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Drawing graphs with right angle crossings ☆

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ABSTRACT

Cognitive experiments show that humans can read graph drawings in which all edge crossings are at right angles equally well as they can read planar drawings; they also show that the readability of a drawing is heavily affected by the number of bends along the edges. A graph visualization whose edges can only cross perpendicularly is called a *RAC (Right Angle Crossing) drawing*. This paper initiates the study of combinatorial and algorithmic questions related to the problem of computing RAC drawings with few bends per edge. Namely, we study the interplay between number of bends per edge and total number of edges in RAC drawings. We establish upper and lower bounds on these quantities by considering two classical graph drawing scenarios: The one where the algorithm can choose the combinatorial embedding of the input graph and the one where this embedding is fixed.

1. Introduction

The problem of making good drawings of relational data sets is fundamental in several applications. To enhance human understanding the drawing must be readable, that is it must easily convey the structure of the data and of their relationships (see, for example, [10,26,27]).

A tangled rat's nest of a diagram can be confusing rather than helpful. Intuitively, one may measure the "tangledness" of a graph layout by the number of its edge crossings and by the number of its bends along the edges. This intuition has some scientific validity: Experiments by Purchase et al. have shown that performance of humans in path tracing tasks is negatively correlated to the number of edge crossings and to the number of bends in the drawing [34,35,40].

This negative correlation has motivated intense research about how to draw a graph with few edge crossings and small *curve complexity* (i.e., maximum number of bends along an edge). As a notable example we recall the many fundamental combinatorial and algorithmic results about planar or quasi-planar straight-line drawings of graphs (see, for example, [29,30]). However, in many practical cases the relational data sets do not induce planar or quasi-planar graphs and a high number of edge crossings is basically not avoidable, especially when a particular drawing convention is adopted. How to handle these crossings in the drawing remains unanswered.

Recent cognitive experiments of network visualization provide new insights into the classical correlation between edge crossings and human understanding of a network visualization. Huang et al. show that the edge crossings do not inhibit human task performance if the edges cross at a large angle [22,23,25]. In fact, professional graphic artists commonly use large crossing angles in network drawings. For example, crossings in hand drawn metro maps and circuit schematics

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are conventionally at 90° (see, for example, [39]). Also, in the guidelines of the CCITT (Comité Consultatif International Téléphonique et Télégraphique) for drawing Petri nets the following requirement is reported: ''There should be no acute angles where arcs cross'' [9].

This paper initiates the study of combinatorial and algorithmic questions related to the problem of computing drawings of graphs where the edges cross at 90°. Graph visualizations of this type are called *RAC (Right Angle Crossing) drawings*. We study the interplay between the curve complexity and total number of edges in RAC drawings and establish upper and lower bounds on these quantities. It is immediate to see that every graph has a RAC drawing where the edges are represented as simple Jordan curves that are ''locally adjusted'' around the crossings so that they are orthogonal at their intersection points. However, not every graph has a RAC drawing if small curve complexity is required.

We consider two classical graph drawing scenarios: In the *variable embedding setting* the drawing algorithm takes as input a graph G and attempts to compute a RAC drawing of G; the algorithm can choose both the circular ordering of the edges around the vertices and the sequence of crossings along each edge. In the *fixed embedding setting* the input graph G is given along with a fixed ordering of the edges around its vertices and a fixed ordering of the crossings along each edge; the algorithm must compute a RAC drawing of G that preserves these fixed orderings. An outline of our results is as follows.

- In Section 3 we study the combinatorial properties of straight-line RAC drawings in the variable embedding setting. We give a tight upper bound on the number of edges of straight-line RAC drawings. Namely, we prove that straight-line RAC drawings with n vertices can have at most $4n - 10$ edges, and that there exist infinitely many graphs with this number of edges that are straight-line RAC drawable. It might be worth recalling that straight-line RAC drawings are a subset of the quasi-planar drawings, for which the problem of finding a tight upper bound on the edge density is still open (see, for example, [2,3,32]).
- Motivated by the previous result, we study in Section 4 how the edge density of RAC drawable graphs varies with the curve complexity. We show how to compute a RAC drawing whose curve complexity is three for any graph in the variable embedding setting. We also show that this bound on the curve complexity is tight by proving that curve complexity one implies $O(n^{\frac{4}{3}})$ edges and that curve complexity two implies $O(n^{\frac{7}{4}})$ edges.
- In Section 5 we investigate the fixed embedding setting. In contrast with the results for the variable embedding setting, we show that in the fixed embedding setting the curve complexity of a RAC drawing may no longer be constant. Namely, we establish an $\Omega(n^2)$ lower bound on the curve complexity in this scenario. The embedded graphs constructed for establishing this bound have the properties that the number of crossings between any two edges is bounded by a constant (independent of n). We also show that if any two edges cross at most k times, it is always possible to compute a RAC drawing with $O(kn^2)$ curve complexity. This last result implies that the quadratic bound in n is tight if we restrict to those embeddings such that the number of crossings between any two edges is bounded by a constant.

Preliminary definitions and basic properties of RAC drawings are given Section 2. Conclusions and open problems can be found in Section 6.

We remark that, right after the ideas of this paper were disseminated (they have been mentioned in an invited talk by Eades at ISAAC 2008 [18], informally communicated to the attendees of the Bertinoro Workshop on Graph Drawing 2009,[1] and presented at WADS 2009 [13]), the study of RAC drawings and of its variants has been receiving increasing interest. Angelini et al. study upward RAC drawings and specific sub-families of non-planar graphs that are RAC drawable with few bends per edge [4]. Arikushi et al. prove linear upper bounds to the number of edges of poly-line RAC drawings [7], thus improving the sub-quadratic bounds presented in this paper. RAC drawings of bipartite graphs are studied in [11,14]. Argyriou et al. prove that deciding whether a graph G admits a straight-line RAC drawing is $\mathcal{NP}$-hard [6]. Relationships between straight-line RAC drawings and 1-planar drawings are described in [20]. The advantages of drawing planar graphs with right angle crossings are investigated by van Kreveld [38]. Relaxations of RAC drawings have been studied by several authors. Namely, drawings where edge crossings form angles of at least α (for some fixed constant $0 < \alpha < \pi/2$), have been independently studied by Di Giacomo et al. [12] and by Dujimović et al. [17]. For this kind of drawing, Di Giacomo et al. prove bounds and trade-offs on the area requirement and number of bends, while Dujimović et al. give bounds on the number of edges. In [17] an alternative proof of the upper bound $4n - 10$ to the number of edges of a straight-line RAC drawing is also presented; this proof is based on charging techniques and on a case analysis similar to the one used in this paper. Drawings where edge crossings form angles of exactly α (for some fixed constant $0 < \alpha < \pi/2$) are studied by Ackerman et al. [1]; they prove that these drawings always have a linear number of edges. Finally, algorithms and systems for computing drawings with good crossing angle resolution are described in [5,15,16,19,24,28].

2. Preliminaries

We recall some basic definitions about graph drawing and graph planarity. For more details see [10]. Let G be a graph. A *drawing* of G is a geometric representation of G in the plane such that each vertex is drawn as a distinct point of the plane and each edge is drawn as a simple Jordan curve between the points representing its end-vertices. A *poly-line drawing* of

1 http://www.diei.unipg.it/~bwgd09/.

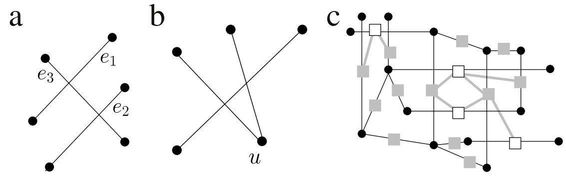

Fig. 1. Basic properties of straight-line RAC drawings: (a) Two edges that cross a common edge must be parallel. (b) There cannot be an edge that crosses two edges incident to the same vertex. (c) The crossing graph is bipartite.

G is such that each edge is drawn as a chain of (straight-line) segments. In a poly-line drawing of G, a point shared by two distinct segments of an edge is called a *bend*. A *straight-line drawing* is a poly-line drawing with no bends. A *planar drawing* of G is a drawing with no edge crossing. A graph G is said to be planar if it admits a planar drawing. A planar drawing of G divides the plane into topologically connected regions, called *faces*. Exactly one of these faces is unbounded and it is called the *external face* of the drawing; the other faces are called the *internal faces* of the drawing.

Let G be any non-planar graph. The *crossing number* of G is the minimum number of edge crossings in a drawing of G, and it is denoted by $cr(G)$. The following bound on $cr(G)$ for any graph G with n vertices and m edges has been proved by Pach et al. [31].

Lemma 1 (*[31]*). $cr(G) \geq \frac{1}{31.1}\frac{m^3}{n^2} - 1.06n$.

A *Right Angle Crossing drawing* (*RAC drawing* for short) of G is a poly-line drawing D of G such that any two crossing segments are orthogonal. Throughout the paper we study RAC drawings such that no edge is self-intersecting and any two edges cross a finite number of times. We also assume that all graphs are simple, that is, they contain neither multiple edges nor self-loops.

The *curve complexity* of D is the maximum number of bends along an edge of D. A *straight-line RAC drawing* has curve complexity zero. We give some simple properties of straight-line RAC drawings that will be used as basic tools throughout the paper.

Property 1. *Let D be a straight-line RAC drawing and let e_1, e_2, e_3 be three edges of D such that e_3 crosses both e_1 and e_2. Then e_1 is parallel to e_2 (see Fig. 1(a)).*

Property 2. *Let D be a straight-line RAC drawing and let u be a vertex of D. There is no edge that crosses two edges incident to u (see Fig. 1(b)).*

Let G be a graph and let D be a straight-line RAC drawing of G; the *crossing graph* $G^*(D)$ of D is the intersection graph of the (open) edges of D. That is, the vertices of $G^*(D)$ are the edges of D, and two vertices of $G^*(D)$ are adjacent in $G^*(D)$ if they cross in D. The following lemma is a consequence of Property 1.

Lemma 2. *The crossing graph of a straight-line RAC drawing is bipartite.*

Proof. Suppose that e is an edge of a straight-line RAC drawing D. From Property 1, every edge in the same connected component as e of $G^*(D)$ is either parallel to e or orthogonal to e. This division into parallel and orthogonal edges forms a bipartition of $G^*(D)$ (see, e.g., Fig. 1(c)). □

3. Straight-line right angle crossing drawings

A *quasi-planar drawing* of a graph G is a drawing of G where no three edges are pairwise crossing [3]. If G admits a quasi-planar drawing it is called a *quasi-planar graph*. Quasi-planar graphs are sometimes called *3-quasi-planar graphs* in the literature.

Lemma 3. *Straight-line RAC drawings are a proper subset of the quasi-planar drawings.*

Proof. By Property 1, in a straight-line RAC drawing there cannot be any three mutually crossing edges. Hence a straight-line RAC drawing is a quasi-planar drawing. The subset is proper because in a quasi-planar drawing edge crossings may not form right angles. □

Quasi-planar drawings have been the subject of intense studies devoted to finding an upper bound on their number of edges as a function of their number of vertices (extremal problems of this type are generically called *Turán-type problems* in combinatorics and in discrete and computational geometry [30]). Agarwal et al. prove that quasi-planar drawings have $O(n)$ edges where n denotes the number of the vertices [3]. This result is refined by Pach, Radoičić, and Tóth, who prove that the number of edges of a quasi-planar drawing is at most $65n$ [32]. This upper bound is further refined by Ackerman and Tardos, who prove that straight-line quasi-planar drawings have at most $6.5n - 20$ edges [2]; this upper bound is tight up to some additive constant.

The main result of this section is a tight upper bound on the number of edges of straight-line RAC drawings with a given number of vertices.

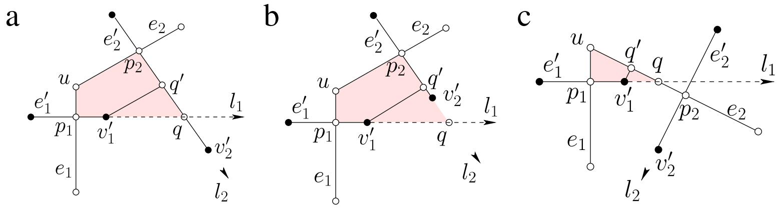

Fig. 2. Illustration of the proof of Lemma 4. (a)–(b) Case 1. (c) Case 2. In all cases, polygon P is represented by a filled region.

Let E be the set of the edges of a straight-line RAC drawing D. Based on Lemma 2 we can partition E into two subsets E_1 and E_2, such that no two edges in the same set cross. We refine this bipartition by dividing E into three subsets as follows: (i) a *red* edge set E_r, whose elements have no crossings; a red edge corresponds to an isolated vertex of $G^*(D)$, (ii) a *blue* edge set $E_b = E_1 - E_r$, and (iii) a *green* edge set $E_g = E_2 - E_r$. We call this partition a *red–blue–green partition* of E. Let $D_{rb} = (V, E_r \cup E_b)$ denote the subgraph of D consisting of the red and blue edges, and let $D_{rg} = (V, E_r \cup E_g)$ denote the subgraph of D consisting of the red and green edges. Graphs D_{rb} and D_{rg} are also called the *red–blue graph* and *red–green graph* induced by D, respectively.

Since only blue and green edges can cross each other in D, it follows that both the red–blue and the red–green are planar embedded graphs. Therefore, each of them has a number of edges that is less than or equal to $3n - 6$, and so a straight-line RAC drawing has at most $6n - 12$ edges. However, to get a tight upper bound $4n - 10$ we need to count more precisely.

Let G be a graph that has a straight-line RAC drawing. We say that G is *RAC maximal* if any graph obtained from G by adding an extra edge does not admit a straight-line RAC drawing. The next lemma gives an important property of RAC maximal graphs that will be used to prove the main theorem of this section.

Lemma 4. *Let G be a RAC maximal graph, let D be any straight-line RAC drawing of G, and let D_{rb} and D_{rg} be the red–blue and red–green graphs induced by a red–blue–green partition of the edges of D, respectively. Every internal face of D_{rb} and every internal face of D_{rg} contains at least two red edges. Also, D_{rb} and D_{rg} have the same external face, whose edges are all red edges.*

Proof. Consider first the (not necessarily simple) polygon $P(D)$ formed by the sequence of vertices, crossing points, and edge segments encountered walking on the external contour of D (i.e., the contour delimiting D). $P(D)$ must be a convex polygon, otherwise it would be possible to add at least an extra red edge to the convex-hull of $P(D)$ between two vertices of $P(D)$ that are also vertices of G; this contradicts the fact that G is RAC maximal. Since the segments of a convex polygon cannot cross, it follows that $P(D)$ is formed only by vertices and edges of G, and all its edges are red edges (because they do not cross). This immediately implies that $P(D)$ coincides with the external face of D_{rb} and of D_{rg}.

We now concentrate on the internal faces of D_{rb} and of D_{rg}. Let f be an internal face of the red–blue graph (the proof is the same for the internal faces of the red–green graph). The boundary of f is a polygon (not necessarily simple) and it must have at least three vertices with an interior angle smaller than 180°. Let u be any of these vertices, and let e_1 and e_2 be the two edges incident to u on the boundary of f that form an angle smaller than 180° inside f (refer also to Fig. 2). We claim that at least one of e_1 and e_2 must be a red edge.

Assume by contradiction that both e_1 and e_2 are blue edges. This implies that there are green edges that cross both e_1 and e_2. Denote by e'_1 the closest green edge to u that crosses e_1 (recall that all the green edges crossing e_1 are parallel). Similarly, denote by e'_2 the closest green edge to u that crosses e_2. Also denote by p_1 the crossing point between e_1 and e'_1, and denote by p_2 the crossing point between e_2 and e'_2. Finally, let l_1 be the semi-line with origin p_1 that contains the part of e'_1 inside f, and let l_2 be the semi-line with origin p_2 that contains the part of e'_2 inside f. Denote by v'_1 the end-vertex of e'_1 that lies on l_1 and by v'_2 the end-vertex of e'_2 that lies on l_2. Note that, v'_1 and v'_2 may coincide. Without loss of generality, assume that the distance between u and p_1 is not greater than the distance between u and p_2. Since the angle formed by e_1 and e_2 in f is smaller than 180° and since e'_1 and e'_2 are orthogonal to e_1 and e_2, respectively, then two cases are possible:

Case 1: l_1 and l_2 intersect in a point q in such a way that $P = (u, p_1, q, p_2)$ is a convex quadrilateral (see Figs. 2(a) and (b)); if $v'_1 = v'_2$ then these vertices also coincide with q.

Case 2: l_1 intersects e_2 (before intersecting l_2) in a point q in such a way that $P = (u, p_1, q)$ is a triangle; this case may happen only if the angle formed by e_1 and e_2 in f is smaller than 90° (see Fig. 2(c)).

We say that a vertex of D is *visible* from u if it can be connected to u by a straight-line segment that does not cross any other edge of D. We show that both in Case 1 and in Case 2 there exists a vertex w inside P or on the boundary of P that is visible from u, thus contradicting the hypothesis that G is RAC maximal. Indeed, if such a vertex w existed it would be possible to add the extra red edge (u, w) to D.

Consider Case 1 first: Since e'_1 and e'_2 cannot cross each other, at least one of v'_1 and v'_2 (possibly both of them) must lie on the poly-line (p_1, q, p_2). Assume first that only one of the vertices v'_1 and v'_2 lies on the poly-line (p_1, q, p_2), say v'_1, as in Fig. 2(a). Consider the convex polygon $P' = (u, p_1, v'_1, q', p_2)$, where q' is the intersection point between e_2 and the line orthogonal to e'_2 through v'_1 (if $v'_1 = v'_2$, then $q = q'$). Clearly P' is contained in P. Denote by S the set of edges that intersect the interior of P'. We claim that there exists at least one vertex w inside P' or on the boundary of P' that is visible from u, and we prove it by induction on $|S|$. We first observe that segments (u, p_1) and (u, p_2) cannot be crossed because, by hypothesis,

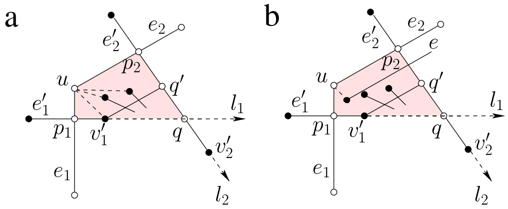

Fig. 3. Illustration of the induction argument in the proof of Lemma 4. (a) The vertices of W' are those connected to u by dashed segments; (b) adding one extra edge e that "covers" all vertices of W' creates a new vertex that is visible from u.

e_1' is the closest green edge to u that crosses e_1 and e_2' is the closest green edge to u that crosses e_2. Also, D cannot have an edge that crosses both segments (p_1, v_1') and (p_2, q'), because these segments are not parallel. Finally, any edge that crosses (p_1, v_1') must be orthogonal to this edge and thus, by construction, it cannot cross the line segment (v_1', q'); similarly, any edge that crosses (p_2, q') must be orthogonal to this edge and thus, by construction, it cannot cross (v_1', q'), unless it overlaps with (v_1', q'). It follows that, if S is not empty, any edge of S has at least one of its end-vertices in the interior of P'. If $|S| = 0$, then v_1' is visible from u and we are done. Suppose now that the claim is true for $|S| = k \geq 1$ and assume that $|S| = k + 1$. Let e be any edge of S. Denote by $W' = \{w_1', w_2', \ldots, w_h'\}$ the set of vertices inside P' or on the boundary of P' that are visible from u when all edges of S except e are in the drawing; also let W be set of vertices inside P' or on the boundary of P' that are visible from u when all edges of S are in the drawing. If $W \cap W'$ is not empty, the claim is true and we are done. If $W \cap W'$ is empty, then edge e must cross all segments (u, w_i') $(1 \leq i \leq h)$, and e has at least one end-vertex in the interior of P'; such a vertex is visible from u (see, e.g., Fig. 3).

If both v_1' and v_2' lie on the poly-line (p_1, q, p_2), as in Fig. 2(b), we further distinguish between two different situations. If segment (v_1', v_2') forms angles of at least 90° with segments (p_1, v_1') and (p_2, v_2'), then consider the convex polygon $P' = (u, p_1, v_1', v_2', p_2)$. Otherwise, it should be possible to define a segment that either connects v_1' to a point q' on e_2' in such a way that (v_1', q') is orthogonal to e_2', or connects v_2' to a point q' on e_1' in such a way that (v_2', q') is orthogonal to e_1'. With the same arguments as the previous sub-case, one can prove that there exists at least one vertex inside P' or on the boundary of P' that is visible from u.

Consider now Case 2: Since e_1' cannot cross e_2 (because e_1 and e_2 are adjacent edges and cannot be parallel), v_1' lies on the boundary of triangle P. Let (v_1', q') the segment such that q' is a point on e_2' and (v_1', q') is orthogonal to e_2', and consider the convex polygon $P' = (u, p_1, v_1', q')$. Denote by S the set of edges that intersect the interior of P'. If S is not empty, each edge of S must have at least one of its end-vertices inside P', because (u, p_1) and (u, q') cannot be crossed. With the same induction technique on $|S|$ as the one used for Case 1, it can be proved that there exists at least one vertex inside P' or on the boundary of P' that is visible from u.

It follows that each vertex u having an angle smaller than 180° inside f has an incident red edge on the boundary of f. Since there are at least three such vertices and since each pair of these vertices cannot share more than one edge of the boundary of f, then f has at least two red edges. □

We are now ready to prove the main result of this section.

Theorem 1. *A straight-line RAC drawing with $n \geq 4$ vertices has at most $4n - 10$ edges. Also, for any $k \geq 3$ there exists a straight-line RAC drawing with $n = 3k - 5$ vertices and $4n - 10$ edges.*

Proof. Let G be a RAC maximal graph with $n \geq 4$ vertices and m edges. Let D be a straight-line RAC drawing of G. Denote by E_r, E_b, E_g the red–blue–green partition of the edges of D and let $m_r = |E_r|$, $m_b = |E_b|$, $m_g = |E_g|$. Assume (without loss of generality) that $m_g \leq m_b$. Of course $m = m_r + m_b + m_g$.

Denote by f_{rb} the number of faces in D_{rb}, and let λ be the number of edges of the external face of D_{rb} (which coincides with the external face of D_{rg}). From Lemma 4 we have that D_{rb} has $f_{rb} - 1$ faces with at least two red edges and one face (the external one) with λ red edges. Also, since every edge is shared by at most two faces, we have

$$m_r \geq f_{rb} - 1 + \lambda/2. \tag{1}$$

Graph D_{rb} is not necessarily connected, but Euler's formula guarantees that

$$m_r + m_b \leq n + f_{rb} - 2. \tag{2}$$

Substituting the inequality (1) into (2) we deduce that

$$m_b \leq n - 1 - \lambda/2. \tag{3}$$

Since D_{rg} has the same external face as D_{rb} we have

$$m_r + m_g \leq 3n - 3 - \lambda. \tag{4}$$

Also, from $m = m_r + m_g + m_b$, we can sum the Inequalities (3) and (4) to obtain

$$m \leq 4n - 4 - 3\lambda/2. \tag{5}$$

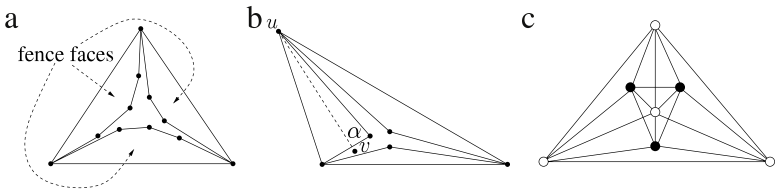

Fig. 4. (a) Fence faces. (b) Triangular fence faces; the dashed edge is a green edge. (c) A straight-line RAC drawing with $n = 7$ vertices and $m = 4n - 10$ edges.

Observe that if $\lambda \geq 4$ then Inequality (5) implies $m \leq 4n - 10$. Thus we need only consider the case when the external face is a triangle, that is, $\lambda = 3$.

Suppose that $\lambda = 3$; consider the (at least one and at most three) internal faces of D_{rb} that share an edge with the external face, as in Fig. 4(a). We call these faces the *fence faces* of D_{rb}. Since we are assuming that $n > 3$ then there is at least one internal vertex. Also, since the graph is RAC maximal, it is impossible that every internal vertex is an isolated vertex. Hence every fence face has at least one internal edge.

Two sub-cases are possible: (i) there exists a fence face that has more than three edges; (ii) every fence face consists of exactly three edges.

(i) If one of the fence faces has more than three edges, then D_{rb} is a planar graph in which at least one face has at least four edges; this implies that

$$m_r + m_b \leq 3n - 7. \tag{6}$$

Since we have assumed that $m_g \leq m_b$, Inequality (6) implies that

$$m_r + m_g \leq 3n - 7. \tag{7}$$

Summing Inequalities (3) (with $\lambda = 3$) and (7) yields $m \leq 4n - 19/2$, which implies $m \leq 4n - 10$ because m is an integer.

(ii) If all the fence faces are triangles there are exactly three fence faces. We show that all the edges of at least two of these faces are red. Suppose that a fence face has one blue edge. This implies that this edge must be crossed by a green edge (u, v). By Property 2 two edges incident on a common vertex cannot be crossed by a third edge. From this fact and since the external face is red, it follows that (u, v) cannot cross another edge of the fence face. Therefore (u, v) must be incident to one vertex of the external face, as in Fig. 4(b). Now (u, v) crosses at an angle of 90°, and so the interior angle α of the triangle that it crosses is less than 90°. However, the sum of the interior angles of the three fence faces is at least 360°. Thus at most one of the three triangles can have an interior angle less than 90°, and so at least two of the fence faces cannot have an edge crossing. Thus at least two of the fence faces have three red edges. Also, the external face has three red edges, and so the drawing has at least three faces in which all three edges are red. It follows that the number of red edges is bounded from below:

$$m_r \geq f_{rb} - 3 + (3 \cdot 3)/2 = f_{rb} + 3/2. \tag{8}$$

Substituting (8) into (2), we deduce that $m_b \leq n - 7/2$, and thus $m \leq 4n - 19/2$, which again implies $m \leq 4n - 10$.

We now prove the second part of the theorem, that is, that for each even integer $k \geq 3$, there exists a RAC maximal graph G_k with $n = 3k - 5$ vertices and $4n - 10$ edges. Graph G_k is constructed as follows (refer to Fig. 4(c) for an illustration where $k = 4$). Start from an embedded maximal planar graph with k vertices and add to this graph its dual planar graph without the face-node corresponding to the external face (in Fig. 4(c) the primal graph has white vertices and the dual graph has black vertices). Also, for each face-node u, add to G_k three edges that connect u to the three vertices of the face associated with u.

A result by Brightwell and Scheinermann about representations of planar graphs and of their duals guarantees that G_k admits a straight-line RAC drawing [8]. More precisely, Brightwell and Scheinermann show that every 3-connected planar graph G can be represented as a collection of circles, a circle for each vertex and a circle for each face. For each edge e of G, the four circles representing the two end-points of e and the two faces sharing e meet at a common point, and the vertex-circles cross the face-circles at right angles. This implies that the union of G and its dual (without the face-node corresponding to the external face) has a straight-line drawing such that the primal edges cross the dual edges at right angles.

Since the number of face-nodes is $2k - 5$, then G_k has $n = 3k - 5$ vertices. The number of edges of G_k is given by $m = (3k - 6) + 3(2k - 5) + 3k - 9$, and hence $m = 12k - 30 = 4n - 10$. □

4. Poly-line right angle crossing drawings

Motivated by Theorem 1, in the attempt to compute RAC drawings of dense graphs we relax the constraint that the edges be drawn as straight-line segments. In this section we study how the edge density of RAC drawable graphs varies with the curve complexity in the variable embedding setting.

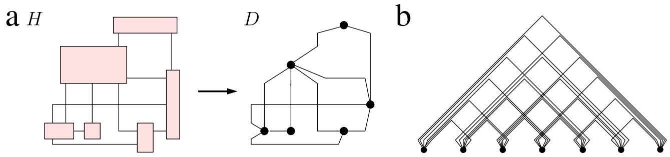

Fig. 5. Two different techniques for constructing a RAC drawing with curve complexity 3: (a) From an orthogonal drawing with box vertices and curve complexity one; (b) placing all the vertices on a line and using exactly three bends per edge.

Lemma 5. *Every graph has a RAC drawing with at most three bends per edge.*

Proof. Papakostas and Tollis describe an algorithm to compute an *orthogonal drawing H* of *G* with at most one bend per edge and such that each vertex is represented as a box [33]. We recall that an orthogonal drawing is such that each edge is drawn as a chain of horizontal and vertical segments. Of course, in an orthogonal drawing any two crossing segments are perpendicular. To get a RAC drawing *D* from *H* it is sufficient to replace each vertex-box with a point placed inside it and to use at most two extra bends per edge to connect the centers to the boundaries of the boxes (see e.g., Fig. 5(a)).

We also observe that an alternative technique that directly constructs a RAC drawing with three bends per edge is to place all vertices along a horizontal line in an arbitrary order, and use slopes of 45 degrees for the middle edge segments, as shown in Fig. 5(b), where a RAC drawing of the complete graph K_7 is depicted. □

Lemma 5 naturally raises the question about whether three bends are not only sufficient but sometimes necessary. This question has a positive answer as we are going to show with the following lemmas.

Let *D* be a poly-line drawing of a graph *G*. An *end-segment* in *D* is an edge segment incident to a vertex. An edge segment in *D* that is not an end-segment is called an *internal segment*. Note that the end points of an internal segment are bends in *D*.

Lemma 6. *Let D be a RAC drawing of a graph G. For any two vertices u and v in G, there are at most two crossings between the end-segments incident to u and the end-segments incident to v.*

Proof. Each crossing between an end-segment incident to *u* and an edge segment incident to v in *D* occurs on the circle *C* whose diameter is the line segment $\overline{uv}$. If there are more than two such points, then at least two crossings occur inside one of the two half circles of *C* defined by $\overline{uv}$. It follows that two line segments meet at an angle larger than 90°, and the drawing is not a RAC drawing. □

Lemma 7. *Let D be a RAC drawing of a graph G with n vertices. Then the number of crossings between all end-segments is at most $n(n-1)$.*

Proof. It follows from Lemma 6 by considering that the number of distinct pairs of vertices is $n(n-1)/2$. □

Lemma 8. *Let D be a RAC drawing and let s be any edge segment of D. The number of end-segments crossed by s is at most n.*

Proof. If *s* crosses more than *n* end-segments in *D*, then there are two of these segments incident to the same vertex, which is impossible by Property 2. □

Lemmas 6–8 and Lemma 1 are the ingredients to show that not all graphs admit a RAC drawing with curve complexity two.

Lemma 9. *A RAC drawing with n vertices and curve complexity two has $O(n^{\frac{7}{4}})$ edges.*

Proof. Let *D* be a RAC drawing with at most two bends per edge. We prove that the number *m* of edges of *D* is $m \le 36n^{\frac{7}{4}}$. Assume by contradiction that $m > 36n^{\frac{7}{4}}$ for arbitrarily large values of *n*. From Lemma 1, the number of crossings in *D* is at least $\frac{1}{31.1}\frac{m^3}{n^2} - 1.06n$. There are at most $3m$ edge segments in *D* because every edge has at most two bends; it follows that there is at least one edge segment *s* with at least $\frac{1}{93.3}\frac{m^2}{n^2} - 0.36\frac{n}{m}$ crossings. For each vertex *u*, at most one end-segment of an edge incident to *u* can cross *s*. Hence, there are at most *n* edges (u, v) that cross *s* in an end-segment of (u, v). This implies that the number m' of edges whose internal segments cross *s* is such that:

$$m' \ge \frac{1}{93.3}\frac{m^2}{n^2} - 0.36\frac{n}{m} - n. \tag{9}$$

From our assumption that $m > 36n^{\frac{7}{4}}$, we can replace *m* on the right hand side of Eq. (9) with $36n^{\frac{7}{4}}$ to obtain $m' > 13.89n^{\frac{3}{2}} - 0.01n^{-\frac{3}{4}} - n$. Since $0.01n^{-\frac{3}{4}} < 1$, it follows that $m' > 13,89n^{\frac{3}{2}} - (n+1)$. Also, since $2n^{\frac{3}{2}} \ge n+1$ (for every $n \ge 1$), it follows that:

$$m' > 11.89n^{\frac{3}{2}}. \tag{10}$$

Let D' be a sub-drawing of *D* consisting of m' edges that cross *s* with an internal segment, as well as the vertices incident to these edges. Let n' be the number of vertices in D'. Using Lemma 1 applied to D', the number of crossings in D' is at least

$\frac{1}{31.1}\frac{m'^3}{n'^2} - 1.06n'$. However, the internal segments of edges in D' are all parallel (since they all cross s at an angle of 90°). Thus, all crossings in D' involve two end-segments. From Lemmas 7 and 8, there are at most $n'(n'-1)+m'n'$ such crossings. Hence, it must be

$$n'(n'-1) + m'n' \geq \frac{1}{31.1}\frac{m'^3}{n'^2} - 1.06n'. \tag{11}$$

Since $n' < n$, and since, from Inequality (10), $m' > n-1$, we have that $2m'n \geq n'(n'-1) + m'n'$. From Inequality (11), it must also hold $2m'n \geq \frac{1}{31.1}\frac{m'^3}{n^2} - 1.06n$, that is:

$$n \geq \frac{1}{62.2}\frac{m'^2}{n^2} - 0.53\frac{n}{m'}. \tag{12}$$

From Inequalities (10) and (12) we have $n \geq 2.27n - 0.045n^{-\frac{1}{2}}$, which is however false for any $n \geq 1$, a contradiction. □

The next lemma completes the analysis of the number of edges in poly-line RAC drawings with curve complexity smaller than three.

Lemma 10. *A RAC drawing with n vertices and curve complexity one has $O(n^{\frac{4}{3}})$ edges.*

Proof. Let D be a RAC drawing with at most one bend per edge. D contains end-segments only. Therefore, from Lemma 7, the number of crossings in D is at most $n(n-1)$. Also, from Lemma 1, the number of crossings in D must be at least $\frac{1}{31.1}\frac{m^3}{n^2} - 1.06n$. It follows that: $n(n-1) \geq \frac{1}{31.1}\frac{m^3}{n^2} - 1.06n$, which implies that $n^4 + 0.06n^3 \geq \frac{1}{1.31}m^3$, and then $m < 3.1n^{\frac{4}{3}}$, i.e., $m = O(n^{\frac{4}{3}})$. □

The following theorem summarizes the interplay between curve complexity and edge density of RAC drawings in the variable embedding setting. It is implied by Theorem 1, Lemmas 5, 9 and 10.

Theorem 2. *Let G be a graph with n vertices and m edges.*

(a) *There always exists a RAC drawing of G with at most three bends per edge.*
(b) *If G admits a RAC drawing with straight-line edges then $m = O(n)$.*
(c) *If G admits a RAC drawing with at most one bend per edge then $m = O(n^{\frac{4}{3}})$.*
(d) *If G admits a RAC drawing with at most two bends per edge then $m = O(n^{\frac{7}{4}})$.*

5. Fixed embedding setting

A classical constraint of many algorithms that draw planar graphs is to preserve a given circular ordering of the edges around the vertices, also called a *combinatorial embedding*. In this section we consider similar constraints for RAC drawings. In contrast with Theorem 2, we show that fixed combinatorial embedding constraints may lead to RAC drawings of non-constant curve complexity, while quadratic curve-complexity is always sufficient for any graph and any fixed combinatorial embedding.

Let G be a graph and let D be a drawing of G. Since in a RAC drawing no three edges can cross each other at the same point, we shall only consider drawings whose crossings involve exactly two edges. We denote by $\overline{G}$ the embedded planar graph obtained from D by replacing each edge crossing with a vertex, and we call it a *planar enhancement* of G. A vertex of $\overline{G}$ that replaces a crossing is called a *cross vertex*. Note that, giving a planar enhancement of G corresponds to specifying the number and the ordering of the cross vertices along each edge, the circular clockwise ordering of the edges incident to each vertex, and the ordered sequence of vertices and crossings that form the external contour of G.

Let G be a graph along with a planar enhancement $\overline{G}$ and let D' be a drawing of G. We say that D' *preserves the planar enhancement* $\overline{G}$ if the planar enhancement of G obtained from D' is the same as $\overline{G}$.

The next theorems establish lower and upper bounds for the curve complexity of RAC drawings in the fixed embedding setting.

Theorem 3. *There are infinitely many values of n for which there exists a graph G with n vertices and a planar enhancement $\overline{G}$ such that any RAC drawing preserving $\overline{G}$ has curve complexity $\Omega(n^2)$.*

Proof. Based on a construction of Roudneff in the projective plane [36], Felsner and Kriegel show *simple arrangements of m pseudolines* in the Euclidean plane forming $m(m-2)/3$ triangular faces for infinitely many values of m [21]. (We recall that in a simple arrangement of pseudolines any two pseudolines cross at most once and no three pseudolines cross at the same point.) For each such values of m, let $\mathcal{A}(m)$ be the corresponding arrangement of pseudolines and let $n = 2(\lfloor\sqrt{m}\rfloor + 1)$.

We define G as a simple bipartite graph with n vertices and m edges such that every partition set of G has $\frac{n}{2}$ vertices; note that, the number of distinct pairs $\{u, v\}$ of vertices of G such that u and v belong to distinct partition sets is $\frac{n^2}{4} \geq m$, and hence G can have m distinct edges. We define a planar enhancement of G by constructing a drawing D where each edge uses a portion of a corresponding pseudoline of $\mathcal{A}(m)$. The planar enhancement of G obtained from D is denoted as $\overline{G}$.

The arrangement of pseudolines defined in [21,36] has the following property: There exists a circle $C(m)$ such that all crossings of $\mathcal{A}(m)$ lie inside $C(m)$ and every pseudoline of $\mathcal{A}(m)$ crosses $C(m)$ in exactly two points. Drawing D is defined as follows (see Fig. 6 for a schematic illustration of the construction):

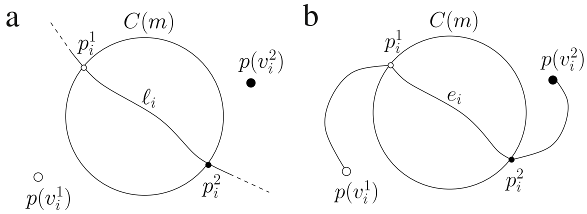

Fig. 6. Schematic illustration of the construction in the proof of Theorem 3: (a) From a pseudoline (b) to an edge.

- Each vertex v of G is drawn as a distinct point $p(v)$ arbitrarily chosen outside $C(m)$.
- Let $\{\ell_1, \ldots, \ell_m\}$ be the pseudolines of $\mathcal{A}(m)$ and let $\{e_1, \ldots, e_m\}$ be the edges of G. Let p_i^1 and p_i^2 be the points of intersection between $C(m)$ and ℓ_i and let $e_i = (v_i^1, v_i^2)$ $(1 \leq i \leq m)$. Edge e_i is drawn as the union of: (i) the portion of ℓ_i inside $C(m)$ that connects p_i^1 with p_i^2; (ii) a simple curve that connects p_i^1 with $p(v_i^1)$ and that does not cross the interior of $C(m)$; (iii) a simple curve that connects p_i^2 with $p(v_i^2)$ and that does not cross the interior of $C(m)$.

Since drawing D maintains all triangular faces of $\mathcal{A}(m)$ and $m = \Theta(n^2)$, it follows that D (and hence $\overline{G}$) has $\Theta(n^4)$ triangular faces inside $C(m)$. Also, the vertices of each triangular face inside $C(m)$ are cross vertices in $\overline{G}$. Therefore, any RAC drawing of G that preserves $\overline{G}$ has at least one bend for each triangular face inside $C(m)$. Hence any RAC drawing of G preserving $\overline{G}$ has $\Omega(n^4)$ bends and curve complexity $\Omega(n^2)$. □

Observe that, in the proof of Theorem 3, the planar enhancement $\overline{G}$ can be constructed in such a way that any two edges cross each other at most three times. Indeed, they cross at most once inside the circle $C(m)$ and they can be routed so that they cross at most twice outside $C(m)$. The next theorem provides an upper bound to the curve complexity of a RAC drawing that preserves a given planar enhancement. This bound is quadratic in n, but also depends on the maximum number of crossings in the drawing associated with the planar enhancement. However, if we restrict to the class of planar enhancements such that two edges cross each other at most three times, the bound of Theorem 3 can be considered a tight bound.

Theorem 4. *Let G be a graph with n vertices and let $\overline{G}$ be a planar enhancement of G obtained from a drawing where any two edges cross at most k times, for some $k \geq 1$. There exists a RAC drawing of G that preserves $\overline{G}$ and that has $O(kn^2)$ curve complexity.*

Proof. Let m be the number of edges of G and let $\overline{n}$ and $\overline{m}$ be the number of vertices and edges of $\overline{G}$, respectively. From the hypothesis that two distinct edges cross at most k times and that an edge cannot cross itself, we have that $\overline{n} \leq n + k(m-1)m$. Namely, every edge of G is subdivided in $\overline{G}$ by at most $k(m-1)$ cross vertices, i.e., it is formed by at most $k(m-1) + 1 = km - k + 1$ edges of $\overline{G}$.

Assume first that G has vertex degree at most four (which of course implies that also $\overline{G}$ has vertices of degree at most four). In this case one can compute a planar orthogonal drawing $\overline{D}$ of $\overline{G}$ with the technique described by Tamassia and Tollis [37]. This technique first computes a visibility representation of the graph, i.e., a planar drawing in which each vertex is drawn as a horizontal segment and each edge is drawn as a vertical segment between its end-vertices (an example of a visibility representation of a planar graph is shown in Fig. 7(a)). Then it replaces each horizontal segment of a vertex v with a point p_v, and connects p_v to the vertical segments representing the incident edges of v, by a local transformation that uses at most two bends per edge around p_v (see, e.g., Fig. 7(b)). Hence an edge can get at most four bends (two for each local transformation around an end-vertex). Therefore, this technique guarantees at most 4 bends per edge. Also, observe that since it is always possible to compute a visibility representation of an embedded planar graph that preserves its planar embedding and since the local transformations do not change this embedding, the technique described above can be applied in such a way that the embedding of $\overline{G}$ is preserved. When cross vertices are replaced by cross points, we get from $\overline{D}$ an orthogonal drawing D of G that preserves $\overline{G}$ and that has at most $4(km - k + 1)$ bends per edge. Since $m < \frac{n^2}{2}$ and since D is a RAC drawing, the theorem follows in this case.

If G has vertices of degree greater than four then we apply a variant of the algorithm of Tamassia and Tollis. Namely, after the computation of a visibility representation of $\overline{G}$ we apply the same transformations as before around the vertices of degree at most four. For a vertex v of degree greater than four we replace the horizontal segment of v with a point p_v, and then locally modify the edges incident to v as shown in Fig. 7(c), by using at most one bend per edge. The drawing $\overline{D}$ obtained in this way is not an orthogonal drawing but it still has at most four bends per edge. By replacing cross vertices with cross points, we still get from $\overline{D}$ a drawing D of G that preserves $\overline{G}$ and that has at most $4(km - k + 1)$ bends per edge. Also, since a cross vertex has degree four in $\overline{D}$, we are guaranteed that D is a RAC drawing, because for these vertices we have applied an orthogonal drawing transformation. □

6. Conclusion and open problems

This paper has studied RAC drawings of graphs, i.e. drawings where edges can cross only at right angles. In fact, many crossings are unavoidable when drawing large graphs and recent perceptual studies have shown that right angle crossings do not have an impact on the readability of a diagram. We have focused on the interplay between edge density and curve

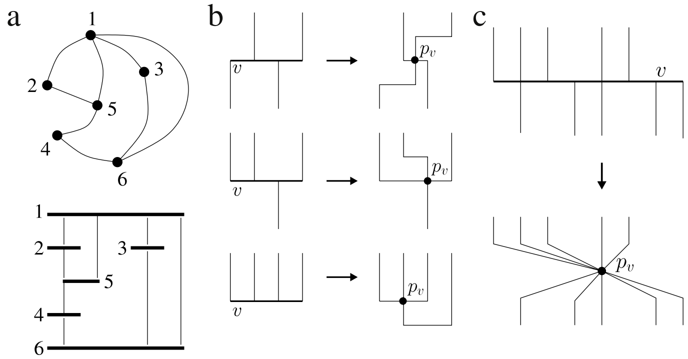

Fig. 7. (a) An embedded planar graph with an embedding preserving visibility representation. Local transformations from a visibility representation to a RAC drawing: (b) for vertices of degree at most four; (c) for vertices of degree greater than four.

complexity of RAC drawings and have proved lower and upper bounds for these quantities. There are several open questions that we consider of interest about RAC drawings. Among them we mention the following, which are strictly related to the results in this paper.

1. By Theorem 1, a straight-line RAC drawing has at most $4n - 10$ edges. Also, as already observed, Argyriou et al. prove that deciding whether a graph G admits a straight-line RAC drawing is $\mathcal{NP}$-hard [6]. One can investigate the following strictly related problems:
   - Characterize the class of graphs having $4n - 10$ edges and admitting a straight-line RAC drawing.
   - What is the complexity of deciding whether a graph with $4n - 10$ edges admits a straight-line RAC drawing?
2. Theorem 2 implies that curve complexity three can be required for computing RAC drawings of non-planar graphs in infinitely many cases. This motivates the investigation of meaningful sub-families of non-planar graphs that can be always drawn with curve complexity one or two. For example, some results on bounded-degree graphs are given in [4].
3. What is the complexity of deciding whether a graph admits a RAC drawing with at most 1 or with at most 2 bends per edges?

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