Graph isomorphism in quasipolynomial time parameterized by treewidth

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Abstract

We extend Babai's quasipolynomial-time graph isomorphism test (STOC 2016) and develop a quasipolynomial-time algorithm for the multiple-coset isomorphism problem. The algorithm for the multiple-coset isomorphism problem allows to exploit graph decompositions of the given input graphs within Babai's group-theoretic framework.

We use it to develop a graph isomorphism test that runs in time $n^{\text{polylog}(k)}$ where n is the number of vertices and k is the minimum treewidth of the given graphs and polylog(k) is some polynomial in $\log(k)$. Our result generalizes Babai's quasipolynomial-time graph isomorphism test.

1 Introduction

The graph isomorphism problem asks for a structure preserving bijection between two given graphs G and H, i.e., a bijection $\varphi:V(G)\to V(H)$ such that $vw\in E(G)$ if and only if $\varphi(v)\varphi(w)\in E(H)$. One central open problem in theoretical computer science is the question whether the graph isomorphism problem can be solved in polynomial time. There are a few evidences that the problem might not be NP-hard. For example, NP-hardness of the problem implies a collapse of the polynomial hierarchy [Sch88]. Moreover, NP-hardness of the graph isomorphism problem would refute the exponential time hypothesis since the problem can be decided in quasipolynomial time [Bab16].

The research of the graph isomorphism problem started with two fundamental graph classes, i.e., the class of trees and the class of planar graphs. In 1970, Zemlyachenko gave a polynomial-time isomorphism algorithm for trees [Zem70]. One year later, Hopcroft and Tarjan extended a result of Weinberg and designed a polynomial-time isomorphism algorithm for planar graphs [HT71],[Wei66]. In 1980, Filotti, Mayer and Miller extended the polynomial-time algorithm to graphs of bounded genus [Mil80],[FM80]¹. The genus is a graph parameter that measures how far away the graph is from being planar.

In Luks's pioneering work in 1982, he gave a polynomial-time isomorphism algorithm for graphs of bounded degree [Luk82]. His group-theoretic approach laid the foundation of many other algorithms that were developed ever since. It turns out that the research in the graph

¹Myrvold and Kocay pointed out an error in Filotti's techniques [MK11]. However, different algorithms have been given which show that the graph isomorphism problem for graphs of bounded genus is indeed decidable in polynomial time [Mil83, Gro00, Kaw15].

isomorphism problem for restricted graph classes was a promising approach in tackling the graph isomorphism problem in general. Shortly after Luks's result, a combinatorial partitioning lemma by Zemlyachenko was combined with Luks's framework. This resulted in an isomorphism algorithm for graphs with n vertices in general that runs in time $2^{\mathcal{O}(\sqrt{n\log n})}$ [ZKT85],[BL83]. This algorithm was the fastest for decades.

In 1983, the seminal work of Robertson and Seymour in graph minors started a new era of graph theory [RS83]. At the same time, Miller extended Luks's group-theoretic framework to hypergraphs [Mil83]. It turned out that the study of general structures such as hypergraphs was also a promising approach in tackling the graph isomorphism problem. In 1991, Ponomarenko could in fact use Miller's hypergraph algorithm to design a polynomial-time isomorphism algorithm for graphs excluding a minor [Pon91].

The work of Robertson and Seymour also rediscovered the notion of treewidth [Die12], a graph parameter that measures how far away the graph is from being a tree. The treewidth parameter was reborn and has been studied ever since. So, researchers went back to the roots and studied the isomorphism problem for graphs of bounded treewidth. In 1990, Bodlaender gave a simple isomorphism-algorithm for graphs of treewidth k with n vertices that runs in time $n^{\mathcal{O}(k)}$ [Bod90]. However, no FPT-algorithm was known, i.e., an isomorphism algorithm with a running time of the form $f(k) \cdot n^{\mathcal{O}(1)}$. The search of a FPT-algorithm occupied researchers over years and this open problem was explicitly stated by several authors [YBdFT99, BCC+06, KM08, KS10, Ota12, BDK12, DF13, GM15]. In 2017, Lokshtanov, Pilipczuk, Pilipczuk and Saurabh finally solved this open problem and designed a FPT-algorithm for the graph isomorphism problem [LPPS17]. Their algorithm runs in time $2^{\mathcal{O}(k^5 \log k)} n^{\mathcal{O}(1)}$ where n is the number of vertices and k is the minimum treewidth of the given graphs.

At the same time, Babai made a breakthrough and designed a quasipolynomial-time algorithm for the graph isomorphism problem in general [Bab16]. His algorithm runs in time $n^{\text{polylog}(n)}$ where n is the number of vertices and polylog(n) is some polynomial in $\log(n)$ (according to Helfgott's analysis the function polylog(n) can chosen to be quadratic in $\log(n)$ [Hel17]). To achieve this result, Babai built on Luks's group-theoretic framework, which actually solves the more general string isomorphism problem. One of the main questions is how to combine Babai's group-theoretic algorithm with the graph-theoretic techniques that have been developed. For example, it is unclear how to exploit a decomposition of the given graphs within Babai's framework since his algorithm actually processes strings rather than graphs.

Recently, Grohe, Neuen and Schweitzer were able to extend Babai's algorithm to graphs of maximum degree d and an isomorphism algorithm was developed that runs in time $n^{\text{polylog}(d)}$ [GNS18]. They suggest that their techniques might be useful also for graphs parameterized by treewidth and conjectured that the isomorphism problem for graphs of treewidth k can be decided in time $n^{\text{polylog}(k)}$.

In [GNSW18], the graph-theoretic FPT-algorithm of Lokshtanov et al. was improved by using Babai's group-theoretic algorithm and the extension given by Grohe et al. as a black box. They decomposed a graph of bounded treewidth into subgraphs with particular properties. They were able to design a faster algorithm that computes the isomorphisms between these subgraphs. However, they pointed out a central problem that arises when dealing with graph decompositions: When the isomorphisms between these subgraphs are already computed, how can they be efficiently merged in order to compute the isomorphisms between the entire graphs? This problem was named as multiple-coset isomorphism problem and is formally defined as follows. Given two sets $J = \{\rho_1 \Delta_1^{\text{Can}}, \ldots, \rho_t \Delta_t^{\text{Can}}\}$ and $J' = \{\rho'_1 \Delta'_1^{\text{Can}}, \ldots, \rho'_t \Delta'_t^{\text{Can}}\}$ where $\rho_i: V \to n, \rho'_i: V' \to n$ are bijections and $\Delta_i^{\text{Can}}, \Delta_i'^{\text{Can}} \leq \text{Sym}([n])$ are permutation groups for all $i \in [t]$, the problem is to decide whether there are bijections $\varphi: V \to V', \pi: [t] \to [t]$ such that

 $\Delta_i^{\operatorname{Can}} = {\Delta'}_{\pi(i)}^{\operatorname{Can}}$ and $\varphi \in \rho_i \Delta_i^{\operatorname{Can}} \rho'_{\pi(i)}$ for all $i \in [t]$. By applying the group-theoretic black box algorithms, they achieved an improved isomorphism test for graphs of treewidth k that runs in time $2^{k \cdot \operatorname{polylog}(k)} n^{\mathcal{O}(1)}$. However, for further improvements, it did not seem to be enough to use the group-theoretic algorithms as a black box only. The question of an isomorphism algorithm that runs in time $n^{\operatorname{polylog}(k)}$ remained open.

In [SW19], the study of the multiple-coset isomorphism problem continued. Rather than using group-theoretic algorithms as a black box, they were able to extend Luks's group-theoretic framework to the multiple-coset isomorphism problem. In order to facilitate their recursion, they introduced the class of combinatorial objects. Their class of combinatorial objects contains hypergraphs, colored graphs, relational structures, explicitly given codes and more. However, the key idea in order to handle the involved structures recursively, was to add so-called labeling cosets to their structures. By doing so, they could combine combinatorial decomposition techniques with Luks's group-theoretic framework. This led to a simply-exponential time algorithm for the multiple-coset isomorphism problem. Although the achieved running time was far away from being quasipolynomial, their result led to improvements of several algorithms. For example, it led to the currently best algorithm for the normalizer problem (a central problem in computational group theory) [Wie20]. However, they were not able to extend also Babai's techniques to their framework and the question of a graph isomorphism algorithm running in time $n^{\text{polylog}(k)}$ remained open.

Our Contribution In this paper, we give a quasipolynomial-time algorithm for the multiple-coset isomorphism problem. This leads to an answer of the conjecture in [GNS18] mentioned above.

Theorem (Theorem 43). The graph isomorphism problem can be decided in time $n^{\text{polylog}(k)}$ where n is the number of vertices and k is the minimum treewidth of the input graphs.

When k = polylog(n), our algorithm runs in time $n^{\mathcal{O}(\log(\log n)^c)}$ (for some constant c) and is significantly faster than Babai's algorithm and existing FPT-algorithms for graphs parameterized by treewidth.

For the present work, we exploit the fact that Babai's algorithm was recently extended to canonization [Bab19]. A canonical labeling of a graph is a function that labels the vertices V of the graph with integers $1, \ldots, |V|$ in such a way that the labeled versions of two isomorphic graphs are equal (rather than isomorphic). The computation of canonical forms and labelings, rather than isomorphism testing, is an important task in the area of graph isomorphism and is especially useful for practical applications. Also the framework given in [SW19] is actually designed for the canonization problem. The present paper is based on these works and our algorithms provide canonical labelings as well. Only the algorithm given in the last section depends on the bounded-degree isomorphism algorithm of Grohe et al. for which no adequate canonization version is known.

The first necessary algorithm that we provide in our work is a simple canonization algorithm for hypergraphs.

Theorem (Theorem 15). Canonical labelings for hypergraphs (V, H) can be computed in time $(|V| + |H|)^{\text{polylog}|V|}$.

There is a simple argument why this algorithm is indeed necessary for our main result. It is well-known that a hypergraph X = (V, H) can be encoded as a bipartite graph $G_X = (V \cup H, E)$ (the bipartite graph G_X has an edge $(v, S) \in E$, if and only if $v \in S$). It is not hard to show that

the treewidth k of this bipartite graph G_X is at most |V|. The bipartite graph G_X uniquely encodes the hypergraph X, in particular, two hypergraphs are isomorphic if and only if their corresponding bipartite graphs are isomorphic. This means that an isomorphism algorithm for graphs of treewidth k running in time $n^{\text{polylog}(k)}$ would imply an isomorphism algorithm for hypergraphs running in time $(|V| + |H|)^{\text{polylog}|V|}$. However, applying Babai's algorithm to the bipartite graph would lead to a running time of $(|V| + |H|)^{\text{polylog}(|V| + |H|)}$. Instead of applying Babai's algorithm to the bipartite graph directly, we decompose the hypergraph and canonize the substructures recursively. To merge the canonical labelings of all subhypergraphs, we use a canonical version of the multiple-coset isomorphism problem. However, for the hypergraph algorithm, it suffices to use Babai's algorithm as a black box only.

Our decomposition technique for hypergraphs can also be used to design a simple canonization algorithm for k-ary relations.

Theorem (Theorem 13). Canonical labelings for k-ary relations $R \subseteq V^k$ can be computed in time $2^{\text{polylog}|V|}|R|^{\mathcal{O}(1)}$.

The algorithm improves the currently best algorithm from [GNS18]. As graphs can be seen as binary relations, our algorithm generalizes the quasipolynomial-time bound for graphs. The achieved running time is the best one can hope for as long as the graph isomorphism problem has no solution better than quasipolynomial time.

Our main algorithm finally solves the multiple-coset isomorphism problem. In fact, the algorithm computes canonical labelings as well.

Theorem (Theorem 22). Canonical labelings for a set J consisting of labeling cosets can be computed in time $(|V| + |J|)^{\text{polylog}|V|}$.

This result is actually of independent interest as it also implies a faster canonization algorithm for the entire class of combinatorial objects (Corollary 35).

To solve this problem, the simple hypergraph canonization algorithm can be used as a subroutine in some places. However, we do not longer use Babai's and Luks's techniques as a black box only. To extend their methods, we follow the route of [SW19] and consider combinatorial objects that allows to combine combinatorial structures with permutation group theory. In particular, we can extend Luks's subgroup reduction and Babai's method and aggregation of local certificates to our framework. All these methods were designed for the string isomorphism problem and need non-trivial extensions when dealing with a set of labeling cosets rather than a string.

Related Work Another extension of Babai's quasipolynomial time algorithm has been independently proposed by Daniel Neuen [Neu20] who provided another algorithm for the isomorphism problem of hypergraphs. However, Neuen can exploit groups with restricted composition factors that are given as additional input in order to speed up his algorithm. This can be exploited in the setting of graphs of bounded Euler genus. He provides a graph isomorphism algorithm that runs in time $n^{\text{polylog}(g)}$ where n is the number of vertices and g is the minimum genus of the given graphs.

On the other hand, his algorithm is not able to handle labeling cosets occurring in the combinatorial structures. In particular, his algorithm is not able to solve the multiple-coset isomorphism problem in the desired time bound, which we require for our isomorphism algorithm for graphs parameterized by treewidth. Moreover, his techniques do not provide canonical labelings.

We hope that both algorithms can be combined to give a faster isomorphism test for the large class of graphs excluding a topological subgraph. This large class of graphs includes the graphs of bounded treewidth, graphs of bounded genus, graphs of bounded degree and graphs excluding a minor. In fact, Grohe and Marx provide a structure theorem which shows that the graph classes mentioned above also characterize graphs excluding a topological subgraph. Informally, they showed that graphs excluding a topological subgraph can be decomposed into almost bounded-degree parts and minor-free parts which in turn can be decomposed into almost-embeddable parts [GM15]. Therefore, we hope that the improved algorithms for the isomorphism problem for bounded-degree graphs and bounded-genus graphs can be combined with our algorithm to exploit the occurring graph decomposition.

Organization of the Paper In Section 3, we show how the multiple-coset isomorphism problem and its canonical version can be reduced to a string canonization problem which in turn can be processed with Babai's algorithm. However, this reduction does not lead to the desired time bound and only works efficiently when the instance is small enough. Section 4 deals with k-ary relations $R \subseteq V^k$ over a vertex set V. We demonstrate how a partitioning technique can be used to reduce the canonization problem of a k-ary relation to instances of small size in each decomposition level. Since we only need to handle small instances at each decomposition level, we can make use of the subroutines given in the previous section. As a result, we obtain a canonization algorithm for k-ary relations that runs in time $2^{\text{polylog}|V|}|R|^{\mathcal{O}(1)}$. In Section 5, we extend our technique to hypergraphs and so-called coset-labeled hypergraphs. The algorithm for coset-labeled hypergraphs is used as a subroutine in our main algorithm given in the next section. In Section 6, we finally present our main algorithm which canonizes a set of labeling cosets and solves the multiple-coset isomorphism problem. Our main algorithm is divided into five subroutines. In the first subroutine, we extend the partitioning technique to families of partitions. The second subroutine extends Luks's subgroup reduction to our framework. The third subroutine reduces to the barrier configuration which can be characterized by a giant representation. The fourth and fifth subroutine extend Babai's method and aggregation of local certificates to our framework. In Section 7, a straightforward application of the multiple-coset isomorphism problem leads to an isomorphism algorithm that runs in time $n^{\text{polylog}(k)}$ where n is the number of vertices and k is the treewidth of the given graphs.

2 Preliminaries

We recall the framework given in [SW19].

Set Theory For an integer t, we write [t] for $\{1, \ldots, t\}$. For a set S and an integer k, we write $\binom{S}{k}$ for the k-element subsets of S and 2^S for the power set of S.

Group Theory The composition of two functions $f: V \to U$ and $g: U \to W$ is denoted by fg and is defined as the function that first applies f and then applies g. The symmetric group on a set V is denoted by $\operatorname{Sym}(V)$ and the symmetric group of degree $t \in \mathbb{N}$ is denoted by $\operatorname{Sym}(t)$. In the following, let $G \subseteq \operatorname{Sym}(V)$ be a group. The index of a subgroup $H \subseteq G$ is denoted by (G: H). The setwise stabilizer of $A \subseteq V$ in G is denoted by $\operatorname{Stab}_G(A) := \{g \in G \mid g(a) \in A \text{ for all } a \in A\}$. The pointwise stabilizer of $A \subseteq V$ in G is denoted by $G_{(A)} := \{g \in G \mid g(a) = a \text{ for all } a \in A\}$. Analogously, the stabilizer of a vertex $v \in V$ in G is denoted by $G_{(v)} := G_{(\{v\})}$. A set $A \subseteq V$ is called G-invariant if $\operatorname{Stab}_G(A) = G$. A set $v^G := \{g(v) \mid g \in G\}$ is called G-orbit of $v \in V$.

The G-orbit partition of V is the partition of V in which each part is a G-orbit (for some $v \in V$). partition. A group $G \leq \operatorname{Sym}(V)$ is called transitive if V is one single G-orbit. A partition of $V = V_1 \cup \ldots \cup V_t$ is called G-invariant if each part V_i and each $g \in G$ it holds $V_i^g := \{g(v) \mid v \in V_i\} \in G$. For transitive groups G, the G-invariant partitions of V are also called block systems for G. A group $G \leq \operatorname{Sym}(V)$ is called primitive if there are no non-trivial block systems for G.

Labeling Cosets A labeling of a set V is a bijection $\rho: V \to \{1, \dots, |V|\}$. A labeling coset of a set V is a set of bijections Λ such that $\Lambda = \Delta \rho = \{\delta \rho \mid \delta \in \Delta\}$ for some subgroup $\Delta \leq \operatorname{Sym}(V)$ and some labeling $\rho: V \to \{1, \dots, |V|\}$. We write $\operatorname{Label}(V)$ to denote the labeling coset $\operatorname{Sym}(V)\rho = \{\sigma \rho \mid \sigma \in \operatorname{Sym}(V)\}$ where ρ is an arbitrary labeling of V. Analogous to subgroups, a set $\Theta \tau$ is called a labeling subcoset of $\Delta \rho$, written $\Theta \tau \leq \Delta \rho$, if the labeling coset $\Theta \tau$ is a subset of $\Delta \rho$.

Hereditarily Finite Sets and Combinatorial Objects Inductively, we define hereditarily finite sets, denoted by HFS(V), over a ground set V.

- A vertex $v \in V$ is an atom and a hereditarily finite set $v \in HFS(V)$,
- a labeling coset $\Delta \rho \leq \text{Label}(V)$ is an atom and a hereditarily finite set $\Delta \rho \in \text{HFS}(V)$,
- if $X_1, \ldots, X_t \in HFS(V)$, then also $\mathcal{X} = \{X_1, \ldots, X_t\} \in HFS(V)$ where $t \in \mathbb{N} \cup \{0\}$, and
- if $X_1, \ldots, X_t \in \mathrm{HFS}(V)$, then also $\mathcal{X} = (X_1, \ldots, X_t) \in \mathrm{HFS}(V)$ where $t \in \mathbb{N} \cup \{0\}$.

A (combinatorial) object is a pair (V, \mathcal{X}) consisting of a ground set V and a hereditarily finite set $\mathcal{X} \in \mathrm{HFS}(V)$. The ground set V is usually apparent from context and the combinatorial object (V, \mathcal{X}) is identified with the hereditarily finite set \mathcal{X} . The set $\mathrm{Objects}(V)$ denotes the set of all (combinatorial) objects over V. The transitive closure of an object \mathcal{X} , denoted by $\mathrm{TClosure}(\mathcal{X})$, is defined as all objects that recursively occur in \mathcal{X} . All labeling cosets that occur in \mathcal{X} are succinctly represented via generating sets. The encoding size of an object \mathcal{X} can be chosen polynomial in $|\mathrm{TClosure}(\mathcal{X})| + |V| + t_{\max}$ where t_{\max} is the maximal length of a tuple in $\mathrm{TClosure}(\mathcal{X})$.

Ordered Objects An object is called *ordered* if the ground set V is linearly ordered. The linearly ordered ground sets that we consider are always subsets of natural numbers with their standard ordering "<". An object is *unordered* if V is a usual set (without a given order). Partially ordered objects in which some, but not all, atoms are comparable are not considered.

Lemma 1 ([SW19]). There is an ordering "<" on pairs of ordered objects that can be computed in polynomial time.

Applying Functions to Unordered Objects Let V be an unordered ground set and let V' be a ground set that is either ordered or unordered. The image of an unordered object $\mathcal{X} \in \text{Objects}(V)$ under a bijection $\mu: V \to V'$ is an object $\mathcal{X}^{\mu} \in \text{Objects}(V')$ that is defined as follows.

- $v^{\mu} \coloneqq \mu(v)$,
- $(\Delta \rho)^{\mu} := \mu^{-1} \Delta \rho$,
- $\{X_1, \dots, X_t\}^{\mu} := \{X_1^{\mu}, \dots, X_t^{\mu}\}$ and
- $(X_1,\ldots,X_t)^{\mu} := (X_1^{\mu},\ldots,X_t^{\mu}).$

Isomorphisms and Automorphisms of Unordered Objects The set of all isomorphisms from an object $\mathcal{X} \in \text{Objects}(V)$ and to an object $\mathcal{X}' \in \text{Objects}(V')$ is denoted by $\text{Iso}(\mathcal{X}; \mathcal{X}') := \{\varphi : V \to V' \mid \mathcal{X}^{\varphi} = \mathcal{X}'\}$. The set of all automorphisms of an object \mathcal{X} is denoted by $\text{Aut}(\mathcal{X}) := \text{Iso}(\mathcal{X}; \mathcal{X})$. Both isomorphisms and automorphisms are defined for objects that are unordered only.

For two unordered sets V and V', the set $\operatorname{Iso}(V;V')$ is also used to denote the set of all bijections from V to V'. This notation indicates and stresses that both V and V' have to be unordered. Additionally, it is used in a context where $\varphi \in \operatorname{Iso}(V;V')$ is seen as an isomorphism $\varphi \in \operatorname{Iso}(\mathcal{X};\mathcal{X}^{\varphi})$.

Induced Groups and Labeling Cosets In the following, let $\mathcal{X} \in \text{Objects}(V)$ be a set and $\Delta \leq \text{Aut}(\mathcal{X}) \leq \text{Sym}(V)$ be a group consisting of automorphisms of \mathcal{X} . For a permutation $\delta \in \Delta$, we define the permutation induced on \mathcal{X} , denoted by $\delta[\mathcal{X}]$, as the permutation that maps $X \in \mathcal{X}$ to $\delta[\mathcal{X}](X) := X^{\delta} \in \mathcal{X}$. We define the group Δ induced on \mathcal{X} , denoted by $\Delta[\mathcal{X}] \leq \text{Sym}(\mathcal{X})$, as the group consisting of the elements $\delta[\mathcal{X}] \in \text{Sym}(\mathcal{X})$ for $\delta \in \Delta$. Similarly, for a labeling ρ of V, we define the labeling ρ induced on \mathcal{X} , denoted by $\rho[\mathcal{X}] : \mathcal{X} \to \{1, \dots, |\mathcal{X}|\}$, as the labeling that orders the elements in \mathcal{X} according to the ordering "<" from Lemma 1, i.e., $\rho(X_i) < \rho(X_j)$ if and only if $X_i^{\rho} < X_j^{\rho}$. Furthermore, for a given labeling cosets $\Delta \rho \leq \text{Label}(V)$, we define the induced labeling coset on \mathcal{X} , denoted by $(\Delta \rho)[\mathcal{X}] \leq \text{Label}(\mathcal{X})$, as $\Delta[\mathcal{X}]\rho[\mathcal{X}]$.

Generating Sets and Polynomial-Time Library For the basic theory of handling permutation groups given by generating sets, we refer to [Ser03]. Indeed, most algorithms are based on strong generating sets. However, given an arbitrary generating set, the Schreier-Sims algorithm is used to compute a strong generating set (of size quadratic in the degree) in polynomial time. In particular, we will use that the following tasks can be performed efficiently when a group is given by a generating set.

- 1. Given a vertex $v \in V$ and a group $G \leq \text{Sym}(V)$, the Schreier-Sims algorithm can be used to compute the pointwise stabilizer $G_{(v)}$ in polynomial time.
- 2. Given a group $G \leq \text{Sym}(V)$, a subgroup that has a polynomial time membership problem can be computed in time polynomial in the index and the degree of the subgroup.
- 3. Let $S = \Delta_1 \rho_1, \ldots, \Delta_t \rho_t \leq \text{Label}(V)$ be a sequence of labeling cosets of V. We write $\langle S \rangle$ for the smallest labeling coset Λ such that $\Delta_i \rho_i \subseteq \Lambda$ for all $i \in [t]$. Given a representation for S, the coset $\langle S \rangle$ can be computed in polynomial time. Furthermore, the computation of $\langle S \rangle$ is isomorphism invariant w.r.t. S, i.e., $\varphi^{-1} \langle S \rangle = \langle \varphi^{-1} S \rangle$ for all bijections $\varphi : V \to V'$.

Definition 2 ([SW19]). Let \mathcal{C} be an isomorphisms-closed class of unordered objects, i.e., for all $\mathcal{X} \in \mathcal{C}$ over a ground set V and all bijections $\varphi : V \to V'$ it holds that $\mathcal{X}^{\varphi} \in \mathcal{C}$. A canonical labeling function CL is a function that assigns each unordered object $\mathcal{X} \in \mathcal{C}$ a labeling coset $\mathrm{CL}(\mathcal{X}) = \Lambda \leq \mathrm{Label}(V)$ such that:

- (CL1) $CL(\mathcal{X}) = \varphi CL(\mathcal{X}^{\varphi})$ for all $\varphi \in Iso(V; V')$, and
- (CL2) $CL(\mathcal{X}) = Aut(\mathcal{X})\pi$ for some (and thus for all) $\pi \in CL(\mathcal{X})$.

In this case, the labeling coset Λ is also called a *canonical labeling* for \mathcal{X} .

Lemma 3 ([SW19], Object Replacement Lemma). Let $\mathcal{X} = \{X_1, \dots, X_t\}$ be an object and let CL and CL_{Set} be canonical labeling functions. Define $\mathcal{X}^{\text{Set}} := \{\Delta_1 \rho_1, \dots, \Delta_t \rho_t\}$ where $\Delta_i \rho_i := \text{CL}(X_i)$ is a canonical labeling for $X_i \in \mathcal{X}$. Assume that $X_i^{\rho_i} = X_j^{\rho_j}$ for all $i, j \in [t]$. Then, $\text{CL}_{\text{Object}}(\mathcal{X}) := \text{CL}_{\text{Set}}(\mathcal{X}^{\text{Set}})$ defines a canonical labeling for \mathcal{X} .

3 Handling Small Objects via String Canonization

We consider the canonical labeling problem for a pair $(E, \Delta \rho)$ consisting of an edge relation $E \subseteq V^2$ and a labeling coset $\Delta \rho \le \text{Label}(V)$.

Problem 4. Compute a function CL_{Graph} with the following properties:

Input $(E, \Delta \rho) \in \text{Objects}(V)$ where $E \subseteq V^2$, $\Delta \rho \leq \text{Label}(V)$ and V is an unordered set.

Output A labeling coset $\operatorname{CL}_{\operatorname{Graph}}(E, \Delta \rho) = \Lambda \leq \operatorname{Label}(V)$ such that:

- (CL1) $\operatorname{CL}_{\operatorname{Graph}}(E, \Delta \rho) = \varphi \operatorname{CL}_{\operatorname{Graph}}(E^{\varphi}, \varphi^{-1} \Delta \rho) \text{ for all } \varphi \in \operatorname{Iso}(V; V').$
- (CL2) $\operatorname{CL}_{\operatorname{Graph}}(E, \Delta \rho) = \operatorname{Aut}((E, \Delta \rho))\pi$ for some (and thus for all) $\pi \in \Lambda$.

The automorphism group of $(E, \Delta \rho)$ is precisely $\operatorname{Aut}((E, \Delta \rho)) = \{\delta \in \Delta \mid (v, w) \in E \iff (\delta(v), \delta(w)) \in E\}$. For $\Delta \rho = \operatorname{Label}(V)$, this is exactly the canonical labeling problem for directed graphs. However, for labeling cosets $\Delta \rho \leq \operatorname{Label}(V)$ in general, the problem is equivalent to the string canonization problem (this can be shown by defining a string $\mathfrak{x} : V^2 \to \{0,1\}$ with positions V^2 such that $\mathfrak{x}((v,w)) = 1$ if and only if $(v,w) \in E$).

Theorem 5 ([Bab19]). A function CL_{Graph} for Problem 4 can be computed in time $2^{\text{polylog}|V|}$.

The next problem can be seen as a canonical intersection-problem for labeling cosets.

Problem 6. Compute a function CL_{Int} with the following properties:

Input $(\Theta \tau, \Delta \rho) \in \text{Objects}(V)$ where $\Theta \tau, \Delta \rho \leq \text{Label}(V)$ and V is an unordered set.

Output A labeling coset $CL_{Int}(\Theta\tau, \Delta\rho) = \Lambda \leq Label(V)$ such that:

(CL1) $\operatorname{CL}_{\operatorname{Int}}(\Theta\tau,\Delta\rho) = \varphi \operatorname{CL}_{\operatorname{Int}}(\varphi^{-1}\Theta\tau,\varphi^{-1}\Delta\rho) \text{ for all } \varphi \in \operatorname{Iso}(V;V').$

(CL2) $CL_{Int}(\Theta\tau, \Delta\rho) = (\Theta \cap \Delta)\pi$ for some (and thus for all) $\pi \in \Lambda$.

Lemma 7. A function CL_{Int} solving Problem 6 can be computed in time $2^{polylog|V|}$.

Proof. It is know that this problem reduces to graph canonization in polynomial time [SW19].

Next, we define the central problem of this paper which is introduced in [GNSW18],[SW19]. This problem is a canonical version of the multiple-coset isomorphism problem.

Problem 8. Compute a function CL_{Set} with the following properties:

Input $J \in \text{Objects}(V)$ where $J = \{\Delta_1 \rho_1, \dots, \Delta_t \rho_t\}, \Delta_i \rho_i \leq \text{Label}(V)$ for all $i \in [t]$ and V is an unordered set.

Output A labeling coset $CL_{Set}(J) = \Lambda \leq Label(V)$ such that:

- (CL1) $\operatorname{CL}_{\operatorname{Set}}(J) = \varphi \operatorname{CL}_{\operatorname{Set}}(J^{\varphi}) \text{ for all } \varphi \in \operatorname{Iso}(V; V').$
- (CL2) $CL_{Set}(J) = Aut(J)\pi$ for some (and thus for all) $\pi \in \Lambda$.

The automorphism group of J is precisely $\operatorname{Aut}(J) = \{ \sigma \in \operatorname{Sym}(V) \mid \exists \psi \in \operatorname{Sym}(t) \forall i \in [t] : \sigma^{-1}\Delta_i \rho_i = \Delta_{\psi(i)} \rho_{\psi(i)} \}$. We explain why this problem is the central problem when dealing with graph decompositions.

The Intuition Behind this Central Problem We want to keep this subsection as simple as possible and do not want to introduce tree decompositions yet. For our purpose, we consider a simplified formulation of a graph decomposition. In this subsection, a graph decomposition of a graph G = (V, E) is a family of subgraphs $\{H_i\}_{i \in [t]}$ that covers the edges of the entire graph, i.e., $E(G) = E(H_1) \cup ... \cup E(H_t)$. We say that a graph decomposition is defined in an isomorphism-invariant way if for two isomorphic graphs G, G' the decompositions $\{H_i\}_{i \in [t]}, \{H'_i\}_{i \in [t]}$ are

defined in such a way that each isomorphism $\varphi \in \text{Iso}(G; G')$ also maps each subgraph H_i of the decomposition of G to a subgraph H'_j of the decomposition of G'. In particular, such a decomposition has to be invariant under automorphisms of the graph.

Assume we have given a graph G for which we can construct a graph decomposition $\{H_i\}_{i\in[t]}$ in an isomorphism-invariant way and our task is the computation of a canonical labeling for G. A priori, it is unclear how to exploit our graph decomposition. In a first step, we could compute canonical labelings $\Delta_i \rho_i := \mathrm{CL}(H_i)$ for each subgraph H_i recursively. The central question is how to merge these labeling cosets $\Delta_i \rho_i$ for H_i in order to obtain a canonical labeling $\Delta \rho$ for the entire graph G.

The easy case occurs when all subgraphs H_i, H_j are pairwise non-isomorphic. In this case, the subgraphs cannot be mapped to each other and indeed $\operatorname{Aut}(G) = \operatorname{Aut}(H_1) \cap \ldots \cap \operatorname{Aut}(H_t)$. Therefore, the computation of $\Delta \rho$ reduces to a canonical intersection-problem. In fact, the algorithm from Lemma 7 can be used to compute canonical labelings $\Delta_{ij}\rho_{ij} := \operatorname{CL}_{\operatorname{Int}}(\Delta_i\rho_i, \Delta_j\rho_j)$ with $\Delta_{ij} = \Delta_i \cap \Delta_j = \operatorname{Aut}(H_i) \cap \operatorname{Aut}(H_j)$. By an iterated use of that canonical intersectionalgorithm, we can finally compute $\Delta \rho$ with $\Delta = \Delta_1 \cap \ldots \cap \Delta_t = \operatorname{Aut}(H_1) \cap \ldots \cap \operatorname{Aut}(H_t) = \operatorname{Aut}(G)$. Actually, the order in which we "intersect" the canonical labelings $\Delta_i \rho_i$ does matter and we need to be careful in order to ensure isomorphism invariance (CL1). (For example, there might be a canonical labeling function with $\operatorname{CL}_{\operatorname{Int}}((\operatorname{Label}(V), \Delta \rho)) = \Delta \rho$ and $\operatorname{CL}_{\operatorname{Int}}((\Delta \rho, \operatorname{Label}(V))) = \Delta \rho \pi$ for $\Delta \rho \neq \operatorname{Label}(V)$ where π is a permutation of $\{1, \ldots, |V|\}$ that swaps 1 and 2 and fixes all other elements. Clearly, $\operatorname{CL}_{\operatorname{Int}}$ can be extended to a canonical labeling function satisfying (CL1) and (CL2). However, $\operatorname{CL}_{\operatorname{Int}}((\operatorname{Label}(V), \Delta \rho)) \neq \operatorname{CL}_{\operatorname{Int}}((\Delta \rho, \operatorname{Label}(V)))$).

Let us consider the second extreme case in which all subgraphs H_i , H_j are pairwise isomorphic. In such a case, we have that $\operatorname{Aut}(G) = \{\sigma \in \operatorname{Sym}(V) \mid \exists \psi(t) \forall i \in [t] : \sigma \in \operatorname{Iso}(H_i; H_{\psi(i)})\}$. Equivalently, we have that $\operatorname{Aut}(G) = \operatorname{Aut}(\{\Delta_1\rho_1, \ldots, \Delta_t\rho_t\})$. Therefore, by the definition of Problem 8, the canonical labeling $\Delta \rho \coloneqq \operatorname{CL}_{\operatorname{Set}}(\{\Delta_1\rho_1, \ldots, \Delta_t\rho_t\})$ defines a canonical labeling for the entire graph G. Alternatively, one can use object replacement (Lemma 3) which intuitively says that for the purpose of canonization the subgraphs H_i can be replaced with their labeling cosets $\Delta_i \rho_i$. This also shows that $\Delta \rho \coloneqq \operatorname{CL}_{\operatorname{Set}}(\{\Delta_1\rho_1, \ldots, \Delta_t\rho_t\})$ define a canonical labeling for the entire graph G. Roughly speaking, Problem 8 can be seen as the task of merging the given labeling cosets.

The mixed case in which some (but not all) subgraphs H_i , H_j are isomorphic can be handled by a mixture of the above cases.

The main algorithm (Theorem 22) solves Problem 8 in a running time of $(|V| + |J|)^{\text{polylog}|V|}$. In Section 7, we apply this problem to graphs G with n vertices of treewidth k. In fact, we are able to bound $|V| \le k$ and $|J| \le n$ in this application which leads to the desired running time of $n^{\text{polylog}(k)}$.

But first of all, we give a simple algorithm that has a weaker running time which is quasipolynomial in |V| + |J|.

 $\textbf{Lemma 9.} \ \textit{A function } \text{CL}_{\text{Set}} \ \textit{solving Problem 8 can be computed in time } (|V| + |J|)^{\text{polylog}(|V| + |J|)}.$

The proof is similar to the proof of Lemma 23 in the arXiv version of [GNSW18]. By increasing the permutation domain V by a factor |J|, Problem 8 can actually be reduced to a graph canonization problem. For the sake of completeness, we give the detailed proof in Appendix A.

We consider the canonization problem for combinatorial objects.

Problem 10. Compute a function CL_{Object} with the following properties:

```
Input
               \mathcal{X} \in \text{Objects}(V) where V is an unordered set.
```

A labeling coset $CL_{Object}(\mathcal{X}) = \Lambda \leq Label(V)$ such that: Output $\mathrm{CL}_{\mathrm{Object}}(\mathcal{X}) = \varphi \, \mathrm{CL}_{\mathrm{Object}}(\mathcal{X}^{\varphi}) \text{ for all } \varphi \in \mathrm{Iso}(V; V').$ (CL1)(CL2) $\mathrm{CL}_{\mathrm{Object}}(\mathcal{X}) = \mathrm{Aut}(\mathcal{X})\pi$ for some (and thus for all) $\pi \in \Lambda$.

For an object $\mathcal{X} \in \text{Objects}(V)$, let $t_{\text{max}}(\mathcal{X})$ be the size of the largest set involved in \mathcal{X} , i.e., $t_{\max}(v) = 0$ and $t_{\max}(\Delta \rho) = 0$ for vertices $v \in V$ and labeling cosets $\Delta \rho \leq \text{Label}(V)$ and inductively $t_{\max}((X_1,\ldots,X_s)) = \max_{i\in[s]} t_{\max}(\mathcal{X}_i)$ and $t_{\max}(\{X_1,\ldots,X_s\}) = \max\{\max_{i\in[s]} t_{\max}(\mathcal{X}_i),s\}$. It is known that canonical labeling for combinatorial objects (on a ground set V) reduces to canonical labeling for instances of Problem 6 (on the same ground set V) and instances of Problem 8 (on the same ground set V and of size t_{max}) in polynomial time [SW19]. Therefore, Problem 8 is a central problem when canonizing combinatorial objects in general.

Corollary 11. A function $\mathrm{CL}_{\mathrm{Object}}$ solving Problem 10 can be computed in time $2^{\mathrm{polylog}(|V|+t_{\mathrm{max}})}n^{\mathcal{O}(1)}$ where n is the input size (as defined in the preliminaries) and $t_{max} \le n$ is the size of the largest set involved in \mathcal{X} .

A later algorithm (Corollary 35) shows that canonical labelings for combinatorial objects can actually be computed in time $n^{\text{polylog}|V|}$ (or more precise $(|V| + t_{\text{max}})^{\text{polylog}|V|} n^{\mathcal{O}(1)}$).

Canonization of k-ary Relations

In this section, we consider the canonization problem for k-ary relations. As graphs can be seen as binary relations, this problem clearly generalizes the graph canonization problem.

Problem 12. Compute a function $\operatorname{CL}_{\operatorname{Rel}}$ with the following properties: Input $R \in \operatorname{Objects}(V)$ where $R \subseteq V^k$ for some $k \in \mathbb{N}$ and V is an unordered set.

A labeling coset $\operatorname{CL}_{\operatorname{Rel}}(R) = \Lambda \leq \operatorname{Label}(V)$ such that: Output

(CL1) $\operatorname{CL}_{\operatorname{Rel}}(R) = \varphi \operatorname{CL}_{\operatorname{Rel}}(R^{\varphi})$ for all $\varphi \in \operatorname{Iso}(V; V')$.

 $\operatorname{CL}_{\operatorname{Rel}}(R) = \{ \sigma \in \operatorname{Sym}(V) \mid (x_1, \dots, x_k) \in R \iff (\sigma(x_1), \dots, \sigma(x_k)) \in R \} \pi \text{ for some }$ (CL2)(and thus for all) $\pi \in \Lambda$.

One way to canonize k-ary relations is by using a well-known reduction to the graph canonization problem [Mil79]. Alternatively, the algorithm from Corollary 11 for combinatorial objects in general could also be applied to k-ary relations. However, both approaches lead to a running time that is quasipolynomial in |V|+|R|, i.e., $2^{\text{polylog}(|V|+|R|)}$. In this section, we will give a polynomial-time reduction to the canonization problem for combinatorial objects which are of input size polynomial in |V| (which does not depend on |R|). With this reduction, we obtain an improved algorithm that runs in time $2^{\text{polylog}|V|}|R|^{\mathcal{O}(1)}$. Our bound improves the currently best algorithm from [GNS18]. Moreover, our time bound is also optimal (when measured in |V| and |R|) as long as the graph isomorphism problem can not be solved faster than quasipolynomial time.

Partitions An (unordered) partition of a set $\mathcal{X} \in \text{Objects}(V)$ is a set $\mathcal{P} = \{P_1, \dots, P_p\}$ such that $\mathcal{X} = P_1 \cup \ldots \cup P_p$ where $\emptyset \neq P_i \subseteq \mathcal{X}$ for all $P_i \in \mathcal{P}$. In the algorithms that follow, our constructions can lead to "partitions" with a non-empty part. In such a case, we implicitly forget about the empty set in the partition. We say that \mathcal{P} is the singleton partition if $|\mathcal{P}| = 1$ and we say that \mathcal{P} is the partition into singletons if $|P_i| = 1$ for all $P_i \in \mathcal{P}$. A partition \mathcal{P} is called trivial if \mathcal{P} is the singleton partition or the partition into singletons.

The Partitioning Technique We suggest a general technique for exploiting partitions. In this setting, we assume that we are given some object $\mathcal{X} \in \text{Objects}(V)$ for which we can construct a partition $\mathcal{P} = \{P_1, \dots, P_p\}$ in an isomorphism-invariant way such that $2 \leq |\mathcal{P}| \leq 2^{\text{polylog}|V|}$. The goal is the computation of a canonical labeling for \mathcal{X} by using an efficient recursion.

Using recursion, we compute a canonical labeling $\Delta_i \rho_i$ for each part $P_i \subseteq \mathcal{X}$ recursively (assumed that we can define a partition for each part again). So far, we computed canonical labelings for each part $P_i \subseteq \mathcal{X}$ independently. The main idea is to use our central problem (Problem 8) to merge all these labeling cosets. Let us restrict our attention to the case in which the parts $P_i, P_j \in \mathcal{P}$ are pairwise isomorphic. In this case, we define the set $\mathcal{P}^{\text{Set}} := \{\Delta_i \rho_i \mid P_i \in \mathcal{P}\}$ consisting of the canonical labelings $\Delta_i \rho_i$ for each part. Moreover, by object replacement (Lemma 3), a canonical labeling for \mathcal{P}^{Set} defines a canonical labeling for \mathcal{P} as well. A canonical labeling for \mathcal{P} in turn defines a canonical labeling for \mathcal{X} since we assume the partition to be defined in an isomorphism-invariant way. Therefore, it is indeed true that a canonical labeling for \mathcal{P}^{Set} would define a canonical labeling for \mathcal{X} . For this reason, we can use the algorithm from Lemma 9 to compute a canonical labeling for \mathcal{P}^{Set} . The algorithm runs in the desired time bound since $|\mathcal{P}^{\text{Set}}| = |\mathcal{P}| \leq 2^{\text{polylog}|V|}$ is bounded by some quasipolynomial.

Let us consider the number of recursive calls $R(\mathcal{X})$ of this approach for a given object \mathcal{X} . Since we recurse on each part $P_i \in \mathcal{P}$, we have a recurrence of $R(\mathcal{X}) = 1 + \sum_{P_i \in \mathcal{P}} R(P_i)$ leading to at most $|\mathcal{X}|^{\mathcal{O}(1)}$ recursive calls. The running time for one single recursive call is bounded by $2^{\text{polylog}|V|}$. For this reason, the total running time is bounded by $2^{\text{polylog}|V|}|\mathcal{X}|^{\mathcal{O}(1)}$.

Theorem 13. A function CL_{Rel} solving Problem 12 can be computed in time $2^{\text{polylog}|V|}|R|^{\mathcal{O}(1)}$.

Proof. An algorithm for $CL_{Rel}(R)$:

If $|R| \le 1$:

Compute and return $\Lambda := CL_{Object}(R)$ using Corollary 11.

 \triangleright Since the size of the largest set involved in R is the set R itself, the algorithm from Corollary 11 runs in time $2^{\text{polylog}|V|}$.

If $|R| \geq 2$:

 \triangleright In this case, it is possible to define a partition \mathcal{P} of R in an isomorphism-invariant way. We will use this partition for a recursion as described in the partitioning technique.

Let r be the first position in which R differs, i.e., the smallest $r \in [k]$ such that there are $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in R$ with $x_r \neq y_r$.

Define an (unordered) partition $\mathcal{P} := \{P_v \mid v \in V\}$ of $R = \bigcup_{v \in V} P_v$ where $P_v := \{(x_1, \dots, x_k) \in R \mid x_r = v\}$.

 \triangleright By the choice of $r \in [k]$, this is not the singleton partition. On the other side, the size $|\mathcal{P}| \leq |V|$ is obviously bounded by a quasipolynomial in |V|.

Compute $\Delta_v \rho_v := \operatorname{CL}_{\operatorname{Rel}}(P_v)$ for each subrelation $P_v \in \mathcal{P}$ recursively.

Define $\mathcal{P}^{\text{Set}} := \{ (\Delta_v \rho_v, v) \mid P_v \in \mathcal{P} \}.$

 \triangleright We define an ordering according to the isomorphism type of the subrelations $P_v, v \in V$. Define an ordered partition $\mathbb{P} := (\mathcal{P}_1, \dots, \mathcal{P}_p)$ of $\mathcal{P}^{\text{Set}} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_p$ such that:

 $P_v^{\rho_v} < P_w^{\rho_w}$, if and only if $(\Delta_v \rho_v, v) \in \mathcal{P}_i$ and $(\Delta_w \rho_w, w) \in \mathcal{P}_j$ for some $i, j \in [p]$ with i < j. Compute $\Lambda_i := \mathrm{CL}_{\mathrm{Set}}(\mathcal{P}_i)$ for each $\mathcal{P}_i, i \in [p]$ using Lemma 9.

 \triangleright Since $|\mathcal{P}_i| \leq |\mathcal{P}^{\text{Set}}| = |V|$, the algorithm from Lemma 9 runs in the desired time bound, i.e., $2^{\text{polylog}|V|}$.

Compute and return $\Lambda := CL_{Object}((\Lambda_1, \dots, \Lambda_p))$ using Corollary 11.

 \triangleright Since $(\Lambda_1, ..., \Lambda_p)$ is a tuple consisting of atoms, no set is involved in this object. Therefore, the algorithm from Corollary 11 also runs in the desired time bound, i.e., $2^{\text{polylog}|V|}$.

(CL1.) We consider the Case $|R| \geq 2$. Assume we have R^{φ} instead of R as an input. We obtain the partition \mathcal{P}^{φ} instead of \mathcal{P} . By induction, we compute $\varphi^{-1}\Delta_v\rho_v$ instead of $\Delta_v\rho_v$ and obtain $(\mathcal{P}^{\text{Set}})^{\varphi}$ instead of \mathcal{P}^{Set} . By (CL1) of CL_{Set}, we obtain $\varphi^{-1}\Lambda_i$ instead of Λ_i . By (CL1) of CL_{Object}, we obtain $\varphi^{-1}\Lambda$ instead of Λ , which was what we wanted to show.

(CL2.) We consider the Case $|R| \geq 2$. We return $\Lambda = \mathrm{CL}_{\mathrm{Object}}((\Lambda_1, \ldots, \Lambda_p))$. By object replacement (Lemma 3), the labeling coset Λ defines a canonical labeling for $(\mathcal{P}_1, \ldots, \mathcal{P}_p)$ as well. The ordered partition $\mathbb{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_p)$ of $\mathcal{P}^{\mathrm{Set}}$ is defined in an isomorphism-invariant way and therefore Λ defines a canonical labeling for $\mathcal{P}^{\mathrm{Set}}$. Since \mathbb{P} orders $\mathcal{P}^{\mathrm{Set}}$ according to the isomorphism type of the subrelations the object replacement lemma (Lemma 3) implies that Λ defines a canonical labeling for \mathcal{P} as well. The (unordered) partition $\mathcal{P} = \{P_v \mid v \in V\}$ of R is defined in an isomorphism-invariant way and therefore Λ defines a canonical labeling for R, which was what we wanted to show.

(Running time.) We claim that the number of recursive calls N(R) is at most $T := |R|^2$. By induction, it can be seen that

$$N(R) = 1 + \sum_{v \in V} N(P_v) \overset{\text{induction}}{\leq} 1 + \sum_{v \in V} |P_v|^2 \leq T.$$

We consider the running time of one single recursive call. The algorithm from Corollary 11 runs in time $2^{\text{polylog}|V|}$. Therefore, the total running time is bounded by $2^{\text{polylog}|V|}|R|^{\mathcal{O}(1)}$.

5 Canonization of Hypergraphs

In this section, we consider hypergraphs and later so-called coset-labeled hypergraphs.

Problem 14. Compute a function CL_{Hyper} with the following properties:

Input $H \in \text{Objects}(V)$ where $H = \{S_1, \dots, S_t\}, S_i \subseteq V$ for all $i \in [t]$ and V is an unordered set.

Output A labeling coset $CL_{Hyper}(H) = \Lambda \leq Label(V)$ such that:

- (CL1) $\operatorname{CL}_{\operatorname{Hyper}}(H) = \varphi \operatorname{CL}_{\operatorname{Hyper}}(H^{\varphi}) \text{ for all } \varphi \in \operatorname{Iso}(V; V').$
- (CL2) $\operatorname{CL}_{\operatorname{Hyper}}(H) = \{ \sigma \in \operatorname{Sym}(V) \mid S \in H \iff S^{\sigma} \in H \} \pi \text{ for some (and thus for all)}$ $\pi \in \Lambda.$

We want to extend the previous partitioning technique to hypergraphs. However, for hypergraphs a non-trivial isomorphism-invariant partition $H = H_1 \cup ... \cup H_s$ of the edge set does not always exist, e.g., the hypergraph $(V, \{S \subseteq V \mid |S| = 2\})$ does not have a non-trivial partition of the edge set that is preserved under automorphisms. Therefore, we can not apply the partitioning technique to this setting. For this reason, we introduce a generalized technique in order to solve this problem. This generalized technique results in a slightly weaker time bound of $(|V| + |H|)^{\text{polylog}|V|}$ (where the dependency on |H| is not polynomial). Indeed, it is an open problem whether the running time for the hypergraph isomorphism problem can be improved to $2^{\text{polylog}|V|} \cdot |H|^{\mathcal{O}(1)}$ [Bab18].

Covers A cover of a set $\mathcal{X} \in \text{Objects}(V)$ is a set $\mathcal{C} = \{C_1, \dots, C_c\}$ such that $\mathcal{X} = C_1 \cup \dots \cup C_c$ where $\emptyset \neq C_i \subseteq \mathcal{X}$ for all $C_i \in \mathcal{C}$. In contrast to a partition, the sets C_i, C_j are not necessarily disjoint for $i \neq j$. We say that \mathcal{C} is the singleton cover if $|\mathcal{C}| = 1$ and we say that \mathcal{C} is the cover into singletons if $|C_i| = 1$ for all $C_i \in \mathcal{C}$. A cover \mathcal{C} of \mathcal{X} is called sparse if $|C_i| \leq \frac{1}{2}|\mathcal{X}|$ for all $C_i \in \mathcal{C}$.

The Covering Technique Extending the partitioning technique, we suggest a technique to handle covers. In this setting, we assume that we have given some object $\mathcal{X} \in \text{Objects}(V)$ for which we can define a cover $\mathcal{C} = \{C_1, \dots, C_c\}$ in an isomorphism-invariant way. Also here, we assume that $2 \leq |\mathcal{C}| \leq 2^{\text{polylog}|V|}$. The goal is the computation of a canonical labeling of \mathcal{X} using an efficient recursion.

First, we reduce to the setting in which \mathcal{C} is a sparse cover of \mathcal{X} . This can be done as follows. We define $C_i^* \coloneqq C_i$ if $|C_i| \le \frac{1}{2}|\mathcal{X}|$ and we define $C_i^* \coloneqq \mathcal{X} \setminus C_i$ if $|C_i| > \frac{1}{2}|\mathcal{X}|$. By definition, we ensured that $|C_i^*| \le \frac{1}{2}|\mathcal{X}|$ for all $i \in [c]$. Let $\mathcal{X}^* \coloneqq \bigcup_{i \in [c]} C_i^*$. Next, we consider two cases.

If $\mathcal{X}^* \subseteq \mathcal{X}$, then we have found a non-trivial partition $\mathcal{X} = \mathcal{X}^* \cup \mathcal{X}^\circ$ where $\mathcal{X}^\circ := \mathcal{X} \setminus \mathcal{X}^*$. We proceed analogously as in the partitioning technique explained in Section 4.

Otherwise, if $\mathcal{X}^* = \mathcal{X}$, then $\mathcal{C}^* \coloneqq \{C_1^*, \dots, C_c^*\}$ is also a cover of \mathcal{X} . But more importantly, the cover \mathcal{C}^* is indeed sparse. In the case of a sparse cover, we also proceed analogously as in the partition technique explained in Section 4. However, the key difference of the covering technique compared to the partitioning technique lies in the recurrence for the number of recursive calls since the sets $C_i^*, C_j^* \in \mathcal{C}^*$ are not necessarily pairwise disjoint. The recurrence we have is $R(\mathcal{X}) = 1 + \sum_{C_i^* \in \mathcal{C}^*} R(C_i^*)$. By using that $|\mathcal{C}^*| = |\mathcal{C}| \le 2^{\text{polylog}|V|}$ and that $|C_i^*| \le \frac{1}{2}|\mathcal{X}|$, we obtain at most $|\mathcal{X}|^{\text{polylog}|V|}$ recursive calls. This is exactly the reason why the algorithm for relations is faster than the algorithm for hypergraphs.

Theorem 15. A function CL_{Hyper} for Problem 14 can be computed in time $(|V| + |H|)^{\operatorname{polylog}|V|}$.

Proof. An algorithm for $CL_{Hyper}(H)$:

If $|H| \le 1$:

Compute and return $CL_{Object}(H)$ using Corollary 11.

 \triangleright Since H consists of at most one hyperedge, the largest set involved in H is bounded by |V|. Therefore, the algorithm from Corollary 11 runs in time $2^{\text{polylog}|V|}$.

If $|H| \geq 2$:

▷ In this case, it is possible to define a cover of the hypergraph H in an isomorphism-invariant way.

Define a cover $C := \{C_v \mid v \in V\}$ of $H = \bigcup_{v \in V} C_v$ where $C_v := \{S_i \in H \mid v \in S_i\}$.

 \triangleright Since $|H| \ge 2$, this is not the singleton cover. On the other side, the size $|\mathcal{C}| \le |V|$ is obviously bounded by a quasipolynomial in |V|. However, the cover might not be sparse. Next, we want to find a sparse cover.

Define
$$C_v^* := \begin{cases} C_v, & \text{if } |C_v| \le \frac{1}{2}|H| \\ H \setminus C_v, & \text{otherwise if } |C_v| > \frac{1}{2}|H|. \end{cases}$$

Define $H^* = \bigcup_{v \in V} C_v^*$

If $H^* \subseteq H$:

 \triangleright In this case, we found an ordered partition of H and proceed with the partitioning technique.

Define an ordered partition $\mathcal{H} = (H^*, H^\circ)$ of H where $H^\circ := H \setminus H^*$.

Compute $\Lambda_1 := CL_{Hyper}(H^*)$ recursively.

Compute $\Lambda_2 := \mathrm{CL}_{\mathrm{Hyper}}(H^{\circ})$ recursively.

 \triangleright Next, we combine the two labeling cosets by using a canonical intersection-problem. Compute and return $\Lambda := \mathrm{CL}_{\mathrm{Object}}((\Lambda_1, \Lambda_2))$ using Lemma 7 or Corollary 11.

If $H^* = H$:

 \triangleright In this case, we found a sparse cover of H and proceed with the covering technique.

Define a sparse cover $C^* := \{C_v^* \mid v \in V\}$ of $H = \bigcup_{v \in V} C_v^*$.

Compute $\Delta_v \rho_v := \mathrm{CL}_{\mathrm{Hyper}}(C_v^*)$ for each subhypergraph $C_v^* \in \mathcal{C}^*$ recursively.

Define $C^{*Set} := \{ (\Delta_v \rho_v, v) \mid C_v^* \in C^* \}.$

 \triangleright We define an ordering according to the isomorphism type of the subhypergraphs $C_v^* \in \mathcal{C}^*$.

Define an ordered partition $\mathbb{P} := (\mathcal{P}_1, \dots, \mathcal{P}_p)$ of $\mathcal{C}^{*\text{Set}} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_p$ such that: $(C_v^*)^{\rho_v} < (C_w^*)^{\rho_w}$, if and only if $(\Delta_v \rho_v, v) \in \mathcal{P}_i$ and $(\Delta_w \rho_w, w) \in \mathcal{P}_j$ for some $i, j \in [p]$ with i < j.

Compute $\Lambda_i := CL_{Set}(\mathcal{P}_i)$ for each $\mathcal{P}_i, i \in [p]$ using Lemma 9.

 \triangleright Since $|\mathcal{P}_i| \leq |\mathcal{P}^{\text{Set}}| = |V|$, the algorithm from Lemma 9 runs in the desired time bound, i.e., $2^{\text{polylog}|V|}$.

Compute and return $\Lambda := \mathrm{CL}_{\mathrm{Object}}((\Lambda_1, \ldots, \Lambda_p))$ using Corollary 11.

 \triangleright Since $(\Lambda_1, \ldots, \Lambda_p)$ is a tuple consisting of atoms, no set is involved in this object. Therefore, the algorithm from Corollary 11 also runs in the desired time bound, i.e., $2^{\text{polylog}|V|}$.

The proof for conditions (CL1) and (CL2) is similar to the proof of Theorem 13. (Running time.) We claim that the number of recursive calls R(H) is at most $T := |H|^{2\log_2 |V|}$. For the case $H^* \nsubseteq H$, we have that

$$R(H) = 1 + R(H^*) + R(H^\circ)$$

induction
 $\leq 1 + |H^*|^{2\log_2|V|} + |H^\circ|^{2\log_2|V|} \leq T$

For the case $H^* = H$, we have that

$$\begin{split} R(H) &= 1 + \sum_{v \in V} R(C_v^*) \\ &\stackrel{\text{induction}}{\leq 1 + \sum_{v \in V}} |C_v^*|^{2\log_2 |V|} \\ &\leq 1 + |V| \frac{|H|^{2\log_2 |V|}}{|V|^2} \leq T \qquad \qquad \text{(using } |C_v^*| \leq \frac{1}{2} |H|). \end{split}$$

We consider the running time of one single recursive call. The algorithm from Corollary 11 runs in time $2^{\text{polylog}|V|}$. Therefore, the total running time is bounded by $(|V| + |H|)^{\text{polylog}|V|}$.

Giants, Johnsons and Cameron Groups The groups $\operatorname{Alt}(V)$ and $\operatorname{Sym}(V)$ are called *giants*. The groups $\operatorname{Alt}(V)[\binom{V}{s}]$ and $\operatorname{Sym}(V)[\binom{V}{s}]$ (the alternating group and the symmetric group on V acting on the s-element subsets of V) are called *Johnson groups* where $s \leq \frac{1}{2}|V|$. The size |V| is called the *Johnson parameter*. A group $\Delta \leq \operatorname{Sym}(V)$ is called a Cameron group, if $V = \binom{W}{s}^k$ for some set W and some integers $k \geq 1 \leq s \leq \frac{1}{2}|W| > 2$ and $(\operatorname{Alt}(W)[\binom{W}{s}])^k \leq \Delta \leq \operatorname{Sym}(W)[\binom{W}{s}] \wr \operatorname{Sym}(k)$ (primitive wreath product action). Additionally, we require that the induced homomorphism $h: \Delta \to \operatorname{Sym}(k)$ is transitive and that $s \neq \frac{1}{2}|W|$. These additional requirements ensure that Cameron groups are primitive.

Composition-Width For a group $\Delta \leq \operatorname{Sym}(V)$, the composition-width of Δ , denoted as $\operatorname{cw} \Delta$, is the smallest integer k such that all composition factors of Δ are isomorphic to a subgroup of $\operatorname{Sym}(k)$.

Proposition 16. Let $\Delta \leq \operatorname{Sym}(\mathcal{X})$ be a primitive group on a set \mathcal{X} with $\operatorname{cw} \Delta \leq d$. Then at least one of the following is true.

- 1. $|\Delta| \in |\mathcal{X}|^{\mathcal{O}(\log d)}$, or
- 2. $d! < |\mathcal{X}|$, or
- 3. there is a sparse cover $C = \{C_1, \dots, C_c\}$ of $\mathcal{X} = C_1 \cup \dots \cup C_c$ with $2 \leq |C| \leq d^3$ which is Δ -invariant.

Moreover, there is a polynomial-time algorithm that determines one of the options that is satisfied and in case of the third option computes the corresponding cover C of X.

Proof. The well known O'Nan-Scott Theorem classifies primitive groups into the following types: I. Affine Groups, II. Almost Simple Groups, III. Simple Diagonal Action, IV. Product Action, V. Twisted Wreath Product Action. For groups $\Delta \leq \operatorname{Sym}(\mathcal{X})$ of Type I, III or V it holds that $|\Delta| \in |\mathcal{X}|^{\mathcal{O}(\log d)}$ [GNS18].

Assume that Δ is of Type II. Then, $|\Delta| \in |\mathcal{X}|^{\mathcal{O}(\log d)}$ or Δ is permutationally isomorphic to a Johnson group with parameter $|V| \leq d$ [GNS18]. We identify $\mathcal{X} = {V \choose s}$. We define a cover $\mathcal{C} := \{C_v \mid v \in V\}$ of $\mathcal{X} = {V \choose s}$ where $C_v := \{X \in {V \choose s} \mid v \in X \subseteq V\}$. Observe that $|\mathcal{C}| = |V| \leq d$. Moreover, this cover is sparse and Δ -invariant.

Assume that $\Delta \leq \operatorname{Sym}(\mathcal{X})$ is of Type IV. Then, $|\Delta| \in |\mathcal{X}|^{\mathcal{O}(\log d)}$ or Δ is a subgroup of a Cameron group $P \wr \Psi$ where P is a Johnson group with parameter $|V| \leq d$ and $\Psi \leq \operatorname{Sym}(k)$ is transitive with $\operatorname{cw} \Psi \leq d$. We identify $\mathcal{X} = \binom{V}{s}^k$. We define a cover $\mathcal{C} \coloneqq \{C_{v,i} \mid v \in V, i \in [k]\}$ of $\mathcal{X} = \binom{V}{s}^k$ where $C_{v,i} \coloneqq \{(X_1, \ldots, X_k) \in \binom{V}{s}^k \mid v \in X_i \subseteq V\}$. Again, \mathcal{C} is sparse and Δ -invariant. Observe that $|\mathcal{C}| \leq |V| \cdot k$. Since $|\mathcal{X}| = \binom{|V|}{s}^k$, it follows that $k \leq \log_2 |\mathcal{X}|$. Furthermore, we can assume that $d! \geq |\mathcal{X}|$ because otherwise Option 2 of the Lemma holds. Therefore, $\log_2 |\mathcal{X}| \leq d \log_2(d)$. This leads to $|\mathcal{C}| \leq d \cdot k \leq d^3$.

Canonical Generating Sets A canonical generating set can be seen as a unique encoding of a group $\Delta^{\operatorname{Can}} \leq \operatorname{Sym}(V^{\operatorname{Can}})$ over a linearly ordered set $V^{\operatorname{Can}} = \{1, \dots, |V|\}$.

Lemma 17 ([AGvM⁺18], Lemma 6.2, [GNSW18], Lemma 21 arXiv version). There is a polynomial-time algorithm that, given a group $\Delta^{\operatorname{Can}} \leq \operatorname{Sym}(\{1,\ldots,|V|\})$ via a generating set, computes a generating set for $\Delta^{\operatorname{Can}}$. The output only depends on $\Delta^{\operatorname{Can}}$ (and not on the given generating set).

The applications of canonical generating sets to our framework are discussed in [SW19]. Assume that we want to use an algorithm A as a black box in our framework which gets as input an encoding of a permutation group $\Delta \leq \operatorname{Sym}(V)$ and produces some output $A(\Delta) \in \operatorname{Objects}(V)$. For example, the algorithm from Proposition 16 gets as input a group $\Delta \leq \operatorname{Sym}(V)$ and might produce a cover \mathcal{C} of V. Another example could be an algorithm that gets as input a group $\Delta \leq \operatorname{Sym}(V)$ and produces a minimal block system \mathcal{B} for Δ . When designing a canonization algorithm, it is important that the subroutines that are used behave in an isomorphism-invariant way. That means that for all bijections $\varphi: V \to V'$ the algorithm satisfies $A(\Delta^{\varphi}) = A(\Delta)^{\varphi}$. We can achieve this as follows. We ensure that black box algorithms are applied to groups $\Delta^{\operatorname{Can}} \leq \operatorname{Sym}(V^{\operatorname{Can}})$ over the linearly ordered set $V^{\operatorname{Can}} = \{1, \ldots, |V|\}$ only. The benefit is that isomorphisms $\varphi: V \to V'$ act trivially on ordered groups, i.e., $(\Delta^{\operatorname{Can}})^{\varphi} = \Delta^{\operatorname{Can}}$. For this reason, it remains to ensure that $A(\Delta^{\operatorname{Can}})$ only depends on $\Delta^{\operatorname{Can}}$ (and not on the representation of

 Δ^{Can}). Here, we use canonical generating sets to represent a group uniquely. We will use this trick in the proof of Lemma 19.

After we consider canonization problem for explicitly given structures such as k-ary relations and hypergraphs, we will now turn back to sets J consisting of implicitly given labeling cosets.

Problem 18. Compute a function CL_{SetSet} with the following properties:

Input $(J, L, \alpha, \Delta \rho) \in \text{Objects}(V)$ where $J = \{\Delta_1 \rho_1, \dots, \Delta_t \rho_t\}, L = \{\Lambda_1, \dots, \Lambda_t\}, \Delta_i \rho_i, \Lambda_i \leq \text{Label}(V), \alpha : J \to L \text{ is a function with } \alpha(\Delta_i \rho_i) = \Lambda_i, \Delta \rho \leq \text{Label}(V) \text{ and } V \text{ is an unordered set. We require that } \Delta \leq \text{Aut}(J).$

Output A labeling coset $CL_{SetSet}(J, L, \alpha, \Delta \rho) = \Lambda \leq Label(V)$ such that:

- (CL1) $\operatorname{CL}_{\operatorname{SetSet}}(J, L, \alpha, \Delta \rho) = \varphi \operatorname{CL}_{\operatorname{SetSet}}(J^{\varphi}, L^{\varphi}, \alpha^{\varphi}, \varphi^{-1} \Delta \rho) \text{ for all } \varphi \in \operatorname{Iso}(V; V').$
- (CL2) $CL_{SetSet}(J, L, \alpha, \Delta \rho) = Aut((J, L, \alpha, \Delta \rho))\pi$ for some (and thus for all) $\pi \in \Lambda$.

In fact, a function $\alpha: J \to L$ can be seen as a set consisting of pairs $(\Delta_i \rho_i, \alpha(\Delta_i \rho_i))$ is an object in our framework. In this problem, we assume that for each $\Delta_i \rho_i \in J$ a second labeling coset $\Lambda_i \in L$ is given. Moreover, we assume that the group $\Delta \leq \operatorname{Sym}(V)$ already permutes the set of labeling cosets $J = \{\Delta_1 \rho_1, \ldots, \Delta_t \rho_t\}$, i.e., $\Delta \leq \operatorname{Aut}(J)$. The automorphisms of the instance $(J, L, \alpha, \Delta \rho)$ are all permutations $\delta \in \Delta$ such that if $(\Delta_i \rho_i)^{\delta} = \Delta_j \rho_j$, then δ also maps the corresponding labeling coset Λ_i to the corresponding labeling coset Λ_j . Formally, this means $\operatorname{Aut}((J, L, \alpha, \Delta \rho)) = \{\delta \in \Delta \mid \forall i, j \in [t] : (\Delta_i \rho_i)^{\delta} = \Delta_j \rho_j \Longrightarrow \Lambda_i^{\delta} = \Lambda_j\}$

Lemma 19. A function CL_{SetSet} solving Problem 18 can be computed in time $(|V|+|J|)^{polylog|V|}$.

Proof. An algorithm for $CL_{SetSet}(J, L, \alpha, \Delta \rho)$:

If $|J| \le 1$:

Compute and return $\mathrm{CL}_{\mathrm{Object}}((J, L, \alpha, \Delta \rho))$ using Corollary 11.

 \triangleright Since $\Delta \leq \operatorname{Aut}(J)$, the group Δ induces a permutation group $\Delta[J] \leq \operatorname{Sym}(J)$.

If $\Delta[J]$ is intransitive:

▶ We proceed with the partitioning technique.

Define an ordered partition $\mathcal{J} := (J_1, J_2)$ of $J = J_1 \cup J_2$ where J_1 is the $\Delta[J]$ -orbit such that J_1^{ρ} is minimal w.r.t. to the ordering " \prec " from Lemma 1.

Define an ordered partition $\mathcal{L} := (L_1, L_2)$ of $L = L_1 \cup L_2$ where $L_i := J_i^{\alpha}$ for both i = 1, 2.

Compute $\Lambda_1 := \mathrm{CL}_{\mathrm{SetSet}}(J_1, L_1, \alpha|_{J_1}, \Delta \rho)$ recursively.

Compute $\Lambda_2 := \mathrm{CL}_{\mathrm{SetSet}}(J_2, L_2, \alpha|_{J_2}, \Delta \rho)$ recursively.

Compute and return $\Lambda := CL_{Object}((\Lambda_1, \Lambda_2))$ using Lemma 7 or Corollary 11.

If $\Delta[J]$ is transitive:

We want to find a cover by using Proposition 16. However, the lemma requires a group that is primitive. For this reason, we will define a minimal block system on which Δ acts as a primitive permutation group. Moreover, we do not want that the cover found by Proposition 16 depends on the representation of Δ . For this reason, we use the trick of canonical generating sets and apply the lemma to a group on a linearly ordered set.

Define $V^{\text{Can}} := \{1, ..., |V|\}.$

Define $\Delta^{\operatorname{Can}} := (\Delta \rho)^{\rho} = \rho^{-1} \Delta \rho \leq \operatorname{Sym}(V^{\operatorname{Can}}).$

Define $J^{\operatorname{Can}} := J^{\rho} \in \operatorname{Objects}(V^{\operatorname{Can}})$.

 $ightharpoonup Both \ \Delta^{\operatorname{Can}} \ and \ J^{\operatorname{Can}} \ do \ not \ depend \ on \ the \ choice \ of \ the \ representative \
ho \ of \ \Delta
ho.$

Compute a minimal block system $\mathcal{B}^{\operatorname{Can}} \coloneqq \{B_1^{\operatorname{Can}}, \dots, B_b^{\operatorname{Can}}\}$ for $\Delta^{\operatorname{Can}}[J^{\operatorname{Can}}]$ acting on $J^{\operatorname{Can}} = B_1^{\operatorname{Can}} \cup \dots \cup B_b^{\operatorname{Can}}$.

Apply the algorithm from Proposition 16 to the primitive group $\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}] \leq \operatorname{Sym}(\mathcal{B}^{\operatorname{Can}})$ of composition-width at most d := |V|.

 \triangleright By using a canonical generating set from Lemma 17 for $\Delta^{\operatorname{Can}}$, we ensure that the output of that algorithm only depends on $\Delta^{\operatorname{Can}}$ (and not on the representation of $\Delta^{\operatorname{Can}}$). Depending on the cases 1-3 of Proposition 16, we do the following.

 $|f|\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]| < |\mathcal{B}^{\operatorname{Can}}|^{\mathcal{O}(\log_2|V|)}$.

 \triangleright In this case, the group $\Delta \leq \operatorname{Sym}(V)$ acting on the block system is small enough to iterate over all permutations of the blocks.

Define $\Psi^{\operatorname{Can}} := \operatorname{Stab}_{\Delta^{\operatorname{Can}}}(B_1^{\operatorname{Can}}, \dots, B_b^{\operatorname{Can}}) \leq \operatorname{Sym}(V^{\operatorname{Can}}).$ Decompose $\Delta^{\operatorname{Can}}$ into left cosets of $\Psi^{\operatorname{Can}}$ and write $\Delta^{\operatorname{Can}} = \bigcup_{\ell \in [s]} \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}}.$

 \triangleright This composition can be computed in time polynomial |V| and in the index s= $|\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]| \leq |\mathcal{B}^{\operatorname{Can}}|^{\mathcal{O}(\log_2|V|)}.$

Compute $\Theta_{\ell} \tau_{\ell} \coloneqq \mathrm{CL}_{\mathrm{SetSet}}(J, L, \alpha, \rho \delta_{\ell}^{\mathrm{Can}} \Psi^{\mathrm{Can}})$ for each $\ell \in [s]$ recursively.

 \triangleright The multiplicative cost of the recursion corresponds to the index s of $\Psi^{\operatorname{Can}}$ in $\Delta^{\operatorname{Can}}$, which is bounded by $s \leq |\mathcal{B}^{\operatorname{Can}}|^{\mathcal{O}(\log_2 |V|)}$.

Define $\widehat{J} := \{ \Theta_{\ell} \tau_{\ell} \mid \ell \in [s] \}.$

 \triangleright We collect the canonical labelings $\Theta_{\ell}\tau_{\ell}$ leading to minimal canonical forms of the

Define $\widehat{J}_{\min} := \arg\min_{\Theta_{\ell, \tau_{\ell} \in \widehat{I}}} (J, L, \alpha, \Delta \rho)^{\tau_{\ell}} \subseteq \widehat{J}$ where the minimum is taken w.r.t. the ordering "<" from Lemma 1.

Return $\Lambda := \langle J_{\min} \rangle$.

 \triangleright This is the smallest coset containing all labeling cosets in J_{\min} as defined in the preliminaries. The correctness proof for (CL2) is given below the algorithm.

If $|V|! < |\mathcal{B}^{\operatorname{Can}}|$:

 \triangleright This case can actually not occur. Since $\Delta^{\operatorname{Can}} \leq \operatorname{Sym}(V^{\operatorname{Can}})$, it follows that $|\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]| \leq |\Delta^{\operatorname{Can}}| \leq |V|!$. Since $\Delta^{\operatorname{Can}}$ is transitive on $\mathcal{B}^{\operatorname{Can}}$, it follows that $|\mathcal{B}^{\operatorname{Can}}| \leq |\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]|$. Therefore, $|\mathcal{B}^{\operatorname{Can}}| \leq |V|!$.

If there is a sparse cover $\mathcal{C}^{\operatorname{Can}}_{\mathcal{B}}$ of $\mathcal{B}^{\operatorname{Can}}$ with $2 \leq |\mathcal{C}^{\operatorname{Can}}_{\mathcal{B}}| \leq |V|^3$ which is $\Delta^{\operatorname{Can}}$ -invariant:

ightharpoonup We proceed with the covering technique. Observe that the cover we found so far is a cover for $\mathcal{B}^{\operatorname{Can}}$ (rather than a cover for J^{Can}). However, we can easily define a cover for J^{Can} as well by taking unions of blocks.

Define a sparse cover $C_i^{\text{Can}} := \{C_1^{\text{Can}}, \dots, C_c^{\text{Can}}\}\ \text{of}\ J^{\text{Can}} = C_1^{\text{Can}} \cup \dots \cup C_c^{\text{Can}}\ \text{where}$ $C_i^{\text{Can}} := \bigcup C_{\mathcal{B},i}^{\text{Can}} \subseteq J^{\text{Can}}\ \text{for each}\ C_{\mathcal{B},i}^{\text{Can}} \in \mathcal{C}_{\mathcal{B}}^{\text{Can}}.$

 \triangleright In the next step, we define the cover corresponding to J.

Define a sparse cover $\mathcal{C} := \{C_1, \dots, C_c\}$ of $J = C_1 \cup \dots \cup C_c$ such that $C_i^{\rho} = C_i^{\operatorname{Can}}$ for each $C_i^{\operatorname{Can}} \in \mathcal{C}^{\operatorname{Can}}$.

 \triangleright Observe that \mathcal{C} does not depend on the choice of the representative ρ of $\Delta\rho$ and is defined in an isomorphism-invariant way.

 \triangleright Next, we will recurse on the cover C.

For each $C_i \in \mathcal{C}$ do:

Define $C_{i^*}^{\operatorname{Can}} \in \mathcal{C}^{\operatorname{Can}}$ be the minimal (w.r.t. "<") image of C_i under $\Delta \rho$.

Define $\Delta_{C_i}^{\iota} \rho_{C_i} := \{ \lambda \in \Delta \rho \mid C_i^{\lambda} = C_{i^*}^{\operatorname{Can}} \}.$

 \triangleright The labeling coset $\Delta_{C_i}\rho_{C_i}$ is essentially a canonical labeling for $(C_i,\Delta\rho)$. Moreover, $\Delta_{C_i} \rho_{C_i} \leq \Delta \rho$ can be computed in polynomial time since the index $(\Delta : \Delta_{C_i})$ is bounded by $|\mathcal{C}| \leq |V|^3$.

Compute $\Theta_i \tau_i := \mathrm{CL}_{\mathrm{SetSet}}(C_i, C_i^{\alpha}, \alpha|_{C_i}, \Delta_{C_i}, \rho_{C_i})$ recursively.

Define $C^{\text{Set}} := \{ \Theta_i \tau_i \mid C_i \in C \}.$

 \triangleright We define an ordering according to the isomorphism type of $C_i \in \mathcal{C}$.

Define an ordered cover $\mathbb{C} := (\mathcal{C}_1, \dots, \mathcal{C}_{c'})$ of $\mathcal{C}^{\text{Set}} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_{c'}$ such that:

 $(C_k, C_k^{\alpha}, \alpha|_{C_k}, \Delta_{C_k} \rho_{C_k})^{\tau_k} \prec (C_{k'}, C_{k'}^{\alpha}, \alpha|_{C_{k'}}, \Delta_{C_{k'}} \rho_{C_{k'}})^{\tau_{k'}}, \text{ if and only if } \Theta_k \tau_k \in \mathcal{C}_i \text{ and } \Theta_{k'} \tau_{k'} \in \mathcal{C}_j \text{ for some } i, j \in [c'] \text{ with } i < j.$

Compute $\Lambda_i := \mathrm{CL}_{\mathrm{Set}}(\mathcal{C}_i)$ for each $\mathcal{C}_i, i \in [c']$ using Lemma 9.

 \triangleright Since $|C_i| \le |C^{\text{Set}}| = |V|^3$, the algorithm from Lemma 9 runs in the desired time bound, i.e., $2^{\text{polylog}|V|}$.

Compute and return $\Lambda := CL_{Object}((\Lambda_1, \dots, \Lambda_{c'}))$ using Corollary 11.

 \triangleright Since $(\Lambda_1, \ldots, \Lambda_{c'})$ is a tuple consisting of atoms, no set is involved in this object. Therefore, the algorithm from Corollary 11 runs in the desired time bound, i.e., $2^{\text{polylog}|V|}$.

Condition (CL1) holds as usual.

(CL2.) We have to show that $\Lambda = \operatorname{Aut}((J, L, \alpha, \Delta \rho))\pi$ for $\pi \in \Lambda$. In the intransitive case, we have that $\Lambda_i = \operatorname{Aut}((J_i, L_i, \alpha|_{J_i}, \Delta \rho))\pi_i$ for both i = 1, 2 by induction. Then, Condition (CL2) follows from Condition (CL2) of Lemma 7.

Consider the case in which Option 1 of Proposition 16 holds. The inclusion $\operatorname{Aut}((J, L, \alpha, \Delta \rho))\pi \subseteq \Lambda$ already follows from the isomorphism invariance (Condition (CL1)) of this algorithm, i.e., $\operatorname{CL}(\mathcal{X}) = \sigma \operatorname{CL}(\mathcal{X}) = \sigma \operatorname{CL}(\mathcal{X})$ for $\sigma \in \operatorname{Aut}(\mathcal{X})$ implies that $\operatorname{Aut}(\mathcal{X})\pi \subseteq \operatorname{CL}(\mathcal{X})$ for some $\pi \in \operatorname{CL}(\mathcal{X})$. We have to show the reversed inclusion. By induction, we have that $\Theta_{\ell} = \operatorname{Aut}((J, L, \alpha, \rho \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}})) \subseteq \operatorname{Aut}((J, L, \alpha, \Delta \rho))$. Therefore, we also have the inclusion $\Lambda = (\widehat{J}_{\min}) \subseteq \operatorname{Aut}((J, L, \alpha, \Delta \rho))\pi$.

The cover case (Option 3) is similar to the recursion in the algorithm of Theorem 13.

(Running time.) Let $k := \operatorname{orb}_{J^{\operatorname{Can}}}(\Delta^{\operatorname{Can}})$ be the size of the largest $\Delta^{\operatorname{Can}}[J^{\operatorname{Can}}]$ -orbit. Let $c \in \mathbb{N}$ be the constant from Proposition 16 that is hidden in the \mathcal{O} -notation in the exponent. We claim that the maximum number of recursive calls $R(J, \Delta^{\operatorname{Can}})$ is at most $T := k^{4c \log_2 |V|} |J|^2$. In the intransitive case, this is easy to see by induction:

$$R(J, \Delta^{\operatorname{Can}}) = 1 + R(J_1, \Delta^{\operatorname{Can}}) + R(J_2, \Delta^{\operatorname{Can}})$$
induction
 $\leq 1 + k^{4c \log_2 |V|} (|J_1|^2 + |J_2|^2) \leq T.$

We consider the transitive case in which Option 1 of Proposition 16 holds. Since $\Delta[J]$ is transitive, it holds $k = |J^{\operatorname{Can}}|$. The recursive calls are done for the subgroup $\Psi^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$ of index $s \leq |\mathcal{B}^{\operatorname{Can}}|^{c\log_2|V|}$. Moreover, we reduce orbit size for the recursive calls and have $\operatorname{orb}_{J^{\operatorname{Can}}}(\Psi^{\operatorname{Can}}) \leq \frac{|J^{\operatorname{Can}}|}{|\mathcal{B}^{\operatorname{Can}}|}$. This leads to the recurrence

$$\begin{split} R(J,\Delta^{\operatorname{Can}}) &= 1 + s \cdot R(J,\Psi^{\operatorname{Can}}) \\ &\overset{\operatorname{induction}}{\leq} 1 + |\mathcal{B}^{\operatorname{Can}}|^{c \log_2 |V|} \cdot \left(\frac{|J^{\operatorname{Can}}|}{|\mathcal{B}^{\operatorname{Can}}|}\right)^{4c \log_2 |V|} |J|^2 \leq T. \end{split}$$

In the cover case, we obtain

$$R(J, \Delta^{\operatorname{Can}}) = 1 + \sum_{C_i \in \mathcal{C}} R(C_i, \Delta_{C_i}^{\operatorname{Can}})$$

$$\stackrel{\operatorname{induction}}{\leq 1 + \sum_{C_i \in \mathcal{C}}} |C_i|^{4c \log_2 |V|} |J|^2$$

$$\leq 1 + |V|^3 \frac{|J^{\operatorname{Can}}|^{4c \log_2 |V|}}{|V|^4} |J|^2 \leq T \qquad \text{(using } |\mathcal{C}| \leq |V|^3 \text{ and } |C_i| \leq \frac{1}{2} |J^{\operatorname{Can}}|).$$

We consider the running time of one single recursive call. The algorithm from Corollary 11 runs in time $2^{\text{polylog}|V|}$. Therefore, the total running time is bounded by $(|V|+|J|)^{\text{polylog}|V|}$.

We consider coset-labeled hypergraphs, which were introduced in [GNSW18]. A coset-labeled hypergraph is essentially a hypergraph for which a labeling coset is given for each hyperedge. This problem generalizes the canonical labeling problem for hypergraphs, but is not that general as Problem 8.

Problem 20. Compute a function $CL_{SetHyper}$ with the following properties:

Input $(H, L, \alpha) \in \text{Objects}(V)$ where $H = \{S_1, \dots, S_t\}$, $L = \{\Lambda_1, \dots, \Lambda_t\}$, $S_i \subseteq V, \Lambda_i \leq \text{Label}(V)$ for all $i \in [t]$, $\alpha : H \to L$ is a function with $\alpha(S_i) = \Lambda_i$ and V is an unordered set.

Output A labeling coset $CL_{SetHyper}(H, L, \alpha) = \Lambda \leq Label(V)$ such that:

- (CL1) $\operatorname{CL}_{\operatorname{SetHyper}}(H, L, \alpha) = \varphi \operatorname{CL}_{\operatorname{SetHyper}}(H^{\varphi}, L^{\varphi}, \alpha^{\varphi}) \text{ for all } \varphi \in \operatorname{Iso}(V; V').$
- (CL2) $\operatorname{CL}_{\operatorname{SetHyper}}(H, L, \alpha) = \{ \sigma \in \operatorname{Sym}(V) \mid \exists \psi \in \operatorname{Sym}(t) \forall i \in [t] : (S_i, \Lambda_i)^{\sigma} = (S_{\psi(i)}, \Lambda_{\psi(i)}) \} \pi \text{ for some (and thus for all) } \pi \in \Lambda.$

Remember that we already have an algorithm that canonizes hypergraphs. Therefore, the previous lemma implies that we can also canonize hypergraphs for which a labeling coset is given for each hyperedge.

Lemma 21. A function $CL_{SetHyper}$ for Problem 20 can be computed in time $(|V| + |H|)^{polylog|V|}$.

Proof. Assume we are given an instance $(H, L, \alpha : H \to L)$. First, we compute a canonical labeling $\Delta \rho := \operatorname{CL}_{\operatorname{Hyper}}(H)$ using Theorem 15. Let $J := \{\Delta_1 \rho_1, \dots, \Delta_t \rho_t\}$ where $\Delta_i \rho_i$ is a canonical labeling for S_i for each $i \in [t]$. The set J is polynomial-time computable since each Δ_i is a direct product of two symmetric groups $\operatorname{Sym}(S_i)$ and $\operatorname{Sym}(V \setminus S_i)$. We define $\alpha_J : J \to L$ by setting $\alpha_J(\Delta_i \rho_i) := \alpha(S_i) = \Lambda_i$. Observe that $\Delta = \operatorname{Aut}(H) = \operatorname{Aut}(J)$. We compute and return the canonical labeling $\Lambda := \operatorname{CL}_{\operatorname{SetSet}}(J, L, \alpha_J, \Delta \rho)$ using Lemma 19.

6 Canonization of Sets and Objects

We recall the central problem that we want to solve.

Problem 8. Compute a function CL_{Set} with the following properties:

Input $J \in \text{Objects}(V)$ where $J = \{\Delta_1 \rho_1, \dots, \Delta_t \rho_t\}, \ \Delta_i \rho_i \leq \text{Label}(V)$ for all $i \in [t]$ and V is an unordered set.

Output A labeling coset $CL_{Set}(J) = \Lambda \leq Label(V)$ such that:

- (CL1) $\operatorname{CL}_{\operatorname{Set}}(J) = \varphi \operatorname{CL}_{\operatorname{Set}}(J^{\varphi}) \text{ for all } \varphi \in \operatorname{Iso}(V; V').$
- (CL2) $\operatorname{CL}_{\operatorname{Set}}(J) = \operatorname{Aut}(J)\pi$ for some (and thus for all) $\pi \in \Lambda$.

Giant Representations A homomorphism $h: \Delta \to \operatorname{Sym}(W)$ is called a *giant representation* if the image of Δ under h is a giant, i.e., $\operatorname{Alt}(W) \le h(\Delta) \le \operatorname{Sym}(W)$

Theorem 22. A function CL_{Set} solving Problem 8 can be computed in time $(|V| + |J|)^{\text{polylog}|V|}$.

Proof Outline For the purpose of recursion, our main algorithm CL_{Set} needs some additional input parameters. The input of the main algorithm is a tuple $(J, A, \Delta^{\text{Can}}, g^{\text{Can}})$ consisting of the following input parameters.

• *J* is a set consisting of labeling cosets,

- $A \subseteq V$ is a subset which is Δ_i -invariant for all $\Delta_i \rho_i \in J$,
- $\Delta^{\operatorname{Can}} \leq \operatorname{Sym}(V^{\operatorname{Can}})$ is a group over the linearly ordered set $V^{\operatorname{Can}} = \{1, \dots, |V|\}$, and
- $g^{\text{Can}}:\Delta^{\text{Can}}\to \text{Sym}(W^{\text{Can}})$ is a giant representation where $W^{\text{Can}}=\{1,\dots,|W^{\text{Can}}|\}$ is a linearly ordered set.

We will define the additional parameters besides J in an isomorphism-invariant way. The additional parameters are used for recursion and can provide information, however, canonical labelings for an instance $(J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ correspond to canonical labelings for J.

Initially, we set A := V and we let $g^{\text{Can}} := \bot$ be undefined. Furthermore, we require three properties that hold for our input instance:

- (A) $(\Delta_i \rho_i)|_{V \setminus A} = (\Delta_i \rho_i)|_{V \setminus A}$ for all $\Delta_i \rho_i, \Delta_i \rho_i \in J$, and
- (B) for all $\Delta_i \rho_i \in J$ it holds that $(\Delta_i \rho_i)^{\rho_i} = \Delta^{\operatorname{Can}}$ and there is a subset $A^{\operatorname{Can}} \subseteq \{1, \dots, |V|\}$ such that for all $\Delta_i \rho_i \in J$ it holds that $A^{\rho_i} = A^{\operatorname{Can}}$.
- (g) if $g^{\operatorname{Can}} \neq \bot$ (i.e., g^{Can} is defined), then $g^{\operatorname{Can}} : \Delta^{\operatorname{Can}} \to \operatorname{Sym}(W^{\operatorname{Can}})$ is a giant representation where $|W^{\operatorname{Can}}| > 2 + \log_2 |V|$ and $|W^{\operatorname{Can}}|$ is greater than some absolute constant and $\Delta^{\operatorname{Can}}$ is transitive on A^{Can} and $\Delta^{\operatorname{Can}}_{(A^{\operatorname{Can}})} \le \ker(g^{\operatorname{Can}})$ (the pointwise stabilizer of A^{Can} in $\Delta^{\operatorname{Can}}$).

With the initial choice of A := V Property (A) holds. Initially, $q^{\operatorname{Can}} := \bot$ is undefined and therefore Property (g) also holds. Furthermore, we can assume that Property (B) holds, otherwise we can define an *ordered* partition of J and recurse on that, i.e.,

If Property (B) is not satisfied:

Define $A_i^{\operatorname{Can}} := A^{\rho_i}$ for some $\Delta_i \rho_i \in J$.

 \triangleright We will define an ordered partition of J according to the ordering "<" from Lemma 1

that is defined on the elements $(A_i^{\operatorname{Can}}, \Delta_i^{\operatorname{Can}})$. Define an ordered partition $\mathcal{J} := (J_1, \dots, J_s)$ of $J = J_1 \cup \dots \cup J_s$ such that: $(A_i^{\operatorname{Can}}, \Delta_i^{\operatorname{Can}}) < (A_j^{\operatorname{Can}}, \Delta_j^{\operatorname{Can}})$, if and only if $\Delta_i \rho_i \in J_p$ and $\Delta_j \rho_j \in J_q$ for some $p, q \in [s]$

Recursively compute $\Lambda_i := \mathrm{CL}_{\mathrm{Set}}(J_i, A, \Delta^{\mathrm{Can}}, g^{\mathrm{Can}})$ for each $i \in [s]$.

Return $\Lambda := CL_{Object}((\Lambda_1, \dots, \Lambda_s))$ using Corollary 11.

 \triangleright Since there is no set involved in the tuple $(\Lambda_1, \ldots, \Lambda_s)$, the algorithm from Corollary 11 runs in time $2^{\text{polylog}|V|}(|V|+|J|)^{\mathcal{O}(1)}$.

Property (B) also implies that A can be defined out of J in an isomorphism-invariant way. In particular, Aut(J, A) = Aut(J).

The Measurement of Progress By $\operatorname{orb}_{A^{\operatorname{Can}}}(\Delta^{\operatorname{Can}})$, we denote the size of the largest $\Delta^{\operatorname{Can}}$ orbit on A^{Can} . Let $\delta(g^{\operatorname{Can}}) = 1$ if g^{Can} is defined and let $\delta(g^{\operatorname{Can}}) = 0$ if $g^{\operatorname{Can}} = \bot$ is undefined. We will show that the number of recursive calls $R(J, A, \Delta^{\operatorname{Can}}, q^{\operatorname{Can}})$ of our main algorithm is at most

$$T := 2^{\log_2(|V|+2)^3(2 \cdot \log_2(|V|+4) \cdot \log_2|J| + 2 \cdot \log_2(\operatorname{orb}_{A^{\operatorname{Can}}}(\Delta^{\operatorname{Can}}))} \cdot |J|^2 \cdot |A| \cdot |V|^{2-2\delta(g^{\operatorname{Can}})}. \tag{T}$$

The function looks quite complicated, but there are only a few properties that are of importance. We list these properties. First, observe that $T \leq (|V| + |J|)^{\text{polylog}|V|}$. Moreover, if we can show that the number of recursive calls R of our main algorithm satisfies the recurrences listed below, then it holds that $R \leq T$. We will allow the following types of recursions for the main algorithm which we refer to as *progress*.

• We split J while preserving $\Delta^{\operatorname{Can}}$ and $g^{\operatorname{Can}},$ i.e.,

$$R(J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}}) = 1 + \sum_{i \in [s]} R(J_i, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}}),$$
 (Linear in J)

where $J = J_1 \cup \ldots \cup J_s$.

• We reduce the size of A while preserving $\Delta^{\operatorname{Can}}$ and g^{Can} = \bot , i.e.,

$$R(J, A, \Delta^{\operatorname{Can}}, \bot) = 1 + R(J, A', \Delta^{\operatorname{Can}}, \bot),$$
 (Linear in A)

where |A'| < |A|.

• At a multiplicative cost of $2^{\log_2(p) + \log_2(|V|)^4}$, we divide the size |J| by p and at a multiplicative cost of $2^{\log_2(|V|)^4}$, we reduce the size of |J| to p while resetting the other parameters A := V and $g := \bot$, i.e.,

$$R(J, A, \Delta^{\text{Can}}, g^{\text{Can}}) = 1 + 2^{\log_2(p) + \log_2(|V|)^4} \cdot R(J', A, \Delta^{\text{Can}}, g^{\text{Can}}) + 2^{\log_2(|V|)^4} \cdot R(J'', V, \Delta^{\text{Can}}, \bot),$$
(In J)

where $|J'| \leq \frac{1}{p}|J|$ and $|J''| \leq p$ for some $p \in \mathbb{N}$ with 1 .

• At a multiplicative cost of $2^{\log_2(|V|)^3}$, we halve the size of the largest Δ^{Can} -orbit while resetting $g^{\text{Can}} := \bot$, i.e.,

$$R(J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}}) = 1 + 2^{\log_2(|V|)^3} \cdot R(\widehat{J}, A, \Psi^{\operatorname{Can}}, \bot),$$
 (In $\Delta^{\operatorname{Can}}$)

where $|\widehat{J}| \leq |J|$ and $\operatorname{orb}_{A^{\operatorname{Can}}}(\Psi^{\operatorname{Can}}) \leq \frac{1}{2}\operatorname{orb}_{A^{\operatorname{Can}}}(\Delta^{\operatorname{Can}})$.

• At a multiplicative cost of |V|, we find a giant representation, i.e.,

$$R(J, A, \Delta^{\operatorname{Can}}, \bot) = 1 + |V| \cdot R(\widehat{J}, A, \Psi^{\operatorname{Can}}, g^{\operatorname{Can}}),$$
 (In g^{Can})

where $|\widehat{J}| \leq |J|$ and $\operatorname{orb}_{A^{\operatorname{Can}}}(\Psi^{\operatorname{Can}}) \leq \operatorname{orb}_{A^{\operatorname{Can}}}(\Delta^{\operatorname{Can}})$ and g^{Can} is defined.

The main algorithm calls the subroutines REDUCETOJOHNSON, PRODUCECERTIFICATES and AGGREGATECERTIFICATES described in Lemma 28, Lemma 32 and Lemma 34, respectively. These subroutines in turn use the subroutines RECURSEONPARTITION and REDUCETOSUBGROUP given in Lemma 23 and Lemma 25. We will ensure that progress is achieved whenever the main algorithm is called recursively. See Figure 1 for a flowchart diagram.

Equipartitions and Partition Families An equipartition is a partition \mathcal{P} in which all parts $P_i \in \mathcal{P}$ have the same size $|P_i|$. A partition family of $\mathcal{X} \in \text{Objects}(V)$ is a family $\mathbb{P} := \{\mathcal{P}_k\}_{k \in K}$ where each member $\mathcal{P}_k = \{P_{k,1}, \dots, P_{k,p_k}\}$ is a partition of $\mathcal{X} = P_{k,1} \cup \dots \cup P_{k,p_k}$. A partition family \mathbb{P} is called trivial if all partitions $\mathcal{P}_k \in \mathbb{P}$ are trivial. The notion of partition families generalizes the notion of covers. More precisely, for each cover $\mathcal{C} = \{C_1, \dots, C_c\}$ of \mathcal{X} we can define a partition family $\mathbb{P} := \{\mathcal{P}_i\}_{i \in [c]}$ by setting $\mathcal{P}_i := \{C_i, \mathcal{X} \setminus C_i\}$ for each $i \in [c]$. In this case, we say that \mathbb{P} is induced by \mathcal{C} .

The next lemmas shows that we can exploit partition families $\{\mathcal{P}_k\}_{k\in K}$ of J algorithmically.

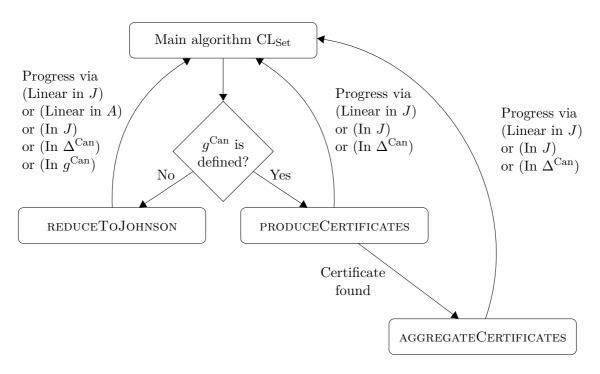


Figure 1: Flowchart of the algorithm for Theorem 22.

Lemma 23. There is an algorithm RECURSEONPARTITION that gets a input a pair $(\mathcal{X}, \mathbb{P})$ where $\mathcal{X} = (J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ is a tuple for which Property (A), (B) and (g) hold and $\mathbb{P} = \{\mathcal{P}_k\}_{k \in K}$ is a non-trivial partition family. In time $2^{\operatorname{polylog}(|V|+|K|)}$, the algorithm reduces the canonical labeling problem of the instance \mathcal{X} to canonical labeling of either

- 1. two instances $(J_1, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ and $(J_2, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ with $|J_1| + |J_2| = |J|$, or
- 2. |K|p-many instances $(J_{k,i}, A, \Delta^{\operatorname{Can}}, g)$ of size $|J_{k,i}| \leq \frac{1}{p}|J|$ and to additionally |K|-many instances $(J_k, V, \Delta_k^{\operatorname{Can}}, \bot)$ of size $|J_k| \leq p$ for some $p \in \mathbb{N}$ with 1 .

In case that |K| is quasipolynomially bounded (or more precisely, bounded by $2^{\log_2(|V|)^4}$) the the lemma facilitates a recursion that leads to progress (In J).

In the following, we sketch the idea how to exploit a partition family.

The Partition-Family Technique Extending the covering technique, we suggest a technique for handling partition families that we use to prove Lemma 23. In this setting, we assume that we are given a set $J \in \text{Objects}(V)$ consisting of labeling cosets for which we can define a non-trivial partition family $\mathbb{P} = \{\mathcal{P}_k\}_{k \in K}$ in an isomorphism-invariant way. We do not require any bound on the size of the partitions \mathcal{P}_k . The goal is the computation of a canonical labeling of J using an efficient recursion.

Let $\mathbb{P}' := \{ \mathcal{P}_k \in \mathbb{P} \mid \mathcal{P}_k \text{ is non-trivial} \}$ be the non-empty set of non-trivial partitions. We can assume that $\mathbb{P}' = \mathbb{P}$, otherwise we continue with $\mathbb{P} := \mathbb{P}'$. We distinguish between two cases.

Case 1: There is a partition $\mathcal{P}_k = \{P_{k,1}, \dots, P_{k,p_k}\} \in \mathbb{P}$ that is an equipartition of J. Again, we assume each $\mathcal{P}_k \in \mathbb{P}$ is an equipartition, otherwise consider the partition family $\mathbb{P} \coloneqq \{\mathcal{P}_k \in \mathbb{P} \mid \mathcal{P}_k \text{ is an equipartition}\}$. Moreover, we can assume that all parts have the same size $|P_{k,i}|$ even across all equipartitions, otherwise we would consider a subset $\mathbb{P} \coloneqq \arg\min_{\mathcal{P}_k \in \mathbb{P}} |\mathcal{P}_k|$. Now, we

use recursion and compute a canonical labeling $\Theta_{k,i}\tau_{k,i}$ for each part $P_{k,i}\subseteq J$. For simplicity, we assume that all parts $P_{k,i}$ and $P_{k,j}$ are isomorphic. Let $\mathcal{P}_k^{\text{Set}} := \{\Theta_{k,i}\tau_{k,i} \mid P_{k,i} \in \mathcal{P}_k\}$. By object replacement (Lemma 3), a canonical labeling for $\mathcal{P}_k^{\text{Set}}$ also defines a canonical labeling for \mathcal{P}_k . To compute a canonical labeling $\Theta_k \tau_k$ for $\mathcal{P}_k^{\text{Set}}$, we use recursion again.

Next, we compute a canonical labeling $\Theta \tau$ for J. We choose the canonical labelings which lead to a minimal canonical form for J. More precisely, let $K := \arg\min_{\Theta_k \tau_k} J^{\tau_k}$ where the minimum is taken w.r.t. the ordering "<" from Lemma 1. We let $\Theta\tau$ be the labeling coset that is generated by all $\Theta_k \tau_k$ for $k \in K$.

We analyze the recurrence of this approach. Let $p \in \mathbb{N}$ be the size $p = |\mathcal{P}_k|$ which is uniform over all partitions $\mathcal{P}_k \in \mathbb{P}$. We have |K|p-many recursive calls for instances $P_{k,i}$ of size $\frac{1}{n}|J|$. After that, we have |K|-many recursive calls for instances $\mathcal{P}_k^{\text{Set}}$ of size p. In case that $|K| \leq 2^{\text{polylog}|V|}$, this recurrence is progress via (In J).

Case 2: Each partition $\mathcal{P}_k \in \mathbb{P}$ is not an equipartition. If a partition \mathcal{P}_k is not an equipartition of J, then \mathcal{P}_k induces a non-trivial ordered partition $\widetilde{\mathcal{P}}_k := (P_k^1, \dots, P_k^{|J|})$ of J where $P_k^x := \bigcup_{P_{k,i} \in \mathcal{P}_k, |P_{k,i}| = x} P_{k,i}$ for $x \in \{1, \dots, |J|\}$. Moreover, let $P_k^* := P_k^x$ where $x \in \mathbb{N}$ is the smallest number such that $1 \leq |P_k^x| \leq \frac{1}{2}|J|$. We define $J^* := \bigcup_{P_k \in \mathbb{P}} P_k^*$.

In the case in which $J^* \nsubseteq J$, we found a non-trivial ordered partition of $J = J^* \cup J \setminus J^*$ and proceed with the partitioning technique. This will lead to progress via (Linear in J).

In the other case in which $J^* = J$, we found a cover of $J = \bigcup_{\mathcal{P}_k \in \mathbb{P}} P_k^*$ and proceed with the covering technique. In case that $|K| \leq 2^{\text{polylog}|V|}$, this will ensure progress via (In J).

Proof of Lemma 23. An algorithm for RECURSEONPARTITION($J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}}, \mathbb{P}$): $\forall We simplify to the all partitions \mathcal{P}_k \in \mathbb{P} \text{ are non-trivial.}$ Define $\mathbb{P} := \{ \mathcal{P}_k \in \mathbb{P} \mid \mathcal{P}_k \text{ is non-trivial} \}.$

If there is a partition $\mathcal{P}_k \in \mathbb{P}$ that is an equipartition:

 \triangleright We simplify to the case in which all $\mathcal{P}_k \in \mathbb{P}$ are equipartitions.

Define $\mathbb{P} := \{ \mathcal{P}_k \in \mathbb{P} \mid \mathcal{P}_k \text{ is an equipartition} \}.$

 \triangleright We simplify to the case in which $|\mathcal{P}_k|$ are equal for all $\mathcal{P}_k \in \mathbb{P}$.

Define $\mathbb{P} := \arg\min_{\mathcal{P}_k \in \mathbb{P}} |\mathcal{P}_k|$.

 \triangleright Now, there is a number $p \in \mathbb{N}$ such that $p = |\mathcal{P}_k|$ for all partitions $\mathcal{P}_k \in \mathbb{P}$.

For each $\mathcal{P}_k \in \mathbb{P}$ do:

 \triangleright We show how to compute a canonical labeling $\Theta_k \tau_k$ for the pair (J, \mathcal{P}_k) for each partition $\mathcal{P}_k \in \mathbb{P}$. Roughly speaking, the instance (J, \mathcal{P}_k) can be seen as an individualization of J obtained by individualizing one partition $\mathcal{P}_k \in \mathbb{P}$. Recursively compute $\Theta_{k,i}\tau_{k,i} \coloneqq \mathrm{CL}_{\mathrm{Set}}(P_{k,i},A,\Delta^{\mathrm{Can}},g^{\mathrm{Can}})$ for each part $P_{k,i} \in \mathcal{P}_k$.

- \triangleright We have a multiplicative cost of $|\mathbb{P}| \cdot p$ and recursive instances of size $|P_{k,i}| = |J|/p$. Define $\mathcal{J}_k^{\text{Set}} := \{ (\Theta_{k,i} \tau_{k,i}, P_{k,i}^{\tau_{k,i}}) \mid P_{k,i} \in \mathcal{P}_k \}.$
- $riangleright In previous algorithms, we computed a canonical labeling for <math>\mathcal{J}_k^{\mathrm{Set}}$ by using Corollary 11. However, in this case, the size $|\mathcal{J}_k^{\text{Set}}| = p$ might not be bounded by a quasipolynomial. For this reason, we use a recursive approach to compute a canonical labeling for $\mathcal{J}_k^{\mathrm{Set}}$. First, we define an ordering according to the isomorphism type of the parts $P_{k,i} \in \mathcal{P}_k$.

Define an ordered partition $\mathbb{J}_k^{\text{Set}} := (\mathcal{J}_{k,1}^{\text{Set}}, \dots, \mathcal{J}_{k,m_k}^{\text{Set}})$ of $\mathcal{J}_k^{\text{Set}} = \mathcal{J}_{k,1}^{\text{Set}} \cup \dots \cup \mathcal{J}_{k,m_k}^{\text{Set}}$ such

 $P_{k,i}^{\tau_{k,i}} < P_{k,j}^{\tau_{k,j}}, \text{ if and only if } (\Theta_{k,i}\tau_{k,i}, P_{k,i}^{\tau_{k,i}}) \in \mathcal{J}_{k,p}^{\text{Set}} \text{ and } (\Theta_{k,j}\tau_{k,j}, P_{k,j}^{\tau_{k,j}}) \in \mathcal{J}_{k,q}^{\text{Set}} \text{ for some } P_{k,j}^{\tau_{k,j}} = P_{k,j}^{\tau_{k,j}}$ $p < q \in [m_k].$

Define $\pi_1(\mathcal{J}_{k,\ell}^{\operatorname{Set}}) \coloneqq \{\Theta_{k,i}\tau_{k,i} \mid (\Theta_{k,i}\tau_{k,i}, P_{k,i}^{\tau_{k,i}}) \in \mathcal{J}_{k,\ell}^{\operatorname{Set}}\}$ for each $\ell \in [m_k]$.

 \triangleright The ordering ensures that Property (B) holds for all instances $\pi_1(\mathcal{J}_{k,p}^{\mathrm{Set}})$ for some group $\Theta_{k,p}^{\mathrm{Can}}$.

Recursively compute $\Lambda_{k,\ell} := \mathrm{CL}_{\mathrm{Set}}(\pi_1(\mathcal{J}_{k,\ell}^{\mathrm{Set}})), V, \Theta_{k,\ell}^{\mathrm{Can}}, \bot)$ for each $\ell \in [m_k]$.

For our running time T(|J|) given in (T), we have that $\sum_{\ell \in [m_k]} T(|\mathcal{J}_{k,\ell}^{\operatorname{Set}}|) \leq T(\sum_{\ell \in [m_k]} |\mathcal{J}_{k,\ell}^{\operatorname{Set}}|) = T(|\mathcal{J}_k^{\operatorname{Set}}|)$. Therefore, in the worst case, we have $m_k = 1$ and $|\mathcal{J}_{k,1}^{\operatorname{Set}}| = |\mathcal{J}_k^{\operatorname{Set}}| = p$. Therefore, we have a multiplicative cost of $|\mathbb{P}| \cdot m_k = |\mathbb{P}|$ and recursive instances of size p.

Compute $\Theta_k \tau_k := \mathrm{CL}_{\mathrm{Set}}((\Lambda_{k,1}, \dots, \Lambda_{k,m_k}))$ using Corollary 11.

 \triangleright Observe that $\Theta_k \tau_k$ is a canonical labeling for (J, \mathcal{P}_k) .

Define $\mathbb{P}^{\text{Set}} := \{ \Theta_k \tau_k \mid \mathcal{P}_k \in \mathbb{P} \}.$

▷ To obtain a canonical labeling Λ for J for the given $\Theta_k \tau_k$ we will proceed as follows. We compare the canonical forms J^{τ_k} that we obtain for each individualized partition $\mathcal{P}_k \in \mathbb{P}$. Then, we collect the canonical labelings leading to a minimal canonical form w.r.t. "<". Define $\mathbb{P}^{\text{Set}}_{\min} := \arg\min_{\Theta_k \tau_k \in \mathbb{P}^{\text{Set}}} J^{\tau_k} \subseteq \mathbb{P}^{\text{Set}}$ where the minimum is taken w.r.t. the ordering "<" from Lemma 1.

Return $\Lambda := \langle \mathbb{P}_{\min}^{\text{Set}} \rangle$.

 \triangleright This is the smallest coset that contains all labeling cosets in $\mathbb{P}^{\text{Set}}_{\min}$ as defined in the preliminaries. The correctness proof for (CL2) is similar to the (CL2)-proof of Lemma 19.

 \triangleright Now, each partition $\mathcal{P}_k \in \mathbb{P}$ of J is not an equipartition.

For each $\mathcal{P}_k \in \mathbb{P}$ do:

Define an ordered partition $\widetilde{\mathcal{P}}_k := (P_k^1, \dots, P_k^{|J|})$ of J where $P_k^x := \bigcup_{P_{k,i} \in \mathcal{P}_k, |P_{k,i}| = x} P_{k,i}$ for $x \in \{1, \dots, |J|\}$.

Define $P_k^* := P_k^x \subseteq J$ where $x \in \mathbb{N}$ is the smallest number such that $1 \le |P_k^x| \le \frac{1}{2}|J|$.

Define $P^* := \bigcup_{\mathcal{P}_k \in \mathbb{P}} P_k^*$.

If $P^* \nsubseteq J$:

 \triangleright We found an ordered partition of J and proceed with the partitioning technique.

Define an ordered partition $\mathcal{P} = (P^*, P^\circ)$ of $J = P^* \cup P^\circ$ where $P^\circ \coloneqq J \setminus P^*$.

 \triangleright The partition is non-trivial since P^* is non-empty by the definition of each part $P_k^* \subseteq J$.

Recursively compute $\Lambda_1 := CL_{Set}(P^*, A, \Delta^{Can}, g^{Can})$.

Recursively compute $\Lambda_2 := CL_{Set}(P^{\circ}, A, \Delta^{Can}, g^{Can})$.

 \triangleright We have that $|P^*| + |P^\circ| = |J|$ and therefore Option 1 of Lemma 23 is satisfied.

Compute and return $\Lambda := CL_{Object}((\Lambda_1, \Lambda_2))$ using Lemma 7 or Corollary 11.

 \triangleright The algorithm from Lemma 7 and Corollary 11 runs in time $2^{\text{polylog}|V|}$.

If $P^* = J$:

 \triangleright We found a sparse cover of J and proceed with the covering technique.

Define a sparse cover $C := \{C_k \mid \mathcal{P}_k \in \mathbb{P}\}\ \text{of}\ J = \bigcup_{k \in K} C_k \text{ where } C_k := P_k^*.$

For each $C_k \in \mathcal{C}$, compute $\Theta_k \tau_k := \mathrm{CL}_{\mathrm{Set}}(C_k, A, \Delta^{\mathrm{Can}}, g^{\mathrm{Can}})$ recursively.

 \triangleright We have a multiplicative cost of $|\mathbb{P}|$ and recursive instances of size $|C_k| \leq \frac{1}{2}|J|$ and therefore Option 2 of Lemma 23 is satisfied.

Define $\mathbb{P}^{\text{Set}} := \{\Theta_k \tau_k \mid \mathcal{P}_k \in \mathbb{P}\}.$

Define an ordered cover $\mathbb{C} := (\mathcal{C}_1, \dots, \mathcal{C}_c)$ of $\mathbb{P}^{\text{Set}} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_c$ such that:

 $(C_k)^{\tau_k} < (C_{k'})^{\rho_{k'}}$, if and only if $\Theta_k \tau_k \in \mathcal{P}_i$ and $\Theta_{k'} \tau_{k'} \in \mathcal{P}_j$ for some $i, j \in [c]$ with i < j.

 \triangleright In fact, \mathcal{C} might not be a partition since $\Theta_k \tau_k = \Theta_{k'} \tau_{k'}$ for $k \neq k'$ might hold.

Compute $\Lambda_i := CL_{Set}(C_i)$ for each $C_i, i \in [c]$ using Lemma 9.

 \triangleright Since $|\mathcal{C}_i| \le |\mathbb{P}^{\text{Set}}| = |\mathbb{P}|$, the algorithm from Lemma 9 runs in time $2^{\text{polylog}(|V|+|\mathbb{P}|)}$.

Compute and return $\Lambda := CL_{Object}((\Lambda_1, ..., \Lambda_c))$ using Corollary 11.

 \triangleright Since $(\Lambda_1, ..., \Lambda_c)$ is a tuple consisting of atoms, no set is involved in this object. Therefore, the algorithm from Corollary 11 runs in time $2^{\text{polylog}|V|}(|V| + |\mathbb{P}|)^{\mathcal{O}(1)}$.

Relative Minimal Base Size Recall that the pointwise stabilizer of a subset $X \subseteq V$ in a group $\Delta \leq \operatorname{Sym}(V)$ is denoted by $\Delta_{(X)}$. The *minimal base size* of a group $\Delta \leq \operatorname{Sym}(V)$ relative to a subgroup $\Psi \leq \Delta$, denoted by $\operatorname{rb}(\Delta, \Psi)$, is the smallest cardinality |X| among all subsets $X \subseteq V$ such that $\Delta_{(X)} \leq \Psi$.

Example 24. We give some examples.

- 1. The minimal base size of Δ is defined as $b(\Delta) := rb(\Delta, 1)$ where $1 \le Sym(V)$ denotes the trivial group. It can easily be seen that $rb(\Delta, \Psi) \le b(\Delta) \le \log_2 |\Delta|$.
- 2. Let $\Psi := \operatorname{Stab}_{\Delta}(A)$ where $A \subseteq V$. We show that $\operatorname{rb}(\Delta, \Psi) \leq \log_2(\Delta : \Psi)$. We assume that $\Psi < \Delta$, otherwise $\operatorname{rb}(\Delta, \Psi) = 0 = \log_2(\Delta : \Psi)$. Since the Ψ -orbit partition is a refinement of the Δ -orbit, there is a Ψ -orbit U and a Δ -orbit W with $U \not\subseteq W$ and $|U| \leq \frac{1}{2}|W|$. Let $v \in U$. It holds that $(\Delta : \Psi) \cdot (\Psi : \Psi_{(v)}) = (\Delta : \Delta_{(v)}) \cdot (\Delta_{(v)} : \Psi_{(v)})$. Moreover, $(\Psi : \Psi_{(v)}) = |U|$ and $(\Delta : \Delta_{(v)}) = |W|$. Therefore, $(\Delta_{(v)} : \Psi_{(v)}) \leq \frac{1}{2}(\Delta : \Psi)$. By induction on the index, it holds that $\operatorname{rb}(\Delta, \Psi) \leq \operatorname{rb}(\Delta, \Delta_{(v)}) + \operatorname{rb}(\Delta_{(v)}, \Psi_{(v)}) \leq 1 + \log_2(\Delta_{(v)} : \Psi_{(v)}) \leq 1 + \log_2(\frac{1}{2}(\Delta : \Psi)) = \log_2(\Delta : \Psi)$.
- 3. Let $\Psi \coloneqq \operatorname{Stab}_{\Delta}(B_1, \dots, B_b)$ where $\mathcal{B} \coloneqq \{B_1, \dots, B_b\}$ is a partition of $V = B_1 \cup \dots \cup B_b$. We show that $\operatorname{rb}(\Delta, \Psi) \leq 2 \cdot \log_2(\Delta : \Psi)$. Let $\Theta \coloneqq \operatorname{Aut}(\mathcal{B}) \cap \Delta$. Since $\Psi \leq \Theta \leq \Delta$, it follows that $\operatorname{rb}(\Delta, \Psi) \leq \operatorname{rb}(\Delta, \Theta) + \operatorname{rb}(\Theta, \Psi)$. By the definition of Θ , for each $\theta \in \Theta$ and each $B \in \mathcal{B}$ it holds that $B^{\theta} \in \mathcal{B}$ and therefore B^{θ} is equal to B or disjoint from B (if Θ would be transitive on V, then B is a block system for Θ). Therefore, fixing a point $v \in B \in \mathcal{B}$ also fixes the set $B \in \mathcal{B}$, i.e., $\Theta_{(v)}[\mathcal{B}] \leq \Theta[\mathcal{B}]_{(B)}$ for all $v \in B \in \mathcal{B}$. Let $\mathcal{X} \subseteq \mathcal{B}$ and $X \coloneqq \bigcup_{B \in \mathcal{X}} B \subseteq V$ and assume $\Theta[\mathcal{B}]_{(X)} \leq \Psi[\mathcal{B}]$. Then, $\Theta_{(X)}[\mathcal{B}] \leq \Theta[\mathcal{B}]_{(X)} \leq \Psi[\mathcal{B}] = 1[\mathcal{B}]$ and thus $\Theta_{(X)} \leq \Psi$. This gives $\operatorname{rb}(\Theta, \Psi) \leq \operatorname{rb}(\Theta[\mathcal{B}], \Psi[\mathcal{B}])$. Moreover, $\operatorname{rb}(\Theta[\mathcal{B}], \Psi[\mathcal{B}]) = \operatorname{b}(\Theta[\mathcal{B}]) \leq \log_2 |\Theta[\mathcal{B}]| = \log_2(\Theta : \Psi)$. Next, we show $\operatorname{rb}(\Delta, \Theta) \leq 2 \cdot \log_2(\Delta : \Theta)$. Consider the permutation groups $\Theta[V^2]$ and $\Delta[V^2]$ induced on V^2 . Fixing two points v, w in the domain of Δ also fixes the point (v, w) in the domain of $\Delta[V^2]$, i.e., $\Delta_{(\{v, w\})}[V^2] \leq \Delta[V^2]_{((v, w))}$ for all $v, w \in V$. Moreover, $\Delta_{(X)}[V^2] \leq \Theta[V^2]$ implies that $\Delta_{(X)} \leq \Theta$ for all $X \subseteq V$. Therefore, $\operatorname{rb}(\Delta, \Theta) \leq 2 \cdot \operatorname{rb}(\Delta[V^2], \Theta[V^2])$. Let $A \coloneqq \{(v, w) \in V^2 \mid \{v, w\} \subseteq B_i \text{ for some } B_i \in \mathcal{B}\}$. Then, $\Theta[V^2] = \operatorname{Stab}_{\Delta[V^2]}(A)$. Therefore, $2 \cdot \operatorname{rb}(\Delta[V^2], \Theta[V^2]) \leq 2 \cdot \log_2(\Delta[V^2]) = 2 \cdot \log_2(\Delta[V^2])$.
- 4. Let $\Psi := \text{Alt}(V) \leq \Delta := \text{Sym}(V)$. This is an example where the relative base size is large compared to the index of the subgroup. It is easy to see that $\text{rb}(\Delta, \Psi) = |V| 1$.

The next lemma facilitate a subgroup reduction, similar as in Luke's framework. The multiplicative cost of this recursion corresponds to the index of the subgroup.

Lemma 25. There is an algorithm REDUCETOSUBGROUP that gets as input a pair $(\mathcal{X}, \Psi^{\operatorname{Can}})$ where $\mathcal{X} = (J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ is a tuple for which Property (A), (B) and (g) hold and $\Psi^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$ is a subgroup. Let $c_{\operatorname{ind}} \coloneqq (\Delta^{\operatorname{Can}} : \Psi^{\operatorname{Can}})$ and $c_{\operatorname{rb}} \coloneqq \operatorname{rb}(\Delta^{\operatorname{Can}}, \Psi^{\operatorname{Can}})$. In time polynomial in the input and output size, the algorithm either

- 1. finds a non-trivial partition family $\mathbb{P} = \{\mathcal{P}_k\}_{k \in K}$ of J with $|K| \leq c_{\text{ind}} \cdot |V|^{c_{\text{rb}}}$, or
- 2. reduces the canonical labeling problem of \mathcal{X} to the canonical labeling problem of c_{ind} -many instances $(\widehat{J}_i, A, \Psi^{\operatorname{Can}}, \bot)$ with $|\widehat{J}_i| \leq |J|$ for $i \in [c_{\operatorname{ind}}]$.

In contrast to Luks's subgroup reduction, the present reduction splits all labeling cosets in Jsimultaneously. We describe the idea of this algorithm.

Intuition of the Subgroup Recursion We consider the decomposition into left cosets of Δ^{Can} = $\bigcup_{\ell \in [s]} \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}$ and define $\widehat{J} := \{ \rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \mid i \in [t], \ell \in [s] \}$. Surprisingly, we can show that $\operatorname{Aut}(\widehat{J}) = \operatorname{Aut}(J)$. This means that a canonical labeling for \widehat{J} defines a canonical labeling for J as well and vice versa. Therefore, the first idea that comes to mind would be a recursion on the instance $(\widehat{J}, A, \Psi^{\operatorname{Can}}, \bot)$. However, there are two problems when recursing on \widehat{J} . First, the instance \widehat{J} does not necessarily satisfy Property (A). To ensure, Property (A) for the recursive instance, one could reset A := V, but this would not lead to the desired recursion. Second, it holds that $|\widehat{J}| > |J|$ (assumed that $\Psi^{\text{Can}} < \Delta^{\text{Can}}$ is a proper subgroup). Also this blow-up in the instance size would not lead to the desired recursion. The given subroutine is designed to fix exactly these two problems. In particular, we construct a decomposition of $\widehat{J} = \widehat{J_1} \cup \ldots \cup \widehat{J_r}$ such that $r \leq c_{\text{ind}}$ and $|\widehat{J}_i| \leq |J|$ and such that Property (A) holds for each instance $(\widehat{J}_i, A, \Psi^{\text{Can}}, \bot)$.

Proof of Lemma 25. An algorithm for REDUCETOSUBGROUP $(J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}}, \Psi^{\operatorname{Can}})$:

Decompose $\Delta^{\operatorname{Can}} = \bigcup_{\ell \in [s]} \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}$ into left cosets of $\Psi^{\operatorname{Can}}$

Define $\widehat{J} := \{ \rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \mid i \in [t], \ell \in [s] \}.$

 \triangleright We claim that $\operatorname{Aut}(\widehat{J}) = \operatorname{Aut}(J)$. It is not difficult to see that $\operatorname{Aut}(J) \subseteq \operatorname{Aut}(\widehat{J})$ since \widehat{J} is defined in an isomorphism-invariant way. On the other side, let $\sigma \in \operatorname{Aut}(\widehat{J})$. Therefore, for each labeling coset $\rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}$ there is a labeling coset $\rho_{i'} \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}$ such that $(\rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}})^{\sigma} = \rho_{i'} \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}}$ or equivalently $\sigma \in \rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \delta_{\ell'}^{\operatorname{Can}} \Phi_{i'}^{\operatorname{Can}}$. In particular, $\sigma \in \rho_i \Delta_{\ell'}^{\operatorname{Can}} \rho_{i'}^{\operatorname{Can}}$ or equivalently $(\rho_i \Delta_{\ell'}^{\operatorname{Can}})^{\sigma} = \rho_{i'} \Delta_{\ell'}^{\operatorname{Can}}$. Therefore, $\sigma \in \operatorname{Aut}(J)$.

Let $X^{\operatorname{Can}} = (x_1^{\operatorname{Can}}, \dots, x_{c_{\operatorname{rb}}}^{\operatorname{Can}}) \in (V^{\operatorname{Can}})^{c_{\operatorname{rb}}}$ be the minimal (w.r.t. the ordering "<") tuple such that $\Delta_{(x_1^{\operatorname{Can}}, \dots, x_{c_{\operatorname{rb}}}^{\operatorname{Can}})}^{\operatorname{Can}} \leq \Psi^{\operatorname{Can}}$.

We say that $X \in V^{c_{\text{rb}}}$ identifies the subcoset $\rho_i \delta_\ell^{\text{Can}} \Psi^{\text{Can}} \leq \Delta_i \rho_i$ if $X^{\rho_i \delta_\ell^{\text{Can}} \psi^{\text{Can}}} = X^{\text{Can}}$ for some $\psi^{\operatorname{Can}} \in \Psi^{\operatorname{Can}}$.

 $\forall We \ claim \ that \ each \ X \in V^{c_{\mathrm{rb}}} \ identifies \ at \ most \ one \ subcoset \ \rho_{i}\delta_{\ell}^{\mathrm{Can}}\Psi^{\mathrm{Can}} \leq \Delta_{i}\rho_{i} \ of \ each \ \Delta_{i}\rho_{i} \in J. \ Assume \ that \ X \in V^{c_{\mathrm{rb}}} \ identifies \ both \ \rho_{i}\delta_{\ell}^{\mathrm{Can}}\Psi^{\mathrm{Can}}, \rho_{i}\delta_{\ell'}^{\mathrm{Can}}\Psi^{\mathrm{Can}} \leq \Delta_{i}\rho_{i} \ for \ some \ \Delta_{i}\rho_{i} \in J. \ We \ show \ that \ \ell = \ell'. \ There \ are \ \psi^{\mathrm{Can}}, \psi^{\mathrm{Can}'} \in \Psi^{\mathrm{Can}} \ such \ that \ X^{\rho_{i}\delta_{\ell}^{\mathrm{Can}}\psi^{\mathrm{Can}}} = X^{\mathrm{Can}} = X^{\rho_{i}\delta_{\ell'}^{\mathrm{Can}}\psi^{\mathrm{Can}'}}. \ This \ implies \ (\delta_{\ell}^{\mathrm{Can}}\psi^{\mathrm{Can}})^{-1}\delta_{\ell'}^{\mathrm{Can}}\psi^{\mathrm{Can}'} \in \Delta_{(\{x_{1}^{\mathrm{Can}}, \dots, x_{c_{\mathrm{rb}}}^{\mathrm{Can}}\})}^{\mathrm{Can}} \leq \Psi^{\mathrm{Can}} \ and \ therefore$ $(\delta_{\ell}^{\operatorname{Can}})^{-1}\delta_{\ell'}^{\operatorname{Can}}\in\Psi^{\operatorname{Can}}\ \ and\ \ thus\ \delta_{\ell}^{\operatorname{Can}}=\delta_{\ell'}^{\operatorname{Can}}.$

Define an (unordered) partition $\widehat{\mathcal{J}} \coloneqq \{\widehat{J}_1, \dots, \widehat{J}_r\}$ of $\widehat{J} = \widehat{J}_1 \cup \dots \cup \widehat{J}_r$ such that: $\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}}, \rho_{i'} \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_k$ for some $\widehat{J}_k \in \widehat{\mathcal{J}}$, iff $(\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}})|_{V \setminus A} = (\rho_{i'} \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}})|_{V \setminus A}$ and there is a tuple $X \in V^{c_{\text{rb}}}$ that identifies both $\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}}$ and $\rho_{i'} \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}}$.

 $\quad \triangleright \ \textit{As already observed}, \ each \ X \in V^{c_{\text{rb}}} \ \ \textit{identifies at most one subcoset} \ \rho_i \delta_{\ell}^{\text{Can}} \Psi^{\text{Can}} \leq \Delta_i \rho_i \ \ \textit{of} \ \Delta_i \rho_i.$ For this reason $|\widehat{J}_k| \leq |J|$ for each $\widehat{J}_k \in \widehat{\mathcal{J}}$. On the other side, $|\widehat{\mathcal{J}}| \leq c_{\mathrm{ind}} \cdot |V|^{c_{\mathrm{rb}}}$.

Define a cover $\mathcal{C} := \{C_1, \dots, C_r\}$ of $J = C_1 \cup \dots \cup C_r$ such that: $\Delta_i \rho_i \in C_k$ if there are $\ell \in [s]$ such that $\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_k$.

Define $\mathbb{P} := \{\mathcal{P}_k\}_{k \in [r]}$ as partition family induced by \mathcal{C} , i.e., $\mathcal{P}_k := \{P_{k,1}, P_{k,2}\}$ where $P_{k,1} := C_k$ and $P_{k,2} := J \setminus C_k$ for $k \in [r]$.

If \mathbb{P} is non-trivial: Return \mathbb{P} .

▷ In this case, Option 1 of Lemma 25 is satisfied.

If there is $\mathcal{P}_k \in \mathbb{P}$ that is the partition into singletons:

Return $\Lambda := CL_{Object}(J)$ using Corollary 11.

- \triangleright Since $\mathcal{P}_k \in \mathbb{P}$ is the partition into singletons and has size $|\mathcal{P}_k| \le 2$, it follows that $|J| \le 2$.
- ▷ Now, each $\mathcal{P}_k \in \mathbb{P}$ is the singleton partition. This means that for each $\widehat{J}_k \in \widehat{\mathcal{J}}$ and each $\Delta_i \rho_i \in J$ there is a subcoset $\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}} \leq \Delta_i \rho_i$ that is contained in \widehat{J}_k . The same argument that shows $\operatorname{Aut}(\widehat{J}) \leq \operatorname{Aut}(J)$ also shows that $\operatorname{Aut}(\widehat{J}_k) \leq \operatorname{Aut}(J)$ for each $\widehat{J}_k \in \widehat{\mathcal{J}}$. Roughly speaking, this means that \widehat{J}_k can be seen as an individualization of \widehat{J} .

Compute $\Theta_k \tau_k := \operatorname{CL}_{\operatorname{Set}}(\widehat{J}_k, A, \Psi^{\operatorname{Can}}, \bot)$ for each $\widehat{J}_k \in \mathcal{J}$ recursively.

▷ In this case, we satisfy Option 2 of Lemma 25.

Define $\widehat{\mathcal{J}}^{\text{Set}} := \{ \Theta_k \tau_k \mid \widehat{J}_k \in \widehat{\mathcal{J}} \}.$

 $ightharpoonup We collect the canonical labelings <math>\Theta_k \tau_k$ of \widehat{J}_k leading to minimal canonical forms of the input. Define $\widehat{\mathcal{J}}_{\min}^{\operatorname{Set}} \coloneqq \arg\min_{\Theta_k \tau_k \in \widehat{\mathcal{J}}^{\operatorname{Set}}} J^{\tau_k} \subseteq \widehat{\mathcal{J}}^{\operatorname{Set}}$ where the minimum is taken w.r.t. the ordering "<" from Lemma 1.

Return $\Lambda := \langle \widehat{\mathcal{J}}_{\min}^{\operatorname{Set}} \rangle$.

 \triangleright This is the smallest coset containing all labeling cosets in $\widehat{\mathcal{J}}_{\min}$ as defined in the preliminaries. The correctness proof for (CL2) is similar to the (CL2)-proof of Lemma 19.

Theorem 26 ([Bab15], Theorem 3.2.1.). Let $\Delta \leq \operatorname{Sym}(V)$ be a primitive group of order $|\Delta| \geq |V|^{1+\log_2|V|}$ where |V| is greater than some absolute constant. Then Δ is a Cameron group and has a normal subgroup N of index at most |V| such that N has a system of imprimitivity on which N acts as a Johnson group. Moreover, N and the system of imprimitivity in question can be found in polynomial time.

Lemma 27. Let $N \leq \Delta \leq \operatorname{Sym}(V)$ be the group from Theorem 26. Then, $\operatorname{rb}(\Delta, N) \leq \log_2 |V|$.

Proof. As $\Delta \leq \operatorname{Sym}(V)$ is a Cameron group, we have $(\operatorname{Alt}(W)[\binom{W}{s}])^k \leq \Delta \leq \operatorname{Sym}(W)[\binom{W}{s}] \geq \operatorname{Sym}(k)$. We identify $V = \binom{W}{s}^k$. We have an induced homomorphism $h : \Delta \to \operatorname{Sym}(k)$. It follows from the proof of Theorem 26 that $N = \ker(h)$. For each $i \in [k]$, we choose two points $A_i = (a_1, \ldots, a_k), B_i = (b_1, \ldots, b_k) \in \binom{W}{s}^k$ such that $a_i = b_i$ and $a_i \neq b_j$ for $i \neq j$. We define $X := \bigcup_{i \in [k]} \{A_i, B_i\}$. Observe that $|X| = 2k \leq 2 \frac{\log_2 |V|}{\log_2 |W|} \leq \log_2 |V|$. We claim that $\Delta_{(X)} \leq N$. Observe that $h(\Delta_{(\{A_i, B_i\})}) \leq \operatorname{Sym}(k)_{(i)}$ for all $i \in [k]$. Therefore, $h(\Delta_{(X)}) \leq \operatorname{Sym}(k)_{(\{1, \ldots, k\})} = 1$ and thus $\Delta_{(X)} \leq \ker(h) = N$.

Lemma 28. There is an algorithm REDUCETOJOHNSON that gets as input an instance $(J, A, \Delta^{\operatorname{Can}}, \bot)$ for which Property (A), (B) and (g) hold. In time $(|V| + |J|)^{\operatorname{polylog}|V|}$, the algorithm reduces the canonical labeling problem of $(J, A, \Delta^{\operatorname{Can}}, \bot)$ to canonical labeling of either

- (progress (Linear in J)) two instances $(J_1, A, \Delta^{\operatorname{Can}}, \bot)$ and $(J_2, A, \Delta^{\operatorname{Can}}, \bot)$ with $|J_1| + |J_2| = |J|$, or
- (progress (Linear in A)) one instance $(J, A', \Delta^{\operatorname{Can}}, \bot)$ with |A'| < |A|, or
- (progress (In J)) $2^{\log_2 p + \log_2(|V|)^4}$ -many instances $(J_{k,i}, A, \Delta^{\operatorname{Can}}, \bot)$ of size $|J_{k,i}| \le \frac{1}{p}|J|$ and to additionally $2^{\log_2(|V|)^4}$ -many instances $(J_k, V, \Delta_k^{\operatorname{Can}}, \bot)$ of size $|J_k| \le p$ for some $p \in \mathbb{N}$ with 1 , or

- $(progress\ (\text{In}\ \Delta^{\text{Can}}))\ 2^{\log_2(|V|)^3}$ -many instances $(\widehat{J}_i, A, \Psi^{\text{Can}}, \bot)$ with $|\widehat{J}_i| \le |J|$ and such that $\operatorname{orb}_{A^{\text{Can}}}(\Psi^{\text{Can}}) \le \frac{1}{2}\operatorname{orb}_{A^{\text{Can}}}(\Delta^{\text{Can}})$, or
- $(progress\ (\text{In}\ g^{\text{Can}}))\ |V|$ -many instances $(\widehat{J_i}, A, \Psi^{\text{Can}}, g^{\text{Can}})$ where $|\widehat{J_i}| \le |J|$ and such that $\operatorname{orb}_{A^{\text{Can}}}(\Psi^{\text{Can}}) \le \operatorname{orb}_{A^{\text{Can}}}(\Delta^{\text{Can}})$ and g^{Can} is defined.

Intuition of the Johnson Reduction First of all, we want to reduce to the case in which all $\Delta_i \leq \operatorname{Sym}(V)$ are transitive on $A \subseteq V$. To achieve transitivity, Babai's algorithm uses Luks's idea of orbit-by-orbit processing. However, the orbit-by-orbit recursion is a tool that is developed for strings and needs a non-trivial adaption when dealing with a set of labeling cosets J. To achieve transitivity, the present algorithm uses an adaption of the orbit-by-orbit recursion that was developed in [SW19]. In the transitive case, we proceed similarly to Babai's algorithm. First, we define a block system $\mathcal{B}^{\operatorname{Can}}$ on which $\Delta^{\operatorname{Can}}$ acts primitively. If the primitive group acting on $\mathcal{B}^{\operatorname{Can}}$ is small, we use the subgroup reduction from Lemma 25 to reduce to a subgroup $\Psi^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$ that is defined as the kernel of that action. In case that the primitive group is large, we use Cameron's classification of large primitive groups which implies that the primitive group by using the subgroup reduction from Lemma 25. The Johnson group (acting on subsets of a set W^{Can}) in turn can be used to define a giant representation $g^{\operatorname{Can}} : \Delta^{\operatorname{Can}} \to \operatorname{Sym}(W^{\operatorname{Can}})$.

Proof of Lemma 28. An algorithm for REDUCEToJohnson $(J, A, \Delta^{\operatorname{Can}}, \bot)$:

If |A| is smaller than some absolute constant:

Return $CL_{Object}(J)$ using Corollary 11.

▷ We claim that Property (A) and (B) imply that |J| is smaller than some absolute constant. By Property (B), it holds that $\Delta_i \rho_i = \rho_i \Delta^{\operatorname{Can}}$ for all $\Delta_i \rho_i \in J$. Let $\Lambda := \{\lambda \in \operatorname{Label}(V) \mid \lambda|_{V \setminus A} = \rho_1|_{V \setminus A}\}$. By definition, |Λ| ≤ |A|!. By Property (A), for all $\rho_i \Delta^{\operatorname{Can}}$ there is a representative $\rho_i^* \in \rho_i \Delta^{\operatorname{Can}}$ with $\rho_i^* \in \Lambda$. The representatives ρ_i^*, ρ_j^* for $i \neq j$ are pairwise distinct since otherwise $\rho_i \Delta^{\operatorname{Can}} = \rho_i^* \Delta^{\operatorname{Can}} = \rho_j^* \Delta^{\operatorname{Can}} = \rho_j \Delta^{\operatorname{Can}}$. Therefore, |J| = |{\rho_1^*, ..., \rho_t^*}| ≤ |Λ| ≤ |A|! which proves the claim. Therefore, the algorithm from Corollary 11 runs in constant time.

If Δ_i is intransitive on A for some (and because of (B) for all) $\Delta_i \rho_i \in J$:

Define $A^{\operatorname{Can}^*} \nsubseteq A^{\operatorname{Can}}$ as the $\Delta^{\operatorname{Can}}$ -orbit on A^{Can} that is minimal w.r.t. the ordering "<" from Lemma 1.

For each $\Delta_i \rho_i \in J$, define $A_i^* \subseteq A$ as the Δ_i -orbit such that $(A_i^*)^{\rho_i} = A^{\operatorname{Can}^*}$.

Define an (unordered) partition $\mathcal{P} := \{P_1, \dots, P_p\}$ of $J = P_1 \cup \dots \cup P_p$ such that:

 $\Delta_i \rho_i, \Delta_j \rho_j \in P_\ell$ for some $P_\ell \in \mathcal{P}$, if and only if $A_i^* = A_j^*$.

If P is non-trivial:

 \triangleright The singleton $\{\mathcal{P}\}$ can be seen as a non-trivial partition family consisting of one single partition.

Return $\Lambda := \text{RECURSEOnPartition}(J, A, \Delta^{\text{Can}}, \bot, \{\mathcal{P}\})$ using Lemma 23.

 \triangleright Since $|\{\mathcal{P}\}| = 1$, we make progress (In J) or (Linear in J).

If \mathcal{P} is the partition into singletons, i.e., $A_i^* \neq A_j^*$ for all $\Delta_i \rho_i \neq \Delta_j \rho_j \in J$:

 \triangleright In this case, we can define a coset-labeled hypergraph (H, J, α) .

Define the hypergraph $H := \{A_1^*, \dots, A_t^*\}.$

Define $\alpha: H \to J$ by setting $\alpha(A_i^*) := \Delta_i \rho_i$ for each $A_i^* \in H$.

Return $\Lambda := CL_{SetHyper}(J, H, \alpha)$ using Lemma 21.

 \triangleright The algorithm from Lemma 21 runs in time $(|V| + |J|)^{\text{polylog}|V|}$.

If \mathcal{P} is the singleton partition, i.e., $A_i^* = A_i^*$ for all $\Delta_i \rho_i, \Delta_j \rho_j \in J$:

Define $A^* := A_i^*$ for some $\Delta_i \rho_i \in J$.

 \triangleright The set A^* is well-defined and does not depend on the choice of $\Delta_i \rho_i \in J$.

Define $\Lambda_i := (\Delta_i \rho_i)|_{V \setminus A^*}$ for each $\Delta_i \rho_i \in J$.

Define an (unordered) partition $Q := \{Q_1, \dots, Q_q\}$ of $J = Q_1 \cup \dots \cup Q_q$ such that:

 $\Delta_i \rho_i, \Delta_j \rho_j \in Q_\ell$ for some $Q_\ell \in \mathcal{Q}$, if and only if $\Lambda_i = \Lambda_j$.

If Q is non-trivial:

 \triangleright The singleton $\{Q\}$ can be seen as a non-trivial partition family consisting of one single partition.

Return $\Lambda := \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, \bot, \{Q\})$ using Lemma 23.

 \triangleright Since $|\{Q\}| = 1$, we make progress (In J) or (Linear in J).

If Q is the singleton partition, i.e., $\Lambda_i = \Lambda_j$ for all $\Delta_i \rho_i, \Delta_j \rho_j \in J$:

Recurse and return $\Lambda := CL_{Set}(J, A^*, \Delta^{Can}, \bot)$.

 \triangleright By definition of the partition, Property (A) also holds with $A^* \nsubseteq A$ in place of A. We have progress (Linear in A).

If Q is the partition into singletons, i.e., $\Lambda_i \neq \Lambda_j$ for all $\Delta_i \rho_i \neq \Delta_j \rho_j \in J$:

Define $\Delta_i^{\circ} := \Delta_i[V \setminus A^*] \times \operatorname{Sym}(A^*) \ge \Delta_i$ and define $J^{\circ} := \{\Delta_1^{\circ} \rho_1, \dots, \Delta_t^{\circ} \rho_t\}$.

 \triangleright Since Q is the partition into singletons, $|J^{\circ}| = |J|$.

Define $\Delta^{\circ \operatorname{Can}} := \rho_i^{-1} \Delta_i^{\circ} \rho_i$ for some $i \in [t]$.

Recursively compute $\Delta \rho \coloneqq \mathrm{CL}_{\mathrm{Set}}(J^{\circ}, A \setminus A^{*}, \Delta^{\circ \mathrm{Can}}, \bot)$.

 \triangleright We claim that Property (A) holds for this instance with $A \setminus A^*$ in place of A. Observe that $V \setminus (A \setminus A^*) = (V \setminus A) \cup A^*$. Since Δ_i° is a direct product, we can consider both direct factors separately and obtain $(\Delta_i \rho_i)|_{V \setminus A} = (\Delta_i \rho_i)|_{V \setminus A}$ and $(\Delta_i \rho_i)|_{A^*} = \operatorname{Sym}(A^*) \rho_i|_{A^*} = \operatorname{Sym}(A^*) \rho_i|_{A^*} = (\Delta_i \rho_i)|_{A^*} \text{ for all } \Delta_i \rho_i \neq \Delta_i \rho_i \in$ J. Since $A \setminus A^* \subseteq A$, we have progress (Linear in A).

Define $\alpha: J^{\circ} \to J$ by setting $\alpha(\Delta_i^{\circ} \rho_i) := \Delta_i \rho_i$ for each $\Delta_i^{\circ} \rho_i \in J^{\circ}$.

Return $\Lambda := \mathrm{CL}_{\mathrm{SetSet}}(J^{\circ}, J, \alpha, \Delta \rho)$ using Lemma 19.

 \triangleright The algorithm from Lemma 19 runs in time $(|V| + |J|)^{\text{polylog}|V|}$.

If Δ_i is transitive on A for some (and because of (B) for all) $\Delta_i \rho_i \in J$:

▷ We reduce the group to the primitive case.

Compute a minimal block system for $\mathcal{B}^{\operatorname{Can}} = \{B_1^{\operatorname{Can}}, \dots, B_b^{\operatorname{Can}}\}$ for $\Delta^{\operatorname{Can}}$ acting on A^{Can} . \triangleright By using a canonical generating set from Lemma 17 for $\Delta^{\operatorname{Can}}$, we can ensure that the block system B^{Can} only depends on $\Delta^{\operatorname{Can}}$ (and not on the representation of $\Delta^{\operatorname{Can}}$). Observe that $\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}] \leq \operatorname{Sym}(\mathcal{B}^{\operatorname{Can}})$ is a primitive group.

If $\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]$ is smaller than or equal to $|V|^{3+\log_2|V|}$:

Define $\Psi^{\operatorname{Can}} := \operatorname{Stab}_{\Delta^{\operatorname{Can}}}(B_1^{\operatorname{Can}}, \dots, B_b^{\operatorname{Can}}).$

ightharpoonup The group can be computed using a membership test as stated in the preliminaries. Apply ReduceToSubgroup $(J, A, \Delta^{\operatorname{Can}}, \bot, \Psi^{\operatorname{Can}})$ using Lemma 25.

If REDUCETOSUBGROUP returns a non-trivial partition family \mathbb{P} :

Return $\Lambda := \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, \bot, \mathbb{P}).$

 $> \textit{It holds that } |\mathbb{P}| \leq c_{\text{ind}} \cdot |V|^{c_{\text{rb}}} \textit{ where } c_{\text{ind}} \coloneqq (\Delta^{\operatorname{Can}} : \Psi^{\operatorname{Can}}) \leq |V|^{3 + \log_2 |V|}. \quad \textit{As}$ in Example 24.3, we have $c_{\rm rb} \coloneqq {\rm rb}(\Delta^{\rm Can}, \Psi^{\rm Can}) \le 2 \cdot \log_2(c_{\rm ind})$. This leads to progress (In J) or (Linear in J).

If REDUCETOSUBGROUP reduces to c_{ind} -many instances $(\widehat{J}_i, A, \Psi^{\text{Can}}, \bot)$:

Recurse on these c_{ind} -many instances $(\widehat{J}_1, A, \Psi^{\text{Can}}, \bot), \dots, (\widehat{J}_{c_{\text{ind}}}, A, \Psi^{\text{Can}}, \bot)$ as

suggested by the subroutine.

▷ We analyze the recurrence. The multiplicative cost is $c_{\text{ind}} = (\Delta^{\text{Can}} : \Psi^{\text{Can}}) \le |V|^{3+\log_2|V|}$. Moreover, $\text{orb}_{A^{\text{Can}}}(\Psi^{\text{Can}}) = |A^{\text{Can}}|/|\mathcal{B}^{\text{Can}}| \le \frac{1}{2}\text{orb}_{A^{\text{Can}}}(\Delta^{\text{Can}})$. This leads to progress (In Δ^{Can}).

If $\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]$ is greater than $|V|^{3+\log_2|V|}$:

 \triangleright Since $|\mathcal{B}^{\operatorname{Can}}|! \ge |\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]| > |V|^{3+\log_2|V|} \ge |A|^{3+\log_2|A|}$ and |A| is greater than some absolute constant we can apply Theorem 26. It follows that $\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]$ is a Cameron group. Next, we will reduce the group to the Johnson case.

Define $N^{\operatorname{Can}} \supseteq \Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}] \subseteq \operatorname{Sym}(\mathcal{B}^{\operatorname{Can}})$ as the subgroup of index at most $b \subseteq |V|$ which has a system of imprimitivity on which N^{Can} acts as a Johnson group as in Theorem 26.

Define $\Psi^{\operatorname{Can}} \subseteq \Delta^{\operatorname{Can}} \subseteq \operatorname{Sym}(V^{\operatorname{Can}})$ as the corresponding normal subgroup for which $\Psi^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}] = N^{\operatorname{Can}}$ holds.

▷ Also $\Psi^{\operatorname{Can}} \trianglelefteq \Delta^{\operatorname{Can}}$ is of index at most $b \leq |V|$ and has a system of imprimitivity on which it acts as a Johnson group. By using a canonical generating set from Lemma 17 for $\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]$, we can ensure that N^{Can} and $\Psi^{\operatorname{Can}}$ only depend on $\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]$ (and not on the representation of $\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]$).

Apply REDUCETOSUBGROUP $(J, A, \Delta^{\operatorname{Can}}, \bot, \Psi^{\operatorname{Can}})$ using Lemma 25.

If REDUCETOSUBGROUP returns a non-trivial partition family \mathbb{P} :

Return $\Lambda := \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, \bot, \mathbb{P}).$

▷ It holds that $|\mathbb{P}| \leq c_{\text{ind}} \cdot |V|^{c_{\text{rb}}}$ where $c_{\text{ind}} \coloneqq (\Delta^{\text{Can}} : \Psi^{\text{Can}}) \leq b \leq |V|$. By Lemma 27, it follows that $c_{\text{rb}} \coloneqq \text{rb}(\Delta^{\text{Can}}, \Psi^{\text{Can}}) \leq \log_2 |V|$. This leads to progress (In J) or (Linear in J).

If REDUCETOSUBGROUP reduces to b-many instances $(\widehat{J}_i, A, \Psi^{\operatorname{Can}}, \bot)$:

Now, there is a system of imprimitivity on which Ψ^{Can} acts as a Johnson group and therefore there is a homomorphism $h^{\text{Can}}: \Psi^{\text{Can}} \to \text{Sym}(W^{\text{Can}})[\binom{W^{\text{Can}}}{s}]$.

Define $g^{\operatorname{Can}}: \Psi^{\operatorname{Can}} \to \operatorname{Sym}(W^{\operatorname{Can}})$ as the giant representation obtained from h^{Can} whose image is acting on W^{Can} (rather than acting on subsets of W^{Can}).

 \triangleright Since $|\Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]| > |V|^{3+\log_2|V|}$, it follows from the proof of Theorem 26 that $|W^{\operatorname{Can}}| > 2 + \log_2|V|$.

Recurse on the *b*-many instances $(\widehat{J}_1, A, \Psi^{\operatorname{Can}}, g^{\operatorname{Can}}), \dots, (\widehat{J}_b, A, \Psi^{\operatorname{Can}}, g^{\operatorname{Can}}).$

Description Description Description Description Description Observe that the algorithm recurses on the instances $(\widehat{J}_i, A, \Psi^{\operatorname{Can}}, g^{\operatorname{Can}})$ rather than $(\widehat{J}_i, A, \Psi^{\operatorname{Can}}, \bot)$. We analyze the recurrence. We have a multiplicative cost of at most $b \leq |V|$ and recursive instances where g^{Can} is defined. This leads to progress (In g^{Can}).

Definition 29 ([Bab16]). Let $\Delta \leq \operatorname{Sym}(V)$ and let $g: \Delta \to \operatorname{Sym}(W)$ be a giant representation. We say that $v \in V$ is affected by g if g does not map $\Delta_{(v)}$, the pointwise stabilizer of v in Δ , onto a giant, i.e., it does not hold $\operatorname{Alt}(W) \leq g(\Delta_{(v)}) \leq \operatorname{Sym}(W)$. A set $S \subseteq V$ consisting of affected points is called affected set.

Theorem 30 ([Bab16], Theorem 6). Let $\Delta \leq \operatorname{Sym}(V)$ be a permutation group and let k denote the length of the largest Δ -orbit of V. Let $g: \Delta \to \operatorname{Sym}(W)$ be a giant representation. Let $U \subseteq V$ denote the set of all elements of V that are not affected by g. Then the following holds.

- 1. (Unaffected Stabilizer Theorem) Assume $|W| > \max\{8, 2 + \log_2 k\}$. Then g maps $G_{(U)}$, the pointwise stabilizer of U in G, onto Alt(W) or Sym(W) (so $g: G_{(U)} \to Sym(W)$ is still a giant representation). In particular, $U \subsetneq V$ (at least one element is affected).
- 2. (Affected Orbits Lemma) Assume $|W| \ge 5$. If S is an affected Δ -orbit, i.e., $S \cap U = \emptyset$, then $\ker(g)$ is not transitive on S; in fact, each orbit of $\ker(g)$ in S has length at most |S|/|W|.

Definition 31 (Certificates of Fullness). A group $G \leq \text{Sym}(V)$ is called *certificate of fullness* for an instance $(J, A, \Delta^{\text{Can}}, g^{\text{Can}})$ if

- 1. $G \leq \operatorname{Aut}(J)$,
- 2. $G^{\operatorname{Can}} := G^{\rho_i} \leq \Delta^{\operatorname{Can}}$ does not depend on the choice of $\Delta_i \rho_i \in J$, and
- 3. $g^{\text{Can}}:G^{\text{Can}}\to \operatorname{Sym}(W^{\text{Can}})$ is still a giant representation.

Lemma 32. There is an algorithm PRODUCECERTIFICATES that gets a input an instance $(J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ for which Property (A), (B) and (g) hold where g^{Can} is defined. In time $(|V| + |J|)^{\operatorname{polylog}|V|}$, the algorithm reduces the canonical labeling problem of $(J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ to canonical labeling of either

- (progress (Linear in J)) two instances $(J_1, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ and $(J_2, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ with $|J_1| + |J_2| = |J|$, or
- (progress (In J)) $2^{\log_2 p + \log_2(|V|)^4}$ -many instances $(J_{k,i}, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ of size $|J_{k,i}| \leq \frac{1}{p}|J|$ and to additionally $2^{\log_2(|V|)^4}$ -many instances $(J_k, V, \Delta_k^{\operatorname{Can}}, \bot)$ of size $|J_k| \leq p$ for some $p \in \mathbb{N}$ with 1 , or
- $(progress\ (\text{In}\ \Delta^{\text{Can}}))\ 2^{\log_2(|V|)^3}$ -many instances $(\widehat{J}_i, A, \Psi^{\text{Can}}, \bot)$ with $|\widehat{J}_i| \le |J|$ and such that $\operatorname{orb}_{A^{\text{Can}}}(\Psi^{\text{Can}}) \le \frac{1}{2}\operatorname{orb}_{A^{\text{Can}}}(\Delta^{\text{Can}})$, or
- (Fullness certificate) finds a certificate of fullness $G \leq \text{Sym}(V)$ for the input instance.

Intuition of the Certificate Producing Algorithm We describe the idea of the algorithm. The algorithm picks a subset $T^{\operatorname{Can}} \subseteq W^{\operatorname{Can}}$ of logarithmic size. We call this set T^{Can} a canonical test set. Next, we define the group $\Delta_T^{\operatorname{Can}} \subseteq \Delta^{\operatorname{Can}}$ which stabilizes T^{Can} in the image under g^{Can} . By doing so, we can define a giant representation $g_T^{\operatorname{Can}} : \Delta_T^{\operatorname{Can}} \to \operatorname{Sym}(T^{\operatorname{Can}})$. Let $S^{\operatorname{Can}}, U^{\operatorname{Can}} \subseteq V^{\operatorname{Can}}$ be set of elements affected and unaffected by g_T^{Can} , respectively. We have a technical difference in our algorithm in contrast to Babai's method. In Babai's method of local certificates, he processes a giant representation $g:\Delta\to\operatorname{Sym}(W)$ and considers multiple test sets $T\subseteq W$ (one test set for each subset of logarithmic size). In our framework, we define the giant representation for a group $\Delta^{\operatorname{Can}}$ over a linearly ordered set V^{Can} . This allows us to choose one single (canonical) test set $T^{\operatorname{Can}}\subseteq W^{\operatorname{Can}}$ only. Here, canonical means that the subset is chosen minimal with respect to the ordering "<". However, when we translate the ordered structures V^{Can} to unordered structures over V, we implicitly consider multiple test sets and giant representations. More precise, by applying inverses of labelings in $\Delta_i \rho_i \in J$ to the ordered group $\Delta_T^{\operatorname{Can}} \subseteq \operatorname{Sym}(V^{\operatorname{Can}})$, we obtain a set of groups over V, i.e., $\{\lambda_i \Delta_T^{\operatorname{Can}} \lambda_i^{-1} \mid \lambda_i \in \Delta_i \rho_i\}$. Similarly, we can define a set of giant representations $\{(g_T^{\operatorname{Can}})^{\lambda_i^{-1}} \mid \lambda_i \in \Delta_i \rho_i\}$ (where $(g_T^{\operatorname{Can}})^{\lambda_i^{-1}}(\delta_i) := g_T^{\operatorname{Can}}(\lambda_i^{-1}\delta_i\lambda_i)$ for $\delta_i \in \Delta_i$) and a set of affected points $H_i := \{S \subseteq V \mid S^{\lambda_i} = S^{\operatorname{Can}} \text{ for some } \lambda_i \in \Delta_i \rho_i\}$. Therefore, when

dealing over unordered structures, we need to consider multiple groups and homomorphisms. It becomes even more complex, since we are dealing with a set J consisting of labeling cosets rather than one single group only. In fact, we obtain a set of affected point sets H_i for each labeling coset $\Delta_i \rho_i \in J$. However, it turns out that the hardest case occurs when $H_i = H_j$ for all $\Delta_i \rho_i, \Delta_j \rho_j \in J$. Roughly speaking, we will apply the following strategy.

We restrict each labeling coset in J to some set of affected points $S \in H_i$ and define a set of local restrictions J_S^* that ignore the vertices outside S. The precise definition of J_S^* is given in the algorithm. Intuitively, the algorithms tries to analyze the labeling cosets locally.

Case 1: The local restrictions J_S^* are pairwise distinct. In this case, we canonize the local restrictions J_S^* recursively. Observe that a canonical labeling $\Delta \rho$ for J_S^* does not necessarily define a canonical labeling for J. However, we can define a function $\alpha: J_S^* \to J$ that assigns each local restriction its corresponding labeling coset $\Delta_i \rho_i \in J$. This function is well-defined since we assumed the local restrictions to be pairwise distinct. Now, we can use the algorithm from Lemma 19 to canonize the instance $(J_S^*, J, \alpha, \Delta \rho)$.

Case 2: Some local restrictions in J_S^* are pairwise different and some local restrictions in J_S^* are pairwise equal. In this case, we can define a non-trivial partition of J in the following way. We say that two labeling cosets $\Delta_i \rho_i$, $\Delta_j \rho_j$ are in the same part, if and only if the corresponding local restrictions in J_S^* coincide. Actually, this leads to a family of partitions since we obtain one partition for each choice of an affected set $S \in H_i$. We exploit this partition family by recursing using the subroutine RECURSEONPARTITION from Lemma 23.

Case 3: The local restrictions J_S^* are pairwise equal. In this case, it is possible to find automorphisms $G_S \leq \operatorname{Sym}(V)$ of J which fix the unaffected points $V \setminus S$. In fact, we can find such automorphisms for all choices of $S \in H_i$, otherwise we are in a situation of a previous case. Finally, we consider the group of automorphisms $G \leq \operatorname{Aut}(J)$ generated by all G_S for $S \in H_i$. We can show that G is indeed a certificate of fullness.

Proof of Lemma 32. An algorithm for PRODUCE CERTIFICATES $(J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$:

Let $g^{\operatorname{Can}}:\Delta^{\operatorname{Can}}\to\operatorname{Sym}(W^{\operatorname{Can}})$ be the giant representation.

 \triangleright By Property (g), the set A^{Can} is an orbit, $|W^{\operatorname{Can}}| > 2 + \log_2 |V| \ge 2 + \log_2 |A^{\operatorname{Can}}|$, $|W^{\operatorname{Can}}|$ is greater than some absolute constant and $\Delta^{\operatorname{Can}}_{(A^{\operatorname{Can}})} \le \ker(g^{\operatorname{Can}})$. By the Unaffected Stabilizer Theorem 30, and since $\Delta^{\operatorname{Can}}_{(A^{\operatorname{Can}})} \leq \ker(g^{\operatorname{Can}})$, at least one element in A^{Can} is affected by g^{Can} . Define Π^{Can} as the kernel of g^{Can} .

 \triangleright By the Affected Orbits Lemma 30, the orbits of Π^{Can} on A^{Can} have size at most $|A^{\operatorname{Can}}|/|W^{\operatorname{Can}}|$. Define $T^{\operatorname{Can}} \coloneqq \{1, \dots, 3 + \lfloor \log_2 |V| \rfloor\} \subseteq W^{\operatorname{Can}}$.

Define $g_T^{\text{Can}}: \Delta_T^{\text{Can}} \to \text{Sym}(T^{\text{Can}})$ as the giant representation that is obtained by restricting the image of g^{Can} .

 \triangleright By the Unaffected Stabilizer Theorem 30, at least one element in V^{Can} is affected by g_T^{Can} . Moreover, since we assume that $\Delta_{(A^{\operatorname{Can}})}^{\operatorname{Can}} \leq \ker(g^{\operatorname{Can}})$, it follows that at least one element in A^{Can} is affected by g_T^{Can} . Decompose $V^{\operatorname{Can}} \coloneqq S^{\operatorname{Can}} \cup U^{\operatorname{Can}}$ where S^{Can} contains the points affected by g_T^{Can} and where

 $U^{\operatorname{Can}} \coloneqq V^{\operatorname{Can}} \setminus S$ contains the unaffected points.

- $\triangleright By \ the \ Unaffected \ Stabilizer \ Theorem \ 30, \ g_T^{\operatorname{Can}}: \Delta_{T,(U^{\operatorname{Can}})}^{\operatorname{Can}} \to \operatorname{Sym}(T^{\operatorname{Can}}) \ is \ still \ a \ giant$
- $\forall We \ have \ the \ subgroup \ chain \ \Pi^{\operatorname{Can}}, \Delta^{\operatorname{Can}}_{T,(U^{\operatorname{Can}})} \leq \Delta^{\operatorname{Can}}_{T} \leq \Delta^{\operatorname{Can}} \leq \operatorname{Sym}(V^{\operatorname{Can}}). \ However, \ \Pi^{\operatorname{Can}}$ and $\Delta_{T(I/Can)}^{Can}$ might be incomparable under the subgroup relation.

Define $\Psi^{\operatorname{Can}} := \operatorname{Stab}_{\Delta^{\operatorname{Can}}}(S^{\operatorname{Can}}).$

 $\triangleright Observe \ that \ \Delta_T^{\operatorname{Can}} \leq \Psi^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}.$

Decompose $\Delta^{\operatorname{Can}} = \bigcup_{\ell \in [s]} \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}$ into left cosets of $\Psi^{\operatorname{Can}}$.

 $\quad \ \, \vdash \ \, This \,\, can \,\, be \,\, done \,\, in \,\, time \,\, polynomial \,\, in \,\, |V| \,\, and \,\, (\Delta^{\operatorname{Can}}:\Psi^{\operatorname{Can}}) \leq |V|^{3+\log_2|V|}.$

Define the hypergraph $H_i := \{ S \subseteq V \mid S^{\rho_i \delta_\ell^{\operatorname{Can}}} = S^{\operatorname{Can}} \text{ for some } \ell \in [s] \}$ for each $\Delta_i \rho_i \in J$.

▷ The hypergraph H_i can be seen as the preimages of affected points for each $\Delta_i \rho_i \in J$. By definition of Ψ^{Can} , the hypergraph H_i does not depend on the choice of the representative ρ_i of $\Delta_i \rho_i$. However, H_i might depend on the choice of the labeling coset $\Delta_i \rho_i \in J$. We want to reduce to the case in which $H_i = H_j$ for all $\Delta_i \rho_i, \Delta_j \rho_j \in J$.

Define an (unordered) partition $\mathcal{P} := \{P_1, \dots, P_p\}$ of $J = P_1 \cup \dots \cup P_p$ such that:

 $\Delta_i \rho_i, \Delta_j \rho_j \in P_\ell$ for some $P_\ell \in \mathcal{P}$, if and only if $H_i = H_j$.

If \mathcal{P} is non-trivial:

 \triangleright The singleton $\{\mathcal{P}\}$ can be seen as a non-trivial partition family consisting of one single partition.

Compute and return $\Lambda := \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, g^{\text{Can}}, \{\mathcal{P}\})$ using Lemma 23. $\triangleright Since \ |\{\mathcal{P}\}| = 1, \ we \ make \ progress \ (\text{In } J) \ or \ (\text{Linear in } J).$

If \mathcal{P} is the partition into singletons, i.e., $H_i \neq H_j$ for all $\Delta_i \rho_i \neq \Delta_j \rho_j \in J$:

▷ It holds that $|H_i| \le (\Delta^{\operatorname{Can}} : \Psi^{\operatorname{Can}}) \le |V|^{3+\log_2|V|}$. We want to use the hypergraphs H_i to define a partition family of J.

Define $K := \{(k_1, k_2) \mid k_1, k_2 \subseteq V, |k_1|, |k_2| \le c\}$ as the set of pairs of subsets of V of size at most $c := \log_2(|V|^{3 + \log_2|V|})$.

 \triangleright Observe that $|K| \le 2^{\log_2(|V|)^4}$ since $|V| \ge |A|$ is greater than some absolute constant.

We say that $(k_1, k_2) \in K$ is compatible with a set $S \subseteq V$ if $k_1 \subseteq S$ and $k_2 \subseteq V \setminus S$.

We say that $(k_1, k_2) \in K$ identifies the hyperedge $S \in H_i$ in the hypergraph H_i if (k_1, k_2) is compatible with S and (k_1, k_2) is not compatible with each $S' \in H_i$ with $S' \neq S$.

- ▷ We claim that for each hypergraph H_i there is a $k \in K$ that identifies a hyperedge in H_i . Let H_i be a hypergraph with $\log_2(|H_i|) \le c$. We prove the claim by induction on $|H_i|$. If $|H_i| = 1$, then $(\emptyset, \emptyset) \in K$ identifies the hyperedge in H_i . Assume that $|H_i| \ge 2$. Let $v \in V$ such that the partition $\{H_{i,v}, H_{i,\overline{v}}\}$ of H is non-trivial where $H_{i,v} := \{S \in H_i \mid v \in S\}$ and $H_{i,\overline{v}} := \{S \in H_i \mid v \notin S\}$. Assume that $1 \le |H_{i,v}| \le \frac{1}{2}|H_i|$ and therefore $\log_2(|H_{i,v}|) \le c - 1$. By induction, there is a $k = (k_1, k_2)$ with $|k_1|, |k_2| \le c - 1$ that identifies a hyperedge $S \in H_{i,v}$ in $H_{i,v}$. Therefore, $(k_1 \cup \{v\}, k_2) \in K$ identifies the hyperedge $S \in H_i$ in H_i . The other case in which $1 \le |H_{i,\overline{v}}| \le \frac{1}{2}|H_i|$ is analogous.
- \triangleright We reduce to the case in which there is a $k \in K$ that identifies a hyperedge in each hypergraph H_i .

Define a cover $C := \{C_k \mid k \in K\}$ of $J = \bigcup C_k$ such that:

 $\Delta_i \rho_i \in C_k$ if $k \in K$ identifies a hyperedge $S \in H_i$ in the hypergraph H_i .

Define $\mathbb{P} := \{\mathcal{P}_k\}_{k \in K}$ as partition family induced by \mathcal{C} , i.e., $\mathcal{P}_k := \{P_{k,1}, P_{k,2}\}$ where $P_{k,1} := C_k$ and $P_{k,2} := J \setminus C_k$ for $k \in K$.

If \mathbb{P} is non-trivial:

Return $\Lambda \coloneqq \texttt{RECURSEONPARTITION}(J, A, \Delta^{\texttt{Can}}, g^{\texttt{Can}}, \mathbb{P})$ using Lemma 23.

 \triangleright Since $|\mathbb{P}| = |K| \le 2^{\log_2(|V|)^4}$, we make progress (In J) or (Linear in J).

If there is a partition $\mathcal{P}_k \in \mathbb{P}$ that is the partition into singletons:

Return $\Lambda := CL_{Object}(J)$ using Corollary 11.

 \triangleright Since $\mathcal{P}_k \in \mathbb{P}$ is the partition into singletons and has size $|\mathcal{P}_k| \le 2$, it follows that $|J| \le 2$.

 \triangleright Therefore, there is a singleton partition $\mathcal{P}_k \in \mathbb{P}$. This means that there is a $k \in K$ that identifies a hyperedge in each hypergraph H_i . We simplify to the case in which each $k \in K$ identifies a hyperedge in each hypergraph H_i .

Define $K := \{k \in K \mid \mathcal{P}_k = \{J\} = \{C_k\} \text{ is the singleton partition } \}$.

 \triangleright Now, each $k \in K$ identifies a hyperedge in each hypergraph H_i . By definition, each $k \in K$ identifies exactly one hyperedge $S \in H_i$ in each hypergraph H_i .

Define $E_k := \{S \mid k \text{ identifies the hyperedge } S \in H_i \text{ in some hypergraph } H_i\}$ for each $k \in K$. $\triangleright By$ definition, $|E_k \cap H_i| = 1$ for all $k \in K$ and all hypergraphs H_i .

Define a partition family $\mathbb{Q} := \{Q_k\}_{k \in K}$ of $J = Q_{k,1} \cup \ldots \cup Q_{k,q_k}$ where $Q_k := \{Q_{k,1}, \ldots, Q_{k,q_k}\}$ such that:

 $\Delta_i \rho_i, \Delta_j \rho_j \in Q_{k,x}$ for some $Q_{k,x} \in Q_k$, if and only if $E_k \cap H_i = E_k \cap H_j$.

If Q is non-trivial:

Return $\Lambda := \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, g^{\text{Can}}, \mathbb{Q})$ using Lemma 23.

 $\triangleright Since |\mathbb{Q}| = |K| \le 2^{\log_2(|V|)^4}$, we make progress (In J) or (Linear in J).

If there is a partition $Q_k \in \mathbb{Q}$ that is the singleton partition:

 \triangleright This means that there is a $k \in K$ such that $E_k \cap H_i = E_k \cap H_j$ for all hypergraphs.

Define $S := \{ S \in E_k \cap H_i \mid Q_k \in \mathbb{Q} \text{ is the singleton partition} \}.$

 \triangleright We have that $S \subseteq H_i$ for all hypergraphs H_i . The case that $S = H_i$ for all hypergraphs cannot occur since we are in a situation with $H_i \neq H_j$ for all $\Delta_i \rho_i \neq \Delta_j \rho_j \in J$.

Define $H'_i := H_i \setminus \mathcal{S}$ for all hypergraphs H_i .

Go to the outer case with H'_i in place of H_i .

 \triangleright Again, we have $H'_i \neq H'_j$ for all $\Delta_i \rho_i \neq \Delta_j \rho_j \in J$.

If all partitions $Q_k \in \mathbb{Q}$ are partitions into singletons:

 \triangleright This means that for all $k \in K$, the sets $E_k \cap H_1, \ldots, E_k \cap H_t$ are pairwise distinct.

For each $k \in K$ do:

 \triangleright We will compute a canonical labeling for (J,k). We will define a coset-labeled hypergraph.

Define a function $\alpha_k : E_k \to J$ by setting $\alpha_k(S) := \Delta_i \rho_i$ for $\{S\} = E_k \cap H_i$.

 \triangleright This is well-defined, since $|H_i \cap E_k| = 1$ and the sets $E_k \cap H_1, \dots, E_k \cap H_t$ are pairwise distinct.

Compute $\Theta_k \tau_k := \mathrm{CL}_{\mathrm{SetHyper}}(E_k, J, \alpha_k)$ using Lemma 21.

 \triangleright The algorithm from Lemma 21 runs in time $(|V| + |J|)^{\text{polylog}|V|}$.

Define $K^{\text{Set}} := \{\Theta_k \tau_k \mid k \in K\}.$

 \triangleright We collect the canonical labelings $\Theta_k \tau_k$ leading to minimal canonical forms of the input.

Define $K_{\min}^{\text{Set}} \coloneqq \arg\min_{\Theta_k \tau_k \in K^{\text{Set}}} J^{\tau_k} \subseteq K^{\text{Set}}$ where the minimum is taken w.r.t. the ordering "<" from Lemma 1.

Return $\Lambda := \langle K_{\min}^{\text{Set}} \rangle$.

 \triangleright This is the smallest coset containing all labeling cosets in K_{\min}^{Set} as defined in the preliminaries. The correctness proof for (CL2) is similar to the (CL2)-proof of Lemma 19.

 \triangleright Now, the partition \mathcal{P} is the singleton partition, i.e., $H_i = H_j$ for all $\Delta_i \rho_i, \Delta_j \rho_j \in J$.

Define $H := H_i$ for some $\Delta_i \rho_i \in J$.

 \triangleright This does not depend on the choice of $\Delta_i \rho_i \in J$.

For each $S \in H$, define a representative $\lambda_{i,S} \in \Delta_i \rho_i$ such that $S^{\lambda_{i,S}} = S^{\operatorname{Can}}$.

Define $\widehat{J} := \{ \rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \mid i \in [t], \ell \in [s] \} = \{ \lambda_{i,S} \Psi^{\operatorname{Can}} \mid i \in [t], S \in H \}.$

 \triangleright It follows that $\operatorname{Aut}(\widehat{J}) = \operatorname{Aut}(J)$ using the same argument as in the REDUCETOSUBGROUP subroutine.

Define $\widehat{J}_S := \{\lambda_{i,S} \Psi^{\operatorname{Can}} \mid i \in [t]\}$ for each $S \in H$ and define $\widehat{\mathcal{J}} := \{\widehat{J}_S \mid S \in H\}$.

▶ We claim that $\operatorname{Aut}(\widehat{J}_S) \leq \operatorname{Aut}(J)$ for all $\widehat{J}_S \in \widehat{\mathcal{J}}$. For all $\widehat{J}_S \in \widehat{\mathcal{J}}$ and all $\Delta_i \rho_i \in J$ there is a subcoset $\lambda_{i,S} \Psi^{\operatorname{Can}} \leq \Delta_i \rho_i$ in \widehat{J}_S . This proves the claim with the same argument as in the REDUCETOSUBGROUP subroutine.

REDUCETOSUBGROUP subroutine. Define a partition family $\mathbb{P} \coloneqq \{\mathcal{P}_S\}_{S \in H}$ of $J = P_{S,1} \cup \ldots \cup P_{S,p_S}$ where $\mathcal{P}_S \coloneqq \{P_{S,1}, \ldots, P_{S,p_S}\}$

such that:

 $\Delta_i \rho_i, \Delta_j \rho_j \in P_{S,\ell}$ for some $P_{S,\ell} \in \mathcal{P}_S$, if and only if $(\lambda_{i,S} \Psi^{\operatorname{Can}})|_{V \setminus A} = (\lambda_{j,S} \Psi^{\operatorname{Can}})|_{V \setminus A}$.

If \mathbb{P} is non-trivial:

Compute and return $\Lambda := \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, g^{\text{Can}}, \mathbb{P})$ using Lemma 23. \triangleright We have that $|\mathbb{P}| = |H| \le (\Delta^{\text{Can}} : \Psi^{\text{Can}}) \le |V|^{3 + \log_2 |V|}$ and therefore we make progress (In J) or (Linear in J).

If there is a partition $\mathcal{P}_S \in \mathbb{P}$ that is the partition into singletons:

Return $CL_{Object}(J)$ using Corollary 11.

- \triangleright Since \mathcal{P}_S is the partition into singletons and $|\mathcal{P}_S| \le (\Delta^{\operatorname{Can}} : \Psi^{\operatorname{Can}}) \le |V|^{3+\log_2|V|}$ is bounded, it follows that $|J| \le |V|^{3+\log_2|V|}$. Therefore, the algorithm from Corollary 11 runs in time $2^{\operatorname{polylog}|V|}$.
- ightharpoonup Now, all partitions $\mathcal{P}_S \in \mathbb{P}$ are singleton partitions. This means that Property (A) holds for each instance $\widehat{J}_S \in \widehat{\mathcal{J}}$. In the next steps, we analyze the sets $\lambda_{i,S}\Psi^{\operatorname{Can}}$ locally. More precisely, we consider the restrictions $(\lambda_{i,S}\Psi^{\operatorname{Can}})|_S$. We consider different cases depending on whether these local restrictions coincide or not. We define the following partition family.

Define a partition family $\mathbb{Q} := \{Q_S\}_{S \in H}$ of $J = Q_{S,1} \cup \ldots \cup Q_{S,q_S}$ where $Q_S := \{Q_{S,1}, \ldots, Q_{S,q_S}\}$ such that:

 $\Delta_i \rho_i, \Delta_j \rho_j \in Q_{S,\ell}$ for some $Q_{S,\ell} \in \mathcal{Q}_S$, if and only if $(\lambda_{i,S} \Psi^{\operatorname{Can}})|_S = (\lambda_{j,S} \Psi^{\operatorname{Can}})|_S$.

If \mathbb{Q} is non-trivial:

Compute and return $\Lambda \coloneqq \texttt{RECURSEONPARTITION}(J, A, \Delta^{\texttt{Can}}, g^{\texttt{Can}}, \mathbb{Q})$ using Lemma 23.

 \triangleright We have that $|\mathbb{Q}| = |H| \le (\Delta^{\operatorname{Can}} : \Psi^{\operatorname{Can}}) \le |V|^{3 + \log_2 |V|}$ and therefore we make progress (In J) or (Linear in J).

If there is a partition $Q_S \in \mathbb{Q}$ that is the partition into singletons:

▷ This means that there is $S \in H$ such that $(\lambda_{i,S}\Psi^{\operatorname{Can}})|_{S} \neq (\lambda_{j,S}\Psi^{\operatorname{Can}})|_{S}$ are pairwise distinct for all $\Delta_{i}\rho_{i}, \Delta_{j}\rho_{j} \in J$. We simplify to the case in which \mathcal{Q}_{S} is the partition into singletons for all $\mathcal{Q}_{S} \in \mathbb{Q}$.

Define $H := \{ S \in H \mid \mathcal{Q}_S \in \mathbb{Q} \text{ is the partition into singletons} \}.$

 \triangleright Now, for all $S \in H$ the local restrictions are pairwise distinct.

For each $S \in H$ do:

 \triangleright We will compute a canonical labeling for (J, S).

Define $\Psi^{*\operatorname{Can}} := \Psi^{\operatorname{Can}}[S^{\operatorname{Can}}] \times \operatorname{Sym}(V^{\operatorname{Can}} \setminus S^{\operatorname{Can}}) \ge \Psi^{\operatorname{Can}}.$

Define $\widehat{J}_{S}^{*} := \{\lambda_{i,S} \Psi^{*\operatorname{Can}} \mid \lambda_{i,S} \Psi^{\operatorname{Can}} \in \widehat{J}_{S}\}.$

 \triangleright Since Q_S is the partition into singletons, it follows that $|\widehat{J}_S^*| = |\widehat{J}_S|$.

Define $A_S := A \cap S$.

 \triangleright Since Property (A) holds for \widehat{J}_S with A_S in place of A, it follows that Property (A) also holds for the instance \widehat{J}_S^* (with A_S in place of A).

Define $\Theta^{\operatorname{Can}} \leq \Delta_T^{\operatorname{Can}} \leq \Psi^{\operatorname{Can}}$ to be the kernel of $g_T^{\operatorname{Can}} : \Delta_T^{\operatorname{Can}} \to \operatorname{Sym}(T^{\operatorname{Can}})$.

 $ightharpoonup Observe that all points in <math>A_S^{\operatorname{Can}} \subseteq S^{\operatorname{Can}}$ are affected by g_T^{Can} . By the Affected Orbit Lemma 30, the $\Theta^{\operatorname{Can}}$ -orbits of A_S^{Can} have size at most $|A_S^{\operatorname{Can}}|/|T^{\operatorname{Can}}|$.

Define $\Theta^{*\operatorname{Can}} := \operatorname{Stab}_{\Psi^{*\operatorname{Can}}}(A_1^{\operatorname{Can}}, \dots, A_a^{\operatorname{Can}})$ be the stabilizer of those orbits. Let $c_{\operatorname{ind}} := (\Psi^{*\operatorname{Can}} : \Theta^{*\operatorname{Can}})$ and let $c_{\operatorname{rb}} := \operatorname{rb}(\Psi^{*\operatorname{Can}}, \Theta^{*\operatorname{Can}})$.

Apply the subroutine REDUCEToSubgroup $(\widehat{J}_{S}^{*}, A_{S}, \Psi^{*\operatorname{Can}}, \Theta^{*\operatorname{Can}})$ using Lemma 25.

> We consider two cases depending on which option of Lemma 25 is satisfied for the subroutine REDUCEToSUBGROUP.

If for all $S \in H$ the subroutine reduces to c_{ind} -many instances with the subgroup Θ^{*Can} :

For each $S \in H$, define $\Delta_S \rho_S := \text{REDUCEToSUBGROUP}(\widehat{J}_S^*, A_S, \Psi^{*Can}, \Theta^{*Can})$.

 \triangleright We analyze the recurrence. We have a multiplicative cost of $H \cdot c_{ind} \leq \binom{|V|}{|T^{Can}|}$. $|T^{\operatorname{Can}}|! \leq |V|^{|T^{\operatorname{Can}}|} \leq |V|^{3+\log_2|V|}$ and recursive instances with $\operatorname{orb}_{A^{\operatorname{Can}}}(\Theta^{\star \operatorname{Can}}) \leq$ $|A_S^{\operatorname{Can}}|/|T^{\operatorname{Can}}| \le \frac{1}{2}|A^{\operatorname{Can}}|$. Therefore, we make progress (In $\Delta^{\operatorname{Can}}$).

For each $S \in H$, define $\alpha_S : \widehat{J}_S^* \to \widehat{J}_S$ by setting $\alpha_S(\lambda_{i,S}\Psi^{*\operatorname{Can}}) := \lambda_{i,S}\Psi^{\operatorname{Can}}$.

 $\triangleright Observe\ that\ \Delta_S \leq \operatorname{Aut}(\widehat{J}_S^*).$

For each $S \in H$, compute $\Theta_S \tau_S \coloneqq \operatorname{CL}_{\operatorname{SetSet}}(\widehat{J}_S^*, \widehat{J}_S, \alpha_S, \Delta_S \rho_S)$ using Lemma 19. \triangleright The algorithm from Lemma 19 runs in time $(|V| + |\widehat{J}_S^*|)^{\operatorname{polylog}|V|}$.

Define $H^{\text{Set}} := \{ \Theta_S \tau_S \mid S \in H \}.$

 \triangleright We collect the canonical labelings $\Theta_S \tau_S$ leading to minimal canonical forms of the input.

Define $H_{\min}^{\text{Set}} \coloneqq \arg\min_{\Theta_S \tau_S \in H^{\text{Set}}} J^{\tau_S} \subseteq H^{\text{Set}}$ where the minimum is taken w.r.t. the ordering "<" from Lemma 1.

Return $\Lambda := \langle H_{\min}^{\text{Set}} \rangle$.

 \triangleright This is the smallest coset containing labeling cosets in $H^{
m Set}_{
m min}$ as defined in the preliminaries. The correctness proof for (CL2) is similar to the (CL2)-proof of

If for some $S \in H$ the subroutine returns a non-trivial partition family $\widehat{\mathbb{P}}_S$ of \widehat{J}_S^* :

> We simplify to the case in which we have a non-trivial partition family for all $S \in H$.

Define $H := \{ S \in H \mid \text{ the subroutine returns a partition family } \widehat{\mathbb{P}}_S \text{ for } \widehat{J}_S^* \}.$

 \triangleright The partition family $\widehat{\mathbb{P}}_S$ for \widehat{J}_S also induces a partition family \mathbb{P}_S of J.

For each $S \in H$, define a non-trivial partition family $\mathbb{P}_S := \{ \mathcal{P}_S \mid \widehat{\mathcal{P}}_S \in \widehat{\mathbb{P}}_S \}$ of J where $\mathcal{P}_S \coloneqq \{P_S \mid \widehat{P}_S \in \widehat{\mathcal{P}}_S\} \text{ such that: } \Delta_i \widehat{\rho_i} \in P_S, \text{ if and only if } \lambda_{i,S} \Psi^*^{\operatorname{Can}} \in \widehat{P}_S.$

> By taking a union, we combine all partition families into one single partition family

Define a non-trivial partition family $\mathbb{P} := \bigcup_{S \in H} \mathbb{P}_S$ of J.

Return $\Lambda := \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, g^{\text{Can}}, \mathbb{P}).$

- \triangleright We analyze the recurrence. In this case $|\mathbb{P}| \leq |H| \cdot |\mathbb{P}_S| \leq |H| \cdot c_{\mathrm{ind}} \cdot |V|^{c_{\mathrm{rb}}}$. We have $c_{\mathrm{ind}} \leq |V|^{3+\log_2|V|}$ and by Example 24.3, we have $c_{\mathrm{rb}} \leq 2 \cdot \log_2(c_{\mathrm{ind}})$. In total, we have $|\mathbb{P}| \leq 2^{\log_2(|V|)^4}$ which leads to progress (In J) or (Linear in J).
- \triangleright Now it holds that Q_S is the singleton partition for each $S \in H$. This means that the local restrictions pairwise coincide. More precisely, this means that $(\lambda_{i,S}^{-1}\lambda_{j,S})[S^{\operatorname{Can}}] \in \Psi^{\operatorname{Can}}[S^{\operatorname{Can}}]$ for all $\lambda_{i,S} \Psi^{\operatorname{Can}}, \lambda_{i,S} \Psi^{\operatorname{Can}} \in \widehat{J}_S$.

For each $S \in H$ do:

 $> Now, \ we \ compute \ automorphisms \ G_S \leq \operatorname{Aut}(\widehat{J}_S) \leq \operatorname{Aut}(J) \ for \ each \ \widehat{J}_S \in \widehat{\mathcal{J}}.$ Define $G_S^{\operatorname{Can}} := \langle (\Delta_{T,(U^{\operatorname{Can}})}^{\operatorname{Can}})^{\Psi^{\operatorname{Can}}} \rangle \leq \Psi^{\operatorname{Can}}, \ \text{the normal closure of} \ \Delta_{T,(U^{\operatorname{Can}})}^{\operatorname{Can}} \ \text{in} \ \Psi^{\operatorname{Can}}.$

Define $G_S := \lambda_{i,S} G_S^{\operatorname{Can}} \lambda_{i,S}^{-1} \leq \Delta_i \leq \operatorname{Sym}(V)$ for some $\Delta_i \rho_i \in J$.

- \triangleright We claim that G_S depends neither on the choice of $\Delta_i \rho_i \in J$ nor on the choice of the representative $\lambda_{i,S} \in \Delta_i \rho_i$. First, we show that G_S does not depend on the choice of the representative $\lambda_{i,S} \in \Delta_i \rho_i$. Let $\lambda'_{i,S} \in \Delta_i \rho_i$ be a second representative. Observe that $\lambda_{i,S}^{-1}\lambda_{i,S}' \in \Psi^{\operatorname{Can}}$ and since $G_S^{\operatorname{Can}} \subseteq \Psi^{\operatorname{Can}}$ the permutation $\lambda_{i,S}^{-1}\lambda_{i,S}'$ normalizes G_S^{Can} . Equivalently, this means that $\lambda_{i,S}G_S^{\operatorname{Can}}\lambda_{i,S}^{-1} = \lambda_{i,S}'G_S^{\operatorname{Can}}\lambda_{i,S}'^{-1}$ which was what we wanted to show. We show that G_S does not depend on the choice of $\Delta_i \rho_i \in J$. Let $\Delta_j \rho_j \in J$. We have that $(\lambda_{i,S}^{-1} \lambda_{j,S})[S^{\operatorname{Can}}] \in \Psi^{\operatorname{Can}}[S^{\operatorname{Can}}]$. and since $G_S^{\operatorname{Can}}[S^{\operatorname{Can}}] \subseteq \Psi^{\operatorname{Can}}[S^{\operatorname{Can}}]$ the permutation $(\lambda_{i,S}^{-1}\lambda_{j,S})[S^{\operatorname{Can}}]$ normalizes $G_S^{\operatorname{Can}}[S^{\operatorname{Can}}]$. Moreover, the permutation $(\lambda_{i,S}^{-1}\lambda_{j,S})[U^{\operatorname{Can}}]$ obviously normalizes $G_S^{\operatorname{Can}}[U^{\operatorname{Can}}] = \Delta_{T,(U^{\operatorname{Can}})}^{\operatorname{Can}}[U^{\operatorname{Can}}] = 1$. In total, $\lambda_{i,S}^{-1}\lambda_{j,S}$ normalizes G_S^{Can} or equivalently $\lambda_{i,S}G_S^{\text{Can}}\lambda_{i,S}^{-1} = \lambda_{j,S}G_S^{\text{Can}}\lambda_{i,S}^{-1}$
- $| In \ particular, \ G_S \leq \operatorname{Aut}(\widehat{J}_S) \leq \operatorname{Aut}(J).$ $| In \ particular, \ G_S^{\lambda_{i,S}} = G_S^{\operatorname{Can}} \ for \ all \ \lambda_{i,S} \Psi^{\operatorname{Can}} \in \widehat{J}.$
- \triangleright In the next step, we will consider the group of automorphisms G generated by all groups G_S and show that G is a certificate of fullness.

Define $G \leq \operatorname{Sym}(V)$ as the group generated by all G_S for all $S \in H$.

- $\forall We \ claim \ that \ G^{\rho_i} = G^{\operatorname{Can}} \ for \ all \ \Delta_i \rho_i \in J. \ Let \ \Delta_i \rho_i \in J. \ We \ have \ G^{\rho_i} = \rho_i^{-1} \langle \lambda_{i,S} G_S^{\operatorname{Can}} \lambda_{i,S}^{-1} |$
- $S \in H \rangle \rho_i = \langle \delta_\ell^{\operatorname{Can}} G_S^{\operatorname{Can}} \delta_\ell^{\operatorname{Can}^{-1}} \mid \ell \in [s] \rangle = \langle (G_S^{\operatorname{Can}})^{\Delta^{\operatorname{Can}}} \rangle = G^{\operatorname{Can}}.$ $\triangleright \text{ We claim that } g^{\operatorname{Can}} : G^{\operatorname{Can}} \to \operatorname{Sym}(W^{\operatorname{Can}}) \text{ is a giant representation. Since } G^{\operatorname{Can}} \preceq \Delta^{\operatorname{Can}}, \text{ it follows that } g^{\operatorname{Can}}(\Delta_{T,(U^{\operatorname{Can}})}^{\operatorname{Can}}) \preceq g^{\operatorname{Can}}(G^{\operatorname{Can}}) \preceq g^{\operatorname{Can}}(\Delta^{\operatorname{Can}}). \text{ Moreover, each non-trivial normal subgroup of the giant } g^{\operatorname{Can}}(\Delta^{\operatorname{Can}}) \text{ is a giant as well.}$

Return the certificate of fullness \hat{G} .

Automorphism Lemma For an object $\mathcal{X} \in \text{Objects}(V)$ and a group $G \leq \text{Sym}(V)$, we define $\mathcal{X}^G := \{ \mathcal{X}^g \mid g \in G \} \in \mathrm{Objects}(V)$

Lemma 33 (Automorphism Lemma). Let $\mathcal{X} \in \text{Objects}(V)$ be an object, let $G \leq \text{Sym}(V)$ be a group and let CL be a canonical labeling function. Assume that $Aut(\mathcal{X}) \leq Aut(\mathcal{X}^G)$. Then, $\mathrm{CL}_{\mathrm{Object}}(\mathcal{X}^G) \coloneqq G\mathrm{CL}(\mathcal{X})$ defines a canonical labeling for \mathcal{X}^G .

Proof. We claim that $\operatorname{Aut}(\mathcal{X}^G) = G\operatorname{Aut}(\mathcal{X})$. The inclusion $G\operatorname{Aut}(\mathcal{X}) \leq \operatorname{Aut}(\mathcal{X}^G)$ follows by the assumption Conversely, we show $\operatorname{Aut}(\mathcal{X}^G) \leq G\operatorname{Aut}(\mathcal{X})$. Let $\sigma \in \operatorname{Aut}(\mathcal{X}^G)$. Therefore, $\mathcal{X}^{\sigma^{-1}} = \mathcal{X}^{g^{-1}}$ for some $g \in G$. This implies $g^{-1}\sigma \in \operatorname{Aut}(\mathcal{X})$ and thus $\sigma \in G\operatorname{Aut}(\mathcal{X})$.

Lemma 34. There is an algorithm AGGREGATE CERTIFICATES that gets as input a pair (\mathcal{X}, G) where $\mathcal{X} = (J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ is a tuple for which Property (A), (B) and (g) hold where g^{Can} is defined and $G \leq \operatorname{Sym}(V)$ is a fullness certificate. In time $(|V| + |J|)^{\operatorname{polylog}|V|}$, the algorithm reduces the canonical labeling problem of X to canonical labeling of either

- (progress (Linear in J)) two instances $(J_1, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ and $(J_2, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ with $|J_1| + |J_2| = |J|$, or
- $(progress \text{ (In } J)) \ 2^{\log_2 p + \log_2(|V|)^4}$ -many instances $(J_{k,i}, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ of size $|J_{k,i}| \leq \frac{1}{p}|J|$ and to additionally $2^{\log_2(|V|)^4}$ -many instances $(J_k, V, \Delta_k^{\operatorname{Can}}, \bot)$ of size $|J_k| \le p$ for some $p \in \mathbb{N}$ with 1 , or

• $(progress\ (\text{In}\ \Delta^{\text{Can}}))\ 2^{\log_2(|V|)^3}$ -many instances $(\widehat{J}_i, A, \Psi^{\text{Can}}, \bot)$ with $|\widehat{J}_i| \le |J|$ and such that $\operatorname{orb}_{A^{\operatorname{Can}}}(\Psi^{\operatorname{Can}}) \leq \frac{1}{2} \operatorname{orb}_{A^{\operatorname{Can}}}(\Delta^{\operatorname{Can}}).$

Intuition of Certificate Aggregation We describe the overall strategy of this subroutine. Let us consider the less technical case in which $g^{\text{Can}}(G^{\text{Can}})$ is the symmetric group (rather than the alternating group). In this case, it holds that $G^{\text{Can}}\Psi^{\text{Can}} = \Delta^{\text{Can}}$ where Ψ^{Can} is the kernel of g^{Can} . Similarly to the REDUCETOSUBGROUP and PRODUCECERTIFICATES subroutine, we consider the decomposition of $\Delta^{\operatorname{Can}} = \bigcup_{\ell \in [s]} \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}$ into left cosets of the kernel and define $\widehat{J} := \{ \rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \mid i \in [t], \ell \in [s] \}. \text{ Again, we have } \operatorname{Aut}(\widehat{J}) = \operatorname{Aut}(J). \text{ The key observation is that } G \text{ is transitive on } \widehat{J} \text{ since } (\rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}})^{g^{-1}} = \rho_i g^{\rho_i} \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \text{ for all } g \in G \text{ and } G^{\operatorname{Can}} \Psi^{\operatorname{Can}} = \Delta^{\operatorname{Can}}.$

First, consider an easy case in which $J = \{\Delta_1 \rho_1\}$ consists of one single labeling coset. In this case, we have a set of automorphisms G acting transitively on the subcosets $\widehat{J} = \{\rho_1 \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \mid$ $\ell \in [s]$. Moreover, each subcoset satisfies $\operatorname{Aut}(\rho_1 \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}) \leq \operatorname{Aut}(J)$ and can be seen as an individualization of J. This means, we can choose (arbitrarily) a subcoset $\rho_1 \delta_{\ell}^{\text{Can}} \Psi^{\text{Can}} \leq \Delta_1 \rho_1$ and recurse on that. Since the automorphisms in G can map each subcoset to each other subcoset it does not matter which subcoset we choose. By recursing on one single subcoset only, we can measure significant progress. At the end, we return $G\widehat{\Lambda}$ where $\widehat{\Lambda}$ is a canonical labeling for the (arbitrarily) chosen subcoset and G is the group of automorphisms (acting transitively on the set of all subcosets).

However, the situation becomes more difficult when dealing with more labeling cosets J = $\{\Delta_1 \rho_1, \dots, \Delta_t \rho_t\}$ for $t \geq 2$. The first idea that comes to mind is the following generalization. We choose (arbitrarily) some $\ell \in [s]$ and define the set of subcosets $\widehat{J}_{\ell} := \{\rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \mid i \in [t]\} \subseteq \widehat{J}$. The set \widehat{J}_{ℓ} contains exactly one subcoset $\rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \leq \Delta_i \rho_i$ of each $\Delta_i \rho_i \in J$. However, the partition $\widehat{\mathcal{J}} := \{\widehat{J_\ell} \mid \ell \in [s]\}$ might not be G-invariant and G might not be transitive on it. The goal of the algorithm is to find a suitable partition $\widehat{\mathcal{J}} := \{\widehat{J}_1, \dots, \widehat{J}_r\}$ of the subcosets \widehat{J} on which G is transitive.

Proof of Lemma 34. An algorithm for AGGREGATE CERTIFICATES $(J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}}, G)$: Define $\Pi^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$ as the kernel of $g^{\operatorname{Can}} : \Delta^{\operatorname{Can}} \to \operatorname{Sym}(W^{\operatorname{Can}})$.

Define $M^{\operatorname{Can}} := \operatorname{Sym}(W^{\operatorname{Can}})_{(\{3,\dots,|W^{\operatorname{Can}}|\})}$, the pointwise stabilizer of all points excluding $1, 2 \in \mathbb{N}$. Define $\Psi^{\operatorname{Can}} := q^{\operatorname{Can}^{-1}}(M^{\operatorname{Can}}) \leq \Delta^{\operatorname{Can}}$.

 \triangleright It holds that $\Pi^{\operatorname{Can}} \leq \Psi^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$, where the former subgroup relation is of index 2. Moreover, $G^{\operatorname{Can}}\Psi^{\operatorname{Can}} = \Delta^{\operatorname{Can}}$

We consider (but not compute) the decomposition of $\Delta^{\operatorname{Can}} = \bigcup_{\ell \in [s]} \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}$ into left cosets and define $\widehat{J}\coloneqq\{\rho_i\delta_\ell^{\operatorname{Can}}\Psi^{\operatorname{Can}}\mid \Delta_i\rho_i\in J, \ell\in[s]\}.$

 \triangleright This decomposition is for the analysis only and its computation is not part of the algorithm.

If $\Pi^{\operatorname{Can}}[V^{\operatorname{Can}} \smallsetminus A^{\operatorname{Can}}] < \Delta^{\operatorname{Can}}[V^{\operatorname{Can}} \smallsetminus A^{\operatorname{Can}}]$ is a subgroup of index greater than 2:

Define the homomorphism $h: \Delta^{\operatorname{Can}} \to \Delta^{\operatorname{Can}}[V^{\operatorname{Can}} \setminus A^{\operatorname{Can}}]$ by restricting the image to $V^{\operatorname{Can}} \smallsetminus A^{\operatorname{Can}}$.

Define $N^{\operatorname{Can}} \coloneqq \ker(h) \leq \Delta^{\operatorname{Can}}$ as the kernel of the homomorphism h.

 \triangleright We claim that $N^{\operatorname{Can}} \leq \Pi^{\operatorname{Can}}$. Since $\Pi^{\operatorname{Can}}, N^{\operatorname{Can}} \subseteq \Delta^{\operatorname{Can}}$, we have that $\Pi^{\operatorname{Can}} \subseteq \Omega^{\operatorname{Can}}$ $\Pi^{\operatorname{Can}} N^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$. Observe that $\Delta^{\operatorname{Can}}/\Pi^{\operatorname{Can}}$ is isomorphic to a giant and all normal subgroups of a giant with index greater than 2 are trivial. By assumption, $(\Delta^{\operatorname{Can}}:\Pi^{\operatorname{Can}}N^{\operatorname{Can}}) \geq (h(\Delta^{\operatorname{Can}}):h(\Pi^{\operatorname{Can}}N^{\operatorname{Can}})) > 2.$ By the Correspondence Theorem, $\Pi^{\text{Can}} N^{\text{Can}} = \Pi^{\text{Can}}$ which proves the claim.

We consider (but not compute) the decomposition $\widehat{\mathcal{J}}\coloneqq\{\widehat{J}_1,\ldots,\widehat{J}_r\}$ of $\widehat{J}=\widehat{J}_1\cup\ldots\cup\widehat{J}_r$ such that:

 $\rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}, \rho_{i'} \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_k \text{ for some } \widehat{J}_k \in \widehat{\mathcal{J}}, \text{ if and only if } (\rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}})|_{V \setminus A} \text{ equals } (\rho_{i'} \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}})|_{V \setminus A}.$

We claim that $|\widehat{J}_k| = |J|$ and $\operatorname{Aut}(\widehat{J}_k) \leq \operatorname{Aut}(J)$. We show a stronger statement, i.e., for each $k \in [r]$, $\Delta_i \rho_i \in J$ there is exactly one $\ell \in [s]$ such that $\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_k$. Property (A) implies that for all $k \in [r]$, $\Delta_i \rho_i \in J$ there is at least one $\ell \in [s]$ such that $\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_k$. On the other side, let $\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}}$, $\rho_i \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_k$. By definition of \widehat{J}_k , it holds that $((\delta_\ell^{\operatorname{Can}})^{-1} \delta_{\ell'}^{\operatorname{Can}})[V^{\operatorname{Can}} \setminus A^{\operatorname{Can}}] \in \Psi^{\operatorname{Can}}[V^{\operatorname{Can}} \setminus A^{\operatorname{Can}}]$ or equivalently $h((\delta_\ell^{\operatorname{Can}})^{-1} \delta_{\ell'}^{\operatorname{Can}}) \in h(\Psi^{\operatorname{Can}})$. Therefore, $(\delta_\ell^{\operatorname{Can}})^{-1} \delta_{\ell'}^{\operatorname{Can}} \in \Psi^{\operatorname{Can}} N^{\operatorname{Can}} = \Psi^{\operatorname{Can}}$. Thus, $\ell = \ell'$ which proves the claim.

claim. Define $\widehat{J}_0 := \widehat{J}_k$ for some arbitrarily chosen $k \in [r]$ (which can depend on the choice of $k \in [r]$).

▷ To compute \widehat{J}_0 , one can use the Schreier-Sims algorithm as follows. First, we pick $\rho_1\Psi^{\operatorname{Can}}$ and define \widehat{J}_0 as the part \widehat{J}_k such that $\rho_1\Psi^{\operatorname{Can}} \in \widehat{J}_k$. Then, we compute the coset $\Delta'_i \coloneqq (\Delta_i \rho_i \rho_1^{-1})_{(V \setminus A)}$ (which is non-empty) and pick an element $\delta' \in \Delta_i$ for each $\Delta_i \rho_i \in J$. Then, $\delta' \rho_1 \Psi^{\operatorname{Can}} \le \Delta_i \rho_i$ and $\delta' \rho_1 \Psi^{\operatorname{Can}}|_{V \setminus A} = \rho_1 \Psi^{\operatorname{Can}}|_{V \setminus A}$ and therefore $\delta' \rho_1 \Psi^{\operatorname{Can}}$ also belongs to \widehat{J}_0 . Therefore, we can compute the entire set \widehat{J}_0 . We claim that G is transitive on \widehat{J} and therefore $(\widehat{J}_0)^G = \widehat{\mathcal{J}}$. This follows from the fact that G is transitive on \widehat{J} and that $\widehat{\mathcal{J}}$ is an automorphism-invariant partition of \widehat{J} .

Compute $\widehat{\Lambda} := \mathrm{CL}_{\mathrm{Set}}(\widehat{J}_0, A, \Psi^{\mathrm{Can}}, \bot)$ recursively.

 $> As \ already \ observed, \ it \ holds \ that \ {\rm orb}_{A^{\rm Can}}(\Psi^{\rm Can}) \leq 2 \cdot {\rm orb}_{A^{\rm Can}}(\Pi^{\rm Can}) \leq 2|A^{\rm Can}|/|W^{\rm Can}| \leq \frac{1}{2} {\rm orb}_{A^{\rm Can}}(\Delta^{\rm Can}). \ This \ leads \ to \ progress \ ({\rm In} \ \Delta^{\rm Can}).$

Return $\Lambda := G\widehat{\Lambda}$.

 \triangleright Since $(\widehat{J_0})^G = \widehat{\mathcal{J}}$, it follows that $\operatorname{Aut}(\Lambda) = \operatorname{Aut}(\widehat{\mathcal{J}}) = \operatorname{Aut}(\widehat{\mathcal{J}}) = \operatorname{Aut}(J)$ by Lemma 33.

If $\Pi^{\operatorname{Can}}[V^{\operatorname{Can}} \setminus A^{\operatorname{Can}}] \leq \Delta^{\operatorname{Can}}[V^{\operatorname{Can}} \setminus A^{\operatorname{Can}}]$ has index 1 or 2:

For each $\Delta_i \rho_i \in J$, define $\Pi_i := \rho_i \Pi^{\operatorname{Can}} \rho_i^{-1} \leq \Delta_i \leq \operatorname{Sym}(V)$.

▷ The group Π_i does not depend on the representative of $\Delta_i \rho_i$, because the kernel $\Pi^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$ is a normal subgroup.

For each $\Delta_i \rho_i \in J$ define the Π_i -orbit partition $\mathcal{B}_i := \{B \subseteq A \mid B \text{ is a } \Pi_i\text{-orbit}\}\ \text{of } A.$

Define an (unordered) partition $\mathcal{P} := \{P_1, \dots, P_p\}$ of $J = P_1 \cup \dots \cup P_p$ such that:

 $\Delta_i \rho_i, \Delta_j \rho_j \in P_\ell$ for some $P_\ell \in \mathcal{P}$, if and only if $\mathcal{B}_i = \mathcal{B}_j$.

If P is a non-trivial partition:

Return $\Lambda := \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, g^{\text{Can}}, \{\mathcal{P}\})$ using Lemma 23.

 \triangleright We have $|\{\mathcal{P}\}| = 1$ which leads to progress (In J) or (Linear in J).

If \mathcal{P} is the partition into singletons:

 \triangleright This means that $\mathcal{B}_i \neq \mathcal{B}_j$ for all $\Delta_i \rho_i \neq \Delta_j \rho_j \in J$. We will define a non-trivial cover \mathcal{C} .

Define a cover $C := \{C_{vw} \mid (v, w) \in A^2\}$ of $J = \bigcup_{(v,w) \in A^2} C_{vw}$ such that:

 $\Delta_i \rho_i \in C_{vw}$, if and only if $\{v, w\} \subseteq B$ for some $B \in \mathcal{B}_i$.

Define $\mathbb{P} := \{\mathcal{P}_{vw}\}_{(v,w)\in A^2}$ as partition family induced by \mathcal{C} , i.e., $\mathcal{P}_{vw} := \{P_{vw,1}, P_{vw,2}\}$

where $P_{vw,1} := C_{vw}$ and $P_{vw,2} := J \setminus C_{vw}$ for $(v, w) \in A^2$.

Return $\Lambda := \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, g^{\text{Can}}, \mathbb{P})$ using Lemma 23.

 \triangleright We have $|\mathbb{P}| = |A^2| \le |V|^2$ which leads to progress (In J) or (Linear in J).

 \triangleright Now, the partition \mathcal{P} is the singleton partition. This means that $\mathcal{B}_i = \mathcal{B}_j$ for all $\Delta_i \rho_i, \Delta_j \rho_j \in J$.

Define $\mathcal{B} := \mathcal{B}_i$ for some $\Delta_i \rho_i \in J$.

 \triangleright The partition \mathcal{B} does not depend on the choice of $\Delta_i \rho_i \in J$.

Define an (unordered) partition $\mathcal{Q} := \{Q_1, \dots, Q_q\}$ of $J = Q_1 \cup \dots \cup Q_q$ such that: $\Delta_i \rho_i, \Delta_j \rho_j \in Q_\ell$ for some $Q_\ell \in \mathcal{Q}$, if and only if $(\Delta_i \rho_i)[\mathcal{B}] = (\Delta_j \rho_j)[\mathcal{B}]$.

If Q is a non-trivial partition:

Compute and return $\Lambda \coloneqq \text{RECURSEONPARTITION}(J, A, \Delta^{\text{Can}}, g^{\text{Can}}, \{Q\}).$

 \triangleright We have $|\{Q\}| = 1$ which leads to progress (In J) or (Linear in J).

If Q is a partition into singletons:

- \triangleright This means that $\Delta_i \rho_i[\mathcal{B}] \neq \Delta_j \rho_j[\mathcal{B}]$ for all $\Delta_i \rho_i \neq \Delta_j \rho_j \in J$.
- ▶ We will use the following strategy. We individualize a labeling coset $\Delta_1 \rho_1 \in J$ at a multiplicative cost of |J|. Then, we choose arbitrarily a subcoset $\rho_1 \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}} \leq \Delta_1 \rho_1$ (no multiplicative cost since the group of automorphisms G is transitive on the set of all possible chosen subcosets). Again, we individualize a subcoset $\Gamma_k := \rho_1 \delta_\ell^{\operatorname{Can}} \psi_k^{\operatorname{Can}} \Pi^{\operatorname{Can}} \leq \rho_1 \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}}$ at a multiplicative cost of 2. With respect to the individualized subcoset Γ_k , we can define a linear ordering on J and solve the canonization problem without further recursive calls.

For each $\Delta_i \rho_i \in J$ do:

 \triangleright We will compute a canonical labeling for $(J, \Delta_i \rho_i)$.

We consider (but not compute) $\widehat{J}_i := \{ \rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \mid \ell \in [s] \}$ of $\Delta_i \rho_i$.

Define $\Gamma_{i,0} := \rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J_i}$ for some arbitrarily chosen $\ell \in [s]$ (which can depend on the choice of $\ell \in [s]$).

 \triangleright We will compute a canonical labeling for $(J, \Gamma_{i,0})$. Again, G is transitive on \widehat{J}_i and therefore $\Gamma_{i,0}^G = \widehat{J}_i$.

Compute the decomposition of $\Psi^{\operatorname{Can}} = \psi_1^{\operatorname{Can}} \Pi^{\operatorname{Can}} \cup \psi_2^{\operatorname{Can}} \Pi^{\operatorname{Can}}$ into left cosets. Decompose $\Gamma_{i,0} = \Gamma_{i,0,1} \cup \Gamma_{i,0,2}$ where $\Gamma_{i,0,2} \coloneqq \rho_i \delta_\ell^{\operatorname{Can}} \psi_k^{\operatorname{Can}} \Pi^{\operatorname{Can}} \le \Delta_i \rho_i$ for k = 1, 2. For each k = 1, 2 do:

 \triangleright We will compute a canonical labeling for $(J, \Gamma_{i,0,k})$.

Compute $\Theta_{i,0,k,j}\tau_{i,0,k,j} := \text{CL}_{\text{Set}}(\Gamma_{i,0,k}, \Delta_j \rho_j)$ using Lemma 7 or Corollary 11 for each $\Delta_j \rho_j \in J$.

Rename indices [t] such that: $(\Gamma_{i,0,k}, \Delta_1 \rho_1)^{\tau_{i,0,k,1}} < \ldots < (\Gamma_{i,0,k}, \Delta_t \rho_t)^{\tau_{i,0,k,t}}$.

 $\begin{array}{ll} \triangleright \ \ We \ \ claim \ \ that \ \ the \ \ ordering \ \ is \ strict. \ \ Assume \ that \ (\Gamma_{i,0,k},\Delta_{j}\rho_{j})^{\tau_{i,0,k,j}} = \\ (\Gamma_{i,0,k},\Delta_{j'}\rho_{j'})^{\tau_{i,0,k,j'}}. \qquad On \ \ the \ \ one \ \ side, \ \ \Gamma_{i,0,k}^{\tau_{i,0,k,j}} = \ \Gamma_{i,0,k}^{\tau_{i,0,k,j'}} \ \ implies \\ \tau_{i,0,k,j'}\tau_{i,0,k,j}^{-1}[\mathcal{B}] = 1 \ \ and \ \ on \ \ the \ \ other \ \ side \ (\Delta_{j}\rho_{j})^{\tau_{i,0,k,j}} = (\Delta_{j'}\rho_{j'})^{\tau_{i,0,k,j'}} \\ implies \ \tau_{i,0,k,j'}\tau_{i,0,k,j}^{-1}[\mathcal{B}] \in \rho_{j'}\Delta^{\operatorname{Can}}\rho_{j}^{-1}[\mathcal{B}]. \ \ Since \ \Delta_{j}\rho_{j}[\mathcal{B}] \neq \Delta_{j'}\rho_{j'}[\mathcal{B}] \ \ for \ \ \ all \ \Delta_{j}\rho_{j} \neq \Delta_{j'}\rho_{j'} \in J, \ \ it \ follows \ \ that \ j=j' \ \ which \ \ proves \ \ the \ \ claim. \end{array}$

Define $\Theta_{i,0,k}\tau_{i,0,k} := \mathrm{CL}_{\mathrm{Object}}((\Delta_1\rho_1,\ldots,\Delta_t\rho_t))$ using Corollary 11.

 \triangleright Observe that $\Theta_{i,0,k}\tau_{i,0,k}$ defines a canonical labeling for $(J,\Gamma_{i,0,k})$.

$$\text{Define } \Theta_{i,0} \coloneqq \begin{cases} \langle \Theta_{i,0,1} \tau_{i,0,1}, \Theta_{i,0,2} \tau_{i,0,2} \rangle, & \text{if } J^{\tau_{i,0,1}} = J^{\tau_{i,0,2}} \\ \Theta_{i,0,k} \tau_{i,0,k}, & \text{if } J^{\tau_{i,0,k}} \prec J^{\tau_{i,0,3-k}} \, . \end{cases}$$

 \triangleright Observe that $\Theta_{i,0}\tau_{i,0}$ defines a canonical labeling for $(J,\Gamma_{i,0})$.

Define $\Theta_i \tau_i := G\Theta_{i,0} \tau_{i,0}$.

 \triangleright We claim that $\Theta_i \tau_i$ defines a canonical labeling for $(J, \Delta_i \rho_i)$. By Lemma 33, we have that $\Theta_i \tau_i$ defines a canonical labeling for $(J, \widehat{J_i})$ since $\Gamma_{i,0}^G = \widehat{J_i}$. Moreover, $\widehat{J_i}$ is an isomorphism-invariant partition of $\Delta_i \rho_i$ which proves the claim.

 \triangleright Next, we compute a canonical labeling Λ for J.

Define $J^{\text{Set}} := \{ \Theta_i \tau_i \mid \Delta_i \rho_i \in J \}$

 \triangleright We collect the canonical labelings $\Theta_i \tau_i$ leading to minimal canonical forms of the input.

Define $J_{\min}^{\text{Set}} \coloneqq \arg\min_{\Theta_i \tau_i \in J^{\text{Set}}} J^{\tau_i} \subseteq J^{\text{Set}}$ where the minimum is taken w.r.t. the ordering "<" from Lemma 1.

Return $\Lambda := \langle J_{\min}^{\text{Set}} \rangle$.

 \triangleright This is the smallest coset containing labeling cosets in J_{\min}^{Set} as defined in the preliminaries. The correctness proof for (CL2) is similar to the (CL2)-proof of Lemma 19.

This means that $\Delta_i \rho_i[\mathcal{B}] = \Delta_i \rho_i[\mathcal{B}]$ for all \triangleright Now, Q is the singleton partition. $\Delta_i \rho_i, \Delta_j \rho_j \in J.$

Define $\mathcal{B}^{\operatorname{Can}} := \{B_1^{\operatorname{Can}}, \dots, B_b^{\operatorname{Can}}\}$ as the Π^{Can} -orbit partition of A^{Can} .

 \triangleright By definition, $\mathcal{B}^{\rho_i} = \mathcal{B}^{\operatorname{Can}}$ for all $\Delta_i \rho_i \in J$.

Define the homomorphism $h: \Delta^{\operatorname{Can}} \to \Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}].$

Define $N^{\operatorname{Can}} \coloneqq \ker(h) \leq \Delta^{\operatorname{Can}}$ as the kernel of the homomorphism h.

 \triangleright Again, we claim that $N^{\operatorname{Can}} \leq \Pi^{\operatorname{Can}}$. Since $\Pi^{\operatorname{Can}}, N^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$, we have that $\Pi^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$ $\Pi^{\operatorname{Can}} N^{\operatorname{Can}} \leq \Delta^{\operatorname{Can}}$. Observe that $\Delta^{\operatorname{Can}}/\Pi^{\operatorname{Can}}$ is isomorphic to a giant and all normal subgroups of a giant with index greater than 2 are trivial. We have $(\Delta^{\operatorname{Can}}:$ $\Pi^{\operatorname{Can}}N^{\operatorname{Can}}) \geq (h(\Delta^{\operatorname{Can}}):h(\Pi^{\operatorname{Can}}N^{\operatorname{Can}})) = |h(\Delta^{\operatorname{Can}})|.$ Since $\Delta^{\operatorname{Can}}$ is transitive on A^{Can} (Property (g)) it holds that $h(\Delta^{\operatorname{Can}}) = \Delta^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]$ is transitive on $\mathcal{B}^{\operatorname{Can}}$ and therefore $h(\Delta^{\operatorname{Can}})| \geq |\mathcal{B}^{\operatorname{Can}}|$. By the Affected Orbits Lemma 30, each $B_i^{\operatorname{Can}} \in \mathcal{B}^{\operatorname{Can}}$ has size at most $|A^{\operatorname{Can}}|/|W^{\operatorname{Can}}|$ and therefore $|\mathcal{B}^{\operatorname{Can}}| \geq |W^{\operatorname{Can}}|$. Moreover, $|W^{\operatorname{Can}}| \geq 3$ is assumed the be greater than some absolute constant. By the Correspondence Theorem, $\Pi^{\operatorname{Can}} N^{\operatorname{Can}} = \Pi^{\operatorname{Can}}$ which proves the claim.

We consider (but not compute) the decomposition $\widehat{\mathcal{J}} := \{\widehat{J}_1, \dots, \widehat{J}_r\}$ of $\widehat{J} = \widehat{J}_1 \cup \dots \cup \widehat{J}_r$

 $\rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}, \rho_{i'} \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_k \text{ for some } \widehat{J}_k \in \mathcal{J}, \text{ iff } \rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}[\mathcal{B}] = \rho_{i'} \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}}[\mathcal{B}].$

 \triangleright Again, we claim that $|\widehat{J}_k| = |J|$ and $\operatorname{Aut}(\widehat{J}_k) \leq \operatorname{Aut}(J)$. We show a stronger statement, $i.e., \ for \ each \ k \in [r], \Delta_i \rho_i \in J \ \ there \ \ is \ \ at \ \ exactly \ \ one \ \ell \in [s] \ \ such \ \ that \ \ \rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J_k}.$ Because of $\Delta_i \rho_i[\mathcal{B}] = \Delta_j \rho_j[\mathcal{B}]$, it holds for all $k \in [r], \Delta_i \rho_i \in J$ there is at least one $\ell \in [s]$ such that $\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_k$. On the other side, let $\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}}, \rho_i \delta_{\ell'}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_k$ \widehat{J}_k . By definition of \widehat{J}_k , it holds $((\delta_\ell^{\operatorname{Can}})^{-1}\delta_{\ell'}^{\operatorname{Can}})[\mathcal{B}^{\operatorname{Can}}] \in \Psi^{\operatorname{Can}}[\mathcal{B}^{\operatorname{Can}}]$ or equivalently $h((\delta_\ell^{\operatorname{Can}})^{-1}\delta_{\ell'}^{\operatorname{Can}}) \in h(\Psi^{\operatorname{Can}})$. Therefore, $(\delta_\ell^{\operatorname{Can}})^{-1}\delta_{\ell'}^{\operatorname{Can}} \in \Psi^{\operatorname{Can}}N^{\operatorname{Can}} = \Psi^{\operatorname{Can}}$. Thus, $\ell = \ell'$ which proves the claim.

Define $\widehat{J}_0 := \widehat{J}_k$ for some arbitrarily chosen $k \in [r]$ (which can depend on the choice of $k \in [r]$).

 $Again, G is transitive on \widehat{\mathcal{J}} and therefore (\widehat{J}_0)^G = \widehat{\mathcal{J}}.$ Define $K := \{(\rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}})|_{V \setminus A} \mid \rho_i \delta_\ell^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_0\}.$

If |K| = 1:

 \triangleright In this case, Property (A) is satisfied for J_0 .

Compute $\widehat{\Lambda} := \mathrm{CL}_{\mathrm{Set}}(\widehat{J_0}, A, \Psi^{\mathrm{Can}}, \bot)$ recursively.

 $\triangleright As \ before, \ it \ holds \ that \ \operatorname{orb}_{A^{\operatorname{Can}}}(\Psi^{\operatorname{Can}}) \leq \frac{1}{2}\operatorname{orb}_{A^{\operatorname{Can}}}(\Delta^{\operatorname{Can}}).$ This leads to progress (In $\Delta^{\operatorname{Can}}$).

Return $\Lambda := G\widehat{\Lambda}$.

 \triangleright Since $(\widehat{J_0})^G = \widehat{\mathcal{J}}$, follows that $\operatorname{Aut}(\Lambda) = \operatorname{Aut}(\widehat{\mathcal{J}}) = \operatorname{Aut}(\widehat{\mathcal{J}}) = \operatorname{Aut}(J)$ by Lemma 33.

 \triangleright Actually, we are in a case in which |K| = 2 since we are still in the case in which $\Pi^{\operatorname{Can}}[V^{\operatorname{Can}} \smallsetminus A^{\operatorname{Can}}] \leq \Delta^{\operatorname{Can}}[V^{\operatorname{Can}} \smallsetminus A^{\operatorname{Can}}] \ \ has \ \ index \ \ 1 \ \ or \ \ 2.$

For both $k \in K$, define $\widehat{J}_{0,k} := \{ \rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}} \in \widehat{J}_0 \mid (\rho_i \delta_{\ell}^{\operatorname{Can}} \Psi^{\operatorname{Can}}) \mid_{V \setminus A} = k \}.$

 \triangleright Now, Property (A) is satisfied for both $\widehat{J}_{0,k}$.

Compute $\Theta_k \tau_k := \mathrm{CL}_{\mathrm{Set}}(\widehat{J}_{0,k}, A, \Psi^{\mathrm{Can}}, \bot)$ recursively for both $k \in K$. Again, $\mathrm{orb}_{A^{\mathrm{Can}}}(\Psi^{\mathrm{Can}}) \leq \frac{1}{2}\mathrm{orb}_{A^{\mathrm{Can}}}(\Delta^{\mathrm{Can}})$. The multiplicative cost is 2 which leads

to progress (In $\Delta^{\operatorname{Can}}$).

Define $\widehat{\mathcal{J}}_0^{\operatorname{Set}} := \{ (\Theta_k \tau_k, (\widehat{J}_{0,k})^{\tau_k}) \mid k \in K \}.$

Compute $\widehat{\Lambda} := \mathrm{CL}_{\mathrm{Object}}(\widehat{\mathcal{J}}_0^{\mathrm{Set}})$ using Corollary 11.

 \triangleright By Lemma 3, it follows that $\widehat{\Lambda}$ defines a canonical labeling for \widehat{J}_0 .

Return $\Lambda := G\widehat{\Lambda}$.

 \triangleright Since $(\widehat{J_0})^G = \widehat{\mathcal{J}}$, follows that $\operatorname{Aut}(\Lambda) = \operatorname{Aut}(\widehat{\mathcal{J}}) = \operatorname{Aut}(\widehat{\mathcal{J}}) = \operatorname{Aut}(J)$ by Lemma 33.

We have all tools together to give the algorithm for Theorem 22.

Proof of Theorem 22. An algorithm for $\text{CL}_{\text{Set}}(J, A, \underline{\Delta}^{\text{Can}}, g^{\text{Can}})$:

If Property (B) is not satisfied:

We recurse as described in the beginning of this section.

If $q^{\operatorname{Can}} = \bot$ is undefined:

Recurse and return $\Lambda := \text{REDUCEToJOHNSON}(J, A, \Delta^{\text{Can}}, \bot)$ using Lemma 28.

If g^{Can} is defined:

Apply the subroutine PRODUCE CERTIFICATES $(J, A, \Delta^{\operatorname{Can}}, g^{\operatorname{Can}})$ using Lemma 32.

If the subroutine returns a certificate of fullness $G \leq \operatorname{Sym}(V)$:

Return $\Lambda := AGGREGATE CERTIFICATES(J, A, \Delta^{Can}, g^{Can}, G)$ using Lemma 34.

If the subroutine finds a canonical labeling Λ using recursion:

Return Λ .

(Running time.) The number of recursive calls of the algorithm CL_{Set} is bounded $T \leq (|V| +$ |J|) polylog |V| where T is the function given in (T). Also each recursive call takes time bounded in $(|V| + |J|)^{\operatorname{polylog}|V|}$.

By improving the running time of Problem 8, we also obtain an improved version of Corollary 11.

Corollary 35. Canonical labelings for combinatorial objects can be computed in time $n^{\text{polylog}|V|}$ where n is the input size and V is the ground set of the object.

7 Isomorphism of Graphs Parameterized by Treewidth

Graph Theory We write $N_G(v) := \{w \in V(G) \mid \{v, w\} \in E(G)\}$ to denote the (open) neighborhood of $v \in V(G)$ in a graph G. We also write $N_G(S) := \bigcup_{v \in S} N_G(v)$ to denote the (open) neighborhood of a subset $S \subseteq V(G)$. We write G[U] to denote the subgraph induced by $U \subseteq V(G)$ in G.

Definition 36 (Tree Decomposition). A tree decomposition of a graph G is a pair (T,β) where T is a tree and $\beta: V(T) \to 2^{V(G)}$ is a function that assigns each node $t \in V(T)$ a subset $\beta(t) \subseteq V(G)$, called bag, such that:

(T1) for each vertex $v \in V(G)$, the induced subtree $T[\{t \in V(T) \mid v \in \beta(t)\}]$ is non-empty and connected, and

(T2) for each edge $e \in E(G)$, there exists $t \in V(T)$ such that $e \subseteq \beta(t)$.

The sets $\beta(s) \cap \beta(t)$ for $\{s,t\} \in E(T)$ are called the *adhesion sets*. The *width* of a tree decomposition T is equal to its maximum bag size decremented by one, i.e. $\max_{t \in V(T)} |\beta(t)| - 1$. The *treewidth* of a graph, denoted by tw G, is equal to the minimum width among all its tree decompositions.

Separations and Separators Let G = (V, E) be a graph and let $v, w \in V(G)$. A pair (A, B) is called a (v, w)-separation if $A \cup B = V(G)$ and $v \in A \setminus B, w \in B \setminus A$ and there are no edges with one vertex in $V(G) \setminus A$ and the other vertex in $V(G) \setminus B$. In this case, $A \cap B$ is called a (v, w)-separator. A separator $A \cap B$ is called clique separator if $A \cap B$ is a clique in G. Among all (v, w)-separations (A, B) with minimal $|A \cap B|$ there is a unique separation (A^*, B^*) with an inclusion minimal A^* . In this case, $S_{v,w} := A^* \cap B^*$ is called the leftmost minimal (v, w)-separator. It is known that $S_{v,w}$ can be computed in polynomial time using the Ford-Fulkerson algorithm.

Improved Graphs The k-improvement of a graph G is the graph G^k obtained from G by connecting every pair of non-adjacent vertices v, w for which there are more than k pairwise internally vertex disjoint paths connecting v and w^2 . The separability of a graph G, denoted by $\operatorname{sep} G$, is the smallest integer k such that $G^k = G$. Equivalently, $\operatorname{sep} G$ equals the maximum size $|S_{v,w}|$ of a leftmost minimal separator among all non-adjacent vertices $v, w \in V(G)$.

The next lemma says that one can k-improve a graph for some $k \ge \operatorname{tw} G$ and reduce the separability of that graph while preserving the treewidth of that graph.

Lemma 37 ([LPPS17]). Let G be a graph and $k \in \mathbb{N}$.

- 1. There is a polynomial-time algorithm that for a given (G,k) computes G^k .
- 2. It holds that $(G^k)^k = G^k$ and therefore $\operatorname{sep} G^k \leq k$.
- 3. Every tree decomposition of G of width at most k is also a tree decomposition of G^k and therefore $\operatorname{tw} G \leq k$ implies $\operatorname{tw} G^k = \operatorname{tw} G$.

The next theorem says that one can decompose a graph into clique-separator-free graphs. By possibly introducing new bags, we can assume that the adhesion sets inside each bag are either pairwise equal or pairwise distinct. This ensures the third property in the following theorem.

Theorem 38 ([Lei93],[ES16]). Let G be a graph. There is an algorithm that, given a graph G, computes a tree decomposition (T,β) with the following properties.

- 1. For every $t \in V(T)$ the graph $G[\beta(t)]$ is clique-separator free,
- 2. each adhesion set of (T,β) is a clique in G, and
- 3. for each bag $\beta(t)$ either the adhesion sets are all equal and $|\beta(t)| \leq (\operatorname{tw} G) + 1$ or the adhesion sets are pairwise distinct.

The algorithm runs in polynomial time and the output of the algorithm is isomorphism-invariant.

 $^{^{2}}$ In [LPPS17], a slightly different notion of improvement is used where an edge is also added when there are exactly k pairwise internally vertex disjoint paths connecting non-adjacent vertices.

We make use of the bounded-degree graph isomorphism algorithm given by Grohe, Neuen and Schweitzer. In fact, they proved a stronger statement and designed a string isomorphism algorithm for groups of bounded composition-width. This implies the following result.

Theorem 39 ([GNS18]). Let G_1, G_2 be two graphs and let $\Delta \varphi \leq \text{Iso}(V(G_1); V(G_2))$. There is an algorithm that, given a triple $(G_1, G_2, \Delta \varphi)$, computes the set of isomorphisms $\text{Iso}(G_1; G_2) \cap \Delta \varphi$ in time $|V(G_1)|^{\text{polylog}(\text{cw }\Delta)}$.

We give an isomorphism algorithm for the clique-separator-free graphs. The algorithm uses the ideas from [GNSW18].

Lemma 40. Let G_1, G_2 be two clique-separator-free graphs. There is an algorithm that, given a pair (G_1, G_2) , computes the set of isomorphisms $\operatorname{Iso}(G_1; G_2)$ in time $|V(G_1)|^{\operatorname{polylog}(\operatorname{tw} G_1 + \operatorname{sep} G_1)}$. Moreover, there is a vertex $v_1 \in V(G_1)$ such that $\operatorname{cw} \operatorname{Aut}(G_1)_{(v_1)} \leq \max(\operatorname{tw} G_1, \operatorname{sep} G_1)$.

Proof. Let minDeg $G_i := \min_{v \in V(G_i)} |N_{G_i}(v)|$ be the minimal degree among all vertices. It is well-known that minDeg $G_i \le \operatorname{tw} G_i$ for all graphs G_i . Let $\mathcal{S}_i := \{N_{G_i}(v) \subseteq V(G_1) \mid v \in V(G_i), |N_{G_i}(v)| = \min \operatorname{Deg} G_i\}$ be the non-empty set of minimal size neighborhoods for both i = 1, 2. We assume $|\mathcal{S}_1| = |\mathcal{S}_2|$, otherwise we reject isomorphism. Since G_i does not have clique separators, it follows that each $S_i \in \mathcal{S}_i$ is not a clique for both i = 1, 2. Since $\operatorname{Iso}(G_1; G_2) = \bigcup_{S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2} \operatorname{Iso}(G_1, S_1; G_2, S_2)$, it suffices to compute the isomorphisms from G_1 to G_2 that map S_1 to S_2 for all possible choices of $S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2$.

We give an algorithm that gets as input $(G_1, S_1, G_2, S_2, \Delta \varphi)$ where $S_i \subseteq V(G_i)$ is not a clique for both i = 1, 2 and $\Delta \varphi \ge \text{Iso}(G_1, S_1; G_2, S_2)[S_1]$ with $\text{cw } \Delta \le \text{max}(\text{tw } G_1, \text{sep } G_1)$. The algorithm outputs $\text{Iso}(G_1, S_1; G_2, S_2)$. Initially, we call the algorithm for some $S_1 \in S_1, S_2 \in S_2$ and $\Delta \varphi := \text{Sym}(S_1) \varphi$ for some bijection $\varphi : S_1 \to S_2$.

An algorithm for $Iso_{Basic}(G_1, S_1, G_2, S_2, \Delta\varphi)$:

If $S_1 \subseteq V(G_1)$:

Let $S_i' := S_i \cup \bigcup_{v,w \in S_i, \{v,w\} \notin E(G_i)} S_{v,w}$ for both i = 1, 2. We claim that $S_i' \not\supseteq S_i$ for both i = 1, 2. Let $Z_i \subseteq V(G_i)$ be the vertex set of a connected component of $G_i - S_i$. Since G_i does not have clique separators, it follows that $N_{G_i}(Z_i)$ is not a clique. Therefore, there are $v, w \in N_{G_i}(Z_i) \subseteq S_i$ with $\{v, w\} \notin E(G_i)$. Moreover, there is a path from v to w with all internally vertices lying in Z_i . Therefore, $S_{v,w} \cap Z_i \neq \emptyset$ and thus $S_i' \supseteq S_i \cup (S_{v,w} \cap Z_i) \supseteq S_i$. Observe that $|S_{v,w}| \le \operatorname{sep} G_i$ for all $v, w \in S_i$ and both i = 1, 2.

First, we ensure for all $\varphi \in \Delta \varphi$ that $S_{v,w}$ and $S_{\varphi(v,w)}$ have the same cardinality. To do so, we define an edge relation $X_i^k := \{(v,w) \mid |S_{v,w}| = k\}$ for each $k \leq n := |V(G_i)|$ and both i = 1, 2. We compute $\Delta \varphi := \text{Iso}(X_1^1, \ldots, X_1^n; X_2^1, \ldots, X_2^n) \cap \Delta \varphi$ using Theorem 39.

Second, we define a wreath product with $\operatorname{Sym}(S_{v,w})$ and Δ . More precisely, we define $\widehat{S}_i := S_i \cup \bigcup_{v,w \in S_i, \{v,w\} \notin E(G_i)} \widehat{S}_{v,w}$ where $\widehat{S}_{v,w} := S_{v,w} \times \{(v,w)\}$ is a disjoint copy of $S_{v,w}$ for both i = 1, 2. We define $\widehat{\Delta}\widehat{\varphi} \leq \operatorname{Iso}(\widehat{S}_1; \widehat{S}_2)$ as $\{\widehat{\varphi} : \widehat{S}_1 \to \widehat{S}_2 \mid \widehat{\varphi}[S_1] \in \Delta\varphi, \forall v, w \in S_i : \widehat{\varphi}(\widehat{S}_{v,w}) = \widehat{S}_{\widehat{\varphi}(v),\widehat{\varphi}(w)}\}$. Observe that $\operatorname{cw}\widehat{\Delta} \leq \max(\max_{v,w \in S_1} |S_{v,w}|, \operatorname{cw}\Delta) \leq \max(\operatorname{tw} G_1, \operatorname{sep} G_1)$.

Third, we define the group $\Delta'\varphi' \geq \operatorname{Iso}(G_1, S_1'; G_2, S_2')[S_1']$ by identifying the corresponding vertices. More precisely, we define an edge relation $X_i := \{((s, v, w), (s, v', w')) \in \widehat{S}_{v, w} \times \widehat{S}_{v', w'}\} \cup \{((s, v, w), s) \in \widehat{S}_{v, w} \times S_i\}$ for both i = 1, 2. Observe that S_i' can be identified with the equivalence classes of X_i for both i = 1, 2. Now, compute $\Delta'\varphi' := (\operatorname{Iso}(X_1; X_2) \cap \widehat{\Delta}\widehat{\varphi})[S_1']$ using Theorem 39.

Finally, we compute and return Iso_{Basic} $(G_1, S'_1, G_2, S'_2, \Delta'\varphi')$ recursively.

If $S_1 = V(G_1)$:

Compute and return $\operatorname{Iso}(G_1; G_2) \cap \Delta \varphi$ using Theorem 39.

(Running time.) The number of recursive calls is bounded by $|V(G_1)|$. In each call, we use the algorithm from Theorem 39 which runs in time $|V(G_1)|^{\text{polylog(tw}\,G_1+\text{sep}\,G_1)}$.

With the above algorithm it is possible to compute the isomorphisms between the clique-separator-free parts of the decomposition from Theorem 38. The adhesion sets (which are the intersections between two clique-separator-free graphs) are cliques in the graph. The next lemma is used in order to respect the adhesion sets of the clique-separator-free parts. Also this lemma uses an idea similar to [GNSW18], Lemma 14 arXiv version.

Lemma 41. Let G_1, G_2 be two clique-separator-free graphs and let $H_1 \subseteq 2^{V(G_1)}, H_2 \subseteq 2^{V(G_2)}$ be sets that contain cliques in the graphs G_1, G_2 , respectively. There is an algorithm that, given a tuple (G_1, H_1, G_2, H_2) , computes the set of isomorphisms $\operatorname{Iso}(G_1, H_1; G_2, H_2)$ in time $|V(G_1)|^{\operatorname{polylog}(\operatorname{tw} G_1 + \operatorname{sep} G_1)}$.

Proof. In the first step, we define a cover capturing all cliques. More precisely, we claim that there is a function $\alpha: V(G) \to K$ where $K \subseteq 2^{V(G)}$ such that

- 1. $|S| \le (\operatorname{tw} G) + 1$ for all $S \in K$, and
- 2. if $C \subseteq V(G)$ is a clique, then there is a $S \in K$ with $C \subseteq S$.

Moreover, this function is polynomial-time computable and is defined in an isomorphism-invariant way. This can be shown by induction on |V(G)|. If $|V(G)| \le 1$ the statement is trivially true. So assume $|V(G)| \ge 2$. Let minDeg $G := \min_{v \in V(G)} |N_G(v)|$ be the minimal degree among all vertices. It is well-known that minDeg $G \le \operatorname{tw} G$. Let $U := \arg\min_{v \in V(G)} |N_G(v)| \subseteq V(G)$ be the non-empty set of vertices of minimal degree. Let $G' := G[V(G) \setminus U]$ and let $\alpha' : V(G') \to K'$ be the function obtained inductively. Now, define $\alpha : V(G) \to K$ as $\alpha(v) := \alpha'(v)$ if $v \in V(G')$ and as $\alpha(v) := N_G(v) \cup \{v\}$ if $v \in U$. It easily follows that $|\alpha(v)| \le \operatorname{tw} G_1 + 1$. Moreover, if $C \subseteq V(G)$ is a clique, then either $C \subseteq V(G')$ or there is a vertex $v \in C \cap U$. In the latter case, $C \subseteq N_G(v) \cup \{v\} = \alpha(v)$. This proves the claim. We will use this claim in order to compute the isomorphisms between the instances.

First of all, we compute the isomorphism $\Delta \varphi := \text{Iso}(G_1; G_2)$ between the two graphs G_1 and G_2 using Lemma 40.

It remains to respect the hyperedges. We compute $\alpha_i : V(G_i) \to K_i$ for both i = 1, 2. We define hypergraphs $H_{v_i} := \{S \in H_i \mid S \subseteq \alpha_i(v_i)\} \subseteq H_i$ for $v_i \in V(G_i)$ and both i = 1, 2. For each pair of vertices $(v_1, v_2) \in V(G_1) \times V(G_2)$, we compute the isomorphisms $\Delta_{v_1} \varphi_{v_1, v_2} := \text{Iso}(H_{v_1}; H_{v_2})$ (seen as hypergraphs on $\alpha_1(v_1), \alpha_2(v_2)$, respectively) using Theorem 15. Since $|\alpha_1(v_1)| \le (\text{tw } G_1) + 1$, the algorithm runs in the desired time bound.

First, we ensure for all $\varphi \in \Delta \varphi$ that the hypergraphs H_{v_1} and $H_{\varphi(v_1)}$ are isomorphic. To do so, we define a vertex-colored graph X_i that colors a vertex $v_i \in V(X_i)$ according to the isomorphism type of H_{v_i} for both i=1,2. We compute $\Delta \varphi \coloneqq \operatorname{Iso}(X_1; X_2) \cap \Delta \varphi$ using Theorem 39. To analyze the running time of the algorithm from Theorem 39, we observe that $\operatorname{cw} \Delta$ is not necessarily bounded by $\max(\operatorname{tw} G_1, \operatorname{sep} G_1)$. However, there is a point $v_1 \in V(G_1)$ such that $\operatorname{cw} \Delta_{(v_1)} \le \max(\operatorname{tw} G_1, \operatorname{sep} G_1)$. By applying Theorem 39 to each coset of the subgroup $\Delta_{(v_1)}$, we still achieve a running time of $|V(G_1)|^{\operatorname{polylog}(\operatorname{tw} G_1 + \operatorname{sep} G_1)}$.

Second, we define a wreath product $\Delta_{v_1}\varphi_{v_1,v_2}$ with $\Delta\varphi$. More precisely, we define $U_i := V(G_i) \cup \bigcup_{v \in V(G_i)} S_v$ where $S_v := \alpha_i(v) \times \{v\}$ is a disjoint copy of $\alpha_i(v)$ for both i = 1, 2. We define

 $\widehat{\Delta}\widehat{\varphi} \leq \operatorname{Iso}(U_1; U_2) \text{ as } \{\widehat{\varphi} : U_1 \to U_2 \mid \widehat{\varphi}[V(G_1)] \in \Delta \varphi, \forall v_1 \in V(G_1) \exists \varphi_{v_1} \in \Delta_{v_1} \varphi_{v_1, \widehat{\varphi}(v_1)} : \widehat{\varphi}(u, v_1) = \emptyset\}$ $(\varphi_{v_1}(u),\widehat{\varphi}(v_1))$. Again, there is a point $v_1 \in V(G_1)$ such that $\operatorname{cw}\widehat{\Delta}_{(v_1)} \leq \max((\operatorname{tw} G_1) + \operatorname{tw} G_1)$ 1, sep G_1), which can be seen as follows. Consider the homomorphism h that restricts $\widehat{\Delta}_{(v_1)}$ to the set $V(G_1)$. By construction, the image of h is $\Delta_{(v_1)}$ which composition-width is bounded by $\max(\operatorname{tw} G_1, \operatorname{sep} G_1)$. Moreover, the kernel of h has orbits bounded by $|S_v| = |\alpha(v)| \le (\operatorname{tw} G_1) + 1$ for some $v \in V(G_1)$, which gives us the bound.

Third, we define the group $\Delta \varphi := \text{Iso}(G_1, H_1; G_2, H_2)$ by identifying the corresponding vertices. More precisely, we define an edge relation $X_i := \{((u,v),(u,v')) \in S_v \times S_{v'}\} \cup \{((u,v),u) \in S_v \times S_v \times$ $S_v \times V(G_i)$ for both i = 1, 2. Observe that V_i can be identified with the equivalence classes of X_i for both i=1,2. Then, we compute and return $\Delta \varphi := (\operatorname{Iso}(X_1; X_2) \cap \widehat{\Delta \varphi})[V(G_1)]$ using Theorem 39. By applying Theorem 39 to each coset of $\Delta_{(v)} \leq \Delta$, we still achieve a running time of $|V(G_1)|^{\operatorname{polylog}(\operatorname{tw} G_1 + \operatorname{sep} G_1)}$.

We recall the coset-labeled hypergraphs from Lemma 21. A coset-labeled hypergraph is a tuple (V, H, J, α) where H is a set of hyperedges $S_i \subseteq V$ and J is a set of labeling cosets $\Delta_i \varphi_i \leq \text{Label}(V)$ and $\alpha: H \to J$ is a function with $\alpha(S_i) = \Delta_i \varphi_i$.

Lemma 42 ([GNSW18], Lemma 9, see [Neu19]). Let $\mathcal{H}_1 = (V_1, H_1, J_1, \alpha_1), \mathcal{H}_1 = (V_2, H_2, J_2, \alpha_2)$ be two coset-labeled hypergraphs and let $\Delta \varphi \leq \operatorname{Iso}(H_1; H_2)$ be a coset that maps the hyperedges in H_1 to hyperedges in H_2 . There is an algorithm that, given a triple $(\mathcal{H}_1, \mathcal{H}_2, \Delta \varphi)$, computes the set of isomorphisms $\operatorname{Iso}(\mathcal{H}_1; \mathcal{H}_2) \cap \Delta \varphi$ in time $(|V_1| + |H_1|)^{\operatorname{polylog}(\operatorname{cw} \Delta)}$.

Theorem 43. Let G_1, G_2 be two connected graphs. There is an algorithm that, given a pair (G_1,G_2) , computes the set of isomorphisms $\operatorname{Iso}(G_1;G_2)$ in time $|V(G_1)|^{\operatorname{polylog}(\operatorname{tw} G_1)}$.

Proof. It is known that the treewidth can be approximated (up to a logarithmic factor) in polynomial time [Ami01]. Let $k \in (\operatorname{tw} G_1)^{\mathcal{O}(1)}$ be such an upper bound on the treewidth of G_1 . We compute the k-improved graphs G_i^k using Lemma 37 for both i = 1, 2. We compute the tree decompositions (T_i, β_i) from Theorem 38 for the k-improvements G_i^k for both i = 1, 2. In particular, (T_i, β_i) is also a tree decomposition for G_i for both i = 1, 2. Let (T_i, β_i, r_i) denote the tree decomposition of G_i rooted at $r_i \in V(T_i)$ for both i = 1, 2. It suffices to give an algorithm for rooted tree decompositions. We give an algorithm that gets as input $(\widehat{\mathcal{X}}_1, \widehat{\mathcal{X}}_2)$ where $\widehat{\mathcal{X}}_i = (\widehat{G}_i, T_i, \beta_i, r_i, S_i)$ for both i = 1, 2 and outputs the set of isomorphisms Iso $(\widehat{\mathcal{X}}_1; \widehat{\mathcal{X}}_2)$.

An algorithm for $\text{Iso}_{\text{Tree}}(\widehat{\mathcal{X}}_1, \widehat{\mathcal{X}}_2)$: Define $G_i := \widehat{G}_i[\beta_i(r_i)]$ as the induced subgraph for both i = 1, 2.

Define $Y_i := \widehat{G}_i^k[\beta_i(r_i)]$ as the induced subgraph of the k-improved graph \widehat{G}_i^k for both i = 1, 2.

Define $C_i := \{t_{i,1}, \dots, t_{i,|C_i|}\} \subseteq V(T_i)$ as the set of children of r_i for both i = 1, 2.

Define $H_i := \{\beta_i(r_i) \cap \beta_i(t) \mid t \in C_i\}$ as the set of adhesion sets of the root r_i for both i = 1, 2.

Let $(T_{i,t}, \beta_{i,t})$ be the tree decomposition of the subtree rooted at $t \in C_i$ and let $\widehat{G}_{i,t}$ be the subgraph of \widehat{G}_i corresponding to $(T_{i,t}, \beta_{i,t})$.

Define $\widehat{\mathcal{X}}_{i,t} := (\widehat{G}_{i,t}, T_{i,t}, \beta_{i,t}, t, \beta_i(r_i) \cap \beta_i(t))$ for all $t \in C_i$ and both i = 1, 2.

Recursively, compute $\widehat{\Delta}_{t_1}\widehat{\varphi}_{t_1,t_2} := \operatorname{Iso}_{\operatorname{Tree}}(\widehat{\mathcal{X}}_{1,t_1},\widehat{\mathcal{X}}_{2,t_2})$ for all $t_1 \in C_1, t_2 \in C_2$.

Define $\Delta_{t_1}\varphi_{t_1,t_2} := (\widehat{\Delta}_{t_1}\widehat{\varphi}_{t_1,t_2})[V(G_i)]$ as the set of isomorphisms restricted to the root bags for all $t_1 \in C_1, t_2 \in C_2$.

Define $\rho_1: V(G_2) \to \{1, \dots, |V(G_2)|\}$ as some arbitrary labeling.

For each $t \in C_1 \cup C_2$, we choose a representative $t^* \in C_2$ in the isomorphism class, i.e., it holds that $\mathcal{X}_{i,t} \cong \mathcal{X}_{1,t^*}$ are isomorphic and that for two isomorphic $\mathcal{X}_{1,t_1} \cong \mathcal{X}_{2,t_2}$ it holds that $\mathcal{X}_{1,t_1^*} = \mathcal{X}_{1,t_2^*}$. Define $J_i := \{\Lambda_t \mid t \in C_i\}$ where $\Lambda_t := \Delta_t \varphi_{t,t^*} \rho_1$ for both i = 1, 2.

If the adhesion sets in H_i are all equal for both i = 1, 2:

Assign each representative $t^* \in C_2$ a different natural number $k(t^*) \in \mathbb{N}$ and assign each $t \in C_i$ the number $m(t) := |\{t' \in C_i \mid t'^* = t^* \text{ and } \Lambda_t = \Lambda_{t'}\}|.$

Define a function $\alpha_i: J_i \to \mathbb{N}$ that assigns each $\Lambda_t \in J_i$ a pair $(k(t^*), m(t)) \in \mathbb{N}^2$ for both i = 1, 2.

- \triangleright The number $k(t^*)$ encodes the isomorphism type of the subgraph corresponding to $t \in C_i$ and the number m(t) encodes the multiplicity of Λ_t in its isomorphism class.
- \triangleright We claim that $\operatorname{Iso}(G_1, J_1, \alpha_1; G_2, J_2, \alpha_2) = \operatorname{Iso}(\widehat{\mathcal{X}}_1; \widehat{\mathcal{X}}_2)[V(G_1)].$ Iso $(G_1, J_1, \alpha_1; G_2, J_2, \alpha_2)$. Therefore, for each pair in $\Lambda_{t_1} \in J_1$ there is a pair $\Lambda_{t_2} \in J_2$ such that $\Lambda_{t_1}^{\varphi} = \Lambda_{t_2}$. Since $\alpha_1^{\varphi} = \alpha_2$, it follows that $\mathcal{X}_{1,t_1} \cong \mathcal{X}_{2,t_2}$ and thus $t_1^* = t_2^*$. Therefore, $\varphi^{-1}\Delta_{t_1}\varphi_{t_1,t_1^*}\rho_1 = \Lambda_{t_1}^{\varphi} = \Lambda_{t_2} = \Delta_{t_2}\varphi_{t_2,t_1^*}\rho_1$. Equivalently, for all $\Lambda_{t_1} \in J_1$ there is a $\Lambda_{t_2} \in J_2$ such that $\varphi \in \Lambda_{t_1} \Lambda_{t_2}^{-1} = \varphi_{t_1,t_1^*} \Delta_{t_1^*} \varphi_{t_2,t_1^*}^{-1} = \operatorname{Iso}(\widehat{\mathcal{X}}_{1,t_1}; \widehat{\mathcal{X}}_{2,t_2})[V(G_1)].$ In other words, there is a function $\psi : J_1 \to J_2$ such that for all $\Lambda_{t_1} \in J_1$ it holds that $\varphi \in \Lambda_{t_1} \psi(\Lambda_{t_1})^{-1}$. Moreover, $\psi : J_1 \to J_2$ is injective (and thus bijective), which can be seen as follows. Assume that $\psi(\Lambda_{t_1}) = \psi(\Lambda_{t'_1}) = \Lambda_{t_2}$. Then, $\varphi \in$ $\operatorname{Iso}(\widehat{\mathcal{X}}_{1,t_1};\widehat{\mathcal{X}}_{2,t_2})[V(G_1)] \cap \operatorname{Iso}(\widehat{\mathcal{X}}_{1,t_1'};\widehat{\mathcal{X}}_{2,t_2})[V(G_1)]. \quad Therefore, \ \operatorname{Iso}(\widehat{\mathcal{X}}_{1,t_1};\widehat{\mathcal{X}}_{1,t_1'})[V(G_1)]$ contains the identity, which means that $\Lambda_t, \Lambda_{t'}$ intersect non-trivially. Since $\mathcal{X}_{1,t}, \mathcal{X}_{1,t'}$ are isomorphic, $\Lambda_t, \Lambda_{t'}$ are cosets of the same group and thus $\Lambda_t = \Lambda_{t'}$, which shows that ψ is bijective. Since $\psi(\Lambda_{t_1}) = \Lambda_{t_2}$ implies $m(t_1) = m(t_2)$, there is a bijective function $\widetilde{\psi}: C_1 \to C_2 \text{ with } \Lambda_{\widetilde{\psi}(t_1)} = \psi(\Lambda_{t_1}). \text{ Therefore, } \varphi \in \operatorname{Iso}(\widehat{\mathcal{X}}_1; \widehat{\mathcal{X}}_2)[V(G_1)].$

Compute $\Delta \varphi := \text{Iso}(G_1, J_1, \alpha_1; G_2, J_2, \alpha_2)$ using Corollary 35.

 \triangleright By the properties of the decomposition from Theorem 38, it holds that $|V(G_1)|$ = $|\beta_1(r_1)| \le (\operatorname{tw} G_1^k) + 1 = (\operatorname{tw} G_1) + 1 \le k + 1.$ Moreover, $|J_1| \le |V(T_1)| \in |V(\widehat{G}_1)|^{O(1)}$. For this reason, the algorithm from Corollary 35 runs in time $|V(\widehat{G}_1)|^{\text{polylog}(k)}$.

If the adhesion sets in H_i are pairwise distinct for both i = 1, 2:

Define a function $\alpha_i: H_i \to J_i$ that assigns each adhesion set $\beta_i(r_i) \cap \beta_i(t_i) \in H_i$ the coset $\Lambda_{t_i} \in J_i$ for both i = 1, 2.

 $ightharpoonup Again, it holds that Iso(G_1, J_1, \alpha_1; G_2, J_2, \alpha_2) = Iso(\widehat{\mathcal{X}}_1; \widehat{\mathcal{X}}_2)[V(G_1)].$

Compute $\Delta \varphi := \text{Iso}(Y_1, H_1; Y_2, H_2)$ using Lemma 41.

 \triangleright The lemma can be applied since Theorem 38 ensures that the adhesion sets in H_1, H_2 are cliques in Y_1, Y_2 , respectively. By Lemma 37, it holds that $sep Y_1, tw Y_1 \le k$. For this reason, the algorithm from Lemma 41 runs in time $n^{\text{polylog}(k)}$.

Compute $\Delta \varphi := \text{Iso}(V(\mathring{G}_1), H_1, J_1, \alpha_1; V(G_2), H_2, J_2, \alpha_2) \cap \Delta \varphi$ using Lemma 42. $\triangleright We \ have \ |H_1| \le |V(T_1)| \in |V(\widehat{G}_1)|^{\mathcal{O}(1)}$. Furthermore, there is a point $v_1 \in V(G_1)$ such that $\operatorname{cw} \Delta_{(v_1)} \leq \max(\operatorname{sep} Y_1, \operatorname{tw} Y_1 \leq k) \leq k$. For this reason, the algorithm runs in time $|V(\widehat{G}_1)|^{\operatorname{polylog}(k)}$.

Compute $\Delta \varphi := \text{Iso}(G_1; G_2) \cap \Delta \varphi$ using Theorem 39.

- \triangleright The algorithm from Theorem 39 runs in time $|V(\widehat{G}_1)|^{\operatorname{polylog}(k)}$.
- \triangleright In both cases, we found the isomorphisms restricted to the root bag, i.e., $\Delta \varphi$ = $\operatorname{Iso}(\widehat{\mathcal{X}}_1;\widehat{\mathcal{X}}_2)[V(G_1)].$

We define $\Delta \widehat{\varphi} := \operatorname{Iso}(\widehat{\mathcal{X}}_1; \widehat{\mathcal{X}}_2)$.

▷ This can be computed as follows. We consider the homomorphism $h: \widehat{\Delta} \to \Delta$ that maps $\widehat{\delta} \in \widehat{\Delta}$ to $\widehat{\delta}[V(G_1)] \in \Delta$. First, we explain how to compute the kernel $\widehat{K} := \ker(h) \leq \widehat{\Delta}$. The pointwise stabilizers $\widehat{\Theta}_{t_1} := (\widehat{\Delta}_{t_1})_{(\beta_1(r) \cap \beta_1(t_1))}$ for $t_1 \in C_1$ are polynomial-time computable. Let $\widehat{\Theta}'_{t_1} \leq \operatorname{Sym}(V(\widehat{G}_1))$ be the group that acts like $\widehat{\Theta}_{t_1}$ on $\widehat{V}(G_{i,t_1})$ and fixes all points in $V(\widehat{G}_1) \setminus \widehat{V}(G_{i,t_1})$. Then, the kernel $\widehat{K} \leq \operatorname{Sym}(V(\widehat{G}_1))$ is the group generated by all groups $\widehat{\Theta}'_{t_1} \leq \operatorname{Sym}(V(\widehat{G}_1))$ for $t_1 \in C_1$. Next, we can compute a subgroup $\widehat{\Theta} \leq \widehat{\Delta}$ with $\widehat{\Theta}[V(G_1)] = \Delta$ by extending each generator of Δ in an arbitrary way. In the same way, we can compute an isomorphism $\widehat{\varphi} \in \widehat{\Delta} \widehat{\varphi}$ with $\widehat{\varphi}[V(G_1)] = \varphi$. Finally, we define $\widehat{\Delta}$ as the group generated by the groups $\widehat{K}, \widehat{\Theta} \leq \widehat{\Delta}$.

8 Outlook and Open Questions

One could ask the question whether our isomorphism algorithm for graphs can be improved to a FPT-algorithm that runs in time $2^{\text{polylog}(k)}n^{\mathcal{O}(1)}$ where n is the number of vertices and k is the maximum treewidth of the given graphs. There are various reasons why this might be difficult. One reason is that our approach would require a FPT-algorithm for the isomorphism problem of graphs of maximum degree d that runs in time $2^{\text{polylog}(d)}n^{\mathcal{O}(1)}$. However, it is an open question whether any FPT-algorithm for the graph isomorphism problem parameterized by maximum degree exists. Another reason is that an algorithm for graphs running in time $2^{\text{polylog}(k)}n^{\mathcal{O}(1)}$ would imply an isomorphism algorithm for hypergraphs (V, H) running in time $2^{\text{polylog}|V|}|H|^{\mathcal{O}(1)}$. It is also an open question, whether such a hypergraph isomorphism algorithm exists [Bab18]. If this were indeed the case, one could hope for an improvement of our canonization algorithm for a set J consisting of labeling cosets that runs in time $2^{\text{polylog}|V|}|J|^{\mathcal{O}(1)}$.

Recently, Babai extended his quasipolynomial-time algorithm to the canonization problem for graphs [Bab19]. With Babai's result, it is a natural question whether the bounded-degree isomorphism algorithm of [GNS18] extends to canonization as well. The present isomorphism algorithm for graphs parameterized by treewidth should then be amenable to canonization as well.

Another question that arises is about permutation groups $G \leq \operatorname{Sym}(V)$. The canonical labeling problem for permutation groups is of great interest because it also solves the normalizer problem. In our recent work, we gave a canonization algorithm for explicitly given permutation groups running in time $2^{\mathcal{O}(|V|)}|G|^{\mathcal{O}(1)}$ [SW19]. Recently, the framework was extended to permutation groups that are implicitly given and the running time was improved to $2^{\mathcal{O}(|V|)}$ [Wie20]. The present work implies a canonization algorithm running in time $(|V| + |G|)^{\operatorname{polylog}|V|}$. An important question is whether the present techniques can be combined with the canonization techniques for implicitly given permutation groups to obtain a canonization algorithm running in time $2^{\operatorname{polylog}|V|}$.

Finally, we ask whether the isomorphism problem can be solved in time $n^{\text{polylog}(|V(H)|)}$ where n is the number of vertices and H is an excluded topological subgraph H of the given graphs. Even for excluded minors H, we do not have such an algorithm.

A Proof of Lemma 9

Proof of Lemma 9. Let $U := V \cup V_1 \cup \ldots \cup V_t$ where $V_i := V \times \{i\}$. The sets V_i can be seen as disjoint copies of V. Let $\Delta_i^{\operatorname{Can}} := \rho_i^{-1} \Delta_i \rho_i$ for all $\Delta_i \rho_i \in J$ (this is well defined and does not depend on the representative ρ_i of $\Delta_i \rho_i$). We define a labeling coset $\Delta_U \rho_U \leq \operatorname{Label}(U)$. Informally, the

labeling coset $\Delta_U \rho_U$ orders the set of components $\{V_1, \ldots, V_t\}$ according to the ordering "<" defined on $\Delta_i^{\operatorname{Can}}$ for $i \in [t]$, and it orders the component V_i according to a labeling in $\Delta_i \rho_i$. More formally, we define $\Delta_U \rho_U \coloneqq \{\lambda_U \in \operatorname{Label}(U) \mid \lambda_U \mid_V \in \operatorname{Label}(V) \text{ and } \exists \lambda_1 \in \Delta_1 \rho_1 \ldots \exists \lambda_t \in \Delta_t \rho_t \exists k_1, \ldots, k_t \in \mathbb{N} \forall i, j \in [t], v \in V : \Delta_i^{\operatorname{Can}} < \Delta_j^{\operatorname{Can}} \Longrightarrow k_i < k_j \text{ and } \lambda_U(v,i) = \lambda_i(v) + k_i \cdot |V| \}.$ We define a graph G = (U, E) which identifies the vertices in V with their corresponding copies in V_i for all $i \in [t]$. More formally, we define $E = \{(v, (v, i)) \mid v \in V, (v, i) \in V_i, i \in [t]\} \subseteq U^2$. We compute $\Theta_U \tau_U \coloneqq \operatorname{CL}_{\operatorname{Graph}}(E, \Delta_U \rho_U)$ using Theorem 5. We claim that the labeling coset $\Delta \rho \coloneqq (\Theta_U \tau_U)[V]$ induced on V defines a canonical labeling for J.

(CL1.) Assume we have J^{φ} instead of J as an input. We have to show that the algorithm outputs $\varphi^{-1}\Delta\rho$ instead of $\Delta\rho$. The group $\Delta_i^{\operatorname{Can}}$ does not depend on φ since $\rho_i^{-1}\Delta_i\rho_i = (\varphi^{-1}\rho_i)^{-1}\varphi^{-1}\Delta_i\rho_i$. By construction, we obtain $\varphi_U^{-1}\Delta_U\rho_U, E^{\varphi_U}$ instead of $\Delta_U\rho_U, E$ where φ_U is a bijection with $\varphi_U|_V = \varphi$. By (CL1) of $\operatorname{CL}_{\operatorname{Graph}}$, we obtain $\varphi^{-1}\Theta_U\tau_U$ instead of $\Theta_U\tau_U$. Finally, we obtain $(\varphi_U^{-1}\Theta_U\tau_U)[V^{\varphi}] = \varphi^{-1}\Delta\rho$ instead of $\Delta\rho$.

(CL2.) We have to show that $\Delta = (\operatorname{Aut}(E) \cap \Delta_U)|_V = \operatorname{Aut}(J)$. The inclusion $\operatorname{Aut}(J) \subseteq \Delta$ follows from (CL1) of this reduction. We thus need to show the reversed inclusion $(\operatorname{Aut}(E) \cap \Delta_U)|_V \subseteq \operatorname{Aut}(J)$. So let $\sigma_U \in \operatorname{Aut}(E) \cap \Delta_U$. Since $\sigma_U \in \Delta_U$, it follows that there are $\lambda_1, \ldots, \lambda_t, \lambda_1', \ldots, \lambda_t'$ and $k_1, \ldots, k_t, k_1', \ldots, k_t' \in \mathbb{N}$ such that for all $i \in [t], (v, i) \in V_i$ it holds that $\sigma_U(v, i) = (\lambda_j'^{-1}(\lambda_i(v) + k_i \cdot |V| - k_j' \cdot |V|), j)$ for some $j \in [t]$. It must hold that $k_i = k_j'$ and therefore $\sigma_U(v, i) = (\lambda_j'^{-1}(\lambda_i(v)), j)$. In particular, $j \in [t]$ only depends on the choice of $i \in [t]$ (and not on the choice of $v \in V_i$). Therefore, there is a $\psi \in \operatorname{Sym}(t)$ such that for all $(v, i) \in U$ it holds that $\sigma_U(v, i) = (w, \psi(i))$ for some $w \in V$. Since $\sigma_U \in \operatorname{Aut}(E)$, it follows that for all $i \in [t]$ there are $\lambda_i \in \Delta_i \rho_i = \rho_i \Delta_i^{\operatorname{Can}}$ and $\lambda_{\psi(i)} \in \Delta_{\psi(i)} \rho_{\psi(i)} = \rho_{\psi(i)} \Delta_{\psi(i)}^{\operatorname{Can}}$ such that $\sigma_U|_V = \lambda_i \lambda_{\psi(i)}^{-1}$. Since $\Delta_i^{\operatorname{Can}} = \Delta_{\psi(i)}^{\operatorname{Can}}$ this is equivalent to $(\sigma_U|_V)^{-1} \Delta_i \rho_i = \Delta_{\psi(i)} \rho_{\psi(i)}$. This implies $\sigma_U|_V \in \operatorname{Aut}(J)$. \square

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