

In this case we say the game has a value $V \equiv V^{\pm}$.

Corollary 3.2 is an immediate consequence of the uniqueness of viscosity solutions; note that $H^{\pm}$ satisfy (1.5).

Proof[3] (of Theorem 3.1.). We will prove V^- is a viscosity solution of (I^-); the proof for V^+ is similar.

Assume $\phi \in C^1$ and $V^- - \phi$ attains a local maximum at some point $(t_0, x_0) \in (0, T) \times \mathbb{R}^m$. We must show

$$\phi_t(t_0, x_0) + H^-(t_0, x_0, D\phi(t_0, x_0)) \geqslant 0. \tag{3.1}$$

Suppose to the contrary this inequality fails; then

$$\phi_t(t_0, x_0) + H^-(t_0, x_0, D\phi(t_0, x_0)) \leqslant -\theta < 0 \tag{3.2}$$

for some $\theta > 0$.

Set

$$\delta = \delta_n = \frac{T - t_0}{n},$$

where n is a positive integer to be selected later.

The principle of optimality for δ-games [8, p. 128] states

$$\sup_{\Gamma_\delta} \inf_{\Delta^\delta} \left\{ V_\delta(t^0 + \varepsilon, x_\delta(t_0 + \varepsilon)) - V_\delta(t_0, x_0) + \int_{t_0}^{t_0 + \varepsilon} h(s, x_\delta(s), y_\delta(s), z^\delta(s))\, ds \right\} = 0, \tag{3.3}$$

[3] This proof is in part modelled on Friedman [8, pp. 127–131].

where $\varepsilon = k\delta$ for some positive integer k, $t_0 + \varepsilon \leqslant T$, Γ_δ and Δ^δ are δ-strategies on the interval $t_0 \leqslant s \leqslant t_0 + \varepsilon$, $(y_\delta(\cdot), z^\delta(\cdot))$ is the outcome of $(\Gamma_\delta, \Delta^\delta)$, and $x_\delta(\cdot)$ is the solution of (ODE) for $t_0 \leqslant s \leqslant t_0 + \varepsilon$ corresponding to y_δ, z^δ and the initial condition $x_\delta(t_0) = x_0$.

Now by Lemma 3.3(a), proved below, it follows from (3.2) that for all sufficiently small δ and ε there exists an upper δ-strategy Δ^δ, such that for any lower δ-strategy Γ_δ

$$\int_{t_0}^{t_0+\varepsilon} \phi_t(s, x_\delta(s)) + f(s, x_\delta(s), y_\delta(s), z^\delta(s)) \cdot D\phi(s, x_\delta(s)) \\ + h(s, x_\delta(s), y_\delta(s)\, z^\delta(s))\, ds \leqslant -\frac{\theta}{2}\varepsilon. \tag{3.4}$$

But

$$\begin{aligned} &\phi(t_0 + \varepsilon, x_\delta(t_0 + \varepsilon)) - \phi(t_0, x_0) \\ &\quad = \int_{t_0}^{t_0+\varepsilon} \frac{d}{ds}\phi(s, x_\delta(s))\, ds \qquad (3.5) \\ &\quad = \int_{t_0}^{t_0+\varepsilon} \phi_t(s, x_\delta(s)) + f(s, x_\delta(s), y_\delta(s), z^\delta(s)) \cdot D\phi(s, x_\delta(s))\, ds, \end{aligned}$$

by (ODE). In addition, since $V^- - \phi$ has a local maximum at (t_0, x_0), we have

$$\begin{aligned} &V^-(t_0 + \varepsilon, x_\delta(t_0 + \varepsilon)) - V^-(t_0, x_0) \\ &\quad \leqslant \phi(t_0 + \varepsilon, x_\delta(t_0 + \varepsilon)) - \phi(t_0, x_0), \end{aligned}$$

provided ε is small enough. This estimate, (3.4), and (3.5) together imply

$$\begin{aligned} &V^-(t_0 + \varepsilon, x_\delta(t_0 + \varepsilon)) - V^-(t_0, x_0) \\ &\quad + \int_{t_0}^{t_0+\varepsilon} h(s, x_\delta(s), y_\delta(s), z^\delta(s))\, ds \leqslant -\frac{\theta}{2}\varepsilon. \end{aligned}$$

Thus

$$\begin{aligned} \sup_{\Gamma_\delta} \inf_{\Delta^\delta} \Big\{ &V^-(t_0 + \varepsilon, x_\delta(t_0 + \varepsilon)) - V^-(t_0, x_0) \\ &+ \int_{t_0}^{t_0+\varepsilon} h(s, x_\delta(s), y_\delta(s), z^\delta(s))\, ds \Big\} \leqslant -\frac{\theta}{2}\varepsilon \end{aligned}$$

for all sufficiently small δ and ε. Now

$$V_{\delta_n}(t, x) \to V^- \qquad \text{as} \quad n \to \infty,$$

uniformly for (t, x) contained in compact subsets of $[0, T) \times \mathbb{R}^m$, $\delta_n = (T-t)/n$. Thus we may choose some small $\varepsilon > 0$ and then choose $\delta = \delta_n = (T-t_0)/n$, $\varepsilon = k_n \delta$, so that

$$\sup_{\Gamma_\delta} \inf_{\Delta^\delta} \left\{ V_\delta(t_0+\varepsilon, x_\delta(t_0+\varepsilon)) - V_\delta(t_0, x_0) \right.$$
$$\left. + \int_{t_0}^{t_0+\varepsilon} h(s, x_\delta(s), y_\delta(s), z^\delta(s))\, ds \right\} \leqslant -\frac{\theta}{4}\varepsilon.$$

This contradicts the optimality principle (3.3) and thereby proves (3.1).

Now assume $V^- - \phi$ attains a local minimum at (t_0, x_0). We must prove

$$\phi_t(t_0, x_0) + H^-(t_0, x_0, D\phi(t_0, x_0)) \leqslant 0, \tag{3.6}$$

and will therefore assume on the contrary that

$$\phi_t(t_0, x_0) + H^-(t_0, x_0 D\phi(t_0, x_0)) \geqslant \theta > 0 \tag{3.7}$$

for some $\theta > 0$.

Then there exists for all sufficiently small $\delta = (T-t)/n$ and $\varepsilon = k\delta$ a lower δ-strategy Γ_δ, such that for each δ-strategy Δ^δ we have

$$\int_{t_0}^{t_0+\varepsilon} \phi_t(s, x_\delta(s)) + f(s, x_\delta(s), y_\delta(s), z^\delta(s)) \cdot D\phi(s, x_\delta(s))$$
$$+ h(s, x_\delta(s), y_\delta(s), z^\delta(s))\, ds \geqslant \frac{\theta}{2}\varepsilon; \tag{3.8}$$

this is Lemma 3.3(b). Since

$$V^-(t_0+\varepsilon, x_\delta(t_0+\varepsilon)) - V^-(t_0, x_0)$$
$$\geqslant \phi(t_0+\varepsilon, x_\delta(t_0+\varepsilon)) - \phi(t_0, x_0)$$

if ε is small enough, we may reason as above to arrive at the contradiction

$$\sup_{\Gamma_\delta} \inf_{\Delta^\delta} \left\{ V_\delta(t_0+\varepsilon, x_\delta(t_0+\varepsilon)) - V_\delta(t_0, x_0) \right.$$
$$\left. + \int_{t_0}^{t_0+\varepsilon} h(s, x_\delta(s), y_\delta(s), z^\delta(s))\, ds \right\} \geqslant \frac{\theta}{4}\varepsilon$$

for some sufficiently small δ, $\varepsilon > 0$. ∎

LEMMA 3.3. *Assume* $\phi \in C^1$.

(a) *If* ϕ *satisfies* (3.2), *then there exists for all sufficiently small* δ *and* ε *an upper* δ*-strategy* Δ^δ *such that* (3.4) *holds for each lower* δ*-strategy* Γ_δ.

(b) *If ϕ satisfies* (3.7), *then there exists for all sufficiently small δ and ε a lower δ-strategy Γ_δ that* (3.8) *holds for each upper δ-strategy Δ^δ.*

Proof. Set

$$\Lambda(t, x, y, z) \equiv \phi_t(t, x) + f(t, x, y, z) \cdot D\phi(t, x) + h(t, x, y, z).$$

(a) According to (3.2) we have

$$\max_{y \in Y} \min_{z \in Z} \Lambda(t_0, x_0, y, z) \leqslant -\theta < 0. \tag{3.9}$$

This implies that for each $y \in Y$ there exists $z \in Z$ such that

$$\Lambda(t_0, x_0, y, z) \leqslant -\theta.$$

As Λ is uniformly continuous, in fact

$$\Lambda(t_0, x_0, \eta, z) \leqslant -\tfrac{3}{4}\theta$$

for all $\eta \in B(y, r) \cap Y$ and some $r = r(y) > 0$. Since Y is compact, there thus exist distinct points $y_1, \ldots, y_n \in Y$, $z_1, \ldots, z_n \in Z$, and $r_1, \ldots, r_n > 0$ such that

$$Y \subset \bigcup_{i=1}^{n} B(y_i, r_i)$$

and

$$\Lambda(t_0, x_0, \eta, z_i) \leqslant -\frac{3\theta}{4} \qquad \text{for all} \quad \eta \in B(y_i, r_i) \cap Y, \quad i = 1, \ldots, n.$$

Define ϕ: $Y \to Z$ by setting

$$\phi(y) = z_k$$

whenever $y \in B(y_k, r_k) \backslash \bigcup_{i=1}^{k-1} B(y_i, r_i)$. Then

$$\Lambda(t_0, x_0, y, \phi(y)) \leqslant -\frac{3\theta}{4}$$

and in fact

$$\Lambda(t, x(t), y, \phi(y)) \leqslant -\frac{\theta}{2} \tag{3.10}$$

for all $y \in Y$, $t_0 \leqslant t \leqslant t_0 + \varepsilon$, and any solution $x(\cdot)$ of (ODE) (with initial condition $x(t_0) = x$, and any controls $y(\cdot)$, $z(\cdot)$), provided $\varepsilon > 0$ is fixed to be sufficiently small.

It is easy to check that $\phi(y(t))$ is a measurable mapping for any measurable function $y(t)$.

Now choose any sufficiently small $\delta = \delta_n = (T - t_0)/n$, $\varepsilon = k\delta$, and let $\Gamma_\delta = (\Gamma_{\delta,1}, ..., \Gamma_{\delta,k})$ be a lower δ-strategy on $[t_0, t_0 + \varepsilon]$. Write

$$y_1 = y_1(t) = \Gamma_{\delta,1} \qquad (t \in I_1)$$

and then define

$$z_1 = z_1(t) = \phi(y_1(t)) \equiv \Delta^{\delta,1}(y_1) \qquad (t \in I_1).$$

If

$$y_2 = y_2(t) = \Gamma_{\delta,2}(y_1, z_1) \qquad (t \in I_2)$$

we set

$$z_2 = z_2(t) = \phi(y_2(t)) \equiv \Delta^{\delta,2}(y_1, z_1, y_2) \qquad (t \in I_2).$$

We continue in this way to define

$$z_j = z_j(t) = \phi(y_j(t)) = \Delta^{\delta,j}(y_1, z_1, \ldots, y_j) \qquad (t \in I_j)$$

for $j = 3, \ldots, k$. Now let $(y_\delta(\cdot), z^\delta(\cdot))$ be the outcome of $(\Gamma_\delta, \Delta^\delta)$ and $x_\delta(\cdot)$ the corresponding solution of (ODE). In light of (3.10), we have

$$\Lambda(t, x_\delta(t), y_\delta(t), z^\delta(t)) \leqslant -\frac{\theta}{2}$$

for all $t_0 \leqslant t \leqslant t_0 + \varepsilon$, and so (3.4) holds.

(b) Next assume that (3.7) is valid:

$$\max_{y \in Y} \min_{z \in Z} \Lambda(t_0, x_0, y, z) \geqslant \theta > 0. \tag{3.11}$$

Then there exists some $y^* \in Y$ such that for all $z \in Z$,

$$\Lambda(t_0, x_0, y^*, z) \geqslant \theta.$$

Since Λ is uniformly continuous we have also

$$\Lambda(t, x(t), y^*, z) \geqslant \frac{\theta}{2} \tag{3.12}$$

for all $z \in Z$, $t_0 \leqslant t \leqslant t_0 + \varepsilon$, and any solution $x(\cdot)$ of (ODE) (with initial condition $x(t_0) = x$ and any controls $y(\cdot). z(\cdot)$) povided $\varepsilon > 0$ is fixed to be small enough.

Now select any sufficiently small $\delta = \delta_n = (T - t_0)/n$, $\varepsilon = k\delta$, and define the lower δ-strategy $\Gamma_\delta = (\Gamma_{\delta,1}, \ldots, \Gamma_{\delta,k})$ on $[t_0, t_0 + \varepsilon]$ by setting

$$\Gamma_{\delta,j} \equiv y^* \qquad (j = 1, \ldots, k).$$

Let Δ^δ be any upper δ-strategy on $[t_0, t_0 + \varepsilon]$, $(y_\delta(\cdot), z^\delta(\cdot))$ the outcome of $(\Gamma_\delta, \Delta^\delta)$, and $x_\delta(\cdot)$ the corresponding solution of (ODE). In view of (3.12) we have

$$\Lambda(t, x_\delta(t), y_\delta(t), z^\delta(t)) \geqslant \frac{\theta}{2}$$

for all $t_0 \leqslant t \leqslant t_0 + \varepsilon$, and so (3.8) is valid. ∎

4. An Application: Approximation by Games with Lipschitz Controls

Consider now the differential game

$$\text{(ODE}'\text{)} \qquad \begin{cases} \dfrac{dx(s)}{ds} = f(s, x(s), y(s), z(s)) & (t < s \leqslant T), \\[2mm] \dfrac{dy(s)}{ds} = u(s) & (t < s \leqslant T), \\[2mm] \dfrac{dz(s)}{ds} = v(s) & (t < s \leqslant T), \\[2mm] x(t) = x, \qquad y(t) = y, \qquad z(t) = z, \end{cases}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, and $z \in \mathbb{R}^l$ are given. Here

$$u(\cdot)\colon [t, T] \to U^M \equiv \{u \in \mathbb{R}^k \mid |u| \leqslant M\}$$

and

$$v(\cdot)\colon [t, T] \to V^L \equiv \{v \in \mathbb{R}^l \mid |v| \leqslant L\}$$

are the controls, and $(x(\cdot), y(\cdot), z(\cdot))$ the corresponding response of the system.

The associated payoff is

$$\text{(P}'\text{)} \qquad P(u(\cdot), v(\cdot)) \equiv g(x(T)) + \int_t^T h(s, x(s), y(s), z(s))\, ds.$$

We will assume

$$
\begin{aligned}
&|f(t,x,y,z)| \leqslant C_3, \\
&|f(t,x,y,z)-f(\hat{t},\hat{x},\hat{y},\hat{z})| \leqslant C_3(|t-\hat{t}|+|x-\hat{x}|+|y-\hat{y}|+|z-\hat{z}|),
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
&|g(x)| \leqslant C_3, \\
&|g(x)-g(\hat{x})| \leqslant C_3|x-\hat{x}|,
\end{aligned} \tag{4.2}
$$

and

$$
\begin{aligned}
&|h(t,x,y,z)| \leqslant C_3, \\
&|h(t,x,y,z)-h(\hat{t},\hat{x},\hat{y},\hat{z})| \leqslant C_3(|t-\hat{t}|+|x-\hat{x}|+|y-\hat{y}|+|z-\hat{z}|),
\end{aligned} \tag{4.3}
$$

for some constant C_3 and all $0 \leqslant t,\ \hat{t} \leqslant T,\ x,\ \hat{x} \in \mathbb{R}^m,\ y,\ \hat{y} \in \mathbb{R}^k,\ z,\ \hat{z} \in \mathbb{R}^l$.

In this situation we have

$$
\begin{aligned}
H^+&(t,x,y,z,p,q,r) \\
&= \min_{|v|\leqslant L}\ \max_{|u|\leqslant M}\ \{f(t,x,y,z)\cdot p + u\cdot q + v\cdot r + h(t,x,y,z)\} \\
&= f(t,x,y,z)\cdot p + M|q| - L|r| + h(t,x,y,z)
\end{aligned}
$$

and

$$
\begin{aligned}
H^-&(t,x,y,z,p,q,r) \\
&= \max_{|u|\leqslant M}\ \min_{|v|\leqslant L}\ \{f(t,x,y,z)\cdot p + u\cdot q + v\cdot r + h(t,x,y,z)\} \\
&= f(t,x,y,z)\cdot p + M|q| - L|r| + h(t,x,y,z)
\end{aligned}
$$

for all $0 \leqslant t \leqslant T,\ x,\ p \in \mathbb{R}^m,\ y,\ q \in \mathbb{R}^k,\ z,\ r \in \mathbb{R}^l$. Hence

$$H^+ \equiv H^-,$$

so that Corollary 3.2 implies the differential game to have a value

$$V^{M,L} = V^{M,L}(t,x,y,z).$$

Notice that

$$|V^{M,L}| \leqslant C_3(T+1) \qquad \text{by (4.2), (4.3).} \tag{4.4}$$

Now from Theorem 3.1 we deduce

THEOREM 4.1. *$V^{M,L}$ is the (unique) viscosity solution of*

$$
(\mathrm{I}^{M,L}) \quad
\begin{cases}
V_t^{M,L} + f(t,x,y,z)\cdot D_x V^{M,L} + M|D_y V^{M,L}| - L|D_z V^{M,L}| \\
\qquad + h(t,x,y,z) = 0 \qquad (0 \leqslant t \leqslant T,\ (x,y,z) \in \mathbb{R}^{m+k+l}). \\
V^{M,L}(T,x,y,z) = g(x).
\end{cases}
$$

Our problem (ODE′), (P′) is a model of a differential game in which the players y and z are constrained to choose Lipschitz controls $y(\cdot)$ and $z(\cdot)$ (the Lipschitz constant for y is M and for z is L). Notice that control of the derivatives $u(\cdot)$ and $v(\cdot)$ is equivalent to direct control of $y(\cdot)$ and $z(\cdot)$ in this case.

Now it seems clear heuristically that if $M, L \to \infty$, that is, if the Lipschitz constraints on the derivatives of $y(\cdot)$ and $z(\cdot)$ are relaxed, then our problem (ODE′), (P′) should in some sense converge to a problem like that studied in Section 3 and therefore $V^{M,L}$ should converge to the solution of some PDE similar to (I^+) or (I^-).

Barron in [1, 2] has formulated and proved via game theory techniques various assertions of this kind. As an application of the new viscosity solution methods we present here a new (and simpler) proof of one of the principal results from [1]. First we need

LEMMA 4.1. *Under the above assumptions $V^{M,L}$ is uniformly Lipschitz and so differentiable* a.e., *with the estimate*

$$|V_t^{M,L}|,\ |D_x V^{M,L}|,\ |D_y V^{M,L}|,\ |D_z V^{M,L}| \leqslant C_4, \tag{4.5}$$

for some constant C_4, independent of M and L. *In particular, $V^{M,L}$ solves* $(\mathrm{I}^{M,L})$ a.e.

Proof. Let us temporarily suppose that f, g, h are smooth functions satisfying (4.1)–(4.3). Fix $\varepsilon > 0$ and set

$$|w|_\varepsilon \equiv (|w|^2 + \varepsilon)^{1/2}.$$

Consider now this (backwards) parabolic approximation to $(\mathrm{I}^{M,L})$:

$$(\mathrm{I}^\varepsilon) \qquad \begin{cases} V_t^\varepsilon + f \cdot D_x V^\varepsilon + M\,|D_y V^\varepsilon|_\varepsilon - L\,|D_z V^\varepsilon|_\varepsilon + h + \varepsilon\,\Delta V^\varepsilon = 0, \\ V^\varepsilon(x, y, z, T) = g(x). \end{cases}$$

According to standard PDE theory there exists a unique smooth solution $V^\varepsilon = V^{\varepsilon,M,L}$ of (I^ε). We will prove the stated bounds (4.5) for V^ε and later send $\varepsilon \to 0$.

Define

$$W \equiv e^{\lambda t}(|D_x V^\varepsilon|^2 + 1),$$

where $\lambda > 0$ is a constant to be selected later. Then

$$\begin{aligned} W_t &= \lambda e^{\lambda t}(|D_x V^\varepsilon|^2 + 1) + 2e^{\lambda t} V_{x_i}^\varepsilon V_{x_i t}^\varepsilon, \\ W_{x_j} &= 2e^{\lambda t} V_{x_i}^\varepsilon V_{x_i x_j}^\varepsilon \qquad (1 \leqslant j \leqslant m), \\ W_{y_j} &= 2e^{\lambda t} V_{x_i}^\varepsilon V_{x_i y_j}^\varepsilon \qquad (1 \leqslant j \leqslant k), \\ W_{z_j} &= 2e^{\lambda t} V_{x_i}^\varepsilon V_{x_i z_j}^\varepsilon \qquad (1 \leqslant j \leqslant l), \end{aligned}$$

and

$$\Delta W = 2e^{\lambda t}(V^{\varepsilon}_{x_i} \Delta V^{\varepsilon}_{x_i} + |DD_x V^{\varepsilon}|^2).$$

Set

$$a_j \equiv \frac{V^{\varepsilon}_{y_j}}{(|D_y V^{\varepsilon}|^2 + \varepsilon)^{1/2}} \qquad (1 \leqslant j \leqslant k)$$

and

$$b_j \equiv \frac{V^{\varepsilon}_{z_j}}{(|D_z V^{\varepsilon}|^2 + \varepsilon)^{1/2}} \qquad (1 \leqslant j \leqslant l).$$

Next differentiate (I^{ε}) with respect to x_i:

$$\begin{aligned} V^{\varepsilon}_{x_i t} + f \cdot D_x V^{\varepsilon}_{x_i} + Ma \cdot D_y V^{\varepsilon}_{x_i} - Lb \cdot D_z V^{\varepsilon}_{x_i} + \varepsilon \Delta V^{\varepsilon}_{x_i} \\ = -h_{x_i} - f_{x_i} D_x V^{\varepsilon} \qquad (1 \leqslant i \leqslant m). \end{aligned} \tag{4.6}$$

Thus we may compute

$$\begin{aligned} LW &\equiv W_t + f \cdot D_x W + Ma \cdot D_y W - Lb \cdot D_z W + \varepsilon \Delta W \\ &= \lambda e^{\lambda t}(|D_x V^{\varepsilon}|^2 + 1) + 2e^{\lambda t}(\varepsilon\, |DD_x V^{\varepsilon}|^2 + \varepsilon V^{\varepsilon}_{x_i} \Delta V^{\varepsilon}_{x_i}) \\ &\quad + 2e^{\lambda t} V^{\varepsilon}_{x_i}(V^{\varepsilon}_{x_i t} + f \cdot D_x V^{\varepsilon}_{x_i} + Ma \cdot D_y V^{\varepsilon}_{x_i} - Lb \cdot D_z V^{\varepsilon}_{x_i}) \\ &\geqslant e^{\lambda t}(\lambda\, |D_x V^{\varepsilon}|^2 + \lambda - 2V^{\varepsilon}_{x_i}(-h_{x_i} - f_{x_i} \cdot D_x V^{\varepsilon})) \qquad \text{(by (4.6))} \\ &> 0, \end{aligned}$$

provided $\lambda = \lambda(C_3)$ is large enough. Consequently the maximum principle implies that

$$\sup_{x,y,z,t} W(x, y, z, t) = \sup_{x,y,z} W(x, y, z, T) = e^{\lambda T}(\|D_x g\|^2 + 1).$$

As the latter quantity is bounded according to (4.3), we have obtained global bounds on W and thus on $|D_x V^{\varepsilon}|$, independently of L, M, ε.

Estimates for $|D_y V^{\varepsilon}|$, $|D_z V^{\varepsilon}|$, and $|V^{\varepsilon}_t|$ have similar proofs once the bound for $|D_x V|$ is established. Note in particular that the estimate for $|V^{\varepsilon}_t|$ depends on $|D_x g| + \varepsilon\, |\Delta g|$.

Since the bounds above are independent of $\varepsilon > 0$, there exists $\varepsilon_j \searrow 0$ and a function $\hat{V}$ such that

$$V^{\varepsilon_j} \to \hat{V} \qquad \text{locally uniformly.}$$

Now according to arguments like those in [4, 5], $\hat{V}$ is the (unique) viscosity solution of ($\mathrm{I}^{M,L}$). Hence $\hat{V} = V^{M,L}$ and in fact

$$V^{\varepsilon} \to V^{M,L} \qquad \text{locally uniformly.}$$

Furthermore $V^{M,L}$ inherits the uniform gradient bounds, so that (4.4) holds for some $C_4 = C_4(C_3, T)$, independent of M, L. We conclude the proof by approximating, if necessary f, g, and h by smooth functions satisfying (4.1)–(4.3). ∎

Remark. Under assumption (4.9) below, game theoretic arguments imply the additional estimates

$$|D_y V^{M,L}| \leqslant \frac{C_5}{L}, \qquad |D_z V^{M,L}| \leqslant \frac{C_5}{M}$$

for some constant $C_5 = C_5(C_3, L)$ (cf. [12]). See also Barron [2] for a game theoretic proof of Lemma 4.1.

We now propose to investigate the limit of $V^{M,L}$ as $M, L \to \infty$. For this let us take

$$Y \equiv \{y \in \mathbb{R}^k \mid 0 \leqslant y_j \leqslant 1, j = 1,..., k\}, \tag{4.7}$$

$$Z \equiv \{z \in \mathbb{R}^l \mid 0 \leqslant z_j \leqslant 1, j = 1,..., l\}. \tag{4.8}$$

We will assume in addition to (4.1)–(4.3) that

$$\begin{gathered} f \text{ and } h \text{ are periodic, with period 1, as functions} \\ \text{of } y_j (1 \leqslant j \leqslant k) \text{ and } z_j (1 \leqslant j \leqslant l). \end{gathered} \tag{4.9}$$

This implies, by uniqueness, that $V^{M,L}$ is 1-periodic in y and z as well.

THEOREM 4.2. *Under the above hypotheses,*

$$\lim_{L \to \infty} \lim_{M \to \infty} V^{M,L} = V^+ \tag{4.10}$$

and

$$\lim_{M \to \infty} \lim_{L \to \infty} V^{M,L} = V^-, \tag{4.11}$$

where V^+ and V^- are the upper and lower values associated with the differential game (ODE), (P), *for Y, Z defined by* (4.7), (4.8).

Proof. We will prove that

$$W \equiv \lim_{L \to \infty} \lim_{M \to \infty} V^{M,L}$$

exists and that W is the (unique) viscosity solution of (I^+). Thus $W = V^+$ by Theorem 3.1. The proof for V^- is similar.

First, in view of the estimates (4.4) and (4.5) there exist $M_j \to \infty$ and a uniformly bounded Lipschitz function V^L such that

$$V^{M_j,L} \to V^L \qquad \text{locally uniformly.} \tag{4.12}$$

Furthermore equation $(\mathrm{I}^{M,L})$, which holds a.e., implies also

$$M\,|D_y V^{M,L}| \leqslant |V_t^{M,L}| + |f|\,|D_x V^{M,L}| + L\,|D_z V^{M,L}| + |h|,$$

so that

$$|D_y V^{M_j,L}| \leqslant \frac{C}{M_j}.$$

Thus

$$D_y V^L = 0 \qquad \text{a.e.}$$

and therefore $V^L = V^L(t, x, z)$ does not depend on y.

We claim that V^L is the (unique) viscosity solution of

$$(\mathrm{I}^L) \qquad \begin{cases} V_t^L + \max_{y \in Y} \{f \cdot D_x V^L + h\} - L\,|D_z V^L| = 0, \\ V^L(T, x, z) = g(x). \end{cases}$$

To prove this let us first suppose ϕ is a C^1 function of (t, x, z) and that $V^L - \phi$ attains a local maximum at (t_0, x_0, z_0), $0 < t_0 < T$. We may assume in fact that this is a strict local maximum (cf. [5]). Owing then to (4.12), for all sufficiently large j there exist (t_j, x_j, y_j, z_j) (with $y_j \in Y$) such that $V^{M_j,L} - \phi$ has a local maximum at (t_j, x_j, y_j, z_j) and $(t_j, x_j, z_j) \to (t_0, x_0, z_0)$ as $j \to \infty$. Since $V^{M_j,L}$ is the viscosity solution of $(\mathrm{I}^{M_j,L})$, we have

$$\phi_t + f \cdot D_x\phi + M_j\,|D_y\phi| - L\,|D_z\phi| + h \geqslant 0$$

at (t_j, x_j, y_j, z_j). In addition $D_y\phi \equiv 0$, as ϕ depends only on (t, x, z). Hence

$$\phi_t + \max_{y \in Y} \{f \cdot D_x\phi + h\} - L\,|D_z\phi| \geqslant 0$$

at (t_j, x_j, z_j). We let $j \to \infty$ to conclude

$$\phi_t + \max_{y \in Y} \{f \cdot D_x\phi + h\} - L\,|D_z\phi| \geqslant 0$$

at (t_0, x_0, z_0), as required.

Conversely, suppose for ϕ as above that $V^L - \phi$ attains a strict local minimum at some point (t_0, x_0, z_0). Choose any $y_0 \in \operatorname{int} Y$ and select also an auxiliary function $\zeta\colon Y \to \mathbb{R}$ so that

$$\zeta \text{ has compact support in int } Y, \qquad 0 \leqslant \zeta \leqslant 1,$$

$$\zeta(y_0) = 1, \qquad \zeta(y) < 1 \qquad \text{for all} \quad y \neq y_0.$$

Set

$$\psi \equiv \phi + \zeta$$

and note that $V^L - \psi$ has a strict local minimum at (t_0, x_0, y_0, z_0). Hence for all large enough j,

$$V^{M_j,L} - \phi \qquad \text{has a local minimum at} \qquad (t_j, x_j, y_j, z_j)$$

and

$$(t_j, x_j, y_j, z_j) \to (t_0, x_0, y_0, z_0) \qquad \text{as} \quad j \to \infty.$$

Since $V^{M_j,L}$ is the viscosity solution of $(\mathrm{I}^{M_j,L})$ it follows that

$$\phi_t + f \cdot D_x\phi + M_j |D_y\zeta| - L\,|D_z\phi| + h \leqslant 0$$

at (t_j, x_j, y_j, z_j). Discard the term involving $D_y\zeta$ above and then send $j \to \infty$ to find

$$\phi_t + f \cdot D_x\phi + h - L\,|D_z\phi| \leqslant 0$$

at (t_0, x_0, y_0, z_0). As $y_0 \in \operatorname{int} Y$ was arbitrary we thus obtain

$$\phi_t + \max_{y \in Y} \{f \cdot D_x\phi + h\} - L\,|D_z\phi| \leqslant 0$$

at (t_0, x_0, z_0), as required.

This completes the proof that V^L is the viscosity solution of (I^L). By uniqueness of this solution we have in fact

$$\lim_{M \to \infty} V^{M,L} = V^L \qquad \text{locally uniformly.}$$

The rest of the proof is similar. Notice

$$|V^L|,\ |V^L_t|,\ |D_x V^L|,\ |D_z V^L| \leqslant C_5,$$

so that there exist $L_j \to \infty$ and a uniformly bounded Lipschitz function W such that

$$V^{L_j} \to W \qquad \text{locally uniformly.}$$

Since V^L solves (I^L) a.e. we have

$$L\,|D_z V^L| \leqslant |V^L_t| + |f|\,|D_x V^L| + |h|,$$

so that

$$|D_z V^{L_j}| \leqslant \frac{C}{L_j}.$$

Thus

$$D_z W = 0 \qquad \text{a.e.}$$

and so $W = W(t, x)$ does not depend on z (or y). We claim that W is the unique viscosity solution of

$$(\mathrm{I}^+) \qquad \begin{cases} W_t + \min_{z \in Z} \max_{y \in Y} \{f \cdot D_x W + h\} = 0, \\ \qquad\qquad\qquad W(T, x) = 0 \end{cases}$$

It will then follow that $W = V^+$.

So let ϕ be a C^1 function of (t, x) and suppose $W - \phi$ attains a strict local minimum at some point (t_0, x_0). Then for all j large enough, there exist (t_j, x_j, z_j) (with $z_j \in Z$) such that $V^{L_j} - \phi$ has a local minimum at (t_j, x_j, z_j) and $(t_j, x_j) \to (t_0, x_0)$ as $j \to \infty$. It follows because V^{L_j} is the viscosity solution of (I^{L_j}) that

$$\phi_t + \max_{y \in Y} \{f \cdot D_x \phi + h\} - L\,|D_z \phi| \leqslant 0$$

at (y_j, x_j, z_j). But $D_z \phi \equiv 0$ and so

$$\phi_t + \min_{z \in Z} \max_{y \in Y} \{f \cdot D_x \phi + h\} \leqslant 0$$

at (t_j, x_j). We let $(t_j, x_j) \to (t_0, x_0)$ to obtain the required estimate

$$\phi_t + \min_{z \in Z} \max_{y \in Y} \{f \cdot D_x \phi + h\} \leqslant 0$$

at (t_0, x_0).

On the other hand suppose $W - \phi$ has a strict local maximum at (t_0, x_0). Pick any $z_0 \in \operatorname{int} Z$ and select an auxiliary function $\zeta\colon Z \to \mathbb{R}$ so that

$$\zeta \text{ has compact support in } \operatorname{int} Z, \qquad 0 \leqslant \zeta \leqslant 1,$$
$$\zeta(z_0) = 1, \qquad \zeta(z) < 1 \qquad \text{for all} \quad z \neq z_0.$$

Define

$$\psi \equiv \phi - \zeta$$

and note that $W - \psi$ has a strict local maximum at (t_0, x_0, z_0). Here $V^{L_j} - \psi$ has a local maximum at some point (t_j, x_j, z_j) if j is large enough and $(t_j, x_j, z_j) \to (t_0, x_0, z_0)$. Thus

$$\phi_t + \max_{y \in Y} \{f \cdot D_x \phi + h\} - L\,|D_z \zeta| \geqslant 0$$

at (t_j, x_j, z_j). We next send $j \to \infty$ to obtain

$$\phi_t + \max_{y \in Y} \{f \cdot D_x \phi + h\} \geqslant 0$$

at (t_0, x_0, z_0); so that—as $z_0 \in \operatorname{int} Z$ was arbitrary—

$$\phi_t + \min_{z \in Z} \max_{y \in Y} \{f \cdot D_x \phi + h\} \geqslant 0$$

at (t_0, x_0).

The last inequality completes the proof that W is the viscosity solution of (I^+). Hence $W = V^+$, and so in fact

$$\lim_{L \to \infty} V^L = \lim_{L \to \infty} \lim_{M \to \infty} V^{M,L} = V^+. \quad \blacksquare$$

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