Noetherian quotients of the algebra of partial difference polynomials and Gröbner bases of symmetric ideals ☆

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A R T I C L E   I N F O

A B S T R A C T

In this paper we develop a Gröbner bases theory for ideals of partial difference polynomials with constant or non-constant coefficients. In particular, we introduce a criterion providing the finiteness of such bases when a difference ideal contains elements with suitable linear leading monomials. This can be explained in terms of Noetherianity of the corresponding quotient algebra. Among these Noetherian quotients we find finitely generated polynomial algebras where the action of suitable finite dimensional commutative algebras and in particular finite abelian groups is defined. We obtain therefore a consistent Gröbner bases theory for ideals that possess such symmetries.

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1. Introduction

The theory of difference algebras (see the books [10,20,27] and references therein) was introduced in the 1930s by the mathematician Joseph Fels Ritt at the same time as the

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theory of differential algebras. Indeed, for a quite long time, difference algebras have attracted less interest among researchers in comparison with differential ones despite the fact that numerical integration of differential equations relies on solving finite difference equations. The rapid development of symbolic computation and computer algebra in the last decade of the previous century gave rise to rather intensive algorithmic research in differential algebras and to the creation of sophisticated software as the *diffalg* library [6], implementing the Rosenfeld–Gröbner algorithm and included in MAPLE and the package DIFFERENTIALTHOMAS [3]. At the same time, except for algorithmization and implementation in MAPLE of the shift algebra of linear operators [9] as a part of the package ORE_ALGEBRA and the Maple package LDA [17], practically nothing has been developed in computer algebra in relation to difference algebras. Nevertheless, in the last few years, the number of applications of the theory and the methods of difference algebras has increased fastly. For instance, it turned out that difference Gröbner bases may provide a very useful algorithmic tool for the reduction of multiloop Feynman integrals in high energy physics [13], for automatic generation of finite difference approximations to partial differential equations [15,26] and for the consistency analysis of these approximations [14,16]. Relevant research has been developed also in the context of linear functional systems [28,30,31]. In addition to these natural applications, another source of interest for difference algebras consists in the notion of "letterplace correspondence" [21,22,24] which transforms non-commutative computations for presented groups and algebras into analogue computations with ordinary difference polynomials. As a result of all this use, a number of computer algebra packages implementing involutive and Buchberger's algorithms for computing difference Gröbner bases have been developed (see [14,17,23] and reference therein). A major drawback in these computations, as for the differential case, is that such bases may be infinite owing to non-Noetherianity of the algebra of difference polynomials. In fact, if X is a finite set and Σ denotes a multiplicative monoid isomorphic to $(\mathbb{N}^r, +)$ then the algebra of difference polynomials is by definition the polynomial algebra P in the infinite set of variables $X \times \Sigma$. Then, to provide the termination of the procedures computing Gröbner bases in P at least in some significant cases, we propose in this paper essentially two solutions. One consists in defining an appropriate grading for P that allows finite truncated computations for difference ideals $J \subset P$ generated by a finite number of homogeneous elements. For monomial orderings of P that are compatible with such a grading this implies a criterion, valid also for the non-graded case, which is able to certify the completeness of a finite Gröbner basis computed on a finite number of variables of P. After the algebra of partial difference polynomials and its Gröbner bases are introduced in Sections 2 and 3, this approach is described in Section 4 and an illustrative example based on the approximation of the Navier–Stokes equations is given in Section 5. A second solution to the termination problem consists in requiring that the difference ideal J contains elements with suitable linear leading monomials which corresponds to have the Noetherian property for the quotient algebra P/J. Some similar ideas appeared for the differential case in [8,32]. One finds this second approach in Section 6. It is interesting to note that a relevant class of such Noetherian quotient

algebras is given by polynomial algebras P' in a finite number of variables which are under the action of a tensor product of a finite number of finite dimensional algebras generated by single elements. These finite dimensional commutative algebras include for instance group algebras of finite abelian groups and hence, as a by-product of the theory of difference Gröbner bases, one obtains a theory for Gröbner bases of ideals of P' that are invariant under the action of such groups or algebras (see also [20,29]). These ideas are presented in Section 7 and a simple application is described in Section 8. Finally, in Section 9 one finds conclusions and hints for further developments of this research.

2. Algebras of difference polynomials

In this section we introduce the algebras of partial difference polynomials as freely generated objects in a suitable category of commutative algebras that are invariant under the action of a monoid isomorphic to $\mathbb{N}^r$ (the monoid of partial shift operators). This is a natural viewpoint since in the formal theory of partial difference equations the unknown functions and their shifts are assumed to be algebraically independent. Note that one has a similar situation with the theory of algebraic equations where the algebras of polynomials are free objects in the category of commutative algebras.

Let $\Sigma = \langle \sigma_1, \ldots, \sigma_r \rangle$ be a free commutative monoid which is finitely generated by the elements σ_i. We denote Σ in the multiplicative way with 1 as the identity element. One has clearly that $(\Sigma, \cdot)$ is isomorphic to the additive monoid $(\mathbb{N}^r, +)$ by the mapping $\sigma_1^{\alpha_1} \cdots \sigma_r^{\alpha_r} \mapsto (\alpha_1, \ldots, \alpha_r)$. Let K be a field and denote by $\mathrm{End}(K)$ the monoid of ring endomorphisms of K. We say that Σ *acts on K* or equivalently that K is a Σ-*field* if there exists a monoid homomorphism $\rho : \Sigma \to \mathrm{End}(K)$. In this case, for all $\sigma \in \Sigma$ and $c \in K$ we denote $\sigma \cdot c = \rho(\sigma)(c)$. Starting from now, we always assume that K is a Σ-field. We say that K is a *field of constants* if Σ acts trivially on K, that is, $\sigma \cdot c = c$, for any $\sigma \in \Sigma$ and $c \in K$.

Let A be a commutative K-algebra. We say that A is a Σ-*algebra* if there is a monoid homomorphism $\rho' : \Sigma \to \mathrm{End}(A)$ extending $\rho : \Sigma \to \mathrm{End}(K)$, that is, $\rho'(\sigma)(c) = \rho(\sigma)(c)$, for all $\sigma \in \Sigma$ and $c \in K$. To simplify notations, for any $\sigma \in \Sigma$ and $a \in A$ we put $\sigma \cdot a = \rho'(\sigma)(a)$. Let B be a K-subalgebra of a Σ-algebra A. We call B a Σ-*subalgebra* of A if $\Sigma \cdot B = \{\sigma \cdot b \mid \sigma \in \Sigma, b \in B\} \subset B$. In the same way, if I is an ideal of A such that $\Sigma \cdot I \subset I$ then we call I a Σ-*ideal* of A. Let B be a K-subalgebra of A and let $X \subset B$ be a subset. If B is the subalgebra generated by $\Sigma \cdot X$ then B coincides clearly with the smallest Σ-subalgebra of A containing X. In this case, we say that B is the Σ-subalgebra which is Σ-*generated by X* and we denote it as $K[X]_\Sigma$. In a similar way, if $X \subset I \subset A$ is the ideal generated by $\Sigma \cdot X$ then one has that I is the smallest Σ-ideal of A containing X. Then, we say that I is the Σ-ideal which is Σ-*generated* by X and we make use of notation $I = \langle X \rangle_\Sigma$. We also say that X is a Σ-*basis* of I.

Let A, B be Σ-algebras and let $\varphi : A \to B$ be a K-algebra homomorphism. We call φ a Σ-*homomorphism* if $\varphi(\sigma \cdot a) = \sigma \cdot \varphi(a)$, for all $\sigma \in \Sigma$ and $a \in A$. In the category of Σ-algebras one can define free objects as follows. Let X be a set and denote $x(\sigma)$ each

element (x, σ) of the product set $X(\Sigma) = X \times \Sigma$. Define $P = K[X(\Sigma)]$ the K-algebra of polynomials in the commuting variables $x(\sigma) \in X(\Sigma)$. For any element $\sigma \in \Sigma$, consider the ring endomorphism $\bar{\sigma} : P \to P$ such that

$$cx(\tau) \mapsto (\sigma \cdot c)x(\sigma\tau)$$

for all $c \in K$ and $x(\tau) \in X(\Sigma)$. Clearly, we have a monoid homomorphism $\rho : \Sigma \to \operatorname{End}(P)$ such that $\rho(\sigma) = \bar{\sigma}$, for any $\sigma \in \Sigma$. By definition of $\bar{\sigma}$, one has that ρ extends to P the action of Σ on the base field K, that is, P is a Σ-algebra. Note that the homomorphism ρ is in fact an injective map. The following result states that P is a free object in the category of Σ-algebras.

Proposition 2.1. *Let A be a Σ-algebra and let $f : X \to A$ be any map. Then, there exists a unique Σ-algebra homomorphism $\varphi : P \to A$ such that $\varphi(x(1)) = f(x)$, for all $x \in X$.*

Proof. A K-algebra homomorphism $\varphi : P \to A$ is clearly defined by putting $\varphi(x(\sigma)) = \sigma \cdot f(x)$, for any $x \in X$ and $\sigma \in \Sigma$. Then, one has that $\varphi(\sigma \cdot cx(\tau)) = \varphi((\sigma \cdot c)x(\sigma\tau)) = (\sigma \cdot c)\varphi(x(\sigma\tau)) = (\sigma \cdot c)(\sigma\tau \cdot f(x)) = \sigma \cdot (c(\tau \cdot f(x))) = \sigma \cdot (c\varphi(x(\tau))) = \sigma \cdot \varphi(cx(\tau))$, for all $c \in K$, $x \in X$ and $\sigma, \tau \in \Sigma$. In other words, the mapping $\varphi : P \to A$ is a Σ-algebra homomorphism and owing to $x(\sigma) = \sigma \cdot x(1)$, it is clearly the unique one such that $\varphi(x(1)) = f(x)$, for all $x \in X$. $\quad\square$

Definition 2.2. We call $P = K[X(\Sigma)]$ the *free Σ-algebra generated by X*. In fact, P is Σ-generated by the subset $X(1) = \{x(1) \mid x \in X\}$.

Note that if A is any Σ-algebra which is Σ-generated by X one has that A is isomorphic to the quotient P/J where $J \subset P$ is the Σ-ideal containing all Σ-algebra relations satisfied by the elements of X. In other words, there is a surjective Σ-algebra homomorphism $\varphi : P \to A$ such that $x(1) \mapsto x$ $(x \in X)$ and one defines $J = \operatorname{Ker}\varphi$.

We are ready now to make the link with the formal theory of partial difference equations. Let K be a field of functions in the variables $t_1, \ldots, t_r$ and fix $h_1, \ldots, h_r$ some parameters (mesh steps). Assume we may define the action of Σ on K by putting for all $\sigma = \prod_i \sigma_i^{\alpha_i} \in \Sigma$ and for any function $f \in K$

$$\sigma \cdot f(t_1, \ldots, t_r) = f(t_1 + \alpha_1 h_1, \ldots, t_r + \alpha_r h_r) \in K.$$

For instance, one can consider the field of rational functions $K = F(t_1, \ldots, t_k)$ over some field F and $h_1, \ldots, h_r \in F$. Consider now a finite set of unknown functions $u_i = u_i(t_1, \ldots, t_r)$ $(1 \leq i \leq n)$ that are assumed to be K-algebraically independent together with the shifted functions $\sigma \cdot u_i = u_i(t_1 + \alpha_1 h_1, \ldots, t_r + \alpha_r h_r)$, for any $\sigma = \prod_i \sigma_i^{\alpha_i} \in \Sigma$. If $X = \{x_1, \ldots, x_n\}$ and if we denote $x_i(\sigma) = \sigma \cdot u_i$ then the free Σ-algebra $P = K[X(\Sigma)]$ is by definition the *algebra of partial difference polynomials*. In particular, if K is a field of constants then the difference polynomials of P are said to be *with constant coefficients*.

Moreover, one uses the term *ordinary difference* when $r = 1$. Note that in the literature one finds the notation $P = K\{X\}$ that emphasizes the role of X as (free) Σ-generating set of the algebra P. According to the notations we have introduced for the Σ-algebras one may write also $P = K[X]_\Sigma$. In fact, we prefer $P = K[X(\Sigma)]$ to mean that P is the usual polynomial algebra defined for some special set of variables $X(\Sigma)$ which is invariant under the action of the monoid Σ, that is, $\Sigma \cdot X(\Sigma) \subset X(\Sigma)$. In the theory of algebraic equations we have that systems of algebraic equations correspond to bases of ideals of the polynomial algebra. In a similar way, one has that systems of partial difference equations corresponds to Σ-bases of Σ-ideals of P which are also called *partial difference ideals*. Note that Σ and therefore $X(\Sigma)$ is an infinite set which implies that P is not a Noetherian algebra. Then, one has that the Σ-ideals have bases and even Σ-bases which are generally infinite.

3. Gröbner bases of difference ideals

In this section we introduce a Gröbner basis theory for the algebra of partial difference polynomials by extending what has be done in [23] for the case of constant coefficients. Note that the concept of difference Gröbner basis has arisen also in [14,17,22].

Definition 3.1. Let $\prec$ be a total ordering on the set $M = \mathrm{Mon}(P)$ of all monomials of P. We call $\prec$ a *monomial ordering of P* if the following properties are satisfied:

(i) $\prec$ is a multiplicatively compatible ordering, that is, if $m \prec n$ then $tm \prec tn$, for any $m, n, t \in M$;
(ii) $\prec$ is a well-ordering, that is, every non-empty subset of M has a minimal element.

It is clear that in this case one has also that

(iii) $1 \prec m$, for all $m \in M$, $m \neq 1$.

Even if the variables set $X(\Sigma)$ is infinite, by Higman's Lemma [19] the polynomial algebra $P = K[X(\Sigma)]$ can be always endowed with a monomial ordering.

Proposition 3.2. *Let $\prec$ be a total ordering on M which verifies the properties (i), (iii) of Definition 3.1. If $\prec$ induces a well-ordering on the variables set $X(\Sigma) \subset M$, then $\prec$ is a well-ordering also on M and hence $\prec$ is a monomial ordering of P.*

Note now that the monomials set M is invariant under the action of Σ, that is $\Sigma \cdot M \subset M$, because the same happens to the variables set $X(\Sigma)$. Clearly, we have to require that a monomial ordering respects this key property for defining Gröbner bases of Σ-ideals of P which are ideals that are Σ-invariant. In other words, one has to introduce the following notion.

Definition 3.3. Let $\prec$ be a monomial ordering of P. We call $\prec$ a *monomial Σ-ordering of P* if $m \prec n$ implies that $\sigma \cdot m \prec \sigma \cdot n$, for all $m, n \in M$ and $\sigma \in \Sigma$.

Note that if $\prec$ is a monomial Σ-ordering of P then one has immediately that $\sigma \cdot m \succeq m$, for all $m \in M$ and $\sigma \in \Sigma$. Examples of such orderings can be easily constructed in the following way. Let $Q = K[\sigma_1, \ldots, \sigma_r]$ be the polynomial algebra in the variables σ_j and therefore $\Sigma = \mathrm{Mon}(Q)$. Moreover, let $K[X] = K[x_1, \ldots, x_n]$ be the polynomial algebra in the variables x_i. Fix a monomial ordering $<$ for Q and a monomial ordering $\prec$ for $K[X]$. For any $\sigma \in \Sigma$, denote $X(\sigma) = \{x_i(\sigma) \mid x_i \in X\}$. Clearly $P(\sigma) = K[X(\sigma)]$ is a subalgebra of P which is isomorphic to $K[X]$ and hence it can be endowed with the monomial ordering $\prec$. Since $X(\Sigma) = \bigcup_{\sigma \in \Sigma} X(\sigma)$, one can define a block monomial ordering for $P = K[X(\Sigma)]$ obtained by $<$ and $\prec$.

Proposition 3.4. *Let $m, n \in M$ be any pair of monomials. Clearly, we can factorize these monomials as $m = m_1 \cdots m_k$, $n = n_1 \cdots n_k$ where $m_i, n_i \in M(\delta_i) = \mathrm{Mon}(P(\delta_i))$ $(\delta_i \in \Sigma)$ and $\delta_1 > \ldots > \delta_k$ $(k \geq 1)$. Note explicitly that some of the factors m_i, n_i may be eventually equal to 1. We define $m \prec' n$ if and only if there is $1 \leq i \leq k$ such that $m_j = n_j$ when $j < i$ and $m_i \prec n_i$. Then, $\prec'$ is a monomial Σ-ordering of P.*

Proof. For all $\sigma \in \Sigma$, one has that $\sigma \cdot m = m_1' \cdots m_k'$ where $m_i' = \sigma \cdot m_i \in M(\sigma \delta_i)$ and $\sigma \delta_1 > \ldots > \sigma \delta_k$ because $<$ is a monomial ordering of Q. Assume $m \prec' n$, that is, $m_j = n_j$ for $j < i$ and $m_i \prec n_i$. Clearly, one has also that $m_j' = n_j'$. Moreover, by definition of the monomial ordering $\prec$ on all subalgebras $P(\sigma) \subset P$ we have that $m_i \prec n_i$ if and only if $m_i' \prec n_i'$. We conclude that $\sigma \cdot m \prec' \sigma \cdot n$. $\square$

Example 3.5. Fix $n = 2$ and $r = 3$, that is, let $X = \{x, y\}$ and $\Sigma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$. To simplify the notation of the variables in $X(\Sigma)$, we identify Σ with the additive monoid $\mathbb{N}^3$, that is, we put $X(\Sigma) = \{x(i, j, k), y(i, j, k) \mid i, j, k \geq 0\}$. By Proposition 3.4, a monomial Σ-ordering is defined for $P = K[X(\Sigma)]$ once two monomial orderings are given for $Q = K[\sigma_1, \sigma_2, \sigma_3]$ and $K[X] = K[x, y]$. Consider for instance the degree reverse lexicographic ordering $<$ on Q $(\sigma_1 > \sigma_2 > \sigma_3)$ and the lexicographic ordering $\prec$ on $K[X]$ $(x \succ y)$. One has that $<$ orders the blocks of variables $X(i, j, k) = \{x(i, j, k), y(i, j, k)\}$ in the following way

$$\ldots > \{x(2,0,0), y(2,0,0)\} > \{x(1,1,0), y(1,1,0)\} > \{x(0,2,0), y(0,2,0)\}$$
$$> \{x(1,0,1), y(1,0,1)\} > \{x(0,1,1), y(0,1,1)\} > \{x(0,0,2), y(0,0,2)\}$$
$$> \{x(1,0,0), y(1,0,0)\} > \{x(0,1,0), y(0,1,0)\} > \{x(0,0,1), y(0,0,1)\}$$
$$> \{x(0,0,0), y(0,0,0)\}.$$

Moreover, the ordering $\prec$ is defined for each subalgebra $K[x(i, j, k), y(i, j, k)]$. The resulting block monomial ordering for P (which is a Σ-ordering by Proposition 3.4) is therefore the lexicographic ordering with

$$\ldots \succ x(2,0,0) \succ y(2,0,0) \succ x(1,1,0) \succ y(1,1,0) \succ x(0,2,0) \succ y(0,2,0)$$
$$\succ x(1,0,1) \succ y(1,0,1) \succ x(0,1,1) \succ y(0,1,1) \succ x(0,0,2) \succ y(0,0,2)$$
$$\succ x(1,0,0) \succ y(1,0,0) \succ x(0,1,0) \succ y(0,1,0) \succ x(0,0,1) \succ y(0,0,1)$$
$$\succ x(0,0,0) \succ y(0,0,0).$$

From now on, we assume that P is endowed with a monomial Σ-ordering $\prec$. Let $f = \sum_i c_i m_i \in P$ with $m_i \in M$ and $0 \neq c_i \in K$. If $m_k = \max_{\prec}\{m_i\}$ then we denote as usual $\mathrm{lm}(f) = m_k$, $\mathrm{lc}(f) = c_k$ and $\mathrm{lt}(f) = c_k m_k$. Since $\prec$ is a Σ-ordering, one has that $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot \mathrm{lm}(f)$ and therefore $\mathrm{lc}(\sigma \cdot f) = \sigma \cdot \mathrm{lc}(f)$, $\mathrm{lt}(\sigma \cdot f) = \sigma \cdot \mathrm{lt}(f)$, for all $\sigma \in \Sigma$. If $G \subset P$ then we denote $\langle G \rangle = \{\sum_i f_i g_i \mid f_i \in P, g_i \in G\}$, that is, $\langle G \rangle$ is the ideal of P generated by G. Moreover, recall that $\langle G \rangle_\Sigma = \langle \Sigma \cdot G \rangle = \{\sum_i f_i(\delta_i \cdot g_i) \mid \delta_i \in \Sigma, f_i \in P, g_i \in G\}$ is the Σ-ideal which is Σ-generated by G, that is, it is the smallest Σ-ideal of P containing G. We call G a Σ-basis of $\langle G \rangle_\Sigma$. Finally, we put $\mathrm{lm}(G) = \{\mathrm{lm}(f) \mid f \in G, f \neq 0\}$ and we denote $\mathrm{LM}(G) = \langle \mathrm{lm}(G) \rangle$.

Proposition 3.6. *Let $G \subset P$. Then $\mathrm{lm}(\Sigma \cdot G) = \Sigma \cdot \mathrm{lm}(G)$. In particular, if I is a Σ-ideal of P then $\mathrm{LM}(I)$ is also a Σ-ideal.*

Proof. Since P is endowed with a Σ-ordering, one has that $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot \mathrm{lm}(f)$, for any $f \in P$, $f \neq 0$ and $\sigma \in \Sigma$. Then, $\Sigma \cdot \mathrm{lm}(I) = \mathrm{lm}(\Sigma \cdot I) \subset \mathrm{lm}(I)$ and therefore $\mathrm{LM}(I) = \langle \mathrm{lm}(I) \rangle$ is a Σ-ideal. $\square$

Definition 3.7. Let $I \subset P$ be a Σ-ideal and $G \subset I$. We call G a *Gröbner Σ-basis* of I if $\mathrm{lm}(G)$ is a Σ-basis of $\mathrm{LM}(I)$. In other words, $\mathrm{lm}(\Sigma \cdot G) = \Sigma \cdot \mathrm{lm}(G)$ is a basis of $\mathrm{LM}(I)$, that is, $\Sigma \cdot G$ is a Gröbner basis of I as an ideal of P.

Since P is not a Noetherian algebra, in general its Σ-ideals have infinite (Gröbner) Σ-bases. Note that one has a similar situation for the free associative algebra and its ideals and this case is strictly related with the algebra of ordinary difference polynomials owing to the notion of "letterplace correspondence" [21,22,24]. See also the comprehensive Bergman's paper [4] where the theory of Gröbner bases (he did not use this name) is provided for both commutative and non-commutative algebras in full generality, that is, without any assumption about Noetherianity. In Section 6 we will prove in fact the existence of a class of Σ-ideals containing finite Gröbner Σ-bases. Note that the existence of such finite Σ-bases, which are also called "difference Gröbner bases", has been already proved for the linear case in [14,17].

Let now $f, g \in P$, $f, g \neq 0$ and put $\mathrm{lt}(f) = cm$, $\mathrm{lt}(g) = dn$ with $m, n \in M$ and $c, d \in K$. If $l = \mathrm{lcm}(m, n)$ one defines the *S-polynomial* $\mathrm{spoly}(f, g) = (l/cm)f - (l/dn)g$.

Proposition 3.8. *For all $f, g \in P$, $f, g \neq 0$ and for any $\sigma \in \Sigma$ one has that $\sigma \cdot \mathrm{spoly}(f, g) = \mathrm{spoly}(\sigma \cdot f, \sigma \cdot g)$.*

Proof. Note that $\mathrm{lt}(\sigma \cdot f) = (\sigma \cdot c)(\sigma \cdot m)$, $\mathrm{lt}(\sigma \cdot g) = (\sigma \cdot d)(\sigma \cdot n)$ with $\sigma \cdot m, \sigma \cdot n \in M$ and $\sigma \cdot c, \sigma \cdot d \in K$. Since Σ acts on the variables set $X(\Sigma)$ by injective maps, if $l = \mathrm{lcm}(m, n)$

then $\sigma \cdot l = \mathrm{lcm}(\sigma \cdot m, \sigma \cdot n)$ and therefore we have

$$\sigma \cdot \mathrm{spoly}(f, g) = \sigma \cdot \left(\frac{l}{cm} f - \frac{l}{dn} g \right)$$

$$= \frac{\sigma \cdot l}{(\sigma \cdot c)(\sigma \cdot m)} \sigma \cdot f - \frac{\sigma \cdot l}{(\sigma \cdot d)(\sigma \cdot n)} \sigma \cdot g = \mathrm{spoly}(\sigma \cdot f, \sigma \cdot g). \qquad \square$$

In the theory of Gröbner bases one has the following important notion.

Definition 3.9. Let $f \in P$, $f \neq 0$ and $G \subset P$. If $f = \sum_i f_i g_i$ with $f_i \in P$, $g_i \in G$ and $\mathrm{lm}(f) \succeq \mathrm{lm}(f_i) \mathrm{lm}(g_i)$ for all i, we say that f *has a Gröbner representation with respect to G.*

Note that if $f = \sum_i f_i g_i$ is a Gröbner representation then $\sigma \cdot f = \sum_i (\sigma \cdot f_i)(\sigma \cdot g_i)$ is also a Gröbner representation, for any $\sigma \in \Sigma$. In fact, from $\mathrm{lm}(f) \succeq \mathrm{lm}(f_i) \mathrm{lm}(g_i)$ it follows that $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot \mathrm{lm}(f) \succeq (\sigma \cdot \mathrm{lm}(f_i))(\sigma \cdot \mathrm{lm}(g_i)) = \mathrm{lm}(\sigma \cdot f_i) \mathrm{lm}(\sigma \cdot g_i)$, for all indices i. Finally, if $\sigma = \prod_i \sigma_i^{\alpha_i}$, $\tau = \prod_i \sigma_i^{\beta_i} \in \Sigma = \langle \sigma_1, \ldots, \sigma_r \rangle$ we define $\gcd(\sigma, \tau) = \prod_i \sigma_i^{\gamma_i}$ where $\gamma_i = \min(\alpha_i, \beta_i)$. For the Gröbner Σ-bases of P we have the following characterization.

Proposition 3.10 (*Σ-criterion*). *Let G be a Σ-basis of a Σ-ideal $I \subset P$. Then, G is a Gröbner Σ-basis of I if and only if for all $f, g \in G$, $f, g \neq 0$ and for any $\sigma, \tau \in \Sigma$ such that $\gcd(\sigma, \tau) = 1$ and $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$, the S-polynomial $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ has a Gröbner representation with respect to $\Sigma \cdot G$.*

Proof. Recall that G is a Gröbner Σ-basis if and only if $\Sigma \cdot G$ is a Gröbner basis of I. By Buchberger's criterion [7] or by Bergman's diamond lemma [4] this happens if and only if the S-polynomials $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ have a Gröbner representation with respect to $\Sigma \cdot G$, for all $f, g \in G$, $f, g \neq 0$ and $\sigma, \tau \in \Sigma$. By the product criterion (see for instance [18]) we may restrict ourselves to considering only S-polynomials such that $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$ since $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot \mathrm{lm}(f)$ and $\mathrm{lm}(\tau \cdot g) = \tau \cdot \mathrm{lm}(g)$. Then, let $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ be any such S-polynomial and put $\delta = \gcd(\sigma, \tau)$ and therefore $\sigma = \delta \sigma'$, $\tau = \delta \tau'$ with $\sigma', \tau' \in \Sigma$, $\gcd(\sigma', \tau') = 1$. One has that $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g) = \delta \cdot \mathrm{spoly}(\sigma' \cdot f, \tau' \cdot g)$ owing to Proposition 3.8. Note now that if $\mathrm{spoly}(\sigma' \cdot f, \tau' \cdot g) = h = \sum_\nu f_\nu (\nu \cdot g_\nu)$ ($\nu \in \Sigma$, $f_\nu \in P$, $g_\nu \in G$) is a Gröbner representation with respect to $\Sigma \cdot G$ then also $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g) = \delta \cdot h = \sum_\nu (\delta \cdot f_\nu)(\delta \nu \cdot g_\nu)$ is a Gröbner representation because $\prec$ is a Σ-ordering of P. We conclude that the S-polynomials to be checked for Gröbner representations may be restricted to the ones satisfying both the conditions $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$ and $\gcd(\sigma, \tau) = 1$. $\square$

From the above result one obtains a variant of Buchberger's procedure based on the "Σ-criterion" $\gcd(\sigma, \tau) = 1$ which is able to compute Gröbner Σ-bases. A standard routine that one needs in this method is the following one (Algorithm 3.1).

Algorithm 3.1 Reduce.

Input: $G \subset P$ and $f \in P$.
Output: $h \in P$ such that $f - h \in \langle G \rangle$ and $h = 0$ or $\mathrm{lm}(h) \notin \mathrm{LM}(G)$.
$h := f$;
while $h \neq 0$ and $\mathrm{lm}(h) \in \mathrm{LM}(G)$ **do**
    choose $g \in G$, $g \neq 0$ such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(h)$;
    $h := h - (\mathrm{lt}(h)/\mathrm{lt}(g))g$;
end while;
return h.

Note that even if G may consist of an infinite number of polynomials, the set of their leading monomials dividing $\mathrm{lm}(h)$ is always a finite one. In other words, the "choose" instruction in the above routine can be actually performed. Moreover, although the polynomial algebra $P = K[X(\Sigma)]$ is infinitely generated, the existence of monomial orderings for P provides clearly the termination. By Proposition 3.10 one obtains the correctness of Procedure 3.2 for enumerating a Gröbner Σ-basis of a Σ-ideal having a finite Σ-basis.

Procedure 3.2 SigmaGBasis.

Input: H, a finite Σ-basis of a Σ-ideal $I \subset P$.
Output: G, a Gröbner Σ-basis of I.
$G := \{g \in H \mid g \neq 0\}$;
$B := \{(f, g) \mid f, g \in G\}$;
while $B \neq \emptyset$ **do**
    choose $(f, g) \in B$;
    $B := B \setminus \{(f, g)\}$;
    for all $\sigma, \tau \in \Sigma$ such that $\gcd(\sigma, \tau) = 1$, $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$ **do**
       $h := \mathrm{REDUCE}(\mathrm{spoly}(\sigma \cdot f, \tau \cdot g), \Sigma \cdot G)$;
       if $h \neq 0$ **then**
          $B := B \cup \{(g, h), (h, h) \mid g \in G\}$;
          $G := G \cup \{h\}$;
       end if;
    end for;
end while;
return G.

For this procedure we do not have general termination owing to non-Noetherianity of the algebra P. In fact, even if we assume that the Σ-ideal $I \subset P$ has a finite Σ-basis, this may be not true for its initial Σ-ideal $\mathrm{LM}(I)$, that is, I may have no finite Gröbner Σ-basis. In the next section, after introducing suitable monomial Σ-orderings of P we will give an algorithm which is able to compute in a finite number of steps a finite Gröbner Σ-basis whenever this exists. Note anyway that in the above procedure all instructions can be actually performed. In particular, for any pair of elements $f, g \in G$ and for all $\sigma, \tau \in \Sigma$ there are only a finite number of S-polynomials $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ satisfying both the criteria $\gcd(\sigma, \tau) = 1$ and $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$. A proof is given by the arguments contained in Proposition 4.8 of the next section. Observe that the case $f = g$ has to be considered whenever $\sigma \neq \tau$. Finally, note that the chain criterion (see for instance [18]) can be added to SigmaGBasis to shorten the number of S-polynomials that have to be reduced. In fact, we can view this procedure as a variant of the classical Buchberger's one applied to the basis $\Sigma \cdot H$ of the ideal I where Proposition 3.10 provides

the additional "Σ-criterion" to avoid useless pairs. In other words, this is one way to actually implement the procedure SIGMAGBASIS (see [23]) in any commutative computer algebra system.

In the following sections we propose two possible solutions for providing termination to SIGMAGBASIS. First, we introduce a grading on P that is compatible with the action of Σ which implies that the truncated variant of this procedure with homogeneous input stops in a finite number of steps. Another approach consists in obtaining finite Gröbner Σ-bases when elements with suitable linear leading monomials belong to the given Σ-ideal I. More precisely, we obtain the Noetherian property for a certain class of (quotient) Σ-algebras P/I.

4. Grading and truncation

A useful grading for the free Σ-algebra P can be introduced in the following way. Consider the set $\hat{\mathbb{N}} = \mathbb{N} \cup \{-\infty\}$ endowed with the binary operations max and $+$. Clearly $(\hat{\mathbb{N}}, \max, +)$ is a commutative semiring which is also idempotent since $\max(d, d) = d$, for all $d \in \hat{\mathbb{N}}$. Moreover, for any $\sigma = \prod_i \sigma_i^{\alpha_i} \in \Sigma$ we put $\deg(\sigma) = \sum_i \alpha_i$.

Definition 4.1. Let $\mathrm{ord} : M \to \hat{\mathbb{N}}$ be the unique mapping such that

(i) $\mathrm{ord}(1) = -\infty$;
(ii) $\mathrm{ord}(mn) = \max(\mathrm{ord}(m), \mathrm{ord}(n))$, for all $m, n \in M$;
(iii) $\mathrm{ord}(x_i(\sigma)) = \deg(\sigma)$, for any variable $x_i(\sigma) \in X(\Sigma)$.

Then, the map ord is a monoid homomorphism from $(M, \cdot)$ to $(\hat{\mathbb{N}}, \max)$. We call ord the *order function* of P.

More explicitly, if $m = x_{i_1}(\delta_1)^{\alpha_1} \cdots x_{i_k}(\delta_k)^{\alpha_k} \in M = \mathrm{Mon}(P)$ is any monomial different from 1 ($x_{i_l}(\delta_l) \in X(\Sigma)$ and $\alpha_l > 0$, for each $1 \leq l \leq k$) we have that

$$\mathrm{ord}(m) = \max\big(\deg(\delta_1), \ldots, \deg(\delta_k)\big).$$

Example 4.2. Let $X = \{x, y\}$ and $\Sigma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$. As in Example 3.5, denote $X(\Sigma) = \{x(i, j, k), y(i, j, k) \mid i, j, k \geq 0\}$. If we consider the monomial

$$m = y(1, 1, 0)^2 x(1, 0, 1) x(1, 0, 0)^3 y(0, 0, 0)^4$$

then $\mathrm{ord}(m) = 2$.

Let $P_d = \langle m \in M \mid \mathrm{ord}(m) = d \rangle_K \subset P$, that is, P_d is the K-subspace of P generated by all monomials having order equal to d. A polynomial $f \in P_d$ is called *ord-homogeneous* and we denote $\mathrm{ord}(f) = d$. By property (ii) of Definition 4.1 one has clearly that $P = \bigoplus_{d \in \hat{\mathbb{N}}} P_d$ is a grading of the algebra P over the commutative monoid $(\hat{\mathbb{N}}, \max)$.

Proposition 4.3. *The following properties hold for the order function:*

(i) $\operatorname{ord}(\sigma \cdot m) = \deg(\sigma) + \operatorname{ord}(m)$, *for any* $\sigma \in \Sigma$ *and* $m \in M$;
(ii) $\operatorname{ord}(\operatorname{lcm}(m, n)) = \operatorname{ord}(mn) = \max(\operatorname{ord}(m), \operatorname{ord}(n))$, *for all* $m, n \in M$. *Therefore, if* $m \mid n$ *then* $\operatorname{ord}(m) \leq \operatorname{ord}(n)$.

Proof. If $m = 1$ then $\operatorname{ord}(\sigma \cdot m) = \operatorname{ord}(m) = -\infty = \deg(\sigma) + \operatorname{ord}(m)$. If otherwise $m = x_{i_1}(\delta_1)^{\alpha_1} \cdots x_{i_k}(\delta_k)^{\alpha_k}$ then $\sigma \cdot m = x_{i_1}(\sigma \delta_1)^{\alpha_1} \cdots x_{i_k}(\sigma \delta_k)^{\alpha_k}$ and hence $\operatorname{ord}(\sigma \cdot m) = \max(\deg(\sigma \delta_1), \dots, \deg(\sigma \delta_k)) = \deg(\sigma) + \max(\deg(\delta_1), \dots, \deg(\delta_k)) = \deg(\sigma) + \operatorname{ord}(m)$. To prove (ii) it is sufficient to note that the order of a monomial does not depend on the exponents of the variables occurring in it. $\square$

Definition 4.4. An ideal $I \subset P$ is called ord-*graded* if $I = \sum_d I_d$ with $I_d = I \cap P_d$. Note that if I is in addition a Σ-ideal then by (i) of Proposition 4.3 one has that $\sigma \cdot I_d \subset I_{\deg(\sigma)+d}$, for any $\sigma \in \Sigma$ and $d \in \hat{\mathbb{N}}$.

Let $f, g \in P$, $f \neq g$ be any pair of ord-homogeneous elements. Then, the S-polynomial $h = \operatorname{spoly}(f, g)$ is also ord-homogeneous and by (ii) of Proposition 4.3 one has that $\operatorname{ord}(h) = \max(\operatorname{ord}(f), \operatorname{ord}(g))$. If $\operatorname{ord}(f), \operatorname{ord}(g) \leq d$ for some $d \in \mathbb{N}$, we have therefore that $\operatorname{ord}(h) \leq d$ which implies the following result.

Proposition 4.5 *(Termination by truncation). Let $I \subset P$ be an* ord-*graded Σ-ideal and let $d \in \mathbb{N}$. Assume there is an* ord-*homogeneous Σ-basis $H \subset I$ such that $H_d = \{f \in H \mid \operatorname{ord}(f) \leq d\}$ is a finite set. Then, there exists also an* ord-*homogeneous Gröbner Σ-basis G of I such that G_d is a finite set. In other words, if one uses for* SIGMAGBASIS *a selection strategy of the S-polynomials based on their orders then the d-truncated variant of* SIGMAGBASIS *with input H_d terminates in a finite number of steps.*

Proof. In the procedure SIGMAGBASIS one computes a subset G of a Gröbner basis $G' = \Sigma \cdot G$ obtained by applying Buchberger's procedure to the basis $H' = \Sigma \cdot H$ of the ideal I. Moreover, Proposition 4.3 implies that the set H' and hence G' consists of ord-homogeneous elements. Define hence $H'_d = \{\sigma \cdot f \mid \sigma \in \Sigma, f \in H, \deg(\sigma) + \operatorname{ord}(f) \leq d\}$. Note that $\Sigma_d = \{\sigma \in \Sigma \mid \deg(\sigma) \leq d\}$ is clearly a finite set and by hypothesis we have that H_d is also a finite one. We conclude that $H'_d \subset \Sigma_d \cdot H_d$ is a finite set. Denote now by Y_d the finite set of variables of P occurring in the elements of H'_d and define the subalgebra $P_{(d)} = K[Y_d] \subset P$. In fact, the d-truncated variant of SIGMAGBASIS computes a subset of a Gröbner basis of the ideal $I_{(d)} \subset P_{(d)}$ generated by H'_d. The Noetherianity of the finitely generated polynomial algebra $P_{(d)}$ provides then termination. $\square$

Note that this result implies an algorithmic solution to the ideal membership for finitely generated ord-graded Σ-ideals. Another consequence of the grading defined by the

order function is that one has a criterion, also in the non-graded case, for verifying that a Σ-basis computed by the procedure SIGMAGBASIS using a finite number of variables of P is a complete finite Gröbner Σ-basis, whenever this basis exists. This is of course important because actual computations can be only performed over a finite number of variables.

Definition 4.6. Let $\prec$ be a monomial Σ-ordering of P. We say that $\prec$ is *compatible with the order function* if $\mathrm{ord}(m) < \mathrm{ord}(n)$ implies that $m \prec n$, for all $m, n \in M$.

Proposition 4.7. *Denote by $\prec$ the monomial Σ-ordering of P defined in Proposition 3.4 and let $<$ be the monomial ordering of $Q = K[\sigma_1, \ldots, \sigma_r]$ which is used to define $\prec$. Assume that $<$ is compatible with the function $\deg$, that is, $\deg(\sigma) < \deg(\tau)$ implies that $\sigma < \tau$, for any $\sigma, \tau \in \Sigma$. Then, one has that $\prec$ is compatible with the function ord.*

Proof. Let $m = m_1 \cdots m_k$, $n = n_1 \cdots n_k$ be any pair of monomials of P, where $m_i, n_i \in M(\delta_i)$ $(\delta_i \in \Sigma)$ and $\delta_1 > \ldots > \delta_k$ (hence $\deg(\delta_1) \geq \ldots \geq \deg(\delta_k)$). Assume $m \prec n$, that is, there is $1 \leq i \leq k$ such that $m_j = n_j$ when $j < i$ and $m_i \prec n_i$. If $i > 1$ or $m_i \neq 1$ one has clearly $\mathrm{ord}(m) = \mathrm{ord}(n) = \deg(\delta_1)$. Otherwise, we conclude that $\mathrm{ord}(m) \leq \deg(\delta_1) = \mathrm{ord}(n)$. $\square$

As before, we denote $\Sigma_d = \{\sigma \in \Sigma \mid \deg(\sigma) \leq d\}$.

Proposition 4.8 *(Finite Σ-criterion). Assume that P is endowed with a monomial Σ-ordering compatible with the order function. Let $G \subset P$ be a finite set and define the Σ-ideal $I = \langle G \rangle_\Sigma$. Moreover, denote $d = \max\{\mathrm{ord}(\mathrm{lm}(g)) \mid g \in G, g \neq 0\}$. Then, G is a Gröbner Σ-basis of I if and only if for all $f, g \in G$, $f, g \neq 0$ and for any $\sigma, \tau \in \Sigma$ such that $\gcd(\sigma, \tau) = 1$ and $\gcd(\sigma \cdot \mathrm{lm}(f), \tau \cdot \mathrm{lm}(g)) \neq 1$, the S-polynomial $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g)$ has a Gröbner representation with respect to the finite set $\Sigma_{2d} \cdot G$.*

Proof. Let $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g) = h = \sum_\nu f_\nu(\nu \cdot g_\nu)$ be a Gröbner representation with respect to $\Sigma \cdot G$, that is, $\mathrm{lm}(h) \succeq \mathrm{lm}(f_\nu)(\nu \cdot \mathrm{lm}(g_\nu))$, for all ν. We want to bound the degree of the elements $\nu \in \Sigma$ occurring in this representation. Put $m = \mathrm{lm}(f)$, $n = \mathrm{lm}(g)$ and hence $\mathrm{lm}(\sigma \cdot f) = \sigma \cdot m$, $\mathrm{lm}(\sigma \cdot g) = \sigma \cdot n$. By the product criterion one has that $u = \gcd(\sigma \cdot m, \tau \cdot n) \neq 1$, that is, there is a common variable $x_i(\sigma\alpha) = x_i(\tau\beta)$ dividing u where $x_i(\alpha)$ divides m and $x_i(\beta)$ divides n. Therefore $\sigma\alpha = \tau\beta$ and we have that $\deg(\alpha) \leq \mathrm{ord}(m) \leq d$ and $\deg(\beta) \leq \mathrm{ord}(n) \leq d$. From $\sigma\alpha = \tau\beta$ and the Σ-criterion $\gcd(\sigma, \tau) = 1$ it follows that $\sigma \mid \beta$, $\tau \mid \alpha$ and hence $\deg(\sigma), \deg(\tau) \leq d$. If $v = \mathrm{lcm}(\sigma \cdot m, \tau \cdot m)$ then we have that $\mathrm{ord}(v) = \max(\deg(\sigma) + \mathrm{ord}(m), \deg(\tau) + \mathrm{ord}(n)) \leq 2d$. Clearly $v \succ \mathrm{lm}(h) \succeq \nu \cdot \mathrm{lm}(g_\nu)$ and therefore $2d \geq \mathrm{ord}(v) \geq \deg(\nu) + \mathrm{ord}(\mathrm{lm}(g_\nu)) \geq \deg(\nu)$. In other words, we have that all elements ν belong to Σ_{2d}, that is, $\mathrm{spoly}(\sigma \cdot f, \tau \cdot g) = \sum_\nu f_\nu(\nu \cdot g_\nu)$ is in fact a Gröbner representation with respect to the set $\Sigma_{2d} \cdot G$. $\square$

Under the assumption of a Σ-ordering compatible with the order function and for Σ-ideals that admit finite Gröbner Σ-bases, by the above criterion one obtains an algorithm to compute such a basis in a finite number of steps. In fact, this can be obtained as an adaptative procedure that keeps the bound $2d$ for the degree of the elements of Σ applied to the generators, constantly updated with respect to the maximal order d of the leading monomials of the current generators. In other words, if we denote by SIGMAGBASIS(H, d) the variant of the procedure SIGMAGBASIS(H) when one substitutes Σ with Σ_d, then we have Algorithm 4.1.

Algorithm 4.1 SigmaGBasis2.

Input: H, a finite Σ-basis of a Σ-ideal $I \subset P$ such that $\mathrm{LM}(I)$ has also a finite Σ-basis.
Output: G, a finite Gröbner Σ-basis of I.
$G := \{g \in H \mid g \neq 0\}$;
$d' := -\infty$;
$d = \max\{\mathrm{ord}(\mathrm{lm}(g)) \mid g \in G\}$;
while $d' < 2d$ **do**
    $d' = 2d$;
    $G := $ SIGMAGBASIS(G, d');
    $d = \max\{\mathrm{ord}(\mathrm{lm}(g)) \mid g \in G\}$;
end while;
return G.

Of course, the above algorithm may be refined to avoid a complete recomputation at each step.

5. An illustrative example

In this section we apply the procedure SIGMAGBASIS to an example arising from the discretization of a well-known system of partial differential equations. Consider the unsteady two-dimensional motion of an incompressible viscous liquid of constant viscosity which is governed by the following system

$$\begin{cases} u_x + v_y = 0, \\[2mm] u_t + uu_x + vu_y + p_x - \dfrac{1}{\rho}(u_{xx} + u_{yy}) = 0, \\[2mm] v_t + uv_x + vv_y + p_y - \dfrac{1}{\rho}(v_{xx} + v_{yy}) = 0. \end{cases}$$

The last two nonlinear equations are the Navier–Stokes equations and the first linear equation is the continuity one. Equations are given in the dimensionless form where (u, v) represents the velocity field and the function p is the pressure. The parameter ρ denotes the Reynolds number. For defining a finite difference approximation of this system one has therefore to fix $X = \{u, v, p\}$ and $\Sigma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ since all functions are trivariate ones. To simplify the notation of the variables in $X(\Sigma)$, we identify Σ with the additive monoid $\mathbb{N}^3$ and we denote $P = K[X(\Sigma)] = K[u(i, j, k), v(i, j, k), p(i, j, k) \mid i, j, k \geq 0]$.

The base field K is the field of rational numbers. The approximation of the derivatives of the function u is given by the following formulas (forward differences)

$$u_x \approx \frac{u(x+h,y,t) - u(x,y,t)}{h} = \frac{u(1,0,0) - u(0,0,0)}{h},$$

$$u_y \approx \frac{u(x,y+h,t) - u(x,y,t)}{h} = \frac{u(0,1,0) - u(0,0,0)}{h},$$

$$u_t \approx \frac{u(x,y,t+h) - u(x,y,t)}{h} = \frac{u(0,0,1) - u(0,0,0)}{h},$$

$$u_{xx} \approx \frac{u(x+2h,y,t) - 2u(x+h,y,t) + u(x,y,t)}{h^2} = \frac{u(2,0,0) - 2u(1,0,0) + u(0,0,0)}{h^2},$$

$$u_{yy} \approx \frac{u(x,y+2h,t) - 2u(x,y+h,t) + u(x,y,t)}{h^2} = \frac{u(0,2,0) - 2u(0,1,0) + u(0,0,0)}{h^2}$$

where h is a parameter (mesh step). One has similar approximations for the derivatives of the functions v, p. If we put $H = \rho h$ then the Navier–Stokes system is approximated by the following system of partial difference equations

$$
\begin{cases}
f_1 := u(1,0,0) + v(0,1,0) - u(0,0,0) - v(0,0,0) = 0, \\
f_2 := -u(2,0,0) - u(0,2,0) + 2u(1,0,0) + 2u(0,1,0) - 2u(0,0,0) \\
\quad + H\big(p(1,0,0) + u(0,0,1) - p(0,0,0) - u(0,0,0)^2 \\
\quad - \big(1 + v(0,0,0) - u(1,0,0)\big)u(0,0,0) + u(0,1,0)v(0,0,0)\big) = 0, \\
f_3 := -v(2,0,0) - v(0,2,0) + 2v(1,0,0) + 2v(0,1,0) - 2v(0,0,0) \\
\quad + H\big(p(0,1,0) + v(0,0,1) - p(0,0,0) - v(0,0,0)^2 \\
\quad + \big(v(1,0,0) - v(0,0,0)\big)u(0,0,0) - \big(1 - v(0,1,0)\big)v(0,0,0)\big) = 0.
\end{cases}
$$

We encode this system as the Σ-ideal $I = \langle f_1, f_2, f_3 \rangle_\Sigma \subset P$ and we want to compute a (hopefully finite) Gröbner Σ-basis of I. We may want to have such a basis to check for the "strong-consistency" [14] of the finite difference approximation that we are using. In fact, this property is necessary for inheritance at the discrete level of the algebraic properties of the differential equations. For instance, in [1] we have compared the numerical behavior of three different finite difference approximations of the Navier–Stokes equations where just one of them is strongly consistent. The computational experiments have confirmed the superiority of the strongly consistent approximation. In the limit when the mesh steps go to zero, the elements in the difference Gröbner basis of the finite difference approximation under consideration become differential polynomials. Then, the strong consistency holds if and only if the latter polynomials belong to the radical differential ideal generated by the polynomials in the input differential equations. Note that this membership test can be done algorithmically by using the *diffalg* library [6] or the differential Thomas decomposition [3].

To perform SIGMAGBASIS, we fix now the degree reverse lexicographic ordering on the polynomial algebra $K[\sigma_1, \sigma_2, \sigma_3]$ ($\sigma_1 > \sigma_2 > \sigma_3$) and the lexicographic ordering

on $K[u, v, p]$ $(u \succ v \succ p)$. By Proposition 3.4 one obtains then a (block) monomial Σ-ordering for P which is in fact the lexicographic ordering such that

$$
\begin{aligned}
\ldots &\succ u(2,0,0) \succ v(2,0,0) \succ p(2,0,0) \succ u(1,1,0) \succ v(1,1,0) \succ p(1,1,0) \\
&\succ u(0,2,0) \succ v(0,2,0) \succ p(0,2,0) \succ u(1,0,1) \succ v(1,0,1) \succ p(1,0,1) \\
&\succ u(0,1,1) \succ v(0,1,1) \succ p(0,1,1) \succ u(0,0,2) \succ v(0,0,2) \succ p(0,0,2) \\
&\succ u(1,0,0) \succ v(1,0,0) \succ p(1,0,0) \succ u(0,1,0) \succ v(0,1,0) \succ p(0,1,0) \\
&\succ u(0,0,1) \succ v(0,0,1) \succ p(0,0,1) \succ u(0,0,0) \succ v(0,0,0) \succ p(0,0,0).
\end{aligned}
$$

Note that this ordering is compatible with the order function and hence Proposition 4.8 is applicable to certify completeness of a Gröbner Σ-basis computed over some finite set of variables $\{u(i,j,k), v(i,j,k), p(i,j,k) \mid i+j+k \leq d\}$.

With respect to the monomial ordering assigned to P, the leading monomials of the Σ-generators of I are $\mathrm{lm}(f_1) = u(1,0,0)$, $\mathrm{lm}(f_2) = u(2,0,0)$, $\mathrm{lm}(f_3) = v(2,0,0)$. Since $\sigma_1 \cdot \mathrm{lm}(f_1) = \mathrm{lm}(f_2)$, by interreducing f_2 with respect to the set $\Sigma \cdot \{f_1, f_3\}$ we obtain the element

$$
\begin{aligned}
f_2' := {}& v(1,1,0) - u(0,2,0) - v(1,0,0) \\
&+ 2u(0,1,0) - v(0,1,0) - u(0,0,0) + v(0,0,0) \\
&+ H\big(p(1,0,0) + u(0,0,1) - p(0,0,0)\big) \\
&- \big(1 + v(0,1,0)\big)u(0,0,0) + u(0,1,0)v(0,0,0)\big)
\end{aligned}
$$

whose leading monomial is $\mathrm{lm}(f_2') = v(1,1,0)$. Owing to the Σ-criterion, the only S-polynomial to consider is then $\mathrm{spoly}(\sigma_1 \cdot f_2', \sigma_2 \cdot f_3)$ whose reduction with respect to $\Sigma \cdot \{f_1, f_2', f_3\}$ leads to the new element

$$
\begin{aligned}
f_4 := {}& p(2,0,0) + p(0,2,0) - 2\big(p(1,0,0) + p(0,1,0) - p(0,0,0)\big) \\
&- 2u(0,1,0)^2 - v(0,2,0)v(1,0,0) - u(0,0,0)^2 + 2v(0,0,0)^2 \\
&+ \big(3u(0,1,0) - 2v(1,0,0) + v(0,1,0) - u(0,2,0) + v(0,0,0)\big)u(0,0,0) \\
&- \big(3v(0,1,0) + u(0,2,0) + v(1,0,0)\big)v(0,0,0) \\
&+ \big(2v(1,0,0) - 2v(0,1,0) + u(0,2,0)\big)u(0,1,0) \\
&+ \big(2v(1,0,0) + u(0,2,0) + v(0,2,0)\big)v(0,1,0) \\
&+ H\big(\big(u(0,1,0) + v(0,1,0)\big)p(0,0,0) - \big(u(0,1,0) + v(1,0,0)\big)u(0,0,1) \\
&- p(1,0,0)v(0,1,0) - p(1,0,0)u(0,1,0) - \big(v(0,1,0) + 1\big)u(0,0,0)^2 \\
&+ \big(p(1,0,0) - p(0,0,0) + u(0,0,1) + v(0,1,0) \\
&+ \big(u(0,1,0) - v(1,0,0) - 1\big)v(0,0,0) \\
&+ \big(v(0,1,0) + 1\big)u(0,1,0) + v(0,1,0)^2\big)u(0,0,0) + u(0,1,0)v(0,0,0)^2 \\
&+ \big(p(1,0,0) - p(0,0,0) + u(0,0,1) - u(0,1,0)v(0,1,0) - u(0,1,0)^2\big)v(0,0,0)\big).
\end{aligned}
$$

The leading monomial of this difference polynomial is $\mathrm{lm}(f_4) = p(2, 0, 0)$ and no more S-polynomials have to be considered. We conclude that the set $\{f_1, f_2', f_3, f_4\}$ is a (finite) Gröbner Σ-basis of the Σ-ideal $I \subset P$. Since we make use of a monomial Σ-ordering for P, this is equivalent to say that $\Sigma \cdot \{f_1, f_2', f_3, f_4\}$ is a Gröbner basis of the ideal I and this can be verified also by applying the classical Gröbner bases routines to a proper truncation of the basis $\Sigma \cdot \{f_1, f_2, f_3\}$. In fact, because the maximal order in the input generators is 2, by Proposition 4.8 it is reasonable to bound initially the order of the variables of P to 4 or 5. Even if it is not the case in this example, observe that the maximal order in the elements of a Gröbner Σ-basis may grow during the computation. Therefore, as a general strategy, we suggest to bound the variables order to a value which is reasonably greater than the double of the input maximal order. The computing time for obtaining a Gröbner basis of I with the implementation in Maple of Faugère's F4 algorithm amounts to 20 seconds for order 4 and 5 hours for order 5 on our laptop Intel Core 2 Duo at 2.10 GHz with 8 GB RAM. By the procedure SIGMAGBASIS that we implemented in the Maple language as a variant of Buchberger's one (see [23]), the computing time for a Gröbner Σ-basis of I is instead 0 seconds for order 4 and 3 seconds for order 5 since just two reductions are needed. In other words, this speed-up is due to the Σ-criterion which decreases drastically the number of S-polynomial reductions which sometimes are very time-consuming. Note finally that the verification method of the property of strong consistency applied to the computed difference Gröbner basis shows that the finite difference approximation $\{f_1, f_2, f_3\}$ of the Navier–Stokes equations satisfies this property.

6. A Noetherianity criterion

As already noted, a critical feature of the algebra of partial difference polynomials $P = K[X(\Sigma)]$ is that some of its Σ-ideals are not only infinitely generated as ideals but also infinitely Σ-generated. One finds an immediate counterexample for $\Sigma = \langle \sigma \rangle$, that is, in the ordinary difference case. In fact, for some fixed variable $x_i \in X$ one has clearly that the ideal $I = \langle x_i(1)x_i(\sigma), x_i(1)x_i(\sigma^2), \ldots \rangle_\Sigma$ has no finite Σ-basis. For any $x_i \in X$ and for all $\sigma^j, \sigma^k \in \Sigma$ we have that $\sigma^k \cdot x_i(\sigma^j) = x_i(\sigma^{k+j})$ and one can identify σ^k with the shift map $f_k : \mathbb{N} \to \mathbb{N}$ such that $f_k(j) = k + j$ which is a strictly increasing one. It is interesting to note that if we consider the larger monoid $\mathrm{Inc}(\mathbb{N})$ of all strictly increasing maps $f : \mathbb{N} \to \mathbb{N}$ acting on P as $f \cdot x_i(\sigma^j) = x_i(\sigma^{f(j)})$ then one has that P is $\mathrm{Inc}(\mathbb{N})$-Noetherian [2]. In other words, any $\mathrm{Inc}(\mathbb{N})$-ideal of P has a finite $\mathrm{Inc}(\mathbb{N})$-basis. We may say hence that the monoid Σ is "too small" to provide Σ-Noetherianity.

One way to solve this problem is to consider suitable quotients of the algebra of partial difference polynomials where Noetherianity and a fortiori Σ-Noetherianity is restored. A similar approach is used for the free associative algebra which is also non-Noetherian where the concepts of "algebras of solvable type, PBW algebras, G-algebras", etc naturally arise (see for instance [25]).

6.1. Countably generated algebras

We start now with a general discussion for (commutative) algebras generated by a countable set of elements. Let $Y = \{y_1, y_2, \ldots\}$ be a countable set and denote $P = K[Y]$ the polynomial algebra with variables set Y. Since P is a free algebra, all algebras generated by a countable set of elements are clearly isomorphic to quotients $P' = P/J$, where J is some ideal of P. To control the cosets in P', a standard approach consists in defining a normal form modulo J associated to a monomial ordering of P. Subsequently, let $\prec$ be a monomial ordering of P such that $y_1 \prec y_2 \prec \ldots$.

Definition 6.1. Put $M = \text{Mon}(P)$ and denote $M'' = M \setminus \text{lm}(J)$. Moreover, define the K-subspace $P'' = \langle M'' \rangle_K \subset P$. The elements of M'' are called *normal monomials modulo J* (*with respect to $\prec$*). The polynomials in P'' are said to be *in normal form modulo J*.

Since P is endowed with a monomial ordering, by a standard argument based on the algorithm REDUCE applied for the set J one obtains the following result.

Proposition 6.2. *A K-linear basis of the algebra P' is given by the set $M' = \{m + J \mid m \in M''\}$.*

Definition 6.3. Let $f \in P$. Denote $\text{NF}(f)$ the unique element of P'' such that $f - \text{NF}(f) \in J$. In other words, one has $\text{NF}(f) = \text{REDUCE}(f, J)$. We call $\text{NF}(f)$ the *normal form of f modulo J* (*with respect to $\prec$*).

By Proposition 6.2, one has that the mapping $f + J \mapsto NF(f)$ defines a linear isomorphism between $P' = P/J$ and $P'' = \langle M'' \rangle_K$. An algebra structure is defined hence for P'' by imposing that such a mapping is also an algebra isomorphism, that is, we define $f \cdot g = \text{NF}(fg)$, for all $f, g \in P''$. Then, we have a complete identification of M' with M'' and P' with P'', that is, we identify cosets with normal forms together with their algebra structures. We will make use of this from now on. We define hence the set of *normal variables*

$$Y' = Y \cap M' = Y \setminus \text{lm}(J).$$

Clearly, normal variables depend strictly on the monomial ordering one uses in P.

Proposition 6.4 *(Noetherianity criterion). Let P be endowed with a monomial ordering. If the set of normal variables Y' is finite then P' is a Noetherian algebra.*

Proof. It is sufficient to note that all normal monomials are products of normal variables and therefore the quotient algebra $P' = P/J$ is in fact generated by the set Y'. If Y'

is finite then P' is a finitely generated (commutative) algebra and hence it satisfies the Noetherian property. $\square$

We need now to introduce the notion of Gröbner basis for the ideals of $P' = P/J$. After the identification of cosets with normal forms, recall that $M' = M \setminus \mathrm{lm}(J)$ and $P' = \langle M' \rangle_K$ is a subspace of P endowed with multiplication $f \cdot g = \mathrm{NF}(fg)$, for all $f, g \in P'$. Then, all ideals $I' \subset P'$ have the form $I' = I/J = \{\mathrm{NF}(f) \mid f \in I\}$, for some ideal $J \subset I \subset P$. Note that $\mathrm{NF}(f) \in I$ for any $f \in I$, which implies that in fact $I' = I \cap P'$. Since the quotient algebra P'/I' is isomorphic to P/I and Gröbner bases give rise to K-linear bases of normal monomials for the quotients, one introduces the following definition.

Definition 6.5. Let $I' = I \cap P'$ be an ideal of P' where I is an ideal of P containing J. Moreover, consider $G' \subset I'$. We call G' a *Gröbner basis* of I' if $G' \cup J$ is a Gröbner basis of I.

Let $G \subset P$. Recall that $\mathrm{LM}(G)$ denotes the ideal of P generated by the set $\mathrm{lm}(G) = \{\mathrm{lm}(g) \mid g \in G, g \neq 0\}$.

Proposition 6.6. *Let I' be an ideal of P' and let $G' \subset I'$. Then, the set G' is a Gröbner basis of I' if and only if $\mathrm{LM}(G') = \mathrm{LM}(I')$.*

Proof. Let $J \subset I \subset P$ be an ideal such that $I' = I \cap P'$. Assume $\mathrm{LM}(G') = \mathrm{LM}(I')$. Let $f \in I$ and denote $f' = \mathrm{NF}(f)$. If $\mathrm{lm}(f) \notin \mathrm{LM}(J)$ then clearly $\mathrm{lm}(f) = \mathrm{lm}(f')$. Moreover, since $\mathrm{lm}(f') \in \mathrm{LM}(I') \subset \mathrm{LM}(G')$ one has that $\mathrm{lm}(f) = \mathrm{lm}(f') = m\,\mathrm{lm}(g')$, for some $m \in M$, $g' \in G'$. We conclude that $G' \cup J$ is a Gröbner basis of I. Suppose now that the latter condition holds. Since $G' \subset I'$, we have clearly that $\mathrm{LM}(G') \subset \mathrm{LM}(I')$. Let now $f' \in I' \subset I$. Then, there is $m \in M$, $g \in G' \cup J$ such that $\mathrm{lm}(f') = m\,\mathrm{lm}(g)$. Since $\mathrm{lm}(f') \in M'$ then also $\mathrm{lm}(g) \in M'$ and hence $g \in G'$. We conclude that $\mathrm{LM}(G') = \mathrm{LM}(I')$. $\square$

Proposition 6.7. *Assume that the set of normal variables $Y' = Y \cap M'$ is finite. Then, any monomial ideal $I = \langle I \cap M' \rangle \subset P$ has a finite basis.*

Proof. It is sufficient to invoke Dickson's Lemma (see for instance [11]) for the ideal I which is generated by normal monomials that are products of a finite number of normal variables. $\square$

Corollary 6.8. *If Y' is a finite set then any ideal $I' \subset P'$ has a finite Gröbner basis.*

Proof. According to Proposition 6.6, consider the ideal $\mathrm{LM}(I') \subset P$ which is generated by the set of normal monomials $\mathrm{lm}(I')$. Then, it is sufficient to apply Proposition 6.7 to this ideal. $\square$

It is clear that if G is any Gröbner basis of an ideal $J \neq P$ then $Y' = Y \setminus \mathrm{lm}(G)$. Note that Y is a countable set. Thus, if Y' is finite and hence $P' = K[Y']$ is a Noetherian algebra then G needs to be an infinite set. In general, such a Gröbner basis cannot be computed but this may be possible when P' is a Σ-algebra owing to the notion of Gröbner Σ-basis.

6.2. Σ-algebras

From now on, we assume again that $P = K[X(\Sigma)]$ is the algebra of partial difference polynomials. Let $J \subset P$ be a Σ-ideal and define the quotient Σ-algebra $P' = P/J$. As an algebra, we have clearly that P' is generated by the cosets $x_i(\sigma) + J$, for all $x_i(\sigma) \in X(\Sigma)$. Moreover, P' is a Σ-algebra which is Σ-generated by the cosets $x_i(1) + J$, for any $x_i(1) \in X(1)$. In fact, J is the Σ-ideal containing all Σ-algebra relations satisfied by such generators.

Let P be endowed with a monomial Σ-ordering $\prec$ and define, as in Subsection 6.1, the set $M' \subset M = \mathrm{Mon}(P)$ of all normal monomials and the set $X(\Sigma)' = X(\Sigma) \cap M'$ of all normal variables. After the identification of cosets with normal forms, we have that P' is an algebra generated by $X(\Sigma)'$ because normal monomials are products of normal variables. One has also the following result.

Proposition 6.9. *The Σ-algebra P' is Σ-generated by $X(1)' = X(1) \cap M'$.*

Proof. It is sufficient to show that $X(\Sigma)' \subset \Sigma \cdot X(1)'$. The set of non-normal variables $X(\Sigma) \setminus X(\Sigma)' = X(\Sigma) \cap \mathrm{lm}(J)$ is clearly invariant under the action of Σ. Therefore, if $x_i(1)$ is not a normal variable then $x_i(\sigma) = \sigma \cdot x_i(1)$ is also not a normal one. In other words, if $x_i(\sigma)$ is a normal variable then $x_i(1)$ is also such a variable and one has that $x_i(\sigma) = \sigma \cdot x_i(1)$. $\square$

To provide the Noetherian property to the quotient algebra $P' = P/J$ by means of Proposition 6.4 one has the following key result.

Proposition 6.10 *(Finiteness criterion). The set of normal variables $X(\Sigma)'$ is finite if and only if for all $1 \leq i \leq n$, $1 \leq j \leq r$ one has that $x_i(\sigma_j^{d_{ij}}) \in \mathrm{lm}(J)$, for some integers $d_{ij} \geq 0$.*

Proof. Put $x_i(\Sigma) = \{x_i(\sigma) \mid \sigma \in \Sigma\}$ and denote $x_i(\Sigma)' = x_i(\Sigma) \cap X(\Sigma)'$, for any $i = 1, 2, \ldots, n$. We have then to characterize when $x_i(\Sigma)'$ is a finite set. Consider the polynomial algebra $Q = K[\sigma_1, \ldots, \sigma_r]$ and a monomial ideal $I \subset Q$. It is well-known (see for instance [11], Ch. 5, §3, Th. 6) that the quotient algebra Q/I is finite dimensional if and only if there are integers $d_j \geq 0$ such that $\sigma_j^{d_j} \in I$, for all $j = 1, 2, \ldots, r$. It follows that $x_i(\Sigma)'$ is a finite set if and only if there exist integers $d_{ij} \geq 0$ such that $x_i(\sigma_j^{d_{ij}}) \in \mathrm{lm}(J)$, for all indices i, j. $\square$

Corollary 6.11 *(Termination by membership). Let $J \subset P$ be a Σ-ideal such that for all $1 \leq i \leq n$, $1 \leq j \leq r$ there are integers $d_{ij} \geq 0$ such that $x_i(\sigma_j^{d_{ij}}) \in \mathrm{lm}(J)$. Then J has a finite Gröbner Σ-basis.*

Proof. Denote $I = \langle x_i(\sigma_j^{d_{ij}}) \mid 1 \leq i \leq n, 1 \leq j \leq r \rangle_\Sigma$ and $L = \mathrm{LM}(J)$. Then, we have that $I \subset L$ and the ideal $L/I \subset P/I$ has a finite basis owing to Proposition 6.7 and Proposition 6.10. In other words, the Σ-ideal L has a finite Σ-basis given by the finite Σ-basis of I together with the finite basis of L/I. $\quad\square$

Note that the above result is not a necessary condition for finiteness of Gröbner Σ-bases. Consider for instance the example presented in Section 5 of [23]. Nevertheless, Corollary 6.11 guarantees termination of the procedure SigmaGBasis when a complete set of variables $x_i(\sigma_j^{d_{ij}})$ for all i, j, occurs as leading monomials of some elements of the Gröbner Σ-basis at some intermediate step of the computation. In other words, reaching this condition ensures that SigmaGBasis will definitely stop at some later step. Of course, if the elements $f_{ij} \in P$ such that $\mathrm{lm}(f_{ij}) = x_i(\sigma_j^{d_{ij}})$ belong to the input Σ-basis of a Σ-ideal $J \subset P$ then we know in advance that all properties of Noetherianity and termination are provided for the quotient $P' = P/J$. One may have that such polynomials are themselves a Gröbner Σ-basis of J and this happens in particular in the monomial case, that is, when $J = \langle x_i(\sigma_j^{d_{ij}}) \mid 1 \leq i \leq n, 1 \leq j \leq r \rangle_\Sigma$, for some $d_{ij} \geq 0$. For all $d \geq 0$, define therefore

$$J^{(d)} = \left\langle x_i(\sigma) \mid 1 \leq i \leq n, \deg(\sigma) = d+1 \right\rangle_\Sigma \supset \left\langle x_i\left(\sigma_j^{d+1}\right) \mid 1 \leq i \leq n, 1 \leq j \leq r \right\rangle_\Sigma$$

and put $J^{(-\infty)} = \langle X(1) \rangle_\Sigma = \langle X(\Sigma) \rangle$. If $P = \bigoplus_{d \in \hat{\mathbb{N}}} P_d$ is the grading of P defined by the order function then the subalgebra $P^{(d)} = \bigoplus_{i \leq d} P_i \subset P$ is clearly isomorphic to the quotient $P/J^{(d)}$ and hence it can be endowed with the structure of a Σ-algebra. Then, to make use of the following filtration of subalgebras

$$K = P^{(-\infty)} \subset P^{(0)} \subset P^{(1)} \subset \ldots \subset P$$

to perform concrete computations with Gröbner Σ-bases as explained in Section 4 corresponds to work progressively modulo the Σ-ideals

$$\left\langle X(\Sigma) \right\rangle = J^{(-\infty)} \supset J^{(0)} \supset J^{(1)} \supset \ldots \supset 0$$

providing the finite set of normal variables $X(\Sigma_d) = \{x_i(\sigma) \mid 1 \leq i \leq n, \deg(\sigma) \leq d\}$ and hence the Noetherian property for each quotient $P/J^{(d)}$ isomorphic to $P^{(d)}$. In other words, termination by truncation is essentially a special instance of termination by membership. Another interesting case is the ordinary one, that is, when $\Sigma = \langle \sigma \rangle$. In this case, any set of polynomials $f_1, \ldots, f_n \in P$ such that $\mathrm{lm}(f_i) = x_i(\sigma^{d_i})$ $(d_i \geq 0)$ is a Gröbner Σ-basis since all S-polynomials trivially reduce to zero according to the product criterion.

To motivate the last result of this section, let us consider the following problem. Assume that K is a field of constants and let V be a finite dimensional K-vector space. Denote by $\mathrm{end}_K(V)$ the algebra of K-linear endomorphisms of V and let $Q' \subset \mathrm{end}_K(V)$ be a subalgebra generated by r commuting endomorphisms. Since $Q = K[\sigma_1, \ldots, \sigma_r]$ is the free commutative algebra with r generators, one has a K-algebra homomorphism $Q \to \mathrm{end}_K(V)$ sending the σ_i onto the generators of Q', that is, V is a Q-module. Consider now the (Noetherian) polynomial algebra R whose variables are a K-linear basis of V. In other words, V is the subspace of linear forms of R or equivalently R is the symmetric algebra on V. Define $\mathrm{End}_K(R)$ the monoid of K-algebra endomorphisms of R. Since $\Sigma = \mathrm{Mon}(Q)$, we can extend the action of Σ on V to a monoid homomorphism $\Sigma \to \mathrm{End}_K(R)$, that is, R is a Σ-algebra. Because $P = K[X(\Sigma)]$ is a free Σ-algebra, there is a suitable set $X = \{x_1, \ldots, x_n\}$ and a Σ-ideal $J \subset P$ such that R is isomorphic to the quotient Σ-algebra $P' = P/J$. Since Q acts linearly over V, one has that J is Σ-generated by linear polynomials. Then, in the following result we analyze from the perspective of Proposition 6.4 and Proposition 6.10 the easiest case for a linear Σ-ideal providing the Noetherian property to the quotient Σ-algebra. In Section 7 we will show that this case corresponds to have the finite dimensional commutative algebra Q' decomposable as the tensor product of r cyclic subalgebras. This happens in particular if Q' is the group algebra of a finite abelian group and one application of this specific case is given in Section 8.

Proposition 6.12. *Let K be a field of constants and consider the linear polynomials $f_{ij} = \sum_{0 \leq k \leq d_{ij}} c_{ijk} x_i(\sigma_j^k) \in P$ where $c_{ijk} \in K$ and $c_{ijd_{ij}} = 1$, for all $1 \leq i \leq n$, $1 \leq j \leq r$. Then $\mathrm{lm}(f_{ij}) = x_i(\sigma_j^{d_{ij}})$ and the set $\{f_{ij}\}$ is a Gröbner Σ-basis.*

Proof. Since $X(\Sigma)$ is endowed with a Σ-ordering, one has that $x_i(\sigma_j^k) \prec x_i(\sigma_j^l)$ if $k < l$ and hence $\mathrm{lm}(f_{ij}) = x_i(\sigma_j^{d_{ij}})$. Then, the only S-polynomials to be considered are

$$s = \mathrm{spoly}\big(\sigma_q^{d_{iq}} \cdot f_{ip}, \sigma_p^{d_{ip}} \cdot f_{iq}\big) = \sum_{0 \leq k < d_{ip}} c_{ipk} x_i\big(\sigma_q^{d_{iq}} \sigma_p^k\big) - \sum_{0 \leq l < d_{iq}} c_{iql} x_i\big(\sigma_p^{d_{ip}} \sigma_q^l\big),$$

for all $1 \leq i \leq n$ and $1 \leq p < q \leq r$. By reducing s with polynomials $\sigma_p^k \cdot f_{iq}$ and $\sigma_q^l \cdot f_{ip}$ one obtains

$$s' = -\sum_{0 \leq k < d_{ip},\, 0 \leq l < d_{iq}} c_{ipk} c_{iql} x_i\big(\sigma_q^l \sigma_p^k\big) + \sum_{0 \leq l < d_{iq},\, 0 \leq k < d_{ip}} c_{iql} c_{ipk} x_i\big(\sigma_p^k \sigma_q^l\big) = 0. \qquad \square$$

Note explicitly that the assumption that K is a field of constants is necessary in the above result. In fact, if Σ acts on K in a non-trivial way then generally

$$s' = -\sum_{0 \leq k < d_{ip},\, 0 \leq l < d_{iq}} \big(\sigma_q^{d_{iq}} \cdot c_{ipk}\big)\big(\sigma_p^k \cdot c_{iql}\big) x_i\big(\sigma_q^l \sigma_p^k\big)$$

$$+ \sum_{0 \leq l < d_{iq},\, 0 \leq k < d_{ip}} \big(\sigma_p^{d_{ip}} \cdot c_{iql}\big)\big(\sigma_q^l \cdot c_{ipk}\big) x_i\big(\sigma_p^k \sigma_q^l\big) \neq 0.$$

7. A Noetherian Σ-algebra of special interest

From now on we assume that K is a field of constants. We define the ideal $J = \langle f_{ij} \rangle_{\Sigma} \subset P$ where $f_{ij} = \sum_{0 \le k \le d_{ij}} c_{ijk} x_i(\sigma_j^k)$ ($c_{ijk} \in K$, $c_{ijd_{ij}} = 1$), for any $1 \le i \le n$, $1 \le j \le r$. We want to describe the (Noetherian) Σ-algebra $P' = P/J$. To simplify notations and since they are interesting in themselves, we consider separately the cases when $r = 1$ and $n = 1$.

First assume that $r = 1$, that is, $\Sigma = \langle \sigma \rangle$ and hence $P' = P/J$ where $J = \langle f_1, \dots, f_n \rangle_{\Sigma}$ with $f_i = \sum_{0 \le k \le d_i} c_{ik} x_i(\sigma^k)$ ($c_{ik} \in K$, $c_{id_i} = 1$). Define $Q = K[\sigma]$ the algebra of polynomials in the single variable σ and denote $g_i = \sum_{0 \le k \le d_i} c_{ik} \sigma^k \in Q$. Moreover, put $d = \sum_i d_i$ and let $V = K^d$. Finally, consider the $d \times d$ block-diagonal matrix

$$A = A_1 \oplus \dots \oplus A_n = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & A_n \end{pmatrix}$$

where each block A_i is the companion matrix of the polynomial g_i, that is,

$$A_i = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_{i0} \\ 1 & 0 & \dots & 0 & -c_{i1} \\ 0 & 1 & \dots & 0 & -c_{i2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{id-1} \end{pmatrix}.$$

Note that A has all entries in the base field K and it can be considered as the Frobenius normal form of a $d \times d$ matrix provided that $g_1 \mid \dots \mid g_n$. Recall that any square matrix is similar over the base field to its Frobenius normal form, that is, we are considering any K-linear endomorphism of V. Then, the monoid Σ or equivalently the algebra Q acts linearly over the vector space V by means of the representation $\sigma^k \mapsto A^k$. If $\{v_q\}_{1 \le q \le d}$ is the canonical basis of V, we denote $x_i(\sigma^k) = v_q$ where $q = \sum_{j < i} d_j + k + 1$ for all $1 \le i \le n$, $0 \le k < d_i$. We have hence $x_i(\sigma^k) = A^k x_i(1) = \sigma^k \cdot x_i(1)$. In other words, for the Q-module V one has the decomposition $V = \bigoplus_i V_i$ where V_i is the cyclic submodule generated by $x_i(1)$ and annihilated by the ideal $\langle g_i \rangle \subset Q$. Denote now by R the (Noetherian) polynomial algebra generated by the finite set of variables $X(\Sigma)' = \{x_i(\sigma^k) \mid 1 \le i \le n, 0 \le k < d_i\}$, that is, V coincides with the subspace of linear forms of R. Then, one extends the action of the monoid $\Sigma = \langle \sigma \rangle$ to the polynomial algebra R in the natural way that is by putting, for all $k \ge 0$ and $x_i(\sigma^j) \in X(\Sigma)'$

$$\sigma^k \cdot x_i(\sigma^j) = A^k x_i(\sigma^j).$$

Denote by $\mathrm{end}_K(P)$ the algebra of all K-linear mappings $P \to P$ and define by $\mathrm{End}_K(P)$ the monoid of K-algebra endomorphisms of P. Note that the representation

$\rho : \Sigma \to \mathrm{End}_K(P)$ can be extended linearly to $\bar{\rho} : Q \to \mathrm{end}_K(P)$. Then, one has that $f_i = \sum_k c_{ik} x_i(\sigma^k) = \sum_k c_{ik} \sigma^k \cdot x_i(1) = g_i \cdot x_i(1)$, for all $i = 1, 2, \ldots, n$.

Proposition 7.1. *If $\Sigma = \langle \sigma \rangle$ then the Σ-algebras P', R are Σ-isomorphic.*

Proof. By Proposition 6.12 we have that the set $\{f_i\}$ is a Gröbner Σ-basis of the Σ-ideal $J \subset P$ and it is clear that the set of normal variables modulo J is exactly $X(\Sigma)' = \{x_i(\sigma^k) \mid 1 \leq i \leq n, 0 \leq k < d_i\}$. Moreover, since $R \subset P$ and $f_i = g_i \cdot x_i(1)$ one has that $\mathrm{NF}(x_i(\sigma^k)) = \mathrm{NF}(\sigma^k \cdot x_i(1)) = A^k x_i(1)$, for all $k \geq 0$ and $x_i(1) \in X(1)'$. □

Note that R is Σ-generated by the set $X(1)' = \{x_i(1) \mid 1 \leq i \leq n, d_i > 0\}$. Since P is a free Σ-algebra, a surjective Σ-algebra homomorphism $\varphi : P \to R$ is defined such that

$$x_i(1) \mapsto \begin{cases} x_i(1) & \text{if } d_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the above result states that the Σ-ideal $\mathrm{Ker}\,\varphi \subset P$ of all Σ-algebra relations satisfied by the generating set $X(1)' \cup \{0\}$ of R is exactly J.

Assume now that $n = 1$, that is, $X = \{x\}$ and $\Sigma = \langle \sigma_1, \ldots, \sigma_r \rangle$. Then $P' = P/J$ where $J = \langle f_1, \ldots, f_r \rangle_{\Sigma}$ with $f_j = \sum_{0 \leq k \leq d_j} c_{jk} x(\sigma_j^k)$ ($c_{jk} \in K$, $c_{jd_j} = 1$). Define $Q = K[\sigma_1, \ldots, \sigma_r]$ the algebra of polynomials in the variables σ_j and denote $g_j = \sum_{0 \leq k \leq d_j} c_{jk} \sigma_j^k \in Q$. One has clearly that $f_j = g_j \cdot x(1)$. As before, we consider the companion matrix A_j of the polynomial g_j in the single variable σ_j. If $d = \prod_j d_j$ then the monoid $\Sigma = \Sigma_1 \times \cdots \times \Sigma_r$ ($\Sigma_j = \langle \sigma_j \rangle$), that is, the algebra $Q = Q_1 \otimes \cdots \otimes Q_r$ ($Q_j = K[\sigma_j]$) acts linearly over the space $V = K^d$ by means of the representation

$$\sigma_1^{k_1} \cdots \sigma_r^{k_r} \mapsto A_1^{k_1} \otimes \cdots \otimes A_r^{k_r},$$

where $A_1^{k_1} \otimes \cdots \otimes A_r^{k_r}$ denotes the Kronecker product of the matrices $A_j^{k_j}$. In other words, the Q-module V is the tensor product $V = V_1 \otimes \cdots \otimes V_r$ where V_j is the cyclic Q_j-module defined by the representation $\sigma_j^k \mapsto A_j^k$. If $\{v_{k_1} \otimes \cdots \otimes v_{k_r}\}_{1 \leq k_j \leq d_j}$ is the canonical basis of V, we put $x(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) = v_{k_1+1} \otimes \cdots \otimes v_{k_r+1}$, for all $1 \leq j \leq r$, $0 \leq k_j < d_j$. One has then

$$x(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) = (A_1^{k_1} \otimes \ldots \otimes A_r^{k_r}) x(1) = (\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \cdot x(1),$$

that is, V is a cyclic module generated by $x(1)$. Denote now by R the polynomial algebra generated by the finite set of variables $X(\Sigma)' = \{x(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \mid 1 \leq j \leq r, 0 \leq k_j < d_j\}$, that is, V is the subspace of linear forms of R. Again, we extend the action of the monoid $\Sigma = \langle \sigma_1, \ldots, \sigma_r \rangle$ to the polynomial algebra R by putting, for all $k_1, \ldots, k_r \geq 0$ and $x(\sigma) \in X(\Sigma)'$

$$(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \cdot x(\sigma) = (A_1^{k_1} \otimes \cdots \otimes A_r^{k_r}) x(\sigma).$$

Proposition 7.2. *If $X = \{x\}$ then P', R are Σ-isomorphic.*

Proof. Assume $d \neq 0$, that is, $d_j \neq 0$ for all j. Again, by Proposition 6.12 one has that the set $\{f_j\}$ is a Gröbner Σ-basis of $J \subset P$ and the set of normal variables modulo J is clearly $X(\Sigma)' = \{x(\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \mid 1 \leq j \leq r, 0 \leq k_j < d_j\}$. Moreover, because $R \subset P$ and $f_j = g_j \cdot x(1)$ we obtain that, for all $k_1, \ldots, k_r \geq 0$

$$\mathrm{NF}\big(x(\sigma_1^{k_1} \cdots \sigma_r^{k_r})\big) = \mathrm{NF}\big((\sigma_1^{k_1} \cdots \sigma_r^{k_r}) \cdot x(1)\big) = \big(A_1^{k_1} \otimes \ldots \otimes A_r^{k_r}\big)x(1).$$

Finally, if $d = 0$ then $P' = R = K$. $\square$

Note that for $d \neq 0$ one has that R is Σ-generated by the element $x(1)$. Then, the above result implies that the Σ-ideal $J \subset P$ coincides with the ideal of Σ-algebra relations satisfied by the generator $x(1)$, that is, it is the kernel of the Σ-algebra epimorphism $P \to R$ such that $x(1) \mapsto x(1)$.

Consider finally the general case for the Σ-algebra $P' = P/J$ where $J = \langle f_{ij} \rangle_\Sigma$ and $f_{ij} = \sum_{0 \leq k \leq d_{ij}} c_{ijk} x_i(\sigma_j^k)$ with $c_{ijk} \in K$, $c_{ijd_{ij}} = 1$, for all $1 \leq i \leq n$ and $1 \leq j \leq r$. By combining the previous results, one may conclude that such a structure arises from the Q-module $V = K^d$ where $d = \sum_{1 \leq i \leq n} \prod_{1 \leq j \leq r} d_{ij}$ and the representation is given by the mapping

$$\prod_j \sigma_j^{k_j} \mapsto \bigoplus_i \bigotimes_j A_{ij}^{k_j}$$

where A_{ij} is the companion matrix of the polynomial $g_{ij} = \sum_{0 \leq k \leq d_{ij}} c_{ijk} \sigma_j^k$. In other words, we have that $V = \bigoplus_i \bigotimes_j V_{ij}$ where V_{ij} is the cyclic Q_j-module annihilated by the ideal $\langle g_{ij} \rangle \subset Q_j$. By denoting $x_i(1)$ the generator of the Q-module $\bigotimes_j V_{ij}$, we obtain that P' is isomorphic to the Σ-algebra $R = K[X(\Sigma)']$ where $X(\Sigma)' = \{x_i(\sigma_1^{k_{i1}} \cdots \sigma_r^{k_{ir}}) \mid 1 \leq i \leq n, 1 \leq j \leq r, 0 \leq k_{ij} < d_{ij}\}$ is the canonical basis of the space V. Then, one has that $J = \langle f_{ij} \rangle_\Sigma$ is exactly the Σ-ideal of Σ-algebra relations satisfied by generating set $X(1)' \cup \{0\}$ of R.

8. Another example

A long-lasting problem in Gröbner bases theory is about the possibility to accord the definition and the computation of such bases to some form of symmetry, typically defined by groups, which one may have on the generators or on the ideal itself of some polynomial algebra (see for instance [5,12]). The main objection against this possibility is that monomial orderings cannot be defined consistently with the group action which implies that the symmetry disappears in the Gröbner basis. In fact, if the symmetry is defined by a monoid Σ isomorphic to $\mathbb{N}^r$ we have found that the notion of Σ-ideal perfectly accords with monomial orderings and Gröbner bases. Moreover, in the previous section we have shown that by means of the notion of quotient Σ-algebra and the

corresponding Gröbner bases tools one can deal with symmetries defined by suitable finite dimensional commutative algebras. Among them one finds group algebras of finite abelian groups and therefore this section is devoted to such a case. In other words, we will show that Gröbner bases of ideals having a finite abelian group symmetry can be "tamed" by means of Σ-algebras and their quotients.

We fix now a setting that has been recently considered in [29]. Note that in our approach all computations can be performed over any field (of constants) but in [29] the base field is required to contain roots of unity. Fix $r = 1$, that is, $\Sigma = \langle \sigma \rangle$ and $Q = K[\sigma]$. Consider $\mathbb{S}_d$ the symmetric group on d elements and let $\gamma \in \mathbb{S}_d$ be any permutation. Denote $\Gamma = \langle \gamma \rangle \subset \mathbb{S}_d$ the cyclic subgroup generated by γ. Moreover, let $\gamma = \gamma_1 \cdots \gamma_n$ be the cycle decomposition of γ and denote by d_i the length of the cycle γ_i. Consider the polynomial algebra $R = K[x_i(\sigma^j) \mid 1 \leq i \leq n, 0 \leq j < d_i]$ and identify the subset $\{x_i(1), \ldots, x_i(\sigma^{d_i-1})\}$ with the support of the cycle γ_i. Define $\mathrm{Aut}_K(R)$ the group of K-algebra automorphisms of R. Clearly R is a Γ-algebra, that is, there is a (faithful) group representation $\rho' : \Gamma \to \mathrm{Aut}_K(R)$. Consider now the polynomials $g_i = \sigma^{d_i} - 1 \in Q$ and define the $d \times d$ block-diagonal matrix

$$
A = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix}
$$

where each block A_i is the companion matrix of the polynomial g_i which is the permutation matrix

$$
A_i = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.
$$

If we order the variables of R as $x_1(1), \ldots, x_1(\sigma^{d_1-1}), \ldots, x_n(1), \ldots, x_n(\sigma^{d_n-1})$ then the representation ρ' is defined as $\gamma^k \cdot x_i(\sigma^j) = A^k x_i(\sigma^j)$, for all i, j, k. In other words, by Proposition 7.1 one has that R is a Σ-algebra isomorphic to $P' = P/J$ where $J = \langle f_1, \ldots, f_n \rangle_\Sigma$ and $f_i = g_i \cdot x_i(1) = x_i(\sigma^{d_i}) - x_i(1) \in P$. Consider now a Γ-ideal (equivalently a Σ-ideal) $L' = \langle h_1, \ldots, h_m \rangle_\Gamma \subset R$ and define the Σ-ideal $L = \langle h_1, \ldots, h_m, f_1, \ldots, f_n \rangle_\Sigma \subset P$. Note that Γ-ideals are called "symmetric ideals" in [29]. According with Definition 6.5 and the identification of R with the quotient P' one has that $G' \subset L'$ is a Gröbner Γ-basis (equivalently Σ-basis) of L' if by definition $G' \cup \{f_1, \ldots, f_n\}$ is a Gröbner Σ-basis of L. In practice, the computation of G' is obtained by the algorithm SIGMAGBASIS which terminates owing to Corollary 6.11.

To illustrate the method we fix now $\gamma = (12345678) \in \mathbb{S}_8$ and $K = \mathbb{Q}$. To simplify the variables notation we identify Σ with $\mathbb{N}$, that is, $R = K[x(0), x(1), \ldots, x(7)]$. Consider the following Γ-ideal of R

$$
\begin{aligned}
L' &= \left\langle x(0)x(2) - x(1)^2, x(0)x(3) - x(1)x(2) \right\rangle_\Gamma \\
&= \left\langle x(0)x(2) - x(1)^2, x(1)x(3) - x(2)^2, x(2)x(4) - x(3)^2, x(3)x(5) - x(4)^2, \right. \\
&\qquad x(4)x(6) - x(5)^2, x(5)x(7) - x(6)^2, x(7)^2 - x(0)x(6), x(1)x(7) - x(0)^2, \\
&\qquad x(0)x(3) - x(1)x(2), x(1)x(4) - x(2)x(3), x(2)x(5) - x(3)x(4), \\
&\qquad x(3)x(6) - x(4)x(5), x(4)x(7) - x(5)x(6), x(6)x(7) - x(0)x(5), \\
&\qquad \left. x(0)x(7) - x(1)x(6), x(2)x(7) - x(0)x(1) \right\rangle.
\end{aligned}
$$

Note that $x(0)x(2) - x(1)^2$, $x(1)x(3) - x(2)^2$, $x(0)x(3) - x(1)x(2)$ are well-known equations of the twisted cubic in $\mathbb{P}^3$. Define now $f = x(8) - x(0) \in P$ and hence $R = P' = P/J$ where $J = \langle f \rangle_\Sigma$. Then, a Gröbner Γ-basis (or Σ-basis) of L' is obtained by computing a Gröbner Σ-basis of the ideal

$$
L = \left\langle x(0)x(2) - x(1)^2, x(0)x(3) - x(1)x(2), f \right\rangle_\Sigma \subset P.
$$

Fix for instance the lexicographic monomial ordering on P (hence on R) with $x(0) \prec x(1) \prec \ldots$ which is clearly a Σ-ordering. The usual minimal Gröbner basis of L' consists of 54 elements whose leading monomials are

$$
x(7)^2, x(6)x(7),
$$
$$
x(0)x(2) \to x(1)x(3) \to x(2)x(4) \to x(3)x(5) \to x(4)x(6) \to x(5)x(7),
$$
$$
x(0)x(3) \to x(1)x(4) \to x(2)x(5) \to x(3)x(6) \to x(4)x(7), x(2)x(7),
$$
$$
x(1)x(7), x(0)x(7), x(6)^3, x(0)x(4)^2 \to x(1)x(5)^2 \to x(2)x(6)^2,
$$
$$
x(0)^2x(4) \to x(1)^2x(5) \to x(2)^2x(6) \to x(3)^2x(7), x(0)^2x(6), x(0)x(6)^2,
$$
$$
x(1)x(6)^2, x(1)^2x(6), x(3)^2x(4) \to x(4)^2x(5) \to x(5)^2x(6),
$$
$$
x(4)x(5)^2 \to x(5)x(6)^2, x(0)x(1)x(6), x(0)x(4)x(5) \to x(1)x(5)x(6),
$$
$$
x(0)x(5)x(6), x(1)x(2)x(6), x(2)^4 \to x(3)^4 \to x(4)^4 \to x(5)^4, x(0)^3x(5),
$$
$$
x(0)x(5)^3, x(2)^3x(3), x(2)x(3)^3 \to x(3)x(4)^3, x(0)^2x(5)^2, x(0)^2x(1)x(5),
$$
$$
x(2)^2x(3)^2, x(1)^2x(2)^3, x(1)^4x(2)^2, x(1)^6x(2), x(1)^8.
$$

The arrow between two monomials means that a monomial can be obtained by the previous one by means of the Σ-action. Then, the minimal Gröbner Γ-basis of L' has just 32 elements and their leading monomials are

$$x(7)^2, x(6)x(7), x(0)x(2), x(0)x(3), x(2)x(7), x(1)x(7), x(0)x(7), x(6)^3,$$
$$x(0)x(4)^2, x(0)^2x(4), x(0)^2x(6), x(0)x(6)^2, x(1)x(6)^2, x(1)^2x(6), x(3)^2x(4),$$
$$x(4)x(5)^2, x(0)x(1)x(6), x(0)x(4)x(5), x(0)x(5)x(6), x(1)x(2)x(6), x(2)^4,$$
$$x(0)^3x(5), x(0)x(5)^3, x(2)^3x(3)x(2)x(3)^3, x(0)^2x(5)^2, x(0)^2x(1)x(5),$$
$$x(2)^2x(3)^2, x(1)^2x(2)^3, x(1)^4x(2)^2, x(1)^6x(2), x(1)^8.$$

In other words, our approach based on Σ-compatible structures is able to define appropriately a Gröbner basis that generates a group invariant ideal up to the group action and this basis is actually more compact than the usual Gröbner basis. The elements of the minimal Gröbner Γ-basis of L' are the following ones

$$x(7)^2 - x(0)x(6), x(6)x(7) - x(0)x(5), x(0)x(2) - x(1)^2, x(0)x(3) - x(1)x(2),$$
$$x(2)x(7) - x(0)x(1), x(1)x(7) - x(0)^2, x(0)x(7) - x(1)x(6), x(6)^3 - x(0)x(5)^2,$$
$$x(0)x(4)^2 - x(2)x(3)^2, x(0)^2x(4) - x(1)^2x(2), x(0)^2x(6) - x(0)x(1)x(5),$$
$$x(0)x(6)^2 - x(1)x(5)x(6), x(1)x(6)^2 - x(0)^2x(5), x(1)^2x(6) - x(0)^3,$$
$$x(3)^2x(4) - x(0)x(1)^2, x(4)x(5)^2 - x(0)x(1)x(5), x(0)x(1)x(6) - x(2)^2x(3),$$
$$x(0)x(4)x(5) - x(0)^2x(1), x(0)x(5)x(6) - x(3)x(4)^2,$$
$$x(1)x(2)x(6) - x(0)x(4)x(5), x(2)^4 - x(0)^4, x(0)^3x(5) - x(3)^3x(4),$$
$$x(0)x(5)^3 - x(3)x(4)^3, x(2)^3x(3) - x(0)^3x(1), x(2)x(3)^3 - x(0)x(1)^3,$$
$$x(0)^2x(5)^2 - x(2)^2x(3)^2, x(0)^2x(1)x(5) - x(3)^2x(4)^2, x(2)^2x(3)^2 - x(0)^2x(1)^2,$$
$$x(1)^2x(2)^3 - x(0)^5, x(1)^4x(2)^2 - x(0)^6, x(1)^6x(2) - x(0)^7, x(1)^8 - x(0)^8.$$

We have computed these elements by applying the algorithm SigmaGBasis to the Σ-ideal $L \subset P$ in the same way as for the example in Section 5. For details about different strategies to implement this method we refer to [23].

9. Conclusions and further directions

In this paper we showed that a viable theory of Gröbner bases exists for the algebra of partial difference polynomials which implies that one can perform symbolic (formal) computations for systems of partial difference equations. In fact, we prove that such Gröbner bases can be computed in a finite number of steps when truncated with respect to an appropriate grading or when they contain elements with suitable linear leading monomials. Precisely, since the algebras of difference polynomials are free objects in the category of Σ-algebras where Σ is a monoid isomorphic to $\mathbb{N}^r$, we obtained the latter result as a Noetherianity criterion for a class of finitely generated Σ-algebras. Among such Noetherian Σ-algebras one finds polynomial algebras in a finite number of variables where a tensor product of a finite number of algebras generated by single matrices acts

over the subspace of linear forms. Considering that such commutative tensor algebras include group algebras of finite abelian groups one obtains that there exists a consistent Gröbner basis theory for ideals of finitely generated polynomial algebras that are invariant under such groups. In our opinion, this represents an interesting step in the direction of development of computational methods for ideals or algebras that are subject to group or algebra symmetries.

As for further developments, we may suggest that the study of important structures related to Gröbner bases like Hilbert series and free resolutions should be developed in the perspective that their definition and computation has to be consistent to the symmetry one defines eventually on a polynomial algebra. An important work in this direction is contained in [20]. Finally, the problem of studying conditions providing Σ-Noetherianity (instead of simple Noetherianity) for finitely generated Σ-algebras is also an intriguing subject.

Acknowledgments

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