Comptes Rendus
Mathématique
0000, 1, n° 0, p. 000-000
DOI unassigned yet

On the exponential generating function of labelled trees

Sur la série génératrice des arbres étiquetés

Alin Bostan [a] **and Antonio Jiménez-Pastor** [*, b]

[a] Inria, 1 rue Honoré d'Estienne d'Orves 91120 Palaiseau, France

[b] Johannes Kepler University Linz, Doctoral Program Computational Mathematics, DK15.

E-mails: Alin.Bostan@inria.fr (A. Bostan), ajpastor@risc.uni-linz.ac.at (A. Jiménez-Pastor).

Abstract. We show that the generating function of labelled trees is not D^{∞}-finite.

Résumé. Nous montrons que la série génératrice des arbres étiquetés n'est pas D^{∞}-finie.

Keywords. Combinatorics, generating functions, differential equations.

2020 Mathematics Subject Classification. 05A15,12H05,34A34.

Funding. This research was partially funded by the Austrian Science Fund (FWF): W1214-N15, project DK15. It was also supported in part by the ANR DeRerumNatura project, grant ANR-19-CE40-0018 of the French *Agence Nationale de la Recherche*.

Electronic supplementary material. Supplementary material for this article is supplied as a separate archive available from the journal's website under article's URL or from the author.

This article is a draft (not yet accepted!)

Version française abrégée

Nous montrons que la série génératrice exponentielle des arbres étiquetés, $T(x) = \sum_{n\geq 1} \frac{n^{n-1}}{n!} x^n$, n'est pas D^{∞}-finie. En particulier, cela implique que, bien que $T(x)$ vérifie des équations différentielles non-linéaires, ces dernières ne peuvent pas être « trop simples ». En particulier, $T(x)$ n'est pas un quotient de deux fonctions D-finies (vérifiant des équations différentielles à coefficients polynomiaux), et plus généralement, $T(x)$ ne vérifie aucune équation différentielle linéaire à coefficients des fonctions D-finies. La preuve repose ultimement sur un résultat de théorie de Galois différentielle. Plusieurs questions ouvertes sont proposées, dont une sur la nature de la série génératrice ordinaire des arbres étiquetés, $\sum_{n\geq 1} n^{n-1} x^n$.

---

[*] Corresponding author.

ISSN (electronic) : 1778-3569 https://comptes-rendus.academie-sciences.fr/mathematique/

1. Context and main result

A formal power series $f(x) = \sum_{n\geq 0} a_n x^n$ in $\mathbb{C}[[x]]$ is called *differentially finite*, or simply D-*finite* [22], if it satisfies a *linear* differential equation with polynomial coefficients in $\mathbb{C}[x]$. Many generating functions in combinatorics and many special functions in mathematical physics are D-finite [2, 9].

DD-finite series and more generally D^n-finite series are larger classes of power series, recently introduced in [12]. DD-finite power series satisfy linear differential equations, whose coefficients are themselves D-finite power series. One of the simplest examples is $\tan(x)$, which is DD-finite (because it satisfies $\cos(x) f(x) - \sin(x) = 0$), but is not D-finite (because it has an infinite number of complex singularities, a property which is incompatible with D-finiteness). Another basic example is the exponential generating function of the Bell numbers B_n, which count partitions of $\{1, 2, \ldots, n\}$, namely:

$$B(x) := \sum_{n\geq 0} \frac{B_n}{n!} x^n. \tag{1}$$

Indeed, it is classical [9, p. 109] that $B(x) = e^{e^x - 1}$, therefore $B(x)$ is DD-finite. On the other hand, $B(x)$ is not D-finite: this can be proved either analytically (using the too fast growth of $B(x)$ as $x \to \infty$), or purely algebraically (using [21], and the fact that the power series e^x is not algebraic).

More generally, given a differential ring R, the set of *differentially definable* functions over R, denoted by $\mathrm{D}(R)$, is the differential ring of formal power series satisfying linear differential equations with coefficients in R. In particular, $\mathrm{D}(\mathbb{C}[x])$ is the ring of D-finite power series, $\mathrm{D}^2(\mathbb{C}[x]) := \mathrm{D}(\mathrm{D}(\mathbb{C}[x]))$ is the ring of DD-finite power series, and $\mathrm{D}^n(\mathbb{C}[x]) := \mathrm{D}(\mathrm{D}^{n-1}(\mathbb{C}[x]))$ is the ring of D^n-finite power series. We say that a power series $f(x) \in \mathbb{C}[[x]]$ is D^∞-*finite* if there exists an n such that $f(x)$ is D^n-finite.

It is known [13] that D^n-finite power series form a strictly increasing sequence of sets and that any D^∞-finite power series is *differentially algebraic*, in short D-*algebraic*, that is, it satisfies a differential equation, possibly *non-linear*, with polynomial coefficients in $\mathbb{C}[x]$. This class, as well as its complement (of *D-transcendental* series), are quite well studied [20, 23].

Let now $(t_n)_{n\geq 0} = (0, 1, 2, 9, 64, 625, 7776, \ldots)$ be the sequence whose general term t_n counts *labelled rooted trees* with n nodes. It is well known that $t_n = n^{n-1}$, for any n. This beautiful and non-trivial result is usually attributed to Cayley [6], although an equivalent result had been proved earlier by Borchardt [4], and even earlier by Sylvester, see [3, Chapter 4]. Due to the importance of the combinatorial class of trees, and to the simplicity of the formula, Cayley's result has attracted a lot of interest over the time, and it admits several different proofs, see e.g., [15, §4] and [1, §30]. One of the more conceptual proofs goes along the following lines (see [9, §II. 5.1] for details). Let

$$T(x) := \sum_{n\geq 0} \frac{t_n}{n!} x^n \tag{2}$$

be the exponential generating function of the sequence $(t_n)_n$. The class $\mathcal{T}$ of all rooted labelled trees is definable by a *symbolic equation* $\mathcal{T} = \mathcal{Z} \star \mathrm{SET}(\mathcal{T})$ reflecting their recursive definition, where $\mathcal{Z}$ represents the atomic class consisting of a single labelled node, and $\star$ denotes the labelled product on combinatorial classes. This symbolic equation provides, by syntactic translation, an implicit equation on the level of exponential generating functions:

$$T(x) = x\,\mathrm{e}^{T(x)}, \tag{3}$$

which can be solved using *Lagrange inversion*

$$t_n = n! \cdot [x^n]\, T(x) = n! \cdot \left(\frac{1}{n} [z^{n-1}] (e^z)^n \right) = n^{n-1}. \tag{4}$$

From (3), it follows easily that $T(x)$ is D-algebraic and satisfies the non-linear equation

$$x(1-T(x))T'(x) = T(x),$$

and from there, that the sequence $(t_n)_{n\geq 0}$ satisfies the non-linear recurrence relation

$$t_{n+1} = \frac{n+1}{n} \cdot \sum_{i=1}^{n} \binom{n}{i} t_i t_{n-i+1}, \quad \text{for all } n \geq 1.$$

This recurrence can also be proved using (4), by taking $y = n$, $x = w = 1$ in Abel's identity [11]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x+wk)^{k-1}(y-wk)^{n-k},$$

and then by isolating the term $k = n$ in the resulting equality.

On the other hand, it is known that the power series $T(x)$ is not D-finite, see [10, Theorem 7], or [8, Theorem 2]. This raises the natural question whether $T(x)$ is DD-finite, or D^n-finite for some $n \geq 2$. Our main result is that this is not the case:

Theorem 1. *The power series $T(x) = \sum_{n\geq 1} \frac{n^{n-1}}{n!} x^n$ in (2) is not D^∞-finite.*

To our knowledge, this is the first explicit example of a natural combinatorial generating function which is provably D-algebraic but not D^∞-finite. In particular, Theorem 1 implies that $T(x)$ is not equal to the quotient of two D-finite functions, and more generally, that it does not satisfy any linear differential equation with D-finite coefficients.

2. Proof of the main result

Our proof of Theorem 1 builds upon the following recent result by Noordman, van der Put and Top.

Theorem 2 ([17]). *Assume that $u(x) \in \mathbb{C}[[x]] \setminus \mathbb{C}$ is a solution of $u' = u^3 - u^2$. Then u is not D^∞-finite.*

The proof of Theorem 2 is based on two ingredients. The first one is a result by Rosenlicht [19] stating that any set of non-constant solutions (in any differential field) of the differential equation $u' = u^3 - u^2$ is algebraically independent over $\mathbb{C}$ (see also [17, Prop. 7.1]); the proof is elementary. The second one [17, Prop. 7.1] is that any non-constant power series solution of an autonomous first-order differential equation with this independence property cannot be D^∞-finite; the proof is based on differential Galois theory.

Proof of Theorem 1. We will use Theorem 2 and a few facts about the (principal branch of the) Lambert W function, satisfying $W(x) \cdot e^{W(x)} = x$ for all $x \in \mathbb{C}$.

Recall [7] that the Taylor series of W around 0 is given by

$$W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n = x - x^2 + \tfrac{3}{2}x^3 - \tfrac{8}{3}x^4 + \tfrac{125}{24}x^5 - \cdots.$$

In other words, our $T(x)$ and $W(x)$ are simply related by $W(x) = -T(-x)$.

The function defined by this series can be extended to a holomorphic function defined on all complex numbers with a branch cut along the interval $(-\infty, -\frac{1}{e}]$; this holomorphic function defines the principal branch of the Lambert W function.

We can substitute $x \mapsto e^{x+1}$ in the functional equation for $W(x)$ obtaining then

$$W(e^{x+1})e^{W(e^{x+1})} = e^{x+1},$$

or, renaming $Y(x) = W(e^{x+1})$, we have a new functional equation: $Y(x)e^{Y(x)-1} = e^x$. From this equality it follows by logarithmic differentiation that $Y'(x)\cdot(1+Y(x)) = Y(x)$.

Take now $U(x) := \frac{1}{1+Y(x)} = \frac{1}{2} - \frac{1}{8}x + \frac{1}{64}x^2 + \frac{1}{768}x^3 + \cdots$. We have that

$$U'(x) = \frac{-Y'(x)}{(1+Y(x))^2} = \frac{-Y(x)}{(1+Y(x))^3} = U(x)^3 - U(x)^2.$$

By Theorem 2, $U(x)$ is not D^∞-finite. By closure properties of D^∞-finite functions (see [13, Theorem 4] and [12, §3]), it follows that $Y(x)$ is not D^∞-finite either.

To conclude, note that by definition, for real x in the neighborhood of 0, we have $W(x) = Y(\log(x) - 1)$, and by Theorem 10 in [13], it follows that $W(x)$ and $T(x)$ are not D^∞-finite either, proving Theorem 1. □

3. Open questions

The class of D-finite power series is closed under Hadamard (term-wise) product. This is false for D^∞-finite power series; for instance, Klazar showed in [14] that the ordinary generating function $\sum_{n\geq 0} B_n x^n$ of the Bell numbers is not differentially algebraic, contrary to its exponential generating function (1), which is DD-finite.

Moreover, it was conjectured by Pak and Yeliussizov [18, Open Problem 2.4] that this is an instance of a more general phenomenon.

Conjecture 3 ([18, Open Problem 2.4]). *If for a sequence $(a_n)_{n\geq 0}$ both ordinary and exponential generating functions $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} a_n \frac{x^n}{n!}$ are D-algebraic, then both are D-finite. (Equivalently, $(a_n)_{n\geq 0}$ satisfies a linear recurrence with polynomial coefficients in n.)*

This conjecture has been recently proven for large (infinite) classes of generating functions [5]. However, the very natural example of the generating function for labelled trees escapes the method in [5].

We therefore leave the following as an open question.

Open question 1. *Is the power series $\sum_{n\geq 1} n^{n-1} x^n$ D^∞-finite? Is it at least differentially algebraic?*

According to Conjecture 3, the answer should be “no” for both questions in Open question 1.

Another natural question concerns the generating function for partition numbers:

$$\sum_{n\geq 0} p_n x^n := \prod_{n\geq 1} \frac{1}{1-x^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \cdots,$$

which is known to be differentially algebraic [16].

Open question 2. *Is it true that $\sum_{n\geq 0} p_n x^n$ is not D^∞-finite?*

One may also ask for the nature of the exponential variant of the generating function for partition numbers.

Open question 3. *Is the power series $\sum_{n\geq 0} \frac{p_n}{n!} x^n$ D^∞-finite, or at least differentially algebraic?*

Acknowledgements We would like to warmly thank the referees for their careful reading and for their constructive remarks.

References

[1] M. Aigner and G. M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, fourth edition, 2010.

[2] G. E. Andrews, R. Askey, and R. Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[3] N. L. Biggs, E. K. Lloyd, and R. J. Wilson. *Graph theory. 1736–1936*. The Clarendon Press, Oxford University Press, New York, second edition, 1986.

[4] C. W. Borchardt. Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante. *J. Reine Angew. Math.*, 57:111–121, 1860.

[5] A. Bostan, L. Di Vizio, and K. Raschel. Differential transcendence of Bell numbers and relatives — a Galois theoretic approach, 2020. In preparation.

[6] A. Cayley. A theorem on trees. *Q. J. Math.*, 23:376–378, 1889.

[7] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert W function. *Adv. Comput. Math.*, 5(4):329–359, 1996.

[8] P. Flajolet, S. Gerhold, and B. Salvy. On the non-holonomic character of logarithms, powers, and the nth prime function. *Electron. J. Combin.*, 11(2):Article 2, 16, 2004/06.

[9] P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.

[10] S. Gerhold. On some non-holonomic sequences. *Electron. J. Combin.*, 11(1):Research Paper 87, 8, 2004.

[11] J. L. W. V. Jensen. Sur une identité d'Abel et sur d'autres formules analogues. *Acta Math.*, 26(1):307–318, 1902.

[12] A. Jiménez-Pastor and V. Pillwein. A computable extension for D-finite functions: DD-finite functions. *J. Symbolic Comput.*, 94:90–104, 2019.

[13] A. Jiménez-Pastor, V. Pillwein, and M. F. Singer. Some structural results on D^n-finite functions. *Adv. in Appl. Math.*, 117:102027, 29, 2020.

[14] M. Klazar. Bell numbers, their relatives, and algebraic differential equations. *J. Combin. Theory Ser. A*, 102(1):63–87, 2003.

[15] L. Lovász. *Combinatorial problems and exercises*. North-Holland Publishing Co., Amsterdam, second edition, 1993.

[16] A. M. Mian and S. Chowla. The differential equations satisfied by certain functions. *J. Indian Math. Soc. (N.S.)*, 8:27–28, 1944.

[17] M. P. Noordman, M. van der Put, and J. Top. Combinatorial autonomous first order differential equations, 2019. Preprint, https://arxiv.org/abs/1904.08152v1.

[18] I. Pak. Complexity problems in enumerative combinatorics. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures*, pages 3153–3180. World Sci. Publ., Hackensack, NJ, 2018.

[19] M. Rosenlicht. The nonminimality of the differential closure. *Pacific J. Math.*, 52:529–537, 1974.

[20] L. A. Rubel. A survey of transcendentally transcendental functions. *Amer. Math. Monthly*, 96(9):777–788, 1989.

[21] M. F. Singer. Algebraic relations among solutions of linear differential equations. *Trans. Amer. Math. Soc.*, 295(2):753–763, 1986.

[22] R. P. Stanley. Differentiably finite power series. *European J. Combin.*, 1(2):175–188, 1980.

[23] J. van der Hoeven. Computing with D-algebraic power series. *Appl. Algebra Engrg. Comm. Comput.*, 30(1):17–49, 2019.