LINEAR EQUATIONS FOR UNORDERED DATA VECTORS IN $[\mathcal{D}]^k \to \mathbb{Z}^d$.

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Abstract. Following a recently considered generalisation of linear equations to unordered-data vectors and to ordered-data vectors, we perform a further generalisation to data vectors that are functions from k-element subsets of the unordered-data set to vectors of integer numbers. These generalised equations naturally appear in the analysis of vector addition systems (or Petri nets) extended so that each token carries a set of unordered data. We show that nonnegative-integer solvability of linear equations is in nondeterministic exponential time while integer solvability is in polynomial time.

1. Introduction.

Diophantine linear equations. The solvability problem for systems of linear Diophantine equations is defined as follows: given a finite input set of d-dimensional integer vectors $\mathcal{I} = \{\mathbf{v}_1 \dots \mathbf{v}_m\} \subseteq \mathbb{Z}^d$, and a target vector $\mathbf{v} \in \mathbb{Z}^d$, we ask if there is a solution $(n_1 \dots n_m)$ such that

$$n_1 \cdot \mathbf{v}_1 + \ldots + n_m \cdot \mathbf{v}_m = \mathbf{v}. \tag{1.1}$$

Restricting solutions $n_1 \dots n_m$ to the set $\mathbb{Z}$ of integers or the set $\mathbb{N}$ of nonnegative integers, we may speak of $\mathbb{Z}$-solvability and $\mathbb{N}$-solvability, respectively. The former problem is in **P-Time**, while the latter is equivalent to *integer linear programming*, a well-known **NP**-complete problem [Kar72].

This paper is a continuation of a line of research that investigates generalisations of the solvability problem to data vectors [HLT17, HL18] i.e. on a high level of abstraction, to vectors indexed by orbit-finite sets instead of finite ones (for an introduction to orbit-finite sets, also known as *sets with atoms*, see [BKL11, BKLT13]). In the simplest setting, given a fixed countable infinite set $\mathcal{D}$ of data values, a *data vector* is a function $\mathbf{a}\colon \mathcal{D} \to \mathbb{Z}^d$. Addition and scalar multiplication are defined pointwise i.e. $(\mathbf{a}+\mathbf{a}')(\alpha) = \mathbf{a}(\alpha) + \mathbf{a}'(\alpha)$

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for every $\alpha \in \mathcal{D}$. The solvability problems over input sets $\mathcal{I}$ of data vectors are defined analogously, with the important difference that the input set $\mathcal{I}$ of data vectors is the closure, under data permutations, of a finite set of data vectors. Given such an infinite, but finite up to data permutation set $\mathcal{I}$, and a target data vector $\mathbf{v}$, we ask if there are data vectors $\mathbf{v}_1 \dots \mathbf{v}_m \in \mathcal{I}$ and numbers $n_1 \dots n_m$, such that the equality (1.1) is satisfied.

Example 1.1. We consider data vectors $\mathcal{D} \to \mathbb{Z}$. For every two different data values $\delta, \varepsilon \in \mathcal{D}$, let $\mathbf{v}_{\delta\varepsilon}(x) = 1$ if $x = \delta$ or $x = \varepsilon$, and $\mathbf{v}_{\delta\varepsilon}(x) = 0$ otherwise. Let $\mathcal{I} = \{\mathbf{v}_{\delta\varepsilon} \colon \mathcal{D} \to \mathbb{Z} \mid \delta, \varepsilon \in \mathcal{D}, \delta \neq \varepsilon\}$. The set $\mathcal{I}$ is closed under data permutation. Indeed, for any data permutation (bijection) $\pi \colon \mathcal{D} \to \mathcal{D}$ and any $\mathbf{v}_{\delta\varepsilon} \in \mathcal{I}$ we have $\mathbf{v}_{\delta\varepsilon} \circ \pi = \mathbf{v}_{\pi^{-1}(\delta)\pi^{-1}(\varepsilon)}$ which is also an element of $\mathcal{I}$.

Finally, let $\mathbf{v}_\beta(\beta) = 2$ and $\mathbf{v}_\beta(x) = 0$ for $x \neq \beta$, where $\beta \in \mathcal{D}$ is some fixed data value. On the one hand, this instance admits a $\mathbb{Z}$-solution, since $\mathbf{v}_\beta$ is presentable as

$$\mathbf{v}_\beta = \mathbf{v}_{\beta\delta} + \mathbf{v}_{\beta\varepsilon} - \mathbf{v}_{\delta\varepsilon},$$

for any two different data values δ, ε different than β. On the other hand, there is no $\mathbb{N}$-solution, as there is no similar presentation of $\mathbf{v}_\beta$ in terms of data vectors $\mathbf{v}_{\delta\varepsilon}$ that uses nonnegative coefficients. Simply, every vector that is a sum of of data vectors from the family $\mathcal{I}$ must be strictly positive for at least two data values. Furthermore, if $\mathbf{v}_\beta(\beta) = 3$ instead of 2, then there is no $\mathbb{Z}$-solution, too. Indeed, notice that $\sum_{\alpha \in \mathcal{D}} \mathbf{w}(\alpha)$ is always an even number if $\mathbf{w}$ is a sum of data vectors from the family $\mathcal{I}$. □

The above simple example is covered by the theory developed in [HLT17]. Here, we extend the results from [HLT17] to data vectors in $[\mathcal{D}]^k \to \mathbb{Z}^d$, where $[\mathcal{D}]^k$ stands for the k-element subsets of $\mathcal{D}$. The complexity of analysis of data vectors in $[\mathcal{D}]^k \to \mathbb{Z}^d$ can already be observed for $k = 2$ and $d = 1$.

Example 1.2. Consider data vectors in $[\mathcal{D}]^2 \to \mathbb{Z}$ where $[\mathcal{D}]^2 = \{\{\delta, \varepsilon\} \colon \delta, \varepsilon \in \mathcal{D},\ \delta \neq \varepsilon\}$. You can think of them as weighted graphs with vertices labelled with elements of $\mathcal{D}$. Suppose the set $\mathcal{I}$ is a set of triangles with weights of all edges equal to 1 i.e.

$$\mathbf{v}_{\gamma\delta\varepsilon}(x) = \begin{cases} 1 & \text{if } x \in \{\{\gamma, \delta\}, \{\delta, \varepsilon\}, \{\varepsilon, \gamma\}\} \\ 0 & \text{otherwise} \end{cases}$$

and $\mathcal{I} = \{\mathbf{v}_{\gamma\delta\varepsilon} \colon [\mathcal{D}]^2 \to \mathbb{Z} \mid \gamma, \delta, \varepsilon \in \mathcal{D}, \delta \neq \varepsilon \neq \gamma \neq \delta\}$ (see Figure 1).

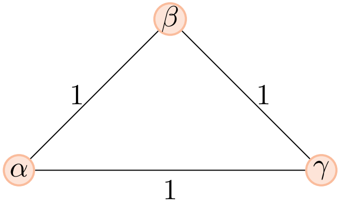


Figure 1: Graph representing the data vector $\mathbf{v}_{\alpha\beta\gamma}$.

The set $\mathcal{I}$ is closed under data permutations, indeed for any data permutation π and any $\mathbf{v}_{\gamma\delta\varepsilon} \in \mathcal{I}$ the data vector $\mathbf{v}_{\gamma\delta\varepsilon} \circ \pi = \mathbf{v}_{\pi^{-1}(\gamma)\pi^{-1}(\delta)\pi^{-1}(\varepsilon)} \in \mathcal{I}$. Finally, we want to know if

$\mathcal{I}$ and the following target data vector $\mathbf{v}_{\gamma\delta}$ admits a $\mathbb{Z}$-solution

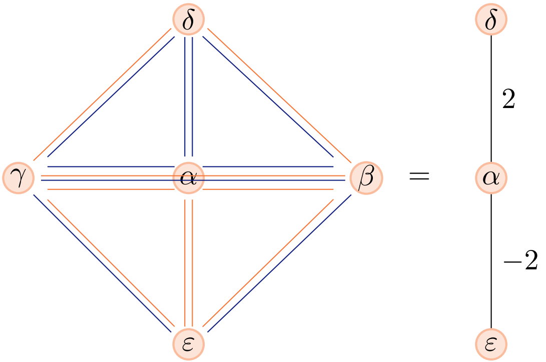
$$\mathbf{v}_{\gamma\delta}(x) = \begin{cases} 6 & \text{if } x = \{\gamma, \delta\} \\ 0 & \text{otherwise} \end{cases}$$

($\mathbf{v}_{\gamma\delta}$ is a single edge with weight 6). The answer is yes, but it is not trivial,

$$\mathbf{v}_{\gamma\delta} = (\mathbf{v}_{\beta\delta\gamma} - \mathbf{v}_{\beta\delta\alpha} + \mathbf{v}_{\delta\gamma\varepsilon} - \mathbf{v}_{\delta\varepsilon\alpha} + \mathbf{v}_{\beta\alpha\varepsilon} - \mathbf{v}_{\beta\gamma\varepsilon}) + (\mathbf{v}_{\delta\beta\gamma} - \mathbf{v}_{\gamma\beta\alpha} + \mathbf{v}_{\delta\varepsilon\gamma} - \mathbf{v}_{\gamma\varepsilon\alpha} + \mathbf{v}_{\varepsilon\beta\alpha} - \mathbf{v}_{\varepsilon\beta\delta}) + 2\mathbf{V}_{\alpha\delta\gamma}$$

where α, β, and ε are any data values different than γ, δ.

The idea behind the above sum is presented in Figure 2.



First we construct a gadget $\mathbf{g}_{\delta\alpha\varepsilon}$ as presented on the left. Blue color denotes addition and orange subtraction of an edge. $\mathbf{g}_{\alpha\delta\varepsilon} = \mathbf{v}_{\alpha\beta\delta} - \mathbf{v}_{\alpha\beta\varepsilon} + \mathbf{v}_{\alpha\gamma\delta} - \mathbf{v}_{\alpha\gamma\varepsilon} + \mathbf{v}_{\gamma\beta\varepsilon} - \mathbf{v}_{\gamma\beta\delta}$ Next, we use such gadgets combined with triangles in the following way $\mathbf{g}_{\gamma\delta\alpha} + \mathbf{g}_{\delta\gamma\alpha} + 2\mathbf{v}_{\alpha\gamma\delta} = \mathbf{v}_{\gamma\delta}$.

Figure 2: Idea behind construction in Example 1.2.

If we ask about $\mathbb{N}$-solvability then the answer is no. If we add triangles then the number of nonzero edges will be greater than 2, so there is no way of reaching $\mathbf{v}_{\gamma\delta}$. But, what if we change 6 to 3? The answer is postponed to Section 3. □

1.1. **Related work and our contribution.** The above-discussed, simplest extension of the $\mathbb{Z}$-solvability problem to data vectors of the form $\mathcal{D} \to \mathbb{Z}^d$ is in **P-Time**, and the $\mathbb{N}$-solvability is **NP**-complete [HLT17]. Further known results concern the more general case of *ordered* data domain $\mathcal{D}$ [HL18]. For ordered data we assume that the set of data forms a dens linear order and the set of data permutations is restricted to the set of order preserving bijections $\mathcal{D} \to \mathcal{D}$. In the ordered data case the $\mathbb{Z}$-solvability problem remains in **P-Time**, while the complexity of the $\mathbb{N}$-solvability is equivalent to the reachability problem of vector addition systems with states (VASS), or Petri nets, and hence Ackermann-complete [Ler21, CO21]. The increase of complexity caused by the order in data is thus remarkable. An example of a result that builds on top of [HLT17] is [GSAH19], where the continuous reachability problem for unordered data nets is shown to be in **P-Time**. The question if the continuous reachability problem is solvable for the ordered data domain remains open. It is particularly interesting due to [BFHRV10], where the coverability problem in timed data nets [AN01] is proven to be interreducible with the coverability problem in ordered data nets.

In this paper, we perform a further generalisation to k-element subsets of unordered data i.e. we consider data vectors of the form $[\mathcal{D}]^k \to \mathbb{Z}^d$, where $[\mathcal{D}]^k$ stands for the k-element subsets of $\mathcal{D}$. We prove two main results: first, for every fixed $k \geq 1$ the complexity of the $\mathbb{Z}$-solvability problem again remains polynomial. Second, we present a **NExp-Time** algorithm for the $\mathbb{N}$-solvability problem. This is done by an improvement of techniques developed in [HLT17]. Namely, we nontrivially extend Theorems 11 and 15 from [HLT17].

- To address $\mathbb{N}$-solvability, we reprove Theorem 11 from [HLT17] in a more general setting (the proof is slightly modified). Next, we combined it with new idea to obtain a reduction to the (easier) $\mathbb{Z}$-solvability, witnessing a nondeterministic exponential blowup.
- Our approach to $\mathbb{Z}$-solvability is an extension of Theorem 15 from [HLT17]. Precisely, if we reformulate the $\mathbb{Z}$-solvability question in terms of (weighted) hypergraphs, then there is a natural way to lift the characterisation of $\mathbb{Z}$-solvability proposed in Theorem 15 [HLT17]. The main contribution of this paper is a new tool-box developed to prove the lifted theorem 15 from [HLT17]. The new characterisation is easily checkable in polynomial time, resulting in the algorithm.

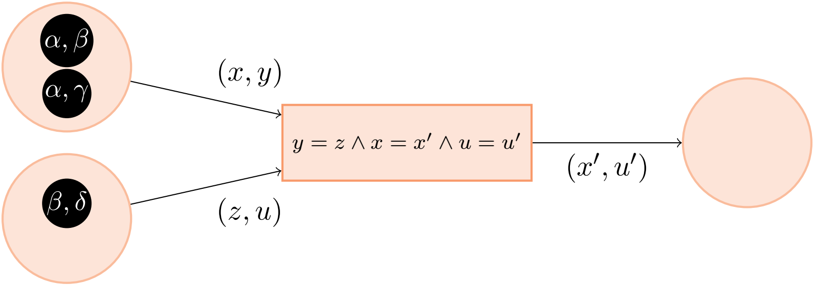
No analogous of Theorems 11 and 15 from [HLT17] appear in [HL18] or are otherwise known, thus we are pessimistic about applicability of the studied here approach in case of the ordered data domain.

As we comment in Conclusions, we believe that elaboration of the techniques of this paper allows also tackling the case of tuples of unordered data.

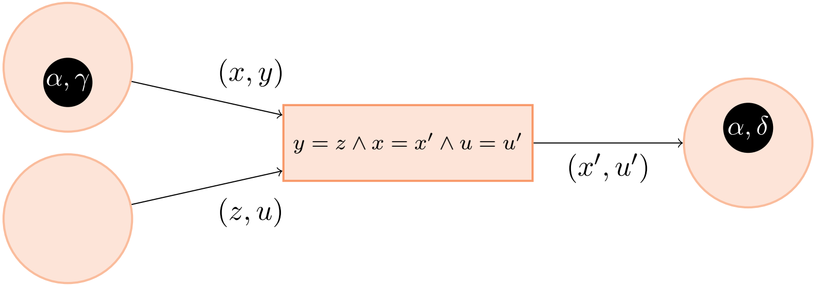
Motivation. Our motivation for this research is two-fold. On the one hand, from a foundational research perspective, our results are a part of a wider research program aiming at lifting computability results in finite-dimensional linear algebra to its orbit-finite-dimensional counterpart. Up to now, research has been focused on understanding solvability [HLT17, HL18], but there are other natural questions about definitions of bases, dimension, linear transformations, etc. On the other hand, we are interested in the analysis of systems with data and solvability may be a useful tool. We highlight three areas where understanding of $\mathbb{N}/\mathbb{Z}$-solvability may be crucial for further development.

Unordered Data nets reachability/coverability [LNO$^+$08]. The model can be seen as a special type of Coloured Petri nets [JvdAB$^+$13]. Data nets are an extension of Petri nets where every token caries a tuple of data values. In addition, each transition is equipped with a Boolean formula that connects data of tokens that are consumed and data of tokens that are produced (for unordered data the formula may use $=$ and $\neq$). To fire a transition we take a valuation satisfying the formula and according to it we remove and produce tokens.

Example 1.3. Consider the following simple net with 3 places and one transition. The initial marking has 3 tokens each with two data values.



If we valuate $x = \alpha, y = \beta, z = \beta, u = \delta, x' = \alpha, u' = \delta$ then we may fire the transition and get the new marking.



If we had chosen another valuation $x = \alpha, y = \beta, z = \beta, u = \gamma, x' = \alpha, u' = \gamma$ then the formula would hold, but we would lack tokens that can be consumed so we could not fire the transition with this valuation.

The reachability and coverability questions can be formulated as usual for Petri nets. It is not hard to imagine that some workflow or a flow of data through a program can be modelled with data nets. Unfortunately, in this richer model the reachability and coverability problems are undecidable [Las16] already for $k = 2$. For $k = 1$ i.e. data vectors in $\mathcal{D} \to \mathbb{Z}^d$, the status of reachability is unknown and coverability is decidable but known to be Ackermann-hard [LT17]. This is not a satisfying answer for engineers and in this case we should look for over and under approximations of the reachability relation, or for some techniques that will help in the analysis of industrial cases. One of the classic over-approximations of the Petri nets reachability relation is so-called integer reachability or Marking Equation in [DE95] Lemma 2.12, where the number of tokens in some places may go negative during the run. It can be encoded as integer programming and solved in **NP**. Its analogue for data nets can be stated as $\mathbb{N}$-solvability over an input set of data vectors, where the set is closed under data permutations.

Here, we should mention that the integer reachability is a member of a wider family of algebraic techniques for Petri nets. We refer to [STC96] for an exhaustive overview of linear-algebraic and integer-linear-programming techniques in the analysis of Petri nets. The usefulness of these techniques is confirmed by multiple applications including, for instance, recently proposed efficient tools for the coverability problem of Petri nets [GLS16, BFHH16].

π-calculus. Another formalism close to unordered data nets are ν-nets (unordered data nets additionaly equipped with an operation of creating a new datum that is not present in the current configuration of the net; in other words, a transition may force creation of data that are globally fresh/unique). In [Ros10], Rosa-Velardo observes that they are equivalent to so-called *multiset rewriting with name binding* systems which, as he showed, are a formalism equivalent to π-calculus. Thus, one may try to transfer algebraic techniques for data nets to π-calculus. This is a long way, but there is no possibility to start it without a good understanding of integer solutions of linear equations with data.

Here, it is worth mentioning that in π-calculus we use constructs like $\bar{c}\langle y\rangle.P$ which send a datum y trough the channel c, thus the sending operation is parameterized with a pair of data values i.e. the name of the channel and the data value. Thus, one cannot expect that already existing results for data vectors $\mathcal{D} \to \mathbb{Z}^d$ [HLT17] will be sufficient and we will need at least theory for $\mathcal{D}^2 \to \mathbb{Z}^d$, for example, to count messages that are sent and received. In fact even $\mathcal{D}^2 \to \mathbb{Z}^d$ may be not enough. In [Ros10], the fundamental concept for the encoding are derivatives, which essentially are terms of bounded depth labelled with data values. In his encoding, Rosa-Velardo represent each derivative with a token, so each token has to carry all data that are needed to identify the derivative. For a given process definition, he produces a ν-net with tokens with bounded but arbitrary high number of data values. Thus, if one wants to use linear algebra with data to describe some properties of the produced ν-net, he will have to work with data vectors in $\mathcal{D}^k \to \mathbb{Z}^d$, for k grater than 2.

Parikh's theorem. Finally, we may try to lift Parikh's theorem from context-free grammars and finite automata to context-free grammars with data [CK98] and register automata [KF94]. It is not clear to what extent it is possible but there are some promising results [HJLP21]. If we want to use this lifted Parikh's theorem then we have to work with semilinear sets with data and be able to check things like membership or nonemptiness of the intersection. Here, one more time techniques to solve systems of linear equations with data will be inevitable.

Outline. In Section 2 we introduce the setting and define the problems. Next, in Section 3 we provide the polynomial-time procedure for the $\mathbb{Z}$-solvability problem: the hypergraph reformulation and an effective characterisation of hypergraph solvability. In Section 4 for pedagogical reasons we present the proof of the characterisation if data vectors are restricted to $[\mathcal{D}]^2 \to \mathbb{Z}^d$. After this, in Sections 5, 6, 7, 8, 9, 10 we provide the full proof of the characterisation. The proof follows the same steps as the proof of the case $[\mathcal{D}]^2 \to \mathbb{Z}^d$, but is much more involved at the technical level. Next, in Section 11 we present a reduction from $\mathbb{N}$- to $\mathbb{Z}$-solvability. Finally, Section 12 concludes this work.

2. Linear equations with data.

In this section, we introduce the setting of linear equations with data and formulate our results. For a gentle introduction of the setting, we start by recalling classical linear equations.

Let $\mathbb{Z}$ and $\mathbb{N}$ denote integers and nonnegative integers, respectively. Classical linear equations are of the form

$$a_1x_1 + \ldots + a_mx_m = a,$$

where $x_1 \ldots x_m$ are variables (unknowns), and $a_1 \ldots a_m \in \mathbb{Z}$ are integer coefficients. For a finite system $\mathcal{U}$ of such equations over the same variables $x_1 \ldots x_m$, a solution of $\mathcal{U}$ is a vector $(n_1 \ldots n_m) \in \mathbb{Z}^m$ such that the valuation $x_1 \mapsto n_1 \ldots, x_m \mapsto n_m$ satisfies all equations in $\mathcal{U}$. It is well known that *integer solvability problem* ($\mathbb{Z}$-solvability problem) i.e. the question whether $\mathcal{U}$ has a solution $(n_1 \ldots n_m) \in \mathbb{Z}^m$, is decidable in **P-Time**. In the sequel we are often interested in nonnegative integer solutions $(n_1 \ldots n_m) \in \mathbb{N}^m$, but one may consider also other solution domains than $\mathbb{N}$. It is well known that the *nonnegative-integer solvability problem* ($\mathbb{N}$-solvability problem) of linear equations i.e. the question whether $\mathcal{U}$ has a nonnegative-integer solution, is NP-complete (for hardness see [Kar72]; NP-membership is a consequence of [Pot91]). The complexity remains the same for other natural variants of this problem, for instance, for inequalities instead of equations (a.k.a. integer linear programming). The $\mathbb{O}$-solvability problem (where $\mathbb{O} \in \{\mathbb{Z}, \mathbb{N}\}$) is equivalently formulated as follows: for a given finite set of coefficient vectors $\mathcal{I} = \{\mathbf{v}_1 \ldots \mathbf{v}_m\} \subseteq \mathbb{Z}^d$ and a target vector $\mathbf{v} \in \mathbb{Z}^d$ (we use bold font to distinguish vectors from other elements), check whether $\mathbf{v}$ is an $\mathbb{O}$-Sums of $\mathcal{I}$ i.e.

$$\mathbf{v} \in \mathbb{O}\text{-}\textsc{Sums}(\mathcal{I}) = \{n_1 \cdot \mathbf{v}_1 + \ldots + n_m \cdot \mathbf{v}_m \mid n_1 \ldots n_m \in \mathbb{O}\}. \tag{2.1}$$

The dimension d corresponds to the number of equations in $\mathcal{U}$.

Data vectors. Linear equations can be naturally extended with data. In this paper, we assume that the data domain $\mathcal{D}$ is a countable infinite set, whose elements are called *data values*. The bijections $\rho\colon \mathcal{D} \to \mathcal{D}$ are called *data permutations*. For a set $\mathcal{X}$ and $k \in \mathbb{N}$, by $[\mathcal{X}]^k$ we denote the set of all k-element subsets of $\mathcal{X}$ (called *k-sets* in short). Data permutations *lift naturally* to k-sets of data values: $\rho(\{\alpha_1 \ldots \alpha_k\}) = \{\rho(\alpha_1) \ldots \rho(\alpha_k)\}$.

Fix a positive integer $k \geq 1$. A *data vector* is a function $\mathbf{v}\colon [\mathcal{D}]^k \to \mathbb{Z}^d$ such that $\mathbf{v}(x) = \mathbf{0} \in \mathbb{Z}^d$ for all but finitely many $x \in [\mathcal{D}]^k$. (Again, we use bold font to distinguish data vectors from other elements.) We call the numbers k and d the *arity* and the *dimension* of $\mathbf{v}$, respectively.

The vector addition and scalar multiplication are lifted to data vectors pointwise: $(\mathbf{v} + \mathbf{w})(x) \overset{\text{def}}{=} \mathbf{v}(x) + \mathbf{w}(x)$, and $(c \cdot \mathbf{v})(x) \overset{\text{def}}{=} c\mathbf{v}(x)$. Further, for a data permutation $\pi\colon \mathcal{D} \to \mathcal{D}$ by $\mathbf{v} \circ \pi$ we mean the data vector defined as follows $(\mathbf{v} \circ \pi)(\{\alpha_1, \alpha_2 \ldots \alpha_k\}) \overset{\text{def}}{=} \mathbf{v}(\{\pi(\alpha_1), \pi(\alpha_2) \ldots \pi(\alpha_k)\})$ for all $\alpha_1, \alpha_2 \ldots \alpha_k \in \mathcal{D}$. It is the natural lift of normal function composition. For a set $\mathcal{I}$ of data vectors we define

$$\textsc{Perm}(\mathcal{I}) = \{\mathbf{v} \circ \pi \mid \mathbf{v} \in \mathcal{I}, \pi \text{ is a data permutation}\}.$$

A data vector $\mathbf{v}$ is said to be an $\mathbb{O}$*-permutation sum* of a finite set of data vectors $\mathcal{I}$ if (we deliberately overload the symbol $\mathbb{O}$-$\textsc{Sums}(\square)$ and use it for data vectors, while in (2.1) it is used for (plain) vectors)

$$\mathbf{v} \in \mathbb{O}\text{-}\textsc{Sums}(\textsc{Perm}(\mathcal{I})) = \{n_1 \cdot \mathbf{v}_1 + \ldots + n_m \cdot \mathbf{v}_m \mid \mathbf{v}_1 \ldots \mathbf{v}_m \in \textsc{Perm}(\mathcal{I}), n_1 \ldots n_m \in \mathbb{O}\}. \tag{2.2}$$

We investigate the following decision problems (for $\mathbb{O} \in \{\mathbb{Z}, \mathbb{N}\}$):

$\mathbb{O}$-SOLVABILITY	
INPUT:	A finite set $\mathcal{I}$ of data vectors and a target data vector $\mathbf{v}$, all of the same arity and dimension.
OUTPUT:	Is $\mathbf{v}$ an $\mathbb{O}$-permutation sum of $\mathcal{I}$?

The insightful reader may notice that, in the motivating examples we consider $\mathbb{O}$-sums of an input set of data vectors that is closed under data permutations. Here, we switch to the $\mathbb{O}$-solvability problem, which is defined by $\mathbb{O}$-permutation sum. But observe that, to formalise expressibility by $\mathbb{O}$-sums as a problem we have to provide a finite representation of the input set. That is why as an input to the problem we take a finite set of data vectors $\mathcal{I}$, and we consider $\mathbb{O}$-sums of its closure under data permutations i.e. of the set $\textsc{Perm}(\mathcal{I})$. This is why we consider $\mathbb{O}$-permutation sums instead of pure $\mathbb{O}$-sums.

For complexity estimations we assume binary encoding of numbers appearing in the input to all decision problems discussed in this paper. Our main results are the following complexity bounds:

Theorem 2.1. *For every fixed arity $k \in \mathbb{N}$, the $\mathbb{Z}$-solvability problem is in* **P-Time**. *(The dependency on k is exponential).*

Theorem 2.2. *For every fixed arity $k \in \mathbb{N}$, the $\mathbb{N}$-solvability problem is in* **NExp-Time**.

For the special case of the solvability problems when the arity $k = 1$, the **P-Time** and **NP** complexity bounds, respectively, have been shown in [HLT17]. Thus, according to Theorem 2.1, in the case of $\mathbb{Z}$-solvability the complexity remains polynomial for every

$k > 1$. In the case of $\mathbb{N}$-solvability the **NP**-hardness carries over to every $k > 1$, and hence a complexity gap remains open between **NP** and **NExp-Time**.

3. Proof of Theorem 2.1.

We start by reformulating the problem in terms of (undirected) weighted uniform hypergraphs (Lemma 3.2 below).

Fix a positive integer $k \geq 1$. By a *k-hypergraph* we mean a pair $\mathbb{H} = (V, \mu)$ where $V \subset \mathcal{D}$ is a finite set called vertices and $\mu\colon [V]^k \to \mathbb{Z}^d$ is a weight function (when k is not relevant we skip it and write a hypergraph). As before, we call the numbers k and d the arity and the dimension of $\mathbb{H}$, respectively. When $k = 2$, we speak of *graphs* instead of hypergraphs. (Note however that the (hyper)graphs we consider are always *weighted*, with weights from $\mathbb{Z}^d$.)

Because vertices are data then instead of usual u, v for vertices we will use Greek letters $\alpha, \beta, \gamma, \delta, \varepsilon \ldots$.

The set of hyperedges is then defined as

$$\textsc{Edges}(\mathbb{H}) = \{e \in [V]^k \mid \mu(e) \neq \mathbf{0}\}.$$

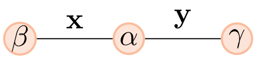
When $\alpha \in e$ for $\alpha \in V$ and $e \in \textsc{Edges}(\mathbb{H})$, we say that the vertex α is *adjacent* with the hyperedge e. The degree of α is the number of hyperedges adjacent with α. We call vertices of degree 0 *isolated*. Two hypergraphs are *isomorphic* if there is a bijection between their sets of vertices that preserves the value of the weight function. Two hypergraphs are *equivalent* if they are isomorphic after removing their isolated vertices. For a set $\mathcal{H}$ of hypergraphs, by $\textsc{Eq}(\mathcal{H})$ we denote the set of all hypergraphs equivalent to ones from $\mathcal{H}$. If $\mathcal{H} = \{\mathbb{H}\}$ then instead of $\textsc{Eq}(\{\mathbb{H}\})$ we write $\textsc{Eq}(\mathbb{H})$.

Scalar multiplication and addition are defined naturally for hypergraphs. First, for $c \in \mathbb{Z}$ and a hypergraph $\mathbb{H} = (V, \mu)$, let $c \cdot \mathbb{H} \overset{\text{def}}{=} (V, c \cdot \mu)$. Second, given two hypergraphs $\mathbb{G} = (W, \widetilde{\mu})$ and $\mathbb{H} = (V, \mu)$ of the same arity k and dimension d, we first add isolated vertices to both hypergraphs to make their vertex sets equal to the union $W \cup V$, thus obtaining $\mathbb{G}' = (W \cup V, \widetilde{\mu}')$ and $\mathbb{H}' = (W \cup V, \mu')$ with the accordingly extended weight functions $\widetilde{\mu}', \mu'\colon [(W \cup V)]^k \to \mathbb{Z}^d$, and then define $\mathbb{G} + \mathbb{H} \overset{\text{def}}{=} (W \cup V, \widetilde{\mu}' + \mu')$. Using these operations we define $\mathbb{O}$-sums of a family $\mathcal{H}$ of hypergraphs of the same arity and dimension (again, we overload the symbol $\mathbb{O}$-$\textsc{Sums}(\square)$ further and use it for hypergraphs):

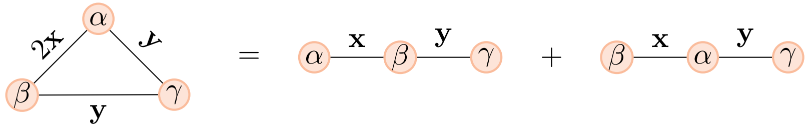
$$\mathbb{O}\text{-}\textsc{Sums}(\mathcal{H}) = \{c_1 \cdot \mathbb{H}_1 + \ldots + c_l \cdot \mathbb{H}_l \mid c_1 \ldots c_l \in \mathbb{O}, \mathbb{H}_1 \ldots \mathbb{H}_l \in \mathcal{H}\}.$$

We say that a hypergraph $\mathbb{H}$ is a $\mathbb{O}$-sum of $\mathcal{H}$ *up to equivalence* if $\mathbb{H}$ is a $\mathbb{O}$-sum of hypergraphs equivalent to elements of $\mathcal{H}$: $\mathbb{H} \in \mathbb{O}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}))$.

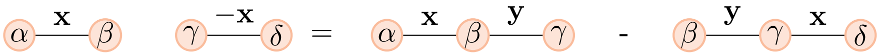
Example 3.1. We illustrate the $\mathbb{Z}$-sums in arity $k = 2$ i.e. using graphs. Consider the following graph $\mathbb{G}$ consisting of 3 vertices and 2 edges:



Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ be arbitrary vectors. Here, there are two examples of graphs which can be presented as a $\mathbb{Z}$-sum of $\{\mathbb{G}\}$ up to equivalence, using a sum of two graphs equivalent to $\mathbb{G}$:



and a difference of two such graphs:



We are now ready to formulate the *hypergraph* $\mathbb{Z}$*-sum problem*, to which $\mathbb{Z}$-solvability is going to be reduced:

HYPERGRAPH $\mathbb{Z}$-SUM

INPUT: A finite set $\mathcal{H}$ of hypergraphs and a target hypergraph $\mathbb{H}$, all of the same arity and dimension.

OUTPUT: Is $\mathbb{H}$ a $\mathbb{Z}$-sum of $\mathcal{H}$ up to equivalence?

Lemma 3.2. *The* $\mathbb{Z}$*-solvability problem reduces in logarithmic space to the hypergraph* $\mathbb{Z}$*-sum problem. The reduction preserves the arity and dimension.*

Proof. The reduction encodes each data vector $\mathbf{v}$ by a hypergraph $\mathbb{H} = (V, \mu)$, where

$$V \quad = \quad \bigcup \{x \in [\mathcal{D}]^k \mid \mathbf{v}(x) \neq \mathbf{0}\}$$

and μ is the restriction of $\mathbf{v}$ to $[V]^k$. In this way, a set $\mathcal{I}$ of data vectors and a target data vector $\mathbf{v}$ are transformed into a set of $\mathcal{H}$ hypergraphs and a target hypergraph $\mathbb{H}$ such that $\mathbf{v}$ is a $\mathbb{Z}$-permutation sum of $\mathcal{I}$ if, and only if, $\mathbb{H}$ is a $\mathbb{Z}$-sum of $\mathcal{H}$ up to equivalence. □

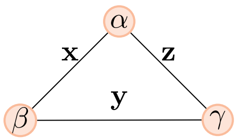
Thus, from now on, we concentrate on solving the hypergraph $\mathbb{Z}$-sum problem. One may ask why we perform such a reduction. The reasons are of pedagogical nature, namely graphs are more convenient for examples and proofs by pictures.

As the next step, we formulate our core technical result (Theorem 3.4). In the theorem we state that the hypergraph $\mathbb{Z}$-sum problem is equivalent to a local $\mathbb{Z}$-sum problem, defined in the following paragraph. It is easy to design a polynomial time algorithm for the local $\mathbb{Z}$-sum problem. This gives us the proof of Theorem 2.1. Let $\mathbb{H} = (V, \mu)$ be a hypergraph. For a set $\mathcal{X} \subset \mathcal{D}$ we consider $\{e \in \textsc{Edges}(\mathbb{H}) \mid \mathcal{X} \subseteq e\}$ — the set of all hyperedges that include $\mathcal{X}$ — and define the weight of $\mathcal{X}$ as the sum of weights of all these edges:

$$\Lambda_{\mathcal{X}}(\mathbb{H}) \overset{\text{def}}{=} \sum_{e \in \textsc{Edges}(\mathbb{H}), \mathcal{X} \subseteq e} \mu(e).$$

In particular when $\mathcal{X} \not\subseteq V$ then $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mathbf{0}$. Further, $\Lambda_{\emptyset}(\mathbb{H})$ is the sum of weights of all hyperedges of $\mathbb{H}$. When $\mathcal{X} = \{\alpha\}$, then $\Lambda_{\mathcal{X}}(\mathbb{H})$ is the sum of weights of all hyperedges adjacent with α (we write Λ_α instead of $\Lambda_{\{\alpha\}}$). Finally, when the cardinality of $\mathcal{X}$ equals k i.e. $|\mathcal{X}| = k$, $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mu(\mathcal{X})$ is the weight of a hyperedge $\mathcal{X}$.

Example 3.3. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^d$ be arbitrary vectors. As an illustration, consider the graph $\mathbb{G}$ on the left below, with the weights of chosen subsets of its vertices listed on the right:



$$\Lambda_{\{\alpha,\beta\}}(\mathbb{G}) = \mathbf{x}$$
$$\Lambda_{\beta}(\mathbb{G}) = \mathbf{x} + \mathbf{y}$$
$$\Lambda_{\gamma}(\mathbb{G}) = \mathbf{z} + \mathbf{y}$$
$$\Lambda_{\emptyset}(\mathbb{G}) = \mathbf{x} + \mathbf{y} + \mathbf{z}.$$

Weights of sets are important as they form a family of homomorphisms from hypergraphs to $\mathbb{Z}^d$. Namely, for any subset of vertices $\mathcal{X}$ we have that

$$\Lambda_{\mathcal{X}}(\mathbb{H} + \mathbb{H}') = \Lambda_{\mathcal{X}}(\mathbb{H}) + \Lambda_{\mathcal{X}}(\mathbb{H}')$$

and

$$c\Lambda_{\mathcal{X}}(\mathbb{H}) = \Lambda_{\mathcal{X}}(c\mathbb{H}), \text{ for any } c \in \mathbb{Z}.$$

This allows us to design a partial test for the hypergraph $\mathbb{Z}$-sum problem. If there is a set of vertices $\mathcal{X}$ such that $\Lambda_{\mathcal{X}}(\mathbb{H})$ cannot be expressed as a $\mathbb{Z}$-sum of vectors in $\Lambda_{\mathcal{X}}(\mathrm{E}\textsc{q}(\mathcal{H}))$ then $\mathbb{H} \notin \mathbb{Z}\text{-}\textsc{Sums}(\mathrm{E}\textsc{q}(\mathcal{H}))$. This motivates the next definition.

We say that a k-hypergraph $\mathbb{H} = (V, \mu)$ is *locally* a $\mathbb{Z}$-sum of a family $\mathcal{H}$ of hypergraphs if for every $\mathcal{X} \subseteq V$ of cardinality $|\mathcal{X}| \leq k$, its weight $\Lambda_{\mathcal{X}}(\mathbb{H})$ is a $\mathbb{Z}$-sum of weights of $|\mathcal{X}|$-element subsets of vertex sets of hypergraphs from $\mathcal{H}$

$$\Lambda_{\mathcal{X}}(\mathbb{H}) \in \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_{\mathcal{X}'}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}, \mathcal{X}' \subseteq V', |\mathcal{X}'| = |\mathcal{X}|\}). \tag{3.1}$$

Note that we only consider hypergraphs $\mathbb{H}' \in \mathcal{H}$, and we do not need to consider hypergraphs $\mathbb{H}' \in \mathrm{E}\textsc{q}(\mathcal{H})$ equivalent to ones from $\mathcal{H}$ as

$$\{\Lambda_{\mathcal{X}'}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}, \mathcal{X}' \subseteq V', |\mathcal{X}'| = |\mathcal{X}|\} = \{\Lambda_{\mathcal{X}}(\mathbb{H}') \mid \mathbb{H}' \in \mathrm{E}\textsc{q}(\mathcal{H})\}.$$

Theorem 3.4. *The following conditions are equivalent, for a finite set $\mathcal{H}$ of hypergraphs and a hypergraph $\mathbb{H}$, all of the same arity and dimension:*

(1) $\mathbb{H}$ *is a $\mathbb{Z}$-sum of* $\mathrm{E}\textsc{q}(\mathcal{H})$*;*
(2) $\mathbb{H}$ *is locally a $\mathbb{Z}$-sum of $\mathcal{H}$.*

Before we embark on proving the result (in the next section) we first discuss how it implies Theorem 2.1. Recall that the arity k is fixed. Instead of checking if $\mathbb{H} = (V, \mu)$ is a $\mathbb{Z}$-sum of $\mathrm{E}\textsc{q}(\mathcal{H})$, the algorithm checks if $\mathbb{H}$ is *locally* a $\mathbb{Z}$-sum of $\mathcal{H}$. Observe that the condition (3.1) amounts to solvability of a (classical) system of linear equations. Therefore, the algorithm tests $\mathbb{Z}$-solvability of a system of the corresponding $d \cdot (1 + |V| + |[V]^2| + \ldots |[V]^k|)$ linear equations, d for every subset $\mathcal{X} \subseteq V$ of the cardinality at most k. The number of equations is exponential in k, but due to fixing k it is polynomial in the input hypergraphs $\mathbb{H}$ and $\mathcal{H}$. Thus, Theorem 2.1 is proved once we prove Theorem 3.4.

Example 3.5. Let us continue Example 1.2. For the target we have that $\Lambda_{\emptyset}(\mathbf{v}_{\gamma\delta}) = \Lambda_{\gamma}(\mathbf{v}_{\gamma\delta}) = \Lambda_{\delta}(\mathbf{v}_{\gamma\delta}) = \Lambda_{\{\gamma,\delta\}}(\mathbf{v}_{\gamma\delta}) = 6$, and for the triangle we have $\Lambda_{\emptyset}(\mathbf{v}_{\delta\gamma\varepsilon}) = 3$, $\Lambda_{x}(\mathbf{v}_{\delta\gamma\varepsilon}) = 2$ for any $x \in \{\delta, \gamma, \varepsilon\}$, and $\Lambda_{\{x,y\}}(\mathbf{v}_{\delta\gamma\varepsilon}) = 1$ for any $\{x, y\} \subset \{\delta, \gamma, \varepsilon\}$.

As $6 = 3 + 3$ and $6 = 2 + 2 + 2$ and $6 = 1 + 1 + 1 + 1 + 1 + 1$ we see that the target is locally a $\mathbb{Z}$-sum of the triangle. Hence the target is a $\mathbb{Z}$-sum of the triangle up to equivalence. Moreover, if we change 6 to a smaller positive number then the target graph will not be a $\mathbb{Z}$-sum of the triangle up to equivalence. For example if, we change 6 to 3, then $\mathbf{v}_{\gamma\delta}$ is not a $\mathbb{Z}$-sum of the triangles as $\Lambda_{\delta}(\mathbf{v}_{\gamma\delta}) = 3$ is not divisible by 2 i.e. the weight of any single vertex of the triangle.

4. Proof of Theorem 3.4 (the case for $k = 2$).

The implication $1 \implies 2$ is immediate. Indeed, suppose $\mathbb{H} = (V, \mu) = c_1 \cdot \mathbb{H}_1 + \ldots + c_l \cdot \mathbb{H}_l$, where $c_i \in \mathbb{Z}$ and $\mathbb{H}_i \in \mathrm{E}\textsc{q}(\mathcal{H})$. Let $\mathcal{X} \subseteq V$ be a subset of cardinality $|\mathcal{X}| \leq k$. We have

$$\Lambda_{\mathcal{X}}(\mathbb{H}) = c_1 \cdot \Lambda_{\mathcal{X}}(\mathbb{H}_1) + \ldots + c_l \cdot \Lambda_{\mathcal{X}}(\mathbb{H}_l),$$

which implies, for some hypergraphs $\mathbb{H}'_i = (V_i, \mu_i) \in \mathcal{H}$ equivalent to $\mathbb{H}_i$, and subsets $\mathcal{Y}_i \subseteq V_i$ of cardinality $|\mathcal{Y}_i| = |\mathcal{X}|$, the following equality holds

$$\Lambda_{\mathcal{X}}(\mathbb{H}) = c_1 \cdot \Lambda_{\mathcal{Y}_1}(\mathbb{H}'_1) + \ldots + c_l \cdot \Lambda_{\mathcal{Y}_l}(\mathbb{H}'_l).$$

As $\mathcal{X}$ was chosen arbitrarily, this shows that $\mathbb{H}$ is locally a $\mathbb{Z}$-sum of $\mathcal{H}$.

The proof of the converse implication $2 \implies 1$ is much more involved. The case for arity $k = 1$ is considered in [HLT17]. Here, for pedagogical reasons, we provide a simplified version of the proof for arity $k = 2$ i.e. for (undirected) graphs. Consequently, we speak of edges instead of hyperedges. We recall that the graphs we consider are actually $\mathbb{Z}^d$-*weighted* graphs.

Let $\mathbb{H} = (V, \mu)$ be a graph and assume that $\mathbb{H}$ is locally a $\mathbb{Z}$-sum of $\mathcal{H}$. We are going to demonstrate that $\mathbb{H}$ is equivalent to a $\mathbb{Z}$-sum of $\textsc{Eq}(\mathcal{H})$. Since we work with arity $k = 2$, the assumption amounts to the following conditions, for every vertex $\alpha \in V$ and every edge $e \in \textsc{Edges}(\mathbb{H})$:

$$\Lambda_{\emptyset}(\mathbb{H}) \in \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_{\emptyset}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}\}) \tag{4.1}$$

$$\Lambda_{\alpha}(\mathbb{H}) \in \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_{\alpha'}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}, \alpha' \in V'\}) \tag{4.2}$$

$$\Lambda_{e}(\mathbb{H}) \in \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_{e'}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}, e' \in \textsc{Edges}(\mathbb{H}')\}). \tag{4.3}$$

Claim 4.1. W.l.o.g. we may assume that $\mathbb{H} = (V, \mu)$ satisfies $\Lambda_{\emptyset}(\mathbb{H}) = \mathbf{0}$.

Proof. Indeed, due to the assumption (4.1) and due the following equality (note that the symbol $\mathbb{Z}\text{-}\textsc{Sums}(\square)$ applies to vectors on the left, and to graphs on the right)

$$\mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_{\emptyset}(\mathbb{H}') \mid \mathbb{H}' \in \mathcal{H}\}) = \{\Lambda_{\emptyset}(\mathbb{H}') \mid \mathbb{H}' \in \mathbb{Z}\text{-}\textsc{Sums}(\mathcal{H})\}, \tag{4.4}$$

there is a graph $\mathbb{H}' = (V', \mu') \in \mathbb{Z}\text{-}\textsc{Sums}(\mathcal{H})$ with $\Lambda_{\emptyset}(\mathbb{H}') = \Lambda_{\emptyset}(\mathbb{H})$. Therefore $\mathbb{H} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}))$ if, and only if, $\mathbb{H} - \mathbb{H}' \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}))$, and hence we can replace $\mathbb{H}$ by $\mathbb{H} - \mathbb{H}'$. Note that, if $\mathbb{H}$ is locally a $\mathbb{Z}$-sum of $\mathcal{H}$ then $\mathbb{H} - \mathbb{H}'$ is also locally a $\mathbb{Z}$-sum of $\mathcal{H}$. □

We proceed in two steps. We start by defining a class of particularly simple graphs, called $\mathbb{H}$-*simple* graphs, and argue that $\mathbb{H}$ is a $\mathbb{Z}$-sum of these graphs (Lemma 4.4 below). Then we prove that every $\mathbb{H}$-simple graph is representable as a $\mathbb{Z}$-sum of $\mathcal{H}$ up to equivalence (Lemma 4.10 below). We may compose these two lemmas because of Lemma 4.2. By this we get that $\mathbb{H}$ is a $\mathbb{Z}$-sum of $\mathcal{H}$ up to equivalence.

Lemma 4.2. *For a family of k-hypergraphs $\mathcal{G}$ and k-hypergraphs $\mathbb{G}, \mathbb{G}_1 \ldots \mathbb{G}_l$, all of the same dimension, If $\mathbb{G}_1 \ldots \mathbb{G}_l \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{G}))$ and $\mathbb{G} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\{\mathbb{G}_1 \ldots \mathbb{G}_l\}))$ then $\mathbb{G} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{G}))$.*

(Lemma 4.2 is expressible, more succinctly, as

$$\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{G})))) = \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{G})).)$$

Proof. This is simply because of three trivial facts:

(1) $\textsc{Eq}(\mathbb{G}_i + \mathbb{G}_j) \subseteq \textsc{Eq}(\mathbb{G}_i) + \textsc{Eq}(\mathbb{G}_j)$ where $\mathbb{G}_i, \mathbb{G}_j$ are any two k-hypergraphs of the same dimension, and the second plus is the Minkowski sum.

(2) $\mathbb{Z}\text{-}\textsc{Sums}(\mathcal{F}) = \mathbb{Z}\text{-}\textsc{Sums}(\mathbb{Z}\text{-}\textsc{Sums}(\mathcal{F}))$, for any $\mathcal{F}$ a family of k-hypergraphs of the same dimension.

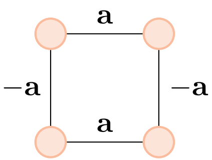
(3) $\textsc{Eq}(\mathcal{F}) = \textsc{Eq}(\textsc{Eq}(\mathcal{F}))$, for any $\mathcal{F}$ a family of k-hypergraphs of the same dimension.

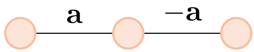
Now,

$$\begin{aligned} \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H})))) \subseteq && \text{because of 1} \\ \mathbb{Z}\text{-}\textsc{Sums}(\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\textsc{Eq}(\mathcal{H})))) = && \text{because of 2 and 3} \\ \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H})). && \end{aligned}$$

The inclusion in the opposite direction is trivial. □

Definition 4.3. For every vector $\mathbf{a} \in \mathbb{Z}^d$ we define an $\mathbf{a}$-*edge simple graph* $\mathbb{S}_{\mathbf{a}}^{\bullet-\bullet}$ (shown on the left) and an $\mathbf{a}$-*vertex simple graph* $\mathbb{S}_{\mathbf{a}}^{\bullet}$ (shown on the right):





We do not specify names of vertices, as we will consider these graphs up to equivalence. We call both types of graphs *simple graphs.*

Let

$$\mathcal{E}_{\mathbb{H}} = \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_e(\mathbb{H}) \mid e \in \textsc{Edges}(\mathbb{H})\}) \subseteq \mathbb{Z}^d \tag{4.5}$$

and

$$\mathcal{V}_{\mathbb{H}} = \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_\alpha(\mathbb{H}) \mid \alpha \in V\}) \subseteq \mathbb{Z}^d. \tag{4.6}$$

Now, let $\mathcal{S}_{\mathbb{H}}^{\bullet} \overset{\text{def}}{=} \{\mathbb{S}_{\mathbf{a}}^{\bullet} \mid \mathbf{a} \in \mathcal{V}_{\mathbb{H}}\}$ and $\mathcal{S}_{\mathbb{H}}^{\bullet-\bullet} \overset{\text{def}}{=} \{\mathbb{S}_{\mathbf{a}}^{\bullet-\bullet} \mid \mathbf{a} \in \mathcal{E}_{\mathbb{H}}\}$.

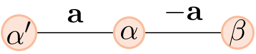
Lemma 4.4. $\mathbb{H} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}))$.

Proof. Suppose $\mathbb{H} = (V, \mu)$. The proof of the lemma is done in steps.

Claim 4.5. There is a $\mathbb{G} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet}))$ such that for any vertex $\alpha \in \mathcal{D}$ it holds that $\Lambda_\alpha(\mathbb{G} + \mathbb{H}) = \mathbf{0}$.

We will use it to further simplify our problem, as $\mathbb{H} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}))$ if, and only if, $\mathbb{G} + \mathbb{H} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet} \cup \mathcal{S}_{\mathbb{H}}^{\bullet-\bullet}))$.

Proof of the claim. Let α, α' be two vertices not in V and let $\mathcal{F} \subseteq \mathcal{S}_{\mathbb{H}}^{\bullet}$ be the family of all $\mathbb{S}_{\mathbf{a}}^{\bullet}$ simple graphs (depicted below) where $\mathbf{a} = \Lambda_\beta(\mathbb{H})$ for $\beta \in V$.



We define $\mathbb{G} = \sum_{\mathbb{F} \in \mathcal{F}} \mathbb{F}$. As $\mathcal{F} \subset \textsc{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet})$, the graph $\mathbb{G}$ is in $\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}_{\mathbb{H}}^{\bullet}))$. Moreover,

(1) for every $\beta \in V$ it holds that $-\Lambda_\beta(\mathbb{H}) = \Lambda_\beta(\mathbb{G})$;
(2) $\Lambda_\alpha(\mathbb{G}) = \mathbf{0}$ as for each $\mathbb{F} \in \mathcal{F}$, $\Lambda_\alpha(\mathbb{F}) = \mathbf{0}$;
(3) $\Lambda_{\alpha'}(\mathbb{G}) = \mathbf{0}$ as $\Lambda_{\alpha'}(\mathbb{G}) = \sum_{\beta \in V} \Lambda_\beta(\mathbb{H}) = 2\Lambda_\emptyset(\mathbb{H}) = \mathbf{0}$, where the second equality reflects the fact that every edge has two ends and the last equality is due to Claim 4.1.

As Λ_x is a homomorphism we get that $\Lambda_x(\mathbb{H} + \mathbb{G}) = \mathbf{0}$ for every $x \in V \cup \{\alpha, \alpha'\}$. The same holds for other vertices as they are isolated. □

Because of the previous claim w.l.o.g we may restrict our self to the following case.

Claim 4.6. We assume that $\mathbb{H}$ has the following property for every $\beta \in V$, it holds that $\Lambda_\beta(\mathbb{H}) = \mathbf{0}$.

Here, there is one issue that should be discussed. Suppose $\mathbb{H}' = \mathbb{H} + \mathbb{G}$ as in Claim 4.5. The issue is that $\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}} = \mathcal{S}^{\bullet}_{\mathbb{H}'} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}'}$ may not hold. So if we prove the lemma for $\mathbb{H}'$ and $\mathcal{S}^{\bullet}_{\mathbb{H}'} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}'}$ it does not necessarily carry to $\mathbb{H}$ and $\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}$. Fortunately, it is sufficient for us if $\mathbb{Z}\text{-}\textsc{Sums}(\mathcal{S}^{\bullet}_{\mathbb{H}'} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}'}) \subseteq \mathbb{Z}\text{-}\textsc{Sums}(\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}})$. The last inclusion holds. Indeed, for every $\beta \in V'$ it holds that $\Lambda_\beta(\mathbb{H}')$ is a sum of weights of vertices in $\mathbb{H}$ thus $\mathbb{Z}\text{-}\textsc{Sums}(\mathcal{S}^{\bullet}_{\mathbb{H}'}) \subseteq \mathbb{Z}\text{-}\textsc{Sums}(\mathcal{S}^{\bullet}_{\mathbb{H}})$ and similarly $\mathbb{Z}\text{-}\textsc{Sums}(\mathcal{S}^{\bullet-\bullet}_{\mathbb{H}'}) \subseteq \mathbb{Z}\text{-}\textsc{Sums}(\mathcal{S}^{\bullet-\bullet}_{\mathbb{H}})$. Thus, we do not loose generality because of the proposed restriction.

Claim 4.7. There is a graph $\mathbb{G}' \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}))$ such that $\mathbb{G}' = (V', \mu')$, $\mathbb{H} + \mathbb{G}'$ has at most 3 nonisolated vertices, and $\Lambda_\beta(\mathbb{H} + \mathbb{G}') = \mathbf{0}$ for every $\beta \in V \cup V'$.

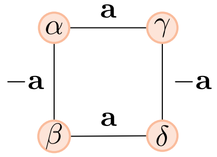
We will use it to further simplify our problem, as $\mathbb{H} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}))$ if, and only if, $\mathbb{H} + \mathbb{G} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}))$.

Proof of the claim. We construct $\mathbb{G}'$ gradually as a sum of a sequence of graphs $\mathbb{G}'_1, \mathbb{G}'_2 \ldots \mathbb{G}'_{last}$. The sequence $\mathbb{G}'_i$ is constructed in parallel with a sequence $\mathbb{H}_i$ where $\mathbb{H}_0 = \mathbb{H}$ and $\mathbb{H}_{i+1} = \mathbb{H}_i + \mathbb{G}'_{i+1}$.

The main property of the sequence $\mathbb{H}_i$ is that $\mathbb{H}_i > \mathbb{H}_{i+1}$ in some well-founded quasi-order on graphs. Suppose that, there is *ALG* an algorithm, that takes as an input $\mathbb{H}_{i-1}$ and produces the next $\mathbb{G}'_i \in \textsc{Eq}(\mathcal{S}^{\bullet-\bullet}_{\mathbb{H}})$. The precondition of the algorithm *ALG* is that $\mathbb{H}_{i-1}$ has more than 3 nonisolated vertices. Now, due to the well-foundedness of the quasi order, the sequence $\mathbb{G}'_i$ is finite. Further, the graph $\mathbb{H}_{last}$ has at most 3 nonisolated vertices because of the precondition of *ALG*. To this end, we need to define the order on graphs and provide the algorithm *ALG*.

Order on graphs. We assume an arbitrary total order $<$ on vertices V. We lift the order to an order on edges $\{\alpha, \beta\}$. We define it as the lexicographic order on pairs (α, β) satisfying $\alpha > \beta$. Finally, the order is extended to a quasi-order on graphs: $\mathbb{G} < \mathbb{G}'$ iff $e < e'$, where e and e' are the largest edges in $\mathbb{G}$ and $\mathbb{G}'$, respectively.

The algorithm *ALG*. Suppose $\{\alpha, \beta\}$ is the largest edge in the graph $\mathbb{H}_i$ and that $\mathbb{H}_i$ has at least 4 nonisolated vertices. Observe that $\Lambda_\alpha(\mathbb{H}_i) = \mathbf{0}$. Indeed, $\Lambda_\alpha(\mathbb{H}_i) = \Lambda_\alpha(\mathbb{H} + \sum_{j=1}^{i} \mathbb{G}'_j) = \Lambda_\alpha(\mathbb{H}) + \sum_{j=1}^{i} \Lambda_\alpha(\mathbb{G}'_j) = \mathbf{0} + \sum_{j=1}^{i} \Lambda_\alpha(\mathbb{G}'_j)$ but each $\mathbb{G}'_j$ belongs to $\textsc{Eq}(\mathcal{S}^{\bullet-\bullet}_{\mathbb{H}})$ and by the definition of edge simple graphs $\Lambda_\varepsilon(\mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}) = \mathbf{0}$ for every vertex ε. As a consequence there must be at least one vertex $\gamma \notin \{\alpha, \beta\}$ such that (α, γ) is an edge in $\mathbb{H}_i$. As $\mathbb{H}_i$ has at least 4 nonisolated vertices, there is $\delta \notin \{\gamma, \alpha, \beta\}$ a nonisolated vertex in $\mathbb{H}_i$. We define $\mathbb{G}'_{i+1}$ as follows



for $\mathbf{a} = \Lambda_{\{\alpha,\beta\}}(\mathbb{H}_i)$.

Observe that edges $(\alpha, \gamma), (\beta, \delta), (\gamma, \delta)$ are smaller than the edge (α, β) and $(\mathbb{H}_i + \mathbb{G}'_{i+1})(\{\alpha, \beta\}) = \mathbf{0}$ so $\mathbb{H}_i + \mathbb{G}'_{i+1} < \mathbb{H}_i$. □

After usage of the claim we get a graph $\mathbb{H}' \overset{\text{def}}{=} \mathbb{H} + \mathbb{G}'$. We know that $\mathbb{H} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}))$ if, and only if, $\mathbb{H}' \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}))$. The graph $\mathbb{H}'$ has at most 3 nonisolated vertices, $\Lambda_\emptyset(\mathbb{H}') = \mathbf{0}$, and $\Lambda_\alpha(\mathbb{H}') = \mathbf{0}$ for any vertex α.

Suppose that the set of nonisolated vertices of $\mathbb{H}'$ is a subset of $\{\alpha,\beta,\gamma\}$. We may write the following system of equations:

$$\begin{aligned}\Lambda_\alpha(\mathbb{H}') &= \Lambda_{\{\alpha,\beta\}}(\mathbb{H}') + \Lambda_{\{\alpha,\gamma\}}(\mathbb{H}') = \mathbf{0}\\ \Lambda_\beta(\mathbb{H}') &= \Lambda_{\{\alpha,\beta\}}(\mathbb{H}') + \Lambda_{\{\beta,\gamma\}}(\mathbb{H}') = \mathbf{0}\\ \Lambda_\gamma(\mathbb{H}') &= \Lambda_{\{\alpha,\gamma\}}(\mathbb{H}') + \Lambda_{\{\beta,\gamma\}}(\mathbb{H}') = \mathbf{0}\end{aligned} \tag{4.7}$$

The only solution is $\Lambda_{\{\alpha,\beta\}}(\mathbb{H}') = \Lambda_{\{\beta,\gamma\}}(\mathbb{H}') = \Lambda_{\{\alpha,\gamma\}}(\mathbb{H}') = \mathbf{0}$. So $\mathbb{H}'$ is the empty graph. In consequence, $\mathbb{H} = -\sum_{i=1}^{last}\mathbb{G}_i \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}^{\bullet}_{\mathbb{H}} \cup \mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}))$, which completes the proof of Lemma 4.4. □

Representation of ℍ-simple graphs

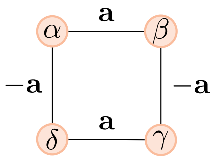
As the second step, we prove that every $\mathbb{H}$-simple graph is representable as a $\mathbb{Z}$-sum of $\mathcal{H}$ up to equivalence (Lemma 4.10). We start with two preparatory lemmas.

Lemma 4.8. *Let $e \in \textsc{Edges}(\mathbb{G})$ for a graph $\mathbb{G} = (V',\mu')$, and let $\Lambda_e(\mathbb{G}) = \mathbf{a}$. Then $\mathbb{S}^{\bullet-\bullet}_{\mathbf{a}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\{\mathbb{G}\}))$.*

Proof. Let $e = \{\alpha,\beta\} \subseteq V'$, $\Lambda_e(\mathbb{G}) = \mathbf{a}$, and let $\gamma,\delta \notin V'$ be two additional fresh vertices outside of V'. We consider three graphs $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_{12}$ which differ from $\mathbb{G}$ only by:

- replacing α with γ (in case of $\mathbb{G}_1$),
- replacing β with δ (in case of $\mathbb{G}_2$),
- replacing both α and β with γ and δ, respectively (in case of $\mathbb{G}_{12}$).

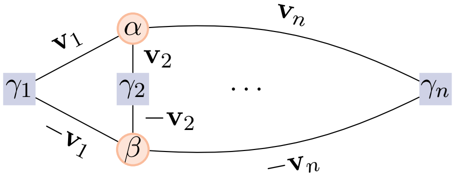
Clearly all the three graphs are equivalent to $\mathbb{G}$. We claim that $\mathbb{G} - \mathbb{G}_1 - \mathbb{G}_2 + \mathbb{G}_{12}$ yields:



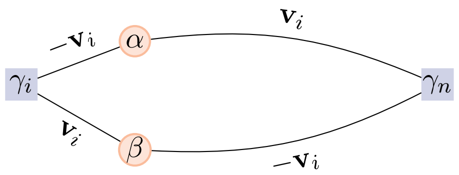
that is a graph equivalent to $\mathbb{S}^{\bullet-\bullet}_{\mathbf{a}}$. Indeed, the above graph operations cancel out all edges nonadjacent to the four vertices $\alpha,\beta,\gamma,\delta$, as well as all edges adjacent to only one of them. The two horizontal $\mathbf{a}$-weighted edges originate from $\mathbb{G}$ and $\mathbb{G}_{12}$, while the two remaining vertical ones originate from $-\mathbb{G}_1$ and $-\mathbb{G}_2$. □

Lemma 4.9. *Let $\alpha \in V'$ be a vertex of a graph $\mathbb{G} = (V',\mu')$, and let $\Lambda_\alpha(\mathbb{G}) = \mathbf{a}$. Then $\mathbb{S}^{\bullet}_{\mathbf{a}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\{\mathbb{G}\}))$.*

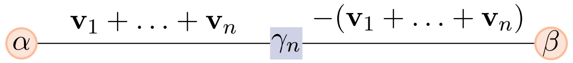
Proof. Let a graph $\mathbb{G}'$ differ from $\mathbb{G}$ only by replacing α with a fresh vertex $\beta \notin V'$. The graph $\mathbb{G}-\mathbb{G}'$ has the following shape (with blue square vertices representing the set $V'\backslash\{\alpha\} = \{\gamma_1,\gamma_2\ldots\gamma_n\}$):



Relying on Lemma 4.8, we add to the graph $\mathbb{G}-\mathbb{G}'$ the following graphs equivalent to $\mathbb{S}^{\bullet-\bullet}_{\mathbf{v}_i}$, for $i = 1,2\ldots n-1$:



This results in collapsing all blue vertices into one:


α
$\mathbf{v}_1 + \ldots + \mathbf{v}_n$
γ_n
$-(\mathbf{v}_1 + \ldots + \mathbf{v}_n)$
β

Since $\mathbf{a} = \mathbf{v}_1 + \ldots + \mathbf{v}_n$, the resulting graph is equivalent to $\mathbb{S}^{\bullet}_{\mathbf{a}}$, as required. □

Lemma 4.10. *Every* $\mathbb{H}$*-simple graph is a* $\mathbb{Z}$*-sum of* $\mathcal{H}$ *up to equivalence:* $\mathcal{S}^{\bullet}_{\mathbb{H}} \subseteq \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}\}))$ *and* $\mathcal{S}^{\bullet-\bullet}_{\mathbb{H}} \subseteq \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}\}))$.

Proof. Let $\mathbb{H} = (V, \mu)$. We have to prove that for any $\mathbf{a} \in \mathcal{V}_{\mathbb{H}}$ ($\mathcal{V}_{\mathbb{H}}$ is defined in Equation 4.6) it holds that $\mathbb{S}^{\bullet}_{\mathbf{a}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}))$ and that for any $\mathbf{b} \in \mathcal{E}_{\mathbb{H}}$ ($\mathcal{E}_{\mathbb{H}}$ is defined in Equation 4.5) it holds that $\mathbb{S}^{\bullet-\bullet}_{\mathbf{b}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}))$. We prove only $\mathbb{S}^{\bullet}_{\mathbf{a}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}))$ as the second proof is the almost same. Because of Lemma 4.2 we know that

$$\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H})))) = \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H})).$$

Thus, because of Lemma 4.9 it is sufficient to prove that there is a graph $\mathbb{G}_{\mathbf{a}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}))$ such that $\Lambda_\alpha(\mathbb{G}_{\mathbf{a}}) = \mathbf{a}$.

As $\mathbf{a} \in \mathcal{V}_{\mathbb{H}}$ we know that

$$\mathbf{a} = z_1 \cdot \mathbf{a}_1 + \ldots + z_l \cdot \mathbf{a}_l, \tag{4.8}$$

where $z_i \in \mathbb{Z}$ and $\mathbf{a}_i$ is $\Lambda_{\beta_i}(\mathbb{H})$ for some $\beta_i \in V$. By the assumption (4.2), for every i we know

$$\mathbf{a}_i \in \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_{\gamma_i}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{H}, \gamma_i \in V'\}).$$

we may use the equivalence relation and rewrite it as follows

$$\mathbf{a}_i \in \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_\alpha(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \textsc{Eq}(\mathcal{H})\}).$$

Thus

$$\mathbf{a}_i = z_{i,1}\Lambda_\alpha(\mathbb{H}'_{i,1}) + z_{i,2}\Lambda_\alpha(\mathbb{H}'_{i,2}) + \ldots + z_{i,h(i)}\Lambda_\alpha(\mathbb{H}'_{i,h(i)}) \text{ and further}$$

$$\mathbf{a} = \sum_{i=1}^{l} z_i \left(z_{i,1}\Lambda_\alpha(\mathbb{H}'_{i,1}) + z_{i,2}\Lambda_\alpha(\mathbb{H}'_{i,2}) + \ldots + z_{i,h(i)}\Lambda_\alpha(\mathbb{H}'_{i,h(i)}) \right)$$

Λ_α is a homomorphism, so

$$\mathbf{a} = \Lambda_\alpha \left(\sum_{i=1}^{l} z_i \left(z_{i,1}\mathbb{H}'_{i,1} + z_{i,2}\mathbb{H}'_{i,2} + \ldots + z_{i,h(i)}\mathbb{H}'_{i,h(i)} \right) \right)$$

so we define

$$\mathbb{G}_{\mathbf{a}} \stackrel{\text{def}}{=} \sum_{i=1}^{l} z_i \left(z_{i,1}\mathbb{H}'_{i,1} + z_{i,2}\mathbb{H}'_{i,2} + \ldots + z_{i,h(i)}\mathbb{H}'_{i,h(i)} \right).$$

The construction of elements of $\mathcal{S}^{\bullet-\bullet}_{\mathbb{H}}$ relays on Lemma 4.8 instead of Lemma 4.9. □

Combining Lemmas 4.4 and 4.10 we know that $\mathbb{H}$ is a $\mathbb{Z}$-sum of $\mathbb{H}$-simple graphs, each of which in turn is a $\mathbb{Z}$-sum of $\mathcal{H}$ up to equivalence. By Lemma 4.2, $\mathbb{H}$ is a $\mathbb{Z}$-sum of $\mathcal{H}$ up to equivalence, as required. The proof of Theorem 3.4, for graphs, is thus completed.

5. Proof of Theorem 3.4 (the outline).

Now, we are ready to prove the implication $2 \implies 1$ from Theorem 3.4 in full generality. The proof is split into five parts. We want to mimic the approach presented in Section 4, thus we need to generalise a few things:

(1) In Section 6 we build an algebraic background to be able to solve systems of equations that in the general case correspond to Equations 4.7.
(2) In Section 7 we introduce simple hypergraphs that generalise graphs defined in Definition 4.3.
(3) In Section 8 we show that our target hypergraph is a $\mathbb{Z}$-sum of simple hypergraphs. This generalises Lemma 4.4.
(4) In Section 9 we prove that simple hypergraphs are $\mathbb{Z}$-sums of $\mathrm{Eq}(\mathcal{H})$. This generalises Subsection Representation of $\mathbb{H}$-simple graphs i.e. Lemmas 4.8, 4.9, and 4.10.
(5) Finally, in Section 10 we complete the proof of Theorem 3.4.

6. Reduction matrices.

In this section we introduce a notion of reduction matrices and prove a key lemma about their rank, Lemma 6.3. They are $0, 1$ matrices related to adjacency matrices of Kneser graphs. We recall that by x-set we mean a set with x elements.

Definition 6.1. Let $a \in \mathbb{N}$ and $\mathcal{A}$ be an a-set. For $a \geq b \geq c$, $a, b, c \in \mathbb{N}$, we define a matrix $\overline{\lceil a, b, c \rceil}$ as follows:

(1) it has $\binom{a}{b}$ columns and $\binom{a}{c}$ rows,
(2) columns and rows are indexed with b-element subsets of $\mathcal{A}$ and c-element subsets of $\mathcal{A}$, respectively,
(3) $\overline{\lceil a, b, c \rceil}[\mathcal{C}, \mathcal{B}] = 1$ if $\mathcal{C} \subseteq \mathcal{B}$ and 0 otherwise, where $\mathcal{C}$ is an index of a row and $\mathcal{B}$ is an index of a column.

The definition is up to reordering of rows and columns.

Example 6.2. Suppose $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$.

(1) $$[\overline{\lceil 4,3,1 \rceil}] = \left[\begin{array}{ccccc} & \{\gamma,\beta,\alpha\} & \{\delta,\beta,\alpha\} & \{\delta,\gamma,\alpha\} & \{\delta,\gamma,\beta\} \\ \{\alpha\} & 1 & 1 & 1 & 0 \\ \{\beta\} & 1 & 1 & 0 & 1 \\ \{\gamma\} & 1 & 0 & 1 & 1 \\ \{\delta\} & 0 & 1 & 1 & 1 \end{array}\right]$$ up to a permutation of rows and columns.

(2) $$[\overline{\lceil 4,2,1 \rceil}] = \left[\begin{array}{ccccccc} & \{\beta,\alpha\} & \{\gamma,\alpha\} & \{\gamma,\beta\} & \{\delta,\alpha\} & \{\delta,\beta\} & \{\delta,\gamma\} \\ \{\alpha\} & 1 & 1 & 1 & 0 & 0 & 0 \\ \{\beta\} & 1 & 0 & 0 & 1 & 1 & 0 \\ \{\gamma\} & 0 & 1 & 0 & 1 & 0 & 1 \\ \{\delta\} & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$ up to a permutation of rows and columns.

The below lemma expresses the important property of reduction matrices.

Lemma 6.3. *Any* $\overline{\lceil 2k+1, k+1, k\rceil}$ *reduction matrix has maximal rank.*

The proof relies on a result from the spectral theory of Kneser graphs.

Definition 6.4. Let $\mathcal{A}$ be an a-set. The *Kneser graph* $K_{a,c}$ is the graph whose vertices are c-element subsets of $\mathcal{A}$, and where two vertices x, y are adjacent if, and only if, $x \cap y = \emptyset$.

The *adjacency matrix* of a graph $(\mathcal{X}, E)$ is a $\mathcal{X} \times \mathcal{X}$-matrix M such that $M[x, y] = 1$ if there is an edge between x and y and 0 otherwise.

Theorem 6.5 ([GR01] Theorem 9.4.3)**.** *All eigenvalues of the adjacency matrix for a Kneser graph are nonzero.*

Corollary 6.6. *The rank of the adjacency matrix for a Kneser graph is maximal.* □

Proof of Lemma 6.3. We relabel columns of $\overline{\lceil 2k+1, k+1, k\rceil}$ in the following way. If a column is labelled with the set $\mathcal{B} \subset \mathcal{A}$ we relabel it to $\mathcal{A} \setminus \mathcal{B}$. Now, both rows and columns are labelled with k-subsets of $\mathcal{A}$. We denote the relabelled matrix by M'. On the one hand, observe that if a k-set $\mathcal{C}$ is a subset of a $(k+1)$-set $\mathcal{B}$ then $\mathcal{C} \cap (\mathcal{A} \setminus \mathcal{B}) = \emptyset$, in which case $M'[\mathcal{C}, \mathcal{A} \setminus \mathcal{B}] = 1$. On the other hand if a k-set $\mathcal{C}$ is not a subset of a $(k+1)$-set $\mathcal{B}$ then $\mathcal{C} \cap (\mathcal{A} \setminus \mathcal{B}) \neq \emptyset$, and $M'[\mathcal{C}, \mathcal{A} \setminus \mathcal{B}] = 0$. So we see that M' is the adjacency matrix of the Kneser graph $K_{2k+1,k}$. But , because of Corollary 6.6, it has maximal rank, and the same holds for M as relabelling of columns does not change the rank of the matrix. □

7. Simple hypergraphs.

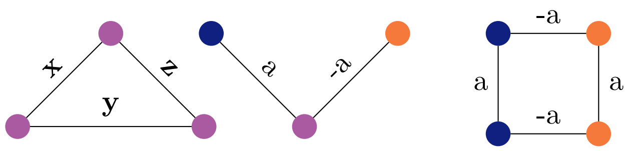
Definition 7.1. Let $\mathbb{H} = (V, \mu)$ be a k-hypergraph. We say that it is *m-isolated* if for any subset $\mathcal{X} \subseteq V$ such that $|\mathcal{X}| \leq m$ it holds that $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mathbf{0}$.

Definition 7.2. Let $0 \leq m \leq k$ and $\mathbf{a} \in \mathbb{Z}^d$. We call a k-hypergraph $\mathbb{G} = (V', \mu')$ $(m, \mathbf{a})$-*simple* if there exist pairwise disjoint sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that

(1) $V' = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.
(2) $\mathcal{A} = \{\alpha_1, \alpha_2 \dots \alpha_m\}$ and $\mathcal{B} = \{\beta_1, \beta_2 \dots \beta_m\}$ are m-sets.
(3) $|\mathcal{C}| = 2(k-m) - 1$ or $\mathcal{C} = \emptyset$ if $m = k$.
(4) $\forall \mathcal{X} \subseteq \mathcal{A} \cup \mathcal{B}$, such that $\mathcal{X} = \{x_1, x_2 \dots x_m\}$, where $x_i \in \{\alpha_i, \beta_i\}$, the equality $\Lambda_{\mathcal{X}}(\mathbb{G}) = (-1)^{|\mathcal{B} \cap \mathcal{X}|} \cdot \mathbf{a}$ holds.
(5) For any other $\mathcal{X} \subseteq V'$ such that $|\mathcal{X}| = m$ the equality $\Lambda_{\mathcal{X}}(\mathbb{G}) = \mathbf{0}$ holds.
(6) $\mathbb{G}$ is $(m-1)$-isolated.

This definition is hard so we analyse it using examples, and explain the required properties. Also, observe that contrary to $\mathbb{S}^{\bullet}_{\mathbf{a}}$ and $\mathbb{S}^{\bullet-\bullet}_{\mathbf{a}}$ $(m, \mathbf{a})$-simple hypergraphs are not fully specified up to isomorphism. Further, it is not clear that they exist for all $(m, \mathbf{a})$. In the example below we comment on this as well

Example 7.3. From the left to the right: a $(0, \mathbf{a})$-simple 2-hypergraph $\mathbb{G}_0$, where $\mathbf{a} = \mathbf{x} + \mathbf{y} + \mathbf{z}$; a $(1, \mathbf{a})$-simple 2-hypergraph $\mathbb{G}_1$; a $(2, \mathbf{a})$-simple 2-hypergraph $\mathbb{G}_2$. The sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are marked with colours.



(1) If $m = 0$ then $\mathcal{A}, \mathcal{B}$ are empty, Property 2 states that $\mathcal{C}$ has 3 vertices, Property 4 that $\Lambda_{\emptyset}(\mathbb{G}_0) = \mathbf{a}$, Properties 5 and 6 are trivial.

(2) If $m = 1$ then $|\mathcal{A}| = |\mathcal{B}| = |\mathcal{C}| = 1$, Properties 4 and 5 provide 3 equations

$$\mathbf{a} = \Lambda_{\text{🔵}}(\mathbb{G}_1) = \Lambda_{\{\text{🔵},\text{🟣}\}}(\mathbb{G}_1) + \Lambda_{\{\text{🔵},\text{🟠}\}}(\mathbb{G}_1) \tag{7.1}$$

$$-\mathbf{a} = \Lambda_{\text{🟠}}(\mathbb{G}_1) = \Lambda_{\{\text{🟠},\text{🟣}\}}(\mathbb{G}_1) + \Lambda_{\{\text{🔵},\text{🟠}\}}(\mathbb{G}_1) \tag{7.2}$$

$$\mathbf{0} = \Lambda_{\text{🟣}}(\mathbb{G}_1) = \Lambda_{\{\text{🟠},\text{🟣}\}}(\mathbb{G}_1) + \Lambda_{\{\text{🔵},\text{🟣}\}}(\mathbb{G}_1). \tag{7.3}$$

It is not hard to derive from them that $\mathbf{a} = \Lambda_{\{\text{🔵},\text{🟣}\}}(\mathbb{G}_1)$, $-\mathbf{a} = \Lambda_{\{\text{🟠},\text{🟣}\}}(\mathbb{G}_1)$, and $\mathbf{0} = \Lambda_{\{\text{🔵},\text{🟠}\}}(\mathbb{G}_1)$. Property 6 states that $\Lambda_{\emptyset}(\mathbb{G}_1) = \mathbf{0}$.

(3) If $m = 2$ then $|\mathcal{A}| = |\mathcal{B}| = 2$ and $\mathcal{C} = \emptyset$. Conditions 4 and 5 define weights of all edges, the Property 4 is responsible for edges with nonzero weights and 5 for the edges with the weight $\mathbf{0}$. Property 6 says that the weight of $\emptyset$ and weights of single vertices are $\mathbf{0}$.

□

Remark 7.4. Note that $(m, \mathbf{a})$-simple hypergraphs are not defined uniquely. For example, the $(0, \mathbf{a})$-simple hypergraph from the example above is not fully defined, as for any $\mathbf{x}$ and $\mathbf{y}$ we may choose appropriate $\mathbf{z} = \mathbf{a} - \mathbf{x} - \mathbf{y}$. The existence of all simple hypergraphs is proven later (we show how to construct them).

Now, why we characterise simple hypergraphs using such complicated Properties? In the following we justify our design. The most important property of simple hypergraph is the last one. It implies the following lemma:

Lemma 7.5. *Let $\mathbb{G} = (V', \mu')$ be a k-hypergraph and $\mathbb{S}^m_{\mathbf{a}}$ be an $(m, \mathbf{a})$-simple k-hypergraph. Then for any $\mathcal{X} \subseteq V', |\mathcal{X}| < m$ we have $\Lambda_{\mathcal{X}}(\mathbb{G}) = \Lambda_{\mathcal{X}}(\mathbb{G} + \mathbb{S}^m_{\mathbf{a}})$.*

Proof. Indeed, $\Lambda_{\mathcal{X}}$ is a homomorphism and $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}) = \mathbf{0}$ because of Property 6 of Definition 7.2. □

Property 6, by this lemma, allows for the following structure of the proof of Theorem 3.4. We gradually simplify the target hypergraph by adding $(i, \mathbf{a})$-simple hypergraphs for growing i. In the step i we start from an $(i-1)$-isolated hypergraph. Using $(i, \mathbf{a})$-simple hypergraphs we simplify it to a hypergraph with $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mathbf{0}$ for all $\mathcal{X} \subset V$ with exactly i vertices. The property 6 guaranties that while we perform the step i, we do not ruin our work from previous steps i.e. we reach an i-isolated hypergraph. Eventually, we reach a hypergraph with all the weights equal to $\mathbf{0}$ i.e. the empty hypergraph.

Now, let us justify the design of Properties 4 and 5. We already mentioned that using $(i, \mathbf{a})$-simple hypergraphs we want to reduce to $\mathbf{0}$ all weights of sets in $[V]^i$; thus it is good to keep weights on the level i as simple as possible i.e. $\mathbf{a}, \mathbf{0}, -\mathbf{a}$.

The designs of Properties 1-3 is a trade-off between two things:

- we want to have simple hypergraphs as small as possible in terms of number of vertices and number of nonzero hyperedges,
- for the proof of Theorem 3.4, we need that simple hypergraphs, from which we construct the target hypergraph $\mathbb{H}$, may be constructed from hypergraphs in the family $\mathcal{H}$.

Now, that Properties 1-6 are commented, we may go back to the proof of the Theorem 3.4.

We introduce two other notions to work with families of simple hypergraphs.

Definition 7.6. Let $\mathcal{G}$ be a family of k-hypergraphs, for every $\mathbb{G} \in \mathcal{G}$ we denote the set of its vertices by $V_{\mathbb{G}}$. A family of k-hypergraphs $\mathcal{S}$ is a *simplification of the family* $\mathcal{G}$ if for every $0 \leq m \leq k$ and every $\mathbf{g}$ in $\mathbb{Z}$-sums of $\{\Lambda_{\mathcal{X}}(\mathbb{G}) \mid \mathbb{G} \in \mathcal{G}, \mathcal{X} \in [V_{\mathbb{G}}]^m\}$ it contains an $(m, \mathbf{g})$-simple k-hypergraph.

Definition 7.7. The family $\mathcal{S}$ is *self-simplified* if $\mathcal{S}$ is a simplification of $\mathcal{S}$. For a family of k-hypergraphs $\mathcal{H}$, *a family* $\mathcal{S}$ *is simple for* $\mathcal{H}$ if $\mathcal{S}$ is self-simplified and is a simplification of $\mathcal{H}$.

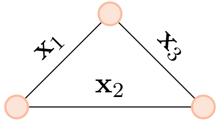
Remark 7.8. To produce a self-simplified family, it suffices to design an algorithm $alg(\mathcal{S})$ that produces a simplification of its input. The following family is self-simplified

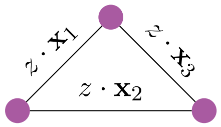
$$\bigcup_{i \in \mathbb{N}} alg^{i+1}(\mathcal{H}) \text{ where } alg^i \text{ is i-th iteration of the } alg.$$

Note that the produced family is not necessarily finite.

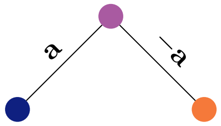
Example 7.9. The picture below presents an example of a simple family $\mathcal{G}$ for the graph $\mathbb{H}$. $z \in \mathbb{Z}$ and the sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are marked with colours.

$\mathbb{H} =$

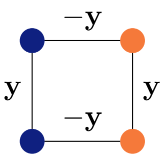




for all $z \in \mathbb{Z}$.



for all $\mathbf{a}$ in $\mathbb{Z}$-Sums$(\{\mathbf{x}_1 + \mathbf{x}_2,\ \mathbf{x}_2 + \mathbf{x}_3,\ \mathbf{x}_3 + \mathbf{x}_1\})$.



for all $\mathbf{y}$ in $\mathbb{Z}$-Sums$(\{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3\})$.

The family is infinite. Depicted graphs represent $(0, z(\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3))$-simple hypergraphs, $(1, \mathbf{a})$-simple hypergraphs, $(2, \mathbf{y})$-simple hypergraphs. To see that the family is self-simplified we observe the few following facts:

- $\{\Lambda_{\emptyset}(\mathbb{G}) \mid \mathbb{G} \in \mathcal{G}\} = \{z(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) \mid z \in \mathbb{Z}\} = \mathbb{Z}\text{-Sums}(\{z(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) \mid z \in \mathbb{Z}\})$. Indeed, $\Lambda_{\emptyset}(\mathbb{G}) = \mathbf{0}$ for $\mathbb{G}$ of the second or the third type of graphs. So the family contains all required $(0, \square)$-simple graphs (triangles).
- $\{\Lambda_{\alpha}(\mathbb{G}) \mid \mathbb{G} = (V_{\mathbb{G}}, \mu_{\mathbb{G}}) \in \mathcal{G}, \alpha \in V_{\mathbb{G}}\} = \{\mathbf{a} \mid \mathbf{a} \in \mathbb{Z}\text{-Sums}(\{\mathbf{x}_1 + \mathbf{x}_2,\ \mathbf{x}_2 + \mathbf{x}_3,\ \mathbf{x}_3 + \mathbf{x}_1\})\}$. This is because of,
  - $\Lambda_{\alpha}(\mathbb{G}) = \mathbf{0}$ for any $\mathbb{G} = (V_{\mathbb{G}}, \mu_{\mathbb{G}})$ of the third type of graphs and any $\alpha \in V_{\mathbb{G}}$.
  - $\Lambda_{\alpha}(\mathbb{G}) = \mathbf{0}$ for $\mathbb{G}$ of the second type of graphs and α being the middle vertex.
  - it is easy to check that $\Lambda_{\alpha}(\mathbb{G}) \in \{\mathbf{a} \mid \mathbb{Z}\text{-Sums}(\{\mathbf{x}_1 + \mathbf{x}_2,\ \mathbf{x}_2 + \mathbf{x}_3,\ \mathbf{x}_3 + \mathbf{x}_1\})\}$ also for other cases.

Thus, the family contains all required $(1, \mathbf{a})$-simple graphs (the second type).

- $\{\Lambda_{\{\alpha,\beta\}}(\mathbb{G}) \mid \mathbb{G} = (V_{\mathbb{G}}, \mu_{\mathbb{G}}) \in \mathcal{G}, \{\alpha, \beta\} \in [V_{\mathbb{G}}]^2\} = \{\mathbf{a} \mid \mathbb{Z}\text{-}\textsc{Sums}(\{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3\})\}$. This is also easy to check.
  Because of $\mathbb{Z}\text{-}\textsc{Sums}(\mathbb{Z}\text{-}\textsc{Sums}(\mathcal{X})) = \mathbb{Z}\text{-}\textsc{Sums}(\mathcal{X})$ for any set $\mathcal{X}$, we see that all needed simple graphs are elements of the family $\mathcal{G}$. Thus, the family contains all required $(2, \mathbf{y})$-simple graphs (the square type).

Lemma 7.10. *Let $\mathcal{S}$ be a family simple for a k-hypergraph $\mathbb{H}$. Suppose $\mathbb{G} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}))$ then the family $\mathcal{S}$ is simple for $\{\mathbb{H} + \mathbb{G}\}$.*

Proof. As $\mathcal{S}$ is self-simplified, we only need prove that $\mathcal{S}$ is a simplification of $\{\mathbb{H} + \mathbb{G}\}$. Let $\mathbb{G} = (V_{\mathbb{G}}, \mu_{\mathbb{G}})$, $\mathbb{H} = (V_{\mathbb{H}}, \mu_{\mathbb{H}})$, and $m \le k$. Let $\mathbf{g} \in \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_{\mathcal{X}}(\mathbb{H} + \mathbb{G}) \mid \mathcal{X} \in [(V_{\mathbb{H}} \cup V_{\mathbb{G}})]^m\})$. We have to prove that $\mathcal{S}$ contains an $(m, \mathbf{g})$-simple graph. Suppose that $\mathbf{g} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 \ldots + c_l\mathbf{a}_l$ where $\mathbf{a}_i \in \{\Lambda_{\mathcal{X}}(\mathbb{H} + \mathbb{G}) \mid \mathcal{X} \in [(V_{\mathbb{H}} \cup V_{\mathbb{G}})]^m\}$ and $c_i \in \mathbb{Z}$. By definition, if in $\mathcal{S}$ there are $(i, \mathbf{a})$-simple hypergraphs and $(i, \mathbf{b})$-simple hypergraphs then in $\mathcal{S}$ there is an $(i, \mathbf{a}+\mathbf{b})$-simple hypergraph too. Thus, it is sufficient to prove that in $\mathcal{S}$ there are $(m, \mathbf{a}_i)$-simple hypergraphs for $\mathbf{a}_i \in \{\Lambda_{\mathcal{X}}(\mathbb{H} + \mathbb{G}) \mid \mathcal{X} \in [(V_{\mathbb{H}} \cup V_{\mathbb{G}})]^m\}$. This is also trivial as $\Lambda_{\mathcal{X}}(\mathbb{H}+\mathbb{G}) = \Lambda_{\mathcal{X}}(\mathbb{H}) + \Lambda_{\mathcal{X}}(\mathbb{G})$ and there are $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{H}))$-simple and $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{G}))$-simple hypergraphs in $\mathcal{S}$. The first one is in $\mathcal{S}$ because $\mathcal{S}$ is a family simple for $\{\mathbb{H}\}$ and the second is in $\mathcal{S}$ because $\mathcal{S}$ is self-simplified. □

8. Expressing $\mathbb{H}$ with simple hypergraphs

Our goal in this section is to prove Theorem 8.2.

Definition 8.1. For a hypergraph $\mathbb{H} = (V, \mu)$ its support is V. We denote it $\textsc{support}(\mathbb{H})$. Let $\mathbb{H} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{H}))$. We say that V' *supports* $\mathbb{H}$ *for the family* $\mathcal{H}$ if there is a solution to the following equation

$$\mathbb{H} = \sum_{i>0} a_i \mathbb{H}_i \text{ where } a_i \in \mathbb{Z}, \mathbb{H}_i \in \textsc{Eq}(\mathcal{H}),\ \text{and } \textsc{support}(\mathbb{H}_i) \subseteq V' \text{ for all } i.$$

For the second part of the definition, one should think that V' may include all supports of the elements of the sum. This naming is motivated by the literature where support is used for objects but also for sets of objects, expressions build from objects e.t.c. For the convenience of the reader in the following text whenever we speak about support of the hypergraph we use $\textsc{support}()$ symbol and all occurrences of word support in the text are used as " . . . supports . . . for a family . . . ".

Theorem 8.2. *Let $\mathbb{H} = (V, \mu)$ be a k-hypergraph. Further, let $\mathcal{S}$ be a simple family for $\mathbb{H}$. Then $\mathbb{H} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathcal{S}))$. Moreover, if $|V| > 2k - 1$ then V supports $\mathbb{H}$ for the family $\mathcal{S}$.*

The proof of this theorem requires a few definitions and lemmas stated below. We start with them and then we prove the theorem while proofs of lemmas are postponed.

Let us recall (Definitiont 7.1) that, a k-hypergraph $\mathbb{H} = (V, \mu)$ is *m-isolated* if for any subset $\mathcal{X} \subseteq V$ such that $|\mathcal{X}| \le m$ it holds that $\Lambda_{\mathcal{X}}(\mathbb{H}) = \mathbf{0}$.

Remark 8.3. If $\mathbb{H}$ is a k-isolated k-hypergraph, then $\mathbb{H}$ is equivalent to the empty hypergraph i.e. it is a union of isolated vertices.

Definition 8.4. Let $\mathbb{H} = (V, \mu)$ be a k-hypergraph. We say that it is *pre m-isolated* if the following two conditions are satisfied:

- $\mathbb{H}$ is $(m-1)$-isolated,
- there is $\mathcal{X} \subseteq V$ a set of vertices such that $|\mathcal{X}| \leq 2m-1$ and for any $\mathcal{Y} \in [V]^m$ such that $\mathcal{Y} \not\subseteq \mathcal{X}$ it holds that $\Lambda_{\mathcal{Y}}(\mathbb{H}) = \mathbf{0}$.

Lemma 8.5. *If $\mathbb{H}$ is a k-hypergraph that is pre m-isolated then it is m-isolated.*

Lemma 8.6. *Let $\mathbb{H} = (V, \mu)$ be an m-isolated k-hypergraph and $\mathcal{S}$ be a simple family for $\mathbb{H}$. Then there is a hypergraph $\mathbb{G} \in \mathbb{Z}$-*SUMS*$(\text{EQ}(\mathcal{S}))$, such that $\mathbb{H} + \mathbb{G}$ is pre $(m+1)$-isolated. Moreover, if $|V| > 2k-1$ then V supports $\mathbb{G}$ for the family $\mathcal{S}$.*

Proof of Theorem 8.2. We assume that $|V| > 2k-1$, as otherwise we extend $\mathbb{H}$ with a few isolated vertices.

Suppose we have a sequence of hypergraphs $\mathbb{G}_i \in \mathbb{Z}$-SUMS$(\text{EQ}(\mathcal{S}))$ which satisfies following properties.

- For every $j \leq k$ the hypergraph $\mathbb{H} - \sum_{i=0}^{j} \mathbb{G}_i$ is j-isolated.
- For every $i \leq k$ the set V supports $\mathbb{G}_i$ for the family $\mathcal{S}$.

We define $\mathbb{G}$ as $\sum_{i=0}^{k} \mathbb{G}_i$. Now, observe that

- $\mathbb{G} \in \mathbb{Z}$-SUMS$(\text{EQ}(\mathcal{S}))$ as each $\mathbb{G}_i$ is in $\mathbb{Z}$-SUMS$(\text{EQ}(\mathcal{S}))$.
- SUPPORT$(\mathbb{G}) \subseteq V$ as SUPPORT$(\mathbb{G}_i) \subseteq V$ for each $i \leq k$.
- V supports $\mathbb{G}$ for the family $\mathcal{S}$ as V supports $\mathbb{G}_i$ for the family $\mathcal{S}$ for each $i \leq k$.
- $\mathbb{H} = \mathbb{G}$ up to equivalence. Because of the first property we know that $\mathbb{H} - \mathbb{G}$ is a k-isolated k-hypergraph, and every k-isolated k-hypergraph is the empty hypergraph, because of Remark 8.3.

Thus, if we have the sequence $\mathbb{G}_i$ then we are done. Now, we show how the sequence $\mathbb{G}_i$ may be constructed. The construction is via induction on i. As $\mathcal{S}$ is a simple family for $\mathbb{H}$ then there is $\mathbb{G}_0 \in \text{EQ}(\mathcal{S})$ and SUPPORT$(\mathbb{G}_0) \subset V$ such that $\mathbb{H} - \mathbb{G}_0$ is 0-isolated. This creates the induction base.

For the inductive step we reason as follows. First we observe that because of Lemma 7.10 the family $\mathcal{S}$ is simple for $\mathbb{H} - \sum_{j=0}^{i} \mathbb{G}_j$. Thus, we may use Lemma 8.6 for the hypergraph $\mathbb{H} - \sum_{j=0}^{i} \mathbb{G}_j$ and the family $\mathcal{S}$. As a consequence we get $\mathbb{G}_{i+1} \in \mathbb{Z}$-SUMS$(\text{EQ}(\mathcal{S}))$ such that $(\mathbb{H} - \sum_{j=0}^{i} \mathbb{G}_j) - \mathbb{G}_{i+1}$ is pre $(i+1)$-isolated. Moreover, V supports $\mathbb{G}_{i+1}$ for the family $\mathcal{S}$. Further, Lemma 8.5 implies that $(\mathbb{H} - \sum_{j=0}^{i} \mathbb{G}_j) - \mathbb{G}_{i+1}$ is $(i+1)$-isolated.

Note that V supports $\mathbb{G}_{i+1}$ for the family $\mathcal{S}$ thus V supports $(\mathbb{H} - \sum_{j=0}^{i+1} \mathbb{G}_j)$ for the family $\mathcal{S}$, too. This ends the inductive step. □

Proof of Lemma 8.5.

Before we prove Lemma 8.5 we prove a simple lemma about the $\Lambda_{\mathcal{X}}$ functions.

Lemma 8.7. *Suppose $\mathbb{H} = (V, \mu)$ is a k-hypergraph, $\mathcal{X} \in [V]^m$, and $m \leq l \leq k$. Let $\mathcal{F} = \{\mathcal{Y} \in [V]^l \mid \mathcal{X} \subseteq \mathcal{Y}\}$, be the family of l-element supersets of $\mathcal{X}$. Then:*

$$\sum_{\mathcal{Y} \in \mathcal{F}} \Lambda_{\mathcal{Y}}(\mathbb{H}) = \binom{k-m}{l-m} \Lambda_{\mathcal{X}}(\mathbb{H}).$$

Proof. Let us recall definition of $\Lambda_{\mathcal{X}}(\mathbb{H})$. It is the sum of all edges e in $\mathbb{H}$ such that $\mathcal{X} \subseteq e$. Let e be a hyperedge in $\mathbb{H}$ such that $\mathcal{X} \subseteq e$. It suffices to prove that $\mu(e)$ appears the same number of times on both sides of the equation. On the right side e is added $\binom{k-m}{l-m}$ times, as $\mu(e)$ appears once in the expression $\Lambda_{\mathcal{X}}(\mathbb{H})$. On the left side the number of times when $\mu(e)$ is added is equal to the number of l-element supersets of $\mathcal{X}$ that are included in e. This is because $\mu(e)$ appears once in the expression $\Lambda_{\mathcal{Y}}(\mathbb{H})$ if $\mathcal{Y} \subseteq e$. But this is equal to $\binom{|e|-|\mathcal{X}|}{l-|\mathcal{X}|} = \binom{k-m}{l-m}$, as required. □

The main tool in the proof of Lemma 8.5 is reduction matrices defined in Section 6.

Proof of Lemma 8.5. Let $\mathcal{X}$ be a set of vertices as in Definition 8.4. We need to prove that for any $\mathcal{X}' \subseteq \mathcal{X}$ such that $|\mathcal{X}'| = m$ it holds that $\Lambda_{\mathcal{X}'}(\mathbb{H}) = \mathbf{0}$.

Let $\mathcal{Y}_1, \mathcal{Y}_2 \ldots \mathcal{Y}_n$ be all the $(m-1)$-subsets of $\mathcal{X}$ and let $\mathcal{X}'_1, \mathcal{X}'_2 \ldots \mathcal{X}'_{n'}$ be all the m-subsets of $\mathcal{X}$.

Because of Lemma 8.7, we know that, for any $\mathcal{Y}_i$, the equation

$$\sum_{\mathcal{X}'_j \supset \mathcal{Y}_i} \Lambda_{\mathcal{X}'_j}(\mathbb{H}) = \binom{k-m}{m-(m-1)} \cdot \Lambda_{\mathcal{Y}_j}(\mathbb{H})$$

holds. But $\mathbb{H}$ is $m-1$ isolated so $\Lambda_{\mathcal{Y}_j}(\mathbb{H}) = \mathbf{0}$.

This system of equation may be rewritten in matrix form

$$Cu = \mathbf{0} \text{ where}$$

$$u \overset{\text{def}}{=} \begin{bmatrix} \Lambda_{\mathcal{X}'_1}(\mathbb{H}) \\ \Lambda_{\mathcal{X}'_2}(\mathbb{H}) \\ \vdots \\ \Lambda_{\mathcal{X}'_n}(\mathbb{H}) \end{bmatrix}$$

and C is the matrix with $\binom{|\mathcal{X}|}{m} = \binom{2m-1}{m}$ columns indexed with m-element subsets of the set $\mathcal{X}$, $\binom{|\mathcal{X}|}{m-1} = \binom{2m-1}{m-1}$ rows indexed with $(m-1)$-element subsets of the set $\mathcal{X}$, such that each individual entry represents inclusion between the index of the row and the index of the column. So up to permutation of rows and columns C is the $\overline{\lceil 2m-1, m, m-1\rceil}$ matrix ($\overline{\lceil \bullet, \bullet, \bullet\rceil}$ is defined in Definition 6.1).

But according to Lemma 6.3 the rank of the matrix C is maximal, which implies $u = \mathbf{0}$ is the only solution of the system of equations. Thus, $\Lambda_{\mathcal{X}'_j}(\mathbb{H}) = \mathbf{0}$ for any $j \leq n'$ and consequently $\mathbb{H}$ is m-isolated. □

Proof of Lemma 8.6.

The proof of Lemma 8.6 requires some preparation.

Definition 8.8. Let $\mathbb{F} = (W, \mu')$ is a k-hypergraph. Suppose $\mathcal{X} \subset W, |\mathcal{X}| < k$. We define $\mathbb{F}_{|-\mathcal{X}} \overset{\text{def}}{=} (W \setminus \mathcal{X}, {\mu'}_{|-\mathcal{X}})$, where ${\mu'}_{|-\mathcal{X}}(e)$ is a function from $[(W-\mathcal{X})]^{k-|\mathcal{X}|}$ to $\mathbb{Z}^d$ and is defined as follows

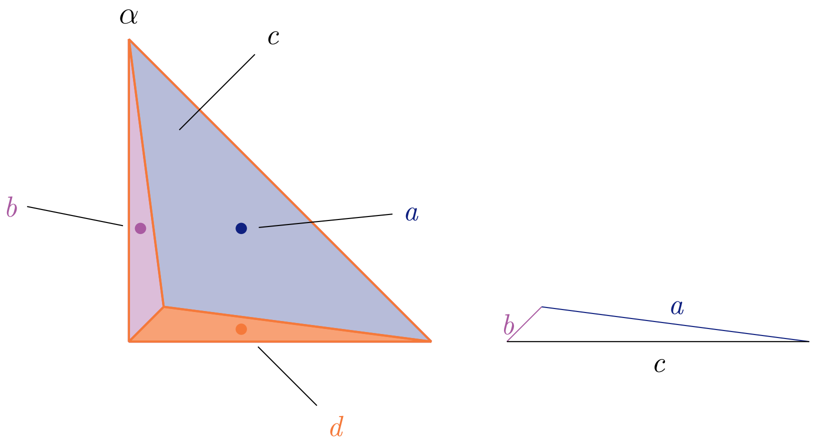
$${\mu'}_{|-\mathcal{X}}(e) \overset{\text{def}}{=} \mu'(e \cup \mathcal{X}) \text{ where } e \in [(W-\mathcal{X})]^{k-|\mathcal{X}|}.$$

The above operation is called $\mathcal{X}$-*cut* of $\mathbb{F}$.

For $\mathcal{X} \cap W = \emptyset$ we define the reverse operation called *enriching* $\mathbb{F}$ with $\mathcal{X}$. It is denoted by

$\mathbb{F}_{|+\mathcal{X}}$ and its effect is the minimal in the sense of inclusion[1] $(k+|\mathcal{X}|)$-hypergraph $\mathbb{F}'$ such that $\mathbb{F}'_{|-\mathcal{X}} = \mathbb{F}$.

If $\mathcal{X} = \{\alpha\}$ is a singleton, then we simplify the notation $\mathbb{F}_{|-\alpha} \overset{\text{def}}{=} \mathbb{F}_{|-\{\alpha\}}$ and $\mathbb{F}_{|+\alpha} \overset{\text{def}}{=} \mathbb{F}_{|+\{\alpha\}}$.



Example 8.9.

On the left side there is a 3-hypergraph $\mathbb{F}$ and on the right side there is a 2-hypergraph $\mathbb{F}_{|-\alpha}$. We use letters a, b, c, d to name corresponding hyperedges. The orange bottom hyperedge d disappears as it does not contain α. $\mathbb{F}_{|-\alpha_{|+\alpha}}$ would look like $\mathbb{F}$ but without the orange hyperedge, as the enriching operation takes the minimal hypergraph among all such that cutting α returns $\mathbb{F}_{|-\alpha}$.

Lemma 8.10. *Let $\mathbb{F} = (W, \mu')$ be a k-hypergraph and $\mathcal{X} \in [W]^m$ for $m < k$. Then for any nonempty set $\mathcal{Y} \subseteq W$ such that $\mathcal{Y} \cap \mathcal{X} = \emptyset$ it holds that $\Lambda_{\mathcal{Y}\cup\mathcal{X}}(\mathbb{F}) = \Lambda_{\mathcal{Y}}(\mathbb{F}_{|-\mathcal{X}})$. In particular, if $\mathbb{F}$ is l-isolated and $m < l$ then $\mathbb{F}_{|-\mathcal{X}}$ is $(l-m)$-isolated.*

Proof. In the equation below, e and e' are indexes, while $W, \mathcal{Y}$ and $\mathcal{X}$ are fixed.

$$\begin{aligned}
&\Lambda_{\mathcal{Y}\cup\mathcal{X}}(\mathbb{F}) = \\
&\sum_{e\in[W]^k, \mathcal{Y}\cup\mathcal{X}\subseteq e} \mu'(e) = \\
&\sum_{e\in[W]^k, \mathcal{Y}\cup\mathcal{X}\subseteq e} \Lambda_{e\setminus\mathcal{X}}(\mathbb{F}_{|-\mathcal{X}}) = \\
&\sum_{(e\setminus\mathcal{X})[W\setminus\mathcal{X}]^{k-|\mathcal{X}|}, \mathcal{Y}\subseteq(e\setminus\mathcal{X})} \Lambda_{e\setminus\mathcal{X}}(\mathbb{F}_{|-\mathcal{X}}) = \\
&\sum_{e'\in[W\setminus\mathcal{X}]^{k-|\mathcal{X}|}, \mathcal{Y}\subseteq e'} \Lambda_{e'}(\mathbb{F}_{|-\mathcal{X}}) = \\
&\Lambda_{\mathcal{Y}}(\mathbb{F}_{|-\mathcal{X}})
\end{aligned}$$

□

[1] we say that a hypergraph includes a second hypergraph if the set of vertices of the first hypergraph includes the set of vertices of the second one and every hyperedge of the second hypergraph is also a hyperedge of the first hypergraph.

Proof of Lemma 8.6. Without loss of generality, we assume that $|V| > 2k-1$. Indeed, if $|V| \leq 2k-1$ then we extend $\mathbb{H}$ with a few isolated vertices.

In the proof, instead of constructing $\mathbb{G}$ directly, we build a finite sequence of hypergraphs $\mathbb{H}_0 \dots \mathbb{H}_{last}$, such that:

- $\mathbb{H}_0 = \mathbb{H}$,
- $\mathbb{H}_{last}$ is pre $(m+1)$-isolated,
- there is a well-quasi order $>$ on hypergraphs such that $\mathbb{H}_i > \mathbb{H}_{i+1}$ for each $0 \leq i < last$, and
- each $\mathbb{H}_{i+1} = \mathbb{H}_i - \mathbb{S}_i$ where $\mathbb{S}_i$ is a $(m+1, \square)$-simple hypergraph such that $\mathbb{S}_i \in \text{EQ}(\mathcal{S})$ and $\text{SUPPORT}(\mathbb{S}_i) \subseteq V$.

Intuitively, we build a sequence of improvements $\mathbb{S}_i$ and consequently we will set $\mathbb{G}$, the hypergraph that we need to build, to be $-\sum_i \mathbb{S}_i$.

First, we define the well-founded quasi-order on hypergraphs, then we present a single improvement i.e. how to obtain $\mathbb{H}_{i+1}$ from $\mathbb{H}_i$, finally we show that $\mathbb{H}_{i+1} < \mathbb{H}_i$ in the defined quasi-order and that if the hypergraph cannot be improved anymore then it is pre $(m+1)$-isolated.

Order. First, we impose an arbitrary well-founded linear order on the set of all vertices. For two sets of vertices $\mathcal{X}_1$ and $\mathcal{X}_2$, we write $\mathcal{X}_1 < \mathcal{X}_2$ if there is a bijection $f \colon \mathcal{X}_1 \to \mathcal{X}_2$ such that $\alpha \leq f(\alpha)$ for every $\alpha \in \mathcal{X}_1$. Finally, for two hypergraphs $\mathbb{H} = (V, \mu)$ and $\mathbb{H}' = (V', \mu')$ we write $\mathbb{H}' \leq_{m+1} \mathbb{H}$ if for every $\mathcal{X}' \in [V']^{m+1}$ such that $\Lambda_{\mathcal{X}'}(\mathbb{H}') \neq \mathbf{0}$ there is a set $\mathcal{X} \in [V]^{m+1}$ such that $\Lambda_{\mathcal{X}}(\mathbb{H}) \neq \mathbf{0}$ and $\mathcal{X}' \leq \mathcal{X}$.

The strict inequality $<_{m+1}$ holds if $\mathbb{H}' \leq_{m+1} \mathbb{H}$ but $\mathbb{H}' \not\geq_{m+1} \mathbb{H}$.

Assumption 8.11. *From now on until the end of this proof, whenever we enumerate vertices of some set $\{\alpha_1, \alpha_2 \dots \alpha_n\}$, we assume that $\alpha_i < \alpha_j$ for $i < j$.*

Improvement. We define the improvement for $\mathbb{H}_i$. Let $\mathcal{F} \subset [\text{SUPPORT}(\mathbb{H}_i)]^{m+1}$ be the family of all sets such that for every set $\mathcal{L} \in \mathcal{F}$ it holds that $\Lambda_{\mathcal{L}}(\mathbb{H}_i) \neq \mathbf{0}$. Note that, if $m+1 = k$ then $\mathcal{F}$ is the set of hyperedges of $\mathbb{H}_i$.

Suppose $\mathcal{L} \in \mathcal{F}$ is maximal in $\mathcal{F}$ and there is a set $\mathcal{L}' \in [V]^{m+1}$ disjoint from $\mathcal{L}$ such that $\mathcal{L}' < \mathcal{L}$. Let $\mathcal{L} = \{\alpha_1, \alpha_2 \dots \alpha_{m+1}\}$. Let $\mathbb{S}_i \in \text{EQ}(\mathcal{S})$ be an $(m+1, \Lambda_{\mathcal{L}}(\mathbb{H}_i))$-simple hypergraph such that $\text{SUPPORT}(\mathbb{S}_i) \subseteq V$, and such that (using the notation of Definition 7.2) $\mathcal{A} = \mathcal{L}$ and $\mathcal{B} = \mathcal{L}' = \{f^{-1}(\alpha_1), f^{-1}(\alpha_2) \dots f^{-1}(\alpha_{m+1})\}$, where f witnesses $\mathcal{L}' < \mathcal{L}$. The hypergraph $\mathbb{S}_i$ exists because of Lemma 7.10.

Let $\mathbb{H}_{i+1} \overset{\text{def}}{=} \mathbb{H}_i - \mathbb{S}_i$. We claim $\mathbb{H}_{i+1} <_{m+1} \mathbb{H}_i$. First, we show inequality, next we show that it is strict. For the inequality we take any subset $\widetilde{\mathcal{L}} \in [V]^{m+1}$ such that $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{H}_{i+1}) \neq \mathbf{0}$ and we find a subset $\widetilde{\mathcal{L}}' \in [V]^{m+1}$ such that $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{H}_i) \neq \mathbf{0}$ and $\widetilde{\mathcal{L}} \leq \widetilde{\mathcal{L}}'$. We have to consider two cases:

- $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{H}_{i+1}) \neq \mathbf{0}$ because of $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{H}_i) \neq \mathbf{0}$ and
- $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{H}_{i+1}) \neq \mathbf{0}$ because of $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{S}_i) \neq \mathbf{0}$.

In the first case $\widetilde{\mathcal{L}}' = \widetilde{\mathcal{L}}$. In the second case $\widetilde{\mathcal{L}}' = \mathcal{L}$. We observe that $\widetilde{\mathcal{L}} \leq \mathcal{L}$ as $\widetilde{\mathcal{L}} = \{x_1, x_2 \dots x_{m+1}\}$ where $x_i \in \{\alpha_i, f^{-1}(\alpha_i)\}$ and $\mathcal{L} = \{\alpha_1, \alpha_2 \dots \alpha_{m+1}\}$ which is the greatest set (with respect to the order $<$) among such sets.

Now, to prove strictness of the inequality we observe that $\Lambda_{\mathcal{L}}(\mathbb{H}_{i+1}) = \mathbf{0}$ and that there is no set $\widetilde{\mathcal{L}}$ such that $\mathcal{L} \leq \widetilde{\mathcal{L}}$ and $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{H}_{i+1}) \neq 0$, so $\mathbb{H}_i \not\leq \mathbb{H}_{i+1}$. The formal proof requires the same case analysis as the proof of the nonstrict inequality. In the first case, we use maximality of $\mathcal{L}$ in the family $\mathcal{F}$. In the second case, we use the fact that $\mathcal{L}$ is is the greatest set (with respect to the order $<$) among possible sets.

Reduced form. We call a hypergraph *reduced* if one cannot improve it any further. If we consequently improve a given hypergraph then eventually we reach $\mathbb{H}_{last}$ which is reduced. Indeed, every improvement goes down in the quasi-order on hypergraphs and the quasi-order is trivially well-founded. What remains to prove is the following claim.

Claim 8.12. A reduced hypergraph is pre $(m+1)$-isolated.

Proof of the claim. First observe that $\mathbb{H}_{last}$ is m-isolated as $\mathbb{H}$ and each $\mathbb{S}_i$ are m-isolated (by Definition 7.2 every $(m+1, \square)$-simple hypergraph is m-isolated).

Observe that $\text{SUPPORT}(\mathbb{H}_{last}) \subseteq V$. Suppose $V = \{\alpha_1, \alpha_2 \dots \alpha_{|V|}\}$. Let $\mathcal{F} \subseteq [V]^{m+1}$ be the family of sets of vertices such that for any $\mathcal{L} \in \mathcal{F}$ we have $\Lambda_{\mathcal{L}}(\mathbb{H}_{last}) \neq \mathbf{0}$ and $\mathcal{L}$ contains a vertex with index greater than $2m+1$. Our goal is to prove that $\mathcal{F}$ is empty; this implies the claim.

We prove it by contradiction. Suppose $\mathcal{F} \neq \emptyset$. For any set $\mathcal{L} \in \mathcal{F}$, $\mathcal{L} = \{\alpha_{i_1}, \alpha_{i_2} \dots \alpha_{i_{m+1}}\}$ we introduce $\mathcal{Y}_{\mathcal{L}} \subseteq \mathcal{L}$ the maximal (in the sense of inclusion) set of vertices $\{\alpha_{i_j}, \alpha_{i_{j+1}} \dots \alpha_{i_{m+1}}\}$ such that $i_j \geq 2j$, $i_{j+1} \geq 2(j+1)$, $\dots i_{m+1} \geq 2(m+1)$. By definition of the family $\mathcal{F}$, for every $\mathcal{L} \in \mathcal{F}$ the set $\mathcal{Y}_{\mathcal{L}}$ is nonempty.

Let $\mathcal{L} \in \mathcal{F}$ be a set such that $\mathcal{Y}_{\mathcal{L}}$ is maximal in the terms of size i.e. for every $\widetilde{\mathcal{L}} \in \mathcal{F}$ it holds that $|\mathcal{Y}_{\mathcal{L}}| \geq |\mathcal{Y}_{\widetilde{\mathcal{L}}}|$. We show the contradiction by proving $\Lambda_{\mathcal{L}}(\mathbb{H}_{last}) = \mathbf{0}$.

Observe that $\mathcal{L} \neq \mathcal{Y}_{\mathcal{L}}$ as in this case $\mathbb{H}_{last}$ would not be reduced.

Let $\mathbb{G}' \overset{\text{def}}{=} (\mathbb{H}_{last})_{|-\mathcal{Y}_{\mathcal{L}}}$ (recall Definition 8.8). As $\mathcal{L} \neq \mathcal{Y}_{\mathcal{L}}$ we know that $\Lambda_{\mathcal{L}}(\mathbb{H}_{last}) = \Lambda_{\mathcal{L} \setminus \mathcal{Y}_{\mathcal{L}}}(\mathbb{G}')$. So, if we prove that $\mathbb{G}'$ is $(|\mathcal{L}| - |\mathcal{Y}_{\mathcal{L}}|)$-isolated then $\mathbf{0} = \Lambda_{\mathcal{L} \setminus \mathcal{Y}_{\mathcal{L}}}(\mathbb{G}) = \Lambda_{\mathcal{L}}(\mathbb{H}_{last})$ and we have the contradiction with the assumption that $\Lambda_{\mathcal{L}}(\mathbb{H}_{last}) \neq \mathbf{0}$.

So what remains, is to prove the following claim.

Claim 8.13. $\mathbb{G}'$ is $(|\mathcal{L}| - |\mathcal{Y}_{\mathcal{L}}|)$-isolated.

Let $m' = (|\mathcal{L}| - |\mathcal{Y}_{\mathcal{L}}|)$. It suffices to prove that $\mathbb{G}'$ is pre m'-isolated. Then because of Lemma 8.5 the hypergraph $\mathbb{G}'$ is m'-isolated.

Note that $\mathbb{G}'$ is $m'-1$-isolated which is inherited from $\mathbb{H}_{last}$, because of Lemma 8.10. So it remains to analyse weights of m'-subsets of V. Our goal is to show that, for every $\widetilde{\mathcal{L}} \in [V]^{m'}$ if $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{G}') \neq \mathbf{0}$ then it is a subset of $\{\alpha_1 \dots \alpha_{2m'-1}\}$.

Let us take $\widetilde{\mathcal{L}} = \{\alpha_{i'_1}, \alpha_{i'_2} \dots \alpha_{i'_{m'}}\}$, such that $i'_{m'} \geq 2m'$. We prove that $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{G}') = \mathbf{0}$. Let us consider the set $\check{\mathcal{L}} = \widetilde{\mathcal{L}} \cup \mathcal{Y}_{\mathcal{L}}$. We claim that $|\mathcal{Y}_{\mathcal{L}}| < |\mathcal{Y}_{\check{\mathcal{L}}}|$. To see this we observe that in the set $\check{\mathcal{L}}$, because of the definition of the set $\mathcal{Y}_{\mathcal{L}}$, there is at least one element not smaller than $\alpha_{2(m+1)}$, at least two elements not smaller than α_{2m}... at least $|\mathcal{Y}_{\mathcal{L}}|$ elements not smaller than $\alpha_{2(m'+1)}$. Moreover, because $i'_{m'} \geq 2m'$, there are at least $|\mathcal{Y}_{\mathcal{L}}|+1$ elements not smaller than $\alpha_{2m'}$. This implies that $(m+1)-(m'-1) \leq |\mathcal{Y}_{\check{\mathcal{L}}}|$ where $|\mathcal{Y}_{\mathcal{L}}| = (m+1)-m'$.

Because of maximality of $\mathcal{Y}_{\mathcal{L}}$, we conclude that $\Lambda_{\check{\mathcal{L}}}(\mathbb{H}_{last}) = \mathbf{0}$, thus $\Lambda_{\check{\mathcal{L}} \setminus \mathcal{Y}_{\mathcal{L}}}(\mathbb{H}_{last|-\mathcal{Y}_{\mathcal{L}}}) = \Lambda_{\widetilde{\mathcal{L}}}(\mathbb{G}') = \mathbf{0}$. Therefore for every m'-set of vertices $\widetilde{\mathcal{L}}$ in $[V]^{m'}$, such that $\Lambda_{\widetilde{\mathcal{L}}}(\mathbb{G}') \neq \mathbf{0}$, it is a subset of $\{\alpha_1, \alpha_2 \dots \alpha_{2m'-1}\}$. This ends the proof that $\mathbb{G}'$ is pre m'-isolated.

Finally, as it was written earlier, from Lemma 8.5 we derive that $\mathbb{G}'$ is m'-isolated. □

9. The construction of simple hypergraphs.

The whole section is devoted to proving the following theorem:

Theorem 9.1. *Suppose $\mathbb{G}$ is a k-hypergraph. Then there is $\mathcal{S}'$ a simple family for $\{\mathbb{G}\}$. Moreover $\mathcal{S}' \subseteq \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$.*

Proof. We want to construct the family $\mathcal{S}'$. To achieve this, because of Remark 7.8, it is sufficient to propose an algorithm that for any k-hypergraph $\mathbb{G}$ produces $\mathcal{S}$ a simplification of $\{\mathbb{G}\}$ such that $\mathcal{S} \subseteq \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$. Existence of such an algorithm is a consequence of the following lemmas.

Lemma 9.2. *Let $\mathbb{G}$ be a k-hypergraph and $0 \leq m \leq k$. Suppose there are $(m, \mathbf{a})$-simple and $(m, \mathbf{b})$-simple hypergraphs that are elements of $\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$. Then $(m, \mathbf{a}+\mathbf{b})$-simple hypergraph is also a member of $\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$.*

Lemma 9.3. *Let $\mathbb{G} = (V', \mu')$ be a k-hypergraph and $\mathcal{X} \in [V']^m$ where $m \leq k$. Let $\mathbf{a} = \Lambda_{\mathcal{X}}(\mathbb{G})$. Then there is a $(m, \mathbf{a})$-simple k-hypergraph $\mathbb{S}^m_{\mathbf{a}}$ such that $\mathbb{S}^m_{\mathbf{a}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$.*

Indeed, using this two lemmas we can produce all elements in the simplification. Suppose, $m \leq k$ and $\mathbf{a} \in \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_{\mathcal{X}}(\mathbb{G}) \mid \mathcal{X} \in [V']^m\})$. We have to show that $(m, \mathbf{a})$-simple hypergraph is in $\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$. Suppose, $\mathbf{a} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 \ldots c_l\mathbf{a}_l$ where $\mathbf{a}_i \in \{\Lambda_{\mathcal{X}}(\mathbb{G}) \mid \mathcal{X} \in [V']^m\}$ and $c_i \in \mathbb{Z}$. Because of Lemma 9.2 it suffices to prove that $(m, \mathbf{a}_i)$-simple hypergraphs are in $\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$. But this is exactly Lemma 9.3. So it remains to provide proofs of lemmas 9.2 and 9.3. The proof of Lemma 9.2 is easy so we start from it and then we concentrate on the much more complicated proof of Lemma 9.3. □

Proof of Lemma 9.2. Let $\mathbb{S}^m_{\mathbf{a}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$ be the $(m, \mathbf{a})-$simple hypergraph and $\mathbb{S}^m_{\mathbf{b}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$ be the $(m, \mathbf{b})-$simple hypergraph. We can write

$$\mathbb{S}^m_{\mathbf{a}} = \sum_i a_i \mathbb{G}_i \text{ where } a_i \in \mathbb{Z} \text{ and } \mathbb{G}_i \in \textsc{Eq}(\mathbb{G})$$

$$\mathbb{S}^m_{\mathbf{b}} = \sum_j b_{\mathbf{w}} \mathbb{G}_j \text{ where } b_{\mathbf{w}} \in \mathbb{Z} \text{ and } \mathbb{G}_j \in \textsc{Eq}(\mathbb{G})$$

We recall the notation used in Definition 7.2 of simple hypergraphs; the vertices of $\mathbb{S}^m_{\mathbf{a}}$ can be split into $\mathcal{A}_{\mathbb{S}^m_{\mathbf{a}}}, \mathcal{B}_{\mathbb{S}^m_{\mathbf{a}}}, \mathcal{C}_{\mathbb{S}^m_{\mathbf{a}}}$ and similarly vertices of $\mathbb{S}^m_{\mathbf{b}}$ are in $\mathcal{A}_{\mathbb{S}^m_{\mathbf{b}}}, \mathcal{B}_{\mathbb{S}^m_{\mathbf{b}}}, \mathcal{C}_{\mathbb{S}^m_{\mathbf{b}}}$. According to Definition 7.2, the elements of sets $\mathcal{A}_{\mathbb{S}^m_{\mathbf{a}}}, \mathcal{B}_{\mathbb{S}^m_{\mathbf{a}}}$ are paired, formally there is a bijection $f : \mathcal{A}_{\mathbb{S}^m_{\mathbf{a}}} \to \mathcal{B}_{\mathbb{S}^m_{\mathbf{a}}}$, similarly elements of sets $\mathcal{A}_{\mathbb{S}^m_{\mathbf{b}}}$ and $\mathcal{B}_{\mathbb{S}^m_{\mathbf{b}}}$ are paired, formally there is a bijection $f' : \mathcal{A}_{\mathbb{S}^m_{\mathbf{b}}} \to \mathcal{B}_{\mathbb{S}^m_{\mathbf{b}}}$.

Then there is a bijection π between vertices that transfers $\mathcal{A}_{\mathbb{S}^m_{\mathbf{a}}}$ to $\mathcal{A}_{\mathbb{S}^m_{\mathbf{b}}}$, $\mathcal{B}_{\mathbb{S}^m_{\mathbf{a}}}$ to $\mathcal{B}_{\mathbb{S}^m_{\mathbf{b}}}$, $C_{\mathbb{S}^m_{\mathbf{a}}}$ to $\mathcal{C}_{\mathbb{S}^m_{\mathbf{b}}}$, and that it preserves the pairings i.e. $\pi(f(\alpha_i)) = f'(\pi(\alpha_i))$ and $\pi(f^{-1}(\beta_i)) = f'^{-1}(\pi(\beta_i))$ for every $\alpha_i \in \mathcal{A}_{\mathbb{S}^m_{\mathbf{a}}}$ and $\beta_i \in \mathcal{B}_{\mathbb{S}^m_{\mathbf{a}}}$. Thus,

$$\mathbb{S}^m_{\mathbf{a}} + \mathbb{S}^m_{\mathbf{b}} \circ \pi = \sum_i a_i \mathbb{G}_i + (\sum_j b_{\mathbf{w}} \mathbb{G}_j) \circ \pi = \sum_i a_i \mathbb{G}_i + \sum_j b_{\mathbf{w}} (\mathbb{G}_j) \circ \pi$$

But this mean that $\mathbb{S}^m_{\mathbf{a}} + \mathbb{S}^m_{\mathbf{b}} \circ \pi \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$. It is not hard to see that $\mathbb{S}^m_{\mathbf{a}} + \mathbb{S}^m_{\mathbf{b}} \circ \pi$ is an $(m, \mathbf{a}+\mathbf{b})$-simple hypergraph. □

We may proceed to the proof of Lemma 9.3. We start with an operator that is used in the following proofs.

Definition 9.4. Suppose $\mathbb{G} = (V', \mu')$ is a k-hypergraph and $\alpha \in V', \alpha' \notin V'$ are two vertices. Let $\sigma_\alpha \colon V' \cup \{\alpha'\} \to V' \cup \{\alpha'\}$ such that:

$$\sigma_\alpha(x) \overset{\text{def}}{=} \begin{cases} \alpha' & \text{when } x = \alpha \\ \alpha & \text{when } x = \alpha' \\ x & \text{otherwise} \end{cases}$$

A *swap of* α *and* α' *in* $\mathbb{G}$ is defined as $\tau_{\{\alpha,\alpha'\}}(\mathbb{G}) \overset{\text{def}}{=} \mathbb{G} \circ \sigma_\alpha$.

The proof of Lemma 9.3 is by induction on k. We encapsulate the most important steps of the proof in four lemmas. Lemma 9.5 is an auxiliary lemma. Lemma 9.6 forms the induction base. Lemmas 9.7 and 9.8 cover the induction step.

Lemma 9.5. *Let* $\mathbb{G}$ *be a* k*-hypergraph and* $\mathbb{S}^m_{\mathbf{a}}$ *be an* $(m, \mathbf{a})$*-simple* k*-hypergraph. Suppose,* $\alpha, \alpha' \notin \textsc{support}(\mathbb{S}^m_{\mathbf{a}})$ *are two vertices and* $\mathbb{S}^m_{\mathbf{a}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}_{|-\alpha}))$. *Then:*

(1) $\mathbb{S}^{m+1}_{\mathbf{a}} = \mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha} - \mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha'}$ *is an* $(m+1, \mathbf{a})$*-simple* $(k+1)$*-hypergraph,*
(2) $\mathbb{S}^{m+1}_{\mathbf{a}} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{G}))$.

Proof. We start by showing Point 1. Let $\textsc{support}(\mathbb{S}^m_{\mathbf{a}}) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are as in the definition of $(m, \mathbf{a})$-simple hypergraph (Definition 7.2). Let $\mathcal{A} = \{\alpha_1, \alpha_2 \ldots \alpha_m\}$ and $\mathcal{B} = \{\beta_1, \beta_2 \ldots \beta_m\}$. We define $\mathcal{A}' \overset{\text{def}}{=} \{\alpha_1, \alpha_2 \ldots \alpha_m, \alpha_{m+1} = \alpha\}$ and $\mathcal{B}' \overset{\text{def}}{=} \{\beta_1, \beta_2 \ldots \beta_m, \beta_{m+1} = \alpha'\}$. To show that $\mathbb{S}^{m+1}_{\mathbf{a}}$ is $(m+1, \mathbf{a})$-simple we split $\textsc{support}(\mathbb{S}^m_{\mathbf{a}}) \cup \{\alpha, \alpha'\}$ to $\mathcal{A}', \mathcal{B}'$, and $\mathcal{C}$ and we verify Properties 1 to 7. Properties 1 to 3 are trivial. Properties 4 to 6 speak about function $\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}})$ where:

- Property 4. $\mathcal{X}$ contains exactly one vertex from every pair α_i, β_i for $0 < i \leq m+1$. Here, there are two cases: $\alpha \in \mathcal{X}$ or $\alpha' \in \mathcal{X}$. We consider only one of them as the second one is similar. Suppose that $\alpha' \in \mathcal{X}$. Then $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha}) = \mathbf{0}$ so
$$\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}}) = -\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha'}) = -\Lambda_{\mathcal{X} \setminus \{\alpha\}}(\mathbb{S}^m_{\mathbf{a}}).$$
Thus, $\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}}) = -1 \cdot (-1)^{|\mathcal{X} \cap \mathcal{B}|}\mathbf{a} = (-1)^{|\mathcal{X} \cap \mathcal{B}|}\mathbf{a}$, as required.
- Property 5. $\mathcal{X}$ has $m+1$ elements but it is not one of the sets considered in Property 4. In this case $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha}) = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha'}) = \mathbf{0}$, thus $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}) = \mathbf{0}$, as required.
- Property 6. $|\mathcal{X}| \leq m$. We consider four cases:
  (1) α and α' belong to the set $\mathcal{X}$,
  (2) α belongs to the set $\mathcal{X}$, but α' does not,
  (3) α' belongs to the set $\mathcal{X}$, but α does not,
  (4) both α, α' do not belong $\mathcal{X}$.
  - In the first case, α is not a vertex of $\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha'}$ and α' is not a vertex of $\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha}$. So, $\mathbf{0} = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha}) = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha'})$. Thus, $\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}}) = \mathbf{0}$, as required.
  - In the second case, $\mathbf{0} = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha'})$ and $\Lambda_{\mathcal{X} \setminus \{\alpha\}}(\mathbb{S}^m_{\mathbf{a}}) = \mathbf{0}$ as $|\mathcal{X} \setminus \{\alpha\}| < m$. Thus, $\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}}) = \mathbf{0}$, as required.
  - The third case is almost the same as second.
  - In the fourth case, $\Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha}) = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha'})$, so $\Lambda_{\mathcal{X}}(\mathbb{S}^{m+1}_{\mathbf{a}}) = \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha}) - \Lambda_{\mathcal{X}}(\mathbb{S}^m_{\mathbf{a}}{}_{|+\alpha'}) = \mathbf{0}$, as required.

We proceed to the proof of Point 2. Let $\mathbb{S}^m_{\mathbf{a}} = \sum_i a_i \mathbb{G}_i$ where $a_i \in \mathbb{Z}$, $\mathbb{G}_i \in \textsc{Eq}(\mathbb{G}_{|-\alpha})$, and additionaly $\alpha' \notin \bigcup_i \textsc{support}(\mathbb{G}_i) \cup \textsc{support}(\mathbb{G})$. Suppose π_i are bijections such that π_i are identity on $\{\alpha, \alpha'\}$ and $\mathbb{G}_i = \mathbb{G}_{|-\alpha} \circ \pi_i$. The bijections π_i are well defined as sets of vertices of $\mathbb{G}_{|-\alpha}$ and $\mathbb{G}_i$ do not contain neither α nor α'. Because of Point 1, we may write the following equation

$$\mathbb{S}_{\mathbf{a}}^{m+1} = \sum_i a_i(\mathbb{G}_{|-\alpha} \circ \pi_i)_{|+\alpha} - a_i(\mathbb{G}_{|-\alpha} \circ \pi_i)_{|+\alpha'}.$$

π_i is the identity on α, α' , so.

$$\mathbb{S}_{\mathbf{a}}^{m+1} = \sum_i a_i\mathbb{G}_{|-\alpha_{|+\alpha}} \circ \pi_i - a_i\mathbb{G}_{|-\alpha_{|+\alpha'}} \circ \pi_i.$$

To prove Point 2 it suffices to prove that for each i, it holds that

$$a_i\mathbb{G}_{|-\alpha_{|+\alpha}} \circ \pi_i - a_i\mathbb{G}_{|-\alpha_{|+\alpha'}} \circ \pi_i = a_i\mathbb{G} \circ \pi_i - a_i\mathbb{G}' \circ \pi_i \tag{9.1}$$

where $\mathbb{G}' = \tau_{\{\alpha,\alpha'\}}(\mathbb{G})$. We simplify further

$$\mathbb{G}_{|-\alpha_{|+\alpha}} \circ \pi_i - \mathbb{G}_{|-\alpha_{|+\alpha'}} \circ \pi_i = \mathbb{G} \circ \pi_i - \mathbb{G}' \circ \pi_i \tag{9.2}$$

To prove Equation 9.2 we have to show that for every k-subset $\textsc{support}(\mathbb{G}_{|-\alpha_{|+\alpha}} \circ \pi_i) \cup \textsc{support}(\mathbb{G}_{|-\alpha_{|+\alpha'}} \circ \pi_i) \cup \textsc{support}(\mathbb{G} \circ \pi_i) \cup \textsc{support}(\mathbb{G}' \circ \pi_i)$ (for every potential hyperedge) the weight of that subset on both sides of Equation 9.2 is the same. $\textsc{support}(\mathbb{G}_{|-\alpha_{|+\alpha}} \circ \pi_i) \cup \textsc{support}(\mathbb{G}_{|-\alpha_{|+\alpha'}} \circ \pi_i) \cup \textsc{support}(\mathbb{G} \circ \pi_i) \cup \textsc{support}(\mathbb{G}' \circ \pi_i) = {\pi_i}^{-1}(\textsc{support}(\mathbb{G}) \cup \textsc{support}(\mathbb{G}'))$, so we formalise this as a following equation. For any $\mathcal{X} \subseteq \pi_i(\textsc{support}(\mathbb{G}) \cup \textsc{support}(\mathbb{G}'))$ we prove

$$\Lambda_{\mathcal{X}}\left(\mathbb{G}_{|-\alpha_{|+\alpha}} \circ \pi_i - \mathbb{G}_{|-\alpha_{|+\alpha'}} \circ \pi_i\right) = \Lambda_{\mathcal{X}}\left(\mathbb{G} \circ \pi_i - \mathbb{G}' \circ \pi_i\right). \tag{9.3}$$

We consider 4 cases depending on whether $\alpha, \alpha' \in \mathcal{X}$.

- If $\mathcal{X}$ does not contain α and α' then $\Lambda_{\mathcal{X}}(\mathbb{G} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G}' \circ \pi_i)$, so the right side is equal to $\mathbf{0}$. On the left side $\Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha_{|+\alpha}} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha_{|+\alpha'}} \circ \pi_i)$ so the left side is equal to $\mathbf{0}$. .
- If $\mathcal{X}$ contains only α then $\Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha_{|+\alpha}} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G} \circ \pi_i)$ and $\Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha_{|+\alpha'}} \circ \pi_i) = \mathbf{0} = \Lambda_{\mathcal{X}}(\mathbb{G}' \circ \pi_i)$, so the equality holds.
- The case where $\mathcal{X}$ contains only α' is similar.
- If $\mathcal{X}$ contains both α and α' then $\mathbf{0} = \Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha_{|+\alpha}} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G}_{|-\alpha_{|+\alpha'}} \circ \pi_i) = \Lambda_{\mathcal{X}}(\mathbb{G}' \circ \pi_i)$. So both sides of Equation 9.3 are $\mathbf{0}$.

Thus, the equality 9.2 holds for every $\mathcal{X}$. □

Lemma 9.6. [From [HLT17], Th 15.] *Let $\mathbb{G} = (V', \mu')$ be a 1-hypergraph. Then for any $\mathcal{X} \subseteq V'$ and $|\mathcal{X}| \leq 1$ there is a $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{G}))$-simple 1-hypergraph $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})}$, such that $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} \in \mathbb{Z}$-*Sums*(*Eq*($\mathbb{G}$)).*

Proof. We consider two cases (i) $|\mathcal{X}| = 1$ and (ii) $|\mathcal{X}| = 0$.

The first case. Let $\mathcal{X} = \{\alpha\}$, $\mathbf{a} = \Lambda_\alpha(\mathbb{G})$, and $\alpha' \notin V'$. We define $\mathbb{S}^\bullet_{\mathbf{a}} \overset{\text{def}}{=} \mathbb{G} - \tau_{\{\alpha,\alpha'\}}(\mathbb{G})$. For any vertex $\beta \neq \alpha, \alpha'$ we have $\Lambda_\beta(\mathbb{S}^\bullet_{\mathbf{a}}) = \mathbf{0}$. So $\mathbb{S}^\bullet_{\mathbf{a}}$ has two nonisolated vertices $\{\alpha, \alpha'\}$ and satisfies all properties of a $(1, \mathbf{a})$-simple 1-hypergraph.

The second case. Let $\Lambda_\emptyset(\mathbb{G}) = \mathbf{b}$, $\alpha' \notin V'$. Further, let $\mathbb{S}^\bullet_{\mu'(\beta)} = \mathbb{G} - \tau_{\{\beta,\alpha'\}}(\mathbb{G})$ for any $\beta \in V'$. Then we put $\mathbb{S}^0_{\mathbf{b}} \overset{\text{def}}{=} \mathbb{G} - \left(\sum_{\beta \in V'} \mathbb{S}^\bullet_{\mu'(\beta)}\right)$. Indeed, it is a hypergraph with only one nonisolated vertex α' and $\Lambda_{\alpha'}(\mathbb{S}^0_{\mathbf{b}}) = -\sum_{\beta \in V'} \Lambda_{\alpha'}(\mathbb{S}^\bullet_{\mu'(\beta)}) = \Lambda_\emptyset(\mathbb{G})$, as required. □

Lemma 9.7. *Let $k \in \mathbb{N}$. Suppose that Theorem 9.1 holds if restricted to k-hypergraphs. Let $\mathbb{G} = (V', \mu')$ be a $(k+1)$-hypergraph. Then for any nonempty $\mathcal{X} \subseteq V'$ such that $0 < |\mathcal{X}| \leq k+1$ there is a $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{G}))$-simple $(k+1)$-hypergraph $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})}$, such that $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} \in \mathbb{Z}$-*Sums*$(\mathrm{E}_{\mathrm{Q}}(\mathbb{G}))$.*

Proof of Lemma 9.7. We show how to construct the hypergraph $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})}$. Let $\alpha \in \mathcal{X}$. Consider $\mathbb{G}_{|-\alpha}$ and a set $\mathcal{X}' = \mathcal{X} \setminus \{\alpha\}$. Observe, $\Lambda_{\mathcal{X}'}(\mathbb{G}_{|-\alpha}) = \Lambda_{\mathcal{X}}(\mathbb{G})$. Because of the assumption we know that there is $\mathbb{S}^{|\mathcal{X}'|}_{\Lambda_{\mathcal{X}}(\mathbb{G})}$ a $(|\mathcal{X}'|, \Lambda_{\mathcal{X}}(\mathbb{G}))$-simple k-hypergraph, such that $\mathbb{S}^{|\mathcal{X}'|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} \in \mathbb{Z}$-Sums(Eq($\mathbb{G}_{|-\alpha}$)). Now, because of Lemma 9.5, for some $\alpha' \notin V'$, the $(k+1)$-hypergraph $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} = \mathbb{S}^{|\mathcal{X}'|}_{\Lambda_{\mathcal{X}}(\mathbb{G})_{|+\alpha}} - \mathbb{S}^{|\mathcal{X}'|}_{\Lambda_{\mathcal{X}}(\mathbb{G})_{|+\alpha'}}$ is $(|\mathcal{X}|, \Lambda_{\mathcal{X}}(\mathbb{G}))$-simple (point 1) and $\mathbb{S}^{|\mathcal{X}|}_{\Lambda_{\mathcal{X}}(\mathbb{G})} \in \mathbb{Z}$-Sums(Eq($\mathbb{G}$)) (point 2). □

Lemma 9.8. *Let $k \in \mathbb{N}$. Suppose that Theorem 9.1 holds if restricted to k-hypergraphs. Let $\mathbb{G}$ be a $(k+1)$-hypergraph and $\Lambda_{\emptyset}(\mathbb{G}) = \mathbf{a}$. Then there is a $(0, \mathbf{a})$-simple $(k+1)$-hypergraph $\mathbb{S}^0_{\mathbf{a}}$ such that $\mathbb{S}^0_{\mathbf{a}} \in \mathbb{Z}$-Sums(Eq($\mathbb{G}$)).*

Proof of Lemma 9.8. First, observe that $\mathbb{S}^0_{\mathbf{a}}$ has to satisfy only two properties: it has $2k+1$ vertices and the sum of weights of all hyperedges equals $\mathbf{a}$.

Thus, the lemma is a consequence of a procedure (defined below) that takes a $(k+1)$-hypergraph $\mathbb{F}$ with l vertices and if $l > 2k+1$ then it returns a $(k+1)$-hypergraph $\mathbb{F}'$ such that $\Lambda_{\emptyset}(\mathbb{F}) = \Lambda_{\emptyset}(\mathbb{F}')$, $|\textsc{support}(\mathbb{F}')| < l$, and $\mathbb{F}' \in \mathbb{Z}$-Sums(Eq($\mathbb{F}$)). We start from a hypergraph $\mathbb{G}$. We apply the procedure until we produce a hypergraph with no more than than $2k+1$ vertices. It is the required simple hypergraph. We are guaranteed to finish, as with each application the number of vertices decreases. Moreover, each application of the procedure does not change $\Lambda_{\emptyset}$, so we know that $\Lambda_{\emptyset}$ of the produced graph is equal to $\mathbf{a}$, as required.

The procedure. We pick any α in $\textsc{support}(\mathbb{F})$. Our goal is to construct a $(k+1)$-hypergraph $\mathbb{F}'$ such that $\mathbb{F}' \in \mathbb{Z}$-Sums(Eq($\mathbb{F}$)), $\textsc{support}(\mathbb{F}') \subseteq \textsc{support}(\mathbb{F}) \setminus \{\alpha\}$ and $\Lambda_{\emptyset}(\mathbb{F}') = \Lambda_{\emptyset}(\mathbb{F})$. As the first step, we construct a $(k+1)$-hypergraph $\mathbb{K}$ such that

(1) $\mathbb{K} \in \mathbb{Z}$-Sums(Eq($\mathbb{F}$))
(2) $\Lambda_{\emptyset}(\mathbb{K}) = \mathbf{0}$.
(3) For every $\mathcal{X}$, a set of vertices containing α, we have that $\Lambda_{\mathcal{X}}(\mathbb{F}) = \Lambda_{\mathcal{X}}(\mathbb{K})$.
(4) $\textsc{support}(\mathbb{K}) \subseteq \textsc{support}(\mathbb{F})$.

Then, $\mathbb{F}' = \mathbb{F} - \mathbb{K}$. The first property guaranties that $\mathbb{F}' \in \mathbb{Z}$-Sums(Eq($\{\mathbb{F}\}$)). Because of the second property $\Lambda_{\emptyset}(\mathbb{F}') = \Lambda_{\emptyset}(\mathbb{F}) - \mathbf{0} = \Lambda_{\emptyset}(\mathbb{F})$. The third property is responsible for deleting all hyperedges containing α and the fourth property for not adding any new vertex.

We define $\mathbb{K}$ as follows. Let $\mathcal{S}$ be a simple family for $\mathbb{F}_{|-\alpha}$. By $\hat{\mathcal{S}}$ we denote the set of all k-hypergraphs $\mathbb{S}$ such that $\textsc{support}(\mathbb{S}) = \textsc{support}(\mathbb{F}) \setminus \{\alpha\}$ and $\mathbb{S} \in$ Eq($\mathcal{S}$). Because of Theorem 8.2 we know that $\mathbb{F}_{|-\alpha} \in \mathbb{Z}$-Sums(Eq($\mathcal{S}$)) is supported by $\textsc{support}(\mathbb{F}) \setminus \{\alpha\}$ i.e.

$$\mathbb{F}_{|-\alpha} = \sum_i a_i {\mathbb{S}^*}_i \text{ where } a_i \in \mathbb{Z} \text{ and } {\mathbb{S}^*}_i \in \hat{\mathcal{S}}. \tag{9.4}$$

Thus,

$$\mathbb{F}_{|-\alpha_{|+\alpha}} = \sum_i a_i {\mathbb{S}^*}_{i|+\alpha}. \tag{9.5}$$

We define $\mathbb{K}$ as

$$\mathbb{K} \stackrel{\text{def}}{=} \mathbb{F}_{|-\alpha_{|+\alpha}} - \sum_i a_i {\mathbb{S}^*}_{i|+\alpha'_i} = \sum_i a_i({\mathbb{S}^*}_{i|+\alpha} - {\mathbb{S}^*}_{i|+\alpha'_i}) \tag{9.6}$$

where each α'_i is any vertex in the set $\textsc{support}(\mathbb{F}) \setminus (\textsc{support}({\mathbb{S}^*}_i) \cup \{\alpha\})$. Observe, that $|\textsc{support}({\mathbb{S}^*}_i) \cup \{\alpha\}| \leq 2k+1 < |\textsc{support}(\mathbb{F})|$ as $|\textsc{support}(\mathbb{F})| > 2k+1$, so α'_i can be always picked.

$\mathbb{K}$ satisfies the required properties:

(1) $\mathbb{K} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{F}))$. It is sufficient to show that ${\mathbb{S}^*}_{i|+\alpha} - {\mathbb{S}^*}_{i|+\alpha'_i} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{F}))$. Because of the assumption that Theorem 9.1 holds for k-hypergraphs we know that ${\mathbb{S}^*}_i \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{F}_{|-\alpha}))$. If we combine this with Lemma 9.5 (point 2) then we get that each hypergraph $({\mathbb{S}^*}_{i|+\alpha} - {\mathbb{S}^*}_{i|+\alpha'_i})$ is in $\mathbb{Z}\text{-}\textsc{Sums}(\textsc{Eq}(\mathbb{F}))$.

(2) $\Lambda_\emptyset(\mathbb{K}) = \mathbf{0}$. Because of Lemma 9.5 (point 1) we know that ${\mathbb{S}^*}_{i|+\alpha} - {\mathbb{S}^*}_{i|+\alpha'_i}$ is an element of a simple family for $\mathbb{F}$ and it is $(l,\cdot)$-simple for $l > 0$. Thus, $\Lambda_\emptyset({\mathbb{S}^*}_{i|+\alpha} - {\mathbb{S}^*}_{i|+\alpha'_i}) = \mathbf{0}$ and $\Lambda_\emptyset(\mathbb{K}) = \mathbf{0}$ as $\mathbb{K}$ is a sum of hypergraphs ${\mathbb{S}^*}_{i|+\alpha} - {\mathbb{S}^*}_{i|+\alpha'_i}$.

(3) $\Lambda_{\mathcal{X}}(\mathbb{K}) = \Lambda_{\mathcal{X}}(\mathbb{F})$ for any $\mathcal{X}$ such that $\alpha \in \mathcal{X}$. Indeed,

$$\Lambda_{\mathcal{X}}(\mathbb{K}) = \sum_i \Lambda_{\mathcal{X}}({\mathbb{S}^*}_{i|+\alpha}) - \Lambda_{\mathcal{X}}({\mathbb{S}^*}_{i|+\alpha'_i}) \quad \text{(by Equation 9.6)}$$

but each ${\mathbb{S}^*}_{i|+\alpha'_i}$ does not contain α thus

$$\Lambda_{\mathcal{X}}(\mathbb{K}) = \sum_i \Lambda_{\mathcal{X}}({\mathbb{S}^*}_{i|+\alpha}) = \Lambda_{\mathcal{X}}(\mathbb{F}_{|-\alpha_{|+\alpha}}) \quad \text{(by Equation 9.5).}$$

But, according to Definition 8.8 (of the $\cdot_{|-\alpha}$ operation) and Lemma 8.10 we know that for any set of vertices $\mathcal{Y} \subseteq \textsc{support}(\mathbb{F}) \setminus \{\alpha\}$:

$$\Lambda_{\mathcal{Y}\cup\{\alpha\}}(\mathbb{F}) = \Lambda_{\mathcal{Y}}(\mathbb{F}_{|-\alpha}) = \Lambda_{\mathcal{Y}\cup\{\alpha\}}(\mathbb{F}_{|-\alpha_{|+\alpha}}). \tag{9.7}$$

Thus, for $\mathcal{X} = \mathcal{Y} \cup \{\alpha\}$ we get $\Lambda_{\mathcal{X}}(\mathbb{K}) = \Lambda_{\mathcal{X}}(\mathbb{F})$.

(4) $\textsc{support}(\mathbb{K}) \subseteq \textsc{support}(\mathbb{F})$. Indeed, for every ${\mathbb{S}^*}_i$ it holds that $\textsc{support}({\mathbb{S}^*}_{i|+\alpha}) \subseteq \textsc{support}(\mathbb{F})$ and $\textsc{support}({\mathbb{S}^*}_{i|+\alpha_i}) \subseteq \textsc{support}(\mathbb{F})$.

□

Proof of Lemma 9.3. The proof is by induction on k. The base for the induction is given by Lemma 9.6. The induction step is given by Lemmas 9.7 and 9.8. □

10. The proof of Theorem 3.4 itself.

Theorems 9.1 and 8.2 are proven, so now, the proof of Theorem 3.4 is simple. Let us recall its statement.

Theorem 3.4 *The following conditions are equivalent, for a finite set $\mathcal{H}$ of hypergraphs and a hypergraph $\mathbb{H}$, all of the same arity and dimension:*

(1) *$\mathbb{H}$ is a $\mathbb{Z}$-sum of* $\textsc{Eq}(\mathcal{H})$*;*

(2) *$\mathbb{H}$ is locally a $\mathbb{Z}$-sum of $\mathcal{H}$.*

Proof. From Point 1 to Point 2. $\mathbb{H} \in \mathbb{Z}$-SUMS(EQ($\mathcal{H}$)) i.e. $\mathbb{H} = \sum_i a_i \mathbb{G}_i$ where $a_i \in \mathbb{Z}$ and $\mathbb{G}_i \in$ EQ($\mathcal{H}$), thus $\Lambda_{\mathcal{X}}(\mathbb{H}) = \sum_i a_i \Lambda_{\mathcal{X}}(\mathbb{G}_i)$, for any set $\mathcal{X} \in V$. The above holds for any set $\mathcal{X} \subseteq V$ thus $\mathbb{H}$ is locally a $\mathbb{Z}$-sum of $\mathcal{H}$.

From Point 2 to Point 1. If $\mathbb{H}$ is locally a $\mathbb{Z}$ sum of $\mathcal{H}$ then for any set $\mathcal{X} \subseteq V$ there is a k-hypergraph $\mathbb{G}_{\mathcal{X}}$ such that $\Lambda_{\mathcal{X}}(\mathbb{H}) = \Lambda_{\mathcal{X}}(\mathbb{G}_{\mathcal{X}})$ and $\mathbb{G}_{\mathcal{X}} \in \mathbb{Z}$-SUMS(EQ($\mathcal{H}$)). Because of Theorem 9.1 we conclude that there is a family simple for $\{\mathbb{H}\}$ such that its elements are in $\mathbb{Z}$-SUMS(EQ($\mathcal{H}$)). Now, we can apply Theorem 8.2 together with Lemma 4.2 and conclude that $\mathbb{H} \in \mathbb{Z}$-SUMS(EQ($\mathcal{H}$)). □

11. Proof of Theorem 2.2.

Before we prove the theorem, let us recall its statement.
Theorem 2.2 *For every fixed arity $k \in \mathbb{N}$, the $\mathbb{N}$-solvability problem is in* **NExp-Time**.

We prove Theorem 2.2 by showing that in **NExp-Time** it is possible to reduce $\mathbb{N}$-solvability to $\mathbb{Z}$-solvability. The produced instance of $\mathbb{Z}$-solvability is of exponential size, and can be solved in **Exp-Time** because of Theorem 2.1.

Before we start, we need to recall some facts about solution of systems of linear equations.

Hybrid linear sets.

Definition 11.1. A set of vectors is called *hybrid linear* if it is the smallest set that includes a finite set $\mathcal{B}$, called a base, and that is closed under the addition of elements from a finite set $\mathcal{P}$, called periods.

Theorem 11.2 ([Pot91])**.** *Let M be a $d \times m$-matrix with integer entries and $\mathbf{y} \in \mathbb{Z}^d$. The set of nonnegative integer solutions of linear equations*

$$M \cdot \mathbf{x} = \mathbf{y}$$

is a hybrid linear set. The base $\mathcal{B}$ *and* periods $\mathcal{P}$ *are as follows:*

- *Base: it is the set of minimal, in the pointwise sense, solutions of $M \cdot \mathbf{x} = \mathbf{y}$.*
- *Periods: is the set of minimal nontrivial solutions of*

$$M \cdot \mathbf{x} = \mathbf{0}. \tag{11.1}$$

In the paper [Pot91], Pottier provides bounds on the norms of $\mathcal{B}$ and $\mathcal{P}$. We present these bounds next.

For a vector $\mathbf{v} \in \mathbb{Z}^m$ we introduce two norms: the infinity norm $\|\mathbf{v}\|_\infty$ and the norm one $\|\mathbf{v}\|_1$, defined as follows:

- $\|\mathbf{v}\|_\infty \overset{\text{def}}{=} max\{|\mathbf{v}[i]| \mid \text{ for } 1 \leq i \leq m\}$.
- $\|\mathbf{v}\|_1 \overset{\text{def}}{=} \sum_{i=1}^{m} |\mathbf{v}[i]|$.

Let $\mathcal{U}$ be a finite family of data vectors. We extend definitions of norms to families of data vectors. Let $\|\mathcal{U}\|_\infty \overset{\text{def}}{=} max\{\|\mathbf{v}\|_\infty \mid \mathbf{v} \in \mathcal{U}\}$ and $\|\mathcal{U}\|_{1,\infty} \overset{\text{def}}{=} max\{\|\mathbf{v}\|_1 \mid \mathbf{v} \in \mathcal{U}\}$. Also, for a $d \times m$-matrix M we introduce $\|M\|_{1,\infty} \overset{\text{def}}{=} \|\mathcal{M}\|_{1,\infty}$ where $\mathcal{M}$ is the set of columns of the matrix M.

Lemma 11.3 ([Pot91]). *Let $M \cdot \mathbf{x} = \mathbf{y}$ be a system of linear equations such that M is a $d \times m$-matrix. Then the set of solutions in $\mathbb{N}^m$ is the hybrid linear set described by the base $\mathcal{B}$ and the set of periods $\mathcal{P}$ such that:*

- $\mathcal{B}, \mathcal{P} \subset \mathbb{N}^m$,
- $|\mathcal{P}| \leq d \cdot m$
- $\|\mathcal{B}\|_\infty, \|\mathcal{P}\|_\infty \leq (\|M\|_{1,\infty} + \|\mathbf{y}\|_\infty + 2)^{d+m}$.

Definition 11.4. Let $\mathcal{U}$ be a family of vectors. A vector $\mathbf{x} \in \mathcal{U}$ is *reversible* in a family of vectors $\mathcal{U}$ if $-\mathbf{x} \in \mathbb{N}\text{-}\textsc{Sums}(\mathcal{U})$. We call vectors that are not reversible *nonreversible.*

Thus, from Equation 11.1 and Lemma 11.3 we conclude:

Lemma 11.5. *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two finite sets of vectors in $\mathbb{Z}^d$ such that every vector in $\mathcal{U}_2$ is nonreversible in $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$. Suppose $\mathbf{y} \in \mathbb{N}\text{-}\textsc{Sums}(\mathcal{U})$. For any solution:*

$$\mathbf{y} = \sum_{\mathbf{v} \in \mathcal{U}_1} a_\mathbf{v} \cdot \mathbf{v} + \sum_{\mathbf{w} \in \mathcal{U}_2} b_\mathbf{w} \cdot \mathbf{w} \tag{11.2}$$

where $a_\mathbf{v}, b_\mathbf{w} \in \mathbb{N}$, it holds that

$$\sum_{\mathbf{w} \in \mathcal{U}_2} b_\mathbf{w} \leq |\mathcal{U}_2| \cdot \ (\|\mathcal{U}\|_{1,\infty} + \|\mathbf{y}\|_\infty + 2)^{d+|\mathcal{U}|}$$

i.e. $\sum_{\mathbf{w} \in \mathcal{U}_2} b_\mathbf{w}$ is bounded exponentially.

Proof of Lemma 11.5. The set of solutions of Equation 11.2 is hybrid linear, and given by some $\mathcal{B}, \mathcal{P} \subset \mathbb{N}^{|\mathcal{U}_1 \cup \mathcal{U}_2|}$. Every period is a solution of the equation

$$\mathbf{0} = \sum_{\mathbf{v} \in \mathcal{U}_1} a'_\mathbf{v} \cdot \mathbf{v} + \sum_{\mathbf{w} \in \mathcal{U}_2} b'_\mathbf{w} \cdot \mathbf{w}. \tag{11.3}$$

From the definition of reversibility we get that in any solution of Equation 11.3 all $b'_\mathbf{w}$ are equal to 0. Thus, in every solution of Equation 11.2, the sum $\sum_{\mathbf{w} \in \mathcal{U}_2} b_\mathbf{w}$ is bounded by $|\mathcal{U}_2| \cdot \|\mathcal{B}\|_\infty \leq |\mathcal{U}_2| \cdot \ (\|\mathcal{U}\|_{1,\infty} + \|\mathbf{y}\|_\infty + 2)^{d+|\mathcal{U}|}$, where the last inequality is given by Lemma 11.3. □

The main part of the proof of Theorem 2.2.

The idea of this reduction is as follows. We use $\mathcal{I}$ for a finite family of data vectors in $[\mathcal{D}]^k \to \mathbb{Z}^d$ and $\mathbf{v}$ for a target vector. Similarly to Definition 11.4 reversibility can be defined for data vectors. Namely, a data vector $\mathbf{y}$ is reversible in a set of data vectors $\mathcal{I}$ if $-\mathbf{y} \in \mathbb{N}\text{-}\textsc{Sums}(\textsc{Perm}(\mathcal{I}))$.

The schema of the proof may be followed on the diagram in Figure 3. Next, we define a homomorphism from data vectors $[\mathcal{D}]^k \to \mathbb{Z}^d$ to vectors in $\mathbb{Z}^d$. The homomorphism is defined in such a way that it has a property that for every data vector $\mathbf{y} \in \mathcal{I}$ it holds that $h(\mathbf{y})$ is reversible in $h(\mathcal{I})$ if, and only if, $\mathbf{y}$ is reversible in $\mathcal{I}$. Now, the given instance of $\mathbb{N}$-solvability problem for data vectors $[\mathcal{D}]^k \to \mathbb{Z}^d$ (called the first problem) is transformed, via the homomorphism, to the $\mathbb{N}$-solvability for vectors in $\mathbb{Z}^d$ i.e. system of linear equations (called the second problem). The homomorphic image of any solution to the first problem is also a solution to the second problem. Because of Lemma 11.3, there is a bound on the number of appearances of nonreversible vectors in any solution to the second problem. This bound can be then transferred back through the homomorphism. This provides us with

$$\begin{tikzcd}
\mathbf{v} \in \mathbb{N}\text{-}\textsc{Sums}(\textsc{Perm}(\mathcal{I}))? \arrow[r, "h"] \arrow[d, dash, "\text{a solution}"] & h(\mathbf{v}) \in \mathbb{N}\text{-}\textsc{Sums}(h(\textsc{Perm}(\mathcal{I})))? \arrow[d, dash, "\text{a solution}"] \\
\begin{array}{l} \mathbf{v} = \sum_{\mathbf{v}'} a_{\mathbf{v}'}\mathbf{v}' + \sum_{\mathbf{w}} b_{\mathbf{w}}\mathbf{w} \\ \text{where} \\ a_{\mathbf{v}'} \in \mathbb{N}, b_{\mathbf{w}} \in \mathbb{N} \\ \mathbf{v}' \text{ are data vectors reversible} \\ \text{in } \textsc{Perm}(\mathcal{I}) \\ \mathbf{w} \text{ are data vectors nonre-} \\ \text{versible in } \textsc{Perm}(\mathcal{I}) \end{array} \arrow[r, "\begin{array}{c} h \text{ preserves} \\ \text{reversibility} \end{array}"] & \begin{array}{l} h(\mathbf{v}) = \sum_{\mathbf{v}'} a_{\mathbf{v}'} h(\mathbf{v}') + \sum_{\mathbf{w}} b_{\mathbf{w}} h(\mathbf{w}) \\ \text{where } a_{\mathbf{v}'} \in \mathbb{N}, b_{\mathbf{w}} \in \mathbb{N} \\ h(\mathbf{v}') \text{ are data vectors reversible in} \\ h(\textsc{Perm}(\mathcal{I})) \\ h(\mathbf{w}) \text{ are data vectors nonreversible} \\ \text{in } h(\textsc{Perm}(\mathcal{I})) \end{array} \arrow[d, dash, "\text{the bound}"] \\
\begin{array}{l} \sum_{\mathbf{w}} b_{\mathbf{w}} \quad \text{is} \\ \text{bounded} \\ \text{exponentially} \end{array} & \text{Lemma 11.3: exponential bound on } \sum_{\mathbf{w}} b_{\mathbf{w}} \arrow[l, "\text{transfer back}"']
\end{tikzcd}$$

Figure 3: The diagram of the proof of Theorem 2.2.

the bound on the number of appearances of nonreversible data vectors in any solution to the first problem. With the bound, we can guess the nonreversible part of the solution to the first problem. What remains is to verify that our guess is correct. To do this we have to show that the target minus the guessed nonreversible part can be expressed using the reversible data vectors. But this is exactly $\mathbb{Z}$-solvability. Indeed, if data vectors can be reversed then we can subtract them freely.

Definition 11.6. Let $\mathbf{y}\colon [\mathcal{D}]^k \to \mathbb{Z}^d$ be a data vector. The *data projection* of $\mathbf{y}$ is defined as $\mathfrak{P}(\mathbf{y}) \overset{\text{def}}{=} \sum_{x \in [\mathcal{D}]^k} \mathbf{y}(x)$. It is well-defined as $\mathbf{y}$ is almost everywhere equal to $\mathbf{0}$. For a set of data vectors $\mathcal{I}$ we define $\mathfrak{P}(\mathcal{I}) \overset{\text{def}}{=} \{\mathfrak{P}(\mathbf{y}) \mid \mathbf{y} \in \mathcal{I}\}$ to be the *data projection* of $\mathcal{I}$.[2]

Proposition 11.7. *A data projection is a homomorphism from the group of data vectors with addition to $\mathbb{Z}^d$.* □

Definition 11.8. For a data vector $\mathbf{y} \in [\mathcal{D}]^k \to \mathbb{Z}^d$ its support denoted $\textsc{support}(\mathbf{y}) \subset \mathcal{D}$ is the set $\{\alpha \mid \exists x \in [\mathcal{D}]^k$ such that $\alpha \in x$ and $\mathbf{y}(x) \neq \mathbf{0}\}$. We say that the vector $\mathbf{y}$ is supported by a set $V \subset \mathcal{D}$ if $\textsc{support}(\mathbf{y}) \subseteq V$. The definition may be lifted to sets $\mathcal{I}$ of data vectors $\textsc{support}(\mathcal{I}) \overset{\text{def}}{=} \bigcup_{\mathbf{y} \in \mathcal{I}} \textsc{support}(\mathbf{y})$.

In the following definitions and lemmas we use notation introduced in the definition of the $\mathbb{O}$-solvability problem (Section 2), so $\mathbf{v}$ is a single data vector and $\mathcal{I}$ is a finite set of data vectors.

Definition 11.9. Let $V \overset{\text{def}}{=} \textsc{support}(\mathcal{I}) \cup \textsc{support}(\mathbf{v})$. By Π we denote the set of all data permutations π such that for any $\alpha \notin V$ we have $\pi(\alpha) = \alpha$. For a data vector $\mathbf{y} \in \mathcal{I} \cup \{\mathbf{v}\}$ we define the *smoothing* operator, $smooth(\mathbf{y}) \overset{\text{def}}{=} \sum_{\pi \in \Pi} \mathbf{y} \circ \pi$.

[2] Data projection $\mathfrak{P}$ is equal to a transformation that takes a data vector and transforms it into a hypergraph (the transformation described in Section 3) composed with $\Lambda_{\emptyset}(\square)$.

Example 11.10. Suppose $\mathbf{v}$ is a data vector $\mathcal{D} \to \mathbb{Z}^2$ as follows: $\alpha = [1,2], \beta = [1,3]$, and $\gamma = [0,0]$ for any $\gamma \in \mathcal{D} \setminus \{\alpha, \beta\}$. Let $\mathcal{I} = \{\mathbf{v}\}$. Then $V = \{\alpha, \beta\}$ and $smooth(\mathbf{v})(\alpha) = smooth(\mathbf{v})(\beta) = [2,5]$ and $smooth(\mathbf{v})(\gamma) = [0,0]$ for any $\gamma \in \mathcal{D} \setminus \{\alpha, \beta\}$.

Lemma 11.11. *Let k, V be as defined above. There is a constant c depending on k and $|V|$, such that for any $x \in [V]^k$ and any data vector $\mathbf{y} \in \mathcal{I} \cup \{\mathbf{v}\}$ the equality $smooth(\mathbf{y})(x) = \mathfrak{P}(\mathbf{y}) \cdot c$ holds.*

Proof. It holds because of symmetry of the formula defining the smooth operator. We do not need to calculate the value of c. □

Lemma 11.12. *$\mathbf{y} \in \mathcal{I}$ is reversible in* Perm$(\mathcal{I})$ *if, and only if, $\mathfrak{P}(\mathbf{y})$ is reversible in $\mathfrak{P}(\mathcal{I})$.*

Proof. $\Longrightarrow$ Trivial as $\mathfrak{P}$ is a homomorphism.
$\Longleftarrow$ Let V and Π be as in Definition 11.9. We list elements of $\mathcal{I} = \{\mathbf{v}_1 \ldots \mathbf{v}_m\}$. By M we denote the $1 \times m$-matrix with entries in $\mathbb{Z}^d$ such that $M[1][j] = \mathfrak{P}(\mathbf{v}_j)$.

From the assumptions we know that there is a vector $\mathbf{x} \in \mathbb{N}^m$ such that

$$M \cdot \mathbf{x} = -\mathfrak{P}(\mathbf{y}). \tag{11.4}$$

Let c be the constant from Lemma 11.11. We multiply both sides of Equation 11.4 by c getting

$$c \cdot M \cdot \mathbf{x} = -c \cdot \mathfrak{P}(\mathbf{y}).$$

But from this and Lemma 11.11 we conclude that for any $z \in [V]^k$

$$\left(\sum_{\mathbf{v}_i \in \mathcal{I}} (smooth(\mathbf{v}_i) \cdot \mathbf{x}[i]) \right)(z) = -smooth(\mathbf{y})(z)$$

and it follows that

$$\sum_{\mathbf{v}_i \in \mathcal{I}} (smooth(\mathbf{v}_i) \cdot \mathbf{x}[i]) = -smooth(\mathbf{y}).$$

Using the definition of smoothing we rewrite further

$$\sum_{\mathbf{v}_i \in \mathcal{I}} (smooth(\mathbf{v}_i) \cdot \mathbf{x}[i]) + \sum_{\pi \in \Pi \setminus \{identity\}} \mathbf{y} \circ \pi = -\mathbf{y}.$$

As $\mathbf{y} \in \mathcal{I}$ we see that we expressed $-\mathbf{y}$ as a sum of elements in Perm$(\mathcal{I})$. □

Corollary 11.13. *For a given vector $\mathbf{y}$ in $\mathcal{I}$, in* **P-Time** *we can answer if $\mathbf{y}$ is nonreversible in* Perm$(\mathcal{I})$. *The complexity depends polynomially from k and d.*

Proof of Corollary 11.13. Let $\mathbf{y} \in \mathcal{I}$. The data vector $\mathbf{y}$ is reversible in Perm$(\mathcal{I})$ if, and only if, there is $l \in \mathbb{N}$ such that $-l \cdot \mathfrak{P}(\mathbf{y}) \in \mathbb{N}$-Sums$(\mathfrak{P}(\mathcal{I}))$. Indeed, it is sufficient to add $(l-1)\mathfrak{P}(\mathbf{y})$. Thus, the question about reversibility is equivalent to the question of whether $\mathfrak{P}(\mathbf{y}) \in \mathbb{Q}_+$-Sums$(\mathfrak{P}(\mathcal{I}))$, where $\mathbb{Q}_+$ stands for nonnegative rationals. The last question is known as the linear programming problem and is known to be solvable in **P-Time** [Kha79, CLS19]. □

Lemma 11.14. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ form the partition of the set $\mathcal{I}$ such that $\mathcal{I}_1$ is the set of data vectors reversible in* $\textsc{Perm}(\mathcal{I})$ *and $\mathcal{I}_2$ is the set of data vectors nonreversible in* $\textsc{Perm}(\mathcal{I})$*. Suppose $\mathbf{v}$ can be expressed as a sum*

$$\mathbf{v} = \left(\sum_{\mathbf{y}\in\textsc{Perm}(\mathcal{I}_1)} a_{\mathbf{y}}\mathbf{y}\right) + \left(\sum_{\mathbf{w}\in\textsc{Perm}(\mathcal{I}_2)} b_{\mathbf{w}}\mathbf{w}\right) \textit{ where } a_{\mathbf{y}}, b_{\mathbf{w}} \in \mathbb{N}. \tag{11.5}$$

Then the sum $\sum_{\mathbf{w}\in\textsc{Perm}(\mathcal{I}_2)} b_{\mathbf{w}}$ is bounded exponentially, precisely $\sum_{\mathbf{w}\in\textsc{Perm}(\mathcal{I}_2)} b_{\mathbf{w}} \leq |\mathfrak{P}(\mathcal{I}_2)|\cdot (\|\mathfrak{P}(\mathcal{I})\|_{1,\infty} + \|\mathfrak{P}(\mathbf{v})\|_\infty + 2)^{d+|\mathcal{I}|}$.

Proof. The proof is based on the bound for a solution in $\mathbb{N}$ of a system of linear equations (Lemma 11.5). By Π we denote the set of data permutations (bijections $\mathcal{D} \to \mathcal{D}$). We can apply the homomorphism $\mathfrak{P}$ to both sides of Equation 11.5 and get

$$\mathfrak{P}(\mathbf{v}) = \mathfrak{P}\left(\sum_{\mathbf{y}\in\textsc{Perm}(\mathcal{I}_1)} a_{\mathbf{y}}\mathbf{y}\right) + \mathfrak{P}\left(\sum_{\mathbf{w}\in\textsc{Perm}(\mathcal{I}_2)} b_{\mathbf{w}}\mathbf{w}\right)$$

and further

$$\mathfrak{P}(\mathbf{v}) = \left(\sum_{\mathbf{y}\in\textsc{Perm}(\mathcal{I}_1)} a_{\mathbf{y}}\mathfrak{P}(\mathbf{y})\right) + \left(\sum_{\mathbf{w}\in\textsc{Perm}(\mathcal{I}_2)} b_{\mathbf{w}}\mathfrak{P}(\mathbf{w})\right) \tag{11.6}$$

$$\mathfrak{P}(\mathbf{v}) = \left(\sum_{\mathbf{y}\in\mathcal{I}_1} \mathfrak{P}(\mathbf{y}) \sum_{\pi\in\Pi} a_{\mathbf{y},\pi}\right) + \left(\sum_{\mathbf{w}\in\mathcal{I}_2} \mathfrak{P}(\mathbf{w}) \sum_{\pi\in\Pi} b_{\mathbf{w},\pi}\right)$$

Where $\sum_{\pi\in\Pi} a_{\mathbf{y},\pi}$ and $\sum_{\pi\in\Pi} b_{\mathbf{w},\pi}$ are obtained by grouping elements of the sums in Equations 11.6 by $\mathfrak{P}(\mathbf{y})$ and $\mathfrak{P}(\mathbf{w})$, respectively.

But, because of Lemma 11.12 we know that $\mathfrak{P}$ preserves reversibility so the bound on the sum $\sum_{\mathbf{w}\in\textsc{Perm}(\mathcal{I}_2)} b_{\mathbf{w}} = \sum_{\mathbf{w}\in\mathcal{I}_2}\sum_{\pi\in\Pi} b_{\mathbf{w},\pi}$ can be taken from Lemma 11.5, namely $\sum_{\mathbf{w}\in\textsc{Perm}(\mathcal{I}_2)} b_{\mathbf{w}} \leq |\mathfrak{P}(\mathcal{I}_2)|\cdot\ (\|\mathfrak{P}(\mathcal{I})\|_{1,\infty} + \|\mathfrak{P}(\mathbf{v})\|_\infty + 2)^{d+|\mathcal{I}|}$. □

Theorem 11.15. *Let $\mathcal{I}_1, \mathcal{I}_2$ be a partition of $\mathcal{I}$ into elements reversible and nonreversible in* $\textsc{Perm}(\mathcal{I})$*, respectively. Suppose, $s_{max} \overset{\text{def}}{=} max(\{|\textsc{support}(\mathbf{w})| \mid \mathbf{w} \in \mathcal{I}_2\})$. Let $\widetilde{V} \subset \mathcal{D}$ be a set such that $\mathbf{v}$ is supported by $\widetilde{V}$ and*

$$|\widetilde{V}| = |\textsc{support}(\mathbf{v})| + s_{max}\cdot\left(|\mathfrak{P}(\mathcal{I}_2)|\cdot\ (\|\mathfrak{P}(\mathcal{I})\|_{1,\infty} + \|\mathfrak{P}(\mathbf{v})\|_\infty + 2)^{d+|\mathcal{I}|}\right). \tag{11.7}$$

Finally, let $\widetilde{\mathcal{I}_2} \subset \textsc{Perm}(\mathcal{I}_2)$ be the set of all data vectors in $\textsc{Perm}(\mathcal{I}_2)$ that are supported by $\widetilde{V}$.

Then $\mathbf{v} \in \mathbb{N}$-$\textsc{Sums}(\textsc{Perm}(\mathcal{I}))$ if, and only if,

$$\mathbf{v} = \sum_{\mathbf{y}\in\textsc{Perm}(\mathcal{I}_1)} a_{\mathbf{y}}\mathbf{y} + \sum_{\mathbf{w}\in\widetilde{\mathcal{I}_2}} b_{\mathbf{w}}\mathbf{w} \textit{ where } a_{\mathbf{y}}, b_{\mathbf{w}} \in \mathbb{N}.$$

Note that the second sum is only over vectors supported by $\widetilde{V}$.

Proof. Implication from right to left is trivial. Implication from left to right is a consequence of Lemma 11.14. Suppose $\mathbf{v} = \sum_{\mathbf{y}\in\textsc{Perm}(\mathcal{I}_1)} a'_{\mathbf{y}}\mathbf{y} + \sum_{\mathbf{w}\in\textsc{Perm}(\mathcal{I}_2)} b'_{\mathbf{w}}\mathbf{w}$. Let $\widehat{V}$ be the union of supports of $\mathbf{w} \in \textsc{Perm}(\mathcal{I}_2)$ that appear with nonzero coefficients in the sum above. Observe

$|\widehat{V}| + |\textsc{support}(\mathbf{v})| \leq |\widetilde{V}|$ due to Lemma 11.14 and the definition of $|\widetilde{V}|$. Let π be a data permutation that is the identity on $\textsc{support}(\mathbf{v})$ and injects $\widehat{V}$ into $\widetilde{V}$. Then, $\mathbf{v} = \mathbf{v} \circ \pi^{-1} = \sum_{\mathbf{y} \in \textsc{Perm}(\mathcal{I}_1)} a'_{\mathbf{y}} \mathbf{y} \circ \pi^{-1} + \sum_{\mathbf{w} \in \textsc{Perm}(\mathcal{I}_2)} b'_{\mathbf{w}} \mathbf{w} \circ \pi^{-1}$, as required (since $\mathbf{w} \circ \pi^{-1} \in \widetilde{\mathcal{I}_2}$). □

Proof of Theorem 2.2. Because of Theorem 11.15, in **NExp-Time** it is possible to construct an exponential size instance of $\mathbb{Z}$-solvability with data problem, that has a solution if, and only if, the original problem has a solution. Precisely, we ask if $\mathbf{v} - \sum_{\mathbf{w} \in \widetilde{\mathcal{I}_2}} b_{\mathbf{w}} \mathbf{w} \in \mathbb{Z}\text{-}\textsc{Sums}(\textsc{Perm}(\mathcal{I}_1))$, where $\widetilde{\mathcal{I}_2}$ is defined in Theorem 11.15 . According to Theorem 2.1 it can be solved in **Exp-Time**. Thus, the algorithm works in **NExp-Time**. □

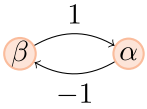
Remark 11.16. The presented reduction works in the same way for a more general class of data vectors of the form $\mathcal{D}^k \to \mathbb{Z}^d$.

12. Conclusions and future work.

We have shown **Exp-Time** and **P-Time** upper bounds for the $\mathbb{N}$- and $\mathbb{Z}$-solvability problems over sets (unordered tuples) of data, respectively, by (1) reducing the former problem to the latter one (with nondeterministic exponential blowup), (2) reformulating the latter problem in terms of (weighted) hypergraphs, and (3) solving the corresponding hypergraph problem by providing a local characterisation testable in polynomial time.

The characterisation of $\mathbb{Z}$-solvability provided by Theorem 3.4 identifies several simple to test properties that all together are equivalent to $\mathbb{Z}$-solvability. Each of the properties is independent of others and may be used as a partial test for nonreachability in data nets. It is not clear if in every industrial application we should use all of them.

The proposed characterisation does not work for directed structures, as all Λ morphisms of the following nontrivial graph are $\mathbf{0}$.



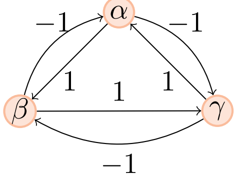
Thus, we should somehow refine the characterisation for data vectors going from k-tuples to d-dimensional vectors of integers, $\mathcal{D}^k \to \mathbb{Z}^d$. For example, for directed graphs the condition stated in Equation 4.2 can be formulated as follows: For every vertex α we define $\overrightarrow{\Lambda_\alpha}(\mathbb{H}) = (out, in)$ where out is the sum of all edges in $\mathbb{H}$ leaving α and in is the sum all edges in $\mathbb{H}$ entering α. Now, instead of

$$\Lambda_\alpha(\mathbb{H}) \in \mathbb{Z}\text{-}\textsc{Sums}(\{\Lambda_{\alpha'}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{I}, \alpha' \in V'\})$$

we have

$$\overrightarrow{\Lambda_\alpha}(\mathbb{H}) \in \mathbb{Z}\text{-}\textsc{Sums}(\{\overrightarrow{\Lambda_{\alpha'}}(\mathbb{H}') \mid \mathbb{H}' = (V', \mu') \in \mathcal{I}, \alpha' \in V'\}) \tag{12.1}$$

This is still not sufficient because of the following example with all weights equal to $\mathbf{0}$:



To deal with this example we need to define another invariant, namely for every pair of vertices α, β the value of $\Lambda_{\{\alpha,\beta\}}$ is a pair of numbers: sum of two edges between vertices α

and β, and the absolute value of the difference of the weights of the two edges. Unfortunately, it is not clear how to generalise such invariants for hypergraphs.

Another approach to a definition of Λ for directed hypergraphs is to linearly order vertices and group edges depending on their orientation with respect to the order. Calculating Λ we add edges within the groups. For example, if the order is $\beta < \alpha < \gamma$ then Λ_α of the triangle above is equal to $[-2, 2]$. In this approach the weight depends on the orientation, which does not look as a desired solution.

Although we do not have the theorem which states that the $\mathbb{Z}$-solvability is equivalent to some lifted version of local characterisation, the local characterizations may be useful. Observe that any of the described invariants is homomorphisms from data vectors in $\mathcal{D}^k \to \mathbb{Z}^d$ to $\mathbb{Z}^{d \cdot n}$ for some $n \in \mathbb{N}$ depending on the homomorphism (for example in case of the (out, in) weight the value $n = 2$). Thus, we identified a family of new heuristics that can be used in the analysis of data nets.

We also should mention that although in the proof of Theorem 2.1 we use data vectors with image in $\mathbb{Z}^d$ all proofs work for any Abelian groups in which the system of equations 4.7 has a unique solution. In the general case, it corresponds to the Abelian group in which Lemma 6.3 holds. Lemma 6.3 speaks about maximality of the rank of some specific family of matrices.

We leave two interesting open problems. First, concerning the complexity of $\mathbb{N}$-solvability, we leave a gap between the **NP** lower bound and our **NExp-Time** upper bound. Currently, available methods seem not suitable for closing this gap. Second, the line of research we establish in this paper calls for continuation, in particular for investigation of the solvability problems over the *ordered* tuples of data. Because of Remark 11.16 we know that the problematic bit is a generalisation of the Theorem 2.1 In our opinion, the solution for this case should be within reach by further developing techniques proposed in this paper.

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