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A DECIDABLE SUBCLASS OF THE MINIMAL GÖDEL CLASS WITH IDENTITY

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The minimal Gödel class with identity (MGCI) is the class of closed, prenex quantificational formulas whose prefixes have the form $\forall x_1 \forall x_2 \exists x_3$ and whose matrices contain arbitrary predicate letters and the identity sign "$=$", but contain no function signs or individual constants. The MGCI was shown undecidable (for satisfiability) in 1983 [Go2]; this both refutes a claim of Gödel's [Gö, p. 443] and settles the decision problem for all prefix-classes of quantification theory with identity.

In this paper, we show the decidability of a natural subclass of the MGCI.[1] The formulas in this subclass can be thought of as exploiting only half of the power of the existential quantifier. That is, since an MGCI formula has prefix $\forall x_1 \forall x_2 \exists x_3$, in general its truth in a model requires for any elements a and b, the existence of both a witness for $\langle a, b\rangle$ and a witness for $\langle b, a\rangle$. The formulas we consider demand less: they require, for any elements a and b, a witness for the *unordered* pair $\{a, b\}$, that is, a witness either for $\langle a, b\rangle$ or for $\langle b, a\rangle$.

More precisely, the *subminimal Gödel class with identity* (SGCI) is the class of MGCI formulas whose matrices have the form $K \vee K^*$, where K^* is obtained from K by interchanging the variables x_1 and x_2. If $F = \forall x_1 \forall x_2 \exists x_3 H$ is in the SGCI, and if a, b, and c are elements of a structure $\mathfrak{A}$, then $\mathfrak{A} \models H[a, b, c]$ iff $\mathfrak{A} \models H[b, a, c]$. Thus, if $\mathfrak{A} \models H[a, b, c]$ we may call c a witness for $\{a, b\}$; and F will be true in $\mathfrak{A}$ iff there exists a witness for each unordered pair of elements.

In §1 we give a condition sufficient for an MGCI formula's having a finite model. Using this condition, in §2 we show that the SGCI is finitely controllable; that is, every satisfiable formula in the SGCI has a finite model. (Finite controllability implies decidability.) Our proof yields a computable function that bounds the size of the smallest finite model of an SGCI formula; but this function is not primitive recursive. We do not know whether this bound is optimal. Finally, in §3, we show that our proof can easily be extended to yield the finite controllability of the class of formulas $F \wedge \forall y \exists z_1 \cdots \exists z_n J$, where F is in the SGCI and J is quantifier-free.

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[1] We obtained this result in 1980, three years before the undecidability of the MGCI was established.

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§1. Let $F = \forall x_1 \forall x_2 \exists x_3 H$ be a formula in the MGCI. We may assume that the matrix H implies $x_3 \neq x_1 \wedge x_3 \neq x_2$, for if H does not, we may replace it with $x_3 \neq x_1 \wedge x_3 \neq x_2 \wedge (H \vee H_1 \vee H_2)$, where H_δ is obtained from H by replacing x_3 with x_δ; the resulting formula will be equivalent to the original formula over all universes of cardinality > 2. Given F, we use $\mathfrak{A}$ and $\mathfrak{B}$ to range over structures appropriate to the language of F, and use A and B for the universes of these structures. By a k-type we shall always mean a quantifier-free k-type. Thus, for any $\mathfrak{A}$, if $a_1, \ldots, a_k \in A$ and $C \subseteq A$, then the *k-type over C realized by $a_1, \ldots, a_k$ in $\mathfrak{A}$*, abbreviated $\mathrm{tp}_{\mathfrak{A}}(a_1, \ldots, a_k; C)$, is the set of quantifier-free formulas J such that $\mathfrak{A} \models J[a_1, \ldots, a_k]$, where J is constructed from predicate letters of F, the identity sign, variables among $x_1, \ldots, x_k$, and constants representing the members of C. Note that if C is finite, then a k-type over C is essentially a finite object, since it is completely determined by its atomic formulas.

The core of the undecidability proof in [Go2] is the construction of a formula in the MGCI that is satisfiable but has no finite models. In every model $\mathfrak{A}$ for this formula there exist elements $a_1, a_2, \ldots$ such that $\mathrm{tp}_{\mathfrak{A}}(a_1; \varnothing)$ is realized in $\mathfrak{A}$ only by a_1 and, for each $k \geq 1$, $\mathrm{tp}_{\mathfrak{A}}(a_{k+1}; \{a_1, \ldots, a_k\})$ is realized in $\mathfrak{A}$ only by a_{k+1}. In fact, a satisfiable MGCI formula F will have a finite model unless every model for it contains such elements, as is implied by the following proposition:

(‡) *Suppose there is a model $\mathfrak{A}$ for F and a finite $C \subseteq A$ such that every 1-type over C that is realized by an element of $A-C$ is realized by at least two elements of $A-C$. Then F has a finite model.*

(Indeed, proposition (‡) holds for any F in the full Gödel class with identity, i.e., any F with prefix $\forall x_1 \forall x_2 \exists x_3 \cdots \exists x_n$.)

To prove the finite controllability of the SGCI we shall exploit a proposition like (‡) but stronger. Let C be a set and t a k-type over C. For $i_1, \ldots, i_j \leq k$, let $[t \mid i_1 \cdots i_j]$ be the unique j-type t' over C with the following property: for any structure $\mathfrak{A}$ with $C \subseteq A$ and any $a_1, \ldots, a_k \in A$, if $\mathrm{tp}_{\mathfrak{A}}(a_1, \ldots, a_k; C) = t$ then $\mathrm{tp}_{\mathfrak{A}}(a_{i_1}, \ldots, a_{i_j}; C) = t'$.

For any $\mathfrak{A}$ and any $C \subseteq A$, a set Q of consistent 2-types over C is said to be *closed* (with respect to $\mathfrak{A}$) iff

(1) for all $a, b \in A$ there exists $t \in Q$ with $[t \mid 1] = \mathrm{tp}_{\mathfrak{A}}(a; C)$ and $[t \mid 2] = \mathrm{tp}_{\mathfrak{A}}(b; C)$;

(2) for each $t \in Q$, $[t \mid 1]$ and $[t \mid 2]$ are realized in $\mathfrak{A}$; and

(3) for each $t \in Q$ there exists a consistent 3-type s over C such that $[s \mid 1\,2] = t$, the matrix H of F is a member of s and, for all $i, j \in \{1, 2, 3\}$, $[s \mid ij] \in Q$.

A 1-type t_0 over C is *replicable* iff there exists a closed set Q of 2-types over C that contains a t with $[t \mid 1] = [t \mid 2] = t_0$ and $x_1 \neq x_2 \in t$.

Note that the set T of all 2-types over C that are realized in $\mathfrak{A}$ is closed. (Obviously clauses (1) and (2) are satisfied. If $t = \mathrm{tp}_{\mathfrak{A}}(a, b; C) \in T$, then the 3-type s demanded by clause (3) may be taken to be $\mathrm{tp}_{\mathfrak{A}}(a, b, e; C)$ for any e such that $\mathfrak{A} \models H[a, b, e]$; such an e exists since $\mathfrak{A} \models F$.) Thus if a 1-type over C is realized by a least two members of $A-C$, it is replicable.

LEMMA 1. *Suppose there is a model $\mathfrak{A}$ for F and a finite set $C \subseteq A$ such that every 1-type over C that is realized by a member of $A-C$ is replicable. Then F has a finite model.*

PROOF. Let $P = \{\mathrm{tp}_{\mathfrak{A}}(a; C) \mid a \in A-C\}$, and let Q be the union of all sets of 2-types over C that are closed (with respect to $\mathfrak{A}$). Note that Q is closed and that $T \subseteq Q$, where T is the set of 2-types over C realized in $\mathfrak{A}$. Let $k > 0$, $U_k = P \times \{1, \ldots, k\}$, and

$B_k = C \cup U_k$. We assume fixed some linear ordering of U_k. Let Φ_k be the set of structures $\mathfrak{B}$ with universe B_k such that

(a) for each $b \in U_k$, $\mathrm{tp}_{\mathfrak{B}}(b; C) = \pi_1 b$, and

(b) for all $b_1, b_2 \in B_k$, $\mathrm{tp}_{\mathfrak{B}}(b_1, b_2; C) \in Q$.

Φ_k is nonempty. Indeed, suppose $\mathfrak{B}$ is any structure over B_k obtained thus: let $\mathrm{tp}_{\mathfrak{B}}(b; C) = \pi_1 b$ for each b in U_k; if K is an atomic formula whose arguments include at least three distinct members of U_k, choose at random whether $\mathfrak{B} \models K$; and, for all distinct b_1 and b_2 in U_k with b_1 earlier than b_2 (in the assumed linear ordering), choose any $t \in Q$ with $[t \mid \delta] = \pi_1 b_\delta$, $\delta = 1, 2$, and let $\mathrm{tp}_{\mathfrak{B}}(b_1, b_2; C) = t$. The existence of such a t is guaranteed by clause (1) of the definition of "closed" if $\pi_1 b_1 \neq \pi_1 b_2$, and by the replicability of $\pi_1 b_1$ if $\pi_1 b_1 = \pi_1 b_2$. It follows that $\mathfrak{B} \in \Phi_k$. Indeed, condition (a) is obviously fulfilled, as is (b) whenever b_1 and b_2 are distinct elements of U_k with b_1 earlier than b_2. If b_1 and b_2 are distinct elements of U_k with b_2 earlier than b_1, then $\mathrm{tp}_{\mathfrak{B}}(b_1, b_2; C) = [t \mid 2\,1]$ for some $t \in Q$; and by clause (3), $[t \mid 2\,1] \in Q$ whenever $t \in Q$. Finally, if $b_1 = b_2$ or if at least one of b_1, b_2 is in C, then condition (a) implies that $\mathrm{tp}_{\mathfrak{B}}(b_1, b_2; C) \in T$. Thus $\mathfrak{B}$ fulfills condition (b) for all b_1 and b_2.

We show that for sufficiently large k, some member of Φ_k is a model for F. We consider Φ_k as a probability space with $\mathrm{Prob}[\mathfrak{B} = \mathfrak{B}_0] = 1/|\Phi_k|$ for each fixed $\mathfrak{B}_0 \in \Phi_k$. We show that there is an $\varepsilon < 1$ not dependent on k such that, for all b_1 and b_2 in B_k,

$$\text{(I)} \qquad \mathrm{Prob}[\mathfrak{B} \models \neg \exists x_3 H[b_1, b_2]] \leq \varepsilon^{k-2}.$$

From this it follows at once that

$$\mathrm{Prob}[\mathfrak{B} \models \neg \forall x_1 \forall x_2 \exists x_3 H] \leq (|B_k|)^2 \varepsilon^{k-2}.$$

Now $|B_k| = |C| + |P| \cdot k$. Hence, for all sufficiently large k, $(|B_k|)^2 \varepsilon^{k-2}$ will be less than 1. For such k, then, $\mathrm{Prob}[\mathfrak{B} \models \neg F] < 1$, so that some member of Φ_k will be a model for F.

To prove (I), let b_1 and b_2 be any members of B_k, and let $\mathfrak{B}$ be a random member of Φ_k. By (b), since Q is closed, there exists a consistent 3-type s over C such that $H \in s$, $[s \mid 1\,2] = \mathrm{tp}_{\mathfrak{B}}(b_1, b_2; C)$, and $[s \mid i\,j] \in Q$ for all $i, j \in \{1,2,3\}$.

Case 1. For some $c \in C$, $x_3 = c \in s$. Since s is consistent, it follows that $\mathrm{tp}_{\mathfrak{B}}(b_1, b_2, c; C) = s$. Hence $\mathfrak{B} \models H[b_1, b_2, c]$, so that

$$\mathrm{Prob}[\mathfrak{B} \models \neg \exists x_3 H[b_1, b_2]] = 0.$$

Case 2. For each $c \in C$, $x_3 \neq c \in s$. Let d be any element of U_k such that $d \neq b_1$, $d \neq b_2$, and $\pi_1 d = [s \mid 3]$. We show that, for suitably specified $\varepsilon < 1$,

$$\text{(II)} \qquad \mathrm{Prob}[\mathrm{tp}_{\mathfrak{B}}(b_1, b_2, d; C) \neq s] \leq \varepsilon.$$

Since $[s \mid \delta\,3] \in Q$ for $\delta = 1, 2$, if $b_\delta \notin C$ then $\mathrm{Prob}[\mathrm{tp}_{\mathfrak{B}}(b_\delta, d; C) = [s \mid \delta\,3]] \geq 1/|Q|$. If $b_\delta \in C$, then since $\mathrm{tp}_{\mathfrak{B}}(d; C) = [s \mid 3]$ and s is consistent, $\mathrm{Prob}[\mathrm{tp}_{\mathfrak{B}}(b_\delta, d; C) = [s \mid \delta\,3]] = 1$. Moreover if $b_1 = b_2$ or if $b_1 \in C$ or $b_2 \in C$, then $\mathrm{tp}_{\mathfrak{B}}(b_1, d; C) = [s \mid 1\,3]$ and $\mathrm{tp}_{\mathfrak{B}}(b_2, d; C) = [s \mid 2\,3]$ together imply $\mathrm{tp}_{\mathfrak{B}}(b_1, b_2, d; C) = s$. If $b_1 \neq b_2$ and $b_1, b_2 \notin C$, then for each atomic formula J containing constants from C and the variables x_1, x_2, x_3,

$$\mathrm{Prob}[\mathfrak{B} \models J[b_1, b_2, d] \text{ iff } J \in s] = 1/2.$$

In any case, then,

$$\mathrm{Prob}[\mathrm{tp}_{\mathfrak{B}}(b_1, b_2, d; C) = s] \geq (1/|Q|)^2(1/2)^r$$

where r is the number of such atomic formulas J. Let $\varepsilon = 1 - (1/|Q|)^2(1/2)^r$; then (II) follows.

There are at least $k - 2$ distinct members d of U_k such that $d \neq b_1$, $d \neq b_2$, and $\pi_1 d = [s|3]$. Moreover, for distinct such d the events $[\mathrm{tp}_{\mathfrak{B}}(b_1, b_2, d; C) \neq s]$ are independent. Thus (II) implies

$$\mathrm{Prob}[\mathrm{tp}_{\mathfrak{B}}(b_1, b_2, d; C) \neq s \text{ for all such } d] \leq \varepsilon^{k-2}.$$

Now if $\mathrm{tp}_{\mathfrak{B}}(b_1, b_2, d; C) = s$ then $\mathfrak{B} \models H[b_1, b_2, d]$, so that $\mathfrak{B} \models \exists x_3 H[b_1, b_2]$. Hence

$$\mathrm{Prob}[\mathfrak{B} \models \neg\exists x_3 H[b_1, b_2]] \leq \varepsilon^{k-2},$$

and (I) is proved.[2] $\square$

Note that the k needed so that Φ_k must contain a model for F can be calculated elementarily from ε, $|C|$, and $\cdot|P|$. Moreover, ε and $|P|$ can be bounded elementarily in F and $|C|$. Hence, the size of the smallest finite model for F is an elementary function of F and $|C|$.

Lemma 1 implies that a satisfiable formula F in the MGCI has no finite model only if each model $\mathfrak{A}$ for F contains an infinite sequence $a_1, a_2, \ldots$ of elements such that $\mathrm{tp}_{\mathfrak{A}}(a_1; \varnothing)$ is not replicable, and, for each k, $\mathrm{tp}_{\mathfrak{A}}(a_{k+1}; \{a_1, \ldots, a_k\})$ is not replicable. Thus, to show that F does have a finite model, it suffices to pick some model $\mathfrak{A}$ of F and show that for any sequence $a_1, a_2, \ldots$ of its elements there is a k such that $\mathrm{tp}_{\mathfrak{A}}(a_k; \{a_1, \ldots, a_k\})$ is replicable. Moreover, if this k can be bounded by some function φ of F, then the set C required for Lemma 1 will have cardinality $\leq \varphi(F)$. This will then yield a function elementary in φ that bounds the size of the smallest finite model for F.

§2. THEOREM. *The subminimal Gödel class with identity is finitely controllable.*

To highlight the central strategy of the argument, we first prove the theorem nonconstructively. Following that, we give the more intricate argument needed to calculate the size of finite models. Let $F = \forall x_1 \forall x_2 \exists x_3 H$ be a satisfiable formula in the SGCI.

NONCONSTRUCTIVE PROOF. Since F is satisfiable, it has an $\aleph_1$-saturated model $\mathfrak{A}$ [CK, p. 216]. Pick any infinite sequence of members of A; for notational convenience, we identify this sequence with $1, 2, 3, \ldots$. By Ramsey's theorem there exist integers $r_1 < r_2 < r_3 < \cdots$ such that

(*) $\mathrm{tp}_{\mathfrak{A}}(r_j; \{1, \ldots, r_i - 1\}) = \mathrm{tp}_{\mathfrak{A}}(r_k; \{1, \ldots, r_i - 1\})$ for all i, j, k with $i < j < k$.

For let f be defined on triples $\langle p, q, r \rangle$ with $p < q < r$ thus: $f(p, q, r) = 0$ if

$$\mathrm{tp}_{\mathfrak{A}}(q; \{1, \ldots, p-1\}) = \mathrm{tp}_{\mathfrak{A}}(r; \{1, \ldots, p-1\}),$$

[2] The proof of Lemma 1 is a slight variant of the random models argument introduced in [GS] to obtain a straightforward proof of the finite controllability of the Gödel class without identity.

and $f(p,q,r) = 1$ otherwise. It suffices to let $\{r_1, r_2, r_3, \ldots\}$ be an infinite set that is homogeneous for f.

For each i let $R_i = \{1, \ldots, r_i - 1\}$. We show the existence of an integer m such that $\mathrm{tp}_{\mathfrak{A}}(r_{m+1}; R_{m+1})$ is replicable.

By (*) and the $\aleph_1$-saturatedness of $\mathfrak{A}$, there exist distinct d, $e \in A$ such that $\mathrm{tp}_{\mathfrak{A}}(d; R_i) = \mathrm{tp}_{\mathfrak{A}}(e; R_i) = \mathrm{tp}_{\mathfrak{A}}(r_{i+1}; R_i)$ for each $i \geq 1$. We shall define a set Q of 2-types over R_{m+1} by altering certain 2-types realized in $\mathfrak{A}$ so as to make d and e into "replicas" of r_{m+1}. Let $d_0, d_1, \ldots$ and $e_0, e_1, \ldots$ be two sequences of members of A specified as follows:

$$\begin{aligned} d_0 &= e_0 = \text{a witness in } \mathfrak{A} \text{ for } \{d, e\}, \\ d_{i+1} &= \text{a witness in } \mathfrak{A} \text{ for } \{d, d_i\}, \\ e_{i+1} &= \text{a witness in } \mathfrak{A} \text{ for } \{e, e_i\}. \end{aligned}$$

We truncate these sequences at the first place, if any, at which a member lies in $\bigcup R_j$. That is, if there is an i such that $d_i \in \bigcup R_j$ then let κ be the least such i and let k be the least integer such that $d_\kappa \in R_k$; and if there is no such i let $\kappa = \omega$ and let $k = 1$. Similarly, if there is an i such that $e_i \in \bigcup R_j$ then let λ be the least such i and let l be the least integer such that $e_\lambda \in R_l$; and if there is no such i let $\lambda = \omega$ and let $l = 1$. Finally, let $m = \max(k, l)$.

If t is any n-type over R_{m+1} and $i \leq n$, let $\psi_i(t)$ be the n-type over R_{m+1} that is like t except that $[\psi_i(t) \mid i] = \mathrm{tp}_{\mathfrak{A}}(r_{m+1};\ R_{m+1})$. Thus, t and $\psi_i(t)$ agree on all atomic formulas except those whose only variable is x_i. Now let Q be the set of 2-types over R_{m+1} that contains:

(a) $\mathrm{tp}_{\mathfrak{A}}(a, b; R_{m+1})$ for all $a, b \in A$;
(b) $\psi_1(\mathrm{tp}_{\mathfrak{A}}(d, d_i, R_{m+1}))$ and $\psi_2(\mathrm{tp}_{\mathfrak{A}}(d_i, d;\ R_{m+1}))$ for each $i < \kappa$;
(c) $\psi_1(\mathrm{tp}_{\mathfrak{A}}(e, e_i; R_{m+1}))$ and $\psi_2(\mathrm{tp}_{\mathfrak{A}}(e_i, e; R_{m+1}))$ for each $i < \lambda$;
(d) $\psi_1\psi_2(\mathrm{tp}_{\mathfrak{A}}(d, e; R_{m+1}))$ and $\psi_1\psi_2(\mathrm{tp}_{\mathfrak{A}}(e, d; R_{m+1}))$.

We claim that Q is closed. This implies that $\mathrm{tp}_{\mathfrak{A}}(r_{m+1}; R_{m+1})$ is replicable, since Q contains the 2-type $t = \psi_1\psi_2(\mathrm{tp}_{\mathfrak{A}}(d, e; R_{m+1}))$ and we have $[t \mid 1] = [t \mid 2] = \mathrm{tp}_{\mathfrak{A}}(r_{m+1}; R_{m+1})$ and $x_1 \neq x_2 \in t$.

That clauses (1) and (2) of the definition of "closed" are satisfied by Q is evident. Now let $t \in Q$. If $t = \mathrm{tp}_{\mathfrak{A}}(a, b; R_{m+1})$ for $a, b \in A$, then the existence of a 3-type s as required by clause (3) follows from the fact that $\mathfrak{A} \models F$, as was mentioned in §1. If $t = \psi_1(\mathrm{tp}_{\mathfrak{A}}(d, d_i; R_{m+1}))$ for $i < \kappa$, then let

$$s = \psi_1(\mathrm{tp}_{\mathfrak{A}}(d, d_i, d_{i+1}; R_{m+1})).$$

Since $\mathrm{tp}_{\mathfrak{A}}(d; R_m) = \mathrm{tp}_{\mathfrak{A}}(r_{m+1}; R_m)$, s and $\mathrm{tp}_{\mathfrak{A}}(d, d_i, d_{i+1}; R_{m+1})$ differ only on formulas that contain constants from $R_{m+1} - R_m$. By the choice of d_{i+1}, $\mathfrak{A} \models H[d, d_i, d_{i+1}]$, and $H[d, d_i, d_{i+1}]$ contains no constants from $R_{m+1} - R_m$, since either $d_{i+1} \notin \bigcup R_j$ or else $d_{i+1} \in R_m$. Hence $H \in s$. Moreover, if $i + 1 < \kappa$ then it is immediate that $[s \mid j\,j'] \in Q$ for all $j, j' \in \{1, 2, 3\}$. If $i + 1 = \kappa$, this is immediate in all cases but $\{j, j'\} = \{1, 3\}$. But here $d_{i+1} = q$ for some $q < r_m$; since $\mathrm{tp}_{\mathfrak{A}}(d; R_m) = \mathrm{tp}_{\mathfrak{A}}(r_{m+1}; R_m)$,

$$[s \mid 1\,3] = \mathrm{tp}_{\mathfrak{A}}(r_{m+1}, q; R_{m+1}),$$

which by (a) is a member of Q. Thus s satisfies the requirements of clause (3).

Similarly, we may let

$$s = \psi_1(\mathrm{tp}_{\mathfrak{A}}(e, e_i, e_{i+1}; R_{m+1})) \quad \text{when } t = \psi_1(\mathrm{tp}_{\mathfrak{A}}(e, e_i; R_{m+1})),$$

and

$$s = \psi_1\psi_2(\mathrm{tp}_{\mathfrak{A}}(d, e, d_0; R_{m+1})) \quad \text{when } t = \psi_1\psi_2(\mathrm{tp}_{\mathfrak{A}}(d, e; R_{m+1})).$$

The t in Q not so far treated are identical to $[t' \mid 2\,1]$ for some t' that has been shown to satisfy clause (3). Since F is in the SGCI, if a 3-type s over R_{m+1} contain H, then so does the 3-type $[s \mid 2\,1\,3]$. Hence if s satisfies the requirements of clause (3) with respect to t', then $[s \mid 2\,1\,3]$ satisfies them with respect to t. Thus clause (3) is satisfied in all cases. □

CONSTRUCTIVE PROOF. We show how to replace the use of the infinite sequence $r_1, r_2, \ldots$ by the use of a finite sequence $r_1, r_2, \ldots, r_N$. For each k let $\chi(k)$ be larger than the number of 2-types in the language of F over a set of cardinality $k - 1$. (We may take $\chi(k) = 2^{\gamma k}$ for a suitable constant γ that depends polynomially on F.) Let $\mathfrak{A}$ be any model for F, and let $1, 2, 3, \ldots$ be an infinite sequence of members of A. Let $r_1 < r_2 < \cdots < r_N$ be a sequence, with $N \geq \chi(r_2) + \chi(r_{\chi(r_2)})$, that has the following property:

$$(**) \qquad \mathrm{tp}_{\mathfrak{A}}(r_j; R_i) = \mathrm{tp}_{\mathfrak{A}}(r_k; R_i) \quad \text{whenever } i < j < k \leq N,$$

where $R_i = \{1, \ldots, r_i - 1\}$ for each i. (As we have seen, that there is such a sequence can be shown nonconstructively. Latter on, though, we show how to calculate a bound on r_N.)

Let $d = r_{N-1}$, $e = r_N$. Let $d_0, d_1, \ldots$ and $e_0, e_1, \ldots$ be two sequences of witnesses defined as in the nonconstructive proof.

LEMMA. *There exist integers m, κ, and λ, with $1 \leq m < N - 1$, such that*

(i) *for each $i < \kappa$, $d_i \notin R_{m+1}$, and either $d_\kappa \in R_m$ or else there exists $p < \kappa - 1$ such that* $\mathrm{tp}_{\mathfrak{A}}(d, d_{\kappa-1}; R_{m+1}) = \mathrm{tp}_{\mathfrak{A}}(d, d_p; R_{m+1})$; *and*

(ii) *for each $i < \lambda$, $e_i \notin R_{m+1}$, and either $e_\lambda \in R_m$ or else there exists $q < \lambda - 1$ such that* $\mathrm{tp}_{\mathfrak{A}}(e, e_{\lambda-1}; R_{m+1}) = \mathrm{tp}_{\mathfrak{A}}(e, e_p; R_{m+1})$.

Before proving the lemma, we show that if m is as in the lemma then $\mathrm{tp}_{\mathfrak{A}}(r_{m+1}; R_{m+1})$ is replicable. Let Q be the set of 2-types over R_{m+1} defined as in the nonconstructive proof. It suffices to show that Q is closed. The verification of this proceeds exactly as before, except that there are two new cases for clause (3), namely, $t = \psi_1(\mathrm{tp}_{\mathfrak{A}}(d, d_{\kappa-1}; R_{m+1}))$ when $d_\kappa \notin R_m$, and $t = \psi_1(\mathrm{tp}_{\mathfrak{A}}(e, e_{\lambda-1}; R_{m+1}))$ when $e_\lambda \notin R_m$. In the former case, there exists $p < \kappa - 1$ such that

$$\mathrm{tp}_{\mathfrak{A}}(d, d_{\kappa-1}; R_{m+1}) = \mathrm{tp}_{\mathfrak{A}}(d, d_p; R_{m+1}):$$

hence $t = \psi_1(\mathrm{tp}_{\mathfrak{A}}(d, d_p; R_{m+1}))$, so that (3) follows for t from its being satisfied for $\psi_1(\mathrm{tp}_{\mathfrak{A}}(d, d_p; R_{m+1}))$. In the latter case there exists $q < \lambda - 1$ such that

$$\mathrm{tp}_{\mathfrak{A}}(e, e_{\lambda-1}; R_{m+1})) = \mathrm{tp}_{\mathfrak{A}}(e, e_q; R_{m+1}),$$

so again (3) follows. Thus clause (3) is satisfied in all cases.

PROOF OF THE LEMMA. *Case* 1. For all $i < \chi(r_2) - 1$, $d_i \notin R_2$ and $e_i \notin R_2$. By the choice of χ and the pigeonhole principle, there exist p, χ, q, and λ with

$0 \leq p < \kappa - 1 < \chi(r_2) - 1$ and $0 \leq q < \lambda - 1 < \chi(r_2) - 1$ such that

$$\mathrm{tp}_{\mathfrak{A}}(d, d_{\kappa-1}; R_2) = \mathrm{tp}_{\mathfrak{A}}(d, d_p; R_2) \quad \text{and} \quad \mathrm{tp}_{\mathfrak{A}}(e, e_{\lambda-1}; R_2) = \mathrm{tp}_{\mathfrak{A}}(e, e_q; R_2).$$

Thus the lemma holds for $m = 1$.

Case 2. For some $i < \chi(r_2) - 1$, d_i or e_i is in R_2. Let k be the least such i; to fix ideas, suppose $d_k \in R_2$. Let j be the least integer, $0 \leq j \leq k$, such that $d_j \in R_{2+(k-j)}$. If $j = 0$, then, since $d_0 = e_0$ and $N > \chi(r_2) + 1$, the lemma holds for $m = 2 + k$ and $\kappa = \lambda = 0$. Assume $j > 0$, and let $l = 2 + (k - j)$. Note that $l < \chi(r_2)$.

Subcase i. For each $i < \chi(r_{l+1}) - 1$, $e_i \notin R_{l+1}$. Then by the pigeonhole principle there exist q and λ, $0 \leq q < \lambda - 1 < \chi(r_{l+1}) - 1$, such that

$$\mathrm{tp}_{\mathfrak{A}}(e, e_{\lambda-1}; R_{l+1}) = \mathrm{tp}_{\mathfrak{A}}(e, e_1; R_{l+1}).$$

Hence the lemma holds for $m = l$, $\kappa = j$, and this λ.

Subcase ii. For some $i < \chi(r_{l+1}) - 1$, $d_i \in R_{l+1}$. By the pigeonhole principle there is an m, $l + 1 \leq m \leq l + j + i$, such that none of $c_0, \ldots, c_{j-1}, d_1, \ldots, d_{i-1}$ is in $R_{m+1} - R_m$. Note that $l + j + i = 2 + k + i < \chi(r_2) + \chi(r_{\chi(r_2)}) - 1 \leq N - 1$. Hence the lemma holds for m, for $\kappa =$ the least $q \leq j$ such that $d_q \in R_m$, and for $\lambda =$ the least $p \leq i$ such that $d_p \in R_m$. □

It remains only to compute a bound on r_N. This can be done using the methods of [KS]; we briefly indicate how to apply them.[3] Given any $\mathfrak{A}$ and any sequence $1, \ldots, m$ of members of A, we may take $\mathfrak{A}$ to induce a tree on $\{1, \ldots, m\}$ with the following property: if $\langle p_1, \ldots, p_l \rangle$ is a branch in this tree, then

$$\mathrm{tp}_{\mathfrak{A}}(p_j; \{1, \ldots, p_i - 1\}) = \mathrm{tp}_{\mathfrak{A}}(p_k; \{1, \ldots, p_i - 1\})$$

for $1 \leq i < j < k \leq l$; moreover, an integer q is an immediate successor of p_l in the tree, $p_l < q \leq m$, iff

$$\mathrm{tp}_{\mathfrak{A}}(q; \{1, \ldots, p_{l-1} - 1\}) = \mathrm{tp}_{\mathfrak{A}}(p_l; \{1, \ldots, p_{l-1} - 1\})$$

and for no $r < q$ is

$$\mathrm{tp}_{\mathfrak{A}}(r; \{1, \ldots, p_l - 1\}) = \mathrm{tp}_{\mathfrak{A}}(q; \{1, \ldots, p_l - 1\}).$$

(Thus the number of immediate successors of p_l is $< \chi(p_l)$.) Now if $\langle r_1, \ldots, r_N \rangle$ is a branch of the tree and $N \geq \chi(r_2) + \chi(r_{\chi(r_2)})$, then (**) is satisfied and we are done. Hence it suffices to compute an m such that any such tree on $\{1, \ldots, m\}$ contains such a branch. By the proof of [KS, Theorem 5.6], it will suffice if $\{1, \ldots, m\}$ is $(\omega^{\omega+\omega+1} + \omega^3 + c)$-large, where c is a constant polynomially calculable from F. Moreover, by [KS, Theorem 4.8], it more than suffices to take $m = \varphi_{\omega+\omega+2}(c)$, where $\varphi_{\omega+\omega+2}$ is the function at level $\omega + \omega + 2$ of the Wainer hierarchy [W].

As a result of this and the remark that concluded §1, we may conclude that there is an elementary function η such that, for each F in the SGCI, if F is satisfiable then F has a model of cardinality $\leq \varphi_{\omega+\omega+2}(\eta(F))$. This bound is not primitive recursive.

[3] A crude bound can be obtained by directly finitizing the use of Ramsey's theorem made in the nonconstructive proof. We are grateful to Robert Solovay for telling us that the finer bound given in this paragraph is sufficient.

We do not know whether the size of smallest models of SGCI formulas can be bounded primitive recursively, nor whether there exists a primitive recursive decision procedure for the SGCI.[4] If in fact no such primitive recursive bound exists, then the SGCI would be the first known natural class of quantificational formulas that is finitely controllable but cannot be so bounded.

§3. Let $\mathscr{C}$ be the class of formulas $F \wedge G$, where F is in the SGCI and $G = \forall y \exists z_1 \cdots \exists z_n K$, K quantifier-free. The above proof is easily extended to yield the finite controllability of $\mathscr{C}$. The definitions of closed set of 2-types and of replicable 1-type remain as in §1; that is, no account is taken of the formula G. The following strengthened version of Lemma 1 can then be shown:

If there exist a model $\mathfrak{A}$ for $F \wedge G$ and a finite set $C \subseteq A$ such that every 1-type over C realized by a member of $A - C$ is replicable, then $F \wedge G$ has a finite model.

Given this, the argument of §2 can then proceed without any change. To prove the strengthened lemma, it suffices to add a further argument to the proof in §1, so as to show that for some $v < 1$ not dependent on k and each $b \in B_k$,

$$\text{(III)} \qquad \operatorname{Prob}[\mathfrak{B} \models \neg \exists z_1 \cdots \exists z_n K[b]] \leq v^m,$$

where $m = [(k-1)/n]$. From (III) and (I) it follows that

$$\operatorname{Prob}[\mathfrak{B} \models (\neg F \vee \neg G)] \leq (|B_k|)^2 \varepsilon^{k-2} + (|B_k|)v^m.$$

For sufficiently large k, the quantity on the right is less than 1; for such k, then, Φ_k contains a model for $F \wedge G$.

To prove (III), let $b \in B_k$. Then there exists $a \in A$ with $\operatorname{tp}_{\mathfrak{A}}(a; C) = \operatorname{tp}_{\mathfrak{B}}(b; C)$. Since $\mathfrak{A} \models J$, there exist $a_1, \ldots, a_n \in A$ such that $\mathfrak{A} \models K[a, a_1, \ldots, a_n]$. Let $t = \operatorname{tp}_{\mathfrak{A}}(a, a_1, \ldots, a_n; C)$. Now if $a_1, \ldots, a_n \in C \cup \{a\}$, then there will exist $b_1, \ldots, b_n \in C \cup \{b\}$ such that

$$\operatorname{Prob}[\operatorname{tp}_{\mathfrak{B}}(b, b_1, \ldots, b_n; C) = t] = 1,$$

so that $\operatorname{Prob}[\operatorname{tp}_{\mathfrak{B}}(b, b_1, \ldots, b_n; C) \neq t] = 0$. If at least one of $a_1, \ldots, a_n$ is not in $C \cup \{a\}$, then there will exist at least m distinct n-tuples $\langle b_1, \ldots, b_n \rangle$ of members of B_k such that for different n-tuples $\langle b_1, \ldots, b_n \rangle$ among these the events

$$[\operatorname{tp}_{\mathfrak{B}}(b, b_1, \ldots, b_n; C) \neq t]$$

are independent, and for each n-tuple $\langle b_1, \ldots, b_n \rangle$,

$$\operatorname{Prob}[\operatorname{tp}_{\mathfrak{B}}(b, b_1, \ldots, b_n; C) = t] \geq (1/|Q|)^q (1/2)^r,$$

where $q = n(n+1)/2$, and r is the number of atomic formulas containing constants from C and at least three distinct variables among $y, z_1, \ldots, z_n$. Let

$$v = 1 - (1/|Q|)^q (1/2)^r.$$

Since $\mathfrak{B} \models \neg \exists z_1 \cdots \exists z_n K[b]$ implies that $\operatorname{tp}_{\mathfrak{B}}(b, b_1, \ldots, b_n; C) \neq t$ for any $b_1, \ldots, b_n$, (III) follows.

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[4] The encoding of [Go1] that yields the nonexistence of a primitive recursive decision procedure for the MGCI apparently cannot be carried out using just formulas in the SGCI.

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