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The Liberalized δ-Rule in Free Variable Semantic Tableaux

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Abstract. In this paper we have a closer look at one of the rules of the tableau calculus presented by Fitting [4], called the δ-rule. We prove that a modification of this rule, called the δ^+-rule, which uses fewer free variables, is also sound and complete. We examine the relationship between the δ^+-rule and variations of the δ-rule presented by Smullyan [9]. This leads to a second proof of the soundness of the δ^+-rule. An example shows the relevance of this modification for building tableau-based theorem provers.

Key words. Mechanical theorem proving, tableau calculus.

1. Introduction

The most popular version of the proof procedure that is usually called analytic tableaux or semantic tableaux is due to Raymond Smullyan [9] and goes back to Beth and Hintikka. Semantic tableaux have recently experienced a renewed interest by AI researchers, since their closeness to the semantic definitions of logical operators makes the basic system easily adjustable to a wide scope of nonstandard logics. For example, in [3] tableaux are extended to cover first-order modal logic and to many-valued logics in [5]. Areas of application include natural language processing [2], nonmonotonic reasoning [1,7], and logic programming [6], just to name a few. The present paper is concerned only with quantifier rules in classical predicate logic, but the results are equally applicable to nonstandard first-order tableaux systems.

We assume that the reader is familiar with the method semantic tableaux (if not, excellent introductions can be found in [9] and [4]). Let us just recall that Smullyan introduced *unified notation*, a classification scheme for logical operators (and thus for tableaux rules) that makes definitions and proofs clearer and much more compact. According to this scheme, there are four types of operators, namely, α (conjunctive propositional), β (disjunctive propositional), γ (universal quantifiers), and δ (existential quantifiers) with corresponding rules. Semantic tableaux for classical logic come in two versions, signed and unsigned, from which we choose the latter. We have summarized γ- and δ-type formulas in Table I.

Table I. γ- And δ-type formulas.

γ	$\gamma(t)$
$(\forall x)\phi(x)$	$\phi(t)$
$\neg(\exists x)\phi(x)$	$\neg\phi(t)$
δ	$\delta(t)$
$\neg(\forall x)\phi(x)$	$\neg\phi(t)$
$(\exists x)\phi(x)$	$\phi(t)$

2. Free-Variable Tableaux

In Smullyan's formulation the γ-rule requires one to substitute an arbitrary but fixed term[1] for the quantified variable; see Table II. Since this 'guess' may be wrong, the γ-rule may have to be applied again and again on the same universal-type formula in a tableau proof. Obviously, this indeterminism can make proofs very long, and it is a natural idea to postpone the instantiation in a γ-rule until more information on the instance actually needed has been collected. We know of two approaches in the literature where this has been expressed formally [4,8]. We concentrate on the former, which we assume the reader to be familiar with.

For convenience, we have given the free quantifier rules from [4] in Table III.

The proviso of the δ-rule ensures that the introduced Skolem term is new on the branch constructed so far, even when the free variables are instantiated later during the proof. Thus it can be safely given an appropriate meaning in order to preserve satisfiability of tableaux after δ-rule applications.

Let us henceforth call the tableau system with these rules the *free* version and the old one the *ground* version.

3. Liberalized δ-Rules

Both versions of tableaux systems, free and ground, have essentially the same

Table II. Ground tableaux rules for quantified formulas.

$\dfrac{\gamma}{\gamma(t)}$	$\dfrac{\delta}{\delta(t)}$
where t is any ground term.	where t is a ground term not occurring on the current branch.

Table III. Free tableaux rules for quantified formulas.

$\dfrac{\gamma}{\gamma(x)}$	$\dfrac{\delta}{\delta(f(x_1,\ldots,x_n))}$
where x is a free variable.	where $x_1,\ldots,x_n$ are the free variables occurring on the current branch and f a new function symbol.

proviso in the δ-rule: under any substitutions, the introduced term has to be absolutely new on the current branch.

Careful inspection of tableaux proofs shows that this proviso is somewhat stronger than is actually needed, and we are led to the formulation of the *liberalized free δ-rule* stated in Table IV.

We will refer to this rule as δ^+. To show the possible advantage of a system using δ^+ over one using δ, we provide an example of a tableau proof using Fitting's δ-rule:

(1) $\neg(\exists x)(((\forall z)p(z)) \vee \neg p(x))$
(2) $\neg(((\forall z)p(z)) \vee \neg p(X))$
(3) $\neg(\forall z)p(z)$
(4) $\neg\neg p(X)$
(5) $p(X)$
(6) $\neg p(f(X))$
(7) $\neg(((\forall z)p(z)) \vee \neg p(Y))$
(8) $\neg(\forall z)p(z)$
(9) $\neg\neg p(Y)$
(10) $p(Y)$
closed by (10) and (6)

Line (6) is obtained from line (3) by Fitting's δ-rule. It is not possible to close the tableau by using lines (6) and (5). Only after a second application of the γ-rule on the formula in line (1) resulting in line (7) can closure be obtained. A closer look at this proof reveals that it is in fact the shortest possible proof using the δ-rule. The same root formula yields the following tableau using the δ^+-rule:

(1) $\neg(\exists x)(((\forall z)p(z)) \vee \neg p(x))$
(2) $\neg(((\forall z)p(z)) \vee \neg p(X))$
(3) $\neg(\forall z)p(z)$
(4) $\neg\neg p(X)$
(5) $p(X)$
(6) $\neg p(c)$
closed by (5) and (6)

Since the system using the free δ-rule is complete, a system using the liberalized

Table IV. Liberalized free tableaux rules for quantified formulas.

$\dfrac{\gamma}{\gamma(x)}$	$\dfrac{\delta}{\delta(f(x_1,\ldots,x_n))}$
where x is a free variable.	where $x_1,\ldots,x_n$ are the free variables occurring in δ and f is a new function symbol.

free rule δ^+ will also be complete. The problem thus lies in proving correctness of the δ^+-rule. A first proof is given in the following section.

4. A Soundness Proof

The soundness proof for the free δ-rule in [4], as it stands, does not carry over for the tableau system using the δ^+-rule. We present a proof proceeding along a different line.

We consider a tableau T that may contain free variables. Usually we think of the free variables as being introduced by the γ-rule.

We define a tableau T to be **satisfiable** if there is a structure $\mathcal{M}$ such that for any tuple $\bar{a}$ of elements from $\mathcal{M}$ there is a branch B in T such that $(\mathcal{M}, \bar{a}) \models B$.

Since in the process of tableau extension Skolem functions are added, it is necessary to pay attention to the signature of languages and structures. The signature of the structure $\mathcal{M}$ in the above definition consists exactly of all constant, function, and predicate symbols occurring in formulas of T. The length of the tuple $\bar{a}$ is the number of different free variables appearing in T. A branch B is considered here as a set of formulas and $(\mathcal{M}, \bar{a}) \models B$ means $(\mathcal{M}, \bar{a}) \models \phi$ for all ϕ in B.

LEMMA 3.1. *If T is a satisfiable tableau and T' is obtained from T by one application of a tableau rule, then T' is also satisfiable.*

Proof. The proof proceeds by cases, according to which tableau rule is applied to obtain T' from T.

β-rule. Let B be a branch in T, $\phi \vee \psi$ a formula in B. T' is obtained from T by adding ϕ to B to obtain the branch B_1 and also adding ψ to B to obtain the branch B_2. Now assume T is satisfiable. There is then a structure $\mathcal{M}$ such that for all tuples $\bar{a}$ from $\mathcal{M}$, we have $(\mathcal{M}, \bar{a}) \models T$. We claim that for the same structure $\mathcal{M}$ we have for all tuples $\bar{a}$ from $\mathcal{M}$ also $(\mathcal{M}, \bar{a}) \models T'$. Let $\bar{a}$ be a fixed tuple. We need to show $(\mathcal{M}, \bar{a}) \models T'$. By assumption there is some branch B_0 in T with $(\mathcal{M}, \bar{a}) \models B_0$. If B_0 is different from B, then B_0 is also a branch in T' and we are through. Now consider the case that $B_0 = B$. Since in particular $(\mathcal{M}, \bar{a}) \models \phi \vee \psi$ is true, we have either $(\mathcal{M}, \bar{a}) \models \phi$ or $(\mathcal{M}, \bar{a}) \models \psi$. In the first case $(\mathcal{M}, \bar{a}) \models B_1$ follows, while in the second case $(\mathcal{M}, \bar{a}) \models B_2$ holds true. In both cases we get $(\mathcal{M}, \bar{a}) \models T'$.

α-rule. Similar to the β-rule and left to the reader.

γ-rule. Let B be a branch in T, $(\forall x)\phi$ a formula in B. T' is obtained from T by adding $\phi(x)$ to B to obtain the branch B'. By assumption there is a structure $\mathcal{M}$ such that for all tuples $\bar{a}$ from $\mathcal{M}$ we have $(\mathcal{M}, \bar{a}) \models T$. We claim that for the same structure $\mathcal{M}$ we have for all tuples $\langle \bar{a}, b \rangle$ from $\mathcal{M}$ also $(\mathcal{M}, \langle \bar{a}, b \rangle) \models T'$. The tableau T' contains the new free variable x, and the element b in the tuple $\langle \bar{a}, b \rangle$ is used as an interpretation for x. We fix some tuple $\langle \bar{a}, b \rangle$ and let B_0 be a branch in T such that $(\mathcal{M}, \bar{a}) \models B_0$ holds true. The only interesting case is when $B_0 = B$. In this case we have in particular $(\mathcal{M}, \bar{a}) \models (\forall x)\phi$. By definition of the relation $\models$ this implies $(\mathcal{M}, \langle \bar{a}, b \rangle) \models \phi(x)$ and therefore also $(\mathcal{M}, \langle \bar{a}, b \rangle) \models B'$.

δ^+-rule. Let B be a branch in T, $(\exists x)\phi$ a formula in B. T' is obtained from T by adding $\phi(f(y_1,\dots,y_k))$ to B to obtain the branch B'. Here f is a new function symbol, and $y_1,\dots,y_k$ are all free variables in the formula $(\exists x)\phi$. By assumption on $\mathcal{M}$ we have $(\mathcal{M},\bar{a}) \models T$ is true for all tuples $\bar{a}$ from $\mathcal{M}$. Again we concentrate on the case that we have $(\mathcal{M},\bar{a}) \models B$ for the particular branch B. We will define a structure $\mathcal{M}'$ such that for all tuples $\bar{a}$ from $\mathcal{M}'$ $(\mathcal{M}',\bar{a}) \models T'$ will be true. The structure $\mathcal{M}'$ will be the same as $\mathcal{M}$, except that the new function symbol f will be interpreted in $\mathcal{M}'$. Let $b_1,\dots,b_k$ be arbitrary elements of $\mathcal{M}$. By assumption $(\mathcal{M},b_1,\dots,b_k) \models (\exists x)\phi$ is true. We pick a witnessing element c from $\mathcal{M}$ such that $(\mathcal{M},b_1,\dots,b_k,c) \models \phi(y_1,\dots,y_k,x)$ and set

$$f^{\mathcal{M}'}(b_1,\dots,b_k) = c.$$

This definition of $\mathcal{M}'$ will certainly make $(\mathcal{M}',\bar{a}) \models \phi(f(y_1,\dots,y_k))$ true for all $\bar{a}$ in $\mathcal{M}$. Since all other formulas in the branch B' do not contain the function symbol f, their truth will not be altered by passing from $\mathcal{M}$ to $\mathcal{M}'$. Thus $(\mathcal{M}',\bar{a}) \models B'$, and we are finished with the δ^+-case. ■

LEMMA 3.2. *Let T be a satisfiable tableau and τ a substitution that associates with every free variable in T a term in the language of T; then $T\tau$ is also satisfiable.*

Proof. By hypothesis for all tubles $\bar{a}$ we have $(\mathcal{M},\bar{a}) \models T$. We claim that for the same structure, we have for all tuples $\bar{b}$ $(\mathcal{M},\bar{b}) \models T\tau$. For definiteness let $x_1,\dots,x_n$ be the free variables in T and $y_1,\dots,y_m$ the free variables in $T\tau$. The length of $\bar{a}$ is thus n and the length of $\bar{b}$ is m. To prove the above claim we consider a given tuple $\bar{b}$. Let a_i be the interpretation of the term $x_i\tau$ in the structure $(\mathcal{M},\bar{b})$, and $\bar{a} = a_1,\dots,a_n$. The substitution lemma tells us that for any formula $\phi(x_1,\dots,x_n)$, $(\mathcal{M},\bar{b}) \models \phi(x_1\tau,\dots,x_n\tau)$ is true if and only if $(\mathcal{M},\bar{a}) \models \phi(x_1,\dots,x_n)$ is true. Thus $(\mathcal{M},\bar{a}) \models T$ implies $(\mathcal{M},\bar{b}) \models T\tau$. ■

As usual, we call a tableau T *closed* when every branch contains a complementary pair of formulas, and *atomically closed* when every branch contains a complementary pair of literals.

LEMMA 3.3. *If T is a closed tableau whose root is labeled by the closed formula $\neg\phi$, then ϕ is universally valid.*

Proof. If T is closed, then T cannot be satisfiable. By Lemmas 3.1 and 3.2 the root cannot be satisfiable, its negation is therefore universally valid. ■

5. Regular Sequences

The proof given in the preceding section is purely model theoretic and does not give any insight into how the interpretations must actually be selected. In this section we give a *syntactical proof* that δ^+ is sound under the mild restriction that free variable substitutions are delayed until all expansion rules have been applied. We will extend an idea already contained in [9]. Smullyan observed and proved that the liberalized ground δ-rule shown in Table V and which we will call the δ^*-rule in the following is still sound.

The key notion in the soundness proof in [9] is that of a *regular sequence* of certain formulas.[2]

DEFINITION 4.1. (*Regular Sequence* [9]). *Regular Sequences* are the finite sequences of formulas that are constructed according to the following rules:

1. $\emptyset$ is a regular sequence.
2. If R is regular, so is $\langle R, \gamma \supset \gamma(t)\rangle$, where t is any ground term.
3. If R is regular, so is $\langle R, \delta \supset \delta(t)\rangle$, provided t is ground and does not occur in R or in δ.

In the above definition $\supset$ denotes material implication. The following fact about regular sequences can be proved:

LEMMA 4.2 ([9]). *If S is a satisfiable set of closed formulas and R a regular sequence and no term that has been introduced in a δ-formula in R occurs in S, then $R \cup S$ is also satisfiable.*

In particular, any regular sequence itself is satisfiable (just take $S = \emptyset$ in the lemma above).

Now consider the sequence of applications of quantifier rules in a tableau proof T. It is easy to see that each of these applications could be substituted by an axiom of the form $Q \supset Q(t)$ (where Q is the premise and $Q(t)$ the conclusion of the used instance of the rule) and some applications of purely propositional rules. Let us denote the sequence of these formulas in the order they were introduced in T with R_T.

Since satisfiability of R_T depends only on the set of formulas in R_T, it is sufficient that R_T can be rearranged into a regular sequence. The proviso of the (strict) δ-rule ensures that R_T (or any ground substitution thereof in the free version) is already a

Table V. Liberalized ground tableaux rules for quantified formulas.

$\dfrac{\gamma}{\gamma(t)}$ where t is any ground term.	$\dfrac{\delta}{\delta(t)}$ where t is ground and either not occurring on the current branch or the following conditions hold simultaneously: 1. t does not occur in δ. 2. t has not been previously introduced by a δ–rule. 3. No t' that has been previously introduced by a δ–rule occurs in $\delta(t)$.

regular sequence. The δ^*-rule is defined such as to ensure that a sequence R_T produced under δ^* can always be reordered accordingly.

We argue that regular sequences can be used to yield an alternative proof of the correctness of δ^+. The first idea that comes to mind is to boil down the proof to ground rule applications and then use Smullyan's proof for δ^*. The proof for δ^* is worked with *quasi-regular sequences*: each sequence generated under δ^* is by definition a quasi-regular sequence. Then Smullyan showed that any quasi-regular sequence can be rearranged into a regular one.

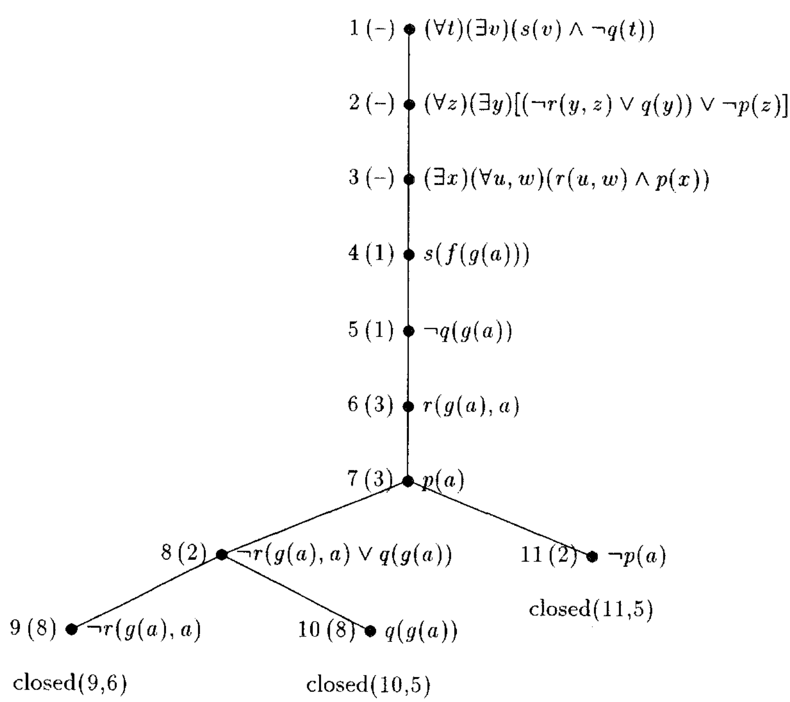
DEFINITION 4.3 (*Quasi-Regular Sequence* [9]). We call a finite sequence of regular formulas $\langle Q_1 \supset Q_1(t_1), \ldots, Q_n \supset Q_n(t_n)\rangle$, *quasi-regular* iff for each $0 < i < n$ either

- t_{i+1} does not occur in any earlier term

or

- t_{i+1} has not been introduced earlier by a δ-rule and for no $j \leqslant i$ t_j occurs in δ_{i+1}.

But the following example of a tableau proof relying on δ^+ shows that this won't work:[3]



Extracting the sequence R_T from the proof yields the sequence (the Skolem terms introduced by the rule applications are boxed and γ-rules have been omitted):[4]

$$\begin{aligned}\langle(\exists v)(s(v) \wedge \neg q(g(a))) &\supset (s(\boxed{\mathrm{f(g(a))}}) \wedge \neg q(g(a))),\\ (\exists x)(\forall u, w)(r(u, w) \wedge p(x)) &\supset (\forall u, w)(r(u, w) \wedge p(\boxed{\mathrm{a}})),\\ (\exists y)[(\neg r(y, a) \vee q(y)) \vee \neg p(a)] &\supset [(\neg r(\boxed{\mathrm{g(a)}}, a) \vee q(\boxed{\mathrm{g(a)}})) \vee \neg p(a)]\rangle\end{aligned}$$

This sequence is not quasi-regular, since $\boxed{\mathrm{g(a)}}$, which is introduced in the third expression, occurs already in the first and $\boxed{\mathrm{a}}$, which is introduced in the second expression, occurs also in the third. On the other hand, we observe that in spite of this fact the sequence can easily be reordered into a regular one: just shift the first expression to the very end.

We draw three consequences from this. First, quasi-regularity is not necessary for regularity. Second, δ^+ for ground cases is not the same as δ^*, for some examples it is stronger.[5] And third, we cannot easily modify Smullyan's proof for our claim.

Before we can give our proof we need another fact: In [4, p. 179f.] it is shown that if there exists a tableau proof of some sentence X, then there exists also a tableau proof where only one application of the substitution rule in the very last step is required. It is obvious that this holds for δ^+-proofs as well; hence we will assume without loss of generality that we deal exclusively with proofs of the latter kind. Now we turn to the proof that ground instances of δ^+-generated sequences are regular. We begin with a definition that enables us to describe regular sequences as certain relations on the set of formulas in δ^+-generated sequences.

DEFINITION 4.4 (*Critical Occurrence*). Let $S = \{\delta_1 \supset \delta_1(t_1), \ldots, \delta_k \supset \delta_k(t_k)\}$ be the set of δ-type formulas in a δ^+-generated sequence. We define a binary relation $\curvearrowright$ on S as follows:

$$\delta_i \supset \delta_i(t_i) \curvearrowright \delta_j \supset \delta_j(t_j) \text{ iff } t_i \text{ occurs in } \delta_j$$

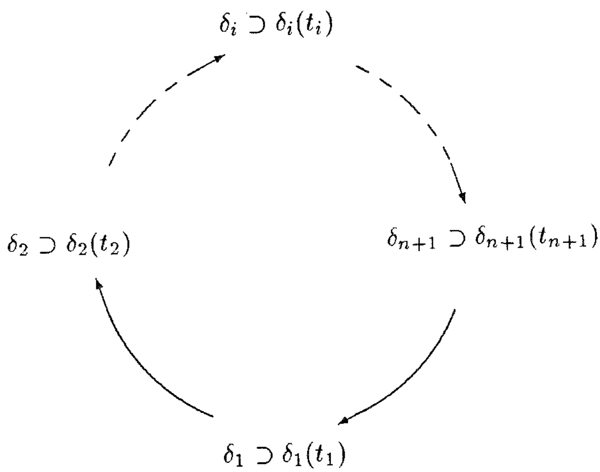
for arbitrary $1 \leqslant i, j \leqslant k$. We say that t_i *occurs critically* in δ_j.

LEMMA 4.5. *Let S be the set of δ-formulas in a δ^+-generated sequence. Then $\curvearrowright$ contains no cycles on S.*

Proof. We prove by induction that for no finite n the relation $\curvearrowright$ contains a cycle of length n in S. Since S is finite, this is obviously sufficient to prove the claim.

$n = 1$. In this case there must be a $\delta_i \supset \delta_i(t_i) \in S$ such that t_i occurs in δ_i. This is clearly impossible by the definition of δ^+.

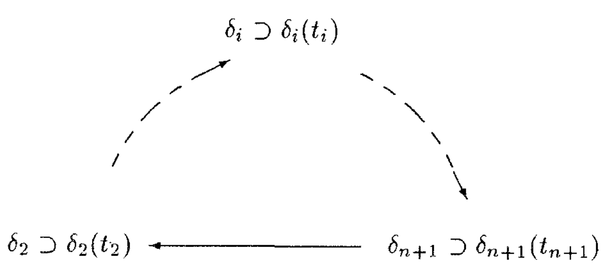
$n \to n + 1$. Assume there is a cycle of length $n + 1$. Then we have for some $\{\delta_1 \supset \delta_1(t_1), \ldots, \delta_{n+1} \supset \delta_{n+1}(t_{n+1})\} \subseteq S$ the following picture:



First let us note that there are only two ways in which a term t_{i-1} that occurs critically in some δ_i can be introduced into δ_i. Either δ_i is a proper subformula of $\delta_{i-1}(t_{i-1})$, or t_{i-1} was substituted into a free variable occurring in δ_i. In the latter case, since we assume substitutions to be delayed in the proof, this variable was still uninstantiated at application time of δ_i and therefore, by definition of δ^+, t_{i-1} must occur as a subterm of t_i. According to this analysis we speak of the relation $\curvearrowright$ between two members of S to be either of *subformula type* or of *free variable type.*

We observe further that at least one of the relations between the formulas in the cycle must be of free variable type, since a formula cannot be a proper subformula of itself. Let us assume without loss of generality that this happens with $\delta_{n+1} \supset \delta_{n+1}(t_{n+1})$ and $\delta_1 \supset \delta_1(t_1)$. But then our argument above yields that t_{n+1} must be a subterm of t_1.

Now, since $\delta_1 \supset \delta_1(t_1) \curvearrowright \delta_2 \supset \delta_2(t_2)$ holds, t_1 and thus t_{n+1} must occur in δ_2. Hence, we can construct a cycle of length n as follows, contradicting the induction hypothesis.



■

LEMMA 4.6. *Let S be the set of δ-type formulas in a δ^+-generated sequence. Then the following are equivalent:*

1. *$\curvearrowright$ contains no cycles on S.*
2. *S can be arranged as a regular sequence.*

Proof. Suppose $\curvearrowright$ contains no cycles on S. Then we apply a standard argument that is used to linearize DAGs.

Let $\mathscr{S}$ be the DAG with nodes from S that is induced by $\curvearrowright$. The following algorithm yields the desired regular sequence S^*.

$S^* := \langle\,\rangle$;
while $\mathscr{S} \neq \emptyset$ do
  Let $\delta \supset \delta(t)$ be an element without predecessor from $\mathscr{S}$;
    % $\delta \supset \delta(t)$ exists since $\mathscr{S}$ is acyclic.
  $\mathscr{S} := \mathscr{S} - \delta \supset \delta(t)$;
  $S^* := \langle \delta \supset \delta(t), S^* \rangle$
od

Termination is obvious, since $\mathscr{S}$ is finite. By definition of $\curvearrowright$ and since $\delta \supset \delta(t)$ is chosen as a minimal element in $\curvearrowright$, the elements that are already in S^* do not occur critically in δ. Moreover, the leading symbol of t is unique and all other subterms are already present in δ, so there can be no critical occurrence in the whole formula $\delta \supset \delta(t)$. Thus the regularity of S^* is a loop invariant, which proves the first part.

To see the converse, assume that $\curvearrowright$ contains cycles on S, and observe that no member of a cycle can be on the right to all other members of the cycle. Thus already the members of the cycle cannot be ordered into a regular sequence. ∎

THEOREM 4.7. *For any tableau proof T using δ^+, any δ^+-generated sequence R_T obtained from T can be reordered such that it becomes a regular sequence.*

Proof. Immediate by the preceding lemmas, if one observes that the γ-type formulas can be placed arbitrarily after the δ-part has been made regular. ∎

THEOREM 4.8 (*Soundness*). *If a sentenxe X has a closed tableau T constructed with δ^+, then X is not satisfiable.*

Proof. Without loss of generality assume that T is ground. We transform T as before into a closed tableau T' for $\{X\} \cup R_T$ that contains only applications of propositional rules, where R_T is the δ^+-generated sequence of T. Soundness of the propositional case yields that the set $\{X\} \cup R_T$ is unsatisfiable. Theorem 4.7 tells us that R_T can be arranged into a regular sequence; and since X was a sentence, we can apply Lemma 4.2 and conclude that $\{X\}$ is unsatisfiable. ∎

Notes

[1]Smullyan did not include function symbols in his first-order language, so in his case constants were the only ground terms. We assure the reader that in the extended language all results are still valid and the proofs may be adopted without any problems.
[2]Again, the following definitions and results contain the necessary modifications of Smullyan's work to cover function symbols. All required results carry over without any problems.
[3]We omitted some intermediate steps and show the tableau after closing substitutions were made. The formulas are numbered on their very left. The bracketed numbers indicate the parent formula.
[4]In the following we will restrict the term *δ^+-generated sequence* to ground instances of expressions of such sequences.

[5]Since the leading functor in the Skolem function of the δ^+ rule must *always* be new (in contrast to the δ^* rule), there are trivial cases when δ^* is stronger.

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