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ON THE DECISION PROBLEM FOR TWO-VARIABLE FIRST-ORDER LOGIC

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Abstract. We identify the computational complexity of the satisfiability problem for FO^2, the fragment of first-order logic consisting of all relational first-order sentences with at most two distinct variables. Although this fragment was shown to be decidable a long time ago, the computational complexity of its decision problem has not been pinpointed so far. In 1975 Mortimer proved that FO^2 has the *finite-model property*, which means that if an FO^2-sentence is satisfiable, then it has a finite model. Moreover, Mortimer showed that every satisfiable FO^2-sentence has a model whose size is at most doubly exponential in the size of the sentence. In this paper, we improve Mortimer's bound by one exponential and show that every satisfiable FO^2-sentence has a model whose size is at most exponential in the size of the sentence. As a consequence, we establish that the satisfiability problem for FO^2 is NEXPTIME-complete.

§1. Introduction. Once the satisfiability problem for first-order logic was shown to be undecidable [12, 55], logicians embarked on an ambitious project aiming to delineate the boundary between decidable and undecidable fragments of first-order logic. In this project, the main focus was on prefix classes and prefix-vocabulary classes, that is, on collections of first-order sentences in prenex normal form defined by imposing restrictions on the quantifier prefix or by imposing restrictions on both the quantifier prefix and the vocabulary of function and relation symbols. For example, the AEA class consists of all relational (i.e., without function symbols) first-order sentences with quantifier prefix of the form $\forall\exists\forall$. After toiling on this project for almost fifty years, researchers were finally able to identify the dividing line between decidability and undecidability for all prefix-vocabulary classes of first-order formulas [15, 41, 23, 6]. Moreover, an effort was made to pinpoint the computational complexity of the decision problem for the decidable classes [6, 18, 24, 25, 37, 42].

A different way to obtain syntactic fragments of first-order logic is to partition the formulas according to the number of their variables. More precisely, k-variable first-order logic FO^k consists of all relational first-order formulas containing at most k different individual variables, $k \geq 1$. These fragments were introduced by Henkin [30], who investigated certain aspects

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of their proof theory. In recent years, both k-variable first-order logics and k-variable infinitary logics gained popularity in the context of finite-model theory, where they have been the focus of extensive study, since the number of variables is considered a logical resource in descriptive complexity theory and since logics with fixpoint constructs can be viewed as effective fragments of k-variable infinitary logics (see [9, 14, 31, 33, 34, 35, 38, 39, 46, 56]). Note that the class AEA mentioned above is contained in FO^3. Since the satisfiability problem for the AEA class is undecidable (see [6, 41]), it follows that FO^3, even without equality, is undecidable. This motivates the study of the decision problem for FO^2, as FO^2 and FO^3 may very well be on opposite sides of the boundary between the decidable and the undecidable.

Modal logic provides another motivation for studying the complexity of the decision problem for FO^2. Modal logic can be described succinctly as the logic of necessity and possibility, of "must be" and "may be". Note that one should not take "necessity" and "possibility" literally, as their meaning may be adapted to the situation at hand. For example, "necessarily" can mean "according to the laws of physics", or "according to my beliefs", or even "after the program terminates". For this reason, in recent years modal logic has been applied to numerous areas of computer science, including artificial intelligence [7, 44], program verification [48, 47], hardware verification [5, 51], database theory [10, 13, 43], and distributed computing [8, 28]. The attractiveness of modal logic for formal reasoning stems to a large degree from the fact that *propositional modal logic* is decidable in a very robust way, as has been amply demonstrated (see [11, 17, 29, 40, 49, 54, 58]). The robust decidability of propositional modal logic is actually rather surprising. In spite of the adjective "propositional", it is well understood that propositional modal logic is essentially a fragment of first-order logic, where the modalities $\Box$ ("necessarily") and $\Diamond$ ("possibly") are intrinsically universal and existential quantifiers, respectively [1, 2]. What, then, makes propositional modal logic so robustly decidable? To answer this question, we have to take a close look at propositional modal logic as a first-order logic. A careful examination reveals that propositional modal logic can in fact be viewed as a fragment of FO^2 without equality (see [20, 3]). Thus, a decidability result for FO^2 would explain the decidability of propositional modal logic. Moreover, a result identifying the precise complexity of FO^2 would also provide an upper bound for the computational complexity of propositional modal logic and several of its variants (see [57]).

It should be noted that the presence or absence of equality may cause the boundary between decidability and undecidability to shift. The most striking instance of this phenomenon is the Gödel class, that is, the class of relational first-order sentences with quantifier prefix of the form $\exists^*\forall\forall\exists^*$ (a string consisting of an arbitrary number of existential quantifiers, followed by precisely two universal quantifiers, followed by an arbitrary number of existential quantifiers). Gödel [21], Kalmár [36], and Schütte [52] showed independently that this class is decidable, provided no occurrence of the equality symbol is allowed in the sentences of the class. In a second paper [22], Gödel also established that this class has the *finite model property*: every satisfiable sentence in this class has a finite model (this property is often referred to as *finite controllability*). At the end of this paper Gödel claimed, without substantiation, that his proof persists in the presence of equality. This claim, however, was refuted by Goldfarb [23], who established that even the *minimal* Gödel class, with quantifier prefix of the form $\forall\forall\exists$, becomes undecidable once equality is allowed.

The presence or absence of equality may also affect the computational complexity of the satisfiability problem for decidable classes. A case in point is the Ackermann class, which consists of all relational first-order sentences with quantifier prefix of the form $\exists^*\forall\exists^*$. Indeed, the satisfiability problem for the Ackermann class without equality is EXPTIME-complete [41, 18], whereas the same problem for the Ackermann class with equality is NEXPTIME-complete [37]. A more dramatic example is provided by the Rabin class, which consists of all first-order sentences with arbitrary quantifier prefix, one unary function symbol, and an arbitrary number of unary relation symbols (but no function or relation symbols of higher arity). The satisfiability problem for this class without equality is NEXPTIME-complete. On the other hand, the same class with equality is decidable, but not elementary recursive, that is, the time complexity of the decision problem exceeds any constant number of iterations of the exponential function (see [6]).

For certain other classes, however, the presence of equality makes no essential difference. Consider, for example, the Bernays-Schönfinkel-Ramsey class, which consists of all relational first-order sentences with quantifier prefix of the form $\exists^*\forall^*$. The satisfiability problem for this class without equality was shown to be decidable by Bernays and Schönfinkel [4]; moreover, Ramsey [50] extended this result[1] to the case with equality. Lewis [42] showed that the satisfiability problem for this class without equality is NEXPTIME-complete; it is easy to see that the same holds true for the case with equality.

The first decidability result for FO^2 was obtained by Scott [53], who showed that the decision problem for FO^2 can be reduced to that of the Gödel class. Since, as mentioned above, only the Gödel class without equality is decidable, Scott's reduction yields the decidability of FO^2 without equality, but does

[1]In fact, Ramsey proved a much stronger result, namely, that the spectrum of every such sentence is either finite or cofinite. It was for the proof of this result that Ramsey developed his celebrated combinatorial theorems.

not cover the case of FO^2 with equality. The full class FO^2 was considered by Mortimer [45]. He proved that this class is decidable by showing that it has the *finite model property*. An analysis of his proof shows that he actually established a *bounded model property* for FO^2: if a FO^2-sentence φ is satisfiable, then it is satisfiable in a model whose size is at most *doubly exponential* in the length of ϕ. It follows that satisfiability of FO^2 with equality is decidable in nondeterministic *doubly exponential* time, since to check whether a FO^2-sentence φ with equality is satisfiable we simply guess a finite structure $\mathbf{A}$ of size at most doubly exponential in the length of φ and verify that $\mathbf{A} \models \varphi$.

In this paper we take a closer look at the decision problem for FO^2, with the aim of pinpointing its computational complexity and, thus, contributing a missing part to the complexity-theoretic analysis of the decidable fragments of first-order logic. The main result of this paper is that the satisfiability problem for FO^2 with equality is NEXPTIME-complete. In particular, this new upper bound for the satisfiability problem for FO^2 with equality improves Mortimer's upper bound by one exponential. To obtain this improvement, we revisit Scott's reduction and observe that in fact it reduces FO^2 to a *proper* fragment of the Gödel class. This fragment, which we call the *Scott class*, consists of all first-order sentences with equality that are conjunctions of sentences with quantifier prefixes of the form $\forall\forall$ and $\forall\exists$. We show that by refraining from converting these sentences to prenex normal form (and viewing them as sentences in the Gödel class) we can realize a significant decrease in complexity. Specifically, we establish an *exponential model property* for the Scott class: if a sentence φ in this class is satisfiable, then it is satisfiable in a model whose size is at most exponential in the size of φ.

The lower bound for the complexity of the satisfiability problem for FO^2 follows from results of Fürer [19], who, building on earlier work by Lewis [42], established that the satisfiability problem of $\forall\forall \wedge \forall\exists$ first-order sentences without equality (that is, sentences that are a conjunction of a single $\forall\forall$ sentence without equality and a single $\forall\exists$ sentence without equality) is NEXPTIME-hard. Thus, equality makes no difference to the complexity of the satisfiability problem for FO^2.

§2. First-order logic with a fixed number of variables. We consider first-order logic FO with equality over a fixed relational vocabulary. The k-*variable first-order logic* FO^k consists of all formulas of FO with at most k distinct individual variables. Thus,

$$FO = \bigcup_{k=1}^{\infty} FO^k .$$

The expressive power of the logics FO^k, $k \geq 1$, on graphs $\mathbf{G} = (V, E)$ is usually illustrated by the fact that for any $n \geq 1$ the property "there is a path of length n from x to y" is expressible by a formula $p_n(x, y)$ of FO^3. Indeed, put $p_1(x, y) \equiv E(x, y)$ and assume, by induction on n, that $p_{n-1}(x, y)$ is a formula of FO^3 asserting that "there is a path of length n from x to y". Then the desired formula $p_n(x, y)$ is

$$(\exists z)[E(x, z) \wedge (\exists x)(x = z \wedge p_{n-1}(x, y))].$$

Note that any formula in prenex normal form that is equivalent to $p_n(x, y)$ requires at least $n + 1$ variables, while $p_n(x, y)$ uses only the variables x, y, and z (the variables x and z have many occurrences in $p_n(x, y)$). Next, consider the property "there are at least n distinct elements", which, in general, can not be expressed by any first-order sentence with fewer than n variables. If, however, we restrict ourselves to linear orders $\mathbf{P} = (P, <)$, then for every $n \geq 1$ there is a sentence χ_n of FO^2 asserting "there are at least n distinct elements". For example, χ_4 is the sentence

$$(\exists x)(\exists y)[x < y \wedge (\exists x)(y < x \wedge (\exists y)(x < y))].$$

§3. **FO^2 and Scott's reduction.** The satisfiability problem for FO^2 was first studied by Scott [53], who showed that it can be reduced to the satisfiability problem for the *Gödel class*, that is, the class of sentences with quantifier prefix of the form $\forall\forall\exists^*$.

Suppose that ϕ is a sentence in FO^2 with individual variables x and y. Let s be the *size* of ϕ, that is, the length of a string encoding ϕ over some fixed alphabet. Before describing Scott's reduction, we want to remove all relation symbols of arity bigger than 2. More precisely, given ϕ as above, we will construct in polynomial time an FO^2-sentence ϕ' with the following properties:

(1) ϕ is satisfiable if and only if ϕ' is satisfiable. Furthermore, for every finite model of ϕ' there is a finite model of ϕ of the same cardinality.
(2) Every relation symbol occurring in ϕ' has arity at most 2.
(3) ϕ' contains $O(s/\log s)$ different relation symbols and is of size $O(s)$.

Consider an atomic subformula $R(v_1, \ldots, v_n)$ of ϕ, where R is an n-ary relation symbol occurring in ϕ and $n > 2$. Note that each variable v_i is either x or y. For each such subformula of ϕ, we introduce a new relation symbol $R^{(v_1,\ldots,v_n)}$. If both x and y are among the variables $v_1, \ldots, v_n$, then $R^{(v_1,\ldots,v_n)}$ has arity 2; in this case, we replace every occurrence of the atomic formula $R(v_1, \ldots, v_n)$ in ϕ by the atomic formula $R^{(v_1,\ldots,v_n)}(x, y)$. If each variable v_i is the variable x (respectively, the variable y), then $R^{(v_1,\ldots,v_n)}$ has arity 1; in this case, we replace every occurrence of the atomic formula $R(v_1, \ldots, v_n)$ in ϕ by the atomic formula $R^{(v_1,\ldots,v_n)}(x)$ (respectively, by the atomic formula $R^{(v_1,\ldots,v_n)}(y)$). Let $\phi^\dagger$ be the sentence resulting from these substitutions. Since (when coded by a fixed alphabet) a formula of length s can contain only $O(s/\log s)$ distinct atomic formulas, it follows that $\phi^\dagger$

has $O(s/\log s)$ different relation symbols and is of size $O(s)$. To complete the construction of ϕ', we must append certain conjuncts to $\phi^\dagger$ asserting that certain atomic formulas involving the new relation symbols $R^{(v_1,\dots,v_n)}$ are equivalent to each other. For example, if the atomic formulas $R(x,y,x)$ and $R(y,x,y)$ occur in ϕ, then we must append a conjunct asserting that

$$(\forall x)(\forall y)(R^{(x,y,x)}(x,y) \leftrightarrow R^{(y,x,y)}(y,x)).$$

Similarly, if the atomic formulas $R(x,x,x)$ and $R(x,y,x)$ occur in ϕ, then we must add a conjunct asserting that

$$(\forall x)(R^{(x,x,x)}(x) \leftrightarrow R^{(x,y,x)}(x,x)).$$

For each atomic subformula $R(v_1,\dots,v_n)$ of ϕ such that both variables x and y are among the variables v_i, we consider the atomic formula $R(w_1,\dots,w_n)$ obtained from $R(v_1,\dots,v_n)$ by replacing every occurrence of x by y, and every occurrence of y by x. If $R(w_1,\dots,w_n)$ happens to be a subformula of ϕ as well, then we append the FO^2-sentence

$$(\forall x)(\forall y)(R^{(v_1,\dots,v_n)}(x,y) \leftrightarrow R^{(w_1,\dots,w_n)}(y,x)).$$

Finally, for every relation symbol R occurring in ϕ, we first consider all atomic subformulas $R(v_1,\dots,v_n)$ of ϕ in which R occurs and then append an FO^1-sentence asserting that all atomic formulas of the form $R^{(v_1,\dots,v_n)}(x,x)$ (or of the form $R^{(v_1,\dots,v_n)}(x)$) are equivalent to each other. This sentence is written as a cycle of implications to avoid a quadratic blow-up. For example, if $R(x,x,x)$, $R(x,y,x)$, $R(x,x,y)$ is a list of all atomic formulas of ϕ in which R occurs, then we append the FO^1-sentence

$$\begin{aligned}&(\forall x)((R^{(x,x,x)}(x) \rightarrow R^{(x,y,x)}(x,x)) \wedge \\ &\qquad (R^{(x,y,x)}(x,x) \rightarrow R^{(x,x,y)}(x,x)) \wedge (R^{(x,x,y)}(x,x) \rightarrow R^{(x,x,x)}(x))).\end{aligned}$$

Let ϕ' be the FO^2-sentence obtained by first appending the above sentences as conjuncts to $\phi^\dagger$ and then converting the resulting sentence to an equivalent one built using $\wedge$, $\neg$, and $\forall$ only. It is now easy to verify that ϕ' has the the desired properties that were listed earlier. In particular, ϕ' is of size $O(s)$.

Next, we describe Scott's reduction. Let ϕ' be a sentence of FO^2. For each subformula ψ of ϕ', we introduce a new relation symbol Q_ψ; the arity of Q_ψ is equal to the number of free variables in ψ, which means that it is 0, 1, or 2. Intuitively, Q_ψ represents the relation containing all tuples that satisfy ψ. We now need to "axiomatize" this intuition. Thus, for each subformula $\psi(\mathbf{v})$, where $\mathbf{v}$ is the tuple of free variables in ψ, we introduce a sentence θ_ψ of the form

$$\forall \mathbf{v}(Q_\psi(\mathbf{v}) \leftrightarrow \theta'_\psi(\mathbf{v})),$$

where θ'_ψ is as follows:

(1) If ψ is an atomic formula, then θ'_ψ is ψ.
(2) If ψ is of the form $\alpha \wedge \beta$, then θ'_ψ is $Q_\alpha(\mathbf{v}) \wedge Q_\beta(\mathbf{v})$.
(3) If ψ is of the form $\neg\alpha$, then θ'_ψ is $\neg Q_\alpha(\mathbf{v})$.
(4) If ψ is of the form $\forall v\alpha$, then θ'_ψ is $\forall v Q_\alpha(\mathbf{v})$.

Note that in the first three clauses θ_ψ has quantifier prefix $\forall$ or $\forall\forall$, while in the last one θ_ψ is (equivalent to) a conjunction of a sentence of quantifier prefix $\forall\forall$ with a sentence of quantifier prefix $\forall\exists$. Let $\Theta_{\phi'}$ be the conjunction of the sentences θ_ψ for all subformulas ψ of ϕ'. Finally, let ϕ^* be the sentence $Q_{\phi'} \wedge \Theta_{\phi'}$. It is not hard to verify that the following holds.

PROPOSITION 3.1 ([53]). *ϕ' is satisfiable if and only if ϕ^* is satisfiable. Furthermore, for every finite model of ϕ^* there a finite model of ϕ' of the same cardinality.*

Note that if ϕ' is of size s, then ϕ^* contains $O(s)$ different relation symbols and is of size $O(s\log(s))$. Indeed ϕ' may contain up to $O(s)$ subformulae, so we need $O(s)$ relation symbols. The extra $\log(s)$ factor in the size of ϕ^* is due to the fact that we need a name (an index) of size $O(\log(s))$ for these different relation symbols.

Suppose now that we combine Scott's reduction with the previous reduction, so that, given a FO^2-sentence ϕ, we apply Scott's reduction to the FO^2-sentence ϕ' produced by the first reduction. Thus, given an FO^2-sentence ϕ, we obtain in polynomial time a sentence ϕ^* with the following properties:

(1) ϕ is satisfiable if and only if ϕ^* is satisfiable. Furthermore, for every finite model of ϕ^* there is a finite model of ϕ of the same cardinality.
(2) Every relation symbol occurring in ϕ^* has arity at most 2.
(3) If s is the size of ϕ, then ϕ^* contains $O(s)$ different relation symbols and has size $O(s\log(s))$.
(4) ϕ^* is a conjunction of sentences with quantifier prefixes of the form $\forall\forall$ or $\forall\exists$.

Scott observed that if the sentence ϕ^* is brought into prenex normal form, it has a quantifier prefix of the form $\forall\forall\exists^*$. In view of this, he concluded that the satisfiability problem for FO^2 is decidable, since it is reducible to that of the Gödel class. At that time it had not been detected yet that, contrary to Gödel's claim, his decidability proof does not persist in the presence of equality. Thus, Scott's proof covers only FO^2 *without* equality. As mentioned in the introduction, Mortimer [45] established that FO^2 with equality has the finite model property, which implies that the satisfiability problem for FO^2 with equality is decidable. Actually, Mortimer's proof shows that every satisfiable FO^2 sentence with equality has a finite model whose size is doubly exponential in the size of the sentence. This yields, in turn, a nondeterministic doubly exponential algorithm for the satisfiability problem for FO^2 with equality.

In what follows, we re-examine Scott's reduction and demonstrate that it is useful even for FO^2 with equality. The key idea is to refrain from converting ϕ^* to prenex formal form.

§4. **The Scott class.** Let ϕ be a sentence of FO^2 with equality. We noted above that the sentence ϕ^* in Proposition 3.1 can actually be written as a conjunction of sentences with quantifier prefix $\forall\forall$ or $\forall\exists$. We call the class of such first-order sentences the *Scott class*. Since a conjunction of $\forall\forall$ sentences is equivalent to a single $\forall\forall$ sentence, we may assume that every sentence θ in the Scott class is of the form

$$(\forall x)(\forall y)\alpha(x,y) \wedge \bigwedge_{i=1}^{m}(\forall x)(\exists y)\beta_i(x,y),$$

where $\alpha(x,y)$ and $\beta_i(x,y)$, $1 \le i \le m$, are quantifier-free formulas. Moreover, we may assume that for every $i \le m$ it is the case that $\beta_i(x,y) \models x \neq y$, since for every formula $\chi(x,y)$ and every structure $\mathbf{A}$ with at least two elements

$$\mathbf{A} \models (\forall x)(\exists y)\chi(x,y) \longleftrightarrow (\forall x)(\exists y)(x \neq y \wedge (\chi(x,x) \vee \chi(x,y))).$$

The main result of this paper is that the satisfiability problem for the Scott class is solvable in nondeterministic exponential time. This result is obtained by establishing an *exponential model* property for the Scott class, that is, every satisfiable sentence in the Scott class has a finite model whose cardinality is at most exponential in the size of the sentence. Although this bound improves the bound in Mortimer [45] by one exponential, it turns out that the proof is actually simpler than Mortimer's. Our construction requires a delicate handling of the *types* that are realized by elements and pairs of elements in models of sentences in the Scott class. We start with the relevant definitions.

DEFINITION 4.1. Let σ be a relational vocabulary.

• If $\mathbf{x} = (x_1, \dots, x_k)$ is a sequence of variables, then an *k-type $t(\mathbf{x})$ in the variables $\mathbf{x}$ over σ* is a maximal consistent set of atomic and negated atomic formulas (including equalities) over the vocabulary σ in the variables $x_1, \dots, x_k$. We often view a type as a quantifier-free formula over σ that is the conjunction of its elements.

• Let $t(x_1, \dots, x_k)$ be a k-type and let $\phi(x_1, \dots, x_k)$ be a quantifier-free formula in the variables $x_1, \dots, x_k$. We say that *t satisfies ϕ* if ϕ is true under the truth assignment that assigns true to an atomic formula precisely when it is a member of t.

• Let $\mathbf{A}$ be a structure over the vocabulary σ and let $\mathbf{a} = (a_1, \dots, a_k)$ be a sequence of elements from the universe A of $\mathbf{A}$. The *type* $t_{\mathbf{a}}$ of $\mathbf{a}$ on $\mathbf{A}$ is the unique k-type $t(z_1, \dots, z_k)$ that the sequence $\mathbf{a}$ satisfies in $\mathbf{A}$, under the assignment $z_i \to a_i$, $1 \leq i \leq k$. We say that a sequence $\mathbf{a}$ *realizes* a type t on a structure $\mathbf{A}$ if $t_{\mathbf{a}} = t$. ⊣

Suppose that we are attempting to construct a model of a sentence θ in the Scott class over a vocabulary σ. As every relation symbol in σ has arity at most 2, to describe a σ-structure $\mathbf{A}$ suffices to first define its universe A and then specify the 1-types and 2-types realized by elements and pairs of elements from A. Since θ may contain equalities, it is conceivable that θ asserts that certain 1-types are realized by at most one element. For example, θ may assert (among other things) that $(\exists y)P(y) \wedge (\forall x)(\forall y)(P(x) \wedge P(y) \rightarrow x = y)$, which implies that in every model of θ if $t(z)$ is a 1-type containing the atomic formula $P(z)$, then $t(z)$ is realized by at most one element. Such elements are special and for this reason we reserve a special name for them.

Definition 4.2. Let $\mathbf{A}$ be a structure and a an element of the universe A of $\mathbf{A}$. We say that a is a *king* in $\mathbf{A}$ if a is the only element of A that realizes the 1-type t_a of a on $\mathbf{A}$. ⊣

In general, the potential presence of kings creates obstructions in constructing models of a sentence, as conflicts may arise when one attempts to assign a 2-type to a pair of elements such that one of the elements in the pair is a king. For example, consider the sentence

$$(\forall x)(\exists y)(t(y) \wedge E(x, y)) \wedge (\forall x)(\exists y)(t(y) \wedge \neg E(x, y)).$$

One can construct a model of this sentence by choosing for every element a two different elements b_1 and b_2 of type $t(y)$, and stipulating that $E(a, b_1)$ and $\neg E(a, b_2)$ hold. This construction, however, can not be carried out if the sentence contains additional conjuncts implying that the type $t(y)$ is realized by at most one element. It should also be pointed out that in certain cases the presence of kings can be exploited to establish that the class under consideration does not have the finite model property and, furthermore, that the satisfiability problem for it is undecidable. Indeed, Goldfarb's [23] proof of the undecidability of the Gödel class with equality involves an essential use of kings. We now show that in the case of the Scott class the complications caused by the kings can be overcome, provided the kings are treated with "proper care and respect".

Theorem 4.3. *Let θ be a sentence in the Scott class. If θ is satisfiable, then it has a finite model with at most $3s2^r$ elements, where s is the size of θ and r is the number of relation symbols occurring in θ.*

PROOF. As stated earlier, we may assume that the sentence θ is of the form

$$(\forall x)(\forall y)\alpha(x,y) \wedge \bigwedge_{i=1}^{m}(\forall x)(\exists y)\beta_i(x,y),$$

where $\alpha(x,y)$ and $\beta_i(x,y)$, $1 \le i \le m$, are quantifier-free formulas and for every $i \le m$ it is the case that $\beta_i(x,y) \models x \neq y$. Let $\mathbf{A}$ be a model of θ, let K be the set of all kings in $\mathbf{A}$, and let $P = \{t_a : a \in A\}$ be the set of all 1-types realized in $\mathbf{A}$. We will show that θ has a finite model of size at most

$$(m+1)|K| + 3m(|P| - |K|),$$

where $|K|$ and $|P|$ stand for the cardinalities of the sets K and P.

Since $\mathbf{A} \models \bigwedge_{i=1}^{m}(\forall x)(\exists y)\beta_i(x,y)$, there exist Skolem functions $g_i : A \to A$, $1 \le i \le m$, such that for every $a \in A$ we have that $\mathbf{A} \models \bigwedge_{i=1}^{m} \beta_i(a, g_i(a))$. Let

$$C = K \cup \left(\bigcup_{i=1}^{m}\{g_i(k) : k \in K\}\right)$$

be the *royal court*, that is the set consisting of the kings and the values of the Skolem functions on the kings. Note that C may be empty, since after all $\mathbf{A}$ may be a "republic" in which kings do not exist. In any case, $|C| \le |K| + m|K| = (m+1)|K|$. Let Q be the set of all 1-types realized by the kings on $\mathbf{A}$, let $n = |P| - |Q| = |P| - |K|$ be the cardinality of the set $P - Q$, and let $t_1, \ldots, t_n$ be an enumeration of all members of $P - Q$. For every $i \le m$ and every $j \le n$, let d_{ij}, e_{ij}, and f_{ij} be distinct new objects that are not members of the universe of $\mathbf{A}$. We will construct a finite model $\mathbf{B}$ of θ with universe the set $B = C \cup D \cup E \cup F$, where

$$\begin{aligned} D &= \{d_{ij} : 1 \le i \le m, 1 \le j \le n\}, \\ E &= \{e_{ij} : 1 \le i \le m, 1 \le j \le n\}, \\ F &= \{f_{ij} : 1 \le i \le m, 1 \le j \le n\}. \end{aligned}$$

The high-level description of the construction is as follows:

• The structure $\mathbf{B}$ will have exactly the same kings as $\mathbf{A}$, since for all we know θ may logically imply that certain 1-types are realized by exactly one element.

• To guarantee that $\mathbf{B} \models (\forall x)(\forall y)\alpha(x,y)$, we will make sure that every pair of elements of B is assigned a 2-type realized by some pair of elements in $\mathbf{A}$.

• We also have to guarantee that every element of B has *Skolem witnesses* for the formulas $\beta_i(x,y)$, $1 \le i \le m$, that is, for every element b of B and every $i \le m$ there is an element b_i of B such that $\mathbf{B} \models \beta_i(b, b_i)$. This turns out to be the most subtle part of the construction. The kings will have members of C as their Skolem witnesses, whereas members of $C - K$ will have members of C or members of D as Skolem witnesses. For the remaining members of B, Skolem witnesses will be provided in a circular manner: members of D will have kings or members of E as Skolem witnesses; members of E will have kings or members of F as Skolem witnesses; finally, members of F will have kings or members of D as Skolem witnesses. Moreover, we will make sure that for every b in B if b_i and b_j are two of its Skolem witnesses and neither b_i nor b_j is a member of C, then $b_i \neq b_j$. In turn, this will make it possible to assign 2-types to pairs of elements of B without creating any conflicts.

It is now time to spell out the formal details of the construction of **B**. For this, we must describe the assignment of 1-types and 2-types on **B**.

• Every member of C is equipped with its 1-type in **A**, and every pair of distinct elements of C is equipped with its 2-type in **A**. Consequently, the substructure $\mathbf{B}|C$ of **B** generated by C coincides with the substructure $\mathbf{A}|C$ of **A** generated by C.

• For every $i \leq m$ and every $j \leq n$, each of the elements d_{ij}, e_{ij}, f_{ij} is equipped with t_j as its 1-type. Note that these two steps of the construction ensure that **A** and **B** have exactly the same kings.

• The assignment of 2-types on pairs of elements of B will follow the assignment of Skolem witnesses to every member b of B.

(1) If b is a king, then its Skolem witnesses are already provided by members of the royal court C. Indeed, in this case we have that $\mathbf{A}|C \models \bigwedge_{i=1}^{m}(\exists y)\beta_i(b,y)$ and, consequently, $\mathbf{B} \models \bigwedge_{i=1}^{m}(\exists y)\beta_i(b,y)$.

(2) Let b be a member of $C - K$. For every $i \leq m$, consider the value $g_i(b)$ of the Skolem function g_i on b; thus, $\mathbf{A} \models \beta_i(b, g_i(b))$. If $g_i(b) \in C$, then $\mathbf{B} \models \beta_i(b, g_i(b))$ and so $g_i(b)$ can serve as a Skolem witness of b for the formula $\beta_i(x,y)$. If $g_i(b) \notin C$, then its type $t_{g_i(b)}$ is a member of $P - Q$ and, consequently, $t_{g_i(b)} = t_j$ for some $j \leq n$. In this case, we assign the element d_{ij} as the Skolem witness of b for the formula $\beta_i(x,y)$. Moreover, we equip the pair (b, d_{ij}) with the 2-type $t_{(b,g_i(b))}$ of the pair $(b, g_i(b))$ on **A**. Note that no conflicts arise in assigning 2-types, as none of the elements of D is used twice as a Skolem witness of b, and the 2-type assigned is consistent with the 1-type of d_{ij}.

(3) Let b be a member of D, which means that there is an $i \leq m$ and a $j \leq n$ such that $b = d_{ij}$. Moreover, b realizes the 1-type t_j on **B**. Let a be an element of A such that the 1-type t_a of a on **A** is equal to t_j. For every $i \leq m$, let $g_i(a)$ be the value of the Skolem function g_i on a; thus, $\mathbf{A} \models \beta_i(a, g_i(a))$. We now distinguish two cases. If $g_i(a)$ is a king, then we assign $g_i(a)$ as the Skolem witness of b for the formula $\beta_i(x,y)$. Moreover, we equip the pair $(b, g_i(b))$ with the 2-type $t_{(a,g_i(a))}$ of the pair $(a, g_i(a))$ on **A**. Note that this is consistent with the assignment of 1-types on **B** and that no conflicts arise,

since so far the pair $(b, g_i(a))$ has not been assigned a 2-type on **B**. If $g_i(a)$ is not a king, then its type on **A** is a member of $P - Q$ and, therefore, it is equal to some type t_l, $l \leq n$. In this case, we assign the element e_{il} as the Skolem witness of b for the formula $\beta_i(x, y)$. Moreover, we equip the pair (b, e_{ij}) with the 2-type $t_{(a, g_i(a))}$ of the pair $(a, g_i(a))$ on **A**. Note again that this assignment is consistent with the assignments of 1-types on **B** and that no conflicts arise in assigning 2-types, as none of the elements of E is used twice as a Skolem witness of b.

(4) We repeat twice the previous step, first with the pair (E, F) in place of the pair (D, E), and then with the pair (F, D) in place of the pair (E, F).

(5) Upon completion of the above steps, every element of B has been assigned Skolem witnesses for the formulas $\beta_i(x, y)$, $i \leq m$. It is conceivable, however, that not every pair of elements of B has been assigned a 2-type. If (b, b') is such a pair, simply choose a pair (a, a') of elements of A such that the 1-type of a (respectively, of a') on **A** coincides with the 1-type of b (respectively, of b') on **B** and equip the pair (b, b') with the 2-type $t_{(a,a')}$ of the pair (a, a') on **A**. The construction of **B** is now complete.

Note that every 1-type and every 2-type realized in **B** is also realized in **A**. Since $\mathbf{A} \models (\forall x)(\forall y)\alpha(x, y)$, it follows that $\mathbf{B} \models (\forall x)(\forall y)\alpha(x, y)$. Moreover, $\mathbf{B} \models \bigwedge_{i=1}^{m}(\forall x)(\exists y)\beta_i(x, y)$, since the construction guarantees that every member of B has Skolem witnesses for the formulas $\beta_i(x, y)$, $i \leq m$. Consequently, **B** is a model of θ. Moreover, as promised earlier, the universe B of **B** has cardinality $|B| = |C| + 3m(|P| - |K|) \leq (m+1)|K| + 3m(|P| - |K|)$, and $m \leq s$. Note that $|K| \leq |P| \leq 2^r$, where r is the number of relation symbols that occur in θ and s is the size of θ. Thus, $|B| \leq 3s2^r$.⊣

It is perhaps worth pointing out that if, instead of using $3m$ copies of every 1-type in $P - Q$, we had attempted to build a model of θ using $2m$ copies of every 1-type in $P - Q$, then the construction would have met with serious obstacles. Indeed, suppose we take $C \cup D \cup E$ as the universe of **B** and attempt to use members of E as Skolem witnesses for members of D, and vice versa use members of D as Skolem witnesses for members of E. Then conflicts may arise in assigning 2-types, as we may have an element d of D and an element e of E such that d and e serve as Skolem witnesses of each other, but different 2-types are required each time to satisfy some of the formulas $\beta_i(x, y)$, $i \leq m$.

§5. The decision problem for FO^2.

THEOREM 5.1. FO^2 *has the exponential model property: there is a constant* c *such that every satisfiable* FO^2*-sentence* ϕ *has a model of cardinality at most* 2^{cs}*, where* s *is the size of* ϕ.

PROOF. Given an FO^2-sentence ϕ, we can reduce it in polynomial time to a sentence ϕ^* in the Scott class such that ϕ is satisfiable if and only if ϕ^* is satisfiable. Moreover, for every finite model of ϕ^* there is a finite model of ϕ of the same cardinality. As shown earlier, if ϕ is of size s, then ϕ^* is of size $O(s \log s)$ and has at most s different relation symbols. By Theorem 4.3, if ϕ^* has a model, then it has a model of cardinality $O(s \log s\ 2^s) = 2^{O(s)}$. ⊣

DEFINITION 5.2. For any function t from positive integers to positive integers, $\mathrm{NTIME}(t(s))$ is the class of all decision problems that can be solved by a non-deterministic Turing machine in time $t(s)$, where s is the size of the input. We denote by NEXPTIME the union, taken over all polynomials p, of the classes $\mathrm{NTIME}(2^{p(s)})$.

A decision problem A is NEXPTIME-*complete* if it is in NEXPTIME and, moreover, every problem in NEXPTIME can be reduced to A in polynomial time. ⊣

In what follows, the quantity s always denotes the size of the given input sentence.

THEOREM 5.3. *The satisfiability problem for the Scott class is in* NTIME $(2^{O(s/\log s)})$. *Further, the satisfiability problem for* FO^2 *is in* $\mathrm{NTIME}(2^{O(s)})$.

PROOF. Let ϕ be a sentence of length s in the Scott class (with equality) with r distinct relation symbols. Since these relation symbols have distinct names, coded over a fixed alphabet, it follows that $r = O(s/\log s)$. By Theorem 4.3, to check whether ϕ is satisfiable, it suffices to guess a structure $\mathbf{A}$ of cardinality at most $O(s2^r) = 2^{cs/\log s}$ (for some fixed constant c), and to verify that $\mathbf{A} \models \phi$. Given that the relation symbols in ϕ are at most binary, a structure of this cardinality can be represented by a string of length $2^{O(s/\log s)}$. Finally it is obvious that the verification that $\mathbf{A} \models \phi$ can be done in time $2^{O(s/\log s)}$. This proves the claim for the Scott class.

The complexity bound for FO^2 follows immediately from the reduction to the Scott class, as explained in Section 3. ⊣

A matching lower bound for the satisfiability problem of the Scott class (even without equality) follows from a result of Fürer [19], who, building on earlier work by Lewis [42], established that the satisfiability problem for $\forall\forall \wedge \forall\exists$ sentences has a lower complexity bound of the form $\mathrm{NTIME}(2^{ds/\log s})$ for some positive constant d. To prove this, Fürer described a log-space reduction that maps any instance x of a decision problem A in $\mathrm{NTIME}(2^n)$ to an $\forall\forall \wedge \forall\exists$-sentence ϕ of size $O(n \log n)$ (where n is the size of x) such that ϕ is satisfiable if and only if $x \in A$. In fact, ϕ is without equality and contains only monadic relation symbols.

This lower bound of course also applies to FO^2 and, together with Theorem 5.3 implies the following completeness result.

COROLLARY 5.4. *The satisfiability problem for* FO^2 *with or without equality is* NEXPTIME-*complete.*

Note, however, that there is a small gap between upper and lower complexity bounds for FO^2, which comes from the increase of the formula length from s to $O(s \log s)$ in the reduction of FO^2 to the Scott class. It is not clear whether this can be avoided, since FO^2-sentences of length s may very well have $\Theta(s)$ nested quantifiers. Thus, while we do know that the satisfiability problem for FO^2 is in $\mathrm{NTIME}(2^{O(s)})$, we do not know whether it is hard for $\mathrm{NTIME}(2^{O(s)})$, since this class is closed under linear reductions, but not under polynomial reductions.[2]

We conclude by discussing the decision problem for certain extensions of FO^2. The exponential model property of Theorem 5.1 and the complexity bound of Theorem 5.3 survive (with the same proof) if constant symbols are allowed in the underlying vocabulary. In this case, the 1-types and 2-types also reflect the relationship of the elements with the constants. The constants themselves are of course kings. Taking this into account the proof of Theorem 4.3 goes through without problems. Nevertheless, these results do not extend to vocabularies containing function symbols of positive arity. Indeed, it is known that already the satisfiability problems of FO^1 with equality and only two unary function symbols, or of FO^2 without equality and just one unary function symbol, are undecidable (see [6]).

It should also be pointed out that the class $\forall\forall\forall \wedge \forall\exists$ does not have the finite model property. Indeed, one can easily construct an *infinity axiom* (that is, a satisfiable formula without a finite model) in this class by expressing, for instance, that a binary relation R is a linear order without a maximal element. Moreover, the satisfiability problem for the class $\forall\forall\forall \wedge \forall\exists$ is undecidable (see [41]). Thus, the Scott class is situated very close to the boundary of decidability/undecidability, as well as to that of finite model property/infinity axioms.

Finally, we note that the decidability result for FO^2 (but not the finite model property) can be extended to FO^2 with *counting quantifiers* [27]. On the other hand, for certain other natural extensions of FO^2 decidability fails [26].

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