THEOREMS ON DEDUCIBILITY.

(SECOND PAPER.)*

BY C. H. LANGFORD.

Questions of deducibility arise especially in connection with sets of defining properties for types of order. A set of properties $p_1, \cdots, p_n$ may be related to a class of properties Q in such a way that, if q be any member of Q, one of the properties q or $\sim q$ follows from $p_1, \cdots, p_n$ jointly. A previous paper has been concerned with the class of all first-order functions which can be formulated on the base K, R_2; and sets of defining properties for three types of dense series have been studied in relation to this class of functions.† It has been shown that each of these sets of properties is sufficient to determine the truth-value of every first-order function on K, R_2, in the sense that one or the other of every pair of mutually contradictory functions on this base follows from the set. The present paper will be concerned with this same class of functions, namely, the class of all first-order functions on K, R_2, in relation to a set of properties for discrete series, which have a first but no last element, and which are such that every element but one has an immediate predecessor. It will be shown that every first-order function on K, R_2 has its truth-value determined by this set.

In the first paper, we have had occasion to show that every first-order function is equivalent to some function having one of the forms

$$(1)\qquad (x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) :. (\exists x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) : \cdots f(x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, \cdots)$$

or

$$(2)\qquad (\exists x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) :. \quad (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) : \cdots f(x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, \cdots),$$

in which f is an elementary function. It has also been pointed out that any elementary function f, on K, R_2, can be expanded with respect to K, R, I,‡ in such a way that f is a disjunctive function of conjunctive functions of the primary propositional constituents on K, R, I, and such that, if p be any primary propositional constituent on K, R, or I, formed of the variables $x_1, \cdots$ in f, either p or $\sim p$ occurs in each conjunctive function.

* Presented to the American Mathematical Society, December 28, 1926, received January 29, 1927.

† Some theorems on deducibility, Annals of Math., vol. 28 (1926), p. 16.

‡ Identity.

Accordingly, any first-order function on K, R_2 can be expressed in one or the other of these forms, and in such a way that the elementary function f is in expanded form with respect to K, R, I.

The following properties characterize linear discrete series which have a first but no last element and in which every element but one has an immediate predecessor.

1. $(x) . \sim Rxx.$
2. $(x, y) : x, y \epsilon K . x \neq y .\supset. Rxy \vee Ryx.$
3. $(x, y) : x, y \epsilon K . x \neq y .\supset. \sim Rxy \vee \sim Ryx.$
4. $(x, y, z) :. x \neq y . y \neq z . x \neq z . x, y, z \epsilon K .\supset: Rxy . Ryz .\supset. Rxz.$
5. $(x) ::: (\exists y) ::. (z) :: x \epsilon K :.\supset:. y \epsilon K . Rxy :. z \epsilon K :\supset: Rzx . Rzy .\vee. Ryz . Rxz .\vee. x = z \vee y = z.$
6. $(\exists x) :. (y) : x \epsilon K : y \epsilon K . x \neq y .\supset. Rxy.$

These properties are especially relevant to elements belonging to the class K; properties 7, 8 take account of elements not in K.

7. $(x, y) :. \sim x \epsilon K .\vee. \sim y \epsilon K :\supset: \sim Rxy.$
8. $(\exists x_1, x_2, \cdots) . x_1 \neq x_2 \cdots \sim x_1 \epsilon K . \sim x_2 \epsilon K \cdots.$

The properties of the set 1–8 will be studied by means of a subsidiary set of properties, 1′–8′, on the base K, K′, R_2.

1′. $(x) : x \epsilon K \vee x \epsilon K' .\supset. \sim Rxx.$

2′. $(x, y) : x \epsilon K . y \epsilon K . x \neq y .\supset. Rxy \vee Ryx.$

3′. $(x, y) : x \epsilon K . y \epsilon K . x \neq y .\supset. \sim Rxy \vee \sim Ryx.$

4′. $(x, y, z) :. x \epsilon K . y \epsilon K . z \epsilon K . x \neq y . y \neq z . x \neq z :\supset: Rxy . Ryz .\supset. Rxz.$

5′. $(x) ::: (\exists y) ::. (z) :: x \epsilon K :.\supset:. y \epsilon K . Rxy :. z \epsilon K :\supset: Rxz . Ryz .\vee. Rzx . Rzy .\vee. x = z \vee y = z.$

6′. $(\exists x) : (y) . x \epsilon K . y \epsilon K \supset Rxy.$

7′. $(x, y) : x \epsilon K' \vee y \epsilon K' .\supset. \sim Rxy.$

8′. $(\exists x_1, x_2, \cdots) . x_1 \neq x_2 \cdots x_1 \epsilon K' . x_2 \epsilon K' \cdots.$

Since there are at least two elements in K, it follows from 2′, 7′ that $\sim (\exists x) . x \epsilon K . x \epsilon K'$. K and K′ are non-overlapping classes ($(x) : x \epsilon K . \supset . \sim x \epsilon K'$), and it is, of course, easy to show that K has at least n elements. The relation of the set 1′–8′ to the set 1–8 is seen when the value $\sim$ K is assigned to K′; that is, when $(x) : x \epsilon K .\equiv. \sim x \epsilon K'$. In this case, 1′–8′ reduces to 1–8; the hypothesis in 1′ becomes logically necessary, and so vanishes; and the property $\sim (\exists x) . x \epsilon K . x \epsilon K'$ is certifiable on logical grounds alone. The set 1′–8′ places no restriction whatever on elements which are neither in K nor in K′. In the arguments which follow, the set on K, K′, R_2 will be under consideration; and then the results established will be applied, in particular, to the set on K, R_1.

Just as any elementary function on K, R_2 can be expressed in expanded form with respect to K, R, I, so any elementary function on K, K′, R_2 can be expressed in expanded form with respect to K, K′, R, I; so that every first-order function on K, K′, R_2 can be expressed so as to involve a single complex quantifier applied to an elementary function in expanded form with respect to K, K′, R, I. In what follows we shall not be concerned with functions which place any restriction on elements which are neither in K nor in K′. Let $F(x_1, \cdots, x_\mu, y_1, \cdots, y_\nu)$ be a function on K, K′, R_2 which involves a single complex quantifier, such that the variables $x_1, \cdots, x_\mu$ are quantified universally and the variables $y_1, \cdots, y_\nu$ are quantified particularly; and let f be the elementary function in F, expressed in expanded form with respect to K, K′, R, I. Denote by f' an elementary function derived from f by omitting from f all those alternatives in which at least one of the variables $x_1, \cdots, x^\mu, y_1, \cdots, y_\nu$ is such that it is assigned both to $\sim$K and to $\sim$K′. Let K″ be the class such that $(x) : x \epsilon \mathrm{K}'' \;.\!\equiv\!.\; \sim x \epsilon \mathrm{K} \,.\, \sim x \epsilon \mathrm{K}'$; then the three classes K, K′, K″ are mutually exclusive and severally exhaustive. Consider the elementary function

$$\text{(a)} \qquad f'(x_1, \cdots, x_\mu, y_1, \cdots, y_\nu) \;.\mathbf{v}.\; x_1 \epsilon \mathrm{K}'' \;.\mathbf{v}.\; \cdots \;.\mathbf{v}.\; x_\mu \epsilon \mathrm{K}'';$$

either the function f' holds, or one of the elements $x_1, \cdots, x_\mu$ is neither in K nor in K′. In an obvious sense, (a) is about elements in K or K′ only; and if F is to place no restriction on elements in K″, f must be equivalent to the function (a). When K″ is null, that is, when $(x) : x \epsilon \mathrm{K} \;.\!\equiv\!.\; \sim x \epsilon \mathrm{K}'$, the alternatives $x_1 \epsilon \mathrm{K}'', \cdots, x_\mu \epsilon \mathrm{K}''$ are impossible, and so vanish; so that f reduces to f'.

If $\mathrm{R}(a_1 a_2) \,.\, \mathrm{R}(a_2 a_3) \,.\, \mathrm{R}(a_1 a_3)$, then we may write simply $\mathrm{R}(a_1 a_2 a_3)$; and in general, $\mathrm{R}(a_1 a_2 \cdots)$ is to mean that any two elements a_i, a_j which occur in the order from left to right are such that $\mathrm{R}(a_i a_j)$ holds. In view of 1′, if none of the elements $a_1, \cdots$ belongs to K″, and if $\mathrm{R}(a_1 a_2 \cdots)$ holds, then $a_1, \cdots$ must all be distinct. If $a_1, a_2, \cdots$ are in K and $\mathrm{R}(a_1 a_2 \cdots)$ holds, then any ordered dyad of distinct elements, $a_i a_j$, which can be formed of the elements $a_1, \cdots$ and which is not asserted in $\mathrm{R}(a_1 a_2 \cdots)$, is such that $\mathrm{R}(a_j a_i)$ fails. We may denote "$\mathrm{R}(a_1 a_2 \cdots)$, and R fails for every other ordered dyad of distinct elements" by $\mathrm{R}'(a_1 a_2 \cdots)$. If $a_1, a_2, \cdots$ are in K, they are also such that $\sim \mathrm{R}(a_1 a_1) \,.\, \sim \mathrm{R}(a_2 a_2) \cdots$. We may denote "$\mathrm{R}'(a_1 a_2 \cdots)$ holds, and $\sim \mathrm{R}(a_1 a_1) \,.\, \sim \mathrm{R}(a_2 a_2) \cdots$" by $\mathrm{R}''(a_1 a_2 \cdots)$.

If $b_1, b_2, \cdots$ are distinct and belong to K′, then $\sim R(b_1 b_2) . \sim R(b_2 b_1) \cdots \sim R(b_1 b_1) . \sim R(b_2 b_2) \cdots$, from 1′, 7′. $S(b_1 b_2 \cdots)$ is to mean "$b_1, b_2, \cdots \epsilon K'$, and R fails for every dyad of distinct or identical elements among $b_1, \cdots$, and $b_1, \cdots$ are all distinct."

If $a_1, a_2, \cdots$ are in K and $b_1, b_2, \cdots$ are in K′, then R fails for every ordered dyad which can be formed of one element from the set $a_1, \cdots$ and one from the set $b_1, \cdots$, from 7′; and we may express the failure of R for these dyads by writing $T(a_1 a_2 \cdots; b_1 b_2 \cdots)$.

It has been shown in the first paper, what is immediately obvious, that, as a consequence of 1′–4′, any m distinct elements, all in K, are such that for some permutation of these elements, $a_1, \cdots, a_m$, $R''(a_1 \cdots a_m)$ holds. But for any n distinct elements, $b_1, \cdots, b_n$, all in K′, $S(b_1 \cdots b_n)$ holds. Accordingly, any $m+n$ elements, m in K and n in K', are such that, for some permutation of the m elements in K,

$$R''(a_1 \cdots a_m) . S(b_1 \cdots b_n) . T(a_1 \cdots a_m;\ b_1 \cdots b_n).$$

When n vanishes this expression reduces to $R''(a_1 \cdots a_m)$, and when m vanishes it reduces to $S(b_1 \cdots b_n)$.

Since

$$(3) \qquad (x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}})\, (\exists x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) \cdots f(x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, \cdots)$$

is equivalent to

$$(4) \quad \sim(\exists x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}})\, (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) \cdots \sim f(x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, \cdots),$$

if it is established that every function of the form (3) has its truth-value determined by 1′–8′, then every function of the form (4) has its truth-value determined, and conversely; for the truth-value of a function determines the truth-value of its contradictory. It is sufficient to establish deducibility in at least one of these cases.

If f is of the form (a), $\sim f$ is

$$\sim f'(x_1, \cdots, x_\mu, y_1, \cdots, y_\nu) . \sim x_1 \epsilon K'' \cdots \sim x_\mu \epsilon K'',$$

in which $\sim f'$ is equivalent to a disjunctive function which involves all those alternatives on $x_1, \cdots, x_\mu, y_1, \cdots, y_\nu$ which are not among the alternatives in f'. But every alternative which involves at least one of the variables $x_1, \cdots, x_\mu$ assigned to K″ fails; so that these alternatives vanish. If the elementary function which results be denoted by f'', then f'' involves all those alternatives on $x_1, \cdots, x_\mu, y_1, \cdots, y_\nu$ in which every variable is assigned to K or to K′ and which do not occur in f', and also every possible alternative in which at least one of the variables $y_1, \cdots, y_\nu$ are assigned to K″. Since all possible alternatives of the kind in question are involved in this latter set, and since there are at least n elements in K and at least n elements in K′, the disjunctive function formed of all these latter alternatives, when it occurs in a function of the form (2), is equivalent to

$$y_1 \epsilon K'' \vee \cdots \vee y_\nu \epsilon K''.$$

In a function of the form (2), $y_1, \cdots, y_\nu$ will, of course, be quantified universally. It follows that any function of the form (1), in which the elementary function is of the form (a), is such that its contradictory can be expressed in the form (2), and such that the elementary function is of the form (a); and conversely. In dealing with functions of this kind, it is sufficient to establish deducibility for at least one of every pair of contradictory functions.

It will be shown that every singly quantified first-order function on K, K′, R_2, in which the elementary function can be expressed in the form (a), has its truth-value determined by properties 1′–8′. Any singly quantified first-order function has one of the forms

$$(x_1, \cdots, x_n) \, . f(x_1, \cdots, x_n) \text{ or } (\exists x_1, \cdots, x_n) \, . f(x_1, \cdots, x_n);$$

and we may confine attention to functions of the second kind. Note that in functions of the second kind in which the elementary function is of the form (a), $f = f'$. Every set of elements in K is such that, for some permutation, $R''(a_1 \, a_2 \cdots)$; and every set of elements in K′ is such that $S(b_1 \, b_2 \cdots)$. If the function is to hold, f' must involve at least one alternative; and any alternative in f' which should involve conditions on identity which are contradictory ($x = y \, . \, x \neq y$) is impossible; and any alternative which should involve a condition on identity, $x = y$, where $x \epsilon K$ and $y \epsilon K'$, must fail, since these classes have no members in common. All of these alternatives may be discarded without altering the truth-value of the function. In the function so reconstituted, let p by any alternative. In general, some of the variables which occur in p must take the same value, due to a condition on identity. In each set of variables all taking the same value, drop all of the variables but one. Call the alternative which results p'; so that in p' all variables take distinct values. If p' is of the form

$$R''(a_1 \, a_2 \cdots) \, . \, S(b_1 \, b_2 \cdots) \, . \, T(a_1 \, a_2 \cdots; \, b_1 \, b_2 \cdots),$$

then it follows from 1′–8′ that $(\exists x_1, \cdots, x_n) \, . f(x_1, \cdots, x_n)$ holds; so that, if there is at least one alternative p which reduces to this form, the function follows from the set. Clearly, the condition is also necessary, since every set of distinct elements must have this form. Derivatively, if we assign to K′ the value $\sim$K, it is clear that *every singly quantified first-order function on* K, R_2 *has its truth-value determined by properties* 1–8.

We wish to establish a similar result in the case of n-tuply quantified functions; and this will require a somewhat detailed preliminary analysis of some properties of general propositions, which is, however, not without interest on its own account. Consider the elementary function

$$(5) \qquad f(\cdots, x_1^{(i)}, \cdots, x_n^{(i)}, \cdots),$$

where f is in expanded form with respect to K, K′, R, I; and let p_j, p_k $(j \neq k)$ be any two alternatives in f. If we discard all of the variables from p_j, p_k except $x_1^{(i)}, \cdots, x_n^{(i)}$, the resulting functions of $x_1^{(i)}, \cdots, x_n^{(i)}$ on K, K′, R, I may be the same or different. Let the alternatives in f be collected into sets $P_1, \cdots, P_t$, such that $P_i = p_{i\alpha} \vee \cdots \vee p_{is}$, where $p_{i\alpha}, \cdots, p_{is}$ are all those alternatives in f involving some one and the same function of $x_1^{(i)}, \cdots, x_n^{(i)}$ on K, K′, R, I. Then $f = P_1 \vee \cdots \vee P_t$; so that we may express an n-tuply quantified function in the form

$$(6) \qquad \cdots (\exists x_1^{(i)}, \cdots, x_n^{(i)}) \cdots . P_1 \vee \cdots \vee P_t.$$

This function is equivalent to

$$(7) \qquad \cdots : (\exists x_1^{(i1)}, \cdots, x_n^{(i1)}) \cdots P_1 .\vee. \cdots .\vee. (\exists x_1^{(it)}, \cdots, x_n^{(it)}) \cdots P_t,$$

since no set of elements $a_1, \cdots, a_n$, substituted for $x_1^{(i)}, \cdots, x_n^{(i)}$, which can satisfy $x_1^{(i)}, \cdots, x_n^{(i)}$ in P_j, can also satisfy these variables in P_k, if $j \neq k$. In view of the relation of (6) to (7), if F be a function which involves a single complex quantifier, and which is such that the elementary function f is in expanded form with respect to K, K′, R, I, then F can be expressed as a function which is such that, (i) if t be any variable to which the applicative "every" attaches, and if we consider all those alternatives in which t occurs, and let $y_1, \cdots, y_\lambda$ be the variables in those alternatives of wider scope than t, then $y_1, \cdots, y_\lambda$ involve one and the same function on K, K′, R, I in all of these alternatives; and (ii), if u be any variable to which the applicative "some" attaches, and if we consider all those alternatives in which u occurs, and let $y_1, \cdots, y_\mu$ be the variables in these alternatives which are not of narrower scope than u, which of course includes u and those variables having the same scope as u, then $y_1, \cdots, y_\mu$ involve one and the same function on K, K′, R, I in all of these alternatives.*

* There are, of course, variables which do not occur in any alternative in which u occurs; the scopes of these variables are independent of the scope of u, and they are not in question in (i) and (ii).

For want of a better term, a function expressed in this way will be said to be in analytic form. It will be seen that a function

$$(8) \qquad (\exists x_1, \cdots, x_m) \cdots f(x_1, \cdots, x_m, \cdots),$$

when expressed in analytic form, breaks up into an elementary disjunctive function,

$$(9) \quad (\exists x_1^{(1)}, \cdots, x_m^{(1)}) \cdots f_1(x_1^{(1)}, \cdots, x_m^{(1)}, \cdots) \,.\mathbf{V}. \cdots .\mathbf{V}.\, (\exists x_1^{(t)}, \cdots, x_m^{(t)}) \cdots f_t(x_1^{(t)}, \cdots, x_m^{(t)}, \cdots);$$

and since the truth-value of an elementary function is determined if the truth-values of the constituent functions are determined, we may confine attention to functions having the form of the alternatives in (9).

Let $f(\cdots, u_1, \cdots, u_n, \cdots)$ be an elementary function on K, K′, R_2 in expanded form with respect to K, K′, R, I (which is not of the form (a)), and let p be any alternative in f. For each pair of variables x, y in f, p involves $x\mathrm{I}y$ or it involves $\sim x\mathrm{I}y$, and not both; so that p contains, as a part, a conjunctive function of $u_1, \cdots, u_n$ on I. This function will, in general, assert identity for some pairs of variables and non-identity for others; and in general, some of the alternatives $p_1, \cdots, p_s$ in f will involve the same function of $u_1, \cdots, u_n$ on identity, and others will involve different functions in this respect. The alternatives in f can be collected into sets, such that any two alternatives belonging to the same set involve the same function of $u_1, \cdots, u_n$ on identity, and any two alternatives belonging to different sets involve different functions of these variables on identity; so that f can be expressed in the form $f_1, \cdots, f_r$, in which f_i is a disjunctive function involving all those and only those alternatives belonging to one and the same set. We may denote the function of $u_1, \cdots, u_n$ on I which occurs in f_1 by $\mathbf{I}_1$ and that which occurs in f_r by $\mathbf{I}_r$; so that $\mathbf{I}_1, \cdots, \mathbf{I}_r$ are all different.

Consider a function,

$$(10) \qquad (\exists x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) : (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) \,.\, F_1 \mathbf{V} \cdots \mathbf{V} F_t,$$

in analytic form, in which in general F_i is not an elementary function. The variables $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}$ may, of course, be supposed to involve the same function on K, K′, R, I throughout $F_1, \cdots, F_t$. We may collect $F_1, \cdots, F_t$ into sets, determined by the function of $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}$ on I which occurs in them. It is clear that only one function of these variables on identity can occur in any function F_i. Then $F_1 \mathbf{V} \cdots \mathbf{V} F_t$ can be expressed in the form $F_1' \mathbf{V} \cdots \mathbf{V} F_r'$, in which F_j' is a disjunctive function made up of functions from among $F_1, \cdots, F_t$, and such that each alternative which occurs in F_j involves the same function of $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}$ on I. Let the functions of $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}$ on I which occur in $F_1', \cdots, F_r'$ be denoted by $\mathrm{I}_1, \cdots, \mathrm{I}_r$ respectively. Then (10) is equivalent to

$$(11)\quad (\exists x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}) : (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) . F_1' \vee \sim \mathrm{I}_1 : \cdots : (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) . F_r' \vee \sim \mathrm{I}_r .$$

Similarly, consider any constituent of (11) of the form

$$(12)\qquad (x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}) . F_j' \vee \sim \mathrm{I}_j,$$

in which F_j' will, in general, be of the form

$$(13)\quad \begin{array}{l} (\exists x_1^{\mathrm{III}(\alpha)}, \cdots, x_p^{\mathrm{III}(\alpha)}) : (x_1^{\mathrm{IV}(\alpha)}, \cdots, x_q^{\mathrm{IV}(\alpha)}) . F_{j_{\alpha 1}} \vee \cdots \vee F_{j_{\alpha a}} : \vee : \cdots : \vee : \\ (\exists x_1^{\mathrm{III}(\varepsilon)}, \cdots, x_p^{\mathrm{III}(\varepsilon)}) : (x_1^{\mathrm{IV}(\varepsilon)}, \cdots, x_q^{\mathrm{IV}(\varepsilon)}) . F_{j_{\varepsilon 1}} \vee \cdots \vee F_{j_{\varepsilon k}} . \end{array}$$

In each of the alternatives

$$(14)\quad (\exists x_1^{\mathrm{III}(\gamma)}, \cdots, x_p^{\mathrm{III}(\gamma)}) : (x_1^{\mathrm{IV}(\gamma)}, \cdots, x_q^{\mathrm{IV}(\gamma)}) . F_{j_{\gamma 1}} \vee \cdots \vee F_{j_{\gamma h}} \qquad (h = a, \cdots, k;\ \gamma = \alpha, \cdots, \varepsilon),$$

any one of the functions $F_{j_{\gamma i}}$ $(i = 1, \cdots, h)$ may or may not be an elementary function, and is such that the elementary alternatives which occur in it involve the same function of $x_1^{\mathrm{I}}, \cdots, x_m^{\mathrm{I}}, x_1^{\mathrm{II}}, \cdots, x_n^{\mathrm{II}}, x_1^{\mathrm{III}(\gamma)}, \cdots, x_p^{\mathrm{III}(\gamma)}, x_1^{\mathrm{IV}(\gamma)}, \cdots, x_q^{\mathrm{IV}(\gamma)}$ on K, K′, R, I. This follows from the fact that (10) is in analytic form. And so, as previously, $F_{j_{\gamma 1}}, \cdots, F_{j_{\gamma h}}$ may be collected into sets; and this analysis may be continued until all universal variables have been taken into account. In a function expressed in this way, if $t_1, \cdots, t_\nu$ be any apparent variables all having the same scope, and if $s_1, \cdots, s_\mu$ be those variables of wider scope than $t_1, \cdots, t_\nu$, such that $t_1, \cdots, t_\nu$ are within their scope, then any two alternatives within the scope of $t_1, \cdots, t_\nu$ involve the same function of $s_1, \cdots, s_\mu, t_1, \cdots, t_\nu$ on identity.

It follows from these considerations that, if we have a function in analytic form which is, further, expressed in such a way that any set of variables in the function involve the same function on identity in every alternative in which they all occur, then there is an equivalent function obtained by taking any two variables x, y, such that the function asserts $x \mathrm{I} y$, and discarding that variable which has the narrower scope, or, in case they have the same scope, that which occurs first in the quantifier. This is possible for the reason that, if the function involves $x \mathrm{I} y$ in one alternative, it involves $x \mathrm{I} y$ in every alternative in which both x and y occur; and that, if x is of wider scope than y, or if they have the same scope, x occurs in every alternative in which y occurs. y may, accordingly, be discarded; which carries with it every elementary constituent $z \neq y$,

$y \varepsilon \mathrm{K}$, $\mathrm{R}zy$, $\cdots$ in which y occurs. This applies also, of course, to any function on identity $\mathrm{I}j$ in which y occurs. In a function expressed in this way, distinct variables denote distinct elements.

The foregoing arguments are concerned with a function in which the elementary function is not of the form (a); and we wish to extend these results to a function in which the elementary function is of this form. Consider a function

$$
\begin{aligned}
(15) \quad & \cdots (x_1^{(i)}, \cdots, x_\lambda^{(i)}) (\exists x_1^{(j)}, \cdots, x_\mu^{(j)}) (x_1^{(k)}, \cdots, x_\nu^{(k)}) \cdots . P_1 \vee \cdots \vee P_t \\
& \cdots \vee x_1^{(i)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(i)} \varepsilon \mathrm{K}'' \vee x_1^{(k)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\nu^{(k)} \varepsilon \mathrm{K}'' \cdots,
\end{aligned}
$$

and express (15) so that

$$\cdots \vee x_1^{(i)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(i)} \varepsilon \mathrm{K}''$$

shall be outside the scope of $(\exists x_1^{(j)}, \cdots, x_\mu^{(j)}) (x_1^{(k)}, \cdots, x_\nu^{(k)}) \cdots$,

$$
\begin{aligned}
(16) \quad & \cdots (x_1^{(i)}, \cdots, x_\lambda^{(i)}) : (\exists x_1^{(j)}, \cdots, x_\mu^{(j)}) (x_1^{(k)}, \cdots, x_\nu^{(k)}) \cdots . P_1 \\
& \vee \cdots \vee P_t \vee x_1^{(k)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\nu^{(k)} \varepsilon \mathrm{K}'' \cdots .\vee. \cdots x_1^{(i)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(i)} \varepsilon \mathrm{K}''.
\end{aligned}
$$

This function is equivalent to

$$
\begin{aligned}
& \cdots (x_1^{(i)}, \cdots, x_\lambda^{(i)}) :. (\exists x_1^{(j1)}, \cdots, x_\mu^{(j1)}) (x_1^{(k1)}, \cdots, x_\nu^{(k1)}) \cdots . P_1 \vee x_1^{(k1)} \varepsilon \mathrm{K}'' \vee \\
(17) \quad & \cdots \vee x_\nu^{(k1)} \varepsilon \mathrm{K}'' \cdots .\vee. \cdots .\vee. (\exists x_1^{(jt)}, \cdots, x_\mu^{(jt)}) (x_1^{(kt)}, \cdots, x_\nu^{(kt)}) \cdots . P_t \\
& \vee x_1^{(kt)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\nu^{(kt)} \varepsilon \mathrm{K}'' \cdots :\vee: \cdots \vee x_1^{(i)} \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(i)} \varepsilon \mathrm{K}''.
\end{aligned}
$$

In view of the relation of (16) to (17), any function can be expressed in such a way that, if we exclude alternatives of the form $x \varepsilon \mathrm{K}''$, properties i, ii (page 464) characterize the function. We shall say, in this case also, that the function is in analytic form.

Consider any constituent of a function in analytic form,

$$
\begin{aligned}
(18) \quad & (x_1, \cdots, x_\lambda) : F_1(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee \cdots \vee F_t(\cdots, x_1, \cdots, x_\lambda, \cdots) \\
& .\vee. \ x_1 \varepsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \varepsilon \mathrm{K}'',
\end{aligned}
$$

where $F_1, \cdots, F_t$ are such that, in each of them, every alternative involves the same function of $x_1, \cdots, x_\lambda$, together with every variable of wider scope, on K, K′, R, I. The functions $F_1, \cdots, F_t$ may be grouped according to the functions of $x_1, \cdots, x_\lambda$ on I which their alternatives involve, and in such a way that any two alternatives belonging to functions in the same group involve the same function of $x_1, \cdots, x_\lambda$ on identity, and any two alternatives belonging to functions in different groups involve different functions of $x_1, \cdots, x_\lambda$ on identity. The groups of functions themselves form disjunctive functions; and we may denote these disjunctive functions by $F_1', \cdots, F_s'$. Then the function just given may be written

$$(19)\quad (x_1, \cdots, x_\lambda) : F_1'(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee \cdots \vee F_s'(\cdots, x_1, \cdots, x_\lambda, \cdots) \\ .\vee.\ x_1 \epsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \epsilon \mathrm{K}'',$$

which is equivalent to

$$(12')\quad \begin{array}{l} (x_1, \cdots, x_\lambda) : F_1'(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee x_1 \epsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \epsilon \mathrm{K}'' \\ .\vee. \cdots .\vee.\ F_s'(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee x_1 \epsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \epsilon \mathrm{K}''. \end{array}$$

The constituents of this function,

$$F_j'(\cdots, x_1, \cdots, x_\lambda, \cdots) \vee x_1 \epsilon \mathrm{K}'' \vee \cdots \vee x_\lambda \epsilon \mathrm{K}'', \qquad j = 1, \cdots, s,$$

are such that, any set of elements substituted for $x_1, \cdots, x_\lambda$ satisfies at most one of them or at least all of them. If none of the elements which are substituted are in K″, and they satisfy one of the F''s, then at most one of the F''s can be satisfied; if one of the elements substituted is in K″, every function is satisfied. It follows that the function is equivalent to

$$(13')\quad \begin{array}{l} (x_1^{(1)}, \cdots, x_\lambda^{(1)}) : F_1'(\cdots, x_1^{(1)}, \cdots, x_\lambda^{(1)}, \cdots) \vee x_1^{(1)} \epsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(1)} \epsilon \mathrm{K}'' .\vee. \sim \mathrm{I}_1 \\ :. \cdots :.\ (x_1^{(s)}, \cdots, x_\lambda^{(s)}) : F_s'(\cdots, x_1^{(s)}, \cdots, x_\lambda^{(s)}, \cdots) \vee x_1^{(s)} \epsilon \mathrm{K}'' \vee \cdots \vee x_\lambda^{(s)} \epsilon \mathrm{K}'' .\vee. \sim \mathrm{I}_s. \end{array}$$

When a function in analytic form has been expanded in this way, it may also be expressed in such a way that distinct variables require distinct values, as we have shown.

There are certain conditions under which the scope of a variable to which the applicative "some" attaches can be increased, without altering the truth-value of the function. To use a simple illustration, consider an elementary alternative $\mathrm{R}''(abx)$, in which x is of wider scope than a, and b of wider scope then x; and let a, b be particular and x universal. The variable a is separated from the universal variable, within whose scope it falls, by a particular variable of still wider scope. Now a particular variable of wider scope has the force of "some one and the same", and a particular variable of narrower scope has the force of "some one or other". Then, in respect of this alternative alone, a can be given a wider scope than x; for, in view of 1′–8′,

$$(\exists b) :. (x) : (\exists a) . \mathrm{R}''(bx) \supset \mathrm{R}''(abx) :.\equiv:. \\ (\exists b) :. (\exists a) : (x) \,\dot{:}\, \mathrm{R}''(bx) \supset \mathrm{R}''(abx).$$

Note that it is indifferent whether x is universal, or whether it is assigned to K, as in this illustration; and that if this relation among a, b, x

holds in every alternative in which a occurs, the scope of a can be increased.

Let F be a function in analytic form whose variables of widest scope are particular, and in which the elementary function is of the form (a). Express F in such a way that the function on identity of any set of universal variables having the same scope is constant, and such that distinct variables of interdependent scopes require distinct values. Let the variables of widest scope in the function be $x_1, \cdots, x_m, x_1^{\mathrm{I}}, \cdots, x_n^{\mathrm{I}}$, where $x_1; \cdots, x_m$ are assigned to K, and $x_1^{\mathrm{I}}, \cdots, x_n^{\mathrm{I}}$ to K$'$. Now if $t_1, \cdots, t_p, t$ are variables in the function having interdependent scopes, and if $t_1, \cdots, t_p$ are of wider scope than t, then the function of $t_1, \cdots, t_p$ on K, K$'$, R, I is constant throughout every alternative in which t occurs. We are to confine attention to those alternatives in F which are such that, if z be any universal variable in one of them, then $\mathrm{R}(x_i z)$, $i = 1, \cdots, m$, are asserted in the alternative—every universal variable follows all of the variables $x_1, \cdots, x_m$. In these alternatives, consider any variable u_1 which is, (i) particular, (ii) other than $x_1, \cdots, x_m$, (iii) asserted to precede some variable $x_1, \cdots, x_m$, (iv) not within the scope of any particular variable, other than $x_1, \cdots, x_m$, which is asserted to precede some variable $x_1, \cdots, x_m$. Then u_1 may be given the same scope as $x_1, \cdots, x_m$. In general, u_1 now has within its scope variables of which it was independent; but since it does not occur in any alternative in which these variables occur, it remains in effect independent of them. In like manner, u_2 may be given the same scope as $x_1, \cdots, x_m, u_1$, and so on, until no variable of this kind remains in the function. Let these particular variables be $u_1, \cdots, u_k$; so that $u_1, \cdots, u_k$, although they have the widest scope in F, do not in general occur in every alternative—the function is not expanded with respect to them. Among themselves, some have independent scopes, and some are within the scopes of others; two of them may be independent and yet be within the scope of some one other.

Consider F before any of the variables $u_1, \cdots, u_k$ have been assigned a wider scope; and let u_1 be given the widest scope in F. Now there may not be any element u_1, as demanded in the alternatives in which u_1 occurs; but the mere existence-demand can, in any case, be satisfied by identifying u_1 with, say, x_1. Let the function obtained by identifying u_1 with x_1 in every alternative be denoted by F_1. Then F_1 implies F. Again, in every alternative in which u_1 does not occur, assign it to K and assign to it the relations to $x_1, \cdots, x_m$, in terms of R, I, which it has in those alternatives in which it does occur; and call this function F_2. Then F_2 implies F. But F implies $F_1 \vee F_2$; so that F has been separated into two functions; but in each of these functions there remain only variables

$u_2, \cdots, u_k$, corresponding to $u_1, \cdots, u_k$ in F, to be given an increased scope. In F_1, u_1 may be discarded, being identical with x_1; so that u_2 may be given the widest scope in F_1, and F_1 expressed in the form $F_{11} \vee F_{12}$, in the manner just indicated. In F_2, give u_2 the widest scope; and identify u_2 with x_1 in every alternative, and call this function F_{21}. Then form another function by giving u_2 the relation to $x_1, \cdots, x_m$, in every alternative, which it has in those alternatives in which it occurs. But in this case a new situation arises; u_1 is among the variables of widest scope in F_2. If u_2 is within the scope of u_1, or has the same scope, give u_2 the relation to u_1, in terms of R, I, in every alternative, which it has in those alternatives in which u_2 occurs. Here we get $F_2 \;.\!\equiv\!.\; F_{21} \vee F_{22}$, without difficulty. Suppose that u_1 and u_2 are of independent scope, but that, as they occur among $x_1, \cdots, x_m$, they are separated by at least one of these variables. In this case the relations of u_1, u_2 are determined by their relations to $x_1, \cdots, x_m$, and by the properties of serial relations; and in this case also, we have $F_2 \;.\!\equiv\!.\; F_{21} \vee F_{22}$. Suppose that u_1, u_2 both fall between some pair of neighboring terms among $x_1, \cdots, x_m$, or both precede all these terms; then Ru_1u_2 may be true, or Ru_2u_1, or $u_1 \mathrm{I} u_2$. Call the functions corresponding to these three cases F_{22}, F_{23}, F_{24}. Then $F_2 \;.\!\equiv\!.\; F_{21} \vee F_{22} \vee F_{23} \vee F_{24}$. The generalization is obvious. Now every alternative into which F is resolved involves $u_1, \cdots, u_k$ with widest scope; but there may be some conditions of identity among $u_1, \cdots, u_k$. In this case, discard all but one of any set of variables which are identified. Then any function F^{I}, into which F is resolved, is such that, (i) any two variables having interdepedent scopes require distinct values, and in particular, all variables of widest scope require distinct values among themselves, and distinct from the values of all other variables; (ii) the variables of widest scope are particular; (iii) these variables exhibit a completely determinate function on K, K′, R, I, the same in every alternative; (iv) in general, the relation of $u_1, \cdots, u_k$, in terms of R, I, to variables other than $x_1, \cdots, x_m$, $x_1^{\mathrm{I}}, \cdots, x_n^{\mathrm{I}}$ is not determinate in certain alternatives; but it is determinate in every alternative in which every variable t, assigned to K and not of widest scope, follows $x_1, \cdots, x_m$. For let x_m be such that $\mathrm{R}x_ix_m$ $(i = 1, \cdots, m-1)$; then $\mathrm{R}u_jx_m$ $(j = 1, \cdots, k)$; so that $\mathrm{R}u_jt$. If we should discard from F^{I} all alternatives except those in which every variable t follows $x_1, \cdots, x_m$, F^{I} would be in analytic form, and, further expressed in such a way that distinct variables require distinct values.

We wish to show that the truth-value of F^{I} is determined by properties 1′–8′. Let the degree of quantification of F, when expressed in expanded form, be n; then the degree of quantification of F^{I}, in expanded form, cannot be more than n, though it can be less. We wish to show that, if the truth-value of every function of degree $n-1$ is determined, the truth-value of F^{I} is determined; and we may, of course, assume that F^{I} is of degree n. Let $x_1, \cdots, x_m$ be such that $\mathrm{R}''(x_1, \cdots, x_m)$; let $a_1, \cdots, a_m$ be distinct elements in K, such that $\mathrm{R}''(a_1, \cdots, a_m)$, and let $b_1, \cdots, b_n$ be distinct elements in K′; substitute $a_1, \cdots, a_m, b_1, \cdots, b_n$ for $x_1, \cdots, x_m, x_1^{\mathrm{I}}, \cdots, x_n^{\mathrm{I}}$, and call the resulting function F_1^{I}. Discard from F_1^{I} all those elementary alternatives in which some universal variable is asserted to precede a_m, and call this function F_1^{II}; so that $F_1^{\mathrm{II}} \supset F_1^{\mathrm{I}} \supset F^{\mathrm{I}}$. Let $\mathrm{R}_\alpha x_1 x_2$ mean $\mathrm{R}x_1x_2 \,.\, \mathrm{R}a_mx_1 \,.\, \mathrm{R}a_mx_2$; $x \varepsilon \mathrm{K}_\alpha$ mean $x\varepsilon \mathrm{K} \,.\, \mathrm{R}a_mx$; $x\varepsilon\mathrm{K}'_\alpha$ mean $x\varepsilon\mathrm{K}' \,.\, x \neq b_1, \cdots, b_n$. It is clear that K_α, K'_α, R_α have all of the properties 1′–8′; so that the truth-value of every function of degree $n-1$ on K_α, K'_α, R_α is determined. Form F_1^{III} from F_1^{II} by discarding $a_1, \cdots, a_m, b_1, \cdots, b_n$; and form $F_{1\alpha}^{\mathrm{III}}$ from F_1^{III} by substituting K_α, K'_α, R_α for K, K′, R. Then $F_{1\alpha}^{\mathrm{III}}$ is of degree $n-1$. Choose $a_1, \cdots, a_m$ so that they are the first m elements in K. Then $F_{1\alpha}^{\mathrm{III}} \supset F_1^{\mathrm{III}} \supset F_1^{\mathrm{II}}$*; and it is to be noted that, if $F_{1\alpha}^{\mathrm{III}}$ follows from 1′–8′ for one choice of x_m, it follows for every choice.

We wish to show that $\sim F_{1\alpha}^{\mathrm{III}} \supset \sim F^{\mathrm{I}}$. F^{I} entails that, for some $x_1, \cdots, x_m$, F_1^{I}; and we may show that $\sim F_{1\alpha}^{\mathrm{III}} \supset \sim F_1^{\mathrm{I}}$ for every $x_1, \cdots, x_m$. F_1^{I} differs from $F_{1\alpha}^{\mathrm{III}}$ only in that certain elementary conjunctive functions, occuring in functions f', do not occur in $F_{1\alpha}^{\mathrm{III}}$, and in that alternatives of the form $t\varepsilon\mathrm{K}''$ are replaced by $t\varepsilon\mathrm{K}'' \vee \mathrm{R}tx_m$. These elementary conjunctive functions are such as involve $\mathrm{R}tx_m$, for some universal variable t. But any elementary conjunctive function which involves $\mathrm{R}tx_m$ can be introduced into $F_{1\alpha}^{\mathrm{III}}$, since every function of this kind is implicit in the alternative $t\varepsilon\mathrm{K}'' \vee \mathrm{R}tx_m$. For $t\varepsilon\mathrm{K}'' \vee \mathrm{R}tx_m$ can be expanded into $t\varepsilon\mathrm{K}'' \vee \cdots$, in which a disjunctive function, consisting of all possible elementary functions which involve $\mathrm{R}tx_m$, replaces $\mathrm{R}tx_m$. If we expand every alternative of the form $t\varepsilon\mathrm{K}'' \vee \mathrm{R}tx_m$, which occurs in $F_{1\alpha}^{\mathrm{III}}$, in this way, then $F_{1\alpha}^{\mathrm{III}}$ is identical with F_1^{I} in every respect, except that, in general, $F_{1\alpha}^{\mathrm{III}}$ involves some elementary conjunctive functions which do not occur in F_1^{I}; so that $\sim F_{1\alpha}^{\mathrm{III}} \supset \sim F_1^{\mathrm{I}}$. It follows that every first-order function on K, K″, R_2 has its truth-value determined by properties 1′–8′; and derivatively, that *every first-order function on the base* K, R_2 *has its truth-value determined by properties* 1–8.

---

* See property (i), p. 470.

HARVARD UNIVERSITY.