F. WILLIAM LAWVERE
dell'Università di Perugia

METRIC SPACES, GENERALIZED LOGIC, AND CLOSED CATEGORIES

(Conferenza tenuta il 30 marzo 1973) *

SUNTO. — In questo articolo viene rigorosamente sviluppata l'analogia fra $\mathrm{dist}(a, b) + \mathrm{dist}(b, c) \geq \mathrm{dist}(a, c)$ e $\hom(A, B) \otimes \hom(B, C) \to \hom(A, C)$, giungendo a numerosi risultati generali sugli spazi metrici, come conseguenza di una « logica pura generalizzata » i cui « valori di verità » sono scelti in una arbitraria categoria chiusa.

INTRODUCTION.

It is a banality that all mathematical structures of a given kind constitute the objects of a category; the sequence: elements/structures/categories thus has led some people to attempt to characterize the philosophical significance of the theory of categories as that of a « third level of abstraction ». But the theory of categories actually penetrates much more deeply than that attempted characterizaion would suggest toward summing up the essence of mathematics. The kinds of structures which actually arise in the practice of geometry and analysis are far from being « arbitrary », and indeed in this paper we will investigate a particular case of the way in which logic should be specialized to take account of this experience of non-arbitrariness, as concentrated in the thesis that *fundamental* structures are themselves categories. Two cases of this thesis have been known for 30 years; an ordered set (often called poset) is a category in which for any ordered pair of objects there is at most one morphism from the first to the second, while a group is a category in which there is just one object and in which every morphism is a isomorphism. That the thesis has non-vacuous implications for these two cases follows from the facts that when the general idea

* Pervenuta in Redazione il 16 novembre 1973.

of functor between categories is restricted to the special categories (that is to the posets, respectively to the groups) it agrees with the correct idea of morphism between these fundamental structures (i.e. with order-preserving map, respectively group homomorphism) and that a functor from one of the special categories to some category (e.g. that of vector spaces) is itself an important structure (a direct system of vector spaces, respectively a linear group-representation; moreover the correct morphism between *these* structures are then just the natural transformations). Two further cases of the thesis have been developed in the past 10 years: functorial semantics, in which categories with special properties are identified as theories and special functors as interpretations or models, and the theory of topoi, in which certain categories correspond to (usefully generalized) topological spaces.

By taking account of a certain natural generalization of category theory within itself, namely the consideration of strong categories whose hom-functors take their values in a given « closed category » $\mathcal{V}$ (not necessarily in the category $\mathcal{S}$ of abstract sets), we will show below that it is possible to regard a metric space as a (strong) category and that moreover by specializing the constructions and theorems of general category theory we can deduce a large part of *general* metric space theory. The theory of closed categories (and of strong categories valued in them) was originally developed to deal with more complicated examples such as compactly generated topological spaces, Banach spaces, differential graded modules, etc., and moreover some of the publications on the subject seem forbiddingly technical to the beginner. I hope that this article can also be read as an introduction to closed categories on the basis of the guiding example of metric spaces considered as strong categories valued in the closed category of nonnegative real quantities.

Since closed categories are just what is suficient to have a reasonable theory of strong categories, we consider some examples of the latter first in order to bring out the elementary nature of the analogy on which the present work is based. In a *metric space* X, we will denote by $X(a, b)$ the non-negative real quantity (we will allow the value ∞) of X-distance from the point a to the point b. Then the laws satisfied by X are greater-than relations

$$\begin{aligned} X(a,b) + X(b,c) &\geq X(a,c) \\ 0 &\geq X(a,a)\,. \end{aligned}$$

(see below for remarks on the possible non-symmetry of the metric).

In a *category* X, we will denote by $X(a, b)$ the abstract set of X-morphisms from the object a to the object b. Then the composition law and specification of identity morphisms for X are mappings

$$\begin{array}{rcl} X(a,b) \times X(b,c) & \longrightarrow & X(a,c) \\ 1 & \longrightarrow & X(a,a) \end{array}$$

(which are subject to associativity and unity axioms which may be expressed as commutative diagrams of mappings betwen abstract sets, using elementary properties of the cartesian product $\times$ of abstract sets and of the one-element set 1). In a *poset* X, we will denote by $X(a, b)$ the truth-value of the X-dominance of the element a over the element b. Then the transitivity and reflexivity laws for X are entailments

$$\begin{array}{rcl} X(a,b) \wedge X(b,c) & \vdash & X(a,c) \\ \text{true} & \vdash & X(a,a) \,. \end{array}$$

If K is a commutative ring, then in a K-*additive category* X we will denote by $X(a, b)$ the K-module of X-morphisms from the object a to the object b. Then the composition and identity laws for X are K-linear mappings

$$\begin{array}{rcl} X(a,b) \otimes X(b,c) & \longrightarrow & X(a,c) \\ K & \longrightarrow & X(a,a) \end{array}$$

(again subject to associativity and unity axioms which may be expressed by commutative diagrams of K-linear mappings of K-modules, using elementary properties of the K-tensor product $\otimes$ of K-modules and of the K-module K). Thus we are led to consider that a greater-than-or-equal-to relation between nonnegative real quantities is analogous to a K-linear mapping between K-modules, since both are morphisms of possible hom-values for categories X, but in two different closed categories $\mathcal{V}$. Similarly, the sum of quantities is analogous to the tensor product of modules, both because they play the same role in the structure of a $\mathcal{V}$-valued category and also because they satisfy the same « elementary properties » (within $\mathcal{V}$ itself) of functoriality (i.e. monotonicity in the case of quantities), of associativity and commutativity up to $\mathcal{V}$-isomorphism (i.e. up to equality in the case of quantities), and of having a unit object K (the zero quantity). We have the table

X	hom-values for X	composition law and identity law for X	domain of composition law for X	domain of identity law for X
metric space	nonnegative real quantities	$\geqq$	sum	zero
category	abstract sets	mapping	cartesian product	one element set
poset	truth values	entailment	conjunction	true
$\mathcal{V}$-valued category	objects in $\mathcal{V}$	morphism in $\mathcal{V}$	« tensor » product in $\mathcal{V}$	unit object K for tensor product in $\mathcal{V}$

The associativity and unity axioms which the composition law and identity law of a $\mathcal{V}$-based category X must satisfy are automatic in the case of metric spaces or posets since *all* diagrams in $\mathcal{V}$ commute if $\mathcal{V} =$ reals or $\mathcal{V} =$ truth-values.

None of our results in this paper will depend on the additional Frechet axioms:

$$\begin{aligned} \text{if } X(a, b) &= 0 \text{ then } a = b \\ X(a, b) &< \infty \\ X(a, b) &= X(b, a) . \end{aligned}$$

The first of these is not very natural from the categorical viewpoint since it corresponds to requiring that isomorphic objects are equal; passage to the quotient can be avoided (as it *must* be for ordinary categories) by employing *equivalence* (broader than isomorphism) of strong categories. Allowing ∞ among the quantities is precisely analogous to including the empty set among abstract sets, and it is done for similar reasons of completeness; a metric space can be analyzed as a structured system of metric spaces with finite distances by considering the equivalence relation defined by « $X(a, b) < \infty$ and $X(b, a) < \infty$ ». The non symmetry is the more serious generalization, and moreover occurs in many naturally arising examples, such as $X(a, b) =$ work required to get from a to b in mountainous region X. Also within analysis itself a naturally arising metric is often non-symmetric but traditionally symmetrized by one of the two procedures

$$X(a, b) + X(b, a)$$
$$\max(X(a, b), X(b, a))$$

which in fact could be applied to any metric; we mention three common examples where this nonsymmetry exists. If M is any Boolean algebra equipped with an outer measure, then

$$M(a, b) \underset{\text{def}}{=} M(b - a)$$

(where $b - a = b \cap a'$ in the Boolean algebra and $M(c)$ denotes the measure of c) defines a metric space in which

$$0 \geq M(a, b) \text{ iff } a \supseteq b$$

almost everywhere.

(For more about this example see the section below on closed functors).

If K is any convex set we may define a metric on it by

$$K(a, b) = \inf_{f: a \to b} \{-\log(\alpha_f)\}$$

where $f: a \to b$ means that $f \in K$ with a on the open segment from f to b, with α_f then denoting the $0 < \alpha < 1$ with $a = (1 - \alpha) f + \alpha b$. (The proof of the triangle inequality follows from the fact that K is actually a « normed category », since if $g: b \to c$ we can define $fg: a \to c$ by

$$fg = \frac{1 - \alpha_f}{1 - \alpha_f \alpha_g} f + \frac{\alpha_f (1 - \alpha_g)}{1 - \alpha_f \alpha_g} g,$$

note that this is associative and that $\alpha_{fg} = \alpha_f \alpha_g$, and adjoin identities formally). The notion of « normed category » can also be related to the (nonsymmetric) Hausdorff metric

$$2^X(A, B) = \sup_{a \in A} \inf_{b \in B} X(a, b)$$

for subsets A, B of a metric space X: let $f: A \to B$ mean that f is any mapping from A to B and define

$$|f| = \sup_{a \in A} X(a, af);$$

then by the axiom of choice

$$2^X(A, B) = \inf_{f: A \to B} |f|$$

and the fundamental property of a normed category

$$|f| + |g| \geq |fg|$$

leads to a proof of the triangle inequality. We will leave as an exercise for the reader to define a closed category $\mathcal{S}(\mathbf{R})$ such that « normed categories » are just $\mathcal{S}(\mathbf{R})$-valued categories and a « closed functor » $\inf: \mathcal{S}(\mathbf{R}) \to \mathbf{R}$ which induces the passage from any « normed category » to a metric space with the same objects. Another approach to the Hausdorff metric will be evident from the discussion below of the comprehension scheme. The canonical symmetrization procedure actually applies to categories valued in any given closed category $\mathcal{V}$; in general, symmetry of a $\mathcal{V}$-valued category X has to mean (not a property but) a given structure consisting of $\mathcal{V}$-isomorphisms

$$\sigma_{ab}: X(a, b) \to X(b, a)$$

subject to suitable coherence axioms, and the first canonical procedure (specializing to the sum in the case of a metric space) is

$$\operatorname{sym}(X)(a, b) = X(a, b) \otimes X(b, a).$$

We still have not, in this introduction, touched on the property of closed categories for which they are called « closed ». Basically, closed categories are closed with respect to the operation of forming the hom of two objects, so that $\mathcal{V}$ itself is a fundamental example of a $\mathcal{V}$-category. The internal Hom (a, c) in $\mathcal{V}$ is related to the internal tensor product in $\mathcal{V}$ by adjointness so that there is a natural one-to-one correspondence between the $\mathcal{V}$-morphisms

$$b \to \operatorname{Hom}(a, c)$$

and the $\mathcal{V}$-morphisms

$$a \otimes b \to c.$$

Thus in the closed category $\mathbf{R} = [0; \infty]$ in which metric spaces are valued

$$\operatorname{Hom}(a, c) = \begin{cases} c - a & \textit{if} \quad c \geq a \\ 0 & \textit{if} \quad a \geq c \end{cases}$$

so that the Hom-$\otimes$ adjointness reduces (if we denote by « minus » this truncated subtraction) to the if-and-only-if

$$\frac{b \geq c - a}{a + b \geq c},$$

whereas in the closed category $\mathcal{S}$ in which ordinary categories are valued, it reduces to the rule of lambda conversion

$$\frac{B \to C^A}{A \times B \to C}$$

and in the closed category **2** in which posets are valued, it reduces essentially to modus ponens and the « deduction theorem »

$$\frac{\beta \vdash \alpha \Rightarrow \gamma}{\alpha \wedge \beta \vdash \gamma}$$

i.e. the internal Hom for truth-values is implication. We will also assume that our closed categories $\mathcal{V}$ have cartesian products and coproducts over arbitrary index sets as well as equalizers and co-equalizers, i.e. that they are complete and cocomplete. From general properties of adjoint functors it follows that $\otimes$ preserves direct limits in each variable separately, while

$$\operatorname{Hom}(\varinjlim a_i, c) \cong \varprojlim \operatorname{Hom}(a_i, c)$$

$$\operatorname{Hom}(a, \varprojlim c_j) \cong \varprojlim \operatorname{Hom}(a, c_j)$$

For example in $\mathbf{R}$, $\varprojlim$ means sup and $\varinjlim$ means inf; in particular 0 is the empty $\varprojlim$ and ∞ is the empty $\varinjlim$ so that

$$\begin{aligned} a + \infty &= \infty \\ c - \infty &= 0 \\ 0 - a &= 0 . \end{aligned}$$

This completeness of $\mathcal{V}$ itself will be necessary for most of our general constructions. We will not consider (categorical) completeness of $\mathcal{V}$-valued categories; but on the other hand we will see that completeness in the *Cauchy* sense does have a meaning for categories valued in any closed $\mathcal{V}$.

The general constructions of functor categories, free categories, left and right Kan extensions, and « discrete » fibrations reduce in the very special case $\mathcal{V} = \mathbf{2}$ to higher types, transitive closure of a relation, existential and universal quantification, and the principle of set abstraction. Like the operation of implication mentioned above, the position of all these constructions in the general scheme as well as their fundamental properties of transformation are uniquely determined by adjointness. Since logic signifies formal relationships which are general in character, we may more precisely identify logic with this scheme of interlocking adjoints and then observe that all of logic applies *directly* to structures valued in an arbitrary closed category $\mathcal{V}$ (not only to structures valued in truth-values). For example, in quantitative logic (the case $\mathcal{V} = \mathbf{R}$ with which we will be mainly concerned in this paper) the isometric embedding of a metric space into a space of functions with sup metric is the application of the exactly same principle of logic which in the case $\mathcal{V} = \mathbf{2}$ gives Dedekind's representation of a poset by order ideals, and the upper and lower integrals of a real function are precisely cases of the generalized universal and existential quantification. This vast generalization is quite compatible with the specialization of logic called for in the first paragraph: although for any given $\mathcal{V}$ we could consider « arbitrary » $\mathcal{V}$-valued structures, there is one type of such structure which is of first importance, namely for $\mathcal{V}$ respectively truth-values, quantities, abstract sets, abelian groups, the structure of respectively poset, metric space, category, additive category (a very natural generalization of ring) is the generally useful first approximation possible with $\mathcal{V}$-valued logic for analyzing various problems; it even seems that there is a natural second approximation, namely the structure of a « rigidly $\mathcal{V}$-closed $\mathcal{V}$-category » which in the four cases mentioned specializes *roughly* to partially ordered abelian group, normed abelian group, rigidly closed category, and (in the additive case) to a common generalization of the category of locally free modules on an algebraic space and the category of finite-dimensional representations of an algebraic group. Detailed discussion of this second approximation awaits further investigation, as does the extension of the results of this paper to an arbitrary base topos, i.e. the extension from the *constant* abstract sets which we consider here to continuously *variable* sets as «sets» of points for metric spaces, as index «sets» for products, etc.

Of course the really deep results in a subject depend very much on the particularity of that subject and the results we offer here in the field of metric spaces, taken individually, will justly appear shallow to those with any experience. Indeed for me the surprising aspect was that methods originally devised to deal with quite different fields of algebra and geometry could yield any significant known theorems at all (for example the known theorem on extension of Lipschitz maps in section three). But there are many particularities, for example the special role of quadratic metrics, which I do not see how could be a result of « generalized logic ».

1. - Closed categories, strong categories, strong functors, closed functors.

In this paper we use the term « closed category » as short for « bicomplete symmetric monoidal closed category ». That is, a closed category $\mathcal{V}$ has equalizers, coequalizers, set-indexed products and coproducts plus a « monoidal » structure, which is symmetric and closed. A monoidal structure is a given functor

$$\mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$$

which is symmetric

$$u \otimes v \cong v \otimes u$$

and associative

$$u \otimes (v \otimes w) \cong (u \otimes v) \otimes w$$

and has a unit object k satisfying

$$k \otimes v \cong v \cong v \otimes k\,;$$

more precisely, the symmetry, associativity, and unitary isomorphisms are in general required to be given, natural, and «coherent», but since these conditions are automatic in the particular examples we consider, we do not emphasize them (MacLane). That the monoidal structure is closed means that we are further given a functor

$$\mathcal{V}^{op} \times \mathcal{V} \xrightarrow{\text{Hom}} \mathcal{V}$$

and two natural transformations

$$u \xrightarrow{\lambda} \text{Hom}\,(a, a \otimes u)$$

$$a \otimes \text{Hom}\,(a, v) \xrightarrow{\epsilon} v$$

such that the two processses

$$a \otimes u \xrightarrow{f} v \rightsquigarrow u \xrightarrow{\lambda} \mathrm{Hom}(a, a \otimes u) \xrightarrow{\mathrm{Hom}(a,f)} \mathrm{Hom}(a, v)$$

$$u \xrightarrow{g} \mathrm{Hom}(a, v) \rightsquigarrow a \otimes u \xrightarrow{a \otimes g} a \otimes \mathrm{Hom}(a, v) \xrightarrow{\epsilon} v$$

are inverse bijections,

$$\downarrow \uparrow \ \frac{a \otimes u \to v}{u \to \mathrm{Hom}(a, v)} .$$

Thus in particular there is a natural bijection

$$\frac{a \to v}{k \to \mathrm{Hom}(a, v)}$$

between $\mathcal{V}$-morphism $a \to v$ and $\mathcal{V}$-morphisms from the unit object to the object $\mathrm{Hom}(a, v)$ for any two objects a, v of $\mathcal{V}$, in terms of which the natural transformation ϵ behaves as an « evaluation » morphism, and two successive applications of appropriate instances of evaluation corresponds to a $\mathcal{V}$-morphism

$$\mathrm{Hom}(a, b) \otimes \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$$

which represents a « strong » form of composition. There is moreover a natural double-dualization morphism

$$u \to \mathrm{Hom}(\mathrm{Hom}(u, b), b)$$

for any fixed object b.

The simplest non-trivial example **2** of a closed category, which serves as the values for classical logic, has two objects false and $k =$ true and three morphisms

$$\begin{array}{lcl} \text{false} & \vdash & \text{false} \\ \text{false} & \vdash & \text{true} \\ \text{true} & \vdash & \text{true} \end{array}$$

with conjunction and implication as the tensor and Hom. Conjunction is of course symmetric and associative and since

$$u \vdash a \Longrightarrow (a \wedge u)$$

$$a \wedge (a \Longrightarrow v) \vdash v$$

always, we have the bijection

$$\frac{a \wedge u \vdash v}{u \vdash (a \Longrightarrow v)}$$

i.e. for any assignments of false, true to the three variables a, u, v either both or neither of these entailments exist. In particular

$$\frac{a \vdash v}{\text{true} \vdash a \Longrightarrow v}$$

and

$$(a \Longrightarrow b) \wedge (b \Longrightarrow c) \vdash a \Longrightarrow c$$

and

$$u \vdash (u \Longrightarrow b) \Longrightarrow b \, .$$

A somewhat more general and « dual » example is as follows: let $\mathcal{M}$ be any system of subsets of a given finite set which contains the empty subset $k = \emptyset$ and which is closed with respect to finite unions and with respect to set-theoretic difference; for $a, v \in \mathcal{M}$ let $a \to v$ mean that $a \supseteq v$, and let $a \otimes u$ mean $a \cup u$. Then $\mathcal{M}$ is a closed category since

$$\frac{a \cup u \supseteq v}{u \supseteq v - a}$$

(Even if $\mathcal{M}$ is not finite, our condition of bicompleteness can often be achieved by considering e.g. measurable sets *modulo* null sets).

Our central example **R** has as objects all non-negative real quantities (including ∞), as morphisms $a \to v$ the greater-than-or-equal-to relations $a \geq v$, and as « tensor » $a \otimes u = a + u$ the sum of quantities. Then Hom is forced to be truncated subtraction

$$\operatorname{Hom}(a, v) = \begin{cases} v - a & v \geq a \\ 0 & a \geq v \end{cases}$$

(denoted simply as subtraction) where in particular

$$\begin{aligned} \infty - \infty &= 0 \\ \infty - a &= \infty \quad \mathit{if} \quad a \neq \infty \\ v - \infty &= 0 \end{aligned}$$

(bicompleteness holds since

$$\prod_{i \in I} v_i = \sup_{i \in I} v_i$$

$$\sum_{i \in I} a_i = \inf_{i \in I} a_i$$

and equalizers and coequalizers are trivial). We have in particular

$$\frac{a \geq v}{0 \geq v - a}$$

$$(b - a) + (c - b) \geq c - a$$

$$u \geq b - (b - u) \, .$$

The « original » example of a closed category is the category $\mathcal{S}$ of abstract sets and mappings with the (unique) closed structure in which tensor means cartesian product, $k = 1$ is a one-element set, and $\mathrm{Hom}(a, v)$ is the (abstract) set of (indices for) all the mappings $a \to v$.

Given a closed category $\mathcal{V}$, a strong category valued in $\mathcal{V}$, or simply a $\mathcal{V}$-category X is any structure consisting of a specified set of X-objects $a, b, c \ldots$ together with the assignment of an object $X(a, b)$ of $\mathcal{V}$ to every ordered pair of X-objects $\langle a, b \rangle$, the assignment of a $\mathcal{V}$-morphism

$$X(a, b) \otimes X(b, c) \xrightarrow{\mu_{abc}} X(a, c)$$

to every ordered triple $\langle a, b, c \rangle$ of X-objects, and the assignment of a $\mathcal{V}$-morphism

$$k \xrightarrow{\eta_a} X(a, a)$$

to every X-object a, subject to the conditions that the following diagrams in $\mathcal{V}$ always commute

$$\begin{tikzcd}
X(a,b) \otimes (X(b,c) \otimes X(c,d)) \arrow[rr, "X(a,b) \otimes \mu_{bcd}"] \arrow[d, "\mathcal{V} - \text{assoc} \cong"'] & & X(a,b) \otimes X(b,d) \arrow[dd, "\mu_{abd}"] \\
(X(a,b) \otimes X(b,c)) \otimes X(c,d) \arrow[d, "\mu_{abc} \otimes X(c,d)"'] & & \\
X(b,c) \otimes X(c,d) \arrow[rr, "\mu_{bcd}"'] & & X(a,d)
\end{tikzcd}$$

$$\begin{tikzcd}
X(a,b) \arrow[r, "\mathcal{V}\text{—unitary}\,\cong"] \arrow[d, "\mathcal{V}\text{—unitary}\,\cong"'] & k \otimes X(a,b) \arrow[r, "\eta_a \otimes X(a,b)"] & X(a,a) \otimes X(a,b) \arrow[dd, "\mu_{abd}"] \\
X(a,b) \otimes k \arrow[d, "X(a,b) \otimes \eta_b"'] \arrow[drr, "\mathrm{id}"] & & \\
X(a,b) \otimes X(b,b) \arrow[rr, "\mu_{abb}"'] & & X(a,b)
\end{tikzcd}$$

Thus an $\mathcal{S}$-category is just an ordinary (small) category-whereas if $\mathcal{V} = Ab$ the category of abelian groups then an Ab-category with one object is just an ordinary ring (and more general Ab-categories arise in linear algebra just as naturally as do rings, e.g. *all* finite rectangular matrices over a given field form an Ab-category with the natural numbers as objects). The associativity and identity axioms (i.e. the above commutative diagrams) are (like coherence) automatic in case $\mathcal{V}$ itself is a poset, and hence, as claimed in the introduction, a **2**-category is an arbitrary poset while an **R**-category is an arbitrary (generalized) metric space.

Every $\mathcal{V}$-category X has an opposite X^{op} with the same objects and units but with

$$X^{op}(a, b) \;=\; X(b, a)$$

and

$$\begin{tikzcd}
X^{op}(a,b) \otimes X^{op}(b,c) \arrow[rr, "\mu^{op}_{abc}"] \arrow[d, equal] & & X^{op}(a,c) \arrow[d, equal] \\
X(b,a) \otimes X(c,b) \arrow[r, phantom, "\cong"] & X(c,b) \otimes X(b,a) \arrow[r, "\mu_{cba}"'] & X(c,a)
\end{tikzcd}$$

If we define also

$$\mathrm{sym}\,(X)(a, b) = X(a, b) \otimes X(b, a)$$

with

$$\begin{tikzcd}
\mathrm{sym}\,(X)(a,b) \otimes \mathrm{sym}\,(X)(b,c) \arrow[r, "\mu^{\mathrm{sym}}_{abc}"] \arrow[d, equal] & \mathrm{sym}\,(X)(a,c) \arrow[dd, equal] \\
(X(a,b) \otimes X(b,a)) \otimes (X(b,c) \otimes X(c,b)) \arrow[d, "\cong"'] & \\
(X(a,b) \otimes X(b,c)) \otimes (X(c,b) \otimes X(b,a)) \arrow[r, "\mu_{abc} \otimes \mu_{cba}"'] & X(a,c) \otimes X(c,a)
\end{tikzcd}$$

and

$$\begin{tikzcd}
k \arrow[r, "\eta_a^{\mathrm{sym}}"] \arrow[d, "\cong"'] & \mathrm{sym}(X)(a, a) \arrow[d, equal] \\
k \otimes k \arrow[r, "\eta_a \otimes \eta_a"] & X(a, a) \otimes X(a, a)
\end{tikzcd}$$

then $\mathrm{sym}(X)$ is a $\mathcal{V}$-category isomorphic with its opposite in an object-preserving manner.

In case the objects of $\mathcal{V}$ can be indexed by a set (which by bicompleteness actually forces $\mathcal{V}$ to have a poset as its underlying category) then $\mathcal{V}$ itself is an example of a $\mathcal{V}$-category (more generally any small part of $\mathcal{V}$ becomes a $\mathcal{V}$-category) by setting

$$\mathcal{V}(a, b) = \mathrm{Hom}(a, b)$$

Thus $\mathbf{2}$ is an example of a poset, while $\mathbf{R}$ is a (highly non-symmetric) example of a metric space. But $\mathrm{sym}(\mathbf{R})$ is the usual metric on the reals.

If X and Y are two $\mathcal{V}$-categories, then by a $\mathcal{V}$-functor $X \xrightarrow{f} Y$ is meant any structure consisting of mapping of the objects of X into the objects of Y together with an assignment of a $\mathcal{V}$-morphism

$$X(a, b) \xrightarrow{f_{ab}} Y(fa, fb)$$

to every ordered pair $\langle a, b \rangle$ of X-objects, subject to the commutativity of the following diagrams in $\mathcal{V}$.

$$\begin{tikzcd}
X(a, b) \otimes X(b, c) \arrow[r, "\mu^X_{abc}"] \arrow[d, "f_{ab} \otimes f_{bc}"'] & X(a, c) \arrow[d, "f_{ac}"] \\
Y(fa, fb) \otimes Y(fb, fc) \arrow[r, "\mu^Y_{fa\,fb\,fc}"'] & Y(fa, fc)
\end{tikzcd}$$

$$\begin{tikzcd}
 & X(a, a) \arrow[dd] \\
k \arrow[ur, "\eta^X_a"] \arrow[dr, "\eta^Y_{fa}"'] & \\
 & Y(fa, fa)
\end{tikzcd}$$

Thus an $\mathcal{S}$-functor is just an ordinary functor between (small) categories, an Ab-functor is just an additive functor (e.g. a ring homomorphism in case X has only one object) while a **2** functor (satisfying

$$X(a, b) \vdash Y(fa, fb)\ )$$

is an arbitrary order-preserving mapping from the poset X to the poset Y, and an **R**-functor (satisfying

$$X(a, b) \geq Y(fa, fb))$$

is an arbitrary distance-decreasing map (Lipschitz map of Lipschitz constant ≤ 1) from the metric space X to the metric space Y.

If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are $\mathcal{V}$-functors we can define $(gf)_{ab}$ as the composition

$$X(a, b) \xrightarrow{f_{ab}} Y(fa, fb) \xrightarrow{g_{fa,fb}} Z(gfa, gfb)$$

in $\mathcal{V}$ to obtain a $\mathcal{V}$-functor $X \xrightarrow{gf} Z$ and thus a category $\mathcal{V}$-Cat whose objects and morphisms are all the $\mathcal{V}$-categories and $\mathcal{V}$-functors. In the next section we will see that $\mathcal{V}$-Cat (e.g. the category of metric spaces and distance-decreasing maps) is itself a closed category.

A morphism between closed categories $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$ is basically the concentrated expression of a process which takes every $\mathcal{V}$-category X into a $\mathcal{W}$-category ΦX with the same objects, every $\mathcal{V}$-functor f into a $\mathcal{W}$-functor Φf and in general interprets $\mathcal{V}$-valued category-theory as $\mathcal{W}$-valued category-theory; in view of many examples it is too restrictive to require that Φ « strictly » preserves the tensor product and Hom, so we adopt the following definition. A closed functor is a triple consisting of a functor $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$, a $\mathcal{W}$-morphism

$$k_{\mathcal{W}} \xrightarrow{\varphi_0} \Phi\left(k_{\mathcal{V}}\right)$$

and a natural transformation

$$\Phi(u) \underset{\mathcal{W}}{\otimes} \Phi(v) \xrightarrow{\varphi_{uv}} \Phi(u \underset{\mathcal{V}}{\otimes} v)$$

subject to compatibility between themselves and with the given symmetry, associativity and unitary isomorphisms in $\mathcal{V}$ and $\mathcal{W}$ (Eilenberg-Kelly).

For example, any locally small closed category $\mathcal{V}$ has a canonical closed functor

$$\mathcal{V} \xrightarrow{V} \mathcal{S}$$

to the category of abstract sets defined by

$$V(u) = \text{set of all } \mathcal{V}\text{-morphisms } k \to u$$
$$1 \xrightarrow{\varphi_0} V(k) = \text{the identity } \mathcal{V}\text{-morphism } k \to k$$

with

$$V(u_1) \times V(u_2) \xrightarrow{\varphi_{u_1 u_2}} V(u_1 \otimes u_2)$$

the mapping taking any ordered pair $k \xrightarrow{f_1} u_1$, $k \xrightarrow{f_2} u_2$ into

$$k \xrightarrow{\cong} k \otimes k \xrightarrow{f_1 \otimes f_2} u_1 \otimes u_2 .$$

If $\mathcal{M}$ is a Boolean ring of sets made into a closed category with union as tensor as above then a closed functor

$$\mathcal{M} \xrightarrow{M} \mathbf{R}$$

is any order-preserving real-valued function on $\mathcal{M}$ satisfying

$$0 = M(\emptyset)$$
$$M(a) + M(b) \geq M(a \cup b)$$

i.e. an « outer measure ». The closed functors $\mathbf{R} \xrightarrow{\lambda} \mathbf{R}$ are just the « subadditive » order-preserving functions, i.e.

$$u_1 \geq u_2 \implies \lambda\, u_1 \geq \lambda\, u_2$$
$$0 = \lambda\, 0$$
$$\lambda\, u_1 + \lambda\, u_2 \geq \lambda\,(u_1 + u_2) .$$

If $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$ is any closed functor and X any $\mathcal{V}$-category, then

$$(\Phi X)(a, b) = \Phi(X(a, b))$$

$$\begin{tikzcd}
(\Phi X)(a,b) \underset{\mathcal{W}}{\otimes} (\Phi X)(b,c) \arrow[r, "(\Phi\,\mu)_{abc}"] \arrow[d, "\varphi"] & (\Phi X)(a,c) \arrow[d, equal] \\
\Phi(X(a,b) \underset{\mathcal{V}}{\otimes} X(b,c)) \arrow[r, "\Phi(\mu_{abc})"'] & \Phi(X(a,c))
\end{tikzcd}$$

$$\begin{tikzcd}
k_{\mathcal{W}} \arrow[rr, dashed, "(\Phi\eta)_a"] \arrow[dr, "\varphi_0"'] & & (\Phi X)(a,a) \\
& \Phi(k_{\mathcal{V}}) \arrow[ur, "\Phi(\eta_a)"'] &
\end{tikzcd}$$

defines a $\mathcal{W}$-category ΦX with the same set of objects as X. Moreover if $X \xrightarrow{f} Y$ is any $\mathcal{V}$-functor by defining Φf to be the same mapping as f on objects and $(\Phi f)_{ab} = \Phi(f_{ab})$ we obtain a $\mathcal{W}$ — functor $\Phi X \to \Phi Y$, and moreover this process is compatible with composition of $\mathcal{V}$-functors (and more). For example, any $\mathcal{V}$-category X has an underlying $\mathcal{S}$-category VX because of the closed functor $\mathcal{V} \xrightarrow{V} \mathcal{S}$.

In particular if $\mathcal{V}$ is small and $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$ is a closed functor, then $\Phi\ \mathcal{V}$ is a $\mathcal{W}$-category. As an example, any Boolean ring of sets $\mathcal{M}$ equipped with an outer measure $\mathcal{M} \xrightarrow{M} \mathbf{R}$ becomes a metric space.

If $\mathbf{R} \xrightarrow{\lambda} \mathbf{R}$ is a closed functor X and Y are metric spaces, then an $\mathbf{R}$-functor

$$\lambda\ X \xrightarrow{f} Y$$

is just a function satisfying

$$\lambda\,(X(a,b)) \geq Y(fa, fb)\,.$$

Since multiplication by a given constant is subadditive and thus a closed functor, we see that the study of (a very natural generalization of) arbitrary Lipschitz mappings $X \to Y$ is naturally incorporated into the functorial set-up.

A special class of closed categories are those in which the tensor product is actually the categorical product, which amounts to

$$\frac{u \longrightarrow v_1 \otimes v_2}{u \longrightarrow v_1\,,\ u \longrightarrow v_2}$$

further « rule of inference of propositional Logic »; such closed categories are usually called cartesian closed. $\mathcal{S}$ and **2** are cartesian closed, while Ab and **R** are not. If a category admits a cartesian closed structure, then that structure is essentially unique. Ab does not admit a cartesian closed structure, but the underlying category of **R** does admit the cartesian closed structure $\mathbf{R}_{cart}$ in which

$$u \otimes v \quad = \quad \max\,(u\,,\,v)$$

$$\mathrm{Hom}\,(a, v) \;=\; \begin{cases} v & v > a \\ 0 & a \geq v \end{cases}$$

An $\mathbf{R}_{cart}$-category is just an *ultrametric space*, so that all of our general results may be reinterpreted as applying to ultrametric spaces. Moreover, the identity mapping $\mathbf{R}_{cart} \to \mathbf{R}$ is a closed functor (i.e. $a + v \geq \max(a, v)$) which induces the inclusion

$$\mathbf{R}_{cart}\text{-Cat} \to \mathbf{R}\text{-Cat}$$

of ultrametric spaces into all metric spaces.

2. - Functor Categories, Yoneda embedding, adequacy, comprehension scheme.

If A and X are two $\mathcal{V}$-categories then there is a $\mathcal{V}$-category $A \times X$ whose objects are the ordered pairs $\langle a, x\rangle$ of objects and whose hom-values are

$$(A \times X)\,(\langle a, x\rangle\,,\,\langle a', x'\rangle) \;=\; A\,(a\,,a') \times X\,(x, x')$$

the last being the cartesian product (e.g. max. in the case of **R**) in $\mathcal{V}$. But if $\mathcal{V}$ is not cartesian closed, there is another more important $\mathcal{V}$-category structure on the same objects, defined by

$$(A \otimes X)\,(\langle a, x\rangle,\langle a', x'\rangle) \;=\; A\,(a\,,a') \otimes X\,(x, x')\,.$$

This gives, for example, the l_1-style metric on the product of two metric spaces. The unit k for this tensor product has one object $*$ with $k(*\,,*) = k$.

This tensor product of $\mathcal{V}$-categories always has a « Hom » adjoint to it, making $\mathcal{V}$-cat itself into a closed category. This Hom always has a concrete interpretation in terms of « strong natural transformations »; we are going to denote it by Y^A, and its objects are just all the $\mathcal{V}$-functors $A \to Y$. Recall that in the usual $\mathcal{S}$-valued case, a natural transformation $f_1 \to f_2$ (where $A \xrightarrow{f_1, f_2} Y$) is a family of Y-morphisms indexed by the objects of A, more exactly, an element of the set $\prod_{a\in A} Y(f_1 a, f_2 a)$ subject to equations indexed by the elements of the $A(a, b)$. In the $\mathcal{V}$-valued case, we define directly (not the set of but) the $\mathcal{V}$-object of natural transformations $Y^A(f_1, f_2)$ by the equalizer in $\mathcal{V}$ of the two morphisms

$$\prod_a \rightrightarrows \prod_{a,b}$$

in

$$Y^A(f_1, f_2) \to \prod_{a\in A} Y(f_1 a, f_2 a) \rightrightarrows \prod_{a,b\in A} \operatorname{Hom}(A(a,b),\, Y(f_1 a, f_2 b))$$

whose constructions we leave to the reader. A $\mathcal{V}$-morphism $k \to Y^A(f_1, f_2)$ may then be identified with a strong natural transformation $f_1 \to f_2$. In case $\mathcal{V}$ is itself a poset, as with $\mathcal{V} = \mathbf{R}$, the two morphisms are already equal so that $Y^A(f_1, f_2) = \prod_{a\in A} Y(f_1 a, f_2 a)$ in such a case. Since $\prod$ in $\mathbf{R}$ means sup, we deduce that

$$Y^A(f_1, f_2) = \sup_{a\in A} Y(f_1 a, f_2 a)$$

in the case of metric spaces, i.e. that the sup metric on the space of 1-Lipschitz maps is a special case of the general notion of $\mathcal{V}$-natural transformations. The reader should be able to verify that there is a bijection

$$\frac{A \otimes X \to Y}{X \to Y^A}$$

between the two indicated sets of $\mathcal{V}$-functors at least in the case $\mathcal{V} = \mathbf{R}$.

Of special interest is the case $\mathcal{V}A^{op}$, because of the existence of the basic Yoneda embedding

$$A \to \mathcal{V}^{A^{op}}$$

which is the $\mathcal{V}$-functor whose value at an object a of A is the $\mathcal{V}$-functor $A^{op} \to \mathcal{V}$ defined by

$$a' \rightsquigarrow A(a', a)$$

and which works on A-morphism objects precisely by use of A-composition. Saying that a $\mathcal{V}$-functor $A \xrightarrow{f} Y$ is $\mathcal{V}$-full-and-faithful iff each of the $\mathcal{V}$-morphisms

$$A(a, b) \xrightarrow{f_{ab}} Y(fa, fb)$$

is actually an isomorphism, we have the important *Yoneda Lemma* (Eilenberg-Kelley): The Yoneda embedding is, for any closed $\mathcal{V}$ and any $\mathcal{V}$-category A, a $\mathcal{V}$-full-and-faithful $\mathcal{V}$-functor.

As a simple example, note that for $\mathcal{V} = \mathbf{2}$, an order-preserving map $A^{op} \to \mathbf{2}$ is equivalent to an order-ideal in A, and the Yoneda embedding is simply Dedekind's representation of a poset by its principal ideals. In the case $\mathcal{V} = \mathcal{S}$ or $\mathcal{V} = k$-Modules, the Yoneda embedding is often (especially when A has one object) called the regular representation of A, and Yoneda's Lemma includes Cayley's theorem on representing an abstract group by transformations. For a metric space A and for each point a, the function assigning to any point a' its distance to a is a distance decreasing function, and Yoneda's Lemma states that assigning to each a the just-described function is an *isometric* embedding of A into the space $\mathbf{R}\, A^{op}$ where the last is equipped with the sup metric.

More generally, given any $\mathcal{V}$-functor $A \to X$, we can consider the Yoneda representation of X restricted to A, i.e. the composite $\mathcal{V}$-functor

$$X \to \mathcal{V}^{X^{op}} \xrightarrow{\mathcal{V}^{i^{op}}} \mathcal{V}^{A^{op}}$$

which assigns to each x the functor $A^{op} \to \mathcal{V}$ defined by $a' \rightsquigarrow X(ia', x)$. In case this restricted representation is still $\mathcal{V}$-full-and-faithful on all X, we say after Isbell that i is *$\mathcal{V}$-adequate*, or in case i is an inclusion that A is a $\mathcal{V}$-adequate subcategory of X. This concept is the basis of much representation theory of categories, especially for algebraic categories and topoi, since it often happens that a quite small category A is adequate in a quite large one X. (We hasten to point out that this paragraph is meaningful even when $\mathcal{V}$ is not small; for example $\mathcal{V}\, A^{op}$ can be interpreted in terms of « modules » as in the following section). For example, if $\mathcal{V}$ = abelian groups and A is any ring, then A (considered as a $\mathcal{V}$-category with only one object) is adequate in any category X of A-modules, no matter how large.

In particular, to say that a subspace A of a metric space X is adequate is to say that for any two points x_1, x_2 of X, the inequality

$$X(x_1, x_2) \geq \sup_{a \in A} [X(a, x_2) - X(a, x_1)]$$

is actually an equality, i.e. that for any $d > 0$ there exists $a \in A$ with

$$X(a, x_1) + X(x_1, x_2) \leq X(a, x_2) + d .$$

For example, the unit circle is adequate in the unit disc (this simple example was pointed out to me by Prof. Isbell). A more restricted notion is that of $\mathcal{V}$-density, by which we mean (here differing in terminology with some authors) that $i_* o i^* \cong 1_X$ in the sense of the next section on bimodules; this reduces in the case of metric spaces to the requirement that

$$X(x_1, x_2) = \inf_{a \in A} [X(x_1, ia) + X(ia, x_2)] .$$

Proposition. - If a distance-decreasing map $A \xrightarrow{i} X$ of metric spaces is $\mathbf{R}$-dense, then it is $\mathbf{R}$-adequate.

Proof. - We always have

$$X(a, x_1) + X(x_1, x_2) \leq X(a, x_1) + X(x_1, a) + X(a, x_2) .$$

But by taking $x_2 = x_1$ in the definition of density, we see that $X(a, x_1) + X(x_1, a) \leq d$ *f*or suitable $a \in A$, as required.

Define a particular metric space A whose points are the natural numbers and for which

$$A(n, m) = \begin{cases} \infty & n \neq m \\ 0 & n = m \end{cases} .$$

Then $\mathbf{R}\, A^{op}$ is just the space of all sequences of nonnegative reals, with the (nonsymmetric) sup metric. Say that a metric space X is *separable* iff there exists an $\mathbf{R}$-dense map $A \rightarrow X$. Then

Corollary. - Any separable metric space X can be isometrically embedded in the space of all sequences of non-negative reals with the sup metric.

To discuss the « comprehension scheme » we will limit ourselves to those closed categories $\mathcal{V}$ in which $K = 1$, i.e. in which the unit object for the tensor product is also the terminal object of $\mathcal{V}$; thus for this form of the comprehension scheme, $\mathcal{V}$ may be cartesian closed, e.g. $\mathcal{S}$ or $\mathbf{2}$, but more generally e.g. $\mathbf{R}$ satisfies this condition, as does the category of so-called « affine modules » over a given commutative ring (but not the usual category of modules). This has the effect that for any $\mathcal{V}$-category E, there is a canonical « augmentation » $E(e, e') \to K$. Thus we may simply consider the category $\mathcal{V}$-Cat$/B$ of all $\mathcal{V}$-categories equipped with with a $\mathcal{V}$--functor with codomain B, and compare it with the category $\mathcal{V}^B$ of all $\mathcal{V}$-functors with domain B and the fixed codomain $\mathcal{V}$. Namely, given any $E \xrightarrow{p} B$, we define $\varphi_p : B \to \mathcal{V}$ by the coequalizer diagram in $\mathcal{V}$

$$\sum_{e,e'} E(e, e') \otimes B(p(e'), b) \rightrightarrows \sum_{e \in E} B(p(e), b) \to \varphi_p(b)$$

where one of the two morphisms in induced by the functor p followed by composition in B, while the other is induced by the augmentation. In the case $\mathcal{V} = \mathbf{R}$, this simply means that for any distance-decreasing map p from a metric space E to the fixed metric space B, we define a real-valued function on B by

$$\varphi_p(b) = \inf_e B(p(e), b)$$

i.e. the distance from the image of p to the variable point b.

In the case $\mathcal{V} = \mathbf{2}$, φ_p is the order-preserving map $B \to \mathbf{2}$ defined by

$$\varphi_p(b) = \text{true} \quad \text{iff } \exists\, e\, [p(e) \text{ dominates } b \text{ in } B]$$

In the case $\mathcal{V} = \mathcal{S}$, $\varphi_p(b)$ may be interpreted as the set of components of the category p/b whose objects are pairs e, β with $p(e) \xrightarrow{\beta} b$ in B and whose morphisms are morphisms $e \xrightarrow{\xi} e'$ in E such that $\beta = p(\xi)\,\beta'$ in B.

The « comprehension scheme » then refers to the right adjoint of the functor

$$\mathcal{V}\text{—}Cat/B \to \mathcal{V}^B$$

defined by $p \rightsquigarrow \varphi_p$. Namely, given any $B\varphi \to \mathcal{V}$, define a category $\{B|\varphi\}$ whose objects are pairs $\langle b, x \rangle$ such that $k \xrightarrow{x} \varphi(b)$ in $\mathcal{V}$

and for which $\{B|\varphi\}$ $(\langle b, x\rangle, \langle b', x'\rangle)$ is defined by the following pullback diagram in $\mathcal{V}$

$$\begin{array}{ccc}
\{B\mid\varphi\}\ (\langle b,x\rangle,\langle b',x'\rangle) & \xrightarrow{\quad\quad n \quad\quad} & k \\
\pi\ xx' \downarrow & & \downarrow x' \\
B(b,b') \cong k\otimes B(b,b') & \xrightarrow[x\otimes id]{} \varphi(b)\otimes B(b,b') \longrightarrow & \varphi(b')
\end{array}$$

where $\varphi(b)\otimes B(b, b')\to\varphi(b')$ is the « action » of B on φ adjoint to the functoriality of φ [note that $\mathcal{V}$-functors with codomain $\mathcal{V}$ always have an equivalent interpretation as « right modules » over the domain category]. The morphism n is unique since we have assumed $k=1$, but without that assumption it would provide the $\mathcal{V}$-category $\{B|\varphi\}$ with an « augmentation » structure in addition to the structural functor π. In the case $\mathcal{V}=\mathcal{S}$, $\{B|\varphi\}$ is sometimes called the category of all elements of φ, and the functor $\{B|\varphi\}\xrightarrow{\pi}B$ is the discrete fibration corresponding to the set-valued functor φ. In case $\mathcal{V}=\mathbf{2}$

$$\{B\mid\varphi\} = \{\, b\in B\mid\varphi(b) = \text{true}\,\}$$

(accounting for the terminology « comprehension scheme ») and we have of course

$$\frac{Im(p)\subseteq\{B\mid\varphi\}}{\varphi_p\vdash\varphi}$$

But in the case $\mathcal{V}=\mathbf{R}$, we have for any « quantity-valued propositional function » φ on the metric space B that

$$\{B\mid\varphi\} = \{\, b\in B\mid 0 = \varphi(b)\}$$

and for any $E\xrightarrow{p}B$,

$$\frac{Im(p)\subseteq\{B\mid\varphi\}}{\inf B(p(e),b)\leq\varphi(b),\ \text{all } b}\ .$$

To what extent are objects in one of the categories $\mathcal{V}$-Cat$/B$, $\mathcal{V}^B$ « equivalent » to objects in the other via this adjoint pair? For $\mathcal{V}=\mathcal{S}$, $\mathbf{2}$, $\mathbf{R}$ respectively, $E\xrightarrow{p}B$ must be respectively a discrete fibration, the inclusion of an order-ideal, the inclusion of a *closed* subset in order to have $p\cong\{|\varphi_p\}$. On the other side, for $\mathcal{V}=\mathbf{R}$ there are distance decreasing maps $B\to\mathbf{R}$ *not* of the form « distance from a certain closed set », but for $\mathcal{V} = \mathcal{S}$ or $\mathcal{V} = \mathbf{2}$ *every* $\varphi \in \mathcal{V}^B$ is of the form φ_p.

3. - Bimodules, Kan quantification, Cauchy completeness.

For any $\mathcal{V}$-category Y, we have the canonical Yoneda embedding $Y \to \mathcal{V}^{Y^{op}}$ which means in particular that we can consider that the concept of a $\mathcal{V}$-functor $X \to \mathcal{V}^{Y^{op}}$ is a generalization of the concept of a $\mathcal{V}$-functor $X \to Y$. Such a generalized $\mathcal{V}$-functor $X \mapsto Y$ is equivalent to a $\mathcal{V}$-functor $Y^{op} \otimes X \xrightarrow{\varphi} \mathcal{V}$, and may be considered as a « $\mathcal{V}$-valued relation » from X to Y, $\varphi(y, x)$ being the « truth-value of the φ-relatedness of y to x ». In particular, every $\mathcal{V}$-functor $X \xrightarrow{f} Y$ thus yields a $X \overset{f_*}{\mapsto} Y$ defined by

$$f_*(y, x) = Y(y, f(x))$$

but also yields $\quad Y \overset{f^*}{\mapsto} X$ defined by

$$f^*(x, y) = Y(f(x), y) .$$

We first give an alternate description, without recourse to the notion of $\mathcal{V}$-functor, of such $\mathcal{V}$-valued relations as *bimodules*. We are using, by the way, the notational convention that inside a $\mathcal{V}$-category composition is written from left to right, while composition of $\mathcal{V}$-functors and of bimodules (= « relations ») is written from right to left.

If X, Y are $\mathcal{V}$-categories, a *bimodule* $X \overset{\varphi}{\mapsto} Y$ (also called a right X, left Y-module) consists of a family $\varphi(y, x)$ of objects of $\mathcal{V}$ indexed by the objects of Y and X together with morphisms

$$Y(y', y) \otimes \varphi(y, x) \to \varphi(y', x)$$

$$\varphi(y, x) \otimes X(x, x') \to \varphi(y, x')$$

in $\mathcal{V}$ which behave as « actions » in the sense that the five axioms (commutative diagrams in $\mathcal{V}$) of X-unity, X-associativity, Y-unity, Y-associativity, and mixed associativity (= commutativity of X-action with Y-action) hold. Thus in case $\mathcal{V} = Ab$, this is just the usual notion of bimodule suitably extended to « rings » X and Y with more than one object. In case $\mathcal{V} = \mathcal{S}$, bimodules are sometimes called « profunctors » or « distributeurs ». In case $\mathcal{V} = \mathbf{R}$, a bimodule $X \overset{\varphi}{\mapsto} Y$ between two metric spaces is just a real-valued function on the product $Y \times X$ satisfying

$$\inf_y \left[Y(y', y) + \varphi(y, x) \right] \geq \varphi(y', x)$$

$$\inf_x \left[\varphi(y, x) + X(x, x') \right] \geq \varphi(y, x') .$$

The latter is clearly equivalent to putting a metric on the sum $X + Y$ extending the given metrics and having $d(x, y) = \infty$ for $x \in X$, $y \in Y$, and indeed this alternate mode of description works generally, recalling that the infinite quantity corresponds to the empty coproduct in $\mathcal{V}$.

If $X \overset{\varphi}{\mapsto} Y \overset{\psi}{\mapsto} Z$ are bimodules, then composition $X \overset{\psi \circ \varphi}{\mapsto} Z$ is defined by the following coequalizer diagram in $\mathcal{V}$

$$\sum_{y_1, y_2} \psi(z, y_1) \otimes Y(y_1, y_2) \otimes \varphi(y_2, x) \rightrightarrows \sum_{y \in Y} \psi(z, y) \otimes \varphi(y, x) \longrightarrow (\psi \circ \varphi)(z, x)$$

Thus in case $\mathcal{V}$ is itself a poset, $\psi \circ \varphi$ reduces to a « matrix product »

$$\sum_{y \in Y} \psi(z, y) \otimes \varphi(y, x)$$

where the sigma denotes coproduct in $\mathcal{V}$, but in general we take the quotient of the matrix product modulo the discrepancy between the two actions of Y in the middle, just as in the familiar case $\mathcal{V} = Ab$ where one often writes more explicitly

$$\psi \circ \varphi = \psi \underset{Y}{\otimes} \varphi .$$

In particular for $\mathcal{V} = \mathbf{2}$

$$(\psi \circ \varphi)(z, x) = \exists\, y \left[\psi(z, y) \wedge \varphi(y, x) \right]$$

is the usual relational product, while for $\mathcal{V} = \mathbf{R}$ we have

$$(\psi \circ \varphi)(z, x) = \inf \left[\psi(z, y) + \varphi(y, x) \right] .$$

It can be verified in general that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are $\mathcal{V}$-functors, then

$$(gf)_* \cong g_* \circ f_*$$

where isomorphism of bimodules has an obvious sense; also 1_X defined by

$$1_X(x, x') = X(x, x')$$

plays the role of identity with respect to composition with any bimodules (not only those of the form f_*). Further, the composition of bimodules is associative up to (coherent) isomorphism, but in general not commutative. Note that the endobimodules of the one-object category k are in correspondence with the objects of $\mathcal{V}$, with the composition of these reducing to the tensor product in $\mathcal{V}$.
The concept Hom, right adjoint to $\otimes$ in $\mathcal{V}$ has two extensions to mixed Hom of bimodules, right adjoint on opposite sides to the composition of bimodules. That is, given any three $\mathcal{V}$-categories X, Y, Z and a bimodule $X \overset{\varphi}{\mapsto} Z$, there are two universal problems: given any $X \overset{\alpha}{\mapsto} Y$, there is a $Y \mapsto Z$ denoted by $\mathrm{Hom}^X(\alpha, \varphi)$ which is « best » in the sense that for any bimodule $Y \overset{\beta}{\mapsto} Z$ there is a natural bijection of morphisms of bimodules

$$\frac{\beta \longrightarrow \mathrm{Hom}^X(\alpha, \varphi)}{\beta \circ \alpha \longrightarrow \varphi}$$

likewise for given β there is a « best α », denoted by $\mathrm{Hom}_Z(\beta, \varphi)$ satisfying

$$\frac{\alpha \longrightarrow \mathrm{Hom}_Z(\beta, \varphi)}{\beta \circ \alpha \longrightarrow \varphi}$$

for any α.

Explicitly, $\mathrm{Hom}^X(\alpha, \varphi)$ is defined by an equalizer diagram in $\mathcal{V}$

$$\mathrm{Hom}^X(\alpha, \varphi)(z, y) \longrightarrow \prod_{x \in X} \mathrm{Hom}(\alpha(y, x)\varphi, (z, x) \rightrightarrows \prod_{x_1, x_2} \mathrm{Hom}(X(x_1, x_2), \mathrm{Hom}(\alpha(y, x_1), \varphi(z, x_2)))$$

and similarly $\mathrm{Hom}_Z(\beta, \varphi)(y, x)$ is defined by an equalizer condition as a subobject of

$$\prod_{z \in Z} \mathrm{Hom}(\beta(z, y), \varphi(z, x)).$$

For *some* bimodules $X \overset{\alpha}{\mapsto} Y$, the operation of composing with α, $\beta \rightsquigarrow \beta \circ \alpha$, has not only the right adjoint $\varphi \rightsquigarrow \mathrm{Hom}^X(\alpha, \varphi)$ but has also a *left* adjoint. The typical such bimodule α is one of the form

$\alpha = f_*$, where $X \xrightarrow{f} Y$ is a $\mathcal{V}$-functor; presently we will also discuss the extent to which « all » such bimodules α are induced by functors f.

Lemma. - For any $\mathcal{V}$-functor $X \xrightarrow{f} Y$ and any $\mathcal{V}$-bimodule $Y \overset{\beta}{\mapsto} Z$, there is a natural isomorphism

$$\beta \circ f_* = \mathrm{Hom}^Y(f^*, \beta)$$

of bimodules $X \mapsto Z$.

Proof. - One verifies that both sides are naturally isomorphic to the bimodule whose typical component is

$$\beta(z, f(x)) .$$

Theorem. - For $\alpha = f_*$, composition $\beta \rightsquigarrow \beta \circ \alpha$ has as left adjoint the operation of composing with f^*

$$\varphi \rightsquigarrow \varphi \circ f^*$$

Proof. -

$$\frac{\dfrac{\varphi \circ f^* \longrightarrow \beta}{\varphi \longrightarrow \mathrm{Hom}^Y(f^*, \beta)}}{\varphi \longrightarrow \beta \circ f_*}$$

Corollary. - (Kan quantification). For any $\mathcal{V}$-functor $X \xrightarrow{f} Y$, the functor « composition with f »

$$\mathcal{V}^Y \xrightarrow{\mathcal{V}^f} \mathcal{V}^X$$

has both left and right adjoints. On $\varphi \in \mathcal{V}^X$, the left adjoint gives as a result the functor $\varphi \circ f^*$, whose value at y is the quotient of

$$\sum_X \varphi(x) \otimes Y(f(x), y)$$

« modulo the distinction between the two actions of X in the middle », while the right adjoint gives as a result the functor $\mathrm{Hom}^X(f_*, \varphi)$ whose value at y is that subobject of

$$\prod_X \mathrm{Hom}(Y(y, f(x)), \varphi(x))$$

« on which the two actions of X agree ». In the special case that f

is $\mathcal{V}$-full-and-faithful, $\mathcal{V}^f$ is also full and faithful, or equivalently the two Kan quantifications are actually *extensions*, in the sense that

$$(\varphi \circ f^*) \circ f_* \cong \varphi$$
$$\mathrm{Hom}^X(f_*, \varphi) \circ f_* \cong \varphi$$

for any φ.

Proof. - The corollary is for the most part just a summary of the preceding discussion for the special case $Z = k$ (Indeed the corollary remains valid if $\mathcal{V}$ is replaced by $\mathcal{V}^{Z^{op}}$ for any (small) $\mathcal{V}$-category Z).

The assertion that, under the assumption that f is $\mathcal{V}$-full-and-faithful, first « extending » φ along f (in either of the two adjoint ways) and then « restricting » along f gives again φ, is proved in very much the same way as the Lemma above.

Corollary. - Let $X \xrightarrow{f} Y$ be an isometric embedding of a metric space X into a metric space Y. Then any Lipschitz function $X \xrightarrow{\varphi} \mathbf{R}$ can be extended to Y, with the same Lipschitz constant λ. In fact, among all extensions there is a largest one and a smallest one, given respectively by

$$\overline{\varphi}(y) = \inf_X \, [\varphi(x) + \lambda \; Y(f(x), y)]$$
$$\underline{\varphi}(y) = \sup_X \, [\varphi(x) - \lambda \; Y(y, f(x))] .$$

Proof. - Apply the preceding discussion to the diagram

$$\begin{tikzcd}
\lambda X \arrow[r, "\lambda f"] \arrow[dr, "\varphi"'] & \lambda Y \arrow[d, dashed, shift left] \arrow[d, dashed, shift right] \\
& \mathcal{V}
\end{tikzcd}$$

noting that λf as a function is the same as the inclusion f.

As an example of the last corollary, we could take for X the space of nonnegative step functions on a probability space S and for Y the space of all nonnegative functions with the natural sup metric, (so that $Y(y_2, y_1) = 0$ iff $y_1 \geq y_2$), and consider as φ the elementary integral; thus in general we might call y φ-integrable if $\overline{\varphi}(y) \cong \underline{\varphi}(y)$.

The (non standard) name Kan quantification was suggested by the case $\mathcal{V} = \mathbf{2}$, in which the adjointness rules reduce to the usual rules of inference for quantification, φ being thought of as property or relation, i.e.

$$(\varphi \circ f^*)(z, y) \equiv \exists\, x\,[\varphi(z, x) \wedge f(x) \geq y]$$

$$\mathrm{Hom}^X(f_*, \varphi)(z, y) \equiv \forall\, x\,[y \geq f(x) \Rightarrow \varphi(z, x)]$$

But for other choices of $\mathcal{V}$, induced representations and relatively free universal algebras can be shown to arise as special cases of these two constructions.

The essential property of a bimodule of the form f_* is that there exists another bimodule f^* which is right adjoint to it, in the sense that there are morphisms

$$1_X \longrightarrow f^* \circ f_*$$

$$f_* \circ f^* \longrightarrow 1_Y$$

satisfying the usual two adjunction equations.

Proposition. - In order that a metric space Y be Cauchy-complete, it is necessary and sufficient that every adjoint pair of bimodules

$$X \underset{f^*}{\overset{f^*}{\rightleftarrows}} Y$$

be induced by a $\mathcal{V}$-functor $X \xrightarrow{f} Y$.

Proof. - It suffices to consider $X = 1$ and show that an adjoint pair of $\mathbf{R}$-valued bimodules $1 \leftrightarrows Y$ is essentially just a point in the completion of Y. But adjointness just means that

$$X(x, x') \geq \inf_y \,[f^*(x, y) + f^*(y, x')]$$

$$\inf_X \,[f_*(y, x) + f^*(x, y')] \geq Y(y, y')$$

in addition to the bimodule property for each of f_*, f^*. In case $X = 1$, we have then

$$0 = \inf_y \,[f^*(y) + f_*(y)]$$

$$f_*(y) + f^*(y') \geq Y(y, y').$$

Thus for each n we can choose y_n satisfying, for example

$$f^*(y_n) + f_*(y_n) \leq \frac{1}{n}$$

and then

$$Y(y_n, y_m) \leq \frac{1}{n} + \frac{1}{m}$$

so that we have a Cauchy sequence, and any other choice y'_n satisfying the same condition would have

$$Y(y_n\, y'_n) \leq \frac{2}{n}$$

i.e. would be an equivalent Cauchy sequence. Conversely, any equivalence class of Cauchy sequences yields an adjoint pair of bimodules by the definition

$$f^*(y) = \lim_{n \to \infty} Y(y_n, y)$$

$$f_*(y) = \lim_{n \to \infty} Y(y, y_n).$$

It can be shown that the suggested definition of « $\mathcal{V}$-Cauchy-completeness » means in case $\mathcal{V} = Ab$ that a « point of the completion » of an additive category Y is simply any finitely generated projective module over Y, while in case $\mathcal{V} = \mathcal{S}$, it means that Y is Cauchy-complete iff all idempotents in Y split in Y.

4. - Free $\mathcal{V}$-categories.

Defining a $\mathcal{V}$-graph to be a pair consisting of any set X and any $X \times X$-indexed family of objects of $\mathcal{V}$, and a morphism $\langle X, \gamma \rangle \xrightarrow{f} \langle Y, \delta \rangle$ to be any pair consisting of a mapping $X \to Y$ and any family

$$\gamma(x, x') \xrightarrow{f_{x,x'}} \delta(f(x), f(x'))$$

of morphisms in $\mathcal{V}$, we have an obvious forgetful functor

$$\mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Graph}.$$

This forgetful functor has a left-adjoint called taking the free $\mathcal{V}$-category generated by a $\mathcal{V}$-graph. Explicitly, the free category has as objects the vertices of the generating graph, and as hom from x to x' has the coproduct in $\mathcal{V}$

$$\sum_{x_1 \dots, x_n} \gamma(x, x_1) \otimes \gamma(x_1, x_2) \otimes \dots \otimes (x_n, x')$$

over all finite sequences $x_1 \dots x_n$ of vertices. For example with $\mathcal{V} = k$-modules, this formula contains the construction of the tensor algebra over a vector space, or with $\mathcal{V} = \mathcal{S}$, we have as a special case the word algebra ($=$ free monoid) over a given set. For the case $\mathcal{V} = \mathbf{R}$, an $\mathbf{R}$-graph is just an arbitrary assignment γ of quantities to pairs of points, and in the « free metric space » over such, the distance from x to x' is

$$\inf_{x_1 \dots, x_n} [\gamma(x, x_1) + \gamma(x_1, x_2) + \dots\dots + \gamma(x_n, x')]$$

the well-known « least-cost » distance.

The adjointness of the free $\mathcal{V}$-category construction contains the essence of the notion of recursion, especially when one considers it in relation with bimodules, where it leads for example to the iteration of endobimodules.

5. - Further remarks.

We already remarked that the composition of bimodules is similar to matrix multiplication, and indeed the analogy with linear algebra goes further. For example, if bimodules rather than $\mathcal{V}$ functors are considered as the « morphisms » between $\mathcal{V}$-categories, then $A^{op} \otimes Y$ plays the role of « Hom ». Either further developing that remark in the case $A = Y$, or proceding directly, it is natural to define, for any endobimodule $A \xrightarrow{\alpha} A$, $Tr(\alpha)$ to be the object of $\mathcal{V}$ defined by coequalizer

$$\sum_{a, b \in A} A(b, a) \otimes \alpha(a, b) \rightrightarrows \sum_{a \in A} \alpha(a, a) \longrightarrow Tr(\alpha) .$$

The injections $\alpha(a, a) \to Tr(\alpha)$, denoted by tr_a, are a natural generalization of many examples of classical constructions such as trace of endomorphisms or Lefschitz numbers (when $\alpha = 1_A$) as has been verified by A. Kock. In case A is a metric space and $\alpha = f_*$ where f is a distance decreasing endomap,

$$Tr(\alpha) = \inf_a A(a, fa)$$

showing that the vanishing of the trace is related to the existence of fixed points. It seems likely that there may be theorems holding for more general $\mathcal{V}$, relating the trace of an endobimodule to an infinite iteration of it, which would extend the Banach fixed-point theorem.

SUMMARY. — The analogy between $\mathrm{dist}(a, b) + \mathrm{dist}(b, c) \geq \mathrm{dist}(a, c)$ and $\hom(A, B) \otimes \hom(B, C) \to \hom(A, C)$ is rigorously developed to display many general results about metric spaces as consequences of a « generalized pure logic » whose « truth-values » are taken in an arbitrary closed category.

REFERENCE

MACLANE S., *Categories for the Working Mathematician.* Springer 1972.

EILENBERG S. & KELLY G. M., *Closed Categories, in Proceedings of La Jolla Conference on Categorical Algebra.* Springer 1966.

BENABOU JEAN, *Les Distributeurs*, Rapport n° 33, janvier 1973, Seminaires de Mathématiques Pure, Institut de Mathématiques, Université Catholique de Louvain (multigraphed).