THE JOURNAL OF SYMBOLIC LOGIC
Volume 53, Number 1, March 1988

ALFRED TARSKI AND DECIDABLE THEORIES

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Any list of Alfred Tarski's achievements would mention his decision procedure for real-closed fields. He proved a number of other less publicized decidability results too. We shall survey these results. After surveying them we shall ask what Tarski had in mind when he proved them. Today our emphases and concepts are sometimes different from those of Tarski in the early 1930s. Some of these changes are the direct result of Tarski's own fundamental work in model theory during the intervening years.

Tarski's work on decidable theories is important not just for the individual decidability theorems themselves. His method for all these decidability results was elimination of quantifiers, and he systematically used this method to prove a range of related theorems about completeness and definability. He also led several of his students to do important work using this same method. Tarski's use of quantifier elimination has had a deep and cumulative influence on model theory and the logical treatment of algebraic theories.

We thank Solomon Feferman, Steven Givant, Haragauri Gupta, Yuri Gurevich. Angus Macintyre, Gregory Moore, Robert Vaught and the referee for helpful discussions and comments. Also we thank Madame Maria Mostowska and Roman Murawski for sending us material from Polish libraries.

§1. The theorems. (a) *Linear orders.* In 1927 C. H. Langford published a pair of papers [1927], [1927a] in which he gave complete axiom systems for the following kinds of linear order: (1) dense without endpoints, (2) dense with first element but no last element, (3) dense with first and last elements, and (4) $(\omega, <)$. Langford showed that in each of these four cases we can settle the truth of any first-order sentence of the language of orderings. In other words, the theory of each of these four cases is *decidable.*

Langford's method in each case was to show that on the basis of the axiom system, any first-order sentence containing a quantifier could be transformed into an equivalent sentence with fewer quantifiers. This was Langford's version of a procedure which had been introduced by Löwenheim (cf. §3 of his [1915]) and polished by Skolem [1919]. At least in Skolem's case, it consisted of an algorithm for reducing formulas to equivalent formulas in a "basic" form. The procedure was

Received January 12, 1987; revised July 10, 1987.

0022-4812/88/5301-0003/$02.60

known as "elimination of quantifiers". But here we shall call it *effective quantifier elimination* for emphasis, because today people often speak of quantifier elimination in contexts where there is no algorithm. (See §2(a) below for a closer analysis.)

Tarski proved his first decidability results in 1926–1928 (cf. [35a + 36d, p. 383 of the English translation]). They were based on this work of Langford. Using effective quantifier elimination, Tarski showed:

THEOREM 1. *If T is the theory of dense linear orders, then for any first-order sentence ϕ in the language of T we can compute a Boolean combination of the sentences "There is a first element" and "There is a last element" which is equivalent to ϕ modulo T.* [35a + 36d, p. 374 of the English translation]

THEOREM 2. *Let T be the theory of linear orders where every element except the first element has an immediate predecessor, and every element except the last element has an immediate successor. Then for any first-order sentence ϕ in the language of T we can compute a Boolean combination of the sentences "There is a first element", "There is a last element", and "There are at most n elements" (for positive integers n) which is equivalent to ϕ modulo T.* [35a + 36d, p. 376]

In the Appendix to [35a + 36d], Tarski used Theorem 1 to list all the complete theories of dense linear orders: they are the theories of the orders η, $1 + \eta$, $\eta + 1$, and $1 + \eta + 1$, where η is the natural ordering of the rational numbers. He also gave the corresponding result for the linear orders of Theorem 2; in this case there are infinitely many complete theories, namely the theories of finite orders and the theories of the four orders ω, ω^*, $\omega^* + \omega$ and $\omega + \omega^*$ (where ω is the natural ordering of the natural numbers and ω^* is its inverse).

Tarski and his student Andrzej Mostowski developed the outline of a proof of the decidability of the theory of well-orderings in the late 1930s. This theory is semantically described as the set of sentences true in all well-ordered structures. By the summer of 1939, they had reached a clear idea of the basic formulas needed for a quantifier-elimination argument. Proceeding in the (by then) usual way, they intended among other things to prove the adequacy of certain axiom systems and to classify the complete extensions of the theory. Decidability is a by-product of such activity. A great many technical details remained to be worked out.

Their collaboration was interrupted by World War II. Tarski had been visiting in the United States at the time of the outbreak of the war, and so escaped the occupation of Poland. Mostowski, and Tarski's family, were less fortunate, although all survived the war unharmed. Despite the difficulties of communication, Tarski and Mostowski each began to work out the technical details.

Mostowski's notes on the proof were destroyed in Warsaw in 1944, and Tarski simply lost his in the course of the many moves he made during the war. When they finally were able to meet again after the war, they had to completely reconstruct everything. They published the abstract [49ªe], and made tentative plans to recreate the detailed proof. However, they did not follow through on this beyond the point of writing down the specification of the basic formulas.

In 1964, Tarski assigned to one of us (Doner, then his student) a summer's task of working out some of these details. This was done in a rough form, but no further progress was made until 1975. At that time, Mostowski and Tarski met and made definite plans to finish their paper, but fate again intervened with Mostowski's death soon afterward. Subsequently, Tarski invited Doner to take up the work and join as a third coauthor. The final result, containing a number of results not mentioned in the 1948 abstract, appeared as [78].

(b) *Boolean algebras.* In [49ᵃc] Tarski announced that he had a decision procedure for the theory of Boolean algebras, and he used it to classify all the complete first-order theories of Boolean algebras in terms of countably many algebraic invariants. Each invariant was expressible by a single first-order sentence. In [48ᵐ, p. 1] Tarski mentions that he found this decision procedure in 1940.

Tarski never published either the decision procedure or a proof of the classification. But it was by effective quantifier elimination, and one can reconstruct its essentials from the invariants described in Tarski's abstract [49ᵃc]. Vaught has told us that his notes from Tarski's seminar "Arithmetical properties of algebras" at Berkeley in 1948–1949 contain a complete presentation of Tarski's argument. (It is hardly surprising that Tarski left items unpublished, in the light of the vast number of results that he proved, and the high standards of clarity and thoroughness that he adopted in his publications.)

Today we have model-theoretic proofs of the classification, by Yu. L. Ershov [1964] (cf. Ershov [1980, Chapter 2]) and by H. J. Keisler (in §5.5 of Chang and Keisler [1973]). The arguments of Ershov and Keisler do imply that the theory of Boolean algebras is decidable, but unlike that of Tarski, they do not give a primitive recursive decision procedure. Ershov's paper classified the complete theories of distributive relatively complemented lattices, which include Boolean algebras as a special case.

Already in [35a + 36d, p. 378f], Tarski has published the analogue of Theorem 2 for atomic Boolean algebras. There is a small historical puzzle here. In [35a + 36d] Tarski went on to claim that if we drop the assumption of atomicity, there are just countably many completions of the axioms of "the algebra of logic", and that he had known this since 1928. He seems to mean that there are just countably many complete first-order theories of Boolean algebras. But it is hard to see how he could have proved this result without having the full classification.

(c) *Geometry.* According to Presburger [1930, p. 95, footnote], in 1927–1928 Tarski proved the completeness of a set of axioms for the concepts "b lies between a and c" and "a is as far from b as c is from d" on a straight line. Presumably Tarski did this by effective quantifier elimination. Tarski [48ᵐ, footnote 4] described his decision method for the geometry of the straight line—presumably he meant this result quoted by Presburger—as a "partial result tending in the same direction" as his later theorem on real-closed fields. (Cf. Szczerba [1986].)

(d) *Fields.* In his abstract [49ᵃd], Tarski briefly discussed the theories of algebraically closed fields and of real-closed fields. The abstract describes in algebraic language each class of the form: all models of some complete first-order theory T which are algebraically closed (or real-closed) fields. Tarski lists the classes separately according to whether or not T can be a finite theory. He remarks that he found a decision procedure for the theory of algebraically closed fields, and he speaks of his classification as "resulting" from this procedure. He also reports that he reached a decision procedure for real-closed fields by extending Sturm's theorem "to arbitrary systems of algebraic equations and inequalities in many unknowns".

He concludes that the theory of real-closed fields is "consistent and complete", and he spells out that any two models of this theory must be elementarily equivalent.

In fact Tarski had already published a full account of the results on real-closed fields in his monograph [48^{m}] (couched in terms of the field of real numbers—real-closed fields appear only in the footnotes). He had found this decision method in 1930 (cf. p. 2 of [48^{m}]). (It seems a fair inference that he knew the results on algebraically closed fields at that time too; in §3.3 of [67^{m}], written in 1939, he sketched a syntactic argument which deduces the decidability of the theory of complex numbers from that of the theory of reals.) Apart from the mention of decidability, the emphases of the abstract put it very much in line with the last dozen pages of [35a + 36d], which describe results that he proved in 1926–1930.

In [31] Tarski reported that he had a complete set of axioms for the first-order theory of the reals with the primitive symbols 1, $\leq$ and $+$. He added that he had a "mechanical method" for determining whether a given sentence is provable or disprovable from these axioms. The paper contains a sketch of the quantifier elimination procedure for this theory, and it describes those relations on the reals which are first-order definable in this vocabulary. (This is Theorem 1 in [31].) In fact it is clear from Note 6 of [67^{m}] that by 1928 Tarski had a quantifier elimination procedure for the theory of infinite divisible ordered abelian groups.

Tarski's quantifier elimination theorem for real-closed fields has turned out to be an astonishingly fruitful mathematical result. See van den Dries [1988] for a full discussion.

(e) *Other theories.* One of us (Doner) wrote to a number of people who had worked with Tarski to ask if they knew of any decidability results that Tarski had left unpublished. It seems that nobody did. But it may be appropriate to mention some decision results which were proved by students writing dissertations under Tarski's supervision.

In the year 1927/28 Tarski gave a course of lectures on first-order theories, and during the course he presented a set of axioms for the arithmetic of the natural numbers with addition but not multiplication. These axioms are now known as *Presburger arithmetic.* Tarski proposed the problem of showing that the axioms were complete. M. Presburger solved this problem in 1928 and published the solution in his Master's thesis [1930]. His method was effective quantifier elimination.

Wanda Szmielew [1949], [1955] gave a decision procedure for the theory of abelian groups. Again she used effective quantifier elimination. Just as Tarski had done for Boolean algebras, she classified all complete first-order theories of abelian groups by a set of algebraic invariants which were expressible by first-order sentences.

In 1965, Doner proved the decidability of the weak second-order theory of two successors, using the technique of tree automata. Although Doner was Tarski's student at the time, Tarski had not suggested this line of research. Shortly after learning about Doner's result, Tarski suggested several improvements, which contributed much to the final version of Doner's paper [1970]. The most significant of these concerned the weak second-order theory of locally free algebras when only unary operators are present. Mal′cev [1962] had earlier proved that the elementary theory of locally free algebras was decidable, and Tarski had just shown the undecidability of the corresponding weak second-order theory when there is at least one more-than-unary operation. He conjectured that the case of unary operations only would be decidable and that Doner's result might be applicable to the proof. This turned out to be the case. Tarski communicated his undecidability result in the abstract [66[a]], which was immediately followed in the same journal by an abstract presenting Doner's decidability result.

§2. Discussion. (a) *Uniform methods.* As McNulty has recorded [1986], Tarski made a major contribution to the study of *undecidable* theories by working out a systematic approach to proofs of undecidability. Tarski's approach was highly original. It combined notions from recursion theory with the more model-theoretic idea of interpreting one theory in another. The outcome of Tarski's work was a small collection of powerful theorems of the form: any theory with such-and-such features must be undecidable.

Did Tarski make a similar contribution to proofs of *decidability*? Apparently not. He was content to use a method that was already well known when he began his work in the field—namely, effective quantifier elimination. In [35a + 36d, p. 374] he described it as "a frequently used method, which consists in reducing the sentences to normal form and successively eliminating the quantifiers".

There is a clear description of this method in §1.5 of Chang and Keisler [1973]; but let us draw out some points to help our discussion. Let T be a first-order theory. We say that a set Φ of formulas is a *set of basic formulas* for T (or more briefly a *basic set*) if, using the axioms of T, one can prove that every formula ϕ of the language is equivalent to a Boolean combination ϕ^* of formulas in Φ, which has only the same free variables as ϕ.

To analyse T by the method of quantifier elimination, one would look for a basic set for T. Every axiom system has at least one basic set, namely the set of all formulas of the language. But one would try to find a better basic set than that. There is no exact criterion for a "good" basic set, but one would hope for a basic set Φ with at least the following three properties. (1) It should be reasonably small and irredundant. (2) Every formula in Φ should have some straightforward mathematical meaning. (3) There should be an algorithm for reducing every formula ϕ to its corresponding ϕ^*. ((3) is precise; when we have it we can claim to have effective quantifier elimination for T. (1) and (2) are more a matter of judgement.) In the best cases we have one thing more: (4) an algorithm which tells us, given any basic sentence ψ, either that ψ is provable or that it is refutable from T. Given (3) and (4), we have both a completeness proof and a decision procedure for the theory T.

It seems that Tarski found this method quite adequate for his purposes and did not seek to generalize it. It was his method of choice, and he was a forceful advocate of it on many occasions. Solomon Feferman has told us that in Berkeley, Tarski used to stress that the heart of a proof of decidability for a theory T must always be something specific to the kind of mathematics that T is concerned with. To elaborate Tarski's point: the main effort goes into finding a good basic set. This usually involves a combination of two things: first, thoughts about the structure of models of T, which should suggest possible basic sets; and second, syntactic experiments to check whether a particular set will work. Both these ingredients are bound to depend very heavily on the particular theory T. As Feferman says, Tarski "thought no general method would help, since syntactic quantifier elimination is often quite intricate".

Today we have several rather general model-theoretic methods for proving decidability. For example, Shoenfield [1971] describes and illustrates a technique that gives some quick and intuitive proofs of decidability for first-order theories. In our language, he give a criterion which determines whether the set of all quantifier-free formulas is a basic set. The criterion by itself does not yield usable decision procedures. But being model-theoretic rather than syntactic, it can lead one smoothly from structure theory to a good basic set. (Abraham Robinson [1958] had earlier given a criterion closely related to Shoenfield's.)

Ferrante and Rackoff [1979] describe a back-and-forth argument that gives decision procedures with good upper bounds on their computational complexity. Their method is closely related to the technique of "neighbourhood systems" which Mekler [1984] uses to give fast and elegant proofs of some of the results of Doner, Mostowski and Tarski [78], together with some decidability results in stationary logic.

There are also some general techniques that have led to decidability results where more syntactic methods failed. One of these is the Feferman-Vaught technique (Feferman and Vaught [1959]), which is good for monadic second-order theories (cf. Gurevich [1985]). This technique only applies directly to classes of structure which admit suitable sum or product decompositions. But indirectly one can apply it to many other classes by means of interpretations. The paper of Feferman and Vaught is partly based on separate work of the two authors when they were students of Tarski.

Practical computers became available some twenty years after Tarski's early work on decidable theories. It would have been a bonus if his quantifier elimination procedures could have been used as feasible computer algorithms. Unfortunately the results of complexity theory have been less than generous here. For Presburger arithmetic there is a superexponential lower bound on time complexity of any decision procedure, and for real-closed fields there is an exponential lower bound; cf. Fischer and Rabin [1974]. (But see §3E of van den Dries [1988] for further comments on practical algorithms for real-closed fields.)

In a way, Tarski's contribution to the *technique* of quantifier elimination was managerial rather than mathematical. He did not invent it, and other people had already pointed out several things that could be done with it. What he did do was to present it as a paradigm of how a logician should study an axiomatic theory. He and his students adopted this paradigm to study a wide range of important theories, and to answer a wide range of important questions about these theories. The result was a valuable accumulation of detailed information for future workers to draw on.

In the remainder of this essay we shall study Tarski's paradigm. What did Tarski take to be the vital features of an axiomatic theory that can be captured by quantifier elimination? When one uses quantifier elimination, what should one be looking for? What are the questions most worth asking? During the early 1930s, Tarski wrote a number of methodological papers setting out the aims and central notions of logic as he saw it; we shall ask how quantifier elimination is related to these aims and notions.

(b) *Definable relations.* Suppose that we have a structure M and a set T of first-order sentences which are true in M. Suppose also that we have found a basic set Φ for the theory T. Then we can give an exact description of those relations on M which are definable by first-order formulas (with or without parameters). Namely, they are the Boolean combinations of the relations which are defined by formulas in Φ.

Apparently the first author to make this point was Tarski in [31]. In that paper Tarski introduced his decision procedure for the theory of the reals, but in the guise of a way of describing the first-order definable relations on the reals.

Tarski was greatly interested in the notion of a definable relation in a structure. In fact his paper [31] was mainly about how to make this notion precise. His joint paper [31a] with Kuratowski showed that every definable relation in the field of reals is a projective set. His essay [35c] discussed the general theory of definability (see Vaught's account in [1986]). There is a clear link between definable relations and cylindrical algebras too (cf. Monk [1986]). Nevertheless Tarski's announcements of decidability or completeness results rarely mentioned this aspect of them; it was evidently not his main target.

Perhaps it is worth mentioning here that the Boolean algebras of definable relations on a structure, which Tarski first introduced, are a central object of interest in model-theoretic stability theory. More precisely, the Stone space of such an algebra is a space of complete types. In his work on the calculus of systems, Tarski was very interested in the cardinalities of these spaces. See for example p. 370 of the English edition of [35a + 36d]; any reader who works through the proofs of the examples at the end of that paper will see how one can use effective quantifier elimination to calculate these cardinalities in some important cases. Stability theory (above all Morley [1965] and Shelah [1978]) shows that the cardinalities of the Stone spaces connected with a theory T are intimately related to the number of models of T, and to the structural analysis of these models, regardless of whether we have a procedure for eliminating quantifiers. Very recently it was found that Tarski's description in [31] of the definable sets in the field of reals and the additive group of reals (cf. §1(d) above) is exactly what one needs to develop an analogue of stability theory for some structures with a linear ordering (Pillay and Steinhorn [1986]).

(c) *Effective decidability.* It was already clear to Tarski by 1930 that effective quantifier elimination could give

> a mechanical method which enables us to decide in each particular case whether a given sentence (of order 1) is provable or disprovable.

This quotation is from [31, p. 134 of the English edition]. The topic is the theory of the reals with primitive symbols $+$, $\leq$ and 1. Presburger [1930] remarked in the first paragraph of his paper that his completeness proof gave

> ein Verfahren, das bei einer vorgelegten arithmetischen Aussage des von uns betrachtenen Gebietes die Entscheidung darüber erlaubt, ob sie einen arithmetischen Satz vorstellt. [... a procedure which allows us, when we are

presented with an arithmetical statement belonging to the domain under consideration, to decide whether the statement represents an arithmetical theorem.]

To this he added a footnote referring to Hilbert and Ackermann [1928] for the notion of decidability.

Today we tend to speak of all the results of Tarski discussed in this survey as "decidability results"—this is the aspect that comes first to our minds. But a closer look at Tarski's writings suggests strongly that effective decision procedures were nowhere near the head of Tarski's own priorities in the period before the war.

In his papers published before World War II, he rarely mentions effective decidability. There is no mention of it in papers such as [30c], [30e] and [35a + 36d], in which Tarski laid out a broad program for studying axiomatic theories. Apart from the passage from [31] just quoted, we could find only a reference (in his abstract [39ᵃ]) to:

> The "effective" character of all positive proofs of completeness so far given—not only the problem of completeness but also the decision problem is solved in the positive sense for all the deductive systems mentioned above.

and a barely relevant remark about decidability in the intuitionistic calculus [38h].

Besides these passages, the nearest he comes is in [30e, p. 93], together with the parallel passage on p. 390 of his joint paper with Lindenbaum [36b]. Here he introduces the notion of the *decision domain* of a set A of sentences. The decision domain of A is the set of all sentences of the appropriate language "which are either consequences of A or which, when added to A, yield an inconsistent set of sentences". Tarski defines a *decision-definite* or *complete* set of sentences to be one whose decision domain is the set of all sentences of the language.

In both [30e] and [36b], Tarski adds in footnotes that his concepts are the same as those of Fraenkel [1928, pp. 347–354]; cf. also Note 1 of [67ᵐ]. Fraenkel's discussion is fairly lengthy, but a salient point seems to be that a system of axioms has "Entscheidungsdefinitheit" (decision-definiteness) if every sentence of the appropriate language can be either proved or refuted from the axioms, by a finite proof in a formal proof calculus.

We shall say more in a moment about this notion of "decision-definiteness". But first we need to examine just what it means to say that a theory is "decidable".

The place to begin is a later work of Tarski, namely his paper *A general method in proofs of undecidability* in the monograph [53ᵐ]. In this paper Tarski gives a precise definition of "decidable theory". Tarski restricts himself to theories in a first-order language in "standard formalization". The *valid sentences* of a theory T are those which are deducible from the axioms of T. (Actually Tarski gives a more general notion of "valid" than this, but the difference is irrelevant for our purposes.) Tarski calls the theory T *decidable* if the set of all its valid sentences is recursive. This definition of "decidable theory" is now completely standard.

Tarski notes that if a theory is complete and has a recursive set of axioms, then it is decidable. He refers to Kleene [1943] for a proof; the essential point is that if both the set of valid sentences and the set of negations of valid sentences are recursively enumerable, then the set of valid sentences is recursive. The following algorithm determines whether a sentence ϕ is valid: list one by one all the proofs from the axioms of the theory, until you reach either a proof of ϕ or a proof of $\neg\phi$. Eventually you will reach one or the other, since the theory is complete.

Of course an algorithm like this is likely to be hopelessly unpractical: in general there is no guarantee that it will give any answers in our lifetime, even if we offer it very short sentences ϕ. By contrast the decision algorithms that Tarski found by quantifier elimination were always perfectly practical. Each step was clearly a step towards a solution. For short sentences there was every hope of getting a Yes or a No in a reasonably short time, and even for long sentences one could compute an upper bound on how long the calculation would take.

This distinction between practical and unpractical algorithms is hardly precise, but the difference is real enough. Today many writers emphasize that their algorithms are practical by pointing out that they are, say, primitive recursive or elementary. These are exact notions, but they do not quite make the distinction between practical and unpractical: even a primitive recursive algorithm might take a billion years to solve the smallest problem. But in any case these refinements of recursiveness were not available to Tarski in the early 1930s.

All the decidability results of Tarski that we listed in §1 above had practical algorithms. We know of one place where Tarski proved decidability simply by proving recursiveness. It occurs in a footnote of a paper of Verena Huber Dyson [1964]. The theorem states a general criterion for decidability; Dyson uses it to deduce among other things that every recursively axiomatizable theory with only finitely many complete extensions is decidable.

To return from the 1960s to Tarski's work in the period around 1930: how should we understand his reference to a mechanical decision method in [31]? At that date, did he have in mind a notion of "mechanical method" which included the unpractical algorithms mentioned above? We suppose not, for two main reasons.

The first reason is the naive meaning of the words. Recursion theory apart, if someone offers you a "mechanical method for testing X", you do not expect to be given a method which may not be able to test anything whatever in less than any given number of centuries.

The second reason brings us back to the "decision-definite" theories of [30e] and [36b]. Recalling his references to Fraenkel, it certainly seems that Tarski had in mind some proof system of a standard kind, which would guarantee that the set of proofs is recursively enumerable. In [36b] the theories under discussion are recursively axiomatized, because in fact they are finite. In [30e] the theories are arbitrary, but Tarski has picked out the finitely axiomatizable theories as an interesting special case.

So in modern terminology, a decision-definite theory is complete and (at least in [36b]) recursively axiomatized. There should be a theorem of the form, "If T is decision-definite then T is decidable by a mechanical decision method." If Tarski had such a theorem, it would have been highly appropriate for him to mention it in at least one of [30e], [35a + 36d] or [36b]. But in fact these papers contain no hint of any thoughts in this direction. (Neither do pages 347–354 of Fraenkel [1928].) We shall see in a moment that the first hints of this theorem appear in a monograph [67$^{\mathrm{m}}$] which Tarski completed in 1939.

Turning for a moment to the other logicians of this period, we find that Tarski's predecessors in quantifier elimination rarely mentioned decidability at all. Langford did not speak of decision procedures, either in his papers [1927] and [1927a] or in Lewis and Langford [1932]. He stated his results in the form: given any first-order sentence ϕ of the appropriate language, either ϕ follows from the axioms or $\neg\phi$ follows from them. When Skolem [1928] reported Langford's results, he described an effective decision procedure without mentioning that it was one.

In fact no author in the 1920s seems to have considered the question whether one can effectively decide which sentences are consequences of a particular set of mathematical axioms. True, many writers talked about the "decision problem". But they always meant the problem of deciding which sentences are logically valid or provable. The text of Hilbert and Ackermann [1928] is an example; even when Ackermann extended the section (III.12) on "The decision problem" for the 1938 edition, he said nothing about decision problems for axiomatic theories.

As far as we know, the first logicians to refer in print to a "decision method" for determining the consequences of a nonlogical system of axioms were Presburger [1930], Tarski [31] and Skolem [1930]. (To quote §3 of Skolem [1930], "Man könnte natürlich die Elimination gewisser Variablen benutzen um ein Verfahren zu bilden, nach dem man entscheiden könnte, ob die Aussage wahr ist oder nicht.") Apart from two brief mentions by Gödel (pp. 240ff and 367 of Gödel [1986]) of the results of Presburger and Skolem, we have not found any further examples of this use of the notion before Tarski [39ª]—the idea was simply not in the air.

At the same time there were mathematicians in another tradition who were working on decision methods for particular parts of mathematics. Grete Hermann's work on effective bounds in algebra [1926] is the classic example. She was writing in the school of Kronecker and Hensel. Tarski's early papers never mention the work or aims of this school.

We draw the conclusion that whatever Tarski was aiming to do with complete theories between the mid-1920s and World War II, it was not primarily to prove decidability results.

Even for a while after the war, Tarski was sometimes reticent on decidability. The Tarski-Mostowski abstract [49ªe] illustrates the point. For a long time, if one enquired about the decidability of the theory of well-orderings, this abstract was cited as the original source. Yet it contains no mention of decidability, nor does it even suggest the existence of an algorithm. Quantifier elimination is not mentioned. Only those familiar with the proof from other sources—seminars, private communications, etc.—would have counted decidability among the immediate consequences.

Nevertheless there is a very noticeable change of emphasis in Tarski's writings after the war. On the first page of Chapter 1 of [53ᵐ] he describes the decision problem as "one of the central problems of contemporary metamathematics". Already one can see a move in this direction if one looks at the abstracts [49ªc] and [49ªd]. Tarski opens each of these abstracts by saying he has found a decision procedure, and he goes on to say that various consequences follow from the procedure. This is a reversal of his style in the 1930s. Thus in the earlier abstract [39ª] he described his results on completeness, and only added at the end that the proofs involve solving the decision problem.

The paper [67^{m}] sheds some light on the evolution of Tarski's thinking on decision problems. This was perhaps his very last paper of the pre-war period. It was written in 1939, and was to have been published in *Actualités Scientifiques et Industrielles* (published by Hermann & C^{ie}, Paris) as the first fascicule of the series *Métalogique et métamathématique*. The paper reached the stage of page proofs, but publication was interrupted by wartime developments. The printers' plate were destroyed, and ultimately Tarski was left with only two copies of the page proofs. In 1967 these proofs were reproduced and given a limited distribution. More widely available was the French translation in [74^{m}]. In [67^{m}], Tarski has rather more to say about effective decidability than in his earlier papers:

> It should be emphasized that the proofs sketched below have (like all proofs of completeness hitherto published) an «effective» character in the following sense: it is not merely shown that every statement of a given theory is, so to speak, in principle provable or disprovable, but at the same time a procedure is given which permits every such statement actually to be proved or disproved by the means of proof of the theory. By the aid of such a proof not only the problem of completeness but also the *decision problem* is solved for the given system in a positive sense[11]. In other words, our results show that it is possible to construct a machine which would provide the solution of every problem in elementary algebra and geometry (to the extent described above).

Also quite germane to our discussion is the following remark in Note 11 of [67^{m}]:

> It is possible to defend the standpoint that in all cases in which a theory is tested with respect to its completeness the essence of the problem is not in the mere proof of completeness but in giving a decision procedure (or in the demonstration that it is impossible to give such a procedure).

The contrast between the introductory remarks to [48^{m}] and those to the earlier [67^{m}] is striking. The 1940 paper speaks mostly of the completeness problem, with a few paragraphs on decidability and the algorithmic nature of the work (although this is more than in the earlier papers). On the other hand, the introduction to [48^{m}] focuses almost exclusively on the decidability aspect. So it appears that Tarski's remarks in [67^{m}] reflect a transition in his conception of the results he had earlier obtained: from describing them as completeness proofs to an emphasis on the algorithmic decidability.

It is not hard to guess what led Tarski to this new perspective. Prior to the development of recursion theory in the mid-1930s, one would not state a theorem in the form "there is an algorithm for so-and-so." Not only was there no precise meaning for the term "algorithm", but few if any mathematicians thought there ever would be. So a statement claiming the existence of an algorithm would lack the precision one expects of a theorem. Typically, an algorithm would be given as part of the proof of something else, and perhaps the author would comment in the text that the proof actually provided a workable procedure.

All this changed with the developments in recursion theory in the mid-1930s and the promulgation of Church's Thesis in Church [1936]. Now one had precise and articulate notions of "algorithm" and "decidable theory". Tarski recognized this immediately and began to explore the ramifications for his own work on completeness problems. We find an interesting hint of his thought along these lines in Note 37 to [67^m], where he is formulating a notion of essential incompleteness:

> ... a (consistent) deductive system is essentially incomplete if it cannot be enlarged to form a consistent and complete system in which the class of provable statements can be constructively defined, i.e. for which a decision procedure ... can be given.
>
> An exact definition of the concepts mentioned (which is, for instance, clearly indispensable if it is desired to give an exact proof that a given system is essentially incomplete) presents certain difficulties. It should, however, be possible to overcome these difficulties with the help of the recently defined concept of *general recursiveness* (cf. Church [1936]). In any case it suffices for the present purpose to assume that "constructively definable" means as much as "general recursive" and that accordingly a system is essentially incomplete if it is impossible to enlarge it to form a complete system in which the class of provable statements is general recursive.

A second but related reason for Tarski's new perspective on his work on completeness problems was that he had become keenly interested in proofs of undecidability, and he published a cluster of undecidability results during the years 1949–1953 (cf. McNulty [1986]).

Tarski himself offered a third reason in the Preface to the 1951 edition of his monograph [48^m]. He said that he was now presenting his decision method for the theory of the reals "in a systematic and detailed way, thus bringing to the fore the possibility of constructing an actual decision machine". He hinted that this emphasis "reflected the specific interests which the RAND Corporation found in the results". The age of computers had begun. (Tarski's Foreword to [67^m], written in 1966, also mentions this shift of emphasis between [67^m] and [48^m]. Cf. van den Dries' [1988] comments on Tarski's changing emphases in his work on real-closed fields.)

Once the study of algorithms (i.e., recursive functions) achieved a stature of its own, then decidability could be considered interesting in its own right. In fact, it became fashionable to emphasize the algorithmic decidability aspects of work that earlier might have been described as related to "completeness problems". The extent to which Tarski himself ever adopted this point of view, or the related one quoted above from Note 11 of [67^m], is problematic. We should enter a caveat here: Tarski had many interests, not all of which would be expressed in a given paper. The fact that he emphasized completeness in one paper and decidability in a later one does not necessarily imply a judgement on the merits of the two topics—perhaps it was just a matter of the point he wanted to make at the time. Whatever his attitude in the immediate post-war period may have been, as soon as decidability results began to be proved by essentially semantical methods, bypassing the detailed analysis of formal expressibility, Tarski was quick to point out how much more information was provided in the earlier syntactically oriented proofs. Tarski's own final view on this topic appears in a paragraph which he contributed to the joint paper [78] with Doner and Mostowski:

> It seems to us that the elimination of quantifiers, whenever it is applicable to a theory, provides us with direct and clear insight into both the syntactical structure and the semantical content of that theory—indeed, a more direct and clearer insight than the modern more powerful methods to which we referred above. In consequence, the procedure of eliminating quantifiers in a theory may prove valuable as a heuristic source and deductive base for a many-sided metamathematical study of this theory leading to substantial results not involving decidability. While recent research into the complexity of algorithms has deprived the pure decidability results of some of their luster, the interest in other results obtained through the elimination of quantifiers remains unaffected.

(d) *Complete extensions and elementary equivalence.* One theme keeps appearing in Tarski's discussions of theories, from the 1930s onwards. Tarski himself used different terminology at different times. But in modern terms the theme is this: given a mathematically interesting first-order theory T, how can we classify the deductively closed extensions of T, and in particular those which are complete? (Cf. [30c] for an early statement and [49^{a}a] for a later one.)

This idea seems to be new with Tarski. Unlike the question of decidability, it was clearly at the front of Tarski's mind throughout the 1930s and beyond. Unlike decidability in the early thirties, it poses quite precise and concrete mathematical questions. We can tentatively trace some stages in its development.

To begin, during the years 1926–1930 Tarski showed that a number of theories allowed elimination of quantifiers. He must very soon have become aware of a point that had eluded Langford: when one eliminates a quantifier, the resulting formula may depend on whether some basic sentence is true or false. (For example, if T is the theory of dense linear orders, then the result of eliminating quantifiers from a formula will depend on whether there are a first element and a last element.) Because of this, the formula will generally reduce, not to a quantifier-free formula, but to a Boolean combination of quantifier-free formulas and basic sentences—just as in Theorem 1 above.

The second step, on Tarski's own account [35a + 36d], was to express this realization in terms of the calculus of systems. (In other words, he now had a metamathematical setting for discussing theories. The calculus of systems gave Tarski a uniform metamathematical framework which embraced both quantifier elimination and his work with Lindenbaum on sentential calculi. Cf. Blok and Pigozzi [1988].)

In modern terminology, Tarski had reached the following conclusions. If T is a theory with quantifier elimination as above, and T' is a deductively closed theory extending T, then T' is completely determined once we know which Boolean combinations of the invariant sentences are in T'. So we have a classification of these theories T'; in particular we can often determine which of these theories T' are finitely axiomatizable. If T' is required to be complete, the classification is simpler:

we only need to know which invariant sentences are in T'. Tarski had reached this viewpoint by the time he wrote [35a + 36d], and very likely some years earlier (cf. [30e]).

The third stage was to make the switch from "classifying complete theories extending T" to "classifying models of T up to elementary equivalence". In the Appendix of [35a + 36d] Tarski noted that the complete description of a model of T is always a complete theory extending T, and deduced the connection with classification of structures up to elementary equivalence. He records that he was only "able to state these facts in a correct and precise form" after his work on the characterization of definability and truth in 1929–1930.

Curiously this Appendix omits the converse result: that every complete extension of T describes a model of T. Of course this follows at once (for countable first-order languages) from Gödel's completeness theorem, which Tarski certainly knew. There is no hint in [35a + 36d] that classifying complete extensions of a first-order theory is always equivalent to classifying models of the theory up to elementary equivalence. For that we have to wait until Tarski's abstract [49^{a}a]. This small point is a revealing indicator of how emphases have changed.

In the 1950s Tarski's interests broadened to more general questions of model theory. He began to ask whether structures can *always* be classified up to elementary equivalence in some purely mathematical way, or in other words, whether elementary equivalence itself can be characterized by structural properties. Vaught [1986] has described where this question led.

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