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Journal of Symbolic Computation 49 (2013) 78–94

Contents lists available at SciVerse ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Dimension-dependent bounds for Gröbner bases of polynomial ideals

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ARTICLE INFO

Article history:
Received 10 December 2010
Accepted 22 July 2011
Available online 22 December 2011

Keywords:
Gröbner basis
Degree bound
Polynomial ideal
Ideal dimension
Regular sequence
Noether normalization
Cone decomposition

ABSTRACT

Given a basis F of a polynomial ideal I in $\mathbb{K}[x_1, \ldots, x_n]$ with degrees $\deg(F) \leq d$, the degrees of the reduced Gröbner basis G w.r.t. any admissible monomial ordering are known to be double exponential in the number of indeterminates in the worst case, i.e. $\deg(G) = d^{2^{\Theta(n)}}$. This was established in Mayr and Meyer (1982) and Dubé (1990).

We modify both constructions in order to give worst case bounds depending on the ideal dimension proving that $\deg(G) = d^{n^{\Theta(1)}2^{\Theta(r)}}$ for r-dimensional ideals (in the worst case).

1. Introduction

The degrees of Gröbner bases play a crucial role for the complexity of Gröbner basis computation and thus for many computational problems of commutative algebra and algebraic geometry. A prominent example is the complexity of the membership problem.

Mayr and Meyer (1982) proved that the membership problem in commutative semigroups is exponential space hard. Since commutative semigroups can be embedded into binomial ideals, this lower complexity bound also applies to the membership problem of polynomial ideals (over fields). This implies that the degree of a reduced Gröbner basis can be double exponential in the degree of an arbitrary ideal basis, cf. Möller and Mora (1984) or Huynh (1986). Later, Yap (1991) improved the construction and obtained a degree bound with a smaller constant in the second exponent.

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doi:10.1016/j.jsc.2011.12.018

Theorem 1 (Yap, 1991). *Fix an admissible monomial ordering $\prec$. Then there is a family of ideals $I_n \subseteq \mathbb{K}[x_1, \dots, x_n]$, for $n \in \mathbb{N}$, generated by $O(n)$ polynomials F_n of degrees bounded by d such that each Gröbner basis G_n of I_n has a maximal degree of at least*

$$\deg(G_n) \geq d^{2^{(1/2-\varepsilon)n}} \text{ for any } \varepsilon > 0 \text{ and sufficiently large } d, n \in \mathbb{N}.$$

Dubé (1990) showed an upper degree bound for Gröbner bases which is double exponential in the number of variables. The bound is larger than the lower bounds by only a constant factor in the second exponent.

Theorem 2 (Dubé, 1990). *Let I be an ideal in $\mathbb{K}[x_1, \dots, x_n]$ generated by polynomials $f_1, \dots, f_s$ of maximal degree d and fix an admissible monomial ordering. Then the reduced Gröbner basis G of I has a degree bounded by*

$$\deg(G) \leq 2\left(\frac{d^2}{2} + d\right)^{2^{n-1}}.$$

Kühnle and Mayr (1996) used this result to show that the membership problem in polynomial ideals is solvable in exponential space. Hence, the membership problem in binomial ideals is as hard as the membership problem in arbitrary polynomial ideals.

This paper is a revised and extended version of Mayr and Ritscher (2010). We investigate the Gröbner basis degree depending on the ideal dimension and establish lower and upper bounds which are single exponential in the number of variables and double exponential in the ideal dimension.

The construction for the upper degree bound is based upon Dubé's cone decompositions. The construction is improved by embedding a homogeneous regular sequence into the ideal whose length equals the ideal height. The cone decomposition is then computed for the space of normal forms of the ideal spanned by the regular sequence. The longer the regular sequence, the smaller the cone decomposition will be. Like Dubé's, we compute an exact cone decomposition and bound its Macaulay constants inductively (and therefore the maximal cone degree). This computation is reduced to a special case, greatly simplifying Dubé's proof. For this reduction, we prove that the Macaulay constants of the exact cone decomposition only depend~~s~~ on the degrees of the input polynomials, the number of variables, and the ideal dimension. Then we choose a simple monomial ideal which has the same parameters, construct an exact cone decomposition, and bound the Macaulay constants. Combining everything, we prove the following bound:

Theorem 3. *Let $\mathbb{K}$ be an infinite field and $I \subsetneq \mathbb{K}[x_1, \dots, x_n]$ be an ideal of dimension r generated by homogeneous polynomials $F = \{f_1, \dots, f_s\}$ of degrees $d_1 \geq \dots \geq d_s$. Then degree of the reduced Gröbner basis is bounded by*

$$\deg(G) \leq 2\left[\frac{1}{2}(d_1 \cdots d_{n-r} + d_1)\right]^{2^{r-1}}.$$

This bound holds for homogeneous ideals only. However, the inhomogeneous case can be reduced to the homogeneous case using the effective Noether normalization by Dickenstein et al. (1991). The upper bound for the inhomogeneous case is slightly larger than the upper bound for the homogeneous case, however still of the same form as our lower bound.

Theorem 4. *Let I be an ideal of dimension r in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ over an infinite field $\mathbb{K}$, let I be generated by polynomials $F = \{f_1, \dots, f_s\}$ of degrees $d_1 \geq \dots \geq d_s$, and fix some admissible monomial ordering. Then the degree of the reduced Gröbner basis G is bounded by*

$$\deg(G) \leq 2\left[\frac{1}{2}\left((d_1 \cdots d_{n-r})^{2(n-r)} + d_1\right)\right]^{2^r}.$$

Simply exponential (in n) for every fixed dimension r.

In order to prove the lower degree bound, we combine the Mayr–Meyer ideals (resp. Yap's improved versions) with a well-known worst case example for zero-dimensional ideals. This enforces the choice of a particular monomial ordering. The resulting bound is given by

Theorem 5. *There are a monomial ordering and a family of ideals $I_{r,n} \subseteq \mathbb{K}[x_1, \dots, x_n]$ of dimension at most r, for all $r, n \in \mathbb{N}$ such that $r \le n$, which are generated by $O(n)$ polynomials $F_{r,n}$ of degrees bounded by d such that each Gröbner basis $G_{r,n}$ has a maximal degree of at least*

$$\deg(G_{r,n}) \ge d^{(n-r)2^{(1/2-\varepsilon)r}} \quad \textit{for any } \varepsilon > 0 \textit{ and sufficiently large } d, r \in \mathbb{N}.$$

2. Preliminaries

2.1. Notation

In this chapter, we define the notation used throughout the paper. For a more detailed introduction into polynomial algebra, the reader may consult the text books Cox et al. (1992) and Cox et al. (2005) or Kreuzer and Robbiano (2000) and Kreuzer and Robbiano (2005).

Let $\mathbb{K}[x_1, \dots, x_n]$ denote the ring of polynomials in the variables $X = \{x_1, \dots, x_n\}$. Each polynomial $f \neq 0$ permits a unique representation $f = f_0 + \cdots + f_d$ such that $f_d \neq 0$ and f_k are homogeneous polynomials of (total) degree k, the so-called *homogeneous components* of f. A set $S \subseteq \mathbb{K}[x_1, \dots, x_n]$ is called *homogeneous* if, for every polynomial $f \in S$, its homogeneous components f_k are also elements of S.

Throughout the paper, we assume some arbitrary but fixed *admissible* monomial ordering $\prec$. Since the monomial ordering will be fixed, we will not keep track of it in the notation. The *leading monomial* (w.r.t. to the fixed monomial ordering) is denoted by $\mathrm{lm}(f)$ and we write $\mathrm{lm}(I)$ for the ideal $\langle \mathrm{lm}(f) : f \in I \rangle$ for any ideal I.

$\langle f_1, \dots, f_s \rangle$ denotes the *ideal* $I = \left\{ \sum_{i=1}^{s} a_i f_i : a_i \in \mathbb{K}[x_1, \dots, x_n] \right\}$ generated by polynomials $f_1, \dots, f_s \in \mathbb{K}[x_1, \dots, x_n]$. A finite set G is a *Gröbner basis* of I if $\langle G \rangle = I$ and $\langle \mathrm{lm}(G) \rangle = \mathrm{lm}(I)$.

$\mathrm{nf}_I(h)$ denotes the *normal form* of $h \in \mathbb{K}$ which, for a fixed monomial ordering, is the unique irreducible polynomial in the coset $h + I$. The set of all normal forms is denoted by N_I. Since the normal forms are unique, the sum $\mathbb{K}[x_1, \dots, x_n] = I \oplus N_I$ is direct.

2.2. Hilbert functions

Let $T \subseteq \mathbb{K}[x_1, \dots, x_n]$ be a homogeneous vector space and

$$T_z = \{f \in T : f \text{ homogeneous with } \deg(f) = z \text{ or } f = 0\}$$

the homogeneous polynomials of T of degree z. The *Hilbert function* of T is defined as

$$\mathrm{HF}_T(z) = \dim_{\mathbb{K}}(T_z),$$

i.e. the vector space dimension of T_z over the field $\mathbb{K}$. It is well-known that, for large values of z, the Hilbert functions $\mathrm{HF}_I(z)$ and $\mathrm{HF}_{N_I}(z)$ of a homogeneous ideal I and its normal forms N_I are polynomials. These polynomials, known as *Hilbert polynomials*, will be denoted by $\mathrm{HP}_I(z)$ and $\mathrm{HP}_{N_I}(z)$, respectively. The minimal degree z_0 such that $\mathrm{HF}_I(z) = \mathrm{HP}_I(z)$ (or, equivalently, $\mathrm{HF}_{N_I}(z) = \mathrm{HP}_{N_I}(z)$) for all $z \ge z_0$ is called *(Hilbert) regularity* $\mathrm{reg}(I) = z_0$.

Since all reductions of homogeneous polynomials are degree-invariant, it easily follows from the dimension theorem for direct sums that

$$\mathrm{HF}_{\mathbb{K}[x_1,\dots,x_n]}(z) = \mathrm{HF}_I(z) + \mathrm{HF}_{N_I}(z).$$

Since we will have to deal with ideal dimensions and vector space dimensions, we will write $\dim(I)$ for the former and $\dim_{\mathbb{K}}(T)$ for the latter in order to avoid confusion.

The *Hilbert series* of a homogeneous vector space T is defined as

$$\mathrm{HS}_T(y) = \sum_{z \ge 0} \mathrm{HF}_T(z) y^z.$$

2.3. Ideal dimension

The *(affine) dimension* of a homogeneous ideal $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ can be defined in many equivalent ways (cf. Cox et al. (1992), Section 9). We will use two of them which turn out to be the most suitable for our purpose.

The first definition of the ideal dimension uses the Hilbert polynomial:

$$\dim(I) = \deg(\mathrm{HP}_{N_I}) + 1$$

with $\deg(0) = -1$. By adding 1 to the degree of the Hilbert polynomial, we obtain the *affine* dimension. This is adequate since we present an affine construction and therefore the affine dimension provides the right intuition.

The second definition is a specialized version of the transcendence degree for polynomial ideals. A set of variables $U \subseteq X$ is called independent w.r.t. an ideal I if $I \cap \mathbb{K}[U] = \{0\}$. Then the ideal dimension is the maximal cardinality of an independent set:

$$\dim(I) = \max\{\#U : U \subseteq X, I \cap \mathbb{K}[U] = \{0\}\}$$

where $\#U$ denotes the cardinality of the set U.

2.4. Homogenization

The *homogenization* of a polynomial f of degree d w.r.t. a new variable x_0 is defined by ${}^hf = x_0^d f_0 + x_0^{d-1} f_1 + \cdots + f_d$. The converse operation, the *dehomogenization*, is the substitution of x_0 with 1 and denoted by ${}^df(x_0, \ldots, x_n) = f(1, x_1, \ldots, x_n)$ for $f \in \mathbb{K}[x_0, \ldots, x_n]$. Finally, the homogenization of an ideal I is defined by ${}^hI = \left\langle {}^hf : f \in I \right\rangle$.

The homogenization of an ideal basis unfortunately does not, in general, generate the homogenization of the ideal. Still, both ideals are tightly connected, which can be expressed by saturation w.r.t. the homogenization indeterminate.

Lemma 6. *Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$. Then*

$$ {}^hI = \left\langle {}^hf_1, \ldots, {}^hf_s \right\rangle : x_0^\infty.$$

Proof. See Kreuzer and Robbiano (2005), Corollary 4.3.8. □

It is well-known that any admissible monomial ordering $\prec$ on $\mathbb{K}[x_1, \ldots, x_n]$ induces an admissible monomial ordering $\prec$ on $\mathbb{K}[x_0, \ldots, x_n]$ such that ${}^d\mathrm{lm}(f) = \mathrm{lm}({}^df)$ for all homogeneous $f \in \mathbb{K}[x_0, \ldots, x_n]$ (cf. Kreuzer and Robbiano (2005), proposition 3.4.14).

Lemma 7. *Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ and $\prec$ be a monomial ordering on $\mathbb{K}[x_1, \ldots, x_n]$. Then any Gröbner basis G of $J = \left\langle {}^hf_1, \ldots, {}^hf_s \right\rangle \subseteq \mathbb{K}[x_0, \ldots, x_n]$ w.r.t. $\prec$ yields a Gröbner basis ${}^dG$ of I.*

Proof. By Lemma 6, ${}^dG \subseteq I$. Since J is homogeneous, it suffices to consider homogeneous polynomials. Therefore we get ${}^d\mathrm{lm}(J) = \mathrm{lm}({}^dJ)$ and

$$\mathrm{lm}(I) = \mathrm{lm}({}^d(J : x_0^\infty)) = \mathrm{lm}({}^dJ) = {}^d\mathrm{lm}(J) = \left\langle {}^d\mathrm{lm}(G) \right\rangle = \left\langle \mathrm{lm}({}^dG) \right\rangle. \quad \square$$

2.5. Regular sequences

algebraically closed?

A sequence $(g_1, \ldots, g_t)$ with $g_k \in \mathbb{K}[x_1, \ldots, x_n]$ is called *regular* if

(1) g_k is a non-zerodivisor in $\mathbb{K}[x_1, \ldots, x_n]/\langle g_1, \ldots, g_{k-1} \rangle$, for all $k = 1, \ldots, t$, and
(2) $\langle g_1, \ldots, g_t \rangle \neq \mathbb{K}[x_1, \ldots, x_n]$.

Regular sequences are nice tools for various reasons. If $(g_1, \ldots, g_t)$ is a regular sequence, $\dim \langle g_1, \ldots, g_t \rangle = n - t$. For homogeneous sequences, moreover, the Hilbert function only depends on few parameters.

Lemma 8. *Let $(g_1, \ldots, g_t)$ be a homogeneous sequence in $\mathbb{K}[x_1, \ldots, x_n]$ with degrees $d_1, \ldots, d_t$, fix an arbitrary monomial ordering, and let $J = \langle g_1, \ldots, g_t \rangle$. Iff $(g_1, \ldots, g_t)$ is regular, N_J has the Hilbert series*

$$\mathrm{HS}_{N_J}(y) = \frac{\prod_{i=1}^{t}(1 - y^{d_i})}{(1-y)^n}.$$

In this case, its Hilbert function HP_{N_J} only depends on n, t, and $d_1, \ldots, d_t$, and the regularity is $\mathrm{reg}(J) = d_1 + \cdots + d_t - n + 1$.

Proof. See Kreuzer and Robbiano (2005), Sections 5.2B and 5.4B. □

It is well-known, that most sequences $(g_1, \ldots, g_t)$ of length $t \le n$ are regular. But there is a result which is in some sense stronger and of crucial importance to our construction. Given an arbitrary ideal I of dimension r, one can "approximate" I by a regular sequence of length $n - r$ which is completely contained in the ideal. As a nice giveaway, the degrees of the sequence are bounded by the degrees of arbitrary generators of I.

Lemma 9. *Let $\mathbb{K}$ be an infinite field and $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ an ideal generated by homogeneous polynomials $f_1, \ldots, f_s$ with degrees $d_1 \ge \cdots \ge d_s$ such that $\dim(I) \le r$. Then there are a strictly decreasing sequence $s \ge j_1 > \cdots > j_{n-r} \ge 1$ and homogeneous $a_{k,i} \in \mathbb{K}[x_1, \ldots, x_n]$ such that*

$$g_k = \sum_{i=j_k}^{s} a_{k,i} f_i \quad \text{for } k = 1, \ldots, n - r$$

form a homogeneous regular sequence, $\dim \langle f_{j_k}, \ldots, f_s \rangle = n - k$, and $\deg(g_k) = d_{j_k}$.

Proof. Analogous to Schmid (1995), Lemma 2.2. □

2.6. Noether normalization

Dickenstein et al. (1991) used Bézout's theorem and the bound from Kollar (1988) for the radical membership in order to proof a representation bound depending on the ideal dimension. Their proof can be slightly improved using the degree bound by Jelonek (2005). The proof is given in the Appendix.

Theorem 10 (Noether Normalization)**.** *Let $\mathbb{K}$ be an infinite field and $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal of dimension r generated by polynomials $f_1, \ldots, f_s$ of degrees $d_1 \ge \cdots \ge d_s$. Then there is an invertible linear change of coordinates*

$$\sigma : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[x_1, \ldots, x_n],\, x_i \mapsto a_{i,1}x_1 + \cdots + a_{i,n}x_n$$

with $a_{i,j} \in \mathbb{K}$ for $i, j = 1, \ldots, n$ such that $\{x_1, \ldots, x_r\}$ is a maximal independent set w.r.t. $\sigma(I)$ and, for each $i = r + 1, \ldots, n$, there is a polynomial $h_i \in \sigma(I) \cap \mathbb{K}[x_1, \ldots, x_i]$ which is monic in x_i, i.e. $\deg_{x_i}(h_i) = \deg(h_i) > 0$. Then $\sigma(I)$ is said to be in Noether position. The degrees can be bounded by $\deg(h_i) \le (d_1 \cdots d_{n-r})^2$.

Proof. Slight improvement of Dickenstein et al. (1991), Section 1. □

3. Upper bound

3.1. Cone decompositions

In Dubé (1990), cone decompositions were introduced as irredundant finite representation of vector spaces $T \subseteq \mathbb{K}[x_1, \ldots, x_n]$ that are generated by monomials (e.g. (leading) monomial ideals and sets of normal forms). By using cones, one can represent large spaces that have a very nice Hilbert function.

Let $h \in \mathbb{K}[x_1, \ldots, x_n]$ and $U \subseteq X$. Then a *cone* is a set $C = \mathbf{C}(h, U) = h \cdot \mathbb{K}[U]$, h is called *vertex* of the cone, the degree of the cone is $\deg(C) = \deg(h)$, and its dimension is $\dim(C) = \#U$.

Lemma 11. *Let C be a cone in* $\mathbb{K}[x_1, \dots, x_n]$. *If* $\dim(C) = 0$,

$$\mathrm{HF}_C(z) = \begin{cases} 0 & \textit{for } z \neq \deg(C) \\ 1 & \textit{for } z = \deg(C) \end{cases},$$

otherwise

$$\mathrm{HF}_C(z) = \begin{cases} 0 & \textit{for } z < \deg(C) \\ \binom{z-\deg(C)+\dim(C)-1}{\dim(C)-1} & \textit{for } z \geq \deg(C) \end{cases}.$$

Proof. See Dubé (1990), Section 2.4. □

If a subspace T of $\mathbb{K}[x_1, \dots, x_n]$ is a direct sum of cones $T = \mathbf{C}(h_1, U_1) \oplus \cdots \oplus \mathbf{C}(h_t, U_t)$, then $P = \{\mathbf{C}(h_1, U_1), \dots, \mathbf{C}(h_t, U_t)\}$ is called *cone decomposition* of T. The degree of a cone decomposition is $\deg(P) = \max\{\deg(C) : C \in P\}$. The part of positive dimension is denoted by $P^+ = \{C \in P : \dim(C) > 0\}$.

Since only cones with homogeneous vertices are considered, $T = T_1 \oplus \cdots \oplus T_t$ implies $\mathrm{HF}_T = \sum_{C \in P} \mathrm{HF}_C$ and $\mathrm{HP}_T = \sum_{C \in P^+} \mathrm{HP}_C$.

We will consider arbitrary cone decompositions later. In order to construct (as a set) finite cone decompositions, we have to prefer low degrees for cones with large dimensions.

Definition 12. A cone decomposition P is *q-standard* for some $q \in \mathbb{N}$ if

(1) $C \in P^+$ implies $\deg(C) \geq q$ and
(2) for each $C \in P^+$ and each $q \leq d \leq \deg(C)$, there exists a cone $C' \in P$ such that $\deg(C') = d$ and $\dim(C') \geq \dim(C)$.

Note that P is q-standard for all $q \in \mathbb{N}$ iff $P^+ = \emptyset$. Otherwise it can be q-standard for at most one q, namely the minimum of the degrees of cones in P^+. Furthermore, the union of q-standard decompositions is q-standard, again.

Cones can be split into direct sums of smaller cones. For a cone $C = \mathbf{C}(h, U)$ in $\mathbb{K}[x_1, \dots, x_n]$ with $U = \{u_1, \dots, u_t\}$, the *fan* of the cone C is defined as

$$\mathbf{F}(C) = \{\mathbf{C}(h, \emptyset)\} \cup \{\mathbf{C}(u_i h, \{u_1, \dots, u_i\} : i = 1, \dots, t)\}.$$

Then $C = \bigoplus_{C' \in \mathbf{F}(C)} C'$. The definition of the fan, however, is not unique since it depends on a choice of an order of the elements of U. This will not matter in the following.

Lemma 13 (Dubé, 1990)**.** *Every q-standard cone decomposition P of a vector space T in* $\mathbb{K}[x_1, \dots, x_n]$ *may be refined into a* $(q+1)$*-standard cone decomposition Q of T with* $\deg(P) \leq \deg(Q)$ *and* $\deg(P^+) \leq \deg(Q^+)$.

Proof. It suffices to replace the cones of positive dimension and minimal degree by their fans. □

Dubé's genius idea was to connect the degree of cone decompositions to the Gröbner basis degree. He gave a constructive proof of the following lemma.

Lemma 14 (Dubé, 1990)**.** *Let I be a homogeneous ideal in* $\mathbb{K}[x_1, \dots, x_n]$ *and fix an admissible monomial ordering* $\prec$. *Then there is a* 0*-standard cone decomposition P of* N_I *such that the degree of the reduced Gröbner basis G of I w.r.t.* $\prec$ *is bounded by* $\deg(G) \leq \deg(P) + 1$.

Proof. See Dubé (1990), Section 4. □

For the mathematical analysis, standard cone decompositions have too many degrees of freedom. A worst-case construction can help here.

Definition 15. A q-standard cone decomposition P is *q-exact* if $\deg(C) \neq \deg(C')$ for all $C \neq C' \in P^+$.

Since q-exact cone decompositions are also q-standard, the cones of higher degrees have lower dimensions, i.e. , $C, C' \in P$, $\deg(C) > \deg(C')$ implies $\dim(C) \leq \dim(C')$. The computation of exact cone decompositions is pretty easy — simply replace cones which contradict the definition by their fans. The interesting part is the proof of the termination.

Note that the algorithm is a reformulation of SHIFT and EXACT in Dubé (1990) and does essentially the same.

Algorithm 1: `Shift`(P)

Data: q-standard cone decomposition P of T
Result: q-exact cone decomposition Q of T
$Q \leftarrow P$
for $d \leftarrow q, \ldots, \deg(Q^+)$ **do**
  $S \leftarrow \{C \in Q^+ : \deg(C) = d\}$
  while $\#S > 1$ **do**
    Choose $C \in S$ with minimal dimension $\dim(C)$.
    $S \leftarrow S \setminus \{C\}$
    $Q \leftarrow Q \setminus \{C\} \cup \mathbf{F}(C)$
  end
end
return Q.

Lemma 16 (Dubé, 1990)**.** *Every q-standard cone decomposition P of a vector space T in $\mathbb{K}[x_1, \ldots, x_n]$ may be refined into a q-exact cone decomposition Q of T with $\deg(P) \leq \deg(Q)$ and $\deg(P^+) \leq \deg(Q^+)$.*

Proof. The claim is that `Shift` always terminates and returns a cone decomposition $Q = $ `Shift`(P) with the desired properties. First consider correctness. It is obvious from the code and the definition of the fan that $S = \{C \in Q^+ : \deg(C) = d\}$ after each while-loop. Since no cones with degree smaller than d are added to Q, in the end Q^+ contains at most one cone per degree. Since a cone C with minimal dimension is chosen from S, Q is q-standard at any time by induction. Hence it is q-exact on termination.

The proof of termination involves a potential function on Q. Let $v \in \mathbb{N}^n$ be the vector with entries $v_i = \#\{C \in Q^+ : \deg(C) \geq d, \dim(C) = n + 1 - i\} - 1$ for $i = 1, \ldots, n$. It counts the number of cones that still have to be processed grouped by their dimensions. Within the for-loop, the first non-zero entry of v stays the same, as each fan $\mathbf{F}(C)$ contains exactly one cone of dimension $\dim(C)$ and none with higher dimension. When d is increased, the first non-zero entry of v decreases by 1 since Q is q-standard at any time.

Finally, $\deg(P) \leq \deg(Q)$ and $\deg(P^+) \leq \deg(Q^+)$ are obvious from the construction and Q. □

Definition 17. Let P be a q-exact cone decomposition in $\mathbb{K}[x_1, \ldots, x_n]$. If $P^+ = \emptyset$, let $q = 0$. Then the *Macaulay constants* of P are defined as

$$a_k = \max\left(\{q\} \cup \left\{\deg(C) + 1 : C \in P^+, \dim(C) \geq k\right\}\right) \quad \text{for } k = 0, \ldots, n+1.$$

Note that $a_0 = \deg(P) + 1$, $a_1 = \deg(P^+) + 1$, and $a_{n+1} = q$. Now the degree of a reduced Gröbner basis was bounded by the degree of a particular 0-standard cone decomposition Q in Lemma 14. If we refine Q to a 0-exact cone decomposition P by Lemma 13, this fulfills the inequality $\deg(Q) \leq \deg(P)$. Hence it suffices to bound the Macaulay constants (actually a_0) of P in order to get a bound of the Gröbner basis degree.

In the following, the Hilbert polynomial of an exact cone decomposition will be expressed by the Macaulay constants. In the next paragraph, it will be discussed how to extend a cone decomposition such that the Hilbert function of the resulting cone decomposition is known thus yielding an approach for the calculation of the Macaulay constants.

Lemma 18 (Dubé, 1990)**.** *Let $q \geq 1$, P be a q-exact cone decomposition of a vector space T in $\mathbb{K}[x_1, \ldots, x_n]$, and $a_0, \ldots, a_{n+1}$ be the Macaulay constants of P. Then*

$$\mathrm{HP}_T(z) = \binom{z - a_{n+1} + n}{n} - 1 + \sum_{i=1}^{n} \binom{z - a_i + i - 1}{i}.$$

Proof. See Dubé (1990), Section 7. □

The formula provided by Lemma 18 still can be terrifying in computations, especially if the Macaulay constants are to be determined from a complicated Hilbert function. We avoid such computations and by reduction to a special case. The essential insight is provided by the following lemma.

Lemma 19 (*Dubé, 1990*)**.** *Let P be a q-exact cone decomposition of a homogeneous subspace T of $\mathbb{K}[x_1, \dots, x_n]$ for some $q \geq 1$. Then the Macaulay constants $a_1, \dots, a_{n+1}$ are uniquely determined by HP_T and q.*

Proof. See Dubé (1990), Section 7. □

As mentioned before, an arbitrary cone decomposition will be extended to the cone decomposition of a space whose Hilbert polynomial is known. Since regular sequences have nice Hilbert functions and capture the dimension of the ideal, they were chosen for this construction. Lemmas 19 and 8 combine to

Corollary 20. *Let J be an ideal in $\mathbb{K}[x_1, \dots, x_n]$ generated by a homogeneous regular sequence $g_1, \dots, g_t$ of degrees $d_1, \dots, d_t$ and fix any admissible monomial ordering. Then the Macaulay constants $a_1, \dots, a_{n+1}$ of (any q-exact cone decomposition of) N_J only depend on q, n, t, and $d_1, \dots, d_t$. If P is a q-exact decomposition of a vector space $T \subseteq \mathbb{K}[x_1, \dots, x_n]$ with $\mathrm{HF}_T = \mathrm{HF}_{N_J}$, the Macaulay constants of P and N_J are the same.*

Note that a_0 is explicitly excluded from the above result. It will be shown later how to overcome this using the bounds on the other Macaulay constants.

3.2. A new decomposition

Up to now, we have seen how to construct a cone decomposition of a space of normal forms. In order to bound the Macaulay constants of a homogeneous ideal $I = \langle f_1, \dots, f_s \rangle$, Dubé uses the decomposition

$$\mathbb{K}[x_1, \dots, x_n] = I \oplus N_I$$

and cone decompositions of I and N_I. He employs the following decomposition:

$$I = \langle f_1 \rangle \oplus \bigoplus_{i=2}^{s} f_i \cdot N_{\langle f_1, \dots, f_{i-1} \rangle : f_i}. \tag{1}$$

The Hilbert functions of $\mathbb{K}[x_1, \dots, x_n]$ and $\langle f_1 \rangle$ are easily determined, and, for all other summands, we have seen how to calculate exact cone decompositions. The drawback is that in Dubé's construction, the Macaulay constants achieve their worst case bound in the zero-dimensional case. Therefore a different decomposition is necessary.

Looking back at Dubé's paper, the key to improvement can be found in Dubé (1990), Corollary 5.2. Instead of calculating a cone decomposition of I, he separates the cone $\mathbf{C}(f_1, X)$ from the cone decomposition as in (1) and thereby improves the final bound slightly. Dealing with arbitrary non-trivial ideals, this is the best that can be done. But restricting to ideals I of a certain dimension r, this decomposition can be improved using an embedded regular sequence $g_1, \dots, g_t$ whose length equals the height of the ideal. The following is a generalization of Dubé (1990), Lemma 5.1.

Lemma 21. *Let I be an ideal in $\mathbb{K}[x_1, \dots, x_n]$ generated by homogeneous polynomials $g_1, \dots, g_t, f_1, \dots, f_s$ and let $J = \langle g_1, \dots, g_t \rangle \subseteq I$. For a fixed admissible monomial ordering $\prec$,*

$$I = J \oplus \bigoplus_{i=1}^{s} f_i \cdot N_{J_{i-1} : f_i} \tag{2}$$

where $J_k = \langle g_1, \dots, g_t, f_1, \dots, f_k \rangle$ for $k = 0, \dots, s$.

Proof. In order to prove this, use induction to show

$$J_k = J \oplus \bigoplus_{i=1}^{k} f_i \cdot N_{J_{i-1}:f_i} \quad \text{for } k = 0, \ldots, s. \tag{3}$$

Then the equality $I = J_s$ yields the stated result.

The "$\supseteq$"-inclusion of (3) is clear since $f_1, \ldots, f_s \in I$ and $J \subseteq I$. For the other inclusion, the case $k = 0$ is trivial. So assume $k > 0$ and prove

$$J_k = J_{k-1} \oplus \left(f_k \cdot N_{J_{k-1}:f_k}\right).$$

Let $f \in J_k$ and thus

$$f = h + a \cdot f_k \quad \text{with } h \in J_{k-1}, a \in \mathbb{K}[x_1, \ldots, x_n].$$

Rewriting

$$a = (a - \mathrm{nf}_{J_{k-1}:f_k}(a)) + \mathrm{nf}_{J_{k-1}:f_k}(a)$$

yields

$$a \cdot f_k \in f_k \cdot (J_{k-1} : f_k) + f_k \cdot N_{J_{k-1}:f_k}.$$

Since $f_k \cdot (J_{k-1} : f_k) \subseteq J_{k-1}$, one gets $f \in J_{k-1} + f_k \cdot N_{J_{k-1}:f_k}$. It remains to show that the sum is direct. For any $k = 0, \ldots, s$, assume $h \in J_{k-1} \cap (f_k \cdot N_{J_{k-1}:f_k})$ and therefore $h = a \cdot f_k$ for some $a \in N_{J_{k-1}:f_k}$. However $h \in J_{k-1}$ implies $a \in J_{k-1} : f_k$ and thus $a = 0$ and $h = 0$ follow. $\square$

The decomposition (2) will be used for constructing a cone decomposition complementing J starting from a cone decomposition of N_I. Since I is the ideal whose Gröbner basis shall be bounded, it will be important to make sure that the maximal degrees of cones do not decrease in order to be able to apply Lemma 14 later.

Lemma 22. *Let I be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ which is generated by homogeneous polynomials $g_1, \ldots, g_t$, $f_1, \ldots, f_s$ and fix an admissible monomial ordering. Furthermore let $J = \langle g_1, \ldots, g_t \rangle \subseteq I$ and $d = \max\{\deg(f_i) : i = 1, \ldots, s\}$. Then any 0-standard cone decomposition Q of N_I may be completed to a d-exact cone decomposition P of a vector space $T \subseteq \mathbb{K}[x_1, \ldots, x_n]$ with $\mathrm{HF}_T = \mathrm{HF}_{N_J}$ such that $\deg(Q) \leq \deg(P)$.*

Proof. By Lemma 14, one can construct a 0-standard cone decomposition Q_k of $N_{J_{k-1}:f_k}$ with $J_k = \langle g_1, \ldots, g_t, f_1, \ldots, f_k \rangle$ for each $k = 1, \ldots, s$. Then $f_k \cdot Q_k$ is a $\deg(f_k)$-standard cone decomposition of $f_k \cdot N_{J_{k-1}:f_k}$. By Lemma 13, $Q, Q_1, \ldots, Q_s$ can be refined to d-standard cone decompositions $\tilde{Q}, \tilde{Q}_1, \ldots, \tilde{Q}_s$. Since

$$\mathbb{K}[x_1, \ldots, x_n] = J \oplus \bigoplus_{i=1}^{s} f_i \cdot N_{J_{i-1}:f_i} \oplus N_I,$$

the union

$$Q' = \tilde{Q} \cup \tilde{Q}_1 \cup \cdots \cup \tilde{Q}_s$$

is a d-standard cone decomposition of $T = \bigoplus_{i=1}^{s} f_i \cdot N_{J_{i-1}:f_i} \oplus N_I$ and $\mathrm{HP}_T = \mathrm{HP}_{N_J}$ is obvious. By Lemma 16, Q' can be refined to a d-exact cone decomposition P of T. None of the operations decreases the degree of the cone decomposition, so $\deg(Q) \leq \deg(P)$. $\square$

By Lemma 19, all Macaulay constants of a d-exact cone decomposition P of a vector space T complementing J except $a_0 = \deg(P) + 1$ are determined by the Hilbert polynomial. The actual purpose of this construction, however, is to bound $\deg(P)$ (see Lemma 14). This can be realized using the regularity of the ideal (which is known for a homogeneous complete intersection) in order to bridge the gap between a_1 and a_0.

Lemma 23. *Let I be a homogeneous ideal in $\mathbb{K}[x_1, \ldots, x_n]$ and fix an admissible monomial ordering $\prec$. If P is a q-exact cone decomposition of a vector space $T \subseteq \mathbb{K}[x_1, \ldots, x_n]$ with $\mathrm{HF}_T = \mathrm{HF}_{N_I}$ and corresponding Macaulay constants $a_0, \ldots, a_{n+1}$,*

$$a_0 \leq \max\{a_1, \operatorname{reg}(I)\}.$$

Proof. Since $\prec$ is graded, the Hilbert function of $\mathbb{K}[x_1, \ldots, x_n] = I \oplus N_I$ can be computed as sum of the Hilbert functions of I and N_I. The latter can be expressed using the Hilbert functions of the cones of P. Since $\mathrm{HF}_{\mathbb{K}[x_1,\ldots,x_n]}(z) = \mathrm{HP}_{\mathbb{K}[x_1,\ldots,x_n]}(z)$ for all $z \in \mathbb{Z}$ and, by definition of the regularity, $\mathrm{HF}_I(z) = \mathrm{HP}_I(z)$ for all $z \geq \operatorname{reg}(I)$, we obtain, for $\max\{a_1, \operatorname{reg}(I)\} \leq z < a_0$,

$$\begin{aligned}\#\{C \in P : \dim(C) = 0, \deg(C) = z\} &= \mathrm{HF}_{N_I}(z) - \mathrm{HP}_{N_I}(z) \\ &= (\mathrm{HF}_{\mathbb{K}[x_1,\ldots,x_n]}(z) - \mathrm{HF}_I(z)) - (\mathrm{HP}_{\mathbb{K}[x_1,\ldots,x_n]}(z) - \mathrm{HP}_I(z)) = 0.\end{aligned}$$

Thus there are no cones with degree greater or equal $\max\{a_1, \operatorname{reg}(I)\}$ which implies the statement. □

Applying Lemma 23 to a homogeneous complete intersection and using Lemma 8, one obtains

Corollary 24. *Let J be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by a homogeneous regular sequence $(g_1, \ldots, g_t)$ with degrees $d_1, \ldots, d_t$ and fix an admissible monomial ordering. If P is a q-exact cone decomposition of a vector space $T \subseteq \mathbb{K}[x_1, \ldots, x_n]$ with $\mathrm{HP}_T = \mathrm{HP}_{N_J}$ and corresponding Macaulay constants $a_0, \ldots, a_{n+1}$, then*

$$a_0 \leq \max\{a_1, d_1 + \cdots + d_t - n + 1\}.$$

Actually now everything is clear — at least for the homogeneous case. Let $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal of dimension r generated by polynomials $f_1, \ldots, f_s$. Using Lemma 9, one obtains a homogeneous regular sequence $(g_1, \ldots, g_{n-r})$ in I (with the same degrees). Let $J = \langle g_1, \ldots, g_{n-r}\rangle$ be the ideal generated by the regular sequence. With Lemmas 14 and 22, one can compute an exact cone decomposition P of a vector space $T \subseteq \mathbb{K}[x_1, \ldots, x_n]$ with $\mathrm{HF}_T = \mathrm{HP}_{N_J}$ such that $\deg(P) + 1$ is a bound of the Gröbner basis degree of I. By Corollary 24, it suffices to determine the Macaulay constants $a_1, \ldots, a_{n+1}$ of P. This can be done – as Dubé originally did – by comparing the Hilbert polynomials of $\mathbb{K}[x_1, \ldots, x_n]$ and $J \oplus N_J$. However, the calculations are somewhat cumbersome.

The clue for avoiding this trouble is the reduction to a special case. Remember Corollary 20: the Macaulay constants of P only depend on a few some constants. Thus it suffices to calculate them once (for each set of parameters) — for a special case with an easy structure.

Theorem 25. *Let $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal of dimension r generated by homogeneous polynomials $g_1, \ldots, g_{n-r}, f_1, \ldots, f_s$ where $(g_1, \ldots, g_{n-r})$ is a regular sequence of degrees $d_1, \ldots, d_{n-r}$ and $d = \max\{\deg(f_i) : i = 1, \ldots, s\}$ and fix an admissible monomial ordering. If Q is a 0-standard cone decomposition of N_I, then*

$$\deg(Q) \leq \max\left\{\deg(P^+), d_1 + \cdots + d_{n-r} - n\right\}$$

where P is a d-exact cone decomposition of N_J and $J = \left\langle x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}}\right\rangle$.

Proof. Let $\tilde{I} = \langle g_1, \ldots, g_{n-r}\rangle$. By Lemma 22, one can extend any 0-standard cone decomposition Q of N_I to a d-exact cone decomposition $\tilde{Q}$ of a vector space T with $\mathrm{HP}_T = \mathrm{HP}_{N_{\tilde{I}}}$ and degree $\deg(\tilde{Q}) \geq \deg(Q)$. Let $a_0, \ldots, a_{n+1}$ be the Macaulay constants of $\tilde{Q}$. By Corollary 24, $\deg(\tilde{Q}) = a_0 - 1 \leq \max\{a_1 - 1, d_1 + \cdots + d_{n-r} - n\}$. However, the Macaulay constants $a_1, \ldots, a_{n+1}$ of $\tilde{Q}$ only depend on $d, n, n - r$, and the degrees $d_1, \ldots, d_{n-r}$ as proved in Corollary 20. The ideal $J = \left\langle x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}}\right\rangle$ is obviously a r-dimensional ideal generated by a homogeneous regular sequence with the same degrees. Thus any d-exact cone decomposition P of N_J (which exists by Lemmas 14, 13 and 16) has the same Macaulay constants (except a_0) and thus $\deg(Q^+) = \deg(P^+) = a_1 - 1$. □

Example 26. It is very surprising that the Macaulay constants are independent of the ideal, but only depend on the degrees of the generators and the dimension. For verification, consider the very simple ideal $J = \langle x^2 \rangle$ with dimension $\dim(I) = 2$ in the ring $\mathbb{K}[x, y, z]$. This ideal is a complete intersection of the form in Theorem 25. Using the concepts and algorithms from Dubé (1990), one can obtain a 2-exact cone decomposition P of N_J. Due to its size, only the cones of positive dimension are listed:

$$\left\{\mathbf{C}(xz, \{y, z\}), \mathbf{C}(z^3, \{y, z\}), \mathbf{C}(y^2z^2, \{y\}), \mathbf{C}(xy^4, \{y\}), \mathbf{C}(y^6, \{y\}), \mathbf{C}(y^6z, \{y\})\right\}.$$

Now let $I = \langle x^2 - xy, xy + xy \rangle$ which also is a homogeneous ideal of dimension $\dim(I) = 2$. One can embed the complete intersection $I' = \langle x^2 - xy \rangle$ into I and then compute a cone decomposition Q of a vector space T which complements I'. This cone decomposition extends a cone decomposition of N_I:

$$Q^+ = \left\{\mathbf{C}(y^2, \{y, z\}), \mathbf{C}(xy^2 + xyz, \{y, z\}), \mathbf{C}(yz^3, \{z\}), \right.$$
$$\left. \mathbf{C}(z^5, \{z\}), \mathbf{C}(xz^5, \{z\}), \mathbf{C}(xyz^5 + xz^6, \{z\})\right\}.$$

Both P and Q are 2-exact cone decompositions with the same parameters n, r, d and thus – as expected – have the same Macaulay constants:

$$a_1 = 8, \qquad a_2 = 4, \qquad a_3 = 2.$$

3.3. Bounding the Macaulay constants

By Theorem 25, it remains to bound the Macaulay constant a_1 of a d-exact cone decomposition of N_J for the ideal $J = \left\langle x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}} \right\rangle$ in $\mathbb{K}[x_1, \ldots, x_n]$, which will be fixed for the remainder of this section. Note that this is a monomial ideal for which all monomial orderings are equivalent. Hence, the monomial ordering will not be mentioned in the following lemmas.

The special shape of this ideal allows to dramatically simplify the calculations compared to the proof in Dubé's paper which does not make any assumption on the ideal. Nevertheless, the obtained bound will apply to any ideal by the preceding considerations (Theorem 25).

From $r = \dim(J) = \deg(\mathrm{HP}_J) + 1$, one immediately deduces:

Lemma 27. *Let $J = \left\langle x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}} \right\rangle$ be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ and $a_0, \ldots, a_{n+1}$ the Macaulay constants of a d-exact cone decomposition P of N_J. Then*

$$a_n = \cdots = a_{r+1} = d.$$

In the following, the construction of a d-exact cone decomposition for J will be presented. On the way, bounds for the remaining Macaulay constants will be derived. First it is necessary to determine N_J. The following observation is obvious.

Corollary 28. *Let $J = \left\langle x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}} \right\rangle$ be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$. Then the space of normal forms of J may be decomposed into the direct product*

$$N_J = T_r \times \mathbb{K}[x_{n-r+1}, \ldots, x_n],$$

where the vector space T_r is given by

$$T_r = \mathrm{span}_{\mathbb{K}}\left\{x^\alpha \in \mathbb{K}[x_1, \ldots, x_{n-r}] : \alpha \in \mathbb{N}^n, \alpha_i < d_i \text{ for } i = 1, \ldots, n - r\right\}. \tag{4}$$

The construction of the cone decomposition will be inductive. It will appear crucial that, in each step, the part of the normal forms which is not covered by already constructed cones has a form similar to N_J. We assume that it is the direct product of a finite vector space T_k generated by monomials and a polynomial subring. Thus the (vector space) dimension of T_k determines the number of cones of the highest dimension.

Lemma 29. *Let $T_k \subseteq \mathbb{K}[x_1, \ldots, x_{n-k}]$ be a vector space generated by monomials and P_k a cone decomposition of $T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n]$. Then P_k has exactly $\dim_{\mathbb{K}}(T_k)$ cones of dimension k.*

Proof. For $k = 0$, the statement is obvious. For $k \geq 1$, the key is to look at the Hilbert polynomials. Consider a monomial basis $\{b_1, \ldots, b_s\}$ of T_k. Thus

$$T_k \times \mathbb{K}[x_{n-k+1}, \cdots, x_n] = b_1\mathbb{K}[x_{n-k+1}, \ldots, x_n] \oplus \cdots \oplus b_s\mathbb{K}[x_{n-k+1}, \ldots, x_n]$$

and

$$\mathrm{HP}_{T_k \times \mathbb{K}[x_{n-k+1},\ldots,x_n]}(z) = \sum_{i=1}^{s} \binom{z - \deg(b_i) + k - 1}{k - 1}.$$

On the other hand, one can compute the Hilbert polynomial from the cone decomposition P_k:

$$\mathrm{HP}_{T_k \times \mathbb{K}[x_{n-k+1},\ldots,x_n]}(z) = \sum_{C \in P_k{}^+} \binom{z - \deg(C) + \dim(C) - 1}{\dim(C) - 1}.$$

Now compare the coefficients of z^{k-1} of both representations of the Hilbert polynomial. Since $P_k{}^+$ only contains cones of dimension at most k, this yields

$$\sum_{i=1}^{s} \frac{1}{(k-1)!} = \sum_{\substack{C \in P_k{}^+ \\ \dim(C) = k}} \frac{1}{(k-1)!}$$

and thus $\#\{C \in P_k{}^+ : \dim(C) = k\} = s = \dim_{\mathbb{K}}(T_k)$. □

Looking at the explicit formula (4) for T_r, one obtains $\dim(T_r) = d_1 \cdots d_{n-r}$ and thus

Corollary 30. *Let $J = \left\langle x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}} \right\rangle$ be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ and $a_0, \ldots, a_{n+1}$ the Macaulay constants of a d-exact cone decomposition P of N_J. Then*

$$a_r = d_1 \cdots d_{n-r} + d.$$

Now turn to the actual construction of a d-exact cone decomposition of N_J. In each induction step, the number of cones can be determined by Lemma 29. This also fixes $a_{k-1} - a_k$.

Lemma 31. *Let $J = \left\langle x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}} \right\rangle$ be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$. Then, for any $d \geq 2$ and $k = 0, \ldots, r$, there exist a d-exact cone decompositions P_k and a finite-dimensional subspace $T_k \subseteq N_J \cap \mathbb{K}[x_1, \ldots, x_{n-k}]$ which have a monomial basis such that*

$$N_J = (T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n]) \oplus \bigoplus_{C \in P_k} C.$$

Let $a_0, \ldots, a_{n+1}$ be the Macaulay constants of P_0. Then $a_{k-1} \leq \frac{1}{2}a_k^2$ for $k = 2, \ldots, r$.

Proof. The proof will be in two steps, first the construction of $P_0, \ldots, P_r$ and then the computation of a bound for $a_1, \ldots, a_r$. The induction starts with $k = r$. Let $P_r = \emptyset$ (which is a d-exact cone decomposition) and define T_r as in (4). Then all requirements are fulfilled. Now $P_{r-1}, \ldots, P_0$ and $T_{r-1}, \ldots, T_0$ will be constructed inductively such that all cones in $P_{k-1} \setminus P_k$ have dimension k for $k = 1, \ldots, r$.

Let $1 \leq k \leq r$. By induction, P_k and T_k exist such that

$$N_J = (T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n]) \oplus \bigoplus_{C \in P_k} C.$$

For the induction, we will construct $T_{k-1} \subseteq T_k$ and $P_{k-1} \supseteq P_k$. Keep in mind that $P_{k-1} \setminus P_k$ will be the subset of a cone decomposition of $T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n]$ containing all the cones of dimension k. Thus, by Lemma 29, $P_{k-1} \setminus P_k$ must contain exactly $\dim_{\mathbb{K}}(T_k)$ cones of dimension k. $P_k \subseteq P_0$ is already constructed and contains all cones of dimension larger than k. Hence $a_n, \ldots, a_{k+1}$ are fixed. Since P_{k-1} shall be d-exact, the cones of dimension k must have the degrees $a_{k+1}, a_{k+1}+1, a_{k+1}+2, \ldots$. Let $\{b_1, \ldots, b_s\}$ be a monomial basis of T_k with $\deg(b_1) \leq \cdots \leq \deg(b_s)$ and choose

$$C_i = b_i x_{n-k+1}^{a_{k+1}+i-\deg(b_i)-1} \mathbb{K}[x_{n-k+1}, \ldots, x_n] \quad \text{for } i = 1, \ldots, s.$$

It is easy to see that $C_i \subseteq T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n]$, $\deg(C_i) = a_{k+1} + i - 1$, and $\dim(C_i) = k$. Thus $P_{k-1} = P_k \cup \{C_1, \ldots, C_s\}$ is a d-exact cone decomposition. Since $T_k \subseteq \mathbb{K}[x_1, \ldots, x_{n-k}]$, furthermore

$$T_k \times \mathbb{K}[x_{n-k+1}, \ldots, x_n] = C_1 \oplus \cdots \oplus C_s \oplus (T_{k-1} \times \mathbb{K}[x_{n-k+2}, \ldots, x_n])$$

where

$$\begin{aligned} T_{k-1} &= \operatorname{span}_{\mathbb{K}}\left\{b_i x_{n-k+1}^e : i = 1, \ldots, s, e = 0, \ldots, a_{k+1} + i - \deg(b_i) - 2\right\} \\ &\subseteq \mathbb{K}[x_1, \ldots, x_{n-k+1}]. \end{aligned}$$

The above formula also implies $\dim_{\mathbb{K}}(T_{k-1}) < \infty$. Moreover

$$N_J = (T_{k-1} \times \mathbb{K}[x_{n-k+2}, \ldots, x_n]) \oplus \bigoplus_{C \in P_{k-1}} C.$$

Hence the existence of the cone decompositions $P_0, \ldots, P_r$ follows by induction. Now we turn to the Macaulay constants. By the Corollaries 27 and 30, the first Macaulay constants equal $a_n = \cdots = a_{r+1} = d$ and $a_r = d_1 \cdots d_{n-r} + d$.

Now let $2 \leq k \leq r$ and prove $a_{k-1} \leq \frac{1}{2} a_k^2$ inductively. By the construction of P_{k-2}, we have $a_{k-1} - a_k = \#\{C \in P_{k-2} : \dim(C) = k-1\} = \dim(T_{k-1})$. Thus we can write $a_{k-1} = \dim(T_{k-1}) + a_k$ and bound the right-hand side.

From the explicit formula for T_{k-1}, we can compute its dimension as

$$\dim_{\mathbb{K}}(T_{k-1}) = \sum_{i=1}^{s} (a_{k+1} + i - \deg(b_i) - 1) \leq \sum_{i=1}^{s} (a_{k+1} + i - 1) = s a_{k+1} + \frac{1}{2} s(s-1).$$

Since $s = \dim_{\mathbb{K}}(T_k) = a_k - a_{k+1}$, the induction hypothesis and $a_{k+1} \geq d \geq 2$ imply

$$\begin{aligned} \dim_{\mathbb{K}}(T_{k-1}) + a_k &\leq (a_k - a_{k+1}) a_{k+1} + \frac{1}{2}(a_k - a_{k+1})(a_k - a_{k+1} - 1) + a_k \\ &= \frac{1}{2}\left(a_k^2 - a_{k+1}^2 + a_k + a_{k+1}\right) \leq \frac{1}{2}\left(a_k^2 - a_{k+1}^2 + \frac{1}{2} a_{k+1}^2 + a_{k+1}\right) \leq \frac{1}{2} a_k^2. \quad \square \end{aligned}$$

Lemma 31 yields a d-exact cone decomposition P_0 that represents N_J up to a finite-dimensional vector space T_0. Let $\{b_1, \ldots, b_s\}$ be a monomial basis of T_0. Then the union $P = P_0 \cup \{\mathbf{C}(b_i, \emptyset) : i = 1, \ldots, s\}$ is a d-exact cone decomposition of N_J with Macaulay constants which fulfill the bounds of Corollary 30 and Lemma 31.

Corollary 32. *Let $J = \left\langle x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}} \right\rangle$ be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ and $a_0, \ldots, a_{n+1}$ the Macaulay constants of a d-exact cone decomposition P of N_J. Then*

$$a_k \leq 2\left[\frac{1}{2}(d_1 \cdots d_{n-r} + d)\right]^{2^{r-k}} \qquad \text{for } k = 1, \ldots, r.$$

From the construction, one can even verify that $a_0 = a_1$ as predicted by Lemma 23. This concludes the proof for the homogeneous case as a_1 bounds the Gröbner basis degree.

Theorem 33. *Let $\mathbb{K}$ be an infinite field and $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal of dimension r generated by homogeneous polynomials $F = \{f_1, \ldots, f_s\}$ of degrees $d_1 \geq \cdots \geq d_s$. Then the degree of the reduced Gröbner basis for every monomial ordering is bounded by*

$$\deg(G) \leq 2\left[\frac{1}{2}(d_1 \cdots d_{n-r} + d_1)\right]^{2^{r-1}}.$$

Proof. W.l.o.g. one can assume $d_1 \geq 2$. Otherwise the Gröbner basis degree would be bounded by 1 trivially. Let Q be a 0-standard cone decomposition of N_I as in Lemma 14. Then $\deg(G) \leq \deg(Q) + 1$. By Lemma 9, one can embed a regular sequence $g_1, \ldots, g_{n-r}$ of degrees $d_1, \ldots, d_{n-r}$ in I. Then Theorem 25 bounds $\deg(Q) + 1$ by the Macaulay constant a_1 of a d_1-exact cone decomposition of $J = \left\langle x_1^{d_1}, \ldots, x_{n-r}^{d_{n-r}} \right\rangle$. Corollary 32, finally, gives a bound on a_1. Since this bound is greater than $d_1 + \cdots + d_{n-r} - n + 1$, the stated bound holds. $\square$

Dubé's bound can be immediately derived from this theorem. Since the bound is trivial for the empty ideal, one can assume $r \leq n - 1$. Since the bound is monotonous in r, this implies

$$\deg(G) \leq 2\left[\frac{1}{2}(d_1 + d_1)\right]^{2^{n-2}} = 2d_1^{2^{n-2}}.$$

3.4. *The inhomogeneous case*

In Mayr and Ritscher (2010) we have wrongly applied directly this approach to dimension dependent bounds. Unfortunately the transfer from homogeneous to inhomogeneous case in not so straightforward as it is shown by following example:

Example 34. Consider the regular sequence $(f_1, \ldots, f_s)$ in the ring $\mathbb{K}[x, y, z_1, \ldots, z_s]$ given by $f_k = x^{s-k+1}y^k - z_k^s$ for $k = 1, \ldots, s$. The ideal generated by this sequence has dimension $\dim \langle f_1, \ldots, f_s \rangle = 2$. The homogenization of the sequence w.r.t. a new indeterminate z_0, ${}^hf_k = x^{s-k+1}y^k - z_k^s z_0$, is not regular for $s > 2$. This is obvious since $\left\langle {}^hf_1, \ldots, {}^hf_s \right\rangle \subseteq \langle x, z_0 \rangle$, and thus $\dim \left\langle {}^hf_1, \ldots, {}^hf_s \right\rangle \geq s + 1$.

It seems inviting to replace each f_i by a linear combination of the original polynomials in order to preserve the maximal degree. However, in this example no cancellation can occur when taking linear combinations. Thus the highest degree terms of the linear combinations of $f_1, \ldots, f_s$ all be multiples of x and thus the homogenizations of the linear combinations will be contained in the ideal $\langle x, z_0 \rangle$.

Given an ideal I generated by polynomials $f_1, \ldots, f_s$, Dubé considers the homogeneous ideal $\tilde{I} = \left\langle {}^hf_1, \ldots, {}^hf_s \right\rangle$. By Lemma 7, the dehomogenization of a Gröbner basis G of $\tilde{I}$ yields a Gröbner basis of I. Since the dehomogenization only might decrease the degrees, it suffices to bound $\deg(G)$.

This approach, however, does not transfer directly to the dimension-dependent bounds. As Example 34 shows, $\dim(\tilde{I})$ might be much larger than $\dim(I)$ such that the benefit of the presented construction vanishes. A possibility to avoid this using polynomials from the Noether normal form will be presented in the following. Unfortunately, the resulting bound will be slightly weaker than in the homogeneous case.

Lemma 35. *Let $\mathbb{K}$ be an infinite field and $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal of dimension r generated by polynomials $f_1, \ldots, f_s$ of degrees $d_1 \geq \cdots \geq d_s$. Then there are polynomials $g_1, \ldots, g_{n-r} \in I$ such that $\dim \left\langle {}^hg_1, \ldots, {}^hg_{n-r} \right\rangle = \dim(I)$ and $\deg(g_i) \leq (d_1 \cdots d_{n-r})^2$ for $i = 1, \ldots, n - r$.*

Proof. By Theorem 10, there are an invertible linear change of variables

$$\sigma : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[x_1, \ldots, x_n], x_k \mapsto a_{k,1}x_1 + \cdots + a_{k,n}x_n$$

with $a_{i,j} \in \mathbb{K}$ for $i, j = 1, \ldots, n$ and $h_i \in I \cap \mathbb{K}[x_1, \ldots, x_i]$ such that $0 < \deg(h_i) = \deg_{x_i}(h_i) \leq (d_1 \cdots d_{n-r})^2$ for $i = r+1, \ldots, n$. Consider ${}^hh_{r+1}, \ldots, {}^hh_n$. These polynomials form a regular sequence of length $n - r$ and thus generate an ideal of height $n - r$. Now apply σ^{-1} and let $g_i = \sigma^{-1}(h_{r+i})$ for $i = 1, \ldots, n - r$. Since the height of an ideal is invariant under changes of variables and ${}^hg_i = \sigma^{-1}({}^hh_{r+i})$, the ideal $\left\langle {}^hg_1, \ldots, {}^hg_{n-r} \right\rangle$ has height $n - r$, too. $\square$

Instead of considering $\tilde{I} = \left\langle {}^hf_1, \ldots, {}^hf_s \right\rangle$, the polynomials ${}^hg_1, \ldots, {}^hg_{n-r}$ from Lemma 35 will be adjoined yielding $K = \left\langle {}^hg_1, \ldots, {}^hg_{n-r}, {}^hf_1, \ldots, {}^hf_s \right\rangle$. Since ${}^hg_1, \ldots, {}^hg_{n-r} \in {}^hI$, Lemma 7 still applies. Since the ideal of this homogeneous sequence has the minimal possible dimension r, ${}^hg_1, \ldots, {}^hg_{n-r}$ is a regular sequence and therefore one can use Theorem 25.

Theorem 36. *Let $\mathbb{K}$ be an infinite field and $I \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal of dimension r generated by polynomials $F = \{f_1, \ldots, f_s\}$ of degrees $d_1 \geq \cdots \geq d_s$. Then the degree of the reduced Gröbner basis is bounded by*

$$\deg(G) \leq 2\left[\frac{1}{2}\left((d_1 \cdots d_{n-r})^{2(n-r)} + d_1\right)\right]^{2^r}.$$

Proof. Let ${}^h g_1, \ldots, {}^h g_{n-r}$ be the polynomials from Lemma 35 with degrees $\tilde{d}_i = \deg(g_i) \leq (d_1 \cdots d_{n-r})^2$ for $i = 1, \ldots, n-r$, and $K = \left\langle {}^h g_1, \ldots, {}^h g_{n-r}, {}^h f_1, \ldots, {}^h f_s \right\rangle$. Because of the inclusions $\left\langle {}^h g_1, \ldots, {}^h g_{n-r} \right\rangle \subseteq K \subseteq {}^h I$, $K \subseteq \mathbb{K}[x_0, \ldots, x_n]$ is a $(r+1)$-dimensional ideal in $n+1$ variables. Let Q be a 0-standard cone decomposition of N_K as in Lemma 14. Then $\deg(G) \leq \deg(Q) + 1$ by Lemma 6. On the other hand, Theorem 25 bounds $\deg(Q)$ by the Macaulay constant $a_1 - 1$ of a d_1-exact cone decomposition of $J = \left\langle x_1^{\tilde{d}_1}, \ldots, x_{(n+1)-(r+1)}^{\tilde{d}_{(n+1)-(r+1)}} \right\rangle$ in the ring $\mathbb{K}[x_0, \ldots, x_n]$. Corollary 32, finally, gives a bound on a_1. Since this bound is greater than $\tilde{d}_1 + \cdots + \tilde{d}_{(n+1)-(r+1)} - (n+1) + 1$,

$$\deg(G) \leq 2\left[\frac{1}{2}\left(\tilde{d}_1 \cdots \tilde{d}_{(n+1)-(r+1)} + d_1\right)\right]^{2^r} \leq 2\left[\frac{1}{2}\left(\left((d_1 \cdots d_{n-r})^2\right)^{n-r} + d_1\right)\right]^{2^r}. \quad \square$$

One can simplify the bound of Theorem 36 to $\deg(G) \leq 2\left[\frac{1}{2}\left(d^{2(n-r)^2} + d\right)\right]^{2^r}$ for $d = \max\{d_1, \ldots, d_s\}$.

4. Lower bound

Remember the Mayr–Meyer construction respectively Theorem 1. To the best of the authors' knowledge, there is little known about the dimension of these ideals. Only for the Mayr–Meyer ideals there is a lower bound for the dimension which is linear in the number of variables (cf. Möller and Mora, 1984). For the following statement, it would be more interesting to have an upper bound. Though the trivial bound given by the number of variables suffices if the constants are irrelevant. The trick for obtaining a lower bound for the Gröbner basis degree is a combination with the well-known construction for the zero-dimensional case.

Example 37. Fix a graded admissible monomial ordering $\prec$ and consider the ideal $I_r \subseteq \mathbb{K}[x_1, \ldots, x_r]$ constructed by Yap. By Theorem 1, the degree of the Gröbner basis G_r of I_r is bounded by $\deg(G_r) \geq d^{2^{(1/2-\varepsilon)r}}$ for any $\varepsilon > 0$ and sufficiently large $d, n \in \mathbb{N}$. Now let $k \in \{1, \ldots, r\}$ (which is to be determined later) and define the ideal

$$I_{r,n} = I_r + \left\langle x_k - x_{r+1}^d, x_{r+1} - x_{r+2}^d, \ldots, x_{n-1} - x_n^d \right\rangle,$$

let $\pi : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[x_1, \ldots, x_r]$ be the projection which sends $x_{r+1}, \ldots, x_n$ to 1, and $\pi' : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[x_{r+1}, \ldots, x_n]$ analogously. Consider the block ordering $\prec'$ on $\mathbb{K}[x_1, \ldots, x_n]$ defined by $x^\alpha \prec' x^\beta$ iff $\pi(x^\alpha) \prec \pi(x^\beta)$ or $\pi(x^\alpha) = \pi(x^\beta)$ and $\pi'(x^\alpha) \prec_{\text{lex}} \pi'(x_\beta)$ for all $x^\alpha, x^\beta \in \mathbb{K}[x_1, \ldots, x_n]$. Here $\prec_{\text{lex}}$ denotes the lexicographic monomial ordering with $x_{r+1} \succ_{\text{lex}} \ldots \succ_{\text{lex}} x_n$.

It is well-known that the set of leading monomials of an irredundant Gröbner basis equals the set of minimally reducible monomials, i.e. the w.r.t. the ideal reducible monomials whose divisors are irreducible. Since $\prec$ is graded, the minimally reducible monomials have degrees up to the degree of the Gröbner basis. By the pigeon hole principle, there is some $k \in \{1, \ldots, r\}$ such that there is a w.r.t. I_r minimally reducible monomial $x^\alpha \in \mathbb{K}[x_1, \ldots, x_r]$ with $\deg_{x_k}(x^\alpha) \geq \frac{1}{r} d^{2^{(1/2-\varepsilon)r}}$. Moreover, $x^\alpha \in \mathbb{K}[x_1, \ldots, x_r]$ is minimally irreducible w.r.t. I_r iff $x^\alpha x_k^{-\alpha_k} x_n^{\alpha_k d^{n-r}}$ is minimally irreducible w.r.t. $I_{r,n}$. Hence, the degree of any Gröbner basis $G_{r,n}$ of $I_{r,n}$ w.r.t. $\prec'$ is bounded by $\deg(G_{r,n}) \geq \frac{1}{r} d^{2^{(1/2-\varepsilon)r}} d^{n-r}$.

Finally note that $\dim(I_{r,n}) \leq \dim(I_{r,n}) \leq r$. This follows easily since any w.r.t. $I_{r,n}$ independent set can contain at most one of the variables $x_k, x_{r+1}, \ldots, x_n$.

Theorem 38. *There are a monomial ordering and a family of ideals $I_{r,n} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ of dimension at most r for $r, n \in \mathbb{N}$ with $r \leq n$ which are generated by $O(n)$ polynomials $F_{r,n}$ of degrees bounded by d such that each Gröbner basis $G_{r,n}$ has a maximal degree of at least*

$$\deg(G_{r,n}) \geq d^{(n-r)2^{(1/2-\varepsilon)r}} \quad \text{for any } \varepsilon > 0 \text{ and sufficiently large } d, r \in \mathbb{N}.$$

This result is weaker than the one by Yap since it only works for special monomial orderings. Most likely, this cannot be avoided since the degree bounds for Gröbner bases of zero-dimensional ideals depend on the monomial ordering (cf. Lazard, 1983).

Appendix. Deferred proofs

In the following, we prove Theorem 10. We assume (without restating) that the reader is familiar with varieties, Zariski closure, the Closure Theorem (cf. Cox et al., 1992, Section 3.2), the degree of a variety as defined in Heintz (1983), and Bézout's Theorem (see Heintz, 1983, Theorem 1).

The main difference to Dickenstein et al. (1991) is the following, improved bound for the radical membership problem.

Theorem 39 (Jelonek, 2005)**.** *Let I be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by polynomials $F = \{f_1, \ldots, f_s\}$ of degrees $d_1 \geq \cdots \geq d_s$. Then*

$$\sqrt{I}^k \subseteq I \quad \text{for some } k \leq \begin{cases} d_1 \cdots d_s & \text{if } 1 \leq s \leq n \\ d_1 \cdots d_{n-1} d_s & \text{if } 1 < n \leq s \\ d_s & \text{if } n = 1. \end{cases}$$

Proof. See Jelonek (2005), Theorem 1.3. Note that the field does *not* have to be algebraically closed. This can be seen by a standard reasoning which reduces $f^k \stackrel{?}{\in} I$ to a system of linear equations. This has a solution over the algebraic closure $\overline{\mathbb{K}}$ iff it is solvable over $\mathbb{K}$. $\square$

Proof of Theorem 10. By the definition of the ideal dimension, there is a w.r.t. I maximal independent set of cardinality $r = \dim(I)$. Hence, permuting the variables with σ one can assume $\sigma(I) \cap \mathbb{K}[x_1, \ldots, x_r] = \{0\}$. The rest of the construction is by induction.

Let $r < k \leq n$, assume there are $h_i \in \sigma(I) \cap \mathbb{K}[x_1, \ldots, x_i]$ monic in x_i for $i = k+1, \ldots, n$, and construct a polynomial $h_k \in \sigma(I)$ and an invertible change of coordinates $\sigma' = \tilde{\sigma} \circ \sigma$ such that $\tilde{\sigma}(h_i) \in \mathbb{K}[x_1, \ldots, x_i]$ is monic in x_i for $i = k, \ldots, n$ and $\{x_1, \ldots, x_r\}$ is independent modulo $\sigma'(I)$.

By Schmid (1995), Lemma 2.2 (the inhomogeneous version of Lemma 9), there is a complete intersection $J \subseteq I$ generated by polynomials $g_1, \ldots, g_{n-r}$ of degrees $d_1 \geq \cdots \geq d_{n-r}$. Hence, $\{x_1, \ldots, x_r\}$ is a maximal independent set w.r.t. $\sigma(J)$ and the ideal $K = \sigma(J) \cap \mathbb{K}[x_1, \ldots, x_r, x_k] \neq \{0\}$. This implies $\dim(\sqrt{K}) = \dim(K) = r$ as ideal of $\mathbb{K}[x_1, \ldots, x_r, x_k]$ (and thus of height 1). Any ideal of height 1 is principal and the degree of the square-free part of the generator equals the degree of the variety. By the Closure Theorem, $\overline{\pi(\mathbf{V}(\sigma(J)))} = \mathbf{V}(K)$ where π is the projection onto the coordinates $\{1, \ldots, r, k\}$. Applying Lemma 2 and Theorem 1 from Heintz (1983), we obtain

$$\deg(\mathbf{V}(K)) \leq \deg(\mathbf{V}(\sigma(J))) \leq d_1 \cdots d_{n-r}.$$

Thus there is $0 \neq h \in \sqrt{K} = \sqrt{\sigma(J)} \cap \mathbb{K}[x_1, \ldots, x_r, x_k]$ with $\deg(h) \leq d_1 \cdots d_{n-r}$. Since $\{x_1, \ldots, x_r\}$ is independent w.r.t. $\sigma(J)$, $\deg_{x_k}(h) > 0$. Finally, by Corollary 39, $h^e \in \sigma(J) \subseteq \sigma(I)$ for some $e \leq d_1 \cdots d_{n-r}$. This yields $h_k = h^e \in \sigma(J) \cap \mathbb{K}[x_1, \ldots, x_r, x_k]$ with $\deg(h_k) \leq (d_1 \cdots d_{n-r})^2$ and $\deg_{x_k}(h_k) > 0$.

For the construction of $\tilde{\sigma}$, let $\tilde{h}$ be the homogeneous component of h_k of highest degree. Since $\tilde{h} \neq 0$ is homogeneous and $\mathbb{K}$ is infinite, there are values $y_1, \ldots, y_r \in \mathbb{K}$ such that $\tilde{h}(y_1, \ldots, y_r, 1) \neq 0$. Define $\tilde{\sigma}$ by $\tilde{\sigma}(x_i) = x_i + y_i x_k$ for $i = 1, \ldots, r$ and $\tilde{\sigma}(x_i) = x_i$ for $i = r+1, \ldots, n$ which certainly is invertible. Then $\tilde{\sigma}(h_i) \in \mathbb{K}[x_1, \ldots, x_i]$ for all $i = k, \ldots, n$. Moreover, $\deg_{x_k}(\tilde{h}) = \deg(\tilde{h})$ shows that $\tilde{\sigma}(h_k)$ is monic in x_k. Since $\deg_{x_i}(h_i) = \deg(h_i)$, we have $\tilde{\sigma}(x_i) = x_i$ for $i = k+1, \ldots, n$, and $\tilde{\sigma}$ preserves the total degree, $\tilde{\sigma}(h_i)$ is also monic in x_i.

It remains to show that $\{1, \dots, x_r\}$ is independent w.r.t. $\sigma'(I)$. Assume to the contrary $0 \neq f \in \sigma'(I) \cap \mathbb{K}[x_1, \dots, x_r]$. Then $\tilde{\sigma}^{-1}(f) \in \sigma(I)$. The inverse of the coordinate change is defined by $\tilde{\sigma}^{-1}(x_i) = x_i$ for $i \neq k$ and $\tilde{\sigma}^{-1}(x_k) = x_k - \sum_{i=1}^{r} y_i x_i$ and hence $\tilde{\sigma}^{-1}(f) \in \mathbb{K}[x_1, \dots, x_r]$. This contradicts the assumption that $\{1, \dots, x_r\}$ is independent w.r.t. $\sigma(I)$. $\square$

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