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Formal Laurent series in several variables

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Abstract

We explain the construction of fields of formal infinite series in several variables, generalizing the classical notion of formal Laurent series in one variable. Our discussion addresses the field operations for these series (addition, multiplication, and division), the composition, and includes an implicit function theorem.

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1. Introduction

The purpose of this article is twofold. In the first part (Sections 2–4), we explain how to construct fields of formal Laurent series in several variables. This part has an expository flavour. The construction we present is not new; similar constructions can already be found in the literature. However, the justification of their validity is usually kept brief or more abstract than necessary. We have found it instructive to formulate the arguments in a somewhat more concrete and expanded way, and we include these proofs here in the hope that this may help to demystify and popularize the use of formal Laurent series in several variables. The results in the second part (Sections 5–6) seem to be new. We discuss there the circumstances under which we can reasonably define the composition of multivariate

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formal Laurent series, and we present a version of the implicit function theorem applicable to multivariate formal Laurent series.

Recall the situation of a single variable. The set $\mathbb{K}[[x]]$ of formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with coefficients in some field $\mathbb{K}$ forms an integral domain together with the usual addition and multiplication. Such a series $f(x)$ admits a multiplicative inverse $g(x) \in \mathbb{K}[[x]]$ if and only if $a_0 \neq 0$ (see, e.g., [14,10]). If $f(x)$ is any nonzero element of $\mathbb{K}[[x]]$, then not all its coefficients are zero, and if e is the smallest index such that $a_e \neq 0$, then we have $f(x) = x^e h(x)$ for some $h(x) \in \mathbb{K}[[x]]$ which admits a multiplicative inverse. The object $x^{-e}h(x)^{-1}$ qualifies as a multiplicative inverse of $f(x)$. In the case of a single variable, we may therefore define $\mathbb{K}((x))$ as the set of all objects $x^e h(x)$ where e is some integer and $h(x)$ is some element of $\mathbb{K}[[x]]$. Then $\mathbb{K}((x))$ together with the natural addition and multiplication forms a field. This is the field of formal Laurent series in the case of one variable.

The case of several variables is more subtle. The set $\mathbb{K}[[x, y]]$ of formal power series $f(x, y) = \sum_{n,k=0}^{\infty} a_{n,k} x^n y^k$ in two variables x, y with coefficients in $\mathbb{K}$ also forms an integral domain, and it remains true that an element $f(x, y) \in \mathbb{K}[[x, y]]$ admits a multiplicative inverse if and only if $a_{0,0} \neq 0$. But in general, it is no longer possible to write an arbitrary power series $f(x, y)$ in the form $f(x, y) = x^{e_1} y^{e_2} h(x, y)$ where $h(x, y) \in \mathbb{K}[[x, y]]$ admits a multiplicative inverse in $\mathbb{K}[[x, y]]$. As an example, consider the series $f(x, y) = x + y = x^1 y^0 + x^0 y^1 \in \mathbb{K}[[x, y]]$. If we want to write $f(x, y) = x^{e_1} y^{e_2} h(x, y)$ for some $h(x, y) \in \mathbb{K}[[x, y]]$, we have $h(x, y) = x^{1-e_1} y^{-e_2} + x^{-e_1} y^{1-e_2}$. In order for $h(x, y)$ to have a nonzero constant term, we can only choose $(e_1, e_2) = (1, 0)$ or $(e_1, e_2) = (0, 1)$, but for these two choices, $h(x, y)$ is $1 + x^{-1}y$ or $xy^{-1} + 1$, respectively, and none of them belongs to $\mathbb{K}[[x, y]]$.

There are at least three possibilities to resolve this situation. The first and most direct way is to consider fields of iterated Laurent series [17, Chapter 2], for instance the field $\mathbb{K}((x))((y))$ of univariate Laurent series in y whose coefficients are univariate Laurent series in x. Clearly this field contains $\mathbb{K}[[x, y]]$, and the multiplicative inverse of $x + y$ in $\mathbb{K}((x))((y))$ is easily found via the geometric series to be

$$\frac{1}{x+y} = \frac{1/x}{1-(-y/x)} = \sum_{n=0}^{\infty} (-1)^n x^{-n-1} y^n.$$

Of course, viewing $x + y$ as an element of $\mathbb{K}((y))((x))$ leads to a different expansion.

The second possibility is more abstract. This construction goes back to Malcev [11] and Neumann [13] (see [15,17] for a more recent discussion). Start with an abelian group G (e.g., the set of all power products $x_1^{i_1} \cdots x_p^{i_p}$ with exponents $i_1, \ldots, i_p \in \mathbb{Z}$ and the usual multiplication) and impose on the elements of G some order $\preccurlyeq$ which respects multiplication (see Section 3 below for definitions and basic facts). Define $\mathbb{K}((G))$ as the set of all formal sums

$$a = \sum_{g \in G} a_g g$$

with $a_g \in \mathbb{K}$ for all $g \in G$ and the condition that their supports $\operatorname{supp}(a) := \{ g \in G \mid a_g \neq 0 \}$ contain no infinite strictly $\preccurlyeq$-decreasing sequence. If addition and multiplication of such series are defined in the natural way, it can be shown that $\mathbb{K}((G))$ is a field (cf. Theorem 5.7 in [13] or Corollary 3.1–11 in [17]).

The third possibility is more geometric and goes back to MacDonald [12]. He considers formal infinite sums of terms of the form $a_{i_1,\ldots,i_p} x_1^{i_1} \cdots x_p^{i_p}$ where the exponent vectors $(i_1, \ldots, i_p)$ are constrained to some fixed cone $C \subseteq \mathbb{R}^p$. It turns out that for every cone C not containing a line, these series form a ring (Theorem 10 below; see Section 2 below for definitions and basic facts concerning cones). MacDonald shows using a multivariate generalization of the Newton–Puiseux method that for every polynomial $f(x_1, \ldots, x_p, y) \in \mathbb{K}[x_1, \ldots, x_p, y]$ one can find a cone such that the corresponding ring contains a series $g(x_1, \ldots, x_p)$ (possibly with fractional exponents) such that $f(x_1, \ldots, x_p, g(x_1, \ldots, x_p)) = 0$. The rings of MacDonald are not fields, but Aroca, Cano and Jung [2,3] observe that a field can be obtained by taking the union of all the rings for some suitable collection of shifted cones (similar to Theorem 15 below). Again allowing fractional exponents, Aroca et al. show that the fields constructed in this way are even algebraically closed. Their elements can thus be considered as the natural multivariate generalizations of Puiseux series.

The construction we give below is, in a sense, a mixture of the approach by Malcev and Neumann on the one hand, and MacDonald–Aroca–Cano–Jung on the other hand. Our goal was to keep the geometric intuition inherent to the latter while at the same time avoiding any technical considerations related to Newton polygons. Our construction is more specific than Malcev–Neumann's in that we do not consider arbitrary groups as carriers of the series, and it is more specific than MacDonald–Aroca–Cano–Jung's in that we do not consider rational exponents. Our series are thus formal infinite sums of terms of the form $a_{i_1,\ldots,i_p} x_1^{i_1} \cdots x_p^{i_p}$ where $(i_1, \ldots, i_p)$ ranges over (some suitable subset of) $\mathbb{Z}^p$. A need to reason about such series arises for instance in lattice path counting (see, e.g., [6] and references given there), in Ehrhart's theory of counting integer points in polytopes (see, e.g., [5] and references given there), or in MacMahon's theory of integer partitions (see, e.g., [1] and references given there). We want to promote them as natural multivariate generalization of the notion of formal Laurent series.

2. Cones

In general, a *cone* $C \subseteq \mathbb{R}^p$ is a set with the property that whenever $\mathbf{u} \in C$ and $c \geq 0$, then $c\mathbf{u} \in C$. The cones we consider here have the following special properties.

Definition 1. A cone $C \subseteq \mathbb{R}^p$ is called

1. *finitely generated* if there exist $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^p$ such that
$$C = \{ z_1\mathbf{v}_1 + z_2\mathbf{v}_2 + \cdots + z_n\mathbf{v}_n \mid z_1, z_2, \ldots, z_n \geq 0 \}.$$
In this case $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is called a *generating set* for C.
2. *rational* if it is finitely generated and has a generating set
$$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{Z}^p.$$
3. *line-free* if for every $\mathbf{v} \in C \setminus \{\mathbf{0}\}$ we have $-\mathbf{v} \notin C$.

Since we will be only considering rational finitely generated cones in this article, we drop these attributes from now on and only say “cone”. With this convention, cones are obviously closed, they obviously all contain $\mathbf{0}$, and they are obviously unbounded or equal to $\{\mathbf{0}\}$. It is also easy to see that all cones are convex (i.e., for all $\mathbf{u}, \mathbf{v} \in C$ and for all $c \in [0, 1]$ we have $c\mathbf{u} + (1 - c)\mathbf{v} \in C$ as well), and that $\mathbf{u}, \mathbf{v} \in C$ implies $\mathbf{u} + \mathbf{v} \in C$. Finally, when C, D are cones, then so is $C + D = \{\, \mathbf{u} + \mathbf{v} \mid \mathbf{u} \in C, \mathbf{v} \in D \,\}$. The following facts are less obvious, but also well-known.

Proposition 2. *Let $K \subseteq \mathbb{R}^p$ be a closed and convex set.*

1. *K is unbounded if and only if there exist $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$ with $\mathbf{v} \neq \mathbf{0}$ such that $\mathbf{u} + c\mathbf{v} \in K$ for all $c \geq 0$ (i.e., K contains a ray).*
2. *Let $\mathbf{w} \in \mathbb{R}^p$ and $R = \{\, c\mathbf{w} \mid c \geq 0 \,\}$. Then for all $\mathbf{u}, \mathbf{v} \in K$ we have $\mathbf{u} + R \subseteq K \iff \mathbf{v} + R \subseteq K$.*

Proof. See statements 1 and 2 in Section 2.5 of Grünbaum [9]. □

In order to give a meaning to an operation (e.g., multiplication) for formal infinite series, we will ensure that every coefficient of the result (e.g., the product) depends only on finitely many coefficients of the operands (e.g., the factors). For some of the operations defined below, it turns out that this property can be shown using the following two lemmas.

Lemma 3. *Let $C \subseteq \mathbb{R}^p$ be a cone and $A \subseteq \mathbb{R}^p$ be a closed and convex set with $C \cap A = \{\mathbf{0}\}$. Then for every $\mathbf{a} \in \mathbb{R}^p$, the set $C \cap (\mathbf{a} + A)$ is bounded.*

Proof. Fix $\mathbf{a} \in \mathbb{R}^p$ and set $K = C \cap (\mathbf{a} + A) \subseteq \mathbb{R}^p$. Assume that K is unbounded. Since C and A are closed and convex, K is also closed and convex, and Proposition 2(1) implies the existence of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^p$ with $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{u} + c\mathbf{v} \in K$ for all $c \geq 0$. We show that $\mathbf{v} \in C \cap A = \{\mathbf{0}\}$ in order to arrive at a contradiction.

Indeed, with $c = 0$ it first follows that $\mathbf{u} \in K \subseteq C$. Since also $\mathbf{0} \in C$, it follows from Proposition 2(2) that $c\mathbf{v} \in C$ for all $c \geq 0$. In particular $\mathbf{v} \in C$.

Similarly, Proposition 2(2) applied to the convex set $\mathbf{a} + A$ and the points $\mathbf{u} \in K \subseteq \mathbf{a} + A$ and $\mathbf{a} \in \mathbf{a} + A$ imply $\mathbf{a} + c\mathbf{v} \in \mathbf{a} + A$ for all $c \geq 0$. Therefore $\mathbf{a} + \mathbf{v} \in \mathbf{a} + A$, and finally $\mathbf{v} \in A$. □

Lemma 4. *Let $C \subseteq \mathbb{R}^p$ be a line-free cone and $S \subseteq C \cap \mathbb{Z}^p$. Then there exists a finite subset $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ of S such that $S \subseteq \bigcup_{i=1}^n (\mathbf{s}_i + C)$.*

Proof. If C is the cone generated by the unit vectors (i.e., $C \cap \mathbb{Z}^p = \mathbb{N}^p$), then this is the classical Dickson Lemma [8,4].

The general case is reduced to this situation as follows. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{Z}^p$ be a set of generators of C. Then each $\mathbf{s} \in S$ can be written as $\mathbf{s} = s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k$ for some nonnegative $s_1, \ldots, s_k \in \mathbb{Q}$. Setting $n_i := \lfloor s_i \rfloor$ $(i = 1, \ldots, k)$, we have $\mathbf{s} = n_1\mathbf{v}_1 + \cdots + n_k\mathbf{v}_k + \mathbf{c}$ for some $\mathbf{c} \in \mathbb{Z}^p$ which is a linear combination of the $\mathbf{v}_i$ with coefficients in $[0, 1]$.

Since $\{z_1\mathbf{v}_1 + \cdots + z_k\mathbf{v}_k : z_1, \ldots, z_k \in [0, 1]\}$ is a bounded set, its intersection with $\mathbb{Z}^p$ is finite, say $\{\mathbf{c}_1, \ldots, \mathbf{c}_\ell\}$. For a fixed vector $\mathbf{c}$, let $N_{\mathbf{c}} \subseteq \mathbb{N}^k$ be the set of all vectors $(n_1, \ldots, n_k) \in \mathbb{N}^k$ such that $n_1\mathbf{v}_1 + \cdots + n_k\mathbf{v}_k + \mathbf{c} \in S$. Then by the original Dickson Lemma, for each of these sets $N_{\mathbf{c}}$ there is a finite subset $B_{\mathbf{c}} \subseteq N_{\mathbf{c}}$ such that for all $(n_1, \dots, n_k) \in N_{\mathbf{c}}$ there exists $(b_1, \dots, b_k) \in B_{\mathbf{c}}$ with $(n_1, \dots, n_k) \in (b_1, \dots, b_k) + \mathbb{N}^k$, viz. $n_i - b_i \geq 0$ for $i = 1, \dots, k$. Then, since C is a cone, we also have $n_1\mathbf{v}_1 + \cdots + n_k\mathbf{v}_k + \mathbf{c} \in b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k + \mathbf{c} + C$.

It finally follows that the finite set $\bigcup_{i=1}^{\ell}\{b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k + \mathbf{c}_i : (b_1, \dots, b_k) \in B_{\mathbf{c}_i}\}$ has the desired property. □

3. Orders

Definition 5. A total order $\preccurlyeq$ on $\mathbb{Z}^p$ is called *additive* if for all $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{Z}^p$ we have

$$\mathbf{i} \preccurlyeq \mathbf{j} \Longrightarrow \mathbf{i} + \mathbf{k} \preccurlyeq \mathbf{j} + \mathbf{k}.$$

An additive order $\preccurlyeq$ is called *compatible* with a cone $C \subseteq \mathbb{R}^p$ if $\mathbf{0} \preccurlyeq \mathbf{k}$ for all $\mathbf{k} \in C \cap \mathbb{Z}^p$.

We take the freedom to write $\mathbf{i} \succcurlyeq \mathbf{j}$ instead of $\mathbf{j} \preccurlyeq \mathbf{i}$, and $\mathbf{i} \precneqq \mathbf{j}$ instead of $\mathbf{i} \preccurlyeq \mathbf{j} \wedge \mathbf{i} \neq \mathbf{j}$, and similar shorthand notations.

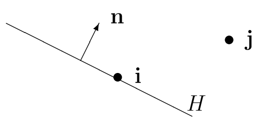
The additivity of an order $\preccurlyeq$ implies that when $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^p$ are such that $\mathbf{v}, \mathbf{w} \succcurlyeq \mathbf{0}$, then also $a\mathbf{v} + b\mathbf{w} \succcurlyeq \mathbf{0}$ for every nonnegative a, b. Note that this is not only true for integers a, b but also for any rational numbers a, b for which $a\mathbf{v} + b\mathbf{w} \in \mathbb{Z}^p$. The reason is that for an additive order we have $\mathbf{v} \succcurlyeq \mathbf{0} \iff d\mathbf{v} \succcurlyeq \mathbf{0}$ for every positive integer d, which allows us to clear denominators.

Example 6. 1. Let $\mathbf{n} = (n_1, \dots, n_p) \in \mathbb{R}^p$ be some vector whose components are linearly independent over $\mathbb{Q}$. For $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^n$, define

$$\mathbf{i} \preccurlyeq_{\mathbf{n}} \mathbf{j} \iff \mathbf{i} \cdot \mathbf{n} \leq \mathbf{j} \cdot \mathbf{n},$$

where $\cdot$ refers to the standard scalar product on $\mathbb{R}^p$. Then $\preccurlyeq_{\mathbf{n}}$ is an additive order.

Geometrically, $\mathbf{i} \preccurlyeq_{\mathbf{n}} \mathbf{j}$ can be interpreted as follows. The affine hyperplane $H = \mathbf{i} + \mathbf{n}^{\perp}$ divides $\mathbb{R}^p$ into two open half spaces, one towards the direction to which $\mathbf{n}$ points and one towards the opposite direction. We have $\mathbf{i} \preccurlyeq_{\mathbf{n}} \mathbf{j}$ if and only if $\mathbf{j}$ belongs to the half space in the direction of $\mathbf{n}$.



The requirement that the coordinates of $\mathbf{n}$ be linearly independent over $\mathbb{Q}$ ensures that $\preccurlyeq_{\mathbf{n}}$ is indeed a total order, for if $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^p$ are such that $\mathbf{i} \preccurlyeq_{\mathbf{n}} \mathbf{j}$ and $\mathbf{j} \preccurlyeq_{\mathbf{n}} \mathbf{i}$, then $\mathbf{n} \cdot \mathbf{i} = \mathbf{n} \cdot \mathbf{j}$, so $\mathbf{n} \cdot (\mathbf{i} - \mathbf{j}) = 0$, and hence, since the coordinates of $\mathbf{i}$ and $\mathbf{j}$ are integers, $\mathbf{i} = \mathbf{j}$.

If C is a line-free cone and $\mathbf{n} \in C$, then $\preccurlyeq_{\mathbf{n}}$ is compatible with C. Moreover, it follows from Lemma 3 that for every $\mathbf{i} \in \mathbb{Z}^p$ there exist only finitely many $\mathbf{j} \in C \cap \mathbb{Z}^p$ such that $\mathbf{j} \preccurlyeq_{\mathbf{n}} \mathbf{i}$. This need not be the case for every additive order, as shown in the following example.

2. For $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^p$, the lexicographic order is defined by letting $\mathbf{i} \preccurlyeq_{lex} \mathbf{j}$ if and only if $\mathbf{i} = \mathbf{j}$ or the leftmost nonzero coordinate of the vector $\mathbf{i} - \mathbf{j}$ is negative. This is an additive order.

If C is a cone which contains no vector $(i_1, \dots, i_p)$ where any of the coordinates $i_1, \dots, i_{p-1}$ is negative, then $\preccurlyeq_{lex}$ is compatible with C. With this order, it may happen that for a fixed $\mathbf{i} \in \mathbb{Z}^n$ there are infinitely many $\mathbf{j} \in C \cap \mathbb{Z}^p$ with $\mathbf{j} \preccurlyeq_{lex} \mathbf{i}$. For instance, if $C \subseteq \mathbb{R}^2$ contains $(1, 0)$ and $(0, 1)$, then $(u, 0) \preccurlyeq_{lex} (\mathbf{0}, \mathbf{1})$ for every $u \in \mathbb{N}$.

However, it is still true that $\preccurlyeq_{lex}$ is a well-founded order on $C \cap \mathbb{Z}^n$. We show in Lemma 8 that this is true for every additive order.

The following two lemmas contain the key properties regarding cones and additive orders which we will use below. The first of them is straightforward, and the second is a reformulation of Lemma 4.

Lemma 7. *Let $C, D \subseteq \mathbb{R}^p$ be cones and let $\preccurlyeq$ be an additive order on $\mathbb{Z}^p$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of generators for C.*

1. *If C is compatible with $\preccurlyeq$, then C is line-free.*
2. *C is compatible with $\preccurlyeq$ if and only if $\mathbf{v}_i \succcurlyeq \mathbf{0}$ for all i.*
3. *If C, D are compatible with $\preccurlyeq$, then $C + D$ is also compatible with $\preccurlyeq$.*

Proof. 1. If $C = \{\mathbf{0}\}$, then C is trivially line-free. If $C \neq \{\mathbf{0}\}$, take some $\mathbf{v} \in C \setminus \{\mathbf{0}\}$. Then $\mathbf{v} \succcurlyeq \mathbf{0}$, because C is compatible with $\preccurlyeq$. Then $-\mathbf{v} \preccurlyeq \mathbf{0}$, because $\preccurlyeq$ is an additive order. Since $\mathbf{v} \neq \mathbf{0}$, also $-\mathbf{v} \neq \mathbf{0}$, and hence $-\mathbf{v} \notin C$.
2. The direction "$\Rightarrow$" is clear because $\mathbf{v}_i \in C$ for all i. The direction "$\Leftarrow$" follows from the observation made after Definition 5 that $\mathbf{v}, \mathbf{w} \succcurlyeq \mathbf{0}$ and $a, b \geq 0$ implies $a\mathbf{v} + b\mathbf{w} \succcurlyeq \mathbf{0}$.
3. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be a generating set for D. Since $C + D$ is generated by $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_m$, the claim follows from part 2. □

Lemma 8. *Let $\preccurlyeq$ be an additive order and C be a cone. If $\preccurlyeq$ is compatible with C, then $\preccurlyeq$ is a well-founded order on $C \cap \mathbb{Z}^p$, i.e., every strictly decreasing sequence $\mathbf{a}_1 \succnsim \mathbf{a}_2 \succnsim \mathbf{a}_3 \succnsim \cdots$ of elements in $C \cap \mathbb{Z}^p$ terminates, or equivalently, every subset of $C \cap \mathbb{Z}^p$ contains a $\preccurlyeq$-minimal element.*

Proof. Let $S \subseteq C \cap \mathbb{Z}^p$. By Lemma 4, there exists a finite subset $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ of S such that $S \subseteq \bigcup_{i=1}^n (\mathbf{s}_i + C)$. From the assumption that C is compatible with $\preccurlyeq$ it follows that when $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^p$ are such that $\mathbf{v} \in \mathbf{w} + C$, then $\mathbf{w} \preccurlyeq \mathbf{v}$. Therefore, the $\preccurlyeq$-minimum of the finite set $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ is also the minimum of S whose existence was to be shown. □

4. Construction

Let $\mathbb{K}$ be a field and $x_1, \ldots, x_p$ be indeterminates. We consider formal infinite series of the form

$$f(\mathbf{x}) := f(x_1, \ldots, x_p) = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$$

where the sum runs over all $\mathbf{k} = (k_1, \ldots, k_p) \in \mathbb{Z}^p$, the $a_{\mathbf{k}}$ are elements of $\mathbb{K}$, and $\mathbf{x}^{\mathbf{k}}$ is a short-hand notation for $x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$.

These objects form a vector space over $\mathbb{K}$ together with the natural addition and scalar multiplication, for if

$$f(\mathbf{x}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \quad \text{and} \quad g(\mathbf{x}) = \sum_{\mathbf{k}} b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$$

then the coefficient of $\mathbf{x^k}$ in $uf(\mathbf{x})+vg(\mathbf{x})$ is simply $u\,a_\mathbf{k}+v\,b_\mathbf{k}$, which is clearly an element of $\mathbb{K}$ for any fixed $u, v \in \mathbb{K}$.

Multiplication is more delicate. In the natural definition

$$f(\mathbf{x})g(\mathbf{x}) := \sum_\mathbf{k}\left(\sum_\mathbf{i} a_\mathbf{i} b_{\mathbf{k}-\mathbf{i}}\right)\mathbf{x^k},$$

the inner sum ranges over infinitely many elements $a_\mathbf{i} b_{\mathbf{k}-\mathbf{i}}$ of $\mathbb{K}$, which is not meaningful in general. To make this summation finite, we restrict the attention to series $f(\mathbf{x}) = \sum_\mathbf{k} a_\mathbf{k}\mathbf{x^k}$ whose support $\operatorname{supp} f(\mathbf{x}) := \{\,\mathbf{k} \in \mathbb{Z}^p \mid a_\mathbf{k} \neq 0\,\}$ is contained in a fixed line-free cone.

Definition 9. Let $C \subseteq \mathbb{R}^p$ be a line-free cone. Then we define the set

$$\mathbb{K}_C[\![\mathbf{x}]\!] := \{\, f(\mathbf{x}) \mid \operatorname{supp} f(\mathbf{x}) \subseteq C\,\}.$$

Using Lemma 3, it can be shown that every coefficient in the product of two elements of $\mathbb{K}_C[\![\mathbf{x}]\!]$ is determined by a sum with only finitely many nonzero terms. The support of the product is again contained in C, as is the support of the sum of two elements of $\mathbb{K}_C[\![\mathbf{x}]\!]$. Therefore, we have the following theorem.

Theorem 10. $\mathbb{K}_C[\![\mathbf{x}]\!]$ *together with the natural addition and multiplication forms a ring.*

Proof. To see that multiplication is well defined, we need to show that for every $\mathbf{k} \in \mathbb{Z}^p$ there exist only finitely many $\mathbf{i} \in \mathbb{Z}^p$ such that $\mathbf{i} \in C$ and $\mathbf{k} - \mathbf{i} \in C$. Since C is line-free, we have $C \cap -C = \{\mathbf{0}\}$. We can therefore apply Lemma 3 to C, $A = -C$ and $\mathbf{a} = \mathbf{k}$, and obtain that $C \cap (\mathbf{k} - C)$ is bounded. A bounded subset of $\mathbb{R}^p$ can only contain finitely many points with integer coordinates, so $C \cap (\mathbf{k} - C) \cap \mathbb{Z}^p$ is finite, and this is what was to be shown.

To see that $\mathbb{K}_C[\![\mathbf{x}]\!]$ is closed under multiplication, consider some $\mathbf{k} \in \mathbb{Z}^p$. In order for the coefficient of some term $\mathbf{x^k}$ in the product of two elements of $\mathbb{K}_C[\![\mathbf{x}]\!]$ to be nonzero, there must be at least one $\mathbf{i} \in C$ such that $\mathbf{k} - \mathbf{i} \in C$ as well. Since C is a cone and cones are closed under addition, $\mathbf{k} \in C$.

Closure under addition is obvious, and it is also obvious that the neutral elements 0 and $1 = \mathbf{x^0}$ belong to $\mathbb{K}_C[\![\mathbf{x}]\!]$. □

When C is the cone consisting of all vectors with nonnegative components, $\mathbb{K}_C[\![\mathbf{x}]\!]$ is the usual ring $\mathbb{K}[\![\mathbf{x}]\!]$ of formal power series in $x_1, \ldots, x_p$. This ring is an integral domain, and a series $f(\mathbf{x}) \in \mathbb{K}[\![\mathbf{x}]\!]$ admits a multiplicative inverse if and only if its constant term is different from zero. Both properties generalize to rings $\mathbb{K}_C[\![\mathbf{x}]\!]$ for arbitrary line-free cones C. The proof ideas are the same as for the usual formal power series ring $\mathbb{K}[\![\mathbf{x}]\!]$.

Theorem 11. *If* $C \subseteq \mathbb{R}^p$ *is a line-free cone, then* $\mathbb{K}_C[\![\mathbf{x}]\!]$ *is an integral domain.*

Proof. Let $f(\mathbf{x}) = \sum_\mathbf{k} a_\mathbf{k}\mathbf{x^k}$ and $g(\mathbf{x}) = \sum_\mathbf{k} b_\mathbf{k}\mathbf{x^k}$ be two nonzero elements of $\mathbb{K}_C[\![\mathbf{x}]\!]$. This means both $\operatorname{supp} f(\mathbf{x})$ and $\operatorname{supp} g(\mathbf{x})$ are nonempty. Let $h(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$. We show that $h(\mathbf{x})$ is not zero, i.e., that $\operatorname{supp} h(\mathbf{x})$ is not empty.

Fix some additive order $\preccurlyeq$ on $\mathbb{Z}^p$ which is compatible with C and let $\mathbf{m} := \min_\preccurlyeq (\operatorname{supp} f(\mathbf{x}) + \operatorname{supp} g(\mathbf{x}))$. If $\mathbf{u} \in \operatorname{supp} f(\mathbf{x})$ and $\mathbf{v} \in \operatorname{supp} g(\mathbf{x})$ are such that $\mathbf{u} + \mathbf{v} =$

$\mathbf{m}$, then we necessarily have $\mathbf{u} = \min_{\preccurlyeq} \operatorname{supp} f(\mathbf{x})$ and $\mathbf{v} = \min_{\preccurlyeq} \operatorname{supp} g(\mathbf{x})$, because $\min_{\preccurlyeq} \operatorname{supp} f(\mathbf{x}) \precneqq \mathbf{u}$ or $\min_{\preccurlyeq} \operatorname{supp} g(\mathbf{x}) \precneqq \mathbf{v}$ would imply

$$\mathbf{m} \preccurlyeq \min_{\preccurlyeq} \operatorname{supp} f(\mathbf{x}) + \min_{\preccurlyeq} \operatorname{supp} g(\mathbf{x}) \precneqq \mathbf{u} + \mathbf{v}.$$

Therefore, the coefficient of $\mathbf{x}^{\mathbf{m}}$ in $h(\mathbf{x})$ is

$$\sum_{\mathbf{i}} a_{\mathbf{i}} b_{\mathbf{m}-\mathbf{i}} = \sum_{\substack{\mathbf{i} \in \operatorname{supp} f(\mathbf{x}) \\ \mathbf{j} \in \operatorname{supp} g(\mathbf{x}) \\ \mathbf{i}+\mathbf{j}=\mathbf{m}}} a_{\mathbf{i}} b_{\mathbf{j}} = a_{\mathbf{u}} b_{\mathbf{v}} \neq 0,$$

because $a_{\mathbf{u}} \neq 0$ and $b_{\mathbf{v}} \neq 0$. □

Theorem 12. *Let $C \subseteq \mathbb{R}^p$ be a line-free cone and $f(\mathbf{x}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \in \mathbb{K}_C[[\mathbf{x}]]$. Then there exists $g(\mathbf{x}) \in \mathbb{K}_C[[\mathbf{x}]]$ with $f(\mathbf{x})g(\mathbf{x}) = 1$ if and only if $a_{\mathbf{0}} \neq 0$.*

Proof. Assume that $a_{\mathbf{0}} = 0$. We show that no multiplicative inverse of $f(\mathbf{x})$ exists. Indeed, if $g(\mathbf{x}) = \sum_{\mathbf{k}} b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ is any element of $\mathbb{K}_C[[\mathbf{x}]]$ then the coefficient of $\mathbf{x}^{\mathbf{0}}$ in the product $f(\mathbf{x})g(\mathbf{x})$ is $a_{\mathbf{0}} b_{\mathbf{0}} = 0$, while for a multiplicative inverse we would need $a_{\mathbf{0}} b_{\mathbf{0}} = 1$.

Assume now $a_{\mathbf{0}} \neq 0$. We show that a multiplicative inverse of $f(\mathbf{x})$ exists. Fix an additive order $\preccurlyeq$ compatible with C. Let $g(\mathbf{x}) = \sum_{\mathbf{k}} b_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ be a series with undetermined coefficients $b_{\mathbf{k}}$. Set $b_{\mathbf{0}} = 1/a_{\mathbf{0}}$, which is possible because $a_{\mathbf{0}} \neq 0$. Then the constant term of $f(\mathbf{x})g(\mathbf{x})$ is 1, regardless of the values of the other $b_{\mathbf{k}}$. We now show by noetherian induction on $\mathbf{k}$ that there is a unique way to choose the coefficients $b_{\mathbf{i}}$ for $0 \precneqq \mathbf{i} \preccurlyeq \mathbf{k}$ such that the coefficient of $\mathbf{x}^{\mathbf{i}}$ in $f(\mathbf{x})g(\mathbf{x}) - 1$ is equal to 0 for all $\mathbf{i} \preccurlyeq \mathbf{k}$.

Assume as induction hypothesis that this is true for all $\mathbf{i}$ with $\mathbf{i} \precneqq \mathbf{k}$. Then for the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $f(\mathbf{x})g(\mathbf{x})$ we have

$$\sum_{\mathbf{i}} a_{\mathbf{i}} b_{\mathbf{k}-\mathbf{i}} = \sum_{\mathbf{0} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{k}} a_{\mathbf{i}} b_{\mathbf{k}-\mathbf{i}} = a_{\mathbf{0}} b_{\mathbf{k}} + \sum_{\mathbf{0} \precneqq \mathbf{i} \preccurlyeq \mathbf{k}} a_{\mathbf{i}} b_{\mathbf{k}-\mathbf{i}}.$$

Since $a_{\mathbf{0}} \neq 0$ and all the $b_{\mathbf{k}-\mathbf{i}}$ on the right hand side are uniquely determined by induction hypothesis, we can (and have to) take $b_{\mathbf{k}} = -a_{\mathbf{0}}^{-1} \sum_{\mathbf{0} \precneqq \mathbf{i} \preccurlyeq \mathbf{k}} a_{\mathbf{i}} b_{\mathbf{k}-\mathbf{i}}$. With this (and only this) choice, the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $f(\mathbf{x})g(\mathbf{x})$ will be zero, as desired. □

In the univariate case, if the constant term of some nonzero series $f(x) \in \mathbb{K}[[x]]$ is zero, we can write $f(x) = x^e h(x)$ for some $e \in \mathbb{N}$ and $h(x) \in \mathbb{K}[[x]]$ with $h(0) \neq 0$. Then $h(x)$ has a multiplicative inverse and we find $x^{-e} h(x)^{-1}$ as the multiplicative inverse of $f(x)$ if we allow terms with negative exponents. Defining formal Laurent series via $\mathbb{K}((x)) := \bigcup_{e \in \mathbb{Z}} x^e \mathbb{K}[[x]]$ therefore already leads to a field.

In the multivariate case, it is not always possible to write a given $f(\mathbf{x}) \in \mathbb{K}_C[[\mathbf{x}]]$ in the form $f(\mathbf{x}) = \mathbf{x}^{\mathbf{e}} h(\mathbf{x})$ for some $h(\mathbf{x}) \in \mathbb{K}_C[[\mathbf{x}]]$, as already illustrated in the introduction. But in cases where this is not possible, we can still write $f(\mathbf{x})$ in the desired form if we allow $h(\mathbf{x})$ to belong to an enlarged ring $\mathbb{K}_{C'}[[\mathbf{x}]]$ for a suitably chosen line-free cone C' containing the original cone C. Then $h(\mathbf{x})$ has a multiplicative inverse in this ring by Theorem 12, and we can regard $\mathbf{x}^{-\mathbf{e}} h(\mathbf{x})^{-1}$ as the multiplicative inverse of $f(\mathbf{x})$.

Example 13. Consider the power series $f(x, y) = x + y \in \mathbb{K}[[x, y]]$ from the introduction. (C is the cone generated by the two unit vectors $(1, 0)$ and $(0, 1)$ here.) This series can also be viewed as an element of $\mathbb{K}_{C'}[[x, y]]$, where C' is the cone generated by $(1, 0)$ and $(-1, 1)$. Then we have $f(x, y) = x^1 y^0 h(x, y)$ with $h(x, y) = 1 + x^{-1} y^1 \in \mathbb{K}_{C'}[[x, y]]$. In this ring, $h(x, y)$ has a multiplicative inverse, and therefore we can regard $x^{-1} y^0 h(x, y)^{-1}$ as the multiplicative inverse of $f(x, y)$.

If a collection of rings is such that for any two rings R_1, R_2 from the collection, the collection contains some other ring R_3 with $R_1 \subseteq R_3$ and $R_2 \subseteq R_3$, and if for any two rings R_1, R_2 from the collection, the respective addition and multiplication of these rings coincide on $R_1 \cap R_2$, then the union over all the rings from the collection forms again a ring in a natural way.

We can therefore make the following definition.

Definition 14. Let $\preccurlyeq$ be an additive order on $\mathbb{Z}^p$. Then we define the sets

$$\mathbb{K}_{\preccurlyeq}[[\mathbf{x}]] := \bigcup_{C \in \mathcal{C}} \mathbb{K}_C[[\mathbf{x}]] \quad \text{and} \quad \mathbb{K}_{\preccurlyeq}((\mathbf{x})) := \bigcup_{\mathbf{e} \in \mathbb{Z}^p} \mathbf{x}^{\mathbf{e}} \mathbb{K}_{\preccurlyeq}[[\mathbf{x}]],$$

where $\mathcal{C}$ is the set of all cones $C \subseteq \mathbb{R}^p$ which are compatible with $\preccurlyeq$.

Theorem 15. *If $\preccurlyeq$ is an additive order on $\mathbb{Z}^p$, then $\mathbb{K}_{\preccurlyeq}[[\mathbf{x}]]$ is a ring and $\mathbb{K}_{\preccurlyeq}((\mathbf{x}))$ is a field.*

Proof. To see that $\mathbb{K}_{\preccurlyeq}[[\mathbf{x}]]$ is a ring, consider two rings $\mathbb{K}_{C_1}[[\mathbf{x}]], \mathbb{K}_{C_2}[[\mathbf{x}]]$ from the collection, i.e., consider two cones C_1, C_2 that are compatible with $\preccurlyeq$. By Lemma 7, the cone $C_3 := C_1 + C_2$ is also compatible with $\preccurlyeq$, so $\mathbb{K}_{C_3}[[\mathbf{x}]]$ also appears in the union. Furthermore, it is clear that addition and multiplication of the rings $\mathbb{K}_{C_1}[[\mathbf{x}]]$ and $\mathbb{K}_{C_2}[[\mathbf{x}]]$ agree on $\mathbb{K}_{C_1}[[\mathbf{x}]] \cap \mathbb{K}_{C_2}[[\mathbf{x}]]$. This shows that $\mathbb{K}_{\preccurlyeq}[[\mathbf{x}]]$ is well-defined as a ring.

To see that $\mathbb{K}_{\preccurlyeq}((\mathbf{x}))$ is a field, consider two nonzero elements $f(\mathbf{x}), g(\mathbf{x}) \in \mathbb{K}_{\preccurlyeq}((\mathbf{x}))$. We show that their sum, their product, and the multiplicative inverse of $f(\mathbf{x})$ also belong to $\mathbb{K}_{\preccurlyeq}((\mathbf{x}))$. Let $A, B \subseteq \mathbb{R}^p$ be cones compatible with $\preccurlyeq$ and let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^p$ be such that $f(\mathbf{x}) = \mathbf{x}^{\mathbf{a}} a(\mathbf{x})$ and $g(\mathbf{x}) = \mathbf{x}^{\mathbf{b}} b(\mathbf{x})$ for some $a(\mathbf{x}) \in \mathbb{K}_A[[\mathbf{x}]]$ and $b(\mathbf{x}) \in \mathbb{K}_B[[\mathbf{x}]]$. Then $f(\mathbf{x})g(\mathbf{x})$ belongs to $\mathbf{x}^{\mathbf{a}+\mathbf{b}} \mathbb{K}_{A+B}[[\mathbf{x}]]$ and $f(\mathbf{x}) + g(\mathbf{x})$ belongs to $\mathbf{x}^{\min_{\preccurlyeq}(\mathbf{a},\mathbf{b})} \mathbb{K}_C[[\mathbf{x}]]$ where C is the cone generated by a generating set of A, a generating set of B, and the single vector $\max_{\preccurlyeq}(\mathbf{a}, \mathbf{b}) - \min_{\preccurlyeq}(\mathbf{a}, \mathbf{b}) \succcurlyeq \mathbf{0}$. Note that $A + B$ and C are compatible with $\preccurlyeq$ by Lemma 7. It follows that $\mathbb{K}_{\preccurlyeq}((\mathbf{x}))$ is closed under addition and multiplication.

As for the multiplicative inverse, let $f(\mathbf{x}) \neq 0$ and $\mathbf{e} := \min_{\preccurlyeq} \operatorname{supp} f(\mathbf{x})$. This minimum exists by Lemma 8 and because $\operatorname{supp} f(\mathbf{x})$ is nonempty for nonzero $f(\mathbf{x})$. Let $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\} \subseteq \operatorname{supp} f(\mathbf{x}) \subseteq A$ be a finite set such that $\operatorname{supp} f(\mathbf{x}) \subseteq \bigcup_{i=1}^n (\mathbf{s}_i + A)$. Such a finite set exists by Lemma 4. Let C be the cone generated by a generating set of A, and $\mathbf{s}_1 - \mathbf{e}, \ldots, \mathbf{s}_n - \mathbf{e}$. By the choice of $\mathbf{e}$, we have $\mathbf{s}_i - \mathbf{e} \succcurlyeq \mathbf{0}$ for all i, so by Lemma 7, the cone C is compatible with $\preccurlyeq$. Now we can write $f(\mathbf{x}) = \mathbf{x}^{\mathbf{e}} h(\mathbf{x})$ for some $h(\mathbf{x}) \in \mathbb{K}_C[[\mathbf{x}]]$ with nonzero constant term. By Theorem 12 there exists a multiplicative inverse $h(\mathbf{x})^{-1} \in \mathbb{K}_C[[\mathbf{x}]] \subseteq \mathbb{K}_{\preccurlyeq}[[\mathbf{x}]]$. Hence $f(\mathbf{x})^{-1} = \mathbf{x}^{-\mathbf{e}} h(\mathbf{x})^{-1}$ belongs to $\mathbb{K}_{\preccurlyeq}((\mathbf{x}))$, as claimed. $\square$

Example 16. Consider the univariate polynomial $f(x) = 1 + x$. The only two additive orders on $\mathbb{Z}$ are the natural order and its reverse.

With respect to the natural order $\leq$, the smallest exponent of $f(x)$ is 0, so $f(x)$ has a multiplicative inverse in $\mathbb{K}_{\leq}[[x]] = \mathbb{K}[[x]]$. Its coefficients can be determined following the proof of Theorem 12, the result being

$$f(x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots.$$

Let now $\leq^{-1}$ denote the reversed order, i.e., $i \leq^{-1} j \iff j \leq i$. Then the smallest exponent of $f(x)$ with respect to this order is 1. Write $f(x) = x(1 + x^{-1})$. The smallest exponent of $1 + x^{-1}$ with respect to $\leq^{-1}$ is 0, so this series does have a multiplicative inverse in $\mathbb{K}_{\leq^{-1}}[[x]] = \mathbb{K}[[x^{-1}]]$. Its first terms are $1 - x^{-1} + x^{-2} - x^{-3} + \cdots$. Consequently, the multiplicative inverse of $f(x)$ in $\mathbb{K}_{\leq^{-1}}((x))$ reads

$$f(x)^{-1} = x^{-1} - x^{-2} + x^{-3} - x^{-4} + \cdots.$$

More generally, the various possible series expansions of a multivariate rational function $r(\mathbf{x}) = u(\mathbf{x})/v(\mathbf{x}) \in \mathbb{K}(\mathbf{x})$ can be obtained as follows. An exponent vector $\mathbf{e} \in \operatorname{supp} v(\mathbf{x}) \subseteq \mathbb{Z}^p$ qualifies as a minimal element if there exists an affine hyperplane $H \subseteq \mathbb{R}^p$ which contains $\mathbf{e}$ and which is furthermore such that all other elements of $\operatorname{supp} v(\mathbf{x})$ belong to the same of the two open half spaces defined by H. Geometrically, these points $\mathbf{e}$ are the corner points in the convex hull of $\operatorname{supp} v(\mathbf{x})$. For each such corner point $\mathbf{e}$, the cone C generated by the elements of $\operatorname{supp} v(\mathbf{x}) - \mathbf{e}$ is line-free, and there exists a series expansion of $r(\mathbf{x})$ in $\mathbf{x}^{-\mathbf{e}}\mathbb{K}_C[[\mathbf{x}]]$.

The coefficients in these series expansions all satisfy a multivariate linear recurrence equation with constant coefficients, which can be read off from the denominator polynomial. In the univariate case, also the converse is true: every sequence satisfying a linear recurrence with constant coefficients is the coefficient sequence of a series expansion of a rational function. The latter implication is no longer valid in the case of several variables. As worked out by Bousquet-Mélou and Petkovšek [7], a multivariate power series whose coefficient sequence satisfies a linear recurrence equation with constant coefficients need not be rational, not even algebraic, not even differentially algebraic.

5. Composition

Our next goal is to understand the composition of multivariate Laurent series. In order to formulate the results, it is convenient to adopt the following notation. If $f(\mathbf{x}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ is any series, then for any fixed $\mathbf{k} \in \mathbb{Z}^p$ we write $[\mathbf{x}^{\mathbf{k}}]f(\mathbf{x}) := a_{\mathbf{k}}$ for the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $f(\mathbf{x})$. Furthermore, if $\preccurlyeq$ is an additive order and $f(\mathbf{x})$ a nonzero series, we call $\operatorname{lexp}_{\preccurlyeq} f(\mathbf{x}) := \min_{\preccurlyeq} \operatorname{supp} f(\mathbf{x}) \in \mathbb{Z}^p$ the leading exponent of $f(\mathbf{x})$, and $\operatorname{lt}_{\preccurlyeq} f(\mathbf{x}) := \mathbf{x}^{\operatorname{lexp}_{\preccurlyeq} f(\mathbf{x})}$ the leading term. We may omit the subscript $\preccurlyeq$ when the order is clear from the context.

For two univariate formal power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x)$, it is natural to define the composition $f(g(x))$ as the power series $\sum_{k=0}^{\infty} a_k g(x)^k$. The latter expression is meaningful provided that $g(0) = 0$ because in this case, $g(x) = xh(x)$ for some power series $h(x)$, and $g(x)^k = x^k h(x)^k$ has zero coefficients for all terms of degree less than k. Therefore, for every $n \in \mathbb{N}$ the coefficient of x^n in $\sum_{k=0}^{\infty} a_k g(x)^k$ is in fact the coefficient of x^n in the finite sum $\sum_{k=0}^{n} a_k g(x)^k$. Neumann [13, Theorem 4.7] and Xin [17, Theorem 3-1.7] prove generalized versions of this criterion for compositions $f(g)$ where $f(x) \in \mathbb{K}[[x]]$ is a univariate power series in the usual sense and g is a Malcev–Neumann series.

We are interested here more generally in compositions $f(g_1(\mathbf{x}), \dots, g_m(\mathbf{x}))$ where $f(\mathbf{y})$ and the $g_i(\mathbf{x})$ are formal Laurent series in several variables as defined above. In order to formally define them, fix an additive order $\preccurlyeq$ on $\mathbb{Z}^p$, let U be any set and consider a function $c\colon U \to \mathbb{K}_{\preccurlyeq}((\mathbf{x}))$ with the following two properties.

1. For all $\mathbf{k} \in \mathbb{Z}^p$, the set $\{u \in U \mid \mathbf{k} \in \operatorname{supp} c(u)\}$ is finite.
2. There exists a line-free cone $C \subseteq \mathbb{R}^p$ such that $\operatorname{supp} c(u) \subseteq C$ for all $u \in U$.

We can then define $h(\mathbf{x}) := \sum_{u\in U} c(u)$ as the unique element of $\mathbb{K}_C[[\mathbf{x}]]$ whose coefficient of $\mathbf{x}^{\mathbf{k}}$ is equal to $\sum_{u\in U} [\mathbf{x}^{\mathbf{k}}]c(u)$ for all $\mathbf{k} \in \mathbb{Z}^p$. The first requirement ensures that this sum is finite for every $\mathbf{k} \in \mathbb{Z}^p$, and the second one ensures that the support of $h(\mathbf{x})$ is contained in C.

Composition of Laurent series can be viewed as a special case of the construction just described: let $\leq$ be an additive order on $\mathbb{Z}^q$ and $\preccurlyeq$ an additive order on $\mathbb{Z}^p$, let $f(\mathbf{y}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{y}^{\mathbf{k}} \in \mathbb{K}_{\leq}((\mathbf{y}))$ and $g_1(\mathbf{x}), \dots, g_q(\mathbf{x}) \in \mathbb{K}_{\preccurlyeq}((\mathbf{x}))$, let $U := \operatorname{supp} f(\mathbf{y}) \subseteq \mathbb{Z}^q$ and define

$$c\colon U \to \mathbb{K}_{\preccurlyeq}((\mathbf{x})), \qquad c(\mathbf{k}) := a_{\mathbf{k}} g_1(\mathbf{x})^{k_1} \cdots g_q(\mathbf{x})^{k_q}.$$

Then the composition $f(g_1(\mathbf{x}), \dots, g_q(\mathbf{x})) \in \mathbb{K}_{\preccurlyeq}((\mathbf{x}))$ is defined as the sum $\sum_{u\in U} c(u)$, provided that this sum exists in the sense defined before.

The main result of this section is the following sufficient condition for the existence of the composition.

Theorem 17. *Let $C \subseteq \mathbb{R}^q$ be a line-free cone and $f(\mathbf{y}) \in \mathbb{K}_C[[\mathbf{y}]]$. Let $\preccurlyeq$ be an additive order on $\mathbb{Z}^p$ and $g_1(\mathbf{x}), \dots, g_q(\mathbf{x}) \in \mathbb{K}_{\preccurlyeq}((\mathbf{x}))\setminus\{0\}$. Let $M \in \mathbb{Z}^{p\times q}$ be the matrix whose i-th column consists of the leading exponent $\operatorname{lexp}(g_i(\mathbf{x}))$ $(i = 1, \dots, q)$. Let $C' \subseteq \mathbb{R}^p$ be a cone containing $MC := \{M\mathbf{x} \mid \mathbf{x} \in C\} \subseteq \mathbb{R}^p$ as well as $\operatorname{supp}\big(g_i(\mathbf{x})/\operatorname{lt}(g_i(\mathbf{x}))\big)$ for $i = 1, \dots, q$. Suppose that $C \cap \ker M = \{\mathbf{0}\}$ and that C' is line-free. Then $f(g_1(\mathbf{x}), \dots, g_q(\mathbf{x}))$ is well-defined and belongs to the ring $\mathbb{K}_{C'}[[\mathbf{x}]]$.*

Proof. We show (1) that for every fixed $\mathbf{k} \in \mathbb{Z}^p$ there are only finitely many $(u_1, \dots, u_q) \in \mathbb{Z}^q$ with $\mathbf{k} \in \operatorname{supp}\big(g_1(\mathbf{x})^{u_1} \cdots g_q(\mathbf{x})^{u_q}\big)$, and (2) that $\operatorname{supp}\big(g_1(\mathbf{x})^{u_1} \cdots g_q(\mathbf{x})^{u_q}\big) \subseteq C'$ for all $(u_1, \dots, u_q) \in \operatorname{supp} f(\mathbf{y})$.

For the second requirement, first observe that the $g_i(\mathbf{x})/\operatorname{lt}(g_i(\mathbf{x}))$ are elements of $\mathbb{K}_{C'}[[\mathbf{x}]]$ with nonzero constant term. Therefore, by Theorems 10 and 12, also $\big(\frac{g_1(\mathbf{x})}{\operatorname{lt}(g_1(\mathbf{x}))}\big)^{u_1} \cdots \big(\frac{g_q(\mathbf{x})}{\operatorname{lt}(g_q(\mathbf{x}))}\big)^{u_q}$ is an element of $\mathbb{K}_{C'}[[\mathbf{x}]]$ for every choice $(u_1, \dots, u_q) \in \mathbb{Z}^p$. Second, because of $MC \subseteq C'$, the exponent vector of the term $\operatorname{lt}(g_1(\mathbf{x}))^{u_1} \cdots \operatorname{lt}(g_q(\mathbf{x}))^{u_q}$ belongs to C' for every $(u_1, \dots, u_q) \in \operatorname{supp} f(\mathbf{y}) \subseteq C$, so the term itself belongs to $\mathbb{K}_{C'}[[\mathbf{x}]]$. Using once more that $\mathbb{K}_{C'}[[\mathbf{x}]]$ is a ring, it follows that $\operatorname{supp}(g_1(\mathbf{x})^{u_1} \cdots g_q(\mathbf{x})^{u_q}) \subseteq C'$ for every $(u_1, \dots, u_q) \in \operatorname{supp} f(\mathbf{y})$.

For the first requirement, let $\mathbf{k} \in \mathbb{Z}^p$. Then by Lemma 3 with $\mathbf{k}$, C' and $-C'$ playing the roles of $\mathbf{a}$, C and A, there are only finitely many $\mathbf{n} \in \mathbb{Z}^p$ such that $\mathbf{k} \in \mathbf{n} + C'$. For some fixed $\mathbf{n} \in \mathbb{Z}^p$, consider the set $\{\mathbf{u} \in C \cap \mathbb{Z}^q \mid M\mathbf{u} = \mathbf{n}\}$. If this set is empty, it is trivially finite. If not, fix an element $\mathbf{w}$ from the set. Then every other element $\mathbf{u}$ of the set can be written as $\mathbf{u} = \mathbf{w} + \mathbf{v}$ for some $\mathbf{v} \in \ker M$: if $\mathbf{u}, \mathbf{u}'$ are two elements of the set, then $M\mathbf{u} = M\mathbf{u}' = \mathbf{n}$, so $\mathbf{u} - \mathbf{u}' \in \ker M$. Therefore we can write $\{\mathbf{u} \in C \mid M\mathbf{u} = \mathbf{n}\} =$

$C \cap (\mathbf{w} + \ker M)$. By Lemma 3 with $\mathbf{w}$, C and $\ker M$ playing the roles of $\mathbf{a}$, C and A, it follows that the set contains only finitely many integer points.

Altogether, we have shown that for all $\mathbf{k} \in \mathbb{Z}^p$ there are only finitely many $\mathbf{u} \in C \cap \mathbb{Z}^q$ such that $\mathbf{k} \in M\mathbf{u} + C'$. The claim follows because

$$\operatorname{supp}\left(g_1(\mathbf{x})^{u_1} \cdots g_q(\mathbf{x})^{u_q}\right) \subseteq M\mathbf{u} + C'$$

by definition of M and C'. $\square$

Example 18. 1. The classical condition for the composition of two power series in a single variable is contained as a special case in Theorem 17. In this case, C and C' are the positive halfline and M is a 1×1-matrix whose single entry is positive if and only if $g(0) = 0$ if and only if $C \cap \ker M = \{0\}$.

2. Consider a power series $f(x) \in \mathbb{K}[[x^{-1}]]$ with negative exponents (i.e., C is the negative halfline) and let $g(x) \in \mathbb{K}((x)) \setminus \{0\}$ be a usual formal Laurent series. Then M is a 1×1-matrix whose single entry is the smallest nonzero exponent appearing in $g(x)$. If this is negative, then MC is the positive halfline, and since also $\operatorname{lt}(g(x))^{-1} g(x) \in \mathbb{K}[[x]]$, we can take for C' the positive halfline. Therefore $f(g(x))$ is well-defined. For example, for

$$f(x) = 1 - 2x^{-1} + 3x^{-2} - 4x^{-3} + 5x^{-4} - 6x^{-5} + \cdots \in \mathbb{Q}[[x^{-1}]]$$

and

$$g(x) = x^{-2} + x^{-1} + 1 + x + x^2 + x^3 + \cdots \in \mathbb{Q}((x))$$

we have

$$f(g(x)) = 1 - 2x^2 + 2x^3 + 3x^4 - 6x^5 - x^6 + 12x^7 + \cdots \in \mathbb{Q}((x)).$$

3. As an example with several variables, let $C \subseteq \mathbb{R}^2$ be the cone generated by $\binom{-1}{1}$ and $\binom{0}{1}$, and let $f(x, y) \in \mathbb{K}_C[[x, y]]$. Let $\preccurlyeq$ be the lexicographic order with $x \preccurlyeq y$, and let $g_1(x, y) = x + y$, $g_2(x, y) = 1/(x + y) = x^{-1} - x^{-2}y + x^{-3}y^2 + \cdots \in \mathbb{K}_{\preccurlyeq}((x, y))$. Then $\operatorname{lt}(g_1(x, y)) = x = x^1y^0$ and $\operatorname{lt}(g_2(x, y)) = x^{-1} = x^{-1}y^0$, so

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

The kernel of M is the vector space generated by $\binom{1}{1}$, therefore $C \cap \ker M = \{\mathbf{0}\}$.

MC is the cone generated by $M\binom{-1}{1} = \binom{-2}{0}$ and $M\binom{0}{1} = \binom{-1}{0}$, and the supports of

$$g_1(x, y)/\operatorname{lt}(g_1(x, y)) = 1 + x^{-1}y^2,$$
$$g_2(x, y)/\operatorname{lt}(g_2(x, y)) = 1 - x^{-1}y + x^{-2}y^2 + \cdots$$

belong to the cone generated by $\binom{-1}{2}$ and $\binom{-1}{1}$. We can therefore take for C' the cone generated by $\binom{-1}{2}$ and $\binom{-1}{0}$. Note that this is indeed a line-free cone.

In conclusion, the composition $f(g_1(x, y), g_2(x, y))$ is well-defined.

In Theorem 17, no restrictions are imposed on the series $f(\mathbf{y}) \in \mathbb{K}_C[[\mathbf{y}]]$ into which the $g_i(\mathbf{x})$ are plugged: if the theorem allows the composition of some fixed set of $g_i(\mathbf{x}) \in \mathbb{K}_{\preccurlyeq}((\mathbf{x}))$ into some fixed element $f(\mathbf{y})$ of $\mathbb{K}_C[[\mathbf{y}]]$, then it also allows the composition of these $g_i(\mathbf{x})$ into any other element of $\mathbb{K}_C[[\mathbf{y}]]$. We can therefore consider a map $\Phi\colon \mathbb{K}_C[[\mathbf{y}]] \to \mathbb{K}_{\preccurlyeq}((\mathbf{x}))$ which to every $f(\mathbf{y}) \in \mathbb{K}_C[[\mathbf{y}]]$ assigns the composition $f(g_1(\mathbf{x}), \ldots, g_q(\mathbf{x})) \in \mathbb{K}_{\preccurlyeq}((\mathbf{x}))$. We show next that this map preserves the ring structure, a fact that is not too surprising but also not entirely trivial.

Theorem 19. *Let $C \subseteq \mathbb{R}^q$ be a line-free cone, $\preccurlyeq$ an additive order on $\mathbb{Z}^p$, and $g_1(\mathbf{x}), \ldots, g_q(\mathbf{x}) \in \mathbb{K}_{\preccurlyeq}((\mathbf{x})) \setminus \{0\}$. Let $M \in \mathbb{Z}^{p\times q}$ and $C' \subseteq \mathbb{R}^p$ be defined as in Theorem 17, and assume, also as in Theorem 17, that $C \cap \ker M = \{\mathbf{0}\}$ and that C' is line-free. Then the map*

$$\Phi\colon \mathbb{K}_C[[\mathbf{y}]] \to \mathbb{K}_{C'}[[\mathbf{x}]], \qquad f(\mathbf{y}) \mapsto f(g_1(\mathbf{x}), \ldots, g_q(\mathbf{x}))$$

is a ring homomorphism.

Proof. It is clear that $\Phi(1) = 1$, and it is also easy to verify that $\Phi(a(\mathbf{y}) + b(\mathbf{y})) = \Phi(a(\mathbf{y})) + \Phi(b(\mathbf{y}))$ for all $a(\mathbf{y}), b(\mathbf{y}) \in \mathbb{K}_C[[\mathbf{y}]]$. We show the case of multiplication in some more detail.

Let $a(\mathbf{y}) = \sum_{\mathbf{n}} a_{\mathbf{n}}\mathbf{y}^{\mathbf{n}}$, $b(\mathbf{y}) = \sum_{\mathbf{n}} b_{\mathbf{n}}\mathbf{y}^{\mathbf{n}} \in \mathbb{K}_C[[\mathbf{y}]]$. Then $\Phi(a(\mathbf{y}))$, $\Phi(b(\mathbf{y}))$, and $\Phi(a(\mathbf{y})b(\mathbf{y}))$ all belong to $\mathbb{K}_{C'}[[\mathbf{x}]]$. We show that for all $\mathbf{n} \in \mathbb{Z}^p \cap C'$ we have $[\mathbf{x}^{\mathbf{n}}]\Phi(a(\mathbf{y})b(\mathbf{y})) = [\mathbf{x}^{\mathbf{n}}]\Phi(a(\mathbf{y}))\Phi(b(\mathbf{y}))$.

As shown in the proof of Theorem 10, for every $\mathbf{k} \in \mathbb{Z}^p \cap C'$, the set

$$I_{\mathbf{k}} := \{\, \mathbf{i} \in \mathbb{Z}^p \mid \mathbf{i} \in C' \text{ and } \mathbf{k} - \mathbf{i} \in C' \,\} \subseteq \mathbb{Z}^p \cap C'$$

is finite. Furthermore, as shown in the proof of Theorem 17, for every $\mathbf{i} \in \mathbb{Z}^p$, the set

$$U_{\mathbf{i}} := \{\, (u_1, \ldots, u_q) \mid \mathbf{i} \in \operatorname{supp}(g_1(\mathbf{x})^{u_1} \cdots g_q(\mathbf{x})^{u_q}) \,\} \subseteq \mathbb{Z}^q$$

is finite.

Now fix an arbitrary $\mathbf{n} \in \mathbb{Z}^p \cap C'$ and let $U := \bigcup_{\mathbf{i}\in I_{\mathbf{n}}} U_{\mathbf{i}} \subseteq \mathbb{Z}^q$. Then U is finite and we have $U_{\mathbf{n}} \subseteq U$ because $\mathbf{0} \in I_{\mathbf{n}}$. By the definition of multiplication and composition, and because $[\mathbf{x}^{\mathbf{n}}]g_1(\mathbf{x})^{k_1} \cdots g_q(\mathbf{x})^{k_q} = 0$ whenever $(k_1, \ldots, k_q) \notin U$, we can write

$$[\mathbf{x}^{\mathbf{n}}]\Phi(a(\mathbf{y})b(\mathbf{y})) = [\mathbf{x}^{\mathbf{n}}]\sum_{\mathbf{k}}\left(\sum_{\mathbf{i}} a_{\mathbf{k}-\mathbf{i}}b_{\mathbf{i}}\right)g^{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{k}\in U}\left(\sum_{\mathbf{i}\in I_{\mathbf{k}}} a_{\mathbf{k}-\mathbf{i}}b_{\mathbf{i}}\right)[\mathbf{x}^{\mathbf{n}}]g^{\mathbf{k}}(\mathbf{x}),$$

where $g^{\mathbf{k}}(\mathbf{x})$ is a shorthand notation for $g_1(\mathbf{x})^{k_1} \cdots g_q(\mathbf{x})^{k_q}$. Furthermore, for $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^p \cap C'$ with $\mathbf{i} + \mathbf{j} = \mathbf{n}$, we can write

$$[\mathbf{x}^{\mathbf{i}}]\Phi(a(\mathbf{y})) = [\mathbf{x}^{\mathbf{i}}]\sum_{\mathbf{k}} a_{\mathbf{k}}g^{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{k}\in U_{\mathbf{i}}} a_{\mathbf{k}}[\mathbf{x}^{\mathbf{i}}]g^{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{k}\in U} a_{\mathbf{k}}[\mathbf{x}^{\mathbf{i}}]g^{\mathbf{k}}(\mathbf{x})$$

and

$$[\mathbf{x}^{\mathbf{j}}]\Phi(b(\mathbf{y})) = [\mathbf{x}^{\mathbf{j}}]\sum_{\mathbf{k}} b_{\mathbf{k}}g^{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{k}\in U_{\mathbf{j}}} b_{\mathbf{k}}[\mathbf{x}^{\mathbf{j}}]g^{\mathbf{k}}(\mathbf{x}) = \sum_{\mathbf{k}\in U} b_{\mathbf{k}}[\mathbf{x}^{\mathbf{j}}]g^{\mathbf{k}}(\mathbf{x}).$$

Consequently,

$$\begin{aligned}
[\mathbf{x^n}]\,\Phi(a(\mathbf{y}))\,\Phi(b(\mathbf{y})) &= \sum_{\mathbf{j}\in I_\mathbf{n}}\big([\mathbf{x^{n-j}}]\,\Phi(a(\mathbf{y}))\big)\big([\mathbf{x^j}]\,\Phi(b(\mathbf{y}))\big)\\
&= \sum_{\mathbf{j}\in I_\mathbf{n}}\left(\sum_{\mathbf{k}\in U} a_\mathbf{k}[\mathbf{x^{n-j}}]g^\mathbf{k}(\mathbf{x})\right)\left(\sum_{\mathbf{k}\in U} b_\mathbf{k}[\mathbf{x^j}]g^\mathbf{k}(\mathbf{x})\right)\\
&= \sum_{\mathbf{j}\in I_\mathbf{n}}\sum_{\mathbf{k}\in U}\sum_{\mathbf{i}\in I_\mathbf{k}} a_\mathbf{k-i}\,[\mathbf{x^{n-j}}]g^\mathbf{k-i}(\mathbf{x})\,b_\mathbf{i}\,[\mathbf{x^j}]g^\mathbf{i}(\mathbf{x})\\
&= \sum_{\mathbf{k}\in U}\sum_{\mathbf{i}\in I_\mathbf{k}}\left(a_\mathbf{k-i}b_\mathbf{i}\sum_{\mathbf{j}\in I_\mathbf{n}}[\mathbf{x^{n-j}}]\,g^\mathbf{k-i}(\mathbf{x})\,[\mathbf{x^j}]g^\mathbf{i}(\mathbf{x})\right)\\
&= \sum_{\mathbf{k}\in U}\sum_{\mathbf{i}\in I_\mathbf{k}} a_\mathbf{k-i}b_\mathbf{i}\,[\mathbf{x^n}]g^\mathbf{k}(\mathbf{x}) = [\mathbf{x^n}]\,\Phi(a(\mathbf{y})b(\mathbf{y})),
\end{aligned}$$

where in the fifth step we have used $g^\mathbf{k-i}(\mathbf{x})g^\mathbf{i}(\mathbf{x}) = g^\mathbf{k}(\mathbf{x})$ and the definition of multiplication. All the performed operations are legitimate because all the sums involved are finite. □

One of the consequences of Theorem 19 is an alternative way to determine the coefficients of a multiplicative inverse of a series $h(\mathbf{x}) \in \mathbb{K}_C[[\mathbf{x}]]$ with $[\mathbf{x^0}]h(\mathbf{x}) = 1$. For the univariate series $f(y) = \sum_{k\ge0} y^k \in \mathbb{K}[[y]]$ we know $(1-y)f(y) = 1$. Applying Theorem 19 to $g(\mathbf{x}) = 1 - h(\mathbf{x})$ gives $\Phi(1-y)\,\Phi(f(y)) = 1$, so

$$h(\mathbf{x})^{-1} = \sum_{k\ge0}\big(1-h(\mathbf{x})\big)^k.$$

Therefore, in order to determine the coefficient of some term $\mathbf{x^e}$ in $h(\mathbf{x})^{-1}$ we can simply choose a term order $\preccurlyeq$ compatible with C and sum up all the powers $(1-h(\mathbf{x}))^k$ for which $k\operatorname{lexp}(h(\mathbf{x})) \preccurlyeq \mathbf{e}$. The coefficient of $\mathbf{x^e}$ in $h(\mathbf{x})^{-1}$ is then equal to the coefficient of $\mathbf{x^e}$ in this sum.

6. Equations

Finally, we consider the question under which circumstances an equation $f(\mathbf{x}, y) = 0$ can be solved for y in some field of multivariate Laurent series. The results below are variants of the implicit function theorem answering this question. For better readability, we have split the derivation into two theorems, the first serving as a lemma used in the proof of the second. The proof of Theorem 20 follows closely one of the many proofs of the classical implicit function theorem [16]. In Theorem 21 we then relax the hypothesis by making use of the fact that $\mathbb{K}_\preccurlyeq((\mathbf{x}))$ is a field.

Theorem 20. *Let $C \subseteq \mathbb{R}^p$ be a line-free cone, and let*

$$f(\mathbf{x}, y) = \sum_{k=0}^{\infty} a_k(\mathbf{x})y^k \in \mathbb{K}_C[[\mathbf{x}]][[y]]$$

be such that $[\mathbf{x^0}]a_0(\mathbf{x}) = 0$ and $a_1(\mathbf{x}) = 1$. Then there exists exactly one $g(\mathbf{x}) \in \mathbb{K}_C[[\mathbf{x}]]$ with $[\mathbf{x^0}]g(\mathbf{x}) = 0$ and $f(\mathbf{x}, g(\mathbf{x})) = 0$.

Proof. First observe that the composition $f(\mathbf{x}, g(\mathbf{x}))$ is legitimate for every $g(\mathbf{x}) \in \mathbb{K}_C[[\mathbf{x}]]$ whose constant term is zero. For $g(\mathbf{x}) = 0$ this is obvious, and for $g(\mathbf{x}) \neq 0$ it follows from Theorem 17 as follows. Regard $f(\mathbf{x}, y)$ as an element of $\mathbb{K}_{C\times H}[[\mathbf{x}, y]]$, where $H \subseteq \mathbb{R}$ denotes the positive halfline. Note that $C \times H \subseteq \mathbb{R}^{p+1}$ is a line-free cone. Taking $g_1(\mathbf{x}) = x_1, \ldots, g_p(\mathbf{x}) = x_p$, and $g_{p+1}(\mathbf{x}) = g(\mathbf{x})$, we have $M = (I, \mathbf{e}) \in \mathbb{Z}^{p\times(p+1)}$ where I is the identity matrix of size p and $\mathbf{e} = \operatorname{lexp}(g(\mathbf{x}))$. Since H is generated (as cone) by 1, $\ker M$ is generated (as vector space) by $(\mathbf{e}, -1)$, and $\mathbf{e}$ belongs to C, we have $\ker M \cap (C \times H) = \{\mathbf{0}\}$, as required. Because of Theorem 15, there exists a cone $C' \subseteq \mathbb{R}^p$ containing C and $\operatorname{supp}(g(\mathbf{x})/\operatorname{lt}(g(\mathbf{x})))$, and since $\mathbf{e} \in C$ implies $M(C \times H) = C$, this cone C' also contains MC, as required.

Turning to the claim of the theorem, fix some additive order $\preccurlyeq$ on $\mathbb{Z}^p$ which is compatible with C'. Consider an ansatz $g(\mathbf{x}) = \sum_{\mathbf{k}} b_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} \in \mathbb{K}_C[[\mathbf{x}]]$ with $b_{\mathbf{0}} = 0$ and otherwise undetermined coefficients $b_{\mathbf{k}}$. We show by noetherian induction that there is precisely one way of choosing the coefficients $b_{\mathbf{k}}$ such that $[\mathbf{x}^{\mathbf{n}}]f(\mathbf{x}, g(\mathbf{x})) = 0$ for all $\mathbf{n} \succcurlyeq \mathbf{0}$.

Let $\mathbf{n} \succcurlyeq \mathbf{0}$ and suppose as induction hypothesis that the claim is true for every $\mathbf{k} \in \mathbb{Z}^p$ with $\mathbf{0} \preccurlyeq \mathbf{k} \precnsim \mathbf{n}$. The coefficient of $\mathbf{x}^{\mathbf{n}}$ in $f(\mathbf{x}, g(\mathbf{x}))$ is

$$b_{\mathbf{n}} + [\mathbf{x}^{\mathbf{n}}]a_0(x) + \sum_{k=2}^{\infty}[\mathbf{x}^{\mathbf{n}}]a_k(\mathbf{x})g(\mathbf{x})^k.$$

The terms $[\mathbf{x}^{\mathbf{n}}]a_k(\mathbf{x})g(\mathbf{x})^k$ only depend on coefficients $b_{\mathbf{k}}$ with $\mathbf{k} \precnsim \mathbf{n}$, because $\operatorname{lexp}(a_k(\mathbf{x})) \succcurlyeq \mathbf{0}$ and $k \geq 2$ and $\operatorname{lexp}(g(\mathbf{x})) \succnsim \mathbf{0}$ together imply

$$\operatorname{lexp}(a_k(\mathbf{x})) + \mathbf{k} + (k-1)\operatorname{lexp}(g(\mathbf{x})) \succnsim \mathbf{n}$$

for every $\mathbf{k} \succcurlyeq \mathbf{n}$, and the expression on the left hand side denotes the smallest possible exponent vector for which the corresponding coefficient may depend on $b_{\mathbf{k}}$. By assumption, the coefficients $b_{\mathbf{k}}$ for $\mathbf{k} \precnsim \mathbf{n}$ are uniquely determined, and hence in order to have $[\mathbf{x}^{\mathbf{n}}]f(\mathbf{x}, g(\mathbf{x})) = 0$, there is one and only one choice for $b_{\mathbf{n}}$, as claimed. $\square$

Theorem 21. *Let $\preccurlyeq$ be an additive order on $\mathbb{Z}^p$, let $C \subseteq \mathbb{R}^p$ be a cone compatible with $\preccurlyeq$, and let*

$$f(\mathbf{x}, y) = \sum_{n=0}^{\infty} a_n(\mathbf{x})y^n \in \mathbb{K}_C[[\mathbf{x}]][[y]]$$

be such that $a_1(\mathbf{x}) \neq 0$, $\operatorname{lexp}(a_1(\mathbf{x})) \precnsim \operatorname{lexp}(a_0(\mathbf{x}))$, and $\operatorname{lexp}(a_1(\mathbf{x})) \preccurlyeq \operatorname{lexp}(a_n(\mathbf{x}))$ for all $n \in \mathbb{N}$ with $a_n(\mathbf{x}) \neq 0$. Then there exists exactly one $g(\mathbf{x}) \in \mathbb{K}_{\preccurlyeq}[[\mathbf{x}]]$ with $[\mathbf{x}^{\mathbf{0}}]g(\mathbf{x}) = 0$ and $f(\mathbf{x}, g(\mathbf{x})) = 0$.

Proof. Because of Theorem 11, we have $f(\mathbf{x}, g(\mathbf{x})) = 0$ if and only if $u(\mathbf{x})f(\mathbf{x}, g(\mathbf{x})) = 0$ for every $u(\mathbf{x}) \in \mathbb{K}_{\preccurlyeq}((\mathbf{x})) \setminus \{0\}$. It is therefore sufficient to prove the theorem for

$$\tilde{f}(\mathbf{x}, y) := \sum_{k=0}^{\infty} \tilde{a}_k(\mathbf{x})y^k := a_1(\mathbf{x})^{-1} f(\mathbf{x}, y) \in \mathbb{K}_{\preccurlyeq}((x))[[y]]$$

in place of $f(\mathbf{x}, y)$. We show that $\tilde{f}(\mathbf{x}, y)$ satisfies the requirements of Theorem 20. To do so, we need to show that $[\mathbf{x}^{\mathbf{0}}]\tilde{a}_0(\mathbf{x}) = 0$, $\tilde{a}_1(\mathbf{x}) = 1$, and that there is some line-free cone $\tilde{C} \subseteq \mathbb{R}^p$ such that $\operatorname{supp}\tilde{a}_k(\mathbf{x}) \subseteq \tilde{C}$ for all $k \geq 0$.

Since $\tilde{a}_k(\mathbf{x}) = a_1(\mathbf{x})^{-1}a_k(\mathbf{x})$ for all $k \in \mathbb{N}$ by definition, it is immediate that $\tilde{a}_1(\mathbf{x}) = 1$, and that $\tilde{a}_k(\mathbf{x}) = 0$ for every $k \in \mathbb{N}$ with $a_k(\mathbf{x}) = 0$. Furthermore, $\operatorname{lexp}(a_1(\mathbf{x})) \not\preccurlyeq \operatorname{lexp}(a_0(\mathbf{x}))$ implies $\operatorname{lexp}(\tilde{a}_0(\mathbf{x})) \not\preccurlyeq \mathbf{0}$, which in turn implies $[\mathbf{x}^{\mathbf{0}}]a_0(\mathbf{x}) = 0$. For $k \geq 2$ with $a_k(\mathbf{x}) \neq 0$, we have by assumption that $\operatorname{lexp}(\tilde{a}_k(\mathbf{x})) = \operatorname{lexp}(a_k(\mathbf{x})) - \operatorname{lexp}(a_1(\mathbf{x})) \succcurlyeq \mathbf{0}$. Lemma 4 applied to $S := \{\operatorname{lexp}(a_k(\mathbf{x})) \mid k \geq 2 \text{ with } a_k(\mathbf{x}) \neq 0\}$ yields a finite subset $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ of S such that $S \subseteq \bigcup_{i=1}^{n}(\mathbf{s}_i + C)$. Let $\tilde{C}$ be the cone generated by C, some $\preccurlyeq$-compatible cone containing $\operatorname{supp}(a_1(\mathbf{x})^{-1}\operatorname{lt}(a_1(\mathbf{x})))$, and $\mathbf{s}_1 - \operatorname{lexp}(a_1(\mathbf{x})), \ldots, \mathbf{s}_n - \operatorname{lexp}(a_1(\mathbf{x}))$. Then $\tilde{C}$ is finitely generated, compatible with $\preccurlyeq$ (hence also line-free; cf. Lemma 7), and contains $\operatorname{supp} \tilde{a}_k(\mathbf{x})$ for all $k \geq 2$. Therefore, by Theorem 20, there exists exactly one $g(\mathbf{x}) \in \mathbb{K}_{\tilde{C}}[[\mathbf{x}]]$ with $\tilde{f}(\mathbf{x}, g(\mathbf{x})) = 0$. Since Theorem 20 still applies if we replace $\tilde{C}$ by any larger cone which is compatible with $\preccurlyeq$, it follows that there is exactly one $g(\mathbf{x}) \in \mathbb{K}_{\preccurlyeq}[[\mathbf{x}]]$ with $\tilde{f}(\mathbf{x}, g(\mathbf{x})) = 0$, as was to be shown. □

The main restriction in the above theorems is that we only allow positive powers of y in $f(\mathbf{x}, y)$. We may equivalently allow only negative powers of y, but we have not been able to come up with a version of the implicit function theorem that is applicable to series $f(\mathbf{x}, y) \in \mathbb{K}_C[[\mathbf{x}, y]]$ where $C \subseteq \mathbb{R}^{p+1}$ is such that its projection to the last coordinate is the full real line. Note that there is no such restriction, not even implicitly, in Theorem 17: it may well be possible that $f(\mathbf{x}, g(\mathbf{x}))$ can be formed even when $f(\mathbf{x}, y)$ contains infinitely many positive and negative powers of y. On the other hand, the following examples show that for such $f(\mathbf{x}, y)$ there may be more than one solution $g(\mathbf{x})$ with $f(\mathbf{x}, g(\mathbf{x})) = 0$, or no solution at all. This indicates that a naive generalization of the implicit function theorem to such series will be false.

Example 22.
- Consider the series

$$f(x, y) = \sum_{n=1}^{\infty}(-y)^n + \sum_{n=1}^{\infty} x^{2n}y^{-n}.$$

This series belongs to $\mathbb{K}_C[[x, y]]$ where $C \subseteq \mathbb{R}^2$ is the cone generated by $\binom{0}{1}$ and $\binom{2}{-1}$. Because of $[x^0y^0]f(x, y) = 0$ and $[x^0y^1]f(x, y) = -1 \neq 0$, we might expect that some suitable version of the implicit function theorem guarantees the existence of a unique series $g(x) \in \mathbb{K}[[x]]$ with $f(x, g(x)) = 0$. However, it turns out that there are two different solutions:

$$g_1(x) = x^2 + x\sqrt{1 + x^2} = x + x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^5 + \cdots \in \mathbb{K}[[x]]$$

$$\text{and} \quad g_2(x) = x^2 - x\sqrt{1 + x^2} = -x + x^2 - \frac{1}{2}x^3 + \frac{1}{8}x^5 + \cdots \in \mathbb{K}[[x]],$$

where $\sqrt{1 + x^2} = \sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n x^{2n}$.

- Now consider the series

$$f(x, y) = \sum_{n=1}^{\infty}(-y)^n + 2\sum_{n=1}^{\infty} x^{2n}y^{-n},$$

which belongs to the same ring $\mathbb{K}_C[[x, y]]$ as before.

Suppose that there is a nonzero $g(x)$ with $f(x, g(x)) = 0$. If x^e is the leading term of $g(x)$, then in the notation of Theorem 17, we have $M = (1\ e) \in \mathbb{Z}^{1\times 2}$, and MC is the cone generated by e and $2 - e$ in $\mathbb{R}$. In order for this cone to be line-free, we must either have $e \geq 0$ and $2 - e \geq 0$ or $e \leq 0$ and $2 - e \leq 0$. The only candidates for $e \in \mathbb{Z}$ are therefore $e = 0$ or $e = 1$ or $e = 2$.

But $e = 0$ would imply $\binom{0}{1} \in C \cap \ker M$, so this case is excluded. Likewise, $e = 2$ would imply $\binom{2}{-1} \in C \cap \ker M$, so this case is excluded as well and the only remaining possibility for a solution $g(x)$ is that its leading term is x^1 if we want to use Theorem 17 to secure the existence of $f(x, g(x))$.

Make an ansatz $g(x) = a_1 x + \cdots \in \mathbb{K}[[x]]$ for the leading coefficient a_1 of $g(x)$. Then $g(x)^n = a_1^n x^n + \cdots$ and $g(x)^{-n} = a_1^{-n} x^{-n} + \cdots$ for all $n \in \mathbb{N}$. Therefore, equating coefficients of x^1 in

$$f(x, g(x)) = \sum_{n=1}^{\infty} (-g(x))^n + 2 \sum_{n=1}^{\infty} x^{2n} g(x)^{-n} \stackrel{!}{=} 0$$

forces $-a_1 + 2a_1^{-1} = 0$, viz. $a_1^2 = 2$. Depending on the ground field $\mathbb{K}$, this equation may or may not have a solution. For example, if $\mathbb{K} = \mathbb{Q}$, no such a_1 exists, and hence no $g(x) \in \mathbb{Q}[[x]]$ with $f(x, g(x)) = 0$ exists.

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