

# Computing the Maximum Degree of Minors in Skew Polynomial Matrices

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## Abstract

Skew polynomials, which have a noncommutative multiplication rule between coefficients and an indeterminate, are the most general polynomial concept that admits the degree function with desirable properties. This paper presents the first algorithms to compute the maximum degree of the Dieudonné determinant of a  $k \times k$  submatrix in a matrix  $A$  whose entries are skew polynomials over a skew field  $F$ . Our algorithms make use of the discrete Legendre conjugacy between the sequences of the maximum degrees and the ranks of block matrices over  $F$  obtained from coefficient matrices of  $A$ . Three applications of our algorithms are provided: (i) computing the dimension of the solution spaces of linear differential and difference equations, (ii) determining the Smith–McMillan form of transfer function matrices of linear time-varying systems and (iii) solving the “weighted” version of noncommutative Edmonds’ problem with polynomial bit complexity. We also show that the deg-det computation for matrices over sparse polynomials is at least as hard as solving commutative Edmonds’ problem.

**Keywords:** skew polynomials, Dieudonné determinant, matrix expansion, discrete Legendre conjugacy, differential equations, difference equations, Smith–McMillan form, Edmonds’ problem

## 1 Introduction

Let  $R$  be a (unitary) ring endowed with an automorphism  $\sigma: R \rightarrow R$  and a  $\sigma$ -derivation  $\delta: R \rightarrow R$ , that is, an additive map satisfying  $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$  for all  $a, b \in R$ . The *skew polynomial ring*, or the *Ore polynomial ring* due to Ore [43] over  $(R, \sigma, \delta)$  in indeterminate  $s$ , which is denoted by  $R[s; \sigma, \delta]$ , is a polynomial ring over  $R$  with the usual addition and a twisted multiplication defined by the commutation rule

$$sa = \sigma(a)s + \delta(a) \quad (1)$$

for all  $a \in R$ . Elements in  $R[s; \sigma, \delta]$  are called *skew polynomials* over  $(R, \sigma, \delta)$ . This paper deals only with skew polynomial rings with  $R$  being a skew (not necessarily commutative) field  $F$ .

The usual polynomial ring  $F[s]$  over  $F$  is trivially a skew polynomial ring with  $\sigma = \text{id}$  and  $\delta = 0$ . A typical nontrivial example of a skew polynomial ring is the ring  $\mathbb{C}(t)[\partial; \text{id}, ']$  of differential operators, where  $\mathbb{C}(t)$  is the rational function field over the set  $\mathbb{C}$  of complex numbers and  $': \mathbb{C}(t) \rightarrow \mathbb{C}(t)$  is the usual differentiation. Another example of a skew polynomial ring is the ring  $\mathbb{C}(t)[S; \tau, 0]$  of shift operators, where  $\tau: \mathbb{C}(t) \rightarrow \mathbb{C}(t)$  is defined by  $f(t) \mapsto f(t+1)$  for  $f \in \mathbb{C}(t)$ . In this way, skew polynomial rings naturally arise as an algebraic abstraction of the rings of the differential and shift (difference) operators. Since both sides of (1) are of “degree

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one” with respect to  $s$ , the degree  $\deg p$  of a skew polynomial  $p \in F[s; \sigma, \delta]$  is well-defined. Skew polynomial rings (over a skew field) are known to be the most general concept of polynomials that (i) admits the degree function with desired properties, e.g.,  $\deg pq = \deg p + \deg q$ , and (ii) has the skew field  $F(s; \sigma, \delta)$  of fractions, called the skew (rational) function field [13, 19].

A *skew polynomial matrix*  $A$  of degree  $\ell$  over  $(F, \sigma, \delta)$  is a matrix over  $F[s; \sigma, \delta]$  in which the maximum degree of an entry is  $\ell$ . This paper addresses the problem of computing

$$d_k(A) := \max\{\deg \text{Det } A[I, J] \mid |I| = |J| = k\} \quad (2)$$

for given  $k$ , where  $A[I, J]$  denotes the submatrix of  $A$  indexed by a row set  $I$  and a column set  $J$ , and  $\text{Det}$  denotes the *Diudonné determinant*, which is a noncommutative generalization of the usual determinant defined for matrices over skew fields [12, 14]. While the value of  $\text{Det } A[I, J]$  is no longer in  $F[s; \sigma, \delta]$ , its degree is well-defined [13, 46].

## 1.1 Motivating Applications

Our motivation of computing  $d_k(A)$  (especially  $d_n(A) = \deg \text{Det } A$  for square matrices of size  $n$ ) is threefold. The first motivation is the application to linear differential and difference equations. Consider a system of linear (ordinary) differential equations

$$A_\ell y + A_{\ell-1} \frac{dy}{dt} + \cdots + A_0 \frac{d^\ell y}{dt^\ell} = 0 \quad (3)$$

for an  $n$ -dimensional unknown vector  $y(t)$ , where  $A_0, \dots, A_\ell \in \mathbb{C}(t)^{n \times n}$ . Using the differential operator  $\partial$ , the equation (3) is rewritten as

$$(A_\ell + A_{\ell-1}\partial + \cdots + A_0\partial^\ell)y = 0. \quad (4)$$

The coefficient matrix  $A := A_\ell + A_{\ell-1}\partial + \cdots + A_0\partial^\ell$  of (4) is a skew polynomial matrix in  $\partial$  over  $(\mathbb{C}(t), \text{id}, ')$ . If  $A_0, \dots, A_\ell \in \mathbb{C}^{n \times n}$ , then  $A$  can be regarded as a (usual) polynomial matrix over  $\mathbb{C}(t)$  and the classical Chrystal’s theorem [9] guarantees that  $\deg \det A$  coincides with the dimension of the solution space of (4). Taelman [46] showed that Chrystal’s theorem holds for general  $A$  by replacing  $\det$  with  $\text{Det}$ . Here the solution space is considered over the *Picard–Vessiot extension* of (4), that is, an extension of  $(\mathbb{C}(t), \text{id}, ')$  in which all the possible solutions of (4) exist; see Section 4 for more rigorous description. Similarly, a system of linear difference equations

$$A_\ell y(t) + A_{\ell-1}y(t+1) + \cdots + A_0y(t+\ell) = 0 \quad (5)$$

with  $A_0, \dots, A_\ell \in \mathbb{C}(t)^{n \times n}$  can be written as  $Ay = 0$ , where  $A := A_\ell + A_{\ell-1}S + \cdots + A_0S^\ell \in \mathbb{C}(t)[S; \tau, 0]^{n \times n}$  is a skew polynomial matrix over  $(\mathbb{C}(t), \tau, 0)$ . In this paper we show that the dimension of the solution space  $V$  of (5) coincides with

$$\dim V = \deg \text{Det } A - \text{ord } \text{Det } A \quad (6)$$

over an “adequate” field extension of  $\mathbb{C}(t)$ . Here,  $\text{ord } \text{Det } A$ , which is the lowest degree of a term in  $\deg A$  in the commutative case, is the dual concept of  $\deg \text{Det } A$  defined for matrices over skew polynomial rings with  $\delta = 0$  and is calculated from  $\deg \text{Det}$  of a skew polynomial matrix obtained from  $A$ . Therefore, an algorithm for computing  $\deg \text{Det } A$  can be used to determine the dimension of the solution spaces of linear differential/difference equations, which is a fundamental problem in computer algebra systems.

The second motivation comes from control theory, which is related to the first one but has a slightly different context. In classical control theory, polynomial matrices over  $\mathbb{C}$  (or rational function matrices in general) appear as *transfer function matrices* of linear time-invariant systems. The *Smith–McMillan form* is a canonical form of a rational function matrix  $A \in \mathbb{C}(s)^{n \times n}$

and is determined from the sequence  $\delta_1(A), \dots, \delta_n(A)$ . For a polynomial matrix  $A$  of degree one, called a *matrix pencil*, this sequence is used to compute the *Kronecker canonical form* of  $A$ . These canonical forms have important significance in theoretical and numerical analyses of linear time-invariant systems [29, 40]. Kamen [30] extended transfer function matrices to linear time-varying systems by employing skew polynomial matrices over  $(F, \text{id}, \delta)$  with  $F$  being a (commutative) field such as  $\mathbb{C}(t)$ . Their Smith–McMillan form is established by Bourlés–Marinescu [6]. Even for time-varying systems, the Smith–McMillan form provides important structural implications and information such as the index of linear time-varying differential equations [6, 16] and the existence of a proper solution in the exact model-matching problem [37]. A recent result [32] indicates that  $\deg \text{Det}$  of skew polynomial matrices obtained from nonlinear systems coincides with the order of the minimum state-space realization of the systems.

Our third motivation comes from combinatorial optimization and combinatorial matrix theory. For a polynomial matrix  $A$  over a field, it is well-known that  $d_k(A)$  is bounded by the maximum weight of a matching of size  $k$  in an edge-weighted bipartite graph associated with  $A$ . Based on this relation, Murota’s *combinatorial relaxation* algorithm [39] computes  $d_k(A)$  by iteratively solving a maximum weight matching problem. Hirai [23] indicated that the deg-det computation of certain types of polynomial matrices corresponds to solving a weighted linear matroid intersection problem and a weighted linear matroid parity problem. These are natural “weighted analog” of the relation between the rank computation of constant matrices and (unweighted) combinatorial optimization problems observed by Edmonds [15] and Lovász [36].

On the field of computational complexity, the noncommutative algebra has gained attention in the recent exploration of Edmonds’ problem. In 1967, Edmonds [15] posed a question whether there exists a polynomial-time algorithm to compute the rank of a *linear matrix*  $B$  over a field  $K$ , which is in the form

$$B = B_0 + B_1x_1 + \dots + B_mx_m,$$

where  $B_0, B_1, \dots, B_m \in K^{n \times n}$  and  $x_1, \dots, x_m$  are commutative symbols. Here, the rank of  $B$  is in the sense of the field  $K(x_1, \dots, x_m)$  of rational functions in  $x_1, \dots, x_m$  over  $K$ . In this paper, we refer to  $s$  as an indeterminate and to  $x_1, \dots, x_m$  as symbols to distinguish them. While the Schwartz–Zippel lemma [44] provides a simple randomized algorithm for this problem if  $|K|$  is large enough [36], no deterministic polynomial-time algorithm is known; the existence of such an algorithm would imply nontrivial circuit complexity lower bounds [28, 48]. Recent studies [17, 21, 24] address the noncommutative version of Edmonds’ problem. This is a problem of computing the *noncommutative rank* (nc-rank) of  $B$ , which is the rank defined by regarding  $x_1, \dots, x_m$  as pairwise noncommutative, i.e.,  $x_i x_j \neq x_j x_i$  if  $i \neq j$ . In this way,  $B$  is viewed as a matrix over the free ring  $K\langle x_1, \dots, x_m \rangle$  generated by noncommutative symbols  $x_1, \dots, x_m$ . Then the nc-rank of  $B$  is precisely the rank of  $B$  over the skew field  $K\langle\!\langle x_1, \dots, x_m \rangle\!\rangle$ , called a *free skew field*, which is the quotient of  $K\langle x_1, \dots, x_m \rangle$  defined by Amitsur [2]. We call a linear matrix over  $K$  having noncommutative symbols a *noncommutative linear matrix* (nc-linear matrix) over  $K$ . The recent studies [17, 21, 24] revealed that noncommutative Edmonds’ problem is deterministically tractable. For the case where  $K$  is the set  $\mathbb{Q}$  of rational numbers, Garg et al. [17] proved that Gurvit’s *operator scaling algorithm* [20] deterministically computes the nc-rank of  $B$  in  $\text{poly}(n, m)$  arithmetic operations on  $\mathbb{Q}$ . Algorithms over general field  $K$  were later given by Ivanyos et al. [24] and Hamada–Hirai [21] exploiting the min-max theorem established for nc-rank. In [20] and [24] applied to the case of  $K = \mathbb{Q}$ , bit-lengths of intermediate numbers are proved to be bounded by a polynomial of the input bit-length.

As a weighted analog of the nc-rank computation, Hirai [23] introduced the following *weighted noncommutative Edmonds’ problem* (WNEP):

### Weighted Noncommutative Edmonds’ Problem (WNEP)

**Input** :  $A = A_\ell + A_{\ell-1}s + \dots + A_0s^\ell \in K\langle x_1, \dots, x_m \rangle[s]^{n \times n}$ , where  $A_d = A_{d,0} + A_{d,1}x_1 + \dots + A_{d,m}x_m \in K\langle x_1, \dots, x_m \rangle^{n \times n}$  is a nc-linear matrix for  $d = 0, \dots, \ell$ .

**Output:**  $\deg \text{Det } A$ .

Here  $s$  commutes any element in  $K\langle x_1, \dots, x_m \rangle$ . We call the matrix  $A$  of the input of WNEP a *noncommutative linear polynomial matrix* (nc-linear polynomial matrix). Hirai [23] formulated the dual problem of WNEP as a minimization of an *L-convex function* on a *uniform modular lattice*, and gave an algorithm based on the steepest gradient descend. Hirai's algorithm uses  $\text{poly}(n, m, \ell)$  arithmetic operations on  $K$  while no bit-length bound has been given for  $K = \mathbb{Q}$ .

## 1.2 Our Contributions

In this paper, we provide the first algorithm to compute  $d_k$  of skew polynomial matrices over  $(\mathbb{F}, \sigma, \delta)$  with  $\mathbb{F}$  being a skew field. Instead of skew polynomial matrices, we deal with  $A = A_0 + A_1 s^{-1} + \dots + A_\ell s^{-\ell} \in F[s; \sigma, \delta]^{n \times n'}$  to make our theorems and algorithm simple;  $d_k$  of a skew polynomial matrix  $A_\ell + A_{\ell-1} s + \dots + A_0 s^\ell$  is obtained by adding  $\ell k$  to  $d_k(A)$ .

Our algorithm is based on a method, called the *matrix expansion*, that constructs a  $\mu \times \mu$  block matrix  $\Omega_\mu(A) \in F^{\mu n \times \mu n'}$  obtained by iteratively applying  $\sigma^{-1}$  and  $\delta$  to the coefficient matrices of  $A$ . Through the Smith–McMillan form which we extend to general skew function fields, it is shown that the sequences of  $(d_0(A), d_1(A), \dots, d_r(A))$  with  $r := \text{rank } A$  and  $(\omega_0(A), \omega_1(A), \dots)$  with  $\omega_\mu(A) := \text{rank } \Omega_\mu(A)$  are concave and convex, respectively. In addition, they are in the relation of the *discrete Legendre conjugate*, that is, they satisfy

$$d_k(A) = \min_{\mu \geq 0} (\omega_\mu(A) - k\mu) \quad (0 \leq k \leq r), \quad (7)$$

$$\omega_\mu(A) = \max_{0 \leq k \leq r} (d_k(A) + k\mu) \quad (\mu \geq 0). \quad (8)$$

The Legendre conjugacy is an important duality relation on discrete convex and concave functions treated in *discrete convex analysis* [41]. These formulas (7) and (8) are “ultimate” generalization of the work on matrix pencils over fields by Murota [42] and on polynomial matrices over  $\mathbb{C}$  by Moriyama–Murota [38]. To prove them, we need equalities that connect  $d_k(A)$  and  $\omega_\mu(A)$ . On this point Murota [42] and Moriyama–Murota [38] depend on the results of Iwata–Shimizu [25] and Tan–Pugh [47], respectively. These results, however, are hard to extend to general skew polynomial matrices because the result of Iwata–Shimizu [25] is based on the Kronecker canonical form which is established only for matrix pencils and Tan–Pugh [47] makes use of the algebraic closedness of  $\mathbb{C}$ . Instead of them, we present a short connection between  $d_k(A)$  and  $\omega_\mu(A)$  through the identity

$$\Omega_\mu(A) \Omega_\mu(B) = \Omega_\mu(AB), \quad (9)$$

which is an extension of an identity given by Van Dooren et al. [51] on rational function matrices over  $\mathbb{C}$  in the context of control theory.

The conjugacy formula (7) reduces the computation of  $d_k(A)$  to a discrete convex optimization problem. In this problem, the objective function is evaluated by computing the rank of the block matrix  $\Omega_\mu(A)$  over  $F$  and its minimization can be efficiently done by the binary search. We also show that the problem has a minimizer no more than  $\ell r$ . The computational cost of our algorithm is summarized in the following main theorem:

**Theorem 1.1.** *Let  $A \in F[s; \sigma, \delta]^{n \times n'}$  be a skew polynomial matrix of degree  $\ell$  over  $(F, \sigma, \delta)$  with  $F$  being a skew field. If  $r := \text{rank } A$  is known, for  $k = 0, \dots, r$ , we can compute  $d_k(A)$  in  $O(\ell^2 n n' r^2 (T_- + T_\sigma + T_\delta) + \log \ell r \cdot \text{RO}(\ell n r, \ell n' r))$  time, where  $T_-$  is the time of the subtraction on  $F$ ,  $T_\sigma$  is the time to apply  $\sigma^{-1}$ ,  $T_\delta$  is the time to apply  $\delta$ , and  $\text{RO}(n, n')$  is the time to compute the rank of an  $n \times n'$  matrix over  $F$ .*

Moreover, from (7), we derive the following formulas with respect to  $r$  and  $d_r(A)$ :

$$r = \omega_{\ell n^* + 1}(A) - \omega_{\ell n^*}(A), \quad (10)$$

$$d_r(A) = \omega_{\ell r}(A) - \ell r^2, \quad (11)$$

where  $n^* := \min\{n, n'\}$ . These formulas provide quite simple algorithms to compute  $r$  and  $d_r(A)$ ; what we need is only the rank computation over  $F$ .

**Theorem 1.2.** *Suppose the same setting as Theorem 1.1. Then there exists an algorithm to compute  $r$  in  $O(\ell^2 nn' n^{*2}(T_- + T_\sigma + T_\delta) + \text{RO}(\ell nn^*, \ell n' n^*))$  time. In addition, if  $r$  is given, we can compute  $d_r(A)$  in  $O(\ell^2 nn' r^2(T_- + T_\sigma + T_\delta) + \text{RO}(\ell nr, \ell n' r))$  time.*

If the arithmetic operations on  $F$  are performed in constant time (e.g. finite fields), the rank of an  $n \times n$  matrix over  $F$  can be obtained by the standard Gaussian elimination in  $O(n^3)$  time (or more efficient algorithms that run in  $O(n^\omega)$  time are available [5], where  $2 \leq \omega < 3$  is the matrix multiplication exponent [34]). Fraction-free Gaussian elimination algorithms [3, 15] enable us to compute the rank of matrices over  $\mathbb{Q}$  and  $\mathbb{Q}[t]$  with polynomial bit complexity, where we assume that a polynomial  $p \in \mathbb{Q}[t]$  is encoded as the array of coefficients of length  $\deg p + 1$ . For a matrix  $A$  over the differential operator ring  $\mathbb{Q}[t][\partial; \text{id}, ']$  and the shift operator ring  $\mathbb{Q}[t][S; \tau, 0]$  over  $\mathbb{Q}[t]$ , the bit-length of  $\Omega_\mu(A)$  is bounded by a polynomial of the bit-length of  $A$ . Hence  $\delta_k(A)$  can be computed in polynomial number of bit operations. This can be applied to the computation of the dimension of the solution spaces of linear differential and difference equations with coefficients in  $\mathbb{Q}[t]$ , as well as the Smith–McMillan form of transfer function matrices in time-varying systems. Here the formula (6) for the linear difference equation (5) is also our contribution. We remark that it is difficult for the combinatorial relaxation algorithm [39] to achieve the same bit complexity because it iteratively performs the Gaussian elimination on the same matrix and thus the magnitude of its entries might swell.

Our algorithm can also be used to solving WNEP. Suppose that  $A \in K\langle x_1, \dots, x_m \rangle[s]^{n \times n}$  is a nc-linear polynomial matrix. In the case of usual polynomial rings, the expanded matrix  $\Omega_\mu(A)$  is built just by arranging the coefficients in  $A$ . Hence  $\Omega_\mu(A)$  is an nc-linear matrix over  $K$ , whose rank can be computed in  $\text{poly}(n, m)$  arithmetic operations on  $K$  by [17, 21, 24]. Furthermore, in the case of  $K = \mathbb{Q}$ , the bit-length of  $\Omega_\mu(A)$  is a polynomial of the bit-length of  $A$  since each block of  $\Omega_\mu(A)$  is just a copy of some coefficient matrix of  $A$ . Therefore, by using the rank computation algorithms [17, 24] for nc-linear matrices with bit-length bounds, we obtain the following:

**Theorem 1.3.** *WNEP is deterministically solvable in  $\text{poly}(n, m, \ell)$  arithmetic operations on  $K$ . In addition, if  $K = \mathbb{Q}$ , the bit-lengths of intermediate numbers are bounded by a polynomial of the input bit-length.*

In view of combinatorial optimization, our algorithm is regarded as a pseudo-polynomial time algorithm since the running time depends on a polynomial of the maximum exponent  $\ell$  of  $s$  instead of  $\text{poly}(\log \ell)$ . Thus it is natural to try to solve the following problem:

### Sparse Degree of Determinant (SDD)

**Input :**  $A = A_1 s^{w_1} + \dots + A_m s^{w_m} \in K[s]^{n \times n}$ , where  $0 \leq w_1 \leq \dots \leq w_m$  are integers.

**Output:**  $\deg \det A$ .

Indeed, it is shown in this paper that (commutative) Edmonds' problem is reducible to SDD as follows:

**Theorem 1.4.** *If there exists a deterministic algorithm to solve SDD over a field  $K$  in  $\text{poly}(n, m, \log w_m)$  arithmetic operations on  $K$ , then Edmonds' problem over  $K$  can be deterministically solved in  $\text{poly}(n, m)$  arithmetic operations on  $K$ .*

Since giving a deterministic polynomial-time algorithm for Edmonds' problem has still been open for more than half a century, Theorem 1.4 implies that SDD would also be a quite challenging problem.

We lastly claim that our algorithms can also be applied to matrices over the multivariate version of skew polynomial rings, called *iterated skew polynomial rings*. This is a polynomial

ring

$$S_m := F[s_1; \sigma_1, \delta_1][s_2; \sigma_2, \delta_2] \cdots [s_m; \sigma_m, \delta_m]$$

in indeterminates  $s_1, \dots, s_m$  over a skew field  $F$ . Here each  $S_i := F[s_1; \sigma_1, \delta_1] \cdots [s_i; \sigma_i, \delta_i]$  is the skew polynomial ring over  $(S_{i-1}, \sigma_i, \delta_i)$  with  $S_0 := F$ , where  $\sigma_i$  is an automorphism of  $S_{i-1}$  and  $\delta_i$  is a  $\sigma_i$ -derivation of  $S_{i-1}$ . Iterated skew polynomial rings arise from partial differential equations. Consider computing the rank of  $A \in S_m^{n \times n}$  whose degree in  $s_i$  is  $\ell_i$  for  $i = 1, \dots, d$ . By the rank formula (10), this can be reduced to the rank computation of  $\Omega_{\ell_m n}(A) \in S_{m-1}^{\ell_m n^2 \times \ell_m n^2}$  and  $\Omega_{\ell_m n+1}(A) \in S_{m-1}^{(\ell_m n+1)n \times (\ell_m n+1)n}$ . Using (10) again, the rank computation of these two matrices can be reduced to the rank computation of four matrices over  $S_{m-2}$  of size  $O(\ell_{m-1} \ell_m^2 n^4)$ . Iterating this operation  $m$  times, we reach to the rank computation of  $2^m$  matrices over  $F$  of size  $O(\ell_1 \ell_2^2 \cdots \ell_m^{2^{m-1}} n^{2^m})$ , which is a polynomial of  $\ell_1, \dots, \ell_m$  and  $n$  if  $m$  is regarded as a constant. The same argument is valid for the computation of  $\deg \text{Det } A$  and  $\delta_k(A)$ , where the degree is with respect to  $s_m$  (or we can compute the total degree of  $\text{Det}$  by adding a new indeterminate), since it is also reduced to the rank computation over  $S_{m-1}$ .

### 1.3 Related Work

In computer algebra, algorithms were proposed for computing various kinds of canonical forms of a skew polynomial matrix  $A \in F[s; \sigma, \delta]^{n \times n}$  such as the *Jacobson normal form* [35], the *Hermite normal form* [18], the *Popov normal form* [31] and their weaker form called a *row-reduced form* [1, 4]. One can use these algorithms to calculate  $\deg \text{Det } A$  since it is immediately obtained from the canonical forms of  $A$ . In particular, algorithms of Beckermann et al. [4] for a row-reduced form, Giesbrecht–Kim [18] for the Hermite normal form and Khochtali et al. [31] for the Popov normal form run in the polynomial number of bit operations for matrices over  $\mathbb{Q}[t][\partial; \text{id}, ']$  and  $\mathbb{Q}[t][S; \tau, 0]$ . Their algorithms solve systems of linear equations over  $\mathbb{Q}[t]$  whose coefficient matrices are variants of expanded matrices  $\Omega_\mu(A)$  by the name of “linearized matrices” [31] and “striped Krylov matrices” [4].

In comparison with these algorithms, our algorithm for computing  $\deg \text{Det } A$  based on the formula (11) is much more simple. In addition, our algorithm is also advantageous in that it requires only the rank computation rather than linear equation solving. While the most algorithms for computing the rank and solving linear equations depend on elimination methods, for nc-linear polynomial matrices, only the rank computation is available [17, 21, 24] because nc-linear polynomials are essentially multivariate polynomials, which can have exponentially many terms with respect to the degree. Hence our algorithm is the first one that can be applied to solving WNEP.

### 1.4 Organization

The rest of this paper is organized as follows. Section 2 provides preliminaries on matrices over skew fields, skew polynomial rings and skew function fields. Section 3 describes our proposed algorithms after introducing the matrix expansion and the Legendre conjugacy. Section 4 gives deg-det type formulas for the dimension of the solution spaces of linear differential and difference equations using a unified framework called  *$\sigma$ -differential equations*. Finally, Section 5 shows a reduction of SDD to Edmonds’ problem.

## 2 Matrices and Skew Polynomials

Let  $\mathbb{Z}$  denote the set of integers and  $\mathbb{N}$  denote the set of nonnegative integers. For  $n \in \mathbb{N}$ , we denote the set  $\{1, 2, \dots, n\}$  by  $[n]$  and  $\{0, 1, 2, \dots, n\}$  by  $[0, n]$ . All the rings are assumed to have the multiplicative identity.



## 2.1 Matrices over Skew Fields

A *skew field*, or a *division ring* is a ring  $F$  such that every nonzero element has a multiplicative inverse in  $F$ . A right  $F$ -module is especially called a *right  $F$ -vector space*. The *dimension* of a right  $F$ -vector space  $V$  is defined as the rank of  $V$  as a module, that is, the size of any basis of  $V$ . A left  $F$ -vector space and its dimension are also defined in the same way. The usual facts from linear algebra on independent sets and generating sets in vector spaces are valid even on skew fields [33].

We denote by  $F^{n \times n'}$  the set of all  $n \times n'$  matrices over  $F$  for  $n, n' \in \mathbb{N}$ . A square matrix  $A \in F^{n \times n}$  is said to be *nonsingular* if there exists  $B \in F^{n \times n}$  such that  $AB = I_n$ , which is equivalent to  $BA = I_n$ , where  $I_n$  is an identity matrix of size  $n$ . If  $A$  is nonsingular, such a matrix  $B$  is unique and is denoted by  $A^{-1}$ . Note that the  $0 \times 0$  matrix is nonsingular. A square matrix is *singular* if it is not nonsingular. The *rank* of a matrix  $A \in F^{n \times n'}$  is the dimension of the right  $F$ -vector space spanned by column vectors of  $A$ , and is equal to the dimension of the left  $F$ -vector space spanned by row vectors of  $A$ . We denote the rank of  $A$  by  $\text{rank } A$ . By definition, it holds  $\text{rank } BAC = \text{rank } A$  for nonsingular  $B \in F^{n \times n}$  and  $C \in F^{n' \times n'}$ . It is observed that a square matrix  $A \in F^{n \times n}$  is nonsingular if and only if  $\text{rank } A = n$ . The rank of  $A \in F^{n \times n'}$  is equal to the minimum  $r \in \mathbb{N}$  such that there exists a decomposition  $A = BC$  by some  $B \in F^{n \times r}$  and  $C \in F^{r \times n'}$  [11]. Here we give another characterization of the rank, which is well-known on the commutative case.

**Proposition 2.1.** *The rank of a matrix  $A \in F^{n \times n'}$  over a skew field  $F$  is equal to the maximum  $r \in \mathbb{N}$  such that  $A$  has a nonsingular  $r \times r$  submatrix. In addition,  $A$  has a nonsingular  $k \times k$  submatrix for all  $k = 0, \dots, r$ .*

*Proof.* We first show the latter part. For  $k = 0, \dots, \text{rank } A$ , we can take a column subset  $J \subseteq [n']$  of cardinality  $k$  such that the column vectors of  $A[[n], J]$  are linearly independent. Since  $\text{rank } A[[n], J] = k$ , there must be  $I \subseteq [n]$  of cardinality  $k$  such that the row vectors of  $A[I, J]$  is linearly independent. Then  $A[I, J]$  is a  $k \times k$  nonsingular submatrix of  $A$  due to  $\text{rank } A[I, J] = k$ .

The former part is shown as follows. Let  $r \in \mathbb{N}$  be the maximum size of a nonsingular submatrix of  $A$ . It holds  $\text{rank } A \leq r$  by the latter part of the claim. To show  $\text{rank } A \geq r$ , take an  $r \times r$  nonsingular submatrix  $A[I, J]$  of  $A$ . Since  $\text{rank } A[I, J] = r$ , the set of column vectors of  $A$  indexed by  $J$  is linearly independent. Thus we have  $\text{rank } A \geq r$ .  $\square$

Next, we define the *Dieudonné determinant* [14] for nonsingular square matrices over a skew field  $F$ . We first introduce a decomposition of matrices needed to define the Dieudonné determinant.

**Lemma 2.2** (Bruhat decomposition [12, Theorem 2.2 in Section 11.2]). *A square matrix  $A \in F^{n \times n}$  over a skew field  $F$  can be decomposed as  $A = LDP$ , where  $L \in F^{n \times n}$  is lower unitriangular,  $D \in F^{n \times n}$  is diagonal,  $P \in F^{n \times n}$  is a permutation matrix and  $U \in F^{n \times n}$  is upper unitriangular. In addition,  $DP$  is uniquely determined.*

Here, a (lower and upper) unitriangular matrix is a (lower and upper) triangular matrix whose diagonal entries are 1. Since each row and column in  $DP$  has at most one nonzero entry, the uniqueness of  $DP$  implies that of  $D$ . In addition, since  $L, P$  and  $U$  are nonsingular,  $A$  and  $D$  have the same rank, which is equal to the number of nonzero entries in  $D$ . Thus  $P$  is also unique if  $A$  is nonsingular.

Let  $F_{\text{ab}}^\times := F^\times / [F^\times, F^\times]$  be the abelization of the multiplicative subgroup  $F^\times = F \setminus \{0\}$  of  $F$ , where  $[F^\times, F^\times] := \langle aba^{-1}b^{-1} \mid a, b \in F^\times \rangle$  is the commutator group of  $F^\times$ . Consider a Bruhat decomposition  $A = LDP$  of a nonsingular matrix  $A \in F^{n \times n}$ . The *Dieudonné determinant*  $\text{Det } A$  of  $A$  is an element of  $F_{\text{ab}}^\times$  defined by

$$\text{Det } A := \text{sgn}(P)e_1e_2 \cdots e_n \bmod [F^\times, F^\times],$$

where  $\text{sgn}(P) \in \{-1, +1\}$  is the sign of the permutation  $P$  and  $e_1, \dots, e_n$  are the diagonal entries of  $D$ . If  $F$  is commutative, the Diudonné determinant coincides with the usual determinant since  $[F^\times, F^\times] = \{1\}$  and  $F_{\text{ab}}^\times = F$ . The Dieudonné determinant of a nonsingular triangular matrix is the product of its diagonal entries modulo  $[F^\times, F^\times]$  since it has a trivial Bruhat decomposition. The Dieudonné determinant of the  $0 \times 0$  matrix is defined to be the identity of  $F_{\text{ab}}^\times$ .

As the usual determinant, the Diudonné determinant enjoys the following identities.

**Proposition 2.3** ([14]). *Let  $F$  be a skew field. Then the following identities hold:*

- (1)  $\text{Det } AB = \text{Det } A \text{Det } B$  for nonsingular  $A, B \in F^{n \times n}$ .
- (2)  $\text{Det} \begin{pmatrix} A & O \\ O & B \end{pmatrix} = \text{Det } A \text{Det } B$  for nonsingular  $A \in F^{n \times n}$  and  $B \in F^{n' \times n'}$ , where blocks in  $O$  represent zero matrices of appropriate size.

Indeed, (2) is clear from the definition of the Diudonné determinant since a Bruhat decomposition of  $\begin{pmatrix} A & O \\ O & B \end{pmatrix}$  can be expressed by combining those of  $A$  and  $B$ . For (1) see also [12, Theorem 2.6 in Section 11.2].

## 2.2 Skew Polynomials

Let  $R$  be a ring,  $\sigma: R \rightarrow R$  an automorphism of  $R$  and  $\delta: R \rightarrow R$  a  $\sigma$ -derivation on  $R$ . We call a triple  $(R, \sigma, \delta)$  a  $\sigma$ -differential ring due to Bronstein [7]. A  $\sigma$ -differential field (skew field) is a  $\sigma$ -differential ring with  $R$  being a field (resp. skew field). A  $\sigma$ -differential ring (field, skew field) with  $\sigma = \text{id}$  is simply called a *differential ring* (resp. *field*, *skew field*). Similarly a  $\sigma$ -differential ring (field, skew field) with  $\delta = 0$  is called a *difference ring* (resp. *field*, *skew field*).

Recall that the skew polynomial ring  $R[s; \sigma, \delta]$  over a  $\sigma$ -differential ring  $(R, \sigma, \delta)$  is defined by the commutation rule (1). Here we give more examples of skew polynomial rings as follows; see also [10].

**Example 2.4.** Let  $F$  be a skew field.

- (1) The usual polynomial ring  $F[s]$  over  $F$  is trivially a skew polynomial ring  $F[s; \text{id}, 0]$ .
- (2) Let  $': F(t) \rightarrow F(t)$  be the usual differentiation defined by  $t' := 1$ . Then  $F(t)[\partial; \text{id}, ']$  is the ring of differential operators over  $F(t)$ . A skew polynomial ring  $F[t][\partial; \text{id}, ']$  is particularly called the *Weyl algebra* over  $F[t]$ . Similarly we can consider the differential operator rings over the field  $F((t))$  of formal Laurent series, the field  $\mathbb{C}(\{t\})$  of convergent Laurent series over  $\mathbb{C}$  and the field of meromorphic functions on an open connected subset of  $\mathbb{C} \cup \{\infty\}$ .
- (3) Let  $\tau: F(t) \rightarrow F(t)$  be an automorphism defined by  $\tau(f(t)) := f(t+1)$  for  $f(t) \in F(t)$ . Then  $F(t)[S; \tau, 0]$  is the ring of shift operators over  $F(t)$ . In addition, define a  $\tau$ -derivative  $\delta: F(t) \rightarrow F(t)$  by  $\delta(f(t)) := f(t+1) - f(t)$  for  $f(t) \in F(t)$ . Then  $F(t)[\Delta; \tau, \delta]$  is the ring of difference operators over  $F(t)$ .
- (4) For  $q \in F \setminus \{0, 1\}$ , define an automorphism  $\sigma_q: F(t) \rightarrow F(t)$  by  $\sigma_q(f(t)) := f(qt)$  and a  $\sigma_q$ -derivative  $\delta_q: F(t) \rightarrow F(t)$  by  $\delta_q(f(t)) := (f(qt) - f(t))/((q-1)t)$  for  $f(t) \in F(t)$ . Then  $F(t)[D_q; \sigma_q, \delta_q]$  is the ring of  $q$ -differential operators over  $F(t)$ .  $\square$

Let  $(F, \sigma, \delta)$  be a  $\sigma$ -differential skew field. Applying the commutation rule (1) repeatedly, any nonzero skew polynomial  $p \in F[s; \sigma, \delta]$  can be uniquely written as  $p = a_\ell + a_{\ell-1}s + \dots + a_0s^\ell$  for some  $\ell \in \mathbb{N}$  and  $a_0, \dots, a_\ell \in F$  with  $a_0 \neq 0$ . The *degree*  $\deg p$  of  $p$  is defined by  $\deg p := \ell$ . Define  $\deg 0 := -\infty$ . Then the minus of the degree enjoys the *valuation* property. Here, in general, a *valuation* of a ring  $R$  is a function  $v: R \rightarrow \mathbb{Z} \cup \{+\infty\}$  such that



(V1)  $v(a) = +\infty$  if and only if  $a = 0$ ,

(V2)  $v(a + b) \geq \min\{v(a), v(b)\}$ ,

(V3)  $v(ab) = v(a) + v(b)$

for all  $a, b \in R$ . In the present case of the degree function,  $\deg(p + q) \leq \max\{\deg p, \deg q\}$  and  $\deg pq = \deg p + \deg q$  for  $p, q \in F[s; \sigma, \delta]$  are important properties.

A skew polynomial ring  $F[s; \sigma, \delta]$  is a principal (right and left) ideal domain (PID), i.e., all the (right and left) ideals are generated by one element. A matrix over a PID is said to be *unimodular* if it is invertible over the PID. The *Smith normal form* is a well-known normal form for matrices over a commutative PID under transformations by unimodular matrices. Jacobson [27] generalized the Smith normal form to matrices over a noncommutative PID, called the *Jacobson normal form*. Recall from [27] that a regular (non-zero-divisor) element  $p \in R$  in a ring  $R$  is called a *total divisor* of a regular element  $q \in R$  if  $RqR \subseteq Rp \cap pR$  holds.

**Proposition 2.5** (Jacobson normal form [27, Theorem 16 in Chapter 3]; see [12, Theorem 2.1 in Section 9.1]). *Let  $R$  be a right and left PID and  $A \in R^{n \times n'}$  a matrix of rank  $r$ . There exist unimodular matrices  $U \in R^{n \times n}$ ,  $W \in R^{n' \times n'}$  and  $e_1, \dots, e_r \in R \setminus \{0\}$  such that  $e_i$  is a total divisor of  $e_{i+1}$  for  $i = 1, \dots, r - 1$  and*

$$UAW = \begin{pmatrix} \text{diag}(e_1, \dots, e_r) & O \\ O & O \end{pmatrix}. \quad (12)$$

## 2.3 Skew Functions

Let  $(F, \sigma, \delta)$  be a  $\sigma$ -differential skew field. It is known that the skew polynomial ring  $F[s; \sigma, \delta]$  is a (right and left) *Ore domain*, i.e., for each  $p, p' \in F[s; \sigma, \delta] \setminus \{0\}$  there exist  $q, q', r, r' \in F[s; \sigma, \delta] \setminus \{0\}$  such that  $pq = p'q'$  and  $rp = r'p'$ . This property enables  $F[s; \sigma, \delta]$  to have a (right and left) *Ore quotient ring*, which is a skew field of fractions whose every element  $f$  is expressed as  $f = pq^{-1} = q'^{-1}p'$  for some  $p, p' \in F[s; \sigma, \delta]$  and  $q, q' \in F[s; \sigma, \delta] \setminus \{0\}$ . This skew field is called the *skew (rational) function field* over  $(F, \sigma, \delta)$ , and is denoted by  $F(s; \sigma, \delta)$ . Elements in  $F(s; \sigma, \delta)$  are called *skew (rational) functions*. See [12, Section 9.1] and [19, Chapter 6] for the detail of Ore domains and Ore quotient rings. The degree on  $F[s; \sigma, \delta]$  is uniquely extended to a valuation (with sign reversed) on  $F(s; \sigma, \delta)$  by  $\deg f := \deg p - \deg q$  for  $f = pq^{-1} \in F(s; \sigma, \delta)$  with  $p \in F[s; \sigma, \delta]$  and  $q \in F[s; \sigma, \delta] \setminus \{0\}$ ; see [13, Proposition 9.1.1]. We call a skew function  $f \in F(s; \sigma, \delta)$  *proper* if  $\deg f \leq 0$ .

A *skew Laurent series field* over  $(F, \sigma, \delta)$  in  $s^{-1}$  is the set of formal power series over  $F$  in the form of

$$f = \sum_{d=-\ell}^{\infty} a_d s^{-d} \quad (13)$$

for some  $\ell \in \mathbb{Z}$  and  $a_{-\ell}, a_{-\ell+1}, \dots \in F$ . This skew field has the natural addition and a multiplication defined by (1) and

$$s^{-1}a = \sum_{d=0}^{\infty} (-1)^d ((\sigma \circ \delta)^d \circ \sigma)(a) s^{-(d+1)} \quad (14)$$

for  $a \in F$ , where  $\circ$  is the composition and  $(\sigma \circ \delta)^d$  denotes the  $d$ th iterate of  $\sigma \circ \delta$ . The multiplication rule (14) is determined so that  $ss^{-1}a = a$ . Then the skew function field  $F(s; \sigma, \delta)$  is embedded in the skew Laurent series field in  $s^{-1}$  [11, Proposition 7.1]. Namely, any skew function  $f \in F(s; \sigma, \delta)$  can be uniquely expanded in the form of (13).

**Proposition 2.6.** *Let  $f \in F(s; \sigma, \delta)$  be a nonzero skew function over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . Express  $f$  as  $f = \sum_{d=-\ell}^{\infty} a_d s^{-d}$  for some  $\ell \in \mathbb{Z}$  and  $a_d, a_{d+1}, \dots \in F$  with  $a_{-\ell} \neq 0$ . Then  $\deg f = \ell$  holds.*

*Proof.* Write  $f$  as  $f = pq^{-1}$  for some  $p, q \in F[s; \sigma, \delta] \setminus \{0\}$ . Let  $b$  be the coefficient of  $s^m$  in  $q$  with  $m := \deg q$ . Then the term of the highest degree in  $fq$  is  $a_\ell \sigma^\ell(b) s^{\ell+m}$ . This implies  $\deg p = \deg fq = \ell + m$  and hence  $\deg f = \ell$ .  $\square$

We next consider matrices over  $F(s; \sigma, \delta)$ , called *skew (rational) function matrices*. For a square skew function matrix  $A \in F(s; \sigma, \delta)^{n \times n}$ , define

$$\deg \text{Det } A := \begin{cases} \deg f & (A \text{ is nonsingular}), \\ -\infty & (A \text{ is singular}), \end{cases}$$

where  $f \in F(s; \sigma, \delta)^\times$  is any representative of  $\text{Det } A \in F(s; \sigma, \delta)_{\text{ab}}^\times$  for nonsingular  $A$ . This is well-defined since all commutators have degree zero. The  $\deg \text{Det}$  of a triangular matrix over  $F(s; \sigma, \delta)$  is equal to the sum of the degrees of its diagonal entries. Note that  $\deg \text{Det}$  of the  $0 \times 0$  matrix is 0.

We describe properties on the degree of the Dieudonné determinant, which are a part of axioms and properties of a *matrix valuation* (with min and max reversed) in the sense of [13, Section 9.3]. For this we shall define the *determinantal sum* of matrices following [13, Section 4.3]. Let  $A, B \in F(s; \sigma, \delta)^{n \times n}$  be matrices which are identical except for their first columns. The *determinantal sum* of  $A$  and  $B$  with respect to the first column is an  $n \times n$  matrix over  $F(s; \sigma, \delta)$  whose the first column is the sum of those of  $A$  and  $B$ , and other columns are the same as  $A$  and  $B$ . The determinantal sums with respect to other columns and rows are also defined. We denote the determinantal sum of  $A$  and  $B$  (with respect to an appropriate column or row) by  $A \nabla B$ .

**Proposition 2.7** ([13, Section 9.3]). *Let  $(F, \sigma, \delta)$  be a  $\sigma$ -differential skew field. Then the following hold:*

- (1)  $\deg \text{Det } AB = \deg \text{Det } A + \deg \text{Det } B$  for  $A, B \in F(s; \sigma, \delta)^{n \times n}$ .
- (2)  $\deg \text{Det} \begin{pmatrix} A & O \\ O & B \end{pmatrix} = \deg \text{Det } A + \deg \text{Det } B$  for  $A \in F(s; \sigma, \delta)^{n \times n}$  and  $B \in F(s; \sigma, \delta)^{n' \times n'}$ .
- (3)  $\deg \text{Det}(A \nabla B) \leq \max\{\deg \text{Det } A, \deg \text{Det } B\}$  for  $A, B \in F(s; \sigma, \delta)^{n \times n}$  such that  $A \nabla B$  is defined. The equality is attained if  $\deg \text{Det } A \neq \deg \text{Det } B$ .

Cohn [13] described a proof of Proposition 2.7 (3) only for the determinantal sum with respect to columns but the row version can be proved in the same way; see [23, Section A.3].

Using the  $\deg \text{Det}$  notion, Giesbrecht–Kim [18] proved that a skew polynomial matrix  $A \in F[s; \sigma, \delta]^{n \times n}$  is unimodular if and only if  $\deg \text{Det } A = 0$ . Here we give the proof along with the third equivalent condition which we use later. Define an *elementary matrix*  $E_n(i_1, i_2; f) \in R^{n \times n}$  over a ring  $R$  as a unitriangular matrix whose  $(i_1, i_2)$ th entry ( $i_1 \neq i_2$ ) is  $f \in R$  and other nondiagonals are zero.

**Proposition 2.8** (see [18, Theorem 4.6]). *Let  $A \in F[s; \sigma, \delta]^{n \times n}$  be a square skew polynomial matrix over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . Then the following are equivalent:*

- (1)  $A$  is unimodular.
- (2)  $\deg \text{Det } A = 0$ .
- (3)  $A$  is the product of permutation matrices, elementary matrices over  $F[s; \sigma, \delta]$  and diagonal matrices whose diagonal entries are in  $F^\times$ .

*Proof.* Suppose (3). Then (1) follows from the fact that matrices of three types in (3) are unimodular and the product of unimodular matrices are unimodular again. Also (2) holds by Proposition 2.7 (1) and the fact that  $\deg \text{Det}$  of these matrices are zero.

Conversely we show (3) from (1) or (2). Let  $D = UAW$  be the Jacobson normal form of  $A$ . The skew polynomial ring  $F[s; \sigma, \delta]$  is a (right and left) Euclidean domain, i.e., for  $a, b \in F[s; \sigma, \delta]$  with  $b \neq 0$  there exist  $q, q', r, r' \in F[s; \sigma, \delta]$  such that  $a = bq + r = q'b + r'$  and  $\deg r, \deg r' < \deg b$ . It is shown in the proof of [12, Theorem 2.1 in Section 9.1] that the Jacobson normal form of matrices over right and left Euclidean domain can be constructed by multiplying matrices satisfying (3). Thus we can assume that  $U$  and  $W$  satisfies (3). Suppose (1). Since  $U, A$  and  $W$  are unimodular,  $D$  is also unimodular. Then  $D$  must be a diagonal matrix whose diagonal entries are in  $F^\times$ . Now we have  $A = U^{-1}DW^{-1}$ , which satisfies (3). Next suppose (2). Since  $\deg \text{Det}$  of matrices satisfying (3) is zero, we have  $\deg \text{Det } D = \deg \text{Det } A = 0$  by Proposition 2.7 (1). Thus  $J$  must be a diagonal matrix whose diagonal entries are in  $F^\times$ , which implies (3).  $\square$

Recall the notation  $d_k(A)$  in (2) for a skew polynomial matrix  $A \in F[s; \sigma, \delta]^{n \times n'}$ . We naturally extend  $d_k(A)$  to a skew function matrix  $A \in F(s; \sigma, \delta)^{n \times n'}$ . Note that  $d_1(A)$  is the maximum degree of an entry in  $A$  and we call  $d_1(A)$  the *degree* of  $A$ . Similarly to (13), a skew function matrix  $A$  can be uniquely expanded as

$$A = \sum_{d=-\ell}^{\infty} A_d s^{-d} \quad (15)$$

for some  $\ell \in \mathbb{Z}$  and  $A_{-\ell}, A_{-\ell+1}, \dots \in F(s; \sigma, \delta)^{n \times n'}$ . By Proposition 2.6,  $A_{-\ell} \neq O$  implies  $d_1(A) = \ell$ .

## 2.4 Smith–McMillan Form

Let  $(F, \sigma, \delta)$  be a  $\sigma$ -differential skew field. A skew function matrix is said to be *proper* if its degree is nonpositive. A square skew function matrix is said to be *biproper* if it is proper, nonsingular and its inverse is also proper. We abbreviate proper and biproper skew function matrices as proper and biproper matrices, respectively. It is easy to see from the valuation properties that the product of proper matrices are proper. From this, the product of biproper matrices are biproper again. A *biproper transformation* is a transformation of a skew function matrix  $A \in F(s; \sigma, \delta)^{n \times n'}$  in the form  $A \mapsto SAT$ , where  $S \in F(s; \sigma, \delta)^{n \times n}$  and  $T \in F(s; \sigma, \delta)^{n' \times n'}$  are biproper matrices.

Under biproper transformations, we can establish a canonical form of function matrices, called the *Smith–McMillan form*. This is well-known for matrices over  $\mathbb{C}(s)$  as the *Smith–McMillan form at infinity* [40, 52] in the context of control theory. Bourlés–Marinescu [6, Definition 2] and Hirai [23, Proposition 2.9] independently extended the Smith–McMillan form to skew function matrices over a differential field  $(F, \text{id}, \delta)$  and to rational function matrices over a skew field  $F$ , respectively. Here we formulate the Smith–McMillan form for general skew function matrices.

**Proposition 2.9** (Smith–McMillan form). *Let  $A \in F(s; \sigma, \delta)^{n \times n'}$  be a skew function matrix of rank  $r$  over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . There exist biproper matrices  $S \in F(s; \sigma, \delta)^{n \times n}$ ,  $T \in F(s; \sigma, \delta)^{n' \times n'}$  and integers  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_r$  such that*

$$SAT = \begin{pmatrix} \text{diag}(s^{\alpha_1}, \dots, s^{\alpha_r}) & O \\ O & O \end{pmatrix}. \quad (16)$$

*The integer  $\alpha_i$  is uniquely determined by*

$$\alpha_i = d_i(A) - d_{i-1}(A) \quad (17)$$

*for  $i \in [r]$ . In particular,  $d_k(A)$  is invariant under biproper transformations for  $k \in [0, r]$ .*

*Proof.* We show by induction on  $k$  that

(\*) for  $k \in [0, r]$ , there exist biproper matrices  $S \in F(s; \sigma, \delta)^{n \times n}$ ,  $T \in F(s; \sigma, \delta)^{n' \times n'}$  and integers  $\alpha_1 \geq \dots \geq \alpha_k$  such that

$$SAT = \begin{pmatrix} \text{diag}(s^{\alpha_1}, \dots, s^{\alpha_k}) & O \\ O & B \end{pmatrix}, \quad (18)$$

where  $B \in F(s; \sigma, \delta)^{(n-k) \times (n'-k)}$  is a matrix with degree at most  $\alpha_i$ .

Note that  $B$  is a zero matrix if and only if  $k = r$  due to  $\text{rank } B = \text{rank } SAT - k = r - k$ . Then the statement (\*) for  $k = r$  immediately implies the former part of the proposition.

If  $k = 0$ , (\*) trivially holds. Suppose (\*) for some  $k \in [0, r-1]$ . We perform biproper transformations on  $SAT$  in (18) as follows. Let  $\alpha_{k+1}$  be the degree of  $B$ . It holds  $\alpha_k \geq \alpha_{k+1}$  by the inductive assumption. Fixing the top left  $k \times k$  submatrix of  $SAT$ , multiply permutation matrices  $P$  and  $Q$  to the left and right of  $SAT$  so that the  $(k+1)$ st diagonal entry of  $\tilde{A} := PSATQ$  has degree  $\alpha_{k+1}$ . Permutation matrices are clearly biproper. Now  $\tilde{A}$  is in the form

$$\tilde{A} = \begin{pmatrix} \text{diag}(s^{\alpha_1}, \dots, s^{\alpha_k}) & 0 & O \\ 0 & b & v \\ O & u & * \end{pmatrix},$$

where  $b \in F(s; \sigma, \delta)$ ,  $u \in F(s; \sigma, \delta)^{(n-k-1) \times 1}$ ,  $v \in F(s; \sigma, \delta)^{1 \times (n'-k-1)}$ , the block in “\*” indicates some matrix and blocks in 0 indicate row or column zero vectors of appropriate dimension. Then we can eliminate  $u$  and  $v$  by multiplying

$$U := \begin{pmatrix} I_k & 0 & O \\ 0 & 1 & 0 \\ O & -ub^{-1} & I_{n-k-1} \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} I_k & 0 & O \\ 0 & 1 & -b^{-1}v \\ O & 0 & I_{n'-k-1} \end{pmatrix}$$

from the left and right of  $\tilde{A}$ , respectively. These matrices  $U$  and  $V$  can be represented as products of elementary matrices over  $F(s; \sigma, \delta)$ . Note that an elementary matrix  $E_n(i_1, i_2; f)$  with distinct  $i_1, i_2 \in [n]$  and  $f \in F(s; \sigma, \delta)$  is biproper if and only if  $f$  is proper since  $E_n(i_1, i_2; f)^{-1} = E_n(i_1, i_2; -f)$ . For this,  $U$  and  $V$  are biproper due to the maximality of the degree of  $b$ . In addition, the degree of the bottom right  $(n-k-1) \times (n'-k-1)$  submatrix  $\tilde{B}$  of  $U\tilde{A}V$  is at most  $\alpha_{i+1}$ . Write  $b$  as  $b = cs^{\alpha_{i+1}}$  with  $\deg c = 0$ . Let  $C \in F(s; \sigma, \delta)^{n \times n}$  be a biproper diagonal matrix having  $c^{-1}$  for the  $(k+1)$ st diagonal entry and 1 for other diagonals. Then we have

$$CU\tilde{A}V = CUPSATQV = \begin{pmatrix} \text{diag}(s^{\alpha_1}, \dots, s^{\alpha_k}) & 0 & O \\ 0 & s^{\alpha_{k+1}} & 0 \\ O & 0 & \tilde{B} \end{pmatrix},$$

which implies (\*) for  $i+1$ .

Next we show (17). Let  $D$  be the diagonal matrix (16) obtained from  $A$  by the above construction. From the order of  $\alpha_i$ , we have  $\alpha_i = d_i(D) - d_{i-1}(D)$  for all  $i \in [r]$ . Therefore it suffices to show that  $d_k$  is invariant throughout the above procedure for  $k = 0, \dots, r$ . It is easy to see that  $d_k$  does not change by multiplying the permutation matrices  $P$ ,  $Q$  and the diagonal matrix  $C$  with diagonal entries of degree zero from Proposition 2.7 (1). We next consider how  $d_k$  changes by multiplying an elementary matrix  $E_n(i_1, i_2; f)$  with distinct  $i_1, i_2 \in [n]$  and proper  $f \in F(s; \sigma, \delta)$  to a matrix  $A \in F(s; \sigma, \delta)^{n \times n'}$  from the left. This corresponds to the operation of adding the  $i_1$ th row multiplied by  $q$  to the  $i_2$ th row. Put  $A' := E_n(i_1, i_2; f)A$  and consider a submatrix with row index set  $I \subseteq [n]$  and column index set  $J \subseteq [n']$  of cardinality  $k$ . If  $i_2 \notin I$ , then  $A'[I, J] = A[I, J]$ . If  $i_1, i_2 \in I$ , then  $A'[I, J] = EA[I, J]$  for some elementary matrix  $E$

of size  $k$ , which implies  $\deg \text{Det } A'[I, J] = \deg \text{Det } A[I, J]$  as  $\deg \text{Det } E = 0$ . Consider the case where  $i_1 \notin I \ni i_2$ . In this case, we have

$$A'[I, J] = A[I, J] \nabla (FA[I', J]),$$

where  $I' := (I \cup \{i_1\}) \setminus \{i_2\}$  and  $F$  is a diagonal matrix having  $a$  for the diagonal entry corresponding to the  $i_1$ th row and 1 for other diagonals. From Proposition 2.7 (3), it holds

$$\begin{aligned} \deg \text{Det } A'[I, J] &\leq \max\{\deg \text{Det } A[I, J], \deg \text{Det } FA[I', J]\} \\ &= \max\{\deg \text{Det } A[I, J], \deg \text{Det } A[I', J] + \deg a\}. \end{aligned} \quad (19)$$

Since  $a$  is proper,  $\deg \text{Det } A'[I, J]$  is at most  $d_k(A)$ . Suppose  $d_k(A) = \deg \text{Det } A[I, J]$ . If  $d_k(A) > \deg \text{Det } A[I', J] + \deg a$ , the equality of (19) is attained. If  $d_k(A) = \deg \text{Det } A[I', J] + \deg a$ , then  $d_k(A) = \deg \text{Det } A[I', J]$  due to  $\deg a \leq 0$  and  $\deg \text{Det } A[I', J] \leq d_k(A)$ , and we have  $\deg \text{Det } A[I', J] = \deg \text{Det } A'[I', J]$  from  $i_2 \notin I'$ . Hence we have  $d_k(A') = d_k(A)$  in all cases. The proof for the right multiplication of elementary matrices is the same.  $\square$

Solving (17) for  $d_k(A)$ , we obtain

$$d_k(A) = \sum_{i=1}^k \alpha_i \quad (20)$$

for  $k \in [0, r]$ . This is a key identity that connects  $d_k(A)$  and the Smith–McMillan form of  $A$ . It is worth mentioning that all  $\alpha_i$  are nonpositive for a proper matrix  $A$  since  $\alpha_1$  is equal to the degree  $d_1(A)$  of  $A$  by (17).

We next give an upper bound on  $d_k$  of a skew function matrix using the Smith–McMillan form and a lower bound on  $d_k$  of a skew polynomial matrix using the Jacobson normal form.

**Proposition 2.10.** *Let  $A \in F(s; \sigma, \delta)^{n \times n'}$  be a skew function matrix over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . For  $k \in [0, n^*]$  with  $n^* := \min\{n, n'\}$ , the following hold:*

- (1)  $d_k(As^\ell) = d_k(A) + \ell k$  for  $\ell \in \mathbb{Z}$ .
- (2)  $d_k(A) \leq \ell k$ , where  $\ell$  is the degree of  $A$ .
- (3)  $d_k(A) > -\infty$  if and only if  $k \leq \text{rank } A$ . In addition, if  $A$  is a polynomial matrix, then  $d_k(A) \geq 0$  for  $k \leq \text{rank } A$ .

*Proof.* (1) follows from the fact that for any  $k \times k$  submatrix  $A[I, J]$  of  $A$ , it holds

$$\begin{aligned} \deg \text{Det } A[I, J]s^\ell &= \deg \text{Det}(A[I, J] \cdot s^\ell I_k) \\ &= \deg \text{Det } A[I, J] + \deg \det s^\ell I_k \\ &= \deg \text{Det } A[I, J] + \ell k. \end{aligned}$$

(2) Let  $\alpha_1, \dots, \alpha_k$  be the exponents of the Smith–McMillan form of a nonsingular  $k \times k$  submatrix  $A[I, J]$  of  $A$ . Then the claim follows from  $\deg \text{Det } A[I, J] = \alpha_1 + \dots + \alpha_k$  and  $\ell \geq \alpha_1 \geq \dots \geq \alpha_k$ .

(3) The former part follows from Proposition 2.1. Suppose that  $A$  is a skew polynomial matrix and consider its  $k \times k$  nonsingular submatrix  $A[I, J]$ . Let  $J := UA[I, J]V$  be the Jacobson normal form of  $A[I, J]$ , where  $U, V \in F[s; \sigma, \delta]^{k \times k}$  are unimodular matrices. Since  $\deg \text{Det}$  of  $U$  and  $V$  are zero by Proposition 2.8, it holds  $\deg \text{Det } A[I, J] = \deg \text{Det } J \geq 0$ .  $\square$

Equivalent conditions for proper matrices to be biproper are established as follows. Whereas the first three conditions are similar to those of Proposition 2.8 for unimodular matrices, the last new one, which is based on the expansion (15) of proper matrices, is crucial for our algorithm.

**Lemma 2.11.** *Let  $A \in F(s; \sigma, \delta)^{n \times n}$  be a square proper matrix over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . Then the following are equivalent:*

- (1)  *$A$  is biproper.*
- (2)  *$\deg \text{Det } A = 0$ .*
- (3)  *$A$  is the product of permutation matrices, proper elementary matrices over  $F(s; \sigma, \delta)$  and diagonal matrices whose diagonal entries are of degree zero.*
- (4) *The coefficient matrix  $A_0$  of  $s^0$  in the expansion (15) of  $A$  is nonsingular.*

*Proof.* (1)  $\Rightarrow$  (4). Put  $B := A^{-1}$  and expand  $A$  and  $B$  as  $A = A_0 + \tilde{A}s^{-1}$  and  $B = B_0 + \tilde{B}s^{-1}$ . Then it holds

$$I_n = AB = (A_0 + \tilde{A}s^{-1})(B_0 + \tilde{B}s^{-1}) = A_0B_0 + A_0\tilde{B}s^{-1} + \tilde{A}s^{-1}B_0 + \tilde{A}s^{-1}\tilde{B}s^{-1}. \quad (21)$$

Here the degree of matrices in the right hand side of (21) other than the first term  $A_0B_0$  is at most  $-1$ . Hence it must hold  $I_n = A_0B_0$ , which implies that  $A_0$  is nonsingular.

(4)  $\Rightarrow$  (2). Let  $D = SAT$  be the Smith–McMillan form of  $A$  with  $D = \text{diag}(s^{\alpha_1}, \dots, s^{\alpha_n})$  and  $D_0, S_0, A_0, T_0$  the coefficient matrices of  $s^0$  in the expansions of  $D, S, A, T$ , respectively. By the same argument as (21), it holds  $D_0 = S_0A_0T_0$ . If  $\deg \text{Det } A < 0$ , i.e.,  $\alpha_k < 0$  for some  $k \in [n]$ , then  $A_0$  must be singular.

(2)  $\Rightarrow$  (3). Let  $D = SAT$  be the Smith–McMillan form of  $A$  with  $D = \text{diag}(s^{\alpha_1}, \dots, s^{\alpha_n})$ . We can see from the proof of Proposition 2.9 that  $S$  and  $T$  can be taken as the product of matrices of three types in (3). Since  $\deg \text{Det}$  of matrices satisfying (3) is zero,  $\deg \text{Det } D = \deg \text{Det } A = 0$ . By  $0 = \deg \text{Det } D = \alpha_1 + \dots + \alpha_n$  and  $0 \geq \alpha_1 \geq \dots \geq \alpha_n$ , all  $\alpha_k$  must be zero and thus  $D = I_n$ . Hence  $A = S^{-1}T^{-1}$ , which satisfies (3).

(3)  $\Rightarrow$  (1) follows from the fact that matrices of three types in (3) are biproper.  $\square$

## 2.5 Order of Skew Functions over Difference Skew Fields

Let  $p = a_0 + a_1s + \dots + a_\ell s^\ell \in F[s; \sigma, \delta]$  be a skew polynomial over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . The *order* of  $p$ , which is denoted by  $\text{ord } p$ , is the maximum  $d \in [0, \ell]$  such that  $a_d \neq 0$ . The order of 0 is set to be  $+\infty$ . Over the polynomial ring  $F[s]$ , it is well-known that the order is the dual concept of the degree in the sense that it serves as a valuation of  $F[s]$ . This is, however, not the case for general skew polynomial rings. An easy counterexample is the following: if we take  $a \in F$  with  $\delta(a) \neq 0$ , then  $\text{ord } sa = \text{ord}(\sigma(a)s + \delta(a)) = 0$  but  $\text{ord } s + \text{ord } a = 1$ , which violates (V3). Nevertheless, we can easily confirm that the order is a valuation of  $F[s; \sigma, \delta]$  if and only if  $\delta = 0$ . In this case, as mentioned in Section 2.3, the order on  $F[s; \sigma, 0]$  can be uniquely extended to a valuation of  $F(s; \sigma, 0)$  by setting  $\text{ord } f := \text{ord } p - \text{ord } q$  for  $f = pq^{-1} \in F(s; \sigma, 0)$  with  $p \in F[s; \sigma, 0]$  and  $q \in F[s; \sigma, 0] \setminus \{0\}$  [13, Proposition 9.1.1]. In the rest of this section, we investigate the relation between the degree and the order on a skew function field  $F(s; \sigma, 0)$  over a difference skew field  $(F, \sigma, 0)$ , which will be used in analysis of linear difference equations in Section 4.

For a skew polynomial ring  $F[s; \sigma, 0]$  over a difference skew field  $(F, \sigma, 0)$ , consider a skew function field  $F(t; \sigma^{-1}, 0)$  in indeterminate  $t$  over another difference skew field  $(F, \sigma^{-1}, 0)$ . Define a map  $\varphi: F[s; \sigma, 0] \rightarrow F(t; \sigma^{-1}, 0)$  by

$$\varphi(a_0 + a_1s + \dots + a_\ell s^\ell) := a_0 + a_1t^{-1} + \dots + a_\ell t^{-\ell} \quad (22)$$

for  $a_0, \dots, a_\ell \in F$ . Clearly it holds  $\text{ord } p = -\deg \varphi(p)$ .

**Lemma 2.12.** *Let  $(F, \sigma, 0)$  be a difference skew field. The map  $\varphi: F[s; \sigma, 0] \rightarrow F(t; \sigma^{-1}, 0)$  defined by (22) is an injective ring homomorphism.*

*Proof.* The conditions  $\varphi(1) = 1$  and  $\varphi(p \pm q) = \varphi(p) \pm \varphi(q)$  for  $p, q \in F[s; \sigma, 0]$  are clear. To show  $\varphi(pq) = \varphi(p)\varphi(q)$ , it suffices to check  $\varphi(as^i bs^j) = \varphi(as^i)\varphi(bs^j)$  for  $a, b \in F$  and  $i, j \in \mathbb{N}$ . This follows from

$$\varphi(as^i bs^j) = \varphi(a\sigma^i(b)s^{i+j}) = a\sigma^i(b)t^{-(i+j)} = at^{-i}bt^{-j} = \varphi(as^i)\varphi(bs^j)$$

as required. The injectivity of  $\varphi$  is clear.  $\square$

Since  $\varphi$  is an injective homomorphism to a skew field by Lemma 2.12, it uniquely extends to a ring homomorphism from  $F(s; \sigma, 0)$  to  $F(t; \sigma^{-1}, 0)$  [19, Proposition 6.3]. For a skew function  $f = pq^{-1} \in F(s; \sigma, 0)$  with  $p \in F[s; \sigma, 0]$  and  $q \in F[s; \sigma, 0] \setminus \{0\}$ , it holds

$$\text{ord } f = \text{ord } p - \text{ord } q = -\deg \varphi(p) + \deg \varphi(q) = -\deg \varphi(p)\varphi(q)^{-1} = -\deg \varphi(f). \quad (23)$$

Let  $A \in F(s; \sigma, 0)^{n \times n}$  be a square skew function matrix over  $(F, \sigma, 0)$ . Since all the commutators of  $F(s; \sigma, 0)^\times$  have order zero, we can define  $\text{ord Det } A$  by

$$\text{ord Det } A := \begin{cases} \text{ord } f & (A \text{ is nonsingular}), \\ +\infty & (A \text{ is singular}), \end{cases}$$

where  $f \in F(s; \sigma, \delta)^\times$  is any representative of  $\text{Det } A \in F(s; \sigma, \delta)_{\text{ab}}^\times$  for nonsingular  $A$ . Similar to  $\deg \text{Det}$ , the  $\text{ord Det}$  of a triangular matrix over  $F(s; \sigma, \delta)$  is equal to the sum of the orders of its diagonal entries. We extend  $\varphi$  to a skew function matrix  $A = (A_{i,j})_{i,j} \in F[s; \sigma, 0]^{n \times n'}$  by  $\varphi(A) := (\varphi(A_{i,j}))_{i,j} \in F[t; \sigma^{-1}, 0]^{n \times n'}$ .

**Theorem 2.13.** *For a square skew function matrix  $A \in F(s; \sigma, 0)^{n \times n}$  over a difference skew field  $(F, \sigma, 0)$ , it holds  $\text{ord Det } A = -\deg \text{Det } \varphi(A)$ .*

*Proof.* Let  $A = LDPU$  be the Burhat normal form of  $A$ , where  $L$  and  $U$  are unitriangular,  $D$  is diagonal and  $P$  is a permutation matrix. Since  $\varphi$  is a homomorphism, it holds

$$\varphi(A) = \varphi(L)\varphi(D)\varphi(P)\varphi(U). \quad (24)$$

Here  $\varphi(L)$  and  $\varphi(U)$  are unitriangular,  $\varphi(D)$  is diagonal and  $\varphi(P) = P$  is a permutation matrix again. Hence (24) is a Burhat decomposition of  $\varphi(A)$ . If  $A$  is singular, then  $D$ ,  $\varphi(D)$  and  $\varphi(A)$  must be singular. Thus  $\text{ord Det } A = +\infty$  coincides with the minus of  $\deg \text{Det } \varphi(A) = -\infty$ . Suppose that  $A$  is nonsingular and let  $f_1, \dots, f_n \in F(s; \sigma, 0)$  denote the diagonal entries of  $D$ . By using (23), we obtain

$$\text{ord Det } A = \sum_{i=1}^n \text{ord } f_i = -\sum_{i=1}^n \deg \varphi(f_i) = -\deg \text{Det } \varphi(A)$$

as required.  $\square$

Through Theorem 2.13, a bunch of properties and algorithms for  $\deg \text{Det}$  can be brought into  $\text{ord Det}$ . The order version of  $d_k(A)$ , which is denoted by  $\zeta_k(A)$  in [25, 26], can also be obtained in a natural way.

### 3 Computing the Maximum Degree of Minors

In this section, we describe algorithms to compute  $d_k$  and the rank of a skew polynomial matrix  $A = \sum_{d=0}^{\ell} A_{\ell-d} s^d \in F[s; \sigma, \delta]^{n \times n'}$  over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . Instead of  $A$ , we deal with a proper matrix obtained from  $A$  by

$$As^{-\ell} = \sum_{d=0}^{\ell} A_d s^{-d} \in F(s; \sigma, \delta)^{n \times n'}. \quad (25)$$

The value of  $d_k(A)$  can be recovered from that of (25) through Proposition 2.10 (1).

Section 3.1 introduces *matrix expansion* which is our key tool. Section 3.2 connects the sequence of  $d_k$  to the rank of expanded matrices via the *Legendre conjugacy*. Making use of them, we give algorithms in Section 3.3.



### 3.1 Matrix Expansion

For a proper matrix  $A \in F(s; \sigma, \delta)^{n \times n'}$  and  $i, d \in \mathbb{N}$ , let  $A_d^{(i)} \in F^{n \times n'}$  be the coefficient matrix of  $s^d$  in the expansion (15) of  $s^{-i}A$ . Namely, for  $\mu \in \mathbb{N}$ , the matrix  $s^{-i}A$  is written as

$$s^{-i}A = \sum_{d=0}^{\infty} A_d^{(i)} s^{-d}.$$

Note that  $A_d^{(i)} = O$  for  $d < i$  as the degree of  $s^{-i}A$  is at most  $-i$ . For  $\mu \in \mathbb{N}$ , we define the  $\mu$ th-order expanded matrix  $\Omega_\mu(A)$  of  $A$  as the following  $\mu n \times \mu n'$  block matrix

$$\Omega_\mu(A) := \begin{pmatrix} A_0^{(0)} & A_1^{(0)} & A_2^{(0)} & \cdots & \cdots & A_{\mu-1}^{(0)} \\ O & A_1^{(1)} & A_2^{(1)} & A_3^{(1)} & & \vdots \\ \vdots & O & A_2^{(2)} & A_3^{(2)} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & A_{\mu-1}^{(\mu-3)} \\ \vdots & & & \ddots & A_{\mu-2}^{(\mu-2)} & A_{\mu-1}^{(\mu-2)} \\ O & \cdots & \cdots & \cdots & O & A_{\mu-1}^{(\mu-1)} \end{pmatrix} \in F^{\mu n \times \mu n'}.$$

Then expanded matrices satisfy the identity (9), which was originally given in [51] for rational function matrices over  $\mathbb{C}$ .

**Lemma 3.1.** *Let  $A \in F(s; \sigma, \delta)^{n \times n'}$  and  $B \in F(s; \sigma, \delta)^{n' \times n''}$  be proper matrices over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . Then it holds (9) for  $\mu \in \mathbb{N}$ .*

*Proof.* Fix  $i = 0, \dots, \mu - 1$  and let  $s^{-i}A = \sum_{d=0}^{\infty} A_d^{(i)} s^{-d}$  be the expansion of  $s^{-i}A$ . Similarly, for  $d = 0, \dots, \mu - 1$ , let  $s^{-d}B = \sum_{j=0}^{\infty} B_j^{(d)} s^{-j}$  be the expansion of  $s^{-d}B$ . Then it holds

$$s^{-i}AB = \left( \sum_{d=0}^{\infty} A_d^{(i)} s^{-d} \right) B = \sum_{d=0}^{\infty} A_d^{(i)} \left( \sum_{j=0}^{\infty} B_j^{(d)} s^{-j} \right) = \sum_{j=0}^{\infty} \left( \sum_{d=0}^j A_d^{(i)} B_j^{(d)} \right) s^{-j}, \quad (26)$$

where the inner sum of the last term stops at  $d = j$  by  $B_j^{(d)} = O$  for  $j < d$ . The equality (26) implies that the coefficient matrix of  $s^{-j}$  in the expansion of  $s^{-i}AB$  is

$$\sum_{d=0}^j A_d^{(i)} B_j^{(d)} = \sum_{d=0}^{\mu-1} A_d^{(i)} B_j^{(d)}$$

for  $j < \mu$ , which is equal to the  $(i+1, j+1)$ st entry of  $\Omega_\mu(A)\Omega_\mu(B)$ .  $\square$

Let  $\omega_\mu(A)$  denote the rank of  $\Omega_\mu(A)$ . The following lemma claims that  $\omega_\mu(A)$  coincides with that of the Smith–McMillan form of  $A$ .

**Lemma 3.2.** *Let  $A \in F(s; \sigma, \delta)^{n \times n'}$  be a proper matrix over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . Then it holds  $\omega_\mu(A) = \omega_\mu(D)$  for  $\mu \in \mathbb{N}$ , where  $D$  is the Smith–McMillan form of  $A$ .*

*Proof.* Let  $S \in F(s; \sigma, \delta)^{n \times n}$  and  $T \in F(s; \sigma, \delta)^{n' \times n'}$  be biproper matrices such that  $SAT = D$ . From Lemma 3.1, we have

$$\omega_\mu(D) = \text{rank } \Omega_\mu(SAT) = \text{rank } \Omega_\mu(S)\Omega_\mu(A)\Omega_\mu(T).$$

For  $i \in \mathbb{N}$ , let  $S_i^{(i)}$  be the coefficient matrix of  $s^{-i}$  in the expansion of  $s^{-i}S$ . Then  $S_i^{(i)}$  is equal to the coefficient matrix of  $s^0$  in the expansion of  $s^{-i}Ss^i$ . Now  $s^{-i}Ss^i$  is biproper by  $(s^{-i}Ss^i)^{-1} = s^{-i}S^{-1}s^i$ . Thus  $S_i^{(i)}$  is nonsingular from Lemma 2.11. Since  $\Omega_\mu(S)$  is a block triangular matrix having  $S_i^{(i)}$  for the  $(i+1)$ st diagonal block, it is nonsingular. Similarly  $\Omega_\mu(T)$  is nonsingular. Therefore we have  $\omega_\mu(D) = \omega_\mu(A)$ .  $\square$

Let  $0 \geq \alpha_1 \geq \dots \geq \alpha_r$  be the exponent of the Smith–McMillan form of a proper matrix  $A \in F(s; \sigma, \delta)^{n \times n'}$  with  $\text{rank } A = r$ . Put

$$N_d := |\{i \in [r] \mid -\alpha_i \leq d\}| \quad (27)$$

for  $d \in \mathbb{N}$ . Lemma 3.2 leads us to the following lemma; a similar result based on the Kronecker canonical form is also known for matrix pencils over a field [25, Theorem 2.3].

**Lemma 3.3.** *Let  $A \in F(s; \sigma, \delta)^{n \times n'}$  be a proper matrix over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . For  $\mu \in \mathbb{N}$ , it holds*

$$\omega_\mu(A) = \sum_{d=0}^{\mu-1} N_d, \quad (28)$$

where  $N_d$  is defined in (27).

*Proof.* Let  $D$  be the Smith–McMillan form of  $A$  and  $D_d^{(i)} \in F^{n \times n'}$  the coefficient matrix of  $s^d$  in the expansion of  $s^{-i}D$  for  $i, d \in \mathbb{N}$ . Since entries of  $D$  are zero or monomials in  $s^{-1}$  with coefficient 1,  $D$  commutes  $s^{-i}$ . This implies  $D_d^{(i)} = D_{d-i}^{(0)} =: D_{d-i}$  for  $d \geq i$ . Now  $\Omega_\mu(D)$  is in the form

$$\Omega_\mu(D) = \begin{pmatrix} D_0 & D_1 & D_2 & \cdots & \cdots & D_{\mu-1} \\ O & D_0 & D_1 & D_2 & & \vdots \\ \vdots & O & D_0 & D_1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & D_2 \\ \vdots & & & \ddots & D_0 & D_1 \\ O & \cdots & \cdots & \cdots & O & D_0 \end{pmatrix}. \quad (29)$$

Let  $\alpha_1, \dots, \alpha_r$  be the exponents of the Smith–McMillan form  $D$ , where  $r := \text{rank } A$ . The  $i$ th diagonal entry of  $D_d$  is 1 if  $i \leq r$  and  $\alpha_i = -d$ , and 0 otherwise. Thus from (29), each row and column in  $\Omega_\mu(D)$  has at most one nonzero entry. Hence  $\omega_\mu(D)$ , which is equal to  $\omega_\mu(A)$  by Lemma 3.2, is equal to the number of nonzero entries in  $\Omega_\mu(D)$ . It is easily checked that the  $(\mu - d)$ th block row of  $\Omega_\mu(D)$  contains  $N_d$  nonzero entries for  $d = 0, \dots, \mu - 1$ .  $\square$

The equality (28) is a key identity that connects  $\omega_\mu(A)$  and the Smith–McMillan form of  $A$ . We remark that (28) can be rewritten as

$$N_d = \omega_{d+1}(A) - \omega_d(A) \quad (30)$$

for  $d \in \mathbb{N}$ .

### 3.2 Legendre Conjugacy of $d_k(A)$ and $\omega_\mu(A)$

Let  $A \in F(s; \sigma, \delta)^{n \times n'}$  be a proper matrix of rank  $r$  and  $\alpha_1 \geq \dots \geq \alpha_r$  the exponents of the Smith–McMillan form of  $A$ . Put  $d_k := d_k(A)$  for  $k = 0, \dots, r$ . From  $\alpha_k \geq \alpha_{k+1}$  and (17), an inequality  $d_{k-1} + d_{k+1} \leq 2d_k$  holds for all  $k \in [r - 1]$ . In addition, for  $\mu \in \mathbb{N}$  put  $\omega_\mu := \omega_\mu(A)$  and define  $N_\mu$  by (27). From  $N_{\mu-1} \leq N_\mu$  and (30), we have  $\omega_{\mu-1} + \omega_{\mu+1} \geq 2\omega_\mu$  for all  $\mu \geq 1$ . These two inequalities for  $d_k$  and  $\omega_\mu$  indicate the *concavity* of  $d_k$  and the *convexity* of  $\omega_\mu$  in the following sense.

A (discrete) function  $f: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{+\infty\}$  is said to be *convex* if

$$f(x-1) + f(x+1) \geq 2f(x)$$

for all  $x \in \mathbb{Z}$ . We call a function  $g: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty\}$  *concave* if  $-g$  is convex. An integer sequence  $(a_k)_{k \in K}$  indexed by  $K \subseteq \mathbb{Z}$  can be identified with a function  $\check{a}: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{+\infty\}$  by letting  $\check{a}(k)$

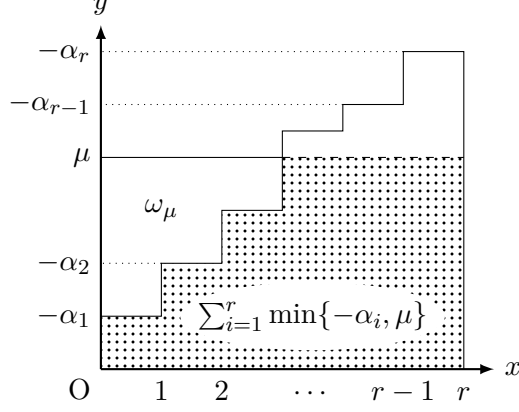


Figure 1: Graphic explanation of (32).

be  $a_k$  if  $k \in K$  and  $+\infty$  otherwise. We can also identify  $a$  with  $\hat{a}: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty\}$  defined by  $\hat{a}(k) := a_k$  if  $k \in K$  and  $\hat{a}(k) := -\infty$  otherwise. In this way we identify  $(d_0, d_1, \dots, d_r)$  and  $(\omega_0, \omega_1, \omega_2, \dots)$  with discrete functions  $\check{d}: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty\}$  and  $\hat{\omega}: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{+\infty\}$ , respectively. From the argument in the previous paragraph,  $(d_0, d_1, \dots, d_r)$  is concave and  $(\omega_0, \omega_1, \omega_2, \dots)$  is convex.

Let  $f: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{+\infty\}$  be a function such that  $f(x) \in \mathbb{Z}$  for some  $x \in \mathbb{Z}$ . The *concave conjugate* of  $f$  is a function  $f^\circ: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty\}$  defined by

$$f^\circ(y) := \inf_{x \in \mathbb{Z}} (f(x) - xy)$$

for  $y \in \mathbb{Z}$ . Similarly for a function  $g: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty\}$  with  $g(y) \in \mathbb{Z}$  for some  $y \in \mathbb{Z}$ , the *convex conjugate* of  $g$  is a function  $g^\bullet: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{+\infty\}$  given by

$$g^\bullet(x) := \sup_{y \in \mathbb{Z}} (g(y) + xy)$$

for  $x \in \mathbb{Z}$ . The maps  $f \mapsto f^\circ$  and  $g \mapsto g^\bullet$  are referred to as the *concave* and *convex discrete Legendre transform*, respectively. In general  $f^\circ$  is concave and  $g^\bullet$  is convex. If  $f$  is convex and  $g$  is concave,

$$(f^\circ)^\bullet = f, \quad (g^\bullet)^\circ = g \quad (31)$$

hold. Hence the Legendre transformation establishes a one-to-one correspondence between discrete convex and concave functions. See [41] for details of discrete convex/concave functions and their Legendre transform.

Indeed, as explained in Section 1, the sequences of  $d_k$  and  $\omega_\mu$  are in the relation of Legendre conjugate. This can be shown from the key identities (20) and (28) that connect  $d_k(A)$  and  $\omega_\mu(A)$  through the Smith–McMillan form of  $A$ .

**Theorem 3.4.** *Let  $A \in F(s; \sigma, \delta)^{n \times n'}$  be a proper matrix of rank  $r$  over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . Then (7) and (8) hold.*

*Proof.* Put  $d_k := d_k(A)$  for  $k = 0, \dots, r$  and  $\omega_\mu := \omega_\mu(A)$  for  $\mu \in \mathbb{N}$ . Since  $(d_0, d_1, \dots, d_r)$  is concave and  $(\omega_0, \omega_1, \omega_2, \dots)$  is convex, (7) and (8) are equivalent by (31). We show (8) as follows.

First we give an equality

$$\omega_\mu = r\mu - \sum_{i=1}^r \min\{-\alpha_i, \mu\} \quad (32)$$

for  $\mu \in \mathbb{N}$ , where  $\alpha_1 \geq \dots \geq \alpha_r$  are the exponents of the Smith–McMillan form of  $A$ . Figure 1 graphically shows this equality. Let  $x$  and  $y$  be the coordinates along the horizontal and vertical axes in Figure 1, respectively. For  $i = 1, \dots, r$ , the height of the dotted rectangle with  $i-1 \leq x < i$  is  $\min\{-\alpha_i, \mu\}$ . Hence the area of the dotted region is equal to  $\sum_{i=1}^r \min\{-\alpha_i, \mu\}$ . In addition, the width of the white rectangle with  $d \leq y < d+1$  is equal to  $N_d$  for  $d = 0, \dots, \mu-1$ , where  $N_d$  is defined by (27). Hence the area of the white stepped region is equal to  $N_0 + \dots + N_{\mu-1} = \omega_\mu$  by (28). Now we have (32) since the sum of the areas of these two regions is  $r\mu$ .

Substituting (20) into the right hand side of (8), we have

$$\max_{0 \leq k \leq r} (d_k + k\mu) = \max_{0 \leq k \leq r} \sum_{i=1}^k (\alpha_i + \mu) = \sum_{i=1}^{k^*} \alpha_i + k^* \mu, \quad (33)$$

where  $k^*$  is the maximum  $0 \leq k \leq r$  such that  $\alpha_k + \mu \geq 0$ . Since  $\min\{-\alpha_i, \mu\}$  is  $-\alpha_i$  if  $i \leq k^*$  and  $\mu$  if  $i > k^*$ , it holds

$$\sum_{i=1}^r \min\{-\alpha_i, \mu\} = -\sum_{i=1}^{k^*} \alpha_i + (r - k^*)\mu. \quad (34)$$

From (33) and (34), we have

$$\max_{0 \leq k \leq r} (d_k + k\mu) = r\mu - \sum_{i=1}^r \min\{-\alpha_i, \mu\},$$

in which the right hand side is equal to  $\omega_\mu$  by (32).  $\square$

### 3.3 Algorithm Description

Let  $A = A_0 + A_1 s^{-1} + \dots + A_\ell s^{-\ell} \in F(s; \sigma, \delta)^{n \times n'}$  be a proper matrix (25) of rank  $r$ . In this section, we first describe an algorithm to compute  $d_k(A)$  for  $k = 0, \dots, r$  under the assumption that we know the value of  $r$ . Later we give a faster algorithm for  $d_r(A)$  and an algorithm for  $r$  by showing (10) and (11).

The expression (32) of  $d_k(A)$  indicates that  $d_k(A)$  is equal to the optimal value of an minimization problem with objective function

$$f_k(\mu) := \omega_\mu(A) - k\mu. \quad (35)$$

Since  $f_k$  is convex, it is minimized by the minimum  $\mu$  such that  $f_k(\mu+1) - f_k(\mu) \geq 0$ . This can be found by the binary search in  $O(\log M)$  evaluations of  $f_k$ , where  $M$  is an upper bound on a minimizer of  $f_k$ . The following lemma claims that we can adopt  $\ell r$  as this upper bound.

**Lemma 3.5.** *Let  $A = \sum_{d=0}^{\ell} A_d s^{-d} \in F(s; \sigma, \delta)^{n \times n'}$  be the proper matrix (25) of rank  $r$ . Then the following hold:*

- (1) *The exponents  $\alpha_1, \dots, \alpha_r$  of the Smith–McMillan form of  $A$  satisfy  $0 \geq \alpha_1 \geq \dots \geq \alpha_r \geq -\ell r$ .*
- (2) *For  $k \in [0, r]$ , the function  $f_k$  defined by (35) has a minimizer  $\mu^* \in \mathbb{N}$  satisfying  $0 \leq \mu^* \leq \ell r$ .*

*Proof.* The claims are trivial if  $r = 0$ . Suppose  $r \geq 1$ .

(1) It suffices to show  $\alpha_r \geq -\ell r$ . Since  $A$  is proper,  $\delta_{r-1}(A)$  is nonpositive. In addition, since  $As^\ell$  is a polynomial matrix of rank  $r$ , we have  $0 \leq \delta_r(As^\ell) = \delta_r(A) + \ell r$  by Proposition 2.10 (1) and (3). Thus  $\alpha_r = \delta_r(A) - \delta_{r-1}(A) \geq -\ell r$  holds.

(2) From Lemma 3.3, the objective function  $f_k$  can be written as

$$f_k(\mu) = \sum_{d=0}^{\mu-1} (N_d - k)$$

for  $\mu \in \mathbb{N}$ . Hence  $f_k$  is minimized by the maximum  $\mu \in \mathbb{N}$  such that  $N_\mu + k < 0$ . Note that such  $\mu$  exists since  $f_k$  has the minimum value. From the definition (27) of  $N_d$ , it holds  $N_d = N_{-\alpha_r}$  for all  $d \geq -\alpha_r$ . Hence  $f_k$  has a minimizer less than or equal to  $-\alpha_r$ , which is at most  $\ell r$  by (1).  $\square$

The process of evaluating  $f_k$  is decomposed into the construction of the expanded matrix  $\Omega_\mu(A)$  and the computation of its rank. Since the  $\ell r$ th-order expanded matrix  $\Omega_{\ell r}(A)$  contains  $\Omega_\mu(A)$  as a submatrix for all  $0 \leq \mu \leq \ell r$ , it suffices to construct  $\Omega_{\ell r}(A)$  at the beginning of the algorithm once. Then  $\omega_\mu(A) = \text{rank } \Omega_\mu(A)$  can be computed by using rank computation algorithms for matrices over  $F$ .

Each block  $A_d^{(i)}$  of expanded matrices can be computed according to the following recursive formula, which is essentially equivalent to (14). To describe this we shall extend  $\sigma$ ,  $\delta$  and the inverse  $\sigma^{-1}$  of  $\sigma$  to matrices over  $F$ : for  $B = (B_{i,j})_{i,j} \in F^{n \times n'}$ , let  $\sigma(B) := (\sigma(B_{i,j}))_{i,j}$ ,  $\delta(B) := (\delta(B_{i,j}))_{i,j}$  and  $\sigma^{-1}(B) := (\sigma^{-1}(B_{i,j}))_{i,j}$ . Note that the commutative rule (1) of skew polynomials can be extended to matrices as  $sB = \sigma(B)s + \delta(B)$  for  $B \in F^{n \times n'}$ .

**Lemma 3.6.** *Let  $A \in F[s; \sigma, \delta]^{n \times n'}$  be a proper matrix over a  $\sigma$ -differential skew field  $(F, \sigma, \delta)$ . For  $i, d \in \mathbb{N}$ , let  $A_d^{(i)} \in F^{n \times n'}$  be the coefficient matrix of  $s^{-d}$  in the expansion of  $s^{-i}A$ . Then  $A_d^{(i)}$  satisfies the following recurrence formula*

$$A_d^{(i)} = \begin{cases} O & (d = 0), \\ \sigma^{-1}(A_{d-1}^{(i-1)} - \delta(A_{d-1}^{(i)})) & (d \geq 1) \end{cases} \quad (36)$$

for  $i, d \in \mathbb{N}$  with  $i \geq 1$ .

*Proof.* For  $i, \mu \geq 1$ , consider the expansion  $\sum_{d=0}^{\infty} A_d^{(i)} s^{-d}$  of  $s^{-i}A$ . By the commutative rule of skew polynomials, it holds

$$\begin{aligned} s^{-(i-1)}A &= s(s^{-i}A) = \sum_{d=0}^{\infty} sA_d^{(i)} s^{-d} \\ &= \sum_{d=0}^{\infty} (\sigma(A_d^{(i)})s + \delta(A_d^{(i)})) s^{-d} \\ &= \sigma(A_0^{(i)})s + \sum_{d=0}^{\infty} (\sigma(A_{d+1}^{(i)}) + \delta(A_d^{(i)})) s^{-d}. \end{aligned} \quad (37)$$

The equality (37) is in the form of the expansion of  $s^{-(i-1)}A$ . Thus we have  $\sigma(A_0^{(i)}) = O$  and  $\sigma(A_{d+1}^{(i)}) + \delta(A_d^{(i)}) = A_d^{(i-1)}$ , which imply (36).  $\square$

Algorithm 1 shows the entire procedure to compute  $d_k$  of a skew polynomial matrix  $A \in F[s; \sigma, \delta]^{n \times n'}$ . Note that the value of  $f_k(a)$  in the last line of Algorithm 1 is equal to  $d_k$  not of  $A$  but of the proper matrix  $As^{-\ell}$ ; hence we add  $\ell k$  according to Proposition 2.10 (1). Thus Theorem 1.1 is obtained.

Finally, we show the formula (10) of  $\text{rank } A$  and (11) of  $d_r(A)$  for a proper matrix  $A$  in (25). These formulas naturally yield efficient algorithms to compute them.

**Lemma 3.7.** *Let  $A = \sum_{d=0}^{\ell} A_d s^{-d} \in F[s; \sigma, \delta]^{n \times n'}$  be the proper matrix (25) of rank  $r$ . Then it holds (10) and (11).*

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**Algorithm 1** Algorithm to compute  $d_k(A)$ 


---

**Input** : skew polynomial matrix  $A := \sum_{d=0}^{\ell} A_d s^d \in F[s; \sigma, \delta]^{n \times n'}$ , its rank  $r$  and  $k \in \{0, \dots, r\}$

**Output**:  $d_k(A)$

```

1:  $A_d^{(0)} := A_d$  for  $0 \leq d \leq \ell$  and  $A_d^{(0)} := O$  for  $\ell < d \leq \ell r - 1$  ▷ Compute  $A_d^{(i)}$ 
2: for  $i = 1$  to  $\ell r - 1$  do
3:    $A_d^{(i)} := O$  for  $0 \leq d < i$ 
4:   for  $d = i$  to  $\ell r - 1$  do
5:      $A_d^{(i)} := \sigma^{-1}(A_{d-1}^{(i-1)} - \delta(A_{d-1}^{(i)}))$ 
6:  $a \leftarrow 0, b \leftarrow \ell r$  ▷ Binary search on  $\{a, a+1, \dots, b\}$ 
7: while  $a < b$  do
8:    $c \leftarrow \lfloor (a+b)/2 \rfloor$ 
9:   if  $f_k(c+1) - f_k(c) < 0$  then ▷  $f_k$  is defined by (35)
10:     $a \leftarrow c+1$ 
11:  else
12:     $b \leftarrow c$ 
13: return  $f_k(a) + \ell k$ 

```

---

*Proof.* We first show (10). It holds  $\omega_{\ell n^*+1}(A) - \omega_{\ell n^*}(A) = N_{\ell n^*}$  by (30). Since  $-\alpha_i$  is at most  $\ell r \leq \ell n^*$  for all  $i \in [r]$  by Lemma 3.5 (1), we have  $r = N_{\ell n^*}$ .

Next we show (11). From (28) and (7), it holds

$$d_r(A) = \min_{\mu \geq 0} \sum_{d=0}^{\mu-1} (N_d - r). \quad (38)$$

Since  $N_0 \leq N_1 \leq \dots \leq N_{\ell r} = N_{\ell r+1} = \dots = r$  by Lemma 3.5 (1), the minimum value of the right hand side of (38) is attained by  $\mu = \ell r$ . Thus we have (11).  $\square$

From (11) and Proposition 2.10 (1), for a skew polynomial matrix  $A \in F[s; \sigma, \delta]^{n \times n'}$  of degree  $\ell$  and rank  $r$ , we have

$$d_r(A) = d_r(As^{-\ell}) + \ell r = \omega_{\ell r}(As^{-\ell}) - \ell r(r-1).$$

This agrees with the result of Henrion–Ševšek [22] for  $\deg \det$  of polynomial matrices over  $\mathbb{C}$ .

For skew polynomial matrices over a difference skew field, Theorem 2.13 and Lemma 3.7 provide an algorithm to compute  $\text{ord Det}$  in the same time complexity as  $\deg \text{Det}$ .

## 4 Application to Linear Differential and Difference Equations

In this section, we provide  $\deg\text{-det}$  type formulas for the dimension of the solution spaces of a linear differential equation (3) and a linear difference equation (5). Taelman [46] gave a formula for the differential case and thus our aim is the formula (6) for the difference case; we show it in a different manner from [46]. To integrally describe both the differential and difference equations, we use  $\sigma$ -*differential equations* introduced by Bronstein [8]. In this section we assume that all the fields are of characteristic zero. In addition, for a  $\sigma$ -differential ring  $(R, \sigma, \delta)$ , we also refer to  $R$  as a  $\sigma$ -differential ring in place of  $(R, \sigma, \delta)$  when  $\sigma$  and  $\delta$  are clear from the context.

### 4.1 $\sigma$ -Differential Equations

Let  $(R, \sigma, \delta)$  be a commutative  $\sigma$ -differential ring. A *constant* of a  $\sigma$ -differential ring  $(R, \sigma, \delta)$  is an element  $a \in R$  such that  $\sigma(a) = a$  and  $\delta(a) = 0$ . The set of all constants of  $(R, \sigma, \delta)$

is denoted by  $\text{Const}_{\sigma,\delta}(R)$  or by  $\text{Const}(R)$  if  $\sigma$  and  $\delta$  are clear from the context. It is easily checked that  $\text{Const}(R)$  is a subring of  $R$ , and if  $R$  is a field, so is  $\text{Const}(R)$ .

An additive map  $\theta: R \rightarrow R$  is said to be *pseudo-linear* if it satisfies  $\theta(ab) = \sigma(a)\theta(b) + \delta(a)b$  for all  $a, b \in R$ . We see the set of additive maps on  $R$  as the endomorphism ring of the additive group of  $R$  and it contains  $R$  as a subring by identifying  $a \in R$  with a map  $b \mapsto ab$  for  $b \in R$ . For a pseudo-linear map  $\theta$  and  $a, b \in R$ , we have

$$(\theta a)(b) = \theta(ab) = \sigma(a)\theta(b) + \delta(a)b = (\sigma(a)\theta + \delta(a))(b),$$

which means  $\theta a = \sigma(a)\theta + \delta(a)$ . Hence  $\theta$  meets the commutation rule (1) of the skew polynomial ring  $R[s; \sigma, \delta]$  over  $(R, \sigma, \delta)$ . From this, the substitution  $p \mapsto p(\theta)$  of a pseudo-linear map  $\theta$  into the indeterminate  $s$  in a skew polynomial  $p = p(s)$  is well-defined; it is a ring endomorphism from  $R[s; \sigma, \delta]$  to the ring of additive maps on  $R$ .

A (scalar) linear  $\sigma$ -differential equation over  $R$  is an equation for  $y \in R$  in the form of

$$a_\ell \theta^\ell(y) + a_{\ell-1} \theta^{\ell-1}(y) + \cdots + a_1 \theta(y) + a_0 y = 0, \quad (39)$$

where  $a_0, \dots, a_\ell \in R$  and  $\theta: R \rightarrow R$  is a pseudo-linear map. The equation (39) can be written as  $p(\theta)(y) = 0$  by using a skew polynomial  $p(s) := a_\ell s^\ell + \cdots + a_1 s + a_0 \in R[s; \sigma, \delta]$ . We call  $\theta$  in (39) the  $\sigma$ -*differential operator* of (39). If  $\sigma = \text{id}$  and  $\theta = \delta$ , then  $\sigma$ -differential equations coincide with the usual linear differential equations. Similarly, if  $\delta = 0$  and  $\theta = \sigma$ , then  $\sigma$ -differential equations are the usual linear difference equations.

Suppose that  $R$  is a field  $F$ . Indeed, any  $\sigma$ -differential equation over a  $\sigma$ -differential field is essentially either a (usual) differential or difference equation. This follows from the following two facts: (i) an additive map  $\theta: F \rightarrow F$  is pseudo-linear if and only if it is in the form of  $\gamma\sigma + \delta$  for some  $\gamma \in F$  [7, Lemma 5], and (ii) if  $\sigma \neq \text{id}$  then there exists  $\alpha \in F$  such that  $\delta = \alpha(\sigma - \text{id})$  [8, Lemma 1]. Therefore a pseudo-linear map  $\theta$  can be written as  $\theta = \delta + \gamma$  if  $\alpha = \text{id}$  and as  $\theta = (\alpha + \gamma)\sigma + \alpha$  if  $\sigma \neq \text{id}$ . By expanding  $\theta^d$  for  $d = 1, \dots, \ell$  using these equations, any  $\sigma$ -differential equation  $p(\theta)(y) = 0$  is represented as  $q(\delta)(y) = 0$  for some  $q \in F[s; \text{id}, \delta]$  if  $\sigma = \text{id}$  and as  $q'(\sigma)(y) = 0$  for some  $q' \in F[s; \sigma, 0]$  if  $\sigma \neq \text{id}$ . A typical example of this reduction is the replacement of the difference operator in a difference equation by the shift operator. Therefore, even though we are treating general  $\sigma$ -differential equations over a  $\sigma$ -differential field, it suffices to consider only differential equations ( $\theta = \delta$ ) over a differential field and difference equations ( $\theta = \sigma$ ) over a difference field. Nonetheless, we make use of the notion of  $\sigma$ -differential equations whenever possible since it provides a useful framework unifying differential and difference equations.

Consider a  $\sigma$ -differential equation  $p(\theta)(y) = 0$  over a  $\sigma$ -differential field  $F$ . The *solution space*  $V$  of the equation is defined by  $V := \{v \in F \mid p(\theta)(v) = 0\}$ . It is easily checked that  $V$  is a vector space over  $C := \text{Const}(F)$ . Now our concern is how large  $\dim_C V$  is. For differential equations the following inequality holds.

**Lemma 4.1** ([50, Lemma 1.10]). *Let  $V$  be the solution space of a differential equation  $p(\delta)(y) = 0$  over a differential field  $(F, \text{id}, \delta)$  with  $p \in F[s; \text{id}, \delta] \setminus \{0\}$ . Then it holds  $\dim_C V \leq \deg p$ , where  $C := \text{Const}(F)$ .*

In the difference case,  $p(\sigma)(y) = 0$  and  $(\sigma p(\sigma))(y) = 0$  have the same solution space  $V$  since  $\sigma$  is an automorphism. From this we see that  $\deg p$  does not nicely serve as an upper bound on  $\dim_C V$ . In this case, the value of the degree minus the order is invariant under the multiplication of  $s$  to  $p$ , where the order is defined in Section 2.5. Indeed  $\deg p - \text{ord } p$  provides an upper bound on  $\dim_C V$  as follows.

**Lemma 4.2** ([1, Theorem 6], [45, Corollary 4.9]). *Let  $V$  be the solution space of a difference equation  $p(\sigma)(y) = 0$  over a difference field  $(F, \sigma, 0)$  with  $p \in F[s; \sigma, 0] \setminus \{0\}$ . Then it holds  $\dim_C V \leq \deg p - \text{ord } p$ , where  $C := \text{Const}(F)$ .*



We remark that the statement of [45, Corollary 4.9] is in the setting of a matrix difference equation  $\sigma(y) = Ay$  for  $y \in F^n$  with nonsingular  $A \in F^{n \times n}$ . Lemma 4.2 is obtained by setting  $A$  as the companion matrix of  $s^{-k}p$ , where  $k := \text{ord } p$ .

We next consider extending differential and difference fields in order for differential and difference equations to have the maximum possible number ( $= \deg p$  in the differential and  $\deg p - \text{ord } p$  in the difference case) of linearly independent solutions. This is analogous to the situation of extending a field to its algebraic closure in order for  $n$ th-order algebraic equations to have  $n$  solutions. Let  $(F, \text{id}, \delta)$  be a differential field. A commutative differential ring  $(R, \text{id}, \bar{\delta})$  is called a *differential extension* of  $F$  if  $F$  is a subring of  $R$  and  $\bar{\delta}$  coincide with  $\delta$  on  $F$ . A differential equation  $p(\delta)(y) = 0$  over  $F$  is naturally extended to a differential equation  $p(\bar{\delta})(y) = 0$  over  $R$ . We call a differential extension  $R$  of  $F$  *adequate* due to [1] if (i)  $C := \text{Const}(R)$  is a field and (ii) for any differential equation  $p(\delta)(y) = 0$  over  $F$ , its extension  $p(\bar{\delta})(y) = 0$  to  $R$  has the solution space  $V$  such that  $\dim_C V = \deg p$ . If  $\text{Const}(F)$  is algebraically closed, then there exists an adequate extension  $R$  of  $F$  such that  $\text{Const}(R) = \text{Const}(F)$ , called the *universal (differential) Picard–Vessiot ring* of  $F$  [50, Section 3.2]. In addition, any differential field  $F$  of characteristic zero has a difference extension whose constant field is the algebraic closure of  $\text{Const}(F)$  [1]; see also [50, Exercise 1.5, 2:(c),(d), 3:(c)]. Therefore, there always exists an adequate extension of any differential field of characteristic zero.

An extension of difference fields is similarly defined as follows. Let  $(F, \sigma, 0)$  be a difference field. A *difference extension* of  $F$  is a commutative difference ring  $(R, \sigma, 0)$  such that  $F$  is a subring of  $R$  and  $\bar{\sigma}$  coincide with  $\sigma$  on  $F$ . This induces an extension of a difference equation  $p(\sigma)(y) = 0$  over  $F$  to a difference equation  $p(\bar{\sigma})(y) = 0$  over  $R$ . A difference extension  $R$  of  $F$  is said to be *adequate* if (i)  $C := \text{Const}(R)$  is a field and (ii) for any difference equation  $p(\sigma)(y) = 0$  over  $F$ , its extension  $p(\bar{\sigma})(y) = 0$  to  $R$  has the solution space  $V$  such that  $\dim_C V = \deg p - \text{ord } p$ . If  $\text{Const}(F)$  is algebraically closed, there exists an adequate extension  $R$  of  $F$  such that  $\text{Const}(R) = \text{Const}(F)$ , called the *universal (difference) Picard–Vessiot ring* of  $F$  [49, Section 1.4]. Indeed, for any difference field  $F$  of characteristic zero, an adequate difference extension  $R$  can be easily constructed [1, Proposition 4], while  $\text{Const}(R) = \text{Const}(F)$  is no longer guaranteed.

## 4.2 Matrix $\sigma$ -Differential Equations

We generalize Lemmas 4.1 and 4.2 to simultaneous differential and difference equations. Let  $(R, \sigma, \delta)$  be a commutative  $\sigma$ -differential ring. A pseudo-linear map  $\theta: R \rightarrow R$  is naturally extended to  $R^n$  by  $\theta(a) := (\theta(a_i))_{i \in [n]}$  for  $a = (a_i)_{i \in [n]} \in R^n$ . An  $\ell$ th-order  $n$ -dimensional matrix  $\sigma$ -differential equation over  $R$  is an equation for  $y \in R^n$  in the form of

$$A_\ell \theta^\ell(y) + A_{\ell-1} \theta^{\ell-1}(y) + \cdots + A_1 \theta(y) + A_0 y = 0, \quad (40)$$

where  $\theta$  is a pseudo-linear map and  $A_0, \dots, A_\ell \in R^{n \times n}$  with  $A_\ell \neq O$ . Using a skew polynomial matrix  $A(s) := A_\ell s^\ell + \cdots + A_1 s + A_0 \in R[s; \sigma, \delta]^{n \times n}$ , the equation (40) is expressed as  $A(\theta)(y) = 0$ , where  $A(\theta)$  is regarded as an additive map on  $R^n$ . The *solution space*  $V$  of a matrix  $\sigma$ -differential equation  $A(\theta)(y) = 0$  over a  $\sigma$ -differential field  $F$  is defined by  $V := \{v \in F^n \mid A(\theta)(v) = 0\}$ , which forms a vector space over  $C := \text{Const}(F)$ . Using the Jacobson normal form established in Proposition 2.5, we give an equality for  $\dim_C V$  as follows.

**Lemma 4.3.** *Let  $A \in F[s; \sigma, \delta]^{n \times n}$  be a square skew polynomial matrix over a  $\sigma$ -differential field  $(F, \sigma, \delta)$  and  $V$  the solution space of a matrix  $\sigma$ -differential equation  $A(\theta)(y) = 0$  for  $y \in F^n$  with  $\sigma$ -differential operator  $\theta$ . In addition, for  $i \in [n]$ , let  $p_i \in F[s; \sigma, \delta]$  be the  $i$ th diagonal entry of the Jacobson normal form (12) of  $A$  and  $V_i$  the solution space of a scalar  $\sigma$ -differential equation  $p_i(\theta)(y_i) = 0$  for  $y_i \in F$ . Then it holds*

$$\dim_C V = \sum_{i=1}^n \dim_C V_i, \quad (41)$$

where  $C := \text{Const}(F)$ .

*Proof.* Let  $V'$  be the solution space of a matrix  $\sigma$ -differential equation  $D(\theta)(y) = 0$ , where  $D := \text{diag}(p_1, \dots, p_n)$  is the Jacobson normal form of  $A$ . Since  $D$  is diagonal,  $V'$  is the direct sum of  $V_1, \dots, V_n$  and thus we have  $\dim_C V' = \sum_{i=1}^n \dim_C V_i$ . Hence our goal is to show  $\dim_C V = \dim_C V'$ .

Let  $U, W \in F[s; \sigma, \delta]^{n \times n}$  be unimodular matrices such that  $A = UDW$ . Then  $A(\theta)(y) = 0$  is written as

$$(U(\theta)D(\theta)W(\theta))(y) = U(\theta)(D(\theta)(W(\theta)(y))) = 0. \quad (42)$$

Since  $U$  and  $W$  are unimodular,  $U^{-1}(\theta)$  and  $W^{-1}(\theta)$  are well-defined. From  $U^{-1}(\theta)(U(\theta)(a)) = (U^{-1}U)(\theta)(a) = a$  for all  $a \in F^n$ , we can see that  $U(\theta)$  (and  $W(\theta)$ ) are bijective as additive maps on  $F^n$ . Thus (42) is equivalent to  $D(\theta)(W(\theta)(y)) = 0$ . Since  $W(\theta)$  is bijective, it holds

$$V' = \{W(\theta)(v) \mid v \in V\}.$$

Now  $V$  and  $V'$  are isomorphic as linear spaces over  $C$  because  $W(\theta)$  is a bijective linear map on  $C$  by  $\theta(cv) = \sigma(c)\theta(v) + \delta(c)v = c\theta(v)$  for any  $c \in C$  and  $v \in F^n$ . Hence we have  $\dim_C V = \dim_C V'$ .  $\square$

As a corollary of Lemma 4.3, one can show that the finiteness of  $\dim_C V$  is characterized by the nonsingularity of  $A$  under the assumption that the field extension  $F/C$  is infinite. We remark that if  $F$  is a differential field then  $F/C$  is finite if and only if  $\delta \neq 0$  because all the algebraic elements in  $F$  over  $C$  are constants [50, Exercise 1.5, 2:(c)] and any finite field extension is algebraic.

**Corollary 4.4.** *Let  $(F, \sigma, \delta)$  be a  $\sigma$ -differential field such that the field extension  $F/C$  is infinite, where  $C := \text{Const}(F)$ . Consider a matrix  $\sigma$ -differential equation  $A(\theta)(y) = 0$  with  $A \in F[s; \sigma, \delta]^{n \times n}$  and  $\sigma$ -differential operator  $\theta$ . Then its solution space is of finite dimension over  $C$  if and only if  $A$  is nonsingular.*

*Proof.* Let  $p_1, \dots, p_n \in F[s; \sigma, \delta]$  be the diagonal entries of the Jacobson normal form of  $A$  and  $V_i$  the solution space of a scalar  $\sigma$ -differential equation  $p_i(\theta)(y_i) = 0$  for  $i \in [n]$ . If  $A$  is nonsingular, all  $p_i$  are nonzero. Then  $\dim_C V_i$  is finite for all  $i \in [n]$  from Lemmas 4.1 and 4.2. Note that any  $\sigma$ -differential equation can be converted into a differential or difference equation as explained above. Hence  $V$  is of finite dimension by (41). Conversely, suppose that  $A$  is singular. Then  $p_n$  must be zero, which implies  $V_n = F$ . From the assumption on the extension degree of  $F/C$ , the dimension of  $V_n$  (and thus of  $V$  by (41)) over  $C$  is infinite.  $\square$

Consider a matrix differential equation  $A(\delta)(y) = 0$  with nonsingular  $A \in F[s; \sigma, \delta]^{n \times n}$ . As a matrix generalization of Lemma 4.1, Taelman [46] showed that the dimension of the solution space of  $A(\delta)(y) = 0$  is bounded by  $\deg \text{Det } A$  and is tight over adequate differential extensions.

**Theorem 4.5** ([46, Corollary 2.2]). *Let  $V$  be the solution space of a matrix differential equation  $A(\delta)(y) = 0$  over a differential field  $(F, \text{id}, \delta)$  with nonsingular  $A \in F[s; \text{id}, \delta]^{n \times n}$ . Then it holds*

$$\dim_C V \leq \deg \text{Det } A,$$

where  $C := \text{Const}(F)$ . For an adequate differential extension  $(R, \text{id}, \bar{\delta})$  of  $F$ , the solution space  $\bar{V}$  of the extended differential equation  $A(\bar{\delta})(y) = 0$  to  $R$  attains

$$\dim_{\bar{C}} \bar{V} = \deg \text{Det } A,$$

where  $\bar{C} := \text{Const}(R)$ .

Taelman [46] proved Theorem 4.5 in a coordinate-free way by using the notion of *differential modules*. Indeed, the same theorem can also be obtained from Lemma 4.3 and the fact that  $\deg \text{Det}$  of unimodular matrices are zero by Proposition 2.8. In this manner we give the difference version of Theorem 4.5 as follows.

**Theorem 4.6.** *Let  $V$  be the solution space of a matrix difference equation  $A(\sigma)(y) = 0$  over a difference field  $(F, \sigma, 0)$  with nonsingular  $A \in F[s; \sigma, 0]^{n \times n}$ . Then it holds*

$$\dim_C V \leq \deg \text{Det } A - \text{ord } \text{Det } A, \quad (43)$$

where  $C := \text{Const}(F)$ . For an adequate difference extension  $(R, \bar{\sigma}, 0)$  of  $F$ , the solution space  $\bar{V}$  of the extended difference equation  $A(\bar{\sigma})(y) = 0$  to  $R$  attains

$$\dim_{\bar{C}} \bar{V} = \deg \text{Det } A - \text{ord } \text{Det } A, \quad (44)$$

where  $\bar{C} := \text{Const}(R)$ .

*Proof.* Let  $UAW = D := \text{diag}(p_1, \dots, p_n)$  be the Jacobson normal form of  $A$  with unimodular matrices  $U, W \in F[s; \sigma, 0]^{n \times n}$  and  $p_1, \dots, p_n \in F[s; \sigma, 0]$ . Let  $V_i$  be the solution space of a scalar difference equation  $p_i(\sigma)(y_i) = 0$  for  $i \in [n]$ . From Lemmas 4.2 and 4.3, it holds

$$\dim_C V = \sum_{i=1}^n \dim_C V_i \leq \sum_{i=1}^n \deg p_i - \sum_{i=1}^n \text{ord } p_i.$$

Since  $\deg \text{Det}$  of unimodular matrices are zero by Proposition 2.8, we have

$$\sum_{i=1}^n \deg p_i = \deg \text{Det } D = \deg \text{Det } U + \deg \text{Det } A + \deg \text{Det } W = \deg \text{Det } A.$$

We also have  $\sum_{i=1}^n \text{ord } p_i = \text{ord } \text{Det } A$  as well. Note that the order of a unimodular matrix is zero because Proposition 2.8 provides a decomposition of unimodular matrices into matrices of order zero: permutation matrices, elementary matrices and diagonal matrices whose diagonal entries are in  $F^\times$ . Hence (43) holds. On the adequate extension  $R$  or  $F$ , the equality (44) is attained from  $\dim_{\bar{C}} \bar{V}_i = \deg p_i - \text{ord } p_i$  for all  $i \in [n]$ , where  $\bar{V}_i$  is the solution space of the extended difference equation  $p_i(\bar{\sigma})(y_i) = 0$ .  $\square$

From Theorems 4.5 and 4.6, the computation of the dimension of the solution spaces of matrix differential and difference equations over adequate extensions is reduced to the rank computation over  $F$  by our algorithm.

## 5 Reducing SDD to Edmonds' Problem

Consider Sparse Degree of Determinant (SDD) problem over a field  $K$ . In this section, we prove Theorem 1.4 by converting an instance  $B = B_0 + B_1x_1 + \dots + B_mx_m \in K[x_1, \dots, x_m]^{n \times n}$  of Edmonds' problem into an instance of SDD preserving their rank. This is achieved by the following simple conversion

$$A := B_0s^{w_0} + B_1s^{w_1} + \dots + B_ms^{w_m} \in K[s]^{n \times n} \quad (45)$$

with  $w_0 := 0$  and  $w_k := (n+1)^{k-1}$  for  $k \in [m]$ .

**Lemma 5.1.** *Let  $B = B_0 + \sum_{k=1}^m B_kx_k \in K[x_1, \dots, x_m]^{n \times n}$  be a linear matrix over a field  $K$  and  $A = \sum_{k=0}^m B_ks^{w_k} \in K[s]^{n \times n}$  the polynomial matrix defined by (45). Then it holds  $\text{rank } A = \text{rank } B$ .*

*Proof.* We show that  $A$  is nonsingular if and only if  $B$  is nonsingular. This implies  $\text{rank } A = \text{rank } B$  by applying it to their submatrices.

Since  $\det B$  is a polynomial in  $x_1, \dots, x_m$  of degree at most  $n$ , we can express  $\det B$  as

$$\det B = \sum_{(d_1, \dots, d_m) \in D} c_{d_1, \dots, d_m} x_1^{d_1} \dots x_m^{d_m},$$

where

$$D := \{(d_1, \dots, d_m) \in [0, n]^m \mid d_1 + \dots + d_m \leq n\}$$

and some  $c_{d_1, \dots, d_m} \in K$  for  $(d_1, \dots, d_m) \in D$ . Then  $\det B = 0$  is equivalent to  $c_{d_1, \dots, d_m} = 0$  for all  $(d_1, \dots, d_m) \in D$ . Since  $A$  is obtained from  $B$  by substituting  $s^{w_1}, \dots, s^{w_m}$  into  $x_1, \dots, x_m$ , respectively, we have

$$\det A = \sum_{(d_1, \dots, d_m) \in D} c_{d_1, \dots, d_m} s^{d_1 w_1 + \dots + d_m w_m}.$$

Now the map  $h: D \rightarrow \mathbb{N}$  defined by  $h(d_1, \dots, d_m) := d_1 w_1 + \dots + d_m w_m$  is injective by the definition of  $w_k$ . Hence  $\det A = 0$  is also equivalent to  $c_{d_1, \dots, d_m} = 0$  for all  $(d_1, \dots, d_m) \in D$ .  $\square$

By  $\log w_m = (m-1) \log(n+1) = O(\text{poly}(n, m))$ , Lemma 5.1 implies that an algorithm to compute  $\text{rank } A$  in  $\text{poly}(n, m, \log w_m)$  arithmetic operations can compute  $\text{rank } B$  in  $\text{poly}(n, m)$  operations. In addition, since an algorithm to compute  $\deg \text{Det } A$  can check the nonsingularity of  $A$ , it can also be used to compute  $\text{rank } A$  by iteratively applying it to submatrices of  $A$  in order from small ones. Hence Theorem 1.4 holds.

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