ON A THEOREM OF R. JUNGEN

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Let us recall the following elementary result in the theory of analytic functions in one variable. power series

THEOREM (R. JUNGEN [7]). *If a is rational and b algebraic their Hadamard product c is algebraic; if, further, b is rational, c also is rational.*

For several variables, Jungen's proof shows that the theorem is still true for the Bochner-Martin [2] Hadamard product. It does not hold for the Cameron-Martin [3] and for the Haslam-Jones [6] Hadamard products. In this note we give a version of Jungen's theorem which is valid for a restricted interpretation of the notions involved when a and b are formal power series in a finite number of noncommuting variables.

1. **Notations.** Let R be a fixed not necessarily commutative ring with unit 1. For any finite set Z, $F(Z)$ is the free monoid generated by Z and $R_{\text{pol}}(Z)$ is the free module on $F(Z)$ over R. An element a of $R_{\text{pol}}(Z)$ will usually be written in the form $a=\sum\{(a,f)\cdot f: f\in F(Z)\}$ where the coefficients (a,f) are in R; $R_{\text{pol}}(Z)$ is graded in the usual manner and $\pi_n a=\sum\{(a,f)\cdot f: f\in F(Z),\ \deg f\leqq n\}$. We identify R with $\pi_0 R_{\text{pol}}(Z)$. $R_{\text{pol}}(Z)$ is also a ring with product $aa' =\sum\{(a,f')(a',f'')\cdot f: f,f',f''\in F(Z),\ f=f'f''\}$. Z^* $R\langle Z^*\rangle$

It is well known (cf., e.g., [4; 3]) that these notions extend to the ring $R(Z)$ of the formal power series (with coefficients in R) in the noncommuting variables $z\in Z$; $R(Z)$ is topologized in the same manner as a ring of commutative formal power-series and $aa' =\lim_{n,n'\to\infty}(\pi_n a)(\pi_{n'}a')$. Any $b\in R^*(Z)=\{a\in R(Z):\pi_0 a=0\}$ has a quasi-inverse $(-b)^*=\lim_{n\to\infty}\sum_{n'<n}(-b)^{n'}$. If a is invertible, $a^{-1}=(1+b^*)(\pi_0 a^{-1})$ where $b=-(\pi_0 a^{-1})(a-\pi_0 a)\in R^*(Z)$. We shall say that $S^*\subset R^*(Z)$ is *rationally closed* if $r,\ r'\in R,\ b,\ b'\in S^*$ imply $rb+b'r',\ bb',\ b^*\in S^*$. If this is so, the set of those elements a of $R(Z)$ such that $a-\pi_0 a\in S^*$ is a ring containing the inverses of its invertible elements. $R\langle\langle\Sigma^*\rangle\rangle$

DEFINITION 1. $R^*_{\text{rat}}(X)$ is the least rationally closed subset (of $R(X)$) containing X. → Nonterminals

Now let $Y=\{y_j\}$ be a set of a finite number M of new variables and $R^M(X\cup Y)$ (resp. $R^M_{\text{pol}}(X\cup Y)$) the cartesian product of M copies

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of the R-module $R(X \cup Y)$ (resp. $R^M_{\text{pol}}(X \cup Y)$). For each $q = (q_1, \cdots, q_m) \in R^M(X \cup Y)$, $\pi_n q = (\pi_n q_1, \cdots, \pi_n q_m)$. If $q \in R^{*M}(X \cup Y)$ (i.e., if $\pi_0 q = 0$) let λ_q be the homomorphism of the monoid $F(X \cup Y)$ into the multiplicative monoid structure of $R(X \cup Y)$ that is induced by $\lambda_q x = x$ if $x \in X$ and $\lambda_q y_j = q_j$ if $y_j \in Y$. Since $\pi_0 q = 0$, λ_q can be extended to an endomorphism of the R-module $R(X \cup Y)$ by $\lambda_q a = \sum \{(a, f)\lambda_q f : f \in F(X \cup Y)\}$; also, $\lambda_q p = (\lambda_q p_1, \cdots, \lambda_q p_M)$ for any $p \in R^M(X \cup Y)$.

We shall say that $p \in R^{*M}(X \cup Y)$ is a *proper system* if $(p_j, y_{j'}) = 0$ for all $j, j' \leqq M$. Then, if $q \in R^{*M}(X)$, $\lambda_q p \in R^{*M}(X)$ and $\pi_{n+1}\lambda_q p = \pi_{n+1}\lambda_{\pi_n q} p$ for all n. Consider now the infinite sequence $p(0) = 0$, $p(1) = \lambda_{p(0)} p, \cdots, p(m+1) = \lambda_{p(m)} p, \cdots$. Trivially, $\pi_{m'} p(m') = \pi_{m'} p(m' + m'') \in R^{*M}(X)$ for $m' = 0$ and all m''. If these relations hold for $m' \leqq m$, they still hold for $m+1$ because

$$\pi_{m+1} p(m+1) = \pi_{m+1}\lambda_{p(m)} p = \pi_{m+1}\lambda_{\pi_m p(m)} p = \pi_{m+1}\lambda_{\pi_m p(m+m'')} p$$
$$= \pi_{m+1}\lambda_{p(m+m'')} p = \pi_{m+1} p(m+1+m'').$$

Hence, $p(\infty) = \lim_{m \to \infty} p(m)$ exists and it satisfies $p(\infty) \in R^{*M}(X)$, $\pi_0 p(\infty) = 0$, $p(\infty) = \lambda_{p(\infty)} p$. In fact, $p(\infty)$ is the only element to satisfy these equations because if $\pi_0 p' = 0$ and $p' = \lambda_{p'} p$, any relation $\pi_m p(\infty) = \pi_m p'$ implies $\pi_{m+1} p' = \pi_{m+1}\lambda_{\pi_m p'} p = \pi_{m+1}\lambda_{\pi_m p(\infty)} p = \pi_{m+1} p(\infty)$. For this reason we call $p(\infty)$ *the solution* of p.

def. of algebraic power series

DEFINITION 2. $R^*_{\text{alg}}(X)$ is the least subset (of $R^*(X)$) that contains every coordinate of the solution of any proper system having its coordinates in $R^*_{\text{pol}}(X \cup Y)$.

(REMARK. It can easily be shown that $R^*_{\text{alg}}(X)$ is rationally closed and that it contains every coordinate of the solution of any proper system having its coordinates in $R^*_{\text{alg}}(X \cup Y)$.)

DEFINITION 3. For any

$$a, b \in R(X), \quad a \odot b = \sum \{(a, f)(b, f) \cdot f : f \in F(X)\}.$$

Hadamard product

2. Main result.

Property 2.1. The element a of $R^*(X)$ belongs to $R^*_{\text{rat}}(X)$ if and only if there exists a finite integer $N \geqq 2$ and a homomorphism μ of $F(X)$ into the multiplicative monoid of $R^{N \times N}$ (the ring of the $N \times N$ matrices with entries in R) such that $a = \sum \{\mu f_{1,N} \cdot f : f \in F(X)\}$ (abbreviated as $\sum \mu f_{1,N} \cdot f$).

PROOF. (1) *The condition is necessary.* This is trivial if $a = \pi_1 a$. Hence it suffices to show that for any $r, r' \in R$, $a = \sum \mu f_{1,N} \cdot f$ and $a' = \sum \mu' f_{1,N'} \cdot f$ one can construct suitable homomorphisms giving $ra + a'r'$, aa' and a^*. This is done below, defining the homomorphisms by their restriction to X.

Addition. Let $N''=N+N'+2$ and $\mu''x\in R^{N''\times N''}$ defined for each $x\in X$ by

$$\mu''x_{i,1} = \mu''x_{N'',i} = 0 \qquad \text{for } 1 \leqq i \leqq N'';$$

$$\mu''x_{1,i+1} = r\mu x_{1,i} \ \text{ and } \ \mu''x_{i+1,N''} = \mu x_{i,N} \qquad \text{for } 1 \leqq i \leqq N;$$

$$\mu''x_{1,i+N+1} = \mu' x_{1,i} \ \text{ and } \ \mu''x_{i+N+1,N''} = \mu' x_{i,N'}\cdot r' \qquad \text{for } 1 \leqq i \leqq N';$$

$$\mu''x_{i,i'} = \text{the direct sum of } \mu x \text{ and } \mu' x \qquad \text{for } 2 \leqq i, i' \leqq N''-1;$$

$$\mu''x_{1,N''} = r\mu x_{1,N} + \mu' x_{1,N'} r'.$$

The verification is trivial.

Product. Let $N''=N+N'$ and define $\nu f\in R^{N''\times N''}$ for each $f\in F(X)$ by $\nu f_{i,i'}=\mu f_{i,N}$ if $f\neq 1$, $1\leqq i\leqq N$, $i'=N+1$; $\nu f_{i,i'}=0$, otherwise. Then, if $\mu''x=\bar{\mu}x+\nu x$ where $\bar{\mu}x$ is the direct sum of μx and $\mu' x$, one has for each $f = x^{(1)}x^{(2)}\cdots x^{(n)}$, $\mu''f = \bar{\mu}f + \sum\{\bar{\mu}f'\nu x^{(j)}\bar{\mu}f'': f'x^{(j)}f'' = f\}$. Since $\nu f x^{(j)}=\bar{\mu}f\nu x^{(j)}$ and $(\nu f'''\bar{\mu}f'')_{1,N''}=0$ when $f''=1$, one has $\mu''f_{1,N''}=\sum\{(\mu f'_{1,N})(\mu' f''_{1,N'}): f'f''=f\}$. Hence, $\sum\mu''f_{1,N''}\cdot f=aa'$.

Quasi-inverse. Let $N''=N$ and define $\nu f\in R^{N\times N}$ for each $f\in F(X)$ by $\nu f_{i,i'}=\mu f_{i,N}$ if $f\neq 1$, $1\leqq i\leqq N$, $i'=1$; $\nu f_{i,i'}=0$, otherwise. Then $\mu''x = \mu x + \nu x$ and since $\mu f\nu x = \nu f x$ identically one has $\mu''f = \sum \nu f^{(1)}\nu f^{(2)}\cdots \nu f^{(k)}\mu f^{(k+1)}$ where the summation is over all the factorisations $f=f^{(1)}f^{(2)}\cdots f^{(k+1)}$ of f in an arbitrary number of factors. The $(1, N)$ entry of any of these products is zero unless all its factors are different from 1 and under this condition, it is equal to $\mu f^{(1)}_{1,N}\,\mu f^{(2)}_{1,N}\cdots \mu f^{(k+1)}_{1,N}$. Hence, $\sum\mu''f_{1,N}\cdot f=\sum_{n>0}a^n=a^*$ and the first part of the proof is completed.

(2) *The condition is sufficient.* We say that the proper system p is *linear* if for each $j\leqq M$, $p_j=q_{j,0}+\sum_{j'}q_{j,j'}y_{j'}$ where all the q's belong to $R^*_{\text{rat}}(X)$ and we verify that all coordinates of the solution of such a system belong to $R^*_{\text{rat}}(X)$.

This is trivial if $M=1$ because $p(\infty)=(1-q_{1,1})^{-1}q_{1,0}(=(1+q^*_{1,1})q_{1,0})$. If it is true for $M'<M$ it is still true for M. Indeed, because $p(\infty)_M=(1-q_{M,M})^{-1}(q_{M,0}+\sum_{j<M}q_{M,j'}p(\infty)_{j'})$, the proper linear system p' defined by $p'_j = p_j - q_{j,M}y_M + q_{j,M}p_M$ for $j<M$ and $p'_M=(1-q_{M,M})^{-1}(p_M - q_{M,M}y_M)$ is such that $p(\infty)=p'(\infty)$. Since its first $M-1$ coordinates do not involve y_M the result follows from the induction hypothesis.

Now, given a homomorphism μ of $F(X)$ into $R^{M\times M}$, the M elements $a_j=\sum\{\mu f_{j,M}\cdot f: f\in F(X),\ f\neq 1\}$ are such that $(a_j,\ xf)=\sum_{j'}\mu x_{j,j'}(a_{j'},f)$. Hence $(a_1,\cdots,a_M)$ is the solution of the linear proper system such that $q_{j,0}=\sum\{\mu x_{j,M}\cdot x: x\in X\}$, $q_{j,j'}=\sum\{\mu x_{j',j}\cdot x: x\in X\}$ for each j, j' and 2.1 is proved.

We now consider two subrings R' and R'' of R that commute element-wise.

Property 2.2. If $a=\sum \mu' f_{1,N}\cdot f\in R'^*_{\mathrm{rat}}(X)$ where μ' is a homomorphism into $R'^{N\times N}$ and if $b=p(\infty)_1\in R''^*_{\mathrm{alg}}(X)$ where the proper system p has its coordinates in $R''^*_{\mathrm{pol}}(X\cup Y)$, then $a\odot b\in R^*_{\mathrm{alg}}(X)$. If, further, $b\in R''^*_{\mathrm{rat}}(X)$ then $a\odot b\in R^*_{\mathrm{rat}}(X)$.

PROOF. We verify first the case of $b\in R''^*_{\mathrm{rat}}(X)$, i.e., of $b=\sum \mu'' f_{1,N''}\cdot f$ for some N'' and μ''. Then $a\odot b=\sum(\mu'\otimes\mu'')f_{1,NN'}\cdot f$ where the kroneckerian product $\mu'\otimes\mu''$ is a homomorphism of $F(X)$ into $R^{NN''\times NN''}$ because R' and R'' commute and the result is proved.

For the general case we denote by $K(Z)$ for any set Z the ring of the $N\times N$ matrices with entries in $R(Z)$. We shall have to consider several homomorphisms of module $\sigma\colon R^M(Z')\to K^M(Z'')$ where Z' and Z'' are two finite sets. In each case σ is defined by a mapping $Z'\to K(Z'')$ which is extended in a natural fashion to a homomorphism of the monoid $F(Z')$ into the multiplicative structure of $K(Z'')$. Then for each

$$a=(a_1,\cdots,a_M)\in R^M(Z'),\quad \sigma a_j=\sum\{(a_j,g)\cdot\sigma g\colon g\in F(Z')\}$$

and $\sigma a=(\sigma a_1,\cdots,\sigma a_M)$.

More specifically, $\mu\colon R^M(X)\to K^M(X)$ is induced by a mapping $\mu\colon X\to K(X)$ such that the entries of each μx belong to $R'^*(X)$.

For each $q\in R''^{*M}(X)$, $\lambda_{\mu q}\colon R(X\cup Y)\to K^M(X)$ is induced by $\lambda_{\mu q}f=\mu f$ if $f\in F(X)$ and $\lambda_{\mu q}y_j=\mu q_j$ if $y_j\in Y$. Hence, since R' and R'' commute element-wise, $\mu\lambda_q g=\lambda_{\mu q}g$ for each $g\in F(X\cup Y)$ (with λ_q as previously defined). Consequently, $\mu\lambda_q p=\lambda_{\mu q}p$ for any $p\in R''^M(X\cup Y)$.

Let now $Z=\{z_{j,i,i'}\}(1\leqq j\leqq M;\ 1\leqq i,\ i'\leqq N)$, a set of $M\times N\times N$ new variables and $\nu\colon R^M(X\cup Y)\to K^M(X\cup Z)$ induced by $\nu f=\mu f$ if $f\in F(X)$, $\nu y_j=$the $N\times N$ matrix with entries $z_{j,i,i'}$ if $y_j\in Y$. Also $\lambda_{\nu q}\colon R(X\cup Z)\to R(X)$ is induced by $\lambda_{\nu q}f=f$ if $f\in F(X)$ and $\lambda_{\nu q}z_{j,i,i'}=(\nu q_j)_{i,i'}$ if $z_{j,i,i'}\in Z$. We extend $\lambda_{\nu q}$ to a homomorphism $K^M(X\cup Z)\to K^M(X)$ by defining $\lambda_{\nu q}\boldsymbol{m}$ for any $\boldsymbol{m}\in K(X\cup Z)$ as the $N\times N$ matrix with entries $\lambda_{\nu q}(\boldsymbol{m}_{i,i'})$.

Because R' and R'' commute, $\lambda_{\mu q}g=\lambda_{\nu q}\nu g$ for each $g\in F(X\cup Y)$ and, consequently, $\lambda_{\mu q}p=\lambda_{\nu q}\nu p$ for each $p\in R''^{*M}(X\cup Y)$. Hence, if p is a proper M-dimensional system with coordinates in $R''^*(X\cup Y)$ we have $\mu p(\infty)=\mu\lambda_{p(\infty)}p=\lambda_{\mu p(\infty)}p$. Since μ and ν coincide on $R''^{*M}(X)$, we have also $\mu p(\infty)=\nu p(\infty)=\lambda_{\mu p(\infty)}p=\lambda_{\nu p(\infty)}\nu p$.

However, the $M\times N\times N$ elements $p'_{j,i,i'}=(\nu p_j)_{i,i'}$ all belong to $R^*(X\cup Z)$ and they constitute a proper system p' of dimension MN^2. Thus, by construction, $(\mu p(\infty)_j)_{i,i'}=p'(\infty)_{j,i,i'}$ identically. If, further, $p \in R''^{*M}_{\text{pol}}(X \cup Y)$ all the entries appearing in νp belong to $R^*_{\text{pol}}(X \cup Z)$ and then finally $(\mu p(\infty)_j)_{i,i'} \in R^*_{\text{alg}}(X)$.

This completes the proof because

$$a \odot b = \sum \{(b, f)\mu' f_{1,N} \cdot f : f \in F(X)\}$$
$$= \sum \{(b, f)\mu f_{1,N} : f \in F(X)\} = \mu b_{1,N}$$

where for each $x \in X$, μ is defined by $\mu x_{i,i'} = \mu' x_{i,i'} \cdot x$.

REMARK 1. Definitions 1, 2, and 3 and the computations of this section used only the structure of monoid of the additive groups considered. Hence, the results are still valid when an arbitrary *semi-ring* S is taken in place of R. For S consisting of two Boolean elements, Jungen's theorem and its special case for b rational have been obtained in a different form by Y. Bar-Hillel, M. Perles and E. Shamir [1] (also by S. Ginsburg and G. F. Rose [5]) and by S. Kleene [8] respectively as by-products of more sophisticated theories.

REMARK 2. Let $R = C$, the field of complex numbers; and p a proper system of dimension M. Introducing $4M$ new symbols z_j and replacing each y_j by $z_{4j} + iz_{4j+1} - z_{4j+2} - iz_{4j+3}$ in the p_js we can deduce from p a new system of dimension $4M$ in which all the coefficients are non-negative real numbers and whose solution is simply related to $p(\infty)$.

Assume now that $p \in C^{*M}_{\text{pol}}(X \cup Y)$ has only real non-negative coefficients and denote by α a homomorphism of $C_{\text{pol}}(X \cup Y)$ into C. Because of the assumption that $(p_j, y_{j'}) = (p_j, 1) = 0$, identically, we can find an $\epsilon > 0$ such that $|\alpha p_j| < \epsilon$ for all j when $|\alpha x| \leqq \epsilon$ and $|\alpha y| \leqq 2\epsilon$ for all $x \in X$ and $y \in Y$. Since the sequence $\alpha p(0)$, $\alpha p(1)$, $\cdots$, $\alpha p(n)$, $\cdots$ is monotonically increasing it converges to a finite solution (cf., e.g., [10]). surjection

Hence, the canonical epimorphism of $C_{\text{pol}}(X \cup Y)$ onto the ring of the ordinary (commutative) polynomials can be extended to an epimorphism of $C_{\text{alg}}(X)$ onto the ring of the Taylor series of the algebraic functions.

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