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MEAN COST CYCLICAL GAMES

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We study the mean cost cyclical game in a more general setting than that in Gurvitch et al. (1988) and Karzanow and Lebedev (1993). The game is played on a directed graph and generalizes the single source shortest path problem, the minimum mean cycle problem (see Karp 1978), and the ergodic extension of matrix games (Moulin 1976). We prove the existence of a solution in uniform stationary strategies and present an algorithm for finding such optimal strategies. In fact, our algorithm is an extension of the algorithms due to Gurvitch et al. (1988) and Karzanow and Lebedev (1993), which were proved to be finite, but exponential in the worst case. We prove that all these algorithms are pseudopolynomial.

1. Introduction. Let $G = (V, E)$ be a directed graph with vertex set V containing n vertices and arc set E containing m arcs. We assume that G may have loops but has no multiple arcs and dead ends, i.e., vertices without leaving arcs. These assumptions are introduced for notational convenience only. Let $E(v)$ denote the set of arcs leaving $v \in V$, and let $d_G(v) = |E(v)|$ be the *outdegree* of v.

For each vertex $v \in V$, a set $\mathcal{F}(v)$ of subsets of $E(v)$ is given. It is supposed that $E(v) \notin \mathcal{F}(v)$. Note that $F(v)$ may be exponentially large in $d_G(v)$. In interesting applications, however, $F(v)$ is usually described combinatorially and can be given by use of an efficient membership procedure, e.g., when $F(v)$ is given by a system of linear inequalities. In what follows we assume that there is a subroutine (or membership oracle) which, for $v \in V$ and $A \subseteq E(v)$, decides whether A is in $\mathcal{F}(v)$, in time $O(T_{\mathcal{F}})$.

Let c be an integer-valued *cost function* on arcs of G, i.e., each arc $(v, w) \in E$ has an integer *cost* $c(v, w)$. We call the triple $(G, \{\mathcal{F}(v) : v \in V\}, c)$ a *game network*. The vertices are *positions* and the arcs are *moves* in the game. There are two players, called *Short* and *Long*. Starting from a specified *origin* $s \in V$, a *chip* is moved step by step along arcs of G. Suppose, at a current moment, the chip is at a position $v \in V$. First, Long forbids a subset $S \in \mathcal{F}(v)$ of arcs leaving v. After that Short chooses a nonforbidden arc $(v, w) \in E(v)\backslash S$ to move the chip to the position w.

The players have complete information about all moves and the game lasts infinitely long. The *payoff function* $f(c, P)$ depends on the cost function and the path $P = (s = v_0, v_1, \cdots)$ passed by the chip. In the *mean cost* game we define

$$(1) \qquad f(c, P) = \liminf_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} c(v_i, v_{i+1}).$$

The interests of the players are strictly opposed in the sense that Long wins $f(c, P)$ and Short loses this value. Later we shall show that the mean cost game is *cyclical*, i.e., the players, playing optimally, after a finite number of moves, can repeatedly move the chip along a simple cycle of G.

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The special case of the mean cost game when, for each vertex v, the set system $\mathcal{F}(v)$ consists of all the sets of cardinality not greater than $k(v)$, was studied by Karzanow and Lebedev (1993). Here, for $v \in V$, $k(v)$ is a nonnegative integer and is interpreted as the maximal number of moves that Long can forbid at the position v. When, for each vertex $v \in V$, $k(v)$ is either 0 or $d_G(v) - 1$, we get the game which was studied by Ehrenfeucht and Mycielski (1979) and Gurvitch et al. (1988).

On the other hand the following simple transformation, proposed by an anonymous referee, allows us to reduce our general model to the model from Gurvitch et al. (1988). Let us fix a vertex $v \in V$ and denote the complements of sets from $\mathcal{F}(v)$ by $S_1, \ldots, S_q$. Then let us connect v with new vertices $v_1, \ldots, v_q$ and, if $(v, w) \in S_i$, connect v_i with w. Now we set $k(v) = q - 1$ and $k(v_i) = 0$, $i = 1, \ldots, q$. Obviously, the obtained cyclical game is of the type studied by Gurvitch et al. (1988) and it is equivalent to the initial game of the general type. This transformation, of course, can be exponential. Therefore the algorithm from Gurvitch et al. (1988) cannot be applied effectively for solving cyclical games of the general form. However, we could use this construction to prove our main theorem (Theorem 1) starting from its analog in Gurvitch et al. (1988).

Next, we will formulate the mean cost cyclical game as a two-player zero-sum game. Without loss of generality, it can be assumed that, for any vertex $v \in V$, $\mathrm{IS}(v) = (E(v), \mathcal{F}(v))$ is an independence system. Recall that an *independence system* (*hereditary family*) IS is specified by an ordered pair $\mathrm{IS} = (X, \mathcal{F})$, where X is a finite set, called the *ground set*, and $\mathcal{F}$ is a set of subsets of X, called the *independent sets*, that satisfy the *independence axioms*:

(i0) $\varnothing \in \mathcal{F}$;

(i1) $A \in \mathcal{F}$, $B \subseteq A$ imply $B \in \mathcal{F}$.

A set, which is not independent, is called *dependent*. The maximal (by inclusion) independent sets are known as *bases*. A *circuit* is a minimum (by inclusion) dependent set. For an independence syste IS, if it is not specified directly, we denote by $\mathcal{B}(\mathrm{IS})$ and $\mathcal{C}(\mathrm{IS})$ its sets of bases and circuits, respectively.

For $v \in V$, let $\mathcal{B}(v) \stackrel{\text{def}}{=} \mathcal{B}(\mathrm{IS}(v))$, $\mathcal{C}(v) \stackrel{\text{def}}{=} \mathcal{C}(\mathrm{IS}(v))$. A *stationary strategy* σ (respectively, τ) of Short (Long), at each position $v \in V$, specifies a choice of a circuit $\sigma(v) \in \mathcal{C}(v)$ (base $\tau(v) \in \mathcal{B}(v)$). The payoff $f(s;\, \sigma,\, \tau)$ is defined to be the minimum of $f(c, P)$ over all s-paths in $G_{\sigma,\tau}$, where the set of arcs, leaving $v \in V$ in $G_{\sigma,\tau}$, is $\sigma(v)\backslash\tau(v)$. A simple reasoning (see also Gurvitch et al. 1988) can be applied to prove that $f(s;\, \sigma,\, \tau) = \mu(s, G_{\sigma,\tau}, c)$. Here $\mu(s, G, c)$ denotes the minimum, over all cycles Γ in G reachable from s, of the *mean cost* of Γ, defined to be the total arc cost $c(\Gamma)$ of Γ divided by the number of arcs it contains. As we can see later, for any starting position $s \in V$, the following discrete minimax identity holds:

$$\nu(s) \stackrel{\text{def}}{=} \min_{\sigma} \max_{\tau} f(s;\, \sigma,\, \tau) = \max_{\tau} \min_{\sigma} f(s;\, \sigma,\, \tau). \tag{2}$$

Equality (2) means that any pair (σ_0, τ_0) of min-max strategies is a solution to the two-person zero-sum game in which the payoff function is $f(s;\, \sigma,\, \tau)$; the pure strategies available to Short and Long are their stationary strategies; Short plays to minimize the payment; Long plays to maximize it. It turns out that there exists a solution (σ_0, τ_0) such that, for any $v \in V$, $|\sigma_0(v)\backslash\tau_0(v)| = 1$. In addition, optimal strategies σ_0, τ_0 can be chosen *uniformly*, i.e., (σ_0, τ_0) is a solution to the game for any starting position s. Of course, the *values* $\nu(s) = f(s;\, \sigma_0, \tau_0)$ of distinct positions s may be different. Now it must be clear that the pair (σ_0, τ_0) determines the optimal policy of each player in the mean cost cyclical game: at a position $v \in V$, Long forbids all arcs from $\tau_0(v)$, and Short moves the chip along the only arc from $\sigma_0(v)\backslash\tau_0(v)$.

Identity (2) also states that playing first and revealing one's strategy is not a disadvantage.

It motivates the formulation of the following *dual rules* of playing the mean cost cyclical game. At a position v, Short chooses a circuit $\sigma(v) \in \mathscr{C}(v)$. After that, his opponent forbids any subset of $|\sigma(v)| - 1$ arcs from $\sigma(v)$. The chip is moved along the only nonforbidden arc from $\sigma(v)$.

2. The main theorem. To state our main theorem, we need some notations. A *price function* is a vertex labeling $p : V \to \mathbb{R}$. For a price function p, the *reduced cost* of an arc (v, w) is $c_p(v, w) = c(v, w) + p(v) - p(w)$. This notion, which originates in the theory of linear programming, is crucial for many network problems. We note that, for any path P from s to t,

$$c_p(P) = c(P) + p(s) - p(t),$$

where $c(P)$ is the cost of P, defined to be the sum of its arc costs. Therefore, after substituting c for c_p, the mean cost of any infinite path is not changed. Thus, substitution c for c_p preserves optimal strategies, if such exist, in the mean cost game.

A real-valued function g given on a finite set X can be viewed as a vector in the vector space $\mathbb{R}^X$; therefore, we use the commonly accepted notations, $\|g\| \overset{\text{def}}{=} (\Sigma_{x\in X}\,(g(x))^2)^{1/2}$ and $\|g\|_\infty \overset{\text{def}}{=} max_{x\in X}|g(x)|$, for the Euclidean l_2-norm and l_∞-norm.

Let IS $= (X, \mathscr{F})$ be an independence system. Assume that elements x of the ground set X have been given weights $w(x)$. Let us denote the value $\max_{A\in\mathscr{F}} \min_{x\in X\setminus A} w(x)$ by ext(IS, w). Since it is generally accepted that the minimum over the empty set is equal to infinity, we define ext(IS, w) $= \infty$ in the case when $X \in \mathscr{F}$. We justify this notation as follows. If $\mathscr{F}$ consists of all the subsets of X of cardinality not greater than k, then ext(IS, w) is the $(k + 1)$st smallest value among the numbers $\{w(x) : x \in X\}$. If $k = 0$ (the empty set is the only member of $\mathscr{F}$), ext(IS, w) $= \min_{x\in X} w(x)$. When $k = |X| - 1$, ext(IS, w) $= \max_{x\in X} w(x)$. We shall use the short notation ext(v, c) instead of ext(IS(v), c).

Edmonds and Fulkerson (1970) proved the following dual relation:

$$\max_{B\in\mathscr{B}(\mathrm{IS})} \min_{x\in X\setminus B} w(x) = \min_{C\in\mathscr{C}(\mathrm{IS})} \max_{x\in C} w(x). \tag{3}$$

We call the base, constructed by the greedy algorithm (see Figure 1), a *w-greedy base*. One can easily check that ext(IS, w) $= w(x_j)$, where x_j is the first element discarded (not included into the constructing base) by the greedy algorithm. It is also clear that a w-greedy base B and any circuit $C \subset \{x_1, \dots, x_j\}$ (*w-greedy circuit*) are solutions to the problems on the left- and right-hand sides of (3), respectively.

Our main result is the following theorem which is a generalization of the corresponding theorems in Gurvitch et al. (1988) and Karzanow and Lebedev (1993).

THEOREM 1. *Let $(G, \{\mathscr{F}(v) : v \in V\}, c)$ be a game network with integral costs. There exists a rational-valued price function p on V such that, for $(v, w) \in E$,*

(a) ext(v, c_p) $=$ ext(w, c_p) *if* $c_p(v, w) =$ ext(v, c_p);

0. LIST THE ELEMENTS OF X BY NONDECREASING WEIGHTS, I.E.,
   $X = \{x_1, \dots, x_Q\}$, $w(x_1) \le w(x_2) \le \cdots \le w(x_Q)$.
   SET $B = \varnothing$.
1. FOR $I = 1, \dots, Q$,
   IF $B \cup x_i \in \mathscr{F}$, LET $B := B \cup x_i$.

FIGURE 1. Greedy algorithm.

(b) $\text{ext}(v, c_p) \leq \text{ext}(w, c_p)$ *if* $c_p(v, w) > \text{ext}(v, c_p)$;
(c) $\text{ext}(v, c_p) \geq \text{ext}(w, c_p)$ *if* $c_p(v, w) < \text{ext}(v, c_p)$.
Furthermore, for $v \in V$,
(d) $0 \leq p(v) \leq (n - 1)(2\|c\|_\infty + 1)$ *and its denominator is at most* n.

We defer the proof of Theorem 1 to §5. In the rest of this section we shall show that Theorem 1 can be viewed as an optimality criterion for the mean cost game.

Call a price function p, satisfying all the conditions of Theorem 1, *canonical* for the game network $(G, \{\mathcal{F}(v) : v \in V\}, c)$. From Theorem 1, one can easily derive the following statement.

COROLLARY 1. *Let p be a canonical price function for a game network $(G, \{\mathcal{F}(v) : v \in V\}, c)$, $\nu_1 < \nu_2 < \cdots < \nu_r$ be all the different numbers among $\{\text{ext}(v, c_p) : v \in V\}$. For $i = 1, \ldots, r$, define $V_i = \{v \in V : \text{ext}(v, c_p) = \nu_i\}$. Then, for $v \in V_i$, $E(v, \bigcup_{q=1}^{i-1} V_q)$ is independent and $E(v, \bigcup_{q=1}^{i} V_q)$ is dependent sets of $IS(v)$.*

We call a stationary strategy σ (τ) of Short (Long) a *c-greedy strategy* if each of its components $\sigma(v)$ ($\tau(v)$) is a c-greedy base (c-greedy circuit). Theorem 1 and Corollary 1 imply that a pair (σ_0, τ_0) of greedy strategies with respect to c_p, where p is a canonical price function, is a solution to the mean cost game. Indeed, let $s \in V_i$ be a starting position. It follows from Theorem 1 that all the arcs of any s-path in G_{σ_0,τ_0} have the same reduced cost $\text{ext}(s, c_p) = \nu_i$. Therefore, $f(s; \sigma_0, \tau_0) = \text{ext}(s, c_p)$. By Corollary 1, for any strategy σ of Short, G_{σ,τ_0} has no arc leaving V_i and entering $\bigcup_{j=1}^{i-1} V_j$. Moreover, for any vertex $v \in \bigcup_{j=i}^{r} V_j$, G_{σ,τ_0} has no arc leaving v of cost less than ν_i. Therefore, $f(s; \sigma_0, \tau_0) \leq f(s; \sigma, \tau_0)$. Similarly, one can check that, for any strategy τ of Long, $f(s; \sigma_0, \tau) \leq f(s; \sigma_0, \tau_0)$. Thus $\nu(s) = \text{ext}(s, c_p)$ is the value of s. It also follows from the description of the greedy algorithm that, for any $v \in V$, $|\sigma_0(v)\backslash\tau_0(v)| = 1$.

In light of Theorem 1, we can regard the goal of the mean cost game problem as the exhibition of a canonical price function. In this paper we present an algorithm for finding such a function. In fact, our algorithm is an extension of the algorithms due to Gurvitch et al. (1988) and Karzanow and Lebedev (1993), which were proved to be finite, but exponential in the worse case. We prove that all these algorithms are pseudopolynomial. It should be noted that, for a special case of the problem (even more special than the model in Gurvitch et al. 1988), a pseudopolynomial algorithm was also developed by Zwick and Paterson (1996).

3. Cyclical game duality.

A given *primal* game network $(G, \{\mathcal{F}(v) : v \in V\}, c)$ can be associated with another game network, called its dual. Although cyclical game duality is not used until §5, it gives an insight into the problem.

First, we introduce the notion of a dual independence system. For a given *primal* independence system $\text{IS} = (X, \mathcal{F})$, there is a *dual* independence system $\text{IS}^* = (X, \mathcal{F}^*)$, in which each independent set is the complement of a dependent set of IS, i.e., $\mathcal{F}^* = \{A \subseteq X: \bar{A} \notin \mathcal{F}\}$. Note that each circuit in IS is the complement of a base in IS* and vice versa.

REMARK. This definition of the dual independence system is not generally accepted. Usually the dual independence system is defined to have bases which are the complements of primal bases.

LEMMA 1. *Let $IS = (X, \mathcal{F})$ be an independence system, w be a weight function on X. Then* $\text{ext}(\mathcal{F}, w) = -\text{ext}(\mathcal{F}^*, -w)$.

PROOF. In view of relation (3), we have

$$\begin{aligned}\operatorname{ext}(\mathcal{F}, w) &= \min_{C \in \mathscr{C}(\mathrm{IS})} \max_{x \in C} w(x)\\ &= \min_{B \in \mathscr{B}(\mathrm{IS}^*)} \max_{x \notin B} w(x)\\ &= -\max_{B \in \mathscr{B}(\mathrm{IS}^*)} \min_{x \notin B} -w(x)\\ &= -\operatorname{ext}(\mathcal{F}^*, -w). \quad \square\end{aligned}$$

Let IS $= (X, \mathcal{F})$ be an independence system and A be a proper subset of X. The independence system IS$\backslash A$ (IS *delete* A) is specified by $(X\backslash A, \mathcal{F}\backslash A)$, where

$$\mathcal{F}\backslash A = \{Y \subseteq X\backslash A : Y \in \mathcal{F}\}.$$

For $A \in \mathcal{F}$, the independence system IS$/A$ (IS *contract* A) is specified by $(X\backslash A, \mathcal{F}/A)$, where

$$\mathcal{F}/A = \{Y \subseteq X\backslash A : Y \cup A \in \mathcal{F}\}.$$

LEMMA 2. *Let IS* $= (X, \mathcal{F})$ *be an independence system. If* $A \in \mathcal{F}$, *then* $(IS/A)^* = IS^*\backslash A$. *If* $A \in \mathcal{F}^*$, *then* $(IS\backslash A)^* = IS^*/A$.

PROOF. Let $A \in \mathcal{F}$ and $I \in \mathcal{F}((\mathrm{IS}/A)^*)$. Then $(X\backslash A)\backslash I \notin \mathcal{F}/A$, or, by definition of the contract minor, $((X\backslash A)\backslash I) \cup A \notin \mathcal{F}$. Since $I \subseteq X\backslash A$, $X\backslash I \notin \mathcal{F}$. Hence, by duality, we have $I \in \mathcal{F}^*$ and, consequently, $I \in \mathcal{F}^*\backslash A$.

Now let $A \in \mathcal{F}^*$ and $I \in \mathcal{F}((\mathrm{IS}\backslash A)^*)$. Then $(X\backslash A)\backslash I \notin \mathcal{F}\backslash A$, which implies $X\backslash(I \cup A) \notin \mathcal{F}$. By duality, $I \cup A \in \mathcal{F}^*$, or, $I \in \mathcal{F}^*/A$. $\square$

Next, let us recall the dual rules of playing the cyclical game. At a position v, Short chooses a circuit $C(v) \in \mathscr{C}(v)$. After that, Long forbids any subset of $|C(v)| - 1$ arcs from $C(v)$, or, what is just the same, chooses an arc $(v, w) \in C(v)$ to move the chip from v to w. Using the notion of duality for independence systems, we can reformulate these rules as follows. At a position v, Short forbids a base $B(v) \in \mathscr{B}^*(v)$. After that, Long chooses a not forbidden arc (v, w) to move the chip from v to w.

Notice that the payoff is an antisymmetric function of c, i.e., $f(c, P) = -f(-c, P)$. Therefore, if we substitute c for $-c$ and interchange the players' names, we get the equivalent game playing by the primal rules. So, we call the game network $(G, \{\mathcal{F}^*(v) : v \in V\}, -c)$ a *dual* game network to $(G, \{\mathcal{F}(v) : v \in V\}, c)$. Accordingly, the mean cost cyclical game on the dual game network is called *dual* to the game playing on the primal network. It is evident from this definition that the duality is *reflexive*, i.e., the dual of the dual is the primal.

THEOREM 2. *If a price function p is canonical for a primal game network* $(G, \{\mathcal{F}(v) : v \in V\}, c)$, *then* $-p$ *is canonical for the dual game network* $(G, \{\mathcal{F}^*(v) : v \in V\}, -c)$, *and vice versa.*

PROOF. Immediately follows from the definitions of the dual independence system and the dual game network by elementary reasoning. $\square$

4. A decomposition algorithm. In this section we study the following decomposition problem: given a game network $(G, \{\mathcal{F}(v) : v \in V\}, c)$, find the partition $(X, \bar{X})$ such that

procedure *Decompose*$(G, \{\mathcal{F}(v) : v \in V\}, c, p, X)$
Input: game network $(G, \{\mathcal{F}(v) : v \in V\}, c)$.
Output: subset $X \subseteq V$, price function p.
0. (*Initialization*) For $v \in V$, set $p(v) := 0$.
1. (*Main loop*) Repeat Steps 1.1 and 1.2 until $\epsilon = \infty$.
   1.1. Set $X := V_p^<$ and repeat Step 1.1.1 until applicable.
      1.1.1. Select a vertex $v \in \bar{X}$ such that
      $E_p^{\leq}(v, X) \cup E_p^{<}(v, \bar{X}) \notin \mathcal{F}(v)$ and add it to X.
   1.2. Compute
      $\epsilon_1 := \max\{\epsilon' : \mathrm{ext}(v, c_{p+\epsilon'\chi^X}) \leq 0,\ v \in X\}$,
      $\epsilon_2 := \max\{\epsilon' : \mathrm{ext}(v, c_{p+\epsilon'\chi^X}) \geq 0,\ v \in \bar{X}\}$,
      $\epsilon := \min\{\epsilon_1, \epsilon_2\}$.
      If $\epsilon < \infty$, set $p := p + \epsilon\chi^X$.
2. Return X and p.

end.

FIGURE 2. The decomposition algorithm.

$$\nu^+(v) < 0 \quad \text{for all } v \in X, \quad \text{and} \quad \nu^-(v) \geq 0 \quad \text{for all } v \in \bar{X}. \tag{4}$$

Here $\nu^+(v) \stackrel{\text{def}}{=} \min_\sigma \max_\tau f(v; \sigma, \tau)$ and $\nu^-(v) \stackrel{\text{def}}{=} \max_\tau \min_\sigma f(v; \sigma, \tau)$. It should be noted that, until identity (2) is proved, we cannot assert that $\nu^+(v) = \nu^-(v)$. Therefore, we call $\nu^+(v)$ and $\nu^-(v)$, respectively, the *upper* and the *lower value* of a position v.

To start, let us introduce some notations. For $X, Y \subseteq V$, let $E(X, Y)$ be the set of arcs outgoing X and incoming Y. We note that $E(v) = E(v, V)$. Here and in the sequel we denote singleton sets by omitting set braces.

Let p be a price function. For $v \in V$ and $X \subseteq V$, we use the following notations:

$$E_p^<(X, Y) \stackrel{\text{def}}{=} \{(v, w) \in E(X, Y) : c_p(v, w) < 0\},$$

$$E_p^\leq(X, Y) \stackrel{\text{def}}{=} \{(v, w) \in E(X, Y) : c_p(v, w) \leq 0\},$$

$$E_p^\geq(X, Y) \stackrel{\text{def}}{=} \{(v, w) \in E(X, Y) : c_p(v, w) \geq 0\},$$

$$V_p^< \stackrel{\text{def}}{=} \{v \in V : \mathrm{ext}(v, c_p) < 0\},$$

$$V_p^> \stackrel{\text{def}}{=} \{v \in V : \mathrm{ext}(v, c_p) > 0\}.$$

Let X be a finite set. The *characteristic function* of a subset $A \subseteq X$ is defined by the rule

$$\chi^A(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \bar{A}. \end{cases}$$

Here $\bar{A}$ denotes, as is customary, the complement of A with respect to the whole set X.

Our decomposition algorithm, called *Decompose*, is described in Figure 2. Starting from the zero price function p, it repeats Steps 1.1 and 1.2 until the partition $(X, \bar{X})$ satisfies (4). At Step 1.1, the algorithm computes the set X such that, for any starting position $s \in X$, Short has a forced strategy which, regardless of actions of Long, moves the chip along arcs of nonpositive reduced costs to a position in $V_p^<$. Moreover, for each $v \in X \backslash V_p^<$, Long is able to forbid all arcs from $E^<(v, X)$; therefore, for any $s \in X$, the subgraph $G_p^{\geq}(X)$ contains a path from s to $V_p^<$. (Here $G(Y) \stackrel{\text{def}}{=} (Y, E(Y, Y))$ is the subgraph of G induced by a vertex set $Y \subseteq V$.) Next, at Step 1.2, the algorithm increases prices of all vertices from X by the same amount ϵ. The variable ϵ is to indicate the maximum value which can be chosen, consistent with the following conditions:

$$\text{ext}(v, c_{p+\epsilon\chi}x) \le 0 \quad \text{for all } v \in X, \qquad \text{ext}(v, c_{p+\epsilon\chi}x) \ge 0 \quad \text{for all } v \in \bar{X}.$$

The next properties of *Decompose* immediately follow from the description of this procedure and the above remark.

LEMMA 3. *Throughout the running of Decompose, the following properties hold*:

(a) $\epsilon \ge 1$;

(b) *if at some time during Decompose* $\text{ext}(v, c_p) = 0$ *for some vertex* v, *then this equality continues to hold until the method halts*;

(c) *if, for some vertex* v, $\text{ext}(v, c_p) > 0$, *then* $p(v) = 0$;

(d) *if, for some vertex* v, $\text{ext}(v, c_p) < 0$, *then* $p(v) = \max_{w \in V} p(w)$;

(e) *if, for some integers* a *and* b, $a \le \text{ext}(v, c) \le b$ *for all* $v \in V$, *then* $a \le \text{ext}(v, c_p) \le b$ *for all* $v \in V$;

(f) $\text{ext}(v, c_p) \le 0$ *for all* $v \in X$, *and* $\text{ext}(v, c_p) \ge 0$ *for all* $v \in \bar{X}$;

(g) *for each* $v \in X$, *in* $G_p^{\ge}(X)$ *there exists a path from* v *to* $V_p^<$.

The next simple lemma is crucial for the analysis of the decomposition procedure.

LEMMA 4. *Let* p *be a price function and* $k \ge 1$ *be an integer. If* $p(w) - p(v) > k\|c\|_\infty$, *then in* $G_p^{\ge} \stackrel{\text{def}}{=} (V, E_p^{\ge}(V, V))$ *there is no path from* v *to* w *of length* k.

PROOF. Assume by way of contradiction that $p(w) - p(v) > k\|c\|_\infty$ and there exists a path $P = (v = v_0, v_1, \ldots, v_k = w)$ in $G_p^{\ge}$ from v to w. Then we have

$$\begin{aligned} 0 &\le \sum_{i=0}^{k-1} c_p(v_i, v_{i+1}) \\ &= \sum_{i=0}^{k-1} c(v_i, v_{i+1}) + p(v) - p(w) \\ &\le k\|c\|_\infty + p(v) - p(w). \end{aligned}$$

This is a contradiction. □

Next we prove the correctness of our decomposition algorithm and estimate its complexity.

THEOREM 3. *The decomposition algorithm in Figure 2 runs in* $O(n^2 T_{\mathcal{F}} \|c\|_\infty \|d_G\|^2)$ *time. When it halts, the following properties hold*:

(a) $\text{ext}(v, c_p) \le 0$ *and* $\nu^+(v) < 0$ *for all* $v \in X$;

(b) $\text{ext}(v, c_p) \ge 0$ *and* $\nu^-(v) \ge 0$ *for all* $v \in \bar{X}$;

(c) $0 \le p(v) < (n-1)^2\|c\|_\infty$ *for all* $v \in V$.

PROOF. Suppose that the method terminates. To prove (a) and (b), according to Lemma 4, it suffices to show that the partition $(X, \bar{X})$ satisfies (4). Since after the last iteration $\epsilon_1 = \infty$, it follows that $E_p^{\le}(v, X) \notin \mathcal{F}(v)$ for all $v \in X$. Moreover, for $v \in V_p^<$, $E_p^<(v, X) \notin \mathcal{F}(v)$. Thus, starting from any position $s \in X$, Short is able to move the chip along arcs of nonpositive reduced costs to some position $t \in V_p^<$ and further move it along an arc (t, w) of negative reduced cost. This means that $\nu^+(v) < 0$ for all $v \in X$, which proves (a). Now let us prove (b). By the rule the set X is constructed and since $\epsilon_2 = \infty$, we obtain $E_p^<(v, \bar{X}) \cup E(v, X) \in \mathcal{F}(v)$ for all $v \in \bar{X}$. The last condition entails that, regardless of actions of Short, Long is able to retain the chip inside $\bar{X}$ by moving it along arcs of nonnegative reduced costs. This indicates that, for $v \in \bar{X}$, $\nu^-(v) \geq 0$.

Next we bound the number of iterations of the algorithm. We define the *potential* $\Phi(p)$ of the current price function p by the formula $\Phi(p) \stackrel{\text{def}}{=} \sum_{v \in V} \min\{\rho(v), (n-1)\|c\|_\infty\}$, where $\rho(v) = \max_{w \in V} p(w) - p(v)$ (cf. (d) in Lemma 3). Since $\rho(v) = 0$ for at least one vertex v, $0 \leq \Phi(p) \leq (n-1)^2\|c\|_\infty$. We claim that each iteration, except the last one, increases the potential by at least $\epsilon \geq 1$ (the proof of this claim follows below). Therefore, there can be no more than $O(n^2\|c\|_\infty)$ iterations in total. At the same time, by Lemma 3(d), each but the last iteration increases the maximal price value exactly by ϵ. Since initially, for $p \equiv 0$, $\max_{v \in V} p(v) = \Phi(p) = 0$, during the execution of *Decompose* we have $\max_{v \in V} p(v) \leq \Phi(p) \leq (n-1)^2\|c\|_\infty$, which proves statement (c).

Consider any but the last iteration and let p be the price function just before this iteration. Since the iteration is not the last, $\epsilon < \infty$, $X \neq \varnothing$, and $X \neq V$. As the algorithm proceeds, the values $\rho(v)$ do not decrease. Therefore, we prove the above claim, if we show that the set $\bar{X}$ contains a vertex v such that $\rho(v) + \epsilon \leq (n-1)\|c\|_\infty$. By Lemmas 4 and 3 (items (d) and (g)), $\rho(v) \leq (|X| - 1)\|c\|_\infty$ for all $v \in X$. Consider two possible cases. If $\epsilon = \epsilon_1$, then, for some arc $(v, w) \in E_p^<(X, \bar{X})$,

$$\epsilon = -c_p(v, w) = -c(v, w) - p(v) + p(w) = -c(v, w) + \rho(v) - \rho(w)$$

and

$$\rho(w) + \epsilon = -c(v, w) + \rho(v) \leq \|c\|_\infty + (|X| - 1)\|c\|_\infty$$

$$= |X| \cdot \|c\|_\infty \leq (n-1)\|c\|_\infty.$$

If $\epsilon = \epsilon_2$, then $\epsilon = c_p(v, w)$ for some arc $(v, w) \in E_p^>(\bar{X}, X)$; therefore,

$$\rho(v) + \epsilon = c(v, w) - \rho(w) \leq \|X\| \cdot \|c\|_\infty \leq (n-1)\|c\|_\infty.$$

Next we analyze the complexity of a particular iteration. To implement Step 1.1, we maintain a list L of vertices v such that $v \in \bar{X}$ and $I(v) = E_p^=(v, X) \cup E_p^<(v, \bar{X}) \notin \mathcal{F}(v)$. This allows us to select a vertex v at Step 1.1.1 in $O(1)$ time. We also maintain the sets $I(v)$ for $v \in \bar{X}$. Every time some vertex x leaves $\bar{X}$, it suffices to make at most d' verifications and corrections in the list L and sets $I(v)$, where d' is the indegree of x. Thus, Step 1.1 takes $O(n^2 T_{\mathcal{F}})$ time if implemented properly. Since, for $v \in V$, the problem

$$\max\{\epsilon : \text{ext}(v, c_{p+\epsilon_X x}) \leq 0\}$$

can be solved in $O((d_G(v))^2 T_{\mathcal{F}})$ time, Step 1.2 takes $O(T_{\mathcal{F}}\|d_G\|^2)$ time. Thus the overall time complexity of the algorithm is $O(n^2\|c\|_\infty T_{\mathcal{F}}\|d_G\|^2)$. This completes the proof. □

The complexity of our decomposition procedure depends on the magnitudes of the costs; this may cause some difficulties in estimating the complexity of several consecutive calls of *Decompose*. More precisely, we will call *Decompose* first for some cost function c, and after that for c_p, where p is the price function returned by the first call. In general, $\|c_p\|_\infty$ may be as large as $\|c\|_\infty + \|p\|_\infty$. Therefore, in view of Theorem 3, the time complexity of the second

procedure *Reduce*(G, c, p, p')
Input: graph G, cost function c, and price function p.
Output: price function p'.
0. (*Initialization*) Set $X := \varnothing$.
1. (*Main loop*) Repeat Steps 1.1 and 1.2 until $X = V$.
    1.1. Compute
        $\epsilon_1 := \min_{(v,w)\in E(X,\bar{X})} - c_p(v, w)$, $\epsilon_2 := \min_{(v,w)\in E(\bar{X},X)} c_p(v, w)$,
        $\epsilon_3 := \min_{v\in\bar{X}} p(v)$, $\epsilon := \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$.
        Set $p := p - \epsilon\xi^{\bar{X}}$.
    1.2. Repeat Step 1.2.1 until applicable.
        1.2.1. Select a vertex $v \in \bar{X}$ such that
            $p(v) \leq |X|\|c\|_\infty$ and add it to X.
2. Return $p' = p$.

end.

FIGURE 3. The price reduction algorithm.

call of *Decompose* may be considerably higher than the complexity of the first call. The simple procedure in Figure 3, called *Reduce*, given a digraph G, cost function c, and price function p, computes another price function p' which shares some properties with p, and, what is important, its magnitude depends only on G and c.

THEOREM 4. *Let* $(G, \{\mathcal{F}(v) : v \in V\}, c)$ *be a game network, and* p *be an integral price function. Then in time* $O(nm)$ *one can compute an integral price function* p' *such that the following properties hold*:

(a) $0 \leq p'(v) \leq (n - 1)\|c\|_\infty$ *for every* $v \in V$;

(b) *for any arc* $(v, w) \in E$, *if* $c_p(v, w) \leq 0$, *then* $c_{p'}(v, w) \leq 0$, *and if* $c_p(v, w) \geq 0$, *then* $c_{p'}(v, w) \geq 0$.

PROOF. We show that the output price function p' of the algorithm in Figure 3 obeys conditions (a) and (b).

Let X be the set constructed at Step 1.2, and let $(v, w) \in E(X, \bar{X})$. Then $p(v) \leq (|X| - 1)\|c\|_\infty$ and $p(w) > |X|\|c\|_\infty$. Therefore,

$$c_p(v, w) = c(v, w) + p(v) - p(w)$$

$$< \|c\|_\infty + (|X| - 1)\|c\|_\infty - |X|\|c\|_\infty = 0.$$

The proof that, for $(v, w) \in E(\bar{X}, X)$, $c_p(v, w) > 0$ is similar.

We have just proved that both $E_p^{\geq}(X, \bar{X})$ and $E_p^{\leq}(\bar{X}, X)$ are empty. This implies that $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Since $p(v) > |X|\|c\|_\infty > 0$ for all $v \in \bar{X}$, $\epsilon_3 > 0$ as well. The choice of ϵ ensures that substitution of p for $p' = p - \epsilon\chi^{\bar{X}}$ cannot make a negative arc cost positive and vice versa. That is why p' obeys (b).

To complete the proof, it suffices to notice that each iteration of the algorithm joins to X at least one vertex. Thus, there can be at most n iterations. Each of them takes $O(m)$ time in the worse case. □

5. A mean cost game algorithm.

In this section we give a constructive proof of Theorem 1 by proposing an algorithm which solves the mean cost game.

For a rational number r, let den(r) denote the denominator of r. For a given integer n, let us denote by $\mathbb{Q}_n$ the set of all rational numbers with the denominators at most n. We will say that an interval $[\alpha, \beta)$, where $\alpha, \beta \in \mathbb{Q}_n$, is *small with respect to* n, if (α, β) contains no number from $\mathbb{Q}_n$.

LEMMA 5. *Let* $(G, \{\mathcal{F}(v) : v \in V\}, c)$ *be a game network*, $[\alpha, \beta)$ *be a small interval with* *respect to n. Suppose, for each* $v \in V$, $\alpha \le \nu^-(v) \le \nu^+(v) < \beta$. *Then in time* $O(n^4 T_{\mathcal{F}} \|c\|_\infty \|d_G\|^2)$ *one can find a price function p, such that, for every* $v \in V$,

(a) $\mathrm{ext}(v, c_p) = \alpha$;

(b) $0 \le p(v) \le 2(n-1)\|c\|_\infty$ *and* $den(p(v)) = den(\alpha)$.

PROOF. First, we claim that, for $v \in V$, $\nu^-(v) = \nu^+(v) = \nu(v) = \alpha$. Let us fix $s \in V$. Let σ_1, τ_1 be a pair of the minmax strategies, i.e.,

$$\nu^+(s) = \min_\sigma \max_\tau f(s;\, \sigma, \tau) = f(s;\, \sigma_1, \tau_1) = \mu(s, G_{\sigma_1,\tau_1}, c).$$

Since c is integral, then $\mu(s, G_{\sigma_1,\tau_1}, c) \in \mathbb{Q}_n$. Therefore, $\nu^+(s) = \alpha$. Similarly, one can prove that $\nu^-(s) = \alpha$ as well.

Let $\gamma = \mathrm{den}(\alpha)$ and $c' = \gamma(c - \alpha\chi^E)$. Since $|\alpha| \le \|c\|_\infty$, we have $\|c'\|_\infty \le 2\gamma\|c\|_\infty$. Now we call in turn *Decompose*$(G, \{\mathcal{F}(v) : v \in V\}, c', q', X)$ and *Reduce*(G, c, q', q); according to Theorems 3 and 4, this takes $O(2\gamma n^2 T_{\mathcal{F}} \|c\|_\infty \|d_G\|^2)$ time. It also follows from these theorems that $X = \varnothing$ and

$$\mathrm{ext}(v, c'_q) \ge 0 \quad \text{and} \quad 0 \le q(v) \le 2\gamma(n-1)\|c\|_\infty \quad \text{for all } v \in V.$$

Furthermore,

$$\|c'_q\|_\infty \le \|c'\|_\infty + \|q\|_\infty \le 2\gamma\|c\|_\infty + 2\gamma(n-1)\|c\|_\infty = 2\gamma n\|c\|_\infty.$$

Next we call *Decompose*$(G, \{\mathcal{F}^*(v) : v \in V\}, -c'_q, l, X)$; now this call takes $O(2\gamma n^3 T_{\mathcal{F}} \|c\|_\infty \|d_G\|^2)$ time. By Lemma 1,

$$\mathrm{ext}(\mathcal{F}^*(v), -c'_q) = -\mathrm{ext}(\mathcal{F}(v), c'_q) = -\mathrm{ext}(v, c'_q) \le 0.$$

Moreover, by duality, the value of each position in the game on the network $(G, \{\mathcal{F}^*(v) : v \in V\}, -c'_q)$ is equal to zero. Hence, Theorem 3 entails that $X = V$ and therefore, for all $v \in V$,

$$\mathrm{ext}(v, c'_{q-l}) = -\mathrm{ext}(\mathcal{F}^*(v), -c'_{q-l}) = -\mathrm{ext}(\mathcal{F}^*(v), (-c'_q)_l) = 0. \tag{5}$$

Now we perform *Reduce*$(G, c', q - l, p')$ and set $p = (1/\gamma)p'$. It follows from (5) and Theorem 4 that, for all $v \in V$, $\mathrm{ext}(v, c_p) = \alpha$ and

$$0 \le p(v) = \frac{1}{\gamma} p'(v) \le \frac{1}{\gamma}(n-1)\|c'\|_\infty \le 2(n-1)\|c\|_\infty. \quad \square$$

Given a game network $\mathcal{N} = (G, \{\mathcal{F}(v) : v \in V\}, c)$, our strategy for finding a canonical price function for $\mathcal{N}$ is essentially as follows. Suppose we know that, for every $v \in V$, $\alpha \le \nu^-(v) \le \nu^+(v) < \beta$, where $\alpha, \beta \in \mathbb{Q}_n$. Initially, we can set $\alpha = \min_{v\in V} \mathrm{ext}(v, c)$ and $\beta = \max_{v \in V} \mathrm{ext}(v, c) + 1$. If the interval $[\alpha, \beta)$ is small with respect to n, then we search for a canonical price function p (according to Lemma 5, such a function exists). Otherwise, let $\gamma \in \mathbb{Q}_n$ be the nearest number to the middle point $(\alpha + \beta)/2$ of $[\alpha, \beta)$. Calling *Decompose*$(G, \{\mathcal{F}(v) : v \in V\}, c', p, X)$, where $c' = \mathrm{den}(\gamma)(c - \gamma\chi^E)$, we find the partition $(X, \bar{X})$ such that

$$\alpha \le \nu^-(v) \le \nu^+(v) < \gamma \quad \text{for all } v \in X,$$

procedure $MCG(G, \{\mathcal{F}(v) : v \in V\}, c, \alpha, \beta, p)$
Input: game network $(G, \{\mathcal{F}(v) : v \in V\}, c)$;
$\alpha, \beta \in \mathbb{Q}_n$, $\alpha \leq \nu^-(v) \leq \nu^+(v) < \beta$ for all $v \in V$.
Output: canonical price function p.

1. Find $\gamma \in \mathbb{Q}_n$ nearest to $(\alpha + \beta)/2$.
   If $\gamma = \alpha$, proceed to Step 2. Otherwise, go to Step 3.
2. (*Search for a canonical price function* (*see the proof of Lemma* 5))
   Set $c' := \mathrm{den}(\alpha)(c - \alpha\chi^E)$ and perform
   $Decompose(G, \{\mathcal{F}(v) : v \in V\}, c', q', X)$, $Reduce(G, c', q', q)$,
   $Decompose(G, \{\mathcal{F}^*(v) : v \in V\}, -c'_q, l, X)$, $Reduce(G, c', q - l, p')$.
   Return $p := p'/\mathrm{den}(\alpha)$.
3. (*Decomposition step*)
   Call $Decompose(G, \{\mathcal{F}(v) : v \in V\}, \mathrm{den}(\gamma)(c - \gamma\chi^E), p, X)$.
   If $X \neq \varnothing$, call $MCG(G(X), \{\mathcal{F}(v)\backslash E(v, \bar{X}) : v \in X\}, c, \alpha, \gamma, p')$.
   If $X \neq V$, call $MCG(G(\bar{X}), \{\mathcal{F}(v)/E(v, X) : v \in \bar{X}\}, c, \gamma, \beta, p'')$.
   For $v \in X$, set $p(v) = p'(v) + (n - |X|)(2\|c\|_\infty + 1)$.
   For $v \in \bar{X}$, set $p(v) := p''(v)$.
   Return p.

end.

FIGURE 4. The mean cost game algorithm.

and

$$\gamma \leq \nu^-(v) \leq \nu^+(v) < \beta \quad \text{for all } v \in \bar{X}.$$

Assume for the moment that we know canonical price functions $p' : X \to \mathbb{Q}_n$ for the game network $\mathcal{N}' = (G(X), \{\mathcal{F}(v)\backslash E(v, \bar{X}) : v \in X\}, c)$, and $p'' : \bar{X} \to \mathbb{Q}_n$ for the game network $\mathcal{N}'' = (G(\bar{X}), \{\mathcal{F}(v)/E(v, X) : v \in \bar{X}\}, c)$. (To simplify notations, the restriction of c to any subset $Y \subseteq E$ is also denoted by c.) If either $X = V$ or $X = \varnothing$, then either $p = p'$ or $p = p''$ is a canonical price function for $\mathcal{N}$. Otherwise, we define the price function $p : V \to \mathbb{Q}_n$ by the rule: $p(v) = p''(v)$ if $v \in \bar{X}$, $p(v) = p'(v) + (n - |X|)(2\|c\|_\infty + 1)$ if $v \in X$. To complete the description of the algorithm, we note that the price function p' (resp., p'') can be found by recursively applying the above arguments to the game network $\mathcal{N}'$ and interval $[\alpha, \gamma)$ (resp., $\mathcal{N}''$ and $[\gamma, \beta)$). Figure 4 provides a detailed description of this algorithm.

THEOREM 5. *The algorithm in Figure* 4 *is correct and runs in* $O(n^3\|c\|_\infty(n + \log(\|c\|_\infty))T_{\mathcal{F}}\|d_G\|^2)$ *time.*

PROOF. To prove the correctness, by induction on the size of the interval $[\alpha, \beta)$, we show that the price function p, constructed by the method, is canonical for $\mathcal{N}$, i.e., it satisfies conditions (a)–(d) of Theorem 1. If the interval $[\alpha, \beta)$ is small with respect to n, then, because of Lemma 5, p is canonical for $\mathcal{N}$. Further, if either $p = p'$ or $p = p''$, then p is canonical by the inductive hypothesis. Now let us assume that X is a proper subset of V. Since, by induction, p' and p'' are canonical, we have

$$0 \leq p'(v) \leq (|X| - 1)(2\|c\|_\infty + 1) \quad \text{for all } v \in X,$$

$$0 \leq p''(v) \leq (n - |X| - 1)(2\|c\|_\infty + 1) \quad \text{for all } v \in \bar{X};$$

therefore, $0 \leq p(v) \leq (n - 1)(2\|c\|_\infty + 1)$ for all $v \in V$, and condition (d) is valid.

By the definition of p, conditions (a)–(d) may be violated only for arcs from $E(X, \bar{X}) \cup E(\bar{X}, X)$. In view of Theorem 3, for all $v \in X$, $w \in \bar{X}$, the next relation holds:

$$(6) \qquad \nu(v) = \operatorname{ext}(\mathscr{F}(v)\backslash E(v, \bar{X}), c_{p'}) < \nu(w) = \operatorname{ext}(\mathscr{F}(w)/E(w, X), c_{p''}).$$

For all $(v, w) \in E(X, \bar{X})$, we have

$$\begin{aligned} c_p(v, w) &= c(v, w) + p(v) - p(w) \\ &\geq -\|c\|_\infty + (n - |X|)(2\|c\|_\infty + 1) - (n - |X| - 1)(2\|c\|_\infty + 1) \\ &= \|c\|_\infty + 1. \end{aligned}$$

Similarly, it is proved that $c_p(v, w) < -\|c\|_\infty - 1$ for all $(v, w) \in E(\bar{X}, X)$. Consequently, for all $v \in X$, $w \in \bar{X}$, $\operatorname{ext}(v, c_p) = \operatorname{ext}(\mathscr{F}(v)\backslash E(v, \bar{X}), c_{p'})$ and $\operatorname{ext}(w, c_p) = \operatorname{ext}(\mathscr{F}(w)/E(w, X), c_{p''})$. This fact along with (6) implies the validity of conditions (a)–(d).

We have not yet discussed the complexity of the algorithm. Each call of *Decompose* either partitions the vertex set (type 1) or almost halves the interval $[\alpha, \beta)$ (type 2). Clearly, the number of calls of the first type is at most n. If $\beta - \alpha \leq 1/n^2$, then the interval $[\alpha, \beta)$ is small with respect to n. Therefore, $O(\log(n\|c\|_\infty))$ is a bound on the number of calls of the second type. Since we always call *Decompose* with respect to the original cost function c, by Theorem 3, each such a call takes $O(n^2\|c\|_\infty T_{\mathscr{F}}\|d_G\|^2)$ time. Thus, Step 3 takes $O(n^2(n + \log\|c\|_\infty)\|c\|_\infty T_{\mathscr{F}}\|d_G\|^2)$ time in total.

Now let us estimate the complexity of Step 2. Let it be carried out k $(k \leq n)$ times, respectively, for subgraphs $G(V_i)$, $i = 1, \ldots, k$. By Lemma 5, Step 2 can be carried out in $O(|V_i|^4\|c\|_\infty T_{\mathscr{F}}\|d_{G_i}\|^2)$ time. Since $\sum_{i=1}^k |V_i|^4 \sum_{v \in V_i} \|d_{G_i}\|^2 \leq n^4\|d_G\|^2$, the overall complexity of Step 2 is $O(n^4\|c\|_\infty T_{\mathscr{F}}\|d_G\|^2)$.

Therefore, the overall complexity of the algorithm is $O(n^3(n + \log\|c\|_\infty)\|c\|_\infty T_{\mathscr{F}}\|d_G\|^2)$. $\square$

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