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Iterated open neighborhood graphs and generalizations

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ABSTRACT

The open neighborhood graph of an undirected graph G is the graph that is defined on the same vertex set as G in which two vertices are adjacent, if they have a common neighbor in G. We analyze the graphs obtained by repeatedly applying the open neighborhood graph construction. We show that stab(G), the number of iterations required for the process to stabilize, is at most 2 larger than the logarithm of the diameter of G rounded up. We also show that for graphs that eventually become a clique, the number of iterations is at least the logarithm of the diameter of G rounded up. That is $\lceil \log_2(\text{diam}(G)) \rceil \leq \text{stab}(G) \leq \lceil \log_2(\text{diam}(G)) \rceil + 2$. These bounds are tight.

We also consider the process of repeatedly forming H-neighborhood graphs. For a graph H with two distinguished vertices, the H-neighborhood graph of G is the graph defined on the same vertex set as G in which two vertices are adjacent if they form the distinguished vertices in a (not necessarily induced) subgraph of G isomorphic to H. If the distinguished vertices of H are adjacent, then the number of iterations for the process to stabilize is at most linear in the number of edges of G. If the distinguished vertices of H are not required to be adjacent, we show that the number of iterations may be exponential. To prove this we show that there is a graph H for which the process of forming H-neighborhood graphs can simulate Conway's Game of Life.

1. Introduction

The *open neighborhood graph* $N(G)$ of an undirected graph G is the graph that is defined on the same vertex set as G in which two vertices are adjacent, if they have a common neighbor in G. Open neighborhood graphs were introduced and characterized in [1]. The name open neighborhood graph originates from the fact that two vertices are adjacent in $N(G)$ if and only if their open neighborhoods in G intersect.

Following the work of Sonntag and Teichert [19], we analyze the behavior of the graph sequence obtained by repeatedly forming the open neighborhood graph. We determine the number of iterations that is required for the iterated process to stabilize. We consider two kinds of stabilization: In the first kind we say the process has stabilized if the graph of the next iteration is identical to the graph of the current iteration. In the second kind we only require the graph of the next iteration to be isomorphic to the current one.

Another name used for the open neighborhood graph is 2-step (or 2-path) graph. Both names, open neighborhood graph and 2-step graph, hint at different generalizations. The (p)-neighborhood graph of a graph G is the graph defined on the vertices of G in which two vertices are adjacent if they have at least p common neighbors in G. The m-step (or m-path) graph is again defined on the same vertex set, but here two vertices are adjacent if they are joined by a path of length exactly m in G. Unifying both generalizations we also analyze iterated H-neighborhood graphs: For a graph H with two distinguished vertices, the *H-neighborhood graph* $N_H(G)$ of G is the graph on the same vertex set as G in which two vertices are adjacent if they form the distinguished vertices in a (not necessarily induced) subgraph of G isomorphic to H.

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Results. We first analyze the behavior of the graphs obtained by iterating the open neighborhood construction (Section 2). Exoo and Harary [5] show that a graph G eventually becomes a clique, if and only it is a connected non-bipartite graph that is not an induced cycle of odd length at least 5. Sonntag and Teichert [19] analyze the number of iterations required for such a graph to become a clique. Following this line of research, we show that this number is, up to an additive difference of 2, the binary logarithm of the diameter of the graph rounded up. We also provide graph families that show that the additive gap of 2 between our lower and upper bound cannot be avoided. In other words, we show that our bounds are tight. Our analysis of the upper bound graph family also proves the lower bound half of an equality conjectured in [19]. We continue by analyzing the mortality of graphs under the iterated application of the open neighborhood construction. That is, we determine which graphs eventually become an independent set.

For the generalization, H-neighborhood graphs, the observable behavior under the iterated process depends on whether the distinguished vertices of H are adjacent (see Section 3). If the distinguished vertices of H are adjacent, the process stabilizes after a number of iterations at most linear in the number of edges G. We also show that there are examples with quadratically many edges for which this linear number of iterations is required. If the distinguished vertices of H are not adjacent, periodic behavior is possible. Furthermore, the stabilization time may be exponential in the number of vertices of G. We show this by constructing a graph H and an infinite grid-like graph G for which the process of forming H-neighborhood graphs simulates the Game of Life, a cellular automaton known to be Turing-complete.

Related work. In the literature the terminology is inconsistent, not only for open neighborhood graphs but also for the various generalizations and related concepts that have been investigated. Various properties of open neighborhood graphs and variants have been analyzed.

A typical type of question considered asks which graphs have an open neighborhood graph with a specified property. To this end, graphs with Hamiltonian open neighborhood graphs have been studied [17]. Likewise, the property of having an interval open neighborhood graph and more general an interval (p)-neighborhood has been studied [14]. Under the name common neighbor graph, open neighborhood graphs have been used to study the injective chromatic number [8].

Yet another name for open neighborhood graphs is competition graph. The term competition graph, originally defined for directed graphs, is frequently used for undirected graphs, since the open neighborhood graph of a graph G is the competition graph of the symmetric orientation of G. The generalized notion corresponding to (p)-neighborhood graphs is the notion of p-competition graphs. They have been studied in [11,12].

A concept similar to but different from the open neighborhood graph is that of the square of a graph. In the square, two vertices are adjacent if they are of distance at most 2 in the original graph. Iterating the squaring procedure yields graph powers, which have also been investigated extensively [9]. The square of a graph is also called the closed neighborhood graph, since two vertices are adjacent in the square, if and only if in the original graph their closed neighborhoods intersect. The characterization of open neighborhood graphs preformed in [1] has been generalized to m-step graphs in [4] (there called m-path graph). The paper also contains a characterization of n-powers. We do not analyze the process of iterated squaring in this paper. However, we remark that for iterated graph powers monotonicity arguments similar to those in the first part of Section 3 apply. This limits the complexity of the iteration process and makes its analysis easier.

In all the previously mentioned graph modifications, the resulting graph is defined on the same vertex set as the original graph. For various graph constructions, such as constructions that replace vertices or edges with other graph gadgets, this is not the case. Researchers have analyzed various graph properties under repeated application of such constructions. Graph substitutions for example are considered in [16]. For iterated line graphs the analyzed properties include their connectivity [18] and the growth of the maximum degree [10]. Limits of iterated H-line graphs, a line graph analogue of the H-neighborhood graph, have been studied in [3]. For another variant of the line graph construction, the so-called Gallai graphs, mortality under iterated application has been studied in [13].

2. Stabilization times of open neighborhood graphs

For a graph G let $N(G)$ be the open neighborhood graph of G, i.e., the graph on the same vertex set as G, for which two vertices share an edge if and only if there is a path of length two between them in the original graph. Iteratively, define $N^{k+1}(G)$ as $N(N^k(G))$. Define $\text{stab}(G)$ to be the smallest integer k such that $N^k(G) = N^{k+1}(G)$. If no such integer exists we let $\text{stab}(G) = \infty$. Since the stabilization time of a disconnected graph is the maximum stabilization time over all components, we will focus on connected graphs.

We summarize some immediate consequences of the definition of $N(G)$: A clique of size at least 3 in G forms a clique in $N(G)$ and thus it is a clique in $N^k(G)$ for every $k \in \mathbb{N}$. If G is bipartite, then $N(G)$ is disconnected. If G is an induced cycle of odd length at least 5, then $N(G)$ is isomorphic but not identical to G. These and various other basic properties are also listed in [19].

Throughout the section we repeatedly consider projections of walks: If $w = v_0, \dots, v_{2n}$ is a walk of even length in G, we call the sequence of vertices $w' = v_0, v_2, \dots, v_{2n}$ of $N(G)$ the *projection* of w. If w' is a projection of a walk w we call w a *lift* of w'. Note that the projection is not necessarily a walk in $N(G)$ since it may have consecutively repeated vertices. We call such a sequence, i.e., a sequence where one is allowed to stay at a vertex instead of traversing an edge, a *lazy* walk. However, every lazy walk can be shortened into a walk and even into a path between its endpoints, by removing repeated vertices and loops.

For the analysis of stabilization times of graphs we require the following lemma.

Lemma 1. *Let v_1, v_2, v_3 be a triangle of a connected graph G. Suppose there is a path from v to v_1 of length at most $d > 0$. Then*

- *$N^{\lceil \log_2(d) \rceil}(G)$ contains the edges (v, v_2) and (v, v_3) or*
- *$N^{\lceil \log_2(d) \rceil}(G)$ contains the edge (v, v_1).*

Proof. We prove the lemma by induction on d. The induction base for $d = 1$ is trivial. Let $d > 1$ and suppose the statement has been shown for all $\widehat{d} < d$ and for all graphs. We now show it for d. We thus suppose there is a path of length $0 < d' \leq d$ from v to v_1. If d' is even, then the path of length d' in G projects in $N(G)$ to a path of length $d'/2$ starting in v and ending in v_1. Thus, by induction, $N^{\lceil \log_2 d' \rceil}(G) = N^{\lceil \log_2 d' \rceil - 1}(N(G)) = N^{\lceil \log_2 (d'/2) \rceil}(N(G))$ contains the edges (v, v_2) and (v, v_3) or the edge (v, v_1). If d' is odd, then in G there is a walk of length $d' + 1$ from v to v_2, as well as such a walk from v to v_3. Thus, in $N(G)$ there exist two lazy walks of length at most $\lceil (d'+1)/2 \rceil = \lceil d'/2 \rceil$ starting in v. One of them ends in v_2 and the other ends in v_3. Removing multiple occurrences of vertices on the walks we get that, in $N(G)$, there exist two paths of length at most $\lceil d'/2 \rceil$ starting in v and ending in v_2 and v_3. If one of the paths has length 0 then $v \in \{v_1, v_2, v_3\}$ and the statement of the lemma is obvious. Otherwise, we can apply the inductive assumption twice on the graph $N(G)$ with v_1 replaced by v_2 and v_3 respectively. We then conclude that $N^{\lceil \log_2 d' \rceil}(G) = N^{\lceil \log_2 d' \rceil - 1}(N(G)) = N^{\lceil \log_2 \lceil d'/2 \rceil \rceil}(N(G))$ contains the edges (v, v_2) and (v, v_1) or the edge (v, v_3) and it also contains the edges (v, v_3) and (v, v_1) or the edge (v, v_2). In any case, it contains the edges (v, v_3) and (v, v_2) or it contains the edge (v, v_1) (amongst other edges). $\square$

As a corollary of the lemma we show that a connected non-bipartite graph that is not an induced odd cycle stabilizes into a clique. This fact has first been shown in [5] and can also be found in [19].

Corollary 2 (*Exoo and Harary [5]*). *For any finite connected non-bipartite graph G that is not an induced odd cycle of length at least 5, there exists an integer $i \in \mathbb{N}$ such that $N^i(G)$ is a clique.*

Proof. If G is a connected non-bipartite graph that is not an odd cycle, then G has a triangle or a vertex of degree at least 3. In either case $N(G)$ has a triangle. The graph G contains an odd cycle C and the vertices on the cycle are connected in $N(G)$. Thus, since for every vertex in G there is a path of even length to some vertex of C, the graph $N(G)$ is connected. The lemma implies now that for sufficiently large j in $N^j(G)$ every pair of vertices has a common neighbor in the triangle. $\square$

We now turn to the main result of this section, namely, bounds on the stabilization time in terms of the diameter of the graph. We first remark that the proof of Lemma 1 can be adapted to show that if G is a connected graph that has a triangle then $\operatorname{stab}(G) \leq \lceil \log_2(\operatorname{diam}(G)) \rceil + 1$. We can directly use this to show that $\operatorname{stab}(G) \leq \lceil \log_2(\operatorname{diam}(G)) \rceil + 3$ for a non-bipartite, connected graph G that is not an odd cycle. However, to show a tight upper bound we need further details.

Theorem 3. *For any finite connected non-bipartite graph G that is not an odd cycle we have*

$$\lceil \log_2(\operatorname{diam}(G)) \rceil \leq \operatorname{stab}(G) \leq \lceil \log_2(\operatorname{diam}(G)) \rceil + 2.$$

Proof. (1). "$\lceil \log_2(\operatorname{diam}(G)) \rceil \leq \operatorname{stab}(G)$": For this we use the fact that G stabilizes into a clique. I.e., if G has n vertices then $N^{\operatorname{stab}(G)}(G) = K_n$. It suffices to show that $\operatorname{diam}(N(G)) \geq \operatorname{diam}(G)/2$, whenever G is a connected non-bipartite graph that is not an odd cycle. Suppose G fulfills these assumptions. Then there are two vertices v_1 and v_2 of distance $\operatorname{diam}(G)$. In $N(G)$ the two vertices are of distance at least $\operatorname{diam}(G)/2$, since every path in $N(G)$ lifts to a walk in G of at most twice the length.

(2). "$\operatorname{stab}(G) \leq \lceil \log_2(\operatorname{diam}(G)) \rceil + 2$": Let C be a shortest odd cycle in G. Such a cycle exists, since G is not bipartite. Note that C is of length at most $2\operatorname{diam}(G) + 1$. Since G is connected and not an odd cycle, the cycle C contains a vertex u of degree at least 3. Let u_1 and u_2 be the two neighbors of u on the cycle, and x a neighbor of u not on the cycle (see Fig. 1). Further suppose that v is a vertex different from u_1, u_2 and x.

Claim. *The graph $N^{\lceil \log_2(\operatorname{diam}(G)) \rceil + 1}(G)$ contains two of the edges (v, u_1), (v, u_2) and (v, x).*

Consider the two vertices w_1 and w_2 furthest away from u on the cycle, say w_1 is closer to u_1 than w_2. Note that $d(w_1, u_1) \leq \operatorname{diam}(G) - 1$ and $d(w_1, u_2) \leq \operatorname{diam}(G)$, where $d(\cdot, \cdot)$ is the distance function between vertices. Likewise $d(w_2, u_2) \leq \operatorname{diam}(G) - 1$ and $d(w_2, u_1) \leq \operatorname{diam}(G)$. Also note that for both $i \in \{1, 2\}$ one of the numbers $d(w_1, u_i)$ and $d(w_2, u_1)$ is even and the other one is odd. Which of the numbers is even depends on the length of C modulo 4.

For an integer k denote by $\operatorname{even}(k)$ the smallest even integer $i \geq k$. We first show that there is a path of even length at most $\operatorname{even}(\operatorname{diam}(G))$ in $N(G)$ from v to u_1 as well as a path of at most this length from v to u_2:

In G there is a path p_1 of length at most $\operatorname{diam}(G)$ from v to w_1 and one such path p_2 from v to w_2. We distinguish two cases depending on the parity of the length of the paths p_i.

Case (1): Suppose the paths p_1 and p_2 both have even length. For each $i \in \{1, 2\}$ there is a $j \in \{1, 2\}$, such that there exists a path from u_i to w_j of even length at most $\operatorname{diam}(G)$. Combining this path with p_j gives a walk from u_i to v of even length at most $2 \cdot \operatorname{diam}(G)$.

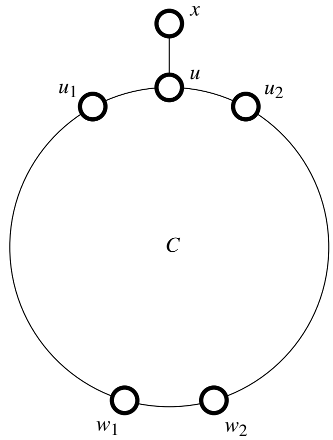

Fig. 1. The figure depicts the shortest odd cycle C and various vertices considered in the proof of Theorem 3. The parity of the distances between the vertices u_i and w_j depends on the length of C modulo 4.

Case (2): Now suppose one of the walks, p_1 say, is of odd length $l_1 \leq \text{diam}(G)$. For some choice of $j_1 \in \{1, 2\}$ the shortest path from w_1 to u_{j_1} is of odd length at most $\text{diam}(G)$. The path that continues this shortest path via u_{j_1} to u_{j_2} (where j_2 is chosen such that $\{j_1, j_2\} = \{1, 2\}$) is of odd length $l_2 \leq \text{diam}(G) + 2$. Since $l_1 < \text{even}(\text{diam}(G))$ and $l_2 < \text{even}(\text{diam}(G)) + 2$, we get that the walk that continues p_1 via u_{j_1} to u_{j_2} is of even length at most $l_1 + l_2 \leq 2\,\text{even}(\text{diam}(G))$. As in Case (1) for each $i \in \{1, 2\}$ there is a walk of even length at most $2\,\text{even}(\text{diam}(G))$ from v to u_i.

Recall that any walk of length $2d$ in G projects to a (lazy) walk in $N(G)$ which can be shortened to a path no longer than d. Thus in either case we conclude that there are paths in $N(G)$ from v to u_1 and to u_2 of length at most $\text{even}(\text{diam}(G))$.

In $N(G)$, the vertices u_1, u_2 and x form a triangle, and to each of the vertices u_1 and u_2 there is a path from vertex v of length at most $\text{even}(\text{diam}(G))$. The claim now follows from Lemma 1 by using the fact that $\lceil \log_2(\text{even}(\text{diam}(G))) \rceil = \lceil \log_2(\text{diam}(G)) \rceil$ whenever $\text{diam}(G) > 1$.

We finish the proof by applying the claim: By the arguments above $N^{\lceil \log_2(\text{diam}(G)) \rceil + 1}(G)$ contains at least two of the edges (v, u_1), (v, u_2) and (v, x) for all vertices v different from u_1 and u_2. Since u_1, u_2 and x form a triangle, two of the edges also exist for $v \in \{u_1, u_2, x\}$. Thus, in $N^{\lceil \log_2(\text{diam}(G)) \rceil + 1}(G)$, any two vertices v and v' have a common neighbor, and therefore, in $N^{\lceil \log_2(\text{diam}(G)) \rceil + 2}(G)$, they share an edge. □

Corollary 4. *For any finite graph G and any integer i that fulfills $i \geq 2 + \log_2(\text{diam}(C))$ for all connected components C of G, the graphs $N^i(G)$ and $N^{i+1}(G)$ are isomorphic.*

Proof. It suffices to show the statement for a connected component C of G. We show the statement by induction on $\text{diam}(C)$. The base case is a component C with $\text{diam}(C) \leq 1$, which is a clique. Thus $\text{stab}(C) \leq 1$. For the induction step, let C be a connected component of G. If C is an odd cycle then C is isomorphic to $N(C)$. If C is bipartite, then $N(C)$ consists of exactly two components C_1 and C_2. For $i \in \{1, 2\}$ we have $\text{diam}(C_i) \leq \text{diam}(C)/2$, and the statement follows by induction. Finally if C is not bipartite and contains a vertex of degree at least 3 then the corollary follows from Theorem 3. □

We now show that both inequalities of Theorem 3 are tight. Figs. 2 and 3 depict the two types of graphs for which the bounds are attained. Let $C_qP_s = C_q \oplus P_{s+1}$ be the graph on $q + s$ vertices obtained by gluing a path of length $s + 1$ at one of its endpoints to a cycle of length q. These graphs were considered in [19].

Theorem 5. *Let i be an integer larger than 2.*

1. *For the graph $G = P_i \times P_2$, i.e. the strong product of a path of length $i - 1$ with an edge, we have* $\text{stab}(G) = \lceil \log_2(\text{diam}(G)) \rceil$.
2. *For the graph $C_qP_s = C_q \oplus P_{s+1}$ with $q \geq 3$ odd and $s \geq 2$ we have* $\text{stab}(C_qP_s) \geq 1 + \lceil \log_2(q + s - 2) \rceil$.
   In particular for $s = 2$ and $q \equiv 1 \bmod 4$ we get $\text{stab}(C_qP_2) = 2 + \lceil \log_2(\text{diam}(C_qP_2)) \rceil$.

Proof. For the first part it suffices to observe that every pair of vertices in $P_i \times P_2$, is joined by a path of length $2^{\lceil \log_2(\text{diam}(G)) \rceil}$.

For the second part, let u, v be the following two vertices: If s is even, then u is the vertex on the path furthest away from the cycle. If s is odd then u is the neighbor of the vertex furthest away from the cycle. In both cases v is the unique vertex at distance 2 from u.

Recursively, a walk $p_1, \ldots, p_t$ in $N^i(G)$ is called feasible, if the walk is of length 1 or if there is a positive integer ℓ such that the walk is of length $t - 1 = 2^\ell$, $p_k \neq p_{k+2}$ whenever k is odd, and the walk $p_1, p_3, p_5, \ldots, p_t$ is feasible in $N^{i+1}(G)$. By induction, two vertices v_1, v_2 are adjacent in $N^i(G)$ if and only if there is a feasible walk of length 2^i from v_1 to v_2.

We now argue that the shortest feasible walk from u to v of length greater than 2 in C_qP_s has length at least $2s - 2 + 2 \cdot q - 2\delta(s)$, where $\delta(s) = 1$ if s is odd, and $\delta(s) = 0$ otherwise. Let P be a feasible walk from u to v, then P traverses every

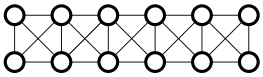

Fig. 2. The depicted graph $P_6 \times P_2$ is an example of the graphs used to show that the lower bound of Theorem 3 is tight.

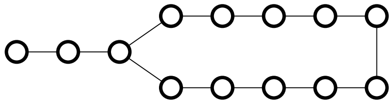

Fig. 3. The graph $C_{11}P_2$ is an example of the graphs used to show that the upper bound of Theorem 3 is tight.

edge of the cycle: Indeed, suppose this is not the case. Removal of an edge of the cycle leaves us with a bipartite graph G' with the following properties: In $N(G')$ vertices u and v are adjacent and the component that contains u and v is bipartite. Thus in no further iteration are u and v adjacent, which contradicts the existence of the feasible walk P not containing every edge of the cycle.

By parity, any walk of even length from u to v has to circle the cycle an even number of times. Since the sum of the distances of v to the cycle and u to the cycle is $2s-2-2\delta(s)$, this shows that P has length at least $2s-2-2\delta(s)+2\cdot q$. Thus $\operatorname{stab}(C_qP_s) \geq$

$$\lceil \log_2(2s-2-2\delta(s)+2q)\rceil = 1+\lceil \log_2(s+q-1-\delta(s))\rceil = 1+\lceil \log_2(s+q-2)\rceil.$$

To prove the additional remark concerning the special case of $s=2$ and $q \equiv 1 \mod 4$ first note that if $q \equiv 1 \mod 4$ then $\lceil \log_2(q)\rceil = \lceil \log_2(q+3)\rceil$. It now suffices to conclude that $\operatorname{stab}(C_qP_2) \geq 1+\lceil \log_2(2+q-2)\rceil = 1+\lceil \log_2(q)\rceil = 1+\lceil \log_2(q+3)\rceil = 2+\lceil \log_2(q/2+3/2)\rceil = 2+\lceil \log_2(2+(q-1)/2)\rceil = 2+\lceil \log_2(\operatorname{diam}(C_qP_2))\rceil$. $\square$

We remark that in [19] it is conjectured that $\operatorname{stab}(C_qP_s) = 1+\lceil \log_2(q+s-2)\rceil$ whenever q is odd and $s \geq 1$. The second part of the previous theorem establishes the lower bound of the equality for $s \geq 3$.

We say a graph is *mortal* if for some $i \in \mathbb{N}$ the graph $N^i(G)$ contains no edges.

Theorem 6. *A graph is mortal if and only if it is a disjoint union of cycles of two-power length and paths.*

Proof. If a graph G contains a vertex of degree larger than 2, then the neighbors of this vertex form a triangle in $N(G)$ and therefore G is not mortal. Thus a mortal graph is a union of paths and cycles. All paths are mortal, and it thus suffices to analyze mortality of cycles. If a cycle C has two-power length, then $N(C)$ is the disjoint union of two cycles of two-power length or $N(C)$ is the disjoint union of two edges. Thus C is mortal. If C is not of two-power length then either C has odd length, and is thus not-mortal, or $N(C)$ is the disjoint union of two cycles both not of two-power length. Thus, by induction cycles that are not of two-power length are not mortal, which concludes the theorem. $\square$

3. Stabilization times of generalized neighborhood graphs

Definition 7. Let H be a graph with two distinguished vertices a and b. For a graph G, we define $N_H(G)$, *the H-neighborhood graph of G*, to be the graph on the vertex set $V(G)$ in which two vertices v_1 and v_2 are adjacent, if there is a subgraph of G isomorphic to H, such that v_1 and v_2 correspond to a and b respectively. If H and G are vertex-colored graphs, we require the subgraph of G isomorphic to H to respect these colors. The *iterated H-neighborhood graph $N_H^i(G)$* is defined by the recursion $N_H^{i+1}(G) = N_H(N_H^i(G))$.

We call the graph H in the definition the *replacement graph*. With this definition the competition graph of G is the graph $N_{P_3}(G)$ where P_3 is the path of length 2 with distinguished end vertices a and b. More general the m-path graph is the graph N_{P_m} where P_m is the path of length $m-1$ with distinguished end vertices. The p-competition graph of G can also be expressed as H-neighborhood graph. It is the graph $N_{K_{2,p}}(G)$, where in the complete bipartite graph $K_{2,p}$ the vertices of the partition class of size 2 are the distinguished vertices. As before we define $\operatorname{stab}_H(G)$ to be the smallest integer k such that $N_H^k(G) = N_H^{k+1}(G)$.

We first analyze the case where the distinguished vertices of H are adjacent. In this case $\operatorname{stab}_H(G)$ is at most as large as the number of edges of G, and in particular finite.

Lemma 8. *If the distinguished vertices a and b of H are adjacent, then* $\operatorname{stab}_H(G) \leq m$ *for any graph G with at most m edges.*

Proof. In iteration i, a pair of vertices a and b can only be adjacent in the H-neighborhood graph, if the vertices are adjacent in iteration $i-1$. Thus the set of edges is monotonically decreasing with respect to inclusion. Once the set does not decrease, stabilization has been reached. $\square$

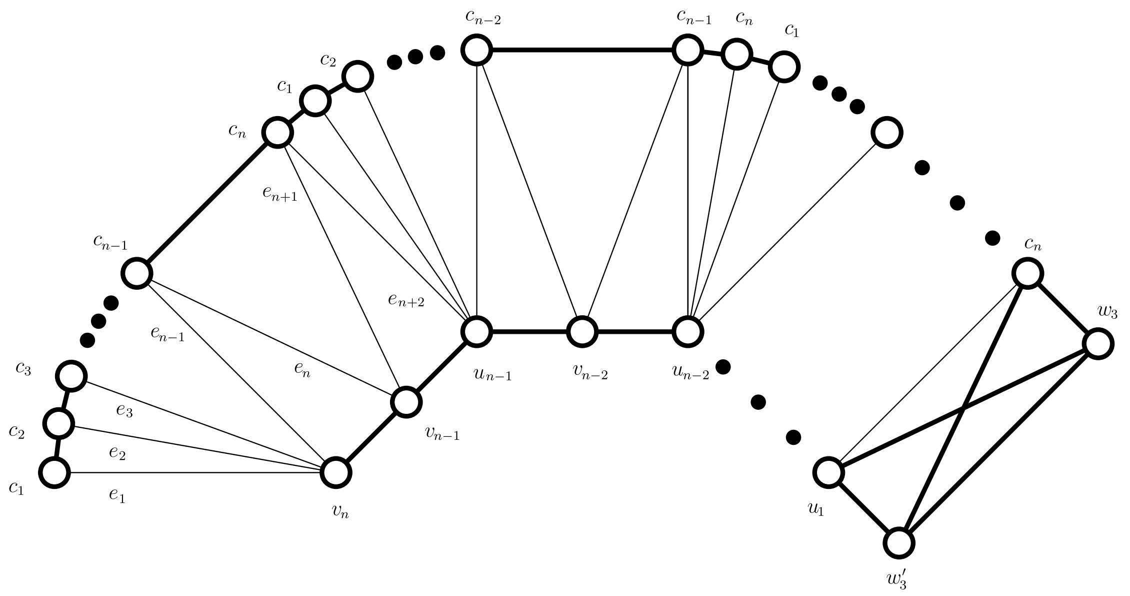

Fig. 4. A graph with stabilization time $\Omega(n)$ for the replacement graph $H = K_2 * I_2$. The vertices w_1, w'_1, w_2 and w'_2, which ensure that the non-relevant edges (thick lines) do not vanish, have been omitted in the figure. The construction ensures that the relevant edges (thin lines) disappear one by one in each iteration. The graph G_n with $\mathrm{stab}(G_n) \in \Omega(n^2)$ constructed in the proof of Theorem 9 is obtained from the depicted graph by identifying cycle vertices equally labeled by c_i.

The next theorem shows that this upper bound on the stabilization time cannot be improved since there is a graph H for which stabilization may require $\Omega(n^2)$ iterations on graphs with n vertices. By letting H be the diamond, we can obtain a family of graphs for which in each iteration exactly one edge vanishes (see Fig. 4). This graph family has a stabilization time in $\Omega(n)$. The following theorem shows how to fold the graph by identifying vertices in order to obtain a graph family with stabilization time in $\Omega(n^2)$.

Theorem 9. *Let H be the diamond, i.e., the graph $K_2 * I_2$ obtained by joining an edge with an independent set of size two. Let the two vertices that originally form the K_2 be the two distinguished vertices in H. Then*

$$\max\{\mathrm{stab}_H(G) \mid G \text{ is a graph of size } n\} \in \Omega(n^2).$$

Proof. For $n \in \mathbb{N} \geq 3$ we construct a graph G_n on $\Theta(n)$ vertices for which $\mathrm{stab}_H(G_n) \in \Omega(n^2)$. We start the construction with a cycle C consisting of vertices $c_1, \ldots, c_n$, such that c_i is adjacent to c_j if $|i - j| \equiv 1 \bmod n$.

We add vertices $u_1, \ldots, u_n$. Each u_i is adjacent to all vertices of the cycle except c_i. We then add vertices $v_1, \ldots, v_{n-1}$ and add edges such that v_i is adjacent to u_i, u_{i+1}, c_i and c_{i+1}. These added vertices u_i and v_i that are not on the cycle are called hubs.

Finally we add two adjacent vertices w_1 and w'_1 adjacent to every vertex of the cycle C, two adjacent vertices w_2 and w'_2 adjacent to every hub and two adjacent vertices w_3 and w'_3 which are adjacent to c_n and to u_1. A lift of the graph G_n is depicted in Fig. 4.

We call an edge *relevant*, if one of its end points is on the cycle and the other end point is a hub. Note that, with the exception of triangles involving the relevant edge $\{c_n, u_1\}$, every triangle that contains a relevant edge contains exactly two relevant edges. Also note that, due to the addition of the vertices w_i and w'_i, every non-relevant edge is contained in a clique of size 4. Thus, such an edge is contained in $N^i_H(G_n)$ for all $i \geq 0$. Likewise, due to the addition of w_3 and w'_3, the edge $e_t = \{c_n, u_1\}$ is contained in $N^i_H(G_n)$ for all $i \geq 0$.

We define an ordering of the hubs by alternating between u_i and v_i with decreasing index $u_n, v_{n-1}, u_{n-1}, v_{n-2}, \ldots, u_1$. Using this ordering we order the relevant edges $e_1, \ldots, e_t$ iteratively by setting $\{c_1, u_n\}$ as first edge. The edge that follows an edge $\{c_i, w\}$ for some hub w is the edge $\{c_{i+1}, w\}$ (where the index $i+1$ is taken modulo n) if this edge exists. If this edge does not exist, the next edge is defined as $\{c_i, w'\}$ where w' is the hub that follows w in the ordering of hubs just defined. We thus obtain the following ordering:

$$\{c_1, u_n\}, \{c_2, u_n\}, \ldots, \{c_{n-1}, u_n\}, \{c_{n-1}, v_{n-1}\}, \{c_n, v_{n-1}\}, \{c_n, u_{n-1}\}, \{c_1, u_{n-1}\}, \ldots, \{c_n, u_1\}.$$

For this ordering, we observe the following property: For $1 < i < t$ the relevant edge e_i is contained in two triangles. One of the triangles involves edge e_{i-1} and one of the triangles involves edge e_{i+1}. Thus, edge e_i is contained in a diamond in some graph $N^t_H(G_n)$ if and only if the edges e_{i-1}, e_i and e_{i+1} are present in the graph. We conclude by induction: edge e_i is contained in $N^i_H(G_n)$ but not in $N^{i+1}_H(G_n)$. Since there is a quadratic number of relevant edges, this shows the theorem. □

To avoid the use of the two distinguished vertices a and b, we could consider the use of an edge-transitive replacement graph H. By definition an edge-transitive graph is a graph such that for every two edges e_1 and e_2 there is an automorphism mapping e_1 to e_2. Thus, distinguishing different pairs of adjacent vertices in such a graph will result in equivalent behavior with respect to stabilization times. However, if any two adjacent vertices in an edge-transitive graph H are chosen as distinguished vertices, then the stabilization time is at most 2:

Lemma 10. *If H is edge-transitive and a and b are adjacent, then* $\mathrm{stab}_H(G) \leq 2$ *for any graph G.*

Proof. By definition $N_H(G)$ is the graph of all edges that are contained in a copy of H in G. Every edge in $N_H(G)$ is thus contained in a copy of H in $N_H(G)$. □

We now focus on the case in which the distinguished vertices of H are not adjacent. Already for the symmetric case we obtain graphs with periodic patterns and therefore $\mathrm{stab}_H(G) = \infty$: Indeed, if H is a symmetric self-complementary graph, by distinguishing two non-adjacent vertices and setting $G = H$ we obtain a periodic pattern with period 2. Infinitely many self-complementary symmetric graphs exist. They are described in [15].

This indicates that the case of a and b non-adjacent yields more complicated behavior. Indeed, in the rest of this section we will prove that there exists a replacement graph H, for which the computation of the function N_H^i is equivalent to the computational behavior of a Turing machine.

To relate our replacement construction to Turing machines we use the concept of grid-like graphs. Intuitively, a graph is grid-like if, up to finitely many edges, it is obtained by placing a gadget to every position of $\mathbb{Z}^2$ and then connecting the gadgets in a manner that respects the underlying grid structure. More formally, we say a graph G is *grid-like* if $\mathbb{Z}^2$ acts freely and cocompactly by automorphisms on a graph obtained from G by inserting and deleting a finite number of edges. In other words, the altered graph has finitely many orbits under the $\mathbb{Z}^2$ action and is of bounded degree. A set E of pairs of vertices is compatible with this grid structure of G if E is preserved by the $\mathbb{Z}^2$ action and has finitely many orbits. We will simply say that E is compatible. Note that grid-like graphs are infinite and that a Turing machine of course cannot be simulated on a finite graph.

Theorem 11. *There is a finite graph H and a grid-like graph G with the following properties: There is a polynomial time reduction that associates with every Turing machine configuration T a graph obtained by altering finitely many edges of G in a compatible set of vertex pairs, such that the graph associated with the configuration after running T for i steps is $N_H^i(G)$.*

In the rest of this section, we show the theorem by reducing the Game of Life to our model. The construction involves repeated modification of the involved replacement graphs.

Road map for the proof of Theorem 11. The proof proceeds in several stages. Each of these stages corresponds to a lemma we prove subsequently:

1. We start with a more general model that allows special vertex pairs and colored vertices in G. Only for the special vertex pairs a modification of the edges is performed. Moreover, we allow different types of special vertex pairs. For each type i, an edge is present in the next iteration, if the vertex pair forms the distinguished vertex pair in a copy of one the replacement graphs $H_{1,i}, \ldots, H_{\ell_i,i}$. (Lemma 13)
2. We then reduce the problem to a problem where only one replacement graph is used for every type of distinguished vertex pair. (Lemma 15)
3. In the next step, we reduce our problem to the case where only one type of special vertex pair is present. (Lemma 16)
4. We further reduce to a problem that involves only uncolored graphs. (Lemma 17)
5. Finally we reduce to the case in which all vertex pairs are subject to the replacement construction, as opposed to restricting to special vertex pairs. (Lemma 18)

Conway's Game of Life. We briefly recall the Game of Life (see [6]). In the Game of Life cells are arranged in a 2-dimensional infinite grid in the plane. A cell can either be dead or alive. The neighbors of a cell are the 8 cells that are adjacent horizontally, vertically or diagonally. The game starts with a seed that assigns a state to each cell as either being dead or alive and then proceeds in iterations (usually called ticks). In an iteration, the state of each cell is determined by the cell's previous state and the previous states of its neighbors: A cell that was alive in iteration i continues to live in iteration $i + 1$ if at least 2 and at most 3 of its neighbors had been alive in iteration i. The cell dies otherwise. A cell that is dead in iteration i becomes alive in iteration $i + 1$ if exactly 3 of its neighbors are alive in iteration i and remains dead otherwise.

We will show that the H-neighborhood construction can be used to simulate the Game of Life. Since the Game of Life can simulate Turing machines [2] this allows us to prove Theorem 11.

Definition 12. Suppose $k \in \mathbb{N}$ is a positive integer. For each $1 \leq i \leq k$ let $\mathcal{H}_i = \{H_{1,i}, \ldots, H_{\ell_i,i}\}$ be a set of finite vertex-colored replacement graphs. Further let G be a vertex-colored graph and let $E_1, E_2, \ldots, E_k \subseteq \binom{V}{2}$ be disjoint sets of pairs of vertices. The *restricted* $(\mathcal{H}_1, \ldots, \mathcal{H}_k)$*-neighborhood graph* with respect to the given vertex pairs and replacement graphs, denoted $N[E_1, \mathcal{H}_1; E_2, \mathcal{H}_2; \ldots; E_k, \mathcal{H}_k](G)$, is the graph defined on the vertex set $V(G)$ with edge set

$$E = (E(G) \setminus (E_1 \cup E_2 \cup \cdots \cup E_k)) \cup \bigcup_{1 \leq i \leq k} \bigcup_{1 \leq j \leq \ell_i} E(N_{H_{j,i}}(G)) \cap E_i.$$

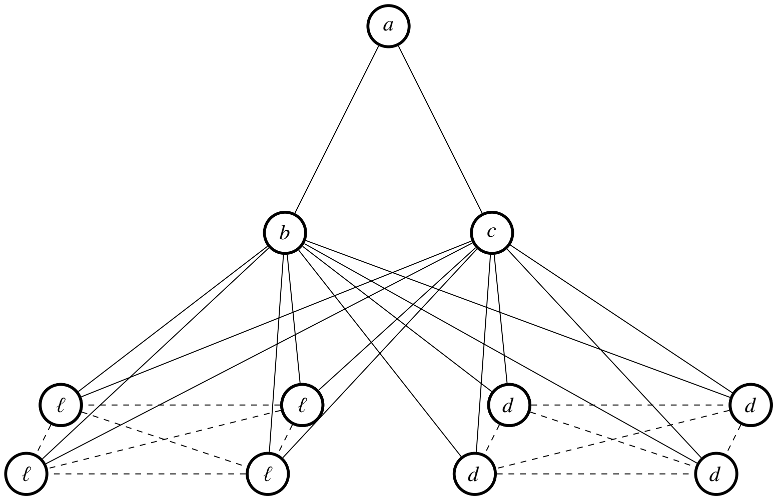

Fig. 5. The graph gadget that is attached to every vertex of Z^{2*} in order to construct the grid-like graph G.

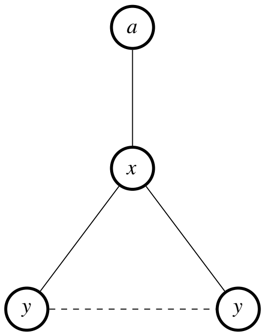

Fig. 6. The basic building block $H_{x,y,\delta}$ used to assemble the replacement graphs $H^t_{x,x'}$. The variables $x \in \{b, c\}$ and $y \in \{\ell, d\}$ serve as parameters determining the colors of some vertices as depicted. The two vertices of color y may serve as distinguished vertices, as indicated by the dashed line. Their adjacency depends on the parameter $\delta \in \{0, 1\}$. They are adjacent if and only if δ is equal to 1.

The *iterated restricted* $(\mathcal{H}_1, \dots, \mathcal{H}_k)$*-neighborhood graph* is defined by the recursion

$$N^{i+1}[E_1, \mathcal{H}_1; \dots; E_k, \mathcal{H}_k](G) = N[E_1, \mathcal{H}_1; \dots; E_k, \mathcal{H}_k]\left(N^i[E_1, \mathcal{H}_1; \dots; E_k, \mathcal{H}_k](G)\right).$$

Let Z^{2*} be the 2-dimensional grid with diagonals, i.e., the graph on the vertex set $\mathbb{Z}^2$ where two vertices $(i, j), (i', j') \in \mathbb{Z}^2$ are adjacent, if $\max\{|i - i'|, |j - j'|\} = 1$.

The graph G is formed by gluing to every vertex of Z^{2*} a copy of the graph depicted in Fig. 5 along the top vertex. (This construction is sometimes called the rooted product [7].) We call the elements of $\mathbb{Z}^2$ *positions*. Thus, at every position exactly one copy of the graph gadget is attached.

A pair of vertices in the same copy of the gadget that are both of color ℓ or both of color d (dashed edges in Fig. 5) is called an indicator. Indicators may be present or not, depending on whether they form an edge in the graph or not.

We call an indicator that consists of vertices of color ℓ a live indicator. An indicator corresponding to vertices of color d is called a dead indicator. Let E_ℓ and E_d be the set of live and dead indicators respectively. A position $(i, j) \in \mathbb{Z}^2$ is said to be live, if all its live indicators are present, and all its dead indicators are not. It is dead, if all its dead indicators are present, but no live indicator is present. It will be clear later, that no mixture of these states arises, if the initial configuration does not have a mixture of these states. The graph in which all indicators are dead thus corresponds to the graph $G + E_d$, i.e., G altered by adding edges to connect all pairs in E_d. By construction, the graphs G and $G + E_d$ are grid-like and the sets E_ℓ and E_d are compatible with both graphs.

To design the replacement graphs we use the graph shown in Fig. 6 as basic building block. We call the vertex of $H_{x,y,\delta}$ of color a and degree 1 the *top vertex*.

The Graph $H^t_{x,x'}$ with $x, x' \in \{\ell, d\}$ and $t \in \{0, \dots, 8\}$ is obtained by first gluing a copy of $H_{b,x,0}$ and a copy of $H_{c,x',1}$ along their top vertices and adding t copies of $H_{c,\ell,1}$ and $8 - t$ copies of $H_{c,d,1}$. We add an edge from all top vertices of these 8 graphs to the identified top vertex of the two previously added graphs. An example is shown in Fig. 7. The two vertices colored x in the copy of $H_{b,x,0}$ become the distinguished vertices.

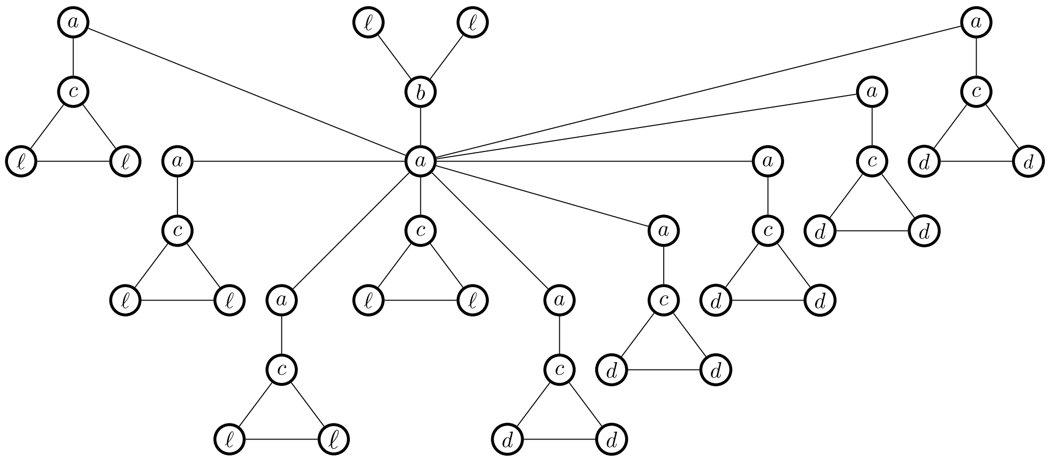

Fig. 7. The graph $H^3_{\ell,\ell}$ obtained by first gluing together $H_{b,\ell,0}$ and $H_{c,\ell,1}$ along their top vertices (depicted near the middle) and then adding 3 copies of $H_{c,\ell,1}$ (left) and 5 copies of $H_{c,d,1}$ (right). The two distinguished vertices are the vertices of color ℓ at the top. In G a live indicator forms the distinguished pair of vertices in a copy of $H^3_{\ell,\ell}$ if and only if the position of the indicator is alive, 3 of its neighbors are alive and 5 of its neighbors are dead.

In G, an indicator of color x forms the distinguished pair of vertices in a copy of $H^t_{x,x'}$ if and only if the following two properties hold:

1. The position of the indicator is alive if $x' = \ell$ and dead if $x' = d$.
2. There are t neighboring positions that are alive and $8 - t$ neighboring positions that are dead.

The graphs $H^t_{x,x'}$ are sufficiently powerful to model the behavior of an arbitrary cellular automaton for which liveliness decisions for a cell only depend on the status the cell itself and on the number of live and dead neighbors.

Lemma 13. *There are two sets of graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ such that the following holds: If the live and dead edges in the graph G are set up in correspondence to a situation s in the Game of Life, then $N[E_\ell, \mathcal{H}_1; E_d, \mathcal{H}_2](G)$ is set up in correspondence to the situation obtained from s by performing one iteration of the Game of Life.*

Proof. In the Game of Life, for each cell the rule whether the cell is dead or alive in the next iteration depends only on the status of the cell itself and on the number of neighbors that are alive. Since the graphs $H^t_{x,x'}$ can model an arbitrary status of a position itself and of the number of neighboring positions that are alive, the desired sets $\mathcal{H}_1$ and $\mathcal{H}_2$ can be chosen as sets of graphs of the form $H^t_{x,x'}$. $\square$

To avoid unsuitable replacement graphs, we restrict ourselves to a certain class: We call a graph H a *reasonable replacement graph* if H is finite and connected, the distinguished vertices v_a and v_b are not adjacent, there is an automorphism that interchanges v_a and v_b, neither v_a nor v_b is a cut-vertex and there is no 2-cut in H different from $\{v_a, v_b\}$.

In order to increase their connectivity, we alter the graph G and all graphs H' in $\mathcal{H}_1$ or $\mathcal{H}_2$ by forming the products $G \times K_3$ and $H' \times K_3$, i.e., for each graph we form the strong product with K_3, the complete graph on three vertices. In each graph, every vertex is assigned the color of its natural projection to the first factor. In $G \times K_3$, the set of live indicators E'_ℓ is the set of vertex pairs that project to a live indicator G. Likewise, the set of dead indicators E'_d is the set of vertex pairs that project to a dead indicator in G. In the products $H' \times K_3$ the distinguished vertices are chosen as some pair of vertices that project to the pair of distinguished vertices of the first factor. By construction, Lemma 13 also holds for these altered graphs. Note that the graphs $H' \times K_3$ are reasonable replacement graphs.

Lemma 14. *There are two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of reasonable replacement graphs such that the following holds: If the live and dead edges in the graph G are set up in correspondence to a situation s in the Game of Life, then $N[E'_\ell, \mathcal{H}_1; E'_d, \mathcal{H}_2](G)$ is set up in correspondence to the situation obtained from s by performing one iteration of the Game of Life.*

We now show that by altering G we can achieve that $\mathcal{H}_1$ and $\mathcal{H}_2$ only contain one graph.

Lemma 15. *Given sets of colored replacement graphs $\mathcal{H}_1$ and $\mathcal{H}_2$, a grid-like colored graph G and disjoint compatible sets of pairs of vertices $E_1, E_2 \subseteq \binom{V}{2}$, there are colored graphs G', H_1, H_2, with G' grid-like and with H_1 and H_2 reasonable replacement graphs, such that G is an induced subgraph of G' and for all $i \in \mathbb{N}$ the subgraph of $N^{i+1}[E_1, \{H_1\}; E_2, \{H_2\}](G')$ that is induced by the vertices of G coincides with the graph $N^{i+1}[E_1, \mathcal{H}_1; E_2, \mathcal{H}_1](G)$.*

Proof. We show how to replace the set $\mathcal{H}_1$ by the set $\{H_1\}$ consisting only of one graph H_1. The set $\mathcal{H}_2$ can be replaced analogously by repeating the construction. For each $H \in \mathcal{H}_1$ add a vertex of a previously unused color x that is adjacent to the distinguished vertices of H and all vertices of H that are adjacent to one of the distinguished vertices of H. Call the obtained graph H_{+x}.

Suppose $v_a(H_{+x})$ and $v_b(H_{+x})$ denote the distinguished vertices of H_{+x}. Let H_1 be the graph obtained in the following way: We first form the disjoint union of all graphs H_{+x}. We then identify all vertices $v_a(H_{+x})$ to a vertex v_a and identify all vertices $v_b(H_{+x})$ to a vertex v_b. Note that H_1 is a reasonable replacement graph. Also note that the number of connected components of $H_1 - \{v_a, v_b\}$ is exactly $|\mathcal{H}_1|$.

Let $\overline{H_1}$ be the graph obtained as follows: Starting from H_1 we replace all vertices of color x by $|\mathcal{H}_1| - 1$ other vertices of color x. Each of the new vertices is adjacent to every vertex that has a neighbor of color x in H_1. We alter G in the following way: For every pair $\{u, v\} \in E_1$ add a vertex colored x that is adjacent to $\{u, v\}$ and to all vertices that are neighbors of u or v. Additionally, to every pair $\{u, v\} \in E_1$ attach a copy of the graph $\overline{H_1}$ by gluing one of the distinguished vertices to u and the other one to v. Call the resulting graph G'. By construction G' is grid-like.

Note that the $\overline{H_1}$-attachment to a pair $\{u, v\}$ can only be used to cover $|\mathcal{H}_1| - 1$ of the components of $H_1 - \{v_a, v_b\}$ since it only contains $|\mathcal{H}_1| - 1$ vertices of color x. This implies, since $H_1 - \{v_a, v_b\}$ has $|\mathcal{H}_1|$ connected components, that a pair $\{u, v\}$ in G is contained in a copy of some graph in $\mathcal{H}_1$ if and only if it is contained in a copy of H_1 in G'. Also note that, due to the absence of 2-cuts other than the distinguished vertices, the attachment does not interfere with edges of E_2 and their containment in copies of some graph in $\mathcal{H}_2$: Indeed, since reasonable replacement graphs do not have 2-cuts other than their distinguished vertices, it is not possible that parts of a replacement graph are part of a graph attached to G' at two vertices, unless the two attachment vertices are the distinguished vertices of the graph.

The modifications we performed to G and the graphs in $\mathcal{H}_1$ in order to obtain G' and H_1 are independent of edges in E_1 or E_2 being present or not. This implies that applying the construction to the restricted open neighborhood graph of G with respect to $\{H_1\}$ and $\mathcal{H}_2$ gives exactly the same graph as applying the construction first and then forming the restricted open neighborhood graph. Thus, we conclude by induction that for all $i \in \mathbb{N}$, the subgraph of $N^{i+1}[E_1, \{H_1\}; E_2, \mathcal{H}_2](G')$ that is induced by the vertices of G coincides with the graph $N^{i+1}[E_1, \mathcal{H}_1; E_2, \mathcal{H}_2](G)$.

As explained at the beginning of the proof, repeating the construction for $\mathcal{H}_2$ allows us to replace $\mathcal{H}_2$ with $\{H_2\}$ and we obtain the lemma. $\square$

Lemma 16. *Given colored, reasonable replacement graphs H_1, H_2, a grid-like colored graph G and disjoint compatible sets of pairs of vertices $E_1, E_2 \subseteq \binom{V}{2}$, there are colored graphs G', H, with G' grid-like and with H a reasonable replacement graph, such that G is an induced subgraph of G' and the subgraph of $N^{i+1}[E_1 \cup E_2, \{H\}](G')$ that is induced by the vertices of G coincides with the graph $N^{i+1}[E_1, \{H_1\}; E_2, \{H_2\}](G)$.*

Proof. Let a_i and b_i be the distinguished vertices of H_i. Let H be the graph obtained from the disjoint union of H_1 and H_2 by identifying a_1 with a_2 and b_1 with b_2. Note that H is a reasonable replacement graph.

For each pair $\{u, v\}$ in E_1 hang a copy of H_2 from the vertices u and v. Likewise for each pair $\{u, v\}$ in E_2 hang a copy of H_1 from the vertices u and v. Call the resulting graph G'. By construction G' is grid-like.

On the one hand, if a pair $\{u, v\}$ in E_1 forms the distinguished vertices in a copy of H_1 in G then by construction the pair forms the distinguished vertices in a copy of H in G'.

On the other hand, if a pair $\{u, v\}$ in E_1 does not form the distinguished vertices in a copy of H_1 in G then the pair does not form the distinguished vertices in a copy of H in G': Indeed, suppose that $\{u, v\}$ are the two vertices that correspond to the distinguished vertices of a copy of H in G'. If in the copy of H parts of the graph H that originate from H_1 are mapped to the attached parts (which should correspond to parts of H_2), then the parts of the graph H that originate from H_2 are mapped somewhere else. By swapping the images of these parts, we can rectify the situation and eventually find a copy that maps all parts originating from H_1 to vertices of G.

Analogously we can show that a pair $\{u, v\}$ in E_2 forms the distinguished vertices in a copy of H_2 in G if and only if the pair forms the distinguished vertices in a copy of H in G'.

Similarly to the proof of the previous lemma, the modifications we have performed on the graph G are independent of edges in E_1 or E_2 being present or not. Thus, by induction, the subgraph of $N^{i+1}[E_1 \cup E_2, \{H\}](G')$ that is induced by the vertices of G coincides with the graph $N^{i+1}[E_1, \{H_1\}; E_2, \{H_2\}](G)$. $\square$

Lemma 17. *Given a reasonable colored replacement graph H, a grid-like colored graph G and a compatible set of pairs of vertices $E \subseteq \binom{V(G)}{2}$, there are uncolored graphs G', H', with G' grid-like and with H' a reasonable replacement graph, and a set of compatible pairs of vertices $E' \subseteq \binom{V(G')}{2}$ such that an uncolored version of G is an induced subgraph of G' and the subgraph of $N^{i+1}[E', \{H'\}](G')$ that is induced by the vertices of G coincides with the uncolored version of the graph $N^{i+1}[E, \{H\}](G)$.*

Proof. Let $G'' = G \times K_3$ be the strong product of G with the clique of size 3. Analogously define $H'' = H \times K_3$. Color the vertices of G'' and H'' according to their projection in G and H. Without loss of generality, we assume that the vertices of G (respectively H) are a subset of the vertices of G'' (respectively H'') such that the natural projection on this set is the identity. The distinguished vertices of H are thus also vertices of H''. We let these two vertices also be the distinguished vertices of H''.

Suppose there are m colors that are used in total among the graphs G and H. Number the colors $1, \dots, m$. Let $F_1, \dots, F_m$ be 4-connected uncolored finite graphs with the following properties: They do not contain each other as subgraphs and do not appear in the uncolored variants of $G'' + E = (V(G''), E(G'') + E)$ or H''. Furthermore we require that each F_i has 3 distinguished vertices, and the automorphism group restricted to these three vertices is the entire permutation group S_3. To each triple $\{v_1, v_2, v_3\}$ in G'' of vertices that project to the same vertex in G of color $x \in \{1, \dots, m\}$ say, attach a copy of the graph F_x along its distinguished vertices to the triple $\{v_1, v_2, v_3\}$. Analogously attach to each triple $\{v_1, v_2, v_3\}$ of H'' of color x the graph F_x.

Call uncolored versions of the resulting graphs G' and H'. Note that H' is a reasonable replacement graph, since H'' and all graphs F_i are 3-connected. Also note that G' is grid-like.

Since the copies of the graphs F_i are 4-connected, but only attached via three vertices to the rest of the graph, and since they do not contain each other as subgraphs, the only copies of a graph F_i in G' are the copies that have been explicitly attached. Let E' be the set of pairs of vertices that project to a pair in E.

By construction, if in G a vertex pair of E forms the distinguished vertices of a copy of H, then any pre-image of the vertex pair under the projection is contained in a copy of H' in G'. Also, for any copy of H' in G', its subgraph induced by the vertices of G forms a properly colored copy of H: Indeed, by their connectivity the nodes of the attached graphs F_i of H' must be mapped to the copies of F_i attached to G. All vertices of the H'' part of the copy of H' in G' must be vertices of G'', since these are the only vertices that are attached to a copy of the appropriate F_i.

As in the previous lemmas, the modifications we performed are independent of edges in E being present or not. Thus by induction, the subgraph of $N^{i+1}[E', \{H'\}](G')$ that is induced by the vertices of G coincides with the uncolored version of the graph $N^{i+1}[E, \{H\}](G)$. □

Lemma 18. *Given a reasonable uncolored replacement graph H, a grid-like uncolored graph G and a compatible set of pairs of vertices $E \subseteq \binom{V(G)}{2}$, there are uncolored graphs G', H', with G' grid-like, such that G is an induced subgraph of G' and the subgraph of $N^{i+1}[\binom{V(G')}{2}, \{H\}](G')$ that is induced by the vertices of G coincides with the graph $N^{i+1}[E, \{H\}](G)$. Additionally, for $\{u, v\} \notin \binom{V(G)}{2}$ we have that the pair $\{u, v\}$ is an edge in $N^{i+1}[\binom{V(G')}{2}, \{H\}](G')$, if and only if it is an edge in G'.*

Proof. Let F be some reasonable replacement graph that is neither a subgraph of H nor of $G + E$ (e.g., a sufficiently large clique). For every pair in E attach a copy of F to the pair. Also form H' from H by attaching a copy of F along the distinguished vertices. Note that H' is a reasonable replacement graph. Let d be a number larger than every clique in G and larger than the number of vertices in H' and at least 3.

In the modified graph G, for every edge not in E attach a copy of K_d to the two vertices. Call the resulting graph G'. Note that G' is grid-like.

Since the graph F is a reasonable replacement graph (and thus does not have 2-cuts other than the distinguished vertices) and since no copy of F exists in G or H, in the graph $G + E$, no pair of non-adjacent vertices that is not in E forms the distinguished vertices in a copy of H'. Also, due to the attachment of the clique, in $G - E$, every pair of adjacent vertices that is not in E forms the distinguished vertices in a copy of H'. Note that this also holds for the edges of the attached cliques themselves.

As in the previous lemmas, the modifications we performed are independent of edges in E being present or not. We thus conclude the lemma. □

Finally, Lemmas 13–18 allow us to proof Theorem 11.

Proof of Theorem 11. It is known that the Game of Life is Turing-complete [2]. We apply Lemma 14 to a graph G and sets of reasonable replacements graphs that simulates the Game of Life. The graphs fulfill the assumptions of Lemma 15. Applying Lemmas 15–18 in this order yields the theorem. □

Corollary 19. *There exist a finite graph H and a positive constant $c > 0$ such that for every integer n there is a finite graph G_n on n vertices with $\mathrm{stab}_H(G_n) \geq c^n$.*

Proof. The corollary follows from Theorem 11 by noting that, since G is grid-like, G can be wrapped around onto the torus, and by noting that there are patterns in the Game of Life on the torus that die out after a number of steps that is exponential in the total number of cells. □

Note that Theorem 11 not only implies that there are graphs with exponential stabilization time, but it also shows hardness of various computational tasks concerning stabilization numbers.

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