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Aperiodic tilings

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Dedicated with admiration and affection to Hillel Furstenberg on his 60th birthday

Let X be a Riemannian manifold. An *X-tiling system*, or simply a *tiling system* where X is understood, is a finite collection $\mathscr{T}$ of colored compact subsets of X called *tiles* together with a finite set of rules controlling which finite (up to a bounded size) configurations of adjacent tiles are allowed ("admissible"). We shall consider only tiles C which are homeomorphic to balls in X. A *tiling* ω is a partition (tessellation) of X of the form $X = \cup C_i$ where each C_i is an isometric copy of some tile of the set $\mathscr{T}$ so that the interior of any two of these tiles C_i are disjoint. Each of the C_i's will be colored in such a way that all the finite configurations appearing in the tiling ω are admissible.

A tiling will be called *periodic* if it factors via a tiling of a compact quotient manifold $M = \Gamma\backslash X$ of X (where $\Gamma < \mathrm{Is}(X)$ is a torsion free uniform lattice). A tiling will be called *weakly periodic* if it factors via a tiling of a proper quotient manifold $M = Z\backslash X$ of X (where $Z < \mathrm{Is}(X)$ is any nontrivial discrete torsion free subgroup of $\mathrm{Is}(X)$).

A tiling system is called *aperiodic* if it admits no periodic tilings of X. It will be called *strongly aperiodic* if it admits no weakly periodic tiling of X.

Naturally, the most extensively studied tiling systems are the Euclidean ones, i.e., when the space X is $\mathbb{R}^n$.

There has been a lot of interest in the construction of aperiodic Euclidean tiling systems. The question of their existence (for $n = 2$) was raised by Wang in the context of certain (un)decidability questions. The first example of an aperiodic Euclidean tiling system was given by Berger [Ber]. We refer to Radin [Rad] for a discussion and survey of aperiodic tiling systems. See also the works of Kenyon [Ke1,2] and Thurston [Th].

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We shall be interested mostly with the case when X is a symmetric space $X = G/K$ where G is a semisimple Lie group without compact factors, different from $SL_2(\mathbb{R})$ and $K < G$ is a maximal compact subgroup. The study of such X-tiling systems is less developed. Robinson [Rob] have discussed some undecidability questions related to tilings of the hyperbolic plane. Penrose [Pen] gave an example of a single tile which tiles H^2 only aperiodically. Block and Weinberger [BW] showed how to use a certain cohomology theory to produce aperiodic tilings of X as above (and much more generally for any "nonamenable manifold" X). One may interpret Penrose example as a special case of the type of tiling systems given by Block and Weinberger. Penrose remarks that the aperiodic tiling system he constructed for the hyperbolic plane is not strongly aperiodic.

We describe here a construction of certain X-tiling systems. In particular we obtain:

(1) an aperiodic X-tiling system for $X = G/K$ where G is a semisimple Lie group without compact factors and different from $SL_2(\mathbb{R})$. The construction is based on the use of two non-commensurable uniform lattices in G combined with Mostow rigidity.

(2) In the case of $X = G/K$ with G being either $\mathrm{Sp}(n, 1)$, $n \geqq 2$, F_4^{-20} or a noncompact simple Lie group of rank at least two we shall construct a strongly aperiodic tiling system. Here the construction is based on the use of two uniform lattices which are disjoint even after arbitrary conjugations and the rigidity results of Pierre Pansu and of Bruce Kleiner and Bernhard Leeb.

The construction

Let G be a semisimple Lie group without compact factors different form $SL_2(\mathbb{R})$. Let $\Gamma, \Lambda < G$ be two torsion free uniform lattices.

Let $\mathscr{F}$ and $\mathscr{P}$ be fundamental polytopes for Γ and Λ respectively. The tiles of the tiling system are going to be polytopes isometric to $\mathscr{F}$ each of them colored by one of finitely many colors. We may put small "bumps" on the boundary of $\mathscr{F}$ to force any tessellation of X by its copies to be isomorphic to the one obtained by $\{\gamma\mathscr{F} \colon \gamma \in \Gamma\}$.

Associated with the polytope $\mathscr{P}$ we have a finite symmetric set of generators of Λ such that each generator $\alpha \in S$ is associated with a certain face L of $\mathscr{P}$ so that $\alpha^{-1}L$ is again a face of $\mathscr{P}$. Further Λ is finitely presented and we have a finite presentation $\Lambda = \langle S | \mathfrak{R} \rangle$ where $\mathfrak{R} \subset S^*$ is a finite set of relations.

The colors: Consider the fundamental polytope $\mathscr{F}$. Fix a cover $\mathbf{C}$ of it by a finite number of balls of radius $\varepsilon > 0$. Now consider a tessellation of (part of) X by copies of $\mathscr{P}$. I.e., a tessellation corresponding to Λ. $\mathscr{F}$ is covered by some copies of $\mathscr{P}$. Each of the balls in the cover $\mathbf{C}$ will be colored white if it is contained in one of the copies of $\mathscr{P}$ and grey if it is not (i.e.. it intersects several copies of $\mathscr{P}$). Grey balls will come in several (finitely many) shades of grey, each shade of grey corresponding to the copy of the (several) pair of

faces of $\mathscr{P}$ intersected by the specific ball. Fix also in each ball of the cover a finite net of dots. For a shade of grey corresponding to a pair of faces L_1 and L_2 we shall mark the dots of the fixed net by L_1 or L_2 according to whether they lie in the interior of the copy of the polytope whose boundary intersected with the ball is contained in L_1 or L_2. In case a dot is on the boundary of a polytope we leave it unmarked. The net has to be rich enough so that always some dots are marked. This induces also a coloring of $\mathscr{F}$ and there are only finitely many possible colors we obtain this way.
Construction of a prototiling: Let $\mathscr{T} = \{\gamma\mathscr{F}: \gamma \in \Gamma\}$ be a tessellation of X by copies of $\mathscr{F}$. Let $\mathscr{S} = \{\alpha\mathscr{P}: \alpha \in \Lambda\}$ be a tessellation of X by copies of $\mathscr{P}$. Color each of the tiles $\gamma\mathscr{F}$ by one of the above colors with respect to the tessellation $\mathscr{S}$. Denote by $\mathscr{T}_{\mathscr{S}}$ the resulting tiling of the space X. We shall use the "prototiling" $\mathscr{T}_{\mathscr{S}}$ to define our tiling system. Let $R > 0$ be fixed (we shall specify how to choose R later). A tessellation of X by colored copies of $\mathscr{F}$ is admissible if for every ball of radius R the configuration of colored tiles in it appears somewhere in $\mathscr{T}_{\mathscr{S}}$. Clearly for any fixed R we obtain this way a tiling system. Denote by Ω both the tiling system and the collection of all admissible tilings. Clearly Ω is non empty.

We shall show that certain properties of the pair of lattices Γ and Λ give rise to some aperiodicity properties of the resulting tiling system.

A complex and a graph

Consider a tiling ω of X ("tiling" always means "admissible tiling"). Looking at it we see white islands (connected components of the union of the white balls) surrounded by a connected grey region. Each white component may be embedded in a copy of the polytope $\mathscr{P}$ so that the neighbouring grey balls intersect its boundary in the appropriate faces. Note that the possible such copies of $\mathscr{P}$ are necessarily located very close to one another, the maximal distance between them is controlled by the original ε we choose. Observe that, for a sufficiently small fixed $\varepsilon > 0$, the notion of two white islands being adjacent is well defined. Moreover the adjacency is via a well defined face of $\mathscr{P}$. Recall that each face of $\mathscr{P}$ corresponds to a generator $\alpha \in S$ of Λ.

Consider the polytope $\mathscr{P}$ as a geometrical realization of a combinatorial complex $\mathscr{P}$. We may define a structure of a cell complex on X by choosing for each white island a map (not necessarily an isometry!) from $\mathscr{P}$ onto a small neighbourhood of the island in such a way that the maps chosen for adjacent islands are compatible and the two are adjacent via the image of the appropriate face. Denote the resulting complex by $\mathscr{C}_\omega$.

Note that by choosing above a sufficiently large R the resulting complex has locally, i.e. in a combinatorial ball of radius R_0 (for a fixed R_0 to be determined later) the same combinatorial structure as the complex $\mathscr{C}_{st}$ of the tessellation $\{\alpha\mathscr{P}: \alpha \in \Lambda\}$. Associated with the resulting complex we have also a graph, denoted by $\mathscr{Y}_\omega$ whose vertices corresponds to the islands – equivalently to the cells of the complex – and two adjacent islands/cells are connected by an edge. The edges being labelled each by the generator from S associated with the corresponding face of the cell. This graph is naturally an oriented one and we have each edge appearing with both orientations with labels α and α^{-1} respectively.

Lemma 1. *For any tiling $\omega \in \Omega$ the graph $\mathscr{Y} = \mathscr{Y}_\omega$ defined above is isomorphic to the Cayley graph $\mathscr{X} = \mathscr{X}(\Lambda, S)$ of Λ with respect to S.*

Proof. Let $\omega \in \Omega$ be an admissible tiling and let $\mathscr{Y} = \mathscr{Y}_\omega$ denote the associated graph defined above. Both $\mathscr{X}$ and $\mathscr{Y}$ are connected $|S|$-regular graphs. The outgoing edges at each vertex are in one to one correspondence with the set of generators S. Fix a vertex $y_0 \in \mathscr{Y}$ and the vertex e of $\mathscr{X}$. Define a map $\varphi : \mathscr{X} \to \mathscr{Y}$ by mapping e to y_0 and then extending the map in a label preserving way. Namely for a vertex $x \in \mathscr{X}$ consider a path $f_1, f_2, \ldots, f_n$ from e to x and let the edge f_i be labelled by a generator $s_i \in S$. We have a unique path $g_1, g_2, \ldots, g_n$ in $\mathscr{Y}$ starting at y_0 such that each edge g_i is labelled by s_i. Define $\varphi(x)$ to be the end vertex of g_n. One has to check that this is a well defined map, i.e. that it does not depend on the choice of the path $f_1, f_2, \ldots, f_n$. Since $\Lambda = \langle S | \mathfrak{R} \rangle$ is finitely presented it follows that every closed path in the Cayley graph $\mathscr{X} = \mathscr{X}(\Lambda, S)$ may be tessellated by closed paths corresponding to elements (relations) of $\mathfrak{R}$. As $\mathfrak{R}$ is finite we may choose the constant R_0 above so that for each relation in $\mathfrak{R}$ the corresponding path from any vertex of $\mathscr{Y}$ "lives" in a ball of radius at most R_0. Hence, since $\mathscr{Y}$ and $\mathscr{X}$ are locally isomorphic, the corresponding path is closed. Thus for any closed path $f_1, f_2, \ldots, f_m$ in $\mathscr{X} = \mathscr{X}(\Lambda, S)$ the corresponding path in $\mathscr{Y}$ is also closed and $\varphi : \mathscr{X} \to \mathscr{Y}$ is a well defined map. Next we define in a similar way a map $\psi : \mathscr{Y} \to \mathscr{X}$. We have to verify that ψ is indeed well defined. Let $\mathfrak{g} = (g_1, g_2, \ldots, g_m)$ be a closed path in $\mathscr{Y}$, the edge g_i being labelled by $s_i \in S$. Let $\mathfrak{f} = (f_1, f_2, \ldots, f_m)$ be the corresponding path starting at e on $\mathscr{X}$ with each f_i being labelled by s_i. Since the graph Y is embedded in the simply connected space X and $\mathfrak{g}$ is a closed loop in X we have a disk $D \subset X$ whose boundary is exactly $\mathfrak{g}$. Recall the structure of a complex on X given by $\mathscr{C}_\omega$. The disk D intersects a finite number of cells of this complex. Denote the subcomplex made of these cells by $\mathscr{C}'$. The intersection of the disk D with the various cells of $\mathscr{C}'$ partition D into finitely many domains, each domain being a connected component of the intersection of D with some cell of $\mathscr{C}'$. Let r be the number of these domains. We argue that the path $\mathfrak{f}$ is a closed path in $\mathscr{X}$ by induction on r. For small enough r the complex $\mathscr{C}'$ has combinatorial diameter smaller than R_0 and the assertion follows from the local isomorphism of the combinatorial structure of the complex $\mathscr{C}_\omega$ with that of $\mathscr{C}_{\mathrm{st}}$. For larger r we may choose a path in D connecting two points on its boundary which partition it into two parts each having smaller number of domains. Using this path we can represent the path $\mathfrak{g}$ as the "sum" of two paths (with some common part which cancels). Each of the two paths bounds a disk which is tessellated by less than r domains and hence we may apply induction.

This implies that the map ψ is indeed well defined and clearly we have $\psi \circ \varphi = \mathrm{id}$, $\varphi \circ \psi = \mathrm{id}$. □

Note that the proof combined with the fact that the complexes $\mathscr{C}_{\mathrm{st}}$ and $\mathscr{C}_\omega$ are locally the same actually shows that:

Corollary 1. *For any tiling $\omega \in \Omega$ the complexes $\mathscr{C}_{\mathrm{st}}$ and $\mathscr{C}_\omega$ are isomorphic.*

Aperiodic tiling systems

Theorem 1. *Let X be the symmetric space associated with a semisimple Lie group G which has no compact factor and different from $\mathrm{SL}_2(\mathbb{R})$. Let the pair of lattices Γ and Λ used for the construction described above be irreducible and such that for no $g \in G$ the lattices $g^{-1}\Gamma g$ and Λ are commensurable. The tiling system Ω admits only aperiodic tilings of X.*

Proof. Suppose that Ω had a periodic tiling $\omega \in \Omega$ of X. I.e., for some finite index subgroup Γ' of Γ we would have a (finite) tiling ω' of the compact manifold $M = \Gamma' \backslash X$ such that its lift to a tiling of X via the natural covering map $\pi : X \to M$ is ω. Note that in this case we obtain an action of $\Gamma' = \pi_1(M)$ on the graph $\mathscr{Y}_\omega$ as a group of automorphisms preserving the labelling of the edges by the elements of S. Since by Lemma 1 $\mathscr{Y}_\omega$ is isomorphic to the Cayley graph $\mathscr{X} = \mathscr{X}(\Lambda, S)$ whose group of label preserving automorphisms is exactly Λ we obtain an injective homomorphism $\tau : \Gamma' \to \Lambda$. Moreover note that since M is compact it follows that the image $\Lambda' = \tau(\Gamma')$ is a finite index subgroup of Λ. However applying Mostow rigidity we conclude that the isomorphism τ is given by some conjugation $\tau(\gamma) = g^{-1}\gamma g$ for some $g \in G$. This however contradicts the assumption that Γ and Λ are not commensurable even after conjugation. □

Strongly aperiodic tiling systems

We turn now to show how to obtain via the above construction a strongly aperiodic tiling system. We shall assume throughout this section that the group G is one of the following possibilities:

(1) $\mathrm{Sp}(n, 1)$, $n \geqq 2$ (here X is a quaternionic hyperbolic space $\mathbb{H}\mathrm{H}^n$).
(2) F_4^{-20} (here X is the Cayley hyperbolic plane $\mathbb{C}\mathrm{aH}^2$).
(3) a simple noncompact Lie group of rank at least two.

Let $\Gamma, \Delta < G$ be two uniform lattices in G. Let Ω be a tiling system constructed for the given pair of lattices as above.

We shall need the following lemma:

Lemma 2. *Corresponding to any tiling $\omega \in \Omega$ there is a map $\psi_\omega : X \to X$ such that*

(1) *ψ_ω is a homeomorphism and a quasiisometry.*

(2) *When we view X in the domain of ψ_ω as the geometric realization of the complex $\mathscr{C}_{\mathrm{st}}$ and the target X as the geometric realization of $\mathscr{C}_\omega$ then the map ψ_ω induces an isomorphism of the complexes $\mathscr{C}_{\mathrm{st}}$ and $\mathscr{C}_\omega$.*

(3) *There exists a unique isometry $\phi_\omega : X \to X$ which is at bounded distance from ψ_ω. I.e., for some constant C we have for all $x \in X$*

$$\mathrm{d}_X(\psi_\omega(x), \phi_\omega(x)) < C$$

where d_X denotes the metric on X.

Proof. By Corollary 1 we have a map $\tilde{\psi}_\omega : \mathscr{C}_{\mathrm{st}} \to \mathscr{C}_\omega$ which identifies the two complexes. It is now clear that we can find a geometric realization $\psi_\omega : X \to X$ of $\tilde{\psi}_\omega : \mathscr{C}_{\mathrm{st}} \to \mathscr{C}_\omega$ which is a homeomorphism giving the required quasi-isometry. This proves assertions 1 and 2. Assertion 3 follows from 1, using the deep results of Pansu [Pa] and of Kleiner and Leeb [KL] (see also Eskin and Farb [EF]) which assert that for the symmetric spaces considered in this subsection every quasiisometry is at a bounded distance from an isometry. The uniqueness of ϕ_ω is clear. □

Applying this lemma together with Corollary 1 we obtain:

Proposition 1. *Let $\omega \in \Omega$ be a tiling which is the covering of a tiling ω' of a proper quotient M of X. M is of the form $M = D\backslash X$ where D is a subgroup of (a conjugate of) Γ. Then for some $h \in \mathrm{Is}(X)$ we have $h^{-1}Dh < \Lambda$.*

Proof. Consider the complex $\mathscr{C}_\omega$. The subgroup $D < \Gamma$ acts on this complex as a group of automorphisms preserving the complex structure as well as the "colors" or "types" of the faces of the various cells. As Λ is exactly the group of type preserving automorphisms of the (combinatorial) complex $\mathscr{C}_{\mathrm{st}}$ which is isomorphic to $\mathscr{C}_\omega$ it follows that D is naturally isomorphic to a subgroup D' of Λ, denote by $f : D \to D'$ this isomorphism. Moreover, observe that the quasiisometry ψ_ω intertwines the action of D on $\mathscr{C}_\omega$ with the action of $D' = f(D)$ on the isomorphic complex $\mathscr{C}_{\mathrm{st}}$. By Lemma 2 (3) we have an isometry $\phi = \phi_\omega \in \mathrm{Is}(X)$ which is at bounded distance of ψ_ω. Applying the map ϕ to the tiling ω we obtain a new tiling which we shall, by abuse of notation, continue to denote by ω. Hence we shall also use D to denote the old subgroup D conjugated by the isometry ϕ. After this "renaming" we shall have a new quasiisometry ψ_ω (the old one composed with the inverse of ϕ) which is at bounded distance from the identity, i.e., for every $x \in X$ we have $\mathrm{d}_X(\psi_\omega(x), x) < C$ for some constant $C > 0$. As before this quasiisometry ψ_ω intertwines the action of D with that of $D' < \Lambda$ (we shall continue to denote by $f : D \to D'$ the canonical isomorphism). We claim that now the map f is actually the identity. Indeed, let $d \in D$ and $d' = f(d) \in D'$. Consider the isometry $g = d^{-1}d'$. Since the map ψ_ω intertwines the actions of d on $\mathscr{C}_\omega$ and the action of d' on $\mathscr{C}_{\mathrm{st}}$ and since ψ_ω is at bounded distance from the identity it follows that for every $x \in X$ we have $\mathrm{d}_X(x, g(x)) < 2C$. This however implies that $g = \mathrm{id}$ and hence that $d = f(d)$. □

This proposition immediately implies:

Theorem 2. *Let X be a symmetric space associated with a group G which is either* $\mathrm{Sp}(n,1)$, $n \geqq 2$, F_4^{-20} *or a simple noncompact Lie group of rank at least two. Let the pair of lattices Γ and Λ used for the construction described above be such that for no $g \in \mathrm{Is}(X)$ the lattices $g^{-1}\Gamma g$ and Λ intersect nontrivially. Then the tiling system Ω is strongly aperiodic.* □

Pairs of uniform lattices which do not share nontrivial elements belonging to the same conjugacy class do exist. We illustrate here a construction of such pairs of lattices in $\mathrm{SL}_n(\mathbb{R})$. Similar constructions yield such pairs also for orthogonal and symplectic groups. Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field and $\mathcal{O}_K \subset K$ be its ring of integers. Let $\underline{H}$ be the algebraic group defined over K defined by:

$$\underline{H} = \{(A,B)\colon A,B \in M_n,\ AA^{\mathrm{tr}} - \sqrt{d}BB^{\mathrm{tr}} = I,\ BA^{\mathrm{tr}} - AB^{\mathrm{tr}} = 0\}$$

(alternatively, one may write the equations defining $\underline{H}$ as $(A+\eta B)(A^{\mathrm{tr}} - \eta B^{\mathrm{tr}}) = I$ where $\eta^2 = \sqrt{d}$, this identification gives also the multiplication in $\underline{H}$). Let σ be the nontrivial Galois automorphism of K. Note that $\underline{H}(\mathbb{R})$ is isomorphic to $\mathrm{SL}_n(R)$, whereas $\underline{H}^\sigma(\mathbb{R})$ is isomorphic to $\mathrm{SU}(n)$. ($\underline{H}^\sigma$ corresponds to $(A+\tilde{\eta}B)(A^{\mathrm{tr}} - \tilde{\eta}B^{\mathrm{tr}}) = I$ where $\tilde{\eta}^2 = \sigma(\sqrt{d}) = -\sqrt{d}$). Thus $\underline{H}(\mathcal{O}_K)$ is a uniform lattice in $\mathrm{SL}_n(\mathbb{R})$. Let $\Gamma < \underline{H}(\mathcal{O}_K)$ be a finite index torsion free subgroup. Choosing a different real quadratic extension L of $\mathbb{Q}$ we may construct in a similar way a torsion free uniform lattice Λ in $\mathrm{SL}_n(\mathbb{R})$. We claim that there is no $\gamma \in \Gamma$ different from the identity e which is conjugate to an element of Λ. Indeed consider the characteristic polynomial $p(x)$ of an element $\gamma \in \Gamma$. It is a standard polynomial over $\mathcal{O}_K$. Applying the Galois automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ to γ we obtain an element $\sigma(\gamma)$ which lies in a compact group $\underline{H}^\sigma(\mathbb{R})$ isomorphic to $\mathrm{SU}(n)$. If the element γ were conjugate to an element of the lattice Λ then the coefficients of its characteristic polynomial were in $\mathbb{Z} = \mathcal{O}_K \cap \mathcal{O}_L$. This however would imply that both γ and $\sigma(\gamma)$ have the same characteristic polynomial. In particular we would have that $p(x) \in \mathbb{Z}[x]$ is a standard polynomial such that all of its roots are on the unit circle. Hence all its roots are roots of unity implying that γ is a torsion element which is impossible (note that the lattice Γ being a uniform lattice contains no unipotent elements).

Tiling affine buildings

Let X be the affine building associated with a semisimple Chevalley group G over a local field. See [Bro], [Ron] for a general introduction to buildings. An X-tiling system is defined in exactly the same way as for a manifold X. Given a pair of torsion free uniform lattices $\Gamma, \Delta < G$ one can imitate the construction carried out above for X being the symmetric space of a semisimple real Lie group for the present case. It turns out however that the situation here is actually simpler. Observe that one may choose the fundamental domains $\mathcal{F}$ and $\mathcal{P}$ for Γ and Δ to be finite complexes made of cells of the building. Associate with each of the finitely many faces of $\mathcal{P}$ a different color and color the $1/100$ neighbourhood of each face in the interior of $\mathcal{P}$ by the corresponding color (where the metric is scaled so that the minimal distance between vertices of the affine building X is 1). Use some other finite set of colors to color also the interior of the finite complex $\mathcal{F}$ in an analogous way. Now consider a fixed tessellation of the building X by Γ translates of $\mathcal{F}$ and "draw" on it also a tessellation of the building by Δ translates of $\mathcal{P}$. This is our "prototiling". Unlike the case of a symmetric space X, in the present case the "painted" copies of $\mathcal{F}$ we obtain this way belong to a finite set – the set of tiles.

Consider the tessellation of X by the Δ translates of $\mathcal{P}$. Note that each face of a translate $g_0\mathcal{P}$ may be a face of several (finitely many) other translates $g\mathcal{P}$, $g \in \Delta$. To any (ordered) pair of adjacent Δ translates of $\mathcal{P}$ corresponds an element of a finite set of generators of Δ. Moreover this generator is determined by the colors of the $1/100$-neighbourhoods of the common face in the corresponding translates.

The set of adjacency rules is defined in the same way as above: Choose some fixed radius R large enough to capture a set of defining relations of Δ as well as a set of defining relations for Γ. The set of allowed configurations is those which in every ball of radius R agree with a configuration appearing in our prototiling.

Theorem 3. *Let X be the affine building associated with a simple Chevalley group G over a local field having rank greater than* 1. *Let $\Gamma, \Delta < G$ be uniform lattices and Ω a tiling system constructed as above for them. For a tiling $\omega \in \Omega$ let $G_\omega = \{g \in G : g\omega = \omega\}$. Then for some $h \in \mathrm{Is}(X)$ we have $G_\omega = \Gamma \cap h^{-1}\Delta h$.*

Proof. Observe that the analogoues of Lemmas 1,2 and Corollary 1 hold also when X is an affine building rather than a symmetric space. Observe that here the assertion of Lemma 2 may be strengthened and the map $\psi_\omega : X \to X$ associated with a tiling $\omega \in \Omega$ is actually an isometry. It follows as in the proof of Proposition 1 that $G_\omega = \Gamma \cap h^{-1}\Delta h$ where we let $h = \psi_\omega^{-1}$. □

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