Physica 25D (1987) 173–180
North-Holland, Amsterdam

EFFECTS OF PHASE SPACE DISCRETIZATION ON THE LONG-TIME BEHAVIOR OF DYNAMICAL SYSTEMS

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Received 3 February 1986
Revised manuscript received 12 June 1986

We investigate the effects that roundoff errors have on discrete-time dynamical systems. Computer-generated orbits terminate in loops whose lengths scale with the machine precision in a universal way. We present arguments to explain this fact.

1. Introduction

There is a growing interest in understanding the qualitative behavior of dynamical systems. Computer modeling of discrete-time systems has played an important role in mediating between experimental situations and theoretical concepts [1–11]. These models can be adapted to imitate certain desired aspects of real systems: ergodicity, mixing, sensitive dependence on initial conditions, subharmonic bifurcation etc. Computer simulations, however, have to be evaluated with utmost care when studying the long-time behavior of a specific model. One reason is that the iteration process often amplifies the rounding errors to the point where they obscure the true behavior of the system under study. Another reason is that, since digital computers are finite-state machines, they effectively discretize the continuous phase space of the model and, therefore, orbits fall into loops of length $L \leq N$ where N is the number of phase cells. In some cases the orbit terminates in a fixed point even though, without truncation error, the dynamical system can be shown to possess no stable fixed points at all.

A model qualifies as being insensitive to computer precision if the typical length L of a periodic orbit is of the order N or Δ^{-1} where Δ is the size of the phase cell. In this case the defects caused by roundoff errors can be minimized efficiently by resorting to higher precision. Unfortunately, many nonlinear systems appear to be sensitive in the sense that they obey a scaling law

$$L_{\max} \sim N^{\varepsilon} \sim \Delta^{-\varepsilon}, \tag{1.1}$$

with $0 < \varepsilon < 1$.

In this paper we address the problem of determining the exponent ε for iterated maps of the interval, $f_\mu(x) = 1 - \mu|x|^z$, $-1 \leq x \leq 1$ where μ and z are treated as parameters. Our results are:

1) Suppose $\mu = 2$. Then $\varepsilon = 1/z$. A prominent example is provided by the quadratic Chebyshev polynomial $1 - 2x^2$ which is ergodic (even mixing). Here $\varepsilon = \frac{1}{2}$ so that the maximal periodic orbit has length $O(\sqrt{N})$ rather than $O(N)$. Notice that the exact system has no stable periodic orbit.

2) Suppose $\mu = \mu_c$ where μ_c is the limit point for the sequence of subharmonic bifurcations (formally the parameter value for which there is a stable orbit of period 2^∞). Then

$$\varepsilon = \frac{\log 2}{z \log \alpha(z)}, \tag{1.2}$$

where $\alpha(z)$ is the Feigenbaum constant. For any

$z \geq 1$ the right hand side of eq. (1.2) assumes values smaller than 0.38165305... .

The general lesson to be drawn from these results is that the maximal period does not grow as N (the naive picture) but grows at a considerably slower rate. That is, systems supposed to exhibit "deterministic chaos" show unavoidable order when implemented on a digital computer [4].

The paper is organized as follows: First we describe a numerical experiment. Then we provide some theoretical arguments to explain the numerical results. Finally, we briefly look at the Hénon map as an example of a two-dimensional system.

2. The numerical experiment

Let I be an interval of the real line and let f: $I \to I$ be a map. We are interested in the nth iterate f^n applied to some initial point $x_0 \in I$. It should be clear that a computer does not really iterate f but a different function $\hat{f}$ due to its finite numerical precision. Let us define the local error by

$$\Delta = \sup_{x \in I} |f(x) - \hat{f}(x)| . \tag{2.1}$$

To be explicit as possible we choose a hypothetical

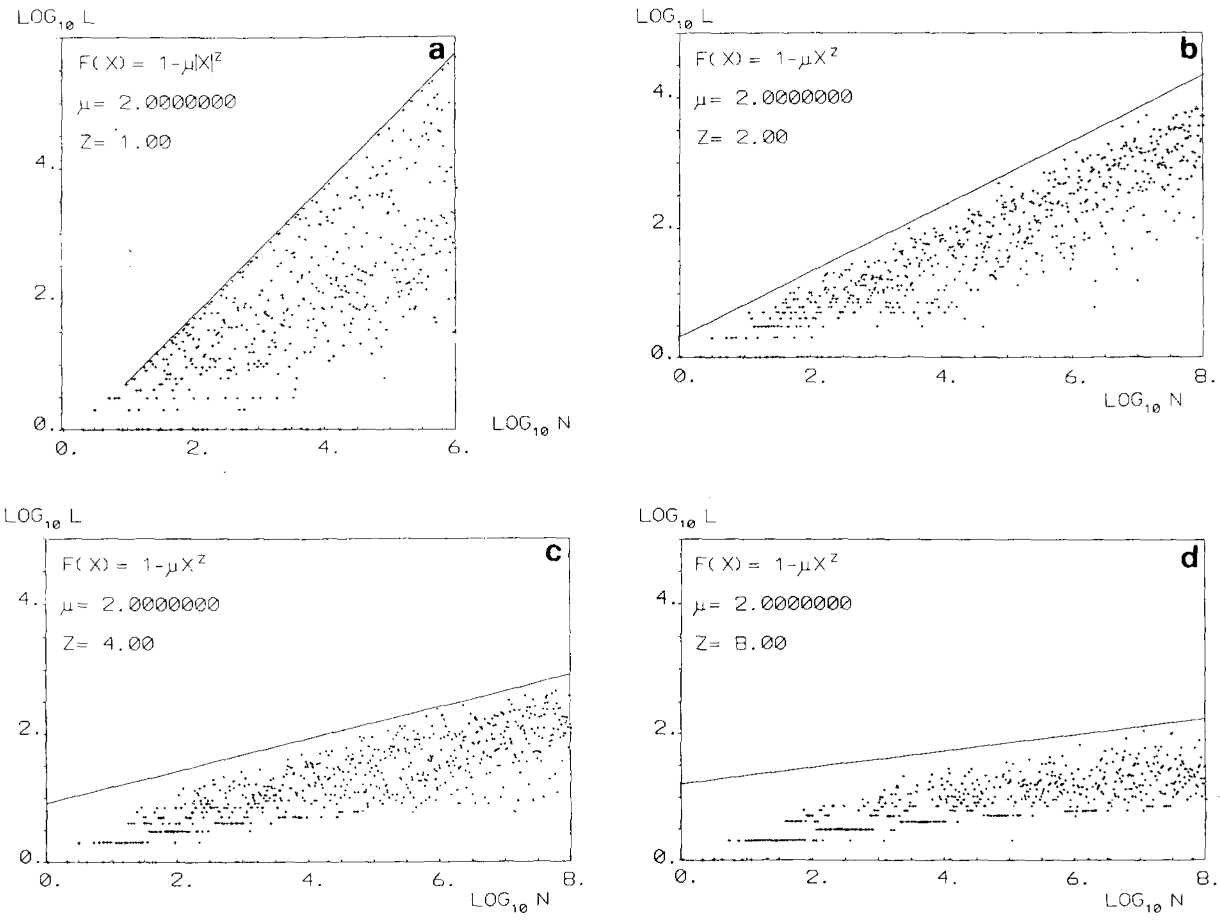

Fig. 1. Length of orbits, L, versus number of phase space cells, $N = 2/\Delta$, for the discretized dynamical system $x_{n+1} = \Delta[\Delta^{-1} f(x_n)]$, [ ]: integer part. $f(x) = 1 - 2|x|^z$, $z = 1$ (a), 2 (b), 4 (c), 8 (d). The initial points are chosen randomly.

computer characterized by

$$\hat{f}(x) = \Delta\left[\Delta^{-1} f(x)\right]. \tag{2.2}$$

Here the bracket $[x]$ stands for the integer part of a real number x. We stress the fact that the distorted dynamical system as defined by the new function $\hat{f}$ is strictly deterministic and, therefore, is not representative for models with noise. The influence that noise might have on a dynamical system is a different matter and has been studied by several authors [1–3].

We realize at once that our hypothetical computer divides the phase space I into N cells of size Δ and, since f simply permutes these cells, any orbit under $\hat{f}$ will ultimately fall into a loop with some period $L \leq N$ depending on the initial point x_0. If we choose x_0 randomly, L will be selected according to some unknown probability distribution. In order to study the details of this distribution we performed the following numerical experiment using various maps f of the interval $[-1, 1]$:

1) Set the initial precision at $\Delta = 2$;
2) lower the present value of Δ and set $N = 2\Delta^{-1}$;
3) use a random number generator to pick some initial point x_0 from $[-1, 1]$;
4) iterate x_0 according to $x_{n+1} = \Delta[\Delta^{-1} f(x_n)]$ until $x_{n+L} = x_n$. Plot $\log L$ against $\log N$. Then check if $\Delta \geq 10^{-8}$ and return to step 2.

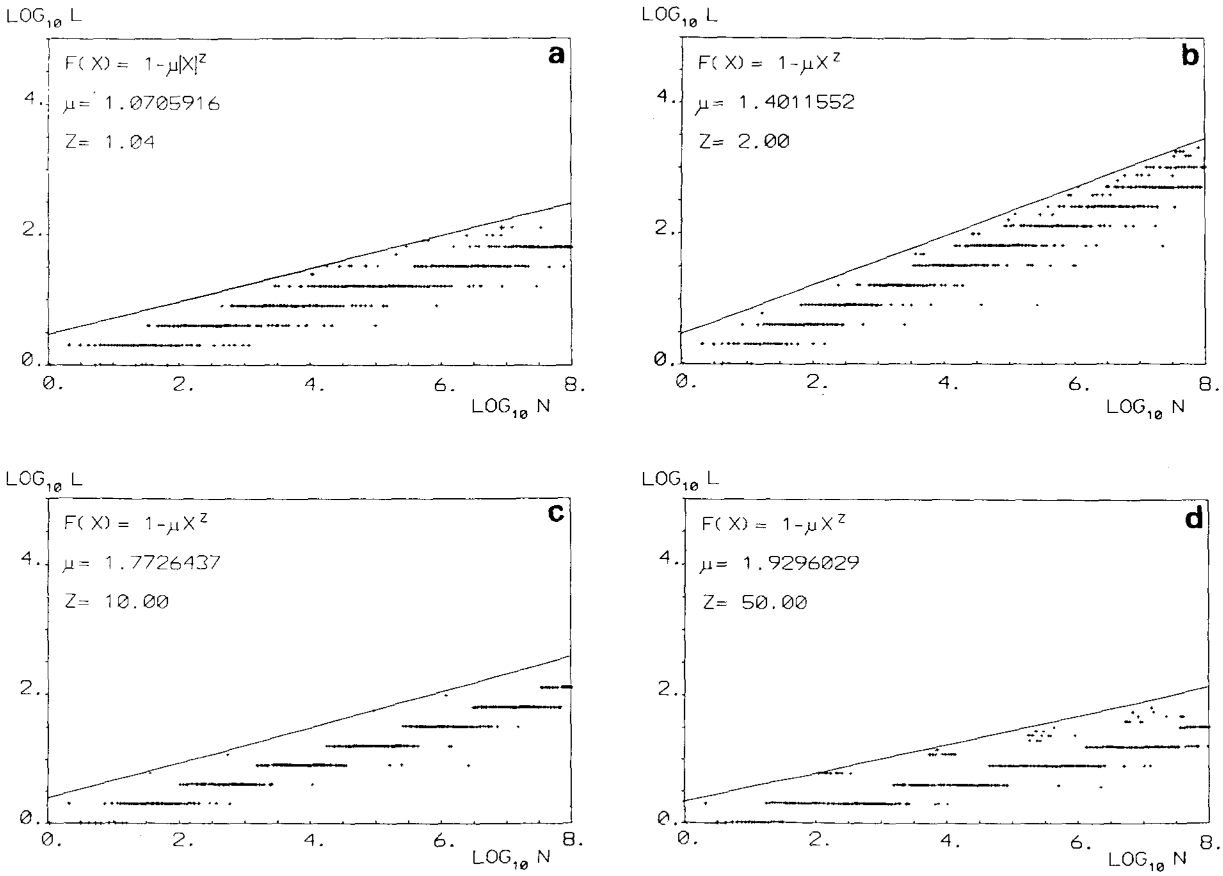

Fig. 2. L versus N for the maps $f(x) = 1 - \mu_c(z)|x|^z$, $z = 1.04$ (a), 2 (b), 10 (c), 50 (d), where $\mu_c(z)$ is the critical point.

The results of this experiment are shown in figs. 1–5. The doubly logarithmic plots for functions of the type $f(x)=1-\mu|x|^z$ fall into two classes. Either the plotted values for $\log L$ appear to be randomly scattered (fig. 1 where $\mu=2$) or the plot shows patterns of self-organization (fig. 2 where $\mu \approx \mu_c$). But in all cases considered the plotted points avoid the domain above some straight line and, hence, satisfy a condition of the type $L \le N^\varepsilon$. It is this law that we claim has universal significance. The straight lines in figs. 1 and 2 are predictions that follow from the discussion in sections 3 and 4. Empirically, the exponent ε is strictly smaller than 1 provided $z>1$. If μ is in the neighborhood of μ_c where any orbit is expected to be attracted by a loop of length 2^∞, we observe that the computer-generated periods are predominantly of the form 2^n $(n=0,1,2,\dots)$. As the number of phase cells N is increased, some period 2^n fades away to give place to a new period 2^{n+1}, the whole process reminding us of the subharmonic bifurcation. Notice, however, that we do not change the parameter μ but the number of phase cells, N.

3. Ergodic systems

As a prototype of an ergodic system let us consider the quadratic Chebyshev polynomial $f(x)=1-2x^2$ for which the invariant measure is known,

$$\mu(S)=\pi^{-1}\int_S(1-x^2)^{-1/2}\,\mathrm{d}x \tag{3.1}$$

$(S\subset[-1,1])$. Suppose we follow an orbit, $x_{n+1}=f(x_n)$, and record the number of recurrences of the event S,

$$k_n=\#\{k\le n\colon x_k\in S\}. \tag{3.2}$$

Then, for almost any starting point x_0, $\lim n^{-1}k_n=\mu(S)$. We choose Δ sufficiently small compared to $|S|$, the size of S, and replace $f(x)$ by the approximate function $\Delta[\Delta^{-1}f(x)]$. Then any orbit falls into some loop. Let $\mathscr{L}$ be the loop whose period L is largest and let $\mathscr{L}$ and S have $N(S)$ points in common. It is still a reasonable assumption that

$$N(S)=L\mu(S). \tag{3.3}$$

Comparing different intervals S of fixed size $|S|=k\Delta$ we observe that $N(S)$ assumes its maximum, N_Δ, if S is next to either $x=1$ or $x=-1$. From (3.1)

$$N_\Delta=L\pi^{-1}(2k\Delta)^{1/2}, \tag{3.4}$$

provided Δ is small $(k\Delta\ll 1)$. As $|S|$ and Δ tend to zero at the same rate (i.e., k stays fixed), it seems both plausible and empirically correct that N_Δ approaches a constant to the effect that L increases at the rate $\Delta^{-1/2}$. It should be stressed that our argument relies on a scaling hypothesis.

By a change of variables the Chebyshev system can be transformed into

$$f(x)=1-2|x|,\quad -1\le x\le 1, \tag{3.5}$$

such that the invariant measure becomes

$$\mu(S)=\tfrac{1}{2}\int_S \mathrm{d}x=\tfrac{1}{2}|S|. \tag{3.6}$$

Following the same line of reasoning as before we predict $L\sim\Delta^{-1}$ for the maximal length of loops under $\Delta[\Delta^{-1}f(x)]$. The lesson to be learned is this: conjugated systems may have a totally different behavior when implemented on a finite-state machine.

The results can be extended to include ergodic systems like $f(x)=1-2|x|^z$ $(z\ge 1)$. Though an explicit expression of the invariant measure μ is known only for certain exponents z, it may be shown [5] that quite generally

$$\int_{1-x}^{1}\mu(\mathrm{d}x)=\mathrm{O}(x^{1/z}),\quad x\to 0. \tag{3.7}$$

In fact, this type of scaling behavior is universal, i.e., it applies to any ergodic map of the interval $[-1,1]$ with a single extremal point of the type $|x|^z$ and which is symmetric under $x \to -x$. Eq. (3.7) expresses the behavior of the invariant measure at the boundary $x=1$. Under further assumptions on f (see [5]) the same can be stated for $x=-1$. In any of these cases we predict $L \sim \Delta^{-1/z}$.

4. Critical systems

Suppose f is a (S-unimodal) map of the interval $[-1,1]$ depending on some parameter μ and some exponent z characterizing the behavior at the maximum such that $\mu = \mu_c(z)$ separates the chaotic regime from the period-doubling regime. It is known that, at the critical value μ_c and if $\alpha = \alpha(z)$ is Feigenbaum's constant,

$$\lim_{k\to\infty} (-\alpha)^k f^{2^k}(\alpha^{-k}x) = g(x), \tag{4.1}$$

where $g(x)$ is a universal function [6]. At μ_c, the Ljapunov exponent

$$\lambda = \lim_{k\to\infty} k^{-1} \log|\mathrm{d}f^k(x)/\mathrm{d}x| \tag{4.2}$$

vanishes and, therefore, errors do not propagate exponentially.

To any orbit $x_{n+1} = f(x_n)$ we associate a distance

$$d_n = \min_{0\le i<j\le 2^n} |x_{N+i} - x_{N+j}|, \tag{4.3}$$

where we take N sufficiently large. Since almost all orbits ultimately fall onto the attractor, d_n gives us the precision necessary to separate 2^n successive points on the attractor. For almost all initial points x_0 and large n,

$$d_{n+1} = \alpha^{-z} d_n, \tag{4.4}$$

due to the fact that we encounter the minimal distance in the vicinity of $x=1$ [7].

Let f_Δ be the machine version of f (at μ_c) depending on the arithmetic precision Δ which we treat as a parameter such that $\lim_{\Delta\to 0} f_\Delta = f$. It seems reasonable to assume that orbits under f_Δ fall onto the attractor of f up to errors of the order Δ which has also been confirmed by our numerical experiment as described in section 2 where periods of length 2^n dominate.

For a certain range, say $\Delta_n \ge \Delta \ge \Delta_{n+1}$, the orbits under f_Δ are likely to merge into a terminal loop with period 2^n. For the loop to be visible we must have that $d_n > \Delta$ and, thus, we expect Δ_n and d_n to essentially represent the same number. As we decrease Δ such that the precision passes the critical value Δ_{n+1} we observe a period dou-

Table I
ε versus the exponent z that characterizes the universality class of maps of the interval

z	$\varepsilon = \dfrac{1}{z \log_2 \alpha}$
1	0
1.02	0.217616
1.04	0.2514358
1.06	0.2737467
1.08	0.2904330
1.1	0.3036497
1.2	0.34350515
1.4	0.37326129
1.6	0.38106323
1.6922	0.38165305
1.8	0.38104479
2	0.37775615
3	0.35203565
4	0.33012808
5	0.3136615
6	0.3010604
8	0.2831245
10	0.2709537
15	0.252612
20	0.242281
25	0.23560
30	0.23090
35	0.22740
40	0.22469
45	0.22250
50	0.22073
⋮	⋮
∞	0.204

bling. After k bifurcations, the loop has length $L = 2^{n+k}$ while the precision has reached $\Delta_{n+k} = \alpha^{-kz}\Delta_n$. This in turn implies that, as k is varied, $L\Delta^{\varepsilon}$ stays constant where

$$\varepsilon^{-1} = z \log_2 \alpha(z). \tag{4.5}$$

We studied the dependence of Feigenbaum's constant α on z by a high precision numerical analysis and then calculated the numerical values for ε (table I) according to (4.5). We found good agreement with the results of section 2. We also found a peculiar behavior: ε is not a monotone function of z but reaches a maximal value of

$$\varepsilon_0 = 0.38165305\ldots$$

at $z_0 = 1.6922\ldots$. In a sense, the constant ε_0 is more universal than Feigenbaum's constant α, since, irrespective of z, the length of the maximal periodic orbit is bounded by cN^{ε_0} where N is the total number of phase cells.

5. Other systems

So far we have discussed dynamical systems that are either critical ($\mu = \mu_c$) or ergodic ($\mu = 2$). To state precisely what happens if a system is supercritical is not an easy task. If, however, the system is subcritical ($\mu < \mu_c$) we observe the following (fig. 3). Each time the precision Δ passes a certain value Δ_n a period doubling $2^{n-1} \to 2^n$ occurs until a critical value for the precision, Δ_k, is reached beyond which no further bifurcation of the period 2^k takes place. Obviously, $L = 2^k$ is the stable period of the system at $\Delta = 0$ and at the given value of the parameter μ. As $\mu \to \mu_c$ we may argue (like in section 4) that, while k increases, $2^k\Delta_k^{\varepsilon}$ stays constant where $\varepsilon^{-1} = z\log_2\alpha$. On the other hand, the difference $\mu_c - \mu$ is of the order δ^{-k}. This in turn produces the result

$$\Delta_k = \text{const}\,|\mu_c - \mu|^{\beta}, \quad \beta = \frac{z \log \alpha}{\log \delta}, \tag{5.1}$$

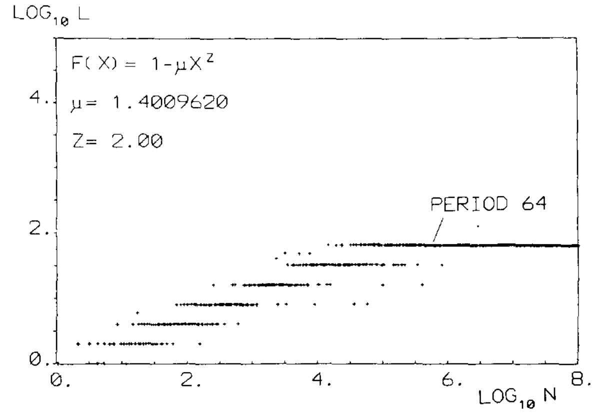

Fig. 3. L versus N for the subcritical map $f(x) = 1 - \mu x^2$, $\mu < \mu_c$. The exact system has a superstable orbit of period 64.

which involves the Feigenbaum constants δ and α simultaneously. Eq. (5.1) provides an estimate of the precision necessary to obtain reliable results from a computer simulation. At this point we would like to mention that the transient behavior of subcritical discretized maps can also be described by universal scaling relations as was recently discovered by Huberman and Wolff [14, 15].

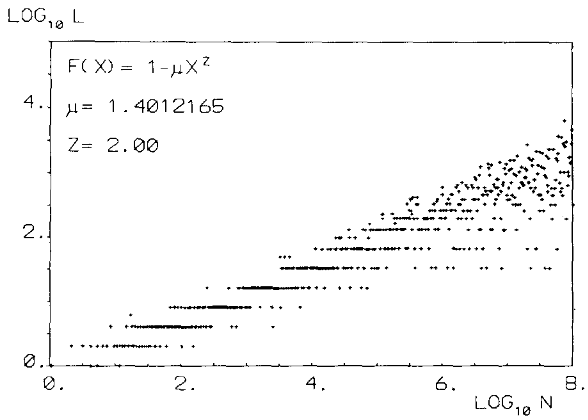

Fig. 4. L versus N for the supercritical map $f(x) = 1 - \mu x^2$, $\mu > \mu_c$; μ is a Misiurewicz point.

We have also checked the significance of eq. (5.1) when μ is slightly above μ_c (fig. 4) so that the system is supercritical. The observation here is that, for $\Delta > \Delta_k$, periods of the type 2^n occur but suddenly disappear at Δ_k. Only if $\Delta < \Delta_k$, the chaotic behavior of the system is revealed.

We have obtained similar results for maps $1 - \mu|x|^z$ where we have set μ close to some value μ_T that marks a tangent bifurcation. Take, for instance, $z = 2$ and $\mu_T = 1.75$. If $\mu = \mu_T + 10^{-7}$, the system has a stable orbit of period 3. Extensive numerical experiments (see fig. 5) have confirmed that this period is concealed unless the precision Δ has passed a critical value Δ_c. By a scaling argument analogous to that presented in section 4 we obtain the estimate

$$\Delta_c = \text{const}\,|\mu - \mu_T|^{1/z}.$$

One may suspect that similar results apply to n-dimensional systems. Here we want to touch on some extension to cases where $n = 2$. As an example we have chosen the Hénon map [8]

$$x_{n+1} = y_n + 1 - a x_n^2,$$
$$y_{n+1} = b x_n,$$

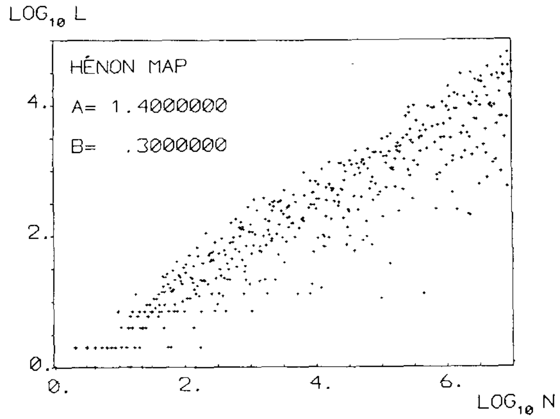

Fig. 6. L versus N for the Hénon map $T(x, y) = (1 - ax^2 + y, bx)$, $a = 1.4$, $b = 0.3$. The exact system is assumed to have a strange attractor.

with parameters $a = 1.4$ and $b = 0.3$. Since this system has a strange attractor, one expects to observe an erratic behavior of computer-generated orbits. However, due to the finite arithmetic precision, orbits again fall into loops whose periods we have recorded and plotted in fig. 6. We have chosen the same precision Δ for x and y and random initial points $x_0, y_0 \in [0, 1]$. For convenience, we have worked with the variable $N = 2\Delta^{-1}$

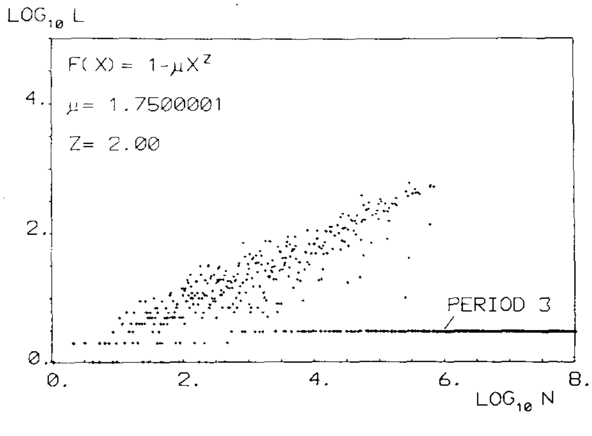

Fig. 5. L versus N for the supercritical map $f(x) = 1 - \mu x^2$, $\mu > \mu_T$, where μ_T is the critical point of tangent bifurcation. The exact system has a stable orbit of period 3.

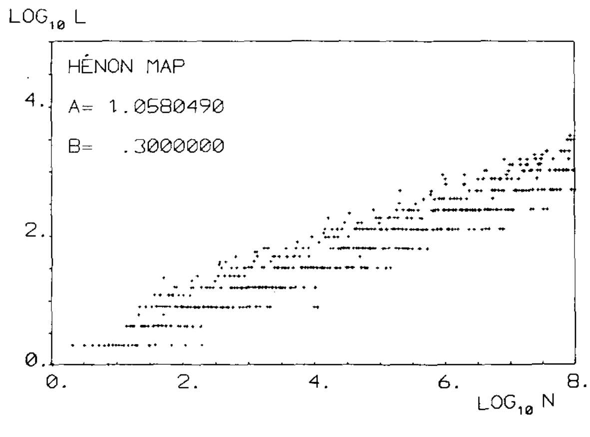

Fig. 7. L versus N for the Hénon map with the critical parameters $a = 1.058049$, $b = 0.3$, where bifurcations accumulate.

so that the phase space appears to be divided into $O(N^2)$ cells. Again, the maximal observed period has $O(N^\varepsilon)$ length and, surprisingly enough, ε is much smaller than 1. At present we have no explanation of this fact. See, however, the related results obtained by Levy [12].

We have also tested the Hénon model for other values of the parameters: $a = 1.058049$ and $b = 0.3$. This case has been investigated before [9, 10]. The parameters were chosen such that the long time behavior is determined by Feigenbaum's universal function, i.e., the function $g(x)$ as defined by eq. (4.1) were $z = 2$. Indeed, the numerical experiment generates periodic orbits of length 2^n (fig. 7) which resemble those obtained in the one-dimensional case (fig. 2b). We take this as a hint that orbits fall onto the attractor though they are perturbed by numerical errors. For a general discussion of small (random) perturbations see the basic article by Ruelle [13].

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