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The petit topos of globular sets

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Abstract

There are now several definitions of weak ω-category [1,2,5,19]. What is pleasing is that they are not achieved by ad hoc combinatorics. In particular, the theory of higher operads which underlies Michael Batanin's definition is based on globular sets. The purpose of this paper is to show that many of the concepts of [2] (also see [17]) arise in the natural development of category theory internal to the petit [1] topos **Glob** of globular sets. For example, higher spans turn out to be internal sets, and, in a sense, trees turn out to be internal natural numbers.

MSC: 18D05

1. Globular objects and ω-categories

A globular set is an infinite-dimensional graph. To formalize this, let **G** denote the category whose objects are natural numbers and whose only non-identity arrows are

$$\sigma_m, \tau_m : m \to n \quad \text{for all } m < n$$

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[1] The distinction between toposes that are "space like" (or *petit*) and those which are "category-of-space like" (or *gros*) was investigated by Lawvere [9,10]. The gros topos of reflexive globular sets has been studied extensively by Michael Roy [12].

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such that the triangles

$$\begin{tikzcd} k \arrow[rr, "\beta_k"] \arrow[dr, "\beta_k"'] & & n \\ & m \arrow[ur, "\alpha_m"'] & \end{tikzcd}$$

commute for all $k < m < n$ and all $\alpha, \beta \in \{\sigma, \tau\}$. Clearly $\mathbf{G}$ is generated by the subgraph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} 2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} 3 \underset{\tau_3}{\overset{\sigma_3}{\rightrightarrows}} \cdots.$$

A functor $X : \mathbf{G}^{\mathrm{op}} \to \mathscr{X}$ is called a *globular object of* $\mathscr{X}$. That is, a globular object X consists of a sequence $(X_n)_{n \geq 0}$ of objects X_n together with arrows $s_n, t_n : X_{n+1} \to X_n$ such that $s_n \circ s_{n+1} = s_n \circ t_{n+1}$ and $t_n \circ s_{n+1} = t_n \circ t_{n+1}$. Each globular object X gives a diagram

$$\underset{t_3}{\overset{s_3}{\rightrightarrows}} X_3 \underset{t_2}{\overset{s_2}{\rightrightarrows}} X_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} X_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} X_0$$

in $\mathscr{X}$. For $m < n$, we write $s_m, t_m : X_n \to X_m$ for the value of $X : \mathbf{G}^{\mathrm{op}} \to \mathscr{X}$ at the arrows $\sigma_m, \tau_m : m \to n$, respectively. There is a category $\mathrm{Glob}\,\mathscr{X} = [\mathbf{G}^{\mathrm{op}}, \mathscr{X}]$ of globular objects where the arrows are natural transformations.

Suppose $\mathscr{X}$ is a category with pullbacks. For $m < n$, define the object $X_n \underset{m}{\times} X_n$ to be the following pullback.

$$\begin{tikzcd} X_n \underset{m}{\times} X_n \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_n \arrow[d, "t_m"] \\ X_n \arrow[r, "s_m"'] & X_m \end{tikzcd}$$

An *ω-category in* $\mathscr{X}$ is a globular object X together with, for all $m < n$, arrows

$$\#_m : X_n \underset{m}{\times} X_n \to X_n, \quad i_m : X_m \to X_n$$

such that the diagram

$$\begin{tikzcd} X_n \underset{m}{\times} X_n \arrow[r, shift left=2, "p_2"] \arrow[r, "\#_m" description] \arrow[r, shift right=2, "p_1"'] & X_n \arrow[r, shift left=2, "s_m"] \arrow[r, shift right=2, "t_m"'] & X_m \arrow[l, "i_m" description] \end{tikzcd}$$

is the truncation of the nerve of a category in $\mathscr{X}$ and, for all $m < k < n$, the arrows $\#_m, i_m$ are functors in $\mathscr{X}$ for the category structures on $X_n \underset{m}{\times} X_n$, X_n, X_m determined by $\#_k, i_k$. An *ω-functor* $f : X \to Y$ between ω-categories X, Y in $\mathscr{X}$ is an arrow in $\mathrm{Glob}\,\mathscr{X}$ which commutes with all the $\#_m$ and i_m. There is a category $\mathrm{Omcat}\,\mathscr{X}$ of ω-categories in $\mathscr{X}$ where the arrows are ω-functors. Write $\Phi : \mathrm{Omcat}\,\mathscr{X} \to \mathrm{Glob}\,\mathscr{X}$ for the forgetful functor.

Write **Set** for the category of sets and **set** for the category of small sets; we intend that **set** should be a category in **Set**. We write **n** for a representative set of cardinality n. Put

$$\mathbf{glob} = \mathrm{Glob}\boldsymbol{set},\ \mathbf{omcat} = \mathrm{Omcat}\boldsymbol{set},\ \mathbf{Glob} = \mathrm{Glob}\boldsymbol{Set},\ \mathbf{Omcat} = \mathrm{Omcat}\boldsymbol{Set}.$$

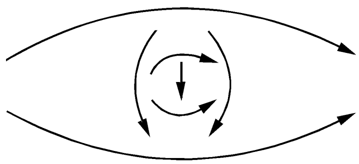
The forgetful functor $\Phi : \mathbf{Omcat} \to \mathbf{Glob}$ can be regarded as a globular object

$$\cdots \underset{t_2}{\overset{s_2}{\rightrightarrows}} \Phi_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} \Phi_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} \Phi_0$$

in the functor category $[\mathbf{Omcat}, \mathbf{Set}]$. Each of the functors $\Phi_n : \mathbf{Omcat} \to \mathbf{Set}$ is representable:

$$\Phi_n \cong \mathbf{Omcat}(\mathbf{2}_n, -).$$

Here is a picture of $\mathbf{2}_4$.



So we obtain a coglobular object

$$\mathbf{2}_* : \quad \mathbf{2}_0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} \mathbf{2}_1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} \mathbf{2}_2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} \mathbf{2}_3 \underset{\tau_3}{\overset{\sigma_3}{\rightrightarrows}} \cdots$$

in **Omcat**. For each ω-category A, we have an isomorphism of globular sets

$$\Phi(A) \cong \mathbf{Omcat}(\mathbf{2}_*, A).$$

Since $\Phi(A)$ is naturally equipped with the ω-category structure of A, it follows, using the Yoneda Lemma, that $\mathbf{2}_*$ becomes a co-ω-category in **Omcat**.

An ω-functor $f : A \to X$ is said to have *discrete fibres* when, for all $a \in A_n$, if $f_n(a) \in X_n$ is an identity for the composition $\#_k$ on X_n, then a is an identity for the composition $\#_k$ on A_n. Write $D(\mathbf{omcat}/X)$ for the full subcategory of the slice category $\mathbf{omcat}/X$ consisting of the ω-functors $f : A \to X$ with discrete fibres. We obtain a pseudo-functor (=homomorphism of bicategories)

$$D(\mathbf{omcat}/-) : \mathbf{Omcat}^{\mathrm{op}} \to \mathbf{Cat}$$

by defining it on each arrow $h : Y \to X$ via pullback $h^* : D(\mathbf{omcat}/X) \to D(\mathbf{omcat}/Y)$ along h.

2. The internalization of a topos

We now make some general remarks about toposes – particularly presheaf toposes. The important property distinguishing a finitely complete category $\mathscr{E}$ as a topos is the existence of the *power object* $\mathscr{P}X$ for each object X: the power object is characterized by a natural isomorphism

$$\mathscr{E}(Y, \mathscr{P}X) \cong \mathrm{Rel}(\mathscr{E})(X, Y).$$

With the existence of a natural numbers object, intuitionistic mathematics can be developed in $\mathscr{E}$. We can also do some category theory by looking at $\mathrm{Cat}(\mathscr{E})$; but this restricts us to "small categories". If $\mathscr{E}$ is a Grothendieck topos, we can avail ourselves of a bigger universe and expand $\mathscr{E}$ to the category $\mathscr{E}'$ of sheaves on the same site with values in the bigger universe. We are then interested in, for each $X \in \mathscr{E}$, a category $\mathscr{S}X$ in $\mathscr{E}'$ with a pseudo-natural equivalence

$$\mathscr{E}'(Y, \mathscr{S}X) \simeq \mathrm{Span}(\mathscr{E})(X, Y).$$

Of course, just as $\mathscr{P}X = \Omega^X$, where Ω is the subobject classifier $\mathscr{P}\mathbf{1}$ and $\mathbf{1}$ is the terminal object of $\mathscr{E}$, we also have

$$\mathscr{S}X = \mathscr{S}^X$$

where $\mathscr{S} = \mathscr{S}\mathbf{1}$. We call the category $\mathscr{S}$ in $\mathscr{E}'$ the *internalization of* $\mathscr{E}$; there is a pseudo-natural equivalence

$$\mathscr{E}'(Y, \mathscr{S}) \simeq \mathscr{E}/Y.$$

Consider the case of a presheaf topos $\mathscr{E} = [\mathscr{C}^{\mathrm{op}}, \mathbf{set}]$. We can find $\mathscr{P}X$ by using the Yoneda Lemma:

$$\begin{aligned}(\mathscr{P}X)(U) &\cong [\mathscr{C}^{\mathrm{op}}, \mathbf{set}](\mathscr{C}(-, U), \mathscr{P}X)\\ &\cong \mathrm{Rel}([\mathscr{C}^{\mathrm{op}}, \mathbf{set}])(X, \mathscr{C}(-, U))\\ &\cong [(U\#X)^{\mathrm{op}}, \mathbf{2}],\end{aligned}$$

where $\mathbf{2} = \{0, 1\}$ with the usual order, and $U\#X$ is the category (used in [13,15]) whose objects are pairs $(r : V \to U, x \in XV)$ and whose arrows $f : (r, x) \to (r', x')$ are arrows $f : V \to V'$ in $\mathscr{C}$ such that $r' \circ f = r$ and $X(f)(x') = x$.

In this presheaf case, $\mathscr{E}' = [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ and we can likewise discover $\mathscr{S}X$ to be given by

$$(\mathscr{S}X)(U) \cong [(U\#X)^{\mathrm{op}}, \mathbf{set}] \quad (\simeq [\mathscr{C}^{\mathrm{op}}, \mathbf{set}]/X \times \mathscr{C}(-, U)).$$

In particular, [2]

$$\mathscr{S} = [(\mathscr{C}/-)^{\mathrm{op}}, \mathbf{set}],$$

[2] The idea of replacing **2** in set theory by **set** in category theory as the object of truth values is due to Lawvere; for example, see [8].

where $\mathscr{C}/-$ is functorial in the blank via composition. In [14], the author has called $\mathscr{S}$ the *gross internal full subcategory of* $[\mathscr{C}^{\mathrm{op}}, \mathbf{set}]$: internal full subcategories of $[\mathscr{C}^{\mathrm{op}}, \mathbf{set}]$ are precisely the subfunctors of $\mathscr{S} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ which land in $\mathbf{cat}$.

If $A : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ is a (pointwise) locally small category in $\mathscr{E}'$, there is a hom arrow

$$\mathrm{Hom}_A : A^{\mathrm{op}} \times A \to \mathscr{S}$$

given by $(\mathrm{Hom}_A)_U(a,b)(r : V \to U) = (AV)((Af)a, (Af)b) \in \mathbf{set}$.

We write $\mathrm{el}(X)$ for the category of elements of the functor $X : \mathscr{C}^{\mathrm{op}} \to \mathbf{set}$. In fact, we have $U\#X = \mathrm{el}(X \times \mathscr{C}(-, U))$.

Proposition. *The unique* (*up to isomorphism*) *extension of* $\mathscr{S} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ *to a limit-preserving functor* $[\mathscr{C}^{\mathrm{op}}, \mathbf{set}]^{\mathrm{op}} \to \mathbf{Cat}$ *is the composite*

$$[\mathscr{C}^{\mathrm{op}}, \mathbf{set}]^{\mathrm{op}} \xrightarrow{\mathrm{el}} \mathbf{cat}^{\mathrm{op}} \xrightarrow{(-)^{\mathrm{op}}} \mathbf{cat}^{\mathrm{op}} \xrightarrow{[-,\mathbf{set}]} \mathbf{Cat}.$$

Proof. It is well known that $[\mathscr{C}^{\mathrm{op}}, \mathbf{set}]^{\mathrm{op}}$ is the small-limit completion of $\mathscr{C}^{\mathrm{op}}$, so such an extension does exist. We must see that it is indeed the asserted composite. For this we must see that there is an isomorphism.

$$\begin{tikzcd}
\mathscr{C}_{\mathrm{op}} \arrow[rr, "\mathrm{Yon}^{\mathrm{op}}"] \arrow[dr, "\mathscr{S}"'] & \arrow[d, phantom, "\overset{\cong}{\Leftarrow}"] & {[C^{\mathrm{op}}, \mathbf{set}]^{\mathrm{op}}} \arrow[dl, "{[\mathrm{el}(-)^{\mathrm{op}}, \mathbf{set}]}"] \\
& \mathbf{Cat} &
\end{tikzcd}$$

which exhibits $[\mathrm{el}(-)^{\mathrm{op}}, \mathbf{set}]$ as a right Kan extension of $\mathscr{S}$ along the Yoneda embedding $Y\mathrm{on}^{\mathrm{op}}$. Beginning with the formula for the value of the right Kan extension at $X \in [\mathscr{C}^{\mathrm{op}}, \mathbf{set}]^{\mathrm{op}}$, we have the calculation

$$\int_U [[\mathscr{C}^{\mathrm{op}}, \mathbf{set}]^{\mathrm{op}}(X, Y\mathrm{on}U), \mathscr{S}U] \cong \int_U [XU, [(\mathscr{C}/U)^{\mathrm{op}}, \mathbf{set}]]$$
$$\cong \left[\left(\int^U \mathscr{C}/U \times XU\right)^{\mathrm{op}}, \mathbf{set}\right] \cong [\mathrm{el}(X)^{\mathrm{op}}, \mathbf{set}],$$

where we leave the calculation

$$\mathrm{el}(X) \cong \int^U \mathscr{C}/U \times XU$$

as an exercise; or see [14]. $\square$

Remark. We hasten to point out that there is a well-known *equivalence* of categories

$$[\mathrm{el}(X)^{\mathrm{op}}, \mathbf{set}] \simeq [\mathscr{C}^{\mathrm{op}}, \mathbf{set}]/X \ (=\mathscr{E}/X);$$

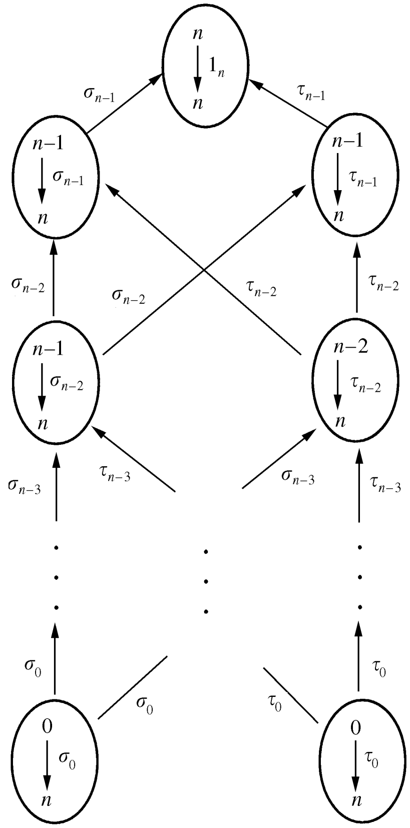


Fig. 1.

but the right-hand side is only *pseudo-functorial* in X using pullback along arrows. The left-hand side is functorial in X as described in the Proposition; the displayed equivalence is pseudo-natural in X. [3]

3. Higher-dimensional spans as internalized globular sets

We now return to globular sets: $\mathscr{E} = [\mathbf{G}^{\mathrm{op}}, \mathbf{set}] = \mathbf{glob}$. We begin by identifying the globular category $\mathscr{S} : \mathbf{G}^{\mathrm{op}} \to \mathbf{Cat}$. First this requires the identification of $\mathbf{G}/n$ for all $n \geq 0$. A generating subgraph of $\mathbf{G}/n$ is displayed in Fig. 1.

Then $\mathscr{S}_n = [(\mathbf{G}/n)^{\mathrm{op}}, \mathbf{set}]$ is the *category of n-spans* in the sense of Batanin [2]. An object S of $\mathscr{S}_n$ is a diagram in **set** as shown in Fig. 2. In particular, $\mathscr{S}_0 = \mathbf{set}$ and $\mathscr{S}_1$

[3] Incidentally, this goes some way towards explaining why the pseudo-functor $X \mapsto \mathscr{E}/X$ is taken as the internalization of a topos $\mathscr{E}$ in parametrized (= indexed = locally internal) category theory based on $\mathscr{E}$.

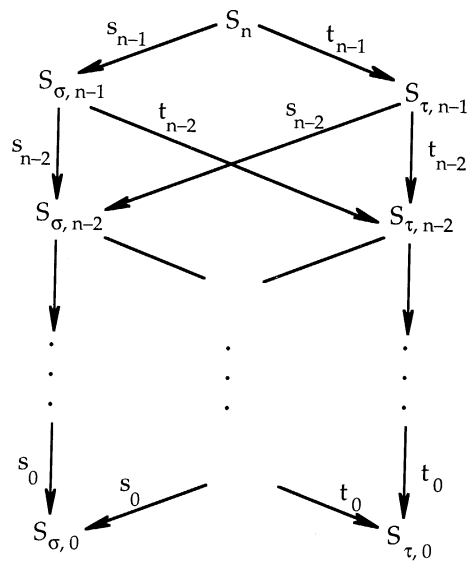
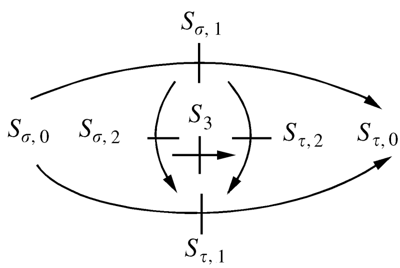


Fig. 2.

is the category of diagrams

$$X \xleftarrow{s} S \xrightarrow{t} Y$$

in **set**. The n-span S can also be regarded as a globular cell of spans; for example, for $n = 3$, we have



where $S_{\alpha,r} : S_{\sigma,r-1} \nrightarrow S_{\tau,r-1}$ is shorthand for the span $(s_{r-1}, S_{\alpha,r}, t_{r-1})$ from $S_{\sigma,r-1}$ to $S_{\tau,r-1}$. From this globular description it is clear what the functors

$$s_{n-1}, t_{n-1} : \mathscr{S}_n \longrightarrow \mathscr{S}_{n-1}$$

must be: we obtain $s_{n-1}(S)$ from S by excising the spans S_n and $S_{\tau,n-1}$ from the diagram for S, and we obtain $t_{n-1}(S)$ from S by excising S_n and $S_{\sigma,n-1}$. This describes

$$\mathscr{S} \in [\mathbf{G}^{\mathrm{op}}, \mathbf{Cat}] = \mathrm{Cat}(\mathscr{E}');$$

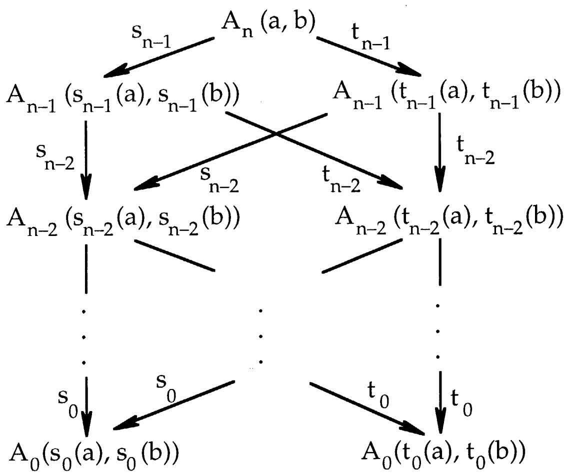


Fig. 3.

it is the globular category which plays the role of the category of small sets for globular category theory: we have the pseudo-natural equivalence

$$[\mathbf{G}^{\mathrm{op}}, \mathbf{Set}](X, \mathscr{S}) \simeq \mathbf{glob}/X.$$

Every locally small globular category A has a *hom globular functor*

$$\mathrm{Hom}_A : A^{\mathrm{op}} \times A \to \mathscr{S}$$

which was described in [2]; for $a, b \in A_n$, the n-span $(\mathrm{Hom}_A)_n(a,b)$ is shown in Fig. 3. This leads to an "internal Yoneda embedding"

$$\mathrm{Yon}_A : A \to [A^{\mathrm{op}}, \mathscr{S}]$$

and a "Yoneda structure" in the sense of [18].

Let $\mathbf{1}$ denote the terminal globular set. A *global element* $x : \mathbf{1} \to X$ of a globular set X is just an arrow from $\mathbf{1}$ to X in **Glob**; it amounts to, for all $n \geq 0$, a choice of element $x_n \in X_n$ such that

$$x_n = s_n(x_{n+1}) = t_n(x_{n+1}).$$

Batanin [2] calls this a *globular element of* X. We have the "global sections" functor

$$\Gamma : \mathbf{Glob} \to \mathbf{Set}$$

given by $\Gamma X = \mathbf{Glob}(\mathbf{1}, X)$; Γ preserves all limits. A *global element of* $\mathscr{S}$ *is precisely a globular set*; so

$$\Gamma\mathscr{S} = \mathbf{glob} \in \mathbf{Cat}.$$

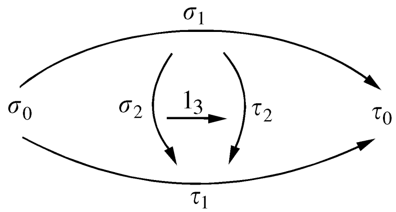
4. The rich structure on higher spans

Apart from the usual virtues[4] of the "internal category of sets" in a topos, the globular category $\mathscr{S}$ has an especially rich structure: it is equivalent to an ω-category in **Cat**. We shall now explain this.

For general universal reasons, the underlying functor Φ : **omcat** $\to$ **glob** has a left adjoint Ψ : **glob** $\to$ **omcat**; indeed, Φ is monadic. The representable globular set $\mathbf{G}(-,n)$ is given by

$$\mathbf{G}(m,n) = \begin{cases} \{1_n\} & \text{for } m = n, \\ \{\sigma_m, \tau_m\} & \text{for } m < n, \\ \emptyset & \text{for } m > n, \end{cases}$$

for example, $\mathbf{G}(-,3)$ is the globular set below.



Since there are no k-composable pairs in the globular set $\mathbf{G}(-,n)$, the free ω-category on $\mathbf{G}(-,n)$ is obtained by simply throwing in identity cells. Hence, or otherwise, we see that

$$\Psi\mathbf{G}(-,n) \cong \mathbf{2}_n.$$

Indeed, we have an invertible natural transformation in the triangle below.

$$\begin{tikzcd} \mathbf{G} \arrow[rr, "2_*"] \arrow[dr, "\mathrm{Yon}"'] & \arrow[d, phantom, "\cong"] & \mathbf{omcat} \\ & \mathbf{glob} \arrow[ur, "\Phi"'] & \end{tikzcd}$$

The adjunction $\Psi \dashv \Phi$ induces an adjunction

$$\begin{tikzcd} \mathbf{glob}/X \arrow[r, shift left=1.5ex, "\Psi'"] \arrow[r, phantom, "\perp"] & \mathbf{omcat}/\Psi X, \arrow[l, shift left=1.5ex, "\Phi'"] \end{tikzcd}$$

where Ψ' takes $f : A \to X$ to $\Psi f : \Psi A \to \Psi X$ and Φ' is obtained by applying Φ and pulling back along the unit $X \to \Phi\Psi X$ for $\Psi \dashv \Phi$. As usual, Ψ' is fully faithful; its full image is $D(\mathbf{omcat}/\Psi X)$. So we have the components

$$\Psi' : \mathbf{glob}/X \simeq D(\mathbf{omcat}/\Psi X)$$

[4] Such as internal cocompleteness.

of a pseudo-natural equivalence which fits into a triangle of pseudo-functors as follows.

$$\begin{tikzcd}
\mathbf{glob}^{\mathrm{op}} \arrow[rr, "\Psi^{\mathrm{op}}"] \arrow[dr, "\mathbf{glob}/-"'] & \simeq & \mathbf{omcat}^{\mathrm{op}} \arrow[dl, "D(\mathbf{omcat}/-)"] \\
& \mathbf{Cat} &
\end{tikzcd}$$

In the remark of Section 2 we learned that the pseudo-functor

$$\mathbf{glob}/- : \mathbf{Glob} \to \mathbf{Cat}$$

is equivalent to a limit-preserving functor. But the functor Ψ preserves all colimits, so the pushouts

$$\begin{tikzcd}
\mathbf{G}(-,k) \arrow[r, "\mathbf{G}(-,\sigma_k)"'] \arrow[d, "\mathbf{G}(-,\tau_k)"'] & \mathbf{G}(-,n) \arrow[d] \\
\mathbf{G}(-,n) \arrow[r] & \mathbf{G}(-,n) \underset{k}{+} \mathbf{G}(-,n)
\end{tikzcd}$$

are taken by Ψ to the precise pushouts

$$\begin{tikzcd}
\mathbf{2}_k \arrow[r, "\sigma_k"'] \arrow[d, "\tau_k"'] & \mathbf{2}_n \arrow[d] \\
\mathbf{2}_n \arrow[r] & \mathbf{2}_n \underset{k}{+} \mathbf{2}_n
\end{tikzcd}$$

in **omcat** which occur in the structure of the co-ω-category $\mathbf{2}_*$. It follows that the pseudo-functor $D(\mathbf{omcat}/-)$ takes $\mathbf{2}_*$ to a globular category equivalent to an ω-category in **Cat**; but we have the equivalences

$$\begin{aligned}
D(\mathbf{omcat}/\mathbf{2}_*) &\simeq D(\mathbf{omcat}/\Psi \circ \mathrm{Yon}_{\mathbf{G}}) \\
&\simeq \mathbf{glob}/\mathrm{Yon}_{\mathbf{G}} \\
&\simeq [(\mathbf{G}/-)^{\mathrm{op}}, \mathbf{set}] \\
&\simeq \mathscr{S}.
\end{aligned}$$

We have proved:

Proposition. *The globular category $\mathscr{S}$ of higher spans is equivalent to an ω-category in the* 2*-category* **Cat**.

It is easy to work through to find the explicit definition of the functorial operations

$$\#_k : \mathscr{S}_n \underset{k}{\times} \mathscr{S}_n \to \mathscr{S}_n;$$

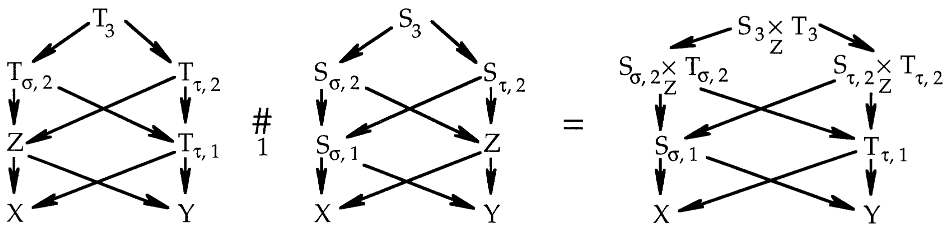


Fig. 4.

they are given by pullback. For example, with $n=3$, $k=1$ and $(T,S)\in\mathscr{S}_3\underset{1}{\times}\mathscr{S}_3$, we form the composite $T\#_kS$ as shown in Fig. 4.

A globular category together with the compositions transported from an equivalent ω-category in **Cat** is called a *monoidal globular category*. For an explicit definition of this structure, see [2,3], or [17]. In fact, ω-categories in **Cat** are *strict* monoidal globular categories. There is a coherence theorem [2] to show that every abstractly defined monoidal globular category is equivalent to a strict one.

For any category $\mathscr{K}$ we can form the globular category $\mathrm{Sp}(\mathscr{K})$ of higher spans in $\mathscr{K}$ by

$$\mathrm{Sp}(\mathscr{K})_n=[(\mathbf{G}/n)^{\mathrm{op}},\mathscr{K}].$$

We have a Yoneda embedding $\mathrm{Yon}:\mathrm{Sp}(\mathscr{K})\to[\mathscr{K}^{\mathrm{op}},\mathscr{S}]$ equal to the composite

$$\mathrm{Sp}(\mathscr{K})_n=[(\mathbf{G}/n)^{\mathrm{op}},\mathscr{K}]\xrightarrow{[1,\mathrm{Yon}]}[(\mathbf{G}/n)^{\mathrm{op}},[\mathscr{K}^{\mathrm{op}},\mathbf{Set}]]\cong[\mathscr{K}^{\mathrm{op}},\mathscr{S}_n].$$

The monoidal globular category structure on $\mathscr{S}$ passes pointwise to $[\mathscr{K}^{\mathrm{op}},\mathscr{S}]$ and this restricts along Yon to $\mathrm{Sp}(\mathscr{K})$ if and only if $\mathscr{K}$ has pullbacks. Of course $\mathrm{Sp}(\mathbf{set})=\mathscr{S}$. For any category $\mathscr{H}$ with pushouts, we can consider

$$\mathrm{Cosp}(\mathscr{H})=\mathrm{Sp}(\mathscr{H}^{\mathrm{op}})^{\mathrm{op}}$$

equipped with the monoidal globular structure arising from pushouts in $\mathscr{H}$. So

$$\mathrm{Cosp}(\mathscr{H})_n=[(\mathbf{G}/n)^{\mathrm{op}},\mathscr{H}^{\mathrm{op}}]^{\mathrm{op}}=[\mathbf{G}/n,\mathscr{H}]$$

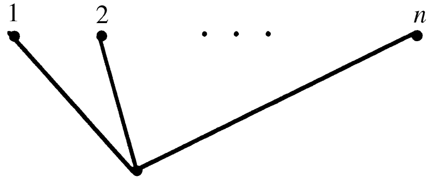
and the compositions

$$\#_k:\ \mathrm{Cosp}(\mathscr{H})_n\underset{k}{\times}\mathrm{Cosp}(\mathscr{H})_n\to\mathrm{Cosp}(\mathscr{H})_n$$

are given by pushouts in $\mathscr{H}$.

5. Trees as generalized natural numbers

It is a banality that 1-stage trees are natural numbers; identify $n \in \mathbb{N}$ with the tree:



So there is an obvious sense in which trees are generalized natural numbers. However, there is a deeper sense.

Write $\Sigma\mathbb{N}$ for the category with one object 0 and with the natural numbers as arrows $n : 0 \to 0$; composition is addition. The natural numbers are determined as a coequalizer

$$1 \rightrightarrows \mathbf{2}_1 \longrightarrow \Sigma\mathbb{N}$$

in **cat**. This shows how the monoid $\mathbb{N}$ arises in the interaction between directed graphs and categories.

Trees arise in the interaction between globular sets and ω-categories. Define the ω-category **Tree** to be the colimit in **omcat** of the diagram

$$\mathbf{2}_*\colon\ \mathbf{2}_0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} \mathbf{2}_1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} \mathbf{2}_2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} \mathbf{2}_3 \underset{\tau_3}{\overset{\sigma_3}{\rightrightarrows}} \cdots .$$

Proposition. **Tree** *is the* ω*-category of higher trees in the sense of Batanin* [2].

Proof. A cocone from the diagram $\mathbf{2}_*$ with vertex an ω-category A amounts to a globular element of A. So $\mathbf{omcat}(\mathbf{Tree}, A) \cong \mathbf{glob}(\mathbf{1}, A)$. So **Tree** is the free ω-category $\Psi\mathbf{1}$ on the terminal globular set $\mathbf{1}$. It is shown in [2] that $\Psi\mathbf{1}$ is the ω-category of higher trees. $\square$

Just as each natural number n can be assigned a set $\mathbf{n}$ of cardinality n, each tree T can be assigned a globular set $|T|$ (denoted by T^* in [2]) of "cardinality" T. We shall conceptually describe this process. The category **glob** of globular sets has pushouts formed pointwise. So we can form the monoidal globular category Cosp(**glob**) as described at the end of Section 4. There is a distinguished globular element of Cosp(**glob**) whose nth component is the composite

$$\mathbf{G}/n \xrightarrow{\text{dom}} \mathbf{G} \xrightarrow{\text{Yon}_{\mathbf{G}}} \mathbf{glob}.$$

This determines a global element $\mathbf{1} \to \mathrm{Cosp}(\mathbf{glob})$, and hence, by the freeness of **Tree**, determines a monoidal globular functor

$$\|-\|\colon\ \mathbf{Tree} \to \mathrm{Cosp}(\mathbf{glob});$$

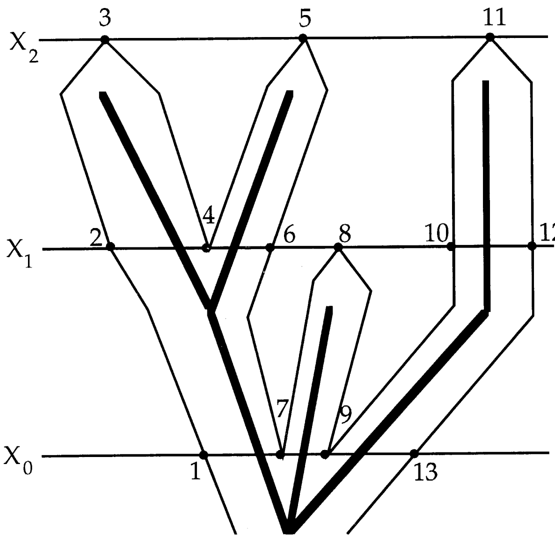


Fig. 5.

the value $||T||$ of this monoidal globular functor at the tree T looks like this:

$$\cdots \rightrightarrows |s_{n-3}T| \underset{\tau_{n-3}}{\overset{\sigma_{n-3}}{\rightrightarrows}} |s_{n-2}T| \underset{\tau_{n-2}}{\overset{\sigma_{n-2}}{\rightrightarrows}} |s_{n-1}T| \underset{\tau_{n-1}}{\overset{\sigma_{n-1}}{\rightrightarrows}} |T|.$$

It is possible to characterize those globular sets which are isomorphic to $|T|$ for some tree T. We first recall the *solid triangle* $x \blacktriangleleft y$ defined for a parity complex [16] but which also makes sense on the set of element of all dimensions in a globular set X. Define the relation $x \prec y$ for $x \in X_n$ if and only if either $x = s_n(y)$ or $t_{n-1}(x) = y$. Then $\blacktriangleleft$ is the reflexive transitive closure of $\prec$.

A *globular cardinal* is a finite globular set X for which the solid triangle order is a linear order. To recapture a tree T from such an X, we obtain the set $T(n)$ (of vertices of height n) from the set X_n by identifying $x, y \in X_n$ when the $\blacktriangleleft$-open interval (x, y) is non-empty and contains only elements of dimension greater than n. Fig. 5 illustrates the relationship between the tree T (shown in heavy lines) and the globular set X (whose points are the black dots numbered according to the solid triangle order).

Definition. A functor $F : \mathscr{A} \to \mathscr{X}$ is called *a parametric right adjoint* (*p.r.a.*) when there exists an object N of $\mathscr{X}$ and cocone $\rho : F \Rightarrow N$ such that the functor $R : \mathscr{A} \to \mathscr{X}/N$, given by $RA = (\rho_A : FA \to N)$, has a left adjoint. We call N the *parametrizing object*. Such a functor F preserves whatever connected limits exist in $\mathscr{A}$. If $\mathscr{A}$ has a terminal object $\mathbf{1}$ then F is parametrically representable iff the induced functor $R : \mathscr{A} \to \mathscr{X}/F\mathbf{1}$ has a left adjoint; in this case, $N = F\mathbf{1}$ and $\rho_A = F(A \to \mathbf{1})$.

In particular, for a cocomplete $\mathscr{A}$, a functor $F:\mathscr{A}\to\mathbf{set}$ is a p.r.a. if and only if it is isomorphic to a coproduct of representables:

$$FA \cong \sum_{n\in N} \mathscr{A}(K_n, A).$$

(The usefulness of this concept in higher category theory was pointed out by Johnson in his thesis [6]; also see [4] where the concept is called "familial representability".) More generally, a functor $F:\mathscr{A}\to[\mathscr{C}^{\mathrm{op}},\mathbf{set}]$ is a p.r.a. if and only if there exist functors $N:\mathscr{C}^{\mathrm{op}}\to\mathbf{set}$, $K:\ \mathrm{el}(N)\to\mathscr{A}$ and a natural isomorphism

$$(FA)U \cong \sum_{n\in NU} \mathscr{A}(K(U,n), A).$$

A good example of a p.r.a. functor is $(-)^*:\ \mathbf{set}\to\mathbf{set}$ which takes a set S to the set S^* of words in the alphabet S; here the parametrizing set N is the set of natural numbers and $K_n=\mathbf{n}$. We have a similar result for trees.

Recall that $\Phi:\ \mathbf{omcat}\to\mathbf{glob}$ denotes the forgetful functor which we have pointed out is monadic. Write $M:\ \mathbf{glob}\to\mathbf{glob}$ for the underlying endofunctor of the monad generated by Φ and its left adjoint Ψ. (Batanin uses the notation D_s for M.) Note that $M\mathbf{1}=\mathbf{Tree}$.

Proposition. *The functor* $M:\ \mathbf{glob}\to\mathbf{glob}$ *is a p.r.a. with parametrizing object* $\mathbf{Tree}$.

Proof. The category $\mathrm{el}(\mathbf{Tree})$ has objects pairs (n,T) where T is a tree of height n; as generators we have the arrows $\sigma_{n-1},\tau_{n-1}\ :\ (n-1, s_{n-1}(T))\to(n,T)$. So we see that $\|-\|$ can be regarded as a functor $\|-\|:\mathrm{el}(\mathbf{Tree})\to\mathbf{glob}$ taking (n,T) to $|T|$ (which is to be the functor K in the description of p.r.a. functors into a presheaf category). By examining Batanin's description of the the free ω-category functor Ψ as given in [2,3,17], we see that there is a natural isomorphism

$$M(X)_n=\Phi\Psi(X)_n \cong \sum_{T\in\mathbf{Tree}_n} \mathbf{glob}(|T|, X). \qquad \Box$$

It follows that $M:\mathbf{glob}\to\mathbf{glob}$ preserves pullbacks.

6. Collections and analytic functors

Batanin [2] defines the category $\mathbf{coll}$ of collections to be $\mathbf{Glob}(\mathbf{Tree},\mathscr{S})$; in the context of Section 5, this reinforces the view that collections are generalized sequences of sets. Yet we have equivalences of categories:

$$\mathbf{Glob}(\mathbf{Tree},\mathscr{S}) \simeq \mathbf{glob}/\mathbf{Tree} \simeq [\mathrm{el}(\mathbf{Tree})^{\mathrm{op}},\mathbf{set}].$$

Here we prefer to think of collections as objects of $\mathbf{glob}/\mathbf{Tree}$; that is, as globular sets A with an augmentation $\gamma: A\to\mathbf{Tree}$. It is of course useful to know that $\mathbf{coll}$ is equivalent to a presheaf category.

Consider the functor $\Theta : \mathbf{coll} \to [\mathbf{glob}, \mathbf{glob}]$ given by $\Theta(A) = F$ where

$$\begin{tikzcd}
FX \arrow[r] \arrow[d] & A \arrow[d, "\gamma"] \\
MX \arrow[r, "M\tau"] & \mathbf{Tree}
\end{tikzcd}$$

is a pullback which is functorial in $X \in \mathbf{glob}$; for an arrow $f \colon A \to B$ in **coll**, the natural transformation $\Theta(f) : F \to G$ is determined by the following diagram in which both squares are pullbacks.

$$\begin{tikzcd}
FX \arrow[r, "\Theta(f)_X"'] \arrow[d] & GX \arrow[r] \arrow[d] & MX \arrow[d, "M\tau"] \\
A \arrow[r, "f"'] & B \arrow[r, "\gamma"'] & \mathbf{Tree}
\end{tikzcd}$$

Note that Θ is faithful since $F\mathbf{1} = A$. From simple properties of pullbacks, it is clear that a natural transformation $\theta : \Theta(A) \to \Theta(B)$ is of the form $\Theta(f)$ for some $f : A \to B$ in **coll** if and only if θ is *cartesian*[5] in the sense of [4].

We now introduce the notion of analytic functor relevant to higher operads: for ordinary operads the relevant notion appears in [7]; recall that an ordinary operad in the sense of May [11] is a monoid for the substitution tensor product of "species of structure" in the sense of Joyal.

Definition. A functor $F :$ **glob** $\to$ **glob** is called *analytic* when there is a globular arrow $\gamma : A \to \mathbf{Tree}$, called the *coefficient collection of* F, such that $F \cong \Theta(A)$.

We conclude with two results which hold when any p.r.a. cartesian[6] monad such as M is taken as base.

Proposition. *The composite of analytic endofunctors on* **glob** *is analytic. The identity endofunctor on* **glob** *is analytic.*

Proof. Let $F, G :$ **glob** $\to$ **glob** be analytic with coefficient collections A, B, respectively. Consider the collection $B \circ A$ given by the middle row of the following diagram in which the bottom square is a pullback.

[5] This means that all the squares that express naturality of the family of arrows θ_X, $X \in \mathbf{glob}$, are pullbacks.

[6] A monad is called cartesian when it is a monad in the 2-category **Cart** of categories with pullback, functors which preserve pullback, and cartesian natural transformations.

$$\begin{tikzcd}
GFX \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & MFX \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & MMX \arrow[r, "\mu_X"'] \arrow[d, "MM\tau"] \arrow[dr, phantom, "\text{p.b.}"] & MX \arrow[d, "M\tau"] \\
B \circ A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & MA \arrow[r] \arrow[d, "M\tau"'] & M\mathbf{Tree} \arrow[r, "\mu_1"'] & \mathbf{Tree} \\
B \arrow[r, "\gamma"] & \mathbf{Tree} & &
\end{tikzcd}$$

The middle square at the top is a pullback since M preserves pullback and A is the coefficient collection for F. The top right square is a pullback since the multiplication μ for the monad M is a cartesian natural transformation (see [3]). The vertical composite of the two left squares is a pullback since B is the coefficient collection for G. So the top left square is a pullback. So $B \circ A$ is the coefficient collection for GF.

The identity endofunctor has coefficient collection given by the component $\eta_1 : \mathbf{1} \to \mathbf{Tree}$ of the unit η for the monad M (using that η is cartesian). $\square$

Indeed, the construction $B \circ A$ in the above proof provides a monoidal structure on the category **coll** of collections such that the functor $\Theta : \mathbf{coll} \to [\mathbf{glob}, \mathbf{glob}]$ is tensor preserving (= strong monoidal). *Batanin's higher operads* are exactly the monoids for this "substitution" tensor product of collections; they precisely give the analytic cartesian monads on **glob**.

Proposition. *Every analytic endofunctor on* **glob** *is a p.r.a.*

Proof. For any collection $\gamma : A \to \mathbf{Tree}$, the functor $\gamma^* : \mathbf{glob}/M\mathbf{1} \to \mathbf{glob}/A$ has a left adjoint; so the composite $\mathbf{glob} \to \mathbf{glob}/A$ of γ^* with $\mathbf{glob} \to \mathbf{glob}/M\mathbf{1}$ has a left adjoint. So the analytic functor with coefficient collection A is a p.r.a. $\square$

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