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Regular sets of infinite message sequence charts☆

Dietrich Kuske*

Institut für Algebra, Technische Universität Dresden, D-01062 Dresden, Germany

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Abstract

This paper resumes the study of regular sets of message sequence charts (MSC) initiated by Henriksen et al. [Technical Report, BRICS RS-99-52, 1999]. Differently from their results, we consider infinite MSCs. It is shown that for bounded sets of infinite MSCs, the notions of recognizability, axiomatizability in monadic second order logic, and acceptance by a deterministic message passing automaton with Muller acceptance condition coincide. We furthermore characterize the expressive power of first order logic and of its extension by modulo-counting quantifiers over bounded infinite MSCs. In order to prove our results, we exhibit a new connection to the theory of Mazurkiewicz traces using relabeling techniques.

Keywords: Message sequence charts; Distributed automata; Monadic second order logic; Mazurkiewicz traces

1. Introduction

Message sequence charts (MSCs) form a popular visual formalism used in the software development. In its simplest incarnation, an MSC depicts the desired exchange of messages and corresponds to a single partial-order execution of the system. Several methods to specify sets of MSCs have been considered, among them MSC-graphs or high-level MSCs (HMSCs) that generate sets of MSCs by concatenating "building blocks", (Büchi-)automata that accept the linear extensions of MSCs, logics like monadic second order logic, and message passing automata, a distributed automata model. In general,

☆ The results of this paper were obtained while the author was affiliated with the University of Leicester. He presented them at STACS02 and they appeared as an extended abstract in [24].

* Fax: +49-351-463-34235.
E-mail address: kuske@math.tu-dresden.de.

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these formalisms have different expressive power, translations between them (when possible) have been considered, e.g., in [1,20,21,30,33].

The collection of MSCs generated by a "bounded" [2] or, equivalently, "locally synchronized" [19,32] MSC-graph is studied in [2,32]. In particular, it is observed that the collection of MSCs generated by a locally synchronized MSC-graph can be represented as a regular string language. Based on this observation, Henriksen et al. [19] study sets of MSCs whose linear extensions form a regular string language. I will call these sets of MSCs "recognizable". The first observation by Henriksen et al. states that any recognizable set of MSCs is bounded: there is a constant B such that all channels can be restricted to size B without disallowing any sequential execution. The notion of recognizability has proven to be robust and fruitful in different settings like strings, trees, Mazurkiewicz traces and other classes of partial orders (both finite and infinite). The robustness is reflected by the fact that there are alternative definitions of recognizable sets of objects using combinatorial, algebraic, or logical methods: in all these settings, recognizable sets can be presented by finite-state devices, by congruences of finite index, or by sentences of monadic second order logic. Even more, natural subclasses of monadic second order logic correspond to natural classes of congruences and finite-state devices (e.g., first-order logic corresponds to groupfree syntactic monoids and to counterfree automata). The main results in [19] show similar equivalences for sets of bounded finite MSCs. In particular, they prove the equivalence of the following three concepts for sets K of bounded finite MSCs:

(A) The set K can be accepted by a nondeterministic message passing automaton.
(B) There is a sentence φ of the monadic second order logic such that K is the set of bounded MSCs that satisfy φ.
(C) The set of linear extensions of K can be accepted by a finite deterministic automaton.

This result was sharpened in [31] where it is shown that deterministic message passing automata suffice. Here a message passing automaton is a collection of finite state machines that communicate via reliable FIFO-channels. They are provided with global accepting conditions and are therefore not deadlock-free. For some natural sets K of MSCs, Genest et al. [18] developed deadlock-free implementations by message passing automata. In all the papers mentioned so far, MSCs specify the *existence* of some communication, but not the *content* of the message sent. In the same way, in this work, the implementation can choose the message content, although it is not allowed to exchange more or fewer messages than specified. The setting in [1,30] is different in as far as there, the message content cannot be changed by the implementation.

The main focus of this paper is the extension of the results from [19,31] to sets of infinite MSCs. These infinite MSCs occur naturally as executions of systems that are not meant to stop, e.g., distributed operating systems or telecommunication networks.

In the first part, we investigate the expressive power of a variant of message passing automata that is capable of accepting infinite MSCs. To this aim, we extend message-passing automata by a Muller-acceptance condition. It is shown that for a set of bounded possibly infinite MSCs K, the following statements are equivalent (Theorem 3.16).

(A′) The set K is accepted by a finite deterministic message passing automaton with Muller-acceptance condition.
(B′) There is a sentence φ of the monadic second order logic such that K is the set of bounded possibly infinite MSCs that satisfy φ.

In the second part, we will extend the equivalence between (B) and (C). We will also consider two fragments of monadic second order logic, namely first-order logic FO and its extension by modulo-counting quantifiers FO + MOD(n) [35]. We describe the expressive power of these logics in the spirit of Büchi's theorem: for a set of bounded possibly infinite MSCs K, the following statements are equivalent (Theorems 4.8, 4.18, and 4.19)

(B$''$) The set K is axiomatizable by a sentence of monadic second order logic (of the logic FO + MOD (n), first-order logic, resp.) relative to all possibly infinite MSCs.

(C$''$) The set of linear extensions of K is recognizable (n-solvable, aperiodic, resp.).

The proof of the implication (A$'$) $\rightarrow$ (B$'$) is an obvious variant of similar proofs for finite automata for words (cf. [37]), asynchronous automata for traces [38], or asynchronous cellular automata for pomsets without autoconcurrency [14]. Mukund et al. proved the implication (B$'$) $\rightarrow$ (A$'$) for finite MSCs. In order to do so, they had to reprove several results from the theory of Mazurkiewicz traces in the more complex realm of MSCs. Differently, my proof for infinite MSCs uses asynchronous mappings for Mazurkiewicz traces directly (cf. [12] for surveys on this theory). These mappings are applicable since any bounded set of MSCs can be represented as a set of traces up to an easy relabeling. This constitutes a newly discovered relation between Mazurkiewicz traces and MSCs that differs fundamentally from those used e.g. in [32] for the investigation of race conditions and confluence properties and in [19] for some undecidability results. This new observation has in my opinion several nice aspects: (1) it simplifies the proof, (2) it also results in smaller message passing automata for finite MSCs, and (3) it highlights the similarity of MSCs and Mazurkiewicz traces and the unifying role that Mazurkiewicz traces can play in the theory of distributed systems. This last point is also stressed by the fact that similar proof techniques have been used, e.g., in [3,13,14,23,25,29].

Using the same relation between message sequence charts and Mazurkiewicz traces, one can also show that a set of bounded MSCs is recognizable if and only if it is axiomatizable in monadic second order logic (cf. Remarks 4.3 and 4.10). Under some restrictions, this simple proof idea works for the extension of first-order logic by modulo-counting quantifiers as well; but it does not seem to be applicable when investigating first-order logic. The proof we actually use is outlined as follows. The proof of the implication (B$''$) $\rightarrow$ (C$''$) relies on a first-order interpretation of a bounded MSC in any of its linearisations. This allows the use of results from [5,26,35] that characterize the expressive power of the logics in question for infinite words. The proof (C$''$) $\rightarrow$ (B$''$) for *finite* MSCs from [19] uses a first-order interpretation of the lexicographically least linear extension of t in the finite MSC t. This proof method does not extend to the current setting since in general no linear extension of order type ω can be defined in an infinite MSC. To overcome this problem, we use ideas from [36] by choping an infinite MSC into its finite and its infinite part. It turns out that the infinite part is the disjoint union of infinite posets to which the "classical" method from [19] is applicable.

On the one hand, we show a close relation between monadic second order logic and message passing automata. The relation between sequential Büchi-automata and monadic second order logic is shown to be even closer since natural fragments of the logic lead to natural restrictions of the class of Büchi-automata. It is not clear whether these logical fragments are mirrored in the setting of message passing automata as well.

2. Basic definitions: MSCs and message passing automata

Let $\mathcal{P}$ be a finite set of processes which communicate with each other through messages via reliable FIFO-channels. Let Σ be the set of communication actions $p!q$ and $q?p$ for $p, q \in \mathcal{P}$ distinct. The action

$p!q$ is to be read as "p sends to q" and $q?p$ is to be read as "q receives from p". Hence $p\theta q$ is performed by the process p, denoted $\mathrm{proc}(p\theta q) = p$. Following [19], we shall not be concerned with the internal actions of the processes which is no essential restriction since the results of this paper can be extended to deal with internal actions. We will also not consider the actual messages that are sent and received (see [1,30] for work that deals with this aspect).

A Σ-labeled poset is a structure $t = (V, \leqslant, \lambda)$ where $(V, \leqslant)$ is a partially ordered set, $\lambda : V \to \Sigma$ is a mapping, $\lambda^{-1}(\sigma) \subseteq V$ is linearly ordered for any $\sigma \in \Sigma$, and any $v \in V$ dominates a finite set. For $v \in V$, we write $\mathrm{proc}(v)$ as a shorthand for $\mathrm{proc} \circ \lambda(v)$. A set $X \subseteq V$ is an *(order) ideal* in t provided $v \leqslant w \in X$ implies $v \in X$ for any $v, w \in V$. Any element $v \in V$ defines a *principal ideal* $\downarrow v = \{u \in V \mid u \leqslant v\}$. The dual notion of an ideal is that of an (order) filter: a set $X \subseteq V$ is an *(order) filter* in t provided $w \geqslant v \in X$ implies $w \in X$ for any $v, w \in V$. Let $X \subseteq V$. Then $t\restriction_X = (X, \leqslant \cap (X \times X), \lambda\restriction_X)$ is the restriction of t to X. For $A \subseteq \Sigma$, we will abbreviate $t\restriction_{\lambda^{-1}(A)}$ by $t\restriction_A$.

The covering relation $\prec\!\!\!\cdot \subseteq V^2$ is defined by $u \prec\!\!\!\cdot\, w$ iff $u < w$ and, for any $v \in V$ with $u \leqslant v < w$, we have $u = v$. In order to define message sequence charts, for a Σ-labeled poset t, we define the relations $\sqsubseteq_{\mathcal{P}}$ and $\sqsubseteq$ as follows:

- $v \sqsubseteq_{\mathcal{P}} w$ iff $\mathrm{proc}(v) = \mathrm{proc}(w)$ and $v \leqslant w$.
- $v \sqsubseteq w$ iff $\lambda(v) = p!q$, $\lambda(w) = q?p$, and $|{\downarrow v} \cap \lambda^{-1}(p!q)| = |{\downarrow w} \cap \lambda^{-1}(q?p)|$ for some $p, q \in \mathcal{P}$ distinct.

The idea is that $\sqsubseteq_{\mathcal{P}}$ describes the linear order of the events executed by the sequential processes. The relation $\sqsubseteq$ (which is not transitive) describes matching send and receive events.

Definition 2.1. A *message sequence chart or MSC* for short is a Σ-labeled poset $t = (V, \leqslant, \lambda)$ satisfying

- $\leqslant = (\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^{\star}$,
- $\mathrm{proc}^{-1}(p) \subseteq V$ is linearly ordered for any $p \in \mathcal{P}$, and
- $|\lambda^{-1}(p!q)| = |\lambda^{-1}(q?p)|$ for any $p, q \in \mathcal{P}$ distinct.

The MSC t is *B-bounded* for some $B \in \mathbb{N}$ if, for any $v \in V$, we have $|{\downarrow v} \cap \lambda^{-1}(p!q)| - |{\downarrow v} \cap \lambda^{-1}(q?p)| \leqslant B$. By MSC^{∞}, we denote the set of all message sequence charts while MSC denotes the set of finite MSCs. Furthermore, MSC_B and MSC_B^{∞} denote the sets of B-bounded (finite) MSCs. Finally, $\downarrow$MSC denotes the set of order ideals in finite MSCs, and $\downarrow\mathrm{MSC}_B^{\infty}$, etc. are defined similarly.

We call a set K of ideals in MSCs *bounded* if there exists $B \in \mathbb{N}$ with $K \subseteq \downarrow\mathrm{MSC}_B^{\infty}$. Let w be a finite or infinite word over Σ. We write w_i for the ith letter of w and $w\restriction_{\leqslant i} = w_0 w_1 \ldots w_i$ for the prefix of w of length $i + 1$ where $0 \leqslant i < |w|$. For $A \subseteq \Sigma$, let $w\restriction_A$ denote the restriction of w to the letters from A. Furthermore, $|w|$ denotes the length of w and $|w|_a$ the number of occurrences of the letter $a \in \Sigma$ in the word w; note that $|w| = \infty$ and $|w|_a = \infty$ are possible. The word w is *proper* if, for any $0 \leqslant i < |w|$ and any $p, q \in \mathcal{P}$ distinct, we have $|w\restriction_{\leqslant i}|_{p!q} \geqslant |w\restriction_{\leqslant i}|_{q?p}$. From a proper word w, we can define the Σ-labeled poset $\mathrm{MSC}(w) = (V, \preceq, \lambda_w)$ as follows:

- $V = \mathrm{dom}(w) = \{i \in \mathbb{N} \mid 0 \leqslant i < |w|\}$,
- $\lambda_w(i) = w_i$, and
- $\preceq = R^{\star}$ where $(i, j) \in R$ iff $i < j$ and either $\mathrm{proc}(w_i) = \mathrm{proc}(w_j)$ or $w_i = p!q$, $w_j = q?p$ and $|w\restriction_{\leqslant i}|_{p!q} = |w\restriction_{\leqslant j}|_{q?p}$.

If w is finite and proper, then $\mathrm{MSC}(w)$ is always an ideal in an MSC (see below). This is not the case for infinite proper words as, e.g., in the partial order $\mathrm{MSC}(p!r\ (q!r\ r?q)^{\omega})$ the process r can never receive the message sent by p. The reason is that process r (which should eventually receive the message from p) has to execute infinitely many actions.[1]

Proposition 2.2. *Let w be a proper word. Then* $\mathrm{MSC}(w)$ *is an ideal in some MSC iff*

$$|w|_{q?p} < |w|_{p!q} \implies \{i \in \mathrm{dom}(w) \mid \mathrm{proc}(w_i) = q\} \text{ is finite}$$

for any distinct processes p and q.

Proof. First let $\mathrm{MSC}(w)$ be an ideal in the MSC $t = (V, \leqslant, \lambda)$. By contradiction, assume that the process q executes infinitely many events in $\mathrm{MSC}(w)$ and that $|w|_{q?p} < |w|_{p!q}$. Since $\mathrm{MSC}(w)$ is an ideal in t, the MSC t cannot execute more events at process q than $\mathrm{MSC}(w)$. Thus, $|\lambda^{-1}(q?p)| < |\lambda^{-1}(p!q)|$, i.e., t is no MSC, a contradiction.

For the other implication, fix some linear order $\leqslant$ on the set of processes $\mathcal{P}$. To define t, let

$$V = \mathrm{dom}(w) \cup \bigcup_{p,q \in \mathcal{P} \text{ distinct}} \{(p, q, i) \mid 0 \leqslant i < |w|_{p!q} - |w|_{q?p}\}$$

where we assume $\infty - \infty = 0$ and $\infty - i = \infty$ for $i \in \mathbb{N}$. For $x \in \mathrm{dom}(w)$, let $\lambda(x) = w_x$, and for $x = (p, q, i) \in V \setminus \mathrm{dom}(w)$ set $\lambda(x) = q?p$. Finally, define $\trianglelefteq = R^*$ where $(x, y) \in R$ holds for some $x, y \in V$ iff one of the following clauses holds:

(1) $x, y \in \mathrm{dom}(w)$, $x < y$, and we have
  1.1 $\mathrm{proc}(w_x) = \mathrm{proc}(w_y)$, or
  1.2 $w_x = p!q$, $w_y = q?p$, and $|w{\restriction}_{\leqslant x}|_{p!q} = |w{\restriction}_{\leqslant y}|_{q?p}$.

(2) $x, y \in V \setminus \mathrm{dom}(w)$. Then $x = (p, q, i)$ and $y = (p', q', j)$ for some processes p, q, p' and q' and for some natural numbers i, j. In this case, we let $(x, y) \in R$ iff
  2.1 $p = p'$ and $i < j$, or
  2.2 $p = p'$, $i = j$, and $q < q'$.

(3) $x \in \mathrm{dom}(w)$, $y = (p, q, i)$, and we have
  3.1 $w_x = p!q$ and $|w{\restriction}_{\leqslant x}|_{p!q} = |w|_{q?p} + i + 1$, or
  3.2 $\mathrm{proc}(w_x) = q$.

Then let $t = (V, \trianglelefteq, \lambda)$.

The binary relation R is acyclic: the relation $R \cap \mathrm{dom}(w)^2$ is contained in the linear order of natural numbers and therefore acyclic. Similarly, the restriction of R to $V \setminus \mathrm{dom}(w)$ is contained in a lexicographic order on $\mathcal{P} \times \mathcal{P} \times \mathbb{N}$ and therefore acyclic. Since there are no edges from $V \setminus \mathrm{dom}(w)$ into $\mathrm{dom}(w)$, the relation R is indeed acyclic. Hence, $\trianglelefteq = R^*$ is a partial order on V.

Now let $x, y \in V$ be distinct with $\mathrm{proc}(x) = \mathrm{proc}(y)$. Then $(x, y) \in R$ or $(y, x) \in R$ by the clauses 1.1, 2, or 3.2. Hence the set $\{x \in V \mid \mathrm{proc}(x) = p\}$ is linearly ordered by $\trianglelefteq$ for any $p \in \mathcal{P}$.

For $y \in V$, let $\downarrow y$ be the principal ideal generated by y in $t = (V, \trianglelefteq, \lambda)$. If $y \in \mathrm{dom}(w)$, then $\downarrow y \subseteq \{x \in \mathrm{dom}(w) \mid x \leqslant y\}$. Hence, in this case, $\downarrow y$ is finite. Now let $y = (p, q, j) \in V \setminus \mathrm{dom}(w)$. Then $\downarrow y$ contains only finitely many elements (p', q', i) from $V \setminus \mathrm{dom}(w)$ since all these elements satisfy $i \leqslant j$. We show that there are only finitely many positions $x \in \mathrm{dom}(w)$ with $x \trianglelefteq y$. For any such

[1] The author thanks one of the anonymous referees for this example that marked a mistake in the submitted version of this paper. The following proposition was added to circumvent problems arising from this mistake.

x, there exist $x' \in \mathrm{dom}(w)$ and $(p', q', i) \in V \setminus \mathrm{dom}(w)$ with $x \trianglelefteq x' R(p', q', i) \trianglelefteq (p, q, j)$. Above, we saw that there are only finitely many elements (p', q', i) with $(p', q', i) \trianglelefteq (p, q, j)$ and that, for any $x' \in \mathrm{dom}(w)$, there are only finitely many elements $x \in V$ with $x \trianglelefteq x'$. Thus, it suffices to show that for any $(p', q', i) \in V \setminus \mathrm{dom}(w)$, there are only finitely many $x' \in \mathrm{dom}(w)$ with $x' R(p', q', i)$. For any such x', we have $\mathrm{proc}(x') = q'$ or $\lambda(x') = p'!q'$ and $|w{\restriction}_{\leqslant x'}|_{p'!q'} = |w|_{q'?p'} + i + 1$. Since $(p', q', i) \in V \setminus \mathrm{dom}(w)$, there are only finitely many positions $x' \in \mathrm{dom}(w)$ with $\mathrm{proc}(x') = q'$. Thus $\downarrow y$ is finite for any $y \in V$. So far, we showed that $t = (V, \trianglelefteq, \lambda)$ is a Σ-labeled poset.

Since none of the elements of $V \setminus \mathrm{dom}(w)$ is labeled by a send-action, we get

$$\begin{aligned}|\lambda^{-1}(p!q)| &= |w|_{p!q} = |w|_{q?p} + (|w|_{p!q} - |w|_{q?p}) \\ &= |\{x \in \mathrm{dom}(w) \mid \lambda(x) = q?p\}| + |\{x \in V \setminus \mathrm{dom}(w) \mid \lambda(x) = q?p\}| \\ &= |\lambda^{-1}(q?p)|\end{aligned}$$

for any two distinct processes p and q.

Recall that $x \sqsubseteq_{\mathcal{P}} y$ iff $(x, y) \in R^*$ and $\mathrm{proc}(x) = \mathrm{proc}(y)$. Since R^* is acyclic, this is equivalent to $(x, y) \in R$ and $\mathrm{proc}(x) = \mathrm{proc}(y)$, i.e., $x \sqsubseteq_{\mathcal{P}} y$ holds iff we have 1.1, 2, or 3.2.

In what follows, $\downarrow x$ is the principal ideal generated by x in the partially ordered set $(V, \trianglelefteq, \lambda)$. Recall that $x \sqsubseteq y$ iff there are two distinct processes p and q such that $\lambda(x) = p!q$, $\lambda(y) = q?p$, and $|\downarrow x \cap \lambda^{-1}(p!q)| = |\downarrow y \cap \lambda^{-1}(q?p)|$. This means in particular $x \in \mathrm{dom}(w)$. But then $\downarrow x \subseteq \mathrm{dom}(w)$ and therefore $|\downarrow x \cap \lambda^{-1}(p!q)| = |w{\restriction}_{\leqslant x}|_{p!q}$.

To show the analogous statement for $\downarrow y \cap \lambda^{-1}(q?p)$, we distinguish the cases $y \in \mathrm{dom}(w)$ and $y \notin \mathrm{dom}(w)$. If $y \in \mathrm{dom}(w)$, we get $\downarrow y \subseteq \mathrm{dom}(w)$ and therefore $|\downarrow y \cap \lambda^{-1}(q?p)| = |w{\restriction}_{\leqslant y}|_{q?p}$. Hence, for $y \in \mathrm{dom}(w)$, we have $x \sqsubseteq y$ iff 1.2 holds. If, alternatively, $y \notin \mathrm{dom}(w)$, then there is $j \in \mathbb{N}$ with $y = (p, q, j)$, and we have

$$\begin{aligned}\downarrow y \cap \lambda^{-1}(q?p) &= \{z \in \mathrm{dom}(w) \mid w_z = q?p\} \cup \{(p, q, i) \in V \setminus \mathrm{dom}(w) \mid 0 \leqslant i \leqslant j\} \\ &= |w|_{q?p} + j + 1.\end{aligned}$$

Thus, in this case $x \sqsubseteq y$ iff 3.1 holds. In other words, we showed $x \sqsubseteq y$ iff 1.2 or 3.1 holds. Hence R is the union of $\sqsubseteq_{\mathcal{P}}$ and $\sqsubseteq$ implying $\trianglelefteq = R^* = (\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^*$. Thus, indeed, $t = (V, \trianglelefteq, \lambda)$ is an MSC. Since in R, there are no edges from $\mathrm{dom}(w)$ into $V \setminus \mathrm{dom}(w)$, the set $\mathrm{dom}(w)$ is an order ideal in t, i.e., $\mathrm{MSC}(w)$ is an ideal in some MSC. $\square$

A proper word w is *complete* provided $|w|_{p!q} = |w|_{q?p}$ for $p, q \in \mathcal{P}$. Any complete proper word w satisfies the condition of the proposition above; hence $\mathrm{MSC}(w)$ is an ideal in an MSC. But even more, $\mathrm{MSC}(w)$ is a message sequence chart. The converse implication holds as well: if w is a proper word such that $\mathrm{MSC}(w)$ is an MSC, then w is complete.

Now let $t = (V, \leqslant, \lambda)$ be some Σ-labeled poset. Then $t' = (V, \preceq, \lambda)$ is an *order extension* of t provided $\leqslant \subseteq \preceq$, it is a *linear extension* if in addition $\preceq$ is a linear order. Let $(V, \preceq, \lambda)$ be a linear extension of t. Since principal ideals and antichains in t are finite, the linear order $(V, \preceq)$ is a wellorder. Let $\mathrm{Lin}_\omega(t)$ be the set of all linear extensions of t of order type at most ω. Then we can understand the elements of $\mathrm{Lin}_\omega(t)$ as finite or infinite words over Σ.

The proper word w is B-bounded if, for any $0 \leqslant i < |w|$, we have $|w{\restriction}_{\leqslant i}|_{p!q} - |w{\restriction}_{\leqslant i}|_{q?p} \leqslant B$.

Lemma 2.3. *Let $t \in {\downarrow}\mathrm{MSC}^\infty$ and $w \in \mathrm{Lin}_\omega(t)$. Then w is proper, $t = \mathrm{MSC}(w)$, and w satisfies*

$$|w|_{q?p} < |w|_{p!q} \implies \{i \in \mathrm{dom}(w) \mid \mathrm{proc}(w_i) = q\} \text{ is finite}$$

for any distinct processes p and q. If, furthermore, t is B-bounded, then the word w is B-bounded.

Proof. We consider w as a linear extension $(V, \preceq, \lambda)$ of $t = (V, \leqslant, \lambda)$. Let $v \in V$. Then the set $X = \{x \in V \mid x \preceq v\}$ is an ideal in t since w is an order extension of t. Hence $|X \cap \lambda^{-1}(p!q)| \geqslant |X \cap \lambda^{-1}(q?p)|$, i.e., the word w is proper. Now suppose $|w|_{q?p} < |w|_{p!q}$. Let $s \in \mathrm{MSC}^\infty$ be an MSC such that t is an ideal in s. Since $|\lambda^{-1}(q?p)| < |\lambda^{-1}(p!q)|$, there is a $q?p$-node v in s which does not belong to t. In s, this node v dominates only finitely many nodes. Hence t and therefore w contain only finitely many nodes labeled by some σ with $\mathrm{proc}(\sigma) = q$. Note that $(x, y) \in R$ (cf. page 83) iff $(x, y) \in (\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)$ (cf. page 83). Hence, indeed, $t = \mathrm{MSC}(w)$.

Finally let t be B-bounded and let $v \in V$. We have to show that

$$n = |\{x \in V \mid x \preceq v, \lambda(x) = p!q\}| - |\{x \in V \mid x \preceq v, \lambda(x) = q?p\}| \leqslant B.$$

Let v' be the maximal position (wrt. $\preceq$) preceding v with $\mathrm{proc}(v') = p$. Then

$$n \leqslant |\{x \in V \mid x \preceq v', \lambda(x) = p!q\}| - |\{x \in V \mid x \preceq v', \lambda(x) = q?p\}|.$$

Since $\mathrm{proc}(v') = p$, we have $\{x \in V \mid x \preceq v', \lambda(x) = p!q\} = {\downarrow}v' \cap \lambda^{-1}(p!q)$ and $\{x \in V \mid x \preceq v', \lambda(x) = q?p\} \supseteq {\downarrow}v' \cap \lambda^{-1}(q?p)$ (where ${\downarrow}v'$ is taken in $(V, \leqslant, \lambda)$). Hence we can continue

$$\begin{aligned} &\leqslant |{\downarrow}v' \cap \lambda^{-1}(p!q)| - |{\downarrow}v' \cap \lambda^{-1}(q?p)| \\ &\leqslant B \end{aligned}$$

since t is B-bounded. □

Note that an MSC t is B-bounded iff any proper word w with $\mathrm{MSC}(w) = t$ is B-bounded. On the other hand, there are proper B-bounded words w for which $\mathrm{MSC}(w)$ is not B-bounded (e.g., $w = (p!q\, q?p)^\omega$).

Message passing automata, the automata model that we consider, reflect the concurrent behavior of an MSC. It is similar to asynchronous cellular automata from the theory of Mazurkiewicz traces.

A *message passing automaton* is a structure $\mathcal{A} = ((\mathcal{A}_p)_{p\in\mathcal{P}}, \Delta, s^{in}, F)$ where

(1) Δ is a finite set of messages,
(2) each component $\mathcal{A}_p$ is of the form $(S_p, \rightarrow_p)$ where
- S_p is a finite set of local states,
- $\rightarrow_p \subseteq S_p \times \Sigma_p \times \Delta \times S_p$ where $\Sigma_p = \{\sigma \in \Sigma \mid \mathrm{proc}(\sigma) = p\}$ is a local transition relation,

(3) $s^{in} \in \prod_{p\in\mathcal{P}} S_p$ is the initial global state, and
(4) $F \subseteq \prod_{p\in\mathcal{P}} S_p$ is a set of accepting global states.

Let $(s, a, m, s') \in \rightarrow_p$ be a local transition of process p. Suppose a is a send action, i.e., $a = p!q$ for some process q. Then the transition (s, a, m, s') denotes that the process p can perform the action $a = p!q$ in state s; it changes its local state to s' and sends a message m into the FIFO-channel from process p to process q. Now suppose that $a = p?q$ is a receive action. Then the transition (s, a, m, s') denotes that the process p can change its local state from s to s' when reading the message m from the channel that leads from q to p.

Let $t = (V, \leqslant, \lambda)$ be an ideal in an MSC and let $\mathcal{A}$ be a message passing automaton. Let furthermore $r : V \rightarrow \bigcup_{p\in\mathcal{P}} S_p$ be a mapping and $v \in V$. We define a second mapping $r^- : V \rightarrow \bigcup_{p\in\mathcal{P}} S_p$: if there is $u < v$ with $\text{proc}(u) = \text{proc}(v)$, let u be maximal with this property and let $r^-(v)$ denote $r(u)$. If v is the minimal event performed by the process $\text{proc}(v)$, let $r^-(v) = s^{in}_{\text{proc}(v)}$. Then $r^-(v)$ denotes, for any $v \in V$, the local state of $\text{proc}(v)$ before executing the event v.

A *run* of $\mathcal{A}$ on t is a pair of mappings $r : V \to \bigcup_{p\in\mathcal{P}} S_p$ and $m : V \to \Delta$ that satisfies the following for any $v \in V$:

(1) If $\lambda(v) = p!q$, then there is a transition $(r^-(v), p!q, m(v), r(v))$ in $\to_p$.
(2) Now let $\lambda(v) = p?q$. Since t is an ideal in an MSC, there is a unique matching node $u \in V$ with $u \sqsubseteq v$. We require that $m(u) = m(v)$ and $(r^-(v), p?q, m(v), r(v)) \in\to_p$.

Let $t = (V, \leqslant, \lambda)$ be a finite ideal in an MSC and let (r, m) be a run of the message passing automaton $\mathcal{A}$ on t. For $p \in \mathcal{P}$ let $f_p = s^{in}_p$ if the process p does not perform any action in t. Otherwise, let $f_p = r(v)$ where v is the last event performed by the process p. The run r is *successful* if the tuple f is an accepting state in F. A finite ideal t in an MSC is *accepted* by a message passing automaton $\mathcal{A}$ if there is a successful run of $\mathcal{A}$ on t. A set K is *accepted by $\mathcal{A}$ relative to* $X \subseteq {\downarrow}\text{MSC}$ if, for any $t \in X$, $t \in K$ iff t is accepted by $\mathcal{A}$.

A message passing automaton is *deterministic* if

- $(s, p!q, m_1, s_1), (s, p!q, m_2, s_2) \in\to_p$ imply $s_1 = s_2$ and $m_1 = m_2$
- $(s, p?q, m, s_1), (s, p?q, m, s_2) \in\to_p$ imply $s_1 = s_2$.

For any MSC t, a deterministic MPA has at most one run (r, m) on t.

A *Muller message passing automaton* is a structure $\mathcal{A} = ((\mathcal{A}_p)_{p\in\mathcal{P}}, \Delta, s^{in}, \mathcal{S})$ where

(1) $((\mathcal{A}_p)_{p\in\mathcal{P}}, \Delta, s^{in}, \emptyset)$ is a deterministic message passing automaton and
(2) $\mathcal{S} \subseteq \prod_{p\in\mathcal{P}} 2^{S_p}$ is a *Muller acceptance condition*.

Let (r, m) be a run of the Muller message passing automaton $\mathcal{A} = ((\mathcal{A}_p)_{p\in\mathcal{P}}, \Delta, s^{in}, \mathcal{S})$ on $t = (V, \leqslant, \lambda) \in {\downarrow}\text{MSC}^\infty$ (formally, (r, m) is a run of the MPA $((\mathcal{A}_p)_{p\in\mathcal{P}}, \Delta, s^{in}, \emptyset)$). Furthermore, let $p \in \mathcal{P}$ be some process. Then this process can be idle in t, or it can execute finitely many or infinitely many actions. In the first case, let $X_p = \{s^{in}_p\}$. Otherwise, let $X_p \subseteq S_p$ be the set of all $s \in S_p$ such that, for any $v \in V$ with $\text{proc}(v) = p$, there exists $w \in V$ with $v \leqslant w$, $\text{proc}(w) = p$, and $r(w) = s$. In other words, X_p is the set of states assumed infinitely often by the process p (if this process performs infinitely many events), and the final state of this process otherwise. The run r is *successful* provided $(X_p)_{p\in\mathcal{P}} \in \mathcal{S}$.

3. Deterministic message passing automata and monadic second order logic

Formulas of the monadic second order language MSO over Σ involve first order variables $x, y, z\ldots$ for nodes and set variables $X, Y, Z, \ldots$ for sets of nodes. They are built up from the atomic formulas $\lambda(x) = \sigma$ for $\sigma \in \Sigma$, $x \leqslant y$, and $x \in X$ by means of the boolean connectives $\neg, \wedge$ and the quantifier $\exists$ (both for first order and for set variables). A *first order formula* is a formula without set variables. Formulas without free variables are called sentences. The satisfaction relation $t \models \varphi$ between Σ-labeled posets t and formulas φ is defined canonically with the understanding that first-order variables range over elements of V and set variables over subsets of V. Let L be a set of Σ-labeled posets. A set $K \subseteq L$ is *MSO-* or *monadically axiomatizable relative to* L iff there is a sentence φ such that $K = \{t \in L \mid t \models \varphi\}$.

Example 3.1. Let K denote the set of all finite MSCs containing a σ-node x that dominates an even number of $p!r$-labeled nodes y. We show that this set is monadically axiomatizable relative to MSC. Let φ denote the following sentence

$$\begin{aligned}
&\exists x \exists Y, E\,(\lambda(x) = \sigma \wedge \forall y(y \in Y \leftrightarrow (\lambda(y) = p!r \wedge y \leqslant x)) \\
&\qquad\qquad \wedge \forall e(e \in E \rightarrow e \in Y) \\
&\qquad\qquad \wedge \forall e, e''((e, e'' \in Y \wedge e < e'' \wedge \forall e'(e' \in Y \wedge e < e' \leqslant e'' \rightarrow e' = e'')) \\
&\qquad\qquad\qquad \rightarrow (e \in E \leftrightarrow e'' \notin E)) \\
&\qquad\qquad \wedge \exists e(e \in E \wedge \forall y(y \in Y \rightarrow e \leqslant y)) \\
&\qquad\qquad \wedge \neg \exists e(e \in E \wedge \forall y(y \in Y \rightarrow y \leqslant e)))
\end{aligned}$$

The first line expresses that x is labeled by σ and that Y is the set of all $p!r$-labeled nodes that precede x, i.e., the remaining lines should express that Y contains an even number of elements. The second line says that E is a subset of Y. The third line says that from any two consecutive elements of Y, precisely one belongs to E. This ensures that E contains every other element of Y. Finally, the last two lines express that the least element of Y belongs to E but the maximal one does not. Hence, indeed, a finite MSC satisfies the sentence φ iff it belongs to K.

A variant of this example is the set of all finite MSCs containing a σ-node x such that the number of $p!r$-labeled nodes y dominated by x is congruent n modulo B – it can be monadically axiomatized using the ideas from the example above. Later on, we will use that similar statements on the cardinality of some sets modulo B can be expressed in MSO. In particular, we can express that $|{\downarrow}u \cap \lambda^{-1}(p!q)| \equiv |{\downarrow}v \cap \lambda^{-1}(q?p)| \operatorname{mod} B$ by saying that both sets contain 0, or both sets contain 1, ... or both sets contain $B-1$ elements (modulo B).

Example 3.2. Let K denote the set of all MSCs of the form

$$\mathrm{MSC}((p!q)^n (p!q')(q'?p)(q'!q)(q?q')(q?p)^m)$$

for some $m, n \in \mathbb{N}$ with $m \leqslant n$. Due to the factor $(p!q')(q'?p)(q'!q)(q?q')$, K consists of linear orders, only. This set can be accepted by an MPA relative to ${\downarrow}$MSC: the process p performs any sequence from $(p!q)^\star(p!q')$ and stops in an accepting state. The process q' performs $(q'?p)(q'!q)$ before moving to an accepting state. Finally, the process q performs any sequence from $(q?q')(q?p)^\star$ and accepts. The only ideals in MSCs that allow a run of this MPA correspond to the words $(p!q)^n(p!q')(q'?p)(q'!q)(q?q')(q?p)^m$ with $m \leqslant n$ (since the read actions $q?p$ can only be performed if there is a message in the channel from p to q). But this set is not monadically axiomatizable since, as a set of words, it is not regular [5].

The example above shows that in general the expressive power of MPAs and that of monadic second order logic differ. The reason is that an MPA is actually a device with infinitely many internal states since a priori the channels are not bounded. I do not know whether there is a monadically axiomatizable set that cannot be accepted by any MPA. When investigating this question, it is natural to ask whether one can (effectively) complement MPAs. Again, I do not know the answer.

In this section, we relate the expressive power of MPAs and of MSO for *bounded* sets of MSCs, a setting where the problems described so far do not occur.

Remark 3.3. An important tool in our investigations are interpretations of one structure in another. In our context of labeled partial orders, the idea is as follows (cf. [22, Chapter 5] for a more detailed exposition): let Σ and Γ be two finite alphabets of labels. Furthermore, let K_Σ and K_Γ be sets of

Σ- and Γ-labeled posets. Now consider an FO-formula α with one free variable x. Then, for any $t = (V, \leqslant, \lambda) \in K_\Sigma$, the formula α defines a set $\alpha^t = \{v \in V \mid t \models \alpha(v)\}$. Similarly, a FO-formula β with free variables x and y defines a binary relation $\beta^t = \{(v, w) \in V^2 \mid t \models \beta(v, w)\}$ on V. Finally, let γ_a be a FO-formula with one free variable for $a \in \Gamma$. Suppose that for any $t \in K_\Sigma$, the structure (α^t, β^t) is a partial order and that $(\gamma_a^t)_{a\in\Gamma}$ is a decomposition of V. Then the tuple $(\alpha, \beta, (\gamma_a)_{a\in\Gamma})$ determines a Γ-labeled poset $(\alpha^t, \beta^t, (\gamma_a^t)_{a\in\Gamma})$. In this case, the mapping $t \mapsto (\alpha^t, \beta^t, (\gamma_a^t)_{a\in\Gamma})$ from K_Σ into K_Γ is an FO-interpretation. Now suppose $L_\Gamma \subseteq K_\Gamma$ is FO-axiomatizable relative to K_Γ. Then one can effectively construct an FO-sentence that axiomatizes the set

$$L_\Sigma = \{t \in K_\Sigma \mid (\alpha^t, \beta^t, (\gamma_a^t)_{a\in\Gamma}) \in L_\Gamma\}.$$

If L_Γ is FO+MOD(n)-axiomatizable (see below), then L_Σ is FO+MOD(n)-axiomatizable as well.

A more general notion is that of an MSO-interpretation (known as MSO-transduction [8] in theoretical computer science) where the formulas α, β, and γ_a are allowed to be MSO-formulas (Courcelle's MSO-transductions are more general, but we only need this restricted case). For an MSO-interpretation, the MSO-axiomatizability of L_Γ implies the MSO-axiomatizability of L_Σ.

Proposition 3.4. *Let $K \subseteq {\downarrow}\mathrm{MSC}_B$ be accepted by a message passing automaton $\mathcal{A}$ relative to ${\downarrow}\mathrm{MSC}_B$. Then K is monadically axiomatizable relative to ${\downarrow}\mathrm{MSC}_B$.*

Proof. The proof is an obvious variant of similar proofs for finite automata for words (cf. [37]), asynchronous automata for traces [38], or asynchronous cellular automata for pomsets without autoconcurrency [14]. The only difficulty is to monadically define the relation $\sqsubseteq$ in $t = (V, \leqslant, \lambda) \in {\downarrow}\mathrm{MSC}_B$. Let $u, v \in V$ be nodes with $\lambda(u) = p!q$ and $\lambda(v) = q?p$. Then $u \sqsubseteq v$ iff $|{\downarrow}u \cap \lambda^{-1}(p!q)| = |{\downarrow}v \cap \lambda^{-1}(q?p)|$. Since t is B-bounded, this is equivalent to $|{\downarrow}u \cap \lambda^{-1}(p!q)| \equiv |{\downarrow}v \cap \lambda^{-1}(q?p)| \bmod B$, $u < v$, and there are at most B actions labeled $p!q$ in the interval $[u, v] = \{x \in V \mid u \leqslant x \leqslant v\}$. Since this is expressible in monadic second order logic, the relation $\sqsubseteq$ can be defined. The rest of the proof are the standard arguments. □

We will extend results from [19,31] to infinite MSCs. Since the proofs rely on the theory of Mazurkiewicz traces, we first investigate the relation between these traces and MSCs.

3.1. The key observation

A *dependence alphabet* is a pair (Γ, D) where Γ is a finite set and $D \subseteq \Gamma^2$ is a reflexive and symmetric dependence relation. A *trace over* (Γ, D) is a Γ-labeled partial order $(V, \leqslant, \lambda')$ such that

- $(\lambda'(x), \lambda'(y)) \notin D$ whenever $x, y \in V$ are incomparable, and
- $(\lambda'(x), \lambda'(y)) \in D$ whenever $x \longrightarrow\!\!< y$.

The set of all traces over (Γ, D) is denoted by $\mathbb{R}(\Gamma, D)$, the set $\mathbb{M}(\Gamma, D)$ comprises the finite traces.

The key observation that is announced by the title of this section is that any MSO-axiomatizable and B-bounded set of MSCs is the "relabeling" of a monadically axiomatizable set of traces over a suitable dependence alphabet. The bound B influences the chosen dependence alphabet as defined in the following paragraph.

For a positive integer $B \in \mathbb{N}$, let $\Gamma = \Sigma \times \{0, 1, \ldots, B-1\}$. On this alphabet, we define a dependence relation D as follows: (σ_1, n_1) and (σ_2, n_2) are dependent iff

(1) $\text{proc}(\sigma_1) = \text{proc}(\sigma_2)$, or
(2) $\{(\sigma_1, n_1), (\sigma_2, n_2)\} = \{(p!q, n), (q?p, n)\}$ for some $p, q \in \mathcal{P}$ and $0 \leqslant n < B$.
For $t = (V, \leqslant, \lambda) \in {\downarrow}\text{MSC}_B^\infty$, we define a new Γ-labeling λ' by

$$\lambda'(v) = (\lambda(v), |{\downarrow} v \cap \lambda^{-1}\lambda(v)| \bmod B),$$

i.e., the first component of the label is the old label and the second counts modulo B the number of occurrences of the same action in the past of v. We then define $\text{tr}(t) = (V, \leqslant, \lambda')$. The following three lemmata show that $\{\text{tr}(t) \mid t \in {\downarrow}\text{MSC}_B^\infty\}$ is a first-order axiomatizable set of traces in $\mathbb{R}(\Gamma, D)$.

Lemma 3.5. *Let $t = (V, \leqslant, \lambda) \in {\downarrow}\text{MSC}_B^\infty$ be an ideal in an MSC. Then $\text{tr}(t)$ is a trace over (Γ, D).*

Proof. Let $v_1, v_2 \in V$ with $\lambda(v_i) = p_i\theta_i q_i$. Suppose $v_1 \longrightarrow\!\!\!< v_2$. Since the partial order $\leqslant$ is the transitive and reflexive closure of $\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq$, there are two possibilities:
(1) If $p_1 = p_2$, we obtain immediately $(\lambda'(v_1), \lambda'(v_2)) \in D$.
(2) If $p_1 \neq p_2$, then $v_1 \sqsubseteq v_2$, i.e., $\lambda(v_1) = p_1!p_2$, $\lambda(v_2) = p_2?p_1$, and $|{\downarrow} v_1 \cap \lambda^{-1}(p_1!p_2)| = |{\downarrow} v_2 \cap \lambda^{-1}(p_2?p_1)|$. Hence $(\lambda'(v_1), \lambda'(v_2)) \in D$.

Now suppose that v_1 and v_2 are incomparable. Then $p_1 \neq p_2$. By contradiction, we assume $(\lambda'(v_1), \lambda'(v_2)) \in D$. Then, w.l.o.g., $\lambda(v_1) = p!q$, $\lambda(v_2) = q?p$, and

$$|{\downarrow} v_1 \cap \lambda^{-1}(p!q)| \equiv |{\downarrow} v_2 \cap \lambda^{-1}(q?p)| \bmod B. \qquad (\star)$$

Since $v_1 \nleqslant v_2$ and $\lambda(v_1) = p!q$, we have $|{\downarrow} v_1 \cap \lambda^{-1}(p!q)| > |{\downarrow} v_2 \cap \lambda^{-1}(p!q)| \geqslant |{\downarrow} v_2 \cap \lambda^{-1}(q?p)|$. By ($\star$), this implies $|{\downarrow} v_1 \cap \lambda^{-1}(p!q)| - |{\downarrow} v_2 \cap \lambda^{-1}(q?p)| \geqslant B$. Since, on the other hand, $v_2 \nleqslant v_1$ and $\lambda(v_2) = q?p$, we also have $|{\downarrow} v_1 \cap \lambda^{-1}(q?p)| < |{\downarrow} v_2 \cap \lambda^{-1}(q?p)|$. Hence

$$\begin{aligned} B &\leqslant |{\downarrow} v_1 \cap \lambda^{-1}(p!q)| - |{\downarrow} v_2 \cap \lambda^{-1}(q?p)| \\ &< |{\downarrow} v_1 \cap \lambda^{-1}(p!q)| - |{\downarrow} v_1 \cap \lambda^{-1}(q?p)|. \end{aligned}$$

But this contradicts our assumption that t is B-bounded. $\square$

We will consider the following three properties of a trace $s \in \mathbb{R}(\Gamma, D)$. Note that they can be expressed in first-order logic:
(I) $s\restriction_{\{(p!q,n),(q?p,n)\}}$ is a prefix of $((p!q, n)(q?p, n))^\omega$ for $p, q \in \mathcal{P}$ and $0 \leqslant n < B$.
(II) $s\restriction_{\{(\sigma,n) \mid 0 \leqslant n < B\}}$ is a prefix of $((\sigma, 1)(\sigma, 2) \dots (\sigma, B-1)(\sigma, 0))^\omega$ for $\sigma \in \Sigma$.
(III) If $v, w \in V$ with $v \longrightarrow\!\!\!< w$, then $\text{proc} \circ \lambda'(v) = \text{proc} \circ \lambda'(w)$ or $\lambda'(v) = (p!q, n)$ and $\lambda'(w) = (q?p, n)$ for some $p, q \in \mathcal{P}$ and $0 \leqslant n < B$.

Lemma 3.6. *Let $t = (V, \leqslant, \lambda) \in {\downarrow}\text{MSC}_B^\infty$. Then $s = \text{tr}(t)$ satisfies I, II, and III.*

Proof. Let $t \in \text{MSC}_B^\infty$ and let t' be an ideal in t. Then $\text{tr}(t')$ is an ideal in $\text{tr}(t)$. Hence it suffices to prove the lemma for B-bounded MSCs.

Property II is immediate by the definition of $\text{tr}(t)$. Next we show property I by contradiction. Since the nth sending precedes the nth receiving on the channel from p to q in t, the word $s\restriction_{\{(p!q,n),(q?p,n)\}}$ starts with $(p!q, n)$. Assume $s\restriction_{\{(p!q,n),(q?p,n)\}}$ contains a factor $(q?p, n)(q?p, n)$. Then there are $v_1, v_2 \in V$ with $\lambda'(v_i) = (q?p, n)$, $v_1 < v_2$, and there is no $u \in V$ with $v_1 < u < v_2$ and $\lambda'(u) = (p!q, n)$. Since t is an MSC, there exists $u \in V$ with $\lambda(u) = p!q$ and $|{\downarrow} u \cap \lambda^{-1}(p!q)| = |{\downarrow} v_2 \cap \lambda^{-1}(q?p)|$. Hence

$\lambda'(u) = (p!q, n)$ and therefore $v_1 \not< u < v_2$, i.e., $u < v_1$ since $\lambda'(u)$ and $\lambda'(v_1)$ are dependent. Now we consider the ideal $\downarrow u$:

$$\begin{aligned}|\downarrow u \cap \lambda^{-1}(p!q)| - |\downarrow u \cap \lambda^{-1}(q?p)| &= |\downarrow v_2 \cap \lambda^{-1}(q?p)| - |\downarrow u \cap \lambda^{-1}(q?p)| \\ &> |\downarrow v_2 \cap \lambda^{-1}(q?p)| - |\downarrow v_1 \cap \lambda^{-1}(q?p)|\end{aligned}$$

since $u < v_1$ and $\lambda(v_1) = q?p$. Since $|\downarrow v_2 \cap \lambda^{-1}(q?p)| > |\downarrow v_1 \cap \lambda^{-1}(q?p)|$ are congruent modulo B, we get $|\downarrow u \cap \lambda^{-1}(p!q)| - |\downarrow u \cap \lambda^{-1}(q?p)| > B$, contradicting the B-boundedness of t.

We also have to consider the case

$$s\restriction_{\{(p!q,n),(q?p,n)\}} \in ((p!q, n)(q?p, n))^{\star}(p!q, n)(p!q, n)W$$

for some word W: Similarly to above, there are $v_1, v_2 \in V$ with $\lambda'(v_i) = (p!q, n)$, $v_1 < v_2$, and there is no $w \in V$ with $v_1 < w < v_2$ and $\lambda'(w) = (q?p, n)$. Since t is an MSC, there exists $w \in V$ with $\lambda(w) = q?p$ and $|\downarrow w \cap \lambda^{-1}(q?p)| = |\downarrow v_1 \cap \lambda^{-1}(p!q)|$. Hence in particular $v_1 < w \not< v_2$, i.e., $v_2 < w$. Now we consider the ideal $\downarrow v_2$:

$$\begin{aligned}&|\downarrow v_2 \cap \lambda^{-1}(p!q)| - |\downarrow v_2 \cap \lambda^{-1}(q?p)| \\ &\quad > |\downarrow v_2 \cap \lambda^{-1}(p!q)| - |\downarrow w \cap \lambda^{-1}(q?p)| \text{ since } v_2 < w \text{ and } \lambda(w) = q?p \\ &\quad = |\downarrow v_2 \cap \lambda^{-1}(p!q)| - |\downarrow v_1 \cap \lambda^{-1}(p!q)|\end{aligned}$$

again contradicting the B-boundedness of t.

It remains to show that s satisfies property III: Let $v, w \in V$ with $v \longrightarrow\!\!< w$ and $\mathrm{proc} \circ \lambda'(v) \neq \mathrm{proc} \circ \lambda'(w)$. Since $\leqslant = (\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^{\star}$, this implies $v \sqsubseteq w$ and therefore $\lambda'(v) = (p!q, n)$ and $\lambda'(w) = (q?p, n)$ for some $p, q \in \mathcal{P}$ and $0 \leqslant n < B$. $\square$

Lemma 3.7. *Let $s \in \mathbb{R}(\Gamma, D)$ be a trace satisfying I, II and III. Then $s \in \mathrm{tr}[\downarrow \mathrm{MSC}_B^{\infty}]$.*

Proof. Let $s = (V, \leqslant, \lambda')$ and define $\lambda(v) = \pi_1 \circ \lambda'(v)$ for $v \in V$ where π_1 is the projection to the first component. Then $t = (V, \leqslant, \lambda)$ is a Σ-labeled poset. From II, we obtain immediately $\mathrm{tr}(t) = s$. So it remains to be shown that t is an ideal in a B-bounded MSC:

- Since $\gamma, \delta \in \Gamma$ are dependent whenever $\mathrm{proc}(\gamma) = \mathrm{proc}(\delta)$, the set $\lambda'^{-1} \circ \mathrm{proc}^{-1}(p)$ is linearly ordered in the trace s. Hence $\sqsubseteq_{\mathcal{P}}$ is a linear order on $\lambda^{-1}(\{p\theta q \mid q \in \mathcal{P}, \theta \in \{?, !\}\})$ for any $p \in \mathcal{P}$.
- Let $v \sqsubseteq w$, i.e., $\lambda(v) = p!q$, $\lambda(w) = q?p$, and $|\downarrow v \cap \lambda^{-1}(p!q)| = |\downarrow w \cap \lambda^{-1}(q?p)|$. Then, by II, $\lambda'(v) = (p!q, n)$ and $\lambda'(w) = (q?p, n)$ for some n, and $|\downarrow v \cap \lambda'^{-1}\lambda'(v)| = |\downarrow w \cap \lambda'^{-1}\lambda'(w)|$. Since the labels $(p!q, n)$ and $(q?p, n)$ occur alternatingly in s, statement I implies $v < w$. Hence $(\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^{\star} \subseteq \leqslant$.
- Let $v \longrightarrow\!\!< w$. Then $(\lambda'(v), \lambda'(w)) \in D$. If $\mathrm{proc} \circ \lambda'(v) = \mathrm{proc} \circ \lambda'(w)$, then $(v, w) \in \sqsubseteq_{\mathcal{P}}$. Otherwise, by III, $\lambda'(v) = (p!q, n)$ and $\lambda'(w) = (q?p, n)$ for some $p, q \in \mathcal{P}$ distinct and $0 \leqslant n < B$. Hence, by I, $|\downarrow v \cap \lambda^{-1}(p!q)| = |\downarrow w \cap \lambda^{-1}(q?p)|$, i.e., $v \sqsubseteq w$. Thus, we showed $\leqslant \subseteq (\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^{\star}$.
- Suppose, by contradiction, that t is not B-bounded. Then there exist $v \in V$ and $p, q \in \mathcal{P}$ with $|\downarrow v \cap \lambda^{-1}(p!q)| - |\downarrow v \cap \lambda^{-1}(q?p)| > B$. Hence, by II, there is n with $0 \leqslant n < B$ such that $|\downarrow v \cap \lambda'^{-1}(p!q, n)| - |\downarrow v \cap \lambda'^{-1}(q?p, n)| > 1$. But this contradicts I. $\square$

Proposition 3.8. *Let $K \subseteq \downarrow \mathrm{MSC}_B^{\infty}$. Then K is monadically axiomatizable relative to $\downarrow \mathrm{MSC}_B^{\infty}$ if and only if $\mathrm{tr}[K] \subseteq \mathbb{R}(\Gamma, D)$ is monadically axiomatizable relative to $\mathbb{R}(\Gamma, D)$.*

Proof. First, let $\mathrm{tr}[K]$ be monadically axiomatizable relative to $\mathbb{R}(\Gamma, D)$. In MSO, we can write a formula $\varphi_n(v)$ with one free variable such that $t \models \varphi_n(v)$ iff $|{\downarrow} v \cap \lambda^{-1}\lambda(v)| \bmod B = n$ for any $t = (V, \leqslant, \lambda) \in {\downarrow}\mathrm{MSC}_B^\infty$ and $v \in V$ (cf. Example 3.1). Hence $\mathrm{tr}(t)$ can be MSO-interpreted in t. Since $\mathrm{tr}[K]$ is monadically axiomatizable relative to $\mathbb{R}(\Gamma, D)$, the set K is MSO-axiomatizable relative to ${\downarrow}\mathrm{MSC}_B^\infty$. Conversely, let K be monadically axiomatizable. By Lemma 3.5, $\mathrm{tr}[{\downarrow}\mathrm{MSC}_B^\infty]$ is a set of traces. Note that the properties I, II, and III are expressible by a first-order sentence. Hence the set $\mathrm{tr}[{\downarrow}\mathrm{MSC}_B^\infty]$ is first-order axiomatizable relative to $\mathbb{R}(\Gamma, D)$ by Lemmas 3.6 and 3.7. Since the labeling of $t \in {\downarrow}\mathrm{MSC}_B^\infty$ can be recovered from the labeling of $\mathrm{tr}(t)$, we can FO-interpret t in $\mathrm{tr}(t)$. Hence $\mathrm{tr}[K]$ is monadically axiomatizable relative to $\mathbb{R}(\Gamma, D)$. $\square$

So far, we transformed any monadically axiomatizable set of bounded MSCs into a monadically axiomatizable set of traces. In order to make use of this transformation, we need the following definitions and results from the theory of Mazurkiewicz traces.

Let $t = (V, \leqslant, \lambda')$ be a trace over (Γ, D) and let $\Delta \subseteq \Gamma$. Then $\partial_\Delta(t)$ is the least ideal in t such that the complementary filter does not contain any Δ-labeled vertex. Let $a \in \Gamma$. Then $D(a) = \{b \in \Gamma \mid (a, b) \in D\}$. Furthermore, ta is the unique trace $(V \dot\cup \{\star\}, \leqslant', \rho)$ with $ta\restriction_V = t$, $\rho(\star) = a$, and $\star \in \max(ta)$. A mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ is *asynchronous* if, for any $\Delta_1, \Delta_2 \subseteq \Gamma$, any $\gamma \in \Gamma$, and any $t \in \mathbb{M}(\Gamma, D)$,
(1) $\mu(\partial_{\Delta_1 \cup \Delta_2}(t))$ is completely determined by $\mu(\partial_{\Delta_1}(t))$, $\mu(\partial_{\Delta_2}(t))$, and the sets Δ_1 and Δ_2, and
(2) $\mu(\partial_\gamma(t\gamma))$ is completely determined by $\mu(\partial_{D(\gamma)}(t))$ and the letter γ.

Theorem 3.9 (cf. [6, 17, 38]). *Let (Γ, D) be a dependence alphabet and $L \subseteq \mathbb{M}(\Gamma, D)$. Then L is monadically axiomatizable if, and only if, there exists an asynchronous mapping μ into some finite set such that $L = \mu^{-1}\mu[L]$.*

This result was used to construct a deterministic asynchronous cellular automaton that accepts a given recognizable language of finite traces. Diekert and Muscholl [11] use the same concept of an asynchronous mapping to construct a deterministic asynchronous cellular automaton with Muller acceptance condition that accepts a given recognizable set of infinite traces. In order to state their result, we need some more notations.

Let (Γ, D) be a dependence alphabet and let $t = (V, \leqslant, \lambda') \in \mathbb{R}(\Gamma, D)$ be a trace. For $v \in V$ let $\mu'(v) = \mu(t\restriction_{\downarrow v})$, i.e., μ' maps the nodes of V to the set A. Now let $\gamma \in \Gamma$ such that $\lambda'^{-1}(\gamma)$ is not empty. Then $\lambda'^{-1}(\gamma)$ is linearly ordered. We let $\mu_\gamma^\infty(t) \subseteq A$ denote the set of all $a \in A$ such that $\lambda'^{-1}(\gamma) \cap \mu'^{-1}(a)$ is cofinal in $\lambda'^{-1}(\gamma)$, i.e., any element of $\lambda'^{-1}(\gamma)$ is dominated by some element from $\lambda'^{-1}(\gamma) \cap \mu'^{-1}(a)$.

If $\lambda'^{-1}(\gamma)$ is infinite, then it is order-isomorphic to ω. Hence $\mu_\gamma^\infty(t)$ is the set of all $a \in A$ for which there are infinitely many nodes $v \in V$ with $\lambda'(v) = \gamma$ and $\mu(t\restriction_{\downarrow v}) = a$. If $\lambda'^{-1}(\gamma)$ is finite and non-empty, then $\mu_\gamma^\infty(t) = \{\mu(\partial_\gamma(t))\}$. Therefore, we define in the remaining case, i.e., if $\lambda'^{-1}(\gamma) = \emptyset$, $\mu_\gamma^\infty(t) = \{\mu(\partial_\gamma(t))\}$ (note that $\partial_\gamma(t)$ is the empty trace $\emptyset$ in this case).

Theorem 3.10 (cf. [11, 17]). *Let (Γ, D) be a dependence alphabet and $L \subseteq \mathbb{R}(\Gamma, D)$ be MSO-axiomatizable. Then there exists a finite set A, a set $\mathcal{T} \subseteq \prod_{\gamma \in \Gamma} 2^A$ of Γ-tuples of subsets of A, and an asynchronous mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ such that for $t \in \mathbb{R}(\Gamma, D)$, we have:*

$$t \in L \iff (\mu_\gamma^\infty(t))_{\gamma \in \Gamma} \in \mathcal{T}.$$

This theorem is not stated in this form in [11], but it can be extracted from the proof of [11, Theorem 4.2].

3.2. The construction of deterministic message passing automata

Above, we associated to any monadically axiomatizable subset of $\downarrow\mathrm{MSC}_B$ a monadically axiomatizable set of traces. By Theorem 3.9, we therefore get an asynchronous mapping. Next, we construct a message passing automaton from an asynchronous mapping.

Proposition 3.11. *Let $\mu : \mathbb{M}(\Gamma, D) \to A$ be some asynchronous mapping into a finite set A. Then there exists a deterministic message passing automaton $\mathcal{A}$ with local state space S and a function $f : S \to A$ with the following properties*:

- *any $t \in \downarrow\mathrm{MSC}_B^\infty$ admits some run (r, m) of $\mathcal{A}$.*
- *$f(s_p^{in}) = \mu(\emptyset)$ for any $p \in \mathcal{P}$ (where $\emptyset$ denotes the empty trace).*
- *for the run (r, m) of $\mathcal{A}$ on $t = (V, \leqslant, \lambda) \in \downarrow\mathrm{MSC}_B^\infty$, we have $f(r(v)) = \mu(\mathrm{tr}(t\restriction_{\downarrow v}))$ for any $v \in V$.*

Proof. We construct the MPA with local state space $S = A \times (\{0, 1, \dots, B-1\}^{|\mathcal{P}|})^2$ and message set A. Then $f(a, \mathrm{snd}, \mathrm{rcv}) = a$ for $(a, \mathrm{snd}, \mathrm{rcv}) \in S$. A run (r, m) on $t = (V, \leqslant, \lambda) \in \downarrow\mathrm{MSC}_B^\infty$ will (for $v \in V$ with $\mathrm{proc}(v) = p$) satisfy:

$$\begin{aligned} r(v) &= (\mu(\mathrm{tr}(t\restriction_{\downarrow v})), \mathrm{snd}, \mathrm{rcv}) \text{ where}\\ \mathrm{snd}(q) &= |{\downarrow v} \cap \lambda^{-1}(p!q)| \bmod B \text{ and}\\ \mathrm{rcv}(q) &= |{\downarrow v} \cap \lambda^{-1}(p?q)| \bmod B \text{ for } q \in \mathcal{P}\end{aligned}$$

and

$$m(v) = \begin{cases} \mu(\mathrm{tr}(t\restriction_{\downarrow v})) & \text{if } \lambda(v) \text{ is a send-action}\\ \mu(\mathrm{tr}(t\restriction_{\downarrow u})) & \text{if } \lambda(v) \text{ is a receive-action and } u \text{ is the matching send-event.}\end{cases}$$

To achieve this, we let the tuple $((a, \mathrm{snd}, \mathrm{rcv}), \tau, b, (a', \mathrm{snd}', \mathrm{rcv}'))$ from $S \times \Sigma \times A \times S$ be a transition iff one of the following two conditions holds:

(1) $\tau = p!q$, $b = a'$, $\mathrm{rcv} = \mathrm{rcv}'$, and

$$\mathrm{snd}'(r) = \begin{cases} \mathrm{snd}(r) + 1 \bmod B & \text{if } r = q\\ \mathrm{snd}(r) & \text{otherwise.}\end{cases}$$

Furthermore, in this case, we require the existence of some finite trace s with $\partial_{D(\tau,\mathrm{snd}'(q))}(s) = s$, $\mu(s) = a$, and $a' = \mu(s(\tau, \mathrm{snd}'(q)))$.

(2) $\tau = q?p$, $\mathrm{snd} = \mathrm{snd}'$, and

$$\mathrm{rcv}(r) = \begin{cases} \mathrm{rcv}(r) + 1 \bmod B & \text{if } r = p\\ \mathrm{rcv}(r) & \text{otherwise.}\end{cases}$$

Furthermore, in this case, we require the existence of some finite trace s with $\partial_{D(\tau,\mathrm{rcv}'(p))}(s) = s$, $a = \mu(\partial_{D(\tau,\mathrm{rcv}'(p))\setminus\{(p!q,\mathrm{rcv}'(p))\}}(s))$, $b = \mu(\partial_{(p!q,\mathrm{rcv}'(p))}(s))$, and $a' = \mu(s(\tau, \mathrm{rcv}'(p)))$.

The initial state s^{in} is given by $s_p^{in} = (\mu(\emptyset), \mathrm{snd}, \mathrm{rcv})$ with $\mathrm{snd}(q) = \mathrm{rcv}(q) = 0$ for all $q \in \mathcal{P}$.

To convince ourselves that the MPA defined this way is deterministic we have to inspect the transitions. The only potential cause of non-determinism is the trace s whose existence is required in both cases above. We start with the first case. Thus, assume s_1 and s_2 are traces with $\partial_{D(\tau,\mathrm{snd}'(q))}(s_i) = s_i$ and $\mu(s_i) = a$ for $i = 1, 2$. The second condition on asynchronous mappings implies $\mu(s_1(\tau, \mathrm{snd}'(p))) = \mu(s_2(\tau, \mathrm{snd}'(p)))$, thus, the state $(a', \mathrm{snd}', \mathrm{rcv}')$ and therefore the message b are completely determined by the state $(a, \mathrm{snd}, \mathrm{rcv})$ and the action $p!q$. In case of a receive-event, let again s_1 and s_2 be traces with $\partial_{D(\tau,\mathrm{rcv}'(p))}(s_i) = s_i$, $a = \mu(\partial_{D(\tau,\mathrm{rcv}'(p))\setminus\{(p!q,\mathrm{rcv}'(p))\}}(s_i))$ and $b = \mu(\partial_{(p!q,\mathrm{rcv}'(p))}(s_i))$. By the first condition on the asynchronous mapping μ, we obtain $\mu(\partial_{D(\tau,\mathrm{rcv}'(p))}(s_1)) = \mu(\partial_{D(\tau,\mathrm{rcv}'(p))}(s_2))$. Now the second condition on μ implies $\mu(s_1(\tau, \mathrm{rcv}'(p))) = \mu(s_2(\tau, \mathrm{rcv}'(p)))$. In other words, the state $(a, \mathrm{snd}, \mathrm{rcv})$, the action $q?p$, and the message b determine the state $(a, \mathrm{snd}, \mathrm{rcv})$ completely. Hence, indeed, the MPA defined above is deterministic, i.e., it has at most one run on any ideal in an MSC.

Now let $t = (V, \leqslant \lambda)$ be an ideal in a B-bounded MSC. Define the mappings $r : V \to S$ and $m : V \to A$ by the above requirements on r and m. To check that this is a run, let $v \in V$ be arbitrary and define $s = t\restriction_{\downarrow v\setminus\{v\}}$. With these settings, one can indeed verify that (r, m) is a run. Hence the MPA has a unique run on any $t \in \downarrow\mathrm{MSC}_B^\infty$ and this run satisfies $f(r(v)) = \mu(\mathrm{tr}(t\restriction_{\downarrow v}))$ for any $v \in V$. $\square$

3.3. The finite case

Recall that Mukund et al. [31] showed that any B-bounded and MSO-axiomatizable set of MSCs can be accepted by a deterministic MPA relative to MSC_B. A preliminary version of it, claiming the existence of a *non*deterministic MPA was shown in [21]. The following lemma gives a slight extension of this result to ideals in finite MSCs.

Lemma 3.12 (cf. [31, Theorem 3.2])**.** *Let $K \subseteq \downarrow\mathrm{MSC}_B$ be monadically axiomatizable. Then there is a deterministic message passing automaton that accepts K relative to $\downarrow\mathrm{MSC}_B$.*

Proof. By Proposition 3.8, $L = \mathrm{tr}[K] \subseteq \mathbb{M}(\Gamma, D)$ is monadically axiomatizable relative to $\mathbb{M}(\Gamma, D)$. By Theorem 3.9, there exists an asynchronous mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ into some finite set A such that $L = \mu^{-1}\mu[L]$. Let $\mathcal{A}$ denote the deterministic message passing automaton from Proposition 3.11. Let $t \in \downarrow\mathrm{MSC}_B$. Then there is precisely one run of $\mathcal{A}$ on t. From the final state reached by this automaton, we can read of $\mu(\partial_a(\mathrm{tr}(t)))$ for $a \in \Gamma$. Since μ is asynchronous, we can determine $\mu(\mathrm{tr}(t))$, i.e., whether $\mathrm{tr}(t)$ belongs to L or not. But this is the case iff $t \in K$. $\square$

The rest of this section aims at showing that the set K above is accepted by a deterministic MPA not only relative to $\downarrow\mathrm{MSC}_B$, but relative to all ideals in finite MSCs, i.e., relative to the set $\downarrow$MSC. In the light of the above lemma, it therefore remains to show that $\downarrow\mathrm{MSC}_B$ can be accepted by a deterministic MPA relative to $\downarrow$MSC. This is achieved by the following two lemmas.

Lemma 3.13. *The set $\downarrow\mathrm{MSC}_B$ is MSO-axiomatizable relative to $\downarrow$MSC.*

Proof. Let $p, q \in \mathcal{P}$ be distinct processes. For $t = (V, \leqslant, \lambda) \in \downarrow\mathrm{MSC}$, we write $\mathrm{cont}(v)$ for $|\downarrow v \cap \lambda^{-1}(p!q)| - |\downarrow v \cap \lambda^{-1}(q?p)|$. Note that $t \in \downarrow\mathrm{MSC}^\infty$ is B-bounded iff (for any pair of processes p and q) we have $\mathrm{cont}(v) \leqslant B$ for all $v \in V$. We show that the set of ideals in MSCs satisfying $\mathrm{cont}(v) \leqslant B$ for $v \in V$ is MSO-axiomatizable; the result follows since $\mathcal{P}$ is finite.

Let $X_0, X_1, \ldots, X_B$ be set variables. Then we can express in MSO that the following holds for any node v in an ideal in a MSC:

(1) If v is minimal and $\lambda(v) = p!q$, then $v \in X_1$. If v is minimal, but $\lambda(v) \neq p!q$, then $v \in X_0$.

(2) If v is not minimal, then there exists a node u with $u \longrightarrow\!\!< v$. Suppose $u \in X_n$. If $\lambda(v) \notin \{p!q, q?p\}$, then $v \in X_n$. If $\lambda(v) = q?p$ and $n > 0$, then $v \in X_{n-1}$. If $\lambda(v) = p!q$ and $n < B$, then $v \in X_{n+1}$.

Let $\mu(X_0, \ldots, X_B)$ be an MSO-formula with free variables expressing that the above conditions hold for any node v. Then we can write a formula $\mu'(X_0, \ldots, X_B)$ expressing that the tuple satisfies μ and that the sets $X_0, X_1, \ldots, X_B$ are minimal satisfying μ. But this is equivalent to

$$X_n = \{v \in V \mid \mathrm{cont}(v) = n \text{ and } \forall u < v : \mathrm{cont}(u) \leqslant B\}$$

Hence, all nodes v satisfy $\mathrm{cont}(v) \leqslant B$ iff V is the union of the sets X_n; i.e., an ideal t in an MSC belongs to $\downarrow\mathrm{MSC}_B^\infty$ iff it satisfies

$$\exists X_0, X_1, \ldots, X_B \left(\mu'(X_0, \ldots, X_B) \wedge \forall v \left(\bigvee_{0 \leqslant n \leqslant B} v \in X_n \right) \right). \quad \square$$

Lemma 3.14. *Let $B > 1$. The set $\downarrow\mathrm{MSC}_{B-1}$ can be accepted by a deterministic message passing automaton relative to $\downarrow\mathrm{MSC}$.*

Proof. Let

$$H = \{t \in \downarrow\mathrm{MSC} \mid \text{ any proper ideal in } t \text{ belongs to } \downarrow\mathrm{MSC}_{B-1}, \text{ but } t \notin \downarrow\mathrm{MSC}_{B-1}\}.$$

By Lemma 3.13, H is an MSO-axiomatizable subset of $\downarrow\mathrm{MSC}_B$. Hence $\mathrm{tr}[H] \subseteq \mathbb{M}(\Gamma, D)$ is monadically axiomatizable, i.e., there exists an asynchronous mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ into some finite set that recognizes $\mathrm{tr}[H]$ (Theorem 3.9). Let $\mathcal{A}$ be the deterministic MPA from Proposition 3.11 with state space S and let $f : S \to A$ be the mapping such that for the run (r, m) of $\mathcal{A}$ on $t = (V, \leqslant, \lambda) \in \downarrow\mathrm{MSC}_B^\infty$, we have $f(r(v)) = \mu(\mathrm{tr}(t\restriction_{\downarrow v}))$ for any $v \in V$. Let $\mathcal{A}'$ be the MPA obtained from $\mathcal{A}$ by deleting all local states s with $f(s) \in \mu[\mathrm{tr}[H]]$. Furthermore, any tuple of local states is accepting in $\mathcal{A}'$, i.e., the MPA $\mathcal{A}'$ accepts as soon as it has a run.

We first show that $\mathcal{A}'$ accepts all elements of $\downarrow\mathrm{MSC}_{B-1}$. So assume $t \in \downarrow\mathrm{MSC}_{B-1}$. Since $\downarrow\mathrm{MSC}_{B-1} \subseteq \downarrow\mathrm{MSC}_B$, the MPA $\mathcal{A}$ has a unique run (r, m) on t. Furthermore, for any node v of t, we have $t\restriction_{\downarrow v} \notin H$. Hence $f(r(v)) \notin \mu[\mathrm{tr}[H]]$, i.e., the pair (r, m) is actually a run of the restricted automaton $\mathcal{A}'$.

Next, let $t \in \downarrow\mathrm{MSC} \setminus \downarrow\mathrm{MSC}_{B-1}$ and suppose that $\mathcal{A}'$ has a run (r, m) on t. Then (r, m) is also a run of the MPA $\mathcal{A}$. Since $t \notin \downarrow\mathrm{MSC}_{B-1}$, there is a minimal node v in t with $t\restriction_{\downarrow v} \notin \downarrow\mathrm{MSC}_{B-1}$. But this implies $t\restriction_{\downarrow v} \in H$. Hence $f(r(v)) = \mu(\mathrm{tr}(t\restriction_{\downarrow v})) \in \mu[\mathrm{tr}[H]]$. But this means that $r(v)$ is not a state of the restricted automaton $\mathcal{A}'$, a contradiction. $\square$

Now we can prove the first sharpening of the result from [31].

Theorem 3.15. *Let $K \subseteq \downarrow\mathrm{MSC}_B$. Then K is MSO-axiomatizable relative to $\downarrow\mathrm{MSC}$ iff there exists a deterministic message passing automaton that accepts L relative to $\downarrow\mathrm{MSC}$.*

Proof. Let K be MSO-axiomatizable relative to $\downarrow\mathrm{MSC}$. By Lemma 3.12, there exists a deterministic message passing automaton $\mathcal{A}_1$ such that $t \in \downarrow\mathrm{MSC}_B$ is accepted by $\mathcal{A}_1$ iff it belongs to K.

By Lemma 3.14, $\downarrow\mathrm{MSC}_B$ can be accepted relative to $\downarrow\mathrm{MSC}$ by a deterministic MPA. Hence K is the intersection of two sets that are acceptable by deterministic MPAs relative to $\downarrow\mathrm{MSC}$.

The other implication is Proposition 3.4. $\square$

3.4. *The infinite case*

Theorem 3.16. *Let $K \subseteq \downarrow\mathrm{MSC}_B^\infty$. Then K is MSO-axiomatizable relative to $\downarrow\mathrm{MSC}^\infty$ iff there exists a Muller message passing automaton that accepts K relative to $\downarrow\mathrm{MSC}^\infty$.*

Proof. Similarly to Proposition 3.4, one can show that any Muller MPA corresponds to a sentence of MSO.

Now let $K \subseteq \downarrow\mathrm{MSC}_B^\infty$ be MSO-axiomatizable relative to $\downarrow\mathrm{MSC}^\infty$. Since one can FO-interpret $t \in \downarrow\mathrm{MSC}_B^\infty$ in the trace $\mathrm{tr}(t)$, the set $\mathrm{tr}[K] \subseteq \mathbb{R}(\Gamma, D)$ is MSO-axiomatizable relative to $\mathbb{R}(\Gamma, D)$. By Theorem 3.10, there exists an asynchronous mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ into some finite set A and a set $\mathcal{T} \subseteq \prod_{p\in\mathcal{P}} 2^A$ that accept $\mathrm{tr}[K] \subseteq \mathbb{R}(\Gamma, D)$.

Before we apply Proposition 3.11, we enrich the asynchronous mapping μ by some more information. First, there is an asynchronous mapping $\mu_h : \mathbb{M}(\Gamma, D) \to A_h$ into some finite set such that for any $s \in \mathbb{M}(\Gamma, D)$, $\gamma \in \Gamma$ and $\Delta_1, \Delta_2 \subseteq \Gamma$, the values $\mu_h(\partial_{\Delta_1}(s))$ and $\mu_h(\partial_{\Delta_2}(s))$ determine whether $\partial_\gamma\partial_{\Delta_1}(s) \leqslant \partial_\gamma\partial_{\Delta_2}(s)$ or not (cf. [9, Cor. 2.4.11]). Hence the following mapping $\overline{\mu} : \mathbb{M}(\Gamma, D) \to A \times A_h \times A^\Gamma$ is asynchronous:

$$\overline{\mu}(s) = (\mu(s), \mu_h(s), (\mu(\partial_\gamma(s)))_{\gamma\in\Gamma}).$$

To this asynchronous mapping, we apply Proposition 3.11. So let $\overline{\mathcal{A}}$ be the MPA constructed there with state space S and mapping $f : S \to A \times A_h \times A^\Gamma$. Now let $t \in \downarrow\mathrm{MSC}_B^\infty$, $\sigma \in \Sigma_p$, and $0 \leqslant n \leqslant B$. Then there exists a unique run (r, m) of $\overline{\mathcal{A}}$ on t. We show that we have

$$\mu^\infty_{(\sigma,n)}(\mathrm{tr}(t)) = \{a \in A \mid \exists (b, c, (d_\gamma)_{\gamma\in\Gamma}) \in f[X_p] : d_{(\sigma,n)} = a\} \tag{$\star$}$$

From this, the theorem follows: the tuple $(X_p)_{p\in\mathcal{P}}$, determines the tuple $\mu^\infty_{(\sigma,n)}(\mathrm{tr}(t))$ and therefore whether $\mathrm{tr}(t)$ is accepted by the asynchronous mapping μ, i.e., whether $t \in K$.

The proof of ($\star$) uses a case distinction according to whether the process p is idle in t, performs finitely or infinitely many actions.

If the process p is idle in t, then $X_p = \{s_p^{in}\}$ by definition. Hence $f[X_p] = \{f(s_p^{in})\} = \{\overline{\mu}(\emptyset)\}$. With $\overline{\mu}(\emptyset) = (b, c, (d_\gamma)_{\gamma\in\Gamma})$, we get $d_{(\sigma,n)} = \mu\partial_{(\sigma,n)}(\emptyset) = \mu(\emptyset)$. Since p is idle in t, we get $\lambda'^{-1}(\sigma, n) = \emptyset$ and therefore $\mu^\infty_{(\sigma,n)}(\mathrm{tr}(t)) = \{\mu(\emptyset)\}$ which proves ($\star$) in this simple case.

Next assume that process p performs finitely many events in t. Then there exists a maximal node $v \in V$ with $\mathrm{proc}(v) = p$ and we have $X_p = \{r(v)\}$. Let $f(r(v)) = (b, c, (d_\gamma)_{\gamma\in\Gamma})$. From Proposition 3.11, we get $(b, c, (d_\gamma)_{\gamma\in\Gamma}) = \overline{\mu}(\mathrm{tr}(t)\restriction_{\downarrow v})$ and therefore $d_{(\sigma,n)} = \mu\partial_{(\sigma,n)}(\mathrm{tr}(t)\restriction_{\downarrow v})$. Since $\mathrm{proc}(v) = \mathrm{proc}(\sigma) = p$ and $v = \max \mathrm{proc}^{-1}(p)$, we get $\partial_{(\sigma,n)}(\mathrm{tr}(t)\restriction_{\downarrow v}) = \partial_{(\sigma,n)}(\mathrm{tr}(t))$. Thus the right-hand side of ($\star$) equals $\{\mu\partial_{(\sigma,n)}(\mathrm{tr}(t))\}$. Since $\mathrm{proc}^{-1}(p)$ is finite, its subset $\lambda'^{-1}(\sigma, n)$ is finite. Hence, by definition, $\mu^\infty_{(\sigma,n)}(\mathrm{tr}(t)) = \{\mu\partial_{(\sigma,n)}(\mathrm{tr}(t))\}$ which equals the right-hand side. Thus, we proved ($\star$) in this slightly more involved case.

Now let $\text{proc}^{-1}(p)$ be an infinite subset of V. This case splits into the subcases $\lambda^{-1}(\sigma)$ finite and infinite. First, let $\lambda^{-1}(\sigma)$ be infinite. If $a \in \mu^{\infty}_{(\sigma,n)}(\text{tr}(t))$, then there exist nodes $v_i \in V$ with $v_i < v_{i+1}$, $\lambda'(v_i) = (\sigma, n)$, and $\mu(\text{tr}(t)\lceil_{\downarrow v_i}) = a$ for all $i \in \mathbb{N}$. Since S is finite, we can assume $r(v_i) = s$ for some fixed $s \in S$. Hence $s \in X_p$. Hence, for $i \in \mathbb{N}$, we have $\overline{\mu}(\text{tr}(t)\lceil_{\downarrow v_i}) = f(s) \in f[X_p]$. With $f(s) = (b, c, (d_\gamma)_{\gamma\in\Gamma})$, we get $d_{(\sigma,n)} = \mu\partial_{(\sigma,n)}(\text{tr}(t)\lceil_{\downarrow v_i})$ which equals $\mu(\text{tr}(t)\lceil_{\downarrow v_i}) = a$ since $\lambda'(v_i) = (\sigma, n)$. Hence a belongs to the right-hand side of ($\star$) which proves the inclusion $\subseteq$. Conversely, let a be an element of the right-hand side. Then there exist $(b, c, (d_\gamma)_{\gamma\in\Gamma}) \in A \times A_h \times A^\Gamma$ and $v_i \in V$ with $d_{(\sigma,n)} = a$, $v_i < v_{i+1}$, $\text{proc}(v_i) = p$, and $f(r(v_i)) = (b, c, (d_\gamma)_{\gamma\in\Gamma})$. Since $\lambda'^{-1}(\sigma, n)$ is nonempty, we can assume that there is $w \in V$ with $\lambda'(w) = (\sigma, n)$ and $w \leqslant v_i$. For $i \in \mathbb{N}$, let v_i' be the maximal node below v_i with $\lambda'(v_i') = (\sigma, n)$. Since $\lambda'^{-1}(\sigma, n)$ is infinite, we can w.l.o.g. assume $v_i' < v_{i+1}'$ for all $i \in \mathbb{N}$. But then

$$\mu(\text{tr}(t)\lceil_{\downarrow v_i'}) = \mu\partial_{(\sigma,n)}(\text{tr}(t)\lceil_{\downarrow v_i}) = d_{(\sigma,n)} = a.$$

Hence we found infinitely many nodes v with $\lambda'(v) = (\sigma, n)$ and $\mu(\text{tr}(t)\lceil_{\downarrow v}) = a$. This implies $a \in \mu^{\infty}_{(\sigma,n)}(\text{tr}(t))$. Thus, we showed ($\star$) in this penultimate case.

Finally, let $\text{proc}^{-1}(p)$ be infinite and $\lambda^{-1}(\sigma)$ be finite. Let v be the maximal node in V with $\lambda'(v) = (\sigma, n)$. Then $\mu^{\infty}_{(\sigma,n)}(\text{tr}(t)) = \{\mu\partial_{(\sigma,n)}(\text{tr}(t))\} = \{\mu(\text{tr}(t)\lceil_{\downarrow v})\}$. Now let $s \in X_p$ be arbitrary. Since proc^{-1} (p) is infinite, there exist $v_i \in V$ with $v_i < v_{i+1}$, $\text{proc}(v_i) = p$, and $r(v_i) = s$ for all $i \in \mathbb{N}$. We assume w.l.o.g. $v < v_0$. Let $f(s) = (b, c, (d_\gamma)_{\gamma\in\Gamma})$. Then $d_{(\sigma,n)} = \mu\partial_{(\sigma,n)}(\text{tr}(t)\lceil_{\downarrow v_i}) = \mu(\text{tr}(t)\lceil_{\downarrow v})$ for any $i \in \mathbb{N}$. Thus, we showed that $\mu(\text{tr}(t)\lceil_{\downarrow v})$ is the only element of the right-hand side of ($\star$). Since the same holds for the left-hand side, ($\star$) is shown in this last case as well.

Thus, $t \in {\downarrow}\text{MSC}^{\infty}_{B}$ is accepted iff it belongs to K. Using the same arguments as in Lemma 3.14, ${\downarrow}\text{MSC}^{\infty}_{B}$ can be accepted by a Muller MPA relative to ${\downarrow}\text{MSC}^{\infty}$. Thus, K is the intersection of two sets acceptable by a Muller MPA relative to ${\downarrow}\text{MSC}^{\infty}$ which finishes the proof. $\square$

4. Logics and Büchi automata

Message passing automata execute MSCs in a distributed way. For specification purposes, one can also describe sets of MSCs by Büchi automata with a global state space that execute MSCs sequentially. In this section, we consider the expressive power of these global Büchi automata and compare it with that of logical calculi like monadic second order logic.

Let $\varphi(x)$ be a first-order formula with one free variable. Since Σ-labeled posets have width at most $|\Sigma|$, for any $m, n \in \mathbb{N}$ with $m < n$, one can express in monadic second order logic that the set of witnesses of φ consists of m mod n elements (cf. Example 3.1). If we add only this modulo-counting ability to first-order logic, we obtain the logic FO + MOD(n): formulas of the logic FO + MOD(n) are built up from the atomic formulas $\lambda(x) = a$ and $x \leqslant y$ by the connectives $\wedge$ and $\neg$ and the first-order quantifier $\exists$ and $\exists^m$ for $0 \leqslant m < n$. A Σ-labeled poset $t = (V, \leqslant, \lambda)$ satisfies $\exists^m \varphi(x)$ if the number of nodes $v \in V$ such that $t \models \varphi(v)$ is finite and congruent m mod n.

To stress that a set of Σ-labeled posets K is axiomatizable relative to X by a sentence of FO, FO + MOD(n), or MSO, we will speak of first-order, FO + MOD(n)-, or monadically axiomatizable sets.

Before we consider these logics, we deal with first-order logic exclusively. Therefore, in this section, notions like “axiomatizable”, “definable” and “interpret” mean “FO-axiomatizable”, “FO-definable”, “FO-interpret” etc. (if not stated otherwise).

Recall that a set L of finite words over Σ is recognizable (by a finite deterministic automaton) iff there exists a finite monoid S and a homomorphism $\eta : \Sigma^{\star} \to S$ such that $L = \eta^{-1}\eta[L]$. The set L is aperiodic if S can be assumed to be aperiodic (i.e., groupfree). Finally, we call L n-solvable (for some $n \in \mathbb{N}$) if we can assume that any group in S is solvable and has order dividing some power of n. Similarly, one can define recognizable, n-solvable, and aperiodic sets of words in Σ^{∞}: $L \subseteq \Sigma^{\infty}$ is recognizable iff there exists a finite monoid S and a homomorphism $\eta : \Sigma^{\star} \to S$ such that, for any $u_i, v_i \in \Sigma^{\star}$ with $\eta(u_i) = \eta(v_i)$ and $u_0u_1u_2 \ldots \in L$, we get $v_0v_1v_2 \ldots \in L$. The set L is said to be n-solvable or aperiodic if the monoid S can be assumed to satisfy the corresponding conditions.

Definition 4.1. A set $K \subseteq {\downarrow}\mathrm{MSC}^{\infty}$ is *recognizable*, *n-solvable*, or *aperiodic* if $\mathrm{Lin}_{\omega}[K] = \bigcup_{t \in K} \mathrm{Lin}_{\omega}(t)$ is recognizable, n-solvable, or aperiodic, respectively.

Henriksen et al. show that any recognizable set K of finite MSCs is bounded. This does not hold for ideals in finite MSCs since, *e.g.*, the set $\{\mathrm{MSC}((p!q)^n) \mid n \in \mathbb{N}\}$ is recognizable but not bounded. But for infinite MSCs, their proof goes through verbatim:

Proposition 4.2 (cf. [21, Proposition 3.2])**.** *Let $K \subseteq \mathrm{MSC}^{\infty}$ be recognizable. Then K is bounded, i.e., there exists a positive integer B such that $K \subseteq \mathrm{MSC}^{\infty}_B$.*

Remark 4.3. Let $K \subseteq {\downarrow}\mathrm{MSC}^{\infty}_B$ be MSO-axiomatizable relative to ${\downarrow}\mathrm{MSC}^{\infty}$. Then, by Proposition 3.8, the set $\mathrm{tr}[K] \subseteq \mathbb{R}(\Gamma, D)$ is MSO-axiomatizable. Hence, by results of Ebinger and Muscholl [17], the set $\mathrm{Lin}_{\omega}(\mathrm{tr}[K])$ is a recognizable word language in Γ^{∞}. Let S be a finite monoid recognizing $\mathrm{Lin}_{\omega}(\mathrm{tr}[K])$. Then the set $\mathrm{Lin}_{\omega}(K)$ can be recognized by a wreath product of S and $(\mathbb{Z}_B)^{\Sigma}$. Hence K is recognizable.

If $n \in \mathbb{N}$ is divisible by B, then one can easily show Proposition 3.8 for the logic $\mathrm{FO} + \mathrm{MOD}(n)$. Following the ideas in [15] and using results from [35], one can also show that $\mathrm{Lin}_{\omega}(\mathrm{tr}[K])$ is n-solvable. Since n-solvable monoids are closed under wreath-products with $\mathbb{Z}_m$ for $m|n$, we obtain that in this case K is n-solvable.

This line of proof cannot be used for $\mathrm{FO} + \mathrm{MOD}(n)$ if B does not divide n and therefore in particular not for first-order logic and aperiodic sets.

4.1. Axiomatizable sets are recognizable

The sets of recognizable, n-solvable, and aperiodic word languages have been characterized in terms of fragments of monadic second order logic. Words over the alphabet Σ can be considered as Σ-labeled (linear) posets in the natural way: the word $w \in \Sigma^{\infty}$ determines the structure $(\mathrm{dom}(w), \leqslant, \lambda_w)$ and *vice versa* where $\leqslant$ is the usual linear order on the natural numbers. In this sense, we can say “$w \models \varphi$” where φ is a sentence.

We first prepare the proof of Proposition 4.7 that shows how to interpret $\mathrm{MSC}(w)$ in a proper and B-bounded word w.

Lemma 4.4. *For $p, q \in \mathcal{P}$ and $B, n \in \mathbb{N}$, let $L_{p,q,B,n}$ be the set of all finite words $w \in \Sigma^{\star}$ satisfying*
(1) *any prefix v of w satisfies $0 \leqslant |v|_{p!q} - |v|_{q?p} \leqslant B$, and*
(2) $|w|_{p!q} - |w|_{q?p} = n$.
There exists a first order sentence $\beta_{p,q,B,n}$ such that, for $w \in \Sigma^{\star}$,

$$w \models \beta_{p,q,B,n} \iff w \in L_{p,q,B,n}.$$

Proof. The set $L = L_{p,q,B,n}$ is easily seen to be recognizable by an automaton with $B+1$ states that keeps track of the difference $|v|_{p!q} - |v|_{q?p}$. Now let $u, v, w \in \Sigma^{\star}$ with $uv^{B+1}w \in L$. By (1), $|v|_{p!q} = |v|_{q?p}$ which implies that $uv^{B+2}w$ satisfies (1) and (2), i.e., belongs to L. If, conversely, $uv^{B+2}w \in L$, we can argue similarly to infer $uv^{B+1}w \in L$. Hence L is aperiodic and therefore first-order axiomatizable by [28]. $\square$

Proposition 4.5. *The sets*

$$\{w \in \Sigma^{\star} \mid w \text{ is } B\text{-bounded, proper, and } \mathrm{MSC}(w) \in {\downarrow}\mathrm{MSC}_B\},$$
$$\{w \in \Sigma^{\star} \mid w \text{ is } B\text{-bounded, complete, and } \mathrm{MSC}(w) \in {\downarrow}\mathrm{MSC}_B\},$$
$$\{w \in \Sigma^{\infty} \mid w \text{ is } B\text{-bounded, proper, and } \mathrm{MSC}(w) \in {\downarrow}\mathrm{MSC}_B^{\infty}\}, \text{ and}$$
$$\{w \in \Sigma^{\infty} \mid w \text{ is } B\text{-bounded, complete, and } \mathrm{MSC}(w) \in {\downarrow}\mathrm{MSC}_B^{\infty}\}$$

are first-order axiomatizable relative to $\Sigma^{\star}$ and Σ^{ω}, respectively.

Proof. A finite word $u \in \Sigma^{\star}$ is B-bounded and proper iff, for any $p, q \in \mathcal{P}$, there exists $n \leqslant B$ such that $u \in L_{p,q,n,B}$. Hence the set of B-bounded and proper finite words is

$$K_1 = \bigcap_{p,q\in\mathcal{P}} \bigcup_{0\leqslant n\leqslant B} L_{p,q,B,n}.$$

By Lemma 4.4, there exists a first-order sentence $\varkappa_1$ that axiomatizes K_1 relative to all finite words over Σ. By Proposition 2.2, $\mathrm{MSC}(u) \in {\downarrow}\mathrm{MSC}$ for any *finite* proper word. Thus, K_1 equals the first set in the proposition. Similarly, the second set equals

$$K_2 = \bigcap_{p,q\in\mathcal{P}} L_{p,q,B,0}$$

and is therefore first-order axiomatizable.

An infinite word is B-bounded and proper iff any of its finite prefixes is in K_1, i.e., iff it satisfies $\forall x(\varkappa_1\restriction_{\leqslant x})$ where $\varkappa_1\restriction_{\leqslant x}$ is obtained from $\varkappa_1$ by restricting any quantification to positions left of x. Next we show that the set of B-bounded and proper words u satisfying $\mathrm{MSC}(u) \in {\downarrow}\mathrm{MSC}$ is first-order axiomatizable relative to all B-bounded and proper words. Recall that by Proposition 2.2, an B-bounded and proper word $u \in \Sigma^{\infty}$ satisfies $\mathrm{MSC}(u) \in {\downarrow}\mathrm{MSC}$ iff

$$|u|_{q?p} < |u|_{p!q} \implies \{i \in \mathrm{dom}(u) \mid \mathrm{proc}(u_i) = q\} \text{ is finite}$$

for any distinct processes p and q. Note that $|u|_{q?p} < |u|_{p!q}$ iff $u = u_1u_2$ with $|u_2|_{q?p} = 0$ and $u_1 \notin L_{p,q,B,0}$. Hence $|u|_{q?p} < |u|_{p!q}$ iff

$$u \in (\Sigma^* \setminus L_{p,q,B,0}) \cdot \{u_2 \in \Sigma^\infty \mid |u_2|_{q?p} = 0\}.$$

The two factors of this set are first-order axiomatizable and therefore starfree. Hence the product is starfree and therefore first-order axiomatizable. On the other hand, $\{i \in \mathrm{dom}(u) \mid \mathrm{proc}(u_i) = q\}$ is finite iff u satisfies

$$\exists x \forall y \left(x < y \to \bigwedge_{\sigma \in \Sigma_q} \neg(\lambda(y) = \sigma \right).$$

Thus, the set of B-bounded and proper words $u \in \Sigma^\infty$ that satisfy $\mathrm{MSC}(u) \in {\downarrow}\mathrm{MSC}$ is first-order axiomatizable relative to all words in Σ^∞.

An infinite word u is B-bounded and complete iff it is B-bounded, proper, and satisfies for any $p, q \in \mathcal{P}$:

- there are infinitely many occurrences of $q?p$ in u or
- there is a finite prefix $v \in L_{p,q,B,0}$ of u and the corresponding suffix does not contain any occurrence of $p!q$.

By Lemma 4.4 and by what we saw above, this can be expressed in first-order logic. Hence the set of B-bounded and complete words in Σ^ω is first-order axiomatizable. Since any such word u satisfies $\mathrm{MSC}(u) \in {\downarrow}\mathrm{MSC}$, this completes the proof. □

Lemma 4.6. *Let $B \in \mathbb{N}$. There exists a first-order formula $\chi(x, y)$ such that for all B-bounded and proper words $w \in \Sigma^\infty$ and all $0 \leqslant i, j < |w|$, we have: $w \models \chi(i, j)$ iff $w_i = p!q$, $w_j = q?p$, and $|w{\restriction_{\leqslant i}}|_{p!q} = |w{\restriction_{\leqslant j}}|_{q?p}$ for some $p, q \in \mathcal{P}$.*

Proof. For $0 \leqslant n \leqslant B$, $p, q \in \mathcal{P}$, let $\beta_{p,q,B,n}$ be the formula from Lemma 4.4 and set

$$\chi_{p,q,n}(x, y) = (\lambda(x) = p!q) \wedge (\lambda(y) = q?p) \wedge x \leqslant y \wedge (\beta_{p,q,B,n}{\restriction_{\leqslant x}}) \wedge$$
$$\text{“the interval } [x, y] \text{ contains precisely } n \text{ positions labelled } q?p\text{”}$$

Then a B-bounded and proper word w satisfies $\chi_{p,q,n}(i, j)$ iff

(1) the letters w_i and w_j are $p!q$ and $q?p$, respectively,
(2) $|w{\restriction_{\leqslant i}}|_{p!q} - |w{\restriction_{\leqslant i}}|_{q?p} = n$, and
(3) $|w{\restriction_{\leqslant j}}|_{q?p} = |w{\restriction_{\leqslant i}}|_{q?p} + n$.

The combination of the second and third statement is equivalent to $|w{\restriction_{\leqslant i}}|_{p!q} = |w{\restriction_{\leqslant j}}|_{q?p}$. Setting $\chi(x, y) = \bigvee_{n=0}^{B} \bigvee_{p,q \in \mathcal{P}} \chi_{p,q,n}(x, y)$ finishes the proof. □

Proposition 4.7. *Let $B \in \mathbb{N}$. There exists a first-order formula $\varphi(x, y)$ such that for any B-bounded and proper word $w \in \Sigma^\infty$, we have*

$$(\mathrm{dom}(w), \varphi^w, \lambda_w) = \mathrm{MSC}(w)$$

where $\varphi^w = \{(i, j) \mid 0 \leqslant i, j < |w|, w \models \varphi(i, j)\}$.

Proof. The result follows since $R^{\star} = R^{2|\mathcal{P}|}$ (cf. page 83) and since R is first-order definable by the preceding lemma. $\square$

Theorem 4.8. *Let* $K \subseteq {\downarrow}\mathrm{MSC}^{\infty}$ *be axiomatizable in* $\begin{Bmatrix} (1) & \text{MSO} \\ (2) & \text{FO} + \text{MOD}(n) \\ (3) & \text{FO} \end{Bmatrix}$ *and bounded. Then* $\mathrm{Lin}_{\omega} K \subseteq \Sigma^{\infty}$ *is* $\begin{Bmatrix} (1) & recognizable \\ (2) & n{-}solvable \\ (3) & aperiodic \end{Bmatrix}$.

Proof. By Proposition 4.5, the set L of B-bounded and proper words u satisfying $\mathrm{MSC}(u) \in {\downarrow}\mathrm{MSC}$ is axiomatizable in first-order logic relative to Σ^{∞}. By Proposition 4.7, we can FO-interpret $\mathrm{MSC}(u)$ in u for any $u \in L$. Hence the set of all $u \in L$ with $\mathrm{MSC}(u) \in K$ is axiomatizable in MSO/FO + MOD(n)/FO whenever K is axiomatizable in the corresponding logic. Now [4,26,35] imply the statement of the theorem. $\square$

Remark 4.9. Büchi proved his result on the relation between finite automata and monadic second order logic for words in order to show that the monadic second order theory of ω is decidable. Since all our constructions are effective, we obtain similarly that for any B, the monadic second order theory of ${\downarrow}\mathrm{MSC}_B^{\infty}$ is decidable. I mention this fact only in passing since a stronger result has been shown by Madhusudan and Meenakshi. A possibly infinite MSC is *existentially B-bounded* if there exists at least one linearisation that is B-bounded. Madhusudan and Meenakshi [27] showed that the monadic second order theory of these existentially B-bounded MSCs is decidable. They used automata theoretic methods. Since these existentially B-bounded MSCs have bounded tree width, their result can also be derived from investigations by Courcelle [7].

4.2. Recognizable sets are axiomatizable

Henriksen et al. [21] showed that any recognizable set K of *finite* MSCs is axiomatizable in monadic second order logic relative to MSC_B for some B. Their proof strategy follows an idea from [17]: The set of linear extensions of K is a recognizable subset of $\Sigma^{\star}$. Hence, by [5], there is a monadic sentence φ that axiomatizes the linear extensions relative to $\Sigma^{\star}$. A canonical linear extension of a finite MSC t can be defined in t by a first-order formula, i.e., this canonical linear extension can be interpreted in t. Hence there is a monadic sentence φ' that is satisfied by t iff the canonical linear extension of t satisfies φ, i.e., K is monadically axiomatizable. Since the canonical linear order is defined by a first-order formula, this proof can also be used to show that any aperiodic/n-solvable set of finite MSCs is first-order/FO + MOD(n) axiomatizable.

Let $t = (V, \leqslant, \lambda)$ be a Σ-labeled poset. If V is finite, then any linear extension of t can be identified with a finite word over Σ. Now suppose V is infinite, *e.g.*, the disjoint union of two ω-chains. Then there is a linear extension that cannot be seen as an ω-word since it is order-isomorphic to $\omega + \omega$. Recall that $\mathrm{Lin}_{\omega}(t)$ is the set of all finite or ω-words that are linear extensions of t. Clearly, since all principal ideals in t are finite, $\mathrm{Lin}_\omega(t)$ is nonempty. Nevertheless, the proof idea from [17,19] cannot be used for infinite MSCs since there is no *definable* linear extension of the MSC $((p!q)(q?p)(p'!q')(q'?p'))^\omega$ of order type ω.[2]

Remark 4.10. Let $K \subseteq \downarrow\mathrm{MSC}_B^\infty$ be recognizable by the finite monoid S. Then the set $\mathrm{Lin}_\omega(\mathrm{tr}[K])$ is recognizable by the direct product of S with some aperiodic monoid (since $\mathrm{tr}[\downarrow\mathrm{MSC}_B^\infty]$ is first-order axiomatizable and therefore aperiodic). By [17], the set $\mathrm{tr}[K]$ is therefore monadically axiomatizable. Hence, by Proposition 3.8, the set K is monadically axiomatizable relative to $\downarrow\mathrm{MSC}^\infty$.

We can argue similarly if K is n-solvable with $B|n$ (analogously to Remark 4.3). But, again, for $B\not| n$, these ideas do not work since the interpretation of $\mathrm{tr}(t)$ in t needs the counting abilities of $\mathrm{FO} + \mathrm{MOD}(n)$.

To overcome this problem, we will chop an MSC into pieces, consider these pieces independently, and combine the results obtained for them.

Before we start this programme, we need an extension of Mezei's theorem to languages of infinite words and therefore some more notation. By $u \sqcup\!\sqcup v \subseteq \Sigma^\infty$ we denote the set of all shuffles of the words u and v. Then $K \sqcup\!\sqcup L$ is the set of shuffles of elements of K and L. A pair (s, e) of elements of a finite monoid S is *linked* if $s \cdot e = s$ and $e \cdot e = e$. Two linked pairs (s, e) and (s', e') are *conjugated* if there exist $x, y \in S$ with $e = xy$, $e' = yx$, and $s' = sx$. Since conjugacy is an equivalence relation [34, Proposition II.3.6], we can speak on conjugacy classes. Now let $\eta : \Sigma^\star \to S$ be a surjective homomorphism. Then two linked pairs (s, e) and (s', e') are conjugated iff $\eta^{-1}(s)(\eta^{-1}(e))^\omega \cap \eta^{-1}(s')(\eta^{-1}(e'))^\omega \neq \emptyset$ (cf. [34, Corollary II.3.8]). A set $L \subseteq \Sigma^\infty$ is *weakly recognized by* η if $L = \bigcup \eta^{-1}(s)(\eta^{-1}(e))^\omega$ where the union is taken over all linked pairs (s, e) in S with $\eta^{-1}(s)(\eta^{-1}(e))^\omega \cap L \neq \emptyset$. If L is recognized by the morphism η, then η weakly recognizes L [34, Proposition VI.5.2]. The converse is false in general, but the following weaker assertion holds. A finite monoid S' is an *aperiodic extension* of a monoid S if there is a surjective homomorphism $\eta : S' \to S$ such that $\eta^{-1}(f)$ is an aperiodic semigroup for any idempotent element $f \in S$. Note that aperiodic extensions of finite/n-solvable/aperiodic monoids are finite/n-solvable/aperiodic. If η weakly recognizes L, then there exists a homomorphism into some aperiodic extension of S that recognizes L [34, Theorem VI.5.10, Proposition VI.5.18].

Theorem 4.11. *Let $\Sigma = \Sigma_1 \dot\cup \Sigma_2$ be an alphabet. Let $L \subseteq \Sigma^\infty$ be recognized by a surjective homomorphism onto $(S, \cdot)$ and suppose that $u_1 \sqcup\!\sqcup u_2 \cap L \neq \emptyset$ implies $u_1 \sqcup\!\sqcup u_2 \subseteq L$ for any $u_i \in \Sigma_i^\infty$ $(i = 1, 2)$. Then L is a finite union of sets $L_1 \sqcup\!\sqcup L_2$ where $L_i \subseteq \Sigma_i^\infty$ is recognized by an aperiodic extension of $(S, \cdot)$.*

Proof. Let $\eta : \Sigma^\star \to (S, \cdot)$ be a homomorphism that recognizes L. Then $\eta_i = \eta\restriction_{\Sigma_i^\star}$ is a homomorphism from $\Sigma_i^\star$ into $(S, \cdot)$. For a linked pair $(s, e) \in S$, let $\overline{\eta}(s, e) = \eta^{-1}(s)\eta^{-1}(e)^\omega$ and define $\overline{\eta}_i(s, e)$ similarly. Furthermore, let $P_\eta(L) = \{(s, e) \in S^2 \mid (s, e)$ is a linked pair and $\overline{\eta}(s, e) \cap L \neq \emptyset\}$. We call a nonempty set $X \subseteq \Sigma^\infty$ *η-minimal* if it is a minimal set weakly recognized by the homomorphism η.

Now we prove the theorem by a sequence of claims.

[2] Suppose the first-order formula $\varphi(x, y)$ defines a linear extension of this partial order of order type ω. Then there exist arbitrary large natural numbers k and ℓ such that the kth $p!q$-event precedes the ℓth $p'!q'$-event. But then for all $m > k$, the mth $p!q$-event precedes the ℓth $p'!q'$-event since first order logic cannot distinguish between these $p!q$-events.

Claim 1. *A nonempty set $X \subseteq \Sigma^\infty$ is η-minimal iff $P_\eta(X)$ is a conjugacy class and X is weakly recognized by η.*

Proof of Claim 1. Suppose X is η-minimal. Let $(s, e), (s', e')$ be conjugated linked pairs with $(s, e) \in P_\eta(X)$. Since X is weakly recognized by η, this implies $\overline{\eta}(s, e) \subseteq X$. Since (s, e) and (s', e') are conjugated, we get $\emptyset \neq \overline{\eta}(s, e) \cap \overline{\eta}(s', e') \subseteq X \cap \overline{\eta}(s', e')$. Hence $P_\eta(X)$ is closed under conjugation.

Now let C be some conjugacy class with $Y = \overline{\eta}[C]$. We show that Y is weakly recognized by η: let $(s', e') \in P_\eta(Y)$. Then there exists $(s, e) \in C$ with $\overline{\eta}(s, e) \cap \overline{\eta}(s'e') \neq \emptyset$. Hence (s, e) and (s', e') are conjugated, i.e., $(s, e) \in C$. In other words, $P_\eta(Y) = C$, i.e., Y is weakly recognized by η.

Hence, for any η-minimal set X, there is a conjugacy class C with $C \subseteq P_\eta(X)$ and $\emptyset \neq \overline{\eta}[C] \subseteq \overline{\eta}[P_\eta(X)] = X$. Since $\overline{\eta}[C]$ is weakly recognized by η, this implies $X = \overline{\eta}[C]$, i.e., $P_\eta(X) = C$.

Let, conversely, C be a conjugacy class and $\emptyset \neq X \subseteq \overline{\eta}[C]$ be weakly recognized by η. Since $P_\eta(X)$ is closed under conjugation, this implies $P_\eta(X) = C$, i.e., $\overline{\eta}[C] = X$. Hence $\overline{\eta}[C]$ is η-minimal. $\square$

Claim 2. *Let (s_i, e_i) be linked pairs s.t. $(\overline{\eta}_1(s_1, e_1) \sqcup\!\sqcup \overline{\eta}_2(s_2, e_2)) \cap L \neq \emptyset$. Then $\overline{\eta}_1(s_1, e_1) \sqcup\!\sqcup \overline{\eta}_2(s_2, e_2) \subseteq L$.*

Proof of Claim 2. Since $(\overline{\eta}_1(s_1, e_1) \sqcup\!\sqcup \overline{\eta}_2(s_2, e_2)) \cap L \neq \emptyset$, there are $u_i \in \overline{\eta}_i(s_i, e_i)$ with $u_1 \sqcup\!\sqcup u_2 \cap L \neq \emptyset$ and therefore $u_1 \sqcup\!\sqcup u_2 \subseteq L$ by our assumption on L. Since $u_i \in \overline{\eta}_i(s_i, e_i)$, there are $u_i^j \in \Sigma_i^\star$ for $j \in \omega$ such that $u_i = u_i^0 u_i^1 u_i^2 \ldots$, $\eta_i(u_i^0) = s_i$, and $\eta_i(u_i^j) = e_i$ for $j > 0$. Then $\prod_{j \in \omega}(u_1^j u_2^j) \in u_1 \sqcup\!\sqcup u_2 \subseteq L$.

Now let $v_i^j \in \Sigma_i^\star$ for $j \in \omega$ such that $\eta_i(v_i^0) = s_i$ and $\eta_i(v_i^j) = e_i$ for $j > 0$, i.e., $\eta_i(v_i^j) = \eta_i(u_i^j)$ for $j \in \omega$. Since η recognizes L and $\prod_{j \in \omega}(u_1^j u_2^j) \in L$, we obtain $\prod_{j \in \omega}(v_1^j v_2^j) \in L$, i.e., $v_1 \sqcup\!\sqcup v_2 \cap L \neq \emptyset$ for $v_i = \prod_{j \in \omega} v_i^j$. Hence, by the assumption on L, we have $v_1 \sqcup\!\sqcup v_2 \subseteq L$. Since $v_i \in \overline{\eta}(s_i, e_i)$ are arbitrary elements, this implies $\overline{\eta}_1(s_1, e_1) \sqcup\!\sqcup \overline{\eta}_2(s_2, e_2) \subseteq L$. $\square$

Claim 3.

$$L = \bigcup \{L_1 \sqcup\!\sqcup L_2 \mid L_i \subseteq \Sigma_i^\infty \ \eta_i\text{-minimal}, L_1 \sqcup\!\sqcup L_2 \cap L \neq \emptyset\}.$$

Proof of Claim 3. Let $u \in L$ and u_i be the projection of u to Σ_i. Then, by [34, Proposition II.3.3], $P_{\eta_i}(\{u_i\})$ is not empty. Hence, by claim 1, there are η_i-minimal sets L_i such that $u_i \in L_i$. But then $u \in L_1 \sqcup\!\sqcup L_2$ and $L_1 \sqcup\!\sqcup L_2 \cap L \neq \emptyset$. Hence we showed the inclusion $\subseteq$.

Now let $L_i \subseteq \Sigma_i^\infty$ be η_i-minimal and suppose $u_i, w_i \in L_i$ with $u_1 \sqcup\!\sqcup u_2 \cap L \neq \emptyset$. By claim 1, there exist conjugated linked pairs (s_i, e_i) and (s_i', e_i') such that $u_i \in \overline{\eta}_i(s_i, e_i)$ and $w_i \in \overline{\eta}_i(s_i', e_i')$. Hence $\overline{\eta}_1(s_1, e_1) \sqcup\!\sqcup \overline{\eta}_2(s_2, e_2) \cap L \neq \emptyset$. Then claim 2 implies $\overline{\eta}_1(s_1, e_1) \sqcup\!\sqcup \overline{\eta}_2(s_2, e_2) \subseteq L$. Furthermore, there exist $v_i \in \overline{\eta}_i(s_i, e_i) \cap \overline{\eta}_i(s_i', e_i')$ by [34, Corollary II.3.8]. Since $\overline{\eta}_1(s_1, e_1) \sqcup\!\sqcup \overline{\eta}_2(s_2, e_2) \subseteq L$, in particular, $v_1 \sqcup\!\sqcup v_2 \subseteq L$ and therefore $\overline{\eta}_1(s_1', e_1') \sqcup\!\sqcup \overline{\eta}_2(s_2', e_2') \cap L \neq \emptyset$. By claim 2, this implies $\overline{\eta}_1(s_1', e_1') \sqcup\!\sqcup \overline{\eta}_2(s_2', e_2') \subseteq L$ and therefore in particular $w_1 \sqcup\!\sqcup w_2 \subseteq L$. $\square$

Now we can prove the theorem: Since $(S, \cdot)$ is finite, there are only finitely many sets weakly recognizable by η_i. Hence, by Claim 3, L is a finite union of sets $L_1 \sqcup\!\sqcup L_2$ with $L_i \subseteq \Sigma_i^\infty$ weakly recognizable by $(S, \cdot)$. By [34, VI.5.10 and VI.5.18], L_i is recognizable by a finite aperiodic extension of $(S, \cdot)$. $\square$

Above, we saw that linear extensions of an MSC need not be of order type ω. We now start to characterize some Σ-labeled posets for which we need not worry about this.

The poset t is *directed* if it does not contain two disjoint nonempty filters.

Now let $t = (V, \leqslant, \lambda) \in {\downarrow}\mathrm{MSC}^{\infty}$. Then $\mathrm{alph}(t) = \lambda[V]$ and $\mathrm{alphInf}(t) = \{\sigma \in \Sigma \mid \lambda^{-1}(\sigma) \text{ is infinite}\}$. Let Y be the largest filter in t with $\lambda[Y] \subseteq \mathrm{alphInf}(t)$ and let $X = V \setminus Y$. Then the *finitary part of t* is defined by $\mathrm{Fin}(t) = t{\restriction}_X$ and the *infinitary part of t* by $\mathrm{Inf}(t) = t{\restriction}_Y$. Note that $\mathrm{Fin}(t)$ is an ideal in a finite MSC while in general $\mathrm{Inf}(t)$ is only a Σ-labeled poset.

Let $E \subseteq \Sigma^2$ contain all pairs $\sigma, \tau \in \Sigma$ with $\mathrm{proc}(\sigma) = \mathrm{proc}(\tau)$ or $\{\sigma, \tau\} = \{p!q, q?p\}$ for some $p, q \in \mathcal{P}$. Then (Σ, E) is an undirected graph.

Lemma 4.12. *Let $t = (V, \leqslant, \lambda) \in {\downarrow}\mathrm{MSC}_B^{\infty}$ and $A = \mathrm{alphInf}(t)$. Let $(A_i)_{1\leqslant i\leqslant n}$ be the connected components of the graph (A, E). Then the Σ-labeled posets $t_i = \mathrm{Inf}(t){\restriction}_{A_i}$ are directed for $1 \leqslant i \leqslant n$, and $\mathrm{Inf}(t) = \dot{\bigcup}_{1\leqslant i\leqslant n} t_i$.*

Proof. We first show that t_i is directed: By contradiction, assume X and Y to be disjoint nonempty filters in t_i. Then they involve disjoint instances, i.e., $\mathrm{proc}[\lambda[X]] \cap \mathrm{proc}[\lambda[Y]] = \emptyset$. Since $A_i = \mathrm{alphInf}(t_i)$ is connected, there exist $p, q \in \mathcal{P}$ with $p!q \in \lambda[X]$ and $q?p \in \lambda[Y]$ (or *vice versa*). Since Y is a filter and $\lambda^{-1}(q?p)$ is a linear order of order type at most ω, the set $\lambda^{-1}(q?p) \setminus Y$ is finite. Furthermore ${\downarrow}x \cap \lambda^{-1}(q?p) \subseteq \lambda^{-1}(q?p) \setminus Y$ for $x \in X$ since X and Y are disjoint and Y is a filter. Since, on the other hand, X contains infinitely many nodes x with $\lambda(x) = p!q$, there is one $x \in X$ with $|{\downarrow}x \cap \lambda^{-1}(p!q)| - |{\downarrow}x \cap \lambda^{-1}(q?p)| > B$, contradicting our assumption that t be B-bounded. Hence, indeed, t_i is directed.

Next we show that $\mathrm{Inf}(t)$ is the disjoint union of the pomsets t_i: Since $A = \bigcup_{1\leqslant i\leqslant n} A_i$, the carrier set of $\mathrm{Inf}(t)$ is the disjoint union of the carrier sets of t_i. Now let x be a node in t_{i_1} and z a node in t_{i_2} such that $x \leqslant z$. Since $t \in {\downarrow}\mathrm{MSC}^{\infty}$, we get $x(\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^{\star} z$. Hence there exist $y_1, y_2, \ldots, y_m$ with $x = y_1$, $(y_i, y_{i+1}) \in \sqsubseteq_{\mathcal{P}} \cup \sqsubseteq$, and $y_m = z$. Since $x \leqslant y_i$ and x is a node in $\mathrm{Inf}(t)$, the label $\lambda(y_i)$ belongs to $\mathrm{alphInf}(t)$. Furthermore, $(\lambda(y_i), \lambda(y_{i+1}))$ is an edge in (Σ, E). Hence $\lambda(x)$ and $\lambda(z)$ belong to the same connected component of (A, E) implying $i_1 = i_2$. $\square$

Lemma 4.13. *Let $t = (V, \leqslant, \lambda) \in {\downarrow}\mathrm{MSC}_B^{\infty}$ be directed. Then any linear extension of t is of order type at most ω.*

Proof. By contradiction, assume there is a linear extension $\preceq$ of t of order type $> \omega$. Then there are $x_i, z_0 \in V$ with $x_1 \prec x_2 \prec x_3 \cdots \prec z_0$. Since ${\downarrow}z_0$ is finite in t, there is $i_1 \in \mathbb{N}$ such that, for $j \geqslant i_1$, we have $z_0 \parallel x_j$. Since t is directed, there exists $z_1 \in V$ with $z_0, x_{i_1} \leqslant z_1$. Since $\preceq$ is an order extension of $\leqslant$, this implies $x_i \prec z_0 \prec z_1$ for all $i \in \mathbb{N}$. Since ${\downarrow}z_1$ is finite in t, we again find $i_2 \in \mathbb{N}$ with $x_j \parallel z_1$ for $j \geqslant i_2$. From $x_{i_1} \leqslant z_i$, it follows $i_1 < i_2$. Inductively, we find an increasing sequence $i_1 < i_2 < i_3 \ldots$ of natural numbers and an increasing sequence $z_1 < z_2 < z_3 \ldots$ of elements of V such that, for all $i, j \in \mathbb{N}$, $z_i \not\leqslant x_{i_j}$, and $x_{i_j} \leqslant z_i$ iff $j < i$. For $j \in \mathbb{N}$, let $y_j = x_{i_j}$.

Since ${\downarrow}z_1$ is finite and $z_1 \parallel y_i$ for $i > 1$, there is a minimal element b_1 satisfying $y_1 \leqslant b_1 \leqslant z_1$ that is incomparable with all y_i for $i > 1$. Since $y_1 < y_2$, we get $y_1 < b_1$. Hence there is a_1 with $y_1 \leqslant a_1 \longrightarrow\!\!\!< b_1 \leqslant z_1$. Since b_1 was chosen minimal, a_1 lies below one (and therefore almost all) y_i. Choose $j_1 > 1$ with $a_1 < y_{j_1}$. We can inductively choose a_i, b_i and $j_i \in \mathbb{N}$ satisfying $y_{j_i} \leqslant a_i \longrightarrow\!\!\!< b_i \leqslant z_{j_i}$, $a_i \leqslant y_{j_{i+1}}$, and $b_i \parallel a_j$ for all $j > i$. We can, without loss of generality, assume $\lambda(a_i) = \sigma$ and $\lambda(b_i) = \tau$ for all j since Σ is finite. Since $b_1 \parallel a_2$, we get $\mathrm{proc}(\sigma) \neq \mathrm{proc}(\tau)$. Hence there are $p, q \in \mathcal{P}$ such that $\sigma = p!q$, $\tau = q?p$, and $|{\downarrow}a_i \cap \lambda^{-1}(\sigma)| = |{\downarrow}b_i \cap \lambda^{-1}(\tau)|$ for $i \in \mathbb{N}$. But this implies $|{\downarrow}a_{B+1} \cap \lambda^{-1}(\sigma)| - |{\downarrow}a_{B+1} \cap \lambda^{-1}(\tau)| > B$, contradicting our assumption on t to be B-bounded. $\square$

Lemma 4.14. *There exists a first-order formula φ with two free variables such that for any Σ-labeled poset $t = (V, \leqslant, \lambda)$, the structure* $\mathrm{canLE}(t) := (V, \varphi^t, \lambda)$ *is a linear extension of t where $\varphi^t = \{(v, w) \in V^2 \mid t \models \varphi(v, w)\}$.*

Proof. The existence of such a formula was shown independently in [16,17] in more restricted settings, but the proofs go through in the current setting as well (cf. also [10] for a comprehensive proof). □

We call the linear extension $\mathrm{canLE}(t)$ the *canonical linear extension of t*. Now suppose $t \in \downarrow\mathrm{MSC}_B^\infty$ and let $\mathrm{alphInf}(t) = A$. Then, by Lemma 4.12, $\mathrm{Inf}(t)$ is the disjoint union of the Σ-labeled posets $t_i = \mathrm{Inf}(t)\restriction_{A_i}$ for A_i a connected component of A. We show that $\mathrm{canLE}(t_i)$ is of order type at most ω: let $V_i \subseteq V$ comprise all nodes from t_i and let $X = \downarrow V_i = \{x \in V \mid \exists y \in V_i : x \leqslant y\}$. Then $t\restriction_X \in \downarrow\mathrm{MSC}_B^\infty$. Since, by Lemma 4.12, t_i is directed, so is $t\restriction_X$. Hence, by Lemma 4.13, any linear extension of X is of order type at most ω. Since any order extension of t_i can be extended to an order extension of X, the canonincal linear extension $\mathrm{canLE}(t_i)$ is of order type at most ω. Hence $\mathrm{canLE}(t_i) \in \mathrm{Lin}_\omega(t_i) \subseteq \Sigma^\infty$ is a possibly infinite word. We define $\mathrm{NFInf}(t) = \sqcup\!\sqcup_{1 \leqslant i \leqslant n} \mathrm{canLE}(t_i) \subseteq \Sigma^\infty$.

Definition 4.15. Let $t \in \downarrow\mathrm{MSC}_B^\infty$, $A = \mathrm{alphInf}(t)$ and let $\$ \notin \Sigma$. Then $\mathrm{NF}(t) = \mathrm{canLE}(\mathrm{Fin}(t)) \cdot \{\$\} \cdot \mathrm{NFInf}(t)$ is the set of normal forms of t. For $K \subseteq \downarrow\mathrm{MSC}_B^\infty$, we set $\mathrm{NF}[K] = \bigcup_{t \in K} \mathrm{NF}(t)$.

Since $\mathrm{NFInf}(t)$ is a subset of Σ^ω, we get $\mathrm{NF}(t) \subseteq (\Sigma \cup \$)^\infty$. Furthermore, the restriction of any word in $\mathrm{NF}(t)$ to Σ is a linear extension of t, i.e., $\mathrm{NF}(t) \subseteq \mathrm{Lin}_\omega(t) \sqcup\!\sqcup \{\$\}$. Next we show that $\mathrm{NF}[\downarrow\mathrm{MSC}_B^\infty]$ is first-order axiomatizable:

Lemma 4.16. *The set* $\mathrm{NF}[\downarrow\mathrm{MSC}_B^\infty]$ *is first-order axiomatizable relative to* $(\Sigma \cup \{\$\})^\infty$.

Proof. The set $\Sigma^\infty \sqcup\!\sqcup \{\$\}$ is first-order axiomatizable relative to $(\Sigma \cup \{\$\})^\infty$. By Proposition 4.5, the set L of B-bounded and proper words in $w \in \Sigma^\infty$ with $\mathrm{MSC}(w) \in \downarrow\mathrm{MSC}^\infty$ is first-order axiomatizable. Hence, it suffices to show that $\mathrm{NF}[\downarrow\mathrm{MSC}_B^\infty]$ is first-order axiomatizable relative to $L \sqcup\!\sqcup \{\$\}$. By Proposition 4.7, there is a first-order formula $\varphi(x, y)$ such that $(\mathrm{dom}(w), \varphi^w, \lambda_w) = \mathrm{MSC}(w)$ for any word $w \in L$. In $t = \mathrm{MSC}(w)$, we can define the nodes that belong to $\mathrm{Fin}(t)$ and the connected components t_i of $\mathrm{Inf}(t)$. Furthermore we can define the canonical linear extension of $\mathrm{Fin}(t)$ and t_i that belong to $\Sigma^\star$ and Σ^ω, respectively. Now let $u_1 \$ u_2$ be a word in $L \sqcup\!\sqcup \{\$\}$. By what we saw so far, we can express that u_1 is the canonical linear extension of $\mathrm{Fin}(\mathrm{MSC}(u_1 u_2))$ and that u_2 is an element of $\mathrm{NFInf}(\mathrm{MSC}(u_1 u_2))$. Hence, indeed, $\mathrm{NF}[\downarrow\mathrm{MSC}_B^\infty]$ is first-order axiomatizable. □

Let $u \in \Sigma^\infty$ and $k \in \mathbb{N}$. Then the *k-first-order theory of u* is the set of first-order sentences of quantifier depth at most k that are satisfied by the word u. A set of first-order sentences of quantifier depth at most k is a *complete k-first-order theory* if it is the k-first-order theory of some word $u \in \Sigma^\infty$. Since, up to logical equivalence, there are only finitely many first-order sentences of quantifier depth at most k, there are only finitely many complete k-first-order theories. Furthermore, each complete k-first-order theory T is characterized by one first-order sentence γ_T of quantifier depth k, i.e., for any word $u \in \Sigma^\infty$, we have $u \models \gamma$ for all $\gamma \in T$ iff $u \models \gamma_T$ (cf. [22, Theorem 3.3.2]). For notational convenience, we will identify the characterizing sentence γ_T and the complete k-first-order theory T.

Let $K \subseteq {\downarrow}\mathrm{MSC}_B^\infty$ be aperiodic. Then $\mathrm{Lin}_\omega[K]$ is first-order axiomatizable relative to Σ^∞. Let $k \geqslant 2$ be the least integer such that $\mathrm{Lin}_\omega[K] \sqcup\!\sqcup \{\$\}$ and $\mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$ are first-order axiomatizable relative to $(\Sigma \cup \{\$\})^\infty$ by a sentence of quantifier depth at most k. Let T be a complete k-first-order theory and $A \subseteq \Sigma$. Then

$$K_{T,A} := \{t \in K \mid \mathrm{canLE}(\mathrm{Fin}(t)) \models T, \mathrm{alphInf}(t) = A\}$$

and

$$X_{T,A} := \mathrm{NF}[K_{T,A}].$$

Lemma 4.17. *In the first order language of $(\Sigma \cup \{\$\})$-labeled linear orders with one constant c, there exists a sentence φ of quantifier depth k such that $(v, c) \models \varphi$ iff $v \in X_{T,A}$ and $\lambda(c) = \$$ for $v \in (\Sigma \cup \{\$\})^\infty$.*

Proof. The sentence $\forall x(\lambda(x) = \$ \leftrightarrow x = c)$ is satisfied by v iff $v = u\$u'$ with $u \in \Sigma^\star$ and $u' \in \Sigma^\infty$ and $c = |u|$. But then $u\$u'$ belongs to $X_{T,A}$ iff $u \models T$, $\mathrm{alphInf}(u\$u') = A$, $u\$u' \in \mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$, and $uu' \in \mathrm{Lin}_\omega[K]$. Note that $u \models T$ iff $(u\$u', |u|) \models T{\restriction}_{<c}$. Since A is finite, $\mathrm{alphInf}(uu') = A$ can be expressed by a sentence of quantifier depth 2. By Lemma 4.16 and by our choice of k, $\mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$ is first-order axiomatizable by a sentence of quantifier depth k. Similarly, $\mathrm{Lin}_\omega[K] \sqcup\!\sqcup \{\$\}$ is first-order axiomatizable by a sentence of quantifier depth k. $\square$

Theorem 4.18. *Let $K \subseteq {\downarrow}\mathrm{MSC}_B^\infty$ be aperiodic. Then K is first-order axiomatizable relative to ${\downarrow}\mathrm{MSC}^\infty$.*

Proof. Let T be a complete k-first-order theory and let $A \subseteq \Sigma$. In order to prove the theorem, it suffices to show that $K_{T,A}$ is first-order axiomatizable. By Lemma 4.17, the set $\{(u\$u', |u|) \mid u\$u' \in X_{T,A}\}$ is axiomatizable by a sentence of quantifier depth k in the extended language. Hence $X_{T,A}$ is first-order axiomatizable (by a sentence of quantifier depth $k + 1$) and therefore aperiodic [26]. Let X denote the set of finite words $u \in \Sigma^\star$ with $u \models T$ and let $Y = \bigcup_{u \in X} (u\$)^{-1} X_{T,A} = \{v \in \Sigma^\infty \mid \exists u \in X : u\$v \in X_{T,A}\}$. This set is aperiodic since $X_{T,A}$ is aperiodic. We show $X_{T,A} = X\$Y$: the inclusion $\subseteq$ is trivial. So let $u \in X$ and $u' \in Y$. Then there is $v \in X$ with $v\$u' \in X_{T,A}$. Since $v \in X$, we have $v \models T$, i.e., the k-first-order theories of v and u coincide. Hence, by a standard application of Ehrenfeucht-Fraïssé-games (cf. [22]), the k-first-order theories of $(v\$u', |v|)$ and $(u\$u', |u|)$ coincide as well. Hence, by Lemma 4.17, $v\$u' \in X_{T,A}$ implies $u\$u' \in X_{T,A}$, i.e., $X_{T,A} = X\$Y$.

By the definition of $X_{T,A}$, this implies $X = \{\mathrm{canLE}(\mathrm{Fin}(t)) \mid t \in K_{T,A}\}$ and $Y = \bigcup_{t \in K_{T,A}} \mathrm{NFInf}(t)$. Let $(A_i)_{1 \leqslant i \leqslant n}$ be the connected components of (A, E), let $u \in Y$, and let u_i be the restriction of u to A_i. Then there exists $t \in K_{T,A}$ such that $\mathrm{Inf}(t) = \dot{\bigcup}_{1 \leqslant i \leqslant n} t_i$ and $u_i = \mathrm{canLE}(t_i)$. Hence $\sqcup\!\sqcup_{1 \leqslant i \leqslant n} u_i \subseteq Y$. Hence we can apply Theorem 4.11 which implies

$$Y = \bigcup_{1 \leqslant j \leqslant m} \sqcup\!\sqcup_{1 \leqslant i \leqslant n} Y_i^j$$

with $Y_i^j \subseteq A_i^\omega$ aperiodic. Let Y_i^j be axiomatized by the first-order sentence ν_i^j relative to Σ^∞ [26]. Furthermore, let φ axiomatize X relative to $\Sigma^\star$. There are certainly first-order formulas that define $\mathrm{Fin}(t)$ and $\mathrm{Inf}(t)$ for $t \in {\downarrow}\mathrm{MSC}_B^\infty$. By Lemma 4.14, we can interpret the canonical linear extension in $\mathrm{Fin}(t)$ and therefore express whether $\mathrm{canLE}(\mathrm{Fin}(t)) \models \varphi$. Similarly, we can define the connected component t_i of $\mathrm{Inf}(t)$. Since t_i is directed, any linear extension of t_i is of order type at most ω by Lemmas 4.12 and

4.13. Since we can interpret the canonical linear extension in t_i, we can express whether this canonical linear extension satisfies ν_i^j. $\square$

There is an obvious definition of a complete k-FO + MOD(n)- and of a complete k-MSO-theory. Using results from [5,35] in place of [26], one obtains the following result along the lines of the above proofs:

Theorem 4.19. *Let $K \subseteq \downarrow\mathrm{MSC}_B^\infty$ be recognizable (n-solvable). Then K is monadically axiomatizable ($\mathrm{FO} + \mathrm{MOD}(n)$-axiomatizable, respectively) relative to $\downarrow\mathrm{MSC}^\infty$.*

Appendix

Let $K \subseteq \downarrow\mathrm{MSC}_B$ be recognizable. We want to compute the number of local states of the deterministic message passing automaton that accepts K: Let $n = |\mathcal{P}|$ be the number of processes. Let m be the number of states of the minimal automaton accepting $\mathrm{Lin}_\omega[K]$.

First, we determine the size of the syntactic monoid of $\mathrm{tr}[\downarrow\mathrm{MSC}_B]$: To check property I for a given triple (p, q, n) with $p, q \in \mathcal{P}$ and $0 \leqslant i < B$, a monoid of size 6 suffices. To check property II for $\sigma \in \Sigma$, we can use a monoid of size $B^2 + 2$. Finally, to check property III, we have to keep track of the minimal and maximal nodes in a trace, i.e., $4^{|\Gamma|} + 1 = 4^{n^2 B} + 1$ monoid elements suffice. Hence, altogether, the syntactic monoid of $\mathrm{tr}[\downarrow\mathrm{MSC}_B]$ contains at most

$$6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1)$$

elements, and that for $\mathrm{tr}[K]$ at most

$$m^m \cdot 6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1).$$

By [6,11], the set of traces $\mathrm{tr}[K]$ can be accepted by an asynchronous mapping into a set of size

$$\begin{aligned}&(|\Gamma| + 1)^{|\Gamma|^2} \cdot m^m \cdot 6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1) \cdot 2^{|\Gamma|}\\ &\quad = (n^2 B + 1)^{(n^2 B)^2} \cdot m^m \cdot 6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1) \cdot 2^{n^2 B}\end{aligned}$$

Hence, the deterministic message passing automaton that we constructed has

$$(n^2 B + 1)^{(n^2 B)^2} \cdot m^m \cdot 6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1) \cdot 2^{n^2 B} \cdot B^{2n^2}$$

local states. But this is in $2^{m \log m + O((n^2 B)^2 \log(n^2 B))}$. Recall that the construction from [31] needed $2^{2^{O(n^2 B)} m \log m}$ local states. Thus, with growing buffer size B, the construction from [31] is less efficient than the one we presented.

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