Winning Infinite Games in Finite Time

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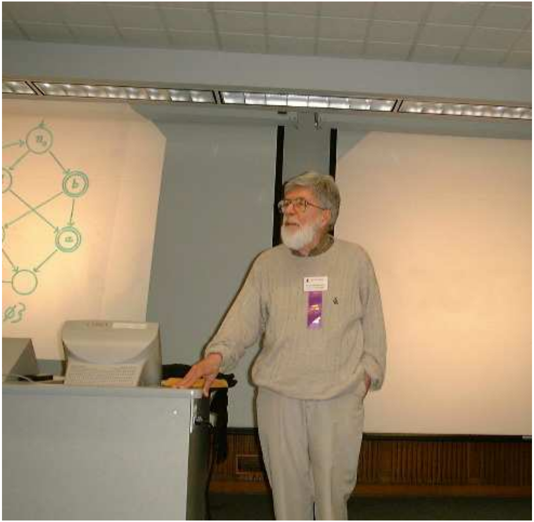
Hausmusik



R. McNaughton

The Problem

Given a Muller game
with the collection $\mathcal{F}$ of "winning loops" for Player 2
play this like a card game in the evening

... and of course go to sleep at some time.

Question: How can one terminate a play after finite time, declaring correctly the winner?

This is trivial for parity games:

Terminate a play when a vertex v is repeated the first time, and declare the winner according to the maximal color seen between the two visits of v.

We pursue the question for Muller games.

From McNaughton's Report (1965)

An infinite game is quite far from any real game situation, if winning and losing must wait until each player has completed an actual infinite number of plays. Such a concept, however, does have application to some real-life game situations; namely to those in which the players at some point are able to see what happens if the game were to last infinitely long, agree on the prognosis, and terminate with the conviction that there is no sense in playing any further. A rather commonplace example of such a situation is a draw in checkers. It is not difficult to imagine that a game conceived as an infinite game might actually be played and enjoyed; but the actual play would always be terminated at some point at which one player admits that the other has a winning strategy. In my opinion any such game would probably be a finite-state infinite game. For, as will be shown, one of the players of such a game always has a winning strategy which is not only effectively determinable but is never too much more complicated than the game as presented.

Scoring

- A Muller game $(G, \mathcal{F}_1, \mathcal{F}_2)$ consists here of an arena $G = (V, V_1, V_2, E)$ and a partition $(\mathcal{F}_1, \mathcal{F}_2)$ of $\mathrm{Pow}V$.
- Player i wins play ϱ iff $\mathrm{Inf}(\varrho) \in \mathcal{F}_i$.
- Strategies, winning strategies, winning regions are defined as before.

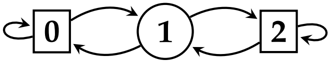
McNaughton's approach: Count for each loop F how often the loop F (as a set) was completely traversed without interruptions.

Call this number at time t of a play the score for F at time t.

McNaughton (2000): The winner of a Muller game is the player who first can reach score $n!$ for one of his winning loops F.

A Muller Game

Example



- $\mathcal{F}_2 = \{\{0,1,2\},\{0\},\{2\}\}$
- $\mathcal{F}_1 = \{\{0,1\},\{1,2\}\}$

Player 2 has a winning strategy: alternate between 0 and 1 (requires two memory states).

Scoring Functions

For $F \subseteq V$ define $\mathrm{Sc}_F\colon V^+ \to \mathbb{N}$:

$$\mathrm{Sc}_F(w) = \max\{k \mid \text{exist words } x_1, \cdots, x_k \in V^+ \text{ s.t. } x_1 \cdots x_k \text{ suffix of } w \text{ and } \mathrm{Occ}(x_i) = F \text{ for all } i\}$$

where $\mathrm{Occ}(w) = \{v \in V \mid \exists j \text{ s.t. } w_j = v\}$.

$\mathrm{Sc}_F(w) = k$ iff all of F is visited k consecutive times

Example:

w	0	0	1	1	0	0	1	2	0	1	2	0	2
$\mathrm{Sc}_{\{0,1\}}$	0	0	1	1	2	2	3	0	0	1	0	0	0
$\mathrm{Sc}_{\{0,1,2\}}$	0	0	0	0	0	0	0	1	1	1	2	2	2

Accumulator Functions

For $F \subseteq V$ define $Acc_F : V^+ \rightarrow 2^F$:

$Acc_F(w)$ contains vertices of F seen since last increase or reset of Sc_F

Example:

w	0	0	1	1	0	0	1	2
$Sc_{\{0,1\}}$	0	0	1	1	2	2	3	0
$Acc_{\{0,1\}}$	$\{0\}$	$\{0\}$	$\emptyset$	$\{1\}$	$\emptyset$	$\{0\}$	$\emptyset$	$\emptyset$
$Sc_{\{0,1,2\}}$	0	0	0	0	0	0	0	1
$Acc_{\{0,1,2\}}$	$\{0\}$	$\{0\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\emptyset$

Finite-time Muller Games

Two properties of the scoring functions (informal versions):

1. **If you play long enough (i.e., $k^{|G|}$ steps), some score value will be high (i.e., k).**
2. **At most one score value can increase at a time.**

A **finite-time Muller game** **has the format $(G, \mathcal{F}_1, \mathcal{F}_2, k)$ with a threshold $k \geq 3$, and the following conditions:**

- **Players move a token through the arena.**
- **Stop play w as soon as score of k is reached for the first time.**
- **There is a unique F such that $\mathrm{Sc}_F(w) = k$.**
- **Player i wins w iff $F \in \mathcal{F}_i$.**

Results

Fearnley, Zimmermann (2010, a GASICS cooperation):

Let $k \geq 3$. The winning regions in a Muller game $(G, \mathcal{F}_1, \mathcal{F}_2)$ and in the finite-time Muller game $(G, \mathcal{F}_1, \mathcal{F}_2, k)$ coincide.

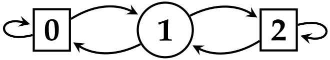
Stronger statement, which implies the theorem:

On his winning region, Player i can prevent her opponent from ever reaching a score of 3 for every set $F \in \mathcal{F}_{1-i}$.

We obtain two "reductions": Muller game to..

1. **..reachability game on unravelling up to score 3 (doubly-exponential blowup)**
2. **..safety game: see next slides.**

“Reducing“ Muller games to Safety Games

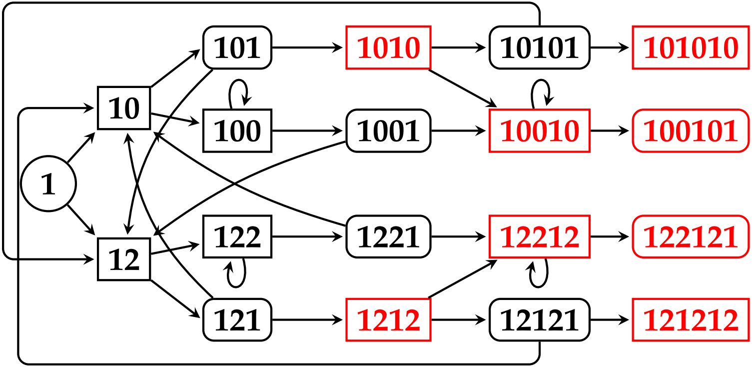


- $\mathcal{F}_2 = \{\{0,1,2\},\{0\},\{2\}\}$
- $\mathcal{F}_1 = \{\{0,1\},\{1,2\}\}$

Idea: keep track of Player 1's scores and avoid $\mathrm{Sc}_F = 3$ for $F \in \mathcal{F}_1$.

- Ignore scores of Player 2.
- Identify plays having the same scores and accumulators for Player 1.
- $w =_{\mathcal{F}_1} w'$ iff
  $\forall F \in \mathcal{F}_1\colon\ \mathrm{Sc}_F(w) = \mathrm{Sc}_F(w')$ and $Acc_F(w) = Acc(w')$
- Build unravelling of $=_{\mathcal{F}_1}$-equivalence classes up to score 3 for Player 1.

Safety Game Graph



Standard Game Reductions

A classical game reduction transforms a complicated game $\mathcal{G}$ to simpler game $\mathcal{G}'$:

Every play in $\mathcal{G}$ is mapped (continuously) to play in $\mathcal{G}'$ that has the same winner.

Solving $\mathcal{G}'$ yields

- both winning regions of $\mathcal{G}$ and
- corresponding finite-state winning strategies for both players.

Muller games cannot be reduced to safety games.

Otherwise we would reduce the Borel level of Muller-recognizable ω-languages ($B(\Pi_2)$) to Π_1.

Results

1. Player i wins the Muller game from v iff she wins the safety game from $[v]_{=_{\mathcal{F}_1}}$.
2. Player 2's winning region in the safety game can be turned into finite-state winning strategy for her in the Muller game.
3. Size of the safety game: $(n!)^3$.

(Neider, Rabinovich, Zimmermann, GandALF 2011)

Remarks:

- Size of parity game in LAR-reduction $n!$. But: simpler algorithms for safety games.
- 2. does not hold for Player 1.
- The reduction is unilateral and not player-symmetric as in the classical sense.

Conclusion

- **Convincing the referee that one can win the game is not the same as winning the game.**
- **One can transform the winner-deciding strategy into a genuine winning strategy.**
  This gives an alternative approach to strategy construction.
- **Task: Study the interplay between symmetric and unilateral game reductions.**

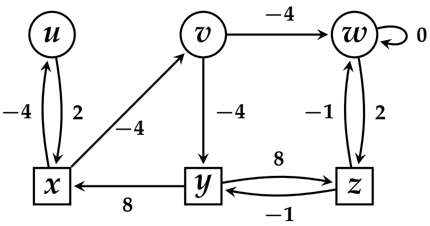
Perspective: Quantitative Aspects

Quantitative Games

- The games studied so far were win-lose games.
- In quantitative games a value is associated to each play.
- Usually, one player tries to maximize and the other player tries to minimize the value.

Other quantitative aspects deals with the economic shape of strategies (e.g., minimization of memory).

A Mean Payoff Game



For a finite play $v_0 \cdots v_n$ we are interested in the mean value

$$\frac{1}{n} \cdot \sum_{i=0}^{n-1} r(v_i, v_{i+1})$$

In the limit, Player 0 tries to maximize and Player 1 tries to minimize this value.

Mean Payoff Game – Formal

A mean payoff game is of the form $\mathcal{G} = (Q, Q_0, E, r)$ where (Q, Q_0, E) is a finite game graph as we know it, and

$$r : E \to \mathbb{Z}$$

is a function assigning a reward to each edge.

As usual the players built up a play $\pi = v_0 v_1 v_2 \cdots$ where

- Player 0 tries to maximize
$$r_0(\pi) := \liminf_{n\to\infty} \left(\frac{1}{n} \cdot \sum_{i=0}^{n-1} r(v_i, v_{i+1}) \right)$$
- Player 1 tries to minimize
$$r_1(\pi) := \limsup_{n\to\infty} \left(\frac{1}{n} \cdot \sum_{i=0}^{n-1} r(v_i, v_{i+1}) \right)$$

Strategies

Strategies for Player i are as before mappings $\sigma : V^*V_i \to V$.

For two strategies σ and τ of Player 2 and Player 1, respectively, and a starting vertex v we denote by $\pi_{\sigma,\tau,v}$ the unique play starting in v and played according to σ and τ.

The Player 0 value of the game from v is

$$\mathrm{val}_0(v) := \sup_\sigma \inf_\tau r_0(\pi_{\sigma,\tau,v}),$$

the Player 1 value of the game from v is

$$\mathrm{val}_1(v) := \inf_\tau \sup_\sigma r_1(\pi_{\sigma,\tau,v}),$$

where σ ranges over Player 0 strategies and τ over Player 1 strategies.

Remark. $\mathrm{val}_0(v) \leq \mathrm{val}_1(v)$

Determinacy of Mean Payoff Games

Theorem (Ehrenfeucht-Mycielski, Zwick-Paterson)

For each finite mean payoff game there are positional strategies σ^* and τ^* for Player 0 and Player 1, respectively, such that for each vertex v

$$\begin{aligned}
\mathrm{val}_0(v) &= \sup_\sigma \inf_\tau r_0(\pi_{\sigma,\tau,v}) \\
&= \inf_\tau r_0(\pi_{\sigma^*,\tau,v}) \\
&= \sup_\sigma r_1(\pi_{\sigma,\tau^*,v}) \\
&= \inf_\tau \sup_\sigma r_1(\pi_{\sigma,\tau,v}) \\
&= \mathrm{val}_1(v)
\end{aligned}$$

The decision problem "Given a finite mean payoff game and a vertex v, is $\mathrm{val}(v) > 0$?" belongs to NP$\cap$co-NP.

From Parity Games to Mean Payoff Games

Theorem. For each parity game $\mathcal{G}$ one can construct a mean payoff game $\mathcal{G}'$ over the same game graph such that for each vertex v Player 0 has a winning strategy in $\mathcal{G}$ from v iff $\mathrm{val}(v) \geq 0$ in $\mathcal{G}'$.

Construction:

- Let n be the number of vertices of $\mathcal{G}$.
- Let (u, v) be an edge of $\mathcal{G}$ and p be the color of u.
- Define $r(u, v) := \begin{cases} n^p & \text{if } p \text{ is even} \\ -n^p & \text{if } p \text{ is odd} \end{cases}$

An Application: Request-Response Games

Request-Response Games

Over a game graph $G = (V, E)$ introduce

"request sets" sets $Rqu_1, \ldots, Rqu_k \subseteq V$

"response" sets $Rsp_1, \ldots, Rsp_k \subseteq V$

RR-condition:

$$\bigwedge_{i=1}^{k} \forall s(Rqu_i(s) \rightarrow \exists t \ (s < t \wedge Rsp_i(t)))$$

Standard solution via a reduction to Büchi games.

Measuring Quality of Solution

- **Linear Penalty model:**
  For each moment of waiting (for each RR-condition) pay 1 unit

- **Quadratic Penalty model:**
  For the i-th moment of waiting pay i units

Activation of i-th condition in a play ϱ is a visit to Rqu_i such that all previous visits to Rqu_i are already matched by an Rsp_i-visit.

Values of Plays and Strategies

For both linear and quadratic penalty define:

- $w_\varrho(n) =$ **sum of penalties in** $\varrho(0)\ldots\varrho(n)$ **divided by number of activations**

  "average penalty sum per activation"

- $w(\varrho) = \limsup_{n\to\infty} w_\varrho(n)$

Given a strategy σ **for controller and a strategy** τ **for adversary**

- $\varrho(\sigma,\tau) :=$ **the play induced by** σ **and** τ
- $w(\sigma) := sup_\tau\, w(\varrho(\sigma,\tau))$

Call σ **optimal if there is no other strategy with smaller value.**

On the Quadratic Penalty

For the linear penalty model, a finite-state optimal strategy does not exist in general.

For the quadratic penalty function one can decide whether a RR-game is won by controller and in this case one can compute a finite-state optimal winning strategy.

(Horn, Th., Wallmeier, ATVA 2008)

Proof ingredients:

- **It suffices to consider strategies with value $\leq M$ (induced by bounded waiting time of standard solution).**
- **Conversely: For strategies with value $\leq M$ one can assume bounded waiting time.**
- **Reduction to mean-payoff games.**

Concluding Remarks

What to take home?

A Fascinating Field

Infinite two-person games are an intriguing subject:

It offers

- **interesting automata theoretic constructions**
- **connections with set theory and logic**
- **many open questions, even from the early papers,**
- **promising perspectives regarding quantitative aspects**

For example, here is an open question from Büchi-Landweber 1969:

Understand the space of all winning strategies of a game, in order to be able to pick the "best" ones.

A Question of 1969

$Zt' = H[Xt, Yt, Zt]$.

(α) If $Zt' \in \{s_1, \ldots, s_n\}$, let i be the first such that $Zt' = s_i$. Then,
$Vt' = [A_1, s_1, h_1, \ldots, A_i, A_i(s_i), h_i]$
$kt' = h_1$.

(13) (β) If $Zt' \in P_{kt-1}[A_1, s_1, \ldots, A_n, s_n]$ but not (α), let j be the first such that $Zt' \in A_j \cap R_{kt-1}[A_1, s_1, \ldots, A_j, s_j]$ ($Zt' \in R_{kt-1}[\quad]$ if $j = 0$). Then,
$Vt' = [A_1, s_1, h_1, \ldots, A_j, s_j, h_j]$
$kt' = kt-1$.

(γ) If $Zt' \in Q_{kt-1}[A_1, s_1, \ldots, A_n, s_n]$ but neither (α) nor (β), let B be the first in the chosen order of U such that, $Zt' \in B \subset A_n$ and $\bigwedge_{u \in B} u \in R_{kt-1}[A_1, s_1, \ldots, A_n, s_n, B, B(u)]$. Then,
$Vt' = [A_1, s_1, h_1, \ldots, A_n, s_n, h_n, B, B(Zt'), kt-1]$
$kt' = kt-1$.

PROBLEM. Modify the recursions (13) to a schema with parameters which, by proper additional specifications for the parameters, will yield any given deterministic operator which solves $\mathfrak{C}(X, Y)$ for Y. Do the same for (16) and solutions of $\mathfrak{C}(X, Y)$ for X.

Summarizing

Church's Problem is far from closed:

- **Even for the (classical) infinite two-person games, we have not yet understood completely how to construct strategies that are "good" — and it is even less clear how to handle multiple optimization criteria.**
- **For the connection between games and logic, a central question is to better understand the relation between definability of games and strategies.**
- **In particular:**
  Is there a compositional framework of strategy construction which reflects the structure of the (logical) specifications and works without the detour through automata theory (algorithmic theory of labelled graphs)?