The spectrum problem and Parikh's theorem

For R. Parikh at his 70th birthday

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by

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Outline

- From Kalmár to Grzegorczyk
- The spectrum problem
- Approach I: Recursion Theory
- Interlude: **Parikh's Theorem**
- Approach II: Complexity Theory
- Approach III: Restricted vocabularies
- Approach IV: Structures of bounded width

László Kalmár 1905-1976

In this talk the **Kalmár-Csillag class**, introduced in 1943, of low-complexity recursive functions, the

Elementary functions $\mathcal{E}$

play a certain rôle.

$\mathcal{E}$ is the smallest class of functions $f : \mathbb{N} \to \mathbb{N}$, containing successor, addition, difference, and which is closed under substitution, indexed sums

$$f(\bar{y}, x) = \sum_{z}^{x} g(\bar{y}, z)$$

and products

$$h(\bar{y}, x) = \prod_{z}^{x} g(\bar{y}, z)$$

$\bar{\mathcal{E}}$ is the class of sets $X \subseteq \mathbb{N}$, with their characteristic functions in $\mathcal{E}$.

Complexity Theory 1950-65

In the early days of Computability Theory, the discipline was called RECURSION theory.

Complexity was characterized by the complexity of the

RECURSION SCHEMES.

Complexity in terms of Turing machines emerged in 1960ff as the child of Automata Theory and Recursion Theory.

The Grzegorczyk hierarchy, I

A. Grzegorczyk (1953) defined classes of sets of natural numbers of **low complexity**, the so called Grzegorczyk hierarchy $\mathcal{E}^i$.

$\mathcal{E}^i$ is the smallest class of functions containing the successor and g_n, which is closed under substitution and bounded primitive recursion.

We put

$$g_0(x,y) = y + 1, \qquad g_1(x,y) = x + y, \qquad g_2(x,y) = x \cdot y$$

and for $n \geq 2$

$$g_{n+1}(x,0) = 1, \qquad g_{n+1}(x,y+1) = g_n(x, g_{n+1}(x,y))$$

This definition is a modification due to **R.W. Ritchie** (1963).

The Grzegorczyk hierarchy, II

$\bar{\mathcal{E}}^i$ is the class of sets $X \subseteq \mathbb{N}$, with their characteristic functions in $\mathcal{E}^i$.

Theorem:(Grzegorczyk 1953)

$\mathcal{E}^i \subseteq \mathcal{E}^{i+1}$ and $\bar{\mathcal{E}}^i \subseteq \bar{\mathcal{E}}^{i+1}$.

The hierarchy $\mathcal{E}^i$ is strict.

The hierarchy $\bar{\mathcal{E}}^i$ is strict for $i \geq 2$.

$\bar{\mathcal{E}}^3 = \bar{\mathcal{E}}$

Germany 1950 - 1970

After WW II the center of German Logic was
Münster in the West and **Greifswald** in the East.

The protagonists were **G. Asser** in Greifswald, and
G. Hasenjäger, **W. Markwald** and **H. Scholz** in Münster.

B. Trakhtenbrot had just proven that the set of all First Order sentences true in all finite models is undecidable.

So what can we say about finite models?

Spectrum of a formula

Let τ be a vocabulary, i.e., set of relation and function symbols.
Let ϕ be a sentence in some logic over a vocabulary τ.
Unless otherwise stated the logic will be first order logic $FOL(\tau)$.

The *spectrum* $spec(\phi)$ of ϕ is the set of finite cardinalities (viewed as a subset of $\mathbb{N}$), in which ϕ has a model.

Examples:

$\{p^q : p$ is a prime $, q \in \mathbb{N}\}$ is a spectrum of the sentence describing a field.

Are the Fibonacci numbers a spectrum ?

The Spectrum problem

In 1952 **Scholz** asked: (Characterization)
What are the spectra of sentences of first order logic FOL?

What kind of an answer do we expect?

Spectra are computable, can we say more?

In 1955 **Asser** asked: (Complement)
Is the complement $\mathbb{N} - spec(\phi)$ also a spectrum of some FOL-sentence?

Here the answer is yes or no.

The first has various solutions for FOL and SOL.
For FOL and $MSOL$, the complemetation problem is still open.

Ramsey's Theorem

Theorem: (Ramsey 1931)
The spectrum of an $\forall\exists$-sentence of FOL is finite or cofinite.

Actually, Ramsey proved his famous combinatorial theorem as a lemma to obtain this result.

For the finite model property of $\forall\exists$-sentences of FOL, and the decidability of their satisfiability, the combinatorial theorem is not needed.

Spectra of formal languages

Let L be a language (a set of words) over a finite alphabet Σ.

$Spec(L)$ is the set $\{\ell(w) \in \mathbb{N} : w \in L\}$.

For $a \in \Sigma$, $\ell_a(w)$ is the number of occurences of a in w.
$Spec_a(L)$ is the set $\{\ell_a(w) \in \mathbb{N} : w \in L\}$.

For $\Sigma_0 \subseteq \Sigma$ with $\Sigma_0 = \{a_1, \ldots, a_k\}$ we define $Spec_{\Sigma_0}(L)$ to be the set

$$\{(\ell_{a_1}(w), \ell_{a_2}(w), \ldots, \ell_{a_k}(w)) \in \mathbb{N}^k : w \in L\}.$$

We call $Spec_{\Sigma_0}(L)$ the **multi-spectrum of L with respect to Σ_0.**

Semi-linear sets

A set $A \subseteq \mathbb{N}^k$ is called **linear** if there are $a, b_1, \ldots, b_k \in N^k$ such that

$$A = \{a + n_1 b_1 + \ldots + n_k b_k \mathbb{N}^k : n_i \in \mathbb{N}, 1 \leq i \leq k\}$$

A set $A \subseteq \mathbb{N}^k$ is called **semi-linear** if it is a finite union of linear sets.

For $k = 1$ the semi-linear sets are just the ultimately periodic sets.

Parikh's Theorem

Theorem: (R. Parikh, 1961, published JACM 1966)
If L is a context-free language over an alphabet Σ, then for every $\Sigma_0 \subseteq \Sigma$ the multi-spectrum $Spec_{\Sigma_0}$ is a semi-linear set.

B. Courcelle has generalized this to context-free vertex replacement graph grammars in 1995.

Using the Büchi-Elgot-Trakhtenbrot characterization of regular languages, we get

Corollary: If L is definable in $MSOL$, then for every $\Sigma_0 \subseteq \Sigma$ the multi-spectrum $Spec_{\Sigma_0}$ is a semi-linear set.

We shall see at the end a generalization of this theorem to multi-spectra of an extension of $MSOL$ with modular counting qunatifiers.

Asser, 1955 and A. Mostowski, 1956

In 1955-56 **G. Asser** and **A. Mostowski** published the first papers on the Spectrum Problem, relating spectra to the Grzegorczyk hierarchy.

They showed that

Theorem:(Asser 1956, and Mostowski 1956)

Every $X \in \bar{\mathcal{E}}^2$ is a first order spectrum.

Every first order spectrum X is in $\bar{\mathcal{E}}^3$ (which is the Kalmár-Csillag $\mathcal{E}$).

The complement problem for SOL

We note that for SOL-sentences Asser's problem has a trivial solution:

If $\tau = \{R_1, \ldots, R_n\}$ and $\phi \in SOL(\tau)$ put $\psi = \exists R_1, \ldots, R_n \phi$. Then

$$spec(\phi) = \{n : \{0, 1, \ldots n-1\} \models \psi\}$$

hence the complement is defined by $\neg\psi$.

Closure properties, I

Let X and Y be two first order spectra.
Put

$$X + Y = \{x + z : x \in X, y \in Y\}$$

and

$$X * Y = \{x \cdot z : x \in X, y \in Y\}.$$

Observation:
The following are also first order spectra:

$$X \cap Y, X \cup Y, X + Y, X * Y$$

Closure properties, II:

Rudimentary sets

Definition:(Smullyan 1961, Bennett 1962)
A set of integers is *rudimentary* (in RUD), if it can be defined in FOL with addition, multiplication and bounded quantification.

Proposition:(Bennett 1962)
Let $X \in RUD$. Then X is a first order spectrum.

Rudimentary sets

Let $UN = \{1^n : n \in \mathbb{N}\}$ the set of unary words.
Put $\mathbf{P}_1 = \mathbf{P} \cap UN$.

Observation:
$RUD \subseteq \mathcal{E}^2$ and $RUD, \mathcal{E}^2$ and $\mathbf{P}_1$ are closed under complement.

Theorem:(R.W. Ritchie 1963)
A set $X \subseteq \mathbb{N}$ is in $\mathcal{E}^2$ iff X is recognizable by a linear bounded automaton.

Corollary: $RUD \subseteq \mathcal{E}^2 \subseteq EXPTIME$,
hence, if numbers are written in unary, $RUD \subseteq \mathcal{E}^2 \subseteq \mathbf{P}_1$.

It is open whether any of these inclusions are strict.

Bennett's doctoral thesis, 1962

Bennett's thesis is unpublished but available via library services.

It is one of the remarkable early texts anticipating later developments. in finite model theory, definability theory and complexity theory.

It contains a characterization (and various definitions) of rudimentary sets and, although lacking the terminology of time and space bounded Turing machines, many of the results concerning spectra, which were formulated and proved in more modern language after 1970.

Theorem:(Bennett 1962)
Let $X \subseteq \mathbb{N}$ with $\{1^n : n \in X\} \in \mathcal{E}^2$.
Then both X and $\mathbb{N} - X$ are first order spectra.

Corollary:

Many sets from arithmetic are spectra.

The primes, the Fibonacci numbers, the perfect numbers, etc.

The Münster School, 1972

Let $exp_0(n) = n$, and $exp_{i+1} = 2^{exp_i}$.

D. Rödding (1937-1984) and **H. Schwichtenberg** proved several results on the Spectrum Problem in 1969 and 1972.

Finally, they proved that all sets in $DSPACE(exp_n)$ are spectra of order $n+1$.

They did not consider non-deterministic complexity classes.

Jones and Selman, 1972

It is the merit of **Jones** and **Selman** to be the first to **publish** a result on first order spectra which relates spectra to

non-deterministic time bounded Turing machines.

In particular, they noted that if $\mathbf{P} = \mathbf{NP}$, the complement of a first order spectrum is also a first order spectrum. Then converse remains open.

But the relationship to non-deterministic complexity classes was independently found also by **Christen** and **Fagin**.

The theses of Fagin and Christen, 1973-1974

Ron Fagin's thesis (1973, UC Berkely, R. Vaught) is treasure of results introducing also generalized spectra which had wide impact on what is now called descriptive complexity and finite model theory.

Most of our knowledge about spectra till about 1985 and, to some extent far beyond that, is contained in the published papers emanating from Fagin's thesis.

Claude Christen's thesis (1974, ETH Zürich, E. Specker) remains unpublished, and only a small part was published in German.

Christen discovered all his results independently, and only in the late stage of his work his attention was drawn to Bennett's work and the paper of **Jones** and **Selman**. It turned out that most of his independently found results were already in print or published by Fagin after completion of Christen's thesis.

Claude Christen, born 1943, joined the faculty of CS at the University of Montreal in 1976 and died there, a full professor, prematurely, April 10, 1994.

First order spectra and $\mathbf{NP}$

Let $UN = \{1^n : n \in \mathbb{N}\}$ the set of unary words.
Put $\mathbf{NP}_1 = \mathbf{NP} \cap UN$ and $\mathbf{PH}_1 = \mathbf{PH} \cap UN$.

Theorem:
Jones and Selman 1974, Fagin 1974, Christen 1974

If $X \in \mathbf{P}_1$ the both X and $\mathbb{N} - X$ are first order spectra.
$X \subseteq \mathbb{N}$ is a first order spectrum iff $\{1^n : n \in X\} \in \mathbf{NP}_1$.
$X \subseteq \mathbb{N}$ is a second order spectrum iff $\{1^n : n \in X\} \in \mathbf{PH}_1$.

What about fragments of Second Order Logic?

What about Monadic Second Order Logic?

Generalized first order spectra and $\mathbf{NP}$

Given a first order sentence $\phi(\tau)$ and $\tau_0 \subseteq \tau$ we denote by $spec_{\tau_0}(\phi)$ the class of finite τ_0-structures which can be expanded to models of ϕ.

Classes of finite τ_0-structures of the form $spec_{\tau_0}(\phi)$ are called **generalized spectra** or **projective classes**.
If $\tau - \tau_0$ consists of unary predicates only, it is called unary.

Theorem (Fagin 1974): A class of finite τ_0-structures K is a generalized spectrum iff $K \in \mathbf{NP}$.

We note that unary generalized spectra are not closed under complement (Fagin 1975, Hajek 1975, Ajtai and Fagin 1990).

First order reductions

First order reductions (translations schemes) were introduced first in **Tarski, Mostowski, Robinson 1953**
and further developed by **Rabin 1965**.

Lovasz, Gacs 1977 used them to show that certain generalized spectra are $\mathbf{NP}$-complete via first order reductions.

Dahlhaus 1983 showed that SAT, CLIQUE and
DHC (directed hamiltonian cycle) are such generalized spectra.

Immerman 1982 and **Vardi 1982** used the same technique to characterize the generalized spectra in $\mathbf{P}$.

DESCRIPTIVE COMPLEXITY THEORY
was born out of the
SPECTRUM PROBLEM

Higher order spectra, I

Kalmar introduced in 1943 the class of *elementary functions* $\mathcal{E}$, which coincides with $\mathcal{E}^3$ of the Grzegorczyk hierarchy.

Recall: Let $exp_0(n) = n$, and $exp_{i+1} = 2^{exp_i}$.

Theorem: (Bennett 1962, Christen 1974)
The higher order spectra are exactly the sets $X \subseteq \mathbb{N}$ with characteristic function in $\mathcal{E}$.

Theorem: (Christen 1974)
$X \subseteq \mathbb{N}$ (in binary) is an $(n+1)$-order spectrum iff it is accepted in $NTIME(exp_{n+1}(cn))$ for some constant c.

Higher order spectra, II

Here we look at second order logic SOL^k where all the second order variables are of arity at most k.

Let $X \subseteq \mathbb{N}$ and put $2^{X^k} = \{n \in \mathbb{N} : 2^{n^k} \in X\}$.

Theorem (More and Olive 1997)

A set X is a spectrum of a sentence in SOL^k
iff 2^{X^k} is rudimentary.

Spectra for unary relations

Proposition (Löwenheim 1915, Fagin 1975):

Let $\tau = \{U_1, \ldots, U_n\}$ consist of unary relation symbols only and $\phi \in MSOL(\tau)$. Then the spectrum of ϕ is finite or co-finite.

Proof:
Use quantifier elimination or Ehrenfeucht-Fraïssé games.

Note:

The even numbers are a spectrum of a sentence with one unary function.

Fixing the vocabulary, I

The empty vocabulary: Equality only.

Observation:
Every finite or co-finite set $X \subseteq \mathbb{N}$ is a first order spectrum for a sentence with equality only.

Conclusion:
If τ consists of a finite (possibly empty) set of unary relation symbols, the $MSOL(\tau)$-spectra are exactly all finite and cofinite subsets of $\mathbb{N}$.

Remark:
Every SOL-spectrum is also an SOL-spectrum over equality only.

Ultimately periodic sets

Definition:
A set $X \subseteq \mathbb{N}$ is *ultimately periodic* if there are $a, p \in \mathbb{N}$ such that for each $n \geq a$ we have that $n \in X$ iff $n + p \in X$.

Observation:
Every ultimately periodic set $X \subseteq \mathbb{N}$ is a first order spectrum for a sentence with one unary function and equality only.

Fixing the vocabulary, II
One unary function symbol.

Theorem:
Durand, Fagin, Loescher 1997,
Gurevich, Shelah 2003

Let ϕ be a sentence of $MSOL(\tau)$ where τ consists of finitely many unary relation symbols, one unary function and equality only.

Then $spec(\phi)$ is ultimately periodic.

If the function is a finite successor function, this can be viewed as a special case of Parikh's theorem.

Fixing the vocabulary, III
One binary relation symbol.

We denote by BIN the set of first order spectra with one binary relation symbol and equality, and with $BIN^1 \subseteq BIN$ the set of spectra of a symmetric, irreflexive relation (simple graphs).

Theorem:(Fagin 1974)

BIN^1 is closed under complement iff
the complement of every first order spectrum is also a spectrum.

In fact, there is $X \in BIN^1$ such $\mathbb{N} - X \in BIN$ iff the complement of every first order spectrum is also a spectrum.

Fixing the vocabulary, IV

One binary relation symbol.

Open problem:(Fagin 1974)
Is every first order spectrum in BIN?

Theorem:(Fagin 1975)

For every first order spectrum X there is $k \in \mathbb{N}$ such that $X^k = \{n^k : n \in X\} \in BIN$.

Fixing the vocabulary, V
Two and more unary function symbols.

We denote by $\mathcal{U}_i$ the set of first order spectra using at most i-many unary function symbols.

Clearly $\mathcal{U}_0 = \mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \ldots$

Theorem:(Loescher 1997)
The set $\{n^2 : n \in \mathbb{N}\} \in \mathcal{U}_2 - \mathcal{U}_1$,
hence the inclusion $\mathcal{U}_1 \subset \mathcal{U}_2$ is proper.

Theorem:(Durand and Ranaivoson 1996)
There is a first order spectrum over two unary function symbols which, written in unary, is $\mathbf{NP}$-complete.

The arity hierarchy, I

We denote by $\mathcal{F}_i$ the set of first order spectra using a finite set of i-ary relation symbols, and put $\mathcal{F} = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$.

Clearly $\mathcal{F}_0 = \mathcal{F}_1 \subseteq BIN \subseteq \mathcal{F}_2 \subseteq \ldots \mathcal{F}_3 \subseteq \ldots$ and $\mathcal{F}_1 \subseteq \mathcal{U} \subseteq \mathcal{F}_2$.

Theorem:(Lynch 1982)
Let $X \subseteq \mathbb{N}$ be recognizable in $NTIME(n^d)$ then $X \in \mathcal{F}_d$.

The arity hierarchy, II

Problem: Is the arity hierarchy strict?

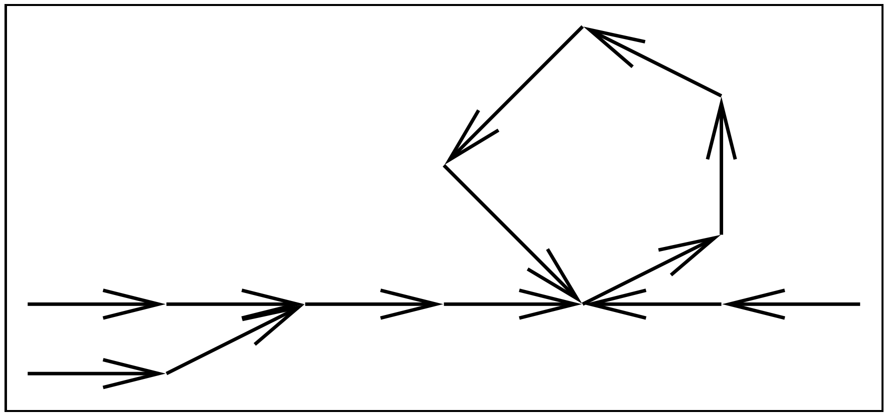
Fagin noted

Theorem:(Fagin 1975)

If $\mathcal{F}_k = \mathcal{F}_{k+1}$ then the arity hierarchy collapses
with $\mathcal{F}_k = \mathcal{F}_m$ for every $m \geq k$.

An observation

The structures which have only one unary function consist of disjoint unions of components of the form



They look like forests where the roots are replaced by a cycle. The unary predicates are just colours attached to the nodes.

Tree width, I

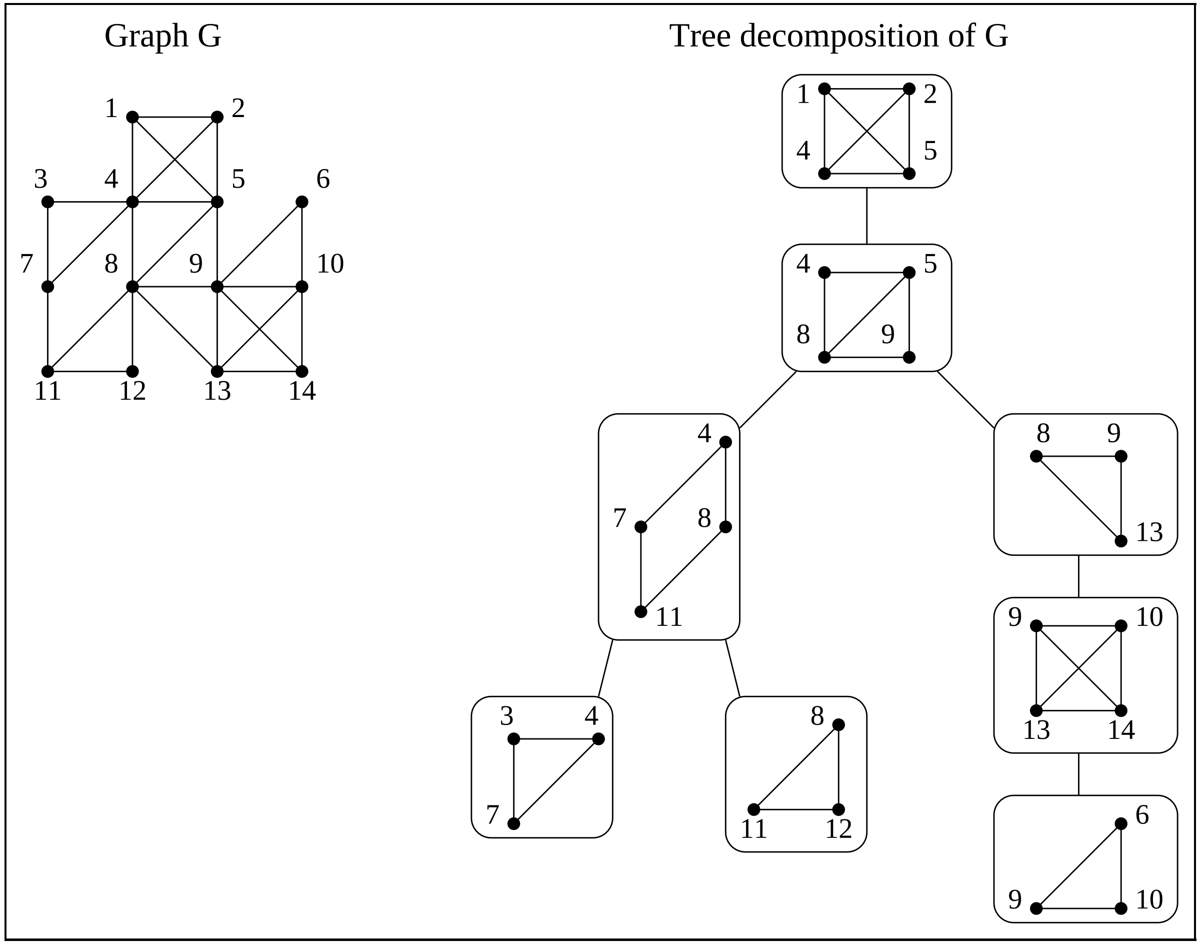
A k-tree decomposition of a graph $G = (V, E)$ is a pair $(\{X_i \mid i \in I\}, T = (I, F))$ with $\{X_i \mid i \in I\}$ a family of subsets of V, one for each node of T, and T a tree such that

1. $\bigcup_{i \in I} X_i = V$.
2. for all edges $(v, w) \in E$ there exists an $i \in I$ with $v \in X_i$ and $w \in X_i$.
3. for all $i, j, k \in I$: if j is on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$.
4. for all $i \in I$, $|X_i| \leq k + 1$.

A graph G is of *tree-width at most* k if G has a k-tree decomposition.

A class of graphs K is a $TW(k)$-class iff all its members have tree width at most k.

Tree width, picture



Tree width of one unary function

Proposition (Exercise):

Let ϕ be a sentence of $MSOL(\tau)$ where τ consists of finitely many unary relation symbols, one unary function (represented as a binary relation) and equality only.

Then all the finite models of ϕ have tree width at most 2.

Bounded tree width of $Mod(\phi)$

Let $CMSOL$ denote the extension of $MSOL$ by modular counting quantifiers. $CMSOL \subseteq SOL$.

Theorem: (Fischer and Makowsky, 2003)

Let ϕ be an $CMSOL$ sentence and $k \in \mathbb{N}$.
Assume that all the models of ϕ are in $TW(k)$.
Then $spec(\phi)$ is ultimately periodic.

Clique width vs tree width

Trees have tree width 1.

Cliques K_n have tree width $n-1$.

Courcelle , Engelfriet and Rozenberg (1993) introduced the clique width of graphs, which was extended to relational structures by **Courcelle** and **Makowsky**(2002).

Every class of bounded tree width has also bounded clique width. The converse is not true.

Spectra and clique width

Theorem: (Fischer and Makowsky, 2003)

Let ϕ be an $CMSOL$ sentence and $k \in \mathbb{N}$.
Assume that all the models of ϕ are in $CW(k)$.
Then $spec(\phi)$ is ultimately periodic.

Remarks on the proofs

The proof of the Fischer-M. theorems is rather involved.

It uses

- A reduction of $CMSOL$-definable classes of bounded tree (clique) width to a class of labeled trees definable in $MSOL$.
- A pumping lemma for $MSOL$-definable classes of labeled trees.

The technique can be further extended to

- multi-spectra describing the size of finitely many definable subsets of the structure,
- Guarded Second Order Logic, and
- various more general notions of bounded width of finite structures, such as rank-width, patch-width, etc.

Thank you for your attention!