Measure, Π^0_1-classes and complete extensions of PA

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As it is well known many classical constructions in recursion theory can be viewed as simple forcing arguments. In particular, sets which force their jump are used in many of them and thus they are closely related to the concept of 1-genericity. For instance, Friedberg completeness criterion, constructions showing various structural properties for initial segments of degrees of the form $\mathcal{D}(\leq \underset{\sim}{a})$ where $\underset{\sim}{a}$ is sufficiently high degree (like $\geq \underset{\sim}{0}'$ or $\notin GL_2$) etc.

When studying properties of degrees of complete extensions of Peano arithmetic (PA) the situation is different. In fact, there is a degree of a complete extension of PA which does not bound any 1-generic degree. Constructions using a complete extension of PA as an oracle must, in general, use other arguments and cannot rely on forcing the jump, i.e. on 1-genericity. One of the convenient tools in this case is the use of Π^0_1-classes.

It is the aim of this paper to show properties of Π^0_1-classes of a special type and by means of these classes to show some results on degrees of complete extensions of PA.

Our notation and terminology are standard. In particular, a string is a finite sequence of 0's and 1's. Strings may also be viewed as functions from finite initial segments of $N = \{0,1,\dots\}$ into $\{0,1\}$. Letters such as σ, τ, ρ are reserved for strings. By $\tau \supseteq \sigma$ we denote that τ extends σ and for any $A \subseteq N$ by $\sigma \subseteq A$ we denote that the characteristic function of A extends σ. We often identify subsets of N with their characteristic functions. By $\tau \prec \sigma$ we denote that τ lexicographically precedes σ. lth (σ) is the length of σ and $\sigma * \tau$ is the string which results from concatenating σ and τ. We analogically use $\sigma * A$ for $A \subseteq N$. We apply notions of recursion theory to strings via Gödel-numbering.

By μ we denote the usual product measure on $\{0,1\}^N$.

For $A \subseteq N$, $n \in N$, let $(A)_n = \{x : \langle n, x \rangle \in A\}$ and let $|A|$ denote the cardinality of A.

Let Φ_e^A be the e-th recursive (partial) function using as oracle the set A and let Φ_e be $\Phi_e^{\emptyset}$. W_e is the domain of Φ_e. W_e^s denotes W_e enumerated to stage s. $f(x)\downarrow$ denotes that f(x) is defined and $f(x)\uparrow$ denotes that f(x) is undefined.

Let $Ext(\sigma)$ denote the class of all sets A for which $\sigma \subsetneq A$ and for $A \subseteq N$ let Ext(A) be the union of $Ext(\sigma)$ for $\sigma \in A$.

We use the usual notion of a tree, i.e. a tree is a function T from strings to strings such that $\tau \subseteq \sigma$ and $T(\sigma)\downarrow$ implies $T(\tau)\downarrow$ and $T(\tau) \subseteq T(\sigma)$, and if one of $T(\sigma * 0)$, $T(\sigma * 1)$ is defined then both are defined and they are incompatible.

A class $\mathcal{A}$ of subsets of N is a Σ_1^0-class and $x \in N$ is a Σ_1^0-index of $\mathcal{A}$ if $\mathcal{A} = Ext(W_x)$. A class of subsets of N is a Π_1^0-class and $x \in N$ is its Π_1^0-index if this class is the complement (in $\{0, 1\}^N$) of $Ext(W_x)$. Any Π_1^0-class can be thought as the set of all infinite branches of a recursive tree.

If a degree contains a set or a function with a certain property we say that the degree itself has that property.

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§ 1. NAP - sets and NAP - degrees

We shall first concentrate ourselves on sets out of all effectively small classes.

Any class containing only one set has measure zero but the question is what is the effectiveness of the measure. It leads to the following definition.

Definition. 1) A class $\mathcal{A}$ (of subsets of N) is of Σ_1^0-measure zero if there is a recursive sequence of Σ_1^0-indices of Σ_1^0-classes $\mathcal{B}_0, \mathcal{B}_1, \ldots$ such that

$$\forall n\ (\mu(\mathcal{B}_n) < 2^{-n}) \quad \text{and} \quad \mathcal{A} \subseteq \bigcap_n \mathcal{B}_n.$$

2) A set $B \subseteq N$ is called non-approximable in measure (a NAP-set) if the class $\{B\}$ is not of Σ_1^0-measure zero.

Remark. Reals corresponding to arithmetical NAP-sets (especially to those recursive in $\emptyset'$) and their role in constructive mathematical analysis were studied by O. Demuth. A partial survey can be found e.g. in Demuth & Kučera (1979) or in Demuth (1982).

P. Martin-Löf (1970) proved that there is a recursive sequence of Σ^0_1-indices of Σ^0_1-classes which is a universal approximating sequence for all classes of Σ^0_1-measure zero, i.e. the following.

Theorem 1. There is a recursive sequence of Σ^0_1-indices of Σ^0_1-classes $\mathcal{U}_0, \mathcal{U}_1, \ldots$ such that
1) $\mathcal{U}_0 \supseteq \mathcal{U}_1 \supseteq \ldots$
2) $\forall n\ (\mu(\mathcal{U}_n) < 2^{-n})$
3) for any class $\mathcal{A}$ of Σ^0_1-measure zero $\quad \mathcal{A} \subseteq \bigcap_n \mathcal{U}_n$.

Sketch of the proof. For every $n \in N$ we shall construct an r.e. set U_n such that $\mathcal{U}_n = \mathrm{Ext}(U_n)$.
Let n be given. For each $e, e > n$, we ask whether $\Phi_e(j)\downarrow$ for $j=0,\ldots,e$. Whenever this is the case and $x = \Phi_e(e)$ we take Σ^0_1-class with Σ^0_1-index x, i.e. the class $\mathrm{Ext}(W_x)$, and contribute for any $s \in N$ all elements of W^s_x to the constructed set U_n under the condition $\mu(\mathrm{Ext}(W^s_x)) < 2^{-e}$. $\square$

Notation. 1) By $U_0, U_1, \ldots$ and $\mathcal{U}_0, \mathcal{U}_1, \ldots$ we shall denote the r.e. sets and the Σ^0_1-classes described in the above proof. By $\mathcal{P}_0, \mathcal{P}_1, \ldots$ we shall denote the Π^0_1-classes which are the complements of the classes $\mathcal{U}_0, \mathcal{U}_1, \ldots$ correspondingly.
2) By NAP we shall denote the class of all NAP-sets.

The following facts follow immediately from the previous theorem.

Theorem 2. 1) $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \ldots$.
2) For any $n \in N$ the Π^0_1-class $\mathcal{P}_n$ contains only NAP-sets.
3) The class NAP is a Σ^0_2-class, namely $\mathrm{NAP} = \bigcup_n \mathcal{P}_n$.
4) The class NAP has measure one.

Although the class NAP is a Σ^0_2-class the class of NAP-degrees is at the same time a class of degrees of members of some Π^0_1-class.

Lemma 3. $\deg(\mathrm{NAP}) = \deg(\mathcal{P}_n)$ for any $n \in N$.

Proof. Let $n \in N$ be given.
It is easy to construct a recursive sequence of Σ^0_1-indices of Σ^0_1-classes $\mathcal{U}_n^{(0)}, \mathcal{U}_n^{(1)}, \ldots$ where
$U_n^{(0)} = U_n$, $U_n^{(k+1)} = \{\tau * \sigma : \tau \in U_n^{(k)} \ \&\ \sigma \in U_n\}$
and $\mathcal{U}_n^{(k)} = \mathrm{Ext}(U_n^{(k)})$.
Obviously $\mu(\mathcal{U}_n^{(k)}) < 2^{-n.(k+1)}$ for any $k \in N$.
Let now A be a NAP-set. Then, by the definition of a NAP-set, for

some $j \in N$ $A \notin \mathcal{U}_n^{(j)}$. Let k be the least such j.
If k=0 then $A \in \mathcal{P}_n$. If $k > 0$ then $A \in \mathcal{U}_n^{(k-1)}$ and $A \notin \mathcal{U}_n^{(k)}$. Hence, there is a string $\sigma \in U_n^{(k-1)}$ such that $\sigma \subseteq A$ and for any string $\tau \in U_n$ $\sigma * \tau \nsubseteq A$. Let B be the set for which $\sigma * B = A$. Then sets A and B are of the same degree and $B \notin \mathcal{U}_n$. Thus $B \in \mathcal{P}_n$. We conclude $\deg(\mathrm{NAP}) \subseteq \deg(\mathcal{P}_n)$. The other inclusion is obvious. □

The proof shows even more.

Corollary. If $\mathcal{A}$ is a Π_1^0-class for which $\mu(\mathcal{A}) > 0$, then $\deg(\mathrm{NAP}) \subseteq \deg(\mathcal{A})$.

The following lemma is obvious.

Lemma 4. For any NAP-set A and $n \in N$ the set $(A)_n$ is again a NAP-set.

We now give another useful information about components of NAP-sets.

Theorem 5. For any NAP-set A its components $(A)_0, (A)_1, \ldots..$ are strongly independent, i.e. for no i $(A)_i$ is recursive in $\{ \langle j,k \rangle : \langle j,k \rangle \in A \ \& \ j \neq i \}$.

Proof. By the method of the proof that the class of minimal degrees has measure zero (Sacks (1963)) it can be proved that any set A for which for some i $(A)_i \leq_T \{ \langle j,k \rangle : \langle j,k \rangle \in A \ \& \ j \neq i \}$ has Σ_1^0-measure zero. □

It follows as in Sacks (1963) that if $\underset{\sim}{a}$ is a NAP-degree then every countable partially ordered set can be embedded in $\mathcal{D}(\leq \underset{\sim}{a})$.

We shall now show that all NAP-sets are bi-effectively immune in a uniform way for any class $\mathcal{P}_n$.

Theorem 6. 1) Any NAP-set is bi-effectively immune.
2) There are recursive functions f,g such that for every $k \in N$, $x \in N$
$A \in \mathcal{P}_k \ \& \ W_x \subseteq A \Rightarrow |W_x| \leq f(k,x)$,
$A \in \mathcal{P}_k \ \& \ W_x \subseteq \overline{A} \Rightarrow |W_x| \leq g(k,x)$.

Proof. Evidently 2) implies 1). Let us prove 2).
Let $\mathcal{B}_{x,n}$ denote a class of sets A for which
$|W_x| \geq n+1$ & (A contains the first n+1 elements (in a standard enumeration) of W_x).
For any $x \in N$, $n \in N$ $\mathcal{B}_{x,n}$ is obviously a Σ_1^0-class and $\mu(\mathcal{B}_{x,n}) < 2^{-n}$. For any $k \in N$, $x \in N$ we can effectively find $e \in N$, $e > k$, such that

$\Phi_e(j)$ is a Σ^0_1-index of $\mathcal{B}_{x,j}$ for all $j \in N$. We have $\mathcal{B}_{x,e} \subseteq \mathcal{U}_e$, $\mathcal{U}_e \subseteq \mathcal{U}_k$ and hence $A \in \mathcal{P}_k \Rightarrow A \notin \mathcal{B}_{x,e}$.
We conclude $W_x \subseteq A$ & $A \in \mathcal{P}_k \Rightarrow |W_x| \leq e$. An analogical argument for sets $\overline{A}$, $A \in \mathcal{P}_k$, completes the proof. □

Definition. Let us call a function h to be fixed point free (an FPF-function) if for every $x \in N$ $W_x \neq W_{h(x)}$. Let FPF denote the class of all FPF-functions.

It is well known that to any effectively immune set A we can find an FPF-function recursive in A. Since the class of FPF-degrees is obviously closed upwards we have the following corollary.

Corollary 1. $\deg(\mathrm{NAP}) \subseteq \deg(\mathrm{FPF})$.

We shall show later that these two classes of degrees are not the same.

It should be mentioned that the following statement has been known but it is difficult to trace who first proved it.

Corollary 2. The class deg(FPF) has measure one.

By a theorem of Arslanov (1977) the only r.e. degree having below an FPF-degree is $\underset{\sim}{0}'$. Thus we have.

Corollary 3. $\underset{\sim}{0}'$ is the only r.e. degree $\underset{\sim}{e}$ for which there is a NAP-degree $\underset{\sim}{a}$, $\underset{\sim}{a} \leq \underset{\sim}{e}$.

Remark. Since $\mathcal{P}_n$ is a Π^0_1- class for any $n \in N$ it follows by lemma 3 and by Jockusch & Soare (1972a),(1972b) that $\underset{\sim}{0}'$ is a NAP-degree and that there are NAP-degrees $\underset{\sim}{a}$, $\underset{\sim}{a} < \underset{\sim}{0}'$.

There is a natural question whether the class of NAP-degrees is closed upwards. We shall see later that this is not the case by which we shall also show that $\deg(\mathrm{NAP}) \neq \deg(\mathrm{FPF})$. On the other hand, however, the class of NAP-degrees contains the upper cone of degrees which are above $\underset{\sim}{0}'$.

Theorem 7. $\deg(\mathrm{NAP}) \supseteq \{\underset{\sim}{a} : \underset{\sim}{a} \geq \underset{\sim}{0}'\}$.

There are several methods for coding information into a NAP-set. We chose one which is useful also in another context. The following simple lemma plays an important role in our method. Especially we shall use its corollary.

<u>Lemma 8.</u> 1) There is a recursive function f such that for any string σ and $n \in N$ if $\mathcal{P}_n \cap \mathrm{Ext}(\sigma) \neq \emptyset$, then $\mu(\mathcal{P}_n \cap \mathrm{Ext}(\sigma)) \geq f(\sigma, n)$.
2) There is a recursive function g such that for any string σ and $n \in N$, $x \in N$ if $(\mathcal{P}_n \cap \mathrm{Ext}(\sigma)) - \mathrm{Ext}(W_x) \neq \emptyset$, then $\mu((\mathcal{P}_n \cap \mathrm{Ext}(\sigma)) - \mathrm{Ext}(W_x)) \geq g(\sigma, n, x)$.

<u>Proof</u>. 1) Let, for each $n \in N$, $U_n = W_{y_n}$ and therefore $\mathcal{U}_n = \mathrm{Ext}(W_{y_n})$.
For any string σ and $n \in N$ we construct $e \in N$, $e > n$, such that for all $j \in N$ a) $\Phi_e(j)\downarrow$ iff there is some $s \in N$ for which $\mu(\mathrm{Ext}(\sigma) - \mathrm{Ext}(W^s_{y_n})) < 2^{-j}$,
b) if $\Phi_e(j)\downarrow$ and t is the least s witnessing it, then $\Phi_e(j)$ is a Σ^0_1-index of the Σ^0_1-class $\mathrm{Ext}(\sigma) - \mathrm{Ext}(W^t_{y_n})$.
Then $\Phi_e(e)\downarrow$ implies $\mathrm{Ext}(\sigma) \subseteq \mathcal{U}_n$, i.e. $\mathrm{Ext}(\sigma) \cap \mathcal{P}_n = \emptyset$.
This proves 1). 2) can be proved by the same method. $\square$

<u>Notation</u>. 1) For any string σ and $x \in N$ if $\mathcal{A}$ is the Π^0_1-class with Π^0_1-index x and if $\mathcal{A} \cap \mathrm{Ext}(\sigma) \neq \emptyset$, then by Left (σ, x) or Right(σ, x) we denote the left-most or right-most infinite branch of $\mathcal{A} \cap \mathrm{Ext}(\sigma)$ (in the lexicographic ordering).
2) Let p_0 be a Π^0_1-index of the class $\mathcal{P}_0$.

Let us remember that in any nonempty Π^0_1-class the left-most and right-most infinite branches are of r.e. degree and hence they are recursive in $\emptyset'$ (Jockusch & Soare(1972a)). Thus, by the corollary 3 of theorem 6, whenever $\mathcal{P}_0 \cap \mathrm{Ext}(\sigma) \neq \emptyset$ then Left(σ, p_0) and Right(σ, p_0) are of the degree $\underset{\sim}{0}'$.

Immediately from the previous lemma we have.

<u>Corollary</u>. 1) There is a recursive function F such that for any string σ $\mathcal{P}_0 \cap \mathrm{Ext}(\sigma) \neq \emptyset$ implies $\mu x(\mathrm{Left}(\sigma, p_0)(x) \neq \mathrm{Right}(\sigma, p_0)(x)) \leq F(\sigma)$. ($\mu x$ = the least x)
2) There is a recursive function G such that for any string σ and $y \in N$ if $\mathcal{A}$ is the Π^0_1-class with Π^0_1-index y then $(\mathcal{A} \subseteq \mathcal{P}_0$ and $\mathcal{A} \cap \mathrm{Ext}(\sigma) \neq \emptyset)$ implies $\mu x(\mathrm{Left}(\sigma, y)(x) \neq \mathrm{Right}(\sigma, y)(x)) \leq G(\sigma, y)$.

<u>Proof of the theorem 7</u>. We shall construct a total tree T recursive in $\emptyset'$ such that
1) all of the infinite branches of T are NAP-sets,
2) for any set A of degree $\geq \underset{\sim}{0}'$ $\deg(T(A)) = \deg(A)$.
We shall use the Π^0_1-class $\mathcal{P}_0$.

The idea is given already $T(\sigma)$ to take $T(\sigma * 0)$ and $T(\sigma * 1)$ as some incompatible beginnings of $\mathrm{Left}(T(\sigma),p_0)$ and of $\mathrm{Right}(T(\sigma),p_0)$ so that from any of them we can recursively decide whether it is a beginning of $\mathrm{Left}(T(\sigma),p_0)$ or of $\mathrm{Right}(T(\sigma),p_0)$. We shall arrange our construction so that for every string σ

(1) $\quad \mathrm{Ext}(T(\sigma)) \cap \mathcal{P}_0 \neq \emptyset$.

Let F be the recursive function from the previous corollary.
Let $T(\emptyset) = \emptyset$. Assume $T(\sigma)$ has been defined so that (1) holds. We can find recursively in $\emptyset'$ strings τ_0, τ_1 such that $T(\sigma) \subsetneq \tau_i$ and $\mathrm{lth}(\tau_i)=F(T(\sigma))$ for $i=0,1$ and $\tau_0 \subsetneq \mathrm{Left}(T(\sigma),p_0)$ and $\tau_1 \subsetneq \mathrm{Right}(T(\sigma),p_0)$. Strings τ_0, τ_1 are, because of their length, incompatible. Let $T(\sigma * i) = \tau_i$ for $i=0,1$. (1) remains valid for these strings (instead of $T(\sigma)$).
The tree T is obviously recursive in $\emptyset'$, total and its infinite branches are NAP-sets (even from the class $\mathcal{P}_0$).
For every set A $\quad T(A) \leqslant_T A \oplus \emptyset'$ and hence $\emptyset' \leqslant_T A$ implies $T(A) \leqslant_T A$. It remains only to show that for any set A $\quad A \leqslant_T T(A)$. Suppose we already know σ, $\mathrm{lth}(\sigma)=n$, $\sigma \subseteq A$. From $T(A)$ by means of the recursive function F we compute $T(\sigma)$ and then the length m of both $T(\sigma * 0)$, $T(\sigma * 1)$. Let τ be a string such that $\mathrm{lth}(\tau)=m$, $\tau \subseteq T(A)$.
Although we cannot (in general) compute both $T(\sigma * 0)$, $T(\sigma * 1)$ we can decide whether $\tau = T(\sigma * 0)$ or $\tau = T(\sigma * 1)$. We can do that because questions whether τ is a beginning of the left-most or right-most infinite branch of $\mathcal{P}_0 \cap \mathrm{Ext}(T(\sigma))$, i.e. whether

$$\forall \rho\, (\mathrm{lth}(\rho)=\mathrm{lth}(\tau)\ \&\ T(\sigma) \subseteq \rho\ \&\ \rho \prec \tau \Rightarrow \mathrm{Ext}(\rho) \subseteq \mathcal{U}_0) \quad \text{or}$$
$$\forall \rho\, (\mathrm{lth}(\rho)=\mathrm{lth}(\tau)\ \&\ T(\sigma) \subseteq \rho\ \&\ \tau \prec \rho \Rightarrow \mathrm{Ext}(\rho) \subseteq \mathcal{U}_0)$$

are Σ^0_1-questions and just for one of them there is the true answer. Thus we can decide whether $\sigma * 0 \subsetneq A$ or $\sigma * 1 \subsetneq A$. This completes the proof. $\square$

§2. Degrees $\gg \underset{\sim}{0}$

Following Simpson (1977) by $\underset{\sim}{a} \gg \underset{\sim}{0}$ we denote that any nonempty Π^0_1-class (of subsets of N) has a member of degree $\leq \underset{\sim}{a}$ or, equivalently, that there is a complete extension of PA of degree $\underset{\sim}{a}$. The relation $\underset{\sim}{a} \ll \underset{\sim}{b}$ is defined in the analogical way (by relativization).

We shall first recall some known facts, see Simpson (1977) and Jockusch & Soare (1972a), (1972b).

The class of all complete extensions of PA is a Π^0_1-class. We shall denote it by $\mathcal{PA}$. The class $\{\underset{\sim}{a} : \underset{\sim}{a} \gg \underset{\sim}{0}\}$ is closed upwards, i.e. it forms an upper cone. For any degrees $\underset{\sim}{a}$, $\underset{\sim}{b}$ if $\underset{\sim}{a} \ll \underset{\sim}{b}$ then there is a degree $\underset{\sim}{c}$ such that $\underset{\sim}{a} \ll \underset{\sim}{c} \ll \underset{\sim}{b}$. The degree $\underset{\sim}{0}'$ is the only r.e. degree $\underset{\sim}{a}$ for which $\underset{\sim}{a} \gg \underset{\sim}{0}$. The class $\{\underset{\sim}{a} : \underset{\sim}{a} \gg \underset{\sim}{0}\}$ has measure zero. Let us recall also the compactness of the space $\{0,1\}^N$.

We shall now list some easy facts for which we omit the proof.

1) The intersection of two Π^0_1-classes is a Π^0_1-class.

2) For any $z \in N$, $x \in N$ $\{A : \Phi^A_z(x)\uparrow\}$ is a Π^0_1-class.

3) The question whether a Π^0_1-class is nonempty is, in fact, a Π^0_1-question and thus can be answered recursively in $\emptyset'$.

4) If $\mathcal{A}$ is a Π^0_1-class and $z \in N$ such that Φ^A_z is total and takes only values 0,1 for every $A \in \mathcal{A}$, then

a) the image of $\mathcal{A}$ by Φ^X_z denoted by $\Phi_z[\mathcal{A}]$, i.e. the class $\{B : B = \Phi^A_z \text{ for some } A \in \mathcal{A}\}$, is again a Π^0_1-class,

b) if $\mathcal{B}$ is a Π^0_1-class then the inverse image of $\mathcal{B}$ in $\mathcal{A}$ by Φ^X_z, i.e. the class $\{A : A \in \mathcal{A} \,\&\, \Phi^A_z \in \mathcal{B}\}$, is again a Π^0_1-class.

Furthermore, the above facts hold uniformly with respect to Π^0_1-indices.

We shall first return to the question whether the class of NAP-degrees forms an upper cone. This is not the case, moreover, it even does not contain the class $\{\underset{\sim}{a} : \underset{\sim}{a} \gg \underset{\sim}{0}\}$.

<u>Theorem 9</u>. There is a degree $\underset{\sim}{a}$, $\underset{\sim}{a} \gg \underset{\sim}{0} \,\&\, \underset{\sim}{a} \leq \underset{\sim}{0}^{(2)}$, such that for any NAP-degree $\underset{\sim}{b}$, $\underset{\sim}{b} \leq \underset{\sim}{a}$, there is no degree $\underset{\sim}{c}$, $\underset{\sim}{c} \gg \underset{\sim}{0} \,\&\, \underset{\sim}{c} \leq \underset{\sim}{b}$ (and hence $\underset{\sim}{b} < \underset{\sim}{a}$).

If $\underset{\sim}{a}$ is such degree then the classes of NAP-degrees and $\{\underset{\sim}{c} : \underset{\sim}{c} \gg \underset{\sim}{0}\}$ restricted to $\mathcal{D}(\leq \underset{\sim}{a})$ are disjoint. Since below any degree $\underset{\sim}{a}$, $\underset{\sim}{a} \gg \underset{\sim}{0}$, there is a NAP-degree, we have the following.

<u>Corollary 1</u>. The class of NAP-degrees does not form an upper cone.

<u>Corollary 2</u>. $\deg(\mathrm{NAP}) \nsubseteq \deg(\mathrm{FPF})$.

<u>Proof of the theorem 9</u>. The proof combines the method of hyperimmune free degree construction with some facts concerning measure. We have to construct recursively in $\emptyset^{(2)}$ a complete extension T of PA such that for any $z \in N$ if Φ^T_z is total and is a NAP-set, say B, then for any $v \in N$ Φ^B_v is not a complete extension of PA. This condition can be simplified because of lemma 3. We can replace in the above "if Φ^T_z is total and is a NAP-set" equivalently by "if $\Phi^T_z \in \mathcal{P}_0$"

In what follows we shall suppose, without loss of generality, that for any set A and $z \in N$ Φ_z^A takes only values 0,1.
We shall construct recursively in $\emptyset^{(2)}$ a sequence of Π_1^0-indices of nonempty Π_1^0-classes $\mathcal{A}_0, \mathcal{A}_1, \ldots$ such that $\mathcal{PA} \supseteq \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \ldots$ and $\bigcap_n \mathcal{A}_n$ will contain only one set. This set will have the required properties.
Let $\mathcal{A}_0 = \mathcal{PA}$ and let $n \in N$. Assume that a Π_1^0-index of Π_1^0-class $\mathcal{A}_n$ has been constructed.
Step n+1. Let $n = \langle z, v \rangle$.
Case 1. There is an $x \in N$ such that $\mathcal{A}_n \cap \{A : \Phi_z^A(x)\uparrow\} \neq \emptyset$.
Let $\mathcal{A}_{n+1} = \mathcal{A}_n \cap \{A : \Phi_z^A(x)\uparrow\}$.
Case 2. For any $A \in \mathcal{A}_n$ Φ_z^A is total and $\Phi_z[\mathcal{A}_n] \cap \mathcal{P}_0 = \emptyset$.
Let $\mathcal{A}_{n+1} = \mathcal{A}_n$.
Case 3. For any $A \in \mathcal{A}_n$ Φ_z^A is total and $\Phi_z[\mathcal{A}_n] \cap \mathcal{P}_0 \neq \emptyset$.
In this case we have to consider two further subcases. First, let $\mathcal{B} = \mathcal{P}_0 \cap \Phi_z[\mathcal{A}_n]$.
Subcase 3a. There is an $x \in N$ such that $\mathcal{B} \cap \{B : \Phi_v^B(x)\uparrow\} \neq \emptyset$.
Let $\mathcal{A}_{n+1} = \{A : A \in \mathcal{A}_n \;\&\; \Phi_z^A \in \mathcal{P}_0 \;\&\; \Phi_v^B(x)\uparrow$ (B stands for Φ_z^A)$\}$.
Subcase 3b. For any $B \in \mathcal{B}$ Φ_v^B is total.
Since $\mathcal{B} \subseteq \mathcal{P}_0$ and $\mathcal{B} \neq \emptyset$ we have by the corollary of lemma 3 $\deg(\mathcal{B}) \supseteq \deg(\mathrm{NAP})$ and thus the measure of the class $\deg(\mathcal{B})$ is one. Hence, the measure of the upper cone of degrees generated by degrees of $\Phi_v[\mathcal{B}]$ is one, too. On the other hand, the measure of the class $\{\underset{\sim}{a} : \underset{\sim}{a} \gg \underset{\sim}{0}\}$ is zero. Hence, there must be a string σ such that $\mathrm{Ext}(\sigma) \cap \mathcal{PA} = \emptyset$ and $\mathrm{Ext}(\sigma) \cap \Phi_v[\mathcal{B}] \neq \emptyset$. Let
$\mathcal{A}_{n+1} = \{A : A \in \mathcal{A}_n \;\&\; \Phi_z^A \in \mathcal{P}_0 \;\&\; \Phi_v^B \in \mathrm{Ext}(\sigma)$ (B stands for Φ_z^A)$\}$.
Of course, in all cases (subcases) $\mathcal{A}_{n+1}$ is a nonempty Π_1^0-class and $\mathcal{A}_{n+1} \subseteq \mathcal{A}_n$. Since we can determine recursively in $\emptyset^{(2)}$ which case (subcase) applies and after this we can find recursively in $\emptyset^{(2)}$ (even in $\emptyset'$) a corresponding witness (whenever needed) we can obviously find recursively in $\emptyset^{(2)}$ a Π_1^0-index of $\mathcal{A}_{n+1}$. It is easy to verify that $\bigcap_n \mathcal{A}_n$ contains only one set, say T. We conclude that $T \leq_T \emptyset^{(2)}$ and that T has all the required properties. $\square$

We see that the technique of coding information used in the proof of theorem 7 does not work in a general case. On the other hand its modification will allow us to prove the cupping property of degrees $\underset{\sim}{a} \gg \underset{\sim}{0}$.

A degree $\underset{\sim}{a}$ is said to have the cupping property if for every degree $\underset{\sim}{b} > \underset{\sim}{a}$ there exists $\underset{\sim}{c} < \underset{\sim}{b}$ such that $\underset{\sim}{a} \cup \underset{\sim}{c} = \underset{\sim}{b}$.

Friedberg, in fact, showed that $\underset{\sim}{0}'$ has the cupping property. More generally, Jockusch & Posner (1978) showed that every degree $\notin GL_2$ has the cupping property and Jockusch (1980) showed it for every 2-generic degree.

We shall now show that every degree $\underset{\sim}{a} \gg \underset{\sim}{0}$ has the cupping property. It should be mentioned that there are degrees $\underset{\sim}{a} \gg \underset{\sim}{0}$ which are neither 2-generic nor $\notin GL_2$ (i.e. which are in GL_2). It follows from the fact that every nonempty Π^0_1-class has a member A of hyper-immune free degree such that $A^{(2)} \equiv_T \emptyset^{(2)}$ (Jockusch & Soare (1972b)). Therefore there is a hyper-immune free degree $\underset{\sim}{a} \gg \underset{\sim}{0}$ which is in GL_2 and which, of course, cannot be 2-generic (even no 1-generic degree is hyper-immune free). These facts indicate that we cannot use the usual techniques of proofs connected with 1-genericity, i.e. with forcing the jump. We shall use instead a technique based on some Π^0_1-classes. Although it would be possible to use the Π^0_1-class $\mathcal{PA}$ and its subclasses we prefer to work with Π^0_1-classes of positive measure like $\mathcal{P}_0$ and its subclasses. It seems to be more instructive and it can be also used in another context.

<u>Theorem 10</u>. Every degree $\underset{\sim}{a} \gg \underset{\sim}{0}$ has the cupping property.

<u>Proof</u>. The idea is, given $A \in \underset{\sim}{a}$, $\underset{\sim}{a} \gg \underset{\sim}{0}$, to construct a total tree T recursive in A such that

1) the degrees of infinite branches of T avoid the upper cone of degrees above $\underset{\sim}{a}$,

2) for any set B, $A \leq_T B$ implies $B \equiv_T T(B) \oplus A$.

In what follows we shall suppose, without loss of generality, that for any set A and $z \in N$ Φ^A_z takes only values 0,1.

We shall first illustrate the method for the case $\underset{\sim}{a} = \underset{\sim}{0}'$ and then we indicate how it can be adapted for the general case $\underset{\sim}{a} \gg \underset{\sim}{0}$.

We shall construct a total tree T recursive in $\emptyset'$ satisfying 1) and 2) (with $\underset{\sim}{0}'$ and $\emptyset'$ instead of $\underset{\sim}{a}$ and A) and a total function S recursive in $\emptyset'$ which associates to every string σ a Π^0_1-index of some nonempty Π^0_1-class, denoted by $\mathcal{A}_\sigma$, such that $\mathrm{Ext}(T(\sigma)) \supseteq \mathcal{A}_\sigma$. Furthermore, $\mathcal{A}_{\sigma * i} \subseteq \mathcal{A}_\sigma$ will hold for every string σ and i = 0,1.

Let $T(\emptyset) = \emptyset$ and let $S(\emptyset) = p_0$ (a Π^0_1-index of $\mathcal{P}_0$).

Let $T(\sigma)$, $S(\sigma)$, $\mathrm{lth}(\sigma) = n$ be given. We shall define $T(\sigma * i)$ and $S(\sigma * i)$ for i = 0,1.

We claim that there is an $x \in N$ such that

$$(2) \qquad \mathcal{A}_\sigma \cap \{E : \Phi^E_n(x)\uparrow \text{ or } \Phi^E_n(x)\downarrow \neq \emptyset'(x)\} \neq \emptyset.$$

Suppose (2) does not hold. Then for every $x \in N$ $\mathcal{A}_\sigma \cap \{E : \Phi_n^E(x)\uparrow\} = \emptyset$, i.e. for any $E \in \mathcal{A}_\sigma$ Φ_n^E is total and, at the same time, the Π_1^0-class $\Phi_n[\mathcal{A}_\sigma]$ contains only one element, namely $\emptyset'$. But then $\emptyset'$ would be recursive which is not the case. Our claim is proved. Moreover, there is a recursive function yielding from $S(\sigma)$ and n an upper bound for an x witnessing(2).
Let $x_0 \in N$ be the first x for which (2) holds and let

$$\mathcal{B} = \mathcal{A}_\sigma \cap \{E : \Phi_n^E(x_0)\uparrow \text{ or } \Phi_n^E(x_0)\downarrow \neq \emptyset'(x_0)\} .$$

It is easy to verify that $\mathcal{B}$ is a Π_1^0-class. Furthermore, we can find recursively in $\emptyset'$ a Π_1^0-index of $\mathcal{B}$, say k, from n and from $S(\sigma)$ (a Π_1^0-index of $\mathcal{A}_\sigma$). In fact, by means of $\emptyset'$ we can decide the validity of (2) for every $x \in N$ and thus find x_0.
As in the proof of theorem 7 we take $T(\sigma * 0)$ and $T(\sigma * 1)$ as some incompatible beginnings of $\mathrm{Left}(T(\sigma),k)$ and $\mathrm{Right}(T(\sigma),k)$. It is important that the length of these beginnings is computed recursively from $T(\sigma)$ and k. More precisely, we can find recursively in $\emptyset'$ strings τ_0, τ_1 such that $T(\sigma) \subseteq \tau_i$, $\mathrm{lth}(\tau_i) = G(T(\sigma),k)$ $i = 0,1$, $\tau_0 \subseteq \mathrm{Left}(T(\sigma),k)$, $\tau_1 \subseteq \mathrm{Right}(T(\sigma),k)$, where G is the recursive function from the corollary of lemma 8.
Let $T(\sigma * i) = \tau_i$ and let $S(\sigma * i)$ be a Π_1^0-index of the Π_1^0-class $\mathcal{B} \cap \mathrm{Ext}(T(\sigma * i))$ for $i = 0,1$.
It is clear that T is a total tree recursive in $\emptyset'$ and S a total function recursive in $\emptyset'$. Furthermore, the degrees of infinite branches of T avoid the upper cone of degrees above $\underset{\sim}{0}'$. (We do not need at this moment that they are NAP-degrees).
It remains to show that for any set B $\emptyset' \leq_T B$ implies $B \equiv_T T(B) \oplus \emptyset'$.
First, $\emptyset' \leq_T B$ obviously implies $T(B) \oplus \emptyset' \leq_T B$.
Second, it is also clear that for any set B $B \leq_T T(B) \oplus \emptyset'$.
This completes the proof for $\underset{\sim}{a} = \underset{\sim}{0}'$.
It remains to show that this proof can be relativized to any $\underset{\sim}{a} \gg \underset{\sim}{0}$.
Informal description. In the above construction we build some finite objects (like strings, indices of classes) from some other finite objects by procedures which can be described in the language of PA. Hence, we can view this construction as a construction in the standard model of PA where we use in fact the true answers on Σ_1^0-questions or Π_1^0-questions.
Given a degree $\underset{\sim}{a} \gg \underset{\sim}{0}$ we can take a complete extension A of PA of degree $\underset{\sim}{a}$ and consider a (countable) model M of A.
We now want to carry out our construction in such a way that we use the answers from A, i.e. consistent answers which correspond to the

situation in the model, instead of the true answers and verify that we succeed.
It can also be viewed as a construction in M giving us a tree T_M in M from which we, at the end, take only the standard part. This part should have the required properties. Thus, at least, we have to be careful to keep ourselves throughout the whole standard part of the construction in the standard initial segment of M and not to go to nonstandard elements.
Our method of the construction of T and S described for $\underset{\sim}{a} = \underset{\sim}{0}'$ allows us to use this way, i.e. to use answers from A instead of the arithmetical truth. Let us look at the main parts.
i) Let us check our claim, i.e. (2). In M for standard σ and $S(\sigma)$ (n is standard since $n = \mathrm{lth}(\sigma)$) we obtain a standard witness x of (2) otherwise $\emptyset'$ of the model M (the truth for Σ^0_1-sentences in M) would be definable by a Δ^0_1-formula. Moreover, for standard σ we can effectively find a standard upper bound for a witness of (2) by a simple diagonal argument as in case $\underset{\sim}{a} = \underset{\sim}{0}'$ (but we shall not use this fact). Thus, for any (standard) string σ there is an $x \in N$ such that A proves (2) - modulo an obvious reformulation.
ii) From $S(\sigma) \in N$ and $n \in N$ we can find by means of A a Π^0_1-index $k \in N$ of the corresponding class $\mathcal{B}$.
iii) Corollary of lemma 8 is proved by a simple diagonal argument which works (mutatis mutandis) in PA. In other words, the recursive function G works in the model M, too.
In this way we obtain a total tree which is recursive in A. (T is in fact the standard part of a tree T_M constructed inside M.)
It remains to show that the tree T has properties 1),2).
If we use the provability in PA of true Σ^0_1-sentences (of PA) we easily see that T has the property 1), i.e. the degrees of infinite branches of T avoid the upper cone of degrees above $\underset{\sim}{a}$ (in fact, above a little less degree, namely the degree of "$A \cap (\Sigma^0_1$-formulas) "). The property 2) is then obvious.
The formal description is tedious and it would be, perhaps, interesting to carry out it in details if we wanted to know how large fragment of PA is in fact needed. $\square$

Remark. It would be possible to work with the class $\mathcal{PA}$ instead of our class $\mathcal{P}_0$. In this case, however, we should take an additional care to ensure in the general case of degrees $\gg \underset{\sim}{0}$ e.g. an analogue of the corollary of lemma 8. This is possible but we find it less convenient.

Let us turn to another property of degrees. Following Simpson (1977) a degree $\underset{\sim}{a}$ is said to be cuppable if for every nonzero degree $\underset{\sim}{b} < \underset{\sim}{a}$ there is a degree $\underset{\sim}{c} < \underset{\sim}{a}$ such that $\underset{\sim}{b} \cup \underset{\sim}{c} = \underset{\sim}{a}$.

Posner & Robinson (1981) showed that $\underset{\sim}{0}'$ is cuppable. Posner (1977) proved a stronger result that every degree $a \in GH_1$ is cuppable, see Lerman (1983). The proofs of these results are based again on forcing the jump. As we already know this method cannot be used, in general, in the case of degrees $\gg \underset{\sim}{0}$. Thus, there is an interesting question whether degrees $\gg \underset{\sim}{0}$ are cuppable but the answer is not known.

Problem. Is every degree $\underset{\sim}{a} \gg \underset{\sim}{0}$ cuppable?

The following theorem illustrates this problem by showing a special behaviour of degrees $\gg \underset{\sim}{0}$.

Theorem 11. There are degrees $\underset{\sim}{a} \gg \underset{\sim}{0}$ and $\underset{\sim}{b}$, $\underset{\sim}{0} \neq \underset{\sim}{b} < \underset{\sim}{a}$, such that for every degree $\underset{\sim}{c}$ $\quad \underset{\sim}{c} \gg \underset{\sim}{0}$ & $\underset{\sim}{c} \leq \underset{\sim}{a}$ implies $\underset{\sim}{b} \leq \underset{\sim}{c}$ (i.e. all degrees $\gg \underset{\sim}{0}$ below $\underset{\sim}{a}$ are above $\underset{\sim}{b}$).

Proof. Let us take a degree $\underset{\sim}{a} \gg \underset{\sim}{0}$ which is hyper-immune free (Jockusch & Soare (1972b)) and let $A \in \underset{\sim}{a}$. Let (B_1, B_2) be a pair of disjoint recursively inseparable r.e. sets such that $\overline{B_1 \cup B_2}$ is hyper-immune (obtained, e.g. by splitting a maximal set).
Let $\mathcal{A}$ be a Π^0_1-class of all sets separating this pair. Let us take a set B from $\mathcal{A}$ which is recursive in A and let $\underset{\sim}{b} = \deg(B)$.
We shall show that the degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the required properties. We claim that the symmetric difference of any two sets from $\mathcal{A}$ which are recursive in A is finite. Otherwise, there would be an infinite subset of hyper-immune set $\overline{B_1 \cup B_2}$ which is recursive in A. But it is not possible since $\deg(A)$ is a hyper-immune free degree.
Given a degree $\underset{\sim}{c}$, $\underset{\sim}{c} \gg \underset{\sim}{0}$ & $\underset{\sim}{c} \leq \underset{\sim}{a}$, there is a set $E \in \mathcal{A}$ such that $\deg(E) \leq \underset{\sim}{c}$ and hence $\deg(E) \leq \underset{\sim}{a}$. By our claim $E \in \underset{\sim}{b}$. □

Let us consider as a special type of the above problem degrees $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as in the preceding theorem. Is there a degree $\underset{\sim}{d} < \underset{\sim}{a}$ such that $\underset{\sim}{b} \cup \underset{\sim}{d} = \underset{\sim}{a}$?

As a final remark, let us mention briefly another result. As it is known the method of forcing the jump allows to prove that for any sufficiently high degree $\underset{\sim}{a}$ the initial segment of degrees $\mathcal{D}(\leq \underset{\sim}{a})$ does not form a lattice. More precisely, Jockusch(1980) showed it for every 1-generic degree and thus, by Jockusch & Posner(1978), for every degree $\notin GL_2$. There is again a question whether this holds even for every degree $\gg \underset{\sim}{0}$. The author would like to announce that this is really the case, i.e. for no degree $\underset{\sim}{a} \gg \underset{\sim}{0}$ $\mathcal{D}(\leq \underset{\sim}{a})$ is a lattice. The proof (to appear in a paper in preparation) uses the following technique. Given a degree $\underset{\sim}{a} \gg \underset{\sim}{0}$ and a set $A \in \underset{\sim}{a}$ we construct sets C,D recursive in A in such a way that

1) we take (uniformly in A) a sequence of sets $B_0, B_1, \ldots\ldots$ such that $\deg(B_0) \ll \deg(B_1) \ll \ldots\ldots \ll \underset{\sim}{a}$, and $\underset{\sim}{0} \ll \deg(B_0)$,

2) we relativize the concept of a NAP-set and work with relativized Π^0_1-classes of relativized NAP-sets,

3) whenever $\Phi^C_n = \Phi^D_n$ and they are total we have that there is a set E recursive in both C,D (and also in B_{n+1}) but not recursive in Φ^C_n ; here we use such facts like Fubini's theorem.

It should be pointed out that we do not construct an exact pair for an a priori given sequence of degrees.

The preceding sketch shows that the relativization of the concept of a NAP-set is very useful and can be used in more complicated constructions.

Remark. The described technique for constructions concerning degrees $\gg \underset{\sim}{0}$ is especially useful if there are only few demands upon where constructed degrees should be located. Of course, it cannot work in solving such problems like the problem raised by Jockusch and Simpson whether every degree $\gg \underset{\sim}{0}$ bounds a minimal degree.

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