On Proof Theory in Computer Science

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1 Introduction

The subject logic in computer science should entail proof theoretic applications. So the question arises whether open problems in computational complexity can be solved by advanced proof theoretic techniques. In particular, consider the complexity classes **NP**, **coNP** and **PSPACE**. It is well-known that **NP** and **coNP** are contained in **PSPACE**, but till recently precise characterization of these relationships remained open. Now [2], [3] (see also [4]) presented proofs of corresponding equalities **NP** = **coNP** = **PSPACE**. These results were obtained by appropriate proof theoretic tree-to-dag compressing techniques to be briefly explained below. But let us first recall basic definitions of complexity classes involved.

1.1 Complexity classes

Recall standard definitions of the complexity classes **NP**, **coNP** and **PSPACE**. A given language $L \subseteq \{0,1\}^*$ is in **NP**, resp. **coNP**, if there exists a polynomial p and a polynomial-time TM M such that for every $x \in \{0,1\}^*$:

$$x \in L \Leftrightarrow \left(\exists u \in \{0,1\}^{p(|x|)}\right) M(x,u) = 1$$
resp.
$$x \in L \Leftrightarrow \left(\forall u \in \{0,1\}^{p(|x|)}\right) M(x,u) = 1$$
(coNP)

That is to say, a given $x \in \{0,1\}^*$ is in L iff M's execution on input (x,u) provides output 1 for some (resp. every) $u \in \{0,1\}^*$ of the length $|u| \leq p(|x|)$, where p and M are determined by x. Note that \mathbf{coNP} is complementary to \mathbf{NP} (and vice versa), i.e. $L \in \mathbf{coNP} \Leftrightarrow L \notin \mathbf{NP}$. However it is unclear a priori whether symmetric difference $(\mathbf{NP} \setminus \mathbf{coNP}) \cup (\mathbf{coNP} \setminus \mathbf{NP})$ is empty or not, as $\operatorname{card}\left(\{0,1\}^{p(|x|)}\right)$ is exponential in x. In the former case we'll have $\mathbf{NP} = \mathbf{coNP}$, which seems more natural and/or plausible, as it reflects an idea of logical equivalence between model theoretical (re: \mathbf{NP}) and proof theoretical (re: \mathbf{coNP}) interpretations of non-deterministic polynomial-time computability.

Now $L \subseteq \{0,1\}^*$ is in **PSPACE** if there exists a polynomial p and a TM M such that for every input $x \in \{0,1\}^*$, the total number of non-blank locations that occur during M's execution on x is at most p(|x|), and $x \in L \Leftrightarrow M(x) = 1$. Thus **PSPACE** requires polynomial upper bounds only on the space – but not time – of entire computation. It is well-known (and not hard to prove) that

PSPACE = coPSPACE, while NP and coNP are contained in PSPACE. It is unclear a priori whether at least one of NP, coNP is a proper subclass of PSPACE. It is clear, however, that the assumption NP = PSPACE implies NP = coNP (via PSPACE = coPSPACE).

1.2 Logic and proof systems

Classical propositional logic provides natural interpretations of **NP** and **coNP**. Namely, the well-known propositional satisfiability and validity problems SAT and VAL are, respectively, **NP**- and **coNP**-complete. That is, an L canonically encoding the set of satisfiable (resp. valid) propositional formulas is universal for the whole class **NP** (resp. **coNP**). However, classical proof systems usually correlated with VAL are less helpful for the comparison **NP** vs **coNP**, as the size of conventional proofs of tautologies x use to be exponential in |x|. It seems that minimal and intuitionistic propositional logics [7], [10] provide us with more suitable refinements . Recall that the minimal logic is determined by the axioms $\alpha \to (\beta \to \alpha)$, $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ and rule modus $\alpha \to \beta$

ponens $\boxed{\frac{\alpha \quad \alpha \to \beta}{\beta}}$ in standard Hilbert-style formalism whose vocabulary

includes propositional variables and propositional connective ' \rightarrow ' (α , β , γ , etc. denote corresponding formulas). The intuitionistic logic extends minimal one by adding one propositional constant \perp (falsity) and new axiom $\perp \rightarrow \alpha$. It is wellknown that there are polynomial-size validity-preserving embeddings of formulas in classical into intuitionistic and intuitionistic into minimal logic, respectively. Apart from Hilbert-style formalism, proof systems for minimal and intuitionistic logic include Gentzen-style sequent calculus (SC) and Gentzen-Prawitz-style natural deductions (ND). Both admit two well-known proof-optimization: cut elimination in SC and normalization in ND. Cut elimination approach provides sound and complete systems of inferences without cut rule that is equivalent to the modus ponens. Inferences in the resulting *cutfree* SC systems satisfy a sort of subformula property (: all premise formulas occur as (sub)formulas in the conclusions), which enables better proof search strategies. We can also assume that the heights of cutfree proofs (derivations) are linear in the weights of conclusions, although such constrain is not obvious for intuitionistic and/or minimal logic, see [6], [2]. In ND, the normalization allows to use just normal proofs that are known to satisfy weak subformula property (: every formula occurring in a maximal thread occurs as (sub)formula in the conclusion), see [9]. However, there are no polynomial upper bounds on the heights of arbitrary normal ND.

These optimizations have been elaborated for standard tree-like versions of SC and ND. Note that tree-like approach can't provide polynomial upper bounds on the size of resulting proofs. To achieve this goal we formalize another idea of horizontal compression. That is, in a given tree-like proof we wish to merge all nodes labeled with identical objects (sequents or formulas) occurring on the same level so that in the compressed daq-like proof every level will contain

mutually different objects. In the case of SC even the compressed polynomial-height dag-like proofs still would be too large due to possibly exponential number of distinct sequents occurring in it. Now consider a "short" normal tree-like ND whose height is polynomial in the weight of conclusion. By the weak subformula property we observe that the total weight of distinct (sub)formulas occurring in it is polynomial in the weight of conclusion. As ND proofs operate with single formulas (not sequents!), compressing this tree-like ND proof will provide us with a desired polynomial-weight dag-like deduction. However, such compressed dag-like deduction requires a modified notion of provability, as merging different occurrences of identical formulas appearing as conclusions in the same level of deduction might require a new separation rule (S)

$$(S): \frac{\overbrace{\alpha \quad \cdots \quad \alpha}^{n \text{ times}}}{\alpha} \ (n \text{ arbitrary})$$

whose identical premises are understood disjunctively: "if at least one premise is proved then so is the conclusion" (in contrast to ordinary inferences: "if all premises are proved then so are the conclusions"). The notion of provability is modified accordingly such that proofs are locally correct deductions assigned with appropriate sets of closed threads that satisfy special conditions of local coherency (in contrast to ordinary local correctness, the local coherency is not verifiable in polynomial time). These locally coherent threads are inherited by the underlying closed tree-like threads. The required "small" polynomial-weight proof now arises by collapsing (S) to plain repetitions

$$(R): \frac{\alpha}{\alpha}$$

with respect to the appropriately chosen premises of (S). The choice is made non-deterministically using the set of locally coherent threads in question.

Keeping this in mind consider the **NP**-complete Hamiltonian graph problem and let ρ be a purely implicational formula expressing in standard form that a given (simple and directed) graph G has no Hamiltonian cycles. We observe that the canonical tree-like proof search for ρ in the minimal ND with standard inferences

yields a normal tree-like proof ∂ whose height is polynomial in |G| (and hence $|\rho|$), provided that G is non-Hamiltonian. Since ∂ is normal, it will obey the requested polynomial upper bounds in question, and hence the weight of its daglike compression will be polynomially bounded, as desired. Summing up, for any given non-Hamiltonian graph G there is some polynomial-weight dag-like ND

refutation of the existence of Hamiltonian cycles in G. Note that polynomial-weight ND proofs (tree- or dag-like) have polynomial-time certificates ([3]: Appendix), while the non-hamiltoniancy of simple and directed graphs is **coNP**-complete. Hence **coNP** is in **NP**, which yields **NP** = **coNP**.

To handle our main assertion $\mathbf{NP} = \mathbf{PSPACE}$ we recall that the validity of the minimal logic under consideration is known to be \mathbf{PSPACE} -complete. Moreover, every minimal tautology is provable in Hudelmaier's cutfree SC (abbr.: HSC) for minimal logic [6] by a tree-like derivation whose height is linear in the weight of conclusion. Furthermore, straightforward ND interpretation of such tree-like input in HSC yields corresponding "short" (though not necessarily normal) tree-like proof in ND for minimal logic whose total weight of distinct (sub)formulas is polynomial in the weight of conclusion [2]. Now the latter tree-like ND proof is horizontally compressible to a polynomial-weight dag-like proof by the same method as sketched above with respect to "short" normal ND. This yields $\mathbf{NP} = \mathbf{PSPACE}$. A more detailed presentation is as follows.

2 Survey of proofs

2.1 Basic tree-like and dag-like ND

Our basic ND calculus for minimal logic, NM_→, includes two basic inferences

$$(\rightarrow I): \frac{[\alpha]}{\beta \atop \alpha \rightarrow \beta}, \qquad (\rightarrow E): \frac{\alpha \quad \alpha \rightarrow \beta}{\beta}$$

and one auxiliary repetition rule $(R):\frac{\alpha}{\alpha}$, where $[\alpha]$ in $(\to I)$ indicates that all α -leaves occurring above β -node exposed are discharged assumptions (cf. [9]). In NM $_{\to}$, tree-like deductions are understood as finite rooted at most binary-branching trees whose nodes are labeled with purely implicational formulas $(\alpha, \beta, \gamma, \text{ etc.})$ that are ordered according to the inferences exposed, as usual in proof theory, whereas dag-like deductions are the analogous finite rooted dags. Thus for any node x in a tree-like deduction ∂ , the set of all nodes occurring below x in ∂ is linearly ordered. This constrain is lacking in dag-like deductions. Note that in dag-like NM $_{\to}$ deductions, all nodes can have at most two premises (= children), but arbitrary many conclusions (= parents) 1 , whereas the latter is forbidden in the tree-like case.

Definition 1 A given (whether tree- or dag-like) NM_{\rightarrow} -deduction ∂ proves its root-formula ρ (abbr.: $\partial \vdash \rho$) iff every maximal thread connecting the root with a leaf labeled α is closed (= discharged), i.e. it contains a ($\rightarrow I$) with conclusion $\alpha \rightarrow \beta$, for some β . A purely implicational formula ρ is valid in minimal logic

 $^{^{1}\}mathrm{This}$ follows from standard conditions of the local correctness.

iff there exists a tree-like NM_{\rightarrow} -deduction ∂ that proves ρ ; such ∂ is called a proof of ρ .

Remark 2 Tree-like constraint in the definition of validity is inessential.

That is, for any dag-like $\partial \in \mathrm{NM}_{\to}$ with root-formula ρ , if $\partial \vdash \rho$ then ρ is valid in minimal logic. Because any given dag-like ∂ can be unfolded into a tree-like deduction ∂' by straightforward thread-preserving top down recursion. To this end every node $x \in \partial$ with n > 1 distinct conclusions should be replaced by n distinct but identically labeled nodes $x_1, \dots, x_n \in \partial'$ to be connected with corresponding single conclusions. This operation obviously preserves the closure of threads, i.e. $\partial \vdash \rho$ infers $\partial' \vdash \rho$.

Formal verification of the assertion $\partial \vdash \rho$ is simple, as follows – whether for tree-like or, generally, dag-like ∂ . Every node $x \in \partial$ is assigned, by top-down recursion, a set of assumptions A(x) such that:

- 1. $A(x) := \{\alpha\}$ if x is a leaf labeled α ,
- 2. A(x) := A(y) if x is the conclusion of (R) with premise y,
- 3. $A(x) := A(y) \setminus \{\alpha\}$ if x is the conclusion of $(\to I)$ with label $\alpha \to \beta$ and premise y,
- 4. $A(x) := A(y) \cup A(z)$ if x is the conclusion of $(\to E)$ with premises y, z.

This easily yields

Lemma 3 Let $\partial \in \text{NM} \to (whether tree- or dag-like)$. Then $\partial \vdash \rho \Leftrightarrow A(r) = \emptyset$ holds with respect to standard set-theoretic interpretations of " \cup " and " \setminus " in A(r), where r and ρ are the root and the root-formula of ∂ , respectively. Moreover, $A(r) \stackrel{?}{=} \emptyset$ is verifiable by a deterministic TM in $|\partial|$ -polynomial time, where by $|\partial|$ we denote the weight of ∂ .

Proof. The equivalence easily follows by induction on the height of ∂ . The second assertion is completely analogous to the well-known polynomial-time decidability of the circuit value problem. \blacksquare

Definition 4 Tree-like NM $_{\rightarrow}$ -deduction ∂ with the root-formula ρ is called polynomial, resp. quasi-polynomial, if its weight (= total number of symbols), resp. height plus total weight of distinct formulas, is polynomial in the weight of conclusion, $|\rho|$.

Theorem 5 Any quasi-polynomial tree-like proof $\partial \vdash \rho$ can be compressed into a polynomial dag-like proof $\partial^* \vdash \rho$.

The mapping $\partial \hookrightarrow \partial^*$ is obtained by a two-folded horizontal compression $\partial \hookrightarrow \partial^b \hookrightarrow \partial^*$, where ∂^b is dag-like deduction in the following modified ND that extends NM $_{\rightarrow}$ by the *separation* rule (S), cf. Introduction.

2.2 ND with the separation rule

Recall that the separation rule (S)

$$(S): \frac{\overbrace{\alpha \cdot \cdots \cdot \alpha}^{n \text{ times}}}{\alpha} (n \text{ arbitrary})$$

is understood disjunctively: "if at least one premise is proved then so is the conclusion" (in contrast to ordinary inferences: "if all premises are proved then so are the conclusions"). Let NM_{\rightarrow}^{\flat} extend NM_{\rightarrow} by adding a new inference (S). The notion of provability in NM_{\rightarrow}^{\flat} is modified as follows. To begin with, for any NM_{\rightarrow}^{\flat} deduction ∂ we modify our basic definition of the set of assignments $\{A(x): x \in \partial\}$ by adding to old recursive clauses 1–4 (see above) a new clause 5 with new separation symbol \S :

5.
$$A(x) = \otimes (A(y_1), \dots, A(y_n))$$
 if x is the conclusion of (S) with premises y_1, \dots, y_n .

Having this done we stipulate

Definition 6 For any given (whether tree- or dag-like) deduction $\partial \in \mathrm{NM}_{\rightarrow}^{\downarrow}$ with root r and root-formula ρ , ∂ is called a modified proof of ρ (abbr.: $\partial \vdash^{\flat} \rho$) if A(r) reduces to \emptyset (abbr.: $A(r) \rhd \emptyset$) by standard set-theoretic interpretations of " \cup ", " \setminus " and nondeterministic disjunctive valuations $(t_1, \cdots, t_n) := t_i$, for any chosen $i \in \{1, \cdots, n\}$. Obviously $\partial \vdash^{\flat} \rho \Leftrightarrow \partial \vdash \rho$ holds for every separation-free ∂ .

Lemma 7 For any ∂ as above, if $\partial \vdash^{\flat} \rho$ then ρ is valid in minimal logic. Moreover, the assertion $\partial \vdash^{\flat} \rho$ can be confirmed by a nondeterministic TM in $|\partial|$ -polynomial time.

Proof. The former assertion reduces to its trivial NM_{\to} case (see above). For suppose that $A(r) \rhd \emptyset$ holds with respect to a successive nondeterministic valuation of the occurrences $\ \odot$. This reduction determines a successive bottom-up thinning of ∂ that results in a "cleansed" (S)-free subdeduction $\partial_0 \in \mathrm{NM}_{\to}^{\flat}$. Thus $A(r) \rhd \emptyset$ in ∂ implies $A(r) = \emptyset$ in ∂_0 . Since (S) does not occur in ∂_0 anymore, we have $\partial_0 \in \mathrm{NM}_{\to}$, and hence $\partial_0 \vdash \rho$ holds by Lemma 3. So by previous considerations with regard to NM_{\to} we conclude that ρ is valid in minimal logic, which can be confirmed in $|\partial|$ -polynomial time, as required (see Lemma 3).

2.3 Horizontal compression with cleansing

In the sequel for any natural deduction ∂ we denote by $h(\partial)$ and $\phi(\partial)$ the height of ∂ and the total weight of the set of distinct formulas occurring in ∂ , respectively. Now we are prepared to explain proof of Theorem 5. For any tree-like NM_{\rightarrow} proof ∂ of ρ let $\partial' \in NM_{\rightarrow}$ be its horizontal compression defined

by bottom-up recursion on $h(\partial)$ such that for any $n \leq h(\partial)$, the n^{th} horizontal section of ∂^b is obtained by merging all nodes with identical formulas occurring in the n^{th} horizontal section of ∂ . The inferences in ∂' are naturally inherited by the ones in ∂ . Obviously ∂' is a dag-like (not necessarily tree-like anymore) deduction with the root formula ρ . However, ∂' need not preserve the local correctness with respect to basic inferences $(\to I)$, $(\to E)$, (R). For example, a compressed multipremise configuration

$$(\rightarrow I, E): \frac{\beta \qquad \gamma \qquad \gamma \rightarrow (\alpha \rightarrow \beta)}{\alpha \rightarrow \beta}$$

that is obtained by merging identical conclusions $\alpha \to \beta$ of

$$(\to I): \frac{\beta}{\alpha \to \beta}$$
 and $(\to E): \frac{\gamma \qquad \gamma \to (\alpha \to \beta)}{\alpha \to \beta}$

is not a correct inference in NM $_{\rightarrow}$. To overcome this trouble we upgrade ∂' to a modified deduction ∂^{\flat} that separates such multiple premises using appropriate instances of the separation rule (S). For example, $(\rightarrow I, E)$ as above should be replaced by this NM $^{\flat}_{\rightarrow}$ -correct configuration

$$(S): \frac{(\rightarrow I): \frac{\beta}{\alpha \rightarrow \beta} \quad (\rightarrow E): \frac{\gamma}{\alpha \rightarrow \beta}}{\alpha \rightarrow \beta}$$

This ∂^{\flat} is a locally correct dag-like (not necessarily tree-like anymore) deduction in NM $_{\rightarrow}^{\flat}$ with the root formula ρ . Moreover ∂^{\flat} is polynomial as $\left|\partial^{\flat}\right| \leq 2 \left|\partial'\right|$ and $\left|\partial'\right| \leq h\left(\partial\right) \times \phi\left(\partial\right)$. However, we can't claim that ∂^{\flat} proves ρ because arbitrary maximal dag-like threads in ∂^{\flat} can arise by concatenating different segments of different threads in ∂ , which can destroy the required closure condition (cf. Definition 1). On the other hand, we know that all threads in ∂ are closed, so let \mathcal{F}^{\flat} be the dag-like image in ∂^{\flat} of these tree-like threads under the mapping $\partial \hookrightarrow \partial^{\flat}$. We observe that \mathcal{F}^{\flat} satisfies the following three conditions of local coherency, where $n := h\left(\partial^{\flat}\right)$ and for any (maximal bottom-up) thread $\Theta = [r = x_0, \cdots, x_n] \in \mathcal{F}^{\flat}$ and $i \leq n$ we let $\Theta \upharpoonright_{x_i} := [x_0, \cdots, x_i]$.

- 1. \mathcal{F}^{\flat} is dense in ∂^{\flat} , i.e. $(\forall u \in \partial^{\flat}) (\exists \Theta \in \mathcal{F}^{\flat}) (u \in \Theta)$.
- 2. Every $\Theta \in \mathcal{F}^{\flat}$ is closed, i.e. its leaf-formula $\alpha(x_n)$ is discharged in Θ .
- 3. \mathcal{F} preserves $(\rightarrow E)$, i.e.

$$(\forall \Theta \in \mathcal{F}^{\flat}) (\forall u \in \Theta) (\forall v \neq w \in Child_{\partial^{\flat}}(u) : v \in \Theta)$$

$$(\exists \Theta' \in \mathcal{F}^{\flat}) (w \in \Theta' \land \Theta \upharpoonright_{u} = \Theta' \upharpoonright_{u}).$$

In the sequel we call any \mathcal{F}^{\flat} satisfying conditions of local coherency the fundamental set of threads (abbr.: fst) in ∂^{\flat} .

Lemma 8 Let ∂^{\flat} be any given locally correct dag-like NM^{\flat}_{\rightarrow} -deduction with root-formula ρ that is supplied with a fst \mathcal{F}^{\flat} . Then ρ has a modified dag-like proof $\partial^* \subseteq \partial^{\flat}$.

Proof. We show that \mathcal{F}^{\flat} determines successive left-to-right (S)-eliminations $(S)(A(y_1), \dots, A(y_n)) \hookrightarrow A(y_i)$ inside A(r) leading to a desired reduction $A(r) \triangleright$ \emptyset (see basic notations in 2.2). These eliminations together with a suitable sub-fst $\mathcal{F}_0^{\flat} \subseteq \mathcal{F}^{\flat}$ arise as follows by bottom-up recursion along \mathcal{F}^{\flat} . Let x be a chosen lowest conclusion of $(\to E)$ in ∂^{\flat} , if any exists. By the density of \mathcal{F}^{\flat} , there exists $\Theta \in \mathcal{F}^{\flat}$ with $x \in \Theta$; so let $\Theta \in \mathcal{F}_0^{\flat}$. Let y and z be the two children of x and suppose that $y \in \Theta$. By the third, $(\to E)$ -preserving fst condition there exists a $\Theta' \in \mathcal{F}^{\flat}$ with $z \in \Theta'$ and $\Theta \upharpoonright_x = \Theta' \upharpoonright_x$; so let $\Theta' \in \mathcal{F}_0^{\flat}$ be the corresponding "upgrade" of Θ . If $z \in \Theta$ then let $\Theta' := \Theta$. Note that $\Theta|_x$ determines substitutions $A(u) = \mathfrak{S}(A(v_1), \dots, A(v_n)) := A(v_i)$ in all parents of the (S)-conclusions uoccurring in both Θ and Θ' below x, if any exist, and thereby all \mathfrak{S} -eliminations $A(u) \hookrightarrow A(v_i)$ in the corresponding subterms of A(r). The same procedure is applied to the nodes occurring in Θ and Θ' between x and the next lowest conclusions of $(\to E)$; this yields new closed threads $\Theta'', \Theta''', \dots \in \mathcal{F}_0^{\flat} \subseteq \mathcal{F}^{\flat}$ and \mathfrak{S} -eliminations in the corresponding initial fragments of A(r). We keep doing this recursively until the list of remaining S-occurrences in $\Theta \in \mathcal{F}_0^{\flat}$ is empty. The final "cleansed" \odot -free conversion of A(r) is represented by a set of formulas that easily reduces to \emptyset by ordinary set-theoretic interpretation of the remaining operations " \cup " and " \setminus ", since every $\Theta \in \mathcal{F}_0^b$ involved is closed. The correlated "cleansed" deduction ∂^* obtained by substituting corresponding instances of (R) for thus eliminated (S) is a locally correct dag-like deduction of ρ in the (S)-free fragment of $\mathrm{NM}^{\flat}_{\to}$, and hence it belongs to NM_{\to} . Moreover the set of maximal threads in ∂^* is uniquely determined by the remaining rules $(R), (\to I), (\to E)$. By the definition these "cleansed" maximal threads are all included in \mathcal{F}^{\flat} thus being closed with respect to $(\to I)$. This yields a desired reduction $A(r) > \emptyset$, i.e. $A(r) = \emptyset$, in ∂^* . Hence ∂^* proves ρ in NM_{\rightarrow} . Obviously ∂^* is a subdeduction of ∂^{\flat} .

Operation $\partial^{\flat} \hookrightarrow \partial^*$ is also referred to as horizontal cleansing.

Corollary 9 By Lemma 7, the assertion of the lemma implies that ρ is valid in minimal logic. Actually ∂^* involved is separation-free, which yields $\partial^* \vdash \rho$.

This completes our proof of Theorem 5 and together with Lemma 3 yields

Corollary 10 Any given ρ is valid in minimal logic iff there exists a polynomial dag-like proof ∂^* of ρ , in NM_{\rightarrow} . Moreover, the assertion $\partial^* \vdash \rho$ can be confirmed by a deterministic TM in $|\rho|$ -polynomial time.

 $^{^2}$ These threads may be exponential in number, but our nondeterministic algorithm runs on the polynomial set of nodes.

2.4 Consequences for computational complexity

2.4.1 Case NP vs coNP

Since normal ND proofs satisfy weak subformula property, we have

Lemma 11 Any given normal tree-like NM_{\rightarrow} -proof ∂ of ρ whose height $h(\partial)$ is polynomial in $|\rho|$ is quasi-polynomial.

Let P be a chosen NP-complete problem and purely implicational formula ρ be valid iff P has no positive solution. In particular, let P be the Hamiltonian graph problem and ρ express in standard way that a given graph G has no Hamiltonian cycles. Suppose that the canonical proof search of ρ in NM $_{\rightarrow}$ yields a normal tree-like proof ∂ whose height is polynomial in |G| (and hence $|\rho|$), provided that G is non-Hamiltonian. Then by the last lemma ∂ will be polynomially bounded. That is, we argue as follows.

Lemma 12 Let P be the Hamiltonian graph problem and purely implicational formula ρ express that a given graph G has no Hamiltonian cycles. There exists a normal tree-like NM_{\rightarrow} -proof of ρ such that $h(\partial)$ is polynomial in |G| (and hence $|\rho|$), provided that G is non-Hamiltonian.

Recall that polynomial ND proofs (whether tree- or dag-like) have polynomial-time certificates, while the non-hamiltoniancy of simple and directed graphs is **coNP**-complete. Hence Corollary 10 yields

Corollary 13 NP = coNP holds true.

So it remains to prove Lemma 12. To this end, consider a simple 3 directed graph $G = \langle V_G, E_G \rangle$, $card(V_G) = n$. A Hamiltonian path (or cycle) in G is a sequence of nodes $\mathcal{X} = v_1 v_2 \dots v_n$, such that, the mapping $i \mapsto v_i$ is a bijection of $[n] = \{1, \dots, n\}$ onto V_G and for every 0 < i < n there exists an edge $(v_i, v_{i+1}) \in E_G$. The (decision) problem whether or not there is a Hamiltonian path in G is known to be NP-complete (cf. e.g. [1]). If the answer is YES then G is called Hamiltonian. In order to verify that a given sequence of nodes \mathcal{X} , as above, is a Hamiltonian path it will suffice to confirm that:

- 1. There are no repeated nodes in \mathcal{X} ,
- 2. No element $v \in V_G$ is missing in \mathcal{X} ,
- 3. For each pair $\langle v_i v_i \rangle$ in \mathcal{X} there is an edge $(v_i, v_i) \in E_G$.

It is readily seen that the conjunction of 1, 2, 3 is verifiable by a deterministic TM in n-polynomial time. Consider a natural formalization of these conditions (cf. e.g. [1]) in propositional logic with one constant \perp (falsum) and three connectives ' \wedge ', ' \vee ', ' \rightarrow '.

³A simple graph has no multiple edges. For every pair of nodes (v_1, v_2) in the graph there is at most one edge from v_1 to v_2 .

Definition 14 For any $G = \langle V_G, E_G \rangle$, $card(V_G) = n > 0$, as above, consider propositional variables $X_{i,v}$, $i \in [n]$, $v \in V_G$. Informally, $X_{i,v}$ should express that vertex v is visited in the step i in a path on G. Define propositional formulas A - E as follows and let $\alpha_G := A \wedge B \wedge C \wedge D \wedge E$.

- 1. $A = \bigwedge_{v \in V} (X_{1,v} \vee \ldots \vee X_{n,v})$ (: every vertex is visited in X).
- 2. $B = \bigwedge_{v \in V} \bigwedge_{i \neq j} (X_{i,v} \to (X_{j,v} \to \bot))$ (: there are no repetitions in X).
- 3. $C = \bigwedge_{i \in [n]} \bigvee_{v \in V} X_{i,v}$ (: at each step at least one vertex is visited).
- 4. $D = \bigwedge_{v \neq w} \bigwedge_{i \in [n]} (X_{i,v} \to (X_{i,w} \to \bot))$ (: at each step at most one vertex is visited).
- 5. $E = \bigwedge_{(v,w)\notin E} \bigwedge_{i\in[n-1]} (X_{i,v} \to (X_{i+1,w} \to \bot))$ (: if there is no edge from v to w then w can't be visited immediately after v).

Thus G is Hamiltonian iff α_G is satisfiable. Denote by SAT_{Cla} the set of satisfiable formulas in classical propositional logic and by $TAUT_{Int}$ the set of tautologies in the intuitionistic one. Then the following conditions hold: (1) G is non-Hamiltonian iff $\alpha_G \notin SAT_{Cla}$, (2) G is non-Hamiltonian iff $\neg \alpha_G \in$ $TAUT_{Cla}$, (3) G is non-Hamiltonian iff $\neg \alpha_G \in TAUT_{Int}$. Glyvenko's theorem yields the equivalence between (2) to (3). Hence G is non-Hamiltonian iff there is an intuitionistic proof of $\neg \alpha_G$. Such proof is called a certificate for the nonhamiltoniancy of G. [11] (also [5]) 9presented a translation from formulas in full propositional intuitionistic language into the purely implicational fragment of minimal logic whose formulas are built up from \rightarrow and propositional variables. This translation employs new propositional variables q_{γ} for logical constants and complex propositional formulas γ (in particular, every $\alpha \vee \beta$ and $\alpha \wedge \beta$ should be replaced by $q_{\alpha\vee\beta}$ and $q_{\alpha\wedge\beta}$, respectively) while adding implicational axioms stating that q_{γ} is equivalent to γ . For any propositional formula γ , let γ^* denote its translation into purely implicational minimal logic in question. Note that $size(\gamma^*) \leq size^3(\gamma)$. Now $\gamma \in TAUT_{Int}$ iff γ^* is provable in the minimal logic. Moreover, it follows from [11], [5] that for any normal ND proof ∂ of γ there is a normal proof ∂_{\to} of γ^{\star} in the corresponding ND system for minimal logic, NM_{\rightarrow} , such that $h(\partial_{\rightarrow}) = \mathcal{O}(h(\partial))$. Thus in order to prove Lemma 12 it will suffice to establish

Claim 15 G is non-Hamiltonian iff there exists a normal intuitionistic tree-like ND proof of $\alpha_G \to \bot$, i.e. $\neg \alpha_G$, whose height is polynomial in n.

Proof. Straightforward (see [4] for details). ■ This completes proofs of Lemma 12 and Corollary 13.

2.4.2 Case NP vs PSPACE

In the sequel we consider standard language $\mathcal{L}_{\rightarrow}$ of minimal logic whose formulas $(\alpha, \beta, \gamma, \rho \text{ etc.})$ are built up from propositional variables (p, q, r, etc.) using

one propositional connective ' \rightarrow '. The sequents are in the form $\Gamma \Rightarrow \alpha$ whose antecedents, Γ , are viewed as multisets of formulas; sequents $\Rightarrow \alpha$, i.e. $\emptyset \Rightarrow \alpha$, are identified with formulas α .

Recall that HSC for minimal logic, LM $_{\rightarrow}$, includes the following axioms (MA) and inference rules (MI1 $_{\rightarrow}$), (MI2 $_{\rightarrow}$), (ME $_{\rightarrow}$ P), (ME $_{\rightarrow}$) in the language $\mathcal{L}_{\rightarrow}$ (the constraints are shown in square brackets).

$$\begin{array}{ll} (\mathrm{M}A): & \Gamma, p \Rightarrow p \\ \hline \\ (\mathrm{M}I1 \rightarrow): & \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} & [(\nexists \gamma): (\alpha \rightarrow \beta) \rightarrow \gamma \in \Gamma] \\ \hline \\ (\mathrm{M}I2 \rightarrow): & \frac{\Gamma, \alpha, \beta \rightarrow \gamma \Rightarrow \beta}{\Gamma, (\alpha \rightarrow \beta) \rightarrow \gamma \Rightarrow \alpha \rightarrow \beta} \\ \hline \\ (\mathrm{M}E \rightarrow P): & \frac{\Gamma, p, \gamma \Rightarrow q}{\Gamma, p, p \rightarrow \gamma \Rightarrow q} & [q \in \mathrm{VAR} \left(\Gamma, \gamma\right), p \neq q] \\ \hline \\ (\mathrm{M}E \rightarrow \rightarrow): & \frac{\Gamma, \alpha, \beta \rightarrow \gamma \Rightarrow \beta}{\Gamma, (\alpha \rightarrow \beta) \rightarrow \gamma \Rightarrow q} & [q \in \mathrm{VAR} \left(\Gamma, \gamma\right)] \\ \hline \end{array}$$

Claim 16 LM $_{\rightarrow}$ is sound and complete with respect to minimal propositional logic and tree-like deducibility. Any given formula ρ is valid in the minimal logic iff sequent $\Rightarrow \rho$ is provable in LM $_{\rightarrow}$ by a quasi-polynomial tree-like deduction.

Proof. Easily follows from [6] (see [2] for details).

Lemma 17 For any valid purely implicational formula ρ there exists a quasi-polynomial tree-like proof $\partial \vdash \rho$ in NM_{\rightarrow} .

Proof. This ∂ is a straightforward interpretation in NM $_{\rightarrow}$ of a proof in LM $_{\rightarrow}$ that must exist by the validity of ρ (see [2] for details).

Recall that the validity problem in minimal logic is **PSPACE**-complete [11], [12]. Together with Theorem 5 and Corollary 12 this yields

Corollary 18 NP = PSPACE holds true.

Corollary 19 The satisfiability and validity problems in quantified boolean logic (QBL) are both NP-complete, since corresponding PSPACE-completeness is well-known (see e.g. [1], [8]). Moreover $\mathbf{BQP} \subseteq \mathbf{NP}$ holds, where \mathbf{BQP} is the class of problems computable in quantum polynomial time. This follows from the known inclusion $\mathbf{BQP} \subseteq \mathbf{PSPACE}$ (cf. [1]).

Conclusion 20 (PSPACE paradise) Denote by \mathbb{U} the universe of solvable computational problems and let $\mathbb{V} := \mathbf{PSPACE} \subsetneq \mathbb{U}$ be the proper subuniverse consisting of problems solvable in polynomial space.

⁴This is a slightly modified, equivalent version of the corresponding purely implicational and \perp -free subsystem of Hudelmaier's intuitionistic calculus LG, cf. [6]. The constraints $q \in VAR(\Gamma, \gamma)$ are added just for the sake of transparency.

Thus V contains all problems whose solutions are polynomially admissible with respect to the space used (regardless of the required time). Loosely speaking, V is the world of problems solvable in the material world of sufficiently big computers, without any time restriction. It is known that V preserves basic propositional operations and non-deterministic provability and includes \mathbf{BQP} (cf. e.g. [1], [8]).

Let $\mathbb{W} := \mathbf{NP} \subseteq \mathbb{V} \subsetneq \mathbb{U}$ be another subuniverse consisting of problems that are potentially solvable in polynomial time. Now Corollary 18 shows that $\mathbb{W} = \mathbb{V}$, i.e. any given problem $X \in \mathbb{V}$ (in particular $X \in \mathbf{BQP}$) is in fact solvable by some deterministic polynomial-time $TM\ M_X$. Hence all problems in \mathbb{V} are polynomially admissible with respect to both space and time used. To paraphrase Hilbert's famous quotation: in \mathbb{V} there is no polynomial-time ignorabimus. One can ask whether M_X can be obtained from X by a polynomial-time algorithm. The answer in NO, provided that $\mathbf{P} \neq \mathbf{NP}$.

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