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Parametric corecursion

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Abstract

This paper gives a treatment of substitution for "parametric" objects in final coalgebras, and also presents principles of definition by corecursion for such objects. The substitution results are coalgebraic versions of well-known consequences of initiality, and the work on corecursion is a general formulation which allows one to specify elements of final coalgebras using systems of equations. One source of our results is the theory of hypersets, and at the end of this paper we sketch a development of that theory which calls upon the general work of this paper to a very large extent and particular facts of elementary set theory to a much smaller extent.

1. Introduction

This paper has two overall goals. The first is a general theory of substitution and corecursion having to do with final coalgebras. To give an example of the kind of phenomena we have in mind, consider any set S and form the functor F on sets defined by $Fa = S \times a \times a$. F is defined on functions in the usual way. Let Tr^S be the set of infinite binary trees with nodes labeled from S. Then with the appropriate operations, Tr^S is a final F-coalgebra. That is, modulo coding, $Tr^S = S \times Tr^S \times Tr^S$. The finality allows us to define functions into Tr^S.

Now take a set X which we would like to think of as "variables". Form the functor F_X defined by $F_X(a) = F(X + a)$. Here $+$ is the disjoint union, and again F_X is defined on functions in the usual way. Let Tr_X^S be a final coalgebra. We think of the elements of Tr_X^S as "parametric trees". In such trees, the elements of X may appear as leaf nodes, and such nodes are not also labeled by elements of S. What this means is that we have a *substitution principle*: for every function $f : X \to Tr^S$ there is a unique $[\![f]\!] : Tr_X^S \to Tr^S$ which acts like substitution. Of course, we need to spell out what this

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means in detail, prove the uniqueness, etc. One of our results, Lemma 2.14 below, does exactly this.

Here is another phenomenon which interests us: A function $e : X \to Tr^S_X$ can be thought of as a system of equations. For example, fix $p, q \in S$, and suppose $X = \{x, y\}$ $e(x) = \langle p, x, y\rangle$, $e(y) = \langle q, y, x\rangle$. Then there is a natural notion of a *solution of e*. The solution would be a map $s : X \to Tr^S$ In this case, $s(x)$ and $s(y)$ would be trees with roots labeled p and q, respectively. For every node n labeled p in either of them, the left child of n would be labeled p and the right q; if n were labeled q, then the opposite conditions would obtain. But these details are less important than the overall theme: a function $e : X \to Tr^S_X$ gives rise to $s : X \to Tr^S$. And again, we want to state what the notion of a solution comes to. (Here we can just use the notation above to say that $s = [\![s]\!] \circ e$.) We also want to work as generally as possible. That is, we want to use the notions of coalgebras of functors, morphisms of coalgebras, and final coalgebras rather than specific results about functors like F. Our main results in this direction are Theorems 2.11 and 2.17. These give a foundation for corecursive definitions with parameters in the sense that they are general results which cover all of the cases of the phenomenon which we know of.

The second goal of the paper is to make connections to the theory of hypersets. Readers familiar with that theory might sense the motivation from that subject for this paper. (Nevertheless, we do not mention that subject in either the general development or in our examples.) Section 5 shows how a good part of the development of *vicious circles* [3], for example, can be unified and simplified by using the much more general work that we do in this paper. One particular focus of our study is the notion of a *uniform functor* on sets, introduced by Turi and Rutten [11] and Turi [9]. We generalize their notion slightly, prove the main result about such functors (that their greatest fixed points are final coalgebras), and we also use our earlier work to check that many functors of interest are uniform. Taken together, the work of this paper shows that much of the extant results on hypersets can be obtained fairly easily using coalgebra and just a little set theory.

Background. Comprehensive background on coalgebras may be found in [8]. We only need the basic definitions and the relevant facts concerning our example functors. At some points we shall use standard concepts of set theory, and these are reviewed in Section 5. Background on hypersets may be found in [1, 3].

2. Substitution and corecursion

2.1. Groundwork: the flattening lemma

Before turning to our main results, we present a very general fact which lies at the heart of most of the results of this paper. It is a reformulation of the "flattening" technique which originated in the study of non-well-founded sets, in the books of Aczel [1] and Barwise and Etchemendy [2].

In this paper, C denotes a category with a designated coproduct operation $+$, and $F : C \to C$ is an endofunctor. If a and b are objects of C, then we have injections $\mathsf{inl} : a \to a+b$ and $\mathsf{inr} : b \to a+b$. When we use subscripts on these injections, we have in mind a special meaning that we introduce in Section 2.2. If the context forces a unique reading, then we prefer not to subscript these injections. If $f : a \to c$ and $g : b \to c$, then we have a unique $\langle f, g\rangle : a + b \to c$ such that $f = \langle f, g\rangle \circ \mathsf{inl}$ and $g = \langle f, g\rangle \circ \mathsf{inr}$. If $f : a \to b$ and $g : c \to d$, then we also have $f + g : a + c \to b + d$ given in the obvious way.

Lemma 2.1 (Flattening). *Let $\psi : c \to Fc$ be a final coalgebra, let $g : a \to Fa$ be an F-coalgebra, and let $g^* : a \to c$ be the unique homomorphism. Consider any $f : b \to F(a+b)$. Then there is a unique map $f' : b \to c$ such that $\psi \circ f' = F\langle g^*, f'\rangle \circ f$:*

$$\begin{tikzcd} b \arrow[r, "f"] \arrow[d, "f'"'] & F(a+b) \arrow[d, "F\langle g^*, f'\rangle"] \\ c \arrow[r, "\psi"'] & Fc \end{tikzcd}$$

Proof. We "flatten" f to $\langle F\mathsf{inl} \circ g, f\rangle : a + b \to F(a + b)$. The point is that unlike f, this is an F-coalgebra. So let $s : a + b \to c$ be the unique homomorphism. Now

$$\begin{aligned} \psi \circ (s \circ \mathsf{inl}) &= Fs \circ (F\mathsf{inl} \circ g) \\ &= F(s \circ \mathsf{inl}) \circ g. \end{aligned}$$

So by finality, $s \circ \mathsf{inl} = g^*$. Let $f' = s \circ \mathsf{inr}$. Then

$$\begin{aligned} \psi \circ f' &= \psi \circ s \circ \mathsf{inr} \\ &= Fs \circ f \\ &= F\langle g^*, f'\rangle \circ f. \end{aligned}$$

It follows that f' has the property in the statement of this lemma.

For the uniqueness assertion, suppose that f'' satisfies the equation for f'. Then $\langle g^*, f''\rangle$ would be a coalgebra morphism from $a + b$ to c. By finality, $\langle g^*, f''\rangle = s = \langle g^*, f'\rangle$. So $f'' = f'$. $\square$

Remark. I am grateful to Daniele Turi for pointing out that the Flattening Lemma is a special case of (the dual of) Theorem 5.1 in [10]. In fact, their result is stronger than the Flattening Lemma. It is used as a starting point for the an approach to *recursion in parameters*, generalizing the use of parameters in, e.g., primitive recursion. The Flattening Lemma is at the heart of our formulation of parametric corecursion. So it is interesting that the two studies have a common generalization.

Example 2.2. Working with non-well-founded sets allows one to solve systems of equations for sets such as $x = \{x, y\}$, $y = \emptyset$. These systems are called *flat* because the right-hand sides of the equations are always subsets of the set of variables which occur on the left-hand sides. Indeed, the Antifoundation Axiom *AFA* just says that such flat systems have unique solutions. Once one starts to work with *AFA* it quickly becomes necessary to solve systems which are not flat, such as

$$
\begin{aligned}
x &= \{0, y, z\},\\
y &= \{\Omega, z\},\\
z &= \{1, y\}.
\end{aligned} \tag{1}
$$

Here 0 is the empty set $\emptyset$, $1 = \{0\}$ and $\Omega = \{\Omega\}$. The standard way to solve this kind of system is to "flatten" it by introducing a new variables, say u, v, and w (for 0, Ω, and 1, respectively). Then one modifies and expands (1) to

$$
\begin{aligned}
x &= \{u, y, z\}, & u &= \emptyset,\\
y &= \{v, z\}, & v &= \{v\},\\
z &= \{w, y\}, & w &= \{u\}.
\end{aligned} \tag{2}
$$

Then (2) is flat, so it has a solution by *AFA*. The solution is a function from $\{x, y, z, u, v, w\}$ to sets, and its restriction to $\{x, y, z\}$ is the solution to the original system (1). Chapter 8 of [3] presents general results on how to flatten systems of equations and justification for the steps mentioned above. [1]

The Flattening Lemma 2.1 generalizes this kind of construction. To see how it works in this example, we would first note that we are working with the power set functor $\mathscr{P}$ on the category *Class* of classes. (Background on *Class* may be found in Section 5. The reason why we need to move from sets to classes is that we want to have a final coalgebra for $\mathscr{P}$. Working with sets alone we cannot have any final coalgebras, by Cantor's Theorem. But if we move to classes, and if we assume *AFA*, then it turns out that $\langle V, \mathsf{id}\rangle$ is a final coalgebra for $\mathscr{P}$.) Flat systems of set equations are coalgebras for $\mathscr{P}$. The coproduct $+$ is the usual disjoint union of sets. We take a and c to be V, and also ψ, g, and g^* will be the identity. We also take $b = \{x, y, z\}$, and f given by

$$
\begin{aligned}
f(x) &= \{\langle 0,0\rangle, \langle 1, y\rangle, \langle 1,z\rangle\},\\
f(y) &= \{\langle 0,\Omega\rangle, \langle 1,z\rangle\},\\
f(z) &= \{\langle 0,1\rangle, \langle 1, y\rangle\}.
\end{aligned}
$$

Then the Flattening Lemma gives some f' defined on b with the property that $f' = F\langle \mathsf{id}_V, f'\rangle \circ f$. This means that

$$f'(x) = \{0, f'(y), f'(z)\},$$

[1] The proof Theorem 8.1 of [3] contains an error. One way to fix things would be to call on the Flattening Lemma 2.1. So the work of this paper gives a slightly different approach to the material of Chapter 8 of [3]. We continue this development in Section 5, where we also develop much of Chapters 16 and 17 of [3].

$$f'(y) = \{\Omega, f'(z)\},$$
$$f'(z) = \{1, f'(y)\}.$$

and this f' is what we would mean by the solution to (1). So the point of this example is that the Flattening Lemma generalizes work on flattening systems of set equations.

2.2. Substitution in parametric corecursion systems

We next give a coalgebraic version of the concept of *substitution*. As it usually appears, substitution is an easy consequence of initiality or recursion. Here is the kind of formulation we have in mind. Let Σ be any signature, that is a set of function symbols with given arities. Then Σ determines a functor $F : Set \to Set$. The set T_Σ of Σ-terms is an initial algebra. Moreover, for each set X, we can consider the derived functor F_X, given by $a \mapsto F(X + a)$ on objects and in the usual way on morphisms. We also consider and its initial algebra $T_\Sigma(X)$. Let the initial algebra maps for T_Σ and $T_\Sigma(X)$ be ε and ε_X, respectively. Now the initiality gives us the following principle: for every map $g : X \to T_\Sigma$ there is a unique $\bar{g} : T_\Sigma(X) \to T_\Sigma$ with the property that $\varepsilon \circ F\langle g, \bar{g}\rangle = \bar{g} \circ \varepsilon_X$.

In this section, we study substitution operations defined on *final coalgebras*. The basic idea is the same as the one mentioned just above, except that in contrast to $\bar{g}$, we cannot define the function we need by *recursion*. Instead, we appeal to finality. We say that the analog of $\bar{g}$ will be defined by *corecursion*.

As we mentioned in the Introduction, we need to assume that all functors mentioned have final coalgebras. Our assumptions lead to the definition below. First, some notation. For all objects c of C, F_c denotes the functor $d \mapsto F(c + d)$. If $g : d \to d'$, then $F_c g = F(\mathsf{id}_c + g)$.

Definition. A *parametric corecursion system* is a structure

$$C = \langle C, +, F, c, \psi, a \mapsto \langle \bar{a}, \varphi_a \rangle \rangle$$

such that C is a category, $+$ is a coproduct operation on C, $F : C \to C$ is a functor, $\psi : c \to Fc$ is a final F-coalgebra, and for all a, $\varphi_a : \bar{a} \to F(a+\bar{a})$ is a final F_a-coalgebra.

The idea here is that $\bar{a}$ represents a final F-coalgebra built on top of the elements of a, considered as parameters. Returning to the opening of this paper, recall that Tr^S_X is the final coalgebra for $F_X a = S \times (X + a) \times (X + a)$. This means that

$$Tr^S_X \cong S \times (X + Tr^S_X) \times (X + Tr^S_X). \tag{3}$$

We read (3) as a specification of Tr^S_X. X here appears as a *set of parameters* in several senses: whatever structure X may have is ignored in (3); the elements of X seem to be "building blocks" for elements of Tr^S_X; and any function $f : X \to Y$ should "extend" to a function between Tr^S_X and Tr^S_Y.

Working in a parametric corecursion system just means that we have a category as before (with a fixed coproduct operation, and with an endofunctor F that has a final coalgebra). It also means that we also assume that each functor F_a has a final coalgebra. The point of including the association $a \mapsto \langle \bar{a}, \psi \rangle$ in the structure is that final coalgebras are only determined up to isomorphism. So to use the notation unambiguously, we incorporate a specific operation in the structure.

The definition of a parametric corecursion system is intended to be quite weak. The point of it is that on the basis of the few assumptions needed, one can get all of the remaining results of this section. Another point concerns *examples* of the concept.

We henceforth assume that we are working with a parametric corecursion system. (In Section 5 we shall exhibit a large number of examples in detail, and these include most of the functors of interest on sets. But until then, one should read the theory here based on examples presented informally, such as the one we have been discussing concerning trees.)

One final piece of notation: For each object a, we have coproduct injections $\mathsf{inl}_a : a \to a + \bar{a}$ and $\mathsf{inr}_a : \bar{a} \to a + \bar{a}$. Subscripted injections in this paper will always be used in this sense.

Here are more details concerning the example mentioned in the Introduction.

Example 2.3. Let $C = Set$, and let $F(a) = S \times a \times a$, where S some set, say the set $\{\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots\}$ of letters of the Roman alphabet. (The use of this particular set is of no consequence, of course.) Let $T = T^S$ be the set of trees with the property that each node is labeled with an element of S and also has exactly two children. Let $i : T \to F(T)$ be the map which takes a tree t to the triple consisting of the label on the root, the left subtree, and the right subtree. Then $\langle T, i \rangle$ is a final coalgebra. Further, let a be any set. Then $\bar{a}$ may be taken to be the set of trees with the property that each node is labeled with an element of S, each node has exactly two children, and these children are either elements of a (and have no children) or are again elements of $\bar{a}$. Moreover, we understand that enough coding machinery is provided that the elements of a are distinguished from all of the trees considered. One can work this out for this particular F in a straightforward way, beginning with set theory. But it is also possible to use some set theory, in particular, the theory of hypersets, to supply the details in an automatic way. We show how to do this in Section 5. Our development there not only gives a foundation for this example, but it also is an *application* of the results of this section.

Here is our substitution principle:

Lemma 2.4 (Substitution). *Let $f : a \to b + \bar{b}$. Then there is a unique $[f] : \bar{a} \to \bar{b}$ so that*

$$F\langle f, \mathsf{inr}_b \circ [f] \rangle \circ \varphi_a = \varphi_b \circ [f]. \tag{4}$$

$$\begin{tikzcd}
\bar{a} \arrow[r, "\varphi_a"] \arrow[d, "{[f]}"'] & F(a + \bar{a}) \arrow[d, "{F\langle f,\, \mathsf{inr}_b \circ [f] \rangle}"] \\
\bar{b} \arrow[r, "\varphi_b"'] & F(b + \bar{b})
\end{tikzcd}$$

Proof. For this proof, let $r:(b+\bar{b})+\bar{a}\to b+(\bar{b}+\bar{a})$ be the obvious rearrangement. Then $f+\mathsf{id}_{\bar{a}}:a+\bar{a}\to(b+\bar{b})+\bar{a}$, so $r\circ(f+\mathsf{id}_{\bar{a}}):a+\bar{a}\to b+(\bar{b}+\bar{a})$. Consider $e:\bar{a}\to F_b(\bar{b}+\bar{a})$, given by

$$e=F(r\circ(f+\mathsf{id}_{\bar{a}}))\circ\varphi_a.$$

By the Flattening Lemma, there is a unique $[f]:\bar{a}\to\bar{b}$ such that $\varphi_b\circ[f]=F_b\langle\mathsf{id}_{\bar{b}},[f]\rangle\circ e$. But

$$\begin{aligned}F_b\langle\mathsf{id}_{\bar{b}},[f]\rangle\circ e&=(\mathsf{id}_b+\langle\mathsf{id}_{\bar{b}},[f]\rangle)\circ F(r\circ(f+\mathsf{id}_{\bar{a}}))\circ\varphi_a\\&=F\langle f,\mathsf{inr}_b\circ[f]\rangle\circ\varphi_a.\end{aligned}$$

This shows that $[f]$ satisfies (4).

For the uniqueness, let g be such that $F\langle f,\mathsf{inr}_b\circ g\rangle\circ\varphi_a=\varphi_b\circ g$. By the calculation above, $F_b\langle\mathsf{id}_{\bar{b}},g\rangle\circ e=\varphi_b\circ g$. So by the uniqueness part of the Flattening Lemma, $g=[f]$. $\square$

Example 2.5. Continuing with Example 2.3, suppose that $a=\{0,1\}$ and $b=\{2,3,4\}$. Suppose that $f:a\to b+\bar{b}$ is defined by $f(0)=2$, and $f(1)$ is the unique tree t such that $t=\langle\mathtt{k},t,4\rangle$. Then $[f]$ is the extension of f to $\bar{a}$. For example, let $u\in\bar{a}$ be $\langle\mathtt{q},0,\langle\mathtt{h},1,1\rangle\rangle$. Then $[f](u)=\langle\mathtt{q},2,\langle\mathtt{h},t,t\rangle\rangle$. It is natural to write $[f](u)$ as $u[f]$, especially to those who use this kind of notation for term substitution. Note that $[f]$ works by finality ("corecursion"), not by recursion in the usual sense.

Remark. In contrast with the Flattening Lemma, I am not aware of any previous formulation of (the dual of) our result on substitution. The same holds for our results on parametric corecursion.

2.3. Additional structure, and an alternative formulation of substitution

At this point, we mention some of the substitution as formulated in Lemma 2.4. Lemmas 2.6–2.9 will be used in Section 5 (so they can be ommitted until then). After doing this, we want to point out a different formulation of our overall framework that fits better with the structure here but has slight disadvantages for other purposes.

Lemma 2.6. *Consider* $\mathsf{inl}_a:a\to a+\bar{a}$. *Then* $[\mathsf{inl}_a]=\mathsf{id}_{\bar{a}}$.

Lemma 2.7. *Let* $f:a\to b+\bar{b}$ *and* $g:b\to c+\bar{c}$. *Then* $[g]\circ[f]=[\langle g,\mathsf{inr}_c\circ[g]\rangle\circ f]$.

To prove Lemma 2.6, we only need to check that $\mathsf{id}_{\bar{a}}$ works for $[\mathsf{inl}_a]$ in (4). This is an easy consequence of functoriality. For Lemma 2.7, we check that $[g]\circ[f]$ has the defining property of $[\langle g,\mathsf{inr}_c\circ[g]\rangle\circ f]$. This takes a bit of routine calculation with the injections.

We extend the object mapping $a\mapsto\bar{a}$ to an endofunctor $L:C\to C$ by taking $f:a\to b$ to $[\mathsf{inl}_b\circ f]:\bar{a}\to\bar{b}$.

Lemma 2.8. *L is a functor.*

Proof. Preservation of identities is Lemma 2.6. And Lemma 2.7 gives the preservation of composition:

$$\begin{aligned} L(g \circ f) &= [\mathsf{inl}_c \circ g \circ f] \\ &= [\langle \mathsf{inl}_c \circ g, \mathsf{inr}_c \circ [\mathsf{inl}_c \circ g]\rangle \circ \mathsf{inl}_b \circ f] \\ &= [\mathsf{inl}_c \circ g] \circ [\mathsf{inl}_b \circ f] \\ &= Lg \circ Lf. \qquad \square \end{aligned}$$

We next define a natural transformation $\mu : L^2 \to L$. For this, note that for all a we have a final F_a-coalgebra $\varphi_a : \bar{a} \to F(a + \bar{a})$. It follows that we also have a final $F_{\bar{a}}$-coalgebra $\varphi_{\bar{a}} : \bar{\bar{a}} \to F(\bar{a} + \bar{\bar{a}})$. Now consider $\mathsf{inr}_a : \bar{a} \to a + \bar{a}$. Using Lemma 2.4, we get $[\mathsf{inr}_a] : \bar{\bar{a}} \to \bar{a}$. We set μ_a to be this map.

Lemma 2.9. *$\mu : L^2 \to L$ is a natural transformation.*

Proof. Let $f : a \to b$. We need to check that $[\mathsf{inl}_b \circ f] \circ [\mathsf{inr}_a] = [\mathsf{inr}_b] \circ [\mathsf{inl}_{\bar{b}} \circ [\mathsf{inl}_b \circ f]]$. For this, we use Lemma 2.7. Both sides equal $[\mathsf{inr}_b \circ [\mathsf{inl}_b \circ f]]$. $\square$

In another direction, we construct a monad M on C, described as a Kleisli triple $\langle M, \mathsf{unit}, -^\star \rangle$ as follows: $Ma = a + \bar{a}$, $\mathsf{unit}_a = \mathsf{inl}_a$, and for each $f : a \to Mb$, $f^\star : Ma \to Mb$ is $\langle f, \mathsf{inr}_b \circ [f] \rangle$.

Lemma 2.10. *$\langle M, \mathsf{unit}, -^\star \rangle$ is a Kleisli triple.*

Proof. We need to check that $\mathsf{unit}_a^\star = \mathsf{id}_{Ma}$, $f^\star \circ \mathsf{unit}_a = f$, and $f^\star \circ g^\star = (f^\star \circ g)^\star$. The first is by Lemma 2.6, the second by the definition of $f^\star$, and the last comes from Lemma 2.7 as follows:

$$\begin{aligned} f^\star \circ g^\star &= \langle f, \mathsf{inr}_b \circ [f] \rangle \circ \langle g, \mathsf{inr}_c \circ [g] \rangle \\ &= \langle f^\star \circ g, \mathsf{inr}_b \circ [f] \circ [g] \rangle \\ &= \langle f^\star \circ g, \mathsf{inr}_c \circ [f^\star \circ g] \rangle \\ &= (f^\star \circ g)^\star. \qquad \square \end{aligned}$$

At this point we want to mention an alternative formulation of all of our work up until this point. Given an object a of C, we define F_a to be the functor $d \mapsto F(a+d)$. From this we have the final F_a-coalgebra $\varphi_a : \bar{a} \to F(a + \bar{a})$. Now instead of this, we can also consider the functor F^a given by $d \mapsto a + F(d)$. Then we can re-define the notion of a parametric corecursion system to require choices of final coalgebras for the derived functors of this form. Let us use the notation $\omega_a : \underline{a} \to a + F\underline{a}$ for a final F^a-coalgebra. It is not hard to see that $a + \bar{a}$ and $\underline{a}$ are isomorphic, and so one can easily translate between results on the two kinds of derived functors. For example, Lemma 2.4 on substitution now can be recast as follows: Let $f : a \to \underline{b}$. Then there is a unique $[f] : \underline{a} \to \underline{b}$ such that $\langle f \circ \omega_b, \mathsf{inr} \circ F[f]\rangle \circ \omega_a = \omega_b \circ [f]$. This kind of reformulation also leads to a simpler presentation of the related Kleisli triple. Indeed, we can take $\langle M, \mathsf{unit}, -^{\star}\rangle$ to be given by $Ma = \underline{a}$, $\mathsf{unit}_a = \omega_a^{-1} \circ \mathsf{inl}$, and $f^{\star} = [f]$.

The reason why we did not work with this reformulation is that some of our results become slightly harder to present with it. These include Theorem 2.11 and also some of the work of Section 5. Once again, the differences between the two approaches are fairly minor and in other contexts one may well want to make a choice which differs from what we do here.

2.4. Corecursion

We now come to the main foundational result of the paper. Theorem 2.11 gives us the notion of a solution to a system of parametric equations into final coalgebras.

Theorem 2.11 (Parametric corecursion). *Let $f : a \to \overline{a+b}$. Then there is a unique $f^{\dagger} : a \to \overline{b}$ so that $f^{\dagger} = [\langle \mathsf{inr}_b \circ f^{\dagger}, \mathsf{inl}_b\rangle] \circ f$:*

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "f^{\dagger}"'] & \overline{a+b} \arrow[d, "{[\langle \mathsf{inr}_b \circ f^{\dagger}, \mathsf{inl}_b\rangle]}"] \\
 & \overline{b}
\end{tikzcd}
$$

Proof. Let $j : (a+b) + \overline{a+b} \to b + (a + \overline{a+b})$ be the obvious rearrangement. Then $Fj \circ \varphi_{a+b} : \overline{a+b} \to F_b(a + \overline{a+b})$. It follows that

$$\langle Fj \circ \varphi_{a+b} \circ f, Fj \circ \varphi_{a+b}\rangle : a + \overline{a+b} \to F_b(a + \overline{a+b}).$$

Call this map e; it is an F_b-coalgebra. By finality, we have $s : a + \overline{a+b} \to \bar{b}$ so that the diagram below commutes:

$$
\begin{tikzcd}
a + \overline{a+b} \arrow[r, "e"] \arrow[d, "s"'] & F_b(a + \overline{a+b}) \arrow[d, "F_b(s)"] \\
\bar{b} \arrow[r, "\varphi_b"'] & F_b(\bar{b})
\end{tikzcd}
\tag{5}
$$

Let $\mathsf{inl} : a \to a + \overline{a+b}$ and $\mathsf{inr} : \overline{a+b} \to a + \overline{a+b}$ be the injections. Let $f^{\dagger} = s \circ \mathsf{inl}$.

Claim. $[\langle \mathsf{inr}_b \circ f^{\dagger}, \mathsf{inl}_b\rangle] = s \circ \mathsf{inr}$.

For the proof, an easy calculation shows that

$$
\begin{aligned}
(\mathsf{id}_b + s) \circ j &= \langle\langle \mathsf{inr}_b \circ s \circ \mathsf{inl}, \mathsf{inl}_b\rangle, \mathsf{inr}_b \circ s \circ \mathsf{inr}\rangle \\
&= \langle\langle \mathsf{inr}_b \circ f^{\dagger}, \mathsf{inl}_b\rangle, \mathsf{inr}_b \circ s \circ \mathsf{inr}\rangle.
\end{aligned}
$$

Then applying F,

$$F_b(s) \circ Fj = F(\langle\langle \mathsf{inr}_b \circ f^\dagger, \mathsf{inl}_b\rangle, \mathsf{inr}_b \circ s \circ \mathsf{inr}\rangle).$$

Now we use (5) and the definition of e to calculate

$$\begin{aligned}\varphi_b \circ s \circ \mathsf{inr} &= F_b(s) \circ Fj \circ \varphi_{a+b}\\ &= F(\langle\langle \mathsf{inr}_b \circ f^\dagger, \mathsf{inl}_b\rangle, \mathsf{inr}_b \circ s \circ \mathsf{inr}\rangle) \circ \varphi_{a+b}.\end{aligned}$$

We appeal to the uniqueness part of Lemma 2.4, taking $\langle \mathsf{inr}_b \circ f^\dagger, \mathsf{inl}_b\rangle$ for f. We see that indeed, $[\langle \mathsf{inr}_b \circ f^\dagger, \mathsf{inl}_b\rangle] = s \circ \mathsf{inr}$. This establishes our claim.

Let us write $[\langle \mathsf{inr}_b \circ f^\dagger, \mathsf{inl}_b\rangle] = s \circ \mathsf{inr}$ as f^*. Then the solution condition of (5) specializes to the following equations:

$$\begin{aligned}\varphi_b \circ f^\dagger &= F_b(s) \circ Fj \circ \varphi_{a+b} \circ f,\\ \varphi_b \circ f^* &= F_b(s) \circ Fj \circ \varphi_{a+b}.\end{aligned}$$

From these, $\varphi_b \circ f^* \circ f = \varphi_b \circ f^\dagger$. Now by Lambek's Lemma, all final coalgebra maps are isomorphisms and are therefore monic. This implies that $f^* \circ f = f^\dagger$, as desired.

As for the uniqueness of $f^\dagger$, suppose that $g^\dagger$ were another map with the same property as $f^\dagger$. Again, let

$$g^* = [\langle \mathsf{inr}_b \circ g^\dagger, \mathsf{inl}_b\rangle].$$

Then $g^* = s \circ \mathsf{inr} = f^*$. Hence $g^\dagger = g^* \circ f = f^* \circ f = f^\dagger$.

Example 2.12. Continuing with Examples 2.3 and 2.5 concerning trees, again let $a = \{0,1\}$ and $b = \{2,3,4\}$. Let $f : a \to \overline{a+b}$ be given as follows: $f(0) = \langle \mathtt{q}, 2, 1\rangle$, and $f(1) = \langle \mathtt{y}, 0, 0\rangle$. Then $f^\dagger : a \to \bar{b}$ would be given by $f^\dagger(0) = \langle \mathtt{q}, 2, f^\dagger(1)\rangle$ and $f^\dagger(1) = \langle \mathtt{q}, f^\dagger(1), f^\dagger(1)\rangle$. Note that the values of $f^\dagger$ belong to $\bar{b}$. Theorem 2.11 is a general result guaranteeing that parametric systems of equations have unique solutions.

Example 2.13. We return to the set-theoretic examples, assuming *AFA* and working with $F = \mathscr{P}$ on the category of classes. Here is what Theorem 2.11 says in this setting. Let X and Y be any sets, and let $e : X \to \overline{X+Y}$. The intuition here is that e takes elements of X to sets which are built from elements of $X + Y$. Then the theorem tells us that there will be a solution to this system $e^\dagger$; the condition of the theorem is exactly what one would mean by the solution of a system of equations. The important points of the formulation are that subsitution is involved, and also that the codomain of $e^\dagger$ is $\overline{Y}$. That is, the range of the solution can be take to be sets built just from the elements of Y. For those familiar with [3], this result is essentially the "Solution Lemma Lemma".

There are several reasons why a results-like Theorem 2.11 can be thought of as a co*recursion* theorem. One is that $f^\dagger$ satisfies a recursion-like condition. (However, this is also true of the functions $[f]$ from earlier.) Another has to do with the formal properties of the operation $f \mapsto f^\dagger$ given in Theorem 2.11. It can be shows that this association gives rise to an iterative algebraic theory in the sense of Bloom and Ésik [4]. (This is equivalent to having a model of the FLR_0-fragment of Moschovakis' formal language of recursion FLR_1, a logical system studied in [5]). The overall point is that the equational properties of $f \mapsto f^\dagger$ are the same as those of the canonical form of recursive definitions, where f is a simultaneous system of monotone functions on a directed-complete partial order, and $f^\dagger$ is interpreted as least fixed point of f. These matters are discussed further in [6].

2.5. Variations

Theorem 2.11 began with $f : a \to \overline{a+b}$. In case the codomain of f is $\bar{a}$, then we expect to get a similar recursion result. Indeed, in case b is an initial object, $\overline{a+b}$ would be isomorphic to $\bar{a}$, and $\bar{b}$ would be a final F-coalgebra. So we might expect to get a recursion result involving $f^\dagger : \bar{a} \to c$, where c is an arbitrary final F-coalgebra. We obtain such a result in Theorem 2.17. Before that, we need a parallel result concerning substitution:

Lemma 2.14. *Let $\psi : c \to Fc$ be a final F-coalgebra. Let $f : a \to c$. Then there is a unique $[\![f]\!] : \bar{a} \to c$ such that $\psi \circ [\![f]\!] = F\langle f, [\![f]\!]\rangle \circ \varphi_a$.*

The proof of this is again by flattening. This time we begin by turning f into $g : a + \bar{a} + c \to F(a + \bar{a} + c)$. Then we use finality. Since the details are quite similar to those of Lemma 2.4, we shall not give them.

Proposition 2.15. *Let $f : a \to c$ and $g : a \to \bar{a}$. Then $[\![[\![f]\!] \circ g]\!] = [\![f]\!] \circ [\mathsf{inr}_a \circ g]$.*

Proof. $[\![f]\!] \circ [\mathsf{inr}_a \circ g]$ satisfies the recursion equation for $[\![[\![f]\!] \circ g]\!]$. $\square$

In the results below, we recall the functor $L = L_F$ from Section 2.3.

Proposition 2.16. *Let $f : a \to c$. Then $[\![f]\!] = [\![\mathsf{id}_c]\!] \circ Lf$.*

Here is the parallel result to the Parametric Corecursion Theorem 2.11 for corecursion.

Theorem 2.17. *Suppose that $f : a \to \bar{a}$. Then there is a unique map $f^\dagger : a \to c$ such that $f^\dagger = [\![f^\dagger]\!] \circ f$. Moreover, $[\![f^\dagger]\!] = [\![\mathsf{id}_c]\!] \circ L f^\dagger$.*

The first part of this result is another flattening argument. The second is an immediate consequence of Proposition 2.16.

3. The effects of natural transformations

In this section, we consider two parametric corecursion systems using the same category and coproduct, say

$$C_F = \langle C, +, F, c, \omega, a \mapsto \langle \bar{a}, \varphi_a \rangle \rangle,$$
$$C_G = \langle C, +, G, d, \chi, a \mapsto \langle \hat{a}, \psi_a \rangle \rangle.$$

We also consider a natural transformation $\eta : F \to G$. The goal is to compare corecursions using the two functors F and G, using η and other maps derived from it.

Example 3.1. Suppose that C is the category of classes, and $+$ is the usual pairing operation. Let S be a fixed class. Then we get two functors F and G by

$$Fa = S \times a \qquad Ga = \mathscr{P}\mathscr{P}(S + a).$$

These work on morphisms in the usual way. For a natural transformation, we take η_a to be

$$\langle s, c \rangle \mapsto \{\{s\}, \{s, c\}\}.$$

We are suppressing the injections here. All that we are doing here is noticing that the usual formulation of ordered pairs in set theory amounts to a natural transformation between the functors above.

We are interested in this section in the following kind of question. Suppose that $a = \{0, 1\}$, $b = \{2, 3\}$ and let t and u be fixed elements of S. Then $\bar{b}$ is the set of streams which are either infinite sequences of elements of S, or else finite sequences which end in an element of b. Similar remarks apply to $\overline{a + b}$, of course.

Let $f : a \to \overline{a + b}$ be given by $f(0) = \langle t, u, 3 \rangle$, and $f(1) = \langle t, 1 \rangle$. (We are ignoring the injections into $a + b$ to save a bit on the notation.) Then by Theorem 2.11, we have $f^\dagger : a \to \bar{b}$ which solves this system: so $f^\dagger(0) = \langle t, u, 3 \rangle$, and $f^\dagger(1) = \langle t, t, \ldots \rangle$.

Now there is another way to solve this system e. We hint at the method, and the details are the content of this section. Instead of $f : a \to \overline{a + b}$ we want to consider a closely related $g : a \to \overline{a + b}$. This g is constructed from f and η. Now the point is that g leads to a solution $g^\dagger$. Actually, since we changed functors, we will change notation and write $f^\uparrow$ instead of $g^\dagger$. There should be some relation between $f^\dagger$ and $f^\uparrow$: after all, the natural transformation amounts to a re-coding, and the intuition is that re-coding should respect the solution of recursion equations. And the main result of this is section is a formulation of exactly this fact (Lemma 3.4).

The following result is a prototype for the rest of this section.

Lemma 3.2. *Let $j : c \to d$ be the final G-coalgebra morphism for $\eta_c \circ \omega : c \to Gc$. Consider an arbitrary F coalgebra $e : a \to Fa$. Let $e^* : a \to c$ be the final F-coalgebra morphism for e. Then the final G-coalgebra morphism for $\eta_a \circ e$ is $j \circ e^*$.*

For the proof, one only needs to show that $j \circ e^*$ *is* a G-coalgebra morphism for $\eta_a \circ e$.

Returning to our development, for each object a we have final F_a- and G_a-coalgebras

$$\varphi_a : \bar{a} \to F(a + \bar{a}) \quad \text{and} \quad \psi_a : \hat{a} \to G(a + \hat{a}).$$

Note that $\eta_{a+\bar{a}} \circ \varphi_a : \bar{a} \to G(a + \bar{a})$. So by finality of $\bar{a}$, there is a map $j_a : \bar{a} \to \bar{a}$ such that $G_a j_a \circ \eta_{a+\bar{a}} \circ \varphi_a = \psi_a \circ j_a$.

$$\begin{tikzcd}
\bar{a} \arrow[r, "\varphi_a"] \arrow[d, "j_a"'] & F(a+\bar{a}) \arrow[r, "\eta_{a+\bar{a}}"] & G(a+\bar{a}) \arrow[d, "G_a j_a"] \\
\hat{a} \arrow[rr, "\psi_a"'] & & G(a+\hat{a})
\end{tikzcd}$$

Suppose next that $f : a \to b + \bar{b}$. By Lemma 2.4, we have $[f] : \bar{a} \to \bar{b}$. Write f' for $\langle \mathsf{id}_b, j_b \rangle \circ f$. Then $f' : a \to b + \hat{b}$. So we have $\langle f' \rangle : \hat{a} \to \hat{b}$. We use a different style of braces to remind ourselves that a different parametric corecursion system is used.

Lemma 3.3. *In the notation above*, $\langle f' \rangle \circ j_a = j_b \circ [f]$.

Proof. Here is a sketch: one considers

$$\bar{a} \to F(a + \bar{a}) \to F((b + \hat{b}) + \bar{a}) \to F(b + (\hat{b} + \bar{a})) \to G(b + (\hat{b} + \bar{a})).$$

The maps are φ_a, $F(f' + \mathsf{id}_{\bar{a}})$, an easy rearrangement, and $\eta_{b+(\hat{b}+\bar{a})}$. By flattening, we can obtain a map $\hat{b} + \bar{a} \to G(b + (\hat{b} + \bar{a}))$. So by finality, we get a unique G_b-morphism into $\langle \hat{b}, \psi_b \rangle$. We check that both $\langle \mathsf{id}, \langle f' \rangle \circ j_a \rangle$ and $\langle \mathsf{id}, j_b \circ [f] \rangle$ are such morphisms. $\square$

Finally, we consider the effect of natural transformations on recursion. Let $f : a \to \overline{a + b}$. By Theorem 2.11, there is a unique $f^\dagger : a \to \bar{b}$ so that

$$f^\dagger = [\langle \mathsf{inr}_b \circ f^\dagger, \mathsf{inl}_b \rangle] \circ f.$$

Note that $j_{a+b} \circ f : a \to \widehat{a + b}$. So in addition we have a unique $f^\uparrow : a \to \widehat{b}$ so that

$$f^\uparrow = [\langle \mathsf{inr}_b \circ f^\uparrow, \mathsf{inl}_b \rangle] \circ j_{a+b} \circ f.$$

Lemma 3.4. $f^\uparrow = j_b \circ f^\dagger$.

Proof. Let $g : a + b \to b + \bar{b}$ be $\langle \mathsf{inr}_b \circ f^\dagger, \mathsf{inl}_b \rangle$. Note that the triangle below commutes, by Lemma 2.4. In the notation before Lemma 3.3, $g' : a + b \to b + \widehat{b}$ is

$$\langle \mathsf{id}_b, j_b \rangle \circ \langle \mathsf{inr}_b \circ f^\dagger, \mathsf{inl}_b \rangle = \langle \mathsf{inr}_b \circ f^\dagger, j_b \circ \mathsf{inl}_b \rangle.$$

Then Lemma 3.3 shows that the square below commutes.

$$\begin{tikzcd}
\overline{a} \arrow[r, "f"] \arrow[dr, "f^{\dagger}"'] & \overline{a+b} \arrow[r, "j_{a+b}"] \arrow[d, "{[g]}"] & \widehat{a+b} \arrow[d, "\langle g' \rangle"] \\
& \overline{b} \arrow[r, "j_b"'] & \hat{b}
\end{tikzcd}$$

Since $j_b \circ f^{\dagger}$ satisfies the equation which $f^{\uparrow}$ alone satisfies, $j_b \circ f^{\dagger} = f^{\uparrow}$. $\square$

Example 3.5. Let $Fa = S \times a \times a$ as before, and let $Ga = S \times a$. The final coalgebra for G is the set of (infinite) streams over S. For each a, the final coalgebra for G_a also allows the streams to end in elements of a. There are two natural transformations from F to G, and we will consider η_a given by $\langle x, a_1, a_2 \rangle \mapsto \langle x, a_2 \rangle$. For all a, j_a takes a tree to a stream by giving the rightmost branch. In Example 2.12 we considered $f : a \to \overline{a+b}$ given by $f(0) = \langle \mathrm{q}, 2, 1 \rangle$ and $f(1) = \langle \mathrm{y}, 0, 0 \rangle$. Let us call $j_{a+b} \circ f$ by the name g. Then $g(0) = \langle \mathrm{q}, 1 \rangle$ and $g(1) = \langle \mathrm{y}, 0 \rangle$. Then $g^{\dagger}$ is what we called $f^{\uparrow}$. For example, $g^{\dagger}(1)$ would be the stream $(\mathrm{y}, \mathrm{q}, \mathrm{y}, \mathrm{q}, \ldots)$. Lemma 3.4 predicts that this will be the rightmost branch of $f^{\dagger}(1)$. Looking back at our calculation of $f^{\dagger}(1)$ from Example 2.12, we see that this is the case.

With this section, we conclude our general theory. The remainder of this paper is devoted to two connections of our work with other studies. First, in Section 4, we study operations on final coalgebras defined in terms of substitution. And in Section 5, we apply our results to give a coalgebraic presentation of the theory of hypersets.

4. Operation on final coalgebras

The results of this section are not used in the sequel. Also, we need some assumptions that go beyond those of the rest of this paper. Instead of formulating the weakest possible assumptions, we shall just assume that in this section, C is *Set*, the usual category of sets. What we shall use is that C has a final object, and all limts, and also that for each natural number n, the set a^n (an object of C) corresponds to $Hom(n, a)$, the set a^n of functions from n to a.

Fix a functor F and a final F-coalgebra $\langle c, \psi \rangle$. Pavlović [7] is interested in operations $\delta : c^n \to c^n$ which have unique fixed points. His paper discusses two conditions on operations which guarantee the existence and uniqueness of fixed points. The simplest condition is that of being *prefixing*. To simplify things a bit, we work with $n = 1$. An operation $\delta : c \to c$ is *prefixing* if there is a natural transformation $\eta : 1 \to F$ such that $\varepsilon \circ \eta_c = \delta$, where $\varepsilon = \psi^{-1}$ is the inverse of the coalgebra structure ψ.

Example 4.1. Consider the case of $C = Set$ with $Fa = \{0,1\} \times a$. Then F^* is the set of streams of 0's and 1's. We identify streams with infinite sequences, and we use α to denote such a sequence. The natural transformations $\eta : 1 \to F$ correspond to the elements of $F1 \cong \{0,1\}$. The prefixing operations are just those of the form $\alpha \mapsto (0, \alpha_0, \alpha_1, \ldots)$ and $\alpha \mapsto (1, \alpha_0, \alpha_1, \ldots)$.

Pavlović also introduced a class of operations on c called *guarded operations*. A general point to note is that his work differs from ours in that he assumed that the final coalgebra was obtained as a limit of some appropriate kind, and then he used an approach closer to least fixed points or initial algebras. Our purpose here is to point out that the results on prefixing operations hold generally, and they can be proved using only the concept of finality.

In fact, we can isolate another class of operations on final coalgebras. Let $\psi : c \to Fc$ be a final coalgebra for a functor F. We apply the results before to classes $\bar{n}$, where n is a natural number. It is natural to think of an element of $\bar{n}$ as a possibly non-wellfounded term using the variables $x_0, \ldots, x_{n-1}$. (We develop this point of view at length in [6].)

Definition. Every $w : n \to \bar{n}$ determines a function $f_w : c^n \to c^n$ in the following way. Let $a : n \to c$, and consider $[\![a]\!] : \bar{n} \to c$ from Lemma 2.14. We define $f_w(a)$ to be $[\![a]\!] \circ w : n \to c$. We call f_w the *substitutive operation determined by* w. And $f : c^n \to c^n$ is *substitutive* if there is some w such that $f = f_w$.

The substitutive operations include the possibility of longer prefixes in front. Returning to the situation of Example 4.1, $\alpha \mapsto (1, 1, 0, 1, \alpha_0, \alpha_1, \ldots)$ is substitutive. These, together with the constants, are all of the substitutive operations.

Proposition 4.2. *Every substitutive operation on c^n has a unique fixed point.*

Proof. By Theorem 2.17, let $w^\dagger : n \to c$ be the unique map such that $w^\dagger = [\![w^\dagger]\!] \circ w$. Then $f_w(w^\dagger) = [\![w^\dagger]\!] \circ w = w^\dagger$. And if a is such that $[\![a]\!] \circ w = f_w(a) = a$, then $a = w^\dagger$, by the uniqueness of Theorem 2.17. $\square$

Incidentally, it is easy to check that the substitutive operations are closed under composition. For suppose that $w, v : n \to \bar{n}$. By Proposition 2.15, $[\![[\![a]\!] \circ w]\!] = [\![a]\!] \circ [\mathsf{inr}_n \circ w]$. Let $x = [\mathsf{inr}_n \circ w] \circ v$. So for all $a : n \to c$, $f_x(a) = [\![a]\!] \circ [\mathsf{inr}_n \circ w] \circ v = [\![[\![a]\!] \circ w]\!] \circ v = f_v(f_w(a))$.

We are able to characterize the prefixing operations in terms of substitutions. In the statement below, let $k : F1 \to \bar{1}$ be $\varphi_1^{-1} \circ F\mathsf{inl}_1$.

Proposition 4.3. *An operation $\delta : c \to c$ is prefixing iff there is some $w : 1 \to F1$ such that $\delta = f_{k \circ w}$. In particular, every prefixing operation on c is substitutive.*

Proof. Suppose first that δ is prefixing, via η. Let $w = \eta_1$. For each $a : 1 \to c$, all of the diagrams below commute:

$$\begin{tikzcd} 1 \arrow[r, "\eta_1 = w"] \arrow[d, "a"'] & F1 \arrow[r, "F\mathsf{inl}_1"] \arrow[d, "Fa"] & F(1+\bar{1}) \arrow[r, "\varphi_1^{-1}"] \arrow[d, "F\langle a, [\![a]\!]\rangle"] & \bar{1} \arrow[d, "[\![a]\!]"] \\ c \arrow[r, "\eta_c"'] & F(c) \arrow[r, "1"'] & F(c) \arrow[r, "\varepsilon"'] & c \end{tikzcd}$$

The first commutes by naturality, the second by functoriality, and the last by the defining property of $[\![a]\!]$. Recall that for all $a \in c$, $f_{k \circ w}(a)$ is $[\![a]\!] \circ k \circ w$. The figure shows that this is $\varepsilon \circ \eta_c \circ a = \delta \circ a$. That is, $f_{k \circ w} = \delta$.

Going the other way, let $w : 1 \to F1$ be so that $\delta = f_{k \circ w}$. We define η by: for all sets A, η_A is $b \mapsto Fb(w)$. (That is, for $b \in A$, consider $b : 1 \to A$, then $Fb : F1 \to FA$, and finally $Fb \circ w : 1 \to FA$. This gives an element of FA.) The naturality of η is easily verified. And then the diagram above shows that $\delta = f_{k \circ w} = \varepsilon \circ \eta_c$. $\square$

Pavlović [7] is mainly concerned with the *guarded* operations on final coalgebras. We shall not repeat the definition here, but we note that in this example, the guarded operations properly contain the substitutive ones. For example, the operation

$$\alpha \mapsto (0, \alpha_0, 0, \alpha_1, 0, \alpha_2, 0, \ldots)$$

is guarded but not substitutive. This shows, in fact, that there are uncountably many guarded operations on streams.

Incidentally, we believe that all substitutive operations should be guarded. This might require working with the assumption that the final coalgebra be obtained by some iterative limit process, and indeed the result might not follow from our general work. In any case, at this time the matter is still open.

5. Coalgebraic treatment of hypersets

In this final section, we show the extent to which the theory of hypersets can be developed on the basis of final coalgebras. We also give in some detail examples of parametric corecursion systems, as we have defined them in Section 2.2.

We take the main goal of the theory of hypersets to provide for a set theory with the resources to handle circular phenomena directly, and to forge tools that will be useful in handling those phenomena. The standard axioms of set theory include the Foundation Axiom, and this tends to complicate analyses of some circular phenomena. This is the overall motivation for changing the axioms of set theory.

Experience with circular phenomena suggests that it is useful to model them using final coalgebras. So we take one of the goals of the mathematical theory of hypersets to be to prove that many naturally occurring functors do have final coalgebras.

The general results of this paper, when specialized to *Set*, imply that final coalgebras exist for many functors; in fact, they are the strongest results we know of which do this. The results here also give a good deal of the theory of hypersets. Precisely, all of the main results on substitution and corecursion follow from the main results so far. Further, previous studies such as [3] used urelements throughout, and the work here shows that this is not necessary. We did not cover bisimulation in this paper (though here the coalgebraic approach certainly applies). Indeed, for results pertaining to bisimulation one needs functors which preserve weak pullbacks, and we did not need this assumption in our work.

The category Class: In work on set theory, one often has to work with classes. While not objects in set theory itself, it is often convenient to work informally in a theory where classes are first-class objects. We do this by moving from the usual category *Set* of sets to the larger category *Class* of classes. Here the objects are (definable) classes, the morphisms are (definable) operations $f : a \to b$ on classes which are *set continuous* in the sense that

$$f(a) = \bigcup\{f(a_0) : a_0 \subseteq a \text{ is a set}\}.$$

We work with all the usual definitions from set theory. These include that of the Kuratowski ordered pair $\langle a,b\rangle = \{\{a\},\{a,b\}\}$. We identify functions with sets of such ordered pairs which satisfy the standard functionality condition. If f is a function and $a \subseteq dom(f)$, then $f[a]$ is the *image*, $\{f(b) : b \in a\}$. We also set $0 = \emptyset$, $1 = \{\emptyset\}$, and then we define the disjoint union of sets a and b by

$$a + b = (\{0\} \times a) \cup (\{1\} \times b).$$

Note that this means that inl is $a' \mapsto \langle 0,a'\rangle$ and inr is $b' \mapsto \langle 1,b'\rangle$.

Let $\mathscr{P} : Set \to Set$ be the usual power set functor. $\mathscr{P}$ takes a function $f : a \to b$ to give the function $\mathscr{P}f : \mathscr{P}(a) \to \mathscr{P}(b)$ given by $a_0 \mapsto f[a_0]$. $\mathscr{P}$ has no fixed points in the sets, but we can extend it to *Class*. For a class c, $\mathscr{P}c$ is the class of all sub*sets* of c. For example, if V is the class of all sets, $\mathscr{P}V = V$. For $f : c \to d$ and all subsets $c_0 \subseteq c$, $(\mathscr{P}f)c_0 = f[c_0]$.

Definition. If $a \subseteq b$, the *inclusion map* $i_{a,b} : a \to b$ has $i_{a,b}a' = a'$ for all $a' \in a$. When $b = V$, we generally omit it from the notation. For example, $i_V = \mathsf{id}_V$. $F : Class \to Class$ is *standard* if whenever $a \subseteq b$, then $Fa \subseteq Fb$, and also $Fi_{a,b} = i_{Fa,Fb}$.

The power set functor is standard, as are all the constant functors. If F and G are standard, so are $F \circ G$, $F + G$, and $F \times G$.

Proposition 5.1. *Let F be standard.*

1. *Let* $f : A \to B$, *let* $A_0 \subseteq A$, *and suppose that* $B_0 \subseteq B$ *is such that* $f[A_0] \subseteq B_0$. *Let* $g : A_0 \to B_0$ *be obtained by restricting* f. *Then for all* $x \in F(A_0)$, $(Ff)x = (Fg)x$.
2. *If* $f : A \to B$, *then* $(Ff)[F(A)] \subseteq F(f[A])$.

Proof. For (1), let $i : A_0 \hookrightarrow A$ and $j : B_0 \hookrightarrow B$. So $j \circ g = f \circ i$. By functoriality, $Fj \circ Fg = Ff \circ Fi$. And by standardness, Fj and Fi are inclusions. This implies our result.

For (2), we apply (1) with $A = A_0$ and $B_0 = f[A]$. By the result of (1), $(Ff)[F(A)] = (Fg)[F(A)]$. But as $Fg : F(A) \to F(B_0)$, so $(Fg)[F(A)] \subseteq F(B_0)$. □

Lemma 5.2. *Let* F *be standard*:
1. *Being monotone*, F *has a least fixed point* F_* *and a greatest fixed point* F^*. *These may be proper classes.*
2. $F^* = \bigcup\{a \in V : a \subseteq Fa\}$.
3. $\langle F^*, \mathsf{id}\rangle$ *is a final* F*-coalgebra iff for every class* b *and every* $e : b \to Fb$, *there exists a unique* $s : b \to V$ *such that* $s = i_{FV} \circ Fs \circ e$.

Proof. All parts are "standard" except perhaps for (3). Consider $e : b \to Fb$ and some associated s. Let $c = s[b]$ be the image of b under s. Suppose the condition mentioned in the second part of (3). Proposition 5.1.2 shows that $Fs[Fb] \subseteq F(s[b]) = Fc$. Our condition in (3) implies that $c \subseteq Fs[Fb]$, and so we see that $c \subseteq Fc$. By the monotonicity of F, we have $c \subseteq F^*$. Let $t : b \to c$ be such that $i_c \circ t = s$. Then $in_{c,F^*} \circ t$ is a coalgebra morphism from $\langle b, e\rangle$ to $\langle F^*, \mathsf{id}\rangle$. The uniqueness part of finality follows from the observation that if f is a coalgebra morphism from $\langle b, e\rangle$ to $\langle F^*, \mathsf{id}\rangle$, and if $c = f[b]$, then $c \subseteq F(c)$.

On the other hand, if $\langle F^*, \mathsf{id}\rangle$ is final, then for every $e : b \to Fb$ we associate $i_{F^*} \circ e^*$, where $e^* : b \to F^*$ is the final F-coalgebra morphism for e. The uniqueness is as above. □

Definition. The *Anti-Foundation Axiom* (*AFA*) is the assertion that for every set b and every $e : b \to \mathscr{P}b$, there exists a unique $s : b \to V$ such that $s = \mathscr{P}s \circ e$. The map s is called the *solution* to the *system* e.

Lemma 5.3 (Turi [9]). *AFA is equivalent to the assertion that* $\langle V, i_{V,\mathscr{P}V}\rangle = \langle V, \mathsf{id}_V\rangle$ *is a final* $\mathscr{P}$*-coalgebra.*

Proof. This is not just a trivial application of Lemma 5.2.3, since the latter statement reads "for all *classes* b" while present statement only uses sets. Nevertheless, one can show that set form implies the class form. Here is the argument in brief: Let b be a class, and let $e : b \to \mathscr{P}b$. Define s by $s(x) = y$ iff there is some (set) subsystem $e' : b' \to \mathscr{P}b'$ of e such that $x \in b'$, and such that the solution s' of e' has $s'(x) = y$. One has to show that s is well defined and total, and that it is the only solution to e. □

In the lemma below, $\mathscr{P}_a^*$ is the greatest fixed point of the monotone operator $\mathscr{P}_a$.

Lemma 5.4. *Assume AFA. For every set or class a, $\langle \mathscr{P}_a^*, \mathsf{id}\rangle$ is a final $\mathscr{P}_a$-coalgebra.*

Proof. Let $e : b \to \mathscr{P}(a+b)$. We want to use Lemma 5.2.3, so we need to show that there is a unique $s : b \to V$ such that $s = i_{\mathscr{P}_a V} \circ \mathscr{P}_a s \circ e$.

We may assume that a is transitive, since if necessary we may replace a by its transitive closure. Thus, we have an inclusion $i_{a,\mathscr{P}(a)} : a \to \mathscr{P}(a)$, and the corresponding final coalgebra morphism is i_a. By the Flattening Lemma, there is a unique $s : b \to V$ such that $i_{V,\mathscr{P}V} \circ s = \mathscr{P}\langle i_a, s\rangle \circ e$. Now $\langle i_a, s\rangle = \langle i_a, \mathsf{id}_V\rangle \circ (\mathsf{id}_a + s)$. So there is a unique s such that

$$\begin{aligned} i_{V,\mathscr{P}V} \circ s &= \mathscr{P}\langle i_a, \mathsf{id}_V\rangle \circ \mathscr{P}(\mathsf{id}_a + s) \circ e \\ &= i_{\mathscr{P}(a+V),V} \circ \mathscr{P}_a s \circ e. \end{aligned} \tag{6}$$

We used standardness in the last step. Now any s which satisfies (6) also satisfies $s = i_{\mathscr{P}_a V} \circ \mathscr{P}_a s \circ e$, since the other maps are inclusions, and conversely, any s with this later property satisfies (6). $\square$

At this point we can give our first example of a parametric corecursion system. It is

$$\langle \mathit{Class}, +, \mathscr{P}, V, \mathsf{id}_V, a \mapsto \langle \bar{a}, \mathsf{id}_{\bar{a}}\rangle\rangle, \tag{7}$$

where for all classes a, $\bar{a} = \mathscr{P}_a^*$.

Now that we have this parametric corecursion system, we can apply the results and notation of Sections 2 and 3 to it. That is, we read those sections again, with $C = \mathit{Class}$ and $F = \mathscr{P}$. We shall apply those results in Section 5.1 below. We should remark that when we write $\bar{a}$ we mean $\mathscr{P}_a^*$, the largest set satisfying $\bar{a} = \mathscr{P}(a + \bar{a})$. We sometimes denote this by La. We do this especially when we want to also apply the L to morphisms; the definition once again is that for $f : a \to b$, $Lf = [\mathsf{inl}_b \circ f]$. Finally, we generally omit mention of all of the identity maps to save on notation.

5.1. Uniform functors

We would like to have many further examples of parametric corecursion systems based on *Class*. For this, we study a *uniformity condition* on endofunctors F which guarantees that the greatest fixed point F^*, together with the identity on it, will be a final coalgebra. This condition below is the cleanest and most generally applicable such condition that we know of.

Definition. $F : \mathit{Class} \to \mathit{Class}$ is *uniform* if there is a natural transformation $\eta : F \to L$ such that $[\![\mathsf{id}_V]\!] \circ \eta_V = i_{FV}$.

There are several intuitions at work here. One can certainly study the subject formally, and then the goal would be for the definition of uniformity to be the weakest one for which Theorem 5.7 and Propositions 5.8 and 5.9 hold. In fact, the definition above is the weakest that we know of which does guarantee the desired results. However, there are some deeper intuitions. While not needed for the development below, the do shed some light on what is happening.

As a way of entering the subject, consider the functors and natural transformation in Example 3.1. What we want first of all is a functor G on *Class* which is big enough so that for all "natural" functors F there is a natural transformation $\eta : F \to G$. We take G to be the functor L based on the parametric corecursion system in (7). Once again, the way L works is that for each set or class a, La is the largest class $\bar{a}$ such that $\bar{a} = \mathscr{P}(a + \bar{a})$. This class includes a copy of every set (as we will see explicitly below), and in fact it may be thought of as the universe of sets built by taking the elements of a as atoms (urelements).

Note in particular that $\bar{V} = \mathscr{P}(V + \bar{V})$ is the class of all sets which can be built from *arbitrary* sets as atoms, and with features that distinguish internal sets as either atoms or other elements of $\bar{V}$. (The distinguishing features are the tags 0 and 1 involved in the paring operations that stand behind the coproduct $+$.) Recall that $[\![\mathsf{id}_V]\!]$ is given in Lemma 2.14: it is the unique map $[\![\mathsf{id}_V]\!] : \bar{V} \to V$ such that $[\![\mathsf{id}_V]\!] = \mathscr{P}\langle \mathsf{id}_V, [\![\mathsf{id}_V]\!] \rangle \circ \varphi_V$. This map $[\![\mathsf{id}_V]\!]$ serves to erase all of these distinguishing features.

The requirement on uniformity is that $[\![\mathsf{id}_V]\!] \circ \eta_V = i_{FV}$. This says that if we encode FV as a subclass of $\bar{V}$ and then collapse back to V via $[\![\mathsf{id}_V]\!]$, we have an inclusion. The reason why we want to do any encoding has to do with co-recursion: given $e : a \to Fa$, we want to use get a solution satisfying an appropriate recursion principle. There is no evident way to do this without extra maps. We use η to get a related map $e' : a \to \bar{a}$. Having this, we can use Theorem 2.17.

Our definition of uniformity is a modification of the one found in [9, 11]. The difference is that the original did not use $L = L_{\mathscr{P}}$ but rather the well-founded version of this functor, taking a set a to the *smallest* set a^+ such that $a^+ = \mathscr{P}(a + a^+)$. So our definition above is satisfied by more functors than the original definition. Functors which use non-wellfounded sets, such as $F(a) = \Omega \times a$, or even the constant functor Ω, will be uniform in our sense but not in the original sense. Nevertheless, the senses are close enough that the original argument is what is behind the proof of Theorem 5.7 below.

Two last remarks on the definition. First, it is easy to show from uniformity that for all classes a, $[\![i_a]\!] \circ \eta_a = i_{Fa}$. The proof uses naturality of η and also Proposition 2.16. And finally, the definition of unformity is delicate in the sense that there are naturally isomorphic functors F and G with the property that F is uniform but G is not. Indeed, we can take $F = \mathsf{id}_{Class}$ and $G = F + 0$ (see Proposition 5.9 below).

Example 5.5. We consider the power set functor. Recall that for every set a we have $\varphi_a : \bar{a} \to \mathscr{P}(a + \bar{a})$. We set $\eta_a : \mathscr{P}a \to \bar{a}$ to be $\varphi_a^{-1} \circ \mathscr{P}\mathsf{inl}_a$. The naturality is easy to check, and the uniformity property follows from $[\![\mathsf{id}_V]\!] \circ \varphi_V^{-1} \circ \mathscr{P}\mathsf{inl}_V = \psi^{-1} \circ \mathscr{P}\mathsf{id}_V = i_{\mathscr{P}V} \circ \mathsf{id}_{\mathscr{P}V} = i_{\mathscr{P}V}$.

Example 5.6. For each set w, let G_w be the constant functor with value w. To define η, we introduce some machinery that will be useful in other examples. For all sets a, consider $V \to \mathscr{P}V \to \mathscr{P}(a+V)$. This is an F_a-coalgebra. Let $k_a : V \to \bar{a}$ be the final coalgebra morphism. Then is is easy to check two things:

1. $[\![i_a]\!] \circ k_a = \mathsf{id}_V$.
2. For all $f : a \to b$, $Lf \circ k_a = k_b$.

Fact (1) is shown by checking that $[\![i_a]\!] \circ k_a$ is a coalgebra morphism from V to itself, and (2) by showing that $Lf \circ k_a$ satisfies the equation defining k_b.

We set $\eta_a : w \to \bar{a}$ to be $w' \mapsto k_a w'$. Then the naturality and uniformity facts for η follow from facts (2) and (1), respectively.

We will show below that the uniform functors are closed under composition, and then use this to show the uniformity of most of the usual functors on sets. (An exception would the identity I, but then $I^* = V$ would not be a final I-coalgebra anyway.) This will give us more examples of parametric corecursion systems besides the one in (7). It is important that in this section, we apply the theory developed in this paper to the parametric corecursion system of (7) and only to this system. For example, L will denote $L_{\mathscr{P}}$ in the rest of this section, and $[f]$ and $[\![f]\!]$ will be used relative to the power set functor as well.

Theorem 5.7 is the main justification for the concept of uniform functors on sets. It shows that assuming *AFA*, the greatest fixed points of uniform functors are the final coalgebras.

The first result of this type is due to Aczel [1]. It stated that if a functor F on sets satisfied a condition called *uniformity on maps*, then the greatest fixed point of F is a final coalgebra. Aczel's condition was reformulated in [9, 11] where it was called *uniformity*. Again, the result is that for uniform functors, the greatest fixed point is a final coalgebra. (Incidentally, Barwise and Moss [3] prove yet another result of this type, but it is for operators on sets which need not be the object parts of functors.) Our condition is more applicable than the previous conditions, since the constant functors come out as uniform in our sense. The proof of Theorem 5.7 below is inspired by the work in [9, 11].

Theorem 5.7. *Assume AFA. Let F be standard and uniform. Then the greatest fixed point F^* gives a final F-coalgebra $\langle F^*, \mathsf{id}\rangle$.*

Proof. Let $e : a \to Fa$. Then $\eta_a \circ e : a \to \bar{a}$. By Lemma 5.3, $\langle V, \mathsf{id}_V\rangle$ is a final $\mathscr{P}$-coalgebra. By Theorem 2.17, there is a unique $s = (\eta_a \circ e)^\dagger : a \to V$ such that $s = [\![s]\!] \circ \eta_a \circ e$. By Proposition 2.16, $[\![s]\!] = [\![\mathsf{id}_V]\!] \circ Ls$. So there is a unique s such that

$$\begin{aligned} s &= [\![\mathsf{id}_V]\!] \circ Ls \circ \eta_a \circ e \\ &= [\![\mathsf{id}_V]\!] \circ \eta_V \circ Fs \circ e \\ &= i_{FV} \circ Fs \circ e. \end{aligned}$$

We used the naturality of η and the uniformity condition. And now we conclude from Lemma 5.2.3 that $\langle F^*, \mathsf{id}\rangle$ is indeed a final F-coalgebra.

We conclude this paper with a few results that substantiate the claim that most functors of interest in set theory are uniform. Propositions 5.8 and 5.9 are new (though they surely hold for other notions of uniformity).

Proposition 5.8. *The composition of uniform functors is uniform.*

Proof. Let $F : V \to V$ be uniform via η, and let $G : V \to V$ be uniform via ρ. We claim that $F \circ G$ is uniform via δ, where

$$\delta_a = \mu_a \circ \rho_{\bar a} \circ F\eta_a$$

In this expression, $\mu_a = [\mathsf{inr}_a] : \bar{\bar a} \to \bar a$ is from Section 2.3; see Lemma 2.9. The naturality of δ follows from the naturality of ρ, η, and μ, and from the functoriality of F.

We need to check that $[\![\mathsf{id}_V]\!] \circ [\mathsf{inr}_V] \circ \rho_{\bar V} \circ F\eta_V = i_{FGV}$. By Propositions 2.15 and 2.16,

$$[\![\mathsf{id}_V]\!] \circ [\mathsf{inr}_V] = [\![[\![\mathsf{id}_V]\!]]\!] = [\![\mathsf{id}_V]\!] \circ L[\![\mathsf{id}_V]\!].$$

(Incidentally, $[\mathsf{inr}_V] \neq L[\![\mathsf{id}_V]\!]$. For example, $\emptyset \in \bar V$, so that $\{\mathsf{inl}_{\bar V}\emptyset\} \in \bar{\bar V}$. Also $[\![\mathsf{id}_V]\!]\emptyset = \emptyset$. Thus

$$L[\![\mathsf{id}_V]\!]\{\mathsf{inl}_{\bar V}\emptyset\} = [\mathsf{inl}_V \circ [\![\mathsf{id}_V]\!]]\{\mathsf{inl}_{\bar V}\emptyset\} = \{\mathsf{inl}_V[\![\mathsf{id}_V]\!]\emptyset\} = \{\mathsf{inl}_V\emptyset\}.$$

On the other hand, $[\mathsf{inr}_V]\{\mathsf{inl}_{\bar V}\emptyset\} = \{\mathsf{inr}_V\emptyset\}$.)

We conclude the proof by examining the diagram below:

$$\begin{tikzcd}
FGV \arrow[r, "F_{\eta V}"] \arrow[dr, "i_{FGV,\,FV}"'] & F\overline{V} \arrow[r, "\rho_{\overline{V}}"] \arrow[d, "F\,[\![\mathsf{id}_V]\!]"] & \overline{\overline{V}} \arrow[d, "L\,[\![\mathsf{id}_V]\!]"] \\
 & FV \arrow[r, "\rho_V"] \arrow[dr, "i_{FV}"'] & \overline{V} \arrow[d, "L\,[\![\mathsf{id}_V]\!]"] \\
 & & V
\end{tikzcd}$$

The square commutes by naturality, and the lower triangle by uniformity of ρ. The upper triangle is the uniformity statement for η, except that we have applied F throughout; we also used standardness to see that Fi_{GV} is the inclusion of FGV into FV. The point is that the composition of the diagonal maps is i_{FGV}, as desired. $\square$

We have already seen that the power set functor is uniform, as is each constant functor. Here are some further closure conditions on the uniform functors. Together with what we have seen, they lead to the conclusion that most functors of interest in set theory are uniform.

Proposition 5.9. *Let F and G be uniform, and let w be any class*:
1. *$F + G$ is uniform.*
2. *$F \times G$ is uniform.*
3. *The functor $a \mapsto w + a$ is uniform.*
4. *The functor $a \mapsto a^w$ is uniform.*

Proof. The point here is that we can use the machinery at hand to faithfully reproduce the specific functions used in the Kuratowski pair and the resulting coproduct.

Here are the details on $F+G$. Let F be uniform via η and G via ρ. Let δ be defined so that $\delta_a : Fa + Ga \to \bar{a}$ is given by

$$\delta_a\langle 0,x\rangle = \{\mathsf{inr}_a k_a\{0\}, \mathsf{inr}_a\{\mathsf{inr}_a k_a 0, \mathsf{inr}_a \eta_a x\}\},$$

$$\delta_a\langle 1,y\rangle = \{\mathsf{inr}_a k_a\{1\}, \mathsf{inr}_a\{\mathsf{inr}_a k_a 1, \mathsf{inr}_a \rho_a y\}\},$$

where $x \in Fa$ and $y \in Ga$.

To check that δ is a natural transformation, we first make a few remarks. Let $f : a \to b$, and consider some $x \in Fa$. The naturality of η implies that $Lf \circ \eta_a = \eta_b Ff$. Further, for all d, $Lfk_a(d) = k_b(d)$, by our work in Example 5.6:

$$\begin{aligned}
Lf\delta_a\langle 0,x\rangle &= Lf\{\mathsf{inr}_a k_a\{0\}, \mathsf{inr}_a\{\mathsf{inr}_a k_a 0, \mathsf{inr}_a \eta_a x\}\} \\
&= \{\mathsf{inr}_b Lfk_a\{0\}, \mathsf{inr}_b Lf\{\mathsf{inr}_a k_a 0, \mathsf{inr}_a \eta_a x\}\} \\
&= \{\mathsf{inr}_b k_b\{0\}, \mathsf{inr}_b\{\mathsf{inr}_b Lfk_a 0, \mathsf{inr}_a Lf\eta_a x\}\} \\
&= \{\mathsf{inr}_b k_b\{0\}, \mathsf{inr}_b\{\mathsf{inr}_b k_b 0, \mathsf{inr}_b \eta_b Ffx\}\} \\
&= \delta_b\langle 0, Ffx\rangle \\
&= \delta_b(F+G)f\langle 0,x\rangle.
\end{aligned}$$

Similar results obtain for all $y \in Ga$, and this essentially verifies the naturality. To check that δ is uniform, use the definition of $[\![\mathsf{id}_V]\!]$ to calculate that for all $x \in FV$,

$$\begin{aligned}
[\![\mathsf{id}_V]\!]\delta_V\langle 0,x\rangle &= [\![\mathsf{id}_V]\!]\{\mathsf{inr}_V k_V\{0\}, \mathsf{inr}_V\{\mathsf{inr}_V k_V 0, \mathsf{inr}_V \eta_V x\}\} \\
&= \{[\![\mathsf{id}_V]\!]k_V\{0\}, [\![\mathsf{id}_V]\!]\{\mathsf{inr}_V k_V 0, \mathsf{inr}_V \eta_V x\}\} \\
&= \{\{0\}, \{[\![\mathsf{id}_V]\!]k_V 0, [\![\mathsf{id}_V]\!]\eta_V x\}\} \\
&= \{\{0\}, \{0,x\}\} \\
&= \langle 0,x\rangle.
\end{aligned}$$

Similar results obtain for all $y \in GV$. This concludes the verification that $F + G$ is uniform.

The details concerning the other parts are similar. □

Corollary 5.10. *Let F be standard and uniform. Then the following is a parametric corecursion system*:

$$\langle Class, +, F, F^*, \mathsf{id}_{F^*}, a \mapsto \langle \bar{a}, \mathsf{id}_{\bar{a}} \rangle,$$

where for all classes a, $\bar{a} = F_a^*$.

Proof. The point is that for every class a, F_a is standard (easily), and uniform by Propositions 5.9 and 5.8.

From classes to sets: Nearly all of the results of this section hold for the category *Set* of sets. The main change is that Lemma 5.2 no longer holds, since monotone operators on sets need not have fixed points. In particular, $\mathscr{P}$ has no fixed points among the sets, and so $\mathscr{P}$ will not have a final coalgebra. However, for each infinite cardinal κ, let the functor $\mathscr{P}_\kappa$ take each set to the set of its subsets of size $<\kappa$. Then each $\mathscr{P}_\kappa$ does have fixed points and indeed, the greatest fixed points give final coalgebras. Moreover, it is straightforward to propose analogs of uniformity for these functors, and then Theorem 5.7 and the rest of the results of this section will hold. These give final coalgebras in *Set* for many functors.

Further remarks: Readers familiar with *Vicious Circles* ([3], henceforth *VC*) will know that the presentation of this paper differs in that it works in a pure set theory, while *VC* begins with a theory involving urelements. This paper is not the place for a full discussion of the foundational and pedagogical issues involved, but we do have some comments. On a technical level, the use of urelements in *VC* comes from the use of functions of the form $e : a \to a \cup b$. The union operation does not lead to the kind of theory presented in this paper, and working in a set theory with urelements is an alternative approach. In this paper, we have used the disjoint union $a + b$ instead. The disadvantage of this approach is that specific examples tend to have a lot of injections and are therefore difficult to read.

The advantage is that the many of the mathematical results can be presented more efficiently. To translate between the results here and those of *VC*, one take the special case of C as *Set*, F as $\mathscr{P}$, and c as V, and then take $\bar{a}$ to be $V_{afa}[a]$ for each class a of urelements. The work here covers a good part of the theory of substitution, Chapter 8 of *VC* (the General Solution Lemma is Theorem 2.17). The Solution Lemma Lemma 16.5 and its versions in Chapter 18 are essentially applications of Theorem 2.11.

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