A QUANTUM COMPLEXITY APPROACH TO THE KIRCHBERG EMBEDDING PROBLEM

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Abstract. The Kirchberg Embedding Problem (KEP) asks if every C*-algebra embeds into an ultrapower of the Cuntz algebra $\mathcal{O}_2$. Motivated by the recent refutation of the Connes Embedding Problem using the quantum complexity result MIP*=RE, we establish two quantum complexity consequences of a positive solution to KEP. Both results involve almost-commuting strategies to nonlocal games.

1. Introduction

The recent landmark quantum complexity result known as $\mathrm{MIP}^* = \mathrm{RE}$ [12] yielded a negative solution to a famous problem in the theory of von Neumann algebras, namely the **Connes Embedding Problem** (CEP). CEP asks if every tracial von Neumann algebra embeds into a tracial ultrapower of the hyperfinite II_1 factor. The negative solution to the CEP can be used to give a negative solution to an analogous problem in the theory of C*-algebras known as the **Blackadar-Kirchberg Problem** (or **MF Problem**), which asked if every stably finite C*-algebra embeds into an ultrapower of the universal UHF algebra (see [10, Proposition 6.1]).

The Blackadar-Kirchberg Problem can be viewed as the "finite" C*-algebra analog of CEP. In this paper, we consider the "infinite" C*-algebra analog of CEP known as the **Kirchberg Embedding Problem** (KEP). KEP asks if every C*-algebra embeds into an ultrapower of the Cuntz algebra $\mathcal{O}_2$.[1] The Kirchberg Embedding Problem was studied model theoretically by the first author and Sinclair in [11]. In that paper, KEP is shown to be equivalent to the statement that there is a C*-algebra that is both nuclear and **existentially closed**. Further model-theoretic equivalents of KEP were given by the first author in [9], where

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[1]Incidentally, one might ask if there is an "infinite", that is, type III, von Neumann algebra analog of CEP. For example, one might ask if every von Neumann algebra embeds with expectation into the Ocneanu ultrapower of the hyperfinite III_1 factor $\mathcal{R}_\infty$. In [1], Ando, Haagerup, and Winslow prove that this question is equivalent to CEP itself (and thus has a negative answer).

it was shown that KEP is equivalent to the statement that $\mathcal{O}_2$ is the enforceable C*-algebra.

The goal of this note is to record two quantum complexity consequences of a positive solution to KEP in the spirit of how MIP*=RE relates to CEP. We believe (perhaps even conjecture) that one (or both) of these quantum complexity statements is (are) false and thus view our results as evidence that KEP has a negative solution.

In the remainder of this introduction, we informally state our two main results. Precise definitions and statements will follow in subsequent sections.

Typically, quantum strategies for nonlocal games do not allow the players (henceforth referred to as Alice and Bob) to communicate. To implement this formally, one assumes that the measurement operators for the two players commute, which allows one to proclaim that the two players measured their part of the quantum state at the same time. In this paper, we will consider quantum strategies which allow the players to communicate "a little bit." Such an idea seems to have been first considered by Ozawa [13] and then later by Coudron and Vidick [6] and Coudron and Slofstra [5]. In all three of these papers, this slight communication between the players was indicated by the fact that the commutator of any measurement for Alice and any measurement for Bob has "small" operator norm; we refer to these as δ-op-almost commuting strategies, where δ is an upper bound on the operator norm of all such commutators. Our second main result indeed uses this formalism for a small amount of communication between the two players. However, our first main result quantifies the small amount of communication by simply asking that the state applied to the commutator of the two measurements be small, where the state in question is the one that the two players share according to their strategy; we refer to these as δ-st-almost commuting strategies[2]. Once again, δ is an upper bound on the state applied to all such commutators. Quantum mechanically, this is also a very natural notion of almost-commuting strategy, since one might only care about a small amount of communication between the players with respect to the state that is being used to compute the quantum correlations.

Traditionally, given a nonlocal game $\mathfrak{G}$, $\mathrm{val}^{co}(\mathfrak{G})$ (resp. $\mathrm{val}^{*}(\mathfrak{G})$) are the maximal expected probability of winning the game $\mathfrak{G}$ when allowed to use any quantum commuting strategy (resp. any finite-dimensional quantum commuting strategy). In this paper, we also consider the two different kinds of almost-commuting analogs of these values. Indeed, for any positive real number δ, we will consider the values $\mathrm{val}^{co}_{\delta,\mathrm{op}}(\mathfrak{G})$ (resp. $\mathrm{val}^{*}_{\delta,\mathrm{op}}(\mathfrak{G})$) to be the maximal expected probability of winning the game $\mathfrak{G}$ when allowed to use any δ-op-almost

[2]This will be defined precisely in the next section

commuting strategies (resp. all finite-dimensional δ-op-commuting strategies). One defines $\mathrm{val}^{co}_{\delta,st}(\mathfrak{G})$ and $\mathrm{val}^{*}_{\delta,st}(\mathfrak{G})$ similarly, this time using δ-st-almost commuting strategies.

A consequence of MIP*=RE is a negative solution to **Tsirelson's Problem**. That is, there is a nonlocal game $\mathfrak{G}$ such that $\mathrm{val}^*(\mathfrak{G}) \neq \mathrm{val}^{co}(\mathfrak{G})$. We can now state our first main result, which shows that if KEP holds, then the almost-commuting with respect to states version of Tsirelson's problem would actually have a positive answer for any choice of "amount of communication":

Theorem 1. *Suppose that KEP holds. Then* $\mathrm{val}^{co}_{\delta,st}(\mathfrak{G}) = \mathrm{val}^{*}_{\delta,st}(\mathfrak{G})$ *for all* $\delta > 0$ *and all nonlocal games* $\mathfrak{G}$*.*

Our second result concerns the almost-commuting version of the complexity class MIP^{co}. The complexity class MIP^{co} consists of all languages L such that there is an effective assignment from sequeunces of bits z to nonlocal games $\mathfrak{G}$ for which: if $z \in L$, then $\mathrm{val}^{co}(\mathfrak{G}_z) = 1$, while if $z \notin L$, then $\mathrm{val}^{co}(\mathfrak{G}_z) \leq \frac{1}{2}$. In [12], the authors pose the question of whether MIP^{co} coincides with the complexity class coRE consisting of all problems whose complement is recurseively enumerable. (That all languages in MIP^{co} are contained in coRE is well-known; the authors are partial to the model-theoretic proof given in [10, Theorem 7.3].) We show that if KEP holds, then the almost-commuting with respect to operator-norm version of this problem has a negative answer:

Theorem 2. *If KEP holds, then for every computable function* $\delta : \mathbb{N} \to [0, 1]$*, every language in* $\mathrm{MIP}^{co}_{\delta,op}$ *is recursively enumerable.*

The complexity class $\mathrm{MIP}^{co}_{\delta,op}$ will be defined below, but essentially it says that a language L belongs to $\mathrm{MIP}^{co}_{\delta,op}$ if there is an effective assignment from sequences of bits z to nonlocal game $\mathfrak{G}_z$ such that: if $z \in L$, then $\mathrm{val}^{co}_{\delta(|z|),op}(\mathfrak{G}_z) = 1$, while if $z \notin L$, then $\mathrm{val}^{co}_{\delta(|z|),op}(\mathfrak{G}_z) \leq \frac{1}{2}$.

We note a few things about Theorem 2. First, one can show that every language in $\mathrm{MIP}^{co}_{\delta,op}$ is coRE (regardless of the truth of KEP), whence the theorem implies that every language in $\mathrm{MIP}^{co}_{\delta,op}$ is actually decidable provided that KEP holds. Second, assuming KEP, every language in $\mathrm{MIP}^{co}_{\delta,st}$, the obvious state-version of $\mathrm{MIP}^{co}_{\delta,op}$, is also decidable; this follows from Theorem 1. Finally, the class $\mathrm{MIP}^{co}_{\delta,op}$ differs from the class $\mathrm{MIP}^{co}_{\delta}$ introduced by Coudron and Slofstra in [5]; in particular, every language in their class is decidable without any KEP assumption.

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2. Preliminaries

A **nonlocal game with** n **questions and** k **answers** is a pair $\mathfrak{G} = (\pi, D)$, where π is a probability distribution on $[n]$ and $D : [n] \times [n] \times [k] \times [k] \to \{0, 1\}$ is called the **decision predicate for the game**. Here, $[n] := \{1, \ldots, n\}$ and analogously for $[k]$. We also refer to the pair (n, k) as the **dimensions** of $\mathfrak{G}$. We view two players, henceforth referred to as Alice and Bob, playing $\mathfrak{G}$ as follows: a pair of questions $(x, y) \in [n] \times [n]$ is randomly chosen according to π and then Alice and Bob somehow respond with a pair of answers $(a, b) \in [k] \times [k]$; they win the game if $D(x, y, a, b) = 1$ and otherwise they lose the game.

In order to describe their strategies for playing $\mathfrak{G}$, we need the notion of POVMs. Recall that a **positive operator-valued measure** or **POVM** on a Hilbert space $\mathcal{H}$ is a finite collection $A_1, \ldots, A_k$ of positive operators on $\mathcal{H}$ such that $A_1 + \cdots + A_k = I$. We refer to k as the **length** of the POVM. More generally, one can use the same definition to define a POVM in any C^*-algebra.

For each k, let $\varphi_k(X)$ denote the formula

$$\max\left(\max_{1\leq i\leq k}\inf_{Z_i}\|Z_i^* Z_i - X_i\|, \|\sum_{i=1}^{k} X_i - I\|\right)$$

in the k variables $X = (X_1, \ldots, X_k)$. The following lemma is easy but will be used throughout the paper:

Lemma 3. *For each $\epsilon > 0$ and $k \geq 1$, there is $\delta > 0$ such that: for any C^*-algebra C and any elements $A_1, \ldots, A_k$ from the unit ball of C, if $\varphi_k(A_1, \ldots, A_k) < \delta$, then there is a POVM $B_1, \ldots, B_k$ in C such that $\max_{1\leq i\leq k} \|A_i - B_i\| < \epsilon$.*

We use POVMs to define strategies for nonlocal games. First suppose that $\mathfrak{G}$ is a nonlocal game with dimensions (n, k) and C is a C^*-algebra. A $\mathfrak{G}$**-measurement in** C is a tuple $A := (A^x)_{x\in[n]}$ of POVMs in C, each of which has length k. Of course, the notion of a $\mathfrak{G}$-measurement in C only depends on the dimensions of the nonlocal game, but the terminology will prove useful in the sequel. Thus, corresponding to each possible question and answer pair $(x, a) \in [n] \times [k]$, we will have a positive element $A^x_a \in C$, and for each $x \in [n]$, we have $\sum_{a\in[k]} A^x_a = I$.

A $\mathfrak{G}$**-strategy in** C is a tuple $\sigma := (A, B, \phi)$, where A and B are $\mathfrak{G}$-measurements in C and $\phi \in S(C)$ is a state on C. Given a $\mathfrak{G}$-strategy σ in C as above, we define the corresponding correlation matrix $p_\sigma \in [0, 1]^{n^2k^2}$ by $p_\sigma(a, b|x, y) = \phi(A^x_a \bullet B^y_b)$, where, for any $A, B \in C$, we define $A \bullet B := \frac{1}{2}(A^{1/2}BA^{1/2} + B^{1/2}AB^{1/2})$.[3] The intuition behind this definition is that if Alice and Bob play $\mathfrak{G}$ according to the strategy σ, then upon receiving the question pair (x, y), they both measure their portion of the state ϕ using their POVMs A^x and B^y; since we are not assuming that these measurements commute, we take the average of the results obtained from when Alice measures first and from when Bob measures first. Consequently, $p_\sigma(a, b|x, y)$ is the probability that they answer the question pair (x, y) with the answer pair (a, b) when using the strategy σ.

Of course, if each A^x_a and B^y_b commute, the above definition degenerates to the usual situation of calculating $p_\sigma(a, b|x, y) = \phi(A^x_a B^y_b)$ and we call the strategy σ **commuting**. As discussed in the introduction, there are two natural ways of generalizing the commuting situation to the almost commuting situation. Fix a positive real number $\delta > 0$. We call the strategy σ **δ-op-almost commuting** if $\sum_{a,b\in[k]} \|[A^x_a, B^y_b]\| < \delta$ for all $(x, y) \in [n] \times [n]$. On the other hand, we call the strategy σ **δ-state-almost commuting** if $\sum_{a,b\in[k]} \phi([A^x_a, B^y_b]) < \delta$ for all $(x, y) \in [n] \times [n]$.[4] Note that the notion of a δ-op-commuting strategy depends only on the pair of $\mathfrak{G}$-measurements, whence it makes sense to say that a pair of such measurements is a δ-op-almost commuting pair. It is clear that any δ-op-almost commuting strategy is also a δ-state-almost commuting strategy.[5]

Given a $\mathfrak{G}$-strategy σ in C, we define the **value of $\mathfrak{G}$ when playing according to** σ to be the expected value Alice and Bob have of winning the game when playing according to σ, that is,

$$\mathrm{val}(\mathfrak{G}, \sigma) := \sum_{x,y} \pi(x, y) \sum_{a,b} D(x, y, a, b) p_\sigma(a, b|x, y).$$

In the sequel, it will behoove us to define, for every pair (A, B) of $\mathfrak{G}$-measurements in C, the element

$$\mathfrak{G}(A, B) := \sum_{x,y} \mu(x, y) \sum_{a,b} D(x, y, a, b)(A^x_a \bullet B^y_b).$$

With this notation, we have $\mathrm{val}(\mathfrak{G}, \sigma) = \phi(\mathfrak{G}(A, B))$.

[3]This notation seems to have first been considered by Ozawa in [13].

[4]We thank Thomas Vidick for pointing out that this is the correct notion of state-almost commuting. Our original definition was simply that $\phi([A^x_a, B^y_b]) < \delta$ for all $(x, y) \in [n] \times [n]$ and all $(a, b) \in [k] \times [k]$. However, this leads to a trivial notion when the answer set for the game is quite large.

[5]We could alternately have defined $\mathfrak{G}$ to be a δ-op-almost commuting strategy if $\|[A^x_a, B^y_b]\| < \delta$ for all $(x, y) \in [n] \times [n]$ and all $(a, b) \in [k] \times [k]$ without falling into a trivial definition as with the state-dependent version. However, uniformizing the two definitions implies that δ-op-almost commuting strategies are δ-state-almost commuting strategies, which is a desirable feature for what is to follow.

If one considers the supremum of $\mathrm{val}(\mathfrak{G},\sigma)$ as σ ranges over all commuting $\mathfrak{G}$-strategies in $\mathcal{B}(\mathcal{H})$ (with no dimension restriction on $\mathcal{H}$), one obtains the **commuting value of** $\mathfrak{G}$, denoted $\mathrm{val}^{co}(\mathfrak{G})$. If one instead considers the supremum to be over commuting $\mathfrak{G}$-strategies in $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ restricted to be finite-dimensional, one arrives at the **entangled value of** $\mathfrak{G}$, denoted $\mathrm{val}^*(\mathfrak{G})$. Of course $\mathrm{val}^*(\mathfrak{G}) \leq \mathrm{val}^{co}(\mathfrak{G})$ for all nonlocal games $\mathfrak{G}$. Tsirelson's problem asked if $\mathrm{val}^*(\mathfrak{G}) = \mathrm{val}^{co}(\mathfrak{G})$ for all nonlocal games $\mathfrak{G}$; a consequence of MIP*=RE is a negative solution to Tsirelson's problem. That is, there is some nonlocal game $\mathfrak{G}$ for which $\mathrm{val}^*(\mathfrak{G}) < \mathrm{val}^{co}(\mathfrak{G})$.

We can of course define almost-commuting analogs of these quantum game values. Indeed, given $\delta > 0$, we define $\mathrm{val}^{co}_{\delta,op}(\mathfrak{G})$ (resp. $\mathrm{val}^{co}_{\delta,st}(\mathfrak{G})$) to be the supremum of $\mathrm{val}(\mathfrak{G},\sigma)$ as σ ranges over all δ-op-almost commuting (resp. δ-state-almost commuting) $\mathfrak{G}$-strategies in $\mathcal{B}(\mathcal{H})$.[6] By restricting our attention to finite-dimensional Hilbert spaces $\mathcal{H}$, we arrive at the values $\mathrm{val}^*_{\delta,op}(\mathfrak{G})$ and $\mathrm{val}^*_{\delta,st}(\mathfrak{G})$.

The following inequalities are clear for any nonlocal game $\mathfrak{G}$ and any $\delta > 0$:

- $\mathrm{val}^*(\mathfrak{G}) \leq \mathrm{val}^*_{\delta,op}(\mathfrak{G}) \leq \mathrm{val}^*_{\delta,st}(\mathfrak{G})$.
- $\mathrm{val}^{co}(\mathfrak{G}) \leq \mathrm{val}^{co}_{\delta,op}(\mathfrak{G}) \leq \mathrm{val}^{co}_{\delta,st}(\mathfrak{G})$.
- $\mathrm{val}^*_{\delta,op}(\mathfrak{G}) \leq \mathrm{val}^{co}_{\delta,op}(\mathfrak{G})$.
- $\mathrm{val}^*_{\delta,st}(\mathfrak{G}) \leq \mathrm{val}^{co}_{\delta,st}(\mathfrak{G})$.

We end this preliminary section with some notation and terminology concerning ultraproducts of C*-algebras. Throughout this paper, $\mathcal{U}$ denotes a nonprincipal ultrafilter on $\mathbb{N}$. Given a C*-algebra C, its **ultrapower with respect to** $\mathcal{U}$, denoted $C^{\mathcal{U}}$, is the quotient of the Banach algebra $\ell^\infty(\mathbb{N}, C)$ of all uniformly norm-bounded sequences from C by the ideal of elements $(A_m)_{m\in\mathbb{N}}$ for which $\lim_{\mathcal{U}} \|A_m\| = 0$. It is well-known that $C^{\mathcal{U}}$ is a C*-algebra once again. Given $(A_m)_{m\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, C)$, we denote its coset in $C^{\mathcal{U}}$ by $(A_m)_{\mathcal{U}}$. If $\phi_m \in S(C)$ is a state on C for each $m \in N$, we let the **ultraproduct state** (also known in the literature as **limit state**) $\phi := (\phi_m)_{\mathcal{U}}$ on C be defined by $\phi((A_m)_{\mathcal{U}}) := \lim_{\mathcal{U}} \phi_m(A_m)$.

3. Proof of Theorem 1

Our proof of Theorem 1 requires a couple of preliminary lemmas. The first lemma is probably well-known (very similar to Proposition 3.4.2 in [7]), but we include a proof for completeness. Below, $\mathrm{sp}(A)$ denotes the spectrum of the element A.

[6] Since states on concretely represented C*-algebras can always be extended to states on the $\mathcal{B}(\mathcal{H})$ containing them, we can replace $\mathcal{B}(\mathcal{H})$ with an arbitrary C*-algebra C without changing this definition.

Lemma 4. *Suppose that* C *is a separable* C^**-algebra,* $A \in C^{\mathcal{U}}$ *is self-adjoint, and* $r \in \sigma(a)$*. Then there are self-adjoint elements* $A_m \in C$ *such that* $A = (A_m)_{\mathcal{U}}$ *and such that* $r \in \mathrm{sp}(A_m)$ *for* $\mathcal{U}$*-many* $m \in \mathbb{N}$*.*

Proof. First write $A = (B_m)_{\mathcal{U}}$ with $B_m \in A$ self-adjoint for each $m \in \mathbb{N}$. It suffices to verify that $\lim_{\mathcal{U}} d(r, \mathrm{sp}(B_m)) = 0$. Indeed, if this is the case, then for $\mathcal{U}$-many $m \in \mathbb{N}$, there is $s_n \in \mathrm{sp}(B_m)$ such that $\lim_{\mathcal{U}}(r_m - s_s) = 0$. Setting $A_m := B_m - (s_m - r_m) \cdot 1$, we have that A_m is self-adjoint, $A = (A_m)_{\mathcal{U}}$, and $r_m \in \mathrm{sp}(A_m)$ for $\mathcal{U}$-many $m \in \mathbb{N}$.

Suppose now, towards a contradiction, that there is $\epsilon > 0$ such that $d(r, \mathrm{sp}(B_m)) \geq \epsilon$ for $\mathcal{U}$-many $m \in \mathbb{N}$. By functional calculus, we can define $B'_m := (B_m - r)^{-1}$ and we note that $\|B'_m\| \leq \frac{1}{\epsilon}$. Setting $B' := (B'_m)_{\mathcal{U}}$, we have $B' = (A - r)^{-1}$, contradicting the fact that $r \in \mathrm{sp}(A)$. □

We thank Aaron Tikuisis for providing us with an easier proof of the following lemma. For the proof, we recall that for any self-adjoint element A of a C^*-algebra C, we have $\mathrm{sp}(A) = \{\phi(A) \ : \ \phi \in S(C)\}$.

Lemma 5. *For any* C^**-algebra* C*, any separable* C^**-subalgebra* D *of* $C^{\mathcal{U}}$*, and any state* ϕ *on* D*, there are states* ϕ_m *on* C *such that* $\phi = (\phi_m)_{\mathcal{U}}|D$*.*

Proof. By a standard argument (see, for example, [14, Theorem 8]), it suffices to show that the ultraproduct states on $C^{\mathcal{U}}$ are weak*-dense in $S(C^{\mathcal{U}})$. Suppose, towards a contradiction, that this is not the case. Let X denote the weak*-closure of the set of ultraproduct states in $S(C^{\mathcal{U}})$ and take $\phi \in S(C^{\mathcal{U}}) \setminus X$. By the Hahn-Banach Separation Theorem, there is a self-adjoint element $A \in C^{\mathcal{U}}$ such that $\psi(A) < 1$ for all $\psi \in X$ and yet $\phi(A) > 1$. Set $r := \phi(A) \in \mathrm{sp}(A)$. By Lemma 4, we may take a sequence $(A_m)_{m \in \mathbb{N}}$ such that $r \in \mathrm{sp}(A_m)$ for all $n \in \mathbb{N}$ and such that $A = (A_m)_{\mathcal{U}}$. Take $\psi_m \in S(C)$ such that $\psi_m(A_m) = r$. Setting $\psi := (\psi_m)_{\mathcal{U}} \in X$, we have that $\psi(A) = r$, a contradiction. □

We need one reminder before giving the proof of Theorem 1. If M is a von Neumann algebra and $\rho \in S(M)$ is a faithful, normal state on M, then we can define the norm $\|\cdot\|_\rho^\#$ on M by $\|A\|_\rho^\# := \sqrt{\rho(A^*A) + \rho(AA^*)}$. It is a standard fact that $\|\cdot\|_\rho^\#$ defines the strong*-topology on the unit ball of M.

We are now ready for the proof of Theorem 1:

Proof of Theorem 1. Fix $\delta > 0$ and a nonlocal game $\mathfrak{G}$. Set $r := \mathrm{val}^{co}_{\delta,st}(\mathfrak{G})$. Fix $\epsilon > 0$ and take a δ-st-almost commuting $\mathfrak{G}$-strategy $\sigma_1 := (A, B, \eta)$ in $\mathcal{B}(\mathcal{H})$ such that $\mathrm{val}(\mathfrak{G}, \sigma_1) > r - \epsilon$. Let C be the C^*-algebra generated by the coordinates of A and B and let η continue to denote the restriction of the original state η to C. By KEP and Lemma 5, there is a sequence $(\phi_m)_{m \in \mathbb{N}}$ of states on $\mathcal{O}_2$ and an embedding

$g : C \hookrightarrow \mathcal{O}_2^{\mathcal{U}}$ such that the restriction of the ultraproduct state $(\phi_n)_{\mathcal{U}}$ to $g(C)$ is the state η (viewed as a state on $g(C)$ in the obvious way). By Lemma 3, we may assume, without loss of generality, that there are $\mathfrak{G}$-measurements $A^{(m)}$ and $B^{(m)}$ in $\mathcal{O}_2$ such that, for all $x \in [n]$ and $a \in [k]$, we have $g(A_a^x) = ((A^{(m)})_a^x)_{\mathcal{U}}$ and $g(B_a^x) = ((B^{(m)})_a^x)_{\mathcal{U}}$. Take $m \in \mathbb{N}$ such that, setting $\bar{A} := A^{(m)}$, $\bar{B} := B^{(m)}$, $\phi := \phi_m$, and $\sigma_2 := (\bar{A}, \bar{B}, \phi)$, we have that σ_2 is a δ-st-almost commuting $\mathfrak{G}$-strategy in $\mathcal{O}_2$ for which $\mathrm{val}(\mathfrak{G}, \sigma_2) > r - 2\epsilon$.

Since $\mathcal{O}_2$ is simple and antiliminal, the type III factor states are dense in $S(\mathcal{O}_2)$ (see [2, Section 3]). Take a type III factor state $\psi \in S(\mathcal{O}_2)$ such that ϕ and ψ are sufficiently close on the elements of $\bar{A}$ and $\bar{B}$ so that, setting $\sigma_3 := (\bar{A}, \bar{B}, \psi)$, we have that σ_3 is a δ-st-almost commuting $\mathfrak{G}$-strategy in $\mathcal{O}_2$ for which $\mathrm{val}(\mathfrak{G}, \sigma_3) > r - 3\epsilon$. Let $\pi_\psi : \mathcal{O}_2 \to \mathcal{B}(L^2(\mathcal{O}_2, \psi))$ denote the GNS representation corresponding to ψ and let M denote $\pi_\psi(\mathcal{O}_2)''$, which is a type III factor. Moreover, since $\mathcal{O}_2$ is nuclear, by [3] and [4], we have that M is a hyperfinite type III factor. Let ψ also denote the restriction of the GNS state to M. Since the faithful normal states on M are dense in the set of normal states on M, we may find a faithful normal state ρ on M which approximates ψ well enough so that, setting $\sigma_4 := (\pi_\psi(\bar{A}), \pi_\psi(\bar{B}), \rho)$, we have that σ_4 is a δ-st-almost commuting $\mathfrak{G}$-strategy in M for which $\mathrm{val}(\mathfrak{G}, \sigma_4) > r - 4\epsilon$. Since M is hyperfinite, by the $\|\cdot\|_\rho^\#$-version of Lemma 3, there is a matrix algebra $M_m(\mathbb{C})$ contained in M and $\mathfrak{G}$-measurements $\tilde{A}$ and $\tilde{B}$ in $M_m(\mathbb{C})$ such that, setting $\sigma_5 := (\tilde{A}, \tilde{B}, \rho)$, we have that σ_5 is a δ-st-almost commuting $\mathfrak{G}$-strategy in $M_m(\mathbb{C})$ for which $\mathrm{val}(\mathfrak{G}, \sigma_5) > r - 5\epsilon$. It follows that $r - 5\epsilon < \mathrm{val}^*_{\delta,\mathrm{st}}(\mathfrak{G})$. The result follows by letting ϵ tend to 0. $\square$

Theorem 1 lends itself to an intriguing attempt to refute KEP. Indeed, recall that in [12], the authors construct an effect map $T \mapsto \mathfrak{G}_T$ from Turing machines to nonlocal games such that: if T halts, then $\mathrm{val}^*(\mathfrak{G}_T) = 1$ while if T does not halt, then $\mathrm{val}^*(\mathfrak{G}_T) \leq \frac{1}{2}$. Suppose, in addition, that this mapping may be chosen with the property that for each T that does not halt, there is a positive number $\delta_T > 0$ such that $\mathrm{val}^*_{\delta_T,\mathrm{st}}(\mathfrak{G}_T) < 1$. We claim then that KEP has a negative solution. Indeed, we claim that under this assumption and a positive solution to KEP, we could decide the halting problem, yielding a contradiction. To see this, one can effectively start computing brute force lower bounds on $\mathrm{val}^*(\mathfrak{G}_T)$ and if one ever sees that $\mathrm{val}^*(\mathfrak{G}_T) > \frac{1}{2}$, one knows that T halts. On the other hand, one can start running machines to see if $\mathrm{val}^{co}_{\frac{1}{n},\mathrm{st}}(\mathfrak{G}_T) \leq 1 - \frac{1}{n}$. (This can be done, for example, using the Completeness Theorem argument given in [10, Theorem 7.3].) Since KEP holds, the above assumption implies that one such machine will eventually return a positive answer and thus we could conclude that T does not halt.

4. Proof of Theorem 2

We begin this section by precisely defining the complexity class $\mathrm{MIP}^{\mathrm{co}}_{\delta,\mathrm{op}}$:

Definition 6. Fix a computable function $\delta : \mathbb{N} \to [0,1]$. We say that a language L belongs to $\mathrm{MIP}^{\mathrm{co}}_{\delta,\mathrm{op}}$ if there is an efficient mapping from sequences of bits z to nonlocal games $\mathfrak{G}_z$ such that:

- If $z \in L$, then $\mathrm{val}^{\mathrm{co}}_{\delta(|z|),\mathrm{op}}(\mathfrak{G}_z) = 1$.
- If $z \notin L$, then $\mathrm{val}^{\mathrm{co}}_{\delta(|z|),\mathrm{op}}(\mathfrak{G}_z) \leq \frac{1}{2}$.

Theorem 2 will follow from a more general result. We first need to introduce some terminology. We say that a separable C^*-algebra C is **locally universal** if every separable C^*-algebra embeds into some (equiv. any) ultrapower of C. Thus, the KEP asks if $\mathcal{O}_2$ is locally universal. Locally universal C^*-algebras exist in abundance; see, for example, [9].

Let C be a C^*-algebra. Given a subset X of the unit ball of C, we let $\langle X\rangle$ be the smallest subset of C containing X and closed under rational rounded combinations (that is, linear combinations of the form $\alpha x + \beta y$, where $\alpha, \beta \in \mathbb{Q}(i)$ are such that $|\alpha| + |\beta| \leq 1$), multiplication, and adjoint. We say that X **generates** C if $\langle X\rangle$ is dense in the unit ball of C. A **presentation** of C is a pair $C^\# := (C, (a_n)_{n\in\mathbb{N}})$, where $\{a_n : n \in \mathbb{N}\}$ is a subset of the unit ball of C that generates C. Elements of the sequence $(a_n)_{n\in\mathbb{N}}$ are referred to as **special points** of the presentation while elements of $\langle\{a_n : n \in \mathbb{N}\}\rangle$ are referred to as **rational points** of the presentation. Finally, we say that C has a **computable presentation** if there is a presentation $C^\#$ of C for which there is an algorithm such that, upon input of a rational point $p \in C^\#$ and $k \in \mathbb{N}$, returns a rational number q such that $|\|p\| - q| < 2^{-k}$.

The following theorem will imply Theorem 2, as we will explain afterwards:

Theorem 7. *If there is a locally universal C^*-algebra C that has a computable presentation, then for every computable function $\delta : \mathbb{N} \to [0,1]$, every language in $\mathrm{MIP}^{\mathrm{co}}_{\delta,\mathrm{op}}$ is recursively enumerable.*

Proof. Suppose that C is a locally universal C^*-algebra with a computable presentation $C^\#$. Fix a computable function $\delta : \mathbb{N} \to [0,1]$ and suppose that L belongs to $\mathrm{MIP}^{\mathrm{co}}_{\delta,\mathrm{op}}$. Here is the algorithm that shows that L is recursively enumerable. Suppose that one inputs the sequence of bits z. Set $\mathfrak{G} := \mathfrak{G}_z$. Start enumerating pairs of $\delta(|z|)$-almost commuting $\mathfrak{G}$-measurements (A, B) in C that consist only of rational points of $C^\#$; this is possible since the presentation is computable. If (A, B) is such a $\delta(|z|)$-op-almost commuting pair, approximate $\|\mathfrak{G}(A, B)\|$ to within error $\frac{1}{4}$. If this approximation exceeds $\frac{1}{2}$, then declare that $z \in L$.

Here is why the algorithm works. We first show that if $z \in L$, then the algorithm will tell us so. As above, set $\mathfrak{G} := \mathfrak{G}_z$. Fix $\epsilon > 0$ small enough and let $\sigma := (A, B, \phi)$ be a $\delta(|z|)$-op-almost commuting $\mathfrak{G}$-strategy in $\mathcal{B}(\mathcal{H})$ such that $\mathrm{val}(\mathfrak{G}, \sigma) > 1 - \epsilon$. It follows that $\|\mathfrak{G}(A, B)\| > 1 - \epsilon$. Let D be the C*-algebra generated by the coordinates of A and B and consider an embedding of D into $C^{\mathcal{U}}$. It follows from Lemma 3 that there are $\delta(|z|)$-op-almost commuting $\mathfrak{G}$-measurements $\bar{A}$ and $\bar{B}$ in C for which $\|\mathfrak{G}(\bar{A}, \bar{B})\| > 1 - 2\epsilon$. Without loss of generality, one can assume that the coordinates of $\bar{A}$ and $\bar{B}$ are rational points of $C^{\#}$. If ϵ is small enough, then approximating $\|\mathfrak{G}(\bar{A}, \bar{B})\|$ to within error $\frac{1}{4}$ will exceed $\frac{1}{2}$. Thus, our algorithm will eventually tell us that $z \in L$.

We now check that the algorithm makes no mistakes, that is, if the algorithm tells us that $z \in L$, then in fact z does belong to L. If the algorithm tells us that (A, B) is a $\delta(|z|)$-op-almost commuting pair of $\mathfrak{G}$-measurements in C for which $\|\mathfrak{G}(A, B)\| > \frac{1}{2}$, then there will be some state ϕ on C such that $\phi(\mathfrak{G}(A, B)) > \frac{1}{2}$. Setting $\sigma := (A, B, \phi)$, we see that $\mathrm{val}^{co}_{op,\delta}(\mathfrak{G}) > \frac{1}{2}$. Consequently, $z \in L$, as desired. □

By a recent result of Alec Fox [8], $\mathcal{O}_2$ has a computable presentation. Theorem 2 follows from this fact and Theorem 7.

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