

On Positivity and Minimality for Second-Order Holonomic Sequences

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Abstract

An infinite sequence $(u_n)_{n \in \mathbb{N}}$ of real numbers is *holonomic* (also known as *P-recursive* or *P-finite*) if it satisfies a linear recurrence relation with polynomial coefficients. Such a sequence is said to be *positive* if each $u_n \geq 0$, and *minimal* if, given any other linearly independent sequence $(v_n)_{n \in \mathbb{N}}$ satisfying the same recurrence relation, the ratio u_n/v_n converges to 0.

In this paper, we focus on holonomic sequences of rational numbers satisfying a second-order recurrence $a(n)u_n = b(n)u_{n-1} + c(n)u_{n-2}$, where each coefficient $a, b, c \in \mathbb{Q}[n]$ is a degree-1 polynomial with rational coefficients. Subject to certain conjectures in number theory due to Kontsevich, Zagier, and Zudilin, we establish the decidability of the Positivity and Minimality Problems for such sequences.

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1 Introduction

Holonomic sequences (also known as *P-recursive* or *P-finite* sequences) are infinite sequences of real (or complex) numbers that obey a linear recurrence relation with polynomial coefficients. The earliest and best-known example is the Fibonacci sequence, given by Leonardo of Pisa in the 12th century¹; more recently, Apéry famously made use of certain holonomic sequences obeying the recurrence relation

$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1} \quad (n \geq 1) \quad (1.1)$$

to prove that $\zeta(3) := \sum_{n=1}^{\infty} n^{-3}$ is irrational [2]. Holonomic sequences now form a vast subject in their own right, with numerous applications in mathematics and other sciences; see, for instance, the monographs [7, 6] or the seminal paper [26].

There is a voluminous amount of literature devoted to the study of identities and asymptotic behaviour for holonomic sequences. However, as noted by Kauers and Pillwein, “in contrast, [...] almost no algorithms are available for inequalities” [11]. Indeed, even for C-finite sequences, the Positivity Problem (i.e., whether every term of a given sequence is non-negative) is only known to be decidable at low orders, and there is strong evidence that the problem is mathematically intractable in general [17, 19]; see also [10, 14, 17, 18]. For holonomic sequences that are not C-finite, virtually no decision procedures currently exist for Positivity, although several partial results and heuristics are known; see, e.g., [15, 11, 16, 25, 20, 21].

Another extremely important property of holonomic sequences is *minimality*; a sequence $(u_n)_{n \in \mathbb{N}}$ is minimal if, given any other linearly independent sequence $(v_n)_{n \in \mathbb{N}}$ satisfying the same recurrence relation, the ratio u_n/v_n converges to 0. Minimal holonomic sequences play a crucial rôle, among others, in numerical calculations and asymptotics, as noted for example in [9, 8, 5]—see also references therein. Unfortunately, there is also ample evidence that determining algorithmically whether a given holonomic sequence is minimal is a very challenging task, for which no satisfactory solution is at present known to exist. Indeed, one of our main results reduces Positivity to Minimality for a certain class of holonomic sequences—see Thm. 1.1 below.

In this paper, we focus on holonomic sequences of rational numbers satisfying a second-order recurrence

$$a(n)u_n = b(n)u_{n-1} + c(n)u_{n-2} \quad (n \geq 2), \quad (1.2)$$

where each coefficient $a, b, c \in \mathbb{Q}[n]$ is an affine (i.e., degree-1) polynomial with rational coefficients. Subject to certain conjectures in number theory due to Kontsevich, Zagier, and Zudilin, we establish the decidability of the Positivity and Minimality Problems for such sequences.

More precisely, we summarise the main contributions of the present paper as follows.

► **Theorem 1.1.** For holonomic sequences as given above, Positivity reduces to Minimality: the existence of an algorithm for deciding Minimality implies the existence of an algorithm for deciding Positivity.

► **Theorem 1.2.** Let (u_n) be a rational solution to recurrence (1.2) with given initial values $u_0, u_1 \in \mathbb{Q}$. Assuming Conjectures A.1 (Zudilin) and A.2 (Kontsevich-Zagier), it is decidable whether (u_n) is minimal.

¹ In fact, the Fibonacci sequence obeys a linear recurrence relation with *constant* coefficients; it therefore belongs to a particularly simple subclass of holonomic sequences, so called *C-finite*.

2 Preliminaries

By a *holonomic sequence* we mean a rational sequence $(u_n)_{n \in \mathbb{N}}$ satisfying a non-trivial linear recurrence relation $p_0(n)u_n + p_1(n)u_{n-1} + \dots + p_\ell(n)u_{n-\ell} = 0$ with fixed polynomial coefficients $p_k(n) \in \mathbb{Q}[n]$. A solution sequence (u_n) is then entirely determined by its initial values $u_0, \dots, u_{\ell-1} \in \mathbb{Q}$ and such a sequence is said to have *order* ℓ . In the literature such sequences are alternatively called *P-recursive* or *P-finite*.

Herein, unless stated otherwise, we restrict our consideration to second-order holonomic sequences with affine coefficients satisfying the recurrence relation (1.2). We shall write $a(n) = \alpha_1 n + \alpha_0$, $b(n) = \beta_1 n + \beta_0$, and $c(n) = \gamma_1 n + \gamma_0$. A (computable) shift of the terms in the sequence (u_n) permits us to make the following assumptions without loss of generality. First, we can assume that $\text{sign}(a(n))$, $\text{sign}(b(n))$ and $\text{sign}(c(n))$ are constant. Second, $\text{sign}(a(n)) = 1$ (for otherwise we can swap the signs of $b(n)$ and $c(n)$ as appropriate). Finally, we may assume that $\alpha_0 > \alpha_1$.

Let $n \in \mathbb{N}$. The n th characteristic polynomial $p(n, \cdot)$ of the recurrence relation in (1.2) is given by $p(n, x) = a(n)x^2 - b(n)x - c(n)$. The n th characteristic roots of $p(n, \cdot)$ are equal to

$$\frac{b(n) \pm \sqrt{b(n)^2 + 4a(n)c(n)}}{2a(n)}.$$

The sequences of characteristic roots converge as $n \rightarrow \infty$ to

$$\frac{\beta_1 \pm \sqrt{\beta_1^2 + 4\alpha_1\gamma_1}}{2\alpha_1}.$$

We call the limits the *characteristic roots* of relation (1.2). Throughout we use the letters λ and Λ to indicate the roots. When the roots are not equal in modulus, our convention is to label the sequences $\lambda_n \rightarrow \lambda$ and $\Lambda_n \rightarrow \Lambda$ so that $|\lambda| < |\Lambda|$. When the characteristic roots are equal in modulus the assignment is not important to us. For the single repeated root case we use the letter λ .

In this paper we are interested in the following sequence phenomena. A rational sequence (u_n) is *positive* if $u_n \geq 0$ for all $n \in \mathbb{N}$. We note our definition is typical of terminology used in the literature and accept the convention that *positive* is taken to mean *non-negative*. Let V be the vector space of real solutions of the recurrence relation (1.2). A solution sequence $(u_n) \in V$ is *minimal* if for every sequence $(v_n) \in V$ that is linearly independent of (u_n) , we have $\lim_{n \rightarrow \infty} u_n/v_n = 0$.

In this note we will frequently consider the sequence (x_n) of successive ratios of terms of a solution sequence (u_n) in order to determine whether (u_n) is positive. Let us take $x_n = u_n/u_{n-1}$ for each $n \in \mathbb{N}$. The term x_n is well-defined only when $u_n \neq 0$. The difficult cases when discussing positivity arise when $\text{sign}(b(n)) = -\text{sign}(c(n))$. So let us assume we are working in such a case. Suppose that a solution (u_n) is not identically zero, but $u_n = 0$ for some $n \in \mathbb{N}$. Then, by inspection, we find that either $u_{n+1} < 0$ or $u_{n+2} < 0$. It follows that we can assume the sequence (x_n) is well-defined, for otherwise we can detect when a solution (u_n) is not a positive sequence.

► **Example 2.1.** Suppose that the sequence (u_n) satisfies the following recurrence relation

$$a(n)u_n = (a(n) - c(n))u_{n-1} + c(n)u_{n-2}$$

$$b(n) = a(n) - c(n)$$

with initial values u_0 and u_1 . We shall also assume that $a(n) > -c(n)$ and $c(n) < 0$ for each $n \in \mathbb{N}$. Let us define the constant $\eta = \sum_{j=1}^{\infty} \prod_{k=2}^j c(k)/a(k)$ and set $1/\eta = 0$ if this infinite

$$\frac{x(m) = \frac{u(m)}{u(m-1)}}{\frac{u(m)}{u(m-1)}} = \frac{\frac{b(m)}{a(m)} \cdot u(m-1) + \frac{c(m)}{a(m)} u(m-2)}{u(m-1)} = \frac{b(m)}{a(m)} + \frac{c(m)}{a(m)} \cdot \frac{1}{x(m-1)}$$

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series diverges. We are interested in the constant $\mu = 1 - 1/\eta$. Using a telescopic argument, one can show that (u_n) is positive if and only if $u_1/u_0 \geq \mu$. We thus call μ the critical ratio of the above recurrence as it determines whether a solution sequence is positive.

Given $s \in \{2, 3, \dots\}$, consider the following recurrence relation

$$n^s u_n = (n^s + (n-1)^s) u_{n-1} - (n-1)^s u_{n-2}$$

with polynomial, rather than linear, coefficients—we note the above setup holds in this example. One can show that $\mu(s) = 1 - 1/\zeta(s)$ where $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. Euler proved that if k is a positive integer then $\zeta(2k)$ is a rational multiple of π^{2k} and so it follows that $\zeta(2k)$ is transcendental. The arithmetic study of the values of $\zeta(2k+1)$ is a major undertaking. For example, Apéry's constant $\zeta(3)$ is irrational and research has shown that infinitely many values of $\zeta(2k+1)$ are irrational [23].

In this paper we talk more generally about a critical ratio μ for a recurrence relation of the form (1.2). In the case that a critical ratio μ exists, determining whether a given solution is positive is tantamount to computing μ as one can compare the initial ratio u_1/u_0 to μ . Since u_1/u_0 is rational, when μ is irrational we need only approximate μ arbitrarily closely. We note however that this is not, in general, a simple task.

3 A Relation Between Positivity and Minimality

Positivity and minimality are two problems that seem unrelated at first glance. However, in many cases, the decidability of those two problems are intertwined. In this section, we develop the relations between the two problems. The main result of this section is Theorem 1.1.

Before we prove the theorem, let us recall standard results from the theory of continued fractions. Much of this presentation follows closely the exposition in [3, Chapter 9] and [24].

3.1 Background on Continued Fractions

Given two rational sequences (p_n) and (q_n) (and assuming that $q_n \neq 0$ for each $n \in \mathbb{N}$), consider the following *irregular continued fraction* construction

$$p_0 + \frac{q_1}{p_1 + \frac{q_2}{p_2 + \frac{q_3}{\ddots + \frac{q_n}{p_n}}}} =: p_0 + \left[\frac{q_1}{p_1} \right] + \left[\frac{q_2}{p_2} \right] + \dots + \left[\frac{q_n}{p_n} \right].$$

We will frequently use Pringsheim's notation (on the right-hand side) to concisely express continued fractions. We also consider *infinite continued fractions* and use the notation

$$p_0 + \left[\frac{q_1}{p_1} \right] + \left[\frac{q_2}{p_2} \right] + \dots \tag{3.1}$$

to indicate the obvious infinite construction. Let $r_n := p_0 + \left[\frac{q_1}{p_1} \right] + \left[\frac{q_2}{p_2} \right] + \dots + \left[\frac{q_n}{p_n} \right]$ be the n th *convergent* of the continued fraction (3.1). If the sequence (r_n) of convergents has a limit α , we call α the *value* of the continued fraction. We will abuse terminology by saying the infinite continued fraction converges to α .

Let us shed light on the relation between second-order recurrence relations and continued fractions by introducing the sequences $(y_n)_{n=-1}^{\infty}$ and $(z_n)_{n=-1}^{\infty}$ defined by the following linear recurrence equations

$$\begin{cases} y_n = p_n y_{n-1} + q_n y_{n-2}, & \text{and} \\ z_n = p_n z_{n-1} + q_n z_{n-2}, \end{cases} \quad (3.2)$$

for $n \in \{1, 2, \dots\}$ with initial conditions $y_{-1} = 1, y_0 = p_0, z_{-1} = 0$, and $z_0 = 1$. It can be shown that $r_n = y_n/z_n$ and $y_n z_{n-1} - y_{n-1} z_n = (-1)^{n-1} \prod_{\ell=1}^n q_\ell$ for each $n \in \mathbb{N}$. Thus the sequence of convergents (r_n) is uniquely determined by the numerators (y_n) and denominators (z_n) that solve the pair of recurrence relations in (3.2) and conversely the sequence of convergents (r_n) of a continued fraction generate sequences (y_n) and (z_n) satisfying (3.2).

By a simple telescoping argument we can express the n th convergent (r_n) as follows:

$$r_n = p_0 + \sum_{\ell=1}^n \frac{(-1)^{\ell-1} \prod_{k=1}^{\ell} q_k}{z_\ell z_{\ell-1}}$$

and so

$$p_0 + \frac{q_1}{p_1} + \frac{q_2}{p_2} + \dots = p_0 + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1} \prod_{k=1}^{\ell} q_k}{z_\ell z_{\ell-1}}.$$

We deduce that the above infinite continued fraction converges if and only if the infinite series with summands $(-1)^{\ell-1} \prod_{k=1}^{\ell} q_k / (z_\ell z_{\ell-1})$ converges.

The duality between continued fractions and three-term linear recurrences is fundamental to our characterisation of positivity in the work that follows. We make use of the following well-known result, relating continued fractions to minimality, due to Pincherle (see also Theorem 1.1 in [9]).

► **Theorem 3.1** (Pincherle [22]). The continued fraction $\frac{q_1}{p_1} + \frac{q_2}{p_2} + \dots$ converges if and only if the recurrence relation

$$u_{n+1} = p_n u_n + q_n u_{n-1}, \quad q_n \neq 0, \quad n \in \{1, 2, \dots\}$$

has a minimal solution (v_n) with $v_0 \neq 0$. Moreover, if $\frac{q_1}{p_1} + \frac{q_2}{p_2} + \dots$ converges then

$$\frac{v_n}{v_{n-1}} = \frac{q_n}{p_n} + \frac{q_{n+1}}{p_{n+1}} + \dots$$

provided that $v_n \neq 0$ for each $n \in \mathbb{N}$.

3.2 A Critical Ratio

Consider the linear recurrence (1.2). In the case that $\text{sign}(b(n)) = \text{sign}(c(n))$ then deciding the positivity of a solution sequence (u_n) is trivial. It is the interesting behaviour in the case that $\text{sign}(b(n)) = -\text{sign}(c(n))$ that requires significantly more work to determine positivity of a solution. In this subsection we consider the case where $\text{sign}(b(n)) = -\text{sign}(c(n))$ and treat the two cases, where $\text{sign}(b(n)) = -1$ and $\text{sign}(b(n)) = 1$ separately.

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$$b(n) < 0 \text{ and } c(n) > 0$$

In this case, defining the sequence of ratios $x_n := u_{n+1}/u_n$ for all $n \in \mathbb{N}$, we rewrite recurrence (1.2) into

$$x_n = -p(n+1) + \frac{q(n+1)}{x_{n-1}} \quad (3.3)$$

where $p(n) = -b(n)/a(n)$ and $q(n) = c(n)/a(n)$ for all $n \in \mathbb{N}$ such that $x_{n-1} \neq 0$. Here the n th convergent r_n of the associated continued fraction is given by

$$r_n = \frac{q(2)}{p(2)} + \frac{q(3)}{p(3)} + \cdots + \frac{q(n+1)}{p(n+1)}.$$

We note the shift in the indexing follows from the indices given in (3.3).

In the next proposition we link the notions of positivity for a solution sequence (u_n) to the limit of the sequence of convergents (r_n) .

► **Proposition 3.2** (Appendix C). *The sequence of convergents (r_n) has a limit μ , called the critical ratio. Moreover, assuming $u_0 > 0$ and $u_1 > 0$ the three following statements are equivalent:*

1. (u_n) is positive;
2. (u_n) is minimal;
3. $u_1/u_0 = \mu$.

$$b(n) > 0 \text{ and } c(n) < 0$$

In this case, we have

$$x_n = p(n+1) - \frac{q(n+1)}{x_{n-1}} \quad (3.4)$$

where $p(n) = b(n)/a(n)$ and $q(n) = -c(n)/a(n)$ for all $n \in \mathbb{N}$ such that $x_{n-1} \neq 0$. In this case we take $r_n := \frac{q(2)}{p(2)} - \frac{q(3)}{p(3)} + \cdots - \frac{q(n+1)}{p(n+1)}$. The positivity of the sequence (u_n) can be obtained by comparing x_0 to the sequence of convergents.

► **Lemma 3.3.** *Suppose that the sequence (r_n) is increasing. Then $x_{n+1} > 0$ if and only if $x_0 > r_{n+1}$.*

Proof. For $i, k \in \mathbb{N}$, let $r_k^{(i)} = \frac{q(i+2)}{p(i+2)} - \frac{q(i+3)}{p(i+3)} + \cdots - \frac{q(k+1)}{p(k+1)}$ if $i < k$, and $r_k^{(i)} = 0$ otherwise. Since (r_k) is increasing, we have that $r_k^{(i)}$ is positive for each $k \in \mathbb{N}$ and $i < k$. For $i \in \{0, \dots, n\}$, we have the following equivalences:

$$x_i > r_{n+1}^{(i)} \Leftrightarrow x_{i+1} > p(i+2) - \frac{q(i+2)}{r_{n+1}^{(i)}} \Leftrightarrow x_{i+1} > r_{n+1}^{(i+1)}.$$

Hence $x_{n+1} > 0$ if and only if $x_0 > r_{n+1}^0 = r_{n+1}$. ◀

► **Corollary 3.4.** *Suppose that the sequence (r_n) is increasing. Then for all $n \in \mathbb{N}$ we have $x_1, \dots, x_{n+1} > 0$ if and only if $x_0 > \max(r_1, \dots, r_{n+1})$.*

Once again, this result holds for arbitrary polynomials $a(n)$, $b(n)$ and $c(n)$. If they are all affine, the limits $p := \lim_{n \rightarrow \infty} p(n)$ and $q := \lim_{n \rightarrow \infty} q(n)$ both exist and the convergence of $p(n)$ and $q(n)$ to their respective limits is monotonic.

Take the discriminant of the recurrence $\Delta(n) := p(n)^2 - 4q(n)$. As before, we shall assume that $\text{sign}(\Delta(n))$ is constant.

► **Proposition 3.5** (Appendix C). *Suppose that $\Delta(n) \geq 0$ for all $n \in \mathbb{N}$. Then for some effectively computable shift of (1.2) the sequence (r_n) is strictly increasing and convergent.*

Sketch of proof. Let (z_n) be the solution to recurrence (1.2) with initial conditions $z_0 = 0$ and $z_1 = 1$. We define $\nu_n := z_{n+1}/z_n$ for all $n \in \mathbb{N}_+$ such that $z_n \neq 0$. We can show by induction that, for all $n \in \mathbb{N}_+$, $\nu_n > 0$. By the discussion preceding Lemma C.1, we have that

$$r_n = \sum_{\ell=0}^n \frac{q_2 q_3 \cdots q_{\ell+2}}{z_{\ell+2} z_{\ell+1}} = \sum_{\ell=0}^n \frac{q_2 q_3 \cdots q_{\ell+2}}{\nu_1^2 \cdots \nu_{\ell}^2 \nu_{\ell+1}}.$$

Thus, by positivity of (ν_n) , the sequence (r_n) is strictly increasing. Moreover, from Corollary 3.4, the sequence (z_n) being positive, (r_n) is bounded above. Therefore (r_n) is convergent. ◀

Combining Proposition 3.5, Corollary 3.4, and Theorem 3.1 and letting μ denote the limit of the sequence (r_n) , we obtain:

► **Corollary 3.6.** *Suppose that $u_0 > 0$ and that $\Delta(n) \geq 0$ for all $n \in \mathbb{N}$. Then $u_n > 0$ for all $n \in \mathbb{N}$ if and only if $u_1/u_0 \geq \mu$. Moreover, a sequence satisfying $u_1/u_0 = \mu$ is a minimal solution.*

As shown by the following result, the difficult case to decide positivity once again arises when the initial ratio is the critical ratio μ .

► **Proposition 3.7** (Appendix D). *Suppose that $\Delta(n) > 0$ for all $n \in \mathbb{N}$, then one can detect whether $u_1/u_0 > \mu$.*

Finally, we have to consider the case where $\Delta(n) < 0$ for all $n \in \mathbb{N}$. When the coefficients in (1.2) are affine, we have the following:

► **Proposition 3.8** (Appendix E). *Consider the recurrence relation (1.2) and suppose that $\Delta(n) < 0$ for all $n \in \mathbb{N}$. There are no positive solutions nor minimal solutions to (1.2).*

Combining Proposition 3.2, Corollary 3.6, Proposition 3.8, and Proposition 3.7, we find a proof of Theorem 1.1. The rest of this note focuses on the decidability of whether a rational sequence is minimal.

4 Minimal Solutions

In this section we consider the problem of deciding whether a solution to the recurrence relation (1.2) is minimal. Throughout this part we assume that a minimal solution exists for the recurrence (1.2). It follows by Corollary 3.6 combined with Proposition 3.8, that the characteristic roots λ, Λ , with $|\Lambda| \geq |\lambda|$, of the recurrence relation are real numbers. In the case that $|\lambda| = |\Lambda|$, we in fact have a single repeated root (since $\beta_1 \neq 0$ we are in the case that $b(n)$ is positive and $c(n)$ is negative), and we denote this root by λ . If $\Lambda < 0$, we consider the recurrence relation $a(n)u_n = -b(n)u_{n-1} + c(n)u_{n-2}$ instead; (u_n) is a solution

to this recurrence if and only if $(-1)^n u_n$ is a solution to (1.2). Minimality is clearly preserved in this transformation, and the characteristic roots of the former recurrence are $-\lambda, -\Lambda$.

We first characterise minimal solutions in terms of the radius of convergence of the generating series of a solution (u_n) . We then study analytic properties of the corresponding generating function. With the appropriate tools at hand, we proceed to prove the main result of this section.

4.1 Minimality and Convergence

Let $\mathcal{G}(x) = \sum_{n=0}^{\infty} u_n x^n$ be a generating function of the solution (u_n) to recurrence (1.2) and let R be its radius of convergence. Then \mathcal{G} converges uniformly for $|x| < R$.

We now characterise minimal solutions to a class of recurrences (1.2) in terms of the limiting behaviour of \mathcal{G} and its derivatives.

► **Lemma 4.1** (Appendix B). *Let (u_n) be a non-trivial solution to the recurrence (1.2), and assume that the characteristic roots λ and Λ are real. Let $\alpha := \alpha_0/\alpha_1$, $\beta := 2\beta_0/\beta_1$, and $\gamma := \gamma_0/\gamma_1$.*

1. *If $|\Lambda| > |\lambda|$, then (u_n) is a minimal solution if and only if the generating function \mathcal{G} and all of its derivatives converge at Λ^{-1} .*
2. *If $\Lambda = \lambda$ and $\beta > \alpha + \gamma$, then (u_n) is a minimal solution if and only if the generating function \mathcal{G} converges at λ^{-1} .*
3. *If $\Lambda = \lambda$, $\beta = \alpha + \gamma$, and $\alpha > \gamma + 1$, then (u_n) is a minimal solution if and only if any of the antiderivatives $\int \mathcal{G}(x) dx = \sum_{n=1}^{\infty} u_{n-1} n^{-1} x^n + c$ of \mathcal{G} converge at λ^{-1} .*

The proof of the above lemma is based on detailed asymptotic analysis of the behaviour of solutions to the recurrence (1.2) performed in Appendix B. The results are a straightforward application of the work of Kooman [13].

4.2 Generating Function Analysis

Let (u_n) be a solution to the recurrence (1.2). We continue our generating function analysis in this subsection in order to establish the characterisation of minimal solutions in Lemma 4.1. Our approach is to give an integral expression for a generating function and to do this we setup an ordinary differential equation. The generating function \mathcal{F} we consider is the Frobenius series given by $\mathcal{F}(x) = \sum_{n=0}^{\infty} u_n x^{n+\alpha}$ wherever the series converges and $\alpha := \alpha_0/\alpha_1$. It is clear that \mathcal{F} and \mathcal{G} share the same analytic properties. We choose to use \mathcal{F} when considering ordinary differential equations rather than \mathcal{G} as this removes a singularity at the origin.

By the recurrence relation (1.2), we have

$$\sum_{n=2}^{\infty} a(n) u_n x^{n+\alpha} = \sum_{n=2}^{\infty} b(n) u_{n-1} x^{n+\alpha} + \sum_{n=2}^{\infty} c(n) u_{n-2} x^{n+\alpha}.$$

Observe now that $\text{for } a(n) = \alpha_1 \cdot n + \alpha_0$

$$\sum_{n=2}^{\infty} a(n) u_n x^{n+\alpha} = \alpha_1 x \sum_{n=2}^{\infty} (n + \alpha) u_n x^{n+\alpha-1} = \alpha_1 x \mathcal{F}'(x) - (\alpha_0 + \alpha_1) u_1 x^{\alpha+1} - \alpha_0 u_0 x^{\alpha}.$$

In a similar fashion one can write

$$\sum_{n=2}^{\infty} b(n) u_{n-1} x^{n+\alpha} = \beta_1 x^2 \mathcal{F}'(x) + (\beta_0 + (1 - \alpha)\beta_1) x \mathcal{F}(x) - (\beta_0 + \beta_1) u_0 x^{\alpha+1},$$

and

$$\sum_{n=2}^{\infty} c(n)u_{n-2}x^{n+\alpha} = \gamma_1 x^3 \mathcal{F}'(x) + (\gamma_0 + (2-\alpha)\gamma_1)x^2 \mathcal{F}(x).$$

We associate the following first-order differential equation to the recurrence relation in (1.2)

$$\mathcal{F}'(x) + p(x)\mathcal{F}(x) = q(x),$$

where

$$p(x) = \frac{(\alpha_0\gamma_1 - (\gamma_0 + 2\gamma_1)\alpha_1)x + \alpha_0\beta_1 - \alpha_1(\beta_0 + \beta_1)}{-\gamma_1\alpha_1(x - 1/\lambda)(x - 1/\Lambda)}$$

and

$$q(x) = \frac{(\alpha_0 + \alpha_1)u_1 - (\beta_0 + \beta_1)u_0 + \alpha_0 u_0 x^{-1}}{\alpha_1 - \beta_1 x - \gamma_1 x^2} x^\alpha.$$

Observe now that both p and q are integrable over an interval containing 0, under our assumption that $\alpha > 1$. Standard methods then yield a solution to the differential equation. Namely, by noting that $\mathcal{F}(0) = 0$ as $\alpha > 0$, we have the solution in an interval containing 0 as follows:

$$\mathcal{F}(x) = \frac{\int_0^x \mathcal{I}(t)q(t) dt}{\mathcal{I}(x)}, \quad (4.1)$$

where $\mathcal{I}(x)$ is the integrating factor.

We shall proceed under the assumption that the characteristic roots λ and Λ are distinct and real, so that $\Lambda > |\lambda|$. The case $\Lambda = \lambda$ is then sketched thereafter.

It is then straightforward to verify that $p(x) = \frac{\mathcal{A}(\lambda)}{x-1/\Lambda} - \frac{\mathcal{A}(\Lambda)}{x-1/\lambda}$, where

$$\mathcal{A}(y) = \frac{\alpha_1(\alpha_1(\gamma_0 + 2\gamma_1) - \alpha_0\gamma_1)y - \gamma_1(\alpha_1(\beta_0 + \beta_1) - \alpha_0\beta_1)}{\gamma_1\alpha_1^2(\lambda - \Lambda)}, \quad (4.2)$$

and, since $\Lambda > 0$, that the integrating factor is given by

$$\mathcal{I}(x) = (1/\Lambda - x)^{\mathcal{A}(\lambda)} |x - 1/\lambda|^{-\mathcal{A}(\Lambda)}.$$

Thus substitution into (4.1) gives

$$\mathcal{F}(x) = \frac{|x - 1/\lambda|^{\mathcal{A}(\Lambda)}}{(1/\Lambda - x)^{\mathcal{A}(\lambda)}} \int_0^x \frac{(1/\Lambda - t)^{\mathcal{A}(\lambda)}}{|t - 1/\lambda|^{\mathcal{A}(\Lambda)}} q(t) dt.$$

We are interested in the analytic properties of $\mathcal{F}(x)$ for $x \in [0, 1/\Lambda)$ and especially the limiting behaviour of $\mathcal{F}(x)$ as $x \rightarrow 1/\Lambda^-$. Our aim is to identify when \mathcal{F} and all its derivatives exist when $x \rightarrow 1/\Lambda$. Note that \mathcal{F} is differentiable in the half-open interval $[0, \frac{1}{\Lambda})$. In the proof that follows, we shall assume that $\lambda > 0$. The other case is analogous.

Let us write $\mathcal{I}(t)q(t) = (1/\Lambda - t)^{\mathcal{A}(\lambda)-1}Q(t)$. The functions $x^\alpha = (1/\Lambda - (1/\Lambda - x))^\alpha$, and $(1/\lambda - x)^{-\mathcal{A}(\Lambda)} = (1/\lambda - 1/\Lambda + (1/\Lambda - x))^{-\mathcal{A}(\Lambda)}$ can be expressed as power series that converge uniformly in some neighbourhood of $1/\Lambda$. Thus Q is analytic in a neighbourhood of $1/\Lambda$ as a product of such functions, and we write $Q(x) = \sum_{n=0}^{\infty} T_n(1/\Lambda - x)^n$ for all x in some open interval containing $1/\Lambda$. (We set $T_n = 0$ for $n < 0$.)

► **Proposition 4.2.** *A rational solution (u_n) is minimal if and only if*

$$\int_0^{\frac{1}{\Lambda}} \left(\mathcal{I}(t)q(t) - r(t) \right) dt = \sum_{n=0}^{\lfloor -\mathcal{A}(\lambda) \rfloor - 1} \frac{-T_n}{n + \mathcal{A}(\lambda)} \frac{1}{\Lambda^{n+\mathcal{A}(\lambda)}} - D$$

where $r(t) = \sum_{n=0}^{\lfloor -\mathcal{A}(\lambda) \rfloor} T_n (1/\Lambda - t)^{n+\mathcal{A}(\lambda)-1}$, and $D = T_{|\mathcal{A}(\lambda)|} \log(\frac{1}{\Lambda})$ if $-\mathcal{A}(\lambda) \in \mathbb{N}$, otherwise $D = T_{\lfloor -\mathcal{A}(\lambda) \rfloor} (\lfloor -\mathcal{A}(\lambda) \rfloor + \mathcal{A}(\lambda))^{-1} \Lambda^{-\lfloor -\mathcal{A}(\lambda) \rfloor - \mathcal{A}(\lambda)}$.

Proof. Let $\varepsilon > 0$ be a real number such that $1/\Lambda - \varepsilon \in I$. Then, for all $x \in [1/\Lambda - \varepsilon, 1/\Lambda)$ we have

$$\begin{aligned} \int_{\frac{1}{\Lambda} - \varepsilon}^x \mathcal{I}(t)q(t) dt &= \int_{\frac{1}{\Lambda} - \varepsilon}^x (1/\Lambda - t)^{\mathcal{A}(\lambda)-1} Q(t) dt = \int_{\frac{1}{\Lambda} - \varepsilon}^x \sum_{n=0}^{\infty} T_n (1/\Lambda - t)^{n+\mathcal{A}(\lambda)-1} dt \\ &= G(x) - G(1/\Lambda - \varepsilon), \end{aligned}$$

where

$$G(t) = \begin{cases} \sum_{\substack{n=0 \\ n \neq |\mathcal{A}(\lambda)|}}^{\infty} \frac{-T_n}{n + \mathcal{A}(\lambda)} (1/\Lambda - t)^{n+\mathcal{A}(\lambda)} - T_{|\mathcal{A}(\lambda)|} \log(1/\Lambda - t) & \text{if } -\mathcal{A}(\lambda) \in \mathbb{N}, \text{ and} \\ \sum_{n=0}^{\infty} \frac{-T_n}{n + \mathcal{A}(\lambda)} (1/\Lambda - t)^{n+\mathcal{A}(\lambda)} & \text{otherwise.} \end{cases}$$

Let $C_\varepsilon = \int_0^{\frac{1}{\Lambda} - \varepsilon} \mathcal{I}(t)q(t) dt - G(1/\Lambda - \varepsilon)$. We shall complete the proof assuming $-\mathcal{A}(\lambda) \notin \mathbb{N}$. We highlight the changes required for the other case.

Now suppose that $x \in [1/\Lambda - \varepsilon, 1/\Lambda)$ for a sufficiently small $\varepsilon > 0$. Using the above observations and notation we find that

$$\begin{aligned} \mathcal{F}(x) &= \mathcal{I}(x)^{-1} \left(\int_0^{\frac{1}{\Lambda} - \varepsilon} \mathcal{I}(t)q(t) dt + \int_{\frac{1}{\Lambda} - \varepsilon}^x \mathcal{I}(t)q(t) dt \right) = \mathcal{I}(x)^{-1} (C_\varepsilon + G(x)) \\ &= C_\varepsilon \frac{(x - 1/\Lambda)^{\mathcal{A}(\lambda)}}{(x - 1/\Lambda)^{\mathcal{A}(\lambda)}} + \sum_{n=0}^{\infty} \frac{-T_n (1/\Lambda - x)^{\mathcal{A}(\lambda)}}{n + \mathcal{A}(\lambda)} (1/\Lambda - x)^n. \end{aligned} \quad (4.3)$$

Let us consider the limit of $\mathcal{F}(x)$ as $x \rightarrow 1/\Lambda -$. We consider two cases. First, if $C_\varepsilon = 0$ then $\mathcal{F}(x)$ and each of its derivatives converge as $x \rightarrow 1/\Lambda -$ by (4.3). Second, suppose that $C_\varepsilon \neq 0$. Here there are two further subcases. If $\mathcal{A}(\lambda) > 0$, then $\mathcal{F}(x) \rightarrow \text{sign}(C_\varepsilon)\infty$ by (4.3). If $\mathcal{A}(\lambda) \leq 0$, then we conclude that the derivative $\mathcal{F}^{(\lfloor -\mathcal{A}(\lambda) \rfloor + 1)}(x)$ diverges as $x \rightarrow 1/\Lambda -$.² By Lemma 4.1, we have $u_1/u_0 = \mu$ if and only if $C_\varepsilon = 0$.

We now complete the proof by relating C_ε to the claim. Let now

$$r(t) = \sum_{n=0}^m T_n (1/\Lambda - t)^{n+\mathcal{A}(\lambda)-1}, \quad \text{and} \quad R(\varepsilon) = \sum_{n=m+1}^{\infty} \frac{-T_n}{n + \mathcal{A}(\lambda)} \varepsilon^{n+\mathcal{A}(\lambda)}$$

with $m = \lfloor -\mathcal{A}(\lambda) \rfloor$. Observe that

$$\begin{aligned} G(1/\Lambda - \varepsilon) &= \sum_{n=0}^{\infty} \frac{-T_n}{n + \mathcal{A}(\lambda)} \varepsilon^{n+\mathcal{A}(\lambda)} = \sum_{n=0}^m \frac{-T_n}{n + \mathcal{A}(\lambda)} \varepsilon^{n+\mathcal{A}(\lambda)} + R(\varepsilon) \\ &= \int_0^{\frac{1}{\Lambda} - \varepsilon} r(t) dt + \sum_{n=0}^m \frac{-T_n}{n + \mathcal{A}(\lambda)} \frac{1}{\Lambda^{n+\mathcal{A}(\lambda)}} + R(\varepsilon). \end{aligned}$$

² In the case that $-\mathcal{A}(\lambda) \in \mathbb{N}$ the result also holds. This is because the analogous expression to (4.3) contains the term $T_{|\mathcal{A}(\lambda)|} (1/\Lambda - x)^{\mathcal{A}(\lambda)} (1/\Lambda - x)^{-\mathcal{A}(\lambda)} \log(1/\Lambda - x)$.

Thus, we have

$$\begin{aligned} C_\varepsilon &= \int_0^{\frac{1}{\Lambda}-\varepsilon} \mathcal{I}(t)q(t) dt - G(1/\Lambda - \varepsilon) \\ &= \int_0^{\frac{1}{\Lambda}-\varepsilon} \mathcal{I}(t)q(t) - r(t) dt - \sum_{n=0}^m \frac{-T_n}{n + \mathcal{A}(\lambda)} \frac{1}{\Lambda^{n+\mathcal{A}(\lambda)}} - R(\varepsilon). \end{aligned}$$

Observe that C_ε is, in fact, independent of ε : this can be checked by differentiating C_ε with respect to ε . Note also that $\lim_{\varepsilon \rightarrow 0+} R(\varepsilon) = 0$. Taking the limit $\varepsilon \rightarrow 0+$ on both sides and rearranging yields

$$\int_0^{\frac{1}{\Lambda}} \mathcal{I}(t)q(t) - r(t) dt = C_\varepsilon + \sum_{n=0}^m \frac{-T_n}{n + \mathcal{A}(\lambda)} \frac{1}{\Lambda^{n+\mathcal{A}(\lambda)}},$$

from which the result follows. ◀

► **Remark 4.3.** When $\mathcal{A}(\lambda) > 0$ we find an interesting characterisation of μ in terms of Gauss's hypergeometric function. For $|x| < 1$, the hypergeometric function ${}_2F_1(a, b; c; x)$ is defined by the power series

$${}_2F_1(a, b; c; x) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n$$

where $(a)_n := a(a+1) \cdots (a+n-1)$ if $n \in \{1, 2, \dots\}$, and $(a)_0 := 1$ is the rising factorial or Pochhammer symbol. We necessarily require $(c)_n$ is non-zero for each n . In this setting, a solution (u_n) is minimal if and only if $\int_0^{\frac{1}{\Lambda}} \mathcal{I}(t)q(t) dt$ vanishes. By linearity, we may write the integral in the form

$$\ell_1 \int_0^{\frac{1}{\Lambda}} t^\alpha (1/\Lambda - t)^{c-\alpha-1} (1/\lambda - t)^{-a} dt + \ell_0 \int_0^{\frac{1}{\Lambda}} t^{\alpha-1} (1/\Lambda - t)^{c-\alpha-1} (1/\lambda - t)^{-a} dt,$$

where $\ell_1 = \frac{1}{-\gamma_1 \alpha_1} ((\alpha_0 + \alpha_1)u_1 - (\beta_0 + \beta_1)u_0)$, $\ell_0 = \frac{\alpha_0 u_0}{-\gamma_1 \alpha_1}$, $a = \mathcal{A}(\Lambda) + 1$, and $c = \mathcal{A}(\lambda) + \alpha$. After a change of variables ($t = \Lambda u$), we find the equivalent expression

$$\frac{\lambda^a \ell_1}{\Lambda^c} \int_0^1 u^\alpha (1-u)^{c-\alpha-1} (1-\lambda u/\Lambda)^{-a} du + \frac{\lambda^a \ell_0}{\Lambda^{c-1}} \int_0^1 u^{\alpha-1} (1-u)^{c-\alpha-1} (1-\lambda u/\Lambda)^{-a} du.$$

By assumption, $c > \alpha > 0$. Hence the integral $\int_0^{1/\Lambda} \mathcal{I}(t)q(t) dt$ is equal to

$$\begin{aligned} &\frac{\lambda^a \ell_1}{\Lambda^c} \frac{\Gamma(\alpha+1)\Gamma(c-\alpha)}{\Gamma(c+1)} {}_2F_1\left(\begin{matrix} a, \alpha+1 \\ c+1 \end{matrix}; \frac{\lambda}{\Lambda}\right) + \frac{\lambda^a \ell_0}{\Lambda^{c-1}} \frac{\Gamma(\alpha)\Gamma(c-\alpha)}{\Gamma(c)} {}_2F_1\left(\begin{matrix} a, \alpha \\ c \end{matrix}; \frac{\lambda}{\Lambda}\right) \\ &= \frac{\lambda^a}{\Lambda^c} \frac{\Gamma(\alpha)\Gamma(c-\alpha)}{\Gamma(c)} \left(\frac{\ell_1 \alpha}{c} {}_2F_1\left(\begin{matrix} a, \alpha+1 \\ c+1 \end{matrix}; \frac{\lambda}{\Lambda}\right) + \ell_0 \Lambda {}_2F_1\left(\begin{matrix} a, \alpha \\ c \end{matrix}; \frac{\lambda}{\Lambda}\right) \right) \end{aligned}$$

by Euler's integral representation (see, e.g., [1, Theorem 2.2.1]). It is now clear that (u_n) is a minimal solution if and only if

$$\frac{{}_2F_1(a, \alpha; c; \lambda/\Lambda)}{{}_2F_1(a, \alpha+1; c+1; \lambda/\Lambda)} = -\frac{\alpha \ell_1}{c \ell_0 \Lambda}.$$

The two functions ${}_2F_1(a, \alpha, \cancel{c}, \cancel{c+1}; \lambda/\Lambda)$ and ${}_2F_1(a, \alpha+1; c+1; \lambda/\Lambda)$ evaluated at the same argument λ/Λ are contiguous—their parameters differ by integers. Relations between contiguous hypergeometric functions have been the subject of study in number theory since Gauss's seminal work in this area. For example, a connection to continued fractions is well-known. We direct the interested reader to [1, §2.5].

We briefly sketch arguments used in the case of a simple repeated characteristic root. The approach is identical to the case where the roots are distinct. We shall state the results without proofs; the complete treatment can be found in Appendix F.

When the characteristic polynomial has a repeated root λ , this root is necessarily rational. To simplify the notation, we normalise the recurrence relation as follows. For a solution (u_n) , we define a sequence $v_n = \lambda^{-n}u_n$. Then (v_n) satisfies the recurrence

$$(n + \alpha)v_n = (2n + \beta)v_{n-1} - (n + \gamma)v_{n-2}, \quad (4.4)$$

where $\alpha = \alpha_0/\alpha_1$, $\beta = 2\beta_0/\beta_1$, and $\gamma = \gamma_0/\gamma_1$. Work in Appendix F concentrates efforts on characterising minimality for this recurrence. The characterisation then carries over to the original recurrence relation. We obtain the following integral formula for \mathcal{F} :

$$\mathcal{F}(x) = (1 - x)^{-A_1} \exp(A_2(x - 1)^{-1}) \int_0^x (1 - t)^{A_1} \exp(A_2(1 - t)^{-1}) q(t) dt \quad (4.5)$$

in terms of the constants $A_1 = \gamma + 2 - \alpha$, $A_2 = \alpha - \beta + \gamma$. This representation gives a complete characterisation of minimality as follows:

► **Proposition 4.4.** *If $A_2 < 0$, then (v_n) is a minimal solution to (4.4) if and only if $\int_0^1 (1 - t)^{A_1} \exp(A_2(1 - t)^{-1}) q(t) dt = 0$. If $A_2 = 0$, then (v_n) is a minimal solution to (4.4) if and only if $v_1/v_0 = \min\{1, \frac{\gamma+2}{\alpha+1}\}$.*

4.3 Decidability of Minimal Solutions

Subject to conjectures in number theory, we can determine whether a rational solution to recurrence (1.2) is minimal.

Proof of Theorem 1.2. Let λ and Λ be the characteristic roots of the recurrence (1.2). If they are not real numbers, then no minimal solution exists as a consequence of Proposition 3.8. Furthermore, if the roots are real but irrational, then by Conjecture A.1, (u_n) is not minimal, since (u_n) is a sequence of rational numbers.

We are thus left with the case that the roots are rational numbers. Assume first that $|\Lambda| > |\lambda|$. By Proposition 4.2 the minimality of a solution (u_n) is characterized by a certain integral being equal to an explicitly given constant. Since we assumed that λ and Λ are rational, the parameters $\mathcal{A}(\lambda)$ and $\mathcal{A}(\Lambda)$ are also rational. Therefore, the quantity on the left-hand side is an integral of an algebraic function over a semi-algebraic set, and thus a *period* in the sense of Kontsevich and Zagier (see Appendix A). The quantity on the right-hand side is a sum of algebraic numbers plus, possibly, the logarithm of an algebraic number, which is again a period. Subject to Conjecture A.2, it is decidable whether the two periods are equal.

We now deal with the case that the roots have equal modulus. Since we assume that $\beta_1 \neq 0$, we must have a single repeated root λ . Let us normalise the sequence (u_n) to a solution (v_n) of (4.4). It suffices to decide whether (v_n) is a minimal solution. We consider three subcases separately.

First, suppose that $\beta - \alpha - \gamma > 0$. Then, by Proposition 4.4, (v_n) is a minimal solution to (4.4) if and only if $\int_0^1 (1 - t)^{A_1} \exp(A_2(1 - t)^{-1}) q(t) dt$ vanishes. The integral represents an exponential period. Thus, by the generalisation of Conjecture A.2, it is decidable whether it equals 0.

Second, suppose that $\beta - \alpha - \gamma < 0$. Then for sufficiently large $n \in \mathbb{N}$,

$$\Delta(n) = (2n + \beta)^2 - 4(n + \alpha)(n + \gamma) = \beta^2 - 4\alpha\gamma + 4(\beta - \alpha - \gamma)n < 0.$$

By Proposition 3.8, the recurrence has no minimal solutions.

Finally, suppose that $\beta = \alpha + \gamma$. By Proposition 4.4, we have that (v_n) is a minimal solution to (4.4) if and only if $v_1/v_0 = \min\{1, \frac{\gamma+2}{\alpha+1}\}$, which is decidable. This covers all the required cases. ◀

References

- 1 George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1999. doi:10.1017/CB09781107325937.
- 2 R. Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, (61):11–13, 1979. Luminy Conference on Arithmetic.
- 3 Jonathan Borwein, Alf van der Poorten, Jeffrey Shallit, and Wadim Zudilin. *Neverending fractions*, volume 23 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, Cambridge, 2014. *An introduction to continued fractions*. doi:10.1017/CB09780511902659.
- 4 George Chrystal. *Algebra: An elementary text-book for the higher classes of secondary schools and for colleges*. 6th ed. Chelsea Publishing Co., New York, 1959.
- 5 Alfredo Deaño, Javier Segura, and Nico M. Temme. Computational properties of three-term recurrence relations for Kummer functions. *J. Computational Applied Mathematics*, 233(6):1505–1510, 2010.
- 6 Graham Everest, Alfred J. van der Poorten, Igor E. Shparlinski, and Thomas Ward. *Recurrence Sequences*, volume 104 of *Mathematical surveys and monographs*. American Mathematical Society, 2003.
- 7 Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.
- 8 W. Gautschi. Minimal solutions of three-term recurrence relations and orthogonal polynomials. *Mathematics of Computation*, 36(154), 1981.
- 9 Walter Gautschi. Computational aspects of three-term recurrence relations. *SIAM Rev.*, 9:24–82, 1967. doi:10.1137/1009002.
- 10 V. Halava, T. Harju, and M. Hirvensalo. Positivity of second order linear recurrent sequences. *Discrete Appl. Math.*, 154(3):447–451, 2006. doi:10.1016/j.dam.2005.10.009.
- 11 Manuel Kauers and Veronika Pillwein. When can we detect that a p-finite sequence is positive? In Wolfram Koepf, editor, *Symbolic and Algebraic Computation, International Symposium, ISSAC 2010, Munich, Germany, July 25–28, 2010, Proceedings*, pages 195–201. ACM, 2010.
- 12 Maxim Kontsevich and Don Zagier. *Periods*. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.
- 13 Robert-Jan Kooman. An asymptotic formula for solutions of linear second-order difference equations with regularly behaving coefficients. *Journal of Difference Equations and Applications*, 13(11):1037–1049, 2007. doi:10.1080/10236190701414462.
- 14 V. Laohakosol and P. Tangsupphathawat. Positivity of third order linear recurrence sequences. *Discrete Appl. Math.*, 157(15):3239–3248, 2009. doi:10.1016/j.dam.2009.06.021.
- 15 Lily Liu. Positivity of three-term recurrence sequences. *Electron. J. Combin.*, 17(1):Research Paper 57, 10, 2010. URL: http://www.combinatorics.org/Volume_17/Abstracts/v17i1r57.html.
- 16 M. Mezzarobba and B. Salvy. Effective bounds for P-recursive sequences. *J. Symbolic Comput.*, 45(10):1075–1096, 2010. doi:10.1016/j.jsc.2010.06.024.
- 17 J. Ouaknine and J. Worrell. Positivity problems for low-order linear recurrence sequences. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 366–379. ACM, New York, 2014. doi:10.1137/1.9781611973402.27.
- 18 Joël Ouaknine and James Worrell. Ultimate positivity is decidable for simple linear recurrence sequences. In *Automata, Languages, and Programming - 41st International Colloquium, ICALP*

- 2014, Copenhagen, Denmark, July 8-11, 2014, *Proceedings, Part II*, volume 8573 of *Lecture Notes in Computer Science*, pages 330–341. Springer, 2014.
- 19 Joël Ouaknine and James Worrell. On linear recurrence sequences and loop termination. *SIGLOG News*, 2(2):4–13, 2015.
- 20 Veronika Pillwein. Termination conditions for positivity proving procedures. In Manuel Kauers, editor, *International Symposium on Symbolic and Algebraic Computation, ISSAC’13, Boston, MA, USA, June 26-29, 2013*, pages 315–322. ACM, 2013.
- 21 Veronika Pillwein and Miriam Schussler. An efficient procedure deciding positivity for a class of holonomic functions. *ACM Comm. Computer Algebra*, 49(3):90–93, 2015.
- 22 Salvatore Pincherle. Delle funzioni ipergeometriche, e di varie questioni ad esse attinenti. *Giornale di Matematiche di Battaglini*, 32:209–291, 1894.
- 23 Tanguy Rivoal. La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs. *Comptes Rendus de l’Académie des Sciences - Series I - Mathematics*, 331(4):267–270, 2000. doi:10.1016/S0764-4442(00)01624-4.
- 24 Michel Waldschmidt. Continued fractions. École de recherche CIMPA-Oujda. Théorie des Nombres et ses Applications. Unpublished.
- 25 Ernest X. W. Xia and X. M. Yao. The signs of three-term recurrence sequences. *Discrete Applied Mathematics*, 159(18):2290–2296, 2011.
- 26 Doron Zeilberger. A holonomic systems approach to special functions identities. *Journal of Computational and Applied Mathematics*, 32(3):321–368, 1990.
- 27 V. V. Zudilin. [Some remarks on linear forms containing Catalan’s constant](#). volume 3, pages 60–70. 2002. Dedicated to the 85th birthday of Nikolai Mikhaïlovich Korobov (Russian).

A Conjectures in Number Theory

A.1 A Rationality Conjecture for Holonomic Sequences

The following conjecture is due to Zudilin [27]: 2002

► **Conjecture A.1.** Consider the second-order linear recurrence relation

$$u_{n+1} = s(n)u_n + t(n)u_{n-1}$$

with coefficients $(s(n))$ and $(t(n))$ that are rational functions having finite limits as $n \rightarrow \infty$. Assume that the characteristic roots λ and Λ associated with the recurrence satisfy $0 < |\lambda| < |\Lambda|$. Suppose that there exist two rational linearly independent solutions (v_n) and (w_n) satisfying $v_{n+1}/v_n \sim \lambda$ and $w_{n+1}/w_n \sim \Lambda$ as $n \rightarrow \infty$.

Then λ and Λ are rational numbers.

A.2 A Decidability Conjecture for Periods

2005 Kontsevich and Zagier's seminal paper [12] defines a *period* as

a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients over domains in \mathbb{R}^n given by polynomial inequalities with rational coefficients.

The set of periods \mathcal{P} form a countable sub-algebra of \mathbb{C} and it is easily seen that $\overline{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C}$. Two initial examples are:

$$\log(\alpha) = \int_1^\alpha \frac{1}{x} dx \quad \text{with } \alpha \in \overline{\mathbb{Q}} \quad \text{and} \quad \pi = \int_{x^2+y^2 \leq 1} dx dy.$$

Given two algebraic numbers α and β , the problem of determining whether $\alpha = \beta$ algorithmically is known to be decidable. The decidability of the equality of two periods; that is, a decision procedure determining whether two periods—given by two explicit integrals—are equal is currently open. (*A priori* there are many different integral representations for a complex number.) However, if the result in the next conjecture, see [12, Conjecture 1], by Kontsevich and Zagier is true then it would imply that equality of periods is also decidable.

► **Conjecture A.2.** Suppose that a period has two integral representations. Then one can pass from one representation to the other using only the linearity of the integral, a change of variables, and Stokes's formula such that any intermediate step preserves the property that all functions and domains of integration are algebraic with coefficients in $\overline{\mathbb{Q}}$.

The following notion of an exponential period was introduced in [12] to extend the definition of period to a strictly larger set containing e . An *exponential period*

is an absolutely convergent integral of the product of an algebraic function with the exponent of an algebraic function, over a real semi-algebraic set, where all the required polynomials here have algebraic coefficients.

It is conjectured by Kontsevich and Zagier in [12] that Conjecture A.2 generalises for exponential periods.

B Asymptotics of Second-Order Holonomic Sequences

The aim of this section is to establish asymptotic behaviours of solutions to the second-order holonomic recurrence relation (1.2) with affine coefficients:

► **Proposition B.1.** *Suppose that recurrence (1.2) has real characteristic roots λ and Λ . Let $\alpha = \alpha_0/\alpha_1$, $\beta = 2\beta_0/\beta_1$, and $\gamma = \gamma_0/\gamma_1$. The recurrence relation has two linearly independent solutions $(u_n^{(1)})$ and $(u_n^{(2)})$ in the following cases.*

1. If $|\Lambda| > |\lambda|$, then

$$u_n^{(1)} \sim \lambda^n n^{-\mathcal{A}(\Lambda)-1}; \quad u_n^{(2)} \sim \Lambda^n n^{\mathcal{A}(\lambda)-1},$$

where $\mathcal{A}(y)$ is as defined in (4.2).

2. If $\Lambda = \lambda$ and $\beta > \alpha + \gamma$, then

$$u_n^{(1)} \sim \lambda^n n^{\frac{1}{4}(1-2\alpha+2\gamma)} e^{2\sqrt{(\beta-\alpha-\gamma)n}}; \quad u_n^{(2)} \sim \lambda^n n^{\frac{1}{4}(1-2\alpha+2\gamma)} e^{-2\sqrt{(\beta-\alpha-\gamma)n}}.$$

3. If $\Lambda = \lambda$, $\beta = \alpha + \gamma$, and $\alpha \neq \gamma + 1$, then

$$u_n^{(1)} \sim \lambda^n n^{1-\alpha+\gamma}; \quad u_n^{(2)} \sim \lambda^n.$$

4. If $\Lambda = \lambda$, $\beta = \alpha + \gamma$, and $\alpha = \gamma + 1$, then

$$u_n^{(1)} \sim \lambda^n; \quad u_n^{(2)} \sim \lambda^n \log n.$$

The above result follows by applying methods of Kooman [13] applied in the affine case. In order to do so, we modify the recurrence relation at hand. We first rewrite the recurrence relation as

$$u_{n+2} = p_n u_{n+1} - q_n u_n, \tag{B.1}$$

where $p_n = b(n+2)/a(n+2)$ and $q_n = -c(n+2)/a(n+2)$. By our assumption we have $p_n \neq 0$ for any $n \geq -1$. Setting $u_n = w_n \prod_{j=0}^{n-2} p_j/2$ for $n \geq 2$ (with $w_0 = p_{-1}/2u_0$ and $w_1 = u_1$), we note that the sequence (w_n) satisfies the recurrence

$$w_{n+2} = 2w_{n+1} - \frac{4q_n}{p_n p_{n-1}} w_n. \tag{B.2}$$

► **Lemma B.2.** *For each solution (u_n) to the recurrence (B.1), there is a corresponding solution (w_n) to (B.2) such that $u_n \sim w_n (\beta_1/2\alpha_1)^n n^{\beta/2-\alpha}$.*

Proof. Given the solution (u_n) , we consider the solution (w_n) to recurrence (B.2) given by $u_n = w_n \prod_{j=0}^{n-2} \frac{p_j}{2}$. First, we have

$$\prod_{j=0}^{n-2} \frac{p_j}{2} = \prod_{j=2}^n \frac{\beta_1(j + \frac{\beta_0}{\beta_1})}{2\alpha_1(j + \alpha)} = \frac{\Gamma(\alpha+2)\Gamma(\beta/2+n+1)}{\Gamma(\beta/2+2)\Gamma(\alpha+n+1)} \left(\frac{\beta_1}{2\alpha_1}\right)^{n-1},$$

Second, by Stirling's formula,

$$\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-1/2} \quad \text{as } \operatorname{Re}(x) \rightarrow \infty.$$

The desired asymptotic formula follows. ◀

We now inspect the asymptotic formulae of solutions to (B.2). For n sufficiently large, we may express $4q_n/(p_n p_{n-1})$ as a Laurent series around $1/n$ as follows:

$$\frac{-4c(n)a(n-1)}{b(n)(b(n-1))} = \frac{-4(\gamma_1 + \frac{\gamma_0}{n})(\alpha_1 + \frac{\alpha_0 - \alpha_1}{n})}{(\beta_1 + \frac{\beta_0}{n})(\beta_1 + \frac{\beta_0 - \beta_1}{n})} = -A - Bn^{-1} - Cn^{-2} + \mathcal{O}(n^{-3}).$$

Here $A = 4\frac{\alpha_1\gamma_1}{\beta_1^2}$, $B = A(\alpha - \beta + \gamma)$, and $C = A(\alpha(\gamma - \beta + 1) + \beta(\frac{3}{4}\beta - \gamma - \frac{1}{2}))$. Note that $1 + A \geq 0$, since $\beta_1^2 + 4\alpha_1\gamma_1 \geq 0$ as we assume the characteristic roots are real. Thus the recurrence relation for (w_n) can be written as

$$w_{n+2} - 2w_{n+1} + (1 - (1 + A) - Bn^{-1} - Cn^{-2} + \mathcal{O}(n^{-3}))w_n = 0. \quad (\text{B.3})$$

We have the following asymptotic formulae for solutions to (B.2) in terms of the values of A , B and C .

► **Lemma B.3.** *Suppose that $1 + A \geq 0$, then there exist two asymptotically inequivalent solutions $(w_n^{(1)})$ and $(w_n^{(2)})$ to (B.2) as follows.*

1. If $A > -1$, then

$$w_n^{(1)} \sim (1 + \sqrt{1 + A})^n n^{\frac{B}{2\sqrt{1+A}+2(1+A)}}; \quad w_n^{(2)} \sim (1 - \sqrt{1 + A})^n n^{\frac{B}{-2\sqrt{1+A}+2(1+A)}}.$$

2. If $A = -1$ and $B > 0$, then

$$w_n^{(1)} \sim n^{1/4-B/2} e^{2\sqrt{Bn}}; \quad w_n^{(2)} \sim n^{1/4-B/2} e^{-2\sqrt{Bn}}.$$

3. If $A = -1$ and $B = 0$, and $C \neq -1/4$, then

$$w_n^{(1)} \sim n^{(\alpha-\gamma)/2}; \quad w_n^{(2)} \sim n^{1-(\alpha-\gamma)/2}.$$

4. If $A = -1$ and $B = 0$, and $C = -1/4$, then

$$w_n^{(1)} \sim \sqrt{n}; \quad w_n^{(2)} \sim \sqrt{n} \log n.$$

Proof. This is immediate from work by Kooman in [13, Example 1]. Kooman's cases are given in terms of a parameter a . The result in (1) corresponds to case $a = 0$, (2) to case $a = -1$, and (3) and (4) to case $a = -2$.

We remark that, for part (3), when $A = -1$ and $B = 0$ (i.e., $\beta_1^2 + 4\alpha_1\gamma_1 = 0$, and $\beta = \gamma + \alpha$), the parameter C is given by $C = \frac{1}{4}((\alpha - (\gamma + 1))^2 - 1)$. The polynomial $x^2 - x - C$ alluded to in [13, Theorem 1(2)] has roots $(\alpha - \gamma)/2$ and $1 - (\alpha - \gamma)/2$. ◀

We shall now prove Proposition B.1 by combining the above two lemmas.

Proof of Proposition B.1. For each solution to recurrence (B.2), there is a corresponding solution to recurrence (1.2). The discussion of the asymptotic behaviour of two solutions is given in Lemma B.2. We prove the theorem point-by-point.

1. Since $1 + A > 0$, there are two characteristic roots Λ and λ with $|\Lambda| > |\lambda|$. Observe that

$$(1 \pm \sqrt{1 + A}) \frac{\beta_1}{2\alpha_1} = \frac{\beta_1 \pm \text{sign}(\beta_1)\sqrt{\beta_1^2 + 4\alpha_1\gamma_1}}{2\alpha_1}.$$

Thus Λ and λ are equal to $(1 \pm \sqrt{1 + A})\beta_1/(2\alpha_1)$ with the appropriate sign choice. Then $\frac{B}{1+A+\sqrt{1+A}} + \frac{\beta}{2} - \alpha = \mathcal{A}(\lambda) - 1$ and $\frac{B}{1+A-\sqrt{1+A}} + \frac{\beta}{2} - \alpha = -\mathcal{A}(\Lambda) - 1$. The claim then follows.

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For the cases below, we note that the assumption $A = -1$ is equivalent to $\beta_1^2 + 4\alpha_1\gamma_1 = 0$, i.e., we have a repeated characteristic root $\lambda = \beta_1/2\alpha_1$.

2. Suppose that $\beta > \alpha + \gamma$. It follows that $B = -(\alpha - \beta + \gamma) > 0$. The claim follows from the fact that $1 - 2B + 2\beta - 4\alpha = 1 - 2\alpha + 2\gamma$.
3. Here $\beta = \alpha + \gamma$ corresponds to $B = 0$ in Lemma B.3. The assumption that $\alpha \neq \gamma + 1$ corresponds to $C \neq -1/4$ in Lemma B.3 (this follows immediately from the formulation of C in the proof of Lemma B.3). We note that $\beta - 2\alpha = \gamma - \alpha$.
4. The assumption that $\beta = \alpha + \gamma$ and $\alpha = \gamma + 1$ corresponds to $C = -1/4$ in the Lemma B.3. The claim follows from the fact that $\gamma - \alpha = -1$.

Thus we have the desired result. \blacktriangleleft

We are in the position to prove Lemma 4.1:

Proof of Lemma 4.1.

1. Suppose that (u_n) is a minimal solution. Then $u_n \sim C\lambda^n n^{\nu_1}$ for some constants $C \neq 0$ and $\nu_1 \in \mathbb{R}$ by Proposition B.1(1). By the Cauchy–Hadamard theorem for power series, the radius of convergence of \mathcal{G} is $1/|\lambda|$. It follows that \mathcal{G} and each of its derivatives, are absolutely continuous at $1/\Lambda$. Suppose, for the contrapositive, that (u_n) is not minimal. Then $u_n \sim C\lambda^n n^{\nu_2}$ for some $\nu_2 \in \mathbb{R}$. In this case we deduce that the radius of convergence of \mathcal{G} is $1/|\Lambda|$. Observe now that $u_n/\Lambda^n \sim Cn^{\nu_2}$, and that $\mathcal{G}^{(\ell)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-\ell)!} u_n x^n$. Thus, for $\ell > -\nu_2$, the series representation of $\mathcal{G}^{(\ell)}(1/\Lambda)$ diverges by comparison to the harmonic series. This concludes the proof of part (1).
2. In part (2), we note that by Proposition B.1(2), a minimal solution (u_n) satisfies

$$u_n \sim C\lambda^n n^{\frac{1}{4}(1-2\alpha+2\gamma)} e^{-2\sqrt{\beta-\alpha-\gamma}\sqrt{n}},$$

while any other linearly independent solution (v_n) satisfies

$$v_n \sim C'\lambda^n n^{\frac{1}{4}(1-2\alpha+2\gamma)} e^{2\sqrt{\beta-\alpha-\gamma}\sqrt{n}}.$$

3. We note that a minimal solution (u_n) satisfies $u_n \sim C\lambda^n n^{1-\alpha+\gamma}$, while any other linearly independent solution (v_n) satisfies $v_n \sim C'\lambda^n$ by Proposition B.1(3). When evaluated at $1/\lambda$, the summand in $\int \mathcal{G}(x) dx$ is asymptotically equal to $Cn^{\gamma-\alpha}$ if and only if (u_n) is a minimal solution. Otherwise the summand is asymptotically equal to C'/n . In the former case, since $\gamma - \alpha < -1$, the series converges. In the latter case, we find the series diverges by comparison to the harmonic series. \blacktriangleleft

C Results for Section 3

C.1 Proof of Proposition 3.2

We make use of the following well-known result for convergents of a continued fraction (cf. [24]).

► **Lemma C.1.** *Suppose that the sequences (p_n) and (q_n) are strictly positive. Then the sequence of convergents given by $r_n = \frac{q_1}{p_1} + \frac{q_2}{p_2} + \dots + \frac{q_n}{p_n}$ satisfies*

$$r_2 < r_4 < \dots < r_{2m} < \dots < r_{2m+1} < \dots < r_3 < r_1.$$

Moreover, Chrystal [4, Part II, Chapter XXXIV, §14] gives the following sufficient condition for convergence.

► **Lemma C.2.** *Suppose that (p_n) and (q_n) are sequences of positive real numbers. Then the infinite continued fraction $\frac{q_1}{p_1} + \frac{q_2}{p_2} + \cdots$ converges if the series $\sum_{k=2}^{\infty} \frac{p_{k-1}p_k}{q_k}$ diverges.*

In the next lemma we link the positivity of a term in a solution sequence (u_n) to terms in the sequence of convergents (r_n) .

► **Lemma C.3.** *Let $n \in \mathbb{N}$ and suppose that $x_0, \dots, x_n > 0$. For all even $n \in \mathbb{N}$ we have $x_{n+1} \geq 0$ if and only if $x_0 \leq r_{n+1}$ and for all odd $n \in \mathbb{N}$ we have $x_{n+1} \geq 0$ if and only if $x_0 \geq r_{n+1}$.*

Proof. Suppose that n is even. Repeated application of the recurrence relation for (x_n) gives the following chain of equivalences:

$$\begin{aligned} x_{n+1} > 0 &\Leftrightarrow x_n \leq \frac{q(n+2)}{p(n+2)}, \\ &\Leftrightarrow x_{n-1} \geq \frac{q(n+1)}{p(n+1)} + \frac{q(n+2)}{p(n+2)}, \\ &\vdots \\ &\Leftrightarrow x_0 \leq \frac{q(2)}{p(2)} + \frac{q(3)}{p(3)} + \cdots + \frac{q(n+2)}{p(n+2)}. \end{aligned}$$

The case that n is odd follows in the same manner. ◀

As a consequence of Lemma C.1, the subsequences (r_{2n}) and (r_{2n+1}) both converge, with limits ℓ_1 and ℓ_2 respectively. Thus, from Lemma C.3, (u_n) is positive if and only if $\ell_1 \leq x_0 \leq \ell_2$. The difficulty of deciding positivity therefore comes from the computation of the exact values of ℓ_1 and ℓ_2 .

This result holds even if the coefficients $a(n)$, $b(n)$, and $c(n)$ are not affine. However, when $p := \lim_{n \rightarrow \infty} p(n)$ and $q := \lim_{n \rightarrow \infty} q(n)$ both exist and are strictly positive, which happens for instance with affine coefficients, then $\ell_1 = \ell_2$ according to Lemma C.2. We deduce that there is a unique value of x_0 for which the sequence (u_n) is positive: the *critical ratio* $\mu = \ell_1 = \ell_2$. From Theorem 3.1, the solution sequences starting with the critical ratio are in fact the minimal solutions of the recurrence.

C.2 Proof of Proposition 3.5

Let (z_n) be the solution to recurrence (1.2) with initial conditions $z_0 = 0$ and $z_1 = 1$. We define $\nu_n := z_{n+1}/z_n$ for all $n \in \mathbb{N}_+$ such that $z_n \neq 0$. We claim that, for all $n \in \mathbb{N}_+$, $\nu_n > 0$. Using the discussion preceding Lemma C.1, we have that

$$r_n = \sum_{\ell=0}^n \frac{q_2 q_3 \cdots q_{\ell+2}}{z_{\ell+2} z_{\ell+1}} = \sum_{\ell=0}^n \frac{q_2 q_3 \cdots q_{\ell+2}}{\nu_1^2 \cdots \nu_{\ell+1}^2}.$$

Thus, by our claim, the sequence (r_n) is strictly increasing. Moreover, from Corollary 3.4, the sequence (z_n) being positive, (r_n) is bounded above. Therefore (r_n) is convergent.

We now have to establish the claim. We consider two cases: when $(q(n))$ is decreasing and when it is increasing. First, suppose that $(q(n))$ is decreasing. We show by induction

that $\nu_n \geq \sqrt{q(n+2)}$ for all $n \in \mathbb{N}_+$. For the base case we have $\nu_1 = p(2) \geq 2\sqrt{q(2)}$. For the induction step we have

$$\nu_{n+1} \geq 2\sqrt{q(n+2)} - \sqrt{q(n+2)} = \sqrt{q(n+2)} \geq \sqrt{q(n+3)}.$$

where the first inequality holds due to $p(n+2)^2 \geq 4q(n+2)$ and $\nu_n \geq \sqrt{q(n+2)}$. This completes the induction.

Now suppose that $q(n)$ is increasing. Then $q(n) = q \cdot (1 - \varepsilon_1/a(n))$ for some $\varepsilon_1 \geq 0$ and $p(n) = p \cdot (1 - \varepsilon_2/a(n))$ for some $\varepsilon_2 \in \mathbb{R}$. But since $p(n) \geq 2\sqrt{q(n)}$ for all n , we must either have $p > 2\sqrt{q}$ or $p = 2\sqrt{q}$ and $\varepsilon_2 \leq \varepsilon_1/2$. In either case, we have $p(n) \geq 2\sqrt{q} \cdot (1 - \varepsilon_1/2a(n))$ for all n sufficiently large. By shifting the indices in (1.2), we can assume that the latter inequality holds for all n and, moreover, that $4q(2) \geq q$.

We show by induction on n that $z_n \geq \sqrt{q}$ for all $n \in \mathbb{N}_+$. For the base case we have $z_1 = p(2) \geq 2\sqrt{q(2)} \geq \sqrt{q}$. For the induction step we have

$$z_{n+1} = p(n+2) - \frac{q(n+2)}{z_n} > 2\sqrt{q} \left(1 - \frac{\varepsilon_1}{2a(n)}\right) - \sqrt{q} \left(1 - \frac{\varepsilon_1}{a(n)}\right) = \sqrt{q}.$$

This completes the induction. This implies that $z_n > 0$ for all $n \in \mathbb{N}_+$, establishing the claim.

D Testing the Initial Ratio

The goal of this section is to prove Proposition 3.7. That is, suppose we are given a solution sequence (u_n) to the recurrence relation (1.2) with real characteristic roots and its associated sequence of consecutive ratios (x_n) . We determine a computable threshold for (x_n) that the sequence crosses if and only if it starts above $x_0 \geq \mu$. An upper bound on the number of steps needed to cross the threshold could be computed, but this number would depend on the distance $|u_1/u_0 - \mu|$, which is unknown. The threshold depends on the monotonicity of the associated sequences of n th characteristic roots.

D.1 Monotonicity and the Characteristic Roots

Consider the recurrence relation (1.2). Recall our earlier definition of the n th characteristic polynomial $p(n, \cdot)$ for the recurrence relation (1.2). For each $x \in \mathbb{R}$, $p(n, x) = a(n)x^2 - b(n)x - c(n)$. For a fixed $n \in \mathbb{N}$, $p(n, \cdot)$ has two real roots λ_n and Λ_n . In this section we break with convention and take $\lambda_n \leq \Lambda_n$.

► **Lemma D.1.** *The sequences (λ_n) and (Λ_n) are eventually monotonic.*

Proof. The polynomial Δ is either the zero polynomial or there exists an $N \in \mathbb{N}$ such that for each $n > N$, $\Delta(n) > 0$. In the former case $\lambda_n = b(n)/(2a(n))$ for each $n \in \mathbb{N}$, and so (λ_n) is eventually monotonic when Δ is the zero polynomial. Let us concentrate on the latter case. Straightforward algebraic manipulations show that

$$\begin{aligned} \lambda_{n+1} - \lambda_n &= \frac{2c(n)}{b(n) + \sqrt{\Delta(n)}} - \frac{2c(n+1)}{b(n+1) + \sqrt{\Delta(n+1)}} \\ &= \frac{2(\gamma_0\beta_1 - \gamma_1\beta_0) + 2(c(n)\sqrt{\Delta(n+1)} - c(n+1)\sqrt{\Delta(n)})}{(b(n) + \sqrt{\Delta(n)})(b(n+1) + \sqrt{\Delta(n+1)})}. \end{aligned}$$

For sufficiently large $n \in \mathbb{N}$, the denominator $(b(n) + \sqrt{\Delta(n)})(b(n+1) + \sqrt{\Delta(n+1)})$ is positive by assumption. It is clear that

$$c(n)\sqrt{\Delta(n+1)} - c(n+1)\sqrt{\Delta(n)} = \frac{c(n)^2\Delta(n+1) + c(n+1)^2\Delta(n)}{c(n)\sqrt{\Delta(n+1)} + c(n+1)\sqrt{\Delta(n)}} \rightarrow \pm\infty$$

as $n \rightarrow \infty$, where the sign depends on the parameters in the recurrence relation. It is then clear that the sequence with terms $\text{sign}(\lambda_{n+1} - \lambda_n)$ is eventually constant. Hence the sequence (λ_n) is eventually monotonic in this case too. An analogous approach can be used to show that (Λ_n) is eventually monotonic. \blacktriangleleft

D.2 A Threshold for Positivity

For a given solution sequence (u_n) of the recurrence (1.2), we have the following results for positivity. Without loss of generality, we shall assume that the sequence (λ_n) is monotonic. We first consider the case where (λ_n) is non-increasing. In the following result we see that, under certain conditions, if there exists $k \in \mathbb{N}$ such that $x_k \geq \lambda_k$ then the sequence (u_n) is positive.

► **Proposition D.2.** *Suppose that (λ_n) is non-increasing, that there exists an $k \in \mathbb{N}$ such that $x_k \geq \lambda_k$ and that $u_0, u_1, \dots, u_k > 0$, then (u_n) is positive.*

Proof. From the assumptions and the recurrence relation for (x_n) , we obtain the following inequalities:

$$a(k+1)x_{k+1} \geq b(k+1) + c(k+1)/\lambda_k \geq a(k+1)\lambda_{k+1}$$

and so $x_{k+1} \geq \lambda_{k+1}$. It follows by induction that $x_n \geq \lambda_n > \lambda$ for all $n > k$. Thus (u_n) is positive. \blacktriangleleft

We obtain a similar threshold for positivity when (λ_n) is increasing.

► **Proposition D.3.** *Suppose that (λ_n) is increasing, $\lambda < \Lambda$, there exists $k \in \mathbb{N}$ such that $x_k \geq \lambda$ and that $u_0, u_1, \dots, u_k > 0$, then (u_n) is positive.*

Proof. We can assume without loss of generality that $\lambda < \Lambda_n$. It can be shown that $\lambda \in [\lambda_n, \Lambda_n)$, and so we have $a(n)\lambda \leq b(n) + c(n)/\lambda$. From this result and the existence of $k \in \mathbb{N}$ such that $x_k \geq \lambda$, we have

$$a(k+1)x_{k+1} \geq b(k+1) + c(k+1)/\lambda \geq a(k+1)\lambda$$

and so $x_{k+1} \geq \lambda$. It follows by induction that $x_n \geq \lambda$ for all $n > k$. Thus (u_n) is positive. \blacktriangleleft

The case when we have a single repeated characteristic root λ is more involved.

► **Proposition D.4.** *Suppose that (u_n) is a solution to recurrence (1.2) and the recurrence has a single repeated characteristic root. Let us assume that (λ_n) is increasing, there exists an $k \in \mathbb{N}$ such that $x_k \geq \sqrt{-c(k+1)/a(k+1)}$ or $x_k \geq b(k)/(2a(k))$, and $u_0, u_1, \dots, u_k > 0$. Then (u_n) is positive.*

Proof. Consider the constant $C = \alpha_0\beta_1 - \alpha_1\beta_0$. We start with the case $C < 0$. In this case the sequence with terms given by $b(n)/a(n)$ is decreasing as

$$\frac{b(n+1)}{a(n+1)} - \frac{b(n)}{a(n)} = \frac{C}{a(n)a(n+1)},$$

and additionally $b(n)/(2a(n)) \geq \lambda$. We obtain the following inequalities using our assumption on $k \in \mathbb{N}$:

$$x_{k+1} \geq \frac{b(k+1)}{a(k+1)} - \frac{\sqrt{-a(k+1)c(k+1)}}{a(k+1)} \geq \frac{b(k+1)}{2a(k+1)} \geq \lambda.$$

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The result in this case follows similarly to the method outlined in Proposition D.3.

Consider the case $C \geq 0$. Let us show that for all $n \geq k + 1$ we have $x_n \geq b(n)/(2a(n))$. We outline the inductive step of the proof. Suppose that $n \geq k + 1$ and assume the inductive hypothesis holds for n . Then

$$x_{n+1} \geq \frac{b(n+1)}{2a(n+1)} + \frac{b(n+1)b(n) + 4c(n+1)a(n)}{2b(n)a(n+1)} \geq \frac{b(n+1)}{2a(n+1)} + \frac{\beta_1\beta_0 + 4\alpha_0\gamma_1}{2b(n)a(n+1)}.$$

As $C \geq 0$, we have that $\beta_0 \geq \beta_1\alpha_0/\alpha_1$. Thus we obtain

$$x_{n+1} \geq \frac{b(n+1)}{2a(n+1)} + \frac{\beta_1\beta_0 + 4\alpha_0\gamma_1}{2b(n)a(n+1)} \geq \frac{b(n+1)}{2a(n+1)},$$

which concludes the induction step. It follows that the sequence (x_n) —and thus (u_n) —remains positive. \blacktriangleleft

Let (μ_n) denote the solution to the recurrence relation (1.2) with initial ratio μ . We have the following:

► **Lemma D.5.** *Let (u_n) be a rational solution to recurrence (1.2) and (x_n) the sequence of consecutive ratios. Suppose that there exists $\varepsilon > 0$ such that $x_0 > \mu + \varepsilon$. Then for each $n \in \mathbb{N}$, we have the following results.*

1. *If (λ_n) is non-increasing, then either $x_n > \mu_n + \varepsilon$ or $x_n \geq \lambda_n$.*
2. *If (λ_n) is increasing, then one of the following occurs: $x_n > \mu_n + \varepsilon$, $x_n \geq \lambda$, $x_n \geq b(n)/(2a(n))$ or $x_n \geq \sqrt{-c_{n+1}/a_{n+1}}$.*

Proof.

1. Suppose that (λ_n) is non-increasing. We proceed by induction. The base case is given by hypothesis. If $x_n \geq \lambda_n$, then, as in the proof of Proposition D.2, $x_{n+1} \geq \lambda_{n+1}$. Similarly, if $x_n \geq \lambda_{n+1}$, we have

$$a(n+1)x_{n+1} = b(n+1) + \frac{c(n+1)}{x_n} \geq b(n+1) + \frac{c(n+1)}{\lambda_{n+1}} \geq a(n+1)\lambda_{n+1}$$

and so $x_{n+1} \geq \lambda_{n+1}$. Otherwise, we have the following inequalities:

$$x_{n+1} - \mu_{n+1} = \frac{c(n+1)}{x_n a(n+1)} - \frac{c(n+1)}{\mu_n a(n+1)} > \frac{-c(n+1)\varepsilon}{a(n+1)x_n \mu_n} > \frac{-c(n+1)\varepsilon}{a(n+1)\lambda_{n+1}^2} > \varepsilon.$$

The last inequality holds since $p(n+1, \sqrt{-c(n+1)/a(n+1)}) < 0$ when $\Delta(n+1) \geq 0$.

2. Suppose that (λ_n) is increasing. The respective inductive proofs for the $x_n \geq \lambda$, $x_n \geq b(n)/2a(n)$ and $x_n \geq \sqrt{-c(n+1)/a(n+1)}$ follow by Propositions D.3 and D.4. Otherwise, as before, we have:

$$x_{n+1} - \mu_{n+1} > \frac{-c(n+1)\varepsilon}{a(n+1)x_n \mu_n} \geq \varepsilon.$$

The proof is complete. \blacktriangleleft

In Summary, since the sequence (μ_n) converges to λ , it follows from Lemma D.5 that the positivity of a solution sequence (u_n) is determined by one the threshold crossings given in Propositions D.2, D.3 and D.4. One can thus detect that $x_0 > \mu$ by computing an initial number of terms in the sequence (x_n) . On the one hand, this algorithm is guaranteed to terminate. On the other hand though the number of steps does not have an upper bound.

E Complex Characteristic Roots

We study the recurrence relation $a(n)u_n = b(n)u_{n-1} + c(n)u_{n-2}$ under the assumption that the discriminant $\Delta(n) = b(n)^2 + 4a(n)c(n)$ of the recurrence relation is negative for each $n \in \mathbb{N}$. We do not lose any generality by making this assumption: in the case that $(\Delta(n))$ is eventually negative we can shift the indices. It immediately follows that $\text{sign}(c(n)) = -1$, as $\text{sign}(a(n)) = 1$. Our aim is to establish Proposition 3.8. We first show using elementary methods no positive solutions exist in this setup. We then proceed to show that no minimal solution exists by analysing the asymptotics of the solutions in a manner similar to Appendix B.

E.1 No Positive Solutions

Our main goal is to prove the following proposition.

► **Proposition E.1.** *Suppose that for every $n \in \mathbb{N}$ the discriminant $\Delta(n)$ of the recurrence (1.2) is negative. Then no rational solution (u_n) of this recurrence is positive.*

In fact, our methods generalise to polynomials having degree larger than one. We relax the degree 1 assumption on the polynomials a, b, c in recurrence (1.2). Herein, we assume that $\deg(a) = d \geq 1$ has degree $d \geq 1$ and that $\deg(b), \deg(c) \leq d$. We shall use the notation $a(n) = \sum_{j=0}^d \alpha_j n^j$, $b(n) = \sum_{j=0}^d \beta_j n^j$, and $c(n) = \sum_{j=0}^d \gamma_j n^j$. As before, we shall assume that the polynomials have constant sign and that $\text{sign}(a(n)) = 1$. We use λ_n and Λ_n to denote the roots of the polynomial $p(n, x) = a(n)x^2 - b(n)x - c(n)$. The root sequences converge as $n \rightarrow \infty$ to the characteristic roots

$$\frac{\beta_d \pm \sqrt{\beta_d^2 + 4\alpha_d \gamma_d}}{2\alpha_d},$$

which we denote by λ and Λ . We shall assume, without loss of generality, that $\Delta(n) = b(n)^2 + 4a(n)c(n) < 0$ for each $n \in \mathbb{N}$. It follows that $\gamma_d < 0$. Thus the n th characteristic roots λ_n, Λ_n are a complex conjugate pair for each $n \in \mathbb{N}$.

Given a solution (u_n) , we define the sequence (x_n) by $x_n = u_{n+1}/u_n$. Notice that the sequence satisfies $a(n+1)x_n = b(n+1) + c(n+1)/x_{n-1}$. For each $n \in \mathbb{N}$, consider the function $f_n: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ given by $f_n(x) = a(n)x - b(n) - c(n)/x$. Observe that f_n is continuous and has no real roots since $p(n, x)$ has no such roots. Furthermore, we have $\lim_{x \rightarrow 0^+} f_n(x) = \lim_{x \rightarrow \infty} f_n(x) = \infty$. We thus conclude that for each $n \in \mathbb{N}$, f_n is a strictly positive function on $(0, \infty)$ as it has no real roots.

► **Lemma E.2.** *We have that $x_n = x_{n-1} - f_n(x_{n-1})/a(n)$ for each $n \in \{1, 2, \dots\}$. Moreover, we have $x_n = x_0 - \sum_{j=1}^n f_j(x_{j-1})/a(j)$. Thus, if (x_n) is a positive sequence, then it is strictly decreasing.*

Proof. Substitution shows that $f_n(x_{n-1}) = a(n)(x_{n-1} - x_n)$, and further we have

$$x_n - x_0 = \sum_{j=1}^n x_j - x_{j-1} = - \sum_{j=1}^n f_j(x_{j-1})/a(j).$$

It is now evident that the sequence (x_n) is strictly decreasing since both $a(j)$ and $f_j(x_{j-1})$ are strictly positive for each $j \in \mathbb{N}$ when assuming that (x_n) is positive. ◀

In the following proposition we give sufficient conditions for the sequence (x_n) to contain a negative element. Under such conditions we deduce that (u_n) is not positive.

► **Proposition E.3.** Suppose that (u_n) is a solution to the recurrence relation

$$a(n)u_n = b(n)u_{n-1} + c(n)u_{n-2}$$

with $d = \deg(a) = \deg(c) \geq \deg(b)$. Assume further that $\Delta(n) < 0$. We have the following:

1. If $\beta_d^2 + 4\alpha_d\gamma_d < 0$, then (x_n) contains a negative element.
2. If $\beta_d^2 + 4\alpha_d\gamma_d = 0$, and $x_0 \leq \sqrt{-\gamma_d/\alpha_d}$ then (x_n) contains a negative element.

Proof. Assume, for the sake of contradiction that (x_n) contains only positive elements. Then by Lemma E.2, (x_n) is a strictly decreasing sequence and so either (x_n) decreases without bound or converges to a limit. In the former case, the sequence necessarily attains a negative value, which is a contradiction. In the latter case, we note that a limit of (x_n) is a solution to the limiting recurrence relation $\alpha_d x - \beta_d - \gamma_d/x = 0$, which has solutions λ and Λ . In the case $\beta_d^2 + 4\alpha_d\gamma_d < 0$ neither λ nor Λ is real and so (x_n) must decrease until it attains a negative value. In the case $\beta_d^2 + 4\alpha_d\gamma_d = 0$, the roots coincide so that $\lambda = \Lambda = \sqrt{-\gamma_d/\alpha_d}$. Thus if $\beta_d^2 + 4\alpha_d\gamma_d = 0$ and, in addition, $x_0 \leq \sqrt{-\gamma_d/\alpha_d}$, then (x_n) cannot converge and so decreases until reaching a negative element. ◀

We have the immediate consequence:

► **Corollary E.4.** For a recurrence relation as in Proposition E.3(1), no positive solutions exist.

In the remainder of this section we restrict our attention to the recurrence relation (1.2) which has affine coefficients rather than general polynomial coefficients. Subject to this restriction, we can improve Proposition E.3 to obtain a proof of Proposition E.1—we do not require the additional assumption that $x_0 \leq \sqrt{-\gamma_d/\alpha_d}$ when $\beta_d^2 + 4\alpha_d\gamma_d = 0$.

In the work that follows it is convenient to write $f_n(x) = g_0(x)/x + g_1(x)n/x$ where $g_j(x)/x = \alpha_j x - \beta_j - \gamma_j/x$ for $j = 0, 1$. Before we prove Proposition E.1, we make a few analytic observations about the function $f_j(x_{j-1}) = g_0(x_{j-1})/x_{j-1} + g_1(x_{j-1})j/x_{j-1}$. Since $\deg(a) = \deg(b) = \deg(c) = 1$ and $\Delta(n) < 0$, then necessarily $\gamma_1 < 0$. We note that the two functions $g_0(x)/x$ and $g_1(x)/x$ are differentiable and non-negative in the domain $\{x \in \mathbb{R} : x > 0\}$. To show that the functions are non-negative we note that

$$\lim_{x \rightarrow 0^+} g_k(x)/x = \lim_{x \rightarrow \infty} g_k(x)/x = \infty,$$

$g_0(x)/x$ has no real roots (for otherwise $p(0, \cdot)$ possesses the same real root), and $g_1(x)/x$ has a single real root if $\beta_1^2 + 4\alpha_1\gamma_1 = 0$ and no root otherwise. By continuity, it follows that there exists an $\varepsilon_0 > 0$ such that for all $x \in \mathbb{R}$, $g_0(x)/x > \varepsilon_0$.

Proof of Proposition E.1. Assume that the sequence (u_n) is positive, so that the sequence (x_n) is positive. Since $g_0(x)/x$ and $g_1(x)/x$ are both non-negative on the domain $\{x \in \mathbb{R} : x > 0\}$ and, in addition, there exists an $\varepsilon_0 > 0$ such that $g_0(x)/x > \varepsilon_0$, we have that $f_j(x_{j-1}) = g_0(x_{j-1})/x_{j-1} + g_1(x_{j-1})j/x_{j-1} > \varepsilon_0$ too. We combine this uniform bound and Lemma E.2 to obtain

$$x_n = x_0 - \sum_{j=1}^n \frac{f_j(x_{j-1})}{a(j)} \leq x_0 - \sum_{j=1}^n \frac{\varepsilon_0}{\alpha_1 j + \alpha_0}.$$

By comparison to the harmonic series we deduce that the sequence (x_n) decreases without bound and so attains a negative value contrary to what was assumed. It follows that (u_n) is not positive. ◀

► **Example E.5** (Comment on Propositions E.3 and E.1). Consider the recurrence equation

$$n^3 u_n = 2n^3 u_{n-1} - (n^3 + 1)u_{n-2}. \quad (\text{E.1})$$

It is straightforward to check that there is a single repeated characteristic root $\lambda = 1$ and $\Delta(n) < 0$ for each $n \in \mathbb{N}$.

By Proposition E.3(2), if $x_0 \leq 1$ then (u_n) is not positive. However, this condition is not necessary. Suppose that (v_n) is a solution to recurrence (E.1) such that $v_0 = 1$ and $v_1 = 17/16$. Then $v_1/v_0 > 1$, but $v_2 = 1$. By Proposition E.3(2), (v_{n+1}) is not positive, and so (v_n) is not positive.

Let us consider the solution (u_n) to equation (E.1) with initial conditions $u_0 = 1$ and $u_1 = 3/2$. We will show inductively that for each $j \in \{1, 2, \dots\}$, $0 \leq u_j - u_{j-1} \leq u_1 - u_0$. From this claim, the solution (u_n) is increasing and since $u_0 = 1$, we deduce that (u_n) is positive. By induction,

$$u_n - u_{n-1} = u_{n-1} - u_{n-2} - \frac{u_{n-2}}{n^3} = u_1 - u_0 - \sum_{k=0}^{n-2} \frac{u_k}{(k+2)^3} \leq u_1 - u_0.$$

Then, by substitution, $u_k = u_0 + \sum_{j=1}^k (u_j - u_{j-1}) \leq u_0 + k(u_1 - u_0)$. Hence

$$u_n - u_{n-1} \geq u_1 - u_0 - \sum_{k=0}^{n-2} \frac{u_0 + k(u_1 - u_0)}{(k+2)^3}.$$

We collect like terms together to obtain

$$u_n - u_{n-1} \geq u_1 \left(1 - \sum_{k=0}^{n-2} \frac{k}{(k+2)^3} \right) + u_0 \left(-1 + \sum_{k=0}^{n-2} \frac{k-1}{(k+2)^3} \right).$$

Let us prove that the right-hand side of the last inequality is non-negative for each $n \in \mathbb{N}$ —in which case (u_n) is increasing—when $u_0 = 1$ and $u_1 = 3/2$. For each $n \in \{2, 3, \dots\}$,

$$\frac{1 - \sum_{k=0}^{n-2} \frac{k-1}{(k+2)^3}}{1 - \sum_{k=0}^{n-2} \frac{k}{(k+2)^3}} \leq \frac{1 + \frac{1}{8} - \frac{1}{64}}{1 - \sum_{k=0}^{\infty} \frac{k}{(k+2)^3}} = \frac{213}{32(12\zeta(3) - \pi^2)} < 1.462.$$

It follows that (u_n) is increasing if $u_0 = 1$ and $u_1 = 3/2$, and so (u_n) is positive.

E.2 No Minimal Solutions

We now prove the second result in Proposition 3.8. The result is as follows. Given a recurrence relation where $\Delta(n) < 0$ for sufficiently large $n \in \mathbb{N}$, no solution sequence is minimal. We note that the normalisation process in Appendix B is completely general and so we apply Lemma B.2 in our current case. Recall, by Lemma B.2, that there is a minimal solution (u_n) to recurrence (1.2) if and only if there is minimal solution (w_n) to recurrence (B.2). It thus suffices to show that, subject to the assumption that $\Delta(n) < 0$ for all sufficiently large $n \in \mathbb{N}$, there are no minimal solutions to the recurrence (B.2).

Let us write recurrence (B.2) in the form of (B.3). The proof of the following result is, like before, based on Kooman's results in [13].

► **Proposition E.6.**

1. Suppose that $A < -1$. Then recurrence (B.3) has two linearly independent solutions satisfying

$$w_n^{(1)} \sim (1 + \sqrt{1+A})^n n^{\frac{B}{2\sqrt{1+A}+2(1+A)}}; \quad w_n^{(2)} \sim (1 - \sqrt{1+A})^n n^{\frac{B}{-2\sqrt{1+A}+2(1+A)}}.$$

2. Suppose that $A = 0$ and $B < 0$. Then recurrence (B.3) has two linearly independent solutions satisfying

$$w_n^{(1)} \sim n^{\frac{1}{4}-B/2} e^{2\sqrt{B}n}; \quad w_n^{(2)} \sim n^{\frac{1}{4}-B/2} e^{-2\sqrt{B}n}.$$

► **Corollary E.7.** Given recurrence (B.3), suppose that $\Delta(n) < 0$ for all sufficiently large n . Then no solution to the recurrence is minimal.

Proof. Observe that $\Delta(n) = \beta_1^2 ((1+A)n^2 + (\beta + A(\alpha + \gamma))n + \beta^2/4 + A\alpha\gamma)$. It is now evident that $\Delta(n) < 0$ for sufficiently large $n \in \mathbb{N}$ if and only if $A < -1$ or $A = -1$ and $B < 0$. Notice that in the event $A = -1$ and $B = 0$, we have $\Delta(n) = (\alpha - \gamma)^2 \geq 0$. We consider the following two cases.

1. Suppose that $A < -1$. Let $1 + \sqrt{1+A} = \lambda$ and $\Lambda = \bar{\lambda} = 1 - \sqrt{1+A}$. We use the shorthand $\sigma \pm i\tau$ with $\sigma, \tau \in \mathbb{R}$ for $\frac{B}{2(1+A) \pm 2\sqrt{1+A}}$. We first show that $w_n^{(1)}$ and $w_n^{(2)}$ are asymptotically incomparable. Assume the contrary, so that $\lim_{n \rightarrow \infty} w_n^{(1)}/w_n^{(2)} = c$ for some $c \in \mathbb{C}$. Then, by Proposition E.6, we have

$$c = \lim_{n \rightarrow \infty} \frac{w_n^{(1)}}{w_n^{(2)}} = \lim_{n \rightarrow \infty} \frac{\lambda^n n^{\sigma+i\tau}}{\Lambda^n n^{\sigma-i\tau}} = \lim_{n \rightarrow \infty} \left(\frac{\lambda}{\Lambda} \right)^n n^{2\tau i}.$$

Observe now that $\rho_n := \left(\frac{\lambda}{\Lambda} \right)^n n^{2\tau i}$ has modulus 1 for each $n \in \mathbb{N}$, implying that $|c| = 1$. In particular, $c \neq 0$. We have now reached the following contradiction:

$$1 = \frac{c}{c} = \lim_{n \rightarrow \infty} \frac{\rho_{n+1}}{\rho_n} = \lim_{n \rightarrow \infty} \frac{\lambda}{\Lambda} \left(\frac{n+1}{n} \right)^{2\tau i} = \frac{\lambda}{\Lambda}.$$

We now show that there exist no minimal solutions. Indeed, take any solution (w_n) defined by $w_n = aw_n^{(1)} + bw_n^{(2)}$ for some $a, b \in \mathbb{C}$. Without loss of generality, we assume that $a \neq 0$. Then $w_n^{(2)}$ and w_n are linearly independent, but

$$\frac{w_n}{w_n^{(2)}} = a \frac{w_n^{(1)}}{w_n^{(2)}} + b$$

which does not converge to 0 due to our choice of a . Thus there are no minimal solutions.

2. Suppose that $A = -1$ and $B < 0$. Again, by Proposition E.6, we find

$$\frac{w_n^{(1)}}{w_n^{(2)}} \sim e^{4\sqrt{|B|}ni}.$$

This ratio clearly does not converge, which shows that the two solutions are asymptotically incomparable. We conclude the claim similar to the previous case. ◀

F A Characterisation of Minimality: Repeated Characteristic Root

In this section we assume that relation (1.2) has a single repeated characteristic root $\lambda = \beta_1/2\alpha_1$. This situation occurs when $\beta_1^2 = -4\alpha_1\gamma_1$. We necessarily have that $\gamma_1 < 0$ as we assume throughout that $\alpha_1 > 0$. Let us introduce the normalised sequence (v_n) with terms given by $v_n = \lambda^{-n}u_n$. Observe that (v_n) is a rational sequence if and only if (u_n) is a rational sequence. We omit the proof of the following lemma.

► **Lemma F.1.** *If (u_n) is a solution to the recurrence (1.2) with repeated characteristic root λ , then (v_n) satisfies the recurrence*

$$(n + \alpha)v_n = (2n + \beta)v_{n-1} - (n + \gamma)v_{n-2}, \quad (\text{F.1})$$

where $\alpha = \alpha_0/\alpha_1$, $\beta = 2\beta_0/\beta_1$, and $\gamma = \gamma_0/\gamma_1$.

We make the following three observations. First, each of the parameters α , β , and γ in (F.1) are positive because $\text{sign}(a(n))$, $\text{sign}(b(n))$, and $\text{sign}(c(n))$ are constant. Second, (u_n) is a minimal solution of (1.2) if and only if (v_n) is a minimal solution of (F.1). Third, let μ_u and μ_v denote the critical ratios of (1.2) and (F.1), respectively. Then it is clear that $\mu_u = \lambda\mu_v$. In the remainder of this section we identify minimal solutions of (F.1).

Using the same approach discussed in Subsection 4.2, we associate both a generating function \mathcal{F} given by $\mathcal{F}(x) = \sum_{n=0}^{\infty} v_n x^{n+\frac{\alpha_0}{\alpha_1}}$ and the differential equation

$$\mathcal{F}'(x) + \left(\frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} \right) \mathcal{F}(x) = q(x)$$

to recurrence (F.1). Here $A_1 = \gamma + 2 - \alpha$, $A_2 = \alpha - \beta + \gamma$, and

$$q(x) = \frac{(\alpha + 1)u_1 + \alpha u_0 x^{-1} - (\beta + 2)u_0}{(x-1)^2} x^\alpha.$$

Thus for $|x| < 1$ we have the following integral equation:

$$\mathcal{F}(x) = |x-1|^{-A_1} \exp\left(\frac{A_2}{x-1}\right) \int_0^x |t-1|^{A_1} \exp\left(\frac{A_2}{1-t}\right) q(t) dt. \quad (\text{F.2})$$

► **Lemma F.2.** *Suppose that $A_2 < 0$. Then (v_n) is a minimal solution of (F.1) if and only if $\int_0^1 (1-t)^{A_1} \exp(A_2(1-t)^{-1}) q(t) dt = 0$.*

Proof. Let us assume that $A_2 = \alpha - \beta + \gamma < 0$. Then it is clear that $\lim_{t \rightarrow 1-} (1-t)^{A_1} \exp(A_2(1-t)^{-1}) q(t) = 0$. Thus the following limit exists

$$\lim_{x \rightarrow 1-} \int_0^x (1-t)^{A_1} \exp\left(\frac{A_2}{1-t}\right) q(t) dt = \int_0^1 (1-t)^{A_1} \exp\left(\frac{A_2}{1-t}\right) q(t) dt.$$

By Lemma 4.1 (2), we have that $\mathcal{F}(x)$ converges as $x \rightarrow 1-$ if and only if (v_n) is a minimal solution. It is clear that $\mathcal{F}(x)$ is defined at $x = 1$ only if the integral vanishes. For otherwise, the divergence of $|x-1|^{-A_1} \exp(A_2(x-1)^{-1})$ as $x \rightarrow 1-$ would force $\mathcal{F}(x)$ to diverge too. This concludes the proof. ◀

We shall resume our inspection of $\mathcal{F}(x)$ in the interval $[0, 1]$ in the case that $A_2 = 0$ in the above, so that the expression for \mathcal{F} simplifies to

$$\mathcal{F}(x) = (1-x)^{-A_1} \int_0^x (1-t)^{A_1} q(t) dt. \quad (\text{F.3})$$

We proceed as in the case of distinct roots, namely, we observe that the integrand may be written in the form $\mathcal{I}(x)q(x) = (1-x)^{A_1-2}Q(x)$, where Q is analytic in an open interval I containing 1. In fact, we may write $Q(x) = \sum_{n=0}^{\infty} T_n(1-x)^n$ for all $x \in [0, 2)$.

► **Lemma F.3.** *Assume that $\alpha > \gamma + 1$, i.e., $A_1 < 1$. Then (v_n) is a minimal solution of (F.1) if and only if $v_1/v_0 = \frac{\gamma+2}{\alpha+1}$.*

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Proof. Let $x \in [0, 1)$. Then

$$\begin{aligned} \int_0^x \mathcal{I}(t)q(t) dt &= \int_0^x (1-t)^{A_1-2} \sum_{n=0}^{\infty} T_n(1-t)^n dt \\ &= \int_0^x \sum_{n=0}^{\infty} T_n(1-t)^{n+A_1-2} dt \\ &= G(x) - G(0), \end{aligned}$$

where

$$G(x) = \sum_{\substack{n=0 \\ n \neq 1-A_1}}^{\infty} \frac{-T_n}{n+A_1-1} (1-x)^{n+A_1-1} + g(x).$$

In the above $g(x) = \chi_{\mathbb{N}}(1-A_1) \log(1-x)^{T_1-A_1}$ where $\chi_{\mathbb{N}}$ is the characteristic function of \mathbb{N} . We now find

$$\begin{aligned} \mathcal{F}(x) &= (1-x)^{-A_1} (G(x) - G(0)) \\ &= (1-x)^{-A_1} \sum_{\substack{n=0 \\ n \neq 1-A_1}}^{\infty} \frac{T_n}{n+A_1-1} (1-x)^{n+A_1-1} + \frac{g(x)}{(1-x)^{A_1}} - \frac{G(0)}{(1-x)^{A_1}} \\ &= \frac{1}{1-x} \sum_{\substack{n=0 \\ n \neq 1-A_1}}^{\infty} \frac{T_n}{n+A_1-1} (1-x)^n + \frac{g(x)}{(1-x)^{A_1}} - \frac{G(0)}{(1-x)^{A_1}}. \end{aligned}$$

Consider the antiderivative of \mathcal{F} given by

$$\frac{T_0}{1-A_1} \log(1-x) + \sum_{\substack{n=0 \\ n \neq 1-A_1}}^{\infty} \frac{T_n}{(n+A_1-1)(n+1)} (1-x)^{n+1} + h(x) + G(0)(1-x)^{1-A_1},$$

where

$$h(x) = \begin{cases} 0 & \text{if } 1-A_1 \notin \mathbb{N}, \text{ and} \\ -T_{1-A_1} \frac{(1-x)^{1-A_1} ((1-A_1) \log(1-x) - 1)}{(1-A_1)^2} & \text{otherwise.} \end{cases}$$

Since $1-A_1 > 0$, it is clear that the above antiderivative of \mathcal{F} converges as $x \rightarrow 1-$ if and only if $T_0 = 0$. However,

$$\begin{aligned} T_0 = Q(1) &= \lim_{x \rightarrow 1-} q(x)(x-1)^2 \\ &= \lim_{x \rightarrow 1-} ((\alpha+1)u_1 - (\alpha+\gamma+2)u_0)x^\alpha + \alpha u_0 x^{\alpha-1} \\ &= (\alpha+1)u_1 - (\gamma+2)u_0. \end{aligned}$$

The claim now follows by Lemma 4.1(3). ◀

Proof of Proposition 4.4. The first claim follows from Lemma F.2. Assume then that $A_2 = 0$. In this case we have $(n+\alpha) = (2n+\beta) - (n+\gamma)$, implying that constant sequence $(1, 1, \dots)$ is a solution to the recurrence. We shall compare other solutions to this one. In the case $\alpha > \gamma+1$, we have that (v_n) is a minimal solution if and only if $v_1/v_0 = \frac{\gamma+2}{\alpha+1}$ by the above lemma. In the case that $\alpha \leq \gamma+1$, the sequence $(1, 1, \dots)$ is minimal by Proposition B.1(3) and (4). Thus (v_n) is a minimal solution if and only if $v_1/v_0 = 1$. Combining these two cases gives the claim. ◀