# DISTRIBUTIVE LAWS

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The usual distributive law of multiplication over addition, $(x_0 + x_1)(y_0 + y_1) \to x_0y_0 + x_0y_1 + x_1y_0 + x_1y_1$, combines the mathematical structures of abelian groups and monoids to produce the more interesting and complex structure of rings. From the point of view of "triples," a distributive law provides a way of interchanging two types of operations and making the functorial composition of two triples into a more complex triple.

The main formal properties and different ways of looking at distributive laws are given in §1. §2 is about algebras over composite triples. These are found to be objects with two structures, and the distributive law or interchange of operations appears in its usual form as an equation which the two types of operations must obey. §3 is about some frequently-occurring diagrams of adjoint functors which are connected with distributive laws. §4 is devoted to Examples. There is an Appendix on compositions of adjoint functors.

I should mention that many properties of distributive laws, some of them beyond the scope of this paper, have also been developed by Barr, Linton and Manes. In particular, one can refer to Barr's paper "Composite cotriples" in this volume. Since Barr's paper is available, I omitted almost all references to cotriples.

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One general fact about triples will be used. If $\varphi : \mathbb{S} \to \mathbb{T}$ is a map of triples in $\underline{A}$, the functor $\underline{A}^{\varphi}: \underline{A}^{\mathbb{S}} \leftarrow \underline{A}^{\mathbb{T}}$ usually has a left adjoint, for which there is a coequalizer formula:

$$ASF^{T} \underset{A\bar{\varphi}}{\overset{\sigma F^{T}}{\rightrightarrows}} AF^{T} \longrightarrow (A,\sigma)\otimes_{\mathbb{S}} F^{T} \quad .$$

Here $(A,\sigma)$ is an $\mathbb{S}$ - algebra and the coequalizer is calculated in $\underline{A}^{\mathbb{T}}$ . The natural operation $\bar{\varphi}$ of $\mathbb{S}$ on $F^T$ is the composition

$$(AST,AS\mu^T) \xrightarrow{\varphi T} (ATT,AT\mu^T) \xrightarrow{\mu^T} (AT,A\mu^T) \text{ .}$$

The notation $(\ )\otimes_{\mathbb{S}}F^T$ for the left adjoint is justifiable. Later on the symbol $\underline{A}^{\varphi}$ is replaced by a Hom notation. The adjoint pair $(\ )\otimes_{\mathbb{S}}F^T$ $,\underline{A}^{\varphi}$ is always tripleable.

1. Distributive laws, composite and lifted triples. A distributive law of $\mathbb{S}$ over $\mathbb{T}$ is a natural transformation $\ell$: TS → ST such that

$$\begin{tikzcd} & T \arrow[dl, "T\eta^S"'] \arrow[dr, "\eta^S T"] & \\ TS \arrow[rr, "\ell"'] & & ST \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "\eta^T S"'] \arrow[dr, "S\eta^T"] & \\ TS \arrow[rr, "\ell"'] & & ST \end{tikzcd}$$

$$\begin{tikzcd} TSS \arrow[r, "\ell S"] \arrow[d, "T\mu^S"'] & STS \arrow[r, "S\ell"] & SST \arrow[d, "\mu^S T"] \\ TS \arrow[rr, "\ell"'] & & ST \end{tikzcd}$$

$$\begin{tikzcd} TTS \arrow[r, "T\ell"] \arrow[d, "\mu^T S"'] & TST \arrow[r, "\ell T"] & STT \arrow[d, "S\mu^T"] \\ TS \arrow[rr, "\ell"'] & & ST \end{tikzcd}$$

commute.

The composite triple defined by $\ell$ is $\mathbb{S}\mathbb{T} = (ST,\eta^S\eta^T,S\ell T.\mu^S\mu^T)$ . That is, the composite functor $ST : \underline{A} \to \underline{A}$, with unit and multiplication

$$\begin{tikzcd}[column sep=small] & S \arrow[dr, "S\eta^T"] & \\ \underline{A} \arrow[ur, "\eta^S"] \arrow[rr] \arrow[dr, "\eta^T"'] & & ST \\ & T \arrow[ur, "\eta^S T"'] & \end{tikzcd} \qquad STST \xrightarrow{S\ell T} \begin{tikzcd}[column sep=small] & STT \arrow[dr, "S\mu^T"] & \\ SSTT \arrow[ur, "\mu^S TT"] \arrow[rr] \arrow[dr, "SS\mu^T"'] & & ST \\ & SST \arrow[ur, "\mu^S T"'] & \end{tikzcd}$$

is a triple in $\underline{A}$ . The units of $\mathbb{S}$ and $\mathbb{T}$ give triple maps

$$S\eta^T : \mathbb{S} \longrightarrow \mathbb{ST} ,$$

$$\eta^S T : \mathbb{T} \longrightarrow \mathbb{ST} .$$

The proofs of these facts are just long naturality calculations. Note that the composite triple should be written $(\mathbb{ST})_\ell$ to show its dependence on $\ell$ , but that is not usually observed.

In addition to the composite triple, $\ell$ defines a <u>lifting</u> of the triple $\mathbb{T}$ into the category of $\mathbb{S}$-algebras. This is the triple $\tilde{\mathbb{T}}$ in $\underline{A}^{\mathbb{S}}$ defined by

$$\tilde{\mathbb{T}} = \begin{cases} \tilde{T} : \quad (A,\sigma)\tilde{T} = (AT, A\ell.\sigma T) , \\ \tilde{\eta} : (A,\sigma)\tilde{\eta} = A\eta : (A,\sigma) \to (A,\sigma)\tilde{T} , \\ \tilde{\mu} : (A,\sigma)\tilde{\mu} = A\mu : (A,\sigma)\tilde{T}\tilde{T} \to (A,\sigma)\tilde{T} . \end{cases}$$

It follows from compatibility of $\ell$ with $\mathbb{S}$ that $A\ell.\sigma T$ is an $\mathbb{S}$-algebra structure, and from compatibility of $\ell$ with $\mathbb{T}$ that $\tilde{\eta},\tilde{\mu}$ are maps of $\mathbb{S}$-algebras.

That $\tilde{\mathbb{T}}$ is a <u>lifting</u> of $\mathbb{T}$ is expressed by the commutativity relations $\tilde{T}U^S = U^S T$ , $\tilde{\eta}U^S = U^S\eta$ , $\tilde{\mu}U^S = U^S\mu$ .

$$\begin{tikzcd} A^{\mathbb{S}} \arrow[r, "\tilde{T}"] \arrow[d, "U^S"'] & \underline{A}^{\mathbb{S}} \arrow[d, "U^S"] \\ \underline{A} \arrow[r, "T"'] & \underline{A} \end{tikzcd}$$

<u>Proposition.</u> Not only do distributive laws give rise to composite triples and liftings, but in fact these three concepts are equivalent:

(1) distributive laws $\ell : TS \to ST$ ,

(2) multiplications $m : STST \to ST$ with the properties : $(\mathbb{S}\mathbb{T})_m = (ST, \eta^S\eta^T, m)$ is a triple in $\underline{A}$, the natural transformations

$$\mathbb{S} \xrightarrow{S\eta^T} \mathbb{S}\mathbb{T} \xleftarrow{\eta^S T} \mathbb{T}$$

are triple maps, and the <u>middle unitary law</u>

$$\begin{tikzcd}
 & ST \arrow[dl, "S\eta^T\eta^S T"'] \arrow[dr, equal] & \\
STST \arrow[rr, "m"'] & & ST
\end{tikzcd}$$

holds,

(3) liftings $\tilde{\mathbb{T}}$ of the triple $\mathbb{T}$ into $\underline{A}^{\mathbb{S}}$ .

<u>Proof</u>. Maps (1) $\to$ (2), (1) $\to$ (3) have been constructed above. It remains to construct their inverses and prove that they are equivalences.

(2) $\to$ (1). Given m, define $\ell$ as the composition

$$TS \xrightarrow{\eta^S TS\eta^T} STST \xrightarrow{m} ST$$

Compatibility of $\ell$ with the units of $\mathbb{S}$ and $\mathbb{T}$ is trivial. As to compatibility with the multiplication in $\mathbb{T}$,

$$\begin{tikzcd}
 & TST \arrow[dr, "\eta TST"] & \\
TTS \arrow[ur, "T\ell"] \arrow[r, "\eta T\eta TS\eta"'] \arrow[d, "\mu S"'] & STSTST \arrow[r, "STm"'] \arrow[d, "mST"'] & STST \arrow[d, "m"] \\
TS \arrow[r, "\eta TS\eta"'] \arrow[rr, bend right, "\ell"'] & STST \arrow[r, "m"'] & ST
\end{tikzcd}
\qquad
\begin{tikzcd}
TST \arrow[d, "\eta TS\eta T"'] \arrow[dr, "\eta TST"] \arrow[dd, bend right=60, "\ell T"'] & \\
STSTT \arrow[r, "STS\mu^T"'] \arrow[d, "mT"'] & STST \arrow[d, "m"] \\
STT \arrow[r, "S\mu^T"'] & ST
\end{tikzcd}$$

commute, the second because $\mathbb{T} \to (\mathbb{S}\mathbb{T})_m$ is a triple map. This reduces the problem to showing that an associative law holds between $\mu^T$ and m:

$$
\begin{tikzcd}
STSTT \arrow[rr, "STS\mu^T"] \arrow[dr, "STST\eta T"] \arrow[ddd, "mT"'] & & STST \arrow[ddd, "m"] \\
& STSTST \arrow[ur, "STm"'] \arrow[d, "mST"] & \\
& STST \arrow[dr, "m"] & \\
STT \arrow[rr, "S\mu^T"] & & ST
\end{tikzcd}
$$

This commutes since $S\mu^T = ST\eta^S T.m$ , as follows from the fact that $\mathbb{T} \to (\mathbb{S}\mathbb{T})_m$ is a triple map, and also the middle unitary law:

$$
\begin{tikzcd}
STT \arrow[r, "S\mu^T"] \arrow[d, "S\eta^T\eta^S T\eta^S T"] \arrow[dd, bend right=60, "ST\eta^S T"'] & ST \arrow[d, "S\eta^T\eta^S T"] \arrow[dd, bend left=60, "ST"] \\
STSTST \arrow[r, "STm"] \arrow[d, "mST"] & STST \arrow[d, "m"] \\
STST \arrow[r, "m"'] & ST
\end{tikzcd}
$$

The proof that $\ell$ is compatible with multiplication in $\mathbb{S}$ is similar; it uses the associative law

$$
\begin{tikzcd}
SSTST \arrow[r, "Sm"] \arrow[d, "\mu^S TST"'] & SST \arrow[d, "\mu^S ST"] \\
STST \arrow[r, "m"'] & ST
\end{tikzcd}
$$

The composition (1) → (2) → (1) is clearly the identity. (2) → (1) → (2) is the identity because of

$$\begin{tikzcd}
STST \arrow[d, "S\eta^S TS\eta^T T"'] \arrow[drr, "S\ell T"] & & \\
SSTSTT \arrow[rr, "SmT"] \arrow[d, "\mu^S TS\mu^T"'] & & SSTT \arrow[d, "\mu^S\mu^T"] \\
STST \arrow[rr, "m"] & & ST
\end{tikzcd}$$

(3) → (1) . If $\tilde{\mathbb{T}}$ is a lifting of $\mathbb{T}$ define $\ell$ as the composition

$$TS \xrightarrow{\eta^S TS} STS = F^S U^S TS = F^S \tilde{T} U^S (FU)^S \xrightarrow{F^S \tilde{T}(\epsilon U)^S} F^S \tilde{T} U^S = (FU)^S T = ST,$$

where the abbreviation $(FU)^S$ stands for $F^S U^S$ , .... (1) → (3) → (1) is the identity.
If we write $\ell \to \tilde{\mathbb{T}} \to \ell'$ , then $\ell' = \ell$ :

$$\begin{tikzcd}
ATS \arrow[r, "A\eta^S TS"] \arrow[ddd, "A\ell"'] & ASTS = (AS, A\mu^S)\tilde{T}(UFU)^S \\
& = (AST, AS\ell . A\mu^S T)(UF)^S U^S \\
& = (ASTS, AST\mu^S)U^S \arrow[d, "AS\ell . A\mu^S T = AF^S \tilde{T}(\epsilon U)^S"'] \\
AST = (AST, AS\ell . A\mu^S T)U^S & \phantom{X}
\end{tikzcd}$$

$$\begin{tikzcd}
& ATS \arrow[dl, "AT\eta^S S"'] \arrow[d, "A\eta^S TS"] & \\
ATSS \arrow[r, dash, "A\ell S"] \arrow[d, "AT\mu^S"'] & ASTS \arrow[r, "AS\ell"] & ASST \arrow[d, "A\mu^S T"] \\
ATS \arrow[rr, "A\ell = A\ell'"'] & & AST
\end{tikzcd}$$

(3) → (1) → (3) is the identity. Let us write $\tilde{\mathbb{T}} \to \ell \to \tilde{\tilde{\mathbb{T}}}$ and prove $\tilde{\mathbb{T}} = \tilde{\tilde{\mathbb{T}}}$.

Any lifting $\tilde{\mathbb{T}}$ of $\mathbb{T}$ can be written $(A,\sigma)\tilde{T} = (AT,(A,\sigma)\tilde{\sigma})$, where $\tilde{\sigma} : U^S TS \to U^S T$ is a natural $\mathbb{S}$-structure on $U^S T$. Restricting the lifting to free $\mathbb{S}$-algebras, $AF^S\tilde{T} = (AST, A\sigma_0)$, where $\sigma_0 = F^S\tilde{\sigma} : STS \to ST$ is a natural $\mathbb{S}$-structure on ST, which in addition satisfies an internal associativity relation involving $\mu^S$ :

$$\begin{tikzcd}
SSTS \arrow[r, "S\sigma_0"] \arrow[d, "\mu^S TS"'] & SST \arrow[d, "\mu^S T"] \\
STS \arrow[r, "\sigma_0"'] & ST
\end{tikzcd}$$

This follows from the fact that $A\mu^S : ASF^S \to AF^S$ is an $\mathbb{S}$-algebra map.

Similarly, write $(A,\sigma)\tilde{\tilde{T}} = (AT,(A,\sigma)\tilde{\tilde{\sigma}})$, $AF^S\tilde{\tilde{T}} = (AST, A\sigma_1)$.

We must show that $\tilde{\sigma} = \tilde{\tilde{\sigma}}$. This is done first for free $\mathbb{S}$-algebras, i.e., $\sigma_0 = \sigma_1$, and then the result is deduced for all $\mathbb{S}$-algebras by means of the canonical epimorphism $\sigma : AF^S \to (A,\sigma)$. If $\tilde{\mathbb{T}} \to \ell$, then $\ell$ is the composition

$$\begin{tikzcd}
ATS \arrow[r, "\eta^S TS"] \arrow[ddddr, "\ell"'] & ASTS = AF^S\tilde{T}U^S F^S U^S \\
& = (AST, A\sigma_0)U^S F^S U^S \\
& = (ASTS, AST\mu^S)U^S \arrow[d, "\sigma_0"] \\
& \phantom{x} \\
& AST = (AST, A\sigma_0)U^S
\end{tikzcd}$$

Now, $(AS, A\mu^S)\tilde{\tilde{T}} = (AST, A\sigma_1)$. But since $\ell \to \tilde{\tilde{\mathbb{T}}}$,

$$\begin{aligned}
(AS, A\mu^S)\tilde{\tilde{T}} &= (AST, AS\ell . A\mu^S T) \\
&= (AST, AS\eta^S TS . AS\sigma_0 . A\mu^S T) \\
&= (AST, A\sigma_0).
\end{aligned}$$

Thus $\sigma_1 = \sigma_0$. Applying $\tilde{\tilde{T}}$ and $\tilde{T}$ to the canonical epimorphism of the free algebra,

$$\begin{tikzcd}
AF^S\tilde{\tilde{T}} = (AST, A\sigma_1) \arrow[r, "\sigma T"] \arrow[d, equal] & (AT,(A,\sigma)\tilde{\tilde{\sigma}}) = (A,\sigma)\tilde{\tilde{T}} \\
AF^S\tilde{T} = (AST, A\sigma_0) \arrow[r, "\sigma T"] & (AT,(A,\sigma)\tilde{\sigma}) = (A,\sigma)\tilde{T}
\end{tikzcd}$$

But $A\eta^{S}T.\sigma T = AT$. A general fact in any tripleable category is that if $f : (A,\sigma) \to (A',\sigma')$, $f : (A,\sigma) \to (A',\sigma'')$, and $f$ is a split epimorphism in $\underline{A}$, then $\sigma' = \sigma''$. Thus $\tilde{\sigma} = \tilde{\tilde{\sigma}}$.

Of course, $\tilde{\tilde{\eta}} = \tilde{\eta}$, $\tilde{\tilde{\mu}} = \tilde{\mu}$, since these are just the unique liftings of $\eta$, $\mu$ into $\underline{A}^{\mathbb{S}}$ and do not depend on $\ell$. Thus $\tilde{\mathbb{T}} = \tilde{\tilde{\mathbb{T}}}$,

q.e.d.

2. <u>Algebras over the composite triple.</u> Let $\ell : TS \to ST$ be a distributive law, and $\mathbb{ST}, \tilde{\mathbb{T}}$ the corresponding composite and lifted triples.

<u>Proposition</u>. Let $(A,\xi)$ be an $\mathbb{ST}$-algebra. Since $\mathbb{S}, \mathbb{T} \to \mathbb{ST}$ are triple maps, the compositions

$$\begin{tikzcd}
AS \arrow[r, "AS\eta^{T}"] \arrow[dr, "\sigma"'] & AST \arrow[d, "\xi"] & AT \arrow[l, "A\eta^{S}T"'] \arrow[dl, "\tau"] \\
 & A &
\end{tikzcd}$$

are $\mathbb{S}$- and $\mathbb{T}$-structures on A, and it turns out that $\sigma$ is "$\ell$-distributive" over $\tau$ :

$$\begin{tikzcd}
ATS \arrow[rr, "\ell"] \arrow[d, "\tau S"'] & & AST \arrow[d, "\sigma T"] \\
AS \arrow[dr, "\sigma"'] & & AT \arrow[dl, "\tau"] \\
 & A &
\end{tikzcd}$$

A $\tilde{\mathbb{T}}$-algebra in $\underline{A}^{\mathbb{S}}$ consists of an $\mathbb{S}$-algebra $(A,\sigma)$ with a $\tilde{\mathbb{T}}$-structure $\tau : (A,\sigma)\tilde{T} \to (A,\sigma)$. Thus $\tau$ must be both a $\mathbb{T}$-structure $AT \to A$ and an $\mathbb{S}$-algebra map. The latter condition is equivalent to $\ell$-distributivity of $\sigma$ over $\tau$. The above therefore defines a functor

$$(\underline{A}^{\mathbb{S}})^{\tilde{\mathbb{T}}} \xleftarrow{\quad \Phi^{-1} \quad} \underline{A}^{\mathbb{S}\mathbb{T}}$$

Finally, the triple induced in $\underline{A}$ by the composite adjoint pair below is exactly the $\ell$-composite $\mathbb{S}\mathbb{T}$. The "semantical comparison functor" $\Phi$ is an isomorphism of categories, with the above $\Phi^{-1}$ as inverse.

$$\begin{tikzcd}
(\underline{A}^{\mathbb{S}})^{\tilde{\mathbb{T}}} \arrow[rr, "\Phi"] \arrow[d, shift left, "U^{\tilde{T}}"] & & \underline{A}^{\mathbb{S}\mathbb{T}} \arrow[ddll, shift left, "U^{ST}"] \\
\underline{A}^{\mathbb{S}} \arrow[u, shift left, "F^{\tilde{T}}"] \arrow[d, shift left, "U^{S}"] & & \\
\underline{A} \arrow[u, shift left, "F^{S}"] \arrow[uurr, shift left, "F^{ST}"] & &
\end{tikzcd}$$

The formula for $\Phi$ is $(A,\sigma,\tau)\Phi = (A,\sigma T.\tau)$ in this context.

<u>Proof.</u> First, distributivity holds between $\sigma,\tau$.

$$\begin{tikzcd}
& ATS \arrow[dl, "\eta TS"'] \arrow[d, "\eta TS\eta"] \arrow[rrr, "\ell"] & & & \\
ASTS \arrow[r, "STS\eta"'] \arrow[d, "\xi S"'] & ASTST \arrow[r, "S\ell T"'] \arrow[d, "\xi ST"'] & ASSTT \arrow[r, "\mu\mu"'] & AST \arrow[d, "\xi"] \\
AS \arrow[r, "S\eta"'] & AST \arrow[rr, "\xi"'] & & A
\end{tikzcd}$$

commutes, so we only need to show that $\xi = \sigma T.\tau$. (This is also the essential part in proving that $\Phi\Phi^{-1} = \Phi^{-1}\Phi = \mathrm{id}$.)

$$\begin{tikzcd}
 & AST \arrow[dl, "S\eta^T T"'] \arrow[d, "S\eta^T\eta^S T"] \arrow[drr, "S\eta^S\eta^T T"] \arrow[drrr, "="] & & & \\
ASTT \arrow[r, "ST\eta T"'] \arrow[d, "\xi T"'] & ASTST \arrow[rr, "S\ell T"'] \arrow[d, "\xi ST"] & & ASSTT \arrow[r, "\mu\mu"'] & AST \arrow[d, "\xi"] \\
AT \arrow[r, "\eta T"'] & AST \arrow[rrr, "\xi"'] & & & A
\end{tikzcd}$$

Now compute the composite adjointness. The formula for $F^{\tilde{T}}$ is $(A,\sigma) \to (AT, A\ell.\sigma T, A\mu^T)$. Thus $F^S F^{\tilde{T}} U^{\tilde{T}} U^S = ST$. Clearly the composite unit is $\eta^S \eta^T$. As for the counit, that is the contraction

$$\begin{tikzcd}
(A,\sigma,\tau)U^{\tilde{T}}U^S F^S F^{\tilde{T}} \arrow[r] \arrow[d, equal] & (A,\sigma,\tau)U^{\tilde{T}}F^{\tilde{T}} \arrow[r] \arrow[d, equal] & (A,\sigma,\tau) \arrow[d, equal] \\
(AST, AS\ell.A\mu^S T, A\mu^T) \arrow[r, "\sigma T"'] & (AT, A\ell.\sigma T, A\mu^T) \arrow[r, "\tau"'] & (A,\sigma,\tau)
\end{tikzcd}$$

The multiplication in a triple induced by an adjoint pair is always the value of the counit on free objects, here objects of the form $AF^S F^{\tilde{T}} = (AST, AS\ell.A\mu^S T, AS\mu^T)$. Thus the multiplication in the induced triple is

$$ASTST \xrightarrow{AS\ell T.A\mu^S TT} ASTT \xrightarrow{AS\mu^T} AST$$

which is exactly that defined by the given distributive law $\ell$. The composite adjointness $\underline{A} \to (\underline{A}^{\mathbb{S}})^{\tilde{\mathbb{T}}} \to \underline{A}$ therefore induces the $\ell$-composite triple $\mathbb{S}\mathbb{T}$.

By the universal formula for $\Phi$ and the above counit formula,

$$\begin{aligned}
(A,\sigma,\tau)\Phi &= (A, (A,\sigma,\tau)((U^{\tilde{T}}\epsilon^S F^{\tilde{T}})\epsilon^{\tilde{T}})U^{\tilde{T}}U^S) \\
&= (A, \sigma T.\tau).
\end{aligned}$$

q.e.d.

3. <u>Distributive laws and adjoint functors.</u> A distributive law enables four pairs of adjoint functors to exist, all of which are tripleable.

$$
\begin{tikzcd}
& \underline{A}^{\mathbb{S}\mathbb{T}} \arrow[dl, shift left, "\widetilde{U^T}"] \arrow[dr, shift left, "\mathrm{Hom}_{\mathbb{T}}(F^{ST},\ )"] & \\
\underline{A}^{\mathbb{S}} \arrow[ur, shift left, "\widetilde{F^T}"] \arrow[dr, shift left, "U^S"] & & \underline{A}^{\mathbb{T}} \arrow[ul, shift left, "()\otimes_{\mathbb{T}} F^{ST}"] \arrow[dl, shift left, "U^T"] \\
& \underline{A} \arrow[ul, shift left, "F^S"] \arrow[ur, shift left, "F^T"] &
\end{tikzcd}
$$

Here $\widetilde{F^T} = \widetilde{F^T}\Phi$ , $\widetilde{U^T} = \Phi^{-1}\tilde{U}^T$ are the liftings of $F^T$, $U^T$ into $\underline{A}^{\mathbb{S}}$ given by the Proposition, §2. $(\ )\otimes_{\mathbb{T}}F^{ST}$ and its adjoint are induced by the triple map $\eta^S T : \mathbb{T} \to \mathbb{S}\mathbb{T}$ as described in the Introduction. $\widetilde{F^T}$ could be written $(\ )\otimes_{\mathbb{S}}F^{ST}$ , of course.

Since the composite underlying $\underline{A}$-object functors $\underline{A}^{\mathbb{S}\mathbb{T}} \to \underline{A}$ are equal, the natural map e described in the Appendix is induced. It is a functorial equality

$$U^S F^T \xrightarrow{\quad = \quad} \widetilde{F^T}.\mathrm{Hom}_{\mathbb{T}}(F^{ST},\ ).$$

The above functorial equality, or isomorphism in general, will be referred to as "distributivity". I now want to demonstrate a converse, to the effect that if an adjoint square is commutative and distributive, then distributive laws hold between the triples and cotriples that are present.

<u>Proposition.</u> Let

$$\begin{tikzcd}
 & \tilde{\underline{B}} \arrow[dl, shift left, "\tilde{U}_1"] \arrow[dr, shift left, "\tilde{U}_0"] & \\
\underline{B}_0 \arrow[ur, shift left, "\tilde{F}_1"] \arrow[dr, shift left, "U_0"] & & \underline{B}_1 \arrow[ul, shift left, "\tilde{F}_0"] \arrow[dl, shift left, "U_1"] \\
 & \underline{A} \arrow[ul, shift left, "F_0"] \arrow[ur, shift left, "F_1"] &
\end{tikzcd}$$

be an adjoint square which commutes by virtue of adjoint natural isomorphisms $u : \tilde{U}_1 U_0 \stackrel{\sim}{\to} \tilde{U}_0 U_1$ , $f : F_1 \tilde{F}_0 \stackrel{\sim}{\to} F_0 \tilde{F}_1$ . Let $\mathbb{T}_0, \mathbb{T}_1, \tilde{\mathbb{G}}_1, \tilde{\mathbb{G}}_0$ be the triples in $\underline{A}$ and cotriples in $\underline{B}$ which are induced. Let $e : U_0 F_1 \to \tilde{F}_1 \tilde{U}_0$, $e' : U_1 F_0 \to \tilde{F}_0 \tilde{U}_1$ be defined as in the Appendix. The adjoint square is <u>distributive</u>(in an asymmetrical sense,0 over1) if e is an isomorphism. Assume this. Then with $\varphi, \psi$ the isomorphisms defined in the proof (and induced by e ), we have that

$$\begin{tikzcd}
T_1 T_0 \arrow[rr, "\ell"] \arrow[d, equal] & & T_0 T_1 \\
F_1 U_1 F_0 U_0 \arrow[r, "F_1 e' U_0"'] & F_1 \tilde{F}_0 \tilde{U}_1 U_0 \arrow[r, "f \tilde{U}_1 U_0"'] & F_0 \tilde{F}_1 \tilde{U}_1 U_0 \arrow[u, "\varphi^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{G}_1\tilde{G}_0 \arrow[rr, "\lambda"] \arrow[d, "\psi^{-1}"'] & & \tilde{G}_0\tilde{G}_1 \arrow[d, equal] \\
\tilde{U}_1U_0F_0\tilde{F}_1 \arrow[r, "uF_0\tilde{F}_1"'] & \tilde{U}_0U_1F_0\tilde{F}_1 \arrow[r, "\tilde{U}_0e'F_1"'] & \tilde{U}_0\tilde{F}_0\tilde{U}_1\tilde{F}_1
\end{tikzcd}$$

are distributive laws of $\mathbb{T}_0$ over $\mathbb{T}_1$ , $\widetilde{\mathbb{G}}_1$ over $\widetilde{\mathbb{G}}_0$ .

If the adjoint square is produced by a distributive law TS $\to$ ST as described at the start of §3 , so that $\mathbb{S}$ corresponds to $\mathbb{T}_0$ and $\mathbb{T}$ to $\mathbb{T}_1$ , then the distributive law given by the above formula is the original one.

<u>Proof.</u> Let

$$\mathbb{T} = \begin{cases} T = F_0\tilde{F}_1\tilde{U}_1U_0 \ : \ \underline{A} \to \underline{A} \\ \eta = \eta_0(F_0\ \tilde{\eta}_1U_0) \ : \underline{A} \to T \\ \mu = F_0\tilde{F}_1((\tilde{U}_1\epsilon_0\tilde{F}_1)\tilde{\epsilon}_1)\tilde{U}_1U_0 \ : TT \to T \end{cases}$$

be the total triple induced by the left hand composite adjointness. e,u induce a natural isomorphism

$$\begin{tikzcd}
T_0T_1 \arrow[rr, "\varphi"] \arrow[d, equal] & & T \arrow[d, equal] \\
F_0U_0F_1U_1 \arrow[r, "F_0eU_1"'] & F_0\tilde{F}_1\tilde{U}_0U_1 \arrow[r, "F_0\tilde{F}_1u^{-1}"'] & F_0\tilde{F}_1\tilde{U}_1U_0
\end{tikzcd}$$

By transfer of structure, any functor isomorphic to a triple also has a triple structure. Thus we have an isomorphism of triples $\varphi : (\mathbb{T}_0\mathbb{T}_1)_m = (T_0T_1,\eta_0\eta_1,m) \to \mathbb{T}$.

Actually, the diagrams in the Appendix show that $\varphi$ transfers units as indicated, and m is the quantity that is defined via the isomorphism. A short calculation also shows that m is middle-unitary. By (1) $\longleftrightarrow$ (2), Proposition, §1 , m is induced by the distributive law $(\eta_0 T_1 T_0 \eta_1)m : T_1 T_0 \to T_0 T_1$ . Now, consider the diagram

$$\begin{tikzcd}
F_1U_1F_0U_0 \arrow[rrr, "F_1e'U_0"] \arrow[d, equal] & & & F_1\tilde{F}_0\tilde{U}_1U_0 \arrow[d, "f\tilde{U}_1U_0"] \\
T_1T_0 \arrow[r, "\eta_0T_1T_0"] \arrow[dr] & T_0T_1T_0 \arrow[r, "\varphi T_0"] \arrow[d, "T_0T_1T_0\eta_1"] & TT_0 = F_0\tilde{F}_1\tilde{U}_1U_0F_0U_0 \arrow[r, "F_0\tilde{F}_1\tilde{U}_1\epsilon_0U_0"] \arrow[d, "TF_0\tilde{\eta}_1U_0"] & F_0\tilde{F}_1\tilde{U}_1U_0 = T \arrow[d, equal] \\
 & T_0T_1T_0T_1 \arrow[r, "\varphi\varphi"'] & TT = F_0\tilde{F}_1\tilde{U}_1U_0F_0\tilde{F}_1\tilde{U}_1U_0 \arrow[r, "\mu"'] & F_0\tilde{F}_1\tilde{U}_1U_0
\end{tikzcd}$$

The upper figure commutes, by expanding $\varphi$ , and naturality. Its top line is $\ell\varphi$ . Thus $\ell = (\eta_0T_1T_0\eta_1)(\varphi\varphi)\mu\varphi^{-1} = (\eta_0T_1T_0\mu_1)m$ is a distributive law.

The proof that $\lambda$ is a distributive law is dual. One defines the total cotriple

$$\mathbb{G} = \begin{cases} \tilde{G} = \tilde{U}_1U_0F_0\tilde{F}_1 & : \tilde{B} \to \underline{\tilde{B}} \\ \tilde{\epsilon} = (\tilde{U}_1\epsilon_0\tilde{F}_1)\tilde{\epsilon}_1 & : \tilde{G} \to \underline{\tilde{B}} \\ \tilde{\delta} = \tilde{U}_1U_0(\eta_0(F_0\tilde{\eta}_1U_0))F_0\tilde{F}_1 & : \tilde{G} \to \tilde{G}\tilde{G} \end{cases}$$

and uses the isomorphism

$$\begin{tikzcd}
\tilde{G}_1\tilde{G}_0 \arrow[d, equal] & & \tilde{G} \arrow[ll, "\psi"'] \arrow[d, equal] \\
\tilde{U}_1\tilde{F}_1\tilde{U}_0\tilde{F}_0 & \tilde{U}_1U_0F_1\tilde{F}_0 \arrow[l, "\tilde{U}_1e\tilde{F}_0"'] & \tilde{U}_1U_0F_0\tilde{F}_1 \arrow[l, "\tilde{U}_1U_0f^{-1}"']
\end{tikzcd}$$

to induce a similar isomorphism of cotriples $\psi : \mathbb{G} \to (\mathbb{G}_1\mathbb{G}_0)_d$ .

Finally, if the original adjoint square is produced by a distributive law $\ell : TS \to ST$, the Proposition, §2, shows that the total triple is the $\ell$-composite $\mathbb{S}\mathbb{T}$ , and $\varphi$ is an identity map.

q.e.d.

One can easily obtain distributive laws of mixed type, for example, $\tilde{\mathbb{T}}_1\mathbb{G}_0 \to \mathbb{G}_0\tilde{\mathbb{T}}_1$.

<u>Remark on structure-semantics of distributive laws.</u> Triples in $\underline{A}$ give rise to adjoint pairs over $\underline{A}$, $\underline{A} \to \underline{A}^{\mathbb{T}} \to \underline{A}$, and adjoint pairs $\underline{A} \to \underline{B} \to \underline{A}$ give rise to triples in $\underline{A}$ . This yields the structure-semantics adjoint pair for triples:

$$\mathrm{Ad}\ \underline{A} \underset{\sigma}{\overset{\check{\sigma}}{\rightleftarrows}} (\mathrm{Trip}\ \underline{A})^{*} \ .$$

This adjoint pair is a reflection ($\sigma\check{\sigma} =$ id.) and the comparison functor $\Phi$ : id. $\to$ $\check{\sigma}\sigma$ is the unit.

Something similar can be done for distributive adjoint situations over $\underline{A}$ and distributive laws.

Define a <u>distributive law in</u> $\underline{A}$ to be a triple $(\mathbb{S},\mathbb{T},\ell)$ where $\mathbb{S},\mathbb{T}$ are triples and $\ell : TS \to ST$ is a distributive law. A <u>map</u> $(\varphi,\psi) : (\mathbb{S},\mathbb{T},\ell) \to (\mathbb{S}',\mathbb{T}',\ell')$ is a pair of triple maps $\mathbb{S} \to \mathbb{S}'$, $\mathbb{T} \to \mathbb{T}'$ which is compatible with $\ell, \ell'$ . Let <u>Dist</u>($\underline{A}$) be this category.

A <u>distributive adjoint situation over</u> $\underline{A}$ means a diagram

$$\begin{tikzcd}
& \tilde{\underline{B}} \arrow[dl, shift left, "\tilde{U}_1"] \arrow[dr, "\tilde{U}_0"] & \\
\underline{B}_0 \arrow[ur, shift left, "\tilde{F}_1"] \arrow[dr, shift left, "U_0"] & & \underline{B}_1 \arrow[dl, shift left, "U_1"] \\
& \underline{A} \arrow[ul, shift left, "F_0"] \arrow[ur, shift left, "F_1"] &
\end{tikzcd}$$

where $(F_0,U_0)$, $(\tilde{F}_1,\tilde{U}_1)$ $(F_1,U_1)$ are adjoint pairs, $\tilde{U}_1\tilde{U}_0 = U_0U_1$, and the natural map $U_0F_1 \to \tilde{F}_1\tilde{U}_0$ is an isomorphism. A map of such adjoint situations consists of functors $\underline{B}_0 \to \underline{B}'_0$, $\underline{B}_1 \to \underline{B}'_1$, $\tilde{\underline{B}} \to \tilde{\underline{B}}'$ commuting with the underlying object functors $\tilde{U}_1$, $U_0$, $\tilde{U}_0$, $U_1$, . . .

Distributive laws give rise to distributive adjoint situations over $\underline{A}$, and vice versa (note that $\ell = (\eta_0 T_1 T_0 \eta_1)m$ and m does not involve $\tilde{F}_0$). Thus we have an adjoint pair

$$\underline{\text{Distributive Adj}}\ \underline{A} \underset{\sigma}{\overset{\check{\sigma}}{\rightleftarrows}} (\underline{\text{Dist}}\ \underline{A})^* \ .$$

The structure functor $\check{\sigma}$ is left adjoint to the semantics functor $\sigma$, $\sigma\check{\sigma} = \text{id.}$, and the unit is a combination of $\Phi$'s. This is the correct formulation of the above Proposition.

4. Examples. (1) Multiplication and addition. Let $\underline{A}$ be the category of sets, let $\$$ be the free monoid triple in $\underline{A}$, and $\mathbb{T}$ the free abelian group triple. Then $\underline{A}^{\$}$ is the category of monoids and $\underline{A}^{\mathbb{T}}$ is the category of abelian groups. For every set X the usual interchange of addition and multiplication

$$\prod_{i=0}^{m} \sum_{j_i=0}^{n_i} x_{ij_i} \longrightarrow \sum_{j_0=0}^{n_0} \cdots \sum_{j_m=0}^{n_m} \prod_{i=0}^{m} x_{ij_i}$$

can be interpreted as a natural transformation $XTS \xrightarrow{\ell} XST$ and is a distributive law of multiplication over addition, that is, of $\$$ over $\mathbb{T}$, in the formal sense.

The composite $\$\mathbb{T}$ is the free ring triple. XST is the polynomial ring Z[X] with the elements of X as noncommuting indeterminates.

The canonical diagram of adjoint functors is:

$$\begin{tikzcd}
& \underline{A}^{\mathbb{S}\mathbb{T}} = \underline{\text{Rings}} \arrow[dl, shift left, "\widetilde{U}^{T}"] \arrow[dr, shift left, "\mathrm{Hom}_{\mathbb{T}}(F^{ST},\ )"] & \\
\underline{A}^{\mathbb{S}} = \underline{\text{Monoids}} \arrow[ur, shift left, "\widetilde{F}^{T}"] \arrow[dr, shift left, "U^{S}"] & & \underline{A}^{\mathbb{T}} = \underline{\text{Abelian groups}} \arrow[ul, shift left, "(\ )\otimes_{\mathbb{T}} F^{ST}"] \arrow[dl, shift left, "U^{T}"] \\
& \underline{A} = \underline{\text{Sets}} \arrow[ul, shift left, "F^{S}"] \arrow[ur, shift left, "F^{T}"] &
\end{tikzcd}$$

$\widetilde{F}^{T}$ is the free abelian group functor lifted into the category of monoids, and is known as the"monoid ring" functor. $\mathrm{Hom}_{\mathbb{T}}(F^{ST},\ )$ is the forgetful functor. If A is an abelian group, the value of the left adjoint, $A\otimes_{\mathbb{T}} F^{ST}$ , is the Z-tensor ring generated by A, namely $Z + A + A\otimes A + \ldots$ . The natural map $U^S F^T \to \widetilde{F}^{T}.\mathrm{Hom}_{\mathbb{T}}(F^{ST},\ )$ is the identity, that is, distributivity holds. Both compositions give the free abelian groups generated by the elements of monoids.

The scheme is : the distributive law $\ell$: TS $\to$ ST produces the adjoint square, which, being distributive (§3) , induces a distributive law $\lambda : \mathbb{G}_{Ab}\mathbb{G}_{Mon} \to \mathbb{G}_{Mon}\mathbb{G}_{Ab}$, where $G_{Mon} = \widetilde{U}^{T}\widetilde{F}^{T}$, $G_{Ab} = \mathrm{Hom}_{\mathbb{T}}(F^{ST},\ )\otimes_{\mathbb{T}} F^{ST}$ . This $\lambda$ is that employed by Barr in his "Composite cotriples", this volume (Theorem(4.6)).

A distributive law ST $\to$ TS would have the air of a universal solution to the problem of factoring polynomials into linear factors. This suggests that the composite TS has little chance of being a triple.

(2) <u>Constants</u>. Any set C can be interpreted as a triple in the category of sets, $\underline{A}$ , via the coproduct injection and folding map $X \to C + X$, $C + C + X \to C + X$ . $\underline{A}^{C+()}$ is the category of sets with C as constants. For example, if C=1, $\underline{A}^{1+()}$ is the category of pointed sets.

Let $\mathbb{T}$ be any triple in $\underline{A}$. A natural map $\ell : C + XT \to (C + X)T$ is defined in an obvious way, using $C\eta$ . $\ell$ is a distributive law of $C + ()$ over $\mathbb{T}$ . The composite triple $C + \mathbb{T}$ has as algebras $\mathbb{T}$-algebras furnished with the set $C$ as constants.

(3) Group actions. Let $\pi$ be a monoid or group. $\pi$ can be interpreted as a triple in $\underline{A}$, the category of sets, via cartesian product:

$$X \xrightarrow{(x,1)} X \times \pi, \qquad X \times \pi \times \pi \xrightarrow{(x,\sigma_1\sigma_2)} X \times \pi.$$

$\underline{A}^{\pi}$ is the category of $\pi$-sets. If $T$ is any functor $\underline{A} \to \underline{A}$, there is a natural map

$$XT \times \pi \xrightarrow{\ell} (X \times \pi)T.$$

Viewing $XT \times \pi$ as a $\pi$-fold coproduct of $XT$ with itself, $\ell$ has the value

$$XT \xrightarrow{(X,\sigma)T} (X \times \pi)T$$

on the $\sigma$-th cofactor, if $(X,\sigma)$ is the map $x \to (x,\sigma)$. If $\mathbb{T} = (T,\eta,\mu)$ is a triple in $\underline{A}$, $\ell$ is a distributive law of $\pi$ over $\mathbb{T}$. The algebras over the composite triple $\pi\mathbb{T}$ are $\mathbb{T}$-algebras equipped with $\pi$-operations. The elements of $\pi$ act as $\mathbb{T}$-homomorphisms.

Example (3) can be combined with (2) to show that any triple $\mathbb{S}$ generated by constants and unary operations has a canonical distributive law over any triple $\mathbb{T}$ in $\underline{A}$. The $\mathbb{S}\mathbb{T}$-algebras are $\mathbb{T}$-linear automata.

(4) No new equations in the composite triple. It is known that if $\mathbb{T}$ is a consistent triple in sets, then the unit $X\eta^T : X \to XT$ is a monomorphism for every $X$. And every triple in sets, as a functor, preserves monomorphisms. Thus if $\mathbb{S}$, $\mathbb{T}$ are consistent triples, and $\ell : TS \to ST$ is a distributive law, then the triple maps $\mathbb{S}, \mathbb{T} \to \mathbb{S}\mathbb{T}$ are monomorphisms of functors.

This means that the operations of $\mathbb{S}$ and of $\mathbb{T}$ are mapped injectively into operations of $\mathbb{S}\mathbb{T}$, and no new equations hold among them in the composite.

The triples excluded as "inconsistent" are the terminal triple and one other:

(a) $XT = 1$ for all $X$,

(b) $oT = o$, $XT = 1$ for all $X \neq o$.

(5) Distributive laws on rings as triples in the category of abelian groups.

Let S and T be rings. S and T can be interpreted as triples $\mathbb{S}$ and $\mathbb{T}$ in the category of abelian groups, $\underline{A}$, via tensor product:

$$A \xrightarrow{a\otimes 1} A\otimes S, \quad A\otimes S\otimes S \xrightarrow{a\otimes s_0 s_1} A\otimes S.$$

$\underline{A}^{\mathbb{S}}$ and $\underline{A}^{\mathbb{T}}$ are the categories of S- and T- modules. The usual interchange map of the tensor product, $\ell : T\otimes S \to S\otimes T$, gives a distributive law of $\mathbb{S}$ over $\mathbb{T}$. This is just what is needed to make the composite $S\otimes T$ into a ring:

$$S\otimes T\otimes S\otimes T \xrightarrow{S\otimes \ell\otimes T} S\otimes S\otimes T\otimes T \xrightarrow{\text{mult.}\,\otimes\,\text{mult.}} S\otimes T.$$

This ring multiplication is the multiplication in the composite triple $\mathbb{S}\mathbb{T}$.

The interchange map is adjoint to a distributive law between the adjoint cotriples Hom(S, ), Hom(T, ). This is a general fact about adjoint triples.

The identities for a distributive law are especially easy to check in this example, as are certain conjectures about distributive laws.

Let $\underline{A}$ be the category of graded abelian groups, and let S and T be graded rings. Then two obvious transpositions of the graded tensor product, $T\otimes S \to S\otimes T$, exist :

$$t\otimes s \to s\otimes t,$$

$$t\otimes s \to (-1)^{\dim s\,.\,\dim t}\, s\otimes t$$

Both are distributive laws of the triple $\mathbb{S} = (\ )\otimes S$ over $\mathbb{T} = (\ )\otimes T$. They give different graded ring structures on $S\otimes T$ and different composite triples $\mathbb{S}\mathbb{T}$.

Finally, note the following ring multiplication in $S\otimes T$:

$$(s_0 \otimes t_0)(s_1 \otimes t_1) = (-1)^{\dim s_0\,.\,\dim t_1}\, s_0 s_1 \otimes t_0 t_1.$$

The maps

$$S \xrightarrow{s\otimes 1} S\otimes T \xleftarrow{1\otimes t} T$$

are still ring homomorphisms, but the "middle unitary" law described in §1 does not hold.

A number of problems are open, in such areas as homology, the relation of composites to tensor products of triples, and possible extension of the distributive law formalism to non-tripleable situations, for example, the following, suggested by Knus-Stammbach:

$$\begin{tikzcd} & \underline{\text{Hopf algebras}/K} \arrow[dl, shift left] \arrow[dr, shift left] & \\ \underline{\text{Groups}} \arrow[ur, shift left, "\tilde{F}_1"] \arrow[dr, shift left] & & \underline{\text{Coalgebras}}/K \arrow[ul, shift left] \arrow[dl, shift left, "U_1"] \\ & \underline{\text{Sets}} \arrow[ul, shift left] \arrow[ur, shift left, "F_1"] & \end{tikzcd}$$

($XF_1 = K(X)$, $CU_1$ = all $c \epsilon C$ such that $\Delta(c) = c \otimes c$, and $\pi\tilde{F}_1$ = the group algebra $K(\pi)$.)

5. Appendix. If there are adjoint pairs of functors

$$\underline{A} \underset{U}{\overset{F}{\rightleftarrows}} \underline{B} \underset{\tilde{U}}{\overset{\tilde{F}}{\rightleftarrows}} \underline{\tilde{B}}\ , \qquad \begin{array}{l} \eta : \underline{A} \to FU,\ \tilde{\eta} : \underline{B} \to \tilde{F}\tilde{U}, \\ \epsilon : UF \to \underline{B},\ \tilde{\epsilon} : \widetilde{UF} \to \underline{\tilde{B}}, \end{array}$$

then there is a composite adjoint pair

$$\underline{A} \underset{\tilde{U}U}{\overset{F\tilde{F}}{\rightleftarrows}} \underline{\tilde{B}}$$

whose unit and counit are

$$\underline{A} \xrightarrow{\eta(F\tilde{\eta}U)} F\tilde{F}\tilde{U}U\ ,\ \tilde{U}UF\tilde{F} \xrightarrow{(\tilde{U}\epsilon\tilde{F})\tilde{\epsilon}} \underline{\tilde{B}}\ .$$

Given adjoint functors

$$\underline{A} \underset{U}{\overset{F}{\rightleftarrows}} \underline{B}\ ,\ \underline{A} \underset{U'}{\overset{F'}{\rightleftarrows}} \underline{B}$$

then the diagrams

$$\begin{tikzcd} F' \arrow[r, "f"] \arrow[d, "\eta F'"'] & F \\ FUF' \arrow[r, "FuF'"'] & FU'F' \arrow[u, "F\epsilon'"'] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "u"] \arrow[d, "U\eta'"'] & U' \\ UF'U' \arrow[r, "UfU'"'] & UFU' \arrow[u, "\epsilon U'"'] \end{tikzcd}$$

establish a 1-1 correspondence between morphisms $u : U \to U'$, $f : F' \to F$. Corresponding morphisms are called <u>adjoint</u>.

Let

$$\begin{tikzcd}
 & \tilde{\underline{B}} \arrow[dl, shift left, "\tilde{U}_1"] \arrow[dr, shift left, "\tilde{U}_0"] & \\
\underline{B}_0 \arrow[ur, shift left, "\tilde{F}_1"] \arrow[dr, shift left, "U_0"] & & \underline{B}_1 \arrow[ul, shift left, "\tilde{F}_0"] \arrow[dl, shift left, "U_1"] \\
 & \underline{A} \arrow[ul, shift left, "F_0"] \arrow[ur, shift left, "F_1"] &
\end{tikzcd}$$

be adjoint pairs of functors, and let $u : \tilde{U}_1U_0 \to \tilde{U}_0U_1$, $f : F_1\tilde{F}_0 \to F_0\tilde{F}_1$ be adjoint natural transformations. Then $u,f$ induce a natural transformation $e$ which plays a large role in §3 :

$$\begin{tikzcd}
U_0F_1\tilde{F}_0\tilde{U}_0 \arrow[r, "U_0f\tilde{U}_0"] & U_0F_0\tilde{F}_1\tilde{U}_0 \arrow[d, "\epsilon_0\tilde{F}_1\tilde{U}_0"] \\
U_0F_1 \arrow[u, "U_0F_1\tilde{\eta}_0"] \arrow[r, "e"] \arrow[d, "\tilde{\eta}_1U_0F_1"'] & \tilde{F}_1\tilde{U}_0 \\
\tilde{F}_1\tilde{U}_1U_0F_1 \arrow[r, "\tilde{F}_1uF_1"'] & \tilde{F}_1\tilde{U}_0U_1F_1 \arrow[u, "\tilde{F}_1\tilde{U}_0\epsilon_1"']
\end{tikzcd}$$

The following diagrams commute:

$$\begin{tikzcd}
 & U_0 \arrow[dl, "U_0\eta_1"'] \arrow[dr, "\tilde{\eta}_1 U_0"] & \\
U_0F_1U_1 \arrow[r, "eU_1"'] & \tilde{F}_1\tilde{U}_0U_1 & \tilde{F}_1\tilde{U}_1U_0 \arrow[l, "\tilde{F}_1 u"]
\end{tikzcd}$$

$$\begin{tikzcd}
U_0F_0\tilde{F}_1 \arrow[dr, "\epsilon_0\tilde{F}_1"'] & U_0F_1\tilde{F}_0 \arrow[l, "U_0 f"'] \arrow[r, "e\tilde{F}_0"] & \tilde{F}_1\tilde{U}_0\tilde{F}_0 \arrow[dl, "\tilde{F}_1\tilde{\epsilon}_0"] \\
 & \tilde{F}_1 &
\end{tikzcd}$$

If in addition adjoint maps $u^{-1} : \tilde{U}_0U_1 \to \tilde{U}_1U_0$ , $f^{-1} : F_0\tilde{F}_1 \to F_1\tilde{F}_0$ are available, they induce a natural map e' with similar unit and counit properties:

$$\begin{tikzcd}
U_1F_0\tilde{F}_1\tilde{U}_1 \arrow[r, "U_1 f^{-1}\tilde{U}_1"] & U_1F_1\tilde{F}_0\tilde{U}_1 \arrow[d, "\epsilon_1\tilde{F}_0\tilde{U}_1"] \\
U_1F_0 \arrow[u, "U_1F_0\tilde{\eta}_1"] \arrow[r, "e'"] \arrow[d, "\tilde{\eta}_0U_1F_0"'] & \tilde{F}_0\tilde{U}_1 \\
\tilde{F}_0\tilde{U}_0U_1F_0 \arrow[r, "\tilde{F}_0u^{-1}F_0"'] & \tilde{F}_0\tilde{U}_1U_0F_0 \arrow[u, "\tilde{F}_0\tilde{U}_1\varepsilon_0"']
\end{tikzcd}$$