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ON LANGUAGES WITH TWO VARIABLES

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In the normal run of logical affairs, every language contains infinitely many variables. Should we consider a language with only finitely many variables, our ability to prove, and our ability to express become severely restricted. Restrictions of the former kind have been studied by HENKIN and MONK (see [2] and [3]). Our concern in this paper is with restrictions of the latter kind; specifically, the existence of "axioms of infinity" i.e. sentences with only infinite models. It is easily seen that, allowing three variables, such a sentence exists—e.g. the axiom for a strict linear ordering without last element. On the other hand,

$$\forall x[g(f(x)) = x] \wedge \forall x P(f(x)) \wedge \exists x \neg P(x)$$

is such a sentence using only one variable, but with function symbols. We prove that these two sentences represent the best results possible—i.e. there is no consistent sentence without function symbols and using two variables which has only infinite models. This answers a question raised by W. HODGES. As a corollary we have SCOTT's result (see [4]) that the theory of a language with two variables and no function symbols is decidable.

Our first result puts the problem in a more manageable form.

Theorem 1. *The following are equivalent:*

(i) *There is a consistent sentence using only two variables and no function symbols which has only infinite models.*

(ii) *There is a consistent sentence with no function symbols which is a boolean combination of sentences in prenex normal form with at most two quantifiers, and has only infinite models.*

Proof. (ii) ⇒ (i). Trivial. (i) ⇒ (ii). It suffices to prove:

Claim. *If θ is a sentence using two variables and no function symbols, there is a sentence ψ, a boolean combination of sentences in prenex normal form with at most two quantifiers s.t. for each model $\mathfrak{A}$,*

$$\mathfrak{A} \models \psi \Rightarrow \mathfrak{A} \models \theta,$$

and

$$\mathfrak{A} \models \theta \textit{ implies there is an expansion } \mathfrak{A}' \textit{ of } \mathfrak{A} \textit{ s.t. } \mathfrak{A}' \models \psi .$$

Proof of claim. By induction on n, the number of quantifiers occurring in θ. If $n = 1$, the claim holds trivially. Suppose θ has n (> 1) quantifiers, and that the claim holds for sentences with less than n quantifiers. Then, since θ uses only two variables (x and y, say), we may without loss of generality choose a subformula $Qx\varphi(x, y)$ of θ where Q is a quantifier, and $\varphi(x, y)$ is quantifier-free. Let R be a predicate symbol not occurring in θ, and let θ_1 be the result of replacing $Qx\varphi(x, y)$ in θ by $R(y)$. Let $\varphi_1 = \forall y(R(y) \leftrightarrow Qx\varphi(x, y))$ — a sentence logically equivalent to a conjunction of sentences in prenex normal form with two quantifiers. Then θ_1 has $n - 1$ quantifiers, and for each model $\mathfrak{A}$,

$$\mathfrak{A} \models \theta_1 \wedge \varphi_1 \Rightarrow \mathfrak{A} \models \theta .$$

Moreover, any model of θ may be expanded to a model of $\theta_1 \wedge \varphi_1$. Using our inductive hypothesis, the claim follows. □

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In [1], EHRENFEUCHT defines a game $G_n(\mathfrak{A}, \mathfrak{B})$, where $n \in \omega$, and $\mathfrak{A}$, $\mathfrak{B}$ are models of the same similarity type. The game is played by two players, and has n moves. Its rules are as follows: On the ith move ($1 \leqq i \leqq n$), player I chooses one of the models, and picks an element from its domain. Player II then picks an element from the domain of the other model. Thus, after n moves, elements $a_1, \ldots, a_n$ will have been picked from dom $\mathfrak{A}$ (the domain of $\mathfrak{A}$), and elements $b_1, \ldots, b_n$ from dom $\mathfrak{B}$, where a_i (resp. b_i) is picked on the ith move. Player II wins just in case for each atomic formula $\theta(x_0, \ldots, x_m)$ of $L(\mathfrak{A})$ (the language of $\mathfrak{A}$), and each $i_0, \ldots, i_m \in \{1, \ldots, n\}$,

$$\mathfrak{A} \models \theta[a_{i_0}, \ldots, a_{i_m}] \Leftrightarrow \mathfrak{B} \models \theta[b_{i_0}, \ldots, b_{i_m}].$$

Player II has a *winning strategy* if he can always win.

The game is of interest to us here because of the following result.

Theorem (EHRENFEUCHT). *If player* II *has a winning strategy in the game* $G_2(\mathfrak{A}, \mathfrak{B})$, *then for each sentence* φ *of* $L(\mathfrak{A})$ *in prenex normal form with at most two quantifiers,*

$$\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi.$$

In proving this result, EHRENFEUCHT does not allow constants or functions to occur in $L(\mathfrak{A})$, but the extension of the theorem to languages with constants (but no functions) is quite straightforward.

Theorem 2. *Let* $\mathfrak{A}$ *be an infinite model of finite similarity type with no individual constants or functions. Then there is a finite model* $\mathfrak{B}$ *s.t. Player* II *has a winning strategy in the game* $G_2(\mathfrak{A}, \mathfrak{B})$.

Proof. Let x, y be distinct variables. Enumerate the predicate letters of $L(\mathfrak{A})$, other than equality, as $\{R_i\}_{1 \leqq i \leqq n}$. Since in $G_2(\mathfrak{A}, \mathfrak{B})$ we are only concerned with relations between two elements at a time, and no individual constants occur in $L(\mathfrak{A})$, we may assume without loss of generality that each R_i ($1 \leqq i \leqq n$) is binary.

Definition. Let $P(x, y)$ be a conjunction of atomic and negated atomic formulae of $L(\mathfrak{A})$, each using just the variables x and y. We call $P(x, y)$ a *complete relation* if $x \neq y$ is a subformula of $P(x, y)$, and for each atomic formula θ of $L(\mathfrak{A})$ using just the variables x and y, exactly one of θ, $\neg\theta$ is a subformula of $P(x, y)$. If $P(x, y)$ is a complete relation, $\mathfrak{C}$ a model, and $a, b \in \operatorname{dom} \mathfrak{C}$, $\langle a, b\rangle$ is said to *realize* $P(x, y)$ in $\mathfrak{C}$ if and only if $\mathfrak{C} \models P[a, b]$. $P(x, y)$ is *realized* in $\mathfrak{C}$ if and only if for some $a, b \in \operatorname{dom} \mathfrak{C}$, $\langle a, b\rangle$ realizes $P(x, y)$ in $\mathfrak{C}$.

Thus at most 2^{4n} distinct complete relations are realized in $\mathfrak{A}$.

Let $\mathfrak{C}$ be a model of the same similarity type as $\mathfrak{A}$, and suppose $a \in \operatorname{dom} \mathfrak{C}$. $X_a^{\mathfrak{C}}$, the *star of a in* $\mathfrak{C}$ is defined to be the set of complete relations $P(x, y)$ s.t. $\mathfrak{C} \models \exists y P[a, y]$. More generally, we define a *star* to be a set of complete relations in $L(\mathfrak{A})$, and define a star X to be *realized* in $\mathfrak{C}$ if for some $a \in \operatorname{dom} \mathfrak{C}$, $X = X_a^{\mathfrak{C}}$.

Thus at most $2^{2^{4n}}$ distinct stars are realized in $\mathfrak{A}$.

Our reason for introducing stars is shown by

Lemma 1. *Let* $\mathfrak{M}$, $\mathfrak{N}$ *be models of the same similarity type as* $\mathfrak{A}$, *both of power greater than one. Then the following are equivalent*:

(i) *For each star* X, X *is realized in* $\mathfrak{M}$ *if and only if* X *is realized in* $\mathfrak{N}$.

(ii) *Player II has a winning strategy in* $G_2(\mathfrak{M}, \mathfrak{N})$.

Proof. (i) $\Rightarrow$ (ii). Suppose that player I picks $a \in \operatorname{dom} \mathfrak{M}$ on his first move. Let player II then pick $b \in \operatorname{dom} \mathfrak{N}$ s.t. $X_a^{\mathfrak{M}} = X_b^{\mathfrak{N}}$ ($\neq \emptyset$). (i) ensures that this can always be done. Suppose that on his second move player I picks $c \in \mathrm{dom}\, \mathfrak{N}$. If $\mathfrak{N} \models b = c$, let player II pick a. Otherwise, let $P(x, y)$ be the complete relation realized by $\langle b, c\rangle$ in $\mathfrak{N}$. Thus $P(x, y) \in X_a^{\mathfrak{M}}$ so $\mathfrak{M} \models \exists y P[a, y]$, and player II can win by picking $d \in \mathrm{dom}\, \mathfrak{M}$ s.t. $\mathfrak{M} \models P[a, d]$. The argument for other cases is similar.

(ii) $\Rightarrow$ (i). Suppose for contradiction that $X_a^{\mathfrak{M}}$ $(\neq \emptyset)$ is not realized in $\mathfrak{N}$, where $a \in \mathrm{dom}\, \mathfrak{M}$. Let player I pick a on his first move. Player II must then pick some $b \in \mathrm{dom}\, \mathfrak{N}$. If for some complete relation $P(x, y)$, $P(x, y) \in X_a^{\mathfrak{M}}$ and $P(x, y) \notin X_b^{\mathfrak{N}}$, let player I pick on his second move some $c \in \mathrm{dom}\, \mathfrak{M}$ s.t. $\mathfrak{M} \models P[a, c]$. It is clear that player player II cannot then win. The argument in other cases is similar. □

The next lemma allows us to replace $\mathfrak{A}$ by a "nicer" model, from which we obtain the finite model $\mathfrak{B}$.

Definition. A star X is *asymmetric* if and only if for each complete relation $P(x, y)$, if $P(x, y) \in X$ then $P(y, x) \notin X$. A star which is not asymmetric we call *symmetric*.

We note that in any model of the same similarity type as $\mathfrak{A}$, an asymmetric star can be realized by *at most one* element.

Lemma 2. *There is a model $\mathfrak{A}'$ s.t.*

(i) *For each star X, X is realized in $\mathfrak{A}'$ if and only if X is realized in $\mathfrak{A}$.*

(ii) *If $P(x, y)$ is a complete relation, $a, b \in \mathrm{dom}\, \mathfrak{A}'$ s.t. $\mathfrak{A}' \models P[a, b]$, and $X_b^{\mathfrak{A}'}$ is symmetric, then there exists $\{a_i\}_{i\in\omega} \subset \mathrm{dom}\, \mathfrak{A}'$ s.t. for each $i \neq j \in \omega$, $\mathfrak{A}' \models a_i \neq a_j \wedge P[a, a_i]$, and $X_{a_i}^{\mathfrak{A}'} = X_b^{\mathfrak{A}'}$.*

Proof. It is easily seen that if $\{\mathfrak{A}_i\}_{i<\lambda}$ is a chain of models such that for each $a \in \mathrm{dom}\, \mathfrak{A}_0$ and each $i < \lambda$, $X_a^{\mathfrak{A}_0} = X_a^{\mathfrak{A}_i}$, then

$$X_a^{\mathfrak{A}_0} = X_a^{\bigcup\{\mathfrak{A}_i : i<\lambda\}}$$

Hence it will suffice to show that for any model $\mathfrak{A}_1$ of the same similarity type as $\mathfrak{A}$, with $a, b \in \mathrm{dom}\, \mathfrak{A}_1$ such that $X_b^{\mathfrak{A}_1}$ is symmetric, if $P(x, y)$ is the complete relation realized by $\langle a, b\rangle$ in $\mathfrak{A}_1$, there is a model $\mathfrak{A}_2 \supset \mathfrak{A}_1$ s.t. $\mathrm{dom}\, \mathfrak{A}_2 - \mathrm{dom}\, \mathfrak{A}_1 = \{c\}$, $X_c^{\mathfrak{A}_2} = X_b^{\mathfrak{A}_1}$, and for each $e \in \mathrm{dom}\, \mathfrak{A}_1$, $X_e^{\mathfrak{A}_1} = X_e^{\mathfrak{A}_2}$. This we now do.

Let $\mathrm{dom}\, \mathfrak{A}_2 = \mathrm{dom}\, \mathfrak{A}_1 \cup \{c\}$ where $c \notin \mathrm{dom}\, \mathfrak{A}_1$. For each $a_1, a_2 \in \mathrm{dom}\, \mathfrak{A}_1$, and each complete relation $P'(x, y)$, assign relations so that $\mathfrak{A}_2 \models P'[a_1, a_2] \Leftrightarrow \mathfrak{A}_1 \models P'[a_1, a_2]$. For each $a_1 \in \mathrm{dom}\, \mathfrak{A}_1 - \{b\}$, and each complete relation $P'(x, y)$, define $\mathfrak{A}_2 \models P'[c, a_1] \Leftrightarrow$ $\Leftrightarrow \mathfrak{A}_1 \models P'[b, a_1]$. Finally, pick a complete relation $Q(x, y)$ s.t. $Q(x, y), Q(y, x) \in X_b^{\mathfrak{A}_1}$ – a symmetric star. Define $\mathfrak{A}_2 \models Q[b, c]$. It is easily checked that $\mathfrak{A}_2$ as constructed is a model satisfying the conditions set out above. □

We now assume that $\mathfrak{A}$ meets condition (ii) of lemma 2. By lemma 1 there is no loss of generality in this assumption, and we proceed to the construction of the finite model. By $|S|$, where S is a set, we shall mean the cardinality of S. For each star X realized in $\mathfrak{A}$, take a set $B(X)$ s.t.

1) If X is symmetric $|B(X)| = (|X| + 1)\, 2^{4n}$,
2) If X is asymmetric $|B(X)| = |X| + 1$,
3) $B(X) \cap \mathrm{dom}\, \mathfrak{A} = \emptyset$.

We assume that this is done in such a way that if X, Y are distinct stars realized in $\mathfrak{A}$, $B(X) \cap B(Y) = \emptyset$. Let $B_0 = \bigcup\{B(X) : X \text{ is realized in } \mathfrak{A}\}$. We build a structure $\mathfrak{B}_1$ on $\mathfrak{B}_0$, in the course of which is defined a set $B \subset B_0$, and a function $F: B \xrightarrow{1-1} \mathrm{dom}\, \mathfrak{A}$. The desired finite model will be $\mathfrak{B} = \mathfrak{B}_1 \restriction B$. Our construction is in three stages.

Stage 1. Enumerate the stars realized in $\mathfrak{A}$ as $\{X_i\}_{1 \leqq i \leqq l}$. Stage 1 consists of l steps: **at step i we define F_i and Δ_i** s.t. F_i is one-to-one, for each $a \in \operatorname{dom} F_i$ (the domain **of F_i) $X^{\mathfrak{A}}_{F_i(a)} = X_i$**, and if $\mathfrak{C} \models \Delta_i$, then $X^{\mathfrak{C}}_a \supseteqq X_i$.

Set $F_0 = \emptyset$, $\Delta_0 = \emptyset$.

Step i (>0).

Case 1. X_i is a symmetric star. Choose $C(X_i) \subset B(X_i)$ s.t. $|C(X_i)| = 2^{4n}$. Choose $\{b_j\}_{1 \leqq j \leqq 2^{4n}} \subset \operatorname{dom} \mathfrak{A}$ s.t. for each $k \neq j \leqq 2^{4n}$, $b_k \neq b_j$, and $X^{\mathfrak{A}}_{b_j} = X_i$. The "niceness" of $\mathfrak{A}$ makes this possible. Choose $g\colon C(X_i) \xrightarrow{1-1} \{b_j\}_{1 \leqq j \leqq 2^{4n}}$, and define

$$F_i = F_{i-1} \cup \{\langle b, g(b)\rangle : b \in C(X_i)\}.$$

Choose $f_{X_i}\colon X_i \times C(X_i) \xrightarrow{1-1} B(X_i) - C(X_i)$, and define

$$\Delta_i = \Delta_{i-1} \cup \{P[b, f_{X_i}(\langle P(x, y), b\rangle)] : b \in C(X_i),\ P(x, y) \in X_i\}.$$

Case 2. X_i is an asymmetric star. Choose $a_{X_i} \in B(X_i)$, and let $C(X_i) = \{a_{X_i}\}$. Let c be the *unique* element of $\operatorname{dom} \mathfrak{A}$ s.t. $X_i = X^{\mathfrak{A}}_c$, and define $F_i = F_{i-1} \cup \{\langle a_{X_i}, c\rangle\}$. Pick $f_{X_i}\colon X_i \xrightarrow{1-1} B(X_i) - C(X_i)$, and define

$$\Delta_i = \Delta_{i-1} \cup \{P[a_{X_i}, f_{X_i}(P(x, y))] : P(x, y) \in X_i\}.$$

It is easily seen that in both cases, Δ_i is a consistent set of sentences, and that F_i is a one-to-one function.

Let $F^1_0 = F_l$, and $\Delta^1_0 = \Delta_l$. Thus F^1_0 is a one-to-one function s.t. if $a \in C(X_i)$ for some $1 \leqq i \leqq l$, then $X^{\mathfrak{A}}_{F^1_0(a)} = X_i$, and if $\mathfrak{C} \models \Delta^1_0$, then $X^{\mathfrak{C}}_a \supseteqq X_i$.

Stage 2. Enumerate $B_0 - \bigcup\{C(X_i) : 1 \leqq i \leqq l\}$ as $\{a_i\}_{1 \leqq i \leqq m}$. Stage 2 consists of m steps. We define $B_1 \ldots B_m$, $F^1_1 \ldots F^1_m$, and $\Delta^1_1 \ldots \Delta^1_m$ so that for $1 \leqq i \leqq j \leqq m$ we have $B_0 \supseteqq B_i \supseteqq B_j$, $F^1_0 \subseteqq F^1_i \subseteqq F^1_j$, and $\Delta^1_0 \subseteqq \Delta^1_i \subseteqq \Delta^1_j$.

Step i. There is precisely one star X s.t. $a_i \in B(X)$. Let b be the unique member of $C(X)$, and $P(x, y) \in X$ the unique complete relation s.t. $P[b, a_i] \in \Delta^1_0$. Thus, since $X_{F^1_0(b)} = X$, $\mathfrak{A} \models \exists y P[F^1_0(b), y]$.

Case 1. In this case, whenever $\mathfrak{A} \models P[F^1_0(b), c]$, $X^{\mathfrak{A}}_c$ is asymmetric. Pick one such c, let $B_i = B_{i-1} - \{a_i\}$, $F^1_i = F^1_{i-1} \cup \{\langle a_i, c\rangle\}$, and let $\Delta^1_i = \Delta^1_{i-1} \cup \{P[b, a_{X^{\mathfrak{A}}_c}]\}$. We note that $\mathfrak{A} \models P[F^1_0(b), F^1_0(a_{X^{\mathfrak{A}}_c})]$.

Case 2. In this case, for some c s.t. $\mathfrak{A} \models P[F^1_0(b), c]$, $X^{\mathfrak{A}}_c$ is symmetric. By the construction of our "nice" model $\mathfrak{A}$, there are infinitely many such c's, all realizing the same (symmetric) star. Pick one, c say s.t. $c \notin \operatorname{rng} F^1_{i-1}$ (the range of F^1_{i-1}). Let $F^1_i = F^1_{i-1} \cup \{\langle a_i, c\rangle\}$, so that $\mathfrak{A} \models P[F^1_0(b), F^1_i(a_i)]$. Pick $g\colon X^{\mathfrak{A}}_c \xrightarrow{1-1} \operatorname{dom} \mathfrak{A}$ s.t. if $P'(x, y) \in X^{\mathfrak{A}}_c$, then $\mathfrak{A} \models P'[c, g(P'(x, y))]$. Also pick $\pi\colon \operatorname{rng} g \xrightarrow{1-1} \bigcup\{C(Y) : Y \text{ is realized in } \mathfrak{A}\}$ s.t.

1) $\pi \circ g(Q(x, y)) = b$ just in case $Q(x, y)$ is $P(y, x)$, and $X^{\mathfrak{A}}_{g(P(y, x))} = X$.

2) For each $d \in \operatorname{rng} g$, $\pi(d) \in C(X^{\mathfrak{A}}_d)$.

Such a π is easily shown to exist since, if Y is symmetric, $|X^{\mathfrak{A}}_c| \leqq |C(Y)|$. We note that if $P'(x, y) \in X^{\mathfrak{A}}_c$, and $\pi \circ g(P'(x, y)) \in C(Y)$, then $P'(y, x) \in Y$. Let $\Delta^1_i = \Delta^1_{i-1} \cup \{P'[a_i, \pi \circ g(P'(x, y))] : P'(x, y) \in X^{\mathfrak{A}}_c\}$, and let $B_i = B_{i-1}$. Condition 1) on π ensures that if $Q[b, a_i] \in \Delta^1_i - \Delta^1_{i-1}$, then $Q(x, y)$ is $P(x, y)$, and condition 2) that if $Q[d, a_i] \in \Delta^1_i - \Delta^1_{i-1}$ for some $d \in \bigcup\{C(X_i) : 1 \leqq i \leqq l\}$, then for some e, $Q[d, e] \in \Delta^1_0$ We show that Δ^1_i is consistent. For case 2 this is obvious. For case 1, if b and $a_{X^{\mathfrak{A}}_c}$ were already related in Δ^1_{i-1}, then for some a_j $(j < i)$, and some $Q(x, y) \in X^{\mathfrak{A}}_c$, $Q[a_{X^{\mathfrak{A}}_c}, b] \in \Delta^1_{i-1}$ and $\mathfrak{A} \models Q[F^1_{i-1}(a_{X^{\mathfrak{A}}_c}), F^1_0(b)]$. But $\mathfrak{A} \models P[F^1_0(b), F^1_0(a_{X^{\mathfrak{A}}_c})]$, so $Q(x, y)$ is $P(y, x)$, and Δ^1_i is consistent. By construction, $F^1_i \upharpoonright B_i$ is one-to-one.

Let $B = B_m$, $F = F^1_m \upharpoonright B$, so that if $\mathfrak{C} \models \Delta^1_m$, then for each $b \in B$, $X^{\mathfrak{C}}_b \supseteqq X^{\mathfrak{A}}_{F(b)}$.

Stage 3. Define $\Delta = \Delta_m^1 \cup \{P[a, b] : P(x, y)$ is a complete relation, $a, b \in B$, $\mathfrak{A} \models P[F(a), F(b)]$, and for no complete relation $Q(x, y)$ are either $Q[a, b]$ or $Q[b, a]$ in $\Delta_m^1\}$. Δ is clearly consistent.

We now define $\mathfrak{B}$. Let dom $\mathfrak{B} = B$, and for each complete relation $P(x, y)$, and each $a, b \in$ dom $\mathfrak{B}$, assign relations so that $\mathfrak{B} \models P[a, b] \Leftrightarrow P[a, b] \in \Delta$. It is easily checked that $\mathfrak{B}$ is a model, and that for each $b \in B$, $X_b^{\mathfrak{B}} = X_{F(b)}^{\mathfrak{A}}$. Hence $\mathfrak{B}$ is a finite model s.t. for each star X, X is realized in $\mathfrak{B}$ if and only if X is realized in $\mathfrak{A}$. $\square$

Finally we remove the restriction on constants in the statement of theorem 2.

Theorem 3. *Let $\mathfrak{A}$ be an infinite model of finite similarity type with no function symbols. Then there is a finite model $\mathfrak{B}$ s.t. for every sentence θ of $L(\mathfrak{A})$ in prenex normal form with at most two quantifiers,*

$$\mathfrak{B} \models \theta \Leftrightarrow \mathfrak{A} \models \theta .$$

Proof. Let L be the finite language consisting of all predicate letters and individual constants of $L(\mathfrak{A})$, and the variables x and y. Enumerate the atomic formulae of L which have no occurrence of y as $\{A_i\}_{1 \leqq i \leqq n}$. Let L' be the language with no individual constants, with variables x and y, and predicate letters $\{A_i'\}_{1 \leqq i \leqq n}$, where if A_i has m occurrences of x, then A_i' is m-ary. Define the L'-structure $\mathfrak{A}'$ by:

$$\text{dom } \mathfrak{A}' = \text{dom } \mathfrak{A}, \quad \text{and for each } \bar{a} \subset \text{dom } \mathfrak{A}, \ \mathfrak{A}' \models A_i'[\bar{a}] \Leftrightarrow \mathfrak{A} \models A_i[\bar{a}].$$

Let $C \subset$ dom $\mathfrak{A}$ be the set of interpretations of constants in $\mathfrak{A}$, and let $S =$ dom $\mathfrak{A} - C$. Define $\mathfrak{A}^+ = \mathfrak{A}' \restriction S$. Thus $\mathfrak{A}^+$ is an infinite model of finite similarity type with no functions or individual constants. By theorem 2, let $\mathfrak{B}^+$ be a finite L'-structure s.t. for each star X in L', X is realized in $\mathfrak{B}^+$ if and only if it is realized in $\mathfrak{A}^+$. Without loss of generality we may assume dom $\mathfrak{B}^+ \cap C = \emptyset$. We define $\mathfrak{B}$ by: dom $\mathfrak{B} =$ dom $\mathfrak{B}^+ \cup C$, where each constant of L is interpreted in $\mathfrak{B}$ as it is in $\mathfrak{A}$. For each $1 \leqq i \leqq n$, and each $\bar{a} \subset$ dom $\mathfrak{B}^+$, define $\mathfrak{B} \models A_i[\bar{a}] \Leftrightarrow \mathfrak{B}^+ \models A_i'[\bar{a}]$. It is easily shown that $\mathfrak{B}$ is a model. Our definitions of complete relation and star carry over to L unchanged, so that each complete relation will contain occurrences of all constants of L.

As remarked before, Ehrenfeucht's theorem holds for L-structures, and it is easily checked that lemma 1 above does also. Hence, to prove our theorem, it suffices to prove:

Claim. 1) *If $a \in C$, then $X_a^{\mathfrak{B}} = X_a^{\mathfrak{A}}$.* 2) *If $a \in$ dom $\mathfrak{A} - C$, $b \in$ dom $\mathfrak{B} - C$, and $X_a^{\mathfrak{A}^+} = X_b^{\mathfrak{B}^+}$, then $X_a^{\mathfrak{A}} = X_b^{\mathfrak{B}}$.*

For it will then follow that for each star X, X is realized in $\mathfrak{B}$ if and only if it is realized in $\mathfrak{A}$.

The proof of this claim is quite straightforward, and we omit it. $\square$

Corollary 1. *Every consistent sentence with no function symbols, and only two variables has a finite model.*

Corollary 2 (Scott [4]). *The theory of a language with no function symbols and two variables is decidable.*

Corollary 3. *Let θ be a sentence with no function symbols and two variables. Then there is $n_\theta \in \omega$ s.t.*

1) *θ has a model if and only if θ has a model of power $\leqq n_\theta$.*
2) *If θ has a model of power $\geqq n_\theta$, then θ has models in each cardinality $\geqq n_\theta$.*
3) *n_θ is primitive recursive in θ.*

Proof. By the claim of theorem 1, let ψ be a boolean combination of sentences with at most two quantifiers, and using no function symbols s.t. for each model $\mathfrak{A}$,

$$\mathfrak{A} \models \psi \Rightarrow \mathfrak{A} \models \theta,$$

and

$$\mathfrak{A} \models \theta \text{ implies there is an expansion } \mathfrak{A}' \text{ of } \mathfrak{A} \text{ s.t. } \mathfrak{A}' \models \psi.$$

The proof of that claim shows that we may assume ψ is primitive recursive in θ.

Let L be the language with variables x, y, and containing just the predicate letters and individual constants occurring in ψ. Suppose L contains m constants and k predicate letters other then equality, and let q be the maximum arity of all these predicate letters. Let $p = \max(3, q)$. Then L has $\leqq (p-2)^m \times k \times (p-2)!$ atomic formulae with only two occurrences of x, and none of y, and hence $\leqq 2^{4k(p-2)^m(p-2)!}$ distinct complete relations. Thus L has $\leqq 2^{2^{4k(p-2)^m(p-2)!}} = n_S$ distinct stars. Let

$$n_\theta = 2^{2^{4k(p-2)^m(p-2)!}} \times 2^{4k(p-2)^m(p-2)!} \times (2^{4k(p-2)^m(p-2)!} + 1) + m.$$

3) is now obvious.

We split the proof of 1) and 2) into four claims.

Claim 1. *If $\mathfrak{A}' \models \psi$ is finite, and $\mathfrak{A}'$ realizes a symmetric star, then there is $\mathfrak{B}' \supset \mathfrak{A}'$ s.t.* (i) $|\mathfrak{B}'| = |\mathfrak{A}'| + 1$, (ii) *For each $a \in \operatorname{dom} \mathfrak{A}'$, $X_a^{\mathfrak{B}'} = X_a^{\mathfrak{A}'}$*, (iii) *$\mathfrak{A}'$ and $\mathfrak{B}'$ realize precisely the same stars.*

Proof. This was proved in lemma 2.

Claim 2. *If $\mathfrak{A}' \models \psi$ and $|\mathfrak{A}'| \geqq n_\theta$, then $\mathfrak{A}'$ realizes a symmetric star.*

Proof. It is sufficient to observe that $n_\theta > n_S$, and that in any L-structure an asymmetric star is realized by at most one element.

Claim 3. *If $\mathfrak{A} \models \theta$, and $|\mathfrak{A}| \geqq n_\theta$, then θ has an infinite model.*

Proof. We may suppose that $\mathfrak{A}$ is finite. Let $\mathfrak{A}'$ be the expansion of $\mathfrak{A}$ to a model of ψ. By claims 1 and 2, choose a chain of L-structures $\{\mathfrak{B}_i\}_{i\in\omega}$ s.t. $\mathfrak{B}_0 = \mathfrak{A}'$, for each $i \in \omega$ $|\mathfrak{B}_{i+1}| = |\mathfrak{B}_i| + 1$, $\mathfrak{B}_i$ and $\mathfrak{B}_{i+1}$ realize precisely the same stars, and for each $a \in$ $\in \operatorname{dom} \mathfrak{B}_i$, $X_a^{\mathfrak{B}_i} = X_a^{\mathfrak{B}_{i+1}}$. It is easy to show that if $\mathfrak{B} = \bigcup\{\mathfrak{B}_i : i \in \omega\}$, then for each star X, X is realized in $\mathfrak{B}$ just in case it is realized in $\mathfrak{A}'$. Thus by lemma 1, and EHRENFEUCHT's theorem, $\mathfrak{B} \models \psi$. Hence $\mathfrak{B}$ is an infinite model of θ. □

Claim 4. *If ψ has an infinite model, then ψ has a model of cardinality $\leqq n_\theta$ realizing a symmetric star.*

Proof. This is just a matter of checking through the proofs of theorems 2 and 3, coupled with the observation that since there are only finitely many stars, any infinite L-structure must realize a symmetric star.

Having proved our claims, 1) and 2) follow easily. □

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