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Reduced Gröbner Bases, Free Difference–Differential Modules and Difference–Differential Dimension Polynomials

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We define a special type of reduction in a free left module over a ring of difference–differential operators and use the idea of the Gröbner basis method to develop a technique that allows us to determine the Hilbert function of a finitely generated difference–differential module equipped with the natural double filtration. The results obtained are applied to the study of difference–differential field extensions and systems of difference–differential equations. We prove a theorem on difference–differential dimension polynomial that generalizes both the classical Kolchin's theorem on dimension polynomial of a differential field extension and the corresponding author's result for difference fields. We also determine invariants of a difference–differential dimension polynomial and consider a method of computation of the dimension polynomial associated with a system of linear difference–differential equations.

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1. Introduction

The efficiency of the classical Gröbner basis methods for the computation of Hilbert polynomials of graded and filtered modules over polynomial rings is well-known. Similar approaches have been explored in differential algebra where the theory of Gröbner bases in free modules over rings of differential operators developed in Mikhalev and Pankratev (1980, 1989), Insa and Pauer (1998) and some other works gives methods of computation of differential dimension polynomials, see Levin and Mikhalev (1987) and Kondrateva *et al.* (1999, Chap. IV).

The concept of the differential dimension polynomial was introduced in Kolchin (1964) where the theorem that lies in the foundation of the dimension theory in differential algebra was proved. Recall that a differential field is a field L considered together with a finite set $\Delta = \{\delta_1, \dots, \delta_m\}$ of mutually commuting derivations of L. It is also called a Δ-field and the set Δ is said to be a basic set of the differential field L. A subfield K of a Δ-field L is called a differential (or Δ-) subfield of L if $\delta(K) \subseteq K$ for any $\delta \in \Delta$. In this case L is said to be a Δ-field extension of K. If K is a Δ-subfield of a Δ-field L and $\Sigma \subseteq L$, then the intersection of all Δ-subfields of L containing K and Σ is the unique Δ-subfield of L containing K and Σ and contained in every Δ-subfield of L containing K and Σ. It is denoted by $K\langle\Sigma\rangle$. If the set Σ is finite, $\Sigma = \{\eta_1, \dots, \eta_p\}$, the Δ-field L is said to be a finitely generated Δ-extension of K. In this case we write $L = K\langle\eta_1, \dots, \eta_p\rangle$.

Let K be a differential field of zero characteristic with a basic set of derivation operators

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$\Delta = \{\delta_1, \dots, \delta_m\}$ and let L be a differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_p\}$. Furthermore, let Θ_Δ denote the free commutative semigroup generated by the set Δ and for any nonnegative integer r, let $\Theta_\Delta(r)$ be the set of all elements $\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \in \Theta_\Delta$ such that $\sum_{i=1}^m k_i \le r$. Then $L = K(\{\theta\eta_j | \theta \in \Theta, 1 \le j \le n\})$ and the fields $L_r = K(\{\theta\eta_j | \theta \in \Theta_\Delta(r), 1 \le j \le n\})$ $(r = 0, 1, 2, \dots)$ form an increasing sequence of subextensions of the field extension L/K. The following classical Kolchin's theorem describes the function $f(r) = \operatorname{trdeg}_K L_r$ and gives some parameters of this function that are differential birational invariants of the Δ-extension L/K.

Theorem 1.1. *With the above notation and conventions, there exists a polynomial $\omega_{\eta|K}(t)$ in one variable t with rational coefficients such that*

(i) $\omega_{\eta|K}(r) = \operatorname{trdeg}_K L_r$ *for all sufficiently large* $r \in \mathbf{Z}$*;*
(ii) $\deg \omega_{\eta|K} \le m$ *and* $\omega_{\eta|K}(t)$ *can be written as* $\omega_{\eta|K}(t) = \sum_{i=0}^m a_i \binom{t+i}{i}$ *where* $a_0, \dots, a_m$ *are integers;*
(iii) *The numbers* $d = \deg \omega_{\eta|K}$*,* a_m *and* a_d *do not depend on the choice of the system of* Δ*-generators* η *of the extension* L/K *(clearly,* $a_d \ne a_m$ *if and only if* $d < m$*, i.e.* $a_m = 0$*). Moreover,* a_m *is equal to the differential transcendence degree of* L *over* K*, i.e. to the maximal number of elements* $\xi_1, \dots, \xi_k \in L$ *such that the set* $\{\theta\xi_i | \theta \in \Theta, 1 \le i \le k\}$ *is algebraically independent over* K*.*

The polynomial $\omega_{\eta|K}(t)$ is called the differential dimensional polynomial of the Δ-field extension L/K associated with the system of Δ-generators $\eta = \{\eta_1, \dots, \eta_p\}$.

Johnson (1969, 1974) showed that the differential dimension polynomial of a differential field extension coincides with the Hilbert polynomial of the filtered module of Kähler differentials associated with the extension. This result allowed to compute differential dimension polynomials using the Gröbner basis technique, as well as involutive basis methods developed in Apel (1995, 1998) and some other works. Various problems of differential algebra involving differential dimension polynomials were studied in Sit (1975, 1978), Johnson and Sit (1979), Levin and Mikhalev (1987) and Kondrateva *et al.* (1999). Note that despite some partial results obtained in Carra Ferro (1989, 1997), the attempt to imitate Gröbner basis methods in the context of differential ideals of a ring of differential polynomials has been unsuccessful to date.

Discussing the problems connected with the differential dimension polynomial one should mention its analytic interpretation. While developing a gravitation theory, Einstein (1953) introduced a concept of the strength of a system of differential equations as a certain function of integer argument associated with the system. Mikhalev and Pankratev (1980) showed that this function coincides with the appropriate differential dimension polynomial and found the strength of some well-known systems of partial differential equations using methods of differential algebra.

Since 1980 the Hilbert polynomial technique has been spread to difference algebra that arises from the study of algebraic difference equations in much the same way as differential algebra arises from the analysis of algebraic differential equations. The basic results, ideas and methods of difference algebra can be found in the classical monograph Cohn (1965) and in the recent work of Put and Singer (1997) that presents the contemporary state of the difference Galois theory.

The concept of the difference dimension polynomial was introduced by the author first for the difference field extensions (Levin, 1978) and then for the inversive difference field extensions and for the difference and inversive difference modules (see Levin, 1980, 1982, 1985).

For the case of difference fields of zero characteristic, the main theorem on the difference dimension polynomial is formulated as follows.

THEOREM 1.2. *Let K be a difference field of zero characteristic with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, i.e. a field K considered together with a finite set σ of mutually commuting automorphisms of this field. Let Γ be the free commutative group generated by the set σ, and for any nonnegative integer r, let $\Gamma(r)$ denote the set of all elements $\gamma = \alpha_1^{k_1} \dots \alpha_n^{k_n} \in \Gamma$ ($k_1, \dots, k_n$ are some integers) such that $\sum_{i=1}^{n} |k_i| \le r$. Furthermore, let L be a difference field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_p\}$. (As a field, $L = K(\{\gamma(\eta_i) | \gamma \in \Gamma, 1 \le i \le p\})$).*

Then there exists a polynomial $\phi_{\eta|K}(t)$ in one variable t with rational coefficients (it is called a difference dimension polynomial of the extension L/K) such that

(i) $\phi_{\eta|K}(r) = \mathrm{trdeg}_K K(\{\gamma(\eta_j) | \gamma \in \Gamma(r), 1 \le j \le p\})$ *for all sufficiently large* $r \in \mathbf{Z}$;

(ii) $\deg \phi_{\eta|K} \le n$ *and the polynomial* $\phi_{\eta|K}(t)$ *can be written as* $\phi_{\eta|K}(t) = \sum_{i=0}^{n} a_i 2^i \binom{t+i}{i}$ *where* $a_0, \dots, a_n$ *are some integers;*

(iii) *The degree d of the polynomial $\phi_{\eta|K}$ and the coefficients a_n and a_d do not depend on the choice of the system of σ-generators η of the extension L/K (clearly, $a_d \ne a_n$ if and only if $d < n$, that is $a_n = 0$). In other words, d, a_n, and a_d are difference birational invariants of the extension. Moreover, the coefficient a_n is equal to the difference transcendence degree of L over K, i.e. to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the set $\{\gamma(\xi_i) | \gamma \in \Gamma, 1 \le i \le k\}$ is algebraically independent over K.*

Difference dimension polynomials play the same role in difference algebra as Hilbert polynomials in commutative algebra or differential dimension polynomials in differential algebra. In particular, the strength of a system of equations in finite differences (defined in the sense of Einstein) coincides with certain difference dimension polynomial. Difference dimension polynomials and their invariants were studied in Balaba (1984), Mikhalev and Pankratev (1989), Levin (1980, 1982, 1985), Pankratev (1989), Kondrateva *et al.* (1999) and some other works.

In this paper we introduce a special type of reduction in a free module over a ring of difference–differential operators and develop the appropriate technique (in the spirit of the Gröbner basis method) that allows us to prove the existence and find an approach to the computation of dimension polynomials in two variables associated with a finitely generated difference–differential field extension. The two main theorems (see Theorem 5.1 and Theorem 5.4) not only generalize Kolchin's and Johnson's results on differential dimension polynomials and the author's theorems on difference characteristic polynomials, but also allow us to develop methods of computation of dimension polynomials associated with systems of algebraic difference–differential equations. The results of the paper give a new tool for the study of systems of algebraic differential equations with delay (in particular, they give a new approach to the study of the strength of such a system in the sense of Einstein, 1953).

2. Preliminaries

Throughout the paper **Z**, **N**, $\mathbf{Z}_-$, and **Q** denote the sets of all integers, all nonnegative integers, all nonpositive integers, and all rational numbers, respectively. By a ring we always mean an associative ring with a unit. Every ring homomorphism is unitary (maps unit onto unit), every subring of a ring contains the unit of the ring. Unless otherwise indicated, by the module over a ring A we mean a left A-module. Every module over a ring is unitary and every algebra over a commutative ring is also unitary.

Let K be a commutative ring and let $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ be sets of derivations and automorphisms of the ring K, respectively, such that any two elements of the set $\Delta \bigcup \sigma$ commute. (In other words, $\beta(\gamma(x)) = \gamma(\beta(x))$ for any $\beta, \gamma \in \Delta \bigcup \sigma$, $x \in K$.) Then K is said to be a difference–differential ring with the basic set of derivations Δ and the basic set of automorphisms σ. This ring is also called a Δ-σ-ring. If a Δ-σ-ring K is a field, it is called a Δ-σ-field (or a difference–differential field with the basic set $\Delta \bigcup \sigma$).

If K_0 is a subfield of a Δ-σ-field K such that $\beta(x) \in K_0$ and $\gamma^{-1}(x) \in K_0$ for any $x \in K_0$, $\beta \in \Delta \bigcup \sigma$, $\gamma \in \sigma$, then K_0 is called a difference–differential (or Δ-σ-) subfield of K. If, in addition, Σ is a subset of K, then the intersection of all Δ-σ-subfields of K containing K_0 and Σ is the unique Δ-σ-subfield of K containing K_0 and Σ and contained in every Δ-σ-subfield of K containing K_0 and Σ. This intersection is denoted by $K_0\langle\Sigma\rangle$. If $K = K_0\langle\Sigma\rangle$ and the set Σ is finite, $\Sigma = \{\eta_1, \dots, \eta_p\}$, then K is said to be a finitely generated Δ-σ-extension of K_0 with the set of Δ-σ-generators $\{\eta_1, \dots, \eta_p\}$. In this case we write $K = K_0\langle\eta_1, \dots, \eta_p\rangle$.

If K is a Δ-σ-ring, then Λ (or $\Lambda(\Delta, \sigma)$, if the basic sets should be specified) will denote the commutative semigroup of elements of the form

$$\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} \tag{2.1}$$

where $k_1, \dots, k_m \in \mathbf{N}$ and $l_1, \dots, l_n \in \mathbf{Z}$. This semigroup contains the free commutative semigroup Θ generated by the set Δ and free commutative group Γ generated by the automorphisms of the set σ. Furthermore, the subset $\{\alpha_1, \dots, \alpha_n, \alpha_1^{-1}, \dots, \alpha_n^{-1}\}$ of Λ will be denoted by σ^*.

By the orders of the element λ of the form (2.1) with respect to the sets Δ and σ we mean the numbers $\mathrm{ord}_\Delta\lambda = \sum_{i=1}^m k_i$ and $\mathrm{ord}_\sigma\lambda = \sum_{j=1}^n |l_j|$, respectively; the number $\mathrm{ord}\,\lambda = \mathrm{ord}_\Delta\lambda + \mathrm{ord}_\sigma\lambda$ is called the order of λ. Furthermore, for every $r, s \in \mathbf{N}$, $\Lambda(r, s)$ will denote the set of all elements $\lambda \in \Lambda$ such that $\mathrm{ord}_\Delta\lambda \le r$ and $\mathrm{ord}_\sigma\lambda \le s$.

An ideal I of a Δ-σ-ring K is said to be a Δ-σ-ideal of K if $\beta(x) \in K$ for any $x \in K, \beta \in \Delta \bigcup \sigma^*$. If $\Sigma \subseteq K$, then the smallest Δ-σ-ideal of K containing Σ is denoted by $[\Sigma]$. Clearly, this ideal is generated (as an ideal in the usual sense) by the set $\{\lambda(x) | \lambda \in \Lambda, x \in \Sigma\}$.

Let K and L be two Δ-σ-rings. A ring homomorphism $f : K \longrightarrow L$ is called a Δ-σ-homomorphism, if $f(\beta(x)) = \beta(f(x))$ for any $x \in K, \beta \in \Delta \bigcup \sigma^*$. In this case $\mathrm{Ker} f$ is a Δ-σ-ideal of K and conversely, if f is a ring epimorphism of a Δ-σ-ring K onto some ring L such that $\mathrm{Ker} f$ is a Δ-σ-ideal of K, then L has a unique structure of a Δ-σ-ring such that f is a Δ-σ-epimorphism. In particular, if I is a Δ-σ-ideal of K, then the factor ring K/I has a unique structure of a Δ-σ-ring such that the natural mapping of K onto K/I is a Δ-σ-epimorphism; in this case K/I is said to be a Δ-σ factor ring of K by I.

Let K be a Δ-σ-field. A subset Σ of some Δ-σ-field extension of K is said to be Δ-σ-algebraically dependent over K, if the set $\{\lambda(x) | \lambda \in \Lambda, x \in \Sigma\}$ is algebraically dependent over K; otherwise, the set Σ is said to be Δ-σ-algebraically independent over K.

If K is a Δ-σ-ring and $Y = \{y_1, \dots, y_q\}$ is a finite set of symbols, then one can consider a polynomial ring $K[\{y_{i,\lambda} | 1 \leq i \leq q, \lambda \in \Lambda\}]$ in a denumerable set of indeterminates $\{y_{i,\lambda}\}$ with the index set $\{1, \dots, q\} \times \Lambda$ as a Δ-σ-ring such that $\beta(y_{i,\lambda}) = y_{i,\beta\lambda}$ for any $\beta \in \Delta \bigcup \sigma^*$. (The elements of $\Delta \bigcup \sigma^*$ act on the coefficients of the polynomials as they act in the ring K.) The Δ-σ-ring obtained is called a ring of Δ-σ-polynomials in Δ-σ-indeterminates $y_1, \dots, y_q$ over K.

If K is a Δ-σ-ring and the semigroup Λ is as above, then an expression of the form $\sum_{\lambda \in \Lambda} a_\lambda \lambda$, where $a_\lambda \in K$ for all $\lambda \in \Lambda$ and only finitely many coefficients a_λ are different from zero, is called a difference–differential (or Δ-σ-) operator over K. Two Δ-σ-operators $\sum_{\lambda \in \Lambda} a_\lambda \lambda$ and $\sum_{\lambda \in \Lambda} b_\lambda \lambda$ are considered to be equal if and only if $a_\lambda = b_\lambda$ for any $\lambda \in \Lambda$.

The set of all Δ-σ-operators over a Δ-σ-ring K can be equipped with a ring structure if one set $\sum_{\lambda \in \Lambda} a_\lambda \lambda + \sum_{\lambda \in \Lambda} b_\lambda \lambda = \sum_{\lambda \in \Lambda} (a_\lambda + b_\lambda)\lambda$, $a(\sum_{\lambda \in \Lambda} a_\lambda \lambda) = \sum_{\lambda \in \Lambda} (a a_\lambda)\lambda$, $(\sum_{\lambda \in \Lambda} a_\lambda \lambda)\mu = \sum_{\lambda \in \Lambda} a_\lambda (\lambda\mu)$, $\delta a = a\delta + \delta(a)$, $\tau a = \tau(a)\tau$ for any Δ-σ-operators $\sum_{\lambda \in \Lambda} a_\lambda \lambda$, $\sum_{\lambda \in \Lambda} b_\lambda \lambda$ and for any elements $a \in K, \delta \in \Delta, \tau \in \sigma^*$, and extend these rules by the distributive law. The ring obtained is called the ring of difference–differential (or Δ-σ-) operators over K, it will be denoted by D (or D_K, if the ring K should be specified).

If $u = \sum_{\lambda \in \Lambda} a_\lambda \lambda$ is a Δ-σ-operator over K, then the orders of u relative to the sets Δ and σ are defined as numbers $\mathrm{ord}_\Delta u = \max\{\mathrm{ord}_\Delta \lambda | a_\lambda \neq 0\}$ and $\mathrm{ord}_\sigma u = \max\{\mathrm{ord}_\sigma \lambda | a_\lambda \neq 0\}$, respectively. The number $\mathrm{ord}\, u = \mathrm{ord}_\Delta u + \mathrm{ord}_\sigma u$ is said to be the order of the Δ-σ-operator u.

In what follows we consider D as a bifiltered ring with the bifiltration $(D_{rs})_{r,s \in \mathbf{Z}}$ such that $D_{rs} = \{u \in D | \mathrm{ord}_\Delta u \leq r, \mathrm{ord}_\sigma u \leq s\}$ for any $r, s \in \mathbf{N}$ and $D_{rs} = 0$, if at least one of the numbers r, s is negative. Obviously, $\bigcup\{D_{rs} | r, s \in \mathbf{Z}\} = D$, $D_{rs} \subseteq D_{r+1,s}$, $D_{rs} \subseteq D_{r,s+1}$ for any $r, s \in \mathbf{Z}$ and $D_{kl}D_{rs} = D_{r+k,s+l}$ for any $r, s, k, l \in \mathbf{N}$.

Definition 2.1. Let D be a ring of Δ-σ-operators over a Δ-σ-ring K. Then a left D-module M is called a difference–differential K-module or a Δ-σ-K-module. In other words, a K-module M is called a Δ-σ-K-module, if the elements of the set $\Delta \bigcup \sigma^*$ act on M in such a way that $\beta(x+y) = \beta(x) + \beta(y)$, $\beta(\gamma(x)) = \gamma(\beta(x))$, $\delta(ax) = a\delta(x) + \delta(a)x$, $\tau(ax) = \tau(a)\tau(x)$, and $\tau(\tau^{-1}(x)) = x$ for any $\beta, \gamma \in \Delta \bigcup \sigma^*$, $\delta \in \Delta$, $\tau \in \sigma^*$, $a \in K$, and $x \in M$.

If K is a Δ-σ-field, a Δ-σ-K-module is called a vector Δ-σ-K-space (or a difference–differential vector space over K).

We say that a Δ-σ-K-module M is finitely generated, if it is finitely generated as a left D-module, i.e. $M = \sum_{i=1}^{q} D x_i$ for some elements $x_1, \dots, x_q \in M$. (Such elements are called generators of the Δ-σ-K-module M.)

Definition 2.2. Let K be a Δ-σ-ring and M a Δ-σ-K-module. A bisequence $(M_{rs})_{r,s \in \mathbf{Z}}$ of vector K-subspaces of the module M is called a *bifiltration* of M if the following three conditions hold:

(i) If $r \in \mathbf{Z}$ is fixed, then $M_{rs} \subseteq M_{r,s+1}$ for all $s \in \mathbf{Z}$ and $M_{rs} = 0$ for all sufficiently small $s \in \mathbf{Z}$. Similarly, if $s \in \mathbf{Z}$ is fixed, then $M_{rs} \subseteq M_{r+1,s}$ for all $r \in \mathbf{Z}$ and $M_{rs} = 0$ for all sufficiently small $r \in \mathbf{Z}$.
(ii) $\bigcup\{M_{rs} | r, s \in \mathbf{Z}\} = M$.
(iii) $D_{kl} M_{rs} \subseteq M_{r+k,s+l}$ for any $r, s \in \mathbf{Z}$; $k, l \in \mathbf{N}$.

The following definition describes a special kind of "good" filtration, in the case of differential modules such filtrations were introduced in Johnson (1969).

Definition 2.3. A bifiltration $(M_{rs})_{r,s\in\mathbf{Z}}$ of a Δ-σ-K-module M is called excellent if every vector K-space M_{rs} $(r, s \in \mathbf{Z})$ is finitely generated and there exist numbers $r_0, s_0 \in \mathbf{Z}$ such that $D_{kl}M_{rs} = M_{r+k,s+l}$ for all $r \geq r_0$, $s \geq s_0$, $k \geq 0$, and $l \geq 0$.

Example 2.1. Let K be a Δ-σ-field and let M be a finitely generated vector Δ-σ-K-space with generators $f_1, \dots, f_q$. Then the vector K-spaces $M_{rs} = \sum_{i=1}^{q} D_{rs}f_i$ $(r, s \in \mathbf{Z})$ form an excellent bifiltration of the module M. This bifiltration will be called a natural bifiltration of M associated with the system of generators $f_1, \dots, f_q$.

Definition 2.4. Let M and N be two Δ-σ-modules over a Δ-σ-ring K. A homomorphism of K-modules $f : M \longrightarrow N$ is called a Δ-σ-homomorphism (or difference–differential homomorphism), if $f(\beta x) = \beta f(x)$ for any $x \in M, \beta \in \Delta \bigcup \sigma^*$. Surjective (respectively, injective or bijective) Δ-σ-homomorphism is called a Δ-σ-epimorphism (Δ-σ-monomorphism or Δ-σ-isomorphism, respectively).

3. Characteristic Polynomials of Subsets of $\mathbf{N}^m$ and $\mathbf{N}^m \times \mathbf{Z}^n$

Definition 3.1. A polynomial $f(t_1, \dots, t_p)$ in p variables $t_1, \dots, t_p$ with rational coefficients is called numerical if $f(r_1, \dots, r_p) \in \mathbf{Z}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{Z}^p$, i.e. there exists a n-tuple $(s_1, \dots, s_p) \in \mathbf{Z}^p$ such that $f(r_1, \dots, r_p) \in \mathbf{Z}$ for all integers $r_1, \dots, r_p \in \mathbf{Z}$ with $r_i \geq s_i$ $(1 \leq i \leq p)$.

It is clear that any polynomial with integer coefficients is numerical. As an example of a numerical polynomial in p variables $(p \geq 1)$ with noninteger coefficients one can consider a polynomial $\binom{t_1}{k_1} \dots \binom{t_p}{k_p}$ where $k_1, \dots, k_p \in \mathbf{N}$ and at least one k_i is greater than 1. (As usual, for any positive integer k, $\binom{t}{k}$ denotes the polynomial $\frac{t(t-1)\dots(t-k+1)}{k!}$ in one variable t; furthermore, we set $\binom{t}{0} = 1$ and $\binom{t}{k} = 0$ for any $k < 0$.)

If $T = t_1^{i_1} \dots t_p^{i_p}$ is a monomial in variables $t_1, \dots, t_p$; then the exponent i_ν is said to be the degree of T with respect to t_ν, it is denoted by $\deg_{t_\nu} T$ $(1 \leq \nu \leq p)$. By the degree (or total degree) of the monomial T we mean the number $\deg T = \deg_{t_1} T + \dots + \deg_{t_p} T$. If $f(t_1, \dots, t_p) = a_1T_1 + \dots + a_dT_d$ is a representation of a numerical polynomial $f(t_1, \dots, t_p)$ as a linear combination of distinct monomials $T_1, \dots, T_d$ with nonzero coefficients, then the degree of $f(t_1, \dots, t_p)$ with respect to t_ν $(1 \leq \nu \leq p)$ and the total degree of this polynomial are defined, respectively, as numbers $\deg_{t_\nu} f = \max\{\deg_{t_\nu} T_j | 1 \leq j \leq d\}$ and $\deg f = \max\{\deg T_j | 1 \leq j \leq d\}$.

The following theorem, proved in Kondrateva *et al.* (1999, Chap. II, Corollary 2.1.5), gives a "canonical" representation of a numerical polynomial.

Theorem 3.1. *Let $f(t_1, \dots, t_p)$ be a numerical polynomial in p variables $t_1, \dots, t_p$ $(p \geq 1)$ and let $\deg f = m$. Then the polynomial $f(t_1, \dots, t_p)$ can be represented in the form*

$$f(t_1, \dots, t_p) = \sum_{\substack{i_1,\dots,i_p \in \mathbf{N} \\ i_1+\dots+i_p \leq m}} a_{i_1\dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p} \tag{3.1}$$

where $a_{i_1\dots i_p}$ are integers uniquely defined by the polynomial $f(t_1, \dots, t_p)$.

In what follows (until the end of the section) we deal with subsets of $\mathbf{N}^m$ and $\mathbf{N}^m \times \mathbf{Z}^n$ where m and n are positive integers. We consider $\mathbf{N}^m$ as an ordered set relative to the product order $\leq$ such that $(i_1, \dots, i_m) \leq (j_1, \dots, j_m)$ if and only if $i_\nu \leq j_\nu$ for $\nu = 1, \dots, m$. (There are no problems with using the same symbol for the product order on $\mathbf{N}^m$ and the natural order on $\mathbf{N}$.)

Let us fix a partition

$$\mathbf{N}_m = \Omega_1 \bigcup \cdots \bigcup \Omega_p \tag{3.2}$$

of the set $\mathbf{N}_m = \{1, \dots, m\}$ into p disjoint subsets ($p \in \mathbf{N}, p \geq 1$), and let us associate with any set $A \subseteq \mathbf{N}^m$ the family of its subsets $\{A(s_1, \dots, s_p) | s_1, \dots, s_p \in \mathbf{N}\}$ such that $A(s_1, \dots, s_p) = \{(i_1, \dots, i_m) \in A | \sum_{\nu \in \Omega_1} i_\nu \leq s_1, \dots, \sum_{\nu \in \Omega_p} i_\nu \leq s_p\}$ for any $s_1, \dots, s_p \in \mathbf{N}$. Furthermore, for any $A \subseteq \mathbf{N}^m$, let V_A denote the set of all elements $v = (v_1, \dots, v_m) \in \mathbf{N}^m$ such that v does not exceed any element of A with respect to the product order. (In other words, $v \in V_A$ if and only if there is no element $(i_1, \dots, i_m) \in A$ such that $i_j \leq v_j$ for $j = 1, \dots, m$.)

The following two theorems proved in Kondrateva *et al.* (1992) generalize the well-known Kolchin result on the numerical polynomials associated with subsets of $\mathbf{N}$ (see Kolchin, 1973, Chap. 0, Lemma 17) and give the explicit formula for the numerical polynomials in p variables associated with a finite subset of $\mathbf{N}^m$.

THEOREM 3.2. *With the above notation, for any set $A \subseteq \mathbf{N}^m$, there exists a numerical polynomial $\omega_A(t_1, \dots, t_p)$ in p variables $t_1, \dots, t_p$ such that*

(i) $\omega_A(r_1, \dots, r_p) = \operatorname{Card} V_A(r_1, \dots, r_p)$ *for all sufficiently large* $(r_1, \dots, r_p) \in \mathbf{N}^p$;
(ii) $\deg \omega_A \leq m$ *and* $\deg_{t_j} \omega_A \leq m_j$ *where m_j denotes the number of elements of the set Ω_j $(j = 1, \dots, p)$;*
(iii) $\deg \omega_A = m$ *if and only if the set A is empty, in this case*

$$\omega_A(t_1, \dots, t_p) = \prod_{j=1}^{p} \binom{t_j + m_j}{m_j};$$

(iv) $\omega_A(t_1, \dots, t_p) = 0$ *if and only if* $(0, \dots, 0) \in A$.

DEFINITION 3.2. The polynomial $\omega_A(t_1, \dots, t_p)$, whose existence is established by Theorem 3.2, is called the characteristic or dimension polynomial of the set A associated with the partition (3.2) of $\mathbf{N}_m$.

THEOREM 3.3. *Let $A = \{a_1, \dots, a_n\}$ be a finite subset of $\mathbf{N}^m$ $(m > 0)$, let $\omega_A(t_1, \dots, t_p)$ be the characteristic polynomial of the set A associated with the partition* (3.2) *of $\mathbf{N}_m$, and let $m_i = \operatorname{Card} \Omega_i$ $(1 \leq i \leq p)$. Furthermore, let $a_i = (a_{i1}, \dots, a_{im})$ $(1 \leq i \leq n)$ and for any $l \in \mathbf{N}$, $0 \leq l \leq n$, let $\Gamma(l, n)$ denote the set of all l-element subsets of the set $\mathbf{N}_n = \{1, \dots, n\}$. Finally, for any $\sigma \in \Gamma(l, n)$, let $\bar{a}_{\sigma k} = \max\{a_{ik} | i \in \sigma\}$ $(1 \leq k \leq m)$ and $b_{\sigma j} = \sum_{h \in \Omega_j} \bar{a}_{\sigma h}$. Then*

$$\omega_A(t_1, \dots, t_p) = \sum_{l=0}^{n} (-1)^l \sum_{\sigma \in \Gamma(l,n)} \prod_{j=1}^{p} \binom{t_j + m_j - b_{\sigma j}}{m_j} \tag{3.3}$$

Now we are going to define dimension polynomials of subsets of $\mathbf{N}^m \times \mathbf{Z}^n$ $(m, n \in \mathbf{N})$. First of all, we introduce some analog of the product order on the set $\mathbf{N}^m \times \mathbf{Z}^n$ as follows. Let us consider the set $\mathbf{Z}^n$ as a union

$$\mathbf{Z}^n = \bigcup_{1 \le j \le 2^n} \mathbf{Z}_j^{(n)} \tag{3.4}$$

where $\mathbf{Z}_1^{(n)}, \ldots, \mathbf{Z}_{2^n}^{(n)}$ are all distinct Cartesian products of n sets each of which is either $\mathbf{N}$ or $\mathbf{Z}_-$. We assume that $\mathbf{Z}_1^{(n)} = \mathbf{N}^n$ and call $\mathbf{Z}_j^{(n)}$ the jth *ortant* of the set $\mathbf{Z}^n$ $(1 \le j \le 2^n)$. Furthermore, we consider $\mathbf{N}^m \times \mathbf{Z}^n$ as a partially ordered set with the order $\trianglelefteq$ such that $(e_1, \ldots, e_m, f_1, \ldots, f_n) \trianglelefteq (e'_1, \ldots, e'_m, f'_1, \ldots, f'_n)$ if and only if $(f_1, \ldots, f_n)$ and $(f'_1, \ldots, f'_n)$ belong to the same ortant $\mathbf{Z}_k^{(n)}$ $(1 \le k \le 2^n)$ and the $(m+n)$-tuple $(e_1, \ldots, e_m, |f_1|, \ldots, |f_n|)$ is less than $(e'_1, \ldots, e'_m, |f'_1|, \ldots, |f'_n|)$ with respect to the product order on $\mathbf{N}^{m+n}$.

The following lemma, proved in Kondrateva *et al.* (1999, Chap. II, Propositions 2.1.7 and 2.1.9), gives some combinatorial formulas that will be used below.

LEMMA 3.1. *For any positive integer m and for any $r \in \mathbf{N}$, let $g_0(m,r)$ and $g(m,r)$ denote the numbers of solutions $(x_1, \ldots, x_m) \in \mathbf{N}^m$ of the equation $x_1 + \cdots + x_m = r$ and the inequality $x_1 + \cdots + x_m \le r$, respectively.*

Furthermore, let $h_0(m,r)$ and $h(m,r)$ denote the numbers of solutions $(y_1, \ldots, y_m) \in \mathbf{Z}^m$ of the equation $|y_1| + \cdots + |y_m| = r$ and the inequality $|y_1| + \cdots + |y_m| \le r$, respectively. Then

$$g_0(m,r) = \binom{m+r-1}{m-1}, \tag{3.5}$$

$$g(m,r) = \binom{m+r}{m}, \tag{3.6}$$

$$h_0(m,r) = \sum_{i=0}^{m} 2^i \binom{m}{i}\binom{r-1}{i-1}, \tag{3.7}$$

$$\begin{aligned} h(m,r) &= \sum_{i=0}^{m} 2^i \binom{m}{i}\binom{r}{i} = \sum_{i=0}^{m} \binom{m}{i}\binom{r+i}{m} \\ &= \sum_{i=0}^{m} (-1)^{m-i} 2^i \binom{m}{i}\binom{r+i}{i}. \end{aligned} \tag{3.8}$$

(As usual, we assume that $\binom{s}{i} = 0$, if $i < 0$ or $i > s$.)

In what follows, for any set $A \subseteq \mathbf{N}^m \times \mathbf{Z}^n$, W_A will denote the set of all elements of $\mathbf{N}^m \times \mathbf{Z}^n$ that do not exceed any element of A with respect to the order $\trianglelefteq$. (Thus, $w \in W_A$ if and only if there is no element $a \in A$ such that $a \trianglelefteq w$.) Furthermore, for any $r, s \in \mathbf{N}$, $A[r,s]$ will denote the set of all elements $x = (x_1, \ldots, x_m, x'_1, \ldots, x'_n) \in A$ such that $\sum_{i=1}^{m} x_i \le r$ and $\sum_{j=1}^{n} |x'_j| \le s$.

Let us fix the partition

$$\mathbf{N}_{m+2n} = \Omega_1 \bigcup \Omega_2 \tag{3.9}$$

of the set $\mathbf{N}_{m+2n} = \{1, \ldots, m+2n\}$ such that $\Omega_1 = \{1, \ldots, m\}$ and $\Omega_2 = \{m+1, \ldots, m+$

$2n\}$. As above, for any set $C \subseteq \mathbf{N}^{m+2n}$ and for any $r, s \in \mathbf{N}$, $C(r,s)$ will denote the set of all elements $(c_1, \dots, c_{m+2n}) \in \mathbf{N}^{m+2n}$ such that $\sum_{i=1}^{m} c_i \leq r$ and $\sum_{i=m+1}^{m+2n} c_i \leq s$.

The following statement gives some properties of the mapping $\rho : \mathbf{N}^m \times \mathbf{Z}^n \longrightarrow \mathbf{N}^{m+2n}$ defined by the condition
$\rho((e_1, \dots, e_m, f_1, \dots, f_n)) = (e_1, \dots, e_m, \max\{f_1, 0\}, \dots, \max\{f_n, 0\}, \max\{-f_1, 0\}, \dots, \max\{-f_n, 0\})$.

Lemma 3.2. *Let A be a subset of $\mathbf{N}^m \times \mathbf{Z}^n$. Then for any $r, s \in \mathbf{N}, \operatorname{Card} A[r,s] = \operatorname{Card} \rho(A)(r,s)$ and $\rho(W_A) = V_{\rho(A)} \bigcap \rho(\mathbf{N}^m \times \mathbf{Z}^n)$.*

Proof. First of all, notice that the mapping ρ is injective. Indeed, if $\rho((e_1, \dots, e_m, f_1, \dots, f_n)) = (a_1, \dots, a_{m+2n})$, then $a_1, \dots, a_{m+2n}$ uniquely determine the coordinates $e_1, \dots, e_m, f_1, \dots, f_n$, since

$$e_i = a_i \, (1 \leq i \leq m), \; |f_j| = \max\{a_{m+j}, a_{m+n+j}\} = a_{m+j} + a_{m+n+j} \qquad (1 \leq j \leq n),$$

and the inequality $a_{m+j} > 0$ implies $f_j > 0$ while the equality $a_{m+n+j} = 0$ implies $f_j \leq 0$. Now, if $\rho((e_1, \dots, e_m, f_1, \dots, f_n)) = (a_1, \dots, a_{m+2n})$, then $\sum_{i=1}^{m} e_i = \sum_{i=1}^{m} a_i$ and $\sum_{j=1}^{n} |f_i| = \sum_{j=1}^{n} (a_{m+j} + a_{m+n+j}) = \sum_{i=m+1}^{m+2n} a_i$, whence $\rho(A[r,s]) = \rho(A)(r,s)$. Furthermore, for any two elements $u, v \in \mathbf{N}^m \times \mathbf{Z}^n$, the inequality $u \trianglelefteq v$ is equivalent to the inequality $\rho(u) \leq \rho(v)$ whence $\rho(W_A) = V_{\rho(A)} \bigcap \rho(\mathbf{N}^m \times \mathbf{Z}^n)$. $\square$

The following definition extends the concept of initial subset of $\mathbf{N}^m$ introduced in Sit (1975) to the case of subsets of $\mathbf{N}^m \times \mathbf{Z}^n$.

Definition 3.3. A set $V \subseteq \mathbf{N}^m$ is called an initial subset of $\mathbf{N}^m$ if the inclusion $v \in V$ implies that $v' \in V$ for any element $v' \in \mathbf{N}^m$ such that $v' \leq v$. Similarly, a set $W \subseteq \mathbf{N}^m \times \mathbf{Z}^n$ is called an initial subset of $\mathbf{N}^m \times \mathbf{Z}^n$ if the inclusion $w \in W$ implies $w' \in W$ for any $w' \in \mathbf{N}^m \times \mathbf{Z}^n$ such that $w' \trianglelefteq w$.

Let us consider the set $\mathbf{N}^m \times \mathbf{N}_q$ $(m, q \in \mathbf{N}, m \geq 1, q \geq 1)$ as an ordered set with respect to the product order $\leq$: $(a_1, \dots, a_m, b) \leq (a'_1, \dots, a'_m, b')$ if and only if $a_i \leq a'_i$ for all $i = 1, \dots, m$ and $b \leq b'$. The proof of the following statement can be found in Kolchin (1964, Chap. 0, Section 17).

Lemma 3.3. *Every infinite sequence of elements of $\mathbf{N}^m \times \mathbf{N}_q$ has an infinite subsequence, strictly increasing relative to the product order, in which every element has the same projection on $\mathbf{N}_q$.*

Lemma 3.4. *A set $V \subseteq \mathbf{N}^m$ is an initial subset of $\mathbf{N}^m$ if and only if there exists a finite set $A \subseteq \mathbf{N}^m$ such that $V = V_A$. Similarly, a set $W \subseteq \mathbf{N}^m \times \mathbf{Z}^n$ is an initial subset of $\mathbf{N}^m \times \mathbf{Z}^n$ if and only if there exists a finite set $B \subseteq \mathbf{N}^m \times \mathbf{Z}^n$ such that $W = W_B$.*

Proof. The first statement of the lemma is proved in Sit (1975). Let us prove the second statement. Clearly, if $B \subseteq \mathbf{N}^m \times \mathbf{Z}^n$, then W_B is an initial subset of $\mathbf{N}^m \times \mathbf{Z}^n$. Conversely, let W be an initial subset of $\mathbf{N}^m \times \mathbf{Z}^n$ and let B be the set of all minimal elements of $(\mathbf{N}^m \times \mathbf{Z}^n) \setminus W$ with respect to the order $\trianglelefteq$. It follows from Lemma 3.3 (where one should set $q = 1$) that the set B is finite. Indeed, if B is infinite, then one of the intersections $B \bigcap \mathbf{Z}_j^{(n)}$ $(1 \leq j \leq 2^n)$ would be infinite. Then the set $B_j =$

$\{(|b_1|, \dots, |b_{m+n}|) \in \mathbf{N}^{m+n} | (b_1, \dots, b_{m+n}) \in B \bigcap \mathbf{Z}_j^{(n)}\}$ would be an infinite subset of $\mathbf{N}^{m+n}$ every two elements of which are incomparable with respect to the product order on $\mathbf{N}^{m+n}$. Since this contradicts Lemma 3.3, we conclude that the set B is finite.

To complete the proof, it remains for us to note that the inclusion $x \in (\mathbf{N}^m \times \mathbf{Z}^n) \setminus W$ is equivalent to the fact that there exists $x' \in B$ such that $x' \trianglelefteq x$, i.e. $x \notin W_B$. It follows that $W = W_B$. □

Lemma 3.5. *Let W be an initial subset of $\mathbf{N}^m \times \mathbf{Z}^n$. Then $\rho(W)$ is an initial subset of $\mathbf{N}^{m+2n}$.*

Proof. By Lemma 3.4, there exists a finite subset $B \subseteq \mathbf{N}^m \times \mathbf{Z}^n$ such that $W = W_B$. Furthermore, by Lemma 3.2, $\rho(W) = \rho(W_B) = V_{\rho(B)} \bigcap \rho(\mathbf{N}^m \times \mathbf{Z}^n)$. Since the intersection of the initial subsets of $\mathbf{N}^{m+2n}$ is, clearly, an initial subset of $\mathbf{N}^{m+2n}$, it remains to note that $\rho(\mathbf{N}^m \times \mathbf{Z}^n)$ is an initial subset of $\mathbf{N}^{m+2n}$. Indeed, an element $(a_1, \dots, a_{m+2n}) \in \mathbf{N}^{m+2n}$ belongs to $\rho(\mathbf{N}^m \times \mathbf{Z}^n)$ if and only if $a_{m+i}a_{m+n+i} = 0$ for $i = 1, \dots, n$. Therefore, if $a = (a_1, \dots, a_{m+2n}) \in \mathbf{N}^{m+2n}$ belongs to $\rho(\mathbf{N}^m \times \mathbf{Z}^n)$, then for any $b \in \mathbf{N}^{m+2n}$, the inequality $b \leq a$ implies $b \in \rho(\mathbf{N}^m \times \mathbf{Z}^n)$. □

Theorem 3.4. *Let A be a subset of $\mathbf{N}^m \times \mathbf{Z}^n$. Then there exists a numerical polynomial $\phi_A(t_1, t_2)$ in two variables t_1 and t_2 with the following properties.*

(i) $\phi_A(r, s) = \operatorname{Card} W_A[r, s]$ *for all sufficiently large* $(r, s) \in \mathbf{N}^2$.
(ii) $\deg \phi_A \leq m + n, \deg_{t_1} \phi_A \leq m$, *and* $\deg_{t_2} \phi_A \leq n$.
(iii) *Let $B = \rho(A) \bigcup \{e_1, \dots, e_n\}$ where e_i $(1 \leq i \leq n)$ is an $(m+2n)$-tuple from $\mathbf{N}^{m+2n}$ whose $(m+i)$th and $(m+n+i)$th coordinates are equal to 1 and all other coordinates are equal to 0. Then $\phi_A(t_1, t_2) = \omega_B(t_1, t_2)$ where $\omega_B(t_1, t_2)$ is the dimension polynomial of the set B (see Definition 3.2) associated with the partition (3.9) of the set $\mathbf{N}_{m+2n}$.*
(iv) *If $A = \emptyset$, then $\deg \phi_A = m+n$. In this case, $\phi_A(t_1, t_2) = \binom{t_1+m}{m} \sum_{i=0}^{n} (-1)^{n-i} 2^i \binom{n}{i} \binom{t_2+i}{i}$.*
(v) $\phi_A(t_1, t_2) = 0$ *if and only if* $(0, \dots, 0) \in A$.

Proof. By Lemma 3.5, $\rho(W_A)$ is an initial subset of $\mathbf{N}^{m+2n}$, and by Lemma 3.4, there exists a set $B \subseteq \mathbf{N}^{m+2n}$ such that $\rho(W_A) = V_B$. Therefore (see Lemma 3.2), $\operatorname{Card} W_A[r, s] = \operatorname{Card} V_B(r, s)$ for all $r, s \in \mathbf{N}$. Applying Theorem 3.2 we obtain that there exists a numerical polynomial $\phi_A(t_1, t_2)$ that satisfies conditions (i) and (ii) of our theorem.

It follows from the proof of Lemma 3.4 that a set C with the property $\rho(W_A) = V_C$ can be chosen as the set of all minimal (with respect to the product order) elements of the set

$$\mathbf{N}^{m+2n} \setminus \rho(W_A) = \mathbf{N}^{m+2n} \setminus \left(V_{\rho(A)} \bigcap \rho(\mathbf{N}^m \times \mathbf{Z}^n)\right)$$
$$= (\mathbf{N}^{m+2n} \setminus V_{\rho(A)}) \bigcup (\mathbf{N}^{m+2n} \setminus \rho(\mathbf{N}^m \times \mathbf{Z}^n)).$$

Furthermore, all minimal elements of the set $\mathbf{N}^{m+2n} \setminus V_{\rho(A)}$ are contained in $\rho(A)$ and any element $a \in \mathbf{N}^{m+2n} \setminus \rho(\mathbf{N}^m \times \mathbf{Z}^n)$ exceeds some $e_i (1 \leq i \leq n)$ with respect to the product order (since $a \notin \rho(\mathbf{N}^m \times \mathbf{Z}^n)$, there exists an index j $(1 \leq j \leq n)$ such that both $(m+j)$th and $(m+n+j)$th coordinates of a are positive). Thus, the set $B = \rho(A) \bigcup \{e_1, \dots, e_n\}$ satisfies the condition $\rho(W_A) = V_B$ whence $\phi_A(t_1, t_2) = \omega_B(t_1, t_2)$.

If $A = \emptyset$, then for any $r, s \in \mathbf{N}$, $W_A[r, s] = \{(e_1, \dots, e_m, f_1, \dots, f_n) \in \mathbf{N}^m \times \mathbf{Z}^n | \sum_{i=1}^{m} e_i \leq r$ and $\sum_{j=1}^{n} |f_j| \leq s\}$. By Lemma 3.1,

$$\mathrm{Card}\, W_A[r, s] = g(m, r)h(n, s) = \binom{r+m}{m} \sum_{i=0}^{n} (-1)^{n-i} 2^i \binom{n}{i} \binom{s+i}{i}.$$

Finally, it remains to note that the inclusion $(0, \dots, 0) \in A$ is equivalent to the equality $W_A = \emptyset$, i.e. to the equality $\phi_A(t_1, t_2) = 0$. $\square$

Definition 3.4. The polynomial $\phi_A(t_1, t_2)$ whose existence is established by Theorem 3.4, is called the **N**-**Z**-characteristic or **N**-**Z**-dimension polynomial of the set $A \subseteq \mathbf{N}^m \times \mathbf{Z}^n$.

4. Reduction in Finitely Generated Free Δ-σ-Modules

Let K be a difference–differential field with a basic set of derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ and a basic set of automorphisms $\sigma = \{\alpha_1, \dots, \alpha_n\}$, let D denote the ring of Δ-σ-operators over K, and let F be a finitely generated free left D-module with free generators $f_1, \dots, f_q$. (Using the "difference–differential" terminology introduced in Section 2, one can say that F is a free difference–differential (or Δ-σ-) K-module with the set of free Δ-σ-generators $\{f_1, \dots, f_q\}$.) Then F can be considered as a vector K-space generated by the set of all elements of the form λf_i $(1 \leq i \leq q)$ where λ is an element of the form (2.1) from the semigroup Λ (we use the notation and conventions of Section 2). This set will be denoted by Λf and its elements will be called *terms*. For any term λf_i, the element λ will be called the *head* of the term.

It is clear that the set Λf is linearly independent over K and every element $f \in D$ has a unique representation as a linear combination of terms:

$$f = a_1 \lambda_1 f_{i_1} + \dots + a_d \lambda_d f_{i_d} \tag{4.1}$$

for some nonzero elements $a_j \in K$ $(1 \leq j \leq d)$ and some distinct elements $\lambda_1 f_{i_1}, \dots, \lambda_d f_{i_d} \in \Lambda f$. We say that the element f contains a term $\lambda_k f_k$, if this term appears in the representation (4.1) with nonzero coefficient.

We define the orders of a term $\lambda f_j \in \Lambda f$ relative to the sets Δ and σ as the appropriate orders of the element λ: $ord_\Delta(\lambda f_j) = ord_\Delta \lambda$ and $ord_\sigma(\lambda f_j) = ord_\sigma \lambda$. The number $ord\,(\lambda f_j) = ord_\Delta(\lambda f_j) + ord_\sigma(\lambda f_j)$ is said to be the order of the term λf_j.

In what follows, we assume that a representation (3.4) of the set $\mathbf{Z}^n$ as a union of 2^n ortants is fixed. Two elements $\lambda_1 = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ and $\lambda_2 = \delta_1^{r_1} \dots \delta_m^{r_m} \alpha_1^{s_1} \dots \alpha_n^{s_n}$ from Λ are called *similar*, if the n-tuples $(l_1, \dots, l_n)$ and $(s_1, \dots, s_n)$ belong to the same ortant of $\mathbf{Z}^n$. Two terms $u = \lambda_1 f_i$ and $v = \lambda_2 f_j$ $(\lambda_1, \lambda_2 \in \Lambda, 1 \leq i, j \leq q)$ are said to be *similar*, if the elements λ_1 and λ_2 are similar.

An element $\lambda_1 = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} \in \Lambda$ is said to be a multiple of an element $\lambda_2 = \delta_1^{r_1} \dots \delta_m^{r_m} \alpha_1^{s_1} \dots \alpha_n^{s_n} \in \Lambda$, if the elements are similar, $r_\nu \leq k_\nu$ for $\nu = 1, \dots, m$, and $|s_\mu| \leq |l_\mu|$ for $\mu = 1, \dots, n$. In this case we write $\lambda_2 | \lambda_1$. It is clear that λ_1 is a multiple of λ_2 if and only if these elements are similar and there exists $\lambda' \in \Lambda$ such that λ' is similar to λ_1 (and λ_2) and $\lambda_1 = \lambda' \lambda_2$.

A term $u = \lambda_1 f_i$ is said to be a multiple of a term $v = \lambda_2 f_j$ $(\lambda_1, \lambda_2 \in \Lambda, 1 \leq i, j \leq q)$

if $i = j$ and λ_1 is a multiple of λ_2. In this case we write $v|u$. (Obviously, $v|u$ if the terms u and v are similar and there exists $\lambda \in \Lambda$ such that λ is similar to λ_1 (and λ_2) and $u = \lambda v$.)

Below we consider two orders $<_\Delta$ and $<_\sigma$ on the set of all terms Λf that are defined as follows: if $u = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} f_i$, $v = \delta_1^{r_1} \dots \delta_m^{r_m} \alpha_1^{s_1} \dots \alpha_n^{s_n} f_j \in \Lambda f$, then $u <_\Delta v$ ($u <_\sigma v$) if and only if the $(m+2n+3)$-tuple $(ord_\Delta u, ord_\sigma u, i, k_1, \dots, k_m, |l_1|, \dots |l_n|, l_1, \dots, l_n)$ is less than $(ord_\Delta v, ord_\sigma v, j, r_1, \dots, r_m, |s_1|, \dots |s_n|, s_1, \dots, s_n)$ (respectively, $(ord_\sigma u, ord_\Delta u, i, k_1, \dots, k_m, |l_1|, \dots |l_n|, l_1, \dots, l_n)$ is less than $(ord_\sigma v, ord_\Delta v, j, r_1, \dots, r_m, |s_1|, \dots |s_n|, s_1, \dots, s_n)$) relative to the lexicographic order on $\mathbf{N}^{m+n+3} \times \mathbf{Z}^n$. It is easy to see that the set Λf is well-ordered with respect to each of the orders $<_\Delta$, $<_\sigma$.

Definition 4.1. Let f be an element of the free Δ-σ-K-module F written in the form (4.1) and let $\lambda_k f_{i_k}$ and $\lambda_l f_{i_l}$ be the greatest terms of the set $\{\lambda_1 f_{i_1}, \dots, \lambda_d f_{i_d}\}$ relative to the orders $<_\Delta$ and $<_\sigma$, respectively. Then the terms $\lambda_k f_{i_k}$ and $\lambda_l f_{i_l}$ are called the Δ-leader and σ-leader of the element f; they are denoted by u_f and v_f, respectively.

Definition 4.2. Let f and g be two elements of the free Δ-σ-K-module F. The element f is said to be reduced (or Δ-reduced) with respect to g if f does not contain any multiple λv_g ($\lambda \in \Lambda$) of the σ-leader v_g such that $ord_\Delta(\lambda u_g) \leq ord_\Delta u_f$. An element $h \in F$ is said to be reduced with respect to a set $\Sigma \subseteq F$, if h is reduced with respect to every element of the set Σ.

Definition 4.3. A subset Σ of the free Δ-σ-K-module F is called autoreduced if every element of Σ is reduced with respect to any other element of this set.

The following statement is the direct consequence of Lemma 3.3.

Lemma 4.1. *Let F be the free Δ-σ-K-module considered above and let S be an infinite sequence of terms from the set Λf. Then there exists an index j $(1 \leq j \leq q)$ and an infinite subsequence $\lambda_1 f_j, \lambda_2 f_j, \dots, \lambda_k f_j, \dots$ of the sequence S such that $\lambda_k | \lambda_{k+1}$ for all $k = 1, 2, \dots$.*

Theorem 4.1. *Every autoreduced subset of the free Δ-σ-K-module F is finite.*

Proof. Let Σ be an autoreduced subset of F. First of all, note that if $f, g \in \Sigma$ and $f \neq g$, then $v_f \neq v_g$. Indeed, since the elements f and g are reduced with respect to each other, the equality $v_f = v_g$ would imply that $ord_\Delta u_g < ord_\Delta u_f$ and $ord_\Delta u_f < ord_\Delta u_g$ at the same time.

Suppose that our autoreduced set Σ is infinite. Then the set $V = \{v_h | h \in \Sigma\}$ is infinite and it does not contain two equal elements. By Lemma 4.1, there exists an infinite sequence

$$v_{h_1}, v_{h_2}, \dots \tag{4.2}$$

of elements of V such that $v_{h_i} | v_{h_{i+1}}$ for all $i = 1, 2, \dots$, i.e. any two elements of the sequence are similar, and $v_{h_{i+1}} = \lambda_i v_{h_i}$ for some element $\lambda_i \in \Lambda$ $(i = 1, 2, \dots)$ which is similar to the head of v_{h_i} (and $v_{h_{i+1}}$). Let $k_i = ord_\Delta v_{h_i}$ and $l_i = ord_\Delta u_{h_i}$ $(i = 1, 2, \dots)$. It is clear that if $l_{i+1} - l_i \geq k_{i+1} - k_i$ for some index i, then h_{i+1} is not reduced with respect to h_i, so we should have $l_{i+1} - l_i < k_{i+1} - k_i$ for all $i = 1, 2 \dots$. Therefore, $l_{i+1} - k_{i+1} < l_i - k_i$

for all $i = 1, 2, \dots$ that contradicts the fact that $l_i \geq k_i$ for $i = 1, 2, \dots$. This completes the proof. □

THEOREM 4.2. *Let $\Sigma = \{g_1, \dots, g_r\}$ be an autoreduced subset of the free Δ-σ-K-module F and let $f \in F$. Then there exists an element $g \in F$ such that $f - g = \sum_{i=1}^{r} Q_i g_i$ for some Δ-σ-operators $Q_1, \dots, Q_r \in D$ and g is reduced with respect to the set Σ.*

PROOF. If f is reduced with respect to Σ, the statement is obvious (one can set $g = f$). Suppose that f is not reduced with respect to Σ. Let u_i and v_i be the leaders of the element g_i relative to the orders $<_\Delta$ and $<_\sigma$, respectively, and let a_i be the coefficient of the term v_i in g_i $(i = 1, \dots, r)$. In what follows, a term w_h, that appears in an element $h \in F$, will be called a Σ-leader of h if w_h is the greatest (with respect to the order $<_\sigma$) term among all terms λv_j (λ is an element of Λ similar to the head of v_j, $1 \leq j \leq r$) that appear in h and satisfy the condition $ord_\Delta(\lambda u_j) \leq ord_\Delta u_h$. (As above, u_h and v_h denote the leaders of the element h relative to the orders $<_\Delta$ and $<_\sigma$, respectively.)

Let w_f be the Σ-leader of the element f and let c_f be the coefficient of w_f in f. Then $w_f = \lambda v_j$ for some $\lambda \in \Lambda$, λ is similar to the head of w_f, and for some j $(1 \leq j \leq r)$ such that $ord_\Delta(\lambda u_j) \leq ord_\Delta u_f$. Without loss of generality we may assume that j corresponds to the maximum (with respect to the order $<_\sigma$) such a σ-leader v_j in the set of all σ-leaders of elements of Σ. Let us consider the element $f' = f - \frac{c_f}{a_j}\lambda g_j$. Obviously, f' does not contain w_f and $ord_\Delta(u_{f'}) \leq ord_\Delta u_f$. Furthermore, f' cannot contain any term of the form $\lambda' v_i$ ($\lambda' \in \Lambda$, λ' is similar to the head of v_i $(1 \leq i \leq r)$), that is greater than w_f (with respect to $<_\sigma$) and satisfies the condition $ord_\Delta(\lambda' u_i) \leq ord_\Delta u_{f'}$. Indeed, if the last inequality holds, then $ord_\Delta(\lambda' u_i) \leq ord_\Delta u_f$, so that the term $\lambda' v_i$ cannot appear in f. This term cannot appear in λg_j either, since $v_{\lambda g_j} = \lambda v_j = w_f <_\sigma \lambda' v_i$ (the first equality is the consequence of the fact that λ is similar to the head of v_j). Thus, $\lambda' v_i$ cannot appear in f', whence the Σ-leader of f' is strictly less (with respect to the order $<_\sigma$) than the Σ-leader of f. Applying the same procedure to the element f' and continuing in the same way, we obtain an element $g \in F$ such that $f - g$ is a linear combination of elements $g_1, \dots, g_r$ with coefficients from D and g is reduced with respect to Σ. This completes the proof. □

The process of reduction described in the proof of the last theorem can be realized by the following algorithm (that can be used for the reduction with respect to any finite set of elements of the free Δ-σ-K-module F). The symbol $lc_\sigma(g)$ $(g \in F)$ denotes the coefficient of the σ-leader v_g of the Δ-σ-operator g, D denotes the ring of Δ-σ-operators over K.

ALGORITHM 4.1. $(f, r, g_1, \dots, g_r; g)$

Input: $f \in F$, a positive integer r, $\Sigma = \{g_1, \dots, g_r\} \subseteq F$ where $g_i \neq 0$
for $i = 1, \dots, r$

Output: Element $g \in F$ and elements $Q_1, \dots, Q_r \in D$ such that
$g = f - (Q_1 g_1 + \dots + Q_r g_r)$ and g is reduced with respect to Σ

Begin

$Q_1 := 0, \dots, Q_r := 0, g := f$

While there exist i, $1 \leq i \leq r$, and a term w, that appears in g with a
nonzero coefficient $c(w)$, such that $v_{g_i} | w$ and
$ord_\Delta(\frac{w}{v_{g_i}} v_{u_i}) \leq ord_\Delta v_g$ **do**

z:= the greatest (with respect to $<_\sigma$) of the terms w that satisfy the above conditions.
j:= the smallest number i for which v_{g_i} is the greatest (with respect to $<_\sigma$) σ-leader of an element $g_i \in \Sigma$ such that $v_{g_i}|z$ and $ord_\Delta(\frac{z}{v_{g_i}}u_{g_i}) \leq ord_\Delta u_g$.
$\lambda_j := \lambda_j + \frac{c(z)z}{lc_\sigma(g_j)v_{g_j}}$
$g := g - \frac{c(z)z}{lc_\sigma(g_j)v_{g_j}}g_j$

End

Definition 4.4. Let f and g be two elements of the free Δ-σ-K-module F. We say that the element f has lower rank than g and write $\mathrm{rk}(f) < \mathrm{rk}(g)$ if either $v_f <_\sigma v_g$ or $v_f = v_g$ and $u_f <_\Delta u_g$. If $v_f = v_g$ and $u_f = u_g$, we say that f and g have the same rank and write $\mathrm{rk}(f) = \mathrm{rk}(g)$.

In what follows, while considering autoreduced subsets of the free Δ-σ-K-module F, we always assume that their elements are arranged in order of increasing rank. (Therefore, if we consider an autoreduced set $\Sigma = \{h_1, \dots, h_r\} \subseteq F$, then $\mathrm{rk}(h_1) < \cdots < \mathrm{rk}(h_r)$.)

Definition 4.5. Let $\Sigma = \{h_1, \dots, h_r\}$ and $\Sigma' = \{h'_1, \dots, h'_s\}$ be two autoreduced subsets of the free Δ-σ-K-module F. An autoreduced set Σ is said to have lower rank than Σ' if one of the following two cases holds:

(1) There exists $k \in \mathbf{N}$ such that $k \leq \min\{r, s\}$, $\mathrm{rk}(h_i) = \mathrm{rk}(h'_i)$ for $i = 1, \dots, k-1$ and $\mathrm{rk}(h_k) < \mathrm{rk}(h'_k)$.
(2) $r > s$ and $\mathrm{rk}(h_i) = \mathrm{rk}(h'_i)$ for $i = 1, \dots, s$. If $r = s$ and $\mathrm{rk}(h_i) = \mathrm{rk}(h'_i)$ for $i = 1, \dots, r$, then Σ is said to have the same rank as Σ'.

Theorem 4.3. *In every nonempty set of autoreduced subsets of the free Δ-σ-K-module F there exists an autoreduced subset of lowest rank.*

Proof. Let Φ be any nonempty set of autoreduced subsets of F. Define by induction an infinite descending chain of subsets of Φ as follows: $\Phi_0 = \Phi$, $\Phi_1 = \{\Sigma \in \Phi_0 | \Sigma$ contains at least one element and the first element of Σ is of lowest possible rank$\}$, $\dots$, $\Phi_k = \{\Sigma \in \Phi_{k-1} | \Sigma$ contains at least k elements and the kth element of Σ is of lowest possible rank$\}$, $\dots$. It is clear that if a set Φ_k is nonempty, then kth elements of autoreduced sets from Φ_k have the same σ-leader v_k and the same Δ-leader u_k. If Φ_k were nonempty for all $k = 1, 2, \dots$, then the set $\{f_k | f_k$ is the kth element of some autoreduced set from $\Phi_k\}$ would be an infinite autoreduced set, and this would contradict Theorem 4.1. Therefore, there is the smallest k such that Φ_k is empty. (Since, $\Phi_0 = \Phi$ is nonempty, $k > 0$.) It is clear that every element of Φ_{k-1} is an autoreduced subset in Φ of lowest rank. □

Definition 4.6. Let N be a Δ-σ-submodule of the free Δ-σ-K-module F (i.e. N is a D-submodule of F where D, as usual, denotes the ring of Δ-σ-operators over K). Then an autoreduced subset of N of lowest rank is called a characteristic set of N.

The following three theorems show that the concept of a characteristic set of a finitely generated Δ-σ-module over a Δ-σ-field can be considered as an analog of the concept of a reduced Gröbner basis of a free module over a polynomial ring. In the case of difference–differential modules we prefer to call this analog "characteristic set" rather than "reduced Gröbner basis", because, first, it is introduced in the same manner as the concept of characteristic set is introduced in differential algebra (see Kolchin, 1973, Chap. 1, Section 10) and, second, it is connected with the two specific "term-orderings" $\leq_\Delta$ and $\leq_\sigma$, while the theory of Gröbner basis works for any term-ordering.

THEOREM 4.4. *Let N be a Δ-σ-submodule of the free Δ-σ-K-module F and let $\Sigma = \{g_1, \ldots, g_r\}$ be a characteristic set of N. Then an element $f \in N$ is reduced with respect to Σ if and only if $f = 0$.*

PROOF. Suppose that f is a nonzero element of N reduced with respect to Σ. If $\mathrm{rk}(f) < \mathrm{rk}(g_1)$, then the autoreduced set $\{f\}$ has lower rank than Σ. If $\mathrm{rk}(g_1) < \mathrm{rk}(f)$ (f and g_1 cannot have the same rank, since f is reduced with respect to Σ), then f and the elements $g \in \Sigma$ that have lower rank than f form an autoreduced set that has lower rank than Σ. In both cases we arrive at the contradiction with the fact that Σ is a characteristic set of N. $\square$

THEOREM 4.5. *Let N be a Δ-σ-submodule of the free Δ-σ-K-module F and let $\Sigma = \{g_1, \ldots, g_r\}$ be a characteristic set of N. Then the elements $g_1, \ldots, g_r$ generate the Δ-σ-module N.*

PROOF. Let f be any element of N. By Theorem 4.2, there exist elements $Q_1, \ldots, Q_r \in D$ and an element $f' \in F$ such that f' is reduced with respect to Σ and $f - f' = \sum_{i=1}^{r} Q_i g_i$. Therefore, $f' \in N$, and Theorem 4.4 shows that $f' = 0$, whence $f = \sum_{i=1}^{r} Q_i g_i$. $\square$

THEOREM 4.6. *Let $\Sigma_1 = \{g_1, \ldots, g_r\}$ and $\Sigma_2 = \{h_1, \ldots, h_s\}$ be two characteristic sets of some Δ-σ-submodule N of the free Δ-σ-K-module F. Furthermore, suppose that the coefficients of the σ-leaders of all elements of these characteristic sets are equal to* 1*. Then $r = s$ and $g_i = h_i$ for all $i = 1, \ldots, r$.*

PROOF. Since Σ_1 and Σ_2 are two autoreduced sets of the same (lowest possible) rank, $r = s$, $u_{g_i} = u_{h_i}$, and $v_{g_i} = v_{h_i}$ for $i = 1, \ldots, r$. Suppose that there exists i, $1 \leq i \leq r$, such that $g_i \neq h_i$. Setting $f_i = g_i - h_i$ we obtain that $v_{f_i} <_\sigma v_{g_i}$ (since the coefficients of v_{g_i} in g_i and h_i are equal to 1), $u_{f_i} \leq_\Delta u_{g_i}$, and f_i is reduced with respect to any element g_j ($1 \leq j \leq r$). Indeed, suppose that f_i contains a multiple λv_{g_j} of some σ-leader v_{g_j} such that $ord_\Delta(\lambda u_{g_j}) \leq ord_\Delta u_{f_i}$ (obviously, f_i is reduced with respect to g_i, so we can assume that $j \neq i$). Then at least one of the elements g_i, h_i must contain λv_{g_j} and $ord_\Delta(\lambda u_{g_j}) \leq ord_\Delta u_{f_i} \leq ord_\Delta u_{g_i} = \mathrm{ord}_\Delta u_{h_i}$ that contradicts the fact that the sets Σ_1 and Σ_2 are autoreduced. Now, Theorem 4.4 shows that $f_i = 0$ whence $g_i = h_i$. This completes the proof. $\square$

THEOREM 4.7. *Let K be a Δ-σ-field, D the ring of Δ-σ-operators over K, and M a finitely generated Δ-σ-K-module with a system of generators $\{h_1, \ldots, h_q\}$. Let F be a free Δ-σ-K-module with a basis $f_1, \ldots, f_q$, and $\pi : F \longrightarrow M$ the natural Δ-σ-epimorphism of F onto M ($\pi(f_i) = h_i$ for $i = 1, \ldots, q$). Furthermore, let $N = \mathrm{Ker}\,\pi$ and let $\Sigma = \{g_1, \ldots, g_d\}$ be a characteristic set of N. Finally, for any $r, s \in \mathbf{N}$, let $M_{rs} = \sum_{i=1}^{q} D_{rs} h_i$*

and U_{rs} denote the set of all terms $w \in \Lambda f$ such that $\mathrm{ord}_\Delta w \leq r$, $\mathrm{ord}_\sigma w \leq s$, *and either w is not a multiple of any v_{g_i} $(1 \leq i \leq d)$ or* $\mathrm{ord}_\Delta(\lambda u_{g_j}) > r$ *for any $\lambda \in \Lambda, g_j \in \Sigma$ such that λ is similar to the head of v_{g_j} and $w = \lambda v_{g_j}$.*

Then $\pi(U_{rs})$ is a basis of the vector K-space M_{rs}.

Proof. Let us prove, first, that every element λh_i $(1 \leq i \leq q,\ \lambda \in \Lambda(r,s))$, that does not belong to $\pi(U_{rs})$, can be written as a finite linear combination of elements of $\pi(U_{rs})$ with coefficients from K (so that the set $\pi(U_{rs})$ generates the K-vector space M_{rs}). Since $\lambda h_i \notin \pi(U_{rs})$, $\lambda f_i \notin U_{rs}$ whence $\lambda f_i = \lambda' v_{g_j}$ for some $\lambda' \in \Lambda$, $1 \leq j \leq d$, such that λ' is similar to the head of v_{g_j} and $\mathrm{ord}_\Delta(\lambda' u_{g_j}) \leq r$. Let us consider the element $g_j = a_j v_{g_j} + \cdots$ $(a_j \in K, a_j \neq 0)$, where dots are placed instead of the other terms that appear in g_j (obviously, those terms are less than v_{g_j} with respect to the order $<_\sigma$). Since $g_j \in N = \operatorname{Ker} \pi$, $\pi(g_j) = a_j \pi(v_{g_j}) + \cdots = 0$, whence $\pi(\lambda' g_j) = a_j \pi(\lambda' v_{g_j}) + \cdots = a_j \pi(\lambda f_i) + \cdots = a_j \lambda h_i + \cdots = 0$, so that λh_i is a finite linear combination with coefficients from K of some elements of the form $\tilde{\lambda} h_k$ $(1 \leq k \leq q)$ such that $\tilde{\lambda} \in \Lambda(r,s)$ and $\tilde{\lambda} f_k <_\sigma \lambda' v_{g_j}$. ($\mathrm{ord}_\sigma \tilde{\lambda} \leq s$, since $\tilde{\lambda} f_k <_\sigma \lambda f_i$ and $\lambda \in \Lambda(r,s)$; $\mathrm{ord}_\Delta \tilde{\lambda} \leq r$, since $\tilde{\lambda} f_k \leq_\Delta u_{\lambda' g_j} = \lambda' u_{g_j}$ and $\mathrm{ord}_\Delta(\lambda' u_{g_j}) \leq r$.) Thus, we can apply the induction on λf_j $(\lambda \in \Lambda, 1 \leq j \leq q)$ with respect to the order $<_\sigma$ and obtain that every element λh_i $(\lambda \in \Lambda(r,s), 1 \leq j \leq q)$ can be written as a finite linear combination of elements of $\pi(U_{rs})$ with coefficients from the field K.

Now, let us prove that the set $\pi(U_{rs})$ is linearly independent over K. Suppose that $\sum_{i=1}^{l} a_i \pi(u_i) = 0$ for some $u_1, \ldots, u_l \in U_{rs}$, $a_1, \ldots, a_l \in K$. Then $h = \sum_{i=1}^{l} a_i u_i$ is an element of N that is reduced with respect to Σ. Indeed, if an element $u = \lambda f_j$ appears in h (so that $u = u_i$ for some $i = 1, \ldots, l$), then either u is not a multiple of any v_{g_j} $(1 \leq j \leq d)$ or $u = \lambda' v_{g_k}$ for some $\lambda' \in \Lambda$, $1 \leq k \leq d$, such that λ' is similar to the head of v_{g_k} and $\mathrm{ord}_\Delta(\lambda' u_{g_k}) > r \geq \mathrm{ord}_\Delta u_h$ (since u_h is one of the elements $u_1, \ldots, u_l$ that lie in U_{rs}). By Theorem 4.4, $h = 0$, whence $a_1 = \cdots = a_q = 0$. This completes the proof of the theorem. □

We conclude this section with a statement that describes a characteristic set of a cyclic Δ-σ-submodule of a finitely generated free Δ-σ-K-module. The proof of this statement is similar to the proof of the appropriate result for difference linear ideals (see Kondrateva *et al.*, 1999, Chap. VI, Corollary 6.5.4).

Theorem 4.8. *Let F be a free Δ-σ-K-module with a finite basis $f_1, \ldots, f_q$ and let N be a Δ-σ-K-submodule of F generated by a single element f. (In other words, $N = Df$ where D is the ring of Δ-σ-operators over K.) Furthermore, let $\preceq$ denote a preorder on F such that $g \preceq h$ if and only if v_h is a multiple of v_g. Then the family of all minimal (with respect to $\preceq$) elements of the set $\{\gamma f | \gamma \in \Gamma\}$ form a characteristic set of the module N.*

Example 4.1. Let F be a free difference–differential module with one free generator f over a difference–differential field K whose basic sets Δ and σ consist of a single derivation operator δ and a single automorphism α, respectively. Let N be a Δ-σ-K-submodule of F generated by the element $g = \alpha\delta f + \alpha^{-2} f$. Then $u_g = \alpha\delta f$, $v_g = \alpha^{-2} f$ and the set $\{\gamma g | \gamma \in \Gamma\}$ has two minimal elements with respect to the preorder $\preceq$: g and $\alpha g = \alpha^2 \delta f + \alpha^{-1} f$. Thus, $\{g, \alpha g\}$ is a characteristic set of the Δ-σ-K-module N.

5. Main Theorems on Difference–Differential Dimension Polynomials

In what follows we keep the notation and conventions of the preceding section.

THEOREM 5.1. *Let K be a Δ-σ-field, D the ring of Δ-σ-operators over K, M a finitely generated Δ-σ-K-module, and $(M_{rs})_{r,s\in\mathbf{Z}}$ an excellent bifiltration of M. Then there exists a numerical polynomial $\psi(t_1,t_2)$ in two variables t_1,t_2 such that* $\deg_{t_1}\psi(t_1,t_2)\le m$, $\deg_{t_2}\psi(t_1,t_2)\le n$, *and* $\psi(r,s)=\dim_K M_{rs}$ *for all sufficiently large* $(r,s)\in\mathbf{N}^2$.

PROOF. Since the bifiltration $(M_{rs})_{r,s\in\mathbf{Z}}$ is excellent, every component M_{rs} is a finitely generated vector K-space and there exist $r_0,s_0\in\mathbf{Z}$ such that $D_{kl}M_{rs}=M_{r+k,s+l}$ for all $r\ge r_0, s\ge s_0, k\ge 0$ and $l\ge 0$ (see Definition 2.3). Let $\{h_1,\dots,h_q\}$ be a basis of the vector K-space $M_{r_0s_0}$. Then the elements $h_1,\dots,h_q$ generate M as a left D-module and $M_{rs}=\sum_{i=1}^{q}D_{r-r_0,s-s_0}h_i$ for all pairs $(r,s)\in\mathbf{Z}^2$ such that $r\ge r_0$ and $s\ge s_0$. Without loss of generality we can assume that $r_0=s_0=0$. (If $\psi(t_1,t_2)$ is a numerical polynomial with the desired properties that corresponds to the case $r_0=0,s_0=0$, then the numerical polynomial $\psi(t_1-r_0,t_2-s_0)$ satisfies the conditions of the theorem in the case of arbitrary $r_0,s_0\in\mathbf{Z}$.) Thus, from now on, we suppose that $M=\sum_{i=1}^{q}Dh_i$ for some elements $h_1,\dots,h_q\in M$ and $M_{rs}=\sum_{i=1}^{q}D_{rs}h_i$ for all $r,s\in\mathbf{Z}$.

Let F be a free Δ-σ-K-module with a basis $f_1,\dots,f_q$, let N be the kernel of the natural epimorphism $\pi: F\longrightarrow M$, and let the set U_{rs} $(r,s\in\mathbf{N})$ be the same as in the conditions of Theorem 4.7. Furthermore, let $\Sigma=\{g_1,\dots,g_d\}$ be a characteristic set of N. By Theorem 4.7, for any $r,s\in\mathbf{N}$, $\pi(U_{rs})$ is a basis of the vector K-space M_{rs}. Therefore, $\dim_K M_{rs}=\operatorname{Card}\pi(U_{rs})=\operatorname{Card}U_{rs}$. (It was shown in the second part of the proof of Theorem 4.7 that the restriction of the mapping π on U_{rs} is bijective.)

Let $U'_{rs}=\{w\in U_{rs}|\ w$ is not a multiple of any element v_{g_i} $(1\le i\le d)\}$ and $U''_{rs}=\{w\in U_{rs}|\ w=\lambda v_{g_j}$ for some g_j $(1\le j\le d)$ and $\lambda\in\Lambda$ such that λ is similar to the head of v_{g_j} and $\operatorname{ord}_\Delta(\lambda u_{g_j})>r\}$. Then $U_{rs}=U'_{rs}\bigcup U''_{rs}$ and $U'_{rs}\bigcap U''_{rs}=\emptyset$, whence $\operatorname{Card}U_{rs}=\operatorname{Card}U'_{rs}+\operatorname{Card}U''_{rs}$. By Theorem 3.4, there exists a numerical polynomial $\phi(t_1,t_2)$ in two variables t_1 and t_2 such that $\phi(r,s)=\operatorname{Card}U'_{rs}$ for all sufficiently large $(r,s)\in\mathbf{N}^2$.

In order to express $\operatorname{Card}U''_{rs}$ in terms of r and s, note that by the combinatorial principle of inclusion and exclusion (see, e.g. Cameron, 1994, Chap. 5, Theorem 5.1), $\operatorname{Card}U''_{rs}$ $(r,s\in\mathbf{Z})$ can be obtained as an alternating sum of the numbers of the form $L(u,v;r,s)=\operatorname{Card}\{w=\lambda v|\lambda\in\Lambda,\lambda$ is similar to the head of v, $\operatorname{ord}_\sigma\lambda\le s-\operatorname{ord}_\sigma v$, and $r-\operatorname{ord}_\Delta u<\operatorname{ord}_\Delta\lambda\le r-\operatorname{ord}_\Delta v\}$ where u and v are some terms. If $u=\delta_1^{k_1}\dots\delta_m^{k_m}\alpha_1^{l_1}\dots\alpha_n^{l_n}f_i$ and $v=\delta_1^{r_1}\dots\delta_m^{r_m}\alpha_1^{s_1}\dots\alpha_n^{s_n}f_j$, then $L(u,v;r,s)=\operatorname{Card}\{(a_1,\dots,a_m,b_1,\dots,b_n)\in\mathbf{N}^{m+n}|\sum_{i=1}^{n}b_i\le s-\sum_{i=1}^{n}s_i$ and $r-\sum_{i=1}^{m}k_i<\sum_{i=1}^{m}a_i\le r-\sum_{i=1}^{m}r_i\}$ for all sufficiently large $(r,s)\in\mathbf{N}^2$.

By Lemma 3.1,

$$L(u,v;r,s)=\binom{s+n-\sum_{i=1}^{n}s_i}{n}\left[\binom{r+m-\sum_{i=1}^{m}r_i}{m}-\binom{r+m-\sum_{i=1}^{m}k_i}{m}\right]$$

for all sufficiently large $(r,s)\in\mathbf{N}^2$.

It follows that there exists a numerical polynomial $\phi_1(t_1,t_2)$ such that $\phi_1(r,s)=\operatorname{Card}U''_{rs}$ for all sufficiently large $(r,s)\in\mathbf{N}^2$ and $\phi_1(t_1,t_2)$ can be written as an alternative sum of the polynomials of the form $\left[\binom{t_1-c_1}{m}-\binom{t_1-c_2}{m}\right]\binom{t_2-c_3}{n}$ $(c_1,c_2,c_3\in\mathbf{Z})$. Therefore, $\deg_{t_1}\phi_1\le m$ and $\deg_{t_2}\phi_1\le n$.

Now, it is clear that the polynomial $\psi(t_1, t_2) = \phi(t_1, t_2) + \phi_1(t_1, t_2)$ satisfies all the conditions of the theorem. □

DEFINITION 5.1. Numerical polynomial $\psi(t_1, t_2)$, whose existence is established by Theorem 5.1, is called a difference–differential (or Δ-σ-) dimension polynomial of the module M associated with the excellent bifiltration $(M_{rs})_{r,s\in\mathbf{N}}$.

EXAMPLE 5.1. Let K be a difference–differential field with a basic set of derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ and a basic set of automorphisms $\sigma = \{\alpha_1, \dots, \alpha_n\}$, and let D be the ring of Δ-σ-operators over K considered with the natural bifiltration $(D_{rs})_{r,s\in\mathbf{Z}}$ introduced in Section 2. Then $(D_{rs})_{r,s\in\mathbf{Z}}$ can be treated as an excellent bifiltration of the Δ-σ-K-module D and one can consider the corresponding Δ-σ-dimension polynomial $\psi_D(t_1, t_2)$. In order to find this polynomial, note that

$$\begin{aligned}\psi_D(r,s) &= \dim_K D_{rs} = \mathrm{Card}\left\{\delta_1^{k_1}\dots\delta_m^{k_m}\alpha_1^{l_1}\dots\alpha_n^{l_n} \in \Lambda | \sum_{i=1}^{m} k_i \le m, \sum_{j=1}^{n} |l_j| \le n\right\}\\ &= g(m,r)h(n,r) = \binom{m+r}{m}\sum_{i=0}^{n}(-1)^{n-i}2^i\binom{n}{i}\binom{r+i}{i}\end{aligned}$$

for all sufficiently large $(r,s) \in \mathbf{N}^2$ whence $\phi_D(t_1,t_2) = \binom{t_1+m}{m}\sum_{i=0}^{n}(-1)^{n-i}2^i\binom{n}{i}\binom{t_2+i}{i}$.

EXAMPLE 5.2. Let K be a difference–differential field whose basic sets Δ and σ consist of a single derivation operator δ and a single automorphism α, respectively. Furthermore, let $M = Dh$ be a cyclic Δ-σ-K-module whose generator h satisfies the defining equation $(\alpha\delta + \alpha^{-2})h = 0$. (As usual, D denotes the ring of Δ-σ-operators over K.) In other words, M is isomorphic to the factor module of a free Δ-σ-K-module F with a free generator f by its Δ-σ-K-submodule $N = D(\delta\alpha + \alpha^{-2})f$. In this case, the proof of Theorem 5.1 shows that the δ-σ-dimension polynomial $\psi_M(t_1, t_2)$ associated with the excellent bifiltration $(D_{rs}h)_{r,s\in\mathbf{Z}}$ of M is determined by the characteristic set $\Sigma = \{g = \delta\alpha f + \alpha^{-2}f, \alpha g = \delta\alpha^2 f + \alpha^{-1}f\}$ of the module N (see Example 4.1). More precisely, $\psi_M(t_1, t_2)$ is determined by the condition

$$\psi_M(r,s) = \mathrm{Card}\,\{\lambda = \delta^i\alpha^j | 0 \le i \le r, |j| \le s \text{ and } \lambda \text{ is a multiple of neither } \alpha^{-2} \text{ nor } \delta\alpha^2\} + \mathrm{Card}\,\{\delta^r\alpha^{-2-l} | 0 \le l \le s-2\} \qquad (*)$$

for all sufficiently large $r, s \in \mathbf{Z}$. If A denotes the set $\{(0,-2),(1,2)\} \subseteq \mathbf{N}\times\mathbf{Z}$ whose elements reflect the the σ-leaders $\alpha^{-2}f$ and $\delta\alpha^2 f$ of the characteristic set Σ, then Theorem 3.4 shows that the first of the two terms in the right-hand part of (*) is equal to $\omega_B(r,s)$ where $\omega_B(t_1,t_2)$ is the characteristic polynomial of the set $B = \rho(A)\bigcup\{(0,1,1)\} = \{(0,0,2),(1,2,0),(0,1,1)\} \subseteq \mathbf{N}^3$ associated with the partition $\{1\}\bigcup\{2,3\}$ of the set $\mathbf{N}_3$. Applying formula (3.3) we obtain that

$$\begin{aligned}\omega_B(t_1,t_2) = &\binom{t_1+1}{1}\binom{t_2+2}{2} - \binom{t_1+1}{1}\binom{t_2}{2} - \binom{t_1}{1}\binom{t_2}{2} - \binom{t_1+1}{1}\binom{t_2}{2}\\ &+\binom{t_1}{1}\binom{t_2-2}{2} + \binom{t_1+1}{1}\binom{t_2-1}{2} + \binom{t_1}{1}\binom{t_2-1}{2} - \binom{t_1}{1}\binom{t_2-2}{2}\\ &= 3t_1 + t_2 + 2.\end{aligned}$$

Since the second term in the right-hand part of (*) is equal to $s-1$, $\psi_M(r,s) = 3r+2s+1$ for all sufficiently large $r, s \in \mathbf{Z}$ whence $\psi_M(t_1,t_2) = 3t_1 + 2t_2 + 1$.

If M is a finitely generated Δ-σ-module over a Δ-σ-field K and $\psi(t_1,t_2)$ is a Δ-σ-dimension polynomial associated with an excellent bifiltration of M, then $\psi(t_1,t_2)$ can be written as

$$\psi(t_1,t_2) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\binom{t_1+i}{i}\binom{t_2+j}{j} \tag{5.1}$$

where $a_{ij} \in \mathbf{Z}$ for all $i = 0,\dots,m;\ j = 0,\dots,n$ (see Theorem 3.1). Let A_ψ denote the set of all pairs $(i,j) \in \mathbf{N}^2$ such that $a_{ij} \neq 0$ in (5.1) and let $\mu(A_\psi) = (\mu_1,\mu_2)$, $\nu(A_\psi) = (\nu_1,\nu_2)$ be the maximal elements of the set A_ψ relative to the lexicographic and reverse lexicographic orders on $\mathbf{N}^2$, respectively.

The following statement gives some invariants of a Δ-σ-dimension polynomial (i.e. numbers that are carried by any Δ-σ-dimension polynomial of a finitely generated Δ-σ-K-module M and that do not depend on the excellent bifiltration of M this polynomial is associated with).

Theorem 5.2. *With the above notation and conventions, the elements $\mu = \mu(A_\psi)$ and $\nu = \nu(A_\psi)$, as well as the coefficients a_{mn}, $a_{\mu_1\mu_2}$ and $a_{\nu_1\nu_2}$, of the Δ-σ-dimension polynomial $\psi(t_1,t_2)$ written in the form* (5.1) *do not depend on the excellent bifiltration of the Δ-σ-K-module M this polynomial is associated with.*

Proof. Let $((M'_{rs}))_{r,s\in\mathbf{Z}}$ be another excellent filtration of the Δ-σ-K-module M, $\psi_1(t_1,t_2) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\binom{t_1+i}{i}\binom{t_2+j}{j}$ ($b_{ij} \in \mathbf{Z}$ for $i = 0,\dots,m; j = 0,\dots,n$) the Δ-σ-dimensional polynomial associated with this filtration, $A_{\psi_1} = \{(i,j) \in \mathbf{N}^2 | 0 \leq i \leq m,\ 0 \leq j \leq n, \text{ and } b_{ij} \neq 0\}$, and $\gamma = (\gamma_1,\gamma_2)$, $\epsilon = (\epsilon_1,\epsilon_2)$ the maximal elements of A_{ψ_1} relative to the lexicographic and reverse lexicographic orders on $\mathbf{N}^2$, respectively. In order to prove the theorem, we should show that $\mu = \gamma$, $\nu = \epsilon$, $a_{mn} = b_{mn}$, $a_{\mu_1\mu_2} = b_{\gamma_1\gamma_2}$, and $a_{\nu_1\nu_2} = b_{\epsilon_1\epsilon_2}$.

It follows from the definition of excellent filtration that there exist elements $r_0, s_0 \in \mathbf{N}$ such that $M_{r_0+i,s_0+j} = D_{ij}M_{r_0s_0}$ and $M'_{r_0+i,s_0+j} = D_{ij}M'_{r_0s_0}$ for all $i,j \in \mathbf{N}$. Let $\{e_1,\dots,e_k\}$ and $\{e'_1,\dots,e'_l\}$ be bases of the vector K-spaces $M_{r_0s_0}$ and $M'_{r_0s_0}$, respectively. Since $\bigcup\{M_{rs} | r,s \in \mathbf{Z}\} = \bigcup\{M'_{rs} | r,s \in \mathbf{Z}\} = M$, there exist positive integers p_1 and p_2 such that $e_1,\dots,e_k \in M'_{r_0+p_1,s_0+p_2}$ and $e'_1,\dots,e'_l \in M_{r_0+p_1,s_0+p_2}$. Then $M_{r_0s_0} \subseteq M'_{r_0+p_1,s_0+p_2}$ and $M'_{r_0s_0} \subseteq M_{r_0+p_1,s_0+p_2}$, therefore $M_{rs} = D_{r-r_0,s-s_0}M_{r_0s_0} \subseteq D_{r-r_0,s-s_0}M'_{r_0+p_1,s_0+p_2} = M'_{r+p_1,s+p_2}$ and $M'_{rs} = D_{r-r_0,s-s_0}M'_{r_0s_0} \subseteq M_{r+p_1,s+p_2}$ for all $(r,s) \in \mathbf{Z}^2$ such that $r \geq r_0, s \geq s_0$. In other words, $\psi(r,s) \leq \psi_1(r+p_1,s+p_2)$ and $\psi_1(r,s) \leq \psi(r+p_1,s+p_2)$ for all sufficiently large $(r,s) \in \mathbf{Z}^2$, namely, for all $(r,s) \in \mathbf{Z}^2$ such that $r \geq r_0$ and $s \geq s_0$. Therefore,

$$\begin{aligned} a_{mn} &= m!n! \lim_{r\to\infty,s\to\infty} \frac{\psi(r,s)}{r^m s^n} \leq m!n! \lim_{r\to\infty,s\to\infty} \frac{\psi_1(r+p_1,s+p_2)}{r^m s^n} \\ &= m!n! \lim_{r\to\infty,s\to\infty} \frac{\psi_1(r,s)}{r^m s^n} = b_{mn} \end{aligned}$$

and similarly $b_{mn} \leq a_{mn}$, so that $b_{mn} = a_{mn}$.

If $a_{mn} \neq 0$, then $(m,n) \in A_\psi$ and $(m,n) \in A_{\psi_1}$ hence $\mu = \nu = \gamma = \epsilon = (m,n)$ and $a_{\mu_1\mu_2} = a_{\nu_1\nu_2} = b_{\gamma_1\gamma_2} = b_{\epsilon_1\epsilon_2} = a_{mn} = b_{mn}$.

Suppose that $a_{mn} = 0$. Then $(\mu_1, \mu_2) \neq (m, n)$, $a_{\mu_1\mu_2} \neq 0$, and the coefficient of the monomial $t_1^{\mu_1} t_2^{\mu_2}$ in the polynomial $\psi(t_1, t_2)$ is equal to $\frac{a_{\mu_1\mu_2}}{\mu_1!\mu_2!}$. Let $s \in \mathbf{N}$, $s \geq s_0$, and let d be a positive integer such that $s^d \geq r_0$. By the choice of the elements μ and γ,

$$\psi(s^d, s) = \frac{a_{\mu_1\mu_2}}{\mu_1!\mu_2!} s^{d\mu_1+\mu_2} + \mathrm{o}(s^{d\mu_1+\mu_2})$$

and

$$\psi_1(s^d, s) = \frac{b_{\gamma_1\gamma_2}}{\gamma_1!\gamma_2!} s^{d\gamma_1+\gamma_2} + \mathrm{o}(s^{d\gamma_1+\gamma_2})$$

for all sufficiently large values of d. (As usual, for any positive integer k, $\mathrm{o}(s^k)$ denotes a polynomial of s whose degree is less than k.) Since

$$\psi(s^d, s) \leq \psi_1(s^d + p_1, s + p_2) = \frac{b_{\gamma_1\gamma_2}}{\gamma_1!\gamma_2!} s^{d\gamma_1+\gamma_2} + o(s^{d\gamma_1+\gamma_2})$$

and

$$\psi_1(s^d, s) \leq \psi(s^d + p_1, s + p_2) = \frac{a_{\mu_1\mu_2}}{\mu_1!\mu_2!} s^{d\mu_1+\mu_2} + \mathrm{o}(s^{d\mu_1+\mu_2})$$

for all $s \geq s_0$, we conclude that $d\mu_1 + \mu_2 = d\gamma_1 + \gamma_2$ for all sufficiently large $d \in \mathbf{N}$ and the coefficients of the power $s^{d\mu_1+\mu_2}$ in $\psi(s^d, s)$ and $\psi_1(s^d, s)$ are equal. Therefore, $\mu_1 = \gamma_1$, $\mu_2 = \gamma_2$ and $a_{\mu_1\mu_2} = b_{\mu_1\mu_2}$. The equalities $\nu_1 = \epsilon_1$, $\nu_2 = \epsilon_2$ and $a_{\nu_1\nu_2} = b_{\nu_1\nu_2}$ can be proved similarly. □

It should be noted that classical Gröbner basis methods of computation of Hilbert polynomials (see, e.g. Becker and Weispfenning, 1993, Chap. 9 and Eisenbud, 1995, Section 15.10) and their analogs for group rings (see Madlener and Reinert, 1998) can be naturally generalized to the case of difference–differential modules. Such a generalization allows us to obtain the following one-dimensional analog of Theorem 5.1 (see Kondrateva *et al.*, 1999, Chap. VI, Theorem 6.7.3).

Theorem 5.3. *Let K be a difference–differential field with a basic set of derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$ and a basic set of automorphisms $\sigma = \{\alpha_1, \ldots, \alpha_n\}$. Let D be the ring of Δ-σ-operators over K and M a finitely generated Δ-σ-K-module with generators $h_1, \ldots, h_q$. Furthermore, for any $r \in \mathbf{N}$, let Λ_r denote the set of all elements $\lambda = \delta_1^{k_1} \ldots \delta_m^{k_m} \alpha_1^{l_1} \ldots \alpha_n^{l_n}$ such that $\sum_{i=1}^m k_i + \sum_{j=1}^n |l_j| \leq r$ and let D_r denote the vector K-subspace of D generated by the set Λ_r. Then there exists a numerical polynomial $\xi(t)$ in one variable t with the following properties.*

(i) *$\xi(r) = dim_K(\sum_{i=1}^q D_r h_i)$ for all sufficiently large $r \in \mathbf{N}$.*
(ii) *$\deg \xi(t) \leq m + n$ and the polynomial $\xi(t)$ can be written as $\xi(t) = \frac{2^n a}{(m+n)!} t^{m+n} + \mathrm{o}(t^{m+n})$. $a \in \mathbf{Z}$ and $\mathrm{o}(t^{m+n})$ denotes a polynomial from $\mathbf{Q}[t]$ whose degree is less than $m + n$).*
(iii) *Integers $d = \deg \xi(t)$, a, and $\Delta^d \xi(t)$ do not depend on the choice of a system of generators of the Δ-σ-K-module M. (As usual, $\Delta^d \xi(t)$ denotes the dth finite difference of $\xi(t)$: $\Delta\xi(t) = \xi(t+1) - \xi(t)$, $\Delta^2\xi(t) = \Delta(\Delta\xi(t))$, etc.)*

The last statement of Theorem 5.3 shows that the polynomial $\xi(t)$ carries some invariants of the module M, i.e. numbers that do not depend on the choice of a system of generators of the Δ-σ-K-module M. It follows from Theorem 5.2 (see also Example 5.2

below) that a Δ-σ-dimension polynomial $\psi(t_1, t_2)$ associated with an excellent bifiltration of M (e.g. with the bifiltration $(\sum_{i=1}^{q} D_{rs}h_i)_{r,s\in\mathbf{Z}}$ where $h_1, \ldots, h_q$ is a system of generators of the Δ-σ-K-module M) carries more invariants than the "one-dimensional" characteristic polynomial $\xi(t)$. From the point of view of the theory of strength of systems of difference–differential equations founded in Einstein (1953) and developed in Mikhalev and Pankratev (1980) and Kondrateva *et al.* (1999), the polynomials $\psi(t_1, t_2)$ and $\xi(t)$ are treated as characteristics of the system of defining equations on the generators of M. In this case, the Δ-σ-dimension polynomial $\psi(t_1, t_2)$ determines the strength of the system with respect to each of the sets of operators Δ and σ while the polynomial $\xi(t)$ determines just the "general" strength of the system with respect to the set $\Delta \bigcup \sigma$.

Remark 1. In Mikhalev and Pankratev (1989) the authors proved Theorem 5.3 and obtained the one-dimensional version of Theorem 3.4 by expressing the appropriate dimension polynomials as Hilbert polynomials of certain ideals of a ring of generalized polynomials. The proof is based on the fact that the ring of Δ-σ-operators over the Δ-σ-field K is isomorphic to the factor ring of the ring of generalized polynomials $K[x_1, \ldots, x_{m+2n}]$ (where $x_i a = a x_i + \delta_i(a)$ $(1 \leq i \leq m)$, $x_{m+j} a = \alpha_j(a) x_{m+j}$ and $x_{m+n+j} a = \alpha_j^{-1}(a) x_{m+n+j}$ $(1 \leq j \leq n)$ for any $a \in K$) by the ideal I generated by the polynomials $x_{m+j} x_{m+n+j} - 1$ $(1 \leq j \leq n)$ that form a Gröbner basis of I. Unfortunately, a similar approach to the two-dimensional case seems to be less fruitful, since the computation of Hilbert polynomials associated with bifiltrations requires homogenization of generalized polynomials, see Kondrateva *et al.* (1999, Section 4.3). Some partial results on such a homogenization were obtained in the monograph (Kondrateva *et al.*, 1999, Chap. IV), however, the technique developed in this paper looks much more effective.

The following result generalizes both the Kolchin theorem on differential dimension polynomial (see Theorem 1.1) and the corresponding author's result for difference field extensions (Theorem 1.2).

Theorem 5.4. *Let K be a difference–differential field of zero characteristic with a basic set of derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$ and a basic set of automorphisms $\sigma = \{\alpha_1, \ldots, \alpha_n\}$, and let $L = K\langle\eta_1, \ldots \eta_p\rangle$ be a finitely generated difference–differential field extension of K with the set of Δ-σ-generators $\eta = \{\eta_1, \ldots, \eta_p\}$. (Recall that, as a field, $L = K(\Lambda\eta_1 \bigcup \cdots \bigcup \Lambda\eta_p)$, where $\Lambda\eta_i$ $(1 \leq i \leq p)$ denotes the set $\{\lambda(\eta_i) | \lambda \in \Lambda\}$).*

Then there exists a numerical polynomial $\chi_{\eta|K}(t_1, t_2)$ in two variables t_1 and t_2 with the following properties.

(i) $\chi_{\eta|K}(r, s) = \mathrm{trdeg}_K K(\Lambda(r, s)\eta_1 \bigcup \cdots \bigcup \Lambda(r, s)\eta_p)$ *for all sufficiently large* $(r, s) \in \mathbf{N}^2$.

(ii) $\deg_{t_1} \chi_{\eta|K} \leq m$ *and* $\deg_{t_2} \chi_{\eta|K} \leq n$, *so that the polynomial* $\chi_{\eta|K}(t_1, t_2)$ *can be written as*

$$\chi_{\eta|K}(t_1, t_2) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \binom{t_1 + i}{i} \binom{t_2 + j}{j}$$

where $a_{ij} \in \mathbf{Z}$ *for all* $i = 0, \ldots, m; j = 0, \ldots, n$.

(iii) *Let* $A_\chi = \{(i, j) \in \mathbf{N}^2 | 0 \leq i \leq m,\ 0 \leq j \leq n,\text{ and } a_{ij} \neq 0\}$ *and let* $\mu = (\mu_1, \mu_2)$, *and* $\nu = (\nu_1, \nu_2)$ *be the maximal elements of the set* A_χ *relative to the lexicographic and reverse lexicographic orders on* $\mathbf{N}^2$, *respectively. Then* μ, ν *and the coefficients* a_{mn},

a_{μ_1,μ_2} and a_{ν_1,ν_2} do not depend on the choice of the system of Δ-σ-generators η of the difference–differential field extension L. (Thus, μ, ν, a_{mn}, a_{μ_1,μ_2} and a_{ν_1,ν_2} are difference–differential birational invariants of the Δ-σ-extension $L = K\langle\eta_1, \dots \eta_p\rangle$).

PROOF. Let $\mathrm{Der}_K L$ denote the vector L-space of all K-linear derivations of the field L into itself (i.e. the set of all derivations $\delta : L \longrightarrow L$ such that $\delta(a) = 0$ for any $a \in K$), and let $\Omega_K(L)$ be the module of differentials associated with the given Δ-σ-field extension, that is a linear L-subspace of the vector L-space $(\mathrm{Der}_K L)^* = \mathrm{Hom}_L(\mathrm{Der}_K L, L)$ generated by the set of all mappings $d\eta$ ($\eta \in L$) such that $d\eta(\delta) = \delta(\eta)$ for any $\delta \in \mathrm{Der}_K L$. As it was shown in Johnson (1969) (see also Kondrateva *et al.*, 1999, Chap. IV, Theorem 6.4.11), $\Omega_K(L)$ can be considered as a differential (with the basic set Δ) vector L-space where the action of elements of Δ on the generators $d\eta(\eta \in L)$ is defined in such a way that $\gamma(d\eta) = d\gamma(\eta)$ for any $\gamma \in \Delta$, $\eta \in L$. At the same time, as was shown in Levin (1980), $\Omega_K(L)$ can be treated as a difference (with the basic set σ) vector L-space such that $\beta(d\eta) = d(\beta(\eta))$ for any $\beta \in \sigma^* = \{\alpha_1, \dots, \alpha_n, \alpha_1^{-1}, \dots, \alpha_n^{-1}\}$, $\eta \in L$. Furthermore (see Johnson, 1969), a set $\Sigma \subseteq L$ is algebraically independent over K if and only if the set $\{d\xi | \xi \in \Sigma\}$ is linearly independent over L.

For any $r, s \in \mathbf{N}$, let $\Omega_K(L)_{rs}$ denote the vector L-subspace of $\Omega_K(L)$ generated by the set $\{d\eta | \eta \in K(\Lambda(r,s)\eta_1 \bigcup \cdots \bigcup \Lambda(r,s)\eta_p)\}$ and let $\Omega_K(L)_{rs} = 0$, if $r, s \in \mathbf{Z}$ and at least one of the numbers r, s is negative. Then $(\Omega_K(L)_{rs})_{r,s\in\mathbf{Z}}$ is an excellent bifiltration of the Δ-L-module $\Omega_K(L)$ and $dim_L \Omega_K(L)_{rs} = \mathrm{trdeg}_K K(\Lambda(r,s)\eta_1 \bigcup \cdots \bigcup \Lambda(r,s)\eta_p)$ for all $r, s \in \mathbf{N}$. Now, applying Theorems 5.1 and 5.2 we obtain all three statements of our theorem. □

DEFINITION 5.2. The numerical polynomial $\chi_{\eta|K}(t_1, t_2)$, whose existence is established by Theorem 5.4, is called a Δ-σ-dimension polynomial of the Δ-σ-field extension L/K associated with set of Δ-σ-generators $\eta = \{\eta_1, \dots, \eta_p\}$.

Theorem 5.4 allows us to assign certain numerical polynomials to any system of linear difference–differential equations with coefficients from a difference–differential (Δ-σ-) field K. Such a system can be written as

$$f_i(y_1, \dots, y_p) = 0 \ (i \in I) \tag{5.2}$$

where $f_i(y_1, \dots, y_p)$ are linear Δ-σ-polynomials in Δ-σ-indeterminates $y_1, \dots, y_p$. Precisely as in the case of differential polynomials (see Kolchin, 1964, Chap. IV, Section 5), one can show that if Q is the Δ-σ-ideal of the ring of Δ-σ-polynomials $K\{y_1, \dots, y_p\}$ generated by the Δ-σ-polynomials $f_i(y_1, \dots, y_p)$ $(i \in I)$, then Q is prime, whence the quotient field L of the factor ring $K\{y_1, \dots, y_p\}/Q$ is a finitely generated Δ-σ-field extension of K: $L = K\langle\eta_1, \dots, \eta_p\rangle$ where η_j $(1 \le j \le p)$ is the image of the Δ-σ-indeterminate y_j under the natural epimorphism $K\{y_1, \dots, y_p\} \longrightarrow K\{y_1, \dots, y_p\}/Q$. In this case, the corresponding Δ-σ-dimension polynomial $\chi_{\eta|K}(t_1, t_2)$ of the extension L/K is said to be a Δ-σ-dimension polynomial of the system (5.2).

If K is a field of differentiable functions of n real variables $x_1, \dots, x_n$, and $h_1, \dots, h_n$ are some constants such that

$$g(x_1, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_n), g(x_1, \dots, x_{i-1}, x_i - h_i, x_{i+1}, \dots, x_n) \in K$$

whenever $g(x_1, \dots, x_n) \in K$, then K can be considered as a difference–differential field whose basic set of derivations Δ consists of the partial differentiations $\partial/\partial x_i$ $(1 \le i \le n)$

and the basic set of automorphisms σ consists of n mappings $\alpha_i : K \to K$ such that

$$(\alpha_i g)(x_1, \dots, x_n) = g(x_1, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_n)$$

($i = 1, \dots, n$). It follows that any system of linear difference–differential equations (also known as a system of differential equations with delay) can be written in the form (5.2) where $y_1, \dots, y_p$ denote the unknown functions. Now, as in the "differential" case (see Mikhalev and Pankratev, 1980), we obtain that the Δ-σ-dimension polynomial of such a system represents the strength of the system in the sense of Einstein (1953).

The last remark shows the importance of the concept of Δ-σ-dimension polynomial of a system of difference–differential equations and methods of computation of Δ-σ-dimension polynomials. The proofs of Theorems 5.4 and 5.1, together with Theorem 3.4 and Algorithms 1 and 2 from Kondrateva *et al.* (1992), provide the idea of such a method. If $L = K\langle \eta_1, \dots, \eta_p \rangle$ is the Δ-σ-field extension corresponding to the system of linear Δ-σ-equations (5.2) with the coefficients from K, then the Δ-σ-dimension polynomial of the system coincides with the appropriate Δ-σ-dimension polynomial $\psi(t_1, t_2)$ of the Δ-σ-K-module of differentials $\Omega_K(L)$ generated, as a difference–differential module, by the elements $d\eta_i$, $1 \le i \le p$ (we use the same notation as in the proof of Theorem 5.4). If N is the kernel of the natural epimorphism $\pi : F \to \Omega_K(L)$ (F is a free Δ-σ-K-module with p free generators) and Σ is a characteristic set of N in the sense of Definition 4.6, then the computation of the Δ-σ-dimension polynomial consists of the computation of certain binomial coefficients and a $\mathbf{N}$-$\mathbf{Z}$-dimension polynomial of a subset of $\mathbf{N}^m \times \mathbf{Z}^n$ (see the proof of Theorem 5.1). By Theorem 3.4(iii), the computation of the $\mathbf{N}$-$\mathbf{Z}$-dimension polynomial can be reduced to the computation of a characteristic polynomial of a subset of $\mathbf{N}^{m+2n}$ (see Definition 3.2) associated with partition (3.9) of the set $\mathbf{N}_{m+2n}$, and the last problem can be solved with the help of Algorithm 1 or 2 from Kondrateva *et al.* (1992).

Example 5.3. Let us determine the Δ-σ-dimension polynomial of the ordinary linear differential equation

$$\delta^a \alpha^b y + \delta^a \alpha^{-b} y + \delta^{a+b} y = 0 \tag{5.3}$$

($a, b \in \mathbf{N}, a \ge 1, b \ge 1$) over some difference–differential field K with the basic set of derivations $\Delta = \{\delta\}$ and basic set of automorphisms $\sigma = \{\alpha\}$.

Let $K\{y\}$ be the ring of Δ-σ-polynomials in one Δ-σ-indeterminate y over K, P the Δ-σ-ideal of $K\{y\}$ generated by the Δ-σ-polynomial $\delta^a \alpha^b y + \delta^a \alpha^{-b} y + \delta^{a+b} y$, η the canonical image of y in the factor ring $K\{y\}/P$, and L the quotient field of this factor ring. Then the Δ-σ-dimension polynomial of equation (5.3) is the Δ-σ-dimension polynomial $\chi_{\eta|K}(t_1, t_2)$ of the Δ-σ-field extension $L = K\langle \eta \rangle$. It follows from the proof of Theorem 5.4 that $\chi_{\eta|K}(t_1, t_2)$ can be found as the Δ-σ-dimension polynomial of the Δ-σ-K-module of differentials $\Omega_K(L) = Dd\eta$ equipped with the bifiltration $(\Omega_K(L)_{rs})_{r,s \in \mathbf{Z}}$. ($D$ denotes the ring of Δ-σ-operators over L, $\Omega_K(L)_{rs}$ is the vector K-space generated by the set $\{\delta^i \alpha^j d\eta | 0 \le i \le r, |j| \le s\}$ ($r, s \in \mathbf{N}$) and $\Omega_K(L)_{rs} = 0$, if at least one of the numbers r, s is negative.)

Let F be a free Δ-σ-L-module with one free generator f, $\pi : F \longrightarrow \Omega_K(L)$ the natural Δ-σ-epimorphism of Δ-σ-L-modules ($f \mapsto d\eta$), and $N = \operatorname{Ker} \pi$. By Kondrateva *et al.* (1992, Chap. VI, Proposition 6.5.5), $N = Dg$ where $g = \delta^a \alpha^b f + \delta^a \alpha^{-b} f + \delta^{a+b} f$. Applying Theorem 4.8 we obtain that the set $\{g, \alpha^{-1} g = \delta^a \alpha^{b-1} f + \delta^a \alpha^{-(b+1)} f + \delta^{a+b} \alpha^{-1} f\}$ is a characteristic set of N. Since $v_g = \delta^a \alpha^b f$, $v_{\alpha^{-1} g} = \delta^a \alpha^{-(b+1)} f$, $u_g = \delta^{a+b} f$, and

$u_{\alpha^{-1}g} = \delta^{a+b}\alpha^{-1}f$, the proof of Theorem 5.1 shows that

$$\begin{aligned}\chi_{\eta|K}(r,s) = \text{Card}\,\{\lambda = \delta^i\alpha^j | 0 \le i \le r, |j| \le s \text{ and } \lambda \text{ is a multiple of neither}\\ \delta^a\alpha^b \text{ nor } \delta^a\alpha^{-(b+1)}\} + \text{Card}\,\{\delta^{a+k}\alpha^{b+l} | r-(a+b) \le k \le r-a, 0 \le l \le s-b\} +\\ \text{Card}\,\{\delta^{a+p}\alpha^{-(b+1)-q} | r-(a+b) \le p \le r-a, 0 \le q \le s-(b+1)\} \qquad (**)\end{aligned}$$

for all sufficiently large $r, s \in \mathbf{Z}$. If A denotes the set $\{(a,b),(a,-(b+1))\} \subseteq \mathbf{N}\times\mathbf{Z}$ whose elements reflect the σ-leaders v_g and $v_{\alpha^{-1}g}$, then Theorem 3.4 shows that the first of the three terms in the right-hand part of (**) is equal to $\omega_B(r,s)$ where $\omega_B(t_1,t_2)$ is the characteristic polynomial of the set $B = \rho(A)\bigcup\{(0,1,1)\} = \{(a,b,0),(a,0,b+1),(0,1,1)\} \subseteq \mathbf{N}^3$ associated with the partition $\{1\}\bigcup\{2,3\}$ of the set $\mathbf{N}_3$. Applying formula (3.3) we obtain that

$$\begin{aligned}\omega_B(t_1,t_2) = &\binom{t_1+1}{1}\binom{t_2+2}{2} - \binom{t_1+1-a}{1}\binom{t_2+2-b}{2}\\ &-\binom{t_1+1-a}{1}\binom{t_2+2-(b+1)}{2}\\ &-\binom{t_1+1}{1}\binom{t_2}{2} + \binom{t_1+1-a}{1}\binom{t_2+2-(2b+1)}{2}\\ &+\binom{t_1+1-a}{1}\binom{t_2+2-(b+1)}{2} + \binom{t_1+1-a}{1}\binom{t_2+2-(b+2)}{2}\\ &-\binom{t_1+1-a}{1}\binom{t_2+2-(2b+1)}{2} = 2bt_1 + 2at_2 + a + 2b - 2ab.\end{aligned}$$

Since the second and third terms in the right-hand part of (**) are equal to $b(s-b+1)$ and $b(s-b)$, respectively, we obtain that $\chi_{\eta|K}(r,s) = 2br + 2(a+b)s + a + 3b - 2ab - 2b^2$ for all sufficiently large $r, s \in \mathbf{Z}$ whence

$$\chi_{\eta|K}(t_1,t_2) = 2bt_1 + 2(a+b)t_2 + a + 3b - 2ab - 2b^2.$$

Considering the Δ-σ-L-module $\Omega_K(L)$ as a filtered module with the one-dimensional filtration $(D_r d\eta)_{r\in\mathbf{Z}}$, we can find the "one-dimensional" characteristic polynomial $\xi(t)$ of equation (5.3) (see Theorem 5.3.) It can be done either by constructing a classical Gröbner basis in the kernel of the natural Δ-σ-epimorphism $F \longrightarrow \Omega_K(L)$ (F is a free filtered left D-module with one generator f and filtration $(D_r f)_{r\in\mathbf{Z}}$) or from a free resolution of the filtered Δ-σ-L-module $\Omega_K(L)$. (Algorithms for these two methods can be found in Mikhalev and Pankratev (1980) and Mikhalev and Pankratev (1989, Chap. 4), respectively.) As it was shown in Kondrateva *et al.* (1999, Chap. VI, Section 6.5), the free resolution of our module $\Omega_K(L)$ is of the form $0 \longrightarrow F^{a+b} \longrightarrow F \longrightarrow \Omega_K(L) \longrightarrow 0$ where F^{a+b} denotes the free left D-module F with the filtration $(D_{r-(a+b)}f)_{r\in\mathbf{Z}}$. Since $\dim_L D_{r-k}f = \text{Card}\,\{\delta^i\alpha^j | i \in \mathbf{N}, j \in \mathbf{Z}, \text{ and } i+|j| \le r-k\} = 2\binom{r+2-k}{2} - \binom{r+1-k}{1}$ for any $k \in \mathbf{Z}$ (see Kondrateva *et al.*, 1999, Chap. VI, (6.7.2)), we obtain that

$$\begin{aligned}\xi(t) &= \left[2\binom{t+2}{2} - \binom{t+1}{1}\right] - \left[2\binom{t+2-(a+b)}{2} - \binom{t+1-(a+b)}{1}\right]\\ &= 2(a+b)t + (a+b)(2-a-b).\end{aligned}$$

Comparing the polynomials $\xi(t)$ and $\chi_{\eta|K}(t_1,t_2)$ we see that the first polynomial carries two parameters that do not depend on the choice of the system of generators of the Δ-σ-field extension $L = K\langle\eta\rangle$, its degree 1 and the leading coefficient $a+b$. At the same time, $\chi_{\eta|K}(t_1, t_2)$ carries three such invariants, its total degree 1, $a+b$, and a. Thus, the Δ-σ-dimension polynomial $\chi_{\eta|K}(t_1, t_2)$ gives both parameters a and b of (5.3) while the "one-dimension" characteristic polynomial $\xi(t)$ gives just the sum of the parameters.

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