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SOME SEQUENCES OF INTEGERS

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Combinatorialists are interested in sequences of integers which count things. We often find that the same sequence counts two families of things with no obvious connection, or that a simple translation connects the answers to two counting problems. In this way, unexpected connections have come to light.

1. The Handbook

What I want to describe is a kind of experimental mathematics, ideal for doing at times when honest thinking is not going well. The requirements are a small computer (pencil and paper suffice, though the calculations are tedious), and Neil Sloane's "A Handbook of Integer Sequences" [15].

This book, a kind of hitch-hikers' guide to the universe $\mathbb{N}^{\mathbb{N}}$, consists mainly of a list of 2372 sequences of nonnegative integers, arranged lexicographically, with an index, references, and notes for users. The main criterion for inclusion of a sequence is that somebody must have found it sufficiently interesting to record it in the literature. The Handbook can be used, then, like a book of tables, using the index to locate a sequence. A more exciting possibility is this. Suppose you find youself in possession of an "unknown" sequence. (This is not an uncommon event; a glance through the Handbook confirms that sequences occur in all provinces of mathematics, and well beyond its frontiers.) If you can locate your sequence in the Handbook, you have both a problem (of showing that your sequence really is the one listed) and a source of information (the references to the sequence). I know of several cases where new results have been discovered this way.

I propose a third way of using the Handbook. There are some naturally-occurring transformations of sequences, two of which I will consider in detail. Finding instances where a known sequence is transformed into another can give rise to new mathematical insights in the way described above. Also any sequence which is transfirmed into a closely-related one gains significance independent of the objects it counts.

Sloane adops the convection that all sequences commence 1, n, when $n > 1$. To ensure this, he deletes "superfluous" leading ones and zeros, and inserts a 1 if necessary. Some valuable information is lost in this way, namely the "natural" starting point of the sequence. But, on the positive side, the weakness of the convention draws our attention to the operation of "shifting" a sequence, which will prove fruitful.

I will refer to known sequences by their number in the Handbook.

2. The background

I will now describe the area from which the sequences of most interest to me arise. There are three main sources.

(i) In combinatorics, sequences come from counting problems: if C is a class of objects of some kind, let x_n be the number of objects of cardinality n in C, up to some well-defined notion of isomorphism. I will refer to $(x_1, x_2, \ldots)$ as the sequence enumerating C.

(ii) Graded algebras. Let $A = F \cdot 1 \oplus V_1 \oplus_2 \oplus \cdots$ be a graded algebra over F – that is, each V_i is a vector space over F, and there is a multiplication on A satisfying $V_i \cdot V_j \subseteq V_{i+j}$. If each V_i is finite-dimensional, we have a sequence $(x_1, x_2, \ldots)$, where $x_n = \dim V_n$. Its generating function $1 + \sum_{n \geq 1} x_n t^n$ is the Poincaré series of A. Another example: it may happen that A is a polynomial ring generated by a set of homogeneous elements (i.e. each lying in V_n for some n); then there is a sequence enumerating generators in V_n.

(iii) Permutation groups. Let G be a permutation group on an infinite set X. Then G acts on the set X_n of n-element subsets of X in an obvious way. Let $f_n(G)$ be the number of G-orbits in X_n. An interesting class consists of those permutation groups G for which $f_n(G)$ is finite for all n. = oligomorphic

(This class has been of interest to model theorists, in view of a celebrated theorem of Svenonius [16]: the countable first-order structure M is ω-categorical (i.e. determined up to isomorphism by first-order axioms and the assumption of countability) if and only if the permutation group $\mathrm{Aut}(M)$ has finitely many orbits on n-subsets of M for all n.)

For such groups, we have a sequence $(f_1, f_2, \ldots)$; we say it is realised by G. It is an interesting problem to characterise sequences realised by groups. Any such sequence must be non-decreasing, and there are gaps in the spectrum of growth rates (Macpherson [13], [14]). Many examples appear in this paper.

In fact, (iii) is a special case of both (i) and (ii). For (i), take a permutation group G on X; it is easy to construct a relational structure M on X such that $G \leq \mathrm{Aut}(M)$, and any isomorphism between finite substructures of M is induced by an automorphism of M – such structures are called *homogeneous* – and even by an element of G. Thus $f_n(G)$ is the number of isomorphism types of finite substructures of M. Furthermore, Fraïssé [9] gave a necessary and sufficient condition for a class of finite structures to be the finite substructures of a homogeneous structure.

For (ii), let $V_n(G)$ be the space of G-invariant functions from X_n to $\mathbb{Q}$, as

$\mathbb{Q}$-vector space. Set

$$A(G) = \mathbb{Q} \cdot 1 \oplus V_1(G) \oplus \cdots,$$ graded algebra associated to G

and define a multiplication as follows: for $f \in V_i(G)$, $g \in V_j(G)$, let $fg \in V_{i+j}(G)$ map the $(i+j)$-set K to

$$fg(K) = \sum \{f(I)g(K \backslash I) : I \subseteq K, |I| = i\}.$$ "convolution product" May convolution?

It is easily checked that $A(G)$ is commutative and associative and that dim $V_n(G) = f_n(G)$ (if this number is finite).

3. The operators

I will be considering the operators S and A defined on the set of sequences of non-negative integers as follows: for $x = (x_n)$, set $Sx = (y_n)$ and $Ax = (z_n)$, where

S:
$$1 + \sum_{n \geq 1} y_n t^n = \prod_{n \geq 1} (1 - t^n)^{-x_n}$$

and

A:
$$1 + \sum_{n \geq 1} z_n t^n = \left(1 - \sum_{n \geq 1} x_n t^n\right)^{-1}.$$ $Ax = \frac{1}{1-x}$ provided $x(0) = 0$

These definitions will look either obvious or artificial, depending on your background. Here is the motivation.

(i) Suppose that x enumerates a class C. Then Sx and Ax each enumerate the class of disjoint unions of members of C where, for Sx, the order of the "component" members of C is unimportant, while for A it is significant. (Think of the process of building a structure, e.g. a graph, from its connected components.)

(ii) Suppose that the graded algebra A is a polynomial ring generated by homogeneous elements, and let x enumerate its generators. Then Sx gives the dimensions of the homogeneous components. The operator A plays a similar role for free associative (non-commutative) algebras.

(iii) We need the concept of wreath product of permutation groups. Let G and H be permuation groups on X and Y respectively. The set on which the wreath product acts is $X \times Y$, regarded as a covering of Y with fibres isomorphic to X (that is, a disjoint union of copies of X indexed by Y). Now the wreath product G *Wr* H is generated by two kinds of permutations: those which permute the fibres according to the action of a member of H on Y, and those which fix every fibre and induce independently-chosen elements of G on the fibres.

Let S denote the symmetric group on an infinite set, and A the group of order-preserving permutations of the rational (or real) numbers. Now, if x is realised by G, then Sx and Ax are realised by G *Wr* S and G *Wr* A respectively.

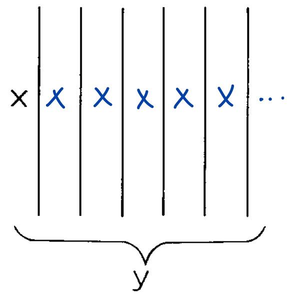

Fig. 1.

The last remark can be translated into either of the other interpretations. Let G Wr H act on $X \times Y$ as above. An n-subset of $X \times Y$ is partitioned by the fibres it meets; the "connected" subsets are those contained in a single fibre, and they are enumerated by $(f_n(G))$. The structure of the set of connected components is governed by H. Also, regardless of the algebraic structure of $A(G)$, $A(G$ Wr $S)$ is always a polynomial ring, and $(f_n(G))$ enumerates its generators. (However, there is no known analogue for G Wr A.)

I remark in passing that some interesting combinatorics, involving Stirling numbers, comes from considering S Wr G instead of G Wr S; see Cameron and Taylor [4].

The remainder of this paper is a sequence of commented examples.

4. The random graph

Erdös and Rényi [7] showed that there is a graph R on a countably infinite set of vertices, which has the following remarkable property: If a countable graph is chosen at random (by considering all pairs of vertices in turn, and for each pair, tossing a coin – fair or biased – to decide whether to join those vertices with an edge), the resulting graph is, with probability 1, isomorphic to R. This property clearly characterises R up to isomorphism. Naturally, R is called "the random graph".

R has many further interesting properties (see Cameron [2]). Two of the most basic are these: Every finite (or countable) graph is embeddable in R, and any isomorphism between finite subgraphs of R extends to an automorphism of R. (In our earlier terminology, R is homogeneous.) Thus the orbits of the group $G = \mathrm{Aut}(R)$ on X_n correspond to n-vertex graphs up to isomorphism, and the sequence realised by G enumerates graphs by number of vertices. This is Sloane #479: 1, 2, 4, 11, 34, 156, 1044, 12346, . . . (Here, as in most other cases, Sloane gives many more terms, enough to fill two lines of text.)

The considerations of the last section show that, if X is the sequence enumerating connected graphs by vertices – i.e. Sloane #649: (1), 1, 2, 6, 21, 112, 853, 11117, . . . – then Sx enumerates all graphs, i.e. $Sx = f(G)$. This might

a graph is a disjoint
union of connected graphs

lead us to guess that the algebra $A(G)$ is a polynomial ring whose generators are enumerated by x. This is true; indeed, the generators are identified with connected graphs under the natural identification of basis vectors with all graphs.

This phenomenon holds much more generally. What is needed is a "good" decomposition of arbitrary objects into connected ones; sufficient conditions can be given. Examples include the pairs $f(G)$ and $f(G\ Wr\ S)$ described earlier, and directed graphs, partially ordered sets, finite topologies, etc. (Sloane ##1133 and 648, 545 and 985, 588 and 1152).

5. The all-1 sequence

Let 1 denote the all-1 sequence. (This is not in the Handbook – Sloane's convention excludes sequences with no term greater than 1 – but it is obviously important. It enumerates sets and totally ordered sets; it gives the dimensions of homogeneous components of a polynomial ring in one variable; and it is realised by both S and A, among others.) What is $S1$? Clearly it enumerates partitions of a set with no structure and with no distinguished order of the parts; in effect, partitions of an integer. Thus, $S1$ is the partition function p (Sloane #244). The definition of S gives the familiar generating function $1 + \sum_{n\geqslant 1} p(n)t^n = \prod_{n\geqslant 1}(1-t^n)^{-1}$. CDA?

If we apply S again and refere to the Handbook, we find Sloane #1019, with the name "functional determinants" and a reference to Cayley [5]. It turns out that Cayley was counting the types of projective transformation over an algebraically closed field, in other words, the Jordan forms of $n \times n$ matrices. There are infinitely many Jordan forms; but, if we neglect the actual eigenvalues and note only whether the eigenvalues in two blocks are equal or not, the number is finite. (For example, all generic matrices, i.e. diagonal matrices with distinct eigenvalues, are identified.) We have a partition of n corresponding to the eigenvalues, and a partition of each part corresponding to the Jordan blocks. The identification with $S^2 1$ is clear.

We turn now to $A1$. As before, this sequence enumerates partitions of n, but

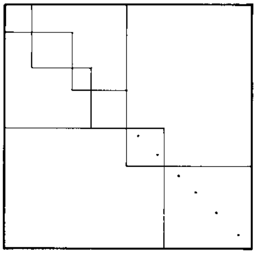

Fig. 2.

now the parts are ordered. For example, $(A1)_3 = 4$, since $3 = 2+1 = 1+2 = 1+1+1$. Experimentally we find that $A1 = (1, 2, 4, 8, 16, \ldots)$ (Sloane #432: powers of 2), proving this is a pleasant exercise. More generally, A^r1 is the sequence of powers of $r+1$. (Sloane lists several of these: ##1129, 1428, 1630, 1765, 1874, 1937, 1992, 2054, 2084, 2107, 2120, 2164, 2182, 2192, 2198).

Further experimentation reveals that $S^{-1}A1$ is Sloane #287: (1,) 1, 2, 3, 6, 9, 18, 30, . . . , called "irreducible polynomials, or necklaces". The first description suggests the proof. There are 2^{n-1} polynomials of degree n over $GF(2)$ with nonzero constant term; each has a unique factorisation into irreducibles, with order unimportant; and all irreducibles $f(t)$ except t occur. We notice, incidentally, that is x enumerates all irreducibles, i.e. $x = (2, 1, 2, 3, 6, 9, \ldots)$ (Sloane #46), then Sx enumerates all polynomials, i.e. $(Sx)_n = 2^n$, which can be regarded as either a shift or a double of #432.

There is a problem here. Sloane #432 is realised by (at least) two quite different groups G. First, we can take $G = H\,Wr A$, where H realises the sequence 1. (Such groups H are called *highly homogeneous*. Both S and A are examples.) Second, we can take a partition of $\mathbb{Q}$ into two dense subsets, and let g be the subgroup of A which preserve the partition (i.e. fix or interchange the subsets). For either group G, if it holds that $A(G)$ is a polynomial ring, then Sloane #287 enumerates the generators. But is $A(G)$ a polynomial ring? Similarly, the subgroup G' of the second G fixing both subsets realises the double of #432, and the same problem arises with #46.

6. The natural numbers

Let x be the sequence $(1, 2, 3, \ldots)$ of natural numbers (Sloane #173). Empirically, Sx is Sloane #1016: planar partitions, and Ax is Sloane #1101: bisection of (i.e. alternate terms of) the Fibonacci sequence. I leave the proofs as exercises.

I do not know whether either #1016 or #1101 is realised by a group. (The Fibonacci numbers themselves, Sloane #256, are realised; for they form the sequence $A(1, 1, 0, 0, \ldots)$, and so are realised by $Z_2\ Wr\ A$. The combinatorial interpretation is that F_n is the number of ways of writing n as an ordered sum of ones and twos.) #173, as it stands, is not realisable; but if we shift this sequence, obtaining $y = (2, 3, 4, 5, \ldots)$, we obtain a realisable sequence. The group in question is $G = S \times S$, acting on the disjoint union of two sets which are orbits of the factors. (The orbit of an n-set is determined by the cardinality r of its intersection with the first G-orbit, and r can be any integer in the range $[0, n]$). One would expect that Sy and Ay would count objects similar to those for Sx and Ax; but neither sequence is in the Handbook. (For the record, the first few terms of Sy and Ay are

2, 6, 14, 33, 70, 149, 298, . . .

$F_n = F_{n-1} + F_{n-2}$;
unordered sum of 1's and 2's?

and

$$2, 7, 24, 82, 280, 956, 3264, \ldots$$

respectively.

7. Self-generating sequences

i.e., recursively defined

I turn now to sequences which are only slightly modified by S or A. No interesting sequence can be wholly unaltered: the only fixed point of A is the zero sequence, and no sequence x with $x_1 > 0$ is fixed by S.

Consider the problems: Which sequences x with $x_1 = 1$ are (a) shifted one place to the left, or (b) doubled (apart from the first term, which is unaltered) by the action of (i) S, (ii) A? There are four distinct problems here. We note that each problem has a unique solution. For

$$(Sx)_n = f_n(x_1, \ldots, x_n)$$

for some function f_n; so the solution to (i) (a) is given by the recurrence

$$x_1 = 1 \cdot x_{n+1} = f_n(x_1, \ldots, x_n).$$

Also, f_n has the form

$$f_n(x_1, \ldots, x_n) = x_n + f'_n(x_1, \ldots, x_{n-1});$$

so the solution to (ii) (a) satisfies the recurrence

$$x_1 = 1,\ x_n = f'_n(x_1, \ldots, x_{n-1}) \quad \text{for } n > 1.$$

The argument for (b) is the same.

Moreover, the solutions are easy to calculate. For example, for (i), use the procedure for evaluating S or A with each output as next input. But can we anticipate the results?

The solution x to (i) (a) should count a class G of structures for which the connected structures on $n + 1$ points correspond to all structures on n points. A little thought shows that rooted trees (Sloane #454) fill the bill. Cayley trees

For (i) (b), we want a class with a bijective correspondence between connected and disconnected objects on n points for $n > 1$. The obvious correspondence to use is complementation of graphs – the complement of a disconnected graph is,

$\partial_t f = t \cdot e^f$

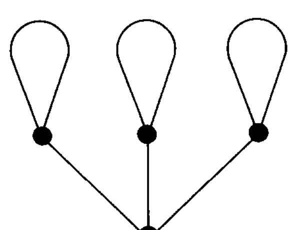

Fig. 3.

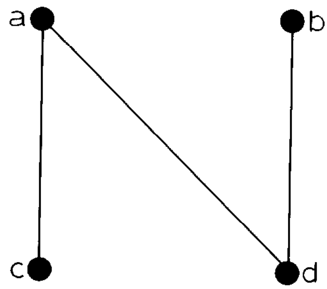

Fig. 4.

after all, connected – and there is a class with the required property. It can be described either as the smallest class containing the 1-vertex graph and closed under complementation and disjoint union, or as the class of *N-free graphs* (graphs containing no path of length 3 as an induced subgraph). The sequence enumerating connected N-free graphs is Sloane #558, and its "double" enumerating all N-free graphs is Sloane #466. These are described as "Series-reduced planted trees" and "Series-parallel networks" respectively. At first glance, it is not clear what these descriptions have to do with one another or with N-free graphs. Answering these questions and realising the sequences by groups has led to some fruitful research on treelike objects (Cameron [3], Covington [6]).

Now consider problem (ii). We would expect the solutions to be ordered analogues of those for (i). Thus, for (ii) (a), we count rooted trees in which the set of branches above the root is ordered, and each branch has the same property. This is just the recursive specification for depth-first search on the tree. In other words, our objects are rooted plane trees. The enumerating sequence is the Catalan numbers (1), 1, 2, 5, 14, 42, 132,. . . (Sloane #577), one of the most celebrated sequences in combinatorial mathematics. (The nth term is $\binom{2n-2}{n-1}/n$.)

The fact that the Catalan numbers form the unique solution to (ii) (a) helps identify them in some of their many guises. For example, consider the number x_n of paths in the plane from (0, 0) to $(2n, 0)$ such that

(i) each step is from (x, y) to $(x+1, y+1)$ or $(x+1, y-1)$;

(ii) apart from the end-points, the path lies strictly above the x-axis.

The sequence Ax enumerates ordered unions of such paths; these are just paths satisfying (i) and (ii) with "strictly above" replaced by "above or on". But any such path from 0 to $2n$ yields, by extending its ends down one step and then translating by (1, 1), a solution to the strict problem, with n replaced by $n+1$. So $(Ax)_n = x_{n+1}$. Since $x_1 = 1$, x is the Catalan sequence.

The sequence A^2x also appears in the Handbook, as #1144 ("Central binomial coefficients"); the nth term is $[\binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n}]$. Why is this? Consider ordered unions of paths satisfying the "weak" specification above, where alternate components are reflected in the horizontal axis. These account for half of all the possible paths (since the first step is upwards). But there are $\binom{2n}{n}$ paths altogether, since we merely have to choose the set of positions where the n upward steps are taken.

This can be viewed another way. Consider the set of paths defined by the "strict" specification and their reflections. These are enumerated by the Catalan numbers doubled (Sloane #128). Ordered unions of these give all possible paths, enumerated by the sequence with nth term $\binom{2n}{n}$ (Sloane #643). I will return to this oddity in the next section.

Note, by the way, that the inverse image under A of the Catalan sequence is Sloane #635. Why?

The unique solution to (ii) (b) enumerates complementary pairs of N-free posets. (A poset is N-free if it has no four elements a, b, c, d with $a > c$, $a > d$, $b > d$, and other pairs incomparable – see Fig. 4. Two posets on the same set are complementary if each pair is comparable in precisely one of the posets.) This sequence is Sloane #1163 ("Dissections of a polygon, or parenthisizing a product"). In fact it is also listed (with a small misprint) as #1170 ("Schröder's second problem"). Once again, explanation proves fruitful.

A question of some interest to statisticians (Bailey [1]) is that of enumerating N-free posets. Let $x_1 = y_1 = z_1 = 1$ and, for $n > 1$, let x_n, y_n, z_n be the numbers of connected, disconnected, and arbitrary N-free posets on n points. Then clearly

$$z_n = x_n + y_n \quad \text{for } n > 1$$

and

$$\overset{\text{set}}{S}x = \overset{\text{sequence}}{A}y = z.$$

From this, the sequences can easily be calculated. We have

$$x = (1,\ 1,\ 3,\ 9,\ 30,\ 103,\ 375, \ldots)$$

$$y = (1,\ 1,\ 2,\ 6,\ 18,\ 64,\ 227, \ldots)$$

$$z = (1,\ 2,\ 5,\ 15,\ 48,\ 167,\ 602, \ldots)$$

Note that the solutions to (i) (b) or (ii) (b), doubled, are lower and upper bounds respectively for z. But I do not have a good asymptotic estimate for z.

The unique sequence x with $x_1 = 2$ which is shifted by the application of A turns out to be twice the solution to (ii) (b)! I shall explain why in the next section.

8. Generating functions and functional equations

Many readers will know, or will have spotted for themselves, that the operator A lends itself readily to analysis by means of generating functions. Given a sequence x, let

$$X(t) = \sum_{n \geq 1} x_n t^n$$

be its generating function. A induces a map on formal power series, which I will denote by α: thus, by definition,

$$(1+\alpha X)=(1-X)^{-1},$$

whence $\boxed{\alpha X = X/(1-X).}$ Now an easy induction shows that $\boxed{\alpha^r X = X/(1-rX).}$ It follows immediately that $r\cdot\alpha^r X=\alpha(rX)$, or, for sequences, $r\cdot A^r x = A(rx)$.

As Patrick McCarthy pointed out to me, for $r=2$ this is an instance of the Feigenbaum-Cvitanović Eq. [8], albeit for functions on $\mathbb{N}^{\mathbb{N}}$ rather than $\mathbb{R}$. He also remarked that A (in its action on generating functions) is formally identical with the solution $f(x)=x/(1-x)$ of the F-C equation

$$f(f(x))=\tfrac{1}{2}f(2x),$$

$$f^2(x) = \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{x}{1-2x} = \frac{1}{2}f(2x)$$

discovered by Hirsch, Nauenberg and Scalapino [10]. For a survey of this area of mathematics, I refer to McCarthy [12].

The case $r=2$ also generalises our observation about Catalan numbers and central binomial coefficients in the last section; it shows that the Catalan numbers were really irrelevant there. But the argument suggests a combinatorial proof of the identity in this case.

Let C be a class of objects enumerated by a sequence x. Then $2x$ enumerates C-objects with an additional distinction into "red" and "blue" objects. Hence $A(2x)$ enumerates ordered unions of C-objects where the points are coloured red and blue so that points in the same component have the same colour. On the other hand, Ax enumerates ordered unions of C-objects, and A^2x ordered unions of these, which we may regard as being coloured alternately red and blue; these account for half of all the general coloured ojects, namely, all those starting with a red component. So $\boxed{2A^2x=A(2x).}$

I do not know a similar proof of the general identity.

Now let M_r denote multiplication by r; let M_r^* denote multiplication of all terms except the first by r; and let T denote the shift one place left. Now, among sequences x with $x_1=1$, each of the following conditions defines a unique one, and in fact they all define the same sequence (for fixed $r\geqslant 1$):

(i) $A^r x = Tx$
(ii) $AM_r x = TM_r x$
(iii) $AM_r^* x = Tx$

(The equivalence of (i) and (ii) is immediate from $AM_r = M_r A^r$ and $TM_r = M_r T$; that of (ii) and (iii) is proved by a generating function argument.)

The unique sequence defined by these conditions for $r=1$ is the Catalan numbers (Sloane #577), of course. For $r=2$, conditions (i) and (iii) imply that $Ax = M_2^* x$ – this was the problem (ii) (b) which characterised Sloane #1163. Now condition (ii) for this sequence, i.e. $A(2x)=T(2x)$, explains the observations right at the end of the last section.

9. Exponentiation and convolution

McCarthy [11, 12] has emphasized the analogy between the Feigenbaum–Cvitanović function, whose self-composition is just a re-scaled version of itself, and the exponential function. We have seen that A is a F-C function; by a delightful coincidence, S is an exponential function!

Let $x \oplus y$ denote the pointwise sum of sequences x and y, and $x \circ y$ their convolution, given by

$$(x \circ y)_n = \sum_{k=0}^{n} x_k y_{n-k},$$

with the convention $x_0 = y_0 = 1$. (Thus $x \circ y$ corresponds to the product of the generating functions $1 + X$ and $1 + Y$ of x and y.) Then it is immediate from the definition of S that $\boxed{S(x \oplus y) = Sx \circ Sy.}$

The operation of convolution ties in naturally with our examples (ii) and (iii). If A and B are graded algebras, the Poincaré series of $A \otimes B$ is the product of those of A and B. (And, if A and B are polynomial rings, then so is $A \otimes B$, generated by the disjoint union of the generating sets for A and B, according to the exponential equation for S.) Also, if G and H act on X and Y, respectively, then $G \times H$ acts on the disjoint union of X and Y, and

$$(f_n(G)) \circ (f_n(H)) = (f_n(G \times H)).$$

A number of sequences in Sloane arise as convolutions of smaller sequences. For example, the kth convolution of 1 has nth term $\binom{n+k-1}{k-1}$ (Sloane ##173, 1002, 1363, 1578, 1719, 1847, 1911, 1976, 2013, 2046, 2073). As a special case, $x \circ 1$ is the sequence of partial sums of x (again with the convention $x_0 = 1$); this accounts for Sloane ##374, 392, 394, 395, 396, 397, 1007, 1050, 1382, 1398. Other examples include Sloane ##128, 525, 533, 535, 536, 537, 1124, 1413, 1600, 1738, 1865.

Three more samples must suffice here.

(i) $S^{-1}1 = (1, 0, 0, \ldots)$; and, if x is the sequence of powers of 2 (#432), then $1 \circ x = 2x$. Thus

$$S^{-1}2x = S^{-1}x \oplus (1, 0, 0, \ldots),$$

as we observed in Section 5.

(ii) Let T be the left shift. Then any sequence satisfies

$$x \circ Ax = 2Ax;$$

and, if $x_1 = 1$, then also

$$Tx \circ Ax = TAx.$$

(These are proved by generating function arguments.)

(iii) Let x be the Catalan sequence (#577), starting 1, 2, 5, ... The familiar convolution property of Catalan numbers shows that

$$x \circ x = Tx.$$

From this, it follows that $Tx \circ Ax = x \circ 2Ax = TAx$. (For $Tx \circ Ax = TAx$ by (ii); the rest follows from the associativity of the convolution $x \circ x \circ Ax$.) These yield such consequences as $2S^{-1}x = S^{-1}Tx$.

10. Final remarks

Apart from the obvious comment that many connections remain to be explored, I want to draw attention to other operators in the class including S and A. If G is any permutation group satisfying our condition that $f_n(G)$ is finite for all n, then there is an operator G on sequences, specified by the rule that, if H realises x, then H *Wr* G realises Gx. Note that G maps the sequence $(1, 0, 0, \ldots)$ to the sequence realised by the group G. An interesting candidate is the group C preserving the cyclic order on the unit circle (or on the set of complex roots of unity). As with any highly homogeneous group C realises the sequence 1, so

$$C(1, 0, 0, \ldots) = (1, 1, 1, \ldots).$$

A short calculation shows that this is equivalent to the identity

$$e^{t/(1-t)} = \prod_{n\geq 1} (1-t^n)^{-\phi(n)/n}.$$

Jaap Seidel encouraged me to consider the operator S^* defined by

$$1 + \sum_{n\geq 1} (S^*x)_n t^n = \prod_{n\geq 1} (1+t^n)^{x_n}.$$

Among its properties are:

(i) If x lists the dimensions of homogeneous components of a graded vector space V, then S^*x lists those of the exterior algebra of V. (Compare S, which does the same job for the symmetric algebra).

(ii) Like S, S^* is an "exponential":

$$S^*(x \oplus y) = S^*x \circ S^*y.$$

(iii) If x enumerates a class C of objects, then S^*x enumerates the class of disjoint unions of objects in C satisfying the "exclusion principle", that is, no two the same. This has some amusing consequences:

(a) S^*1 enumerates partitions with all parts distinct (Sloane #100).

(b) If x enumerates asymmetric connected graphs, then S^*x enumerates asymmetric graphs.

(c) The unique sequence x with $x_1 = 1$ which is shifted by S^* enumerates asymmetric rooted trees.

(iv) The identity $(1+t)\ (1-t^2)^{-1} = (1-t)^{-1}$ has the following consequence. For any sequence x,

$$S^*x \circ (Sx)^\circ = Sx,$$

where y° denotes the sequence obtained by alternating zeros with terms of y.

Note added in proof

As I hoped, my lecture at the conference elicited some further connections from members of the audience.

(i) Several people gave a combinatorial proof of the identity $A(rx) = rA^r x$.

(ii) Concerning Ax, where x is $(1, 2, 3, 4, \ldots)$: Ron Graham and Donald Knuth informed me that that identifying this sequence with the alternate Fibonacci numbers is an exercise in their forthcoming book. (To recapitulate: the n^{th} term is found by expressing n in all possible ways as an ordered sum of positive integers, multiplying the terms in each sum, and adding these products. The problem is to prove the identification without using generating functions). Knuth also pointed out that this sequence counts spanning trees in a fan.

(iii) Some explorers discovered that the other bisection of the Fibonacci sequence, viz. $(1, 2, 5, 13, 34, \ldots)$ is Ay, where $y = (1, 1, 2, 4, \ldots)$ (powers of 2, shifted right and preceded by 1).

(iv) Knuth also remarked that the double of the sequence satisfying (ii) (b) of Section 7 counts permutations which can be produced by a deque (double-ended queue).

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