

COMBINATORIAL RESOLUTION OF SYSTEMS OF DIFFERENTIAL EQUATIONS. IV. SEPARATION OF VARIABLES

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In the context of the combinatorial theory of ordinary differential equations recently introduced by the authors, a concrete interpretation is given to the classical method of separation of variables. This approach is then extended to more general equations and applied to systems of differential equations with forcing terms.

1. Introduction

Classically, the ordinary differential equation

$$y' = dy/dt = f(t)g(y), \quad y(0) = \alpha \quad (1.1)$$

is said to have 'separable variables' since we can write, in an informal manner, (see [4], pp. 12–20)

$$dy/g(y) = f(t) dt \quad (1.2)$$

and integrate on both sides to get

$$\Phi(y) = \int_{\alpha}^y du/g(u) = \int_0^t f(x) dx \quad (1.3)$$

If we now let $\Phi^{(-1)}$ denote the inverse function of Φ , the solution of (1) can be simply written as

$$y(t) = \Phi^{(-1)}\left(\int_0^t f(x) dx\right). \quad (1.4)$$

In this paper we establish the validity of this solution by giving a direct combinatorial interpretation of the function $y = \Phi^{(-1)}(t)$ as the solution of the

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autonomous differential equation

$$y' = g(y), \quad y(0) = \alpha \quad (1.5)$$

and then by showing that the combinatorial solution of (1.1) admits a kind of 'separation of vertices' which implies (1.4). Moreover, we show that this combinatorial separation of variables can be applied to more general differential equations, or systems of differential equations, of the form

$$y = \sum_j f_j(t) g_j(y) \quad (1.6)$$

where j varies over some finite index set J . Examples include the linear equation

$$y' = f(t)y + g(t) \quad (1.7)$$

and equations of control theory such as

$$y' = g_0(y) + u(t)g_1(y) \quad (1.8)$$

where g_0 and g_1 are given and $u(t)$ is a variable function.

In particular we establish by purely combinatorial methods a functional expansion for the solution of (1.6), in terms of iterated integrals, which implies the so-called fundamental formula of Fließ for systems of differential equations with forcing terms ([1, 2]).

As in our previous papers [8, 9], we work in a combinatorial setting where functions are replaced by species of structure over linearly ordered sets (\mathbb{L} -species). The relationship with analysis is obtained by taking generating functions of \mathbb{L} -species, a process which preserves all the usual operations of calculus, including integration, when properly defined on \mathbb{L} -species. The reader is referred to [8], where the combinatorial theory of differential equations has been initiated and which serves as a basis for the present paper, for background and terminology.

2. Combinatorial separation of variables

We first consider the example of the differential equation

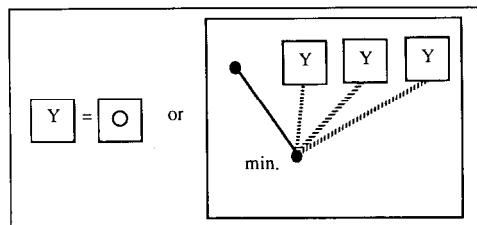
$$Y' = T \cdot Y^3, \quad Y(0) = 1 \quad (2.1)$$

whose analytical solution is given by

$$y(t) = 1/\sqrt{1-t^2}. \quad (2.2)$$

The use of capital letters in (2.1) indicates the fact that we want a combinatorial solution, in terms of \mathbb{L} -species. As shown in [8], there is a unique \mathbb{L} -species $Y = A_{TY^3}(T)$ which satisfies (2.1): the equation can be rewritten in the integral form

$$Y = 1 + \int_0^T XY^3(X) dX \quad (2.3)$$



and interpreted as follows (see Fig. 1), using the standard combinatorial interpretation of the integral (see [8, §2] and Fig. 5):

- there is a unique $A_{T\gamma^3}$ -structure on the empty set, denoted by \circ ;
- any non-empty $A_{T\gamma^3}$ -structure can be canonically identified with the composite structure consisting of
 - (a) the minimum element of the underlying set
 - (b) a singleton point
 - (c) an ordered triple (left, middle, right) of similar $A_{T\gamma^3}$ -structures.

One can then attach the singleton point and the ordered triple of A_{TY} -structures to the minimum element (then called a *fertile point*) as shown in Fig. 1. Iterating this procedure we find that A_{TY} -structures essentially consist of certain *increasing labelled rooted planar trees* (with labels increasing from root to leaves) such as the one illustrated by Fig. 2, on the linearly ordered set $[19] = \{1, 2, \dots, 19\}$.

Let $Y = \text{Ter}(T)$ denote the \mathbb{L} -species of *increasing ternary trees*, that is the solution of the autonomous equation

$$Y' = Y^3, \quad Y(0) = 1. \quad (2.4)$$

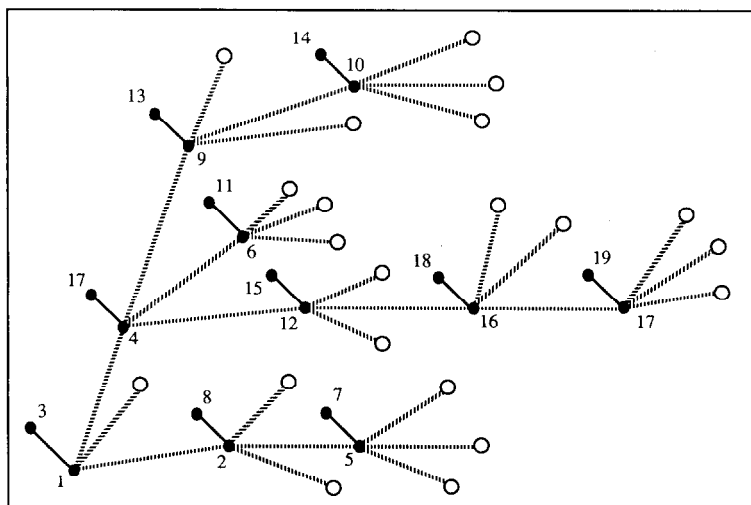


Fig. 2.

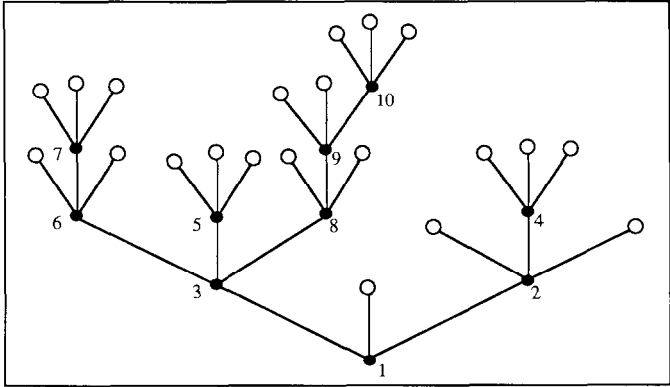


Fig. 3.

They are similar to A_{TY^3} -structures but simpler since there are no singletons attached to the roots; an example is given in Fig. 3.

For $n \geq 0$, let t_n be the number of increasing ternary trees with n internal vertices. It is easily seen that $t_n = (2n - 1)t_{n-1}$, for $n \geq 1$, and that $t_0 = 1$. Thus

$$t_n = (2n - 1) \cdots (3)(1) \quad (2.5)$$

and we obtain the generating function,

$$\begin{aligned} \text{Ter}(t) &= \sum_{n \geq 0} t_n t^n / n! \\ &= \frac{1}{\sqrt{1 - 2t}} \end{aligned} \quad (2.6)$$

by the binomial theorem (see also [8], example 5.5).

A comparison of Figs 2, 3 and 4 shows that an A_{TY^3} -structure can be viewed as an increasing ternary tree where vertices have been replaced by cells containing

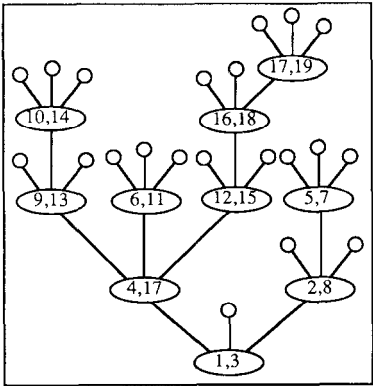


Fig. 4.

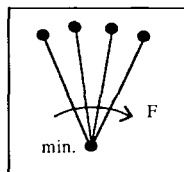


Fig. 5.

(unordered) pairs of elements, these cells being linearly ordered according to their minimum elements. This corresponds to substitution of species: we have

$$A_{TY^3}(T) = \text{Ter}\left(\frac{T^2}{2!}\right) \quad (2.7)$$

where, as usual, equality stands for isomorphism of species.

Using (2.6), we conclude that the generating function for A_{TY^3} is

$$\begin{aligned} A_{TY^3}(t) &= \text{Ter}\left(\frac{t^2}{2}\right) \\ &= \frac{1}{\sqrt{1-t^2}} \end{aligned} \quad (2.8)$$

as expected, and also, by the binomial theorem, that

$$A_{TY^3}(t) = \sum_{n \geq 0} \frac{[(2n-1) \cdots (3)(1)]^2 t^{2n}}{2n!} \quad (2.9)$$

Note that $\frac{1}{2}T^2 = \int_0^T X dX$ and that, more generally, a structure of species $\int F = \int_0^T F(X) dX$ can be represented as in Fig. 5.

We end this section with the basic classical result on the method of separation of variables which is surprisingly simple from a combinatorial point of view. It comes as a natural generalization of the previous example and of example 4.5.a of [8] (the homogeneous linear equation).

Proposition 2.1. *Let $Y = A_{G(Y)}(T)$ be the solution of the autonomous differential equation*

$$\underline{Y' = G(Y)}, \quad Y(0) = Z; \quad (2.10)$$

then the solution $Y = A_{F(T)G(Y)}(T)$ of the equation with separable variables

$$\underline{Y' = F(T)G(Y)}, \quad Y(0) = Z \quad (2.11)$$

is given by

$$A_{F(T)G(Y)}(T) = A_{G(Y)}\left(\int_0^T F(X) dX\right). \quad (2.12)$$

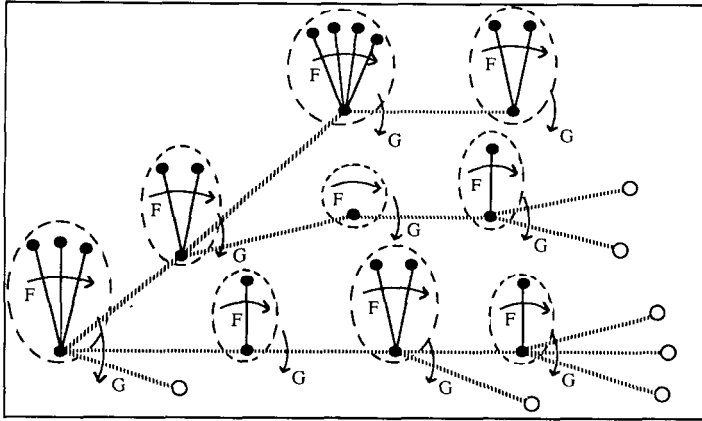


Fig. 6.

Proof. Simply observe that the increasing $F(T)G(Y)$ -enriched arborescences that occur as the canonical solution of (2.11) can be viewed as $A_{G(Y)}(\int F)$ -structures. See Fig. 6. \square

3. Iterated integrals

Iterated integrals such as

$$\int_0^T F(S) \int_0^S G(R) \int_0^R H(X) dX dR dS \quad (3.1)$$

occur in the expansion of product integrals (see [3]) and in the solution of certain systems of differential equations (see e.g. [5], chap. 14). The standard combinatorial interpretation of the integral can be applied to interpret iterated integrals (3.1) as \mathbb{L} -species. One gets enriched increasing arborescences as in Fig. 7.

More generally, given a family $F = \{F_j\}_{j \in J}$ of \mathbb{L} -species, where J is some finite set, and given any word $\alpha \in J^*$, there is an induced iterated integral, denoted by

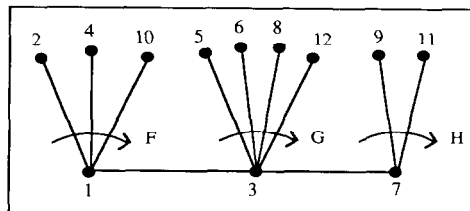


Fig. 7.

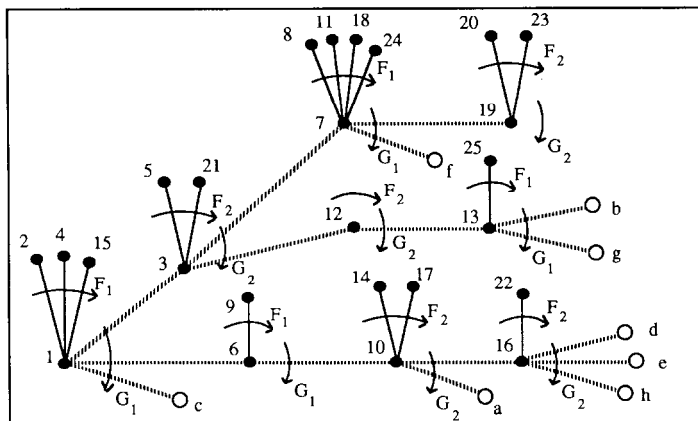


Fig. 8.

$\int_0^T F d\alpha$, which is defined inductively as follows:

If $\alpha = \varepsilon$, the empty word, then

$$\int_0^T F d\varepsilon = 1 \quad (3.2)$$

the 'empty set' species;

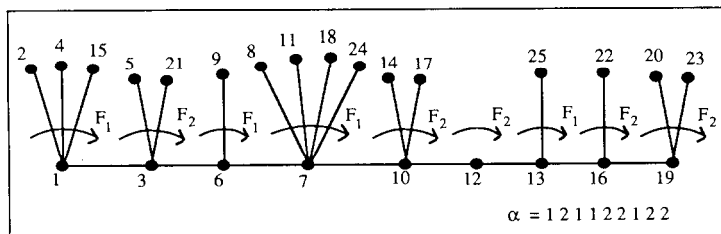
If $\alpha = j\beta$, i.e. $j \in J$ is the first letter of the word α , then

$$\int_0^T F d(j\beta) = \int_0^T F_j(X) \int_0^X F d\beta dX. \quad (3.3)$$

An example of an $\int_0^T F d\alpha$ -structure is given in Fig. 9 (see below), with $\alpha = 1\ 2\ 1\ 1\ 2\ 2\ 1\ 2\ 2$. It is readily seen that an $\int_0^T F d\alpha$ -structure is an assembly of $(\sum_{j \in J} \int_0^T F_j(X) dX)$ -structures aligned in the natural order of their smallest elements. Hence we can state:

Proposition 3.1. For any finite family $\{F_j\}_{j \in J}$ of \mathbb{L} -species, we have

$$\sum_{\alpha \in J^*} \int_0^T F d\alpha = \exp\left(\sum_{j \in J} \int_0^T F_j(X) dX\right). \quad (3.4)$$

Fig. 9. $I(T)$ -structure.

4. A generalized combinatorial separation of variables

The technique of combinatorial separation of variables can also be applied to equations, or systems of equations, of the form

$$Y' = \sum_j F_j(T)G_j(Y), \quad Y(0) = Z \quad (4.1)$$

where j varies over some finite index set J . We will first deal with one simple equation:

$$Y' = F_1(T)G_1(Y) + F_2(T)G_2(Y), \quad Y(0) = Z \quad (4.2)$$

with $J = \{1, 2\}$, and later give appropriate generalizations as well as examples, in Section 5.

Setting $M = M(T, Y) = F_1(T)G_1(Y) + F_2(T)G_2(Y)$, the canonical solution of (4.2) is the \mathbb{L} -species $Y = A_M = A_M(T, Z)$ of so-called *M-enriched increasing arborescences with buds* (buds constitute an extra sort of elements, corresponding to the variable Z ; see §3 of [8]). A typical A_M -structure is represented by Fig. 8 where it is seen that the M -enrichment at each fertile point consists of either an $F_1(T)G_1(Y)$ - or an $F_2(T)G_2(Y)$ -structure.

Now combinatorial separation of variables can be applied as follows: associate with each M -enriched increasing arborescence a pair of structures as illustrated by Figs 9 and 10, for the arborescence of Fig. 8, thus defining two species $I(T)$ and $H(Z)$.

The $I(T)$ -structure is in fact an iterated integral $\int_0^T F d\alpha$ -structure obtained by extracting the $\int F_1$ - and $\int F_2$ -structures of the arborescence and aligning them in their natural order. Moreover, we take note of this relative order in what remains of the arborescence to obtain a structure on buds (see Fig. 10). This is the $H(Z)$ -structure.

It can be seen that the $H(Z)$ -structure comes from the application to Z of a sequence of operators of the type \mathcal{D}_1 or \mathcal{D}_2 , according to the same word α as in the iterated integral; the \mathcal{D}_i 's are 'eclosion operators' (terminology due to G. Labelle; see [6, 7] and [8, §5]) defined, for $j \in J$, by $\mathcal{D}_j = G_j(Z) \partial / \partial Z$ (Fig. 11).

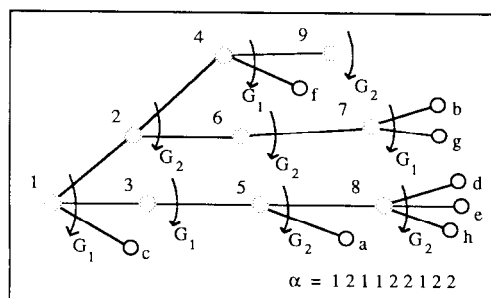


Fig. 10. $H(Z)$ -structure.

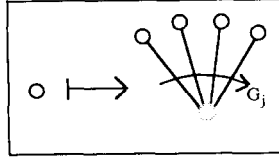


Fig. 11.

For instance the $H(Z)$ -structure of Fig. 10 is a typical structure of the species

$$(\mathcal{D}_2 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \mathcal{D}_2 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \mathcal{D}_1 \circ \mathcal{D}_2 \circ \mathcal{D}_1)(Z). \quad (4.3)$$

Note that we compose operators from right to left so that the order of α is actually reversed. We introduce the notation \mathcal{D}_α for the operators defined inductively by

$$\mathcal{D}_\varepsilon = Id, \quad \mathcal{D}_{(j\beta)} = \mathcal{D}_\beta \circ \mathcal{D}_j. \quad (4.4)$$

To summarize, combinatorial separation of variables associates to any M -enriched increasing arborescence, a word α and a pair of structures of species $\int_0^T F d\alpha$ and $\mathcal{D}_\alpha(Z)$ respectively. This correspondence is clearly reversible and bijective and also valid for an index set J of any finite cardinality. Hence we have:

Proposition 4.1. *Let $Y = A_M(T, Z)$ be the solution of the differential equation*

$$Y' = \sum_{j \in J} F_j(T) G_j(Y), \quad Y(0) = Z \quad (4.5)$$

where $\{F_j\}_{j \in J}$ and $\{G_j\}_{j \in J}$ are given finite families of \mathbb{L} -species. Then we have

$$A_M(T, Z) = \sum_{\alpha \in J^*} \left(\int_0^T F d\alpha \right) (\mathcal{D}_\alpha(Z)) \quad (4.6)$$

and, more generally, for any \mathbb{L} -species Φ ,

$$\Phi(A_M(T, Z)) = \sum_{\alpha \in J^*} \left(\int_0^T F d\alpha \right) (\mathcal{D}_\alpha(\Phi(Z))). \quad (4.7)$$

Proof. Only (4.7) remains to be proved. For any word α , the expression $\int_0^T F d\alpha \mathcal{D}_\alpha$ can be interpreted as an operator on \mathbb{L} -species; it should be clear that when this operator is applied to any $\Phi(Z)$ -structure, the result is a $\Phi(A_M(T, Z))$ -structure. This induces the desired isomorphism. \square

Remark. We now show that formula (4.7) is also valid in the case of a system of differential equations of the form (4.5) where

- Y is a vector of species $Y = (Y_1, \dots, Y_n)$
- each F_j is an \mathbb{L} -species as before,
- each G_j is a vector $(G_{j,1}, \dots, G_{j,n})$ of species of n sorts
- Z is a vector (Z_1, \dots, Z_n) of variables that correspond to n sorts of buds.

The system (4.5) can then be written as

$$Y'_i = \sum_{j \in J} F_j(T) G_{j,i}(Y_1, \dots, Y_n), \quad Y_i(0) = Z_i, \quad i = 1, \dots, n \quad (4.8)$$

As observed in [8, §6], the combinatorial solution of a system of equations is similar to that of a single equation. It consists, in the present case, of enriched increasing arborescences as in Fig. 8, with the following additional features:

A ‘color’, i.e. an element of the set $\{1, \dots, n\}$, is attributed to each fertile point and the structure attached to a fertile point of color i is required to be a $F_i(T)G_{j,i}(Y_1, \dots, Y_n)$ -structure for some $j \in J$. The i th component $Y_i = A_{M,i}(T, Z)$ of the solution $Y = A_M(T, Z)$ consists of structures which are either a single bud of sort i or such colored arborescences whose root has color i .

For each $j \in J$, the ‘eclosion’ operator \mathcal{D}_j is defined by

$$\mathcal{D}_j = \sum_{i=1}^n \frac{G_{j,i}(Z_1, \dots, Z_n) \partial}{\partial Z_i} \quad (4.9)$$

it applies to species $\Phi(Z) = \Phi(Z_1, \dots, Z_n)$ of n sorts of buds as follows: a $\mathcal{D}_j(\Phi(Z))$ -structure is a $\Phi(Z)$ -structure in which, for some color i , a bud of sort i has been replaced by a $G_{j,i}(Z_1, \dots, Z_n)$ -structure.

With this interpretation, formula (4.7) is then again valid. This constitute a combinatorial proof of the ‘fundamental formula’ of Michel Fliess (see [1, 2]) for systems of differential equations with forcing terms, that is systems of the form (4.8), with $J = \{0, 1, \dots, m\}$, $F_0(T) = 1$ and, for $j \geq 1$, $F_j(T) = U_j(T)$ is a variable ‘input’ function.

5. Examples

The linear differential equation

$$Y' = F(T)Y + G(T), \quad Y(0) = Z \quad (5.1)$$

was studied from a combinatorial point of view in [8] and the classical formulas proven. However this is a special case of (4.5), with $J = \{1, 2\}$, $F_1(T) = F(T)$, $F_2(T) = G(T)$, $G_1(Y) = Y$, and $G_2(Y) = 1$, and the application of (4.6) gives an alternate formula for the solution of (5.1) which is closer to the simple combinatorial solution. Indeed, in this case, we have $\mathcal{D}_1 = Z\partial/\partial Z$ and $\mathcal{D}_2 = \partial/\partial Z$ and the only words $\alpha \in J^*$ for which the operator \mathcal{D}_α applied to Z yields a non zero result are

$$1 \ 1 \cdots 1 = 1^k \quad \text{and} \quad 1 \ 1 \cdots 1 \ 2 = 1^k 2, \quad k \geq 0. \quad (5.2)$$

For these words, we have

$$\mathcal{D}_{(1^k)}(Z) = Z \quad \text{and} \quad \mathcal{D}_{(1^k 2)}(Z) = 1. \quad (5.3)$$

Hence we get the following expression (to be compared with formula (4.16) of [8]) for the solution of (5.1).

Proposition 5.1. *The solution $Y = A_{F,G}(T, Z)$ of the linear differential equation (5.1) can be expressed as*

$$A_{F,G}(T, Z) = \sum_{k \geq 0} \int_0^T (F, G) d(1^k) Z + \sum_{k \geq 0} \int_0^T (F, G) d(1^k 2) \quad (5.4)$$

where, for $\alpha \in \{1, 2\}^*$, $\int_0^T (F, G) d\alpha$ denotes the iterated integral associated with α and the family $(F_1, F_2) = (F, G)$.

We now give an application to differential equation with forced entry of the form

$$Y' = G(Y) + U(T), \quad Y(0) = 0 \quad (5.5)$$

and in particular to the equation

$$Y' = aY + bY^2 + U(T), \quad Y(0) = 0 \quad (5.6)$$

that models an electric circuit with a quadratic resistance. $U(T)$ represents a variable 'entry' or 'control' function (the current, in (5.6)), and $Y(T)$, the output (voltage) function. These equations are of the form (4.5), with $J = \{0, 1\}$, $F_0(T) = 1$, $F_1(T) = U(T)$, $G_0(Y) = G(Y)$ and $G_1(Y) = 1$; thus $\mathcal{D}_0 = G(Z) \partial / \partial Z$ and $\mathcal{D}_1 = \partial / \partial Z$. By proposition 4.1, the solution $Y = V(T)$ of the differential equation (5.5) can be expressed as

$$V(T) = \sum_{\alpha \in \{0,1\}^*} \int_0^T (1, U) d\alpha \mathcal{D}_\alpha(Z) |_{Z=0}. \quad (5.7)$$

The evaluation at $Z = 0$ in (5.7) comes from the initial condition $V(0) = 0$.

We see that $V(T)$ depends not only on T but also on the variable function $U(T)$, i.e. $V = V(T, U)$. Thus (5.7) is a functional expansion of $V(T, U)$ in terms of iterated integrals $\int_0^T (1, U) d\alpha$, for words $\alpha \in \{0, 1\}^*$ and the family $(F_0, F_1) = (1, U)$. For the associated generating series, we get an analogous functional expansion

$$V(t, U) = \sum_{\alpha \in \{0,1\}^*} V_\alpha \int_0^t (1, U) d\alpha \quad (5.8)$$

which we call the *Fliess series* of the system 5.5. Much attention is devoted in control theory to the numerical computation of the coefficients V_α in the Fliess series. In particular, Fliess and his school (see [1, 2]) transform (5.5) into an algebraic equation satisfied by the formal power generating series of $\{V_\alpha\}_{\alpha \in J^*}$

$$S = \sum_{\alpha \in \{0,1\}^*} V_\alpha X_\alpha \quad (5.9)$$

in non commuting variables X_0 and X_1 (the monomials X_α are defined recursively

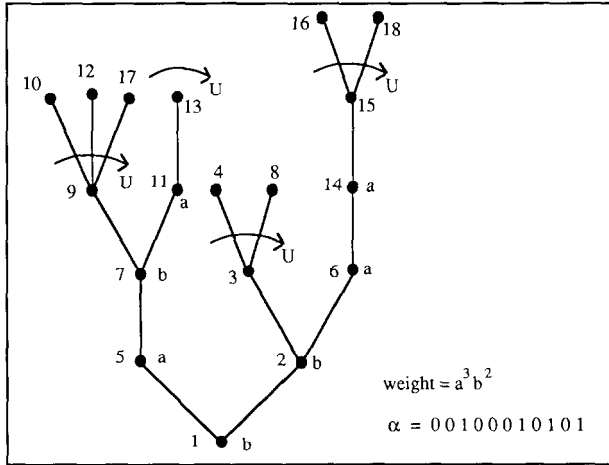


Fig. 12.

by $X_e = 1$ and $X_{j\beta} = X_j X_\beta$). The equation is of the form $S = f(S)$ and can be solved iteratively as a fixed point problem using a programmed computer.

The approach proposed here can greatly simplify the computation (see [2]) of the coefficients V_α since it provides a direct combinatorial interpretation for them. Consider, for example, the equatoin (5.6), where a and b are formal parameters. The combinatorial solution $Y = V(T)$ consists of increasing arborescences with the following properties (see Fig. 12):

- No buds appear, since, by the initial condition, $V(0) = 0$.
- There are two types (0 or 1) of fertile points. Fertile points of type 0 have 1 or 2 sons (left, right) and carry a weight, a or b , accordingly; these sons are themselves fertile. Those of type 1 are simply the roots of $\int U$ -structures; they carry no weight.
- The global weight of the structure is the product of the weights of the fertile points of type 0.

The word $\alpha \in \{0, 1\}^*$ is obtained by reading successively the types of the fertile points in their natural order. Combinatorial separation of variables then gives, on the one hand, a $\int_0^T (1, U) d\alpha$ -structure (Fig. 13) and, on the other hand, a

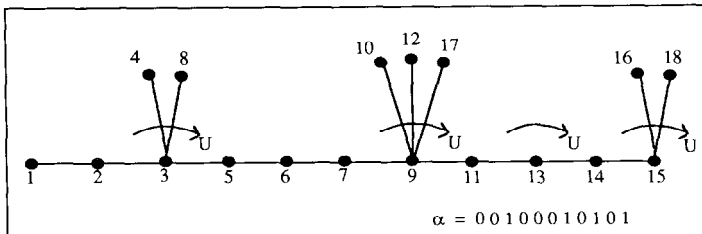


Fig. 13.

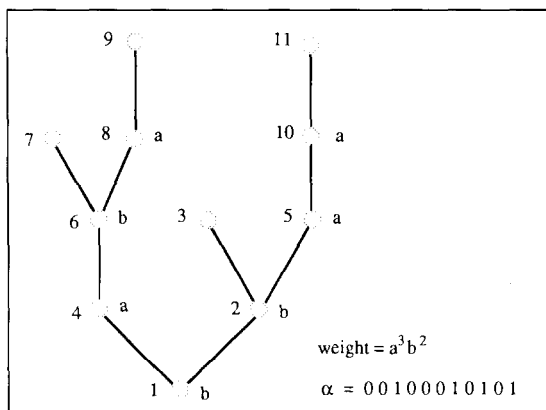


Fig. 14.

so-called ‘weighted increasing 1–2 planar tree’ on $k = |\alpha|$ phantom buds (Fig. 14). The latter has labels corresponding to the order of enclosures, which can be of two kinds:

$$\mathcal{D}_0 = \frac{(aZ + bZ^2)\partial}{\partial Z} \quad \text{and} \quad \mathcal{D}_1 = \frac{\partial}{\partial Z}. \quad (5.10)$$

The word α and the weight $a^i b^j$ are the same as those of the arborescence and can be read on the associated tree.

For $\alpha \in \{0, 1\}^*$, let $P(\alpha)$ be the set of all such 1–2 trees over α and let $P_\alpha(a, b)$ be its generating polynomial:

$$P_\alpha(a, b) = \sum_{p \in P(\alpha)} \text{weight}(p). \quad (5.11)$$

As formula (5.7) shows, $P(\alpha)$ is the set of empty (i.e. no buds) $\mathcal{D}_\alpha(Z)$ -structures, that is

$$P(\alpha) = \mathcal{D}_\alpha(Z)|_{Z=0}. \quad (5.12)$$

In terms of generating functions, we obtain the following:

Corollary 5.2. *The coefficients V_α in the Fliess series (5.8) for the solution $Y = V(T)$ of the equation $Y' = aY + bY^2 + U(T)$, $Y(0) = 0$ are the generating polynomials of 1–2 trees over α :*

$$V_\alpha = P_\alpha(a, b). \quad (5.13)$$

In conclusion, we note that the polynomials $P_\alpha(a, b)$ can be given a more explicit form using standard combinatorial techniques such as Motzkin words and paths and Dyck words (see [10]).

Acknowledgments

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