An algebraic interpretation of the λβK-calculus and a labelled λ-calculus

Jean-Jacques LEVY
IRIA-LABORIA
78150-Rocquencourt
France

Introduction : A wide range of λ-calculus models has been proposed by Scott[10,11] In these interpretations, the interconvertibility relation among λ-expressions is extended by mainly equating the unsolvable terms (i.e. expressions M such that, for any arguments $N_1, N_2, \ldots N_k$, the expression $MN_1N_2\ldots N_k$ has no normal form). This extension has been shown consistent by Barendregt [1] and Wadsworth [13]. Wadsworth [13] showed the adequacy of most of Scott's models from a computational point of view ; more precisely, each expression is equal to the limit of its approximations in these models. We will try to go in the reverse direction, in the first part of this paper, and to define the value of an expression from its set of approximations. Then we prove that, as usual, our interpretation defines (using Milner's words [7]) a congruence on the language of λ-expressions. For this, we follow Welch [14] who stated a conjecture about the completeness, in the reducibility sense, of "inside-out reductions". This conjecture is proved in the second part of this paper by introducing a "labelled λ-calculus", which the author believes to be a useful tool for some λ-calculus problems. The results in this paper are related to the ones in Hyland [4] and Welch [15]. The definition of our interpretation is very similar to the one of Nivat [9] and Vuillemin [12] used for systems of recursively defined functions. Most results appeared in the author's thesis [5].

Syntax : We consider the set Λ of λ-expressions, built from an infinite alphabet V of variables, which is the minimal set containing :

(1)	x	(variable)
(2)	(λxM)	(abstraction)
(3)	(MN)	(application)

where x is in V and M,N are already in Λ. And we will use the standard abbreviationswhere :

$$MNN_1N_2\ldots N_k \text{ stands for } (\ldots(((MN)N_1)N_2)\ldots N_k)$$
$$(\lambda x_1x_2\ldots x_m.M) \quad " \quad " \quad (\lambda x_1(\lambda x_2\ldots(\lambda x_mM)\ldots))$$

and M,N,N_i are expressions in Λ, x_i are variables. We shall also omit the outermost parenthesis of an expression. The usual notions of free and bound variables are assumed defined and we note M[x\N] for the substitution of N for the free occurrences of x in M. We consider only two rules of conversion : the α and β rules. If M derives from M by an α-conversion, we write $M \underset{\alpha}{\to} N$. Similarly we have $M \underset{\beta}{\to} N$, and a reduction (possibly of length zero) using only α-conversion from M to N is written $M \overset{*}{\underset{\alpha}{\to}} N$. Hence we note $M \overset{*}{\underset{\beta}{\to}} N$ and $M \xrightarrow[\alpha,\beta]{*} N$ for β-reduction or any sequence of α and β conversions from M to N. We often forget α-conversions and $M \to N$ or $M \overset{*}{\to} N$ are understood as $M \underset{\beta}{\to} N$ or $M \xrightarrow[\alpha,\beta]{*} N$. Equality must also be considered as equality modulo some α-conversions. We will try to use the usual terminology (residuals, standard, reductions ...) defined in [2,3]. We also make use of the context notation (See [8,13]).

Let us first remark that Λ can also be considered as the smallest set containing :

(i)	$\lambda x.M$	(abstraction)
(ii)	$xM_1M_2\ldots M_n$	(head normal form)
(iii)	$(\lambda x.M)NM_1M_2\ldots M_n$	

if x is a variable and M,N,M_i are expressions of Λ. More generally, a head normal form is any expression of the form $\lambda x_1x_2\ldots x_m.xM_1M_2\ldots M_n$ where $m,n \geq o$ (See[13]). Others expressions are of the form $\lambda x_1x_2\ldots x_m.(\lambda x.M)NM_1M_2\ldots M_n$ and have a head redex $(\lambda x.M)N$. If $M \overset{*}{\to} N$ and N is an abstraction (respectively a head normal form) we say that M has an abstraction form (respectively a head normal form).

Proposition 1 : If M has an abstraction form, then M has a minimal abstraction form $\lambda x.N_0$, i.e. we have $M \overset{*}{\to} \lambda x.N_0$ and, for any $\lambda x.N$ such that $M \overset{*}{\to} \lambda x.N$, we have $\lambda x.N_0 \overset{*}{\to} \lambda x.N$.

Proof : M can be only of form (i) or (iii). In the first case, we have $M = \lambda x.N_0$. Otherwise for any, $\lambda x.N$ such that $M \overset{*}{\to} \lambda x.N$, by the standardization theorem, there is a standard reduction :

$$M = M_0 \overset{R_1}{\to} M_1 \overset{R_2}{\to} M_2 \overset{R_3}{\to} \ldots \overset{R_n}{\to} M_n = \lambda x.N$$

from M to $\lambda x.N$. Let M_k be the first M_i which is an abstraction. Then, since the reduction is standard, the redexes R_j contracted between M_{j-1} and M_j are the head redexes of M_{j-1} for $1 \le j \le k$. So each standard reduction from M to some $\lambda x.N$ has a common initial part :

$$M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \xrightarrow{R_3} \ldots \xrightarrow{R_k} M_k \qquad \square$$

Proposition 2 : If M has a head normal form, then M has a minimal one.

The proof is very similar to the preceding one. In both cases, the minimal form is obtained by contracting head redexes until an expression of the desired form is reached.

Approximations : We still follow Wadsworth [13] and define the direct approximation $\phi(M)$ of an expression M by :

$$\begin{cases} \phi(\lambda x.M) = \lambda x.\phi(M) \\ \phi(xM_1M_2\ldots M_n) = x(\phi(M_1))(\phi(M_2))\ldots(\phi(M_n)) \\ \phi(\lambda x.M)NM_1M_2\ldots M_n) = \Omega \end{cases}$$

where Ω is an extra constant. Basically, $\phi(M)$ is obtained from M by replacing all (outermost) redexes of M by Ω and substituting ΩM by Ω until normal form. If Ω is understood as "undefined", $\phi(M)$ is the information we have from M without contracting its redexes. There is a slight modification from the Wadsworth's definition, because we do not want to identify Ω and $\lambda x.\Omega$.

We define N as $N = \phi(\Lambda)$.Obviously, N is the set of expressions in ω-β normal forms. More precisely N is the minimal set containing :

$$\begin{cases} \Omega \\ \lambda x.a \\ xa_1a_2\ldots a_n \end{cases}$$

if x is a variable and a,a_i are already in N. By considering Ω as a minimal element in N and extending by monotony, we get the following partial order $<$ in N :

$$\begin{cases} \Omega < a \\ \lambda x.a < \lambda x.b \qquad \text{if } a < b \\ xa_1a_2\ldots a_n < xb_1b_2\ldots b_n \qquad \text{if } a_i < b_i \text{ for } 1 \le i \le n \end{cases}$$

where a,b,a_i are expressions of N, x is a variable and $n \ge o$. Here, we must take care of α-conversion and the order $<$ is, in fact, an order between equivalence classes defined on N by the α-interconvertibility. So, if $a \overset{*}{\underset{\alpha}{\rightarrow}} a'$ and $b \overset{*}{\underset{\alpha}{\rightarrow}} b'$, we have $a < b$ iff $a' < b'$. Moreover, we notice that $a<b$ iff there are M,N in Λ such that $\phi(M) = a$, $\phi(N) = b$ and $M \xrightarrow{*} N$.

Proposition 3 : The set N is a semi-lattice where every directed subset (*) is a lattice. More precisely :

1) N has a minimal element Ω.

2) for any pair a,b of elements in N, there exists a greatest lower bound $a \sqcap b$. (meet operation)

3) for any pair a,b of elements in N which are dominated by a common upper bound, there exists a least upper bound $a \sqcup b$. (join operation)

The proof is trivial, and obviously we can give the inductive definitions of $a \sqcup b$ and $a \sqcap b$ (up to some α-conversions as for the definition of < above) :

$$\begin{cases} (\lambda x.a) \sqcap (\lambda x.b) = \lambda x.(a \sqcap b) \\ (x a_1 a_2 \ldots a_n) \sqcap (x b_1 b_2 \ldots b_n) = x(a_1 \sqcap b_1)(a_2 \sqcap b_2)\ldots(a_n \sqcap b_n) \\ a \sqcap b = \Omega \quad \text{otherwise} \end{cases}$$

and

$$\begin{cases} \Omega \sqcup a = a \sqcup \Omega = a \\ (\lambda x.a) \sqcup (\lambda x.b) = \lambda x.(a \sqcup b) \\ (x a_1 a_2 \ldots a_n) \sqcup (x b_1 b_2 \ldots b_n) = x(a_1 \sqcup b_1)(a_2 \sqcup b_2)\ldots(a_n \sqcup b_n) \\ a \sqcup b \quad \text{is not defined otherwise} \end{cases}$$

where x is a variable and a,b,a_i,b_i are expressions of N and $n \geq 0$. The set N is also complete for the $\sqcap$ operation, i.e. each subset X of N has a greatest lower bound $\sqcap X$ in N. Moreover, the order < is well-founded in N and we have no infinite stricly decreasing chains in N.

To any expression M, Wadsworth [13] associates a set of approximations $A(M)$, which is the set of direct approximations of all expressions reducible from M :

$$A(M) = \{\phi(N) \mid M \xrightarrow{*} N\}$$

We briefly review some descriptive properties.

Proposition 4 : The set $A(M)$ of approximations of any λ-expression M is a sublattice of N (with the same meet and join operations than in N)

Proof : We only need to show that $\sqcap$ and $\sqcup$ are closed in $A(M)$. Suppose a,b are in $A(M)$.

(*) If (D,<) is a partial order structure, a directed subset X of D is such that for any a,b in X, there is c in X such that a<c, b<c. (Notion similar to ascending chains if D is denumerable). See Scott [10].

1) $a \sqcap b$ is in $A(M)$ by induction on the size $\|a \sqcap b\|$ of $a \sqcap b$. There are three cases :

1.1. If $a \sqcap b = \Omega$, then $a \neq \Omega$ and $b \neq \Omega$ is impossible, because of propositions 1 and 2. Therefore, $a=\Omega$ or $b=\Omega$ which implies $\phi(M)=\Omega$, and then $a \sqcap b$ is in $A(M)$.

1.2. If $a \sqcap b = \lambda x.c_1$, then $a = \lambda x.a_1$ and $b = \lambda x.b_1$. Hence, M has an abstraction form and, by proposition 1, a minimal one $\lambda x.M_0$. As a_1 and b_1 are in $A(M_0)$, we know by induction that $a_1 \sqcap b_1$ is in $A(M_0)$. Thus, $\lambda x.c_1 = \lambda x.(a_1 \sqcap b_1)$ is in $A(\lambda x.M_0)$, and $a \sqcap b$ is in $A(M)$.

1.3. If $a \sqcap b = xc_1c_2...c_n$, we have the same proof and we use proposition 2.

2) The Church-Rosser theorem shows the existence of c such that $a<c$ and $b<c$. Hence, $a \sqcup b$ is defined and a similar proof, based on an induction on the the size $\|a \sqcup b\|$, shows that $a \sqcup b$ is in $A(M)$. □

Proposition 5 : For any a in $A(M)$, there is a minimal λ-expression N_a such that :

i) $M \overset{*}{\rightarrow} N_a$

ii) $\phi(N_a) = a$

iii) for any N such that $M \overset{*}{\rightarrow} N$ and $a<\phi(N)$, then $N_a \overset{*}{\rightarrow} N$

Proof : by induction on the size of a

1) $a = \Omega$. Then $\phi(M)= \Omega = a$ and $N_a = M$

2) $a = \lambda x.a_1$. Then if $M \overset{*}{\rightarrow} N$ and $a<\phi(N)$, we have $N = \lambda x.N_1$, $\phi(N) = \lambda x.\phi(N_1)$ and $a_1<\phi(N_1)$. By proposition 1, there is a minimum abstraction form $\lambda x.M_0$ of M. Hence $M_0 \overset{*}{\rightarrow} N_1$ and by induction there is N_{a_1} minimum for a_1 and reducible from M_0. Hence, if $N_a = \lambda x.N_{a_1}$, we have :

$$M \overset{*}{\rightarrow} \lambda x.M_0 \overset{*}{\rightarrow} N_a = \lambda x.N_{a_1} \overset{*}{\rightarrow} \lambda x.N_1 = N.$$

and $\phi(N_a) = \lambda x.\phi(N_{a_1}) = \lambda x.a_1 = a.$

3) $a = xa_1a_2...a_n$. Same proof but we need now proposition 2. □

Hence, generalizing propositions 1 and 2, we have a minimal expression N_a for any approximation a. We reach it by head reductions (leftmost outermost reductions) until a head normal form (if necessary) and repeating this process on arguments of the head normal form (when necessary). We notice also that if a,b are in some $A(M)$, then $a<b$ iff for any N', such that $M \overset{*}{\rightarrow} N'$ and

$\phi(N') = b$, there is N such that $M \overset{*}{\to} N$ and $\phi(N) = a$ and $N \overset{*}{\to} N'$.

The interpretation domain : In the set Λ, some λ-expressions have a finite set of approximations ; we call them expressions of finite information. The other expressions have an infinite information and, in order to be able to speak of their value, we will complete the set N of ω-β normal forms, by adding infinite points. Let $\bar{N}$ be the set of all directed subsets of N :

$$\bar{N} = \{ S \mid S \text{ directed, } S \subset N \}$$

We can extend the relation < to $\bar{N}$ by defining for S and S' in $\bar{N}$:

$$S \subseteq S' \text{ iff } \forall a \in S. \exists b \in S'. a<b$$

In order to keep an ordering, we define a quotient set :

$$\hat{N} = \bar{N}/\equiv$$

where :

$$S \equiv S' \text{ iff } S \subseteq S' \subseteq S$$

Hence, if we note by $[S]$ the equivalence class of S in $\hat{N}$, we have in $\hat{N}$:

$$[S] \subseteq [S'] \text{ iff } S \subseteq S'$$

Proposition 6 : The set $\hat{N}$ is a semi-lattice where every directed subset is a lattice. More precisely :

1) $\hat{N}$ has a minimal element $[\{\Omega\}]$
2) any pair of elements $[S],[S']$ in $\hat{N}$ has a greatest lower bound $[S] \sqcap [S']$
3) any pair of elements $[S],[S']$ in $\hat{N}$, which is dominated by a common upper bound, has a least upper bound $[S] \sqcap [S']$.

The proof is obvious and the definitions of $\sqcap$ and $\sqcup$ in $\hat{N}$ are given by :

$$[S] \sqcap [S'] = [\{ a \sqcap b \mid a \in S, b \in S' \}]$$
$$[S] \sqcup [S'] = [\{ a \sqcup b \mid a \in S, b \in S' \}]$$

But $\hat{N}$ has a richer structure. Using Scott's terminology (see for instance [10]), we have :

Proposition 7 : The domain $\hat{N}$ is :

1) complete for directed subsets of $\hat{N}$,
2) continuous,
3) algebraic since $\hat{N}$ admits a denumerable basis of isolated elements $[\{a\}]$ where $a \in N$.

This means that every directed subset X of $\hat{N}$ has a least upper bound $\sqcup X$ and that each element of $\hat{N}$ is the least upper bound of the finite information points $[\{a\}]$ (where $a \in N$) which are below it. The proof follows from the construction of $\hat{N}$ and we skip it. The method we use for the completion of $\hat{N}$ is equivalent to the one of Vuillemin [12].

Interpretation of λ-expressions : We associate to any expression M an element in $\hat{N}$ by the following equation :

$$\mathcal{J}(M) = [A(M)]$$

and we can thus induce a partial preorder on Λ defined by : $M \sqsubseteq M'$ iff $\mathcal{J}(M) \sqsubseteq \mathcal{J}(M')$. By the definition of $\hat{N}$, we have :

$$M \sqsubseteq M' \text{ iff } \forall N \text{ s.t. } M \overset{*}{\to} N.\ \exists N' \text{ s.t. } M' \overset{*}{\to} N' \text{ and } \phi(N) < \phi(N').$$

We write $M \equiv M'$ for $M \sqsubseteq M' \sqsubseteq M$ and we expect the usual properties for our interpretation $\mathcal{J}$. Moreover, if X is a directed subset of λ-expressions, $\sqcup X$ means $\sqcup \mathcal{J}(X)$.

Theorem 1 : The β-rule of conversion is valid in $\mathcal{J}$,i.e. if $M \overset{*}{\to} M'$ then $M \equiv M'$.

Proof : Since $M \overset{*}{\to} M'$, we have $A(M') \subset A(M)$ and then $M' \sqsubseteq M$. Now, suppose $M \overset{*}{\to} N$; then as $M \overset{*}{\to} M'$, we know, by the Church-Rosser theorem, the existence of an N' such that $M \overset{*}{\to} N'$ and $M' \to N'$, and hence $\phi(N) < \phi(N')$. So we have $M \sqsubseteq M'$. □

Let the set Λ be now extended by allowing the constant Ω whenever a free variable is possible. So Λ is now the set of λ-Ω expressions and we consider not only β-reductions but also an ω-rule of conversion defined by replacing any subexpression of the form ΩM by Ω. We note this kind of reduction $N \underset{\omega}{\overset{*}{\to}} N'$. Let C[ ] denote any context (see [8]), i.e a λ-Ω expression with one subexpression missing, and C[M] be the corresponding expression where M stands at the place of the previously missing subexpression.

Proposition 8 : $C[\Omega] \sqsubseteq C[M]$ for any context M and any expression M.

Proof : The set of expressions reducible from C[Ω] is isomorphic to a subset of the one reducible from C[M]. Moreover, $\phi(N[x\backslash\Omega]) < \phi(N[x\backslash M])$ for any λ-Ω expressions M,N. Hence, $C[\Omega] \sqsubseteq C[M]$ by definition of $\sqsubseteq$. □

Theorem 2 : The ω-rule of conversion is valid in $\mathcal{J}$,i.e. if $M \underset{\omega}{\overset{*}{\to}} M'$ then $M \equiv M'$.

Proof : By the above proposition, we already know that $M' \subseteq M$. Now, suppose $M \overset{*}{\rightarrow} N$, we can show easily (by an induction on the pair $< \ell, \ell' >$ if ℓ and ℓ' are the length of the reductions $M \overset{*}{\rightarrow} N$ and $M \overset{*}{\underset{\omega}{\rightarrow}} M'$) the existence of N' such that $N \overset{*}{\underset{\omega}{\rightarrow}} N'$ and $M' \overset{*}{\rightarrow} N'$. Hence $\phi(N) = \phi(N')$ and then $M \subseteq M'$. □

We turn now to the main point of this paper, i.e. we show that, for any context $C[\ ]$, we have $C[M] \subseteq C[M']$ if $M \subseteq M'$. So, using the definition of $\subseteq$, we need to show that if $C[M] \overset{*}{\rightarrow} N$, then there is an N' such that $C[M'] \overset{*}{\rightarrow} N'$ and $\phi(N) < \phi(N')$. But all we know is that, for any approximation of M, we can have a better one for M'. Therefore, in order to compare C[M] and C[M'], we try to point out the approximation of M needed by any reduction from C[M] to some N. That is the Welch' conjecture about inside-out reductions ([14]), which we prove later. Using the Welch's notations, let $C[M] \underset{M}{\overset{*}{\not\rightarrow}} N$ designate any reduction :

$$C[M] = M_0 \overset{R_1}{\rightarrow} M_1 \overset{R_2}{\rightarrow} M_2 \overset{R_3}{\rightarrow} \dots \overset{R_n}{\rightarrow} M_n = N$$

where, for all i ($1 \le i \le n$), the redex R_i contracted between M_{i-1} and M_i is not a residual of a redex internal to the subexpression M in M_0. Similarly, if F is a set of redexes of M_0, we write $M_0 \underset{F}{\overset{*}{\not\rightarrow}} M_n$ if none of the R_i is a residual of a redex of F. Hence, if F is the set of all redexes of M, we have $C[M] \underset{M}{\overset{*}{\not\rightarrow}} N$ iff $C[M] \underset{F}{\overset{*}{\not\rightarrow}} N$. Moreover, let $M[F \backslash \Omega]$ denote the substitution in M of all redexes of F by the constant Ω. Now, we can state what we want. (The proof is postponed until the next section.)

Proposition 9 : For any context $C[\ ]$ and any expression M, if $C[M] \overset{*}{\rightarrow} N$, then there are expressions M' and N', such that $M \overset{*}{\rightarrow} M'$, $N \overset{*}{\rightarrow} N'$ and $C[M'] \underset{M'}{\overset{*}{\not\rightarrow}} N'$.

Lemma 1 : Given a set of redexes F in an expression M, if $M \underset{F}{\overset{*}{}} M'$ and iff F' is the set of the residuals of the redexes of F in M', then $M[F \backslash \Omega] \overset{*}{\rightarrow} M'[F' \backslash \Omega]$.

Proof : by induction on the length of the reduction from M to M', which is of the form $M \overset{R}{\rightarrow} M_1 \underset{F_1}{\overset{*}{\not\rightarrow}} M'$, where the redex R, first contracted, is not one of the redexes of F and F_1 is the set of residuals of the redexes of F in M_1. Then, depending on the relative position of R and F, we obviously have $M[F \backslash \Omega] \overset{*}{\rightarrow} M_1 [F_1 \backslash \Omega]$. □

Lemma 2 : If a,b are in the set N of ω-β normal forms and such that $a<b$, then $C[a] \sqsubseteq C[b]$ for any context $C[\ ]$.

Proof : Corollary of proposition 8, since, if $a<b$, then a matches b except for some Ω's. □

Lemma 3 : For any context $C[\ ]$ and expression M :

$$C[M] \equiv \sqcup \{ C[a] \mid a \in A(M) \}$$

Proof : Let $X = \{C[a] \mid a \in A(M)\}$. By lemma 2, as $A(M)$ is a directed set, X also is directed and $\sqcup X$ exists, since $\hat{N}$ is complete for directed subsets.

First, if a is in $A(M)$, there is an expression N such that $M \xrightarrow{*} N$ and $a = \phi(N)$. So, we have, by proposition 8, $C[a] \sqsubseteq C[N]$ since a matches N except in some Ω's. But, as $M \xrightarrow{*} N$, we have too $C[M] \xrightarrow{*} C[N]$ and $C[M] \equiv C[N]$ by theorem 1. Hence, we get $C[a] \sqsubseteq C[M]$ for any a in $A(M)$. Therefore $C[M]$ is an upper bound of X and $\sqcup X \sqsubseteq C[M]$

Conversely, if $C[M] \xrightarrow{*} N$, there are M' and N' such that $M \xrightarrow{*} M'$, $N \xrightarrow{*} N'$ and $C[M'] \underset{M'}{\overset{*}{\nearrow\!\!\!\!\!\!\rightarrow}} N'$ (by proposition 9). Let F be the set of all redexes of M' ; we have $C[M'] \underset{F}{\overset{*}{\nearrow\!\!\!\!\!\!\rightarrow}} N'$. By lemma 1, if F' is the set of the residuals of the redexes of F in N', we have $C[M'][F\backslash\Omega] \xrightarrow{*} N'[F'\backslash\Omega]$. Moreover, $C[M'][F\backslash\Omega] \underset{\omega}{\xrightarrow{*}} C[\phi(M')]$ and since ω and β-conversions are valid, we get :

$$\phi(N)<\phi(N') = \phi(N'[F'\backslash\Omega]) \sqsubseteq N'[F'\backslash\Omega] \equiv C[M'][F\backslash\Omega] \equiv C[\phi(M')]$$

And then, for any N such that $C[M] \xrightarrow{*} N$, there is an M' such that $M \xrightarrow{*} M'$ and $\phi(N) \sqsubseteq C[\phi(M')]$. Since $C[M] \equiv \sqcup \{ a \mid a \in A(C[M]) \}$, we have $C[M] \sqsubseteq \sqcup X$. □

Theorem 3 : If $M \sqsubseteq M'$, then $C[M] \sqsubseteq C[M']$ for any context $C[\ ]$.

Proof : Since $M \sqsubseteq M'$, for any a in $A(M)$, there is a b in $A(M')$ such that $a<b$. Hence, by lemma 2, $C[a] \sqsubseteq C[b]$. Since $\hat{N}$ is complete, we get :

$$\sqcup \{ C[a] \mid a \in A(M) \} \sqsubseteq \sqcup \{ C[b] \mid b \in A(M') \}$$

and, by lemma 3, $C[M] \sqsubseteq C[M']$. □

Inside-out reductions : In order to show proposition 9, we follow Welch [14] and define an inside-out reduction as any reduction :

$$M = M_0 \overset{R_1}{\to} M_1 \overset{R_2}{\to} M_2 \overset{R_3}{\to} \dots \overset{R_n}{\to} M_n = M'$$

where, for all i,j such that i<j and $1 \le i \le n-1$, $2 \le j \le n$, the redex R_j (contracted between M_{j-1} and M_j) is not a residual of a redex internal to R_i in M_{i-1}. Let $M \xrightarrow[i.o.]{*} M'$ be a notation for any such inside-out reduction from M to M'. Welch conjectured the completeness of inside-out reductions, i.e. if $M \overset{*}{\to} N$, then there exists N' such that $N \overset{*}{\to} N'$ and $M \xrightarrow[i.o.]{*} N'$. So, as pointed out by Welch :

Proposition 10 : If inside-out reductions are complete, then proposition 9 is true.

Proof : Let $C[M] \overset{*}{\to} N_1$, there is N such that $N_1 \overset{*}{\to} N$ and $C[M] \xrightarrow[i.o.]{*} N$, as inside-out reductions are complete. Let us prove, by induction on the size of the context C[ ], that if $C[M] \xrightarrow[i.o.]{*} N$, there is an M' such that $M \to M'$ and $C[M'] \xrightarrow[/M']{*} N$.

1) If C[ ] = [ ], then C[M] = M and M' = C[M'] = N.

2) If $C[\ ] = \lambda x.C_1[\ ]$, the induction works easily.

3) If $C[\ ] = M_1 C_1[\ ]$, as the reduction $C[M] \xrightarrow[i.o.]{*} N$ is inside-out, we have $M_1 \xrightarrow[i.o.]{*} M'_1$, $C_1[M] \xrightarrow[i.o.]{*} N'$ and $M'_1 N' \xrightarrow[/N']{*} N$. Hence, by induction there is M' such that $M \overset{*}{\to} M'$ and $C_1[M'] \xrightarrow[/M']{*} N'$. Thus $M \overset{*}{\to} M'$ and $C[M'] = M_1 C_1[M'] \xrightarrow[/M']{*} N$.

4) If $C[\ ] = C_1[\ ]\ M_1$, we have the same proof. □

A labelled λ-calculus : The problem is now to keep track of the redexes contracted in some reduction $M \overset{*}{\to} N$ in order to be able to reorder them in an inside-out way, and to show the inside-out completeness. We do this by introducing a new set of λ-expressions (Λ') defined on a set of labels L as follows. Let L_0 = { a,b,c ... } be an infinite set of letters. We consider the set L of all strings of characters formed on L_0, with any level of overlining and underlining. So expressions of L are :

$$\begin{cases} a & \text{if } a \in L_0 \\ \alpha\beta & \text{if } \alpha,\beta \in L \\ \overline{\alpha} & \text{"} \quad \text{"} \\ \underline{\alpha} & \text{"} \quad \text{"} \end{cases}$$

and expressions of Λ' are :

$$\begin{cases} x^{\alpha} & \text{if} \quad \alpha \in L \quad \text{and} \quad x \in V \\ (\lambda x.M)^{\alpha} & \text{if} \quad " \quad \text{and} \quad M,N \in \Lambda' \\ (MN)^{\alpha} & \text{if} \quad " \quad " \quad " \end{cases}$$

Thus, the labelled λ-expressions are like usual λ-expressions except that every subexpression has an arbitrary label. This λ-calculus is a generalization of the one of Wadsworth [13] since, instead of considering integers as exponents, we have strings of characters. For any label α and expression M of Λ', we define $\alpha.M$ as :

$$\begin{cases} \alpha.x^{\beta} = x^{\alpha\beta} \\ \alpha.(\lambda x.M)^{\beta} = (\lambda x.M)^{\alpha\beta} \\ \alpha.(MN)^{\beta} = (MN)^{\alpha\beta} \end{cases}$$

and the substitution operation is defined by :

$$\begin{cases} x^{\alpha}[x\backslash N] & = \alpha\circ N \\ y^{\alpha}[x\backslash N] & = y^{\alpha} \\ (\lambda y.M)^{\alpha}[x\backslash N] & = (\lambda y.M[x\backslash N])^{\alpha} \\ (MM')^{\alpha}[x\backslash N] & = (M[x\backslash N]\ M'[x\ N])^{\alpha} \end{cases}$$

where we forget the difficulties due to α-conversion. Then the β-rule is defined (by monotony) from :

$$((\lambda x.M)^{\alpha}\ N)^{\beta} \rightarrow \beta\bar{\alpha}.M[x\backslash\underline{\alpha}\circ N]$$

(We do not care for the precedences between the . and substitution operators because they commute). Furthermore we will allow this reduction iff some predicate $P(\alpha,\beta)$ is verified. So, for instance, using a graph notation for λ-expressions (see Morris [8] where nodes ⓛ and ⓖ corresponds to abstraction and application), we have figure 1 if we suppose $P(\alpha,\beta)$ always true. In fact, we can restrict our attention to λ-expressions labelled by a set L' of labels defined as containing :

$$\begin{cases} a & \text{if} \quad a \in L_0 \\ \alpha\bar{\beta}\gamma & \text{if} \quad \alpha,\beta,\gamma \in L' \\ \alpha\underline{\beta}\gamma & \text{if} \quad " \quad " \end{cases}$$

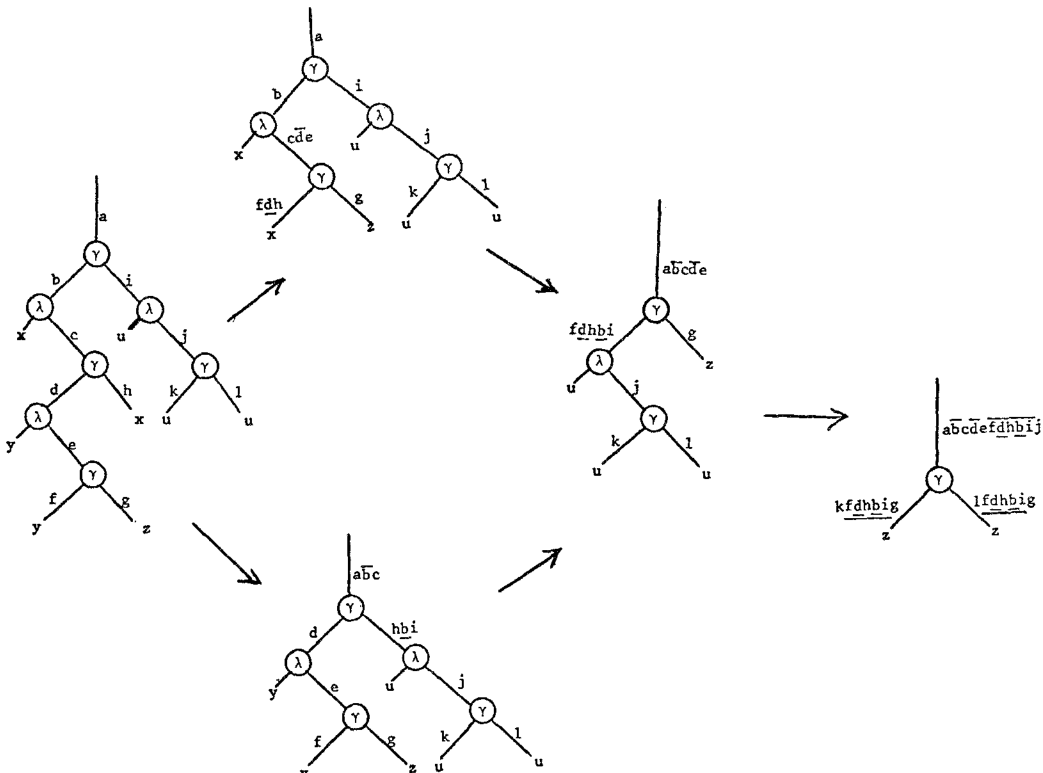

figure 1: Reductions from $((\lambda x.(\lambda y.(y^f z^g)^e)^d x^h)^c)^b(\ u.(u^k u^e)^j)^i)^a$

and it is clear that expressions labelled by L' keep their labels in L' after some β-reductions. We remark too that other λ-calculus languages are obtainable from this one by some homomorphism : for instance Wadsworth's typed λ-calculus and Morris' definition of descendants. Let the heigth $h(\alpha)$ of a label α of L be defined by :

$$\begin{cases} h(a) = o & \text{if } a \in L_0 \\ h(\alpha\beta) = \lceil h(\alpha),\ h(\beta) \rceil & \text{if } \alpha,\beta \in L \\ h(\underline{\alpha}) = h(\bar{\alpha}) = 1 + h(\alpha) & \text{if } \alpha \in L. \end{cases}$$

and let the degree of a redex be the label of its abstraction part. Hence, $\text{degree}(((\lambda x.M)^{\alpha}N)^{\beta}) = \alpha$.

Proposition 1' : The residuals of a redex R have the same degree as R.

Proof : Suppose $M \overset{R}{\to} N$ and R is a redex, in M, of the form $R = ((\lambda x.P)^{\alpha}Q)^{\beta}$. If $S = ((\lambda y.T)^{\gamma}U)^{\delta}$ is another redex of M, we show by cases that residual(s) of S in N have the same degree γ than S in M.

1) If R and S are 2 disjoint expressions, it is obvious.

2) If S is in $\mathcal{R}$, then S is in P or Q and the contraction of R may have only an effect on the external label δ of S.

3) If R is in S, then R is in T or U and the contraction of R has no effect on the degree α of S. □

<u>Proposition 2'</u> : If $P(\alpha,\beta)$ implies $P(\alpha,\gamma\beta)$ for any labels α,β,γ of L, then the β-rule of the labelled calculus is Church-Rosser.

<u>Proposition 3'</u> : If

1) $P(\alpha,\beta)$ implies $P(\alpha,\gamma\beta)$ for any labels α,β,γ of L

2) the set $\{ h(\alpha) \mid P(\alpha,\beta) \text{ is true } \}$ is bounded

then any labelled λ-expression strongly normalizes (i.e. any reduction in this labelled calculus has a finite length).

The proofs of these propositions are given in the appendix, by the usual techniques. We go back to the inside-out completeness and we will use letters as M,N to designate expressions of Λ and U,V for labelled λ-expressions of Λ'.

<u>Theorem 4</u> : If $M \overset{*}{\rightarrow} N$, then there is an expression N' such that $M \xrightarrow[\text{i.o.}]{*} N'$ and $N \overset{*}{\rightarrow} N'$.

<u>Proof</u> : Let U be the labelled λ-expression obtained from M by labelling all the subexpressions of M with a different letter of L_0. We can associate to the reduction $M \overset{*}{\rightarrow} N$, an isomorphic labelled reduction $U \overset{*}{\rightarrow} V$. More precisely, this reduction can be written :

$$U = U_0 \xrightarrow{R_1} U_1 \xrightarrow{R_2} U_2 \xrightarrow{R_3} \dots \xrightarrow{R_n} U_n = V$$

Let us now consider the predicate $P(\alpha,\beta)$ defined on labels by :

$$P(\alpha,\beta) \text{ is true iff } \alpha = \text{degree } (R_i) \text{ for some } i \ (1 \le i \le n)$$

The two assumptions of proposition 3' are verified and, hence, U strongly normalizes. Let V' be the normal form of V, then $V \overset{*}{\rightarrow} V'$, because the Church-Rosser assumption is true and for instance, any innermost reduction reaches the normal form V'. Let

$$U = V_0 \xrightarrow{S_1} V_1 \xrightarrow{S_2} V_2 \xrightarrow{S_3} \dots \xrightarrow{S_m} V_m = V'$$

be such an innermost reduction. (We then have, for all i, degree (S_i) = degree (R_j) for some j between 1 and n). We claim that this reduction is inside-out. Suppose i<j for some i,j between 1 and m and suppose S_j is a residual of a redex S'_j internal to S_i in V_{i-1}. By proposition 1', we have degree (S'_j) = degree (S_j) and then, as the predicate P is true for S_j, P is also true for S'_j. The reduction from U to V' is thus not an innermost reduction and we have a contradiction. Let N' be the λ-expression obtained by erasing the labels of V'. As an isomorphic reduction of Λ corresponds to any labelled reduction, we have $N \xrightarrow{*} N'$ and $M \xrightarrow[\text{i.o.}]{*} N'$. □

In fact, with the same method, we have, if $M \xrightarrow{*} M'$ and $M \xrightarrow{*} M''$, the existence of an N such that $M' \xrightarrow[\text{i.o.}]{*} N$ and $M'' \xrightarrow[\text{i.o.}]{*} N$.

<u>Conclusion</u>: The interpretation $<\mathcal{I},\hat{\mathcal{N}}>$,although strongly inspired by Scott's theory of computation,is purely algebraic.Here,we do not have a definition of application as in Scott [10,11] or Welch [15] .But with the help of the labelled calculus, any expression can be considered as the limit of expressions having a normal form.If we think of λ-expressions as programs, the interpretation $<\mathcal{I},\hat{\mathcal{N}}>$ seems to be the minimal one to consider.Thus we expect that $<\mathcal{I},\hat{\mathcal{N}}>$ is some kind of free interpretation.This is proved by Welch for "continuous semantics",i.e. roughly speaking for interpretations where the Wadsworth theorem is true.Welch did it for his model but his interpretation seems to be equivalent to the one we used here.Hyland [4],who independently considered also the same interpretation, proved that there is an extensional equivalence relation corresponding to equality in $\mathcal{I}$.Furthermore, he showed for any λ-expressions M,M' that $M \underline{\subseteq} M'$ iff $M \subseteq_{P\omega} M'$ where $P\omega$ is Scott's model [11] .Another question is to take into account extensionality and build an algebraic interpretation where the η-rule is valid.This is done by Hyland [4] Finally,the labelled λ-calculus seems interesting in itself [6],since we can capture the history of any reduction in the labels.

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Appendix 1:The Church-Rosser property in the labelled calculus (by the Taït-Martin Löf method)

Let $C(M)$ be defined by:

$$\begin{cases} C(x^{\alpha}) = x^{\alpha} \\ C((\lambda x.M)^{\alpha}) = (\lambda x.C(M))^{\alpha} \\ C(((\lambda x.M)^{\alpha}N)^{\beta}) = \beta\overline{\alpha}.C(M)[x\backslash\underline{\alpha}.C(N)] & \text{if } P(\alpha,\beta) \\ C((MN)^{\alpha}) = (C(M)C(N))^{\alpha} & \text{otherwise} \end{cases}$$

Let $M \to M'$ denote a parallel step of reduction and be defined by the following inference rules and axiom:

-I- $\quad x^{\alpha} \to x^{\alpha}$

-II- $$\frac{M \to M'}{(\lambda x.M)^{\alpha} \to (\lambda x.M')^{\alpha}}$$

-III- $$\frac{M \to M', N \to N'}{(MN)^{\alpha} \to (M'N')^{\alpha}}$$

-IV- $$\frac{M \to M', N \to N'}{((\lambda x.M)^{\alpha}N)^{\beta} \to \beta\overline{\alpha}.M'[x\backslash\underline{\alpha}.N']} \quad \text{if } P(\alpha,\beta)$$

Moreover,we suppose $P(\alpha,\beta)$ implies $P(\alpha,\gamma\beta)$ for any labels α,β,γ.

First,we notice that the associativity of concatenation implies:

$$\alpha.(\beta.M) = \alpha\beta.M$$

Lemma 1: $\quad \alpha.(M[x\backslash N]) = (\alpha.M)[x\backslash N]$

Proof:by cases on M.The only problem is when $M = x^{\beta}$ .Then the associativity of concatenation gives the answer. □

Lemma 2: If $x \neq y$ and x is not free in N',then:

$$M[x\backslash N][y\backslash N'] = M[y\backslash N'][x\backslash N[y\backslash N']]$$

Proof:by induction on the size of M.The only problem is when $M = x^{\alpha}$. Then we apply lemma 1. □

Lemma 3: If $M \to M'$,then $\alpha.M \to \alpha.M'$

Proof:by cases on the rule or axiom used for $M \to M'$ .The only interesting case is when:

$$M = ((\lambda x.M_1)^{\gamma}M_2)^{\beta} \to M' = \beta\overline{\gamma}.M'_1[x\backslash\underline{\gamma}.M'_2]$$

with $M_1 \to M'_1$, $M_2 \to M'_2$ and $P(\gamma,\beta)$.Then:

$$\alpha.M = ((\lambda x.M_1)^{\gamma}M_2)^{\alpha\beta} \quad \text{and} \quad \alpha.M' = \alpha.(\beta\overline{\gamma}.M'_1[x\backslash\underline{\gamma}.M'_2])$$

As $P(\gamma,\beta)$ implies $P(\gamma,\alpha\beta)$,we have by rule IV:

$$\alpha.M = ((\lambda x.M_1)^{\gamma}M_2)^{\alpha\beta} \to \alpha\beta\bar{\gamma}.M_1'[x\backslash\underline{\gamma}.M_2']$$

and by the associativity of concatenation $\alpha.M \to \alpha.M'$.□

Lemma 4: If $M \to M'$ and $N \to N'$,then $M[x\backslash N] \to M'[x\backslash N']$

Proof:by induction on the size of M.

1) M is a variable:

a) $M = x^{\alpha} = M'$ .Then we use lemma 3.

b) $M = y^{\alpha} = M'$ .Then obvious by axiom I.

2) M is not a variable and we have several cases according to the rule used for $M \to M'$ .The only interesting one is when:

$$M = ((\lambda y.M_1)^{\alpha}M_2)^{\beta} \to M' = \beta\bar{\alpha}.M_1'[y\backslash\underline{\alpha}.M_2']$$

with $M_1 \to M_1'$, $M_2 \to M_2'$ and $P(\alpha,\beta)$.Then,ignoring α-conversions,we have:

$$\begin{aligned} M[x\backslash N] &= ((\lambda y.M_1[x\backslash N])^{\alpha}M_2[x\backslash N])^{\beta} \\ M'[x\backslash N'] &= (\beta\bar{\alpha}.M_1'[y\backslash\underline{\alpha}.M_2'])[x\backslash N'] \\ &= \beta\bar{\alpha}.(M_1'[y\backslash\underline{\alpha}.M_2'][x\backslash N']) && \text{(by lemma 1)} \\ &= \beta\bar{\alpha}.(M_1'[x\backslash N'][y\backslash(\underline{\alpha}.M_2')[x\backslash N']]) && \text{(by lemma 2)} \\ &= \beta\bar{\alpha}.(M_1'[x\backslash N'][y\backslash\underline{\alpha}.(M_2'[x\backslash N'])]) && \text{(by lemma 1)} \end{aligned}$$

By induction,we know that $M_1[x\backslash N] \to M_1'[x\backslash N']$ and $M_2[x\backslash N] \to M_2'[x\backslash N']$,and using rule IV,we have $M[x\backslash N] \to M'[x\backslash N']$.□

Lemma 5: If $M \to M'$,then $M' \to C(M)$.

Proof:by induction on the size of M .There are two interesting cases:

1)When:

$$M = ((\lambda x.M_1)^{\alpha}M_2)^{\beta} \to M' = (M_3'M_2')^{\beta}$$

with $(\lambda x.M_1)^{\alpha} \to M_3'$, $M_2 \to M_2'$ and $P(\alpha,\beta)$ .Then it is clear that $M_3' = (\lambda x.M_1')^{\alpha}$ and $M_1 \to M_1'$ .Hence we have by induction $M_1' \to C(M_1)$ and $M_2' \to C(M_2)$ .Then,using rule IV:

$$M' = ((\lambda x.M_1')^{\alpha}M_2')^{\beta} \to \beta\bar{\alpha}.C(M_1)[x\backslash\underline{\alpha}.C(M_2)] = C(M)$$

2)When:

$$M = ((\lambda x.M_1)^{\alpha}M_2)^{\beta} \to M' = \beta\bar{\alpha}.M_1'[x\backslash\underline{\alpha}.M_2']$$

with $M_1 \to M_1'$, $M_2 \to M_2'$ and $P(\alpha,\beta)$ .Then we have by induction $M_1' \to C(M_1)$ and $M_2' \to C(M_2)$ .Hence,by lemma 3: $\underline{\alpha}.M_2' \to \underline{\alpha}.C(M_2)$,and by lemma 4 and 3:

$$M' \to \beta\bar{\alpha}.C(M_1)[x\backslash\underline{\alpha}.C(M_2)] = C(M) \quad .\square$$

Lemma 6: If $M \to M'$ and $M \to M''$, then there is an N such that $M' \to N$ and $M'' \to N$.

Proof:We take $N = C(M)$ and use lemma 5. $\square$

Proposition: If $M \to M'$ and $M \to M''$, then there is an N such that $M' \to N$ and $M'' \to N$. $\square$

Proof:by induction on the sum of length of the reductions $M \to M'$ and $M \to M''$. $\square$

Appendix 2:Strong normalization in the labelled λ-calculus by a method due to D. van Daalen.We suppose:

(1) $P(\alpha,\beta)$ implies $P(\alpha,\gamma\beta)$

(2) $\{h(\alpha) \mid P(\alpha,\beta) \text{ is true}\}$ is bounded

Hence,we have the Church-Rosser property.Let write $\tau(N)$ for the external label of N .So $\tau(x^{\alpha}) = \tau((\lambda x.M)^{\alpha}) = \tau((MN)^{\alpha}) = \alpha$ and we call SN the set of strongly normalizable labelled λ-expressions.

Lemma 1: If $(\ldots((MN_1)^{\beta_1}N_2)^{\beta_2}\ldots N_n)^{\beta_n} \stackrel{*}{\to} (\lambda x.N)^{\alpha}$,then we have $h(\tau(M)) \leq h(\alpha)$.

Proof:by induction on n .If $n = 0$,this is clearly true.Otherwise, we must have:

$$(\ldots((MN_1)^{\beta_1}N_2)^{\beta_2}\ldots N_{n-1})^{\beta_{n-1}} \stackrel{*}{\to} (\lambda y.P)^{\gamma}$$

and:

$$((\lambda y.P)^{\gamma}N_n)^{\beta_n} \to \beta_n\overline{\gamma}.P[y\backslash\underline{\gamma}.N_n] \stackrel{*}{\to} (\lambda x.N)^{\alpha}$$

Hence,we get $h(\tau(M)) \leq h(\gamma)$ by induction.We also have:

$$h(\gamma) < h(\overline{\gamma}) \leq h(\tau(\beta_n\overline{\gamma}.P[y\backslash\underline{\gamma}.N_n])) \leq h(\alpha) \quad .\square$$

Lemma 2: If $M[x\backslash N] \stackrel{*}{\to} (\lambda y.P)^{\alpha}$,we only have two cases:

1) $M \stackrel{*}{\to} (\lambda y.M')^{\alpha}$ and $M'[x\backslash N] \stackrel{*}{\to} P$

or

2) $M \stackrel{*}{\to} M' = (\ldots((x^{\beta}M_1)^{\beta_1}M_2)^{\beta_2}\ldots M_n)^{\beta_n}$ and $M'[x\backslash N] \stackrel{*}{\to} (\lambda y.P)^{\alpha}$

Proof : application of propositions 1 and 2. $\square$

Lemma 3: If M,N are in SN, then $M[x\backslash N]$ is in SN.

Proof : Let m be the upper bound of the set $\{h(\alpha) \mid P(\alpha,\beta) \text{ is true}\}$, which exists by assumption (2), and $prof(M)$ be the maximal length of reductions starting from M in SN .We do an induction on the triple:

$$< h(\tau(N)) - m, prof(M), \|M\| >$$

where $\|M\|$ is the size of M.

The only interesting case is when :

$$M = (M_1 M_2)^{\alpha} \quad \text{and} \quad M_1[x\backslash N] \overset{*}{\to} (\lambda y.P_1)^{\alpha_1}$$

We know by induction that $M_1[x\backslash N]$ and $M_2[x\backslash N]$ are in *SN* ,but we wonder if $((\lambda y.P_1)^{\alpha_1} M_2[x\backslash N])^{\alpha}$ is in *SN* ,i.e. if $M' = \alpha\overline{\alpha_1}.P_1[y\backslash \underline{\alpha_1}.M_2[x\backslash N]]$ is in *SN* .Then lemma 2 tells us there are two subcases:

1) $M_1 \overset{*}{\to} (\lambda y.M_1')^{\alpha_1}$ and $M_1'[x\backslash N] \overset{*}{\to} P_1$ .Then by lemmas 2,3,4 of the Church-Rosser proof,we get :

$$\alpha\overline{\alpha_1}.M_1'[y\backslash \underline{\alpha_1}.M_2][x\backslash N] = \alpha\overline{\alpha_1}.M_1'[x\backslash N][y\backslash \underline{\alpha_1}.M_2[x\backslash N]]$$
$$\overset{*}{\to} \alpha\overline{\alpha_1}.P_1[y\backslash \underline{\alpha_1}.M_2[x\backslash N]] = M'$$

But $M = (M_1 M_2)^{\alpha} \overset{*}{\to} ((\lambda y.M_1')^{\alpha_1} M_2)^{\alpha} \to \alpha\overline{\alpha_1}.M_1'[x\backslash \underline{\alpha_1}.M_2]$ .Hence:

$$\operatorname{prof}(\alpha\overline{\alpha_1}.M_1'[x\backslash \underline{\alpha_1}.M_2]) < \operatorname{prof}(M)$$

and by induction M' is in *SN*.

2) $M_1 \overset{*}{\to} Q_1 = (\ldots((x^{\beta} N_1)^{\beta_1} N_2)^{\beta_2} \ldots N_n)^{\beta_n}$ and $Q_1[x\backslash N] \overset{*}{\to} (\lambda y.P_1)^{\alpha_1}$ .As $M_1[x\backslash N]$ is in *SN* and $M_1[x\backslash N] \overset{*}{\to} Q_1[x\backslash N]$,we have P_1 in *SN* .Moreover:

$$Q_1[x\backslash N] = (\ldots(((\beta.N) N_1')^{\beta_1} N_2')^{\beta_2} \ldots N_n')^{\beta_n}$$

where $N_i' = N_i[x\backslash N]$ for all i.Hence,using lemma 1:

$$h(\tau(N)) \le h(\tau(\beta.N)) \le h(\alpha_1) < h(\underline{\alpha_1}) \le h(\tau(\underline{\alpha_1}.M_2[x\backslash N]))$$

and by induction M' is in *SN* .□

<u>Proposition</u>: If P verifies assumptions (1) and (2), every expression M strongly normalizes.

<u>Proof</u>:by induction on the size of M and application of lemma 3.□

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