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# Topological extension of parity automata

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ABSTRACT

The paper presents a concept of a coloring — an extension of deterministic parity automata. A coloring $K$ is a function $A^* \to \mathbb{N}$ satisfying

$$\forall_{\alpha \in A^\omega} \liminf_{n\to\infty} K(\alpha\restriction_n) < \infty.$$

Every coloring defines a subset of $A^\omega$ by the standard parity condition

$$[K] = \left\{\alpha \in A^\omega\colon \liminf_{n\to\infty} K(\alpha\restriction_n) \equiv 0 \bmod 2\right\}.$$

We show that sets defined by colorings are exactly all $\mathbf{\Delta}^0_3$ sets in the standard product topology on $A^\omega$. Furthermore, when considering natural subfamilies of all colorings, we obtain families $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$, $\mathbf{\Delta}^0_2$, and $\mathcal{BC}(\boldsymbol{\Sigma}^0_1)$. Therefore, colorings can be seen as a characterisation of all these classes with a uniform acceptance condition.
Additionally, we analyse a similar definition of colorings where the lim sup condition is used instead of lim inf. It turns out that such colorings have smaller expressive power — they cannot define all $\mathbf{\Delta}^0_3$ sets.



## 1. Introduction

The Borel hierarchy provides a natural measure of topological complexity of sets. The lowest levels of this hierarchy contain all open ($\boldsymbol{\Sigma}^0_1$) and all closed ($\boldsymbol{\Pi}^0_1$) sets of a given topological space. The successive levels are obtained using countable unions and intersections: $\boldsymbol{\Sigma}^0_{n+1}$ contains all countable unions of sets in $\boldsymbol{\Pi}^0_n$ and $\boldsymbol{\Pi}^0_{n+1}$ contains all countable intersections of sets in $\boldsymbol{\Sigma}^0_n$. $\mathbf{\Delta}^0_n$ denotes the intersection of $\boldsymbol{\Sigma}^0_n$ and $\boldsymbol{\Pi}^0_n$. By $\mathcal{BC}(\boldsymbol{\Sigma}^0_n)$ we denote the class of all Boolean combinations of sets in $\boldsymbol{\Sigma}^0_n$.

One of the ways to find the Borel complexity of a given set is to compare it with known examples of complete sets for various classes. Especially, the first two levels of the hierarchy contain many natural examples of complete sets. For example the $\omega$-regular language "only finitely many letters $b$" is $\boldsymbol{\Sigma}^0_2$-complete.

The classes $\mathbf{\Delta}^0_n$ and $\mathcal{BC}(\boldsymbol{\Sigma}^0_n)$ are more subtle to analyse than $\boldsymbol{\Sigma}^0_n$ and $\boldsymbol{\Pi}^0_n$. When considering some $\boldsymbol{\Sigma}^0_n$ set there is a natural representation of it as a countable union of simpler sets. In the case of $\mathbf{\Delta}^0_n$ we have to consider two representations at the same time — a countable union and a countable intersection. Moreover, there are no complete sets for $\mathbf{\Delta}^0_n$ nor $\mathcal{BC}(\boldsymbol{\Sigma}^0_n)$.

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The paper studies an idea of a coloring which is any function $K : A^* \to \mathbb{N}$ satisfying $\forall_{\alpha\in A^\omega} \liminf_{n\to\infty} K(\alpha\restriction_n) < \infty$. Every coloring defines a subset $[K]$ of $A^\omega$ by the standard parity condition (see (1)): an infinite branch $\alpha$ belongs to $[K]$ if the limes inferior of values of $K$ on $\alpha$ is even.

The following diagram presents two lower levels of the Borel hierarchy (lines between families denote strict inclusions).

$$\begin{tikzcd}
\boldsymbol{\Sigma}^0_1 \arrow[dr, dash] & & & \boldsymbol{\Sigma}^0_2 \arrow[dr, dash] & & \\
& \mathcal{BC}(\boldsymbol{\Sigma}^0_1) \arrow[r, dash] & \boldsymbol{\Delta}^0_2 \arrow[ur, dash] \arrow[dr, dash] & & \mathcal{BC}(\boldsymbol{\Sigma}^0_2) \arrow[r, dash] & \boldsymbol{\Delta}^0_3 \\
\boldsymbol{\Pi}^0_1 \arrow[ur, dash] & & & \boldsymbol{\Pi}^0_2 \arrow[ur, dash] & &
\end{tikzcd}$$

For all families presented in the diagram we propose combinatorial properties such that colorings satisfying a given property define exactly all sets from the given family. In particular, $\boldsymbol{\Delta}^0_2$ sets are those defined by *monotone* (non-decreasing) colorings.

All $\omega$-regular languages are in $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$. Since there are only countably many of them they cannot cover any standard class of topological complexity. In [1] Büchi shows that every $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ set is recognised by a *recursion scheme* with a Müller acceptance condition. A coloring can be seen as a natural model of a *recursion scheme* with parity condition. It extends $\omega$-regular languages in two ways: not only all sets of a given complexity are defined by colorings but also the complexity of the defined sets can be greater ($\boldsymbol{\Delta}^0_3$ instead of $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$).

The simple combinatorial structure of colorings makes them a handy tool in the analysis of $\boldsymbol{\Delta}^0_{n+1}$ and $\mathcal{BC}(\boldsymbol{\Sigma}^0_n)$ sets for $n = 1, 2$. Constructions based on colorings are used in [2] and [3].

One of the results in this paper states that a similar definition of a coloring where the property lim inf is changed into lim sup has smaller expressive power. That is: there exists a $\boldsymbol{\Delta}^0_3$ set that can be recognised by a standard coloring but not by any coloring with the lim sup condition. This can be interpreted in the following way: in the case of parity automata with infinitely many states, the acceptance conditions lim inf and lim sup are not equivalent and lim inf is stronger.

Another interesting difference between the conditions lim inf and lim sup was observed in [4]. This paper analyses parity games with infinitely many priorities. The question of positional determinacy of such games is investigated. The main result states that the lim inf condition admits positional determinacy while the lim sup condition in general does not.

The rest of the paper is organised in the following way. Section 2 presents a connection between colorings and the theory of $\omega$-regular languages. Section 3 contains a definition of colorings and introduces their main properties. At the end of this section the main theorem is stated.

Section 4 analyses connections between colorings and the difference hierarchy. Section 5 contains explicit constructions via colorings of sets in $\boldsymbol{\Delta}^0_3 \setminus \mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ and in $\boldsymbol{\Delta}^0_2 \setminus \mathcal{BC}(\boldsymbol{\Sigma}^0_1)$. The existence of such sets is known from the literature but the proofs usually go through a diagonal argument (see e.g. [5]).

In Section 6 the difference between the lim inf and lim sup conditions is analysed. Finally, Section 7 provides a simple proof of $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ determinacy and in Section 8 we formulate a number of open questions regarding colorings.

## 2. Motivation

The idea of a coloring is motivated by finite state deterministic parity automata. Such automata are one of the equivalent models to Büchi automata used to show decidability of MSO logic over $\omega$ (see [6]).

A finite state deterministic parity automaton is defined as a tuple $\mathcal{A} = \langle A, Q, \delta, \Omega, q_0\rangle$, where

- $A$ is a finite *alphabet*,
- $Q$ is a finite set of objects called *states* of the automaton,
- $\delta$ is a *transition* function $Q \times A \to Q$,
- $\Omega$ is a function $Q \to \mathbb{N}$ mapping states into their *ranks*,
- $q_0$ is an *initial state* of the automaton.

Such an automaton transforms a given word $\alpha \in A^\omega$ into the unique *run* — an infinite sequence $\tau \in Q^\omega$ defined by the following rules: $\tau_0 = q_0$, $\tau_{n+1} = \delta(\tau_n, \alpha_n)$. We say that an automaton $\mathcal{A}$ *accepts* a word $\alpha \in A^\omega$ if

$$\liminf_{n\to\infty} \Omega(\tau_n) \equiv 0 \bmod 2.$$

In other words $\mathcal{A}$ accepts $\alpha$ if the least rank occurring infinitely often in the run $\tau$ on $\alpha$ is even. The set of all words accepted by an automaton is denoted by $\mathrm{L}(\mathcal{A})$. The set $\mathrm{L}(\mathcal{A})$ is often called the *language recognised by* $\mathcal{A}$.

### 2.1. Infinite state automata

From the point of view of computer-science, the key property of automata is that the set $Q$ and the alphabet $A$ are finite, so an automaton can be explicitly represented in the memory of a computer. Nevertheless, we can forget about this restriction and consider automata where $Q$ and $A$ are countable. In that situation, if we add the constraint that for each $\alpha \in A^\omega$ there exists $\liminf_{n\to\infty} \Omega(\tau_n) < \infty$, the resulting object is exactly a coloring — a generic map from finite words into $\mathbb{N}$ with the above additional property.

### 2.2. Borel automata

Another idea of infinite automata is presented in the book [7]. The approach used by the authors differs from the one used in this work. Perrin and Pin consider the so-called *Borel automata*.

**Definition 1.** A *Borel automaton* is a non-deterministic automaton $\mathcal{A} = \langle A, Q, \delta, R\rangle$, where $A$, $Q$ are countable sets, $\delta$ is a transition relation, and $R$ is an arbitrary Borel acceptance condition $R \subseteq Q^\omega$.

The key difference to colorings is that if we consider a coloring as an automaton it has one fixed acceptance condition — the parity of limes inferior of ranks must be even. In the case of Borel automata we can adjust the set $R$. For instance, given a Borel language $L \subseteq A^\omega$ we can consider the automaton that just mimics the input word, put $R := L$, and obtain a trivial Borel automaton recognising exactly the given language $L$.

### 2.3. Automata with advice

The paper [2] analyses the topological complexity of sets of branches of an infinite binary tree that can be defined by an MSO formula. The authors allow a tree to have some extra monadic predicates on nodes. The main result states that the class of languages of branches definable this way is exactly $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$. The fact that every set in $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ is definable as a set of branches is expressed in Theorem 2.1 of the cited paper. The proof of this theorem is inspired by the idea of finite colorings, compare to Proposition 3 here.

## 3. Colorings

In this section we provide definition of a *coloring*. Later on we show closure properties of families of sets defined by colorings. Finally, we prove the relationship between the Borel levels and the colorings of particular kinds.

Let $A$ denote any finite or countable set called an *alphabet*. We assume that $A$ contains at least two letters. We will consider $A^\omega$ — the space of all infinite sequences of letters from $A$ with its natural product topology. If $A$ is finite, $A^\omega$ is homeomorphic to the Cantor set, otherwise $A^\omega$ is homeomorphic to the Baire space.

Let $A^*$ denote the set of all finite words over $A$ ordered by the prefix relation $\preccurlyeq$. The set $A^*$ is downward closed so (as an order) it forms a tree. Let $\epsilon$ denote the empty word — the root of $A^*$. The concatenation of words $u, v$ is denoted by $u \cdot v$ or just by $uv$. For a non-empty word $s \in A^* \setminus \{\epsilon\}$ let $\mathrm{parent}(s) = s\restriction_{|s|-1}$, that is the parent of $s$ in the tree $A^*$. Observe that $A^\omega$ is the set of infinite branches of $A^*$.

**Definition 2.** A *coloring* is a function $K : A^* \to \mathbb{N}$ satisfying the following property

$$\forall_{\alpha\in A^\omega} \liminf_{n\to\infty} K(\alpha\restriction_n) < \infty.$$

A coloring $K$ is *finite* if the set $K(A^*) \subseteq \mathbb{N}$ is finite. The greatest value obtained by a finite coloring is called its *oscillation*.
A coloring $K$ is *monotone* if $K$ is a monotone function — for all words $s \preccurlyeq r$ we have $K(s) \leqslant K(r)$.

Observe that this definition gives us four *kinds* of colorings:

- general,
- finite,
- monotone,
- finite monotone.

**Definition 3.** For a coloring $K$ define the set $[K] \subseteq A^\omega$ as the set of all infinite branches $\alpha$ such that limes inferior of $K$ on $\alpha$ is even:

$$[K] = \Big\{\alpha \in A^\omega\colon \liminf_{n\to\infty} K(\alpha\restriction_n) \equiv 0 \bmod 2\Big\}. \tag{1}$$

Therefore, every coloring induces a subset of $A^\omega$. Of course two colorings may induce the same subset.

**Definition 4.** (See [8] or [5, Section II 21.E].) Let $f : A^\omega \to A^\omega$ be a continuous function and $X, Y \subseteq A^\omega$ be two sets. Assume that $f$ satisfies $f^{-1}(Y) = X$. In that case we say that *$f$ is a continuous reduction of $X$ to $Y$*. The fact that there exists such a reduction is denoted by $X \leqslant_W Y$.

Colorings behave well when considering continuous reductions. It is summarised by the following theorem.

**Theorem 1.** *Let $f : A^\omega \to A^\omega$ be a continuous reduction of a set $X \subseteq A^\omega$ to $[K]$ for some coloring $K$. Then there exists a coloring $K'$ of the same kind as $K$ with the property $[K'] = X$.*

To show this, we use the following lemma.

**Lemma 1.** *(See [5, Proposition 2.6].) Let $f : A^\omega \to A^\omega$ be a continuous function. Then there exists a function $\overline{f} : A^* \to A^*$ such that for every word $\alpha \in A^\omega$:*

(i) $\lim_{n\to\infty} |\overline{f}(\alpha\restriction_n)| = \infty$,
(ii) $\forall_{n\leqslant m}\, \overline{f}(\alpha\restriction_n) \preccurlyeq \overline{f}(\alpha\restriction_m) \prec f(\alpha)$.

**Proof of Theorem 1.** Take $\overline{f} : A^* \to A^*$ as in Lemma 1.

Let $K'(\epsilon) = 0$. Take any word $s \in A^* \setminus \{\epsilon\}$. Let $u = \overline{f}(\mathrm{parent}(s))$ and $v = \overline{f}(s)$. We know that $u \preccurlyeq v$.

Consider the sequence of words $u = w_0 \prec w_1 \prec w_2 \prec \cdots \prec w_i = v$, satisfying $\mathrm{parent}(w_{j+1}) = w_j$. This sequence contains the vertices of $A^*$ on the path from $u$ to $v$. Define

$$K'(s) := \min\bigl\{K(w_0), K(w_1), \ldots, K(w_i)\bigr\}. \tag{2}$$

Now, it is enough to show that for every $\alpha \in A^\omega$ there holds

$$\liminf_{n\to\infty} K'(\alpha\restriction_n) = \liminf_{n\to\infty} K\bigl(f(\alpha)\restriction_n\bigr).$$

This property entails that $K'$ is a coloring of the same kind as $K$ and that $[K'] = f^{-1}([K]) = X$, what was to be shown.

Take any $\alpha \in A^\omega$. Denote by $m = \liminf_{n\to\infty} K(f(\alpha)\restriction_n)$. There exists a number $N \in \mathbb{N}$ such that for any $n > N$ there holds $K(f(\alpha)\restriction_n) \geqslant m$. By assumptions about $\overline{f}$, there exists a number $M \in \mathbb{N}$ such that for all words $s \prec \alpha$ and $|s| > M$ we have $|\overline{f}(s)| > N$.

Therefore, for words $s \prec \alpha$ long enough, the word $u$ in the definition of $K'(s)$ is longer than $N$. For such a word $s$, all values $K(w_0), K(w_1), \ldots, K(w_i)$ considered in Eq. (2) are not less than $m$. Therefore $K'(s) \geqslant m$.

Conversely, for infinitely many numbers $n \in \mathbb{N}$ there holds $K(f(\alpha)\restriction_n) = m$. So, for infinitely many words $s \prec \alpha$ among values $\{K(w_j)\colon 0 \leqslant j \leqslant i\}$ considered in Eq. (2) there appears $m$. Therefore, for infinitely many numbers $n \in \mathbb{N}$ there holds $K'(\alpha\restriction_n) \leqslant m$.

So, by the two previous paragraphs $\liminf_{n\to\infty} K'(\alpha\restriction_n) = m$. □

**Remark 1.** The families of sets defined by colorings of each kind are closed downwards with respect to the order $\leqslant_W$, that is: if $X \leqslant_W Y$ and $Y$ is defined by a coloring of a given kind then $X$ is also defined by a coloring of this kind.

### *3.1. Closure properties*

For each kind of colorings the family of sets defined by them is closed under Boolean combinations. This section provides a proof of this fact for finite colorings. For general colorings the construction is more complicated but the fact follows from other results of this paper, see Remark 3.

**Fact 1.** *For every coloring $K$ there exists a coloring $K'$ of the same kind as $K$ such that*

$$\bigl[K'\bigr] = A^\omega \setminus [K].$$

**Proof.** For a given coloring $K$ consider $K'$ defined by the formula

$$K'(s) = K(s) + 1.$$

Then $K'$ is a coloring of the same kind as $K$ and $[K'] = A^\omega \setminus [K]$. □

Using this fact it is enough to show the closure under intersection. To do that we need one additional technical definition.

**Definition 5.** For a number $n \in \mathbb{N}$ let $\mathrm{parity}(n) = (n \bmod 2) \in \{0, 1\}$.

- Let $S_\wedge : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as follows

$$S_\wedge(n, m) = n + m - \mathrm{parity}(n) \cdot \mathrm{parity}(m).$$

- Let $S_\vee : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as follows

$$S_\vee(n, m) = n - \text{parity}(n) + m - \text{parity}(m) + \text{parity}(n) \cdot \text{parity}(m).$$

It is easy to check that functions $S_\wedge, S_\vee$ are monotone with respect to both coordinates and the following properties hold

$$\begin{aligned}
&m + n - 1 \leqslant S_\wedge(n, m) \leqslant n + m,\\
&m + n - 1 \leqslant S_\vee(n, m) \leqslant n + m,\\
&S_\wedge(n, m) \text{ is even} \quad \Leftrightarrow \quad n \text{ is even and } m \text{ is even},\\
&S_\vee(n, m) \text{ is even} \quad \Leftrightarrow \quad n \text{ is even or } m \text{ is even}.
\end{aligned}$$

**Lemma 2.** *For all finite colorings $K_1$, $K_2$ there exists a finite coloring $K''$ such that*

$$[K''] = [K_1] \cap [K_2].$$

*Moreover, if $K_1$, $K_2$ are monotone, so is $K''$.*

**Proof.** The following construction is very similar to the case of finite parity automata.

Assume that given colorings $K_1$, $K_2$ have oscillations $n_1$, $n_2$ respectively. It is easy to see that any definition in which (for a given word $s \in A^*$) the value $K''(s)$ depends only on values $K_1(s)$, $K_2(s)$ is wrong.

By induction on the length of a word $s \in A^*$ we define a function

$$M_s : \{0, 1, \dots, n_1\} \to \{0, 1, \dots, n_2\}$$

and values $K''(s) \leqslant S_\wedge(n_1, n_2)$. $M_s(n)$ is the least value of $K_2$ since the last occurrence of $n$ as a value of $K_1$, or 0 if $n$ did not occur as a value of $K_1$, see (4) and (5) below.

Let $M_\epsilon \equiv 0$ and $K''(\epsilon) = 0$.

Take a word $s \in A^* \setminus \{\epsilon\}$. Let $r = \text{parent}(s)$. Assume that the function $M_r$ and the value $K''(r)$ are defined. Take:

$$K''(s) := S_\wedge\big(K_1(s), M_r\big(K_1(s)\big)\big), \tag{3}$$

$$M_s\big(K_1(s)\big) := K_2(s), \tag{4}$$

$$M_s(n) := \min\big(M_r(n), K_2(s)\big) \quad \text{for } n \in \{0, 1, \dots, n_1\} \setminus \big\{K_1(s)\big\}. \tag{5}$$

**Example 1.** The following table presents example values of $K_1(s), K_2(s)$ and the resulting values of $M_s(n)$ and $K''(s)$ for $s \leqslant a^4$.

| $s$ | $= \epsilon$ | $a^1$ | $a^2$ | $a^3$ | $a^4$ |
|---|---|---|---|---|---|
| $K_1(s)$ | $= 0$ | 1 | 0 | 0 | 1 |
| $K_2(s)$ | $= 1$ | 4 | 5 | 2 | 6 |
| $M_s(0)$ | $= 0$ | 0 | 5 | 2 | 2 |
| $M_s(1)$ | $= 0$ | 4 | 4 | 2 | 6 |
| $K''(s)$ | $= 0$ | $S_\wedge(1,0)$ | $S_\wedge(0,0)$ | $S_\wedge(0,5)$ | $S_\wedge(1,2)$ |

It is enough to check that $K''$ is a coloring and $[K''] = [K_1] \cap [K_2]$. Take any branch $\alpha \in A^\omega$. Assume that lim inf of values of $K_1$, $K_2$ on $\alpha$ are respectively $m_1$, $m_2$.

**Lemma 3.** *Using the above definitions the following equality holds*

$$\liminf_{n\to\infty} K''(\alpha{\restriction_n}) = S_\wedge(m_1, m_2).$$

**Proof.** Let $H_1 = \{n \in \mathbb{N}\colon\ K_1(\alpha{\restriction_n}) = m_1\}$ and $H_2 = \{n \in \mathbb{N}\colon\ K_2(\alpha{\restriction_n}) = m_2\}$. By the definition of $m_1, m_2$ we know that sets $H_1, H_2$ are infinite. Let $D = \{n \in \mathbb{N}\colon\ M_{\alpha\restriction_n}(m_1) \leqslant m_2\}$. We show that $H_2 \subseteq D$, so in particular $D$ is infinite. Take any $h \in H_2$ and consider the value $M_{\alpha\restriction_h}(m_1)$. No matter whether it is defined by Eq. (4) or (5), there holds $M_{\alpha\restriction_h}(m_1) \leqslant K_2(\alpha{\restriction_h}) = m_2$. Therefore $h \in D$.

Now we show that for infinitely many $n \in \mathbb{N}$ there holds $K''(\alpha\restriction_n) \leqslant S_\wedge(m_1, m_2)$. Take any $d \in D$. Let $h \in H_1$ be the least number in $H_1$ satisfying $h > d$. By the definition of $h$, for all $j \in \{d+1, d+2, \ldots, h-1\}$ $K_1(\alpha\restriction_j) \neq m_1$, therefore by (5)

$$M_{\alpha\restriction_j}(m_1) = \min\bigl(M_{\alpha\restriction_{j-1}}(m_1), K_2(\alpha\restriction_j)\bigr) \leqslant M_{\alpha\restriction_{j-1}}(m_1).$$

The above equation allows us to show (by induction on $j = d, d+1, \ldots, h-1$) that $M_{\alpha\restriction_j}(m_1) \leqslant M_{\alpha\restriction_d}(m_1)$. For $j = h-1 \geqslant d$, using (3), it means that

$$K''(\alpha\restriction_h) = S_\wedge\bigl(m_1, M_{\alpha\restriction_{h-1}}(m_1)\bigr) \leqslant S_\wedge\bigl(m_1, M_{\alpha\restriction_d}(m_1)\bigr) \leqslant S_\wedge(m_1, m_2).$$

What remains is to show that for $s \prec \alpha$ long enough there holds $K''(s) \geqslant S_\wedge(m_1, m_2)$. We know that for some $M \in \mathbb{N}$ and all $m \geqslant M$ there hold $K_1(\alpha\restriction_m) \geqslant m_1$ and $K_2(\alpha\restriction_m) \geqslant m_2$. Let $q = \alpha\restriction_M$. There are at most $n_1 + 1$ numbers $n$ for which $M_q(n) < m_2$, because the domain of $M_q$ has cardinality $n_1 + 1$. Moreover, values assigned to $M_r$ by Eq. (4) are, for $s$ satisfying $q \prec s \prec \alpha$, at least $m_2$.

Define the set of words

$$B = \bigl\{s \prec \alpha\colon\ s \succ q \wedge K''(s) < S_\wedge(m_1, m_2)\bigr\}.$$

We show that the set $B$ is finite. Observe that, by (3) and monotonicity of $S_\wedge$, for every $s \in B$ there holds $M_{\mathrm{parent}(s)}(K_1(s)) < m_2$. Therefore, by Eqs. (4) and (5) we obtain that $M_s(K_1(s)) = K_2(s) \geqslant m_2$ and for $r \succcurlyeq s$ and $r \prec \alpha$ there holds $M_r(K_1(s)) \geqslant m_2$. So, for $s, s' \in B$ and $s \neq s'$, there holds $K_1(s) \neq K_1(s')$. Therefore, the cardinality of the set $B$ is at most $n_1 + 1$. It means that for $s \prec \alpha$ long enough $K''(s) \geqslant S_\wedge(m_1, m_2)$ holds. □

Since $\liminf_{n\to\infty} K''(\alpha\restriction_n) = S_\wedge(m_1, m_2)$, this value is even if and only if $m_1, m_2$ are even. So $\alpha \in [K'']$ if and only if $\alpha \in [K_1] \cap [K_2]$.

By the construction, if $K_1$, $K_2$ are monotone, so is $K''$. □

**Remark 2.** The same construction as above works for the sum of sets e.g. $[K''] = [K_1] \cup [K_2]$ if we use $S_\vee$ instead of $S_\wedge$.

It is worth noticing that if we consider finite colorings $K_1$, $K_2$ then the coloring $K''$ constructed above has oscillation bounded by the sum of oscillations of $K_1$, $K_2$. The same fact holds for finite monotone colorings and in the case of sum instead of intersection.

### *3.2. Upper bounds*

In this section we show upper bounds on the Borel complexity of sets defined by colorings of various kinds.

**Proposition 1.** *For a given finite coloring $K$ there holds $[K] \in \mathcal{BC}(\boldsymbol{\Sigma}^0_2)$. Moreover, if $K$ is monotone, $[K] \in \mathcal{BC}(\boldsymbol{\Sigma}^0_1)$.*

**Proof.** Take a coloring $K$ with values bounded by $N \in \mathbb{N}$. Consider a sequence of sets $U_n \subseteq A^\omega$ defined by the formula

$$U_n = \bigl\{\alpha \in A^\omega\colon\ \exists_{I\in\mathbb{N}}\, \forall_{i>I}\ K(\alpha\restriction_i) \geqslant n\bigr\}.$$

The sequence $U_n$ is decreasing, $U_0 = A^\omega$, $U_{N+1} = \emptyset$, and all sets $U_n$ are $\boldsymbol{\Sigma}^0_2$.

Observe that if $K$ is monotone then $U_n = \{\alpha \in A^\omega\colon\ \exists_{i\in\mathbb{N}}\ K(\alpha\restriction_i) \geqslant n\}$. Therefore, in that case $U_n \in \boldsymbol{\Sigma}^0_1$.

By the definition of $[K]$ we obtain that

$$[K] = \bigcup_{0\leqslant i\leqslant N/2} U_{2i} \setminus U_{2i+1}. \tag{6}$$

This concludes the proof. □

**Proposition 2.** *For a given general coloring $K$ there holds $[K] \in \boldsymbol{\Delta}^0_3$. Moreover, if $K$ is monotone then $[K] \in \boldsymbol{\Delta}^0_2$.*

**Proof.** Since sets defined by (monotone) colorings are closed under complementation, it is enough to show that $[K] \in \boldsymbol{\Sigma}^0_3$ (resp. $\boldsymbol{\Sigma}^0_2$).

The property that $\alpha \in [K]$ can be expressed in the following $\boldsymbol{\Sigma}^0_3$ way

$$\exists_{l,m\in\mathbb{N}\wedge\mathrm{parity}(l)=0}\ \forall_{M>m}\ \bigl(K(\alpha\restriction_M) \geqslant l \wedge \exists_{M'>M} K(\alpha\restriction_{M'}) = l\bigr).$$

The case of monotone colorings is analogous. □

### 3.3. Lower bounds

It turns out that the upper bounds presented in the previous section are tight — every set from the appropriate family is defined by a coloring of an adequate kind.

**Proposition 3.** *For every set $X \in \mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ there exists a finite coloring $K$ such that $[K] = X$. If $X \in \mathcal{BC}(\boldsymbol{\Sigma}^0_1)$ then there exists a finite monotone $K$ with this property.*

**Proof.** Since sets defined by colorings are closed under continuous reductions and Boolean combinations, it is enough to show that some $\boldsymbol{\Sigma}^0_2$-complete set is defined by a finite coloring (resp. a $\boldsymbol{\Sigma}^0_1$-complete set and a finite monotone coloring).

Consider the $\boldsymbol{\Pi}^0_2$-complete set of sequences that contain infinitely many letters $b$

$$F = \left\{\alpha \in A^\omega \colon \forall_n \, \exists_{k>n} \, \alpha(k) = b\right\}.$$

Define the coloring $K(\epsilon) = 0$, $K(s \cdot a) = 1$, and $K(s \cdot b) = 0$. By the definition $\alpha \in [K] \Leftrightarrow \alpha \in F$.

Additionally observe that the oscillation of $K$ is 1.

For the case of finite monotone colorings we consider a $\boldsymbol{\Pi}^0_1$-complete set of sequences not containing $b$ and the coloring $K$ defined by property $K(s) = 1$ if $s$ contains $b$ and $K(s) = 0$ otherwise. □

The above proof can be also expressed in a slightly different way, using the following lemma. The key property is that with every $\boldsymbol{\Pi}^0_2$ set we can bind a set of finite prefixes that are *good* for that set. This fact is presented in [7] and [9].

**Lemma 4.** *(See Theorem III.3.11 from [7].) A language $L \subseteq A^\omega$ is $\boldsymbol{\Pi}^0_2$ if and only if there exists a language of finite words $J \subseteq A^*$ such that $L$ is exactly the set of such words that contain infinitely many elements of $J$ as prefixes.*

**Another proof of Proposition 3.** It is enough to show that every $\boldsymbol{\Pi}^0_2$ language is defined by a coloring. Take such a language $L \subseteq A^\omega$ and consider $J \subseteq A^*$ as in Lemma 4. Consider the coloring $K$ such that $K(s) = 0$ if $s \in J$ and $K(s) = 1$ if $s \notin J$. In that case

$$\alpha \in [K] \iff \text{infinitely many } s \prec \alpha \text{ satisfy } s \in J \iff \alpha \in L. \qquad \square$$

**Corollary 1.** *Finite colorings with oscillation bounded by 1 define exactly all $\boldsymbol{\Pi}^0_2$ sets. The same holds for finite monotone colorings and the family $\boldsymbol{\Pi}^0_1$. Dually, (monotone) colorings of values $\{1, 2\}$ define exactly all $\boldsymbol{\Sigma}^0_2$ (resp. $\boldsymbol{\Sigma}^0_1$) sets.*

**Proof.** By the above construction and the fact that continuous reductions preserve bounds on oscillation. □

**Proposition 4.** *For every set $X \in \boldsymbol{\Delta}^0_3$ there exists a coloring $K$ such that $[K] = X$. If $X \in \boldsymbol{\Delta}^0_2$ then there exists some monotone $K$ with this property.*

**Proof.** Since $X \in \boldsymbol{\Delta}^0_3$ (resp. $\boldsymbol{\Delta}^0_2$) so both $X$ and $A^\omega \setminus X$ are countable unions of $\boldsymbol{\Pi}^0_2$ (resp. $\boldsymbol{\Pi}^0_1$) sets. Let $H_i$ be a sequence of $\boldsymbol{\Pi}^0_2$ (resp. $\boldsymbol{\Pi}^0_1$) sets such that $\bigcup_i H_{2i+1} = X$ and $\bigcup_i H_{2i} = A^\omega \setminus X$. By Corollary 1 there exist colorings $K_i$ with values $\{0, 1\}$ such that $[K_i] = H_i$. Additionally, if $X \in \boldsymbol{\Delta}^0_2$, we can take $K_i$ to be monotone. Let $K(\epsilon) = 0$ and $K(s)$ be the least $i$ such that $K_i(s) = 0$, if it exists. If not, let $K(s)$ be equal to $K(\mathrm{parent}(s))$. It is easy to check that $[K] = X$ and if $X \in \boldsymbol{\Delta}^0_2$ then the coloring $K$ defined this way is monotone. □

**Remark 3.** The family of sets defined by general (monotone) colorings is closed under Boolean combinations.

**Proof.** Using Propositions 2 and 4 the family of sets defined by general colorings equals $\boldsymbol{\Delta}^0_3$. This family is closed under Boolean combinations because it is self-dual and families $\boldsymbol{\Sigma}^0_3$, $\boldsymbol{\Pi}^0_3$ are closed under positive Boolean combinations.

The same argument holds for monotone colorings. □

### 3.4. Characterisation

The facts presented in the previous two sections can be summarised by the following theorem.

**Theorem 2.** *Every cell in the following table represents one kind of colorings. It contains the family of all sets defined by colorings of this kind.*

| Coloring | Oscillation $\leqslant 1$ | Finite | General |
|---|---|---|---|
| monotone | $\boldsymbol{\Pi}^0_1$ | $\mathcal{BC}(\boldsymbol{\Sigma}^0_1)$ | $\boldsymbol{\Delta}^0_2$ |
| general | $\boldsymbol{\Pi}^0_2$ | $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ | $\boldsymbol{\Delta}^0_3$ |

Using this theorem we can see colorings as a combinatorial representation of sets from the selected families. This result is used in Section 5 to show concrete and relatively simple counterexamples — e.g. a set that is in $\mathbf{\Delta}^0_3$ but not in $\mathcal{BC}(\mathbf{\Sigma}^0_2)$.

## 4. Difference hierarchy

Finite colorings and finite levels of the difference hierarchy over $\mathbf{\Sigma}^0_2$ are equivalent. To show that, we first recall the definition of the difference hierarchy. For simplicity it is not a generic definition but the one based on $\mathbf{\Sigma}^0_2$ sets. Analogical facts hold for finite monotone colorings and the difference hierarchy over $\mathbf{\Sigma}^0_1$ with essentially the same proofs.

**Definition 6.** For a given increasing sequence of sets $(F_i)_{i<n} \subseteq \mathbf{\Sigma}^0_2$, define:

- if $\mathrm{parity}(n) = 0$ then

$$D_n\big((F_i)_{i<n}\big) := \bigcup_{i<n,\ \mathrm{parity}(i)=0} (F_{i+1} \setminus F_i).$$

- if $\mathrm{parity}(n) = 1$ then

$$D_n\big((F_i)_{i<n}\big) := F_0 \cup \bigcup_{i<n,\ \mathrm{parity}(i)=1} (F_{i+1} \setminus F_i).$$

Let $D_n$ be the family of all sets $D_n((F_i)_{i<n})$ for all possible increasing sequences of $(F_i)_{i<n} \subseteq \mathbf{\Sigma}^0_2$. The family $D_n$ is called *the nth level of the difference hierarchy*.

For a wider explanation of the difference hierarchy see [10] and Chapter 22.E in the book [5]. The following fact summarises the most important properties.

**Fact 2.** *The definition of $D_\eta$ can be extended to all ordinals $\omega \leqslant \eta < \omega_1$. The families $D_\eta$ form a strictly increasing hierarchy. Moreover, $\bigcup_{\eta<\omega_1} D_\eta = \mathbf{\Delta}^0_3$ and $\bigcup_{\eta<\omega} D_\eta = \mathcal{BC}(\mathbf{\Sigma}^0_2)$.*

The connection between colorings and the difference hierarchy is expressed by the following theorem.

**Theorem 3.** *Sets defined by finite colorings of oscillation bounded by n are exactly complements of sets belonging to the family $D_n$.*

**Proof.** Both directions of this equivalence are based on Eq. (6) in the proof of Proposition 1. In both cases we have to *revert* a given sequence of sets.

Take a coloring $K$ with an even oscillation $N$ (the case of odd $N$ is the same). Write

$$[K] = \bigcup_{0\leqslant i\leqslant N/2} U_{2i} \setminus U_{2i+1}$$

where the sets $U_i$ are defined as in the proof of Proposition 1. The sequence $(U_n)_{0\leqslant n\leqslant N+1}$ is decreasing. Define a new sequence $F_n = U_{N-n}$ for $n = 0, 1, \ldots, N-1$. The sequence $F_n$ is increasing and $D_N(F_n) = A^\omega \setminus [K]$ by the definition.

Conversely, take a set $D \in D_N$ for some $N \geqslant 0$. For simplicity, consider the case of odd $N$, the other case is similar. By the definition

$$D = F_0 \cup (F_2 \setminus F_1) \cup \cdots \cup (F_{N-1} \setminus F_{N-2}),$$

for an increasing sequence $(F_i)_{i<N}$ of $\mathbf{\Sigma}^0_2$ sets. For $n = 0, 1, \ldots, N-1$ let $K_n$ be a coloring with values $\{1, 2\}$ such that $[K_n] = F_{N-1-n}$. For a word $s \in A^*$ define $K(s)$ as the least $n$ such that $K_n(s) = 1$ or $N$ if no such $n$ exists. Take $\alpha \in A^\omega$. Assume that $\liminf_{n\to\infty} K(\alpha\restriction_n) = m$ for some $m \leqslant N$. If $m = 0$ then $\alpha \notin F_{N-1}$ so $\alpha \notin D$. If $m = N$ then $\alpha \in F_0$ so $\alpha \in D$. Otherwise $\alpha \in F_{N-m}$ and $\alpha \notin F_{N-m-1}$ so $\alpha \in D \Leftrightarrow \mathrm{parity}(m) = 1$. So, no matter what the value of $m$ is, we obtain that $\alpha \in D \Leftrightarrow \alpha \notin [K]$. Therefore $D = A^\omega \setminus [K]$. □

This equivalence does loose its strictness for transfinite levels of the difference hierarchy. Currently there is no method known to define a transfinite oscillation of a general coloring. Notice that in the above proof it is important to *revert* orientation of the sequence and then treat indices of sets as values of a coloring. This construction does not have a natural meaning in the context of a transfinite sequence of sets.

Colorings enable us to estimate the growth of oscillation when performing Boolean operations on sets. This estimation carries over to finite levels of the difference hierarchy.

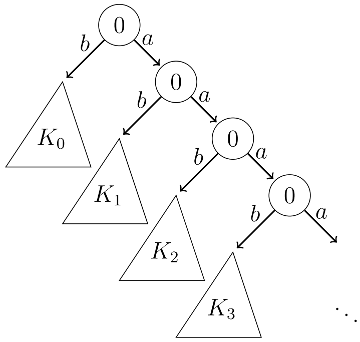


**Fig. 1.** The coloring $K$ obtained as a join of $K_n$'s. Applies to Facts 3 and 4.

**Proposition 5.** *If $X \in D_n$ and $Y \in D_m$ then $X \cap Y$, $X \cup Y \in D_{n+m}$.*

**Proof.** By the construction of Lemma 2 and Theorem 3. □

## 5. Separating examples

In this section we provide examples of sets in $\boldsymbol{\Delta}^0_3 \setminus \mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ and $\boldsymbol{\Delta}^0_2 \setminus \mathcal{BC}(\boldsymbol{\Sigma}^0_1)$. The existence of such sets is well known but the following constructions may be appreciated for their simplicity.

Let us fix the alphabet $A = \{a, b\}$.

**Fact 3.** *There exists a coloring $K$ such that there is no finite coloring $K'$ with the property $[K] = [K']$. (See Fig. 1.)*

**Proof.** The first step of the proof is to construct a sequence of colorings demanding higher and higher oscillations.

For a word $s \in A^*$ of the form $wba^ib$ for $0 \leqslant i \leqslant n$ let $K_n(s) = i$. For all other words $s \in A^*$ let $K_n(s) = n$. By the definition, $K_n$ is a coloring with oscillation $n$.

Assume for contradiction that for some $n$ there exists a coloring $K'$ with oscillation less than $n$ such that $[K'] = [K_n]$. We inductively construct an infinite word $\alpha \in A^\omega$ based on the values of $K'$ on currently constructed prefix of $\alpha$. Let $s_0 = \epsilon$ and $s_1 = b$. Take any $i > 1$ and assume that $s_0 \prec \cdots \prec s_{i-2} \prec s_{i-1}$ are defined. Let $l_1 = |s_{i-2}|$ and $l_2 = |s_{i-1}|$. Let $M$ be the least value in the set

$$\left\{K'(s_{i-1}\restriction_{l_1+1}), K'(s_{i-1}\restriction_{l_1+2}), \ldots, K'(s_{i-1}\restriction_{l_2})\right\}.$$

Take $s_i = s_{i-1}a^{M+1}b$. Let $\alpha \in A^\omega$ be the limit of $s_i$ when $i \to \infty$. It is easy to check that

$$\liminf_{n\to\infty} K_n(\alpha\restriction_n) = \liminf_{n\to\infty} K'(\alpha\restriction_n) + 1.$$

Therefore $\alpha \in [K'] \Leftrightarrow \alpha \notin [K_n]$.

Now we provide the definition of $K$ as a coloring that joins all $K_n$. For a word $s$ of the form $s = a^nbw$ let $K(s) = K_n(w)$ and for other words $s \in A^*$ let $K(s) = 0$. It is easy to see that $K$ is the desired coloring. □

**Fact 4.** *There exists a monotone coloring $K$ such that there is no finite monotone coloring $K'$ with the property $[K] = [K']$.*

**Proof.** The proof is similar but simpler than the above one. Colorings $K_n$ are defined as follows: $K_n(s) = \min(n, |s|_b)$, where $|s|_b$ is the number of occurrences of letter $b$ in a word $s$. Of course $K_n$ is a monotone coloring. Again, $K_n$ has oscillation $n$ and there is no equivalent monotone coloring with smaller oscillation. Therefore, $K$ defined as before, as a *join* of all $K_n$, is monotone and is not equivalent to any finite monotone coloring $K'$. □

## 6. lim inf vs. lim sup

One may ask why in the definition of coloring lim inf is used instead of lim sup. The following section discusses this problem.

**Definition 7.** A function $M : A^* \to \mathbb{N}$, with the property that for each infinite word $\alpha \in A^\omega$ the value $\limsup_{n\to\infty} M(\alpha\restriction_n)$ is finite, is called a max *coloring*.

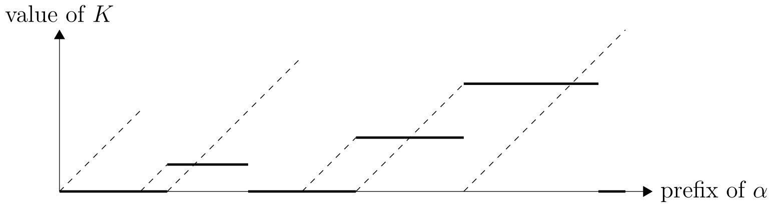


**Fig. 2.** Values of $K$ on prefixes of some $\alpha \in A^\omega$. Dashed line presents the number of trailing $a$'s in a given prefix of $\alpha$.

For a given max coloring $M$ let

$$[M]_{\max} = \left\{\alpha \in A^\omega \colon \limsup_{n\to\infty} M(\alpha\restriction_n) \equiv 0 \bmod 2\right\}.$$

We define finite and monotone max colorings in the same way as in the case of standard colorings.

Similarly to standard colorings, max colorings fall into four kinds: general, finite, monotone, finite monotone. All these kinds except general are equivalent to normal colorings — for a monotone function $\liminf = \limsup$ and if a coloring is finite we can revert the order of its values. So the whole theory for this three kinds of colorings holds for max colorings. Moreover, general max colorings always define $\mathbf{\Delta}^0_3$ sets — the argument is the same as for normal colorings. The only question that remains is whether every $\mathbf{\Delta}^0_3$ set is defined by some max coloring. The answer is negative, the rest of this section is devoted to showing this.

Let us introduce the following notation ($s \in A^*$ is a finite word)

$$[s] = \left\{\alpha \in A^\omega \colon s \prec \alpha\right\}.$$

**Definition 8.** We say that a set $X \subseteq A^\omega$ has the *simplification property* if there exists a word $s \in A^*$ such that $X \cap [s]$ is a $\mathcal{BC}(\mathbf{\Sigma}^0_2)$ subset of $[s]$.

It turns out that the following fact holds.

**Fact 5.** *If $M$ is a* max *coloring then the set $[M]_{\max} \subseteq A^\omega$ has the simplification property.*

**Proof.** We construct an infinite word $\alpha \in A^\omega$ inductively. If this procedure stops, we have found $s \in A^*$. If not, the constructed $\alpha$ is a witness that $M$ is not a max coloring.

Let $s_0 = \epsilon$. In step $i > 0$ take $s_0 \prec s_1 \prec \cdots \prec s_i$ with the inductive property $M(s_0) < M(s_1) < \cdots < M(s_i)$. If there is no value greater than $M(s_i)$ in the subtree with $s_i$ as root, we finish. If there is one, denote it as $s_{i+1}$ and continue.

If the procedure does not stop, the resulting sequence $s_0 \prec s_1 \prec \cdots \prec \alpha \in A^\omega$ has property that $\limsup_{n\to\infty} M(\alpha\restriction_n) = \infty$, a contradiction. So, since $M$ is a max coloring, at some step $i$ the procedure has stopped. Therefore, $M$ is a finite coloring when restricted to the subtree with root $s_i$. So $[M]_{\max} \cap [s_i] \in \mathcal{BC}(\mathbf{\Sigma}^0_2)$. $\square$

Now it is enough to construct a coloring $K$ with the following properties:

- For every finite word $w \in A^*$ and infinite $\alpha \in A^\omega$ there holds $\alpha \in [K] \Leftrightarrow w \cdot \alpha \in [K]$. In other words $[K]$ is *prefix independent*.
- There is no finite coloring $K'$ such that $[K] = [K']$.

If such a coloring $K$ exists then $[K]$ is a $\mathbf{\Delta}^0_3 \setminus \mathcal{BC}(\mathbf{\Sigma}^0_2)$ set which is prefix-independent. So $[K]$ does not have the simplification property and it cannot be defined by any max coloring.

Let $K$ has the following properties (see Fig. 2):

- Let the alphabet $A = \{a, b\}$,
- if a word $s \in A^*$ is of the form $wa^i$ or $wa^ib$ and there is no subword $a^i$ in $w$, then $K(s) = 0$,
- else if $s$ is of the form $wba^ib$ then $K(s) = i$,
- else if $s$ is of the form $wa$ then $K(s) = K(w)$,
- else $s = \epsilon$ and $K(s) = 0$.

It is easy to see that these properties define a function that is a coloring. Moreover, the following easy fact holds.

**Lemma 5.** *For every infinite word $\alpha \in \{a, b\}^\omega$ there are two possibilities*:

1. *$\alpha$ contains $a^i$ for arbitrarily large $i$. Then $K(\alpha\restriction_n)$ equals 0 for infinitely many $n$, so*

$$\liminf_{n\to\infty} K(\alpha\restriction_n) = 0.$$

2. *$\alpha$ does not contain $a^i$, for some $i$. Let $k_n$ denote the number of $a$'s between the $n$th and $(n+1)$th $b$ in $\alpha$. Then*

$$\liminf_{n\to\infty} K(\alpha\restriction_n) = \liminf_{n\to\infty} k_n.$$

**Proposition 6.** *There exists no* max *coloring $M$ such that $[K] = [M]_{\max}$.*

**Proof.** Using Lemma 5 it is easy to see that $[K]$ is prefix-independent. Moreover, using an argument similar to the one used in Fact 3 we can show that there is no finite $K'$ equivalent to $K$. Therefore, the set $[K]$ does not have the simplification property so it is not defined by any max coloring. □

## 7. $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ determinacy

In this section we provide a straightforward proof of $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ determinacy using colorings. The idea is to find a coloring equivalent to a given winning set and treat this coloring as a parity game. A combinatorial result (see [11,12]) shows that such games are determined.

Determinacy of $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ games is a particular case of Borel determinacy proved by Martin ([13], see [14] and [15] for special cases of $\boldsymbol{\Sigma}^0_2$ and $\boldsymbol{\Sigma}^0_3$ determinacy).

Büchi in [16] used an idea similar to finite colorings (called by the author as *marked trees*). A marked tree may be seen as a finite coloring with Muller's acceptance condition, instead of the parity condition proposed here. One of the presented results says that every $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ set is defined by a marked tree. Additionally, Büchi provides a proof of a special form of $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ determinacy based on Davis' $\boldsymbol{\Sigma}^0_3$ determinacy [15].

Another paper of Büchi [1] provides a combinatorial characterisation of $\boldsymbol{\Delta}^0_3$ sets. The representation used there is called a *state recursion* — a way to map a given infinite word $\alpha \in J^\omega$ into an infinite sequence of *states* of the machine. The difference to general colorings is the acceptance condition. In the case of colorings it is the parity lim inf condition of ranks, Büchi uses various more complicated conditions, see e.g. Lemma §5.1. The main goal of [1] is a combinatorial proof of $\boldsymbol{\Delta}^0_3$ determinacy.

Our result about determinacy of $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ sets is less general but also much less technical.

**Definition 9.** For a given set $D \subseteq A^\omega$ let $\Gamma(D)$ denote the following two-player perfect information game (called a *Gale–Stewart game*, see [17]): in $i$th step for $i = 0, 1, \ldots$ player ($i$ mod 2) chooses a letter $a_i \in A$. After infinitely many steps they define an infinite word $\alpha = (a_i)_{i\in\mathbb{N}} \in A^\omega$. Player 0 wins if $\alpha \in D$.

**Definition 10.** We say that a game $\Gamma(D)$ is *determined* if one of the two players has a winning strategy.

It is easy to show that for some $D \subseteq A^\omega$ the game $\Gamma(D)$ is not determined [17].

**Definition 11.** A *parity game* is a two-player perfect information game. The arena is any countable set $V$ of positions partitioned into two subsets $V_0$, $V_1$ belonging to players 0 and 1 respectively. Additionally, there is a transition relation $E \subseteq V^2$ and a function $\Omega : V \to \{0, 1, \ldots, n\}$ for some $n \in \mathbb{N}$. The game starts in a selected initial position $v_0 \in V$. If in step $i$ the game has reached a position $v_{i-1} \in V_j$, it is player $j$'s turn to choose a position $v_i \in V$ such that $(v_{i-1}, v_i) \in E$.

A play $(v_i)_{i\in\mathbb{N}}$ is won by player 0 if

$$\liminf_{i\to\infty} \Omega(v_i) \equiv 0 \bmod 2.$$

The papers [11] and [12] contain direct combinatorial proofs of determinacy of all parity games. One of the important features of the proofs is that they are self contained and relatively simple. In particular, they make no use of topology.

Using determinacy of parity games we can easily show determinacy of $\mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ games. The key tool are colorings.

**Theorem 4.** *For every set $D \in \mathcal{BC}(\boldsymbol{\Sigma}^0_2)$ the game $\Gamma(D)$ is determined.*

**Proof.** Let $K$ be a finite coloring with the property $[K] = D$. Consider the parity game $\Upsilon(K)$ with the arena $V = A^*$, $V_0 = \{s \in A^*\colon\ |s| \equiv 0 \bmod 2\}$ and $V_1 = V \setminus V_0$. Let $E = \{(s,t) \in A^* \times A^*\colon\ s = \mathrm{parent}(t)\}$, $\Omega(s) = K(s)$ and $v_0 = \epsilon$. Since all parity games are determined, so is $\Upsilon(K)$. Any play in $\Upsilon(K)$ defines an infinite word $\alpha \in A^\omega$. A winning strategy in $\Upsilon(K)$ can be seen as a winning strategy in $\Gamma(D)$, the key property is that $(\alpha\restriction_i)_{i\in\mathbb{N}}$ is a winning play in $\Upsilon(K)$ if and only if $\alpha \in D$. □

## 8. Conclusions and open questions

A coloring seems to be a simple and easy-to-use combinatorial object, characterising several lower levels of the Borel hierarchy. Moreover, it can be seen as a topological extension of deterministic parity automata. It enables us to easily construct examples of sets with demanded properties. A combinatorial proof of $\mathcal{BC}(\boldsymbol{\Sigma}_2^0)$ determinacy is another example of usage of colorings. The equivalence of finite levels of the difference hierarchy and finite colorings can be used in both directions: colorings may help to better understand the structure of the difference hierarchy and simultaneously the difference hierarchy is another perspective from which colorings can be seen.

Topics for further investigation include:

- Is it possible to stratify the family of general colorings into an $\omega_1$ hierarchy? If yes, is the hierarchy equivalent to the difference hierarchy?
- How naturally does the idea of a coloring extend to capture higher levels of the Borel hierarchy? Is it possible to keep all good properties of colorings?

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## References

[1] J.R. Büchi, State-strategies for games in $F_{\sigma\delta} \cap G_{\delta\sigma}$, The Journal of Symbolic Logic 48 (4) (1983) 1171–1198.
[2] M. Bojańczyk, D. Niwiński, A. Rabinovich, A. Radziwończyk-Syta, M. Skrzypczak, On the Borel complexity of MSO definable sets of branches, Fundamenta Informaticae 98 (4) (2010) 337–349.
[3] A. Arnold, H. Michalewski, D. Niwiński, On the separation question for tree languages, in: STACS, 2012, pp. 396–407.
[4] E. Grädel, I. Walukiewicz, Postinal determinacy of games with infinitely many priorities, CoRR, arXiv:cs/0610034 [cs.LO].
[5] A. Kechris, Classical Descriptive Set Theory, Springer-Verlag, New York, 1995.
[6] J.R. Büchi, On a decision method in restricted second-order arithmetic, in: Proc. 1960 Int. Congr. for Logic, Methodology and Philosophy of Science, 1962, pp. 1–11.
[7] D. Perrin, J. Éric Pin, Infinite Words: Automata, Semigroups, Logic and Games, Elsevier, 2004.
[8] W. Wadge, Reducibility and determinateness in the Baire space, Ph.D. thesis, University of California, Berkeley, 1983.
[9] W. Thomas, H. Lescow, Logical specifications of infinite computations, in: REX School/Symposium, 1993, pp. 583–621.
[10] K. Kuratowski, Topology, vol. I, Academic Press, New York, 1966.
[11] A.W. Mostowski, Games with forbidden positions, Tech. Rep., University of Gdańsk, 1991.
[12] A. Emerson, C. Jutla, Tree automata, mu-calculus and determinacy, in: FOCS'91, 1991, pp. 368–377.
[13] D.A. Martin, Borel determinacy, Annals of Mathematics 102 (2) (1975) 363–371.
[14] P. Wolfe, The strict determinateness of certain infinite games, Pacific Journal of Mathematics 5 (1955) 841–847.
[15] M. Davis, Infinite games of perfect information, in: Advances in Game Theory, Princeton University Press, Princeton, NJ, 1964, pp. 85–101.
[16] J.R. Büchi, Using determinancy of games to eliminate quantifiers, in: Fundamentals of Computation Theory, in: Lecture Notes in Computer Science, vol. 56, Springer, Berlin/Heidelberg, 1977, pp. 367–378.
[17] D. Gale, F.M. Stewart, Infinite games with perfect information, in: Contributions to the Theory of Games, vol. 2, in: Annals of Mathematics Studies, vol. 28, Princeton University Press, 1953, pp. 245–266.