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IMAGES AND PREIMAGES IN RANDOM MAPPINGS*

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Abstract. We present a general theorem that can be used to identify the limiting distribution for a class of combinatorial schemata. For example, many parameters in random mappings can be covered in this way. In particular, we can derive the limiting distribution of those points with a given number of total predecessors.

Key words. random mappings, combinatorial constructions, limiting distributions

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1. Introduction. By a random mapping $\varphi \in \mathcal{F}_n \subseteq \mathcal{F} = \bigcup_{n\ge 0} \mathcal{F}_n$ we mean an arbitrary mapping $\varphi : \{1,\ldots,n\} \to \{1,\ldots,n\}$ such that every mapping has equal probability n^{-n}. The main purpose of this paper is to obtain limit theorems, when n tends to infinity, for special parameters in random mappings, e.g., for the number of image points. Since every random mapping $\varphi \in \mathcal{F}_n$ has equal probability it suffices to count the number of random mappings $\varphi \in \mathcal{F}_n$ satisfying a special property, e.g., that the number of image points equals k. By dividing this number by n^n we get the probability of interest. In order to get the limit distribution for $n \to \infty$ it is not necessary to know the exact value. We just have to evaluate these numbers asymptotically. We shall show that this can be done by a singularity analysis of a proper bivariate generating function.

It should be noted that some of our limit distributions on random mappings are well known (compare with [4, 16]). But our main goal is to provide a general method to derive such limit theorems. In particular, we use bivariate generating functions and singularity analysis. Especially we are able to characterize the (up to now unknown) limit distribution of the number of those points with a fixed number of total predecessors. It is a Gaussian distribution.

Our basic combinatorial concept is that of labelled combinatorial constructions and the relation to exponential generating functions. A big advantage of such combinatorial constructions is that we can mark a parameter in the constructions which directly leads to a bivariate generating function for the number of objects according to their size and the value of the parameter of interest.

2. Marking in random mappings. Every mapping $\varphi \in \mathcal{F}_n$ can be identified with its functional graph G_φ where $V(G_\varphi) = \{1,\ldots,n\}$ and $E(G_\varphi) = \{(i,\varphi(i)) \,|\, 1 \le i \le n\}$. It is obvious that each component of G_φ consists of a cycle (at least of a loop), and every cyclic point is the root of (labelled) tree (see Figure 1).

Hence we can interpret a mapping $\varphi \in \mathcal{F}$ as a set of cycles of trees. Furthermore, since there is no restriction on their structure, the trees (usually known as Cayley trees) can be recursively described as a root followed by a set of trees:

$$\mathcal{F} = \mathtt{set}(\mathtt{cycle}(\mathcal{T})), \tag{1}$$

$$\mathcal{T} = \circ \cdot \mathtt{set}(\mathcal{T}). \tag{2}$$

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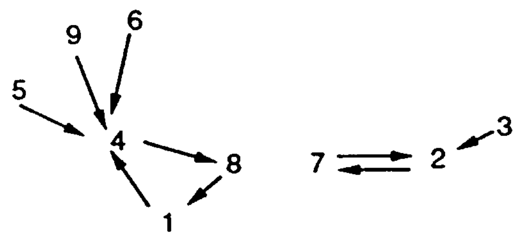

FIG. 1.

Both structures $\mathcal{F}$ and $\mathcal{T}$ fit into the concept of (labelled) combinatorial structures synthesized by Flajolet [11] (see also [17]). Let us give a short description of such structures.

Let C be a combinatorial structure of (labelled) elements, $|c|$ denote the size of $c \in C$, and $c_n = |\{c \in C \mid |c| = n\}|$ denote the number of elements of size n. (Labelled means that there are always $n!$ "isomorphic" elements $c \in C$ of size n which differ by their labels $1, \ldots, n$.) Furthermore we associate a (labelled) combinatorial structure C with the (exponential) generating function

$$\hat{c}(x) = \sum_{c \in C} \frac{x^{|c|}}{(|c|)!} = \sum_{n \geq 0} \frac{c_n}{n!} x^n. \tag{3}$$

The advantage of these generating functions is that there is a correspondence between special combinatorial constructions and special operations with the corresponding generating functions. For example, if the (labelled) combinatorial structure C is the product $C_1 \cdot C_2$, then

$$\hat{c}(x) = \hat{c}_1(x)\hat{c}_2(x). \tag{4}$$

Note that $C_1 \cdot C_2$ is not the set theoretic cartesian product because you have to transform the labelling $\{1, \ldots, k\}$ of $c_1 \in C_1$ and the labelling $\{1, \ldots, n-k\}$ of $c_2 \in C_2$ to a labelling $\{1, \ldots, n\}$ of $c \in C_1 \cdot C_2$. Since there are $k!$ "isomorphic" elements in C_1 and $(n-k)!$ "isomorphic" elements in C_2 we have

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{c_{1,k}}{k!} \frac{c_{2,n-k}}{(n-k)!}$$

according to (4). Hence, if $C = \texttt{set}(C_1)$ then we get

$$\hat{c}(x) = \sum_{m \geq 0} \frac{1}{m!} \hat{c}_1^m(x) = e^{\hat{c}_1(x)}, \tag{5}$$

and if $C = \texttt{cycle}(C_1)$ then we have

$$\hat{c}(x) = \sum_{m \geq 1} \frac{1}{m} \hat{c}_1^m(x) = \log \frac{1}{1 - \hat{c}_1(x)}. \tag{6}$$

Applying this concept to random mappings, the (exponential) generating function

$$\hat{f}(x) = \sum_{n \geq 0} \frac{n^n}{n!} x^n \tag{7}$$

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satisfies

$$\hat{f}(x) = \exp\left(\log \frac{1}{1-\hat{t}(x)}\right) = \frac{1}{1-\hat{t}(x)}, \tag{8}$$

where $\hat{t}(x)$, the generating function for Cayley trees, is given by

$$\hat{t}(x) = xe^{\hat{t}(x)}. \tag{9}$$

A big advantage of such combinatorial constructions is that we can formally mark a parameter in the constructions by a symbol like $[u]$. And this marking directly leads to bivariate generating function for the number of objects according to their size and the value of the parameter of interest.

For example, if we are interested in the number of trees in graphs of random mappings we have to mark the trees in the combinatorial construction

$$\mathcal{F} = \mathtt{set}(\mathtt{cycle}([u]\mathcal{T})).$$

Formally this leads to

$$\hat{f}(x,u) = \exp\left(\log \frac{1}{1-u\hat{t}(x)}\right) = \frac{1}{1-u\hat{t}(x)},$$

which is exactly the generating function $\hat{f}(x,u) = \sum f_{nk} \frac{x^n}{n!} u^k$ of the numbers f_{nk} of random mappings with k trees in the graph representation. (Note that the number of trees is exactly the number of cyclic points.)

Or if we are interested in the number of components, we have to mark

$$\mathcal{F} = \mathtt{set}([u]\mathtt{cycle}(\mathcal{T}))$$

and get

$$\hat{f}(x,u) = \exp\left(u \log \frac{1}{1-\hat{t}(x)}\right) = \frac{1}{(1-\hat{t}(x))^u}.$$

Next we will use this marking method to describe special parameters related to image and preimage points.

Points at distance d to a cycle. First we will discuss preimages of cyclic points. For this purpose let Y_φ denote the cyclic points of a random mapping $\varphi \in \mathcal{F}$ and $d \ge 1$. Specifically, we are interested in $\varphi^{-d}(Y_\varphi) \setminus \varphi^{-d+1}(Y_\varphi)$, i.e., noncyclic points at distance d to the cyclic points.

LEMMA 1. *Let*

$$A_0(x,u) = \frac{1}{1-u}$$

and

$$A_{d+1}(x,u) = A_d(x, xe^u)$$

for $d \ge 1$. Then

$$\hat{f}(x,u) = A_d(x, u\hat{t}(x)) \tag{10}$$

is the (exponential) generating function of random mappings where points contained in $\varphi^{-d}(Y_\varphi) \setminus \varphi^{-d+1}(Y_\varphi)$ *are marked.*

Proof. Let $\hat{t}_d(x,u)$ denote the (exponential) generating function of labelled rooted trees where nodes of distance $d \geq 0$ from the root are marked. Obviously we have

$$\hat{t}_0(x,u) = u\hat{t}(x) \qquad \text{and}$$
$$\hat{t}_{d+1}(x,u) = xe^{\hat{t}_d(x,u)} \quad \text{for } d \geq 0,$$

which directly leads to

$$\hat{f}(x,u) = \frac{1}{1-\hat{t}_d(x,u)} = A_d(x, u\hat{t}(x)). \qquad \square$$

Points with in-degree r. Another interesting parameter is the number of points ν with $|\varphi^{-1}(\{\nu\})| = r$, where $r \geq 0$ is a fixed integer.

LEMMA 2. *Let* $\hat{p}_r(x,u)$ *denote the solution of*

$$\hat{p}_r(x,u) = xe^{\hat{p}_r(x,u)} + (u-1)x\frac{\hat{p}_r(x,u)^r}{r!}. \tag{11}$$

Then

$$\hat{f}(x,u) = \frac{1}{1-\left(xe^{\hat{p}_r(x,u)} + (u-1)x\frac{\hat{p}_r(x,u)^{r-1}}{(r-1)!}\right)} \tag{12}$$

is the (exponential) generating function of random mappings where points ν *with* $|\varphi^{-1}(\{\nu\})| = r$ *are marked.*

Proof. According to the recursive structure of Cayley trees $\mathcal{T} = \circ \cdot \mathtt{set}(\mathcal{T})$, the nodes with in-degree r are those followed by r subtrees. Hence the bivariate generating function for trees with variable u marking nodes with in-degree r satisfies

$$\begin{aligned}\hat{p}_r(x,u) &= x\sum_{m\neq r}\frac{\hat{p}_r(x,u)^m}{m!} + ux\frac{\hat{p}_r(x,u)^r}{r!}\\ &= xe^{\hat{p}_r(x,u)} + (u-1)x\frac{\hat{p}_r(x,u)^r}{r!}.\end{aligned}$$

Now a cyclic point in the functional graph of a random mapping has in-degree r if and only if it has in-degree $r-1$ in the corresponding trees. This proves (12). $\square$

Notice that the expression of the bivariate generating function is simpler if we neglect the edges between cyclic points (i.e., cyclic points are marked if they have in-degree $r+1$, and noncyclic points are marked if they have in-degree r). Actually, we then consider sequences of Cayley trees instead of random mappings.

LEMMA 2′. *The (exponential) generating function of sequences of Cayley trees, where marked nodes are those with in-degree* r*, is*

$$\hat{f}(x,u) = \frac{1}{1-\hat{p}_r(x,u)}\,, \tag{13}$$

where $\hat{p}_r(x,u)$ *is the same as in Lemma* 2.

Points with r*-antecedents.* Finally, we want to count those points where the total number of preimages equals $r \geq 0$.

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LEMMA 3. *Let $\hat{a}_r(x,u)$ denote the solution of*

$$\hat{a}_r(x,u) = xe^{\hat{a}_r(x,u)} + (u-1)t_r x^r, \tag{14}$$

where $t_r = \frac{r^{r-1}}{r!}$. Then

$$\hat{f}(x,u) = \frac{1}{1-\hat{a}_r(x,u)+(u-1)t_r x^r} \exp\left(\frac{x^r}{r}\sum_{m=0}^{r} \frac{r^{r-m}}{(r-m)!}(u^m-1)\right) \tag{15}$$

is the (exponential) generating function of random mappings where points ν with

$$\left|\bigcup_{d\ge 0} \varphi^{-d}(\{\nu\})\right| = r$$

are marked.

Proof. As in the proof of Lemma 2 we first mark the nodes with the total number of preimages r in Cayley trees : a node is marked if and only if it is the root of a tree of total size r (the root is considered to be its own preimage). Hence we get (14), where t_r is the coefficient of x^r in the series expansion

$$\hat{t}(x) = \sum_{n\ge 0} t_n x^n.$$

Since $\hat{t}(x)$ satisfies the functional equation (9), using Lagrange's inversion theorem we get

$$t_n = \frac{1}{n}[y^{n-1}]e^{yn} = \frac{n^{n-1}}{n!}.$$

For cyclic points in a random mapping all points in the corresponding component (of the functional graph) are preimages. Hence for a component with m cyclic points, the bivariate generating function, with u marking the number of points having r preimages, is

$$\frac{\left(xe^{\hat{a}_r(x,u)}\right)^m}{m} + \frac{1}{m}(u^m-1)x^r[v^r]\hat{t}(v)^m,$$

where

$$[v^r]\hat{t}(v)^m = \frac{m}{r}[y^{r-m}]e^{yr} = \frac{mr^{r-m}}{r(r-m)!}$$

is the number of forests composed with m components and of total size r. This directly gives (15). ◻

dth iterate points. It is also interesting to consider $\varphi^d(\{1,\dots,n\})$, the dth iterate image points.

LEMMA 4. *Set*

$$h_0(x) = 0 \qquad \textit{and}$$
$$h_{i+1}(x) = xe^{h_i(x)} \quad \textit{for } i \ge 0,$$

and let $\hat{i}_d(x,u)$ *be the solution of*

$$\hat{i}_d(x,u) = xue^{\hat{i}_d(x,u)} - (u-1)h_d(x). \tag{16}$$

Then

$$\hat{f}(x,u) = \frac{1}{1 - \hat{i}_d(x,u) - (u-1)h_d(x)} \tag{17}$$

is the (exponential) generating function of random mappings where points $\nu \in \varphi^d(\{1,\dots,n\})$ *are marked.*

Proof. Clearly, $h_d(x)$ is the (exponential) generating function of Cayley trees with height $< d$. In Cayley trees, the dth iterate image points are points at distance $\geq d$ from a leaf. Hence, $\hat{i}_d(x,u)$, the bivariate generating function of trees where nodes having a leaf at distance $\geq d$ are marked, satisfies (16). For random mappings, since all cyclic points are dth iterate image points we get

$$\hat{f}(x,u) = \frac{1}{1 - uxe^{\hat{i}_d(x,u)}},$$

which leads to (17). $\square$

Here again, as in the case of points with in-degree r, the expression of the bivariate generating function is simpler if we neglect the edges between cyclic points.

LEMMA 4′. *The (exponential) generating function of random mappings, where the marked points are those at distance* $\geq d$ *from a leaf of their own subtree, is*

$$\hat{f}(x,u) = \frac{1}{1 - \hat{i}_d(x,u)}, \tag{18}$$

where $\hat{i}_d(x,u)$ *is the same as in Lemma* 4.

Direct dth iterate points. The most difficult example (from the combinatorial point of view) is the case of dth iterate image points of nonimage points $\varphi^d(\{1,\dots,n\} \setminus \varphi(\{1,\dots,n\}))$. In other words we will count those nodes that are connected by a (directed) path of length d to a nonimage point. Nevertheless, there is a rather easy subcase where edges between cyclic points are neglected; i.e., the problem can be reduced to a problem inside trees. Although this is only a very small change, it will turn out that the corresponding limiting distributions differ. (And it will also be the case for Lemmas 2 and 4 versus Lemmas 2′ and 4′.)

LEMMA 5. *Set*

$$c_0(x,y) = x \qquad \text{and}$$
$$c_{i+1}(x,y) = x\left(e^y - e^{y - c_i(x,y)}\right) \quad \text{for } i \geq 0,$$

and let $\hat{l}_d(x,u)$ *be the solution of*

$$\hat{l}_d(x,u) = xe^{\hat{l}_d(x,u)} + (u-1)c_d(x, \hat{l}_d(x,u)). \tag{19}$$

Then

$$\tilde{f}(x,u) = \exp\left(\log \frac{1}{1 - \hat{l}_d(x,u)}\right) = \frac{1}{1 - \hat{l}_d(x,u)} \tag{20}$$

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is the (exponential) generating function of random mappings, where marked points are those connected to a leaf by a path of length d which does not contain cyclic edges. (Note that the root of a tree of size 1 is also a leaf.)

Proof. Let $\hat{l}_d(x,u)$ denote the generating function of Cayley trees where nodes having a leaf at distance d are marked. The generating function where leaves are marked is $\hat{l}_0(x,u) = ux + x(e^{l_0(x,u)} - 1)$ and, inductively, $\hat{l}_d(x,u) = y_d(x,u) + xe^{\hat{l}_d(x,u)-y_{d-1}(x,u)}$, where $y_d(x,u)$ is the generating function for trees with a leaf at distance d to the root: $y_d(x,u) = uc_d(x,\hat{l}_d(x,u))$, and for $i = 1,\ldots,(d-1)$, $y_i(x,u) = c_i(x,\hat{l}_d(x,u))$. Since $xe^{\hat{l}_d(x,u)-y_{d-1}(x,u)} = xe^{\hat{l}_d(x,u)} - c_d(x,\hat{l}_d(x,u))$ represents the trees such that the root is not at distance d to a leaf, $\hat{l}_d(x,u)$ satisfies (19) and hence (20). ◻

LEMMA 5′. *Let $c_i(x,y)$ and $\hat{l}_d(x,u)$ be defined as in Lemma 5 and $\hat{f}_d(x,u)$ be the (exponential) generating function of random mappings where points $\nu \in \varphi^d(\{1,\ldots,n\}\setminus \varphi(\{1,\ldots,n\}))$ are marked.*

For $d=0$ and $d=1$ we have

$$\hat{f}_0(x,u) = \frac{1}{1-xe^{\hat{l}_0(x,u)}} \quad \text{and} \quad \hat{f}_1(x,u) = \frac{1}{1-\hat{l}_1(x,u)}. \tag{21}$$

For $d=2$ set $y_1(x,u) = c_1(x,\hat{l}_2(x,u))$, $y_2(x,u) = uc_2(x,\hat{l}_2(x,u))$, and

$$y_{12}(x,u) = ux\left(e^{\hat{l}_2(x,u)} - e^{\hat{l}_2(x,u)-x} - e^{\hat{l}_2(x,u)-y_1(x,u)} + e^{\hat{l}_2(x,u)-y_1(x,u)-x}\right).$$

Then

$$\hat{f}_2(x,u) = \frac{1}{1-\hat{l}_2(x,u)-(u-1)\left(y_1(x,u)(1-y_2(x,u)) - y_{12}(x,u)(1-\hat{l}_2(x,u))\right)}. \tag{22}$$

Proof. The results for $\hat{f}_0(x,u)$ and $\hat{f}_1(x,u)$ are obvious: for $d=0$, cyclic points are not leaves of random mappings and, for $d=1$, edges inside cycles are of no importance. For $d=2$, the situation gets more delicate: in addition to the interpretations of $y_1(x,u)$ and $y_2(x,u)$ observe that $y_{12}(x,u)$ corresponds to those trees having both a leaf at distance 1 to the root and a leaf at distance 2 to the root. (For the sake of shortness we will use the terms y_1-tree (respectively, y_2-tree) for a tree with a leaf at distance 1 (respectively, 2) to the root.) Set

$$w = u(y_1 - y_{12}) + (\hat{l}_2 - y_1 - y_2 + y_{12}) \qquad \text{and}$$
$$s = uy_{12} + (y_2 - y_{12}).$$

Then w corresponds to also marking y_1-trees that are not y_2-trees. In the same way, s corresponds to twice marking y_2-trees that are y_1-trees too. We will show (at the end of the proof) that the generating function

$$\frac{w^m}{m} + \frac{y_2^m}{m} + \frac{1}{m}\sum_{k=1}^{m-1}\sum_{l=1}^{k} A_{mkl}s^l(\hat{l}_2 - y_2)^l w^{m-k-l}y_2^{k-l} \tag{23}$$

corresponds to a cycle of m trees where all nodes having a leaf at distance 2 are marked (A_{mkl}, see below, counts the number of cycles of length m containing k y_2-trees, $(k-l)$

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of which are followed by a y_2-tree). Since

$$\sum_{m\geq 1}\frac{1}{m}\sum_{k=1}^{m-1}\sum_{l=1}^{k} A_{mkl}s^l(\hat{l}_2-y_2)^l w^{m-k-l}y_2^{k-l} = \log\frac{1}{1-y_2-\frac{s(\hat{l}_2-y_2)}{1-w}} - \log\frac{1}{1-y_2},$$

we immediately get (22).

Therefore, it remains to interpret (23). The problem on a cycle is that a y_1-tree forces an additional mark at the next root on the cycle if and only if this next root is not marked, i.e., the corresponding tree is not a y_2-tree. For example, if a cycle of length m contains no y_2-tree, then it is immediately clear that $\frac{1}{m}w^m$ is the correct corresponding function, whereas the case of a cycle containing only y_2-trees the generating function of interest is $\frac{1}{m}y_2^m$. For the remaining cases consider a cycle containing exactly k $(0<k<m)$ y_2-trees such that l $(0<l\leq k)$ of these trees are followed by a tree that is not a y_2-tree. Note that

$$A_{mkl} = \frac{m}{l}\binom{m-k-1}{l-1}\binom{k-1}{l-1} \tag{24}$$

is the number of such arrangements on a (labelled) cycle of length m. In any of these cases the corresponding generating function is $\frac{1}{m}y_2^{k-l}s^l w^{m-k-l}(\hat{l}_2-y_2)^l$. This proves (23). ∎

3. General theorems. Let $c(x)=\sum c_n x^n$ be the generating function of a combinatorial structure and $c(x,u)=\sum c_{nk}x^n u^k$ the bivariate generating function where a parameter of interest has been marked, i.e., $c(x,1)=c(x)$. Now we will be interested in the asymptotic distribution of this parameter in the system of combinatorial objects of size n when n tends to infinity. For this purpose we introduce a sequence of random variables X_n, $n\geq 1$, defined by

$$\mathbf{Pr}[X_n=k]=\frac{c_{nk}}{c_n}=\frac{[x^n u^k]c(x,u)}{[x^n]c(x,1)},$$

where $\mathbf{Pr}$ denotes probability. Now the above problem reduces to finding the limiting distribution of X_n.

An important analytic schema, related to combinatorial constructions "sequence" or "set of cycles," is

$$c(x,u)=\frac{1}{1-a(x,u)}\ .$$

The next three theorems study this schema when $a(x,u)$ has an algebraic singularity $\rho(u)$ of square-root type such that $a(\rho(1),1)=1$. According to further analytic properties of $a(x,u)$, the limiting distribution of X_n is shown to be either Gaussian, Rayleigh, or the convolution of Gaussian and Rayleigh, and in each case the global limit result (convergence of distribution functions) is accompanied by a local limit result (convergence of densities).

Let us first state precisely the general form of the analytic schemas under consideration.

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Hypothesis [H]. Let $c(x,u) = \sum_{n,k} c_{nk} x^n u^k$ be a power series in two variables with nonnegative coefficients $c_{nk} \ge 0$ such that $c(x,1)$ has a radius of convergence of $\rho > 0$.

We suppose that $c(x,u)$ expresses as $c(x,u) = 1/d(x,u)$, where $d(x,u)$ has the local representation

$$d(x,u) = g(x,u) + h(x,u)\sqrt{1 - \frac{x}{\rho(u)}} \tag{25}$$

for $|u-1| < \varepsilon$ and $|x - \rho(u)| < \varepsilon$, $\arg(x - \rho(u)) \ne 0$, where $\varepsilon > 0$ is some fixed real number, and $g(x,u)$, $h(x,u)$, and $\rho(u)$ are analytic functions.

Furthermore, these functions satisfy $g(\rho,1) = 0$, $h(\rho,1) > 0$, and $\rho(1) = \rho$.

In addition, $x = \rho(u)$ is the only singularity on the circle of convergence $|x| = |\rho(u)|$ for $|u-1| < \varepsilon$ and $d(x,u)$, respectively $c(x,u)$, can be analytically continued to a region $|x| < \rho + \delta$, $|u| < 1 + \delta$, $|u-1| > \frac{\varepsilon}{2}$ for some $\delta > 0$.

Under this hypothesis, the limiting distribution of X_n in $c(x,u)$ depends on $\rho'(1)$ and $g_u(\rho(1),1)$, as stated in the following three theorems.

THEOREM 1. *Let $c(x,u)$ be a bivariate generating function satisfying* [H]. *If $\rho(u) = \rho = \text{const}$ for $|u-1| < \varepsilon$ and $g_u(\rho,1) < 0$, then the sequence of random variables X_n defined by*

$$\mathbf{Pr}[X_n = k] = \frac{[x^n u^k] c(x,u)}{[x^n] c(x,1)} \tag{26}$$

has a Rayleigh limiting distribution; *i.e.,*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{R}(\lambda), \tag{27}$$

where $\lambda = \frac{h(\rho,1)^2}{2g_u(\rho,1)^2}$ and $\mathcal{R}(\lambda)$ has density $\lambda x \exp\left(-\frac{\lambda}{2}x^2\right)$ for $x \ge 0$. Expected value and variance are given by

$$\mathbf{E}X_n = \sqrt{\frac{\pi}{2\lambda}}\sqrt{n} + \mathcal{O}(1) \quad \text{and} \quad \mathbf{V}X_n = \left(2 - \frac{\pi}{2}\right)\frac{n}{\lambda} + \mathcal{O}(\sqrt{n}). \tag{28}$$

Moreover, we have the local law

$$\mathbf{Pr}[X_n = k] = \frac{\lambda k}{n}\exp\left(-\frac{\lambda k^2}{2n}\right) + \mathcal{O}((k+1)n^{-\frac{3}{2}}) + \mathcal{O}(n^{-1}) \tag{29}$$

uniformly for all $k \ge 0$.

THEOREM 2. *Let $c(x,u)$ be a bivariate generating function satisfying* [H]. *If $\rho'(1) \ne 0$ and $\alpha = \frac{\partial}{\partial u} g(\rho(u),u)|_{u=1} = 0$, then X_n has a Gaussian limiting distribution*; *i.e.,*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{30}$$

where $\mu = -\rho'(1)/\rho$ and $\sigma^2 = \mu^2 + \mu - \rho''(1)/\rho$. Expected value and variance are given by

$$\mathbf{E}X_n = \mu n + \mathcal{O}(1) \quad \text{and} \quad \mathbf{V}X_n = \sigma^2 n + \mathcal{O}(\sqrt{n}). \tag{31}$$

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Furthermore, there is local law of the form

$$\mathbf{Pr}[X_n = k] = \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{(k-\mu n)^2}{2\sigma^2 n}\right) + \mathcal{O}(n^{-\frac{3}{4}}) \tag{32}$$

uniformly for all $k \geq 0$.

THEOREM 3. *Let $c(x,u)$ be a bivariate generating function satisfying* [H]. *If $\rho'(1) \neq 0$ and $\alpha = \frac{\partial}{\partial u} g(\rho(u),u)|_{u=1} < 0$, then the limiting distribution of X_n is the convolution of a Gaussian and a Rayleigh distribution*; *i.e.,*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1) * \mathcal{R}(\lambda), \tag{33}$$

where $\lambda = \frac{h(\rho,1)^2\sigma^2}{2\alpha^2}$ and μ and σ^2 are defined as in Theorem 2. *Expected value and variance are given by*

$$\mathbf{E}X_n = \mu n - \frac{\sqrt{\pi}\alpha}{h(\rho,1)}\sqrt{n} + \mathcal{O}(1) \quad \text{and} \quad \mathbf{V}X_n = \left(\sigma^2 + \frac{(4-\pi)\alpha^2}{h(\rho,1)^2}\right) n + \mathcal{O}(\sqrt{n}) \tag{34}$$

and there is local law of the form

$$\begin{aligned} \mathbf{Pr}[X_n = k] = {} & \frac{\lambda}{1+\lambda} \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{(k-\mu n)^2}{\sigma^2 n}\right) \\ & + \frac{\lambda}{(1+\lambda)^{\frac{3}{2}}} \frac{k-\mu n}{\sigma^2 n} \exp\left(-\frac{\lambda}{1+\lambda}\frac{(k-\mu n)^2}{\sigma^2 n}\right) \Phi\left(\frac{k-\mu n}{\sqrt{(1+\lambda)\sigma^2 n}}\right) \\ & + \mathcal{O}(n^{-\frac{3}{4}}) \end{aligned} \tag{35}$$

uniformly for all $k \geq 0$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt. \tag{36}$$

(If $\alpha > 0$ then the corresponding Rayleigh distribution is supported on the negative real axis and a similar local law holds.)

Remark. It should be noticed that condition $g(\rho,1) = 0$ in the theorems is not a real restriction. In fact, it turns out that the case $g(\rho,1) = 0$ is the most difficult one, and the limiting distribution for other cases can be found also.

If $g(\rho,1) > 0$ then $c(x,u)$ has a local representation of the form

$$c(x,u) = \frac{1}{g(x,u) + h(x,u)\sqrt{1 - x/\rho(u)}} = G(x,u) - H(x,u)\sqrt{1 - \frac{x}{\rho(u)}}. \tag{37}$$

On the other hand, if $g(\rho,1) < 0$, the algebraic singularity is not the dominating one. Here $d(\overline{\rho},1) = 0$ for $\overline{\rho} < \rho$ and (usually) $d_x(\overline{\rho},1) \neq 0$. Hence, by the Weierstrass preparation theorem, $d(x,u)$ has a local representation of the form $d(x,u) = D(x,u)(1 - x/\overline{\rho}(u))$, where $D(x,u)$ and $\overline{\rho}(u)$ are analytic functions satisfying $D(\overline{\rho},1) \neq 0$, $\overline{\rho}(1) = \overline{\rho}$, and $\overline{\rho}'(1) \neq 0$. Thus

$$c(x,u) = \frac{1/D(x,u)}{1 - \frac{x}{\overline{\rho}(u)}}. \tag{38}$$

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In both cases, (37) and (38), we can apply Bender's theorem [5] (compare also with [6] and [7]) to get asymptotic normality if $\rho'(1) < 0$. (Evaluating the expected value shows that $\rho'(1)$ cannot be positive.)

When $\rho(u) = \text{const}$ (see also [14] for this case), the limiting distribution is Gaussian for $g(\rho, 1) > 0$ and discrete for $g(\rho, 1) < 0$ (for example, there is a derivated geometric law for the schema $c(x, u) = (1 - ua(x))^{-1}$).

Finally, we want to remark that the assumption $\rho(u) = \text{const}$ in Theorem 1 can be weakened to $\rho'(1) = 0$. However, the proof would be a little bit more complicated. Furthermore, no example is known where $\rho'(1) = 0$ but $\rho(u) \neq \text{const}$.

Before proving Theorems 1, 2, 3 (see section 5) we want to discuss why such theorems have some importance in relation to random mappings.

4. Applications to random mappings. In this section we apply our theorems to obtain the limiting distributions for various parameters of random mappings. It should be noted that some of the obtained results are known, but our intention is to provide all the results by applying only one general principle. The underlying point is that the combinatorial specification of random mappings out of Cayley trees, together with the analytic form of the Cayley trees series, imply that all bivariate generating functions $\hat{f}(x, u)$ constructed in section 2 satisfy Hypothesis [H].

4.1. Analytic frame. The basic property is that solutions of functional equations usually have algebraic singularities of square-root type.

PROPOSITION 1. *Let $F(a, x, u)$ be a power series on three variables with nonnegative coefficients and $F(0, 0, 0) = 0$. Suppose that the system of equations*

$$a_0 = F(a_0, x_0, 1), \tag{39}$$

$$1 = F_a(a_0, x_0, 1) \tag{40}$$

has positive solutions $a_0 > 0$, $x_0 > 0$ (which are supposed to be minimal) such that $(a_0, x_0, 1)$ is contained in the region of convergence of $F(a, x, u)$ and that

$$F_x(a_0, x_0, 1) \neq 0 \quad \text{and} \quad F_{aa}(a_0, x_0, 1) \neq 0. \tag{41}$$

Then there exists a unique analytic solution $a = a(x, u) = \sum_{nk} a_{nk} x^n u^k$ of

$$a = F(a, x, u) \tag{42}$$

with nonnegative coefficients $a_{nk} \geq 0$ and $a_{00} = 0$ such that $a(x, u)$ has the local representation

$$a(x, u) = g(x, u) - h(x, u)\sqrt{1 - \frac{x}{\rho(u)}} \tag{43}$$

for $|u - 1| < \varepsilon$ and $|x - \rho(u)| < \varepsilon$, $\arg(x - \rho(u)) \neq 0$, where $g(x, u)$, $h(x, u)$, and $\rho(u)$ are analytic functions that satisfy

$$g(x_0, 1) = a_0, \quad h(x_0, 1) = \sqrt{\frac{2x_0 F_x(a_0, x_0, 1)}{F_{aa}(a_0, x_0, 1)}}, \quad \text{and} \quad \rho(1) = x_0 \tag{44}$$

and $\varepsilon > 0$ is some fixed real number. Furthermore, if there are n_1, n_2, n_3 and $k_1 < k_2 < k_3$ such that $a_{n_1k_1} a_{n_2k_2} a_{n_3k_3} > 0$ and $\gcd(k_3 - k_1, k_2 - k_1) = 1$ and if

$$\gcd\left\{n - l \,:\, \sum_k a_{nk} > 0\right\} = 1,$$

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where

$$l = \min\left\{ m \,:\, \sum_k a_{mk} > 0 \right\},$$

then $x = \rho(u)$ *is the only singularity on the circle of convergence* $|x| = |\rho(u)|$ *for* $|u-1| < \varepsilon$*, and there exists some* $\delta > 0$ *such that* $a(x,u)$ *can be analytically continued in the region* $|x| < x_0 + \delta$*,* $|u| < 1 + \delta$*,* $|u-1| > \frac{\varepsilon}{2}$*.*

The proof of Proposition 1 is a combination of the implicit function theorem and the Weierstrass preparation theorem (cf. [7, 8]).

Now it is easy to see the connection to random mappings. In any of the above combinatorial constructions the solution $a(x,u)$ (satisfying $a(\frac{1}{e},1) = 1$) of a functional equation of the type (42) is used to construct a final generating function that is more or less of the form

$$c(x,u) = \frac{1}{1 - a(x,u)}.$$

Hence, we can directly apply our theorems to obtain the kind of asymptotic distribution we are seeking.

4.2. Distribution of parameters. The examples mentioned in section 2 cover the three types of limiting distributions, Gaussian, Rayleigh, or a convolution of both. It should be noted that in Applications 2, 4, and 5, small structural modifications (neglecting cyclic edges) lead to different limit laws.

For the sake of brevity we will only mention the weak convergence law. However, in all the cases the local law and the asymptotic expansions for mean and variance hold, too.

APPLICATION 1 (see [16]). *Let* X_n *denote the number of noncyclic points at a fixed distance* $d > 0$ *to a cycle in random mappings of size* n*. Then*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{R}(1). \tag{45}$$

Proof. From Proposition 1 it follows that $\hat{t}(x)$ has a local representation of the kind

$$\hat{t}(x) = a(x) - b(x)\sqrt{1 - ex},$$

where $a(x)$ and $b(x)$ are analytic functions around $x_0 = \frac{1}{e}$ with $a(\frac{1}{e}) = 1$ and $b(\frac{1}{e}) = \sqrt{2}$. Furthermore, we can use the Taylor series expansion of

$$A_d(x,u)^{-1} = \sum_{l,k \geq 0} c_{lk}(u-1)^l \left(x - \frac{1}{e}\right)^k,$$

where $c_{00} = 0$ and $c_{10} = -1$ to see that $A_d(x, u\hat{t}(x,u))$ has a representation of the kind

$$A_d(x, u\hat{t}(x,u))^{-1} = g(x,u) + h(x,u)\sqrt{1 - ex},$$

where $g(\frac{1}{e},1) = c_{00} = 0$ and $h(\frac{1}{e},1) = -c_{10}b(\frac{1}{e}) = \sqrt{2}$. Hence, we can apply Theorem 1. □

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APPLICATION 2 (see [4]). *Let $r \ge 0$ be a fixed integer, and let X_n denote the number of points ν with $|\varphi^{-1}(\{\nu\})| = r$ in mappings $\varphi \in \mathcal{F}_n$. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{46}$$

where $\mu = \frac{1}{er!}$ and $\sigma^2 = \mu + (1 + (r-1)^2)\mu^2$.

Proof. Let

$$F(p,x,u) = xe^p + (u-1)x\frac{p^r}{r!}.$$

Then another application of Proposition 1 provides a local representation of

$$\hat{p}_r(x,u) = a(x,u) - b(x,u)\sqrt{1 - \frac{x}{\rho(u)}},$$

where $\hat{p}_r(x,1) = \hat{t}(x)$, and, consequently, $\rho(1) = \frac{1}{e}$, $a(\frac{1}{e},1) = 1$, and $b(\frac{1}{e},1) = \sqrt{2}$. Hence, we obtain

$$\begin{aligned}\hat{f}(x,u) &= \frac{1}{1 - \left(\hat{p}_r(x,u) - (u-1)x\frac{\hat{p}_r(x,u)^r}{r!} + (u-1)x\frac{\hat{p}_r(x,u)^{r-1}}{(r-1)!}\right)}\\ &= \frac{1}{g(x,u) + h(x,u)\sqrt{1 - x/\rho(u)}}\end{aligned}$$

in which $g(\frac{1}{e},1) = 0$, $h(\frac{1}{e},1) = \sqrt{2}$, and

$$\alpha = \frac{\partial}{\partial u} g(\rho(u),u)|_{u=1} = -\frac{\partial}{\partial u} a(\rho(u),u)|_{u=1} + \frac{1}{er!} - \frac{1}{e(r-1)!}.$$

Since $p(u) = a(\rho(u),u) = \hat{p}_r(\rho(u),u)$ satisfies the system of equations

$$\begin{aligned} p(u) &= F(p(u),\rho(u),u),\\ 1 &= F_p(p(u),\rho(u),u),\end{aligned}$$

implicit differentiation gives

$$\begin{aligned}\rho'(1) &= -\frac{F_u(1,\frac{1}{e},1)}{F_x(1,\frac{1}{e},1)} = -\frac{1}{er!},\\ \rho''(1) &= \frac{1}{F_{pp}F_x^3}(F_x^2(F_{pu}^2 - F_{pp}F_{uu}) + F_u^2(F_{px}^2 - F_{pp}F_{xx})\\ &\qquad\qquad -2F_xF_u(F_{px}F_{pu} - F_{pp}F_{xu}))\\ &= -\frac{(r-1)^2}{e^3(r!)^2},\\ p'(1) &= \frac{F_{px}F_u - F_{pu}F_x}{F_{pp}F_x} = \frac{1}{er!} - \frac{1}{e(r-1)!}.\end{aligned}$$

Thus we can apply Theorem 2. ◻

APPLICATION 2′. *Let $r \ge 0$ be a fixed integer and let X_n denote the number of nodes with in-degree r in a sequence of Cayley trees of total size n. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1) * \mathcal{R}(\lambda),$$

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where μ and σ^2 are the same as in Application 2, and $\lambda = \frac{\sigma^2(er!)^2}{(1-r)^2}$. In the special case $r = 1$, the limiting distribution is only Gaussian since $\lambda^{-1} = 0$.

Proof. In this case $\alpha = p'(1)$ is not equal to 0, except for $r = 1$. Hence, the convolution results by Theorem 3. ∎

APPLICATION 3. *Let $r \geq 0$ be a fixed integer, and let X_n denote the number of points ν with*

$$\left| \bigcup_{d \geq 0} \varphi^{-d}(\{\nu\}) \right| = r$$

in mappings $\varphi \in \mathcal{F}_n$. Then

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{47}$$

where $\mu = \frac{r^{r-1}}{r!} e^{-r}$ and $\sigma^2 = \mu - 2r\mu^2$.

Proof. First notice that the analytic factor $\exp(\cdots)$ in (15) has no influence on the parameters of interest α, $\rho'(1)$, and $\rho''(1)$. Therefore, we can neglect it. Hence we can proceed as in the proof on Application 2. Here we have

$$\begin{aligned} \rho'(1) &= -t_r e^{-r-1}, \\ \rho''(1) &= (1+2r) t_r^2 e^{-2r-1}, \\ a'(1) &= t_r e^{-r}. \end{aligned}$$

Consequently, $\alpha = 0$ and we obtain a Gaussian limiting distribution. ∎

APPLICATION 4. *Let $d \geq 0$ be a fixed integer, and let X_n denote the number of points $\nu \in \varphi^d(\{1,\ldots,n\})$ mappings $\varphi \in \mathcal{F}_n$. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1), \tag{48}$$

where $\mu = h_d(\frac{1}{e})$ and $\sigma^2 = \frac{2}{e} h_d'(\frac{1}{e})(1-\mu) - \mu$.

Proof. The proof is almost the same as the proof of Application 2. ∎

We want to mention that the mean value was already determined in [13].

APPLICATION 4$'$. *Let $d \geq 0$ be a fixed integer, and let X_n denote the number of points at distance $\geq d$ from a leaf of their own subtree in random mappings of size n. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1) * \mathcal{R}(\lambda), \tag{49}$$

where μ and σ^2 are the same as in Application 4, and $\lambda = \frac{\sigma^2}{h_d^2(\frac{1}{e})}$.

Proof. In this case, $\alpha = h_d(\frac{1}{e}) \neq 0$. Hence the convolution result. ∎

APPLICATION 5. *Let $d \geq 0$ be fixed and X_n denote the number of nodes that are connected to a leaf by a path of length d containing no cyclic edge in random mappings of size n. Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1) * \mathcal{R}(\lambda), \tag{50}$$

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where $\mu = c_d = c_d(\frac{1}{e}, 1)$, $\sigma^2 = c_d(1 - \frac{2}{e}c_{d,x}) + 2c_d c_{d,y} - c_{d,y}^2$, *and* $\lambda^{-1} = (c_d - c_{d,y})^2/\sigma^2$ $(c_{d,x} = \frac{\partial}{\partial x}c_d(\frac{1}{e}, 1)$, $c_{d,y} = \frac{\partial}{\partial y}c_d(\frac{1}{e}, 1))$. *Since* $\lambda^{-1} = 0$ *for* $d = 1$, *the limiting distribution is only Gaussian in this special case.*

Proof. The proof runs along the same lines as the preceding ones. You only have to apply Theorem 3 since $\alpha = c_d - c_{d,y} \neq 0$ for $d \neq 1$. $\square$

APPLICATION 5′. *Let* $d \in \{0, 1, 2\}$ *be fixed and let* X_n *be the number of points* $x \in \varphi^d(\{1, \ldots, n\} \setminus \varphi(\{1, \ldots, n\}))$ *in mappings* $\varphi \in \mathcal{F}_n$. *Then*

$$\frac{X_n - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0, 1), \tag{51}$$

where μ *and* σ^2 *are as in Application* 5; *i.e.,*

$$c_d(e^{-1}, 1) = \begin{cases} e^{-1} & \textit{for } d = 0, \\ 1 - e^{-e^{-1}} & \textit{for } d = 1 \\ 1 - e^{-(1-e^{-1/e})} & \textit{for } d = 2, \end{cases}$$

and

$$\sigma^2 = \begin{cases} e^{-1} - 2e^{-2} & \textit{for } d = 0, \\ e^{-e^{-1}}\left(1 - 2e^{-1}\right)\left(1 - e^{-e^{-1}}\right) & \textit{for } d = 1, \\ 2e^{-1+e^{-1/e}-e^{-1}} - e^{-1+e^{-1/e}} & \textit{for } d = 2. \\ \quad -2e^{-2+e^{-1/e}-e^{-1}} - e^{-2+2e^{-1/e}-2e^{-1}} & \\ \quad +2e^{-3+2e^{-1/e}-e^{-1}} & \end{cases}$$

Proof. Especially in the case $d = 2$ you have to calculate α very carefully, but in all the cases $\alpha = 0$. $\square$

Note. In the case $d > 2$, the combinatorial description is much more involved. Nevertheless, it may be conjectured that the limiting distribution is still Gaussian.

5. Proof of the theorems. The proofs of Theorems 1, 2, 3 proceed in the following way. First we derive asymptotic expansions for mean value and variance; then we prove a weak limit theorem using characteristic functions and, finally, we establish the corresponding local limit theorem. This procedure seems to be redundant, and in fact it is. But our aim is not only to prove special theorems but to provide an example for a general method to analyze the asymptotic distribution of a parameter in combinatorial constructions.

5.1. Preliminaries. We first list some useful formulae related to Gaussian and Rayleigh distributions.

LEMMA 6. *Let* γ *be a Hankel contour starting from* $+e^{2\pi i}\infty$, *passing around* 0, *and tending to* $+\infty$. *Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-z}}{\sqrt{-z} - is}\,dz = \frac{1}{\sqrt{\pi}}\varphi_{\mathcal{R}}(\sqrt{2}s), \tag{52}$$

where

$$\begin{aligned} \varphi_{\mathcal{R}}(t) &= \int_0^\infty e^{itx} x e^{-x^2/2}\,dx \\ &= 1 + ite^{-t^2/2}\left(\sqrt{\frac{\pi}{2}} - i\int_0^t e^{u^2/2}\,du\right) \end{aligned}$$

denotes the characteristic function of the Rayleigh distribution.

Proof. It suffices to compare the Taylor expansion around $s = 0$. By the Hankel integral representation of $\Gamma(s)^{-1}$ we get

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-z}}{\sqrt{-z}-is} = \frac{1}{2\pi i}\int_\gamma \sum_{n\ge 0}(is)^n(-z)^{-\frac{n+1}{2}}e^{-z}\,dz = \sum_{n\ge 0}\frac{(is)^n}{\Gamma\left(\frac{n+1}{2}\right)}.$$

On the other hand we have

$$\begin{aligned}\varphi_{\mathcal{R}}(t) &= \int_0^\infty \sum_{n\ge 0}(it)^n x^{n+1}e^{-x^2/2}\,dx\\ &= \sum_{n\ge 0}(it)^n 2^{\frac{n}{2}}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma(n+1)}\\ &= \sqrt{\pi}\sum_{n\ge 0}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}\left(\frac{it}{\sqrt{2}}\right)^n,\end{aligned}$$

where we have used the duplication formula for the Γ-function. ◻

LEMMA 7. *Let γ be as in Lemma 6. Then*

$$\frac{1}{2\pi i}\int_\gamma e^{-s\sqrt{-z}-z}\,dz = \frac{s}{2\sqrt{\pi}}e^{-s^2/4} \tag{53}$$

and

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{-z}}{\sqrt{-z}}\,dz = \frac{1}{\sqrt{\pi}}. \tag{54}$$

Proof. (53) and (54) follow immediately from the substitution $z = w^2$. ◻

LEMMA 8. *Let γ be as in Lemma 6 and α, β be real constants. Then*

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_\gamma \frac{e^{i\alpha w - w^2/2 - z}}{\sqrt{-z}-i\beta w}\,dz\,dw \\ &\qquad = \frac{e^{-\alpha^2/2}}{\sqrt{2}(\frac{1}{2}+\beta^2)} - \frac{\sqrt{\pi}\alpha\beta}{(\frac{1}{2}+\beta^2)^{\frac{3}{2}}}\exp\left(-\frac{\alpha^2}{4(\frac{1}{2}+\beta^2)}\right)\Phi\left(-\frac{\alpha\beta}{\sqrt{\frac{1}{2}+\beta^2}}\right).\end{aligned} \tag{55}$$

Proof. Since both sides of (56) can be interpreted as analytic functions in α, β around the real axis, it suffices to prove (56) for the case $\alpha\beta > 0$. In this case we can use the substitutions $z = \frac{u^2}{2}$ and $w = v + i\alpha$ to obtain

$$\frac{e^{-\alpha^2/2}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}(u^2+v^2)}}{\frac{1}{\sqrt{2}}u+\beta v+i\alpha\beta}\,u\,du\,dv.$$

Then we can apply the polar substitution $u = r\cos\varphi$, $v = r\sin\varphi$ to get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}(u^2+v^2)}}{\frac{1}{\sqrt{2}}u+\beta v+i\alpha\beta}\,u\,du\,dv$$

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$$= \int_0^\infty \int_0^{2\pi} \frac{r^2 e^{\frac{1}{2}r^2} \cos\varphi}{r\left(\frac{1}{\sqrt{2}}\cos\varphi + \beta\sin\varphi\right) + i\alpha\beta}\, d\varphi\, dr$$

$$= \int_0^\infty r^2 e^{-\frac{1}{2}r^2} \int_{|z|=1} \frac{\frac{1}{2}(z+z^{-1})}{r\left(\frac{1}{2\sqrt{2}}(z+z^{-1}) + \frac{\beta}{2i}(z-z^{-1})\right) + i\alpha\beta} \frac{dz}{iz}\, dr$$

$$= \int_0^\infty r^2 e^{-\frac{1}{2}r^2} \frac{\sqrt{2}\pi}{r\left(\frac{1}{2}+\beta^2\right)} \left(1 - \frac{\alpha\beta}{\sqrt{\alpha^2\beta^2 + r^2\left(\frac{1}{2}+\beta^2\right)}}\right) dr$$

$$= \frac{\sqrt{2}\pi}{\frac{1}{2}+\beta^2} - \frac{2\pi^{\frac{3}{2}}\alpha\beta}{\left(\frac{1}{2}+\beta^2\right)^{\frac{3}{2}}} \exp\left(\frac{\alpha^2\beta^2}{2\left(\frac{1}{2}+\beta^2\right)}\right) \Phi\left(-\frac{\alpha\beta}{\sqrt{\frac{1}{2}+\beta^2}}\right),$$

where the integral $\int_0^{2\pi} \dots d\varphi$ is solved by using the substitution $z = e^{i\varphi}$ and the residue theorem

$$\int_{|z|=1} \frac{z^2+1}{r(\frac{i}{\sqrt{2}}+\beta)z^2 - 2\alpha\beta z + r(\frac{i}{\sqrt{2}}-\beta)} \frac{dz}{z}$$

$$= 2\pi i \left(\frac{-\frac{i}{\sqrt{2}}-\beta}{r(\frac{1}{2}+\beta^2)} + \frac{\frac{i}{\sqrt{2}}\alpha\beta + \beta\sqrt{\alpha^2\beta^2 + r^2(\frac{1}{2}+\beta^2)}}{r(\frac{1}{2}+\beta^2)\sqrt{\alpha^2\beta^2 + r^2(\frac{1}{2}+\beta^2)}}\right)$$

$$= \frac{\sqrt{2}\pi}{r(\frac{1}{2}+\beta^2)} \left(1 - \frac{\alpha\beta}{\sqrt{\alpha^2\beta^2 + r^2(\frac{1}{2}+\beta^2)}}\right).$$

The residues have to be calculated for

$$z_1 = 0 \quad \text{and for} \quad z_2 = \frac{\alpha\beta - \sqrt{\alpha^2\beta^2 + r^2(\frac{1}{2}+\beta^2)}}{r(\frac{1}{2}+\beta^2)}.$$

This completes the proof of Lemma 8. ∎

5.2. Proof of Theorem 1. We first derive asymptotic expansions for mean value and variance. Since

$$c(x,u) = \frac{1}{g(x,u) + h(x,u)\sqrt{1 - x/\rho(u)}}, \tag{56}$$

we get

$$[x^n]c(x,1) = [x^n]\frac{1}{h(x,1)}\left(1 - \frac{x}{\rho}\right)^{-\frac{1}{2}} + \frac{\rho g_x(\rho,1)}{h(\rho,1)} + \mathcal{O}(\sqrt{1-x/\rho})$$
$$= \frac{\rho^{-n} n^{-\frac{1}{2}}}{h(\rho,1)\sqrt{\pi}}(1 + \mathcal{O}(n^{-1}))$$

and

$$[x^n]c_u(x,1) = [x^n]\left(\frac{-g_u(x,1)}{h(x,1)^2}\left(1 - \frac{x}{\rho}\right)^{-1} + \frac{-h_u(x,1)}{h(x,1)^2}\left(1 - \frac{x}{\rho}\right)^{-\frac{1}{2}}\right)$$
$$= \frac{-g_u(\rho,1)\rho^{-n}}{h(\rho,1)^2}(1 + \mathcal{O}(n^{-\frac{1}{2}})).$$

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Hence,

$$\mathbf{E}X_n = \frac{[x^n]c_u(x,1)}{[x^n]c(x,1)} = \frac{-g_u(\rho,1)}{h(\rho,1)}\sqrt{\pi n} + \mathcal{O}(1).$$

Similarly, we get

$$[x^n]c_{uu}(x,1) = \frac{4g_u(\rho,1)^2}{\sqrt{\pi}h(\rho,1)^3}\rho^{-n}n^{\frac{1}{2}}(1+\mathcal{O}(n^{-\frac{1}{2}}))$$

and

$$\mathbf{V}X_n = \frac{[x^n]c_{uu}(x,1)}{[x^n]c(x,1)} + \mathbf{E}X_n - (\mathbf{E}X_n)^2 = (4-\pi)\frac{g_u(\rho,1)^2}{h(\rho,1)^2}n + \mathcal{O}(n^{\frac{1}{2}}).$$

Next we will determine the characteristic function of $X_n/\sqrt{n}$. Since

$$\mathbf{E}e^{itX_n/\sqrt{n}} = \frac{[x^n]c(x,e^{\frac{it}{\sqrt{n}}})}{[x^n]c(x,1)}, \tag{57}$$

we have to expand $[x^n]c(x,u)$ for $u = e^{it/\sqrt{n}} = 1 + i\frac{t}{\sqrt{n}} + \mathcal{O}(n^{-1})$. For this purpose we will use Cauchy's formula

$$[x^n]c(x,u) = \frac{1}{2\pi i}\int_\Gamma c(z,u)\,\frac{dz}{z^{n+1}} \tag{58}$$

for the following path of integration $\Gamma = \Gamma_1 \cup \Gamma_2$:

$$\Gamma_1 = \left\{z = \rho\left(1+\frac{s}{n}\right)\ :\ s\in\gamma'\right\}, \tag{59}$$

$$\Gamma_2 = \left\{z = \mathrm{R}e^{i\vartheta}\ :\ R = \rho\left|1+\frac{\log^2 n + i}{n}\right|, \arg\left(1+\frac{\log^2 n+i}{n}\right) \le |\vartheta| \le \pi\right\},$$

where $\gamma' = \{s : |s| = 1, \Re s \le 0\} \cup \{s : 0 < \Re s < \log^2 n, \Im s = \pm 1\}$ is the major part of a Hankel contour γ.

First let us concentrate on the path Γ_1. By using the substitution $z = \rho\left(1+\frac{s}{n}\right)$ we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}} &= \frac{\rho^{-n}}{2\pi i}\int_{\gamma'} \frac{e^{-s}(1+\mathcal{O}(s^2n^{-1}))}{g_u(\rho,1)\frac{it}{\sqrt{n}} + h(\rho,1)\sqrt{\frac{-s}{n}} + \mathcal{O}\left(\frac{s}{n}\right)}\,\frac{ds}{n} \\ &= \frac{\rho^{-n}n^{-\frac{1}{2}}}{h(\rho,1)}\int_{\gamma'} \frac{e^{-s}}{\sqrt{-s} + i\frac{g_u(\rho,1)}{h(\rho,1)}t}\,ds + \mathcal{O}(\rho^{-n}n^{-1}).\end{aligned} \tag{60}$$

Since

$$\int_{\gamma\setminus\gamma'} \frac{e^{-s}}{\sqrt{-s}+iCt}\,ds = \mathcal{O}\left(e^{-\log^2 n}\right)$$

we immediately get by (60) and Lemma 6

$$\frac{1}{2\pi i}\int_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}} = \frac{\rho^{-n}n^{-\frac{1}{2}}}{\sqrt{\pi}h(\rho,1)}\varphi_{\mathcal{R}}\left(\frac{-\sqrt{2}g_u(\rho,1)}{h(\rho,1)}t\right) + \mathcal{O}(\rho^{-n}n^{-1}). \tag{61}$$

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Now we make use of the fact that there are $\varepsilon_1 > 0$, $\delta_1 > 0$ such that

$$\max_{|x|=x_1,|\arg x|\ge\vartheta_1} |c(x,u)| = |c(x_1 e^{i\vartheta_1},u)|$$

for $1 \le x_1 \le 1+\delta_1$ and $|u-1| < \varepsilon_1$. You only have to observe that $c_{nk} \ge 0$, that $x = \rho$ is the only singularity on the circle of convergence, and that $c(x,u)$ has the local representation (56). Hence, it follows from

$$\left| c\left(\rho\left(1+\frac{\log^2 n + i}{n}\right), e^{\frac{it}{\sqrt{n}}}\right)\right| = \mathcal{O}\left(n^{\frac12}\log n\right)$$

that

$$\int_{\Gamma_2} c(z,u)\,\frac{dz}{z^{n+1}} = \mathcal{O}\left(n^{\frac12}\log n\, e^{-\log^2 n}\right). \tag{62}$$

Consequently, by (57), (58), (61), and (62),

$$\mathbf{E}e^{itX_n/\sqrt{n}} = \varphi_{\mathcal{R}}\left(\frac{-\sqrt{2}g_u(\rho,1)}{h(\rho,1)}t\right) + \mathcal{O}(n^{-\frac12}).$$

Thus we have proved a weak limit theorem.

In order to prove the local limit theorem we again use Cauchy's formula

$$[x^n u^k]c(x,u) = \frac{1}{(2\pi i)^2}\int_\Gamma\int_\Delta c(z,u)\,\frac{du}{u^{k+1}}\,\frac{dz}{z^{n+1}},$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ is as above (see (59)), and Δ will be properly chosen.

If $z = \rho(1+\frac{s}{n}) \in \Gamma_1$ then the mapping $u \mapsto c(z,u)$ has a polar singularity at $u_0 = 1 + \frac{t_0}{\sqrt{n}}$, where

$$t_0 = -\frac{h(\rho,1)}{g_u(\rho,1)}\sqrt{-s} + \mathcal{O}\left(\frac{s}{n}\right)$$

with residue

$$\frac{1}{g_u(\rho,1)}\left(1+\mathcal{O}\left(\frac{s}{n}\right)\right).$$

Hence, we can transform Δ in a way that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\Delta c(z,u)\,\frac{du}{u^{k+1}}\\
&= -\frac{u_0^{-k-1}}{g_u(\rho,1)}\left(1+\mathcal{O}\left(\frac{s}{n}\right)\right) + \frac{1}{2\pi i}\int_{|u|=1+\varepsilon_2} c(z,u)\,\frac{du}{u^{k+1}}\\
&= \frac{-1}{g_u(\rho,1)}\exp\left(\frac{k}{\sqrt{n}}\frac{h(\rho,1)}{g_u(\rho,1)}\sqrt{-s}\right)\left(1+\mathcal{O}\left(\frac{(k+1)s}{n}\right)\right) + \mathcal{O}\left((1+\varepsilon_2)^{-k}\right).
\end{aligned}$$

Consequently,

$$\frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_\Delta c(z,u)\,\frac{du}{u^{k+1}}\,\frac{dz}{z^{n+1}}$$

$$
\begin{aligned}
&= \frac{-1}{g_u(\rho,1)} \frac{\rho^{-n}}{2\pi i} \int_{\gamma'} \exp\left(-s + \frac{k}{\sqrt{n}} \frac{h(\rho,1)}{g_u(\rho,1)} \sqrt{-s}\right) \left(1 + \mathcal{O}\left(\frac{(k+1)s^2}{n}\right)\right) \frac{ds}{n} \\
&\quad + \mathcal{O}\left(\rho^{-n}(1+\varepsilon_2)^{-k}\right) \\
&= \frac{k}{n^{\frac{3}{2}}} \frac{h(\rho,1)}{g_u(\rho,1)^2} \frac{\rho^{-n}}{2\sqrt{\pi}} \exp\left(-\frac{k^2}{4n} \frac{h(\rho,1)^2}{g_u(\rho,1)^2}\right) + \mathcal{O}\left(\rho^{-n} \frac{k+1}{n^2}\right) \\
&\quad + \mathcal{O}\left(\rho^{-n} \frac{(1+\varepsilon_2)^{-k}}{n}\right).
\end{aligned}
$$

By elementary considerations we obtain

$$
\max_{z \in \Gamma_2, |u|=1} c(z,u) = \mathcal{O}\left(n^{\frac{1}{2}} \log n\right).
$$

Hence, by choosing $\Delta = \{u : |u| = 1\}$ for $z \in \Gamma_2$ we can estimate the remaining integral by

$$
\frac{1}{(2\pi i)^2} \int_{\Gamma_2} \int_{\Delta} c(z,u) \frac{du}{u^{k+1}} \frac{dz}{z^{n+1}} = \mathcal{O}\left(n^{\frac{1}{2}} \log n \, e^{-\log^2 n}\right)
$$

and finally have proved the local limit theorem.

5.3. Proof of Theorem 2. As above we have

$$
[x^n]c(x,1) = \frac{\rho^{-n} n^{-\frac{1}{2}}}{h(\rho,1)\sqrt{\pi}} (1 + \mathcal{O}(n^{-1}))
$$

and from

$$
\begin{aligned}
c_u(x,1) = \frac{-\rho'(1)}{2\rho h(\rho,1)} \left(1 - \frac{x}{\rho}\right)^{-\frac{3}{2}} - \frac{\alpha}{h(\rho,1)^2} \left(1 - \frac{x}{\rho}\right)^{-\frac{1}{2}} \\
+ \mathcal{O}\left(\left(1 - \frac{x}{\rho}\right)^{\frac{1}{2}}\right)
\end{aligned}
$$

we immediately get ($\alpha = 0$)

$$
\begin{aligned}
\mathbf{E} X_n &= \frac{[x^n]c_u(x,1)}{[x^n]c(x,1)} \\
&= -\frac{\rho'(1)}{\rho} n + \mathcal{O}(1) \\
&= \mu\, n + \mathcal{O}(1),
\end{aligned}
$$

and from a little bit more refined analysis we get

$$
\mathbf{V} X_n = \sigma^2 n + \mathcal{O}(\sqrt{n}).
$$

Since

$$
\varphi_{(X_n - \mu n)/\sqrt{\sigma^2 n}}(t) = e^{-it\sqrt{n}\mu/\sigma} \varphi_{X_n}\left(\frac{t}{\sqrt{\sigma^2 n}}\right) = e^{-it\sqrt{n}\mu/\sigma} \frac{[x^n]c(x, e^{it/\sqrt{\sigma^2 n}})}{[x^n]c(x,1)},
$$

we have to determine

$$
[x^n]c(x,u) = \frac{1}{2\pi i} \int_{\Gamma} c(z,u) \frac{dz}{z^{n+1}}
$$

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for $u = e^{it/\sqrt{\sigma^2 n}} = 1 + i\frac{t}{\sqrt{\sigma^2 n}} - \frac{t^2}{2\sigma^2 n} + \mathcal{O}(n^{-2})$ in which we use the following path of integration $\Gamma = \Gamma_1 \cup \Gamma_2$:

$$
\begin{aligned}
&\Gamma_1 = \left\{ z = \rho(u)\left(1+\frac{s}{n}\right) \;:\; s \in \gamma' \right\},\\
(63)\qquad &\Gamma_2 = \left\{ z = Re^{i(\vartheta - \arg(u))} \;:\; R = |\rho(u)|\left|1 + \frac{\log^2 n + i}{n}\right|,\right.\\
&\qquad\qquad \left. \arg\left(1 + \frac{\log^2 n + i}{n}\right) \le |\vartheta| \le \pi \right\}.
\end{aligned}
$$

From

$$\rho(u)^{-n} = \rho^{-n} \exp\left(\sqrt{n}\frac{\mu}{\sigma} - \frac{t^2}{2}\right)\left(1 + \mathcal{O}(n^{-\frac12})\right)$$

and from

$$g(z,u) + h(z,u)\sqrt{1 - z/\rho(u)} = h(\rho,1)\sqrt{\frac{-s}{n}} + \mathcal{O}\left(\frac{s}{n}\right)$$

for $x \in \Gamma_1$, by applying Lemma 7 we directly get

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}}\\
&= \frac{\rho(u)^{-n}}{h(\rho,1)\sqrt{n}}\frac{1}{2\pi i}\int_{\gamma'} \frac{e^{-s}}{\sqrt{-s}}\left(1 + \mathcal{O}\left(\frac{s^2}{n} + \left|\frac{s}{n}\right|\right)\right) ds\\
&= \exp\left(\sqrt{n}\frac{\mu}{\sigma} - \frac{t^2}{2}\right)\frac{\rho^{-n}n^{-\frac12}}{h(\rho,1)\sqrt{\pi}}\left(1 + \mathcal{O}(n^{-\frac12})\right).
\end{aligned}
$$

It remains to estimate the integral on Γ_2. But this can be done as in the proof of Theorem 1 since

$$\max_{z\in\Gamma_2} |c(z,u)| = \mathcal{O}(n^{\frac12}\log n).$$

In order to prove the local law we again use Cauchy's formula

$$[x^n u^k]c(x,u) = \frac{1}{(2\pi i)^2}\int_\Delta\int_\Gamma c(z,u)\,\frac{dz}{z^{n+1}}\,\frac{du}{u^{k+1}},$$

where $\Delta = \{u \;:\; |u| = 1\}$. For $u \in \Delta_1 = \{u = e^{it} \;:\; |t| \le n^{-5/12}\}$ let $\Gamma = \Gamma_1 \cup \Gamma_2$ as in the proof of the weak limit theorem; for $u \in \Delta_2 = \{u = e^{it} \;:\; n^{-5/12} < |t| < \varepsilon\}$ (for some sufficiently small $\varepsilon > 0$) let $\Gamma = \{z \;:\; |z| = \frac12(\rho + |\rho(e^{it})|)\}$; and for $u \in \Delta_3 = \{u = e^{it} \;:\; \varepsilon \le |t| \le \pi\}$ let $\Gamma = \{z \;:\; |z| = \rho(1+\delta)\}$ for some sufficiently small $\delta > 0$.

First, let $u = e^{it} \in \Gamma_1$, i.e., $|t| \le n^{-5/12}$, and $z \in \Gamma_1$. By direct approximation we have

$$g(z,u) + h(z,u)\sqrt{1 - z/\rho(u)} = n^{-\frac12}h(\rho,1)\sqrt{-s}(1 + \mathcal{O}(n^{-\frac13}))$$

(note that $\alpha = 0$ and that $|s| \ge 1$) and

$$z^{-n}u^{-k} = \rho^{-n}e^{-it(k-\mu n) - \frac12 t^2\sigma^2 n - s}(1 + \mathcal{O}(n^{-\frac14})).$$

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Hence, we obtain by using the methods of [12] and a saddle-point-like integration (compare with [9])

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\Delta_1}\int_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}}\,\frac{du}{u^{k+1}}\\
&= \frac{\rho^{-n}}{\sqrt{n}h(\rho,1)}\frac{1}{(2\pi i)^2}\int_{-n^{-5/12}}^{n^{-5/12}}\int_{\gamma'}\frac{e^{-it(k-\mu n)-\frac{1}{2}t^2\sigma^2 n-s}}{\sqrt{-s}}(1+\mathcal{O}(n^{-\frac{1}{4}}))\,ds\,dt\\
&= \frac{\rho^{-n}}{\sqrt{n}h(\rho,1)}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{2\pi\sigma^2 n}}\left(\exp\left(-\frac{(k-\mu n)^2}{2\sigma^2 n}\right)+\mathcal{O}(n^{-\frac{1}{4}})\right).
\end{aligned}$$

Therefore, the proof is finished if the remaining integrals are sufficiently small. As above we have

$$\frac{1}{(2\pi i)^2}\int_{\Delta_1}\int_{\Gamma_2} c(z,u)\,\frac{dz}{z^{n+1}}\,\frac{du}{u^{k+1}} = \mathcal{O}\left(\rho^{-n}n^{\frac{1}{12}}\log n\, e^{-\log^2 n}\right).$$

Next we get

$$\frac{1}{(2\pi i)^2}\int_{\Delta_2}\int_{\Gamma} c(z,u)\,\frac{dz}{z^{n+1}}\,\frac{du}{u^{k+1}} = \mathcal{O}\left(\rho^{-n}e^{-cn^{\frac{1}{6}}}\right)$$

since $|\rho(u)| \ge \rho(1+c_1 n^{-5/6})$ for $u \in \Delta_2$ (and some sufficiently small constants $c, c_1 > 0$). Finally, since $c(z,u)$ is bounded for $u \in \Delta_3$ and $z \in \Gamma$ we obtain

$$\frac{1}{(2\pi i)^2}\int_{\Delta_3}\int_{\Gamma} c(z,u)\,\frac{dz}{z^{n+1}}\,\frac{du}{u^{k+1}} = \mathcal{O}\left(\rho^{-n}(1+\delta)^{-n}\right),$$

which completes the proof of the local theorem.

5.4. Proof of Theorem 3. As in the proof of Theorem 2 we get

$$\mathbf{E}X_n = \mu\, n - \sqrt{\pi}\frac{\alpha}{h(\rho,1)}\sqrt{n} + \mathcal{O}(1)$$

and

$$\mathbf{V}X_n = \left(\sigma^2 + (4-\pi)\frac{\alpha^2}{h(\rho,1)^2}\right)n + \mathcal{O}(\sqrt{n}).$$

Now, if we use the same normalization and the same path of integration (63) as in Theorem 2, by applying Lemma 6 we obtain for $u = e^{it/\sqrt{\sigma^2 n}}$

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_1} c(z,u)\,\frac{dz}{z^{n+1}}\\
&= \frac{\rho(u)^{-n}}{h(\rho,1)\sqrt{n}}\frac{1}{2\pi i}\int_{\gamma'}\frac{e^{-s}}{\sqrt{-s}+i\alpha t/(h(\rho,1)\sigma)}\left(1+\mathcal{O}\left(\frac{s^2}{n}+\left|\frac{s}{n}\right|\right)\right)ds\\
&= \frac{\rho^{-n}n^{-\frac{1}{2}}}{h(\rho,1)\sqrt{\pi}}\exp\left(\sqrt{n}\frac{\mu}{\sigma}-\frac{t^2}{2}\right)\varphi_{\mathcal{R}}\left(\frac{-\sqrt{2}\alpha}{h(\rho,1),\sigma}\right)\left(1+\mathcal{O}(n^{-\frac{1}{2}})\right).
\end{aligned}$$

The remaining integral on Γ_2 can be estimated as above. Hence, we have proved the weak convergence property.

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In order to prove the local law we will proceed as in the proof of Theorem 2. As above we can concentrate on the path of integration $\Delta_1 \times \Gamma_1$. The remaining integrals are negligible. Direct approximation yields

$$\frac{1}{(2\pi i)^2} \int_{\Delta_1} \int_{\Gamma_1} c(z,u) \, \frac{dz}{z^{n+1}} \, \frac{du}{u^{k+1}}$$
$$= \frac{\rho^{-n}}{\sqrt{n} h(\rho,1)} \frac{1}{(2\pi i)^2} \int_{-n^{-5/12}}^{n^{-5/12}} \int_{\gamma'} \frac{e^{-it(k-\mu n) - \frac{1}{2}t^2\sigma^2 n - s}}{\sqrt{-s} + i\frac{\alpha}{h(\rho,1)}\sqrt{n} t} \, (1 + \mathcal{O}(n^{-\frac{1}{4}})) \, ds \, dt.$$

Hence an application of Lemma 8 and easy tail estimates complete the proof of Theorem 3.

6. Conclusions. The main purpose of this paper is to provide general techniques to obtain the limiting distribution of parameters in combinatorial constructions. It is the second paper of a (planned) series of papers [10] devoted to this topic. Theorems 1–3 should be considered as examples of analytic theorems providing a link between combinatorial constructions and their asymptotic distributions. (They seem to be proper theorems to discuss random mappings.) The authors are convinced that the methods presented in the preceding proofs can be used in many other (different) problems. The basic ideas are singularity analysis (introduced by Flajolet and Odlyzko [12]) and saddle-point approximation.

Random mappings are widely and intensively discussed in literature; e.g., in Kolchin's book [15] a probabilistic approach via branching processes is presented, whereas Aldous and Pitman [3] use a completely different probabilistic concept related to Aldous's continuum random trees [1, 2]. Our concept of generating functions goes back to Arney and Bender [4] and to Flajolet and Odlyzko [13]. They could identify many limiting distributions and provided asymptotic expansions for mean and variance. (One gap could be filled by Application 3.) It should be mentioned, too, that Arney and Bender [4] discussed a slightly more general case, namely that the number of immediate predecessors of a point $|\varphi^{-1}(\{\nu\})|$ is not arbitrary but must be contained in a subset D of nonnegative integers; i.e., the corresponding tree function $\hat{t}_D(x)$ satisfies the functional equation

$$\hat{t}_D(x) = x \sum_{n \in D} \frac{\hat{t}_D(x)^n}{n!} = x\phi(\hat{t}_D(x))$$

and the corresponding generating function for those mappings

$$\hat{f}_D(x) = \frac{1}{1 - x\phi'(\hat{t}_D(x))}.$$

From Proposition 1 it follows that if D contains a number ≥ 2 then it has a square-root singularity

$$\hat{t}_D(x) = g_D(x) - h_D(x)\sqrt{1 - x/\rho_D},$$

around $x = \rho_D = t_0/\phi(t_0)$, where $t_0 > 0$ satisfies $t_0\phi'(t_0) = \phi(t_0)$. Hence, $\rho_D\phi'(t_0) = 1$ and we are in a similar situation as in the classical case. Especially we can adapt all the combinatorial constructions to this more general case and obtain analog results by applying our theorems.

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