ON BIRKHOFF'S PFAFFIAN SYSTEMS*

BY

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Introduction

The *Pfaffian systems* to which this paper is devoted were defined and considered for the first time in 1920 by G. D. Birkhoff in the Chicago Colloquium lectures. Later he was led to a consideration of them in his memoir† *Stability and the equations of dynamics* and published in his book‡ *Dynamical Systems* the demonstration of the important results he announced there.

I shall set down later the precise definition of those systems; for the present it is sufficient to say that the Hamiltonian form can be obtained from them by a natural particularization, and to give their fundamental property: namely, of being of the same form after any point transformation of the dependent variables. This last statement must be compared with the well known invariance of the Hamiltonian form under the canonical transformations, and so one is naturally led to the question of the analytic significance of the Pfaffian form, and that is the first point studied here. First of all, any Pfaffian system is related to a differential expression ω so that the choice of a special form for the system is equivalent to the choice of ω. Moreover it is easy to see that any system under consideration admits $\int\omega$ as a relative integral invariant and is precisely the "*characteristic system*" of $\int\omega'$. So without referring to any dynamical problem, I first intend to make as clear as possible the interest attached to the Pfaffian form and the necessity of its use. For the same purpose in the case where a certain number of integrals are known, I give an extension of Poisson's and Lie's theorems and also several other properties relative to the use of the known integrals for reducing the order of the system. At the end of the first section I find a very simple expression for the integral invariant of order equal to the order of the system, and consequently a noteworthy form for the "*last multiplier*"; this last multiplier is the square root of the determinant formed with all the coefficients appearing in the first members in the Pfaffian system.

As an elucidation of the way in which one can be led to the consideration of Pfaffian systems in dynamical problems, I study in the second section the

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† American Journal of Mathematics, vol. 49 (1927), p. 1.

‡ American Mathematical Society Colloquium Publications, vol. IX, 1927.

reduction of the Lagrangian to the Pfaffian form. The entire question depends on a system of simultaneous partial differential equations in the F_i, G_i (which define a point change of variables) and the X_i, Z of the form ω (which determines the Pfaffian system under consideration). Hence a whole class of problems arises when we regard some of the functions F_i, G_i, X_i, Z as known and the others as a solution of the system of simultaneous partial differential equations. In applying this general process I obtain first a reduction to the normal form in a way which presents the advantage of leading directly to the result, and I give afterwards a reduction to the Hamiltonian form.

At the beginning of the third section I briefly review, following there Birkhoff's book quoted above, the most important properties of the Pfaffian systems considered from a dynamical point of view, together with the different problems to which they can be applied. Then, in applying results of the first section, I show how, after the greatest possible reduction, the final order of the dynamical system for the three-body and some similar problems can be obtained in a very simple way.

Finally I consider the "prepared" Pfaffian expression ω which Birkhoff attaches to a Pfaffian system in the vicinity of an equilibrium point, and I study the reduction of ω to a canonical form $\bar{\omega}$, in fact the reduction of ω to the sum of ω and of a perfect differential dE.

This is a problem of fundamental importance from a dynamical point of view, of which Birkhoff* has given a formal solution by a process that does not introduce small divisors. This last fact suggests at once the possibility of establishing the considered reduction by the use of convergent series, but it does not give a demonstration of that possibility. So I was led to consider this problem not only in a formal sense but in an actual sense. For this problem I have obtained a solution which I have shown to be both formal and actual, in the case of two degrees of freedom. Indeed at the end of this paper I demonstrate, when $m=2$, that the considered reduction can always be established by means of analytic functions.†

I

We shall consider systems of differential equations of the following form:

$$\text{(I)} \qquad \sum_{j=1}^{2m} a_{ij}\frac{dX_j}{dt} + \frac{\partial X_i}{\partial t} - \frac{\partial Z}{\partial x_i} = 0 \qquad (i = 1, 2, \cdots, 2m),$$

* American Journal of Mathematics, loc. cit.

† Some of the results of this paper have been briefly indicated in two notes published in the Paris Comptes Rendus, vol. 188, p. 1029 and p. 1144.

where X_i and Z are $2m+1$ functions of the x_i and of t, and

$$a_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}.$$

Such systems give the extremals in the variational problem that can be stated by

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{2m} X_j \frac{dx_j}{dt} + Z \right] dt = 0.$$

Let us introduce now the system

$$\text{(II)} \qquad \sum_{j=1}^{2m} a_{ij} dx_j + \left(\frac{\partial X_i}{\partial t} - \frac{\partial Z}{\partial x_i} \right) dt = 0 \quad (i = 1, 2, \cdots, 2m)$$

from which the former arises when the variables x_i have to be functions of t. The system (II) is very closely connected with the Pfaffian expression

$$\omega = \sum_{j=1}^{2m} X_j dx_j + Z dt,$$

and it is interesting to consider with some attention those connections in order to realize fully the analytic significance of systems (I). For that purpose one can refer to E. Cartan, *Leçons sur les Invariants Intégraux*,* where those connections are to be found among other still more powerful results. Here we are going to enunciate only the properties that seem to be the most important in dynamics and to recall at the same time some definitions that may facilitate a reference to the book cited.

For the system (II) we have a relative integral invariant $\int \omega$ and it is a linear integral invariant. Furthermore to every form ω there corresponds in general a system (II) and only one, admitting $\int \omega$ as a relative integral invariant. It is called the *characteristic system* of $\int \omega$ or the *associated system* of the form ω' (ω' being the bilinear covariant of the form ω).† Hence the systems (II) can be defined as the systems admitting a linear relative integral invariant. If we now suppose the existence of known integrals, we can extend to our systems Poisson's theorem and also Lie's theorem‡ (evidently with the particular case $n=m$ due to Liouville) that gives the reduction of

* Paris, Librairie Hermann, 1922. See particularly chapter XII entitled *Systems admitting a relative integral-invariant.*

† It is the "first Pfaffian system" of Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge, 1927, p. 307, and also, according to Morera, the "associate of the Pfaffian expression ω," Rendiconti, Accademia dei Lincei, vol. 12 (1st sem. 1903), pp. 113–122.

‡ See Levi-Civita e Amaldi, *Lezioni di Meccanica Razionale*, vol. II, part 2, p. 380; Bologna, 1927.

the order of a system when there are given n integrals independent and moreover in involution.

In a more general manner Cartan gives the largest reduction in the order of the system that can be made, when n integrals are known, independent but not necessarily in involution. He shows the way in which this reduction can be obtained and states a theorem giving the number by which the order of the system can be diminished. In particular if ω can be written as an exact differential with respect to the known integrals, the problem can be solved by one quadrature. Finally let us point out that whatever may be the number of integrals given, the order of the system is to be reduced by an even number, so that if we know $2k+1$ integrals, the order of the system will be reduced by at least $2k+2$. It is clear that those properties belong also to the systems (I), if none of the integrals we have to consider is ever $t=C$.

In chapter I of the same book, Cartan shows that the Hamiltonian equations can be characterized as those which admit the integral invariant $\int\omega_\delta$, with $\omega_\delta=\sum p_i\delta q_i-H\delta t$, and he calls this last expression "tenseur quantité de mouvement-énergie." By any change of variables ω_δ will be transformed into a form ω and the Hamiltonian system into a system (I); furthermore, the system (I) will be determined by ω, just as the Hamiltonian system is the only one that belongs to ω_δ. So we can conclude that in their more general form the dynamical systems can be represented by a linear differential expression ω.

It is evident now that every particularization of the form ω will give a particularization for the corresponding system: the Hamiltonian system is to be obtained when ω is taken in the particular form ω_δ. Naturally the question arises whether a different particularization of ω can lead to a system more appropriate to the study of certain problems of dynamics and also whether avoiding the reduction to the Hamiltonian form would not be in some cases a simplification. Birkhoff* opened the way in that direction when he first defined as a *Pfaffian system*† the system (I) corresponding to a form

* Loc. cit.

† To avoid all difficulties let us make the following remarks about the different denominations that are to be used:

(1) Equations in which the first members are linear differential expressions, when not distinguishing any of the independent variables, are known as Pfaff's equations and systems of such equations as *Pfaff's systems*.

(2) The systems studied in this paper were named Pfaffian systems by G. D. Birkhoff; when there might be some confusion they can be designated as "dynamical Pfaffian systems."

(3) In a preceding note, we have already referred to the *first Pfaff's system of a form* ω, so called by Whittaker.

(4) Finally, in the theory of skew-symmetric determinants we have later to deal with expressions spoken of by Cayley as "Pfaffians."

ω in which X_i and Z do not involve t. Further, he uses the name *generalized Pfaffian system* when X_i and Z are periodic in t with the same period.

We will limit ourselves hereafter in this paper to the consideration of the dynamical Pfaffian systems. For those systems $Z=C$ is an integral, and by use of it and by elimination of dt, it is easy to show that any such system of order $2m$ is in fact equivalent to a system of differential equations of order $2m-2$. After this reduction the system obtained is analogous to a Pfaffian system, with the difference, however, that the functions X_i and Z involve t; we can say, in other words, that X_i and Z involve t as in a generalized Pfaffian system but are no longer periodic in that variable.

Furthermore, the Pfaffian form is preserved by any of the following changes of variables:

$$\bar{x}_i = f_i(x_1, x_2, \cdots, x_{2m}) \qquad (i=1, 2, \cdots, 2m),$$
$$\bar{t} = \phi(t),$$

f_i and ϕ being $2m+1$ arbitrary functions. Let us suppose now that we have $2k+1$ integrals not involving t, independent of each other and independent of $Z=C$:

$$F_1(x_1, x_2, \cdots, x_{2m}) = C_1,$$
$$F_2(x_1, x_2, \cdots, x_{2m}) = C_2,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$F_{2k+1}(x_1, x_2, \cdots, x_{2m}) = C_{2k+1}.$$

The change of variables

$$\bar{x}_1 = F_1, \qquad \bar{x}_2 = F_2, \quad \cdots, \ \bar{x}_{2k+1} = F_{2k+1},$$
$$\bar{x}_{2k+2} = x_{2k+2},\ \bar{x}_{2k+3} = x_{2k+3}, \cdots, \ \bar{x}_{2m} = x_{2m}$$

transforms ω into another form $\bar{\omega}$ and $Z=C$ into another integral $\bar{Z}=C$.

The given system in which we know the $2k+1$ integrals is the system that gives the extremals in the variational problem $\delta\int\bar{\omega}=0$ when we have replaced, in $\bar{\omega}$, $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{2k+1}$ by arbitrary constants. If we write this system, we see at once that it can admit solutions only if a certain determinant (characteristic) is 0. This last equality allows us to express one of the variables $\bar{x}_{2k+2}, \cdots, \bar{x}_{2m}$ in terms of the others, and to reduce ω to a form containing only $2m-2k-2$ variables. The system which gives the extremals will be of that order $2m-2k-2$ and still of the same form, i.e., it will admit an integral such as $Z=C$, so, in dealing with this last fact and by elimination of dt, its order can be possibly reduced by two units more.

It is clear that the system we consider now in which $Z=C$ is an integral admits as a relative invariant integral

$$\int \sum_{i=1}^{2m} X_i dx_i.$$

From this can be obtained at once by Stokes' theorem the absolute integral invariant

$$\int\int \sum_{i=1}^{2m} \sum_{j=1}^{2m} a_{ij} dx_i dx_j.$$

It has been known, since Poincaré's *Méthodes Nouvelles de la Mécanique Céleste*, vol. III, No. 247, that from an absolute invariant of order 2 can be deduced a set of others, of orders 4, 6, 8, $\cdots$, successively to the last of order $2m$. In the case $m=2$ by one operation we have the volume invariant

$$\int | a_{ij} |^{1/2} dx_1 dx_2 dx_3 dx_4,$$

where $|a_{ij}|$ is the skew-symmetric determinant of the a_{ij}. It is easy to verify this result by showing that the rational expression $|a_{ij}|^{1/2}$ is the "last multiplier" of the system. This verification suggests that the preceding result may be valid whatever may be the number $2m$ of the independent variables occurring in the system.

That is what we shall now demonstrate. We suppose, for example, that, in a_{ij}, i indicates the rank of the row, j that of the column, and we call A_{ij} the minor corresponding to a_{ij}. The resolution of the system gives $2m$ equations:

$$\frac{dx_i}{dt} = \frac{A_{1i}\dfrac{\partial Z}{\partial x_1} + A_{2i}\dfrac{\partial Z}{\partial x_2} + \cdots + A_{ni}\dfrac{\partial Z}{\partial x_n}}{| a_{ij} |} = P_i.$$

We want to show that $|a_{ij}|^{1/2}$ is the last multiplier, so that

$$\sum_{i=1}^{2m} \frac{\partial(| a_{ij} |^{1/2} P_i)}{\partial x_i} = 0. \tag{1}$$

We shall remark, first, that in the preceding sum all the terms containing second derivatives of Z vanish, because the A_{ij} themselves form a skew-symmetric square array. Let us write

$$f_{ij} = \frac{A_{ij}}{| a_{ij} |^{1/2}}$$

and equate to 0 each of the coefficients of the terms involving

$$\frac{\partial Z}{\partial x_1}, \frac{\partial Z}{\partial x_2}, \cdots, \frac{\partial Z}{\partial x_n}$$

in the relation (1). Thus, we have the $2m$ relations

$$\sum_{j=1}^{2m} \frac{\partial f_{ij}}{\partial x_j} = 0.$$

If one of these relations is satisfied, on account of the symmetry of the indices, the same will be true for all the others. Then we shall limit ourselves to a verification of the first one of these relations.

Each term in this equation involves one second derivative of one of the X_i. This derivative in any arbitrary term can be found by taking for example in f_{1s} $(s\neq 1)$ one factor a_{uv} and in this factor $\partial X_u/\partial x_v$ which is to be derived with respect to x_s. So, an arbitrary term can be written

$$\frac{\partial^2 X_u}{\partial x_v \partial x_s} M,$$

the indices u, v, s taking all the values between 1 and $2m$, but under the following condition: the four numbers 1, s, u, v must all be distinct. The way in which this condition can be brought about is to be explained as follows: first, $s\neq 1$ because $f_{11}=0$; then f_{1s} is the minor corresponding to a_{1s} in the "Pfaffian" $|a_{ij}|^{1/2}$ and in a "Pfaffian" each index can be written only once, so u, v are entirely distinct from 1 and s; moreover, $u\neq v$ because $a_{uu}=0$. We shall now show that there exists one term which is just the negative of the one we have written above; thus they will vanish together, and, since the first term was arbitrary, we can conclude that all the terms vanish.

The first equation with which we are dealing involves the expression $\partial f_{1v}/\partial x_v$ because $v\neq 1$. But f_{1v} surely contains terms in which there is to be found the factor a_{us} because all the numbers 1, s, u, v are distinct. Hence, after differentiation we have a term such as

$$\frac{\partial^2 X_u}{\partial x_s \partial x_v} M'.$$

It remains to show that $M'=-M$. For that purpose we first remark that in the expansion of the whole determinant $|a_{ij}|$, $M|a_{ij}|^{1/2}$ can be considered as the coefficient of the product $a_{1s}a_{uv}$ and $M'|a_{ij}|^{1/2}$ as the coefficient of the product $a_{1v}a_{us}$. Now the truth of the assertion that one of these coefficients ($M|a_{ij}|^{1/2}$ and $M'|a_{ij}|^{1/2}$) is the negative of the other appears at once when we take the expansion of the determinant $|a_{ij}|$ according to Laplace's theorem written in the following form:

$$| a_{ij} | = \sum(-1)^{\nu} \begin{vmatrix} a_{1p} & a_{1q} \\ a_{up} & a_{uq} \end{vmatrix} \begin{vmatrix} a_{2,r} & a_{2,s} & \cdots & a_{2,t} \\ \cdot & \cdot & \cdot & \cdot \\ a_{u-1,r} & a_{u-1,s} & \cdots & a_{u-1,t} \\ a_{u+1,r} & a_{u+1,s} & \cdots & a_{u+1,t} \\ \cdot & \cdot & \cdot & \cdot \\ a_{2m,r} & a_{2m,s} & \cdots & a_{2m,t} \end{vmatrix}$$

$$(p, q, r, s, \cdots, t = 1, \cdots, 2m),$$

where we consider the minors of the second order composed of elements of the rows of rank 1 and u.

So the last of Poincaré's successive invariants can be written

$$\int | a_{ij} |^{1/2} dx_1 dx_2 \cdots dx_{2m}$$

and this form, interesting because of its simplicity, has been directly obtained by a simple verification.

By using exterior calculus we can present this result in a slightly different form. Referring to Cartan (*Leçons sur les Invariants Intégraux*, p. 78) we know that if $\int \Omega'$ is an absolute integral invariant ($\Omega' = \sum a_{ij}[dx_i dx_j]$), $\int (\Omega')^m$ is the absolute integral invariant of order $2m$. Hence what we have established above is equivalent to the formula

(III) $$(\Omega')^m = K \, | a_{ij} |^{1/2} dx_1 \cdot dx_2 \cdots dx_{2m}.$$

This formula can be obtained also by merely using the rules of the exterior calculus. But whichever way one proceeds, one has to deal with the expressions called "Pfaffians"* in the theory of skew-symmetrical determinants, the definition of which can be stated as follows: $\sum(-1)^q a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2m-1} i_{2m}}$, the sum being extended to all the terms that are different, q being the permutation number of $i_1, i_2, \cdots, i_{2m}$.

II

By the use of Hamiltonian systems as a first step it is easy to see that the change of variables

$$x_1 = f_1 \left(q, \frac{\partial L}{\partial q'} \right),$$
$$\cdots\cdots\cdots$$
$$x_{2m} = f_{2m} \left(q, \frac{\partial L}{\partial q'} \right)$$

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* See Cayley, *Recherches ultérieures sur les déterminants gauches*, Crelle's Journal, vol. 55, pp. 299–313, and, for example, T. Muir, *The Theory of Determinants*, vol. II, London, 1911.

Also von Weber, *Vorlesungen über das Pfaff'sche Problem*, Leipzig, Teubner, 1900, where the Pfaffians are mentioned as "Pfaffsche Aggregat."

transforms any Lagrangian system, given by a function $L(q, q')$ in the variables q, q', into a Pfaffian system in the variables $x_1, x_2, \cdots, x_{2m}$. Let us fix our attention for a while on the most general transformations by which we can pass from Lagrangian to Pfaffian systems. Suppose now that by means of the transformation

$$(2) \qquad q_i = F_i(x_1, x_2, \cdots, x_{2m}); \; q_i' = G_i(x_1, x_2, \cdots, x_{2m}) \qquad (i = 1, 2, \cdots, m)$$

the following equality

$$(3) \qquad L(F, G) = \sum_{j=1}^{2m} X_j(x_1, x_2, \cdots, x_{2m})x_j' + Z(x_1, x_2, \cdots, x_{2m})$$

holds as long as the variables x_j satisfy the system

$$(4) \qquad x_j' = \frac{\displaystyle\sum_{k=1}^{2m} A_{kj}\frac{\partial Z}{\partial x_k}}{\Delta} \qquad (j = 1, 2, \cdots, 2m).$$

In these relations (4), x_j, Z are $2m+1$ functions of $x_1, x_2, \cdots, x_{2m}$; Δ is $|a_{ij}|$, the determinant of the

$$a_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i},$$

and A_{kj} is the minor corresponding to the element a_{kj} in that determinant. Furthermore, we know that (4) gives the extremals ($\delta\int L=0$) if we replace L by the second member of the equation (3); so this system (4) will take the place of the Lagrangian equations after the transformation (2). The equation (3) must be satisfied only when we have (4), and for that the necessary and sufficient condition is

$$(5) \qquad L(F, G) - \frac{1}{\Delta} \sum_{j=1}^{2m} \left[X_j \sum_{k=1}^{2m} A_{kj}\frac{\partial Z}{\partial x_k}\right] = Z.$$

Moreover, let us remark that the relations $q_i' = \partial q_i/\partial t$ give, in the new variables,

$$G_i = \sum_{j=1}^{2m} \frac{\partial F_i}{\partial x_j} x_j',$$

which are to be satisfied at the same time as (4); hence, we have in addition to (5) the following m conditions:*

$$(6) \qquad \sum_{j=1}^{2m} \left[\frac{\partial F_i}{\partial x_j} \sum_{k=1}^{2m} A_{kj}\frac{\partial Z}{\partial x_k}\right] = G_i\Delta.$$

* In (5) and (6) the fractions A_{ij}/Δ can be replaced by f_{ji}/Δ which are also rational expressions. The f_{ij} are the minors of the Pfaffian $|\Delta|^{1/2}$ according to the definitions we have already recalled at the end of § I.

The $m+1$ relations (5) and (6) constitute a system which we call (S) relating F_i, G_i, X_i, Z, all functions of $2m$ independent variables x (L being always a given function).

This system (S) in which F_i, G_i are considered as given functions and X_i, Z as the unknown functions is in general completely integrable;* therefore, we conclude that by an appropriate choice of the X_i and Z a Lagrangian system can in general be reduced to a Pfaffian system by use of a transformation (2) in which F_i and G_i are all arbitrary. In a similar way we can assume the X_i, Z given, and then we have a solution in F_i, G_i. *Hence, it follows that we can reduce any Lagrangian system to the Pfaffian system corresponding to a form* ω *arbitrarily chosen.*†

In particular in the case

$$\omega = y_{m+1}dy_1 + y_{m+2}dy_2 + \cdots + y_{2m}dy_m - y_1dy_{m+1} - y_2dy_{m+2} - \cdots - y_mdy_{2m} - 2y_{m+1}dt$$

we can, by the preceding general process, write the given system

$$\frac{dy_1}{dt} = 1, \quad \frac{dy_2}{dt} = 0, \quad \cdots, \quad \frac{dy_{2m}}{dt} = 0.$$

This is the "normal form" considered by Birkhoff,‡ who has shown that any Lagrangian system can be reduced to it in an actual sense by solving equations deduced from a variational principle. The method we give here is purely formal, but on the other hand it leads directly to the normal form, and thus the resolution with respect to dy_1/dt, dy_2/dt, $\cdots$, dy_{2m}/dt of the differential equations of the extremals is rendered unnecessary.

From the most general point of view, we can say that if we consider as known a certain set of functions among the F_i, G_i, X_i, Z, the other functions can be given by the system (S) under the condition that this system admit a solution (with respect to the unknown functions considered). As an example, it is easy to see that we obtain, by use of the system (S), the reduction to the Hamiltonian form when we suppose given the F_i and the X_i in the following way:

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* Indeed the system (S) can, *in general*, be written in such a form that it will appear, according to Riquier's theory (*Les Systèmes d'Equations aux Dérivées Partielles*, Paris, Gauthier-Villars, 1910) to be "orthonome" and "passif," consequently *completely integrable*, i.e., admitting an existence theorem. When we say "*in general*" we may leave out of consideration some exceptional points. It is easy to see that these points are extremely uncommon.

† $\omega = \sum X_j dx_j + Zdt$. See first section.

‡ *Dynamical Systems*, loc. cit., p. 56.

$$F_1 = x_1, \qquad F_2 = x_2, \qquad \cdots, \; F_m = x_m,$$
$$X_1 = x_{m+1}, \quad X_2 = x_{m+2}, \; \cdots, \; X_m = x_{2m},$$
$$X_{m+1} = 0, \qquad X_{m+2} = 0, \qquad \cdots, X_{2m} = 0,$$

the G_i and Z being the only unknown functions.

III

In this section we shall consider the Pfaffian systems formerly defined, from a different point of view, in order to show the use one can make of these systems in dynamical problems. First, for that purpose, we have to recall the fundamental results given for the first time by Birkhoff when he defined and introduced the consideration of those systems. In his book already cited* he demonstrates that in a *formal* sense "the Pfaffian equilibrium problem" can in general be reduced to Hamiltonian form, with the following consequence: "the normal form in the Hamiltonian case serves also in the Pfaffian case." An analogous statement can be made for generalized Pfaffian systems with respect to the "generalized Pfaffian equilibrium problem." According to these results there is to be found in the two following chapters of the same book a study of stability and existence of periodic motions based on the use of Pfaffian as well as Hamiltonian systems. Furthermore let us recall the following remark which shows the significance of the form of the dynamical equations for stability: the Pfaffian systems possess the same property as the Hamiltonian "of automatically fulfilling all of the conditions for complete stability, once the obvious conditions for first order stability are satisfied."

It is not without interest to obtain, from the considerations developed in the first section, the least order to which the system can be reduced in some classical problems of dynamics. The results are of course well known, but we shall give the following demonstration on account of its simplicity. In the problem of three bodies one starts with a Hamiltonian system of the eighteenth order for which one knows the energy integral $H=C$ and nine other integrals (all independent of each other). We can consider the system as Pfaffian, and according to a statement made above,† by the use of the nine integrals we can reduce its order by ten. After this reduction it will be of the same Pfaffian form and still admit the energy integral $H=C$. Then it will be of order 8, still reducible by two units more, by the use of $H=C$ and elimination of t.‡ This result can be generalized for the n-body problem for which

* Pp. 89–94.

† First section, p. 821.

‡ First section, p. 821.

the order of the system can be reduced to $6n-12$. In the case of the "three-body problem in a plane" by an analogous treatment the final order of the system can be made equal to 4.

In his memoir *Stability and the equations of dynamics* G. D. Birkhoff gives a special process which reduces a prepared linear differential form $R_1d\xi_1+R_2d\xi_2+\cdots+S_1d\eta_1+S_2d\eta_2+\cdots+Qdt$ corresponding to a Pfaffian system (in the vicinity of an equilibrium point for this system) to the sum of a canonical form ω and of a perfect differential dE, by the use of a succession of changes of variables such as

$$(C) \qquad \xi_i = \bar{\xi}_i + F_i, \qquad \eta_i = \bar{\eta}_i + G_i,$$

F_i, G_i beginning with terms of at least the second degree.* But the reduction was only considered there in a "formal sense," so the question remained open whether or not it could be accomplished by convergent series. In the rest of this section we shall obtain, for the case of two degrees of freedom, an *actual* transformation, i.e., a change of variables of the form (C) defining the considered reduction by means of analytic functions F_i, G_i. With Birkhoff's notations, R_1, R_2, S_1, S_2, Q are functions of ξ_1, ξ_2, η_1, η_2; R_1, R_2, Q begin with terms of the second order at least, whereas in S_1, S_2 the lower terms are respectively ξ_1, ξ_2. By the change of variables

$$(7) \qquad \begin{aligned} \xi_i &= \bar{\xi}_i, \\ \eta_i &= \bar{\eta}_i + G_i(\bar{\xi}_1, \bar{\xi}_2, \bar{\eta}_1, \bar{\eta}_2) \end{aligned}$$

(G_i beginning with terms of at least the second degree) the preceding form becomes

$$\begin{aligned} \bar{\omega} = {} & d\bar{\xi}_1\left[\bar{S}_1\frac{\partial G_1}{\partial\bar{\xi}_1} + \bar{S}_2\frac{\partial G_2}{\partial\bar{\xi}_1} + \bar{R}_1\right] \\ & + d\bar{\xi}_2\left[\bar{S}_1\frac{\partial G_1}{\partial\bar{\xi}_2} + \bar{S}_2\frac{\partial G_2}{\partial\bar{\xi}_2} + \bar{R}_2\right] \\ & + d\bar{\eta}_1\left[\bar{S}_1\frac{\partial G_1}{\partial\bar{\eta}_1} + \bar{S}_2\frac{\partial G_2}{\partial\bar{\eta}_1} + \bar{S}_1\right] \\ & + d\bar{\eta}_2\left[\bar{S}_1\frac{\partial G_1}{\partial\bar{\eta}_2} + \bar{S}_2\frac{\partial G_2}{\partial\bar{\eta}_2} + \bar{S}_2\right] + \bar{Q}dt. \end{aligned}$$

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* If we consider only the reduction of a Pfaffian system to a Hamiltonian system it is sufficient to reduce $\omega+dE$ to a canonical form $\bar{\omega}$.

Properly choosing $dE=\sum c_i dx_i$ we can be sure that all the coefficients of $\omega^{(2m-2)}$, i.e. of $(\omega')^{m-1}\omega$, vanish only if the same is true of the coefficients of $(\omega')^{m-1}$ and therefore of the coefficients of $(\omega')^m$. The fact that all these coefficients do not vanish in our problem, even in the vicinity of an equilibrium point, is a result of the condition $|a_{ij}|\neq 0$ and of the formula (III). Then the possibility of an actual reduction follows as a consequence of Goursat's theorem stating the existence of *conjugate groups of functions*: *Le Problème de Pfaff*, Paris, Gauthier-Villars, p. 186.

Suppose now that the following system, in which E is any function whatever of ξ_1, ξ_2, η_1, η_2 (we now omit the bars).

$$(8)\qquad \begin{aligned} \frac{\partial E}{\partial \xi_1} &= S_1\frac{\partial G_1}{\partial \xi_1} + S_2\frac{\partial G_2}{\partial \xi_1} + R_1, \\ \frac{\partial E}{\partial \xi_2} &= S_1\frac{\partial G_1}{\partial \xi_2} + S_2\frac{\partial G_2}{\partial \xi_2} + R_2 \end{aligned}$$

admits a solution E, G_1, G_2. Then by subtracting the exact differential

$$\frac{\partial E}{\partial \xi_1}d\xi_1 + \frac{\partial E}{\partial \xi_2}d\xi_2 + \frac{\partial E}{\partial \eta_1}d\eta_1 + \frac{\partial E}{\partial \eta_2}d\eta_2$$

from the preceding sum $\bar{\omega}$ we see it is sufficient to take for G_1, G_2 a solution of (8) in order to make vanish the coefficients of the terms in $d\xi_1$ and $d\xi_2$ in the above expression of $\bar{\omega}$. It remains now only to establish the existence of a solution for the system (8).

Let us change our notation; with

$$E = u_1, \quad G_1 = u_2, \quad G_2 = u_3,$$
$$\xi_1 = x_1, \quad \xi_2 = x_2, \quad \eta_1 = x_3, \quad \eta_2 = x_4,$$

the system (8) can be written

$$\frac{\partial u_1}{\partial x_1} = S_1\frac{\partial u_2}{\partial x_1} + S_2\frac{\partial u_3}{\partial x_1} + R_1, \qquad \frac{\partial u_1}{\partial x_2} = S_1\frac{\partial u_2}{\partial x_2} + S_2\frac{\partial u_3}{\partial x_2} + R_2,$$

S_1, S_2, R_1, R_2 being now expressed by means of the variables x_1, x_2, x_3, x_4 and the two unknown functions u_2, u_3.

Writing the condition of integrability for this system and introducing two new functions by

$$\frac{\partial u_2}{\partial x_1} = u_4, \quad \frac{\partial u_2}{\partial x_2} = u_5,$$

we have

$$(9)\qquad \begin{aligned} &\frac{\partial u_1}{\partial x_1} = S_1u_4 + S_2\frac{\partial u_3}{\partial x_1} + R_1, \quad \frac{\partial u_1}{\partial x_2} = S_1u_5 + S_2\frac{\partial u_3}{\partial x_2} + R_2, \quad \frac{\partial u_2}{\partial x_1} = u_4, \quad \frac{\partial u_2}{\partial x_2} = u_5, \\ &\frac{\partial u_3}{\partial x_1} = \frac{\left[u_4\left(\frac{\partial S_2}{\partial u_2} - \frac{\partial S_1}{\partial u_3}\right) + \frac{\partial S_2}{\partial x_1} - \frac{\partial R_1}{\partial u_3}\right]\frac{\partial u_3}{\partial x_2} - u_4\left(\frac{\partial S_1}{\partial x_2} - \frac{\partial R_2}{\partial u_2}\right) + u_5\left(\frac{\partial S_1}{\partial x_1} - \frac{\partial R_1}{\partial u_2}\right) - \frac{\partial R_1}{\partial x_2} + \frac{\partial R_2}{\partial x_1}}{\frac{\partial S_2}{\partial x_2} - \frac{\partial R_2}{\partial u_3} + u_5\left(\frac{\partial S_2}{\partial u_2} - \frac{\partial S_1}{\partial u_3}\right)}, \\ &\frac{\partial u_5}{\partial x_1} = \frac{\partial u_4}{\partial x_2}. \end{aligned}$$

For the purpose of studying this linear system of simultaneous partial differential equations of the first order we can refer to C. Bourlet's thesis* *Sur les équations aux dérivées partielles simultanées* and especially to the second part of it devoted to the case of linear systems; it is clear then that (9) is of the form

$$\frac{\partial u_i}{\partial x_k} = \overset{ik}{a}_{00} + \sum_{j>i} \overset{ik}{a}_{jk} \frac{\partial u_j}{\partial x_k} + \sum_{h>k} \overset{ik}{a}_{sh} \frac{\partial u_s}{\partial x_h},$$

where the a are functions of the x and u. So this linear system belongs to the class denoted there as "canonique" and moreover it is *completely integrable*, i.e. the two expressions of every "doubly principal derivative" can be shown to be identical by the use of the system itself and of the equations that can be deduced from it by differentiation. If we remark that the coefficients a are in our case analytic in the vicinity of $x_1 = x_2 = x_3 = x_4 = 0$, $u_1 = u_2 = u_3 = u_4 = u_5 = 0$, Theorem VIII of Bourlet can be applied to our system. Hence we conclude that (9) admits a solution analytic near $x_1 = x_2 = x_3 = x_4 = 0$, and for which $u_1 = u_2 = u_3 = u_4 = u_5 = 0$. This solution is to be obtained by taking all the "parametric derivatives" equal to 0. Furthermore in the case where all the parametric derivatives are equal to 0, we observe that for the solution all the first derivatives vanish. Hence the G_i do not involve terms of the first order, and after the change of variables (7) determined by the solution under consideration the function $\overline{Q}$ in $\bar{\omega}$ still begins with terms of the second order at least and we are still in the vicinity of an equilibrium point.

We achieve the reduction to the Hamiltonian form by

$$\eta_i = \bar{\eta}_i,$$

$$S_1 \frac{\partial G_1}{\partial \eta_1} + S_2 \frac{\partial G_2}{\partial \eta_1} + S_1 - \frac{\partial E}{\partial \eta_1} = \bar{\xi}_1, \tag{10}$$

$$S_1 \frac{\partial G_1}{\partial \eta_2} + S_2 \frac{\partial G_2}{\partial \eta_2} + S_2 - \frac{\partial E}{\partial \eta_2} = \bar{\xi}_2.$$

The two last equations can be solved so as to give ξ_1, ξ_2 in the vicinity of the origin, the Jacobian $D(\bar{\xi}_1, \bar{\xi}_2)/D(\xi_1, \xi_2)$ being there equal to 1, since the $\partial^2 E/\partial\eta_i\partial\xi_k$ vanish, as is easy to verify by differentiating (8).

If we pay special attention to the terms of lower degree, the resolution of the equations (10) gives

$$\xi_1 = \bar{\xi}_1 + \cdots,$$
$$\xi_2 = \bar{\xi}_2 + \cdots,$$

* Also Annales de l'Ecole Normale Supérieure. (3), vol. 8 (1891), supplément, p. 1.

and that enables us to state that we remain in the vicinity of an equilibrium point. So by the successive use of the two changes of variables (7) and (10) in which G_1, G_2 are a solution of the system (8), analytic in the vicinity of $x_i=0$, $u_i=0$ we can reduce the expression ω to $\xi_1 d\eta_1+\xi_2 d\eta_2+Qdt$ in which Q begins with terms of the second order at least. In conclusion we have accomplished in a *formal sense*, i.e. *by use of convergent series*, the reduction which is proposed in Birkhoff's memoir. The prepared linear form from which we start has been transformed by a change of variables (C) into the sum of a canonical ω and of a perfect differential dE.†

The way we have proceeded suggests the following remarks:

(1) dE itself vanishes in the vicinity of the point under consideration;

(2) the unknown functions F_i, G_i which determine the change of variables are given by a system of partial differential equations, which system is linear, and completely integrable, and which we have written out entirely;

(3) in the system (9), η_1, η_2 are parametric variables for the function G_1, so every term of the solution G_1 must involve either ξ_1, ξ_2 or both, and

$$G_1 = \xi_1 G_1{}^*(\xi_1\xi_2\eta_1\eta_2) + \xi_2 G_1{}^{**}(\xi_1\xi_2\eta_1\eta_2).$$

In the same way, ξ_2, η_1, η_2 being parametric variables with respect to G_2 for the solution of (9), we can write

$$G_2 = \xi_1 G_2{}^*(\xi_1\xi_2\eta_1\eta_2).$$

It follows that the solution whose existence we have established both in a formal and in an actual sense, is different from the formal solution that can be obtained by a succession of changes of variables according to the general method of Birkhoff.

† This conclusion, the main result of §3, has been extended, during the printing of this paper, to the general case, i.e., for any number of degrees of freedom. Cf. Paris Comptes Rendus, vol. 190, p. 358. The method used for this extension is almost the same as the one explained in detail above.

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