

## ON THE GROWTH OF LINEAR RECURRENCES IN FUNCTION FIELDS

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(Received 21 July 2020; accepted 12 August 2020; first published online 9 November 2020)

### Abstract

Let  $(G_n)_{n=0}^\infty$  be a nondegenerate linear recurrence sequence whose power sum representation is given by  $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$ . We prove a function field analogue of the well-known result in the number field case that, under some nonrestrictive conditions,  $|G_n| \geq (\max_{j=1,\dots,t} |\alpha_j|)^{n(1-\varepsilon)}$  for  $n$  large enough.

2020 *Mathematics subject classification*: primary 11B37; secondary 11J87, 11R58.

*Keywords and phrases*: function fields, linear recurrences,  $S$ -units.

### 1. Introduction

Let  $(G_n)_{n=0}^\infty$  be a nondegenerate linear recurrence sequence with power sum representation  $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$ . This expression makes sense for a sequence  $(G_n)_{n=0}^\infty$  taking values in any field  $K$ ; the characteristic roots  $\alpha_i$  as well as the coefficients of the polynomials  $a_i$  then lie in a finite extension  $L$  of  $K$ . In this paper  $K$  is either a number field or a function field in one variable of characteristic zero. The nondegenerate condition means in the number field case that no ratio  $\alpha_i/\alpha_j$  for  $i \neq j$  is a root of unity and in the function field case that no ratio  $\alpha_i/\alpha_j$  for  $i \neq j$  is contained in the field of constants. In the number field case it is well known that, if  $\max_{j=1,\dots,t} |\alpha_j| > 1$ , then for any  $\varepsilon > 0$  the inequality

$$|G_n| \geq (\max_{j=1,\dots,t} |\alpha_j|)^{n(1-\varepsilon)} \quad (1.1)$$

is satisfied for every sufficiently large  $n$ .

The purpose of this paper is to prove an analogous result in the case of a function field in one variable of characteristic zero. Thus we answer, in the setting we are

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Supported by Austrian Science Fund (FWF): I4406.

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working in, Open Question 3 in [11]; we are grateful to Shparlinski for bringing this paper to our attention. Firstly, we will prove a theorem which states an inequality for an arbitrary valuation in the splitting field  $L$  of the characteristic polynomial belonging to the linear recurrence sequence. Secondly, we will derive a corollary for the special case of polynomial power sums. In this special case the inequality takes a form very similar to (1.1). At this point we make the following observation: Theorem 1 in [6] already implies that under some nondegeneracy conditions the degree of polynomials in a linear recurrence sequence with polynomial roots cannot be bounded and therefore must grow to infinity as  $n$  does. But this theorem does not say how fast it must grow. In the present paper we will give a bound depending on  $n$  for the minimal possible degree of  $G_n$ .

The number field case is often mentioned (see [3, 8] or more recently [1, 2]), but it is not that easy to access a proof of it (see [10] or the formulation in [9]). So we give a complete proof based on results of Evertse and Schmidt in the Appendix. By doing so we contribute to the goal that well-known facts should be fully accessible with proof following van der Poorten's wise statement in [9] that 'all too frequently, the well known is [often] not generally known, let alone known well'.

## 2. Results and notations

Throughout the paper we denote by  $K$  a function field in one variable over  $\mathbb{C}$ . By  $L$  we usually denote a finite algebraic extension of  $K$ . For the convenience of the reader we give a short summary of the notion of valuations that can also be found in [4]. For  $c \in \mathbb{C}$  and  $f(x) \in \mathbb{C}(x)$ , where  $\mathbb{C}(x)$  is the rational function field over  $\mathbb{C}$ , we denote by  $\nu_c(f)$  the unique integer such that  $f(x) = (x - c)^{\nu_c(f)} p(x)/q(x)$  with  $p(x), q(x) \in \mathbb{C}[x]$  such that  $p(c)q(c) \neq 0$ . Further, we write  $\nu_\infty(f) = \deg q - \deg p$  if  $f(x) = p(x)/q(x)$ . These functions  $\nu : \mathbb{C}(x) \rightarrow \mathbb{Z}$  are up to equivalence all valuations in  $\mathbb{C}(x)$ . If  $\nu_c(f) > 0$ , then  $c$  is called a zero of  $f$ , and if  $\nu_c(f) < 0$ , then  $c$  is called a pole of  $f$ , where  $c \in \mathbb{C} \cup \{\infty\}$ . For a finite extension  $K$  of  $\mathbb{C}(x)$  each valuation in  $\mathbb{C}(x)$  can be extended to no more than  $[K : \mathbb{C}(x)]$  valuations in  $K$ . This again gives up to equivalence all valuations in  $K$ . Both in  $\mathbb{C}(x)$  as well as in  $K$  the sum formula

$$\sum_{\nu} \nu(f) = 0$$

holds, where the sum is taken over all valuations in the relevant function field. Valuations have the properties  $\nu(fg) = \nu(f) + \nu(g)$  and  $\nu(f + g) \geq \min(\nu(f), \nu(g))$  for all  $f, g \in K$ . Each valuation in a function field corresponds to a place and vice versa. The places can be thought of as the equivalence classes of valuations. For more information about valuations and places we refer to [13].

For any power sum  $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$  with  $a_j(n) = \sum_{k=0}^{m_j} a_{jk}n^k$  and any valuation  $\mu$  (in a function field  $L/K$  containing the  $\alpha_j$  and the coefficients of the  $a_j$ ) we

have the trivial bound

$$\begin{aligned}
 \mu(G_n) &= \mu(a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n) \geq \min_{j=1,\dots,t} \mu(a_j(n)\alpha_j^n) \\
 &\geq \min_{j=1,\dots,t} \mu(a_j(n)) + \min_{j=1,\dots,t} \mu(\alpha_j^n) \\
 &\geq \min_{j=1,\dots,t} \min_{k=0,\dots,m_j} \mu(a_{jk}n^k) + n \cdot \min_{j=1,\dots,t} \mu(\alpha_j) \\
 &= \min_{\substack{j=1,\dots,t \\ k=0,\dots,m_j}} \mu(a_{jk}) + n \cdot \min_{j=1,\dots,t} \mu(\alpha_j) = \widetilde{C} + n \cdot \min_{j=1,\dots,t} \mu(\alpha_j).
 \end{aligned}$$

Our main result is now the following theorem which gives a bound in the other direction.

**THEOREM 2.1.** *Let  $(G_n)_{n=0}^\infty$  be a nondegenerate linear recurrence sequence taking values in  $K$  with power sum representation  $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$ . Let  $L = K(\alpha_1, \dots, \alpha_t)$  be the splitting field of the characteristic polynomial of that sequence and let  $\mu$  be a valuation on  $L$ . Then there is an effectively computable constant  $C$ , independent of  $n$ , such that, for every sufficiently large  $n$ , the inequality*

$$\mu(G_n) \leq C + n \cdot \min_{j=1,\dots,t} \mu(\alpha_j)$$

*holds.*

For the special case of a linear recurrence sequence of complex polynomials having complex polynomials as characteristic roots we get the following lower bound for the degree of the  $n$ th member of the sequence.

**COROLLARY 2.2.** *Let  $(G_n)_{n=0}^\infty$  be a nondegenerate linear recurrence sequence of polynomials in  $\mathbb{C}[x]$  with power sum representation  $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$  such that  $\alpha_1, \dots, \alpha_t \in \mathbb{C}[x]$ . Then there is an effectively computable constant  $C$ , independent of  $n$ , such that, for every sufficiently large  $n$ , the inequality*

$$\deg G_n \geq n \cdot \max_{j=1,\dots,t} \deg \alpha_j - C$$

*holds.*

In the case of a binary recurrence sequence of polynomials, that is,  $t = 2$  in Corollary 2.2, one can use Mason's function field *abc* theorem (see [7]) to show that the number of distinct zeros of  $G_n$  must go to infinity as  $n$  does. By considering this in slightly more detail, the number of distinct zeros of  $G_n$  can be bounded above (trivially) and below (by means of the *abc* theorem) both by linear polynomials in  $n$ .

It would be interesting to prove a function field variant of Corollary 3.1 in [1]. However, because of Lemma 3.1, which is based on Dirichlet's classical approximation theorem, we are not (yet) able to prove such a statement.

In the proof given in the next section we will apply the following special case of Theorem 1 in [6].

**LEMMA 2.3.** *Let  $K$  be as above and  $L$  be a finite extension of  $K$  of genus  $g$ . Furthermore, let  $\alpha_1, \dots, \alpha_d \in L^*$  with  $d \geq 2$  be such that  $\alpha_i/\alpha_j \notin \mathbb{C}^*$  for each pair of subscripts  $i, j$  with  $1 \leq i < j \leq d$ . Moreover, for every  $i = 1, \dots, d$ , let  $\pi_{i1}, \dots, \pi_{ir_i} \in L$  be  $r_i$  linearly independent elements over  $\mathbb{C}$ . Put*

$$q = \sum_{i=1}^d r_i.$$

*Then, for every  $n \in \mathbb{N}$  such that*

$$\{\pi_{il}\alpha_i^n : l = 1, \dots, r_i, i = 1, \dots, d\}$$

*is linearly dependent over  $\mathbb{C}$ , but no proper subset of this set is linearly dependent over  $\mathbb{C}$ , we have*

$$n \leq C = C(q, g, \pi_{il}, \alpha_i : l = 1, \dots, r_i, i = 1, \dots, d).$$

The proof will also make use of height functions in function fields. Let us therefore define the height of an element  $f \in L^*$  by

$$\mathcal{H}(f) := - \sum_{\nu} \min(0, \nu(f)) = \sum_{\nu} \max(0, \nu(f)),$$

where the sum is taken over all valuations in the function field  $L/\mathbb{C}$ . For every  $z \in L \setminus \mathbb{C}$ ,

$$\begin{aligned} \mathcal{H}(z) &= \sum_{\nu} \max(0, \nu(z)) = \sum_P \max(0, \nu_P(z)) \\ &= \deg \sum_P \max(0, \nu_P(z))P = \deg(z)_0 = [L : \mathbb{C}(z)] = \deg_{\mathbb{C}}(z), \end{aligned}$$

by Theorem I.4.11 in [13], where we have used the fact that all places have degree one since we are working over  $\mathbb{C}$  (instead of the height, one can use  $\deg_{\mathbb{C}}(z) = [L : \mathbb{C}(z)]$  as in [14]). Additionally, we define  $\mathcal{H}(0) = \infty$ . This height function satisfies some basic properties that are listed in the next lemma which is proven in [5].

**LEMMA 2.4.** *Let  $\mathcal{H}$  denote the height on  $L/\mathbb{C}$  as above. Then, for  $f, g \in L^*$ :*

- (a)  $\mathcal{H}(f) \geq 0$  and  $\mathcal{H}(f) = \mathcal{H}(1/f)$ ;
- (b)  $\mathcal{H}(f) - \mathcal{H}(g) \leq \mathcal{H}(f + g) \leq \mathcal{H}(f) + \mathcal{H}(g)$ ;
- (c)  $\mathcal{H}(f) - \mathcal{H}(g) \leq \mathcal{H}(fg) \leq \mathcal{H}(f) + \mathcal{H}(g)$ ;
- (d)  $\mathcal{H}(f^n) = |n| \cdot \mathcal{H}(f)$ ;
- (e)  $\mathcal{H}(f) = 0 \iff f \in \mathbb{C}^*$ ;
- (f)  $\mathcal{H}(A(f)) = \deg A \cdot \mathcal{H}(f)$  for any  $A \in \mathbb{C}[T] \setminus \{0\}$ .

We will also use the following function field analogue of the Schmidt subspace theorem.

**PROPOSITION 2.5 (Zannier [14]).** *Let  $F/\mathbb{C}$  be a function field in one variable and of genus  $g$ . Let  $\varphi_1, \dots, \varphi_n \in F$  be linearly independent over  $\mathbb{C}$  and let  $r \in \{0, 1, \dots, n\}$ .*

Let  $S$  be a finite set of places of  $F$  containing all the poles of  $\varphi_1, \dots, \varphi_n$  and all the zeros of  $\varphi_1, \dots, \varphi_r$ . Put  $\sigma = \sum_{i=1}^n \varphi_i$ . Then

$$\sum_{\nu \in S} (\nu(\sigma) - \min_{i=1, \dots, n} \nu(\varphi_i)) \leq \binom{n}{2} (|S| + 2g - 2) + \sum_{i=r+1}^n \mathcal{H}(\varphi_i).$$

### 3. Proofs

**PROOF OF THEOREM 2.1.** Denote the coefficients of the polynomial  $a_j(n) \in L[n]$  by  $a_{j0}, a_{j1}, \dots, a_{jm_j}$  where  $m_j$  is the degree of  $a_j(n)$ . So

$$a_j(n) = \sum_{k=0}^{m_j} a_{jk} n^k.$$

First assume that the recurrence sequence is of the shape  $G_n = a_1(n)\alpha_1^n$ . Using Lemma 2.4,

$$\begin{aligned} \mu(G_n) &= \mu(a_1(n)) + n\mu(\alpha_1) \leq \mathcal{H}(a_1(n)) + n\mu(\alpha_1) \\ &\leq \sum_{k=0}^{m_1} \mathcal{H}(a_{1k} n^k) + n\mu(\alpha_1) = \sum_{k=0}^{m_1} \mathcal{H}(a_{1k}) + n\mu(\alpha_1). \end{aligned}$$

Thus from now on we can assume that  $t \geq 2$ . Let  $\pi_{j1}, \dots, \pi_{jk_j}$  be a maximal  $\mathbb{C}$ -linear independent subset of  $a_{j0}, a_{j1}, \dots, a_{jm_j}$ . Then we can write the sequence as

$$G_n = \sum_{j=1}^t \left( \sum_{i=1}^{k_j} b_{ji}(n) \pi_{ji} \right) \alpha_j^n$$

with polynomials  $b_{ji}(n) \in \mathbb{C}[n]$ . Since  $a_j(n)$  is not the zero polynomial, there is for each  $j$  at least one index  $i$  such that  $b_{ji}(n)$  is not the zero polynomial. Without loss of generality we can assume that no  $b_{ji}(n)$  is the zero polynomial since otherwise we can throw out all zero polynomials and renumber the remaining terms. It does not matter whether all  $\pi_{ji}$  occur in the sum or not. Moreover, we assume that  $n$  is large enough such that  $b_{ji}(n) \neq 0$  for all  $j, i$ .

Consider as a next step the set

$$M := \{\pi_{ji} \alpha_j^n : i = 1, \dots, k_j, j = 1, \dots, t\}.$$

We intend to apply Lemma 2.3. If  $M$  is linearly dependent over  $\mathbb{C}$ , then we choose a minimal linearly dependent subset  $\widetilde{M}$  of  $M$ , that is, a linearly dependent subset  $\widetilde{M}$  with the property that no proper subset of  $\widetilde{M}$  is linearly dependent. Let  $\widetilde{G}_n$  be the linear recurrence sequence associated with this subset  $\widetilde{M}$ , that is,

$$\widetilde{G}_n = \sum_{j=1}^s \left( \sum_{i=1}^{\widetilde{k}_j} b_{ji}(n) \pi_{ji} \right) \alpha_j^n$$

for  $s \leq t$  and after a suitable renumbering of the summands. Since  $\pi_{j_1}, \dots, \pi_{j_k}$  are  $\mathbb{C}$ -linearly independent we have  $s \geq 2$ . Applying Lemma 2.3 to

$$\widetilde{M} := \{\pi_{ji}\alpha_j^n : i = 1, \dots, \widetilde{k}_j, j = 1, \dots, s\}$$

gives an upper bound for  $n$ . Thus for  $n$  large enough this subset  $\widetilde{M}$  of  $M$  cannot be linearly dependent. Because of the fact that there are only finitely many subsets of  $M$ , for  $n$  large enough the set  $M$  must be linearly independent.

We assume from here on that  $n$  is large enough such that  $M$  is linearly independent. For each fixed  $n$  we have  $b_{ji}(n) \in \mathbb{C}^*$ . Thus the set

$$M' := \{b_{ji}(n)\pi_{ji}\alpha_j^n : i = 1, \dots, k_j, j = 1, \dots, t\}.$$

is linearly independent over  $\mathbb{C}$  and contains for each  $j = 1, \dots, t$  at least one element. Let  $S$  be a finite set of places of  $L$  containing all zeros and poles of  $\alpha_j$  for  $j = 1, \dots, t$  and of the nonzero  $a_{ji}$  for  $j = 1, \dots, t$  and  $i = 1, \dots, m_j$  as well as  $\mu$  and the places lying over  $\infty$ . Now applying Proposition 2.5 yields

$$\sum_{v \in S} \left( \nu(G_n) - \min_{\substack{j=1, \dots, t \\ i=1, \dots, k_j}} \nu(b_{ji}(n)\pi_{ji}\alpha_j^n) \right) \leq \binom{\sum_{j=1}^t k_j}{2} (|S| + 2g - 2) =: C_1$$

and, since each summand in the sum on the left-hand side is nonnegative,

$$\mu(G_n) - \min_{\substack{j=1, \dots, t \\ i=1, \dots, k_j}} \mu(b_{ji}(n)\pi_{ji}\alpha_j^n) \leq C_1.$$

Therefore for all  $j_0 = 1, \dots, t$  and  $i_0 = 1, \dots, k_{j_0}$ ,

$$\begin{aligned} \mu(G_n) &\leq C_1 + \min_{\substack{j=1, \dots, t \\ i=1, \dots, k_j}} \mu(b_{ji}(n)\pi_{ji}\alpha_j^n) \\ &\leq C_1 + \mu(b_{j_0 i_0}(n)\pi_{j_0 i_0}\alpha_{j_0}^n) \\ &= C_1 + \mu(\pi_{j_0 i_0}) + n\mu(\alpha_{j_0}) \\ &\leq C_1 + \max_{\substack{j=1, \dots, t \\ i=0, \dots, m_j, a_{ji} \neq 0}} \mu(a_{ji}) + n\mu(\alpha_{j_0}) \\ &\leq C_1 + \max_{\substack{j=1, \dots, t \\ i=0, \dots, m_j, a_{ji} \neq 0}} \mathcal{H}(a_{ji}) + n\mu(\alpha_{j_0}) \\ &= C_2 + n\mu(\alpha_{j_0}). \end{aligned}$$

Since this holds for all  $j_0 = 1, \dots, t$ ,

$$\mu(G_n) \leq C_2 + n \cdot \min_{j=1, \dots, t} \mu(\alpha_j). \quad \square$$

**PROOF OF COROLLARY 2.2.** We can apply Theorem 2.1 with  $L = K = \mathbb{C}(x)$  and  $\mu = \nu_\infty$ . This yields

$$-\deg G_n = \nu_\infty(G_n) \leq C + n \cdot \min_{j=1, \dots, t} \nu_\infty(\alpha_j) = C - n \cdot \max_{j=1, \dots, t} \deg \alpha_j$$

which immediately implies the inequality in question.  $\square$

## Appendix A. The number field case

In this appendix we will give a proof of the following theorem.

**THEOREM A.1.** *Let  $(G_n)_{n=0}^\infty$  be a nondegenerate linear recurrence sequence taking values in a number field  $K$  and let  $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$  with algebraic integers  $\alpha_1, \dots, \alpha_t$  be its power sum representation satisfying  $\max_{j=1, \dots, t} |\alpha_j| > 1$ . Denote by  $|\cdot|$  the usual absolute value on  $\mathbb{C}$ . Then, for any  $\varepsilon > 0$ , the inequality*

$$|G_n| \geq (\max_{j=1, \dots, t} |\alpha_j|)^{n(1-\varepsilon)}$$

*is satisfied for every sufficiently large  $n$ .*

Note that this result is not effective in the sense that we do not give a bound  $n_0$  such that the inequality is satisfied for all  $n$  greater than  $n_0$ . If we were to look more precisely at the limitations placed on  $n$  in the proof given below, it would be possible to give an (admittedly) rather complicated upper bound on the number of exceptions. This bound would have the following form: if  $n/\log n > B_1$ , then there are at most  $B_2$  values of  $n$  for which the inequality is not valid. Since the explicit constants are not so enlightening we will not calculate them in detail.

From here on  $K$  will denote a number field. In the proof we will need three auxiliary results which are listed below. The first one is a result of Schmidt.

**LEMMA A.2 (Schmidt [12]).** *Suppose that  $(G_n)_{n \in \mathbb{Z}}$  is a nondegenerate linear recurrence sequence of complex numbers, whose characteristic polynomial has  $k$  distinct roots of multiplicity at most  $a$ . Then the number of solutions  $n \in \mathbb{Z}$  of the equation*

$$G_n = 0$$

*can be bounded above by*

$$c(k, a) = e^{(7k^a)^{8ka}}.$$

The second is a result of Evertse. We use the notation

$$\|\mathbf{x}\| = \max_{\substack{k=0, \dots, t \\ i=1, \dots, D}} |\sigma_i(x_k)|$$

with  $\{\sigma_1, \dots, \sigma_D\}$  the set of all embeddings of  $K$  in  $\mathbb{C}$  and  $\mathbf{x} = (x_0, x_1, \dots, x_t)$ . Moreover, we denote by  $O_K$  the ring of integers in  $K$ .

**LEMMA A.3 (Evertse [3]).** *Let  $t$  be a nonnegative integer and  $S$  a finite set of places in  $K$ , containing all infinite places. Then for every  $\varepsilon > 0$  a constant  $C$  exists, depending only on  $\varepsilon, S, K, t$  such that for each nonempty subset  $T$  of  $S$  and every vector  $\mathbf{x} = (x_0, x_1, \dots, x_t) \in O_K^{t+1}$  with*

$$x_{i_0} + x_{i_1} + \cdots + x_{i_s} \neq 0$$

for each nonempty subset  $\{i_0, i_1, \dots, i_s\}$  of  $\{0, 1, \dots, t\}$ ,

$$\left( \prod_{k=0}^t \prod_{v \in S} \|x_k\|_v \right) \prod_{v \in T} \|x_0 + x_1 + \dots + x_t\|_v \geq C \left( \prod_{v \in T} \max_{k=0, \dots, t} \|x_k\|_v \right) \|\mathbf{x}\|^{-\varepsilon}.$$

Furthermore, we will need the following lemma which also can be found in [3].

**LEMMA A.4.** *Suppose  $K$  is a number field of degree  $D$ , let  $f(X) \in K[X]$  be a polynomial of degree  $m$  and  $T$  a nonempty set of primes on  $K$ . Then there exists a positive constant  $c$ , depending only on  $K, f$  such that for all  $r \in \mathbb{Z}$  with  $r \neq 0$  and  $f(r) \neq 0$ ,*

$$\begin{aligned} c^{-1} |r|^{-Dm} &\leq \left( \prod_v \max(1, \|f(r)\|_v) \right)^{-1} \leq \prod_{v \in T} \|f(r)\|_v \\ &\leq \prod_v \max(1, \|f(r)\|_v) \leq c |r|^{Dm}. \end{aligned}$$

**PROOF OF THEOREM A.1.** Since the characteristic roots  $\alpha_j$  of  $G_n$  are algebraic integers we can find a nonzero integer  $z$  such that  $za_j(n)\alpha_j^n$  are algebraic integers for all  $j = 1, \dots, t$  and all  $n \in \mathbb{N}$ . Set  $L = K(\alpha_1, \dots, \alpha_t)$ , the splitting field of the characteristic polynomial of the sequence  $G_n$ . Choose  $S$  as a finite set of places in  $L$  containing all infinite places as well as all places such that  $\alpha_1, \dots, \alpha_t$  are  $S$ -units. Let  $\mu$  be such that  $\|\cdot\|_\mu = |\cdot|$  is the usual absolute value on  $\mathbb{C}$ . In particular,  $\mu \in S$ . Further, define  $T = \{\mu\}$ .

As  $G_n$  is nondegenerate, the sequence  $\widetilde{G}_n = zG_n$  is also nondegenerate. Therefore by Lemma A.2, for  $n$  large enough,

$$za_{j_1}(n)\alpha_{j_1}^n + \dots + za_{j_s}(n)\alpha_{j_s}^n \neq 0$$

for each non-empty subset  $\{j_1, \dots, j_s\}$  of  $\{1, \dots, t\}$ . Thus we can apply Lemma A.3 and get

$$\left( \prod_{j=1}^t \prod_{v \in S} \|za_j(n)\alpha_j^n\|_v \right) |zG_n| \geq C \max_{j=1, \dots, t} |za_j(n)\alpha_j^n| \|\mathbf{z}\mathbf{x}\|^{-\varepsilon}$$

for  $\mathbf{x} = (a_1(n)\alpha_1^n, \dots, a_t(n)\alpha_t^n)$ . Without loss of generality, we can assume that  $|\alpha_1| = \max_{j=1, \dots, t} |\alpha_j|$ . Since  $z$  is a fixed integer and the  $\alpha_j$  are  $S$ -units, we can rewrite this as

$$\begin{aligned} \left( \prod_{j=1}^t \prod_{v \in S} \|a_j(n)\|_v \right) |G_n| &\geq C_1 \max_{j=1, \dots, t} |a_j(n)\alpha_j^n| \|\mathbf{x}\|^{-\varepsilon} \\ &\geq C_1 |a_1(n)\alpha_1^n| \|\mathbf{x}\|^{-\varepsilon} = C_1 |a_1(n)| |\alpha_1|^n \|\mathbf{x}\|^{-\varepsilon}. \end{aligned} \quad (\text{A.1})$$

In preparation for the next step, note that there exists a positive constant  $A$  such that

$$\max_{\substack{j=1, \dots, t \\ i=1, \dots, D}} |\sigma_i(\alpha_j)| \leq A \cdot |\alpha_1|.$$



We decompose  $\varepsilon = \gamma \cdot \delta$  with small  $\delta$  and  $A^\gamma \leq |\alpha_1|$ . This gives the estimates

$$\begin{aligned} \|\mathbf{x}\| &= \max_{\substack{j=1,\dots,t \\ i=1,\dots,D}} |\sigma_i(a_j(n)\alpha_j^n)| = \max_{\substack{j=1,\dots,t \\ i=1,\dots,D}} |\sigma_i(a_j(n))\sigma_i(\alpha_j)^n| \\ &\leq \max_{\substack{j=1,\dots,t \\ i=1,\dots,D}} |\sigma_i(a_j(n))| \cdot \max_{\substack{j=1,\dots,t \\ i=1,\dots,D}} |\sigma_i(\alpha_j)|^n \\ &\leq C_2 n^m \cdot \max_{\substack{j=1,\dots,t \\ i=1,\dots,D}} |\sigma_i(\alpha_j)|^n \leq C_2 n^m A^n |\alpha_1|^n, \end{aligned}$$

with  $m = \max_{j=1,\dots,t} \deg a_j$ , and

$$\|\mathbf{x}\|^\varepsilon \leq C_3 n^{m\varepsilon} A^{\gamma n \delta} |\alpha_1|^{n\varepsilon} \leq C_3 n^{m\varepsilon} |\alpha_1|^{n(\varepsilon+\delta)}.$$

Now we insert this into inequality (A.1), giving

$$\left( \prod_{j=1}^t \prod_{v \in S} \|a_j(n)\|_v \right) |G_n| \geq C_4 |a_1(n)| |\alpha_1|^n n^{-m\varepsilon} |\alpha_1|^{-n(\varepsilon+\delta)} \geq C_5 n^{-m\varepsilon} |\alpha_1|^{n(1-\varepsilon-\delta)}.$$

Applying Lemma A.4 to the product in the brackets on the left hand side gives the bound

$$\prod_{j=1}^t \prod_{v \in S} \|a_j(n)\|_v \leq \prod_{j=1}^t C_6^{(j)} n^{Dm} \leq C_7 n^{tDm}.$$

Altogether, for  $n$  large enough,

$$|G_n| \geq C_8 n^{-tDm-m\varepsilon} |\alpha_1|^{n(1-\varepsilon-\delta)}.$$

Hence, recalling that  $|\alpha_1| = \max_{j=1,\dots,t} |\alpha_j|$ , for  $n$  large enough,

$$|G_n| \geq \left( \max_{j=1,\dots,t} |\alpha_j| \right)^{n(1-\bar{\varepsilon})}.$$

This proves the theorem.  $\square$

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