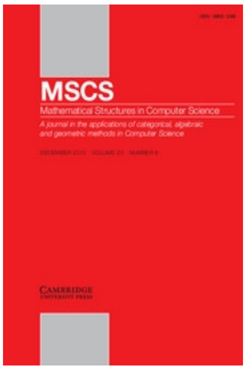

A categorical generalization of Scott domains

JI[Rcaron]Í ADÁMEK

Mathematical Structures in Computer Science / Volume 7 / Issue 05 / October 1997, pp 419 - 443
DOI: 10.1017/S0960129597002351, Published online: 08 September 2000

Link to this article: http://journals.cambridge.org/abstract_S0960129597002351

How to cite this article:
JI[Rcaron]Í ADÁMEK (1997). A categorical generalization of Scott domains. Mathematical Structures in Computer Science, 7, pp 419-443 doi:10.1017/S0960129597002351

Math. Struct. in Comp. Science (1997), *vol.* 7, *pp.* 419–443. Printed in the United Kingdom

A categorical generalization of Scott domains

JIŘÍ ADÁMEK†

Technical University of Braunschweig, P.O. Box 3329, 38023 Braunschweig, Germany
Email: adamek@iti.cs.tu-bs.de

Received 30 October 1995; revised 3 October 1996

Algebraic CPOs naturally generalize to finitely accessible categories, and Scott domains (*i.e.*, consistently complete algebraic CPOs) then correspond to what we call Scott-complete categories: finitely accessible, consistently (co-)complete categories. We prove that the category SCC of all Scott-complete categories and all continuous functors is cartesian closed and provides fixed points for a large collection of endofunctors. Thus, SCC can serve as a basis for semantics of computer languages.

1. Introduction

In categorical logic an important idea is to generalize the classical ordering of propositions

$$x \leqslant y \text{ iff } y \text{ can be proved from } x$$

by giving individual names to proofs, and writing

$$f : x \to y \text{ iff } f \text{ is a proof of } y \text{ from } x.$$

Thus, one uses categories instead of posets. In the present paper we take the first steps in an analogous generalization of posets to categories in Domain Theory. Thus, the ordering of computation stages used there

$$x \sqsubseteq y \text{ iff a further computation leads from } x \text{ to } y$$

is substituted by giving individual names to computations, and writing

$$f : x \to y \text{ iff } f \text{ is a computation leading from } x \text{ to } y.$$

This forms a category in a natural sense, and the concept of Scott domain naturally generalizes to what we call Scott-complete categories. We show that they form a cartesian closed category: the proof of algebraicity of function spaces is based on 'step functors', which generalize the well known step functions $\langle k; l \rangle$ (sending x to l if $k \sqsubseteq x$, otherwise to $\perp$) by observing that $\langle k; l \rangle$ is the composite of $\hom(k, -)$ with the adjoint of $\hom(l, -)$ (see Lemma 1 below). We also show how the fixed-point theory extends to the present generality. In spite of the results achieved, we stress that only the first steps in the theory

† The grant 201/96/0119 of the Grant Agency of Czech Republic is gratefully acknowledged.

have been taken so far, and that the paper does not present any examples not covered by the classical Domain Theory, nor applications of the richer structure. In future work we expect that it will be shown how categorical concepts bring a new and important view to various concepts of Domain Theory. For example, J. Velebil has proved that approximable relations generalize to flat profunctors (private communication). We also expect to show that certain constructions of power domains are best performed in the realm of Scott-complete categories.

Recall that a Scott domain is a partially ordered set that is

(a) algebraic, *i.e.*, it has directed joins and bottom and every element is a directed join of finite (=compact) elements, and
(b) consistently complete, *i.e.*, every nonempty set with an upper bound has a join.

The concept of a finite element in a poset generalizes immediately to that of a *finitely presentable* object of a category $\mathscr{K}$: it is an object A such that $\hom(A, -): \mathscr{K} \to \mathrm{Set}$ preserves directed colimits. That is, if $(K_i \to K)_{i \in I}$ is a directed colimit in $\mathscr{K}$, then every morphism $f: A \to K$ has an essentially unique factorization through one of the morphisms k_i (more precisely: there exists $i \in I$ such that $f = k_i \circ f'$, and if $f = k_i \circ f' = k_i \circ f''$, then f' and f'' are merged by one of the connecting morphisms $K_i \to K_j$, $i \leqslant j$, of the given diagram). And the concept of an algebraic CPO generalizes to that of a *finitely accessible category*, as introduced by Lair (1981) and Makkai and Paré (1989), *i.e.*, a category $\mathscr{K}$ such that

(a) $\mathscr{K}$ has directed colimits, and
(b) $\mathscr{K}$ has a set $\mathscr{A}$ of finitely presentable objects such that every object of $\mathscr{K}$ is a directed colimit of objects in $\mathscr{A}$.

It is well known from domain theory that algebraic CPOs have the fundamental disadvantage that they do not form a cartesian closed category: if A and B are algebraic CPOs, the poset $[A \to B]$ of all continuous maps from A to B, ordered pointwise, need not be algebraic. Several full subcategories of the category of algebraic CPOs and continuous maps have thus been considered (see, for example, Abramsky and Jung (1994)), and one of the most commonly used is that of Scott-domains (Scott 1982); this has a direct generalization to finitely accessible, consistently cocomplete categories with initial objects, as given by the following definition.

Definition 1. A category is called *Scott-complete* if it is finitely accessible and every diagram with a cocone has a colimit.

We denote by SCC the category of all Scott-complete categories and functors that are *continuous*, that is, preserve directed colimits. (Let us remark here that some category theorists prefer working with filtered rather than directed colimits. However, as proved, for example, in Adámek and Rosický (1994), a category has filtered colimits iff it has directed ones, and a functor preserves filtered colimits iff it preserves directed ones.) For Scott-complete categories $\mathscr{K}$ and $\mathscr{L}$ we prove that the category $[\mathscr{K} \to \mathscr{L}]$ of all continuous functors, a full subcategory of $\mathscr{L}^{\mathscr{K}}$, is also Scott-complete. Consequently, the category SCC is cartesian closed.

We also introduce a generalization of the concept of Scott's embedding-projection (Scott 1972). Although the concept is much more technical than in the case of partial orders, the idea remains the same: an embedding-projection pair of continuous functors is an adjoint pair $E \dashv P$ such that $PE = \mathrm{id}$ and the unit of the adjunction is the identity. What also remains the same is the close relationship between directed limits and directed colimits in the category

$$\mathrm{SCC}^{\mathrm{e}}$$

of all Scott-complete categories and all embedding-projection adjunctions. As a consequence, we obtain a strong fixpoint theorem for endofunctors of $\mathrm{SCC}^{\mathrm{e}}$ that are locally continuous – well, more precisely: for locally continuous 2-functors from $\mathrm{SCC}^{\mathrm{e}}$ into itself. Recall that $\mathrm{SCC}^{\mathrm{e}}$ has the structure of a 2-category because for arbitrary two objects $\mathscr{K}$, $\mathscr{L}$ of $\mathrm{SCC}^{\mathrm{e}}$ we have an obvious structure of a category on $\hom(\mathscr{K}, \mathscr{L})$ whose morphisms are natural transformations. Now a 2-functor from $\mathrm{SCC}^{\mathrm{e}}$ maps not only objects (Scott-complete categories) to objects, and morphisms (continuous functors) to morphisms, but also maps natural transformations between those morphisms to natural transformations. Here one can see another step in a direction that the (by now 'classical') theory of recursively defined domains as fixed points of functors has taken. In the category $\mathrm{CPO}^{\mathrm{e}}$, a recursive definition $A ::= T(A)$ is interpreted as follows: T is an object part of an endofunctor of $\mathrm{CPO}^{\mathrm{e}}$, and we take it for granted that there is a corresponding morphism-part turning T into a locally continuous functor $T\colon \mathrm{CPO}^{\mathrm{e}} \to \mathrm{CPO}^{\mathrm{e}}$. Then T has a least fixed point, which is our interpretation of a solution to the original recursive equation. Now in $\mathrm{SCC}^{\mathrm{e}}$ we start, again, with a recursive definition $A ::= T(A)$ and interpret it as an object-part of an endofunctor, but here we also have to consider the morphism-part and the natural-transformation-part of T. If the resulting 2-functor $T\colon \mathrm{SCC}^{\mathrm{e}} \to \mathrm{SCC}^{\mathrm{e}}$ is locally continuous, *i.e.*, the derived functors $\hom(\mathscr{K}, \mathscr{L}) \to \hom(T(\mathscr{K}), T(\mathscr{L}))$ are continuous for all pairs $\mathscr{K}$, $\mathscr{L}$ of Scott-complete categories, then T has a canonical solution, *i.e.*, least-and-largest fixed point, of the given recursive equation. Let us remark that, although in the category $\mathrm{SCC}^{\mathrm{e}}$ we do not consider functors (as morphisms) but only functors up to natural isomorphism, this does not diminish the precision with which fixed points serve as solutions of recursive equations; in fact, all the usual constructions of domains (product, function-space, and so on) do *not* specify endofunctors, but only endofunctors up to natural isomorphism.

We finally characterize sketches that sketch precisely the Scott-complete categories and show which first-order logical theories precisely axiomatize all Scott-complete categories. From that characterization the cartesian closedness of the category SCC can be directly derived from Theorem 7.1.3. of Aageron (1991).

2. Scott-complete categories

Remark 1.

(i) Recall from Makkai and Paré (1989) that a category $\mathscr{K}$ is called *accessible* provided that there exists a regular cardinal λ such that

(a) $\mathscr{K}$ has λ-directed colimits (*i.e.*, colimits over all λ-directed posets)

and

(b) $\mathscr{K}$ has a set $\mathscr{A}$ of λ-presentable objects such that every object of $\mathscr{K}$ is a λ-directed colimit of A-objects.

(ii) A category is called *consistently complete* if each nonempty diagram with a cone has a limit (dually: consistently cocomplete).

Notation 1. We denote by $\mathscr{K}_\lambda$ a full subcategory of $\mathscr{K}$ representing all λ-presentable objects (*i.e.*, such that every λ-presentable object of $\mathscr{K}$ is isomorphic to precisely one object of $\mathscr{K}_\lambda$). As proved in Makkai and Paré (1989), $\mathscr{K}_\lambda$ is a small category.

When $\lambda = \omega$ we call $\mathscr{K}$ a *finitely accessible* category and use

$$\mathscr{K}_{\mathrm{fp}}$$

rather than $\mathscr{K}_\omega$. For every object K of a finitely accessible category, the comma-category $\mathscr{K}_{\mathrm{fp}} \downarrow K$ of all arrows with a domain in $\mathscr{K}_{\mathrm{fp}}$ and the codomain K is filtered. Thus, the canonical diagram $D\colon \mathscr{K}_{\mathrm{fp}} \downarrow K \to \mathscr{K}$, assigning to each arrow its domain, is a filtered diagram (equivalently: a diagram with a directed cofinal subdiagram).

Theorem 1. An accessible category is consistently complete iff it is consistently cocomplete.

Proof. Put $\mathscr{A} = \mathscr{K}_\lambda$ in the above notation.

(1) Assume that $\mathscr{K}$ is consistently complete. Let $D\colon \mathscr{D} \to \mathscr{K}$ be a nonempty diagram with a cocone in $\mathscr{K}$. As proved in Makkai and Paré (1989), there exists a regular cardinal λ such that

(i) $\mathscr{D}$ has less than λ morphisms,
(ii) every object Dd is λ-presentable in $\mathscr{K}$
and
(iii) $\mathscr{K}$ has properties (a), (b) of the above Remark 1; $\mathscr{K}$ is λ-accessible for short.

We prove first that every cocone of D factorizes through a cocone with a codomain in $\mathscr{A}$. In fact, let $(Dd \xrightarrow{c_d} C)$ be a cocone of D. By (b) in Remark 1, we have a λ-directed colimit $(C_i \xrightarrow{q_i} C)_{i\in I}$ with every C_i in $\mathscr{A}$. For each $d \in \mathscr{D}^{\mathrm{obj}}$, since Dd is λ-presentable, there exists $i \in I$ such that c_d factorizes through q_i (say, $c_d = q_i c_d^+$), and moreover, i can be chosen independent of d, since I is λ-directed and the number of all d's is smaller than λ. Analogously, for each $\delta\colon d \to d'$ in $\mathscr{D}$, since $c_d = c_{d'} \cdot D\delta$ and Dd is λ-presentable, there exists $j \geqslant i$ such that the connecting morphism $C_{ij}\colon C_i \to C_j$ fulfils $C_{ij} \cdot c_d^+ = C_{ij} \cdot c_{d'} \cdot D\delta$. Again, j can be chosen independent of δ, since the number of all δ's is smaller than λ. Put $c_d^* = C_{ij} \cdot c_d^+$. Then $(Dd \xrightarrow{c_d^*} C_j)$ is a cocone of D through which the original cocone factorizes:

$$c_d = q_i \cdot c_d^+ = q_j \cdot C_{ij} \cdot c_d^+ = q_j \cdot c_d^*.$$

Denote by $\mathscr{L}$ the category of all cocones of D and their natural transformations. We need to prove that D has a colimit, that is, that $\mathscr{L}$ has an initial object. We have just observed that the (small) set of all cocones with a codomain in $\mathscr{A}$ is weakly initial in $\mathscr{L}$; moreover, this set is nonempty because, by assumption, D has a cocone. By Freyd's Adjoint Functor Theorem, it is sufficient to show that $\mathscr{L}$ has nonempty limits. In fact, for each nonempty diagram $D^*\colon \mathscr{D}^* \to \mathscr{L}$, we observe that the diagram $UD^*\colon \mathscr{D}^* \to \mathscr{K}$, where $U\colon \mathscr{L} \to \mathscr{K}$ is the codomain-functor, has a limit in $\mathscr{K}$: we know that $\mathscr{D}^*$ is nonempty and that UD^* has a cone obtained by choosing an object $d \in \mathscr{D}^{\text{obj}}$ and forming the d-components of cocones. Let $(L \xrightarrow{p_{d^*}} UD^*d^*)$ be a limit of UD^*in $\mathscr{K}$. For each $d \in \mathscr{D}^{\text{obj}}$ we have a cone of UD^* formed, for each $d^* \in (\mathscr{D}^*)^{\text{obj}}$, by all d-components of the cocone D^*d^*. Let $r_d\colon Dd \to L$ be the unique morphism factorizing that cone, then it is easy to verify that $(Dd \xrightarrow{r_d} L)$ is a cocone of D, and that this object of $\mathscr{L}$ together with the morphisms p_{d^*} for $d^* \in (\mathscr{D}^*)^{\text{obj}}$ form a limit of D^* in $\mathscr{L}$. Consequently, $\mathscr{L}$ has an initial object, *i.e.*, a colimit of D.

(2) Let $\mathscr{K}$ be consistently complete. The Yoneda embedding $E\colon \mathscr{K} \to \mathrm{Set}^{\mathscr{A}^{\mathrm{op}}}$ with $EK = \hom(-, K)/\mathscr{A}^{\mathrm{op}}$ is full and faithful. In fact, this is equivalent to saying that $\mathscr{A}$ is a dense category, and this follows from the accessibility of $\mathscr{K}$, see Makkai and Paré (1989).

Let $D\colon \mathscr{D} \to \mathscr{K}$ be a nonempty diagram with a cone. Since $\mathscr{K}$ is λ-accessible, D has a cone

$$(C \to Dd)_{d \in \mathscr{D}^{\text{obj}}} \text{ with } C \in \mathscr{A}.$$

To prove that D has a limit in $\mathscr{K}$, we first form a limit of ED in $\mathrm{Set}^{\mathscr{A}^{\mathrm{op}}}$; say,

$$(F \xrightarrow{f_d} EDd)_{d \in \mathscr{D}^{\text{obj}}}.$$

The functor $F\colon \mathscr{A}^{\mathrm{op}} \to \mathrm{Set}$ can be described as follows: for every object X, FX is the set of all cones of D with the domain X (and the corresponding component of f_d is the d-component of the cones); in particular,

$$FC \neq \varnothing\,.$$

Consequently, the category $\mathscr{P}_f$ of points of F (whose objects are pairs (A, a) with $A \in \mathscr{A}$ and $a \in FA$, and morphisms $f\colon (A, a) \to (B, b)$ are $\mathscr{K}$-morphisms $f\colon A \to B$ with $Ff(b) = a$) is nonempty. The diagram $P_f\colon \mathscr{P}_f \to \mathscr{K}$ given by $(A, a) \mapsto A$ has a cocone: in fact, every object d of the (nonempty) category $\mathscr{D}$ defines a cocone whose (A, a)-component is $a_d\colon A \to Dd$, the d-component of the cone a. Consequently, P_f has a colimit in $\mathscr{K}$, say,

$$((A, a) \xrightarrow{w_{A,a}} C^*) \qquad \text{for all } (A, a) \in \mathscr{P}_f^{\text{obj}}.$$

For each object d of $\mathscr{D}$ let $c_d^*\colon C^* \to Dd$ be the unique factorization of the above cocone, that is,

$$w_{A,a} \cdot c_d^* = a_d \qquad \text{for all } (A, a) \in \mathscr{P}_F^{\text{obj}}, d \in \mathscr{D}^{\text{obj}}.$$

It is easy to verify that $(C^* \xrightarrow{c_d^*} Dd)$ is a cone of D. To see that this is a limit of D, let $(A_0, a_0) \in \mathscr{P}_F^{\text{obj}}$ be another cone. It factorizes through the cone (c_d^*) via w_{A_0,a_0}. It remains to verify the uniqueness of factorization: given $h, h'\colon A \to C^*$ with $c_d^* h = c_d^* h' (= a_d)$ for any d, we will show that $h = h'$. The equation $c_d^* h = c_d^* h'$ guarantees that a coequalizer of h and h' exists in $\mathscr{K}$; say, $k\colon C^* \to B$. In order to prove $h = h'$, we will show that k is a split monomorphism: let $b_d\colon B \to Dd$ be the unique morphism with

$$c_d^* = b_d \cdot k \qquad \text{for } d \in \mathscr{D}^{\text{obj}},$$

and let $b = (b_d)$ be the corresponding cone of D, then we will show that the object (B, b) of $\mathscr{P}_F$ fulfils

$$w_{B,b} \cdot k = \mathrm{id}_{C^*}.$$

It is sufficient to prove that for every object (A, a) of $\mathscr{P}_F$ we have $w_{B,b} \cdot k \cdot w_{A,a} = w_{A,a}$. In fact, we have a morphism

$$k \cdot w_{A,a} \colon (A, a) \to (B, b)$$

of $\mathscr{P}_F$, since for each $d \in \mathscr{D}^{\text{obj}}$

$$\begin{aligned} F(k, w_{A,a})(b_d) &= b_d \; k \; w_{A.a} \\ &= c_d^* \; w_{A,a} \\ &= a_d, \end{aligned}$$

and therefore, the required equality follows from the compatibility of the above limit cone of P_F. □

Corollary 2. A category is Scott-complete iff it is finitely accessible, consistently complete, and has an initial object.

Examples 1.

(1) A poset, considered as a category, is Scott-complete iff it is a Scott domain.

(2) Every locally presentable category in the sense of Gabriel and Ulmer (1971), *i.e.*, every complete, finitely accessible category, is Scott-complete, and has a terminal object. Conversely, every Scott-complete category with a terminal object is locally finitely presentable. Thus, the relationship between locally finitely presentable categories and Scott-complete categories is analogous to that between continuous lattices and Scott-domains.

(3) Scott-complete categories are precisely the free completions under directed colimits of small, finitely consistently cocomplete categories. (This is quite analogous to the fact that Scott-domains are precisely the directed-join completions of conditional semilattices). More precisely:

   (i) Let $\mathscr{K}$ be Scott-complete. Then $\mathscr{K}_{\text{fp}}$ is a small category in which every finite diagram with a cocone has a colimit. In fact, finite colimits of finitely presentable objects are finitely presentable. As proved in Makkai and Paré (1989), $\mathscr{K}$ is a free completion of $\mathscr{K}_{\text{fp}}$ under directed colimits. That is, every functor $F \colon \mathscr{K}_{\text{fp}} \to \mathscr{L}$, where $\mathscr{L}$ has directed colimits, has a continuous extension to $\mathscr{K}$, which is unique up-to natural isomorphism.

   (ii) Conversely, given a small category $\mathscr{A}$ with colimits of all finite diagrams with a cocone, let $\mathscr{K}$ be a free completion of $\mathscr{A}$ with respect to directed colimits. ($\mathscr{K}$ is described in Part 2.C of Adámek and Rosický (1994).) Then $\mathscr{K}$ has both directed colimits and colimits of finite consistent diagrams – thus, $\mathscr{K}$ is consistently cocomplete. Since $\mathscr{A}$ has an initial object, so does $\mathscr{K}$.

3. The cartesian closed category of Scott-complete categories

Definition 2. We define the category SCC to have as objects all Scott-complete categories and as morphisms all continuous (*i.e.*, directed colimits preserving) functors.

Observation 1. There are, essentially, no set-theoretical problems connected with the above definition: since, by Example 1(3), Scott-complete categories are precisely the free completions of small, consistently finitely cocomplete categories, we conclude that

(a) SCC-objects can be coded (up to isomorphism of categories) by small categories; thus, $\mathrm{SCC}^{\mathrm{obj}}$ is a class

(b) SCC-morphisms from $\mathscr{K}$ to $\mathscr{L}$ are fully determined by their restriction to $\mathscr{K}_{\mathrm{fp}}$, thus $\hom_{\mathrm{SCC}}(\mathscr{K},\mathscr{L})$ is a (small) set.

Notation 2. For two Scott-complete categories $\mathscr{K}$ and $\mathscr{L}$ we denote by $[\mathscr{L} \to \mathscr{K}]$ the category of all continuous (*i.e.*, directed-colimits preserving) functors from $\mathscr{L}$ to $\mathscr{K}$ and all natural transformations. Observe that this is equivalent to the category of *all* functors from the small category $\mathscr{L}_{\mathrm{fp}}$ to $\mathscr{K}$, that is,

$$[\mathscr{L} \to \mathscr{K}] \cong \mathscr{K}^{\mathscr{L}_{\mathrm{fp}}}.$$

In fact, by Example 1(3) above, each functor F in $\mathscr{K}_{\mathrm{fp}}$ has an essentially unique extension to a functor F^* in $[\mathscr{L} \to \mathscr{K}]$, and then $F \mapsto F^*$ is an equivalence of the above two categories. We want to prove that $[\mathscr{L} \to \mathscr{K}]$ is a Scott-complete category. This is analogous to the proof that a function-space of two Scott domains is a Scott domain. Whereas the latter proof is based on step functions, our proof will use the following 'step' functors.

Lemma 1. Let $\mathscr{K}$ and $\mathscr{L}$ be Scott-complete categories. Given finitely presentable objects K in $\mathscr{K}$ and L in $\mathscr{L}$, the functor

$$P_{L,K} = F_K \cdot \hom(L,-)\colon \mathscr{L} \to \mathscr{K}$$

where $F_K\colon \mathrm{Set} \to \mathscr{K}$ is a left adjoint of $\hom(K,-)$, is a finitely presentable object of $[\mathscr{L} \to \mathscr{K}]$.

Proof.

(1) $P_{L,K}$ is a continuous functor. In fact, we first observe that the category $\mathscr{K}$ has co-powers $\coprod_M K$ because the discrete diagram of M copies of K has a cocone (with codomain K if $M \neq \varnothing$ and codomain $\perp$ if $M = \varnothing$). Thus, $\hom(K,-)$ has a left adjoint F_K given by $F_K M = \coprod_M K$. Now F_K preserves colimits, and, since L is finitely presentable, $\hom(L,-)$ preserves directed colimits – thus, $P_{L,K}$ is continuous.

(2) The following type of Yoneda lemma holds for all functors Q in $[\mathscr{L} \to \mathscr{K}]$: there is a bijective correspondence between morphisms from $P_{L,K}$ to Q and maps $f\colon K \to QL$, that is,

$$\hom_{\mathscr{K}}(K, QL) \cong \hom_{\mathrm{SCC}}(P_{L,K}, Q).$$

In fact, each $f\colon K \to QL$ induces a natural transformation $f^*\colon P_{L,K} \to Q$ whose map $f^*_X\colon \coprod_{\hom(L,X)} K \to QX$ has the h-component given by

$$(*) \qquad (f^*_X)_h = Qh \cdot f\colon K \to QX \text{ for each } X \in \mathscr{L}^{\mathrm{obj}}, h \in \hom(L,X).$$

Conversely, given any natural transformation $t\colon P_{L,K} \to Q$, there exists a unique $f\colon K \to QL$ with $t = f^*$, viz, the id_L-component of $t_L\colon \coprod_{\hom(L,L)} K \to QL$.

(3) Each $P_{L,K}$ is finitely presentable in the category $[\mathscr{L} \to \mathscr{K}]$. In fact, let D be a directed diagram with a colimit $(R_i \xrightarrow{r_i} R)_{i\in I}$ in $[\mathscr{L} \to \mathscr{K}]$. For each morphism $t\colon P_{L,K} \to R$

we have $t = f^*$ where $f: K \to RL$. Since K is finitely presentable, and

$$(R_i L \xrightarrow{(r_i)_L} RL)_{i \in I}$$

is a directed colimit in $\mathscr{K}$ (recall that $[\mathscr{L} \to \mathscr{K}] \cong \mathscr{K}^{\mathscr{L}_{\mathrm{fp}}}$, thus, directed colimits are formed object-wise in $[\mathscr{L} \to \mathscr{K}]$), we see that f factors essentially uniquely through some $(r_i)_L$. Now, $f = (r_i)_L \cdot g$ is equivalent to $f^* = r_i \cdot g^*$, and thus $t = f^*$ factors essentially uniquely through r_i. □

Theorem 3. A finite product of Scott-complete categories is Scott-complete, and for Scott-complete categories $\mathscr{L}$ and $\mathscr{K}$ the functor category $[\mathscr{L} \to \mathscr{K}]$ is Scott-complete. Thus, SCC is a cartesian closed category.

Proof. The statement about finite products is trivial because in a finite product of categories

(a) colimits are computed coordinate-wise, and
(b) finitely presentable objects are just those with finitely presentable coordinates.

Let $\mathscr{L}$ and $\mathscr{K}$ be Scott-complete categories. Since colimits in $[\mathscr{L} \to \mathscr{K}] \cong \mathscr{K}^{\mathscr{L}_{\mathrm{fp}}}$ are computed object-wise, the category $[\mathscr{L} \to \mathscr{K}]$ has directed colimits and is consistently cocomplete. It remains to find a set $\mathscr{A}$ of finitely presentable objects of $[\mathscr{L} \to \mathscr{K}]$ such that every continuous functor is a directed colimit of functors in $\mathscr{A}$. Let $\mathscr{A}$ be the closure of the set of all step-functors $P_{L,K}$ with K in $\mathscr{K}_{\mathrm{fp}}$ and L in $\mathscr{L}_{\mathrm{fp}}$ under existing finite colimits in $[\mathscr{K} \to \mathscr{L}]$.

Because of the previous lemma, each object of $\mathscr{A}$ is finitely presentable in $[\mathscr{L} \to \mathscr{K}]$. Thus, to conclude the proof, we only have to show that every object R of $[\mathscr{L} \to \mathscr{K}]$ is a directed (or, equivalently, filtered) colimit of a diagram in $\mathscr{A}$. We use the canonical diagram D whose scheme is the comma-category $\mathscr{A} \downarrow R$ (consisting of all $P \xrightarrow{p} R$ with $P \in \mathscr{A}$) and which is given by $D(P \xrightarrow{p} R) = P$. This is a filtered diagram, that is, $\mathscr{A} \downarrow R$ is a filtered category, which follows immediately from the fact that $\mathscr{A}$ is closed under existing finite colimits: given a finite subcategory $\mathscr{B}$ of $\mathscr{A} \downarrow R$, we have that $D(\mathscr{B})$ has a cocone (with codomain R) in $[\mathscr{L} \to \mathscr{K}]$, thus $P = \operatorname{colim} \mathscr{B}$ exists and the canonical map $P \xrightarrow{p} R$ induced by this colimit yields an object of $\mathscr{A} \downarrow R$, giving a cocone to $\mathscr{B}$ in $\mathscr{A} \downarrow R$. It remains to prove that $R = \operatorname{colim} D$ – more precisely, that the canonical cocone $p: D(P \xrightarrow{p} R) \to R$ is a colimit cocone in $[\mathscr{L} \to \mathscr{K}]$. Let $\overline{p}: P \to \overline{R}$ be another cocone. That is, for each morphism $p: P \to R$, a morphism $\overline{p}: P \to \overline{R}$ is given with

$$\overline{p}t = \overline{pt} \qquad \text{for all } t: P' \to P \text{ in } \mathscr{A}. \tag{1}$$

We are to prove that there exists a unique $r: R \to \overline{R}$ such that $\overline{p} = r \cdot p$ for all p. Let us first turn to the uniqueness: it is sufficient to show that r_L is uniquely determined for each $L \in \mathscr{L}_{\mathrm{fp}}$ (since $[\mathscr{L} \to \mathscr{K}]$ is equivalent to $\mathscr{K}^{\mathscr{L}_{\mathrm{fp}}}$). Since $\mathscr{K}$ is finitely accessible, RL is a canonical colimit of the diagram $D_{RL}: \mathscr{K}_{\mathrm{fp}} \downarrow RL \to \mathscr{K}$ assigning to each $K \xrightarrow{k} RL$, $K \in \mathscr{K}_{\mathrm{fp}}$, the value K. Thus, it is sufficient to show how $r_L \cdot k: K \to R\overline{L}$ is determined. Consider the morphism $k^*: P_{L,K} \to R$ of the Yoneda lemma $(*)$ above. It yields a morphism $\overline{k^*}: P_{L,K} \to \overline{R}$ for which there exists a unique map $\overline{k}: K \to \overline{R}L$ in $\mathscr{K}$ with $\overline{k}^* = \overline{k^*}$. From

$r \cdot k^* = \overline{k^*} = \overline{k}^*$ it follows that

$$r_L \cdot k = \overline{k} \qquad \text{for each } k\colon K \to RL, K \in \mathscr{K}_{\mathrm{fp}}.$$

This proves the uniqueness. Now let us show that, conversely, the last property defines $r_L\colon RL \to \overline{R}L$, in other words, that the morphisms $\overline{k}$ form a cocone of the diagram D_{RL}:

$$\begin{tikzcd} K \arrow[rr, "t"] \arrow[dr, "k"'] & & K' \arrow[dl, "k'"] \\ & RL & \end{tikzcd} \qquad \begin{tikzcd} K \arrow[rr, "t"] \arrow[dr, "\overline{k}"'] & & K' \arrow[dl, "\overline{k'}"] \\ & \overline{R}L & \end{tikzcd}$$

From $k' \cdot t = k$, we are to derive $\overline{k'} \cdot t = \overline{k}$. We use $\tilde{t}\colon P_{LK} \to P_{LK'}$ to denote the natural transformation determined by coproducts of copies of t. Then, obviously, $k' \cdot t = k$ implies $(k')^* \cdot \tilde{t} = k^*$. Thus, by (1), $\overline{(k')^*} \cdot \tilde{t} = \overline{k^*}$ or, equivalently, $\overline{k'}^* \cdot \tilde{t} = \overline{k}^*\colon P_{LK} \to R$. By applying this to $X = L$ and considering the id_L-component, we obtain the desired equations $\overline{k'} \cdot t = \overline{k}$. Therefore, the above equations $r_L \cdot k = \overline{k}$ define $r_L\colon RL \to \overline{R}L$ for each $L \in \mathscr{L}_{\mathrm{fp}}$. It remains to prove the naturality, that is, $\overline{R}f \cdot r_L = r_{L'} \cdot Rf$ for every $f\colon L \to L'$ in $\mathscr{L}_{\mathrm{fp}}$. We use the finite accessibility of $\mathscr{K}$ again: it is sufficient to prove that $(\overline{R}f \cdot r_L) \cdot k = (r_{L'} \cdot Rf) \cdot k$ for all $k\colon K \to RL$ with $K \in \mathscr{K}_{\mathrm{fp}}$. In fact, the morphism $f\colon L \to L'$ yields a natural transformation $\widehat{f}\colon P_{L,K} \to P_{L',K}$ where $\widehat{f}_X\colon \coprod_{\hom(L',X)} K \to \coprod_{\hom(L,X)} K$ is given by the maps $\hom(L', X) \to \hom(L, X)$ of composition with f. Obviously, $k^* \cdot \widehat{f} = (Rf \cdot k)^*$, and thus, by (1), we get $\overline{k^*} \cdot \widehat{f} = \overline{(Rf \cdot k)^*}$, that is, $\overline{k}^* \cdot \widehat{f} = \overline{Rf \cdot k}^*$. This implies $\overline{Rf \cdot k} = \overline{R}f \cdot \overline{k}$, and consequently

$$r_{L'} \cdot Rf \cdot k = \overline{Rf \cdot k} = \overline{R}f \cdot \overline{k} = \overline{R}f \cdot r_L \cdot k.$$

Let us now prove that the above natural transformation $r\colon R \to \overline{R}$ fulfils

$$r \cdot p = \overline{p} \text{ for all } p\colon P \to R \text{ with } P \in \mathscr{A}.$$

When $P = P_{L,K}$, this is obvious: we have, again by the above Yoneda lemma $(*)$, a map $k\colon K \to RL$ with $p = k^*$, and then

$$r \cdot p = r \cdot k^* = \overline{k^*} = \overline{p}.$$

Next, the set $\mathscr{C}$ of all functors P such that $r \cdot p = \overline{p}$ holds for all $p\colon P \to R$ is, obviously, closed under existing finite colimits: given a colimit cocone $(P_i \xrightarrow{p_i} P)_{i \in I}$, we only have to prove $r \cdot p \cdot p_i = \overline{p} \cdot p_i$ for each i, assuming $r \cdot p \cdot p_i = \overline{p \cdot p_i}$ – from (1) we get $r \cdot p \cdot p_i = \overline{p \cdot p_i} = \overline{p} \cdot p_i$. Since $\mathscr{C}$ contains all step-functors $P_{L,K}$, it contains all of $\mathscr{A}$, and the proof is concluded. □

4. Embedding-projection adjunctions

An important property of CPOs is the coincidence of directed limits with directed colimits in the category $\mathrm{CPO}^{\mathrm{e}}$ of CPOs and embedding-projection pairs. We will now show that the category $\mathrm{SCC}^{\mathrm{e}}$ of Scott-complete categories and embedding-projection adjunctions also has this property. From a categorical point of view, an embedding-projection pair

$$\mathscr{K} \underset{p}{\overset{e}{\rightleftharpoons}} \mathscr{L}$$

between CPOs is a pair of adjoint functors (*i.e.*, order-preserving maps with $pe(x) \leqslant x$ and $ep(y) \leqslant y$ for all $x \in \mathscr{K}$, $y \in \mathscr{L}$) that are continuous and have their unit of adjunction formed by the identity-transformation (that is, $pe(x) = x$). Analogously, given Scott-complete (or, more generally, finitely accessible) categories $\mathscr{K}$ and $\mathscr{L}$, we can define an embedding-projection adjunction

$$\mathscr{K} \underset{P}{\overset{E}{\rightleftharpoons}} \mathscr{L}$$

as a pair of adjoint continuous functors $E \dashv P$ whose unit of adjunction is $\eta = \mathrm{id}\colon \mathrm{Id}_{\mathscr{K}} \to PE = \mathrm{Id}_{\mathscr{K}}$. There is a technical difficulty here: if we want the category $\mathrm{SCC}^{\mathrm{e}}$ to have directed colimits, we should not distinguish between functors which are naturally isomorphic (because if we do distinguish, we only obtain weaker concepts of directed bicolimit, known from 2-category theory, which we want to 'escape' here). Thus, given an embedding-projection pair

$$\mathscr{K} \underset{P}{\overset{E}{\rightleftharpoons}} \mathscr{L}$$

and given a functor $E'\colon \mathscr{K} \to \mathscr{L}$ naturally isomorphic to E, we identify the given pair with

$$\mathscr{K} \underset{P}{\overset{E'}{\rightleftharpoons}} \mathscr{L};$$

analogously with $P' \cong P$. This makes the definition of the category $\mathrm{SCC}^{\mathrm{e}}$ more technical, but the reward is that

(1) the embedding E uniquely determines the embedding-projection adjunction, and
(2) $\mathrm{SCC}^{\mathrm{e}}$ has directed limits and directed colimits and they canonically coincide.

Concerning (1), one can say that an embedding-projection pair $\mathscr{K} \rightleftharpoons \mathscr{L}$ is nothing other than a choice of a coreflective full subcategory of $\mathscr{L}$ that is finitely accessible and whose coreflector $\mathscr{L} \to \mathscr{K}$ is continuous.

Definition 3.

(1) Let $\mathscr{K}$ and $\mathscr{L}$ be finitely accessible categories. An *embedding-projection adjunction* is a pair

$$E\colon \mathscr{K} \to \mathscr{L} \text{ and } P\colon \mathscr{L} \to \mathscr{K}$$

of continuous functors with $PE = \mathrm{Id}_{\mathscr{K}}$, together with a natural transformation

$$\tau\colon EP \to \mathrm{Id}_{\mathscr{L}}$$

satisfying

$$P\tau = \mathrm{id}_P \text{ and } \tau E = \mathrm{id}_E\ .$$

In other words, an adjoint pair $E \dashv P$ of continuous functors with a unit of adjunction $\mathrm{id}\colon \mathrm{Id}_{\mathscr{K}} \to PE$ and counit of adjunction $\tau\colon EP \to \mathrm{Id}_{\mathscr{L}}$.

(2) Two embedding-projection adjunctions (E, P, τ) and (E', P', τ') from $\mathscr{K}$ to $\mathscr{L}$ are called *isomorphic*, notation

$$(E, P, \tau) \equiv (E', P', \tau'),$$

provided that there exist natural isomorphisms

$$e: E \to E' \text{ and } p: P \to P'$$

with

$$\tau = \tau' \cdot E'p \cdot EP.$$

Notation 3. We denote by

$$\mathrm{SCC}^{\mathrm{e}}$$

the category whose objects are Scott-complete categories and whose morphisms from $\mathscr{K}$ to $\mathscr{L}$ are all isomorphism-classes $[E, P, \tau]: \mathscr{K} \to \mathscr{L}$ of embedding-projection adjunctions $E: \mathscr{K} \to \mathscr{L}$; $P: \mathscr{L} \to \mathscr{K}$; $\tau: EP \to \mathrm{Id}_{\mathscr{L}}$. Composition is defined by $[E', P', \tau'][E, P, \tau] = [E'E, PP', \tau' \cdot (E'\tau P')]$

$$(E'E)(PP') \overset{E'\tau P'}{\to} E'P' \overset{\tau'}{\to} \mathrm{Id}$$

and the identity arrows are $[\mathrm{Id}_{\mathscr{K}}, \mathrm{Id}_{\mathscr{K}}, \mathrm{id}]$.
(We have to verify that the composition is independent of the choice of representatives, that is, if (E, P, τ) is isomorphic to $(\widehat{E}, \widehat{P}, \widehat{\tau})$, then also

$$(E'E, PP', \tau' \cdot (E'\tau P')) \text{ and } (E'\widehat{E}, \widehat{P}P', \tau' \cdot (E'\widehat{\tau}P'))$$

are isomorphic. This is an easy and straightforward computation, which we omit. Analogously, below we also omit the appropriate easy verifications concerning the choice of representatives for embedding-projection adjunctions.)

Remark 2. We will now prove that directed colimits of embedding-projection adjunctions can be computed from directed limits of projections (in the 'category' of all categories). This is quite analogous to directed colimits in $\mathrm{CPO}^{\mathrm{e}}$, see Theorem 2 of (Smyth and Plotkin 1982).
Let D be a directed diagram in $\mathrm{SCC}^{\mathrm{e}}$ indexed by an (up-)directed poset I. That is, for each $i \in I$, a Scott-complete category $\mathscr{K}_i$ is given, and for all $i \leqslant j$ in I, morphisms $[E_{ij}, P_{ij}, \tau_{ij}]: \mathscr{K}_i \to \mathscr{K}_j$ in $\mathrm{SCC}^{\mathrm{e}}$ are given with the obvious compatibility condition. We form a limit

$$P_i: \mathscr{L} \to \mathscr{K}_i \qquad i \in I$$

of the directed diagram of the categories $\mathscr{K}_i$ and the projection functors $P_{ij}: \mathscr{K}_i \to \mathscr{K}_j$ $(i \leqslant j)$. (The category $\mathscr{L}$ can be described in the expected way: objects are collections $(K_i)_{i \in I}$ of objects $K_i \in \mathscr{K}_i^{\mathrm{obj}}$ such that for all $i \leqslant j$ we have $P_j(K_j) = K_j$; morphisms are collections $(f_i)_{i \in I}$ of morphisms $f_i \in \mathscr{K}_i^{\mathrm{mor}}$ such that for all $i \leqslant j$ we have $P_{ij}(f_j) = f_i$. And P_i is the i-th projection.) We claim that

(i) $\mathscr{L}$ is a Scott-complete category and P_i are continuous functors.
(ii) The universal property of the limit yields for each $i \in I$ a unique functor $E_i: K_i \to \mathscr{L}$ with $P_j E_i = E_{ij}$ for all $j \geqslant i$ and a unique natural transformation $\tau_i: E_i P_i \to \mathrm{Id}_{\mathscr{L}}$ with $P_j \tau_i = \tau_{ij} P_j$ for all $j \geqslant i$.

(iii) $[E_i, P_i, \tau_i]: \mathscr{K}_i \to \mathscr{L}$ are morphisms forming a cocone of the given diagram D.
(iv) A colimit of the directed diagram of all functors $E_iP_i: \mathscr{L} \to \mathscr{L}$ $(i \in I)$ and all natural transformations $E_j\tau_{ij}P_j: E_iP_i \to E_jP_j$ $(i \leqslant j)$ in the category $\mathscr{L}^{\mathscr{L}}$ is $\mathrm{Id}_{\mathscr{L}}$ with colimit maps $\tau_i: E_iP_i \to \mathrm{Id}_{\mathscr{L}}$ $(i \in I)$.
(v) Property (iv) implies that the cocone (iii) is a limit of D in $\mathrm{SCC^e}$.

Theorem 4. (Directed colimits in $\mathrm{SCC^e}$). For each directed diagram D in $\mathrm{SCC^e}$ a directed limit of projections coincides with a directed colimit of embedding (both in CAT). A cone $[E_i, P_i, \tau_i]$ of D is a colimit in $\mathrm{SCC^e}$ iff $\operatorname{colim} E_iP_i = \mathrm{Id}$ (more precisely: (iv) above holds).

Remark 3. The proof consists of two parts, the first of which has nothing to do with Scott-completeness (and proceeds analogously to Theorem 2 of Smyth and Plotkin (1982) for CPOs): let $\mathrm{FAC^e}$ denote the category of all finitely accessible categories and isomorphism classes of embedding-projection adjunctions. We first prove Theorem 4 for this larger category, and at the end we show that if the given diagram lives in $\mathrm{SCC^e}$, the colimit remains in $\mathrm{SCC^e}$.

Proof.

Part I. Directed colimits in $\mathrm{FAC^e}$.

Let $(I, \leqslant)$ be a directed poset, let $\mathscr{K}_i$ $(i \in I)$ be finitely accessible categories, and let $[E_{ij}, P_{ij}, \tau_{ij}]: \mathscr{K}_i \to \mathscr{K}_j$ be a compatible system of embedding-projection adjunctions for all $i \leqslant j$ in I.

We first prove all the claims made in Remark 2.

(a) For each $i \in I$ we can define $E_i: \mathscr{K}_i \to \mathscr{L}$ by $P_jE_i = E_{ij}$ for all $j \geqslant i$. In other words, all E_{ij}, $j \geqslant i$, form a cone of the diagram of projections P_{jk} for all $k \geqslant j \geqslant i$. In fact, from $P_{jk}E_{jk} = \mathrm{Id}$ we get

$$E_{ij} = P_{jk}E_{jk}E_{ij} = P_{jk}E_{ik} \; .$$

(b) For each $i \in I$ we can define $\tau_i: E_iP_i \to \mathrm{Id}_{\mathscr{L}}$ by

$$P_j\tau_i = \tau_{ij}P_j \text{ for all } j \geqslant i \; .$$

In other words, we have the compatibility $P_{jk}(\tau_{ik}P_k) = \tau_{ij}P_j$ for all $k \geqslant j \geqslant i$. This follows from $\tau_{ik} = \tau_{jk} \cdot E_{jk}\tau_{ij}P_{jk}$ (see composition in $\mathrm{SCC^e}$), since $P_{jk}\tau_{jk} = \mathrm{id}$ implies $P_{jk}\tau_{jk}P_k = \mathrm{id}$. Thus,

$$P_{jk}\tau_{jk}P_k = P_{jk}E_{jk}\tau_{ij}P_{jk}P_k = \tau_{ij}P_j \; .$$

(c) (E_i, P_i, τ_i) is an embedding-projection adjunction for each $i \in I$. In fact, $P_iE_i = \mathrm{Id}$ because $E_{ii} = \mathrm{Id}$. Also $P_i\tau_i = \mathrm{id}$ because $\tau_{ii}P_i = \mathrm{id}$. The equality $\tau_iE_i = \mathrm{id}$ follows from the fact that

$$P_j(\tau_iE_i) = \tau_{ij}P_jE_i = \tau_{ij}E_{ij} = \mathrm{id} \text{ for all } j \geqslant i \; .$$

(d) Consider the directed diagram of all E_iP_i $(i \in I)$ and all

$$E_i\tau_{ij}P_j: E_iP_i \to E_jP_j \quad \text{for } i \leqslant j$$

in $\mathscr{L}^{\mathscr{L}}$. We prove that the cocone $(E_iP_i \xrightarrow{\tau_i} \mathrm{Id})_{i\in I}$ forms a colimit of that diagram. The cocone is compatible, that is,

$$\tau_j \cdot E_j\tau_{ij}P_j = \tau_i \quad \text{for } i \leqslant j$$

because for each $k \geqslant j$ we have

$$\begin{aligned} P_k(\tau_j \cdot E_j\tau_{ij}P_j) &= \tau_{jk}P_k \cdot E_{jk}\tau_{ij}P_j \\ &= \tau_{jk}P_k \cdot E_{jk}\tau_{ij}P_{jk}P_k \\ &= (\tau_{jk} \cdot E_{jk}\tau_{ij}P_{jk}) \\ &= \tau_{jk}P_k \\ &= P_k\tau_i. \end{aligned}$$

To verify the universal property, it is sufficient to prove that for each $k \in I$, the cocone $(P_kE_iP_i \xrightarrow{P_k\tau_i} P_k)_{i\in I}$ has the corresponding universal property (since colimits in $\mathscr{L}^{\mathscr{L}}$ are formed object-wise). This is obvious, because for the upper-set $\{i \in I\,; i \geqslant k\}$ the P_k-image of the restriction of our diagram is the constant diagram with value P_k:

$$P_k(E_iP_i) = P_{ki}P_iE_iP_i = P_{ki}P_i = P_k \qquad \text{for all } i \geqslant k$$

and

$$\begin{aligned} P_k(E_j\tau_{ij}P_j) &= P_{kj}\tau_{ij}P_j \\ &= P_{ki}(P_{ij}\tau_{ij})P_j \\ &= P_{ki}\,\mathrm{id}\,P_j \\ &= \mathrm{id} \qquad \text{for all } j \geqslant i \geqslant k. \end{aligned}$$

We also have $P_k\tau_i = \mathrm{id}_{P_k}$ for all $i \geqslant k$.

(e) $\mathscr{L}$ has directed colimits, and P_i and E_i are continuous functors. In fact, since $\mathscr{K}_i$ have directed colimits and the connecting functors preserve them, it follows that the functors P_i $(i \in I)$ preserve and, in fact, collectively create, directed colimits. The functors E_i preserve all existing colimits, since E_i is a left adjoint of P_i.

(f) $\mathscr{L}$ is finitely accessible. In fact, the collection $\mathscr{A}$ of all objects E_iX, where $i \in I$ and X is finitely presentable in $\mathscr{K}_i$, is essentially small. Let us verify first that it consists of finitely presentable objects of $\mathscr{L}$. Given a directed colimit $(L_t \xrightarrow{a_t} L)_{t\in I}$ in $\mathscr{L}$ and a morphism $f\colon E_iX \to L$ for some finitely presentable object X of $\mathscr{K}_i$, we have that the morphism $P_if\colon X \to P_iL = \operatorname{colim} P_iL_t$ factors as

$$P_if = P_ia_t \cdot g \qquad \text{for some } g\colon P_iX \to P_iL_t, t \in T$$

and this proves that f factors through a_t:

$$\begin{aligned} f &= f \cdot (\tau_i)_{E_iX} && (\tau_iE_i = \mathrm{id}) \\ &= (\tau_i)_L \cdot E_iP_if && (\text{naturality}) \\ &= (\tau_i)_L \cdot E_iP_ia_t \cdot E_ig && \\ &= a_t \cdot (\tau_i)_{L_t} \cdot E_ig && (\text{naturality}). \end{aligned}$$

Moreover, if $f = a'_t h' = a''_t h''$ for some $h' \colon E_i X \to L_{t'}$ and $h'' \colon E_i X \to L_{t''}$, then from the finite presentability of X and from $P_i a_{t'} \cdot P_i h' = P_i a_{t''} \cdot P_i h''$ we conclude the existence of $t \geqslant t'$, $t \geqslant t''$ and $k \colon X \to P_i L_t$ with

$$k = P_i L_{t't} \cdot P_i h' = P_i L_{t''t} \cdot P_i h''$$

(where $L_{t't} \colon L_{t'} \to L_t$ denotes the connecting morphisms).
Then $h = (\tau_i)_{L_t} \cdot E_i k \colon E_i X \to L_t$ fulfils

$$h = L_{t't} \cdot h' = L_{t''t} \cdot h'' \ .$$

This proves that $E_i X$ is finitely presentable in $\mathscr{L}$. For each object L of $\mathscr{L}$ the canonical diagram $\mathscr{A} \downarrow \mathscr{L} \to \mathscr{L}$ is filtered: given a finite subcategory $\mathscr{C}$ of $\mathscr{A} \downarrow \mathscr{L}$, we first find $i \in I$ such that for each object $E_j X \xrightarrow{f} L$ of $\mathscr{C}$ we have $j \geqslant i$; then that object can be substituted with $E_i(E_{ij}X) \xrightarrow{f} L$ (the proof that $E_{ij}X$ is finitely presentable in $\mathscr{K}_i$ is analogous to the above proof that $E_i X$ is finitely presentable in $\mathscr{L}$), thus, we can assume that all the objects of $\mathscr{C}$ have form $E_i X \xrightarrow{f} L$ for suitable finitely presentable objects X of $\mathscr{K}_i$. We obtain a corresponding finite category of arrows $X \xrightarrow{P_i f} P_i L$ in $\mathscr{K}_i$, and since $P_i L$ is a directed colimit of finitely presentable objects in $\mathscr{K}_i$, there exists $X_0 \xrightarrow{g} P_i L$ with X_0 finitely presentable such that for each $E_i X \xrightarrow{f} L$ in $\mathscr{C}$ we have a factorization

$$\begin{tikzcd}
X \arrow[r, "P_i f"] \arrow[dr, "f'"'] & P_i L \\
 & X_0 \arrow[u, "g"']
\end{tikzcd}$$

in $\mathscr{K}_i$ and thus, a factorization

$$\begin{tikzcd}
E_i \arrow[r, "f"] \arrow[dr, "E_i f'"'] & XL \\
 & E_i X_0 \arrow[u, "(\tau_i)_L \cdot E_i g"']
\end{tikzcd}$$

To prove that L is a canonical colimit of the above canonical diagram, we use the fact that $(E_i P_i L \xrightarrow{(\tau_i)_L} L)_{i \in I}$ is a colimit (see (d)). On the one hand, each arrow $E_i X \xrightarrow{f} L$ with $E_i X$ finitely presentable in $\mathscr{L}$ factorizes through some of the colimit arrows $(\tau_i)_L$, $j \in I$, simply because that colimit is directed. On the other hand, for each $i \in I$ we know, since E_i preserves directed colimits and $\mathscr{K}_i$ is finitely accessible, that $E_i P_i L$ is a directed colimit $(E_i X_s \xrightarrow{f_s} E_i P_i L)$ of arrows with X_s finitely presentable in $\mathscr{K}_i$.

(g) The morphisms $[E_i, P_i, \tau_i]\colon \mathscr{K}_i \to \mathscr{L}$ form a compatible cocone of the given diagram, that is, for $i \leqslant j$ we have

$$E_j \cdot E_{ij} = E_i$$

(since, given $k \geqslant j$, $P_k(E_j E_{ij}) = E_{jk} E_{ij} = E_{ik} = P_k E_i$),

$$P_{ij} \cdot P_j = P_i,$$

and

$$\tau_j \cdot E_j \tau_{ij} P_j = \tau_i \ .$$

In fact, given $k \geqslant j$

$$\begin{aligned} P_k(\tau_j \cdot E_j \tau_{ij} P_j) &= \tau_{jk} P_k \cdot P_k E_k E_{jk} \tau_{ij} P_{jk} P_k \\ &= (\tau_{jk} \cdot E_{jk} \tau_{ij} P_{jk}) P_k \\ &= \tau_{ik} P_k \\ &= P_k \tau_i. \end{aligned}$$

(h) So far we verified (iii) and (iv) of Remark 2. We now prove that this implies that $[E_i, P_i, \tau_i]$ is a colimit cocone of D in $\mathrm{FAC}^{\mathrm{e}}$. Let $[E_i', P_i', \tau_i']\colon \mathscr{K}_i \to \mathscr{L}'$ $(i \in I)$ be a cocone of D in $\mathrm{FAC}^{\mathrm{e}}$.
We define a morphism

$$[E, P, \tau]\colon \mathscr{L} \to \mathscr{L}'$$

as follows:

(i) $E\colon \mathscr{L}' \to \mathscr{L}$ is a directed colimit of the functors $E_i' P_i$ $(i \in I)$ and the natural transformations

$$E_j' \tau_{ij} P_i\colon E_i' P_i \to E_j' P_j (i \leqslant j)$$

in $(\mathscr{L}')^{\mathscr{L}}$. Let

$$\gamma_i\colon E_i' P_i \to E \ (i \in I)$$

denote the colimit cocone. Since $E_i' P_i$ are continuous, so is E.

(ii) $P\colon \mathscr{L} \to \mathscr{L}'$ is defined by $P_i \cdot P = P_i'$ $(i \in I)$. Since P_i' are continuous, so is P. Moreover, $EP = \operatorname{colim} E_i' P_i'$ with colimit cocone $\gamma_i P$ $(i \in I)$.

(iii) $\tau\colon EP \to \mathrm{Id}$ is defined by $\tau = \operatorname{colim} \tau_i'$, that is, $\tau \cdot \gamma_i P = \tau_i'$ $(i \in I)$.

(h1) $[E, P, \tau]$ is a morphism: we have $PE = \mathrm{Id}$ because

$$\begin{aligned} P_k(PE) &= P_k' \operatorname{colim}_{i \geqslant k} E_i' P_i \\ &= \operatorname{colim}_{i \geqslant k} P_k' E_i' P_i \\ &= \operatorname{colim}_{i \geqslant k} P_{ki} P_i' E_i' P_i \\ &= \operatorname{colim}_{i \geqslant k} P_{ki} P_i \\ &= \operatorname{colim}_{i \geqslant k} P_k \\ &= P_k. \end{aligned}$$

Further, $P\tau = \mathrm{id}$ because

$$\begin{aligned} P_k(P\tau) &= P'_k \operatorname{colim}_{i \geqslant k} \tau_i \\ &= \operatorname{colim}_{i \geqslant k} P'_k \tau_i \\ &= \operatorname{colim}_{i \geqslant k} P_{ki}(P'_i \tau_i) \\ &= \operatorname{colim}_{i \geqslant k} \mathrm{id} \\ &= \mathrm{id}. \end{aligned}$$

Finally, $\tau E = \operatorname{colim} \tau_i E'_i P_i = \operatorname{colim} \mathrm{id}_{E'_i P_i} = \mathrm{id}_E$.

(h2) The morphism of (h1) is a factorization of the given cocone, that is,

$$[E'_i, P'_i, \tau'_i] = [E, P, \tau] \cdot [E_i, P_i, \tau_i] \quad (i \in I).$$

In fact, $E'_i = E \cdot E_i$ because

$$\begin{aligned} E \cdot E_i &= \operatorname{colim}_{k \geqslant i} E'_k P_k E_i \\ &= \operatorname{colim}_{k \geqslant i} E'_k P_k E_k E_{ik} \\ &= \operatorname{colim}_{k \geqslant i} E'_k E_{ik} \\ &= \operatorname{colim}_{k \geqslant i} E'_i \\ &= E'_i. \end{aligned}$$

We clearly have $P'_i = P_i P$, and to prove $\tau'_i = \tau \cdot E\tau_i P$, we use $\tau = \operatorname{colim}_{k \geqslant i} \tau_k$ and $E = \operatorname{colim}_{k \geqslant i} E'_k P_k$, as well as

$$\tau_i = \tau_k \cdot E_k \tau_{ik} \cdot P_k \text{ and } t'_i = \tau'_k \cdot E'_k \tau'_{ik} \cdot P'_k$$

(from the compatibility) to get,

$$\begin{aligned} \tau \cdot E\tau_i P &= \operatorname{colim}_{k \geqslant i} \tau'_k \cdot E'_k P_k \tau_i P \\ &= \operatorname{colim}_{k \geqslant i} \tau'_k \cdot E'_k P_k \tau_k P \cdot E'_k P_k E_k \tau_{ik} P_k P \\ &= \operatorname{colim}_{k \geqslant i} \tau'_k \cdot \mathrm{id} \cdot E'_k \tau_{ik} P'_k \\ &= \operatorname{colim}_{k \geqslant i} \tau'_i \\ &= \tau'_i. \end{aligned}$$

(h3) The factorization of (h2) is unique, that is, if a morphism $[E^*, P^*, \tau^*] \colon \mathscr{L} \to \mathscr{L}'$ fulfils

$$[E'_i, P'_i, \tau'_i] = [E^*, P^*, \tau^*] \cdot [E_i, P_i, \tau_i] \quad \text{for all } (i \in I),$$

then $(E, P, \tau) \equiv (E^*, P^*, \tau^*)$. In fact, we have natural isomorphisms

$$\alpha_i \colon E'_i \to E^* E_i$$

and

$$\beta_i \colon P'_i \to P_i P^*$$

with

$$\tau'_i = (\tau^* \cdot E^* \tau_i P^*) E^* E_i \beta_i \cdot \alpha_i P'_i \tag{2}$$

for all $i \in I$. We define natural isomorphisms $\alpha \colon E \to P^*$ and $\beta \colon P \to P^*$ as follows. Since E^* is continuous and $(E_i P_i \xrightarrow{\tau_i} \mathrm{Id})_{i \in I}$ is a directed colimit in $\mathscr{L}^{\mathscr{L}}$, we know that

$(E^*E_iP_i \xrightarrow{E^*\tau_i} E^*)_{i\in I}$ is a directed colimit in $(\mathscr{L}')^{\mathscr{L}}$, and thus compositions with the natural isomorphisms α_iP_i also yield a colimit. Consequently, we have two colimits of $E_i'P_i'$s, and we obtain a unique natural isomorphism

$$\alpha\colon E \to E^* \qquad \text{with } \alpha \cdot \gamma_i = E^*\tau_i \cdot \alpha_iP_i \ (i \in I)\,. \tag{3}$$

Analogously, since $(E_iP_i' \xrightarrow{\tau_iP} P)$ is a directed colimit in $\mathscr{L}^{\mathscr{L}'}$, we have a unique natural transformation

$$\beta\colon P \to P^* \text{ with } \beta \cdot \tau_iP = \tau_iP^* \cdot E_i\beta_i \ (i \in I)\,. \tag{4}$$

This is also a natural isomorphism, whose inverse is defined by $\beta^{-1} \cdot \tau_iP^* = \tau_iP \cdot E_i\beta_i^{-1}$.

We only have to prove

$$\tau = \tau^* \cdot E^*\beta \cdot \alpha P,$$

or equivalently,

$$\tau_i' = \tau^* \cdot E^*\beta \cdot \alpha P \cdot \gamma_iP \qquad (i \in I).$$

Because of (2), it is sufficient to show

$$E^*\tau_iP^* \cdot E^*E_i\beta_i \cdot \alpha_iP_i' = E^*\beta \cdot \alpha P \cdot \gamma_iP$$

and because of (3) this follows from

$$E^*\tau_iP^* \cdot E^*E_i\beta_i = E^*\beta \cdot E^*\tau_iP,$$

and the latter follows from (4). This concludes the proof of $(E, P, \tau) \equiv (E^*, P^*, \tau^*)$.

Part II. Directed colimits in $\mathrm{SCC}^{\mathrm{e}}$.

We will prove that if each category $\mathscr{K}_i$is Scott-complete, then so is $\mathscr{L}$. Let $D\colon \mathscr{D} \to \mathscr{L}$ be a diagram with a cocone in $\mathscr{L}$. For each $i \in I$ the diagram P_iD has a cocone in $\mathscr{K}_i$, thus, it has colimit

$$\operatorname{colim} P_iD = (Dd \xrightarrow{r_{di}} R_i)_{i\in I} \tag{5}$$

Since E_i is a left adjoint, it preserves the above colimit, and we can define, for all $i \leqslant j$ in I, a morphism $r^{ij}\colon E_iR_i \to E_jR_j$ by

$$r^{ij} \cdot E_ir_{di} = E_jr_{dj} \cdot (E_j\tau_{ij}P_j)_{Dd} \quad \text{ for all } d \in \mathscr{D}^{\mathrm{obj}}. \tag{6}$$

This is well defined since the right-hand side is a cocone of E_iP_iD: given a morphism $\delta\colon d \to d'$, in $\mathscr{D}$, we have

$$\begin{aligned}
E_jr_{d'j} \cdot (E_j\tau_{ij}P_j)_{Dd'} \cdot E_iP_iD\delta \quad &= \quad E_j(r_{d'j} \cdot (\tau_{ij})_{P_jDd'} \cdot E_{ij}P_{ij}P_jD\delta)\\
&= \quad E_j(r_{d'j} \cdot P_jD\delta \cdot (\tau_{ij})_{P_jDd})\\
&= \quad E_j(r_{d'j} \cdot (\tau_{ij})_{P_jDd})\\
&= \quad E_jr_{dj} \cdot (E_j\tau_{ij}P_j)_{Dd}.
\end{aligned}$$

The morphisms $r^{ij}\colon E_iR_i \to E_jR_j$ form a directed diagram D^* in $\mathscr{L}$ – denote by $(E_iR_i \xrightarrow{r^i} R)_{i\in I}$ a colimit of D^*. We define, for each $d \in \mathscr{D}^{\mathrm{obj}}$, a morphism $c_d\colon Dd \to R$, by using the above colimit $Dd = \operatorname{colim} E_i P_i Dd$:

$$c_d \cdot (\tau_i)_{Dd} = r^i \cdot E_i r_{di} \qquad \text{for all } d \in \mathscr{D}^{\text{obj}},\ i \in I. \tag{7}$$

This is well defined because the right-hand side is a cocone: for all $i \leqslant j$ in I we have

$$r^i \cdot E_i r_{di} = r^j \cdot r^{ij} \cdot E_i r_{di} = (r^j \cdot E_j r_{dj}) \cdot (E_j \tau_{ij} P_j)_{Dd} \quad \text{by (3).}$$

We will prove that the cocone $(Dd \xrightarrow{c_d} R)_{d \in \mathscr{D}^{\text{obj}}}$ is a colimit of D in $\mathscr{L}$. First, this is indeed a cocone because for each morphism $\delta\colon d \to d'$ in $\mathscr{D}$ we have $c_d = c_{d'} \cdot D\delta$, since for all $i \in I$

$$\begin{aligned}
c_d \cdot (\tau_i)_{Dd} &= r^i E_i r_{di} && \text{by (4)}\\
&= r^i E_i (r_{d'i} \cdot P_i D\delta) && \text{by compatibility of } \operatorname{colim} P_i D\\
&= c_{d'} \cdot (\tau_i)_{Dd'} \cdot E_i P_i D\delta && \text{by (4)}\\
&= c_{d'} \cdot D\delta \cdot (\tau_i)_{Dd} && \text{by naturality of } \tau_i.
\end{aligned}$$

Second, let $(Dd \xrightarrow{c'_d} R')_{d \in \mathscr{D}^{\text{obj}}}$ be another cocone of D in $\mathscr{L}$. For each i there exists a unique morphism $f_i\colon R_i \to P_i R'$ in $\mathscr{K}_i$ with

$$f_i \cdot r_{di} = P_i c'_d \qquad \text{for all } d \in \mathscr{D}^{\text{obj}}. \tag{8}$$

The morphisms $(\tau_i)_{R'} \cdot E_i f_i\colon E_i R_i \to R'$ form a cone of the above diagram D^*, that is, for all $i \leqslant j$ we have

$$(\tau_j)_{R'} E_j f_j \cdot r^{ij} = (\tau_i)_{R'} \cdot E_i f_i\colon E_i R_i \to R'.$$

To verify this, we use the fact that $E_i R_i$ is a colimit of $E_i P_i Dd$'s: for each $d \in \mathscr{D}^{\text{obj}}$ we have

$$\begin{aligned}
(\tau_j)_{R'} E_j f_j \cdot r^{ij} \cdot E_i r_{di} &= (\tau_j)_{R'} \cdot E_j (f_j r_{dj} \cdot (\tau_{ij})_{P_j Dd}) && \text{by (6)}\\
&= (\tau_j)_{R'} \cdot E_j P_j c'_d \cdot (E_j \tau_{ij} P_j)_{Dd} && \text{by (8)}\\
&= c'_d \cdot (\tau_j)_{Dd} \cdot (E_j \tau_{ij} P_j)_{Dd} && \text{by naturality of } \tau_j\\
&= c'_d \cdot (\tau_j)_{Dd} \cdot E_j P_j\ (\tau_i)_{Dd} && \text{by definition of } \tau_i\\
&= c'_d \cdot (\tau_i)_{Dd} \cdot (\tau_j)_{E_i P_i Dd} && \text{by naturality of } \tau_j\\
&= c'_d \cdot (\tau_i)_{Dd} && \text{since } \tau_j E_i = \text{id}\\
&= (\tau_i)_{R'} \cdot E_i P_i c'_d && \text{by naturality of } \tau_{ij}\\
&= (\tau_i)_{R'} \cdot E_i f_i \cdot E_i r_{di} && \text{by (6).}
\end{aligned}$$

Consequently, we can define a morphism $f\colon R \to R'$ by

$$f \cdot r^i = (\tau_i)_{R'} \cdot E_i f_i\colon E_i R_i \to R' \quad \text{for all } i \in I. \tag{9}$$

This is the desired factorisation of the given cocone of D, that is, $f \cdot c_d = c'_d$ for all $d \in \mathscr{D}^{\text{obj}}$: it is sufficient to observe that for each $i \in I$ we have

$$\begin{aligned}
c'_d \cdot (\tau_i)_{Dd} &= (\tau_i)_{R'} \cdot E_i P_i c'_d && \text{by naturality of } \tau_i\\
&= (\tau_i)_{R'} \cdot E_i (f_i r_{di}) && \text{by (8)}\\
&= f \cdot r^i \cdot E_i r_{di} && \text{by (9)}\\
&= (f \cdot c_d) \cdot (\tau_i)_{Dd} && \text{by (8).}
\end{aligned}$$

It remains to prove that f is unique. Given $\overline{f}: R \to R'$ with $\overline{f} \cdot c_d = c'_d$ for all $d \in \mathscr{D}^{\mathrm{obj}}$, we prove that $f = \overline{f}$ by showing that $f \cdot r^i = \overline{f} \cdot r^i$ for all $i \in I$. This follows from the fact that E_i preserves the above colimit of $P_i D$: for each object d in $\mathscr{D}$ we have

$$\begin{aligned}
(\overline{f} \cdot r^i) \cdot E_i r_{di} &= \overline{f} \cdot c_d \cdot (\tau_i)_{Dd} && \text{by (8)} \\
&= f \cdot c_d \cdot (\tau_i)_{Dd} \\
&= (f \cdot r^i) \cdot E_i r_{di} && \text{by (8).}
\end{aligned}$$

□

5. Recursive domain equations

In this section we will show how solutions of equations $X \cong T(X)$ can be obtained for Scott-complete categories X. The idea is quite analogous to that of solving such equations for CPOs, but we have to go one level deeper. In the case of CPOs the given rule $T(X)$ for objects is 'somehow' understood to be a functor, that is, we assume that a rule $T(f)$ for morphisms (continuous functions) f is also given. If, moreover, this rule is locally continuous, that is, $T(\bigsqcup_{n\in\omega} f_n) = \bigsqcup_{n\in\omega} T(f_n)$ for all ω-chains (f_n) of continuous maps with a given domain and codomain, we obtain a locally continuous functor $T: \mathrm{CPO} \to \mathrm{CPO}$, which restricts to a continuous functor $T^{\mathrm{e}} : \mathrm{CPO}^{\mathrm{e}} \to \mathrm{CPO}^{\mathrm{e}}$. The latter has a canonical fixed point, which we declare as 'the' solution of $X \cong T(X)$.

Now for Scott-complete categories we have to extend T from the object part $T(X)$ in two levels: for continuous functors $F: X \to Y$ we need a rule to obtain continuous functors $T(F): T(X) \to T(Y)$. In other words, we extend T to a functor $T: \mathrm{SCC} \to \mathrm{SCC}$. But we also need a rule that, given continuous functors $F_1, F_2: X \to Y$, assigns to each natural transformation $\varphi: F_1 \to F_2$ a natural transformation $T(\varphi): T(F_1) \to T(F_2)$. In other words, we need a 2-functor (see, for example, Borceux (1994)) on the 2-category SCC whose

— objects (0-cells) are Scott-complete categories,
— morphisms (1-cells) are all continuous functors,
and
— 2-cells are all natural transformations.

That is, we now consider SCC as a sub-2-category of the usual 2-category of all categories, all functors and all natural transformations.

Examples 2.

(1) $- \times \mathscr{K}$: for each Scott-complete category $\mathscr{K}$ we extend the object-rule $X \mapsto X \times \mathscr{K}$ to a 2-functor $T: \mathrm{SCC} \to \mathrm{SCC}$ defined by
$T(X) = X \times \mathscr{K}$ on objects X
$T(F) = F \times \mathrm{Id}_{\mathscr{K}}$ on morphisms F
$T(\varphi) = \varphi \times \mathrm{id}$ on natural transformations φ

(2) Lifting $()_\perp$: we define a 2-functor as follows:
$X_\perp$ is the category obtained from the Scott-complete category X by adding a new initial object $\perp$ and adding a unique morphism $\perp \to a$ for each $a \in X^{\mathrm{obj}}$;
$F_\perp$ is the functor extending F by $F_\perp(\perp) = \perp$;
$\varphi_\perp$ is the natural transformation extending φ by the $\perp$-component $\mathrm{id}_\perp$.

(3) Product $\times$: we define a 2-bifunctor $\times\colon \mathrm{SCC} \times \mathrm{SCC} \to \mathrm{SCC}$ by the rule
$\times(X, Y) = X \times Y$ for pairs of objects
$\times(F, G) = F \times G$ for pairs of morphisms
and
$\times(\varphi, \psi) = \varphi \times \psi$ for pairs of natural transformations.

(4) Sum $\oplus$. (This construction is not a categorical coproduct – in fact, SCC does not have coproducts since SCC-objects are required to possess an initial object but SCC-morphisms are not required to preserve initial objects.) We define a 2-bifunctor

$$\oplus\colon \mathrm{SCC} \times \mathrm{SCC} \to \mathrm{SCC}$$

by the rule
$X \oplus Y = (X + Y)_\perp$, a lifting of the disjoint union of X and Y, for pairs of objects;
$F \oplus G = (F + G)_\perp$ for pairs of morphisms.
and
$\varphi \oplus \psi = (\varphi + \psi)_\perp$for pairs of natural transformations.

(5) Function-space $\to$: we define a 2-bifunctor $\to\colon \mathrm{SCC}^{\mathrm{op}} \times \mathrm{SCC} \to \mathrm{SCC}$ (contravariant in the first variable and covariant in the second one) by
$\to(X, Y) = [X \to Y]$, the Scott-complete category of all continuous functors from X to Y (see Part 3), for pairs of objects,
$\to(F, G)\colon [X \to Y] \to [X' \to Y']$, for continuous functors $F\colon X' \to X$, $G\colon Y \to Y'$, is given by $K \mapsto GKF$ on objects $K\colon X \to Y$ and $k \mapsto GkF$ on morphisms $k\colon K \to K'$,
$\to(\varphi, \psi)$, for natural transformations $\varphi\colon F_1 \to F_2$ and $\psi\colon G_1 \to G_2$, has the K-component $\psi * K\varphi$, the Godement-product of ψ and $K\varphi$.

Definition 4. A 2-functor $T\colon \mathrm{SCC} \to \mathrm{SCC}$ is said to be *locally continuous* provided the derived functor from $[\mathscr{K} \to \mathscr{L}]$ to $[T(\mathscr{K}) \to T(\mathscr{L})]$, given by
$F \to T(F)$ on objects $F\colon \mathscr{K} \to \mathscr{L}$
$\varphi \to T(\varphi)$ on morphisms $\varphi\colon F \to F'$,
is continuous for each pair $\mathscr{K}$, $\mathscr{L}$ of SCC-objects.

In other words, a 2-functor T is locally continuous iff for each directed collection of continuous functors $F_i\colon \mathscr{K} \to \mathscr{L}$ $(i \in I)$ we have $T(\operatorname{colim} F_i) = \operatorname{colim} T(F_i)$. Analogously, a 2-bifunctor $T\colon \mathrm{SCC} \times \mathrm{SCC} \to \mathrm{SCC}$ is locally continuous if for every directed collection of continuous functors $F_i\colon \mathscr{K}_1 \to \mathscr{L}_1$ and $G_i\colon \mathscr{K}_2 \to \mathscr{L}_2$, we have $T(\operatorname{colim} F_i, \operatorname{colim} G_i) = \operatorname{colim} T(F_i, G_i)$: more precisely, if the derived functors from $[(\mathscr{K}_1, \mathscr{K}_2) \to (\mathscr{L}_1, \mathscr{L}_2)]$ to $[T(\mathscr{K}_1, \mathscr{K}_2) \to T(\mathscr{L}_1, \mathscr{L}_2)]$ are continuous. And, finally, a 2-bifunctor $T\colon \mathrm{SCC}^{\mathrm{op}} \times \mathrm{SCC} \to \mathrm{SCC}$ is locally continuous if the derived functors from $[(\mathscr{K}_1, \mathscr{K}_2) \to (\mathscr{L}_1, \mathscr{L}_2)]$ to $[T(\mathscr{L}_1, \mathscr{K}_2) \to T(\mathscr{K}_1, \mathscr{L}_2)]$ are continuous.

Example 3. All the 2-functors and 2-bifunctors in Examples 2(1)–(5) above are locally continuous.

Observation 2. Every locally continuous 2-functor $T\colon \mathrm{SCC} \to \mathrm{SCC}$ defines a continuous functor

$$T^{\mathrm{e}}\colon \mathrm{SCC}^{\mathrm{e}} \to \mathrm{SCC}^{\mathrm{e}}$$

as follows:

$$T^{\mathrm{e}}X = TX \quad \text{for all objects } X$$

and

$$T^{\mathrm{e}}[E, P, \tau] = [T(E), T(P), T(\tau)] \qquad \text{for morphisms } [E, P, \tau] .$$

In fact, since 2-functors preserve (vertical and horizontal) composition, it is easy to see that for each embedding-projection adjunction (E, P, τ), the image $(T(E), T(P), T(\tau))$ is also an embedding-projection adjunction and two isomorphic adjunctions have isomorphic images. Thus, T^{e} is a well-defined functor. For each directed diagram D a colimit satisfies (iv) of Remark 2. Since the derived functor from $[\mathscr{L} \to \mathscr{L}]$ to $[T(\mathscr{L}) \to T(\mathscr{L})]$ is continuous, from $\operatorname{colim} E_i P_i = \mathrm{Id}_{\mathscr{L}}$ we conclude $\operatorname{colim} T(E_i)T(P_i) = \mathrm{Id}_{T(\mathscr{L})}$ – by Theorem 4 this implies that T preserves the colimit of D.

Remark 4. We can now conclude that SCC is algebraically compact with respect to locally continuous 2-functors in the sense of P. Freyd (Freyd 1991). Recall that if $T : \mathscr{A} \to \mathscr{A}$ is a functor, a T-*algebra* is a pair (A, a) consisting of an object A and a morphism $a : T(A) \to A$; *homomorphisms* from a T-algebra (A, a) into a T-algebra (A', a') are $\mathscr{A}$-morphisms $f : A \to A'$ with $f \cdot a = a' \cdot Tf$. As proved in Lambek (1968), if (A, a) is an initial T-algebra (initial object of the category of T-algebras and homomorphisms), a is an isomorphism. Thus A solves $X \cong TX$. Dually, a T-coalgebra is a pair (A, a) with $a : A \to T(A)$. By a *canonical solution* of the recursive equation $X \cong T(X)$, we mean an object A and an isomorphism $i : T(X) \to X$ such that both (X, i) is an initial T-algebra and (X, i^{-1}) is a final T-coalgebra. P. Freyd calls a category *categorically compact* if every 'appropriate' endofunctor T has a canonical solution of $X \cong T(X)$. For this, a trivial necessary condition is that the category have a zero-object (one which is initial as well as final) – this is not true in SCC, because morphisms are not supposed to preserve initial objects. However, for the 2-category

$$\mathrm{SCC}_{\perp}$$

of all Scott-complete categories, all strict and continuous functors (*i.e.*, continuous functors preserving initial objects) and all natural transformations, we have the following theorem.

Theorem 5. $\mathrm{SCC}_{\perp}$ is an algebraically compact category with respect to locally continuous 2-functors. That is, every locally continuous 2-functor $T : \mathrm{SCC}_{\perp} \to \mathrm{SCC}_{\perp}$ has a canonical solution of the equation $X \cong T(X)$.

Proof. The one-morphism trivial category $\perp$ is an initial object of $\mathrm{SCC}_{\perp}$. For each locally continuous 2-functor T, the corresponding functor T^{e} is continuous, and thus, it preserves the colimit of the ω-chain $d_n : \mathscr{K}_n \to \mathscr{K}_{n+1}$ defined as follows:

$$\mathscr{K}_0 = \perp \text{ and } \mathscr{K}_{n+1} = T(\mathscr{K}_n); \tag{10}$$

$d_0 = [E_0, P_0, \tau_0] : \perp \to T(\perp)$ is given by the constant functor P_0, the functor E_0 mapping the unique object of $\perp$ to an initial abject of $T(\perp)$, and the obvious natural transformation τ_0; and $d_{n+1} = T^{\mathrm{e}}(D_n) : T(\mathscr{K}_n) \to T(\mathscr{K}_{n+1})$.

It follows that, given a colimit cocone $(E_n^*, P_n^*, \tau_n^*) : \mathscr{K}_n \to \mathscr{L}$ of that chain, we have

(a) $P_n^* : \mathscr{L} \to \mathscr{K}_n$ $(n \in \omega)$ is a limit of the co-chain $\mathscr{K}_0 \leftarrow \mathscr{K}_1 \leftarrow \mathscr{K}_2 \cdots$ and T preserves this limit;

(b) $E_n^* : \mathscr{K}_n \to \mathscr{L}$ ($n \in \omega$) is a colimit of the chain $\mathscr{K}_0 \to \mathscr{K}_1 \to \mathscr{K}_2 \cdots$ and T preserves this colimit.

As proved in Adámek (1974), (b) implies that $\mathscr{L}$ is an initial T-algebra, and by duality, it is a canonical T-algebra. □

Corollary 6. For locally continuous 2-bifunctors $T : \mathrm{SCC}_\perp{}^{\mathrm{op}} \times \mathrm{SCC}_\perp \to \mathrm{SCC}_\perp$, the equation $X \cong T(X, X)$ has a solution.

In fact, by a general procedure presented by P. Freyd (Freyd 1991), a minimal solution of the equation $X \cong T(X, X)$ is obtained as follows: from the compactness of $\mathrm{SCC}_\perp$ it follows that $\mathrm{SCC}_\perp{}^{\mathrm{op}}$, and hence $\mathrm{SCC}_\perp{}^{\mathrm{op}} \times \mathrm{SCC}_\perp$, are algebraically compact. The mixed-variance 2-functor T yields a covariant 2-functor $\widehat{T} : \mathrm{SCC}_\perp{}^{\mathrm{op}} \times \mathrm{SCC}_\perp \to \mathrm{SCC}_\perp{}^{\mathrm{op}} \times \mathrm{SCC}$ given by

$$\widehat{T}(X, Y) = (T(Y, X), T(X, Y)) \quad \text{on objects}$$
$$\widehat{T}(F, G) = (T(G, F), T(F, G)) \quad \text{on morphisms}$$

and

$$\widehat{T}(\varphi, \psi) = (T(\psi, \varphi), T(\varphi, \psi)) \quad \text{on natural transformations.}$$

which is locally continuous if T is. A canonical solution of $(X, Y) \cong \widehat{T}(X, Y)$ then yields a minimal solution of $X \cong T(X, X)$, that is, a solution having an embedding-projection adjunction into any other solution.

Theorem 7. Every locally continuous endofunctor of SCC has a final coalgebra.

Proof. This is quite analogous to the proof of Theorem 5, here we do not get the initial T-algebra, because, in (10), $\mathscr{K}_0$ fails to be initial in SCC. □

6. How Scott-complete categories are sketched and axiomatized

Recall that a finite-limit sketch (or FL-sketch) $\mathscr{S}$ is a small category $\mathscr{A}$ in which a set of finite diagrams with cones is selected. The *category of models* of $\mathscr{S}$ is the full subcategory **Mod** $\mathscr{S}$ of $\mathrm{Set}^{\mathscr{A}}$ consisting of all set functors turning the selected cones to limit cones. It has been shown by Gabriel and Ulmer (1971) that **Mod** $\mathscr{S}$ is a locally finitely presentable category and, conversely, every locally finitely presentable category is *sketchable* by an FL-sketch $\mathscr{S}$ (that is, is equivalent to **Mod** $\mathscr{S}$).

We extend this to sketches for Scott-complete categories. Recall that a mixed sketch, in general, selects cones of some diagrams (to become limit cones in Set) and cocones of some diagrams (to become colimit cocones in Set). Here we restrict the cocones to the empty ones, *i.e.*, to the specification that some objects be mapped to the empty set.

Definition 5. By an $\mathrm{FL}_\perp$-sketch $\mathscr{S}$ is meant a small category $\mathscr{A}$ together with a choice of

(a) a set of finite diagrams with cones, and
(b) a set M of objects.

A *model* of $\mathscr{S}$ is a functor $T : \mathscr{A} \to \mathrm{Set}$ that maps

(a) the selected cones to limit cones in Set, and
(b) each object of M to $\varnothing$.

We call a category *sketchable* by an $\mathrm{FL}_\perp$-sketch $\mathscr{S}$ if it is equivalent to the category **Mod** $\mathscr{S}$ of all models and all natural transformations.

Theorem 8. A category is Scott-complete iff it is sketchable by an $\mathrm{FL}_\perp$-sketch.

Proof.

Sufficiency.

For each $\mathrm{FL}_\perp$-sketch $\mathscr{S}$ we will show that $\mathbf{Mod}\,\mathscr{S}$ is Scott-complete. Denote by $\mathscr{S}_0$ the FL-sketch obtained from $\mathscr{S}$ by forgetting the selection of M. Then $\mathbf{Mod}\,\mathscr{S}_0$ is a locally finitely presentable category closed under directed colimits in $\mathrm{Set}^{\mathscr{A}}$ (because directed colimits commute with finite limits in Set). It is obvious that $\mathbf{Mod}\,\mathscr{S}$ is closed under directed colimits in $\mathbf{Mod}\,\mathscr{S}_0$: if T is a directed colimit of functors T_i, $i \in I$, in $\mathbf{Set}^{\mathscr{A}}$ and if $T_iA = \varnothing$ for all $A \in M$ and $i \in I$, then also $TA = \varnothing$ for all $A \in M$. Further, for each object X in $\mathscr{A}$ such that $\hom(X, A) = \varnothing$ for all $A \in M$, we see that $\hom(X, -)$ is a model of $\mathscr{S}$, and, in fact, $\hom(X, -)$ is a finitely presentable object of $\mathbf{Mod}\,\mathscr{S}$ (since it is finitely presentable in $\mathrm{Set}^{\mathscr{A}}$ and $\mathbf{Mod}\,\mathscr{S}$ is closed under directed colimits in $\mathrm{Set}^{\mathscr{A}}$). Let $\mathscr{B}$ be the closure of the set of all these hom-functors under existing finite colimits in $\mathbf{Mod}\,\mathscr{S}$. Then each object of $\mathscr{B}$ is finitely presentable in $\mathbf{Mod}\,\mathscr{S}$, and we will prove that every object T of $\mathbf{Mod}\,\mathscr{S}$ is a directed colimit of objects in $\mathscr{B}$. In fact, T is a colimit of the diagram $D\colon \mathscr{D} \to \mathbf{Mod}\,\mathscr{S}$, where $\mathscr{D}$ is the comma-category of T with respect to all hom-functors in $\mathrm{Set}^{\mathscr{A}}$; now whenever $t\colon \hom(X, -) \to T$ is a map of $\mathrm{Set}^{\mathscr{A}}$, we have for each $A \in M$ from $TA = \varnothing$ that it follows that $\hom(X, A) = \varnothing$, and thus, $\hom(X, -)$ is a model of $\mathscr{S}$. Each finite subdiagram $D/\mathscr{D}_0\colon \mathscr{D}_0 \to \mathbf{Mod}\,\mathscr{S}$ of D has a colimit in $\mathbf{Mod}\,\mathscr{S}_0$ (since $\mathbf{Mod}\,\mathscr{S}_0$ is cocomplete) and this colimit has a map into T in $\mathbf{Mod}\,\mathscr{S}_0$, from which it, again, follows that $\operatorname{colim} D/\mathscr{D}_0$ is a model of $\mathscr{S}$. We thus obtain a directed diagram of all $\operatorname{colim} D/\mathscr{D}_0 \in \mathscr{B}$ and a colimit of this diagram is T. This proves that $\mathbf{Mod}\,\mathscr{S}$ is finitely accessible. Finally, to show that $\mathbf{Mod}\,\mathscr{S}$ is consistently cocomplete, we observe that for any diagram D in $\mathbf{Mod}\,\mathscr{S}$ with a cocone having a codomain $T \in \mathbf{Mod}\,\mathscr{S}$, we can form a colimit in $\mathbf{Mod}\,\mathscr{S}_0$ and the existence of an arrow from that colimit to T then guarantees that the colimit is a model of $\mathscr{S}$.

Necessity.

For each Scott-complete category $\mathscr{K}$ we will find an $\mathrm{FL}_\perp$-sketch. Recall here that, by a result of Lair (1981), every finitely accessible category can be sketched by a mixed sketch. That is, there exists a triple $\mathscr{S} = (\mathscr{A}, \mathbf{L}, \mathbf{C})$ consisting of a small category $\mathscr{A}$, a specification $\mathbf{L}$ of cones for some diagrams of $\mathscr{A}$ and a specification $\mathbf{C}$ of cocones for some diagrams in $\mathscr{A}$ such that $\mathscr{K}$ is equivalent to $\mathbf{Mod}\,\mathscr{S}$, the category of all functors in $\mathrm{Set}^{\mathscr{A}}$ mapping the specified (co-)cones to (co-)limits. A concrete description of $\mathscr{S}$ has been presented in Adámek and Rosický (1994): start with a set $\mathscr{C}$ of finitely presentable objects of $\mathscr{K}$ such that all objects are directed colimits of objects from $\mathscr{C}$. We consider $\mathscr{C}$ as a full subcategory of $\mathscr{K}$ and we form the Yoneda embedding

$$Y : \mathscr{K}^{\mathrm{op}} \to \mathrm{Set}^{\mathscr{C}}, \qquad YK = \hom(K, -)/\mathscr{C}.$$

For each finite diagram D in $\mathscr{K}$ choose a limit of the diagram $Y \cdot D^{\mathrm{op}}$ in $\mathrm{Set}^{\mathscr{C}}$

$$M_D = \lim Y \cdot D^{\mathrm{op}} \quad \text{in } \mathbf{Set}^{\mathscr{C}},$$

and let C_D denote the canonical colimit cocone expressing M_D as a colimit of hom-functors. The following sketch $\mathscr{S} = (\mathscr{A}, \mathbf{L}, \mathbf{C})$ sketches $\mathscr{K}$:

$$\mathscr{A} = Y(\mathscr{C}^{\mathrm{op}}) \cup \{M_D\,; D \text{ finite diagram in } \mathscr{A}\}$$

(a full subcategory of $\mathrm{Set}^{\mathscr{C}}$), $\mathbf{L}$ are the cones expressing T_D as a limit of D, and $\mathbf{C}$ are the cocones C_D. Now observe that whenever a finite diagram D has a colimit $A = \operatorname{colim} D$ in $\mathscr{K}$, we need not add M_D because $YA \cong \lim YD^{\mathrm{op}}$. Since $\mathscr{K}$ is Scott-complete and has $\perp = \operatorname{colim} \varnothing$, we only need to add M_D for nonempty, inconsistent diagrams D. But for each such D we have $M_D = M_\varnothing$, the constant functor of value $\varnothing$. (In fact, for each D the set $M_D C$ consists of all cones of $Y \cdot D^{\mathrm{op}}$ with the domain YC, that is, all cocones of D with the domain C in $\mathscr{K}$). Thus, if $\mathscr{K}$ has no inconsistent nonempty diagrams, that is, if it is locally finitely presentable, the above sketch is a limit sketch. If, on the other hand, we just have

$$\mathscr{A} = Y(\mathscr{C}^{\mathrm{op}}) \cup \{M_\varnothing\},$$

that is, we add (formally) an initial object $M_\varnothing$ to $Y(\mathscr{C}^{\mathrm{op}})$, and $\mathbf{L}$ consists, besides the FL cones of $Y(\mathscr{C}^{\mathrm{op}})$, of the cones of nonempty, finite, inconsistent diagrams with the domain $M_\varnothing$, while in $\mathbf{C}$ we only have $M_\varnothing = \operatorname{colim} \varnothing$. Consequently, $\mathscr{S}$ is an $\mathrm{FL}_\perp$-sketch. □

Remark 5. Let us recall from Coste (1979) that locally finitely presentable categories are precisely those that can be *axiomatized* by a limit theory T of first-order logic (in some S-sorted signature Σ), that is, that are equivalent to the category

$$\mathrm{Mod}\, T$$

of all models of T and all Σ-homomorphisms. A limit theory is a theory using *limit sentences* only, that is, sentences of the form

$$(\forall x_i : s_i)[\varphi(x_1, \ldots, x_n) \Longrightarrow (\exists y_j : t_j)\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

where φ and ψ are conjunctions of atomic formulae, s_i and t_j are sorts and x_i and y_j are variables of the specified sorts.

Definition 6. A theory in first-order logic is called a *limit-⊥ theory* if each of its sentences is either a limit sentence or a sentence of the form

$$(\forall x : s)[(x : s) \Longrightarrow \text{false}].$$

(The semantics of the latter sentence is: no element has sort s.)

Corollary 9. A category is Scott-complete iff it is axiomatizable by a limit-⊥ theory.

It is sufficient to show how each $\mathrm{FL}_\perp$-sketch is axiomatized: we choose sorts=objects and operations=morphisms, where each morphism $f : a \to b$ is a unary operation-symbol with variable of sort a and result of sort b. The limit specifications of $\mathscr{S}$ can easily be axiomatized by limit sentences, for example, for a discrete diagram with a cone $(a \to a_i)$

(that is, a product-specification) the obvious sentence is

$$(\forall x_1 : a_1, \ldots, x_n : a_n)(\exists ! y : a)(\bigwedge \pi_i(y) = x_i).$$

The set M of objects in $\mathscr{S}$ is axiomatized by the sentences

$$(\forall x : s)[(x = x) \Longrightarrow \text{false}]$$

for each $s \in M$.

Acknowledgement

I am grateful to P. Taylor for an interesting discussion on the topic of my paper, and for pointing out a more general limit-colimit coincidence he has proved in Taylor (1994).

References

Adámek, J. (1974) Free algebras and automata realizations in the language of categories. *Commentationes Mathematicae Universitatis Carolinae* **15** 589–602.

Ageron, P. (1991) *Structure des logiques et logique des structures*, Ph.D. thesis, Paris VII.

Abramsky, S. and Jung, A. (1994) Domain theory. In: Abramsky, S., Gabbay, D. M. and Maibaum, T. S. E. (eds.) *Handbook of Logic in Computer Science*, volume 3, Clarendon Press 1–168.

Adámek, J. and Rosický, J. (1994) *Locally presentable and accessible categories*, Cambridge University Press.

Borceux, F. (1994) *Handbook of categorical algebra I*, Cambridge University Press.

Coste, M. (1979) Localization, spectra and sheaf representation. *Springer-Verlag Lecture Notes in Mathematics* **753** 212–238.

Freyd, P. J. (1991) Algebraically complete categories. In: Carboni, A., Pedicchio, M. C. and Rosolini, G. (eds.) Como Category Theory Conference. *Springer-Verlag Lecture Notes in Mathematics* **1488** 95–104.

Gabriel, P. and Ulmer, F. (1971) Lokal präsentierbare Kategorien. *Springer-Verlag Lecture Notes in Mathematics* **221**.

Lair, C. (1981) Categories modélables et catégories esquissables. *Diagrammes* **6** 1–20.

Lambek, J. (1968) Fixpoint theorem for complete categories. *Math. Z.* **103** 175–195.

Makkai, M. and Paré, R. (1989) Accessible categories. *Contemporary Mathematics* **104**.

Scott, D. S. (1972) Continuous lattices. In: Lawvere, E. (ed.) Toposes, Algebraic Geometry and Logic. *Springer-Verlag Lecture Notes in Mathematics* **274** 97–136.

Scott, D. S. (1982) Domains for denotational semantics. In: Nielson, M. and Schmidt, E. M. (eds.) International Colloquium on Automata, Languages and Programs. *Springer-Verlag Lecture Notes in Computer Science* **140** 577–613.

Smyth, M. B. and Plotkin, G. D. (1982) The category-theoretic solution of recursive domain equations. *SIAM J. Computing* **11** 761–783.

Taylor, P. (1994) The limit-colimit coincidence for categories. Preprint, Imperial College, London.