Complex Tropical Currents, Extremality, and Approximations *

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Abstract

To a tropical p-cycle $V_{\mathbb{T}}$ in $\mathbb{R}^n$, we naturally associate a normal closed and (p,p)-dimensional current on $(\mathbb{C}^*)^n$ denoted by $\mathscr{T}_n^p(V_{\mathbb{T}})$. Such a "tropical current" $\mathscr{T}_n^p(V_{\mathbb{T}})$ will not be an integration current along any analytic set, since its support has the form $\mathrm{Log}^{-1}(V_{\mathbb{T}}) \subset (\mathbb{C}^*)^n$, where Log is the coordinate-wise valuation with $\log(|.|)$. We remark that tropical currents can be used to deduce an intersection theory for effective tropical cycles. Furthermore, we provide sufficient (local) conditions on tropical p-cycles such that their associated tropical currents are "strongly extremal" in $\mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$. In particular, if these conditions hold for the effective cycles, then the associated currents are extremal in the cone of strongly positive closed currents of bidimension (p,p) on $(\mathbb{C}^*)^n$. Finally, we explain certain relations between approximation problems of tropical cycles by amoebas of algebraic cycles and approximations of the associated currents by positive multiples of integration currents along analytic cycles.

1 Introduction

A positive closed current T on a complex manifold X is called extremal in the cone of closed positive currents, if any decomposition of T into a sum of two non-zero positive closed currents, $T = T_1 + T_2$, implies that T_1 and T_2 are positive multiples of T (see Section 2 for an introduction to currents). It was noted in [8] that the cone of positive closed (p,p)-currents is the closed convex envelope of the extremal elements of this cone (endowed the weak topology of currents). Pierre Lelong in [17] showed that integration currents along irreducible analytic cycles are extremal, and also asked whether positive multiples of integration currents along irreducible analytic cycles are the only extremal currents. Subsequently in [8], Jean-Pierre Demailly found an example of an extremal current in $\mathbb{CP}^2$, namely $T_D := dd^c \log\max\{|z_0|, |z_1|, |z_2|\}$,

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which has a support of real dimension 3, and therefore cannot be an integration current along any analytic set. Later on, Eric Bedford noticed that many extremal currents naturally occur in dynamical systems on several complex variables whose supports are in general fractal sets, and therefore not analytic (see [27], [10], [14], [11] and references therein).

Here we attempt to generalize Demailly's example, using the fact that T_D has the extremality property, but in a stronger sense : that for every other normal closed current $\tilde{T}$ of bidimension $(1,1)$ on $\mathbb{CP}^2$, which has the same support as T_D there exists a $\rho \in \mathbb{C}$ such that $\tilde{T} = \rho\, T_D$. Evidently, this **strong extremality** is a property of the supports. Defining $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$, $(z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$, we note that for $n = 2$, the restriction of T_D to $(\mathbb{C}^*)^2$ is just the closed positive current of bidimension $(1,1)$ given by $dd^c \log\max\{1, |z_1|, |z_2|\}$. Thus, the support of this current is $\mathrm{Log}^{-1}(L_{\mathbb{T}})$, where $L_{\mathbb{T}}$ is a tropical line in $\mathbb{R}^2$. Accordingly, we attach to any tropical p-cycle $V_{\mathbb{T}} \subset \mathbb{R}^n$ (Definition 3.3) a normal closed current of bidimension (p,p) with support equal to $\mathrm{Log}^{-1}(V_{\mathbb{T}}) \subset (\mathbb{C}^*)^n$. We refer to such a current as the **tropical current** attached to the tropical p-cycle $V_{\mathbb{T}} \subset \mathbb{R}^n$, denoted by $\mathscr{T}_n^p(V_{\mathbb{T}})$. Such a construction could actually be carried out for any weighted polyhedral p-dimensional complex $\mathcal{P}$ and one will prove that if $\mathcal{P}$ fulfills the balancing condition at each facet (*i.e.* face of codimension one), then closedness of $\mathscr{T}_n^p(\mathcal{P})$ is implied (Theorem 4.8). Moreover, extremality of the tropical currents in the stronger sense is detectable from the combinatorial data of the corresponding tropical cycles ; suppose that a tropical p-cycle $V_{\mathbb{T}} \subset \mathbb{R}^n$ is connected in codimension 1. Also that at each facet W, a set of primitive vectors $\{v_1, \ldots, v_s\}$ which makes the balancing condition hold at W satisfies the following two conditions : first $\{h_W(v_1), \ldots, h_W(v_s)\}$, spans the dual space $W^{\perp}$ as an $\mathbb{R}$-basis, where h_W is the projection along W; second, every proper subset of $\{h_W(v_1), \ldots, h_W(v_s)\}$ is a set of independent vectors ; then the current $\mathscr{T}_n^p(V_{\mathbb{T}})$ stands as a strongly extremal element of (p,p)-dimensional normal closed currents on $(\mathbb{C}^*)^n$, (Theorem 4.8).

Employing tropical geometry, we will also show that tropical currents associated to effective tropical hypersurfaces are of the form $dd^c\,[q \circ \mathrm{Log}\,]$, where $q : \mathbb{R}^n \to \mathbb{R}$ is a tropical polynomial and,

$$dd^c\,[q \circ \mathrm{Log}\,] = \mathscr{T}_n^p(V_{\mathbb{T}}(q))\,,$$

where $V_{\mathbb{T}}(q)$ is the associated (effective) tropical hypersurface (Theorem 5.2). We remark that using a formula of Alexander Rashkovskii in [24] and expansion of Monge-Ampère measure of a tropical polynomial (as in Example 3.20 in [30]), the above relation provides an intersection theory for the effective tropical cycles (Remark 5.4).

In addition we prove (Theorem 6.4) that if for a $V_{\mathbb{T}}$, $\mathscr{T}_n^p(V_{\mathbb{T}})$ is positive and extremal, and $V_{\mathbb{T}}$ is **set-wise** approximable (Definition 6.1) by amoebas of algebraic cycles in $(\mathbb{C}^*)^n$, then $\mathscr{T}_n^p(V_{\mathbb{T}})$ is in the closure (in the sense of currents) of

$$\mathcal{I}^p((\mathbb{C}^*)^n) = \big\{\lambda[Z] : Z \;\; p\text{-dim. irreducible analytic subset in } (\mathbb{C}^*)^n,\, \lambda \geq 0\big\}.$$

This article is structured as follows. In Section 2, we state the preliminaries of the theory of currents. Section 3 is a brief discussion of tropical geometry. In Section 4, we define the tropical current $\mathscr{T}_n^p(\mathcal{P})$ attached to a weighted p-dimensional polyhedral complex $\mathcal{P}$. In this section, we state and prove the main result of this paper about extremality (mentioned above) ; this is done step by step, first in the case $p = 1$, then in the case $p > 1$ assuming $\mathcal{P}$ has a single facet and then finally in the general case of $p > 1$ tropical cycles. In Section 5, we illustrate how these constructions may be used in order to produce extremal currents on complex projective planes, and we remark how an intersection theory can be deduced. In Section 6, we discuss the problems of the approximability of tropical cycles by amoebas and will explain how it could be related to the problem of approximating the tropical currents by analytic cycles. We end this paper with some open problems.

2 Currents

Throughout this paper X is either $(\mathbb{C}^*)^n$, $\mathbb{C}^n$ or $\mathbb{CP}^n$, which are analytic complex manifolds of dimension n. If k, p, q are non-negative integers, possibly $k = \infty$, we denote by $\mathcal{C}^k_{p,q}(X)$ (resp. $\mathcal{D}^k_{p,q}(X)$) the space of differential forms of bidegree (p, q) and of class C^k (resp. with compact support) on X. The elements of $\mathcal{D}^k_{p,q}(X)$ are called test forms.

The space of currents of **order** k and of bidimension (p, q), or equivalently of bidegree $(n - p, n - q)$, is by definition the topological dual space $[\mathcal{D}^k_{p,q}(X)]'$, where $\mathcal{D}^k_{p,q}(X)$ is endowed with the inductive limit topology. $\mathcal{D}^\infty_{p,q}(X)$ (resp. $[\mathcal{D}^\infty_{p,q}(X)]'$) is usually denoted instead by $\mathcal{D}_{p,q}(X)$ (resp. by $\mathcal{D}'_{p,q}(X)$). A current $T \in \mathcal{D}'_{p,q}(X)$ is called **closed** if for every $\alpha \in \mathcal{D}_{p-1,q}(X)$,

$$\langle dT, \alpha\rangle := (-1)^{p+q+1}\langle T, d\alpha\rangle\,, \tag{2.1}$$

vanishes.

An important concept in this theory is positivity.

Definition 2.1. A form $\psi \in \mathcal{C}^0_{p,p}(X)$ is called

- strongly positive, if for all $z \in X$, $\psi(z)$ is in the convex cone generated by (p, p) forms of the type

  $$(i\psi_1 \wedge \bar{\psi}_1) \wedge \cdots \wedge (i\psi_p \wedge \bar{\psi}_p),$$

  where $\psi_j \in \bigwedge^{1,0} T^*_z X$;

- positive, if at every point $z \in X$ and all p-planes F of the tangent space T_zX, the restriction $\psi(z)_{|F}$ is a strongly positive (p,p)-form.

A current $T \in \mathcal{D}'_{p,p}(X)$ is called strongly positive (resp. positive), if

$$\langle T, \psi \rangle \geq 0$$

for every positive (resp. strongly positive) test form $\psi \in \mathcal{D}_{p,p}(X)$. We denote the set of positive (resp. strongly positive) closed currents of bidimension (p,p) by

$$PC^p(X), \quad (\text{resp. } SPC^p(X)).$$

In this paper we are mainly concerned with **extremal** currents. Recall that the **support** of a current is the smallest closed set in the ambient space X such that on its complement the current vanishes, and a current T is called **normal** if T and dT are of order zero. One can see that every closed positive current is normal.

Definition 2.2. A current $T \in PC^p(X)$ (resp. $\in SPC^p(X)$) is called extremal in $PC^p(X)$ (resp. in $SPC^p(X)$) if whenever we have a decomposition $T = T_1 + T_2$ with $T_1, T_2 \in PC^p(X)$ (resp. $\in SPC^p(X)$), then there exist $\lambda_1, \lambda_2 \geq 0$ such that $T = \lambda_1 T_1$ and $T = \lambda_2 T_2$. A closed current $T \in \mathcal{D}'_{p,p}(X)$ of order zero is called strongly extremal, if for any closed current $\tilde{T} \in \mathcal{D}'_{p,p}(X)$ of order zero which has the same support as T, there exists a $\rho \in \mathbb{C}$ such that $T = \rho \tilde{T}$.

Remark 2.3. Note that the extremality properties are invariant under invertible affine linear transformations. Furthermore, strong extremality of a positive (resp. strongly positive) closed current $T \in \mathcal{D}'_{p,p}(X)$ implies extremality $PC^p(X)$ (resp. in $SPC^p(X)$) . In addition, strong extremality can be considered as a "rigidity" property of supports (see also [11]). Therefore, if a normal closed (p,p)-dimensional current T is supported by a set of the form $\text{Log}^{-1}(V_{\mathbb{T}}) \subset (\mathbb{C}^*)^n$ for a tropical p-cycle $V_{\mathbb{T}}$, then the question of the strong extremality of T relies merely on the combinatorial structure of $V_{\mathbb{T}}$.

Let us denote

$$\mathcal{I}^p(X) = \big\{\lambda[Z] : \ \lambda \geq 0, \ Z \subset X, \ p\text{-dimensional irreducible analytic subset } \big\},$$

and by $\mathcal{E}^p(X)$ the set of extremal elements of $SPC^p(X)$. Using the support theorems below, it is not hard to see that ([17], [7])

$$\mathcal{I}^p(X) \subset \mathcal{E}^p(X).$$

2.1 Support theorems

We need to quote two important structure theorems for supports of currents. For a through treatment see [7].

Let $S \subset X$ be a closed C^1 real submanifold of X ($= (\mathbb{C}^*)^n$, $\mathbb{C}^n$ or $\mathbb{CP}^n$). The complex dimension of the holomorphic tangent in S, i.e.

$$\dim_{\mathbb{C}} (T_x S \cap iT_x S) ,$$

is called the **Cauchy-Riemann** dimension of S at x. The maximal dimension

$$\max_{x \in S} \dim_{\mathbb{C}} (T_x S \cap iT_x S)$$

is called the Cauchy-Riemann dimension of S, denoted by $\mathrm{CRdim}\, S$. If this dimension is constant for all $x \in S$, then S is called a Cauchy-Riemann submanifold of X.

The following theorem implies that a complex structure of dimension at least p is needed on the support of a normal current in order to accommodate (p, p) test forms.

Theorem 2.4 (Theorem 2.10 in [7]). *Suppose $T \in \mathcal{D}'_{p,p}(X)$ is a normal current. If the support of T is contained in a real submanifold S of Cauchy-Riemann dimension less than p, then $T = 0$.*

The next theorem about supports permits us to streamline a current if its support is a fiber space.

Theorem 2.5 (Theorem 2.13 in [7]). *Let $S \subset X$ be a Cauchy-Riemann submanifold with Cauchy-Riemann dimension p such that there is a submersion $\sigma : S \to Y$ of class C^1 whose fibers $\sigma^{-1}(y)$ are connected and that for all the points $z \in S$ we have*

$$T_z S \cap i\, T_z S = T_z F_z,$$

where $F_z = \sigma^{-1}(\sigma(z))$ is the fiber of the point z and $T_z S$, $T_z F_z$ are the tangent spaces at z corresponding to S and F_z. Then, for every closed currents T of bidimension (p, p) and of order 0 (resp. positive) with support in S, there exists a unique (resp. positive) Radon measure μ on M such that

$$T = \int_{y \in Y} [\sigma^{-1}(y)] d\mu(y),$$

i.e.

$$\langle T, \psi \rangle = \int_{y \in Y} \Big(\int_{\sigma^{-1}(y)} \psi \Big) d\mu(y) ,$$

for $\psi \in \mathcal{D}_{p,p}(X)$.

3 Tropical cycles

We start off by recalling a definition of tropical curves. Throughout this article a **rational** graph is a finite union of rays and segments in $\mathbb{R}^n$ whose directions have rational coordinates. We call these rays and segments (1-cells) as edges and the endpoints (0-cells) as vertices. Hence a graph Γ is the data $(\mathcal{C}_0(\Gamma), \mathcal{C}_1(\Gamma))$ of the 0-cells and 1-cells. A **primitive** vector is an integral vector such that the greatest common divisor of its components is 1. For each edge e incident to a vertex a there exists a primitive vector v_e which has a representative with support on e pointing away from a. Assume that every edge e of Γ is weighted by a non-zero integer m_e. We say that Γ satisfies the **balancing condition** at a vertex a if

$$\sum_{\{a\} \prec\, e \,\in \mathcal{C}_1(\Gamma)} m_e v_e = 0, \tag{3.2}$$

where the sum is taken over all the edges incident to the vertex a.

Definition 3.1. A tropical curve in $\mathbb{R}^n$ is a weighted rational graph $\Gamma = (\mathcal{C}_0(\Gamma), \mathcal{C}_1(\Gamma))$ which satisfies the balancing condition (3.2) at every vertex $a \in \mathcal{C}_0(\Gamma)$.

In the same spirit, one can define the **tropical p-cycles** in $\mathbb{R}^n$. First, a **polyhedral complex** is a finite set of polyhedra which are joined to each other along common faces. A polyhedral complex is called **rational** if each polyhedron is the intersection of rational half spaces, *i.e.* the half spaces which are given by the inequalities of the form

$$\langle \nu, x \rangle \geq a\,, \quad \text{with given constants } \nu \in \mathbb{Z}^n\,, a \in \mathbb{R}^n\,, \quad \forall x \in \mathbb{R}^n\,.$$

Such a complex is said to be **weighted** if a non-zero integral weight is assigned to each of its p-dimensional cells. Now assume that a $(p-1)$-dimensional face W is adjacent to p dimensional faces $P_1, \ldots, P_s$, $s \geq 2$, which have respective weights $m_1, \ldots, m_s$. Choose a point a in W and respective primitive vectors v_j, $(j = 1, \ldots, s)$, emanating from a inward each P_j. One defines the balancing condition in higher dimensions to be that the sum

$$\sum_{j=1}^{s} m_j\, v_j \quad \text{is parallel to } \ W\,. \tag{3.3}$$

Remark 3.2. Assume that W lies in an affine $(p-1)$-plane H_W and that each V_j lies in an affine p-plane H_{V_j}. One can find a $\mathbb{Z}$-basis $\{w_1, \ldots, w_{p-1}\}$ for $W \cap \mathbb{Z}^n$ (the initial point for these vectors is considered to be a point in W) and extend it to $\{w_1, \ldots, w_{p-1}, v_j\}$ for each $j = 1, \ldots, s$, such that $\{w_1, \ldots, w_{p-1}, v_j\}$ is a $\mathbb{Z}$-basis for $H_{V_j} \cap \mathbb{Z}^n$ and the balancing condition (3.3) is satisfied by the v_j. This simply implies that $\sum_{j=1}^{s} m_j\, v_j$ lies in H_W; in other words every $p \times p$ minor of the $n \times p$ matrix with columns vectors $\big(w_1, \ldots, w_{p-1}, \sum_{j=1}^{s} m_j\, v_j\big)$ vanishes.

Definition 3.3 ([22], [25], [26])**.** A weighted rational polyhedral complex of pure dimension p is called a tropical p-cycle if the balancing condition (3.3) is satisfied at every codimension 1 face. Such a cycle is called effective if every weight is a positive integer.

Therefore, tropical 1-cycles are the tropical graphs. Also, a tropical $(n-1)$-cycle in $\mathbb{R}^n$ is called a tropical hypersurface. To define the effective tropical cycles of codimension 1, one might use **tropical polynomials** which are defined as follows.

Definition 3.4. A tropical Laurent polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is a function of the form

$$(x_1, \dots, x_n) \mapsto \max \big\{ c_{\alpha_1,\dots,\alpha_n} + \alpha_1 x_1 + \dots + \alpha_n x_n \big\}, \tag{3.4}$$

over a finite set of indices, in which α_i, $i = 1, \dots, n$ are integer numbers and $c_{\alpha_1,\dots,\alpha_n}$ are real numbers; we might abbreviate the notation to

$$x \mapsto \max_{\alpha} \big\{ c_\alpha + \langle \alpha, x \rangle \big\}, \tag{3.5}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\langle \ , \ \rangle$ is the usual inner product in $\mathbb{R}^n$.

To justify the above definition one considers the **tropical semi-field** $(\mathbb{T}, \oplus, \odot)$. Where $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, with the operations $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$ for $a, b \in \mathbb{T}$. Then the usual definition of a Laurent polynomial carried with tropical operations instead of the usual ones leads to that of a tropical Laurent polynomial in which the c_α's are coefficients and α_i's the respective degrees. If all $\alpha_i \in \mathbb{Z}^{\geq 0}$, the tropical Laurent polynomial p is said to be a tropical polynomial. The **tropical hypersurface** corresponding to a given tropical Laurent polynomial p is denoted by $V_{\mathbb{T}}(p)$ and defined as the set below

$$V_{\mathbb{T}}(p) = \big\{ x \in \mathbb{R}^n : \text{values of at least two monomials in } p \text{ coincide and maximize at } x \big\},$$

which is basically the corner locus of $x \mapsto p(x)$: the set of points over which the graph of the piece-wise linear convex function p is broken. This set has a rational polyhedral complex structure of pure dimension $n-1$. However, we still need to assign the weights to each of the polyhedra to make it an honest tropical cycle : suppose F is a $(n-1)$-dimensional cell where the monomials $c_{\alpha_j} + \langle \alpha_j, x \rangle$, $\alpha_j \in \mathbb{Z}^n$, $j = 1, \dots, s$ are equal and maximized, then for dimensional reasons, the slopes α_j lie in a line in $\mathbb{Z}^n$; the weight $w(F)$ assigned to F is the maximal lattice length of this line segment connecting the points representing these slopes (the lattice length of a line segment being the number of lattice points on this line minus 1) ; one can check that with such weights $V_{\mathbb{T}}(p)$ satisfies the balancing condition at each facet.

One also defines the **tropical projective space** in the following way.

Definition 3.5. The tropical projective space $\mathbb{TP}^n = \mathbb{T}^n \backslash \{(-\infty)^n\}/\sim$ where $\sim$ is the relation defined by

$$(x_0, \dots, x_n) \sim (\lambda \odot x_0, \dots, \lambda \odot x_n), \quad \lambda \in \mathbb{T}^* = \mathbb{R}\,.$$

Now one considers the tropical cycles in $\mathbb{TP}^n$, which are locally the tropical cycles in open subsets of $\mathbb{R}^n$. Accordingly, the **homogeneous** tropical polynomials and the associated tropical hypersurfaces in $\mathbb{TP}^n$. The theorem below is now classic in tropical geometry.

Theorem 3.6 ([19], [25] for $n = 2$)**.** *Every effective tropical hypersurfaces in $\mathbb{R}^n$ (resp. $\mathbb{TP}^n$) is of the form $V_{\mathbb{T}}(p)$, for a tropical (resp. homogeneous tropical) polynomial p.*

Another important notion to be recalled here is the notion of **amoeba**. Consider

$$\mathrm{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \dots, z_n) \mapsto \left(\frac{\log|z_1|}{\log t}, \dots, \frac{\log|z_n|}{\log t}\right) \tag{3.6}$$

(when $t = \exp(1)$ we drop the subscript).

Definition 3.7 ([12])**.** The amoeba of an algebraic subvariety $V \subset (\mathbb{C}^*)^n$, denoted by $\mathcal{A}_V$, is the set $\mathrm{Log}\,(V) \subset \mathbb{R}^n$.

Given a family $(X_t)_{t\in\mathbb{R}_+}$ of algebraic subvarieties of $(\mathbb{C}^*)^n$, one considers the family of amoebas $\mathrm{Log}_t(X_t) \subset \mathbb{R}^n$. Assume that $\mathrm{Log}_t(X_t)$, as t goes to infinity, converges (with respect to the Hausdorff metrics on compact sets of $\mathbb{R}^n$) to a limit set X ; then X inherits a structure of a tropical cycle, *i.e.* as a set it is a rational polyhedral complex, which can moreover be equipped with positive integer weights to become balanced (see [28]). Therefore, a natural question arises : which effective tropical cycles can be realized as Hausdorff limit of amoebas of a family of algebraic subvarieties of $(\mathbb{C}^*)^n$? We are concerned with a modification of this problem in Section 6 and we relate these approximations to approximations of tropical extremal currents by analytic cycles.

4 Tropical currents

Assume $V_{\mathbb{T}}$ is a tropical p-cycle. We define a current supported on $\mathrm{Log}^{-1}(V_{\mathbb{T}})$ which inherits the respective weights of $V_{\mathbb{T}}$ and then determine whether this current is strongly extremal. We introduce the following abridged notations.

Notation 4.1. *For a complex number ζ and an integral vector $\nu = (\nu_1, \ldots, \nu_m)$ ($m \in \mathbb{N}^*$) we set*

$$\zeta^\nu = (\zeta^{\nu_1}, \ldots, \zeta^{\nu_m}).$$

Moreover for two vectors $\nu = (\nu_1, \ldots, \nu_m)$, $\nu' = (\nu'_1, \ldots, \nu'_m)$

$$\nu \star \nu' := (\nu_1 \nu'_1, \ldots, \nu_m \nu'_m).$$

Recall that a rational p-plane in $\mathbb{R}^n$ means that is given by the equations

$$\langle \nu_i, x \rangle = 0, \quad \nu_i \in \mathbb{Z}^n \,, i \,=\, 1, \ldots, n-p\,.$$

Lemma 4.2. *Suppose H is a rational p-plane in $\mathbb{R}^n$ (which passes the origin), ($1 \leq p \leq n$). Let $B = \big(w_1, \ldots, w_p\big)$ and $B' = \big(w'_1, \ldots, w'_p\big)$ be two $\mathbb{Z}$-basis for $H \cap \mathbb{Z}^n$. Define for any $\gamma \in (\mathbb{S}^1)^n$, the two subsets of $(\mathbb{C}^*)^n$:*

$$Z_B^\gamma := \{\tau_1^{w_1} \star \cdots \star \tau_p^{w_p} \star \gamma = \iota_\gamma^w(\tau) \; ; \tau_1, \ldots, \tau_p \in \mathbb{C}^*\}$$

and

$$Z_{B'}^\gamma = \{\tau_1^{w'_1} \star \cdots \star \tau_p^{w'_p} \star \gamma = \iota_\gamma^{w'}(\tau) \; ; \tau_1, \ldots, \tau_p \in \mathbb{C}^*\}.$$

Then, the integration currents

$$T = [Z_B^\gamma] := (\iota_\gamma^w)_*([(\mathbb{C}^*)^p]), \quad T' = [Z_{B'}^\gamma] := (\iota_\gamma^{w'})_*[(\mathbb{C}^*)^p]$$

coincide.

Proof. The analytic sets Z_B^γ and $Z_{B'}^\gamma$ are equal. We prove that they are analytically isomorphic. Consider B, B' as matrices with the given vectors as columns. There exists $C \in GL(p, \mathbb{Z})$, such that $B\,C = B'$. Set

$$(\tau'_1, \ldots, \tau'_p) = (\tau_1^{c_1} \star \cdots \star \tau_p^{c_p})$$

where $c_1, \ldots, c_p$ are the columns of C. This is an invertible monoidal change of coordinates, and it is easy to see that

$$(\tau'_1)^{w_1} \star \cdots \star (\tau'_p)^{w_p} = \tau_1^{w'_1} \star \cdots \star \tau_p^{w'_p}\,,$$

which concludes the proof. $\square$

Remark 4.3. The sets of the form Z_B^γ, when $\gamma = 1$, are referred to as **toric** sets [29]. They can be understood as zero locus of binomial ideals in $\mathbb{C}^n$. In fact, if $\xi_1, ..., \xi_M$ is a set of primitive generators for $\operatorname{Ker} B^t \cap \mathbb{Z}^n$, such that each ξ_ℓ is split as $\xi_\ell^+ - \xi_\ell^-$, where $\xi_\ell^+ = (\xi_{\ell,1}^+, \ldots, \xi_{\ell,n}^+)$ and $\xi^- = (\xi_{\ell,1}^-, \ldots, \xi_{\ell,n}^-)$ have non-negative components in $\mathbb{Z}^n$ and disjoint supports, then the current $[Z_B^\gamma]_{\mathrm{red}}$ is given by

$$\mathbb{1}_{Z_B^\gamma} \cdot \Big[dd^c \log\Big(\sum_{\ell=1}^M \Big|\prod_{j=1}^n \zeta_j^{\xi_{\ell,j}^+} - \prod_{j=1}^n \gamma_j^{\xi_{\ell,j}} \zeta_j^{\xi_{\ell,j}^-}\Big|^2\Big)\Big]^{n-p}$$

by King's formula (see [7], page 181).

As before let $H = H_0$ be a rational p-plane (passing through 0). One can find a $\mathbb{Z}$-basis for the lattice $L_H := H \cap \mathbb{Z}^n$, $B = \big(w_1, \dots, w_p\big)$. Moreover, B can be completed as $D = \big(w_1, \dots, w_p, u_1, \dots, u_{n-p}\big)$ which stands, as a set, as a $\mathbb{Z}$-basis for $\mathbb{Z}^n$ (if D denotes the matrix of such vectors as columns, one has $\det D = \pm 1$). Note that, if D and D' are two such completions of B, one has

$$D' = D \cdot \begin{bmatrix} \mathrm{Id}_p & 0 \\ K & \widetilde{C} \end{bmatrix}$$

where K and $\widetilde{C}$ are respectively $(n-p, p)$ and $(n-p, n-p)$ matrices with integer coefficients and $\det \widetilde{C} = (\det D)^{-1} \times \det D' = \pm 1$. Fix for the moment a basis B and consider such a completion $D_B = D$ of B. Consider, for each $(\theta_{p+1}, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}$, the set

$$\Delta_{H,D}(\theta) := \{\tau_1^{w_1} \star \cdots \star \tau_p^{w_p} \star e^{2i\pi\theta_{p+1}u_1} \star \cdots \star e^{2i\pi\theta_n u_{n-p}}\,;\, \tau \in (\mathbb{C}^*)^p\}. \tag{4.7}$$

This is a p-dimensional analytic subset of $(\mathbb{C}^*)^n$ which is a toric set of the form $Z_B^{\gamma_u}$. In addition, one can parametrize $S_H := \mathrm{Log}^{-1}(H)$ in the following way :

$$S_H = \Big\{\tau_1^{w_1} \star \cdots \star \tau_p^{w_p} \star e^{2\pi\theta_{p+1}u_1} \star \cdots \star e^{2i\pi\theta_n u_{n-p}}\,;\, \tau \in (\mathbb{C}^*)^p,\ (\theta_{p+1}, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}\Big\}.$$

Therefore each $\Delta_{H,D}(\theta_{p+1}, \dots, \theta_n)$ can be considered as the fiber over $(\theta_{p+1}, ..., \theta_n)$ of the submersion $\sigma_{H,D}$:

$$\begin{array}{c} \tau_1^{w_1} \star \cdots \star \tau_p^{w_p} \star e^{2i\pi\theta_{p+1}u_1} \star \cdots \star e^{2i\pi\theta_n u_{n-p}} \in S_H \\ \Big\downarrow \sigma_{H,D} \\ (\theta_{p+1}\,, \dots, \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}\,. \end{array}$$

We define the positive (p,p) current $T_{H,D}$

$$T_{H,D} = \int_{(\theta_{p+1}, \dots, \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}} \big[\Delta_{H,D}(\theta_{p+1}, \dots, \theta_n)\big]\, d\theta_{p+1} \dots d\theta_n\,. \tag{4.8}$$

If one considers two completions $D = (B, U)$ and $D' = (B, U')$ of B, though the fibers $\Delta_{H,D}$ do vary when D is changed into D' (as well as the integration currents $[\Delta_{H,D}]$)), the sum $T_{H,D}$ does not since $U' = U \cdot \widetilde{C}$ (where $\widetilde{C} \in GL(n-p, \mathbb{Z})$) and the Lebesgue measure on $(\mathbb{R}/\mathbb{Z})^{n-p}$ is preserved under the action of monoidal automorphisms of the torus $(\mathbb{S}^1)^{n-p}$ whose matrix $\widetilde{C}$ of exponents belongs to $GL(n-p, \mathbb{Z})$. As a result, the current $T_{H,D}$ depends only on B and one can write $T_{H,D} = T_{H,D_B} = T_H^{[B]}$ for any completion D_B of B. On the other hand, if U is fixed, it follows from Lemma 4.2 that, if one considers $D = (B, U)$ and $D' = (B', U)$, where B and B' are two lattice basis of L_H, then $[\Delta_{H,D}(\theta)] = [\Delta_{H,D'}(\theta)]$ for any $\theta = (\theta_{p+1}, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}$, hence $T_{H,D} = T_{H,D'}$. Accordingly, $T_{H,D_B} = T_H^{[B]}$ is in fact independent of B, and one defines in such a way a positive current

$$T_H = T_H^{[B]} = T_{H,\{B,U_B\}} = T_{H,D_B}$$

which is independent of the choice of the lattice basis B for $L_H = H \cap \mathbb{Z}^n$ as well as that of its completion $D = D_B = (B, U_B)$ as a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. The support of T_H (considered as a (p,p)-dimensional positive current in $(\mathbb{C}^*)^n$ is clearly $\mathrm{Log}^{-1}(H) = S_H$.

Now assume that $H_a \subset \mathbb{R}^n$ is a rational affine p-plane obtained by translation of a rational p-plane $H = H_0$ via $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Define the linear map

$$\begin{aligned} L_a &: \mathbb{C}^n \to \mathbb{C}^n, \\ z = (z_1, \ldots, z_n) &\mapsto \exp(-a) \star z = (\exp(-a_1)z_1, \ldots, \exp(-a_n)z_n). \end{aligned}$$

Set

$$T_{H_a} := L_a^*(T_H) = \int_{(\theta_{p+1},\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [L_a^{-1}(\Delta_{H,D}(\theta_{p+1}, \ldots, \theta_n))] d\theta_{p+1} \ldots d\theta_n.$$

Accordingly,

$$S_{H_a} = \exp(a) \star S_H,$$

and

$$\Delta_{H_a,D} = \exp(a) \star \Delta_{H,D} \, .$$

It is easily seen that the definition of T_{H_a} is independent of the choice of the base point $a \in H_a$, which makes us ready to propose the following definition.

Definition 4.4. Assume $\mathcal{P}$ is a weighted rational polyhedral complex of pure dimension p. Let $\mathcal{C}_p(\mathcal{P})$ be the family of all p dimensional cells of $\mathcal{P}$. Each $P \in \mathcal{C}_p(\mathcal{P})$ is equipped with a non-zero integral weight m_P and lies in an affine p-plane H_{a_P} which passes through a chosen base point $a_P \in P$. Let

$$\mathscr{T}_P = \mathbb{1}_{\mathrm{Log}^{-1}(\mathrm{int}\ P)}\, T_{H_{a_P}}$$

be the restriction of the positive (p,p)-dimensional current $T_{H_{a_P}}$ (supported by $\mathrm{Log}^{-1}(H_{a_P})$) to $\mathrm{Log}^{-1}(\mathrm{int}\ P) \subset \mathrm{Log}^{-1}(H_{a_P}) \subset (\mathbb{C}^*)^n$. Here $\mathrm{int}(P)$ denotes the relative interior of P in the affine p-plane H_{a_P} . This definition is independent of the chosen base point a_P. We define

$$\mathscr{T}_n^p(\mathcal{P}) = \sum_{P \in \mathcal{C}_p(\mathcal{P})} m_P \, \mathscr{T}_P \, .$$

Obviously, if $\mathcal{P}$ is positively weighted, then $\mathscr{T}_n^p(\mathcal{P})$ is a positive current. In this article we are interested in the case where $\mathcal{P}$ is a tropical cycle $V_{\mathbb{T}}$. In such case, we call $\mathscr{T}_n^p(V_{\mathbb{T}})$ the **tropical current** associated to $\mathcal{P} = V_{\mathbb{T}}$.

Before stating the main theorem of this article, we introduce the following terminology.

Definition 4.5. A set of vectors is said to be linearly sub-independent over a field $\mathbb{K}$ if each proper subset of this set is a set of linearly independent vectors.

Remark 4.6. Suppose that the set of vectors $\{v_1, \dots, v_s\}$ is linearly sub-independent over $\mathbb{R}$ and there exist $a_j, b_j \in \mathbb{C}$, $j = 1, \dots, s$ such that $\sum_{j=0}^{s} a_j v_j = \sum_{j=0}^{s} b_j v_j = 0$. Then there exists a $\rho \in \mathbb{C}$ such that $a_j = \rho\, b_j$ for $j = 1, \dots, s$.

Definition 4.7. A tropical p-cycle $V_{\mathbb{T}} \subset \mathbb{R}^n$ is said to be strongly extremal if

1. $V_{\mathbb{T}}$ is connected in codimension 1 ;
2. each $p-1$ dimensional face (facet) W of $V_{\mathbb{T}}$ is adjacent to exactly $n-p+2$ polyhedra (cells) of dimension p ;
3. for each facet of W of $V_{\mathbb{T}}$, let $\{v_1, \dots, v_{n-p+2}\}$ be the primitive vectors, one in each of the $n-p+2$ polyhedra above, that make the balancing condition hold. Then, the set of their projections along W, $\{h_W(v_1), \dots, h_W(v_{n-p+2})\}$, forms a sub-independent set.

For instance, when $V_{\mathbb{T}} \subset \mathbb{R}^n$ is a tropical 1-cycle, then the strong extremality conditions means that the graph is $(n+1)$-valent at every vertex and the corresponding $(n+1)$- primitive vectors span $\mathbb{R}^n$. It is also clear that for tropical hypersurfaces in the number $n-p+2$ is exactly $n-(n-1)+2 = 3$.

Theorem 4.8. *If $V_{\mathbb{T}} \subset \mathbb{R}^n$ is a tropical p-cycle, then the normal and (p,p)-dimensional tropical current $\mathscr{T}_n^p(V_{\mathbb{T}})$ is closed. If moreover $V_{\mathbb{T}}$ is strongly extremal, then $\mathscr{T}_n^p(V_{\mathbb{T}})$ is strongly extremal in $\mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$.*

In order to make our understanding progressive, we first explore the case of tropical curves ($p = 1$), then that of p-dimensional tropical cycles with a single codimension 1 face, a facet.

4.1 Tropical $(1,1)$-dimensional currents

In this section we study $\mathscr{T}_n^1(\Gamma)$, where Γ is a weighted rational graph. We prove Theorem 4.8 in this case. Suppose an edge e of Γ of weight m_e (spanning the affine line $E \subset \mathbb{R}^n$) is parameterized by

$$t \mapsto t\, v_e + a,$$

where $\{a\}$ ($a \in \mathbb{R}^n$) is one of the vertices of e, $v_e = v_e^{[a\to]} \in \mathbb{R}^n$ is the corresponding (inward) primitive vector for e from the vertex $\{a\}$, and $t \in [0, t_0] \subset \mathbb{R}$ is a real parameter, $t_0 \in [0, +\infty]$; when $t = \infty$ the edge is a ray. We complete $\{v_e\}$ to a basis D_e of the lattice $\mathbb{Z}^n$, say $D_e = (v_e, U_e) = \{v_e, u_1^e, \dots, u_{n-1}^e\}$, that is, if one denotes also D_e as the matrix with columns $v_e^t, (u_1^e)^t, \dots, (u_{n-1}^e)^t$, one has $\det(D_e) = \pm 1$, *i.e.* $D_e \in GL(n, \mathbb{Z})$. We can now define an open subset $S_{e,D_e,a} \subset S_E := \mathrm{Log}^{-1}(E)$ as :

$$\begin{aligned} S_{e,D_e,a} := \Big\{ &\exp(a) \star \tau^{v_e} \star \exp(2i\pi\theta_2 u_1^e) \star \cdots \star \exp(2i\pi\theta_n u_{n-1}^e) \ ; \\ &\tau \in \mathbb{C}^*, \ 1 < |\tau| < \exp(t_0), \ \theta = (\theta_2, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-1} \Big\}. \end{aligned} \tag{4.9}$$

Such an open set $S_{e,D_e,a} \subset S_E$ (considered here as a submanifold with boundary of the manifold S_E with real dimension $n+1$) is injectively foliated over the Cartesian product $(\mathbb{R}/\mathbb{Z})^{n-1}$ through the submersion

$$\begin{array}{c} \exp(a) \star \tau^{v_e} \star \exp(2i\pi\theta_2 u_1^e) \star \cdots \star \exp(2i\pi\theta_n u_{n-1}^e) \in S_{e,D,a} \\ \Big\downarrow {\scriptstyle \sigma_{e,D_e,a}} \\ (\theta_2\,, \dots, \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-1}\,. \end{array}$$

One also denotes as $\boldsymbol{\tau}_{e,D_e,a}$ the parameterization map from $(\mathbb{C}^*)^n$ into itself which is used to get (through its inverse) the submersion $\sigma_{e,D_e,a}$, that is the monoidal map :

$$\boldsymbol{\tau}_{e,D_e,a} \ : (\tau_1, \lambda_2, \dots, \lambda_n) \in (\mathbb{C}^*)^n \mapsto \exp(a) \star \tau_1^{v_e} \star \lambda_2^{u_1^e} \star \cdots \star \lambda_n^{u_{n-1}^e} \in (\mathbb{C}^*)^n.$$

Denote as $\Sigma_{e,D_e,a}$ the cycle

$$\begin{aligned} &\Sigma_{e,D_e,a} := \partial S_{e,D_e,a} \ : \\ &(\theta_1, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^n \mapsto \exp(a) \star \exp(2i\pi\theta_1\, v_e) \star \exp(2i\pi\theta_2\, u_1^e) \star \cdots \star \exp(2i\pi\theta_n u_{n-1}^e) \ . \end{aligned}$$

The support of the cycle $\Sigma_{e,D_e,a}$ equals $\mathrm{Log}^{-1}(\{a\})$. For each $(\theta_2, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-1}$, denote as $\Delta_{e,D_e,a}$ the fiber $\sigma_{e,D_e,a}^{-1}(\{(\theta_2, ..., \theta_n)\})$ of the submersion $\sigma_{e,D_e,a}$ over $(\theta_2, ..., \theta_n)$ and consider the $(1,1)$-dimensional positive current in $(\mathbb{C}^*)^n$ defined as

$$T_{e,D_e,a} := \int_{(\theta_2,\dots,\theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-1}} [\Delta_{e,D_e,a}(\theta_2, ..., \theta_n)]\, d\theta_2 \dots d\theta_n.$$

The current $T_{e,D_e,a}$ is obviously not closed ; but nevertheless, its support is the set $\mathrm{Log}^{-1}(e)$. As we have explained in the beginning of this section, the current $T_{e,D_e,a}$ is independent of the choice of the completion D_e for $\{v_e\}$ because of the invariance of the Lebesgue measure on $(\mathbb{R}/\mathbb{Z})^{n-1}$ under the action of the linear group $GL(n-1,\mathbb{Z})$ (considered in the multiplicative sense). In fact $T_{e,D_e,a}$ depends only on e and stands as the current $\mathscr{T}_e$ obtained as the restriction to the edge e of the positive $(1,1)$-dimensional current T_E (in order to check this point, one can easily reduce the situation up to translation to the case $a=0$). We however keep track of the averaged representation

$$\mathscr{T}_e = T_{e,D_e,a} := \int_{(\theta_2,\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} [\Delta_{e,D_e,a}(\theta_2,...,\theta_n)]\, d\theta_2 \ldots d\theta_n, \tag{4.10}$$

where the average of integration currents $[\Delta_{e,D_e,a}]$ indeed depend on the specified vertex a of e and on the completion D_e of the set $\{v_e\}$, where $v_e = v_e^{[a\to]}$ denotes the primitive (inward) vector spanning E and emanating from its specified vertex a.

Lemma 4.9. *Let ω be a 1-test form on $(\mathbb{C}^*)^n$, with support in a neighborhood of* $\mathrm{Log}^{-1}(\{a\}) \subset (C^*)^n$*, with the restriction*

$$\omega_{|\mathrm{Log}^{-1}(\{a\})} = \sum_{j=1}^{n} \omega_j(t_1,...,t_n)\, dt_j\ .$$

Then

$$\begin{gathered}\langle d\mathscr{T}_e,\omega\rangle = \\ \sum_{j=1}^{n} v_{e,j} \int_{\theta\in(\mathbb{R}/\mathbb{Z})^n} \omega_j\Big(v_{e,1}\theta_1 + \sum_{\ell=1}^{n-1} u^e_{\ell,1}\theta_{\ell+1},\ \ldots, v_{e,n}\theta_1 + \sum_{\ell=1}^{n-1} u^e_{\ell,n}\,\theta_{\ell+1}\Big) d\theta_1\cdots d\theta_n\,.\end{gathered} \tag{4.11}$$

Proof. By definition of differentiation of currents and Stokes' formula, it follows that, for such ω,

$$\begin{aligned}\langle d\mathscr{T}_e,\omega\rangle &:= -\int_{(\theta_2,\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} \big\langle [\Delta_{e,D_e,a}(\theta_2,...,\theta_n)], d\omega\big\rangle\, d\theta_2\ldots d\theta_n \\ &= \int_{(\theta_2,\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} \big\langle [\partial\Delta_{e,D_e,a}(\theta_2,...,\theta_n)], \omega\big\rangle\, d\theta_2\ldots d\theta_n,\end{aligned} \tag{4.12}$$

Note that the induced orientation on boundary of each fiber $\partial\Delta_{e,D_e,a}(\theta_2,\ldots,\theta_n)$ is given by $-d\theta_1$, since this boundary is obtained by letting $\tau_1 = 1$ in (4.9). Moreover, for each fixed $(\theta_2,\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}$, $\partial\Delta_{e,D_e,a}(\theta_2,\ldots,\theta_n)$ can be understood as the image

$$\boldsymbol{\tau}^{(\theta_2,\ldots,\theta_n)}_{e,D_e,a}(\mathbb{R}/\mathbb{Z}) := \boldsymbol{\tau}_{e,D_e,a}\big((\mathbb{R}/\mathbb{Z}),\theta_2,\ldots,\theta_n\big).$$

Therefore,

$$\langle d\mathscr{T}_e,\omega\rangle = \int_{(\theta_2,\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} \int_{\theta_1\in(\mathbb{R}/\mathbb{Z})} \big(\boldsymbol{\tau}^{(\theta_2,\ldots,\theta_n)}_{e,D_e,a}\big)^*(\omega)\,.$$

It is clear that

$$\big(\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\dots,\theta_n)}\big)^*(t_j) = t_j \circ \big(\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\dots,\theta_n)}\big) = v_{e,j}\theta_j + \sum_{\ell=1}^{n-1} u_{\ell,j}^e \theta_{\ell+1}, \tag{4.13}$$

and

$$\big(\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\dots,\theta_n)}\big)^*(dt_j) = d\Big(t_j \circ \big(\boldsymbol{\tau}_{e,D_e,a}^{(\theta_2,\dots,\theta_n)}\big)\Big) = d(v_{e,j}\theta_1 + \sum_{\ell=1}^{n-1} u_{\ell,j}^e \theta_{\ell+1}) = v_{e,j}\, d\theta_1\,,$$

which easily give the result. $\square$

The next lemma relates the balancing condition to closedness of the corresponding currents. We refer the reader for a similar result on "super currents" to [16]. Suppose every edge e of Γ is weighted by a non-zero integer m_e. Then, one has the following lemma.

Lemma 4.10. *Let $\mathcal{P}$ a weighted rational 1-polyedral complex in $\mathbb{R}^n$, $\{a\}$ be one of its vertices and ω be a 1-test form in $(\mathbb{C}^*)^n$ supported in an open neighborhood of $\mathrm{Log}^{-1}(\{a\})$. One has*

$$\langle d\mathscr{T}_n^1(\mathcal{P}), \omega\rangle = \sum_{\{e\in\mathcal{C}_1(\mathcal{P})\,;\,\{a\}\prec e\}} m_e \,\langle d\mathscr{T}_e\,, \omega\rangle = 0 \quad \Longleftrightarrow \quad \sum_{\{e\in\mathcal{C}_1(\mathcal{P})\,;\,\{a\}\prec e\}} m_e v_e^{[a\to]} = 0\,, \tag{4.14}$$

where $\{a\} \prec e$ means that $\{a\}$ is a vertex of the edge e and $v_e^{[a\to]}$ denotes then the inward primitive vector contained in the edge e and pointing away from a ; In particular, the tropical current $\mathscr{T}_n^1(V_{\mathbb{T}})$ attached to a tropical curve $V_{\mathbb{T}}$ is closed.

Proof. To prove the lemma it is enough to check the result for any 1-test form ω in a neighborhood of $\mathrm{Log}^{-1}(\{a\})$ in $(\mathbb{C}^*)^n$ such that $\omega = e^{2i\pi\langle\nu,\theta\rangle}\, d\theta_j$ for some $j \in \{1,\dots,n\}$ and $\nu \in \mathbb{Z}^n$. This follows from the fact that the characters $\theta \mapsto \chi_{n,\nu}(\theta) := e^{2i\pi\langle\nu,\theta\rangle}$ $(\nu \in \mathbb{Z}^n)$ form an orthonormal basis for the Hilbert space $L^2_{\mathbb{C}}((\mathbb{R}/\mathbb{Z})^n, d\theta)$. Then the equivalence stated here follows from the formula (4.11) established in Lemma 4.9. The second claim follows from the fact that the balancing condition is fulfilled at any vertex $\{a\}$ of any tropical curve $V_{\mathbb{T}}$. $\square$

Recall that for a tropical curve $\Gamma \subset \mathbb{R}^n$ strong extremality means $(n+1)$-valency for any vertex $\{a\}$ and sub-independency of the set whose elements are the $(n+1)$ primitive vectors $v_e^{[a\to]}$ ($e \in \mathcal{C}_1(\Gamma)$ such that $\{a\} \prec e$).

Theorem 4.11. *Let $\Gamma \subset \mathbb{R}^n$ be a strongly extremal tropical curve. Then the $(1,1)$-dimensional closed current normal $\mathscr{T}_n^1(\Gamma)$ is strongly extremal in $\mathcal{D}'_{1,1}(\mathbb{C}^*)^n$.*

We first prove first Theorem 4.11 for a tropical curve Γ which has only one vertex.

Lemma 4.12. *Suppose $\Gamma \in \mathbb{R}^n$ is a strongly extremal tropical curve with only one vertex at the origin. Then $\mathscr{T}_n^1(\Gamma)$ is strongly extremal in $\mathcal{D}'_{1,1}(\mathbb{C}^*)^n$.*

Proof. The proof of the lemma is divided into three steps.

Each edge $e \in \mathcal{C}_1(\Gamma)$ is contained in an affine line E. For such a E consider $w = v_e$ the inward primitive vector $w = v_e^{[0\to]}$ initiated from the vertex $\{0\}$, lying in E. We fix an arbitrary completion D_e of $\{v_e\}$ with vectors $u_1^e, ..., u_{n-1}^e$ in $\mathbb{Z}^n$. One has

$$\mathscr{T}_1^n(\Gamma) = \sum_{e\in\mathcal{C}_1(\Gamma)} m_e\, T_{e,D_e,\{0\}}$$

as seen in Subsection 4.1 above.

We assume from now on that $\widetilde{\mathscr{T}}$ is a $(1,1)$-dimensional normal closed current in $(\mathbb{C}^*)^n$ with support equal to that of $\mathscr{T}_1^n(\Gamma)$, *i.e.* $\mathrm{Supp}\,(\widetilde{\mathscr{T}}) = \mathrm{Log}^{-1}(\Gamma)$.

Step 1. For any $e \in \mathcal{C}_1(\Gamma)$, let $\mathcal{U}_e$ be the open subset of $(\mathbb{C}^*)^n$ defined as

$$\mathcal{U}_e := \mathrm{Log}^{-1}\big(\mathbb{R}^n \setminus \bigcup_{\substack{e'\in\mathcal{C}_1(\Gamma)\\ e'\neq e}} |e'|\big).$$

It follows from Theorem 2.5 that, for each $e \in \mathcal{C}_1(\Gamma)$, there is a Radon measure $d\mu_e$ on $(\mathbb{R}/\mathbb{Z})^{n-1}$ such that (as currents in the open subset $\mathcal{U}_e$ of $(\mathbb{C}^*)^n$) :

$$\widetilde{\mathscr{T}}_{|\mathcal{U}_e} = \int\limits_{(\theta_2,...,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} [\Delta_{e,D_e,0}(\theta_2, ..., \theta_n)]\, d\mu_e(\theta_2, ..., \theta_n).$$

Since the normal current $\widetilde{\mathscr{T}}_{|\mathcal{U}_e}$ extends globally as the $(1,1)$-dimensional normal closed current $\widetilde{\mathscr{T}}$ in the whole ambient manifold $(\mathbb{C}^*)^n$, one can certainly define $(1,1)$-dimensional normal current $\widetilde{\mathscr{T}_e}$ in $(\mathbb{C}^*)^n$ as

$$\widetilde{\mathscr{T}_e} := \int\limits_{(\theta_2,...,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-1}} [\Delta_{e,D_e,0}(\theta_2, ..., \theta_n)]\, d\mu_e(\theta_2, ..., \theta_n). \tag{4.15}$$

The support of $\widetilde{\mathscr{T}_e}$ equals $\mathrm{Log}^{-1}(e)$, which implies that all currents $\widetilde{\mathscr{T}_{e'}}$ (for $e' \in \mathcal{C}_1(\Gamma)$ such that e' is distinct from e) vanish in U_e. Hence $\widetilde{\mathscr{T}} = \sum_{e\in\mathcal{C}_1(\Gamma)} \widetilde{\mathscr{T}_e}$ in each U_e. Hence the current $\widetilde{\mathscr{T}} - \sum_{e\in\mathcal{C}_1(\Gamma)} \widetilde{\mathscr{T}_e}$ (which is normal) is supported by $\mathrm{Log}^{-1}(\{0\})$

which equals to the real n-dimensional torus and therefore has Cauchy-Riemann dimension 0. It follows then from Theorem 2.4 that one has the decomposition :

$$\widetilde{\mathscr{T}} = \sum_{e \in \mathcal{C}_1(\Gamma)} \widetilde{\mathscr{T}_e}$$

(as currents this time in the whole ambient space $(\mathbb{C}^*)^n$).

Remark 4.13. Although the current $\mathscr{T}_n^1(\Gamma)$ is not dependent on completions of v_e to lattice bases D_e, the representation in (4.15) is. The representation, indeed depends on the chosen foliation which comes from the completions D_e of v_e to a $\mathbb{Z}$-basis for every edge e of Γ. Therefore as mentioned before, at this point we need to fix a lattice basis for each of the edges of the tropical graph.

Step 2. One can repeat the proof of Lemma 4.9 for each edge $e \in \mathcal{C}_1(\Gamma)$ and use the expression (4.15) of $\widetilde{\mathscr{T}_e}$, in order to get the following result.

Lemma 4.14. *Let ω be a 1-test form on $(\mathbb{C}^*)^n$, with support in a neighborhood of $\operatorname{Log}^{-1}(\{0\})$ with restriction given by*

$$\omega_{|\operatorname{Log}^{-1}(\{0\})} = \sum_{j=1}^{n} \omega_j(t_1, ..., t_n)\, dt_j.$$

Then

$$\langle d\widetilde{\mathscr{T}_e}, \omega\rangle = \sum_{j=1}^{n} v_{e,j} \times$$
$$\int_{\theta \in (\mathbb{R}/\mathbb{Z})^n} \omega_j\Big(v_{e,j}\theta_1 + \sum_{\ell=1}^{n-1} u^e_{\ell,1}\theta_{\ell+1},\ \dots, v_{e,n}\theta_1 + \sum_{\ell=1}^{n-1} u^e_{\ell,n}\,\theta_{\ell+1}\Big)\, d\theta_1 \otimes d\mu_e(\theta_2, ..., \theta_n)\,. \tag{4.16}$$

Step 3. The current $\widetilde{\mathscr{T}} = \sum_{e \in \mathcal{C}_1(\Gamma)} \widetilde{\mathscr{T}_e}$ is closed by hypothesis. We try to fully exploit this property in order to derive information on the measures μ_e, $e \in \mathcal{C}_1(\Gamma)$. To do that, we use the fact that a Radon measure $d\mu$ on the group $(\mathbb{R}/\mathbb{Z})^{n-1}$ is characterized by the complete list of its Fourier coefficients

$$\widehat{\mu}(\nu) = \int_{[0,1]^{n-1}} \chi_{n-1,\nu}(\theta_2, ..., \theta_n)\, d\theta_2 \dots d\theta_n := \int_{[0,1]^{n-1}} \exp(-i\langle \nu, \theta\rangle)\, d\theta_2 \dots d\theta_n$$
$$(\nu \in \mathbb{Z}^{n-1}).$$

Fix $e \in \mathcal{C}_1(\Gamma)$. Let $\omega_\nu^{[1]}$ be a 1-test form on $(\mathbb{C}^*)^n$, with support in a neighborhood of $\operatorname{Log}^{-1}(\{0\})$ such that its restriction is given by

$$(\omega_\nu^{[1]})_{|\operatorname{Log}^{-1}(\{0\})} = \chi_{n,\nu}(t_1, ..., t_n)\, dt_1.$$

After simplifications, (4.16) reduces to the scalar equation :

$$\langle d\,\widetilde{\mathscr{T}_e}, \omega_\nu^{[1]}\rangle = \delta^0_{\langle \nu, v_e\rangle}\,\widehat{\mu}_e\big(-\langle \nu, u_1^e\rangle, \dots, -\langle \nu, u_{n-1}^e\rangle\big)\, v_{e,1} \qquad (4.17)$$

(δ_α^η denotes here the Kronecker's symbol). Since $\widetilde{\mathscr{T}}$ is closed we conclude, after performing the same computations for all e in $\mathcal{C}_1(\Gamma)$, that

$$0 = \langle d\,\widetilde{\mathscr{T}}, \omega_\nu^{[1]}\rangle = \sum_{e\in\mathcal{C}_1(\Gamma)} \langle d\,\widetilde{\mathscr{T}_e}, \omega_\nu^{[1]}\rangle = \sum_{e\in\mathcal{C}_1(\Gamma)} \delta^0_{\langle \nu, v_e\rangle}\,\widehat{\mu}_e\big(-\langle \nu, u_1^e\rangle, \dots, -\langle \nu, u_{n-1}^e\rangle\big)\, v_{e,1}\,. \qquad (4.18)$$

If one performs the same operations when $\omega_\nu^{[1]}$ is replaced by $\omega_\nu^{[j]}$ $(1 \le j \le n)$ such that

$$(\omega_\nu^{[j]})_{|\mathrm{Log}^{-1}(\{0\})} = \chi_{n,\nu}(t_1, ..., t_n)\, dt_j,$$

one gets the vectorial equation

$$\sum_{e\in\mathcal{C}^1(\Gamma)} \delta^0_{\langle \nu, v_e\rangle}\widehat{\mu}_e\big(-\langle \nu, u_1^e\rangle, \dots, -\langle \nu, u_{n-1}^e\rangle\big)\, v_e = 0. \qquad (4.19)$$

Equation (4.19) implies the two following facts :

- Taking $\nu = (0, \dots, 0)$ leads to

  $$\sum_{e\in\mathcal{C}_1(\Gamma)} \widehat{\mu}_e(0, \dots, 0)\, v_e = 0.$$

  Recall that the balancing condition $\sum_{e\in\mathcal{C}^1(\Gamma)} m_e\, v_e = 0$ is also satisfied, it follows from the sub-independency hypothesis (see Remark 4.6) that there exists a complex number ρ such that

  $$\widehat{\mu}_e(0, \dots, 0) = \rho\, m_e \quad \forall\, e \in \mathcal{C}_1(\Gamma).$$

- Let $\ell = (\ell_2, \dots, \ell_n) \neq (0, \dots, 0)$ be an arbitrary non-zero integral vector. Fix $e \in \mathcal{C}_1(\Gamma)$. There exists a unique $\nu_e \in \mathbb{Z}^n$ such that at the same time $\langle \nu_e, v_e\rangle = 0$ and $\langle \nu_e, u_j^e\rangle = -\ell_{j+1}$ for $j = 1, ..., n-1$, since $D_e = \{v_e, u_1^e, ..., u_{n-1}^e\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Since the graph Γ is $(n+1)$-valent and the $(n+1)$-primitive vectors $v_{e'}$ ($e' \in \mathcal{C}_1(\Gamma)$) affinely span the whole $\mathbb{R}^n$, there exists at least one edge $e'[e]$ (distinct from e) of Γ such that $\langle \nu_e, v_{e'[e]}\rangle \neq 0$, thus $\delta^0_{\langle \nu_e, v_{e'[e]}\rangle} = 0$. Therefore, in view of Remark 4.6, all of the coefficients involved in the vectorial equation (4.19) must vanish, as well as $\delta^0_{\langle \nu_e, v_e\rangle}\widehat{\mu}_e(\ell) = \widehat{\mu}_e(\ell)$, that is $\widehat{\mu}_e(\ell) = 0$. Consequently, for every $0 \neq \ell \in \mathbb{Z}^{n-1}$, we have $\widehat{\mu}_e(\ell) = 0$.

It means that every $d\mu_e$ ($e \in \mathcal{C}_1(\Gamma)$) is a Lebesgue measure given by $d\mu_e(\theta_2, ..., \theta_n) = \rho\, m_e\, d\theta_2 \ldots \theta_n$, and therefore, $\widetilde{\mathscr{T}} = \rho \mathscr{T}_n^1(\Gamma)$. This concludes to the strong extremality of $\mathscr{T}_n^1(\Gamma)$. □

Now it is easy to prove the Theorem 4.11.

Proof of Theorem 4.11. Let $\widetilde{\mathscr{T}}$ be a closed $(1,1)$-dimensional normal current with support $\operatorname{Log}^{-1}(|\Gamma|)$. For any vertex a of Γ there is an open neighborhood $\mathcal{V}_a$ of a in $\mathbb{R}^n$ which does not contain any other vertex of the tropical curve Γ. We are thus reduced to the situation of a tropical curve with just one vertex. It follows then from the Lemma 4.12 (the reasoning may be applied locally, in the open set $\operatorname{Log}^{-1}(\mathcal{V}_a)$ instead as in $\operatorname{Log}^{-1}(\mathbb{R}^n) = (\mathbb{C}^*)^n$) that for each vertex $\{a\}$ of Γ there is a complex number ρ_a such that

$$\widetilde{\mathscr{T}}_{|_{\operatorname{Log}^{-1}(\mathcal{V}_a)}} = \rho_a\, \mathscr{T}_n^1(\Gamma)_{|_{\operatorname{Log}^{-1}(\mathcal{V}_a)}}.$$

Similarly for an adjacent vertex $\{b\}$, we can write for some complex number ρ_b

$$\widetilde{\mathscr{T}}_{|_{\operatorname{Log}^{-1}(\mathcal{V}_b)}} = \rho_b\, \mathscr{T}_n^1(\Gamma)_{|_{\operatorname{Log}^{-1}(\mathcal{V}_b)}}.$$

On the other hand, if $\{a\}$ and $\{b\}$ are connected via the edge e, then we have, using the notations from the previous lemma, that

$$\widetilde{\mathscr{T}_e} = \rho_a\, m_e\, \mathscr{T}_e = \rho_b\, m_e\, \mathscr{T}_e$$

(as currents in some open neighborhood of $\operatorname{Log}^{-1}(e \backslash \{a, b\})$ in $(\mathbb{C}^*)^n$). Hence $\rho_a = \rho_b$. Since Γ is strongly extremal and thus connected, one can show that all ρ_a are indeed equal by taking a chain of successive adjacent vertices from $\{a\}$ to an arbitrary other vertex. □

4.2 Tropical (p, p)-dimensional currents

In this section we prove the Theorem 4.8. We start by treating the simplest case, namely when $V_{\mathbb{T}} \subset \mathbb{R}^n$ is a tropical p-cycle with only one facet W. Note that such a hypothesis implies that this facet is in fact an affine $(p-1)$-plane in $\mathbb{R}^n$ and that all p-cells are of the form $[0, \infty[\times v_P + W$ for some primitive inward vector $v_P = v_P^{[W\to]}$. Let us analyze the current $\mathscr{T}_n^p(V_{\mathbb{T}})$ in that particular case. Assume that W (which is here assumed to be the sole facet of $V_{\mathbb{T}}$) passes through the origin and is the common facet of the p-dimensional polyhedra $P_1, \ldots, P_s$, $s \geq 3$, with corresponding weights m_P. We have already shown in the beginning of Section 4 that in the definition of $\mathscr{T}_n^p(V_{\mathbb{T}})$

$$\mathscr{T}_n^p(V_{\mathbb{T}}) = \sum_{P \in \{P_1, \ldots, P_s\}} m_P\, \mathscr{T}_P,$$

is independent of the choice of the base point, and $\mathbb{Z}$-bases for $P \cap \mathbb{Z}^n$ as well as their completions to $\mathbb{Z}$-bases of $\mathbb{Z}^n$. Accordingly, we choose $\{w_1, \ldots, w_{p-1}\}$ a $\mathbb{Z}$-basis for $W \cap \mathbb{Z}^n$ and for each $P \in \{P_1, ..., P_s\}$, we choose the inward primitive vector $v_P = v_P^{[W\to]} \in \mathbb{Z}^n$ pointing inward P from the origin such that $\{w_1, \ldots, w_{p-1}, v_P\}$ is a $\mathbb{Z}$-basis for $H_P \cap \mathbb{Z}^n$ where H_P is the p-plane containing P. Also, the balancing condition means (see Remark 3.2) that every $p \times p$ minor of the $n \times p$ matrix of columns $\big(w_1, \ldots, w_{p-1}, \sum_P m_P v_P\big)$ vanishes. Equivalently, this implies that under the projection along W, $h_W : \mathbb{R}^n \to \mathbb{R}^{n-p+1}$, we have

$$\sum_P m_p\, h_W(v_P) = 0. \tag{4.20}$$

Furthermore, we extend each $\{w_1, \ldots, w_{p-1}, v_P\}$ to

$$D_P = \{w_1, \ldots, w_{p-1}, v_P, u_1^P, \ldots, u_{n-p}^P\}\,,$$

a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Correspondingly, we define the open subset S_{D_P} of $\operatorname{Log}^{-1}(H_P)$ by

$$\begin{aligned} S_{D_P} = \big\{ &(\tau_1^{w_1} \star \cdots \star \tau_{p-1}^{w_{p-1}} \star \tau^{v_P} \star \exp(2i\pi\theta_{p+1}u_1^P) \star \cdots \star \exp(2i\pi\theta_n u_{n-p}^P)\ ; \\ &(\tau_1, ..., \tau_{p-1}, \tau) \in (\mathbb{C}^*)^p,\ |\tau| > 1,\ (\theta_{p+1}, ..., \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}\big\}\,. \end{aligned} \tag{4.21}$$

Each S_{D_P} (which is a $(n+p)$-dimensional real manifold) is injectively foliated over $(\mathbb{R}/\mathbb{Z})^{n-p}$ through the submersion

$$\begin{gathered} \tau_1^{w_1} \star \cdots \star \tau^{w_{p-1}} \star \tau^{v_P} \star \exp(2i\pi\theta_{p+1}u_1^P) \star \cdots \star \exp(2i\pi\theta_n u_{n-p}^P) \in S_{D_P} \\ \Big\downarrow{\scriptstyle \sigma_P} \\ (\theta_{p+1}\,, \ldots, \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}\,. \end{gathered}$$

We again denote the fiber over $(\theta_{p+1}, \ldots, \theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}$, as $\Delta_P(\theta_{p+1}, \ldots, \theta_n)$. Note that the complex dimension of each $\Delta_P(\theta_{p+1}, \ldots, \theta_n)$ is p and that the boundary of such a fiber is a real analytic $(2p-1)$-cycle. Denote by $\boldsymbol{\tau}_P$ the parameterization map

$$\begin{aligned} &\boldsymbol{\tau}_P\ : (\tau, \theta_{p+1}, \ldots, \theta_n) \in (\mathbb{C}^*)^p \times (\mathbb{R}/\mathbb{Z})^{n-p} \mapsto \\ &\tau_1^{w_1} \star \ldots \tau_{p-1}^{w_{p-1}} \star \tau_p^{v_P} \star \exp(2i\pi\theta_{p+1}u_1^P) \star \cdots \star \exp(2i\pi\theta_n u_{n-p}^P) \in (\mathbb{C}^*)^n. \end{aligned}$$

Identifying $\mathbb{C}^{p-1} \times (\mathbb{R}/\mathbb{Z}) \simeq (\mathbb{R}^+)^{p-1} \times (\mathbb{R}/\mathbb{Z})^p$, $\partial\Delta_P(\theta_{p+1}, \ldots, \theta_n)$ (with orientation induced by $-d\theta_p$) can be therefore understood as the image

$$\begin{aligned} &\boldsymbol{\tau}_P\big((\mathbb{R}^+)^{p-1} \times (\mathbb{R}/\mathbb{Z})^p \times \{(\theta_{p+1}, \ldots, \theta_n)\}\big) \\ &\qquad\qquad =: \boldsymbol{\tau}_P^{(\theta_{p+1},\ldots,\theta_n)}\big((\mathbb{R}^+)^{p-1} \times (\mathbb{R}/\mathbb{Z})^p\big). \end{aligned} \tag{4.22}$$

By definition of the tropical current associated to $V_{\mathbb{T}}$,

$$\mathscr{T}_n^p(V_{\mathbb{T}}) = \sum_{P\in C_p(V_{\mathbb{T}})} m_P\, \mathscr{T}_P = \sum_{P\in C_p(V_{\mathbb{T}})} m_P \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_P(\theta_{p+1},...,\theta_n)]\, d\theta_{p+1}\dots d\theta_n\,.$$

Lemma 4.15. *Let $V_{\mathbb{T}}\subset\mathbb{R}^n$ be a p-tropical cycle such that $\mathcal{C}_{p-1}(V_{\mathbb{T}}) = \{W\}\ni\{0\}$. The current $\mathscr{T}_n^p(V_{\mathbb{T}})$, which can then be decomposed as*

$$\mathscr{T}_n^p(V_{\mathbb{T}}) = \sum_{P\in C_p(V_{\mathbb{T}})} m_P \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_P(\theta_{p+1},...,\theta_n)]\, d\theta_{p+1}\dots d\theta_n$$

is a closed (p,p)-dimensional current. Moreover, assume $\mathrm{card}(C_p(V_{\mathbb{T}})) = n-p+2$, *with inward vectors v_P in P, that make the balancing condition hold. If the set of vectors $\{h_W(v_P)\,;\, P\in C_p(V_{\mathbb{T}})\}$, h_W being the projection along W, is linearly sub-independent, then $\mathscr{T}_n^p(V_{\mathbb{T}})$ is strongly extremal.*

Proof. The proof is similar to that of Lemmas 4.10 and 4.12. Consider a $(2p-1)$-test form $\omega_{\eta,\nu}^{[K,J]}$ in $\mathbb{C}^*$ such that in a neighborhood of the $\mathrm{Log}^{-1}(0)\subset\mathrm{Log}^{-1}(W)$ is expressed in polar coordinates $z_j = r_j\, e^{2i\pi t_j}$, $j=1,...,n$, by

$$\omega_{\eta,\nu}^{[K,J]}(r_1,...,r_n,t_1,\dots,t_n) = \eta(r_1,...,r_n)\,\chi_{n,\nu}(t_1,\dots,t_n)\bigwedge_{k\in K} dr_k\wedge\bigwedge_{j\in J} dt_j,$$

where $K\subset J\subset\{1,...,n\}$ with $|K| = p-1\,, |J| = p\,.$ η a test function in $r = (r_1,\dots,r_n)$. Also $\nu\in\mathbb{Z}^n$ and $\chi_{n,\nu}$ denotes as before the character $t=(t_1,\dots,t_n)\mapsto \exp(2i\pi\langle\nu,t\rangle)$ on the torus $(\mathbb{R}/\mathbb{Z})^n$. Thanks to Fourier analysis, looking at the application of $\mathscr{T}_n^p(V_{\mathbb{T}})$ on these forms one can extract all information needed in order to verify closedness as well as extremality of $\mathscr{T}_n^p(V_{\mathbb{T}})$.

By definition of the exterior derivative of a current and the Stokes' formula, taking into account the orientation induced on the boundary of each $\Delta_P(\theta_{p+1},\dots,\theta_n)$,

$$\begin{gathered}\big\langle d\mathscr{T}_n^p(V_{\mathbb{T}}),\omega_{\eta,\nu}^{[K,J]}\big\rangle = \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\big\langle\, d\mathscr{T}_P,\omega_{\eta,\nu}^{[K,J]}\big\rangle = \\ \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}}\ \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\big\langle\partial\Delta_P(\theta_{p+1},\dots,\theta_n),\omega_{\eta,\nu}^{[K,J]}\big\rangle\,.\end{gathered}\tag{4.23}$$

Therefore by (4.22),

$$\begin{gathered}\big\langle d\mathscr{T}_n^p(V_{\mathbb{T}}),\omega_{\eta,\nu}^{[K,J]}\big\rangle = \\ \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}}\ \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\Big(\int_{(\mathbb{R}^+)^{p-1}\times(\mathbb{R}/\mathbb{Z})^p}(\boldsymbol{\tau}_P^{(\theta_{p+1},\dots,\theta_n)})^*\big(\eta(r)\,dr_K\wedge\chi_{n,\nu}(t)\,dt_J\big)\Big)\,.\end{gathered}\tag{4.24}$$

A computation similar to Lemma 4.9 and (4.18) gives the scalar equation

$$\begin{aligned}\left\langle d\mathscr{T}_n^p(V_{\mathbb{T}}), \omega_{\eta,\nu}^{[K,J]}\right\rangle = & \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P \times M_{\eta,K,W} \times \\ & \times \Big(\prod_{\ell=1}^{p-1} \delta^0_{\langle \nu, w_\ell\rangle}\Big) \delta^0_{\langle\nu,v_P\rangle}\, \delta^0_{\langle \nu, u_1^P\rangle} \cdots \delta^0_{\langle \nu, u^P_{n-p}\rangle} \operatorname{Det}_J\big(w_1,\dots,w_{p-1},v_P\big).\end{aligned} \tag{4.25}$$

In the above $M_{\eta,K,W}$ is a constant coming from integration of $\big(\boldsymbol{\tau}_P^{(\theta_{p+1},\dots,\theta_n)}\big)^*\big(\eta(r)\,dr_K\big)$ depending on η, K, W which can be chosen to be 1. $\operatorname{Det}_J\big(w_1,\dots,w_{p-1},v_P\big)$ denotes the $p\times p$ minor of the $n\times p$ matrix $\big(w_1,\dots,w_{p-1},v_P\big)$ corresponding to the rows with indices $j\in J$, this term appears from $\big(\boldsymbol{\tau}_P^{(\theta_{p+1},\dots,\theta_n)}\big)^*\big(dt_J\big)$ in (4.24) (compare to (4.13)). For any $0\neq \nu\in\mathbb{Z}^n$, (4.25) becomes zero. Assuming $\nu=0$, yields

$$\begin{aligned}\left\langle d\mathscr{T}_n^p(V_{\mathbb{T}}), \omega_{\eta,\nu}^{[K,J]}\right\rangle = 0 \quad \text{if and only if} & \\ \operatorname{Det}_J\big(w_1,\dots,w_{p-1}, \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P\, v_P\big) &= 0, \\ \forall\, J\subset\{1,\dots,n\}, |J| &= p\,.\end{aligned}$$

Which implies the equivalence of d-closedness of $\mathscr{T}_n^p(V_{\mathbb{T}})$ and the balancing condition.

For any $P\in\mathcal{C}_p(V_{\mathbb{T}})$, let $\mathcal{U}_P$ be the open subset of $(\mathbb{C}^*)^n$ defined as

$$\mathcal{U}_P := \operatorname{Log}^{-1}\Big(\mathbb{R}^n \setminus \bigcup_{\substack{P'\in\mathcal{C}_p(V_{\mathbb{T}})\\ P'\neq P}} |P'|\Big)$$

Suppose that $\widetilde{\mathscr{T}}$ is a (p,p)-dimensional normal current in $(\mathbb{C}^*)^n$ with support exactly $\operatorname{Log}^{-1}(V_{\mathbb{T}})$. As in the 1-dimensional case, by Theorem 2.5, for any $P\in\mathcal{C}_p(V_{\mathbb{T}})$, there exists a unique Radon measure $d\mu_P$ on $(\mathbb{R}/\mathbb{Z})^{n-p}$ such as (as currents in the open subset $\mathcal{U}_P\subset(\mathbb{C}^*)^n$) one has

$$\widetilde{\mathscr{T}}_{|\mathcal{U}_P} = \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P,D_P,W}(\theta_{p+1},...,\theta_n)]\, d\mu_P(\theta_{p+1},...,\theta_n).$$

Since $\widetilde{\mathscr{T}}_{|\mathcal{U}_P}$ extends globally as the normal closed current to the whole of $(\mathbb{C}^*)^n$, one defines normal currents $\widetilde{\mathscr{T}}_P$ on $(\mathbb{C}^*)^n$ by setting

$$\widetilde{\mathscr{T}}_P := \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P,D_P,W}(\theta_{p+1},...,\theta_n)]\, d\mu_P(\theta_{p+1},...,\theta_n).$$

The normal (p,p)-dimensional current $\widetilde{\mathscr{T}} - \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} \widetilde{\mathscr{T}_P}$, which is supported by $\text{Log}^{-1}(W)$, equals zero for dimension reasons thanks to theorem 2.4, so that one has (as currents in $(\mathbb{C}^*)^n$ this time) the representation (which indeed depends on the chosen foliation):

$$\widetilde{\mathscr{T}} = \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P,D_P,W}(\theta_{p+1},...,\theta_n)]\, d\mu_P(\theta_{p+1},...,\theta_n).$$

Using the fact that $\langle d\widetilde{\mathscr{T}}, \omega_{\eta,\nu}^{[K,J]}\rangle = 0$ for any $\nu\in\mathbb{Z}^n$, any $K\subset J\subset\{1,\dots,n\}$ with $|J| = |K|+1 = p$, any test function η in r with non-zero integral leads to

$$\sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} \Big(\prod_{\ell=1}^{p-1} \delta^0_{\langle\nu,w_\ell\rangle}\Big)\delta^0_{\langle\nu,v_P\rangle}\, \widehat{\mu}_P\big(-\langle\nu,u_1^P\rangle,...,-\langle\nu,u_{n-p}^P\rangle\big)\, \mathrm{Det}_J(w_1,...,w_{p-1},v_P) = 0\,,$$
$$J\subset\{1,\dots,n\},\ |J| = p\,. \quad (4.26)$$

Recall that by hypothesis the set $\{h_W(v_P)\,;\, P\in\mathcal{C}_p(V_{\mathbb{T}})\}$ is linearly sub-independent and spans $W^\perp$ as an $\mathbb{R}$-basis, where h_W is the projection along W. The balancing condition also gives $\sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P h_W(v_P) = 0$.

Similar to the bottom of the proof of Lemma 4.12 we deduce :

- Choosing $\nu = 0$, together with sub-independency implies that there exists a complex number ρ such that $\widehat{\mu}_P(0,...,0) = \rho\, m_P$ by the Remarks 4.6 and 4.20 for every P.

- Now assume $(\ell_{p+1},...,\ell_n)\in\mathbb{Z}^{n-p}$ is any non-zero vector. Since for any P, $\{w_1,...,w_{p-1},v_P,u_1^P,...,u_{n-p}^P\}$ is a lattice basis of $\mathbb{Z}^n$, there exists a unique $\nu_P\in\mathbb{Z}^n\cap W^\perp$ such that at the same time $\langle\nu_P,v_P\rangle = 0$, $\langle\nu_P,u_j^P\rangle = -\ell_{p+j}$ for $j = 1,...,n-p$. However, for at least one $P'\neq P$, $\langle\nu_P,v_{P'}\rangle\neq 0$, and therefore $\delta^0_{\langle\nu_P,v_{P'}\rangle} = 0$ in (4.26). The sub-independency thus implies that $\widehat{\mu}_P(\ell_{p+1},...,\ell_n) = 0$ for every P.

Therefore $d\mu_P(\theta_{p+1}\dots\theta_n) = \rho\, m_P\, d\theta_{p+1}\dots d\theta_n$ for any $P\in\mathcal{C}_p(V_{\mathbb{T}})$. This proves the strong extremality of $\mathscr{T}_n^p(V_{\mathbb{T}})$ and ends the proof of the lemma. □

Proof of Theorem 4.8. Let $V_{\mathbb{T}}$ be a strongly extremal tropical p-cycle. Let P be a p-dimensional cell of the tropical p-cycle $V_{\mathbb{T}}$. The current $\mathscr{T}_P$ defined (in the preliminaries of Section 4) as

$$\mathscr{T}_P := (T_{H_P})_{|\text{Log}^{-1}(\text{int(P)})}$$

coincides with the current

$$\int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P,D_P,W}(\theta_{p+1},\dots,\theta_n)]\, d\theta_{p+1}\dots d\theta_n$$

about any point (in $(\mathbb{C}^*)^n$) which belongs to the $(n+p)$-dimensional real submanifold $\mathrm{Log}^{-1}(\mathrm{int}(P))$, where $\mathrm{int}(P)$ denotes the relative interior of P in the affine p-plane H_P (the argument is again the same as the one which has been invoked in the discussion preceding the Lemma 4.9). About any point $\mathrm{Log}^{-1}(a)$, where a lies in the relative interior (in the affine $(p-1)$-plane H_W) of a given facet W of P, the normal current $\sum_{W\prec P'} \mathscr{T}_{P'}$ coincides with the current

$$\sum_{W\prec P'} \int_{(\theta_{p+1},\dots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{P',D_{P'},W}(\theta_{p+1},\dots,\theta_n)]\, d\theta_{p+1}\dots d\theta_n,$$

By Theorem 2.4. Since closedness of a current can be tested locally, it follows from the argument developed in the proof of Lemma 4.15 that the current

$$\mathscr{T}_n^p(V_{\mathbb{T}}) = \sum_{P\in\mathcal{C}_p(V_{\mathbb{T}})} m_P \mathscr{T}_P$$

is closed in a any compact neighborhood of any point $\mathrm{Log}^{-1}(a)$ in $\mathrm{Log}^{-1}(\mathrm{int}(W))$, W being an arbitrary facet of P, which in turn implies the closedness of $\mathscr{T}_n^p(V_{\mathbb{T}})$, noting that in light of Theorem 2.4 we need not to check the closedness for faces of codimension higher that 1. Suppose now that the for each facet $W\in\mathcal{C}_{p-1}(V_{\mathbb{T}})$, for each $P\in\mathcal{C}_p(V_{\mathbb{T}})$ that shares W as a facet, the projection along W of primitive vectors $v_P^{[\to W]}$ form a linearly sub-independent set with cardinality $n-p+2$. Let $\widetilde{\mathscr{T}}\in\mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ be a normal closed current with support $\mathrm{Log}^{-1}(V_{\mathbb{T}})$. If $W\in\mathcal{C}_{p-1}(V_{\mathbb{T}})$ and a is a point in the relative interior of W in H_W, the argument used in the proof of Lemma 4.15 also shows that there exists some complex number $\rho_{W,a}$ such that, in neighborhood of $\mathrm{Log}^{-1}(a)$ in $(\mathbb{C}^*)^n$, one has $\widetilde{\mathscr{T}} = \rho_{W,a}\mathscr{T}_n^p(V_{\mathbb{T}})$. Obviously, all $\rho_{W,a}$ (for a in the relative interior of an arbitrary facet W of $V_{\mathbb{T}}$) are equal to some complex number ρ_W. This implies that $\mathbb{1}_{\mathrm{Log}^{-1}(\mathrm{int}P)}\,\widetilde{\mathscr{T}} = \rho_W\, m_P\, \mathscr{T}_P$. If $W'\neq W$ is another facet of P we find a complex number $\rho_{W'}$ such that

$$\mathbb{1}_{\mathrm{Log}^{-1}(\mathrm{int}P)}\,\widetilde{\mathscr{T}} = \rho_{W'}\, m_P\, \mathscr{T}_P\,,$$

and $\rho_W = \rho_{W'}$ is imposed. Connectivity of $V_{\mathbb{T}}$ in codimension 1 (as in the final step in the proof of Theorem 4.11) shows that all numbers ρ_W ($W\in\mathcal{C}_{p-1}(V_{\mathbb{T}})$) coincide (note that higher codimensional connectivity is not sufficient). This concludes the proof of the strong extremality of the current $\mathscr{T}_n^p(V_{\mathbb{T}})$. $\square$

5 Tropical currents in $\mathcal{D}'_{p,p}(\mathbb{CP}^n)$

We first show that for a given effective tropical p-cycle $V_{\mathbb{T}}$ in $\mathbb{R}^n$ the closed positive (p,p)-dimensional current $\mathscr{T}_n^p(V_{\mathbb{T}})$ (considered as a current in $(\mathbb{C}^*)^n$) can be extended by zero to a closed positive (p,p)-dimensional current in $\mathbb{CP}^n$.

Lemma 5.1. *For any effective tropical p-cycle $V_{\mathbb{T}}$ in $\mathbb{R}^n$, the positive tropical current $\mathscr{T}_n^p(V_{\mathbb{T}}) \in \mathcal{D}'_{p,p}((\mathbb{C}^*)^n)$ can be extended by zero to $\mathbb{CP}^n$ as a current $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ in $\mathcal{D}'_{p,p}(\mathbb{CP}^n)$. Moreover, if $T \in \mathcal{E}^p((\mathbb{C}^*)^n)$, then also $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}}) \in \mathcal{E}^p(\mathbb{CP}^n)$.*

Proof. Let $(\zeta_1, \ldots, \zeta_n)$ be the coordinates on the complex torus $(\mathbb{C}^*)^n$. Assume $P \in \mathcal{C}_p(V_{\mathbb{T}})$, and without loss of generality that $0 \in \mathrm{int}P$. The current T_{H_P} is expressed as the average

$$T_{H_P} = T_{H_P,D_P} = \int_{(\theta_{p+1},\ldots,\theta_n)\in(\mathbb{R}/\mathbb{Z})^{n-p}} [\Delta_{H_P,D_P}(\theta_{p+1},...,\theta_n)]\, d\theta_{p+1}\ldots d\theta_n$$

(see formula (4.8)). For each $(\theta_{p+1},...,\theta_n) \in (\mathbb{R}/\mathbb{Z})^{n-p}$, the complex p-dimensional analytic variety $\Delta_{H_P,D_P}(\theta_{p+1},...,\theta_n)$ is included in the toric subset of $(\mathbb{C}^*)^n$ defined in the coordinates $(\zeta_1,...,\zeta_n)$ by the set of binomial equations

$$\prod_{j=1}^n \zeta_j^{\xi^+_{\ell,j}} - \prod_{j=1}^n (\gamma_j(\theta, U_P, a))^{\xi_{\ell,j}}\, \zeta_j^{\xi^-_{\ell,j}} = 0, \quad \ell = 1, ..., M_{D_P},$$

where the $\xi_\ell = \xi_\ell^+ - \xi_\ell^-$ form a set of generators for $\mathrm{Ker}_{B^t_P} \cap \mathbb{Z}^n$ and

$$\gamma_j(\theta, U_P, a) = \exp\left(2i\pi(\theta_{p+1}u^P_{1,j} + \cdots + \theta_n u^P_{n-p,j})\right) \in \{\zeta \in \mathbb{C}^*\,;\, |\zeta| = 1\}, \quad j = 1, ..., n.$$

Each integration current $\Delta_{H_P,D_P}(\theta_{p+1},...,\theta_n)$ can then be extended to $\mathbb{CP}^n$ as the integration of the Zariski closure (in $\mathbb{CP}^n$) of the toric subset

$$\Big\{(\zeta_1,...,\zeta_n) \in (\mathbb{C}^*)^n\,;\, \prod_{j=1}^n \zeta_j^{\xi^+_{\ell,j}} - \prod_{j=1}^n (\gamma_j(\theta, U_P, a))^{\xi_{\ell,j}}\, \zeta_j^{\xi^-_{\ell,j}} = 0\Big\}.$$

Since the degree of this projective algebraic variety is bounded independently of $(\theta_{p+1},...,\theta_n)$, the current $\mathscr{T}_n^p(V_{\mathbb{T}})$ has finite mass about any point in $\mathbb{CP}^n \setminus (\mathbb{C}^*)^n$. By the extension theorem of Skoda-El Mir (see [7], page 138), this current can then be trivially extended by 0 as a positive (p,p)-dimensional closed current on $\mathbb{CP}^n$. The last assertion follows from the fact that $\mathscr{T}_n^p(V_{\mathbb{T}})$ and $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ have the same support in the dense open subset $(\mathbb{C}^*)^n \subset \mathbb{CP}^n$ and support of $\bar{\mathscr{T}}_n^p(V_{\mathbb{T}})$ is the closure of support of $\mathscr{T}_n^p(V_{\mathbb{T}})$ in $\mathbb{CP}^n$. $\square$

Let $d = d' + d''$ the usual decomposition of the de Rham (exterior) derivative and $d^c = (d' - d'')/(2i\pi)$, so that $dd^c = (1/i\pi)\, d''d'$. The following theorem gives a simpler representation for tropical currents of bidimension $(n-1, n-1)$ (equivalently of bidegree $(1,1)$).

Theorem 5.2. *Positive tropical currents of bidimension $(n-1, n-1)$ in $(\mathbb{C}^*)^n$ (resp. their extension by zero to $\mathbb{CP}^n$) are exactly the currents of the form $dd^c[p \circ \mathrm{Log}\,]$, where p is a tropical polynomial on $\mathbb{R}^n$ (resp. p is a homogeneous tropical polynomial on $\mathbb{TP}^n$).*

Proof. Observe that for the given tropical polynomial $p : \mathbb{R}^n \to \mathbb{R}$, the two positive closed $(n-1, n-1)$-dimensional currents $dd^c[p \circ \mathrm{Log}\,]$ and $\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p))$ share the same support, $\mathrm{Log}^{-1}(V_{\mathbb{T}}(p))$, in $(\mathbb{C}^*)^n$. In order to show they coincide in $(\mathbb{C}^*)^n$, it is enough to prove they coincide in the open subset $\mathrm{Log}^{-1}\big(\mathbb{R}^n \setminus \bigcup_{\tau \in \mathcal{C}_{n-2}(V_{\mathbb{T}})} |\tau|\big)$ (then they coincide in the whole $(\mathbb{C}^*)^n$ for dimensional reasons thanks to theorem 2.4). Since equality of currents can be tested locally, it is even enough to test such an equality in a neighborhood of $\mathrm{Log}^{-1}(a)$, where a is an arbitrary point in the relative interior of a $(n-1)$-dimensional cell P of the tropical hypersurface $V_{\mathbb{T}}(p)$. By a translation in $\mathbb{R}^n$, one can then assume that $p(x) = \max\{\langle \alpha, x\rangle, 0\}$, where $\alpha \in \mathbb{Z}^n \setminus \{(0, ..., 0)\}$. Let $\alpha = m_\xi\, \xi$, where ξ is a primitive vector in $\mathbb{Z}^n$ and $m_\xi \in \mathbb{N}^*$. Let $B := \{w_1, ..., w_{n-1}\}$ be a $\mathbb{Z}$-basis for $H_P \cap \mathbb{Z}^n$ and consider a completion $D_B := \{w_1, ..., w_{n-1}, u\}$ of B as in the preliminaries of Section 4. For each $\theta \in (\mathbb{R}/\mathbb{Z})$, the toric set $\Delta_{H_P, D_B}(\theta)$ is the $(n-1)$-dimensional (reduced) toric hypersurface in $(\mathbb{C}^*)^n$ defined by the irreducible binomial $\prod_{j=1}^n \zeta_j^{\xi_j^+} - \gamma_u(\theta) \prod_{j=1}^n \zeta_j^{\xi_j^-}$ for some $\gamma_u(\theta) \in \mathbb{S}^1$ (see Remark 4.3).

Let $\overline{\Delta_{H_P, D_B}(\theta)}$ ($\theta \in \mathbb{R}/\mathbb{Z}$) be the Zariski closure of the hypersurface $\Delta_{H_P, D_B}(\theta)$ in $\mathbb{CP}^n$, which is in fact the zero set in $\mathbb{CP}^n$ of homogenization of the above equation. It follows from Crofton's formula (see [7], page 170, or Example (4.6) in [5]) that

$$\deg\big(\overline{\Delta_{H_P, D_B}(\theta)}\big) = \max\Big\{\sum_{j=1}^n \xi_j^+, \sum_{j=1}^n \xi_j^-\Big\} = \int_{\mathbb{CP}^n} \big[\overline{\Delta_{H_P, D_B}(\theta)}\big] \wedge \omega^{n-1}$$

where ω denotes the Kähler form $\omega = dd^c \log \|\ \|$ in $\mathbb{CP}^n$. On the other hand, it is easy to see that, in the weak sense of currents in $(\mathbb{C}^*)^n$,

$$\lim_{m \to \infty} \frac{m_\xi}{m} \log\Big|\prod_{j=1}^n \zeta_j^{m\xi_j} + 1\Big| = p \circ \mathrm{Log}\,,$$

which implies, taking dd^c,

$$\lim_{m \to \infty} \frac{m_\xi}{m}\, dd^c\Big[\log\Big|\prod_{j=1}^n \zeta_j^{m\xi_j} + 1\Big|\Big] = dd^c\, [p \circ \mathrm{Log}\,].$$

It follows that, if one denotes as $\overline{dd^c[p\circ \mathrm{Log}\,]}$ the trivial extension by 0 of the positive closed $(n-1,n-1)$-dimensional current $dd^c[p\circ \mathrm{Log}\,]$ from $(\mathbb{C}^*)^n$ to $\mathbb{CP}^n$, one has

$$\begin{aligned}\int_{\mathbb{CP}^n} \overline{dd^c[p\circ \mathrm{Log}\,]}\wedge \omega^{n-1} = \\ \int_{\mathbb{CP}^n}\Big[\int_{\mathbb{R}/\mathbb{Z}} \overline{[\Delta_{H_P,D_B}(\theta)]}\, d\theta\Big]\wedge \omega^{n-1} = \max\Big\{\sum_{j=1}^n \xi_j^+, \sum_{j=1}^n \xi_j^-\Big\}. \quad (5.27)\end{aligned}$$

Chose now $\xi'\in \mathbb{Z}^n\setminus\{(0,...,0)\}$ and a strictly increasing sequence $(N_k)_{k\geq 1}$ of positive integers such that all tropical $(n-1,n-1)$-hypersurfaces $V_{\mathbb{T}}(p_k)$, where

$$p_k\ : x\in\mathbb{R}^n \mapsto \max\big\{p(x), \langle \xi',x\rangle - N_k\big\} = \max\big\{\langle\xi,x\rangle, 0, \langle \xi',x\rangle - N_k\big\},\quad k\in\mathbb{N}^*,$$

are trivalent. For any relatively compact open subset $\mathcal{V}\subset\mathbb{R}^n$, $p\equiv p_k$ in $\mathcal{V}$ and the currents $dd^c[p\circ\mathrm{Log}\,]$ and $dd^c[p_k\circ \mathrm{Log}\,]$ coincide in $\mathrm{Log}^{-1}(\mathcal{V})$ provided k is large enough (depending on $\mathcal{V}$). Since the current $\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p_k))$ is extremal in $(\mathbb{C}^*)^n$ thanks to Theorem 4.8 ($p=n-1$), there exists, for each such $\mathcal{V}\subset\mathbb{R}^n$ and for any $k>>1$ large enough (depending on $\mathcal{V}$), a strictly positive constant $\rho_{\mathcal{V},k}$ such that one has

$$\begin{aligned}&\big(\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p))\big)_{|\mathrm{Log}^{-1}(\mathcal{V})} = \big(\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p_k))\big)_{|\mathrm{Log}^{-1}(\mathcal{V})} = \\ &= \rho_{\mathcal{V},k}\big(dd^c[p_k\circ\mathrm{Log}\,]\big)_{|\mathrm{Log}^{-1}(\mathcal{V})} = \rho_{\mathcal{V},k}\big(dd^c[p\circ\mathrm{Log}\,]\big)_{|\mathrm{Log}^{-1}(\mathcal{V})}.\end{aligned} \quad (5.28)$$

Taking an exhaustion of $\mathbb{R}^n$ with relatively open subsets $\mathcal{V}_\ell$, $\ell=1,2,...$, such that $\mathcal{V}_\ell\subset\mathcal{V}_{\ell+1}$ for any $\ell\in\mathbb{N}^*$, it follows that all $\rho_{\mathcal{V},k}$ are equal, so that there exists some strictly positive constant ρ such that

$$\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p)) = \rho\, dd^c\,[p\circ\mathrm{Log}\,]$$

(as currents in $(\mathbb{C}^*)^n$). The fact that the normalization constant ρ equals 1 follows from (5.27) since

$$\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p)) = \int_{\theta\in(\mathbb{R}/\mathbb{Z})} [\Delta_{H_P,D_B}(\theta)]\, d\theta$$

(so that the trivial extensions of $\mathscr{T}_n^{n-1}(V_{\mathbb{T}}(p))$ and $dd^c\,[p\,\circ\mathrm{Log}\,]$ to $\mathbb{CP}^n$ share the same total mass as currents in the projective space $\mathbb{CP}^n$ equipped with its Fubini-Study Kähler form).

Regarding the statement for the homogeneous tropical polynomials, observe that extension by zero of $dd^c\,[p\circ\mathrm{Log}\,]$ to $\mathbb{CP}^n$, in the sense of currents, is exactly $dd^c\,[\tilde p\ \circ\ \mathrm{Log}\,]$, where $\tilde p$ is the homogenization of p.

For the converse statement, just note that by Theorem 3.6, every tropical hypersurface $V_{\mathbb{T}}$ can be understood as $V_{\mathbb{T}}(p)$ for a tropical polynomial p, with equality of respective weights. $\square$

By previous theorem and Theorem 4.8 one readily has the following.

Corollary 5.3. *Let p be a homogeneous tropical polynomial defining a tropical hypersurface in $\mathbb{TP}^n$. Then, the positive current $dd^c\,[p\circ \mathrm{Log}\,]$ is in $\mathcal{E}^{n-1}(\mathbb{CP}^n)$ if every facet of the tropical hypersurface associated to p is the common intersection of exactly 3 polyhedra.*

Remark 5.4. Let $V_1,\dots,V_n\subset\mathbb{R}^n$ be the tropical hypersurfaces associated to $p_1,\dots,p_n:\mathbb{R}^n\to\mathbb{R}$, respectively. If these hypersurfaces intersect transversally, then the product

$$\mathscr{T}_n^{n-1}(V_1)\wedge\cdots\wedge\mathscr{T}_n^{n-1}(V_n)=dd^c\,[p_1\circ\mathrm{Log}\,]\wedge\cdots\wedge dd^c\,[p_n\circ\mathrm{Log}\,] \tag{5.29}$$

is well-defined. Using a formula due to A. Rashkovskii (see [24], [23]) one has

$$\frac{1}{n!}\int_{\mathrm{Log}^{-1}(E)} dd^c\,[p_1\circ\mathrm{Log}\,]\wedge\cdots\wedge dd^c\,[p_n\circ\mathrm{Log}\,]=\int_E \tilde{M}(p_1,\dots,p_n), \tag{5.30}$$

where the right hand side is the real mixed Monge-Ampère measure of the Borel set $E\subset\mathbb{R}^n$ corresponding to the convex function $p_1,\dots,p_n$. On the other hand for a tropical polynomial $p:\mathbb{R}^n\to\mathbb{R}$, one can calculate (as in [30] Example 3.20),

$$M(p):=\tilde{M}(p,\dots,p)=\sum_{a\in\,\mathcal{C}_0(V_{\mathbb{T}}(p))}\mathrm{Vol}_n(\{a\}^*)\delta_a\,,$$

where δ_a denotes the Dirac mass at the vertex a of $V_{\mathbb{T}}(p)$, and $\mathrm{Vol}_n(\{a\}^*)$ is the volume of the n-cell dual to $\{a\}$ in a dual decomposition of the Newton polytope of p. This formula therefore, corresponds to a stable intersection, and also gives

$$\tilde{M}(p_1,\dots,p_n)=\sum_{\{a\}\in\,\mathcal{C}_0(V_1\cap\cdots\cap V_n)}\mathrm{Vol}_n(\{a\}^*)\delta_a\,.$$

From which the tropical Bézout's and Bernstein's theorems follow (see [2], [25]). We also remark that in view of (5.29), (5.30) relates the intersection of toric sets to intersection of tropical cycles.

6 Amoebas and approximations of tropical currents

In this section we assume that all the tropical cycles are effective.

Suppose $\{X_t\}_{t\in\mathbb{R}_+}$ is a family of algebraic cycles in $(\mathbb{C}^*)^n$, $V_{\mathbb{T}}$ a tropical cycle, and

$$\lim_{t\to\infty}\mathrm{Log}\,(X_t)=V_{\mathbb{T}}$$

with respect to the Hausdorff metric on compact sets of $\mathbb{R}^n$. As we have mentioned in Section 3, if such a limit exists, this limit inherits the structure of a tropical cycle, the weights of this tropical cycle being related to the degrees of the cycles X_t. The approximation problem considered in [4],[15],[28],[3] deals with approximation of a tropical cycle or a tropical curve by algebraic varieties X_t with equal total degrees. However we are interested here in the problem of approximations of the tropical cycles as sets, which makes the approximation more flexible. For instance, Grigory Mikhalkin's example of a spatial tropical cubic ([20], [28]) of genus 1 is not approximable by amoebas of cubic curves in $(\mathbb{C}^*)^3$ but, as a set, it is approximable by a family of sextic curves, the resulting sextic tropical curve being the cubic tropical curve with doubled weights (see [1]). As for the theory of currents, it is important to know if there are elements of $\mathcal{E}^k(X)$ (with Hodge classes) which are not in $\overline{\mathcal{I}^k(X)}$, where X is a projective variety (see [8], [6]). In this section we relate the approximation problem of tropical cycles by amoebas to the problem of approximating tropical currents by integration currents along analytic cycles with positive coefficients.

Definition 6.1. We call a tropical cycle set-wise approximable if its underlying set is approximable in Hausdorff metric by amoebas of algebraic varieties of any degree.

Remark 6.2. The set-wise approximability for strongly extremal tropical cycles is equivalent to having a multiple (obtained by multiplying the weights) which is approximable by amoebas of algebraic subvarieties of $(\mathbb{C}^*)^n$ with equal degrees.

Example 6.3. Consider the tropical polynomial

$$p : \mathbb{R}^n \to \mathbb{R}, \quad x = (x_1, \dots, x_n) \mapsto \max_{\alpha} \big\{c_\alpha + \alpha_1 x_1 + \cdots + \alpha_n x_n\big\}$$

attached to a finite set of indices $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$. Now, for each $m, l \in \mathbb{N}^*$, consider the polynomial map

$$f_{l,m} : \mathbb{C}^n \to \mathbb{C}, \quad z = (z_1, \dots, z_n) \mapsto \sum_{\alpha} \exp(l\, c_\alpha) z_1^{m\alpha_1} \dots z_n^{m\alpha_n}.$$

It is easy to see that, in the sense of distributions,

$$\lim_{m\to\infty} \frac{1}{m} \log |f_{m,m}(z)| = p \circ \mathrm{Log}\,(z).$$

Poincaré-Lelong equation ([7], page 143) yields (in the sense of currents)

$$\lim_{m\to\infty} \frac{1}{m} [Z_{f_{m,m}}] = dd^c[p \circ \mathrm{Log}\,(z)], \tag{6.31}$$

where $Z_{f_{m,m}}$ denotes the divisor of $f_{m,m}$ with multiplicities taken into account. Moreover

$$\lim_{m\to\infty} \mathrm{Log}\,(\mathrm{Supp}\;[Z_{f_{m,m}}]) = \lim_{m\to\infty} \mathcal{A}_{f_{m,m}} = \lim_{m\to\infty} \frac{1}{m}\mathcal{A}_{f_{m,1}} = V_{\mathbb{T}}(p),$$

where in the third equation "multiplying" the amoeba $\mathcal{A}_{f_{m,1}}$ by $1/m$ means dilating this amoeba by this factor. Therefore, the support of the currents on the left hand side of (6.31) approximates the support of the current on the right hand side and the coefficient $\frac{1}{m}$ makes the total masses equal. Combining this with Theorem 5.2 one has

$$\lim_{m\to\infty} \frac{1}{m}[Z_{f_{m,m}}] = dd^c[p \circ \mathrm{Log}\,(z)] = \mathscr{T}_n^{n-1}(V_{\mathbb{T}})\,. \tag{6.32}$$

Assume now that $V_{\mathbb{T}} \subset \mathbb{R}^n$ is an effective strongly extremal tropical p-cycle. Suppose next that there is a family of algebraic p-cycles $(X_t)_{t>1}$ in $(\mathbb{C}^*)^n$ such that we have the set-wise approximation

$$\lim_{t\to\infty} \mathrm{Log}_{\,t}(X_t) = V_{\mathbb{T}}\,, \tag{6.33}$$

where $\mathrm{Log}_{\,t}(z_1,\dots,z_n) := (\log|z_1|^{1/\log t},\dots,\log|z_n|^{1/\log t})$ for $t>1$. Starting with such a set-wise approximation, we intend to find a sequence of integration currents $\mathcal{I}^p((\mathbb{C}^*)^n)$ that converges to a multiple of $\mathscr{T}_n^p(V_{\mathbb{T}})$.

For every positive integer m, define the proper smooth map

$$\Phi_m : \mathbb{C}^n \to \mathbb{C}^n\ ,\ (z_1,\dots,z_n) \mapsto (z_1^m,\dots,z_n^m) \tag{6.34}$$

and consider the current integration current $\Phi_m^*[X_t] := [\Phi_m^{-1}(X_t)]$. The support of this current is obviously the set

$$\begin{aligned}\Phi_m^{-1}(X_t) &= \big\{(w_1,\dots,w_n)\in(\mathbb{C}^*)^n\,;\,(w_1^m,\dots,w_n^m)\in X_t\big\}\\ &= \Big\{\Big(\exp\big(\frac{2\pi i k_1+\arg(z_1)}{m}\big)|z_1|^{1/m},\dots,\exp\big(\frac{2\pi i k_n+\arg(z_n)}{m}\big)|z_n|^{1/m}\Big)\,,\\ &\qquad\qquad (z_1,\dots,z_n)\in X_t\,,0\le k_j\le m-1\Big\}.\end{aligned} \tag{6.35}$$

Note that as m increases, the set $\{e^{2\pi i k/m}, k=0,\dots,m-1\}$ tends to a dense set in the unit circle $\mathbb{S}^1$. Let $m\ :[1,\infty[\to\mathbb{N}$ be an increasing function tending to infinity when t tends to infinity. Therefore the support of a limit current for any convergent sequence of the form $\big(\lambda_{m(t_k)}[\Phi_{m(t_k)}^{-1}(X_{t_k})/\deg X_t]\big)_k$ such that $(t_k)_k$ tends to $+\infty$, is necessarily of the form $\mathrm{Log}^{\,-1}(V)$ for some closed set $V\subset\mathbb{R}^n$.
On the other hand, if $x=(x_1,\dots,x_n)\in V_{\mathbb{T}}$, then there exists a sequence of points

$$\Big(\zeta_{t_{\nu_k}} = (\zeta_{t_{\nu_k},1},\dots,\zeta_{t_{\nu_k},n})\in C_{t_{\nu_k}}\Big)_k$$

such that

$$\operatorname{Log}_{t_{\nu_k}}(\zeta_{t_{\nu_k}}) \to x,$$

or

$$(|\zeta_{t_{\nu_k},1}|^{1/\log t_{\nu_k}}, \ldots, |\zeta_{t_{\nu_k},n}|^{1/\log t_{\nu_k}}) \to (e^{x_1}, \ldots, e^{x_n})$$

as the sub-sequence $(\nu_k)_k = (\nu_k(x))_k$ tends to $+\infty$. Comparing this with (6.35), if one takes $m \ : t \in [1, +\infty[\mapsto [\log t]$, the integer part of $\log t$, then the support of a limit current for any convergent sequence of the form $\big(\lambda_{m(t_k)}[\Phi^{-1}_{m(t_k)}(X_{t_k})]\big)_k$ such that $(t_k)_k$ tends to $+\infty$ equals necessarily to $V_{\mathbb{T}}$. If one takes $\lambda_m = m^{n-p}$ the family of currents

$$\frac{1}{(m(t))^{n-p}} \frac{1}{\deg X_t} \big[\Phi^*_{m(t)}[X_t]\big], \qquad t > 1$$

is normalized (with degrees all equal to 1). Thanks to Theorem 4.11, any subsequence of it converges towards the same multiple $\lambda \mathscr{T}^p_n(V_{\mathbb{T}})$ $(\lambda > 0)$ of the extremal current $\mathscr{T}^p_n(V_{\mathbb{T}})$. So we have proved the following.

Theorem 6.4. *Assume that the tropical cycle $V_{\mathbb{T}}$ is strongly extremal and set-wise approximable as $\lim_{t\to+\infty} \operatorname{Log}_t(X_t)$ by amoebas of irreducible algebraic p-cycles $(X_t)_{t>1}$ of $(\mathbb{C}^*)^n$. Then there exists $\lambda > 0$ such that*

$$\mathscr{T}^p_n(V_{\mathbb{T}}) = \lambda \lim_{m\to\infty} \frac{1}{m^{n-p}} \Phi^*_m[X_{e^m}].$$

In particular, $\mathscr{T}^p_n(V_{\mathbb{T}}) \in \overline{\mathcal{I}^p\big((\mathbb{C}^*)^n\big)}$.

Remark 6.5. Let $V_{\mathbb{T}}$ be an effective tropical p-cycle. By Theorem 5.1 the current $\mathscr{T}^p_n(V_{\mathbb{T}}) \in SPC^p((\mathbb{C}^*)^n)$ can be extended by zero to $\bar{\mathscr{T}}^p_n(V_{\mathbb{T}}) \in SPC^p(\mathbb{CP}^n)$. As a result, if in the above theorem one approximates $V_{\mathbb{T}}$ by amoebas of irreducible algebraic cycles (= analytic cycles by Chow's theorem) of $\mathbb{CP}^n$ which do not lie entirely in $\{z_0 \cdots z_n = 0\}$ then the theorem also gives $\bar{\mathscr{T}}^p_n(V_{\mathbb{T}}) \in \overline{\mathcal{I}^p\big(\mathbb{CP}^n\big)}$.

The above discussion highlights the following important questions.

Problem 6.6. Are there strongly extremal tropical cycles which are not set-wise approximable?

Problem 6.7 (Converse of Theorem 6.4)**.** Assume $V_{\mathbb{T}}$ is a tropical p-cycle such that $\mathscr{T}^p_n(V_{\mathbb{T}})$ is extremal. Does $\mathscr{T}^p_n(V_{\mathbb{T}}) \in \overline{\mathcal{I}^p\big((\mathbb{C}^*)^n\big)}$ imply that $V_{\mathbb{T}}$ is set-wise approximable by amoebas of algebraic varieties in $(\mathbb{C}^*)^n$?

Problem 6.8. How one could generalize these constructions to "infinite" tropical cycles?

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