A Reachability Algorithm for General Petri Nets Based on Transition Invariants

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Abstract. A new reachability algorithm for general Petri nets is proposed. Given a Petri net with an initial and a target markings, a so called complemented Petri net is created first that consists of the given Petri net and an additional, complementary transition. Thereby, the reachability task is reduced to calculation and investigation of transition invariants (T-invariants) of the complemented Petri net. The algorithm finds all minimal-support T-invariants of the complemented Petri net and then calculates a finite set of linear combinations of minimal-support T-invariants, in which the complementary transition fires only once. Finally, for each T-invariant with a single firing of the complementary transition, the algorithm tries to create a reachability path from initial to target marking or determines that there is no such path.

Keywords: Petri nets, reachability, transition invariants.

1 Introduction

Petri nets are an important formal paradigm for modeling and analysis of discrete event systems. In such systems, a researcher is often interested to know whether the system can transit from one state into another state. In terms of Petri nets, the answer to this question is obtained as a solution of a reachability problem.

The reachability problem in Petri nets is formulated as follows: for any Petri net *PN*, with an initial marking M^0, and for some other marking M, determine whether the relation $M \in R(PN, M^0)$ is true, where $R(PN, M^0)$ is the reachability set of *PN* for its initial marking M^0 [1]. The decidability of the reachability problem has been proved for a number of restricted classes of Petri nets, and there are efficient algorithms for such classes as acyclic Petri nets, marked graphs, and others [2], [3], [4], [5].

It has been shown that the reachability problem is decidable for general Petri nets as well [6]. In practice, two different techniques are used most often to determine the reachability of a marking in generalized Petri nets. The first technique is based on the creation and investigation of a complete or reduced reachability graph. The main drawback of this approach is a state explosion problem. A closely related technique is the use of *stubborn sets* [8]. Unfortunately, generation of minimal or reduced reachability graphs, as is required by this technique, is known to be an NP-hard

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problem [10]. If Petri net has no specific properties like a symmetry or reversibility, the corresponding reduced reachability graph will have almost the same size as that of the full reachability graph [9].

The second technique is based on methods of linear algebra. Given a *pure* Petri net (i.e. a Petri net without self-loops), with sets of transitions T and places P, its structure is represented unambiguously by the *incidence matrix*

$$D = [d(t_i, p_j)] = [d_{ij}], \quad i = 1, 2, \dots, m = |T|, \quad j = 1, 2, \dots, n = |P|, \tag{1}$$

where $d(t_i, p_j) = Post(p_j, t_i) - Pre(p_j, t_i)$, Pre and $Post$ are the input and output functions of the Petri net, with $Pre(p, t) = v$ if there is a directed arc from p to t with the weight v, and $Post(p, t) = v$ if there is an arc from t to p with the weight v.

It is known that a *necessary* condition for reachability of marking M from some other marking M^0 is the existence of a nonnegative integer solution of the matrix equation

$$M = M^0 + FD \tag{2}$$

relative to F, where $F = [f_1, f_2, \dots, f_m]$ is a *firing count vector* [12].

Unfortunately, the existence of a nonnegative integer solution of equation (2) is not a sufficient condition for reachability of marking M from M^0 [1]. The second drawback of this method is that the solution of equation (2) does not contain any information about the order of firings. More than that, this can have infinite number of nonnegative integer solutions, some of which work while others fail, and there is a problem to select working firing count vectors [7].

In our paper [11], linear algebra methods were used for reachability analysis of a particular class of place/transition Petri nets having no transition invariants (T-invariants). Algebraically, T-invariants of a Petri net with incidence matrix D are non-negative integer $(1 \times m)$ vectors F such that $FD = 0$ [13].

This paper generalizes the approach described in [11] for arbitrary place/transition nets, including Petri nets with T-invariants. The existence of T-invariants in Petri nets considerably complicates the reachability analysis. In contrast with the scheme in [11], in the generalized scheme the set of T-invariants for investigation is theoretically infinite. Nevertheless, as will be shown in this paper, it is always possible to effectively limit this set without the loss of reachability information and then to use T-invariants from this finite set for performing a reachability analysis.

2 Notation and Basic Statements

We adopt the notation and basic statements from our paper [11]. In particular, the structure of any (pure) Petri net will be represented by its incidence matrix (1), in which rows correspond to transitions and columns correspond to places, as in [1] and [7].

Let M^0 be an initial marking and M be some other marking of given Petri net PN. If we are interested in reachability of M from M^0 then marking M will be called the *target* marking. It is assumed, throughout the paper, that $M^0 \neq M$.

In the paper, all vectors are considered as row vectors. In particular, markings of PN will be expressed as $(1 \times n)$ row vectors, so that we can write

$$M^0 = [m_1^0, m_2^0, ..., m_n^0] \text{ and } M = [m_1, m_2, ..., m_n], \tag{3}$$

where the ith entry in M^0 and M denotes the number of tokens in place $p_i \in P$.

If marking M is reachable from M^0 in PN, then there exists at least one sequence of markings $\mu = M^0 M^1 \ldots M^r$ with $M^r = M$, and a sequence of firing transitions $\tau = t_{i_1} t_{i_2} \ldots t_{i_r}$, with the two sequences related by the state equation $M^k = M^{k-1} + e[i_k]_m D$, $k = 1, 2, \ldots, r$. Here $e[i_k]_m$ is an $(1 \times m)$ *control vector*, in which m - 1 entries are zero and the i_kth entry is one, indicating that a transition t_{i_k} fires at step k. Sequences μ and τ can be combined to obtain a *reachability path* from marking M^0 to M^r:

$$M^0 \xrightarrow{t_{i_1}} M^1 \xrightarrow{t_{i_2}} \ldots \xrightarrow{t_{i_r}} M^r \tag{4}$$

Its determination is the main problem of reachability analysis. With linear algebra methods, this analysis is usually carried out in two stages. The first stage is computing of one or more firing count vectors satisfying equation (2). The second stage is attempting to find reachability paths corresponding to the computed firing count vectors [27]. At the first stage, it is important to limit the number of firing count vectors, without the loss of reachability information. In the proposed approach, this stage is reduced to the computation of T-invariants of so called complemented Petri net.

Definition 1. For any Petri net PN with incidence matrix D specified by (1), and initial and target markings M^0 and M represented by vectors (3), there exists a unique *complemented* Petri net PN_c that has the same set of places P as PN, the set of transitions $T_c = T \cup \{t_{m+1}\}$, and is described structurally by the incidence matrix

$$D_c = \begin{bmatrix} D \\ \Delta M \end{bmatrix}, \tag{5}$$

where t_{m+1} is an additional, *complementary* transition, and $\Delta M = M^0 - M = [\Delta m_1, \Delta m_2, \ldots, \Delta m_n]$, with $\Delta m_i = m_i^0 - m_i$, $i = 1, 2, \ldots, n$ [11]. ♦

Using the right side of equation (2) with marking M instead of M^0, control vector $e[m + 1]_{m+1}$ instead of F and incidence matrix D_c instead of D, one can get $M + e[m + 1]_{m+1} D_c = M + \Delta M = M^0$. That is, a single firing of the complementary transition in marking M of PN_c results in marking M^0.

It is known that the reproducibility of a firing sequence in a Petri net indicates the existence of one or more T-invariants [13]. Thus the following statement holds.

Statement 1. Given a Petri net PN with the incidence matrix D and an initial marking M^0, a necessary (but generally not sufficient) condition for some other marking $M \neq M^0$ to be reachable from M^0 is the existence of an integer solution of the matrix equation $F_c D_c = 0$ relative to $F_c = [f_1, f_2, \ldots, f_m, f_{m+1}]$, such that $F_c \geq 0$ and $f_{m+1} = 1$. Here D_c is the matrix defined in (5). ♦

In sequel, each T-invariant of the complemented Petri net PN_c having the last entry $f_{m+1} = 1$ will be called a *singular complementary* T-invariant.

The importance of Statement 1 is that the reachability analysis of the *original* Petri net PN can be reduced to the computation and investigation of T-invariants of the complemented Petri net PN_c. One advantage of this reduction is the existence of efficient techniques for the calculation of T-invariants [14], [15], [16]. Algorithms for the calculation of T-invariants are implemented in many Petri net software tools such as INA [17], GreatSPN [18], TimeNET [19], and QPN [20], to mention only a few. Even more important benefit of this reduction is that the space for the search of firing sequences, transforming M^0 into M in the given Petri net, can be effectively limited as will be shown in this paper.

It is known that, in any Petri net with T-invariants, there are *minimal-support T-invariants* which can be used as generators of all T-invariants of the given net [1], [13]. Let $\Phi = \{F_1, F_2, \ldots, F_s\}$ be the set of minimal-support T-invariants of some Petri net consisting of $m = |T|$ transitions, where $F_i = [f_{i1}, f_{i2}, \ldots, f_{im}] \underset{\neq}{>} 0,$ and s is the number of minimal-support T-invariants. We use here, for a vector X, a denotation $X \underset{\neq}{>} 0$ if $X \geq 0$ and $x_i \neq 0$ for some ith entry of X. Each $F_i \in \Phi$ specifies a nonempty subset of transitions $\|F_i\| \subseteq T$ such that $t_j \in \|F_i\|$ if and only if $f_{ij} > 0$, with $\|F_i\| \not\subset \|F_k\|$ and $\|F_k\| \not\subset \|F_i\|$ for every pair of distinct indices $i, k = 1, 2, \ldots, s$. Here $\|F_i\|$ represents the *minimal support* of T-invariant F_i.

Statement 2. In any Petri net the number of minimal-support T-invariants is finite [11].

Statement 3. For any Petri net PN, its complemented net PN_c includes all T-invariants of the original net PN [11].

Statement 4. For every reachability path from an initial marking M^0 to a target marking M of a given Petri net PN, there exists a T-invariant $F = [f_1, f_2, \ldots, f_m, f_{m+1}]$ of the corresponding complemented Petri net PN_c of PN, with $f_{m+1} = 1$. That is, F is a singular complementary T-invariant. ♦

Corollary 1. For any Petri net, with given initial and target markings M^0 and M respectively, all existing reachability paths from M^0 to M are the paths that can be created on the set of singular complementary T-invariants. This corollary is a generalization of the corresponding result for T-invariant-less Petri nets in [11]. ♦

Let $\Phi_c = \{F_1, F_2, \ldots, F_w\}$ be a set of all minimal-support T-invariants of PN_c, where $F_j = [f_{j1}, f_{j2}, \ldots, f_{jm}, f_{j,m+1}] \underset{\neq}{>} 0,$ with $j = 1, 2, \ldots, w$. Notice that, according to the basic property of a T-invariant, each entry in vector F_j may be only a nonnegative integer [13].

Now, depending on the value of the last entry, the minimal-support T-invariants of set Φ_c can be classified into the following three *disjoint* groups:

$$\{F_j \mid f_{j,m+1} = 0,\ j \in I_w\} \tag{6}$$

$$\{F_j \mid f_{j,m+1} = 1,\ j \in I_w\} \tag{7}$$

$$\{F_j \mid f_{j,m+1} > 1,\ j \in I_w\} \tag{8}$$

where $I_w = \{1, 2, \dots, w\}$ is the indexing set of Φ_c. According to Statement 2, each of these groups is finite. Depending on the Petri net and its initial and target markings, some or even all these three groups can be empty.

Without the last, (m+1)th entry, T-invariants of group (6), by Statement 3, are minimal-support T-invariants of the original Petri net *PN*. We will call members of group (6) *non-complementary* minimal-support T-invariants of the complemented Petri net PN_c. Group (7) consists of singular complementary T-invariants. Finally, members of group (8) are nonsingular complementary T-invariants in which the complementary transition fires more than once. Together, members of groups (7) and (8) are called minimal-support *complementary* T-invariants of PN_c.

3 Minimal Singular T-Invariants of a Complemented Petri Net

By Corollary 1, the search for all reachability paths from initial marking M^0 to target marking M in a given Petri net can be carried out only on singular T-invariants of the corresponding complemented Petri net. These include, first of all, minimal-support T-invariants of group (7). However, these are not the only singular T-invariants of the complemented Petri net. Indeed, linear combinations of minimal-support T-invariants of groups (6), (7), and (8) can yield additional singular T-invariants. The number of such combinations is infinite in general. However, there exists a finite set of *minimal* singular T-invariants of the complemented Petri net.

Consider a linearly-combined T-invariant

$$F = [f_1, f_2, \dots, f_m, f_{m+1}] = \sum\nolimits_{j=1}^{w} k_j F_j \tag{9}$$

with rational coefficients k_j, where F_j are minimal-support T-invariants of groups (6), (7) and (8), and w is the number of elements in the three groups. In agreement with Corollary 1, we are looking only for those combined T-invariants F which yield f_{m+1} = 1. Thus, the following constraint must hold for each linear combination F in (9):

$$f_{m+1} = \sum\nolimits_{j=1}^{w} k_j f_{j,m+1} = 1. \tag{10}$$

With $k_j \ge 0$, the product $k_j F_j$ in (9) can be considered as a contribution of firings of transitions of T-invariant F_j to firings of transitions of the combined T-invariant F. On the other hand, a negative coefficient k_j in (9) may be interpreted as a reverse, or backward firing of transitions, corresponding to T-invariant F_j, and this is *not legal* in the normal semantics of Petri nets [21]. Thus, for T-invariants of groups (7) and (8), taking into account (10), their coefficients k_j must be in the following range:

$$0 \le k_j \le 1. \tag{11}$$

That is, for groups (7) and (8), in which $f_{j,m+1} \ge 1$, to satisfy (10) the following inequality must hold:

$$\sum k_j \le 1. \tag{12}$$

However, coefficients k_j for T-invariants of group (6) in (9) may have arbitrary (non-negative) values without affecting the constraint (10). As a particular case, these

T-invariants can be combined in (9) with coefficients $k_j \leq 1$. The case when T-invariants of group (6) can be included into linearly-combined T-invariants (9) with arbitrary large coefficients is considered in Section 6.

The linearly-combined T-invariants (9), with the constraints (10), (11) and (12), are called *minimal singular* T-invariants of the complemented Petri net. As a subset, they include all minimal-support T-invariants of group (7). Minimal singular T-invariants of the complemented Petri net can be found by existing methods of linear algebra [23], [24]. Due to space limitation, we omit the details of the computational procedure.

4 Relation Graph of T-Invariants

In general, each singular T-invariant should be tested for the creation of a reachability path not only *alone*, but also in different linear combinations with non-complementary T-invariants (6), since these T-invariants can "help" the singular T-invariant to become realizable in given initial marking M^0. As will be shown in this section, in general not all non-complementary T-invariants can affect realization of the given singular T-invariant.

Definition 2. Let F be a T-invariant of a Petri net, with the support $\|F\|$. Then $P(F) = \{p_j \mid t_i \in \|F\|, d_{ij} \neq 0\}$ is a set of places of this Petri net *affected* by F when it becomes realizable in some marking. Here, d_{ij} is an element of the incidence matrix(1). ♦

Statement 5. Let F_1 and F_2 be some T-invariants of a Petri net, and let P_1 and P_2 be sets of places affected by F_1 and F_2 respectively. If $P_1 \cap P_2 = \varnothing$, then T-invariants F_1 and F_2 have no *direct* effect on the realizability of each other. ♦

Even if $P_1 \cap P_2 = \varnothing$, T-invariants F_1 and F_2 can *indirectly* affect the realizability of each other through other T-invariants having common affected places with F_1 and F_2. Consider a *relation graph* of T-invariants. Nodes in this graph are T-invariants. Two nodes corresponding to T-invariants F_i and F_j are connected by a non-oriented edge if $P(F_i) \cap P(F_j) \neq \varnothing$, and the corresponding T-invariants F_i and F_j are called *directly connected* T-invariants.

For a Petri net, such a graph generally consists of a number of connected components. A connected component may include complementary and non-complementary T-invariants, or only one type of T-invariants. We say that two T-invariants F_i and F_j can affect realizability of each other if they belong to the same connected component.

The algorithm for determining all connected components of a graph is well known [22]. In our problem, the algorithm will determine a connected component consisting of nodes representing a given singular T-invariant and non-complementary T-invariants. For this purpose, the algorithm will use the incidence matrix of the *original* Petri net and the array of T-invariants.

5 Realization of T-Invariants with Borrowing of Tokens

Let p be a place affected by two T-invariants F_i and F_j in a given Petri net. Assume that, in a given initial marking of the net, F_i is realizable, but F_j can become realizable if place p accumulates r_j tokens during realization of T-invariant F_i. Suppose further that, at some intermediate step during realization of F_i, r_i tokens will be created in place p. If $r_i \geq r_j$ then, by temporary borrowing of r_j tokens in place p, T-invariant F_j becomes realizable and, at the end of its realization, will return the borrowed tokens to place p, so that T-invariant F_i can complete its started realization.

With $r_i < r_j$, T-invariant F_j cannot borrow the necessary number of tokens in place p. However, if T-invariant F_i, after creation of r_i tokens in p at some step of its first realization, can start a new realization before the completion of the first one, then additional r_i tokens will be created in place p, so that this place will now accumulate $2r_i$ tokens. In general, if F_i can start v realizations before the completion of the previous ones, then place p will accumulate vr_i tokens. If, for some v, $vr_i \geq r_j$ then, after borrowing r_j tokens in p, T-invariant F_j becomes realizable. After the completion of its realization, all tokens borrowed by F_j will be returned to place p, and T-invariant F_i can complete all its started realizations.

The possibility of borrowing of tokens among connected T-invariants can be determined with the use of a two-dimensional borrowing matrix G. In this matrix, rows correspond to T-invariants and columns correspond to places of the given Petri net. Formally, $G = [g_{ij}]$, $i = 1, 2, \ldots, s$; $j = 1, 2, \ldots, n$, where s is the number of connected T-invariants and n is the number of places in the net. The elements of matrix G are integers and have the following meaning. If $g_{ij} > 0$ then, for its realization, T-invariant F_i needs to borrow g_{ij} tokens in place p_j affected by some other T-invariants of the considered group. If $g_{ij} < 0$ then T-invariant F_i, at some intermediate step of its *single* realization, creates $|g_{ij}|$ tokens in place p_j. Finally, $g_{ij} = 0$ means that F_i does not affect place p_j.

To create a borrowing matrix, the proposed algorithm will use the incidence matrix of the given original Petri net and a group of connected T-invariants of the corresponding complemented Petri net. Due to a relative simplicity of the underlying procedure and to space limitation, the details of this procedure are omitted.

6 Combining a Singular Complementary T-Invariant with Non-complementary T-Invariants

Denote by F_c a singular T-invariant of some complemented Petri net. It can be a member of group (7) or a minimal T-invariant. If group (6) is not empty, then the following linear combination

$$F = F_c + \sum k_j F_{nc}^j, \tag{13}$$

with coefficients $k_j \geq 0$, is also a valid singular T-invariant, if components of F are nonnegative integers. Here F_{nc}^j is a T-invariant of group (6) connected with F_c.

The expression (13) implies that the singular T-invariant F_c in general should be tested for the creation of a reachability path not only alone, but also in different linear combinations with non-complementary T-invariants (6), since these T-invariants can “help” the non-realizable T-invariant F_c to become realizable in marking M^0.

Without loosing generality, we assume that coefficients k_j in (13) are nonnegative integers. Indeed, if a singular T-invariant F_c is realizable for some non-integer values of coefficients k_j in (13), then it will remain realizable when these coefficient values are replaced by the nearest integer values not less than k_j. The case when $k_j \leq 1$ was considered in Section 3. With integer coefficients $k_j > 1$, the product $k_j F_{nc}^j$ in (13) corresponds to a *multiple* realization of T-invariant F_{nc}^j. A multiple realization is a series of k_j sequential or *interleaved* single realizations. Interleaved realizations of a T-invariant, if they are possible in a given marking, can have a different effect on place marking in comparison with sequential realizations. Consider, for example, a simple Petri net consisting of two transitions t_1, t_2 and one place p that is the output place for t_1 and the input place for t_2. This Petri net has a T-invariant $F = [1, 1]$ realizable in any initial marking of p. In particular, with the zero initial marking, place p will never have more than one token if single realizations of F are strictly sequential as in $t_1t_2t_1t_2t_1t_2$. However, if single realizations of F are interleaved, place p can accumulate an arbitrary large number of tokens at some intermediate step.

In general, the number of valid combinations (13) is infinite. This section describes how to limit the values of coefficients k_j in (13) without the loss of reachability information using the concept of structural boundedness of Petri nets.

It is known [1] that a Petri net is *structurally bounded* if and only if there exists a $(1 \times n)$ vector $Y = [y_1, y_2, \ldots, y_n]$ of positive integers, such that

$$D\,Y^T \leq 0, \tag{14}$$

where D is the $(m \times n)$-incidence matrix of the Petri net with m transitions and n places.

A Petri net is said to be *structurally unbounded* if and only if there exists a $(1 \times m)$ vector of (nonnegative) integers $X = [x_1, x_2, \ldots, x_m] \underset{\neq}{>} 0$, such that

$$D^T X^T = \Delta M^T \tag{15}$$

for some $\Delta M \underset{\neq}{>} 0$, where m is the number of transitions in the Petri net, and ΔM is a $(1 \times n)$ vector of marking increments as a result of firing of all transitions corresponding to vector X.

In a structurally unbounded Petri net, at least one place is structurally unbounded. A place p_i in such a Petri net is said to be *structurally unbounded* if and only if there exists a $(1 \times m)$ vector $X \underset{\neq}{>} 0$ of nonnegative integers, such that

$$D^T X^T = \Delta M^T \text{ for some } \Delta m_i > 0 \text{ in } \Delta M = (\Delta m_1, \Delta m_2, \ldots, \Delta m_i, \ldots, \Delta m_n) \underset{\neq}{>} 0. \tag{16}$$

It is known that, according to Farkas' lemma [1], one of the systems (14) or (15) has solutions. For our problem, we do not need to know all solutions of (14) or (15).

Rather, it is sufficient to find only one, "minimal" solution of (14) or (15). The minimal solutions of (14) or (15) can be found as solutions of integer linear programming (ILP) tasks. For the system (14), the corresponding ILP problem can be formulated as follows:

$$\min \quad z = \sum_{i=1}^{n} y_i, \quad \text{sub. to} \quad DY^T \le 0, \;\; y_i \ge 1, \;\; i = 1,\; 2, ..., n. \tag{17}$$

For the system (15), the corresponding ILP problem is:

$$\min v = \sum_{i=1}^{m} x_i, \text{sub. to} \quad D^T X^T \underset{\neq}{>} 0, \;\; \sum_{i=1}^{m} x_i \ge 1, \;\; x_i \ge 0, \;\; i = 1,\; 2, ..., m. \tag{18}$$

Let us consider the case when the subnet corresponding to F_{nc}^{j} is not structurally bounded and describe how to determine coefficient k_j for F_{nc}^{j} in the linear combination (13). If F_{nc}^{j} and F_c belong to different connected components of the relation graph of T-invariants, then F_{nc}^{j} should be ignored at all, by setting $k_j = 0$ in (13). On the other hand, if F_{nc}^{j} and F_c belong to the same connected component of the relation graph of T-invariants, then the subnet corresponding to F_{nc}^{j} has common places with the subnets corresponding to F_c or other non-complementary T-invariants belonging to the same connected component. Thus, F_{nc}^{j} can affect realizability of F_c, directly or indirectly and therefore should be included in (13) with $k_j > 0$.

Suppose, that F_{nc}^{j} has the support $\{t_1, t_2, \dots, t_l\}$, $l \le m$, and the set of affected places $\{p_1, p_2, \dots, p_q\}$, $q \le n$, where m and n are the numbers of transitions and places in the original Petri net. Assume that F_c, to become realizable, needs to borrow $n_i > 0$ tokens in each place of set

$$\{p_1, p_2, \dots, p_h\}, h \le q, \tag{19}$$

in which F_{nc}^{j} can create tokens during its realization. Then, to facilitate the realize-bility of F_c, F_{nc}^{j} should be included in the linear combination (13) with a positive integer coefficient k_j determined by applying the following steps.

1. Try to solve an ILP problem:

$$\min \quad v = \sum_{i=1}^{l} x_i, \quad \text{sub. to} \quad D^T X^T \ge \Delta M^T, \;\; \sum_{i=1}^{l} x_i \ge 1, \;\; x_i \ge 0, \tag{20}$$

where $\Delta M = [\Delta m_1, \Delta m_2, .., \Delta m_h, \Delta m_{h+1}, \dots, \Delta m_q] = [n_1, n_2, \dots., n_h, 0, \dots, 0]$ is a vector of the desired numbers of tokens which are expected to be created in places (19) as a result of one or more realizations of F_{nc}^{j}, l is the number of transitions in the support of F_{nc}^{j}, and q is the number of places affected by F_{nc}^{j}. In the matrixmultiplication, only those rows and columns of D are used which correspond to the support of F_{nc}^{j} and places affected by F_{nc}^{j}.

2. If, for the specified vector ΔM, the problem (20) has a solution $X^{*} = [x_1^{*}, x_2^{*}, ..., x_l^{*}]$, then components of X^{*} represent the total numbers of firings of respective transitions sufficient to accumulate the desired number of tokens in places of set (19) in a few realizations of F_{nc}^{j}, and $\left\lceil \frac{x_i^{*}}{f_i^{j}} \right\rceil$ is the number of realizations of F_{nc}^{j} to get the necessary number of firings of transition t_i, $i = 1, 2, \ldots, l$. In this case,

$$k_j = \max(\left\lceil \frac{x_i^{*}}{f_i^{j}} \right\rceil \mid i = 1,\ 2,\ ...,\ l). \tag{21}$$

3. If, on the other hand, the problem (20) has no feasible solution then it means that at least one of places in set (19) p_i is structurally bounded and can not accumulate the desired number of tokens Δm_i in multiple realizations of F_{nc}^{j}. In this case, using (16), determine all structurally unbounded places in set (19).

4. Solve the ILP problem (20) simultaneously for all structurally unbounded places found at the previous step, to obtain a solution vector X^{*}. That is, in solving (20), vector ΔM should have nonzero entries n_i only for structurally unbounded places. According to Farkas' lemma, this solution always exists. Then coefficient k_j is determined by the use of expression (21).

In case, when the subnet for F_{nc}^{j} is found to be structurally bounded, then the number of tokens in each of its places is bounded. However, this bound generally depends on realizations of other, connected T-invariants and is not known in advance. For such a subnet, coefficient k_j can be evaluated with the use of the borrowing matrix G computed for F_c and all its connected non-complementary T-invariants, including F_{nc}^{j}. Let, in this matrix, c and j be indexes of rows corresponding to F_{nc}^{j} and F_c, respectively. Then it is *sufficient* to include F_{nc}^{j} in (13) with coefficient k_j, computed as

$$k_j = \sum \left\lceil \frac{g_{ci}}{|g_{ji}|} \right\rceil, \tag{22}$$

where g_{ci} and g_{ji} are entries in the borrowing matrix, and the sum is computed for all pairs $g_{ci} > 0$ and $g_{ji} < 0$. Indeed, with this coefficient, the sufficient number of interleaved realizations of F_{nc}^{j} are *allowed* to accumulate the required numbers of tokens in places which are common for F_c and F_{nc}^{j} and in which T-invariant F_c can borrow them during its realization. In general, coefficient k_j calculated as was described can result in a larger number of realizations of T-invariant F^j_{nc} than is actually necessary. The reason is that other T-invariants in (13) can also create tokens in (19).

After computing all coefficients k_j in (13), an appropriate method can be applied to determine a reachability path for the combined T-invariant F if such a path exists. For this purpose, known computational procedures can be used [11], [27], [28].

7 An Example

This section illustrates the proposed algorithm by an example. The example was tested with a prototype C program that implemented almost all steps of the algorithm. For solving the related ILP problems the interactive system QS was used [26].

For this example, Fig. 1 shows a Petri net, consisting of m = 10 transitions and n = 9 places (recall that, in the corresponding incidence matrix, rows correspond to transitions). The initial and target markings are M^0 = [2, 0, 0, 0, 0, 0, 0, 0, 0] and M = [2, 0, 0, 0, 0, 0, 0, 0, 1], respectively. To get the complemented Petri net, the algorithm appends a row $\Delta M = M^0 - M$ = [0, 0, 0, 0, 0, 0, 0, 0, -1] to the original incidence matrix.

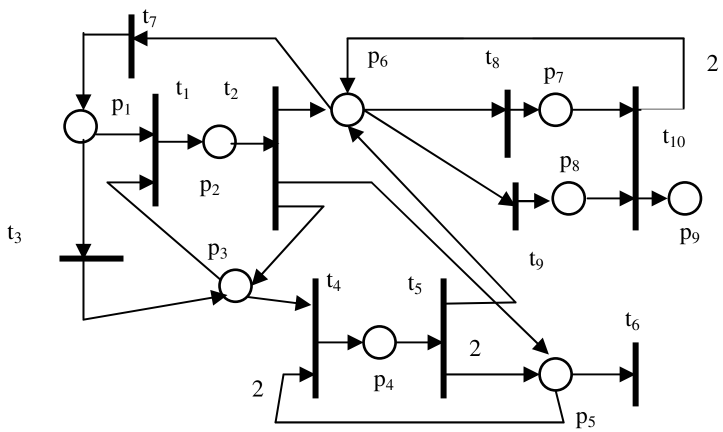

Fig. 1. The example Petri net

Minimal-support T-invariants of the corresponding complemented Petri net are two non-complementary T-invariants F_1 = [0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0] and F_2 = [1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0], and one singular complementary T-invariant F_3 = [0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1], with the sets of affected places $\{p_1, p_3, p_4, p_5, p_6\}$, $\{p_1, p_2, p_3, p_5, p_6\}$ and $\{p_6, p_7, p_8, p_9\}$, respectively. Thus, all these T-invariants are connected. F_3 can become realizable if it borrows tokens in places affected by F_1 and F_2. All these

T-invariants can become realizable if they borrow tokens in some of their common affected places as the borrowing matrix G for this example shows:

		p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
F_1:	Borrows	-1	0	-1	-1	2	-1	0	0	0
F_2:	Borrows	-1	-1	1	0	-1	-1	0	0	0
F_3:	Borrows	0	0	0	0	0	2	-1	-1	-1

Specifically, F_1 needs to borrow two tokens in place p_5, F_2 needs to borrow one token in place p_3, and F_3 borrows two tokens in place p_6. A token borrowed by F_2 in place p_3 can be produced by F_1 in a single realization so that $k_1 = 1$. On the other hand, F_2 is capable, in a single realization, to lend only one token to F_1, but two tokens are necessary in p_5 for F_1. Therefore, F_1 and F_2 can help each other to become realizable. Together, they are capable to produce two tokens in place p_6 to be borrowed by F_3.

The desired number of tokens in p_5 can be accumulated if the subnet corresponding to F_2 is not structurally bounded. To learn this, the algorithm tries to solve an ILP problem (17) for F_2, in the form in which only variables y_1, y_2, y_3, y_5, and y_6 are taken into consideration:

$$\min z = y_1 + y_2 + y_3 + y_5 + y_6,$$

sub. to: $-y_1 + y_2 - y_3 \le 0$, $-y_2 + y_3 + y_5 + y_6 \le 0$, $-y_5 \le 0$, $y_1 - y_6 \le 0$, $y_1, y_2, y_3, y_5, y_6 \ge 1$.

This ILP problem has no feasible solution. Thus, the subnet corresponding to F_2 is not structurally bounded, so that at least one of its affected places is not structurally bounded. We are interested in accumulating two tokens in p_5, so that $\Delta M = [0, 0, 0, 2, 0]$. Therefore, with $l = 4$ and $q = 5$, the algorithm attempts to solve now an ILP problem (20):

$\min v = x_1 + x_2 + x_6 + x_7$, sub. to: $-x_1 + x_7 \ge 0$, $x_1 - x_2 \ge 0$, $-x_1 + x_2 \ge 0$, $x_2 - x_6 \ge 2$, $x_2 - x_7 \ge 0$, $x_1 + x_2 + x_6 + x_7 \ge 1$.

This ILP problem has the optimal (minimal) solution $X^* = [x_1^*,\ x_2^*,\ x_6^*,\ x_7^*] = [2,2,2,0]$. Now, using (21), the algorithm finds that

$$k_2 = \max\left(\left\lceil \frac{x_i^*}{f_{2i}} \right\rceil \mid\ i = 1,\ 2,\ 6,\ 7\right) = 2.$$

Since T-invariant F_2 borrows only one token in place p_3 and this token can be created during a single realization of F_1, it is sufficient to have $k_1 = 1$. Thus, the combined T-invariant is $F = F_1 + 2F_2 + F_3 = [2, 2, 1, 1, 1, 2, 3, 1, 1, 1, 1]$. For it, the algorithm creates a reachability path from M^0 to M consisting of 16 nodes, with the sequence of 15 firing transitions $t_3t_1t_2t_7t_1t_2t_4t_5t_6t_6t_8t_9t_{10}t_7t_7$. This is the shortest path although there exist other paths of this length.

8 Conclusion

A new reachability algorithm for general Petri nets is proposed. For a given original Petri net, the reachability task is reduced to the investigation of T-invariants of the complemented Petri net consisting of the original Petri net and an additional, complementary transition. It is shown that, without the loss of reachability information, the algorithm tries to find a reachability path from the initial marking to the target one using a finite number of T-invariants. During the search for reachability paths, the algorithm needs memory for storing only the reachability path being created.

We did not address, in this paper, complexity aspects of the proposed algorithm. Complexity of some problems of Petri nets was investigated in [25]. We can note only that the algorithm will spend most of its time calculating minimal-support T-invariants, solving ILP problems, and trying to find reachability paths for calculated T-invariants.

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