

Two-variable logic revisited

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Abstract

In this paper we present another proof for the well-known *small model property* of two-variable logic. As far as we know, existing proofs of this property rely heavily on model theoretic concepts. In contrast, ours is combinatorial in nature and uses only a very simple counting argument, which we find intuitive and elegant. We also consider matching lower bounds.

1 Introduction

Two-variable logic (FO^2) is a well known fragment of first-order logic that comes with decidable satisfiability problem. Intuitively, FO^2 is a class of first-order sentences where only two variables x and y are used. It was first proved to be decidable in double-exponential time by Mortimer [11]. The upper bound was later improved to single-exponential by Grädel, Kolaitis and Vardi [4]. Indeed, both Grädel, et. al. and Mortimer proved that FO^2 has the so called *small model property*. That is, if a formula $\varphi \in \text{FO}^2$ is satisfiable, then it is satisfiable by a model with cardinality exponential in the length of φ , or double-exponential in the case of Mortimer's. Both proofs are rather delicate and involve some intricate model theoretic constructions.

In this paper we present another proof for the FO^2 small model property. The bound that we achieve is single-exponential, matching the one by Grädel, et. al. However, our proof is purely combinatorics in nature. We establish a very simple graph-theoretic lemma from which the small model property is a direct implication. Our proof relies on a simple counting argument, and not only do we find it elegant, but it also contains some information about the small model that cannot be deduced from the other proofs. One example is that the spectrum of an FO^2 sentence is either finite or co-finite.

Note that the small model property immediately implies that the satisfiability problem for FO^2 is decidable in NEXPTIME. A matching lower bound (NEXPTIME-hardness) was established by Fürer [2] whose proof is based on the work by Lewis [9]. In this paper we also present another proof for the NEXPTIME-hardness. The main idea is quite similar to the one by Lewis [9], but ours is arguably more direct and transparent. Indeed, we establish that the exponential bound in the small model property is tight by a very simple example, which is then be modified easily to obtain the NEXPTIME-hardness.

Other related works. Two-variable logic has a long interesting history that can be traced back to the works of Scott [13] and Gödel [3]. It is also tightly related to other well known decidable logics such as propositional modal logic [14] and description logics [10]. For more details, we refer the interested reader to the beautiful introduction in Grädel, et. al.'s paper [4]. The notion of first-order spectrum was first introduced by Scholtz [12]. For some recent works

in first-order spectra with bounded variables, including two-variable logic, we refer the reader to the work of Kopczynski and Tan [6–8] and the references therein, as well as the survey by Durand, et. al. [1].

Organization. We establish a simple combinatorial lemma in Section 2. The main results are all presented in Section 3.

2 A simple combinatorial lemma

Let C, D be two disjoint *finite* sets of colors whose elements are called vertex and edge colors, respectively. A (C, D) -graph is a complete, undirected graph (with no simple loop) where the vertices and edges are colored with vertex and edge colors, respectively. A (C, D) -graph is also called a (k, ℓ) -graph, where $|C| = k$ and $|D| = \ell$. Note that a (C, D) -graph may be infinite.

We first introduce a few terminologies. Let G be a (C, D) -graph. We write $col_G(u, v)$ to denote the color of the edge (u, v) in G . For a vertex color c , $c(G)$ denotes the set of vertices in G with color c . A vertex u in G is incident to an edge color d , if there is an edge incident to u with color d . We also say that a vertex u is incident to a pair $(d, c) \in D \times C$, if there is a vertex v with color c and the edge (u, v) has color d . We write $D_{c_1, c_2}(G)$ to denote the set of the edge colors whose two incident vertices are colored with c_1 and c_2 .

A color c is a *king* color (in G), if $|c(G)| = 1$. The vertex with a king color is called a *king* vertex, or a king, for short. We denote by $KC(G)$ the set of king colors in G . Obviously, $|KC(G)|$ is precisely the number of kings in G .

Let v_1, \dots, v_t be the kings in G and let c_1, \dots, c_t be their respective colors. For a non-king vertex u , the *profile* of u is the set $\{(d_1, c_1), \dots, (d_t, c_t)\}$ where each d_j is the color of the edge connecting u and v_j . Intuitively, the profile of u contains the information about the relation between u and each of the kings.

Lemma 2.1 *Let G be a (k, ℓ) -graph, where $k, \ell \geq 3$. Then there is a (k, ℓ) -graph H with the following properties.*

- (a) $KC(G) = KC(H)$.
- (b) For every non-king color $c \in C$, $|c(H)| = k \cdot \ell$.
- (c) For every $c_1, c_2 \in C$, $D_{c_1, c_2}(G) = D_{c_1, c_2}(H)$.
That is, the colors of the edges between any two vertices with colors c_1 and c_2 are the same in both G and H .
- (d) For every non-king vertex u in H , there is a non-king vertex v in G with the same color and profile as u .
- (e) For every (not necessarily different) non-king colors c_1, c_2 , for every edge color $d \in D_{c_1, c_2}(G)$, every vertex $u \in c_1(H)$ is incident to (d, c_2) .

That is, every vertex $u \in c_1(H)$ has the following property: For every $d \in D_{c_1, c_2}(G)$, u is incident to an edge with color d and the other end point of that edge is of color c_2 .

Proof. For every non-king colors c , we pick pairwise disjoint sets Z_1^c, \dots, Z_k^c , where each $|Z_i^c| = \ell$. We let $Z^c = Z_1^c \cup \dots \cup Z_k^c$. The graph H is obtained from G by replacing the vertices in $c(G)$ with Z^c , where all vertices in Z^c are colored with c . The king colors and the edges

between them remain the same as in G . Obviously, at this point (a) and (b) already hold H . We will show how to obtain (c)–(e).

For each non-king color c , we color the edges incident to vertices in Z^c as follows.

Step 1: Color the edges between the king vertices and Z^c .

Let v_1, \dots, v_t be the kings with colors c_1, \dots, c_t , respectively. Since $t \leq k$, we can pick t sets Z_1^c, \dots, Z_t^c . We first color the edges between the kings and $Z_1^c \cup \dots \cup Z_t^c$. For each king $v_i \in \{v_1, \dots, v_t\}$, we do the following.

- We color the edges between v_i and vertices in Z_i^c such that all the colors in $D_{c_i, c}(G)$ are used, which is possible since each $|Z_i^c| = \ell$.
- Note that we only use the colors in $D_{c_i, c}(G)$ for edges between v_i and Z_i^c . So, for every vertex $u \in Z_i^c$, there is a vertex $x \in c(G)$ such that $col_H(u, v_i) = col_G(x, v_i)$. Now we can color the edges between u and the rest of the kings (i.e., kings that are not v_i) so that the profile of u in H is the same as the profile of x in G .

For all the other vertex $u \in Z^c - (Z_1^c \cup \dots \cup Z_t^c)$, we pick a vertex $u' \in Z_1^c \cup \dots \cup Z_t^c$, and color the edges between u' and the kings so that both u' and u have the same profile.

Step 2: Color the edges between vertices in Z^c .

For each $i = 1, \dots, k$, for each vertex $u \in Z_i^c$, we color the edges between u and Z_{i+1}^c so that all the colors in $D_{c, c}(G)$ are used. (When $i = k$, replace Z_{i+1}^c with Z_1^c .)

All the other edges not yet colored can be colored with arbitrary colors from $D_{c, c}(G)$.

Step 3: Color the edges between Z^{c_0} and Z^c , for any non-king color $c_0 \neq c$.

- For each $i = 1, \dots, k$, for each vertex $u \in Z_i^c$, we color the edges between u and $Z_i^{c_0}$ such that all the colors in $D_{c, c_0}(G)$ are used.
- For each $i = 1, \dots, k$, for each vertex $u \in Z_i^{c_0}$, we color the edges between u and Z_{i+1}^c such that all the colors in $D_{c, c_0}(G)$ are used. (When $i = k$, replace Z_{i+1}^c with Z_1^c .)
- All the other edges between Z^c and Z^{c_0} not yet colored can be colored with arbitrary colors from $D_{c, c_0}(G)$.

The edges between any two vertices in H with colors c_1 and c_2 are colored only with colors from $D_{c_1, c_2}(G)$. Thus, (c) holds. That (d) holds is immediate in Step 1. Finally, that (e) holds is immediate in Steps 2 and 3. ■

Remark 2.2 Note that the proof of Lemma 2.1 will still hold, if we replace condition (b) with $|c(H)| = N$ for any $N \geq k \cdot l$.

3 Two-variable logic

In this section we establish the small model property of FO^2 . Matching lower bounds will be presented immediately after. We start by invoking a well known result of Scott [13] which

states that every FO^2 sentence can be converted in polynomial time into the following *Scott normal form*:

$$\Phi := \forall x \forall y \alpha(x, y) \wedge \bigwedge_{i=1}^m \forall x \exists y (\beta_i(x, y) \wedge x \neq y) \quad (1)$$

where α and β_i are all quantifier free. Without loss of generality, we can assume that each β_i is an atomic formula. We also assume that the binary relations used in φ are only β_1, \dots, β_m . Later we will explain how to extend our proof when there are other binary relations.

We let n to be the number of unary predicates used in Φ . By adding redundant predicates, if necessary, we can assume that $n, m \geq 2$. We recall a few standard terminologies. A *1-type* is a maximally consistent set of atomic and negated atomic formulas using only the variable x . A *2-type* is a maximally consistent set of atomic and negated atomic formulas using only the relations β_1, \dots, β_m and variables x, y . A type can be viewed as a quantifier-free formulas that is the conjunction of its elements. The number of 1-types and 2-types are 2^n and 2^{2m} , respectively.

For a structure \mathcal{A} , the *type of an element* $a \in A$ is the unique 1-type π that a satisfies in \mathcal{A} . A 1-type is a *king*, if there is only one element in A that satisfies it. Similarly, the type of a pair $(a, b) \in A \times A$ is the unique 2-type that (a, b) satisfies in \mathcal{A} . A 1-type/2-type is called *realizable* in \mathcal{A} if there is an element/a pair of elements that satisfies it.

A structure $\mathcal{A} \models \varphi$ can be viewed as a complete *directed* graph where the vertex and edge colors are the 1-types and 2-types, respectively. Since the 2-type of (a, b) uniquely determines the 2-type of its reverse (b, a) , we can view \mathcal{A} as *undirected graph*, but the edge colors become pairs (t, \otimes) , where t is a 2-type and $\otimes \in \{\leftarrow, \rightarrow\}$ indicates the “position” of variables x and y . For example, if a pair (a, b) has color (t, \rightarrow) means the 2-type of (a, b) is t . If it has color (t, \leftarrow) , the 2-type of (b, a) is t . Altogether, the number of edge colors now becomes $2 \cdot 2^{2m}$.

Let $\mathcal{A} \models \Phi$, where Φ is a sentence as in (1). Viewing \mathcal{A} as a $(2^n, 2^{2m+1})$ -graph, by Lemma 2.1, there is a structure \mathcal{B} where each non-king 1-type has exactly 2^{2m+n+1} elements. Overall, \mathcal{B} has at most $O(2^{2m+2n})$ elements. Since both the realizable and non-realizable types are preserved in \mathcal{B} , it is immediate that $\mathcal{B} \models \Phi$. This establishes the small model property as stated below.

Theorem 3.1 *Every satisfiable FO^2 sentence in Scott Normal Form is satisfiable by a structure with cardinality $O(2^{2m+2n})$.*

Note that if the sentence Φ uses binary relations other than β_1, \dots, β_m , they appear only in the fragment $\forall x \forall y \alpha(x, y)$ and are already involved in the realizable 2-types (in \mathcal{A}), which are preserved in \mathcal{B} . Thus, our argument above still holds.

Remark 3.2 Our proof also implies that the spectrum of an FO^2 sentence is either finite or co-finite. Recall that the spectrum of a first-order sentence φ is the set of the cardinalities of the finite models of φ . Indeed, let φ be an FO^2 sentence. If it has a model with some non-king 1-types, by Remark 2.2, it has a model with arbitrary large cardinalities, hence, its spectrum is co-finite. Otherwise, in all its models, only king 1-types are realizable, so, its spectrum must be finite.

Next, we will show that the bound is tight. For an integer $n \geq 1$, fix a vocabulary $\tau = \{U_1, \dots, U_n\}$, where each U_i is unary predicate symbol. We can use a 1-type π to encode

an integer $0 \leq N_\pi \leq 2^n - 1$, where $U_i(x) \in \pi$ if and only if bit i in the binary representation of N_π is 1. Consider the following formulas.

$$\begin{aligned}\varphi_{\min}(x) &:= \bigwedge_{i=1}^n \neg U_i(x) & \varphi_{\max}(x) &:= \bigwedge_{i=1}^n U_i(x) \\ \varphi_{\text{suc}}(x, y) &:= \bigvee_{i=1}^n \left(\bigwedge_{j=1}^{i-1} (\neg U_j(y) \wedge U_j(x)) \wedge U_i(y) \wedge \neg U_i(x) \wedge \bigwedge_{j=i+1}^n (U_j(x) \leftrightarrow U_j(y)) \right)\end{aligned}$$

Intuitively, the formulas $\varphi_{\min}(x)$ and $\varphi_{\max}(x)$ define the 1-types that represent 0 and $2^n - 1$, respectively, and $\varphi_{\text{suc}}(x, y)$ define the successor relations on the 1-types, i.e., 1-type of y is 1-type of x plus 1. Consider the following sentence Ψ .

$$\Psi := \exists x \varphi_{\min}(x) \wedge \forall x (\neg \varphi_{\max}(x) \rightarrow \exists y \varphi_{\text{suc}}(x, y)) \quad (2)$$

Note that Ψ uses only unary predicates and does not use equality. Over a fixed alphabet, each U_i can be encoded with a string of length $O(\log n)$. So, the length of Ψ is $O(n^2 \log n)$. Now, for every model $\mathcal{A} \models \Psi$, every 1-type is realizable in \mathcal{A} . Hence, every model of Ψ has cardinality at least 2^n , i.e., exponential in the length of Ψ .

Modifying the technique above, we can easily obtain the NEXPTIME-hardness for FO^2 satisfiability problem. Let $L \subseteq \{0, 1\}^*$ be a language accepted by a non-deterministic Turing machine M in time $O(2^{n^k})$. To avoid clutter, we assume that the working alphabet of M is $\Sigma = \{0, 1, \#\}$, where $\#$ is the blank symbol. We also assume that M runs in time 2^n .

Let Q be the set of states of M and let $\Delta = \Sigma \cup (Q \times \Sigma)$. It is pretty standard that an accepting run of M on a word w of length n can be viewed as a function $F : \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \rightarrow \Delta$, where $\mathbb{Z}_{2^n} = \{0, \dots, 2^n - 1\}$. Intuitively, if $F(i, j) = c \in \Sigma$, it means that in time j , cell i in the tape contains symbol c . If $F(i, j) = (q, b) \in Q \times \Sigma$, it means M is in state q with the head currently reading cell i containing symbol b .

An accepting run $F : \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \rightarrow \Delta$ can be encoded as a structure in the following sense. Let $X_1, \dots, X_n, Y_1, \dots, Y_n$ be $2n$ unary predicates. For each symbol $c \in \Delta$, let R_c be a unary predicate. We say that 1-type π encodes a tuple $(i, j, F(i, j)) \in \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \Delta$, if:

- π encodes (i, j) , i.e., for every $1 \leq t \leq n$: (1) $X_t(x) \in \pi$ if and only if bit t in the binary representation of i is 1, and (2) $Y_t(x) \in \pi$ if and only if bit t in the binary representation of j is 1.
- $R_{F(i, j)}(x) \in \pi$ and $R_{c'}(x) \notin \pi$, for every other $c' \neq F(i, j)$.

We say that a model \mathcal{A} encodes an accepting run $F : \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \rightarrow \Delta$, if for every $(i, j) \in \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, the type that encodes $(i, j, F(i, j))$ is realizable in \mathcal{A} .

Given a word $w \in \{0, 1\}^*$ of length n , we show how to construct an FO^2 sentence Ψ_w in polynomial time such that every model of Ψ_w encodes an accepting run of M on w (if $w \in L$). Basically the sentence Ψ_w is the conjunction of several sentences that “describe” the properties of an accepting run. The construction is not that difficult, but for completeness, we present it in the following paragraphs. Also note that neither the equality symbol nor binary relations are used.

Similar to the above, we first define the formulas $\varphi_{x-\min}(x)$, $\varphi_{x-\max}(x)$, $\varphi_{x-\text{suc}}(x, y)$, $\varphi_{<_x}(x, y)$ and $\varphi_{=_x}(x, y)$ to represent the *minimum*, the *maximum*, the *successor*, the *less than* and the *equality* relations on the X-component in $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, respectively. Similar relations along the

Y-component can be defined, which we denote by $\varphi_{y-\min}(x)$, $\varphi_{y-\max}(x)$, $\varphi_{y-\text{suc}}(x, y)$, $\varphi_{<_y}(x, y)$ and $\varphi_{=y}(x, y)$.

First, we construct an FO^2 sentence Ψ_0 such that every model of Ψ_0 encodes a function $F : \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \rightarrow \Delta$. The construction is a rather straightforward modification of the sentence Ψ in eq. (2), hence, omitted. Some other properties such as: for each $j \in \mathbb{Z}_{2^n}$, there is a unique i such that $F(i, j) \in Q \times \Sigma$ and that an accepting state must appear somewhere, are also straightforward, hence, omitted.

In the following we will focus on the sentences for describing the initial configuration and that the transitions of M are obeyed. Let $w = b_0 b_1 \cdots b_{n-1}$ be the input word, where each $b_i \in \{0, 1\}$. For each $0 \leq t \leq 2^n - 1$, let $c_t \in \Delta$ be the symbol where $c_0 = (q_0, b_0)$, $c_t = b_t$, if $1 \leq t \leq n - 1$ and $c_t = \#$, if $n \leq t \leq 2^n$.

In an accepting run $F : \mathbb{Z}_{2^n-1} \times \mathbb{Z}_{2^n-1} \rightarrow \Delta$ (of M on w), $F(t, 0) = c_t$, for each $0 \leq t \leq 2^n - 1$, i.e., in time 0 the head is reading the first cell and the first n cells contain the input word w . A sentence that describes it can be constructed as follows. First, for each $0 \leq t \leq n$, for a variable $z \in \{x, y\}$, define the formula $\phi_t(z)$ as follows.

$$\begin{aligned} \phi_t(z) &:= R_{c_t}(z) \wedge \exists z' (\varphi_{x-\text{suc}}(z, z') \wedge \phi_{t+1}(z')), & \text{if } 0 \leq t \leq n-1 \\ \phi_t(z) &:= R_{\#}(z) \wedge \forall z' (\varphi_{<_x}(z, z') \rightarrow R_{\#}(z')), & \text{if } t = n \end{aligned}$$

where $z' \in \{x, y\}$ is a variable different from z . Let Ψ_{init} be the following sentence.

$$\Psi_{\text{init}} := \exists x \varphi_{x-\min}(x) \wedge \varphi_{y-\min}(x) \wedge \phi_1(x)$$

Intuitively it states that the types that encode tuples $(t, 0, c_t)$ must be realizable, for each $0 \leq t \leq 2^n - 1$.

To ensure that the transitions of M are obeyed on each step, the labels $F(i-1, j+1)$, $F(i, j+1)$ and $F(i+1, j+1)$ as well as the labels $F(i-1, j)$, $F(i, j)$ and $F(i+1, j)$ must agree according to the transitions of M , for every $(i, j) \in \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$.^{*} To this end, for each $\bar{c} = (c_1, c_2, c_3) \in \Delta^3$, for each variable $z \in \{x, y\}$, define a formula $\varphi_{\bar{c}}(z)$ as follows.

$$\varphi_{\bar{c}}(z) := R_{c_1}(z) \wedge \exists z' (\varphi_{x-\text{suc}}(y, x) \wedge R_{c_2}(y)) \wedge \exists z' (\varphi_{x-\text{suc}}(x, y) \wedge R_{c_3}(y))$$

where $z' \in \{x, y\}$ is a variable different from z . Intuitively, it states that the labels $F(i-1, j)$, $F(i, j)$ and $F(i+1, j)$ are c_1 , c_2 and c_3 , respectively.

For every $\bar{c}, \bar{c}' \in \Delta^3$, we can define a sentence $\Phi_{\bar{c}, \bar{c}'}$ as follows.

$$\forall x \left(\neg \varphi_{x-\min}(x) \wedge \neg \varphi_{x-\max}(x) \wedge \neg \varphi_{y-\max}(x) \wedge \varphi_{\bar{c}}(x) \rightarrow \exists y (\varphi_{y-\text{suc}}(x, y) \wedge \varphi_{\bar{c}'}(y)) \right)$$

To state that the transitions of M are obeyed in the run, we simply conjunct $\Phi_{\bar{c}, \bar{c}'}$ for all appropriate $\bar{c}, \bar{c}' \in \Delta^3$.

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^{*}This is, of course, when (i, j) are not “border points,” i.e., $i \neq 0, 2^n - 1$ and $j \neq 2^n - 1$. When $i = 0$ or $i = 2^n - 1$ or $j = 2^n - 1$, straightforward modifications can be applied.

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