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Generalized power series solutions to linear partial differential equations

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Abstract

Let $\Theta = \mathcal{C}[\mathrm{e}^{-x_1}, \ldots, \mathrm{e}^{-x_n}][\partial_1, \ldots, \partial_n]$ and $\mathcal{S} = \mathcal{C}[x_1, \ldots, x_n][[\mathrm{e}^{\mathcal{C}x_1 + \cdots + \mathcal{C}x_n}]]$, where $\mathcal{C}$ is an effective field and $x_1^{\mathbb{N}} \cdots x_n^{\mathbb{N}} \mathrm{e}^{\mathcal{C}x_1 + \cdots + \mathcal{C}x_n}$ and $\mathcal{S}$ are given a suitable asymptotic ordering $\preccurlyeq$. Consider the mapping $L : \mathcal{S} \to \mathcal{S}^l; f \mapsto (L_1 f, \ldots, L_l f)$, where $L_1, \ldots, L_l \in \Theta$. For $g = (g_1, \ldots, g_l) \in \mathcal{S}_L^l = \operatorname{im} L$, it is natural to ask how to solve the system $Lf = g$. In this paper, we will effectively characterize $\mathcal{S}_L^l$ and show how to compute a so called distinguished right inverse $L^{-1} : \mathcal{S}_L^l \to \mathcal{S}$ of L. We will also characterize the solution space of the homogeneous equation $Lh = 0$.

Keywords: Linear partial differential equation; Asymptotics; Algorithm; Differential algebra; Formal power series; Tangent cone algorithm

1. Introduction

A well-known theorem (Fabry, 1885) states that any linear differential equation over $\mathbb{C}[[z]]$ admits a basis of formal solutions of the form

$$(f_0(\sqrt[p]{z}) + \cdots + f_d(\sqrt[p]{z}) \log^d z) z^\alpha \mathrm{e}^{P(1/\sqrt[p]{z})},$$

with $f_0, \ldots, f_d \in \mathbb{C}[[z]]$, $\alpha \in \mathbb{C}$, $P \in \mathbb{C}[X]$ and $p, d \in \mathbb{N}^>$. This theorem naturally generalizes to the case when $\mathbb{C}$ is replaced by an effective algebraically closed field of coefficients $\mathcal{C}$. If we also replace the coefficients by polynomials in $\mathcal{C}[z]$, then several algorithms exist for

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the computation of a basis of solutions (Malgrange, 1979; Della Dora et al., 1982; van Hoeij, 1997).

There are several directions in which the above theorem may be generalized. In van der Hoeven (1997, 2001, 2006), it is shown how to deal with so called transseries coefficients (a transseries is an object which is constructed from $\mathbb{R}$ or $\mathbb{C}$ and an infinitely large variable x using exponentiation, logarithm and infinite summation). In collaboration with M. Aschenbrenner and L. van den Dries, we are currently working on a generalization to arbitrary asymptotic fields (an asymptotic field is a differential field with a total asymptotic ordering which is "naturally compatible" with the derivation).

In this paper, we will be concerned with the generalization to the case of linear partial differential equations. The asymptotic resolution of systems of such equations can be decomposed into two subproblems: the computation of analogues of the exponential parts $\mathrm{e}^{P(1/\sqrt[p]{z})}$ and the computation of the corresponding coefficients. We intend to deal with the first subproblem in a forthcoming paper and focus on the second subproblem in what follows.

In the case of holonomic systems of linear differential equations, algorithms are known for the computation of formal and convergent generalized series solutions (Saito et al., 2000, Chapter 2) in what the authors call "Nilsson rings" (Nilsson, 1965). On the other extreme, there exists a method (Aroca and Cano, 2001) to find "fractional power series solutions" to a single p.d.e. with coefficients in $\mathcal{C}[[z_1, \ldots, z_n]]$. In this paper, we will search for formal series solutions to consistent systems of linear differential equations in variants of Nilsson rings of the form $\mathcal{C}[\log z_1, \ldots, \log z_n][[z_1^{\mathcal{C}} \cdots z_n^{\mathcal{C}}]]$. One of the major difficulties is to cope with the integrability constraints which arise when considering more than one equation.

In fact, in the continuation of our previous work on transseries, we will rather work with infinitely large variables $x_1, \ldots, x_n$ and series in $\mathrm{e}^{-x_1}, \ldots, \mathrm{e}^{-x_n}$. In this equivalent setting, our linear differential operators belong to $\Theta = \mathcal{C}[\mathrm{e}^{-x_1}, \ldots, \mathrm{e}^{-x_n}][\partial_1, \ldots, \partial_n]$ and we consider series in

$$\mathcal{C}[x][[\mathfrak{E}]] = \mathcal{C}[x_1, \ldots, x_n][[\mathfrak{E}]]$$

where $\mathfrak{E} = \mathrm{e}^{\mathcal{C}x_1 + \cdots + \mathcal{C}x_n}$. More precisely, we assume a total asymptotic ordering $\preccurlyeq$ on $\mathfrak{E}$ and consider so called grid-based series (van der Hoeven, 1997, 2006) with monomials in $\mathfrak{E}$ and coefficients in $\mathcal{C}[x_1, \ldots, x_n]$.

In Sections 2 and 3 we first recall classical algorithms for the computation of "standard bases", which are used to reduce a system of equations like $Lf = g$ with $L \in \Theta$ and $g \in \mathcal{C}[x][[\mathfrak{E}]]$ to suitable normal forms. The first algorithm is a variant of the skew version (Castro, 1984, 1987; Galligo, 1985; Takayama, 1991) of Buchberger's algorithm (Buchberger, 1965, 1985), although we rather compute coherent autoreduced sets in the sense of differential algebra (Rosenfeld, 1959; Boulier, 1994). We also recall Mora's standard cone algorithm (Mora, 1983; Mora et al., 1992). However, we will systematically present them in the setting of p.d.e.s with second members, so the reader might at least want to take a look at the notations. Also, Corollaries 2 and 4 characterize when a system of equations with second members satisfies the necessary integrability constraints which ensure the existence of a solution.

In Section 4, we will start with the study of linear p.d.e.s with constant coefficients in $\mathcal{C}$. It is classical that the resolution of such equations in $\mathfrak{E}$ is equivalent to finding the roots of a set of polynomial equations in $\mathcal{C}[\xi] = \mathcal{C}[\xi_1, \ldots, \xi_n]$. In particular, solution sets in $\mathfrak{E}$ correspond to radical ideals in $\mathcal{C}[\xi]$. More generally, we will show that there exists a correspondence between the solution sets in $\mathcal{S} = \bigoplus_{\mathfrak{e} \in \mathfrak{E}} \mathcal{C}[x]\mathfrak{e}$ and arbitrary ideals in $\mathcal{C}[\xi]$.

An important technique that we will use is the computation of so called "distinguished solutions" to systems of equations with second members. More precisely, given $L = (L_1, \ldots, L_l) \in \mathcal{C}[\partial_1, \ldots, \partial_n]^l$, we may consider L as an operator $L : \mathcal{S} \to \mathcal{S}^l; f \mapsto (L_1 f, \ldots, L_l f)$. Denoting $\mathcal{S}_L^l = \operatorname{im} L$, we will effectively construct a right inverse $L^{-1} : \mathcal{S}_L^l \to \mathcal{S}$ of L. This right inverse is unique with the property that the coefficient of any $\mathfrak{h} \in \mathfrak{H}_L$ in any $f \in \operatorname{im} L^{-1}$ vanishes, where $\mathfrak{H}_L$ denotes the set of dominant monomials of solutions h to $Lh = 0$. Having constructed L^{-1}, we will also show how the space of solutions $\mathcal{H}_L$ to $Lh = 0$ can be obtained from $\mathfrak{H}_L$.

In the last Section 5, we will study the case of linear p.d.e.s with coefficients in $\mathcal{C}[\mathrm{e}^{-x_1}, \ldots, \mathrm{e}^{-x_n}]$ (for effective purposes) and $\mathcal{C}[[\mathfrak{E}]]$ (for theoretical purposes). We will first show how to reduce systems of such equations to suitable asymptotic normal forms. Given a system in normal form, we will next show how to compute a distinguished right inverse in a coefficientwise manner. We will also characterize the set $\mathfrak{H}_L$ in this context and give an explicit "strong basis" for $\mathcal{H}_L$.

Remark 1. Section 5.2 in particular contains a skew version of Mora's tangent cone algorithm. One of the referees pointed us to another such algorithm, which appeared recently (Granger et al., 2005). Besides the fact this alternative algorithm is applied to another problem (ideal membership and the computation of sygyzies), it is also a bit different in spirit: whereas our algorithm uses a twisted version of reduction (which enforces good properties for the ecart), the algorithm in Granger et al. (2005) is based on homogenization.

2. Standard bases for admissible monomial orderings

2.1. Monomial orderings

Consider the "monomial monoid" $\mathfrak{X} = x_1^{\mathbb{N}} \cdots x_n^{\mathbb{N}}$, whose elements are of the form $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, with $\alpha \in \mathbb{N}^n$. A total ordering $\preccurlyeq$ on $\mathfrak{X}$ is called a *monomial ordering*, if it is compatible with the multiplication, i.e. $x^\alpha \preccurlyeq x^\beta \wedge x^{\alpha'} \preccurlyeq x^{\beta'} \Rightarrow x^{\alpha+\alpha'} \preccurlyeq x^{\beta+\beta'}$. It is classical (Robbiano, 1985) that any such an ordering is non-uniquely determined by a finite sequence of vectors $\lambda_1, \ldots, \lambda_l \in \mathbb{R}^n \setminus \{0\}$ and

$$x^\alpha \succ x^\beta \tag{1}$$

$$\Longleftrightarrow \exists i, (\alpha - \beta) \cdot \lambda_1 = \cdots = (\alpha - \beta) \cdot \lambda_{i-1} = 0 \wedge (\alpha - \beta) \cdot \lambda_i > 0. \tag{2}$$

Here $\cdot$ denotes the scalar product. Clearly, the relation (1) allows to extend $\preccurlyeq$ to $x^{\mathbb{Z}^n}$ and even $x^{\mathbb{Q}^n}$. Moreover, this extension is unique so as to preserve the compatibility with the multiplication.

We say that $\preccurlyeq$ is *admissible* if $1 \prec x_i$ for all i. In that case, $\preccurlyeq$ extends the (partial) divisibility ordering $|$ on $\mathfrak{X}$. In particular, from Dickson's lemma, it follows that $\preccurlyeq$ is well-ordered. Given a subset $\mathfrak{G} \subseteq \mathfrak{X}$, we will denote by $\mathfrak{F}_{\mathfrak{G}} = \{\mathfrak{x} \in \mathfrak{X} : \exists \mathfrak{y} \in \mathfrak{G}, \mathfrak{y} | \mathfrak{x}\}$ the final segment of $\mathfrak{X}$ generated by $\mathfrak{G}$ for the divisibility relation. We recall that each final segment is finitely generated.

Let $\mathcal{C}$ be a constant field of characteristic zero. Given a monomial ordering on $\mathfrak{X}$, a non-zero polynomial $f \in \mathcal{C}[x] = \mathcal{C}[x_1, \ldots, x_n]$ and a monomial $\mathfrak{x} \in \mathfrak{X}$, we denote by $f_{\mathfrak{x}}$ the coefficient of $\mathfrak{x}$ in f. We also denote by $\mathfrak{d}_f$ the highest monomial for $\preccurlyeq$ occurring in f and by c_f the corresponding coefficient. We call $\mathfrak{d}_f$ the *dominant monomial* of f, c_f its *dominant coefficient* and $\tau_f = c_f \mathfrak{d}_f$ its *dominant term*. The relation $\preccurlyeq$ naturally extends to $\mathcal{C}[x]$ by $f \preccurlyeq g \Leftrightarrow f = 0 \vee (f \neq 0 \wedge g \neq 0 \wedge \mathfrak{d}_f \preccurlyeq \mathfrak{d}_g)$. We denote by $\asymp$ the equivalence relation associated to $\preccurlyeq$, so that $f \asymp g \Leftrightarrow f \preccurlyeq g \preccurlyeq f$. Similarly, we write $f \sim g$ if $\tau_f = \tau_g$, which is equivalent to $f - g \prec f$.

2.2. Differential polynomials

Let $\mathcal{K}$ be a differential field with derivations $\partial_1, \ldots, \partial_n$ and field of constants $\mathcal{C}$. Given formal variables $F_1, \ldots, F_k$, we denote by $\mathcal{K}\{F_1, \ldots, F_k\}$ the differential algebra of differential polynomials in $F_1, \ldots, F_k$ over $\mathcal{K}$. Any $P \in \mathcal{K}\{F_1, \ldots, F_k\}$ admits a unique decomposition

$$P = P_0 + \cdots + P_d,$$

where each P_i is homogeneous of degree i. We denote by $\mathcal{K}\{F\}_i$ the space of homogeneous polynomials of degree i and $\mathcal{K}\{F\}_{\leqslant i} = \mathcal{K}\{F\}_i \oplus \cdots \oplus \mathcal{K}\{F\}_0$.

Given $P \in \mathcal{K}\{F_1, \ldots, F_q\}^p$ and a tuple $Q \in \mathcal{K}\{F_1, \ldots, F_k\}^q$, the substitution of Q_i for F_i $(i = 1, \ldots, q)$ in P yields a new tuple of differential polynomials in $\mathcal{K}\{F_1, \ldots, F_k\}^p$, called the composition of P and Q, and denoted by $P \circ Q$. If $P \in \mathcal{K}\{F_1, \ldots, F_q\}^p_{\leqslant i}$ and $Q \in \mathcal{K}\{F_1, \ldots, F_k\}^q_{\leqslant j}$, then $P \circ Q \in \mathcal{K}\{F_1, \ldots, F_k\}^p_{\leqslant ij}$. In particular, $\mathcal{K}\{F\}_{\leqslant 1}$ is an algebra for $\circ$ and $\mathcal{R}\{F\}_1 \oplus \mathcal{S}$ is a subalgebra of $\mathcal{K}\{F\}_{\leqslant 1}$ whenever $\mathcal{R}$ and $\mathcal{S}$ are subalgebras of $\mathcal{K}$ with $\mathcal{S} \supseteq \mathcal{R}$. If $P \in \mathcal{K}\{F_1, \ldots, F_k\}_1$, we will denote by $P^1, \ldots, P^k$ the unique elements of $\mathcal{K}\{F\}_1$, such that $P = P^1 \circ F_1 + \cdots + P^k F_k$.

Example 1. If

$$\begin{aligned} P &= \partial_1 \partial_2 F_1 + 3\partial_1 F_1 + 2\partial_2 F_2 \\ Q_1 &= \partial_1 F + \partial_2 F \\ Q_2 &= x_2 \partial_1^2 F \end{aligned}$$

then

$$\begin{aligned} P \circ (Q_1, Q_2) &= \partial_1 \partial_2 Q_1 + 3\partial_1 Q_1 + 2\partial_2 Q_2 \\ &= \partial_1^2 \partial_2 F + \partial_1 \partial_2^2 F + 3\partial_1^2 F + 3\partial_1 \partial_2 F + 2x_2 \partial_1^2 \partial_2 F + 2\partial_1^2 F. \end{aligned}$$

Example 2. If $K, L \in \mathcal{K}[\partial_1, \ldots, \partial_n]$ are differential operators, then

$$(KF) \circ (LF) = (KL)(F).$$

In other words, $\mathcal{K}\{F\}_1$ is isomorphic to $\mathcal{K}[\partial_1, \ldots, \partial_n]$.

In the remainder of this paper, we will only study differential polynomials with second members $P \in \mathcal{R}\{F\}_1 \oplus \mathcal{S}$ with $\mathcal{R}$ and $\mathcal{S}$ as above (and often $\mathcal{R} = \mathcal{S} = \mathcal{K}$). Formally speaking, the monomial monoid $\mathfrak{T} = \{\partial^\alpha F = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} F : \alpha \in \mathbb{N}^n\}$ for $\circ$ is isomorphic to $\mathfrak{X}$. Consequently, the sets $\mathcal{K}[x]$ and $\mathcal{K}\{F\}_1 = \mathcal{K}[\mathfrak{T}]$ are isomorphic as vector spaces (but not necessarily as algebras, except when $\mathcal{K} = \mathcal{C}$). This isomorphism induces natural definitions of $\mathfrak{d}_P$, c_P and τ_P for $P \in \mathcal{K}\{F\}_1$ and of $\preccurlyeq$, $\asymp$, $\sim$ and $|$ on $\mathcal{K}\{F\}_1$. These definitions naturally extend to $\mathcal{K}\{F\}_{\leqslant 1} = \mathcal{K}[\mathfrak{T} \cup \{1_{\mathcal{K}}\}]$, by taking $1_{\mathcal{K}} \prec \mathfrak{t}$ for all $\mathfrak{t} \in \mathfrak{T}$. For instance, $\mathfrak{d}_{\partial_1^2 F + 2\partial_2 F + 3} = \partial_1^2 F$, if $\partial_1 \succ \partial_2 \succ 1$.

In our context of linear differential polynomials with second members, a differential ideal of $\mathcal{K}\{F\}_{\leqslant 1}$ is a $\mathcal{K}$-subvector space which is stable under $\partial_1, \ldots, \partial_n$ (i.e. left composition with $\partial_1 F, \ldots, \partial_n F$). Moreover, if $I \cap \mathcal{K} \neq \{0\}$, then we require that $I = \mathcal{K}\{F\}_{\leqslant 1}$. Any tuple $P = (P_1, \ldots, P_p) \in \mathcal{K}\{F\}^p_{\leqslant 1}$ naturally generates a differential ideal $[P]$. If $[P] \cap \mathcal{K} = \{0\}$, then $[P] = \{A \circ P : A \in \mathcal{K}\{F_1, \ldots, F_p\}_1\}$. When seeing P as a system of equations

$P_1(f) = \cdots = P_p(f) = 0$, where f belongs to any differential $\mathcal{K}$-algebra $\mathcal{S}$, then these equations are equivalent to $\forall A \in [P], A(f) = 0$. In particular, we say that a second system $Q = (Q_1, \ldots, Q_q) \in \mathcal{K}\{F\}_{\leqslant 1}^p$ is *equivalent* to P, if $[P] = [Q]$.

In the sequel, it will be convenient to extend notation for sets to tuples. For instance,

$$(P_1, \ldots, P_p) \cup (Q_1, \ldots, Q_q) := (P_1, \ldots, P_p, Q_1, \ldots, Q_q)$$

and

$$(P_1, \ldots, P_p) \setminus (Q) := (P_1, \ldots, P_{i-1}, P_{i+1} \ldots, P_p)$$

if i is smallest with $P_i = Q$ and

$$(P_1, \ldots, P_p) \setminus (Q) := (P_1, \ldots, P_p)$$

if no such i exists.

2.3. Ritt reduction for linear equations with second members

Assume that we fixed an admissible monomial ordering $\preccurlyeq$ on $\mathfrak{X}$ and denote $\mathcal{L} = \mathcal{K}\{F\}_{\leqslant 1}$. Let $P, Q \in \mathcal{L} \setminus \mathcal{K}$ with $\mathfrak{d}_Q | \mathfrak{d}_P$, so that $\mathfrak{d}_P = \mathfrak{d}_{P,Q} \circ \mathfrak{d}_Q$ for some $\mathfrak{d}_{Q,P} \in \mathfrak{T}$. The *partial reduction* Red (P, Q) of P w.r.t. Q is defined by

$$\text{Red}\,(P, Q) := c_Q P - c_P \mathfrak{d}_{P,Q} \circ Q \prec P \tag{3}$$

Given a system $Q = (Q_1, \ldots, Q_q) \in (\mathcal{L} \setminus \mathcal{K})^q$, let $\mathfrak{F}_Q = \mathfrak{F}_{\{\mathfrak{d}_{Q_1}, \ldots, \mathfrak{d}_{Q_q}\}}$. A *normal form* for $P \in \mathcal{L}$ modulo Q is an $R \in \mathcal{L}$, with $R \in \mathcal{K}$ or $\mathfrak{d}_R \notin \mathfrak{F}_Q$, and such that

$$H \circ P = A \circ Q + R,$$

for certain $H \in \mathcal{K}^{\neq} F$ (where $\mathcal{K}^{\neq} = \{c \in K : c \neq 0\}$) and $A \in \mathcal{K}\{F_1, \ldots, F_q\}_1$ with $A^i \circ Q_i \preccurlyeq P$ for all i. In that case, we write $P \longrightarrow_Q R$. We say that Q is *autoreduced* if $Q_i \longrightarrow_{Q \setminus (Q_i)} Q_i$ for all i. By using partial reductions of P w.r.t. members of Q as long as possible, one obtains a normal form with $H \in c_{Q_1}^{\mathbb{N}} \cdots c_{Q_q}^{\mathbb{N}} F$:

Algorithm NF (P, Q)
Input: $P \in \mathcal{L}$ and $Q \in (\mathcal{L} \setminus \mathcal{K})^q$
Output: a normal form of P modulo Q
while $P \notin \mathcal{K} \wedge \exists i, \mathfrak{d}_{Q_i} | \mathfrak{d}_P$ **do**
  $P :=$ Red (P, Q_i)
return P

Given $P, Q \in \mathcal{L} \setminus \mathcal{K}$, let α, β be such that $\mathfrak{d}_P = \partial^\alpha F$ and $\mathfrak{d}_Q = \partial^\beta F$. Setting $\gamma = \sup(\alpha, \beta) = (\max(\alpha_1, \beta_1), \ldots, \max(\alpha_n, \beta_n))$, $\mathfrak{d}_{P,Q} = \partial^{\gamma - \beta} F$ and $\mathfrak{d}_{Q,P} = \partial^{\gamma - \alpha} F$, the *$\Delta$-polynomial* of P and Q is defined by

$$\Delta_{P,Q} := c_Q \mathfrak{d}_{Q,P} \circ P - c_P \mathfrak{d}_{P,Q} \circ Q. \tag{4}$$

By construction, be have $\Delta_{P,Q} \prec \mathfrak{d}_{Q,P} \circ P \asymp \mathfrak{d}_{P,Q} \circ Q$. We also notice that $\Delta_{P,Q} = \text{Red}\,(P, Q)$, whenever $\mathfrak{d}_Q | \mathfrak{d}_P$. We say that a system $Q = (Q_1, \ldots, Q_q)$ is *coherent* if $\Delta_{Q_i, Q_j} \longrightarrow_Q 0$ for all $1 \leqslant i < j \leqslant q$. A coherent and autoreduced system $Q = (Q_1, \ldots, Q_q)$ will also be called a *standard basis*. Given an arbitrary system $Q \in \mathcal{L}^q$, the following classical algorithm computes a standard basis which is equivalent to Q.

Algorithm SB (Q)
Input: $Q \in \mathcal{L}^q$
Output: a standard basis which is equivalent to Q

$Q' := ()$
while $Q \neq Q'$
  $Q' := Q$
  if $\exists i, Q_i \in \mathcal{K}^{\neq}$ **then return** (1)
  while $\exists i$, NF $(Q_i, Q \setminus (Q_i)) = 0$ **do**
    $Q := Q \setminus (Q_i)$
  if $\exists i, \exists j, i < j \wedge R :=$ NF $(\Delta_{Q_i,Q_j}, Q) \neq 0$ **then**
    $Q := Q \cup (R)$
return Q

Remark 2. If $\mathcal{K} = \mathcal{C}$, then under the natural isomorphism of $\mathcal{C}\{F\}_1$ with $\mathcal{C}[x]$, the notions of partial reduction and Δ-polynomials correspond to reduction and S-polynomials in Buchberger's algorithm (up to details: Buchberger rather takes $S_{P,Q} = c_Q^{-1}(c_Q \mathfrak{d}_Q P - c_P \mathfrak{d}_P Q)$. Also, he not only reduces the dominant term of P in NF , but all terms). Consequently, Buchberger's algorithm for computing a Gröbner basis (Buchberger, 1965, 1985) corresponds to the above algorithm for computing a coherent autoreduced set. Coherent autoreduced sets were first introduced by Rosenfeld (Rosenfeld, 1959) and they are similar (although more effective) to the characteristic sets introduced by Ritt (Ritt, 1950). We opted for Hironaka's name standard bases here (Hironaka, 1964) in view of the generalization in the next section.

2.4. Theoretical properties of standard bases

Consider a standard basis $(Q_1, \ldots, Q_q) \in (\mathcal{L} \setminus \mathcal{K})^q$. Then the reduction of each Δ-polynomial Δ_{Q_i,Q_j} with $i < j$ to zero yields a relation

$$\Delta_{Q_i,Q_j} = c_{Q_j} \mathfrak{d}_{Q_j,Q_i} \circ Q_i - c_{Q_i} \mathfrak{d}_{Q_i,Q_j} \circ Q_j = A^1 \circ Q_1 + \cdots + A^q \circ Q_q,$$

with $A^k \circ Q_k \prec \mathfrak{d}_{Q_j,Q_i} \circ Q_i \asymp \mathfrak{d}_{Q_i,Q_j} \circ Q_j$ for all k. This relation may be rewritten as

$$R_{Q,i,j} \circ Q = 0 \tag{5}$$

with $R_{Q,i,j} \in \mathcal{K}\{F_1, \ldots, F_q\}_1$. We call (5) the *critical relation* for the pair (Q_i, Q_j). Notice that we may regard the set of all critical relations as a tuple $R_Q \in \mathcal{K}\{F_1, \ldots, F_q\}_1^{q(q-1)/2}$.

Lemma 1. *Let $Q = (Q_1, \ldots, Q_q) \in (\mathcal{L} \setminus \mathcal{K})^q$ be a standard basis. Then the $R_{Q,i,j}$ generate the space of all $A \in \mathcal{K}\{F_1, \ldots, F_q\}_1$ with $A \circ Q = 0$. In other words, given $A \in \mathcal{K}\{F_1, \ldots, F_q\}_1$ with $A \circ Q = 0$, there exists a $\Sigma \in \mathcal{K}\{F_1, \ldots, F_{q(q-1)/2}\}_1$ with $A = \Sigma \circ R_Q$.*

Proof. Assume for contradiction that there exists a relation $A \circ Q = 0$ which is not generated by the $R_{Q,i,j}$. We may choose A such that $\mathfrak{t} = \max\{\mathfrak{d}_{A^i \circ Q_i} : A^i \neq 0\}$ is minimal, as well as the number of i with $\mathfrak{t} = \mathfrak{d}_{A^i \circ Q_i}$. Since $(A \circ Q)_{\mathfrak{t}} = 0$, there must be at least two indices i and j with $\mathfrak{t} = \mathfrak{d}_{A^i \circ Q_i} = \mathfrak{d}_{A^j \circ Q_j}$. Using the fact that $\mathfrak{d}_{Q_j,Q_i} \circ \mathfrak{d}_{Q_i}$ divides $\mathfrak{t}$, let $\mathfrak{u} \in \mathfrak{T}$ be such that $\mathfrak{t} = \mathfrak{u} \circ \mathfrak{d}_{Q_j,Q_i} \circ \mathfrak{d}_{Q_i}$, $\lambda = c_{A_i}/c_{\mathfrak{u} \circ (R_{Q,i,j})_i}$ and $\tilde{A} = A - \lambda \mathfrak{u} \circ R_{Q,i,j}$. By construction, $\tilde{A}^i \prec A^i$ and $\tilde{A}^j \preccurlyeq A^j$, so $\tilde{A}^i \circ Q_i \prec \mathfrak{t}$ and $\tilde{A}^j \circ Q_j \preccurlyeq \mathfrak{t}$. For all $k \notin \{i, j\}$, we also have $\tilde{A}^k \sim A^k$, so

$\tilde{A}^k \circ Q_k \sim A^k \circ Q_k$. It follows that the relation $\tilde{A} \circ Q$ is smaller than the original relation $A \circ Q$ in the sense of the minimality hypothesis. This contradiction completes the proof. □

Consider a system $L = (L_1, \ldots, L_l)$ of linear differential polynomials in $\mathcal{K}\{F\}_1$. Given a tuple $g = (g_1, \ldots, g_l) \in \mathcal{K}^l$, we say that g is compatible with L, if for every relation $A \circ L = 0$ with $A \in \mathcal{K}\{F_1, \ldots, F_l\}_1$, we have $A \circ g = 0$. The set of such tuples forms a subvector space of $\mathcal{K}^l$, which we denote by $\mathcal{K}^l_L$.

Corollary 1. *The system $L - g = (L_1 - g_1, \ldots, L_l - g_l)$ with $L_i \in \mathcal{K}\{F\}_1^{\neq}$ and $g_i \in \mathcal{K}$ is a standard basis if and only if L is a standard basis and g is compatible with L.*

Proof. Assume that $L - g$ is a standard basis. Consider $P, Q, R \in \mathcal{K}\{F\}_1 \oplus \mathcal{K}$ with $P_1 \neq 0$ and $Q_1 \neq 0$. Then Red $(P, Q)_1 =$ Red (P_1, Q_1) if $\mathfrak{d}_Q | \mathfrak{d}_P$ and $(\Delta_{P,Q})_1 = \Delta_{P_1,Q_1}$. It follows that L is a standard basis with critical relation $R_{L,i,j} = R_{L-g,i,j}$ for all $i < j$. Given a relation $A \circ L = 0$, Lemma 1 now implies $A = \Sigma \circ R_L = \Sigma \circ R_{L-g}$ for a certain $\Sigma \in \mathcal{K}\{F_1, \ldots, F_l\}_1^{l(l-1)/2}$. We conclude that $A \circ (L - g) = 0$, whence $A \circ g = 0$.

Assume now that L is a standard basis and that g is compatible with L. Then $L - g$ is autoreduced, since $\mathfrak{d}_{L_i - g_i} = \mathfrak{d}_{L_i} \nmid \mathfrak{d}_{L_j} = \mathfrak{d}_{L_j - g_j}$ for all $i \neq j$. Furthermore, for all $i \neq j$, the relation $R_{L,i,j} \circ L = 0$ implies $R_{L,i,j} \circ g = 0$. But the relation $R_{L,i,j} \circ (L - g) = 0$ precisely proves that $\Delta_{L_i - g_i, L_j - g_j}$ reduces to zero modulo $L - g$. Hence $L - g$ is coherent. □

Corollary 2. *Given a standard basis $L \in \mathcal{K}\{F\}_1^l$ and $g \in \mathcal{K}^l$, we have $g \in \mathcal{K}^l_L$ if and only if $R_L(g) = 0$.*

2.5. Canonical forms

Let $P \in \mathcal{L}$ and $Q \in (\mathcal{L} \setminus \mathcal{K})^q$. A *canonical form* for P modulo Q is an $R \in \mathcal{L}$ with

$$H \circ P = A \circ Q + R,$$

for certain $H \in \mathcal{K}^{\neq} F$ and $A \in \mathcal{K}\{F_1, \ldots, F_q\}_1$ with $A^i \circ Q_i \preccurlyeq P$ for all i, and such that $\mathfrak{t} \notin \mathfrak{F}_Q$ for each term $c\mathfrak{t} \in \mathcal{K}\mathfrak{T}$ occurring in R. It is easy to modify NF so that it computes a canonical form R of P modulo Q with $R \preccurlyeq P$:

Algorithm CF (P, Q)
Input: $P \in \mathcal{L}$ and $Q \in (\mathcal{L} \setminus \mathcal{K})^q$
Output: a canonical form of P modulo Q
while $P \notin \mathcal{K} \wedge \exists i, \exists \mathfrak{t} \in \mathfrak{T}, P_{\mathfrak{t}} \neq 0 \wedge \mathfrak{d}_{Q_i} | \mathfrak{t}$ **do**
  Choose $\mathfrak{t}$ highest for $\preccurlyeq$
  $P := c_{Q_i}(P - P_{\mathfrak{t}}\mathfrak{t}) +$ Red $(P_{\mathfrak{t}}\mathfrak{t}, Q_i)$
return P

Lemma 2. *Let $Q = (Q_1, \ldots, Q_q) \in (\mathcal{L} \setminus \mathcal{K})^q$ be a standard basis. Then we have*

$$\mathfrak{d}_{[Q] \setminus \mathcal{K}} = [\mathfrak{d}_Q],$$

where $\mathfrak{d}_{[Q] \setminus \mathcal{K}} := \{\mathfrak{d}_Q : Q \in [Q] \setminus \mathcal{K}\}$ and $\mathfrak{d}_Q := (\mathfrak{d}_{Q_1}, \ldots, \mathfrak{d}_{Q_q})$.

Proof. Assume for contradiction that $P \in [Q]$ is such that $\mathfrak{d}_P \notin [\mathfrak{d}_Q]$. Replacing P by CF (P, Q), we may assume without loss of generality that P is a canonical form w.r.t. Q. Now choose $A \in \mathcal{K}\{F_1, \ldots, F_q\}_1$ with $P = A \circ Q$ such that $\mathfrak{t} = \max\{\mathfrak{d}_{A^i \circ Q_i} : A^i \neq 0\}$ is minimal, in the same sense as in the proof of Lemma 1. Since $\mathfrak{t} \in [\mathfrak{d}_Q]$, we have $P_{\mathfrak{t}} = 0$, so there must be at least two indices i and j with $\mathfrak{t} = \mathfrak{d}_{A^i \circ Q_i} = \mathfrak{d}_{A^j \circ Q_j}$. Setting $\tilde{A} = A - \lambda \mathfrak{u} \circ R_{Q,i,j}$ with the notations from the proof of Lemma 1, $P = \tilde{A} \circ Q$ then yields a more minimal representation for P. This contradiction proves that $\mathfrak{d}_P \in [\mathfrak{d}_Q]$ for all $P \in [Q] \setminus \mathcal{K}$. $\square$

3. Standard bases for tangent cone orderings

In classical polynomial elimination theory, the use of non-admissible monomial orderings allows for the computation in localized rings and completions, such as rings of power series. However, additional care is needed in order to ensure termination. For instance, the naive reduction of x modulo $x - x^2$ would yield an infinite sequence $x, x^2, x^3, \ldots$. The tangent cone algorithm (Mora, 1983; Mora et al., 1992) allows for the computation of standard bases in the case of localizations of polynomial rings.

In this section, we will present the tangent cone algorithm in the differential setting. In all what follows, $\mathcal{K}$ is a differential field with constant field $\mathcal{C}$. Geometrically speaking, elements of $\mathcal{C}[[\partial]] = \mathcal{C}[[\partial_1, \ldots, \partial_n]]$ or localizations of $\mathcal{C}[\partial]$ can still be thought of as operators. For instance, $\mathcal{C}[[\partial]]$ naturally operates on $\mathcal{C}[x]$.

3.1. Definition and properties of the ecart

Let $\mathcal{L} = \mathcal{C}\{F\}_1 \oplus \mathcal{K}$ and let $\preccurlyeq$ be a monomial ordering on $\mathfrak{X}$. Given $P, Q \in \mathcal{L} \setminus \mathcal{K}$, we define $\mathfrak{d}_{P,Q}$, $\mathfrak{d}_{Q,P}$ and $\Delta_{P,Q}$ as in (4). As a special case, Red $(P, Q) = \Delta_{P,Q}$ is given by (3) if $\mathfrak{d}_Q | \mathfrak{d}_P$. Now let $\preccurlyeq^*$ be the opposite ordering of $\preccurlyeq$. Given $P \in \mathcal{K}\{F\}^{\neq}_{\leqslant 1}$, we denote the dominant monomial of P for $\preccurlyeq^*$ by $\mathfrak{d}^*_P$ for and we define c^*_P, τ^*_P, Red $^*(P, Q)$, $\Delta^*_{P,Q}$, etc. in a similar way. We will also write $\mathfrak{w}_P = x^\alpha$ for the element of $\mathfrak{X}$ with $\mathfrak{d}^*_P = \partial^\alpha F$. If $\preccurlyeq$ is admissible, $f \in \mathcal{C}[x]$ and $\partial_i = \partial / \partial x_i$ for all i, then we notice that $P(f) \preccurlyeq f / \mathfrak{w}_P$ (i.e. $P(f) \preccurlyeq \mathfrak{d}_f / \mathfrak{w}_P$ for the natural extension of the ordering $\preccurlyeq$ to $x_1^{\mathbb{Z}} \cdots x_n^{\mathbb{Z}}$). Moreover, if $\mathfrak{w}_P | \mathfrak{d}_f$, then $P(f) \asymp f / \mathfrak{w}_P$.

In the sequel, we will assume that the vectors $\lambda_1, \ldots, \lambda_l$ which determine $\preccurlyeq$ using (1) are all in $\mathbb{Z}^n$. In that case $\preccurlyeq$ is called a *tangent cone ordering*. Notice that it is possible to consider more general tangent cone orderings (Mora et al., 1992), but we have chosen to keep the exposition as simple as possible. Given $n_1, \ldots, n_k \in \mathbb{Z}$, let

$$\mathfrak{T}_{n_1,\ldots,n_k} := \{\partial^\alpha F : \lambda_1 \cdot \alpha = n_1 \wedge \cdots \wedge \lambda_k \cdot \alpha = n_k\}.$$

Given $P \in \mathcal{L} \setminus \mathcal{K}$ with $\mathfrak{d}_P \in \mathfrak{T}_{n_1,\ldots,n_k}$, we denote

$$\tau_{P;k} := \sum_{\mathfrak{t} \in \mathfrak{T}_{n_1,\ldots,n_k}} P_{\mathfrak{t}} \mathfrak{t}.$$

Notice that $\tau_{P;0} = P_1$ and $\tau_{P;l+1} = \tau_P$ (for a dummy λ_{l+1}). Now let n_k and n^*_k be such that $\mathfrak{d}_{\tau_{P;k-1}} \in \mathfrak{T}_{n_1,\ldots,n_{k-1},n_k}$ and $\mathfrak{d}^*_{\tau_{P;k-1}} \in \mathfrak{T}_{n_1,\ldots,n_{k-1},n^*_k}$. Then we have $n_k \geqslant n^*_k$ and we define the *k-th ecart* of P by $E_{P;k} := n_k - n^*_k$. We call $E_P := (E_{P;1}, \ldots, E_{P;l}) \in \mathbb{N}^l$ the *ecart* of P and recall that $\mathbb{N}^l$ is well-ordered by the lexicographical ordering. The definition extends to the case when $P \in \mathcal{K}$ by taking $E_{P;k} = -\infty$ for all k.

Given $P, Q \in \mathcal{L} \setminus \mathcal{K}$, some easy properties of the ecart are

$$E_{\Omega \circ P} = E_P \quad (\Omega \in C\mathfrak{T}) \tag{6}$$
$$E_{P - \tau_P} < E_P$$

Moreover, if $\mathfrak{d}_P = \mathfrak{d}_Q$, then

$$E_{P+Q} \leqslant E_P \vee E_Q = (\max(E_{P;1}, E_{Q;1}), \ldots, \max(E_{P;l}, E_{Q;l})),$$

where the inequality is strict whenever $\tau_P + \tau_Q = 0$. It follows that

$$E_{\Delta_{P,Q}} < E_P \vee E_Q. \tag{7}$$

In particular, if $\mathfrak{d}_Q | \mathfrak{d}_P$, then

$$E_{\mathrm{Red}\,(P,Q)} < E_P \vee E_Q. \tag{8}$$

The following lemma will guarantee the termination of the tangent cone algorithm.

Lemma 3. *Let $p > 0$ and $P_1, P_2, \ldots \in \mathcal{L} \setminus \mathcal{K}$ be such that for all $i \geqslant p$, we have*

(a) $P_{i+1} = \mathrm{Red}\,(P_i, P_{r(i)})$ *for some* $r(i) < i$.
(b) *Whenever* $\mathfrak{d}_{P_q} | \mathfrak{d}_{P_i}$ *for some* $q < i$, *then* $E_{\mathrm{Red}\,(P_i, P_q)} \geqslant E_{P_{i+1}}$.

Then the sequence $P_1, P_2, \ldots$ is finite.

Proof. Assume for contradiction that there exist infinitely many $i \geqslant p$ with $E_{P_i} \leqslant E_{P_{i+1}}$. By Dickson's lemma, we may find two such indices $q < i$ with $\mathfrak{d}_{P_q} | \mathfrak{d}_{P_i}$ and $E_{P_q;1} \leqslant E_{P_i;1}, \ldots, E_{P_q;l} \leqslant E_{P_i;l}$. But then

$$E_{\mathrm{Red}\,(P_i, P_q)} < E_{P_q} \vee E_{P_i} = E_{P_i} \leqslant E_{P_{i+1}},$$

which contradicts our assumption (*b*). It follows that E_{P_i} is strictly decreasing for sufficiently large i. We conclude by the fact that $\mathbb{N}^l$ is well-ordered. □

3.2. The tangent cone algorithm

Given $P, Q \in \mathcal{L} \setminus \mathcal{K}$, a *normal form* for P modulo Q is an $R \in \mathcal{L}$, with $R \in \mathcal{K}$ or $\mathfrak{d}_R \notin \mathfrak{F}_Q$, and such that

$$H \circ P = A \circ Q + R,$$

for certain $H \in \mathcal{C}\{F\}_1$ and $A \in \mathcal{C}\{F_1, \ldots, F_q\}_1$ with $\mathfrak{d}_H = F$ and $A^i \circ Q_i \preccurlyeq P$ for all i. Notice that this notion extends the previous notion of normal forms, since $\mathfrak{d}_H = F \Rightarrow H \in \mathcal{C}^{\neq} F$ if $\preccurlyeq$ is admissible. In our new context, we may use the following algorithm to compute a normal form:

Algorithm NF (P, Q)
Input: $P \in \mathcal{L}$ and $Q \in (\mathcal{L} \setminus \mathcal{K})^q$
Output: a normal form of P modulo Q
while $P \notin \mathcal{K} \wedge \exists i, \mathfrak{d}_{Q_i} | \mathfrak{d}_P$ **do**
  Take i with $\mathfrak{d}_{Q_i} | \mathfrak{d}_P$ such that $E_{\mathrm{Red}\,(P, Q_i)}$ is minimal
  $Q := Q \cup \{P\}$
  $P := \mathrm{Red}\,(P, Q_i)$
return P

Indeed, the sequence $P_1 \succ P_2 \succ \cdots$ of successive values of P during the algorithm fulfills the conditions of Lemma 3, so this sequence is finite. Moreover, using induction, it is easily checked that there exist $A_i \in \mathcal{C}\{F_1, \ldots, F_q\}_1$ and $B_i \in \mathcal{C}\{F\}_1$ and with $P_i = A_i \circ Q + B_i \circ P$ and $\mathfrak{d}_{B_i} = F$ for all i. So the last term of the sequence is indeed a normal form for P modulo Q.

Defining the notions of autoreduced systems, coherent systems and standard bases as in Section 2.3, the same algorithm SB may be used to compute an equivalent standard basis for a given system. Given a standard basis $Q \in \mathcal{L}^l$ and $1 \leqslant i < j \leqslant q$, we have a relation

$$\begin{aligned} H \circ \Delta_{Q_i,Q_j} &= H \circ (c_{Q_j} \mathfrak{d}_{Q_j,Q_i}) \circ Q_i - H \circ (c_{Q_i} \mathfrak{d}_{Q_i,Q_j}) \circ Q_j \\ &= A^1 \circ Q_1 + \cdots + A^q \circ Q_q, \end{aligned}$$

with $\mathfrak{d}_H = F$ and $A^k \circ Q_k \prec \mathfrak{d}_{Q_j,Q_i} \circ Q_i \asymp \mathfrak{d}_{Q_i,Q_j} \circ Q_j$ for all k. As before, we may rewrite this relation as a critical relation of the form $R_{Q,i,j} \circ Q = 0$.

In order to generalize Lemma 1, let $\mathcal{C}\{F\}_1 = \mathcal{C}[[\partial]](F) \supseteq \mathcal{C}\{F\}_1$ be the set of series $Q = \sum_{\mathfrak{t} \in \mathfrak{T}} Q_{\mathfrak{t}} \mathfrak{t}$ with well-ordered support supp $Q = \{\mathfrak{t} \in \mathfrak{T} : Q_{\mathfrak{t}} \neq 0\}$. If $\preccurlyeq$ is admissible, then $\mathcal{C}\{F\}_1$ coincides with $\mathcal{C}\{F\}_1$. If $\preccurlyeq^*$ is admissible then elements of $\mathcal{C}\{F\}_1$ are power series in $\partial_1, \ldots, \partial_n$ applied to F. The set $\mathcal{C}\{F\}_1$ is naturally stable under composition. We denote $\mathcal{C}\{F_1, \ldots, F_q\}_1 = \mathcal{C}[[\partial]](F_1) \oplus \cdots \oplus \mathcal{C}[[\partial]](F_q)$.

Lemma 4. *Let $Q = (Q_1, \ldots, Q_q) \in (\mathcal{L} \setminus \mathcal{K})^q$ be a standard basis. Then the $R_{Q,i,j}$ generate the space of all $A \in \mathcal{C}\{F_1, \ldots, F_q\}_1$ with $A \circ Q = 0$.*

Proof. We have to construct $\Sigma \in \mathcal{C}\{F_{i,j}\}_1$ with $A = \Sigma \circ R_Q$, where $F_{i,j}$ corresponds to the critical relation $R_{Q,i,j}$. For each $1 \leqslant i < j \leqslant q$, let $\mathfrak{s}_{i,j} = \mathfrak{d}_{Q_j,Q_i} \circ \mathfrak{d}_{Q_i} = \mathfrak{d}_{Q_i,Q_j} \circ \mathfrak{d}_{Q_j}$. Writing $\Sigma = \sum_{i<j} \sum_{\mathfrak{t} \in \mathfrak{T}} \mathrm{T}_{\mathfrak{t}}^{i,j} \mathfrak{d}_{\mathfrak{t},\mathfrak{s}_{i,j}}$, let us construct $\mathrm{T}_{\mathfrak{t}}$ by transfinite induction over $\mathfrak{t}$. Given an ordinal α, the induction hypothesis is as follows:

- $\Sigma_{\mathfrak{t}}$ has been constructed for all $\mathfrak{t}$ in a final segment $\mathfrak{F}_{;\alpha}$ of $\mathfrak{T}$ for $\preccurlyeq$.
- $\mathfrak{F}_{;\beta} \subsetneq \mathfrak{F}_{;\alpha}$ for all $\beta < \alpha$.
- Denoting $\Sigma_{;\alpha} = \sum_{i<j} \sum_{\mathfrak{t} \in \mathfrak{F}_{;\alpha}} \mathrm{T}_{\mathfrak{t}}^{i,j} \mathfrak{d}_{\mathfrak{t},\mathfrak{s}_{i,j}}$ and $A_{;\alpha} = A - \Sigma_{;\alpha} \circ R_Q$, we have $A_{;\alpha}^i \circ Q_i \prec \mathfrak{t}$ for all i and $\mathfrak{t} \in \mathfrak{F}_{;\alpha}$.

If $\alpha = 0$ or α is a limit ordinal, then we may take $\mathfrak{F}_{;\alpha} = \bigcup_{\beta<\alpha} \mathfrak{F}_{;\beta}$. If $\alpha = \beta + 1$ and $A_{;\alpha} = 0$, then we are done. So assume that $\alpha = \beta + 1$ and $A_{;\beta} \neq 0$. Let $\mathfrak{t} = \max\{\mathfrak{d}_{A_{;\beta}^i \circ Q_i} : A_{;\beta}^i \neq 0\} \notin \mathfrak{F}_{;\beta}$ and let i be minimal such that $(A_{;\beta}^i \circ Q_i)_{\mathfrak{t}} \neq 0$. Let $\mathrm{T}_{\mathfrak{t}} = \sum_{j>i} \lambda_j F_{i,j}$, with

$$\lambda_j = \frac{(A_{;\beta}^j \circ Q_j)_{\mathfrak{t}}}{(\mathfrak{d}_{\mathfrak{t},\mathfrak{s}_{i,j}} \circ R_{Q,i,j}^j \circ Q_j)_{\mathfrak{t}}},$$

let $\mathfrak{F}_{;\beta} = \{\mathfrak{u} \in \mathfrak{T} : \mathfrak{u} \succcurlyeq \mathfrak{t}\}$ and take $\mathrm{T}_{\mathfrak{u}} = 0$ for all $\mathfrak{u} \succ \mathfrak{t}$ with $\mathfrak{u} \notin \mathfrak{F}_{;\beta}$. By construction,

$$A_{;\alpha}^j \circ Q_j = (A_{;\beta} - \mathrm{T}_{\mathfrak{t}} \circ R_Q)^j \circ Q_j = (A_{;\beta}^j - \lambda_j \mathfrak{d}_{\mathfrak{t},\mathfrak{s}_{i,j}} \circ R_{Q,i,j}^j) \circ Q_j \prec \mathfrak{t}$$

for all $j > i$. Since $(A_{;\alpha} \circ Q)_{\mathfrak{t}} = 0$, it follows that $A_{;\alpha}^i \circ Q_i \prec \mathfrak{t}$ as well. This proves the last induction hypothesis. By transfinite induction, we conclude that there exists an α with $A_{;\alpha} = 0$, whence $A = \Sigma_{;\alpha} \circ R_Q$. □

Consider a system $L = (L_1, \ldots, L_l)$ of linear differential polynomials in $\mathcal{C}\{F\}_1$. Assume also that $\mathcal{C}\{F\}_1$ naturally operates on a subring $\mathcal{R}$ of $\mathcal{K}$ (for instance, we may take $\mathcal{R} = \mathcal{C}[x]$). Given a tuple $g = (g_1, \ldots, g_l) \in \mathcal{R}^l$, we say that g is compatible with L, if for every relation $A \circ L = 0$ with $A \in \mathcal{C}\{F_1, \ldots, F_l\}_1$, we have $A \circ g = 0$. The set of such tuples forms a (strong) subvector space $\mathcal{R}^l_L$ of $\mathcal{R}^l$. The following consequences of the above lemma is proved in a similar way as Corollaries 1 and 2.

Corollary 3. *The system $L - g = (L_1 - g_1, \ldots, L_l - g_l)$ with $L_i \in \mathcal{C}\{F\}_1^{\neq}$ and $g_i \in \mathcal{R}$ is a standard basis if and only if L is a standard basis and g is compatible with L.* $\square$

Corollary 4. *Given a standard basis $L \in \mathcal{C}\{F\}_1^l$ and $g \in \mathcal{R}^l$, we have $g \in \mathcal{R}^l_L$ if and only if $R_L(g) = 0$.* $\square$

Let $P \in \mathcal{C}\{F\}_1 \oplus \mathcal{R}$ and $Q \in (\mathcal{L} \backslash \mathcal{K})^q$. A *canonical form* for P modulo Q is an $R \in \mathcal{C}\{F\}_1 \oplus \mathcal{R}$ with

$$H \circ P = A \circ Q + R$$

for certain $H \in \mathcal{C}\{F\}_1$ and $A \in \mathcal{C}\{F_1, \ldots, F_q\}_1$ with $\mathfrak{d}_H = F$ and $A^i \circ Q_i \preccurlyeq P$ for all i, and such that $\mathfrak{t} \notin \mathfrak{F}_Q$ for each term $c\mathfrak{t} \in \mathcal{C}\mathfrak{T}$ occurring in R. Although we have no algorithm to compute canonical forms, like in Section 2.5, the existence of canonical forms can be proved using a similar transfinite induction as in the proof of Lemma 4. Using another transfinite induction, Lemma 2 also generalizes to the current setting:

Lemma 5. *Let $Q = (Q_1, \ldots, Q_q) \in (\mathcal{L} \setminus \mathcal{K})^q$ be a standard basis. Then $\mathfrak{d}_{[Q] \setminus \mathcal{K}} = [\mathfrak{d}_Q]$.* $\square$

4. Linear differential equations with constant coefficients

In this section, we consider systems $L = (L_1, \ldots, L_l)$ of linear partial differential equations in one unknown F with coefficients in a field of constants $\mathcal{C}$ of characteristic zero. We will consider the resolution of such systems in the algebras

$$\mathcal{R} = \bigoplus_{\xi \in \mathcal{C}^n} \mathcal{C}\mathrm{e}^{\xi \cdot x};$$

$$\mathcal{S} = \bigoplus_{\xi \in \mathcal{C}^n} \mathcal{C}[x]\mathrm{e}^{\xi \cdot x},$$

where $\partial_i x_j = \delta_{i,j}$ (Kronecker symbol). We will first consider homogeneous linear differential equations, but we will also study linear differential equations with second members. In the latter case, we will allow the second members to belong to $\mathcal{R}$ or $\mathcal{S}$. Throughout this section $\preccurlyeq$ stands for an admissible tangent cone ordering on $\mathfrak{X}$.

4.1. Solving $L(f) = 0$ in $\mathcal{R}$

In this section, we will only consider linear p.d.e.s without second members. Let $L \in \mathcal{C}\{F\}_1$ be a homogeneous linear differential polynomial. We may represent L as

$$L = P_L(\partial_1, \ldots, \partial_n)(F),$$

where P_L is a polynomial in $\mathcal{C}[\xi] = \mathcal{C}[\xi_1, \ldots, \xi_n]$. Inversely, each polynomial $P \in \mathcal{C}[\xi]$ gives rise to a homogeneous linear differential polynomial $L_P = P(\partial_1, \ldots, \partial_n)(F) \in \mathcal{C}\{F\}_1$. Denoting $\mathfrak{e}_\xi = \mathrm{e}^{\xi \cdot x}$, we have

$$L(\mathfrak{e}_\xi) = P_L(\xi)\mathfrak{e}_\xi$$

for all $\xi \in \mathcal{C}^n$ and in particular

$$L(\mathfrak{e}_\xi) = 0 \Longleftrightarrow P_L(\xi) = 0.$$

Let $\mathcal{H}_L$ denote the set of all $\mathfrak{e} \in \mathfrak{E} = \mathrm{e}^{\mathcal{C}x_1+\cdots+\mathcal{C}x_n}$ with $L(\mathfrak{e}) = 0$. We have

$$L(f) = 0 \Longleftrightarrow f \in \mathrm{Vec}\,(\mathcal{H}_L)$$

for all $f \in \mathcal{R}$, where Vec $(\mathcal{H}_L)$ denotes the $\mathcal{C}$-vector space generated by $\mathcal{H}_L$. Given $\mathfrak{e} = \mathrm{e}^{\xi\cdot x} \in \mathfrak{E}$, we will denote $\xi_\mathfrak{e} = \xi$.

More generally, given a set $\mathcal{D}$ of homogeneous linear differential polynomials, a subset $\mathcal{H}$ of $\mathfrak{E}$, a subset $\mathcal{I}$ of $\mathcal{C}[\xi_1, \ldots, \xi_n]$ and a subset $\mathcal{V}$ of $\mathcal{C}^n$, we denote

$$\begin{aligned}
&\mathcal{I}_\mathcal{D} = \{P_L \in \mathcal{C}[\xi] | L \in \mathcal{D}\};\\
&\mathcal{D}_\mathcal{I} = \{L_P \in \mathcal{C}\{F\}_1 | P \in \mathcal{I}\};\\
&\mathcal{V}_\mathcal{H} = \{\xi_\mathfrak{e} \in \mathcal{C}^n | \mathfrak{e} \in \mathcal{H}\};\\
&\mathcal{H}_\mathcal{V} = \{\mathfrak{e}_\xi \in \mathfrak{E} | \xi \in \mathcal{V}\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{I}_\mathcal{V} = \{P \in \mathcal{C}[\xi] | \forall \xi \in \mathcal{V} : P(\xi) = 0\};\\
&\mathcal{V}_\mathcal{I} = \{\xi \in \mathcal{C}^n | \forall P \in \mathcal{I} : P(\xi) = 0\};\\
&\mathcal{D}_\mathcal{H} = \{L \in \mathcal{C}\{F\}_1 | \forall \mathfrak{e} \in \mathcal{H} : L(\mathfrak{e}) = 0\};\\
&\mathcal{H}_\mathcal{D} = \{\mathfrak{e} \in \mathfrak{E} | \forall L \in \mathcal{D} : L(\mathfrak{e}) = 0\}.
\end{aligned}$$

Because of the natural isomorphisms

$$\begin{aligned}
&\mathcal{I}_\mathcal{D} \cong \mathcal{D}; \mathcal{D}_\mathcal{I} \cong \mathcal{I};\\
&\mathcal{V}_\mathcal{H} \cong \mathcal{H}; \mathcal{H}_\mathcal{V} \cong \mathcal{V},
\end{aligned}$$

all algebraic geometry properties of the correspondences $\mathcal{I} \mapsto \mathcal{V}_\mathcal{I}$ and $\mathcal{V} \mapsto \mathcal{I}_\mathcal{V}$ induce analogue properties for the correspondences $\mathcal{D} \mapsto \mathcal{H}_\mathcal{D}$ and $\mathcal{H} \mapsto \mathcal{D}_\mathcal{H}$. In particular, Hilbert's Nullstellensatz implies

Theorem 1. *Let $L = (L_1, \ldots, L_l)$ be a coherent and autoreduced system with $L_1, \ldots, L_l \in \mathcal{C}\{F\}_1 \setminus \mathcal{C}F$. If $\mathcal{C}$ is algebraically closed, then L admits a solution $\mathfrak{e} \in \mathfrak{E}$.*

4.2. Solving $L(f) = 0$ in $\mathcal{C}[x]$

Recall that $\preccurlyeq$ stands for an admissible tangent cone ordering on $\mathfrak{X}$. Consider a standard basis $L = (L_1, \ldots, L_l) \in \mathcal{C}\{F\}_1^l$ for $\preccurlyeq^*$. We may regard L as an operator from $\mathcal{C}[x]$ into $\mathcal{C}[x]^l$, whose image is in $\mathcal{C}[x]_L^l$. We denote by $\mathfrak{H}_L$ the set of monomials x^α, such that $\mathfrak{w}_{L_i} \nmid x^\alpha$ for all i. The aim of this section is to construct a right inverse $L^{-1} : \mathcal{C}[x]_L^l \to \mathcal{C}[x]$ of L, which is "distinguished" in the sense that $f_\mathfrak{h} = 0$ for all $f \in \mathrm{im}\, L^{-1}$ and $\mathfrak{h} \in \mathfrak{H}_L$.

The relation $\preccurlyeq$ on $\mathcal{C}[x]$ induces a relation $\preccurlyeq_L$ on $\mathcal{C}[x]^l$ by

$$\begin{aligned}
&(g_1, \ldots, g_l) \preccurlyeq_L (h_1, \ldots, h_l)\\
\Longleftrightarrow\ &\max\{\mathfrak{w}_{L_1}\mathfrak{d}_{g_1}, \ldots, \mathfrak{w}_{L_l}\mathfrak{d}_{g_l}\} \preccurlyeq \max\{\mathfrak{w}_{L_1}\mathfrak{d}_{h_1}, \ldots, \mathfrak{w}_{L_l}\mathfrak{d}_{h_l}\}.
\end{aligned}$$

Whenever $f, \tilde{f} \in \mathcal{C}[x]^{\neq}$ are such that $\mathfrak{d}_f \notin \mathfrak{H}_L$ and $\mathfrak{d}_{\tilde{f}} \notin \mathfrak{H}_L$, it follows that

$$f \preccurlyeq \tilde{f} \Longleftrightarrow L(f) \preccurlyeq_L L(\tilde{f}).$$

Indeed, if $\mathfrak{d}_f \notin \mathfrak{H}_L$, then $L_i(f) \asymp f/\mathfrak{w}_{L_i}$ for at least one i with $\mathfrak{w}_{L_i}|x^{\alpha}$.

Proposition 1. *Given a standard basis $L = (L_1, \ldots, L_l)$ for $\preccurlyeq^*$ and $g \in \mathcal{C}[x]^l_L$, let i be such that $\mathfrak{x} = \mathfrak{d}_{g_i}\mathfrak{w}_{L_i}$ is maximal for $\preccurlyeq$. Then $\tau = (c_{g_i}/c_{L_i(\mathfrak{x})})\mathfrak{x}$ does not depend on the choice of i and $g - L(\tau) \prec_L g$.*

Proof. We will first show that $c_{g_j}/c_{L_j(\mathfrak{x})} = c_{g_i}/c_{L_i(\mathfrak{x})}$ whenever $j \neq i$ is another index with $\mathfrak{x} = \mathfrak{d}_{g_j}\mathfrak{w}_{L_j}$. Let $\Omega_j \in \mathcal{C}\mathfrak{T}$ and $\Omega_i \in \mathcal{C}\mathfrak{T}$ be such that $\Delta^*_{L_i,L_j} = \Omega_j \circ L_i - \Omega_i \circ L_j$ and consider the associated critical relation

$$\Omega_j \circ L_i - \Omega_i \circ L_j = K^1 \circ L_1 + \cdots + K^l \circ L_l,$$

with $\mathfrak{w}_{K^k}\mathfrak{w}_{L_k} \succ \mathfrak{w}_{\Omega_j}\mathfrak{w}_{L_i} = \mathfrak{w}_{\Omega_i}\mathfrak{w}_{L_j}$ for all k. Since g is compatible with L, it follows that

$$\Omega_j(g_i) - \Omega_i(g_j) = K^1(g_1) + \cdots + K^l(g_l).$$

For each k, we have

$$K^k(g_k) \preccurlyeq \frac{\mathfrak{d}_{g_k}}{\mathfrak{w}_{K^k}} \preccurlyeq \frac{\mathfrak{d}_{g_i}\mathfrak{w}_{L_i}}{\mathfrak{w}_{L_k}\mathfrak{w}_{K^k}} \prec \frac{\mathfrak{d}_{g_i}}{\mathfrak{w}_{\Omega_j}} =: \mathfrak{u}.$$

It follows that

$$[\Omega_j(g_i) - \Omega_i(g_j)]_{\mathfrak{u}} = c_{\Omega_j(g_i)} - c_{\Omega_i(g_j)} = 0. \tag{9}$$

Hence

$$\begin{aligned} c_{\Omega_j(\mathfrak{d}_{g_i})}c_{g_i} &= c_{\Omega_i(\mathfrak{d}_{g_j})}c_{g_j} \\ c_{\Omega_j(\mathfrak{d}_{L_i(\mathfrak{x})})}c_{L_i(\mathfrak{x})} &= c_{\Omega_i(\mathfrak{d}_{L_j(\mathfrak{x})})}c_{L_j(\mathfrak{x})} \end{aligned}$$

It follows that

$$\frac{c_{g_i}}{c_{L_i(\mathfrak{x})}}\frac{c_{\Omega_j(\mathfrak{d}_{g_i})}}{c_{\Omega_j(\mathfrak{d}_{L_i(\mathfrak{x})})}} = \frac{c_{g_j}}{c_{L_j(\mathfrak{x})}}\frac{c_{\Omega_i(\mathfrak{d}_{g_j})}}{c_{\Omega_i(\mathfrak{d}_{L_j(\mathfrak{x})})}}.$$

Now $\mathfrak{d}_{g_i}/\mathfrak{d}_{L_i(\mathfrak{x})} = \mathfrak{d}_{g_j}/\mathfrak{d}_{L_j(\mathfrak{x})}$ implies $c_{\Omega_j(\mathfrak{d}_{g_i})}/c_{\Omega_j(\mathfrak{d}_{L_i(\mathfrak{x})})} = c_{\Omega_i(\mathfrak{d}_{g_j})}/c_{\Omega_i(\mathfrak{d}_{L_j(\mathfrak{x})})}$, so we conclude that $c_{g_j}/c_{L_j(\mathfrak{x})} = c_{g_i}/c_{L_i(\mathfrak{x})}$.

It remains to be proved that $g - L(\tau) \prec_L g$, i.e. $g_j - L_j(\tau) \prec \mathfrak{x}/\mathfrak{w}_{L_j}$ for all j. If $\mathfrak{w}_{L_j} \nmid \mathfrak{x}$, then $\mathfrak{d}^*_{L_j}(\tau) = 0$ and for all $\Omega \in \operatorname{supp} L_j$ with $\Omega \prec \mathfrak{d}^*_{L_j}$, we have $\mathfrak{w}_{L_j}\Omega(\tau) \preccurlyeq (\mathfrak{w}_{L_j}/\mathfrak{w}_{\Omega})\tau \prec \mathfrak{x}$. By strong linearity, it follows that $\mathfrak{w}_{L_j}L_j(\tau) \prec \mathfrak{x}$. Furthermore $\mathfrak{w}_{L_j} \nmid \mathfrak{x}$ implies $\mathfrak{d}_{g_j}\mathfrak{w}_{L_j} \prec \mathfrak{x}$, whence $g_j - L_j(\tau) \prec \mathfrak{x}/\mathfrak{w}_{L_j}$. If $\mathfrak{w}_{L_j}|\mathfrak{x}$, then the relation (9) remains valid. Moreover, if $\Xi \in \mathfrak{T}$ is such that $\mathfrak{w}_{\Xi}\mathfrak{w}_{\Omega_i}\mathfrak{w}_{L_j} = \mathfrak{w}_{\Xi}\mathfrak{w}_{\Omega_j}\mathfrak{w}_{L_i} = \mathfrak{x}$, then

$$[(\Xi \circ \Omega_j)(g_i) - (\Xi \circ \Omega_i)(g_j)]_1 = 0$$

and

$$\begin{aligned} [(\Xi \circ \Omega_i)(g_j)]_1 &\in \mathcal{C}^{\neq}(g_j)_{\mathfrak{w}_{\Xi}\mathfrak{w}_{\Omega_i}} = \mathcal{C}^{\neq}(g_j)_{\mathfrak{x}/\mathfrak{w}_{L_j}} \\ [(\Xi \circ \Omega_j)(g_i)]_1 &\in \mathcal{C}^{\neq}(g_i)_{\mathfrak{w}_{\Xi}\mathfrak{w}_{\Omega_j}} = \mathcal{C}^{\neq}(g_i)_{\mathfrak{x}/\mathfrak{w}_{L_i}} \end{aligned}$$

Since $(g_i)_{\mathfrak{x}/\mathfrak{w}_{L_i}} \neq 0$, it follows that $(g_j)_{\mathfrak{x}/\mathfrak{w}_{L_j}} \neq 0$, whence $\mathfrak{x} = \mathfrak{d}_{g_j}\mathfrak{w}_{L_j}$. By construction, we therefore have $g_j - L_j(\tau) \prec \mathfrak{x}/\mathfrak{w}_{L_j}$. $\square$

Given $g \in \mathcal{C}[x]^l_L$, let $\tau_g = \tau$ be the term as in Proposition 1. Now consider the sequence defined by $g_0 = g$ and $g_{i+1} = g_i - L(\tau_{g_i})$. This sequence is finite, since $\tau_{g_0} \succ \tau_{g_1} \succ \cdots$ and $\preccurlyeq$ is a well-ordering on $\mathfrak{X}$. Consequently, $f = \tau_{g_0} + \tau_{g_1} + \cdots \in \mathcal{C}[x]$ is a solution to $L(f) = g$ with $f_{\mathfrak{h}} = 0$ for all $\mathfrak{h} \in \mathfrak{H}_L$. We set $f = L^{-1}(g)$ and call L^{-1} the *distinguished right inverse* of L.

Let $\mathfrak{h} \in \mathfrak{H}_L$. Since $L_i(\mathfrak{h}) \prec \mathfrak{h}/\mathfrak{w}_{L_i}$ for all i, it follows that $L^{-1}(L(\mathfrak{h})) \prec \mathfrak{h}$. Consequently $h = b_{\mathfrak{h}} = \mathfrak{h} - L^{-1}(L(\mathfrak{h}))$ is a solution to $L(h) = 0$ with $\mathfrak{d}_h = \mathfrak{h}$. Inversely, $Lh = 0$ implies $\mathfrak{d}_h \in \mathfrak{H}_L$, since otherwise $\mathfrak{w}_{L_i} | \mathfrak{d}_h$ for some i and $L_i(h) \asymp h/\mathfrak{w}_{L_i} \neq 0$. We claim that the $b_{\mathfrak{h}}$ form a basis for the solution space $\mathcal{H}_L$ of $L(h) = 0$ in $\mathcal{C}[x]$. Indeed, given an arbitrary solution h, consider the sequence defined by $h_0 = h$ and $h_{i+1} = h_i - c_{h_i} b_{\mathfrak{d}_{h_i}}$ as long as $h_i \neq 0$. This sequence is necessarily finite, since $\mathfrak{d}_{h_0} \succ \mathfrak{d}_{h_1} \succ \cdots$ and $\mathfrak{X}$ is well-ordered. Hence, $h = c_{h_0} b_{\mathfrak{d}_{h_0}} + c_{h_1} b_{\mathfrak{d}_{h_1}} + \cdots$. We call $(b_{\mathfrak{h}})_{\mathfrak{h} \in \mathfrak{H}_L}$ the *distinguished basis* of $\mathcal{H}_L$.

We notice that $\mathcal{C}[x] = \mathcal{H}_L \oplus \mathcal{H}_L^\perp$, where $\mathcal{H}_L^\perp = \{f \in \mathcal{C}[x] : \forall \mathfrak{h} \in \mathfrak{H}_L, f_{\mathfrak{h}} = 0\}$, so that $L : \mathcal{C}[x] \to \mathcal{C}[x]^l_L$ decomposes into an isomorphism on $\mathcal{H}_L^\perp$ with left inverse L^{-1} and the zero map on $\mathcal{H}_L$. We also notice that the distinguished right inverse L^{-1} is uniquely determined by the fact that $L^{-1}(L(g)) = g$ for all $g \in \mathcal{C}[x]^l_L$ and $L^{-1}(g) \in \mathcal{H}_L^\perp$. Indeed, assume that $L(f) = g$ and $L(\tilde{f}) = g$ and $\tilde{f}_{\mathfrak{h}} = f_{\mathfrak{h}} = 0$ for all $\mathfrak{h} \in \mathfrak{H}_L$. Then $L(h) = 0$ for $h = \tilde{f} - f$ and $h_{\mathfrak{h}} = 0$ for all $\mathfrak{h} \in \mathfrak{H}_L$. It follows that $h = 0$.

Let us now consider an arbitrary system $L = (L_1, \ldots, L_l) \in \mathcal{C}\{F\}^l_1$. Using the tangent cone algorithm, L may be rewritten into an equivalent system $\tilde{L}$ which is a standard basis. Then the sets $\mathfrak{H}_{\tilde{L}}$ and $\mathcal{H}_{\tilde{L}}^\perp$ are independent from the particular choice of $\tilde{L}$, since $\mathfrak{H}_{\tilde{L}}$ is precisely the set of elements which cannot occur as dominant monomials of elements in $[L]$, by Lemma 5. Consequently, the construction of the distinguished right inverse and the distinguished basis $\mathcal{H}_{\tilde{L}}$ do not depend on the choice of $\tilde{L}$, and we may define $\mathfrak{H}_L = \mathfrak{H}_{\tilde{L}}$, $L^{-1} = \tilde{L}^{-1}$, etc.

4.3. Solving $L(f) = g$ in $\mathcal{S}$

Let us now consider a general system $L \in \mathcal{C}\{F\}^l_1$ as an operator $L : \mathcal{S} \to \mathcal{S}^l_L$. Then L acts "by spectral components" $\mathcal{C}[x]\mathrm{e}^{\xi \cdot x}$. More precisely, given $\mathfrak{e} = \mathrm{e}^{\xi \cdot x} \in \mathfrak{E}$, let $L_{\ltimes \mathfrak{e}}$ be the unique operator such that

$$L_{\ltimes \mathfrak{e}}(f) = \mathfrak{e}^{-1} L(\mathfrak{e} f)$$

for all f. Considering L as an operator in $\mathcal{C}[\partial]^l$, we obtain $L_{\ltimes \mathfrak{e}}$ from L by substituting $\partial_i - \xi_i$ for each ∂_i. Given

$$f = \sum_{\mathfrak{e} \in \mathfrak{E}} f_{\mathfrak{e}} \mathfrak{e},$$

with $f_{\mathfrak{e}} \in \mathcal{C}[x]$, it follows that

$$L(f) = \sum_{\mathfrak{e} \in \mathfrak{E}} L_{\ltimes \mathfrak{e}}(f_{\mathfrak{e}}) \mathfrak{e}.$$

Hence, denoting by $\mathcal{H}_{L_{\ltimes \mathfrak{e}}}$ the solution space of $L_{\ltimes \mathfrak{e}}(\varphi) = 0$ for $\varphi \in \mathcal{C}[x]$, the solution space of $L(f) = 0$ for $f \in \mathcal{S}$ is given by

$$\mathcal{H}_L = \bigoplus_{\mathfrak{e} \in \mathfrak{E}} \mathcal{H}_{L_{\ltimes \mathfrak{e}}} \mathfrak{e}.$$

Denoting by $L_{\ltimes\mathfrak{e}}^{-1}$ the distinguished inverse of $L_{\ltimes\mathfrak{e}}$ as an operator on $\mathcal{C}[x]$, the mapping

$$\begin{aligned} L^{-1} : \mathcal{S}_L^l &\longrightarrow \mathcal{S} \\ g &\longmapsto \sum_{\mathfrak{e}\in\mathfrak{E}} L_{\ltimes\mathfrak{e}}^{-1}(g_{\mathfrak{e}})\mathfrak{e} \end{aligned}$$

is a right inverse of L. Moreover, L^{-1} is unique with the property that

$$\operatorname{im} L^{-1} \subseteq \mathcal{H}^{\perp},$$

where

$$\mathcal{H}_L^{\perp} = \bigoplus_{\mathfrak{e}\in\mathfrak{E}} \mathcal{H}_{L_{\ltimes\mathfrak{e}}}^{\perp}\mathfrak{e}.$$

Remark 3. When extending the total ordering $\preccurlyeq$ on $\mathfrak{X}$ to $\mathfrak{X}\mathfrak{E}$ in any way which preserves spectral components (i.e. if $\mathfrak{e} \prec \mathfrak{f}$, then $\mathfrak{x}\mathfrak{e} \prec \mathfrak{y}\mathfrak{f}$ for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$), the space $\mathcal{H}^{\perp}$ coincides with the set of all $f \in \mathcal{S}$ such that $f_{\mathfrak{d}_h} = 0$ for all $h \in \mathcal{H}_L^{\neq}$; see the next section.

Theorem 2. *Let $\mathscr{L}$ be the set of differential ideals of $\mathcal{C}\{F\}_1$ and let $\mathscr{H}$ the set of subsets of $\mathcal{S}$ which occur as zero-sets of systems $\mathcal{D} \in \mathcal{C}\{F\}_1^l$. Then the correspondences*

$$\begin{aligned} \mathcal{D} \in \mathscr{L} &\longmapsto \mathcal{H}_{\mathcal{D}} = \{h \in \mathcal{S} | \forall L \in \mathcal{D} : L(h) = 0\} \in \mathscr{H} \\ \mathcal{H} \in \mathscr{H} &\longmapsto \mathcal{D}_{\mathcal{H}} = \{L \in \mathcal{C}\{F\}_1 | \forall h \in \mathcal{H} : L(h) = 0\} \in \mathscr{L} \end{aligned}$$

are mutually inverse bijections.

Proof. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two differential ideals with the same set of solutions $\mathcal{H}_{\mathcal{D}_1} = \mathcal{H}_{\mathcal{D}_2}$. Then the differential ideal $\mathcal{D}$ generated by $\mathcal{D}_1$ and $\mathcal{D}_2$ still has the same set of solutions. Assuming for contradiction that $\mathcal{D}_1 \neq \mathcal{D}_2$, the set $\mathcal{D}$ strictly contains $\mathcal{D}_1$ or $\mathcal{D}_2$, say $\mathcal{D}_1$. Now consider the differential ideal $\mathcal{D}_1 : \mathcal{D} = \{L \in \mathcal{C}\{F\}_1 : L \circ \mathcal{D} \subseteq \mathcal{D}_1\}$. By Theorem 1, there exists an $\mathfrak{e} \in \mathfrak{E}$ with $(\mathcal{D}_1 : \mathcal{D})(\mathfrak{e}) = 0$. Since $(\mathcal{D}_1 : \mathcal{D}) \circ \mathcal{D} \subseteq \mathcal{D}_1 \not\ni 1$ and $\mathfrak{H}_{(\mathcal{D}_1:\mathcal{D})_{\ltimes\mathfrak{e}}} \neq \varnothing$ (here $\mathfrak{H}_{(\mathcal{D}_1:\mathcal{D})_{\ltimes\mathfrak{e}}}$ stands for $\mathfrak{H}_{L_{\ltimes\mathfrak{e}}}$, where L is any system which generates $\mathcal{D}_1 : \mathcal{D}$), it follows that $\mathfrak{H}_{\mathcal{D}_{\ltimes\mathfrak{e}}} \neq \mathfrak{H}_{(\mathcal{D}_1)_{\ltimes\mathfrak{e}}}$. But then $\mathcal{H}_{\mathcal{D}_{\ltimes\mathfrak{e}}} \neq \mathcal{H}_{(\mathcal{D}_1)_{\ltimes\mathfrak{e}}}$ and $\mathcal{H}_{\mathcal{D}} \neq \mathcal{H}_{\mathcal{D}_1}$. $\square$

Remark 4. Whereas Hilbert's Nullstellensatz establishes a correspondence between radical ideals and algebraic sets, Theorem 2 yields a correspondence between *any* differential ideal of $\mathcal{C}\{F\}_1$ (which is necessarily radical and even prime) and "linear differentially algebraic" zero-sets in $\mathcal{S}$. Via the isomorphism $\mathcal{C}[\mathfrak{X}] \cong \mathcal{C}\{F\}_1$, arbitrary ideals of $\mathcal{C}[\mathfrak{X}]$ are therefore also in a geometric correspondence with zero-sets of linear differential operators. This provides a geometrical reason why the existence of Ritt–Rosenfeld–Buchberger-type algorithms for the computation with ideals, and not merely radical ideals, is important.

5. Equations with polynomial coefficients

The study of the linear p.d.e.s with coefficients in $\mathcal{C}[x]$ is equivalent to the study of equations with coefficients in $\mathcal{C}[\mathrm{e}^{-x}] = \mathcal{C}[\mathrm{e}^{-x_1}, \ldots, \mathrm{e}^{-x_n}]$ modulo the substitutions $x_i \to \mathrm{e}^{x_i}$, $\delta_i = x_i\partial/\partial x_i \to \partial/\partial x_i$ and multiplication with a suitable $\mathrm{e}^{\alpha\cdot x}$. Since the ordinary partial derivatives preserve the "valuation" in $\mathcal{C}[\mathrm{e}^{-x}]$, it will be more convenient to work with coefficients in $\mathcal{C}[\mathrm{e}^{-x}]$.

Assume that we have fixed an admissible ordering $\preccurlyeq$ on $\mathfrak{X}$, determined by $\lambda_1, \ldots, \lambda_l \in \mathbb{Z}^n$. Assume also that we have fixed a total ordering on $\mathcal{C}$ which gives $\mathcal{C}$ the structure of a totally ordered $\mathbb{Q}$-vector space. Then we also have a natural ordering $\preccurlyeq$ on $\mathfrak{E} = \mathrm{e}^{\mathcal{C}x_1+\cdots+\mathcal{C}x_n}$:

$$\begin{aligned}&\mathrm{e}^{\alpha\cdot x} \prec \mathrm{e}^{\beta\cdot x}\\ \Longleftrightarrow\ &\exists i, (\alpha-\beta)\cdot\lambda_1 = \cdots = (\alpha-\beta)\cdot\lambda_{i-1} = 0 \wedge (\alpha-\beta)\cdot\lambda_i < 0.\end{aligned}$$

A subset $\mathfrak{G}$ of $\mathfrak{E}$ is said to be *grid-based* if there exist $\mathfrak{g}_1,\ldots,\mathfrak{g}_k,\mathfrak{h}\in\mathfrak{E}$ with $\mathfrak{g}_1 \prec 1,\ldots,\mathfrak{g}_k \prec 1$ and $\mathfrak{G} \subseteq \mathfrak{g}_1^{\mathbb{N}}\cdots\mathfrak{g}_k^{\mathbb{N}}\mathfrak{h}$. Given a ring of coefficients $\mathcal{R}$ the set of series $f = \sum_{\mathfrak{e}\in\mathfrak{E}} f_{\mathfrak{e}}\mathfrak{e}$ with grid-based support $\operatorname{supp} f = \{\mathfrak{e}\in\mathfrak{E} : f_{\mathfrak{e}} \neq 0\}$ forms an $\mathcal{R}$-algebra (van der Hoeven, 1997, 2006). We denote this algebra by $\mathcal{R}[[\mathfrak{E}]]$ and its elements are called *grid-based series*. This still goes through for coefficients in $\mathcal{C}\{F\}_1 \oplus \mathcal{C}[x]$, since such operators act by spectral components. In this section, we will consider systems of linear p.d.e.s in $\mathcal{L} = \mathcal{C}\{F\}_1[[\mathfrak{E}]] \oplus \mathcal{C}[x][[\mathfrak{E}]]$ and study their solutions in $\mathcal{S} = \mathcal{C}[x][[\mathfrak{E}]]$.

5.1. Skew standard bases

The admissible orderings $\preccurlyeq$ on $\mathfrak{X}$ and $\preccurlyeq$ on $\mathfrak{E}$ may be combined into a total admissible ordering $\preccurlyeq^\sharp$ on $\mathfrak{X}\mathfrak{E}$ using

$$\mathfrak{x}\mathfrak{e} \preccurlyeq^\sharp_{\mathfrak{X}\mathfrak{E}} \mathfrak{y}\mathfrak{f} \iff \mathfrak{e} \prec_{\mathfrak{E}} \mathfrak{f} \vee (\mathfrak{e} = \mathfrak{f} \wedge \mathfrak{x} \preccurlyeq_{\mathfrak{X}} \mathfrak{y}).$$

Hence, an element $f \in \mathcal{S}$ can also be regarded as a series $f = \sum_{\mathfrak{m}\in\mathfrak{X}\mathfrak{E}} f_{\mathfrak{m}}\mathfrak{m}$ with anti-well-ordered support in $\mathfrak{X}\mathfrak{E}$ (the support is not necessarily grid-based, although we might have required this). Similarly, elements in $\mathcal{C}\{F\}_1[[\mathfrak{E}]]$ can be seen as series with monomials in $\mathfrak{E}\mathfrak{T}$. The ordering $\preccurlyeq^\sharp$ is extended to $\mathcal{L}$ by understanding that $x^n\mathfrak{e} \prec^\sharp \mathfrak{E}\mathfrak{T}$ for all $x^n\mathfrak{e} \in x^{\mathbb{N}}\mathfrak{E}$. We will use $\sharp$ in order to emphasis when a notation should be understood with respect to the relation $\preccurlyeq^\sharp$.

Consider a system $L \in (\mathcal{L}\setminus\mathcal{S})^l$ such that $L_i \asymp 1$ for all i. Given $i < j$ with $\mathfrak{d}^\sharp_{L_i} = \partial^\alpha F$ and $\mathfrak{d}^\sharp_{L_j} = \partial^\beta F$, let $\gamma = \sup(\alpha,\beta)$, $\mathfrak{d}^\sharp_{L_j,L_i} = \partial^{\gamma-\alpha}(F)$, $\mathfrak{d}^\sharp_{L_i,L_j} = \partial^{\gamma-\beta}(F)$ and

$$\Delta_{L_i,L_j} := c^\sharp_{L_j}\mathfrak{d}^\sharp_{L_j,L_i} \circ L_i - c^\sharp_{L_i}\mathfrak{d}^\sharp_{L_i,L_j} \circ L_j.$$

We say that L is a *standard basis* for $\preccurlyeq^\sharp$ if for each $i < j$ there exists a *critical relation*

$$R_{L,i,j} \circ L = \Delta_{L_i,L_j} - A \circ L = 0, \tag{10}$$

where $A \in \mathcal{C}\{F_1,\ldots,F_l\}_1[[\mathfrak{E}]]$ is such that $\mathfrak{d}^\sharp_{A^k} \circ \mathfrak{d}^\sharp_{L_k} \prec^\sharp \mathfrak{d}^\sharp_{L_j,L_i} \circ L_i \asymp^\sharp \mathfrak{d}^\sharp_{L_i,L_j} \circ L_j$ for all k.

Given $f \in \mathcal{S}$ with $f \preccurlyeq 1$ (or $L \in \mathcal{L}$ with $L \preccurlyeq 1$), let us denote $\overline{f} = f_1$ (resp. $\overline{L} = L_1$).

Lemma 6. *Let $L \in \mathcal{C}\{F\}_1[[\mathfrak{E}]]^l$ be a standard basis and let $g \in \mathcal{S}^l_L$ be such that $g \preccurlyeq 1$. Then $\overline{L} \in \mathcal{C}\{F\}^l_1$ is a standard basis and $\overline{g} \in \mathcal{C}[x]^l_{\overline{L}}$.*

Proof. Since $\mathfrak{d}^\sharp_{\overline{L_i}} = \mathfrak{d}^\sharp_{L_i}$ for all i, the system $\bar{L}$ is autoreduced. For all $i < j$, the relation (10) implies

$$\overline{R_{L,i,j}} \circ \overline{L} = \overline{\Delta_{L_i,L_j}} - \overline{A \circ L} = \Delta_{\overline{L_i},\overline{L_j}} - \overline{A} \circ \overline{L} = 0,$$

so $\overline{L}$ is a standard basis for the relations $R_{\overline{L},i,j} = \overline{R_{L,i,j}}$. Now consider a relation $B \circ \overline{L} = 0$. Then we have $B = \Sigma \circ R_{\overline{L}}$ for some $\Sigma \in \mathcal{C}\{F_1,\ldots,F_{l(l-1)/2}\}_1$. Now $\Sigma \circ R_L \circ L = 0$ implies $(\Sigma \circ R_L)(g) = 0$. We conclude that $B(\overline{g}) = \Sigma \circ \overline{R_L}(\overline{g}) = \overline{\Sigma \circ R_L(g)} = 0$, so $\overline{g} \in \mathcal{C}[x]^l_{\overline{L}}$. □

Lemma 7. *Let $L \in \mathcal{L}^l_1$ be a standard basis and $\mathfrak{e} \in \mathfrak{E}$. Then $L_{\ltimes\mathfrak{e}}$ is again a standard basis.*

Proof. Any $P, Q \in \mathcal{C}\{F\}_1[[\mathfrak{E}]]^{\neq}$ satisfy the relations

$$\begin{aligned}(P \circ Q)_{\ltimes \mathfrak{e}} &= P_{\ltimes \mathfrak{e}} \circ Q_{\ltimes \mathfrak{e}}\\ \tau_{P_{\ltimes \mathfrak{e}}} &= \tau_P\\ (\Delta_{P,Q})_{\ltimes \mathfrak{e}} &= \Delta_{P_{\ltimes \mathfrak{e}}, Q_{\ltimes \mathfrak{e}}}\end{aligned}$$

Hence, any critical relation $R_{L,i,j} \circ L$ for L induces a critical relation $(R_{L,i,j})_{\ltimes \mathfrak{e}} \circ L_{\ltimes \mathfrak{e}} = 0$ for $L_{\ltimes \mathfrak{e}}$. So we may take $R_{L_{\ltimes \mathfrak{e}}} = (R_L)_{\ltimes \mathfrak{e}}$. □

5.2. Computation of skew standard bases

Given an arbitrary system $L \in \mathcal{L}^l$, an equivalent standard basis can be "computed" by a variant of Hironaka's infinite division "algorithm". If the dependency of L in $\mathrm{e}^{-x_1}, \ldots, \mathrm{e}^{-x_n}$ is only polynomial, then a fully effective method can be devised, by adapting the algorithms from Section 2.3.

In this subsection *and in this subsection only*, let $\mathfrak{E} = \mathrm{e}^{-\mathbb{N}x_1 - \cdots - \mathbb{N}x_n}$, $\mathcal{R} = \mathcal{C}[\mathrm{e}^{-x}]$, $\mathcal{S} = \mathcal{C}[x][\mathrm{e}^{-x}]$ and $\mathcal{L} = \mathcal{R}\{F\}_1 \oplus \mathcal{S}$. The set $\mathcal{R}\{F\}_1$ is formally isomorphic (as a vector space) to $\mathcal{C}[\partial_1, \ldots, \partial_{2n}](F)$ by sending each e^{-x_i} to ∂_i and ∂_i to ∂_{n+i}. Moreover, the ordering $\preccurlyeq^\sharp$ on $\mathfrak{E}\mathfrak{T}$ corresponds to a tangent cone ordering on $\partial_1^{\mathbb{N}} \cdots \partial_{2n}^{\mathbb{N}} F$. Consequently, the definition of ecart in Section 3.2 transposes to elements in $\mathcal{L} \setminus \mathcal{S}$.

Unfortunately, we do not necessarily have $E_{\Omega \circ P} = E_P$ for $\Omega \in \mathcal{C}\mathfrak{E}\mathfrak{T}$ and $P \in \mathcal{L} \setminus \mathcal{S}$ (for instance $E_{(\partial_1 F) \circ (\mathrm{e}^{-x_1} F)} > 0$). Nevertheless, this relation does hold if $P \asymp 1$. For this reason, we adapt the definition of partial reduction by setting

$$\mathrm{Red}^{\ltimes}(P, Q) := c_Q^\sharp P - c_P^\sharp \mathfrak{d}^\sharp_{P,Q,\ltimes \mathfrak{d}_Q^{-1}} \circ Q$$

for all $P, Q \in \mathcal{L} \setminus \mathcal{S}$ with $\mathfrak{d}_Q^\sharp | \mathfrak{d}_P^\sharp$. Because of the twist, we again have

$$E_{\mathrm{Red}^{\ltimes}(P,Q)} < E_P \vee E_Q.$$

We also notice that $\mathrm{Red}^{\ltimes}$ coincides with the usual partial reduction "up to lower order terms", since $\mathfrak{d}^\sharp_{P,Q,\ltimes \mathfrak{d}_Q^{-1}} \sim \mathfrak{d}^\sharp_{P,Q}$. We obtain the following version of NF :

Algorithm NF (P, Q)
Input: $P \in \mathcal{L}$ and $Q \in (\mathcal{L} \setminus \mathcal{S})^q$
Output: an "asymptotic normal form" of P modulo Q
while $P \notin \mathcal{S} \wedge \exists i, \mathfrak{d}^\sharp_{Q_i} | \mathfrak{d}^\sharp_P$ **do**
  Take i with $\mathfrak{d}^\sharp_{Q_i} | \mathfrak{d}^\sharp_P$ such that $E_{\mathrm{Red}^{\ltimes}(P, Q_i)}$ is minimal
  $Q := Q \cup \{P\}$
  $P := \mathrm{Red}^{\ltimes}(P, Q_i)$
return P

The termination of the modified version of NF is proved in the same way as before. Again, the successive values $P_1, P_2, \ldots$ of P in the algorithm verify relations

$$P_i = A_i \circ Q + (H_i + B_i) \circ P,$$

for certain $H_i \in \mathcal{C}^{\neq} F$, $A_i \in \mathcal{R}\{F_1, \ldots, F_q\}_1$ and $B_i \in \mathcal{R}\{F\}_1$ with $B_i \prec 1$ and $A_i^j \circ Q_j \preccurlyeq^\sharp P$ for all j.

Example 3. Let $P = \mathrm{e}^{-2x_1}\partial_1\partial_2 F$ and $Q = (\mathrm{e}^{-x_1} + \mathrm{e}^{-2x_1})\partial_2 F$. Then

$$\begin{aligned}P_1 &= \mathrm{Red}^{\ltimes}(P, Q) = P - \mathrm{e}^{-x_1}(\partial_1 + 1)Q = -\mathrm{e}^{-3x_1}\partial_1\partial_2 F + \mathrm{e}^{-3x_1}\partial_2 F\\ P_2 &= \mathrm{Red}^{\ltimes}(P_1, P) = P_1 - \mathrm{e}^{-x_1}P = \mathrm{e}^{-3x_1}\partial_2 F\\ P_3 &= \mathrm{Red}^{\ltimes}(P_2, Q) = P_2 - \mathrm{e}^{-2x_1}Q = -\mathrm{e}^{-4x_1}\partial_2 F\\ P_4 &= \mathrm{Red}^{\ltimes}(P_3, P_2) = P_3 + \mathrm{e}^{-x_1}P_2 = 0.\end{aligned}$$

Hence Q divides P, from the asymptotic point of view.

In a similar way, we may define the twisted Δ-polynomial of $P, Q \in \mathcal{L} \setminus \mathcal{S}$ by

$$\Delta^{\ltimes}_{L_i,L_j} := c^{\sharp}_{L_j}\mathfrak{d}^{\sharp}_{L_j,L_i,\ltimes\mathfrak{d}^{-1}_{L_i}} \circ L_i - c^{\sharp}_{L_i}\mathfrak{d}^{\sharp}_{L_i,L_j,\ltimes\mathfrak{d}^{-1}_{L_j}} \circ L_j.$$

Given a system $Q \in (\mathcal{L} \setminus \mathcal{S})^q$, the corresponding algorithm SB now computes an equivalent system $\tilde{Q} \in (\mathcal{L} \setminus \mathcal{S})^{\tilde{q}}$, such that for all $i < j$ we have a relation

$$(H + B) \circ \Delta^{\ltimes}_{\tilde{Q}_i,\tilde{Q}_j} + A \circ \tilde{Q} = 0,$$

where $H \in \mathcal{C}^{\neq}F$, $A \in \mathcal{R}\{F_1, \ldots, F_{\tilde{q}}\}_1$ and $B \in \mathcal{R}\{F\}_1$ are such that $B \prec 1$ and $A^k_i \circ \tilde{Q}_k \preccurlyeq^{\sharp} \mathfrak{d}^{\sharp}_{\tilde{Q}_i,\tilde{Q}_j} \circ \mathfrak{d}^{\sharp}_{\tilde{Q}_j}$ for all k. But $H + B$ admits $(H + B)^{-1} = H^{-1} - H^{-2} \circ B + H^{-3} \circ B \circ B + \cdots$ as inverse in $\mathcal{C}\{F\}_1[[\mathfrak{E}]]$, which leads to the relation

$$\Delta^{\ltimes}_{\tilde{Q}_i,\tilde{Q}_j} + (H + B)^{-1} \circ A \circ \tilde{Q} = 0. \tag{11}$$

Moreover, each $\tilde{Q}_i$ induces an element

$$\hat{Q}_i := \mathfrak{d}^{-1}_{\tilde{Q}_i} \circ \tilde{Q}_i \in \mathcal{C}\{F\}_1[[\mathrm{e}^{\mathbb{Z}x_1+\cdots+\mathbb{Z}x_n}]] \oplus \mathcal{C}[x][[\mathrm{e}^{\mathbb{Z}x_1+\cdots+\mathbb{Z}x_n}]],$$

with $\hat{Q}_i \asymp 1$. When rewriting (11) in terms of $\hat{Q}_i$ and $\hat{Q}_j$, we obtain a critical relation for $\hat{Q}_i$ and $\hat{Q}_j$ in the sense of Section 5.1. Modulo this normalization of the result, SB therefore computes a skew standard basis.

5.3. Theoretical properties of standard bases

Let again $\mathfrak{E} = \mathrm{e}^{\mathcal{C}x_1+\cdots+\mathcal{C}x_n}$, $\mathcal{L} = (\mathcal{C}\{F\}_1 \oplus \mathcal{C}[x])[[\mathfrak{E}]]$ and $\mathcal{S} = \mathcal{C}[x][[\mathfrak{E}]]$. Let $A \in \mathcal{C}\{F_1, \ldots, F_k\}_1[[\mathfrak{E}]]^p$. Using the isomorphism $\mathcal{C}\{F_1, \ldots, F_k\}_1[[\mathfrak{E}]]^p \cong \mathcal{C}\{F_1, \ldots, F_k\}^p_1[[\mathfrak{E}]]$, we observe that $\mathfrak{d}_A$, $\mathrm{supp}\ A$, etc. are well-defined. Given $B \in \mathcal{C}\{F_1, \ldots, F_l\}_1[[\mathfrak{E}]]^q$ it is also convenient to extend the notation $\preccurlyeq$ by setting $A \preccurlyeq B$ if and only if $\mathfrak{d}_A \preccurlyeq \mathfrak{d}_B$.

Lemma 8. *Let $L \in \mathcal{L}^l$ and $A \in \mathcal{C}\{F_1, \ldots, F_l\}_1[[\mathfrak{E}]]$ be such that L is a standard basis for $\preccurlyeq^{\sharp}$ and $A \preccurlyeq 1$. Then there exists a $\Sigma \in \mathcal{C}\{F_1, \ldots, F_{l(l-1)/2}\}[[\mathfrak{E}]]$ with $\Sigma \preccurlyeq 1$ and $A - \Sigma \circ R_L \preccurlyeq A \circ L$. In particular, if $A \circ L = 0$, then there exists a Σ with $A = \Sigma \circ R_L$.*

Proof. Let $\mathfrak{G} = \mathfrak{g}_1^{\mathbb{N}} \cdots \mathfrak{g}_k^{\mathbb{N}} \subseteq \mathfrak{E}$ with $\mathfrak{g}_1 \prec 1, \ldots, \mathfrak{g}_k \prec 1$ be such that $\mathrm{supp}\ L \cup \mathrm{supp}\ R_L \cup \mathrm{supp}\ A \subseteq \mathfrak{G}$. Then $\mathcal{L}_{\mathfrak{G}} := \{P \in \mathcal{L} : \mathrm{supp}\ P \subseteq \mathfrak{G}\}$ is stable under composition. For each $\mathfrak{e} \in \mathfrak{G}$ with $\mathfrak{e} \succ A \circ L$, let us show how to construct $\Sigma_{\mathfrak{e}} \in \mathcal{C}\{F_1, \ldots, F_{l(l-1)/2}\}^b_1$, such that $A^{\succcurlyeq\mathfrak{e}} = A - (\sum_{\mathfrak{f}\succcurlyeq\mathfrak{e}} \mathfrak{f}\Sigma_{\mathfrak{f}}) \circ R_L$ satisfies $A^{\succcurlyeq\mathfrak{e}} \prec \mathfrak{e}$. We use weak induction over $\mathfrak{G}$.

So let $\mathfrak{e} \in \mathfrak{G}$ and assume that $\Sigma_{\mathfrak{f}}$ has been constructed for all $\mathfrak{f} \succ \mathfrak{e}$. Let $A^{\succ\mathfrak{e}} = A - (\sum_{\mathfrak{f}\succ\mathfrak{e}} \mathfrak{f}\Sigma_{\mathfrak{f}}) \circ R_L$. Since $A^{\succ\mathfrak{e}} \prec \mathfrak{f}$ for all $\mathfrak{f} \in \mathfrak{G}$ with $\mathfrak{f} \succ \mathfrak{e}$, and $\mathrm{supp}\ A^{\succ\mathfrak{e}} \subseteq \mathfrak{G}$, we have $A^{\succ\mathfrak{e}} \preccurlyeq \mathfrak{e}$.

Setting $B = \mathfrak{e}^{-1} A^{\succ \mathfrak{e}}$, we have $\overline{B} \circ \overline{L} = 0$, so $\overline{B} = \mathrm{T} \circ \overline{R_L}$ for some $\mathrm{T} \in \mathcal{C}\{F_1, \ldots, F_{l(l-1)/2}\}_1^b$. Taking $\Sigma_{\mathfrak{e}} = \mathrm{T}$, it follows that $A^{\succcurlyeq \mathfrak{e}} = A^{\succ \mathfrak{e}} - \mathfrak{e}\mathrm{T} \circ R_L = A^{\succ \mathfrak{e}} - \mathfrak{e}(A^{\succ \mathfrak{e}})_{\mathfrak{e}} + o(\mathfrak{e}) \prec \mathfrak{e}$.

By induction, we conclude that $\Sigma = \sum_{\mathfrak{f} \succ A \circ L} \mathfrak{f} \Sigma_{\mathfrak{f}}$ is well-defined and we have $A - \Sigma \circ R_L = A^{\succcurlyeq \mathfrak{e}} + o(\mathfrak{e}) \prec \mathfrak{e}$ for all $\mathfrak{e} \succ A \circ L$, so $A - \Sigma \circ R_L \preccurlyeq A \circ L$. □

Corollary 5. *Given a standard basis $L \in \mathcal{C}\{F\}_1[[\mathfrak{E}]]^l$ and $g \in \mathcal{S}^l$, we have $g \in \mathcal{S}_L^l$ if and only if $R_L(g) = 0$.*

Proof. Similar to the proofs of Corollaries 1 and 2. □

Corollary 6. *Assume that $L \in \mathcal{C}\{F\}_1[[\mathfrak{E}]]^l$ is a standard basis for $\preccurlyeq^\sharp$ and let $K \in [L]$ be non-zero. Then $c_K \in [\overline{L}]$.*

Proof. Let A be such that $K = A \circ L$. Modulo division of A by $\mathfrak{d}_A$, we may assume without loss of generality that $A \preccurlyeq 1$. Let Σ be as in the above lemma, so that $\tilde{A} = A - \Sigma \circ R_L \preccurlyeq A \circ L$. In fact, $\tilde{A} \asymp A \circ L$, since $L \preccurlyeq 1$ implies $A \circ L = \tilde{A} \circ L \preccurlyeq \tilde{A}$. We conclude that $c_K = c_{\tilde{A} \circ L} = c_{\tilde{A}} \circ \overline{L} \in [\overline{L}]$. □

5.4. Solving $L(f) = g$ in $\mathcal{S}$

Consider a standard basis $L \in \mathcal{C}\{F\}_1[[\mathfrak{E}]]^l$ for $\preccurlyeq^\sharp$. Given $\mathfrak{e} \in \mathfrak{E}$, we may regard $\overline{L_{\ltimes \mathfrak{e}}}$ as an operator on $\mathcal{C}[x]^l$. We denote

$$\mathcal{H}_L = \{\mathfrak{h}\mathfrak{e} : \mathfrak{e} \in \mathfrak{E}, \mathfrak{h} \in \mathcal{H}_{\overline{L_{\ltimes \mathfrak{e}}}}\};$$
$$\mathcal{H}_L^\perp = \{\mathfrak{h}\mathfrak{e} : \mathfrak{e} \in \mathfrak{E}, \mathfrak{h} \in \mathcal{H}^\perp_{\overline{L_{\ltimes \mathfrak{e}}}}\},$$

and write $\overline{L_{\ltimes \mathfrak{e}}}^{-1}$ for the distinguished right inverse of $\overline{L_{\ltimes \mathfrak{e}}}$.

Proposition 2. *Let $L \in \mathcal{C}\{F\}_1[[\mathfrak{E}]]^l$ be a standard basis for $\preccurlyeq^\sharp$. Then $L : \mathcal{S}^l \mapsto \mathcal{S}_L^l$ admits a unique right inverse L^{-1} such that $L^{-1}(g) \in \mathcal{H}_L^\perp$ for all $g \in \mathcal{S}_L^l$.*

Proof. Let $\mathfrak{G} = \mathfrak{g}_1^{\mathbb{N}} \cdots \mathfrak{g}_k^{\mathbb{N}} \subseteq \mathfrak{E}$ with $\mathfrak{g}_1 \prec 1, \ldots, \mathfrak{g}_k \prec 1$ and $\mathfrak{h} = \mathfrak{d}_g$ be such that $\operatorname{supp} L \subseteq \mathfrak{G}$ and $\operatorname{supp} g \subseteq \mathfrak{G}\mathfrak{h}$. For any $f \in \mathcal{S}$ with $\operatorname{supp} f \subseteq \mathfrak{G}\mathfrak{h}$, it follows that $\operatorname{supp} L(f) \subseteq \mathfrak{G}\mathfrak{h}$. Let us show by well-ordered induction over $\mathfrak{e} \in \mathfrak{G}\mathfrak{h}$ how to construct $f_{\mathfrak{e}} \in \mathcal{H}^\perp_{\overline{L_{\ltimes \mathfrak{e}}}}$ such that $L(f) = g$ for $f = \sum_{\mathfrak{e} \in \mathfrak{G}\mathfrak{h}} f_{\mathfrak{e}} \mathfrak{e}$.

Given $\mathfrak{e} \in \mathfrak{G}\mathfrak{h}$, we assume that $f_{\mathfrak{f}}$ has been constructed for all $\mathfrak{f} \in \mathfrak{G}\mathfrak{h}$ with $\mathfrak{f} \succ \mathfrak{e}$. Denoting $f_{\succ \mathfrak{e}} = \sum_{\mathfrak{f} \succ \mathfrak{e}} f_{\mathfrak{f}} \mathfrak{f}$, we also assume that $g - L(f_{\succ \mathfrak{e}}) \prec \mathfrak{f}$ for all $\mathfrak{f} \in \mathfrak{G}\mathfrak{h}$ with $\mathfrak{f} \succ \mathfrak{e}$. By construction, we first observe that $\operatorname{supp} L(f_{\succ \mathfrak{e}}) \subseteq \mathfrak{G}\mathfrak{h}$, whence $g - L(f_{\succ \mathfrak{e}}) \preccurlyeq \mathfrak{e}$. Now we take $f_{\mathfrak{e}} := \overline{L_{\ltimes \mathfrak{e}}}^{-1}((g - L(f_{\succ \mathfrak{e}}))_{\mathfrak{e}})$, which is well-defined by Lemmas 7 and 6. Setting $f_{\succcurlyeq \mathfrak{e}} = f_{\succ \mathfrak{e}} + f_{\mathfrak{e}} \mathfrak{e} = \sum_{\mathfrak{f} \succcurlyeq \mathfrak{e}} f_{\mathfrak{f}} \mathfrak{f}$, it follows that $L(f_{\succcurlyeq \mathfrak{e}})_{\mathfrak{e}} = L(f_{\succ \mathfrak{e}})_{\mathfrak{e}} + \overline{L_{\ltimes \mathfrak{e}}}(f_{\mathfrak{e}}) = g_{\mathfrak{e}}$. For all $\mathfrak{f} \in \mathfrak{G}\mathfrak{h}$ with $\mathfrak{f} \succ \mathfrak{e}$, we also have $g - L(f_{\succcurlyeq \mathfrak{e}}) = g - L(f_{\succ \mathfrak{e}}) + O(\mathfrak{e}) \prec \mathfrak{f}$. We infer that $g - L(f_{\succcurlyeq \mathfrak{e}}) \prec \mathfrak{e}$. By induction, we obtain a series $f \in \mathcal{H}_L^\perp$ with $\operatorname{supp} f \subseteq \mathfrak{G}\mathfrak{h}$ and $g - L(f) = g - L(f_{\succcurlyeq \mathfrak{e}}) + o(\mathfrak{e}) \prec \mathfrak{e}$ for all $\mathfrak{e} \in \mathfrak{G}\mathfrak{h}$. We conclude that $L(f) = g$. The uniqueness is proved as usual. □

Proposition 3. *Let $L \in \mathcal{C}\{F\}_1[[\mathfrak{E}]]^l$ be a standard basis for $\preccurlyeq^\sharp$. For each $\mathfrak{h} \in \mathcal{H}_L$, let $b_{\mathfrak{h}} = \mathfrak{h} - L^{-1}(L(\mathfrak{h}))$. Then*

$$h = \sum_{\mathfrak{h} \in \mathcal{H}_L} h_{\mathfrak{h}} b_{\mathfrak{h}}$$

for all solutions $h \in \mathcal{S}$ to $L(h) = 0$.

Proof. Setting

$$\pi = \sum_{\mathfrak{h} \in \mathcal{H}_L} h_{\mathfrak{h}} \mathfrak{h},$$

we have

$$\tilde{h} = \sum_{\mathfrak{h} \in \mathcal{H}_L} h_{\mathfrak{h}} b_{\mathfrak{h}} = \pi - L^{-1}(L(\pi)).$$

Now $\tilde{h}_{\mathfrak{h}} = h_{\mathfrak{h}}$ for all $\mathfrak{h} \in \mathcal{H}_L$, by the distinguished property of L^{-1} and the fact that $b_{\mathfrak{h}} = \mathfrak{h} + o(\mathfrak{h})$. Consequently, supp $\tilde{h} - h \subseteq \mathcal{H}_L^{\perp}$ and $L(\tilde{h} - h) = 0$. But this is only possible if $\tilde{h} = h$. $\square$

Let us now consider an arbitrary system $K \in \mathcal{C}\{F\}_1[[\mathfrak{E}]]^k$ and let $L \in \mathcal{C}\{F\}_1[[\mathfrak{E}]]^l$ be an equivalent standard basis. By Corollary 6, we notice that the differential ideal $[\overline{L}]$ does not depend on the particular choice of L, and similarly for the twisted differential ideals $[\overline{L_{\ltimes \mathfrak{e}}}]$. Consequently, the spaces $\mathcal{H}_L$, $\mathcal{H}_L^{\perp}$ and the operator L^{-1} are independent of the particular choice of L. We may therefore define the distinguished right inverse K^{-1} of K by $K^{-1} = L^{-1}$.

Putting everything together from the effective point of view, we have:

Theorem 3. *There exists an algorithm which, given $L \in \mathcal{C}[\mathrm{e}^{-x}]\{F\}_1^l$ and $g \in \mathcal{C}[\mathrm{e}^{-x}]_L^l$, computes the asymptotic expansion of $L^{-1}g$.*

Proof. Using the algorithm from Section 5.2, we start by computing an equivalent standard basis $L := \Sigma \circ L$ for L and make the corresponding change $g := \Sigma \circ g$ for g. We next test whether g is compatible with L using Corollary 5. If so, and assuming that $g \neq 0$, we determine the dominant term $c_g \mathfrak{e}$ of g and compute the dominant term $\overline{L_{\ltimes \mathfrak{e}}}^{-1}(c_g)\mathfrak{e}$ of $f = L^{-1}g$ using the method from Section 4.2. Setting $\tilde{g} = g - L(c_f \mathfrak{e})$ and continuing the same procedure with $\tilde{g}$ instead of g, we obtain the asymptotic expansion of f. $\square$

Remark 5. The theorem still works if we take $g \in \mathcal{C}[\mathfrak{E}]_L^l$, where $\mathcal{C}[\mathfrak{E}] = \bigoplus_{\mathfrak{e} \in \mathfrak{E}} \mathcal{C}\mathfrak{e}$.

Remark 6. Using the technique of Cartesian representations (van der Hoeven, 1997, 2006), it is possible to compute the full expansion of $L^{-1}g$ and not merely the first ω terms (as done by the above algorithm).

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