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ON THE DECIDABILITY OF THE SEQUENCE EQUIVALENCE PROBLEM FOR D0L-SYSTEMS*

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A property, called smoothness, of a family of D0L-systems is introduced. It is shown that the sequence equivalence problem is decidable for every smooth family of D0L-systems. Then a large subfamily of D0L-systems, called simple D0L-systems is shown to be smooth.

0. Introduction

Shortly after the introduction of 0L-systems by Lindenmayer in [8], the question was asked whether the equivalence problem is decidable for these systems [13]. The undecidability of the equivalence problem for (nondeterministic) 0L-systems was shown, e.g., in [1]. The same question for deterministic 0L-systems (D0L-systems) is conjectured to be decidable but remains open; according to the survey paper [12] it is "without any doubt, the most intriguing open mathematical problem around L-systems".

The equivalence problem was shown decidable for some special subclasses of D0L-systems, e.g., [7]. The growth-equivalence problem for D0L-systems was shown to be decidable in [10] as well as the equivalence problem for other types of weak equivalences [9]. It was also shown in [9] that the language equivalence problem for D0L-systems is recursively decidable iff the sequence equivalence problem is recursively decidable and that to resolve the latter it is enough to consider D0L-systems in certain normal form.

The main goal of this paper is to show a sufficient condition for a subfamily of D0L-systems to have recursively decidable the sequence equivalence problem. Our approach is based on the notion of the balance of a string. Consider two D0L-systems $G_i = (\Sigma, h_i, \sigma)$ for $i = 1, 2$. The balance of a string w in Σ^* is the difference of lengths of $h_1(w)$ and $h_2(w)$. We say that a pair of D0L-systems has bounded balance if there exists a constant $c > 0$ such that no prefix of any string generated by these systems has the balance larger than c. Finally, a subfamily of

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D0L-systems is called smooth if every pair of sequence equivalent systems from the subfamily has bounded balance. We close Section 2 by showing that smoothness is a sufficient condition for decidability of the sequence equivalence problem.

In the next section we exhibit an example of a smooth subfamily of D0L-systems, called simple D0L-systems. Intuitively, a D0L-system is simple if every symbol of its alphabet can be obtained (possibly in several steps) from every other symbol of the alphabet. To show that the family of simple D0L-systems is smooth we first demonstrate that for every pair of sequence equivalent simple D0L-systems the balance of a long prefix of a string generated by such systems is "very small" compared with the length of the prefix. Then we strengthen this result by showing that the pair has bounded balance.

We have strong reasons to conjecture that the above approach can be extended to show the decidability of sequence equivalence problem for the family of (all) D0L-systems[1].

1. Prerequisites

The set of non-negative integers is denoted by $\mathbf{N}$. Given an alphabet Σ, Σ^* is the free monoid generated by Σ with the unit ε (empty string); $\Sigma^+ = \Sigma^* - \{\varepsilon\}$.

For $w \in \Sigma^*$ and $a \in \Sigma$, $\#_a(w)$ is the number of occurrences of symbol a in the string w. If $\Sigma = \{a_1, \ldots, a_n\}$ then the vector $(\#_{a_1}(w), \#_{a_2}(w), \ldots, \#_{a_n}(w))$ is called the *Parikh vector* of w and is denoted by $[w]$.

For an integer i let $|i|$ denote the absolute value of i. For w in Σ^*, $|w|$ denotes the length of w; in particular $|\varepsilon| = 0$. For $\alpha \in \mathbf{N}^k$, $\alpha = (\alpha_1, \ldots, \alpha_k)$, let $|\alpha| = \Sigma_{i=1}^k |\alpha_i|$, thus $|[w]| = |w|$ for $w \in \Sigma^*$. For a set Σ let $|\Sigma|$ be the cardinality of Σ.

For $w \in \Sigma^*$, $\min(w) = \{a \in \Sigma\colon a$ occurs in $w\}$.

A *D0L-system* is a 3-tuple $G = (\Sigma, h, \sigma)$ where Σ is an alphabet, h is a homomorphism on Σ^* and axiom σ is in Σ^+.

For D0L-system $G = (\Sigma, h, \sigma)$ the *language generated by G* is defined as $L(G) = \{h^n(\sigma)\colon n \geq 0\}$.

Two D0L-systems $G_i = (\Sigma_i, h_i, \sigma_i)$ for $i = 1, 2$, are called (*sequence*) *equivalent* if $h_1^n(\sigma_1) = h_2^n(\sigma_2)$ for all n in $\mathbf{N}$; we write $G_1 \equiv G_2$. They are *language equivalent* if $L(G_1) = L(G_2)$. The *growth matrix* of D0L system G is defined as in [10]. If $G = (\{a_1, \ldots, a_n\}, h\sigma)$, then $M_{i,j} = \#_{a_i} h(a_j)$ for $1 \leq i, j \leq n$.

For D0L-system $G = (\Sigma, h, \sigma)$ we say that w in Σ^+ is a *G-prefix* (*G*-substring) if w is a prefix (substring) of $h^n(\sigma)$ for some $n \geq 0$.

2. A sufficient condition for decidability of sequence equivalence

Let $G_i = (\Sigma, h_i, \sigma)$ for $i = 1, 2$, be two D0L-systems, and let w be in Σ^*. The *balance of w with respect to* (G_1, G_2) is denoted by $\beta(w)$ and defined as

[1] Note added in proof: The extension has been obtained by the author and I. Fris.

$$\beta(w) = |\,|h_1(w)| - |h_2(w)|\,|.$$

We say shortly *balance of* w if a pair (G_1, G_2) is understood.

Let $G_i = (\Sigma, h_i, \sigma)$ for $i = 1, 2$, be two equivalent D0L-systems and $c \geq 0$. We say that the pair (G_1, G_2) has *c-bounded balance* if $\beta(w) \leq c$ for every G_1-prefix w. We say that (G_1, G_2) has *bounded balance* if it has c-bounded balance for some $c \geq 0$.

We say that a family $\mathcal{F}$ of D0L-systems is *smooth* if every pair of sequence equivalent systems from $\mathcal{F}$ has bounded balance.

Theorem 2.1. *The (sequence) equivalence problem is recursively decidable for every smooth family* $\mathcal{F}$ *of D0L-systems.*

Proof. Clearly, we can restrict ourselves to pairs of D0L-systems from $\mathcal{F}$ with identical alphabets and identical axioms.

We will exhibit two semidecision procedures, one for non-equivalence and the other for equivalence.

(1) The semidecision procedure for non-equivalence is trivial, we compute $h_1^n(\sigma)$ and $h_2^n(\sigma)$ for $n = 0, 1, 2, \ldots,$ and stop with answer "non-equivalent" if $h_1^n(\sigma) \neq h_2^n(\sigma)$ for some n.

(2) Our semiprocedure for equivalence is based on the assumption that $\mathcal{F}$ is smooth, i.e. that a pair of equivalent systems from $\mathcal{F}$ has bounded balance.

Clearly, $h_1^n(\sigma) = h_2^n(\sigma)$ for $n \geq 0$ iff $h_1^n(\sigma) = h_2(h_1^{n-1}(\sigma))$ for $n \geq 0$, iff $h_1(w) = h_2(w)$ for each w in $L(G_1)$. $L(G_1)$ is a D0L-language and therefore also an E0L-language [11].

Now we design a semiprocedure which will check successively for $k = 1, 2, \ldots$ whether the pair (G_1, G_2) has k-bounded balance and whether G_1 and G_2 are sequence equivalent. We already know that to check the sequence equivalence it is enough to check whether $h_1(w) = h_2(w)$ for each $w \in L(G_1)$. The checking of these two properties for a particular k is done as follows:

Let M_k be a deterministic g.s.m. [6] with a "buffer" of length k in its finite control which for any input string w in Σ^* attempts to check (from left to right when reading w) whether $h_1(w) = h_2(w)$. It is obviously possible to do this if G_1 and G_2 have k-bounded balance since we have available a "buffer" of length k (i.e. a buffer able to contain k symbols from Σ). Given input w, our g.s.m. M_k will produce its output as follows:

(i) If the buffer of M_k does not overflow and $h_1(w) = h_2(w)$, then no output is produced (M_k goes into a non-accepting state).

(ii) If M_k finds that $h_1(v) \neq h_2(v)$ for some prefix v of w before its buffer overflows, it stops (in an accepting state) and produces "0".

(iii) Otherwise (buffer overflows) M_k stops (in an accepting state) and produces output "1".

Note. The different outputs in (ii) and (iii) are used to describe an alternative procedure below.

Let T_k be the translation defined by M_k. Clearly, $T_k(L(G_1)) = \emptyset$ iff the pair (G_1, G_2) has bounded balance and $h_1(w) = h_2(w)$ for all $w \in L(G_1)$. By [4] or [3] we can construct an E0L-system S_k such that $L(S_k) = T_k(G_1)$. Finally, it is recursively decidable [2, 11] whether the E0L-language $L(S_k)$ is empty. Therefore, simply enumerate $S_1, S_2, \ldots$ and test each S_k for $L(S_k) = \emptyset$. Clearly $G_1 = G_2$ iff there is k so that $L(S_k) = \emptyset$.

Our semiprocedure must eventually stop if $G_1 \equiv G_2$ since, because $\mathcal{F}$ is smooth, there exists $c > 0$ so that G_1 and G_2 have c-bounded balance. □

Alternative proof of Theorem 2.1. We can drop the first semiprocedure and modify the second into an algorithm (which always halts) as follows:

We again construct the E0L-system S_k successively for $k = 1, 2, \ldots$. For every k, if $L(S_k) = \emptyset$ then stop with answer "$G_1 \equiv G_2$". Otherwise, if $0 \in L(S_k)$, then stop with answer "$G_1 \not\equiv G_2$". Otherwise, increase k and repeat.

We are able to check whether $0 \in L(S_k)$ since the membership problem is decidable for E0L-languages [2, 11].

If $G_1 \equiv G_2$ then the algorithm halts for the same reason as the second procedure above. If $G_1 \not\equiv G_2$, even if the balance is not bounded, there exists a shortest G_1-prefix u such that $h_1(u) \neq h_2(u)$. We need a buffer of at most length $|u|$ to establish $G_1 \not\equiv G_2$ so the algorithm stops, at the latest, at system $S_{|u|k}$. □

3. Simple D0L-systems

Let G be a D0L-system over at least a two letter alphabet with growth matrix M. We say that G is a *simple* D0L-system (SD0L-system) if there exists $k \geq 1$ so that all elements of M^k are nonzero.

Lemma 3.1. *Let $G = (\Sigma, h, \sigma)$ be an SD0L-system. Then G is exponentially growing* [10]. *Moreover, there exist $n_0, d, c_1, c_2 > 0$ so that for all $n \geq n_0$ and every w in Σ^*,*

$$c_1 d^n |w| \leq |h^n(w)| \leq c_2 d^n |w|.$$

Proof. It follows from results in [10]. □

Lemma 3.2. *Let $G_i = (\Sigma, h_i, \sigma)$ be two sequence equivalent SD0L-systems. For each a in Σ and each $\varepsilon > 0$ there exists $n_{a,\varepsilon}$ so that $\beta(h_1^n(a)) \leq \varepsilon\,|h_1^n(a)|$ for all $n \geq n_{a,\varepsilon}$.*

Proof (version due to J. Hammerum). Let M_1 and M_2 be the growth matrices of G_1 and G_2, respectively. Let k be the smallest k such that M_i^k has all nonzero-elements, for $i = 1, 2$. Such k exists since G_1 and G_2 are simple.

Then for all vectors v and all $\varepsilon > 0$ there exists m_0 so that for $m > m_0$ there is a vector t_m and a number d_m so that

$$vM_1^{km} = d_m u + t_m,$$

where $|t_m| < \varepsilon\, |vM_1^{km}|$ and u is the characteristic vector with the largest eigenvalue for M_1^k.

It is easy to establish that M_1 has the same property as M_1^k above. From this follows that for all $a \in \Sigma$ and $\varepsilon > 0$ there exists n_0, so that for all $n > n_0$ there exists a vector t_n and a number d_n so that

$$[h_1^n(a)] = [a]M_1^n = d_n u + t_n,$$

where $|t_n| < \varepsilon\, |h_1^n(a)|$ and u is the characteristic vector with the largest eigenvalue for M_1.

We can prove that

$$\beta(w) \leqslant |[w](M_1 - M_2)|$$

because

$$\begin{aligned}\beta(w) &= |\,|h_1(w)| - |h_2(w)|\,| \\ &\leqslant \sum_{a\in\Sigma} |\#_a(h_1(w)) - \#_a(h_2(w))| \\ &= |[w](M_1 - M_2)|\end{aligned}$$

(one may notice that equality occurs when one of words $h_1(w)$ and $h_2(w)$ is a subword of the other).

Noting that u is a characteristic vector for M_2 if G_1 and G_2 are equivalent the following inequality holds

$$\begin{aligned}\beta(h_1^n(a)) &\leqslant |(d_n u + t_n)(M_1 - M_2)| \\ &\leqslant |d_n u(M_1 - M_2)| + |t_n(M_1 - M_2)| \\ &= |t_n(M_1 - M_2)|\end{aligned}$$

which proves the lemma. □

Lemma 3.3. *Let $G_i = (\Sigma, h_i, \sigma)$ for $i = 1,2$, be two sequence equivalent SD0L-systems. For each $\varepsilon > 0$ there is $n_\varepsilon > 0$ so that for every w in Σ^+ and all $n \geqslant n_\varepsilon$ we have $\beta(h_1^n(w)) \leqslant \varepsilon\, |h_1^n(w)|$.*

Proof. By Lemma 3.2 for each a in Σ and each $\varepsilon > 0$ there is $n_{a,\varepsilon}$ so that for $n \geqslant n_{a,\varepsilon}$ we have $\beta(h_1^n(a)) \leqslant \varepsilon\, |h_1^n(a)|$. Let $n_\varepsilon = \max_{a\in\Sigma}\{n_{a,\varepsilon}\}$ and let $w = a_1 a_2 \ldots a_k$. For $n \geqslant n_\varepsilon$ we have

$$\beta(h_1^n(w)) \leqslant \sum_{i=1}^{k} \beta(h_1^n(a_i)) \leqslant \varepsilon \sum_{i=1}^{k} |h_1^n(a_i)| = \varepsilon\, |h_1^n(w)|. \square$$

Theorem 3.1. *Let $G_i = (\Sigma, h_i, \sigma)$ for $i = 1,2$, be two sequence equivalent SD0L-systems. For each $\varepsilon > 0$ there exists $K_\varepsilon > 0$ so that $\beta(w) \leqslant \varepsilon\, |w|$ for every G_1-prefix w such that $|w| \geqslant K_\varepsilon$.*

Proof. Let w be a prefix of $h_1^n(\sigma)$. We define $t > 0$ and $u_t, v_t, u_{t-1}, v_{t-1}, \ldots, u_1, v_1$ in Σ^* as follows:

(i) Let t be the maximal integer such that $h_1^t(b)$ is a prefix of w where b is the first symbol of $h_1^{n-t}(\sigma)$.

(ii) Let $u_t = h_1^{n-t}(\sigma)$ and let v_t be the longest prefix of $h_1^{n-t}(\sigma)$ such that $h_1^t(v_t)$ is a prefix of w.

(iii) For $i = t-1, \ldots, 0$, u_i is obtained from $h_1^{n-i}(\sigma)$ by removing its prefix $h_1^{t-i}(v_t)\, h_1^{t-i-1}(v_{t-1}) \ldots h_1(v_{i+1})$, and v_i is the longest prefix of u_i such that $h_1^t(v_t) h_1^{t-1}(v_{t-1}) \ldots h_1^i(v_i)$ is a prefix of w ($h_1^0(x) = x$ for each x in Σ^*). Let $w_i = h_1^i(v_i)$ for $i = 0, 1, \ldots, t$. Clearly, $w = w_t w_{t-1} \ldots w_0$.

Note that w_k may be empty for some k.

Given $\varepsilon > 0$ there exists, by Lemma 3.3, $n_{\varepsilon/2}$ so that $\beta(h_1^i(v_i)) \leqslant \frac{1}{2}\varepsilon\, |h_1^i(v_i)|$ for $n_{\varepsilon/2} \leqslant i \leqslant t$, i.e. $\beta(w_i) \leqslant \frac{1}{2}\varepsilon\, |w_i|$ for $n_{\varepsilon/2} \leqslant i \leqslant t$ and, therefore, $\beta(w_t w_{t-1} \ldots w_{n_{\varepsilon/2}}) \leqslant \frac{1}{2}\varepsilon\, |w_t w_{t-1} \ldots w_{n_{\varepsilon/2}}|$. Let $Q = \max_{a \in \Sigma} \beta(a)$. We have $\beta(w_{n_{\varepsilon/2}-1} \ldots w_1) \leqslant Q\, |w_{n_{\varepsilon/2}-1} \ldots w_1|$ and we, clearly, can choose K_ε so that if $|w| \geqslant K_\varepsilon$, then $|w_{n_{\varepsilon/2}-1} \ldots w_1| \leqslant (\frac{1}{2}\varepsilon/Q)\, |w|$; and thus $\beta(w_{n_{\varepsilon/2}-1} \ldots w_1) \leqslant \frac{1}{2}\varepsilon\, |w|$.

Together, we have for w such that $|w| \geqslant K_\varepsilon$,

$$\beta(w) \leqslant \beta(w_t \ldots w_{n_{\varepsilon/2}}) + \beta(w_{n_{\varepsilon/2}-1} \ldots w_1) \leqslant (\tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon)\, |w| = \varepsilon\, |w|. \square$$

Note that only a weaker equivalence than sequence equivalence is used in proofs of Lemma 3.2, Lemma 3.3 and Theorem 3.1, namely the so called Parikh equivalence, see [9].

A *derivation forest* of a string w with respect to a D0L-system G is an obvious modification of the well-known notion of derivation tree for a context-free grammar, where we have an axiom (string of symbols) rather than one starting symbol of a context-free grammar.

Let F be a derivation forest of $h^n(\sigma)$ for some $n > 0$ with respect to the SD0L-system $G = (\Sigma, h, \sigma)$. A path in F from a node on the lowest level (of σ) to a node α on level n is called *the chain of* α. Formally, a chain q is a string in $(\Sigma \times \mathrm{N})^*$ such that if $q = (a_0, k_0) \ldots (a_n, k_n)$, then

(i) $1 \leqslant k_0 \leqslant |\sigma|$,

(ii) $1 \leqslant k_i \leqslant |h(a_{i-1})|$ for $1 \leqslant i \leqslant n$,

(iii) a_{i+1} is the k_{i+1}th symbol in $h(a_i)$ for $0 \leqslant i \leqslant n-1$.

Intuitively, k_0 is the position of a_0 in σ, and for $i \geqslant 1$, k_i determines which of possibly several branches is taken. First components of pairs are clearly redundant but they allow us to state easily the condition (iii).

The string $a_0 \dots a_n$ is called *the trace of chain q*.

A chain q is said to be *periodic with period p and prefix (initial segment)* q_1 if $q = q_1 p^m q_2$ for some $m \geq 2$ and q_2 is a prefix of p.

Chain q is *leftmost (rightmost)* on level i if $k_i = 1$ ($k_i = |h(a_i)|$). Chain q is *fully leftmost (fully rightmost)* if it is leftmost (rightmost) on all levels.

For a node α of derivation forest F, and a specific occurrence of G-substring w, we say that w contains α if α is one of the nodes labeled by symbols from w.

Let q be a periodic chain with prefix r and period p. Then there are cyclically repeating (after each $|p|$ steps) common G-substrings on at least one side of q (see [5, Theorems 11.3 and 11.4]).

Theorem 3.2. *If $G_i = (\Sigma, h_i, \sigma)$ for $i = 1, 2$, are two sequence equivalent SD0L-systems, then the pair (G_1, G_2) has bounded balance.*

Proof. Assume that the pair (G_1, G_2) does not have bounded balance (Assumption 1). Therefore, for every $n_0 > 0$ there must exist n, $n \geq n_0$, and u, v in Σ^* so that $h_1^n(\sigma) = uv$ and the following conditions hold:

(i) $\beta(u) > \beta(w_1)$ for any prefix w_1 of $h_1^j(\sigma)$ where $0 \leq j < n$.

(ii) $\beta(u) \geq \beta(w_2)$ for any prefix w_2 of $h_1^n(\sigma)$.

(iii) $\beta(u) > \beta(w_3)$ for any prefix w_3 of u.

Let F be the derivation forest of G_1 and α be the node in F at the last symbol of prefix u at level n. Let q be the chain of α in F and let α_1 and α_2 be the first two nodes of chain q (from top) such that the label of α_1 is the same as that of α_2, the label of the left neighbor of α_1 is the same as that of the left neighbor of α_2 and the same also holds for the right neighbors. Let the common labels be a, b, c from the left; they, of course, are not necessarily different.

Let the levels of α_1 and α_2 in the derivation forest of G_1 (from top) be r and $r + t$, respectively. Clearly, given an SD0L-system G, there exists a constant C so that $r + t \leq C$ independently on n_0. We only note that first we have the constant $C_1 = |\Sigma| + 1$ with the property that on levels higher than C_1 there is at least one neighbor both to the left and to the right of the node of chain q. This is so since otherwise q would have a fully leftmost (rightmost) initial segment with some symbol occurring at least twice in its trace; therefore, u would be a prefix (v would be a suffix) of $h_1^j(\sigma)$ for some $j < n$, which would be in contradiction with condition (i) implied by Assumption 1.

Let $q = q_1 q_2 q_3$ where q_2 is the section of q between nodes α_1 and α_2. Let q' be the periodical chain defined by $q' = q_1 q_2^j q_4$ where $j > 0$ and q_4 is a proper prefix of q_2, j and q_4 chosen so that the length of q' is the same as the length of q. Informally, we have chosen q' so that it coincides with q up to the second occurrence of abc and then continues periodically. Therefore, there are cyclically repeating longer and longer substrings on both sides of q'; specifically $h_1^{n-t-r}(abc)$ is a common substring of $h_1^{n-t}(\sigma)$ and $h_1^{n}(\sigma)$ which on level n contains node α since chain q goes through node α_2. Moreover, node α is not close to either end of the common substring since α_2 is labeled by the middle symbol b in abc and both $|h_1^m(a)|$ and $|h_1^m(c)|$ are exponentially growing (for growing m) by Lemma 3.1.

Now, let $h_1^{n-t}(\sigma) = u_1xyv_1$ and $h_1^n(\sigma) = u_2xyv_2$ where $xy = h_1^{n-r-t}(abc)$, $u_2x = u$ and $yv_2 = v$, i.e. the node α on level n is at the last symbol of x. We write $u' = u_1x$ and $v' = yv_1$. Clearly, $h_1^{n-r-t}(a)$ is a prefix of x and $h_1^{n-r-t}(c)$ is a suffix of y, therefore from Lemma 3.1 and discussion above it follows that the length of both x and y for growing n is linearly proportional to the length of the whole string $h_1^n(\sigma)$, i.e. there exists constant K, dependent on G_1 only, such that $K|x| \geqslant |h_1^n(\sigma)|$ and $K|y| \geqslant |h_1^n(\sigma)|$. Therefore, it follows from Theorem 3.1 that for each $\varepsilon > 0$ there exists n_0 so that $\beta(u) \leqslant \varepsilon |x|$ and $\beta(u) \leqslant \varepsilon |y|$ where u, x and y are determined by n_0.

Now, we explain first the following step in the proof informally and then we will give the details. Both $h_1^n(\sigma)$ and $h_1^{n-t}(\sigma)$ have y as a substring with node α at the last symbol preceding y on level n. Since the two systems are equivalent both $h_1(y)$ and $h_2(y)$ are substrings of $h_1^{n+1}(\sigma)$ and of $h_1^{n-t+1}(\sigma)$. We recall that both $\beta(u')$ and $\beta(u)$ are "very small" with respect to $|h_1(y)|$. By Assumption 1, $\beta(u') < \beta(u)$, and therefore the relative position of $h_1(y)$ and $h_2(y)$ as substrings of $h_1^{n-t+1}(\sigma)$ is by a "small' shift (with respect to the length of $h_1^{n+1}(\sigma)$ and also of $h_1(y)$) different from the relative position of the same strings as substrings of $h^{n+1}(\sigma)$. Therefore $h_1(y)$ has to have "long" identical prefix and suffix and consequently must be periodic with a period arbitrarily short with respect to its length for large enough n.

Formally, using the notation introduced above, we have:

(1) $h_1^{n-t}(\sigma) = u'yv_1$ and $h_1^n(\sigma) = uyv_2$,

where $\beta(u)$ is strictly maximal up to the level n. Since the systems G_1 and G_2 are equivalent we obtain from (1),

(2) $h_1(u')h_1(y)h_1(v_1) = h_2(u')h_2(y)h_2(v_1)$, and

(3) $h_1(u)h_1(y)h_1(v_2) = h_2(u)h_2(y)h_2(v_2)$.

Without loss of generality we may assume that $|h_1(u')| \geqslant |h_2(u')|$, i.e. $h_1(u') = h_2(u')z'$ for some z' in Σ^*. Therefore by removing prefix $h_2(u')$ on both sides of (2), we have

(4) $z'h_1(y)h_1(v_1) = h_2(y)h_2(v_1)$.

Now we have to consider two cases.

Case A. Let $|h_1(u)| \geqslant |h_2(u)|$, i.e. $h_1(u) = h_2(u)z$ for some z in Σ^*. By Assumption 1, $\beta(u) > \beta(u')$ and thus $|z| > |z'|$. By removing prefix $h_2(u)$ on both sides of (3), we get

(5) $zh_1(y)h_1(v_2) = h_2(y)h_2(v_2)$.

Since $|z| > |z'|$ and $|z|$ is "very small" with respect to $|h_1(y)|$ and $|h_2(y)|$ it follows by comparing (4) and (5) that there exists p in Σ^+ so that $z = z'p$ and $h_1(y) = p^id$ where both p and d are "very small" with respect to $h_1(y)$.

Case B. Let $|h_1(u)| < |h_2(u)|$, i.e. $h_1(u)z = h_2(u)$ for some z in Σ^*, where again $|z|$ is "very small" with respect to $|h_1(y)|$ and $|h_2(y)|$. By removing the prefix $h_2(u)$ from both sides of (3), we obtain

(6) $\delta h_1(v_2) = h_2(y)h_2(v_2)$,

where δ is obtained from $h_1(y)$ by removing prefix z, i.e. $h_1(y) = z\delta$. By comparing (4) and (6), we see that δ is prefix of $z'h_1(y)$ and therefore the string $h_1(y)$ has an identical "very long" prefix and suffix and thus must be periodic, i.e. of the form $h_1(y) = p^j d$, for some p in Σ^+ and d in Σ^*, where both $|p|$ and $|d|$ are "very small" with respect to $|h_1(y)|$.

Thus in both Case A and Case B, $h_1(y)$ has to be of the form $p^j d$ where by choosing n_0 large enough we can make p arbitrarily short with respect to $h_1(y)$ and therefore j arbitrarily large. Since

$$h_1^{n-t+1}(\sigma) = h_1(u_1)h_1(xy)h_1(v_1) = h_2(u_1)h_2(xy)h_2(v_1),$$

$$h_1^{n+1}(\sigma) = h_1(u_2)h_1(xy)h_1(v_2) = h_2(u_2)h_2(xy)h_2(v_2),$$

it is clear that not only $h_1(y)$ has an identical string as both its nontrivial prefix and suffix but that the same holds also for the whole string $h_1(xy)$. Therefore $h_1(xy)$ is periodic, i.e. $h_1(xy) = a_1q^k a_2$ for some "short" a_1, a_2 in Σ^*, q in Σ^+ and "large" k. Since the string xy on level n is, clearly, a substring of $h_1^{t-2}(h_1(xy))$ and $h_1(xy)$ is periodic, also xy must be periodic with a period not longer than $|q|c_2d_1^{t-1}$ where constants c_2, d_1 are determined by Lemma 3.1 independently on n for large enough n. Thus for large enough n the period is still arbitrarily short with respect to the length of xy. Since both x and y are "long" and we can, if necessary, shift the period, we can write

(7) $xy = b_1 r^{k_1+k_2} b_2$

for some b_1, b_2 in Σ^*, r in Σ^+ and $k_1, k_2 \geq 1$ such that

(8) $x = b_1 r^{k_1}$ and $y = r^{k_2} b_2$.

So far we have used only the fact that $\beta(u)$ is strictly maximal up to level n (condition (i)) not yet the conditions (ii) and (iii) implied by Assumption 1. Now we will exploit them by considering two cases:

Case I: Let $\beta(r) = 0$. By (7) and (8), $u = \bar{w}r$ for some $\bar{w}$ in Σ^* and therefore $\beta(u) = \beta(\bar{w})$ which is in contradiction with condition (iii) implied by Assumption 1.

Case II: Let $\beta(r) > 0$. By (7) and (8), we can write $h_1^n(\sigma) = \bar{u}r^2\bar{v}$ for some $\bar{u}, \bar{v}$ in Σ^* such that $\bar{u}r = u$ and $r\bar{v} = v$. Since $\beta(r) > 0$, clearly, either $\beta(\bar{u}) > \beta(u)$ or $\beta(\bar{u}r^2) > \beta(u)$ which is in contradiction with condition (ii) implied by Assumption 1. □

Corollary. *The sequence equivalence problem is decidable for the family of SD0L systems.*

Proof. By Theorems 3.2 and 2.1. □

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