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Characterization of Barriers of Differential Games[1]

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Abstract. In pursuit-evasion games, when a barrier occurs, splitting the state space into capture and evasion areas, in order to characterize this manifold, the study of the minimum time function requires discontinuous generalized solutions of the Isaacs equation. Thanks to the minimal oriented distance from the target, we obtain a characterization by approximation with continuous functions. The barrier is characterized by the largest upper semicontinuous viscosity subsolution of a variational inequality. This result extends the Isaacs semipermeability property.

Key Words. Differential games, barriers, Isaacs equation, viscosity solutions.

1. Introduction

Consider a continuous-time pursuit-evasion game defined by the dynamical equation

$$\dot{x}=f(t,x,u,v), \qquad t\in\mathbb{R}^+, x\in\mathbb{R}^n, \tag{1}$$

and a closed target $\mathcal{T}\subset\mathbb{R}^n$; u and v are the controls of two players (respectively the pursuer and the evader), taking values in U and V, compact sets of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Our aim is to characterize the set $\mathscr{C}$ of initial conditions $(t_0, x_0)\in\mathbb{R}^+\times(\mathbb{R}^n\backslash\mathcal{T})$ from which there exists a strategy of the pursuer guaranteeing that the trajectory reaches the target (maybe asymptotically), whatever his opponent is playing, and the set $\mathscr{E}$ of positions from which the evader is able to escape from the target, as long as he wants.

Since the Isaacs works (see Ref. 1), this problem has been studied extensively in the literature. One way to tackle it is to study the minimax

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hitting time and to characterize the barrier as the locus where the value function jumps from finite to infinite values. This approach requires an appropriate concept of discontinuous generalized solution of the Isaacs equation. With different approximation techniques, Subbotin (see Ref. 2) and Bardi, Bottacin, and Falcone (see Ref. 3) have proposed a unique characterization of discontinuous value functions of differential games. But these solutions are known to coincide with the VREK (Varaiya, Roxin, Elliott, and Kalton) value of the game (see Section 2 for definition) only when it is continuous. Later, Cardaliaguet, Quincampoix, and Saint-Pierre have proposed a characterization of the minimax hitting time function for VREK strategies, in terms of the smallest viscosity supersolution (see Ref. 4). Due to viability techniques (see Ref. 5), their approximation is based on the knowledge of an a priori closed set of constraints on the state space. Here, we propose another way to investigate the problem and provide an approximation with a decreasing sequence of continuous functions, without any restriction on their domain, but under the hypothesis of a certain behavior of the trajectories over infinite horizon; see Hypothesis (H6) in Section 3.

Our approach consists of using the oriented-distance function instead of the capture time,

$$\forall \xi \in \mathbb{R}^n, \qquad d^o(\xi, S) = \begin{cases} d(\xi, S), & \text{if } \xi \notin S \subset \mathbb{R}^n, \\ -d(\xi, \mathbb{R}^n \setminus S), & \text{otherwise,} \end{cases}$$

where d stands for the usual distance function in $\mathbb{R}^n$ and S is a given nonempty subset of $\mathbb{R}^n$. Note that this function is always Lipschitz continuous with respect to ξ. So, for a given initial position (t_0, x_0) and a pair of controls $u(\cdot)$, $v(\cdot)$, the criterion that we consider is a kind of infinite norm of this function,

$$J(t_0, x_0, u(\cdot), v(\cdot)) = \operatorname{es}\inf_{t \geq t_0} \{d^o(x(t), \mathcal{T})\},$$

along a trajectory $x(\cdot)$ solution of

$$\dot{x}(t) = f(t, x(t), u(t), v(t)), \qquad x(t_0) = x_0.$$

Then, we define the value functions

$$V_+(t_0, x_0) = \sup_{\psi \in \Gamma} \inf_{u \in \mathscr{U}} J(t_0, x_0, u, \psi[u]), \tag{2}$$

$$V_-(t_0, x_0) = \inf_{\phi \in \Delta} \sup_{v \in \mathscr{V}} J(t_0, x_0, \phi[v], v), \tag{3}$$

where the class of strategies Γ, Δ, $\mathscr{U}$, $\mathscr{V}$ will be made explicit later. These functions provide exactly the information that we are looking for,

$$\mathscr{C}=\{(\tau,\xi)\in\mathbb{R}^+\times\mathbb{R}^n \mid V_+(\tau,\xi)\leq 0\},$$

$$\mathscr{E}=\{(\tau,\xi)\in\mathbb{R}^+\times\mathbb{R}^n \mid V_-(\tau,\xi)>0\}.$$

When V_+ and V_- coincide, the game is said to have a value

$$V=V_+=V_-;$$

we shall show that it is the case when the Isaacs condition is satisfied. If $\mathscr{C}$ and $\mathscr{E}$ are nonempty sets and the value function is continuous, then $\mathscr{C}$ and $\mathscr{E}$ are separated by a C^0 manifold called barrier,

$$\mathscr{B}=\{(\tau,\xi)\mid V(\tau,\xi)=0\}.$$

As we cannot guarantee the continuity of the value function, we study a modified problem with an integral cost,

$$J^\epsilon(t_0,x_0,u(\cdot),v(\cdot))=\text{es}\inf_{t\geq t_0}\left\{d^o(x(t),\mathscr{T})+\int_{t_0}^{t}\epsilon\,d\tau\right\},\qquad \epsilon>0,$$

that brings continuity to the associated value functions and gets the benefit of the theory of continuous viscosity solutions with mild hypotheses. A characterization of $V=V^{\epsilon=0}$ is then provided, passing to the limit when ϵ tends toward 0.

2. Differential Games with Minimum Cost

Now, we study in a more general framework differential games with dynamics described by (1), but with the criterion

$$\begin{aligned}&J(t_0,x_0,u(\cdot),v(\cdot))\\&=\text{ess}\inf_{t\geq t_0}\left\{M(t,x(t))+\int_{t_0}^{t}L(\tau,x(\tau),u(\tau),v(\tau)\,d\tau\right\},\end{aligned}$$

under the following hypotheses:

(H1) f: $\mathbb{R}^+\times\mathbb{R}^n\times\mathsf{U}\times\mathsf{V}\to\mathbb{R}^n$ is jointly continuous.

For any compact set $\mathscr{K}\subset\mathbb{R}^+\times\mathbb{R}^n$, f has linear growth and is Lipschitz continuous with respect to x: there exists $k\in L^1(\mathbb{R}^+,\mathbb{R}^+)$

and $\mathscr{L} > 0$ such that, for any (t, x), $(t, x') \in \mathscr{X}$, uniformly in (u, v),

$$|f(t, x, u, v)| \leq k(t)(1 + |x|),$$

$$|f(t, x', u, v) - f(t, x, u, v)| \leq \mathscr{L}|x' - x|.$$

(H2) $M: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is K_M-Lipschitz continuous.

(H3) $L: \mathbb{R}^+ \times \mathbb{R}^n \times \mathsf{U} \times \mathsf{V} \to \mathbb{R}$ is K_L-Lipschitz continuous.

(H4) L is uniformly bounded below by $L_0 > 0$.

(H5) M is bounded below by $M_0 > -\infty$.

We use the nonanticipative memory strategies, due to Varaiya, Roxin, Elliott, and Kalton (VREK), the most general class of controls that allows the theory of viscosity solutions,

$$\Delta = \{\phi: \mathscr{V} \to \mathscr{U} \,|\, [\forall t < t',\, v(t) = v'(t)] \Rightarrow [\forall t < t',\, \phi[v](t) = \phi[v'](t)]\},$$

$$\Gamma = \{\psi: \mathscr{U} \to \mathscr{V} \,|\, [\forall t \leq t',\, u(t) = u'(t)] \Rightarrow [\forall t \leq t',\, \psi[u](t) = \psi[u'](t)]\},$$

where $\mathscr{U}$ and $\mathscr{V}$ stand for the open-loop strategies

$$\mathscr{U} = \{u: \mathbb{R}^+ \to \mathsf{U},\ \text{measurable}\},$$

$$\mathscr{V} = \{v: \mathbb{R}^+ \to \mathsf{V},\ \text{measurable}\}.$$

The definitions (2) and (3) of the value functions should now be understood with these strategies. Then, we define an abstract terminal time

$$\bar{t}_{\pm}(t_0, x_0, u(\cdot), v(\cdot)) = \inf_{t \geq t_0} \{t \,|\, V_{\pm}(t, x(t)) = M(t, x(t))\} \text{ or } +\infty,$$

where x is solution of (1), and an abstract target in $\mathbb{R}^+ \times \mathbb{R}^n$,

$$\mathscr{T}_{\pm} = \{(\tau, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n \,|\, V_{\pm}(\tau, \xi) = M(\tau, \xi)\}.$$

Differential games with minimum or maximum cost have already been studied in the literature by Barron (see Ref. 6) and by Barron and Ishii (see Ref. 7), but under the hypothesis that the time interval is fixed and bounded. In the next section, we follow the paper of Evans and Souganidis (see Ref. 8), but adapted to our variable end time.

2.1. Optimality Principle.

Lemma 2.1. For any initial condition $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ and for all $t \geq t_0$, one has

$$V_-(t_0, x_0) \leq \inf_{\phi \in \Delta} \sup_{v \in \mathscr{V}} \left\{ \int_{t_0}^{t} L(\tau, x(\tau), \phi[v](\tau), v(\tau))\, d\tau + V_-(t, x(t)) \right\}, \tag{4}$$

$$V_+(t_0, x_0) \leq \sup_{\psi \in \Gamma} \inf_{u \in \mathscr{U}} \left\{ \int_{t_0}^{t} L(\tau, x(\tau), u(\tau), \psi[u](\tau))\, d\tau + V_+(t, x(t)) \right\}. \tag{5}$$

Proof. Let (t_0, x_0) and $t \geq t_0$ be fixed, and name the quantity

$$S = \inf_{\phi \in \Delta} \sup_{v \in \mathscr{V}} \left\{ \int_{t_0}^{t} L(\tau, x(\tau), \phi[v](\tau), v(\tau))\, d\tau + V_-(t, x(t)) \right\}.$$

Take $\epsilon > 0$; then,

$$\exists \bar{\phi} \in \Delta \text{ s.t. } S \geq \sup_{v \in \mathscr{V}} \left\{ \int_{t_0}^{t} L(\tau, x(\tau), \bar{\phi}[v](\tau), v(\tau)\, d\tau + V_-(t, x(t)) \right\} - \epsilon.$$

On the other hand, for all $x \in \mathbb{R}^n$, by definition

$$\exists \phi_x \in \Delta \quad \text{s.t. } V_-(t, x) \geq \sup_{v \in \mathscr{V}} J(t, x, \phi_x, v) - \epsilon.$$

Consider the controller $\tilde{\phi}$ defined as follows:

$$\tilde{\phi}[v](s): \begin{cases} \bar{\phi}[v](s), & t_0 \leq s < t, \\ \phi_{x(t)}[v](s), & t \leq s \leq \bar{t}_-(t, x(t), \phi_{x(t)}[v], v). \end{cases}$$

Then, for all $v(\cdot) \in \mathscr{V}$, one has

$$\begin{aligned} S &\geq \int_{t_0}^{t} L(\tau, x(\tau), \tilde{\phi}[v](\tau), v(\tau))\, d\tau + J(t, x, \tilde{\phi}, v) - 2\epsilon \\ &\geq \inf_{t_f \geq t} \left\{ \int_{t_0}^{t_f} \mathrm{L}(\tau, \mathrm{x}(\tau), \tilde{\phi}[\mathrm{v}](\tau), \mathrm{v}(\tau))\, \mathrm{d}\tau + \mathrm{M}(\mathrm{t}_f, \mathrm{x}(\mathrm{t}_f)) \right\} - 2\epsilon \\ &\geq \inf_{t_f \geq t_0} \left\{ \int_{t_0}^{t_f} \mathrm{L}(\tau, \mathrm{x}(\tau), \tilde{\phi}[\mathrm{v}](\tau), \mathrm{v}(\tau))\, \mathrm{d}\tau + \mathrm{M}(\mathrm{t}_f, \mathrm{x}(\mathrm{t}_f)) \right\} - 2\epsilon \\ &= \mathrm{J}(\mathrm{t}_0, \mathrm{x}_0, \tilde{\phi}, \mathrm{v}) - 2\epsilon. \end{aligned}$$

Therefore,

$$S \geq \sup_{v \in \mathscr{V}} J(t_0, x_0, \tilde{\phi}, v) - 2\epsilon \geq V_-(t_0, x_0) - 2\epsilon.$$

Since ϵ is arbitrary, (4) follows. The proof for V_+ is similar. □

We shall call a nonanticipative stopping time any function σ of the pair $(u(\cdot), v(\cdot))$ such that

$$\forall (u, v) \in \mathscr{U} \times \mathscr{V}, \qquad \sigma(u, v) \leq \bar{t}_\pm(t_0, x_0, u, v),$$

$$[\forall t < \sigma(u, v), u(t) = u'(t) \text{ and } v(t) = v'(t)] \Rightarrow \sigma(u', v') \geq \sigma(u, v).$$

Proposition 2.1. For any initial condition $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R} \backslash \mathscr{T}_\pm$ and any nonanticipative stopping time σ, one has

$$V_-(t_0, x_0) = \inf_{\phi \in \Delta} \sup_{v \in \mathscr{V}} \left\{ \int_{t_0}^{\sigma(\phi[v], v)} L(\tau, x(\tau), \phi[v](\tau), v(\tau))\, d\tau + V_-(\sigma(\phi[v], v), x(\sigma(\phi[v], v))) \right\},$$

$$V_+(t_0, x_0) = \sup_{\psi \in \Gamma} \inf_{u \in \mathscr{U}} \left\{ \int_{t_0}^{\sigma(u, \psi[u])} L(\tau, x(\tau), u(\tau), \psi[u](\tau))\, d\tau + V_+(\sigma(u, \psi[u]), x(\sigma(u, \psi[u]))) \right\}.$$

Proof. Let us call

$$S = \inf_{\phi \in \Delta} \sup_{v \in \mathscr{V}} \left\{ \int_{t_0}^{\sigma(\phi[v], v)} L(\tau, x(\tau), \phi[v](\tau), v(\tau))\, d\tau + V_-(\sigma(\phi[v], v), x(\sigma(\phi[v], v))) \right\}.$$

The previous lemma provides already one inequality,

$$V_-(t_0, x_0) \leq S.$$

Take $\epsilon > 0$; then, there exists $\bar{\phi} \in \Delta$ such that

$$V_-(t_0, x_0) \geq \sup_{v \in \mathscr{V}} J(t_0, x_0, \bar{\phi}[v], v) - \epsilon. \tag{6}$$

Moreover,

$$S \leq \sup_{v \in \mathcal{V}} \left\{ \int_{t_0}^{\sigma(\bar{\phi}[v],v)} L(\tau, x(\tau), \bar{\phi}[v](\tau), v(\tau))\, d\tau \right.$$
$$\left. + V_-(\sigma(\bar{\phi}[v], v), x(\sigma(\bar{\phi}[v], v))) \right\},$$

and there exists a control $v^1 \in \mathcal{V}$ such that

$$S \leq \int_{t_0}^{\sigma(\bar{\phi}[v^1],v^1)} L(\tau, x(\tau), \bar{\phi}[v^1](\tau), v^1(\tau))\, d\tau$$
$$+ V_-(\sigma(\bar{\phi}[v^1], v^1), x(\sigma(\bar{\phi}[v^1], v^1))) + \epsilon.$$

We have also, coarsely,

$$V_-(\sigma(\bar{\phi}[v^1], v^1), x(\sigma(\bar{\phi}[v^1], v^1)))$$
$$\leq \sup_{v \in \mathcal{V}} J(\sigma(\bar{\phi}[v^1], v^1), x(\sigma(\bar{\phi}[v^1], v^1)), \bar{\phi}[v], v).$$

So, there exists $v^2 \in \mathcal{V}$ such that

$$V_-(\sigma(\bar{\phi}[v^1], v^1), x(\sigma(\bar{\phi}[v^1], v^1)))$$
$$\leq J(\sigma(\bar{\phi}[v^1], v^1), x(\sigma(\bar{\phi}[v^1], v^1)), \bar{\phi}[v^2], v^2) + \epsilon.$$

Since

$$(\sigma(\bar{\phi}[v^1], v^1), x(\sigma(\bar{\phi}[v^1], v^1))) \notin \mathcal{T}_-,$$

we can consider the open-loop controller

$$\bar{v}(s) = \begin{cases} v^1(s), & t_0 \leq s < \sigma(\bar{\phi}[v^1], v^1), \\ v^2(s), & \sigma(\bar{\phi}[v^1], v^1) \leq s \leq \bar{t}_-(\sigma(\bar{\phi}[v^1], v^1), x(\sigma(\bar{\phi}[v^1], v^1)), \bar{\phi}[v^2], v^2); \end{cases}$$

then,

$$S \leq \int_{t_0}^{\sigma(\bar{\phi}[v^1],v^1)} L(\tau, x(\tau), \bar{\phi}[v^1](\tau), v^1(\tau))\, d\tau$$
$$+ J(\sigma(\bar{\phi}[v^1], v^1), x(\sigma(\bar{\phi}[v^1], v^1)), \bar{\phi}[v^2], v^2) + 2\epsilon.$$

Due to the nonanticipativity of σ, one has

$$\sigma(\bar{\phi}[\bar{v}], \bar{v}) \geq \sigma(\bar{\phi}[v^1], v^1),$$

and so,

$$J(t_0, x_0, \bar{\phi}[\bar{v}], \bar{v}) = \int_{t_0}^{\sigma(\bar{\phi}[v^1], v^1)} L(\tau, x(\tau), \bar{\phi}[v^1](\tau), \bar{v}(\tau))\, d\tau + J(\sigma(\bar{\phi}[\bar{v}], \bar{v}), x(\sigma(\bar{\phi}[\bar{v}], \bar{v}], \bar{\phi}[\bar{v}], \bar{v}).$$

We conclude that

$$\begin{aligned} S &\leq J(t_0, x_0, \bar{\phi}[\bar{v}], \bar{v}) + 2\epsilon \\ &\leq \sup_{v \in \mathcal{V}} J(t_0, x_0, \bar{\phi}[v], v) + 2\epsilon \\ &\leq V_-(t_0, x_0) + 3\epsilon, \end{aligned}$$

using (6). The proof for V_+ is similar. □

2.2. Regularity of the Value Functions.

Proposition 2.2. Under Hypotheses (H1)–(H5), the value functions V_+ and V_- are locally Lipschitz continuous on $\mathbb{R}^+ \times \mathbb{R}^n$.

Proof. Notice first that, for any initial condition (t_0, x_0) and for all pair of open-loop strategies $(u, v) \in \mathcal{U} \times \mathcal{V}$, we have

$$J > M_0 + (t_1 - t_0)L_0$$

as soon as the infimum in time of J is not reached before t_1; see Hypotheses (H4) and (H5). We have also

$$M(t_0, x_0) > M_0 + (t_1 - t_0)L_0.$$

So the infimum is necessarily reached before the time $t_0 + (M(t_0, x_0) - M_0)/L_0$.

Fix now $(t_0, \bar{x}_0)$, and consider the neighborhood $\mathcal{O} = B(\bar{x}_0, (M(t_0, \bar{x}_0 - M_0 + R)/K_M)$, for a given $R > 0$. Thanks to the Lipschitz continuity of M, we have

$$|M(t_0, x_0) - M(t_0, \bar{x}_0)| \leq M(t_0, \bar{x}_0) - M_0 + R, \qquad \forall x_0 \in \mathcal{O}.$$

Fix a pair (u, v) and consider $\bar{x}(\cdot)$, $x(\cdot)$ the trajectories with respective initial conditions $\bar{x}_0$ and x_0 in $\mathcal{O}$ on the interval $[t_0, T]$, with

$$T = t_0 + [2(M(t_0, \bar{x}_0) - M_0) + R)/L_0.$$

By a standard Gronwall argument and the Lipschitz continuity of the dynamics, there exists a constant $K > 0$ such that

$$|x(t) - \bar{x}(t)| \leq K|x_0 - \bar{x}_0|, \qquad \forall t \in [t_0, T].$$

Then, thanks to the Lipschitz continuity of M and L, we have also

$$\max_{t\in[t_0,T]} |M(t, x(t)) - M(t, \bar{x}(t))| \leq K_M K |x_0 - \bar{x}_0|,$$

$$\max_{t\in[t_0,T]} \left| \int_{t_0}^{t} L(\tau, x(\tau), u(\tau), v(\tau)) - L(\tau, \bar{x}(\tau), u(\tau), v(\tau))\, d\tau \right|$$

$$\leq K_L K(T - t_0)|x_0 - \bar{x}_0|,$$

and we deduce that

$$|J(t_0, x_0, u, v) - J(t_0, \bar{x}_0, u, v)| \leq K(K_M + K_L(T - t_0))|x_0 - \bar{x}_0|,$$

uniformly in (u, v). We conclude that V_+ and V_- are locally Lipschitz continuous with respect to x.

Let us now study the dependency in time for a given x_0. Once again, for any neighborhood $\mathcal{N} = B(t_0, r)$, $r > 0$, the minimum cost is reached before

$$T = t_0 + r + [M(t_0, x_0) + K_M r - M_0]/L_0,$$

uniformly in (u, v). Take $t_2 > t_1$ in $\mathcal{N}$.

For $\epsilon > 0$, there exists $\phi_1 \in \Delta$ such that

$$\sup_{v\in\mathcal{V}} J(t_1, x_0, \phi_1[v], v) \leq V_-(t_1, x_0) + \epsilon.$$

For any $v \in \mathcal{V}|_{[t_2,T]}$, let us build a strategy $\phi_2[v] = \phi_1[\bar{v}]|_{[t_2,T]}$, where for an arbitrary $z \in \mathrm{V}$,

$$\bar{v}(\tau) = \begin{cases} z, & \tau \in [t_1, t_2), \\ v(\tau), & \tau \in [t_2, T]. \end{cases}$$

Consider $x_1(t)$ and $x_2(t)$ the trajectories solutions of

$$\dot{x}_1 = f(t, x_1, \phi_1[\bar{v}](t), \bar{v}(t)), \qquad x_1(t_1) = x_0, \qquad t \in [t_1, T],$$

$$\dot{x}_2 = f(t, x_2, \phi_2[v](t), v(t)), \qquad x_2(t_2) = x_0, \qquad t \in [t_2, T].$$

By usual o.d.e. arguments, the linear growth and the Lipschitz continuity of f provide the existence of constants $L_1, L_2 > 0$ such that

$$|x_1(t) - x_0| \leq L_1(t_2 - t_1), \qquad \forall t \in [t_1, t_2],$$
$$|x_2(t) - x_1(t)| \leq L_2|x_1(t_2) - x_0|, \qquad \forall t \in [t_2. T],$$

whatever are ϕ_2 and v. Then, the Lipschitz continuity of M and L brings the following inequalities:

$$|M(t, x_2(t)) - M(t, x_1(t))| \leq K_M L_1 L_2 (t_2 - t_1),$$

$$\left| \int_{t_2}^{t} L(\tau, x_2(\tau), \phi_2[v](\tau), v(\tau)) - L(\tau, x_1(\tau), \phi_1[\bar{v}](\tau), \bar{v}(\tau)\, d\tau \right|$$
$$\leq K_L L_1 L_2 (T - t_2)(t_2 - t_1),$$

for any $t \in [t_2, T]$, and we have [the second inequality is due to (H4)]

$$\begin{aligned} &J(t_2, x_0, \phi_2[v], v) \\ &\leq \inf_{t\in[t_2,T]} \left\{ M(t, x_1(t)) + \int_{t_2}^{t} L(\tau, x_1(\tau), \phi_1[\bar{v}](\tau), \bar{v}(\tau))\, d\tau \right\} \\ &\quad + L_1 L_2 (K_M + K_L(T - t_2))(t_2 - t_1) \\ &\leq J(t_1, x_0, \phi_1[\bar{v}], \bar{v}) + \max_{t\in[t_1,t_2]} |M(t, x_1(t)) - M(t_1, x_0)| \\ &\quad + L_1 L_2 (K_M + K_L(T - t_2))(t_2 - t_1) \\ &\leq J(t_1, x_0, \phi_1[\bar{v}], \bar{v}) + \alpha (t_2 - t_1), \end{aligned}$$

with

$$\alpha = K_M L_1 + L_1 L_2 (K_M + K_L (T - t_2)).$$

This inequality is valid for any $v \in \mathscr{V}|_{[t_2,T]}$, and we deduce that, for any $\epsilon > 0$,

$$V_-(t_2, x_0) \leq V_-(t_1, x_0) + \alpha (t_2 - t_1) + \epsilon.$$

For the other inequality, consider $\phi_2 \in \Delta$ such that

$$\sup_{v\in\mathscr{V}} J(t_2, x_0, \phi_2[v], v) \leq V_-(t_2, x_0) + \epsilon.$$

As in Barron (see Ref. 6) and Berkovitz (see Ref. 9), we use a change of variable in time to build a strategy ϕ_1 on $[t_1, T]$:

$$\phi_1[v](t) = \phi_2[\bar{v}](s(t)), \qquad \forall v \in \mathscr{V}|_{[t_1,T]}, \qquad \forall t \in [t_1, T],$$

with

$$s(t) = t_2 + [(T - t_2)/(T - t_1)](t - t_1),$$
$$\bar{v}(s) = v(t_1 + [(T - t_1)/(T - t_2)](s - t_2)), \qquad s \in [t_2, T].$$

Then, if we consider the trajectories $x_1(t)$ and $x_2(t)$ solutions of

$$\dot{x}_1 = f(t, x_1, \phi_1[v](t), v(t)), \qquad x_1(t_1) = x_0, \qquad t \in [t_1, T], \tag{7}$$

$$\dot{x}_2 = f(s, x_2, \phi_2[\bar{v}](s), \bar{v}(s)), \qquad x_2(t_2) = x_0, \qquad s \in [t_2, T], \tag{8}$$

a result from Friedman (see Lemma 2.6.1 in Ref. 10) states that, under Hypothesis (H1), there exists a constant $\bar{L}>0$ such that

$$|x_2(s(t))-x_1(t)|\leq \bar{L}(t_2-t_1), \qquad \forall t\in[t_1, T],$$

uniformly in (u, v). So, as previously, let us write the Lipschitz continuity of f, M, L as

$$|M(t, x_1(t))-M(t, x_2(s(t)))|\leq K_M\bar{L}(t_2-t_1),$$

$$\left|\int_{t_1}^{t} L(\tau, x_1(\tau), \phi_1[v](\tau), v(\tau))-L(\tau, x_2(s(\tau)), \phi_2[\bar{v}](s(\tau)), \bar{v}(s(\tau)))\, d\tau\right|$$

$$\leq K_L\bar{L}(T-t_1)(t_2-t_1),$$

for $t\in[t_1, T]$. With

$$\beta=\bar{L}(K_M+K_L(T-t_1)),$$

we have then, for any $v\in\mathscr{V}$,

$$J(t_1, x_0, \phi_1[v], v)\leq J(t_2, x_0, \phi_2[\bar{v}], \bar{v})+\beta(t_2-t_1),$$

and we conclude that

$$V_-(t_1, x_0)\leq V_-(t_2, x_0)+\beta(t_2-t_1)+\epsilon,$$

whatever is $\epsilon>0$. Hence, V_- is locally Lipschitz continuous with respect to t. The proof for V_+ is similar. □

Remark 2.1. If we require stronger assumptions on the data, we can prove the uniform continuity of the value functions:

(H1b) f is bounded and uniformly continuous with respect to x.

(H5b) M is bounded above by $M_1<+\infty$.

Corollary 2.1. Under Hypotheses (H1)–(H5), (H1b), (H5b), the value functions V_+ and V_- are uniformly Lipschitz continuous.

Proof. The length of the time interval on which the infimum of J is reached is now bounded, uniformly with respect to the initial condition

$$(T-t_0)\leq(M_1-M_0)/L_0.$$

The boundedness and the uniform continuity of f render the constants K, L_1, L_2, $\bar{L}$ in the proof of Proposition 2.2 uniform with respect to (t_0, x_0). So, all the Lipschitz constants made explicit in this latter proof are uniform. □

2.3. Partial Differential Equation of Dynamic Programming. We use the following notations for the Hamiltonian of the system, $H: \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathsf{U} \times \mathsf{V} \to \mathbb{R}$,

$$H(t, x, \lambda, u, v) = \lambda' f(t, x, u, v) + L(t, x, u, v),$$

and optimized Hamiltonians, $H^*_{\pm}: \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$H^*_+(t, x, \lambda) = \min_{u \in \mathsf{U}} \max_{v \in \mathsf{V}} H(t, x, \lambda, u, v),$$

$$H^*_-(t, x, \lambda) = \max_{v \in \mathsf{V}} \min_{u \in \mathsf{U}} H(t, x, \lambda, u, v).$$

Due to the lack of explicit boundary conditions, the Isaacs equation is a variational inequality,

$$\min\{M(t, x) - V(t, x), V_t(t, x) + H^*_{\pm}(t, x, V_x(t, x))\} = 0, \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n. \tag{9}$$

Consider classes of admissible feedback strategies

$$\Phi_d = \{\phi : (t, x, v) \mapsto \phi(t, x, v) \in \mathsf{U}\},$$

$$\Psi = \{\psi : (t, x) \mapsto \psi(t, x) \in \mathsf{V}\},$$

such that:

(a) open-loops are admissible: $\mathcal{U} \subset \Phi_d$ and $\mathcal{V} \subset \Psi$;
(b) Φ_d and Ψ are closed by concatenation; i.e., switching from one strategy in the set to another one, at an intermediate instant of time, is allowed;
(c) $\forall (\phi, \psi) \in \Phi_d \times \Psi$, $\forall x_0$, there exists an unique solution $\dot{x} = f(t, x, \phi(\cdot), \psi(\cdot))$ over $\mathbb{R}^+$, leading to admissible controls $u(\cdot) = \phi(\cdot, x(\cdot), v(\cdot)) \in \mathcal{U}$ and $v(\cdot) = \psi(\cdot, x(\cdot)) \in \mathcal{V}$.

These properties do not uniquely define the pair (Φ_d, Ψ), but it is clear that such classes exist and the above assumptions justify the Isaacs equation; see Ref. 11. We can then give the statement of a smooth verification theorem for the variational inequality (9).

Proposition 2.3. If there exists a function $U \in C^1(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R})$, solution of (9) with H^*_-, then $U = V_-$. Moreover, if there exist admissible controllers $(\phi^*, \psi^*) \in \Phi_d \times \Psi$ such that

$$\phi^*(t, x, v) \in \arg \min_{u \in \mathsf{U}} H(t, x, U_x, u, v),$$

$$\psi^*(t, x) \in \arg \max_{v \in \mathsf{V}} H(t, x, U_x, \phi^*(t, x, v), v),$$

then (ϕ^*, ψ^*) is optimal, and an optimal stopping time is

$$t^* = \inf\{t \geq t_0 \mid U(t, x(t)) = M(t, x(t))\},$$

along the trajectory $x(\cdot)$, solution of $\dot{x} = f(t, x, \phi^*(\cdot), v(\cdot))$ for any $v(\cdot) \in \mathscr{V}$.

Remark 2.2. ψ^* is called a state feedback and ϕ^* a state discriminating feedback.

Remark 2.3. In a similar way, there exists a proposition for V_+ with H_+^* and analogous classes of admissible feedbacks Φ and Ψ_d.

Proof. Let (t_0, x_0) be an initial condition, and let $v(\cdot)$ be any control law played against the strategy ϕ^*. If $U(t_0, x_0) = M(t_0, x_0)$, it is clear that

$$U(t_0, x_0) \geq J(t_0, x_0, \phi^*, v).$$

If $U(t_0, x_0) \neq M(t_0, x_0)$, then as soon as $U(t, x(t)) \neq M(t, x(t))$, we have

$$\begin{aligned} &U_t(t, x(t)) + U_x(t, x(t)) f(t, x(t), \phi^*(t, x(t), v(t)), v(t)) \\ &+ L(t, x(t), \phi^*(t, x(t), v(t)), v(t)) \leq 0 \\ \Rightarrow\ &\mathrm{U}(t, x(t)) + \int_{t_0}^{t} L(\tau, x(\tau), \phi^*(\tau, x(\tau)), v(\tau))\, v(\tau))\, d\tau \\ &\leq U(t_0, x_0). \end{aligned}$$

At time

$$t^* = \inf\{t \mid U(t, x(t)) = M(t, x(t))\},$$

we obtain

$$\begin{aligned} U(t_0, x_0) &\geq M(t^*, x(t^*)) \\ &\quad + \int_{t_0}^{t^*} L(\tau, x(\tau), \phi^*(\tau, x(\tau), v(\tau)), v(\tau))\, d\tau \\ &\geq J(t_0, x_0, \phi^*, v), \end{aligned}$$

whatever is the control $v(\cdot) \in \mathscr{V}$. So,

$$U(t_0, x_0) \geq \sup_{v \in \mathscr{V}} J(t_0, x_0, \phi^*, v) \geq V_-(t_0, x_0).$$

Let now ϕ be an admissible discriminating feedback strategy, and consider an open-loop law

$$\bar{v}(t) \in \arg\max_{v \in \mathrm{V}} H(t, x, U_x(t, x), \phi(t, x, v), v),$$

defined along the trajectory $x(\cdot)$ generated by the pair $(\phi, \bar{v})$. $\bar{v}$ is well defined, and so $(\phi, \bar{v})$ is an admissible pair of strategies. Then one has, $\forall t \geq t_0$,

$$\begin{aligned} &U_t(t, x(t)) + U_x(t, x(t)) f(t, x(t), \phi(t, x(t), \bar{v}(t)), \bar{v}(t)) \\ &\quad + L(t, x(t), \phi(t, x(t), \bar{v}(t)), \bar{v}(t)) \geq 0 \\ &\Rightarrow U(t, x(t)) - U(t_0, x_0) \\ &\quad + \int_{t_0}^{t} L(\tau, x(\tau), \phi(\tau, x(\tau), \bar{v}(\tau)), \bar{v}(\tau))\, d\tau \geq 0 \\ &\Rightarrow M(t, x(t)) - U(t_0, x_0) \\ &\quad + \int_{t_0}^{t} L(\tau, x(\tau), \phi(\tau, x(\tau), \bar{v}(\tau)), \bar{v}(\tau))\, d\tau \geq 0. \end{aligned}$$

So,

$$U(t_0, x_0) \leq J(t_0, x_0, \phi, \bar{v}) \leq \sup_{v \in \mathscr{V}} J(t_0, x_0, \phi, v).$$

These inequalities can be obtained for any admissible discriminating feedback strategy, and so,

$$U(t_0, x_0) \leq \inf_{\phi \in \Phi^d} \sup_{v \in \mathscr{V}} J(t_0, x_0, \phi, v) = V_-(t_0, x_0).$$

If the pair (ϕ^*, ψ^*) is admissible, the cost $U(t_0, x_0)$ is reached, and we conclude that

$$U(t_0, x_0) = \min_{\phi \in \Phi^d} \max_{\psi \in \Psi} J(t_0, x_0, \phi, \psi) = V_-(t_0, x_0).$$

The optimal stopping time t^* satisfies

$$U(t^*, x(t^*)) = M(t^*, x(t^*)).$$

Otherwise, we shall have $U(t^*, x(t^*)) < M(t^*, x(t^*))$ and then there should exist $\bar{t} > t^*$ such that $U(\bar{t}, x(\bar{t})) = M(\bar{t}, x(\bar{t}))$ and $U(t, x(t)) < M(t, x(t))$, $\forall t \in (t_*, \bar{t})$. Thanks to (9), we obtain

$$\begin{aligned} &U(t^*, x(t^*)) \\ &= M(\bar{t}, x(\bar{t})) + \int_{t^*}^{\bar{t}} L(\tau, x(\tau), \phi^*(\tau, x(\tau), \psi^*(\tau, x(\tau))), \psi^*(\tau, x(\tau)))\, d\tau \\ &< M(t^*, x(t^*)), \end{aligned}$$

which contradicts the optimality of t^* for the initial condition $(t^*, x(t^*))$. According to (9), along a trajectory $x^*(\cdot)$ generated by the pair (ϕ^*, ψ^*), the map

$$t \mapsto U(t, x^*(t)) + \int_{t_0}^{t} L(\tau, x^*(\tau), \psi^*(\tau, x^*(\tau))), \psi^*(\tau, x^*(\tau))) \, d\tau$$

is nondecreasing, and as a consequence,

$$t^* = \inf_{t \geq t_0} \{t \mid U(t, x^*(t)) = M(t, x^*(t))\}.$$

Until t^*, we have

$$\begin{aligned} U(t_0, x_0) &= M(t^*, x^*(t^*)) \\ &\quad + \int_{t_0}^{t^*} L(\tau, x^*(\tau), \psi^*(\tau, x^*(\tau))), \psi^*(\tau, x^*(\tau))) \, d\tau \\ &= J(t_0, x_0, \phi^*, \psi^*). \end{aligned}$$

□

2.4. Viscosity Solutions. An upper [resp. lower] semicontinuous function $w: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is said to be a viscosity subsolution [resp. supersolution] of the partial differential equation

$$w_t + h(t, x, w, w_x) = 0,$$

if $\forall (t, x)$, $\forall \phi \in C^1$ such that (t, x) is a local maximum [resp. minimum] of $w - \phi$,

$$\phi_t(t, x) + h(t, x, w(t, x), \phi_x(t, x)) \geq 0 \text{ [resp.} \leq 0].$$

w is said to be a continuous viscosity solution if it is both a subsolution and a supersolution.

Another equivalent definition (see Ref. 12) for w to be a sub[super]solution is the following, $\forall (t, x)$:

$$\forall (\rho_t, \rho_x) \in \partial^+ w(t, x), \qquad \rho_t + h(t, x, w(t, x), \rho_x) \geq 0,$$

$$[\text{resp. } \forall (\rho_t, \rho_x) \in \partial^- w(t, x), \qquad \rho_t + h(t, x, w(t, x), \rho_x) \leq 0],$$

where ∂^- $[\partial^+]$ stands for the Fréchet sub[super]differential, when u is a locally Lipschitz continuous function from $\mathbb{R}^n$ to $\mathbb{R}$,

$$\partial^- u(x) = \{p \in \mathbb{R}^n \mid \liminf_{y \to x} [u(y) - u(x) - \langle p, y - x \rangle] / |y - x| \geq 0\},$$

$$\partial^+ u(x) = -\partial^-(-u)(x).$$

We shall need the following technical lemma, due to Evans and Souganidis (see Lemma 4.3 in Ref. 8).

Lemma 2.2. Consider a function $W \in C^1(\mathbb{R} \times X, \mathbb{R})$ and the quantity

$$\Lambda(t, x, u, v) = W_t(t, x) + W_x(t, x) f(t, x, u, v) + L(t, x, u, v).$$

Under Hypotheses (H1) and (H3), for any $\epsilon > 0$, one can establish the following results:

(i) If (t_0, x_0) is such that $W_t(t_0, x_0) + H_+^*(t_0, x_0, W_x(t_0, x_0)) \leq -\epsilon$, then

$$\exists \eta > 0, \exists u_\epsilon \in \mathcal{U} \text{ s.t.}, \forall \psi \in \Gamma, \forall t \in [t_0, t_0 + \eta[,$$
$$\int_{t_0}^t \Lambda(s, x(s), u_\epsilon(s), \psi[u_\epsilon](s))\, ds \leq (t - t_0)\epsilon/2.$$

(ii) If (t_0, x_0) is such that $W_t(t_0, x_0) + H_+^*(t_0, x_0, W_x(t_0, x_0)) \geq \epsilon$, then

$$\exists \eta > 0, \exists \psi_\epsilon \in \Gamma \text{ s.t.}, \forall u \in \mathcal{U}, \forall t \in [t_0, t_0 + \eta[,$$
$$\int_{t_0}^t \Lambda(s, x(s), u(s), \psi_\epsilon[u](s))\, ds \geq (t - t_0)\epsilon/2.$$

Proposition 2.4. Under Hypotheses (H1)–(H3) and (H5), the following results hold:

(i) If the value function V_+ [resp. V_-] is upper semicontinuous on $\mathbb{R}^+ \times \mathbb{R}^n$, then it is a viscosity subsolution of (9) with H_+^* [resp. H_-^*].
(ii) If the value function V_+ [resp. V_-] is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$, then it is a continuous viscosity solution of (9) with H_+^* [resp. H_-^*].

Proof. Let $(t_0, x_0) \notin \mathcal{T}_+$. Thanks to the upper semicontinuity of V_+, the set $\{(t, x) \mid V(t, x) < M(t, x)\}$ is open, and so,

$$\exists \eta^1 > 0 \text{ s.t.}, \forall t \in [t_0, t_0 + \eta^1[, \forall (u, v) \in \mathcal{U} \times \mathcal{V}, (t, x(t)) \notin \mathcal{T}_+.$$

We can then apply the optimality principle (Proposition 2.1) on the interval $[t_0, t_0 + \eta_1[$,

$$V_+(t_0, x_0) = \sup_{\psi \in \Gamma} \inf_{u \in \mathcal{U}} \left\{ \int_{t_0}^t L(s, x(s), u(s), \psi[u](s))\, ds + V_+(t, x(t)) \right\},$$
$$\forall t \in [t_0, t_0 + \eta^1[. \qquad (10)$$

Let $W \in C^1$ such that (t_0, x_0) is a local maximum of $V_+ - W$. Without loss of generality, we can suppose that $W(t_0, x_0) = V_+(t_0, x_0)$. Then, locally

$W \geq V_+$. From (10), we deduce that

$$\sup_{\psi \in \Gamma} \inf_{u \in \mathscr{U}} \left\{ \int_{t_0}^{t} L(s, x(s), u(s), \psi[u](s))\, ds + W(t, x(t)) - W(t_0, x_0) \right\} \geq 0. \quad (11)$$

On the other hand, if

$$W_t(t_0, x_0) + H_+^*(t_0, x_0, W_x(t_0, x_0)) = -\epsilon < 0,$$

we shall have, according to Lemma 2.2,

$$\exists \eta^2 > 0,\ \exists u_\epsilon \in \mathscr{U} \text{ s.t. } \int_{t_0}^{t} \Lambda(s, x(s), u(s), \psi[u_\epsilon](s))\, ds \leq -(t - t_0)\epsilon/2,$$

$$\forall \psi \in \Gamma,\ \forall t \in [t_0, t_0 + \eta^2[.$$

Let

$$\eta = \min(\eta^1, \eta^2);$$

then

$$\begin{aligned} &\sup_{\psi \in \Gamma} \inf_{u \in \mathscr{U}} \int_{t_0}^{t_0 + \eta} W_t(s, x(s)) + W_x(s, x(s)) \cdot f(s, x(s), u(s), \psi[u](s)) \\ &\qquad\qquad + L(s, x(s), u(s), \psi[u](s))\, ds \\ &= \sup_{\psi \in \Gamma} \inf_{u \in \mathscr{U}} \left\{ \int_{t_0}^{t_0 + \eta} L(s, x(s), u(s), \psi[u](s))\, ds \right. \\ &\qquad \left. + W(t_0 + \eta, x(t_0 + \eta)) - W(t_0, x_0) \right\} \\ &\leq -\eta\epsilon/2 < 0, \end{aligned}$$

which contradicts (11), and so,

$$W_t(t_0, x_0) + H_+^*(t_0, x_0, W_x(t_0, x_0)) \geq 0.$$

If $W \in C^1$ is such that (t_0, x_0) is a local minimum of $V_+ - W$ (this requires the lower semicontinuity of V_+), we proceed in the same way, but we still need the upper semicontinuity of V_+, to show that

$$W_t(t_0, x_0) + H_-^*(t_0, x_0, W_x(t_0, x_0)) \leq 0.$$

Now, if the initial condition is such that $V_+(t_0, x_0) = M(t_0, x_0)$ and $W \in C^1$ admits a local maximum of $V_+ - W$ at (t_0, x_0), we use Lemma 2.1 to obtain the inequality

$$\sup_{\psi\in\Gamma}\inf_{u\in\mathcal{U}}\left\{\int_{t_0}^{t} L(\tau, x(\tau), u(s\tau), \psi[u](\tau))\, ds + W(t, x(t)) - W(t_0, x_0)\right\}\geq 0,$$

$$\forall t\geq t_0,$$

and we continue the proof using Lemma 2.2. If $W\in C^1$ admits a local minimum of $V_+ - W$ at (t_0, x_0), there is nothing to prove. The proof for V_- is similar. □

Corollary 2.2. Under Hypotheses (H1)–(H5), the value functions V_+ [resp. V_-] are continuous viscosity solutions of (9) with H_+^* [resp. H_-^*].

2.5. Existence of a Value.

Proposition 2.5. Under Hypotheses (H1)–(H4), there exists a unique bounded below uniformly continuous viscosity solution of (9) for the Hamiltonian H_+^* [resp. H_-^*] on $\mathbb{R}^+\times\mathbb{R}^n$.

Proof. Take w a bounded below uniformly continuous viscosity solution, and apply the Kruzkov transformation

$$\bar{w}(t, x) = -\exp(-w(t, x));$$

$\bar{w}$ is then a bounded uniformly continuous (BUC) viscosity solution of the new partial differential equation

$$\begin{aligned} g(t, x, \bar{w}, \nabla\bar{w}) = \max\Big\{&\bar{w} + \exp(-M),\\ &-\bar{w}_t + \max_{v\in V}\min_{u\in U} -\bar{w}_x f(t, x, u, v) + \bar{w}L(t, x, u, v)\Big\}\\ &\times\Big[\text{resp. } \max\Big\{\bar{w} + \exp(-M),\\ &-\bar{w}_t + \min_{u\in U}\max_{v\in V} - \bar{w}_x f(t, x, u, v) + \bar{w}L(t, x, u, v)\Big\}\Big] = 0. \end{aligned}$$

To apply the uniqueness result of BUC viscosity solution (see Theorem 2.11, p. 49 in Ref. 12) with the augmented state $\xi=(t, x)\in\mathbb{R}^+\times\mathbb{R}^n$, we check that:

(i) g is uniformly continuous on bounded sets, thanks to (H1)–(H3).
(ii) $\exists C>0$ such that $|g_\xi|\leq C(1+|p|)$, thanks to (H1)–(H3).
(iii) $\exists\gamma$ such that $g_{\bar{w}}\geq\gamma>0$. This last property is due to (H4). □

Corollary 2.3. Under Hypotheses (H1)–(H5), (H1b), (H5b), and when the following Isaacs condition holds:

$$H_-^*(t, x, \lambda)=H_+^*(t, x, \lambda), \qquad \forall(t, x, \lambda)\in\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n, \tag{12}$$

the game has a value; i.e., $V_-=V_+$

Let us stress the importance of Hypothesis (H4). Even when the value function turns out to be continuous, there might not exist a unique BUC viscosity solution, as we shall illustrate now.

Example 2.1. We consider a simple system in the plane, autonomous without controls,

$$\dot{x}=-y,$$

$$\dot{y}=x,$$

with

$$L=0, \qquad M(x, y)=\min(0, x^2+y^2-1).$$

Thanks to the dynamics, the function $t\to x^2(t)+y^2(t)$ is constant along any trajectory for any initial condition. So, the value function is obviously

$$V(x, y)=M(x, y).$$

The associated variational inequality is

$$\min\{-yV_x+xV_y, M(x, y)-V(x, y)\}=0,$$

but one can check that

$$V_n(x, y)=\min(0, (x^2+y^2)^n-1)$$

are also BUC viscosity solutions, for any $n \geq 1$:

(i) $\forall (x, y)$, $V_n(x, y) \leq M(x, y)$.

(ii) V_n is not differentiable exactly when $x^2 + y^2 = 1$, but its Fréchet subdifferentials are

$$\partial^+ V_n(x, y) = 2n[0, 1]\begin{bmatrix} x \\ y \end{bmatrix}, \qquad \partial^- V_n(x, y) = \varnothing,$$

and the viscosity condition is fulfilled:

$$\forall (\rho_x, \rho_y) \in \partial^+ V_n(x, y), \qquad -y\rho_x + x\rho_y \geq 0.$$

(iii) When $x^2 + y^2 \neq 1$, $\partial_x V_n \dot{x} + \partial_y V_n \dot{y} = 0$.

3. Characterization of Pursuit-Evasion Barriers

Now, we drop the integral cost in the criterion and consider time-independent functions M. We shall also assume that the dynamics is separated, quite a usual assumption in pursuit-evasion games:

$$\forall (t, x, u, v) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathsf{U} \times \mathsf{V},$$

$$f(t, x, u, v) = f_P(t, x, u) + f_E(t, x, v).$$

Then, the Isaacs condition (12) is fulfilled. Note that we can also require Hypothesis (H1b) to be fulfilled without loss of generality. If the dynamics is not autonomous, we consider the time t as part of the new state $z = (t, x)$,

$$\dot{z} = \bar{f}(z, u, v) = \begin{bmatrix} 1 \\ f(t, x, u, v) \end{bmatrix}.$$

Then, for an arbitrary $\rho > 0$, we consider the dynamics

$$\tilde{f}(z, u, v) = \begin{cases} \bar{f}(z, u, v), & \text{if } |\bar{f}(z, u, u)| \leq \rho, \\ \rho\bar{f}(z, u, v)/|\bar{f}(z, u, v)|, & \text{if } |\bar{f}(z, u, v)| > \rho. \end{cases}$$

It is clear that replacing f by $\tilde{f}$ does not modify the cost J.

In order to deal with an infinite horizon without imposing any a priori bound on the capture time, we shall require an additional property of the function M with respect to the trajectories, when the time tends toward infinity:

(H6) For all initial conditions $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ and for all strategies $\phi \in \Delta$,

$$\forall \eta > 0, \exists T < +\infty \text{ s.t., } \forall v \in \mathscr{V}, M(x(t)) \geq Mx(T)) - \eta, \forall t > T,$$

where $x(\cdot)$ is solution of (1) with initial condition (t_0, x_0) and controls $(\phi[v], v)$.

Note that this assumption is less restrictive than the requirement of a uniformly bounded terminal time, as the following example illustrates it.

Example 3.1. Consider the following dynamical system in the plane:

$$\dot{x} = 1 - |y|,$$

$$\dot{y} = (1 - |y|)(1 + u + v)/2,$$

with controls $u, v \in [0, 1]$. The target is the segment

$$\mathscr{T} = \{(0, y) | \, |y| \leq 1\}.$$

Take $M(\cdot) = d^o(\cdot, \mathscr{T})$,

$$M(x, y) = \begin{vmatrix} \sqrt{x^2 + (|y| - 1)^2}, & \text{if } |y| \geq 1, \\ |x|, & \text{if } |y| < 1; \end{vmatrix}$$

one can then easily check that Hypothesis (H6) is fulfilled and that the value function is

$$V(x, y) = \begin{cases} (y - x - 1)/\sqrt{2}, & \text{if } y > \max(1 + x, 1 - x), \\ 0, & \text{if } -1 \leq y \leq 1 + x \text{ and } x \leq 0, \\ y - x - 1, & \text{if } y \geq 1 + x \text{ and } |y| \leq 1, \\ x - y + 1, & \text{if } y \leq 1 + x \text{ and } |y| \geq 1, \\ (x - y - 1)/\sqrt{2}, & \text{if } -1 - x \leq y \leq -1, \\ M(x, y), & \text{otherwise.} \end{cases}$$

Then, from the initial condition $(-1, 0)$ and the control laws $u^* = 0$, $v^* = 1$, which are optimal, the target is reached in infinite time.

Without loss of generality, for the characterization of barriers, we can also require Hypotheses (H5) and (H5b) to be fulfilled. Take

$$M(x) = \tilde{d}^o(x, \mathscr{T}) = \max\{\min(d^o(x, \mathscr{T}), R), -R\},$$

where R is any fixed positive number.

Proposition 3.1. Under Hypotheses (H1) and (H6), the game admits an upper semicontinuous value, which is the largest viscosity subsolution V of

$$\min\{\tilde{d}^o(x, \mathcal{T}) - V(t, x), \\ V_t(t, x) + \max_{v \in V} \min_{u \in U} V_x(t, x) \tilde{f}(t, x, u, v)\} = 0, \tag{13}$$

$$[\text{or } \min\{\tilde{d}^o(x, \mathcal{T}) - V(t, x), \\ V_t(t, x) + \min_{u \in U} \max_{v \in V} V_x(t, x) \tilde{f}(t, x, u, v)\} = 0], \tag{14}$$

$\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$, and the barrier is the set

$$\mathcal{B} = \partial\{(\tau, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n \mid V(\tau, \xi) \geqslant 0\}.$$

Furthermore, V can be approximated by a decreasing sequence of continuous functions V^ϵ, which are the value functions of the game with an additional integral term $L = \epsilon$.

Proof. We choose $M(\cdot) = \tilde{d}^o(\cdot, \mathcal{T})$, as explained before. Hypotheses (H2), (H5), (H5b) are then fulfilled. For the criterion

$$J^\epsilon(t_0, x_0, u) = \text{ess} \inf_{t \geq t_0} \left\{ M(x(t)) + \int_{t_0}^{t} \epsilon \, d\tau \right\}, \qquad \epsilon > 0,$$

Hypotheses (H3) and (H4) are also valid, and we can apply Proposition 2.5. So, there exist unique continuous viscosity solutions V_+^ϵ and V_-^ϵ, and furthermore the ϵ-game admits a value $V^\epsilon = V_+^\epsilon = V_-^\epsilon$, thanks to the Isaacs condition; see Corollary 2.3.

It is clear that, as ϵ tends toward 0, V^ϵ is a nonincreasing sequence of continuous functions, bounded below by M_0. So, V^ϵ converges toward an upper semicontinuous function $\bar{V}$. Necessarily,

$$\forall (t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad V_-^0(t_0, x_0),\ V_+^0(t_0, x_0) \leq \bar{V}(t_0, x_0).$$

We are going now to prove that

$$\bar{V}(t_0, x_0) = V_-^0(t_0, x_0) = V_+^0(t_0, x_0).$$

Let us first notice that, for a given trajectory $x(\cdot)$ and a fixed $T \in (t_0, \infty)$, one has

$$\inf_{t \in [t_0, T]} M(x(t)) \geq \inf_{t \geq t_0} \{M(x(t)) + \epsilon(t - t_0)\} - \eta/2, \tag{15}$$

on the condition that

$$\epsilon(T-t_0)\leq \eta/2.$$

Fix an initial condition (t_0, x_0). If $M(x_0)=M_0$, there is nothing to prove. Let $\eta>0$, and consider a strategy $\bar{\phi}\in\Delta$ such that

$$\forall v\in\mathcal{V}, \qquad J^0(t_0, x_0, \bar{\phi}[v], v)\leq V^0_-(t_0, x_0)+\eta/2.$$

Then, according to Hypothesis (H6), there exists $T<+\infty$ such that

$$\forall v\in\mathcal{V}, \qquad \inf_{t\geq T} M(x(t))\geq M(x(T))-\eta/4.$$

But, for any $v\in\mathcal{V}$,

$$\begin{aligned} M(x(T)) &\geq \inf_{t\in[t0,T]} M(x(t)) \\ &\geq \inf_{t\in[t_0,T]} \{M(x(t))+\epsilon(t-t_0)\}-\eta/4, \end{aligned}$$

on the condition that

$$\epsilon(T-t_0)\leq \eta/4.$$

Then, using the property (15), we have

$$\begin{aligned} \forall v\in\mathcal{V}, \quad J^0(t_0, x_0, \bar{\phi}[v], v) &= \min\left\{\inf_{t\in[t_0,T]} M(x(t)), \inf_{t\geq T} M(x(t))\right\} \\ &\geq J^\epsilon(t_0, x_0, \bar{\phi}[v], v)-\eta/2. \end{aligned}$$

So,

$$\begin{aligned} V^0_-(t_0, x_0) &\geq \sup_{v\in\mathcal{V}} J^0(t_0, x_0, \bar{\phi}[v], v)-\eta/2 \\ &\geq \sup_{v\in\mathcal{V}} J^\epsilon(t_0, x_0, \bar{\phi}[v], v)-\eta \\ &\geq V^\epsilon(t_0, x_0)-\eta \\ &\geq \bar{V}(t_0, x_0)-\eta, \end{aligned}$$

for any $\eta>0$ and then

$$V^0_-(t_0, x_0)=\bar{V}(t_0, x_0).$$

For a given $\epsilon>0$, consider now a strategy $\bar{\psi}\in\Gamma$ such that

$$\forall u\in\mathcal{U}, \qquad \bar{V}(t_0, x_0)\leq V^\epsilon(t_0, x_0)\leq J^\epsilon(t_0, x_0, u, \bar{\psi}[u])+\eta/2.$$

Then, from property (15), one has

$$\forall u\in\mathcal{U}, \qquad \bar{V}(t_0, x_0)\leq \inf_{t\in[t_0,T]} M(x(t))+\eta,$$

for any

$$T \le t_0 + \eta/(2\epsilon).$$

Let now ϵ tend toward 0,

$$\forall u \in \mathscr{U}, \qquad \bar{V}(t_0, x_0) \le J^0(t_0, x_0, u, \bar{\psi}[u]) + \eta,$$

and we obtain that

$$\bar{V}(t_0, x_0) \le V_+^0(t_0, x_0) + \eta.$$

So,

$$V_+^0(t_0, x_0) = \bar{V}(t_0, x_0) = V_-^0(t_0, x_0)$$

is upper semicontinuous and, according to Proposition 2.4, it is also a viscosity subsolution of the variational inequality (13).

If H_ϵ^* denotes the Hamiltonian for the criterion J^ϵ, we have

$$\forall (t, x, p) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \qquad H_\epsilon^*(t, x, p) = H_0^*(t, x, p) + \epsilon,$$

so any upper-semicontinuous viscosity subsolution W of (13) is also a viscosity subsolution of (9) with H_ϵ^*. As in the proof of the Proposition 2.5, we use the Kruzkov transformation and a comparison theorem, but this time for semicontinuous bounded solutions [see Theorem 4.3, p. 93 in Ref. 12; the conditions (H1)–(H5) are fulfilled when $\epsilon > 0$], to conclude that

$$\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \qquad W(t, x) \le V^\epsilon(t, x),$$

and passing to the limit when $\epsilon \to 0$, we obtain that V^0 is the largest upper semicontinuous subsolution of (13).

Then, the barrier is the set of positions where the upper-semicontinuous value function jumps between positive and negative values,

$$\mathscr{B} = \{\xi \in \mathbb{R}^+ \times \mathbb{R}^n \mid \liminf_{\zeta \to \xi} V^0(\zeta) \le 0 \text{ and } V^0(\xi) \ge 0\} = \partial\{\xi \mid V^0(\xi) \geqslant 0\}. \qquad \square$$

Hypothesis (H6) requires a certain behavior of the trajectories which cannot always be checked in a straightforward way on the data of the problem. Let us give now some stronger conditions that might be easier to check.

Lemma 3.1. If for any initial condition (t_0, x_0) and $(u, v) \in \mathscr{U} \times \mathscr{V}$, the function $t \mapsto M(x(t))$ is convex and bounded below for $t \in [t_0, +\infty)$, then (H6) is fulfilled.

Proof. Fix (t_0, x_0) and write $x_{u,v}(\cdot)$ for the trajectory solution of (1) when the controls u, v are used. Take $\phi \in \Delta$, $v \in \mathscr{V}$, and consider

$$\mu_v(t) = \min\left\{ \inf_{s \le t} M(x_{\phi[v],v}(s)),\ M(x_{\phi[v],v}(t)) \right\}, \qquad t \in [t_0, +\infty);$$

μ_v are convex, nonincreasing, and bounded functions. Then,

$$\mathscr{M}(t) = \sup_{v \in \mathscr{V}} \left\{ \mu_v(t) - \inf_{s \ge t_0} \mu_v(s) \right\}$$

is also convex, nonincreasing, and tends toward 0 when the time tends towards infinity. So, for a given $\eta > 0$, there exists $T > t_0$ such that $\mathscr{M}(T) < \eta$. If

$$\mu_v(T) = M(x_{\phi[v],v}(T)),$$

then

$$M(x_{\phi[v],v}(t)) \ge \mu_v(t) \ge \mu_v(T) - \eta, \qquad t > T,$$

and (H6) is fulfilled. Otherwise, due to convexity, the function $t \mapsto M(x_{\phi[v],v}(t))$ is necessarily nonincreasing for times larger than T and (H6) is also fulfilled. □

For instance, consider the second-order system:

$$\dot{x} = y,$$

$$\dot{y} = g(x, y, u, v),$$

with $x \mapsto M(x)$ twice differentiable and bounded below. A sufficient condition to ensure the convexity of $t \to M(x(t))$, uniformly with respect to (u, v), is to have

$$d^2M(x(t))/dt^2 = y' M_{xx}(x) y + M_x(x) g(x, y, u, v) \ge 0, \qquad \forall (x, y, u, v).$$

4. Target Contact Conditions

In this section, we consider games with separated and autonomous dynamics, $M = \vec{d}(\cdot, \mathscr{T})$, and $L = 0$; see Section 3. Isaacs introduced in Ref. 1 the semipermeability property that a barrier must satisfy in the smooth case (later generalized by Lewin in Ref. 13 for Lipschitz-continuous manifolds),

$$\min_{u\in U}\max_{v\in V}\vec{n}(x)\cdot f(x,u,v)=\max_{v\in V}\min_{u\in U}\vec{n}(x)\cdot f(x,u,v)=0,$$

where $\vec{n}(x)$ stands for the outer normal to the barrier at x. Outside the target, it is straightforward to check that this condition is equivalent to our Equation (13) at points x such that $V(x)=0$ and V is differentiable ($\vec{n}(x)=V_x(x)$).

To characterize the barrier among all semipermeable surfaces, Isaacs derived some necessary conditions for the contact of the barrier with the target, which he called the limit of the usable part (LUP). We show that the variational inequality (13) is a straightforward way to find these conditions and also to derive second-order ones, accidentally discovered with the isotropic rocket game; see Example 8.3 in Ref. 1.

Proposition 4.1. If the autonomous variational inequality (13) admits a solution V C^1 at a point ξ belonging to $\partial\mathscr{T}$, then we have the following properties:

(i) $V(\xi)=0\Rightarrow V_\xi(\xi)\in\partial^- M(\xi)$.

(ii) $H^*(\xi, V_\xi)>0\Rightarrow V(\xi)=0$. The set of such points is called by Isaacs the nonusable part of the boundary of the target (NUP).

(iii) If ξ is the first contact with the target of a trajectory on the barrier, then

$$\exists p\in\partial M^-(\xi)|p'f^*(\xi, V_\xi(\xi))=0,$$

where $f^*(x,\lambda)=f(x,\bar{\phi}(x,\lambda),\bar{\psi}(x,\lambda))$ and

$$\bar{\phi}(x,\lambda)\in\arg\min_{u\in U}H(x,\lambda,u,\bar{\psi}(x,\lambda)),$$

$$\bar{\psi}(x,\lambda)\in\arg\max_{v\in V}H(x,\lambda,\bar{\phi}(x,\lambda),v).$$

This set of points is called by Isaacs the limit of the usable part of the boundary of the target (LUP).

(iv) If M is C^2 and ξ is the first contact with the target of a trajectory on the barrier, a necessary condition to have $V(\xi)=0$ is that

$$H^*(\xi, M_\xi)=0,$$

$$f^{*\prime}M_{\xi\xi}f^*+M_\xi df^*/dt\geq 0.$$

The set of such points represents the usable part of the LUP [UP(LUP)].

(v) If M and V are C^2, and if the optimal feedbacks $(\phi^*(\xi), \psi^*(\xi))$ are unique, then the points ξ of $\partial\mathcal{T}$ such that

$$V(\xi)=0,$$

$$H^*(\xi, M_\xi)=0,$$

$$f^{*\prime} M_{\xi\xi} f^* + M_\xi df^*/dt = 0$$

are points of the LUP such that $f^*(\xi, M_\xi)$ is tangent to the LUP.

Proof.

(i) When $V(\xi)=M(\xi)=0$, then ξ is a local minimum of $M-V$, by Proposition 2.3, and a necessary condition (see nonsmooth rules calculus, for instance in Ref. 14) is that

$$0\in\partial^-(M-V)(\xi)\Rightarrow V_\xi(\xi)\in\partial^- M(\xi).$$

(ii) By contraposition of Proposition 2.3, we have

$$H^*(\xi, V_\xi)>0\Rightarrow V(\xi)=M(\xi)=0.$$

When M is C^1, we rediscover the concept of nonusable part of the boundary of the target, introduced by Isaacs,

$$\mathrm{NUP}(\mathcal{T})=\{\xi\in\partial\mathcal{T}\,|\,H^*(\xi, M_\xi(\xi))>0\},$$

which is the set of positions from which the evader can force the dynamics to be outer of the target.

(iii) From Proposition 2.3 with $L=0$, when V is C^1, the iso-V manifolds are loci of trajectories with the dynamics

$$\dot{\xi}=f^*(\xi, V_\xi)=f(\xi, \bar{\phi}(\xi, V_\xi), \bar{\psi}(\xi, V_\xi)),$$

until

$$t^*=\inf\{t\,|\,V(\xi(t))=M(\xi(t))\}.$$

For $t\le t^*$, H^* is always equal to zero (see Proposition 2.3), and we rediscover the semipermeability condition for a hypersurface with an outer normal V_ξ,

$$H^*(\xi, V_\xi)=\min_{u\in\mathrm{U}}\max_{v\in\mathrm{V}} V_\xi' f(\xi, u, v)=\max_{v\in\mathrm{V}}\min_{u\in\mathrm{U}} V_\xi' f(\xi, u, v)=0.$$

If ξ is a contact point with the target of a trajectory $x(\cdot)$ on the barrier, then t^* is a minimum of the function $g: t\mapsto M(x(t))$, and so,

$$0\in\partial^- g(t_f)\Rightarrow 0\in\{p'f^*(\xi, V_\xi(\xi))|p\in\partial^- M(\xi)\}.$$

(iv) If M is C^2, a second-order necessary condition for a local minimum is that

$$d^2g/dt^2(t_f) = f^{*\prime}M_{\xi\xi}f^* + M_\xi df^*/dt \geq 0.$$

Calling $\nu = M_\xi^t/|M_\xi|$ the unitary outer normal of the target, when $f^* \neq 0$, we can rewrite this last inequality as follows:

$$(f^{*\prime}[d\nu/d\xi]f^*)/|f^*|^2 \geq -(\nu'[df^*/dt])/|f^*|^2,$$

and recognize the left term as the curvature of the section of $\partial\mathcal{T}$ in the hyperplane (f^*, ν) and the right term as the curvature of the trajectory of $\mathcal{B}$ passing by ξ. When the inequality is strict, the trajectory does not stay upon the target because t^* is the unique local minimum of g.

(v) When M and V are C^2, a second-order sufficient condition for a local minimum of $M - V$ is that

$$M_{\xi\xi} - V_{\xi\xi} > 0,$$

which implies that

$$f^{*\prime}M_{\xi\xi}f^* + M_\xi[df^*/dt] = (\partial[M_\xi f^*]/\partial\xi)f^* \geq (\partial[V_\xi f^*]/\partial\xi)f^*.$$

If $D_\xi H^*(h)$ stands for the directional derivative of H^* in the direction h, the Danskin theorem (see for instance Corollary 2, p. 87 of Ref. 14) brings

$$D_\xi H^*(\xi, V_\xi)(h) = \min_{\bar\phi \in \bar\Phi} \max_{\bar\psi \in \bar\Psi} D_\xi(\xi, H\bar\phi, \bar\psi)(h) = 0, \qquad \forall h \in \mathbb{R}^n,$$

where

$$\bar\Phi(x, \lambda) = \bigcup_{\bar\psi \in \bar\Psi(x, \lambda)} \arg\min_{u \in \mathrm{U}} H(x, \lambda, u, \bar\psi),$$

$$\bar\Psi(x, \lambda) = \bigcup_{\bar\phi \in \bar\Phi(x, \lambda)} \arg\max_{v \in \mathrm{V}} H(x, \lambda, \bar\phi, v).$$

When the optimal controls $(\bar\phi, \bar\psi)$ are unique, H^* is differentiable and we deduce that

$$\begin{aligned} dH^*(\xi, V_\xi)/dt &= (dH^*(\xi, V_\xi)/d\xi)f^* \\ &= [\partial H/\partial\xi(\xi, V_\xi, \bar\phi, \bar\psi)]f^* = 0. \end{aligned}$$

The equality occurs for positions ξ such that

$$M_\xi f^* = 0,$$

$$(\partial[H^*(\xi, M_\xi)]/\partial\xi)f^* = 0,$$

which is equivalent to saying that f^* is tangent to the LUP,

$$M(\xi) = 0,$$

$$H^*(\xi, M_\xi(\xi)) = 0.$$

Notice that, at such positions, a trajectory of the barrier is able to stay on the NUP, when there is no uniqueness of local minima of g in t^*. □

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