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NOTE

AN ASYMPTOTIC EQUIVALENT FOR THE NUMBER OF TOTAL PREORDERS ON A FINITE SET

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In this note, it is shown that, the number of total preorders on a finite set with n elements is equivalent, for n infinite, to $n!/2(\mathrm{Log}\,2)^{n+1}$.

1. A proposition

Let X be a set with n elements and let $P(n)$ denote the number of total preorders on X.

Proposition. *For n infinite, $P(n)$ is equivalent to $n!/2(\mathrm{Log}\,2)^{n+1}$*

2. The proof

The number of total preorders on X with exactly k classes is—obviously—$k!\,S(n,k)$, $S(n,k)$ being the Stirling number of second kind (number of X's partitions into exactly k classes). Let us compute the generating function $F(t)=\sum_{n\geq 0}[P(n)/n!]t^n$.

$$F(t)=\sum_{n\geq 0}\frac{1}{n!}\left(\sum_{k=0}^{n}k!\,S(n,k)\right)t^n$$
$$=\sum_{k\geq 0}k!\left(\sum_{n\geq k}\frac{S(n,k)}{n!}\,t^n\right).$$

It is well known that

$$\sum_{n\geq k}\frac{S(n,k)}{n!}\,t^n=\frac{(e^t-1)^k}{k!}\quad\text{(see [2]).}$$

So

$$F(t)=\sum_{k\geq 0}(e^t-1)^k=\frac{1}{2-e^t}.$$

(convergence for $|t| < \text{Log}\, 2$). $t = \text{Log}\, 2$ is the only singularity of the function $1/(2-e^t)$ and it is a pole of order one. The "Darboux theorem" (see [1, 3]) gives us an asymptotic equivalent for $P(n)$:

$$P(n) = \frac{-n!}{(\text{Log}\, 2)^{n+1}} c_{-1} + o((n-1)!)$$

where c_{-1} is the residue at $t = \text{Log}\, 2$ of $F(t)$.

$$c_{-1} = \lim_{t \to \text{Log}\, 2} \frac{t - \text{Log}\, 2}{2 - e^t} = -\frac{1}{2}.$$

Hence the result.

3. Some remarks

From the generating function

$$F(t) = \sum_{n \geq 0} \frac{P(n)}{n!} t^n = \frac{1}{2 - e^t},$$

we can deduce that:

(1) The derivatives of $F(t)$ satisfying the recurrence:

$$2F^{(n)}(t) - \sum_{p=0}^{n} \binom{n}{p} e^t F^{(n-p)}(t) = 0,$$

we get:

$$P(n) = \sum_{p=1}^{n} \binom{n}{p} P(n-p).$$

(2) The radius of convergence of $\sum_{n \geq 0} [P(n)/n!] t^n$ being Log 2,

$$\lim_{n \to \infty} \frac{nP(n-1)}{P(n)} = \text{Log}\, 2.$$

$nP(n-1)/P(n)$ is the probability for a total preorder to have only one greatest element (when all total preorders are equiprobable).

(3) $$H(t) = \frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n}{n!} t^n$$

being the generating function for Bernoullis numbers, we get

$$F(t) = \frac{-1}{2(t - \text{Log}\, 2)} H(t - \text{Log}\, 2)$$

$$= \frac{-1}{2(t - \text{Log}\, 2)} + \sum_{n=0}^{\infty} \frac{B_{n+1}}{(n+1)!} (t - \text{Log}\, 2)^n$$

(because $B_0=0$): this is the Laurent's expansion of $F(t)$ at $t=\operatorname{Log}2$. From

$$\frac{-1}{2(t-\operatorname{Log}2)}=\frac{1}{2\operatorname{Log}2}\cdot\sum_{p\geq 0}\frac{t^p}{(\operatorname{Log}2)}p$$

and

$$\sum_{n=0}^{\infty}\frac{B_{n+1}}{(n+1)!}(t-\operatorname{Log}2)^n=$$

$$=\sum_{n=0}^{\infty}\frac{B_{n+1}}{(n+1)!}\sum_{p\leq n}\binom{n}{p}t^p\cdot(-1)^{n-p}(\operatorname{Log}2)^{(n-p)}$$

$$=\sum_{p=0}^{\infty}t^p\Big(\sum_{n>p}\frac{B_{n+1}}{(n+1)!}\binom{n}{p}(\operatorname{Log}2)^{n-p}(-1)^{n-p}\Big)$$

it follows that

$$P(n)=\frac{n!}{2(\operatorname{Log}2)^{n+1}}+\sum_{p=0}^{\infty}(-1)^p\frac{B_{n+p+1}}{(n+p+1)p!}(\operatorname{Log}2)^p.$$

$\sum_{p=0}^{\infty}(-1)^p[B_{n+p+1}/(n+p+1)p!](\operatorname{Log}2)^p$ is the value of the o $((n-1)!)$ in the fomula

$$P(n)=\frac{n!}{2(\operatorname{Log}2)^{n+1}}+\mathrm{o}\,((n-1)!).$$

References

[1] G. Darboux, Mémoire sur l'approximation des fonctions de très grands nombres, J.M. Pures Appl. 4 (1878) 5–56 and 377–416.
[2] J. Riordan, An Introduction to Combinatorial Analysis, (Wiley, New York, 1958).
[3] G. Szego, Orthogonal Polynomials, 3rd ed. (Amer. Math. Soc., Providence, RI, 1967).