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# DECIDABILITY OF THE EQUIVALENCE PROBLEM FOR FINITELY AMBIGUOUS FINANCE AUTOMATA

KOSABURO HASHIGUCHI*, KENICHI ISHIGURO and SHUJI JIMBO

*Department of Information Technology, Faculty of Engineering*
*Okayama University, Tsushima, Okayama 700-0082, Japan*
**hasiguti@kiso.it.okayama-u.ac.jp*

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A finance automaton is a sixtuple $\langle \Sigma, Q, \delta, S, F, f \rangle$, where $\langle \Sigma, Q, \delta, S, F \rangle$ is a (nondeterministic) finite automaton, $f : Q \times \Sigma \times Q \to R \cup \{-\infty\}$ is a finance function, $R$ is the set of real numbers and $f(q, a, q') = -\infty$ if and only if $q' \notin \delta(q, a)$. The function $f$ is extended to $f : 2^Q \times \Sigma^* \times 2^Q \to R \cup \{-\infty\}$ by the plus-max principle. For any $w \in \Sigma^*$, $f(S, w, F)$ is the profit of $w$. It is shown that the equivalence problem for finitely ambiguous finance automata is decidable.

*Keywords*: Tropical semiring; transducer; finance automata; plus-max principle.

## 1. Introduction

A finance automaton $\mathcal{A}$ is a sixtuple $\langle \Sigma, Q, \delta, S, F, f \rangle$, where $\langle \Sigma, Q, \delta, S, F \rangle$ is a finite automaton and $f$ is a "finance function". The function $f$ assigns a real number to each atomic transition. In an automaton describing some financial activity of some company, $\Sigma$ can be regarded as a finite set of atomic financial actions of the company, each $q \in Q$ is a financial state of the company, and for each $p, q \in Q$, $a \in \Sigma$ with $q \in \delta(p, a)$, the action of $p \to q$ by $a$ can be understood that when the company is in state $p$, it can perform action $a$ and after performing $a$, the state of the company can be $q$ and if $f(p, a, q) > 0$, then the company gets the profit $f(p, a, q)$, if $f(p, a, q) = 0$, then the profit plus the cost is zero, and if $f(p, a, q) < 0$, then the company paid — $f(p, a, q)$ in performing $a$. $f$ is extended to the set of input words $w$ as a function $P(w, \mathcal{A})$ by the plus-max principle so that $P(w, \mathcal{A})$ is the profit of $w$. If $P(w, \mathcal{A}) = -\infty$, then this means that $w$ is not accepted by $\mathcal{A}$, and if $-\infty < P(w, \mathcal{A}) < 0$, then $-P(w, \mathcal{A})$ is the cost needed for $\mathcal{A}$ to process $w$. If $P(w, \mathcal{A}) > 0$, then $P(w, \mathcal{A})$ is the net profit of $\mathcal{A}$ by processing $w$. Because $\mathcal{A}$ is generally nondeterministic, for each $w \in \Sigma^+$, there exist many transition sequences of states on input $w$. The company may want to know which transition sequence produces the maximum profit. The value is given by $f(S, w, F) = P(w, \mathcal{A})$.

Each financial automaton may be regarded to be a model describing financial activities. A finite automaton $\mathcal{B}$ with a distance function $d$ was introduced in [3], and called a $D$-automaton for short. The function $d$ assigns 0 (zero) or 1 (one) to each atomic transition, and is extended to the set of input words $w$ by the plus-min principle as a function $D(w, \mathcal{B})$ for any input word $w$. If $D(w, \mathcal{B}) = \infty$, then this means that $w$ is not accepted by $\mathcal{B}$. If $D(w, \mathcal{B}) \neq \infty$, then $0 \leq D(w, \mathcal{B})$ and $D(w, \mathcal{B})$ is the cost (or distance) needed for $\mathcal{B}$ to process $w$. $D$-automata play important roles as basic computational machines for solving the representation problems [4] and the problems of determining star height, relative star height, inclusion star height and relative inclusion star height for regular languages [5, 6]. One can see that for any $D$-automaton $\mathcal{B}$ with a distance function $d$, one can construct a finance automaton $\mathcal{A}$ with a finance function $f$ such that for each atomic transition $(p, a, q)$, $f(p, a, q) = -d(p, a, q)$. Then for any input word $w : D(w, \mathcal{B}) = -P(w, \mathcal{A})$. Thus $\mathcal{A}$ and $\mathcal{B}$ may be said to be "isomorphic", and one may say that the class of $D$-automata is a subclass of the class of finance automata.

We remark that a finance automaton $\mathcal{A} = \langle \Sigma, Q, \delta, S, F, f \rangle$ such that the image of $f$ is a subset of $Z \cup \{\infty\}$, $Z =$ the set of integers, corresponds to a rational series with multiplicities in the semiring whose support is $Z \cup \{\infty\}$ and which has two operations, min and $+$. (Recall the above remark about the relation between distance automata and finance automata.) This theory has been studied widely (see [1, 2, 9, 12]). The tropical semiring is the subsemiring of the above semiring with support $N \cup \{+\infty\}$, $N =$ the set of nonnegative integers. Krob [9] showed that the equivalence problem for rational series over the two letter alphabet with multiplicities in the semiring $\mathcal{Z} = (Z \cup \{+\infty\}, \min, +)$ is undecidable, and by suitable transformations, also showed that the equivalnce problem for rational series over the two letter alphabet with multiplicities in the tropical semiring and the equivalence problem for $\{0, 1\}$-automata over the two letter alphabet are both undecidable. A $\{0, 1\}$-automaton is a distance automaton. Thus these results imply that the equivalence problem for distance automata over the two letter alphabet is undecidable, and a fortiori, the equivalence problem for finance automata over the two letter alphabet is undecidable. Now our problem is: for what kind of financial automata, is the equivalence problem decidable?

Let $\mathcal{R}$ denote the semiring which has $R \cup \{-\infty\}$ as support and whose addition $\oplus$ is defined by $a \oplus b = \max\{a, b\}$ and whose product $\otimes$ is defined by $a \otimes b = a + b$. We shall call $\mathcal{R}$ the financial semiring, and often denote it by $\mathcal{R} = (R \cup \{-\infty\}, \max, +)$. We note that each finance automaton $\mathcal{A}$ over an input alphabet $\Sigma$ can be regarded as a rational series over the alphabet $\Sigma$ with multiplicities in the financial semiring $\mathcal{R} = (R \cup \{-\infty\}, \max, +)$ and vice versa. As stated above about the relation between distance automata and finance automata, the tropical semiring can be regarded as a subsemiring of the financial semiring. Thus the class of rational series over the alphabet $\Sigma$ with multiplicities in the tropical semiring can be regarded to be a subclass of finance automata over the input alphabet $\Sigma$.

In this paper, we prove decidability of the equivalence problem for finitely ambiguous finance automata by reducing the problem to the problem of finding integer solutions for simultaneous linear inequalities. We also study several subproblems so that the simpler a subproblem is, the simpler the solution is. The paper consists of five Sections. In Sec. 2, we present notation and definitions about finance automata. In Sec. 3, we present the definition of a union finance automaton which is composed from finitely many deterministic finance automata. In Sec. 4, first we present a proof showing decidability of the equivalence problem of a union finance automaton and a determinintistic finance automaton. Then we show that the inequality problem of a union finance automaton and a deterministic finance automaton is decidable. This immediately implies decidability of the equivalence problem for union finance automata. In Sec. 5, we shall show in almost the same way as in Sec. 4 that the equivalence problem for finitely ambiguous finance automata is decidable. The arguments in Sec. 5 will be presented briefly and the reader should refer to Sec. 4 to understand the content of Sec. 5. In Sec. 5, we present a finitely ambiguous finite transducer which is not equivalent to any union finite transducer (Remark 5.1). The following problem will be left open : for any finitely ambiguous finance automaton $\mathcal{A}$, does there exist a union finance automaton which is equivalnet to $\mathcal{A}$? At present, it seems that there exists a nonderministic finance automaton which is not equivalent to any finitely ambiguous finance automaton because the equivalence problem for arbitrary finance automata is undecidable.

## 2. Finance Automata

An alphabet is a nonempty set of symbols. A word is a finite sequence of symbols from the alphabet. A word of length zero is called the null word and denoted by $\lambda$. $\Sigma^*$ denotes the set of words over an alphabet $\Sigma$, and $\Sigma^+$ denotes the set of nonnull words. For a word, $w = a_1 a_2 \cdots a_n$ ($n \geq 0, a_1, \ldots, a_n \in \Sigma$), $n$ is the length of $w$, and is denoted by $|w|$. The cardinality of a set $A$ is denoted by $|A|$. $R$ denotes the set of real numbers. $R_{-\infty}$ denotes $R \cup \{-\infty\}$. $\varnothing$ denotes the empty set.

**Definition 2.1.** A finance automaton (in short, an $F$-automaton) is a sixtuple $\mathcal{A} = \langle \Sigma, Q, \delta, S, F, f \rangle$, where

$$\Sigma : \text{an input alphabet}, \ Q : \text{a finite set of states}$$
$$\delta : \text{a transition function } \delta : Q \times \Sigma \times Q \to 2^Q$$
$$S \subset Q : \text{the set of initial states}$$
$$F \subset Q : \text{the set of final states}$$
$$f : \text{a finance function } f : Q \times \Sigma \times Q \to R_{-\infty} \,.$$

Thus $\langle \Sigma, Q, \delta, S, F \rangle$ is a finite automaton, and $\delta$ is extended to $\delta : Q \times \Sigma^* \times Q \to 2^Q$ and $\delta : 2^Q \times \Sigma^* \times 2^Q$ as for finite automata. A word $w \in \Sigma^*$ is accepted by $\mathcal{A}$ if $\delta(S, w) \cap F \neq \varnothing$. $f$ satisfies the following:

$$\forall (p, a, q) \in Q \times \Sigma \times Q, \ q \notin \delta(p, a) \Longleftrightarrow f(p, a, q) = -\infty \,.$$

**Definition 2.2.** Let $\mathcal{A} = \langle \Sigma, Q, \delta, S, F, f \rangle$ be an $F$-automaton.

(1) $f$ is extended to $f : Q \times \Sigma^* \times Q \to R_{-\infty}$ and to $f : 2^Q \times \Sigma^* \times 2^Q \to R_{-\infty}$ inductively in the following way.

  (a) For any $p, q \in Q$, $f(p, \lambda, q) = 0$ if $p = q$ and $f(p, \lambda, q) = -\infty$ if $p \neq q$.
  (b) For any $p, q \in Q, w \in \Sigma^*$ and $a \in \Sigma$,

$$f(p, wa, q) = \max\{f(p, w, q') + f(q', a, q) \mid q' \in Q\}\,.$$

  Here, for any $i \in R_{-\infty}$, $\max\{i, -\infty\} = i$ and $i + (-\infty) = -\infty$.
  (c) For any $t, t' \subset Q$ and $w \in \Sigma^*$, $f(t, w, t') = \max\{f(p, w, q) \mid p \in t, q \in t'\}$.

(2) The set of words accepted by $\mathcal{A}$ is denoted by $L(\mathcal{A})$: $L(\mathcal{A}) = \{w \in \Sigma^* \mid \delta(S, w) \cap F \neq \varnothing\}$.
(3) Two $F$-automata $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be $L$-equivalent if $L(\mathcal{A}_1) = L(\mathcal{A}_2)$.
(4) For any $w \in \Sigma^*$, $f(S, w, F)$ is the profit of $w$ (by $\mathcal{A}$). The profit of $w$ will be sometimes denoted by $P(w, \mathcal{A})$.
(5) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two $F$-automata over the input alphabet $\Sigma$.

  (a) If for any $w \in \Sigma^*$: $P(w, \mathcal{A}_1) = P(w, \mathcal{A}_2)$, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be equivalent and we write $A_1 \equiv A_2$.
  (b) If for any $w \in \Sigma^*$: $P(w, \mathcal{A}_1) \geq P(w, \mathcal{A}_2)$, then $\mathcal{A}_1$ is said to be greater than or equal to $\mathcal{A}_2$, and we write $\mathcal{A}_1 \geq \mathcal{A}_2$.

(6) An $F$-automaton $\mathcal{A}$ is deterministic if the following holds.

  For any $q \in Q$, $a \in \Sigma$, $|\delta(q, a)| \leq 1$ and $|S| \leq 1$.
  When $\mathcal{A}$ is deterministic, for any $q \in Q$, $a \in \Sigma$, $q' \in \delta(q, a)$, we write $\delta(q, a) = q'$.

(7) For any $m \geq 1$, $m$ deterministic $F$-automata $\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ $(1 \leq i \leq m)$ are said to be mutually disjoint if for any $j, k$ $(1 \leq j < k \leq m)$, $Q_j \cap Q_k = \varnothing$.

In [11], it is shown that two deterministic $F$-automata $\mathcal{A}_1 = \langle \Sigma, Q_1, \delta_1, \{s_1\}, F_1, f_1 \rangle$ and $\mathcal{A}_2 = \langle \Sigma, Q_2, \delta_2, \{s_2\}, F_2, f_2 \rangle$ are equivalent if and only if for all $w \in \Sigma^*$ with length at most $2 \times |Q_1| \times |Q_2|$, $f(\{s_1\}, w, F_1) = f(\{s_2\}, w, F_2)$.

## 3. Union *F*-Automata

In this section, when given a finite set of mutually disjoint deterministic $F$-automata, we define a union $F$-automaton whose set of states is the union of the sets of states of these given deterministic $F$-automata.

**Definition 3.1.** For $m$ $(m \geq 1)$ mutually disjoint deterministic $F$-automata, $\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ $(1 \leq i \leq m)$, a union $F$-automaton $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m = \langle \Sigma, Q, \delta, S, F, f \rangle$ is defined as follows.

(1) $Q = Q_1 \cup \cdots \cup Q_m$, $S = \{s_1, \ldots, s_m\}$, $F = F_1 \cup \cdots \cup F_m$.
(2) For any $i$ $(1 \le i \le m)$, $q \in Q_i$, $a \in \Sigma$,

$$\delta(q, a) = \delta_i(q, a) \text{ and } f(q, a, \delta(q, a)) = f_i(q, a, \delta_i(q, a))\,.$$

**Proposition 3.1.** *Let* $\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ $(1 \le i \le m)$ *be* $m$ $(m \ge 1)$ *mutually disjoint deterministic* $F$*-automata. Then for any* $w \in \Sigma^*$,

$$P(w, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m) = \max\{P(w, \mathcal{A}_i) \mid 1 \le i \le m\}\,.$$

*Especially* $L(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m) = L(\mathcal{A}_1) \cup \cdots \cup L(\mathcal{A}_m)$.



**Proof.** Let $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m = \langle \Sigma, Q, \delta, S, F, f \rangle$. For any $w \in \Sigma^*$, $P(w, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m) = f(S, w, F)$. By definition, for any $t, t' \subset Q$, $w \in \Sigma^*$, $f(t, w, t') = \max\{f(p, w, q) \mid p \in t, q \in t'\}$. Thus

$$\begin{aligned} f(S, w, F) &= \max\{f(s_i, w, p_{in}) \mid s_i \in S, p_{in} \in F_i, 1 \le i \le m\} \\ &= \max\{P(w, \mathcal{A}_i) \mid 1 \le i \le m\}\,. \end{aligned}$$

Hence $P(w, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m) = \max\{P(w, \mathcal{A}_i) \mid 1 \le i \le m\}$. Now the assertion $L(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m) = L(\mathcal{A}_1) \cup \cdots \cup L(\mathcal{A}_m)$ is also clear. □

## 4. The Equivalence Problem for Union *F*-Automata

In this section, we show the equivalence problem for union $F$-automata is decidable. We shall first present an algorithm for deciding whether or not for $n \ge 1$, a union $F$-automaton $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ and a deterministic $F$-automaton $\mathcal{A}_{n+1}$ are equivalent. Let $n \ge 1$ and for each $1 \le i \le n+1$, let $\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ be a deterministic $F$-automaton.

**Definition 4.1.** A finite deterministic automaton $\Gamma(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}) = \langle \Sigma, Q, \delta, \{s\}, F \rangle$ is defined as follows.

(1) $Q = Q_1 \times \cdots \times Q_{n+1}$, $S = (s_1, \ldots, s_{n+1})$, $F = \{(q_1, \ldots, q_{n+1}) \in Q \mid \text{for some } 1 \le i \le n+1, q_i \in F_i\}$.
(2) For any $a \in \Sigma$, $(q_1, \ldots, q_{n+1}) \in Q_1 \times \cdots \times Q_{n+1}$,

$$\delta((q_1, \ldots, q_{n+1}), a) = (\delta_1(q_1, a), \ldots, \delta_{n+1}(q_{n+1}, a))\,.$$

**Proposition 4.1.** *Assume that* $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n \equiv \mathcal{A}_{n+1}$. *Then for any* $x, z \in \Sigma^*$, $y \in \Sigma^+$, *and* $(q_1, \ldots, q_{n+1}) \in Q$, *if* $\delta(s, x) = \delta(s, xy) = (q_1, \ldots, q_{n+1})$ *and* $\delta(s, xyz) \in F$, *then there exists* $1 \le i \le n$ *for which the following* (1)–(3) *hold.*

(1) $P(xyz, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n) = P(xyz, \mathcal{A}_i) = P(xyz, \mathcal{A}_{n+1})$.
(2) $P(xz, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n) = P(xz, \mathcal{A}_i) = P(xz, \mathcal{A}_{n+1})$.
(3) $f_i(q_i, y, q_i) = \max\{f_j(q_j, y, q_j) \mid 1 \le j \le n\} = f_{n+1}(q_{n+1}, y, q_{n+1})$.

**Proof.** We put $a = \max\{f_i(q_i, y, q_i) \mid 1 \le i \le n\}$ and $b = P(xyz, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n)$. Define two sets $X, Y$ by $X = \{i \mid 1 \le i \le n \text{ and } f_i(q_i, y, q_i) = a\}$, $Y = \{i \mid 1 \le i \le n$ and $P(xyz, \mathcal{A}_i) = b\}$. Since $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n \equiv \mathcal{A}_{n+1}$, $b = P(xyz, \mathcal{A}_{n+1})$. Put $c = \max\{P(xyz, \mathcal{A}_i) \mid i \in X\}$, and define a set $W$ by $W = \{i \in X \mid P(xyz, \mathcal{A}_i) = c\}$. We consider the following two cases.

(1) $b = c$. Then $X \cap Y = W$ and for any $i \in X \cap Y$, $a = f_i(q_i, y, q_i)$ and $c = P(xyz, \mathcal{A}_i) = P(xyz, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n)$. Then $P(xz, \mathcal{A}_i) = c - a = P(xz, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n)$ because $c - a = P(xz, A_{n+1})$. Now one can see that for any $i \in W$, (1)–(3) hold.

(2) $b \ne c$. Clearly $b > c$. Put $d = \max\{f_i(q_i, y, q_i) \mid i \in Y\}$. Since $b > c$, we have $d < a$. For each $k > 1$, consider the word $w = xy^k z$. Then for each $i \in W$, $c + ka = P(xy^{k+1}z, \mathcal{A}_i)$. Since $P(xz, \mathcal{A}_{n+1}) = P(xz, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n)$, we have $P(xz, \mathcal{A}_{n+1}) \ge b - d$. Together with this fact, $a > d$ and $P(xyz, \mathcal{A}_{n+1}) = b$, we have $f_{n+1}(q_{n+1}, y, q_{n+1}) < a$. If we consider a sufficiently large $k$, we would have for each $i \in W$, $P(xy^k z, \mathcal{A}_{n+1}) < P(xy^k z, \mathcal{A}_i) \le P(xy^k z, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n)$. This is a contradiction to $A_1 \cup \cdots \cup \mathcal{A}_n \equiv \mathcal{A}_{n+1}$. Thus the case $b \ne c$ is impossible. □

We shall present a necessary and sufficient condition for $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ and $\mathcal{A}_{n+1}$ to be equivalent.

**Theorem 4.2.** *Let $\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ $(1 \le i \le n+1, n \ge 1)$ be $n+1$ deterministic $F$-automata, and let $\Gamma(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}) = \langle \Sigma, Q, \delta, \{s\}, F \rangle$ be as in Definition* 4.1. *Then the following three conditions are equivalent.*

(1) $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n \equiv \mathcal{A}_{n+1}$.

(2) $L(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_n) = L(\mathcal{A}_{n+1})$ *and for any $w \in L(\mathcal{A}_{n+1})$ with length $\le 3 \cdot (n+1) \cdot (|Q_1| \times \cdots \times |Q_{n+1}| + 1)$, the following* (A) *holds.*

(A): *For any $x, y, z \in \Sigma^*$ and $(q_1, \ldots, q_{n+1}) \in Q$, if $w = xyz$ and $\delta(s, x) = \delta(s, xy) = (q_1, \ldots, q_{n+1})$, then there exists $1 \le i \le n$ such that the following* (2.1)–(2.3) *hold.*

(2.1) $P(xyz, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n) = P(xyz, \mathcal{A}_i) = P(xyz, \mathcal{A}_{n+1})$.

(2.2) $P(xz, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n) = P(xz, \mathcal{A}_i) = P(xz, \mathcal{A}_{n+1})$.

(2.3) $f_i(q_i, y, q_i) = \max\{f_j(q_j, y, q_j) \mid 1 \le j \le n\} = f_{n+1}(q_{n+1}, y, q_{n+1})$.

(3) $L(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_n) = L(\mathcal{A}_{n+1})$ *and for any $w \in L(\mathcal{A}_{n+1})$, Condition* (A) *above holds.*

**Proof.** (1) ⇒ (2) follows from Proposition 4.1. (2) ⇒ (3). Assume (2) holds. We shall prove (3) by induction on $|w|$. Put $c = 3 \cdot (n+1) \cdot (|Q_1| \times \cdots \times |Q_{n+1}| + 1)$. When $|w| \le c$, (A) holds. Let $k > c$, and assume that for $|w| < k$, (A) holds. Consider the case $|w| \ge k$. Assume that for $w = xyz$ and $(q_1, \ldots, q_{n+1}) \in Q$, we have $\delta(s, x) = \delta(s, xy) = (q_1, \ldots, q_{n+1})$ and $\delta(s, xyz) \in F$. Since $|w| > c$, we have $|x| \ge c/3$, or $|y| \ge c/3$ or $|z| \ge c/3$.

First consider the case $|y| \geq c/3$. By the definition of $c$, $y$ has a decomposition $y = y_0 y_1 \cdots y_{n+2}$, $y_0, y_{n+2} \in \Sigma^*$, $y_i \in \Sigma^+$ $(1 \leq i \leq n+1)$ such that for some $(p_1, \ldots, p_{n+1}) \in Q$, $\delta(s, xy_0) = \delta(s, xy_0y_1) = \cdots = \delta(s, xy_0y_1 \cdots y_{n+1}) = (p_1, \ldots, p_{n+1})$. For each $1 \leq i \leq n+1$, put $v_i = y_1 \cdots y_{i-1} y_{i+1} \cdots y_{n+1}$.

Consider the case $i = 1$. Then $v_1 = y_2 y_3 \cdots y_{n+1}$. By the inductive hypothesis, for $x'_1 = xy_0$, $y'_1 = v_1$, $z'_1 = y_{n+2}z$, there exists $\mathcal{A}_{i_1}$ to which (A) holds. $\mathcal{A}_{i_1}$ has the maximum profits at $x'_1 y'_1 z'_1$, $x'_1 z'_1$ and $y'_1$. The set of such $\mathcal{A}_{i_1}$ is denoted by $X_{11}$. Next for $x'_2 = xy_0$, $y'_2 = y_2$, $z'_2 = y_3 y_4 \cdots y_{n+2} z$, there exists $\mathcal{A}_{i_2}$ satisfying (A). The set of such $\mathcal{A}_{i_2}$ is denoted by $X_{12}$. One can see that $X_{11} \subset X_{12}$. Each $\mathcal{A}_j$ belonging to $X_{12}$ has the maximum profits at $x'_2 y'_2 z'_2$, $x'_2 z'_2$ and $y'_2$. Then for $x'_3 = xy_0$, $y'_3 = y_3$, $z'_3 = y_4 y_5 \cdots y_{n+2} z$, there exists $\mathcal{A}_{i_3}$ satisfying (A). $\mathcal{A}_{i_3}$ has the maximum profits at $x'_3 y'_3 z'_3$, $x'_3 z'_3$ and $y'_3$. The set of such $\mathcal{A}_{i_3}$ is denoted by $X_{13}$. Then $X_{13} \supset X_{12}$. By continuing this argument, each $\mathcal{A}_j$ belonging to $X_{11}$ has the maximum profits at $y_2, y_3, y_4, \ldots, y_{n+1}$, and $xy_0 y_{n+2} z$.

In the same way, for each $2 \leq i \leq n+1$, the set $X_{i_1}$ can be defined. Then there exist $1 \leq p < q \leq n+1$ and $1 \leq r \leq n$ such that $A_r \in X_{p_1} \cap X_{q_1}$ holds. This implies $\mathcal{A}_r$ has the maximum profits at $y_1, y_2, \ldots, y_{n+1}$, and $xy_0 y_{n+2} z$ each of which is equal to the corresponding profit by $A_{n+1}$. Then because $\delta(s, x) = \delta(s, xy_0 y_{n+2})$, as in the proof of Proposition 4.1, $A_r$ has the maximum profits also at $xz$ and $y_0 y_{n+2}$. Thus $\mathcal{A}_r$ satisfies (A).

When $|x| \geq c/3$, we consider a decomposition of $x$, $x = x_0 x_1 \cdots x_{n+1} x_{n+2}$ such that $x_0$, $x_{n+2} \in \Sigma^*$ and for each $1 \leq i \leq n+1$, $x_i \in \Sigma^+$ and $\delta(s, x_0) = \delta(s, x_0 x_1 \cdots x_i)$. As in the above case $y \geq c/3$, there exists $1 \leq r \leq n$ such that $A_r$ has the maximum profits at $x_1, \ldots, x_{n+1}, x_0 x_{n+2} yz$ and $w$, respectvely. Then because $\delta(s, x_0 x_{n+2}) = \delta(s, x_0 x_{n+2} y)$, as above, $A_r$ has the maximum profits at $x_0 x_{n+2} z$ and $y$. Then $A_r$ satisfies (A). The case $|z| \geq c/3$ can be handled in the same way. Thus (2) $\Rightarrow$ (3) is proved. (3) $\Rightarrow$ (1) is clear. This completes the proof of Theorem 4.2. □

The following theorem is now clear.

**Theorem 4.3.** *Given $n+1$ deterministic $F$-automata $\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ $(1 \leq i \leq n+1)$, one can decide whether or not $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n \equiv \mathcal{A}_{n+1}$.*

Now we shall proceed to the equivalence problem for union $F$-automata. The following proposition can be proved easily.

**Proposition 4.4.** *For $m + n$ $(m, n \geq 1)$ deterministic $F$-automata $\mathcal{A}_k = \langle \Sigma, Q_k, \delta_k, \{s_k\}, F_k, f_k \rangle$ $(1 \leq k \leq m+n)$, the following two conditions are equivalent.*

(1) $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m \equiv A_{m+1} \cup \cdots \cup A_{m+n}$.
(2) *For each $1 \leq i \leq m$, $\mathcal{A}_i \leq \mathcal{A}_{m+1} \cup \cdots \cup \mathcal{A}_{m+n}$ and for each $m+1 \leq j \leq m+n$, $\mathcal{A}_j \leq \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m$.*

First we shall study the inequality problem for union $F$-automata.

**Definition 4.2.** The following problem is called the inequality problem for union $F$-automata.

Input: $n+1$ $(n \geq 1)$deterministic $F$-automata
$\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ $(1 \leq i \leq n+1)$.
Output: if $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n \geq \mathcal{A}_{n+1}$, then "yes".
if $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n \not\geq \mathcal{A}_{n+1}$, then "no".

From here to Theorem 4.7 below, let $n \geq 1$, and $\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ $(1 \leq i \leq n+1)$ be $n+1$ deterministic $F$-automata. As in Definition 4.1, define the deterministic finite automaton $\Gamma(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}) = \langle \Sigma, Q, \delta, \{s\}, F \rangle$.



**Definition 4.3.** For any $w \in \Sigma^+$, define a deterministic finite automaton $A(w) = \langle \Sigma, Q(w), \delta(w), \{s\}, F(w) \rangle$ as follows.

(1) $Q(w) = \{(p_1, \ldots, p_{n+1}) \in Q \mid$ for some $x, y \in \Sigma^*$, $w = xy$ and $(p_1, \ldots, p_{n+1}) = \delta(s, x)\} = \{\delta(s, x) \in Q \mid x \in \Sigma^*$ and $x$ is a prefix of $w\}$.
(2) $\delta(w)$ is a mapping from $Q(w) \times \Sigma$ to $Q(w)$ such that for any $(p_1, \ldots, p_{n+1}) \in Q(w)$ and any $a \in \Sigma$, the following hold.
   (a) If for some $x, y \in \Sigma^*$, we have $w = xay$ and $(p_1, \ldots, p_{n+1}) = \delta(s, x)$, then
$$\delta(w)((p_1, \ldots, p_{n+1}), a) = \delta((p_1, \ldots, p_{n+1}), a).$$
   (b) Otherwise $\delta(w)((p_1, \ldots, p_{n+1}), a) = \varnothing$.
(3) $F(w) = \{\delta(s, w)\} \cap F$.

**Definition 4.4.** Define a set $S(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$ by: $S(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}) = \{A(w) \mid w \in \Sigma^+\}$.

**Definition 4.5.** For $m \geq 1$ and $1 \leq r \leq m$, put $P_{m,r} = m(m-1)\cdots(m-(r-1))$.

**Definition 4.6.** Define an integer $I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$ as follows, where $m = |Q| \times |\Sigma|$.
$$I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}) = (1 + |Q|)(1 + P_{m,1} + P_{m,2} + \cdots + P_{m,|Q|}).$$

**Definition 4.7.** Let $w \in \Sigma^+$ and $A(w) = \langle \Sigma, Q(w), \delta(w), \{s\}, F(w) \rangle$. For any $p \in Q(w)$, $i \geq 1$, $a_1, \ldots, a_i \in \Sigma$, if $\delta(w)(p, a_1 \cdots a_i) = p$ and either $i = 1$ or for $1 \leq j < k \leq i$, $\delta(w)(p, a_1 \cdots a_j) \neq \delta(w)(p, a_1 \cdots a_k)$, then $(p, a_1 \cdots a_i, p)$ is called a minimal cycle of $A(w)$.

**Lemma 4.5.** *For any $w \in \Sigma^+$, $A(w)$ has at most $P_{m,1} + P_{m,2} + \cdots + P_{m,|Q|}$ minimal cycles. Here $m = |Q| \times |\Sigma|$.*

**Proof.** Let $c = (p, a_1 \cdots a_i, p)$ be a minimal cycle of $A(w)$. If $i = 1$, then the number of such $c$ is at most $P_{m,1} = m$. If $i \geq 2$, then put $t(c) = ((p, a_1), (\delta(p, a_1), a_2), \ldots, (\delta(p, a_1 \cdots a_{i-1}), a_i))$. Since $c$ is a minimal cycle, all $i$ pairs in $t(c)$ are distinct; moreover $i \leq |Q|$. Thus the number of minimal cycles of length $i$ is at most $P_{m,i}$, and the total number of minimal cycles is at most $P_{m,1} + P_{m,2} + \cdots + P_{m,|Q|}$. $\square$



**Proposition 4.6.** *$S(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}) = \{A(w) \mid w \in \Sigma^+$ and $|w| < I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})\}$.*

**Proof.** Put $B = \{A(w) \mid w \in \Sigma^+ \text{ and } |w| < I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})\}$. We shall prove by induction on $|w|$ that for any $w \in \Sigma^+$, $A(w) \in B$. When $|w| < I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$, the assertion is clear. Assume that for $|w| < k+1$, the assertion holds, and consider the case $|w| = k + 1 \geq I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$. Put $m = |Q| \times |\Sigma|$, $q = P_{m,1} + P_{m,2} + \cdots + P_{m,|Q|} + 1$. Since $|w| \geq I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$, there exists a decomposition of $w$, $w = w_1 w_2 \cdots w_q w_{q+1}$, with $|w_i| = |Q| + 1$ $(1 \leq i \leq q)$. Since $|w_i| = |Q| + 1$, in the path $(s, w_1 w_2 \cdots w_q w_{q+1}, \delta(s, w))$ of $A(w)$, there exists a minimal cycle $(p_i, v_i, p_i)$ in each part $w_i$. Here $w_i$ is decomposed as $w_i = x_i v_i y_i$ for $x_i, y_i \in \Sigma^*$. Namely in $A(w)$, there exists a path

$$(s, x_1, p_1, v_1, p_1, y_1 x_2, p_2, v_2, p_2, y_2 x_3, \ldots, y_{q-1} x_q, p_q, v_q, p_q, y_q w_{q+1}, \delta(s, w))\,.$$

The number of minimal cycles is at most $q - 1$ by Lemma 4.5. Thus for some $1 \leq i < j \leq q$, $(p_i, v_i, p_i) = (p_j, v_j, p_j)$. Now by putting $w' = w_1 w_2 \cdots w_{j-1} x_j y_j w_{j+1} \cdots w_{q+1}$, we have $A(w') = A(w)$ by definition of minimal cycles and $A(w)$. By induction, $A(w') \in B$. Hence $A(w) \in B$. $\square$

**Definition 4.8.** For $w \in \Sigma^+ \cap L(\mathcal{A}_{n+1})$, when $A(w)$ has a minimal cycle, define the following (1)–(2), where $MCD$ is named from "minimal cycle decomposition".

(1) $MCD(w)$ denotes the set of all sequences

$$(s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \ldots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$$

which satisfy the following (1)(a) and (b). Each sequence in $MCD(w)$ is called a minimal cycle decomposition of $w$.

  (a) $k \geq 1$ and $w = x_1 y_1 x_2 y_2 \cdots x_k y_k x_{k+1}$
  (b) For each $1 \leq i \leq k$, it holds that $p_i = \delta(s, x_1 y_1 x_2 y_2 \cdots x_{i-1} y_{i-1} x_i)$, $(p_i, y_i, p_i)$ is a minimal cycle of $A(w)$ and $|x_i| \leq |Q|$.

(2) Let $MCD(w) = \{\beta_1, \beta_2, \ldots, \beta_t\}$ $(t \geq 1)$. For each $\beta_i$ $(1 \leq i \leq t)$, define a set of linear inequlities $LIS(w, \beta_i)$ as follows. Let $\beta_i = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \ldots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$. The following set of inequalities over $k$ variables $X_1, X_2, \ldots, X_k$ is $LIS(w, \beta_i)$:

$$X_j \geq 0 \quad (1 \leq j \leq k)$$
$$a_i + b_{i\,1} X_1 + b_{i\,2} X_2 + \cdots + b_{i\,k} X_k < a_{n+1} + b_{n+1,1} X_1 + b_{n+1,2} X_2 + \cdots + b_{n+1,k} X_k$$
$$(1 \leq i \leq n)\,.$$

Here $a_K = P(x_1 x_2 \cdots x_{k+1}, A_K)$ $(1 \leq K \leq n+1)$, and for each $1 \leq g \leq n+1$ and $1 \leq j \leq k$, $b_{g\,j} = f_g(\delta_g(s_g, x_1 x_2 \cdots x_j), y_j, \delta_g(s_g, x_1 x_2 \cdots x_j))$.

**Theorem 4.7.** *The following three conditions are equivalent.*

(1) $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n \geq \mathcal{A}_{n+1}$.

(2) *For each* $w \in \Sigma^+ \cap L(\mathcal{A}_{n+1})$ *with length less than* $I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$, *one of* (2)(a) *or* (b) *holds*:

  (a) *If* $A(w)$ *has no minimal cycle, then* $P(w, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n) \geq P(w, \mathcal{A}_{n+1})$.

  (b) *If* $A(w)$ *has a minimal cycle, then for each minimal cycle decomposition* $\beta \in MCD(w)$ *of* $w$, *the set of simultaneous linear inequalities* $LIS(w, \beta)$ *has no integer solution.*

(3) *For each* $w \in \Sigma^+ \cap L(\mathcal{A}_{n+1})$, *one of* (3)(a) *or* (b) *holds*:

  (a) *If* $A(w)$ *has no minimal cycle, then* $P(w, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n) \geq P(w, \mathcal{A}_{n+1})$.

  (b) *If* $A(w)$ *has a minimal cycle, then for each minimal cycle decomposition* $\beta \in MCD(w)$ *of* $w$, *the set of simultaneous linear inequalities* $LIS(w, \beta)$ *has no integer solution.*

**Proof.** (1) $\Rightarrow$ (3). Assume (1) holds. Consider any $w \in \Sigma^+ \cap L(\mathcal{A}_{n+1})$, If $A(w)$ has no minimaal cycle, then (3) holds. Otherwise assume that $A(w)$ has a minimal cycle, and consider a minimal cycle decomposition of $w$, $\beta = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \ldots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$. By definition of $\beta$, for any $i_1 \geq 0, \cdots, i_k \geq 0$, the following hold.

$$x_1 y_1^{i_1} x_2 y_2^{i_2} \cdots x_k y_k^{i_k} x_{k+1} \in L(\mathcal{A}_{n+1})$$

and

$$P(x_1 y_1^{i_1} x_2 y_2^{i_2} \cdots x_k y_k^{i_k} x_{k+1}, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n) \geq P(x_1 y_1^{i_1} x_2 y_2^{i_2} \cdots x_k y_k^{i_k} x_{k+1}, \mathcal{A}_{n+1}) .$$

From this, one can see that $LIS(w, \beta)$ has no integer solution.

(3) $\Rightarrow$ (1). We shall prove the contraposition. Assume (1) does not hold. There exists $w \in \Sigma^+ \cap L(\mathcal{A}_{n+1})$ such that $P(w, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n) < P(w, \mathcal{A}_{n+1})$. If $A(w)$ has no minimal cycle, then (3)(a) does not hold. If $A(w)$ has a minimal cycle, consider any minimal cycle decomposition of $w$, $\beta = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \ldots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$, and let $LIS(w, \beta)$ be the following simultaneous linear inequalities.

$$X_j \geq 0 \ (1 \leq j \leq k)$$

$$a_i + b_{i\,1} X_1 + b_{i\,2} X_2 + \cdots + b_{i\,k} X_k$$

$$< a_{n+1} + b_{n+1,1} X_1 + b_{n+1,2} X_2 + \cdots + b_{n+1,k} X_k \ (1 \leq i \leq n) .$$

Here $a_i$ and $b_{i\,j}$ are as in Definition 4.8. Since $P(w, \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n) < P(w, \mathcal{A}_{n+1})$, $LIS(w, \beta)$ has an integer solution $X_1 = X_2 = \cdots = X_k = 1$. Hence (3)(b) does not hold.

(3) $\Rightarrow$ (2) is clear. (2) $\Rightarrow$ (3). Assume (2) holds. We shall prove (3) by induction on $|w|$. When $|w| < I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$, the assertion is clear. Assume (3) holds for each $w \in \Sigma^*$ with length at most $k \geq I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}) - 1$. Consider

$w \in \Sigma^*$ with $|w| = k+1 \geq I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$. Since $|w| \geq I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$, $A(w)$ has a minimal cycle. Consider a minimal cycle decomposition of $w$, $\beta = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \ldots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$. It suffices to show the set of simultaneous inequalites $LIS(w, \beta)$ has no integer solution. Let $LIS(w, \beta)$ be of the following form as in Definition 4.8.

$$X_j \geq 0 \ (1 \leq j \leq k)$$

$$\begin{aligned} &a_i + b_{i\,1}X_1 + b_{i\,2}X_2 + \cdots + b_{i\,k}X_k \\ &\quad < a_{n+1} + b_{n+1,1}X_1 + b_{n+1,2}X_2 + \cdots + b_{n+1,k}X_k \ (1 \leq i \leq n)\,. \end{aligned}$$

Assume these simultaneous inequalities have an integer solution $X_j = r_j (1 \leq j \leq k)$. Since $|w| \geq I(\mathcal{A}_1, \ldots, \mathcal{A}_{n+1})$, we have $k > P_{m,1} + P_{m,2} + \cdots + P_{m,|Q|}$ and there exists $1 \leq g < h \leq k$ such that the minimal cycle $(p_g, y_g, p_g)$ and the minimal cycle $(p_h, y_h, p_h)$ are the same. Consider the word

$$v = x_1y_1x_2y_2 \cdots x_gy_gx_{g+1}y_{g+1} \cdots x_{h-1}y_{h-1}x_hx_{h+1}y_{h+1} \cdots x_ky_kx_{k+1}\,.$$

The word $v$ has a minimal cycle decomposition of the form

$$\begin{aligned} \beta' = (&s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \ldots, x_g, p_g, y_g, p_g, x_{g+1}, p_{g+1}, y_{g+1}, p_{g+1}, \ldots, \\ &\times x_{h-1}, p_{h-1}, y_{h-1}, p_{h-1}, A, p_{h+1}, y_{h+1}, p_{h+1}, \ldots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})\,. \end{aligned}$$

Here $A$ is $x_hx_{h+1}$ if $|x_hx_{h+1}| \leq |Q|$, and otherwise is the corresponding part containing a minimal cycle.

We consider the case $A = x_hx_{h+1}$. In the other case, the proof is similar. The set of simultaneous linear inequalities $LIS(v, \beta')$ are of the following form.

$$Y_l \geq 0 \ (1 \leq l \leq k-1)$$

$$\begin{aligned} &c_m + d_{m\,1}Y_1 + d_{m\,2}Y_2 + \cdots + d_{m\,k-1}Y_{k-1} \\ &\quad < c_{n+1} + d_{n+1,1}Y_1 + d_{n+1,2}Y_2 + \cdots + d_{n+1,k-1}Y_{k-1} (1 \leq m \leq n)\,. \end{aligned}$$

By comparing $LIS(w, \beta)$ with $LIS(v, \beta')$, one can see easily that $LIS(v, \beta')$ has the following integer solution.

(i) $Y_j = r_j \ (1 \leq j < h,\ j \neq g)$.
(ii) $Y_g = r_g + r_h$.
(iii) $Y_j = r_{j+1} \ (h \leq j \leq k-1)$.

Since $|v| < |w|$, this is a contradiction to the inductive hypothesis. Thus $LIS(w, \beta)$ has no integer solution. Hence (3) holds. □

By Theorem 4.7, the inequality problem for union $F$-automata can be reduced to the problem of finding integer solutions for simultaneous linear inequalities. The latter problem is decidable (see for example [10]). Thus the following theorem holds.

**Theorem 4.8.** *The inequality problem for union $F$-automata is decidable.*

**Definition 4.9.** The following propblem is called the equivalence problem for union $F$-automata.

Input: $m$ $(m \geq 1)$deterministic $F$-automata
$\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ $(1 \leq i \leq m)$.
$n$ $(n \geq 1)$deterministic $F$-automata
$\mathcal{A}_{m+j} = \langle \Sigma, Q_{m+j}, \delta_{m+j}, \{s_{m+j}\}, F_{m+j}, f_{m+j} \rangle$ $(1 \leq j \leq n)$.
Output: if $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m \equiv \mathcal{A}_{m+1} \cup \cdots \cup \mathcal{A}_{m+n}$, then "yes".
if $A_1 \cup \cdots \cup \mathcal{A}_m \not\equiv \mathcal{A}_{m+1} \cup \cdots \cup \mathcal{A}_{m+n}$, then "no".

**Theorem 4.9.** *The equivalence problem for union $F$-automata is decidable.*

**Proof.** From Proposition 4.4 and Theorem 4.8, the assertion follows immediately. $\square$

## 5. Decidability of the Equivalence Problem for Finitely Ambiguous $F$-Automata

In this section, we shall prove decidability of the equivalence problem for finitely ambiguous $F$-automata (Theorem 5.7). We shall first prove that the problem whether or not an arbitrarily given $F$-automata is finitely ambiguous is decidable. This problem is equivalent to the corresponding problem of finite automata. The result (Theorem 5.4 below) is folklore, but we shall present a proof for readability and because our proof seems new. Then we establish Theorem 5.7, whose proof depends on almost the same ideas concerning minimal cycles as in Sec. 4.

**Notation.** For any language $L \subset \Sigma^*$, $P_r(L)$ denotes the set: $\{x \mid \text{for some } y \in \Sigma^*, xy \in L\}$, i.e. $P_r(L)$ is the set of prefixes of words in $L$.

**Definition 5.1.** For a nondeterministic $F$-automaton $\mathcal{A} = \langle \Sigma, Q, \delta, S, F, f \rangle$ and a word $w = a_1 a_2 \cdots a_m \in L(A)$ $(m \geq 1, a_i \in \Sigma)$, a successful path of $w$ is a path spelling $w$ from a state of $S$ to a state of $F$, i.e.

$$(p_0, a_1, p_1, a_2, p_2, \ldots, p_{m-1}, a_m, p_m).$$

Here $p_i \in Q$ $(0 \leq i \leq m)$, $p_j \in \delta(p_{j-1}, a_j)$ $(1 \leq j \leq m)$, $p_0 \in S$, $p_m \in F$.

**Definition 5.2.** A nondeterministic $F$-automaton $\mathcal{A} = \langle \Sigma, Q, \delta, S, F, f \rangle$ is $k$-ambiguous for $k \geq 1$ if for every $w \in L(\mathcal{A})$, the number of successful paths of $w$ is at most $k$. $\mathcal{A}$ is finitely ambiguous if for some positive integer $k$, it is $k$-ambiguous.

In Theorem 5.3 below, we shall present a necessary and sufficient condition for a nondeterministic $F$-automaton $\mathcal{A} = \langle \Sigma, Q, \delta, S, F, f \rangle$ not to be finitely ambiguous.

**Definition 5.3.** Let $\mathcal{A} = \langle \Sigma, Q, \delta, S, F, f, \rangle$ be an $F$-automaton. $\mathcal{A}$ is said to be forest-like if $|S| \geq 2$ and tree-like if $|S| = 1$. It is said to be trim if for any $q \in Q$, there exist $u, v \in \Sigma^*$ such that $q \in \delta(S, u)$ and $\delta(q, v) \cap F \neq \varnothing$. For each $q \in S$, $(\mathcal{A}, q)$ denotes the $F$-automaton $\langle \Sigma, Q, \delta, \{q\}, F, f \rangle$.

The following proposition is clear.

**Proposition 5.1.** *For any $F$-automaton $\mathcal{A} = \langle \Sigma, Q, \delta, S, F, f\rangle$ with $S = \{q_1, \ldots, q_m\}(m \geq 1)$ and $w \in \Sigma^*$, $P(w, \mathcal{A}) = \max\{P(w, (\mathcal{A}, q_i)) \mid 1 \leq i \leq m\}$.*

Due to the above proposition, from now to Theorem 5.3 below, let $\mathcal{A} = \langle \Sigma, q, \delta, \{s\}, F, f\rangle$ be an arbitrary tree-like trim $F$-automaton. We define an equivalence relation $\approx$ over $\Sigma^*$ as follows. For any $v, w \in \Sigma^*$: $v \approx w$ if and only if for any $p \in Q$, $\delta(p, v) = \delta(p, w)$.

**Definition 5.4.** For any $w \in Pr(L(\mathcal{A}))$, the transition tree $T(w) = \langle V, E, r\rangle$ (for short, T-tree of $w$) is a rooted labelled tree which is defined inductuively as follows.

(1) If $w = \lambda$, then $V$ is a singleton $\{r\}$ and $E = \varnothing$. The label $l(r)$ of $r$ is $(s, \lambda)$, and $r$ is the root and an active leaf of $T(\lambda)$. The state label $st(r)$ of $r$ is $s$.
(2) Let $w = xa$ for $x \in \Sigma^*$ and $a \in \Sigma$. Let $T(x) = \langle V, E, r\rangle$. For each active leaf $t \in V$ of $T(x)$ labelled by $(p, x)(p \in Q)$, introduce the sets of new vertices $V(t, a)$ and new edges $E(t, a)$ by:
   (a) If $\delta(p, a) = \varnothing$, then $V(t, a) = \{(t, \{\})\}, E(t, a) = \{(t, (t, \{\}))\}$, and the label of $(t, \{\})$ is $\{\}$. $(t, \{\})$ is a nonactive leaf of $T(xa)$.
   (b) If $\delta(p, a) = \{q_1, \ldots, q_m\}(m \geq 1)$, then introduce $m$ new vertices, $V(t, a) = \{(t, q_i) \mid 1 \leq i \leq m\}$ and define $E(t, a) = \{(t, (t, q_i)) \mid 1 \leq i \leq m\}$. For each $1 \leq i \leq m$, $(t, q_i)$ is an active leaf of $T(xa)$ and is labelled by $l((t, q_i)) = (q_i, xa)$. The edge $(t, (t, q_i))$ is labelled by $l((t, (t, q_i))) = a$. The state label $st((t, q_i))$ of $(t, q_i)$ is $q_i$.
   (c) Define $T(ta) = \langle V', E', r\rangle$ by: $V\prime = V \cup (\cup_{t \in A} V(t, a))$ and $E' = E \cup (\cup_{t \in A} E(t, a))$, where $A$ is the set of active leaves of $T(x)$.

**Definition 5.5.** Let $w \in L(\mathcal{A}) \cap \Sigma^+$ and $T(w) = \langle V, E, r\rangle$.

(1) A path $P$ (of $T(w)$) is a sequence of vertices $P = (t_1, t_2, \ldots, t_k)$ $(k \geq 2)$ such that $t_i \in V$ $(1 \leq i \leq k)$ and $t_k$ is not a nonactive leaf, and $t_{j+1}$ is a child of $t_j$ in $T(w)$ for $1 \leq j \leq k-1$. The label sequence $l(P)$ of $P$ is the sequence $(l(t_1, t_2), l(t_2, t_3), \ldots, l(t_{k-1}, t_k))$ where $l(t_j, t_{j+1})$ is the label of the edge $(t_j, t_{j+1})$ in $T(w)$ $(1 \leq j \leq k-1)$. $l(t_1, t_2)l(t_2, t_3) \cdots l(t_{k-1}, t_k)$ is the label of $P$, and a factor of $w$. The transition sequence $tr(P)$ of $P$ is the sequence $(p_1, a_1, p_2, a_2, \ldots, p_{k-1}, a_{k-1}, p_k)$ where for $1 \leq i \leq k$, $p_i \in Q$ is the state label of $t_i$, and $a_j \in \Sigma$ is the label $l(t_j, t_{j+1})$, for $1 \leq j \leq k-1$.
(2) A path $P = (t_1, \ldots, t_k)$ $(k \geq 2)$ is a prefix path if $t_1 = r$. It is complete if $k = |w| + 1$, and successful if it is complete and $st(t_k) \in F$. $P$ is a suffix path if it is a "suffix" subpath of some complete path. $SP(\mathcal{A}, w)$ denotes the set of successful paths of $w$.
(3) For any prefix $u$ of $w$, $NL(u, w)$ denotes the number of prefix paths of all successful paths in $T(w)$, whose label is $u$.

The following proposition is clear.

**Proposition 5.2.** (1) *For any* $u$, $w \in \Sigma^*$ *and* $v \in \Sigma^+$ *with* $uvw \in L(\mathcal{A})$ *and* $u \approx uv$ *: if* $NL(u, uvw) < NL(uv, uvw)$, *then for any* $k \geq 1$, *we have* $NL(uv^k, uv^k w) \geq NL(u, uvw) + k$.

(2) *For any* $u$, $w \in \Sigma^*$ *and* $a \in \Sigma$ *with* $uaw \in L(\mathcal{A})$, *we have* $NL(ua, uaw) - NL(u, uaw) \leq |Q| \cdot NL(u, uaw)$.

**Theorem 5.3.** *Let* $\mathcal{A} = \langle \Sigma, Q, \delta, \{s\}, F, f \rangle$ *be as above. The following three conditions are equivalent.*

(1) $\mathcal{A}$ *is not finitely ambiguous.*
(2) *There exist* $u$, $w \in \Sigma^*$ *and* $v \in \Sigma^+$ *such that* $uvw \in L(\mathcal{A})$, $u \approx uv$ *and* $NL(u, uvw) < NL(uv, uvw)$.
(3) *There exist* $u$, $w \in \Sigma^*$ *and* $v \in \Sigma^+$ *such that* $|u|, |v|, |w| \leq 2^{|Q|^2}$, $uvw \in L(\mathcal{A})$, $u \approx uv$ *and* $NL(u, uvw) < NL(uv, uvw)$.

**Proof.** (1) $\Rightarrow$ (2). Assume that $\mathcal{A}$ is not finitely ambiguous. Then there exist $w \in L(\mathcal{A})$ and a decomposition of $w$, $w = u_0 u_1 \cdots u_m u_{m+1}$, such that $m > 2^{|Q|^2}$, and for each $1 \leq i \leq m$, $NL(u_0 u_1 \cdots u_i, w) > NL(u_0 u_1 \cdots u_{i-1}, w)$. Note that $|\{f \mid f : Q \to 2^Q\}| = 2^{|Q|^2}$. Thus there exist $1 \leq i < j \leq m$ such that $u_0 u_1 \cdots u_i \approx u_0 u_1 \cdots u_j$. Thus (2) holds.

(2) $\Rightarrow$ (3). Assume that for $u$, $w \in \Sigma^*$ and $v \in \Sigma^+$, (2) holds. If $|u| > 2^{|Q|^2}$, then there exists a decomposition of $u, u = u_0 u_1 u_2$, such that $u_0, u_2 \in \Sigma^*$, $u_1 \in \Sigma^+$ and $u_0 \approx u_0 u_1$. Now it is clear that for $u_0 u_2$, $v$ and $w$, (2) holds. Thus we may assume that $|u| \leq 2^{|Q|^2}$. Now if $|v| > 2^{|Q|^2}$, then there exists a decomposition of $v$, $v = v_0 v_1 v_2$, such that $v_0$, $v_2 \in \Sigma^*$, $v_1$, $v_0 v_2 \in \Sigma^+$ and $v_0 \approx v_0 v_1$. If $NL(uv_0, uvw) < NL(uv_0 v_1, uvw)$, then for $uv_0$, $v_1$ and $v_2 w$, (2) holds. Otherwise, for $u$, $v_0 v_2$ and $w$, (2) holds. Then, as above, we may assume $|v| \leq 2^{|Q|^2}$. In the same way, we may assume that $|w| \leq 2^{|Q|^2}$. Thus (3) holds.

(3) $\Rightarrow$ (1). Assume (3) holds for $u$, $w \in \Sigma^*$ and $v \in \Sigma^+$. By Proposition 5.2, for any $k \geq 1$, we have $NL(uv^k, uv^k w) > NL(u, uvw) + k$. Thus in $T(uv^k w)$, there exist at least $k$ successful paths. Thus (1) holds. □

**Theorem 5.4.** *It is decidable whether or not any given nondeterministic* $F$*-automaton is finitely ambiguous.*

In the rest of this section, let $\mathcal{A}_i = \langle \Sigma, Q_i, \delta_i, S_i, F_i, f_i \rangle$ $(i = 1, 2)$ be two finitely ambiguous $F$-automata. Let $K_i$ $(i = 1, 2)$ be the smallest integer $k_i$ such that $\mathcal{A}_i$ is $k_i$-ambiguous. One can see that we can obtain $K_i$ $(i = 1, 2)$ effectively by depending on Propositions 5.1 and 5.2 and Theorem 5.3. We say that $\mathcal{A}_i$ is greater than or equal to $\mathcal{A}_j$ $(\{i, j\} = \{1, 2\})$ (write $\mathcal{A}_i \geq \mathcal{A}_j$) if for any $w \in \Sigma^*$, $P(w, \mathcal{A}_i) \geq P(w, \mathcal{A}_j)$. As in the case of union $F$-automata, it suffices to solve the inequality problem, i.e. deciding whether or not $\mathcal{A}_1 \geq \mathcal{A}_2$.

Our arguments for solving this inequality problem are similar to the ones for the inequality problem for union $F$-automata. So we shall present the arguments briefly. First we decide whether or not $L(A_1) \supset L(A_2)$. If $L(A_1) \not\supset L(A_2)$, then the answer is "no". Otherwise consider any $w = a_1 a_2 \cdots a_n \in L(\mathcal{A}_1)$ ($n \geq 1, a_i \in \Sigma$ for $1 \leq i \leq n$). If $w \notin L(\mathcal{A}_2)$, then we have done. Otherwise let $SP(\mathcal{A}_1, w) = \{P_1, P_2, \ldots, P_l\}$ and $SP(\mathcal{A}_2, w) = \{P'_1, P'_2, \ldots, P'_m\}$ be the sets of successful paths of $w$ of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Thus $1 \leq l \leq K_1$ and $1 \leq m \leq K_2$. For each $1 \leq c \leq m$, we shall construct sets of linear inequalities, $LIS(\mathcal{A}_1, w, P'_c, \alpha)$ for many $\alpha$, where each $\alpha$ is a minimal cycle decomposition, and will be explained precisely in the sequel. For each $1 \leq j \leq l$, let $P_j = (p_{j0}, a_1, p_{j1}, a_2, \ldots, p_{jn-1}, a_n, p_{jn})$, and let $P'_c = (p'_0, a_1, p'_1, a_2, \ldots, p'_{n-1}, a_n, p'_n)$. First we construct the product state transition sequence, $pst(\mathcal{A}_1, w, P'_c)$ by: $pst(\mathcal{A}_1, w, P'_c) = (Z_0, a_1, Z_1, a_2, \ldots, Z_{n-1}, a_n, Z_n)$, where for each $0 \leq i \leq n$, $Z_i = (p_{1i}, p_{2i}, \ldots, p_{li}, p'_i)$. As in Sec. 4, a minimal cycle of $pst(\mathcal{A}_1, w, P'_c)$ is a subsequence $(Z_i, a_{i+1}, Z_{i+1}, a_{i+2}, \ldots, Z_{i+k-1}, a_{i+k}, Z_{i+k})$ of $pst(\mathcal{A}_1, w, P'_c)$ such that $k \geq 1$, $Z_i = Z_{i+k}$ and for any $i \leq J < K < i + k$, $Z_J \neq Z_K$. This cycle will be often denoted by $(Z_i, a_{i+1}, a_{i+2}, \ldots, a_{i+k}, Z_i)$ if the context is clear. Then a minimal cycle decomposition of $pst(\mathcal{A}_1, w, P'_c)$ is a sequence, $\alpha = (X_0, x_1, Y_1, y_1, Y_1, x_2, Y_2, y_2, Y_2, \ldots, x_d, Y_d, y_d, Y_d, x_{d+1}, Z_n)$, such that $d \geq 1$, $w = x_1 y_1 x_2 y_2 \cdots x_d y_d x_{d+1}$, $1 \leq |x_i| \leq n(\mathcal{A}_1, w, P'_c)$ ($1 \leq i \leq d+1$) and $(Y_j, y_j, Y_j)$ is a minimal cycle ($1 \leq j \leq d$); here $n(\mathcal{A}_1, w, P'_c)$ is the maximum length of minimal cycles so that $1 \leq n(\mathcal{A}_1, w, P'_c) \leq |Q_1|^l \cdot |Q_2|$. Then we introduce $d$ variables $X_1, X_2, \ldots, X_d$ and construct the following set of linear inequalities $LIS(\mathcal{A}_1, w, P'_c, \alpha)$ as in Sec. 4:

$$X_j \geq 0 \ (1 \leq j \leq d)$$
$$a_i + b_{i1} X_1 + b_{i2} X_2 + \cdots b_{id} X_d$$
$$< a_{l+1} + b_{l+1,1} X_1 + b_{l+1,2} X_2 + \cdots b_{l+1,d} X_d \ (1 \leq i \leq l)$$

where for each $1 \leq i \leq l$, $a_i$ is the profit of $x_1 x_2 \cdots x_{d+1}$ along the path $P_i$ and for each $1 \leq j \leq d$, $b_{ij}$ is the profit of $y_j$ along the path $P_i$, and $a_{l+1}$ and $b_{l+1,j}$ ($1 \leq j \leq d$) are the corresponding profits along the path $P'_c$. Then the number of minimal cycles is bounded by $I(P_1, P_2, \ldots, P_l, P'_c)/(1 + |Q_1|^l \cdot |Q_2|)$, as in Lemma 4.5, where $I(P_1, P_2, \ldots, P_l, P'_c)$ is defined as in Definition 4.6. Then the following proposition is clear from Definitions 4.6 and 4.7, Lemma 4.5.

**Proposition 5.5.** *It holds that* $\max\{I(P_1, P_2, \ldots, P_l, P'_c) \mid P_1, P_2, \ldots, P_l, P'_c$ *are as above for some* $w \in L(\mathcal{A}_1 \cap L(\mathcal{A}_2)\} \leq I(\mathcal{A}_1, \mathcal{A}_2)$. *Here* $I(\mathcal{A}_1, \mathcal{A}_2) = (1 + |Q_1|^{K_1} |Q_2|)(1 + P_{m,1} + P_{m,2} + \cdots + P_{m,m})$ *and* $m = |Q_1|^{K_1} |Q_2||\Sigma|$.

Then as in the proofs of Lemma 4.5 and Theorem 4.7, one can prove the following theorem.

**Theorem 5.6.** *The following two conditions are equivalent.*

(1) $\mathcal{A}_1 \geq \mathcal{A}_2$.
(2) $L(\mathcal{A}_1) \supset L(\mathcal{A}_2)$ *and for each* $w \in \Sigma^+ \cap L(\mathcal{A}_2)$ *with* $|w| \leq I(\mathcal{A}_1, \mathcal{A}_2)$, *if* $SP(\mathcal{A}_1, w) = \{P_1, P_2, \ldots, P_l\}$ *and* $SP(\mathcal{A}_2, w) = \{P'_1, P'_2, \ldots, P'_m\}$, *then for each* $1 \leq c \leq m$, *one of the following holds.*
   (a) *If* $pst(\mathcal{A}_1, w, P'_c)$ *has no minimal cycle, then* $P(w, \mathcal{A}_1) \geq P(w, \mathcal{A}_2)$.
   (b) *If* $pst(\mathcal{A}_1, w, P'_c)$ *has a minimal cycle decomposition* $\alpha$, *then for each such* $\alpha$, *the set of simultaneous linear inequalities* $LIS(\mathcal{A}_1, w, P'_c, \alpha)$ *has no integer solution.*

The following is the main theorem of this paper.

**Theorem 5.7.** *The equivalence problem for finitely ambiguous* $F$*-automata is decidable.*

**Remark 5.1.** Let $\mathcal{B} = \langle \Sigma, \Delta, Q, \delta, S, F, \Gamma \rangle$ be a finite transducer such that $\Sigma = \{a\}$ and $\Delta = \{0, 1\}$ are the input and output alphabets, respectively, $Q = \{p, q\}$ is the set of states, $\delta : Q \times \Sigma \to 2^Q$ is the transition function ($\delta(p, a) = \{p, q\}, \delta(q, a) = \varnothing$), $S = \{p\}$ and $F = \{q\}$ are the sets of initial and final states, respectively, $\Gamma : Q \times \Sigma \times Q \to \Delta$ is the output function ($\Gamma(p, a, p) = 0,\ \Gamma(p, a, q) = 1$). One can see easily that $\mathcal{B}$ cannot be decomposed into a union $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m$ of $m \geq 1$ deterministic finite transducers for any $m \geq 1$ such that $\mathcal{B}$ is equivalent to $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m$.

**Remark 5.2.** In this paper, the following problem is left open: for any finitely ambiguous $F$-automaton $\mathcal{A}$, does there exist a union $F$-automaton which is equivalent to $\mathcal{A}$? To solve this problem, we cannot depend on the set $\{A(w) \mid w \in \Sigma^*\}$ as for union $F$-automata as in Defintion 4.3 for any finitely ambiguos $F$-automaton because in general we do not have $L(\mathcal{A}) = \cup_{w \in \Sigma^*} L(A(w))$. For example, consider the finite transducer $\mathcal{B}$ in Remark 5.1. It is 1-ambiguous, but we need a deterministic $F$-automaton $\mathcal{A}_1$ which is equivalent to $\mathcal{B}$ but is different from $\mathcal{B}$. Thus generally one cannot develop similar arguments for defining a union $F$-automaton as in Definition 4.3. Thus in this paper, the above problem is left open. Because the equivalence problem for arbitrary $F$-automata is undecidable, it seems that there exists a nondeterministic $F$-automaton which is not equivalent to any finitely ambiguous $F$-automaton. This problem is also left open.

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