

Algebraic Solutions of n^{th} Order Linear
Differential Equations

by

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§1. Introduction

Let K be a number field and

$$L = \frac{d^n}{dx^n} + a_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_0(x) \quad (1.1)$$

a linear differential operator with coefficients in $K(x)$. In this paper, we present a procedure to decide in a finite number of steps if all solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$, and, if this is the case, how to produce a basis for the solution space.

When $n = 2$, this problem was considered by Schwartz, Klein, Fuchs and others, yet none of these mathematicians seems to have presented a complete decision procedure.

In [2], Baldassarri and Dwork, building on the work of Klein, give such a procedure. Their method is based on the fact that the only finite subgroups of $\mathrm{PGL}(2, \mathbb{C})$ ($=\mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$) are either cyclic, dihedral, tetrahedral, octahedral or icosahedral and that one can decide, for each of these five types, if a given 2^{nd} order linear operator L has a projectivized monodromy group of that type. L can be assumed to be of Fuchsian type, in which case having a finite monodromy group is equivalent to all solutions of $Ly = 0$ being algebraic over $\mathbb{C}(x)$.

Our algorithm also relies on group theoretic information. We use the following fact due to Jordan: given a finite subgroup G of $\mathrm{GL}(n, \mathbb{C})$, there exists an abelian normal subgroup H of G of index $\leq (49n)^{n^2}$. We use this to show that if all solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$, then for some such solution y , $u = y'/y$ is of degree at most $(49n)^{n^2}$ over

$\mathbb{C}(x)$. The element u satisfies a nonlinear differential equation of order $n-1$ called the Riccati equation. We then deal with the following two problems:

1) Let L be as in (1.1). Decide if the associated Riccati equation has a solution algebraic over $\mathbb{C}(x)$ of degree $\leq (49n)^{n^2}$

2) Given u , algebraic over $\mathbb{C}(x)$, decide if $y'/y = u$ has a solution y algebraic over $\mathbb{C}(x)$.

Problem 2 has been referred to, [1], as the Problem of Abel. A decision procedure to solve this problem was first given by Risch [13] and independently by Baldassarri and Dwork [2]. To solve problem 1, we show that if

$$Y^m + a_{m-1}(x)Y^{m-1} + \dots + a_0(x), \quad a_i(x) \in \mathbb{C}(x)$$

is the minimum polynomial of a solution of the Riccati equation, then we can bound the degrees of the numerators and denominators of the a_i 's. This allows us to reduce problem 1 to a question in classical elimination theory. If L is an irreducible operator (i.e. L cannot be written as the composite of two operators of smaller order) and the associated Riccati equation has a solution u , algebraic over $\mathbb{C}(x)$ such that $y'/y = u$ has a solution also algebraic over $\mathbb{C}(x)$, then all solutions of $Ly = 0$ are algebraic. We give an algorithm to decide if L is irreducible. If it is not, we show how to find a factorization and show how we can reduce the problem to the previous case.

1878

Jordan proves his group theoretic theorem in a paper [5] dealing with algebraic solutions of linear differential equations. He concludes from this theorem that if all solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$, then there exists a homogeneous form $\varphi(Y_1, \dots, Y_n)$ of degree N such that $\varphi(y_1, \dots, y_n)$, where y_1, \dots, y_n is a basis for the solution space of $Ly = 0$, is the radical of a rational function. Furthermore N depends only on n and not on the operator L . Jordan does not show (nor do we see) how this criterion can be used to give a decision procedure. 1887

After the main body of this paper was completed, we found out that our algorithm was essentially discovered by Painlevé in [9]. Painlevé's ideas are most clearly presented by his student Boulanger ([1], p. 92-95). They restricted their attention to the case where $n=3$, L an irreducible operator, and reduced the problem to the Problem of Abel. Painlevé's method can clearly be generalized to the case of general n . The methods for reducing the case of L reducible to the case where one assumes L irreducible were known in the 19th Century and we feel that, had they known the solution of the Problem of Abel, Painlevé and Boulanger could have presented a complete solution of the problem.

We present our version of this algorithm in section 3 of this paper. In section 2, we review some basic facts concerning linear differential operators and lay the groundwork for the algorithm. All relevant definitions have been included, but some proofs have been omitted when they appear in the modern literature. In particular, we have omitted an exposition of

the solution of the Problem of Abel and refer the reader to [2] or [13] for treatments of this. We have tried to found our algorithm on basic principles but where ad hoc techniques suffice (as in the case of Boulanger's version of Proposition 3.1) we have outlined these. Finally, we note that we can weaken our assumption that K is a number field. It is only necessary to assume that K is a constructable field of characteristic zero with a splitting algorithm, as in [11].

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§2. Preliminaries

A) Regular Singular Points: We shall review some standard facts about linear differential equations with regular singular points. The general reference for this section is [10].

Let

$$L = \frac{d^n}{dx^n} + p_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + p_0(x) \quad (2.1)$$

be a linear differential operator with $a_i(x) \in \mathbb{C}(x)$.

Let a be a point on the Riemann Sphere, S^2 . We say that a is a Regular Point of (2.1) if, given any angular sector \mathcal{J} at a and any solution $y(x)$ of $Ly = 0$ which is holomorphic in \mathcal{J} , we have, for some natural number N ,

$$\lim_{x \rightarrow a} (x-a)^N y(x) = 0 \quad \text{if } a \text{ is finite} \quad (2.2)$$

$$\lim_{x \rightarrow \infty} x^{-N} y(x) = 0 \quad \text{if } a = \infty .$$

Fuchs showed that this analytic property is equivalent to an algebraic property.

Criterion of Fuchs ([10], p. 55): Let a be a point on the Riemann Sphere. If a is a finite point, a is a regular

point of (2.1) iff $(x-a)^i p_{n-i}(x)$ is regular at $x = a$ for
 $i = 0, \dots, n-1$. Infinity is a regular point iff $x^i p_{n-i}(x)$
 is regular at $x = \infty$ for $i = 0, \dots, n-1$.

The criterion of Fuchs allows us to effectively check if a point
 is a regular point.

A finite point a on the Riemann Sphere is called a singular point
 of (2.1) if, for some i , $p_i(x)$ has a pole at $x = a$. Infinity is called a
 singular point of (2.1) if, after we make the change of variable $x = \frac{1}{t}$,
 the new operator we get has $t = 0$ as a singular point. A point which is
 not a singular point is said to be an ordinary point. Note that all ordinary
 points are regular. If (2.1) has only regular singular points, we say it is of
Fuchsian type. Noting that functions algebraic over $\mathbb{C}(x)$ satisfy the
 conditions (2.2) at all points on the Riemann Sphere we have the following
 proposition:

Proposition 2.1: Let L be as in (2.1). If $Ly = 0$ has n linearly independent
 solutions, each algebraic over $\mathbb{C}(x)$, then L must be of Fuchsian type.

Given a finite point $x = a$ on the Riemann Sphere, we shall be
 interested in determining if $Ly = 0$, L of Fuchsian type, has a solution
 of the form

$$y(x) = (x-a)^b \sum_{i=0}^{\infty} c_i (x-a)^i \quad (2.3)$$

with b and the c_i 's in \mathbb{C} , $c_0 \neq 0$. If we formally substitute (2.3) for y in $Ly = 0$ we see that the lowest power of $(x-a)$ appearing in the resulting expression is $(x-a)^{b-n}$. The coefficient of this term will be c_0 times a polynomial $I_a(b)$ in powers of b , of degree n , whose coefficients are determined by L . This polynomial is called the indicial polynomial of (2.1) at $x = a$ ([10], p. 62), and we see that a necessary condition that $Ly = 0$ have a solution of the form (2.3) is that b satisfy $I_a(b) = 0$. The roots of I_a are called the exponents of (2.1) at $x = a$. If we transform (2.1) via the substitution $x = \frac{1}{t}$, we can find a polynomial $I_\infty(b)$ such that if the new differential equation has a solution of the form

$$y(t) = t^b \sum_{i=0}^{\infty} c_i t^i, \quad c_0 \neq 0,$$

then $I_\infty(b) = 0$. I_∞ is called the indicial polynomial of (2.1) at infinity and its roots are called the exponents of (2.1) at infinity.

We shall need the following proposition in later sections:

Proposition 2.2: Let L be a linear differential operator of Fuchsian type with singular points $\{a_1, \dots, a_m\}$. If $y(x)$ is a solution of $Ly = 0$ such that $y'/y \in \mathbb{C}(x)$, then y is of the form

Hyperexponential solution

$$g(x) \prod_{i=1}^m (x-a_i)^{e_i}$$

where g is a polynomial. Furthermore, the degree of g and the e_i 's must come from a finite set of values, determined by the exponents of L at its singular points.

Proof. ([14], Sec. 178): Let $\frac{dy}{dx}/y = R(x) \in \mathbb{C}(x)$. Since L is of Fuchsian type, y must satisfy the appropriate growth condition at each point on the Riemann Sphere. This implies that $R(x)$ is of the form

$$\sum \frac{\beta_i}{x - \alpha_i},$$

or, equivalently,

$$y(x) = c \prod (x - \alpha_i)^{\beta_i}.$$

If α_i is not a singular point, $y(x)$ must be regular at α_i and so, in this case, β_i is a natural number. We can therefore write

$$y(x) = g(x) \prod_{i=1}^m (x - a_i)^{e_i},$$

where $g(x) \in \mathbb{C}[x]$.

We now see that for $i = 1, \dots, m$, e_i must be an exponent at a_i and $-(\deg g + \sum_{i=1}^m e_i)$ must be an exponent at infinity. ■

Using this proposition one can decide, for a linear operator L , of Fuchsian type with coefficients in $K(x)$, if $Ly = 0$ has a solution $y(x)$ with $\frac{dy}{dx}/y$ in $\mathbb{C}(x)$. Schlesinger ([14], Sec. 177) also shows how one can decide this question without assuming that L is of Fuchsian type.

B) Monodromy Groups: Let L be a linear differential operator with coefficients in $\mathbb{C}(x)$ and let $\{a_1, \dots, a_n\}$ be the singular points of L on the Riemann Sphere, S^2 . Let a_0 be a point in the complement of $\{a_1, \dots, a_n\}$ and let y_1, \dots, y_n be a basis for the solution space of $Ly = 0$. If γ is any path on $S^2 - \{a_1, \dots, a_n\}$, we can analytically continue each y_i around γ and get a new solution $\overline{y_i}$ of $Ly = 0$. Since each $\overline{y_i}$ must be a linear combination of the y_i 's, the map $(y_1, \dots, y_n) \mapsto (\overline{y_1}, \dots, \overline{y_n})$ is linear and is given by some matrix $A_\gamma \in GL(n, \mathbb{C})$. This matrix only depends on the homotopy class of γ and we get a map $\varphi : \Pi_1(S^2 - \{a_1, \dots, a_n\}) \rightarrow GL(n, \mathbb{C})$. The image of this map \mathcal{G} , is called the Monodromy Group of L . This image depends on a_0 and the choice of y_1, \dots, y_n but is determined up to conjugacy by L . If we let $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$ be the smallest field, closed under differentiation, containing $\mathbb{C}(x)$ and y_1, \dots, y_n , we see that \mathcal{G} acts as a group of automorphisms of $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$.

Proposition 2.3: Let L be a linear differential operator of Fuchsian type.

a) If $f \in \mathbb{C}\langle x, y_1, \dots, y_n \rangle$ is left fixed by the action of \mathcal{G} on $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$, then $f \in \mathbb{C}(x)$.

b) If each y_1, \dots, y_n is algebraic over $\mathbb{C}(x)$, then $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$ is a normal extension of $\mathbb{C}(x)$ and its galois group is isomorphic to the monodromy group of L .

Proof: a) $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$ is the field generated by x, y_1, \dots, y_n and all their derivatives. This observation allows us to conclude that any

function $f \in \mathbb{C}\langle x, y_1, \dots, y_n \rangle$ satisfies the growth condition (2.2). If f is left fixed by the action of \mathcal{G} , then f is single valued on $S^2 - \{a_1, \dots, a_m\}$ and furthermore, has, at worst, poles at a_1, \dots, a_m . Therefore $f \in \mathbb{C}(x)$.

b) We first observe that if each of the y_i are algebraic over $\mathbb{C}(x)$, then $\mathbb{C}\langle x, y_1, \dots, y_n \rangle = \mathbb{C}(x, y_1, \dots, y_n)$, that is, the field $\mathbb{C}(x, y_1, \dots, y_n)$ is already closed under differentiation. If $\overline{y_i}$ is a conjugate of y_i , this $\overline{y_i}$ will also satisfy $Ly = 0$ and so be a linear combination of y_1, \dots, y_n . Therefore $\mathbb{C}(x, y_1, \dots, y_n)$ is a normal extension of $\mathbb{C}(x)$. \mathcal{G} acts as a group of automorphisms of $\mathbb{C}(x, y_1, \dots, y_n)$ and so can be considered a subgroup of the galois group G . By a), \mathcal{G} and G have the same fixed field $\mathbb{C}(x)$, so $\mathcal{G} = G$. ■

We will need the following definition. Let $L(Y) = 0$ be an n^{th} order homogeneous linear differential equation. Let U be a new variable and set $Y' = UY$. If we continue to differentiate this relation we get: $Y'' = UY' + U'Y = U^2Y + U'Y$; $Y''' = 2UU'Y + U^2Y' + U''Y + U'Y' = 3UU'Y + U^3Y + U''Y$; etc. If we replace each $Y^{(i)}$ in $L(Y) = 0$ by the appropriate expression above, and divide by Y , we get a nonlinear differential equation of order $n-1$, $R(U) = 0$, which is called the Riccati equation associated with L . For $n = 2$, if $L = \frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x)$, then $R(U) = U' + p(x)U + U^2 + q(x)$. $R(U)$ has the following property: y is a nonzero solution of $L(Y) = 0$ iff $u = y'/y$ is a solution of $R(U) = 0$.

Hyperexponential solutions of L become rational solutions of R , and viceversa.

Jordan's Theorem ([5], [3], p. 98): Let G be a finite subgroup of $GL(n, \mathbb{C})$. Then G has a normal, abelian subgroup H with index $|G : H| \leq (49n)^{n^2}$.

Proposition 2.4: Let L be a linear differential operator of order n , with coefficients in $\mathbb{C}(x)$. If all solutions of $L(Y) = 0$ are algebraic over $\mathbb{C}(x)$, then the Riccati equation $R(U) = 0$ associated with L has a solution u , algebraic over $\mathbb{C}(x)$ such that $[\mathbb{C}(x, u) : \mathbb{C}(x)] \leq (49n)^{n^2}$.

Proof: Let y_1, \dots, y_n be a basis of the solution space of $L(Y) = 0$. By Proposition 2.3b, $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$ is a normal extension of $\mathbb{C}(x)$. Let G be its galois group, which we consider a subgroup of $GL(n, \mathbb{C})$. Jordan's Theorem then implies that G has an abelian, normal subgroup H with index $|G : H| \leq (49n)^{n^2}$. We can assume that y_1, \dots, y_n were chosen such that H consists of diagonal matrices. For any $\sigma \in H$, we have $\sigma(y_1'/y_1) = y_1'/y_1$. If F is the fixed field of H , we then have $u = y_1'/y_1 \in F$ and $[F : \mathbb{C}(x)] = |G : H| \leq (49n)^{n^2}$. u is the desired solution of the ^{Riccati} Riccati equation. ■

We will need one more proposition whose proof uses the monodromy group. A linear differential operator L , with coefficients in $\mathbb{C}(x)$ is said to be reducible, if there exist linear differential operators L_1 and L_2 , of lower order, with coefficients in $\mathbb{C}(x)$ such that $L = L_1 \circ L_2$. If L is not reducible, it is said to be irreducible.

Proposition 2.5: Let L be an irreducible linear differential operator with coefficients in $\mathbb{C}(x)$. If $Ly = 0$ has a solution which is algebraic over $\mathbb{C}(x)$, then all solutions are algebraic over $\mathbb{C}(x)$.

Proof: Let \mathcal{G} be the monodromy group of L and V the space of all solutions of $Ly = 0$ which are algebraic over $\mathbb{C}(x)$. If n is the order of L , we must show that $\dim V = n$. Assume that this is not the case, and that $\dim V = k < n$. Let y_1, \dots, y_k be a basis of V and let $L_1(Y) = \text{Wr}(Y, y_1, \dots, y_k) / \text{Wr}(y_1, \dots, y_k)$, where Wr denotes the wronskian determinant. We claim that L_1 has coefficients in $\mathbb{C}(x)$. For $\sigma \in \mathcal{G}$, let σ act on the coefficients of L_1 and call this new operator L_1^σ . $L_1^\sigma(Y) = \text{Wr}(Y, \sigma y_1, \dots, \sigma y_k) / \text{Wr}(\sigma y_1, \dots, \sigma y_k) = L_1(Y)$ since σ restricts to a linear transformation on V . Therefore the coefficients of L_1 are fixed by \mathcal{G} and Proposition 2.3a lets us conclude that they are in $\mathbb{C}(x)$.

Considering L_1 and L as noncommutative polynomials in $\frac{d}{dx}$, we can formally divide L by L_1 on the right. In this way we can find an operator L_2 such that $L - L_2 \circ L_1$ is of order less than k . $L - L_2 \circ L_1$ has the k linearly independent solutions y_1, \dots, y_k , so it must be identically zero. This implies that L is reducible, a contradiction. \blacksquare so that $L = L_2 L_1$

Proposition 2.3 shows the connection between the monodromy and galois groups. Although, we shall not use it in what follows, we wish to outline the full story. Let L be a linear operator of order n with coefficients in $\mathbb{C}(x)$. If y_1, \dots, y_n are linearly independent solutions of

$L(Y) = 0$, we can form $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$ as above. Let G be the group of all differential automorphisms of $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$ (i. e. automorphism σ such that $(\sigma\alpha)' = \sigma(\alpha')$ for all $\alpha \in \mathbb{C}\langle x, y_1, \dots, y_n \rangle$), leaving $\mathbb{C}(x)$ fixed. If V is the \mathbb{C} -span of y_1, \dots, y_n , then G acts linearly on V and we get a representation of G in $GL(n, \mathbb{C})$. It is known ([6], [7]) that G is an algebraic subgroup of $GL(n, \mathbb{C})$ and that there is a one to one correspondence between the subfields of $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$ closed under differentiation (differential subfields) and the Zariski closed subgroups of G . Furthermore, the dimension of G equals the transcendence degree of $\mathbb{C}\langle x, y_1, \dots, y_n \rangle$ over $\mathbb{C}(x)$. If L is of Fuchsian type, then G is the Zariski closure of \mathcal{G} , [15]. These facts would allow us to prove the above propositions only using the galois group G of L . For further information about the galois theory of linear differential equations, the reader is referred to [6] and [7].

C) Auxillary Linear Operators: In the decision procedure presented in section 3, we will have to deal with the following problem:

Let L be a linear differential operator of Fuchsian type and $P(Y_1, \dots, Y_k)$ a differential polynomial, both L and P having coefficients in $K(x)$. Bound the degrees of the numerator and denominator of rational functions $R(x) \in \mathbb{C}(x)$ for which there exist solutions y_1, \dots, y_k of $Ly = 0$ such that $P(y_1, \dots, y_k) \neq 0$ and $(P(y_1, \dots, y_k))' / P(y_1, \dots, y_k) = R(x)$.

To solve this problem, we will first produce a linear differential operator L_P of Fuchsian type, such that $L_P(P(y_1, \dots, y_k)) = 0$ for all solutions y_1, \dots, y_k of $Ly = 0$. Proposition 2.2 will then allow us to get the bound on the degrees. Producing L_P is the aim of the next two propositions.

Proposition 2.6: Let L_1 and L_2 be linear differential operators with coefficients in $K(x)$. One can effectively construct linear differential operators L_3, L_4 and L_5 with coefficients in $K(x)$ such that:

- a) The solution space of $L_3y = 0$ is generated by $\{y_1y_2 \mid y_1 \text{ is a solution of } L_1y = 0 \text{ and } y_2 \text{ is a solution of } L_2y = 0\}$,
- b) The solution space of $L_4y = 0$ is generated by $\{y_1 + y_2 \mid y_1 \text{ is a solution of } L_1y = 0 \text{ and } y_2 \text{ is a solution of } L_2y = 0\}$.
- c) The solution space of $L_5y = 0$ is generated by $\{y' \mid y \text{ is a solution of } L_1y = 0\}$.

Furthermore, if L_1 and L_2 are of Fuchsian type then so are L_3, L_4 and L_5 .

Proof: a) Let L_1 be of order n_1 and L_2 be of order n_2 . Let U and V be new indeterminants. If we formally differentiate UV n_1n_2 times we get a system of $n_1n_2 + 1$ equations:

$$\begin{aligned}
 UV &= UV \\
 (UV)' &= U'V + UV' \\
 (UV)^{n_1n_2} &= \sum_{j=0}^{n_1n_2} \binom{n_1n_2}{j} U^{(j)} V^{(n_1n_2-j)}
 \end{aligned} \tag{2.4}$$

Whenever $U^{(i)}$, $i \geq n_1$, occurs, we use the relation $L_1(U) = 0$ (and its derivatives) to replace $U^{(i)}$ with an expression only involving terms $U^{(i)}$ with $i < n_1$. We similarly use $L_2(V) = 0$ to replace the terms $V^{(i)}$, $i \geq n_2$, with expressions only involving $V^{(i)}$, $i < n_2$. In this way, the right hand side of (2.4) gives us $n_1 n_2 + 1$ linear forms in the terms $U^{(i)} V^{(j)}$, $0 \leq i < n_1$, $0 \leq j < n_2$, with coefficients in $K(x)$. These forms must therefore be linearly dependent over $K(x)$. Let k be the smallest natural number such that the first k of these forms are linearly dependent over $K(x)$. We can then find $a_{k-2}(x), \dots, a_0(x)$ in $K(x)$ such that

$$(UV)^{(k-1)} + a_{k-2}(x)(UV)^{(k-2)} + \dots + a_0(x)UV = 0$$

We have therefore found a linear operator L_3 such that $L_3(y_1 y_2) = 0$ for all solutions y_1 of $L_1 y = 0$ and y_2 of $L_2 y = 0$. We will now show that the solution space of L_3 is spanned by $\{y_1 y_2 \mid L_1 y_1 = 0 \text{ and } L_2 y_2 = 0\}$. Assuming that this is not the case, we have that the dimension of the \mathbb{C} span of $\{y_1 y_2 \mid L_1 y_1 = 0 \text{ and } L_2 y_2 = 0\}$ is $s < k - 1$. Let w_1, \dots, w_s be a basis of this space and form

$$\bar{L}y = \frac{\text{Wr}(y, w_1, \dots, w_s)}{\text{Wr}(w_1, \dots, w_s)}$$

Let $\{a_1, \dots, a_r\}$ be the combined singular points of L_1 and L_2 . Each w_i can be continued along any path γ in $S^2 - \{a_1, \dots, a_r\}$ to yield a

new function $\overline{w_i}$. Furthermore, there is a $A_\gamma \in GL(r, \mathbb{C})$ such that

$$\begin{pmatrix} w_1 \\ \vdots \\ w_s \end{pmatrix} = A_\gamma \begin{pmatrix} \overline{w_1} \\ \vdots \\ \overline{w_s} \end{pmatrix}$$

From this we can conclude that the coefficients of $\overline{L}y$ remain invariant when continued analytically along γ . These coefficients are therefore single valued functions on $S^2 - \{a_1, \dots, a_n\}$. Since they can be expressed in terms of solutions of $L_1 y = 0$ and $L_2 y = 0$, they satisfy the Fuchsian growth condition (2.2), and therefore must be in $\mathbb{C}(x)$. We have shown \overline{L} has coefficients in $\mathbb{C}(x)$, and we shall now show how this leads to a contradiction, using the following theorem of Seidenberg [13a]:

Seidenberg's Theorem: Let E be a differential field of functions, meromorphic in some domain $\mathcal{G} \subset \mathbb{C}$, and let F be a differential field containing E . If F is finitely generated over \mathbb{Q} then there exists a field of functions \tilde{F} , meromorphic in some domain $\tilde{\mathcal{G}} \subset \mathcal{G}$ such that

1) $E \subset \tilde{F}$ (i. e. \tilde{F} contains the restriction to $\tilde{\mathcal{G}}$ of all functions in E).

2) There exists a differential isomorphism φ mapping F to \tilde{F}

whose restriction to E is the identity.

Now construct $K\langle x, U, V \rangle = F$, where $K\langle x, U, V \rangle = K(x, U, \dots, U^{(n_1-1)}, V, \dots, V^{(n_2-1)})$ where $x, U, \dots, U^{(n_1-1)}, V, \dots, V^{(n_2-1)}$

are algebraically independent and the derivation is defined by:

$$x' = 1$$

$$(U^{(i-1)})' = U^{(i)} \quad \text{for } i \leq n_1 - 1$$

$$(U^{(n_1-1)})' = -b_{n_1-1} U^{(n_1-1)} - \dots - b_0 U$$

$$(\text{where } L_1 = \frac{d}{dx} \frac{U^{(n_1-1)}}{U^{(n_1-1)}} + b_{n_1-1} \frac{d}{dx} \frac{U^{(n_1-1)}}{U^{(n_1-1)}} + \dots + b_0)$$

$$(V^{(i-1)})' = V^{(i)} \quad \text{for } i \leq n_2 - 1$$

$$(V^{(n_2-1)})' = -c_{n_2-1} V^{(n_2-1)} - \dots - c_0 V$$

$$(\text{where } L_2 = \frac{d}{dx} \frac{V^{(n_2-1)}}{V^{(n_2-1)}} + c_{n_2-1} \frac{d}{dx} \frac{V^{(n_2-1)}}{V^{(n_2-1)}} + \dots + c_0).$$

Note that in $F(\mathbb{C})$, UV satisfies no linear equation over $\mathbb{C}(x)$ of order $< k$. Let $E = K(x)$ and apply Seidenberg's Theorem. In \tilde{F} we find a function $\varphi(UV)$ which does not satisfy $\bar{L}y = 0$. Since $\varphi(UV)$ is in the span of $\{y_1 y_2 \mid L_1 y_1 = 0 \text{ and } L_2 y_2 = 0\}$ this gives us a contradiction. Therefore L_3 is our desired operator.

b) L_4 is formed by differentiating $U + V$ $n_1 + n_2$ times and proceeding as above.

c) To construct L_5 , assume L_1 is of the form

$$L_1 = \frac{d}{dx} \frac{U^{(n_1-1)}}{U^{(n_1-1)}} + a_{n_1-1}(x) \frac{d}{dx} \frac{U^{(n_1-1)}}{U^{(n_1-1)}} + \dots + a_0(x).$$

If $a_0(x) = 0$, let

$$L_5 = \frac{d}{dx} \frac{n_1}{n_1-1} + a_{n_1-1}(x) \frac{d}{dx} \frac{n_1-2}{n_1-2} + \dots + a_1(x).$$

In this case, $L_5(y) = L_1(\int y dx)$ which is zero iff y is the derivative of a solution of $L_1 y = 0$. If $a_0(x) \neq 0$. Let

$$L_5 = \frac{d}{dx} \frac{n_1}{n_1} + a_{n_1-1}(x) \frac{d}{dx} \frac{n_1-1}{n_1-1} + \left(\frac{da_{n_1-1}(x)}{dx} + a_{n_1-2}(x) \right) \frac{d}{dx} \frac{n_1-2}{n_1-2} + \dots + \left(\frac{da_1(x)}{dx} + a_0(x) \right) \\ - \frac{da_0(x)}{dx} (a_0(x))^{-1} \left[\frac{d}{dx} \frac{n_1-1}{n_1-1} + a_{n_1-1}(x) \frac{d}{dx} \frac{n_1-2}{n_1-2} + \dots + a_1(x) \right]$$

If y is any solution of $L_1 y = 0$, then $L_5(y') = (L_1(y))' = 0$. If y_1, \dots, y_{n_1} are n_1 linearly independent solutions of $L_1 y = 0$, then y_1', \dots, y_{n_1}' must also be linearly independent (otherwise $a_0(x) = 0$). This establishes c).

If L_1 and L_2 are of Fuchsian type, then the solutions of L_1 and L_2 satisfy the appropriate growth conditions at each point on the Riemann Sphere. Since the solutions of L_3 , L_4 and L_5 are explicitly given in terms of solutions of L_1 and L_2 , these former linear differential operators will also be of Fuchsian type. ■

Proposition 2.7: Let $P(Y_1, \dots, Y_k)$ be a differential polynomial and L a linear differential operator, both L and P having coefficients in $K(x)$.

We can effectively construct a nonzero linear differential operator L_P such that if y_1, \dots, y_k are any solutions of $Ly = 0$, then $L_P(P(y_1, \dots, y_k)) = 0$. Furthermore, if L is of Fuchsian type, L_P will also be of Fuchsian type.

Proof: The proof proceeds by induction on the complexity of P .

If $P \in K(x)$, $P \neq 0$, let $L_P = \frac{d}{dx} - (P'/P)$. If $P = P_1 + P_2$ or $P = P_1 P_2$ or $P = P_1'$, then apply the induction hypothesis and Proposition 2.6. ■

The idea of associating a linear differential operator with a differential polynomial as in Proposition 2.7 is not new. In ([14], Sec. 167) Schlesinger defines the notion of associated differential equations. To define these with our terminology, let

$$\text{Wr}(Y_1, \dots, Y_k) = \begin{vmatrix} Y_1 & Y_2 & \dots & Y_k \\ Y_1' & Y_2' & \dots & Y_k' \\ \dots & \dots & \dots & \dots \\ Y_1^{(k-1)} & Y_2^{(k-1)} & \dots & Y_k^{(k-1)} \end{vmatrix}$$

be the usual wronskian determinant. If L is an n^{th} order linear differential operator, then $L_{\text{Wr}} y = 0$ is called the $(n-k)^{\text{th}}$ associated linear differential equation derived from L . These associated linear differential equations can be used to decide if a linear operator is reducible and can also be given a geometrical significance ([14], Sec. 169).

D. Linear Differential Equations with Rational Solutions: We will need to know when a system of linear differential equations has a solution in $\mathbb{C}(x)$. The following proposition is a generalization of a result of Risch [12] and the proof is a modification of his proof.

Proposition 2.8: Let $q(x), q_0(x), \dots, q_n(x)$ be in $K(x)$. One can decide in a finite number of steps if

$$L(Y) = q_n(x) \frac{d^n Y}{dx^n} + \dots + q_0(x)Y = q(x) \quad (2.4)$$

has a solution in $\mathbb{C}(x)$ and if so find such a solution.

Proof: Let $y \in \mathbb{C}(x)$ be a solution of $L(Y) = q(x)$ and let $x = c$ be a pole of y . We know that $x = c$ must also be a pole of one of the q_i or q and we shall see how to bound the order of this pole of y in terms of the orders of the poles of the q_i and q . Expanding around $x = c$, we have:

$$y = \frac{A}{(x-c)^\alpha} + \dots$$

$$q_i = \frac{B_i}{(x-c)^{\beta_i}} + \dots \quad i=0, \dots, n$$

$$q = \frac{C}{(x-c)^\gamma} + \dots$$

Substituting in (2.4), we get

$$\begin{aligned} & \frac{(-1)^n \alpha(\alpha+1) \dots (\alpha+n-1) A B_n}{(x-c)^{\alpha+n+\beta_n}} + \dots + \frac{(-1)^{n-1} \alpha(\alpha+1) \dots (\alpha+n-2) A B_{n-1}}{(x-c)^{\alpha+n-1+\beta_{n-1}}} + \\ & + \dots + \dots + \frac{A B_0}{(x-c)^{\alpha+\beta_0}} + \dots = \frac{C}{(x-c)^\gamma} + \dots \end{aligned} \quad (2.5)$$

We now compare the highest powers of $(x-c)^{-1}$. If $\gamma \geq \alpha + i + \beta_i$ for some i , we get a bound on α in terms of the β_i and γ . Therefore we can assume that $\alpha + i + \beta_i > \gamma$ for $i = 1, \dots, n$. Therefore the terms involving the highest power, say m , of $(x-c)^{-1}$ on the left hand side of (2.5) must cancel. Let $\mathcal{J} = \{i_1, \dots, i_k\} \subset \{0, \dots, n\}$ be the set of numbers such that $\alpha + i + \beta_i = m$, that is, those i such that $i + \beta_i = \max_j \{j + \beta_j\}$. Comparing coefficients of $(x-c)^{-m}$ we get

$$\begin{aligned} & (-1)^{i_1} \alpha(\alpha+1) \dots (\alpha+i_1-1) A B_{i_1} + \dots + (-1)^{i_k} \alpha(\alpha+1) \dots (\alpha+i_k-1) A B_{i_k} = 0 \\ \text{or} \\ & (-1)^{i_1} \alpha(\alpha+1) \dots (\alpha+i_1-1) B_{i_1} + \dots + (-1)^{i_k} \alpha(\alpha+1) \dots (\alpha+i_k-1) B_{i_k} = 0 \end{aligned} \quad (2.6)$$

From this latter polynomial, we can bound the size of α .

We therefore now know that if $y \in \mathbb{C}(x)$ is a solution of $L(Y) = 0$, then

$$y(x) = \frac{P(x)}{(x-c_0)^{\alpha_0} \dots (x-c_m)^{\alpha_m}} \quad (2.7)$$

where each c_i is a pole of some q_i or q and each α_i can be determined. We now wish to bound the degree of $P(x)$. Substituting (2.7) in $L(Y) = 0$ and clearing denominators, we can find polynomials $Q_n(x), \dots, Q_0(x), Q(x)$ such that

$$Q_n(x)P^{(n)}(x) + \dots + Q_0(x)P(x) = Q(x) \quad (2.8)$$

Let $P(x) = ax^m + \text{lower order terms}$, $Q_i(x) = b_i x^{m_i} + \text{lower order terms}$ and $Q(x) = cx^\ell + \text{lower order terms}$. Equating coefficients as before will then give us a bound on m . If we then write $P(x) = a_m x^m + \dots + a_0$ and substitute this into (2.8), we will get a system of linear equations which determines the a_i . ■

Let $K(x)[\frac{d}{dx}]$ be the ring of linear operators in $\frac{d}{dx}$, with coefficients in $K(x)$. We then have:

Proposition 2.10: Let A be an $n \times n$ matrix with entries in $K(x)[\frac{d}{dx}]$

and B an $n \times 1$ matrix with entries in $K(x)$. One can decide if there exist y_1, \dots, y_n in $\mathbb{C}(x)$ such that

$$A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = B$$

and if so produce such a set of solutions.

Proof: It is known that the ring $K(x)[\frac{d}{dx}]$ has both a left and a right division algorithm ([10], p. 3).

Using this, one can row and column reduce the matrix A ([10], p. 39; [4], p. 43), that is, one can find invertible matrices U and V with entries in $K(x)[\frac{d}{dx}]$ such that $UAV = D$, D a diagonal matrix. $Y = (y_1, \dots, y_n)^T$ is a solution of $AY = B$ iff $W = V^{-1}Y$ is a solution of $DW = UB$. Therefore, deciding if $AY = B$ has a solution $Y = (y_1, \dots, y_n)^T$, $y_i \in \mathbb{C}(x)$ is equivalent to deciding if $DW = UB$ has a solution $W = (w_1, \dots, w_n)^T$, $w_i \in \mathbb{C}(x)$. Since D is diagonal, we can apply Proposition 2.9. ■

E. Elimination Theory: The decision procedure in section 3 reduces several questions to deciding if a system of polynomial equations have a common solution. In this section we present those facts from elimination theory which will handle this problem.

A set $A \subset \mathbb{C}^n$ is said to be K-constructible if there exist polynomials $p_{ij}(x_1, \dots, x_n)$, $q_i(x_1, \dots, x_n)$, $1 \leq i, j \leq m$, in $K[x_1, \dots, x_n]$ such that $(a_1, \dots, a_n) \in A$ if and only if, for some k , $1 \leq k \leq m$, $p_{k1}(a_1, \dots, a_n) = p_{k2}(a_1, \dots, a_n) = \dots = p_{km}(a_1, \dots, a_n) = 0$ and $q_k(a_1, \dots, a_n) \neq 0$. The polynomials p_{ij} , q_i are called the defining polynomials of A . It is easy to see that the collection of K-constructible sets is precisely the Boolean algebra generated by the K-closed sets in \mathbb{C} (a subset of \mathbb{C}^m is K-closed if it consists of the common zeroes of a set of polynomials with coefficients in K). The main result of classical elimination theory is contained in the following proposition:

Proposition 2.11 ([8], [16]): Let $p : \mathbb{C}^n \rightarrow \mathbb{C}^r$ be the projection map $p(x_1, \dots, x_n) = (x_1, \dots, x_r)$, $r \leq n$. If A is a K -constructible subset of \mathbb{C}^n , then $p(A)$ is a K -constructible subset of \mathbb{C}^r . Furthermore, if we know the defining polynomials of A , we can effectively construct the defining polynomials of $p(A)$.

Given polynomials p_{ij}, q_i defining a K -constructible set $A \subset \mathbb{C}^n$, we can apply Proposition 2.11 to the projection $p : \mathbb{C}^n \rightarrow \mathbb{C}^1$, and get a K -constructible set $p(A) \subset \mathbb{C}^1$. One can then easily decide if $p(A)$ is empty or not. In this way we can decide if any K -constructible set is empty or not. If A is not empty, we can produce a member of A , that is, we can find polynomials $f_1(x_1), f_2(x_1, x_2), \dots, f_n(x_1, \dots, x_n)$ such that:

(i) f_1 is irreducible over K , (ii) if a_1, \dots, a_k are solutions of $f_1(a_1) = f_2(a_1, a_2) = \dots = f_k(a_1, \dots, a_k) = 0$, then $f_{k+1}(a_1, \dots, a_k, x_{k+1})$ is irreducible over $K(a_1, \dots, a_k)$ and (iii) any solution of $f_1(x_1) = \dots = f_n(x_1, \dots, x_n) = 0$ is in A .

In the next section we shall need to know that several sets are constructible.

Proposition 2.12: Let c_1, \dots, c_m, x, y be indeterminants. Let $P(c_1, \dots, c_m, x, y)$ be a monic polynomial in $K(c_1, \dots, c_m, x)[y]$, L_1 and L_2 linear differential operators and $R(c_1, \dots, c_m, x, y)$ a differential polynomial, L_1, L_2 and R having coefficients in $K(c_1, \dots, c_m, x)$. The following sets are K -constructible:

a) The set S_1 of $(\gamma_1, \dots, \gamma_m)$ in \mathbb{C}^m such that:

(i) the denominators of the coefficients of powers of Y

in $P(\gamma_1, \dots, \gamma_m, x, y)$ do not vanish identically.

(ii) $P(\gamma_1, \dots, \gamma_m, x, y)$ is irreducible in $\mathbb{C}(x)[y]$.

b) The subset S_2 of S_1 such that:

(i) The denominators of the coefficients of powers of each

$y^{(i)}$ in $R(\gamma_1, \dots, \gamma_m, x, y)$ do not vanish identically.

(ii) If u is a solution of $P(\gamma_1, \dots, \gamma_m, x, u) = 0$ then

$R(\gamma_1, \dots, \gamma_m, x, u) = 0$.

c) The set S_3 of $(\gamma_1, \dots, \gamma_m)$ in \mathbb{C}^m such that:

(i) The denominators of the coefficients of L_1 and L_2 do not vanish identically.

(ii) There exists a linear differential operator L_3 , with coefficients in $\mathbb{C}(x)$ such that $L_1 = L_3 \circ L_2$.

Proof: a) The set A_1 of $(\gamma_1, \dots, \gamma_m) \in \mathbb{C}^m$ such that the denominators of the coefficients of powers of y do not vanish clearly forms a constructible set. Let $f(c_1, \dots, c_m, x)$ equal the product of the denominators of the coefficients of P and let N equal the total degree of $f(c_1, \dots, c_m, x)P(c_1, \dots, c_m, x, y)$, and M be its degree in y . If $(\gamma_1, \dots, \gamma_m) \in A_1$, and $P(\gamma_1, \dots, \gamma_m, x, y)$ factors, then $f(\gamma_1, \dots, \gamma_m, x)P(\gamma_1, \dots, \gamma_m, x, y)$ equals the product of two polynomials in $\mathbb{C}[x, y]$ each of total degree has than or equal to N and degree in y , less than M . The set of polynomials with these restrictions on

their degrees can be identified with some affine space \mathbb{C}^μ . If $\bar{\alpha} = (\alpha_1, \dots, \alpha_\mu)$ we will denote by $P_{\bar{\alpha}}$ the corresponding polynomial. The set of elements $(\alpha_1, \dots, \alpha_\mu, \beta_1, \dots, \beta_\mu, \gamma_1, \dots, \gamma_m)$ such that $f(\gamma_1, \dots, \gamma_m, \mathbf{x})P(\gamma_1, \dots, \gamma_m, \mathbf{x}, y) = P_{\bar{\alpha}} \cdot P_{\bar{\beta}}$ forms a K -closed subset A_2 of $\mathbb{C}^\mu \times \mathbb{C}^\mu \times \mathbb{C}^m$. If $p : \mathbb{C}^\mu \times \mathbb{C}^\mu \times \mathbb{C}^m \rightarrow \mathbb{C}^m$, then Proposition 2.11 implies that $p(A_2)$ is constructible. Letting A_3 denote its complement, we have that $A_1 \cap A_3$ is our desired constructible set.

b) Let us assume that $P(c_1, \dots, c_m, \mathbf{x}, y)$ is irreducible in $\mathbb{C}(c_1, \dots, c_m, \mathbf{x})[y]$, since otherwise S_1 is empty and b) is trivially true. The field $F = K(c_1, \dots, c_m, \mathbf{x})[y] / (P(c_1, \dots, c_m, \mathbf{x}, y)) = K(c_1, \dots, c_m, \mathbf{x}, u)$ is an algebraic extension of $K(c_1, \dots, c_m, \mathbf{x})$ and the derivation $\frac{d}{dx}$, where $\frac{d}{dx}(c_i) = 0$, on this latter field extends uniquely to F . In F , we have $\frac{du}{dx} = - \left(\frac{\partial P}{\partial x}(c_1, \dots, c_m, \mathbf{x}, u) \right) / \frac{\partial P}{\partial y}(c_1, \dots, c_m, \mathbf{x}, u) = P_1(c_1, \dots, c_m, \mathbf{x}, u)$ where P_1 is some polynomial in u with coefficients in $K(c_1, \dots, c_m, \mathbf{x})$. Similarly we can find polynomials P_i such that $\frac{d^i u}{dx^i} = P_i(c_1, \dots, c_m, \mathbf{x}, u)$. In R , we can formally replace each $y^{(i)}$ with $P_i(c_1, \dots, c_m, \mathbf{x}, y)$. The resulting expression will be an (algebraic) polynomial $R_1(c_1, \dots, c_m, \mathbf{x}, y)$. Dividing this polynomial by $P(c_1, \dots, c_m, \mathbf{x}, y)$, we get a remainder $R_2(c_1, \dots, c_m, \mathbf{x}, y)$. A necessary and sufficient condition that any solution of $P(\gamma_1, \dots, \gamma_m, \mathbf{x}, y) = 0$ be a solution of $R(\gamma_1, \dots, \gamma_m, \mathbf{x}, y) = 0$ is that $R_2(\gamma_1, \dots, \gamma_m, \mathbf{x}, y)$ vanish identically. The set of $(\gamma_1, \dots, \gamma_m) \in S_1$ such that (i) of b) holds and for which $R_2(\gamma_1, \dots, \gamma_m, \mathbf{x}, y)$ is identically zero is a constructible set.

c) Let L_1 be of order r and L_2 of order s . If we formally divide L_1 by L_2 we will find elements A_1, \dots, A_{r-s} in $K(c_1, \dots, c_m, x)$ such that $L_3 = L_1 - \left(\frac{d^{r-s}}{dx^{r-s}} + A_1 \frac{d^{r-s-1}}{dx^{r-s-1}} + \dots + A_{r-s} \right) \circ L_2$ is a linear differential operator of order less than s . L_2 divides L_1 if and only if L_3 vanishes identically. The set of $(\gamma_1, \dots, \gamma_m) \in \mathbb{C}^m$ for which the denominators of the coefficients of L_1 and L_2 do not vanish identically and for which L_3 vanishes identically, is a constructible set. ■

§3. The Decision Procedure

We shall first show how to decide if (1.1) has only solutions algebraic over $\mathbb{C}(x)$, assuming that L is irreducible over $\mathbb{C}(x)$. We then show how to decide if (1.1) is reducible and give a factorization if it is. This will then allow us to deal with the general case. Before we proceed we will need some terminology. Let k be a field. If $\alpha \in k(x)$, say $\alpha = p(x)/q(x)$, $(p, q) = 1$, we define the degree of α ($\deg \alpha$) to be $\max(\deg p, \deg q)$. If u is algebraic over $k(x)$ with minimal polynomial $Y^n + a_{n-1}(x)Y^{n-1} + \dots + a_0(x)$, we define the height of u to be $\max(n, \deg a_{n-1}, \dots, \deg a_0)$. If L is the linear differential operator $\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + p_0(x)$ with $p_i(x) \in k(x)$, we define the height of L to be $\max(n, \deg p_{n-1}, \dots, \deg p_0)$.

Proposition 3.1: Let L be a linear differential operator of Fuchsian type with coefficients in $K(x)$. We can effectively find a natural number N such that if all solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$, then the Riccati equation $R(U) = 0$ associated with L has a solution u , algebraic over $\mathbb{C}(x)$ and of height $\leq N$.

Proof: Proposition 2.4 implies that the Riccati equation $R(U) = 0$ has a solution u , algebraic over $\mathbb{C}(x)$, such that $[\mathbb{C}(x, u) : \mathbb{C}(x)] \leq (49n)^{n^2}$, where n is the order of L .

It is therefore enough to show that for each natural number $m \geq 0$, we can find a natural number N_m such that if u is a solution of $R(U) = 0$ with minimum polynomial $Y_m^m + a_{m-1}(x)Y^{m-1} + \dots + a_0(x)$, then for each

$i, 0 \leq i \leq m-1$, the height of $a_i \leq N_m$. If we have done this, then, letting $M = (49n)^{\frac{n}{2}}$, we set $N = \max(M, N_0, \dots, N_M)$.

Let u be an algebraic solution of $R(U) = 0$ with minimum polynomial $Y_m^m + a_{m-1}Y_{m-1}^{m-1} + \dots + a_0$ and let $u_1 = u, u_2, \dots, u_m$ be the conjugates of u . Each u_i also satisfies $R(u_i) = 0$, so there exist y_1, \dots, y_m , solutions of $Ly = 0$, such that $y'_i/y_i = u_i$. We now note that

$$a_{m-1} = - \sum_{i=1}^m u_i = - \sum_{i=1}^m y'_i/y_i = - \left(\prod_{i=1}^m y_i \right)' / \prod_{i=1}^m y_i. \quad (3.1)$$

Let $P_{m-1}(Y_1, \dots, Y_m) = \prod_{i=1}^m Y_i$. By Proposition 2.7, we can find a linear differential operator L_{m-1} of Fuchsian type such that $L_{m-1}(P_{m-1}(y_1, \dots, y_m)) = 0$. By (3.1), we see that $P_{m-1}(y_1, \dots, y_m)$ is a solution of $L_{m-1}y = 0$ whose logarithmic derivative is in $\mathbb{C}(x)$. Using Proposition 2.2, we can therefore bound the degree of a_{m-1} .

Looking at a_{m-2} , we see that

$$a_{m-2} = \sum_{1 \leq i < j \leq m} u_i u_j = \sum_{1 \leq i < j \leq m} \frac{y'_i y'_j}{y_i y_j} = [2(m-2)! \prod_{i=1}^m y_i]^{-1} \sum y'_i y'_j y'_k \dots y'_m,$$

this latter sum takes over all permutations of $(1, \dots, m)$. Let $P_{m-2} =$

$\sum y'_i y'_j y'_k \dots y'_m$. Proposition 2.7 assures us that we can find a linear

differential operator L_{m-2} of Fuchsian type such that $L_{m-2}(P_{m-2}(y_1, \dots, y_m)) = 0$.

Calculating, we find

$$(P_{m-2}(y_1, \dots, y_m))' / P_{m-2}(y_1, \dots, y_m) = a'_{m-2} / a_{m-2} + a_{m-1} \in \mathbb{C}(x). \quad (3.2)$$

Using Proposition 2.2, we can determine the form of $P_{m-2}(y_1, \dots, y_m)$ and using (3.2) we can bound the height of a_{m-2} .

Continuing in this way we can bound the heights of each of the a_i , $0 \leq i \leq m-1$. ■

Proposition 3.1, together with a solution of Abel's problem, forms the core of the decision procedure. For this reason, we shall outline the Boulanger-Painlevé treatment of this latter proposition.

Boulanger first notes that Jordan's theorem allows one to bound the degree of the minimum polynomial of an algebraic solution of $R(U) = 0$. Given such a minimal polynomial, we can clear the denominators of the coefficients and get a polynomial $b_m Y^m + \dots + b_0$, with the b_i 's in $\mathbb{C}[x]$. We need to bound the degrees of the b_i .

To do this Boulanger shows that if the degree of b_m is less than or equal to r , then the degree of each b_i is less than or equal to $r + m - i$ for $i = 0, \dots, m$. To see this, let s be the maximum of the degrees of b_m, \dots, b_0 . Substituting $X = \frac{1}{x}$ and clearing denominators, we get a new polynomial

$$B_m(X)X^{s-\alpha_m}Y^m + B_{m-1}(X)X^{s-\alpha_{m-1}}Y^{m-1} + \dots + B_0(X)X^{s-\alpha_0}$$

where each α_i is the degree of b_i , one of the $s - \alpha_i$ is zero and none of the B_i have X as a factor. If u_1, \dots, u_m are the roots of $b_m Y^m + \dots + b_0$, then for each i , $u_i = y'_i/y_i$ for some solution y_i of $Ly = 0$. Each u_i will have a pole of order at most 1 at $X = 0$. Therefore, $u_1 u_2 \dots u_m$ will have a pole of order at most m at $X = 0$, $u_1 u_2 \dots u_{m-1} + u_1 \dots u_{m-2} u_m + \dots + u_2 \dots u_m$ will have a pole of order at most $m-1$, etc. Therefore,

$$\alpha_0 - \alpha_n \leq m$$

$$\alpha_1 - \alpha_n \leq m - 1$$

$$\vdots$$

$$\text{or } \alpha_i \leq r + m - i \text{ for } i = 0, \dots, m.$$

We must now bound the degree of b_m . To do this Boulanger examines the zeroes of b_m . These zeroes are either the singular points of the original linear operator or ordinary points which are zeroes of y_1, y_2, \dots or y_m . Let a_1, \dots, a_k be the singular points of L and c_1, \dots, c_ℓ the ordinary points which are zeroes of y_1, \dots or y_m . We know k , and if we also know ℓ , we would be done; the degree of b_m would be at most $m(\ell+k)$. This is because u_1, \dots, u_m have only simple poles at $a_1, \dots, a_k, c_1 \dots c_\ell$ so at such a point b_m will have a zero of order at most m .

To bound ℓ , consider the product $y_1 \dots y_m$. Since $(y_1 \dots y_m)' / y_1 \dots y_m = u_1 + \dots + u_m = b_{m-1}/b_m$, we have

$$y_1 \dots y_m = \frac{\prod_{i=1}^{\ell} (x-a_i)^{\alpha_i} \prod_{i=1}^{\beta_i} (x-c_i)^{\beta_i}}{\prod (x-a'_i)^{\alpha'_i}}$$

where the a_i and the a'_i are singular points, the c_i are regular points, the α_i and α'_i are rational numbers and the β_i are natural numbers. The α_i and α'_i can be bounded in terms of the exponents of L . Moreover one can effectively bound the order of each of the y_i at infinity by some integer r . Since the order of $y_1 \dots y_m$ at infinity will be less than mr , we have

$$\sum_{i=1}^{\ell} \beta_i + \sum \alpha_i - \sum \alpha'_i < mr$$

Therefore,

$$\ell \leq \sum_{i=1}^{\ell} \beta_i \leq mr - \sum \alpha_i + \sum \alpha'_i.$$

This gives the desired bound on ℓ .

Theorem 1: Let L be a linear differential operator with coefficients in $K(x)$, and assume L is irreducible over $\mathbb{C}(x)$. One can decide in a finite number of steps if all solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$. If this is the case, one can produce a polynomial $f(Y)$ in $K(x)[Y]$ such that the solutions of $f(Y) = 0$ span the solution space of $Ly = 0$.

Proof: If all solutions of $Ly = 0$ are algebraic then L must be of Fuchsian type. So, to start the decision procedure, use the Criterion of Fuchs to decide if L is of Fuchsian type. Now assume L is of Fuchsian type. By Proposition 3.1, we can find an integer N such that if all solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$, then the Riccati equation associated with L will have a solution u , algebraic over $\mathbb{C}(x)$, of height $\leq N$. Let $P(c_1, \dots, c_m, x, y)$ be the polynomial $y^N + a_{N-1}(c_1, \dots, c_m, x)y^{N-1} + \dots + a_0(c_1, \dots, c_m, x)$ where each a_i , $0 \leq i \leq N-1$, is the ratio of two polynomials of degree N in x with undetermined coefficients. We insure that the variables appearing as coefficients in the a_i are all distinct and denote these by $\{c_1, \dots, c_m\}$. By Proposition 2.12, we know that the set S of $(\gamma_1, \dots, \gamma_m) \in \mathbb{C}^m$ such that $P(\gamma_1, \dots, \gamma_m, x, y)$ is the minimum polynomial of a solution the Riccati equation $R(U) = 0$ is constructible. We can therefore decide if S is nonempty. Let us assume that it is and let $(\gamma_1, \dots, \gamma_m) \in S$. We must now decide if there is a y , algebraic over $\mathbb{C}(x)$ such that $y'/y = u$ where $P(\gamma_1, \dots, \gamma_m, x, u) = 0$, and if so produce the minimum polynomial of y . This is the Problem of Abel referred to in the introduction. We refer the reader to either [13] or [2] for an exposition of the algorithm which decides this question. If there exists such a y , then $Ly = 0$ will have a solution algebraic over $\mathbb{C}(x)$, and by Proposition 2.5, all solutions will be algebraic over $\mathbb{C}(x)$. Conversely, if $Ly = 0$ has a solution algebraic over $\mathbb{C}(x)$, then the above procedure will yield such a solution. ■

We shall now show how to deal with the possibility of the linear operator being reducible.

Proposition 3.2. Let L be a linear differential operator with coefficients in $K(x)$ and assume L is of Fuchsian type. One can effectively find an integer N such that if L_1 and L_2 are linear differential operators with coefficients in $\mathbb{C}(x)$ such that $L = L_1 \circ L_2$, then the height of L_2 is less than or equal to N .

Proof: Let us assume that there exist two linear differential operators, L_1 and L_2 , with coefficients in $\mathbb{C}(x)$ such that $L = L_1 \circ L_2$. Let L_2 be of order k and let y_1, \dots, y_k be linearly independent solutions of $L_2 y = 0$, which a fortiori, are solutions of $Ly = 0$. We then have

$$\begin{aligned} L_2(Y) &= \frac{Wr(Y, y_1, \dots, y_k)}{Wr(y_1, \dots, y_k)} \\ &= Y^{(k)} + a_{k-1}(x)Y^{(k-1)} + \dots + a_0(x)Y \end{aligned}$$

with each $a_i \in \mathbb{C}(x)$. Furthermore, each a_i is of the form

$$a_i(x) = \frac{P_i(y_1, \dots, y_k)}{W_r(y_1, \dots, y_k)} \quad 0 \leq i \leq k-1$$

where $P_i(Y_1, \dots, Y_k)$ is a differential polynomial with coefficients in \mathbb{Q} ; in fact $P_{k-1}(Y_1, \dots, Y_k) = (Wr(Y_1, \dots, Y_k))'$. By Proposition 2.7, we can find linear differential operators \tilde{L}_i $0 \leq i \leq k-1$ of Fuchsian type such that:

$$\begin{aligned} \tilde{L}_{k-1}(Wr(y_1, \dots, y_k)) &= 0 \\ \tilde{L}_i(P_i(y_1, \dots, y_k)) &= 0 \quad 0 \leq i \leq k-2. \end{aligned}$$

Since

$$\frac{(Wr(y_1, \dots, y_k))'}{Wr(y_1, \dots, y_k)} = a_{k-1} \in \mathbb{C}(x)$$

and

$$\frac{(P_i(y_1, \dots, y_k))'}{P_i(y_1, \dots, y_k)} = \frac{(a_i(x))'}{a_i(x)} + a_{k-1}(x) \in \mathbb{C}(x)$$

for $i = 0, \dots, k-2$,

we can apply Proposition 2.2 and find the forms of $Wr(y_1, \dots, y_k)$, $P_1(y_1, \dots, y_k)$, $\dots, P_{k-2}(y_1, \dots, y_k)$. In this way we can bound the heights of each $a_i(x)$. ■

Theorem 2: Let L be a linear differential operator, with coefficients in $K(x)$, of Fuchsian type. One can decide in a finite number of steps if L is reducible over $\mathbb{C}(x)$, and, if it is, produce linear differential operators L_1 and L_2 such that $L = L_1 \circ L_2$.

Proof: It is enough to show that, given k , $1 \leq k \leq n-1$, one can decide if there exist linear differential operators L_1 and L_2 with coefficients in $\mathbb{C}(x)$ such that the order of $L_1 = n-k$, the order of $L_2 = k$ and $L = L_1 \circ L_2$. Let N be the number found in Proposition 3.2. Let $L_2(c_1, \dots, c_m)$ be the operator

$$\frac{d^k}{dx^k} + a_{k-1}(c_1, \dots, c_m, x) \frac{d^{k-1}}{dx^{k-1}} + \dots + a_0(c_1, \dots, c_m, x)$$

where each $a_i(x)$ is the quotient of two polynomials of degree N with

undetermined coefficients and let $\{c_1, \dots, c_m\}$ be the set of all coefficients appearing in the a_i . By Proposition 2.12, we can decide if there exist $\gamma_1, \dots, \gamma_m$ in \mathbb{C} and a linear operator L_1 such that $L = L_1 \circ L_2(\gamma_1, \dots, \gamma_m)$. We can further produce such $\gamma_1, \dots, \gamma_m$ if they exist. Formal division will then produce L_1 . ■

Schleisinger ([14], Sec. 176) gives a slightly more explicit form of Theorem 2, using the associated differential equations. He also generalizes this theorem by doing away with the hypothesis that L is of Fuchsian type. Our version cuts through some of the technicalities and is sufficient for our purposes. Theorem 2 and the following proposition will yield the final decision procedures.

Proposition 3.3. Let L, L_1, L_2 be linear differential operators with coefficients in $\mathbb{C}(x)$, and of orders $n, n-k$ and k respectively. Assume $L = L_1 \circ L_2$. All solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$ if and only if

a) All solutions of $L_1y = 0$ and all solutions of $L_2y = 0$ are algebraic over $\mathbb{C}(x)$.

b) For some set of linearly independent solutions y_1, \dots, y_{n-k} of $L_1y = 0$, each of the equations $L_2y = y_i$ has a solution algebraic over $\mathbb{C}(x)$.

If a) and b) hold, then $L_2\tilde{y} = y$ has a solution algebraic over $\mathbb{C}(x)$ for any solution y of $L_1y = 0$.

Proof: Let V, V_1, V_2 be the solution spaces of $Ly = 0$, $L_1y = 0$, $L_2y = 0$ respectively. We note that L_2 maps V onto V_1 with kernel V_2 .

If V consists of elements algebraic over $\mathbb{C}(x)$, then a) and b) follow.

Conversely assume that a) and b) are true. Let y_1, \dots, y_{n-k} be as,

in b), $\tilde{y}_{n-k+1}, \dots, \tilde{y}_n$ a basis for V_2 . For each i , $i \leq n-k$, let

\tilde{y}_i be a solution of $L_2 y = y_i$, \tilde{y}_i algebraic over $\mathbb{C}(x)$. $\{\tilde{y}_1, \dots, \tilde{y}_n\}$ is a subset

of V . To see that $\tilde{y}_1, \dots, \tilde{y}_n$ are linearly independent, let $c_i \in \mathbb{C}$ be

such that $\sum_{i=1}^n c_i \tilde{y}_i = 0$. We then have

$$L_2 \left(\sum_{i=1}^n c_i \tilde{y}_i \right) = \sum_{i=1}^{n-k} c_i y_i = 0,$$

so $c_1 = \dots = c_{n-k} = 0$. Therefore $\sum_{i=n-k+1}^n c_i \tilde{y}_i = 0$, so $c_{n-k+1} = \dots = c_n = 0$.

Therefore V is spanned by elements algebraic over $\mathbb{C}(x)$. This implies that all solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$. ■

Theorem 3. Let L be a linear differential operator with coefficients in $K(x)$. One can decide in a finite number of steps if all solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$. If this is the case one can produce a polynomial $f(Y)$ in $\mathbb{C}(x)[Y]$ such that the solutions of $f(Y) = 0$ span the solution space of $Ly = 0$.

Proof: We prove this theorem by induction on the order of L . First use the Criterion of Fuchs to decide if L is of Fuchsian type. If it is not, we can stop, so assume L is of Fuchsian type. By Theorem 2, we can decide if L is irreducible. If it is, apply Theorem 1. If L is reducible, find linear differential operators L_1 and L_2 such that $L = L_1 \circ L_2$.

Using the induction hypothesis we can decide if all solutions of $L_1 y = 0$ and

all solutions of $L_2 y = 0$ are algebraic over $\mathbb{C}(x)$. If this is not the case, then $L y = 0$ will not have all its solutions algebraic over $\mathbb{C}(x)$. Therefore, assume that both $L_1 y = 0$ and $L_2 y = 0$ have only algebraic solutions, and let y_1, \dots, y_{n-k} be linearly independent solutions of $L_1 y = 0$ and y_{n-k+1}, \dots, y_n be linearly independent solutions of $L_2 y = 0$. We now wish to find \tilde{y}_i such that $L_2 \tilde{y}_i = y_i$ for $i=1, \dots, n-k$ and such that each \tilde{y}_i is algebraic over $\mathbb{C}(x)$. We will use the method of variation of parameters to find each \tilde{y}_i . Fix i , $1 \leq i \leq n-k$ and let

$$\tilde{y}_i = \sum_{j=1}^k u_j y_{n-k+j} \quad (3.3)$$

where the u_j satisfy

$$\begin{aligned} u'_1 y_{n-k+1} + u'_2 y_{n-k+2} + \dots + u'_k y_n &= 0 \\ u'_1 y'_{n-k+1} + u'_2 y'_{n-k+2} + \dots + u'_k y'_n &= 0 \\ \vdots & \\ u'_1 y^{(k-2)}_{n-k+1} + u'_2 y^{(k-2)}_{n-k+2} + \dots + u'_k y^{(k-2)}_n &= 0 \\ u'_1 y^{(k-1)}_{n-k+1} + u'_2 y^{(k-1)}_{n-k+2} + \dots + u'_k y^{(k-1)}_n &= y_i \end{aligned} \quad (3.4)$$

or equivalently

$$u'_j = y_i \frac{\text{Wr}(y_{n-k+1}, \dots, \hat{y}_{n-k+j}, \dots, y_n)}{\text{Wr}(y_{n-k+1}, \dots, y_n)} \quad (3.5)$$

where \hat{y}_{n-k+j} denotes that this term is omitted. From (3.4) we can deduce that

$$\tilde{y}_i^{(\ell)} = \sum_{j=1}^k u_j y_{n-k+j}^{(\ell)} \quad \text{for } \ell = 0, \dots, k-1 \quad (3.6)$$

$$L_2 \tilde{y}_i = y_i \quad (3.7)$$

If each u_j is algebraic over $\mathbb{C}(x)$, then \tilde{y}_i will be algebraic over $\mathbb{C}(x)$.

Conversely, if \tilde{y}_i is algebraic over $\mathbb{C}(x)$, then (3.6) and Cramer's rule implies that each u_j is algebraic over $\mathbb{C}(x)$. Therefore, deciding if \tilde{y}_i is algebraic over $\mathbb{C}(x)$ is equivalent to deciding if there exist u_i , as defined in (3.5), algebraic over $\mathbb{C}(x)$. Since the right hand side of (3.5) is algebraic over $\mathbb{C}(x)$, it suffices to solve the following problem: Given α , algebraic over $\mathbb{C}(x)$, decide if there exists a u , algebraic over $\mathbb{C}(x)$ such that $u' = \alpha$, and if this is the case find u . The contents of section 2D allow us to deal with this in the following manner:

If $u' = \alpha$ has a solution algebraic over $\mathbb{C}(x)$, then, letting $\text{Tr}(u)$ denote the trace of u over $\mathbb{C}(x, \alpha)$, we have $(\text{Tr}(u))' = m\alpha$ for some integer m . Therefore, $u' = \alpha$ will have a solution in $\mathbb{C}(x, \alpha)$. Let $P(Y)$ be the minimum polynomial of α over $\mathbb{C}(x)$ and let p be its degree. We want to decide if there exist $a_i(x) \in \mathbb{C}(x)$, $0 \leq i \leq p-1$ such that

$$\left(\sum_{i=0}^{p-1} a_i(x) \alpha^i \right)' = \alpha \quad (3.8)$$

If we expand the left hand side of this equation and reduce mod $P(x)$, we see that (3.8) is equivalent to the a_i satisfying a system of first order equations

$$\sum_i b_{ij} a_i' + \sum_i c_{ij} a_i = d_j \quad j = 0, \dots, p-1$$

where the b_{ij} , c_{ij} and d_j are known. Proposition 2.10 allows us to decide if this system has solutions in $\mathbb{C}(x)$. If it does, Proposition 3.3 says that all solutions of $Ly = 0$ are algebraic over $\mathbb{C}(x)$. Conversely, if all solutions of $Ly = 0$ are algebraic, Proposition 3.3 says that this procedure will produce n linearly independent solutions. ■

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