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POLYNOMIAL ALGORITHMS IN LINEAR PROGRAMMING*

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EXACT algorithms of linear programming, whose labour-content is bounded by a polynomial of the length of the binary form of the problem, are constructed.

Introduction

We consider a system of $m \geqslant 2$ linear inequalities with respect to $n \geqslant 2$ real variables $x_1, \ldots, x_j, \ldots, x_n$:

$$a_{i1}x_1+\ldots+a_{in}x_n \leqslant b_i, \qquad i=1, 2, \ldots, m, \tag{0.1}$$

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with integer coefficients a_{ij}, b_i. Let

$$L = \left[\sum_{i,j=1}^{m,n} \log_2(|a_{ij}|+1) + \sum_{i=1}^{m} \log_2(|b_i|+1) + \log_2 nm \right] + 1 \tag{0.2}$$

be the length of input of the system, i.e. the number of 0's and 1's needed to write the coefficients in binary form.

In Sections 1–4 an algorithm, of polynomial type in L, is constructed for finding whether a system of inequalities (0.1) is compatible or incompatible in R^n. The algorithm requires for its operation a memory of order $O(nm + n^2)$ numbers, each of which has $O(L)$ places when written in fixed point binary form. On these numbers $O(n^3(n+m)L)$ elementary operations $+, -, \times, :, \sqrt{\ }, \max,$ are performed, the required accuracy of performing the operations being $O(L)$ binary places. In other words, we will show that the problem of determining the compatibility of a system of linear inequalities in R^n belongs to the class P of problems, polynomially solvable [1, 2] on determinate Turing machines.

Note. 1. A polynomial algorithm was constructed in [3] for determining the compatibility of a system of linear inequalities, which required $O(n^3(n^2+m)L)$ elementary operations to be performed over $O(nL)$-place numbers, and required a memory of $O(nm + n^2)$ such numbers. Our present algorithm is better than that described in [3], since it operates with $O(L)$-place numbers, fewer operations, and smaller memory.

For the algorithm constructed in Sections 1–4 we estimate the labour content in Section 5 in terms of the dimensionalities n and m and the maximum modulus of the coefficients a_{ij}, b_i. We also estimate in Section 5 the labour content of inequalities with unimodular matrices of constraints.

In Section 6 we consider two other common problems associated with linear inequalities: to find the solution $x^0 \in R^n$ of a compatible system (0.1), and to solve the linear programming problem, i.e. maximization of a linear form with integral coefficients under constraints (0.1). Polynomial algorithms are constructed for solving these problems, and their labour contents are estimated. We also prove in Section 6 the polynomial solvability of linear fractional programming.

Notice that the question was posed in [2] as to the completeness of linear programming in the class of problems NP, polynomially solvable on determinate Turing machines.

In the conclusion, we discuss the possibility of constructing polynomial algorithms for other convex problems.

1. Localization of solutions and measure of incompatibility

Lemma 1

If the system of linear inequalities (0.1) with input length L is compatible, it has a solution belonging to the Euclidean sphere $S = \{x \mid \|x\| \leqslant 2^L\}$.

Proof. If all the $a_{ij} \equiv 0$, the lemma is obvious. Otherwise, it follows from the principle of boundary solutions (see [4], p. 32) that the system has a solution x^0, each component of which has the form $x_j^0=\Delta_j/\Delta$, where $\Delta\neq 0$ is a determinant of the extended matrix of the system

$$\left\| \begin{matrix} a_{11} & \dots & a_{1n} & b_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \dots & a_{mn} & b_m \end{matrix} \right\|, \tag{1.1}$$

and Δ_j are also certain determinants of matrix (1.1), possibly zero. It follows from (0.2) that

$$|\Delta_j|\leqslant \prod_{i,j=1}^{m,n} (|a_{ij}|+1)\times\prod_{i=1}^{m} (|b_i|+1)\leqslant\frac{2^L}{nm} \tag{1.2}$$

and in addition, since the coefficients are integers, $|\Delta| \geqslant 1$. Hence every component of the vector x^0 has modulus not exceeding $2^L/nm$, which proves the lemma.

Let

$$\theta(x)=\max_{i=1,2,\dots,m} \{a_{i1}x_1+\dots+a_{in}x_n-b_i\}$$

be the discrepancy of the system at the point $x \in R^n$. Notice that $\theta(x)\leqslant 0$ if and only if x is a solution of the system.

Lemma 2

If the system of linear inequalities (0.1) with input length L is incompatible, then, given any point x of R^n, the discrepancy $\theta(x)$ exceeds the quantity $\theta(x)\geqslant 2\cdot 2^{-L}$.

Proof. If all the $a_{ij} \equiv 0$, the lemma is obvious. Otherwise, let t be the measure of the incompatibility of the system:

$$t=\min_{x\in R^n}\theta(x)>0.$$

Notice that the minimum with respect to $x \in R^n$ is attained (see [4], p. 401). We know (see [4], pp. 402, 407) that the strictly positive quantity t has the form $t=\Delta/(\Delta_1+\dots+\Delta_m)$, where Δ and Δ_i are determinants of the matrix (1.1). Using estimate (0.2), we obtain

$$t\geqslant(|\Delta_1|+\dots+|\Delta_m|)^{-1}\geqslant\frac{nm}{m}2^{-L}\geqslant 2\cdot 2^{-L},$$

which proves the lemma.

Let θ_S be the minimum of the discrepancy $\theta(x)$ in the sphere $S=\{x\mid \|x\|\leqslant 2^L\}$. By Lemmas 1 and 2, to find whether system (0.1) is compatible, it is sufficient to find the point x in R^n such that

$$\theta(x)\leqslant\theta_S+2^{-L}. \tag{1.3}$$

For, if the system is compatible, then, by Lemma 1, $\theta_S \leqslant 0$, and hence it follows from (1.3) that $\theta(x) \leqslant 2^{-L}$. If the system is incompatible, then, by Lemma 2, given any x we have $\theta(x) \geqslant 2 \cdot 2^{-L}$. Hence, if x satisfies relation (1.3), then either $\theta(x) \leqslant 2^{-L}$ in which case the system is compatible, or else $\theta(x) \geqslant 2 \cdot 2^{-L}$ and the system is incompatible.

In Section 3 we describe a polynomial in L algorithm for finding the required point x, satisfying inequality (1.3). The convergence of the algorithm, which is based on the ideas of Shor's method [5, 6], is proved in Section 4. First, we require some auxiliary constructions.

2. Auxiliary constructions

Some properties of ellipsoids in R^n, $n \geqslant 2$, will be needed later. The ellipsoid E is specified by the pair (x, B), where $x \in R^n$ is the centre of the ellipsoid, and $B = \| b_{\alpha\beta} \|$ is a real $n \times n$ matrix. The ellipsoid is the image, shifted to the point x, of the sphere $\| z \| \leqslant 1$ under the mapping B, i.e.

$$E = \{y \mid y = x + Bz, \ \|z\| \leqslant 1\}. \tag{2.1}$$

In particular, E is non-degenerate if $\det B \neq 0$.

Let E be a non-degenerate ellipsoid, and $r_1, \ldots, r_n$ the lengths of its semi-axis. Let $r(E)$ denote the minimum length of the semi-axis, called the thickness of the ellipsoid. We put

$$\|B\| = \left(\sum_{\alpha,\beta=1}^{n} b_{\alpha\beta}^{2} \right)^{1/2},$$

then the maximum semi-axis will not exceed the norm $\| B \|$, since $\|Bz\| \leqslant \|B\| \, \|z\|$. Hence, from the equation $|\det B| = r_1 r_2 \ldots r_n$ we have

$$\frac{|\det B|}{\|B\|^{n-1}} \leqslant r(E) \leqslant |\det B|^{1/n}. \tag{2.2}$$

Further, let $E \backsim (x, B)$ be an ellipsoid, and $\lambda > 1$ a scalar parameter. Denote by λE the ellipsoid $(x, \lambda B)$, obtained from E by stretching it by the factor λ relative to the centre x. Finally, if $E \backsim (x, B)$ and $E' \backsim (x', B')$ are two ellipsoids, we put $\|E - E'\| = \|x - x'\| + \|B - B'\|$ and we say, in cases when $\|E - E'\| \leqslant \delta$, that ellipsoid E' approximates E with accuracy δ.

Assume that $\|E - E'\| \leqslant \delta$, and let $y \in E$, $y' \in E'$ be two points of ellipsoids E and E', obtained for the same value of the parameter z from the sphere $\| z \| \leqslant 1$, i.e.

$$y = x + Bz, \qquad y' = x' + B'z, \qquad \|z\| \leqslant 1.$$

Then,

$$\|y - y'\| \leqslant \|x - x'\| + \|B - B'\| \, \|z\| \leqslant \delta.$$

Hence, if E' approximates E with accuracy δ, every point y' of E' is obtained by a δ-shift of a point y of E, and vice versa. This assertion also holds for the boundaries of E and E', since these boundaries are images of the sphere $\| z \| = 1$.

Lemma 3

Let $E \infty (x, B)$ be a non-degenerate ellipsoid of thickness $r(E)$, and let $\lambda \in (1.2)$ be a scalar paramter. If the ellipsoid $E' \infty (x', B')$ approximates ellipsoid λE with accuracy δ, where

$$\delta \leqslant (\lambda - 1) r(E), \tag{2.3}$$

then E' wholly contains E (in particular, E' is non-degenerate), and moreover,

$$\text{mes}\, E'/\text{mes}\, E \leqslant (2\lambda - 1)^n, \tag{2.4}$$

where mes E and mes E' are the volumes of E and E' in R^n.

Proof. Let us first prove the inclusion $E \subseteq E'$. For this, it is sufficient to show that the centre of E' lies inside E, while the boundary of E' is outside E. The first claim follows from

$$\|x - x'\| \leqslant \delta \leqslant (\lambda - 1) r(E) \leqslant r(E).$$

To prove the second, we note that the Euclidean distance ρ between the boundaries of E and λE is $\rho = (\lambda - 1) r(E)$ and from (2.3), $\delta \leqslant \rho$. Since the boundary of E' is obtained by a δ-shift of the boundary of λE, then the boundary of E' must lie outside E; hence $E \subseteq E'$.

To prove inequality (2.4), we only have to prove the inclusion $E' \subseteq (2\lambda - 1) E$, the proof is similar to the above. For, the Euclidean distance between the boundaries of λE and $(2\lambda - 1)E$ is also equal to $(\lambda - 1)r(E) = \rho$, so that $\delta \leqslant \rho$. Since the boundary of E' is obtained by a δ-shift of the boundary of λE, the entire boundary of E' lies outside $(2\lambda - 1)E$, whence follows the inclusion $E' \subseteq (2\lambda - 1) E$, and hence inequality (2.4), and the lemma is proved.

Note 2. The lemma also holds for $\lambda \geqslant 2$, but we do not need this fact.

Let $E = \{y \mid y = x + Bz,\ \|z\| \leqslant 1\}$ be a non-degenerate ellipsoid, and R a non-zero n-dimensional vector. Denote by $E_R/2$ the figure (semi-ellipsoid) obtained by the intersection of E with the half-space $R^\tau (y - x) \leqslant 0$ (here and throughout, all vectors are to be understood as columns; T denotes transposition).

The aim of the next constructions is to see how to describe about $E_R/2$ a new ellipsoid $E' \infty (x', B')$ in such a way that, first, the quantities x' and B' can be found analytically from x, B and R, using only approximate calculations, and second, so that E' has as small a volume as possible. We start with the following proposition [5].

Let V be the Euclidean sphere $\| y \| \leqslant 1$, and $V/2$ its "upper" hemisphere $y_1 \geqslant 0$. This hemisphere can be enclosed in the ellipsoid W,

$$\frac{1}{a^2}\left(y_1-\frac{1}{n+1}\right)^2+\frac{1}{c^2}\sum_{j=2}^{n} y_j^2\leqslant 1,$$

with semi-axes $a=n/(n+1)$ and $c=n/(n^2-1)^{1/2}$. Notice that $W\infty(x,B)$ is specified by the centre $x=((n+1)^{-1},0,\dots,0)$ and the matrix $B=\Lambda_n$, where the diagonal $n\times n$ matrix is

$$\Lambda_n=\operatorname{diag}\left(\frac{n}{n+1},\frac{n}{(n^2-1)^{1/2}},\dots,\frac{n}{(n^2-1)^{1/2}}\right). \tag{2.5}$$

From this proposition we have:

Lemma 4

Let $E\infty(x,B)$ be a non-degenerate ellipsoid, and $E_R/2$ its semi-ellipsoid, specified by the non-zero n-dimensional vector R. The semi-ellipsoid $E_R/2$ can be wholly enclosed in the ellipsoid $E^R\infty(x^R,B^R)$,

$$x^R=x-\frac{B\eta}{(n+1)\|\eta\|},\qquad B^R=BF\Lambda_nF^T, \tag{2.6}$$

where the vector $\eta=B^TR$, and F is an orthogonal $n\times n$ matrix whose first column is the vector $\eta/\|\eta\|$.

Proof. In the z coordinates, in which ellipsoid (2.1) is the sphere $\|z\|\leqslant 1$, the inequality $R^T(y-x)\leqslant 0$ has the form $(B^TR)^Tz\leqslant 0$. To find the centre of the new ellipsoid, we have to move in z coordinates from the origin along the direction $-\eta=-B^TR$, a distance $1/(n+1)$, and hence the centre of the new ellipsoid is $z^R=-\eta/(n+1)\|\eta\|$. Returning to y coordinates, we obtain the first of Eqs. (2.6). The second equation is obtained in a similar way. The lemma is proved.

Notice that Eqs. (2.6) were obtained in [6]; the second equation can be written in the equivalent form

$$B^R=\frac{n}{(n^2-1)^{1/2}}\left\{B+\left[\left(\frac{n-1}{n+1}\right)^{1/2}-1\right]\frac{B\hat{\eta}}{\|\eta\|^2}\right\},$$

where $\hat{\eta}$ is a symmetric $n\times n$ matrix with elements $\hat{\eta}_{\alpha\beta}=\eta_\alpha\eta_\beta$. This form of matrix B^R will be used in the next section; for the present, the form (2.6) is more convenient.

From Eqs. (2.6) we have two inequalities which will be useful later:

$$\|x^R\| \leqslant \|x\| + \frac{\|B\|}{n+1}, \qquad \|B^R\| \leqslant \frac{n}{(n^2-1)^{1/2}} \|B\|. \tag{2.7}$$

In addition, we shall need later the relation

$$r(E^R) \geqslant \frac{n}{n+1} r(E) \tag{2.8}$$

between the thicknesses of ellipsoids E and E^R. This relation follows geometrically from the fact that ellpisoid E^R is most strongly "flattened" when the vector R is directed along the shortest semi-axis of E. For the formal proof of (2.8), we only need to note that

$$r(E) = \min_{\|z\|=1} \|Bz\|, \qquad r(E^R) = \min_{\|z\|=1} \|BF\Lambda_n F^{\mathrm{T}} z\|.$$

When the vector z runs over the sphere $\| z \| = 1$, the vector $z_1 = F^{\mathrm{T}} z$ also runs over this sphere, while vector $z_2 = \Lambda_n z_1$ moves outside the sphere $\|z_2\| \geqslant n/(n+1)$, i.e., the vector $z_3 = F z_2$ also moves outside this sphere. Consequently,

$$r(E^R) \geqslant \min_{\|z_3\| = n/(n+1)} \|Bz_3\| = \frac{n}{n+1} \min_{\|z\|=1} \|Bz\| = \frac{n}{n+1} r(E),$$

which gives (2.8).

Equations (2.6) have the drawback that they have to be calculated exactly in order to construct the ellipsoid E^R. For, whatever the accuracy $\delta > 0$ with which Eqs. (2.6) are calculated, i.e. whatever the accuracy of finding the ellipsoid $\|E^R - E'\| \leqslant \delta$, the inclusion $E_R/2 \subseteq E'$ is not guaranteed; all we can say is that $E_R/2$ falls within the Euclidean δ-neighbourhood of E'. To overcome this difficulty, we consider the ellipsoid $\lambda_n E^R$, where $\lambda_n \in (1, 2)$ is a suitably chosen constant, dependent on the dimensionality n. Then, in accordance with Lemma 3, we can indicate for ellipsoid $\lambda_n E^R$ an accuracy $\|\lambda_n E^R - E'\| \leqslant \delta$, with which it is sufficient to perform the calculations in order to satisfy the inclusion $E_R/2 \subseteq E'$. We shall first require two numerical inequalities.

Let $\lambda_n = 1 + 1/16n^2$ and let the determinant of the matrix Λ_n (see (2.5)) be

$$q_n = \det \Lambda_n = \frac{n^n}{n+1} (n^2-1)^{-(n-1)/2}. \tag{2.9}$$

With $n \geqslant 2$, we have the inequalities

$$\frac{n\lambda_n}{(n^2-1)^{1/2}}+\frac{1}{24n^2}\leqslant 2^{1/n^2}, \tag{2.10}$$

$$(2\lambda_n-1)^n q_n \leqslant 2^{-1/2n}. \tag{2.11}$$

We omit the proof of these inequalities.

Lemma 5

Let $E\infty(x, B)$ be a non-degenerate ellipsoid of thickness $r(E)$, and let $E_R/2$ with $R \neq 0$ be its semi-ellipsoid. Further, let $E^R\infty(x^R, B^R)$ be the ellipsoid whose elements are found from Eqs. (2.6), while ellipsoid $\lambda_n E^R$ is obtained by stretching E^R by a factor $\lambda_n=1+1/16n^2$ Then, if ellipsoid $E'\infty(x', B')$ approximates $\lambda_n E^R$ with accuracy δ, where

$$\delta\leqslant\frac{r(E)}{24n^2}, \tag{2.12}$$

then

$$E_R/2\subseteq E^R\subseteq E', \tag{2.13}$$

$$\|x'\|\leqslant\|x\|+\frac{\|B\|}{n}, \qquad \|B'\|\leqslant\|B\|2^{1/n^2}, \tag{2.14}$$

$$\operatorname{mes} E'/\operatorname{mes} E\leqslant 2^{-1/2n}. \tag{2.15}$$

Proof. We proved the inclusion $E_R/2\subseteq E^R$ in Lemma 4. To prove the inclusion $E^R \subseteq E'$, we only need to show, in accordance with Lemma 3, that

$$\delta\leqslant(\lambda_n-1)r(E^R). \tag{2.16}$$

But relation (2.16) follows at once from (2.8) and condition (2.12):

$$\delta\leqslant\frac{r(E)}{24n^2}\leqslant\frac{n+1}{n}\frac{r(E^R)}{24n^2}\leqslant\frac{3}{2}\frac{r(E^R)}{24n^2}=(\lambda_n-1)r(E^R).$$

Hence inclusion (2.13) is proved. Let us now prove the first of inequalities (2.14). Since $\|x'\|\leqslant\|x^R\|+\delta$, while the thickness $r(E)$ is not greater than $\|B\|$, we have from (2.7):

$$\|x'\| \leqslant \|x^R\| + \frac{r(E)}{24n^2} \leqslant \|x\| + \|B\| \left(\frac{1}{n+1} + \frac{1}{24n^2}\right) \leqslant \|x\| + \frac{\|B\|}{n},$$

which proves the first of inequalities (2.14). The second inequality follows from relations (2.7) and (2.10):

$$\|B'\| \leqslant \|\lambda_n B^R\| + \delta \leqslant \lambda_n \|B^R\| + \frac{r(E)}{24n^2} \leqslant$$

$$\leqslant \frac{n\lambda_n}{(n^2-1)^{1/2}} \|B\| + \frac{1}{24n^2} \|B\| \leqslant 2^{1/n^2} \|B\|.$$

It remains to prove (2.15). It can be seen from (2.16) that Lemma 3 can be applied, and the inequality obtained:

$$\operatorname{mes} E'/\operatorname{mes} E^R \leqslant (2\lambda_n - 1)^n.$$

Next, from the second of Eqs. (2.6) and (2.9), we have

$$\operatorname{mes} E^R/\operatorname{mes} E = |\det B^R| / |\det B| = \det \Lambda_n = q_n,$$

and hence

$$\operatorname{mes} E'/\operatorname{mes} E = (\operatorname{mes} E'/\operatorname{mes} E^R)\,(\operatorname{mes} E^R/\operatorname{mes} E) \leqslant (2\lambda_n - 1)^n q_n,$$

so that (2.15) follows from (2.11), and the lemma is proved.

We conclude the present section with a discussion of Lemma 5. Suppose we are given the non-degenerate ellipsoid E, the pair (x, B), and the semi-ellipsoid $E_R/2$, specified by the non-zero vector R. Our lemma asserts that, if we perform approximately the calculations

$$\begin{aligned} x' &\approx x - \frac{B\eta}{(n+1)\|\eta\|}, \\ B' &\approx \left(1 + \frac{1}{16n^2}\right) \frac{n}{(n^2-1)^{1/2}} \left\{B + \left[\left(\frac{n-1}{n+1}\right)^{1/2} - 1\right] \frac{B\hat{\eta}}{\|\eta\|^2}\right\}, \end{aligned} \tag{2.17}$$

where $\eta \neq B^{\mathrm{T}}R$ and $\hat{\eta}$ is the matrix with coefficients $\hat{\eta}_{\alpha\beta} = \eta_\alpha \eta_\beta$, these calculations being performed with an accuracy δ determined from the thickness of E in accordance with inequality (2.12), then we obtain a new ellipsoid $E' \infty (x', B')$, which wholly contains $E_R/2$, and has a volume $2^{-1/2n}$ times the volume of E, while the norms $\| x' \|$, $\| B' \|$ for it do not seriously exceed the norms $\| x \|$ and $\| B \|$ for the initial ellipsoid.

3. Description of the algorithm

Let us describe the polynomial algorithm for determining the compatibility of the system of linear inequalities (0.1), which it is convenient here to write in the form

$$A_i^T x \leqslant b_i, \qquad i=1, 2, \dots, m, \tag{3.1}$$

where A_i are the rows of the system. We can assume without loss of generality that all the $A_i \neq 0$, since if e.g. $A_1 = 0$, then either the first inequality in (3.1) can be removed (when $b_1 \geqslant 0$), or else the system is incompatible (when $b_1 \leqslant -1$).

The algorithm consists of N iterations, with numbers $k = 0, 1, \dots, N$, where and L is the system input. At the k-th iteration, in addition to the initial information, i.e. the integer-valued matrix (1.1), we also have the ellipsoid $E_k \infty (x_k, B_k)$ and the scalar θ_k. "Physically", x_k, B_k, and θ_k are n–, $(n \times n)$–, and 1-dimensional blocks, each scalar element of which is written in binary form on tape M, having $10L$ binary places before the point and $3L$ places after. We shall say that the scalar θ_k goes onto tape M if $|\theta_k| \leqslant 2^{10L}$. Similarly, the blocks x_k and B_k go onto tape M if each scalar element of them goes on. For this, it is sufficient that $\|x_k\|$, $\|B_k\| \leqslant 2^{10L}$.

At the initial iteration, number $k = 0$, we put

$$x_0=0, \qquad B_0=\operatorname{diag}(2^L, \dots, 2^L), \qquad \theta_0 = \max_{i=1,2,\dots,m} \{-b_i\}, \tag{3.2}$$

i.e. the initial ellipsoid E_0 is the same as the sphere $S=\{x \mid \|x\| \leqslant 2^L\}$, while θ_0 is the discrepancy at the centre of the sphere. At the k-th iteration, x_k, B_k, and θ_k satisfy the inequalities

$$\|x_k\| \leqslant \frac{k}{n} 2^{8L}, \qquad \|B_k\| \leqslant 2^{2L+k/n^2}, \qquad |\theta_k| \leqslant 2^{10L}. \tag{3.3}$$

In particular, since $k \leqslant 6n^2L$, it follows from (3.3) that x_k, B_k, and θ_k go onto tapes M. With $k = 0$, inequalities (3.3) are satisfied (see (3.2)), while their satisfaction for other k will be obvious from the description of the algorithm.

At the k-th iteration the algorithm operation starts with the evaluation of

$$\theta(x_k) = \max_{i=1,2,\dots,m} \{A_i^T x_k - b_i\}, \tag{3.4}$$

specifying the discrepancy at the centre of the next ellipsoid E_k. Since

$$\|A_i\| \leqslant 2^L/nm, \qquad |b_i| \leqslant 2^L/nm \tag{3.5}$$

for any $i = 1, 2, \ldots, m$, each of the differences in (3.4) has modulus not exceeding

$$|A_i^T x_k - b_i| \leqslant \|A_i\| \, \|x_k\| + |b_i| \leqslant \frac{6L}{m} 2^{9L} + \frac{2^L}{nm} \leqslant 2^{10L}. \tag{3.6}$$

Hence, since vectors A_i have integral components, each difference $A_i^T x_k - b_i$ can be calculated exactly and put onto M. Consequently, we can find exactly the quantity $\theta(x_k)$, and also the number i_k of the row in which the maximum of (3.4) is reached at the k-th iteration (if there are several such rows, we take any one of them). We then put $\theta_{k+1} = \min\{\theta_k, \theta(x_k)\}$. The exactly calculated θ_{k+1} is the minimal discrepancy in the approximations $x_0, \ldots, x_k$ obtained at the k-th iteration, and by relation (3.6), it satisfies the third of inequalities (3.3) with $k + 1$, since this inequality was true for k.

After these exact calculations, we then find the next ellipsoid E_{k+1}, for which we first find the vector $\eta_k = B_k^T A_{i_k}$. Since A_{i_k} is an integer-valued vector, and, from (3.3) and (3.5), $\|\eta_k\| \leqslant \|B_k\| \, \|A_{i_k}\| \leqslant 2^{9L}$, the vector η_k can be calculated exactly and put onto tapes M. If $\eta_k = 0$, the calculations stop, and the algorithm supplies the recorded value of discrepancy θ_{k+1}. If η_k is non-zero, the next approximations x_{k+1}, B_{k+1} are calculated from relations similar to (2.17):

$$\begin{aligned} x_{k+1} &\approx x_k - \frac{B_k \eta_k}{(n+1)\|\eta_k\|}, \\ B_{k+1} &\approx \left(1 + \frac{1}{16n^2}\right) \frac{n}{(n^2-1)^{1/2}} \left\{ B_k + \left[\left(\frac{n-1}{n+1} \right)^{1/2} - 1 \right] \frac{B_k \hat{\eta}_k}{\|\eta_k\|^2} \right\}, \end{aligned} \tag{3.7}$$

the calculations being performed approximately, to an accuracy

$$\delta \leqslant \frac{2^{-2L}}{24n^2} \tag{3.8}$$

(recall that tape M has $3L$ places after the point). The following remark is important: since $\|x_k\|$, $\|B_k\| \leqslant 2^{10L}$ and matrix $\hat{\eta}_k$ has a specific structure (its $\alpha\beta$-th element is the product of components $\eta_{k\alpha}$ and $\eta_{k\beta}$ of vector η_k), then to perform calculations (3.7) to accuracy (3.8), $O(n^2)$ operations $+, -, \times, :, \sqrt{\ }, \max$, are sufficient; these operations need to be performed to an accuracy of $O(L)$ places.

After computing the ellipsoid $E_{k+1} \backsim (x_{k+1}, B_{k+1})$ we check for satisfaction of the inequalities

$$(k+1) \leqslant 6n^2 L, \qquad \|x_{k+1}\| \leqslant \frac{(k+1)}{n} 2^{8L}, \qquad \|B_{k+1}\| \leqslant 2^{2L+(k+1)/n^2}, \tag{3.9}$$

for which again, $O(n^2)$ operations over $O(L)$-place numbers are sufficient. If one or more inequalities is violated, the algorithm operation stops and the number θ_{k+1} is printed; otherwise, we proceed to the next $(k + 1)$-th iteration. This completes the description of the algorithm.

Before proving that the algorithm is convergent, let us consider some details of its operation. Let $E_0, \ldots, E_N$ be the ellipsoids obtained during operation of the algorithm, and let $r(E_0)=2^L, \ldots, r(E_N)$ be their thicknesses. Let us show that, among ellipsoids E_k, at least one has thickness $r(E_k)\leqslant 2^{-2L}$. Assume the contrary: $r(E_k)>2^{-2L}$ for all $k = 0, 1, \ldots, N$, and let us ask why the algorithm stops at the N-th iteration. It cannot stop because the vector $\eta_N=B_N{}^TA_{i_N}$ vanishes: the integer-valued vector A_{iN} is non-zero, and the matrix B_N is non-degenerate, since E_N has non-zero thickness. Let us show that the algorithm cannot stop because the second or third of inequalities (3.9) is violated. Since $r(E_{N-1})>2^{-2L}$, it follows from (3.8) that the accuracy with which ellipsoid E_N is evaluated from ellipsoid E_{N-1}, satisfies the condition $\delta\leqslant r(E_{N-1})/24n^2$ of Lemma 5, and hence the second and third of inequalities (3.9) hold for N, inasmuch as they hold for $N - 1$, and inasmuch as we have the inequalities (2.14). In short, the algorithm stops because the first of conditions (3.9) is violated and $N = 6n^2L$. Further, it follows from the conditions $r(E_k)>2^{-2L}$ and (3.8) that each of ellipsoids E_{k+1} satisfies the conditions of Lemma 5 relative to the previous ellipsoid E_k, so that inequality (2.15) can be used, and we obtain

$$\operatorname{mes} E_{k+1}/\operatorname{mes} E_k\leqslant 2^{-1/2n}, \qquad k=0, 1, \ldots, N-1,$$

whence, with $N = 6n^2L$, we have

$$|\det B_N|/|\det B_0| =\operatorname{mes} E_N/\operatorname{mes} E_0\leqslant 2^{-3nL}.$$

On now remarking that $\det B_0 = 2^{nL}$, we obtain $|\det B_N|\leqslant 2^{-2nL}$, i.e. by (2.2), $r(E_N)\leqslant 2^{-2L}$. All in all, among ellipsoids $E_0, \ldots, E_N$, there must be one, E_k, with thickness not exceeding 2^{-2L}.

Now let E_t be the first ellipsoid to have a thickness not exceeding 2^{-2L}, i.e.

$$r(E_k)\geqslant 2^{-2L},\ k=0, 1, \ldots, t-1, \qquad r(E_t)\leqslant 2^{-2L} \tag{3.10}$$

(notice that the number t of the iteration at which (3.10) holds for the first time, remains unknown after operation of the algorithm). As we have mentioned, up to the t-th iteration, the conditions of Lemma 5 hold. In particular, if $E_k/2$, $k \leqslant t - 1$, is the semi-ellipsoid, obtained from ellipsoid $E_k=\{x\,|\,x=x_k+B_kz,\ \|z\|\leqslant 1\}$ by cutting out the domain $A_i{}^T(x-x_k)\geqslant 0$, in which the discrepancy certainly exceeds $\theta(x_k)$, then the semi-ellipsoid $E_k/2$ lies wholly in the next ellipsoid E_{k+1}.

4. Proof of convergence

For the proof, it is useful to introduce the concept of the thickness of a geometric figure in R^n. If $A \subset R^n$ is a bounded closed figure, we define its thickness $r(A)$ as the maximum Euclidean sphere which can be wholly contained in A, i.e.

$$r(A)=\max_{x\in A}\min_{y\in\Gamma}\|x-y\|, \tag{4.1}$$

where Γ is the boundary of A. For the case when A is an ellipsoid, definition (4.1) agrees with our earlier definition, since (4.1) gives the minimal semi-axis. Notice that, if $A \subseteq B$, then $r(A) \leqslant r(B)$.

Lemma 6

Let S and V be two Euclidean spheres in R^n of radius r_1 and r_2 respectively, $0 < r_2 < r_1$. If the centre of V belongs to S, then $r(V\cap S) \geqslant r_2/2$.

Proof. We only need to note that a sphere of radius $r_2/2$ can be included in the intersection $V \cap S$.

We now turn to the proof of convergence, i.e. we will show that the minimum of discrepancy θ_{N+1} satisfies the inequality

$$\theta_{N+1}\leqslant\theta_S+2^{-L}, \tag{4.2}$$

where θ_S is the minimum of the discrepancy in the sphere $S=\{x\mid \|x\|\leqslant 2^L\}$. We showed in Section 1 that, in this case, we can judge from the value θ_{N+1} whether or not system (0.1) is compatible. For the proof, we only need to show that

$$\theta_{t+1}\leqslant\theta_S+2^{-L}, \tag{4.3}$$

where t is the number of the iteration at which the thickness of ellipsoid E_k is for the first time less than or equal to 2^{-2L} (see (3.10)), while

$$\theta_{t+1}=\min\ \{\theta(x_0),\ldots,\theta(x_t)\} \tag{4.4}$$

is the minimum value of the discrepancy at the centres of the ellipsoids $E_0=S,\ E_1,\ldots,E_t$.

Proceeding to the proof of (4.3), let $P_k,\ k=0,\ 1,\ldots,t,$ be an open half-space of the form $P_k=\{x\mid A_{i_k}{}^T(x-x_k)>0\}$. Since, at the point $x = x_k$, the maximum of (3.4) is reached in the i_k-th row, then

$$\theta(x)>\theta(x_k),\qquad x\in P_k. \tag{4.5}$$

Denote by S_k the figure resulting from intersection of the sphere S with the union of half-spaces P_l, $l = 0, 1, \ldots, k$ (the figure S_k is non-empty, since a hemisphere is cut from the sphere S even at the initial iteration), and let D_k be the convex, closed, possibly empty figure augmenting S_k to the sphere S:

$$S_k = S \cap [\bigcup_{l=0}^{k} P_l], \qquad S_k \cup D_k = S, \qquad S_k \cap D_k = \varnothing.$$

From (4.5) and (4.4), if S_t is the last of sets S_k, then

$$\theta(x) \geqslant \theta_{t+1}, \qquad x \in S_t. \tag{4.6}$$

If $S_t = S$, i.e. set D_t is empty, then (4.3) follows from (4.6), and the convergence of the algorithm is proved; hence we can confine ourselves to the case when $D_t \neq \phi$. Let us show that, then

$$\theta(x) \geqslant \theta_{t+1} - 2^{-L}, \qquad x \in D_t. \tag{4.7}$$

Since S_t and D_t combine to give the entire sphere S, then, once we have proved (4.7), we again obtain inequality (4.3) on additionally using (4.6), so that the convergence will again be proved.

We shall first show that the set D_t belongs to the ellipsoid E_t. In fact, it follows from the description of the algorithm that $D_0 \subseteq E_1$. To show that, in general, for any $k \leqslant t-1$, we have $D_k \subseteq E_{k+1}$. we shall assume that this inclusion holds for k, and show that it then holds for $k+1$. The set D_{k+1} is obtained from D_k by cutting out a part by the half-space P_{k+1}. Then,

$$D_{k+1} = D_k / P_{k+1} \subseteq E_{k+1} / P_{k+1} = E_{k+1}/2 \subseteq E_{k+2}$$

(see end of Section 3), so that the inclusion $D_k \subseteq E_{k+1}$ is proved by induction. On applying the inclusion for $k = t-1$, we get $D_{t-1} \subseteq E_t$, i.e. all the more, $D_t \subseteq E_t$.

We now turn to the proof of (4.7). Let $\xi \in D_t$ be the point at which the discrepancy reaches its minimum in compactum D_t:

$$\min_{x \in D_t} \theta(x) = \theta(\xi), \qquad \xi \in D_t. \tag{4.8}$$

Further, let V be the Euclidean sphere of radius $r_2 = 4 \cdot 2^{-2L}$ centre the point ξ, and let $W = V \cap D_t$. The boundary of the convex closed figure W in general consists of three mutually intersecting pieces:

the boundary Γ_S, formed by part of the sphere $\| x \| = 2^L$,

the boundary Γ_V, formed by part of the sphere $\| x - \xi \| = r_2$,

the boundary Γ_P, formed by the boundaries of half-spaces P_k, i.e. by all possible planes $A_{i_k}{}^T (x - x_k) = 0$, $k = 0, 1, \ldots, t$.

Let us show that $\Gamma_P \neq \phi$. In fact, if Γ_P is empty, i.e. the set W intersects no hyperplane $A_{i_k}{}^T (x - x_k) = 0$, then W is the same as $V \cap S$, i.e. $W = V \cap D_t = V \cap S$. Then, by Lemma 6, the thickness $r(W)$ exceeds $r_2/2$, i.e. $r(W) \geqslant 2 \cdot 2^{-2L}$. Since $W \subseteq D_t$, then $r(D_t) \geqslant 2 \cdot 2^{-2L}$, which is impossible, since the set D_t belongs to the ellipsoid E_t of thickness $r(E_t) \leqslant 2^{-2L}$.

In short, $\Gamma_P \neq \phi$, i.e. the sphere V intersects at least one hyperplane $A_{i_k}{}^T (x - x_k) = 0$. Let η be a point of the intersection:

$$A_{i_k}^T(\eta - x_k) = 0, \qquad \|\eta - \xi\| \leqslant r_2 = 4\cdot 2^{-2L}. \tag{4.9}$$

From (4.5) we have

$$\theta(\eta) \geqslant \theta(x_k) \geqslant \theta_{t+1}. \tag{4.10}$$

Further, the function $\theta(x)$ satisfies a Lipschitz condition with respect to x with constant $\max\{\|A_i\|\} \leqslant 2^L/nm$, and hence we obtain from (4.9):

$$|\theta(\xi) - \theta(\eta)| \leqslant \frac{2^L}{nm}\|\xi - \eta\| \leqslant \frac{4}{nm} 2^{-L} \leqslant 2^{-L},$$

which, along with (4.8) and (4.10), gives the inequality

$$\min_{x \in D_t} \theta(x) = \theta(\xi) \geqslant \theta(\eta) - 2^{-L} \geqslant \theta_{t+1} - 2^{-L},$$

which is the same as (4.7). On now applying (4.7) jointly with (4.6), we obtain (4.3), and hence (4.2), i.e. the convergence of the algorithm is proved. In accordance with the results of Section 1, either $\theta_{N+1} \leqslant 2^{-L}$, and system (0.1) is compatible, or else $\theta_{N+1} \geqslant 2\cdot 2^{-L}$ and the system is incompatible. Our polynomial algorithm is now constructed.

5. Estimates of the labour of determining the compatibility of a system of linear inequalities

We have constructed above an algorithm for determining the compatibility in R^n of an $n \times m$ system of linear inequalities (0.1) with input length L; it requires for its realization $O(n^3(n+m)L)$ elementary operations $+, -, \times, :, \sqrt{\ }, \max$ over $O(L)$-place numbers, and storage of $O(nm + n^2)$ such numbers.

Notice the following: let $\alpha = \min\{n, m\}$, $\beta = \max\{n, m\}$ be the minimum and maximum dimensionalities. Since system (0.1) is compatible if and only if the dual system is incompatible, i.e. the $m \times (n + m + 1)$ system of equations and inequalities

$$\begin{aligned} &a_{1j}\lambda_1 + \ldots + a_{mj}\lambda_m = 0, \qquad j = 1, 2, \ldots, n,\\ &b_1\lambda_1 + \ldots + b_m\lambda_m = -1,\\ &\lambda_i \geqslant 0, \qquad i = 1, 2, \ldots, m, \end{aligned}$$

whose input is $O(L)$, is incompatible, then, on passing in the case $n \geqslant m$ from the determination of the compatibility of the initial system to the determination of the incompatibility of the dual system, we obtain the following result: to determine the compatibility of system (0.1), we only need $O(\beta\alpha^3 L)$ elementary operations over $O(L)$-place numbers and storage of $O(nm)$ such numbers.

Let us now find estimates for the labour of the algorithm in terms of dimensionalities n, m and the maximum modulus H of the coefficients:

$$|a_{ij}|,\ |b_i| \leqslant H, \qquad i=1, 2, \ldots, m, \quad j=1, 2, \ldots, n.$$

It is clear from the proofs of Lemmas 1 and 2 that, in the estimates of the localization of solutions and the measure of incompatibility, the quantity 2^L can be replaced by $\Delta\,\alpha$, where Δ is the maximum modulus of the determinants of the extended matrix (1.1). Consequently, in the labour estimates we can replace L by $\log \Delta\,\alpha$. Since $\Delta \leqslant (\alpha^{1/2} H)^{\alpha}$, the labour indices are: for determining the compatibility of system (0.1), $O(\beta\alpha^4 \log H\alpha)$ elementary operations are sufficient, over $O(\alpha \log H\,\alpha)$-place numbers, and the storage of $O(nm)$ such numbers.

If we use a model of an abstract computing machine, we can speak about the time and capacity complexities of the algorithm. For instance, if we use the model of equally accessible address machine PAM (see [7], p. 22), then the time complexity of the algorithm of determining the compatibility of system (0.1) is

$$T(n,\ m,\ H) = O(\beta\alpha^5 \log^2 H\alpha),$$

and the capacity complexity is

$$S(n,\ m,\ H) = O(\beta\alpha \log H + \alpha^3 \log H\alpha).$$

Note 3. We have mentioned that, if Δ is the maximum modulus of all possible determinants of matrix (1.1) of the system, then, to find the compatibility of the system, $O(\beta\alpha^3 \log \Delta\alpha)$ elementary operations suffice over $O(\log \Delta\,\alpha)$-place numbers, and storage of $O(nm)$ such numbers. This fact is useful for estimating the labour of problems with unimodular matrices of constraints $\| a_{ij} \|$, and in general, for problems in which Δ increases as a polynomial in H, n, and m. For instance, if we apply this approach to finding the size $\nu = \nu(G)$ of the maximum flow in a mesh G with n nodes, the handling capacities of the arcs of which are not greater than H, and we seek the integer-valued quantity ν by the method of halving of the interval $\nu \in [0,\ Hn^2]$ we obtain an algorithm requiring $O(n^4 \log^2 Hn)$ elementary operations over $O(\log Hn)$-place numbers. This result is worse than some familiar flow algorithms of combinatorial type (see [8], Chapter 1).

6. Polynomial algorithms in linear programming

Let us now consider two other common problems in the field of linear inequalities: to find the solution of system (0.1) when it is compatible; and the problem of linear programming (l.p.), i.e. the maximization of the linear form $c^{\mathrm{T}}x$ with linear coefficients under constraints (0.1):

$$\begin{aligned} &A_i^{\mathrm{T}}x = a_{i1}x_1 + \ldots + a_{in}x_n \leqslant b_i, \qquad i=1, 2, \ldots, m, \\ &c^{\mathrm{T}}x = c_1x_1 + \ldots + c_nx_n \to \max = \psi^*. \end{aligned} \tag{6.1}$$

We will show that polynomial algorithms can be constructed for these problems. Start with the problem of finding the extremal value of the function $c^{\mathrm{T}}x$ in the l.p. problem (6.1). Let

$$L_1 = L + \left[\sum_{j=1}^{n} \log_2(|c_j|+1) \right]$$

be the input length of the l.p. problem, where the input L of linear inequalities (0.1) is specified by expression (0.2). Using the same approach as in Section 1 for systems of linear inequalities, we can show that, if the constraints are compatible and the functional $c^T x$ is bounded, its extremal value is rational and has the form $\psi^* = t/s$, where t and s are relatively prime integers, not exceeding 2^{L_1}. Hence an algorithm T_1, polynomial in L_1, for seeking ψ^* can be constructed as follows. Let T be the earlier constructed algorithm for determining the compatibility of a system of linear inequalities. Using T, we check if constraints (6.1) are compatible; if they are not, then $\psi^* = -\infty$. If they are, we check if the functional is bounded; for this, we use T to determine the compatibility of the constraints in the dual l.p. problem. If the constraints in the dual problem are incompatible, then the functional is not bounded ($\psi^* = +\infty$); otherwise, we pass to finding its extremal values, i.e. the integers t and s, $|t|, |s| \leqslant 2^{L_1}$. For this, by the method of halving the interval $\psi \in [-2^{L_1}, 2^{L_1}]$ and checking at each step by means of T for compatibility of the systems with integer coefficients of the type

$$A_i^T x \leqslant b_i,\ i=1, 2, \ldots, m, \qquad 2^{2L_1+2} c^T x \geqslant 2^{2L_1+2} \psi_k,$$

where ψ_k is the value of the functional in the k-th system, we find after $3L_1 + 2$ steps the approximate value ψ^* to an accuracy 2^{-2L_1-2}. In other words, after $3L_1 + 2$ steps we find the number $\bar{\psi} = \psi_{3L_1+2}$, such that, first, $\bar{\psi}$ has not more than L_1 binary places before, and $2L_1 + 2$ binary places after, the point, and second,

$$|\bar{\psi} - t/s| \leqslant 2^{-2L_1-2}, \qquad |s| \leqslant 2^{L_1}. \tag{6.2}$$

It follows from (6.2) (see [9], p. 43) that the fraction t/s is the best rational approximation and a convergent of the number $\bar{\psi}$, and hence can be found by expanding $\bar{\psi}$ in a continued fraction. In short, to find the exact value $\psi^* = t/s$, we expand the approximate value $\bar{\psi}$ in a continued fraction and seek the (unique) convergent, satisfying relation (6.2). Since the binary form of $\bar{\psi}$ has $O(L_1)$ places, to expand $\bar{\psi}$ in a continued fraction we require at most $O(L_1)$ arithmetic operations over $O(L_1)$-place numbers. The polynomial algorithm T_1 is constructed. We estimate the labour of T_1 in terms of n, m, and H: to find the extremal value of the functional in the l.p. problem, $O(\beta\alpha^5 \log^2 H\alpha)$ elementary operations suffice over $O(\alpha \log H\alpha)$-place numbers, and storage of $O(nm)$ such numbers; as before, α and β are the minimum and maximum of the dimensionality.

Let us turn to the problem of solving a compatible system of linear inequalities (0.1), and construct a suitable solving algorithm T_2, polynomial with respect to its input L. Let system (0.1) have a non-empty set of solutions $X \subseteq R^n$. We replace the first inequality $A_1^T x \leqslant b_1$ in the system by the equation $A_1^T x = b_1$ and find by means of algorithm T if the resulting system is compatible. If it is not, i.e. the hyperplane $A_1^T x = b_1$ does not intersect X, then the first inequality in (0.1) is superfluous and can be dropped without changing X. If the system is compatible, i.e. hyperplane $A_1^T x = b_1$ intersects X, then we fix an equation once and for all in the first row of (0.1). In any case, a new system U_1 of inequalities and equations is obtained with non-empty set of solutions $X_1 \subseteq X$, in which the total number of inequalities and equations has not increased, while the number of inequalities is reduced by unity. For system U_1 we replace the first inequality $A_2^T x \leqslant b_2$ by the equation $A_2^T x = b_2$, and find by means of T if the hyperplane $A_2^T x = b_2$ intersects the set X_1, etc. After m steps we obtain a compatible system of linear equations U_m, every solution of which gives a solution of the initial system of inequalities (0.1).

Next, after checking the linear dependence of rows of U_m on others, we arrange for r linearly independent rows to be left in U_m, where the rank $r \leqslant \alpha \leqslant n$. After this, on equating the components of vector x in turn to zero, and checking the compatibility of the resulting systems, we isolate the $n - r$ components x_j which can be assumed to be zero in the solution. An $r \times r$ non-degenerate system of linear equations is obtained for determining the other components. Methods of solving this system present no difficulties, and hence the characteristics of T_2 are as follows: for solving the system of inequalities (0.1), $O(\beta^2\alpha^4 \log H\alpha)$ elementary operations suffice, over $O(\alpha \log H \alpha)$-place numbers, and the storage of $O(nm)$ such numbers.

Finally, if, in addition to finding the extremal value of the functional ψ^* for the l.p. problem (6.1), we want to find the point at which the extremum is reached, this may be done by successive application of T_1 and T_2. In short we have:

Theorem

The linear programming problem is polynomially solvable.

Notes 4. Algorithms T_1 and T_2 can be constructed in other ways. For instance, using T_1 we can construct T'_2, whereby system (0.1) is solved by successive solution of the l.p. problems consisting in extremization of the j-th component and in substitution of the results in the system in order to reduce the number of unknowns x_j. Conversely, using T_2, we can construct T'_1, whereby the linear functional is extremized by a device consisting in a combination of the forward and dual l.p. problems into a single system of inequalities, and in solving the latter by means of T_2. Algorithms T'_1 and T'_2 may prove preferable to T_1 and T_2, or vice versa, depending on n, m, and H.

5. The polynomial solvability of linear programming implies the polynomial solvability of linear fractional programming (l.f.p.). Consider the l.f.p. problem

$$A_i^{\mathrm{T}} x \leqslant b_i, \qquad i=1, 2, \ldots, m, \quad x \in R^n,$$

$$f(x) = \frac{c^{\mathrm{T}} x + d}{p^{\mathrm{T}} x + q} \to \max = f^*$$

with integral coefficients $a_{ij}, b_i, c_j, p_j, d, q$; there is no need to explain what is meant by the input length L_2 of an l.f.p. problem. We make the usual assumption for l.f.p.: the system of linear constraints is compatible, and in the set X of its solutions, the denominator $p^{\mathrm{T}} x + q$ of the functional does not vanish, say $p^{\mathrm{T}} x + q > 0$. Then, we know [10] that, if the extremum is reached at a finite point, and not on a ray, then we can take as the extremal point an edge point of the polyhedron X, in particular, the point x^0 with coordinates $x_j^0 = \Delta_j/\Delta$, where Δ_j and Δ are the determinants of the matrix of the system of inequalities. Hence, if the extremum is reached at a finite point, then the extremal value $f^* = f(x^0)$ is rational and has the form $f^* = t/s$, where $|t|, |s| \leqslant 2^{2L_2}$; it can be shown that f^* has this form also when a finite extremal value is reached on a ray. In the light of this fact, the following polynomial l.f.p. algorithm can be constructed:

(1) we check with the aid of T, to see if the inequality $f(x) \geqslant 2^{2L_2}+1$; is possible in the set X of admissible points; if it is, the functional is unbounded; otherwise, $f^* < +\infty$;

(2) by the method of halving the interval $f \in [-2^{2L_2}, 2^{2L_2}]$ while checking at each step if the inequality $f(x) \geqslant f_k$ is possible in X, we find after $6L_2 + 2$ steps the approximate value of f^* to an accuracy 2^{-4L_2-2}; on expanding the approximate value in a continued fraction, we find the exact extremal value $f^* = t/s$;

(3) we use T to see if the equation $F(x) = t/s$ is possible in X; if it is, we find the finite extremal point x^0 by means of T_2; if it is not, then there are no finite extremal points.

Conclusion

The polynomial algorithm of linear programming constructed above are based on the method of ellipsoids for preliminary localization of solutions and estimation of the measure of incompatibility. The question arises as to whether a similar approach might not be used to construct polynomial algorithms in other classes of convex, e.g., quadratic, programming problems. If we wish to determine the compatibility in R^n of a class of systems of convex inequalities, the problem can be solved by a polynomial algorithm provided that the following three conditions are satisfied:

1. Localization of the solutions. If the system, with input length L, is compatible, then there is a solution x^0 belonging to the Euclidean sphere $\|x\| \leqslant 2^{P_1(L)}$, where the polynomial $P_1(L)$ is fixed for the entire class of problems (we assume that $L \geqslant n, m$, where n and m are the numbers of variables and constraints).

2. Measure of incompatibility. If the system, with input length L, is incompatible, then its discrepancy in R^n exceeds $2^{-P_2(L)}$, where $P_2(L)$ is another fixed polynomial.

3. Polynomial computability of functions and gradients. If $x \in R^n$ is a rational vector, each component of which has l binary places before and after the point, then the value of each function-constraint and its (generalized) gradient at the point x can be computed in time polynomially dependent on l, L, and $\log(1/\epsilon)$, with accuracy $\epsilon > 0$.

The sufficiency of these three conditions, for constructing a polynomial (in L) algorithm for determining the compatibility of a class of systems of convex inequalities, may be proved by the method of ellipsoids, in the same way as above for systems of linear inequalities.

In particular, using conditions 1–3, we can prove the polynomial solvability of systems of linear inequalities with one convex quadratic constraint, i.e. systems of the type

$$A_i^{\mathrm{T}} x \leqslant b_i, \qquad i = 1, 2, \ldots, m, \quad x \in R^n,$$
$$f(x) = x^{\mathrm{T}} Q x + c^{\mathrm{T}} x + d \leqslant 0,$$

where Q is a symmetric integer-valued positive semi-definite matrix. Hence follows the polynomial solvability of convex quadratic programming: there is a polynomial algorithm of exact minimization of a convex quadratic functional $f(x)$ under linear constraint-inequalities [11].

On the other hand, it is useless to try to use the same approach to construct a polynomial algorithm for determining the compatibility in R^n of a system of convex quadratic inequalities, because conditions 1 and 2 are violated for such systems. For instance, the input of the incompatible system of convex quadratic inequalities

$$2x_1 \leqslant -1,$$

$$x_2+x_1^2\leqslant 0,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$x_{n+1}+x_n^2\leqslant 0,$$

$$x_{n+1}\geqslant 0$$

is polynomial in n, yet at the point with coordinates $x_j = -2^{-2j}$ the discrepancy of the system is so small that it is specified by the two-stage exponential function $\theta(x)=2^{-2^n}$. But if we replace the first inequality $2x_1 \leqslant -1$ by the inequality $x_1 \leqslant -2$, and the last inequality $x_{n+1} \geqslant 0$ is discarded, we obtain a compatible system, all the solutions of which lie outside the sphere $\|x\| \geqslant 2^{2n}$.

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