

# The Extended Theory of Trees and Algebraic (Co)datatypes

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The first-order theory of finite and infinite trees has been studied since the eighties, especially by the logic programming community. Following Djelloul, Dao and Frhworth, we consider an extension of this theory with an additional predicate for finiteness of trees, which is useful for expressing properties about (not just datatypes but also) codatatypes. Based on their work, we present a simplification procedure that determines whether any given (not necessarily closed) formula is satisfiable, returning a simplified formula which enables one to read off all possible models. Our extension makes the algorithm usable for algebraic (co)datatypes, which was impossible in their original work due to restrictive assumptions. We also provide a prototype implementation of our simplification procedure and evaluate it on instances from the SMT-LIB.

## 1 Introduction

Trees play a fundamental role in computer science: syntactic terms can be regarded as finite trees, and operations like matching and unification, which are essential to functional and logic programming languages, can be viewed as solving certain first-order constraints in the structure of finite trees. Furthermore, trees are a model for program schemes, such as higher-order recursion schemes [11], and more generally as computation trees. The structures of finite and infinite trees are also central to the declarative semantics of logic (e.g. [7]) and functional languages (e.g. [15]). Furthermore, they play a role in the verification of programs [12] and in term rewriting systems [5]. The theory of finite and infinite trees was extensively studied by the logic programming community in the eighties. An axiomatization and a decision procedure for these structures was given by Maher in [8, 9].

The structure of trees just consists of what one would normally think of as trees with labeled nodes, except that we allow them to be infinite. Examples of finite and infinite trees are depicted in Fig. 1. The labels for these trees are suggestively named after constructors for common algebraic datatypes because we want to specifically consider applications to the theory of (co)datatypes. In functional programming, two common data structures are natural numbers and linked lists:

```
data nat = zero | succ(pred: nat)
data list = nil | cons(head: nat, tail: list)
```

Inhabitants of these types are naturally viewed as trees: the term `cons(succ(zero), cons(zero, nil))` is shown as a tree in Fig. 1a. In some languages, such as Haskell, datatypes behave in fact more like codatatypes [14], i.e. they can be infinitely nested. For example, the term `let t = cons(zero, cons(succ(zero), t)) in t` corresponds to the infinite tree shown in Fig. 1b.

In this work, we consider the first-order theory of trees, extended with a predicate  $\text{fin}(t)$  for stating finiteness of  $t$ , and propose it as a tool for reasoning about algebraic datatypes and codatatypes. Why not use the theory of (co)datatypes as implemented in many SMT solvers? First, the theory of (co)datatypes is undecidable because selectors (`head`, `tail`) can be applied to the wrong constructor (`nil`) and the standard semantics from the SMT-LIB [2] does not specify the result of such an operation (cf. Section 3).

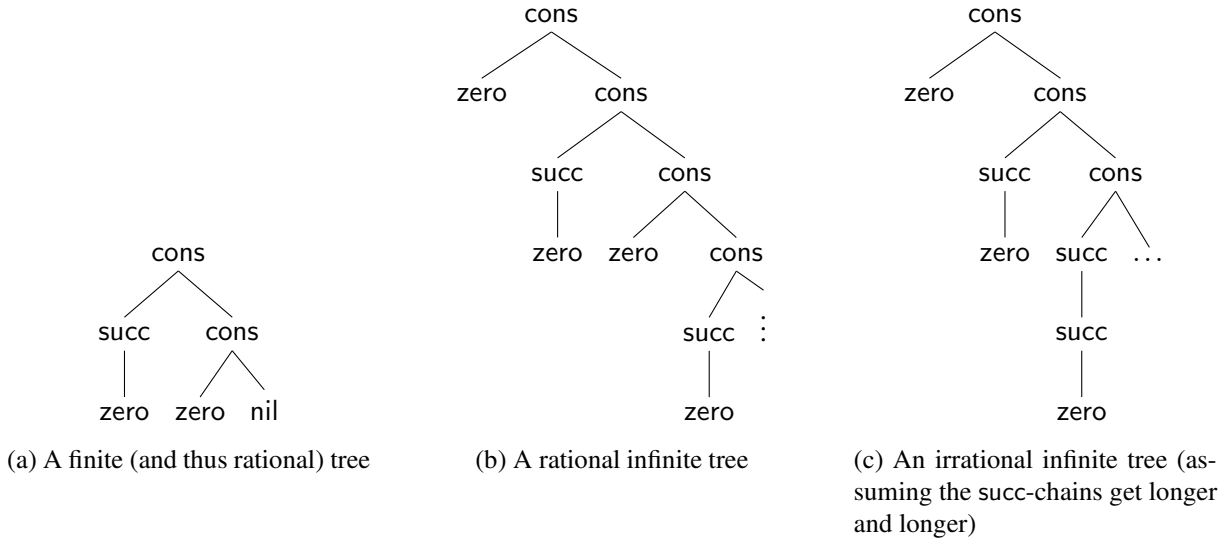


Figure 1: Examples of trees

Secondly, the theory of trees not only allows us to treat datatypes and codatatypes in a uniform way (both are represented by trees, datatypes just require a fin-predicate), but it is even more expressive: it can explicitly state non-finiteness ( $\neg \text{fin}(t)$ ), as well as finiteness only for a proper subformula (cf. Section 3). Codatatypes are often used in mechanized proofs to represent infinite structures [14] and we believe that this increased flexibility can be useful there as well.

This extended theory of trees (with a fin-predicate) was first presented by Djelloul, Dao and Frhwirth in [4], where they also present a complete axiomatization and a decision procedure. However, one of their core assumptions is that there are infinitely many function symbols (i.e. constructors), which renders it unsuitable for algebraic (co)datatypes found in programming languages since those never have infinitely many constructors. In this work, we lift this restriction and allow sorts with finitely many generators. Note that we require at least two generators, however. Sorts with one generator are not hard to support in principle but require a lot of special casing, so we do not discuss them in this work.

**Contributions** Our first contribution is a formal description of the relationship between the theory of algebraic (co)datatypes and the extended theory of trees. We also present a decision procedure for first-order formulae (including quantifiers) in the latter, based on [4]. To the best of our knowledge, no decision procedure for this theory allowing finitely generated sorts was known before. Just like the algorithm in [4], it is not only a decision procedure that outputs “satisfiable” or “unsatisfiable” but instead, it simplifies the given input formula as much as possible, which makes it easy to read off all satisfying valuations of the free variables.

We propose this extended theory of trees as an interesting background theory for constrained Horn clauses. Recently, Ong and Wagner [10] proved that satisfiability of higher-order constrained Horn clauses (HoCHCs) is semi-decidable if the background theory is decidable. Since the extended theory of trees is decidable, it is potentially suitable as a background theory for HoCHCs. In addition, we hope that the existence of a simplification procedure instead of a mere decision procedure will also have useful applications to constrained Horn clauses.

**Outline** The paper is organized as follows. In Section 2, we explain the theory of trees. The following Section 3 introduces (co)datatypes and explores their relationship with the theory of trees. In Section 4, we describe how to check for (finitely generated) sorts with only finitely many finite, respectively infinite, trees. This step is necessary for the extension of the algorithm by Djelloul, Dao and Frhwirth [4] to finitely generated sorts. The extended algorithm is presented in Section 5. Throughout the paper, a sequence of mathematical objects  $x_1, \dots, x_n$  is abbreviated as  $\bar{x}$ . Unless otherwise stated,  $u, v, w, x, y, z$  stand for variables,  $s$  for sorts,  $t$  for terms,  $\phi, \psi$  for logical formulae, and  $f, g$  for constructors and generators.

## 2 Trees

An *ordered tree* is defined as a (potentially infinite) connected directed acyclic graph with a distinguished node  $r$  (the *root*) such that every vertex has exactly one incoming edge, except for  $r$ , which has none. Additionally, for each node, its outgoing edges (and the corresponding nodes) are ordered. Furthermore, each node is labeled with an element of a *label set*, also called a *function symbol* or a *constructor*. For a node  $v$ , its *subtree* rooted at  $v$  is the induced subgraph containing exactly the nodes reachable from  $v$ . A tree is called *finite* if it has finitely many nodes, and *rational* if it has finitely many distinct subtrees. Examples of trees can be found in Fig. 1. These use labels suggestively named *zero*, *succ* (representing natural numbers) and *nil*, *cons* (representing lists). This alludes to the connection with algebraic datatypes mentioned in the introduction and further explored in Section 3.

The logical setting for the theory of trees is many-sorted first-order logic. We have a set of *sorts*  $S$ , a set of *function symbols*  $F$ , a set of *predicate symbols*  $P$  and a countable set of *variables*  $V$ . Each function symbol  $f$  has an *arity*  $f : s_1 \times \dots \times s_n \rightarrow s$  with  $s_1, \dots, s_n, s \in S$  and we say that  $f$  is a *generator* of  $s$ . If  $n = 0$ , we write  $f : s$  and say that  $f$  is a *constant*. The set of generators of  $s$  is called  $F_s$  and we assume that each  $F_s$  is nonempty. We say that  $s$  is *finitely generated* if  $F_s$  is finite, and *singular* if  $F_s$  is a singleton. Similarly, each predicate symbol  $p$  has an arity  $p : s_1 \times \dots \times s_n$  with  $s_1, \dots, s_n \in S$ , and each variable  $v$  has a sort  $v : s$ . For example, the trees in Fig. 1 can be expressed in the first-order language with sorts  $S = \{\text{nat}, \text{list}\}$  and function symbols  $\text{zero} : \text{nat}$ ,  $\text{nil} : \text{list}$ ,  $\text{succ} : \text{nat} \rightarrow \text{nat}$ , and  $\text{cons} : \text{list} \times \text{list} \rightarrow \text{list}$ . In the first-order language of trees, the only predicates are  $P = \{\text{fin}_s : s \mid s \in S\}$ , which state that a given tree of sort  $s$  is finite. We will drop the index  $s$  if there is no ambiguity.

Given such a *signature*  $(S, F, P)$ , a many-sorted *structure*  $\mathcal{A}$  consists of non-empty sets  $s^{\mathcal{A}}$  for each  $s \in S$ , functions  $f^{\mathcal{A}} : s_1^{\mathcal{A}} \times \dots \times s_n^{\mathcal{A}} \rightarrow s^{\mathcal{A}}$  for each function symbol  $f : s_1 \times \dots \times s_n \rightarrow s$  and a predicate  $\text{fin}_s^{\mathcal{A}} \subseteq s^{\mathcal{A}}$  for each predicate symbol  $\text{fin}_s$ . A *valuation* for  $\mathcal{A}$  is a family of mappings  $V_s \rightarrow s^{\mathcal{A}}$ , indexed by  $S$ , where  $V_s$  denotes the variables of sort  $s$ . A *model* of a formula  $\phi$  in  $\mathcal{A}$  is a valuation making  $\phi$  true in  $\mathcal{A}$ .

The structure  $\mathcal{T}$  of trees interprets a signature  $(S, F, P)$  as follows. Each sort  $s$  is interpreted as the set  $s^{\mathcal{T}}$  of trees of sort  $s$ , meaning the trees where the root is labeled with a generator  $f : s_1 \times \dots \times s_n \rightarrow s$  and its children (in order) are roots of subtrees of sorts  $s_1, \dots, s_n$ , respectively. Each function symbol  $f : s_1 \times \dots \times s_n \rightarrow s$  is interpreted as a function  $f^{\mathcal{T}} : s_1^{\mathcal{T}} \times \dots \times s_n^{\mathcal{T}} \rightarrow s^{\mathcal{T}}$  such that  $f^{\mathcal{T}}(t_1, \dots, t_n)$  is the tree with a root labeled  $f$  and subtrees  $t_1, \dots, t_n$ . Each predicate  $\text{fin}_s$  is interpreted as the subset  $\text{fin}_s^{\mathcal{T}} \subseteq s^{\mathcal{T}}$  of finite trees of sort  $s$ . Thus  $\text{fin}_s(t)$  holds in  $\mathcal{T}$  if and only if the interpretation of  $t$  is a finite tree. For example, if  $t_1$  is the term  $\text{cons}(\text{succ}(\text{zero}), \text{cons}(\text{zero}, \text{nil}))$ , depicted in Fig. 1a, then  $\text{fin}(t_1)$  is true in  $\mathcal{T}$ . On the other hand, if  $t_2$  is the unique tree to make  $t_2 = \text{cons}(\text{zero}, \text{cons}(\text{succ}(\text{zero}), t_2))$  true (this tree is shown in Fig. 1b) then  $\text{fin}(t_2)$  is false because this tree is infinite.

We call the *theory* of  $\mathcal{T}$ , i.e. the set of sentences that are true of  $\mathcal{T}$ , the *extended theory of trees* (“extended” because of the additional predicate  $\text{fin}$ ). This theory was first presented by Djelloul, Dao and

Frhwirth [4]. However, they require that each sort have at least one constant generator and infinitely many non-constant generators. This assumption simplifies the treatment of the theory significantly but has the serious drawback that it makes their method unsuitable for algebraic (co)datatypes, which typically have only finitely many constructors (i.e. generators). Therefore, we first take a look at the relation between the two theories.

### 3 Relationship with (Co)Datatypes

The theory of *algebraic (co)datatypes*, also called *(co)inductive datatypes*, is similar to the theory of trees but there are a couple of important differences. For one thing, the set of sorts is partitioned into  $S = S_{dt} \cup S_{ct}$  where  $S_{dt}$  is the set of datatypes and  $S_{ct}$  is the set of codatatypes. The function symbols are partitioned into the constructors  $F_{ctr}$  and the selectors  $F_{sel}$ , and there are no predicate symbols. Each (co)datatype  $\delta$  is equipped with  $m \geq 1$  constructors  $F_{ctr}^\delta = \{C_1, \dots, C_m\}$ . Each constructor  $C_i \in F_{ctr}^\delta$  has an arity  $\delta_1 \times \dots \times \delta_{n_i} \rightarrow \delta$  and is associated with  $n_i$  selectors  $\text{sel}_{C_i}^j : \delta \rightarrow \delta_j$ . Note that for a datatype (resp. codatatype) declaration, all constructor arguments must be datatypes (resp. codatatypes); no mixing is allowed. Such an assumption is common, for example in [14]. Another requirement is that datatypes be well-founded, i.e. one must be able to exhibit a ground term for each datatype. This excludes examples like a datatype *infinite* with a single constructor  $\text{next} : \text{infinite} \rightarrow \text{infinite}$ . However, this is allowed as a *codatatype*.

**Example 3.1.** Consider Booleans and lists. Their declaration in many programming languages looks roughly like this:

```
data bool = True | False
data list = Nil | Cons(head : bool, tail : list)
```

where the selectors are called  $\text{head} := \text{sel}_{\text{Cons}}^1$  and  $\text{tail} := \text{sel}_{\text{Cons}}^2$ . The former extracts the first element of a given list, if it is nonempty, and the latter returns the rest of the list. Hence we have  $F_{ctr}^{\text{bool}} = \{\text{True}, \text{False}\}$ ,  $F_{sel}^{\text{bool}} = \emptyset$ ,  $F_{ctr}^{\text{list}} = \{\text{Nil}, \text{Cons}\}$ , and  $F_{sel}^{\text{list}} = \{\text{head}, \text{tail}\}$ .

**Semantics** Both datatypes and codatatypes are interpreted as *constructor trees*, i.e. trees labeled only with constructors (not selectors). A structure  $\mathcal{D}$  of (co)datatypes interprets a codatatype  $\gamma$  as the set  $\gamma^\mathcal{D}$  of constructor trees of sort  $\gamma$  and a datatype  $\delta$  as the set  $\delta^\mathcal{D}$  of finite constructor trees of sort  $\delta$ . Each constructor  $C : s_1 \times \dots \times s_n \rightarrow s$  is interpreted as the function  $C^\mathcal{D} : s_1^\mathcal{D} \times \dots \times s_n^\mathcal{D} \rightarrow s^\mathcal{D}$  constructing a new tree out of the given ones, with root  $C$ . Each selector  $\text{sel}_C^i : s \rightarrow s_i$  is interpreted as a function  $(\text{sel}_C^i)^\mathcal{D} : s^\mathcal{D} \rightarrow s_i^\mathcal{D}$ , which must satisfy  $(\text{sel}_C^i)^\mathcal{D}(C^\mathcal{D}(t_1, \dots, t_n)) = t_i$  but is not specified on inputs built with the wrong constructor. This semantics is very common and what the SMT-LIB standard specifies [2]. Note that other semantics are possible, however, such as returning a fixed default value if a selector is applied to the wrong constructor [3]. We call the latter the *semantics with default values*. The theory of (co)datatypes is the set of sentences that are true in any structure of (co)datatypes satisfying the above.

**Theorem 3.2.** *The first-order theory of (co)datatypes is undecidable.*

*Proof idea.* The proof is based on the undecidability of formulae with quantifiers in the theory of uninterpreted functions (EUF). Uninterpreted functions  $f : s_1 \times s_2 \rightarrow s$  can be emulated using the following construct:

```
data dummy = c(a1 : s1, a2 : s2) | d(h : s)
```

Then  $h(c(x, y))$  acts like an uninterpreted function  $f(x, y)$  because the selector  $h$  is applied to the wrong constructor  $c$ . For the full proof, refer to the appendix.  $\square$

In the semantics with default values, selectors present much less of a problem: they can simply be eliminated.

**Theorem 3.3.** *In the theory of (co)datatypes with default values, a given formula can be effectively transformed into an equivalent one without selectors.*

*Proof idea.* The idea is to introduce additional variables such that selectors only occur in equations of the form  $x = \text{sel}_C^i(t)$  where  $t$  does not contain selectors and then to rewrite such an equation as

$$(\exists \bar{v}. t = C(\bar{v}) \wedge x = v_i) \vee ((\neg \exists \bar{v}. t = C(\bar{v})) \wedge x = T_C^i)$$

where  $T_C^i$  is the default value for this selector. For the full proof, refer to the appendix.  $\square$

But even in the standard semantics, quantifier-free formulae in the theory of (co)datatypes are decidable [14]. In fact, we can also eliminate selectors from such formulae.

**Theorem 3.4.** *In the theory of (co)datatypes with standard semantics, a quantifier-free formula can be effectively transformed into an equisatisfiable one without selectors (but including quantifiers).*

*Proof idea.* The first step is to introduce additional variables such that selectors only occur in equations of the form  $x = \text{sel}_C^i(t)$  where  $t$  does not contain selectors. For each such equation, we add the conjunct  $\forall \bar{z}. t = C(\bar{z}) \rightarrow z_i = x$ , which ensures that the selector correctly extracts the argument when applied to the right constructor. Furthermore, for each pair of such equations  $x = \text{sel}_C^i(t)$ ,  $x' = \text{sel}_C^i(t')$ , we add the conjunct  $t = t' \rightarrow x = x'$ , which ensures that selectors behave like functions, i.e. return the same result when applied to the same arguments. For the full proof, refer to the appendix.  $\square$

Finally, we show that the theory of trees is enough for selector-free formulae.

**Theorem 3.5.** *A selector-free formula in the theory of (co)datatypes can be effectively transformed into an equisatisfiable formula in the extended theory of trees.*

*Proof idea.* Since in both theories, terms are interpreted as trees, we just have to ensure that datatypes are interpreted as finite trees. Hence, for a datatype  $d$ , existential quantification  $\exists x : d. \phi$  is replaced by  $\exists x : d. \text{fin}(x) \wedge \phi$  and universal quantification  $\forall x : d. \phi$  is replaced by  $\forall x : d. \text{fin}(x) \rightarrow \phi$ . Finally, to ensure equisatisfiability, free variables  $x : d$  require adding the conjunct  $\text{fin}(x)$  to the whole formula. For the full proof, refer to the appendix.  $\square$

The last result raises the question of how the expressiveness of the extended theory of trees compares to selector-free (co)datatypes. The former is, in fact, more expressive because it allows specifying non-finiteness of individuals, such as  $\neg \text{fin}(x)$ . This is impossible in the theory of (co)datatypes since datatypes have only finite values and codatatypes can have finite and infinite values. Additionally, it facilitates specifying finiteness only in parts of the formula, such as in  $(\text{fin}(t) \rightarrow \phi) \vee (\neg \text{fin}(t) \rightarrow \psi)$ , where  $t$  is finite in  $\phi$  but infinite in  $\psi$ . This shows that the extended theory of trees is more powerful than the (selector-free) theory of (co)datatypes.

## 4 Analyzing Finitely Generated Sorts

Having shown how formulae involving (co)datatypes can often be reduced to formulae involving trees, we want to find a decision procedure for the latter based on the work by Djelloul, Dao and Frhwirth [4]. Their algorithm, however, makes the assumption that each sort contains infinitely many non-constant generators and one constant generator. As a consequence, each sort contains infinitely many finite and

false : <i>bool</i>	zero : <i>nat</i>	nil : <i>list</i>
true : <i>bool</i>	succ : <i>nat</i> → <i>nat</i>	cons : <i>nat</i> × <i>list</i> → <i>list</i>
tree1 : <i>inftree</i> → <i>inftree</i>	c1 : <i>bool</i> → <i>d</i>	g1 : <i>bool</i> × <i>bool</i> → <i>t</i>
tree2 : <i>inftree</i> × <i>inftree</i> → <i>inftree</i>	c2 : <i>nat</i> × <i>inftree</i> → <i>d</i>	g2 : <i>bool</i> × <i>nat</i> → <i>t</i>

Figure 2: Generators for the sorts  $S = \{bool, nat, list, inftree, d, t\}$ .

infinitely many infinite trees. This simplifies solving logical formulae: the predicate  $\text{fin}(x)$  can always be made true or false for an appropriate valuation of  $x$ . However, their assumption is obviously not satisfied for sorts arising from (co)datatypes.

Therefore, we consider the setting with finitely generated sorts, where the situation is more complicated. For instance, if  $x$  is of, say, a Boolean sort with only constant generators then the predicate  $\text{fin}(x)$  is always true. Due to these complications, we need to analyze the set of sorts and check for sorts with only finitely many finite or infinite trees.

In the following, we allow sorts with finitely many generators but assume that any sort has at least two generators. As mentioned before, this restriction is not hard to lift in principle but saves us a lot of technical details and space in this paper. Note that sorts with a single non-recursive generator can just be unfolded in the place that they are used.

For a sort  $s$ , denote by  $s_{\text{fin}}$ , respectively  $s_{\text{infin}}$ , the set of finite, respectively infinite, trees of sort  $s$ . Denote by  $S_{0F}$ ,  $S_{FF}$ ,  $S_{0I}$ ,  $S_{1I}$ ,  $S_{FI} \subseteq S$  the sets of sorts with no finite trees, finitely many finite trees, no infinite trees, exactly one infinite tree, and finitely many infinite trees, respectively. In the following, we present algorithms for computing these sets.

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**Algorithm 1** Algorithm computing the sets of sorts containing no finite ( $S_{0F} \subseteq S$ ) or no infinite trees ( $S_{0I} \subseteq S$ ), when given a signature  $(S, F, P)$  as input.

---

```

 $S_{0I} \leftarrow \emptyset$ 
 $S_{0F} \leftarrow S$ 
repeat
   $S_{0I} \leftarrow S_{0I} \cup \{s \in S \mid \forall (g : s_1 \times \dots \times s_n \rightarrow s) \in F_s : \forall i \in \{1, \dots, n\} : s_i \in S_{0I}\}$ 
   $S_{0F} \leftarrow S_{0F} \setminus \{s \in S \mid \exists (g : s_1 \times \dots \times s_n \rightarrow s) \in F_s : \forall i \in \{1, \dots, n\} : s_i \notin S_{0F}\}$ 
until no changes in the last iteration

```

---

**Theorem 4.1.** *Given a signature  $(S, F, P)$ , Algorithm 1 correctly computes the sets  $S_{0F}$  and  $S_{0I}$ .*

*Proof idea.* A sort contains no infinite trees if every generator only takes arguments of sorts containing no infinite trees. A sort contains no finite trees unless some generator takes only arguments of sorts with finite trees. The sets  $S_{0F}$  and  $S_{0I}$  can thus be computed as fixed points, the former a least fixed point, the latter a greatest fixed point. For details, refer to the full proof in the appendix.  $\square$

**Example 4.2.** Consider the sorts and generators in Fig. 2. How would Algorithm 1 act on this input? In the first iteration, it would add *bool* to  $S_{0I}$  because each generator is a constant. At the same time, *bool*, *list*, and *nat* are removed from  $S_{0F}$  because each one has a constant generator. In the next iteration,  $S_{0I}$

---

**Algorithm 2** Algorithm computing the sets of sorts containing only finitely many finite ( $S_{FF}$ ), respectively infinite ( $S_{FI}$ ), trees; and their finite ( $s_{\text{fin}}$ ), respectively infinite ( $s_{\text{infin}}$ ), inhabitants.

---

```

Compute  $S_{0I}$  and  $S_{0F}$  as in Algorithm 1
 $S_{FF} \leftarrow S_{0F}$ 
 $s_{\text{fin}} \leftarrow \emptyset$  for each  $s \in S$ 
 $S_{1I} \leftarrow S \setminus S_{0I}$ 
 $U_s \leftarrow \emptyset$  for each  $s \in S$ 
repeat
  for  $s \in S$  do
     $F_s^{\text{infin}} \leftarrow \{(g : s_1 \times \dots \times s_n \rightarrow s) \in F_s \mid \exists i \in \{1, \dots, n\} : s_i \in S_{0F}\}$ 
    if  $|F_s \setminus F_s^{\text{infin}}| < \infty$  and  $\forall (g : s_1 \times \dots \times s_n \rightarrow s) \in F_s \setminus F_s^{\text{infin}} : \forall i \in \{1, \dots, n\} : s_i \in S_{FF}$  then
       $S_{FF} \leftarrow S_{FF} \cup \{s\}$ 
       $s_{\text{fin}} \leftarrow \{g(r_1, \dots, r_n) \mid (g : s_1 \times \dots \times s_n \rightarrow s) \in F_s \setminus F_s^{\text{infin}}, r_i \in (s_i)_{\text{fin}}\}$ 
    if  $\exists (g : s_1 \rightarrow s) \in F_s : s_1 \in S_{1I} \wedge (\forall (g' : s'_1 \times \dots \times s'_n \rightarrow s) \in F_s \setminus \{g\} : \forall i : s'_i \in S_{0I})$  then
       $U_s \leftarrow \{u_s = g(u_{s_1})\} \cup U_{s_1}$ 
    else
       $S_{1I} \leftarrow S_{1I} \setminus \{s\}$ 
  until no changes in the last iteration
 $S_{FI} \leftarrow S_{0I} \cup S_{1I}$ 
 $s_{\text{infin}} \leftarrow \emptyset$  for each  $s \in S_{0I}$ 
 $s_{\text{infin}} \leftarrow \{u_s\}$  for each  $s \in S_{1I}$ 
repeat
  for  $s \in S$  do
     $F_s^{\text{infin}} \leftarrow \{(g : s_1 \times \dots \times s_n \rightarrow s) \in F_s \mid \exists i \in \{1, \dots, n\} : s_i \notin S_{0I}\}$ 
    if  $|F_s^{\text{infin}}| < \infty$  and  $\forall (g : s_1 \times \dots \times s_n \rightarrow s) \in F_s^{\text{infin}} : \forall i \in \{1, \dots, n\} :$ 
       $s_i \in S_{0I} \vee (s_i \in S_{FI} \wedge (\forall j \in \{1, \dots, n\} \setminus \{i\} : s_j \in S_{FF} \cap S_{FI}))$  then
       $S_{FI} \leftarrow S_{FI} \cup \{s\}$ 
       $s_{\text{infin}} \leftarrow \{g(r_1, \dots, r_n) \mid (g : s_1 \times \dots \times s_n \rightarrow s) \in F_s^{\text{infin}}, r_j \in (s_j)_{\text{fin}} \cup (s_j)_{\text{infin}} \text{ for } j = 1, \dots, n$ 
        such that  $\exists i \in \{1, \dots, n\} : r_i \notin (s_i)_{\text{fin}}\}$ 
  until no changes in the last iteration

```

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stays unchanged but  $d$  is removed from  $S_{0F}$  because it has the generator  $\text{cl}$  whose parameter sort  $\text{bool}$  is not in  $S_{0F}$  anymore. For a similar reason,  $t$  is removed from  $S_{0F}$ . After this point, no more changes happen and we obtain  $S_{0I} = \{\text{bool}\}$  and  $S_{0F} = \{\text{inf tree}\}$ .

Next, we consider the sets  $S_{FF}, S_{1I}, S_{FI}$ . Note that the sort  $\text{nat}$  from the above example has exactly one infinite tree, namely  $\text{succ}(\text{succ}(\dots))$ . For such sorts  $s \in S_{1I}$ , we introduce variables  $u_s$  for their unique infinite tree. For instance,  $u_{\text{nat}} = \text{succ}(u_{\text{nat}})$  describes the unique infinite tree of  $\text{nat}$ . The sort  $t$  has two infinite trees  $\text{g2}(\text{false}, u_{\text{nat}})$  and  $\text{g2}(\text{true}, u_{\text{nat}})$ . Hence to describe all infinite trees of sorts  $s \in S_{FI}$ , we need the variables  $u_s$  for  $s \in S_{1I}$  and their equations, like  $u_{\text{nat}} = \text{succ}(u_{\text{nat}})$ . Algorithm 2 computes all this.

**Theorem 4.3.** *Given a signature  $(S, F, P)$ , Algorithm 2 correctly computes the sets  $S_{FF}$ ,  $S_{1I}$ , and  $S_{FI}$ . Furthermore it computes the set  $s_{\text{fin}}$  (the terms for the finite trees of sort  $s$  for  $s \in S_{FF}$ ), and the set  $s_{\text{infin}}$  (the terms for the infinite trees of sort  $s$  for  $s \in S_{FI}$ ). The latter makes use of the variables  $u_s$  (for  $s \in S_{1I}$ ), standing for the unique infinite tree of  $s$ . The equations that uniquely determine these  $u_s$  are output in  $U_s$ .*

*Proof idea.* Similarly to the previous algorithm, these sets are computed as fixed points. A sort  $s$  has only finitely many finite trees if there is a finite number of generators that only take sorts with finitely many finite trees as arguments ( $F_s \setminus F_s^{\text{infin}}$ ), and the remaining generators ( $F_s^{\text{infin}}$ ) take at least one argument of a sort that contains no finite trees (because such a generator cannot create finite trees). Along the way, the algorithm builds up the set  $s_{\text{fin}}$  from the generators of the former category.

Constructing the set  $S_{FI}$  works similarly, except for the fact that we start the fixed point iteration with  $S_{OI} \cup S_{II}$  instead of the empty set. The reason is that for every sort with finitely many infinite trees, it can be shown (but is nontrivial) that the infinite parts of each such tree are built from the unique infinite trees of the sorts  $S_{II}$ . These sorts with a unique infinite tree are also constructed by fixed point iteration. They can only have a single generator  $g$  that constructs infinite trees and it can only take one argument because otherwise we would have at least two infinite trees since each sort is assumed to have at least two generators.

A sort  $s$  only has finitely many infinite trees if the set of generators constructing infinite trees ( $F_s^{\text{infin}}$ ) is finite, and when picking an arbitrary argument  $i$  of it, this argument allows no infinite trees; or it allows finitely many infinite trees and all the other arguments allow only finitely many trees. This explains the fixed point iteration for  $S_{FI}$  and  $s_{\text{infin}}$ . For details, refer to the full proof in the appendix.  $\square$

**Example 4.4.** Consider again the signature from Fig. 2. How does Algorithm 2 act on it? At the start of the first loop, we have  $S_{FF} = \{\text{inftree}\}$  and  $S_{II} = S$ . In the first iteration,  $\text{bool}$  is added to  $S_{FF}$  because all its generators are constants, and  $\text{bool}_{\text{fin}} = \{\text{false}, \text{true}\}$ . Additionally,  $\text{nat}$  stays in  $S_{II}$  because its generator  $\text{succ}$  satisfies  $\text{nat} \in S_{II}$  and the other generator is constant. Therefore  $U_{\text{nat}} = \{u_{\text{nat}} = \text{succ}(u_{\text{nat}})\}$ . All the other sorts are removed from  $S_{II}$ , either because they don't have a unary generator ( $\text{bool}$ ,  $\text{list}$ ) or there is another generator that allows infinite trees, destroying uniqueness ( $\text{tree2}$  for  $\text{inftree}$ ,  $\text{c2}$  for  $d$ , and  $\text{g2}$  for  $t$ ). In the second iteration,  $d$  is added to  $S_{FF}$  because  $\text{c1} : \text{bool} \rightarrow d$  only constructs finitely many finite trees since  $\text{bool} \in S_{FF}$  and its other generator  $\text{c2}$  constructs only infinite trees. Therefore,  $d_{\text{fin}}$  is set to  $\{\text{c1}(\text{true}), \text{c1}(\text{false})\}$ . After this points, no more changes happen.

At the start of the second loop, we have  $S_{FI} = \{\text{bool}, \text{nat}\}$  and  $\text{nat}_{\text{infin}} = \{u_{\text{nat}}\}$ . In the loop iteration,  $t$  is added to  $S_{FI}$  because we have  $F_t^{\text{infin}} = \{\text{g2}\}$ , which is finite, and its only generator  $\text{g2}$  has the property that its first parameter is  $\text{bool} \in S_{OI}$  and its second parameter is  $\text{nat} \in S_{FI}$  with the additional property that all remaining parameters, i.e.  $\text{bool}$ , are in  $S_{FF} \cap S_{FI}$ . Therefore  $t_{\text{infin}}$  is set to  $\{\text{g2}(\text{false}, u_{\text{nat}}), \text{g2}(\text{true}, u_{\text{nat}})\}$ . After this point, no more changes happen. The algorithm has computed  $S_{FF} = \{\text{inftree}, \text{bool}, d\}$  and  $S_{FI} = \{\text{nat}, t\}$ .

## 5 Simplification Procedure for the Theory of Trees

Having explained how to analyze finitely generated sorts, we can now describe how the simplification procedure from [4] is extended to finitely generated sorts. Before going into detail, we provide a brief outline of this algorithm. The procedure works on special formulae, called *normal formulae*. Any formula can be transformed into an equivalent normal formula, so this is not a restriction. Roughly speaking, the output of our algorithm is a disjunction of *fully simplified* formulae that is equivalent to the original formula. A fully simplified formula makes it easy to read off all its models. The simplification algorithm works similarly to [4], except for the fact that finitely generated sorts sometimes require case splits (also called instantiations) for certain variables (called *instantiateable*). These case splits can be on the finitely many generators of a sort, or on the finitely many (finite or infinite) inhabitants of a sort if it is in  $S_{FF}$  or  $S_{FI}$ . In this section, we focus on these instantiateable variables and case splits because it is the



novel part of our extension of [4]. The full algorithm is described in the appendix. Before we can start with the concept of normal formulae, we first need to define *basic formulae*.

**Definition 5.1.** A *basic formula* is of the form  $(\bigwedge_i v_i = t_i) \wedge (\bigwedge_j \text{fin}(u_j))$  where  $\bar{u}, \bar{v}$  are variables and each  $t_i$  is a variable or a term of the form  $f(\bar{z})$  for a function symbol  $f$  and variables  $\bar{z}$ . Such a formula will be abbreviated by  $\overline{v=t} \wedge \overline{\text{fin}(u)}$ . Given a total order on its free variables, it is called *solved* if (1) the variables  $\bar{u}, \bar{v}$  are distinct and for each equation  $x = y$ , we have  $x > y$ , and (2) if  $\text{fin}(v)$  occurs then the sort of  $v$  contains both finite and infinite trees. A variable  $x_n$  is *reachable* from a variable  $x_0$  if the basic formula contains  $x_0 = t_0 \wedge x_1 = t_1 \wedge \dots \wedge x_{n-1} = t_{n-1}$  where each  $t_i$  contains  $x_{i+1}$ . It is *properly reachable* if  $n > 0$ . The subformulae  $u = t$  and  $\text{fin}(u)$  are considered reachable if  $u$  is.

The variable ordering is important when we consider basic subformulae of larger formulae. Then this ordering ensures that in solved basic formulae, variables bound more deeply inside the whole formula occur on the left-hand side of equations, which is important for the correctness proof. In order to offer some intuition for reachability: if  $y$  is reachable from  $x$ , this means that  $y$  is a subtree of  $x$ . Djelloul, Dao and Frhwirth describe an algorithm to solve a basic formula (rules 1–10 in [4]). In our extended setting, two things have to be changed: if  $\text{fin}(u)$  occurs in the basic formula where  $u : s$  with  $s \in S_{0I}$ , or  $s \in S_{0F}$ , then  $\text{fin}(u)$  is always satisfied and can be removed, or is never satisfied and the basic formula is unsolvable, respectively. This is summarized by the following theorem.

**Theorem 5.2.** *There is an algorithm  $\text{SOLVEBASIC}(\bar{v} = v_0 < \dots < v_n, \alpha)$  (Algorithm 4 in the appendix) that correctly solves basic formulae  $\alpha$ , i.e. it turns  $\alpha$  into an equivalent solved formula (with respect to the given variable ordering) or returns false if none exists.*

Basic formulae are insufficient for the general case but they are an important building block for the concept of *normal formulae*, which can express any first-order formula.

**Definition 5.3.** A *normal formula*  $\phi$  of depth  $d \geq 1$  takes the form  $\neg(\exists \bar{x}. \alpha \wedge \bigwedge_{i=1}^n \phi_i)$  where  $\alpha$  is a basic formula, and each  $\phi_i$  is a normal formula of depth  $d_i$  with  $d = 1 + \max(0, d_1, \dots, d_n)$ .

The simplest normal formula is  $\neg \text{true}$ . As a normal formula allows expressing negation, conjunction, existential quantification and nesting, the following theorem is straightforward to prove [4][Property 4.3.3].

**Theorem 5.4.** *There is an algorithm  $\text{NORMALIZE}(\phi)$  which turns any first-order formula  $\phi$  into a normal one that is equivalent in the theory of trees.*

**Example 5.5.** Consider the formula  $\forall x : \text{nat}. \neg \text{fin}(x) \rightarrow x = \text{succ}(x)$ . It can be rewritten as  $\neg(\exists x : \text{nat}. \neg \text{fin}(x) \wedge \neg(x = \text{succ}(x)))$ , which is a normal formula of depth 2.

Now we come to the main difference with the original algorithm from [4]: our more general setting necessitates case splits (or instantiations) for certain variables. For instance, consider the normal formula  $\phi_1 \equiv \neg(\exists x : \text{list}. \neg(x = \text{nil}) \wedge \neg(\exists y, z. x = \text{cons}(y, z)))$ . If *list* had infinitely many generators, it would always be possible to find a value for  $x$  that is neither *nil* nor *cons*. However, since *list* only has those two generators, no such  $x$  exists and the formula is true. Here our extended algorithm will do a case split on both constructors of *list* (described later in more detail) and realize that neither works.

As another example, consider  $\phi_2 \equiv \neg(\exists x : t. \neg \text{fin}(x) \wedge \neg(x = y) \wedge \neg(x = z))$ . If  $t$  had infinitely many infinite trees, then this would be true because we could always choose a valuation for  $x$  that is different from the free variables  $y$  and  $z$ . Since  $t$  has only two infinite trees, our extended algorithm does a case split on all two infinite trees of  $t$ , instantiating  $x$  with  $\text{g2}(\text{true}, u_{\text{nat}})$  and  $\text{g2}(\text{false}, u_{\text{nat}})$  where  $u_{\text{nat}}$  is the unique tree of sort *nat*, namely  $\text{succ}(\text{succ}(\dots))$ . A similar case occurs with a constraint  $\text{fin}(x)$  where  $x$

only has finitely many finite trees or, in general, if  $x$  has only finitely many trees. This leads us to the definition of an instantiable variable, i.e. a variable that requires a case split.

Note that every normal formula can be transformed into an equivalent one of depth at most 2 by repeatedly applying rule 16 (depth reduction) from [4, section 4.6]. Therefore we can limit our attention to such formulae in the following.

**Definition 5.6** (instantiable variable). Let  $\neg\exists\bar{x}. \alpha \wedge \bigwedge_i \neg(\exists\bar{y}_i. \beta_i)$  be a normal formula of depth at most 2 such that each  $\alpha$  and  $\beta_i$  are solved basic formulae. Let  $\beta_i^*$  be  $\beta_i$  with all conjuncts also occurring in  $\alpha$  removed. Then a variable  $v : s$  that is free in the formula or occurs in  $\bar{x}$  is called *instantiable* if one of the following conditions is satisfied:

1.  $s$  has finitely many generators  $F_s$  and some  $\beta_i^*$  contains  $v = f(\bar{w})$  and  $v$  is not properly reachable from  $v$  in  $\beta_i^*$ , or
2.  $s \in S_{FF} \cap S_{FI}$ , some  $\beta_i^*$  contains  $v$ , and  $\alpha$  contains no equation  $v = t$  for any term  $t$ , or
3.  $s \in S_{FF}$ ,  $\alpha$  contains  $\text{fin}(v)$ , and some  $\beta_i^*$  contains  $v$ , or
4.  $s \in S_{FI}$  and some  $\beta_j^*$  contains only  $\text{fin}()$ -constraints, among them  $\text{fin}(v)$ .

---

**Algorithm 3** Algorithm for finding instantiable variables and their instantiations.

---

```

function FINDINSTANTIATION( $\bar{v}, \neg(\exists\bar{x}. \alpha \wedge \bigwedge_i \neg(\exists\bar{y}_i. \beta_i))$ )
   $\beta_i^* \leftarrow \beta_i$  without the conjuncts occurring in  $\alpha$ 
  for  $u : s \in \bar{v}\bar{x}$  do
    if  $s$  has finitely many generators, and  $u = f(\bar{z})$  occurs in some  $\beta_i^*$ ,
      and  $u$  is not properly reachable from  $u$  in  $\beta_i^*$  then
      return  $\{\exists\bar{z}. u = g(\bar{z}) \mid g \in F_s\}$ 
    if  $s \in S_{FF} \cap S_{FI}$  and  $u$  occurs in some  $\beta_i^*$  and  $\alpha$  contains no  $u = t$  then
      return  $\{\exists u_{s_1}, \dots, u_{s_n}. u = t \wedge \bigwedge_{s \in S_{II}} U_s \mid t \in s_{\text{fin}} \cup s_{\text{infin}}\}$  where  $S_{II} = \{s_1, \dots, s_n\}$ 
    if  $s \in S_{FF}$  and  $u$  occurs in some  $\beta_i^*$ , and  $\text{fin}(u)$  in  $\alpha$  then
      return  $\{u = t \mid t \in s_{\text{fin}}\}$ 
    if  $s \in S_{FI}$  and  $\text{fin}(u)$  occurs in some  $\beta_j^*$  that contains only  $\text{fin}()$ -constraints then
      return  $\{\text{fin}(u)\} \cup \{\exists u_{s_1}, \dots, u_{s_n}. u = t \wedge \bigwedge_{s \in S_{II}} U_s \mid t \in s_{\text{infin}}\}$  where  $S_{II} = \{s_1, \dots, s_n\}$ 
  return none

```

---

Algorithm 3 looks for an instantiable variable (if any) in a normal formula  $\neg(\exists\bar{x}. \alpha \wedge \bigwedge_i \phi_i)$  with free variables  $\bar{v}$  by checking exactly the four conditions from above. If it finds an instantiable variable  $u$ , it returns a set  $I$  of formulae, called *instantiations*. Note that while we write “ $u = t$ ”, which is not a basic formula, in the return value for simplicity, we actually mean an equivalent formula  $\exists\bar{z}. \gamma$  where  $\gamma$  is a basic formula. For instance, by  $u = \text{c1}(\text{true})$ , we mean  $\exists z. u = \text{c1}(z) \wedge z = \text{true}$  for a fresh variable  $z$ . We can use these instantiations to get rid of instantiable variables, as the following theorem explains.

**Theorem 5.7.** Let  $\phi \equiv \neg(\exists\bar{x}. \alpha \wedge \bigwedge_i \phi_i)$  be a normal formula of depth at most 2 with free variables  $\bar{v}$ . Let  $I$  be the result of  $\text{FINDINSTANTIATION}(\bar{v}, \phi)$  from Algorithm 3. If  $I$  is “none”, then there is no instantiable variable. Otherwise, let  $u$  be the first instantiable variable found in  $\text{FINDINSTANTIATION}$ . Then  $\phi$  is equivalent to the following conjunction of normal formulae, in which the variable  $u$  is no longer instantiable:

$$\bigwedge_{(\exists\bar{z}. \psi) \in I} \neg(\exists\bar{x}\bar{z}. \alpha \wedge \psi \wedge \bigwedge_i \phi_i).$$

**Example 5.8.** In the formula  $\phi_1 \equiv \neg(\exists x : \text{list}. \neg(x = \text{nil}) \wedge \neg(\exists y, z. x = \text{cons}(y, z)))$  from above,  $x$  is instantiable because of condition 1. (Note that the reachability check in this condition is required to

avoid infinite loops for recursive equations like  $x = \text{cons}(y, x)$ .) Here `FINDINSTANTIATION` returns  $I = \{x = \text{nil}; \exists y, z. x = \text{cons}(y, z)\}$ . By the above theorem,  $\phi_1$  is equivalent to

$$\neg(\exists x : \text{list}. x = \text{nil} \wedge \neg(x = \text{nil}) \wedge \neg(\exists y, z. x = \text{cons}(y, z))) \\ \wedge \neg(\exists x : \text{list}. y, z. x = \text{cons}(y, z) \wedge \neg(x = \text{nil}) \wedge \neg(\exists y, z. x = \text{cons}(y, z))).$$

Both existential subformulae obviously contain a contradiction, so the whole formula simplifies to  $\neg(\text{false}) \wedge \neg(\text{false})$  and thus true.

**Example 5.9.** In the other formula  $\phi_2 \equiv \neg(\exists x : t. \neg \text{fin}(x) \wedge \neg(x = y) \wedge \neg(x = z))$  from above,  $t$  is instantiable because of condition 4. Algorithm 3 returns the instantiations

$I = \{\text{fin}(x); \exists u_{\text{nat}}. x = \text{g2}(\text{true}, u_{\text{nat}}) \wedge u_{\text{nat}} = \text{succ}(u_{\text{nat}}); \exists u_{\text{nat}}. x = \text{g2}(\text{false}, u_{\text{nat}}) \wedge u_{\text{nat}} = \text{succ}(u_{\text{nat}})\}$ , which means  $x$  is either a finite tree or one of the two infinite trees  $\text{g2}(\text{false}, u_{\text{nat}})$ ,  $\text{g2}(\text{true}, u_{\text{nat}})$  where  $u_{\text{nat}}$  is the unique tree with  $u_{\text{nat}} = \text{succ}(u_{\text{nat}})$ . By the above theorem,  $\phi_2$  is equivalent to

$$\neg(\exists x : t. \text{fin}(x) \wedge \neg \text{fin}(x) \wedge \neg(x = y) \wedge \neg(x = z)) \\ \wedge \neg(\exists x, u_{\text{nat}} : t. x = \text{g2}(\text{false}, u_{\text{nat}}) \wedge u_{\text{nat}} = \text{succ}(u_{\text{nat}}) \wedge \neg \text{fin}(x) \wedge \neg(x = y) \wedge \neg(x = z)) \\ \wedge \neg(\exists x, u_{\text{nat}} : t. x = \text{g2}(\text{true}, u_{\text{nat}}) \wedge u_{\text{nat}} = \text{succ}(u_{\text{nat}}) \wedge \neg \text{fin}(x) \wedge \neg(x = y) \wedge \neg(x = z)).$$

The other parts of the simplification procedure (unchanged from [4]) simplify this to

$$\text{true} \\ \wedge \neg(\exists u_{\text{nat}} : t. u_{\text{nat}} = \text{succ}(u_{\text{nat}}) \wedge \neg(y = \text{g2}(\text{false}, u_{\text{nat}})) \wedge \neg(z = \text{g2}(\text{false}, u_{\text{nat}}))) \\ \wedge \neg(\exists u_{\text{nat}} : t. u_{\text{nat}} = \text{succ}(u_{\text{nat}}) \wedge \neg(y = \text{g2}(\text{true}, u_{\text{nat}})) \wedge \neg(z = \text{g2}(\text{true}, u_{\text{nat}}))),$$

where the variable  $x$  is removed because it is unreachable from the free variables. The resulting formula essentially expresses that  $y$  or  $z$  has to be equal to  $\text{g2}(\text{false}, u_{\text{nat}})$ ; and that  $y$  or  $z$  has to be equal to  $\text{g2}(\text{true}, u_{\text{nat}})$ . In other words, they can only take on those two infinite values and have to be different. Note that the algorithm has not completed at this point yet because  $y$  and  $z$  are now instantiable by condition 1. We skip the following (less interesting) instantiations for space reasons.

At this point, we can introduce the notions of *solved* and *fully simplified* formulae, which make up the output of our extended simplification procedure.

**Definition 5.10.** A normal formula  $\phi \equiv \neg(\exists \bar{x}. \alpha \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i))$ , of depth at most 2, is called *solved* if it satisfies the following properties.

1. Each  $\beta_i$  and  $\alpha$  are solved basic formulae with respect to a variable ordering  $u < v$  if the binding of  $v$  is more deeply nested than  $u$ , i.e.  $u$  is free where  $v$  is bound.
2. The equations of  $\alpha$  are included in every  $\beta_i$ .
3. Each  $\beta_i$  contains at least one conjunct that does not occur in  $\alpha$ .
4. There are no instantiable variables.
5. All the variables  $\bar{x}$  and  $\bar{y}_i$  are reachable from the variables that are free in the subformulae  $\exists \bar{x}. \alpha$  and  $\exists \bar{y}_i. \beta_i$ , respectively.

A formula  $\psi$  is called *fully simplified* if  $\neg\psi$  is a solved normal formula. (This is an extension of the definition of “explicit solved form” in [4].)

**Example 5.11.** The point of fully simplified formulae is that they’re easy to interpret, i.e. it is easy to read of all possible models from them. For instance, consider the fully simplified formula

$$\exists v. x = \text{succ}(v) \wedge v = y \wedge \text{fin}(y) \wedge \neg(\exists w. y = \text{succ}(w) \wedge \text{fin}(w) \wedge \text{fin}(z))$$

Any model has to satisfy  $x = \text{succ}(y)$  and  $y$  has to be finite. To falsify the other part  $\exists w. y = \text{succ}(w) \wedge \text{fin}(w) \wedge \text{fin}(z)$ , there are two options for the free variables  $y$  and  $z$ : (1) instantiate  $y$  with any finite tree with a root other than  $\text{succ}$  and  $z$  with any tree, or (2) instantiate  $y$  with any finite tree and  $z$  with any infinite tree. These are the only two classes of models for the above fully simplified formula. In general, the following holds about fully simplified formulae.

**Theorem 5.12.** *Let  $\phi$  be a fully simplified formula. If  $\phi$  has no free variables then  $\phi \equiv \text{true}$ . Otherwise both  $\phi$  and  $\neg\phi$  are satisfiable in the theory of trees.*

The task of the main algorithm, is, given a formula  $\phi$ , to return a disjunction of fully simplified formulae. Since each individual disjunct allows an easy description of its models, we can describe all possible models of  $\phi$ .

**Theorem 5.13.** *There is an algorithm  $\text{SOLVE}(\phi)$  (Algorithm 5 in the appendix) that, given a formula  $\phi$ , returns true, false, or a disjunction of fully simplified formulae that is equivalent to  $\phi$  in the extended theory of trees. In particular, if  $\phi$  is closed, it returns true or false.*

*Proof idea.* The simplification procedure from [4] does not have to be changed a lot. We use the function  $\text{SOLVEBASIC}$  from Theorem 5.2 to solve basic formulae. Afterwards, the rules 11–14 and 16 from [4] ensure that the result is a disjunction of formulae satisfying conditions (1–3) of Definition 5.10. At this point, we make use of  $\text{FINDINSTANTIATION}$  and Theorem 5.7, to ensure that condition (4) is satisfied. After each such instantiation, the previous rules have to be applied again because conditions (1–3) may have been invalidated. These steps will not, however, invalidate condition (4). The nontrivial proof of the fact that these instantiations terminate can be found in the appendix. Avoiding infinite loops of instantiations is the reason for the complicated condition 1 in Definition 5.6. Finally, Rule 15 of the original algorithm [4] ensures that condition (5) is satisfied as well. The proof that this part is still correct in our more general setting is again nontrivial and can be found in the appendix.  $\square$

**Time complexity** Regarding the performance of our extended algorithm, note that the original algorithm has non-elementary time complexity [4]. In fact, Vorobyov proved that deciding first-order formulae in the (ordinary) theory of trees already has non-elementary time complexity [16], so we cannot hope for an efficient algorithm for the extended theory of trees in the worst case.

**Implementation** In order to evaluate the performance in practice, we created a prototype implementation in Scala. Due to a lack of benchmarks involving formulae of trees, we took the `tests` set of the `QF_DT` (quantifier-free datatypes) suite of the SMT-LIB and transformed each instance to a formula in the extended theory of trees using the results from Section 3. Then we ran our extended simplification procedure on the transformed instances. Over 90% of them completed in less than 1 second and about 5% timed out after 10 seconds (more data in Table 1 in the appendix). While state-of-the-art SMT solvers decide each `QF_DT` instance in a few milliseconds, our transformed instances are considerably harder because they contain quantifiers and can be significantly larger than the original ones. Furthermore, our prototype implementation obviously cannot compete with heavily optimized SMT solvers and leaves a lot of room for improvements: for instance, the normalization of formulae can be optimized, and heuristics for choosing which variable to instantiate (instead of picking the first one) could make a big difference. It nevertheless demonstrates that our algorithm has a reasonable performance on many practical instances. A web interface to our implementation can be found at <http://mjolnir.cs.ox.ac.uk/trees-codata/>.

## 6 Conclusion

We believe that the extended theory of trees is an interesting theory to study because of its decidability and connections with algebraic (co)datatypes. We have explained the complications arising from finitely generated sorts, which are necessary to apply it to (co)datatypes. The fact that we not only provide a decision procedure but a simplification procedure should make it easier to conduct further research on the theory of trees, such as investigating Craig interpolation.

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## A Omitted proofs from Section 3

**Theorem** (Theorem 3.2, repeated). *The first-order theory of (co)datatypes is undecidable.*

*Proof of Theorem 3.2.* The proof works by reduction from Post's correspondence problem. An instance of the problem is given by a finite set of pairs of bit strings  $\{(x_1, y_1), \dots, (x_n, y_n)\}$ , i.e. each  $x_i, y_i \in \{0, 1\}^*$ . A solution to such an instance is a nonempty finite sequence of indices  $i_1, \dots, i_k$  with each  $i_j \in \{1, \dots, n\}$  such that

$$x_{i_1} \dots x_{i_k} = y_{i_1} \dots y_{i_k}.$$

The decision problem is to decide whether such a solution exists, and is famously undecidable [13]. Let  $\{(x_1, y_1), \dots, (x_n, y_n)\}$  be an instance of the problem. In the following, we construct a signature and formula  $\phi$ , which is satisfiable in the theory of datatypes if and only if the instance has a solution.

For the encoding, consider the following datatype declarations.

```
data bool = true | false
data bitstring = e | 0(tail0 : bitstring) | 1(tail1 : bitstring)
data dummy = f(b1 : bitstring, b2 : bitstring) | g(h : bool)
```

We have the usual definition of Booleans (true or false), bit strings are empty (e) or start with a 0 or with a 1, and the dummy datatype is just used to emulate uninterpreted functions:  $h(f(s, s'))$  acts like an unspecified function on the bit strings  $s, s'$  because  $h$  is applied to the wrong constructor and the standard semantics leaves this case unspecified. In this proof,  $\phi$  is constructed in such a way that  $h(f(s, s')) = \text{true}$  holds if  $s, s'$  is a valid pair of bit strings that can be built out of the  $(x_i, y_i)$ .

First, for a bit string  $x$ , we write  $p_x(t)$  for the bit string that prepends  $x$  to  $t$ . For example, if  $x = 101$ ,  $p_x(t)$  stands for  $1(0(1(t)))$ . Consider the following formulae.

$$\begin{aligned} \phi_1 &\equiv \bigwedge_{i=1}^n h(f(p_{x_i}(e), p_{y_i}(e))) = \text{true} \\ \phi_2 &\equiv \forall u, v. h(f(u, v)) = \text{true} \rightarrow \bigwedge_{i=1}^n h(f(p_{x_i}(u), p_{y_i}(v))) = \text{true} \\ \phi_3 &\equiv \exists u. h(f(u, u)) = \text{true} \end{aligned}$$

First,  $\phi_1$  and  $\phi_2$  specify what pairs of bit strings can be constructed out of  $x_i$  and  $y_i$ , and finally  $\phi_3$  specifies that a solution to the instance exists. Hence we claim that  $\phi \equiv \phi_1 \wedge \phi_2 \rightarrow \phi_3$  is valid if and only if the given instance of Post's correspondence problem is solvable.

First, suppose  $\phi$  is valid. Consider the model  $\mathcal{M}$  where  $h$  is interpreted as follows:

$$h^{\mathcal{M}}(z) = \begin{cases} y & \text{if } z = g^{\mathcal{M}}(y) \\ \text{true}^{\mathcal{M}} & \text{if } z = \mathcal{M}(f(p_x(e), p_y(e))) \\ & \text{where } \exists k \geq 1, i_1, \dots, i_k : x = x_{i_1} \dots x_{i_k} \wedge y = y_{i_1} \dots y_{i_k} \\ \text{false}^{\mathcal{M}} & \text{otherwise} \end{cases}$$

In this model,  $\phi_1$  and  $\phi_2$  are satisfied by construction, hence  $\phi_3$  is satisfied as well. This means there is an element  $u \in \text{bitstring}^{\mathcal{M}}$  such that  $h^{\mathcal{M}}(f^{\mathcal{M}}(u, u)) = \text{true}^{\mathcal{M}}$ . By the choice of  $\mathcal{M}$ , this means that  $u = \mathcal{M}(p_x(e))$  such that there are  $i_1, \dots, i_k$  with  $x = x_{i_1} \dots x_{i_k}$  and  $y = y_{i_1} \dots y_{i_k}$ , which means the given instance is solvable.

Conversely, suppose the instance of Post's correspondence problem has a solution  $i_1, \dots, i_k$ . Let  $\mathcal{M}$  be a structure where  $\phi_1 \wedge \phi_2$  is true. Let  $s_j = x_{i_1} \dots x_{i_j}$  and  $t_j = y_{i_1} \dots y_{i_j}$ . Then  $\phi_1$  ensures that  $h(f(p_{s_1}(e), p_{t_1}(e))) = \text{true}$  in  $\mathcal{M}$  and  $\phi_2$  ensures that the induction step works and hence that  $h(f(p_{s_j}(e),$

$p_{t_j}(e)) = \text{true}$  holds for all  $j$ , in particular for  $j = k$ . Set  $u = p_{s_k}(e)$ . Since  $s_k = t_k$  is a solution to the instance, we have  $h(u, u) = \text{true}$  in  $\mathcal{M}$ . Hence  $\phi_3$  also holds in  $\mathcal{M}$ . Therefore  $\phi$  is valid.

Since a formula is valid if and only if its negation is unsatisfiable, this proof also shows that the satisfiability problem of first-order formulae in the theory of datatypes is undecidable.

Note that in the proof, we never used the fact that a *bitstring* is finite, so the same proof works when replacing the above datatype declarations by codatatype declarations.  $\square$

**Theorem** (Theorem 3.3, repeated). *In the theory of (co)datatypes with default values, a given formula can be effectively transformed into an equivalent one without selectors.*

*Proof of Theorem 3.3.* The transformation works in two steps. First, it ensures that selectors don't occur nested but instead only occur in equations of the form  $x = \text{sel}_C^i(t)$  for a variable  $x$ . Afterwards, it replaces such equations by an equivalent formula that doesn't contain selectors.

**Step 1** The first step isolates selectors to simple equations. For every equation  $t = t'$  where one side contains a selector, do the following. If  $t = t'$  is of the form  $x = \text{sel}_C^i(s)$  for a variable  $x$  and a selector-free term  $s$  then there is nothing to do. Otherwise, without loss of generality, assume that  $t$  contains a (nested) selector, so  $t$  can be written as  $r[\text{sel}_C^i(s)/x]$  for a term  $r$  where  $x$  is a fresh variable. Then rewrite  $r[\text{sel}_C^i(s)/x] = t'$  to  $\exists x. r = t' \wedge x = \text{sel}_C^i(s)$ , which is always equivalent in first-order logic. This process is repeated until no more changes can be made.

**Step 2** Now the formula only contains selectors as part of equations of the form  $x = \text{sel}_C^i(s)$  where  $s$  is selector-free. Each such equation can be rewritten as

$$(\exists \bar{v}. s = C(\bar{v}) \wedge x = v_i) \vee ((\neg \exists \bar{v}. s = C(\bar{v})) \wedge x = T_C^i)$$

where  $\bar{v}$  are fresh variables. This is equivalent because

$$\begin{aligned} x = \text{sel}_C^i(s) &\leftrightarrow ((\exists \bar{v}. s = C(\bar{v})) \wedge x = \text{sel}_C^i(s)) \vee (\neg(\exists \bar{v}. s = C(\bar{v})) \wedge x = \text{sel}_C^i(s)) \\ &\leftrightarrow (\exists \bar{v}. s = C(\bar{v}) \wedge x = \text{sel}_C^i(s)) \vee (\neg(\exists \bar{v}. s = C(\bar{v})) \wedge x = \text{sel}_C^i(s)) \\ &\leftrightarrow (\exists \bar{v}. s = C(\bar{v}) \wedge x = v_i) \vee (\neg(\exists \bar{v}. s = C(\bar{v})) \wedge x = T_C^i) \end{aligned}$$

where  $T_C^i$  is the default value for this selector. The first equivalence is simply a case split on  $\exists \bar{v}. s = C(\bar{v})$ , the second one just moves a quantifier outward and the last one uses the definition of selectors.  $\square$

**Theorem** (Theorem 3.4, repeated). *In the theory of (co)datatypes with standard semantics, a quantifier-free formula can be effectively transformed into an equisatisfiable one without selectors (but including quantifiers).*

*Proof of Theorem 3.4.* The transformation works in two steps. The first step is similar to the previous proof and replaces every selector term by an existentially quantified fresh variable. The second step is different from before obviously, but is similar to Ackermann's reduction for uninterpreted functions [1] (see [6, Section 11.2.1] for an exposition).

**Step 1** Let  $\phi$  be the formula to transform. Let  $S$  be a set of equations, initially empty. As long as  $\phi$  contains a selector term, pick one that doesn't contain any nested selector terms. Let that term be  $t \equiv \text{sel}_C^i(s)$ . Replace  $t$  by a fresh variable  $v$  in  $\phi$  and add the equation  $v = \text{sel}_C^i(s)$  to  $S$ . This is repeated until  $\phi$  contains no more selector terms.

**Step 2** Let  $\phi$  be the formula after the first step and  $S = \{v_j = \mathbf{sel}_{C_j}^{i_j}(s_j) \mid j \in \{1, \dots, n\}\}$  the set of equations of selectors. For each  $j = 1, \dots, n$  define the following two formulae

$$\begin{aligned}\psi_j &\equiv \forall \bar{z}. s_j = C_j(\bar{z}) \rightarrow z_i = v_j \\ \chi_j &\equiv \bigwedge_{j'=1}^n s_j = s_{j'} \rightarrow v_j = v_{j'}\end{aligned}$$

where  $\bar{z}$  are fresh variables for each  $j$ . The idea is that the  $\psi_j$ 's specify that  $\mathbf{sel}_C^i(C(\bar{z})) = z_i$  and the  $\chi_j$ 's (which is also part of Ackermann's reduction) ensure functional consistency, i.e. that if the same selector is applied to equal terms then the results should be equal. The latter property is important if the selector  $\mathbf{sel}_C^i$  is applied to a term that is not of the form  $C(\dots)$  because then the result is unspecified but it still has to be consistent. Finally define  $\psi$  to be the following formula

$$\psi \equiv \bigwedge_{j=1}^n (\phi_j \wedge \chi_j).$$

Let  $\phi$  be the original formula and  $\phi'$  be the result of step 1. We claim that  $\psi \wedge \phi'$ , which does not contain selectors anymore, is equisatisfiable with  $\phi$ .

First, suppose that  $\phi$  is satisfiable. Let  $\mathcal{M}$  be a model of  $\phi$ . We modify it to  $\mathcal{M}'$  satisfying

$$\mathcal{M}'(v) = \begin{cases} \mathcal{M}'(\mathbf{sel}_{C_j}^{i_j}(s_j)) & \text{if } v \equiv v_j \\ \mathcal{M}(v) & \text{otherwise} \end{cases}$$

Note that the first case is well-defined since there are no cyclic dependencies between the variables  $\bar{v}$  by the design of Step 1. Then  $\mathcal{M}'$  satisfies each  $\psi_j$  since the interpretation of  $\mathbf{sel}_C^i$  in  $\mathcal{M}$  (and hence  $\mathcal{M}'$ ) has to satisfy the selector axioms. It also satisfies each  $\chi_j$  because  $\mathcal{M}'(\mathbf{sel}_C^i)$  is a function that has to give the same result if applied to the same object. Hence if  $\mathcal{M}'(s_j) = \mathcal{M}'(s_{j'})$  then  $\mathcal{M}'(v_j) = \mathcal{M}'(v_{j'})$ . This shows that  $\mathcal{M}'$  satisfies  $\psi$ .

Furthermore, it is not hard to see that  $\mathcal{M}'$  satisfies  $\phi'$  as well. First of all,  $\mathcal{M}'$  satisfies  $\phi$  because the variables  $v_j$  don't occur in  $\phi$  since they were assumed to be fresh and  $\mathcal{M}'$  agrees with  $\mathcal{M}$  on everything else. Also note that by construction of  $\mathcal{M}'$ , every equation  $v_j = \mathbf{sel}_{C_j}^{i_j}(s_j)$  in  $S$  is true of  $\mathcal{M}'$ . Since each  $\mathbf{sel}_{C_j}^{i_j}(s_j)$  in  $\phi$  is replaced by  $v_j$  in  $\phi'$ , the interpretations of  $\phi$  and  $\phi'$  are the same in  $\mathcal{M}'$ . Hence  $\phi'$  is true of  $\mathcal{M}'$ . Altogether,  $\psi \wedge \phi'$  is satisfiable.

Next, suppose that  $\psi \wedge \phi'$  is satisfiable and let  $\mathcal{M}$  be a model of it. We define a new model  $\mathcal{M}'$  modifying the interpretation of the selectors such that it satisfies:

$$\mathcal{M}'(\mathbf{sel}_C^i)(x) = \begin{cases} y_i & \text{if } x = C^{\mathcal{M}}(y_1, \dots, y_n) \\ \mathcal{M}(v_j) & \text{if } x = \mathcal{M}(s_j) \text{ for some } j \in \{1, \dots, n\} \\ \mathcal{M}(\mathbf{sel}_C^i)(x) & \text{otherwise} \end{cases}$$

It is not immediately clear that this is well-defined. First, we check what happens if there is more than one  $j$  such that  $x = \mathcal{M}(s_j)$ . Suppose  $x = \mathcal{M}(s_j) = \mathcal{M}(s_{j'})$ . Since  $\mathcal{M}$  satisfies  $\psi$  and in particular  $\chi_j$ , we have  $\mathcal{M}(v_j) = \mathcal{M}(v_{j'})$ . Hence it does not matter which  $j$  is chosen.

Next, we check that the first and second case are compatible. Suppose  $x = C^{\mathcal{M}}(y_1, \dots, y_n) = \mathcal{M}(s_j)$ . Since  $\mathcal{M}$  satisfies  $\psi$  and in particular  $\psi_j$ , we have  $y_i = \mathcal{M}(v_j)$ . So the first and second case are not in conflict. Altogether, this shows that  $\mathcal{M}'$  is well-defined.

Furthermore, it is not hard to see that  $\mathcal{M}'$  is a model of  $\phi$ . First of all,  $\mathcal{M}'$  satisfies  $\phi'$  because  $\mathcal{M}$  does,  $\phi'$  does not contain any selectors, and  $\mathcal{M}'$  agrees with  $\mathcal{M}$  on everything else. Also note that by construction of  $\mathcal{M}'$ , every equation  $v_j = \mathbf{sel}_{C_j}^{i_j}(s_j)$  in  $S$  is true of  $\mathcal{M}'$ . Since each  $\mathbf{sel}_{C_j}^{i_j}(s_j)$  in  $\phi$  is replaced



by  $v_j$  in  $\phi'$ , the interpretations of  $\phi$  and  $\phi'$  are the same in  $\mathcal{M}'$ . Hence  $\phi$  is true of  $\mathcal{M}'$ , and thus  $\phi$  is satisfiable, which finishes the proof.  $\square$

**Theorem** (Theorem 3.5, repeated). *A selector-free formula in the theory of (co)datatypes can be effectively transformed into an equisatisfiable formula in the extended theory of trees.*

*Proof of Theorem 3.5.* Without loss of generality, we can assume that the given formula  $\phi$  contains only existential quantifiers, no universal ones, because  $\forall x.\psi$  can be replaced by  $\neg\exists x.\neg\psi$ . Let  $\phi'$  be the result of replacing each  $\exists v_1, \dots, v_n.\psi$  occurring in  $\phi$  with

$$\exists v_1 : s_1, \dots, v_n : s_n. \left( \bigwedge_{i \in \{1, \dots, n\}, s_i \text{ is a datatype}} \text{fin}(v_i) \right) \wedge \psi.$$

Finally let  $u_1 : s_1, \dots, u_n : s_n$  be the free variables of  $\phi'$ . Set

$$\phi'' \equiv \left( \bigwedge_{i \in \{1, \dots, n\}, s_i \text{ is a datatype}} \text{fin}(u_i) \right) \wedge \phi'.$$

Then it is clear that each model of  $\phi$  can be reduced to a model of  $\phi''$  by forgetting the interpretation of the selector functions because the interpretation of datatypes are finite trees, thus satisfying all the additional  $\text{fin}()$ -constraints. Conversely, each model of  $\phi''$  in the theory of trees can be extended to a model of  $\phi$  by picking arbitrary selector functions. The  $\text{fin}()$ -constraints ensure that the interpretation of each datatype variable is a finite tree, so it is, in fact, a model of  $\phi$ .  $\square$

## B Omitted proofs from Section 4

Note that we make the standard assumption that the first-order language of trees is well-founded, meaning that there is no infinite sequence  $s_0, s_1, \dots$  of sorts such that for all  $i \in \mathbb{N}$ , there is a function symbol  $f_i$  of arity  $f_i : \dots \times s_{i+1} \times \dots \rightarrow s_i$ .

**Theorem** (Theorem 4.1, repeated). *Given a signature  $(S, F, P)$ , Algorithm 1 correctly computes the sets  $S_{0F}$  and  $S_{0I}$ .*

*Proof of Theorem 4.1.* The validity of the fixed point computations is implied by the following two lemmas.  $\square$

**Lemma B.1.** *The set  $S_{0I}$  of sorts without infinite trees is the least fixed point of the following function  $f : P(S) \rightarrow P(S)$  where  $P(\cdot)$  denotes the power set:*

$$f(X) := X \cup \{s \in S \mid \forall (g : s_1 \times \dots \times s_n \rightarrow s) \in F_s : \forall i \in \{1, \dots, n\} : s_i \in X\}$$

*Proof.* It is clear that  $S_{0I}$  is a fixed point because for each generator  $g$  each parameter sort must only allow finite trees, i.e. be in  $S_{0I}$ , yielding  $f(S_{0I}) = S_{0I}$ . We claim that any sort  $s$  containing only finite trees of depth  $\leq d$  is included in the set  $f^{d+1}(\emptyset)$ . This is proved inductively. For  $d = 0$ , this means  $s$  contains only constant symbols. Then the condition  $\forall g : s_1 \times \dots \times s_n \rightarrow s \in F_s : \forall i \in \{1, \dots, n\} : s_i \in \emptyset$  is vacuously true and  $s \in f(\emptyset)$ . For  $d \geq 1$ , this means that the argument of each generator has depth at most  $d - 1$ . Hence by the induction hypothesis, all generator arguments have sorts in  $f^d(\emptyset)$ . By the definition of  $f$ , this means that then  $s \in f(f^d(\emptyset))$ , proving the claim. Hence  $S_{0I} = \bigcup_{d \in \omega} f^{d+1}(\emptyset)$ , in other words,  $S_{0I}$  is the least fixed point of  $f$ . Note that the least fixed point is guaranteed to exist by the Knaster-Tarski theorem.  $\square$

**Lemma B.2.** *The set  $S_{0F}$  of sorts without finite trees is the greatest fixed point of the following function  $f : P(S) \rightarrow P(S)$ :*

$$f(X) := X \setminus \{s \in S \mid \exists (g : s_1 \times \cdots \times s_n \rightarrow s) \in F_s : \forall i \in \{1, \dots, n\} : s_i \notin X\}$$

*Proof.* It is clear that  $S_{0F}$  is a fixed point because a sort with a generator  $g$  where all parameter sorts of  $g$  allow finite trees cannot contain only infinite trees. Hence all such sorts must be excluded, which is what  $f$  does. Therefore  $S_{0F} = f(S_{0F})$ . Note again that  $f$  is monotonic, so the greatest fixed point is guaranteed to exist by the Knaster-Tarski theorem. Similarly to the proof of Lemma B.1, it is easy to see by induction that any sort  $s$  containing a finite tree of depth  $d$  is excluded from the set  $f^{d+1}(S)$ . Hence  $S_{0F} = \bigcap_{d \in \omega} f^{d+1}(S)$ , in other words,  $S_{0F}$  is the greatest fixed point of  $f$ .  $\square$

**Theorem** (Theorem 4.3, repeated). *Given a signature  $(S, F, P)$ , Algorithm 2 correctly computes the sets  $S_{FF}$ ,  $S_{1I}$ , and  $S_{FI}$ . Furthermore it computes the set  $s_{\text{fin}}$  (the terms for the finite trees of sort  $s$  for  $s \in S_{FF}$ ), and the set  $s_{\text{infin}}$  (the terms for the infinite trees of sort  $s$  for  $s \in S_{FI}$ ). The latter makes use of the variables  $u_s$  (for  $s \in S_{1I}$ ), standing for the unique infinite tree of  $s$ . The equations that uniquely determine these  $u_s$  are output in  $U_s$ .*

*Proof of Theorem 4.3.* The correctness of the fixed point computation of  $S_{FF}$  is implied by Lemma B.3. An analogous argument verifies the computation of  $s_{\text{fin}}$ , for  $s \in S_{FF}$ . Lemma B.4 shows the correctness of the fixed point computation of  $S_{1I}$  and the  $U_s$ . For the correctness proof of the fixed point computation of  $S_{FI}$ , we need Lemma B.5, which states that the finitely many infinite trees of sorts  $s \in S_{FI}$  are all built from the unique infinite trees  $u_s$  with  $s \in S_{1I}$ . Using this result, Lemma B.6 proves the fixed point computation of  $S_{FI}$  correct. An analogous argument works for the sets  $s_{\text{infin}}$  for  $s \in S_{FI}$ .  $\square$

**Lemma B.3.** *Let  $S_{FF} \subseteq S$  be the set of sorts such that each  $s \in S_{FF}$  has only finitely many finite trees. Let  $F_s^{\text{infin}} = \{(g : s_1 \times \cdots \times s_n \rightarrow s) \in F_s \mid \exists i \in \{1, \dots, n\} : s_i \in S_{0F}\}$  be the set of generators building only infinite trees. Then  $S_{FF}$  is the least fixed point of the following function  $f : P(S) \rightarrow P(S)$ :*

$$f(X) := X \cup S_{0F} \cup \{s \in S \mid |F_s \setminus F_s^{\text{infin}}| < \infty \wedge \forall (g : s_1 \times \cdots \times s_n \rightarrow s) \in (F_s \setminus F_s^{\text{infin}}) : \forall i : s_i \in X\}$$

where  $S_{0F}$  denotes the set of sorts with only infinite trees.

*Proof.* Why is  $S_{FF}$  a fixed point? First,  $F_s^{\text{infin}}$  are generators that can only construct infinite trees. So any other generator (in  $F_s \setminus F_s^{\text{infin}}$ ) can construct at least one finite tree. For there to be only finitely many finite trees in  $s$ , there have to be finitely many of the latter generators and for each such generator, each parameter sort  $s_i$  must have only finitely many finite trees. This explains the definition of the function  $f$ . Note that  $f$  is monotonic, so the least fixed point is guaranteed to exist by the Knaster-Tarski theorem.

In fact, similarly to the proof of Lemma B.1, it is easy to see inductively that if a sort  $s$  has finitely many finite trees of depth at most  $d$  then  $s \in f^{d+1}(\emptyset)$ . Hence  $S_{FF} = \bigcup_{d \in \omega} f^{d+1}(\emptyset)$ , in other words,  $S_{FF}$  is in fact the least fixed point of  $f$ .  $\square$

**Lemma B.4.** *Let  $S_{1I} \subseteq S$  be the set of sorts such that each  $s \in S_{1I}$  has exactly one infinite tree. Then  $S_{1I}$  is the greatest fixed point of the following function  $f : P(S \setminus S_{0I}) \rightarrow P(S \setminus S_{0I})$ :*

$$f(X) := \{s \in X \mid \exists (g : s_1 \rightarrow s) \in F_s : s_1 \in X \wedge (\forall (g' : s'_1 \times \cdots \times s'_n \rightarrow s) \in F_s \setminus \{g\} : \forall i : s'_i \in S_{0I})\}$$

Furthermore, the equations  $U_s$  that uniquely determine the unique infinite inhabitant  $u_s$  of  $s \in S_{1I}$  are given by the least fixed point of

$$f'(X_s) = X_s \cup \{u_s = g(u_{s_1})\} \cup X_{s_1}$$

where  $f'$  maps sets of terms to sets of terms, both indexed by  $s \in S_{1I}$ , and  $g : s_1 \rightarrow s$  is the unique generator with  $s_1 \in S_{1I}$ .

*Proof.* Why is  $S_{II}$  a fixed point? If  $s$  has a unique infinite tree then it must start with some generator  $g \in F_s$ . If  $g$  had more than one parameter then the choice of the other parameter would create at least two infinite inhabitants, contradiction. So  $g$  has only one parameter. Furthermore every other generator  $g'$  can only create finite trees because otherwise we would lose uniqueness of the infinite tree. The function  $f$  removes all sorts from  $X$  that do not satisfy these criteria. Hence  $S_{II}$  is a fixed point.

Conversely, a sort  $s$  in the fixed point must have two distinct infinite trees and they have to differ at some finite depth  $d$ . Then  $s \notin f^{d+1}(S \setminus S_{II})$  because  $f$  removes sorts that have more than one infinite inhabitant and if those two inhabitants differ at depth  $d$  this is detected after at most  $d + 1$  applications of  $f$ . This can be proved by induction, similarly to the proof of Lemma B.1. Hence  $S_{II} = \bigcap_{d \in \omega} f^{d+1}(S \setminus S_{II})$ , in other words,  $S_{II}$  is the greatest fixed point of  $f$ .

Why is  $U_s$  a fixed point of  $f'$ ? The equation  $u_s = g(u_{s_1})$  must be true by the above arguments. In order to describe  $u_{s_1}$  uniquely, we need the equations  $U_{s_1}$  as well. Thus the  $U_s$  are a fixed point of  $f'$ . They are, in fact, the least fixed point because we are interested in the smallest set of equations describing the  $u_s$ .  $\square$

**Lemma B.5.** *Let  $s$  be a sort with at least one but only finitely many infinite trees. Then each infinite tree of  $s$  can be described by a term containing only variables  $u_i : s_i$  representing the unique infinite tree of some sort  $s_i \in S_{II}$ .*

*Proof.* Proof by induction on the number  $\#s_{\text{infin}}$  of infinite trees of sort  $s$ .

$\#s_{\text{infin}} = 1$ : Then  $s$  has a unique infinite tree represented by  $u_s$ .

$\#s_{\text{infin}} \geq 2$ : Let  $a = f(b_1, \dots, b_n)$  be an infinite tree with subtrees  $b_1 : s_1, \dots, b_n : s_n$ . Suppose  $n \geq 2$ . Then without loss of generality, assume that  $b_1$  is an infinite subtree. Since  $s_2$  has at least two generators, the number of infinite inhabitants of  $s_1$  is at most  $\#s_{\text{infin}}/2 < \#s_{\text{infin}}$ . By induction hypothesis,  $b_1$  has the desired form. The same argument works for other infinite subtrees of  $a$ . For each finite subtree, there is a ground term describing it. Hence  $a$  has the desired form.

Next, suppose  $f$  has only one parameter,  $a = f(a_1)$ . If  $s$  has another infinite tree  $a' = f'(a'_1)$  starting with a different function symbol  $f'$  then the sort of  $a_1$  has less than  $\#s_{\text{infin}}$  infinite trees, and the induction hypothesis gives us the desired form for  $a_1$  and thus for  $a$ . Otherwise, all infinite trees of  $s$  start with the same function symbol  $f$ . We can apply the same argument to  $a_1$  and see that all the infinite trees of the sort of  $a_1$  must have the form  $a_1 = f_1(a_2)$ . So there are two cases. Case 1: there is an  $a_n$  such that  $a = f(f_1(\dots f_{n-1}(a_n) \dots))$  and either  $a_n$  has more than one parameter or the sort of  $a_n$  has two infinite trees starting with different generators. Then the induction hypothesis can be applied to  $a_n$  as before, yielding the desired form also for  $a$ . Case 2: there is no such  $a_n$ , meaning that the tree  $a$  is uniquely determined, as an infinite path of unary function symbols. But then  $s$  only contains one infinite tree, contradiction. So this case cannot occur.  $\square$

**Lemma B.6.** *Let  $S_{FI} \subseteq S$  be the set of sorts such that each  $s \in S_{FI}$  has only finitely many infinite trees. Let  $F_s^{\text{infin}} = \{(g : s_1 \times \dots \times s_n \rightarrow s) \mid \exists i \in \{1, \dots, n\} : s_i \notin S_{OI}\}$  be the set of generators that can construct infinite trees. Then  $S_{FI}$  is the least fixed point of the following function  $f : P(S) \rightarrow P(S)$ :*

$$f(X) := X \cup S_{OI} \cup S_{II} \cup \{s \in S \mid |F_s^{\text{infin}}| < \infty \wedge \forall g : s_1 \times \dots \times s_n \rightarrow s \in F_s : \\ \forall i \in \{1, \dots, n\} : s_i \in S_{OI} \vee (s_i \in X \wedge (\forall j \in \{1, \dots, n\} \setminus \{i\} : s_j \in S_{FI} \cap X))\}$$

where  $S_{OI}$  is the set of sorts with only finite trees and  $S_{II}$  is the set of sorts with a unique infinite tree.

*Proof.* Why is  $S_{FI}$  a fixed point? First of all, it is clear that  $S_{OI} \cup S_{II} \subseteq S_{FI}$ . Furthermore, for the sort  $s$  to have finitely many infinite trees, there have to be finitely many generators  $F_s^{\text{infin}}$  that can construct

infinite trees. Additionally, for each such generator  $g : s_1 \times \dots \times s_n \rightarrow s \in F_s^{\text{fin}}$ , there have to be finitely many infinite trees starting with  $g$ .

It is easier to describe the negation of this: If a generator  $g$  starts infinitely many infinite trees, there must be a parameter  $i$  such that  $s_i$  contains infinite trees and one of the following: (1)  $s_i$  containing infinitely many infinite trees or (2) one of the other  $s_j$  containing infinitely many trees. In either case, this leads to infinitely many infinite trees starting with  $g$ . As a formula:  $\exists i \in \{1, \dots, n\} : s_i \notin S_{0I} \wedge (s_i \notin X \vee \exists j \in \{1, \dots, n\} \setminus \{i\} : s_j \notin S_{FF} \cap X)$ . The negation is what is written in the above function definition. This explains why  $S_{FI}$  is a fixed point of  $f$ .

Next, we show that  $S_{FI}$  is the least fixed point. Let  $s$  be a sort with finitely many infinite trees and  $a$  such a tree. By Lemma B.5, there is a term  $t_a$  describing  $a$ , containing only variables  $u_i : s_i$  representing the unique infinite tree of  $s_i$ . We always choose  $t_a$  to be of minimal depth among those terms. By definition of  $f$ , each  $s_i \in f(\emptyset)$ . Let  $a$  be the infinite tree in  $S_{FI}$  such that its corresponding  $t_a$  has maximal depth  $d$ . Then one can see inductively, as in the proof of Lemma B.1, that  $s \in f^{d+1}(\emptyset)$ . Hence  $S_{FI} = \bigcup_{d \in \omega} f^{d+1}(\emptyset)$ , in other words,  $S_{FI}$  is in fact the least fixed point of  $f$ .  $\square$

## C Supplementary material for Section 5

In this section, when talking about reachability in a formula, we mean reachability from the free variables of the formula. We are also going to need the *Unique Solution Axiom* (axiom 3 in [4]), which states that for any sequence of distinct variables  $\bar{z}$  and non-variable terms  $t_i$  containing only the variables  $\bar{x}$  and  $\bar{z}$ , we have

$$\forall \bar{x}. \exists! \bar{z}. \bigwedge_i z_i = t_i.$$

---

**Algorithm 4** Algorithm for solving a basic formula  $\alpha$  with free variables  $v_0 < \dots < v_n$ . The rules 1–10 are taken from [4]. The two rules in blue at the end are new. Throughout,  $u, v, x, y, z$  denote variables,  $f, g$  function symbols and  $t$  terms.

---

**function** SOLVEBASIC( $\bar{v} = v_0 < \dots < v_n, \alpha$ )

**repeat**

**if**  $\alpha$  is  $u = u \wedge \alpha'$  **then**  $\alpha \leftarrow \alpha'$  ▷ Rule 1 (numbering as in [4])

**if**  $\alpha$  is  $u = v \wedge \alpha'$  and  $u < v$  **then**  $\alpha \leftarrow v = u \wedge \alpha'$  ▷ Rule 2

**if**  $\alpha$  is  $v = u \wedge v = t \wedge \alpha'$  and  $u < v$  **then**  $\alpha \leftarrow v = u \wedge u = t \wedge \alpha'$  ▷ Rule 3

**if**  $\alpha$  is  $u = f(\bar{y}) \wedge u = g(\bar{z}) \wedge \alpha'$  and  $f \not\equiv g$  **then return** false ▷ Rule 4

**if**  $\alpha$  is  $u = f(\bar{y}) \wedge u = f(\bar{z}) \wedge \alpha'$  **then**  $\alpha \leftarrow u = f(\bar{y}) \wedge \bar{y} = \bar{z} \wedge \alpha'$  ▷ Rule 5

**if**  $\alpha$  is  $\text{fin}(u) \wedge \text{fin}(u) \wedge \alpha'$  **then**  $\alpha \leftarrow \text{fin}(u) \wedge \alpha'$  ▷ Rule 6

**if**  $\alpha$  is  $v = u \wedge \text{fin}(v) \wedge \alpha'$  and  $u < v$  **then**  $\alpha \leftarrow v = u \wedge \text{fin}(u) \wedge \alpha'$  ▷ Rule 7

**if**  $\alpha$  is  $v = u \wedge \text{fin}(v) \wedge \alpha'$  and  $u < v$  **then**  $\alpha \leftarrow v = u \wedge \text{fin}(u) \wedge \alpha'$  ▷ Rule 8

**if**  $\alpha$  is  $\text{fin}(u) \wedge \alpha'$  and  $u$  is properly reachable from  $u$  **then return** false ▷ Rule 9

**if**  $\alpha$  is  $u = f(\bar{y}) \wedge \text{fin}(u) \wedge \alpha'$  **then**  $\alpha \leftarrow u = f(\bar{y}) \wedge \overline{\text{fin}(\bar{y})} \wedge \alpha'$  ▷ Rule 10

**if**  $\alpha$  is  $\text{fin}(u) \wedge \alpha'$  and  $u : s$  with  $s \in S_{0I}$  **then**  $\alpha \leftarrow \alpha'$  ▷ (\*)

**if**  $\alpha$  is  $\text{fin}(u) \wedge \alpha'$  and  $u : s$  with  $s \in S_{0F}$  **then return** false ▷ (\*)

**until** no changes in the last iteration

**return**  $\alpha$

---

**Theorem** (Theorem 5.2, repeated). *The function  $\text{SOLVEBASIC}(\bar{v} = v_0 < \dots < v_n, \alpha)$  from Algorithm 4 correctly solves basic formulae  $\alpha$ , i.e. it turns  $\alpha$  into an equivalent solved formula (with respect to the given variable ordering) or returns false if none exists.*

*Proof of Theorem 5.2.* Most parts of the algorithm (the numbered rules) are taken from [4] and property (1) of solved basic formulae is proven correct there. The two additional rules (\*) involve variables  $u : s$  where  $s$  is a sort without infinite, respectively finite, trees. Obviously,  $\text{fin}(u)$  is then always, respectively never, satisfied. Therefore property (2) of solved basic formulae holds as well.  $\square$

**Theorem** (Theorem 5.7, repeated). *Let  $\phi \equiv \neg(\exists \bar{x}. \alpha \wedge \bigwedge_i \phi_i)$  be a normal formula of depth at most 2 with free variables  $\bar{v}$ . Let  $I$  be the result of  $\text{FINDINSTANTIATION}(\bar{v}, \phi)$  from Algorithm 3. If  $I$  is “none”, then there is no instantiable variable. Otherwise, let  $u$  be the first instantiable variable found in  $\text{FINDINSTANTIATION}$ . Then  $\phi$  is equivalent to the following conjunction of normal formulae, in which the variable  $u$  is no longer instantiable:*

$$\bigwedge_{(\exists \bar{z}. \psi) \in I} \neg(\exists \bar{x} \bar{z}. \alpha \wedge \psi \wedge \bigwedge_i \phi_i).$$

*Proof of Theorem 5.7.* We first show that in each case, the result  $I$  of the call to  $\text{FINDINSTANTIATION}$  satisfies  $\alpha \rightarrow \bigvee_{\psi \in I} \psi$ .

For the first return statement in  $\text{FINDINSTANTIATION}$ , this is clear because if there are finitely many generators of  $s$  then one of them has to be used to construct a tree of sort  $s$ . For the second return statement, it is clear because  $u$  can have only finitely many values, so if  $I$  contains formulae describing each possible value then the disjunction over all of them must be true. For the third return statement, note that  $\text{fin}(u)$  occurs in  $\alpha$ , so  $u$  has to be finite. Hence  $I$  only contains formulae describing each finite value of  $s$ , and we have  $\text{fin}(u) \rightarrow \bigvee_{\psi \in I} \psi$ . Finally, consider the fourth return statement. The variable  $u$  has to represent either a finite tree, meaning  $\text{fin}(u)$  or one of the finitely many infinite trees in  $s$ . Again, we find that  $\bigvee_{\psi \in I} \psi$  holds.

Since  $\alpha \rightarrow \bigvee_{\psi \in I} \psi$  holds in each case,  $\phi$  is equivalent to:

$$\begin{aligned} & \neg \left( \exists \bar{x}. \alpha \wedge \left( \bigvee_{\psi \in I} \psi \right) \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i) \right) \\ & \leftrightarrow \neg \left( \exists \bar{x}. \alpha \wedge \left( \bigvee_{(\exists \bar{z}. \psi') \in I} (\exists \bar{z}. \psi') \right) \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i) \right) \\ & \leftrightarrow \neg \left( \bigvee_{(\exists \bar{z}. \psi') \in I} \exists \bar{x} \bar{z}. \alpha \wedge \psi' \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i) \right) \\ & \leftrightarrow \bigwedge_{(\exists \bar{z}. \psi') \in I} \neg \left( \exists \bar{x} \bar{z}. \alpha \wedge \psi' \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i) \right) \end{aligned}$$

By the construction of  $I$ , the instantiable variable  $u$  found in the algorithm is no longer instantiable in this transformed formula.  $\square$

**Theorem** (Theorem 5.12, repeated). *Let  $\phi$  be a fully simplified formula. If  $\phi$  has no free variables then  $\phi \equiv \text{true}$ . Otherwise both  $\phi$  and  $\neg\phi$  are satisfiable in the theory of trees.*

*Proof of Theorem 5.12.* The formula  $\phi$  has the form

$$\exists \bar{x}. \alpha \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}_i. \beta_i)$$

First consider the case of no free variables. Then no variable can be reachable in  $\exists \bar{x}. \alpha$ , hence by condition (5) of Definition 5.10,  $\bar{x}$  is empty. This implies that  $\alpha$  is just true because it cannot mention any variables. The same argument applied to each  $\exists \bar{y}_i. \beta_i$  means that  $\bar{y}_i$  is empty and  $\beta_i \equiv \text{true}$ . Since  $\phi$  is fully simplified, each  $\beta_i$  must include a conjunct not occurring in  $\alpha$ . Hence  $I = \emptyset$ . Altogether, we have  $\phi \equiv \text{true}$

If  $\phi$  contains free variables, it is enough to find a valuation for the free variables such that  $\phi$  is true in the theory of trees and another one such that  $\phi$  is false in the theory of trees.

To find a valuation that makes  $\phi$  false, consider the following: If  $\alpha$  contains a free variable  $z$ , it can be made false like this.

- If  $\alpha$  contains  $z = w$ , then  $z > w$  according to the variable ordering since  $\alpha$  is solved. Hence  $w$  is also a free variable and  $\alpha$  can be made false by instantiating  $z$  and  $w$  to different trees.
- If  $z = f(\bar{w})$  occurs in  $\alpha$ , it is enough to instantiate  $z$  to a tree not starting with  $f$  to make  $\alpha$  false, which is always possible because each sort has at least two generators.
- If  $w = t$  with  $t$  containing  $z$  occurs in  $\alpha$ , this equation must be reachable in  $\exists \bar{x}. \alpha$  by condition (5) of Definition 5.10. This means that there is an equation of the form  $z' = \dots$  in  $\alpha$ , with  $z'$  free and  $w$  reachable from  $z'$ . This situation was already handled in one of the previous two cases.
- If  $\text{fin}(z)$  occurs in  $\alpha$ , simply instantiate  $z$  to an infinite tree (which is possible by the modified definition of solved basic formula) to make  $\alpha$  false.

Otherwise,  $\alpha$  contains no free variables, so  $\bar{x}$  is empty and  $\alpha$  is true by the same argument as before. Since  $\phi$  contains a free variable, there must be a  $\beta_i$  that contains a free variable, so is nonempty. Since  $\beta_i$  is a solved basic formula, it is satisfiable by Lemma C.1. Hence there is a valuation of free variables that makes  $\neg \exists \bar{y}_i. \beta_i$  false. Then the same valuation makes  $\phi$  false.

Next, we want to find a valuation making  $\phi$  true. Let  $\beta_i^*$  be  $\beta_i$  with all conjuncts occurring in  $\alpha$  removed. Our goal is to find a valuation of the free variables and  $\bar{x}$  that makes  $\alpha$  true and every  $\exists \bar{y}_i. \beta_i^*$  false (since we cannot make the parts of  $\beta_i$  that also occur in  $\alpha$  false). Let  $\bar{x}_{lhs}$  be the variables from  $\bar{x}$  that occur on the left-hand side of an equation in  $\alpha$ . The valuation for these variables will be picked last because it is uniquely determined by the Unique Solutions Axiom, once the valuation for the other variables is chosen. So the equations of  $\alpha$  are taken care of.

If  $\text{fin}(v)$  occurs in  $\alpha$  then any equation  $v = f(\bar{w})$  occurring in any  $\beta_i$  is automatically false because  $v$  has to be properly reachable from itself (otherwise  $v$  would be instantiable), but then  $v$  cannot be finite. So the only constraints on  $v$  on the left-hand side that we care about in  $\beta_i$  are  $v = w_i$  for other variables  $w_i$ . In this case, each  $w_i$  is also a free variable because  $v > w_i$  by the variable ordering, and the sort of  $v$  has infinitely many finite trees because otherwise  $v$  would be instantiable. Thus it is always possible to find a valuation that contradicts all these finitely many equations of the form  $v = w_i$  by picking a value for  $v$  that is different from the values picked for all the  $w_i$ . This proves that we can always make  $\alpha$  true.

Next, we do a case analysis on the  $\beta_i^*$  that have not been made false yet. By reachability, each  $\beta_i^*$  has to contain  $\text{fin}(v)$  or  $v = t$  for a free variable  $v$ . Then  $v \notin \bar{x}_{lhs}$  by Definition 5.10 and we can assume that  $\text{fin}(v)$  does not occur in  $\alpha$  because this case was already discussed above. For each such free variable  $v$ , we do the following case analysis:

- Suppose there is a  $\beta_i^*$  that contains  $\text{fin}(v)$ . If  $\beta_i^*$  also contains an equation, then the following cases apply and suffice to make it false. So suppose  $\beta_i^*$  only contains fin-constraints. Then the sort of  $v$  has infinitely many infinite trees because otherwise  $v$  would be instantiable. This makes

the following cases work, by restricting the set of possible values for  $v$  to the set of infinite trees. Using such a value also makes  $\text{fin}(v)$ , and thus  $\beta_i^*$ , false as desired.

- Suppose  $v = f(\bar{w})$  occurs in some  $\beta_i^*$ . Then  $v$  must be properly reachable from itself in  $\beta_i$  because otherwise, it would be instantiable. If the sort of  $v$  had only finitely many trees then  $v$  would be instantiable, contradiction. Hence the sort of  $v$  has infinitely many trees. Since the previous cases are already handled, we can assume that the only constraints on  $v$  in all the  $\beta_j^*$ 's are of the form  $v = f(\bar{w})$  with  $v$  properly reachable from itself in  $\beta_j^*$  or  $v = w$ . Since the sort of  $v$  has infinitely many trees, it is possible to contradict all these constraints.
- Suppose  $v = w$  occurs in some  $\beta_i$ . Since the previous cases are already handled, we can assume that the only constraints on  $v$  from the  $\beta_i^*$ 's are of the form  $v = w_i$  for variables  $w_i$ . Then each  $w_i$  is also a free variable because  $v > w_i$  by the variable ordering, and the sort of  $v$  has infinitely many trees because otherwise  $v$  would be instantiable. Thus it is always possible to find a valuation that contradicts all these finitely many equations of the form  $v = w_i$  by picking a value for  $v$  that is different from the values picked for all the  $w_i$ .

This case analysis shows that we can make all the  $\beta_i^*$  false. Thus it is always possible to find a valuation that makes  $\phi$  true, as desired.  $\square$

**Lemma C.1.** *Any solved basic formula is satisfiable.*

*Proof.* Let the basic formula be given by  $\overline{v \equiv t} \wedge \overline{\text{fin}(u)}$ . By the condition (2) of solved basic formulae (Definition 5.1), each  $\bar{u}$  can be given the value of some finite tree. Since the variables  $\bar{v}$  and  $\bar{u}$  are disjoint, the Unique Solution Axiom tells us that  $\exists \bar{v}. \overline{v \equiv t}$  is satisfiable for this valuation of  $\bar{u}$ .  $\square$

**Theorem** (Theorem 5.13, repeated). *Given a formula  $\phi$ , the function  $\text{SOLVE}(\phi)$  from Algorithm 5 returns true, false, or a disjunction of fully simplified formulae that is equivalent to  $\phi$  in the extended theory of trees. In particular, if  $\phi$  is closed, it returns true or false.*

*Proof of Theorem 5.13.* The proof of this is quite involved and will take up the rest of this section. The function  $\text{SOLVE}(\phi)$  first normalizes  $\neg\phi$  and then solves its basic formula. If the latter contains a contradiction,  $\phi$  is unsatisfiable. Otherwise, the function  $\text{SOLVENESTED}$  recursively solves  $\neg\phi$ : it returns a set of solved normal formulae  $\{\psi_1, \dots, \psi_n\}$  such that  $\neg\phi$  is equivalent to  $\bigwedge_{i=1}^n \psi_i$ . It works very similarly to the original algorithm in [4]. The only change is the instantiation step, highlighted in Algorithm 5.

The following lemmas prove the correctness of this change. Lemma C.3 establishes the termination of repeated instantiation steps. The termination of the unchanged parts of the original algorithm is shown in [4]. The fact that the instantiation step is correct was proven in Theorem 5.7 already. Next, Lemma C.4 proves that the properties (1) to (4) of a solved formula (Definition 5.10) are satisfied when  $\text{REMOVEUNREACHABLEPARTS}$  is called. Lemma C.5 proves that the return value of  $\text{REMOVEUNREACHABLEPARTS}$  is correct. By construction, it satisfies property (5) as well, thus it is solved.

Since the return value of  $\text{SOLVENESTED}$  is a set of solved normal formulae  $\{\psi_1, \dots, \psi_n\}$  such that  $\neg\phi$  is equivalent to  $\bigwedge_{i=1}^n \psi_i$ ,  $\phi$  is equivalent to  $\bigvee_{i=1}^n \neg\psi_i$ . In particular, if  $n = 0$  then  $\phi$  is always false. Conversely, if each  $\psi_i$  is  $\neg\text{true}$  then  $\phi$  is always true. In all other cases, we remove subformulae that were duplicated by Rule 12 in  $\text{SOLVENESTED}$  and return the whole disjunction.  $\square$

To prove the termination of repeated instantiations, we need the following concept.

**Definition C.2** (depth). Let  $\alpha$  be a solved basic formula. The *depth* of a variable  $v$  in  $\alpha$ , denoted by  $\text{depth}_\alpha(v)$ , is defined as follows. If  $v$  is properly reachable from itself or doesn't occur on the left-hand side of an equation in  $\alpha$ , its depth is 0. Else if  $v = f(\bar{w})$  occurs in  $\alpha$ , its depth is  $\text{depth}_\alpha(v) := 1 + \max_i(\text{depth}_\alpha(w_i))$ . Else if  $v = w$  occurs in  $\alpha$ , its depth is  $\text{depth}_\alpha(v) := \text{depth}_\alpha(w)$ .

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**Algorithm 5** Extension of Djelloul, Dao, and Frhwirth's algorithm [4] for transforming a normal formula into an equivalent conjunction of solved formulae. The added part is in blue.

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function SOLVE( $\phi$ )
   $\tilde{\phi} \leftarrow \text{NORMALIZE}(\neg\phi)$  ▷ cf. Theorem 5.4
   $\bar{v} \leftarrow$  the free variables of  $\tilde{\phi}$  in some fixed order
   $\tilde{\phi} \leftarrow \text{SOLVEBASIC}(\bar{v}, \tilde{\phi})$  ▷ cf. Algorithm 4
  if  $\tilde{\phi} \equiv \text{false}$  then return false
   $\{\psi_1, \dots, \psi_n\} \leftarrow \text{SOLVENESTED}(\bar{v}, \tilde{\phi})$ 
  if  $n = 0$  then return false
  if each  $\psi_i$  is of the form  $\neg(\text{true})$  then return true
  Let  $\neg(\exists \bar{x}_i. \alpha_i \wedge \bigwedge_{j \in J_i} \neg(\exists \bar{y}_{ij}. \beta_{ij})) \equiv \psi_i$  for each  $i$ 
  remove all conjuncts (of the form  $u = v$  or  $\text{fin}(u)$ ) from each  $\beta_{ij}$  that already occur in  $\alpha_i$ 
  return  $\bigvee_{i=1}^n (\exists \bar{x}_i. \alpha_i \wedge \bigwedge_{j \in J_i} \neg(\exists \bar{y}_{ij}. \beta_{ij}))$ 

function SOLVENESTED( $\bar{v}, \phi$ )
  Let  $\neg(\exists \bar{x}. \alpha \wedge \bigwedge_i \psi_i) \equiv \phi$ 
  for  $i$  do
    add the conjunct  $\alpha$  to the basic formula of  $\psi_i$  ▷ Rule 12 in [4]
     $\psi_i \leftarrow \text{SOLVEBASIC}(\bar{v}\bar{x}, \psi_i)$ 
    if  $\psi_i \equiv \text{false}$  then  $\Psi_i \leftarrow \emptyset$ 
    else
      replace each  $u = t$  in the basic formula of  $\psi_i$  by  $u = s$  if  $u = s$  occurs in  $\alpha$  ▷ Rule 13
       $\Psi_i \leftarrow \text{SOLVENESTED}(\bar{v}\bar{x}, \psi_i)$ 
  return  $\text{SOLVEFINAL}(\bar{v}, \neg(\exists \bar{x}. \alpha \wedge \bigwedge_i (\bigcup_i \Psi_i)))$ 

function SOLVEFINAL( $\bar{v}, \phi$ )
  Let  $\neg(\exists \bar{x}. \alpha \wedge \bigwedge_i \phi_i) \equiv \phi$ 
  if there is an  $i$  such that  $\phi_i \equiv \exists \bar{y}. \alpha$  then return  $\emptyset$  ▷ Rule 14
  if depth of  $\phi$  is 3 then ▷ Rule 16
    Choose a  $j$  such that  $\phi_j$  has depth 2
    Let  $\neg(\exists \bar{y}. \beta \wedge \bigwedge_k \neg(\exists \bar{z}_k. \gamma_k)) \equiv \phi_j$ 
     $\psi \leftarrow \neg(\exists \bar{x}. \alpha \wedge \neg(\exists \bar{y}. \beta) \wedge \bigwedge_{i, i \neq j} \phi_i)$ 
     $\chi_k \leftarrow \neg(\exists \bar{x}\bar{y}\bar{z}_i. \gamma_i \wedge \bigwedge_{i, i \neq j} \phi_i)$ 
    return  $\text{SOLVEFINAL}(\bar{v}, \psi) \cup \bigcup_k \text{SOLVENESTED}(\bar{v}, \chi_k)$ 
   $I \leftarrow \text{FINDINSTANTIATION}(\bar{v}, \phi)$  ▷ cf. Algorithm 3
  if  $I \neq \text{none}$  then
    return  $\bigcup \{ \text{SOLVENESTED}(\bar{v}, \neg(\exists \bar{x}\bar{z}. \alpha \wedge \psi \wedge \bigwedge_i \phi_i)) \mid (\exists \bar{z}. \psi) \in I \}$  ▷ cf. Theorem 5.7
  else
    return  $\{ \text{REMOVEUNREACHABLEPARTS}(\bar{v}, \phi) \}$  ▷ Rule 15 (cf. Algorithm 6 in the appendix)

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**Algorithm 6** Rule 15 from [4], which removes unreachable variables and subformulae of a normal formula of depth at most 2.

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**function** REMOVEUNREACHABLEPARTS( $\bar{v}, \neg(\exists \bar{x}. \alpha \wedge \bigwedge_i \psi_i)$ ) ▷ Rule 15

$\bar{x}' \leftarrow$  the variables from  $\bar{x}$  reachable in  $\alpha$  from the free variables  $\bar{v}$   
   $\alpha' \leftarrow$  the conjuncts of  $\exists \bar{x}. \alpha$  reachable from the free variables  $\bar{v}$   
   $\alpha'' \leftarrow$  the fin-subformulae of  $\exists \bar{x}. \alpha$  unreachable from the free variables  $\bar{v}$   
   $\alpha''' \leftarrow$  the equations of  $\exists \bar{x}. \alpha$  unreachable from the free variables  $\bar{v}$   
   $\bar{x}''' \leftarrow$  the variables of  $\exists \bar{x}. \alpha$  occurring on the LHS of an equation in  $\alpha$ , and unreachable from  $\bar{v}$   
   $\bar{x}'' \leftarrow \bar{x}$  without  $\bar{x}'$  and  $\bar{x}'''$   
  **for**  $i$  **do**  
    Let  $\neg(\exists y_i. \beta_i) \equiv \phi_i$   
     $\beta_i^* \leftarrow \beta_i$  with  $\alpha''$  removed  
     $\bar{y}_i' \leftarrow$  the variables of  $\bar{x}''' \bar{y}_i$  in  $\exists \bar{x}''' \bar{y}_i. \beta_i^*$  reachable from its free variables  
     $\beta_i' \leftarrow$  the conjuncts of  $\exists \bar{x}''' \bar{y}_i. \beta_i^*$  reachable from its free variables  
   $K \leftarrow$  the set of indices  $i$  where no variable of  $\bar{x}''$  occurs in  $\beta_i'$   
  **return**  $\{\neg \exists \bar{x}'. \alpha' \wedge \bigwedge_{i \in K} \neg(\exists \bar{y}_i'. \beta_i')\}$

---

Note that this is well-defined because of the “cycle check” using reachability in the definition.

**Lemma C.3.** *There are only finitely many instantiations (calls to FINDINSTANTIATION that don't return “none”) happening in Algorithm 5. Hence the algorithm terminates.*

*Proof.* For a given normal formula  $\phi \equiv \neg(\exists \bar{x}. \alpha \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i))$  of depth 2 with free variables  $\bar{v}$ , let

$$N(\phi) = (N_k(\phi), \dots, N_1(\phi), N_0(\phi))$$

where  $k$  is the maximal depth of a variable from  $\bar{v}\bar{x}$  in  $\beta_i$ , and

$$N_j(\phi) = |\{v : s \in X \mid \max_i(\text{depth}_{\beta_i}(v)) = j\}|.$$

where  $X$  is the set of instantiable variables.

By the definition of FINDINSTANTIATION, the Instantiation Rule is only applied if  $N(\phi) \neq (0, \dots, 0)$ . We claim that the value of  $N(\phi)$  decreases with respect to lexicographical order in each recursive call of SOLVENORMALIZED after every application of the Instantiation Rule. Note that it was proved in [4] that when FINDINSTANTIATION is called, the normal formula satisfies conditions (1) to (3) of Definition 5.10.

Suppose the variable  $u$  returned by FINDINSTANTIATION was selected because there is an equation  $u = f(\bar{w})$  in  $\beta_i^*$  where  $u$  is not properly reachable from  $u$ . Then  $u$  does not occur on a LHS in  $\alpha$  because of the variable ordering: If  $u = v$  occurred in  $\alpha$ , it would also occur in  $\beta_i$  by condition (2) and  $\beta_i$  would not be solved, violating condition (1). After instantiating  $u$ , meaning adding the equation  $u = g(\bar{z})$ , the resulting basic formula  $\alpha \wedge u = g(\bar{z})$  is therefore solved, so SOLVEBASIC does not change it at all. Next, Rule 12 copies  $\alpha$  into each  $\beta_j$ . If  $\beta_i$  contains  $u = f(\bar{w})$ , this leads to the situation  $u = g(\bar{z}) \wedge u = f(\bar{w})$ . If  $f \neq g$ , this is a conflict and removes  $\beta_i$  from  $\phi$ . Otherwise, that part of  $\beta_i$  is replaced with  $\bar{z} \equiv \bar{w}$ . Given that  $\beta_i$  is a solved basic formula, the only applicable rule in  $\beta_i$  is Rule 2, switching the ordering of  $z_k = w_k$  to  $w_k = z_k$  if  $w_k > z_k$ . If there is another equation  $w_k = t$ , rule 3 will change it to  $w_k = z_k$  and  $z_k = t$ . Afterward, no more rules are applicable, and the resulting formula is solved. Denote the resulting formulae with a prime '. Let  $d = \max_i(\text{depth}_{\beta_i}(v))$ . Then we have  $\text{depth}_{\beta_i'}(z_k) \leq d - 1$  and thus  $N_j(\phi) = N_j(\phi')$  for  $j > d$  and  $N_d(\phi') < N_d(\phi)$ . Hence  $N(\phi') < N(\phi)$  as desired.

Next, suppose the variable  $u$  returned by FINDINSTANTIATION was selected because  $u$  occurs in  $\beta_i^*$  and the sort  $s$  of  $u$  has only finitely many trees. Then by the same arguments as before, new variables  $\bar{z}$

are introduced in  $\phi$  after adding  $\exists \bar{z}. \gamma$  to  $\alpha$ . However, since  $\gamma$  describes a single value for  $u$ , every variable out of  $\bar{z}, u$  occurs on the left-hand side of an equation. Hence  $u$  is no longer instantiable and none of the newly introduced variables  $\bar{z}$  are. Hence the set  $X$  gets smaller and hence each  $N_j(\phi)$  can only decrease. Thus  $N(\phi') < N(\phi)$  for the new formula  $\phi'$  as desired.

Next, suppose the variable  $u$  returned by FINDINSTANTIATION was selected because  $\text{fin}(u)$  occurs in  $\alpha$ ,  $u$  occurs in  $\beta_i^*$  and  $s \in S_{FF}$ . Then the same argument works as in the previous case.

Next, suppose the variable  $u$  returned by FINDINSTANTIATION was selected because  $s \in S_{FI}$  and there is a  $\beta_j^*$  consisting only of  $\text{fin}()$ -constraints, including  $\text{fin}(u)$ . After an instantiation of the form  $\text{fin}(u)$ ,  $\beta_j^*$  does not contain  $\text{fin}(u)$  anymore, so  $u$  is not instantiable anymore. Since no more instantiable variables were introduced, the set  $X$  gets smaller and  $N(\phi)$  decreases. After an instantiation of the form  $\exists \bar{z}. \gamma$  describing an infinite value for  $u$ , the variable  $u$  is also not instantiable anymore. And since all the additional variables  $\bar{z}$  occur on the left-hand side of an equation in  $\exists \bar{z}. \gamma$ , they are not instantiable either. Hence the set  $X$  of instantiable variables gets smaller and  $N(\phi)$  decreases. Therefore the termination of the algorithm follows from the termination of the original algorithm [4].  $\square$

**Lemma C.4.** *In Algorithm 5, when REMOVEUNREACHABLEPARTS is called,  $\phi$  satisfies conditions (1) to (4) of a solved formula from Definition 5.10*

*Proof.* From the proof of correctness of the unmodified algorithm [4], which works the same until the application of the Instantiation Rule, it follows that up until that point,  $\phi$  satisfies conditions (1) to (3). As soon as REMOVEUNREACHABLEPARTS is called, (4) is satisfied because otherwise FINDINSTANTIATION would find a variable violating (4).  $\square$

**Lemma C.5.** *The function REMOVEUNREACHABLEPARTS from Algorithm 6 (Rule 15 in [4]) is still correct in the context of the extended algorithm.*

*Proof.* As the previous lemma states, at the point where Rule 15 is applied,  $\phi \equiv \neg(\exists \bar{x}. \alpha \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i))$  satisfies conditions (1) to (4) of Definition 5.10. As in the algorithm let

- $\bar{x}'$  be the reachable variables of  $\exists \bar{x}. \alpha$ ,
- $\bar{x}'''$  the unreachable variables from  $\bar{x}$  that occur on the LHS of an equation in  $\alpha$ ,
- $\bar{x}''$  the variables from  $\bar{x}$  that are not in  $\bar{x}'\bar{x}'''$ ,
- $\alpha'$  be the reachable conjuncts of  $\exists \bar{x}. \alpha$ ,
- $\alpha''$  the unreachable  $\text{fin}()$ -subformulae of  $\exists \bar{x}. \alpha$ ,
- $\alpha'''$  the unreachable equations of  $\exists \bar{x}. \alpha$ ,
- $\beta_i^*$  the result of removing  $\alpha''$  from  $\beta_i$ ,
- $\bar{y}'_i$  the reachable variables among  $\bar{x}'''\bar{y}_i$  in  $\exists \bar{x}'''\bar{y}_i. \beta_i^*$ ,
- $\beta'_i$  the reachable conjuncts in  $\exists \bar{x}'''\bar{y}_i. \beta_i^*$
- $K \subseteq \{1, \dots, n\}$  the set of indices  $i$  such that  $i \in K$  if and only if no variable of  $\bar{x}''$  occurs in  $\beta'_i$ .

Then the claim is that  $\neg(\exists \bar{x}. \alpha \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i))$  is equivalent to  $\neg(\exists \bar{x}'. \alpha' \wedge \bigwedge_{i \in K} \neg(\exists \bar{y}'_i. \beta'_i))$ .

First note that  $\neg(\exists \bar{x}. \alpha \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i))$  is equivalent to

$$\neg(\exists \bar{x}'. \alpha' \wedge (\exists \bar{x}'' . \alpha'' \wedge (\exists \bar{x}''' . \alpha''' \wedge \bigwedge_i \neg(\exists \bar{y}_i. \beta_i))))$$

because the variables  $\bar{x}''$  can only occur in  $\alpha''$  and the variables  $\bar{x}'''$  can only occur in  $\alpha'''$ . By the Unique Solution Axiom and since  $\alpha'''$  is a solved formula, we have  $\exists! \bar{x}''' . \alpha'''$  in the extended theory of trees. According to Property 3.1.11 from [4], the previous formula is equivalent to

$$\neg(\exists \bar{x}'. \alpha' \wedge (\exists \bar{x}'' . \alpha'' \wedge \bigwedge_i \neg(\exists \bar{x}''' . \alpha''' \wedge \exists \bar{y}_i. \beta_i))).$$

By our variable convention, no variable names conflict, so the innermost existential can be pulled outside:

$$\neg(\exists \vec{x}'. \alpha' \wedge (\exists \vec{x}''. \alpha'' \wedge \bigwedge_i \neg(\exists \vec{x}''' \vec{y}_i. \alpha''' \wedge \beta_i))).$$

By condition (2) of Definition 5.10, the equations of  $\alpha$  are included in each  $\beta_i$ . In particular,  $\alpha'''$  is part of each  $\beta_i$ , which simplifies the formula to

$$\neg(\exists \vec{x}'. \alpha' \wedge (\exists \vec{x}''. \alpha'' \wedge \bigwedge_i \neg(\exists \vec{x}''' \vec{y}_i. \beta_i))).$$

Note that  $\beta_i^* \wedge \alpha'' \leftrightarrow \beta_i \wedge \alpha''$  by definition, so we can use propagate  $\alpha''$  into the innermost existential formulas:  $\alpha'' \wedge \bigwedge_i \neg(\exists \vec{x}''' \vec{y}_i. \alpha'' \wedge \beta_i) \leftrightarrow \alpha'' \wedge \bigwedge_i \neg(\exists \vec{x}''' \vec{y}_i. \alpha'' \wedge \beta_i^*)$  and back out, yielding:

$$\neg(\exists \vec{x}'. \alpha' \wedge (\exists \vec{x}''. \alpha'' \wedge \bigwedge_i \neg(\exists \vec{x}''' \vec{y}_i. \beta_i^*))).$$

Since unreachable parts of a solved basic formula can be removed by the following Lemma C.6, this is equivalent to

$$\neg(\exists \vec{x}'. \alpha' \wedge (\exists \vec{x}''. \alpha'' \wedge \bigwedge_i \neg(\exists \vec{y}_i'. \beta_i'))).$$

Since a variable from  $\vec{x}''$  can only occur in  $\beta_i'$  if  $i \notin K$ , this is equivalent to

$$\neg \left( \exists \vec{x}'. \alpha' \wedge \left( \bigwedge_{i \in K} \neg(\exists \vec{y}_i'. \beta_i') \right) \wedge \left( \exists \vec{x}''. \alpha'' \wedge \bigwedge_{i \notin K} \neg(\exists \vec{y}_i'. \beta_i') \right) \right).$$

As we will see later, the last conjunct is always true, which simplifies the formula to the desired result

$$\neg \left( \exists \vec{x}'. \alpha' \wedge \left( \bigwedge_{i \in K} \neg(\exists \vec{y}_i'. \beta_i') \right) \right).$$

To complete the proof, we have to show that the last conjunct

$$\exists \vec{x}''. \alpha'' \wedge \bigwedge_{i \notin K} \neg(\exists \vec{y}_i'. \beta_i')$$

is always true. For this, it suffices to find valuations for  $\vec{x}''$  satisfying  $\alpha''$  but none of  $\exists \vec{y}_i'. \beta_i'$  for  $i \notin K$ .

Since each  $\beta_i'$  cannot be equal to  $\alpha$  by condition (3) of Definition 5.10, each  $\beta_i'$  contains one of

- $\text{fin}(v)$  for  $v \in \vec{x}''$  and by the construction of  $\beta_i^*$ ,  $\text{fin}(v)$  does not occur in  $\alpha''$ ,
- $v = f(\vec{w})$  for  $v \in \vec{x}''$ ,
- $v = w$  where  $v \in \vec{x}''$  and  $v > w$ , implying  $w \notin \vec{y}_i'$ ,
- $u = t$  where  $v \in \vec{x}''$  occurs in  $t$ . Since it has to be reachable, that means that  $\beta_i'$  contains the conjunction  $\bigwedge_{j=1}^k w_j = t_j$  with  $t_j$  containing  $w_{j+1}$ ,  $w_{k+1} \equiv v$  and  $w_1 \notin \vec{y}_i'$ . Since the case  $w_1 \in \vec{x}''$  was already handled in a previous case, we can assume without loss of generality that  $w_1$  is a free variable.

The goal now is to find a valuation of  $\vec{x}''$  that satisfies  $\alpha''$  but that makes each of the above cases false, thus making  $\neg \exists \vec{y}_i'. \beta_i'$  true. Fix a valuation for the free variables of the formula. Let  $v : s$  be a variable from  $\vec{x}''$ .

- If  $\text{fin}(v)$  occurs in  $\alpha''$  then no  $\beta_i'$  can contain  $v = f(\vec{w})$  because  $v$  is not instantiable and thus  $v$  would have to be properly reachable from itself, contradicting finiteness. If  $s$  only contains finitely many finite trees then  $v$  occurs in no  $\beta_i'$  because  $v$  is not instantiable. Then  $v$  can be given any finite value to make  $\alpha''$  true. Otherwise,  $v$  occurs only in equations of the form  $u = t$  (reachable from some free variable  $w_1$  as seen above) or  $v = w$  in the  $\beta_i'$ . In the former case, to make the equation false, we pick a value for  $v$  that is different from the one that is determined by the fixed value of  $w_1$ . In the latter case, we pick a value  $v$  different from the value of  $w$ . Since  $s$  contains infinitely many finite trees, it is possible to pick one as the value for  $v$ , which contradicts all those finitely many equations.

- If  $\text{fin}(v)$  does not occur in  $\alpha''$  and there is a  $\beta'_j$  containing only  $\text{fin}()$ -constraints, among them  $\text{fin}(v)$ , then since  $v$  is not instantiable, we have  $s \notin S_{FI}$ . Thus there are infinitely many infinite trees of sort  $s$ . Since there are only finitely many equations of the form  $v = w$ , or  $u = t$  with  $t$  containing  $v$  (reachable from some free variable as above), or  $v = t$  with  $v$  properly reachable from itself in the  $\beta'_i$ , it is possible to find a valuation for  $v$  that contradicts all of them, as desired.
- If  $\text{fin}(v)$  does not occur in  $\alpha''$  and there is no  $\beta'_j$  containing only  $\text{fin}()$ -constraints, among them  $\text{fin}(v)$ , then there are two cases. If  $s \in S_{FF} \cap S_{FI}$  then since  $v$  is not instantiable, no  $\beta'_i$  contains  $v$ , and there are no constraints to contradict, or  $\alpha$  contains an equation  $v = t$ , in which case each  $\beta'_i$  also contains the same equation. Hence all the  $\beta'_i$  can be contradicted by picking a value different from  $t$  for  $v$ . Otherwise ( $s \notin S_{FF} \cap S_{FI}$ ), there are infinitely many possible valuations for  $v$  while there are only finitely many constraints of the form  $u = t$  with  $t$  containing  $v$  (reachable from some free variable as above), or  $v = w$ , or  $v = f(\bar{w})$  with  $v$  properly reachable from itself. Hence it is possible to find a valuation for  $v$  that contradicts all of these constraints.

We have shown above that by picking valuations for the variables from  $\bar{x}''$  as described above, each  $\beta'_i$  containing an equation is contradicted by the above valuation. If a  $\beta'_i$  contains only  $\text{fin}()$ -constraints then at least one of those  $\text{fin}(v)$  is contradicted as described above. This means that the above valuations for  $v$  make all the  $\beta'_i$  false, while satisfying  $\alpha''$ , independently of the values of the free variables.

This means that the formula

$$\exists \bar{x}'' . \alpha'' \wedge \bigwedge_{i \notin K} \neg(\exists \bar{y}'_i . \beta'_i)$$

is valid in the extended theory of trees. □

The above proof made use of the following lemma.

**Lemma C.6.** *Let  $\bar{x}$  be a vector of variables and  $\alpha$  a solved basic formula and let  $\bar{x}'$  be reachable variables and  $\alpha'$  be the conjunction of equations and  $\text{fin}()$ -formulae that are reachable in  $\exists \bar{x} . \alpha$ . Then in the theory of trees,  $\exists \bar{x} . \alpha$  is equivalent to  $\exists \bar{x}' . \alpha'$ .*

*Proof.* Let  $\bar{x}''$  be the unreachable variables in  $\exists \bar{x} . \alpha$  that do not occur on the LHS of an equation of  $\alpha$  and  $\bar{x}'''$  be the unreachable variables which do. Similarly, let  $\alpha''$  be the conjunction of unreachable  $\text{fin}()$ -formulae and  $\alpha'''$  be the conjunction of unreachable equations in  $\exists \bar{x} . \alpha$ . By the definition reachability,  $\bar{x}''$  and  $\bar{x}'''$  do not occur in  $\alpha'$ . Hence  $\exists \bar{x} . \alpha$  is equivalent to

$$\exists \bar{x}' . \alpha' \wedge (\exists \bar{x}'' . \alpha'' \wedge (\exists \bar{x}''' . \alpha''')).$$

By the Unique Solution Axiom,  $\exists ! \bar{x}''' . \alpha'''$  holds in the extended theory of trees. Hence the formula simplifies to

$$\exists \bar{x}' . \alpha' \wedge (\exists \bar{x}'' . \alpha'').$$

Since  $\alpha''$  contains only  $\text{fin}()$ -formulae and since by Definition 5.1, they are all satisfiable,  $\exists \bar{x}'' . \alpha''$  is true in the theory of trees as well. Hence the original formula is equivalent to  $\exists \bar{x}' . \alpha'$ , as desired. □

Time to solve	selector semantics			
	standard		default values	
< 1 ms	534	13.35%	197	4.93%
< 10 ms	2241	56.04%	1415	35.38%
< 100 ms	3247	81.20%	3224	80.62%
< 1 s	3659	91.50%	3779	94.50%
< 10 s	3816	95.42%	3929	98.25%
timed out (> 10 s)	183	4.58%	70	1.75%
total	3999	100%	3999	100%

Table 1: Results of the SMT-LIB QF\_DT benchmark suite: the number of benchmarks solved in the specified time limit (wall-clock time). The measurements were made on a notebook computer with an Intel Core i5-8250U CPU and 16 GB of RAM.