Improved upper and lower bounds for Büchi Disambiguation

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Abstract. We present a new ranking based construction for disambiguating non-deterministic Büchi automata and show that the state complexity tradeoff of the translation is in $O(n \cdot (0.76n)^n)$. This exponentially improves the best upper bound (i.e., $4 \cdot (3n)^n$) known earlier for Büchi disambiguation. We also show that the state complexity tradeoff of translating non-deterministic Büchi automata to strongly unambiguous Büchi automata is in $\Omega((n-1)!)$. This exponentially improves the previously known lower bound (i.e. $\Omega(2^n)$). Finally, we present a new technique for proving the already known exponential lower bound of disambiguating automata over finite or infinite words. Our technique is significantly simpler than earlier techniques based on ranks of matrices used for proving disambiguation lower bounds.

1 Introduction

Unambiguous Büchi automata represent an interesting class of automata that are structurally situated between deterministic and non-deterministic Büchi automata, and yet are as expressive as non-deterministic Büchi automata. For notational convenience, we use UBA (resp., NBA) to denote the class of unambiguous (respectively non-deterministic) Büchi automata over words in the rest of this paper. The expressive equivalence of UBA and NBA was first shown by Arnold [1], and later re-proven by Carton and Michel [3] and Kahler and Wilke [6]. The class of automata studied by Carton and Michel have also been called *prophetic* automata by others [4]. Bousquet and Löding [2] showed that language equivalence and inclusion checking can be achieved in polynomial time for a special sub-class of UBA, called strongly unambiguous Büchi automata (or SUBA), which is expressively equivalent to NBA. In later work [5], two other incomparable sub-classes of UBA were also shown to admit polynomial-time language inclusion and equivalence checking. In a recent work, Preugschat and Wilke [10] have described a framework for characterizing fragments of linear temporal logic (LTL). Their characterization relies heavily on the use of prophetic automata and special Ehrenfeucht-Fraïssé games.

Despite a long history of studies on UBA (including several papers in recent years), important questions about disambiguation still remain open. Notable among these are the exact state complexity tradeoffs in translating NBA to language-equivalent UBA or SUBA. The state complexity tradeoff question asks

"Given an n-state NBA, how many states must a language-equivalent UBA (resp., SUBA) have as a function of n?" The present work attempts to address these questions, and makes the following contributions.

1. We show that the NBA to UBA state complexity tradeoff is in $\mathcal{O}(n \cdot (0.76n)^n)$. This is exponentially more succinct than the previously known best tradeoff of $4(3n)^n$ due to Kahler and Wilke [6]. The improved upper-bound is obtained by extending Kupferman and Vardi's ranking function based techniques [8] to the construction of unambiguous automata.
2. We show that the NBA to SUBA state complexity tradeoff is in $\Omega((n-1)!)$. This is exponentially larger than the previously known best lower bound of $\Omega(2^n - 1)$ due to Schmidt [12]. The improved lower bound is obtained by a full-automaton technique [13].
3. We present a new technique for proving the already known $2^n - 1$ NBA to UBA state complexity tradeoff. Our proof generalizes to all common notions of acceptance for finite and infinite words, and is conceptually simpler than the earlier proof based on ranks of matrices, due to Schmidt [12].

2 Notation and Preliminaries

An NBA is a 5-tuple $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$, where Σ is a finite alphabet, Q is a finite set of states, $Q_0 \subseteq Q$ is the set of initial states, $\delta : Q \times \Sigma \to 2^Q$ is the state transition relation and $F \subseteq Q$ is a set of accepting or final states. For notational convenience, we often use (with abuse of notation) $\delta(S, a)$ to denote $\bigcup_{q_j \in S} \delta(q_j, a)$ for $S \subseteq Q$. Given a word $\alpha \in \Sigma^\omega$ (also called an ω-word), let $\alpha(j)$ denote the j^{th} letter of α. By convention, we say that $\alpha(0)$ is the first letter of α. A run ρ of $\mathcal{A}$ over α is an infinite sequence of states $q_0 q_1 q_2 \dots$ such that $q_{i+1} \in \delta(q_i, \alpha(i))$ and $q_i \in Q$ for all $i \geq 0$. Given a run ρ of $\mathcal{A}$ over α, let $\rho(j)$ denote the j^{th} state along ρ. The set $\mathit{inf}(\rho)$ is the set of states of $\mathcal{A}$ that appear infinitely often along ρ. A run ρ is called *final* if $\mathit{inf}(\rho) \cap F \neq \emptyset$; it is called *accepting* if it is final and $q_0 \in Q_0$. An ω-word α is said to be accepted by NBA $\mathcal{A}$ iff there is an accepting run of $\mathcal{A}$ on α. The set of ω-words accepted by $\mathcal{A}$ is the language of $\mathcal{A}$ and is denoted $L(\mathcal{A})$. A state q_s of an NBA is a *principal sink* if the following conditions hold: (i) q_s is non-final, (ii) every state, including q_s, has an outgoing transition on every $a \in \Sigma$ to q_s, and (iii) q_s has no outgoing transitions to any state other than q_s. It is easy to see that every NBA can be converted to a language-equivalent NBA with a principal sink by adding at most one state. Unless otherwise stated, we assume that all NBAs considered in this paper have a principal sink.

Definition 1 (UBA). *A unambiguous Büchi automaton (UBA) $\mathcal{U} = (\Sigma, Q, Q_0, \delta, F)$ is an NBA such that there is at most one accepting run for every $\alpha \in \Sigma^\omega$.*

The *run-DAG* of NBA $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$ over $\alpha \in \Sigma^\omega$ is a directed acyclic graph $G^{\mathcal{A}}_\alpha = (V, E)$ with vertex set $V = \{(q, i) \mid q \in Q, i \in \mathbb{N}\}$ and edge set $E = \{((q, i), (q', i+1)) \mid (q, i) \in V, (q', i+1) \in V, q' \in \delta(q, \alpha(i))\}$. For every $i \in \mathbb{N}$, the set of vertices $\{(q,i) \mid q \in Q\}$ constitutes *level* i of $G_\alpha^\mathcal{A}$. A vertex (q,i) of $G_\alpha^\mathcal{A}$ is called an F-vertex if $q \in F$. Given a vertex (q,i) of $G_\alpha^\mathcal{A}$, a vertex (q,j) is called a *successor* of (q,i) if there is a directed path in $G_\alpha^\mathcal{A}$ from (q,i) to (q,j). The vertex (q,j) is an immediate successor of (q,i) if it is a successor and $j = i+1$. For every natural number $n \geq 1$, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$, $[n]^{odd}$ (resp., $[n]^{even}$) to denote the set of odd (resp., even) integers in $[n]$, and $\langle n \rangle$ to denote the set $[n] \cup \{\infty\}$, where $\infty > j$ for all $j \in [n]$.

2.1 Full rankings

In [8], Kupferman and Vardi showed that given an NBA $\mathcal{A}$ with n states and a word $\alpha \in \Sigma^\omega$, there exists a family of ranking functions that assign ranks in $[2n]$ to the vertices of $G_\alpha^\mathcal{A}$ such that $\alpha \notin L(\mathcal{A})$ iff all runs of $\mathcal{A}$ on α that start from its initial states get trapped in odd ranks. They also described a unique ranking function, henceforth called $\mathsf{r}^{\mathsf{KV}}_{\mathcal{A},\alpha}$, that serves this purpose. Extending their idea, we define a *full-ranking* of $G_\alpha^\mathcal{A} = (V, E)$ to be a function $r : V \to \langle 2n \rangle$ that satisfies the following conditions: (i) for every $(q,i) \in V$, if $r((q,i)) \in [2n]^{odd}$ then $q \notin F$, (ii) for every edge $((q,i),(q',i+1)) \in E$, $r((q',i+1)) \leq r((q,i))$, and (iii) every infinite path in $G_\alpha^\mathcal{A}$ eventually gets trapped in a rank in $\{\infty\} \cup [2n]^{odd}$, with at least one path trapped in ∞ iff $w \in L(\mathcal{A})$. The remainder of the discussion in this section closely parallels that in [8, 7], where ranking based complementation techniques for NBA were described. Nevertheless, we present the definitions and arguments in their entirety below to render the paper self-contained.

For every $\alpha \in \Sigma^\omega$, we define a unique full-ranking, $\mathsf{r}^\star_{\mathcal{A},\alpha}$, of $G_\alpha^\mathcal{A}$ along the same lines as the definition of $\mathsf{r}^{\mathsf{KV}}_{\mathcal{A},\alpha}$ in [8, 7]. Specifically, we define a sequence of DAGs $G_0 \supseteq G_1 \supseteq \ldots$, where $G_0 = G_\alpha^\mathcal{A}$. A vertex v is *finite* in G_i if there are no infinite paths in G_i starting from v, while v is *F-free* in G_i if it is not finite and there is no F-vertex (q,l) that is reachable from v in G_i. The DAGs G_i are now inductively defined as follows.

- For every $i \geq 0$, $G_{2i+1} = G_{2i} \setminus \{(q,l) \mid (q,l) \text{ is finite in } G_{2i}\}$.
- For every $i \geq 0$, if G_{2i+1} has at least one F-free vertex, then $G_{2i+2} = G_{2i+1} \setminus \{(q,l) \mid (q,l) \text{ is F-free in } G_{2i+1}\}$. Otherwise, G_{2i+2} is the empty DAG.

A full-ranking function $\mathsf{r}^\star_{\mathcal{A},\alpha}$ for $G_\alpha^\mathcal{A}$ can now be defined as follows. For every $i \geq 0$, we assign

- $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = 2i$, for every vertex (q,l) that is finite in G_{2i}
- If G_{2i+1} has at least one F-free vertex, then $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = 2i+1$ for every vertex (q,l) that is F-free in G_{2i+1}. Otherwise, $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = \infty$ for every vertex (q,l) in G_{2i+1}.

Using the same arguments as used in [8], it can be shown that if $\mathcal{A}$ has n states, the maximum finite (i.e., non-∞) rank in the range of $\mathsf{r}^\star_{\mathcal{A},\alpha}$ is in $[2n]$. For notational convenience, we will henceforth refer to the above technique for assigning full-ranks to vertices of a run-DAG as $\mathsf{FullRankProc}$.

Analogous to the concept of a *level-ranking* defined in [8], we define a *full-level ranking* as a function $f : Q \rightarrow \langle 2n \rangle \cup \{\bot\}$, such that for every $q \in Q$, if $f(q) \in [2n]^{odd}$, then $q \notin F$. Let FL represent the set of all full-level rankings and $\mathsf{FL}_\infty \subsetneq \mathsf{FL}$ represent the set of all full-level rankings such that $f \in \mathsf{FL}_\infty$ iff $f(q) = \infty$ for some $q \in Q$. Given two full-level rankings g_1 and g_2, and a letter $a \in \Sigma$, g_2 is said to be a *full-cover* of (g_1, a) if for all $q \in Q$ such that $g_1(q) \neq \bot$ and for all $q' \in \delta(q, a)$, $g_2(q') \leq g_1(q)$. A full-ranking r of $G_\alpha^{\mathcal{A}}$ induces a full-level ranking for every level $l \geq 0$ of $G_\alpha^{\mathcal{A}}$ such that all states not in level l of $G_\alpha^{\mathcal{A}}$ are assigned rank $\bot$. It is easy to see that if g and g' are full-level rankings induced by a full-ranking r for levels l and $l+1$ respectively, then g' is a full-cover of $(g, \alpha(l))$. Let $max\_odd(g)$ (resp., $max\_rank(g)$) denote the highest odd rank (resp., highest rank) in the range of full-level ranking g. A full-level ranking g is said to be *tight* if the following conditions hold: (i) $max\_rank(g)$ is in $[2n]^{odd} \cup \infty$, and (ii) for all $i \in [2n]^{odd}$ such that $i \leq max\_rank(g)$, there is a state $q \in Q$ with $g(q) = i$.

The ranking $\mathsf{r}^\star_{\mathcal{A},\alpha}$ has a number of interesting properties that characterize it. These are described in Lemma 1.

Lemma 1. *Let $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$ be an NBA, and $\alpha \in \Sigma^\omega$. Let (q, l) be a vertex in $G_\alpha^{\mathcal{A}}$. For every $l \in \mathbb{N}$ and $q \in Q$, we have the following.*

1. *There exists a level $l^* > 0$, such that all full-level rankings induced by $\mathsf{r}^\star_{\mathcal{A},\alpha}$ for levels $l > l^*$ are tight.*
2. *If (q, l) is not an F-vertex or $\mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) = \infty$, there exists a $q' \in \delta(q, \alpha(l))$ such that $\mathsf{r}^\star_{\mathcal{A},\alpha}((q', l+1)) = \mathsf{r}^\star_{\mathcal{A},\alpha}((q, l))$.*
3. *If (q, l) is an F-vertex with rank $\mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) \in [2n]^{even}$ then there exists a vertex $(q', l+1)$ such that $q' \in \delta(q, \alpha(l))$ and either $\mathsf{r}^\star_{\mathcal{A},\alpha}((q', l+1)) = \mathsf{r}^\star_{\mathcal{A},\alpha}((q, l))$ or $\mathsf{r}^\star_{\mathcal{A},\alpha}((q', l+1)) = \mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) - 1$.*
4. *If $\mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) \neq \infty$, there is no $q' \in \delta(q, \alpha(l))$ such that $\mathsf{r}^\star_{\mathcal{A},\alpha}((q', l+1)) = \infty$.*
5. *If $\mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) \in [2n]^{even}$, every path starting from (q, l) in $G_\alpha^{\mathcal{A}}$ eventually visits a vertex (q', l') such that $1 \leq \mathsf{r}^\star_{\mathcal{A},\alpha}((q', l')) < \mathsf{r}^\star_{\mathcal{A},\alpha}((q, l))$.*
6. *If $\mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) \in [2n]^{odd}$ and $\mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) > 1$, there exists a (q', l') such that (q', l') is an F-vertex reachable from (q, l) in $G_\alpha^{\mathcal{A}}$, and $\mathsf{r}^\star_{\mathcal{A},\alpha}((q', l')) = \mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) - 1$.*
7. *If $\mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) = \infty$, there exists a (q', l') such that (q', l') is an F-vertex reachable from (q, l) in $G_\alpha^{\mathcal{A}}$, and $\mathsf{r}^\star_{\mathcal{A},\alpha}((q', l')) = \infty$.*

Proof. 1. Consider the sequence of DAGs $G_\alpha^{\mathcal{A}} = G_0 = G_1 \supseteq G_2 \supseteq \ldots$ used in $\mathsf{FullRankProc}$ to assign the ranking $\mathsf{r}^\star_{\mathcal{A},\alpha}$ to $G_\alpha^{\mathcal{A}}$. Clearly, if some G_j, $j > 0$ is non-empty then so is every graph $G_{j'}$, for $j' < j$. Let $k = max\_rank(\mathsf{r}^\star_{\mathcal{A},\alpha})$. We will first show that there exists $l^x \geq 0$ such that for all levels $l \geq l^x$ and for all $j \in [2n]^{odd}$ such that $j \leq k$, there is a vertex (q, l) at level l with $\mathsf{r}^\star_{\mathcal{A},\alpha}((q, l)) = j$. We consider two sub-cases below.

(a) $k \in [2n]$: In order for $\mathsf{FullRankProc}$ to assign rank k to any vertex in $G_\alpha^{\mathcal{A}}$, DAG G_k must be non-empty. Hence, by the argument above, G_j is non-empty for $0 \leq j \leq k$. Also, since k is finite every such G_j for j odd must contain an F-free vertex and G_j when j is even must contain a finite vertex. Let $j = 2i+1 \in [2n]^{odd}$ such that $2i+1 \leq k$ (with equality possible only if k is odd). Hence, the graph G_{2i+1} contains at least one F-free vertex, say (q_{2i+1}, l_{2i+1}) When assigning ranks, $\mathsf{FullRankProc}$ therefore assigns rank $2i+1$ to all vertices reachable from (q_{2i+1}, l_{2i+1}) in G_{2i+1}. Since G_{2i+1} is obtained by removing all finite vertices from G_{2i}, all paths in G_{2i+1} are infinite. Therefore, there is at least one vertex reachable from (q_{2i+1}, l_{2i+1}) at every level $l \geq l_{2i+1}$ of G_{2i+1}, which is ranked $2i+1$ by $\mathsf{FullRankProc}$. It follows that for all levels $l \geq \max_{2i+1<k} l_{2i+1}$ and for all $j \in [2n]^{odd}$ such that $j < k$, there is a vertex at level l with rank j. We therefore choose l^x to be the largest of all l_{2i+1} levels, where $2i+1 \leq k$, with equality possible only of k is odd.

(b) $k = \infty$: Hence, there must be a graph G_{2i+1}, for some $i > 0$ such that G_{2i+1} does not contain either finite or F-free vertices. We can now repeat the arguments given for the case when $k \in [2n]$, by considering $k = 2i$. This again gives us the required level l^x.

Suppose, $max\_rank(\mathsf{r}^\star_{\mathcal{A},\alpha}) = k$, where $k \in \infty \cup [2n]^{odd}$ then let $l^* = l^x$. Then all full-level rankings f_l for every level $l > l_{lim}$ satisfy the following - (i) $max\_rank(f_l) = \infty \cup [2n]^{odd}$ (ii) for all $j \in [2n]^{odd}$ such that $j \leq k$, there is a state q such that $f_l(q) = j$. In other words, f_l is a tight full-level ranking. Suppose, $max\_rank(\mathsf{r}^\star_{\mathcal{A},\alpha}) = k$, where $k \in [2n]^{even}$. Then there is a vertex (q,l) at some level l such that $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = k$. But, since k is even the vertex (q,l) must be finite in G_k. Hence, it has only finitely many successors in G_k, all of which are assigned rank k. After a finite number of levels, every successor of (q,l) in $G^{\mathcal{A}}_\alpha$ must have a rank $< k$. Let l_{end} be the level after which $G^{\mathcal{A}}_\alpha$ does not contain any vertices with rank k. But, we have already shown earlier that even when k is even there is a level l^x such that for all levels $l > l^x$ the level rankings induced by $\mathsf{r}^\star_{\mathcal{A},\alpha}$ contain every odd rank $< k$. Let l^* be the larger of l^x and l_{end}. Then all full-level rankings f_l for every level $l > l^*$ satisfy the following - (i) $max\_rank(f_l) = \infty \cup [2n]^{odd}$ (ii) for all $j \in [2n]^{odd}$ such that $j \leq k$, there is a state q such that $f_l(q) = j$. In other words, f_l is a tight full-level ranking.

2. Let (q,l) be a non F-vertex with rank $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = k$, where $k \in \langle 2n \rangle$. Suppose, by way of contradiction assume that there is no immediate successor $(q', l+1)$ of (q,l) in $G^{\mathcal{A}}_\alpha$ with rank k. Then every immediate successor of (q,l) in $G^{\mathcal{A}}_\alpha$ has a rank $< k$. Let $j < k$, where $j \in [2n]$ be the largest rank among all immediate successors of (q,l) and consider the graph G_j. The rank j must be in $[2n]$. All immediate successors of (q,l) with ranks lower than j are not in G_j as they must have been removed by procedure $\mathsf{FullRankProc}$ in graphs G_m, $0 \leq m \leq j$. Hence, in G_j the vertex (q,l) only has immediate successors with rank j. We have the following two cases -

   (a) If j is odd then every immediate successor (q_j, l_j) of (q,l) is F-free in G_j. But, in G_j even (q,l) must be F-free and must get rank $j < k$, a contradiction.
   (b) If j is even then every immediate successor (q_j, l_j) of (q,l) is finite in G_j. But, since only vertices of the type (q_j, l_j) with rank j are immediate

successors of (q,l) in G_j, it must mean that (q,l) is also finite in G_j and gets rank $j < k$, a contradiction.

Let (q,l) be a vertex in $G_\alpha^{\mathcal{A}}$ with rank $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = \infty$. Suppose, by way of contradiction no immediate successor of (q,l) in $G_\alpha^{\mathcal{A}}$ has rank ∞. Then every immediate successor of (q,l) in $G_\alpha^{\mathcal{A}}$ has a rank $< k$. Let $j < \infty$, where $j \in [2n]$ be the largest rank among all immediate successors of (q,l) and consider the graph G_j. The rank j must be in $[2n]$. All immediate successors of (q,l) with ranks lower than j are not in G_j as they must have been removed by procedure $\mathsf{FullRankProc}$ in graphs G_m, $0 < m \leq j$. Hence, in G_j the vertex (q,l) only has immediate successors with rank j. We have the following two cases -

(a) If j is odd then every immediate successor (q_j, l_j) of (q,l) is (a non F-vertex and) F-free in G_j. We again analyze two cases -
   - If (q,l) is not an F-vertex then in G_j, even (q,l) must be F-free and must get rank $j < \infty$, a contradiction.
   - If (q,l) is an F-vertex then once all successors of (q,l) are removed in graph G_j, the vertex (q,l) is finite in G_{j+1} and hence must get the finite (non ∞) even rank $j+1$, a contradiction.

(b) If j is even then every immediate successor (q_j, l_j) of (q,l) is finite in G_j. But, since only vertices of the type (q_j, l_j) with rank j are immediate successors of (q,l) in G_j, it must mean that (q,l) is also finite in G_j and gets rank $j < k$, a contradiction.

Hence there must be a successor $(q', l{+}1)$ of (q,l) such that $\mathsf{r}^\star_{\mathcal{A},\alpha}((q', l{+}1)) = k$.

3. Let $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = k \in [2n]^{even}$ and suppose that every path in $G_\alpha^{\mathcal{A}}$ starting at (q,l) does not visit a vertex with a lower rank. We know that every vertex in $G_\alpha^{\mathcal{A}}$ has infinitely many vertices reachable from it (due to the principal sink). Hence, there must be an infinite path π starting at (q,l) in $G_\alpha^{\mathcal{A}}$ such that every vertex on π is assigned rank k by the ranking $\mathsf{r}^\star_{\mathcal{A},\alpha}$.
   Now consider the graph G_k. Every vertex that is ranked k by $\mathsf{r}^\star_{\mathcal{A},\alpha}$ is finite in G_k. Hence, (q,l) is also finite in G_k. But, we already know that there is an infinite path π starting at (q,l) with every vertex along π ranked k. Hence, the path π is present in G_k. This implies that (q,l) cannot be finite in G_k, a contradiction! Hence every path starting at (q,l) visits a vertex with a rank lower than k.
4. Let (q,l) be an F-vertex with $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = k$, where $k \in [2n]^{even}$. Since $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) \neq \infty$ all immediate successors of (q,l) in $G_\alpha^{\mathcal{A}}$ must have a non-∞ rank $\in [2n]^{even} \cup [2n]^{odd}$. Suppose, by way of contradiction, every immediate successor of (q,l) has rank in $[k-2]$ in $G_\alpha^{\mathcal{A}}$. Let $k' \in [k-2]$ be the largest rank among all successors of (q,l). We analyze the following two cases -
   - k' is odd : Then $k' = k-3$. Consider all immediate successors of (q,l) with rank k' assigned by $\mathsf{r}^\star_{\mathcal{A},\alpha}$. In graph $G_{k'}$, all immediate successors of (q,l) with rank less than k' are not present since they have already been removed earlier. Hence, the only immediate successors of (q,l) in $G_{k'}$ are those that are assigned rank k' by $\mathsf{r}^\star_{\mathcal{A},\alpha}$. These vertices are removed by procedure $\mathsf{FullRankProc}$ in $G_{k'}$ and assigned the rank k'. Hence, in graph

$G_{k'-1}$ (where $k'-1$ is even) the vertex (q,l) is finite and does not have any successors. As a consequence (q,l) should be assigned the even rank $k'-1$, where $k'-1 = k-2 < k$, by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$, a contradiction!
- k' is even: Then $k' = k-2$. Consider all immediate successors of (q,l) with rank k' assigned by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$. In graph $G_{k'}$, all immediate successors of (q,l) with rank less than k' are not present since they have already been removed earlier. Hence, the only immediate successors of (q,l) in $G_{k'}$ are those that are assigned rank k' by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$. But, all vertices that are assigned an even rank k'' by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ are finite in graph $G_{k''}$. Hence, all immediate successors of (q,l) that are assigned rank k' by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ are finite in $G_{k'}$. Together with the fact that (q,l) does not have other immediate successors in $G_{k'}$, the vertex (q,l) itself is finite in $G_{k'}$ and should be assigned the rank $k' = k-2 < k$ by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$, a contradiction!

Hence there must be at least one immediate successor of (q,l) with rank either k or $k-1$.

5. Since, $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q,l)) = k$, where $k \in \langle 2n \rangle^{odd}$, the vertex (q,l) was F-free in G_k. The vertex (q,l) was not F-free in G_{k-2}, else it would have been assigned the odd rank $k-2$ by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$. Hence there was an F-vertex (q',l') reachable from (q,l) in G_{k-2}. Since only F-free vertices are removed in G_{k-2} the vertex (q',l') was not removed in G_{k-2}. But since (q,l) is F-free in l, the vertex (q',l') was removed in G_{k-1} when constructing G_k by procedure $\mathsf{FullRankProc}$. But, if (q',l') was removed by procedure $\mathsf{FullRankProc}$ in G_{k-1}, the ranking $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ must assign it the rank $k-1$. Hence (q',l') is the required F-vertex with rank $k-1$ and reachable from (q,l) in $G^{\mathcal{A}}_{\alpha}$.
6. By definition, for any full-ranking r, for every edge $((q,i),(q',i+1)$ in $G^{\mathcal{A}}_{\alpha}$, we must have $r(q',l+1) \leq r(q,l)$. If $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q,l)) \neq \infty$ then $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q,l)) \in [2n]$. Suppose, $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q,l) = k$. Hence, every immediate successor of (q,l) must have a rank $\leq k$. But, the rank $\infty > j$ for every $j \in [2n]$. Hence, there can be no immediate successor of $(q',l+1) \in \delta(q,l)$ such that $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q',l+1) = \infty$.
7. Since $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q,l)) = \infty$, by construction of ranking $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ by procedure $\mathsf{FullRankProc}$, there is a $G_{k'}$, $k' \in [2n]^{odd}$ such that
   - $G_{k'}$ contains contains every vertex ranked ∞ by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ and hence $G_{k'}$ contains (q,l).
   - There are no F-free vertices in $G_{k'}$, hence there is an F-vertex (q',l'), where $l' > l$ reachable from (q,l) in $G_{k'}$.
   - $G_{k'}$ contains only vertices that are ranked ∞ by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$. Since (q',l) is in $G_{k'}$, $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q',l') = \infty$.

   Thus, if $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q,l) = \infty$ there is an F-vertex (q',l') reachable from (q,l) in $G^{\mathcal{A}}_{\alpha}$ such that $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q',l') = \infty$.

□

Properties 2, 3 and 4 in the above Lemma can be checked by examining consecutive levels of the ranked run-DAG; hence these are *local* properties. In contrast, checking properties 5, 6 and 7 requires examining an unbounded fragment of the ranked run-DAG; hence these are *global* properties.

3 Improved upper bound by rank based disambiguation

The main contribution of this section is a rank based algorithm, called $\mathsf{B\ddot{u}chiDisambiguate}$, that takes as input an NBA $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$ with $|Q| = n$, and constructs a UBA $\mathcal{U} = (\Sigma, Z, Z_0, \delta_U, F_U)$ such that (i) $L(\mathcal{U}) = L(\mathcal{A})$, and (ii) $|Z| \in O(n(0.76n)^n)$. Without loss of generality, we assume that $Q = \{q_0, q_1, \ldots q_{n-1}\}$, $Q_0 = \{q_0\}$ and $q_0 \notin F$. For notational convenience, we use "$\mathcal{A}$-states" (resp., "$\mathcal{U}$-states") to refer to states of $\mathcal{A}$ (resp., states of $\mathcal{U}$) in the following discussion.

3.1 Overview

Drawing motivation from Schewe's work [11], we define a state of $\mathcal{U}$ to be a 4-tuple (f, O, X, i), where $i \in \langle 2n \rangle$, $f : Q \to \langle 2n \rangle$ is a FL_∞ ranking, and O and X are subsets of Q containing $\mathcal{A}$-states that are ranked i by f. Since every state of $\mathcal{U}$ gives a full-level ranking of $\mathcal{A}$, a run of $\mathcal{U}$ gives an infinite sequence of full-level rankings of $\mathcal{A}$ that can be stitched together to potentially obtain a full-ranking of $G^{\mathcal{A}}_\alpha$. The purpose of algorithm $\mathsf{B\ddot{u}chiDisambiguate}$ is to define the transitions and final states of $\mathcal{U}$ in such a way that a run of $\mathcal{U}$ on $\alpha \in \Sigma^\omega$ is accepting iff the full-ranking of $G^{\mathcal{A}}_\alpha$ thus stitched together from the run is $\mathsf{r}^\star_{\mathcal{A},\alpha}$ with at least one run of $\mathcal{A}$ eventually trapped in ∞.

Informally, algorithm $\mathsf{B\ddot{u}chiDisambiguate}$ works as follows. Suppose $\mathcal{U}$ is in state (f, O, X, i) after reading a finite prefix $\alpha(0) \ldots \alpha(k-1)$ of α. On reading the next letter, i.e. $\alpha(k)$, we want $\mathcal{U}$ to non-deterministically guess the full-level ranking, say f', induced by $\mathsf{r}^\star_{\mathcal{A},\alpha}$ at level $k+1$ of $G^{\mathcal{A}}_\alpha$. Every such choice of f' must be a full-cover of $(f, \alpha(k))$ and must satisfy the local properties in Lemma 1. Given f, f' and $\alpha(k)$, the local properties are easy to check, and are used by algorithm $\mathsf{B\ddot{u}chiDisambiguate}$ to filter the full-level rankings that can serve as f'. Once a choice of f' has been made, algorithm $\mathsf{B\ddot{u}chiDisambiguate}$ uses the O-, X- and i-components of the current state (f, O, X, i) to *uniquely* determine the corresponding components of the next state (f', O', X', i'). In doing so, we use a technique reminiscent of that used by Miyano and Hayashi [9], and subsequently by Schewe [11], to ensure that the global properties in Lemma 1 are satisfied by the sequence of full-level rankings corresponding to an accepting run of $\mathcal{U}$. Note that the choice of f' gives rise to non-determinism in the transition relation of $\mathcal{U}$. However, in every step of the run of $\mathcal{U}$ on α, there is only choice of f' that can give rise to $\mathsf{r}^\star_{\mathcal{A},\alpha}$ when the full-level rankings corresponding to the run are stitched together.

A closer inspection of Lemma 1 shows that there are two types of global properties: those that relate to *every* path (property 5), and those that relate to *some* path (e.g., properties 6 and 7). Schewe gave a ranking based construction to enforce properties of the first type in the context of Büchi complementation [11]. We use a similar idea here for enforcing property 5. Specifically, suppose $\mathcal{U}$ is in state (f, O, X, i) after reading $\alpha(0) \ldots \alpha(k-1)$, and suppose f' has been chosen as the full-ranking of level $k+1$ of $G^{\mathcal{A}}_\alpha$. Suppose further that we wish to enforce property 5 for all vertices $(q, k+1)$ in $G^{\mathcal{A}}_\alpha$ where q is assigned an even rank j by f'. To do so, we set i' to j, populate O' with *all* $\mathcal{A}$-states assigned rank j by f', and use the O-components of subsequent $\mathcal{U}$-states along the run to keep track of the successors (in $\mathcal{A}$) of all $\mathcal{A}$-states in O'. During this process, if we encounter an $\mathcal{A}$-state q_k with rank $< j$ in the O-component of a $\mathcal{U}$-state, we know that property 5 is satisfied for all paths ending in q_k. The state q_k is therefore removed from O, and the above process repeated until O becomes empty. The emptying of O signifies that all paths in $G_\alpha^{\mathcal{A}}$ starting from $\mathcal{A}$-states with rank j at level $k+1$ eventually visit a state with rank $< j$. Once this happens, we reset O to $\emptyset$, choose the next (in cyclic order) even rank i and repeat the above process. Using the same argument as used in [11], it can be shown that a run of $\mathcal{U}$ visits a $\mathcal{U}$-state with $O = \emptyset$ and i set to the smallest even rank infinitely often iff the sequence of full-level rankings corresponding to the run satisfies property 5.

A naive way to adapt the above technique to enforce property 6 (resp., property 7) in Lemma 1 is to choose an odd rank (resp., ∞ rank) $i > 1$, populate O' with *a single non-deterministically chosen* $\mathcal{A}$-state assigned rank i by f', and track *a non-deterministically chosen single successor* of this state in the O-components of $\mathcal{U}$-states along the run until we find an $\mathcal{A}$-state that is final and assigned rank $i-1$ (resp., ∞). The problem with this naive adaptation is that the non-deterministic choice of $\mathcal{A}$-state above may lead to multiple accepting runs of $\mathcal{U}$ on α. This is undesirable, since we want $\mathcal{U}$ to be a UBA. To circumvent this problem, we choose the O-component of the next state, i.e. O' in (f', O', X', i'), deterministically, given the current $\mathcal{U}$-state (f, O, X, i) and $\alpha(k)$. Specifically, for every $\mathcal{A}$-state q_r in O, we find the $\alpha(k)$-successors of q_r in $\mathcal{A}$ that are assigned rank i by f', and choose only one of them, viz. the one with the minimum index, to stay in O'. For notational convenience, for $S \subset Q = \{q_0, \ldots q_{n-1}\}$, let $\downarrow S$ denote $\{q_i \mid q_i \in S \text{ and } \forall q_j \in S,\ i \leq j\}$. Then, $O' = \bigcup_{q_r \in O} \downarrow \{q_l \mid q_l \in \delta(q_r, \alpha(k)) \text{ and } f'(q_l) = i\}$.

Choosing O' as above has an undesired consequence: not all $\mathcal{A}$-states that are successors (in $\mathcal{A}$) of some state in O and have rank i may be tracked in the O-components of $\mathcal{U}$-states along the run. This may prevent the technique of Schewe [11] from detecting that property 6 (or property 7) is true in the sequence of full-rankings corresponding to a run of $\mathcal{U}$ on α. To rectify this situation, we use the X-component of $\mathcal{U}$-states as follows. We periodically load the X-component with a single $\mathcal{A}$-state from O, which is then removed from O. *All* successors (in $\mathcal{A}$) of the $\mathcal{A}$-state thus loaded in X are then tracked in the X-components of $\mathcal{U}$-states along the run, until we encounter a final $\mathcal{A}$-state with the desired rank ($i-1$ for property 6, and ∞ for property 7) in X. Once this happens, we empty X, load it with another $\mathcal{A}$-state (specifically, the one with the minimum index), say q_r, from O, remove q_r from O, and repeat the process until both O and X are emptied. When both O and X become empty, we set i to be the next rank i' of interest in cyclic order, load O with all $\mathcal{A}$-states assigned rank i', and repeat the entire process. Extending the reasoning used by Schewe in [11], it can be shown that a run of $\mathcal{U}$ visits a $\mathcal{U}$-state with $O = \emptyset, X = \emptyset$ and i set to the smallest rank of interest infinitely often iff the corresponding sequence of full-level rankings satisfies property 6 (or property 7, as the case may be).

3.2 Our algorithm and its analysis

The pseudocode for algorithm BüchiDisambiguate is given below. Note that the checks for global properties are deferred until all full-level rankings induced by $\mathsf{r}^\star_{\mathcal{A},\alpha}$ have become tight. This is justified by property 1 in Lemma 1. The choice of initial state of $\mathcal{U}$ is motivated by the observation that q_0 is ranked ∞ in the full-level ranking induced by $\mathsf{r}^\star_{\mathcal{A},\alpha}$ at level 0 of $G^{\mathcal{A}}_\alpha$ iff $\alpha \in L(\mathcal{A})$. Finally, algorithm BüchiDisambiguate implements the following optimization when calculating the next $\mathcal{U}$-state (f', O', X', i') from a given $\mathcal{U}$-state (f, O, X, i) and $a \in \Sigma$: if i is odd or ∞ and if $\delta(X, a)$ intersects O', X' is reset to $\emptyset$ instead of being populated with $\delta(X, a)$. This is justified because every $\mathcal{A}$-state in O' must eventually have one of its successors with rank i moved to the X-component of a $\mathcal{U}$-state further down the run, for the run of $\mathcal{U}$ to be accepting.

Algorithm : BüchiDisambiguate
Input: NBA $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$
Output: UBA $\mathcal{U} = (\Sigma, Z, Z_0, \delta_U, F_U)$

- States : $Z = \mathsf{FL}_\infty \times 2^Q \times 2^Q \times \langle 2n \rangle$. Furthermore, if $(f, O, X, i) \in Z$, then $O \subseteq Q$, $X \subseteq Q$, and $f \in \mathsf{FL}_\infty$ is such that $\forall q_j \in O \cup X,\ f(q_j) = i$.
- Initial State: $Z_0 = \{(f, O, X, i) \mid f(q_0) = \infty,\ \ O = X = \emptyset,\ \ i = 1$ and $\forall q \in Q\,(q \neq q_0 \rightarrow f(q) = \perp)\}$.
- Transitions: For every $(f', O', X', i') \in \delta_U((f, O, X, i), a)$, where $a \in \Sigma$, the following conditions hold.
  1. Let $S = \{q_l \mid f(q_l) \neq \perp\}$. For all $q_j \notin \delta(S, a)$, $f'(q_j) = \perp$.
  2. f' is a full-cover of (f, a).
  3. For all $q_j \in Q$ such that $f(q_j) = \infty$, there is a $q_l \in \delta(q_j, a)$ such that $f'(q_l) = f(q_j)$.
  4. For all $q_j \in Q \setminus F$, there is a $q_l \in \delta(q_j, a)$ such that $f'(q_l) = f(q_j)$.
  5. For all $q_j \in Q \cap F$ such that $f(q_j) \in [2n]^{even}$, there is a $q_l \in \delta(q_j, a)$ such that either $f'(q_l) = f(q_j)$ or $f'(q_l) = f(q_j) - 1$.
  6. For all $q_j \in Q$ such that $f(q_j) \neq \infty$, there is no $q_l \in \delta(q_j, a)$ such that $f'(q_l) = \infty$.
  7. In addition, O', X' and i' satisfy the following conditions.
     (a) If f is not a tight full-level ranking, then $O' = X' = \emptyset$, $i' = 1$.
     (b) If $O \cup X \neq \emptyset$, then $i' = i$. Furthermore, the following conditions hold. For notational convenience, let $O'' = \bigcup_{q_j \in O} \downarrow \{q_l \mid q_l \in \delta(q_j, a) \wedge f'(q_l) = i\}$ and let $X'' = \{q_l \mid q_l \in \delta(X, a) \wedge f'(q_l) = i\}$.
        i. If $i = 1$, then $O' = X' = \emptyset$.
        ii. If $i \in [2n]^{odd}$ and $i \neq 1$, then
           a. If $X = \emptyset$, then $X' = \downarrow O''$, $O' = O'' \setminus X'$.
           b. Else if $X'' \cap O'' \neq \emptyset$ or $(\exists q_l \in \delta(X, a) \cap F,\ f'(q_l) = (i - 1))$, then $X' = \emptyset$, $O' = O''$.
           c. Else, $X' = X''$, $O' = O''$.
        iii. If $i \in [2n]^{even}$, then $O' = \{q_l \mid q_l \in \delta(O, a) \wedge f'(q_l) = i\}$, $X' = \emptyset$.

iv. If $i = \infty$, then
   a. If $X = \emptyset$, then $X' = \downarrow O''$, $O' = O'' \setminus X'$.
   b. Else if $X'' \cap O'' \neq \emptyset$ or $X'' \cap F \neq \emptyset$, then $X' = \emptyset$, $O' = O''$.
   c. Else, $X' = X''$, $O' = O''$.

(c) If $O \cup X = \emptyset$, then $X' = \emptyset$. In addition, the following hold.
   i. If $(i = 1)$ then $i' = max\_rank(f')$.
   Else if $(i = \infty)$ then $i' = max(\{j \mid \exists q \in Q, f'(q) = j\} \cap [2n])$.
   Else $i' = i - 1$.
   ii. $O' = \{q_l \mid f'(q_l) = i'\}$.

- $F_U = \{(f, O, X, i) \mid O = X = \emptyset,\ i = 1,\ f$ is a tight full-level ranking$\}$.

End Algorithm : BüchiDisambiguate

3.3 Proof of correctness

Let $\rho = (f_0, O_0, X_0, i_0), (f_1, O_1, X_1, i_1), \ldots$ be an accepting run of the NBA $\mathcal{U}$ constructed using algorithm BüchiDisambiguate. The run ρ *induces* a full-ranking r of $G^{\mathcal{A}}_{\alpha}$ as follows: for every $i \geq 0$, $r(q, i) = k$ iff $f_i(q) = k$ where $k \in \langle 2n \rangle$. Note that if $f_i(q) = \perp$, then q is not reachable in $\mathcal{A}$ from q_0 after reading $\alpha(0) \ldots \alpha(i-1)$.

Lemma 2. *For every vertex (q, l) in $G^{\mathcal{A}}_{\alpha}$, if $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q, l)) \neq \infty$, every path π from (q, l) satisfies the following :*

1. *every vertex (q', l') along π has $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q', l')) \neq \infty$, and*
2. *π visits F-vertices finitely often.*

Proof. 1. Since, $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q, l)) \neq \infty$ it must be the case that $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q, l)) \in [2n]$, say rank k. But, since $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ is a full-ranking every rank in $[2n]$ is smaller than ∞. Also, along every path in $G^{\mathcal{A}}_{\alpha}$ the ranks can never increase. Hence, there can be no successor of (q, l) with a rank greater than $k \in [2n]$. This shows that for every vertex (q', l') along every path π starting at (q, l) we have $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q', l')) \neq \infty$.
2. By the earlier argument there can be no successor of (q, l) with a rank greater than $k \in [2n]$ and hence for every vertex (q', l') along every path π starting at (q, l) we have $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q', l')) \neq \infty$. But, since $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ is a full-ranking every path in $G^{\mathcal{A}}_{\alpha}$ either gets trapped in rank ∞ or some odd rank. But, no path starting at (q, l) can get trapped in rank ∞. Hence, every path starting at (q, l) must get trapped in an odd rank. But, by definition of a full-ranking an F-vertex cannot get an odd rank. Hence, every path π starting at (q, l) must visit F-vertices finitely often.

∎

Lemma 3. *For every vertex (q, l) in $G^{\mathcal{A}}_{\alpha}$, if $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q, l)) = \infty$, there is a path π starting at (q, l) that satisfies the following:*

1. *every vertex (q', l') along π has $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q', l')) = \infty$, and*
2. *π visits F-vertices infinitely often.*

Proof. Let (q, l) be a vertex in $G^{\mathcal{A}}_{\alpha}$ such that $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q, l)) = \infty$. From Property 7 of $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ in Lemma (1), there must be an F-vertex say (q_1, l_1) in $G^{\mathcal{A}}_{\alpha}$ such that (q_1, l_1) is reachable from (q, l) in $G^{\mathcal{A}}_{\alpha}$ and $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q_1, l_1) = \infty$. Similarly, since $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q_1, l_1) = \infty$, applying Property 7 of $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ in Lemma (1) gives a vertex (q_2, l_2) in $G^{\mathcal{A}}_{\alpha}$ such that that (q_2, l_2) is reachable from (q_1, l_1) in $G^{\mathcal{A}}_{\alpha}$ and $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q_2, l_2) = \infty$. Applying this argument ad infinitum we can construct an infinite path $\pi = (q, l) \rightsquigarrow (q_1, l_1) \rightsquigarrow (q_2, l_2) \rightsquigarrow \ldots$ such that for each (q_i, l_i) along the path for $i > 0$, $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q_i, l_i) = \infty$. Since, ranks along a path in $G^{\mathcal{A}}_{\alpha}$ cannot increase every vertex along π must be ranked ∞. The infinite path π is the required path that satisfies the lemma. $\square$

Lemma 4. *For every vertex (q, l) in $G^{\mathcal{A}}_{\alpha}$, if $r((q, l)) \neq \infty$, then every path $\overline{p}$ starting at (q, l) satisfies the following: (i) every vertex (q', l') along $\overline{p}$ has $r((q', l')) \neq \infty$, and (ii) $\overline{p}$ eventually gets trapped in an odd rank and hence visits F-vertices finitely often.*

Proof. Since, $r((q, l)) \neq \infty$ we have $r((q, l)) = k \in [2n]^{even} \cup [2n]^{odd}$. As ranks along a path cannot increase, Step (3) of Algorithm $\mathsf{B\ddot{u}chiDisambiguate}$ cannot ensure the existence of an infinite path starting at (q, l) such that every vertex along the path has a rank ∞. Hence every vertex along every path in $G^{\mathcal{A}}_{\alpha}$ starting starting at (q, l) has a rank in $[2n]^{even} \cup [2n]^{odd}$.

In the construction for $\mathcal{U}$ since successive level rankings are covers of the previous level rankings, ranks along a path never increase. There are finitely many ranks and the smallest rank is 1. Hence, every path eventually gets trapped in a fixed rank k'. We now argue that $k' \notin [2n]^{even}$. Suppose, by way of contradiction let k' be even. Then there is an infinite path π of $\mathcal{A}$ such that the composition of successive level rankings in (f, O, X, i) states along ρ eventually assigns the same even rank k' to states along π.

Since ρ is an accepting run of $\mathcal{U}$, we have $inf(\rho) \cap F_U \neq \emptyset$. Hence, in states of $\mathcal{U}$ along ρ we have $O \cup X = \emptyset$ infinitely often. In our construction $O \cup X = \emptyset$ in Step (7c) for some state, say $z = (f, O, X, i)$ of $\mathcal{U}$. Suppose, $i \in [2n]^{odd}$ for z, where $i > k'$. Then the O-set is populated with states ranked i in the successor of z. Since $O \cup X = \emptyset$ infinitely often we must have a state z' of $\mathcal{U}$ along ρ such that z' follows z and $O \cup X = \emptyset$ in z'. When this happens in the (f, O, X, i) state of $\mathcal{U}$ after z' along ρ the index i is set to $i - 1 \in [2n]^{even}$ and the O-set is populated with states ranked $i - 1$. When $O \cup X = \emptyset$ in a state along ρ again, which must happen, states with succesively lower ranks are added to the O-set. Hence, there must be a (f, O, X, i) state along ρ such that $O \cup X = \emptyset$ and $i = k' - 1$. When this happens, in the next state, say $\hat{z}$ along ρ the index i is set to k' and the O set is loaded with states that are ranked k'. Without loss of generality assume that the states on π ranked k' that are loaded in the O-set belong to the suffix of π when every state along it is ranked k'. Now successive O-sets in successive states along ρ are computed by removing states with a rank less that k'. But, since states along π are also in the O-sets the O-set can never become empty. Hence, a final state of $\mathcal{U}$ with $O \cup X = \emptyset$ cannot be reached after this point. Hence, ρ sees only finitely many final states, a contradiction! Hence, k' must be odd. Since, final states (corresponding to F-vertices in the run-DAG) cannot get an odd rank the run π eventually sees only non final states. Let $\overline{p}$ be the path in $G^{\mathcal{A}}_{\alpha}$ corresponding to π. Then $\overline{p}$ eventually sees only non F-vertices and every vertex along $\overline{p}$ has a non-∞ rank.

Now, since π was chosen to be an arbitrary path, it must mean that every path in $G^{\mathcal{A}}_{\alpha}$ starting at (q, l) sees only non-∞ ranks and sees only finitely many F-vertices. $\square$

Lemma 5. *For every vertex (q, l) in $G^{\mathcal{A}}_{\alpha}$, if $r((q, l)) = \infty$, there is at least one path $\overline{p}$ starting from (q, l) that satisfies the following: (i) every vertex (q', l') along $\overline{p}$ has $r((q', l')) = \infty$, and (ii) $\overline{p}$ visits F-vertices infinitely often.*

Proof. Since ρ is an accepting run of $\mathcal{U}$ we have $inf(\rho) \cap F_U \neq \emptyset$ and consequently in (f, O, X, i) states along ρ we have $O \cup X = \emptyset$ infinitely often. In our construction $O \cup X = \emptyset$ in Step (7c). Suppose, $i = \infty$ for some $z = (f, O, X, i)$ state along run for which $O \cup X = \emptyset$. Then since it is guaranteed that $O \cup X = \emptyset$ for some state z' along ρ such that z' follows z, it must mean that the check in Step (7(b)iv) of Algorithm BüchiDisambiguate must have been successful. This means that there is some state q_1 of $\mathcal{A}$, which is reachable from one of the states initially loaded in O, where q_1 is ranked ∞ and is final.

The fact that successive level rankings are full-covers ensures that there is a path π_{q_1} starting q_1 such that every successor along this path is ranked ∞.

We observe that $O \cup X = \emptyset$ infinitely often and the ranks in the i-component of (f, O, X, i) states decrease down to 1 and then back to ∞. Hence, some state q' ranked ∞ on the path π_{q_1} will eventually be loaded back into the O-set in some state of $\mathcal{U}$ along ρ. But, since $O \cup X = \emptyset$ will be satsified again, there exists a final state ranked ∞ say q_2 reachable from q_1.

Since each successive full-level ranking is a full-cover of the previous full-level ranking it is ensured that there is a path π_{q_2} starting q_2 such that every successor along this path is ranked ∞.

Repeating the above arguments infinitely many times we can construct infinitely many path segments $\pi_{q_1}, \pi_{q_2}, \pi_{q_3}, \pi_{q_4}, \ldots$ such that the infinite composition of these paths $\pi = \pi_{q_1}\pi_{q_2}\pi_{q_3}\ldots$ gives an infinite path π that sees infinitely many final states and every state along π is ranked ∞ by the level rankings. The path π corresponds to a path $\overline{p}$ in $G^{\mathcal{A}}_{\alpha}$ such that every vertex along it is ranked ∞ and sees infinitely many F-vertices. $\square$

Lemma 6. *For every vertex (q, l) in $\mathcal{G}^{\mathcal{A}}_{\alpha}$, $r((q, l)) = \mathsf{r}^{\star}_{\mathcal{A},\alpha}((q, l))$.*

Proof. We prove the claim by induction on the rank i. For a rank $i \in \langle 2n \rangle$ the rank $succ^{j}(i) \in \langle 2n \rangle$ is the j^{th} rank after i and greater than i in the total order $\leq$ on the ranks in $\langle 2n \rangle$. We use the shorthand notation $succ(i)$ for $succ^{1}(i)$ to denote the immediate successor of i greater than i.

Base case: Let $(q,l) \in V_{\mathsf{r}^\star_{\mathcal{A},\alpha},1}$. By definition, (q,l) is F-free in $G^{\mathcal{A}}_{\alpha} = G_0 = G_1$. Suppose $r((q,l)) = m$, where $m > 1$ and $m \in \langle 2n \rangle$, if possible.

Suppose $m \in [2n]^{even}$. If (q,l) has an F-vertex descendant in G_1 then (q,l) is not F-free in G_1 and hence $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) \neq 1$, a contradiction. Hence, suppose (q,l) does not have an F-vertex descendant. Hence, by the constraint embodied in Step (4) of our construction there is an infinite path $\overline{p}$ starting at (q,l) such that every vertex on $\overline{p}$ is not an F-vertex and has the same even rank m. But, for the O-set to become empty it is required that along all paths starting at every vertex with an even rank in $[2n]^{even}$ that appears in the O-set, a vertex with a rank $< m$ is visited. But, due to the presence of the path $\overline{p}$ the O-set is eventually never empty. Hence, $\mathit{inf}(\rho) \cap F_U = \emptyset$, a contradiction. Hence, $r((q,l)) \leq 1$.

Suppose, $m \in [2n]^{odd} \cup \{\infty\}$ and $m > 1$. Then, by the constraints embodied in Steps (3), (4), (5), (6), and Lemma (5) of our disambiguation construction coupled with the fact that $O = \emptyset$ infinitely often, imply that (q,l) has an F-vertex descendant (q',l') in $G^{\mathcal{A}}_{\alpha}$. Therefore, (q,l) is not F-free in G_1 – a contradiction! Hence, $r((q,l)) \leq 1$. Since 1 is the lowest rank in the range of r, we finally have $r((q,l)) = 1$. This shows that $V_{\mathsf{r}^\star_{\mathcal{A},\alpha},1} \subseteq V_{r,1}$.

Now suppose $(q,l) \in V_{r,1}$. By Lemma (4) every path starting at (q,l) in $G^{\mathcal{A}}_{\alpha}$ gets trapped in an odd rank. Since ranks cannot increase along any run in B and 1 is the minimum rank in the range of r implies that all descendants of (q,l) in $G^{\mathcal{A}}_{\alpha}$ are assigned rank 1 by r. Since F-vertices must be assigned even ranks, this implies that (q,l) is F-free in $G^{\mathcal{A}}_{\alpha}$. It follows that $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = 1$. Therefore, $V_{r,1} \subseteq V_{\mathsf{r}^\star_{\mathcal{A},\alpha},1}$. From the above two results, we have $V_{r,1} = V_{\mathsf{r}^\star_{\mathcal{A},\alpha},1}$.
Hypothesis: Assume that $V_{r,j} = V_{\mathsf{r}^\star_{\mathcal{A},\alpha},j}$ for $1 \leq j \leq i$.
Induction : Let $(q,l) \in V_{\mathsf{r}^\star_{\mathcal{A},\alpha},succ(i)}$. Then by the induction hypothesis, (q,l) cannot be in any $V_{r,j}$ for $j \leq i$. Suppose $r((q,l)) = m$, where $m > succ(i)$, if possible. We have the following cases

- $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = succ(i) = \infty$. Since, ∞ is the highest rank in the range of any full-ranking and (q,l) cannot be in any $V_{r,j}$ for $j \leq i$, we must have $r((q,l)) = \infty$.
- $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = succ(i) = k \neq \infty$: Then by Lemma (2) every path starting at (q,l) eventually gets trapped in an odd rank under the ranking $\mathsf{r}^\star_{\mathcal{A},\alpha}$. Hence, by Lemmas (4) and (5), $r((q,l)) \neq \infty$. Hence suppose $r((q,l)) = m \neq \infty$ and let $m > k$ if possible. We have the following cases
  - $m \in [2n]^{odd}$: By Step (7(b)ii) of our construction for B and the fact that the O-set is empty infinitely often there is an F-vertex descendant (q',l') of (q,l) such that $r((q',l')) = m - 1$. Since $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = k$ we have $\mathsf{r}^\star_{\mathcal{A},\alpha}((q',l')) \leq k - 1$ i.e $\mathsf{r}^\star_{\mathcal{A},\alpha}((q',l')) \leq i$. But since $m > k$ we have $m - 1 > i$. But for all ranks $j \leq i$ we have $V_{r,j} = V_{\mathsf{r}^\star_{\mathcal{A},\alpha},j}$ by the induction hypothesis. This leads to a contradiction.
  - $m \in [2n]^{even}$: If (q,l) has an F-vertex descendant (q',l') then $r((q',l')) = m-1$. Since $\mathsf{r}^\star_{\mathcal{A},\alpha}((q,l)) = k$ we have $\mathsf{r}^\star_{\mathcal{A},\alpha}((q',l')) \leq k-1$ i.e $\mathsf{r}^\star_{\mathcal{A},\alpha}((q',l')) \leq i$. But since $m > k$ we have $m - 1 > i$. But for all ranks $j \leq i$ we have $V_{r,j} = V_{\mathsf{r}^\star_{\mathcal{A},\alpha},j}$ by the induction hypothesis. This leads to a contradiction.

Hence, suppose (q,l) does not have an F-vertex descendant (including itself). Hence, by the constraint embodied in Step (4) of our construction there is an infinite path $\overline{p}$ starting at (q,l) such that every vertex on $\overline{p}$ is not an F-vertex and has the same even rank m. But, for the O-set to become empty it is required that along all paths starting at every vertex with an even rank in $[2n]^{even}$ that appears in the O-set, a vertex with a rank $\leq m$ is visited. But, due to the presence of the path $\overline{p}$ the O-set is eventually never empty. Hence, $\mathit{inf}(\rho) \cap F_U = \emptyset$, a contradiction.

Hence, $m \leq succ(i)$. But, since, for all ranks $j \leq i$ $V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},j} \subseteq V_{r,j}$ we must have $m = succ(i)$. As a result, $V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},succ(i)} \subseteq V_{r,succ(i)}$.

Let $(q,l) \in V_{r,succ(i)}$. Then by the induction hypothesis, (q,l) cannot be in any $V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},j}$ for $j \leq i$. Suppose $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q,l)) = m$, where $m > succ(i)$, if possible. We have the following cases

- $r((q,l)) = succ(i) = \infty$. Since, ∞ is the highest rank in the range of any full-ranking and (q,l) cannot be in any $V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},j}$ for $j \leq i$, we must have $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q,l)) = \infty$.
- $r((q,l)) = succ(i) = k \neq \infty$: Then by Lemma (4) every path starting at (q,l) eventually gets trapped in an odd rank under the ranking r. Hence, by Lemmas (2) and (3) we have $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q,l)) \neq \infty$. Hence suppose $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q,l)) = m \neq \infty$ and $m > k$ if possible. We have the following cases
  - $m \in [2n]^{odd}$: By Property (6) of $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ in Lemma (1) there is an F-vertex descendant (q',l') of (q,l) such that $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q',l')) = m-1$. Since $r((q,l)) = k$ we have $r((q',l')) \leq k-1$ i.e $r((q',l')) \leq i$. But since $m > k$ we have $m-1 > i$. But for all ranks $j \leq i$ we have $V_{r,j} = V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},j}$ by the induction hypothesis. This leads to a contradiction as ranking r assigns a rank $\leq i$ to (q',l'), while $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ assigns a rank $> i$ to (q',l').
  - $m \in [2n]^{even}$: If (q,l) has an F-vertex descendant (q',l') then $\mathsf{r}^{\star}_{\mathcal{A},\alpha}((q',l')) = m-1$. Since $r((q,l)) = k$ we have $r((q',l')) \leq k-1$ i.e $r((q',l')) \leq i$. But since $m > k$ we have $m-1 > i$. But for all ranks $j \leq i$ we have $V_{r,j} = V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},j}$ by the induction hypothesis. This leads to a contradiction. Hence, suppose (q,l) does not have an F-vertex descendant. Hence, by Property (2) in Lemma (1) there is an infinite path $\overline{p}$ starting at (q,l) such that every vertex on $\overline{p}$ is not an F-vertex and has the same even rank m. But by Lemma (2) every path starting at (q,l) must get trapped in an odd rank, a contradiction.

Hence, $m \leq succ(i)$. But for all ranks $j \leq i$ we must have $V_{r,succ(i)} \subseteq V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},succ(i)}$. This shows that $V_{r,succ(i)} \subseteq V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},succ(i)}$. Earlier, we showed that $V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},succ(i)} \subseteq V_{r,succ(i)}$. Hence, putting both the results together we have $V_{r,succ(i)} = V_{\mathsf{r}^{\star}_{\mathcal{A},\alpha},succ(i)}$. As a consequence for every vertex (q,l) in $G^{\mathcal{A}}_{\alpha}$ $r((q,l)) = \mathsf{r}^{\star}_{\mathcal{A},\alpha}((q,l))$.

□

Theorem 1. $L(\mathcal{U}) = L(\mathcal{A})$.

Proof. Suppose $\alpha \in L(\mathcal{U})$. Then there exists an accepting run ρ of $\mathcal{U}$ on α such that $\rho(0) = (f_0, O_0, X_0, i_0)$ and ρ visits infinitely many (f, O, X, i) states with $O = X = \emptyset$ and $i = 1$. By construction $\{q_0\} = Q_0$ such that $f_0(q_0) = \infty$. Let r be the ranking induced by ρ. Then $r(q_0, 0) = \infty$. Since, ρ visits infinitely many (f, O, X, i) states with $O = X = \emptyset$ and $i = 1$, by Lemma (6) $r(q, l) = \mathsf{r}^{\star}_{\mathcal{A},\alpha}(q, l)$ for every vertex (q, l) in $G^{\mathcal{A}}_{\alpha}$. Hence, $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q_0, 0) = \infty$. Invoking Lemma (3) on vertex $(q_0, 0)$ we obtain a path π in $G^{\mathcal{A}}_{\alpha}$ starting at $(q_0, 0)$ such that every vertex along π is ranked ∞ and π visits F-vertices infinitely often. Thus, π corresponds to an accepting run of $\mathcal{A}$ on α. Hence, $\alpha \in L(\mathcal{A})$.

Suppose $\alpha \in L(\mathcal{A})$. The state $(f_0, O_0, X_0, i_0) \in Z_0$ is the initial state of $\mathcal{U}$, where $O_0 = X_0 = \emptyset$, $i_0 = 1$ and $f_0(q_0) = \infty$, while $f_0(q) = \perp$ for all $q \neq q_0$. Let f_i be the level ranking corresponding to the ranks assigned by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ to level i of $G^{\mathcal{A}}_{\alpha}$ i.e. $f_i(q) = k$ iff $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q, i) = k$, $k \in \langle 2n \rangle$ and $f_i(q) = \perp$ if (q, i) is not a vertex in $G^{\mathcal{A}}_{\alpha}$. The automaton $\mathcal{U}$ guesses next level rankings f' that satisfies constraints enforced by Steps (2) to (6) of algorithm $\mathsf{BüchiDisambiguate}$. But, by definition, for all $i \geq 0$ the level ranking f_{i+1} corresponding to the ranks assigned by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ at level $i + 1$ of $G^{\mathcal{A}}_{\alpha}$ satisfies these constraints as enumerated in Lemma (1). Hence, $\rho = (f_0, O_0, X_0, i_0), (f_1, O_1, X_1, i_1), \ldots$ is a valid run of $\mathcal{U}$ for appropriate choices of O_j, X_j and i_j and where f_j is a full cover of f_{j-1} for all $j > 0$. It is now left to show that ρ is an accepting run i.e. $O_j = X_j = \emptyset$ and $i_j = 1$ for infinitely many j.

Every path in $G^{\mathcal{A}}_{\alpha}$ gets trapped in either an odd rank or ∞ under the ranking $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$. Since, $\alpha \in L(\mathcal{A})$ the run-DAG $G^{\mathcal{A}}_{\alpha}$ contains an infinite path π that visits infinitely many F-vertices. This fact together with the contrapositive of Lemma (2) implies the existence of a vertex (q, l) for some $q \in Q$ and $l \geq 0$ such that $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q, l) = \infty$. Then, it follows from Lemma (3) that there is a path π'' (possibly different from π) such that every vertex along π' is ranked ∞ by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ and π' visits F-vertices infinitely often. The path π'' can be extended backwards to $(q_0, 0)$, where $\pi''(0) = (q_0, 0)$ and $\mathsf{r}^{\star}_{\mathcal{A},\alpha}(q_0, 0) = \infty$ to give the path π' that begins at $(q_0, 0)$ such that every vertex along π' is ranked ∞ by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$. Since, ρ induces $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ we are left with the following fact.

Fact 1 For every $j \geq 0$ in state (f_j, O_j, X_j, i_j) along ρ, $f_j^{-1}(\infty) \neq \emptyset$.

The automaton $\mathcal{U}$ starts in state $(f_0, \emptyset, \emptyset, 1)$. After level rankings in states along ρ are tight (which they must be after a finite number of levels by condition (1) on $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ proved in Lemma (1)) - if $O_j \cup X_j = \emptyset$ in any state (f_j, O_j, X_j, i_j) along ρ, O', the next O-set is populated with all states that get the rank $i_j - 1$ under the level ranking at level $j+1$. If $O_j \cup X_j \neq \emptyset$ then the next O and X sets are populated according to the procedure in Step (7b) of algorithm $\mathsf{BüchiDisambiguate}$. We now show that in states along ρ, infinitely often $O = X = \emptyset$ and $i = 1$. Let the current state of $\mathcal{U}$ along ρ be (f_j, O_j, X_j, i_j) where $O_j \cup X_j \neq \emptyset$, w.l.o.g. $X_j = \emptyset$ and i_j is as follows.

$i_j = \infty$: Then $X_{j+1} = \downarrow O''$, where O'' is in turn equal to $\bigcup_{q_j \in O_j} \downarrow \{q_l \mid q_l \in \delta(q_j, \sigma) \wedge f_{j+1}(q_l) = i\}$. Also, $O_{j+1} = O'' \setminus X_{j+1}$. Hence, X_{j+1} contains the minimum indexed state of $\mathcal{A}$ that was guessed to have rank ∞ by f_{j+1}. Due to the action of $\downarrow$ and the way O_{j+1} is computed from O_j ($O_{j+1} = O'' \setminus X_{j+1}$), we have $|O_k| < |O_j|$ for all $k > j$ until the set O_k is empty.

The level rankings along ρ correspond to the ranking $\mathsf{r}^\star_{\mathcal{A},\alpha}$ at each level of $G^{\mathcal{A}}_\alpha$. Property (7) of $\mathsf{r}^\star_{\mathcal{A},\alpha}$ in Lemma (1) requires that every vertex (state) with rank ∞ has a path to an F-vertex with rank ∞. Hence, some successor of the state added to X_{j+1} must be an F-vertex with rank ∞. Suppose, X_k for some $k > j+1$ contains a final state q_f of $\mathcal{A}$ corresponding to an F-vertex of $G^{\mathcal{A}}_\alpha$ where $f_k(q_f) = \infty$. Hence, in Step (7(b)iv(b)) the X_{k+1} is set to $\emptyset$. Proceeding thus, minimum indexed states from the O-set are moved to the X-set, where at each such movement the size of the O-set monotonically decreases. This eventually leads to a state (f_l, O_l, X_l, i_l), where $O_l = X_l = \emptyset$ and $i_l = \infty$. Then due to the action of Step (7c) of algorithm BüchiDisambiguate the next state is computed as $(f_{l+1}, O_{l+1}, X_{l+1}, i_{l+1})$, where i_{l+1} is the largest non-∞ rank assigned by f_{l+1} (in $[2n]$), while $X_{l+1} = \emptyset$ and O_{l+1} contains all states that are guessed to have the rank i_{l+1} by f_{l+1}.

$i_j \in [2n]^{odd}$: Then $X_{j+1} = \downarrow O''$, where O'' is in turn equal to $\bigcup_{q_j \in O_j} \downarrow \{q_l \mid q_l \in \delta(q_j, \sigma) \wedge f_{j+1}(q_l) = i\}$. Also, $O_{j+1} = O'' \setminus X_{j+1}$. Hence, X_{j+1} contains the minimum indexed state of $\mathcal{A}$ that was guessed to have rank i_j by f_{j+1}. Due to the action of $\downarrow$ and the way O_{j+1} is computed from O_j ($O_{j+1} = O'' \setminus X_{j+1}$), we have $|O_k| < |O_j|$ for all $k > j$ until the set O_k is empty.

The level rankings along ρ correspond to the ranking $\mathsf{r}^\star_{\mathcal{A},\alpha}$ at each level of $G^{\mathcal{A}}_\alpha$. Property (6) of $\mathsf{r}^\star_{\mathcal{A},\alpha}$ in Lemma (1) requires that every vertex (state) with odd rank h has a path to an F-vertex with even rank $h-1$. Hence, some successor of the state added to X_{j+1} must be an F-vertex with even rank $i_j - 1$. Suppose, X_k for some $k > j+1$ contains a final state of $\mathcal{A}$ corresponding to an F-vertex of $G^{\mathcal{A}}_\alpha$ where $f_k(q_f) = i_j - 1$. Hence, in Step (7(b)ii(b)) the X_{k+1} is set to $\emptyset$. Proceeding thus, minimum indexed states from the O-set are moved to the X-set, where at each such movement the size of the O-set monotonically decreases. This eventually leads to a state (f_l, O_l, X_l, i_l), where $O_l = X_l = \emptyset$ and $i_l = i_j$. Then due to the action of Step (7c) of algorithm BüchiDisambiguate the next state is computed as $(f_{l+1}, O_{l+1}, X_{l+1}, i_{l+1})$, where $i_{l+1} = i_j - 1$, while $X_{l+1} = \emptyset$ and O_{l+1} contains all states that are guessed to have the rank $i_j - 1$ by f_{l+1}.

$i \in [2n]^{even}$: Then O_{j+1} contains successors of states in O_j that have been guessed to have rank i_j by f_{j+1}, while $X_{j+1} = \emptyset$. The level rankings along ρ correspond to the ranking $\mathsf{r}^\star_{\mathcal{A},\alpha}$ at each level of $G^{\mathcal{A}}_\alpha$. Property (5) of $\mathsf{r}^\star_{\mathcal{A},\alpha}$ in Lemma (1) requires that every path beginning at vertex (state) with even rank h eventually visits a vertex (state) with a lower rank. Hence, repeated applications of Step (7(b)iii) to states in the O-set will cause the O-set to eventually become empty. Hence, eventually a state (f_l, O_l, X_l, i_l), where $O_l = X_l = \emptyset$ and $i_l = i_j$ is reached. Then due to the action of Step (7c) of algorithm BüchiDisambiguate the next state is computed as $(f_{l+1}, O_{l+1}, X_{l+1}, i_{l+1})$, where $i_{l+1} = i_j - 1$, while $X_{l+1} = \emptyset$ and O_{l+1} contains all states that are guessed to have the rank $i_j - 1$ by f_{l+1}.

$i_j = 1$: By Step (7(b)i) of algorithm BüchiDisambiguate $O_{j+1} = X_{j+1} = \emptyset$. Hence, the next state is $(f_{j+1}, O_{j+1}, X_{j+1}, i_{j+1})$ where $O_{j+1} = X_{j+1} = \emptyset$ and $i_{j+1} = i_j = 1$, which is a final state of $\mathcal{U}$.

The ranking $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$ is a full-ranking of $G^{\mathcal{A}}_{\alpha}$. Hence, by definition, every path in $G^{\mathcal{A}}_{\alpha}$ gets trapped in either rank ∞ or an odd rank. Also, there is a finite level j after which every level ranking along ρ is tight i.e. every odd rank up to the maximum non-∞ rank is present in the level rankings. Together with Property (2) of Lemma (1), it is eventually the case that every level ranking in (f, O, X, i) states along ρ guesses the ranks ∞ and every odd rank in $[h]^{odd}$, where h is the largest non-∞ rank assigned by $\mathsf{r}^{\star}_{\mathcal{A},\alpha}$.

By the above discussion starting from a state (f_j, O_j, X_j, i_j) along ρ, the automaton $\mathcal{U}$ as it reads the word α eventually reaches a state (f_l, O_l, X_l, i_l), where

- If $i_j = \infty$ then i_l is the largest non-∞ rank assigned by f_l.
- If $i_j \in [2n]^{odd}$ and $i_j > 1$ then $i_l = i_j - 1$.
- If $i_j \in [2n]^{even}$ then $i_l = i_j - 1$.
- If $i_j = 1$ then $l = j + 1$ and $O_l = X_l = \emptyset$ and $i_l = 1$. Also, $i_{l+1} = max\_rank(f_{l+1}) = \infty$ and O_{l+1} contains all states that have been guessed to have rank ∞ by f_{l+1}.

From this it is seen that starting from a state (f, O, X, i) with $i = \infty$, a final state of $\mathcal{U}$, (f', O', X', i'), where $O' = X' = \emptyset$ and $i' = 1$ is eventually visited. But, starting from a state (f', O', X', i'), where $i' = 1$ a state (f, O, X, i) with $i = \infty$ must be visited along ρ. Applying this argument *ad infinitum* we prove that the run ρ visits final states of $\mathcal{U}$ infinitely often. Hence, ρ is an accepting run of $\mathcal{U}$ and $\alpha \in L(\mathcal{U})$.

□

Theorem 2. *The automaton $\mathcal{U}$ is unambiguous.*

Proof. Suppose, if possible, there is a word $\alpha \in \Sigma^{\omega}$ that has two distinct accepting runs ρ_1, ρ_2 in $\mathcal{U}$. By Lemma 6, $r_1((q', l')) = \mathsf{r}^{\star}((q', l')) = r_2((q', l'))$ for every vertex (q', l') in $G^{\mathcal{A}}_{\alpha}$. Let $(f_{1,l}, O_{1,l}, X_{1,l}, i_{1,l})$ and $(f_{2,l} O_{2,l}, X_{2,l}, i_{2,l})$ be the l^{th} states reached along ρ_1 and ρ_2 respectively. We show below by induction on l that $(f_{1,l}, O_{1,l}, X_{1,l}, i_{1,l}) = (f_{2,l}, O_{2,l}, X_{2,l}, i_{2,l})$ for all $l \geq 0$.

Base case : By our construction, $O_{1,0} = O_{2,0} = X_{1,0} = X_{2,0} = \emptyset$, $i_{1,0} = i_{2,0} = 1$. Since $r_1((q_0, 0)) = r_2((q_0, 0))$ as well, it follows that $f_{1,0} = f_{2,0}$, and hence $(f_{1,0}, O_{1,0}, X_{1,0}, i_{1,0}) = (f_{2,0}, O_{2,0}, X_{2,0}, i_{2,0})$.

Hypothesis : Assume the claim is true for $l \geq 0$. Hence, $(f_{1,l}, O_{1,l}, X_{1,l}, i_{1,l}) = (f_{2,l}, O_{2,l}, X_{2,l}, i_{2,l})$.

Induction step : Let $q \in Q$ such that $(q, l+1)$ is a vertex in $G^{\mathcal{A}}_{\alpha}$. Since $r_1((q', l')) = r_2((q', l')) = \mathsf{r}^{\star}_{\mathcal{A},\alpha}((q', l'))$ for every vertex (q', l') in G_{α}, it follows that $r_1((q, l+1)) = \mathsf{r}^{\star}((q, l+1)) = r_2((q, l+1))$. This implies $f_{1,l+1}(q) = \mathsf{r}^{\star}((q, l+1)) = f_{2,l+1}(q)$ for every such $q \in Q$. For every $q' \in Q$ such that

$(q', l+1)$ is not in $G_\alpha^{\mathcal{A}}$, by definition of level rankings $f_{1,l+1}(q') = f_{2,l+1} = \bot$. Hence, $f_{1,l+1} = f_{2,l+1}$. We thus have the following relations: (i) $f_{1,l+1} = f_{2,l+1}$ and (ii) $(f_{1,l}, O_{1,l}, X_{1,l}, i_{1,l}) = (f_{2,l}, O_{2,l}, X_{2,l}, i_{2,l})$. Furthermore, step (7) of our construction of algorithm BüchiDisambiguate uniquely determines the values of $O_{k,l+1}, X_{k,l+1}, i_{k,l+1}$ from values of $f_{k,l}, f_{k,l+1}, O_{k,l}, X_{k,l}, i_{k,l}$, for $k \in 1, 2$. It follows from the above facts that $(f_{1,l+1}, O_{1,l+1}, X_{1,l+1}, i_{1,l+1}) = (f_{2,l+1}, O_{2,l+1}, X_{2,l+1}, i_{2,l+1})$. Hence, $(f_{1,l}, O_{1,l}, X_{1,l}, i_{1,l}) = (f_{2,l}, O_{2,l}, X_{2,l}, i_{2,l})$ for all $l \geq 0$. This, however, contradicts the fact that ρ_1 and ρ_2 are distinct runs. Therefore, ρ_1 and ρ_2 must be the same run of $\mathcal{U}$.

We have thus shown that for every $\alpha \in \Sigma^\omega$, there is at most one accepting run of $\mathcal{U}$. It follows that $\mathcal{U}$ is unambiguous. □

Theorem 3. *The number of states of $\mathcal{U}$ is in $O(n \cdot (0.76n)^n)$.*

Proof. Every state of $\mathcal{U}$ is a (f, O, X, i) tuple, where f is a full-level ranking. We encode (f, O, X, i) tuples as 3-tuples (p, g, i), where $p \in \{0, 1, 2\}$ and g is a modified ranking function similar to that used in [11, 7].

While some states (f, O, X, i) in our construction correspond to tight full-level rankings f, others do not. We first use an extension of the idea in [11] to encode (f, O, X, i) with tight full-level ranking f as a tuple (p, g, i), where $g : Q \to \{1, \ldots, r\} \cup \{-1, -2, -3, \infty\}$ and $r = \mathit{max\_odd}(f)$. This is done as follows. For all $q \in Q$ but $q \notin O \cup X$, we let $g(q) = f(q)$. If $q \in O \cup X$ and $f(q) \notin [r]^{odd}$, then we let $g(q) = -1$ and $i = f(q)$. This part of the encoding is similar to that used in [11]. We extend this encoding to consider cases where $q \in O \cup X$ and $f(q) = k \in [r]^{odd}$.

There are three sub-cases to consider: (i) $O \cup X \neq \{q \mid q \in Q \wedge f(q) = k\}$, (ii) $O \cup X = \{q \mid q \in Q \wedge f(q) = k\}$ and $O \neq \emptyset$, and (iii) $X = \{q \mid q \in Q \wedge f(q) = k\}$. In the first case, we let $p = 0$, $i = k$, $g(q) = -2$ for all $q \in O$ and $g(q) = -3$ for all $q \in X$. Since there exists a state $q' \in Q \setminus (O \cup X)$ with rank k, the range of g contains $k \in [r]^{odd}$ in this case. In the second case, we let $p = 1$, $i = k$, $g(q) = k$ for all $q \in O$ and $g(q) = -3$ for all $q \in X$. Finally, in the third case, we let $p = 2$, $i = k$ and $g(q) = k$ for all $q \in X$. Thus, the range of g contains $k \in [r]^{odd}$ in both the second and third cases as well. Note that the component p in (p, g, i) is used only when $i \in [r]^{odd}$. It is now easy to see that g is always onto one of the sets $A_j \cup \{1, 3, \ldots, r\}$, where A_j is a subset of $\{\infty, -1, -2, -3\}$. The total number of functions of each of the above types is $O(\mathit{tight}(n))$. Since $p \in \{0, 1, 2\}$ following Schewe's analysis [11], the total number of (p, g, i) tuples is upper bounded by $O(n.\mathit{tight}(n)) = O(\mathit{tight}(n+1))$.

Now, let us consider states with non-tight full-level rankings. Our construction ensures that once an odd rank i appears in a level ranking g along a run ρ, all subsequent level rankings along ρ contain every rank in $\{i, i+2, \ldots \mathit{max\_odd}(g)\}$. The O, X and i components in states with non-tight full-level ranking is inconsequential; hence we ignore these. Suppose a state with non-tight level ranking f contains the odd ranks $\{j, \ldots, i-2, i\}$, where $1 < j \leq i$, $i = \mathit{max\_odd}(f)$. To encode this state, we first replace f with a level ranking g as follows. For all $k \in \{j, \ldots, i, i+1\}$ and $q \in Q$, if $f(q) = k$, then $g(q) = k - j + c$, where $c = 0$

if j is even and 1 otherwise. If $f(q) = \infty$, we let $g(q) = \infty$. Effectively, this transforms f to a tight full-level ranking g by shifting all ranks down by $j - c$. The original state can now be represented as the tuple $(p, g, -(j - c))$. Note that the third component of a state represented as (p, g, i) is always non-negative for states with tight full-level ranking, and always negative for states with non-tight full-level ranking. Hence, there is no ambiguity in decoding the state representation. Clearly, the total no. of states with non-tight full-level rankings is $O(n \cdot tight(n)) = O(tight(n + 1))$. Thus, the total count of all (f, O, X, i) states is in $O(tight(n + 1)) = O(n \cdot tight(n))$, where $tight(n) \approx (0.76n)^n$. □

4 An exponentially improved lower bound for NBA-SUBA translation

In this section, we prove a lower bound for the state complexity trade-off in translating an NBA to a strongly unambiguous NBA (SUBA). We start with the definition of SUBA.

Definition 2 (SUBA). *The automaton $\mathcal{A}$ is a* strongly unambiguous Büchi automata *(SUBA) if there is at most one final run for every $\alpha \in \Sigma^\omega$.*

Our proof technique relies on the full automaton technique of Yan [13]. The following definition of full automaton is from [13].

Definition 3 (Full automaton). *A full automaton $\mathcal{A}$ is described by the structure $\mathcal{A} = (\Sigma, Q, I, \delta, F)$ where Q is the set of states, $I \subseteq Q$ is the set of initial states, $\Sigma = 2^{(Q\times Q)}$ is the alphabet and δ is defined as follows: for all $q, q' \in Q, a \in \Sigma, \langle q, a, q' \rangle \in \Delta \Leftrightarrow \langle q, q' \rangle \in a$.*

Thus, a full automaton has a rich alphabet of size $2^{|Q|^2}$, and every automaton with $|Q|$ states has an embedding in a full automaton with the same number of states. An ω-word α over the alphabet Σ of a full automaton corresponds directly to the run-DAG $G^{\mathcal{A}}_\alpha$ of $\mathcal{A}$. Correspondingly, a letter $a \in \Sigma$ represents two successive levels of the run-DAG, and a finite word $w \in \Sigma^*$ represents a finite section of the run-DAG. Details on full automata and the full automata technique can be found in [13].

For purposes of this section, we focus on a special family of full automata $\mathcal{F} = \{\mathcal{A}_n \mid n \geq 2\}$. Automaton $\mathcal{A}_n$ in this family is given by $\mathcal{A}_n = (\Sigma_n, Q_n, I_n, \delta_n, F_n)$, where $Q_n = \{q_0, \ldots, q_{n-1}, q_n\}$ is a set of $n+1$ states, $I_n = \{q_0, \ldots q_{n-2}\}$ is the set of initial states, and $F_n = \{q_n\}$ is the singleton set of final states. We define a Q-ranking for $\mathcal{A}_n$ to be a full-level ranking $r : Q_n \to \langle n - 1 \rangle \cup \{\perp\}$ such that (i) r is a tight full-level ranking, (ii) $r(q_n) = \perp$, (iii) $r(q_{n-1}) = \infty$, and (v)for every $k \in [n - 1]$, $|r^{-1}(k)| = 1$. The total number of Q-rankings of $\mathcal{A}_n$ is easily seen to be $(n - 1)!$.

Let r_1 and r_2 be Q-rankings for $\mathcal{A}_n$. The word w is said to be Q-compatible with (r_1, r_2) if the following conditions are satisfied with r_1 (resp., r_2) as the full-level ranking of states at the first (resp., last) level of w viewed as a finite section of the run-DAG of $\mathcal{A}_n$.

- There is no path from the first level to the last level of w that either starts or ends in q_n.
- There is a path from q_i in the first level of w to q_j in the last level of w iff either $r_1(q_i) > r_2(q_j)$, or $r_1(q_i) = r_2(q_j) \in [n-1]^{odd} \cup \{infty\}$. Such a path is said to be *final* if it visits q_n; otherwise, it is *non-final*.

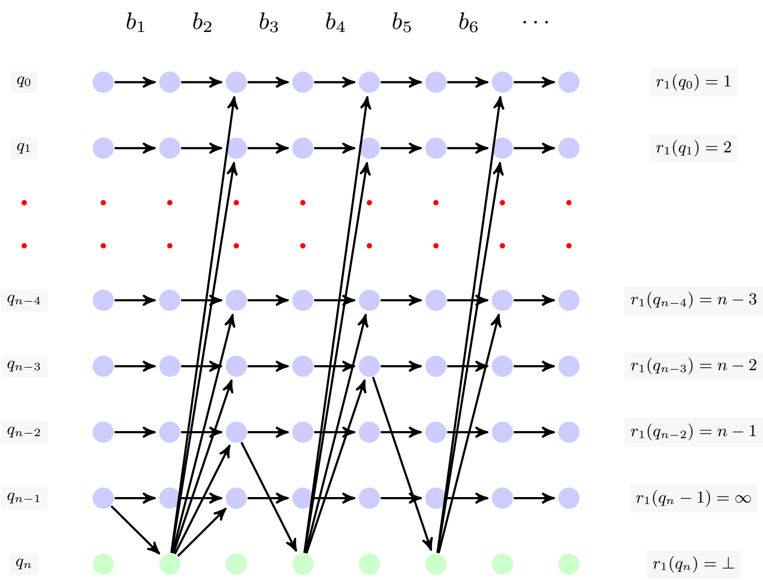

Fig. 1. Construction for w_1

Lemma 7. *For every pair (r_1, r_2) of Q-rankings for $\mathcal{A}_n$, there is a word $w \in \Sigma^*$ that is Q-compatible with (r_1, r_2).*

Proof. We show how to construct w as the concatenation of three words $w_1, w_2, w_3 \in \Sigma^*$. The proof that $w_1.w_2.w_3$ is Q-compatible with (r_1, r_2) follows immediately from their construction, and follows a similar argument as a related proof in [13] (Lemma 2 in [13]).

The word w_1 is given by $b_1 b_2 b_3 \ldots b_{2n}$, where each $b_i \in \Sigma = \wp(Q_n \times Q_n)$ is defined as follows. In the following, we use $Id_{non-final}$ as a shorthand for $\{(q_j, q_j) \mid 0 \leq j < n\}$ for notational convenience.

- $b_1 = Id_{non-final} \cup \{(q_{n-1}, q_n)\}$
- $b_2 = Id_{non-final} \cup \{(q_n, q_j) \mid 0 \leq j < n\}$
- For $1 \leq i \leq n-1$, $b_{2i+1} = Id_{non-final} \cup \{(q_j, q_n) \mid r_1(q_j) = n - i\}$ and $b_{2i+2} = Id_{non-final} \cup \{(q_n, q_j) \mid r_1(q_j) < n - i\}$

The word w_2 consists of the single letter of Σ given by $\{(q_i, q_j) \mid r_1(q_i) = r_2(q_j) \in [n-1]^{odd} \cup \{\infty\}$. The word w_3 is constructed in the same manner as w_1, but with r_2 used in place of r_1.

Lemma 8. *Let r_1, r_2 and r_3 be Q-rankings for $\mathcal{A}_n$. If $w_1 \in \Sigma^*$ is compatible with (r_1, r_2) and $w_2 \in \Sigma^*$ is compatible with (r_2, r_3), then $w_1 w_2$ is compatible with (r_1, r_3).*

The proof follows from Lemma 7 and mimics the proof of a related result in [13] (Lemma 3 in [13]).

We now show a factorial lower bound of the NBA to SUBA state complexity tradeoff. We make use of the following special class of UBA for this purpose. Given a NBA $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$, let $L(\mathcal{A}^{\{q\}})$ denote the set of ω-words accepted by $\mathcal{A}$ starting from state $q \in Q$.

Definition 4 (EUBA). *A UBA $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$ is called a state-exclusive UBA (EUBA) if for every state $q \in Q$ either $L(\mathcal{A}^{\{q\}}) \subseteq L(\mathcal{A})$ or $L(\mathcal{A}^{\{q\}}) \subseteq \Sigma^\omega \setminus L(\mathcal{A})$. In other words, all words accepted starting from q are either in the language of $\mathcal{A}$ or its complement.*

We show below that the NBA to EUBA state complexity tradeoff is in $\Omega((n-1)!)$. Since every SUBA is clearly a EUBA, this gives us the desired lower bound of the NBA to SUBA state complexity tradeoff.

Theorem 4. *Every EUBA that is language equivalent to the full automaton $\mathcal{A}_n$ in the family $\mathcal{F}$ has at least $(n-1)!$ states.*

Proof. Let E_n be a EUBA such that $L(E_n) = L(\mathcal{A}_n)$. Let $r_1, r_2, \ldots, r_{n-1!}$ be the Q-rankings for $\mathcal{A}_n$. Repeating this sequence of full-level rankings infinite number of times, we get an infinite sequence of full-level rankings. Let w_i be the word (as constructed in the proof of Lemma 7) that is Q-compatible with (r_i, r_{i+1}), for $i \in [(n-1)! - 1]$. Let w be the infinite word $(w_1 w_2 \ldots w_{(n-1)!-1})^\omega$.

From the definition of Q-compatibility, there is a final path between q_{n-1} (ranked ∞) at the first level of states in w_i to the same state (ranked ∞) at the last level of states in w_i, for every $i \in [(n-1)! - 1]$. The concatenation of these paths gives a path π' that starts from q_{n-1} and visits the final state q_n infinitely often. Note that we cannot have a path in w that starts from any state other than q_{n-1} and visits q_n infinitely often. This is because a visit to q_n from any other state q_j ($\neq q_{n-1}$) with rank k must necessarily be followed by a visit to a state with rank $< k$. Hence, infinitely many visits to q_n will result in infinitely many rank reductions. This is an impossibility since ranks cannot increase along a path. Hence, paths in w starting from every state other than q_{n-1} can visit final states only finitely often.

Since q_{n-1} is not an initial state of $\mathcal{A}_n$, the path π' considered above is not an accepting run of $\mathcal{A}_n$. Hence, $w \notin L(\mathcal{A}_n)$. Let a be the letter in Σ that consists of only the edge (q_0, q_{n-1}). Hence, the path $q_0\pi'$ is an accepting run of $\mathcal{A}_n$, and $aw \in L(\mathcal{A}_n)$.

Since $L(E_n) = L(\mathcal{A}_n)$, there must be an accepting run ρ of E_n on aw. Let k be the smallest index such that $\rho(k) \in inf(\rho)$ for all $i \geq k$. Suppose further that $s_0 = 0$ and $|w_i| = s_i$ for all $i \in [(n-1)!-1]$, and $s = \sum_{i=0}^{[(n-1)!-1]} s_i$. Let t be the smallest index such that $t \geq k$ and $t = p.s$ for some integer $p > 0$ and let $T = (n-1)! - 1$. Consider the sequence of indices $t, t+n_1.s+s_1, t+n_2.s+s_2, \ldots t+n_T.s+s_T$, where $n_i \geq 0$ and $n_i \geq n_j$ for $1 \leq j < i \leq T$ are such that ρ visits a final state of E_n between conseuctive indices in the sequence. If E_n has fewer than $(n-1)!$ states, there must be a state of E_n that repeats in $\rho(t), \rho(t+n_1.s+s_1), \ldots \rho(t+n_T.s+s_T)$. Let $i, j \in [(n-1)!-1]$ be such that $i < j$ and $\rho(t+n_i.s+s_i) = \rho(t+n_j.s+s_j) = z$, say. Clearly, $z \in inf(\rho)$, ρ visits a final state of E_n between indices $t + n_i.s + s_i$ and $t + n_j.s + s_j$, and $r_i \neq r_j$ for these values of i and j. Let the segment of the word aw between these two occurrences of z be v. Then, there is a final path from z to itself along v; hence z accepts v^ω.

We now show that v^ω is also in $L(E_n)$. Consider the two Q-rankings r_i and r_j obtained above. Clearly, the word v is Q-compatible with (r_i, r_j). Since $r_i \neq r_j$ and both are Q-rankings for $\mathcal{A}_n$, there is a state q_l for $l \in [n-2]$, such that $r_i(q_l) > r_j(q_l)$. By the definition of compatibility, there is a final run along v from state q_l to itself. Since q_l is an initial state of $\mathcal{A}_n$, there exists an accepting run of $\mathcal{A}_n$ on v^ω. Therefore, $v^\omega \in L(\mathcal{A}_n)$; since $L(E_n) = L(\mathcal{A}_n)$, $v^\omega \in L(E_n)$ as well.

Since ρ is an accepting run of E_n, it visits at least one final state of E_n infinitely often. Therefore, there exists a strict suffix w' of w starting from the $(t + n_i.s + s_i)^{th}$ index such that the corresponding run of E_n starting at z sees at least one final state infinitely often. Hence, $w' \in L(E_n^{\{z\}})$. However, the final run of $\mathcal{A}_n$ on w' starts at q_{n-1} since this is the only state ranked ∞ in w. Since q_{n-1} is not an initial state of $\mathcal{A}_n$, $w' \notin L(\mathcal{A}_n)$. Since $L(\mathcal{A}_n) = L(E_n)$, this also implies that $w' \notin L(E_n)$.

Thus, $v^\omega \in L(E_n^{\{z\}})$ and $v^\omega \in L(E_n)$, while $w' \in L(E_n^{\{z\}})$ and $w' \notin L(E_n)$. This contradicts the state-exclusivity property of EUBA. Hence, the number of states of E_n is at least $(n-1)!$. □

Theorem 5. *Every SUBA that is language equivalent to NBA $\mathcal{A}_n$ has at least $n - 1!$ states.*

Proof. The proof follows from the inclusion of the class SUBA within the class EUBA. □

5 A new lower bound proof for disambiguation

We now show an exponential lower bound for the NBA-UBA state complexity trade-off. This lower bound was already known from a result due to Schmidt [12]. However, the technique used by Schmidt involves computing ranks of specially constructed matrices. In contrast, our proof uses the full automata technique of Yan.

Definition 5 (Trim UBA). *Let $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$ be a UBA. $\mathcal{A}$ is* trim *if $L(\mathcal{A}^{\{q\}}) \neq \emptyset$ for every $q \in Q$.*

Every UBA can be transformed to a language equivalent trim UBA simply by removing states from which no word can be accepted. Let $\mathcal{A}$ be a full automaton having 1 initial state, 1 final state, and n other (non-initial and non-final) states. For each non-empty subset S of non-initial and non-final states (there are $2^n - 1$ such subsets), let a_S denote the letter on which we have edges in $\mathcal{A}$ from the initial state to only the states in S. Similarly, let b_S denote the letter on which we have edges in $\mathcal{A}$ from only the states in S to the final state, and also from the final state to itself. It is easy to see that $a_{S_1} b_{S_2}^\omega$ is accepted if and only if $S_1 \cap S_2 \neq \emptyset$. For this proof, we will restrict ourselves to words of the form b_S^ω. Let D be an unambiguous and trim automaton accepting the same language as $\mathcal{A}$.

Theorem 6. *The number of states of D is at least $2^n - 1$.*

Proof. For every state q of D, let $\hat{L}(D^{\{q\}})$ denote the set of words of the form b_S^ω accepted by D, starting from q. Furthermore, for a state q of D, and for $l \in \Sigma^*$, define $L(l, q) = \{w \mid w \in \Sigma^\omega, l \cdot w \in \hat{L}(D^{\{q\}})\}$. If s_1 and s_2 are initial states of D, by definition of unambiguous automata, $\hat{L}(D^{\{s_1\}}) \cap \hat{L}(D^{\{s_2\}}) = \emptyset$. Also if any state s in D has paths to two distinct states r_1 and r_2 that are labeled by the same word $l \in \Sigma^*$, then by definition of unambiguous and trim automata, $\hat{L}(D^{\{r_1\}})$ and $\hat{L}(D^{\{r_2\}})$ must be disjoint. Clearly, $L(l, s)$ is the union of $\hat{L}(D^{\{r\}})$, where r ranges over all the states that have a path in D from s labeled l. It follows that these languages $\hat{L}(D^{\{r\}})$ must be pairwise disjoint, and their union is $L(l, s)$.

For a set T of states of D, define $\hat{L}(D^T) = \bigcup_{s \in T} \hat{L}(D^{\{s\}})$. If we also have $\hat{L}(D^{\{s_1\}})$ and $\hat{L}(D^{\{s_2\}})$ disjoint for all distinct $s_1, s_2 \in T$, then $\hat{L}(D^T)$ equals the XOR of the sets $\hat{L}(D^{\{r\}})$, where r ranges over T. Now for each non-empty set K of the non-initial non-final states of $\mathcal{A}$, consider the set T of states in D that have an edge from an initial state of D on the letter a_K. Then $\hat{L}(D^T)$ is the set of all words of the form b_S^ω, where S is a subset of non-initial non-final states of $\mathcal{A}$ such that $S \cap K \neq \emptyset$. Call this set $M(K)$. Since D is unambiguous, $M(K)$ can be obtained as the XOR of languages $\hat{L}(D^{\{t\}})$, where t ranges over T. We will now show that for each subset S of non-initial non-final states of $\mathcal{A}$, the set containing only the word b_S^ω can be obtained by XORing appropriate languages $\hat{L}(D^{\{t\}})$. Since there are $2^n - 1$ possible subsets S, this shows that by XORing appropriate languages $\hat{L}(D^{\{t\}})$, we can get up to $2^{2^n - 1}$ different sets. This, in turn, implies that the number of distinct values taken by t, i.e. number of states in D, is at least $2^n - 1$.

We will prove the above claim by downward induction on the number of states in S. Suppose S is the set of all n non-initial and non-final states of $\mathcal{A}$. Then the set containing only b_S^ω can be obtained by XORing $M(K)$, where K ranges over all subsets of non-initial non-final states of $\mathcal{A}$. This is because b_S^ω occurs in all $2^n - 1$ (i.e, an odd number) languages $M(K)$, where K ranges over all subsets of non-initial non-final states of $\mathcal{A}$. However, for any other subset S' of non-initial non-final states of $\mathcal{A}$, $b^{\omega}_{S'}$ occurs only in those $M(K)$s where $K \cap S' \neq \emptyset$. The latter is precisely the set of all subsets K excluding those that are disjoint from S', and the number of such subsets is even for $|S'| < n$.

Now suppose we can obtain b^{ω}_S for all S with $|S| > t$. Then we can XOR every $M(K)$ suitably with singleton sets containing b^{ω}_S for $|S| > t$, such that the resulting modified languages $M'(K)$ do not contain any b^{ω}_S for $|S| > t$. Now consider any set S of size t. Then take XOR of the modified languages $A(K)$ obtained above for all subsets K of S. By definition, b^{ω}_S occurs in all of these $M'(K)$, which are odd in count $(2^t - 1)$. For any other set S' of cardinality $\leq t$, its intersection with S is a strict subset of S. It doesn't occur in precisely those $M'(K)$s where K is a non-empty subset of $S \setminus S'$; this, however, is odd in count as $S \setminus S'$ is nonempty. So the sets containing $b^{\omega}_{S'}$ are even in number.

This proves that all singleton sets containing only b^{ω}_S can be obtained for all nonempty subset S of non-initial non-final states of $\mathcal{A}$. As argued above, this implies that D must have at least $2^n - 1$ states. $\square$

6 Summary of results

We now summarize our results on the state space blowup in transforming an NBA to UBA and SUBA. Let $\mathsf{Size}_{NBA:\mathcal{C}}(n)$ denote the worst-case state complexity of an automaton in class $\mathcal{C}$ that accepts the same language as an NBA with n states. Table 1 shows upper and lower bounds of the state complexity trade-off when transforming an arbitrary NBA to an automaton of the specified class $\mathcal{C}$.

Target class ($\mathcal{C}$)	$\mathsf{Size}_{NBA:\mathcal{C}}(n)$ from this paper		$\mathsf{Size}_{NBA:\mathcal{C}}(n)$ from earlier work	
	Lower bound	Upper bound	Lower bound	Upper bound
UBA	$2^n - 1$	$\mathsf{O}(\mathsf{n} \cdot (0.76\mathsf{n})^{\mathsf{n}})$ (Thm 3)	$2^n - 1$ [12]	$4 \cdot (3n)^n$ [6]
SUBA	$\Omega(\mathsf{n} - 1!)$ (Thm 5)	$O((12n)^n)$	$2^n - 1$ [12]	$O((12n)^n$ [3]

Table 1. Comparison of state complexity trade-offs

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