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Iterated pushdown automata and sequences of rational numbers

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Abstract

We introduce a link between automata of level k and tree-structures. This method leads to new decidability results about integer sequences. We also reduce some equality problems for sequences of rational numbers to the equivalence problem for deterministic automata of level k.

Keywords: Iterated pushdown automata; Tree-structure; Integer sequences; Weak arithmetics; Equivalence problems

1. Introduction

The class of pushdown automata of level k (for $k \geq 1$) has been introduced in [21,26] as a generalisation of the automata and grammars of [1,2,20] and has been the object of many further studies: see [27,13,15,18,19,14], and more recently [9,24].

We focus here on some links between these automata and, on one hand, some results in mathematical logics, on the other hand some new classes of sequences of numbers.

We show that the structure of the memory of any pushdown automaton of level k with pushdown alphabet Γ, is logically definable inside the k-fold expansion of the finite structure Γ. This remark enables one to make use of the powerful generalisation of Rabin's tree-theorem [30] over arbitrary tree-structures due to Muchnik [31,28,35,22]. We thus re-obtain some known decidability properties of this class of automata and also obtain some new ones.

We focus then on a class of *integer* sequences recognised (in a suitable sense) by such automata (we denote by $\mathbb{S}_k$ the class of integer sequences recognised by deterministic pushdown automata of level k). This class enjoys nice closure properties and seems quite wide. Level 2 contains the classical *rational* sequences of integers (see [7]).

The decidability results obtained above lead to extensions of the well-known result of Büchi establishing the decidability of the Monadic Second-order Theory of $\langle \mathbb{N}, S \rangle$, the set of natural integers endowed with just the successor function S [8].

Next, we consider the class $\mathcal{F}(\mathbb{S}_k)$ consisting of all the sequences of *rational* numbers which can be decomposed as $\frac{a_n - b_n}{a'_n - b'_n}$ for sequences $a, b, a', b' \in \mathbb{S}_k$. This class enjoys nice closure properties too and generalizes some well-known

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classes of recurrent sequences (or formal power series). The level 3, for example, contains all the so-called P-recurrent sequences of rational numbers, corresponding also to the D-finite formal power series (see [34] for a survey). As a corollary of the above closure properties, the equality problem for two sequences in $\mathcal{F}(S_k)$ reduces to the equivalence problem for deterministic pushdown automata of level k. This establishes a bridge between the algorithmic problems about sequences (treated in [29], for example) and the decision problems about automata (treated in [32], for example).

2. Preliminaries

We introduce here some notation and basic definitions which will be used throughout the text.

2.1. Words

If A is a set, A^* denotes the set of words (finite sequences) over A, ε is the empty word and $A^+ = A^* - \{\varepsilon\}$. The symbol $\bar{a}$ will always denote a letter or the empty word (even in contexts where the alphabet is not designated by the symbol A). For a given word $u \in A^*$, we denote by $|u|$ the length of u.

For $n \geq 0$ we define $A^n = \{u \in A^*. |u| = n\}$, $A^{(n)} = \{u \in A^*. |u| \leq n\}$ and $[n] = \{1, \ldots, n\}$.

2.2. Logics

2.2.1. Monadic Second Order Logic

Let $Sig = \{r_1, \ldots, r_n\}$ be a signature containing relational symbols only, where $(\rho_i, \tau_i) \in \mathbb{N}^2$ is the arity of symbol r_i and
$Var = \{x, y, z, \ldots, X, Y, Z \ldots\}$ be a set of variables, where $x, y, \ldots$ denote first order variables and $X, Y, \ldots$ second order variables.

The set of MSO-formulas over Sig is the smallest set such that:

- $x \in X$ and $Y \subseteq X$ are MSO-formulas for every $x, Y, X \in Var$
- $r(x_1, \ldots x_\rho, X_1 \ldots X_\tau)$ is a MSO-formula for every $r \in Sig$, of arity (ρ, τ) and every first order variables $x_1, \ldots x_\rho \in Var$ and second-order variables $X_1 \ldots X_\tau \in Var$
- if Φ, Ψ are MSO-formulas then $\neg\Phi$, $\Phi \vee \Psi$, $\exists x.\Phi$ and $\exists X.\Phi$ are MSO-formulas.

Let $\mathcal{M} = \langle D_\mathcal{M}, r_1, \ldots, r_n\rangle$ be a structure over the signature Sig, and val a function $val : Var \to D_\mathcal{M} \cup \mathcal{P}(D_\mathcal{M})$ such that for every x, $X \in Var$, $val(x) \in D_\mathcal{M}$ and $val(X) \in \mathcal{P}(D_\mathcal{M})$.

The satisfiability of an MSO-formula in the structure $\mathcal{M}$ with valuation val is then defined by induction on the structure of the formula, in the usual way.

2.2.2. Semantic interpretations

Let $Sig = \{r_1, \ldots, r_n\}$ (resp. $Sig' = \{r'_1, \ldots, r'_m\}$) be some relational signature and $\mathcal{M}$ (resp. $\mathcal{M}'$) be some structure over the signature Sig (resp. Sig').We denote by $\mathcal{L}$ (resp. $\mathcal{L}'$) the set of MSO-formulas over Sig (resp. Sig').

Definition 1 (*Interpretations*). We call an MSO-*interpretation* of the structure $\mathcal{M}$ into the structure $\mathcal{M}'$ every injective map $\varphi : D_\mathcal{M} \to D_{\mathcal{M}'}$ such that,

1. There exists a formula $\Phi'(X) \in \mathcal{L}'$, with one free variable X, which is second order, fulfilling that, for every subset $X_{\mathcal{M}'} \subseteq D_{\mathcal{M}'}$
$$X_{\mathcal{M}'} = \varphi(D_\mathcal{M}) \Leftrightarrow \mathcal{M}' \models \Phi'(X_{\mathcal{M}'}).$$
2. For every $i \in [1, n]$, there exists a formula $\Phi'_i(x_1, \ldots, x_{\rho_i}, X_1, \ldots, X_{\tau_i})$, fulfilling that, for every valuation $val : Var \to D_\mathcal{M} \cup \mathcal{P}(D_\mathcal{M})$
$$(\mathcal{M}, val) \models r_i(x_1, \ldots, x_{\rho_i}, X_1, \ldots, X_{\tau_i}) \Leftrightarrow (\mathcal{M}', \varphi \circ val) \models \Phi'_i(x_1, \ldots, x_{\rho_i}, X_1, \ldots, X_{\tau_i}).$$

(In the definition above, φ denotes also its natural extension to subsets of $D_\mathcal{M}$.) When $D_\mathcal{M} \subseteq D'_\mathcal{M}$ and φ is just the natural injection from $D_\mathcal{M}$ into $D_{\mathcal{M}'}$, we say that $\mathcal{M}$ is MSO-*definable* inside $\mathcal{M}'$.

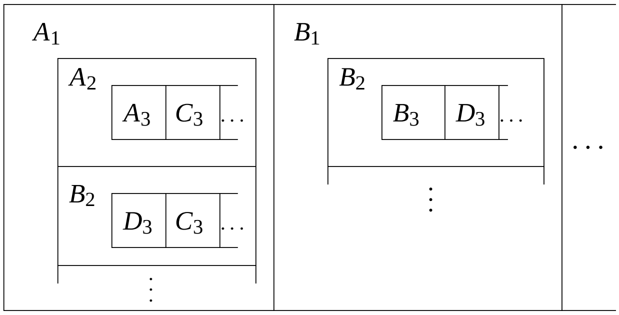

Fig. 1. A $3-$ pds.

Theorem 2. *Suppose that there exists an* MSO-*interpretation of the structure* $\mathcal{M}$ *into the structure* $\mathcal{M}'$. *Then, there exists a computable map from* $\mathcal{L}$ *to* $\mathcal{L}'$: $\Phi \mapsto \hat{\Phi}$ *such that*

$$\mathcal{M}' \models \hat{\Phi} \text{ iff } \mathcal{M} \models \Phi.$$

In particular, if $\mathcal{M}'$ *has a decidable MSO-theory, then* $\mathcal{M}$ *has a decidable MSO-theory too.*

This theorem is proved in [5, Section 3.1 pp. 613–615]. Given two sets S, S' we denote by Π (resp. Π') the projection from $S \times S'$ on its first (resp. second) component S (resp. S').

Definition 3 (*Direct-product*). The direct product $\mathcal{M} \times \mathcal{M}'$ is the structure over the signature $Sig \cup Sig'$ defined by: for every $i \in [1, n]$, $j \in [1, m]$

1. $D_{\mathcal{M}\times\mathcal{M}'} = D_{\mathcal{M}} \times D_{\mathcal{M}'}$
2. $\forall(\bar{x}, \bar{X}) \in D_{\mathcal{M}\times\mathcal{M}'}^{\rho_i} \times (\mathcal{P}(D_{\mathcal{M}\times\mathcal{M}'}))^{\tau_i}$, $(\bar{x}, \bar{X}) \in r_{i,\mathcal{M}\times\mathcal{M}'} \Leftrightarrow \Pi(\bar{x}, \bar{X}) \in r_{i,\mathcal{M}}$
3. $\forall(\bar{x}, \bar{X}) \in D_{\mathcal{M}\times\mathcal{M}'}^{\rho'_j} \times (\mathcal{P}(D_{\mathcal{M}\times\mathcal{M}'}))^{\tau'_j}$, $(\bar{x}, \bar{X}) \in r'_{j,\mathcal{M}\times\mathcal{M}'} \Leftrightarrow \Pi'(\bar{x}, \bar{X}) \in r'_{j,\mathcal{M}'}$.

The following lemma is straightforward, but useful.

Lemma 4. *If* $\mathcal{M}$ *has a decidable MSO-theory and* $\mathcal{M}'$ *is finite, then* $\mathcal{M} \times \mathcal{M}'$ *has a decidable MSO-theory.*

2.3. Automata

The level-k languages have been introduced by Maslov [26] by means of an extension of indexed grammars [1] at level k. Indexed languages defined by Aho admit a characterization by automata: the *nested stack* automata [2]. Maslov gives also a characterization of level-k languages by means of *multilevel stack* automata [27]. Since this time, many other grammars and automata have been devised to define this class of languages. One can cite “level-k grammars” studied by Engelfriet [18] and Damm [13] which are an extension of “macro grammars “ of Fischer [20] and “iterated-pushdown automata” studied by Damm and Goerdt [14], Engelfriet [15,16] and more recently by Knapik et al. [24].

2.3.1. General definitions

Here we shall use the definition of [14] and stick to their notation. Iterated pushdown automata are an extension of classical pushdown automata to a storage structure built iteratively. This storage structure — defined by [21] in the more general setting of *Abstract Families of Automata* — can be described as follows:

- a 1-iterated pushdown store consists of a classical pushdown-list of symbols
- a $(k + 1)$-iterated pushdown store consists of a pushdown-list of pairs (pushdown-symbol, k-iterated pushdown store).

Fig. 1 depicts a typical $3-$ pds (3-iterated pushdown store), as it is represented in [14].

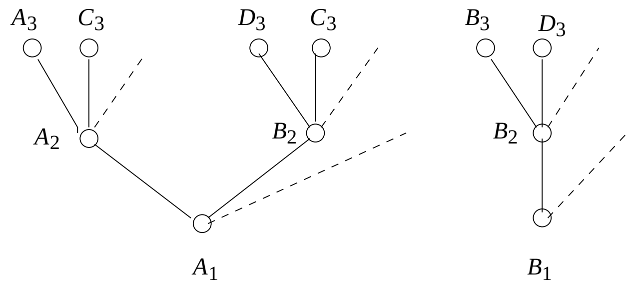

Fig. 2. pds as a planar tree.

Definition 5 (*k-Iterated Pushdown Store*). Let Γ be a set. We define inductively the set of *k-iterated pushdown-stores over* Γ,

$$0-\mathrm{pds}(\Gamma)=\{\varepsilon\} \quad (k+1)-\mathrm{pds}(\Gamma)=(\Gamma[k-\mathrm{pds}(\Gamma)])^{*} \quad \mathrm{it}-\mathrm{pds}(\Gamma)=\bigcup_{k\geq 0} k-\mathrm{pds}(\Gamma).$$

Remark 6.

1. The empty word, ε, belongs to every set $k-\mathrm{pds}(\Gamma)$.
2. It follows that, for every symbol $A\in\Gamma$, $A[\varepsilon]$ belongs to $(k+1)-\mathrm{pds}(\Gamma)$.
3. More generally,
   $0-\mathrm{pds}(\Gamma)\subseteq 1-\mathrm{pds}(\Gamma)\subseteq \ldots k-\mathrm{pds}(\Gamma)\subseteq (k+1)-\mathrm{pds}(\Gamma)\subseteq \ldots$.
4. In the rest of the paper we will often denote by $\cdots AB\cdots$ what should be denoted by $\cdots A[\varepsilon]B\cdots$ (where A,B are letters from Γ). More precisely: inside a word denoting a $k-\mathrm{pds}$, every letter $A\in\Gamma$ followed by a symbol other than "[", means the pds $A[\varepsilon]$.
5. For every $k\in\mathbb{N}$, the set $k-\mathrm{pds}(\Gamma)$ endowed with the concatenation operation is a monoid.

Definition 7. Let $\omega\in k-\mathrm{pds}(\Gamma)$. We say that ω is *atomic* iff, for every $\omega_1,\omega_2\in k-\mathrm{pds}(\Gamma)$, $\omega=\omega_1\cdot\omega_2\Rightarrow(\omega_1=\varepsilon$ or $\omega_2=\varepsilon)$.

In that case we also say that ω is an atom.

One can check that $(k-\mathrm{pds}(\Gamma),\cdot)$ is a free monoid with base $\{\omega\in k-\mathrm{pds}(\Gamma)\mid \omega \text{ is atomic}\}$. Notice that, if ω is an atomic k-pds, it is also an atomic k'-pds for every $k'\geq k$. Every nonempty $\omega\in(k+1)-\mathrm{pds}(\Gamma)$ has a unique decomposition as:

$$\omega=A[\mathrm{flag}]\cdot\mathrm{rest} \tag{1}$$

with $A\in\Gamma$, flag $\in k-\mathrm{pds}(\Gamma)$, and rest $\in(k+1)-\mathrm{pds}(\Gamma)$.

Example 8. Let us denote by ω the $3-\mathrm{pds}$ given in Fig. 1 without the dots. According to Definition 5, we should write:

$$\omega=A_1[A_2[A_3[\varepsilon]C_3[\varepsilon]]B_2[D_3[\varepsilon]C_3[\varepsilon]]]B_1[B_2[B_3[\varepsilon]D_3[\varepsilon]]].$$

According to Remark 6, point 4, we shall (abusively) write:

$$\omega=A_1[A_2[A_3C_3]B_2[D_3C_3]]B_1[B_2[B_3D_3]].$$

The decomposition of ω as a product of atoms is:

$$\omega=A_1[A_2[A_3C_3]B_2[D_3C_3]]\cdot B_1[B_2[B_3D_3]].$$

Its decomposition of the form (1) corresponds to:

$$A=A_1,\quad \mathrm{flag}=A_2[A_3C_3]B_2[D_3C_3],\quad \mathrm{rest}=B_1[B_2[B_3D_3]].$$

Fig. 2 depicts the $3-\mathrm{pds}$ of Example 8, based on a representation of each atom as a planar tree, with labels in Γ.

We now formalize operations allowed on the store.

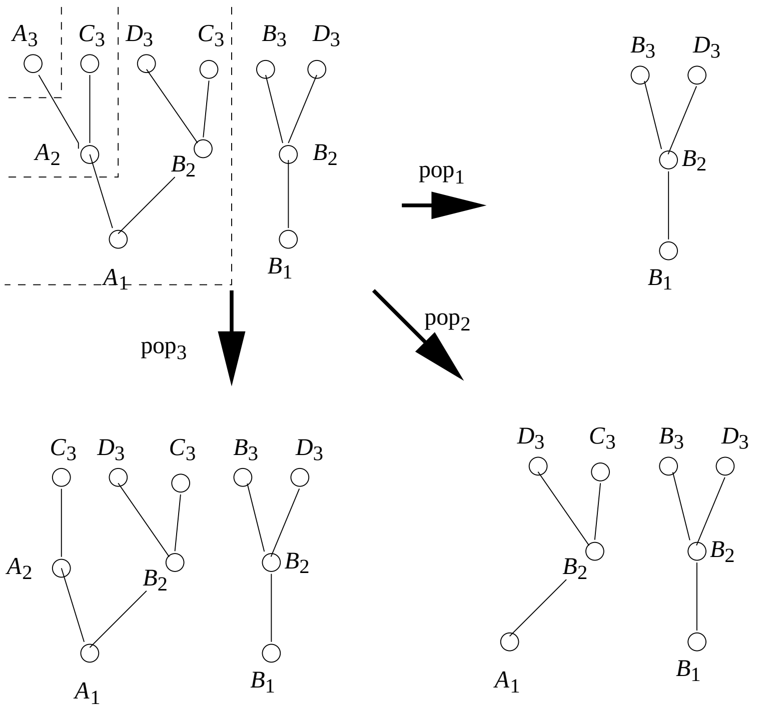

Fig. 3. The pop operations.

Definition 9 (*The Reading Operation*). The map topsyms : $\text{it}-\text{pds}(\Gamma) \to \Gamma^*$ is defined by:

$$\text{topsyms}(\varepsilon) = \varepsilon, \quad \text{topsyms}(A[f]r) = A \cdot \text{topsyms}(f).$$

Example 10. The reading operation, applied on the above example gives:
$\text{topsyms}(\omega) = A_1 A_2 A_3$.

Notice that the word $\text{topsyms}(\omega)$ corresponds to the sequences of labels on the leftmost branch of the leftmost atom of ω (see Fig. 2).

Definition 11 (*The* pop *Operation at Level* j). The map $\text{pop}_j : \text{it}-\text{pds}(\Gamma) \to \text{it}-\text{pds}(\Gamma)$ is defined by:

$$\text{pop}_j(\varepsilon) \text{ is undefined }, \quad \text{pop}_1(A[f]r) = r, \quad \text{pop}_{j+1}(A[f]) = A[\text{pop}_j(f)]r.$$

Example 12. The pop operation, applied on the above example gives:
$\text{pop}_1(\omega) = B_1[B_2[B_3 D_3]]$
$\text{pop}_2(\omega) = A_1[B_2[D_3 C_3]]B_1[B_2[B_3 D_3]]$
$\text{pop}_3(\omega) = A_1[A_2[C_3]B_2[D_3 C_3]]B_1[B_2[B_3 D_3]]$.

(See Fig. 3 for a planar representation).

Definition 13 (*The* push *Operation at Level* j). Let $\gamma = A_1 \dots A_n \in \Gamma^+$. The map $\text{push}_j(\gamma) : \text{it}-\text{pds}(\Gamma) \to \text{it}-\text{pds}(\Gamma)$ is defined by:

$$\text{push}_1(\gamma)(\varepsilon) = \gamma, \quad \text{push}_{j+1}(\gamma)(\varepsilon) \text{ is undefined for } j \geq 1,$$
$$\text{push}_1(\gamma)(A[f]r) = A_1[f]\dots A_n[f]r, \quad \text{push}_{j+1}(\gamma)(A[f]r) = A[\text{push}_j(\gamma)(f)]r.$$

Example 14. The push operation, applied on the above example gives:
$\text{push}_2(AB)(\omega) = A_1[A[A_3 C_3]B[A_3 C_3]B_2[D_3 C_3]]B_1[B_2[B_3 D_3]]$
$\text{push}_3(AB)(\omega) = A_1[A_2[ABC_3]B_2[D_3 C_3]]B_1[B_2[B_3 D_3]]$.

(See Fig. 4 for a planar representation.)

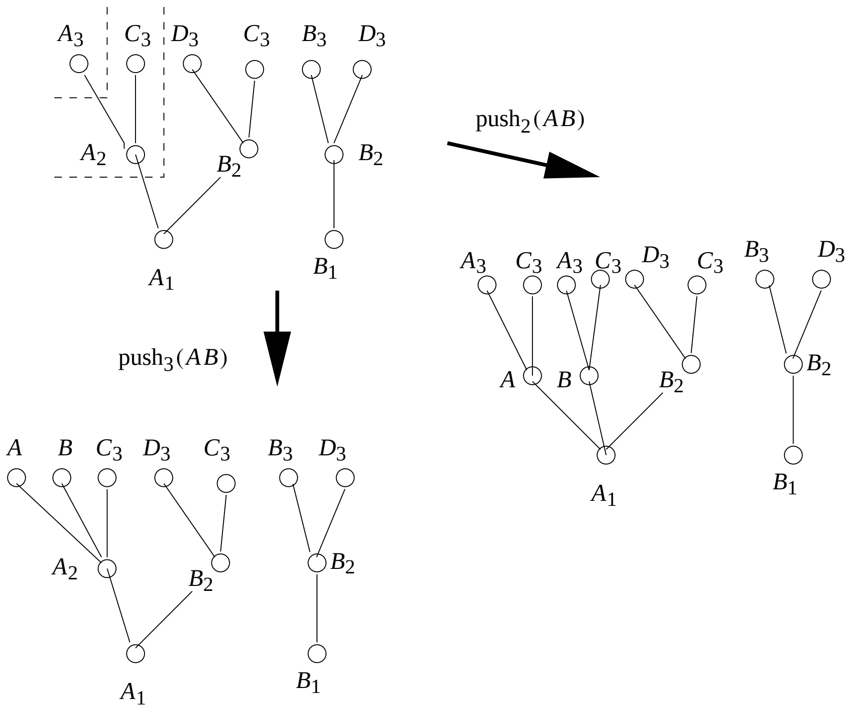

Fig. 4. The push operations.

Definition 15 (*Syntax of k-pdas*). Let $k \geq 1$:

(1) Let $POP = \{\text{pop}_j \mid j \in [k]\}$, $PUSH(\Gamma) = \{\text{push}_j(\gamma) | \gamma \in \Gamma^+,\ j \in [k]\}$, and $TOPSYMS(\Gamma) = \Gamma^{(k)} - \{\varepsilon\}$.

(2) A k-iterated pushdown automaton over a terminal alphabet Σ is a 6-tuple $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, Z)$ where

- Q is a finite set of states, $q_0 \in Q$ denoting the initial state,
- Γ is a finite set of pushdown-symbols with $Z \in \Gamma$ as initial symbol,
- the transition function δ is a map from $Q \times (\Sigma \cup \{\varepsilon\}) \times TOPSYMS(\Gamma)$ into the set of finite subsets of $Q \times (PUSH(\Gamma) \cup POP)$ such that:
  if $(q, \text{push}_j(\gamma)) \in \delta(p, \bar{a}, \gamma)$ then $j \leq |\gamma| + 1$ and if $(q, \text{pop}_j) \in \delta(p, \bar{a}, \gamma)$ then $j \leq |\gamma|$. These conditions avoid cases where operations are undefined.

Definition 16 (*Semantics of k-pdas*). Let $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, Z)$ be some k-pda:

(1) The set of *configurations* of $\mathcal{A}$ is $Con_{\mathcal{A}} = Q \times \Sigma^* \times k - \text{pds}(\Gamma)$.

(2) The *single step relation* $\vdash_{\mathcal{A}}\ \subseteq Con_{\mathcal{A}} \times Con_{\mathcal{A}}$ of $\mathcal{A}$ is defined by
$(p, w, \omega) \vdash_{\mathcal{A}} (q, v, \omega')$ iff
$(q, f) \in \delta(p, \bar{a}, \text{topsyms}(\omega))$, $\bar{a}v = w$ and $\omega' = f(\omega)$.

(3) We denote by $\vdash^*_{\mathcal{A}}$ the reflexive, transitive closure of $\vdash_A$.

(4) The *language accepted by* $\mathcal{A}$ (with empty store) is defined by
$\text{L}(\mathcal{A}) = \{w \in \Sigma^* | \exists q \in Q, (q_0, w, Z) \vdash^*_{\mathcal{A}} (q, \varepsilon, \varepsilon)\}$.

Using standard techniques from automata-theory, one can prove that acceptance by empty store and acceptance by (final states and empty store) define the same class of languages.

Example 17. The following 2-pda $\mathcal{A}$ fulfills: $\mathrm{L}(\mathcal{A}) = \{a^{f(n)} | n \geq 0\}$, where f denotes the Fibonacci's sequence.

$\mathcal{A} = (\{q_0, q_1, q_2\}, \{a\}, \{Z, X_1, X_2, F\}, \delta, q_0, Z)$ with
$\delta(q_0, \varepsilon, Z) = \{(q_0, \mathrm{push}_2(F)),\ (q_0, \mathrm{push}_1(X_2))\}$,
$\delta(q_0, \varepsilon, ZF) = \{(q_0, \mathrm{push}_2(FF)),\ (q_0, \mathrm{push}_1(X_2))\}$,
$\delta(q_0, \varepsilon, X_1F) = \{(q_1, \mathrm{pop}_2)\}$, $\delta(q_0, \varepsilon, X_2F) = \{(q_2, \mathrm{pop}_2)\}$,
$\delta(q_0, a, X_1) = \{(q_0, \mathrm{pop}_1)\}$, $\delta(q_0, a, X_2) = \{(q_0, \mathrm{pop}_1)\}$,
$\delta(q_1, \varepsilon, X_1F) = \{(q_0, \mathrm{push}_1(X_1X_2))\}$, $\delta(q_2, \varepsilon, X_2F) = \{(q_0, \mathrm{push}_1(X_1)\}$,
$\delta(q_1, \varepsilon, X_1) = \{(q_0, \mathrm{push}_1(X_1X_2))\}$, $\delta(q_2, \varepsilon, X_2) = \{(q_0, \mathrm{push}_1(X_1)\}$.

We give an accepting configurations sequence for $a^{f(3)} = a^3$:

$(q_0, a^3, Z[\varepsilon]) \vdash (q_0, a^3, Z[F]) \vdash (q_0, a^3, Z[FF]) \vdash (q_0, a^3, Z[FFF]) \vdash$
$(q_0, a^3, X_2[FFF]) \vdash (q_2, a^3, X_2[FF]) \vdash (q_0, a^3, X_1[FF]) \vdash (q_1, a^3, X_1[F]) \vdash$
$(q_0, a^3, X_1[F]X_2[F]) \vdash (q_1, a^3, X_1[\varepsilon]X_2[F]) \vdash (q_0, a^3, X_1[\varepsilon]X_2[\varepsilon]X_2[F]) \vdash$
$(q_0, a^2, X_2[\varepsilon]X_2[F]) \vdash (q_0, a, X_2[F]) \vdash (q_2, a, X_2[\varepsilon]) \vdash (q_0, a, X_1[\varepsilon]) \vdash$
$(q_0, \varepsilon, \varepsilon)$.

2.3.2. Some basic tools

Let $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, Z)$ be some k-dpda. A *total state* of $\mathcal{A}$ is any pair $(q, \omega) \in Q \times k - \mathrm{pds}(\Gamma)$. A *mode* is a pair $(q, \omega) \in Q \times \Gamma^{(k)}$. Given a configuration $c = (q, u, \omega) \in Con_{\mathcal{A}}$, the total state of c is (q, ω) and the mode of (q, ω) (and of c, as well) is $(q, \mathrm{topsyms}(\omega))$.

2.3.2.1. Derivation. We associate with $\mathcal{A}$ an infinite "alphabet"

$$V_{\mathcal{A}} = \{(p, \omega, q) \mid p, q \in Q, \omega \in k - \mathrm{pds}(\Gamma) - \{\varepsilon\}\}. \tag{2}$$

The set of *productions* associated with $\mathcal{A}$, denoted by $P_{\mathcal{A}}$ is made of the set of all the following rules:
the *transition* rules:

$$(p, \omega, q) \rightarrow_{\mathcal{A}} \bar{a}(p', \omega', q)$$

if $(p, \bar{a}, \omega) \vdash_{\mathcal{A}} (p', \varepsilon, \omega')$ and $q \in Q$ is arbitrary,

$$(p, \omega, q) \rightarrow_{\mathcal{A}} \bar{a}$$

if $(p, \bar{a}, \omega) \vdash_{\mathcal{A}} (q, \varepsilon, \varepsilon)$
the *decomposition* rule:

$$(p, \omega, q) \rightarrow_{\mathcal{A}} (p, \eta, r)(r, \eta', q)$$

if $\omega = \eta \cdot \eta', \eta \neq \varepsilon, \eta' \neq \varepsilon$ and $r \in Q$ is arbitrary. The one-step *derivation* generated by $\mathcal{A}$, denoted by $\rightarrow_{\mathcal{A}}$, is the smallest subset of $(V \cup \Sigma)^* \times (V \cup \Sigma)^*$ which contains $P_{\mathcal{A}}$ and is compatible with left product and right product. Finally, the *derivation* generated by $\mathcal{A}$, denoted by $\rightarrow^*_{\mathcal{A}}$, is the reflexive and transitive closure of $\rightarrow_{\mathcal{A}}$. These notions correspond to the usual notion of *context-free grammar* associated with the following pushdown automaton $\mathcal{A}_1$: this automaton has the pushdown alphabet $\Gamma_1 = \{A[\omega] \mid A \in \Gamma, \omega \in (k-1) - \mathrm{pds}(\Gamma)\}$ and has the transition function

$$\delta_1(q, \bar{a}, A[\omega]) = \{(q', \eta') \in Q \times \Gamma_1^* \mid (q, \bar{a}, A[\omega]) \vdash_{\mathcal{A}} (q', \varepsilon, \eta')\}.$$

Of course, as soon as $k \geq 2$, this pushdown alphabet is infinite, but all the usual properties of the relation $\rightarrow_{\mathcal{A}} = \rightarrow_{\mathcal{A}_1}$ and its links with $\vdash_{\mathcal{A}} = \vdash_{\mathcal{A}_1}$ remain true in this context (see [23, Proof of Theorem 5.4.3, pp. 151–158]). In particular, for every $u \in \Sigma^*, p, q \in Q, \omega \in \Gamma_1^*$

$$(p, \omega, q) \rightarrow^*_{\mathcal{A}} u \Leftrightarrow (p, u, \omega) \vdash^*_{\mathcal{A}} (q, \varepsilon, \varepsilon).$$

The following lemma is useful.

Lemma 18. *Let $p_i, q_i \in Q, \omega_i \in \Gamma_1^*$ for $i \in 1, 2, 3$. The following properties are equivalent:*

(1) $(p_1, \omega_1, q_1) \to_{\mathcal{A}}^* (p_2, \omega_2, q_2)(p_3, \omega_3, q_3)$
(2) *There exist $\omega_2', \omega_3' \in \Gamma_1^*$, such that:*
$(p_1, \varepsilon, \omega_1) \vdash_{\mathcal{A}}^* (p_2, \varepsilon, \omega_2\omega_2'); \quad (q_2, \varepsilon, \omega_2') \vdash_{\mathcal{A}}^* (p_3, \varepsilon, \omega_3\omega_3'); \quad (q_3, \varepsilon, \omega_3') \vdash_{\mathcal{A}}^* (q_1, \varepsilon, \varepsilon)$.

We usually assume that Γ and Q are disjoint, therefore, omitting the commas in (p, ω, q) does not lead to any confusion.

2.3.2.2. Determinism. The automaton $\mathcal{A}$ is said to be *deterministic* iff, for every $q \in Q, \gamma \in \Gamma^{(k)}, \sigma \in \Sigma$

$$\mathrm{Card}(\delta(q, \varepsilon, \gamma)) \leq 1 \text{ and } \mathrm{Card}(\delta(q, \sigma, \gamma)) \leq 1, \tag{3}$$
$$\mathrm{Card}(\delta(q, \varepsilon, \gamma)) = 1 \Rightarrow \mathrm{Card}(\delta(q, \sigma, \gamma)) = 0. \tag{4}$$

2.3.2.3. Terms. Given a denumerable alphabet Γ of pushdown symbols, we introduce another alphabet $\mathcal{U} = \{\Omega, \Omega', \Omega'', \ldots, \Omega_1, \Omega_2, \ldots, \Omega_n, \ldots\}$ of *undeterminates*. We suppose that $\Gamma \cap \mathcal{U} = \emptyset$. We call a *term* of level k over the constant alphabet Γ and the alphabet of undeterminates $\mathcal{U}$, any $T \in k - \mathrm{pds}(\Gamma \cup \mathcal{U})$ such that every occurrence of an undeterminate U in T is a leaf (if we see a pds as a planar tree, as we did in Figs. 2–4); equivalently, every occurrence of $U \in \mathcal{U}$ in T is followed by $[\varepsilon]$, in the rigorous bracketed notation.

We denote by $k - \mathrm{term}(\Gamma \cup \mathcal{U})$ the set of all terms of level k over the constant alphabet Γ and the alphabet of undeterminates $\mathcal{U}$.

We denote an element of $k - \mathrm{term}(\Gamma \cup \mathcal{U})$ by $T[\Omega_1, \Omega_2, \ldots, \Omega_n]$ (resp. $T[\Omega, \Omega', \Omega'']$) provided that the only undeterminates appearing in T belong to $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ (resp. $\{\Omega, \Omega', \Omega''\}$).

Definition 19. Let $T \in k - \mathrm{term}(\Gamma \cup \mathcal{U})$. The term T is said to be

- *linear* iff each undeterminate has at most one occurrence in T.
- *k'-uniform* iff, every occurrence of an undeterminate has level exactly k' (the terms reduced to one undeterminate are thus 1-uniform).
- *standard* iff, T is linear, has exactly one occurrence of each undeterminate $\{\Omega_1, \ldots, \Omega_n\}$ (for some $n \geq 0$) and, for every $1 \leq i \leq j \leq n$, the occurrence of Ω_i is on the left of the occurrence of Ω_j.

Example 20. Let $A, B, C \in \Gamma$. Let us consider the terms:

$$T_1 = A, \quad T_1' = \Omega_2, \quad T = A[B[\Omega_1]]$$
$$T' = A[B[\Omega_2]C[\Omega_1]]C[C[\Omega_2]]; \quad T'' = A[\Omega_1 B[\Omega_3]]C[C[C[\Omega_2]]]$$
$$T''' = A[B[\Omega_1]C[\Omega_2]]C[A[\Omega_3]\Omega_4].$$

T_1 is linear, k-uniform (for every $k \geq 1$) and standard.
T_1' is linear, k-uniform (for $k = 1$ but not for $k \geq 2$) and standard.
T is linear, k-uniform (for $k = 3$ but not for $k \neq 3$) and standard.
T' is not linear, not k-uniform (for every $k \geq 1$) and non-standard.
T'' is linear, not k-uniform (for every $k \geq 1$) and non-standard.
T''' is linear, not k-uniform (for every $k \geq 1$) and standard.

We denote by $k - \mathrm{uterm}(\Gamma \cup \mathcal{U})$ the set of all terms in $k - \mathrm{term}(\Gamma \cup \mathcal{U})$ which are k-uniform.

2.3.2.4. Substitutions. Given $T[\Omega_1, \ldots, \Omega_n] \in k - \mathrm{term}(\Gamma \cup \mathcal{U})$, and $H_1, \ldots, H_n \in k' - \mathrm{term}(\Gamma \cup \mathcal{U})$, we denote by $T[H_1, \ldots, H_n]$ the $(k + k' - 1)$-term obtained by substituting H_i for Ω_i in T. The following "substitution principle" is straightforward and will be widely used in our proofs. Given some ℓ-pda $\mathcal{A}$ over a pushdown alphabet included in Γ, we extend the relations $\to_{\mathcal{A}}^*, \vdash_{\mathcal{A}}^*$ to the pushdown alphabet $\Gamma \cup \mathcal{U}$.

Lemma 21. *Let $\vec{\Omega} = (\Omega_1, \ldots, \Omega_n)$, $T[\vec{\Omega}], T'[\vec{\Omega}] \in k - \mathrm{term}(\Gamma \cup \mathcal{U})$ and $p, q \in Q$. If*

$$(pT[\vec{\Omega}]q) \to_{\mathcal{A}}^* (p'T'[\vec{\Omega}]q')$$

then, for every $\vec{H} \in (k' - \mathrm{term}(\Gamma \cup \mathcal{U}))^n$,

$$(pT[\vec{H}]q) \to^*_{\mathcal{A}} (p'T'[\vec{H}]q').$$

The key idea for this lemma is that, as $\Gamma \cap \mathcal{U} = \emptyset$, the symbols Ω_i can be copied or erased during the derivation, but they cannot *influence* the sequence of rules used in that derivation.

2.3.2.5. Normalized automata. We say that $\mathcal{A}$ is *level-partitioned* iff Γ is the disjoint union of subsets $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ such that, in every transition of $\mathcal{A}$, every occurrence of a letter from Γ_i is at level i. It is easy to transform any k-pushdown automaton $\mathcal{A}$ into another one $\mathcal{B}$ which recognizes the same language and is *level partitioned*. Moreover, if $\mathcal{A}$ is deterministic (resp. counter, counter-deterministic) then $\mathcal{B}$ is deterministic (resp. counter, counter-deterministic).

2.4. Sequences

Let $(\mathbb{Q}, +, \cdot)$ be the field of rational numbers. A *sequence* of rational numbers is any map $u : \mathbb{N} \to \mathbb{Q}$. We denote by $u(n)$ (sometimes also by u_n) the image of the integer n by the map u. Such a sequence u can be also viewed as a formal power series

$$u(X) = \sum_{n=0}^{\infty} u_n X^n.$$

The following operators on series are classical:
E: the *shift* operator

$$(\mathsf{E}u)(n) = u(n+1);\ (\mathsf{E}u)(X) = \frac{u(X) - u(0)}{X}$$

Δ: the difference operator

$$(\Delta u)(n) = u(n+1) - u(n);\ (\Delta u)(X) = \frac{u(X)(1-X) - u(0)}{X}$$

Σ: the summation operator

$$(\Sigma u)(n) = \sum_{j=0}^{n} u(j);\ (\Sigma u)(X) = \frac{u(X)}{1-X}$$

$+$: the sum operator

$$(u+v)(n) = u(n) + v(n);\ (u+v)(X) = u(X) + v(X)$$

$\cdot$: the external product, for every $r \in \mathbb{Q}$

$$(r \cdot u)(n) = r \cdot u(n);$$

$\odot$: the Hadamard product (also called the "ordinary" product)

$$(u \odot v)(n) = u(n) \cdot v(n);$$

$\times$: the convolution product

$$(u \times v)(n) = \sum_{k=0}^{n} u(k) \cdot v(n-k);\ (u \times v)(X) = u(X) \cdot v(X)$$

$\circ$: the sequence composition

$$(u \circ v)(n) = u(v(n));$$

$\bullet$: the series composition

$$(u \bullet v)(X) = \sum_{n=0}^{\infty} u(n) \cdot v(X)^n$$

(this last operation is defined as soon as $v(0) = 0$, which ensures that the family of series $(u(n) \cdot v(X)^n)_{n \geq 0}$ is summable).

The set of all sequences of rational numbers is also denoted by $\mathbb{Q}[[X]]$. The structure $(\mathbb{Q}[[X]], +, \times)$ is a ring, with unit $\mathbb{1} = (1, 0, \ldots, 0, \ldots)$. Given some number $r \in \mathbb{Q}$, we use also the same notation r for the sequence (or series):

$$r = r \cdot \mathbb{1} = (r, 0, \ldots, 0, \ldots)$$

while we use the notation $\frac{r}{1-X}$ for the constant sequence (or series):

$$\frac{r}{1 - X} = (r, r, \ldots, r, \ldots).$$

3. Tree-structures, words and pushdowns

We construct here a connection between the notion of k-iterated pushdown store (recalled in Definition 5) and the structure $\Gamma^{<k>}$ obtained from the alphabet Γ by iterating k times the *tree-structure* operation. This connection allows us to obtain a general decidability result for the computation-graphs of k-pushdown automata (Theorem 40). This prepares the ground for Section 6 where we define a wide class of unary predicates P for which the structure $\langle \mathbb{N}, S, P \rangle$ admits a decidable monadic second order theory.

3.1. Tree-structures

Definition 22 (*Tree-structure*). Let $Sig = (r_1, \ldots)$ be a signature containing only relational symbols. For a structure $M = \langle D_M, r_1, \ldots \rangle$ over the signature Sig one constructs the tree-structure $M^* = \langle D_M^*, son, clone, r_1^*, \ldots \rangle$ over the extended signature $Sig^* = Sig \cup \{son, clone\}$, where D_M^* is the set of all finite sequences of elements of D_M and the relations are defined by:

$$\begin{aligned} son &= \{(w, dw) : \quad w \in D_M^*, \quad d \in D_M\} \\ clone &= \{ddw : \quad w \in D_M^*, \quad d \in D_M\} \\ r^* &= \{(d_1 w, \ldots, d_k w) : \quad w \in D_M^*, \quad (d_1, \ldots, d_k) \in r_M\}, \end{aligned}$$

(for all $r \in Sig$, of arity k).

Example 23. Let $\mathcal{S}_{\{a,b\}} = \langle \{a, b\}, r_a, r_b \rangle$ with $r_a = \{a\}$ and $r_b = \{b\}$.
Then $\mathcal{S}^*_{\{a,b\}} = \langle \{a, b\}^*, r_a^*, r_b^*, son, clone \rangle$, with $r_a^* = \{ua.\ u \in \{a, b\}^*\}$ is the usual complete binary tree structure by the *clone* relation, augmented with the "clone" predicate.

The following lemma is useful.

Lemma 24. *If $\psi(x_1, \ldots, x_n)$ is a* first order *formula over a structure $\mathcal{S}$, then one can effectivly find a first order formula $\psi^*(x_1, \ldots, x_n)$ over $\mathcal{S}^*$ such that $\forall u_1, \ldots, u_n \in D_{\mathcal{S}}^*$*
$\mathcal{S}^* \models \psi^*(u_1, \ldots, u_n)$ *iff* $\exists u \in D^*_{\mathcal{S}}.\ \forall i \in [1, n], u_i = [w_i] \bullet u$ *and* $\mathcal{S} \models \psi(w_1, \ldots, w_n)$.

Such a map $\psi \mapsto \psi^*$ can be defined by induction over the structure of the formulas.

Theorem 25. *For every MSO formula Φ over the signature Sig^* one can effectively find an MSO formula $\widehat{\Phi}$ over the signature Sig such that, for every structure M:*

$$M \models \widehat{\Phi} \quad iff \quad M^* \models \Phi.$$

This theorem was first stated in [31,28] and is completely proved in [35,22]. It implies immediately the

Corollary 26. *The* MSO-*theory of M is decidable iff the* MSO-*theory of M^* is decidable.*

The structure $\mathcal{S}_{\{a,b\}}$ given in Example 23 thus has a decidable MSO-theory, which entails Rabin's theorem on decidability of the MSO-theory of $\langle \{a,b\}^*, r_a^*, r_b^*, son\rangle$.

Iterating twice the "tree-structure" operation, we obtain:
$(\mathcal{S}^*_{\{a,b\}})^* = \langle (\{a,b\}^*)^*, (r_a^*)^*, (r_b^*)^*, son^*, clone^*, son, clone\rangle$, where
$(\{a,b\}^*)^*$ denotes the set of words of words over $\{a,b\}$, i.e. the set of finite sequences of elements of $\{a,b\}^*$.

Definition 27 (*Star Iteration*). Let Σ be a finite set of symbols.

For all $k \geq 0$, we denote by $\Sigma^{<k>}$, $\Sigma^{[k]}$ the sets defined inductively by:

$$\Sigma^{<0>} = \Sigma^{[0]} = \Sigma, \quad \Sigma^{<k+1>} = (\Sigma^{<k>})^*, \quad \Sigma^{[1]} = \Sigma^*, \quad \Sigma^{[k+2]} = (\Sigma^{[k+1]} - \{\epsilon_{k+1}\})^*.$$

Here ε_{k+1} denotes the empty word of $\Sigma^{<k+1>}$, (for $k \geq 0$).

We represent each nonempty word of $\Sigma^{<k+1>}$ as a finite sequence of words of $\Sigma^{<k>}$ between brackets. We denote by $\bullet_{k+1}$ (or $\bullet$ if the level is understood) the concatenation of two words in $\Sigma^{<k+1>}$ defined by:

$$[u_0,\ldots,u_n] \bullet_{k+1} [v_0,\ldots,v_m] = [u_0,\ldots,u_n,v_0,\ldots,v_m]$$

(for all $n, m, k \geq 0, u_i, v_i \in \Sigma^{<k>}$).

Example 28. Let $u = [[[a,b,c],\varepsilon_1,[g,e]]] \bullet_3 [\varepsilon_2] \bullet_3 [[[a]]]$.
Then $u = [[[a,b,c],\varepsilon_1,[g,e]],\ \varepsilon_2,\ [[a]]]$ Notice $u \in \Sigma^{<3>}$ but $u \notin \Sigma^{[3]}$.
Let $v = [[[a,b,c,d],[g,e]],[[a]]]$. The fact that the two components of v are the level-2 words: $[[a,b,c,d],\ [g,e]]$ and $[[a]]$, which are both in $\Sigma^{[2]}$, ensures that $v \in \Sigma^{[3]}$.

Starting with a structure having just Σ as domain, and iterating k times the "tree-structure" operation, we obtain a structure that we name $\mathcal{S}^k_\Sigma$.

Definition 29. Let Σ be a finite alphabet. We define inductively a structure $\mathcal{S}^k_\Sigma$, with domain $\Sigma^{<k>}$ and signature Sig^k as follows:

$$\Sigma^{<0>} = \Sigma; \quad Sig^0 = \{(r_a^0)_{a\in\Sigma}\}; \quad r_a^0 = \{a\}.$$

$$\Sigma^{<k+1>} = (\Sigma^{<k>})^*; \quad Sig^{k+1} = \{(r_a^{k+1})_{a\in\Sigma}\} \cup \{(son_{k+1,i})_{1\leq i\leq k+1}\} \cup \{(clone_{k+1,i})_{1\leq i\leq k+1}\}$$

where, for every $k \in \mathbb{N}$, $i \in [1,k]$

$$\begin{aligned} r_a^{k+1} &= \{[u]\bullet v | u \in \Sigma^{<k>},\ v \in \Sigma^{<k+1>},\ r_a^k(u)\} \\ clone_{k+1,1} &= \{[u]\bullet[u]\bullet v|\ u \in \Sigma^{<k>},\ v \in \Sigma^{<k+1>}\} \\ clone_{k+1,i+1} &= \{[u]\bullet w|\ u \in \Sigma^{<k>},\ v \in \Sigma^{<k+1>}, clone_{k,i}(u)\}. \end{aligned}$$

We will often abbreviate $son_{k,1}$ by son_k or son if the level is understood (idem for $clone_{k,1}$). The following theorem follows immediately from Theorem 25.

Theorem 30. *For every integer $k \geq 0$ and finite alphabet Σ, the structure $\mathcal{S}^k_\Sigma$ has a decidable MSO-theory.*

3.2. k-Pushdowns viewed as k-words

The computations of a $k\text{-}pda$ are naturally expressed in the following structure $\mathcal{P}^k_{\Gamma,n}$.

Definition 31. Let Γ be a finite alphabet and k, n two natural integers. We define the structure $\mathcal{P}^k_{\Gamma,n}$, by:

$$\mathcal{P}^k_{\Gamma,n} = \left\langle k - \mathrm{pds}(\Gamma), (\mathrm{topsyms}_{\bar{A}})_{\bar{A}\in\Gamma^{(k)}}, (\mathrm{pop}_i)_{1\leq i\leq k}, (\mathrm{push}_i(\gamma))_{\substack{1\leq i\leq k\\ \gamma\in\Gamma^{(n)}}}\right\rangle.$$

This structure consists of the set of $k-$ pds over Γ, endowed with all the operations which are used in the definition a k-pda. Here the one-place predicate topsyms$_{\bar{A}}$ corresponds to the set of k-pushdowns with topsymbols $\bar{A}$ and the two-place predicates pop$_i$, push$_i(\gamma)$ are the graphs of the corresponding operations over pushdowns. The integer n stands as an upper bound on the length of the words which are used in the push-operations. Its most usual value is $n = 2$.

This subsection is devoted to the proof of the following

Theorem 32. *For every finite alphabet Γ and integers k, n, there exists a finite alphabet Γ' and an MSO-interpretation*

$$\varphi_k : \mathcal{P}^k_{\Gamma,n} \to \mathcal{S}^k_{\Gamma'}.$$

In other words: the structure of k-pushdowns can be MSO-interpreted into the structure of k-words.

Let us introduce a new alphabet:

$$\Gamma_k = \Gamma \cup (\Gamma \times \Gamma) \cup \cdots \cup \Gamma^k$$

and also some auxiliary predicates over the domain $\Gamma_k^{<k>}$:

$$S_{k,\bar{A}}(u, v), Eps_k(u), Change_{k,\bar{A}}(u, v), Chgleft_{k,A}(u, v), Eq_{k,A}(u), Add_A(u, v)$$

where $A \in \Gamma$, $\bar{A} \in \Gamma_k$ with the following meanings

- $S_{k,\bar{A}}(u, v) \Leftrightarrow r^k_{\bar{A}}(v)$ and $son_{k,k}(u, v)$.
- $Eps_k(u) \Leftrightarrow u = \varepsilon_k$.
- $Change_{k,\bar{A}}(u, v) \Leftrightarrow$ ($u \neq \varepsilon_k$ and v is obtained from u by changing its leftmost symbol into the symbol $\bar{A} \in \Gamma_k$).
- $Chgleft_{k,A}(u, v) \Leftrightarrow$ ($u \neq \varepsilon_k$ and v is obtained from u by changing the leftmost component of its leftmost symbol into the component $A \in \Gamma$).
- $Eq_{k,A}(u) \Leftrightarrow u = [A]_k$, where $[A]_1 = [A]$ and $[A]_{i+1} = [[A]_i] \in \Gamma_k^{<i+1>}$.
- $Add_{k,A}(u, v) \Leftrightarrow v$ is obtained from u by changing its leftmost symbol $\bar{B} \in \Gamma_{k-1}$ into the symbol $(A, \bar{B}) \in \Gamma_k$.

We define, for every $k \geq 1$, an injective monoïd homomorphism

$$\varphi_k : (k - \text{pds}(\Gamma), \cdot) \to (\Gamma_k^{[k]}, \bullet)$$

by: for every $A \in \Gamma$,

$$\varphi_1(A[\varepsilon]) = [A] \tag{5}$$

and for every $k \geq 1$, $A \in \Gamma$, $f \in k - \text{pds}(\Gamma)$:

$$Add_A([\varphi_k(f)], \varphi_{k+1}(A[f])). \tag{6}$$

In other words, $\varphi_{k+1}(A[f])$ is obtained from $\varphi_k(f)$ by putting it into brackets and adding the component A to its leftmost letter. (See the general schema on Fig. 5).

Example 33. Let $\Gamma = \{A, B, C, D, E, F, G\}$, and

$$\begin{aligned}
u &= [[[A, (B, C), C, D], [G, E]], [[(A, C)]]] \\
v &= [[[(B, A), (B, C), C, D], [G, E]], [[(A, C)]]] \\
w &= [[[(C, B, A), (B, C), C, D], [G, E]], [[(A, C)]]] \\
x &= [[[(D, B, A), (B, C), C, D], [G, E]], [[(A, C)]]].
\end{aligned}$$

Then the following predicates hold:
$Change_{(B,A)}(u, v), Add_C(v, w), Chgleft_D(w, x)$.
$\varphi_2(CD[EG]) = [[C], [(D, E), G]]$ and $\varphi_2(D[E]) = [[(D, E)]]$
$\varphi_3(AB[CD[EG]]F[D[E]]) = [[[A]], [[(B, C)], [(D, E), G]], [[(F, D, E)]]]$.

(See a planar representation on Fig. 6.)

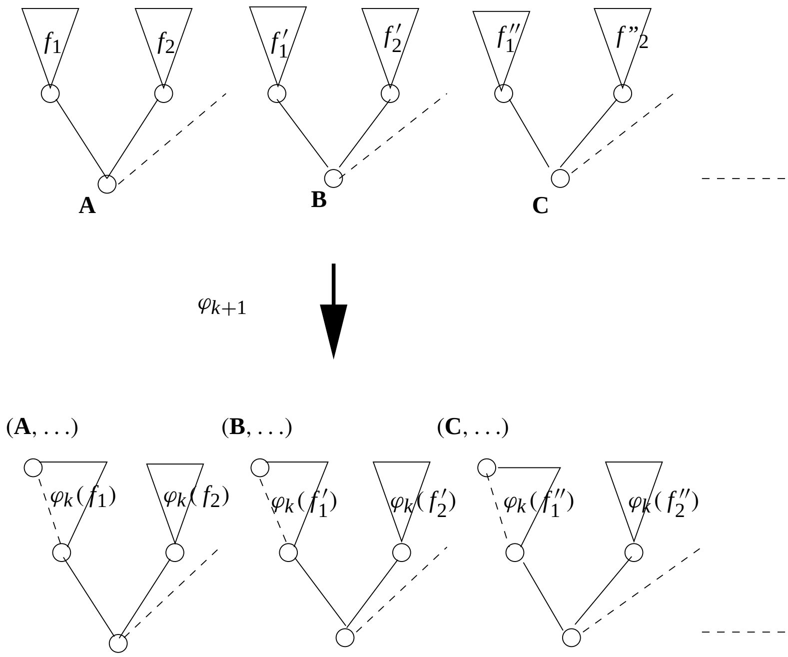

Fig. 5. The map φ_{k+1}.

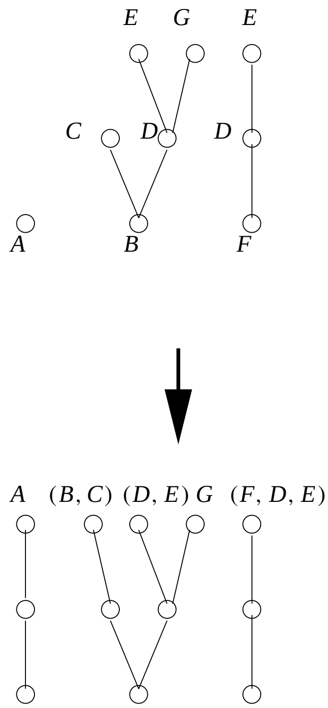

Fig. 6. The map φ_3.

Lemma 34. *The predicates $S_{k,\bar{A}}(u, v)$, $Eps_k(u)$, $Change_{k,\bar{A}}(u, v)$, $Chgleft_{k,A}(u, v)$, $Eq_{k,A}(u)$, $Add_A(u, v)$ are MSO-definable in the structure $\mathcal{S}^k_{\Gamma_k}$.*

Proof.

(1) $S_{k,\bar{A}}(u, v)$ is expressed by: $son_{k,k}(u, v) \wedge r^k_{\bar{A}}(v)$

(2) $Eps_k(u)$ is expressed by: $\neg(\exists x.son_{k,1}(x,u))$
(3) $Change_{k,\bar{A}}(u,v)$ is expressed by: $\exists x.son_{k,k}(x,u) \wedge S^k_{\bar{A}}(x,v)$
(4) $Chgleft_{k,A}(u,v)$ is expressed by:
$\exists x. \bigvee_{\substack{B\in\Gamma \\ \bar{C}\in\Gamma_{k-1}}} S_{k,(B,\bar{C})}(x,u) \wedge S_{k,(A,\bar{C})}(x,v) \bigvee_{B\in\Gamma} S_{k,B}(x,u) \wedge S_{k,A}(x,v)$
(5) $Eq_{1,A}(u)$ is expressed by: $\exists x.Eps_1(x) \wedge S_{1,A}(x,u)$
$Eq_{k+1,A}(u)$ is expressed by: $\exists x.Eps_{k+1}(x) \wedge son(x,u) \wedge Eq^*_{k,A}(u)$
(6) $Add_{k,A}(u,v)$ is expressed by:

$$(Eps_k(u) \wedge Eq_{k,A}(v)) \vee \left(\neg Eps_k(u) \wedge \exists x. \bigvee_{\bar{B}\in\Gamma_{k-1}} S^k_{\bar{B}}(x,u) \wedge S^k_{(A,\bar{B})}(x,v)\right). \quad \square$$

We can now define formulas encoding operations on $k - pds$.

Lemma 35. *For all $\bar{A} \in \Gamma_k \cup \{\varepsilon\}$, $n \geq 0$, one can find an MSO-formula $\theta_{\bar{A}}(u)$ fulfilling for all $\omega \in k - \mathrm{pds}(\Gamma)$:*

$$\mathcal{P}^k_{\Gamma,n} \models \mathrm{topsyms}_{\bar{A}}(\omega) \text{ iff } \mathcal{S}^k_{\Gamma_k} \models \theta_{\bar{A}}(\varphi_k(\omega)).$$

Proof. The formulas $\theta_\varepsilon(u) := Eps_k(u)$ and $\theta_{\bar{A}}(u) := r^k_{\bar{A}}(u)$ (for $\bar{A} \in \Gamma_k$) fulfill the required property. $\square$

Lemma 36. *For all $1 \leq i \leq k$, $n \geq 0$, one can find an MSO-formula $\Pi_{k,i}$ verifying for all $\omega, \omega' \in k - \mathrm{pds}(\Gamma)$:*

$$\mathcal{P}^k_{\Gamma,n} \models \mathrm{pop}_i(\omega,\omega') \text{ iff } \mathcal{S}^k_{\Gamma_k} \models \Pi_{k,i}(\varphi_k(\omega),\varphi_k(\omega')).$$

Construction.

- For $k \geq 1$: $\Pi_{k,1}(u,v) := son(v,u)$
- For $1 \leq i \leq k$: $\Pi_{k+1,i+1}(u,v) := \exists x, y.\ \Pi^*_{k,i}(x,y) \wedge \bigvee_{A\in\Gamma} [Add_A(x,u) \wedge Add_A(y,v)]$.

Proof. We prove by induction on $i \geq 1$, that the property is true for all $k \geq i$.
Basis: if $i = 1$, then $\forall\omega,\ \omega' \in k - pds(\Gamma)$:

$$\mathcal{P}^k_{\Gamma,n} \models \mathrm{pop}_1(\omega,\omega')$$

iff there exists $u_1 \in \Gamma^{[k-1]}$ such that

$$\varphi_k(\omega) = [u_1] \bullet \varphi_k(\omega')$$

iff

$$\mathcal{S}^k_{\Gamma_k} \models \Pi^k_1(\varphi_k(\omega),\varphi_k(\omega')).$$

Induction step: let us assume the property is true for $i \geq 1$.
For all $k \geq i, \omega,\ \omega' \in (k+1) - \mathrm{pds}(\Gamma)$:

$$\mathcal{P}^{k+1}_{\Gamma,n} \models \mathrm{pop}_{i+1}(\omega,\omega')$$

iff $\exists A \in \Gamma,\ f,\ f' \in k - \mathrm{pds}(\Gamma), r \in (k+1) - \mathrm{pds}(\Gamma)$.

$$\omega = A[f]r, \omega' = A[f']r, \mathrm{pop}_i(f,f')$$

iff $\exists A \in \Gamma,\ f,\ f' \in k - \mathrm{pds}(\Gamma), r \in (k+1) - \mathrm{pds}(\Gamma), x, y \in \Gamma^{<k+1>}_{k+1}$.

$$x = [\varphi_k(f)]\varphi_{k+1}(r) \wedge y = [\varphi_k(f')]\varphi_{k+1}(r) \wedge \mathrm{pop}_i(f,f') \wedge Add_A(x,\varphi_{k+1}(\omega)) \wedge Add_A(y,\varphi_{k+1}(\omega'))$$

iff $\exists A \in \Gamma, x, y \in \Gamma^{<k+1>}_{k+1}$.

$$\Pi^*_{k,i}(x,y) \wedge Add_A(x,\varphi_{k+1}(\omega)) \wedge Add_A(y,\varphi_{k+1}(\omega'))$$

(these conditions are sufficient because, as x, y are the leftmost atoms of words in $\mathrm{Im}\varphi_{k+1}$, and as they are related by $\Pi^*_{k,i}$, by the induction hypothesis, some "flag" and "rest" f, r fulfilling the property just above must exist)
iff

$$\mathcal{S}^{k+1}_{\Gamma_{k+1}} \models \Pi_{k+1,i+1}(\varphi_{k+1}(\omega), \varphi_{k+1}(\omega'))$$

(by the inductive definition of $\Pi_{k+1,i+1}$). $\square$

Lemma 37. *For all* $1 \leq i \leq k, \gamma \in \Gamma^+, n \geq |\gamma|$, *one can find an MSO-formula* $\Psi_{k,i}(\gamma)$ *verifying for all* $\omega, \omega' \in k - \mathrm{pds}(\Gamma)$*:*

$$\mathcal{P}^k_{\Gamma,n} \models \mathrm{push}_i(\gamma)(\omega, \omega') \text{ iff } \mathcal{S}^k_{\Gamma_k} \models \Psi_{k,i}(\omega)(\varphi_k(\omega), \varphi_k(\omega')).$$

Construction.

- $\forall k \geq 1, A \in \Gamma, \Psi_{k,1}(A)(u, v) :=$

$$\text{if } Eps_k(u) \text{ then } Eq_{k,A}(v) \text{ else } Chgleft_{k,A}(u, v)$$

- $\forall k \geq 1, A \in \Gamma, \gamma \in \Gamma^+, \Psi_{k,1}(A \cdot \gamma)(u, v) :=$

$$\exists x, y. \Psi_{k,1}(\gamma)(u, x) \wedge son(x, y) \wedge clone(y) \wedge Chgleft_{k,A}(y, v)$$

- $\forall 1 \leq i \leq k, \gamma \in \Gamma^+,$

$$\Psi_{k+1,i+1}(\gamma)(u, v) := \exists x, y. \Psi_{k,i}(\gamma)^*(x, y) \wedge \bigvee_{A \in \Gamma} [Add_A(x, u) \wedge Add_A(y, v)].$$

Proof. We prove the lemma by induction on $(i, k, |\gamma|)$.
Basis: $i = 1, k = 1, |\gamma| = 1$.
This case is obvious.
Induction step 1: $i = 1, k \to k + 1, |\gamma| = 1$.
Let $\gamma = A \in \Gamma, \omega, \omega' \in (k + 1) - \mathrm{pds}(\Gamma), k \geq 1$.
Case 1.1: $\omega = \varepsilon$.

$$\mathcal{P}^{k+1}_{\Gamma,1} \models \mathrm{push}_1(\omega, \omega')$$

iff

$$\omega = \varepsilon, \quad \omega' = A[\varepsilon]$$

iff

$$\varphi_{k+1}(\omega) = \varepsilon_{k+1}, \quad \varphi_{k+1}(\omega') = [A]_{k+1}$$

iff

$$\mathcal{S}^{k+1}_{\Gamma_{k+1}} \models \Psi_{k+1,1}(A)(\varphi_{k+1}(\omega), \varphi_{k+1}(\omega'))$$

(because of the "if" part of formula $\Psi_{k+1,1}$).
Case 1.2: $\omega \neq \varepsilon$.

$$\mathcal{P}^{k+1}_{\Gamma,1} \models \mathrm{push}_1(\omega, \omega')$$

iff $\exists B \in \Gamma, f \in k - \mathrm{pds}(\Gamma), r \in (k + 1) - \mathrm{pds}(\Gamma)$,

$$\omega = B[f]r, \omega' = A[f]r$$

iff $\exists B \in \Gamma, f \in k - \mathrm{pds}(\Gamma), r \in (k + 1) - \mathrm{pds}(\Gamma), x, x' \in \Gamma^{<k+1>}_{k+1}$,

$$Add_B([\varphi_k(f)], x), Add_A([\varphi_k(f)], x'), \varphi_{k+1}(\omega) = x\varphi_{k+1}(r), \omega' = x'\varphi_{k+1}(r)$$

iff

$$Chgleft_{k,A}(\varphi_{k+1}(\omega), \varphi_{k+1}(\omega'),$$

iff

$$\mathcal{S}^{k+1}_{\Gamma_{k+1}} \models \Psi_{k+1,1}(A)(\varphi_{k+1}(\omega), \varphi_{k+1}(\omega'))$$

(because of the "else" part of formula $\Psi_{k+1,1}$).
Induction step 2: $i = 1, k \geq 1, |\gamma| = n + 1, n \geq 1$.
Let $\gamma = A \cdot \gamma'$.

$$\mathcal{P}^{k}_{\Gamma,n+1} \models \mathrm{push}_i(\gamma)(\omega, \omega')$$

iff $\exists B \in \Gamma,\ f \in (k-1) - \mathrm{pds}(\Gamma), r \in k - \mathrm{pds}(\Gamma)$

$$\mathrm{push}_1(\gamma)(\omega, \omega_n) \wedge \omega_n = B[f]r \wedge \omega' = A[f]B[f]r$$

iff $\exists B \in \Gamma,\ f \in (k-1) - \mathrm{pds}(\Gamma), r, \omega_n \in k - \mathrm{pds}(\Gamma), x, y \in \Gamma^{[k]}$

$$x = \varphi_k(B[f]) \bullet \varphi_k(r) \wedge y = \varphi_k(B[f]) \bullet \varphi_k(B[f]) \bullet \varphi_k(r) \wedge$$
$$\mathrm{push}_1(\gamma)(\omega, \omega_n) \wedge \varphi_k(\omega_n) = x \wedge \varphi_k(\omega') = \varphi_k(A[f]) \bullet x$$

iff $\exists f \in (k-1) - \mathrm{pds}(\Gamma), r \in k - \mathrm{pds}(\Gamma), x, y \in \Gamma^{[k]}$

$$\Psi_{k,1}(\gamma)(\varphi_k(\omega), x) \wedge son(x, y) \wedge clone(y) \wedge Chgleft_{k,A}(y, \varphi_k(\omega'))$$

iff

$$\mathcal{S}^{k}_{\Gamma_k} \models \Psi_{k,i}(\omega)(\varphi_k(\omega), \varphi_k(\omega')).$$

Induction step 3: $i \to i + 1, k \to k + 1, |\gamma| = n$ (for $i \geq 1, k \geq 1, n \geq 1$).
Similar to the proof of the previous lemma. □

Lemma 38. *The set $\varphi_k(k - \mathrm{pds}(\Gamma))$ is MSO-definable in $\mathcal{S}^k_{\Gamma_k}$.*

Proof. For $k = 1$, we just have: $\varphi_1(1 - \mathrm{pds}(\Gamma)) = \Gamma_1^{<1>}$. Hence the lemma is true. For $k \geq 2$, $\varphi_k(k - \mathrm{pds}(\Gamma))$ is the smallest set X such that: $\forall v \in \Gamma_k^{<k>}, v \in X$ iff

$$\exists z \in \Gamma^{<k-1>}, y, t, u \in \Gamma^{<k>}.u\ = \epsilon_k \vee \left(y \in X \wedge z \in \varphi_{k-1}((k-1) - \mathrm{pds}(\Gamma)) \wedge t = [z] \wedge \bigvee_{A \in \Gamma} Add_A(t, u) \wedge v = u \bullet y \right).$$

One can thus construct, inductively, an MSO-formula I_k such that, for every $v \in \Gamma_k^{<k>}$,

$$v \in \varphi_k(k - \mathrm{pds}(\Gamma)) \Leftrightarrow \mathcal{S}^k_{\Gamma_k} \models I_k(v). \quad \square$$

By means of Lemmas 35–38, the map φ_k meets all the conditions of Definition 1. This achieves the proof of Theorem 32.

3.3. Computation graph

We show here that the structure induced by the *computation graph* of a given k-pda working on a pushdown alphabet Γ is MSO-interpretable in the structure $\mathcal{P}^k_{\Gamma,n}$. It follows, using the results of previous subsection, that such a computation-graph has always a decidable MSO-theory.

We define below the structure $\mathcal{C}(\mathcal{A})$ (resp. $\mathcal{C}_0(\mathcal{A})$) induced by the computation-graph (resp. the rooted computation-graph) of the automaton $\mathcal{A}$.

Definition 39. Let $\mathcal{A}$ be some k-pda ($k \geq 1$) with terminal alphabet Σ and pushdown alphabet Γ. We define the structures:

$$\mathcal{C}(\mathcal{A}) := \langle V,\ (R_{\bar{a}})_{\bar{a}\in\Sigma\cup\{\varepsilon\}}\rangle$$

with

$$V := Q \times k - \mathrm{pds}(\Gamma), \quad R_{\bar{a}} := \{((p,\omega),(p',\omega')) \in V \times V |\ (p,\bar{a},\omega) \vdash_{\mathcal{A}} (p',\varepsilon,\omega')\}$$

and

$$\mathcal{C}_0(\mathcal{A}) := \langle V_0, (R_{0,\bar{a}})_{\bar{a}\in\Sigma\cup\{\varepsilon\}}, I_0, T_0\rangle$$

with

$$V_0 := \{(p,\omega) \in Q \times k - \mathrm{pds}(\Gamma) |\ (p,\omega) \text{ is accessible from } (q_0, Z)\}, \quad R_{0,\bar{a}} := R_{\bar{a}} \cap V_0 \times V_0,$$
$$I_0 := \{(q_0, Z)\}, \quad T_0 := \{(p,\varepsilon_k) \mid p \in Q\}.$$

Theorem 40. *Both structures* $\mathcal{C}_0(\mathcal{A}), \mathcal{C}(\mathcal{A})$ *have a decidable* MSO-*theory.*

Proof. Let us consider the direct product of the structure $\mathcal{P}^k_\Gamma$ by the finite structure $\mathcal{Q} = \langle Q, (E_q)_{q\in Q}\rangle$, where $E_{q,\mathcal{Q}} = \{q\}$. By Theorems 32, 30 and 2, the structure $\mathcal{P}^k_\Gamma$ has a decidable MSO theory. Applying Lemma 4, the structure $\mathcal{Q} \times \mathcal{P}^k_\Gamma$ has also a decidable MSO theory. As each predicate $R_{\bar{a}}$ is clearly MSO-definable in $\mathcal{Q} \times \mathcal{P}^k_\Gamma$, it follows that the identity map is an MSO-interpretation of the structure $\mathcal{C}(\mathcal{A})$ in the structure $\mathcal{Q} \times \mathcal{P}^k_\Gamma$. The initial total state $s_0 = (q_0, Z)$ and the terminal total states (p, ε_k) are MSO-definable in $\mathcal{Q} \times \mathcal{P}^k_\Gamma$. Therefore, the set $V_0 = \{s \in V | (s_0, s) \in (\bigcup_{\bar{a}\in\Sigma\cup\{\varepsilon\}} R_{\bar{a}})^*\}$ is also MSO-definable (because the transitive closure of a definable binary predicate is also a definable binary predicate). Finally, both structures $\mathcal{C}_0(\mathcal{A}), \mathcal{C}(\mathcal{A})$ are MSO-interpretable inside $\mathcal{Q} \times \mathcal{P}^k_\Gamma$, which ensures that they have decidable MSO-theory. $\square$

Corollary 41 (*Language Problems*). *Let* $k \geq 1$. *Then:*

1. *It is decidable, given a k-pda* $\mathcal{A}$ *over the terminal alphabet* Σ *and a word* $u \in \Sigma^*$ *whether* $u \in \mathrm{L}(\mathcal{A})$.
2. *It is decidable, given a k-pda* $\mathcal{A}$*, whether* $\mathrm{L}(\mathcal{A}) = \emptyset$.
3. *It is decidable, given a deterministic k-pda* $\mathcal{A}$*, whether* $\mathrm{L}(\mathcal{A})$ *is finite.*

Proof. One can easily check that problem 1 (resp. 2) reduces to the validity of some MSO-formula over the structure $\mathcal{C}_0(\mathcal{A})$. Given a deterministic k-pda $\mathcal{A}$, the language $\mathrm{L}(\mathcal{A})$ is infinite iff $\mathcal{C}_0(\mathcal{A})$ admits:

- either a loop with at least one edge labelled by some $\sigma \in \Sigma$, and whose every vertex c_n is co-accessible from the set T_0.
- or an infinite path $(c_0, \bar{a}_1, c_1)\cdots(c_n, \bar{a}_{n+1}, c_{n+1})\cdots$, such that, for infinitely many integers n, $\bar{a}_n \neq \varepsilon$, and every vertex c_n is co-accessible from the set T_0.
  Hence finiteness of $\mathrm{L}(\mathcal{A})$ is expressible in MSO. $\square$

Point (2) of Corollary 41 is stated in [21, p. 12, lines 32–33, crediting Aho–Ullman], in [26, p. 1171, line 28] and fully proved in [13, Theorems 7.8 and 7.17]. In [16, Theorem 7.12 p. 71], the precise complexity of problem 2 is determined: the emptiness problem for k-pda is DTIME($exp_{k-1}(\mathrm{O}(n^2))$)-complete. Point (1) follows easily from point (2) and from the effective closure of k-level languages under intersection with regular languages. Point (3) is established even for non-deterministic automata, but for levels $k \leq 2$ only, in [4, Corollary 5.1], by means of an iteration lemma.

4. Integer sequences

4.1. Sequences defined by automata

We define here a class of *integer sequences* by means of k-pushdown automata. Specifically, we use a slightly restrictive class of k-pdas, the *counter k-pda*. These are an extension of the classical *counter pda* which recognize some words with a memory consisting of natural integers only. We show that the class of integer sequences thus defined is closed under many natural operations (Theorem 72).

Definition 42 (*Counter k-Pushdown Store*). Let Γ be an alphabet with a distinguished symbol $F \in \Gamma$. The set of *k-counter* pushdown stores over Γ, with counter F, is denoted by $k - \mathrm{cpds}(\Gamma)$ and defined by:

$$1 - \mathrm{cpds}(\Gamma) = (F[\varepsilon])^* \quad (k+1) - \mathrm{cpds}(\Gamma) = ((\Gamma - \{F\}) \cdot [k - \mathrm{cpds}(\Gamma)])^*.$$

In other words, the symbol F can appear at level k only and no other symbol can occur at level k.

Definition 43 (*Counter k-pdas*). A k-pda $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, Z)$ is said to be a *counter k-pda*, with counter F, if $\mathcal{A}$ is a k-pda over a pushdown alphabet $\Gamma \supseteq \{F\}$, such that the set of counter pushdown stores over Γ is closed under the computation relation i.e. for every $q, q' \in Q, \omega, \omega' \in k - \mathrm{pds}(\Gamma), u, u' \in X^*$, if $\omega \in k - \mathrm{cpds}(\Gamma)$ and $(q, u, \omega) \vdash_{\mathcal{A}} (q', u', \omega')$ then $\omega' \in k - \mathrm{cpds}(\Gamma)$.

In the rest of the paper we abbreviate "deterministic counter k-pushdown automaton" by k-dcpda.

Example 44. Here is a 3-cpds :

$$A[B[FF]C[F]]B[E[FFF]].$$

Definition 45 (*k-Computable Sequences*). A sequence of natural integers f is called a *k-computable sequence* iff there exists a *k-dcpda* $\mathcal{A}$, over a pushdown alphabet Γ containing at least k different symbols $A_1, A_2, \ldots, A_{k-1}, F$, with counter F, such that, for all $n \geq 0$:

$$(q_0, a^{f(n)}, A_1[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \varepsilon).$$

One denotes by $\mathbb{S}_k$ the set of all k-computable sequences of natural integers.

We show in next lemmas that from any counter automaton $\mathcal{A}$ *computing* a sequence $n \mapsto f(n)$, in the sense of Definition 45, one can derive a non-deterministic k-cpda accepting the language $L = \{a^{f(n)}, n \geq 0\}$ and a deterministic k-cpda recognizing the single infinite word $\prod_{n \geq 0}(a^{f(n)}b)$.

Lemma 46. *For every level-k sequence f, one can construct a non-deterministic k-cpda $\mathcal{A}'$ such that $L(\mathcal{A}') = \{a^{f(n)}, n \geq 0\}$.*

Proof. Let us suppose that $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ is some k-dcpda such that:

$$(q_0, a^{f(n)}, A_1[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \varepsilon).$$

Let us set $\mathcal{A}' = (Q, \{a\}, \Gamma \cup \{Z'\}, \delta \cup \delta', q_0, Z')$ with:
$\delta'(q_0, \varepsilon, Z') = \{(q_0, \mathrm{push}_2(A_2))\}$
for all $2 \leq i \leq k-2$ $\delta'(q_0, \varepsilon, Z'A_2 \ldots A_i) = \{(q_0, \mathrm{push}_{i+1}(A_{i+1}))\}$
$\delta'(q_0, \varepsilon, Z'A_2 \ldots A_{k-1}) = \{(q_0, \mathrm{push}_k(F)), (q_0, \mathrm{push}_1(A_1))\}$
$\delta'(q_0, \varepsilon, Z'A_2 \ldots A_{k-1}F) = \{(q_0, \mathrm{push}_k(FF)), (q_0, \mathrm{push}_1(A_1))\}$.

Then, each accepting computation has the form:
$(q_0, a^{f(n)}, Z') \vdash^*_{\mathcal{A}'} (q_0, a^{f(n)}, Z'[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \vdash_{\mathcal{A}'}$
$(q_0, a^{f(n)}, A_1[\ldots [A_{k-1}[F^n]] \ldots]) \vdash^*_{\mathcal{A}'} (q_0, \varepsilon, \varepsilon)$. □

Lemma 47. *For every level-k sequence f, one can construct a k-dcpda $\mathcal{A}' = (Q', \{a, b\}, \Gamma', \delta', q_0, Z')$ and there exists a sequence $(\omega_n)_{n \in \mathbb{N}}$ of elements of $k - \mathrm{pds}(\Gamma')$ such that: for every $n \geq 0$*

$$\omega_0 = Z' \text{ and } (q_0, a^{f(n)}b, \omega_n) \vdash^*_{\mathcal{A}'} (q_0, \varepsilon, \omega_{n+1}).$$

Proof. Let us suppose that $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ is some k-dcpda such that:

$$\forall n \in \mathbb{N}, (q_0, a^{f(n)}, A_1[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \varepsilon)$$

We define $\mathcal{A}' = (Q \cup \{q_c\}, \{a\}, \Gamma \cup \{Z', C\}, \delta \cup \delta', q_0, Z')$ with:
$\delta'(q_0, \varepsilon, Z') = (q_0, \mathrm{push}_2(A_2))$
for all $2 \leq i \leq k-2$, $\delta'(q_0, \varepsilon, Z'A_2 \ldots A_i) = (q_0, \mathrm{push}_{i+1}(A_{i+1}))$
$\delta'(q_0, \varepsilon, Z'A_2 \ldots A_{k-1}) = (q_0, \mathrm{push}_1(A_1C))$
$\delta'(q_0, b, CA_2 \ldots A_{k-1}) = (q_c, \mathrm{push}_k(F))$

$\delta'(q_0, b, CA_2 \dots A_{k-1}F) = (q_c, \text{push}_k(FF))$
$\delta'(q_c, \varepsilon, CA_2 \dots A_{k-1}F) = (q_0, \text{push}_1(A_1C))$.

One can check, by induction on n, that the $k-$pds

$$\omega_n = A_1[A_2 \dots [A_{k-1}[F^n]] \dots] \cdot C[A_2 \dots [A_{k-1}[F^n]] \dots]$$

for $n \geq 1$, has the required property. □

4.2. Some k-computable sequences

We show in this section that $\mathbb{N}$-rational sequences are 2-computable and sequences that are the solution of a system of polynomial recurrence equations with integer coefficients are 3-computable.

Definition 48 (*$\mathbb{N}$-Rational Sequences*). A sequence $(u_n)_{n\geq 0}$ is $\mathbb{N}$-rational iff there is a matrix M in $\mathbb{N}^{d\times d}$ and two vectors L in $\mathbb{B}^{1\times d}$ and C in $\mathbb{B}^{d\times 1}$ such that $u_n = L \cdot M^n \cdot C$.

Example 49. Let $(u_n)_{n\geq 0}$ be the sequence having the following representation of dimension 2:

$$L = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We set

$$M^n = \begin{pmatrix} u_{1,1}(n) & u_{1,2}(n) \\ u_{2,1}(n) & u_{2,2}(n) \end{pmatrix} \quad \text{then } M^{n+1} = \begin{pmatrix} u_{1,1}(n) + u_{1,2}(n) & 2.u_{1,1}(n) \\ u_{2,1}(n) + u_{2,2}(n) & 2.u_{2,1}(n) \end{pmatrix}$$

and $(u_n)_{n\leq 0}$ is the sequence defined by:

$$u_n = u_{1,1}(n) + u_{1,2}(n)$$

with:

$$\begin{array}{llll} u_{1,1}(0) = 1 & u_{1,1}(n+1) = u_{1,1}(n) + u_{1,2}(n) & u_{1,2}(0) = 0 & u_{1,2}(n+1) = 2.u_{1,1}(n) \\ u_{2,1}(0) = 0 & u_{1,1}(n+1) = u_{2,1}(n) + u_{2,2}(n) & u_{2,2}(0) = 1 & u_{1,2}(n+1) = 2.u_{2,1}(n). \end{array}$$

Proposition 50. *If $(u_n)_{n\geq 0}$ is a $\mathbb{N}$-rational sequence, then $(u_n)_{n\geq 0} \in \mathbb{S}_2$.*

Let us assume u is defined from $L \in \mathbb{B}^{1\times d}$, $M \in \mathbb{N}^{d\times d}$, $C \in \mathbb{B}^{d\times 1}$.

Construction:

Let $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ with: $Q = \{q_0\} \cup \{q_{i,j}, 1 \leq i, j \leq d\}$, $\Gamma = \{F\} \cup \{U_{i,j}, 1 \leq i, j \leq d\}$, and the transition function δ is defined by:

1. $\delta(q_0, a, U_{i,i}) = (q_0, \text{pop}_1)$ and if $i \neq j$, $\delta(q_0, \varepsilon, U_{i,j}) = (q_0, \text{pop}_1)$,
2. $\delta(q_0, \varepsilon, U_{i,j}F) = (q_{i,j}, \text{pop}_2)$,
3. $\delta(q_{i,j}, \varepsilon, U_{i,j}F) = \delta(q_{i,j}, \varepsilon, U_{i,j}) = \left(q_0, \text{push}_1\left(\prod_{1\leq l\leq d} U_{i,l}^{m_{l,j}}\right)\right)$
4. $\delta(q_0, \varepsilon, UF) = \delta(q_0, \varepsilon, U) = \left(q_0, \text{push}_1\left(\prod_{1\leq j\leq d}\left(\prod_{1\leq i\leq d} U_{i,j}^{l_i}\right)^{c_j}\right)\right)$.

Proof. Let us consider the sequences $u_{i,j}$ defined by the following recurrence relations:

$$u_{i,j}(n+1) = \sum_{1\leq l\leq d} u_{i,l}(n).m_{l,j},$$
$$u_{i,i}(0) = 1, \quad \text{and } u_{i,j}(0) = 0 \quad \text{for } i \neq j.$$

Let us show by induction on $n \geq 0$ the following auxiliary property $P(n)$:
$\forall 1 \leq i, j \leq d$,

$$(q_0 U_{i,j}[F^n]q_0) \to^*_{\mathcal{A}} a^{u_{i,j}(n)}. \tag{7}$$

Basis: $n = 0$. Transitions (1) ensure $P(0)$ is true.
Induction step: Let $n \geq 0$ and let us assume $P(n)$.
By a transition (2) followed by a transition (3) and by a decomposition rule:

$$\begin{aligned}(q_0 U_{i,j}[F^{n+1}]q_0) &\to_{\mathcal{A}} (q_{i,j} U_{i,j}[F^n]q_0)\\ &\to_{\mathcal{A}} \left(q_0 \prod_{1\leq l\leq d} (U_{i,l}[F^n])^{m_{l,j}} q_0\right)\\ &\to^*_{\mathcal{A}} \prod_{1\leq l\leq d} (q_0 U_{i,l}[F^n]q_0)^{m_{l,j}}.\end{aligned}$$

By the induction hypothesis:

$$(q_0 U_{i,l}[F^n]q_0) \to^*_{\mathcal{A}} a^{u_{i,l}(n)}.$$

Composing the above derivations we obtain:

$$(q_0 U_{i,j}[F^{n+1}]q_0) \to^*_{\mathcal{A}} \prod_{1\leq l\leq d} (a^{u_{i,l}(n)})^{m_{l,j}} = a^{u_{i,j}(n+1)}.$$

Hence $P(n+1)$ is proved.

Property (7) is thus established.

Let us examine now the sequence u_n. For every $n \geq 0$:

$$u(n) = \sum_{1\leq i\leq d} \sum_{1\leq j\leq d} \ell_i \cdot u_{i,j}(n) \cdot c_j. \tag{8}$$

Applying transition (4) of $\mathcal{A}$ followed by decompositions, we see that:

$$\begin{aligned}(q_0 U[F^n]q_0) &\to^*_{\mathcal{A}} \left(q_0 \left(\prod_{1\leq j\leq d}\left(\prod_{1\leq i\leq d} U_{i,j}[F^n]^{l_i}\right)^{c_j}\right) q_0\right)\\ &\to^*_{\mathcal{A}} \prod_{1\leq j\leq d}\left(\prod_{1\leq i\leq d} (q_0 U_{i,j}[F^n]q_0)^{l_i\cdot c_j}\right)\end{aligned} \tag{9}$$

and from $P(n)$ we deduce that

$$\prod_{1\leq i\leq d} (q_0 U_{i,j}[F^n]q_0)^{l_i\cdot c_j} \to^*_{\mathcal{A}} \prod_{1\leq i\leq d} a^{u_{i,j}(n)\cdot l_i\cdot c_j}. \tag{10}$$

Combining derivations (9) and (10) we obtain, by formula (8):

$$(q_0 U[F^n]q_0) \to^*_{\mathcal{A}} a^{u(n)}. \quad \square$$

Lemma 51. *Let $(u_n)_{n\geq 0}$ be the sequence defined by $u_{n+1} = (u_n)^d$, $d \geq 1$ and $u_0 = c \in \mathbb{N}$. Then $(u_n)_{n\geq 0} \in \mathbb{S}_3$.*

Construction. Let us set $\mathcal{A} = (\{q_0, q_F, q\}, \{a\}, \{F, A\}, \delta, q_0, Z)$ with:

1.1. $\delta(q_0, \varepsilon, AAF) = (q_F, \mathrm{pop}_3)$,
1.2. $\delta(q_F, \varepsilon, AAF) = \delta(q_F, \varepsilon, AA) = (q_0, \mathrm{push}_2(A^d))$,
2.1. $\delta(q_0, \varepsilon, AA) = (q, \mathrm{pop}_2)$,
2.2. for all $\omega \in \Gamma^{(3)}$, $\delta(q, \varepsilon, \omega) = (q_0, \mathrm{push}_1(A^c))$,
3. $\delta(q_0, a, A) = (q_0, \mathrm{pop}_1)$.

Proof. Let Ω be an undeterminate for the automaton $\mathcal{A}$ (we recall it means that $\Omega \notin \{F, A\}$). We prove by induction on n, the following property $\mathbf{P}(n)$:

$$(q_0 A[A[F^n]\Omega]q_0) \to^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^{u_n}.$$

Basis: $n = 0$
By transition (2.1)

$$(q_0 A[A[\varepsilon]\Omega]q_0) \rightarrow_{\mathcal{A}} (q A[\Omega]q_0)$$

using then a transition (2.2) and the definition of $\rightarrow$ we get:

$$(q A[\Omega]q_0) \rightarrow_{\mathcal{A}} (q_0 (A[\Omega])^c q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^c. \tag{11}$$

Thus $\mathbf{P}(0)$ is proved.
Induction step: Let $n \geq 0$ and let us assume $\mathbf{P}(n)$.
Using transition (1.1) we get:

$$(q_0 A[A[F^{n+1}]\Omega]q_0) \rightarrow_{\mathcal{A}} (q_F A[A[F^n]\Omega]q_0) \tag{12}$$

using then transition (1.2) we get:

$$(q_F A[A[F^n]\Omega]q_0) \rightarrow (q_0 A[(A[F^n])^d \Omega]q_0). \tag{13}$$

Let $i \in \mathbb{N}$ and $\omega_i = (A[F^n])^i \Omega$. Substituting this $3 -$ pds ω to the undeterminate Ω in $P(n)$ we obtain:

$$(q_0 A[(A[F^n])^i \Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[(A[F^n])^{i-1} \Omega]q_0)^{u_n}.$$

Composing all these derivations (for $1 \leq i \leq d$) together we obtain:

$$(q_0 A[(A[F^n])^d \Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^{(u_n)^d}. \tag{14}$$

Composing derivations (12)–(14) we obtain

$$(q_0 A[A[F^{n+1}]\Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^{(u_n)^d} = (q_0 A[\Omega]q_0)^{u_{n+1}}$$

i.e. $\mathbf{P}(n+1)$. Transition (3) expressed as a derivation gives:

$$q_0 A[\varepsilon]q_0 \rightarrow_{\mathcal{A}} a$$

we can then conclude that, for every $n \in \mathbb{N}$

$$(q_0 A[A[F^n]]q_0) \rightarrow^*_{\mathcal{A}} a^{u_n}. \quad \square$$

Proposition 52. *Let $P(X) = \sum_{0 \leq i \leq d} a_i X^i$ be a polynomial with coefficients $a_i \in \mathbb{N}$ and $(u_n)_{n \geq 0}$ be the sequence defined by $u_{n+1} = P(u_n)$ and $u_0 = c \in \mathbb{N}$. Then $(u_n)_{n \geq 0} \in \mathbb{S}_3$.*

We use the same ideas as in the proof above.

Construction. Let us set $\mathcal{A} = (\{q_0, q, q_F\}, \{a\}, \{F, A, A_0, \ldots, A_d\}, \delta, q_0, Z)$ with:

1.1. $\delta(q_0, \varepsilon, AAF) = (q_F, \mathrm{pop}_3)$,
1.2. $\delta(q_F, \varepsilon, AAF) = \delta(q_F, \varepsilon, AA) = (q_0, \mathrm{push}_1(A_0^{a_0} A_1))$,
2. $\delta(q_0, \varepsilon, A_0 AF) = \delta(q_0, \varepsilon, A_0 A) = (q_0, \mathrm{pop}_2)$
3. $\delta(q_0, \varepsilon, A_1 AF) = \delta(q_0, \varepsilon, A_1 A) = (q_0, \mathrm{push}_1(A^{a_1} A_2))$,
4. for all $2 \leq i \leq d$, $\delta(q_0, \varepsilon, A_i AF) = \delta(q_0, \varepsilon, A_i A) = (q_0', \mathrm{push}_2(AA))$,
5. for all $2 \leq i < d$, $\delta(q_0', \varepsilon, A_i AF) = \delta(q_0', \varepsilon, A_i A) = (q_0, \mathrm{push}_1(A^{a_i} A_{i+1}))$,
6. $\delta(q_0', \varepsilon, A_d AF) = \delta(q_0', \varepsilon, A_d A) = (q_0, \mathrm{push}_1(A^{a_d}))$,
7.1. $\delta(q_0, \varepsilon, AA) = (q_0, \mathrm{push}_1(B^c))$
7.2. $\delta(q_0, \varepsilon, BA) = (q_1, \mathrm{push}_1(A))$
7.3. $\delta(q_1, \varepsilon, AA) = (q_0, \mathrm{pop}_2)$,
8. $\delta(q_0, a, A) = (q_0, \mathrm{pop}_1)$.

Proof. Let Ω be an undeterminate. Let us show that, for every $n \geq 0$, the following property $\mathbf{P}(n)$ holds:

$$(q_0 A[A[F^n]\Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 \Omega q_0)^{u(n)}.$$

We first check that, for every $n \geq 0$

$$(q_0 A[A[F^{n+1}]\Omega]q_0) \rightarrow^*_{\mathcal{A}} \prod_{i=0}^{d} (q_0 A[(A[F^n])^i \Omega]q_0)^{a_i}. \tag{15}$$

Such a derivation can be detailed as:

$$\begin{array}{llr}
 & (q_0 A[A[F^{n+1}]\Omega]q_0) & \\
\rightarrow_{\mathcal{A}} & (q_F A[A[F^n]\Omega]q_0) & \text{(by a transition (1.1))} \\
\rightarrow_{\mathcal{A}} & (q_0 (A_0[A[F^n]\Omega])^{a_0} A_1[A[F^n]\Omega]q_0) & \text{(by a transition (1.2))} \\
\rightarrow^*_{\mathcal{A}} & (q_0 (A_0[A[F^n]\Omega]q_0)^{a_0} (q_0 A_1[A[F^n]\Omega]q_0) & \text{(by decomposition rule)} \\
\rightarrow_{\mathcal{A}} & (q_0 A[\Omega]q_0)^{a_0} (q_0 A_1[A[F^n]\Omega]q_0) & \text{(by transition (2))} \\
\rightarrow^*_{\mathcal{A}} & (q_0 A[\Omega]q_0)^{a_0} (q_0 (A[A[F^n]\Omega])^{a_1} A_2[A[F^n]\Omega]q_0) & \text{(by (3))} \\
\rightarrow^*_{\mathcal{A}} & (q_0 A[\Omega]q_0)^{a_0} (q_0 (A[A[F^n]\Omega]q_0)^{a_1} (q_0 A_2[A[F^n]\Omega]q_0) & \text{(by decomposition)}
\end{array}$$

$\rightarrow^*_{\mathcal{A}}$ $(q_0 A[\Omega]q_0)^{a_0} \prod_{i=1}^{d} (q_0 (A[(A[F^n])^i \Omega]q_0)^{a_i}$ (using $d - 2$ times (4, 5) and finally (4, 6)). Let us prove by induction $\mathbf{P}(n)$:
Basis: $n = 0$
The following derivation is valid:

$$\begin{array}{rll}
q_0 A[A[\varepsilon]\Omega]q_0 \rightarrow_{\mathcal{A}} & q_0 (B[A[\varepsilon]\Omega])^c q_0 & \text{(by (7.1))} \\
\rightarrow^*_{\mathcal{A}} & (q_0 (B[A[\varepsilon]\Omega]q_0)^c & \text{(by decomposition rules)} \\
\rightarrow^*_{\mathcal{A}} & (q_1 A[A[\varepsilon]\Omega]q_0)^c & \text{(by (7.2))} \\
\rightarrow^*_{\mathcal{A}} & (q_0 A[\Omega]q_0)^c & \text{(by (7.3)).}
\end{array}$$

Inductive step: Let us assume $\mathbf{P}(n)$.
By induction on i, using $\mathbf{P}(n)$ and the substitution $\Omega \leftarrow (A[F^n])^{i-1}\Omega$, we can show that:

$$(q_0 A[(A[F^n])^i \Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^{u(n)^i}.$$

These derivations allow us to obtain:

$$\prod_{i=0}^{d} (q_0 A[(A[F^n])^i \Omega]q_0)^{a_i} \rightarrow^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^{P(u(n))} = (q_0 A[\Omega]q_0)^{u(n+1)}.$$

The combination of derivation (15) with the derivation above proves $\mathbf{P}(n+1)$.

Substituting ε for Ω in property $\mathbf{P}(n)$ and applying transition (8), we may conclude that

$$(q_0 A[A[F^n]]q_0) \rightarrow^*_{\mathcal{A}} a^{u(n)}. \quad \square$$

Proposition 53. *Let $P_i(X_1, \ldots, X_p)$, $(1 \leq i \leq p)$ be polynomials with coefficients in $\mathbb{N}$, $c_1, \ldots, c_i, \ldots, c_p \in \mathbb{N}$ and, u_i, $(1 \leq i \leq p)$ be the sequence defined by $u_i(n+1) = P_i(u_1(n), \ldots, u_p(n))$, and $u_i(0) = c_i$. Then $u_1 \in \mathbb{S}_3$.*

Sketch of proof. Let us suppose that for every $1 \leq i \leq p$,

$$P_i(X_1, \ldots, X_p) = \sum_{j=0}^{\nu_i} a_{i,j} X_1^{d_{i,j,1}} \ldots X_p^{d_{i,j,p}}.$$

Following the same lines as in the proof of Proposition 52, it is possible to construct a 3-dcp automaton satisfying for all $1 \leq i \leq p$

$$(q_0 A[U_i[F^{n+1}]\Omega]q_0) \rightarrow^* \prod_{j=0}^{\nu_i} \left(q_0 A \left[\left(\prod_{\ell=1}^{p} (U_\ell[F^n])^{d_{i,j,\ell}} \right) \cdot \Omega \right] q_0 \right)^{a_{i,j}}$$

and also

$$(q_0 A[U_i[\varepsilon]\Omega]q_0) \to^* (q_0 A[\Omega]q_0)^{c_i}.$$

These derivations imply, by induction on n, that:

$$(q_0, A[U_i[F^n]]q_0) \to^* a^{u_i(n)}. \quad \square$$

4.3. Operations over sequences/automata

In this section, we investigate the closure properties of the classes $\mathbb{S}_k$. It turns out that the union of all the $\mathbb{S}_k$ is closed under classical operations like sum, product, composition, convolution product, and by more complex operations like resolution of system of polynomial recurrence equations with coefficients in $\mathbb{S}_k$.

We conclude the section by Theorem 72 which summarizes the closure properties established so far.

The following technical lemma will be useful for these constructions.

Lemma 54. *Let $f \in \mathbb{S}_{k+1}$, $k \geq 2$ and $\mathcal{A}$ be a $(k+1)$-dcpda over a pushdown alphabet $\Gamma \supseteq \{A_1, A_2, \ldots, A_k, F\}$ fulfilling:* $\forall n \geq 0$,

(P0) $(q_0, a^{f(n)}, A_1[\ldots[A_k[F^n]]\ldots]) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \varepsilon)$.
Then, one can construct a $(k+1)$-dcpda $\mathcal{A}'$ defined on a pushdown-symbols set $\Gamma' \supseteq \Gamma$, containing a special symbol $\bar{A}_1$ and a set of states Q', such that:
(P1) *$\mathcal{A}'$ is level partitioned*
(P2) *for every letter $\Omega \notin \Gamma'$, $(q_0, A_1[A_2[\ldots[A_k[F^n]]\ldots]\Omega], q_0) \to^*_{\mathcal{A}'} (q_0, \bar{A}_1[\Omega], q_0)^{f(n)}$.*
(P3) *δ' has no lefthand side of the form $(q_0, \bar{A}_1 \cdot \omega)$ for any $\omega \in \Gamma'^{(k)}$.*

Remark 55.

1. Let us add to the transitions of $\mathcal{A}'$ the transition: $\delta(q_0, a, \bar{A}_1) = (q_0, \mathrm{pop}_1)$. We thus obtain an automaton $\mathcal{A}''$ fulfilling properties (**P0**), (**P1**), (**P2**).
2. Property (**P3**) makes the automaton $\mathcal{A}'$ "open" to a combination with another automaton: it suffices to add transitions starting from $q_0\bar{A}_1[\omega]$ for some well-chosen ω, and leading to a total state from another deterministic automaton. Property (**P3**) guaranties that the compound automaton thus constructed will be deterministic.

Construction.
First step
From $\mathcal{A}$ it is possible to build another $(k+1)$-dcpda $\mathcal{B}$ fulfilling conditions (**P0**), (**P1**) and the additional condition (**P4**): $\forall q \in Q, \forall A \in \Gamma$, there is no transition with lefthand side (q, A).
The automaton $\mathcal{A}$ can be transformed into a level-partitioned automaton $\mathcal{B}_0$, as explained in Section 2.3.2. One can then transform $\mathcal{B}_0$ by means of adding a "bottom symbol" at level two allowing us to simulate the transitions starting with an empty second level. The resulting automaton $\mathcal{B}$ meets conditions (**P0**), (**P1**), (**P4**).
Second step
Let us suppose now that $\mathcal{A}$ fulfills (**P0**), (**P1**), (**P4**).
Let us set $\mathcal{A}' = (Q', \emptyset, \Gamma', \delta', r_0, Z)$ with
$Q' = Q \cup \{\bar{q} \mid q \in Q\} \cup \{r_0, r_1, r_2, r_3\}$ and $\Gamma' = \Gamma \cup \{\bar{A}_1, B_1, B_2\} \cup \{A^q \mid A \in \Gamma, q \in Q\}$
and δ' consists of the following transitions:

- for the precise symbols $A_1, A_2, \ldots, A_k, F$ used in (**P2**)

  0.1 $\delta'(r_0, \varepsilon, A_1 A_2 \cdots A_k F) = (r_1, \mathrm{push}_1(A_1 B_1))$
  0.2 $\delta'(r_1, \varepsilon, A_1 A_2 \cdots A_k F) = (q_0, \mathrm{push}_2(A_2 B_2))$

- for all $\delta(q, \varepsilon, \omega) = (p, f)$

  1 $\delta'(q, \varepsilon, \omega) = (p, f)$

- for all $A \in \Gamma$, $\delta(q, a, A\omega) = (p, f)$

  2.1 $\delta'(q, \varepsilon, A\omega) = (r_2, \mathrm{push}_1(A_1 A^q))$

2.2 $\delta'(r_0, \varepsilon, A^q\omega) = (\bar{q}, \mathrm{push}_1(A))$
2.3 $\delta'(\bar{q}, \varepsilon, A\omega) = (p, f)$

- for all $L_1, L_2 \in \Gamma, \omega \in \Gamma^{(k-1)}$

2.4 $\delta'(r_2, \varepsilon, L_1 L_2 \omega) = (r_2, \mathrm{pop}_2)$

- for all $\omega \in \Gamma^{(k-1)}$

2.5 $\delta'(r_2, \varepsilon, A_1 B_2 \omega) = (r_0, \mathrm{push}_1(\bar{A}_1)$
2.6 $\delta'(r_3, \varepsilon, \bar{A}_1 B_2 \omega) = (r_0, \mathrm{pop}_2)$

- for all $A \in \Gamma, \omega \in \Gamma^{(k-1)}$

3 $\delta'(q_0, \varepsilon, B_1 A\omega) = (r_0, \mathrm{pop}_1)$.

4.3.0.6. Informal explanations. By (1) $\mathcal{A}'$ mimics all ε-moves of $\mathcal{A}$. Let $\mathcal{A}$ perform a reading-move: $(q, a, \omega_1) \vdash (p, \varepsilon, \eta_1)$. The automaton $\mathcal{A}'$ pushes a symbol A_1, thus reaching a pds $A_1[\omega_2 \Omega] \cdot \omega_1'$. Then $\mathcal{A}'$ erases ω_2 and substitutes the special symbol $\bar{A}_1$ to A_1, thus reaching a pds $\bar{A}_1[\Omega] \cdot \omega_1'$. Rules (2.1,2.2,2.3) are devised so that, starting from state r_0 and pds ω_1', it can simulate the reading move of $\mathcal{A}$, thus reaching (p, ε, η_1).

4.3.0.7. Determinism. Let us check that $\mathcal{A}'$ is deterministic: this amounts to checking that the modes (i.e. the pairs (state, pushdown-word)) of two different types of transitions are disjoint. For the pair (1),(2.1), there is no common mode since $\mathcal{A}$ is deterministic.

For the pair (1),(3), there is no common mode since B_1 is a new pushdown symbol.

For the pair (2.1),(3), the same argument applies.

For the pair (2.4),(2.5), there is no common mode since B_2 is a new pushdown symbol, hence $B_2 \neq L_2$.

For all other pairs, the sets of states on which they apply are disjoint.

4.3.0.8. Conditions (**P1**), (**P3**). As the initial automaton $\mathcal{A}$ is level partitioned, so is $\mathcal{A}'$.

The only lefthand side of transition where $\bar{A}_1$ occurs, is the l.h.s. of (2.6). As its state is different from r_0, condition (**P3**) is fulfilled.

In order to prove that the automaton $\mathcal{A}'$ fulfills property (**P2**) we establish the sequence of Facts 56–61.

Fact 56. *For every* $p, p' \in Q, \omega, \omega' \in (k+1) - \mathrm{pds}(\Gamma')$,
if $(p, \varepsilon, \omega) \vdash^*_{\mathcal{A}} (p', \varepsilon, \omega')$ *then* $(p, \varepsilon, \omega) \vdash^*_{\mathcal{A}'} (p', \varepsilon, \omega')$.

This fact can be deduced from transitions (1). Notice that we consider the possibility that ω, ω' contain occurrences of letters from $\Gamma' - \Gamma$. The relation $\vdash^*_{\mathcal{A}}$ is defined from the transitions of $\mathcal{A}$, but applied to total states in $Q \times (k+1) - \mathrm{pds}(\Gamma')$).

We define the pds:

$$\alpha_n = A_1[A_2[A_3[\cdots[F^n]\cdots]]]; \quad \beta_n = A_2[A_3[\cdots[F^n]\cdots]]; \quad \gamma_n = A_3[\cdots[F^n]\cdots] \tag{16}$$

(if $k = 2$ $\gamma_n = F^n$). We define a map $\tau : (k+1) - \mathrm{pds}(\Gamma) \to (k+1) - \mathrm{pds}(\Gamma' \cup \{\Omega\})$ by: for every atom $A[\omega_2]$, with $\omega_2 \in k - \mathrm{pds}(\Gamma')$

$$\tau(A[\omega_2]) = A[\omega_2 B_2[\gamma_n]\Omega]$$

and for every $\omega = \eta_1 \eta_2 \cdots \eta_\ell$ where η_i are atoms of $(k+1) - \mathrm{pds}(\Gamma)$,

$$\tau(\omega) = \tau(\eta_1)\tau(\eta_2)\cdots\tau(\eta_\ell) B_1[\beta_n].$$

From now on, we call *special* the pds which have the form $\tau(\omega)$ for some $\omega \in (k+1) - \mathrm{pds}(\Gamma)$.

Fact 57. *For every* $p, p' \in Q, u, u' \in a^*, \omega, \omega' \in (k+1) - \mathrm{pds}(\Gamma)$,
if $(p, u, \omega) \vdash^*_{\mathcal{A}} (p', u', \omega')$ *then* $(p, u, \tau(\omega)) \vdash^*_{\mathcal{A}} (p', u', \tau(\omega'))$.

This fact holds because, as $\mathcal{A}$ fulfills (**P4**), the first computation cannot use the information that the list of top-symbols has length one. Hence the first computation is mapped by τ into the second one.

Fact 58. *For every $p, p' \in Q, q \in Q', \omega, \omega'$ special pds,*
if $(p, a, \omega) \vdash_{\mathcal{A}} (p', \varepsilon, \omega')$ *then* $(p, \omega, q) \to^{*}_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0)(p', \omega', q)$.

Let us prove this fact. The pds ω is special, hence, there exist $\omega_1 \in (k+1) - \mathrm{pds}(\Gamma), \omega_2 \in (k) - \mathrm{pds}(\Gamma), \omega_3 \in (k-1) - \mathrm{pds}(\Gamma)$ such that:

$$\omega = A[\omega_2 B_2[\omega_3]\Omega] \cdot \omega_1.$$

The hypothesis of the fact shows that, by a transition (2.1),

$$(p, \varepsilon, \omega) \vdash_{\mathcal{A}'} (r_2, \varepsilon, A_1[\omega_2 B_2[\omega_3]\Omega] \cdot A^q[\omega_2 B_2[\omega_3]\Omega] \cdot \omega_1),$$

hence, for every $q \in Q$,

$$(p, \omega, q) \to_{\mathcal{A}'} (r_2, A_1[\omega_2 B_2[\omega_3]\Omega], r_0) \cdot (r_0, A^q[\omega_2 B_2[\omega_3]\Omega]\omega_1, q). \tag{17}$$

By transitions (2.4)–(2.6)

$$(r_2, A_1[\omega_2 B_2[\omega_3]\Omega], r_0) \to^{*}_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0) \tag{18}$$

and by transitions (2.2) and (2.3)

$$(r_0, A^q[\omega_2 B_2[\omega_3]\Omega]\omega_1, q) \to^{*}_{\mathcal{A}'} (p', \omega', q). \tag{19}$$

Composing the three derivations (17)–(19), we obtain the conclusion of Fact 58.

Fact 59. $(q_0, A_1[\beta_n B_2[\gamma_n]\Omega] B_1[\beta_n], r_0) \to^{*}_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0)^{f(n)} \cdot (q_0, B_1[\beta_n], r_0)$.

Let us show this fact, reformulated as:

$$(q_0, \tau(\alpha_n), r_0) \to^{*}_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0)^{f(n)} \cdot (q_0, \tau(\varepsilon), r_0). \tag{20}$$

By hypothesis

$$(q_0, a^{f(n)}, \alpha_n) \vdash^{*}_{\mathcal{A}} (q_0, \varepsilon, \varepsilon)$$

Fact 57 allows us to deduce that

$$(q_0, a^{f(n)}, \tau(\alpha_n)) \vdash^{*}_{\mathcal{A}} (q_0, \varepsilon, \tau(\varepsilon)).$$

This computation can be factorised into $f(n) + 1$ subcomputations:

$$(p_i, \varepsilon, \tau(\eta_i)) \vdash^{*}_{\mathcal{A}} (s_i, \varepsilon, \tau(\omega_i)); \quad (s_i, a, \tau(\omega_i)) \vdash_{\mathcal{A}} (p_{i+1}, \varepsilon, \tau(\eta_{i+1})) \tag{21}$$

for $0 \le i \le f(n) - 1$ with $p_0 = q_0$ (the initial state of $\mathcal{A}$), and

$$(p_{f(n)}, \varepsilon, \tau(\eta_{f(n)})) \vdash^{*}_{\mathcal{A}} (s_{f(n)}, \varepsilon, \tau(\varepsilon)) \tag{22}$$

with $s_{f(n)} = q_0$. Via Facts 56 and 58, for every $q \in Q'$, the above computations translate into the derivations:

$$(p_i, \tau(\eta_i), q) \to^{*}_{\mathcal{A}'} (s_i, \tau(\omega_i), q) \to_{\mathcal{A}'} (r_0, A_1[\Omega], r_0)(p_{i+1}, \tau(\eta_{i+1}), q) \tag{23}$$

(for $0 \le i \le f(n) - 1$ with $p_0 = q_0$) and

$$(p_{f(n)}, \tau(\eta_{f(n)}, q)) \to^{*}_{\mathcal{A}'} (s_{f(n)}, \tau(\varepsilon), q) = (q_0, \tau(\varepsilon), q). \tag{24}$$

The composition of all derivations (23) and (24), for $q = r_0$, gives derivation (20).

Fact 60. $(r_0, A_1[\beta_n \cdot \Omega], r_0) \to^{*}_{\mathcal{A}'} (q_0, A_1[\beta_n] B_2[\gamma_n]\Omega] B_1[\beta_n], r_0)$.

This is just the grammatical counterpart of transitions (0.1) and (0.2).

Fact 61. $(q_0, B_1[\beta_n], r_0) \to_{\mathcal{A}'} \quad \varepsilon$.

This translates transition (3).

Proof. Let us combine into one derivation the derivations given in Facts 60, 59 and finally Fact 61. We obtain the required derivation:

$$(r_0, A_1[\beta_n \cdot \Omega], r_0) \rightarrow^*_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0)^{f(n)}.$$

Up to a renaming of the states, (**P2**) holds for $\mathcal{A}'$. □

Proposition 62 (*Sum*). *If $f, g \in \mathbb{S}_{k+1}$, $k \geq 2$ then $f + g \in \mathbb{S}_{k+1}$.*

Proof. Let $\mathcal{A}, \mathcal{A}'$ be two $(k+1)$-dcpda computing respectively f and g. We assume

$$q_0 A_1[\beta_n] q_0 \rightarrow^*_{\mathcal{A}} a^{f(n)}; \quad q_0 A'_1[\beta_n] q_0 \rightarrow^*_{\mathcal{A}'} a^{g(n)}.$$

It suffices to construct a $(k+1)$-dcpda $\mathcal{B}$ performing the following computation: starting from total state $r_0 D_1[\beta_n]$, by a push_1 operation it moves to $q_0 A_1[\beta_n] A'_1[\beta_n]$, where q_0 is the common starting (and final) state of $\mathcal{A}, \mathcal{A}'$. Then it simulates $\mathcal{A}$ on $q_0 A_1[\beta_n]$ and finally, it simulates $\mathcal{A}'$ on $q_0 A'_1[\beta_n]$. □

Proposition 63 (*Ordinary Product*). *If $f, g \in \mathbb{S}_{k+1}$, $k \geq 2$ then $f \odot g \in \mathbb{S}_{k+1}$.*

Construction. By Lemma 54, after a suitable choice for the concrete sets of states and pushdown alphabets, we obtain two $(k+1)$-dcpda $\mathcal{A}_1$ and $\mathcal{A}_2$ fulfilling conditions:

(**P1**) $\mathcal{A}_1, \mathcal{A}_2$ are level-partitioned
(**P2.1**) $\forall \Omega \in \mathcal{U}$, $(q_0, A_1[A_2[\ldots[A_k[F^n]]\ldots]\Omega], q_0) \rightarrow^*_{\mathcal{A}_1} (q_0, \bar{A}_1[\Omega], q_0)^{f(n)}$.
(**P2.2**) $\forall \Omega \in \mathcal{U}$, $(q_0, \bar{A}_1[A_2[\ldots[A_k[F^n]]\ldots]\Omega], q_0) \rightarrow^*_{\mathcal{A}_2} (q_0, A'_1[\Omega], q_0)^{g(n)}$.
(**P3.1**) δ_1 has no lefthand side of the form $(q_0, \bar{A}_1 \cdot \omega)$ for any $\omega \in \Gamma_1^{(k)}$.
(**P3.2**) δ_2 has no lefthand side of the form $(q_0, A'_1 \cdot \omega)$ for any $\omega \in \Gamma_2^{(k)}$.
(**P4**) $Q_1 \cap Q_2 = \{q_0\}$.
(**P5**) $\Gamma_1 \cap \Gamma_2 = \{\bar{A}_1, A_2, A_3 \ldots A_k, F\}$.

We consider the $(k+1)$-dcpda $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ where: $Q = Q_1 \cup Q_2 \cup \{r_0\}$, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \{D_1\}$ and δ is the union of $\delta_1 \cup \delta_2$ with the additional transitions:

1.1 $\delta(q_0, \varepsilon, D_1 A_2 \ldots A_k) = (r_0, \text{push}_2(A_2))$
1.2 $\delta(r_0, \varepsilon, D_1 A_2 \ldots A_k) = (q_0, \text{push}_1(A_1))$
2 $\delta(q_0, a, A'_1) = (q_0, \text{pop}_1)$.

Proof. The automaton $\mathcal{A}$ is deterministic:
let us consider $(q_1, \bar{a}, \omega_1)$ the l.h.s. of a rule of δ_1, $(q_2, \bar{b}, \omega_2)$ the l.h.s. of a rule of δ_2. Both can apply on the same total state only if $q_1 = q_2$ and $\omega_1 = \omega_2$. In that case $q_1 = q_2 = q_0$ and $\omega_1 = \omega_2 \in \bar{A}_1 A_2 \cdots A_k \cdot \{F, \varepsilon\}$. But condition **P3.1** makes impossible such an l.h.s. for δ_1. Transition (1.2) is the only one starting from state r_0. No transition from δ_1 uses symbol A'_1 and, by **P3.2**, no transition from δ_2 starts from (q_0, A'_1).

Let us show now that it computes $f \odot g$. For every $n \geq 0$, we keep noting $\beta_n = A_2[\cdots[A_k[F^n]]\cdots]$. The following derivation holds:

$$\begin{array}{llll} (q_0 D_1[\beta_n] q_0) \rightarrow_{\mathcal{A}} & (q_0 A_1[\beta_n \beta_n] q_0) & \text{(by (1.1,1.2))} \\ \rightarrow^*_{\mathcal{A}} & (q_0 \bar{A}_1[\beta_n] q_0)^{f(n)} & \text{(by } \mathbf{P2.1}) \\ \rightarrow^*_{\mathcal{A}} & (q_0 A'_1[\varepsilon] q_0)^{f(n) \cdot g(n)} & \text{(by } \mathbf{P2.2}) \\ \rightarrow^*_{\mathcal{A}} & a^{f(n) \cdot g(n)} & \text{(by (2)).} \end{array}$$ □

Proposition 64. *If $f \in \mathbb{S}_{k+1}$, $k \geq 2$, and g is the sequence defined by $g(0) = c$ and $g(n+1) = f(n).g(n)^d$, $d \geq 1$, then $g \in \mathbb{S}_{k+1}$.*

Proof. There exists a $(k+1)$-dcpda $\mathcal{A}_1 = (Q_1, \{a\}, \Gamma_1, \delta_1, q_0, Z)$ fulfilling conditions (**P1**), (**P2**), (**P3**) stated in Lemma 54. We consider the $(k+1)$-dcpda $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ where: $Q = Q_1 \cup \{r_0, r_1, r_2\}$, $\Gamma = \Gamma_1 \cup \{D_1\}$ and δ is the union of δ_1 with the additional transitions:

0.1 $\delta(q_0, \varepsilon, D_1 A_2 \ldots A_k F) = (r_0, \text{pop}_{k+1})$

0.2 $\delta(r_0, \varepsilon, D_1 A_2 \dots A_k F) = \delta(r_0, \varepsilon, D_1 A_2 \dots A_k) = (r_1, \text{push}_2(A_2^{d+1}))$
0.3 $\delta(r_1, \varepsilon, D_1 A_2 \dots A_k F) = \delta(r_1, \varepsilon, D_1 A_2 \dots A_k) = (q_0, \text{push}_1(A_1))$
0.4 $\delta(q_0, \varepsilon, D_1 A_2 \dots A_k) = (r_2, \text{push}_1(D_1^c))$
0.5 $\delta(r_2, \varepsilon, D_1 A_2 \dots A_k) = (q_0, \text{pop}_2)$
1 $\delta(q_0, \varepsilon, \bar{A}_1 A_2 \dots A_k F) = \delta(q_0, \varepsilon, \bar{A}_1 A_2 \dots A_k) = (q_0, \text{push}_1(D_1))$
2 $\delta(q_0, a, D_1) = (q_0, \text{pop}_1)$.

This automaton is deterministic: δ_1 is a deterministic transition map and, by condition (**P3**), transition (1) does not break this determinism. Moreover transitions (0.i) and (2) cannot interfere with δ_1 (since D_1 is a new letter), and cannot interfere with each other.

In order to show that $\mathcal{A}$ does compute g, we summarize some interesting basic derivations:
By (**P2**):

$$(q_0 A_1[\beta_n \Omega] q_0) \to_{\mathcal{A}}^* (q_0 A_1[\beta_n \Omega] q_0)^{f(n)}$$

Starting rules: using transitions (0.1,0.2,0.3)

$$(q_0 D_1[\beta_{n+1} \Omega] q_0) \to_{\mathcal{A}}^* (q_0 A_1[\beta_n^{d+1} \Omega] q_0)$$

and using transition (0.4), the decomposition rule, and then (0.5):

$$(q_0 D_1[\beta_0 \Omega] q_0) \to_{\mathcal{A}}^* (q_0 D_1[\Omega] q_0)^c$$

Gluing rule: using transition (1), for every $n \geq 0$

$$(q_0 \bar{A}_1[\beta_n \Omega] q_0) \to_{\mathcal{A}}^* (q_0 D_1[\beta_n \Omega] q_0)$$

Ending rule: using transition (2)

$$(q_0 D_1[\varepsilon] q_0) \to_{\mathcal{A}} a.$$

Let us prove, by induction over $n \geq 0$, the following property $\mathbf{P}(n)$:

$$(q_0 D_1[\beta_n \Omega] q_0) \to_{\mathcal{A}}^* (q_0 D_1[\Omega] q_0)^{g(n)}$$

Basis: $n = 0$.
The second starting rule proves $\mathbf{P}(0)$.
Induction step
We exhibit the derivation:

$$\begin{aligned} (q_0 D_1[\beta_{n+1} \Omega] q_0) \to_{\mathcal{A}}^* &\quad (q_0 A_1[\beta_n^{d+1} \Omega] q_0) && \text{(by first starting rule)} \\ \to_{\mathcal{A}}^* &\quad (q_0 \bar{A}_1[\beta_n^d \Omega] q_0)^{f(n)} && \text{(by } \mathbf{P2}\text{)} \\ \to_{\mathcal{A}}^* &\quad (q_0 D_1[\beta_n^d \Omega] q_0)^{f(n)} && \text{(by the gluing rule)}. \end{aligned} \tag{25}$$

Applying d times $\mathbf{P}(n)$ (with accurate substitutions to the undeterminate Ω) we get that:

$$(q_0 D_1[\beta_n^d \Omega] q_0) \to_{\mathcal{A}}^* (q_0 D_1[\Omega] q_0)^{g(n)^d}. \tag{26}$$

Composing derivations (25) and (26) we obtain:

$$(q_0 D_1[\beta_{n+1} \Omega] q_0) \to_{\mathcal{A}}^* (q_0 D_1[\Omega] q_0)^{f(n) \cdot g(n)^d} = (q_0 D_1[\Omega] q_0)^{g(n+1)}.$$

Hence $\mathbf{P}(n+1)$ is proved.

Combining $\mathbf{P}(n)$ with the ending rule, we deduce that, for every $n \geq 0$,

$$(q_0 D_1[\beta_n] q_0) \to_{\mathcal{A}}^* a^{g(n)}. \quad \square$$

Let us notice that, by Propositions 62 and 63, $(\mathbb{S}_k, +, \cdot)$ is a semi-ring. We denote by $P(n, X_1, \dots, X_j, \dots, X_p)$ any element of the semi-ring $\mathbb{S}_k[X_1, \dots, X_j, \dots, X_p]$ to emphasise the fact that the coefficients of P are functions of the integer argument n.

Proposition 65. *Let $k \geq 2$. Let $P_i(n, X_1, \ldots, X_p)$, $1 \leq i \leq p$ be polynomials with coefficients in $\mathbb{S}_{k+1}$ and u_i, for $1 \leq i \leq p$, be sequences defined by $u_i(n+1) = P_i(n, u_1(n), \ldots, u_p(n))$, and $u_i(0) = c_i$. Then $u_1 \in \mathbb{S}_{k+1}$.*

Sketch of proof. Let us suppose that for every $1 \leq i \leq p$,

$$P_i(X_1, \ldots, X_p) = \sum_{j=0}^{\nu_i} a_{i,j}(n) X_1^{d_{i,j,1}} \ldots X_p^{d_{i,j,p}}.$$

Every coefficient $a_{i,j}(n)$ is computed by some $(k+1)$-dcpda $\mathcal{A}_{i,j} = (Q_{i,j}, \{a\}, \Gamma_{i,j}, \delta_{i,j}, q_0, Z_{i,j})$ fulfilling conditions (**P1**), (**P2**), (**P3**) stated in Lemma 54. A suitable choice of the sets involved in the definition of these automata can be made so that, for every $(i, j) \neq (i', j')$

$$Q_{i,j} \cap Q_{i',j'} = \{q_0\}; \quad \Gamma_{i,j} \cap \Gamma_{i',j'} = \{A_2, \ldots, A_k, F\}.$$

We assume that, for every (i, j):

$$Q \supseteq Q_{i,j}; \quad \Gamma \supseteq \Gamma_{i,j}; \quad \delta \supseteq \delta_{i,j};$$

and Γ posesses some additional symbols $U_1, U_2, \ldots, U_p$ of level 2, and A of level 1. Suppose that the transitions allow the following basic derivations:
Coefficient rules:

$$(q_0 A_{i,j}[\beta_n \Omega] q_0) \to^*_{\mathcal{A}_{i,j}} (q_0 \bar{A}_{i,j}[\Omega] q_0)^{a_{i,j}(n)}$$

(this is just condition (**P2**) for the automata $\mathcal{A}_{i,j}$)
Starting rules:

$$(q_0 A[U_i[\gamma_{n+1}]\Omega] q_0) \to^*_{\mathcal{A}} (q_0 A_{i,0}[\beta_n^2 \Omega] A_{i,1}[\beta_n^2 \Omega] \cdots A_{i,\nu_i}[\beta_n^2 \Omega] q_0)$$

and

$$(q_0 A[U_i[\gamma_0]\Omega] q_0) \to^*_{\mathcal{A}} (q_0 A[\Omega] q_0)^{c_i}$$

Gluing rules: for every $n \geq 0$, the gluing rule (**Gij**) is:

$$(q_0 \bar{A}_{i,j}[\beta_n \Omega] q_0) \to^*_{\mathcal{A}} \left(q_0 A \left[\left(\prod_{\ell=1}^{p} U_\ell[\gamma_n]^{d_{i,j,\ell}} \right) \Omega \right] q_0 \right)$$

Ending rule:

$$(q_0 A[\varepsilon] q_0) \to_{\mathcal{A}} a.$$

Let us consider property $\mathbf{P}(n)$ defined by:

$$\forall i \in [1, p], (q_0 A[U_i[\gamma_n]\Omega] q_0) \to^*_{\mathcal{A}} (q_0 A[\Omega] q_0)^{u_i(n)}.$$

Property $\mathbf{P}(n)$ can be proved by induction on n, under the assumption that the coefficient rules, the starting rules, the gluing rules and the ending rules are valid. Leaning on the normalisation property (**P3**), it is possible to add transitions to the union of the $\delta_{i,j}$ in such a way that all these rules are made valid and the automaton $\mathcal{A}$ remains deterministic:

- the different modes of the different lefthand sides of the rules given above are distinct
- it suffices thus to decompose each of these rules into a finite sequence of elementary moves, using disjoint sets of states for the intermediate total states of the different rules, to obtain a *deterministic* cpda. □

Proposition 66. *Let $f \in \mathbb{S}_{k+1}$, $g \in \mathbb{S}_k$, $k \geq 3$, Then, the sequence h defined for all $n \geq 0$ by $h(n) = f(n)^{g(n)}$ belongs to $\mathbb{S}_{k+1}$.*

Proof. Let us proceed as in the proof of Proposition 64: we expose, in a first step, a list of particular derivations (that we call "rules") and prove that these rules are sufficient to compute the required sequence; in a second step, we explain how to construct a deterministic automaton which makes these rules available.

First step

Let $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ be some $(k+1)$-dcpda fulfilling condition (**P1**) stated in Lemma 54. We suppose that $\Gamma \supseteq \{A_1, \bar{A}_1, A_2, \ldots, A_k, F\} \cup \{B_2, \bar{B}_2\}$, where the levels are given by the indices. For every operation f on $k - \mathrm{pds}(\Gamma)$, we define the operation $A_1 \cdot \mathrm{f}$ as:

$$A_1 \cdot \mathrm{push}_i(L) = \mathrm{push}_{i+1}(L),\ A_1 \cdot \mathrm{pop}_i = \mathrm{pop}_{i+1}\,.$$

Let us define: $A_1^{-1}\mathcal{A} = (Q, \{a\}, \Gamma, A_1^{-1}\delta, q_0, Z)$, where

$$A_1^{-1}\delta = \{(q, \bar{a}, \omega, q', \mathrm{f}) \in Q \times \{a, \varepsilon\} \times k - \mathrm{pds}(\Gamma) \times Q \times (PUSH(\Gamma) \cup POP) \mid (q, \bar{a}, A_1\omega, q', A_1\mathrm{f}) \in \delta\}.$$

We suppose that $A_1^{-1}\mathcal{A}$ fulfills condition (**P3**) of Lemma 54, for the state r_0 and the letter of level 1, $\bar{B}_2$ (notice $\bar{B}_2$ has level 2 in $\mathcal{A}$). Let us suppose that $\mathcal{A}$ allows the following basic derivations (where Ω is an undeterminate):

f-computation, D1:

$$(q_0 A_1[A_2[\gamma_n]\Omega]q_0) \to^*_{\mathcal{A}} (q_0\bar{A}_1[\Omega]q_0)^{f(n)}$$

g-computation, D2:

$$(r_0 B_2[\gamma_n\Omega]r_0) \to^*_{A_1^{-1}\mathcal{A}} (r_0\bar{B}_2[\Omega]r_0)^{g(n)}$$

Gluing rule, G21: $\forall \omega_3 \in (k-1) - \mathrm{pds}(\Gamma)$,

$$(r_0 A_1[\bar{B}_2[\omega_3]\Omega]q_0) \to_{\mathcal{A}} (q_0 A_1[A_2[\omega_3]\Omega]q_0)$$

Gluing rule, $G^{(0)}21$:

$$(r_0 A_1[\varepsilon]q_0) \to_{\mathcal{A}} (q_0\bar{A}_1[\varepsilon]q_0)$$

Gluing rule, G12: for every $\omega_3 \in (k-1) - \mathrm{pds}(\Gamma)$ and $L_2 \in \Gamma$, $L_2 \neq \bar{B}_2$,

$$(q_0\bar{A}_1[L_2[\omega_3]\Omega]q_0) \to_{\mathcal{A}} (r_0 A_1[L_2[\omega_3]\Omega]q_0).$$

Ending rule, E: for every $\omega_3 \in (k-1) - \mathrm{pds}(\Gamma)$ and $L_2 \in \Gamma$, $L_2 \neq \bar{B}_2$,

$$(q_0\bar{A}_1[\varepsilon]q_0) \to_{\mathcal{A}} a.$$

The intuition behind these rules is that the gluing rule Gij allows us to connect the end of a computation Di with the beginning of a computation Dj. The special gluing rule, $G^{(0)}21$ handles the case where the computation D2 results in the number 0, leading to the value $f(n)^0 = 1$.(*)[2]

Let us prove by induction over $p \geq 0$ the following property $\mathcal{P}(p)$:

for every $H_p \in k - \mathrm{pds}(\Gamma)$, which does not have $\bar{B}_2$ as leftmost head-symbol, if

$$(r_0 H_p r_0) \to^*_{A_1{}^{-1}\mathcal{A}} (r_0\bar{B}_2[\gamma_n]r_0)^p \tag{27}$$

then

$$(q_0\bar{A}_1[H_p]q_0) \to^*_{\mathcal{A}} (q_0\bar{A}_1[\varepsilon]q_0)^{f(n)^p}. \tag{28}$$

Basis: $p = 0$

We suppose that (27) holds. We then exhibit the derivation:

$$\begin{aligned} & (q_0\bar{A}_1[H_0]q_0) \\ \to_{\mathcal{A}}\ & (r_0 A_1[H_0]q_0) \quad \text{(by rule G12, notice } \bar{B}_2 \text{ is not the leftmost head-symbol)} \\ \to^*_{\mathcal{A}}\ & (r_0 A_1[\varepsilon]q_0) \quad \text{(by hypothesis (27))} \\ \to^*_{\mathcal{A}}\ & (q_0\bar{A}_1[\varepsilon]q_0) \quad \text{(by rule } G^{(0)}21). \end{aligned}$$

[2] we adopt the convention that $0^0 = 1$ in the definition of $h = f^g$.

Induction step:
We suppose that hypothesis (27) is fulfilled by $p+1$ and that $\mathcal{P}(p)$ holds. By means of Lemma 18 we can translate hypothesis (27) into: there exists some $H_p \in k - \mathrm{pds}(\Gamma)$ such that

$$(r_0, \varepsilon, H_{p+1}) \vdash^*_{A_1{}^{-1}\mathcal{A}} (r_0, \varepsilon, \bar{B}_2[\gamma_n]H_p) \text{ and } (r_0 H_p r_0) \to^*_{A_1{}^{-1}\mathcal{A}} (r_0 \bar{B}_2[\gamma_n] r_0)^p.$$

We exhibit the derivation:

$$\begin{aligned} (q_0\bar{A}_1[H_{p+1}]q_0) &\to^*_{\mathcal{A}} (r_0 A_1[H_{p+1}]q_0) && \text{(by rule G12)} \\ &\to^*_{\mathcal{A}} (r_0 A_1[\bar{B}_2[\gamma_n]H_p]q_0) && \text{(by above translation)} \\ &\to^*_{\mathcal{A}} (q_0 A_1[A_2[\gamma_n]H_p]q_0) && \text{(by rule G21)} \\ &\to^*_{\mathcal{A}} (q_0\bar{A}_1[H_p]q_0)^{f(n)} && \text{(by D1).} \end{aligned} \tag{29}$$

Combining this derivation with $\mathcal{P}(p)$, we get:

$$(q_0\bar{A}_1[H_{p+1}]q_0) \to^*_{\mathcal{A}} (q_0\bar{A}_1[\varepsilon]q_0)^{f(n)^{p+1}},$$

(end of the induction).

Let us consider $H = B_2[\gamma_n\gamma_n]$. By derivation $D2$, H fulfills hypothesis (27) for the integer $p = g(n)$. Hence, by $\mathcal{P}(p)$,

$$(q_0\bar{A}_1[B_2[\gamma_n\gamma_n]]q_0) \to^*_{\mathcal{A}} (q_0\bar{A}_1[\varepsilon]q_0)^{f(n)^{g(n)}}. \tag{30}$$

Second step
Let us construct such an automaton $\mathcal{A}$. The sequence $f(n)$ is computed by some $(k+1)$-dcpda $\mathcal{A}_1 = (Q_1, \{a\}, \Gamma_1, \delta_1, q_0, Z)$ fulfilling conditions (**P1**), (**P2**), (**P3**) stated in Lemma 54. As well the sequence $g(n)$ is computed by some k-dcpda $\mathcal{A}_2 = (Q_2, \{a\}, \Gamma_2, \delta_2, r_0, Z)$ fulfilling the same conditions. We suppose that $\Gamma_1 \cap \Gamma_2 = \{A_3, \ldots, A_k, F\}$, where A_i has level i for $\mathcal{A}_1$ (resp. level $i-1$ for $\mathcal{A}_2$) and F has level $k+1$ for $\mathcal{A}_1$ (resp. level k for $\mathcal{A}_2$). Let us define $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, s_0, Z)$ where

$$Q = Q_1 \cup Q_2 \cup \{s_1\}; \quad \Gamma = \Gamma_1 \cup \Gamma_2 \cup \{D_1\};$$

δ is the union of $\delta_1 \cup (A_1 \cdot \delta_2)$ with the following rules:

0.1 $\delta(q_0, \varepsilon, D_1 A_2 \ldots A_k) = \delta(q_0, \varepsilon, A_1 A_2 \ldots A_k F) = (s_1, \mathrm{push}_3(B_2 B_2))$
0.2 $\delta(s_1, \varepsilon, D_1 A_2 \ldots A_k) = \delta(s_1, \varepsilon, D_1 A_2 \ldots A_k F) = (q_0, \mathrm{push}_1(\bar{A}_1))$
2.1.0 $\delta(r_0, \varepsilon, A_1) = (q_0, \bar{A}_1)$
2.1 $\delta(r_0, \varepsilon, A_1\bar{B}_2\omega) = (q_0, \mathrm{push}_1(A_2))$ for every $\omega \in \Gamma^{(k-1)}$
1.2 $\delta(q_0, \varepsilon, \bar{A}_1 L_2\omega) = (r_0, \mathrm{push}_1(A_1))$ for every $\omega \in \Gamma^{(k-1)}$, $L_2 \neq \bar{B}_2$
3 $\delta(q_0, a, \bar{A}_1) = (q_0, \varepsilon)$.

Due to conditions (**P3**) for the initial automata $\mathcal{A}_i$, transitions (2.1),(1.2),(3) do not introduce any non-determinism. Transition (2.1.0) uses a mode (r_0, A_1) which is not used in $A_1 \cdot \delta_2$. Transitions (0.1),(0.2) use a new pushdown symbol. Thus $\mathcal{A}$ is deterministic.

The transitions are chosen so as to make the rules (described in first step) available: D_1 holds by the choice of δ_1, D_2 holds by the choice of δ_2, $G21$ holds by transitions (2.1), $G^{(0)}21$ holds by transition (2.1.0), $G12$ holds by transitions (1.2), E holds by transition (3).

From the initial rules (0.1,0.2), property (30) and the ending rule, we get:

$$(q_0 D_1[\beta_n]q_0) \to^*_{\mathcal{A}} (q_0\bar{A}_1[B_2[\gamma_n\gamma_n]]q_0) \to^*_{\mathcal{A}} a^{f(n)^{g(n)}}. \quad \square$$

Proposition 67 (*Convolution-product*). *Let $f \in \mathbb{S}_{k+1}$ and $g \in \mathbb{S}_k$, for $k \geq 3$. Then $f \times g \in \mathbb{S}_{k+1}$ where $f \times g$ denotes the convolution-product:*

$$(f \times g)(n) = \sum_{m=0}^{n} f(n-m).g(m) \text{ for all } n \in \mathbb{N}.$$

Proof. The proof of this proposition uses the same kind of argument as Proposition 63 concerning the product. We just have to combine the construction given there with a set of rules generating the sequence of pairs $(0, n), (1, n - 1) \ldots (n, 0)$.

First step

Let us suppose we are given a $(k+1)$-dcpda $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ fulfilling condition (**P1**) stated in Lemma 54. We suppose that $\Gamma \supseteq \{D_1, A_1, \bar{A}_1, \hat{A}_1, A'_1, A_2, \ldots, A_k, G, F, E\}$, where the levels are given by the indices, and G, E have level k, F has level $k+1$. As usual the letters $\Omega, \Omega', \Omega''$ used below are undeterminates. The letters G, E will constitute the counters for the sequence g while the letter F will be used in the counters for the sequence f. Let us use the notations: for every $0 \le m \le n$,

$$\beta_n = A_2[\cdots[A_k[F^n]]\cdots]; \quad \gamma_{m,n} = G[F^m]\left(\prod_{j=m+1}^{n} E[F^j]\right)\cdot E[F^n]$$

(notice that $|\gamma_{m,n}| = n - m + 2$),

$$\omega_{k-2}[\Omega] = A_2[\cdots[A_{k-1}[\Omega]]\cdots];$$

Let us suppose that $\mathcal{A}$ allows the following basic derivations:

Initial derivation, D0: for every $n \ge 0$

$$(q_0 D_1[\beta_n \Omega]q_0) \to^*_{\mathcal{A}} (q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{n,n}])^2 \Omega]q_0)$$

f-computation, D1: for every $n \ge 0$

$$(q_0 A_1[\omega_{k-2}[G[F^n]E[\Omega']\Omega'']\Omega]q_0 \to^*_{\mathcal{A}} (q_0 \bar{A}_1[\Omega]q_0)^{f(n)}$$

g-computation, D2: for every $\ell \ge 2, \eta_1, \eta_2, \ldots, \eta_\ell \in F^*$,

$$(q_0 \bar{A}_1[\omega_{k-2}[G[\eta_1]E[\eta_2]\cdots E[\eta_\ell]]\Omega]q_0 \to^*_{\mathcal{A}} (q_0 A'_1[\Omega]q_0)^{g(\ell-2)}$$

pair-generation, D3: for every $1 \le m \le n$

$$(q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{m,n}])^2 \Omega]q_0) \to^*_{\mathcal{A}} (q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{m-1,n}])^2 \Omega]q_0)(q_0 A_1[(\omega_{k-2}[\gamma_{m,n}])^2 \Omega]q_0)$$

D30: for every $0 \le n$

$$(q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{0,n}])^2 \Omega]q_0) \to^*_{\mathcal{A}} (q_0 A_1[(\omega_{k-2}[\gamma_{0,n}])^2 \Omega]q_0)$$

ending rule, E:

$$(q_0 A'_1[\varepsilon]q_0) \to^*_{\mathcal{A}} a.$$

From these rules the following derivations would follow:

$$\begin{aligned}(q_0 D_1[\beta_n]q_0) \to^*_{\mathcal{A}} & \quad (q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{n,n}])^2]q_0) \quad \text{(by rule D0)} \\ \to^*_{\mathcal{A}} & \prod_{m=0}^{n} (q_0 A_1[(\omega_{k-2}[\gamma_{m,n}])^2]q_0) \quad \text{(by rules D3, D30).}\end{aligned} \tag{31}$$

Starting with each factor of this product we derive:

$$\begin{aligned}(q_0 A_1[(\omega_{k-2}[\gamma_{m,n}])^2]q_0) \to^*_{\mathcal{A}} & \quad (q_0 \bar{A}_1[\omega_{k-2}[\gamma_{m,n}]]q_0)^{f(m)} \quad \text{(by rule D1)} \\ \to^*_{\mathcal{A}} & \quad (q_0 A'_1[\varepsilon]q_0)^{g(n-m)\cdot f(m)} \quad \text{(by rule D2).}\end{aligned} \tag{32}$$

Combining the two derivations (31) and (32), we get:

$$\begin{aligned}(q_0 D_1[\beta_n]q_0) \to^*_{\mathcal{A}} & \quad (q_0 A'_1[\varepsilon]q_0)^{\sum_{m=0}^{n} g(n-m)\cdot f(m)} \\ = & \quad (q_0 A'_1[\varepsilon]q_0)^{f \times g(n)}.\end{aligned}$$

Second step
Let us construct such an automaton $\mathcal{A}$. The sequence $f(n)$ is computed by some $(k+1)$-dcpda $\mathcal{A}'_1 = (Q_1, \{a\}, \Gamma_1, \delta_1, q_0, Z)$, where $\Gamma_1 \supseteq \{A_1, \bar{A}_1, A_2, \ldots, A_{k-1}, G, E, F\}$ fulfilling conditions (**P1**), (**P2**), (**P3**) stated in Lemma 54. By a variation of the construction given for Lemma 54, we can build $\mathcal{A}_1$ fulfilling (**P1**), (D1), (**P3**): the main idea is to treat symbol E as if it was a bottom symbol of level k; G plays the role of A_k in Lemma 54.

As well the sequence $g(n)$ is computed by some k-dcpda $\mathcal{A}_2 = (Q_2, \{a\}, \Gamma_2, \delta_2, q_0, Z)$, where $\Gamma_2 \supseteq \{\bar{A}_1, A'_1, A_2, \ldots A_{k-1}, G, E, F\}$, fulfilling conditions (**P1**), (D2), (**P3**). The main idea here, is to start with the automaton $\mathcal{A}'_2$ fulfilling (**P1**), (**P2**), (**P3**), and to replace the single symbol F by two symbols E, G, of level k, playing the same role as F did. We then add a symbol F of level $k+1$ and just "ignore it": every occurrence of F can be changed into ε without any effect on derivation (D2). We choose the alphabets so that $\Gamma_1 \cap \Gamma_2 = \{\bar{A}_1, A_2, \ldots, A_{k-1}, G, E, F\}$.

Let us define $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, s_0, Z)$ where

$$Q = Q_1 \cup Q_2 \cup \{r_1, r_2, r_3, s_1, s_2\}; \quad \Gamma = \Gamma_1 \cup \Gamma_2 \cup \{\hat{A}_1, D_1\};$$

δ is the union of $(\delta_1 \cup \delta_2)$ with the following rules:

0.1 $\delta(q_0, \varepsilon, D_1 A_2 \ldots A_k) = \delta(q_0, \varepsilon, D_1 A_2 \ldots A_k F) = (s_1, \text{push}_k(GE))$
0.2 $\delta(s_1, \varepsilon, D_1 A_2 \ldots A_{k-1} G) = \delta(s_1, \varepsilon, D_1 A_2 \ldots A_{k-1} G F) = (s_2, \text{push}_2(A_2 A_2))$
0.3 $\delta(s_2, \varepsilon, D_1 A_2 \ldots A_{k-1} G) = \delta(s_2, \varepsilon, D_1 A_2 \ldots A_{k-1} G F) = (q_0, \text{push}_1(A_1))$
3.1 $\delta(q_0, \varepsilon, \hat{A}_1 A_2 \ldots A_{k-1} G F) = (r_1, \text{push}_1(\hat{A}_1 A_1))$
3.2 $\delta(r_1, \varepsilon, \hat{A}_1 A_2 \ldots A_{k-1} G F) = (r_2, \text{pop}_2)$
3.3 $\delta(r_2, \varepsilon, \hat{A}_1 A_2 \ldots A_{k-1} G F) = (r_3, \text{pop}_{k+1})$
3.4 $\delta(r_3, \varepsilon, \hat{A}_1 A_2 \ldots A_{k-1} G F) = \delta(r_3, \varepsilon, A_1 A_2 \ldots A_{k-1} G) = (q_0, \text{push}_2(A_2 A_2)$
30 $\delta(q_0, \varepsilon, \hat{A}_1 A_2 \ldots A_{k-1} G) = (q_0, \text{push}_1(A_1))$
4 $\delta(q_0, a, A'_1) = (q_0, \text{pop}_1)$.

Due to conditions (**P3**) for the initial automata $\mathcal{A}_i$, this new automaton $\mathcal{A}$ is still deterministic.

The transitions are chosen so as to make the rules (described in the first step) available: (D0) holds by the choice of rules (0.i), (D1) by the choice of δ_1, (D2) by the choice of δ_2, (D3) by transitions (3.j), (D30) by transition (30) and (D4) by transition (4). We can conclude that $\mathcal{A}$ computes $f \times g$. □

Proposition 68 (Convolution-inverse). *Let $g \in \mathbb{S}_k$, $k \geq 3$, and f be the sequence defined by $f(0) = 1$ and for all $n \geq 0$, $f(n+1) = \sum_{m=0}^{n} f(m) g(n-m)$. Then $f \in \mathbb{S}_{k+1}$.*

Sketch of proof. We use the same notation and follow the same lines as for Proposition 67.
First step Let us suppose we are given a $(k+1)$-dcpda $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$, which is level-partitioned. We suppose that $\Gamma \supseteq \{D_1, A_1, \bar{A}_1, \hat{A}_1, A'_1, A_2, \ldots, A_k, G, F, E, \bar{E}\}$, where this new symbol $\bar{E}$ plays the role of a bottom symbol for the automaton computing g. We call here "blocking pds" every 2-pds $\bar{U}$ of the form $\bar{E}[\omega_1] \cdot \omega_2$, for some ω_i which is a i-pds or $\bar{U} = \varepsilon$. Let us suppose that $\mathcal{A}$ allows the following basic derivations:
Initial derivation, D0: for every $n \geq 0$

$$(q_0 D_1[\beta_n] q_0) \to^*_{\mathcal{A}} (q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{n,n}])^2] q_0)$$

g-computation, D2: for every $\ell \geq 2$, $\eta_1, \eta_2, \ldots, \eta_\ell \in F^*$, and $\bar{U}$, blocking pds

$$(q_0 \bar{A}_1[\omega_{k-2}[G[\eta_1] E[\eta_2] \cdots E[\eta_\ell] \bar{U}] \Omega] q_0 \to^*_{\mathcal{A}} (q_0 A'_1[\Omega] q_0)^{g(\ell-2)}$$

pair-generation, D3: for every $n \geq 0$

$$(q_0 \hat{A}_1[\omega_{k-2}[\gamma_{n+1,n+1} \Omega'] \Omega] q_0) \to^*_{\mathcal{A}} \prod_{m=0}^{n} (q_0 \bar{A}_1[(\omega_{k-2}[\gamma_{m,n} \Omega'])^2 \Omega] q_0)$$

starting pairs, D30:

$$(q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{0,0} \Omega'])^2 \Omega] q_0) \to^*_{\mathcal{A}} (q_0 A'_1[\Omega] q_0)$$

gluing rule, G23: for every $0 \le m \le n$, there exists some blocking pds $\bar{U}$ such that,

$$(q_0 A'_1[\omega_{k-2}[\gamma_{m,n}\Omega']\Omega]q_0) \to^*_{\mathcal{A}} (q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{m,m}\bar{U}\Omega'])^2]\Omega]q_0).$$

ending rule, E:

$$(q_0 A'_1[\varepsilon]q_0) \to^*_{\mathcal{A}} a.$$

Let us prove by induction the following property $\mathbf{P}(n)$: for every $0 \le m \le n$ and every blocking pds $\bar{U}$

$$(q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{m,m}\bar{U}])^2\Omega]q_0) \to^*_{\mathcal{A}} (q_0 A'_1[\Omega]q_0)^{f(m)}.$$

Basis: $\mathbf{P}(0)$ follows from (D30), by substituting $\bar{U}$ for Ω'.
Induction step:

$$\begin{aligned}
(q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{n+1,n+1}\bar{U}])^2]\Omega]q_0) &\to^*_{\mathcal{A}} \prod_{m=0}^{n} (q_0 \bar{A}_1[(\omega_{k-2}[\gamma_{m,n}\bar{U}])^2\Omega]q_0) && \text{(by (D3)}\\
&\to^*_{\mathcal{A}} \prod_{m=0}^{n} (q_0 A'_1[\omega_{k-2}[\gamma_{m,n}\bar{U}]\Omega]q_0)^{g(n-m)} && \text{(by D2)}\\
&\to^*_{\mathcal{A}} \prod_{m=0}^{n} (q_0 \hat{A}_1[(\omega_{k-2}[\gamma_{m,m}\bar{U}_m])^2\Omega]q_0)^{g(n-m)} && \text{(by G23)}\\
&\to^*_{\mathcal{A}} \prod_{m=0}^{n} (q_0 A'_1[\Omega]q_0)^{f(m)\cdot g(n-m)} && \text{(by ind. hyp.)}\\
&= (q_0 A'_1[\Omega]q_0)^{f(n+1)}
\end{aligned}$$

(where all the $\bar{U}_m$ are blocking pds). Using (D0) and (E) we finally obtain:

$$\forall n \in \mathbb{N},\ (q_0 D_1[\beta_n]q_0) \to^*_{\mathcal{A}} a^{f(n)}.$$

Second step:
One can construct an automaton $\mathcal{A}_2$ realizing (D2) and fulfilling also conditions (**P1**, **P3**) of Lemma 54. The other rules can be made valid by a set of transitions, in a way similar to that used in the proof of Proposition 67. □

Remark 69. Let us see the sequence g as a formal power series

$$g = \sum_{n=0}^{\infty} g(n) X^n.$$

Proposition 68 asserts that the series $\frac{1}{1-Xg}$ belongs to $\mathbb{S}_{k+1}$. In other words, the convolution inverse of every formal power series of the form $1 - Xg$, where $g \in \mathbb{S}_k$, belongs to $\mathbb{S}_{k+1}$.

Proposition 70 (*Sequence Composition*). *Let* $k_1 \ge 2$, $k_2 \ge 2$, $f \in \mathbb{S}_{k_1+1}$ *and* $g \in \mathbb{S}_{k_2+1}$, *then* $f \circ g \in \mathbb{S}_{k_1+k_2+1}$.

Construction. By Lemma 54, after a suitable choice for the concrete sets of states and pushdown alphabets, we obtain two $(k+1)$-dcpda $\mathcal{A}_1$ (with counter F) and $\mathcal{A}_2$ (with counter G) fulfilling conditions:

(Q1) $\mathcal{A}_1, \mathcal{A}_2$ are level-partitioned
(Q2.1) $\forall \Omega \in \mathcal{U}$, $(q_0, A_1[A_2[\ldots[A_{k_1}[F^n]]\ldots]\Omega], q_0) \to^*_{\mathcal{A}_1} (q_0, \bar{A}_1[\Omega], q_0)^{f(n)}$.
(Q2.2) $\forall \Omega \in \mathcal{U}$, $(r_0, B_1[B_2[\ldots[B_{k_2}[G^n]]\ldots]\Omega], r_0) \to^*_{\mathcal{A}_2} (r_0, \bar{B}_1[\Omega], r_0)^{g(n)}$.
(Q3.1) δ_1 has no lefthand side of the form $(q_0, \bar{A}_1 \cdot \omega)$ for any $\omega \in \Gamma_1^{(k_1)}$.
(Q3.2) δ_2 has no lefthand side of the form $(r_0, \bar{B}_1 \cdot \omega)$ for any $\omega \in \Gamma_2^{(k_2)}$.
(Q4) $\Gamma_1 \cap \Gamma_2 = \{F\} = \{\bar{B}_1\}$.

We consider the $(k_1 + k_2 + 1)$-dcpda $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, (q_0, r_0), Z)$ where: $Q = Q_1 \times Q_2$, $\Gamma = \Gamma_1 \cup \Gamma_2$ and δ is the union of the two following types of transitions.

4.3.0.9. Transitions inherited from $\mathcal{A}_1$. For all $\delta_1(q_1, \varepsilon, \omega) = \{(q_1', \mathrm{f})\}$, $\omega_1 \in \Gamma_1^{(k_1+1)} - \{\varepsilon\}$,
1 $\delta((q_1, r_0), \varepsilon, \omega_1) = \{((q_1', r_0), \mathrm{f})\}$.

4.3.0.10. Transitions inherited from $\mathcal{A}_2$. For all $\delta_2(r, \varepsilon, \omega_2) = \{(r', \mathrm{f})\}$, $\omega_2 \in \Gamma_2^{(k_2+1)} - \{\varepsilon\}$, $r, r' \in Q_2$, and for all $q_1 \in Q_1$, $\omega_1 \in \Gamma_1^{k_1}$
2 $\delta((q_1, r), \varepsilon, \omega_1 \cdot \omega_2) = \{((q_1, r'), \mathrm{f} + k_1)\}$,
where the notation $\mathrm{f} + k$ means:

- if $\mathrm{f} = \mathrm{pop}_i$ then $\mathrm{f} + k = \mathrm{pop}_{i+k}$
- if $\mathrm{f} = \mathrm{push}_i(\omega)$ then $\mathrm{f} + k = \mathrm{push}_{i+k}(\omega)$.

Proof. Let us prove that the above automaton $\mathcal{A}$ has the required properties.

The fact that the initial automata $\mathcal{A}_i$ are deterministic, entails that no pair of transitions of the same type (1) (resp. (2)) can have the same mode. Now, suppose that there is one transition of type (1) sharing its starting mode with a transition of type (2). We would then have

$$(q, r_0, \omega_1) = (q_1, r, \omega_1' \cdot \omega_2')$$

for some $q, q_1 \in Q_1, r \in Q_2, \omega_1 \in \Gamma_1^{(k_1+1)} - \{\varepsilon\}, \omega_1' \in \Gamma_1^{k_1}, \omega_2' \in \Gamma_2^{(k_2+1)} - \{\varepsilon\}$. By **Q4**, the only possibility for such an equality is that

$$r = r_0, \qquad \omega_2' = F = \bar{B}_1.$$

But, by condition **Q3.2**, there is no transition of δ_2 starting with mode $(r_0, \bar{B}_1)$. Finally, we are sure that $\mathcal{A}$ is a *deterministic* $(k_1 + k_2 + 1)$-dcpda, with counter G.

Let us check now that

$$((q_0, r_0)A_1[\cdots A_{k_1}[B_1[\cdots[B_{k_2}[G^n]]\cdots]]\cdots](q_0, r_0)) \to_{\mathcal{A}}^* ((q_0, r_0)\bar{A}_1[\varepsilon](q_0, r_0))^{g(f(n))}. \tag{33}$$

In order to show such a derivation, we introduce a partial map Φ, from the set of variables $V_{\mathcal{A}}$ (defined in Section 2.3.2 by Eq. (2)) to the set of variables $V_{\mathcal{A}_1}$. Let us define, for every $H \in (k_2 + 1) - \mathrm{pds}(\Gamma_2)$

$$\varphi(H) = F^n \Leftrightarrow (r_0 H r_0) \to_{\mathcal{A}_2}^* (r_0 \bar{B}_1 r_0)^n \tag{34}$$

(hence $\varphi(H)$ is defined exactly for those H such that, $(r_0 H r_0)$ derives (modulo $\mathcal{A}_2$) into $(r_0 \bar{B}_1 r_0)^*$).

For every $T[\Omega_1, \ldots, \Omega_n] \in (k_1 + 1) - \mathrm{uterm}(\Gamma_1 \cup \mathcal{U})$ and every $H_1, \ldots, H_n \in (k_2 + 1) - \mathrm{pds}(\Gamma_2)$, $q \in Q_1$, we define

$$\Phi((q, r_0)T[H_1, \ldots, H_n](q', r_0)) = (qT[\varphi(H_1), \ldots, \varphi(H_n)]q') \tag{35}$$

(hence $\Phi((q, r_0)T[H_1, \ldots, H_n](q', r_0))$ is defined iff, for every Ω_i appearing in T, $H_i \in \mathrm{dom}(\varphi)$). We extend the map Φ over words by setting:

$$\Phi(V_1 V_2 \cdots V_m) = \Phi(V_1)\Phi(V_2)\cdots\Phi(V_m)$$

if for every i, $V_i \in \mathrm{dom}(\Phi)$ and $\Phi(V_1 V_2 \cdots V_m)$ is undefined otherwise.

Lemma 71. *If $U \in \mathrm{dom}(\Phi)$ and $U_1' \in V_{\mathcal{A}_1}^*$ are such that*

$$\Phi(U) \to_{\mathcal{A}_1} U_1'$$

then, there exists a word $U' \in \mathrm{dom}(\Phi)$ such that

$$U \to_{\mathcal{A}}^* U' \quad \& \quad \Phi(U') = U_1'.$$

Let us prove this lemma. It is sufficient to prove it in the case where U is reduced to one variable. Suppose $U = (q, r_0)T[H_1, H_2, \ldots, H_n](q', r_0)$ where $T[\Omega_1, \Omega_2, \ldots, \Omega_n] \in (k_1 + 1) - \mathrm{uterm}(\Gamma_1 \cup \mathcal{U})$, all the H_i belong to $\mathrm{dom}(\varphi)$ and $q, q' \in Q_1$. Without loss of generality we can suppose that T is standard (see Definition 19). We suppose that

$$\Phi(U) \to_{\mathcal{A}_1} U'_1. \tag{36}$$

Let us distinguish three cases, depending on the type of rule used in derivation (36).
Case 1: Decomposition rule.
This means that $T = T' \cdot T''$ and

$$\begin{aligned} qT[\varphi(H_1), \dots, \varphi(H_n)]q' \to_{\mathcal{A}_1} &\ qT'[\varphi(H_1), \dots, \varphi(H_n)]q'' \cdot q''T''[\varphi(H_1), \dots, \varphi(H_n)]q' \\ = &\ U'_1. \end{aligned}$$

In this case

$$U' = (q, r_0)T'[\varphi(H_1), \dots, \varphi(H_n)](q'', r_0) \cdot (q'', r_0)T''[\varphi(H_1), \dots, \varphi(H_n)](q', r_0)$$

fulfills the conclusion of the lemma.
Case 2: A rule which does not use $\varphi(H_1)$, i.e. the leftmost branch of the planar tree $\Phi(U)$ has no common node with the occurrence of $\varphi(H_1)$.

$$qT[\varphi(\vec{H})]q' \to_{\mathcal{A}_1} q''T'[\varphi(\vec{H})]q' = U'_1,$$

for some term T' (which might be non-standard, but belongs to $(k_1 + 1) - \mathrm{uterm}(\Gamma_1 \cup \mathcal{U})$). Let $U' = (q'', r_0)T'[\vec{H}](q', r_0)$. The above remark about T' ensures that $U' \in \mathrm{dom}(\Phi)$, and by a transition of type (1) we have

$$\begin{aligned} (q, r_0)T[\varphi(\vec{H})](q', r_0) \to_{\mathcal{A}_1} &\ (q'', r_0)T[\varphi(\vec{H})](q', r_0) \\ = &\ U'_1. \end{aligned}$$

Case 3: A rule which does use $\varphi(H_1)$.
Subcase 1: Push operation at level $\leq k_1$ and $\varphi(H_1) = \varepsilon$.

$$qT[\varepsilon, \varphi(H_2), \dots, \varphi(H_n)]q' \to_{\mathcal{A}_1} q''T'[\varepsilon, \varphi(H_2), \dots, \varphi(H_n)]q' = U'_1.$$

By a rule of type (1) we also have:

$$(q, r_0)T[\varepsilon, \varphi(H_2), \dots, \varphi(H_n)](q', r_0) \to_{\mathcal{A}} (q'', r_0)T'[\varepsilon, \varphi(H_2), \dots, \varphi(H_n)](q', r_0),$$

hence $U' = (q'', r_0)T'[\varepsilon, H_2, \dots, H_n](q', r_0)$ satisfies the conclusion of the lemma.
Subcase 2: Push operation at level $\leq k_1$ and $\varphi(H_1) = F^{m+1}$.

$$\begin{aligned} qT[F^{m+1}, \varphi(H_2), \dots, \varphi(H_n)]q' \to_{\mathcal{A}_1} &\ q''T'[F^{m+1}, \varphi(H_2), \dots, \varphi(H_n)]q' \\ = &\ U'_1. \end{aligned} \tag{37}$$

By definition of φ, there exists $\hat{H}_1 \in \mathrm{dom}(\varphi)$ such that:

$$(r_0 H_1 r_0) \to^*_{\mathcal{A}_2} (r_0 \bar{B}_1 r_0)(r_0 \hat{H}_1 r_0)$$

and

$$\varphi(\hat{H}_1) = F^m.$$

By rules of type (2) we get

$$(q, r_0)T[H_1, H_2, \dots, H_n](q', r_0) \to^*_{\mathcal{A}} (q, r_0)T[F\hat{H}_1, H_2, \dots, H_n](q', r_0)$$

and then by a rule of type (1) deduced from the rule used in (37):

$$(q, r_0)T[F\hat{H}_1, H_2, \dots, H_n](q', r_0) \to_{\mathcal{A}} (q'', r_0)T'[F\hat{H}_1, H_2, \dots, H_n](q', r_0).$$

Hence $U' = (q'', r_0)T'[F\hat{H}_1, H_2, \dots, H_n](q', r_0)$ satisfies the required conditions.
Subcase 3: Push operation at level $k_1 + 1$.

Subcase 4: Pop operation at level $\leq k_1$.
Subcase 5: Pop operation at level $k_1 + 1$.
These three remaining subcases can be solved in the same way as subcase 2: by a sequence of type (2) moves, $\mathcal{A}$ can make a symbol F appear as leftmost letter of the leftmost block H_1; then a move of type (1) allows us to obtain a suitable U'. The lemma is proved.

Let us prove now derivation (33). We remark that,

$$\Phi(((q_0, r_0)\alpha_{k_1}[\beta_{k_2}[G^n]](q_0, r_0))) = (q_0\alpha_{k_1}[F^{g(n)}]q_0) \rightarrow^*_{\mathcal{A}_1} (q_0\bar{A}_1[\varepsilon]q_0)^{f(g(n))}.$$

Applying Lemma 71 iteratively, we obtain some $U' \in \mathrm{dom}(\Phi)$ such that:

$$((q_0, r_0)\alpha_{k_1}[\beta_{k_2}[G^n]](q_0, r_0)) \rightarrow^*_{\mathcal{A}} U' \quad \& \quad \Phi(U') = (q_0\bar{A}_1[\varepsilon]q_0)^{f(g(n))}.$$

But the only possible value for a pre-image by Φ of $(q_0\bar{A}_1[\varepsilon]q_0)^{f(g(n))}$ is

$$U' = ((q_0, r_0)\bar{A}_1[\varepsilon](q_0.r_0))^{f(g(n))},$$

which proves Proposition 70. □

Let us summarize the closure properties demonstrated in this section.

Theorem 72.

0. *For every $f \in \mathbb{S}_{k+1}$, $k \geq 1$, and every integer $c \in \mathbb{N}$,*
   the sequences Ef (the shift of f), $f + \frac{c}{1-X}$, belong to $\mathbb{S}_{k+1}$;
   if $\forall n \in \mathbb{N}$, $f(n) \geq c$ then $f - \frac{c}{1-X}$ belongs to $\mathbb{S}_{k+1}$;
   the sequence defined by $0 \mapsto c, n+1 \mapsto f(n)$ belongs to $\mathbb{S}_{k+1}$.
1. *For every $f, g \in \mathbb{S}_{k+1}$, $k \geq 1$, the sequence $f + g$ belongs to $\mathbb{S}_{k+1}$.*
2. *For every $f, g \in \mathbb{S}_{k+1}$, $k \geq 2$, $f \odot g$ (the ordinary product), belongs to $\mathbb{S}_{k+1}$ and for every $f' \in \mathbb{S}_{k+2}$, f'^g belong to $\mathbb{S}_{k+2}$.*
3. *For every $f \in \mathbb{S}_{k+1}$, $g \in \mathbb{S}_k$, $k \geq 2$, $f \times g$ (the convolution product) belongs to $\mathbb{S}_{k+1}$.*
4. *For every $g \in \mathbb{S}_k$, $k \geq 2$, the sequence f defined by: $f(0) = 1$ and $f(n+1) = \sum_{m=0}^{n} f(m) \cdot g(n-m)$ (the convolution inverse of $1 - X \times f$) belongs to $\mathbb{S}_{k+1}$.*
5. *For every $f \in \mathbb{S}_k$, $g \in \mathbb{S}_\ell$, $k, l \geq 2$, $f \circ g$ (the sequence composition) belongs to $\mathbb{S}_{k+\ell-1}$.*
6. *For every $k \geq 2$ and for every system of recurrent equations expressed by polynomials in $\mathbb{S}_{k+1}[X_1, \ldots, X_p]$, with initial conditions in $\mathbb{N}$, every solution belongs to $\mathbb{S}_{k+1}$.*

Proof. Point (0) is obvious. Points (2) and (6) have been proved in previous propositions. Points (1), (3)–(5) have been proved in previous propositions, but with the restriction that all sequences involved have a level $k \geq 3$. This is due to the fact that the normal form given in Lemma 54 is proved for level $k \geq 3$ only. For sequences of level 2, one could state a slightly weaker lemma, where $\mathcal{A}'$ is a 2-dpda defined on a pushdown-symbol set $\Gamma' \supseteq \Gamma \cup \{\bar{A}_1, \bar{F}\}$, and a set of states Q', such that:
(**P′1**) $\mathcal{A}'$ is level partitioned , with exactly two distinct symbols of level 2, F, $\bar{F}$
(**P′2.1**) $\forall \Omega \in \mathcal{U}$, $(q_0, A_1[F^n\bar{F}\Omega], q_0) \rightarrow^*_{\mathcal{A}} (q_0, \bar{A}_1[\Omega], q_0)^{f(n)}$,
(**P′2.2**) The only transitions of δ' which have the form $\delta'(q, \varepsilon, L\bar{F}) = (q', \mathrm{pop}_2)$, for some $q, q' \in Q'$, $L \in \Gamma'$, are of the form: $\delta'(q, \varepsilon, \bar{A}_1\bar{F}) = (q_0, \mathrm{pop}_2)$,
(**P3**) δ' has no lefthand side of the form $(q_0, \bar{A}_1 \cdot \omega)$ for any $\omega \in \Gamma' \cup \{\varepsilon\}$.

Conversely, any 2-dpda $\mathcal{A}'$, fulfilling (**P′1**), (**P′2.1**), (**P′2.2**), (**P3**) defines a function f which belongs to $\mathbb{S}_2$. (The idea is that $\bar{F}$ is a blocking symbol, which acts as if it was marking the bottom of the pushdowns at level 2).

Owing to this complementary version of Lemma 54, one can adapt the proofs to the case where some sequences have level 2. □

5. Integer double sequences

We introduce here a notion of *k-computable multiple sequence* $u(n_1, n_2, \ldots, n_r)$. We focus on a particular kind[3] of double sequences $f(m, n)$ wich is needed in the study of simple sequences of rational numbers (see Section 7).

Definition 73. Let $k \geq 2$. The double sequence $f(m, n)$, $0 \leq m \leq n$ is called a *k-computable* double sequence iff, there exists a k-dcp $\mathcal{A}$ such that, for all $0 \leq m \leq n$,

$$(q_0, a^{f(m,n)}, A_1[\ldots A_{k-1}[\gamma_{m,n}]\ldots]) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \varepsilon),$$

where $\gamma_{m,n}$ denotes the 2-pds $\left(\prod_{j=m}^{n-1} A_k[F^j]\right) \cdot \bar{A}_k[F^n]$.

We denote by $\mathbb{S}_k^{(2)}$ the set of level k double sequences.

Lemma 74. *Let $k \geq 2$. Let $b_{1,1}, b_{1,2}, b_{2,1}, b_{2,2}, c_1, c_2 \in \mathbb{S}_{k+1}$. Let us consider the double sequences f_1, f_2 defined by: for every $0 \leq \ell < n$,*

$$\begin{pmatrix} f_1(\ell, n) \\ f_2(\ell, n) \end{pmatrix} = \begin{pmatrix} b_{1,1}(\ell) & b_{1,2}(\ell) \\ b_{2,1}(\ell) & b_{2,2}(\ell) \end{pmatrix} \cdot \begin{pmatrix} f_1(\ell+1, n) \\ f_2(\ell+1, n) \end{pmatrix}$$

and

$$f_1(n, n) = c_1(n), \qquad f_2(n, n) = c_2(n).$$

Then f_1, $f_2 \in \mathbb{S}_{k+1}^{(2)}$.

Proof. We use the notation:

$$\gamma_{m,n} = \left(\prod_{j=m}^{n-1} A_k[F^j]\right) \cdot \bar{A}_k[F^n]; \quad \beta_n = A_2[\ldots A_{k-1}[A_k[F^n]]\ldots]; \quad \omega_{k-2}[\Omega] = A_2[\cdots[A_{k-1}[\Omega]]\cdots];$$

First step:
Let us suppose we are given a $(k+1)$-dcpda $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ which is level partitioned. The alphabet Γ contains at least the following symbols: U_1, U_2 (symbols for f_1, f_2), $D_1, \bar{D}_1, D_2, \bar{D}_2$ (symbols for c_1, c_2), $A_{i,j}, \bar{A}_{i,j}$ (symbols for $b_{i,j}$) and $\{A_1, A_2, \ldots, A_{k-1}, A_k, \bar{A}_k, F\}$. Suppose that the following rules are valid:
Starting rule, D0:

$$(q_0 U_i[\omega_{k-2}[\gamma_{n,n}]\Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 D_i[\beta_n \Omega]q_0)$$

Coefficient rule, D1:

$$(q_0 A_{i,j}[\beta_n \Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 \bar{A}_{i,j}[\Omega]q_0)^{b_{i,j}(n)}$$

Simple sequence rules, D2:

$$(q_0 D_i[\beta_n \Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 \bar{D}_i[\Omega]q_0)^{c_i(n)}$$

Gluing rules, G(i,(i,j)): for $0 \leq m < n$

$$(q_0 U_i[\omega_{k-2}[\gamma_{m,n}]\Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 A_{i,1}[\beta_n \omega_{k-2}[\gamma_{m+1,n}]\Omega]q_0)(q_0 A_{i,2}[\omega_{k-2}[\gamma_{m+1,n}]\Omega]q_0)$$

Gluing rules, G((i,j),j):

$$(q_0 \bar{A}_{i,j}[\Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 U_j[\Omega]q_0)$$

[3] A sytematic and thorough study of multiple sequences would certainly be useful but has been left for future work.

Ending rule, E:

$$(q_0 \bar{D}_i[\varepsilon] q_0) \rightarrow^*_{\mathcal{A}} a.$$

Let us prove, by descending induction over m, that, for every $0 \leq m \leq n$, the following propert $\mathbf{P}(m, n)$ holds:

$$(q_0 U_i[\omega_{k-2}[\gamma_{m,n}]] q_0) \rightarrow^*_{\mathcal{A}} a^{f_i(m,n)}.$$

Basis: $m = n$.
We get the derivation:

$$\begin{array}{lll} (q_0 U_i[\omega_{k-2}[\gamma_{n,n}]] q_0) \rightarrow^*_{\mathcal{A}} & (q_0 D_i[\beta_n] q_0) & \text{(by rule D0)} \\ \rightarrow^*_{\mathcal{A}} & (q_0 \bar{D}_i[\varepsilon] q_0)^{f_i(n,n)} & \text{(by D2)} \\ \rightarrow^*_{\mathcal{A}} & a^{f_i(n,n)} & \text{(by rule E).} \end{array}$$

which establishes $\mathbf{P}(n, n)$.
Induction step: $0 \leq m < n$ and $\mathbf{P}(m+1, n)$ is assumed true. We get the derivation:

$$\begin{array}{lll} & (q_0 U_i[\omega_{k-2}[\gamma_{m,n}]] q_0) & \\ \rightarrow^*_{\mathcal{A}} & (q_0 A_{i,1}[\beta_n \omega_{k-2}[\gamma_{m+1,n}]] q_0)(q_0 A_{i,2}[\omega_{k-2}[\gamma_{m+1,n}]] q_0) & \text{(by rule G(i(i,j)))} \\ \rightarrow^*_{\mathcal{A}} & (q_0 \bar{A}_{i,1}[\beta_n \omega_{k-2}[\gamma_{m+1,n}]] q_0)^{b_{i,1}(n)} (q_0 \bar{A}_{i,2}[\omega_{k-2}[\gamma_{m+1,n}]] q_0)^{b_{i,2}(n)} & \text{(by D1)} \\ \rightarrow^*_{\mathcal{A}} & (q_0 U_1[\omega_{k-2}[\gamma_{m+1,n}]] q_0)^{b_{i,1}(n)} (q_0 U_2[\omega_{k-2}[\gamma_{m+1,n}]] q_0)^{b_{i,2}(n)} & \text{(by G((i,j)j))} \\ \rightarrow^*_{\mathcal{A}} & (a^{f_1(m+1,n)})^{b_{i,1}(n)} (a^{f_2(m+1,n)})^{b_{i,2}(n)} & \text{(by } \mathbf{P}(m+1, n)) \end{array}$$

which establishes $\mathbf{P}(m, n)$.
Second step:
By Lemma 54 there exist automata $\mathcal{B}_{i,j}$ (resp. $\mathcal{C}_i$) computing the sequences $b_{i,j}$ (resp. c_i), and fulfilling conditions (**P1**, **P2**, **P3**). These automata furnish the sets of rules allowing (D1)(D2). As usual we choose the concrete sets of states and pushdown alphabets, in such a way that the only common state between two of them is $\{q_0\}$ and the only common pushdown symbols between two of them are $\{A_2, A_3, \ldots, A_k, \bar{A}_k, F\}$. Each of the rules G((i,j),j) and (D3) can be easily reduced to a single transition. The rules D0 and G(i,(i,j)) can be decomposed into a finite number of transitions, with distinct states for all the intermediate total states. One can check that the modes of the initial total states of the different rules are all distinct. The existence of the deterministic $(k+1)$-dcpda $\mathcal{A}$ follows. □

Lemma 75. *Let $k \geq 2$. Let $b_{1,1}, b_{1,2}, b_{2,1}, b_{2,2}, c, d \in \mathbb{S}_{k+1}$ and $\alpha, \beta \in \mathbb{N}$. Let us consider the double sequences $\overline{f}, \overline{g}$ defined by:*

$$\overline{f}(m, n) = f(0, m, n), \quad \overline{g}(m, n) = g(0, m, n)$$

where the triple sequences f, g fulfill, for every $0 \leq \ell \leq n, 0 \leq m \leq n$

$$\begin{array}{ll} f(n, n, n) & = \alpha \\ g(n, n, n) & = \beta \\ f(\ell, m, n) & = f(\ell+1, m, n) \text{ if } \ell = m, \ell < n \\ g(\ell, m, n) & = g(\ell+1, m, n) \text{ if } \ell = m, \ell < n \\ f(n, m, n) & = c(n) \text{ if } 0 \leq m < n \\ g(n, m, n) & = d(n) \text{ if } 0 \leq m < n \\ f(\ell, m, n) & = b_{1,1}(\ell) f(\ell+1, m, n) + b_{1,2}(\ell) g(\ell+1, m, n) \text{ if } \ell \neq m, \ell < n \\ g(\ell, m, n) & = b_{2,1}(\ell) f(\ell+1, m, n) + b_{2,2}(\ell) g(\ell+1, m, n) \text{ if } \ell \neq m, \ell < n. \end{array}$$

Then, $\overline{f}, \overline{g} \in \mathbb{S}^{(2)}_{k+1}$.

Sketch of proof. Let us represent a triple (ℓ, m, n) where $0 \le \ell \le n$, $0 \le m \le n$ by the following 2-pds $\gamma_{l,m,n}$:

$$\gamma_{\ell,m,n} = \left(\prod_{j=\ell}^{m-1} A_k[F^j]\right) \hat{A}_k[F^m] \left(\prod_{j=m+1}^{n-1} A_k[F^j]\right) \bar{A}_k[F^n] \text{ if } \ell \le m < n,$$

$$\gamma_{\ell,m,n} = \left(\prod_{j=\ell}^{n-1} A_k[F^j]\right) \bar{A}_k[F^n] \text{ if } l \le m < n,$$

$$\gamma_{\ell,m,n} = \left(\prod_{j=\ell}^{n-1} A_k[F^j]\right) A'_k[F^n] \text{ if } \ell \le m = n,$$

where $\hat{A}_k, A'_k$ are two new symbols of level k. The recurrence defining f, g follows essentially the same pattern as in Lemma 74: it is a linear recurrence scheme, where the two variables are ℓ, n while m can be seen as a parameter. These recurrence relations can be translated into rules and finally into a $(k+1)$-dcpda. □

Lemma 76. *If $f(m,n) \in \mathbb{S}_k^{(2)}$, then for every fixed $m_0 \in \mathbb{N}$, $f(m_0, n)$, $f(n,n) \in \mathbb{S}_k$.* ! nice closure under diagonals !

The proof is obvious.

Lemma 77. *If $f(n) \in \mathbb{S}_k$, then the double sequence F defined by: for every $0 \le m \le n$, $F(m,n) = f(n)$ belongs to $\mathbb{S}_k^{(2)}$.*

Lemma 78 (*Ordinary Product*). *Let $f(m,n)$, $g(m,n) \in \mathbb{S}_{k+1}^{(2)}$, $k \ge 2$, then $(f(m,n).g(m,n))_{0 \le m \le n} \in \mathbb{S}_{k+1}^{(2)}$.*

The proof is analogous with that of Proposition 63.

Lemma 79 (*Pseudo-convolution*). *Let $f(m,n) \in \mathbb{S}_k^{(2)}$, $g(m,n) \in \mathbb{S}_{k+1}^{(2)}$, $k \ge 2$. Then the sequence h defined for all $n \ge 0$ by $h(n) = \sum_{0 \le m \le n} f(m,n).g(n-m,n)$ belongs to $\mathbb{S}_{k+1}$.*

The proof is analogous with that of Proposition 67.

6. Application to weak arithmetics

In [17], Elgot and Rabin devise a method for constructing unary predicates P such that the MSO theory of $\langle \mathbb{N}, S, P \rangle$ is decidable (here S denotes the successor relation). Further results in this direction have been established in [33,31, 25,10]. This kind of problem takes place in the more general perspective of studying "weak" arithmetical theories, which possess interesting decidability properties [6].

We use here decidability results on k-pdas in order to demonstrate the decidability of the monadic theory of structures $\langle \mathbb{N}, S, P \rangle$, for a large class of predicates P (Theorems 82 and 92).

6.1. *Extensions of the structure* $\langle \mathbb{N}, S \rangle$

We first consider some graphs having a particular form.

Definition 80 ($\mathbb{N}$-*Graphs*). We call a $\mathbb{N}$-graph, every graph $\mathcal{G} = (V, E)$, labelled over the alphabet $\{a, b, e\}$, such that:

1. $\mathcal{G}$ consists of exactly one path, starting from a vertex v_0, labelled by an infinite word $u_{\mathcal{G}} \in \{a, b, e\}^\omega$
2. The word $u_{\mathcal{G}}$ has infinitely many occurrences of letter a and also infinitely many occurrences of letter b.

Let us denote by $v \xrightarrow{u}_{\mathcal{G}} v'$ the fact that there is a path, labelled by the word u, from vertex v to vertex v', in the graph $\mathcal{G}$. Given a $\mathbb{N}$-graph $\mathcal{G}$, we define an injection $\varphi : \mathbb{N} \to V$ and a predicate $P \subseteq \mathbb{N}$ as follows:

$$\forall v \in V, \forall x \in \{a, b, e\}, (v, x, \varphi(0)) \notin E. \tag{38}$$

$$\forall n \in \mathbb{N}, \exists u \in \{b, e\}^* a, \varphi(n) \xrightarrow{u}_{\mathcal{G}} \varphi(n+1). \tag{39}$$

$$\forall n \in \mathbb{N}, n \in P \Leftrightarrow \exists v \in V, u \in e^* b, \varphi(n) \xrightarrow{u}_{\mathcal{G}} v. \tag{40}$$

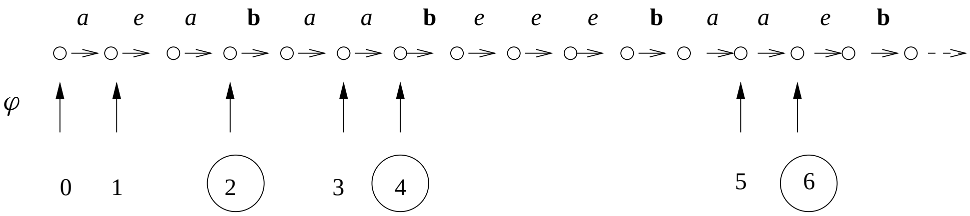

Fig. 7. A $\mathbb{N}$-graph.

(We give an example on Fig. 7; the integers fulfilling P are surrounded by a circle.) The map φ is well-defined because the word $u_{\mathcal{G}}$ has infinitely many occurrences of a. It is injective because $\mathcal{G}$ consists of a path.

Let us consider the structure $\langle V, (R_\alpha)_{\alpha\in\{a,b,e\}}\rangle$ defined by

$$R_\alpha = \{(v, v') \in V \times V, v \xrightarrow{\alpha}_{\mathcal{G}} v'\}.$$

Lemma 81. *Let $\mathcal{G}$ be a $\mathbb{N}$-graph $\mathcal{G}$ and let φ, P be the map (resp. the predicate) defined by* (38) *and* (39) *(resp.* (40)*). Then, the map φ is an MSO-interpretation from $\langle \mathbb{N}, S, P\rangle$ into $\langle V, (R_\alpha)_{\alpha\in\{a,b,e\}}\rangle$.*

(We recall that MSO-interpretations are introduced in Definition 1.)

Proof. For every $v \in V$, $n, m \in \mathbb{N}$, the following equivalences hold:

$$v \in \mathrm{Im}(\varphi) \Leftrightarrow v \text{ has no predecessor or } \exists v' \in V, v' \xrightarrow{a}_{\mathcal{G}} v \tag{41}$$

$$S(n, m) \Leftrightarrow \exists u \in \{b, e\}^* a, \varphi(n) \xrightarrow{u}_{\mathcal{G}} \varphi(m) \tag{42}$$

$$P(n) \Leftrightarrow \exists v \in V, u \in e^* b, \varphi(n) \xrightarrow{u}_{\mathcal{G}} v. \tag{43}$$

The righthand side of equivalence (41) (resp. (42) and (43)) can be expressed under the form $\Phi_1(v)$ (resp. $\Phi_2(\varphi(n), \varphi(m))$, $\Phi_3(\varphi(n))$) for some MSO-formulas Φ_1, Φ_2, Φ_3. Hence φ is an MSO-interpretation. $\square$

Theorem 82. *Let us associate with every sequence $f \in \mathbb{S}_k$ the predicate*

$$P_f = \left\{ \sum_{0\le i\le n} f(i) \mid n \in \mathbb{N} \right\}.$$

Then, the structure $\langle \mathbb{N}, S, P_f\rangle$ has a decidable MSO-theory.

Proof.
Case 1: f is ultimately 0.
In this case the predicate P_f is expressible in the MSO-theory of $\langle \mathbb{N}, S\rangle$. Hence the theorem is true, by Büchi's theorem [8].
Case 2: f is not ultimately 0.
Let us consider the k-dcpda $\mathcal{A}'$ constructed in the proof of Lemma 47. Let $\mathcal{A}''$ be the k-dcpda over the terminal alphabet $\{a, b, e\}$ obtained from $\mathcal{A}'$ by replacing every ε-transition $\delta'(q, \varepsilon, \gamma) = (q', f)$ by a transition $\delta''(q, e, \gamma) = (q', f)$, and copying the transitions which read letter a or b. Let $\mathcal{C}_0(\mathcal{A}'')$ be the structure associated with $\mathcal{A}''$ (see Definition 39). By Lemma 47, $\mathcal{C}_0(\mathcal{A}'')$ admits an infinite path, starting from (q_0, Z') and labelled by an infinite word

$$u = u_0 b u_1 b \cdots b u_n b \cdots \text{ where } |u_n|_a = f(n), \quad u_n \in \{a, e\}^*. \tag{44}$$

Moreover, $\mathcal{C}_0(\mathcal{A}'')$ has no edge going outside of this path, hence it meets condition 1 of Definition 80. The assumption that f is not ultimately 0 and the special form of the word u given in (44) entail that $\mathcal{C}_0(\mathcal{A}'')$ also meets condition 2 of Definition 80, hence it is a $\mathbb{N}$-graph.

The special form of the word u in (44) shows that the predicate P associated with $\mathcal{C}_0(\mathcal{A}'')$ is exactly P_f. By Lemma 81, $\langle \mathbb{N}, S, P_f\rangle$ is MSO-interpretable inside $\mathcal{C}_0(\mathcal{A}'')$, and by Theorem 40 and 2, it follows that $\langle \mathbb{N}, S, P_f\rangle$ has a decidable MSO-theory. $\square$

6.2. Differentiably, k-computable sequences

The particular form of the predicates P_f considered in Theorem 82 leads naturally to the following class of sequences.

Definition 83. Let $k \geq 2$. We define the class $\Sigma\mathbb{S}_k \subseteq \mathbb{N}^{\mathbb{N}}$ as the set

$$\Sigma\mathbb{S}_k = \{\Sigma v \mid v \in \mathbb{S}_k\}.$$

The definition of the operator Σ, as well as other classical definitions about sequences are recalled in Section 2.4.

Fact 84. *Let $k \geq 2$ and $u \in \mathbb{N}^{\mathbb{N}}$. The sequence u belongs to $\Sigma\mathbb{S}_k$ if and only if Δu belongs to $\mathbb{S}_k$.*

This follows easily from point (0) of Theorem 72. After this fact we name "differentiably k-computable sequences" the elements of $\Sigma\mathbb{S}_k$.

Lemma 85. *Let $k \geq 1$ and $U \in \Sigma\mathbb{S}_{k+1}$. Then $\mathsf{E}U \in \Sigma\mathbb{S}_{k+1}$.*

Proof. Suppose that $U \in \Sigma\mathbb{S}_{k+1}$. We notice that $\Delta\mathsf{E}U = \mathsf{E}\Delta U$. Using Fact 84 and stability of $\mathbb{S}_{k+1}$ by shift, we obtain that $\mathsf{E}U \in \Sigma\mathbb{S}_{k+1}$. $\square$

Lemma 86. *Let $k \geq 1$ and $U, V \in \Sigma\mathbb{S}_{k+1}$. Then $U + V \in \Sigma\mathbb{S}_{k+1}$.*

Lemma 87. *Let $k \geq 2$ and $U, V \in \Sigma\mathbb{S}_{k+1}$. Then $U \odot V \in \Sigma\mathbb{S}_{k+1}$.*

Proof. Let $U, V \in \Sigma\mathbb{S}_{k+1}$. The following identity is well-known:

$$\Delta(U \odot V) = \Delta(U \odot \mathsf{E}V) + U \odot (\Delta V).$$

By Theorem 72, the sequences $U, V, \mathsf{E}V$ all belong to $\mathbb{S}_{k+1}$, and the righthand side of the above identity must belong to $\mathbb{S}_{k+1}$. By Fact 84 $U \odot V \in \Sigma\mathbb{S}_{k+1}$. $\square$

Lemma 88. *Let $k \geq 1$ and $U, V \in \Sigma\mathbb{S}_{k+1}$. Then $U \times V \in \Sigma\mathbb{S}_{k+1}$.*

Proof. Let $U = \Sigma u$ and $V = \Sigma v$ for some $u, v \in \mathbb{S}_{k+1}$. Let us transform the expression $\Delta(U \times V)$ into an expression which does not use the operator Δ any more.

$$\begin{aligned}
\Delta(\Sigma u \times \Sigma v) &= \frac{(\Sigma u \times \Sigma v)(1 - X) - \Sigma u(0) \cdot \Sigma v(0)}{X} \\
&= \frac{\frac{u \times v}{1-X} - \frac{u(0) \cdot v(0)}{1-0}}{X} \\
&= \mathsf{E}\left(\frac{u \times v}{1 - X}\right) \\
&= \mathsf{E}\Sigma(u \times v).
\end{aligned}$$

By Theorem 72, the final expression obtained belongs to $\mathbb{S}_{k+1}$, hence $\Sigma u \times \Sigma v$ belongs to $\Sigma\mathbb{S}_{k+1}$. $\square$

Lemma 89. *Let $V \in \Sigma\mathbb{S}_k$, $k \geq 2$, such that $V(0) \geq 1$. Let U be the sequence defined by*

$$U(0) = 1 \text{ and for all } n \geq 0, U(n+1) = \sum_{k=0}^{n} U(k) \cdot V(n-k).$$

Then $U \in \Sigma\mathbb{S}_{k+1}$.

Proof. Let $v \in \mathbb{S}_{k+1}$ such that $V = \Sigma v$. As asserted in Remark 69

$$U = \frac{1}{1 - \frac{Xv}{1-X}}.$$

Let us compute the series ΔU.

$$\Delta U = \Delta\left(\frac{1}{1-\frac{Xv}{1-X}}\right) = \Delta\left(\frac{1-X}{1-X-Xv}\right) = \frac{1}{X}\left[\frac{(1-X)^2}{1-X-Xv} - 1\right]$$

hence

$$\Delta U = \frac{1}{X}\left[\frac{(-X+X^2+Xv)}{1-X-Xv}\right]. \tag{45}$$

Let us compute the series $U \times v - 1$:

$$U \times v - 1 = \frac{(1-X)v}{1-X-Xv} - 1 = \frac{1}{X}\left[\frac{(-X+X^2+Xv)}{1-X-Xv}\right]. \tag{46}$$

From Eqs. (45) and (46) we get the identity:

$$\Delta U = U \times v - 1. \tag{47}$$

By the stability properties established in Theorem 72, U belongs to $\mathbb{S}_{k+1}$ and $U \times v$ belongs to $\mathbb{S}_{k+1}$ too. The hypothesis that $U(0) = 1$ and $V(0) \geq 1$ ensure that $U \times v - 1$ belongs to $\mathbb{S}_{k+1}$. Formula (47) shows that $U \in \Sigma\mathbb{S}_{k+1}$. □

Lemma 90. *Let $k_1 \geq 1, k_2 \geq 1, U \in \Sigma\mathbb{S}_{k_1+1}$ and $V \in \Sigma\mathbb{S}_{k_2+1}$. Then $U \circ V \in \Sigma\mathbb{S}_{k_1+k_2+1}$.*

Sketch of proof. Let $U = \Sigma u, V = \Sigma v$ for some $u \in \mathbb{S}_{k_1+1}, v \in \mathbb{S}_{k_2+1}$. Then $U \circ V = \sum_{k=0}^{k=V(n)} u(k)$. Hence

$$(\Delta(U \circ V))(n) = \sum_{k=V(n)+1}^{k=V(n+1)} u_k.$$

Some (k_1+k_2+1)-dcpda computing $U \circ V$ can be constructed along the following lines.

We suppose, in a first step, that $k_1 \geq 2, k_2 \geq 2$. Let us notice that, by Theorem 72, point (6), V belongs also to $\mathbb{S}_{k_2+1}$. By Lemma 54, there exists a (k_1+1)-dcpda $\mathcal{A}$ over a set of pushdown symbols $\Gamma_{\mathcal{A}} \supseteq \{A_1, \bar{A}_1, A_2, \ldots, A_{k_1}, F\}$, a (k_2+1)-dcpda $\mathcal{B}$ over a set of pushdown symbols $\Gamma_{\mathcal{B}} \supseteq \{B_1, \bar{B}_1, B_2, \ldots, B_{k_2}, G\}$ and a (k_2+1)-dcpda $\mathcal{C}$ over a set of pushdown symbols $\Gamma_{\mathcal{C}} \supseteq \{C_1, \bar{C}_1, C_2, \ldots, C_{k_2}, G\}$, with sets of states $Q_{\mathcal{A}} \ni q_0, Q_{\mathcal{B}}, Q_{\mathcal{C}}$, chosen in such a way that:

$$Q_{\mathcal{B}} \cap Q_{\mathcal{C}} = \{r_0\}$$

$$(q_0, A_1[A_2[\ldots[A_{k_1}[F^n]]\ldots]\Omega], q_0) \to^*_{\mathcal{A}} (q_0, \bar{A}_1[\Omega], q_0)^{u(n)} \tag{48}$$

$$(r_0, B_1[B_2[\ldots[B_{k_2}[F^n]]\ldots]\Omega], r_0) \to^*_{\mathcal{B}} (r_0, \bar{B}_1[\Omega], r_0)^{v(n)} \tag{49}$$

$$(r_0, C_1[C_2[\ldots[C_{k_2}[F^n]]\ldots]\Omega], r_0) \to^*_{\mathcal{C}} (r_0, \bar{C}_1[\Omega], r_0)^{V(n)}. \tag{50}$$

Derivation (49) shows the existence of a sequence $H_1, \ldots, H_i, \ldots, H_{v(n)}$ of elements of $(k_2+1)-\mathrm{pds}(\Gamma_{\mathcal{B}} \cup \{\Omega\})$ fulfilling:

$$H_{v(n+1)} = B_1[B_2[\ldots[B_k[F^{n+1}]]\ldots]\Omega], \quad H_1 = \bar{B}_1[\Omega],$$
$$(r_0 H_{i+1} r_0) \to_{\mathcal{B}} (r_0 \bar{B}_1[\Omega] r_0)(r_0 H_i[\Omega] r_0).$$

By a construction analogous with that of Proposition 70, we obtain a (k_1+k_2+1)-dcpda $\mathcal{D}$, over a pushdown alphabet $\Gamma \supseteq \Gamma_{\mathcal{A}} \cup \Gamma_{\mathcal{B}} \cup \Gamma_{\mathcal{C}} \cup \{\hat{A}_1, D_1, D_2, \ldots, D_{k_2}\}$ and a set of states $Q \supseteq Q_{\mathcal{A}} \times (Q_{\mathcal{B}} \cup Q_{\mathcal{C}})$, making the following rules valid:

argument generation, D1: for every $n \geq 0$

$$(q_0, r_0)\hat{\alpha}_{k_1}[D_1[\cdots[D_{k_2}[F^n]]\cdots]](q_0, r_0) \to^*_{\mathcal{D}} \prod_{i=1}^{v(n+1)} (q_0, r_0)\alpha_{k_1}[H_i \cdot \chi_{k_2}[F^n]](q_0, r_0)$$

$u \circ v$-computation, D2: for every $n \geq 0$, $v(n+1) \geq i \geq 1$

$$(q_0, r_0)\alpha_{k_1}[H_i \cdot \chi_{k_2}[F^n]](q_0, r_0) \rightarrow^*_{\mathcal{D}} ((q_0, r_0)\bar{D}_1[\varepsilon](q_0, r_0))^{u(i+V(n))}$$

where $\alpha_{k_1}[\Omega]$, $\hat{\alpha}_{k_1}[\Omega]$, $\chi_{k_2}[\Omega]$ are the terms

$$\alpha_{k_1}[\Omega] = A_1[A_2[\cdots[A_{k_1}[\Omega]]\cdots]]; \quad \hat{\alpha}_{k_1}[\Omega] = \hat{A}_1[A_2[\cdots[A_{k_1}[\Omega]]\cdots]],$$
$$\chi_{k_2}[\Omega] = C_1[C_2[\cdots[C_{k_2}[\Omega]]\cdots]].$$

Combining (D1) and (D2) we finally obtain:

$$\begin{aligned}(q_0, r_0)\hat{\alpha}_{k_1}[D_1[\cdots[D_{k_2}[F^n]]\cdots]](q_0, r_0) &\rightarrow^*_{\mathcal{D}} \prod_{i=1}^{v(n+1)} ((q_0, r_0)\bar{D}_1[\varepsilon](q_0, r_0))^{u(i+V(n))} \\ &= ((q_0, r_0)\bar{D}_1[\varepsilon](q_0, r_0))^{\Delta(U \circ V)(n)}.\end{aligned}$$

Let us examine now the case where $k_1 = 1$ or $k_2 = 1$. The previous arguments can be adapted, owing to the modification of Lemma 54 described in the proof of Theorem 72. $\square$

Lemma 91. *Let $k \geq 2$. Let $U_1, U_2, \ldots, U_p$ be sequences of integers, $P_1, P_2, \ldots, P_p$ be polynomials in $\Sigma\mathbb{S}_{k+1}[X_1, X_2, \ldots, X_p]$, $c_1, c_2, \ldots, c_p \in \mathbb{N}$ such that: for every $1 \leq i \leq p$*

$$U_i(n+1) = P_i(n, U_1(n), U_2(n), \ldots, U_p(n)) \text{ and } c_i = U_i(0) \leq U_i(1).$$

Then $U_1 \in \Sigma\mathbb{S}_{k+1}$.

Proof. Let U_i, P_i, c_i fulfill the hypothesis of the lemma. Let $a_0(n), a_1(n), \ldots, a_q(n)$ be a sequence enumerating all the coefficients of the polynomials $P_1, P_2, \ldots, P_p$. There exist some polynomials $Q_i \in \mathbb{N}[X_0, \ldots, X_q, X_{q+1}, \ldots, X_{q+p}]$, such that, for every $1 \leq i \leq p$:

$$P_i(n, U_1(n), U_2(n), \ldots, U_p(n)) = Q_i(a_0(n), \ldots, a_q(n), U_1(n), \ldots, U_p(n)).$$

The Euler–MacLaurin formula applied to polynomials Q_i expresses the difference $Q_i(X_0, \ldots, X_{q+p}) - Q_i(Y_0, \ldots, Y_{q+p})$ under the form:

$$\sum_{\bar{k}} \frac{1}{\bar{k}!} \frac{\partial^{\bar{k}} Q_i}{(\partial X_0)^{k_0} \ldots (\partial X_{q+p})^{k_{q+p}}}(Y_0, \ldots, Y_{q+p}) \cdot (X_0 - Y_0)^{k_0} \cdots (X_{q+p} - Y_{q+p})^{k_{q+p}}, \tag{51}$$

where $\bar{k} = (k_1, k_2, \ldots, k_{q+p})$ varies over all the $(q+p)$-tuples with sum $k_1 + k_2 + \cdots + k_{q+p}$ smaller or equal to the degree of Q_i. For every monomial $M = X_0^{d_0} X_1^{d_1} \cdots X_{q+p}^{d_{q+p}}$ the partial derivative

$$\frac{1}{\bar{k}!} \frac{\partial^{\bar{k}} M}{(\partial X_0)^{k_0} \ldots (\partial X_{q+p})^{k_{q+p}}}(Y_0, \ldots, Y_{q+p}),$$

is equal to

$$\binom{d_0}{k_0}\binom{d_1}{k_1} \cdots \binom{d_{q+p}}{k_{q+p}} \cdot Y_0^{d_0 - k_0} Y_1^{d_1 - k_1} \cdots Y_{q+p}^{d_{q+p} - k_{q+p}}. \tag{52}$$

Every partial derivative

$$R_{i,\bar{k}} = \frac{\partial^{\bar{k}} Q_i}{(\partial X_0)^{k_0} \ldots (\partial X_{q+p})^{k_{q+p}}}(Y_0, \ldots, Y_{q+p})$$

is a linear combination, with coefficients in $\mathbb{N}$, of monomials of the form (52), hence it has only non-negative integer coefficients:

$$R_{i,\bar{k}} \in \mathbb{N}[Y_0, \ldots, Y_{q+p}].$$

Let us apply the following substitution to the undeterminates $X_0, \ldots, X_{q+p}, Y_0, \ldots, Y_{q+p}$,

$$X_j \leftarrow a_j(n+1) \text{ for } 0 \leq j \leq q; \quad X_{q+\ell} \leftarrow U_\ell(n+1) \text{ for } 0 \leq \ell \leq p,$$
$$Y_j \leftarrow a_j(n) \text{ for } 0 \leq j \leq q; \quad Y_{q+\ell} \leftarrow U_\ell(n) \text{ for } 0 \leq \ell \leq p.$$

We obtain: $(\Delta U_i)(n+1) =$

$$\sum_{\bar{k}} R_{i,\bar{k}}(a_0(n+1), \ldots, a_q(n+1), U_1(n), \ldots, U_p(n)) \cdot (\Delta\bar{a}(n))^{\bar{k}} \cdot (\Delta\bar{U}(n))^{\bar{k}} \tag{53}$$

where the expression $(\Delta\bar{a})^{\bar{k}}(n)$ means: $(\Delta a_0)^{k_0}(n) \cdots (\Delta a_q)^{k_q}(n)$
and the expression $(\Delta\bar{U})^{\bar{k}}(n)$ means: $(\Delta U_1)^{k_{q+1}}(n) \cdots (\Delta U_p)^{k_{q+p}}(n)$.
By the closure properties established in Theorem 72, every sequence $R_{i,\bar{k}}(a_0(n+1), \ldots, a_q(n+1), U_1(n), \ldots, U_p(n))$ belongs to $\mathbb{S}_{k+1}$. Eq. (53) is thus a system of polynomial equations, with coefficients in $\mathbb{S}_{k+1}$, with initial conditions $U_i(1) - U_i(0) \in \mathbb{N}$ and whose vector of solutions is:

$$((\Delta U_1)(n), \ldots, (\Delta U_p)(n)).$$

By Theorem 72, point (4), all the $(\Delta U_i)(n)$ belong to $\mathbb{S}_{k+1}$, which proves that all the $U_i(n)$ belong to $\Sigma\mathbb{S}_{k+1}$. □

Let us summarize the closure properties demonstrated in this subsection.

Theorem 92.

0. *For every $U \in \Sigma\mathbb{S}_{k+1}$, $k \geq 1$, and every integer $c \in \mathbb{N}$,*
   the sequences EU (the shift of U), $U + \frac{c}{1-X}$ (adding number c to every term), belong to $\Sigma\mathbb{S}_{k+1}$;
   if every $U(n) \geq c$ then $U - \frac{c}{1-X}$ (subtracting number c from every term) belongs to $\Sigma\mathbb{S}_{k+1}$;
   if the number $U(0)$ is greater or equal to c, then the sequence defined by $0 \mapsto c, n+1 \mapsto U(n)$ belongs to $\Sigma\mathbb{S}_{k+1}$.
1. *For every $U, V \in \Sigma\mathbb{S}_{k+1}$, $k \geq 1$, the sequence $U + V$ belong to $\Sigma\mathbb{S}_{k+1}$.*
2. *For every $U, V \in \Sigma\mathbb{S}_{k+1}$, $k \geq 2$, the sequence $U \odot V$ (the ordinary product) belongs to $\Sigma\mathbb{S}_{k+1}$.*
3. *For every $U \in \Sigma\mathbb{S}_{k+1}$, $V \in \Sigma\mathbb{S}_k$, $k \geq 2$, $U \times V$ (the convolution product) belongs to $\Sigma\mathbb{S}_{k+1}$.*
4. *For every $V \in \Sigma\mathbb{S}_k$, $k \geq 2$, such that $V(0) \geq 1$, the sequence U defined by: $U(0) = 1$ and $U(n+1) = \sum_{m=0}^{n} U(m) \cdot V(n-m)$ (the convolution inverse of $1 - XV$) belongs to $\Sigma\mathbb{S}_{k+1}$.*
5. *For every $U \in \Sigma\mathbb{S}_k$, $V \in \Sigma\mathbb{S}_\ell$, $k, l \geq 2$, $U \circ V$ (the sequence composition) belongs to $\Sigma\mathbb{S}_{k+\ell-1}$.*
6. *For every $k \geq 2$, if $U_1(n), \ldots, U_p(n)$ is the vector of solutions of a system of recurrent equations expressed by polynomials in $\Sigma\mathbb{S}_{k+1}[X_1, \ldots, X_p]$, with initial conditions $U_i(0), U_i(1) \in \mathbb{N}$, with $U_i(0) \leq U_i(1)$, then $U_1 \in \Sigma\mathbb{S}_{k+1}$.*

Let us recall that, by Theorem 82, for every sequence $U \in \Sigma\mathbb{S}_{k+1}$, the predicate $P = \{U(n) \mid n \in \mathbb{N}\}$ leads to a structure $\langle \mathbb{N}, S, P \rangle$ which has a decidable Monadic Second Order theory.

7. Sequences of rational numbers

We define here a class of sequences of *rational* numbers that can be described by k-level automata. The results of Section 4 showing that many natural operations over sequences can be translated as operations over automata are carried over this more general situation.

Definition 93. Let $\mathbb{S}$ be a set of sequences of natural integers. We denote by $\mathcal{D}(\mathbb{S})$ the set of sequences $(u_n)_{n\geq 0}$ of the form:

$$u_n = a_n - b_n \quad \text{for all } n \geq 0,$$

for some sequences $a, b \in \mathbb{S}$. We denote by $\mathcal{F}(\mathbb{S})$ the set of sequences $(r_n)_{n\geq 0}$ of the form:

$$r_n = \frac{a_n - b_n}{a'_n - b'_n} \quad \text{for all } n \geq 0,$$

for some sequences $a, b, a', b' \in \mathbb{S}$.

Theorem 94. *Let u, v be sequences of rational numbers in $\mathcal{F}(\mathbb{S}_k)$ (resp. $\mathcal{D}(\mathbb{S}_k)$) for some integer $k \geq 3$. Then the sequences of rational numbers $u + v, u - v, u \odot v$ are in $\mathcal{F}(\mathbb{S}_k)$ (resp. $\mathcal{D}(\mathbb{S}_k)$).*

If v does not vanish, then $\frac{u}{v}$ is in $\mathcal{F}(\mathbb{S}_k)$ too.

Proof. This Theorem 94 is clear since, by Theorem 72, $\mathbb{S}_k$ is closed under sum and ordinary product. □

Let us define the equality problem for sequences in $\mathcal{F}(\mathbb{S}_k)$ as the following algorithmic problem:

INPUT: two sequences $u, v \in \mathcal{F}(\mathbb{S}_k)$,

QUESTION: $u = v$?

i.e. is it true that, $\forall n \in \mathbb{N}, u_n = v_n$?

Corollary 95. *Let $k \geq 3$. The equality problem for sequences in $\mathcal{F}(\mathbb{S}_k)$ reduces to the equivalence problem for deterministic counter k-pushdown automata.*

Proof. Just notice that $\frac{a-b}{a'-b'} = \frac{c-d}{c'-d'}$ iff

$$ac' + bd' + a'd + b'c = ad' + bc' + a'c + b'd. \tag{54}$$

By Theorem 72 each side of this equation is recognized by a single k-dcpda that can be computed from the eight automata defining $a, b, c, d, a', b', c', d'$. Eq. (54) can thus be considered as an instance of the equivalence problem for k-dcpda. □

Let us notice that, by Theorem 94, $(\mathcal{D}(\mathbb{S}_k), +, \cdot)$ and $(\mathcal{F}(\mathbb{S}_k), +, \cdot)$ are rings. We denote by $P(n, X_1, \ldots, X_j, \ldots, X_p)$ any element of $\mathcal{F}(\mathbb{S}_k)[X_1, \ldots, X_j, \ldots, X_p]$ to emphasize the fact that the coefficients of P are functions of the integer argument n.

Theorem 96. *Let $P_i(n, X_1, \ldots, X_j, \ldots, X_p)$, for $1 \leq i \leq p$, be polynomials with coefficients in $\mathcal{F}(\mathbb{S}_k)$ $(k \geq 3)$ and $c_1, c_2, \ldots, c_p \in \mathbb{Q}$. Let u_i, for $1 \leq i \leq p$, be sequences defined by $u_i(n+1) = P_i(n, u_1(n), \ldots, u_j(n), \ldots u_p(n))$, and $u_i(0) = c_i$. Then $u_1 \in \mathcal{F}(\mathbb{S}_k)$.*

Proof. We suppose the sequences $u_1(n), u_2(n), \ldots, u_p(n)$ fulfill the recurrence

$$u_i(n+1) = P_i(n, u_1(n), \ldots, u_j(n), \ldots u_p(n)) \tag{55}$$

for $1 \leq i \leq p, n \in \mathbb{N}$ and

$$u_i(0) = c_i. \tag{56}$$

We prove the theorem in three steps.
Step 1: Case where the $P_i \in \mathcal{D}(\mathbb{S}_k)[X_1, \ldots, X_p], c_i \in \mathbb{Z}$.
Let us consider the polynomial

$$Q_i(n, X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p) = P_i(n, X_1 - Y_1, X_2 - Y_2, \ldots, X_p - Y_p).$$

It can be decomposed as a sum of monomials of the form

$$\epsilon \cdot u_n \cdot X_1^{\alpha_1} Y_1^{\beta_1} X_2^{\alpha_2} Y_2^{\beta_2} \cdots X_p^{\alpha_p} Y_p^{\beta_p}$$

where $\epsilon \in \{+1, -1\}, \alpha_i, \beta_i \in \mathbb{N}, (u_n)_{n \in \mathbb{N}} \in \mathbb{S}_k$. The polynomials Q_i can thus be decomposed as

$$\begin{aligned} &Q_i(n, X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p) \\ &= Q_i^+(n, X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p) - Q_i^-(n, X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p) \end{aligned}$$

with $Q_i^+, Q_i^- \in \mathbb{S}_k[X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p]$. As well,

$$c_i = c_i^+ - c_i^-$$

for some $c_i^+, c_i^- \in \mathbb{N}$. Let us define new sequences $u_i^+(n), u_i^-(n)$ by:

$$u_i^\epsilon(n+1) = Q_i^\epsilon(n, u_1^+(n), u_1^-(n), \ldots, u_j^+(n), u_j^-(n), \ldots, u_p^+(n), u_p^-(n)) \tag{57}$$

for $1 \le i \le p, \epsilon \in \{+, -\}, n \in \mathbb{N}$, and

$$u_i^\epsilon(0) = c_i^\epsilon \tag{58}$$

for $1 \le i \le p, \epsilon \in \{+, -\}$.

From the recurrence (57), the initial conditions (58) and the definition of Q_i, Q_i^ϵ one can see that the sequences $u_i^+ - u_i^-$ are fulfilling recurrence (55) and initial condition (56). Hence $u_1 = u_1^+ - u_1^-$ where $u_1^\epsilon \in \mathbb{S}_k$, which shows that $u_1 \in \mathcal{D}(\mathbb{S}_k)$.

Step 2: Case where the $P_i \in \mathcal{F}(\mathbb{S}_k)[X_1, \dots, X_p]$ and are all homogeneous of the same degree $d \in \mathbb{N}$, $c_i \in \mathbb{Q}$.

As the set $\mathcal{D}(\mathbb{S}_k)$ is closed under the Hadamard product, we can suppose that all the coefficients of the polynomials P_i are sequences of the form

$$\frac{A(n)}{D(n)}$$

for different sequences $A \in \mathcal{D}(\mathbb{S}_k)$ and a single sequence $D \in \mathcal{D}(\mathbb{S}_k)$.

The Eqs. (55) and (56) can then be rewritten as

$$D(n) \cdot u_i(n+1) = R_i(n, u_1(n), \dots, u_j(n), \dots, u_p(n))$$

for $1 \le i \le p, n \in \mathbb{N}$, $R_i \in \mathcal{D}(\mathbb{S}_k)[X_1, \dots, X_p]$, where the R_i are homogeneous of degree d, and

$$u_i(0) = c_i,$$

for $1 \le i \le p$.

Let us define the sequence $F(n)$ by:

$$F(n+1) = D(n) \cdot F(n)^d$$

for all $n \ge 0$, and

$$F(0) = 1.$$

One can check that:

$$F(n+1) \cdot u_i(n+1) = R_i(n, F(n)u_1(n), \dots, F(n)u_j(n), \dots, F(n)u_p(n))$$

for $1 \le i \le p, n \in \mathbb{N}$ and

$$F(0)u_i(0) = c_i.$$

Using step 1 of this proof, we know that both sequences $(F(n) \cdot u_1(n))_{n \ge 0}$ and $(F(n))_{n \ge 0}$ belong to $\mathcal{D}(\mathbb{S}_k)$. It follows that $u_1 \in \mathcal{F}(\mathbb{S}_k)$.

Step 3: General case.

Let $d \ge 0$ be the maximum degree of all polynomials $P_1, \dots, P_p$. Let us introduce a new undeterminate Z and consider the polynomials $Q_i(n, X_1, X_2, \dots, X_p, Z)$ which are homogeneous of degree d and such that

$$Q_i(n, X_1, X_2, \dots, X_p, 1) = P_i(n, X_1, X_2, \dots, X_p).$$

We also introduce the constant sequence

$$u_{p+1}(n) = 1$$

for every $n \ge 0$. One can check that the sequences $u_1(n), u_2(n), \dots, u_p(n), u_{p+1}(n)$ are fulfilling the conditions

$$u_i(n+1) = Q_i(n, u_1(n), \dots, u_j(n), \dots, u_p(n), u_{p+1}(n))$$

for $1 \le i \le p+1, n \in \mathbb{N}$ and

$$u_i(0) = c_i \text{ if } 1 \le i \le p, \quad u_{p+1}(0) = 1.$$

By step 2 of this proof, we can conclude that $u_1 \in \mathcal{F}(\mathbb{S}_k)$. $\square$

Theorem 97. *Let $u \in \mathcal{F}(\mathbb{S}_{k+1})$ and $v \in \mathcal{F}(\mathbb{S}_k)$ for some integer $k \geq 3$. Then the convolution product $u \times v$ belongs to $\mathcal{F}(\mathbb{S}_{k+1})$.*

Proof. Let $a^+, a^-, b^+, b^- \in \mathbb{S}_{k+1}, c^+, c^-, d^+, d^- \in \mathbb{S}_k$ such that

$$u = \frac{a^+ - a^-}{b^+ - b^-}, v = \frac{c^+ - c^-}{d^+ - d^-}.$$

We introduce the auxiliary sequences:

$$b = b^+ - b^-, \quad d = d^+ - d^-; \quad B(n) = \Pi_{\ell=0}^n b(\ell); \quad D(n) = \Pi_{\ell=0}^n d(\ell),$$

and

$$\bar{B}(m, n) = \Pi_{\substack{\ell=0\\ \ell \neq m}}^n b(\ell); \quad \bar{D}(m, n) = \Pi_{\substack{\ell=0\\ \ell \neq m}}^n d(\ell).$$

One can check that, for every $n \in \mathbb{N}$:

$$u \times v(n) = \frac{1}{B(n) \cdot D(n)} \cdot \left(\sum_{m=0}^n a(m)\bar{B}(m, n) \cdot c(n - m)\bar{D}(n - m, n) \right). \tag{59}$$

The product $B(n) \cdot D(n)$ can be decomposed as

$$B(n) \cdot D(n) = (B^+(n)D^+(n) + B^-(n)D^-(n)) - (B^+(n)D^-(n) + B^-(n)D^+(n)) \tag{60}$$

where the B^ϵ fulfill the equations

$$\begin{aligned}
B_2^\epsilon(n, n) &= b^\epsilon(n)\\
B_2^+(\ell, n) &= b^+(\ell)B_2^+(\ell + 1, n) + b^-(\ell)B_2^-(\ell + 1, n)\\
B_2^-(\ell, n) &= b^+(\ell)B_2^-(\ell + 1, n) + b^-(\ell)B_2^+(\ell + 1, n)
\end{aligned}$$

for $0 \leq \ell < n$, and $B^\epsilon(n) = B_2^\epsilon(0, n)$.

It follows from Lemma 74 that $B_2^\epsilon \in \mathbb{S}_{k+1}^{(2)}$ and from Lemma 76 that $B^\epsilon \in \mathbb{S}_{k+1}$. Similarly, $D^\epsilon \in \mathbb{S}_k$.

The double sequence $\bar{B}(m, n)$ can be defined through the following triple sequence:

$$\bar{B}_3(\ell, m, n) = \Pi_{\substack{\ell'=\ell\\ \ell' \neq m}}^n b(\ell')$$

via the formula

$$\bar{B}(m, n) = \bar{B}_3(0, m, n).$$

We can decompose $\bar{B}_3(\ell, m, n)$ as

$$\bar{B}_3(\ell, m, n) = \bar{B}_3^+(\ell, m, n) - \bar{B}_3^-(\ell, m, n)$$

where the $\bar{B}_3^\epsilon$ are fulfilling the equations:

$$\begin{aligned}
\bar{B}_3^+(n, n, n) &= 1\\
\bar{B}_3^-(n, n, n) &= 0\\
\bar{B}_3^\epsilon(\ell, m, n) &= \bar{B}_3^\epsilon(\ell + 1, m, n) \text{ if } \ell = m, \ell < n, \epsilon \in \{+, -\}\\
\bar{B}_3^\epsilon(n, m, n) &= b^\epsilon(n) \text{ if } 0 \leq m < n\\
\bar{B}_3^+(\ell, m, n) &= b^+(\ell)\bar{B}_3^+(\ell + 1, m, n) + b^-(\ell)\bar{B}_3^-(\ell + 1, m, n) \text{ if } \ell \neq m, \ell < n\\
\bar{B}_3^-(\ell, m, n) &= b^-(\ell)\bar{B}_3^+(\ell + 1, m, n) + b^+(\ell)\bar{B}_3^-(\ell + 1, m, n) \text{ if } \ell \neq m, \ell < n.
\end{aligned}$$

It follows from Lemma 75 that $\bar{B}^\epsilon \in \mathbb{S}_{k+1}^{(2)}$. Similarly, $\bar{D}^\epsilon \in \mathbb{S}_k^{(2)}$.

Using now Lemma 77 (simple sequences viewed as double), Lemma 78 (closure under ordinary product) and Lemma 79 (closure under pseudo-convolution), we obtain that the numerator of the righthand side of Eq. (59) belongs to $\mathcal{D}(\mathbb{S}_{k+1})$. By Theorem 72 the set $\mathbb{S}_{k+1}$ is closed under ordinary product and under sum. The decomposition (60) thus shows that the denominator of the righthand side of Eq. (59) belongs to $\mathcal{D}(\mathbb{S}_{k+1})$. Finally, $u \in \mathcal{F}(\mathbb{S}_{k+1})$. □

7.0.0.11. Comparison with other classes. The set of *rational* sequences of rational numbers is a subset of $\mathcal{F}(\mathbb{S}_3)$: such sequences are defined by recurrences of the form (55) and (56) with polynomials P_i of degree 1, with constant rational coefficients; hence, by Theorem 96, they belong to $\mathcal{F}(\mathbb{S}_3)$.

The set of the so-called P-*recurrent* sequences of rationals is a subset of $\mathcal{F}(\mathbb{S}_3)$: by Proposition 50 polynomial functions (with coefficients in $\mathbb{N}$) belong to $\mathbb{S}_2$, hence polynomial functions (with coefficients in $\mathbb{Q}$) belong to $\mathcal{F}(\mathbb{S}_2)$, and by Theorem 96 solutions of equations with polynomial rational coefficients belong to $\mathcal{F}(\mathbb{S}_3)$.

Let us recall that, the set of P-recurrent sequences is closed under Hadamard product [34, Theorem 2.10] and also under convolution product [34, Theorem 2.3]. This last property is not known for the set $\mathcal{F}(\mathbb{S}_3)$. Theorem 97 can be seen as a "weak" closure property in this respect.

8. Related work and perspectives

8.1. Related work

In parallel with our submission to the "2nd logic days" has appeared the work [12]: these authors characterize in [12, Theorem 3] the computation-graphs of k-pda in terms of a hierarchy of graphs called the *Caucal-hierarchy*, which is defined by means of two natural operations on graphs: *unfolding* and *inverse rational substitutions*. Our Theorem 40 follows also from this characterisation, since the two above operations preserve the decidability of MSO (see [11]).

8.2. Perspectives

The classes of sequences (or formal power series) introduced here deserve further study.

1. Other closure properties for $\cup_{k\geq 2}\mathbb{S}_k$ should be investigated. For example we strongly believe that it is closed under the operation $\bullet$ (the series-substitution) and also under recurrent equations with coefficients and also *exponents* in this class. Whether these closure properties would transfer to $\cup_{k\geq 2}\Sigma\mathbb{S}_k$ and whether the closure under operation $\bullet$ would also transfer to $\cup_{k\geq 2}\mathcal{F}(\mathbb{S}_k)$ is an interesting question.
2. Comparison with other classes of sequences defined by means of automata would be interesting too: we think $\cup_{k\geq 2}\mathbb{S}_k$ is included in the class of "residually ultimately periodic" sequences studied in [10] (work in preparation). Comparisons with the various classes considered in [3] should be done too. A combination of these different methods can be hoped for (work in preparation).
3. The notion of *multiple* sequences of level k just sketched in Section 5 should be studied in full generality.
4. Corollary 95 might be used in two directions:
   - we could derive from some undecidability problem for sequences of numbers an undecidability result for deterministic automata of level $k \geq 2$,
   - any progress toward an equivalence algorithm for automata of level $k \geq 2$ could lead to new techniques allowing us to solve equality problems for sequences of numbers.

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References

[1] A.V. Aho, Indexed grammars-an extension of context-free grammars, J. Assoc. Comput. Mach. 15 (1968) 647–671.
[2] A.V. Aho, Nested stack automata, J. Assoc. Comput. Mach. 16 (1969) 383–406.
[3] J.P. Allouche, J. Shallit, Automatic Sequences, Theory, Applications, Generalizations, Cambridge University Press, 2003.
[4] T. Ayashi, On derivation trees of indexed grammars—an extension of the uvwxy-thoerem, Publ. Res. Inst. Math. Sci. 9 (1973) 61–92.

[5] M.O. Rabin, Decidable Theories, in: J. Barwise (Ed.), Handbook of Mathematical Logic, North-Holland, 1977, pp. 595–629.
[6] A. Bès, A survey on arithmetical definability, in: A Tribute to Maurice Boffa, Soc. Math. Belgique, 2002, pp. 1–54.
[7] J. Berstel, C. Reutenauer, Rational Series and their Languages, Springer-Verlag, 1988.
[8] J.R. Büchi, On a decision method in resticted second-order arithmetic, in: Methodology and Philosophy of Science, Stanford Univ. Press, Int. Congr. for Logic, 1960, pp. 1–11.
[9] D. Caucal, On infinite terms having a decidable monadic theory, in: Proceedings MFCS 2002, Springer-Verlag, 2002, pp. 165–176.
[10] O. Carton, W. Thomas, The monadic theory of morphic infinite words and generalisations, in: MFCS'2000, in: LNCS, vol. 1893, 2000.
[11] B. Courcelle, I. Walukiewicz, Monadic second-order logic, graph coverings and unfoldings of transition systems, Ann. Pure Appl. Logic 92 (1) (1998) 35–62.
[12] A. Carayol, S. Wöhrle, The Caucal hierarchy of infinite graphs in terms of logic and higher-order pushdown automata, in: FST TCS 2003: Foundations of Software Technology and Theoretical Computer Science, in: Lecture Notes in Comput. Sci., vol. 2914, Springer, Berlin, 2003, pp. 112–123.
[13] W. Damm, The IO- and OI-hierarchies, TCS 20 (1982) 95–207.
[14] W. Damm, A. Goerdt, An automata-theoretical characterization of the OI-hierarchy, Inform. and Control 71 (1986) 1–32.
[15] J. Engelfriet, Iterated pushdown automata and complexity classes, in: Proc. 15th Annu ACM Sympos, Theory Comput, Assoc. Comp Mach., 1983, pp. 365–373.
[16] J. Engelfriet, Iterated stack automata and complexity classes, Inform. and Comput. 95 (1) (1991) 21–75.
[17] C.C. Elgot, M.O. Rabin, Decidability and undecidability of extensions of second (first) order theory of (generalized) successor, J. Symbolic Logic 31 (2) (1966) 169–181.
[18] J. Engelfriet, G. Slutski, Extended macro grammars and stack controlled machines, J. Comput. System Sci. 29 (1984) 366–408.
[19] J. Engelfriet, H. Vogler, Pushdown machines for the macro tree transducer, TCS 42 (1986) 251–368.
[20] M.J. Fischer, Grammars with macro-like productions, Ph.D. Thesis, Harvard University, 1968. See also: Proc. 9th Symp. on Switching and Automata Theory, 1968, pp. 131–142.
[21] S. Greibach, Full AFL's and nested iterated substitution, Inform. and Control (1970) 7–35.
[22] D. Berwanger, A. Blumensath, The monadic theory of tree-like structures, in: E. Grädel, W. Thomas (Eds.), Automata, Logics and Infinite Games, in: LNCS, vol. 2500, Springer, 2003, pp. 285–301.
[23] M.A. Harrison, Introduction to Formal Language Theory, Addison-Wesley, Reading, MA, 1978.
[24] T. Knapik, D. Niwinski, P. Urzyczyn, Higher-order pushdown trees are easy, in: FoSSaCs 2002, in: LNCS, 2002.
[25] A. Maes, An automata theoretic decidability proof for the first-order theory of $(\mathbb{N}, <, p)$ with morphic predicate p, J. Autom. Lang. Comb. 4 (1999) 229–245.
[26] A.N. Maslov, The hierarchiy of indexed languages of an arbitrary level, Soviet Math. Dokl. 15 (1974) 1170–1174.
[27] A.N. Maslov, Multilevel stack automata, Probl. Pereda. Inf. 12 (1976) 38–43.
[28] A. Muchnik, A.L. Semenov, Automata on infinite objects, monadic theories, and complexity, in: Dagstuhl-Seminar, 1992.
[29] M. Petkovšek, H.S. Wilf, D. Zeilberger, $A = B$, A K Peters Ltd., Wellesley, MA, 1996.
[30] M.O. Rabin, Decidability of second-order theories and automata on infinite trees, Trans. Amer. Math. Soc. 141 (1969) 1–35.
[31] A.L. Semenov, Decidability of monadic theories, in: Mathematical Foundation of Computer Science, in: LNCS, vol. 174, 1984, pp. 162–175.
[32] G. Sénizergues, L(A)=L(B)? a simplified decidability proof, Theoret. Comput. Sci. 281 (2002) 555–608.
[33] D. Siefkes, Decidable extensions of monadic second order successor arithmetic, in: Automatentheorie und Formale Sprachen (Tagung, Math. Forschungsinst., Oberwolfach, 1969), Bibliographisches Inst., Mannheim, 1970, pp. 441–472.
[34] R.P. Stanley, Differentiably finite power series, European J. Combin. 1 (1980) 175–188.
[35] I. Walukiewicz, Monadic second-order logic on tree-like structures, Theoret. Comput. Sci. 275 (1–2) (2002) 311–346.