Induction-Recursion – 20 years later

Anton Setzer

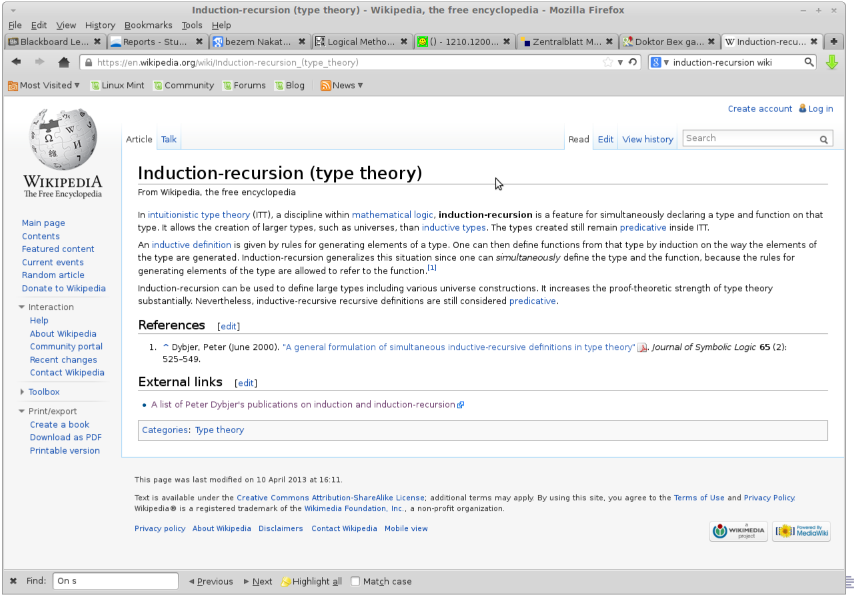
Swansea University, Swansea UK

Gothenburg, Sweden, 5 June 2013

Symposium on Semantics and Logics of Programs
Dedicated to Peter Dybjer on Occasion of his 60th Birthday

Happy Birthday

Wikipedia



Induction-recursion (type theory)

From Wikipedia, the free encyclopedia

In intuitionistic type theory (ITT), a discipline within mathematical logic, **induction-recursion** is a feature for simultaneously declaring a type and function on that type. It allows the creation of larger types, such as universes, than inductive types. The types created still remain predicative inside ITT.

An inductive definition is given by rules for generating elements of a type. One can then define functions from that type by induction on the way the elements of the type are generated. Induction-recursion generalizes this situation since one can *simultaneously* define the type and the function, because the rules for generating elements of the type are allowed to refer to the function.[1]

Induction-recursion can be used to define large types including various universe constructions. It increases the proof-theoretic strength of type theory substantially. Nevertheless, inductive-recursive recursive definitions are still considered predicative.

References [edit]

1. ^ Dybjer, Peter (June 2000). "A general formulation of simultaneous inductive-recursive definitions in type theory". *Journal of Symbolic Logic* **65** (2): 525–549.

External links [edit]

- A list of Peter Dybjer's publications on induction and induction-recursion

Categories: Type theory

Martin-Löf 1972

martinloefIntuitionisticTheoryOfTypes.pdf — martin_loef_72

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An intuitionistic theory of types

Per Martin-Löf
Department of Mathematics, University of Stockholm

The theory of types with which we shall be concerned is intended to be a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book by Bishop 1967. The language of the theory is richer than the language of first order predicate logic. This makes it possible to strengthen the axioms for existence and disjunction. In the case of existence, the possibility of strengthening the usual elimination rule seems first to have been indicated by Howard 1969, whose proposed axioms are special cases of the existential elimination rule of the present theory. Furthermore, there is a reflection principle which links the generation of objects and types and plays somewhat the same role for the present theory as does the replacement axiom for Zermelo-Fraenkel set theory.

An earlier, not yet conclusive, attempt at formulating a theory of this kind was made by Scott 1970. Also related, although less closely, are the type and

- Preprint, 1972, published in 25 years of Constructive Type Theory (1998).
- Introduction of Intuitionistic Type Theory.
- A type theory of inductive definitions.
- In addition a Russell style universe V and a normalisation theorem.

Backhouse “Do it yourself type theory” (1988)

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On the Meaning and Construction of the Rules in Martin-Löf's Theory of Types

Roland Backhouse
Department of Mathematics and Computing Science
University of Groningen
PO Box 800
9700 AV GRONINGEN
The Netherlands

Abstract We describe a method to construct the elimination and computation rules from the formation and introduction rules for a type in Martin-Löf's theory of types. The construction is based on an understanding of the inference rules in the theory as judgements in a pre-theory. The motivation for the construction is to permit disciplined extensions to the theory as well as to have a deeper understanding of its structure.

Reference

Roland Backhouse: **On the meaning and construction of the rules in Martin-Löf's Theory of Types.**
In: A. Avron, R. Harper, F. Honsell, I. Mason, and G. Plotkin (Eds.): Workshop on General Logic. Edinburgh, February 1987.
LFCS, Department of Computer Science, University of Edinburgh, Edinburgh, UK, ECS-LFCS-88-52
pp. 269 – 283, 1988.

Motivation of Backhouse

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equipped to adapt the theory to individual needs

The present work grew out of a feeling of discontent with the theory. On first encounter the universal reaction among computing scientists appears to be that the theory is formidable. Indeed, several have specifically referred to the overwhelming number of rules in the theory. On closer examination, however, the theory betrays a rich structure — a structure that is much deeper than the superficial observation that types are defined by introduction, elimination and computation rules. Once recognised this structure considerably reduces the burden of understanding. And yet, to my knowledge, the structure of the theory has not been properly discussed or documented; Martin-Löf, himself, alludes to the fact that there is a "pattern... in the type forming operations" in the preface to the notes prepared by Giovanni Sambin [ML1], but he does not give a detailed account of the pattern.

So much for the ideological motivations for this paper. At a more practical level it has become increasingly clear to us that there is a need to freely permit *disciplined* extensions to the theory. That the theory is open to extension is a fact that was clearly intended by Martin-Löf. Indeed, it is a fact that has been exploited by several individuals; Nordström, Petersson and Smith [NPS] have extended the theory to include lists, they and Constable et al [Co] have added subset types and Constable et al have introduced quotient types, Nordström has introduced multi-level functions [No], Chisholm has introduced a very special-purpose type of tree structure [Ch] and Dyckhoff [Dy] has defined the type of categories.

Initially we were against such extensions on the grounds that it is often possible to define them in terms of the W-type (for examples see [Kh]), because they add to the complexity of the theory and because they

Dybjer: Schema for Inductive Definitions

- **Peter Dybjer: An inversion principle for Martin-Löf's type theory.**
  Proceedings of the Workshop on Programming Logic. Programming Methodology Group, University of Goteborg and Chalmers University of Technology, **1989.**
- Not yet traced.
- **Peter Dybjer: Inductive sets and families in Martin-Löf's type theory and their set-theoretic semantics**
  In: G Huet and G. Plotkin (Eds): First Workshop on Logical Frameworks. Antibes.
  (Informal proceedings).
  May 1990.
- **Formal proceedings** of that workshop: **1991**.

Thierry Coquand and Christine Paulin

- Another schema:
  Thierry Coquand and Christine Paulin: Inductively defined types
  In Martin-Löf, Per and Mints, Grigori (Eds.): Proceedings of COLOG-88, LNCS 417, 1990, pp. 50 – 66.

Dybjer, Schema of Inductive Definitions

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GH from constable

Inductive Sets and Families in Martin-Löf's Type Theory and Their Set-Theoretic Semantics

Peter Dybjer

Chalmers University of Technology

(Draft)

Abstract

Martin-Löf's type theory is presented in several steps. The kernel is a dependently typed λ-calculus. Then there are schemata for inductive sets and families of sets and for primitive recursive functions and families of functions. Finally, there are set formers (generic polymorphism) and universes. At each step syntax, inference rules, and set-theoretic semantics are given.

Source of Dybjer's Schema

Mainly based on

Per Martin-Löf: **Hauptsatz for the Intuitionistic Theory of Iterated Inductive Definitions**.

In J.E. Fenstad (Ed.): Proceedings of the Second Scandinavian Logic Symposium, Elsevier, 1971, pp. 179 - 216.

Martin-Löf Hauptsatz Article

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HAUPTSATZ FOR THE INTUITIONISTIC THEORY OF ITERATED INDUCTIVE DEFINITIONS

Per MARTIN-LÖF
University of Stockholm

1. **Introduction.**

1.1. The principle of definition by generalized induction, perhaps best exemplified by the definition of the constructive second number class given by Church and Kleene, and the corresponding principle of proof by generalized induction were first formalized by Kreisel 1963. Also, the idea of iterating generalized inductive definitions, as done by Church and Kleene in their definition of the higher constructive number classes, gives rise to a corresponding principle of proof which was first stated as a formal schema by Kreisel 1964 in his proof of the wellordering of Takeuti's 1957 ordinal diagrams of finite order. A complete formulation of a classical theory of generalized inductive definitions iterated along a primitive recursive wellordering was given by Feferman 1969 whose main object was to establish the relation between his theory and certain subsystems of classical analysis.

1.2. In the present paper I shall give a proof theoretical analysis of the intui-

Non-inductive Example: The Σ-Type

Dependent product

- **Formation rule:**

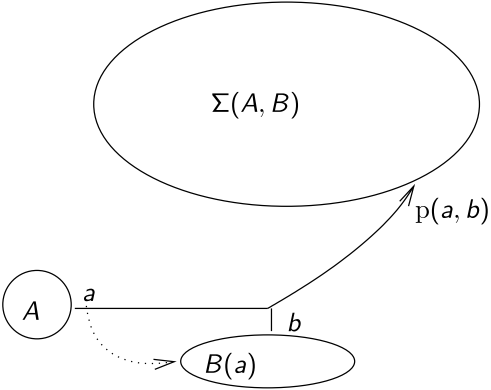
$$\frac{A : \mathrm{Set} \qquad B : A \to \mathrm{Set}}{\Sigma(A, B) : \mathrm{Set}}$$

- **Introduction rule:**

$$\frac{a : A \qquad b : B(a)}{\mathrm{p}(a, b) : \Sigma(A, B)}$$

- **Elimination/equality rule:**
  If we can derive $C(\mathrm{p}(a, b))$ for $a : A$ and $b : B(a)$, then we can derive $C(c)$ for $c : \Sigma(A, B)$.

Visualisation ($\Sigma(A, B)$)



- p has two non-inductive arguments.
- The type of the 2nd argument depends on the 1st argument.

Inductive Example: The $\boxed{\text{W-Type}}$

Lists: $\dfrac{a : A \qquad as : A^*}{(a :: as) : A^*}$

- **Formation rule:**

$$\frac{A : \mathrm{Set} \qquad B : A \to \mathrm{Set}}{\mathrm{W}(A, B) : \mathrm{Set}}$$

- **Introduction rule:**

$$\frac{a : A \qquad b : B(a) \to \mathrm{W}(A, B)}{\underline{\mathrm{sup}(a, b)} : \mathrm{W}(A, B)}$$

label of the root

index of children

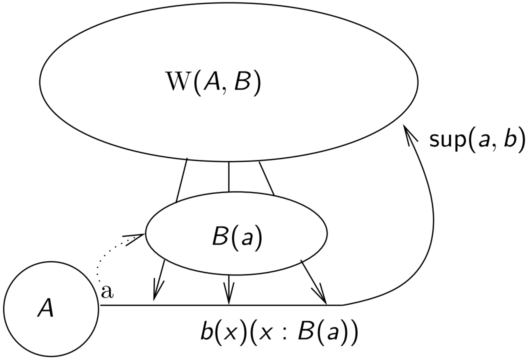
- **Elimination/equality rule:**

  Induction over trees.

Non-dependent version:

$$\frac{a : A \qquad b : B \to T(A, B)}{\mathrm{sup}(a, b) : T(A, B)}$$

Visualisation ($\mathrm{W}(A, B)$)



sup has two arguments

- First argument is non-inductive.
- Second argument is inductive, indexed over $B(a)$.
- $B(a)$ depends on the first argument a.

Observations

- **Inductive Arguments**, **non-inductive arguments**.
- Inductive arguments refer to sets previously defined.
- Non-inductive Arguments refer to elements of the set defined inductively, indexed over a set previously defined.
- Type of **later arguments** can **depend** on previous **non-inductive arguments**.
- **What about dependency on previous inductive arguments?**
- **Universes will answer the question.**

Ingredient 1: Universes à la Tarski

- Universes à la Russell occurred in Martin-Löf 1972.
- **History of universes à la Tarski** is according to an email by by Peter Dybjer on the Agda email List as follows:
  - The universe a la Tarski appeared for the first time, I believe, in the book **Intuitionistic Type Theory** (Bibliopolis) from **1984**. It was based on lectures in Padova given in 1980.
    Previously, universes were a la Russell.
    Aczel had a universe a la Tarski in his **1974/1977** paper about the Interpretation of Martin-Löf Type Theory in a First Order Theory of Combinators.
- So Aczel 1974/77 probably first occurrence of Tarski universes in literature, although they might have been around at that time.
- Peter Aczel private communication: Defining a realisability model forced to have a Tarski style universe.

Ingredient 2: Elimination Rules for Universes

- Martin-Löf mentions,in his 1972 paper the existence of a "principle of (transfinite) induction over V", but rejects it on the grounds that the Russel style universe V should be open in the sense of adding later closure under type constructors
  (p. 7 of the printed version in 25 years of constructive type theory, thanks to Thierry Coquand for pointing this out to AS).
- First formal presentation seems to be in Peter Aczel's 1974/77 paper.

Ingredient 3: Computability Predicate in Martin-Löf 1972

- According Peter Dybjer major inspiration for the principle of induction-recursion.
- Shows that universes are an example of a more general schema.

Quote Computability Predicate Martin-Löf 1972

4.1.1.1. $\varphi_{P(a_1,\ldots,a_n)}$ is the species of normalizable closed terms of type $P(a_1,\ldots,a_n)$.

4.1.1.2. Suppose that φ_A has been defined and that $\varphi_{B[a]}$ has been defined for all closed terms a of type A such that $\varphi_A(a)$. We then define $\varphi_{(\Pi x\in A)B[x]}$ by the following three clauses.

4.1.1.2.1. If $(\lambda x)b[x]$ is a closed term of type $(\Pi x \in A)B[x]$ and $\varphi_{B[a]}(b[a])$ for all closed terms a of type A such that $\varphi_A(a)$, then $\varphi_{(\Pi x\in A)B[x]}((\lambda x)b[x])$.

4.1.1.2.2. A closed normal term of type $(\Pi x \in A)B[x]$ which is not of the form $(\lambda x)b[x]$ satisfies $\varphi_{(\Pi x\in A)B[x]}$.

4.1.1.2.3. If the closed term b of type $(\Pi x\in A)B[x]$ has elimination form and reduces to a term a such that $\varphi_{(\Pi x\in A)B[x]}(a)$, then $\varphi_{(\Pi x\in A)B[x]}(b)$.

First Mentioning of Induction-Recursion

In the slides of Peter Dybjer of a talk
"**A General Formulation of Inductive and Recursive Definitions in Type Theory**"
given at the
EC project meeting: Proof Theory and Computation,
Munich, 28 – 30 May 1992
(Part of the Twinning Project Munich – Leeds – Oslo)
the first definition of the principle of induction-recursion was given:

Complete Slide

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WHAT ABOUT UNIVERSES ?

Universe – reflection : different principle ?

$\mathcal{U}$: set should be inductively defined ...

Universe à là Tarski (Martin-Löf 1984) contains a second component, the "decoding" function

$\mathcal{T} : (\mathcal{U})$ set defined by $\mathcal{U}$-recursion.

$\mathcal{U}$ - INTRODUCTION

$$\frac{a : \mathcal{U} \qquad \begin{matrix}(x : \mathcal{T}^{*}(a)) \\ b(x) : \mathcal{U}\end{matrix}}{\pi(a,b) : \mathcal{U}} \quad , \text{etc} \ldots$$

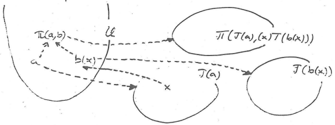
$\mathcal{T}$ - EQUALITY

$$\mathcal{T}(\pi(a,b)) = \Pi(\mathcal{T}(a), (x)\mathcal{T}(b(x))) \quad , \text{etc} \ldots$$

* $\mathcal{T}$ participates in the definition of $\mathcal{U}$:

simultaneous inductive - recursive definition

still "predicative" :



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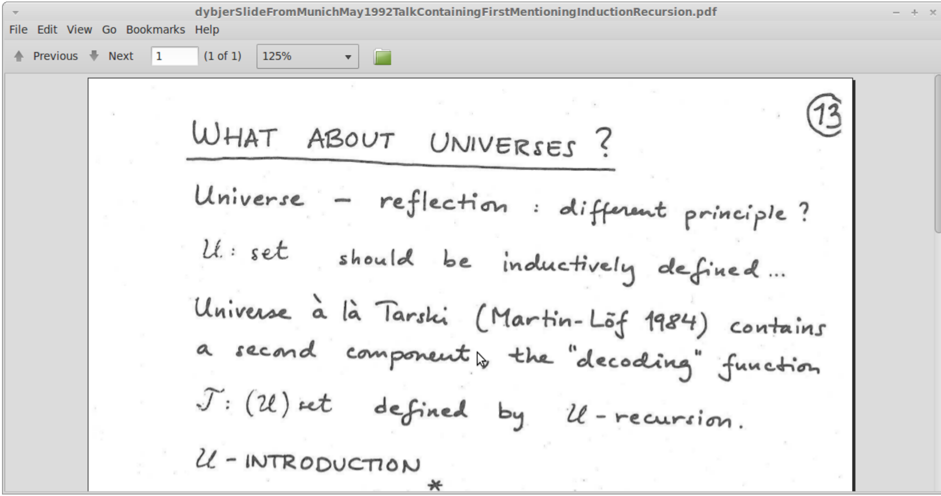

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$\mathcal{U}$ - INTRODUCTION

*

Slide 2

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$\mathcal{T} : (\mathcal{U})$ set defined by $\mathcal{U}$-recursion.

$\mathcal{U}$ - INTRODUCTION

$$\frac{a : \mathcal{U} \qquad \begin{array}{c}(x : \overset{*}{\mathcal{T}}(a))\\ b(x) : \mathcal{U}\end{array}}{\pi(a,b) : \mathcal{U}} \quad , \text{etc} \ldots$$

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<u>simultaneous inductive - recursive</u> definition

still "predicative":

Slide 3

dybjerSlideFromMunichMay1992TalkContainingFirstMentioningInductionRecursion.pdf

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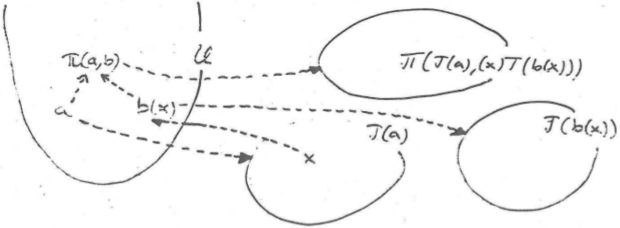
$\mathcal{T}(\pi(a,b)) = \Pi(\mathcal{T}(a),(x)\mathcal{T}(b(x)))$, etc...

* $\mathcal{T}$ participates in the definition of $\mathcal{U}$:

<u>simultaneous inductive-recursive</u> definition

still "predicative":



First Article on Induction-Recursion

Peter Dybjer: Universes and a General Notion of Simultaneous Inductive-Recursive Definition in Type Theory

In Bengt Nordström, Kent Petersson, and Gordon Plotkin: Proceedings of the 1992 workshop on types for proofs and programs, Båstad, June 1992.

First Article Induction-Recursion

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Universes and a General Notion of Simultaneous Inductive-Recursive Definition in Type Theory (Draft)

Peter Dybjer*
Chalmers University of Technology

August 1992

Abstract

In Martin-Löf's type theory we may define new sets (and families of sets) inductively, and new functions by recursion on the way the elements of these sets are generated. The schema for such definitions that we considered before includes all the standard set formers of type theory with the exception of universes.

Here we give an extended schema which also covers universes à la Tarski. These consist of simultaneous inductive definitions of sets of codes for small sets and recursive definitions of decoding functions. This extension is a small modification of the old schema and includes a general formulation of the notion of a simultaneous inductive-recursive definition.

There are several interesting applications of this extension. Here we show how to obtain an external universe hierarchy, where each level in the hierarchy faithfully reflects the previous level. We also show how to obtain the universe constructions of Griffor and Palmgren. These include an internal unfaithful universe hierarchy and a super universe.

Other examples are the construction of Frege structures in type theory, and various constructions relevant to the formalization of type theory inside type theory.

JSL Paper Peter Dybjer (2000)

dybjerjslsimultaneouslyinductiverecursivedefinitions.pdf

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A GENERAL FORMULATION OF SIMULTANEOUS INDUCTIVE-RECURSIVE DEFINITIONS IN TYPE THEORY

PETER DYBJER

Abstract. The first example of a simultaneous inductive-recursive definition in intuitionistic type theory is Martin-Löf's universe à la Tarski. A set U_0 of codes for small sets is generated inductively at the same time as a function T_0, which maps a code to the corresponding small set, is defined by recursion on the way the elements of U_0 are generated.

In this paper we argue that there is an underlying *general* notion of simultaneous inductive-recursive definition which is implicit in Martin-Löf's intuitionistic type theory. We extend previously given schematic formulations of inductive definitions in type theory to encompass a general notion of simultaneous induction-recursion. This enables us to give a unified treatment of several interesting constructions including various universe constructions by Palmgren, Griffor, Rathjen, and Setzer and a constructive version of Aczel's Frege structures. Consistency of a restricted version of the extension is shown by constructing a realisability model in the style of Allen.

Universes

- **Formation rules:**

$$\mathrm{U} : \mathrm{Set} \qquad \mathrm{T} : \mathrm{U} \to \mathrm{Set}$$

- **Introduction and Equality rules:**

$$\widehat{\mathbb{N}} : \mathrm{U} \qquad \mathrm{T}(\widehat{\mathbb{N}}) = \mathbb{N}$$

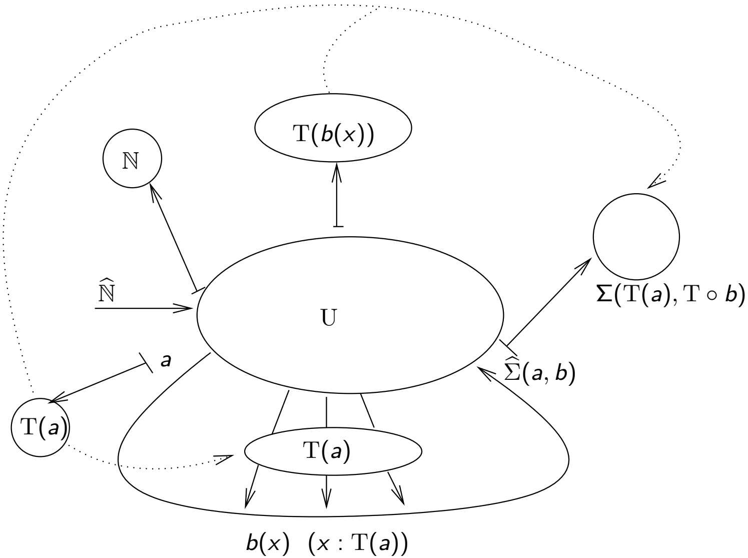
$$\frac{a : \mathrm{U} \qquad b : \mathrm{T}(a) \to \mathrm{U}}{\widehat{\Sigma}(a, b) : \mathrm{U}}$$

$$\mathrm{T}(\widehat{\Sigma}(a, b)) = \Sigma(\mathrm{T}(a), \mathrm{T} \circ b)$$

  Similarly for other type formers (except for U).

- **Elimination/equality rules:** Induction over U.

Visualisation (U)



Analysis

- Elements of U are defined **inductively**,
  while defining $\mathrm{T}(a) : \mathrm{Set}$ for $a : \mathrm{U}$ **recursively**.
- As before we have **inductive Arguments**,
  non-inductive arguments
- Later arguments can depend on
  - **previous non-inductive arguments** (as before),
  - T **applied to previous inductive arguments.**
- Principle can be generalised to $\mathrm{T}(u) : D$ for any type D.
  - E.g. $D = \mathrm{Fam}(\mathrm{Set}) \rightarrow \mathrm{Fam}(\mathrm{Set})$.
    Erik Palmgren's higher order universes.
  - E.g. $D : \mathrm{Set}$.

Closed Formalisation of Induction-Recursion

- **1999 A.S. and Peter Dybjer: A finite axiomatization of inductive-recursive definitions.**
  - Goal was to develop a **finite axiomatisation** of induction-recursion which allows a **proof theoretic analysis.**
  - Observation that in order to introduce a new inductive-recursive definition, a **proof obligation** needs to be fulfilled:
    Proof that the sets used in inductive and non-inductive arguments are sets depending on previous arguments.
  - **Data type of inductive-recursive definitions.**
  - Data type has ingredients of the **Mahlo universe.**

Induction-Recursion and Initial Algebras

- Slight reformulation of closed formalisation of induction-recursion.
- Proof of **equivalence** of **elimination rules** for induction recursion and induction recursion as **initial algebra**.
- Proof that induction recursion reaches the **proof-theoretic strength of at least KPM.** (This does not rely on the data type of induction-recursion).

Indexed Induction Recursion (Peter Dybjer, AS, 2001)

- Extension to **indexed induction-recursion**.
- Difference between **restricted** and **generalised** indexed IR.
  - Restricted means that for each index i you determine the type of constructors for U_i.
  - Generalised defines for each constructor its resulting index. Example: identity type.

Many more Investigations

- E.g. **Bove/Capretta**'s formulation of partial functions as inductive-recursive definitions.
- Use of induction recursion in **generic programming**
- Examples:
  - Recently **Randy Pollack** usage of induction-recursion in his theory of bindings.
  - **Surreal numbers** as an extended inductive-recursive definition and inductive-inductive definition (Forsberg).
- ...

Induction-Induction (Forsberg, AS)

- Induction-Induction means that we define
  - $A : \mathrm{Set}$ inductively,
  - while defining simultaneously $B : A \to \mathrm{Set}$ inductively.
- Extracted from PhD thesis Danielsson (2007) formalising the syntax of inductive definitions.
- Essentially
  - Induction-recursion allows to formulate models of type theory
  - Induction-induction allows to formulate the syntax of type theory.

Defining Syntax using Induction-Induction

- Formulate Syntax of Type Theory inside Type Theory
- Define inductively simultaneously:
  - $\widehat{\mathrm{Context}} : \mathrm{Set}$.
    - $\Gamma : \widehat{\mathrm{Context}}$ represents
      $\Gamma \Rightarrow \mathrm{Context}$.
  - $\widehat{\mathrm{Set}} : \widehat{\mathrm{Context}} \to \mathrm{Set}$.
    - $A : \widehat{\mathrm{Set}}\ \Gamma$ represents
      $\Gamma \Rightarrow A : \mathrm{Set}$.
  - $\widehat{\mathrm{Term}} : (\Gamma : \widehat{\mathrm{Context}}) \to (A : \widehat{\mathrm{Set}}\ \Gamma) \to \mathrm{Set}$.
    - $r : \widehat{\mathrm{Term}}\ \Gamma\ A$ represents
      $\Gamma \Rightarrow r : A$.
  - $\widehat{\mathrm{Set}}_{=} : (\Gamma : \widehat{\mathrm{Context}}) \to (A, B : \widehat{\mathrm{Set}}\ \Gamma) \to \mathrm{Set}$.
    - $p : \widehat{\mathrm{Set}}_{=}\ \Gamma\ A\ B$ represents a derivation of
      $\Gamma \Rightarrow A = B : \mathrm{Set}$.
  - etc.

LICS 2013 (Ghani, Hancock, Malatesta, Forsberg, AS)

- Categorical generalisation.
- Main consideration rule

$$\frac{A : \mathrm{Set} \qquad F : (A \to D) \to \mathrm{IR}_D}{\delta_A(F) : \mathrm{IR}_D}$$

$$\mathbb{F}^{\mathrm{U}}_{\delta_A(F)}(U, T) = (g : A \to U) \times \mathbb{F}^{\mathrm{U}}_{F(T \circ g)}(U, T) : \mathrm{Set}$$

$$\mathbb{F}^{\mathrm{T}}_{\delta_A(F)}(U, T, \langle g, x \rangle) = \mathbb{F}^{\mathrm{T}}_{F(T \circ g)}(U, T, x) : D$$

Fibred Induction-Recursion

- Let $\mathrm{Fam}(D) := (U : \mathrm{Set}) \times (U \to D)$.
- The functor
$$\begin{array}{l} \mathrm{index} : \mathrm{Fam}(D) \to \mathrm{Set} \\ \mathrm{index}(\langle U, T \rangle) = U \end{array}$$
  is a split fibration.
- Replace
  - $\mathrm{index} : \mathrm{Fam}(D) \to \mathrm{Set}$ by an arbitrary split fibration $K : \mathcal{E} \to \mathcal{B}$,
    therefore $\langle U, T \rangle : \mathrm{Fam}(D)$ is replaced by $Q : \mathcal{E}$
  - $A \to D$ by the discrete fibre $|\mathcal{E}_A|$ over A,
  - $T \circ g$ (for $g : A \to U$) by $g^*(Q)$,
  - $F(T \circ a)$ by $F(g^*(Q))$.

New Rule

$$\frac{A : \mathcal{B} \qquad F : |\mathcal{E}_A| \to \mathrm{IR}_K}{\delta_A(F) : \mathrm{IR}_K}$$

$$\mathbb{F}_{\delta_A(F)}(Q) = (g : A \to K\ Q) \times \mathbb{F}_{F(g^*(Q))}(Q)$$

What do we get?

- Generalisation from $\mathrm{Fam}(D)$ to arbitrary fibrations.
- Indexed IR is now a special cases.
- Relational IR (define $U : \mathrm{Set}$ and $T : U \times U \to \mathrm{Set}$) might become an example.
- $\mathbb{F}_\gamma : \mathcal{E}^{\mathrm{sp}} \to \mathcal{E}^{\mathrm{sp}}$ is a functor.
- Existence theorem of initial algebras for $\mathbb{F}_\gamma$.

Conclusion

- Emergence of a generalised schema of inductive definitions from Backhouse to Dybjer 1989/90.
- Emergence of inductive-recursive definitions May 1992.
- Closed formalisation.
- Induction-induction.
- Fibred induction-recursion.

Research Questions

- More practical examples in computing and mathematics.
- Combination of induction-recursion with the Mahlo principle.
- Coinduction-corecursion.