


Reachability in 3-VASS is in Tower

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Abstract

The reachability problem for vector addition systems with states (VASS) has been shown to be ACKERMANN-complete. For every $k \geq 3$, a completeness result for the k -dimensional VASS reachability problem is not yet available. It is shown in this paper that the 3-dimensional VASS reachability problem is in TOWER, improving upon the current best upper bound \mathbf{F}_7 established by Leroux and Schmidt in 2019.

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1 Introduction

Petri net theory has been studied for over half a century. As a model for concurrency and causality, Petri net model finds a wide range of applications in system specification and verification [2, 9]. Two equivalent formulations of Petri nets, vector addition system (VAS) and vector addition system with state (VASS) have been studied for a long time. In vector addition systems configurations are formulated as vectors on non-negative integers. Computation rules of systems are captured by vectors on integers. A computation is a sequence of legal transitions of configurations. The reachability problem asks whether a target configuration is reachable from an initial configuration in a given VAS. The problem holds a central position in the study of VAS, as numerous issues in the areas of language, logic, and concurrency can be effectively reduced to this particular problem [24].

Decidability result of the VASS reachability problem is among the most significant theoretical breakthroughs in computer science. In early 1970s, some decidability results [10, 14] for low dimension VASS has been established. The initial work of Sacerdote and Tenny [23] gives an incomplete proof of the decidability of the problem. In early 1990s Mayr [20] presents a complete decidability proof of VASS reachability. Later Kosaraju [11] and Lambert [12] refined the proof. The algorithm is now referred to as KLMST decomposition. In recent years, Leroux gives another proof of the decidability in a more logic setting using Presburger invariants [15].

In 2015 Leroux and Schmitz obtained the first upper bound for the KLMST algorithm, pointing out that it is in the cubic-Ackermann complexity class \mathbf{F}_{ω^3} [17]. The upper bound was improved to \mathbf{F}_{ω} [18] by Leroux in 2019. The **EXPSpace**-hardness of the problem was shown by Lipton in 1976 [19]. For many years this has been the only lower bound we knew. Until in 2018 Czerwiński came up with the “Amplifier” technique and applied it to obtain a non-elementary lower bound [4]. This was improved to \mathbf{F}_{ω} in 2022, independently by several groups [5, 13, 16]. The Ackermann-completeness of the problem is thus established.

Reachability in fixed dimension VASS has also gathered widespread attention. Completeness results have been established for low dimensional VASS. Haase, Kreutzer, Ouaknine and Worrell showed that reachability of 1-dimensional VASS is NP-complete [8]. In the

2-dimensional case, Boldin, Finkel, Göller, Hasse and McKenzie proved that the problem is PSPACE complete [1] and Englert, Lazić and Totzke pointed out that the problem is NL-complete [7] if unary encoding is used. It has been known from early stage that while reachability sets are semi-linear in the two dimensional case, they are not semi-linear for VASS in three or more dimensions. This stops us from generalizing the proof of the completeness result for 2-VASS to higher dimensional VASS. A completeness characterization of the k -VASS reachability problem, where $k \geq 3$, remains open. Currently it is known that the reachability problem for d -VASS with $d \geq 3$ is in \mathbf{F}_{d+4} . For the lower bound Leroux proved in [16] that reachability in $(2d+4)$ -VASS is \mathbf{F}_d -hard for $d \geq 3$, and Czerwiński proved that reachability in 8-VASS is non-elementary [3, 6]. **For 3-VASS in particular, the best lower bound is PSPACE-hard and the best upper bound is \mathbf{F}_7 .** There remains a substantial gap between the known lower bounds and the known upper bounds for the fixed dimension VASS reachability problem.

A completeness result for 3-VASS is important. In this paper **we prove that the reachability problem for 3-VASS is in TOWER.** Our algorithm incorporates the linear path scheme characterization for 2-VASS to the general KLMST algorithm. We show that a kind of special 3-VASS, to be called effectively 2-dimensional, has the linear path scheme property. As a consequence the upper bound can be improved from \mathbf{F}_7 to \mathbf{F}_3 .

The rest of the paper is organized as follows. Section 2 states the preliminaries. Section 3 recalls the KLMST algorithm. Section 4 reviews the linear path schemes for 2-VASS and extends the technique to the effectively 2-dimensional 3-VASS. Section 5 defines the almost normal KLM sequence and the algorithm for 3-VASS in TOWER, which is the main contribution of the paper. Section 6 makes a few comments.

2 Preliminary

Let \mathbb{N} be the set of natural numbers (nonnegative integers) and \mathbb{Z} the set of integers. Let \mathbb{V} denote the set of variables for nonnegative integers. For $L \in \mathbb{N} \setminus \{0\}$ the notation $[L]$ stands for the set $\{1, \dots, L\}$ and $[L]_0$ for $\{0\} \cup [L]$. For a finite set S let $|S|$ denote the number of element of S . We introduce an infinite number ω with $n < \omega$ for all $n \in \mathbb{N}$ and let $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$ be the extended set of natural numbers. We introduce the partial order \sqsubseteq over \mathbb{N}_ω defined as follows: $x \sqsubseteq y$ whenever $y \in \{x, \omega\}$.

We write \mathbf{m}, \mathbf{n} for d -dimensional vectors in \mathbb{N}^d , \mathbf{u}, \mathbf{v} for vectors in \mathbb{N}_ω^d , and \mathbf{x}, \mathbf{y} for vectors in \mathbb{V}^d . For $i \in [d]$ we write for example $\mathbf{a}(i)$ for the i -th entry of \mathbf{a} . Let $\mathbf{1} = (1, \dots, 1)^\dagger$ and $\mathbf{0} = (0, \dots, 0)^\dagger$, where $(_)^\dagger$ is the transposition operator. We write σ for a finite sequence of vectors and $|\sigma|$ for the length of σ . For $i \in [|\sigma|]$ we write $\sigma[i]$ for the i -th element that appears in σ . The notation $\sigma[i, \dots, j]$ is for $\sigma[i]\sigma[i+1]\dots\sigma[j]$ if $i \leq j$ and is for ϵ if $i > j$.

Recall that the 1-norm $\|\mathbf{m}\|_1$ of \mathbf{m} is $\sum_{i \in [d]} |\mathbf{m}(i)|$. The 1-norm $\|A\|_1$ of an integer matrix A is $\sum_{i,j} |A(i,j)|$, and the 1-norm $\|\mathbf{u}\|_1$ of $\mathbf{u} \in \mathbb{N}_\omega^d$ is defined by $\sum_{i \in [d], \mathbf{u}(i) \neq \omega} |\mathbf{u}(i)|$, ignoring the ω components.

2.1 Non-Elementary Complexity Class

Reachability in VASS is not elementary even in the fixed dimension [5]. To characterize the problem complexity, one needs complexity classes beyond the elementary class. Schmidt introduced an ordinal indexed class of complexity classes $\mathbf{F}_3, \mathbf{F}_4, \dots, \mathbf{F}_\omega, \dots, \mathbf{F}_{\omega^2}, \dots, \mathbf{F}_{\omega^\omega}, \dots$ and showed that many problems arising in theoretical computer science are complete problems in this hierarchy [24]. In the above sequence $\mathbf{F}_3 = \text{TOWER}$ and $\mathbf{F}_\omega = \text{ACKERMANN}$. The class TOWER is closed under elementary reduction and ACKERMANN is closed under primitive

recursive reduction. For the purpose of this paper it suffices to say that TOWER contains all the problems whose space complexity is bounded by tower functions of the form $2^{\dots^{2^n}}\}^{f(n)}$, where $f(n)$ is an elementary function.

The notation $\text{poly}(n)$ will stand for a polynomial bound, and $\text{exp}(n)$ an exponential bound. For simplicity we shall always omit constant factors when making statements about bounds.

2.2 Vector Addition System with States

By a *digraph* we mean a finite directed graph in which multi-edges and self loops are admitted. A *d-dimensional vector addition system with states*, or *d-VASS*, is a labeled digraph $G = (Q, T)$ where Q is the set of vertices and T is the set of edges. The edges are labeled by elements of \mathbb{Z}^d , and the labels are called *displacements*. A *state* is identified to a vertex and a *transition* is identified to a labeled edge. We write o, p, q for states, t and its decorated versions for edges. A transition from p to q labeled \mathbf{t} is denoted by $p \xrightarrow{\mathbf{t}} q$. A VASS often comes with an input state q_{in} and an output state q_{out} , and accordingly it is often specified by $G = (Q, T, q_{in}, q_{out})$.

A path π from p_0 to q_n of $G = (Q, T)$ is a sequence $(p_0, \mathbf{a}_0, q_0), \dots, (p_n, \mathbf{a}_n, q_n) \in T^*$ such that $p_i = q_{i-1}$ for $i \in [n]$. The displacement $\Delta(\pi)$ of π is defined by $\sum_{i \in [n]} \mathbf{a}_i$. If $p = p_0 = q_n$, we call π a *cycle* of G on p . We call π a *complete* path of G if p_0 and q_n are the input state and the output state of G respectively. In the rest of the paper we refer to $G = (Q, T)$, or $G = (Q, T, p_{in}, q_{out})$ if we need to specify the input and output states, either as a graph or as a VASS. The input size $|G|$ of a VASS $G = (Q, T)$ is to be understood as the length of its binary code. Let $|Q|$ be the size of Q and $|T|$ be the size of T . Let $\|T\|$ be the maximal 1-norm of vectors labelled in T .

A Parikh image for $G = (Q, T)$ is a vector in \mathbb{N}^T . We will write ϕ, φ, ψ for Parikh images. The displacement $\Delta(\psi)$ is defined by $\sum_{t=(p, \mathbf{t}, q) \in T} \psi(t) \cdot \mathbf{t}$. For the path π on G , we define $\delta(\pi)$ as the Parikh image of π .

Given a space \mathbb{M} , a *configuration* in \mathbb{M} for the d -VASS $G = (Q, T)$ is a pair $(p, \mathbf{m}) \in Q \times \mathbb{M}$, often abbreviated to $p(\mathbf{m})$ for simplicity. The vector \mathbf{m} is called the *location* of the configuration $p(\mathbf{m})$. For $t = (p, \mathbf{a}, q)$, we write $p(\mathbf{m}) \xrightarrow{t}_{\mathbb{M}} q(\mathbf{n})$ whenever $\mathbf{n} = \mathbf{m} + \mathbf{a}$ and $\mathbf{m}, \mathbf{n} \in \mathbb{M}$. Given a path $\pi = t_1 \dots t_n$ from p and q , we say π is a *run* in \mathbb{M} , written $p(\mathbf{m}) \xrightarrow{\pi}_{\mathbb{M}} q(\mathbf{n})$, if there exist configurations $p_1(\mathbf{m}_1), \dots, p_{n-1}(\mathbf{m}_{n-1})$ in \mathbb{M} such that $p(\mathbf{m}) \xrightarrow{t_1}_{\mathbb{M}} p_1(\mathbf{m}_1) \xrightarrow{t_2}_{\mathbb{M}} \dots \xrightarrow{t_{n-1}}_{\mathbb{M}} p_{n-1}(\mathbf{m}_{n-1}) \xrightarrow{t_n}_{\mathbb{M}} q(\mathbf{n})$. We write $p(\mathbf{m}) \xrightarrow{G}_{\mathbb{M}} q(\mathbf{n})$ for the existence of a run $p(\mathbf{m}) \xrightarrow{\pi}_{\mathbb{M}} q(\mathbf{n})$ in G . We say that π is a *walk* if it is a run in \mathbb{N}^d . We often omit the subscript \mathbb{N}^d when talking about walks, and will assume that $\mathbf{m}, \mathbf{n} \in \mathbb{N}^d$ unless stated otherwise. The reachability problem can be formally stated as follows:

Given a d -VASS $G = (Q, T)$ and two configurations $p(\mathbf{m}), q(\mathbf{n})$, is $p(\mathbf{m}) \xrightarrow{G} q(\mathbf{n})$?

Given a d -VASS $G = (Q, T, p, q)$ and $i \in [d]$, if there exists a function $f_i : Q \rightarrow \mathbb{N}$ such that $f_i(q) = f_i(p) + \mathbf{a}(i)$ for every transition $(p, \mathbf{a}, q) \in T$, then i is said to be *fixed* for G . Let I_G be the set of the fixed dimension of G .

For an edge $t \in T$, let $V_G(t)$ be the vector space $V_G(t) \subseteq \mathbb{Q}^d$ spanned by the displacements of cycles that contain t . Let V_G be the vector space spanned by the displacements of all cycles in G . Let n_G be the dimension of V_G . We say that G is *effectively n_G -dimensional*. Intuitively the space spanned by the edge labels of G is n_G -dimensional if G is effectively n_G -dimensional. We also say G is at most *effectively N_G -dimensional* for all $N_G \geq n_G$.

3 KLMST Algorithm

In this section we recall the well-known KLMST algorithm [18]. Special attention is paid to dimension reduction since that is where our algorithm improves upon the KLMST algorithm in the 3-dimensional case.

3.1 KLM sequence

A KLM sequence in dimension d is a sequence

$$\xi = (\mathbf{u}_0 G_0 \mathbf{v}_0) \mathbf{a}_1 (\mathbf{u}_1 G_1 \mathbf{v}_1) \mathbf{a}_2 \cdots \mathbf{a}_n (\mathbf{u}_n G_n \mathbf{v}_n), \quad (1)$$

where

- $\mathbf{u}_i, \mathbf{v}_i \subseteq \mathbb{N}_\omega^d$ for all $i \in [n]_0$,
- $G_i = (Q_i, T_i, p_i, q_i)$ is a d -VASS for all $i \in [n]_0$, and
- $\mathbf{a}_i \in \mathbb{N}^d$ is the displacement of a transition connecting q_{i-1} to p_i for all $i \in [n]$.

The component $\mathbf{u}_i G_i \mathbf{v}_i$ is denoted by ξ_i . For convenience the transition whose displacement is \mathbf{a}_i is often referred to by \mathbf{a}_i . The size of ξ is $2(d+1)^{d+1}(k + \sum_{i=0}^n (\|\mathbf{x}_i\|_1 + |G_i| + \|\mathbf{y}_i\|_1) + \sum_{i=1}^n \|\mathbf{a}_i\|_1)$, denoted by $|\xi|$. A run ρ of ξ is of the form $\sigma_0 \mathbf{a}_1 \sigma_1 \cdots \mathbf{a}_n \sigma_n$ where σ_i is a path from p_i to q_i in G_i . The run ρ is a *witness* if there exists a sequence of natural number vectors $\mathbf{m}_0, \mathbf{n}_0, \dots, \mathbf{m}_n, \mathbf{n}_n$ such that

- $\mathbf{m}_i \subseteq \mathbf{u}_i, \mathbf{n}_i \subseteq \mathbf{v}_i$ for all $i \in [n]_0$, and
- $p_0(\mathbf{m}_0) \xrightarrow{\sigma_0} q_0(\mathbf{n}_0) \xrightarrow{\mathbf{a}_1} p_1(\mathbf{m}_1) \xrightarrow{\sigma_1} \cdots \xrightarrow{\mathbf{a}_n} p_n(\mathbf{m}_n) \xrightarrow{\sigma_n} q_n(\mathbf{n}_n)$.

Let W_ξ be the set of witnesses of ξ . Whether $q(\mathbf{n})$ is reachable from $p(\mathbf{m})$ in a d -VASS $G = (Q, T)$ is equivalent to asking if there is a witness to the KLM sequence $\xi = \mathbf{m} G' \mathbf{n}$ where $G' = (Q, T, p, q)$.

A remarkable measure on the KLM sequences was proposed by Leroux and Schmitz [18]. The *ranking function* r maps a VASS G onto a $(d+1)$ -dimensional vector $(r_d, r_{d-1}, \dots, r_0)$, where r_k denotes the number of edges t satisfying $\dim V_{G_i}(t) = k$. The rank $r(\xi)$ of the KLM sequence (1) is defined by the summation $r(\xi) = \sum_{i \in [n]} r(G_i)$. It has been shown that the operations of the KLMST algorithm strictly decrease the rank of KLM sequence, hence the termination of the algorithm.

The characteristic system \mathcal{E}_ξ for ξ is a linear Diophantine system that provides an algebraic characterization to the witnesses of ξ . Formally \mathcal{E}_ξ is defined by the following equations:

$$\mathbf{y}_i = \mathbf{x}_i + \sum_{t=(p, \mathbf{a}, q) \in T_i} \phi_i(t) \cdot \mathbf{a}, \quad (2)$$

$$\mathbf{x}_{j+1} = \mathbf{y}_j + \mathbf{a}_j, \quad (3)$$

$$\mathbf{1}_{q_i} - \mathbf{1}_{p_i} = \sum_{t=(p, \mathbf{a}, q) \in T_i} \phi_i(t) \cdot (\mathbf{1}_q - \mathbf{1}_p), \quad (4)$$

$$\mathbf{x}_i \subseteq \mathbf{u}_i, \quad (5)$$

$$\mathbf{y}_i \subseteq \mathbf{v}_i, \quad (6)$$

where $i \in [n]_0$ and $j \in [n-1]_0$. In (4) the notation $\mathbf{1}_q$ for example is an indicator vector whose q -th entry is 1 and whose other entries are 0. The equality (4) is called *Euler Condition*, which guarantees the existence of a path from p_i to q_i in G_i whose Parikh image is ϕ_i . In (5) and (6) the order relation imposes constraints on the entering location and the exit location.

Let \mathbf{h} be a solution to \mathcal{E}_ξ . We denote by $\mathbf{h}(\mathbf{x}_i)$ for instance the restriction of \mathbf{h} to the variables \mathbf{x}_i . The size of \mathbf{h} is defined by $\sum_{i=0}^n (\|\mathbf{h}(\mathbf{x}_i)\|_1 + \|\mathbf{h}(\mathbf{y}_i)\|_1 + \|\mathbf{h}(\phi_i)\|_1)$. Now let

$V_\xi = \bigcup_{i \in [n]_0} (\{\mathbf{x}_i(k) \mid k \in [d]\} \cup \{\mathbf{y}_i(k) \mid k \in [d]\} \cup \{\phi_i(t) \mid t \in T_i\})$, which is the set of the variables in \mathcal{E}_ξ . For each $x \in V_\xi$ let S_x be the set $\{\mathbf{h}(x) \mid \mathbf{h} \text{ is a solution to } \mathcal{E}_\xi\}$. We say that x is *unbounded* if S_x is infinite. It is clear that if \mathcal{E}_ξ is not *satisfiable*, meaning that \mathcal{E}_ξ has no solution, then ξ has no witness. Hence in the following we assume all considered KLM sequences are satisfiable, since we can just throw away the unsatisfiable KLM sequences in the algorithm.

Suppose \mathbf{h}, \mathbf{h}' are solutions to \mathcal{E}_ξ and $\mathbf{h} \leq \mathbf{h}'$. One gets a solution $\mathbf{h}' - \mathbf{h}$ to the homogeneous characteristic system \mathcal{E}_ξ^0 that is defined by the following equations:

$$\mathbf{y}_i^0 = \mathbf{x}_i^0 + \sum_{t=(p,\mathbf{a},q) \in T_i} \phi_i^0(t) \cdot \mathbf{a}, \quad (7)$$

$$\mathbf{x}_{j+1}^0 = \mathbf{y}_j^0, \quad (8)$$

$$\mathbf{0} = \sum_{t=(p,\mathbf{a},q) \in T_i} \phi_i^0(t) \cdot (\mathbf{1}_q - \mathbf{1}_p), \quad (9)$$

$$\mathbf{x}_i^0(k) = 0, \text{ whenever } \mathbf{m}_i[k] \neq \omega, \quad (10)$$

$$\mathbf{y}_i^0(k) = 0, \text{ whenever } \mathbf{n}_i[k] \neq \omega, \quad (11)$$

where $i \in [n]_0$ and $j \in [n-1]_0$ and $k \in [d]$. Let $V_\xi^0 = \bigcup_{i \in [n]_0} (\{\mathbf{x}_i^0(k) \mid k \in [d]\} \cup \{\mathbf{y}_i^0(k) \mid k \in [d]\} \cup \{\phi_i^0(t) \mid t \in T_i\})$. The size of a homogeneous solution is defined similarly. Leroux proved in [18] a convenient result stated next.

► **Theorem 1** ([18]). *There exists a solution \mathbf{h}^0 to \mathcal{E}_ξ^0 of size no more than $|\xi|^{|\xi|-2}$ such that $\mathbf{h}^0(x^0) > 0$ if and only if $x \in V_\xi$ is unbounded. Moreover the sum of the values of the bounded variables is bounded by $|\xi|^{|\xi|-1}$.*

3.2 Standardization

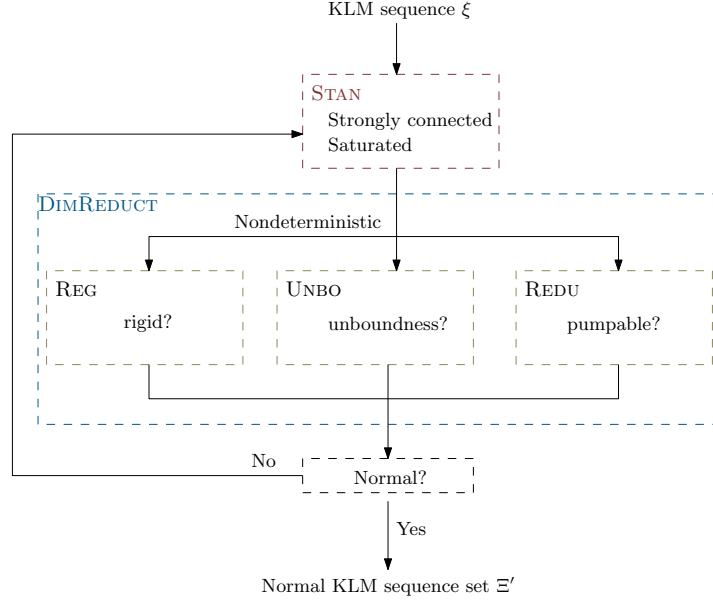
The KLMST algorithm builds on the fact that a KLM sequence can be converted to a ‘good’ one that enjoys a number of properties. Our account of these properties follows [18] closely. A KLM sequence ξ is *strongly connected* if for every $i \in [n]_0$ the graph G_i in ξ_i is strongly connected; it is *saturated* if for every $i \in [n]_0$ and every $j \in [d]$, the equality $\mathbf{m}_i(j) = \omega$, respectively the equality $\mathbf{n}_i(j) = \omega$, implies that $\mathbf{x}_i(j)$, respectively $\mathbf{y}_i(j)$, is unbounded.

A KLM sequence ξ is *standard* if it is both *strongly connected* and *saturated*. Every KLM sequence can be transformed to a set of standard KLM sequences.

► **Theorem 2.** *A set Ξ of standard KLM sequences of smaller rank can be computed from a nonstandard ξ in $\exp(|\xi|^{|\xi|})$ time such that $\mathbf{W}_\xi = \bigcup_{\xi' \in \Xi} \mathbf{W}_{\xi'}$ and $|\xi'| \leq |\xi|^{|\xi|}$ for every $\xi' \in \Xi$.*

Intuitively to decompose a KLM sequence to a set of standard KLM sequences, one finds out the strongly connected components of every G_i and nondeterministically chooses a sequence, and then update the constraints $\mathbf{u}_i, \mathbf{v}_i$ using the solution \mathbf{h}^0 in Theorem 1. Let this process be called STAN. It follows from Theorem 2 that every KLM sequence can be transformed by STAN with an exponential size amplification. Figure 1 displays the overview of KLMST algorithm. The red dashed line box is the decomposition procedure, denoted by STAN. Clearly STAN always terminates.

► **Remark.** In [18] the standard KLM sequences is called *clean* KLM sequences.



■ **Figure 1** The framework of KLMST Algorithm

3.3 Dimension Reduction

This section is dedicated to the second part of the KLMST algorithm, the dimension reduction step. The so-called rigidity is another useful property of KLM sequences. Intuitively the property guarantees that values in a fixed dimension never drop below 0. Formally if for any G_i in ξ and any of its fixed dimension $k \in I_{G_i}$, there exists a function $g : Q_i \rightarrow \mathbb{N}$ such that $g(q) = g(p) + \mathbf{a}(k)$ for all $(p, \mathbf{a}, q) \in G_i$, $g(p_i) \sqsubseteq \mathbf{u}(k)$ and $g(q_i) \sqsubseteq \mathbf{v}(k)$, then ξ is *rigid*.

Transforming a standard KLM sequence ξ to a set of rigid KLM sequences is subtle. A transformation may turn a standard KLM sequence to a non-standard one. The strong connectedness property for example may be destroyed during the transformation procedure. Fortunately if ξ is not rigid, the following theorem ensures that it can be converted to a set of KLM sequences with smaller rank.

► **Theorem 3.** *A set Ξ of KLM sequences of smaller rank can be computed from a standard but not rigid ξ in $\text{poly}(|\xi|)$ time such that $\mathcal{W}_\xi = \bigcup_{\xi' \in \Xi} \mathcal{W}_{\xi'}$ and $|\xi'| \leq |\xi|$ for every $\xi' \in \Xi$.*

The procedure stated in Theorem 3 will be referred to by REG. Rigidity is clearly related to dimension reduction.

► **Lemma 4.** *The d -VASS G is at most effectively $(d - |I_G|)$ -dimensional.*

Proof. Suppose $k \in I_G$, there exists $f_k : Q \rightarrow \mathbb{N}$ such that $f_k(q) = f_k(p) + \mathbf{a}(k)$ for every transition $(p, \mathbf{a}, q) \in T$. Thus $\Delta(\theta)(k) = 0$ for every cycle θ in G . So G is effectively $(d-1)$ -dimensional. The general result follows by induction. ◀

A KLM sequence ξ is said to be *unbounded* if for every $i \in [n]_0$ and every edge t of G_i , the set $S_{\phi_i(t)}$ is unbounded; it is *bounded* otherwise. The unboundedness is completely determined by the solution set to the homogeneous characteristic system. If $S_{\phi_i(t)}$ is finite we may decompose G_i to smaller graphs, none of which containing the edge t .

► **Theorem 5.** *A set Ξ of KLM sequences of smaller rank can be computed from a bounded ξ in $\exp(|\xi|^{|\xi|})$ time such that $\mathcal{W}_\xi = \bigcup_{\xi' \in \Xi} \mathcal{W}_{\xi'}$ and $|\xi'| \leq |\xi|^{|\xi|}$ for every $\xi' \in \Xi$.*

Let's denote by UNBO the procedure guaranteed by the above theorem. It should not come as a surprise that every VASS in every KLM sequence in $\text{UNBO}(\mathbf{uGv})$ is effectively of lower dimension than G . In the present paper the following weaker result is sufficient.

► **Lemma 6.** *Suppose $G = (V, T)$ is effectively k -dimensional. If \mathbf{uGv} is decomposed by UNBO to the sequence $(\mathbf{u}_1 G_1 \mathbf{v}_1) \dots (\mathbf{u}_n G_n \mathbf{v}_n)$, then the number of G_i that remains effectively k -dimensional is no more than $|T| - 1$.*

The output of UNBO could be very long. But most VASS's appearing in the output are in lower dimensional spaces. In Section 5 we take a look at this fact in the 3-dimensional case.

Dimension reduction can also be done if the input KLM sequence does not meet the so-called pumpability condition. Given a d -dimensional $\mathbf{u}_i G_i \mathbf{v}_i$, we consider runs in \mathbb{N}_ω^d by imposing the additional equality $\omega + n = \omega$ for every $n \in \mathbb{N}_\omega$. If there exist $\mathbf{u}'_i, \mathbf{v}'_i$ such that $p_i(\mathbf{u}_i) \xrightarrow{G_i}_{\mathbb{N}_\omega^d} p_i(\mathbf{u}'_i)$ and $q_i(\mathbf{v}'_i) \xrightarrow{G_i}_{\mathbb{N}_\omega^d} q_i(\mathbf{v}_i)$ and $\mathbf{u}'_i > \mathbf{u}_i$ and $\mathbf{v}'_i > \mathbf{v}_i$, and for any dimension j that is not fixed, the strict inequalities $\mathbf{u}'_i(j) > \mathbf{u}_i(j), \mathbf{v}'_i(j) > \mathbf{v}_i(j)$ hold, then $\mathbf{u}_i G_i \mathbf{v}_i$ is *pumpable*; otherwise it is *nonpumpable*. If $\xi_i = \mathbf{u}_i G_i \mathbf{v}_i$ is pumpable for every i , ξ is *pumpable*; it is *nonpumpable* otherwise.

A standard KLM sequence is said to be *normal* if it is rigid, unbounded and pumpable. The pumpability of a KLM sequence can be seen as a form of coverability, and as pointed out in [22], the latter can be verified in $\exp(|\xi|)$ time. The next theorem provides a basic fact for further complexity analysis of the KLMST algorithm [18].

► **Theorem 7.** *Every normal KLM sequence ξ has a witness of length less than $|\xi|^{3|\xi|}$.*

Due to Theorem 7 the goal of KLMST algorithm is to transform a KLM sequence to a set of normal KLM sequences. In what follows we show how to handle a nonpumpable KLM sequence. The core idea is that if $\mathbf{u}_i G_i \mathbf{v}_i$ is nonpumpable, there exists at least one dimension in which values are bounded by $|\xi|$ in all the witnesses. Let such procedure denoted by REDU. The following lemma describes how to handle a nonpumpable KLM sequence.

► **Lemma 8.** *Suppose $\xi = \mathbf{uGv}$ is a d -dimensional standard KLM sequence. If ξ is nonpumpable, then a set of effectively $(d-1)$ -dimensional KLM sequence Ξ can be computed from ξ in $\exp(|\xi|^{d+d^d})$ time such that $W_\xi = \bigcup_{\xi' \in \Xi} W_{\xi'}$ and $|\xi'| \leq |\xi|^{d+d^d}$ for all $\xi' \in \Xi$.*

By composing REG, UNBO and REDU we obtain the second part of the KLMST algorithm called DIMREDUCT, see the blue dashed line box in Figure 1. The algorithm DIMREDUCT deals with standard but not normal KLM sequences. Upon receiving a standard KLM sequence ξ , DIMREDUCT executes one of the followings, whichever is applicable:

1. If ξ is not rigid, apply the REG procedure.
2. If ξ is bounded, apply the UNBO procedure.
3. If ξ is nonpumpable, apply the REDU procedure.

The following theorem states the effect of the DIMREDUCT procedure.

► **Theorem 9.** *DIMREDUCT produces from a standard but not normal ξ a set Ξ of KLM sequences of smaller rank in $\exp(|\xi|^{|\xi|})$ time such that $W_\xi = \bigcup_{\xi' \in \Xi} W_{\xi'}$ and $|\xi'| \leq |\xi|^{|\xi|}$.*

By using the constructions guaranteed by Theorem 2, Theorem 3 and Theorem 9 the KLMST algorithm can be defined as follows:

1. Run STAN to get a set Ξ of standard KLM sequences.
2. Apply DIMREDUCT to get a set Ξ' of KLM sequences with strictly smaller rank.
3. Output $\text{NORM}(\Xi') \cup \bigcup_{\xi' \in \Xi' \text{ is not normal}} \text{KLMST}(\xi')$, where $\text{NORM}(\Xi')$ is the subset of Ξ' that contains all the normal KLM sequences in Ξ' .

The outline of the KLMST algorithm is now complete. Theorem 2 and Theorem 9 are the Cleaning Lemma and Decomposition Lemma of [18].

4 Effectively 2-Dimensional 3-VASS

It has been known for some time that the paths in 2-VASS can be described by linear path schemes. We will point out in this section a non-surprising generalization of this result, that is all paths in an effectively 2-dimensional 3-VASS are describable by linear path schemes. We review in Section 4.1 the characterization of path in terms of the linear path scheme, and carry out the generalization in Section 4.2.

4.1 Linear Path Scheme

A linear path scheme is a regular expression of the form $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_n^* \alpha_n$, where the α 's are paths and the β 's are cycles. Let $\text{src}(\rho)$ be the entry state of α_0 and $\text{tgt}(\rho)$ be the exit state of α_n . The length $|\rho|$ of ρ is defined by $|\alpha_0 \beta_1 \alpha_1 \cdots \beta_n \alpha_n|$. For two configurations $p(\mathbf{m})$ and $q(\mathbf{n})$, $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$ if there exist $k_1, \dots, k_n \in \mathbb{N}$ such that $p(\mathbf{m}) \xrightarrow{\alpha_0 \beta_1^{k_1} \cdots \beta_n^{k_n} \alpha_n} q(\mathbf{n})$. Intuitively ρ demonstrates a path pattern described by a set of cycles. Let $\mathcal{L}(G)$ be the set of the linear path schemes bounded by $2^{c|G|}$ for some constant c . Blondin proved that there exists a constant $c \in \mathbb{N}$ such that every walk in a 2-VASS G can be transformed to a linear path scheme in $\mathcal{L}(G)$.

► **Theorem 10** ([1]). *Given a 2-VASS G and two configurations $p(\mathbf{m}), q(\mathbf{n})$ in \mathbb{N}^2 , $p(\mathbf{m}) \xrightarrow{G} q(\mathbf{n})$ if and only if $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$ for some $\rho \in \mathcal{L}(G)$. Moreover every linear path scheme ρ is characterized by a system \mathcal{E}_ρ of linear Diophantine equations in the sense that \mathcal{E}_ρ has a solution if and only if $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$.*

With Theorem 10 in mind, one introduces the linear path scheme systems. In the following definition $\mathbf{x}, \phi, \mathbf{y}, \{\mathbf{a}_{k,l_k}\}_k, \{\mathbf{b}_{k,l_k}\}_k, \{\mathbf{c}_{k,l_k}\}_k$ are variable vectors.

► **Definition 11.** *Let $\rho = \alpha_0 \beta_1^* \alpha_1 \beta_2^* \cdots \beta_n^* \alpha_n \in \mathcal{L}(G)$ in a d -dimensional VASS G . The linear path scheme system (LPS system) \mathcal{E}_ρ for $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$ consists of the following equations.*

$$\mathbf{x} + \sum_{i=0}^n \Delta(\alpha_i) + \sum_{i=1}^n \phi(\beta_i) \cdot \Delta(\beta_i) = \mathbf{y}, \quad (12)$$

$$\mathbf{x} + \sum_{i=1}^k (\Delta(\alpha_{i-1}) + \phi(\beta_i) \Delta(\beta_i)) + \Delta(\alpha_k[1, \dots, l]) = \mathbf{a}_{k,l}, \quad (13)$$

$$\mathbf{x} + \sum_{i=1}^k (\Delta(\alpha_{i-1}) + \phi(\beta_i) \Delta(\beta_i)) + \Delta(\alpha_k) + \Delta(\beta_{k+1}[1, \dots, l]) = \mathbf{b}_{k+1,l}, \quad (14)$$

$$\mathbf{x} + \sum_{i=1}^{k+1} (\Delta(\alpha_{i-1}) + \phi(\beta_i) \Delta(\beta_i)) - \Delta(\beta_{k+1}[l, \dots, |\beta_{k+1}|]) = \mathbf{c}_{k+1,l}. \quad (15)$$

We will refer to the LPS system defined by the linear path scheme ρ by $\mathbf{y} = \mathcal{E}_\rho(\mathbf{x})$. In the above definition $0 \leq k \leq n$ and $0 \leq l \leq |\alpha_k|$ in (13), $0 \leq k \leq n-1$ and $1 \leq l \leq |\beta_{k+1}|$ in (14) and (15). Suppose \mathbf{f} is a solution to $\mathbf{y} = \mathcal{E}_\rho(\mathbf{x})$. We write for example $\mathbf{f}(\mathbf{x})$ for the part of the solution restricted to \mathbf{x} . The homogeneous LPS system \mathcal{E}_ρ^0 is defined by the following

equations.

$$\mathbf{x}^0 + \sum_{i=1}^n \phi^0(\beta_i) \Delta(\beta_i) = \mathbf{y}^0, \quad (16)$$

$$\mathbf{x}^0 + \sum_{i=1}^k \phi^0(\beta_i) \Delta(\beta_i) = \mathbf{a}_{k,l}^0, \quad (17)$$

$$\mathbf{x}^0 + \sum_{i=1}^k \phi^0(\beta_i) \Delta(\beta_i) = \mathbf{b}_{k+1,l}^0, \quad (18)$$

$$\mathbf{x}^0 + \sum_{i=1}^{k+1} \phi^0(\beta_i) \Delta(\beta_i) = \mathbf{c}_{k+1,l}^0. \quad (19)$$

It will be referred to by $\mathbf{y}^0 = \mathcal{E}_\rho^0(\mathbf{x}^0)$. Let \mathbf{f}^0 be a solution to $\mathbf{y}^0 = \mathcal{E}_\rho^0(\mathbf{x}^0)$. Then $\mathbf{f} + h\mathbf{f}^0$ is a solution to \mathcal{E}_ρ for all $h \in \mathbb{N}$. The following theorem points out that a solution to the equation system $\mathbf{y} = \mathcal{E}_\rho(\mathbf{x})$ corresponds to a walk of the form ρ .

► **Theorem 12.** Suppose $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_n^* \alpha_n$ is a linear path scheme of $G = (Q, T)$, $p = \text{src}(\rho)$ and $q = \text{tgt}(\rho)$. Then \mathbf{f} is a solution to $\mathbf{y} = \mathcal{E}_\rho(\mathbf{x})$ if and only if $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$, where $\mathbf{m} = \mathbf{f}(\mathbf{x})$ and $\mathbf{n} = \mathbf{f}(\mathbf{y})$.

► **Corollary 13.** Suppose $\{G_i = (Q_i, T_i, p_i, q_i)\}_{i \in [n]_0}$ is a family of d -VASS and $\{q_i \xrightarrow{\mathbf{a}_i} p_{i+1}\}_{i \in [n-1]_0}$ is a family of edges. Then $p_0(\mathbf{m}) \xrightarrow{\rho_0 \mathbf{a}_0 \rho_1 \mathbf{a}_1 \cdots \mathbf{a}_{n-1} \rho_n} q_n(\mathbf{n})$ for the linear path schemes ρ_0, \dots, ρ_n for G_0, \dots, G_n respectively and $\mathbf{m}, \mathbf{n} \in \mathbb{N}^d$, if and only if the following equation system has a solution:

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{m}, \\ \mathbf{y}_n &= \mathbf{n}, \\ \mathbf{y}_i &= \mathcal{E}_{\rho_i}(\mathbf{x}_i), \text{ for } i \in [n]_0, \\ \mathbf{x}_{i+1} &= \mathbf{y}_i + \mathbf{a}_i, \text{ for } i \in [n-1]_0. \end{aligned}$$

4.2 Path in Effectively 2-Dimensional 3-VASS

We show in this section that if there are walks between two configurations in an effectively 2-dimensional 3-VASS, the walks can be characterised by linear path schemes.

Let G be an effectively 2-dimensional 3-VASS. A path from a configuration admitted by the VASS may go in a zig-zag fashion, but values in one dimension would not cause the path to cross the current octant. Let $\mathcal{D} \geq 2^{O(|G|)}$ be a big number. Define $\mathbb{D} \stackrel{\text{def}}{=} [\mathcal{D}, +\infty]$. We start with the following theorem, whose proof uses an approach similar to the one in [1].

- **Theorem 14.** Suppose $G = (Q, T)$ is an effectively 2-dimensional 3-VASS and $\mathbf{m}, \mathbf{n} \in \mathbb{D}^3$.
- If $q(\mathbf{m}) \rightarrow_{\mathbb{N}^3} q(\mathbf{n})$, then there exists a linear path scheme ρ with at most two cycles such that $|\rho| \leq 2^{O(|G|)}$ and $q(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$.
 - If $p(\mathbf{m}) \rightarrow_{\mathbb{D}^3} q(\mathbf{n})$, then there exists a linear path scheme ρ with at most $2|Q|$ cycles such that $|\rho| \leq 2^{O(|G|)}$ and $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$.

Theorem 14 is a slight generalization of the main lemma of Blondin *et. al* [1]. It can be proved by a projection technique. We use a triple $(\#_1, \#_2, \#_3)$ to denote an octant Z , where $\#_i \in \{\geq, \leq\}$, and for any $\mathbf{m} \in Z$ it holds that $\mathbf{m}(1)\#_1 0$, $\mathbf{m}(2)\#_2 0$ and $\mathbf{m}(3)\#_3 0$. For instance the octant \mathbb{N}^3 is denoted as (\geq, \geq, \geq) . Notice that $V \cap Z \supseteq \{(0, 0, 0)\}$. The following lemma is one way to describe the projection.

► **Lemma 15.** *Let $V \subseteq \mathbb{Q}^3$ be a 2-dimensional vector space, and let $Z = (\#_1, \#_2, \#_3)$ represent an octant. If the spanned vector space of $V \cap Z$ is still 2-dimensional, then there exist distinct $i, j \in [3]$ such that $\mathbf{m} \in Z$ whenever $\mathbf{m} \in V$, $\mathbf{m}(i) \#_i 0$ and $\mathbf{m}(j) \#_j 0$.*

Next we repeat the argument of [1] to the effectively 2-dimensional 3-VASS. Consider a walk in a region where one dimension is bounded. Similar to what we have done in the DIMREDUCT procedure, values in the bounded dimension are encoded into the states as it were. A state becomes a pair (q, t) for every $q \in Q$ and $t \in [\mathcal{D}]_0$. This operation transforms the effectively 2-dimensional 3-VASS to a 2-VASS, which can be characterized by linear path schemes according to Theorem 10.

► **Theorem 16.** *Let $\mathbb{I}_{\mathcal{D}}$ be one of the regions $[\mathcal{D}] \times \mathbb{N}^2$, $\mathbb{N} \times [\mathcal{D}] \times \mathbb{N}$, $\mathbb{N}^2 \times [\mathcal{D}]$. Suppose G is an effectively 2-dimensional 3-VASS and $\mathbf{m}, \mathbf{n} \in \mathbb{I}_{\mathcal{D}}$. Then $p(\mathbf{m}) \rightarrow_{\mathbb{I}_{\mathcal{D}}} q(\mathbf{n})$ if and only if there exists a linear path scheme ρ with size $|\rho| \leq (\mathcal{D} + |G|)^{O(1)}$ such that $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$.*

We now introduce the following notations $\mathbb{I}_1 = [\mathcal{D}] \times \mathbb{N}^2$, $\mathbb{I}_2 = \mathbb{N} \times [\mathcal{D}] \times \mathbb{N}$, $\mathbb{I}_3 = \mathbb{N}^2 \times [\mathcal{D}]$ and define $\mathbb{L}_{\mathcal{D}} = \mathbb{I}_1 \cup \mathbb{I}_2 \cup \mathbb{I}_3$. The next theorem shows that though a walk in $\mathbb{L}_{\mathcal{D}}$ may cross between \mathbb{I}_1 , \mathbb{I}_2 , \mathbb{I}_3 , it can still be converted to a linear path scheme.

► **Theorem 17.** *Suppose G is an effectively 2-dimensional 3-VASS and $\mathbf{m}, \mathbf{n} \in \mathbb{L}_{\mathcal{D}}$. Then $p(\mathbf{m}) \rightarrow_{\mathbb{L}_{\mathcal{D}}} q(\mathbf{n})$ if and only if there exists a linear path scheme ρ with size $|\rho| \leq (\mathcal{D} + |G|)^{O(1)}$ such that $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$.*

We have now proved that a walk in an effectively 2-dimensional 3-VASS can be converted to a linear path scheme if either the entering location and the exit location are high up in the space or one dimension is restricted. For a general walk we divide the space into two regions:

- $\mathbb{D} \stackrel{\text{def}}{=} (\mathcal{D}_1, +\infty)^3$, and
- $\mathbb{L} \stackrel{\text{def}}{=} [\mathcal{D}_2] \times \mathbb{N}^2 \cup \mathbb{N} \times [\mathcal{D}_2] \times \mathbb{N} \cup \mathbb{N}^2 \times [\mathcal{D}_2]$,

where \mathcal{D}_1 is as large as necessary to validate Theorem 14 and $\mathcal{D}_2 = 2\mathcal{D}_1$. Suppose $p(\mathbf{m}) \rightarrow q(\mathbf{n})$ and consider all the configurations in the walk. Let $q(\mathbf{m}_{q,in})$ and $q(\mathbf{m}_{q,out})$ be the first and the last configuration in the walk that q appears in the intersection $\mathbb{D} \cap \mathbb{L}$. Using this strategy, the walk can be segmented to consecutive sub-paths of two categories:

1. cycling walk from $q(\mathbf{m}_{q,in})$ to $q(\mathbf{m}_{q,out})$ for some $q \in Q$, and
2. walks between the cycles.

Walks in the first category can be converted into linear path schemes by Theorem 14, and those in the second category are in either \mathbb{L} or \mathbb{D} , and can be transformed into linear path schemes by Theorem 14 and Theorem 17.

► **Theorem 18.** *Suppose G is a strongly connected effectively 2-dimensional 3-VASS G and $p(\mathbf{m}), q(\mathbf{n})$ are configurations. If $q(\mathbf{n})$ is reachable from $p(\mathbf{m})$, then there exists a linear path scheme $\rho < \exp(|G|)$ such that $p(\mathbf{m}) \xrightarrow{G} q(\mathbf{n})$.*

To construct a walk from $p(\mathbf{m})$ to $q(\mathbf{n})$, one only has to guess a member in $\mathcal{L}(G)$ and check if the associated LPS system has any solution.

5 Almost Normal 3-Dimensional KLM Sequence

Section 3 and Section 4 have recalled the techniques applicable to the VASS in general and the 2-VASS in particular. In this section we use these approaches to construct an algorithm for 3-VASS. The basic idea is to apply the KLMST algorithm to a 3-dimensional KLM sequence, and then replace every effectively 2-dimensional KLM sequence by its LPS system. The idea works because the majority components generated by the KLMST algorithm are

effectively 2-dimensional and that being a linear path scheme is a property independent of its input/output location.

5.1 The Witness of Almost Normal KLM Sequence

We begin with a natural definition. Let ξ be a KLM sequence of the form (1).

► **Definition 19** (almost normal KLM sequence). *The KLM sequence ξ is almost normal if, for all $k \in [n]_0$, ξ_k is either normal or effectively 2-dimensional.*

To give a more informative characterization of the almost normal KLM sequences, we introduce the composite characteristic system $\mathcal{C}_{\xi, \Lambda}$ and its homogeneous version $\mathcal{C}_{\xi, \Lambda}^0$ indexed by a function Λ from $[n]_0$ to $\mathcal{L}(G)$.

► **Definition 20** (composite characteristic system). *Let $J \subseteq [n]_0$ be the set of the indexes i such that $\mathbf{u}_i G_i \mathbf{v}_i$ is effectively 2-dimensional. Suppose $\Lambda : J \rightarrow \bigcup_{i \in J} \mathcal{L}(G_i)$ is such that $\Lambda(i) \in \mathcal{L}(G_i)$ for all $i \in J$. The composite characteristic system $\mathcal{C}_{\xi, \Lambda}$ for Λ is defined as follows:*

$$\mathbf{y}_i = \mathcal{C}_{\Lambda(i)}(\mathbf{x}_i), \text{ for } i \in J, \quad (20)$$

$$\mathbf{y}_i = \mathbf{x}_i + \sum_{t=(p, \mathbf{a}, q)} \phi_i(t) \cdot \mathbf{a}, \text{ for } i \in [n]_0 \setminus J, \quad (21)$$

$$\mathbf{1}_{q_i} - \mathbf{1}_{p_i} = \sum_{t=(p, \mathbf{a}, q) \in T_i} \phi_i(t)(\mathbf{1}_q - \mathbf{1}_p), \text{ for } i \in [n]_0 \setminus J, \quad (22)$$

$$\mathbf{x}_{i+1} = \mathbf{y}_i + \mathbf{a}_i, \text{ for } i \in [n]_0, \quad (23)$$

$$\mathbf{x}_i \sqsubseteq \mathbf{u}_i, \quad (24)$$

$$\mathbf{y}_i \sqsubseteq \mathbf{v}_i. \quad (25)$$

The homogeneous composite characteristic system $\mathcal{C}_{\xi, \Lambda}^0$ for Λ is defined as follows:

$$\mathbf{y}_i^0 = \mathcal{C}_{\Lambda(i)}^0(\mathbf{x}_i), \text{ for } i \in J \quad (26)$$

$$\mathbf{y}_i^0 = \mathbf{x}_i^0 + \sum_{t=(p, \mathbf{a}, q)} \phi_i^0(t) \cdot \mathbf{a}, \text{ for } i \in [n]_0 \setminus J, \quad (27)$$

$$\mathbf{0} = \sum_{t=(p, \mathbf{a}, q) \in T_i} \phi_i^0(t)(\mathbf{1}_q - \mathbf{1}_p), \text{ for } i \in [n]_0 \setminus J, \quad (28)$$

$$\mathbf{x}_i^0[j] = 0, \text{ for } \mathbf{u}_i[j] \neq \omega, \quad (29)$$

$$\mathbf{y}_i^0[j] = 0, \text{ for } \mathbf{v}_i[j] \neq \omega. \quad (30)$$

For uniformity we keep the form of the variables of $\mathcal{C}_{\xi, \Lambda}$ as $V_\xi = \{(\mathbf{x}_i, \phi_i, \mathbf{y}_i)\}_{i \in [n]_0}$. If $\mathbf{u}_k G_k \mathbf{v}_k$ is effectively 2-dimensional, ϕ_i are the variables in the LPS system $\mathbf{y}_i = \mathcal{C}_{\Lambda(i)}(\mathbf{x}_i)$; otherwise ϕ_i is a Parikh image on T_i . By definition $\mathcal{C}_{\xi, \emptyset}$ is the characteristic system defined in Section 3.1. If $J = \{i_1, \dots, i_k\}$ we also write Λ as a sequence of the linear path schemes $\rho_{i_1} \dots \rho_{i_k}$, where $\rho_{i_t} = \Lambda(i_t)$ for all $t \in [k]$. The almost normal KLM sequences are good in the sense of the following theorem.

► **Theorem 21.** *The almost normal KLM sequence ξ has a witness bounded by $|\xi|^{3|\xi|}$.*

5.2 Generating Almost Normal 3-Dimensional KLM Sequence

We have shown that almost normal KLM sequences have bounded witness. In this section we propose an algorithm that can transform a 3-dimensional KLM sequence to a set of almost normal KLM sequences. The idea of the algorithm is simple. Given a 3-dimensional KLM sequence ξ as in (1), we apply the KLMST decomposition algorithm on ξ . Whenever an effectively 2-dimensional $\xi_k = \mathbf{u}_k G_k \mathbf{v}_k$ is generated, the algorithm guesses a linear path scheme ρ for it and utilize the composite characteristic system. We continue to use ξ_k to denote the k -th component $\mathbf{u}_k G_k \mathbf{v}_k$ of ξ . No further decomposition is applied to $\mathbf{u}_k G_k \mathbf{v}_k$. We remark that it is not really necessary to deal with rigidity explicitly since every effectively 2-dimensional ξ is rigid.

► **Lemma 22.** *Given a 3-VASS $G = (Q, T)$, if I_G is not empty, then G is effectively 2-dimensional.*

The input to the new 3-KLMST decomposition algorithm is a triple (ξ, Λ, J) . Initially both Λ and J are empty, meaning that $\mathcal{C}_{\xi, \Lambda}$ is the characteristic system defined in Section 3.1. In what follows we omit the edges connecting the components since their treatment is routine. Let $\Xi = \{(\xi, \Lambda, J)\}$.

1. Let $(\xi, \Lambda, J) \in \Xi$ be the current triple. For each $j \notin J$ apply the STAN procedure to $\mathbf{u}_j G_j \mathbf{v}_j$. For each $j \in J$ let j' be the new index after decomposition. We then update J to J' and let $\Lambda'(j') = \Lambda(j)$. Let Ξ' be the finite set of triples (ξ', Λ', J') so generated.
2. Let Ξ'' be

$$\bigcup_{(\xi', \Lambda', J') \in \Xi'} \{(\xi'_0 \xi'_1 \dots \xi'_n, \Lambda', J') \mid \xi'_j = \xi_j \text{ if } j \in J, \xi'_j = \text{DIMREDUCT}(\xi_j) \text{ if } j \notin J\}.$$

In the above definition Λ', J' are modified accordingly.

3. For each $(\xi, \Lambda, J) \in \Xi''$, check whether $\mathbf{u}_j G_j \mathbf{v}_j$ is effectively 2-dimensional for every $j \notin J$. Let J_{new} be the index of $\mathbf{u}_j G_j \mathbf{v}_j$ that is effectively 2-dimensional and $j \notin J$, and let Ψ_Λ be

$$\Psi_\Lambda = \{\Lambda' \mid \Lambda'(j) = \Lambda(j) \text{ for some } j \in J, \text{ or } \Lambda'(j) \in \mathcal{L}(G_j) \text{ for some } j \in J_{\text{new}}\}.$$

Let $\Xi = \bigcup_{(\xi, \Lambda, J) \in \Xi''} \{(\xi, \Lambda', J \cup J_{\text{new}}) \mid \Lambda' \in \Psi_\Lambda\}$ be the set of the new triples.

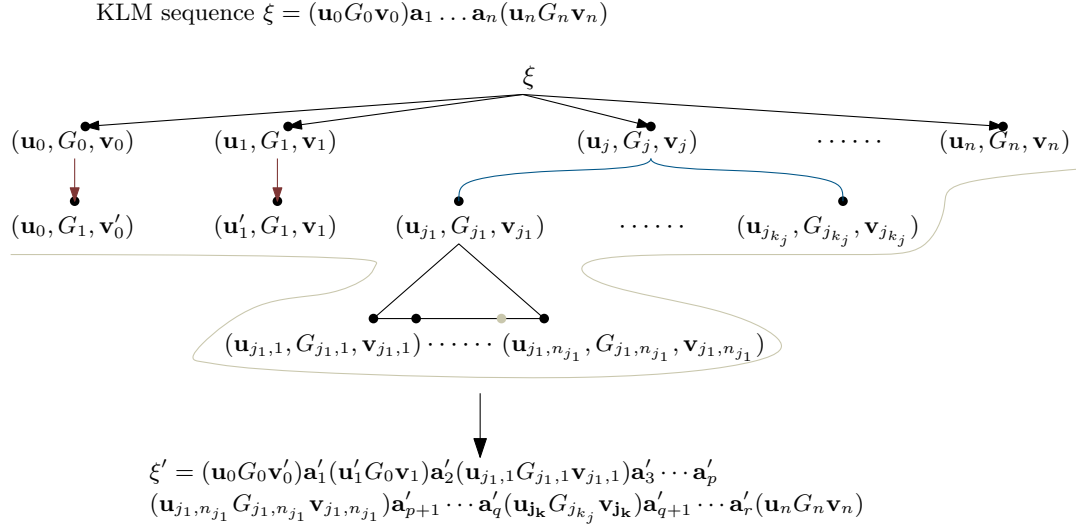
4. If there exists a triple (ξ, Λ, J) in Ξ that is not almost normal, go to Step 1 with (ξ, Λ, J) being the current triple.

We now prove the correctness of the algorithm. For convenience we assume that all the KLM sequence after UNBO is standard. Suppose

$$\Xi = \bigcup_{\xi} \left\{ (\xi, \Lambda, J) \mid \Lambda : J \rightarrow \bigcup_{j \in J} \mathcal{L}(G_j) \right\}.$$

Clearly J is completely determined by ξ , and Λ is a traversal of $\bigcup_{j \in J} \mathcal{L}(G_j)$ throughout ξ . We only need to analyze the property of ξ . The execution of the algorithm can be best explained in terms of tree growing. Take the input 3-dimensional KLM sequence $\xi = \mathbf{u} G \mathbf{v}$ as the root of the tree. This is the initial tree representation. Let n be the number of transitions of G . The ordered sequence of the leaves are the KLM sequence ξ . Here we have omitted the connecting edges \mathbf{a}_j . A leaf $\mathbf{u}_j G_j \mathbf{v}_j$ will be grown in the following fashion.

- $\mathbf{u}_j G_j \mathbf{v}_j$ is decomposed by the STAN procedure. There are two subcases.



■ **Figure 2** The tree representation of ξ' generated from ξ .

- $\mathbf{u}_j G_j \mathbf{v}_j$ is not strongly connected. It is decomposed to say $\mathbf{u}_{j_1} G_{j_1} \mathbf{v}_{j_1} \dots \mathbf{u}_{j_{n_j}} G_{j_{n_j}} \mathbf{v}_{j_{n_j}}$. The leaf has become an internal node with n_j children.
- $\mathbf{u}_j G_j \mathbf{v}_j$ is not saturated. It is decomposed to $\mathbf{u}'_j G_j \mathbf{v}'_j$ with \mathbf{u}_j and/or \mathbf{v}_j modified. The leaf has turned to an internal node with one child $\mathbf{u}'_j G_j \mathbf{v}'_j$.
- $\mathbf{u}_j G_j \mathbf{v}_j$ is decomposed by the DIMREDUCT procedure. There are two subcases.
 - $\mathbf{u}_j G_j \mathbf{v}_j$ is not unbounded. The bounded edges are singled out and a new sequence $\mathbf{u}_{j_1} G_{j_1} \mathbf{v}_{j_1} \dots \mathbf{u}_{j_{n_j}} G_{j_{n_j}} \mathbf{v}_{j_{n_j}}$ is generated. The leaf has become an internal node with n_j children.
 - $\mathbf{u}_j G_j \mathbf{v}_j$ is not pumpable. It is decomposed to $\mathbf{u}'_j G'_j \mathbf{v}'_j$ with an effectively 2-dimensional VASS G'_j . The leaf has turned to an internal node with one child $\mathbf{u}'_j G'_j \mathbf{v}'_j$.

Let $\text{TREE}(\xi)$ be the tree presentation of ξ . For a node $u = \mathbf{u}' G' \mathbf{v}'$ in the tree, we let $\text{EDGE}(u)$ be $|T_i|$ and $\omega(u)$ be the number of ω that appears in \mathbf{u}' and \mathbf{v}' . Notice that for 3-VASS one has $\omega(u) \leq 6$. The depth of u is the distance between u and root, and the depth of tree is the maximal depth of its nodes. Figure 2 provides an illustration of the tree representation. A non-leaf node in the tree may have one child as in the red branch, or many children as in the blue branch. The brown line represents the final sequence ξ' . The following is a simple observation about $\text{TREE}(\xi)$.

▷ **Claim 23.** If $\mathbf{u}_k G_k \mathbf{v}_k$ is effectively 2-dimensional, then it is a leaf of $\text{TREE}(\xi)$.

Claim 23 is the key for a not too large upper bound on the size of $\text{TREE}(\xi)$. A normal ξ_j does not grow. However the normal property could be destroyed during the execution of the algorithm. For instance, let $\mathbf{u}_k = \mathbf{v}_k = (\omega, \omega, \omega)$. If $\mathbf{u}_k G_k \mathbf{v}_k$ is standard and unbounded, it is normal. But if we modify ω by some specific value, it may not be pumpable any more. Such a modification can be caused by neighboring components of the KLM sequence. This is the key observation that leads to the TOWER size upper bound. The following lemma points out that the depth of such a tree is bounded by $6n$.

▶ **Lemma 24.** If a non-leaf node v is a child of a node u , then one of the followings is valid:

- $\text{EDGE}(v) < \text{EDGE}(u)$.
- $\text{EDGE}(v) = \text{EDGE}(u)$ and $\omega(v) < \omega(u)$.

Proof. Let u be a non-leaf node. Suppose u is decomposed by STAN. There are two cases. If u is not strongly connected, then v must be a strongly connected component in u , hence $\text{EDGE}(v) < \text{EDGE}(u)$. If u is not saturated, then v is the only child of u obtained by replacing some ω in \mathbf{x} or \mathbf{y} by a number, which implies $\text{EDGE}(v) = \text{EDGE}(u)$ and $\omega(v) < \omega(u)$. Suppose u is decomposed by DIMREDUCT. There are also two cases. If u is not pumpable, then u is decomposed in the REDU procedure. By Lemma 33 the only child v is effectively 2-dimensional, and by Claim 23 the lemma holds. If u is not unbounded, then for any child v , we have $\text{EDGE}(v) < \text{EDGE}(u)$ since u has some additional bounded edges. \blacktriangleleft

Lemma 24 bounds the depth of $\text{TREE}(\xi)$. We now give a bound on the width of the tree. Thanks to the fact that most decomposed components are effectively 2-dimensional, one has the following lemma.

► **Lemma 25.** *An internal node u in the tree has at most n non-leaf children in $\text{TREE}(\xi)$.*

The composite effect of Lemma 24 and Lemma 25 forces $\text{TREE}(\xi)$ to have both bounded depth and bounded branching number. Hence the following.

► **Lemma 26.** *Let ξ' be generated by the 3-KLMST algorithm. The followings are valid.*

1. *The number of the non-leaf nodes of ξ' is at most n^{6n} .*
2. *The size $|\xi'|$ is bounded by $f^{n^{6n}}(|\xi|)$, where $f(x) \stackrel{\text{def}}{=} x^x$.*
3. *The size of the composite characteristic system \mathcal{E}_ξ is bounded by $\exp(|\xi|)$.*

Proof. The first proposition follows directly from Lemma 24 since the depth is bounded by $6n$ and the number of the non-leaf children is bounded by n . The third proposition is also straightforward since the size of linear path scheme depends on G_i . It suffices to bound $|\xi|$. Notice that when a KLM sequence ξ is fed to STAN or DIMREDUCT, the output will see an exponential growth by $|\xi|^{|\xi|}$. So $|\xi'|$ is bounded by $f^{n^{6n}}(|\xi|)$, where $f(x) \stackrel{\text{def}}{=} x^x$. \blacktriangleleft

We can immediately establish the correctness of 3-KLMST using Lemma 26.

► **Theorem 27.** *For any 3-dimensional KLM sequence ξ , the algorithm terminates and generates a finite set of almost normal 3-dimensional KLM sequence Ξ such that*

1. $W_\xi = \bigcup_{\xi' \in \Xi} W_{\xi'}$, and that
2. *there exists a function $f \in \mathcal{F}_{<3}$ such that $|\xi'| \leq f^{n^{6n}}(|\xi|)$ for any $\xi' \in \Xi$.*

Based on Theorem 27, we finally obtain the main results of this paper.

► **Theorem 28.** *Reachability in 3-VASS is in TOWER.*

6 Conclusion

Our algorithm for the 3-VASS reachability applies the KLMST algorithm to convert an input KLM sequence nondeterministically to a KLM sequence so that every VASS in the output KLM sequence is either a 3-dimensional normal VASS or at most 2-dimensional. The key property the algorithm relies on is that in the lower dimension, the length of a witness is bounded by a function on the size of graph and is independent of the entry/exit locations. The dimension reduction methodology ought to be instructive to the complexity theoretical study of the fixed dimension VASS reachability. This is currently under investigation.

The best lower bound for the 3-VASS reachability problem is only **PSPACE**-hard. There is still a significant gap between the currently known lower bound and upper bound. It remains an open problem whether reachability in 3-VASS is elementary. But it seems unlikely that a more careful analysis of the KLMST algorithm can lead to such a result. Further research is necessary to settle the problem.

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A Integer Programming

We shall need a result in integer linear programming [21]. Let A be an $m \times k$ integer matrix and $\mathbf{x} \in \mathbb{V}^k$. The *homogeneous equation system* of A is given by the linear equation system \mathcal{E} specified by

$$A\mathbf{x} = \mathbf{0}. \quad (31)$$

A nontrivial solution to (31) is some $\mathbf{m} \in \mathbb{N}^k \setminus \{\mathbf{0}\}$ such that $A\mathbf{m} = \mathbf{0}$. The set of solutions form a monoid $(\mathcal{S}, \mathbf{0}, +)$. Since the pointwise ordering \leq is a well quasi order on \mathbb{N}^k , the set \mathcal{S} must be generated by a finite set of nontrivial minimal solutions. This finite set is called the *Hilbert base* of \mathcal{E} , denoted by $\mathcal{H}(\mathcal{E})$. The following important result is proved by Pottier [21], in which r is the *rank* of A .

► **Lemma 29** (Pottier). $\|\mathbf{m}\|_1 \leq (1 + k \cdot \|A\|_1)^r$ for every $\mathbf{m} \in \mathcal{H}(\mathcal{E})$.

Let $\mathbf{r} \in \mathbb{Z}^k$. Nonnegative integer solutions to *equation system*

$$A\mathbf{x} = \mathbf{r} \quad (32)$$

can be derived from the Hilbert base of the homogeneous equation system $A\mathbf{x} - x'\mathbf{r} = \mathbf{0}$. Let $\mathbb{S}^{\mathbf{r}}$ be the finite set of the minimal solutions to $A\mathbf{x} - x'\mathbf{r} = \mathbf{0}$ with $x' = 1$, and $\mathbb{S}^{\mathbf{0}}$ be the finite set of the minimal solutions to $A\mathbf{x} - x'\mathbf{r} = \mathbf{0}$ with $x' = 0$. A solution to (32) is of the form

$$\mathbf{m} + \sum_{i \in [\|\mathbb{S}^{\mathbf{0}}\|]} k_i \mathbf{m}_i,$$

where $\mathbf{m} \in \mathbb{S}^{\mathbf{r}}$, $\mathbf{m}_i \in \mathbb{S}^{\mathbf{0}}$ and k_i is a natural number for each $i \in [\|\mathbb{S}^{\mathbf{0}}\|]$. The following is an immediate consequence of Lemma 29.

► **Corollary 30.** $\|\mathbf{m}\|_1 \leq (1 + k \cdot \|A\|_1 + \|\mathbf{r}\|_1)^{r+1}$ for all $\mathbf{m} \in \mathbb{S}^{\mathbf{r}} \cup \mathbb{S}^{\mathbf{0}}$.

The size of (31) can be defined by $mk \log(\|A\|_1)$, and the size of (32) by $mk \log(\|A\|_1) + \|\mathbf{r}\|_1$. The size of $(1 + k \cdot \|A\|_1 + \|\mathbf{r}\|_1)^{r+1}$ is polynomial. Thus $|\mathbb{S}^{\mathbf{r}}|$ and $|\mathbb{S}^{\mathbf{0}}|$ are bounded by exponentials. In polynomial space a nondeterministic algorithm can guess a solution and check if it is minimal, hence the following.

► **Corollary 31.** Both $\mathbb{S}^{\mathbf{r}}$ and $\mathbb{S}^{\mathbf{0}}$ can be produced in $\text{poly}(n)$ space.

B

 The Redu procedure

We start with the following lemma from [18, 22].

► **Lemma 32** ([22]). *Suppose V is a d -VASS, $\mathbf{u}_0 \in \mathbb{N}_\omega^d$, and $c = |\{i | \mathbf{u}_0(i) \in \mathbb{N}\}|$. If there is a run $p_0(\mathbf{u}_0) \xrightarrow{\mathbf{a}_1}_{\mathbb{N}_\omega^d} p_1(\mathbf{u}_1) \xrightarrow{\mathbf{a}_2}_{\mathbb{N}_\omega^d} \dots \xrightarrow{\mathbf{a}_k}_{\mathbb{N}_\omega^d} p_k(\mathbf{u}_k)$ such that for any $i \in [d]$ some $j \in [k]$ exists such that $\mathbf{u}_j(i) > C^{1+c}$, where C is a natural number satisfying $C \geq |V|$, then there exists a path π such that $p_0(\mathbf{u}_0) \xrightarrow{\pi} p_0(\mathbf{u})$ and $\mathbf{u} \geq C \cdot \mathbf{1}$ with $|\pi| < C^{(c+1)^{c+1}}$.*

Lemma 32 is extremely useful in that it provides an explicit boundary for the values in any unfixed dimension of a nonpumpable KLM sequence. Suppose ξ is nonpumpable. By definition some $\mathbf{u}_j G_j \mathbf{v}_j$ is nonpumpable. Without loss of generality, we may assume that $p_j(\mathbf{m}_j)$ is not pumpable in some i -th dimension. By Lemma 32 every run from $p_j(\mathbf{m}_j)$ to $q_j(\mathbf{n}_j)$ must fall in the region:

$$\mathbb{B}_i \stackrel{\text{def}}{=} \underbrace{\mathbb{N} \times \dots \times \mathbb{N}}_{i-1 \text{ times}} \times [0, B] \times \underbrace{\mathbb{N} \times \dots \times \mathbb{N}}_{d-i \text{ times}}.$$

Here $\mathbf{m}_j, \mathbf{n}_j \in \mathbb{N}^d$ and $B \stackrel{\text{def}}{=} (2|\xi|)^{1+d^d}$. It now becomes clear how to *reduce* the dimension of a nonpumpable ξ_j . Let $G_j^{-i} = (Q_j^{-i}, T_j^{-i})$, where Q_j^{-i}, T_j^{-i} are defined as follows:

$$\begin{aligned} Q_j^{-i} &= \{(p, g) \mid p \in Q_j \text{ and } g \in [B]_0\}, \\ T_j^{-i} &= \left\{ (p, g) \xrightarrow{\mathbf{t}^{-i}} (q, g + \mathbf{t}(i)) \mid p \xrightarrow{\mathbf{t}} q \in T_j, g, g + \mathbf{t}(i) \in [B]_0 \right\}. \end{aligned}$$

In the above definition the notation \mathbf{t}^{-i} stands for the vector obtained from \mathbf{t} by removing the i -th entry, for example $(4, 3, 2, 1)^{-2} = (4, 2, 1)$. The construction of $\xi' = \mathbf{u}'_j G'_j \mathbf{v}'_j$ falls into one of the three categories:

- If $\mathbf{u}_j(i), \mathbf{v}_j(i) \in \mathbb{N}$, then $\mathbf{u}'_j = \mathbf{u}_j, \mathbf{v}'_j = \mathbf{v}_j, G'_j = G_j^{-i}, p'_j = (p_j, \mathbf{u}_j(i)), q'_j = (q_j, \mathbf{v}_j(i))$.
- If $\mathbf{u}_j(i) \in \mathbb{N}$ and $\mathbf{v}_j(i) = \omega$, then $\mathbf{u}'_j = \mathbf{u}_j, G'_j = G_j^{-i}, p'_j = (p_j, \mathbf{u}_j(i)), q'_j = (q_j, r)$ for some $r \in [B]_0$, and \mathbf{v}'_j differs from \mathbf{v}_j only in that $\mathbf{v}'_j(i) = r$.
- If $\mathbf{v}_j(i) \in \mathbb{N}$ and $\mathbf{u}_j(i) = \omega$, then $\mathbf{v}'_j = \mathbf{v}_j, G'_j = G_j^{-i}, p'_j = (p_j, r), q'_j = (q_j, \mathbf{v}_j(i))$ for some $r \in [B]_0$, and \mathbf{u}'_j differs from \mathbf{u}_j only in that $\mathbf{u}'_j(i) = r$.

In the latter two cases our reduction algorithm has to make a guess about r . Let REDU be the above procedure and $\text{REDU}(\xi)$ be the set of all KLM sequences returned by this algorithm. It is worth noting that we actually encode bounded values in one dimension into the states of the VASS. Although the VASS is d -dimensional by definition, it is geometrically $(d-1)$ -dimensional.

► **Lemma 33.** *Let $\xi' = \mathbf{x}' G' \mathbf{y}'$. If G' is d -dimensional, then the VASS produced by $\text{REDU}(\xi')$ is at most effectively $(d-1)$ -dimensional.*

C

 Missing proof in Section 4

► **Theorem 12.** *Suppose $\rho = \alpha_0 \beta_1^* \alpha_1 \dots \beta_n^* \alpha_n$ is a linear path scheme of $G = (Q, T)$, $p = \text{src}(\rho)$ and $q = \text{tgt}(\rho)$. Then \mathbf{f} is a solution to $\mathbf{y} = \mathcal{E}_\rho(\mathbf{x})$ if and only if $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$, where $\mathbf{m} = \mathbf{f}(\mathbf{x})$ and $\mathbf{n} = \mathbf{f}(\mathbf{y})$.*

Proof. We prove the “only if” part, the other implication is clear. We claim that the path defined by a solution \mathbf{f} is a witness of $p(\mathbf{m})$ to $q(\mathbf{n})$. Let $\pi = \alpha_0 \beta_1^{\mathbf{f}(\phi(\beta_1))} \alpha_1 \dots \beta_n^{\mathbf{f}(\phi(\beta_n))} \alpha_n$.

We only need to prove by induction that no configurations in the path fall outside \mathbb{N}^d . This is done by induction. Suppose $p(\mathbf{m}) \xrightarrow{\alpha_0 \beta_1^{\mathbf{f}(\phi(\beta_1))} \alpha_1 \dots \beta_k^{\mathbf{f}(\phi(\beta_k))}} p_k(\mathbf{m}_k)$ holds for k . There are two cases to consider.

■ Since the equation (13) holds, one has, for all $l \in [\alpha_k]$, the following

$$\mathbf{m}_k + \Delta(\alpha_k[1, \dots, l]) = \mathbf{m} + \sum_{i=1}^{k+1} (\Delta(\alpha_{i-1}) + \mathbf{f}(\phi(\beta_i))\Delta(\beta_i)) + \Delta(\alpha_k[1, \dots, l]) = \mathbf{f}(\mathbf{a}_{k,l}) \geq \mathbf{0}.$$

So the run of α_k that starts with $p_k(\mathbf{m}_k)$ never drops below 0.

■ Let the configuration after α_k be $p'_k(\mathbf{m}'_k)$, the equations (14) and (15) show that, for all $l \in [\beta_{k+1}]$, one has the following.

$$\mathbf{m}'_k + \Delta(\beta_{k+1}[1, \dots, l]) = \mathbf{f}(\mathbf{b}_{k+1,l}) \geq \mathbf{0},$$

$$\mathbf{m}'_k + (\mathbf{f}(\phi(\beta_{k+1})) - 1)\Delta(\beta_{k+1}) + \Delta(\beta_{k+1}[1, \dots, l]) = \mathbf{f}(\mathbf{c}_{k+1,l}) \geq \mathbf{0}.$$

The first inequality shows that $p'_k(\mathbf{m}'_k)$ can perform the first cycle β_{k+1} and the second inequality guarantees that $p'_k(\mathbf{m}'_k + (\mathbf{f}(\phi(\beta_{k+1})) - 1)\Delta(\beta_{k+1}))$ can do the last cycle β_{k+1} .

By monotonicity $p_k(\mathbf{m}'_k)$ can perform the cycle β_{k+1} for $\mathbf{f}(\phi(\beta_{k+1}))$ times.

We are done. \blacktriangleleft

► **Lemma 15.** *Let $V \subseteq \mathbb{Q}^3$ be a 2-dimensional vector space, and let $Z = (\#_1, \#_2, \#_3)$ represent an octant. If the spanned vector space of $V \cap Z$ is still 2-dimensional, then there exist distinct $i, j \in [3]$ such that $\mathbf{m} \in Z$ whenever $\mathbf{m} \in V$, $\mathbf{m}(i) \#_i 0$ and $\mathbf{m}(j) \#_j 0$.*

Proof. Without loss of generality let Z be \mathbb{N}^3 . If $V \cap \mathbb{N}^3 = \{(0, 0, 0)\}$ or V is one of the coordinate planes, the lemma is valid. Assume that $V \cap Z \supsetneq \{(0, 0, 0)\}$ and that V is none of the coordinate planes. Suppose V is spanned by $\{(a_1, a_2, a_3), (b_1, b_2, b_3)\}$. Then by Cramer rule V is the hyperplane defined by

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} x + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} y + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} z = 0. \quad (33)$$

Let \mathbf{I}_{xy} , respectively \mathbf{I}_{xz} , \mathbf{I}_{yz} , be the projection on V and the xOy plane, respectively the xOz plane, the yOz plane. Thinking of a point in the space as a direction vector,

$$\begin{aligned} \mathbf{I}_{xy} &= \text{span} \left\{ \left(\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, 0 \right) \right\}, \\ \mathbf{I}_{yz} &= \text{span} \left\{ \left(0, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \right) \right\}, \\ \mathbf{I}_{xz} &= \text{span} \left\{ \left(\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix}, 0, \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \right) \right\}. \end{aligned}$$

Setting $a_1 = b_1 = 1$. If one of the direction vectors is an axis, the lemma holds. Otherwise $a_2 \neq b_2$, $a_3 \neq b_3$ and $\frac{a_2}{b_2} \neq \frac{a_3}{b_3}$. In this case the direction vector can be defined by

$$I_{xy} = \left(1, \frac{a_2 b_3 - a_3 b_2}{b_3 - a_3}, 0 \right), \quad I_{yz} = \left(0, 1, \frac{b_3 - a_3}{b_2 - a_2} \right), \quad I_{xz} = \left(\frac{a_2 - b_2}{a_2 b_3 - a_3 b_2}, 0, 1 \right). \quad (34)$$

Observe that $I_{xy}(2) \cdot I_{yz}(3) \cdot I_{xz}(1) = -1 < 0$, implying that either all the three are less than 0 or one of them is less than 0. Since we are considering $V \cap \mathbb{N}^3$, the first case is impossible.

Suppose $I_{xy}(2), I_{yz}(3) > 0$. Then for all $\mathbf{m} \in V$, the inequalities $\mathbf{m}(1), \mathbf{m}(3) > 0$ implies $\mathbf{m}(2) > 0$. This is because \mathbf{m} can be represented as $cI_{xy} + dI_{yz}$ for some $c, d \in \mathbb{Q}$. Thus

$$\mathbf{m} = cI_{xy} + dI_{yz} = \left(c, \frac{a_2b_3 - a_3b_2}{b_3 - a_3} \cdot c + d, \frac{b_3 - a_3}{b_2 - a_2} \cdot d \right) > \mathbf{0} \quad (35)$$

if and only if $c, d > 0$. That completes the proof. \blacktriangleleft

► **Theorem 14.** Suppose $G = (Q, T)$ is an effectively 2-dimensional 3-VASS and $\mathbf{m}, \mathbf{n} \in \mathbb{D}^3$.

- If $q(\mathbf{m}) \rightarrow_{\mathbb{N}^3} q(\mathbf{n})$, then there exists a linear path scheme ρ with at most two cycles such that $|\rho| \leq 2^{O(|G|)}$ and $q(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$.
- If $p(\mathbf{m}) \rightarrow_{\mathbb{D}^3} q(\mathbf{n})$, then there exists a linear path scheme ρ with at most $2|Q|$ cycles such that $|\rho| \leq 2^{O(|G|)}$ and $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$.

Proof. To prove the first proposition, consider the octant $Z = (\#_1, \#_2, \#_3)$ to which the vector $\mathbf{n} - \mathbf{m}$ belongs. Let the dimension of spanned space of $V_G \cap Z$ be still 2, otherwise the proposition is trivial. By Lemma 15 we may assume that for all $\mathbf{w} \in V_G$, $\mathbf{w}(3) \#_3 0$ whenever $\mathbf{w}(1) \#_1 0$ and $\mathbf{w}(2) \#_2 0$. Let V_G be spanned by $(1, 0, a), (0, 1, b)$ for some $a, b \in \mathbb{Q}$. Let $f : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$ be the projection function defined by $f((x_1, x_2, x_3)) = (x_1, x_2)$. The 2-VASS $G_{xy} = (Q_{xy}, T_{xy})$ is defined by

- $Q_{xy} = Q$, and
- $T_{xy} = \{(p, f(\mathbf{a}), q) \mid (p, \mathbf{a}, q) \in T\}$.

Since $q(\mathbf{m}) \xrightarrow{\pi}_{\mathbb{N}^3} q(\mathbf{n})$ in G for some path π , we have $q(f(\mathbf{m})) \xrightarrow{\pi'}_{\mathbb{N}^2} q(f(\mathbf{n}))$ in G_{xy} for some path π' . We extend f to the map $T \rightarrow T_{xy}$. So we may write for example $q(f(\mathbf{m})) \xrightarrow{f(\pi)}_{\mathbb{N}^2} q(f(\mathbf{n}))$. For fixed state q , the map f is bijective. Clearly we can let $\pi' = f(\pi)$.

Since $f(\mathbf{m}), f(\mathbf{n}) \in \mathbb{D}^2$, there exists a zigzag free linear path scheme ρ_{xy} with at most two cycles such that $|\rho_{xy}| \leq 2^{O(|G|)}$ and $q(f(\mathbf{m})) \xrightarrow{\rho_{xy}} q(f(\mathbf{n}))$. Let ρ_{xy} be $\alpha_0\beta_1^*\alpha_1\beta_2^*\alpha_2$ and let π_{xy} be $\alpha_0\beta_1^{e_1}\alpha_1\beta_2^{e_2}\alpha_2$. Evidently $\Delta(\pi_{xy}) = \Delta(\pi') = \Delta(f(\pi)) = f(\Delta(\pi))$ and $f^{-1}(\beta_i)$ belongs to the same octant. We also need to prove $\Delta(f^{-1}(\pi_{xy})) = \Delta(\pi)$, which boils down to proving the equality in the third dimension. This is obvious since both π and $f^{-1}(\pi_{xy})$ are cycles of G . Consequently,

$$\Delta(f^{-1}(\pi_{xy}))(3) = a\Delta(\pi_{xy})(1) + b\Delta(\pi_{xy})(2) = a\Delta(\pi)(1) + b\Delta(\pi)(2) = \Delta(\pi)(3).$$

Now we prove that $q(\mathbf{m}) \xrightarrow{f^{-1}(\pi_{xy})} q(\mathbf{n})$. This is because of the following two facts.

1. $\mathbf{n} = \mathbf{m} + \Delta(f^{-1}(\pi_{xy}))$.
2. $q(\mathbf{m})$ admits the path $f^{-1}(\pi_{xy})$. Due to the construction of π_{xy} , values in the first two dimensions never go below 0. The third dimension is maintained because of the monotonicity guaranteed by the zigzag free property by Lemma 15 and the following inequality holds for every $i \in [|f^{-1}(\rho_{xy})|]$.

$$\mathbf{m}(3) + \Delta(f^{-1}(\alpha_0\beta_1\alpha_1\beta_2\alpha_2)[1, \dots, i])(3) \geq \mathcal{D} + \Delta(f^{-1}(\alpha_0\beta_1\alpha_1\beta_2\alpha_2)[1, \dots, i])(3) \geq 0.$$

This completes the proof of the first proposition. The second proposition follows the same line of reasoning as in the 2-dimensional case. Assuming that $p(\mathbf{m}) \rightarrow_{\mathbb{D}^3} q(\mathbf{n})$, there exists a path

$$p(\mathbf{m}) \xrightarrow{\alpha_0} q_1(\mathbf{m}_1) \xrightarrow{\beta_1} q_1(\mathbf{m}'_1) \xrightarrow{\alpha_1} \dots q_k(\mathbf{m}_k) \xrightarrow{\beta_k} q_k(\mathbf{m}'_k) \xrightarrow{\alpha_{k+1}} q(\mathbf{n})$$

such that

- $\mathbf{m}_i, \mathbf{m}'_i \in \mathbb{D}^3$, and, for all $i \in [k]$,
- $|\alpha_i| \leq |Q|$ and $q_i(\mathbf{m}'_i)$ is the last configuration in which q_i appears.

This implies $k \leq |Q|$. Applying the first proposition, each β_i can be replaced by a linear path scheme $\alpha_{i,0}\beta_{i,1}^*\alpha_{i,1}\beta_{i,2}^*\alpha_{i,2}$. Thus the path can be replaced by a linear path scheme ρ where $|\rho| \leq 2^{O(|G|)}$, and ρ has no more than $2|Q|$ cycles. This completes the proof. \blacktriangleleft

► **Theorem 17.** *Suppose G is an effectively 2-dimensional 3-VASS and $\mathbf{m}, \mathbf{n} \in \mathbb{L}_{\mathcal{D}}$. Then $p(\mathbf{m}) \rightarrow_{\mathbb{L}_{\mathcal{D}}} q(\mathbf{n})$ if and only if there exists a linear path scheme ρ with size $|\rho| \leq (\mathcal{D} + |G|)^{O(1)}$ such that $p(\mathbf{m}) \xrightarrow{\rho} q(\mathbf{n})$.*

Proof. Consider the vector space V_G . If it is one of the coordinate planes, say the xOy -plane, then a walk within $\mathbb{L}_{\mathcal{D}}$ is also a walk within $\mathbb{N}^2 \times [\mathcal{D}]$. We are done by applying Theorem 16. In the rest of the proof we assume that V_G is not parallel to any coordinate plane. Let

$$\mathbb{J}_{12} = [\mathcal{D}]^2 \times \mathbb{N}, \mathbb{J}_{13} = [\mathcal{D}] \times \mathbb{N} \times [\mathcal{D}], \mathbb{J}_{23} = \mathbb{N} \times [\mathcal{D}]^2.$$

Recall that $\mathbb{I}_1 = [\mathcal{D}] \times \mathbb{N}^2$, $\mathbb{I}_2 = \mathbb{N} \times [\mathcal{D}] \times \mathbb{N}$, $\mathbb{I}_3 = \mathbb{N}^2 \times [\mathcal{D}]$ we clearly have $\mathbb{J}_{ij} = \mathbb{I}_i \cap \mathbb{I}_j$. These are the regions of crossing points. A walk π in $\mathbb{L}_{\mathcal{D}}$ from $p(\mathbf{m})$ to $q(\mathbf{n})$ may pass many crossing points. It is generally of the following form

$$p(\mathbf{m}) \xrightarrow{\pi_1}_{\mathbb{B}_1} p_1(\mathbf{m}_1) \xrightarrow{\pi_2}_{\mathbb{B}_2} p_2(\mathbf{m}_2) \xrightarrow{\pi_3}_{\mathbb{B}_3} \cdots \xrightarrow{\pi_k}_{\mathbb{B}_k} p_k(\mathbf{m}_k) \xrightarrow{\pi_{k+1}}_{\mathbb{B}_{k+1}} q(\mathbf{n}), \quad (36)$$

where

- $\mathbb{B}_i \in \{\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3\}$ all $i \in [k+1]$,
- $\mathbb{B}_{i+1} \neq \mathbb{B}_i$ for all $i \in [k]$, and
- $\mathbf{m}_i \in \{\mathbb{J}_{12}, \mathbb{J}_{13}, \mathbb{J}_{23}\}$ for all $i \in [k]$.

If $k \leq 3 \cdot |Q| \cdot (\mathcal{D} + 1)^2$, then since by Theorem 16 each π_j can be represented by a linear path scheme with length bound $(\mathcal{D} + |G|)^{O(1)}$, the path π itself is then represented by a linear path scheme of length bounded by $(\mathcal{D} + |G|)^{O(1)}$. Next suppose $k > 3 \cdot |Q| \cdot (\mathcal{D} + 1)^2$. By the pigeon hole principle there must exist $i < j$ such that $p_i = p_j$ and at least two of the equalities $\mathbf{m}_i(1) = \mathbf{m}_j(1)$, $\mathbf{m}_i(2) = \mathbf{m}_j(2)$, $\mathbf{m}_i(3) = \mathbf{m}_j(3)$ are valid. Without loss of generality assume that $\mathbf{m}_i(1) = \mathbf{m}_j(1)$ and $\mathbf{m}_i(2) = \mathbf{m}_j(2)$. Let $\mathbf{a} = (0, 0, a) = \mathbf{m}_j - \mathbf{m}_i$ be the displacement of the cycle. By definition $\mathbf{a} \in V_G$. If also $|\{i \mid \mathbf{m}_i \in \mathbb{J}_{13}\}| > |Q| \cdot (\mathcal{D} + 1)^2$, there would be some displacement $(0, b, 0) \in V_G$, contradicting to the assumption that V_G is not a coordinate plane. The argument is also valid for the set $\{i \mid \mathbf{m}_i \in \mathbb{J}_{23}\}$. Therefore

$$|\{i \mid \mathbf{m}_i \in \mathbb{J}_{13}\}| \leq |Q| \cdot (\mathcal{D} + 1)^2, \quad (37)$$

$$|\{i \mid \mathbf{m}_i \in \mathbb{J}_{23}\}| \leq |Q| \cdot (\mathcal{D} + 1)^2. \quad (38)$$

Let's take a look at the sub-walks $\pi_1, \dots, \pi_k, \pi_{k+1}$. We are interested in those sub-walks that are in \mathbb{I}_1 and/or \mathbb{I}_2 . There are three cases.

1. If a walk in \mathbb{I}_1 is of length at least $|Q| \cdot (\mathcal{D} + 1)$, it must contain a sub-walk from some $p'(\mathbf{m}')$ to some $p''(\mathbf{m}'')$ such that $\mathbf{m}'(1) = \mathbf{m}''(1)$. The equality $\mathbf{m}'(2) = \mathbf{m}''(2)$ must also be valid otherwise it again would contradict to the assumption that V_G is not parallel to any of the coordinate planes. This observation implies that if $p_j(\mathbf{m}_j) \xrightarrow{\pi_{j+1}}_{\mathbb{B}_{j+1}} p_{j+1}(\mathbf{m}_{j+1})$ is in \mathbb{I}_1 (i.e. $\mathbb{B}_{j+1} = \mathbb{I}_1$), then π_{j+1} must be in the region

$$[\mathcal{D}] \times [\max\{0, \mathbf{m}_j(2) - \mathcal{D}_1\}, \mathbf{m}_j(2) + \mathcal{D}_1] \times \mathbb{N},$$

where $\mathcal{D}_1 \stackrel{\text{def}}{=} \|T\| \cdot |Q| \cdot (\mathcal{D} + 1)$. In other words π_{j+1} is essentially one dimensional and can be represented by a linear path scheme of length bounded by $(\mathcal{D} + |G|)^{O(1)}$.

2. For the same reason if $p_j(\mathbf{m}_j) \xrightarrow{\pi_{j+1}}_{\mathbb{B}_{j+1}} p_{j+1}(\mathbf{m}_{j+1})$ is in \mathbb{I}_2 , then π_{j+1} must be in the region

$$[\max\{0, \mathbf{m}_j(1) - \mathcal{D}_1\}, \mathbf{m}_j(1) + \mathcal{D}_1] \times [\mathcal{D}] \times \mathbb{N},$$

and π_{j+1} can be represented by a linear path scheme of length bounded by $(\mathcal{D} + |G|)^{O(1)}$.

3. Finally consider a sub-walk $p_i(\mathbf{m}_i) \xrightarrow{\pi_{i+1}}_{\mathbb{B}_{i+1}} p_{i+1}(\mathbf{m}_{i+1}) \xrightarrow{\pi_{i+2}}_{\mathbb{B}_{i+2}} \dots \xrightarrow{\pi_j}_{\mathbb{B}_j} p_j(\mathbf{m}_j)$ of (36), where $\mathbb{B}_{i+1}, \mathbb{B}_{i+2}, \dots, \mathbb{B}_j$ is a proper alternating sequence of $\mathbb{I}_1, \mathbb{I}_2$, and $\mathbf{m}_i, \mathbf{m}_j \in \{\mathbb{J}_{13}, \mathbb{J}_{23}\}$. If $\mathbf{m}_i \in \mathbb{J}_{13}$, then $\mathbf{m}_i(2) < \mathcal{D}_1$; otherwise the sub-walk would be completely in \mathbb{I}_1 . If $\mathbf{m}_i \in \mathbb{J}_{23}$, then $\mathbf{m}_i(1) < \mathcal{D}_1$; otherwise the sub-walk would be completely in \mathbb{I}_2 . It should then be clear that the sub-walk must be in the region

$$([\mathcal{D}] \times [0, 2\mathcal{D}_1] \times \mathbb{N}) \cup ([0, 2\mathcal{D}_1] \times [\mathcal{D}] \times \mathbb{N}),$$

which is essentially one-dimensional. So $p_i(\mathbf{m}_i) \xrightarrow{\pi_{i+1}}_{\mathbb{B}_{i+1}} p_{i+1}(\mathbf{m}_{i+1}) \xrightarrow{\pi_{i+2}}_{\mathbb{B}_{i+2}} \dots \xrightarrow{\pi_j}_{\mathbb{B}_j} p_j(\mathbf{m}_j)$ can be represented by a linear path scheme of length bounded by $(\mathcal{D} + |G|)^{O(1)}$.

In summary the bound in (37) and (38) and the above analysis tell us that (36) is the concatenation of at most $2 \cdot |Q| \cdot (\mathcal{D} + 1)^2 + 1$ linear path schemes, each of length bounded by $(\mathcal{D} + |G|)^{O(1)}$. We are done. \blacktriangleleft

D Missing proof in Section 5

► **Theorem 21.** *The almost normal KLM sequence ξ has a witness bounded by $|\xi|^{3|\xi|}$.*

Proof. Let $J \subseteq [n]_0$ be the index set of G_j that are effectively 2-dimensional. By Theorem 18 for each $j \in J$ there exists a linear path scheme $\rho_j \in \mathcal{L}(G_j)$ such that $p_j(\hat{\mathbf{x}}_j) \xrightarrow{\rho_j} q_j(\hat{\mathbf{y}}_j)$ if $p_j(\hat{\mathbf{x}}_j) \xrightarrow{G_j} q_j(\hat{\mathbf{y}}_j)$. Now let $J = \{j_1, \dots, j_k\}$ and $\Lambda = \rho_{j_1} \dots \rho_{j_k}$ be the linear path scheme sequence. Consider the composite characteristic system $\mathcal{C}_{\xi, \Lambda}$. Let $\mathbf{h} = (\bar{\mathbf{x}}_0, \bar{\phi}_0, \bar{\mathbf{y}}_0) \dots (\bar{\mathbf{x}}_n, \bar{\phi}_n, \bar{\mathbf{y}}_n)$ be a minimal solution to $\mathcal{C}_{\xi, \Lambda}$ and $\mathbf{h}^0 = (\mathbf{x}_0^0, \bar{\phi}_0^0, \mathbf{y}_0^0) \dots (\mathbf{x}_n^0, \bar{\phi}_n^0, \mathbf{y}_n^0)$ be a solution to the homogeneous system $\mathcal{C}_{\xi, \Lambda}^0$. By Theorem 1 these systems render true the following statements.

- $|\mathbf{h}| \leq |\xi|^{|\xi|-1}, |\mathbf{h}_0| \leq |\xi|^{|\xi|-2}$.
- For every $j \notin J$, the followings are valid.
 - For every $t \in T_j$, $\bar{\phi}_j(t) > 0$ and $\bar{\phi}_j^0(t) > 0$.
 - For every $k \in [3]$, $\mathbf{u}_j[k] = \omega$ implies $\bar{\mathbf{x}}_j^0[k] > 0$.
 - For every $k \in [3]$, $\mathbf{v}_j[k] = \omega$ implies $\bar{\mathbf{y}}_j^0[k] > 0$.

Now consider a normal $\xi_k = \mathbf{u}_k G_k \mathbf{v}_k$. By Theorem 7 there exists $r \in \mathbb{N}$ such that

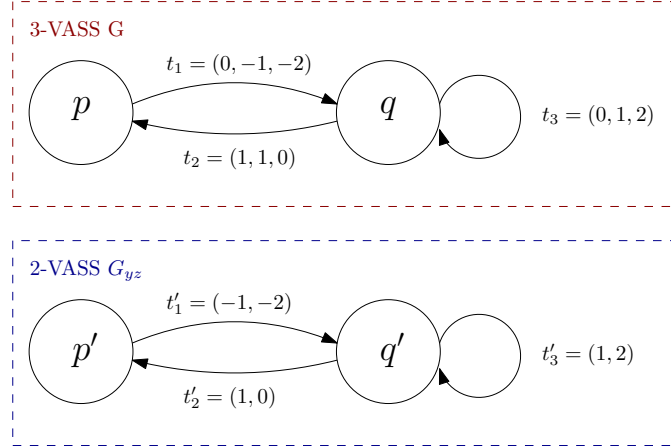
$$p_k(\bar{\mathbf{x}}_k + r\bar{\mathbf{x}}_k^0) \xrightarrow{G_k} q_k(\bar{\mathbf{y}}_k + r\bar{\mathbf{y}}_k^0). \quad (39)$$

Let π_k be the above walk. Clearly $|\pi_k| \leq |\mathbf{h}| + r|\mathbf{h}^0|$. Next suppose $\mathbf{u}_k G_k \mathbf{v}_k$ is effectively 2-dimensional. Let $\tilde{\eta}_k = (\bar{\mathbf{x}}_k, \bar{\phi}_k, \bar{\mathbf{y}}_k) + r(\mathbf{x}_k^0, \bar{\phi}_k^0, \mathbf{y}_k^0) = (\tilde{\mathbf{x}}_k, \tilde{\phi}_k, \tilde{\mathbf{y}}_k)$. Clearly $\tilde{\eta}_k$ is a solution to the linear path scheme system \mathcal{C}_{ρ_k} , that is $\tilde{\mathbf{y}}_k = \mathcal{C}_{\rho_k}(\tilde{\mathbf{x}}_k)$, where $\rho_k = \alpha_0^k(\beta_1^k)^* \alpha_1^k(\beta_2^k)^* \dots (\beta_{n_k}^k)^* \alpha_{n_k}^k$. The walk π_k is of the following form

$$p_k(\tilde{\mathbf{x}}_k) \xrightarrow{\alpha_0^k} \xrightarrow{(\beta_1^k)^* \tilde{\phi}_k(\beta_1^k)} \xrightarrow{\alpha_1^k} \dots \xrightarrow{(\beta_{n_k}^k)^* \tilde{\phi}_k(\beta_{n_k}^k)} \xrightarrow{\alpha_{n_k}^k} q_k(\tilde{\mathbf{y}}_k). \quad (40)$$

In summary there exists a walk $\pi = \pi_0 \pi_1 \dots \pi_n$ that satisfies

- $\delta(\pi_k) = \tilde{\phi}_k$ for all $k \in [n]_0$, and
- $p_0(\tilde{\mathbf{x}}_0) \xrightarrow{\pi} q_n(\tilde{\mathbf{y}}_n)$.



■ **Figure 3** An Example of Effectively 2-dimensional 3-VASS

Since $r \leq |\xi|^{2|\xi|}$, the walk π is bounded by $|\xi|^{3|\xi|}$ in length. We are done. ◀

► **Lemma 25.** *An internal node u in the tree has at most n non-leaf children in $TREE(\xi)$.*

Proof. Let u be a non-leaf node. By the definition of 3-KLMST there are two cases.

- u is decomposed by STAN. If u is not strongly connected, then since the directed graph G with n edges has at most n distinct strongly connected components, the number of the children of u is at most n . If u is not saturated, then u has only one child v .
- u is decomposed by DIMREDUCT. If u is not pumpable, then u is decomposed in the REDU procedure and has only one child. In another case, u is regular but not unbounded. Furthermore, u is effectively 3-dimensional and strongly connected, with a rank of u in the form of $(r, 0, 0, 0)$ where $r < n$. Since the UNBO procedure strictly reduces the rank of u , the generated standard KLM sequence $(\mathbf{u}_{i_1} G_{i_1} \mathbf{v}_{i_1}) \mathbf{a}_{i_1} \dots (\mathbf{u}_{i_j} G_{i_j} \mathbf{v}_{i_j})$ has at most $(r - 1)$ value in the first coordinate, which means that there are at most $(r - 1)$ children that are effectively 3-dimensional. By definition, u has at most $r - 1 < n$ non-leaf nodes.

We are done. ◀

► **Theorem 28.** *Reachability in 3-VASS is in TOWER.*

Proof. By theorem 27, if $q(\mathbf{n})$ is reachable from $p(\mathbf{m})$, the 3-KLMST algorithm will produce from $\xi = \mathbf{u}G\mathbf{v}$ an almost normal KLM sequence ξ' with $\mathbf{u} = \mathbf{m}$ and $\mathbf{v} = \mathbf{n}$. By Theorem 21 there exists a witness π from $p(\mathbf{m})$ to $q(\mathbf{n})$ with $|\pi| \leq |\xi'|^{3|\xi'|}$. Since $|\xi'|$ is bounded by $f^{n^{6n}}(|\xi|)$ where $f(x) \stackrel{\text{def}}{=} x^x$, the size of $|\xi'|$ is TOWER size of the input, so is the size of π . Thus the walk can be guessed nondeterministically in $TOWER(|\xi|)$ time. ◀

E An Example of Effectively 2-Dimensional 3-VASS

Figure 3 shows an example of effectively 2-dimensional 3-VASS G . G contains two states p, q and three transitions $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$. Clearly V_G can be spanned by the following two vectors $(1, 0, -2)$ and $(0, 1, 2)$. Now consider the path from $p(\mathbf{u})$ to $q(\mathbf{v})$, where $\mathbf{u} = (2, 2, 2)$ and $\mathbf{v} = (22, 21, 0)$. It can be generated from a 2-VASS. Let Z be the octant that $\mathbf{v} - \mathbf{u}$ belongs. It's evident that $V_G \cap Z$ is above the yOz plane, which means for any $\mathbf{x} \in V_G$, $\mathbf{x}(2) > 0, \mathbf{x}(3) < 0$ implies $\mathbf{x}(1) > 0$. Now consider the 2-VASS G_{yz} that is project G to the yOz plane. The related projection function is $f(\mathbf{x}) = (\mathbf{x}(2), \mathbf{x}(3))$ and we can conclude

$f^{-1}(\mathbf{x}) = (\mathbf{x}(1), \mathbf{x}(2), 2(\mathbf{x}(2) - 2\mathbf{x}(1)))$. Since there exists the following path π from $p'(f(\mathbf{u}))$ to $q'(f(\mathbf{v}))$:

$$p'(2, 2) \xrightarrow{\mathbf{t}'_1} q'(1, 0) \xrightarrow{(\mathbf{t}'_3)^{20}} q'(21, 40) \xrightarrow{\mathbf{t}'_2} p'(22, 40) \xrightarrow{(\mathbf{t}'_1 \mathbf{t}'_2)^{19}} p'(22, 2) \xrightarrow{\mathbf{t}'_1} q'(21, 0)$$

Then in G there exists a path $f^{-1}(\pi)$ from $p(\mathbf{u})$ to $q(\mathbf{v})$:

$$p(2, 2, 2) \xrightarrow{\mathbf{t}_1} q(2, 1, 0) \xrightarrow{(\mathbf{t}_3)^{20}} q(2, 21, 40) \xrightarrow{\mathbf{t}_2} p(3, 22, 40) \xrightarrow{(\mathbf{t}_1 \mathbf{t}_2)^{19}} p(22, 22, 2) \xrightarrow{\mathbf{t}_1} q(22, 21, 0).$$

F The Bad Case for KLMST in 3-VASS

We give an example of 3-VASS, which can produce k -exponential size even we only decompose them due to the saturation property and unboundedness property. For convenience, in the following we use the same name to denote one node or transition, even they're not in the same VASS.

Fixed an integer k and consider the following 3-VASS G_k in Figure 4: G_k has $3k$ states, with the input state p_1 and the output state r_k . The displacement of the transitions is defined as follows:

- For $i \in [k]$, we have $\mathbf{t}_{i,1} = (-1, 1, 0)$, $\mathbf{t}_{i,2} = (0, 0, 0)$, $\mathbf{t}_{i,3} = (2, -1, 0)$, $\mathbf{t}_4 = (0, 0, -1)$.
- For $i \in [k-1]$, we have $\mathbf{t}_{i,5} = (-1, 0, 1)$ and $\mathbf{t}_{k,5} = (-1, 0, 0)$ for k .
- For $i \in [k-1]$, we have $\mathbf{a}_{2i-1} = (0, 0, 0)$ and $\mathbf{a}_{2i} = (1, 0, 0)$.

Now let $\mathbf{m} = (1, 0, n)$ and $\mathbf{n} = (0, 0, 0)$ where $n > k$. We use the KLMST algorithm to consider whether $p_1(\mathbf{m}) \xrightarrow{G_k} r_k(\mathbf{n})$. Since G_k is not strongly connected, it is equivalent to ask whether the following KLM sequence ξ has a witness:

$$\xi = (\mathbf{u}_0 G_0 \mathbf{v}_0) \mathbf{a}_1 (\mathbf{u}_1 G_1 \mathbf{v}_1) \mathbf{a}_2 (\mathbf{u}_2 G_2 \mathbf{v}_2) \cdots (\mathbf{u}_{2k-1} G_{2k-1} \mathbf{v}_{2k-1}) \mathbf{a}_{2k} (\mathbf{u}_{2k} G_{2k} \mathbf{v}_{2k}).$$

Here:

- $\mathbf{u}_0 = (1, 0, n)$ and for any $i \in [2k]$, $\mathbf{u}_i = (\omega, \omega, \omega)$.
- $\mathbf{v}_0 = (0, 0, 0)$ and for any $i \in [2k]$, $\mathbf{v}_i = (\omega, \omega, \omega)$.
- For $i \in [k]$, we have $G_{2i-1} = H_i$, with the input state p_i and the output state q_i .
- For $i \in [k-1]$, we have $G_{2i} = I_i$ and $G_{2k} = I_k$.

ξ has a witness since there exists a run of the form

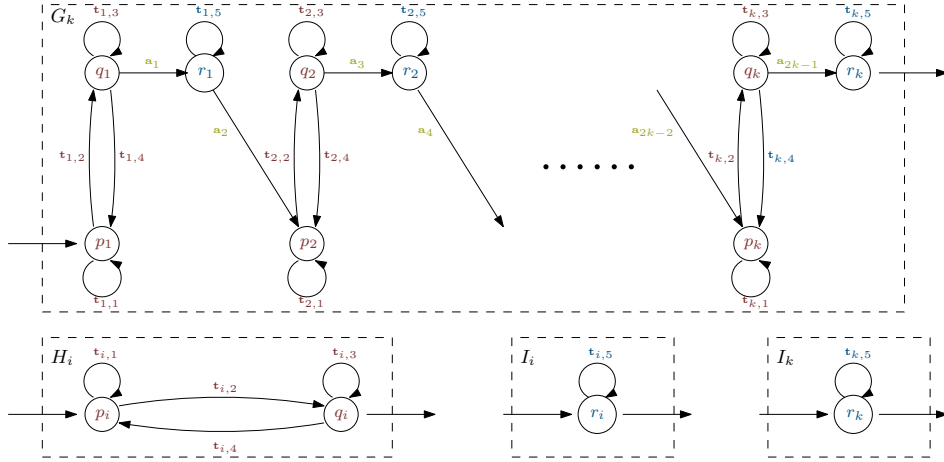
$$\rho = \mathbf{t}_{1,2} \mathbf{a}_1 \mathbf{a}_2 \mathbf{t}_{1,2}, \dots, \mathbf{a}_{2k-2} (\mathbf{t}_{k,2} \mathbf{t}_{k,4})^n \mathbf{t}_{k,2} (\mathbf{t}_{k,5})^{k-1}.$$

Hence ξ is satisfied. Now we consider its decomposition. Let Ξ be the set of the KLM sequence during the decomposition. Since there is no path π in H_0 such that $\Delta(\delta(\pi))(3) > 0$, there doesn't exist one solution h^0 for the homogeneous characteristic system such that $h^0(\mathbf{v}_0)(3) > 0$, which means ξ is not saturated. By the STAN procedure, there exists such a KLM sequence $\xi' \in \Xi$:

$$\xi' = (\mathbf{u}_0 G_0 \mathbf{v}'_0) \mathbf{a}_1 (\mathbf{u}_1 G_1 \mathbf{v}_1) \mathbf{a}_2 (\mathbf{u}_2 G_2 \mathbf{v}_2) \cdots (\mathbf{u}_{2k-1} G_{2k-1} \mathbf{v}_{2k-1}) \mathbf{a}_{2k} (\mathbf{u}_{2k} G_{2k} \mathbf{v}_{2k}).$$

with the only modification of $\mathbf{v}'_0 = (\omega, \omega, 0)$. Now we focus on the first part of ξ' . Due to the value of $\mathbf{v}'_0(3) = 0$, we can conclude that $\mathbf{t}_{0,2}$ is just performed n times. Hence it's not unbounded. By the UNBO procedure, $\mathbf{u}_0 G_0 \mathbf{v}'_0$ would be decomposed to the following KLM sequence:

$$\xi_{1,0} = (\tilde{\mathbf{u}}_0 \tilde{G}_0 \tilde{\mathbf{v}}_0) \tilde{\mathbf{a}}_1 (\tilde{\mathbf{u}}_1 \tilde{G}_1 \tilde{\mathbf{v}}_1) \tilde{\mathbf{a}}_2 (\tilde{\mathbf{u}}_2 \tilde{G}_2 \tilde{\mathbf{v}}_2) \cdots (\tilde{\mathbf{u}}_{2n-1} \tilde{G}_{2n-1} \tilde{\mathbf{v}}_{2n-1}) \tilde{\mathbf{a}}_{2n} (\tilde{\mathbf{u}}_{2n} \tilde{G}_{2n} \tilde{\mathbf{v}}_{2n}).$$



■ **Figure 4** An Example of 3-VASS that using KLMST algorithm would be very complex

Where G_{2i-1} (resp. G_{2i}) is the graph induce by H_0 with the only node p_0 (resp. q_0) and the only transition $\mathbf{t}_{0,1}$ (resp. $\mathbf{t}_{0,3}$). And we have $\tilde{\mathbf{a}}_{2i-1} = (0, 0, 0)$ and $\tilde{\mathbf{a}}_{2i} = (0, 0, -1)$ for every $i \in [k]$. Since every \tilde{G}_{2i} is a one-transition graph and there're no transition $\mathbf{t} \in \mathbb{N}^3 \cup (-\mathbb{N})^3$, $\xi_{1,0}$ is not normal any all. Hence, by guessing all the possible values in $\xi_{1,0}$, finally there exists one KLM sequence ξ_2 in Ξ :

$$\xi_1 = (\mathbf{u}_0 \xi_{1,0} \mathbf{v}_{0,final}) \mathbf{a}_1 (\mathbf{u}_1 G_1 \mathbf{v}_1) \mathbf{a}_2 (\mathbf{u}_2 G_2 \mathbf{v}_2) \cdots (\mathbf{u}_{2k-1} G_{2k-1} \mathbf{v}_{2k-1}) \mathbf{a}_{2k} (\mathbf{u}_{2k} G_{2k} \mathbf{v}_{2k}).$$

where $\mathbf{v}_{0,final} = (2^n, 0, 0)$. Hence by the connection of ξ_2 , we can obtain $\mathbf{u}_1 = (2^n, 0, 0)$. With the same reason as above, $\mathbf{u}_1 G_1 \mathbf{v}_1$ is not normal, hence there exists ξ'_2 in Ξ like the following form:

$$\xi_2 = (\mathbf{u}_0 \tilde{\xi}_1 \mathbf{v}_{1,final}) \mathbf{a}_2 (\mathbf{u}_2 G_2 \mathbf{v}_2) \cdots (\mathbf{u}_{2k-1} G_{2k-1} \mathbf{v}_{2k-1}) \mathbf{a}_{2k} (\mathbf{u}_{2k} G_{2k} \mathbf{v}_{2k}).$$

Here $\mathbf{v}_{1,final} = (0, 0, 2^n)$, and hence $\mathbf{u}_2 = (1, 0, 2^n)$ immediately. Since G_2 is completely the same as G_0 , we can decompose the ξ_2 repeatedly as the following form:

$$\begin{aligned} \xi_3 &= (\mathbf{u}_0 \tilde{\xi}_2 \mathbf{v}_{2,final}) \mathbf{a}_2 (\mathbf{u}_3 G_3 \mathbf{v}_3) \cdots (\mathbf{u}_{2k-1} G_{2k-1} \mathbf{v}_{2k-1}) \mathbf{a}_{2k} (\mathbf{u}_{2k} G_{2k} \mathbf{v}_{2k}) \\ \xi_4 &= (\mathbf{u}_0 \tilde{\xi}_3 \mathbf{v}_{3,final}) \mathbf{a}_2 (\mathbf{u}_3 G_3 \mathbf{v}_3) \cdots (\mathbf{u}_{2k-1} G_{2k-1} \mathbf{v}_{2k-1}) \mathbf{a}_{2k} (\mathbf{u}_{2k} G_{2k} \mathbf{v}_{2k}) \\ &\cdots \\ \xi_{2k-1} &= (\mathbf{u}_0 \tilde{\xi}_{2k-1} \mathbf{v}_{2k-1,final}) \mathbf{a}_{2k} (\mathbf{u}_{2k} G_{2k} \mathbf{v}_{2k}) \end{aligned}$$

where $\mathbf{v}_{2i-2,final} = (2^{\cdots 2^n} \}^i, 0, 0)$ and $\mathbf{v}_{2i-1,final} = (1, 0, 2^{\cdots 2^n} \}^i)$ which makes the ξ_{2k-1} k -expontial size.