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Decidability of Krohn–Rhodes complexity $c = 1$ of finite semigroups and automata

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Abstract

When decomposing a finite semigroup into a wreath product of groups and aperiodic semigroups, complexity measures the minimal number of groups that are needed. Determining an algorithm to compute complexity has been an open problem for almost 60 years. The main result of this paper proves **decidability of Krohn–Rhodes complexity $c = 1$** of finite semigroups and automata. This is achieved by showing the lower bounds in work by Henckell, Rhodes and Steinberg from 2012 is sharp using profinite methods and results of McCammond from 1991 and 2001.

1 Introduction

The Krohn–Rhodes Theorem, originally known as the Prime Decomposition Theorem, is one of the most important theorem of finite semigroup theory. It first appeared in print in [6] and has since appeared in many monographs devoted to finite semigroups and automata. See for example [8, 26, 9, 20]. The theorem states that every finite semigroup S divides, that is, is a quotient of a subsemigroup, of a wreath product of groups and the semigroup U consisting of two right-zeros and an identity. One can choose the groups to be simple groups that divide S . A semigroup is called prime if whenever it divides the wreath product of two semigroups, then it divides one of the factors. The theorem also shows that a semigroup is prime if and only if it is a simple group or is a subsemigroup of U .

It follows that a semigroup is aperiodic, that is, has only trivial subgroups if and only if it divides a wreath product of copies of U . One now defines the **(Krohn–Rhodes) complexity Sc** of the finite semigroup S to be the **least number of groups needed in any wreath product decomposition** of S as a divisor of wreath products of groups and aperiodic semigroups. Complexity was first defined and developed in [7] and is extensively described in [8] and [20].

It is known that there are semigroups of arbitrary complexity. Indeed, the full transformation semigroup T_n on n elements has complexity $n - 1$. In general, any decomposition of a finite semigroup S gives an upper bound for complexity. For example, the Depth Decomposition Theorem [24] gives a decomposition that shows that the complexity of S is at most the length of the longest chain of non-aperiodic \mathcal{J} -classes in S . This bound can be arbitrarily bigger than the actual complexity of S . For example, the complexity of an inverse semigroup is at most 1, but the longest chain of non-aperiodic \mathcal{J} -classes can be arbitrarily large.

Lower bounds are more difficult to find. If the complexity of a semigroup is at least n , then a lower bound must guarantee there is no decomposition of the semigroup with less than or equal to $n - 1$ groups and the collection of all such decompositions is infinite. Early lower bounds used certain subsemigroup chains [22, 23], but proved to be proper lower bounds. See [20, Chapter 4] for examples. A much more sophisticated lower bound was proved in [5]. The main purpose of this paper is to prove

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that a slightly modified version of this lower bound is perfect if the complexity of S is at most 1. The lower bound in [5] uses flows on set-partition lattices. In this paper, we use flows on Rhodes lattices [12] instead. Since the lower bound in [5] is effectively computable, this proves that **if a finite semigroup has complexity 1 is decidable**, thus settling the major open problem of finite semigroup theory, a problem that has been open for 60 years.

The paper is organized as follows. In Section 2 we give the preliminaries and background for the paper. In particular we discuss flows on the Rhodes lattice and its relation to the Presentation Lemma [5, 2]. Section 3 is devoted to a variation of the evaluation semigroup $\text{Eval}(S)$ of a group-mapping semigroup S that plays a big role in [5] and in this paper. We recall basic properties of pointlike functors in Section 4. In addition, we describe the various expansions we use in the paper. Most important are the Karnofsky–Rhodes expansion and the McCammond expansion [20, 15]. We look at geometric semigroup theory in the sense of [15] and define the GST expansion of a finite semigroup. In Section 5, we introduce methods from profinite semigroup theory and prove the main theorem. Besides profinite methods, we also make crucial use of the geometry of the Cayley graph of free aperiodic Burnside semigroups proved by McCammond in [14, 16].

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2 Complexity and the Presentation Lemma

In this section, we review the basics of transformation semigroups and the semilocal theory of semigroups. See [8, Chapter 7-8] and [20, Chapter 4] for more details.

We present a new proof of one direction of the Presentation Lemma based on the notion of flows in the Rhodes Lattice. See [2] and [20] for other formulations and proofs of the Presentation Lemma. See [12] for the basics of the Rhodes lattice and its connection to the Dowling lattice and [5] for the relation of the Presentation Lemma to flows in the Rhodes lattice.

2.1 Actions and products

Given a nonempty set Q (respectively set Q), we denote by $T(Q)$ (or $PT(Q)$) the monoid of all full (respectively partial) transformations on Q , where in the composition $\varphi \circ \psi$ we apply φ first (and so we denote by $q\varphi$ the image of q by φ). In this paper, we use the convention that we treat partial transformations in $PT(Q)$ as full transformations by adjoining a new element \square to Q and setting $q\varphi = \square$ if $q\varphi$ is not defined and $\square\varphi = \square$.

A *right action* of a semigroup S on a set Q is a semigroup morphism from S to $T(Q)$. A right action of a semigroup S on a set Q *by partial mappings* is a semigroup morphism α from S to $PT(Q)$. If this morphism is injective, the action is *faithful*. We use the notation (Q, S) to express that a semigroup S acts faithfully on the right on a set Q by partial mappings. We call (Q, S) a *transformation semigroup*. Usually, we write qs to denote $q(s\alpha)$.

If I is a right ideal of S , then S acts on the right of I by multiplication through the morphism

$$\begin{aligned} \theta : S &\rightarrow T(I) \\ s &\mapsto \theta_s \end{aligned}$$

defined by $a\theta_s = as$ for all $a \in I$.

Let S^{op} denote the opposite semigroup of S . A *left action* of S on Q is a right action β of S^{op} on Q . Usually, we write sq to denote $q(s\beta)$. We define left actions by partial mappings, left actions by multiplication and left transformation semigroups (S, Q) in the obvious way.

Let S and T be two semigroups. Let $\beta : T \rightarrow \mathcal{T}(S)$ be a left action of T on S . The *semidirect product* $S \rtimes_{\beta} T$ is the semigroup defined by considering the multiplication $(s, t)(s', t') = (s(ts'), tt')$ on the set $S \times T$. Usually, we write just $S \rtimes T$.

Next, we define the *wreath product* $(B, S) \wr (Q, T)$ of the two transformation semigroups (B, S) and (Q, T) . The wreath product $(B, S) \wr (Q, T)$ consists of all pairs (f, t) where $t \in T$ and $f : Q \rightarrow S \in S^Q$, the semigroup of all functions from Q to S with pointwise multiplication. The action of (f, t) on $(b, q) \in B \times Q$ is

$$(b, q)(f, t) = (b(qf), qt).$$

This defines the product in the semigroup of the wreath product by

$$(f, t)(f', t') = (f(t \circ f'), tt').$$

This defines the wreath product $(B, S) \wr (Q, T)$ as the transformation semigroup $(B \times Q, S^Q \times T)$. The product in the wreath product above gives $S^Q \times T$ the structure of a semidirect product.

2.2 Actions related to Green's relations

For background on Green's relations and Rees Theorem, see [8] and [20, Chapter 4 and Appendix A].

Let S be a finite semigroup and let J be a \mathcal{J} -class of S . We denote by $L(J)$ the set of all \mathcal{L} -classes contained in J . We simplify the notation to $B = L(J)$. The right action of S on B is defined by partial maps through

$$b \cdot s = \begin{cases} bs & \text{if } bs \subseteq J, \\ \text{undefined} & \text{otherwise,} \end{cases}$$

for $b \in B$ and $s \in S$. In other words, we act on the \mathcal{L} -class b by S if this stays within the \mathcal{J} -class; otherwise the map is not defined. We denote this action by $\lambda_J : S \rightarrow \text{PT}(L(J))$. Note that λ_J is well defined by Green's Lemma [20, Lemma A.3.1]. We denote the semigroup $S\lambda_J$ by $\text{RLM}_J(S)$, called the *right letter mapping semigroup* associated to J .

An ideal of S is called *0-minimal* if it is a minimal nonzero ideal; in particular, in a semigroup without zero we count the minimal ideal as a 0-minimal ideal.

We say that S is *right (left) mapping* if it contains a 0-minimal ideal I such that S acts faithfully on the right (left) of I by multiplication. If S is both right and left mapping, then it is called *generalized group-mapping*. Furthermore, if I is not aperiodic, we say that S is a *group-mapping* semigroup.

In this paper (until the last section), (S, X) will be a fixed group-mapping semigroup with generators X , the \mathcal{L} -classes of its unique 0-minimal \mathcal{J} -class denoted by B , and its maximal subgroup denoted by G . Here we view the generators X as an external set, meaning that S is generated by X if there is a surjective morphism from the free semigroup X^+ to S . The right action on B is denoted by $\text{RLM}(S, X)$. So both $(B, \text{RLM}(S, X))$ and $(G \times B, (S, X))$ are faithful right actions. Both $\text{RLM}(S, X)$ and (S, X) are generated by X . Hence X acts on the right on B and $G \times B$. See also Lemma 2.2.4.

Proposition 2.2.1. [20, Proposition 4.6.22] *Every right or left mapping semigroup has a unique 0-minimal ideal, which is necessarily regular.*

Given a right (left) mapping semigroup S , we denote by I this unique 0-minimal ideal. Write $J = I \setminus \{0\}$. For all $u, v \in J$, it follows that $S^1 u S^1$ and $S^1 v S^1$ are nonzero ideals contained in I . By minimality of I , we have $S^1 u S^1 = I = S^1 v S^1$ and so $u \mathcal{J} v$. Conversely, $u \mathcal{J} v \in J$ implies $u \in J$, hence J is a \mathcal{J} -class of S , called the *distinguished \mathcal{J} -class* of S .

Proposition 2.2.2. [20, Proposition 4.6.35] *Let J be a regular \mathcal{J} -class of a finite semigroup S . Then $\text{RLM}_J(S)$ is a right mapping semigroup with distinguished \mathcal{J} -class $J\lambda_J$. Furthermore, $J\lambda_J$ is aperiodic.*

Assume now that S is a right mapping semigroup. It is routine to check that the right action by multiplication

$$\begin{aligned} \theta : S &\rightarrow \mathbf{T}(I) \\ s &\mapsto \theta_s \end{aligned}$$

induces a right action by partial mappings (by multiplication)

$$\begin{aligned} \theta' : S &\rightarrow \mathbf{PT}(J) \\ s &\mapsto \theta'_s, \end{aligned}$$

where θ'_s is the restriction of θ_s with domain $J \setminus 0\theta_s^{-1}$. Since θ is faithful, θ' is faithful as well.

Assume now that R is an \mathcal{R} -class contained in J . Let θ_s^R be the restriction of θ'_s with domain R . We claim that

$$\begin{aligned} \theta^R : S &\rightarrow \mathbf{PT}(R) \\ s &\mapsto \theta_s^R \end{aligned}$$

is still a faithful right action by partial mappings by multiplication.

First, we note that θ_s^R is well defined. If $u \in R \cap \text{dom}(\theta'_s)$, then $u\theta_s^R = us \leq_{\mathcal{R}} u$ and $u\theta_s^R \mathcal{J} u$. Since S , being a finite semigroup, is stable, we obtain $u\theta_s^R \in R$. Thus θ_s^R is well defined. We refer to [20, Appendix A.2] for basic definitions and properties of stability.

Since each θ_s^R is obtained from θ'_s by restriction, it is still a right action. Suppose now that $\theta_s^R = \theta_t^R$ for some $s, t \in S$. We claim that $\theta'_s = \theta'_t$. Indeed, let $u \in J$. Then there exists some $v \in R \cap L_u$, hence $u = xv$ for some $x \in S^1$. It follows that

$$u\theta'_s = xvs = x(v\theta_s^R) = x(v\theta_t^R) = xvt = u\theta'_t.$$

Thus $\theta'_s = \theta'_t$. Since θ' is faithful, this yields $s = t$. Therefore θ^R is also faithful.

Remark 2.2.3. *We remarked before that $S^1uS^1 = I$ for every $u \in J$. Since J is regular, u has an inverse u' also in J , hence*

$$I = S^1uS^1 = S^1uu'uu'uS^1 \subseteq I^1uI^1 \subseteq I$$

and so I is 0-simple. Since S is finite, it follows from the Rees Theorem that I is up to isomorphism a Rees matrix semigroup of the form $M^0[G, A, B, C]$. We may assume that

- G is a group isomorphic to one of the group \mathcal{H} -classes in J ;
- A is the set of \mathcal{R} -classes in J ;
- B is the set of \mathcal{L} -classes in J ;
- $C = (c_{ba})$ is a $B \times A$ matrix with entries in $G \cup \{0\}$.

Since I is regular, each row or column of C contains nonzero entries.

To understand the right action of S on I , we must understand the corresponding right action of S on $M^0[G, A, B, C]$. Of course, $0s = 0$ for every $s \in S$. And we have seen that $(a, g, b)s \neq 0$ implies $(a, g, b)s \mathcal{R} (a, g, b)$, so the first coordinate will not change. Hence the right action of S on I is all about the action on G and on B . Hence we can arbitrarily choose some $a_1 \in A$, and consider the \mathcal{R} -class $\{a_1\} \times G \times B$, which we simply call $G \times B$.

Lemma 2.2.4. [8, Proposition 2.17] *Let S be a general mapping semigroup acting on the \mathcal{R} -class $G \times B$ of its distinguished \mathcal{J} -class J , isomorphic to $M^0(G, A, B, C)$. Then $(G \times B, S)$ embeds into $(G, G) \wr (B, \text{RLM}_J(S))$.*

Given Lemma 2.2.4, if $(a, g, b) \in J$ and $s \in S$, and if $(a, g, b)s \neq 0$, then we can write

$$(a, g, b)s = (a, g(b)s, bs), \quad (2.2.1)$$

where $b \in B \mapsto bs \in B$ is the partial function $\text{RLM}(s)$, and $(.)s$ is a function $\text{dom}(\text{RLM}(s)) \rightarrow G$. We will use this notation throughout the rest of the paper.

Let $\mu : B \rightarrow G$ be a partial function and let $s \in S$. Define the partial function $\mu^s : Bs \rightarrow G$ by

$$(bs)\mu^s = (b)\mu(b)s \quad \text{for } b \in \text{dom}\mu \cap \text{dom}s. \quad (2.2.2)$$

Note that $\text{dom}(s) = \text{dom}(\text{RLM}(s))$, and that μ^s defines a partial function if and only if

$$bs = b's \quad \text{implies} \quad (b)\mu(b)s = (b')\mu(b')s. \quad (2.2.3)$$

This is called the *cross section condition*.

2.3 Subsets, partitions, cross-sections

Let G be a finite group and let B be a finite set. A *partial partition* of B is a partition of a (possibly empty) subset W of B . Given $W \subseteq B$, we denote by G^W the set of all functions $f : W \rightarrow G$. The group G acts on G^W on the left by $\mu \mapsto^{g\cdot} g\mu$, defined by

$$(g\mu)(x) = g\mu(x)$$

for any $\mu \in G^W$, $g \in G$, and $x \in W$. Then the orbit $G\mu = \{g\mu : g \in G\}$ of μ , in the set of orbits $G^W/G = \{G\mu : \mu \in G^W\}$, is called a *cross section* with domain W .

An **spc** (short for Subset, Partition, Cross Section) over G is a triple (W, π, C) , where the three components are as follows:

- W is any subset of B ;
- π is any partition of W ; let π_1, \dots, π_k be the blocks of π ;
- C is any set of k cross sections with domains respectively π_1, \dots, π_k .

We describe next an alternative formalism. Given a partition π of $W \subseteq B$ and $\mu, \mu' \in G^W$, we define

$$\mu \sim_\pi \mu' \text{ if and only if for every block } \pi_i \text{ of } \pi \text{ there is } g_i \in G \text{ such that } \mu|_{\pi_i} = g_i\mu'|_{\pi_i}.$$

In other words, $G\mu|_{\pi_i} = G\mu'|_{\pi_i}$ for every block π_i of π , where $G\mu|_{\pi_i}$ is the orbit of μ under the action of G .

Then \sim_π is an equivalence relation on G^W . If we denote by $[\mu]_\pi$ the equivalence class of $\mu \in G^W$, then we can view an **spc**-triple as a triple of the form $(W, \pi, [\mu]_\pi)$, where $W \subseteq B$, π is a partition of W , and $\mu \in G^W$.

We define a partial order on the **spc**-triples, based on containment of sets and partitions:

$$(W, \pi, [\mu]_\pi) \leq (W', \pi', [\mu']_{\pi'}) \quad \text{iff}$$

- (1) $W \subseteq W'$,
- (2) every block of π is contained in a (necessarily unique) block of π' ,

$$(3) \quad [\mu'|_W]_\pi = [\mu]_\pi.$$

With this partial order, $\text{SPC}(B, G)$ is a \wedge -semilattice (it is a lattice if and only if G is trivial). Assume now that G is nontrivial. By adding a top element $\hat{1}$ to $\text{SPC}(B, G)$ we obtain the *Rhodes lattice* $Rh_B(G)$, in which this new element $\hat{1}$ is the join of any two elements of $\text{SPC}(B, G)$ not admitting a unique least common upper bound for the partial order defined above. See [12] for more details and connections between Rhodes lattices and Dowling lattices.

2.4 Flows on automata

A (*deterministic partial*) *automaton* is a structure of the form $\mathcal{A} = (Q, X, \delta)$, where Q and X are finite nonempty sets and $\delta : Q \times X \rightarrow Q$ is a partial function. When δ is a total function, \mathcal{A} is called a *complete* deterministic automaton. We usually write qx or $q \cdot x$ instead of $(q, x)\delta$. Thus each $x \in X$ defines a partial function $q \in Q \mapsto qx \in Q$, and we have the corresponding morphism from X^+ into $\text{PT}(Q)$. Let $T_{\mathcal{A}}$ denote its transition monoid, which is the submonoid of $\text{PT}(Q)$ generated by the functions $\cdot x$ for $x \in X$. We say that \mathcal{A} is *aperiodic* if $T_{\mathcal{A}}$ is aperiodic. Recall that an aperiodic finite semigroup is one in which each subgroup is trivial.

Let $\mathcal{A} = (Q, X, \delta)$ be an automaton and let S be a group-mapping semigroup generated by X . Let $I = M^0[G, A, B, C]$ be the 0-minimal ideal of S using the notation of Remark 2.2.3. Recall the operator $\mu \mapsto \mu^s$ defined by (2.2.2).

A *flow on the deterministic automaton \mathcal{A} relative to S* is a map $\xi : Q \rightarrow Rh_B(G)$ satisfying the following conditions for all $q \in Q$ and $x \in X$:

if $q\xi = (W_q, \pi_q, [\mu_q]_{\pi_q})$ and $(qx)\xi = (W_{qx}, \pi_{qx}, [\mu_{qx}]_{\pi_{qx}})$, then

- (F1) $W_q x \subseteq W_{qx}$;
- (F2) if $(\pi_q)_i$ is a block of π_q , then $(\pi_q)_i x \subseteq (\pi_{qx})_j$ for some necessarily unique block $(\pi_{qx})_j$ of π_{qx} ;
- (F3) $(\pi_q)_i \rightarrow (\pi_q)_i x$ is a partial one-to-one function between W_q/π_q and W_{qx}/π_{qx} ;
- (F4) μ_q^x as in (2.2.2) is well defined;
- (F5) $\mu_q^x|_{(\pi_q)_i x} \in G \mu_{qx}|_{(\pi_{qx})_j}$ for each block $(\pi_q)_i$ of π_q .

We also denote such a flow (for a given $q \in Q$ and $x \in X$) by

$$\text{spc} = (W_q, \pi_q, [\mu_q]_{\pi_q}) \xrightarrow{\cdot x} (W_{qx}, \pi_{qx}, [\mu_{qx}]_{\pi_{qx}}) = \text{spc}'. \quad (2.4.1)$$

To show that the concept is well defined, we have to show that whenever $[\nu]_{\pi_q} = [\mu]_{\pi_q}$, conditions (F4) and (F5) hold for ν if and only if they hold for μ .

Write $\pi_q = \{(\pi_q)_1, \dots, (\pi_q)_m\}$. Since $[\nu]_{\pi_q} = [\mu]_{\pi_q}$, for each $i = 1, \dots, m$, there exists some $g_i \in G$ such that $\nu|_{(\pi_q)_i} = g_i \mu|_{(\pi_q)_i}$.

Assume that μ_q^x is well defined. Then by equation (2.2.3) we have

$$bx = b'x \text{ implies } (b)\mu_q (b)x = (b')\mu_q (b')x.$$

If $bx = b'x$, it follows from (F3) that $b, b' \in (\pi_q)_i$ for some $i \in \{1, \dots, m\}$. Hence

$$(b)\nu_q (b)x = g_i (b)\mu_q (b)x = g_i (b')\mu_q (b')x = (b')\nu (b')x.$$

Thus equation (2.2.3) implies that μ_q^x is well defined. The reverse implication follows by symmetry.

Assume now that for each $i = 1, \dots, m$, there exists some $h_i \in G$ such that $\mu_q^x|_{(\pi_q)_i x} = h_i \mu_{qx}|_{(\pi_q)_i x}$. For every $b \in \text{dom } \nu_q \cap (\pi_q)_i = \text{dom } \mu_q \cap (\pi_q)_i$, we have

$$(bx) \nu_q^x = (b\nu_q)(bx) = g_i(b\mu_q)(bx) = g_i(bx \mu_q^x) = g_i h_i bx (\mu_{qx})^x.$$

Hence $\nu_q^x|_{(\pi_q)_i x} = g_i h_i (\nu_{qx}|_{(\pi_q)_i x})$. The reverse implication follows by symmetry and so the concept of flow is well defined.

2.5 Complexity

Given two pseudovarieties \mathbf{V} and \mathbf{W} of finite semigroups, we denote by $\mathbf{V} * \mathbf{W}$ the pseudovariety of finite semigroups generated by all semidirect products $S \rtimes T$, with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. The semidirect product is an associative operation on pseudovarieties.

Let \mathbf{S} , \mathbf{A} and \mathbf{G} denote respectively the pseudovarieties of finite semigroups, aperiodic finite semigroups and finite groups. The complexity pseudovarieties are defined inductively by

$$\mathbf{C}_0 = \mathbf{A}, \quad \mathbf{C}_{n+1} = \mathbf{A} * \mathbf{G} * \mathbf{C}_n \quad (n \geq 0).$$

The Prime Decomposition Theorem, also known as the Krohn–Rhodes Theorem, is given as follows.

Theorem 2.5.1. (Prime Decomposition Theorem) $\mathbf{S} = \bigcup_{n \geq 0} \mathbf{C}_n$.

The complexity function $c : \mathbf{S} \rightarrow \mathbb{N}$ is defined by $Sc = \min\{n \in \mathbb{N} \mid S \in \mathbf{C}_n\}$.

Theorem 2.5.2. (Fundamental Lemma of Complexity) *Let $\varphi : S \rightarrow T$ be a surjective homomorphism of finite semigroups. If φ is injective on every subgroup of S , then $Tc = Sc$.*

Assume that J is a \mathcal{J} -class of a finite semigroup containing a nontrivial subgroup. We define a congruence \equiv_J on S by

$$s \equiv_J t \text{ if } \forall x, y \in J (xsy \in J \Leftrightarrow xty \in J, \text{ and in this case } xsy = xty).$$

The quotient S / \equiv_J is denoted by $\text{GGM}_J(S)$. Note that $\text{GGM}_J(S)$ is a generalized group-mapping semigroup.

If G is a subgroup of S with identity e , then the restriction of the projection $\gamma_{J_e} : S \rightarrow \text{GGM}_{J_e}(S)$ to G is injective. Indeed, if $g, h \in G$ satisfy $g \equiv_{J_e} h$, then $ege = ehe$ and consequently $g = h$. It follows that if G is a non-trivial group, then $\text{GGM}_J(S)$ is a group-mapping semigroup that we denote by $\text{GM}_J(S)$.

Theorem 2.5.3. *Let S be a finite subgroup and let J_1, \dots, J_n be the \mathcal{J} -classes of S containing a nontrivial subgroup. Then $Sc = \max\{(\text{GM}_{J_1}(S))c, \dots, (\text{GM}_{J_n}(S))c\}$.*

Proof. Let $\gamma : S \rightarrow \prod_{i=1}^n \text{GM}_{J_i}(S)$ be the homomorphism defined by $s\gamma = (s\gamma_{J_1}, \dots, s\gamma_{J_n})$. In view of the previous comment, γ is injective on every subgroup of S . It follows from Theorem 2.5.2 that $Sc = S\gamma c$. Now

$$S\gamma c \leq \left(\prod_{i=1}^n \text{GM}_{J_i}(S) \right) c \leq \max\{(\text{GM}_{J_1}(S))c, \dots, (\text{GM}_{J_n}(S))c\}.$$

On the other hand, $\text{GM}_{J_i}(S) \leq S\gamma$ yields $(\text{GM}_{J_i}(S))c \leq S\gamma c$ for $i = 1, \dots, n$. Thus

$$\max\{(\text{GM}_{J_1}(S))c, \dots, (\text{GM}_{J_n}(S))c\} \leq S\gamma c$$

and so $Sc = \max\{(\text{GM}_{J_1}(S))c, \dots, (\text{GM}_{J_n}(S))c\}$. □

We define a group-mapping semigroup to be *pure* of $\text{GMc} = \text{RLMc} + 1$. Theorem 2.5.3 implies that every semigroup S has complexity equal to at least one of its group-mapping images. It follows by induction on the order of S that we can reduce the problem of computing complexity to that of deciding if a group-mapping semigroups is pure or not.

Given $n \geq 1$, we denote by Sym_n the symmetric group on an n -set. We give a version of the Presentation Lemma based on flows as in [5]. For other versions see [20] and [2]. See Section 2.6 for further details and history on the Presentation Lemma.

Theorem 2.5.4. (Presentation Lemma) *Let S be a group-mapping semigroup generated by X acting by multiplication on the right on its 0-minimal ideal $I = M^0[G, A, B, C]$. Write $J = I \setminus \{0\}$ and assume that $(\text{RLM}_J(S))c = n \geq 1$. Then $Sc = n$ if and only if there exists some flow on an automaton $\mathcal{A} = (Q, X, \delta)$ relative to S with transition semigroup $T_{\mathcal{A}}$ of the automaton \mathcal{A} with $T_{\mathcal{A}}c = n - 1$. In this case S divides $(G \wr \text{Sym}_{|B|} \wr T_{\mathcal{A}}) \times \text{RLM}_J(S)$.*

Proof. We prove the direction that states if there exists a flow on an automaton of complexity $n - 1$ and $\text{RLMS}_J(S)c = n$, then $Sc = n$. The converse can be found in [2].

For this direction we restrict to the case $n = 1$. The case $n > 1$ is the obvious generalization.

Let (S, X) be a group-mapping semigroup acting on $G \times B$ with $\text{RLM}(S, X)c \leq 1$. Suppose there exists a flow ξ for an aperiodic automaton $\mathcal{A} = (Q, X, \delta)$. We need to show that $S \prec (G \wr \text{Sym}_{|B|} \wr T_{\mathcal{A}}) \times \text{RLM}_J(S)$.

We use ξ to define a new group-mapping semigroup (S^ξ, X) generated by X . For $q \in Q$, consider the SPC $q\xi = (W_q, \pi_q, [\mu_q]_{\pi_q})$. The right action by $x \in X$ is given by $(qx)\xi = (W_{q \cdot x}, \pi_{q \cdot x}, [\mu_{q \cdot x}]_{\pi_{q \cdot x}})$. We need to show that a direct product decomposition exists as implied by the statement of the presentation lemma. When we write $q\xi = (W_q, \pi_q, [\mu_q]_{\pi_q})$, we pick a fixed arbitrary representative in the orbit of $[\mu_q]$ under left multiplication of G , which is fixed throughout this proof.

S^ξ will be a group-mapping semigroup of total maps. Hence its minimal ideal, which is a Rees matrix semigroup, has no zeros in its matrix. S^ξ acts on $G \times [\bar{b}] \times Q$, where $[\bar{b}] = \{1, 2, \dots, \bar{b}\}$ with \bar{b} the maximal number of blocks in any of the partitions π in the flow $q\xi$. We consider π_q always to have b blocks by adding empty blocks. We number the blocks in some arbitrary but fixed order, fixed throughout this proof. Then by (F3) in Section 2.4, we extend the map $(\pi_q)_i \mapsto (\pi_q)_i \cdot x$ to a permutation of $[\bar{b}]$, denoted by $p_{q, qx}$. This permutation is fixed throughout this proof.

Now we define the right action of S^ξ on $[\bar{b}] \times Q$ by the generator $x \in X$ as follows

$$(j, q)x = ((j)p_{q, qx}, qx) =: (jx, qx).$$

The permutation $p_{q, qx}$ maps blocks of π_q onto blocks of π_{qx} . This action of S^ξ on $[\bar{b}] \times Q$ is clearly in $\text{Sym}_{\bar{b}} \wr (Q, T_{\mathcal{A}})$.

To compute the action of X on $G \times [\bar{b}] \times Q$ define

$$(g, j, q) \cdot x = (g \cdot \tilde{g}(j, q, x), jx, qx),$$

where $\tilde{g}(j, q, x)$ is defined as follows. Let c_j be the cross section of the j -th block of $q\xi$ given by the fixed representative of the j -th block of $[\mu_q]_{\pi_q}$. If this j -th block is empty, $\tilde{g}(j, q, x)$ is arbitrary but fixed throughout this proof. Similarly, let $c_{j'}$ be the cross section of $(qx)\xi$, where $j' = (j)p_{q, qx}$. Then by (F4) and (F5) there exists $\tilde{g}(j, q, x)$ such that

$$c_j \mu^x \leq \tilde{g}(j, q, x) c_{j'}$$

with μ^x as defined in (2.2.2). Altogether this defines the action

$$(g, j, q)x = (g\tilde{g}(j, q, x), jx, qx). \quad (2.5.1)$$

Hence X generates a semigroup acting faithfully on the right on $G \times [\bar{b}] \times Q$, which is a subsemigroup of $G \wr \text{Sym}_{\bar{b}} T_{\mathcal{A}}$ and which is the group-mapping with all total maps. More precisely, this is a right group-mapping semigroup of total maps. We define S^ξ to be the group-mapping image of this semigroup. Its minimal ideal is a Rees matrix semigroup without zero.

Let $\tilde{Q} = G \times [\bar{b}] \times Q$ and consider $(\tilde{Q}, S^\xi, X) \times (B + 0, \text{RLM}(S), X)$. We define a subset \overline{Q} of $\tilde{Q} \times (B + 0)$ as follows:

1. $\tilde{q} \times 0$ is in \overline{Q} for all $\tilde{q} \in \tilde{Q}$;
2. $((g, j, q), b)$ is in \overline{Q} where $b \in (\pi_q)_j \neq \emptyset$.

Define the map $\rho : \overline{Q} \rightarrow G \times B$ by $\tilde{q} \times 0 \mapsto 0$ and $((g, j, q), b) \mapsto (g \cdot (b)c_j, b)$. This provides a morphism of transformation semigroups

$$(\overline{Q}, (S^\xi, X) \times (\text{RLM}(S), X)) \longrightarrow (G \times B + 0, (S, X)).$$

Namely, if bx is not defined, then the morphism condition is satisfied trivially. If bx is defined, then (2.5.1) shows that it is a morphism.

This proves the forward direction of the Presentation Lemma. □

2.6 Historical remarks on the Presentation Lemma and its uses

The first use of cross-sections and implicitly of the Presentation Lemma (Theorem 2.5.4) appeared in Tilson's paper [25]. This paper proved that complexity is decidable for semigroups with at most 2 non-zero \mathcal{J} -classes. In the early 1970s, Rhodes formalized the notion of cross-sections and proved a version of the Presentation Lemma. He used it to give improved lower bounds for complexity in [17]. A number of non-trivial examples that require the Presentation Lemma to compute their complexity are also given in this paper. This has been one of the most important uses of the Presentation Lemma. See [20, Sections 4.14-4.16] and [11] for many illuminating examples.

The first published proof of the Presentation Lemma appeared in the important paper [2]. Rhodes developed the notion of flows on automata in [18]. This was presented in [20, Section 4.14] and with much more detail in [5]. Our treatment of the Presentation Lemma is based on these latter references with a very important difference. The treatment in [20] and [5] is based on flows in the set-partition lattice referred to as $\text{SP}(M, X)$ in [5], while our treatment in this paper is based on flows in the Rhodes lattice $Rh_B(G)$.

In our notation, the set-partition lattice is $Rh_{G \times B}(1)$, the Rhodes lattice on the set $G \times B$ relative to the trivial group. There are important differences between $Rh_{G \times B}(1)$ and $Rh_B(G)$ that were noted in our text and appear in [11, Section 4.1]. In particular, $Rh_{G \times B}(1)$ is a lattice and $Rh_B(G)$ is a quotient lattice. The maximal elements of these two lattices differ. The maximal element of $Rh_B(G)$ has the intended meaning of having arrived at a contradiction to the existence of a flow and plays a crucial role in this paper. There is no such notion in $Rh_{G \times B}(1)$. We strongly prefer the use of $Rh_B(G)$ and cross-sections to describe and prove the results of this paper.

2.7 The Presentation Lemma for inverse semigroups

A group-mapping inverse semigroup is an inverse subsemigroup of a wreath product of partial transformation monoids of the form $G \wr (B, \text{SIM}_B)$, where G is a finite group and B is a finite set; SIM_B is the symmetric inverse monoid on B .

It is convenient to represent the semigroup of $G \wr (B, \text{SIM}_B)$ by partial monomial matrices over G . We assume $B = \{1, \dots, n\}$ for some $n \geq 1$. An $n \times n$ *partial monomial matrix* over a group G is an $n \times n$ matrix with entries in $G \cup \{0\}$ such that each row and each column has at most one non-zero

entry. We identify an element $(g, i) \in G \times B$ with the n -dimensional row vector that has g in position i and 0 in all other positions. We identify the collection of all such vectors with $G \times B$. The usual multiplication of a vector by a matrix then gives the completion (i.e., a transformation semigroup of total functions) of the wreath product $G \wr (B, \text{SIM}_B)$, with the all-0 vector acting as the sink state.

Clearly the group of units of $G \wr (B, \text{SIM}_B)$ is the usual *monomial group* $G \wr (B, \text{Sym}_{|B|})$, where $\text{Sym}_{|B|}$ denotes the symmetric group on the set B . The monomial group consists of the monomial matrices over G , i.e., the $n \times n$ matrices with entries in $G \cup \{0\}$ with exactly one non-zero entry in each row and each column. $G \wr (B, \text{SIM}_B)$ is a generalized group-mapping semigroup. When G is trivial, we identify it with (B, SIM_B) . If G is non-trivial then it is a GM semigroup.

It is not difficult to see that $G \wr (B, \text{SIM}_B)$ has a unique 0-minimal ideal which consists of those partial monomial matrices that have at most one non-zero entry in the matrix. Hence, this 0-minimal ideal is isomorphic to the $n \times n$ Brandt semigroup $B_n(G)$ over G . Thus, a GM inverse semigroup is an inverse subsemigroup of $G \wr (B, \text{SIM}_B)$ that contains all the matrices in $B_n(G)$, for some non-trivial group G . Furthermore, for inverse semigroups, elementary facts from the semilocal theory of finite semigroups (see [8, Chapters 7-8] and [20, Chapter 4]) show that an inverse semigroup S is *right letter mapping* (RLM) if and only if S is GGM over the trivial group, if and only if S is a subsemigroup of some SIM_B containing all partial constant functions.

If $S \subseteq G \wr (B, \text{SIM}_B)$ is a GM inverse semigroup, then the map which replaces each non-zero entry in a matrix $s \in S$ by the identity element gives the map $\text{RLM} : S \rightarrow \text{RLM}(S)$. If s is a partial monomial matrix over G we define $\text{dom}(s) \subseteq B$ to be the set of non-zero rows of s . Thus $\text{dom}(s) = B$ if and only if $s \in G \wr (B, \text{Sym}_{|B|})$, and thus every element s of $G \wr (B, \text{SIM}_B)$ is the restriction to $\text{dom}(s)$ of some element of the group of units. It follows that the semigroup

$$T(S) = \{(s, \tau) \mid \tau \in G \wr (B, \text{Sym}_{|B|}), \tau_{\text{dom}(s)} = s\}$$

is a subdirect product of S and $G \wr (B, \text{Sym}_{|B|})$. Furthermore it is easy to see, via the functorial properties of the $\text{RLM}(\cdot)$ map, that $\text{RLM}(T(S)) = \text{RLM}(S)$ (see [20, Chapter 4.6]). It follows that the function

$$(\tau, \text{RLM}(s)) \in G \wr (B, \text{Sym}_{|B|}) \times \text{RLM}(S) \text{ such that } (s, \tau) \in T(S) \quad \longmapsto \quad s \in S$$

is a well defined surjective morphism. Therefore, we have the following theorem, which is the Presentation Lemma for inverse semigroups; \prec denotes division of transformation semigroups.

Theorem 2.7.1. *Let $S \subseteq G \wr (B, \text{SIM}_B)$ be a GM inverse semigroup. Then*

$$(G \times B, S) \prec G \wr (B, \text{Sym}_{|B|}) \times \text{RLM}(S).$$

We note that a proof of this result follows from the general Presentation Lemma. Since a GM inverse semigroup S generated by X acts by partial bijections on $G \times B$, it follows that the automaton with one state, with trivial semigroup, and one loop for each generator $x \in X$, gives a flow. We send the single state to the SPC whose set is B , and whose partitions blocks are all singletons with the trivial cross-section on each block. The reader can check that this is a flow, and the proof of the Presentation Lemma gives the decomposition in the previous theorem.

For readers familiar with the theory of inverse semigroups, the Presentation Lemma for inverse semigroups is intimately related to the theory of E -unitary semigroups. Indeed by a theorem of McAlister and Reilly [10], every E -unitary inverse semigroup is a subdirect product of a group and a fundamental inverse semigroup. A GGM finite inverse semigroup S is *fundamental* if and only if S is RLM. Indeed the maximal fundamental image of a GM inverse semigroup is its RLM image. See [10] for the theory of E -unitary semigroups.

Let us look more closely at the direct product decomposition given by this theorem.

According to the proof of the Presentation Lemma, the subsemigroup of $G \wr (B, \text{Sym}_{|B|}) \times \text{RLM}(S)$ that maps onto S is

$$T = \{(M, \text{RLM}(s)) \mid s \in S \text{ and } M \text{ is an } n \times n \text{ monomial matrix over } G \text{ whose restriction to the non-zero rows of a partial monomial matrix } s \text{ is equal to } s\};$$

that is, the matrix M as it appears in T is a monomial matrix that extends the partial monomial matrix s .

As a specific example we can take the small monoid S consisting of the group of units $G \wr (B, \text{SIM}_B)$ and the 0-minimal ideal $B_n(G)$. A *small monoid* is by definition a monoid consisting of its group of units and a unique 0-simple ideal. As noted above, $B_n(G)$ consists of all partial monomial matrices that have at most one non-zero entry in the whole matrix. Such a matrix is either the 0-matrix, or it can be encoded by the triple (i, g, j) where $g \in G$ is the unique non-zero entry located in row i and column j . This gives the connection to the usual definition of $B_n(G)$.

$\text{RLM}(S)$ is obtained by changing all non-zero entries of elements of S to 1. By identifying the symmetric inverse monoid on B with the monoid of all partial permutation matrices (i.e., matrices over $\{0, 1\}$ with at most one non-zero entry in each row and column), we find that $\text{RLM}(S)$ is a small monoid whose group of units is the symmetric group $\text{Sym}_{|B|}$ (in the guise of all $n \times n$ permutation matrices), and whose 0-minimum ideal is the set of all partial permutation matrices with at most one non-zero entry (also known as the “matrix units”). Clearly, this 0-minimum ideal is isomorphic to the Brandt semigroup $B_n(\{1\})$.

The semigroup T , lifting S to the subdirect product, consists of three \mathcal{J} -classes: J_0, J_1, J_2 . The class J_0 is the minimal ideal, and is equal to $\{(M, 0) \mid M \in G \wr (B, \text{Sym}_{|B|})\}$. Thus J_0 is isomorphic to $G \wr (B, \text{Sym}_{|B|})$. The class $J_2 = \{(M, \text{RLM}(M)) \mid M \in G \wr (B, \text{Sym}_{|B|})\}$ is the group of units of T and consists of all pairs of a monomial matrix and its permutation matrix image. Clearly J_2 is isomorphic to $G \wr (B, \text{Sym}_{|B|})$.

The class J_1 is the inverse image of $B_n(G)$ under the morphism from T to S . Since the congruence of this morphism is contained in \mathcal{H} (by the general theory), $J_1 \approx B_n(H)$ for some group H , which we now determine. As the unit of H we take an idempotent $e = (I_n, e_1) \in J_1$, where I_n is the identity matrix and e_1 is the $n \times n$ matrix with 1 in position $(1, 1)$ and 0 elsewhere. The idempotent e_1 can also be considered to be an element of S , and then the group $G_{e_1} \approx G$ consists of all partial monomial matrices whose $(1, 1)$ position contains an element of G and all other elements are 0. Thus, $H_e = \{(M, e_1) \mid M \text{ is a monomial matrix that is non-zero in position } (1, 1)\}$. Such a monomial matrix M over G consists of two diagonal blocks: a 1×1 block consisting of an element of G , and an arbitrary $(n-1) \times (n-1)$ monomial matrix over G . Therefore, $H \approx G \times G \wr (\{1, \dots, n-1\}, \text{Sym}_{n-1})$.

The morphism from T to S is as follows: it sends $(M, \text{RLM}(M)) \in J_2$ to M ; it sends $(M, \text{RLM}(M)) \in J_1$ to $\psi(M_1)$, where ψ is the morphism from $B_n(G \times G \wr (\{1, \dots, n-1\}, \text{Sym}_{|B|}))$ to $B_n(G)$, induced by the projection from $G \times G \wr (\{1, \dots, n-1\}, \text{Sym}_{n-1})$ to G ; and it sends all elements of J_0 to 0.

As a generalization we look at S_r , which is the rank- r small monoid consisting of the group of units $G \times G \wr (B, \text{Sym}_{|B|})$ and the \mathcal{J}^0 -class of all partial monomial matrices of rank r . It is easy to see that the 0-minimal ideal is isomorphic to the Brandt semigroup $B_{\binom{n}{r}}(G \wr \text{Sym}_r)$. An analysis as above shows that the lift T consists of three \mathcal{J} -classes: the group of units is $G \times G \wr (B, \text{Sym}_{|B|})$, as is the minimal ideal; and the lift of $B_{\binom{n}{r}}(G \wr \text{Sym}_r)$ is $B_{\binom{n}{r}}(G \wr \text{Sym}_r \times G \wr \text{Sym}_{n-r})$. This latter group is the analogue of the Young subgroup on a two-block partition in $G \wr (B, \text{Sym}_{|B|})$, and is intimately connected to the representation theory of $G \wr \text{SIM}_B$.

3 The evaluation semigroup $\text{Eval}(S)$

We use the notation of Section 2.3 and consider $(W, \pi, [\mu]_\pi) \in \text{SPC}(B, G)$ with a subset $W \subseteq B$, a partition $\pi = (\pi_1, \dots, \pi_k)$, and a cross section $[\mu]_\pi$. In particular, we consider $c = (c_1, \dots, c_k)$ where $c_j : \pi_j \rightarrow G$ is a fixed representative for the orbits of G acting on G^W on the left; G^W is the set of all functions from W to G . Hence $(W, \pi, c) = (W, \pi, c')$ if c_j and c'_j are in the same G -orbit. The top element $\hat{1}$ in the Rhodes lattice $Rh_B(G)$ is called the *contradiction*. The Rhodes lattice $Rh_B(G)$ is fixed in the remainder of this paper.

As in Section 2.2, let (S, X) be a fixed group-mapping semigroup with generators X and distinguished zero-minimal ideal $M^0[G, A, B, C]$ with right action as in (2.2.1). In addition, we assume $\text{RLM}(S, X)c = 1$ (until the last section). Sections 3-5 are devoted to proving that the question whether $(S, X)c = 1$ is decidable. We do this by proving that the lower bound from [5] is an upper bound to complexity as well.

In this section, we adapt the evaluation semigroup of [5] to our purposes by making minor but important changes, especially to the action.

3.1 Preliminaries

The evaluation semigroup $\text{Eval}(S)$ is defined as a collection of order preserving self maps on a certain sublattice of $Rh_B(G)$ closed under composition. The elements on $\text{Eval}(S)$ are closure operators in $Rh_B(G) \times Rh_B(G)$, meaning that they are order-preserving, idempotent and increasing.

Definition 3.1.1. [5, Definition 2.1] *A closure operator on a lattice L is a function $f : L \rightarrow L$ such that for $\ell, \ell_1, \ell_2 \in L$:*

1. (order-preserving) $\ell_1 \leq \ell_2$ implies $\ell_1 f \leq \ell_2 f$;
2. (idempotent) $\ell f^2 = \ell f$;
3. (increasing) $\ell \leq \ell f$.

We will use the Rhodes lattice $Rh_B(G)$ and the product lattice $Rh_B(G) \times Rh_B(G)$, based on the product order in $Rh_B(G)$. If f is a closure operator on $Rh_B(G) \times Rh_B(G)$, its image is a meet-closed subset of $Rh_B(G) \times Rh_B(G)$; the empty meet is included. This meet-closed subset is called the *stable set* of f and is denoted by $\text{stable}(f)$. As meet-closed subsets of $Rh_B(G) \times Rh_B(G)$ are binary relations on $Rh_B(G)$, we can compose them as relations. By [5, Proposition 2.5], these form a subsemigroup of the monoid of all relations on $Rh_B(G)$; we denote this subsemigroup by $C(B, G)$.

If f is a closure operator on $Rh_B(G) \times Rh_B(G)$ with stable set $\text{stable}(f)$ then by [5, Section 2],

$$(\ell_1, \ell_2)f = \bigwedge \{(\bar{\ell}_1, \bar{\ell}_2) : (\bar{\ell}_1, \bar{\ell}_2) \geq (\ell_1, \ell_2) \text{ and } (\bar{\ell}_1, \bar{\ell}_2) \in \text{stable}(f)\}. \quad (3.1.1)$$

Recall that flows have been defined in Section 2.4 and that for $x \in X$, the operator $\cdot x$ on $Rh_B(G)$ was defined in equation (2.4.1).

Lemma 3.1.2. [5, Section 2.2.2] *The set*

$$\{(\text{spc}_1, \text{spc}_2) \in Rh_B(G) \times Rh_B(G) : (\exists x \in X)[\text{spc}_1 \xrightarrow{\cdot x} \text{spc}_2]\}$$

is meet-closed.

See later in this section the example BIRIP (back inject rest in peace) for examples of evaluating the associated closure operation on $Rh_B(G) \times Rh_B(G)$ of Lemma 3.1.2 given by (3.1.1) denoted $\cdot x$.

For any finite semigroup S and any element $s \in S$ there exists $s^\omega \in S$, which is defined to be the unique idempotent in the subsemigroup generated by s . Moreover, letting $n \in \mathbb{N}$ be such that $s^\omega = s^n = s^{2n}$, we define $s^{\omega-1}$ to be s^{2n-1} ; it is easy to prove that $s^{\omega-1}$ is well-defined in every finite semigroup.

Since $C(B, G)$ is a semigroup, for each $f \in C(B, G)$ there exists an element f^ω . The semigroup $C(B, G)$ is closed under another unary operation sending the element f to $f^{\omega+*}$ which we now define.

Definition 3.1.3. For $f \in C(B, G)$ we define $f^{\omega+*}$ in such a way that it has stable pairs $(\overline{\text{spc}}_1, \overline{\text{spc}}_2)$ if and only if $(\overline{\text{spc}}_1, \overline{\text{spc}}_2) \in \text{stable}(f^\omega)$ and $\overline{\text{spc}}_2 f \overline{\text{spc}}_2$.

The condition $(\overline{\text{spc}}_1, \overline{\text{spc}}_2) \in \text{stable}(f^\omega)$ and $\overline{\text{spc}}_2 f \overline{\text{spc}}_2$ can be depicted as follows

$$\overline{\text{spc}}_1 \xrightarrow{f^\omega} \overline{\text{spc}}_2 \curvearrowright f$$

Definition 3.1.4. Let f be a closure operator on $Rh_B(G) \times Rh_B(G)$. Then the back-flow operator \overleftarrow{f} is the closure operator on $Rh_B(G)$ with image the domain of f .

Let $\pi_{\leftarrow}, \pi_{\rightarrow} : Rh_B(G) \times Rh_B(G) \rightarrow Rh_B(G)$ be projections defined by

$$\begin{aligned} (\ell, \ell') \pi_{\leftarrow} &= \ell, \\ (\ell, \ell') \pi_{\rightarrow} &= \ell'. \end{aligned}$$

Proposition 3.1.5. [5, Proposition 2.13] For a closure operator f on $Rh_B(G) \times Rh_B(G)$ and $\ell \in Rh_B(G)$, we have

$$\ell \overleftarrow{f} = \left(\ell \xrightarrow{f} \square \right) \pi_{\leftarrow},$$

where \square is the bottom element of the lattice $Rh_B(G)$. Thus $\ell \overleftarrow{f}$ is the least $\ell' \in Rh_B(G)$ such that $\ell \leq \ell'$ and there is a stable pair (ℓ', ℓ'') .

Definition 3.1.6. For a closure operator f on $Rh_B(G) \times Rh_B(G)$ and $\ell \in Rh_B(G)$, we define an order-preserving map $\overrightarrow{f} : Rh_B(G) \rightarrow Rh_B(G)$, called forward-flow along f , by

$$\ell \overrightarrow{f} = \left(\ell \xrightarrow{f} \square \right) \pi_{\rightarrow},$$

where \square is the bottom element of the lattice $Rh_B(G)$. Thus $\ell \overrightarrow{f}$ is the least ℓ'' such that there is an ℓ' with $\ell \leq \ell'$ and (ℓ', ℓ'') is a stable pair. Note that if $\ell \overleftarrow{f} = \ell$, then $(\ell, \ell \overrightarrow{f})$ is a stable pair.

For the next definitions, L is a lattice.

Definition 3.1.7. [5, Definition 4.26] The flow monoid $\mathbf{M}(L)$ is the smallest subset of closure operators on $L \times L$ satisfying the following axioms:

1. (Identity) The multiplicative identity I of the closure operators on $L \times L$.
2. (Points) For all $x \in X$, free flow along x belongs to $\mathbf{M}(L)$.
3. (Products) If $f_1, f_2 \in \mathbf{M}(L)$, then $f_1 f_2 \in \mathbf{M}(L)$.
4. (Vacuum) If $f \in \mathbf{M}(L)$, then $\overleftarrow{f} \in \mathbf{M}(L)$.
5. $f^{\omega+*}$ is in $\mathbf{M}(L)$.

Definition 3.1.8. [5, Definition 4.29] *The closure operator*

$$V = \bigvee_{f \in \mathbf{M}(L)} \overleftarrow{f}$$

is called the vacuum.

Definition 3.1.9. [5, Definition 4.31] *An element $\ell \in L$ is called V -stable if $\ell V = \ell$. An element $f \in \mathbf{M}(L)$ is called V -stable if $VfV = f$.*

The vacuum V captures the back-flow from all elements of $\mathbf{M}(L)$.

Proposition 3.1.10. [5, Proposition 4.36] *If $\ell \in L$ is V -stable and $f \in \mathbf{M}(L)$, then $\ell \overleftarrow{f} = \ell$.*

3.2 Upstairs and downstairs notation

We introduced the Rhodes lattice in Section 2.3 using subset, partition and cross-sections. Let $(W, \pi, [\mu]_\pi) \in Rh_B(G)$ be an element in the Rhodes lattice. Let $\pi_1, \pi_2, \dots, \pi_k$ be the equivalence classes of π and let $\mu_i = \mu|_{\pi_i} : \pi_i \rightarrow G$ for $i \in \{1, 2, \dots, k\}$. Write $W/\langle \mu_1 | \mu_2 | \dots | \mu_k \rangle$ for $(W, \pi, [\mu]_\pi)$. This is also called the “downstairs notation”.

Example 3.2.1. *Let $G = \{1, -1\} = Z_2$, $W = \{1, 2, 3, 4, 5, 6\}$, and $\pi = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$. Let $\mu : \{1, 2, 3, 4, 5, 6\} \rightarrow Z_2$ be defined by $\mu(1) = \mu(3) = \mu(5) = 1$ and $\mu(2) = \mu(4) = \mu(6) = -1$. Then we write $\{1, 2, 3, 4, 5, 6\}/\langle 1 | -1 | -1 | -1 | 1 | -1 \rangle$.*

The Rhodes lattice without its top element $Rh_B(G) \setminus \{\widehat{1}\}$ can also be identified with the meet subsemilattice of the set-partition lattice $\mathbf{SP}(G \times B)$ as pairs (Y, Π) , where Y is a subset of $G \times B$ and Π is a partition of Y such that

1. $Y = G \times P(Y)$, where $P : G \times B \rightarrow B$ is the projection.
2. Each Π -class is a cross-section. That is, if (g, b) and (h, b) are in Π -class, then $g = h$.
3. If ρ is a Π -class, then so is $g\rho$ for all $g \in G$.

See also [12]. In other words, we obtain the Rhodes semilattice by taking the quotient of $\mathbf{SP}(G \times B)$ consisting of all pairs in which the partition is not a cross-section. See [11, Section 4.1] for further details. This is called the “upstairs notation”. Upstairs notation is used in [5].

Example 3.2.2. *Let us write Example 3.2.1 in upstairs notation. In this case $(Y, \Pi) \in \mathbf{SP}(Z_2 \times B)$, where $Y = Z_2 \times \{1, 2, 3, 4, 5, 6\}$ and the classes of Π are*

$$\begin{aligned} &\{(1, 1)\}, \{(-1, 1)\}, \\ &\{(-1, 2), (1, 3)\}, \{(1, 2), (-1, 3)\}, \\ &\{(-1, 4), (1, 5), (-1, 6)\}, \{(1, 4), (-1, 5), (1, 6)\}. \end{aligned}$$

3.3 Definition of $\text{Eval}(S)$

We first need the notion of well-formed formulae.

Definition 3.3.1. *Let x be an alphabet. A well-formed formula is defined inductively as follows:*

1. *The empty string ε is a well-formed formula.*

2. Each letter $x \in X$ is a well-formed formula.
3. If τ, σ are well-formed formulae, then so is $\tau\sigma$.
4. If τ is a well-formed formula and $\tau \neq \sigma^n$ for some $n > 1$, then $\tau^{\omega+*}$ is a well-formed formula.

The set of well-formed formulae is denoted by $\Omega(X)$.

Next, well-formed formulae are interpreted in $VM(L)V$.

Definition 3.3.2. *Recursively define $\Upsilon : \Omega(X) \rightarrow VM(L)V$ as follows:*

1. Set $\varepsilon\Upsilon = V$.
2. Define $x\Upsilon = VxV$.
3. If Υ is defined on $\tau, \sigma \in \Omega(X)$, then $(\tau\sigma)\Upsilon = \tau\Upsilon\sigma\Upsilon$.
4. If $\tau \in \Omega(X)$ and $\tau \neq \sigma^n$ for some $n > 1$, then $\tau^{\omega+*}\Upsilon = (\tau\Upsilon)^{\omega+*}$.

We now modify the definition of **States** as in [5, Definition 5.3].

Definition 3.3.3. *States is the smallest subset of $Rh_B(G)$ such that:*

1. Singletons $\{b\}/\langle 1 \rangle \in \text{States}$.
2. If τ is a well-formed formula and $\ell \in \text{States}$, then $\ell \xrightarrow{\tau} \in \text{States}$. More concretely, $\ell \xrightarrow{\tau}$ is the least state $\ell' \in Rh_B(G)$ such that (ℓ, ℓ') is a stable pair for τ .

There is a monoid action of $VM(L)V$ on LV by total functions defined by $\ell \cdot f = \ell \xrightarrow{f}$ for $\ell \in LV$ and $f \in VM(L)V$.

Definition 3.3.4. [5, Definition 5.6] *The above action of $VM(L)V$ on LV restricts to an action of $\Omega(X)\Upsilon$ on **States**. The associated faithful transformation monoid $\text{Eval}(S)$ is called the evaluation semigroup.*

3.4 Main result

Theorem 3.4.1. *For (S, X) the fixed group-mapping semigroup with generators X and $\text{RLM}(S, X)c = 1$, we have $(S, X)c = 1$ if and only if $\text{Eval}(S)$ never takes the value contradiction (or $\widehat{1}$).*

Theorem 3.4.1 will be proved in Sections 4-5. Theorem 3.4.1 states that the bound given in [5] is perfect for complexity 1.

3.5 BIRIP example

Consider the matrices

$$M_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with indexing sets $\{1', 3'\}$ and $\{1, 2, 3, 4\}$, respectively. In addition, consider the partial monomial maps

$$x = \begin{cases} 1' \mapsto 3', \\ 3' \mapsto 1', \end{cases} \quad a = \begin{cases} 1 \mapsto 4, \\ 2 \mapsto 3, \end{cases} \quad b = \begin{cases} 1 \mapsto 3, \\ 3 \mapsto 1, \\ 2 \mapsto 4, \\ 4 \mapsto 2, \end{cases} \quad \text{and} \quad t = \begin{cases} 1' \mapsto 1, \\ 3' \mapsto -3. \end{cases}$$

It is routine to check that the maps x, a, b, t satisfy the linked equations [8, Chapter 7] and [20, Chapter 4] with respect to the Rees matrix semigroup S with structure matrix, the sum of M_4 and M_2 over the group $Z_2 = \{1, -1\}$. This implies that the *semigroup* BIRIP generated by x, a, b, t and the Rees matrix semigroup S is a group-mapping semigroup with distinguished ideal equal to S .

Note that $\text{RLM}(\text{BIRIP})$ has the distinguished ideal I over the trivial group with structure matrix $M_4 + M_2$. Since x, a, b, t are partial one-to-one maps, the quotient $\text{RLM}(\text{BIRIP})/I$ embeds into a symmetric inverse semigroup. By the Fundamental Lemma of Complexity 2.5.2, it follows that $\text{RLM}(\text{BIRIP})c = 1$.

Consider the aperiodic automaton \mathcal{A} with three states

$$x \curvearrowright \mathbf{1} \xrightarrow{t} \mathbf{2} \xrightarrow{a} \mathbf{3} \curvearrowright a, b$$

$\begin{matrix} b \\ \downarrow \\ \uparrow \end{matrix}$

where in addition $(\alpha, g, \beta) \in I$ acts as the constant map on $\mathbf{1}, \mathbf{2}, \mathbf{3}$ to 1 if $\beta \in \{1', 3'\}$ and to 3 if $\beta \in \{1, 2, 3, 4\}$.

We claim that

$$\mathbf{1} \times i = \{1', 3'\}/\langle 11 \rangle, \quad \mathbf{2} \times i = \{1, 3\}/\langle 1 - 1 \rangle, \quad \mathbf{3} \times i = \{1, 2, 3, 4\}/\langle 1111 \rangle$$

is a flow (in downstairs notation). Here $\mathbf{1} \times i$ lies in **States** of Definition 3.3.3, because

$$\mathbf{1} \times i = 1'x^{\omega+\star}.$$

Similarly, $\mathbf{2} \times i$ lies in **States**, because

$$\mathbf{2} \times i = (\mathbf{1} \times i)t.$$

Finally, $\mathbf{3} \times i$ lies in **States**, because

$$\mathbf{3} \times i = 1't(ab^{\omega+\star})^{\omega+\star}.$$

The reader should check that $\times i$ is indeed a flow.

Hence by the Presentation Lemma 2.5.4 we have $(S, X)c = 1$.

4 Pointlike functors and geometric semigroup theory

4.1 Pointlike with respect to \mathbf{V}

Let \mathbf{V} be a pseudovariety of finite semigroups.

Definition 4.1.1. *Let S be a finite semigroup. We say that $X \subseteq S$ is pointlike with respect to \mathbf{V} if and only if for all relational morphisms $r : S \rightarrow V \in \mathbf{V}$ there exists $v \in V$ such that $X \subseteq r^{-1}(v)$. Define $\text{Pl}_{\mathbf{V}}(S)$ to be the set of all pointlike $X \subseteq S$.*

Note that $\text{Pl}_{\mathbf{V}}(S)$ satisfies all of the following (not necessarily independent) properties:

1. (points) $\{s\} \in \text{Pl}_V(S)$;
2. (subsets) If $Y \subseteq X$ for $X \in \text{Pl}_V(S)$, then $Y \in \text{Pl}_V(S)$;
3. (products) If $X, Y \in \text{Pl}_V(S)$, then $X \cdot Y \in \text{Pl}_V(S)$, where

$$X \cdot Y = \{xy \in S \mid x \in X, y \in Y\}.$$

Here $\text{Pl}_V(S)$ is a subsemigroup of the semigroup $P(S)$, consisting of all subsets of S under the above multiplication (3).

Here is a summary of some further properties. For further details, see [20]. For point (8), see [1].

Proposition 4.1.2.

1. (push) If $\varphi : S \rightarrow T$ is a surmorphism and $X \in \text{Pl}_V(S)$, then $\varphi(X) \in \text{Pl}_V(T)$.
2. (pull) If $\varphi : S \rightarrow T$ is a surmorphism and $Y \in \text{Pl}_V(T)$, then there exists $X \in \text{Pl}_V(S)$ such that $\varphi(X) = Y$.
3. (monadic) The map

$$\begin{aligned} U : \quad \text{Pl}_V(\text{Pl}_V(S)) &\rightarrow \text{Pl}_V(S) \\ \{X_1, \dots, X_k\} &\mapsto X_1 \cup \dots \cup X_k \end{aligned}$$

is a surmorphism.

4. (injective subsemigroups) We have the following injective inclusions of semigroups

$$S \leq \text{Pl}_V(S) \leq \text{Pl}_V(\text{Pl}_V(S)) \leq \dots$$

5. (injections with respect to relational morphisms) If

$$r : S \rightarrow T \in \mathbf{V}$$

is a relational morphism, then r extends to $\text{Pl}_V(S) \rightarrow T \in \mathbf{V}$ by sending $X \in \text{Pl}_V(S)$ to $r(X)$.

6. (compactness) There exists a relational morphism $r : S \rightarrow T \in \mathbf{V}$ such that $X \in \text{Pl}_V(S)$ if and only if there exists a $T \in \mathbf{V}$ such that $X \subseteq r^{-1}(T)$. We say that r computes $\text{Pl}_V(S)$.
7. (singletons) We have $S \in \mathbf{V}$ if and only if $\text{Pl}_V(S) = \{\emptyset, \{s\} \mid s \in S\}$.
8. (decidability) There exists a pseudovariety \mathbf{V} such that membership of finite semigroups in \mathbf{V} is decidable, but the computation of $\text{Pl}_V(S)$ for arbitrary S is not decidable.

Note that, regarding Proposition 4.1.2 (6), there exist examples (see Example 4.1.3 below), where r computes $\text{Pl}_V(S)$, but r extended to $\text{Pl}_V(S)$ does not compute $\text{Pl}_V(\text{Pl}_V(S))$. How far up the sequence in Proposition 4.1.2 (4) the induced map r computes $\text{Pl}_V^{(k)}(S)$ is a measure for the *tightness* of r .

Example 4.1.3. Let $S = Z_2 = \{1, -1\}$. Then the relational morphism $r : Z_2 \rightarrow 1$ computes $\text{Pl}_{\text{Ap}}(Z_2) = P(Z_2)$. The induced map r is trivially $P(Z_2) \rightarrow \{0, 1\}$, where the empty set is mapped to 0. This implies that $\{\emptyset, Z_2, 1, -1\}$ is in $\text{Pl}_{\text{Ap}}(\text{Pl}_{\text{Ap}}(Z_2))$, if the induced map r computed $\text{Pl}_{\text{Ap}}(\text{Pl}_{\text{Ap}}(Z_2))$. But $\{\emptyset, Z_2, 1, -1\}$ maps onto an aperiodic semigroup by identifying 1 and -1 , so this set is not point-like.

Problem 4.1.4. Let $\text{Pl}_V'(S)$ satisfy $\text{Pl}_V'(S) \subseteq \text{Pl}_V(S)$, Conditions (1)-(3) and properties (1)-(7) of Proposition 4.1.2. The question is does this imply that $\text{Pl}_V'(S) = \text{Pl}_V(S)$?

4.2 Pl_V , Pl_{Ap} and $\text{Eval}(S)$

Let S be a semigroup with regular \mathcal{R} -class R , that is, R contains an idempotent e . Sometimes we require that the maximal subgroup G of R is non-trivial, so that R has Rees coordinates $G \times B$, where B is an index set for the \mathcal{L} -classes of the \mathcal{J} -class of R . The set $\{eX \cap R \mid X \in \text{Pl}_V(S)\}$ is called the *semilocal* version of $\text{Pl}_V(S)$ with respect to R . Note that the semilocal version is independent of the chosen \mathcal{R} -class in the \mathcal{J} -class and which idempotent e is chosen in the following sense. If e and f are \mathcal{J} -related idempotents and $xe\mathcal{L}f$ and $yxe = e$ for some x, y in this \mathcal{J} -class, then the map $\{eX\} \rightarrow \{xeX\}$ is a one-to-one correspondence with inverse $\{fX\} \rightarrow \{yfx\}$ (see [20, Section 4.6]).

Remark 4.2.1. *We will see in this section when S is a group-mapping semigroup that $\text{Eval}(S)$ is very closely related to the semilocal version of $\text{Pl}_{\mathbf{G} \star \mathbf{Ap}}$, where \mathbf{Ap} stands for the pseudovariety of aperiodic semigroups and \mathbf{G} stands for the pseudovariety of finite groups.*

It turns out that we only need to prove the semilocal version of complexity c is decidable, to prove that c is decidable.

4.3 Computing $\text{Pl}_{\text{Ap}}(\text{Eval}(S), \text{Eval}(S))$ via Henckell's construction

From here until the end of Section 5, we only need $\mathbf{V} = \mathbf{Ap}$. Here $\text{Pl}_{\text{Ap}}(S)$ is decidable by Henckell's theorem [3, 20], which gives the important formula:

$\text{Pl}_{\text{Ap}}(S)$ is the smallest subset of $P(S)$ containing singletons, closed under products, and subsets, and

$$X \in \text{Pl}_{\text{Ap}}(S) \text{ implies that } \bigcup \{X^\omega X^k \mid k = 1, 2, \dots, \omega\} \in \text{Pl}_{\text{Ap}}(S). \quad (4.3.1)$$

Note that the last formula is similar to the definition of the $(\omega + \star)$ -operation that we used to define $\text{Eval}(S)$, but here the definition is concerned with the subset lattice and not the Rhodes lattice. Thus the states of $\text{Eval}(S)$ restricted just to sets equals $\text{Pl}_{\text{Ap}}(S)$.

A key property of $\text{Eval}(S)$, which follows from Henckell's Theorem [5, Proposition 4.28], is the following.

Proposition 4.3.1. *If $\{f_1, \dots, f_k\} \in \text{Pl}_{\text{Ap}}(\text{Eval}(S))$, then $f_1 \vee \dots \vee f_k$ (with \vee in the Rhodes lattice) is less than or equal to f for some $f \in \text{Eval}(S)$.*

Remark 4.3.2 ($\text{Pl}_V(S)$ and generators). *Let T be a finite semigroup generated by $Z \subseteq T$, denoted by (T, Z) . Then $\text{Pl}_V(T, Z) := \text{Pl}_V((T, Z))$ only depends on T and not the choice of generators Z . But often we write $\text{Pl}_V(T, Z)$ for the following reason.*

Constructing a relational morphism $r_T : T \rightarrow W \in \mathbf{V}$, which computes $\text{Pl}_V(T)$ (that is, $r_T^{-1}(w)$ belongs to $\text{Pl}_V(T)$ for all $w \in W$), is usually done by finding a right transformation semigroup (Q, W) . For each $\eta \in Z$, one constructs a map $\bar{\eta} : Q \rightarrow Q$ with $\bar{\eta} \in W$, followed by taking the relational submorphism for all pairs $\langle \eta, \bar{\eta} \rangle$. Here we consider $r_T \leq\leq T \times W$ with $\leq\leq$ subdirect product. In other words, we “cut r_T to generators η ”.

Remark 4.3.3. *For each (T, Z) , Henckell effectively constructs (Q_0, A_0, Z) , where Q_0 is a finite set, $A_0 \in \mathbf{Ap}$, and Z generates A_0 , with the relational morphism $r_T : (T, Z) \rightarrow (A_0, Z)$ and flow $f : Q_0 \rightarrow \text{Pl}_{\text{Ap}}(T, Z)$ defined by*

$$q \xrightarrow{\eta} q \cdot \eta \implies (q)f \cdot \bar{\eta} \subseteq (q \cdot \bar{\eta})f,$$

where $\bar{\eta} \in r_T^{-1}(\eta)$. Thus r_T computes $\text{Pl}_{\text{Ap}}(T, Z)$ and establishes the important formula (4.3.1).

A proof of Henckell's theorem can be found in [3] and [20, Chapter 4]. The easiest proof yet of Henckell's theorem is given in [27].

4.4 Geometric semigroup theory (GST)

4.4.1 Karnofsky–Rhodes expansion

In this section, we define the right Cayley graph of a finite semigroup and its Karnofsky–Rhodes expansions. Note that for a semigroup S , we denote by S^1 the semigroup with an identity $\mathbb{1}$ attached (even if S already contains an identity).

Definition 4.4.1 (Right Cayley graph). *Let (S, X) be a finite semigroup S together with a set of generators X . The right Cayley graph $\text{RCay}(S, X)$ of S with respect to X is the rooted graph with vertex set S^1 , root $r = \mathbb{1} \in S^1$, and edges $s \xrightarrow{a} s'$ for all $(s, a, s') \in S^1 \times X \times S^1$, where $s' = sa$ in S^1 .*

A path p in $\text{RCay}(S, X)$ is a sequence

$$p = \left(v_1 \xrightarrow{a_1} \cdots \xrightarrow{a_\ell} v_{\ell+1} \right),$$

where $v_i \in S^1$ are vertices in $\text{RCay}(S, X)$ and $v_i \xrightarrow{a_i} v_{i+1}$ are edges in $\text{RCay}(S, X)$. The endpoint of p is $\tau(p) := v_{\ell+1}$. The length of the path p is $\ell(p) := \ell$, which equals the number of edges. A *simple path* is a path that does not visit any vertex twice. Empty paths are considered simple. A path which starts and ends at the same vertex is called a *circuit*. A circuit that is simple, when the last vertex is removed, is called a *loop*.

Definition 4.4.2 (Transition edges). *An edge $s \xrightarrow{a} s'$ in the right Cayley graph $\text{RCay}(S, X)$ is a transition edge if there is no directed path from s' to s in $\text{RCay}(S, X)$. In other words, there does not exist any sequence $a_1, \dots, a_k \in X$ with $k \geq 1$ such that $s'(a_1 \cdots a_k) = s$.*

Let us now define the Karnofsky–Rhodes expansion of the right Cayley graph (see also [15, Definition 4.15] and [13, Section 3.4]). Let (X^+, X) be the free semigroup with generators X , where X^+ is the set of all words $a_1 \dots a_\ell$ of length $\ell \geq 1$ over X with multiplication given by concatenation. When we write $[a_1 \cdots a_\ell]_S$, we mean the element in S when taking the product in the semigroup with the generators $a_i \in X$.

Definition 4.4.3 (Right Karnofsky–Rhodes expansion). *The right Karnofsky–Rhodes expansion $\text{RKR}(S, X)$ is obtained as follows. Start with the right Cayley graph $\text{RCay}(X^+, X)$. Identify two paths in $\text{RCay}(X^+, X)$*

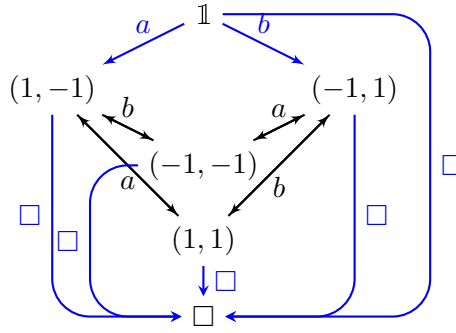
$$p := \left(\mathbb{1} \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_\ell} v_\ell \right) \quad \text{and} \quad p' := \left(\mathbb{1} \xrightarrow{a'_1} v'_1 \xrightarrow{a'_2} \cdots \xrightarrow{a'_{\ell'}} v'_{\ell'} \right)$$

in $\text{RKR}(S, X)$ if and only if the corresponding paths in $\text{RCay}(S, X)$

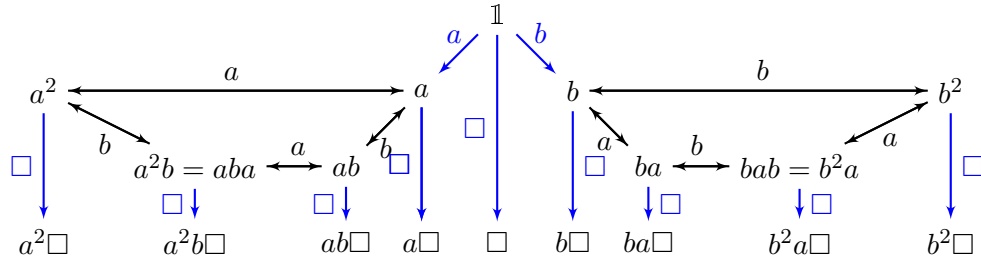
$$[p]_S := \left(\mathbb{1} \xrightarrow{a_1} [v_1]_S \xrightarrow{a_2} \cdots \xrightarrow{a_\ell} [v_\ell]_S \right) \quad \text{and} \quad [p']_S := \left(\mathbb{1} \xrightarrow{a'_1} [v'_1]_S \xrightarrow{a'_2} \cdots \xrightarrow{a'_{\ell'}} [v'_{\ell'}]_S \right),$$

where $v_i = a_1 a_2 \dots a_i$ and $v'_i = a'_1 a'_2 \dots a'_i$, end at the same vertex $[v_\ell]_S = [v'_{\ell'}]_S$ and in addition the set of transition edges of $[p]_S$ and $[p']_S$ in $\text{RCay}(S, X)$ is equal.

Example 4.4.4. *Consider the right Cayley graph of the Klein 4-group $Z_2 \times Z_2$ with zero with generators $\{a, b, \square\}$, where $a = (1, -1)$, $b = (-1, 1)$, and \square is the zero. The right Cayley graph $\text{RCay}(Z_2 \times Z_2 \cup \{\square\}, \{a, b, \square\})$ is*



where all three arrows a, b, \square fix the vertex \square at the bottom. Transition edges are indicated in blue. Double edges mean that right multiplication by the label for either vertex yields the other vertex. The right Karnofsky–Rhodes expansion of this right Cayley graph is given by



where arrows a, b, \square fix all the vertices at the bottom.

Proposition 4.4.5. [19, Proposition 2.15] $\text{RKR}(S, A)$ is the right Cayley graph of a semigroup, also denoted by $\text{RKR}(S, A)$.

Remark 4.4.6. The right Cayley graph and right Karnofsky–Rhodes expansion have left analogues as well, denoted by $\text{LCay}(S, X)$ and $\text{LKR}(S, X)$, where generators $a \in X$ act on the left instead of the right.

4.4.2 The McCammond expansion

Let us now turn to the McCammond expansion [16, 15] of the Karnofsky–Rhodes expansion of the right Cayley graph of (S, X) . Recall that a *simple path* in $\text{RKR}(S, X)$ is a path that does not visit any vertex twice. Empty paths are considered simple.

Definition 4.4.7 (McCammond expansion). The McCammond expansion $\text{Mc} \circ \text{RKR}(S, X)$ of $\text{RKR}(S, X)$ is the graph with vertex set V , which is the set of simple paths in $\text{RKR}(S, X)$. The edges are given by

$$E := \{(p, a, q) \in V \times X \times V \mid \tau(q) = \tau(p)a, \ell(q) \leq \ell(p) + 1, \\ q \text{ is an initial segment of } p \text{ if } \ell(q) \leq \ell(p)\}.$$

In other words, if the path pa in $\text{RKR}(S, X)$ is simple, then $q = pa$. Otherwise $\tau(pa) = v$ is a vertex of p and then q is the initial segment of p up to and including v .

Remark 4.4.8. Note that $\text{Mc} \circ \text{RKR}(S, X)$ has a spanning tree T with the same vertex set as $\text{Mc} \circ \text{RKR}(S, X)$, but only those edges $(p, a, q) \in E$ such that $\ell(q) = \ell(p) + 1$.

Example 4.4.9. The McCammond expansion of $\text{RKR}(S, X)$ of Example 4.4.4 is given in Figure 1.

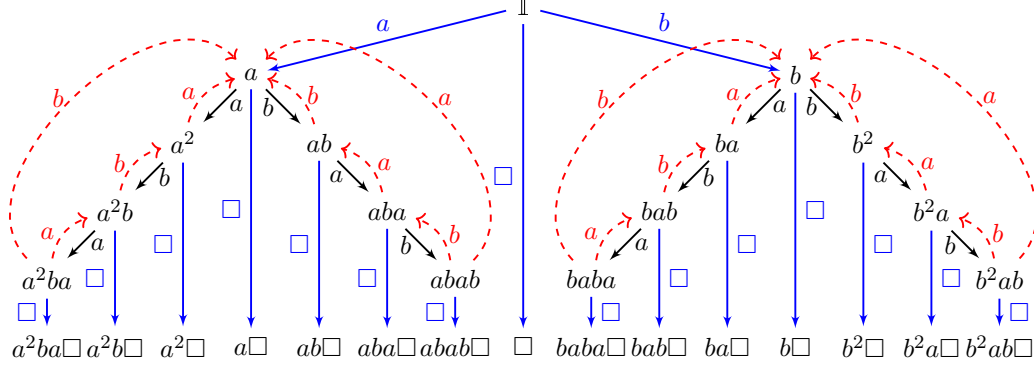


Figure 1: The McCammond expansion of $\text{RKR}(S, X)$ of Example 4.4.4. Transition edges are blue. The edges $(p, a, q) \in E$ with $\ell(q) = \ell(p) + 1$ are solid, whereas the edges with $\ell(q) \leq \ell(p)$ are dashed and red. The spanning tree T is obtained by removing all the dashed red arrows.

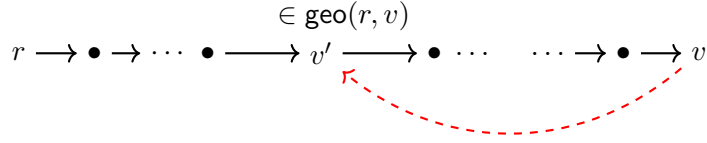


Figure 2: Part of the McCammond tree T (black edges). A global edge is indicated in dashed red.

By Remark 4.4.8, the McCammond expansion $\text{Mc} \circ \text{RKR}(S, X)$ has a spanning tree T . In this tree, the vertices are naturally labeled by the sequence of edge labels in the path from 1 to the vertex. More concretely, if

$$p = \left(1 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \dots \xrightarrow{a_\ell} v_\ell \right)$$

is a path in T , then the vertex v_ℓ is naturally labeled by $a_1 \dots a_\ell$. Hence the corresponding vertex v_ℓ in $\text{Mc} \circ \text{RKR}(S, X)$ has a *normal form* given by $a_1 \dots a_\ell$.

Remark 4.4.8 also ensures that $\text{Mc} \circ \text{RKR}(S, X)$ has the unique simple path property, defined as follows.

Definition 4.4.10 (Unique simple path property). *A rooted graph $(\Gamma, 1)$ with root 1 has the unique simple path property if for each vertex v in Γ there is a unique simple path from the root 1 to v .*

Figure 3 gives a schematic sketch of the McCammond expansion. The edges of the spanning tree T are indicated in black in Figure 3. An edge not in the spanning tree, falling back onto the tree, is indicated by a dashed red line in Figure 3. We call them *bold edges* (as in [15]).

The *geometric semigroup theory (GST) operator* is obtained by first taking the left KR expansion of (S, X) followed by the right KR expansion, followed by the McCammond expansion Mc of [15]. We denote the GST operator by GST . More concretely

$$\text{GST}(S, X) = \text{Mc} \circ \text{RKR} \circ \text{LKR}(S, X).$$

Definition 4.4.11. *For a semigroup S , define $\omega(S)$ as the smallest integer n such that*

$$t^n = (t^n)^2$$

for all $t \in S$.

5 Profinite smoothing

5.1 Statement of the main theorem

The goal of this section is to prove that complexity 1 is decidable. We begin with the main theorem. In this section we use the version of the Evaluation Semigroup that is expounded in Section 3 of this paper. This is slightly different than that of [5] as explained in Section 3. If S is a GM semigroup we write $Sl = 1$ if the modified (for the use of $\text{Eval}(S)$ in place of the Evaluation semigroup of [5]) lower bound from [5] is 1.

Theorem 5.1.1. *Let (S, X) be an X -generated GM semigroup with $\text{RLM}(S)c = 1$. Then $Sc = 1$ if and only if $Sl = 1$.*

5.2 Background in free aperiodic Burnside semigroups

We will give an apparently weaker version of Theorem 5.1.1. First we need some background in free aperiodic Burnside semigroups. We follow the presentation in [16]. We review the material we need in this paper.

Let n and m be positive integers. Let X be a set of cardinality n . We define $B_n(m)$, the free Burnside n -generated aperiodic semigroup of degree m to be the free semigroup with n -generators in the variety of semigroups defined by the identity $x^m = x^{m+1}$. Up to isomorphism, $B_n(m)$ is the quotient of the free semigroup X^+ by the smallest congruence containing $\{(w^m, w^{m+1}) \mid w \in X^+\}$. We will always assume that we have a fixed X generated GM semigroup (S, X) , so that n is fixed throughout the discussion. The variable m will always be an integer $m > 6$ and will vary in the discussion depending on the context.

Let $\Gamma_R(m, X)$ be the right Cayley graph of $B_n(m)$ relative to the generating set X . If X and m are clear from the context we will write Γ for this graph. We are working with semigroup Cayley graphs. So Γ has vertices the elements of $B_n(m)$ and an initial state I not belonging to $B_n(m)$. We identify x with its value in $B_n(m)$ under the canonical morphism from X^+ . The $(X$ labeled) edges of Γ are of the form $s \xrightarrow{x} sx$, where for $s = I$, we define Ix to be x . Notice that this implies that there are no edges that end at I .

An effective construction of Γ solves the word problem for $B_n(m)$. We do this by considering for each $s \in X^+$, the automaton $\Gamma(s)$ that has I as initial state and the value of s in $B_n(m)$ as terminal state and all states that lie on a path from I to s . Then a word t is accepted by $\Gamma(s)$ if and only if $s = t$ as elements of $B_n(m)$. Note that the states of $\Gamma(s)$ are those in the quotient of Γ by the right ideal of elements $t \in B_n(m)$ such that $t <_{\mathcal{R}} s$. Thus the states of $\Gamma(s)$ consist of I , all elements t such that $s \leq_{\mathcal{R}} t$ and a sink state \square representing all the elements in the right ideal above.

One of the very important results of [14, 16] is that for every $s \in X^+$, the automaton $\Gamma(s)$ is a finite automaton. Moreover, $\Gamma(s)$ accepts the word problem for s , that is the language $WP(s) = \{w \in X^+ \mid w\theta = s\}$, where $\theta: X^+ \rightarrow B_n(m)$ is the natural morphism. It follows that $WP(s)$ is a computable regular language for every $s \in B_n(m)$. McCammond also constructs a regular expression for $WP(s)$ that does not use the union operation. For each word w , there is a unique minimal length word $\text{red}(w)$ such that $w\theta = \text{red}(w)\theta \in B_n(m)$. Furthermore if $\text{red}(w) = x_1x_2 \cdots x_k$ for $x_i \in X$, then it is shown in [14, 16] that every directed loop in $\Gamma(w\theta)$ not containing \square passes through at least one of the vertices $(x_1 \cdots x_j)$ for $1 \leq j \leq k$. By definition, no non-empty loop passes through I .

We remark that what we call $\Gamma(s)$, McCammond calls $\text{str}(w)$ in [14, 16]. He constructs this automaton in two ways. The first is a detailed study of the Todd–Coxeter process that gives a sequence of approximating finite automata for $\Gamma(s)$ whose limit (in an appropriate sense) is all of $\Gamma(s)$. The difficult part of the theorem is to prove that this process ends in a finite number of steps. After doing this a Church–Rosser rewriting system is constructed for the word problem. This picks a

canonical representative for each element of $B_n(m)$. It is shown how to compute both the rewriting system and how to use it to effectively construct $\Gamma(s)$ for all $s \in B_n(m)$. See [14, 16].

5.3 Partial flows on automata

In this section, we define the category $\mathcal{C}(X) = \text{Cat}(\text{Partial}, X)$ of automata that we will be working with when discussing flows and partial flows. The objects of $\mathcal{C}(X)$ are finite automata $A = (I, Q, X, \square)$. Here X is the (finite) input alphabet. The states of the automata are $Q \cup \{\square\}$, where Q is a non-empty finite set and \square is an element not belonging to Q called the sink state which is fixed by all $x \in X$. Note that \square is part of the signature of our automata. The element $I \in Q$ is the initial state. We assume that A is a complete automaton, that is, elements of X act as total functions on the state set. In this section we assume that all automata have aperiodic transition semigroups. For brevity, we will write \mathcal{C} for $\mathcal{C}(X)$, since the alphabet X will be fixed throughout our discussion.

Let $A_1 = (I_1, Q_1, X, \square)$ and $A_2 = (I_2, Q_2, X, \square)$ be objects of \mathcal{C} . A morphism between A_1 and A_2 is a (total) function $h: Q_1 \rightarrow Q_2$ such that $I_1 h = I_2$ and for all $p, q \in Q_1, x \in X$, if $px \neq \square$, then $(ph)x = (px)h \neq \square$. We emphasize that morphisms are not defined on \square .

Let $A = (I, Q, X, \square)$ be an automaton and P a subset of Q such that $I \notin P$. Then $A \setminus P = (I, Q \setminus P, X, \square)$ is an automaton in \mathcal{C} and the inclusion map $Q \setminus P \rightarrow Q$ induces an injective morphism from $A \setminus P$ to A . We can consider the former to be a subobject of the latter.

We think of an arrow in an automaton of the form $q \xrightarrow{x} \square$ as actions “not yet defined” and in inductive procedures, \square will be expanded to “not yet defined” states.

Let $A = (I, Q, X, \square)$ be an object of \mathcal{C} . Recall that we have a fixed X -generated GM semigroup (S, X) , with 0-minimal ideal $M^0(A, G, B, C)$ and that $Rh_B(G)$ is the Rhodes lattice with B and group G . We define a *preflow* on A to be any function $h: Q \rightarrow Rh_B(G)$. A *partial flow* on A is a function $f: Q \rightarrow Rh_B(G)$ such that for all $p, q \in Q, x \in X$, if $px \neq \square$, then $(pf) \xrightarrow{x} (pf)x \neq \square$ satisfies the flow condition as defined in Section 2.4. We emphasize that like morphisms, partial flows are not defined on \square . Note that the restriction of a flow to Q is a partial flow and a partial flow with Q an invariant set in A is a flow.

Now we can state a theorem that we will prove is equivalent to Theorem 5.1.1. Recall that the exponent of a finite semigroup S is the least integer m such that s^m is an idempotent for all $s \in S$.

Let $s \in B_n(m)$ and $\theta \in Rh_B(G)$. Consider the preflow f_θ on $\Gamma_m(s)$ that sends I to θ and all other states to the empty SPC, that is, to the bottom of the lattice $Rh_B(G)$. Then there is at most one minimal partial flow f on $\Gamma_m(s)$ such that $If = \theta$. This is because the collection of partial flows is closed under meet. We remark that there may be no partial flow extending this preflow, if the preflow forces us to arrive at the contradiction $\implies \Leftarrow$. Recall the definition of the set of States from Section 3. We will show that the following Theorem is equivalent to Theorem 5.1.1.

Theorem 5.3.1. *Let (S, X) be an X -generated GM semigroup with $\text{RLM}(S)c = 1$. Define m to be $2(\omega + 4) + 1$, where ω is the exponent of the evaluation semigroup $\text{Eval}(S)$. Then $Sc = 1$ if and only if for each $\theta \in \text{States}$, the preflow f_θ extends to a partial flow in $\Gamma_m(s)$ for all $s \in B_{|X|}(m)$.*

5.4 Bottom up definition of operators and applications

We begin by giving the procedure to compute $(\text{spc})t^{\omega+*}$ for $t \in \text{Eval}(S)$ and $\text{spc} \in \text{States}$:

1. **Base case:** Set $\text{spc}_0 = (\text{spc})t^\omega$.
2. **Iteration:** Given spc_j , define spc_{j+1} to be

$$\text{spc}_j \vee \text{spc}_j t \vee \dots \vee \text{spc}_j t^{\omega-1}.$$

Clearly, we have

$$\mathbf{spc}_0 \leq \mathbf{spc}_1 \leq \dots \leq \mathbf{spc}_j \leq \mathbf{spc}_{j+1} \leq \dots$$

3. **Stabilization:** At some point, this sequence stabilizes $\mathbf{spc}_j = \mathbf{spc}_{j+1} =: \mathbf{spc}'$. Then

$$(\mathbf{spc})t^{\omega+*} = \mathbf{spc}'.$$

Note that by construction $\mathbf{spc}_j \leq (\mathbf{spc})t^{\omega+*}$. Furthermore, the pair $(\mathbf{spc}, \mathbf{spc}')$ is stable for $t^{\omega+*}$. At the end of this process, \mathbf{spc} does not change because of the vacuum V of $\text{Eval}(S)$ as discussed in Section 3.

We can summarize the above in the following picture:

$$\mathbf{spc} \xrightarrow{t^{\omega+*}} \mathbf{spc}'.$$

Next we give some applications. Recall the definition of the set of well-formed formulas $\Omega(X)$ in [5]. For a semigroup S , $P(S)$ is the *power semigroup* consisting of all subsets of S , with product $Z_1 Z_2 = \{z_1 z_2 \mid z_i \in Z_i, i = 1, 2\}$ for $Z_1, Z_2 \subseteq S$.

Definition 5.4.1. Define a map $\Lambda: \Omega(X) \rightarrow P(S^{\mathbb{N}})$ recursively as follows. Set $\varepsilon\Lambda = \{\mathbb{1}\}$, where $\mathbb{1}$ is the identity of $S^{\mathbb{N}}$. Put $x\Lambda = \{[x]_{S^{\mathbb{N}}}\}$. If Λ is defined on $\sigma, \tau \in \Omega(X)$, then set $(\sigma\tau)\Lambda = \sigma\Lambda\tau\Lambda$. If σ is not a proper power, define

$$\sigma^{\omega+*}\Lambda = \bigcup_{k \geq 0} (\sigma\Lambda)^{\omega} (\sigma\Lambda)^k.$$

Part (1) of Theorem 5.4.2 is [5, Conjecture 5.14]. Here R is the distinguished \mathcal{R} -class of S . For part (2), given two finite lattices L_1 and L_2 , a map $f: L_1 \rightarrow L_2$ is a **sup** map if the minimal element is mapped to the minimal element and $(x \vee y)f = xf \vee yf$ for all $x, y \in L_1$.

Theorem 5.4.2.

1. If $\tau \in \Omega(X)$ and $Y \subseteq R$, then $Y \cdot \tau = Y(\tau\Lambda)$.
2. In $\text{Eval}(S)$, restricted to sets, each member is a **sup** map.

Proof. The proof of both parts uses the same key observation. If by induction $t \in \text{Eval}(S)$ is a **sup**-map on sets, then $t^{\omega+*}$ is a **sup**-map on sets, because only on the first iteration of the above bottom-up construction do the sets change. \square

5.5 Bottom up definition of flow on a fixed automata

Let $\mathbf{A} = (I, Q, X, \cdot)$ be a finite state complete automaton with state set Q , start state I , input alphabet X and action $\cdot: Q \times X \rightarrow Q$. Let $f: Q \rightarrow Rh_B(G)$ be a flow as defined in Section 2.

Assume that we have applied the GST expansion to \mathbf{A} as described in Section 4.4. Then by [15], \mathbf{A} has a unique directed spanning tree \mathcal{T} . While \mathcal{T} contains all the vertices of Q , \mathbf{A} may have additional edges \mathbf{E} . The edges in \mathbf{E}

$$v \xrightarrow{x} v'$$

have the property that v' lies on the geodesic from the root q_0 to v of the tree \mathcal{T} . The *geodesic* from I to v is denoted by $\text{geo}(I, v)$. The edges in \mathbf{E} are called *global edges*, see [15]. See also Figure 3.

We now define an iterative procedure to compute a flow. The *initial preflow* ℓ_1 is defined by

$$(I)\ell_1 = b_0, (q)\ell_1 = \emptyset, q \neq I.$$

Here $b_0 \in B$ is a fixed element of our GM semigroup (S, X) with 0-minimal ideal $M^0(G, A, B, C)$. We identify $b \in B$ with the SPC $b / < 1 >$. Also \emptyset represents the empty SPC which is the bottom of $Rh_B(G)$.

We now iterate the initial preflow under two moves:

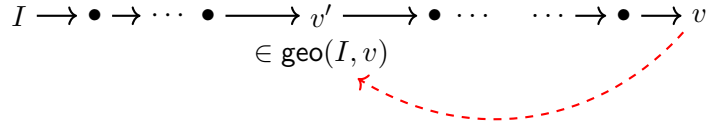


Figure 3: Schematic sketch of part of the McCammond tree \mathcal{T} with black tree edges and a global edge indicated in dashed red.

1. **Move forward:** At q , change $(q \cdot x)\ell$ to $(q)\ell \cdot x \vee (q \cdot x)\ell$.
2. **Move flow back under $\cdot x$:** At q , if $(q)\ell$ has blocks B_1, \dots, B_k , all $B_j \cdot x \neq \emptyset$ and all $B_1 \cdot x, \dots, B_k \cdot x$ form a block of $(q \cdot x)\ell$, then consider the union $B_1 \cup \dots \cup B_k$ and put on the unique cross section so that $\cdot x$ satisfies the flow condition of Section 3. This will be the new $(q \cdot x)\ell$.

We remark that in part (2) above, the unique cross section comes from the Tie-Your-Shoe Lemma [20, Chapter 4]. We also note that the automaton \mathbf{A} has a unique pointwise minimal flow $f: I \rightarrow Rh_B(G)$ with $I(f) = b_0$ or has no flow, since the collection of flows with this initial condition is closed under meet.

If ℓ_j is the preflow at the j -th iteration, then clearly $\ell_j \leq \ell_{j+1}$ pointwise. So either ℓ_{j+1} has the contradiction $\widehat{1}$ as a value or $\ell_j = \ell_{j+1}$ is a flow (which is the unique minimal flow with base point b_0). Indeed, if $\widehat{1}$ is never reached, then $\ell_j = \ell_{j+1}$ is a flow since clearly $(q)\ell \cdot x \leq (q \cdot x)\ell$ by the forward move which is a flow by the flow back under $\cdot x$.

5.6 Relations between flows and partial flows

As above, we have a fixed X -generated GM semigroup (S, X) . Let $\mathbf{A} = (I, Q, X, \cdot)$ be an automaton with aperiodic transition semigroup $S = S(\mathbf{A})$. Let m be the least integer such that $s^m = s^{m+1}$ for all $s \in S$. It is easy to see that m is the exponent of S , that is the least integer such that s^m is an idempotent for each $s \in S$. Recall that if $w \in X^+$, then $\Gamma(m, w)$ is the minimal automaton of the language of words that have the same value as w in the free Burnside semigroup $B_{|X|}(w)$. The following fact follows easily from the definitions.

Fact 5.6.1. *Let \mathbf{A} and \mathbf{B} be automata in the category \mathcal{C} over the same alphabet X . If $h: \mathbf{B} \rightarrow \mathbf{A}$ is an automaton morphism and $f: \mathbf{A} \rightarrow Rh_B(G)$ is a (partial) flow, then $hf: \mathbf{B} \rightarrow Rh_B(G)$ is a (partial) flow. We say that h lifts the (partial) flow f .*

Lemma 5.6.2. *Let $\mathbf{A} = (I, Q, X, \cdot)$ be an aperiodic automaton with exponent m . If there is a flow $f: Q \rightarrow Rh_B(G)$, then for all $w \in X^+$, the automaton $\Gamma(m, w)$ has a partial flow.*

Proof. Recall that the non-sink states of $\Gamma(m, w)$ are the elements of $B_{|X|}(m)$ that are \mathcal{R} greater or equal to $w\theta$, where $\theta: X^+ \rightarrow B_{|X|}(m)$ is the canonical morphism. It follows that if $u, v \in X^+$ are such that $u\theta = v\theta \geq_{\mathcal{R}} w\theta$, then $I'u = I'v$, where I' is the start state of $\Gamma(m, w)$. Since there is a natural morphism $\alpha: B_{|X|}(m) \rightarrow S(\mathbf{A})$, it follows that the map sending u to Iu is a well-defined morphism in \mathcal{C} and therefore by Fact 5.6.1, αf is a partial flow on $\Gamma(m, w)$. \square

5.7 Relational morphisms

In this subsection, we review the basics of relational morphisms between semigroups and between automata. The main references are [20, 21].

Let X be a finite set. We define the category $\mathcal{A}(X)$ to have as objects all isomorphism classes of finite complete trim deterministic automata (I, Q, X, \square) with alphabet X . Note that the morphisms in this category are different from the ones in the previous section. Here Q is the set of states, I is the start state, and \square is the sink state. Trim means that there is a path in the automaton from I to every state $q \in Q$. We also assume that there are no edges ending in I . An important example is the right Cayley automaton of a semigroup presentation $\phi: X^+ \twoheadrightarrow S$, as defined in Section 5.3. Note that the morphism ϕ is surjective. Recall that a morphism between (I_1, Q_1, X, \square) to (I_2, Q_2, X, \square) is a function $f: Q_1 \rightarrow Q_2$ such that $I_1 f = I_2$ and for all $q \in Q, x \in X$ we have $(qx)f = (qf)x$.

Trimness and the preservation of start states imply that there is at most one morphism between any two automata (I_1, Q_1, X, \square) and (I_2, Q_2, X, \square) . Indeed, by trimness every state $q \in Q_1$ is of the form $q = I_1 w$ for some $w \in X^*$ and thus if there is a morphism $f: Q_1 \rightarrow Q_2$, we must have that $qf = (I_1 w)f = (I_1 f)w = I_2 w$. Thus there is a morphism if and only if whenever $I_1 w = I_1 v$ for $v, w \in X^+$, then $I_2 w = I_2 v$ and this morphism is unique.

Let A_1, A_2 be automata in $\mathcal{A}(X)$. It follows that if we define $A_1 \leq A_2$ to mean that there is a surjective morphism from A_2 to A_1 , then we can consider \leq to be a partial order on $\mathcal{A}(X)$. We show now that this poset is a lattice.

Let $A_1 = (I_1, Q_1, X, \square)$ and $A_2 = (I_2, Q_2, X, \square)$ be automata in $\mathcal{A}(X)$. We define $A_1 \vee A_2$ to be the automaton with states $Q = \{(I_1 w, I_2 w) \mid w \in X^*\}$, initial state (I_1, I_2) and action $(q_1, q_2)x = (q_1 x, q_2 x)$ for $(q_1, q_2) \in Q$ and $x \in X$. The projections $\pi_i: Q \twoheadrightarrow Q_i$ for $i = 1, 2$ that send (q_1, q_2) to q_i are surjective morphisms. It is routine to check that $A_1 \vee A_2$ is the join of A_1 and A_2 in the poset $(\mathcal{A}(X), \leq)$. Therefore $(\mathcal{A}(X), \leq)$ is a join semilattice. Since the X automaton 1_X with one state is the minimal element of $(\mathcal{A}(X), \leq)$ and each principal down ideal is finite (by finiteness of states), we can define the meet of two automata to be the join of all elements less than or equal to each of them as usual. Therefore $(\mathcal{A}(X), \leq)$ is a lattice and has finite principal downsets.

We take a closer look at the join in the case of Cayley graphs of X -generated semigroups. Let $\phi_1: X^+ \twoheadrightarrow S_1$ and $\phi_2: X^+ \twoheadrightarrow S_2$ be presentations of the semigroups S_1 and S_2 . Let Γ_1 and Γ_2 be the corresponding right Cayley automata. We define $\phi: X^+ \rightarrow S_1 \times S_2$ by $x\phi = (x\phi_1, x\phi_2)$ for all $x \in X$. We write $\text{Im}(\phi) = S_1 \vee S_2$. Clearly $\Gamma(S_1 \vee S_2) = \Gamma_1 \vee \Gamma_2$.

Note that $S_1 \vee S_2$ is a subdirect product of S_1 and S_2 . We have the following diagram in $\mathcal{A}(X)$

$$\Gamma_1 \xleftarrow{\pi_1} \Gamma_1 \vee \Gamma_2 \xrightarrow{\pi_2} \Gamma_2$$

and corresponding morphisms in the category of X -generated finite semigroups

$$S_1 \xleftarrow{\pi_1} S_1 \vee S_2 \xrightarrow{\pi_2} S_2,$$

where we use the same name for the corresponding projection morphisms. In the language of semigroup theory [20], we have the associated onto relational morphism $r = (\pi_1)_1^{-1} \pi_2: S_1 \twoheadrightarrow S_2$. The following fact is straightforward, but important. We call this *string computation* of the relational morphism r .

Fact 5.7.1. *For all $s_1 \in S_1$ we have $s_1(r) = \{I_2 w \mid w \in X^+, I_1 w = s_1\}$.*

The notion of relational morphism can be generalized to the whole category $\mathcal{A}(X)$. Thus if $A_1 = (I_1, Q_1, X, \square)$ and $A_2 = (I_2, Q_2, X, \square)$ are automata we have the following diagram:

$$Q_1 \xleftarrow{\pi_1} Q_1 \vee Q_2 \xrightarrow{\pi_2} Q_2.$$

Here π_1, π_2 are the projections and $r_{\text{states}} = (\pi_1)_1^{-1} \pi_2: S_1 \twoheadrightarrow S_2$ is a relational morphism between A_1 and A_2 . Let S_i for $i = 1, 2$ be the transition semigroups of A_i . Then one has a companion

relational morphism $r_{\text{sgp}}: S_1 \rightarrow S_2$. See [20]. We then obtain a corresponding diagram of morphisms of transformation semigroups:

$$(Q_1, S_1) \xleftarrow{\pi_1} (Q_1 \vee Q_2, S_1 \vee S_2) \xrightarrow{\pi_2} (Q_2, S_2).$$

In this context, Fact 5.7.1 becomes the following.

Fact 5.7.2. *Let $r_{\text{states}}: Q_1 \rightarrow Q_2$ be the relational morphism defined above. Then*

$$(q_1)r_{\text{states}} = \{I_2w \mid w \in X^*, I_1w = q_1\}.$$

5.8 Profinite methods

We assume that the reader is familiar with the definitions and results of [4]. We use the following standing notation: $\text{Auto}_{\text{top}} = (Q, A, X, q_0)$ is a complete aperiodic automaton with state set Q , start state $q_0 \in Q$, input alphabet X , and aperiodic transition semigroup A ; there are no accept states. We assume that Auto_{top} has a flow, and hence has a minimal flow f_{\min} with $q_0f_{\min} = q_0f$. The letters a, b, c with various subscripts range over strings in X^* , considered as inputs of the automaton Auto_{top} . We also have a relational morphism $r: \Gamma(\text{Eval}(S), \text{Eval}(S)) \rightarrow \text{Auto}_{\text{top}}$. We assume that r computes pointlike subsets of $\text{Eval}(S)$. This means that for each state $q \in Q$, qr^{-1} is a pointlike subset of $\text{Eval}(S)$.

We begin by generalizing [4, Cor. 7.3], that describes the inevitability of the graph below. In [5], graphs have vertices labelled by elements of the semigroup. Here we only label the edges of the graph. In our convention, we start at the initial state \mathcal{I} with an edge going from \mathcal{I} to the next vertex labelled a :

$$\mathcal{I} \xrightarrow{a} \bullet \xrightarrow{b} \bullet \curvearrowright c$$

We now consider the inevitability of the graph, where the loop c in the above graph is replaced by k loops labeled c_1, \dots, c_k . We call this graph $\Gamma_{\{c_1, \dots, c_k\}}$.

Definition 5.8.1 (Inevitable graph). *The three-tuple $(A, B, \{C_1, \dots, C_k\})$ is inevitable with respect to the graph $\Gamma_{\{c_1, \dots, c_k\}}$ and X -generated semigroup (S, X) if for all relational morphisms $r: S \rightarrow A \in \text{Ap}$ there are strings (depending on r) $x_1x_2 \cdots x_a = X_a$ and $y_1y_2 \cdots y_b = X_b$ in X^+ and c_1, \dots, c_k , so that $\mathcal{I}X_aX_bC_j^*$ is in the right Cayley graph of (A, X) and $A \subseteq (\mathcal{I}X_a)r^{-1}$, $B \subseteq (\mathcal{I}X_b)r^{-1}$, and $C_j \subseteq (c_j)r^{-1}$.*

The ordering on the three-tuples is inclusion on each coordinate.

Lemma 5.8.2. *Let $A, B, C_1, \dots, C_k \in \text{Pl}(\text{Eval}(S))$. Let N be the subsemigroup generated by $\{C_1, \dots, C_k\}$ and assume that (C, N) is a stable pair, where C is in the minimal ideal of N . Then $(A, B, \{C_1, \dots, C_k\})$ is a maximal inevitable tuple for the graph $\Gamma_{\{c_1, \dots, c_k\}}$ if and only if one of the following holds:*

1. **(Separation)** $BC = B$ in $\text{Pl}(\text{Eval}(S))$. Equivalently, $(\forall j)[BC_j = B]$.
2. **(Roll Back I)** *There exists $T \in \text{Pl}(\text{Eval}(S))$ such that $ABT = A$ and $C = TB$.*

Picture in Auto_{top} : $\mathcal{I} \xrightarrow{a} \bullet \xrightleftharpoons[b]{t} \bullet \curvearrowright c_{\min}$

3. **(Roll Back II)** *There exist $S, T \in \text{Pl}(\text{Eval}(S))$ such that $A = AST$, $B = (ST)^iS$ for some $i \geq 1$, and $C = TS$.*

Picture in Auto_{top} : $\mathcal{I} \xrightarrow{a} \bullet \xrightleftharpoons[s]{t} \bullet \curvearrowright c_{\min} = ts$

So in (1), C_{\min} right-stabilizes B , and in cases (2) and (3), C_{\min} factors.

Proof. The idea of the proof is to modify the proof of [4, Cor. 7.3] for the graph

$$\mathcal{I} \xrightarrow{a} \bullet \xrightarrow{b} \bullet \curvearrowright c_{\min}$$

where $c_{\min} \in \text{kernel}\langle c_1, \dots, c_k \rangle$. So, $c_{\min} c = c_{\min}$ for all $c \in \langle c_1, \dots, c_k \rangle$.

We first show that any of the conditions (1), (2), (3), implies inevitability of the graph

$$\mathcal{I} \xrightarrow{a} \bullet \xrightarrow{b} \bullet \curvearrowright c_1, \dots, c_k$$

(1) Choose a such that $(a)r^{-1} \supseteq A$ and b such that $(b)r^{-1} \supseteq B$. Note that $C = (c)r^{-1}$ for all $c \in \langle c_1, \dots, c_k \rangle$. Now consider a, b, c in Auto_{top} .

Note that $(b)r^{-1} \cdot (c_{\min})r^{-1} \supseteq B \cdot C = B$, by the hypotheses of (1). Also, $BC_i = (BC)C_i = BC = B$. Furthermore, there are $a \in A, b \in B$ such that $abcc_i = abc$, so (1) is inevitable.

(2) Choose t so that $(t)r^{-1} \supseteq T$ and b so that $(b)r^{-1} \supseteq B$. Consider abt , $b, c = tb$. Then $ABT = A$ implies $(abt)r^{-1} \supseteq A$. Also, $(abt)b(tb) = (abt)bc = ab$. Moreover, there is n such that $(abt)(bt)^n = ab$. So (2) is inevitable.

(3) This is similar to (2).

For the rest of the proof we follow [4, Cor. 7.3] for the graph $\Gamma_{\{c\}}$ above. In the profinite theorem, since $c_{\min} c = c_{\min}$ for any choice of c_{\min} in the minimal ideal of the closed subsemigroup generated by c in the free profinite aperiodic semigroup, we can use (the non-continuous) morphism of [4, Section 4.2] via the obvious generalization to the current situation. The results then follow. \square

5.9 The setup for the proof

In this section, (S, X, ϕ) denotes a GM semigroup S together with a surjective morphism $\phi: X^+ \rightarrow S$. If the morphism ϕ is understood in context, we write (S, X) .

We begin with $(\text{Eval}(S), \text{Eval}(S))$. We construct a relational morphism

$$r: (\text{Eval}(S), \text{Eval}(S)) \rightarrow (S_{\text{Ap}}, \text{Eval}(S)),$$

where S_{Ap} is an aperiodic semigroup with the following properties:

- (a) r computes $\text{Pl}_{\text{Ap}}(\text{Eval}(S), \text{Eval}(S))$. That is, for all $s \in S_{\text{Ap}}$, $sr^{-1} \in \text{Pl}_{\text{Ap}}(\text{Eval}(S))$.
- (b) r computes inevitable pairs of $(\text{Eval}(S), \text{Eval}(S))$. That is, for all $s \in S_{\text{Ap}}$, $(s, \text{Stab}_R(s))r^{-1}$ is a stable pair.
- (c) r computes inevitable solutions for the graph $\Gamma_{\{c_1, \dots, c_k\}}$ of Section 5.8 for each k -tuple of $(\text{Eval}(S), \text{Eval}(S))$ in the sense of [4].

We know that it is decidable to effectively find such an S_{Ap} and relational morphism r . This follows from [4] and Lemma 5.8.2.

We have the relational morphism $r^{-1}: (S_{\text{Ap}}, \text{Eval}(S)) \rightarrow (\text{Eval}(S), \text{Eval}(S))$. This implies that for all $s \in S_{\text{Ap}}$, we have $r^{-1}(s) = \{I_{\text{Eval}(S)}w \mid w \in \text{Eval}(S), I_A w = s\} \in \text{Pl}_{\text{Ap}}(\text{Eval}(S), \text{Eval}(S))$, since we have assumed that r computes the pointlike subsets of $\text{Eval}(S)$.

Recall that we have a fixed GM X -generated semigroup (S, X) and that we can consider X to be a subset of $\text{Eval}(S)$. We restrict the generators of $(A, \text{Eval}(S))$ to A . Let A' be the subsemigroup of A generated by X . We let r'^{-1} be the restriction of the relational morphism.

Remark 5.9.1.

1. For all $s \in S_{\text{Ap}}$, we have $r^{-1}(s) \in \text{Pl}_{\text{Ap}}(\text{Eval}(S)) \cap S$ but not necessarily in $\text{Pl}_{\text{Ap}}(S)$.
2. The join $\bigvee r^{-1}(s)$ over elements in S is not necessarily in $\text{Eval}(S)$, but there exists a not necessarily unique element f in $\text{Eval}(S)$ so that $\bigvee r^{-1}(s) \leq f$ (see [5, Prop. 4.28]). We choose such an element f arbitrarily and denote it $\bigvee r^{-1}(s)$. The map that sends a subset Z of $\text{Eval}(S)$ to $\bigvee Z$ is called the Augmentation map and will be studied in a later section.
3. The next sections will prove that applying the GST expansion to $(S_{\text{Ap}}, X)^{\text{GST}}$ gives a transformation semigroup $(S_{\text{Ap}}, X)^{\text{GST}}$ that has a flow if and only if $Sc = 1$. That is either $(S_{\text{Ap}}, X)^{\text{GST}}$ has a flow or no flow exists and $Sc = 2$. This is the decidability criterion for complexity 1.

5.10 The procedure

We expand our relational morphism to a relational morphism between the GST expansion $(A, X)^{\text{GST}}$ and the right Cayley automaton $\Gamma(S, X)$ of the X -generated GM semigroup S . For ease of notation, we continue to use (A, X) for our aperiodic automaton and we continue to call this relational morphism r^{-1} . We denote this as follows

$$r^{-1}: (S_{\text{Ap}}, X)^{\text{GST}} \rightarrow \Gamma(S, X).$$

Thus r^{-1} is a relational morphism of automata. We also consider it to be a relational morphism between pointed transformation semigroups, where I is the initial point of $(A, X)^{\text{GST}}$ and we choose a fixed initial state $(1, b_0)$ for (S, X) .

We list a number of consequences of having applied the GST expansion to (S_{Ap}, X) :

1. Since we have applied the GST expansion, (S_{Ap}, X) has a unique directed spanning tree \mathcal{T} . In addition, there are possible non-tree edges called **bold** edges in [15]. Each bold edge $e: v \rightarrow w$ has the property that w is on the unique geodesic $\text{geo}(I, v)$.
2. We can pick a finite set of elements $\{t_1, \dots, t_k\}$ of the free semigroup X^+ such that these elements map bijectively onto the tree \mathcal{T} . Recall that if $t \in X^+$, then $\Gamma(t)$ is the subautomaton of (S_{Ap}, X) consisting of all states for which there is a path to a vertex in the image of the path reading t in (S_{Ap}, X) . It follows that (S_{Ap}, X) is the join $\bigvee_{i=1}^k \Gamma(t_i)$.

We now define a sequence of partial flows on (S_{Ap}, X) that we prove under the hypothesis of Theorem 5.3.1 limit to a flow. Each partial flow f_i will be defined on a subautomaton D_i of $(A, X)^{\text{GST}}$ called the domain of f_i . The partial flow f_0 has domain $D_0 = \{I\}$ and sends I to $(1, b_0)$. By induction we assume that D_i is a connected subautomaton of (S_{Ap}, X) that contains all the strongly connected components of each of its vertices and D_i is a proper subset of D_{i+1} .

If D_i has all the states of (S_{Ap}, X) , then f_i is a flow and the procedure ends. Otherwise pick a state $q \in D_i$ and a word $w = x_1 \cdots x_\ell$ with $\ell \geq 1$ with the following properties:

1. None of $qx_1, qx_1x_2, \dots, qx_1x_2 \cdots x_{\ell-1}x_\ell = qw$ is in D_i .
2. There are no loops at the states $qx_1, qx_1x_2, \dots, qx_1x_2 \cdots x_{\ell-1}$.

Since the complement of D_i is also closed under strongly connected components, we are guaranteed that there is at least one such q and w as described above.

We now formulate our inductive hypothesis. Consider the subautomaton of (S_{Ap}, X) called A_k

$$q \xrightarrow{x_1} qx_1 \xrightarrow{x_2} qx_1x_2 \xrightarrow{x_3} \cdots \xrightarrow{x_{\ell-1}} qx_1 \cdots x_{\ell-1} \xrightarrow{x_\ell} qw \curvearrowright c_1, \dots, c_n \quad (5.10.1)$$

Hypothesis 5.10.1. The preflow from the automaton A_k that sends q to qf_k and all other states to the empty SPC extends to a minimal partial flow with $q\tilde{f}_{k+1} = qf_k$.

Assume that Hypothesis 5.10.1 holds for all k . Then if D_k is not all of Q , we let D_{k+1} be D_k together with all the states from the automaton A_k . We define f_{k+1} to be f_k on D_k and \tilde{f}_{k+1} on the states of A_k . Hypothesis 5.10.1 guarantees that f_{k+1} is a partial flow on D_{k+1} . We can then continue until we find a k such that $D_k = Q$ whence our partial flow is a flow.

We have reduced the theorem to proving that Hypothesis 5.10.1 is true for all k . In the next section we do this by defining *witnesses*. A witness is a word $t_k \in X^+$ and the minimal automaton for the word problem $\Gamma(m, t_k)$ which as we have seen is a subautomaton of the Cayley automaton of the free Burnside monoid $B_{|X|}(m)$. We then prove that this witness shows that the Hypothesis 5.10.1 is true if we assume that each $\Gamma(m, t_k)$ has a partial flow.

5.11 Witnesses

In this section we first define witnesses to Hypothesis 5.10.1 in $\text{Pl}_{\text{Ap}}(\text{Eval}(S))$ using profinite methods. In the next section we use McCammond's work [14, 16] to turn them into witnesses in $\text{Eval}(S)$.

Using the notation of Hypothesis 5.10.1, we write SPC_k for qf_k . We want to replace SPC_k by a word $a_k \in X^+$ so that the preflow on $\Gamma(m, a_k)$ that sends $I \rightarrow b_0$ and all other states to the empty SPC extends to a partial flow f with $If \geq \text{SPC}_k$ but has no value equal to $\Rightarrow \Leftarrow$. Therefore we can rewrite the automaton (5.10.1) as follows, where for short, $a = a_k$.

$$I \xrightarrow{a} Ia \xrightarrow{a'} Iaa' \curvearrowright c_1, \dots, c_n \quad (5.11.1)$$

This is the graph whose inevitability we studied in Section 5.8. Using our relational morphism r we obtain the inevitable tuple $(A', B', \{C_1, \dots, C_\ell\})$, where $A' = ar^{-1}, B' = br^{-1}, C_j = c_j r^{-1}$. By abuse of notation, we can consider each $c_j \in X^+$ and thus we have that $c_j \in C_j$. By our assumption on (A, X) , all of these sets are in $\text{Eval}(S)$. We can embed this inevitable tuple into a **maximal** tuple (with respect to inclusion) and we thus assume this tuple to be a maximal tuple. Let \mathcal{V} be the subsemigroup of $\text{Pl}(\text{Eval}(S))$ generated by $\{C_1, \dots, C_\ell\}$. Since (A, X) computes stable pairs, it follows that the minimal ideal $I(\mathcal{V})$ of \mathcal{V} is a left-zero semigroup [4]. Therefore, $CX = C$ for all $C \in I(\mathcal{V}), X \in \mathcal{V}$ and if we pick an element $C \in I(\mathcal{V})$, then (C, \mathcal{V}) is a stable pair in $\text{Pl}(\text{Eval}(S))$ by our assumption that (A, X) computes stable pairs.

Therefore by Lemma 5.8.2 we have one of the conditions: Profinite Separation, Roll Back I, or Roll Back II.

Assume that we are in condition Roll Back I. Then for the inevitable graph (5.11.1), we take $a = 1, a' = a$ and all the loops at Ia . We now repeat the process above to get a new maximal tuple. This is necessarily in case Profinite Separation as there are no loops at I by our convention for automata. A similar argument holds if we are in case Roll Back II. Notice that after rolling back, if necessary, we arrive at a maximal tuple that have the same starting loops $\{C_1, \dots, C_\ell\}$ at the end of a' , but with different base points at the beginning of the tuple. This also works for the element C in the minimal ideal $I(\mathcal{V})$ that we chose, since $CC_j = C$ for all $j = 1, \dots, \ell$. Therefore these rollbacks provide a witness as well. So we can assume that we are in the Profinite Separation case. We call this a *profinite witness* for Hypothesis 5.10.1.

5.12 Witnesses in free aperiodic and free Burnside semigroups

Up to now, we have not considered the value of m . In this section, we determine m and later, using [16], we go to very large m and then descend to our stated m .

Let $m = 2(4 + \omega) + 1$, where ω is the exponent of $\text{Eval}(S)$. Recall that the exponent of a finite semigroup is the least positive integer n such that s^n is an idempotent for each element s . We now use the results of [14, 16] to turn the witness of the previous section into a witness of the form $\Gamma(m, t)$,

where $t \in X^+$ and $\Gamma(m, t)$ is the minimal automaton of the word problem for t in the free Burnside semigroup $B_{|X|}(m)$.

We will need the concept of Well-Defined Formula (WFF) as defined in [5, Definition 5.1]. We note that we have three interpretations of the $\omega + *$ operator on certain semigroups:

1. In the pointlike aperiodic semigroup $\text{Pl}(S)$ of a semigroup S , if $Z \in \text{Pl}(S)$, then $Z^{\omega+*} = Z^\omega(\cup_{i=1}^{\omega-1} Z^i)$. Henckell's Theorem [3] states that $\text{Pl}(S)$ is the smallest subsemigroup of $P(S)$ containing the singleton sets, closed under subset and also the $\omega + *$ operator.
2. If $t \in \text{Eval}(S)$, then $t^{\omega+*} = t^\omega(\bigvee_{i=1}^{\omega-1} t^i)$. Here we regard elements of $\text{Eval}(S)$ as operating on the Rhodes lattice $Rh_B(G)$.
3. In the free Burnside semigroup $B_{|X|}(m)$, $t^{\omega+*} = t^\omega$.

We now show how to obtain a string $t \in X^+$ such that $\Gamma(m, t)$ is a witness for Hypothesis 5.10.1. We begin with the profinite witness that we constructed in the previous section. By definition, if $Z \in \text{Eval}(S)$, then $\overline{\bigvee Z} \in \text{Eval}(S)$.

Definition 5.12.1. *We define the augmentation map*

$$\text{Aug}: \text{Pl}(\text{Eval}(S)) \rightarrow \text{Eval}(S)$$

by $\text{Aug}(Z) = \overline{\bigvee Z}$ (see also Remark 5.9.1).

We now use the theory of free Burnside aperiodic semigroups as expositied in [14, 16] to find a witness for Hypothesis 5.10.1 of the form $\Gamma(m, t)$. We begin with our maximal inevitable tuple (A, B, C) as in Section 5.11. Then by definition, $\overline{\bigvee A} \in \text{Eval}(S)$.

Pick a Well-Formed Formula (WFF) α representing $\overline{\bigvee A}$. It is easy to see that we can use the same WFF that represents $A \in \text{Pl}(\text{Eval}(S))$, which exists by Henckell's Theorem. We now construct a witness $\Gamma(m, t_A)$ as follows:

1. Construct a value $v(\alpha)$ for α as defined in [5, Definition 4.23]. The existence of $v(\alpha)$ is guaranteed by [5, Lemma 4.25]. One notes that the proof of Lemma 4.25 of [5] shows that there is a computable value for α that belongs to the free aperiodic ω -semigroup, $F_{|X|}$ considered in [16].
2. Use the Knuth-Bendix presentation of [16, Section 9] to obtain t_A in the free Burnside semigroup $B_{|X|}(m)$.
3. t_A is known to be the unique minimal length word accepted by $\Gamma(m, t_A)$ [14, 16]. We therefore construct $\Gamma(m, t_A)$ by adding a loop reading w for each word w^m reading a path in the linear automaton of t_A and folding to make the automaton deterministic. By iterating these expansions and folding we are guaranteed to compute $\Gamma(m, t_A)$ in a finite number of steps [14].
4. We compute $\Gamma(m, t_B)$ in a similar fashion.

We emphasize that in the WFFs used above, $t^{\omega+*} = t^\omega$ when we are working in $B_{|X|}(m)$ and is the $\omega + *$ operator in $\text{Eval}(S)$ when working in that semigroup. See [5].

Let (A, B, C_1, \dots, C_j) be the maximal inevitable triple we constructed in Section 5.11. Let $t_B \in B_{|X|}(m)$ be the element constructed above. It follows from the construction of t_B that $Ib \neq \emptyset$, where $br^{-1} \subseteq B$ in the automaton (A^{GST}, X) computing inevitable tuples of $\text{Eval}(S)$. The next Lemma is crucial in our argument.

Lemma 5.12.2. *Using the notation above, for all $b' \in B$ and all words $w = x_1 \dots x_d \in C_i$, for some $1 \leq i \leq j$, we have $Ib'w = Ib' = Ib$ in $\Gamma(m, t_B)$.*

Proof. Since $B \in \text{Pl}(\mathbf{E}^-(S))$, and the semigroup of $\Gamma(m, t_B)$ is aperiodic it follows that $Ib = Ib'$ for all $b' \in B$. We have $BC_i = B$ by profinite separation. So $b'w \in B$ and thus $Ib'w = Ib' = Ib$ as claimed. \square

It follows that if $\Gamma(m, t_B)$ has a partial flow f with $If \geq qf_k$ in the notation of Hypothesis 5.10.1, then $\Gamma(m, t_B)$ is a witness for Hypothesis 5.10.1. Therefore we have the following result.

Lemma 5.12.3. *Assume that for every state $\tau \in \text{States}$ and for every $t \in X^+$, $\Gamma(m, t)$ has a partial flow with $If \geq \tau$, then $(S_{\text{Ap}}, X)^{\text{GST}}$ has a total flow.*

We now show that every $\Gamma(m, t)$ defines a function in $\text{Eval}(S)$. This will allow us to complete the proof that complexity 1 is decidable.

Let $t \in B_{|X|}(m)$. We define a function $f_t: Rh_B(G) \rightarrow Rh_B(G)$ as follows. Let $\text{spc} \in Rh_B(G)$ be such that $(\text{spc})V = \text{spc}$, where V is the Vacuum [5]. Consider the pre-flow $\phi_{\text{spc}}: \Gamma(m, t) \rightarrow Rh_B(G)$ that sends I to spc and all other states to the bottom of $Rh_B(G)$. Let $\overline{\phi_{\text{spc}}}$ to be the minimal partial flow defined by ϕ_{spc} . We define $(\text{spc})f_t = \text{spc}'$, where $\text{spc}' = t\overline{\phi_{\text{spc}}}$.

Theorem 5.12.4. *Let $t \in B_{|X|}(m)$. Then $f_t \in \text{Eval}(S)$.*

Proof. In this proof we assume knowledge of tools and results on free Burnside semigroups and free ω -semigroups, see [16]. In particular, we need the definition of i -length [16, Definition 5.5]. In addition, for the limit terms β_j^ω the maximal rank of β_j can be read on the base $(m-1)$ times in $B_{|X|}(m)$ for $m \geq 6$.

We prove this by double induction on the str-rank and length of an element $t \in B_{|X|}(m)$, see [14, 16]. The shortest words of str-rank 0 are the letters $x \in X$. In this case, it follows immediately from the definition that f_x is free flow by x [5] and is thus a member of $\text{Eval}(S)$. More generally, if $t = wx$ is a word of str-rank 0, where $x \in X$, then w also has string rank w and thus $f_w \in \text{Eval}(S)$ by induction. Clearly, $f_t = f_w f_x$ and we are done in the case that the str-rank is 0.

Assume that we have proved the assertion for all words of str-rank k and let t be a word of str-rank $k+1$ in normal form, so that t is the shortest word in X^+ with the same value in $B_{|X|}(m)$. Thus the length of t is its length as a word in X^+ , see [14, 16]. If $t = wx$, where $x \in X$ labels an edge in $\Gamma(m, t)$ that has a trivial \mathcal{R} -class, then it follows from the definition that $f_t = f_w f_x$. By induction it follows that $f_t \in \text{Eval}(S)$. \square

Lemma 5.12.5. *If $h \in \text{Eval}(S)$ and h comes from a WFF with value $t_h \in X^+$ and f is any flow for GM , then $b_0 h \leq b_0 f$.*

Proof. To prove the claim, we do induction on the WFF, see also [5]. The product is straightforward and the exponent $\alpha^{\omega+\star}$ can be checked by considering powers of α being a multiple of the exponents of $\text{Eval}(S)$. Eventually, the spc of the inductive $b_0 \alpha^j$ is smaller or equal to $b_0 f$. \square

Lemma 5.12.6. *Let $m = 2(4 + \omega) + 1$. Then $\text{str}^m(t) \in \text{Eval}(S)$.*

Proof. We use [14, 16] to find a WFF for $\text{str}^m(t)$, so that it is in $\text{Eval}(S)$ (and it is even deterministic after a sink is added). We induct on the length t via the Knuth–Bendix relations. If $\text{str}^m(t)$ ends in an edge, the claim follows by induction. If $\text{str}^m(t)$ ends in $\beta_N \neq \emptyset$ of [14, 16, Definition 5.5], the claim again follows by induction. If $\text{str}^m(t)$ ends in $\beta_N^{\omega+\star} = \beta_N^\omega$, then the maximal rank of the loop of β_N can be read on the base $m+1$ times by [16, Example 9.14]. We take the last m copies of β_N . Now we can proceed by induction. This gives a WFF exist for $\text{str}^m(t)$. \square

Corollary 5.12.7. *Because the WFF gives a deterministic automaton with a sink, it has a partial flow.*

It follows from Theorem 5.12.4 that f_t is a witness that $\Gamma(m, t)$ has a partial flow f with $If \geq \tau$ for any state τ . Therefore, by Lemma 5.12.3, we have proved that Theorem 5.3.1 is true. This allows us to prove that complexity 1 is decidable.

Theorem 5.12.8. *It is decidable if a finite semigroup S has complexity at most 1.*

Proof. As we know, we can reduce to the case that S is a GM semigroup with $\text{RLM}(S)c = 1$. We have our fixed aperiodic semigroup $(S_{\text{Ap}}, X)^{\text{GST}}$, where X generates S . We now build inductively the domains D_0, D_1, \dots, D_k as in the discussion before Hypothesis 5.10.1. By the work in this section we have provided a witness of the form $f_t \in \text{Eval}(S)$ coming from a computable automaton $\Gamma(m, t)$ [14]. If for each k this witness has a partial flow with initial value greater or equal to the value in Hypothesis 5.10.1, then the complexity of S is 1. Otherwise, if some such witness has the contradiction $\implies \Leftarrow$, then the complexity of S is 2. Since it is clearly decidable if a fixed automaton $\Gamma(m, t)$ has such a partial flow, we are done. \square

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