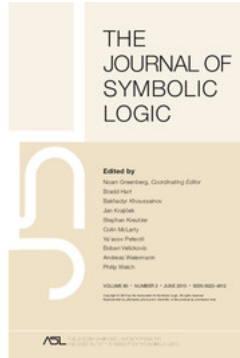

Barendregt H. P.. *The lambda calculus. Its syntax and semantics*. Studies in logic and foundations of mathematics, vol. 103. North-Holland Publishing Company, Amsterdam, New York, and Oxford, 1981, xiv + 615 pp.

E. Engeler

The Journal of Symbolic Logic / Volume 49 / Issue 01 / March 1984, pp 301 - 303
DOI: 10.2307/2274112, Published online: 12 March 2014

THE JOURNAL OF SYMBOLIC LOGIC
Volume 49, Number 1, March 1984

REVIEWS

Numerical cross references are to previous reviews in this JOURNAL, or to *A bibliography of symbolic logic* (this JOURNAL, vol. 1, pp. 121–218), or to *Additions and corrections* to the latter (this JOURNAL, vol. 3, pp. 178–212).

References beginning with a Roman numeral are by volume and page to the place at which a publication has previously been reviewed or listed. When necessary in connection with such references, a third number will be added in parentheses, to indicate position on the page. Such a reference is ordinarily to the publication itself, but when so indicated the reference may be to the review or to both the publication and its review. Thus "XLIII 148" will refer to the review beginning on page 148 of volume 43 of this JOURNAL, or to the publication which is there reviewed; "XLIII 154" will refer to one of the reviews or one of the publications reviewed or listed on page 154 of volume 43, with reliance on the context to show which one is meant; "XLI 701(6)" will refer to the sixth item listed on page 701 of volume 41, i.e., to Russell's *On denoting*; and "XLVI 870(4)" will refer to the fourth item on page 870 of volume 46, i.e., to Frege's *Function and concept*. References such as 247*18*, 534*1* are to the entries so numbered in the *Bibliography*.

H. P. BARENDREGT. ***The lambda calculus. Its syntax and semantics.*** Studies in logic and foundations of mathematics, vol. 103. North-Holland Publishing Company, Amsterdam, New York, and Oxford, 1981, xiv + 615 pp.

By 1930 various attempts had been made to address the "problem of foundations" by a formal axiomatic characterization of the notion of function (Frege's system, von Neumann's set theory, Schönfinkel and Curry's combinators). The lambda calculus owes its birth to this same endeavor. But today its story is best told as that of a piece of mathematics in its own right. This is what the book under review—Barendregt's *magnum opus*—does.

The very sensible notation "$\lambda x.x^2 + 1$" for the function whose evaluation at x yields $x^2 + 1$ immediately gives rise to a little calculus for the handling of such expressions. It has essentially only one rule: $(\lambda x. M)N$ equals the result of substituting N for x in M. Multiple exposures have dulled our sense of excitement at the fact that very innocuous-looking formal systems are sometimes very powerful; this is certainly the case with the lambda calculus. Work over five decades has helped to build up an intuition about the calculus, sorely missing at the beginning, by experimentation and by relating it to other contemporary mathematical developments, in particular to the study of formal systems, theories of computation, and model theory. In all this, it is curious to note the crucial conceptual services that the lambda calculus has rendered to the mainstream of mathematical thought, while leading, on the whole, a rather peripheral existence.

I believe that this book will play an important role in acquainting a larger public with the lambda calculus, even if it is long and does not go into such popular applications as computer science. It is very soundly constructed: In the first 124 pages the material to be covered in the book is introduced and surveyed; this material falls somewhat naturally into four parts which are then covered individually in the main portion of the text.

Calculi of conversion are the subject matter of Chapters 6 through 10. These calculi treat equations $M = N$ between terms in a formal deductive manner and are distinguished by the terms they admit: λK calculus admits all terms formed by λ-abstraction and application, λI-calculus (Church's original system) restricts λ-abstraction to terms that contain free the variable being bound, while $\boldsymbol{CL}_K$ and $\boldsymbol{CL}_I$, versions of Curry's combinatory logic, do not use λ-abstraction at all but instead add axioms providing for objects corresponding to λ-abstraction in the two versions. It is not until some of the main mathematical structures inside these calculi emerge that the relative merits of the four calculi, their fruitful interaction, and their descendants can be presented. In all these conversion calculi it is possible, and usual, to define various fixed point operators, introduce numerals, define (invent, historically) the

0022-4812/84/4901-0028/$03.10

partial recursive functions, and to obtain (discover, historically) rather transparent versions of *undecidability results and recursion theorems. One main point of distinction elaborated here is this:* In λI, a term M is solvable if and only if it has a normal form, i.e., contains no subterm which is a redex $(\lambda x . M)N$ (Barendregt); while in λK, M is solvable if and only if the strictly weaker condition holds that M have a head normal form (Wadsworth). Solvability of a term M means the existence of terms $N_1, \cdots, N_n$ such that for the λ-closure $\bar{M}$ of M the equation $\bar{M}N_1 \cdots N_n = I$ is provable. Head normal forms $\lambda x_1 \cdots x_n . yN_1 \cdots N_m$ are used to structure terms of the lambda calculus into (Böhm) trees, by branching a term M into subtrees $M_1, \cdots, M_m$ if M has head normal form $\lambda x_1 \cdots x_n . yM_1 \cdots M_m$. These trees are useful throughout the book, e.g. for proving a separability theorem (Coppo, Dezani-Ciancaglini, and Ronchi della Rocca) and invertibility (Dezani-Ciancaglini, Bergstra, and Klop), and, in general, for investigating the local (i.e., equational) structure of models (Hyland).

Reduction calculi, covered in Chapters 11 through 15, are derived from conversion calculi by supplementing equational axioms with a direction (of "reduction") and, occasionally, with labels and other extensions of notation. This is by now a widespread method of analysis of computations and proofs, first used by Church and Rosser to analyze provability in the lambda calculus. The Church–Rosser theorem (CR) on the confluence of reductions of a term is proved three times, with subtle enlargements of language and calculi (Schroer, Hyland, and Hindley). Another difference between λK and λI arises: In λI, terms with normal form are always strongly normalizable (i.e., all reduction paths eventually terminate), which is not the case in λK (Church). The main concerns of this part of the book are (a) the standardization of reduction—λ-abstractions occurring to the left of a redex that has been reduced in the reduction path cannot be used in later reduction steps (Curry and Feys); (b) the study of equivalence relations on reduction paths (Hindley, Berry, and Lévy), including an algebraic calculus on reduction paths and leading up to an algorithmic version of standardization (Klop); and (c) a study of reduction strategies (e.g. Curry's leftmost reduction, Gross–Knuth reduction, the recursive CR strategy by Bergstra and Klop, and the perpetual strategy of Barendregt, Bergstra, Klop, and Volken). Reduction strategies can sometimes be useful for showing that a term has no normal form, that two terms are non-convertible, or, more generally, to prove directly some facts about reduction, e.g. a general conservation (of non-normalizability) theorem (for λK). Also, one obtains in this way some facts, in the formal framework, which are model-theoretic in character but useful to know independently of models, for example, that abstraction and application are continuous (in the Böhm-tree topology), and that unsolvable terms are computationally irrelevant: If FM reduces to normal form P for an unsolvable M then $FN = P$ for all N (the genericity lemma of Barendregt). In addition, other notions of reduction are studied in some detail: $\beta\eta$-reduction (for extensional calculi) and $\beta\delta$-reduction (for calculi with externally motivated constants). Here belong a general condition for CR (by Mitschke) and the observation (by Klop) that some very simple extensions of the lambda calculus, while consistent, are not CR (e.g. the additional rule $\delta MM \to \varepsilon$ for new constants δ and ε, and a version of introducing pairing and projecting).

Chapters 16 and 17 deal with theories that expand the lambda calculus. Their concise treatment is one of the motivations behind the structuring of this book (another is the treatment of models in Chapters 18 through 20). Accordingly, an adequate foundation is laid in the preceding 404 pages. The main points are now well motivated; for example to call a theory (which always is a set of equations) sensible, if it extends the theory $\mathscr{H}$ in which all unsolvable closed terms are identified, is itself sensible. $\mathscr{H}$ can be shown consistent, even with extensionality (by using the genericity lemma). There are continuum-many sensible theories. One particular way to define such theories (Morris) is by choosing a class P of terms and equating all closed lambda terms M and N with the property that $C[M] \in P$ if and only if $C[N] \in P$ in all contexts $C[\ ]$. If P *is the set of all solvable terms, the resulting theory* $\mathscr{H}^*$ *is the unique consistent and* complete (i.e., all non-provable equations are inconsistent) common extension of all sensible theories. (This result was originally obtained by Hyland and Wadsworth through the study of models.) $\mathscr{H}^*$ also includes the theory $\mathscr{B}$ which equates all terms with equal Böhm trees, and, more generally, all theories in which no unsolvable term is equated with a solvable one. It is interesting that $\mathscr{H}^*$ satisfies the ω-rule (if $MZ = NZ$ for all closed Z then $M = N$). The validity of the ω-rule is then investigated in subtheories of $\mathscr{H}^*$. The study of recursively enumerable λ-theories (Visser) is introduced and yields, for example, the existence of a λ-theory $\mathscr{T}_r$ for each real r with $\mathscr{T}_r \subsetneq \mathscr{T}_{r'}$ for all $r' > r$.

Non-trivial models of lambda calculi were of course possible (as term models), as soon as formal consistency was proved (which was indeed a striking success of Hilbert's program for these particular formal systems). There is a basic peculiarity in the relation between theories and models here: The theory of a model is not in general a complete theory, as we are used to in model theory. This makes the study of theories versus models particularly fruitful. Models whose theories turn out to have independent characterizations are in fact among the first that were invented: Scott's models D_∞, constructed as limits of complete lattices, have $\mathscr{H}^*$ as their theory (Hyland and Wadsworth); Plotkin and Scott's graph model $P\omega$, constructed on the set of subsets of $\mathbb{N}$, has $\mathscr{B}$ as its theory (Hyland), as does a model constructed with Böhm-like trees. The above models are examples of continuous models and therefore have the property that Curry's paradoxical combinator is the least fixed point operator (Park). They also have the range property, an important generalization of undecidability results (Scott): If φ maps the model into itself and is definable, then the range of φ either has one element or has as many as the model (Wadsworth).

The arrangement of the material in the book reflects, of course, the particular viewpoint of the author and the particular point in time at which the book was written, as does the highlighting of contributions (and contributors): The highest level is having one's photograph at the beginning of one of the five parts of the book, the lowest is inclusion in the exercises. This treatment is on the whole fair in my opinion; to organize a topic of this complexity into a linear hierarchy is no mean task. The exercises help the reader to get his hands on the formalism and give him a feeling of the liveliness of the subject. Quite generally, the choice of notation, if not already uniquely fixed by tradition, is fortunate; for example, $M[x := N]$ represents the result of substituting N for x in M. Exception: In the notation of page 576, x-y-$x = y$ is true because it says "x is substitutible for y in the formula $x = y$." There must have been an exceptional amount of effort to ensure the high quality of text and printing. E. ENGELER

HARTRY H. FIELD. ***Science without numbers. A defence of nominalism.*** Princeton University Press, Princeton 1980, xiii + 130 pp.

One of the surprises about physics is that the theory is not expressed as a body of statements about physical relations between physical objects (between, faster than, warmer than—Field calls this nominalistic language; I will use 'physicalistic'), but rather in terms of numerical functional relationships between mathematical objects (quantities). How much of the ordinary formalism is in fact necessary for the purposes of physics, and why? ***Science without numbers*** significantly extends our knowledge of how physical theory may (in part) be expressed in terms of physical relations and quantification. Field's approach is indeed science without numbers, avoiding, for example, Suppes's postulation of the real numbers as a set-theoretic primitive or Montague's axiomatization of the real numbers as part of the theory. Not that the numbers are far away; the structures Field axiomatizes are to classically described physical structures as Hilbert's synthetic geometry is to Euclidean space defined in terms of a metric on $\mathbb{R}^n$, with representation theorems to show that Field has indeed axiomatized the same physical theory.

Field's ultimate goal is to undercut a philosophical argument attributed to Quine: To the extent that it is essential to physical theory that the statement of the theory involves reference to mathematical objects (real numbers, functions, sets), a philosophical description of the world should incorporate them in the same way in which it incorporates physical theoretical entities (electrons, fields) similarly referred to in the statement of the theory. This conclusion is odious to a nominalist such as Field, who wants to maintain that physical theoretical entities truly exist, in the same sense as the more obvious physical objects (tables, air), but that mathematical objects do not exist in this sense; in particular, that mathematical statements cannot be said to be true or false. Field wants to undercut the Quinean argument by showing, at least to the satisfaction of an opponent who does accept that mathematical entities exist and that mathematical statements are true of them, the *mathematical entities and statements are eliminable from physical theory, in a way that physical theoretical entities and statements referring to them are not.*

Towards the first goal, he expresses a classical continuum mechanics of gravitation in a physicalistic relational language. This is the heart of the book, and is carried out in Chapter 8 on the basis of preliminary work in Chapters 6 and 7.

A model for Field's axioms will be a space-time structure with additional predicates. As a space-time structure it is isomorphic to four-dimensional (globally $\mathbb{R}^4$) "Galilean" space-time, axiomatized in terms of primitives x Simul y, y Bet xz, and xy S-Cong zw. In the Galilean space-time on $\mathbb{R}^4$ (with the ordinary Euclidean distance $d(x, y)$), these are defined by (1) x Simul y iff $x_4 = y_4$; (2) y Bet xz iff $d(x, y) + d(y, z) = d(x, z)$; and (3) xy S-Cong zw iff $d(x, y) = d(z, w)$ and x Simul y and z Simul w. The intended interpretations are simultaneity, betweenness (spatial, if x, y, z are simultaneous, otherwise on an inertial trajectory), and pairwise simultaneous equidistance (as might be verified by transport of a rigid rod). The