Logic II
(LGIC 320 / MATH 571 / PHIL 412)
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Lectures 1–5: Propositional Intuitionistic Logic

$\boxed{\text{Lecture 1, Jan 12}}$

1. Propositional Intuitionistic Calculus

Propositional formulae are built from a countable set of *propositional variables* $\mathrm{Var} = \{p, q, r, \ldots\}$ and the *falsity constant* $\bot$ using three binary connectives: $\to$ *(implication)*, $\wedge$ *(conjunction,* or logical "and"), $\vee$ *(disjunction,* or logical "or").

Note that in this formulation we haven't included *negation* as an official logical operation. Instead of this, $\neg A$ ("not A") is considered as a shortcut for $(A \to \bot)$.

Intuitionistic propositional logic, Int, is defined by the following axioms:

1. $A \to (B \to A)$
2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
3. $(A \wedge B) \to A$
4. $(A \wedge B) \to B$
5. $A \to (B \to (A \wedge B))$
6. $A \to (A \vee B)$
7. $B \to (A \vee B)$
8. $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
9. $\bot \to A$

and one inference rule:

$$\frac{A \quad A \to B}{B}$$

called *modus ponens* ("MP" for short).

Adding the 10th axiom, $A \vee \neg A$ (*tertium non datur,* or the law of excluded middle), to Int yields classical propositional logic, CL.

Note that all these axioms are actually *axiom schemata:* one can substitute arbitrary formulae for the meta-variables A, B, C, obtaining *instances* of axioms. For example, $(p \vee q) \to ((q \to r) \to (p \vee q))$ is an instance of Ax. 1 (with $A = (p \vee q)$ and $B = (q \to r)$).

This is a Hilbert-style calculus. The rules and axioms have clear motivation, but practical derivation can be painful:

Example 1. Derive $E \to E$.

The derivation is as follows:

(1)	$(E \to ((E \to E) \to E)) \to ((E \to (E \to E)) \to (E \to E))$	Ax. 2 with $A = C = E$ and $B = (E \to E)$
(2)	$E \to ((E \to E) \to E)$	Ax. 1 with $A = E$, $B = (E \to E)$
(3)	$(E \to (E \to E)) \to (E \to E)$	MP from (2) and (1)
(4)	$E \to (E \to E)$	Ax. 1 with $A = B = E$
(5)	$E \to E$	MP from (4) and (3)

Formally speaking, a *derivation* is a linearly ordered list of formulae, and each of them is either an instance of an axiom or is obtained from earlier formulae using the MP rule. If there exists a derivation ending with formula B, then B is called *derivable* (denoted by $\vdash_{\mathrm{Int}} B$). We also consider derivations from *hypotheses:* let Γ be a set of formulae, and we allow them to appear in derivations, along with axioms of Int. If B is derivable using Γ, we write $\Gamma \vdash_{\mathrm{Int}} B$.

2. Deduction Theorem

Theorem 1 (Deduction Theorem)**.** *Let Γ be an arbitrary finite set of formulae. Then $\Gamma, A \vdash_{\mathrm{Int}} B$ if and only if $\Gamma \vdash_{\mathrm{Int}} A \to B$.*

Proof. The *if* part is just an application of MP: from Γ we derive $A \to B$, and then combine it with the given A yielding B.

For the *only if* part, proceed by induction on the derivation of B from $\Gamma \cup \{A\}$ in Int. The possible cases for B are as follows.

Case 1: B is an axiom of Int or $B \in \Gamma$. Then B is also derivable from Γ, and we obtain $A \to B$ by applying MP to B and $B \to (A \to B)$ (an instance of Ax. 1).

Case 2: $B = A$. Then $B \to A$ (actually $A \to A$) is derivable, see Example 1.

Case 3: B is obtained from previously derived C and $C \to B$ by MP. Then, by induction, $\Gamma \vdash_{\mathrm{Int}} A \to C$ and $\Gamma \vdash_{\mathrm{Int}} A \to (C \to B)$. Then we proceed as follows:

(1)	$A \to C$	
(2)	$A \to (C \to B)$	
(3)	$(A \to (C \to B)) \to ((A \to C) \to (A \to B))$	an instance of Ax. 2
(4)	$(A \to C) \to (A \to B)$	MP from (2) and (3)
(5)	$A \to B$	MP from (1) and (4)

□

The Deduction Theorem makes deriving much simpler:

Example 2. $\vdash_{\mathrm{Int}} (A \wedge B) \to (B \wedge A)$

By Deduction Theorem (with an empty Γ), it is sufficient to establish $A \wedge B \vdash_{\mathrm{Int}} B \wedge A$. This is done in the following way:

(1)	$A \wedge B$	
(2)	$(A \wedge B) \to A$	an instance of Ax. 3
(3)	A	MP from (1) and (2)
(4)	$(A \wedge B) \to B$	an instance of Ax. 4
(5)	B	MP from (1) and (4)
(6)	$B \to (A \to (B \wedge A))$	an instance of Ax. 5
(7)	$A \to (B \wedge A)$	MP from (5) and (6)
(8)	$B \wedge A$	MP from (3) and (7)

Actually, the Deduction Theorem is an ouverture for another formalism, called the calculus of *natural deduction* (we'll discuss it later).

3. BHK Semantics

Before going further, let's discuss some intuitions on which intuitionistic logic is based. We start with an informal interpretation, called *BHK-semantics* (due to Brouwer, Heyting, and Kolmogorov). Under this interpretation, a formula is considered *valid* ("intuitionistically true"), if it is *justified* by something. The question of what a *justification,* or *witness* actually is, is now left unanswered (there are several approaches, and we'll discuss them later). However, witnessess operate with logical operations in the following way:

- a witness for $A_1 \wedge A_2$ is a pair $\langle u_1, u_2 \rangle$, where u_1 is a witness for A_1 and u_2 is a justification for A_2;
- a witness for $A_1 \vee A_2$ is a pair $\langle i, u \rangle$, where either $i = 1$ and u is a witness for A_1, or $i = 2$ and u is a witness for A_2;
- a witness for $A \to B$ defines a function f that transforms any witness for A into a witness for B (if x justifies A, then $f(x)$ should justify B);
- there is no witness for $\bot$.

It's quite easy to see that all axioms of Int and the MP rule are adequate to BHK. On the other hand, $A \vee \neg A$ isn't: to justify it, you should either justify A or justify $\neg A$. However, there exists statements such that neither A nor $\neg A$ is known to be true. Due to the informal nature of BHK, this doesn't actually show that one can't derive, say, $p \vee \neg p$ in Int. This can be done either by analyzing derivations (but not in a Hilbert-style calculus), or using a formal semantics, such as Kripke's possible worlds semantics.

4. Kripke Semantics

A *Kripke model* is a triple $\mathcal{M} = \langle W, R, v \rangle$, where W a non-empty set of *possible worlds,* R is a preorder (i.e., a reflexive and transitive relation) on W, and $v\colon \mathrm{Var} \times W \to \{0, 1\}$ is the *variable valuation function.* The function v is required to be *monotonic* w.r.t. R: if xRy, then $v(p, x) \leq v(p, y)$ for any $p \in \mathrm{Var}$. In other words, if $v(p, x) = 1$ and xRy, then $v(p, y) = 1$.

By $R(x)$ we denote the set $\{y \mid xRy\}$.

In different worlds, different formulae are considered true. If formula A is true in world x of $\mathcal{M}$, we write $\mathcal{M}, x \Vdash A$; $\Vdash$ is called the *forcing relation* and defined as follows:

- $\mathcal{M}, x \nVdash \bot$ (falsity is never true);
- $\mathcal{M}, x \Vdash p$ iff $v(p, x) = 1$ (truth of variables is prescribed by the v function);
- $\mathcal{M}, x \Vdash A \wedge B$ iff $\mathcal{M}, x \Vdash A$ and $\mathcal{M}, x \Vdash B$ (conjunction is computed classically);
- $\mathcal{M}, x \Vdash A \vee B$ iff $\mathcal{M}, x \Vdash A$ or $\mathcal{M}, x \Vdash B$ (so is disjunction);
- $\mathcal{M}, x \Vdash A \to B$ iff for every $y \in R(x)$ either $\mathcal{M}, y \nVdash A$ or $\mathcal{M}, y \Vdash B$.

These definition is designed (especially in the implication case) to preserve monotonicity of forcing: if $\mathcal{M}, x \Vdash A$ and xRy, then $\mathcal{M}, y \Vdash A$.

If the Kripke model has only one world ($|W| = 1$), then it is a model for classical propositional logic.

Intuitionistic propositional logic is sound w.r.t. Kripke semantics:

Theorem 2. *If $\vdash_{\mathrm{Int}} A$, then for every Kripke model $\mathcal{M} = \langle W, R, v\rangle$ and for every possible world $x \in W$ of this model $\mathcal{M}, x \Vdash A$.*

Proof. In order to prove soundness, one needs to prove two things: (1) if A is an axiom of Int, then $\mathcal{M}, x \Vdash A$; (2) if $\mathcal{M}, x \Vdash A$ and $\mathcal{M}, x \Vdash A \to B$, then $\mathcal{M}, x \Vdash B$ (forcing in $\mathcal{M}$ is closed under application of modus ponens).

The (2) part is easy: if $x \Vdash A \to B$, then for every world $y \in R(x)$ we have either $y \nVdash A$ or $y \Vdash B$. Take $y = x$ (x is in $R(x)$ by reflexivity of R). Then, given $x \Vdash A$, we obtain $x \Vdash B$.

For the (1) part, one needs to check all the 9 axioms. It is time-consuming, but technical. Let's try one of the most complicated axioms, Ax. 2.

We need to prove $x \Vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$. In order to establish that a formula of the form $E \to F$ is true in x, one needs to check that for every $y \in R(x)$ if $y \Vdash E$, then $y \Vdash F$. Consider an arbitrary $y \in R(x)$, such that $y \Vdash A \to (B \to C)$. We need to prove that $y \Vdash (A \to B) \to (A \to C)$. Again, consider an arbitrary $z \in R(y)$, such that $z \Vdash A \to B$. On this turn, we need to show that $z \Vdash A \to C$. Let w be a world from $R(z)$, such that $w \Vdash A$ and finally we need $w \Vdash C$. Now the picture is as follows (we omit arrows that come from transitivity and reflexivity, such as xRx or xRz):

$$\begin{tikzcd}
\bullet\, w \Vdash A \\
\bullet\, z \Vdash A \to B \arrow[u] \\
\bullet\, y \Vdash A \to (B \to C) \arrow[u] \\
\bullet\, x \arrow[u]
\end{tikzcd}$$

By monotonicity, since yRw and zRw, the formulae $A \to (B \to C)$ and $A \to B$ are also true in w. Since modus ponens is applicable for $\Vdash$, we have $w \Vdash B \to C$, $w \Vdash B$, and finally $w \Vdash C$, which is our goal.

Other axioms of Int are checked similarly. We leave it as an exercise. □

Using this soundness theorem, one can prove that a formula is not derivable in Int.

Example 3. $\nvdash_{\mathrm{Int}} p \vee \neg p$

This formula is classically valid, therefore we should use more than one Kripke world to falsify it. Fortunately, two worlds are already sufficient. Let $W = \{x, y\}$, xRy (and, of course, xRx and yRy, but not yRx). Then let $v(p, x) = 0$ and $v(p, y) = 1$.

$$\begin{tikzcd}
\bullet\, y \Vdash p \\
\bullet\, x \nVdash p,\ x \nVdash \neg p \arrow[u]
\end{tikzcd}$$

In this model, neither $x \Vdash p$, nor $x \Vdash \neg p$ (because p is true in $y \in R(x)$). Thus, $p \vee \neg p$ is not true in x and therefore is not derivable in Int.

Lecture 2, Jan 17

5. Kripke Completeness

In this section we prove the converse of Theorem 2, the *completeness theorem.*

Theorem 3. *If a formula is true in every possible world of any Kripke model, then it is derivable in* Int.

We proceed by contraposition. Let A be a formula such that $\nvdash_{\text{Int}} A$. We construct a *countermodel* for A, that is, a model $\mathcal{M}$ that contains a world x, such that $\mathcal{M}, x \nVdash A$. In fact, we'll construct one model, that acts as a countermodel for all non-derivable formulae. This will be the *canonical model* for Int, denoted by $\mathcal{M}_0$.

Definition. A set Γ of formulae is called a *disjunctive theory*, if

1. Γ is *deductively closed*, i.e., if $\Gamma \vdash_{\text{Int}} B$, then $B \in \Gamma$;
2. Γ is *consistent*, i.e., $\Gamma \nvdash_{\text{Int}} \bot$;
3. Γ is *disjunctive*, i.e., if $\Gamma \vdash_{\text{Int}} A \vee B$, then $\Gamma \vdash_{\text{Int}} A$ or $\Gamma \vdash_{\text{Int}} B$.

Definition. The *canonical model* for Int is the model $\mathcal{M}_0 = \langle W_0, R_0, v_0 \rangle$, where

- W_0 is the set of all disjunctive theories, ← restrict those to the subsets of subformulas of the input formula φ
- R_0 is the subset relation ($\Gamma_1 R_0 \Gamma_2 \iff \Gamma_1 \subseteq \Gamma_2$),
- v_0 is defined as follows: $v_0(p, \Gamma) = 1 \iff p \in \Gamma$.

The main property of $\mathcal{M}_0$ is that disjunctive theories, as worlds of $\mathcal{M}_0$, force the same formulae that they derive, as theories over Int:

Lemma 4. $\mathcal{M}_0, \Gamma \Vdash B \iff B \in \Gamma.$

This lemma is sometimes called the *Main Semantic Lemma*.

Now let A be a formula that is not derivable in Int. To prove that $\mathcal{M}_0$ is a countermodel for A, it is sufficient to construct a disjunctive theory that doesn't include A. In classical logic, we would take $\{\neg A\}$ and extend it to a complete (disjunctive) theory. However, in intuitionistic logic, $\{\neg A\}$ could be actually inconsistent:

Example 4. Let $A = p \vee \neg p$. Then (see Example 3) $\nvdash_{\text{Int}} A$. On the other hand, $\vdash_{\text{Int}} \neg\neg(p \vee \neg p)$ (exercise!), and therefore $\neg A \vdash_{\text{Int}} \bot$, i.e., $\{\neg A\}$ is inconsistent.

Still, we need a way to control that A doesn't get accidentally included into the theory while we extend it. So, we consider *pairs* of sets of formulae. Intuitively, in a pair (Γ, Δ) Γ is the *positive* part (actually, the theory), and Δ is the *negative* part (formulae which we want to prevent from being included into Γ).

Definition. A pair (Γ, Δ) is called "*consistent,*" if there are no such $G_1, \ldots, G_n \in \Gamma$ and $D_1, \ldots, D_k \in \Delta$, that

$$\vdash_{\mathrm{Int}} G_1 \wedge \ldots \wedge G_n \to D_1 \vee \ldots \vee D_k.$$

Important particular cases are $n = 0$ and $k = 0$. The empty conjunction is $\top = \neg\bot$, and the empty disjunction is $\bot$. Thus, $(\Gamma, \varnothing)$ is consistent iff Γ is consistent as a theory ($\Gamma \nvdash_{\mathrm{Int}} \bot$), and $(\varnothing, \Delta)$ is consistent iff no disjunction of formulae from Δ is derivable in Int. Also, if (Γ, Δ) is consistent, then $\Gamma \nvdash_{\mathrm{Int}} \bot$.

Consistency means that the negative part doesn't follow from the positive one.

Definition. A consistent pair (Γ, Δ) is called *complete*, if for each formula B either $B \in \Gamma$ or $B \in \Delta$. In other words, complete pairs a consistent pairs of the form $(\Gamma, \mathrm{Fm} - \Gamma)$.

Disjunctive theories and complete pairs are in a one-to-one correspondence:

Lemma 5. *1. If (Γ, Δ) is a complete pair, then Γ is a disjunctive theory.*

2. If Γ is a disjunctive theory, then $(\Gamma, \mathrm{Fm} - \Gamma)$ is a complete pair.

Proof. 1. Since (Γ, Δ) is consistent, then Γ is consistent (as a theory). Let $\Gamma \vdash_{\mathrm{Int}} B$. Then B cannot be in Δ (this would violate consistency: take for $G_1, \ldots, G_n$ the formulae from Γ that occur in the derivation—there is a finite number of them—and apply Deduction Theorem). Therefore, by completeness, $B \in \Gamma$. This means Γ is deductively closed.

Now let $\Gamma \vdash_{\mathrm{Int}} B \vee C$. We need to prove that $\Gamma \vdash_{\mathrm{Int}} B$ or $\Gamma \vdash_{\mathrm{Int}} C$. Suppose the contrary. Then $B, C \in \Delta$. But this violates consistency (take $n = 1$, $k = 2$, $G_1 = B \vee C$, $D_1 = B$, $D_2 = C$). Therefore Γ is disjunctive.

2. We need to show that $(\Gamma, \mathrm{Fm} - \Gamma)$ is consistent (then it is complete by definition). Suppose the contrary: $\vdash_{\mathrm{Int}} G_1 \wedge \ldots \wedge G_n \to D_1 \vee \ldots \vee D_k$. Let $G = G_1 \wedge \ldots \wedge G_n$. Since Γ is deductively closed and of course $\Gamma \vdash_{\mathrm{Int}} G$, $G \in \Gamma$. Then, by Deduction Theorem $\Gamma \vdash_{\mathrm{Int}} D_1 \vee \ldots \vee D_k$. Since Γ is disjunctive, we have $\Gamma \vdash_{\mathrm{Int}} D_i$ for some i (formally, we have to proceed by induction on k). But then $D_i \in \Gamma$. Contradiction. $\square$

Lemma 6. *If (Γ, Δ) is a consistent pair, then there exists a complete pair (Γ', Δ'), such that $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$.*

Proof. Enumerate all formulae: $B_1, B_2, \ldots$, and add them one by one into either Γ or Δ. It is sufficient to show that the next formula B_i can be added to at least one side without making the pair inconsistent. If not, then we have

$$\vdash_{\mathrm{Int}} G_1 \wedge \ldots \wedge G_n \wedge B_i \to D_1 \vee \ldots \vee D_k \qquad \text{and} \qquad \vdash_{\mathrm{Int}} G_1 \wedge \ldots \wedge G_n \to D_1 \vee \ldots \vee D_k \vee B_i.$$

(We can always choose the same G_i's and D_j's, because we can weaken the statements by adding new stuff from Γ and Δ.) Then (exercise!) by Deduction Theorem we can deduce $\vdash_{\mathrm{Int}} G_1 \wedge \ldots \wedge G_n \to D_1 \vee \ldots \vee D_k$. But we suppose that the pair was consistent before adding B_i. Contradiction. $\square$

The process of extending a consistent pair into a complete one is called *saturation.*

Now we're ready to prove Lemma 4.

Proof of Lemma 4. Induction on the structure of B.

1. B is a variable. By definition of v_0.

2. $B = \bot$. Then $\mathcal{M}_0, \Gamma \nVdash \bot$ (by definition of forcing) and $\bot \notin \Gamma$ (since Γ is consistent).

3. $B = B_1 \vee B_2$. Then $\Gamma \Vdash B_1 \vee B_2$ iff $\Gamma \Vdash B_1$ or $\Gamma \Vdash B_2$ iff $B_1 \in \Gamma$ or $B_2 \in \Gamma$ iff $(B_1 \vee B_2) \in \Gamma$. The second step is by induction, and the third one is due to the disjunctiveness of Γ.

4. $B = B_1 \overset{\wedge}{\cancel{\vee}} B_2$. Proceed as in the $\vee$ case. The last step holds since Γ is deductively closed (use axioms for $\wedge$).

5. $B = C \to D$. The most interesting case. Let $(C \to D) \in \Gamma$. Then for any $\Gamma' \in R_0(\Gamma)$ we also have $(C \to D) \in \Gamma'$ (since $R_0 = \subseteq$). Then if $C \in \Gamma'$, then $D \in \Gamma'$ (Γ' is closed under modus ponens). By induction this means that if $\Gamma' \Vdash C$, then $\Gamma' \Vdash D$, for any $\Gamma' \in R_0(\Gamma)$. Therefore, $\Gamma \Vdash C \to D$ (by definition of forcing).

   Now let $(C \to D) \notin \Gamma$. We need to show that $\Gamma \nVdash C \to D$, i.e. to construct such $\Gamma' \in R_0(\Gamma)$ that $\Gamma' \Vdash C$ and $\Gamma' \nVdash D$. By induction this means $C \in \Gamma'$ and $D \notin \Gamma'$. Consider the pair $(\Gamma \cup \{C\}, \{D\})$. This pair is consistent: otherwise $\vdash_{\mathrm{Int}} G_1 \vee \ldots \vee G_n \vee C \to D$, and by Deduction Theorem $\Gamma \vdash_{\mathrm{Int}} C \to D$, and this is not the case by our assumption. Therefore, by Lemma 6 there exists a complete pair (Γ', Δ'), such that $\Gamma \cup \{C\} \subseteq \Gamma'$ and $\{D\} \subseteq \Delta'$. Then Γ' is the disjunctive theory we actually need: $\Gamma \subseteq \Gamma'$ (i.e. $\Gamma R_0 \Gamma'$), $C \in \Gamma'$, and $D \notin \Gamma'$.

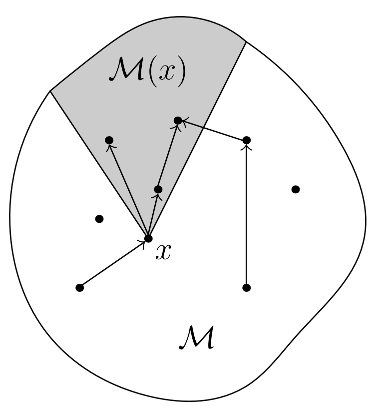
□

Now we can finish the proof of Theorem 3. Let $\nvdash_{\mathrm{Int}} A$. Then the pair $(\varnothing, \{A\})$ is consistent, and by Lemma 6 there exists a complete pair (Γ, Δ), such that $A \in \Delta$. Therefore, $A \notin \Gamma$, and finally $\mathcal{M}_0, \Gamma \nVdash A$ (by Lemma 4).

6. Disjunctive Property

If a Kripke model has a minimal element (i.e., such x_0, that $x_0 R x$ for all $x \in W$, or, in other words, $W = R(x_0)$), then this element is called the *root* of the model.

Since the definition of forcing in a world $x \in W$ depends only on worlds from $R(x)$, the same formulae will remain true in x if we remove all the worlds not from $R(x)$. The part of $\mathcal{M}$ that is left is called the *cone* with root x, and is denoted by $\mathcal{M}(x)$.



Thus, if a formula A is false in a world x of model $\mathcal{M}$, then it is also false in the root of the model $\mathcal{M}(x)$. In other words, if a formula A is not derivable in Int, then there exists a Kripke model with a root such that A is false in its root.

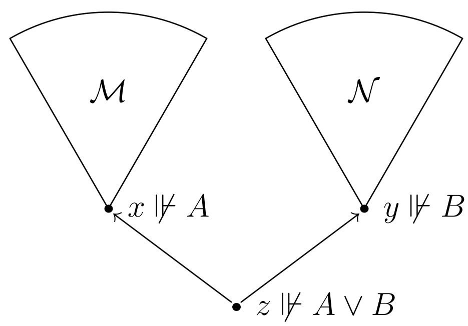
Now we're ready to prove an interesting property of intuitionistic disjunction that supports its BHK understanding:

Theorem 7 (Disjunctive Property)**.** *If* $\vdash_{\mathrm{Int}} A \vee B$*, then* $\vdash_{\mathrm{Int}} A$ *or* $\vdash_{\mathrm{Int}} B$*.*

(The converse also holds trivially, due to the axioms $A \to (A \vee B)$ and $B \to (A \vee B)$.)

Disjunctive property is invalid for CL: for example, $\vdash_{\mathrm{CL}} p \vee \neg p$, but neither $\vdash_{\mathrm{CL}} p$, nor $\vdash_{\mathrm{CL}} \neg p$. In fact, it supports the constructive reading of disjunction: *to prove a disjunction means to choose one of the disjuncts and prove it.*

Proof of Theorem 7. Suppose the contrary: $\nvdash_{\mathrm{Int}} A$ and $\nvdash_{\mathrm{Int}} B$. Then, due to Theorem 3, there exist Kripke models $\mathcal{M}$ and $\mathcal{N}$ and worlds x and y such that $\mathcal{M}, x \nVdash A$ and $\mathcal{N}, y \nVdash B$. As noticed above, we can assume that x is the root of $\mathcal{M}$ and y is the root of $\mathcal{N}$. Also we suppose that the sets of worlds of $\mathcal{M}$ and $\mathcal{N}$ do not intersect. Then we can join these two models in the following way:



We add a new root, z. In order to maintain monotonicity of v, we declare all variables to be false in z. Then, by monotonicity of forcing, $z \nVdash A$ and $z \nVdash B$. Hence, $z \nVdash A \vee B$, and therefore $\nvdash_{\mathrm{Int}} A \vee B$ by Theorem 2. ☐

Disjunctive property actually means that the "empty" theory without any non-logical axioms, namely, $\Theta = \{A \mid \ \vdash_{\mathrm{Int}} A\}$, is a disjunctive theory. Moreover, every disjunctive theory Γ includes Θ (because Γ is deductively closed and therefore includes all theorems of Int). This means that Θ is the root of the canonical model $\mathcal{M}_0$, and the canonical model has the following universality property: $\vdash_{\mathrm{Int}} A$ iff $\mathcal{M}_0, \Theta \Vdash A$ (a formula is derivable in Int if and only if it is true in the root of the canonical model).

Lecture 3, Jan 19

7. Finite Model Property

The canonical model $\mathcal{M}_0$ constructed above is infinite. However, for every formula that is not derivable in Int there exists a *finite* countermodel.

Theorem 8. *A formula is derivable in* Int *if and only if it is true in all finite models.*

Can we build the filtered model directly?

Proof. If $\nvdash_{\mathrm{Int}} A$, then $\mathcal{M}_0, \Theta \nVdash A$. Let $\Phi = \mathrm{SubFm}(A)$ be the set of all *subformulae* of A. Note that Φ is finite. The definition of forcing for A refers only to formulae from Φ, therefore, if two worlds force the same formulae from Φ, we can consider them equivalent and join them into one world.

disjunctive theories

To formalize this idea, we define an equivalence relation on W_0: $x \sim_\Phi y$ iff for any formula $A \in \Phi$ we have $x \Vdash A \iff y \Vdash A$. It is easy to see that $\sim_\Phi$ is indeed an equivalence relation (i.e., it is transitive, reflexive, and symmetric). Now we identify equivalent worlds. This procedure is called *filtration* of the model $\mathcal{M}_0$. We define a new model $\mathcal{M}_0/{\sim_\Phi} = \langle W_0/{\sim_\Phi}, \bar{R}, v\rangle$. The new set of worlds $W_0/{\sim_\Phi}$ is the set of *equivalence classes* of worlds from W_0 w.r.t. $\sim_\Phi$. The equivalence class of $x \in W_0$ is the set $[x]_{\sim_\Phi} = \{y \mid y \sim_\Phi x\}$; $x_1 \sim_\Phi x_2 \iff [x_1]_{\sim_\Phi} = [x_2]_{\sim_\Phi}$. Further we omit the subscript in the notation for $[x]$.

Now, $[x]\bar{R}[y]$ iff $x \Vdash B$ implies $y \Vdash B$ for every $B \in \Phi$. Note that, since in equivalent worlds the same formulae from Φ are true, this definition does not depend on what particular elements we take from $[x]$ and $[y]$: if $[x'] = [x]$ and $[y'] = [y]$, then the implication $x' \Vdash B \Rightarrow y' \Vdash B$ is equivalent to the implication $x \Vdash B \Rightarrow y \Vdash B$.

The new relation $\bar{R}$ is reflexive and transitive by definition.

The new variable valuation, v, is defined as $v(p, [x]) = v_0(p, x)$ for $p \in \Phi$ (for such variables all worlds from $[x]$ have the same v_0 valuation); variables not from Φ are declared to be always false, to maintain monotonicity.

The filtered model $\mathcal{M}_0/{\sim_\Phi}$ is finite (since there is only a finite number of possible valuations for formulae from Φ) and preserves forcing for formulae from Φ:

$$\mathcal{M}_0, x \Vdash B \iff \mathcal{M}_0/{\sim_\Phi}, [x] \Vdash B \qquad\qquad \text{if } B \in \Phi.$$

This statement is checked by induction on the structure of B (exercise!). By applying it to A, we get that $\mathcal{M}_0/{\sim_\Phi}, [\Theta] \nVdash A$, which is our goal. $\square$

Finite model property yields *algorithmic decidability* of intuitionistic propositional logic:

Theorem 9. Int *(more precisely, the set $\Theta = \{A \mid\ \vdash_{\mathrm{Int}} A\}$) is decidable.*

Proof. We run two algorithms in parallel: one generates all possible derivations, trying to prove A; the other generates all possible finite Kripke models, trying to find a countermodel. Due to Theorem 8, one of these algorithms succeeds. Say "yes" if it is the first one, and "no" if it is the second one. $\square$

Lectures 4 & 5, Jan 24, 26

8. Finite-Valued Logics and Intuitionistic Logic

Recall the two-world Kripke model that we used to falsify $p \vee \neg p$: $\overset{\bullet}{\underset{\bullet}{\uparrow}}$. In this frame, each formula A can have three possible valuations:

$$\begin{tikzcd}
\bullet\, y \nVdash A & \bullet\, y \Vdash A & \bullet\, y \Vdash A \\
\bullet\, x \nVdash A \arrow[u] & \bullet\, x \nVdash A \arrow[u] & \bullet\, x \Vdash A \arrow[u]
\end{tikzcd}$$

(The fourth possibility, $x \Vdash A$ and $y \nVdash A$, violates the monotonicity constraint.)

Let's denote these valuations by 0, $1/2$, and 1 respectively. Since the valuation of a complex formula is determined by valuations of its subformulae (maybe in different worlds), we can use "truth tables" instead of the Kripke frame here. For example, if $\bar{v}(A) = 1$ and $\bar{v}(B) = 1/2$, then $\bar{v}(A \to B) = 1/2$: indeed, we have $x \Vdash A$, $y \Vdash A$, $x \nVdash B$, and $y \Vdash B$, therefore $A \to B$ is true in y and false in x. The complete truth tables are as follows[1]:

for $A \to B$

$A \backslash B$	0	$1/2$	1
0	1	1	1
$1/2$	0	1	1
1	0	$1/2$	1

for $A \wedge B$

$A \backslash B$	0	$1/2$	1
0	0	0	0
$1/2$	0	$1/2$	$1/2$
1	0	$1/2$	1

for $A \vee B$

$A \backslash B$	0	$1/2$	1
0	0	$1/2$	1
$1/2$	$1/2$	$1/2$	1
1	1	1	1

Since $\bar{v}(\bot) = 0$ and $\neg A$ is an abbreviation for $(A \to \bot)$, the negation enjoys the following truth table:

A	$\neg A$
0	1
$1/2$	0
1	0

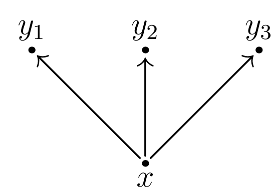
(By the way, thus $p \vee \neg p$ is invalid here, since for $v(p) = 1/2$ we have $\bar{v}(p \vee \neg p) = 1/2 \neq 1$.)

A formula A is a *"3-valued tautology"* if $\bar{v}(A) = 1$ for any valuation of variables (or, in other words, if it is true in any Kripke model based on our two-world frame). Trivially, every formula that is derivable in Int is a 3-valued tautology.

The converse, however, doesn't hold. Consider the formula

$$I_3 = (p_0 \leftrightarrow p_1) \vee (p_0 \leftrightarrow p_2) \vee (p_0 \leftrightarrow p_3) \vee (p_1 \leftrightarrow p_2) \vee (p_1 \leftrightarrow p_3) \vee (p_2 \leftrightarrow p_3).$$

This formula is a 3-valued tautology: we have 4 variables (p_0, p_1, p_2, p_3) and 3 possible truth values, therefore for any valuation v at least two variables, p_i and p_j, receive the same truth value (by the pigeon-hole principle). Then $\bar{v}(p_i \leftrightarrow p_j) = 1$ and $\bar{v}(I_3) = 1$. On the other hand, there is a Kripke model that falsifies I_3. Consider the following frame:



and let p_i be true only in y_i for $i = 1, 2, 3$; p_0 is false in all worlds. Then y_1 falsifies $(p_0 \leftrightarrow p_1)$, $(p_1 \leftrightarrow p_2)$, and $(p_1 \leftrightarrow p_3)$, y_2 falsifies $(p_0 \leftrightarrow p_2)$ and $(p_2 \leftrightarrow p_3)$, and y_3 falsifies $(p_0 \leftrightarrow p_3)$. Hence, all 6 disjuncts are false in x (by monotonicity), and therefore $x \nVdash I_3$ and $\nvdash_{\mathrm{Int}} I_3$.

[1] They correspond to the RM$_3$ logic introduced by B. Sobociński.

We shall generalize this argument to show that Int does not coincide with any finite-valued logic. As a corollary, we establish that there is no finite universal Kripke model or frame for Int (since in a finite frame the set of possible valuations for variables/formulae is also finite).

To do this, we first formulate the notion of a *finite-valued logic* more accurately. A *k-valued semantic frame* is a tuple $\mathcal{F} = \langle V, T, \ominus, \owedge, \ovee, \obot \rangle$, where V is a k-element set of *truth values*, $T \subset V$ is the set of truth values declared as "true", $\obot \in V$ is the interpretation for the falsity constant, and $\ominus, \owedge, \ovee \colon V \times V \to V$ are binary operations on V ("truth tables").

As usually, the valuation function $v \colon \mathrm{Var} \to V$ is defined arbitrarily on variables and then propagated to all formulae:

- $\bar{v}(p) = v(p)$ for $p \in \mathrm{Var}$;
- $\bar{v}\bot = \obot$;
- $\bar{v}(A \to B) = \bar{v}(A) \ominus \bar{v}(B)$;
- $\bar{v}(A \wedge B) = \bar{v}(A) \owedge \bar{v}(B)$;
- $\bar{v}(A \vee B) = \bar{v}(A) \ovee \bar{v}(B)$.

A formula A is a k-valued tautology w.r.t. $\mathcal{F}$ if $\bar{v}(A) \in T$ for any valuation v. The set of all tautologies is the *logic* of $\mathcal{F}$:

$$\mathrm{Log}(\mathcal{F}) = \{A \mid \bar{v}(A) \in T \text{ for all } v \text{ on } \mathcal{F}\}.$$

Note that we don't impose any specific restrictions on $\mathcal{F}$: we don't require $\owedge$ and $\ovee$ to be commutative, associative, and mutually distributive, we don't suppose that $\ominus$ obeys modus ponens, we even allow $\obot$ to belong to T. This enables some degenerate cases: if $T = V$, then $\mathrm{Log}(F)$ includes all formulae and defines the *logic of contradiction*; if $T = \varnothing$, the logic is empty. The more interesting cases include CL (with $V = \{0, 1\}$, $\obot = 0$, and $\ominus$, $\owedge$, $\ovee$ defined by classic truth tables) and a lot of well-known many-valued logics (see the "Many-Valued Logic" article of the Stanford Encyclopedia of Philosophy for examples).

Theorem 10. *There is no such k-valued semantic frame $\mathcal{F}$, that*

$$\{A \mid \ \vdash_{\mathrm{Int}} A\} = \mathrm{Log}(\mathcal{F}).$$

In other words, Int *is not a k-valued logic for any finite k.*

Proof. Suppose the contrary: let Int be the logic of some $\mathcal{F} = \langle V, T, \ominus, \owedge, \ovee, \obot \rangle$.

We call $a \in T$ *useless,* if there are no such k-valued tautology $A \in \mathrm{Log}(\mathcal{F})$ and valuation $v \colon \mathrm{Var} \to V$ that $a = \bar{v}(A)$ (in other words, this element of T is never used for establishing that something is a tautology). Then removing a from T doesn't change the logic. Further (for technical reasons) we suppose that T doesn't include useless elements.

Let

$$I_k = \bigvee_{0 \le i < j \le k} (p_i \leftrightarrow p_j).$$

Now it is sufficient to prove two facts:

1. $I_k \in \mathrm{Log}(\mathcal{F})$;

2. $\nvdash_{\mathrm{Int}} I_k$.

The proof of the second fact is a straightforward generalization of the argument above for I_3 (we construct a Kripke model with a root and k incomparable worlds visible from it, one for each variable $p_1, \ldots, p_k$; p_0 is never true).

The first fact, however, is essentially non-trivial, because truth tables of $\mathcal{F}$ are arbitrary, and it is true only in the presupposition that the logic of $\mathcal{F}$ coincides with Int and that T doesn't contain useless elements. To establish that I_k is a k-valued tautology w.r.t. $\mathcal{F}$, we prove the following two statements:

1. $(a \ominus a) \in T$ for every $a \in V$ (here $b \ominus c$ is a shortcut for $(b \ominus c) \otimes (c \ominus b)$; clearly $\bar{v}(B \leftrightarrow C) = \bar{v}(B) \ominus \bar{v}(C)$);

2. if $a \in T$ or $b \in T$, then $a \otimes b \in T$.

For the first statement we notice that, since $\vdash_{\mathrm{Int}} p \leftrightarrow p$ and the logic of $\mathcal{F}$ is Int, $\bar{v}(p \leftrightarrow p) = v(p) \ominus v(p) \in T$ for any valuation v. Then let $v(p) = a$.

The second statement is a bit trickier. Suppose that $a \in T$ (the $b \in T$ case is symmetric). Since T doesn't contain useless elements, $a = \bar{v}(\tilde{A})$ for some k-valued tautology $\tilde{A}$. Being a k-valued tautology w.r.t. $\mathcal{F}$, $\tilde{A}$ is derivable in Int. Now let q be a fresh variable, so we can define $v(q)$ arbitrarily not affecting the valuation of $\tilde{A}$. Let $v(q) = b$. The formula $\tilde{A} \vee q$ is also derivable in Int (by modus ponens with the $\tilde{A} \to (\tilde{A} \vee q)$ axiom). Hence, $\bar{v}(\tilde{A} \vee q) = \bar{v}(\tilde{A}) \otimes v(q) = a \otimes b \in T$.

Now we've accumulated enough good properties of $\mathcal{F}$ to show that I_k is a k-valued tautology w.r.t. $\mathcal{F}$. Indeed, since we have $k+1$ variables ($p_0, p_1, \ldots, p_k$), at least two of them receive the same truth value: $v(p_i) = v(p_j) = a \in T$. Due to our first statement, $\bar{v}(p_i \leftrightarrow p_j) = a \ominus a \in T$. Then we apply the second statement many times to propagate this to the whole disjunction and get $\bar{v}(I_k) \in T$, therefore $I_k \in \mathrm{Log}(\mathcal{F})$. Contradiction. □

9. Embedding CL into Int

At the first glance, Int is a subsystem of CL (everything provable in Int is also provable in CL, but not vice versa). Using only CL, however, one cannot distinguish intuitionistically valid formulae; in fact, the opposite holds: there are formula translations faithfully mapping into a fragment of Int. We present some of them here.

The **Gödel – Gentzen negative translation** A^N of formula A is defined recursively as follows:

- $p^N = \neg\neg p$ for $p \in \mathrm{Var}$;
- $\bot^N = \bot$;
- $(A \wedge B)^N = A^N \wedge B^N$;
- $(A \vee B)^N = \neg(\neg A^N \wedge \neg B^N)$;
- $(A \to B)^N = A^N \to B^N$.

Theorem 11. *For any formula A,*

$$\vdash_{\mathrm{CL}} A \quad \textit{iff} \quad \vdash_{\mathrm{Int}} A^N.$$

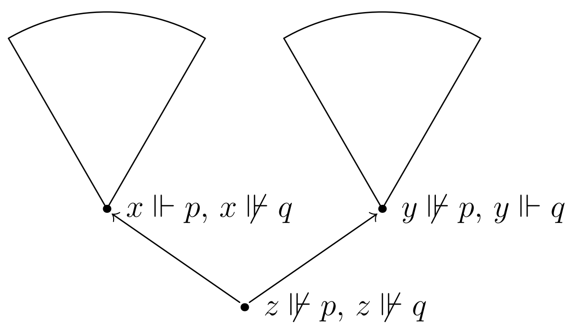
The right-to-left direction is obvious: $\vdash_{\mathrm{Int}} A^N$ implies $\vdash_{\mathrm{CL}} A^N$, and in CL the formulae A^N and A are equivalent, due to the double negation principle and one of de Morgan laws.

For the opposite direction, we proceed by contraposition and use Kripke models. Let $\nvdash_{\mathrm{Int}} A$. Then there exists a countermodel $\mathcal{M}_0$ with root x_0 such that $\mathcal{M}_0, x_0 \nVdash A$. Now we use the following key lemma:

Lemma 12. *Let $\mathcal{M}$ be a model with root x and let B be an arbitrary formula. Then there exists a world y such that any subformula C of B has the same truth value in all worlds from $\mathcal{M}(y)$, and for the formula B itself this truth value coincides with the truth value of B^N in the root world x.*

This lemma, being applied to A and $\mathcal{M}_0$, immediately yields the main result. Since for every subformula of A its truth value is the same for all worlds in the cone $\mathcal{M}_0(y)$, the valuation for these formulae is actually computed classically, according to truth tables. Therefore, since A^N is false in the root world x_0, this valuation assigns "false" to A. Therefore, $\nvdash_{\mathrm{CL}} A$.

In Lemma 12, the positive case, when B^N is true in x, is indeed expected, since the truth of B^N is propagated to the whole model $\mathcal{M}$ by monotonicity, and it looks plausible that B should also be widely true. The negative case, however, is interesting, since for formulae not of the form B^N this generally doesn't hold. For example, consider the following model:



Here $p \vee q$ is false in the root but is true in both cones on top. The Gödel – Gentzen translation for disjunction in de Morgan style rules out such branching situations.

Proof of Lemma 12. Proceed by structural induction on B.

1. $B = p \in \mathrm{Var}$ and $x \Vdash B^N = \neg\neg p$. Then $x \nVdash \neg p$, and therefore there exists a world $y \in R(x)$ such that $y \Vdash p$. By monotonicity, p is true in the whole cone $\mathcal{M}(y)$.

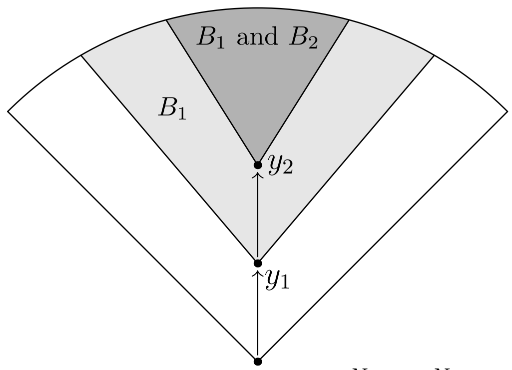
2. $B = p \in \mathrm{Var}$ and $x \nVdash B^N = \neg\neg p$. Then there exists a world y such that $y \Vdash \neg p$. By definition of forcing for negation, p is false in the whole cone $\mathcal{M}(y)$.

3. $B = \bot$ and $x \Vdash B^N = \bot$. Impossible, since $\bot$ is never true.

4. $B = \bot$ and $x \nVdash B^N = \bot$. Take $y = x$: $B = \bot$ is false everywhere and this coincides with the truth value of B^N in the root.

5. $B = B_1 \wedge B_2$ and B^N is true in x. By definition, $B^N = B_1^N \wedge B_2^N$, and both B_1^N and B_2^N are true in x. By induction hypothesis, there exists a world y_1 such that in $\mathcal{M}(y_1)$ for every subformula C of B_1 is either true everywhere or false everywhere, and B_1 itself is true (since $x \Vdash B_1^N$). Now, by monotonicity, $y_1 \Vdash B_2^N$. Therefore we can apply induction hypothesis

once more and obtain a worls $y_2 \in R(y_1)$ such that in the submodel $\mathcal{M}(y_2)$ our statement holds *both* for subformulae of B_1 *and* B_2, and therefore for all subformulae of B. Let $y = y_2$. Since B_1 and B_2 are both true everywhere in $\mathcal{M}(y)$, so is $B = B_1 \wedge B_2$.



6. $B = B_1 \wedge B_2$ and $B^N = B_1^N \wedge B_2^N$ is false in x. Then either B_1^N or B_2^N is false in x. Let it be B_1^N. Apply induction hypothesis to B_1^N and obtain a cone $\mathcal{M}(y_1)$ in our statement holds for all subformulae of B_1, and B_1 itself is false. Now we again go into a subcone $\mathcal{M}(y_2)$ to stabilize truth values for subformulae of B_2. The truth value of B_2 itself doesn't matter, because the falsity of B_1 already falsifies $B = B_1 \wedge B_2$.

7. $B = B_1 \vee B_2$ and $B^N = \neg(\neg B_1^N \wedge \neg B_2^N)$ is true in x. Then $x \nVdash \neg B_1^N \wedge \neg B_2^N$, and therefore either $\neg B_1^N$ or $\neg B_2^N$ is false in x. Let it be $\neg B_1^N$. Then there exists a world $y_1 \in R(x)$ such that $y_1 \Vdash B_1^N$. By induction hypothesis there is a world $y_2 \in R(y)$ such that $y_2 \Vdash B_1$ and in all worlds of $\mathcal{M}(y_2)$ subformulae of B_1 have the same truth value. Applying induction hypothesis once again, we stabilize also subformulae of B_2 in a subcone $\mathcal{M}(y)$ for $y \in R(y_2)$. The truth value of B_2 doesn't matter, because B_1 is sufficient to make $B_1 \vee B_2$ true.

8. $B = B_1 \vee B_2$ and $B^N = \neg(\neg B_1^N \wedge \neg B_2^N)$ is false in x. Then there exists a world $y_1 \in R(x)$ such that $y_1 \Vdash \neg B_1^N \wedge \neg B_2^N$, so *both* $\neg B_1^N$ and $\neg B_2^N$ are true in this world[2]. Now we proceed exactly as in Case 5, applying the induction hypothesis first for B_1^N, then for B_2^N (by monotonicity, $\neg B_2^N$ remains true, therefore B_2^N remains false when going upwards). Thus we obtain a world y such that $\mathcal{M}(y)$ satisfies the statement of the lemma for B_1 and B_2 (and, therefore, for $B_1 \vee B_2$), and $B_1 \vee B_2$ is false in all worlds of $\mathcal{M}(y)$.

9. $B = B_1 \to B_2$ and $B^N = B_1^N \to B_2^N$ is true in x. Consider two subcases:

   - B_1^N is false in x. Then, by induction hypothesis, there exists a cone $\mathcal{M}(y_1)$ such that in all worlds of this cone B_1 is false, and all subformulae of B_1 get the same truth values in all worlds of this cone. Then $B_1 \to B_2$ is true (ex falso) everywhere in $\mathcal{M}(y_1)$. Then we apply the induction hypothesis to B_2 to stabilize truth values of its subformulae. The truth value of B_2 itself doesn't matter, since if B_1 is false, $B_1 \to B_2$ is always true.

[2] This is the crucial difference of the Gödel – Gentzen translation for disjunction from the original disjunction. In Int, if $A \vee B$ is not true, A and B can be falsified in *different* worlds. Here we guarantee that there exists a cone (due to monotonicity) that falsifies A and B *simultaneously.*

- B_1^N is true in x. Then, by monotonicity, it is true everywhere, and so is B_2^N. Now we proceed exactly as in Case 5.

10. $B = B_1 \to B_2$ and $B^N = B_1^N \to B_2^N$ is false in x. Then there exists a world y_1 such that $y_1 \Vdash B_1^N$ and $y_1 \nVdash B_2^N$. Apply the induction hypothesis first to B_2: we get a cone $\mathcal{M}(y_2)$ (where $y_2 \in R(y_1)$), satisfying the statement for B_2 and where B_2 is false in all worlds. By monotonicity, B_1^N is still true in y_2. Applying the induction hypothesis to B_2 now, we get such a world $y \in R(y_2)$ that subformulae of B_1 (and, by previous reasoning, of B_2 also) get the same truth values in all worlds of $\mathcal{M}(y)$, and, moreover, B_1 is true and B_2 is false in these worlds. Thus, in all worlds of $\mathcal{M}(y)$ the formula $B = B_1 \to B_2$ is false.

$\square$

The Gödel – Gentzen negative translation can be generalized to *theories* over CL and Int. For an arbitrary theory (set of formulae) Γ, let $\Gamma^N = \{A^N \mid A \in \Gamma\}$.

Theorem 13. *For any theory Γ and formula B,*

$$\Gamma \vdash_{\mathrm{CL}} B \qquad \textit{iff} \qquad \Gamma^N \vdash_{\mathrm{Int}} B^N.$$

Proof. As in Theorem 11, the implication from right to left is obvious.

Now let $\Gamma \vdash_{\mathrm{CL}} B$. Since the derivation is finite, in this derivation we use only a finite subtheory[3] $\Gamma_0 \subset \Gamma$. Let $\bigwedge \Gamma_0$ be the conjunction of all formulae from Γ_0. Then, applying Deduction Theorem and axioms for $\wedge$, we get

$$\vdash_{\mathrm{CL}} \bigwedge \Gamma_0 \to B.$$

By Theorem 11,

$$\vdash_{\mathrm{Int}} \left(\bigwedge \Gamma_0 \to B\right)^N.$$

Since the Gödel – Gentzen translation commutes with $\wedge$ and $\to$, $(\bigwedge \Gamma_0 \to B)^N$ is graphically equal to $\bigwedge \Gamma_0^N \to B^N$. By applying modus ponens and axioms for $\wedge$, we get $\Gamma_0^N \vdash_{\mathrm{Int}} B^N$, and since $\Gamma_0^N \subset \Gamma^N$, we obtain our goal: $\Gamma^N \vdash_{\mathrm{Int}} B^N$. $\square$

The Gödel – Gentzen negative translation is not the only method of embedding CL into Int. A simpler translation is given by **Glivenko's theorem:**

Theorem 14 (Glivenko)**.** *For any formula A,*

$$\vdash_{\mathrm{CL}} A \qquad \textit{iff} \qquad \vdash_{\mathrm{Int}} \neg\neg A.$$

The proof is left as an exercise (*hint:* use the finite model property).

Glivenko's theorem also yields faithfullness of the following **Kolmogorov double-negation translation:**

- $p^{\neg\neg} = \neg\neg p$;
- $\bot^{\neg\neg} = \neg\neg\bot$;

[3] This is an instance of the *compactness* argument.

- $(A \land B)^{\neg\neg} = \neg\neg(A^{\neg\neg} \land B^{\neg\neg})$;
- $(A \lor B)^{\neg\neg} = \neg\neg(A^{\neg\neg} \lor B^{\neg\neg})$;
- $(A \to B)^{\neg\neg} = \neg\neg(A^{\neg\neg} \to B^{\neg\neg})$;

In this translation, every subformula gets decorated with $\neg\neg$.

Theorem 15. *For any formula A,*

$$\vdash_{\text{CL}} A \qquad \textit{iff} \qquad \vdash_{\text{Int}} A^{\neg\neg}.$$

This is a trivial corollary of Glivenko's theorem, since $A^{\neg\neg} = \neg\neg\tilde{A}$, where $\tilde{A}$ is a formula that is classically equivalent to A. Then we get the following:

$$\vdash_{\text{CL}} A \iff \vdash_{\text{CL}} \tilde{A} \iff \vdash_{\text{Int}} A^{\neg\neg}.$$

Here the second step is due to Glivenko's theorem.

10. Topological Models for Int

Recall the notion of abstract *topological space.* A topological space is a pair $\langle X, \tau\rangle$, where X is a set and $\tau \subset \mathcal{P}(X)$ is a family of subsets of X that are declared as *"open".* The family τ is required to obey the following conditions:

- $\varnothing \in \tau$, $X \in \tau$;
- if $A, B \in \tau$, then $A \cap B \in \tau$ (τ is closed under *finite* intersections);
- if $\mathcal{A}$ is a family of sets from τ, then its union, $\bigcup \mathcal{A}$, also belongs to τ (τ is closed under *arbitrary* unions).

τ is called a *topology* on X. The standard example of a topological space is the Euclidean n-dimensional space $\mathbb{R}^n$ with the standard topology: a set $A \subset \mathbb{R}^n$ is open iff for every point $x \in A$ there exists such $r > 0$ that $\mathrm{B}_r(X) \subset A$, where $\mathrm{B}_r(x)$ is the ball of radius r with its center in x. In other world, a set is open if every its point belongs to it with a *neighbourhood.*

We're going to interpret formulae of Int as subsets of a topological space $\langle X, \tau\rangle$, maintaining the constraint that the valuation of every formula should be an open set. For variables we define the valuation arbitrarily, $v\colon \mathrm{Var} \to \tau$; $\bar{v}(\bot) = \varnothing$. The propagation for conjunction and disjunction is easy:

$$\bar{v}(A \land B) = \bar{v}(A) \cap \bar{v}(B), \qquad \bar{v}(A \lor B) = \bar{v}(A) \cup \bar{v}(B).$$

(Due to the properties of topological spaces, $\bar{v}(A \land B)$ and $\bar{v}(A \lor B)$ also belong to τ.)

For implication one could classically expect $\bar{v}(A \to B) = (X - \bar{v}(A)) \cup \bar{v}(B)$ (in CL, $(A \to B) \equiv (\neg A \lor B)$), but this set could be not an open one. In order to force it to be open, we modify the definition:

$$\bar{v}(A \to B) = \mathrm{In}((X - \bar{v}(A)) \cup \bar{v}(B)).$$

Here $\mathrm{In}(D)$ is the *interior* of a set D, i.e., the maximal open set that is included in D. (More formally, it is the *union* of all open subsets of D, $\mathrm{In}(D) = \bigcup\{E \in \tau \mid E \subset D\}$; by definition, it is also an open set.)

The valuation for negation is computed as follows:

$$\bar{v}(\neg A) = \bar{v}(A \to \bot) = \mathrm{In}((X - \bar{v}(A)) \cup \bar{v}(\bot)) = \mathrm{In}(X - \bar{v}(A)).$$

In other words, negation is interpreted as the interior of the complement.

A formula A is considered *true* under valuation v on a topological space $\langle X, \tau\rangle$, if $\bar{v}(A) = X$.

One can easily see that this interpretation violated the law of excluded middle: indeed, a usual open set A in $\mathbb{R}^n$ (for example, an open ball) has a non-trivial *border* that consists of points that belong neither to A nor to the interior of its complement, $\mathrm{In}(\mathbb{R}^n - A)$. Every neighbourhood of a border point contains points both from A and from its complement.

On the other hands, axioms of Int and the modus ponens rule are valid w.r.t. this interpretation (exercise!). For example, take axiom $A \to (B \to A)$. Then

$$\bar{v}(A \to (B \to A)) = \mathrm{In}((X - \bar{v}(A)) \cup \mathrm{In}((X - \bar{v}(B)) \cup \bar{v}(A))) \supseteq \mathrm{In}((X - \bar{v}(A)) \cup \mathrm{In}(\bar{v}(A))),$$

since In is monotonic (if $A \subseteq B$, then $\mathrm{In}(A) \subseteq \mathrm{In}(B)$. Since $\bar{v}(A)$ is open, it coincides with its interior; then we get $\mathrm{In}((X - \bar{v}(A)) \cup \bar{v}(A)) = \mathrm{In}(X) = X$, thus $\bar{v}(A \to (B \to A)) \subseteq X$. The other inclusion is obvious.

The following completeness theorem was proved by Tarski:

Theorem 16. *For every $n \geq 1$ the following holds: $\vdash_{\mathrm{Int}} A$ iff $\bar{v}(A) = \mathbb{R}^n$ for every valuation v on $\mathbb{R}^n$ with the standard topology.*

We shall prove a weaker result, namely, completeness w.r.t. *arbitrary* topological models. This class is bigger than the class of models on $\mathbb{R}^n$, and finding a countermodel is easier. In fact, we build it from a Kripke model.

Theorem 17. *If $\nvdash_{\mathrm{Int}} A$, then there exists a topological space $\langle X, \tau\rangle$ and a valuation v on it such that $\bar{v}(A) \neq X$.*

Proof. By Theorem 3, there exists a Kripke countermodel for A, $\mathcal{M} = \langle W, R, v\rangle$. We construct a topological space on W in the following way: for any $A \subseteq W$ we declare $A \in \tau$ iff for every $x \in A$ all points from $R(x)$ also belong to A (in other words, open sets are those that are upwardly closed under R). Next, define the topological valuation v_τ: $v_\tau(p_i) = \{x \in W \mid x \Vdash p_i\}$. Due to monotonicity, these sets are open in τ. Moreover, the main semantic lemma holds:

$$\bar{v}_\tau(B) = \{x \in W \mid x \Vdash B\}$$

for every formula B (proved by structural induction).

Since $\mathcal{M}$ is a countermodel for A, there exists such $x_0 \in W$ that $x_0 \nVdash A$. Therefore, $x_0 \notin \bar{v}_\tau(A)$, therefore $\bar{v}_\tau(A) \neq W$. $\square$