Two-way Parikh Automata

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Abstract

Parikh automata extend automata with counters whose values can only be tested at the end of the computation, with respect to membership into a semi-linear set. Parikh automata have found several applications, for instance in transducer theory, as they enjoy decidable emptiness problem.

In this paper, we study two-way Parikh automata. We show that emptiness becomes undecidable in the non-deterministic case. However, it is PSPACE-C when the number of visits to any input position is bounded and the semi-linear set is given as an existential Presburger formula. We also give tight complexity bounds for the inclusion, equivalence and universality problems. Finally, we characterise precisely the complexity of those problems when the semi-linear constraint is given by an arbitrary Presburger formula.

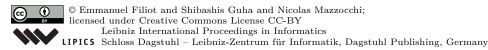
1 Introduction

Parikh automata, introduced in [17], extend finite automata with counters in \mathbb{Z} which can be incremented and decremented, but the counters can only be tested at the end of the computation, for membership in a semi-linear set (represented for instance as an existential Presburger formula). More precisely, transitions are of the form (q, σ, \vec{v}, q') where q, q' are states, σ is an input symbol and $\vec{v} \in \mathbb{Z}^d$ is a vector of dimension d. A word w is accepted if there exists a run ρ on w reaching an accepting state and whose final vector (the componentwise sum of all vectors along ρ) belongs to a given semi-linear set. Parikh automata strictly extend the expressive power of finite automata. For example, the context-free language of words of the form a^nb^n is definable by a deterministic Parikh automaton which checks membership in a^*b^* , counts the number of occurrences of a and b, and at the end tests for equality of the counters, i.e. membership in the linear set $\{(n,n) \mid n \in \mathbb{N}\}$. They still enjoy decidable, NP-C, non-emptiness problem [8].

Parikh automata (PA) have found applications for instance in *transducer* theory, in particular to the equivalence problem of functional transducers on words, and to check structural properties of transducers [9], as well as in answering queries in graph databases [8]. Extensions of Parikh automata with a pushdown stack have been considered in [16] with positive decidability results with respect to emptiness. Two-way Parikh automata with a visibly pushdown stack have been considered in [6] with applications to tree transducers.

In this paper, our objective is to study two-way Parikh automata (2PA), the extension of PA with a two-way input head, where the semi-linear set is given by an existential Presburger formula. For 2PA as well as subclasses such as deterministic 2PA (2DPA), we aim

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at characterizing the precise complexity of their decision problems (membership, emptiness, inclusion, equivalence), and analysing their expressiveness and closure properties.

Contributions Since semi-linear sets are closed under all Boolean operations, it is easily seen that deterministic Parikh automata (DPA) are closed under all Boolean operations. More interestingly, it is also known that, while they strictly extend the expressive power of DPA, unambiguous PA (UPA) are (non-trivially) closed under complement (as well as union and intersection) [2]. We give here a simple explanation to these good closure properties: UPA effectively correspond to 2DPA. Closure of 2DPA under Boolean operations indeed holds straightforwardly due to determinism. The conversion of UPA to 2DPA is however non-trivial, but is obtained by the very same result on word transducers: it is known that unambiguous finite transducers are equivalent to two-way deterministic finite transducers [20], based on a construction by Aho, Hopcroft and Ullman [1], recently improved by one exponential in [7]. Parikh automata can be seen as transducers producing sequences of vectors (the vectors occurring on their transitions), hence yielding the result. The conversion of 2DPA to UPA is a standard construction based on crossing sections, which however needs to be carefully analysed for complexity purposes.

The effective equivalence between 2DPA and UPA indeed entails decidability of the non-emptiness problem for 2DPA. However, given that non-emptiness of PA is known to be NP-C [8], and the conversion of 2DPA to UPA is exponential, this leads to NEXP complexity. By a careful analysis of this conversion and small witnesses properties of Presburger formulas, we show that emptiness of 2DPA, and even bounded-visit 2PA, is actually PSPACE-C. Bounded-visit 2PA are non-deterministic 2PA such that for some natural number k, each position of an input word k is visited at most k times by any accepting computation on k. In particular, 2DPA are always k-visit for k the number of states. If the number k of visits is a fixed constant, non-emptiness is then NP-C, which entails complexity result of [8] for (one-way) PA (by taking k = 1). We show that dropping the bounded-visit restriction however leads to undecidability.

Thanks to the closure properties of 2DPA, we show that the inclusion, universality and equivalence problems are all CONEXP-C. Those problems are known be undecidable for PA [17]. The membership problem of 2PA turns out to be NP-C, just as for (one-way) PA. The CONEXP lower bound holds for one-way deterministic Parikh automata, a result which is also new, to the best of our knowledge.

Finally, we study the extension of two-way Parikh automata with a semi-linear set defined by a Σ_i -Presburger formula, i.e. a formula with a fixed number i of unbounded blocks of quantifiers where the consecutive blocks alternate i-1 times between existential and universal blocks, and the first block is existential. We characterise tightly the complexity of the non-emptiness problem for bounded-visit Σ_i -2PA, as well as the universality, inclusion and equivalence problems for Σ_i -2DPA, in the weak exponential hierarchy [12]. For i>1, we find that the complexity of these problems is dominated by the complexity of checking satisfiability or validity of Σ_i -Presburger formulas. This is unlike the case i=1: the non-emptiness problem for bounded-visit 2PA is PSPACE-C while satisfiability of Σ_1 -formulas is NP-C.

Related work Parikh automata are known to be equivalent to reversal-bounded multicounter machines (RBCM) [15] in the sense that they describe the same class of languages [2]. Two-way RBCM (2RBCM), even deterministic, are known to have undecidable emptiness problem [15]. While, using diophantine equations as in the case of [15], we show that emptiness of 2PA is undecidable, our decidability result for 2DPA contrasts with the undecidability of deterministic

2RBCM. The difference is that 2RBCM can test their counters at any moment during a computation, and not only at the end. Based on the fact that the number of reversals is bounded, deferring the tests at the end of the computation is always possible [15] but non-determinism is needed. Unlike 2DPA, deterministic 2RBCM are not necessarily bounded-visit. A 2DPA can be seen as a deterministic 2RBCM whose tests on counters are only done at the end of a computation.

Two-way Parikh automata on nested words have been studied in [6] where it is shown that under the single-use restriction (a generalisation of the bounded-visit restriction to nested words), they have NEXP-C non-emptiness problem. Bounded-visit 2PA are a particular case of those Parikh automata operating on (non-nested) words. Applying the result of [6] to 2PA would yield a non-optimal NEXP complexity for the non-emptiness problem, as it first goes through an explicit but exponential transformation into a one-way machine with known NP-C non-emptiness problem. Here instead, we rely on a small witness property, whose proof uses a transformation into one-way Parikh automaton, and then we apply a PSPACE algorithm performing on-the-fly the one-way transformation up to some bounded length.

Finally, the emptiness problem for the intersection of n PA was shown to be PSPACE-C in [8]. Our PSPACE-C result on 2PA emptiness generalises this result, as the intersection of n PA can be simulated trivially by a (sweeping) n-bounded 2PA. The main lines of our proof are similar to those in [8], but in addition, it needs a one-way transformation on top of the proof in [8], and a careful analysis of its complexity.

2 Two-way Parikh automata

Two-way Parikh automata are two-way automata extended with weight vectors and a semi-linear acceptance condition. In this section, we first define two-way automata, semi-linear sets and then two-way Parikh automata.

Two-way Automata A two-way finite automaton (2FA for short) A over an alphabet Σ is a tuple (Q,Q_I,Q_H,Q_F,Δ) whose components are defined as follows. We let \vdash and \dashv be two delimiters not in Σ , intended to represent the beginning and the end of the word respectively. The set Q is a non-empty finite set of states partitioned into the set of right-reading states Q^R and the set of left-reading states Q^L . Then, $Q_I \subseteq Q^R$ is the set of initial states, $Q_H \subseteq Q$ is the set of halting states, and $Q_F \subseteq Q_H$ is the set of accepting states. The states belonging to $Q_H \setminus Q_F$ are said to be rejecting. Finally, $\Delta \subseteq Q \times (\Sigma \cup \{\vdash, \dashv\}) \times Q$ is the set of transitions. Intuitively, the reading head of A is always placed in between input positions, a transition from $q \in Q^R$ (resp. $q \in Q^L$) reads the input letter on the right (resp. left) of the head and moves the head one step to the right (resp. left). We also have the following restrictions on the behaviour of the head to keep it in between the boundaries \vdash and \dashv and to ensure the following properties on the initial and the halting states.

- 1. no outgoing transition from a halting state: $(Q_H \times (\Sigma \cup \{\vdash, \dashv\}) \times Q) \cap \Delta = \emptyset$
- 2. the head cannot move left (resp. right) when it is to the left of \vdash (resp. right of \dashv): $(Q^{\mathsf{L}} \times \{\vdash\} \times Q^{\mathsf{L}}) \cap \Delta = \emptyset$ (resp. $(Q^{\mathsf{R}} \times \{\dashv\} \times (Q^{\mathsf{R}} \setminus Q_F)) \cap \Delta = \emptyset$)
- **3.** all transitions leading to a halting state q_H read the delimiter \dashv : $((q, a, q_H) \in \Delta \land q_H \in Q_H) \implies (q \in Q^R \land a = \dashv)$

A configuration $(u^{\mathsf{L}}, p, u^{\mathsf{R}})$ of A on a word $u \in \Sigma^*$ consists of a state p and two words $u^{\mathsf{L}}, u^{\mathsf{R}} \in (\Sigma \cup \{\vdash, \dashv\})^*$ such that $u^{\mathsf{L}}u^{\mathsf{R}} = \vdash u\dashv$. A $run\ \rho$ on a word $u \in \Sigma^*$ is a sequence $\rho = (u_0^{\mathsf{L}}, q_0, u_0^{\mathsf{R}})a_1(u_1^{\mathsf{L}}, q_1, u_1^{\mathsf{R}})\dots a_n(u_n^{\mathsf{L}}, q_n, u_n^{\mathsf{R}})$ alternating between configurations on u and

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letters in $\Sigma \cup \{\vdash, \dashv\}$ such that for all $1 \leq i \leq n$, we have $(q_{i-1}, a_i, q_i) \in \Delta$, and for all $s \in \{\mathsf{L}, \mathsf{R}\}$, if $q_{i-1} \in Q^s$ then $|u_i^s| = |u_{i-1}^s| - 1$. The length of the run ρ , denoted $|\rho|$ is the number of letters appearing in ρ . Here $|\rho| = n$. The run ρ is halting if $q_n \in Q_H$ (and hence $u_n^R = \varepsilon$ by condition 3), initial if $u_0^L = \varepsilon$ and $q_0 \in Q_I$, accepting if it is both initial and halting, and $q_n \in Q_F$; otherwise the run is rejecting. A word u is accepted by A if there exists an accepting run of A on $\vdash u \dashv$, and the language L(A) of A is defined as the set of words it accepts.

An automaton A is said to be one-way (FA) if Q^{L} is empty. A run ρ is said to be k-visit if every input position is visited at most k times in the run ρ , i.e. for $\rho = (u_0^{\mathsf{L}}, q_0, u_0^{\mathsf{R}}) \dots (u_n^{\mathsf{L}}, q_n, u_n^{\mathsf{R}})$, we have $\max\{|P| \mid P \subseteq \{0, \dots, n\} \land \forall i, j \in P, \ u_i^{\mathsf{L}} = u_j^{\mathsf{L}}\} \leq k$. A is said to be k-visit if all its accepting runs are k-visit, and bounded-visit if it is k-visit for some k. Also, A is said to be deterministic if for all $p \in Q$ and all $a \in \Sigma \cup \{\vdash, \dashv\}$ there exists at most one $q \in Q$ such that $(p, a, q) \in \Delta$. Finally, it is unambiguous (denoted by the class 2UFA or UFA depending on whether it is two-way or one-way) if for every input word there exists at most one accepting run. The following proposition is trivial but useful:

▶ **Proposition 2.1.** Any bounded-visit 2FA with n states is k-visit for some $k \leq n$.

Semi-linear Sets Let $d \in \mathbb{N}_{\neq 0}$. A set $L \subseteq \mathbb{Z}^d$ of dimension d is linear if there exist $\vec{v}_0, \ldots, \vec{v}_k \in \mathbb{Z}^d$ such that $L = \{\vec{v}_0 + \sum_{i=1}^k x_i \vec{v}_i \mid x_1, \ldots, x_n \in \mathbb{N}\}$. The vectors $(\vec{v}_i)_{1 \leq i \leq k}$ are the periods and \vec{v}_0 is called the base, forming what we call a period-base representation of L, whose size is $d \cdot (k+1) \cdot \log_2(\mu+1)$ where μ is the maximal absolute integer appearing on the vectors. A set is semi-linear if it is a finite union of linear sets. A period-base representation of a semi-linear set is given by a period-base representation for each of the linear sets it is composed of, and its size is the sum of the sizes of all those representations.

Alternatively, a semi-linear set of dimension d can be represented as the models of a Presburger formula with d free variables. A Presburger formula is a first-order formula built over terms t on the signature $\{0,1,+,\times_2\} \cup X$, where X is a countable set of variables and \times_2 denotes the doubling (unary) function⁴. In particular, Presburger formulas obey the following syntax:

$$\Phi \stackrel{\mathrm{def}}{=} t \leq t \mid \exists x \; \Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \neg \Phi$$

The class of formulas of the form $\exists x_1, \forall x_2 \dots, \Omega_i x_i \ [\varphi]$ where φ is quantifier free and $\Omega \in \{\forall, \exists\}$ is denoted by Σ_i . In particular, Σ_1 is the set of existential Presburger formulas. The size $|\Psi|$ of a formula is its number of symbols. We denote by $\vec{v} \models \varphi$ the fact that a vector \vec{v} of dimension d satisfies a formula φ with d free variables, and that φ is satisfiable is there exists such \vec{v} . We say that φ is valid if it is satisfied by any \vec{v} . It is well-known [11] that a set $S \subseteq \mathbb{Z}^d$ is semi-linear iff there exists an existential Presburger formula ψ with d free variables such that $S = \{\vec{v} \mid \vec{v} \models \psi\}$.

Let $\Sigma = \{a_1, \ldots, a_n\}$ be an alphabet (assumed to be ordered), and $u \in \Sigma^*$, the Parikh image of u is defined as the vector $\mathfrak{P}(u) = (|u|_{a_1}, \ldots, |u|_{a_n})$ where $|u|_a$ denotes the number of times a occurs in u. The Parikh image of language $L \subseteq \Sigma^*$ is $\mathfrak{P}(L) = \{\mathfrak{P}(u)|u \in L\}$. Parikh's theorem states that the Parikh image of any context-free language is semi-linear.

Two-way Parikh automata A two-way Parikh automaton (2PA) of dimension $d \in \mathbb{N}$ over Σ is a tuple $P = (A, \lambda, \psi)$ where $A = (Q, Q_I, Q_H, Q_F, \Delta)$ is a 2FA over Σ , $\lambda : \Delta \to \mathbb{Z}^d$ maps transitions to vectors, and ψ is an *existential* Presburger formula with d free variables, and

⁴ The function \times_2 is syntactic sugar allowing us to have simpler binary encoding of values

is called the acceptance constraint. The value $V(\rho)$ of a run ρ of A is the sum of the vectors occurring on its transitions, with $V(\rho) = 0_{\mathbb{Z}^d}$ if $|\rho| = 0$. A word is accepted by P if it is accepted by some accepting run ρ of A and $V(\rho) \models \psi$. The language L(P) of P is the of words it accepts. The automaton P is said to be one-way, two-way, k-visit, unambiguous and deterministic if its underlying automaton P is so. We define the representation size of P as $|P| = |Q| + |\psi| + |\operatorname{range}(\lambda)| (d \log_2(\mu + 1) + |Q|^2)$ where $\operatorname{range}(\lambda) = {\lambda(t) \mid t \in \Delta}$ and P is the maximal absolute entries appearing in weight vectors of P. Finally two 2PA are equivalent if they accept the same language.

Examples Let $\Sigma = \{a,b,c,\#\}$ and for all $n \in \mathbb{N}$, let $L_n = \{a^k \# u \mid u \in \{b,c\}^* \land k = |\{i \mid 1 \le i \le |u| - n \land u[i] \ne u[i+n]\}|\}$, i.e. k is the number of positions i in u such that the ith letter u[i] mismatches with u[i+n]. For all n, L_n is accepted by the 2DPA of Fig. 1 which has O(n) states, tagged with R or L to indicate whether they are right- or left-reading respectively. On a word w, the automaton starts by reading a^k and increments its counter to store the value k (state q^a). Then, for the first |u| - n positions i of u, the automaton checks whether $u[i] \ne u[i+n]$ in which case the counter is decremented. To do so, it stores $\sigma = u[i]$ in its state, moves n+1 times to the right (states $q_0, q_1^{\sigma}, \ldots, q_n^{\sigma}$), checks whether $u[i+n] \ne u[i]$ (transitions q_n^{σ} to p_1) and decrements the counter accordingly. Then, it moves n times to the left (states p_1 to p_n). Whenever it reads \dashv from states q_j^{σ} , p_j or q_0 , it moves to state q_F and accepts if the counter is zero.

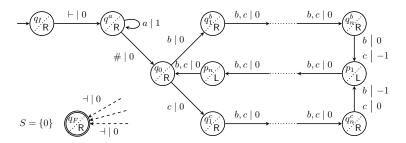


Figure 1 A 2DPA recognising $L_n = \{a^k \# u \mid u \in \{b, c\}^* \land k = |\{i \mid 1 \le i \le |u| - n \land u[i] \ne u[i+n]\}\}$

Our second example shows how to encode multiplication. The language $\{a^n\#a^m\#a^{n\times m}\mid n,m\in\mathbb{N}\}$ is indeed definable by the 2PA of Figure 2 which has dimension 2. When reading a word of the form $a^n\#a^m\#a^\ell$, every accepting run makes p passes over a^n where p is chosen non-deterministically by the choice made on state q_1 on reading #. Along those k passes, the automaton increments the first dimension whenever a is read in a right-to-left pass. It also counts the number of passes in the second dimension. Thus, when entering state q_2 , the sum of the vectors so far is (np,p). Then, on a^m , it decrements the second dimension and on a^ℓ , it decrements the first dimension, and eventually checks that both the counters are equal to zero, which implies that p=m and $\ell=np=nm$. Note that this automaton is not bounded-visit as its number of visits to any position of a^n is arbitrary.

Note that weight vectors are not memorized on transition but into a table and transition only carry a key of this table to refer the corresponding weight vectors

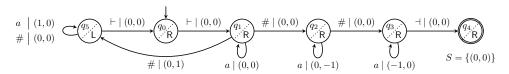


Figure 2 A 2PA recognising $\{a^n \# a^m \# a^{n \times m} \mid n, m \in \mathbb{N}\}$

3 Relating two-way and one-way Parikh automata

In this section, we provide an algorithm which converts a bounded-visit 2PA into a PA defining the same language, through a $crossing\ section$ construction. This technique is folkloric in the literature (see Section 2.6 of [14]) and has been introduced to convert a 2FA into an equivalent FA. Intuitively, the one-way automaton is constructed such that on each position i of the input word, it guesses a tuple of transitions (called $crossing\ section$), triggered by the original two-way automaton at the same position i and additionally checks a local validity between consecutive tuples (called $matching\ property$). A one-way automaton takes $crossing\ sections\ as\ set\ of\ states$. Furthermore, the matching property is defined to ensure that the sequence of $crossing\ sections\ which\ successively\ satisfy\ it,\ correspond\ to\ the\ sequence\ of\ crossing\ sections\ of\ an\ accepting\ two-way\ run.$ Thanks to the commutativity of +, the order in which weights are combined by the two-way automaton does not matter and therefore, transitions of the one-way automaton are labelled by summing the weights of transitions of the crossing section. Formally, we define a crossing section as follows:

▶ Definition 3.1 (crossing section). Let $k \in \mathbb{N}_{\neq 0}$. Consider a k-visit 2PA A over Σ and $a \in \Sigma \cup \{\vdash, \dashv\}$. An a-crossing section is a sequence $c = (p_1, a, q_1) \dots (p_\ell, a, q_\ell) \in \Delta^+$ such that $1 \leq \ell \leq k$, $p_1, q_\ell \in Q^R$ and for all $m \in \{L, R\}$, $p_i \in Q^m \implies p_{i+1} \notin Q^m$. We define the value of c as $V(c) = \sum_{i=1}^{\ell} \lambda(p_i, a, q_i)$, and its length $|c| = \ell$. From the sequence $s = p_1q_2p_3 \dots q_{\ell-1}p_\ell$, the L-anchorage of c is defined by $p_1f(q_2, p_3) \dots f(q_{\ell-1}, p_\ell)$ where $f(q_i, p_{i+1}) = \varepsilon$ if $q_i = p_{i+1}$ and $q_i \in Q^R$, otherwise $f(q_i, p_{i+1}) = q_ip_{i+1}$. The R-anchorage of c is defined dually⁶. Furthermore, c is said to be initial if its L-anchorage is $p_1 \in Q_I$. Dually, c is said to be accepting if its R-anchorage is $q_\ell \in Q_F$.

Given a run ρ of a 2PA over u and a position $1 \le i \le |u|$, the crossing section of ρ at position i is defined as the sequence of all transitions triggered by ρ when reading the ith letter, taken in the order of appearance in ρ . We also define the crossing section sequence $\mathcal{C}(r)$ as the sequence of crossing sections of ρ from position 1 to |u|. Note that the first crossing section is initial and the last crossing section of ρ is accepting if ρ is accepting.

- ▶ Example 3.2. Figure 3, shows a run over the word $\vdash ab \dashv$. Consider the a-crossing section $c = (q_2, a, q_3)(q_3, a, q_4)(q_4, a, q_5)(q_{11}, a, q_{12})(q_{12}, a, q_{13})$. We have that L-anchorage of c is $q_2f(q_4, q_4)f(q_{12}, q_{12}) = q_2$, R-anchorage of c is $f(q_3, q_3)f(q_5, q_{11})q_{13} = q_5q_{11}q_{13}$ and $V(c) = \vec{v}_2 + \vec{v}_3 + \vec{v}_4 + \vec{v}_{11} + \vec{v}_{12}$. Note that, the states of the crossing section do not appear in the anchorage when the run changes its reading direction.
- ▶ Definition 3.3 (matching relation). Consider two crossing sections c_1, c_2 from the same automaton. The matching relation M is defined such that $(c_1, c_2) \in M$ if the R-anchorage of c_1 equals the L-anchorage of c_2 .

From $s = q_1 p_2 \dots q_{\ell-2} p_{\ell-1} q_{\ell}$, we define $f(q_1 p_2) \dots f(q_{\ell-2} p_{\ell-1}) q_{\ell}$ where $f(q_i, p_{i+1}) = \varepsilon$ if $q_i = p_{i+1}$ and $q_i \in Q^{\mathsf{L}}$ otherwise $f(q_i, p_{i+1})$ is the identity

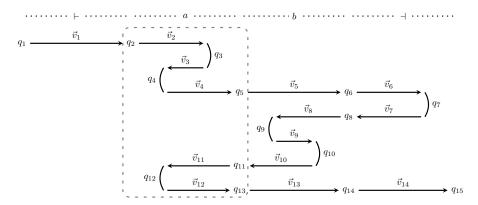


Figure 3 A *a*-crossing section of a run

In general, an arbitrary sequence of crossing sections may not correspond to a run of a two-way automaton, that is a crossing section sequence $s = c_1, \ldots, c_\ell$ such that $\mathcal{C}(r) \neq s$ for all run ρ . Lemma 3.4 shows that the matching property ensures the existence of such a run ρ in the two-way automaton.

- ▶ Lemma 3.4. Consider $s = c_1, \ldots, c_n$ where c_i is an a_i -crossing section such that c_1 is initial, c_n is accepting, and $(c_i, c_{i+1}) \in M$ for all $i \in \{1, \ldots, n-1\}$. Then there exists an accepting two-way run ρ over $a_1 \ldots a_n$ such that $C(\rho) = s$. Moreover, $V(r) = \sum_{i=1}^n V(c_i)$.
- ▶ **Theorem 3.5.** Let $k \in \mathbb{N}_{\neq 0}$. Given a k-visit 2PA P, one can effectively construct a language equivalent PA R that is at most exponentially bigger. Furthermore, if P is deterministic then R is unambiguous.

Proof. Let $P = (A, \lambda, \psi)$ with $A = (Q, Q_I, Q_H, Q_F, \Delta)$ be a k-visit 2PA of dimension d with n = |Q| states. In this proof we show how to construct $R = (B, \omega, \psi)$ where $B = (V, V_I, V_H, V_F, \Gamma)$ is a PA of dimension d having $\mathcal{O}(n^{2k})$ states such that $|\operatorname{range}(\omega)| \leq |\operatorname{range}(\lambda)|^{k+1}$. Note that the formula ψ is the same in both P and R.

To do so, we first consider a symbol \top and extend the relation M such that $(c, \top) \in M$ holds for all accepting crossing section c. Then, we define R as follows:

- \blacksquare V is the set of crossing sections of length at most k
- V_I is the set of initial crossing sections and $V_H = V_F = \{\top\}$
- $\Gamma = \{(c_1, a, c_2) \in V \times \Sigma \cup \{\vdash, \dashv\} \times V \mid (c_1, c_2) \in M \land c_1 \text{ is an } a\text{-crossing section} \}$
- $\omega: (c_1, a, c_2) \mapsto V(c_1)$

Similar to the case of 2FA, a word u is accepted by B if there exists an accepting run of B on $\vdash u \dashv$, and the language L(B) of B is defined as the set of words it accepts. The inclusion $L(R) \subseteq L(P)$ is a direct consequence of Lemma 3.4, while the other direction is based on the following observation: any accepting two-way run ρ has a sequence of crossing sections C(r), consecutively satisfying the matching relation. Note that, the choice of c_2 in a transition (c_1, a, c_2) is non-deterministic in general; but when P is deterministic at most one such choice of c_2 will corresponds to a two-way run ensuring unambiguity. Details can be found in Appendix.

The previous crossing section construction permits to construct a one-way automaton from a bounded-visit two-way one. This construction is exponential in the number of states and in the number of distinct weight vectors. Nevertheless, a close inspection of the proof of Theorem 3.5, reveals that the exponential explosion in the number of distinct weight vectors can be avoided, while preserving the non-emptiness (but not the language).

▶ Lemma 3.6. Let P be a k-visit 2PA. We can effectively construct a PA R with $\mathcal{O}(n^{2k})$ states and such that $L(R) = \emptyset$ iff $L(P) = \emptyset$. Furthermore, R has the same set of weight vectors and the same acceptance constraint as P.

Proof. The construction is the same as in Theorem 3.5 but each transition of the one-way automaton $t = (c_1, a, c_2)$ is split into the following $|c_1|$ consecutive transitions, using a fresh symbol $\# \notin \Sigma$: $c_1 \xrightarrow{a} (t, 1) \xrightarrow{\#} (t, 2) \xrightarrow{\#} \dots (t, |c_1| - 2) \xrightarrow{\#} (t, |c_1| - 1) \xrightarrow{\#} c_2$. The vectors of those transitions are defined as follows. If $c_1[i]$ denotes the *i*th transition of c_1 , then the vector of the first R-transition is the vector of the P-transition $c_1[i]$, and the vector of any R-transition from state (t,i) is the vector of the P-transition $c_1[i+1]$. The two languages are then equal modulo erasing # symbols.

▶ Theorem 3.7. Unambiguous Parikh automata have the same expressiveness as two-way deterministic (even reversible⁷) Parikh automata i.e. UPA = 2DPA. Furthermore, the transformation from one formalism to the other can be done in Exp.

Proof. We only show here UPA \subseteq 2DPA. The opposite direction is given by Theorem 3.5. Let $P = (A, \lambda, \psi)$ be a UPA of dimension d over Σ . Consider the alphabet $\Lambda \subseteq \mathbb{Z}^d$ as the set of vectors occurring on the transitions of P. We can see the automaton A with the morphism λ as an unambiguous finite transducer T defining a function from Σ^* to Λ^* . It is known that any unambiguous letter-to-letter one-way transducer can be transformed into an equivalent letter-to-letter deterministic two-way transducer. This result is explicitly stated in Theorem 1 of [20] which is based on a general technique introduced by Aho, Hopcroft and Ullman [1]⁸. Recently, another technique has been introduced which improves AHU's technique by one exponential [7], and allows to show that any unambiguous finite transducer is equivalent to a reversible two-way transducer exponentially bigger, yielding our result.

4 Emptiness Problem

The emptiness problem asks, given a 2PA, whether the language it accepts is empty. We have seen in Example 2 how to encode the multiplication of two natural numbers encoded in unary. We can generalise this to the encoding of solutions of Diophantine equations as languages of 2PA, yielding undecidability:

▶ **Theorem 4.1.** *The emptiness problem for 2PA is undecidable.*

The proof of this theorem relies on the fact that an input position can be visited an arbitrary number of times, due to non-determinism. If instead we forbid this, we recover decidability. To prove it, we proceed in two steps: first, we rely on the result of the previous section showing that any bounded-visit 2PA can be effectively transformed into some (one-way) PA. This yields decidability of the emptiness problem as this problem is known to be decidable for PA. To get a tight complexity in PSPACE, we analyse this transformation (which is exponential), to get exponential bounds on the size of shortest non-emptiness witnesses. A key lemma is the following, whose proof gathers ideas and arguments that already appeared in [19, 8]. Since the statement was not explicit in those papers, and its proofs relies on arguments that appear at different places, we prove it in Appendix.

deterministic and co-deterministic

⁸ Based on AHU's technique, a similar result was shown in [4] for weighted automata, namely that unambiguous weighted automata over a semiring can be equivalently converted into deterministic two-way weighted automata

▶ **Lemma 4.2.** Let P be a one-way Parikh automaton with n states and γ distinct weight vectors. Then, we can construct an existential Presburger formula $\varphi(x) = \bigvee_{i=1}^{m} \varphi_i(x)$ such that for all $\ell \in \mathbb{N}$, $\varphi(\ell)$ holds iff there exists $w \in L(P) \cap \Sigma^{|\ell|}$. Furthermore, $\log_2(m)$ and each φ_i are $O(\operatorname{poly}(|P|, \log n))$, and can be constructed in time $2^{\mathcal{O}(\gamma^2 \log(\gamma n))}$.

Thanks to the lemma above, we are able to show that the non-emptiness problem for bounded-visit 2PA is PSPACE-C, just as the non-emptiness problem for two-way automata. In some sense, adding semi-linear constraints to two-way automata is for free as long as it is bounded-visit.

▶ **Theorem 4.3.** The non-emptiness problem for bounded-visit 2PA is PSPACE-C. It is NP-C for k-visit 2PA when k is fixed.

- **Proof.** Consider a k-visit 2PA $P=(A,\lambda,\psi)$ of dimension d. We start with the PSPACE membership of the non-emptiness problem for 2DFA. Intuitively, we first want to apply Lemma 3.6 in order to deal with a one-way automaton, and apply then Lemma 4.2 to reduce the non-emptiness problem of the one-way Parikh automaton to the satisfiability of an existential Presburger formula. Nevertheless, we cannot explicitly transform P into a one-way automaton while keeping polynomial space. So, in the sequel, (i) we highlight an upper bound on the smallest witness of non-emptiness and based on it, (ii) we provide an NPSPACE algorithm which decides if there exists such a witness.
- (i) By Lemma 4.2 applied on the PA obtained from Lemma 3.6, there exists an existential Presburger formula $\varphi(\ell) = \bigvee_{i=1}^m \varphi_i(\ell)$ where each φ_i is polynomial in |P|. This formula is satisfiable iff there exists $w \in \Sigma^{|\ell|}$ such that $w \in L(P)$. By Theorem 6 (A) of [21], there exists N exponential in $|\varphi_i|$ such that φ_i is satisfiable iff $\varphi_i(\ell)$ holds for some $0 \le \ell \le N$. Hence, there exists N exponential in |P| such that $\min\{|u| \mid u \in L(P)\} \le N$.
- (ii) The algorithm guesses a witness u of length at most N on-the-fly and a run on it. It controls its length by using a binary counter: as N is exponential in |P|, the memory needed for that counter is polynomial in |P|. The transitions of the one-way automaton obtained from Lemma 3.6 can also be computed on-demand in polynomial space. Eventually, it suffices to check the last state is accepting and the sum $\vec{v} = (v_1, \ldots, v_d)$ of the vectors computed onthe-fly along the run, satisfies the Presburger formula $\psi(x_1, \ldots, x_d)$. To do so, our algorithm constructs a closed formula $\psi^{\vec{v}}$ in polynomial time such that $\psi^{\vec{v}}$ is true iff $\vec{v} \models \psi$. To do so, it hardcodes the values of \vec{v} in ψ by substituting each x_i by a term t_{v_i} of size $(\log_2(v_i))^2$ encoding v_i , by using the function symbol \times_2 . E.g. $t_{13} = \times_2(\times_2(\times_2(1))) + \times_2(\times_2(1)) + 1$. Let us argue that $\psi^{\vec{v}}$ has polynomial size. Let μ be the maximal absolute entry of vectors of P, then $v_i \leq \mu N$, and since N is exponential in |P|, t_{v_i} has polynomial size in |P| and $\log_2(\mu)$. Hence $\psi^{\vec{v}}$ has polynomial size, and its satisfiability can be checked in NP [21].

The lower bound is direct as it already holds for the emptiness problem of deterministic two-way automata, by a trivial encoding of the PSPACE-C intersection problem of n DFA [18].

When k is fixed, then the conversion to a one-way automaton (Lemma 3.6) is polynomial. Then, the result follows from the NP-C result for the non-emptiness of PA [8].

▶ Remark 4.4. In [8], non-emptiness is shown to be polynomial time for PA when the dimension is fixed, the values in the vectors are unary encoded and the semi-linear constraint is period-base represented. As a consequence, for all fixed d, k, the non-emptiness problem for k-visit 2PA with vectors in $\{0,1\}^d$ and a period-base represented semi-linear constraint can be solved in P.

5 Closure properties and comparison problems

Since the class of 2DPA is equivalent to the class of UPA that is known to be closed under Boolean operations [3, 17], we get the closure properties of 2DPA for free, although with non-optimal complexity. We show here that they can be realised in linear-time for intersection and union, and with linear state-complexity for the complement.

▶ Theorem 5.1 (Boolean closure). Let P, P_1, P_2 be 2DPA such that $P = (A, \lambda, \psi)$. One can construct a 2DPA $\overline{P} = (A', \lambda', \psi')$ such that $L(\overline{P}) = \overline{L(P)}$ and the size of A' is linear in the size of A. One can construct in linear-time a 2DPA P_{\cup} (resp. P_{\cap}) such that $L(P_{\cup}) = L(P_1) \cup L(P_2)$ (resp. $L(P_{\cap}) = L(P_1) \cap L(P_2)$).

Proof. Let us start by intersection, assuming $P_i = (A_i, \lambda_i, \psi_i)$ has dimension d_i . The automaton P_{\cap} is constructed with dimension $d_1 + d_2$. Then P_{\cap} first simulates P_1 on the first d_1 dimensions (with weight vectors belonging to $\mathbb{Z}^{d_1} \times \{0\}^{d_2}$), and then, if P_1 eventually reaches an halting state, it stops if it is non-accepting and reject, otherwise it simulates P_2 on the last d_2 dimensions with vectors in $\{0\}^{d_1} \times \mathbb{Z}^{d_2}$, and accepts the word if the word is accepted by P_2 as well. The Presburger acceptance condition is defined as $\psi(\vec{x}_1, \vec{x}_2) = \psi_1(\vec{x}_1) \wedge \psi_2(\vec{x}_2)$. Note that if P_1 never reaches an halting state, then P_{\cap} won't either, so the word is rejected by both automata. It is also a reason why this construction cannot be used to show closure under union: even if P_1 never reaches an halting state, it could well be the case that P_2 accepts the word, but the simulation of P_2 in that case will never be done. However, assuming that P_1 halts on any input, closure under union works with a similar construction. Additionally, we need to keep in some new counter c the information whether P_1 has reached an accepting state: First P_{\cup} simulates P_1 , if P_1 halts in some accepting state, then c is incremented and P_{\cup} halts, otherwise P_{\cup} proceeds with the simulation of P_2 . The formula is then $\psi(\vec{x}_1, \vec{x}_2, c) = (c = 1 \wedge \psi_1(\vec{x}_1)) \vee \psi_2(\vec{x}_2)$.

So, we have closure under union in linear-time as long as P_1 halts on every input. This can be used to show closure under complement, using the following observation: $\overline{L(P)} = \overline{L(A)} \cup L(A, \lambda, \neg \psi)$ and moreover, it is known that 2DFA can be complemented in linear-time into a 2DFA which always halts [10]. The formula $\neg \psi$ is universal since ψ is existential. Then, $\neg \psi$ could be converted into an equivalent existential formula using quantifier elimination [5].

For the closure under union, we use the equality $L(P_1) \cup L(P_2) = \overline{L(P_1)} \cap \overline{L(P_2)}$. It can be done in linear-time because the formulas for $\overline{P_1}$ and $\overline{P_2}$ are universal, and so is the formula for the 2DPA accepting $\overline{L(P_1)} \cap \overline{L(P_2)}$. By applying again the complement construction, we get an existential formula (without using quantifier eliminations).

Thanks to Theorem 5.1 and decidability of non-emptiness for 2DPA, we easily get the decidability of the universality problem (deciding whether $L(P) = \Sigma^*$), the inclusion problem (deciding whether $L(P_1) \subseteq L(P_2)$), and the equivalence problem (deciding whether $L(P_1) = L(P_2)$) for 2DPA. The following theorem establishes tight complexity bounds. It is a consequence of a more general result (Theorem 6.4) that we establish for Parikh automata with arbitrary Presburger formulas in Section 6.

▶ **Theorem 5.2** (Comparison Problems). *The universality, inclusion and equivalence problems are* CONEXP-C *for 2DPA*.

Finally, we study the membership problem which asks given a Parikh automaton P and a word $w \in \Sigma^*$, whether $w \in L(P)$. Hardness was known already for PA [8].

▶ **Theorem 5.3.** *The membership problem for 2PA is* NP-C.

6 Parikh automata with arbitrary Presburger acceptance condition

In this section, we consider Parikh automata where the acceptance constraint is given as an arbitrary Presburger formula, that is, not restricted to existential Presburger formula, and we study the complexity of their decision problems. For all i > 0, a two-way Σ_i -Parikh automaton (Σ_i -2PA for short) is a tuple $P = (A, \lambda, \Psi)$ where A, λ are defined just as for 2PA and $\Psi \in \Sigma_i$. In particular, a Σ_1 -2PA is exactly a 2PA. Similarly, we also define Σ_i -DPA, Σ_i -PA respectively, and their Π_i counterpart (when the formula is in Π_i).

The complexity of Presburger arithmetic has been connected to the weak EXP hierarchy [13, 12] which resides between NEXP and EXPSPACE is defined as $\bigcup_{i>0} \Sigma_i^{\text{EXP}}$ where:

$$\begin{array}{ll} \boldsymbol{\Sigma}_{0}^{P} \overset{\text{def}}{=} \boldsymbol{\Pi}_{0}^{P} \overset{\text{def}}{=} \boldsymbol{P} & \boldsymbol{\Sigma}_{i+1}^{P} \overset{\text{def}}{=} \boldsymbol{N} \boldsymbol{P}^{\boldsymbol{\Sigma}_{i}^{P}} & \boldsymbol{\Pi}_{i+1}^{P} \overset{\text{def}}{=} \boldsymbol{CoN} \boldsymbol{P}^{\boldsymbol{\Sigma}_{i}^{P}} \\ \boldsymbol{\Sigma}_{0}^{\text{EXP}} \overset{\text{def}}{=} \boldsymbol{\Pi}_{0}^{\text{EXP}} \overset{\text{def}}{=} \boldsymbol{EXP} & \boldsymbol{\Sigma}_{i+1}^{\text{EXP}} \overset{\text{def}}{=} \boldsymbol{NEXP}^{\boldsymbol{\Sigma}_{i}^{P}} & \boldsymbol{\Pi}_{i+1}^{\text{EXP}} \overset{\text{def}}{=} \boldsymbol{CoNEXP}^{\boldsymbol{\Sigma}_{i}^{P}} \end{array}$$

Since Lemma 4.2 uses the acceptance constraint as a black box, we can generalise it as follows.

▶ Lemma 6.1. For any fixed $i \in \mathbb{N}_{\neq 0}$, given a Σ_i -PA P with n states and γ distinct weight vectors, we can construct a Σ_i -formula Φ such that for all $\ell \in \mathbb{N}$ we have that $\Phi(\ell) = \bigvee_{j=1}^m \Phi_j(\ell)$ holds iff there exists $w \in L(P) \cap \Sigma^{|\ell|}$. Furthermore, $\log_2(m)$ and size of each Φ_i are $\operatorname{poly}(|P|, \log n)$, and can be constructed in time $2^{\mathcal{O}(\gamma^2 \log(\gamma^n))}$.

Using Lemma 6.1, we can extend Theorem 4.3 to bounded-visit Σ_{i+1} -2PA. Note that the case of Σ_1 -2PA is not covered by the following statement.

▶ **Theorem 6.2.** For any fixed $i \in \mathbb{N}_{\neq 0}$, the non-emptiness problem for bounded-visit Σ_{i+1} -2PA is Σ_i^{Exp} -C.

Proof. For the upper-bound, we show that this problem can be solved by an alternating Turing machine in exponential time, which alternates at most i times between sequences of non-deterministic and universal transitions, starting with non-deterministic transitions. By [12], the satisfiability of Σ_{i+1} -formulas is complete for Σ_i^{Exp} -C. Hence there is an i-alternating machine \mathcal{M} running in exponential time which checks the satisfiability of such formulas. Now, similar to the case of Σ_1 in Theorem 4.3, from a bounded-visit Σ_{i+1} -2PA P one can construct a Σ_{i+1} -formula which is true iff the automaton has a non-empty language. We can do so by applying Lemma 6.1 on the PA obtained from Lemma 3.6. Hence, non-emptiness of a bounded-visit Σ_{i+1} -2PA reduces to satisfiability of a Σ_{i+1} -formula $\Phi(\ell) = \bigvee_{j=1}^m \Phi_j(\ell)$ such that $\log_2(m)$ and the size of each Φ_j are polynomial in |P| and can be constructed in time $2^{\mathcal{O}(\gamma^2 \log(\gamma n))}$. However we cannot construct explicitly Φ , since its size is exponential in |P|. Instead we construct an i- alternating machine \mathcal{M}' that first guesses a disjunct Φ_s and constructs it in exponential time, and then simulates the machine \mathcal{M} on Φ_s . Recall the \mathcal{M} starts with non-deterministic transitions. Thus the machine \mathcal{M}' runs in exponential time, and also performs only i alternations, which provides Σ_i^{Exp} upper bound.

Hardness comes from checking if a Σ_{i+1} -sentence holds true, which is Σ_i^{Exp} -C by [12]. From a Σ_{i+1} -sentence Ψ it suffices to construct a Parikh automaton $P = (A, \lambda, \Psi)$ of dimension 0 such that $L(A) \neq \emptyset$, therefore $L(P) \neq \emptyset$ iff L(P) = L(A) iff Ψ holds.

 $^{^9}$ Lemma 3.6 can be trivially adapted to $\Sigma_i\text{-formula}$ as acceptance condition

▶ Theorem 6.3 (Boolean closure). Let P, P_1, P_2 be Σ_i -2DPA. One can construct in linear time a Π_i -2DPA \overline{P} and two Σ_i -2DPA P_{\cup}, P_{\cap} such that $L(\overline{P}) = \overline{L(P)}, L(P_{\cup}) = L(P_1) \cup L(P_2)$ and $L(P_{\cap}) = L(P_1) \cap L(P_2)$.

Proof. The constructions are the same as in the proof of the case i=1 of Theorem 5.1, using closure under disjunction and conjunction of Σ_i and the fact that negating a Σ_i -formula yields a Π_i -formula.

▶ **Theorem 6.4** (Comparison Problems). For all fixed $i \in \mathbb{N}_{\neq 0}$, the universality, inclusion and equivalence problems for Σ_{i} -2DPA are Π_{i}^{Exp} -C.

Proof. We first prove the upper bound for the most general problem which is inclusion. Let $P_i = (A_i, \lambda_i, \psi_i)$ be a Σ_i -2DPA. Note that $L(P_1) \subseteq L(P_2)$ iff $L(P_1) \cap \overline{L(P_2)} = \varnothing$. So, using Theorem 6.3 we first construct in linear-time a Π_i -2DPA $\overline{P_2} = (A_2', \lambda_2', \Psi_2')$ such that $L(\overline{P_2}) = \overline{L(P_2)}$ and then $P_{\cap} = (A, \lambda, \Psi)$ such that $L(P_{\cap}) = L(P_1) \cap L(\overline{P_2})$. From the construction in Theorem 5.1 generalised to Σ_i -2DPA, recall that the formula Ψ is defined as $\Psi(\vec{x}_1, \vec{x}_2) = \Psi_1(\vec{x}_1) \wedge \Psi_2'(\vec{x}_2)$. Let $\Psi_1(\vec{x}_1) = \exists \vec{y}_1 \forall \vec{y}_2 \dots \Omega \vec{y}_i \ [\varphi_1(\vec{x}_1, \vec{y}_1, \dots, \vec{y}_i)]$, and $\Psi_2'(\vec{x}_2) = \forall \vec{z}_1 \exists \vec{z}_2 \dots \mho \vec{z}_i \ [\varphi_2(\vec{x}_2, \vec{z}_1, \dots, \vec{z}_i)]$ where $\Omega, \mho \in \{\exists, \forall\}$ such that $\Omega \neq \mho$. Hence Ψ is equivalent to the following Σ_{i+1} -formula.

$$\exists \vec{y_1} \forall \vec{z_1} \forall \vec{y_2} \exists \vec{z_2} \exists \vec{y_3} \dots \Omega \vec{z_{i-1}} \vec{y_i} \\ \forall \vec{z_i} \Big[\varphi_1(\vec{x_1}, \vec{y_1}, \dots, \vec{y_i}) \land \varphi_2(\vec{x_2}, \vec{z_1}, \dots, \vec{z_i}) \Big]$$

Finally, emptiness of P_{\cap} can be decided in Π_i^{Exp} by Theorem 6.2.

For the lower bound, we show that the universality problem of Σ_i -DPA is Π_i^{Exp} -hard. This holds even for a fixed number of states and vector values in $\{-1,0,1\}$, showing that the complexity comes from the formula part. From a Σ_i -formula Ψ with d free variables, we construct a Parikh automaton $P=(A,\lambda,\Psi)$ of dimension d over alphabet $\Sigma=\{a_i^+,a_i^-\}_{1\leq i\leq d}$. Any word w over Σ defines a valuation $\mu_w(x_i)=|w|_{a_i^+}-|w|_{a_i^-}$ for all $1\leq i\leq d$. Conversely, any valuation μ can be encoded as a word over Σ . Hence, Ψ holds for all values iff for all $w\in\Sigma^*$, we have $\mu_w\models\Psi$. We construct a deterministic one-way automaton A such that $L(A)=\Sigma^*$ and for all $w\in\Sigma^*$, the value of the run r over w is μ_w . The automaton A has one accepting and initial state q over which it loops and, when reading a_i^+ (resp. a_i^-) it increases dimension i by 1 (resp. by -1).

▶ Remark 6.5. Since a 2DPA is a Σ_1 -2DPA, and the class CONEXP is the same as Π_1^{EXP} , we have that Theorem 6.4 for i=1 is exactly the same as Theorem 5.2.

7 Conclusion

In this paper, we have provided tight complexity bounds for the emptiness, inclusion, universality and equivalence problems for various classes of two-way Parikh automata. We have shown that when the semi-linear constraint is given as a Σ_i -formula, for i>1, the complexity of those problems is dominated by the complexity of checking satisfiability or validity of Σ_i -formulas. We have shown that 2DPA (resp. bounded-visit 2PA) have the same expressive power as unambiguous (one-way) PA (resp. non-deterministic PA). In terms of succinctness, it is already known that 2DFA are exponentially more succinct than FA, witnessed for instance by the family $D_n = \{uu \mid u \in \{0,1\}^* \land |u| = n\}$. However D_n is accepted by a PA with polynomially many states in n, using 2n vector dimensions to store the letters of its input, then checked for equality using the acceptance constraint. We conjecture that 2DPA are exponentially more succinct than PA, witnessed by the language

 L_n of Section 2. We leave as future work the introduction of techniques allowing to prove such results (pumping lemmas), as the dimension and acceptance constraint size has to be taken into account as well, as shown with D_n .

Finally, we plan to extend the pattern logic of [9], which intensively uses (one-way) Parikh automata for its model-checking algorithm, to reason about structural properties of two-way machines, and use two-way Parikh automata emptiness checking algorithm for model-checking this new logic.

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A Section 2: Two-way Parikh automata

Proof of Theorem 3.5 (continued). We prove now that $L(R) \subseteq L(P)$. Consider $u \in L(R)$ and let r be an accepting run of R over u with $s = c_1, \ldots, c_m$ the sequence of states visited by r to reach \top . By Lemma 3.4, there exists an accepting run ρ of P over u such that $C(\rho) = s$. Moreover, $V(\rho) = \sum_{i=1}^{m} V(c_i) = V(r)$. Hence $u \in L(P)$ since P have the same acceptance constraint as R.

We prove now that $L(P) \subseteq L(R)$. Consider $u = a_1 \dots a_m \in L(P)$ and let ρ be an accepting two-way run of P over u with $C(\rho) = c_1, \dots, c_m$ i.e. c_i is the a_i -crossing sections of ρ . Since ρ is accepting then c_1 is initial, c_m is accepting and $(c_i, c_{i+1}) \in M$. Furthermore, the k-visitness of P implies that each c_i have length at most k. So, there exists an accepting run r of R over u which visit the sequence of states c_1, \dots, c_m, \top . Moreover, $V(r) = \sum_{i=1}^m c_i = V(\rho)$. Hence $u \in L(R)$ since R have the same acceptance constraint as P.

We prove now that if P is deterministic then R is unambiguous by contrapositive. Let r_1, r_2 be two distinct accepting runs of R over some word u with s_1 and s_2 be the respective sequences of state states visited by r_1 and r_2 to reach \top . Since $r_1 \neq r_2$ then $s_1 \neq s_2$. By Lemma 3.4, there exist ρ_1, ρ_2 two accepting runs of P over u such that $\mathcal{C}(\rho_1) = s_1$ and $\mathcal{C}(\rho_2) = s_2$. Furthermore $\mathcal{C}(\rho_1) \neq \mathcal{C}(\rho_2)$ implies that $\rho_1 \neq \rho_2$. Hence P is not deterministic.

B Section 4: Emptiness Problem

Proof of Theorem 4.1. We reduce the problem of deciding whether a system of diophantine equations S over a finite set of variables $X = \{x_1, \ldots, x_n\}$ has a solution in \mathbb{N} , which is known to be undecidable. Each equation is of the form $p_1 = p_2$ where p_1, p_2 are polynomials over X, whose coefficient are assumed to be in \mathbb{N} . A valuation ν is mapping $\nu: X \to \mathbb{N}$. We denote by $\nu(p)$ the value of polynomial p under valuation ν . We first explain how to encode the values $\nu(p)$ for all ν as a language and show how to define it with a 2PA.

Given a polynomial p, we let sub(p) all the subpolynomials appearing in p, which is inductively defined by $sub(p_1 + p_2) = sub(p_1) \cup sub(p_2) \cup \{p_1 + p_2\}$, $sub(p_1 \times p_2) = sub(p_1) \cup sub(p_2) \cup \{p_1 \times p_2\}$, $sub(a) = \{a\}$ for $a \in \mathbb{N}$.

Given a polynomial p over $X = \{x_1, \ldots, x_n\}$, we let $\Sigma_p = \{0_x \mid x \in X\} \cup \{1_{p'} \mid p' \in sub(p)\}$ be a finite alphabet. Note that if $p_2 \in sub(p_1)$, then $\Sigma_{p_2} \subseteq \Sigma_{p_1}$.

Given a word $w \in \Sigma_p^*$, we let ν_w the valuation $\nu_w(x) = |0_x|_w$. We say that w is a ν -encoding of p if $\nu = \nu_w$ and $\nu(p) = |1_p|_w$. A language $L \subseteq \Sigma_p^*$ is a good encoding of p if for all ν solution of p, there exists a ν -encoding of p in L and conversely, any $w \in L$ is a ν -encoding of p for some ν . We now show by induction on p that there exists good encoding L_p of p definable by a 2PA A_p .

- 1. if $p = a \in \mathbb{N}$ is a given constant, then we let A_p be a finite automaton accepting any word $w \in \Sigma_a^*$ such that $|1_a|_w = a$. It has a states.
- 2. if $p = p_1 + p_2$, then we let A_{p_1} and A_{p_2} be the two 2PA constructed inductively on p_1 and p_2 , assumed to be of dimension d_1 and d_2 respectively. Then, A_p is constructed as follows: it works on alphabet Σ_p and has dimension $d = d_1 + d_2 + 3$. It first simulates A_{p_1} on the first d_1 dimensions (with vector updates in $\mathbb{Z}^{d_1} \times \{0\}^{d_2+3}$), ignoring letters in $\Sigma_p \setminus \Sigma_{p_1}$ until it reaches an halting state q. If q is rejecting, A_p rejects, otherwise it goes back to the beginning of the word and simulates A_{p_2} on next d_2 dimensions (with updates in $\{0\}^{d_1} \times \mathbb{Z}^{d_2} \times \{0\}^3$), ignoring letters in $\Sigma_p \setminus \Sigma_{p_2}$, until it reaches an halting state q'. If q' rejects, A_p rejects, otherwise it goes back to the beginning, count with a one-way pass the number occurrences of symbols 1_{p_1} , 1_{p_2} and $1_{p_1+p_2}$ respectively in three counters x_{p_1} , x_{p_2}

- and $x_{p_1+p_2}$ corresponding to the last three dimensions The semi-linear condition is then given by the formula $\varphi(x_1,\ldots,x_d)=\varphi_1(x_1,\ldots,x_{d_1})\wedge\varphi_2(x_{d_1+1},\ldots,x_{d_1+d_2})\wedge x_{p_1}+x_{p_2}=x_{p_1+p_2}$. By construction and induction hypothesis, A_p is a good encoding of p.
- 3. if $p=p_1\times p_2$, then A_p is on alphabet Σ_p and has dimension d_1+d_2+4 . Initially it works as $A_{p_1+p_2}$ during the two first phases (simulation of A_{p_1} followed by simulation of A_{p_2}). After those two simulations, A_p enters phase 2, during which it makes k passes over the whole input, where k is chosen non-deterministically (by using the non-determinism of 2PA). On each of these passes, it counts the number of occurrences of symbol 1_{p_2} in some counter x_{mult} (intended at the end to contain the value of $p_1 \times p_2$). At the end of each pass, it increments by one a counter x_{pass} . It non-deterministically decides to move to phase 3 during which it also makes a last pass over the whole input to count the number of occurrences of 1_{p_1} and $1_{p_1 \times p_2}$ in some counters x_{p_1} and $x_{p_1+p_2}$ and accepts. The acceptance formula is then: $\varphi(x_1,\ldots,x_d) = \varphi_1(x_1,\ldots,x_{d_1}) \wedge \varphi_2(x_{d_1+1},\ldots,x_{d_1+d_2}) \wedge x_{pass} = x_{p_1} \wedge x_{mult} = x_{p_1 \times p_2}$. If A_p makes k passes during phase 2 on input k, then we know that the value of k wult is equal to $k \times \nu_w(p_2)$. The formula also requires that $k = \nu_w(p_1)$ which leads to the result.

To encode an equation $p_1 = p_2$, we first construct A_{p_1} and A_{p_2} , then construct a 2PA $A_{p_1=p_2}$ which first simulates A_{p_1} and A_{p_2} on input $w \in (\Sigma_{p_1} \cup \Sigma_{p_2})^*$, and then performs a last pass where it counts the number of occurrences of 1_{p_1} in some counter x_{p_1} , and similarly for 1_{p_2} in some counter x_{p_2} . The final formula also requires that $x_{p_1} = x_{p_2}$. Then $L(A_{p_1=p_2}) \neq \emptyset$ iff there exists a solution to $p_1 = p_2$. It can be easily generalised to a system of equations.

To prove Lemma 4.2 one needs the following result:

▶ Theorem B.1 (Theorem 7.3.1 of [19]). Let A be an NFA with n states over an alphabet Λ of size γ . Then, the Parikh image $\mathfrak{P}(L(A))$ is equal to the semi-linear set $\bigcup_{i=1}^m \{\vec{b_i} + \sum_{j=1}^\gamma x_{i,j} p_{i,j}^- \mid x_{i,j} \in \mathbb{N}\}$ where $m \leq n^{\gamma^2 + 3\gamma + 3} \gamma^{4\gamma + 6}$, $||\vec{b_i}|| \leq n^{3\gamma + 3} \gamma^{4\gamma + 6}$ and $p_{i,j} \in \{0,\ldots,n\}^{\gamma}$. Furthermore, b_i and $p_{i,j}$ can be computed in time $2^{\mathcal{O}(\gamma^2 \log(\gamma n))}$.

Proof of Lemma 4.2. Let $P=(A,\lambda,\psi)$ be a PA of dimension d over Σ with the FA $A=(Q,\Delta,I,F)$. Consider the alphabet $\Lambda\subseteq\mathbb{Z}^d$ the set of vectors occurring on the transitions of P with an arbitrary order defined as $\Lambda=\{\vec{a}_1,\ldots,\vec{a}_\gamma\}$ where $\vec{a}_i\in\mathrm{range}(\lambda)$ and $\gamma=|\mathrm{range}(\lambda)|$. We construct the FA A_λ over Λ from P which takes weight vectors as letters instead of Σ . Formally, $A_\lambda=(Q,\Delta_\lambda,I,F)$ such that $(p,\lambda(p,a,q),q)\in\Delta_\lambda$ iff $(p,a,q)\in\Delta$.

This proof shows the existence of the existential Presburger formula $\varphi(\ell)$ which holds iff $L(P) \neq \varnothing$. To do that we consider $\mathfrak{P}(L(A_{\lambda})) \subseteq \mathbb{N}^{\gamma}$, the Parikh image of $L(A_{\lambda})$, assuming that it can be denoted by the existential Presburger formula ξ . Indeed, $(\tau_1, \ldots, \tau_{\gamma}) \in \mathfrak{P}(L(A_{\lambda}))$ iff there exists an accepting run ρ of A_{λ} which visits each weight vector $\vec{a}_i \in \Lambda$ exactly τ_i times. Now intuitively, from $\tau_1, \ldots, \tau_{\gamma}$ we are able to recover the tuple computed by P at the end of the run ρ using existential Presburger arithmetic. So, in the sequel, (i) we describe how to construct ξ which defines $\mathfrak{P}(L(A_{\lambda}))$ and (ii) from ξ we define the existential Presburger formula $\varphi(\ell)$ which holds iff there exists an accepting run of P of length ℓ .

(i) From Theorem B.1 applied on the FA A_{λ} , there exist m linear sets $L_i = \{\vec{b}_i + \sum_{j=1}^{\gamma} x_{i,j} \vec{p}_{i,j} \mid x_{i,j} \in \mathbb{N}\}$ such that $\mathfrak{P}(L(A_{\lambda})) = \bigcup_{i=1}^{m} L_i$. The linear set L_i can be denoted by the following existential Presburger formula:

$$\xi_i(\vec{\tau}) = \exists \vec{x} \left[\bigwedge_{k=1}^{\gamma} \operatorname{proj}_k(\vec{\tau}) = \operatorname{proj}_k(\vec{b}_i) + \sum_{j=1}^{\gamma} \operatorname{proj}_j(\vec{x}) \times \operatorname{proj}_k(\vec{p}_{i,j}) \right]$$
(1)

Note that, \vec{b}_i and $\vec{p}_{i,j}$ depends on P only and can be computed in time $2^{\mathcal{O}(\gamma^2\log(\gamma n))}$. Also, Theorem B.1 ensures that m is at most $n^{\gamma^2+3\gamma+3}\gamma^{4\gamma+6}$. Moreover, for all $1 \leq i \leq m$, all $1 \leq j \leq \gamma$, we have that $\operatorname{proj}_k(\vec{b}_i) \leq n^{3\gamma+3}\gamma^{4\gamma+6}$ and $\vec{p}_{i,j} \in \{0,\ldots,n\}^{\gamma}$. Since constants are encoded in binary, we have that $|\xi_i|$ is polynomial in γ and logarithmic in n.

(ii) Now, we explain how ξ and the acceptance constraint¹⁰ of P are glued together. Recall that $\Lambda = \{\vec{a}_1, \dots, \vec{a}_\gamma\}$ where $\vec{a}_i \in \mathbb{Z}^d$ is the set of weight vectors of the original Parikh automaton P. The value of each dimension $1 \le k \le d$ at the end of a run can be computed from the number of visits $\tau_1, \dots, \tau_\gamma$ by $c_k = \sum_{j=1}^{\gamma} \tau_j \times \operatorname{proj}_k(\vec{a}_j)$ and the number of transition taken is $\ell = \sum_{j=1}^{\gamma} \tau_j$. Then, we define the formula φ_i as follows.

$$\varphi_{i}(\ell) = \exists \vec{\tau}, \exists \vec{c} \left[\bigwedge \begin{cases} \xi_{i}(\vec{\tau}) \wedge \psi(\vec{c}) \wedge \ell = \sum_{j=1}^{\gamma} \tau_{j} \\ \bigwedge_{k=1}^{d} \operatorname{proj}_{k}(\vec{c}) = \sum_{j=1}^{\gamma} \operatorname{proj}_{j}(\vec{\tau}) \times \operatorname{proj}_{k}(\vec{a}_{j}) \end{cases} \right]$$
(2)

We have that $|\varphi_i| = \mathcal{O}(|\xi_i| + |\psi| + \gamma + d\gamma \log_2(\mu))$ where μ is the maximal absolute value appearing on weight vectors of P i.e. $\mu = \max\{\|\vec{a}_i\| \mid 1 \leq i \leq \gamma\}$. Recall that $|P| = \mathcal{O}(n + |\psi| + (d\log_2(\mu + 1) + n^2) \times \gamma)$. Thus, $|\varphi_i|$ is polynomial in |P|, logarithmic in n and can be computed in time $2^{\mathcal{O}(\gamma^2 \log(\gamma n))}$.

C Section 5: Closure properties and comparison problems

Proof of Theorem 5.3. Given a 2PA $P=(A,\lambda,\psi)$ with $A=(Q,Q_I,Q_F,\Delta)$ and a word $w\in \Sigma^*$, we construct an PA P_w such that $w\in L(P)$ iff $L(P_w)\neq\varnothing$. Intuitively, each state of P_w encodes a configuration that appears in a run of P on input w. We define $P_w=(A_w,\lambda',\psi)$ with $A_w=(Q',Q'_I,Q'_F,\Delta')$ where $Q'=\{(w_1,q,w_2)\mid q\in Q,w_1w_2=\vdash w\dashv\}$, and $Q'_I=\{(\varepsilon,q_0,\vdash w\dashv)\mid q_0\in Q\}, Q'_F=\{(w,q_f,\varepsilon)\mid q_f\in F\}$ and λ' is defined as following partial function for which Δ' is the domain:

$$\bigcup \begin{cases} \left\{ \left((w_1, q_1, aw_2), a, (w_1 a, q_2, w_2) \right) \mapsto \vec{v} \mid (q_1, a, q_2) \in \Delta \land \lambda(q_1, a, q_2) = \vec{v} \land q_1 \in Q^{\mathsf{R}} \right\} \\ \left\{ \left((w_1 a, q_1, w_2), a, (w_1, q_2, aw_2) \right) \mapsto \vec{v} \mid (q_1, a, q_2) \in \Delta \land \lambda(q_1, a, q_2) = \vec{v} \land q_1 \in Q^{\mathsf{L}} \right\} \end{cases}$$

Note that a run of P_w is a (one-way) PA which simulates the sequence of configurations corresponding to a run of P on input w, hence we have $L(P_w) \neq \emptyset$ iff $w \in L(P)$. Non-emptiness and membership are shown to be NP-C for PA in [8] which yields the statement.

¹⁰Lemma 4.2 ψ can be trivially be generalised with an acceptance constraint which belongs to $\mathsf{PA}(\alpha,\beta)$ for some $\alpha,\beta\in\mathbb{N}$