

Canonisation and Definability for Graphs of Bounded Rank Width

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Abstract—We prove that the combinatorial Weisfeiler-Leman algorithm of dimension $(3k + 4)$ is a complete isomorphism test for the class of all graphs of rank width at most k . Rank width is a graph invariant that, similarly to tree width, measures the width of a certain style of hierarchical decomposition of graphs; it is equivalent to clique width.

It was known that isomorphism of graphs of rank width k is decidable in polynomial time (Grohe and Schweitzer, FOCS 2015), but the best previously known algorithm has a running time $n^{f(k)}$ for a non-elementary function f . Our result yields an isomorphism test for graphs of rank width k running in time $n^{O(k)}$. Another consequence of our result is the first polynomial time canonisation algorithm for graphs of bounded rank width.

Our second main result is that fixed-point logic with counting captures polynomial time on all graph classes of bounded rank width.

I. INTRODUCTION

Rank width, introduced by Oum and Seymour [40], [39], is a graph invariant that measures how well a graph can be decomposed hierarchically in a certain style. In this respect, it is similar to the better-known tree width, but where tree width measures the complexity, or width, of a separation in such a hierarchical decomposition in terms of the “connectivity” between the two sides, rank width measures the complexity of a separation in terms of the rank of the adjacency matrix of the edges between the two sides of the separation. This makes rank width (almost) invariant under complementation of a graph and thus relevant for dense graphs, where tree width usually becomes meaningless. Rank width is closely related to clique width, which had been introduced by Courcelle and Olariu [11]: for every graph G it holds that $\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1} - 1$, where $\text{rw}(G)$ denotes the rank width and $\text{cw}(G)$ the clique width of G . This implies that many hard algorithmic problems can be solved efficiently on graphs of bounded rank width (see, for example, [13]), among them all problems definable in monadic second-order logic [10]. Furthermore, graph classes of bounded clique width, or equivalently bounded rank width, are precisely those that can be obtained by means of a monadic second-order transduction over a class of trees [7], [8], [9].

In this paper we study the graph isomorphism problem and the closely related graph canonisation problem as well

as logical definability and descriptive complexity on graph classes of bounded rank width.

Despite Babai’s recent quasipolynomial time algorithm [2], it is still wide open whether the graph isomorphism problem can be solved in polynomial time. Polynomial time algorithms are only known for specific graph classes, among them all classes of bounded degree [37], all classes of bounded tree width [3], [36], all classes excluding a fixed graph as a minor [41], even all classes excluding a fixed graph as a topological subgraph [20], and most recently, graph classes of bounded rank width [23]. This last result was the starting point for our present paper. The running time of the isomorphism test in [23] is $n^{f(k)}$, where n is the number of vertices and k the rank width of the input graph, and f is a *non-elementary* function. Of course this is unsatisfactory. Moreover, the algorithm is extremely complicated, using both advanced techniques from structural graph theory [42], [39], [24] and the group theoretic graph isomorphism machinery [37].

Our first contribution is a simple isomorphism test for graphs of rank width at most k running in time $n^{O(k)}$. Indeed, the algorithm we use is a generic combinatorial isomorphism test known as the Weisfeiler-Leman algorithm [44], [2], [5]. The ℓ -dimensional Weisfeiler Leman algorithm (ℓ -WL) iteratively colours ℓ -tuples of vertices of the two input graphs and then compares the resulting colour patterns. If they differ, we know that the two input graphs are nonisomorphic. If two graphs have the same colour pattern, in general they may still be nonisomorphic [5]. Thus ℓ -WL is not a *complete* isomorphism test for all graphs. However, we prove that it is for graphs of bounded rank width. We say that ℓ -WL *identifies* a graph G if it distinguishes G from every graph H not isomorphic to G .

Theorem I.1. *The $(3k + 4)$ -dimensional Weisfeiler-Leman algorithm identifies every graph of rank width at most k .*

Combining this theorem with a result due to Immerman and Lander on the running time of the WL algorithm, we obtain the following.

Corollary I.2. *Isomorphism of graphs of rank width k can be decided in time $O(n^{3k+5} \log n)$.*

Another way of stating Theorem I.1 is that the *Weisfeiler-Leman (WL) dimension* [18] of graphs of rank width k is at

most $3k+4$. While it is known that many natural graph classes have bounded WL dimension, among them the class of planar graphs [16], [31], classes of bounded genus [17], bounded tree width [19], classes of graphs excluding some fixed graph as a minor [18], and interval graphs [32], [34], all these except for the class of interval graphs are classes of sparse graphs (with an edge number linear in the number of vertices). Our result adds a rich family of classes that include dense graphs to the picture.

Immerman and Lander [30] (also see [5]) showed that ℓ -WL is an equivalence test for $\mathcal{C}^{\ell+1}$, the $(\ell+1)$ -variable fragment of first-order logic with counting. Hence our result can also be read as a definability result.

Corollary I.3. *For every graph G of rank width at most k there is a sentence φ_G of the logic \mathcal{C}^{3k+5} that characterises G up to isomorphism.*

We use this connection to logic in our proof of Theorem I.1, which is based on a characterisation of equivalence in the logic \mathcal{C}^ℓ in terms of an Ehrenfeucht-Fraïssé game, the so-called ℓ -bijective pebble game due to Hella [27].

A *canonisation algorithm* A for a class \mathcal{C} of graphs associates with each graph $G \in \mathcal{C}$ a graph $A(G)$ that is isomorphic to G in such a way that if $G, H \in \mathcal{C}$ are isomorphic then $A(G)$ and $A(H)$ are identical. Clearly, a canonisation algorithm can be used to test if two graphs are isomorphic; the converse is not known. It is known, however, that if a class of graphs has WL dimension at most ℓ then there is a canonisation algorithm for this class running in time $O(n^{\ell+3} \log n)$. Hence as another corollary to Theorem I.1, we obtain the first polynomial time canonisation algorithm for graphs of bounded rank width.

Corollary I.4. *There is a canonisation algorithm for the class of graphs of rank width at most k running in time $O(n^{3k+7} \log n)$.*

The second part of our paper is concerned with descriptive complexity theory. The central open question of the field is whether there is a logic that *captures polynomial time* [6], [25]. Intuitively, this means that all sentences of the logic can be evaluated in polynomial time (by a uniform algorithm) and that all polynomial time decidable properties can be defined in the logic. By the Immerman-Vardi Theorem [28], [43], least-fixed point logic LFP captures polynomial time on ordered structures (that is, structures with a distinguished binary relation that is a linear order of the universe). But for general structures the question is still wide open more than 35 years after it was first raised by Chandra and Harel [6]. The question is related to isomorphism testing and canonisation. Indeed, a polynomial time canonisation algorithm for the class of all graphs would imply that there is a logic capturing polynomial time.

The question for a logic capturing polynomial time, as formulated by Gurevich [25], casts the notion of what constitutes a logic deliberately wide. However, we are not mainly interested in an abstract logic, but in a “nice” logic that conveys some insights on the nature of polynomial time computation. A logic that arguably falls in this category

is *fixed-point logic with counting* FP+C, first proposed by Immerman [28] and later formalised by Grädel and Otto [15] in the form commonly used today. It is known that FP+C does not capture polynomial time [5]. But over the last 10 years it has become clear that the logic is surprisingly powerful. It captures specific polynomial time algorithms such as linear programming [1], and it does capture polynomial time on rich graph classes, including all classes excluding some fixed graph as a minor [18]. Our second main result further broadens the scope of FP+C-definability.

Theorem I.5. *For every k , fixed-point logic with counting FP+C captures polynomial time on the class of all graphs of rank width at most k .*

Technically, this theorem is related to the first and is based on the same graph theoretic ideas, but it is significantly harder to prove. On an abstract level, this can be explained by highlighting an important difference between Theorem I.5 and Corollary I.3, which rephrases Theorem I.1 in terms of logic. Corollary I.3 is a *nonuniform* definability result: for every fixed graph we construct a formula characterising this graph. By contrast, Theorem I.5 requires *uniform* definability: for every polynomial time property we want a unique sentence that defines this property for all graphs of rank width at most k . This means that we have to internalise the construction that is underlying the proof of Theorem I.1 in the logic FP+C.

The paper is organised as follows: after reviewing the necessary preliminaries on rank width, graph isomorphism testing, and the WL algorithm in Section II, in Section III we introduce our technical machinery for dealing with rank decompositions that is underlying the proofs of both theorems. We prove Theorem I.1 in Section IV and Theorem I.5 in Section V, after giving additional background in descriptive complexity theory in Subsection V-A.

Due to space limitations some proofs are omitted in this extended abstract. For the missing proofs we refer to the full version of this paper [21].

II. PRELIMINARIES

A. Graphs

A *graph* is a pair $G = (V, E)$ with vertex set $V = V(G)$ and edge relation $E = E(G)$. In this paper all graphs are finite, simple (no loops or multiple edges), and undirected. We denote edges by $vw \in E(G)$ where $v, w \in V(G)$. The *neighbourhood* of $v \in V(G)$ is denoted by $N(v)$. For $A \subseteq V(G)$ we denote by $G[A]$ the *induced subgraph* of G on A . Also, we denote by $G \setminus A$ the induced subgraph on the complement of A , that is $G \setminus A := G[V(G) \setminus A]$.

An *isomorphism* from a graph G to another graph H is a bijective mapping $\varphi: V(G) \rightarrow V(H)$ which preserves the edge relation, that is $vw \in E(G)$ if and only if $\varphi(v)\varphi(w) \in E(H)$ for all $v, w \in V(G)$. Two graphs G and H are *isomorphic* ($G \cong H$) if there is an isomorphism from G to H . We write $\varphi: G \cong H$ to denote that φ is an isomorphism from G to H .

A (vertex-)coloured graph is a tuple (G, χ) where $\chi: V(G) \rightarrow \mathcal{C}$ is a mapping and \mathcal{C} is a finite set of colours.

Typically the set of colours is just an initial segment $[n] := \{1, \dots, n\}$ of the natural numbers. Isomorphisms between coloured graphs have to respect the colours of the vertices. In this paper, we typically consider coloured graphs also when not explicitly stated. Note that an uncoloured graph may be viewed as a coloured graph where each vertex gets the same colour.

B. Rank Width and Clique Width

In this work, we are interested in graphs of bounded rank width and graphs of bounded clique width. This section formally defines both parameters and describes the basic connections between them.

Rank width is a graph invariant that was first introduced by Oum and Seymour [39] and which measures the width of a certain style of hierarchical decomposition of graphs. Intuitively, the aim is to repeatedly split the vertex set of the graph along cuts of low complexity in a hierarchical fashion. For rank width, the complexity of a cut is measured in terms of the rank of the matrix capturing the adjacencies between the two sides of the cut over the 2-element field \mathbb{F}_2 .

Let G be a graph. For $X, Y \subseteq V(G)$ we define $M(X, Y) \in \mathbb{F}_2^{X \times Y}$ where $(M(X, Y))_{x,y} = 1$ if and only if $xy \in E(G)$. Furthermore $\rho_G(X) = \text{rk}_2(M(X, \bar{X}))$ where $\bar{X} = V(G) \setminus X$ and $\text{rk}_2(A)$ denotes the \mathbb{F}_2 -rank of a matrix A .

A *rank decomposition* of G is a tuple (T, γ) consisting of a binary directed tree T and a mapping $\gamma: V(T) \rightarrow 2^{V(G)}$ such that

- (R.1) $\gamma(r) = V(G)$ where r is the root of T ,
- (R.2) $\gamma(t) = \gamma(s_1) \cup \gamma(s_2)$ and $\gamma(s_1) \cap \gamma(s_2) = \emptyset$ for all internal nodes $t \in V(T)$ with children s_1 and s_2 , and
- (R.3) $|\gamma(t)| = 1$ for all $t \in L(T)$, where $L(T)$ denotes the set of leaves of the tree T .

Note that, instead of giving γ , we can equivalently specify a bijection $f: L(T) \rightarrow V(G)$ (this completely specifies γ by Condition (R.2)). The *width* of a rank decomposition (T, γ) is $\text{wd}(T, \gamma) = \max\{\rho_G(\gamma(t)) \mid t \in V(T)\}$. The rank width of a graph G is $\text{rw}(G) = \min\{\text{wd}(T, \gamma) \mid (T, \gamma) \text{ is a rank decomposition of } G\}$.

Clique width [11] is another measure aiming to describe the structural complexity of a graph, but unlike rank width, it considers the complexity of an algebraic expression defining the graph.

For $k \in \mathbb{N}$ a *k-graph* is a pair (G, lab) where G is a graph and $\text{lab}: V(G) \rightarrow [k]$ is a labelling of vertices. We define the following four operations for *k-graphs*:

- 1) for $i \in [k]$ let \cdot_i denote an isolated vertex with label i ,
- 2) for $i, j \in [k]$ with $i \neq j$ we define $\eta_{i,j}(G, \text{lab}) = (G', \text{lab})$ where $V(G') = V(G)$ and $E(G') = E(G) \cup \{vw \mid \text{lab}(v) = i \wedge \text{lab}(w) = j\}$,
- 3) for $i, j \in [k]$ we define $\rho_{i \rightarrow j}(G, \text{lab}) = (G, \text{lab}') \text{ where}$

$$\text{lab}'(v) = \begin{cases} j & \text{if } \text{lab}(v) = i \\ \text{lab}(v) & \text{otherwise} \end{cases},$$

- 4) for two *k-graphs* (G, lab) and (G', lab') we define $(G, \text{lab}) \oplus (G', \text{lab}')$ to be the disjoint union of the two *k-graphs*.

A *k-expression* t is a well-formed expression in these symbols and defines a *k-graph* (G, lab) . In this case t is a *k-expression* for G . The clique width of a graph G , denoted by $\text{cw}(G)$, is the minimum $k \in \mathbb{N}$ such that there is a *k-expression* for G .

Comparing clique width and rank width, each parameter is bounded in terms of the other.

Theorem II.1 ([39]). *For every graph G it holds that*

$$\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1} - 1.$$

C. The Weisfeiler-Leman Algorithm

The *k-dimensional Weisfeiler-Leman algorithm* is a procedure that, given a graph G and a colouring of the *k*-tuples of the vertices, computes an isomorphism-invariant refinement of the colouring. Let $\chi_1, \chi_2: V^k \rightarrow \mathcal{C}$ be colourings of the *k*-tuples of vertices of G , where \mathcal{C} is some finite set of colours. We say χ_1 *refines* χ_2 ($\chi_1 \preceq \chi_2$) if for all $\bar{v}, \bar{w} \in V^k$ we have

$$\chi_1(\bar{v}) = \chi_1(\bar{w}) \Rightarrow \chi_2(\bar{v}) = \chi_2(\bar{w}).$$

For an integer $k > 1$ and a vertex-coloured graph (G, χ) , we first set $\chi_0^{G,k}: V^k \rightarrow \mathcal{C}$ to be the colouring where each *k*-tuple is coloured by the isomorphism-type of its underlying ordered subgraph. More precisely, $\chi_0^{G,k}(v_1, \dots, v_k) = \chi_0^{G,k}(w_1, \dots, w_k)$ if and only if for all $i \in [k]$ it holds that $\chi(v_i) = \chi(w_i)$ and for all $i, j \in [k]$ it holds $v_i = v_j \Leftrightarrow w_i = w_j$ and $v_i v_j \in E(G) \Leftrightarrow w_i w_j \in E(G)$. Then, we recursively define the colouring $\chi_{i+1}^{G,k}$ by setting $\chi_{i+1}^{G,k}(v_1, \dots, v_k) := (\chi_i^{G,k}(v_1, \dots, v_k); \mathcal{M})$, where \mathcal{M} is a multiset defined as

$$\{ \{ \chi_i^G(\bar{v}[w/1]), \chi_i^G(\bar{v}[w/2]), \dots, \chi_i^G(\bar{v}[w/k]) \} \mid w \in V \}$$

where $\bar{v}[w/i] := (v_1, \dots, v_{i-1}, w, v_{i+1}, \dots, v_k)$.

For $k = 1$ the definition is similar but we iterate only over the neighbours of v_1 , that is the multiset is defined by $\mathcal{M} := \{ \{ \chi_i^{G,1}(w) \mid w \in N(v_1) \} \}$. The initial colouring $\chi_0^{G,1}$ is simply equal to χ , the vertex-colouring of the input graph.

By definition, every colouring $\chi_{i+1}^{G,k}$ induces a refinement of the partition of the *k*-tuples of the graph G with colouring $\chi_i^{G,k}$. Thus, there is some minimal i such that the partition induced by the colouring $\chi_{i+1}^{G,k}$ is not strictly finer than the one induced by the colouring $\chi_i^{G,k}$ on G . For this minimal i , we call the colouring $\chi_i^{G,k}$ the *stable* colouring of G and denote it by $\chi_{(\infty)}^{G,k}$.

For $k = 1$ we will usually omit the index k and write $\chi_{(\infty)}^G$ instead of $\chi_{(\infty)}^{G,k}$. Also, in some cases we will omit the graph G if it is apparent from context and just write $\chi_{(\infty)}$.

For $k \in \mathbb{N}$, the *k-dimensional Weisfeiler-Leman algorithm* takes as input a coloured graph (G, χ) and returns the coloured graph $(G, \chi_{(\infty)}^{G,k})$. This can be implemented in time $O(n^{k+1} \log n)$ [30]. For two graphs G and H , we say that the *k-dimensional Weisfeiler-Leman algorithm distinguishes* G and H if there is some colour c such that the sets $\{\bar{v} \mid \bar{v} \in V^k(G), \chi_{(\infty)}^{G,k}(\bar{v}) = c\}$ and $\{\bar{w} \mid$

$\bar{w} \in V^k(H), \chi_{(\infty)}^{H,k}(\bar{w}) = c\}$ have different cardinalities. We write $G \simeq_k H$ if the k -dimensional Weisfeiler-Leman algorithm does not distinguish between G and H . The k -dimensional Weisfeiler-Leman algorithm *identifies* a graph G if it distinguishes G from every non-isomorphic graph H .

We will not require details about the information computed by the Weisfeiler-Leman algorithm and rather use the following pebble game that is known to capture the same information. Let $k \in \mathbb{N}$. For graphs G, H on the same number of vertices and with vertex colourings χ_G and χ_H , respectively, we define the *bijective k -pebble game* $\text{BP}_k(G, H)$ as follows:

- The game has two players called Spoiler and Duplicator.
- The game proceeds in rounds. Each round is associated with a pair of positions (\bar{v}, \bar{w}) with $\bar{v} \in V(G)^\ell$ and $\bar{w} \in V(H)^\ell$ where $0 \leq \ell \leq k$.
- The initial position of the game is $(((), ()))$ (the pair of empty tuples).
- Each round consists of the following steps. Suppose the current position of the game is $(\bar{v}, \bar{w}) = ((v_1, \dots, v_\ell), (w_1, \dots, w_\ell))$. First, Spoiler chooses whether to remove a pair of pebbles or to play a new pair of pebbles. The first option is only possible if $\ell > 0$ and the latter option is only possible if $\ell < k$. If Spoiler wishes to remove a pair of pebbles he picks some $i \in [\ell]$ and the game moves to position $(\bar{v} \setminus i, \bar{w} \setminus i)$ where $\bar{v} \setminus i = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_\ell)$ ($\bar{w} \setminus i$ is defined in the same way). Otherwise the following steps are performed.

(D) Duplicator picks a bijection $f: V(G) \rightarrow V(H)$.

(S) Spoiler chooses $v \in V(G)$ and sets $w = f(v)$.

The new position is then $((v_1, \dots, v_\ell, v), (w_1, \dots, w_\ell, w))$.

Spoiler wins the play if for the current position $((v_1, \dots, v_\ell), (w_1, \dots, w_\ell))$ the induced graphs are not isomorphic. More precisely, Spoiler wins if there is an $i \in [\ell]$ such that $\chi_G(v_i) \neq \chi_H(w_i)$ or there are $i, j \in [\ell]$ such that $v_i = v_j \not\leftrightarrow w_i = w_j$ or $v_i v_j \in E(G) \not\leftrightarrow w_i w_j \in E(H)$. If the play never ends Duplicator wins.

We say that Spoiler (resp. Duplicator) wins the bijective k -pebble game $\text{BP}_k(G, H)$ if Spoiler (resp. Duplicator) has a winning strategy for the game.

Theorem II.2 ([5], [27]). *Let G, H be two graphs. Then $G \simeq_k H$ if and only if Duplicator wins the pebble game $\text{BP}_{k+1}(G, H)$.*

III. SPLIT PAIRS AND FLIP FUNCTIONS

We first show that the ℓ -dimensional Weisfeiler-Leman algorithm identifies all graphs of rank width at most k for some $\ell \in O(k)$. Let G be a graph of rank width k . On a high level, our approach is similar to the proof of the same result for graphs of bounded tree width [19]. For a set $X \subseteq V(G)$ such that $\rho_G(X) \leq k$ we wish to find a small set of vertices such that pebbling these vertices splits the graph into multiple sets C that can be treated independently. Moreover, each of

these sets C should satisfy that $C \subseteq X$ or $C \subseteq \bar{X}$. As there may be many edges between X and \bar{X} , it is not obvious how to achieve this. In particular, we cannot simply remove a few vertices in order to separate X from \bar{X} . Split pairs and flip function are our way of dealing with this.

Let G be a graph and $X \subseteq V(G)$. For $v, w \in X$ we define $v \approx_X w$ if $N(v) \cap \bar{X} = N(w) \cap \bar{X}$. For $v \in X$ we define the vector $\text{vec}_X(v) = (a_{v,w})_{w \in \bar{X}} \in \mathbb{F}_2^{\bar{X}}$ where $a_{v,w} = 1$ if and only if $vw \in E(G)$. Note that $v \approx_X w$ if and only if $\text{vec}_X(v) = \text{vec}_X(w)$. Moreover, for $S \subseteq X$ we define $\text{vec}_X(S) = \{\text{vec}_X(v) \mid v \in S\}$.

Lemma III.1. *Let $Y \subseteq X \subseteq V(G)$ and suppose $S \subseteq X$ such that $\text{vec}_X(S)$ is linearly independent. Then $\text{vec}_Y(S \cap Y)$ is linearly independent.*

For any set of vectors $S \subseteq \mathbb{F}_2^n$ we denote by $\langle S \rangle$ the linear space spanned by S . A set $B \subseteq \mathbb{F}_2^n$ is a *linear basis* for $\langle S \rangle$ if B is linearly independent and $\langle B \rangle = \langle S \rangle$.

Definition III.2. Let G be a graph and $X \subseteq V(G)$. A pair (A, B) is a *split pair* for X if

- 1) $A \subseteq X$ and $B \subseteq \bar{X}$,
- 2) $\text{vec}_X(A)$ forms a linear basis for $\langle \text{vec}_X(X) \rangle$, and
- 3) $\text{vec}_{\bar{X}}(B)$ forms a linear basis for $\langle \text{vec}_{\bar{X}}(\bar{X}) \rangle$.

Note that $|A| = \rho_G(X)$ and $|B| = \rho_G(X)$. Also observe that if (A, B) is a split pair for X then (B, A) is a split pair for \bar{X} . As a special case the pair (\emptyset, \emptyset) is defined to be a split pair for $X = V(G)$. An *ordered split pair* for X is a pair $(\bar{a}, \bar{b}) = ((a_1, \dots, a_q), (b_1, \dots, b_p))$ such that $(\{a_1, \dots, a_q\}, \{b_1, \dots, b_p\})$ is a split pair for X .

Lemma III.3. *Let G be a graph, $X \subseteq V(G)$ and suppose (A, B) is a split pair for X . Also let $v, w \in X$ such that $N(v) \cap B = N(w) \cap B$. Then $v \approx_X w$. Similarly, $v' \approx_{\bar{X}} w'$ for all $v', w' \in \bar{X}$ such that $N(v') \cap A = N(w') \cap A$.*

Proof. Let $v, w \in X$ and suppose $B = \{b_1, \dots, b_p\}$. Then, for all $i \in [p]$ we have $vb_i \in E(G)$ if and only if $wb_i \in E(G)$. Thus

$$(\text{vec}_{\bar{X}}(b_i))_v = (\text{vec}_{\bar{X}}(b_i))_w,$$

that is, the v -entry of the vector $\text{vec}_{\bar{X}}(b_i)$ coincides with the w -entry. Since $\text{vec}_{\bar{X}}(B)$ forms a linear basis for $\langle \text{vec}_{\bar{X}}(\bar{X}) \rangle$, we conclude that

$$(\text{vec}_{\bar{X}}(v'))_v = (\text{vec}_{\bar{X}}(v'))_w$$

for all $v' \in \bar{X}$. But this means $N(v) \cap \bar{X} = N(w) \cap \bar{X}$ and thus, $v \approx_X w$. The second statement is proved analogously. \square

For a coloured graph $G = (V, E, \chi)$ and a sequence of vertices $\bar{v} = (v_1, \dots, v_\ell) \in V^\ell$ we define the colouring

$$\chi^{\bar{v}}: V \rightarrow \mathbb{N}: v \mapsto \begin{cases} i & \text{if } v = v_i \wedge \forall j > i: v \neq v_j \\ \chi(v) + \ell & \text{otherwise} \end{cases}.$$

Moreover, we denote by $\chi_{(\infty)}^{\bar{v}, G}$ the stable colouring obtained from applying the colour refinement algorithm (i.e. the 1-dimensional Weisfeiler-Leman algorithm) to $(G, \chi^{\bar{v}})$. As be-

fore, we may omit the graph G if it is clear from context and only write $\chi_{(\infty)}^{\bar{u}}$.

Also, to simplify notation, for tuples $\bar{a} = (a_1, \dots, a_k)$ and $\bar{b} = (b_1, \dots, b_\ell)$ we write (\bar{a}, \bar{b}) for the tuple $(a_1, \dots, a_k, b_1, \dots, b_\ell)$ obtained from concatenating \bar{a} and \bar{b} .

Corollary III.4. *Let G be a graph, $X \subseteq V(G)$ and suppose (\bar{a}, \bar{b}) is an ordered split pair for X . Also let $v, w \in X$ such that $\chi_{(\infty)}^{(\bar{a}, \bar{b})}(v) = \chi_{(\infty)}^{(\bar{a}, \bar{b})}(w)$. Then $v \approx_X w$. Similarly, $v' \approx_{\bar{X}} w'$ for all $v', w' \in \bar{X}$ such that $\chi_{(\infty)}^{(\bar{a}, \bar{b})}(v') = \chi_{(\infty)}^{(\bar{a}, \bar{b})}(w')$.*

We need to argue how to actually split the graph into independent parts using split pairs. Similar to the previous corollary, we individualise a split pair and perform the colour refinement algorithm. We claim that this graph consists of independent parts as desired. In order to make these parts visible we consider the concept of a flip function.

Definition III.5. Let $G = (V, E, \chi)$ be a vertex-coloured graph where $\chi: V \rightarrow \mathcal{C}$. A *flip function* for G is a mapping $f: \mathcal{C} \times \mathcal{C} \rightarrow \{0, 1\}$ such that $f(c, c') = f(c', c)$ for all $c, c' \in \mathcal{C}$.

Moreover, for a graph $G = (V, E, \chi)$ and a flip function f we define the flipped graph $G^f = (V, E^f, \chi)$ where

$$E^f = \{vw \mid vw \in E \wedge f(\chi(v), \chi(w)) = 0\} \\ \cup \{vw \mid v \neq w \wedge vw \notin E \wedge f(\chi(v), \chi(w)) = 1\}.$$

For a coloured graph G and a flip function f we let $\text{Comp}(G, f) \subseteq 2^{V(G)}$ be the set of vertex sets of the connected components of G^f . Observe that $\text{Comp}(G, f)$ forms a partition of the vertex set of G .

Lemma III.6. *Let $G = (V, E, \chi)$ be a coloured graph and $X \subseteq V(G)$. Also let (\bar{a}, \bar{b}) be an ordered split pair for X .*

Then there is a flip function f for the graph $G' = (V(G), E(G), \chi_{(\infty)}^{(\bar{a}, \bar{b}), G})$ such that for every $C \in \text{Comp}(G', f)$ it holds that $C \subseteq X$ or $C \subseteq \bar{X}$.

Proof. Let $\bar{u} = (\bar{a}, \bar{b})$. We define the flip function f such that $f(c, c') = 1$ if there are $v \in X$ and $w \in \bar{X}$ such that $vw \in E(G)$ and $\{\chi_{(\infty)}^{\bar{u}}(v), \chi_{(\infty)}^{\bar{u}}(w)\} = \{c, c'\}$. We argue that there are no $v \in X$ and $w \in \bar{X}$ such that vw is an edge in the flipped graph G^f .

Suppose towards a contradiction this statement does not hold, that is, there are $v \in X$ and $w \in \bar{X}$ such that $vw \in E(G^f)$. Let $c = \chi_{(\infty)}^{\bar{u}}(v)$ and $c' = \chi_{(\infty)}^{\bar{u}}(w)$. Then $vw \notin E(G)$, because if $vw \in E(G)$ then $f(c, c') = 1$ and thus $vw \notin E(G^f)$. Moreover, $f(c, c') = 1$, because $vw \notin E(G)$ and $vw \in E(G^f)$. This means that there are $v' \in X$ and $w' \in \bar{X}$ such that $v'w' \in E(G)$ and $\{\chi_{(\infty)}^{\bar{u}}(v'), \chi_{(\infty)}^{\bar{u}}(w')\} = \{c, c'\}$.

Now we distinguish two cases. The first is that $\chi_{(\infty)}^{\bar{u}}(v') = c$ and hence, $\chi_{(\infty)}^{\bar{u}}(w') = c'$. Then $v \approx_X v'$ and $w \approx_{\bar{X}} w'$ by Corollary III.4. But this implies that

$$vw \in E(G) \Leftrightarrow vw' \in E(G) \Leftrightarrow v'w' \in E(G)$$

which is a contradiction.

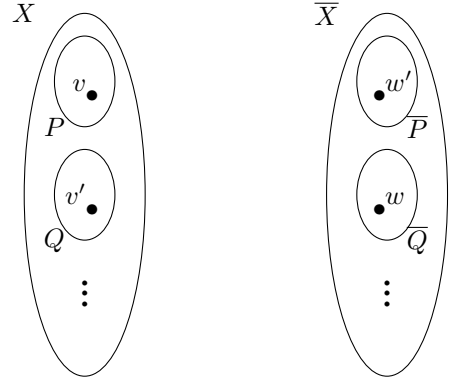


Fig. 1. Visualisation of the sets P, \bar{P}, Q and \bar{Q} from the proof of Lemma III.6.

Let us turn to the second, more complicated, case that $\chi_{(\infty)}^{\bar{u}}(v') = c'$ and $\chi_{(\infty)}^{\bar{u}}(w') = c$. Let $P = (\chi_{(\infty)}^{\bar{u}})^{-1}(c) \cap X$, $\bar{P} = (\chi_{(\infty)}^{\bar{u}})^{-1}(c) \cap \bar{X}$, $Q = (\chi_{(\infty)}^{\bar{u}})^{-1}(c') \cap X$ and $\bar{Q} = (\chi_{(\infty)}^{\bar{u}})^{-1}(c') \cap \bar{X}$. So $v \in P$, $v' \in Q$, $w \in \bar{Q}$ and $w' \in \bar{P}$ (see Figure 1).

Claim 1. Let $y \in P$ and $z \in \bar{Q}$. Then $yz \notin E(G)$.

Proof. We have $v \approx_X y$ and $w \approx_{\bar{X}} z$ by Corollary III.4. Hence,

$$vw \in E(G) \Leftrightarrow vz \in E(G) \Leftrightarrow yz \in E(G). \quad \lrcorner$$

Claim 2. Let $y \in Q$ and $z \in \bar{P}$. Then $yz \in E(G)$.

Proof. We have $v' \approx_X y$ and $w' \approx_{\bar{X}} z$ by Corollary III.4. Hence,

$$v'w' \in E(G) \Leftrightarrow v'z \in E(G) \Leftrightarrow yz \in E(G). \quad \lrcorner$$

Now $|N(v) \cap Q| = |N(v) \cap (Q \cup \bar{Q})| = |N(w') \cap (Q \cup \bar{Q})| \geq |Q|$ by Claim 1 and 2. This means $Q \subseteq N(v)$. In particular, $v \in N(v')$. It follows from Claim 2 that $\bar{P} \subseteq N(v')$. Thus $|N(v') \cap (P \cup \bar{P})| \geq |\bar{P}| + 1$. Since $\chi_{(\infty)}^{\bar{u}}(v') = \chi_{(\infty)}^{\bar{u}}(w) = c'$ we conclude that $|N(w) \cap (P \cup \bar{P})| \geq |\bar{P}| + 1$. But $|N(w) \cap (P \cup \bar{P})| = |N(w) \cap \bar{P}| \leq |\bar{P}|$ by Claim 1. This is a contradiction. \square

To be able to treat the connected components of the flipped graph independently we need to argue that applying a flip function to two graphs neither changes the isomorphism problem nor the effect of the Weisfeiler-Leman algorithm.

Lemma III.7. *Let G, H be two coloured graphs and let f be a flip function for G and H . Also let $\varphi: V(G) \rightarrow V(H)$ be a bijection. Then $\varphi: G \cong H$ if and only if $\varphi: G^f \cong H^f$.*

Proof. Trivial. \square

Lemma III.8. *Let $G = (V, E, \chi)$, $G' = (V', E', \chi')$ be two coloured graphs and let f be a flip function for G and G' . Also let $(\bar{v}, \bar{w}) = ((v_1, \dots, v_k), (w_1, \dots, w_k))$ be a position in the k -bijjective pebble game $\text{BP}_k(G, G')$. Then Spoiler wins*

from (\bar{v}, \bar{w}) in $\text{BP}_k(G, G')$ if and only if Spoiler wins from (\bar{v}, \bar{w}) in $\text{BP}_k(G^f, (G')^f)$.

Proof. A position (\bar{v}, \bar{w}) in the pebble game $\text{BP}_k(G, G')$ is a winning position for Spoiler if and only if it is a winning position for Spoiler in the game $\text{BP}_k(G^f, (G')^f)$. \square

For two colourings $\chi, \chi': V \rightarrow \mathcal{C}$ we write $\chi \equiv \chi'$ if $\chi \preceq \chi'$ and $\chi' \preceq \chi$, that is the partitions induced by the colour classes are the same for both colourings.

Corollary III.9. *Let $G = (V, E, \chi)$ be a coloured graph and let f be a flip function for G . Then $\chi_{(\infty)}^G \equiv \chi_{(\infty)}^{G,f}$ where $\chi_{(\infty)}^{G,f}$ is the stable colouring computed by colour refinement applied to the graph G^f .*

For $\bar{v} = (v_1, \dots, v_k) \in V^k$ and $C \subseteq V$ we define the tuple $\bar{v} \cap C = (v_i)_{i \in I}$ where $I = \{i \in [k] \mid v_i \in C\}$. Also, for a second tuple $\bar{w} = (w_1, \dots, w_\ell) \in V^\ell$, we write $\bar{v} \subseteq \bar{w}$ if $\{v_1, \dots, v_k\} \subseteq \{w_1, \dots, w_\ell\}$.

Corollary III.10. *Let $G = (V, E, \chi)$, $G' = (V', E', \chi')$ be two coloured graphs and let f be a flip function for G and G' . Let $\bar{v} \in V^k$ and $\bar{v}' \in (V')^k$. Let C be a connected component of G^f such that $\chi(u) \neq \chi(w)$ for all $u \in C$, $w \in V \setminus C$, and let C' a connected component of $(G')^f$ such that $\chi'(u') \neq \chi'(w')$ for all $u' \in C'$, $w' \in V' \setminus C'$. Suppose that*

$$(G[C], \chi_{(\infty)}^{\bar{v}, G}) \not\equiv (G'[C'], \chi_{(\infty)}^{\bar{v}', G'}).$$

Let $\bar{w} = \bar{v} \cap C$ and $\bar{w}' = \bar{v}' \cap C'$. Then

$$(G[C], \chi_{(\infty)}^{\bar{w}, G}) \not\equiv (G'[C'], \chi_{(\infty)}^{\bar{w}', G'})$$

or $(G, \chi^{\bar{v}}) \not\equiv_1 (G', (\chi')^{\bar{v}'})$.

IV. WEISFEILER-LEMAN FOR GRAPHS OF BOUNDED RANK WIDTH

In this section we give a proof of Theorem I.1. The basic strategy for the proof is simple. Given two non-isomorphic graphs G and H , where G has rank width at most k , we give a winning strategy for Spoiler in the game $\text{BP}_\ell(G, H)$ for $\ell = 3k + 5$. Spoiler's strategy in the game is to play along a rank decomposition (T, γ) for the graph G . At a specific node $t \in V(T)$ of the rank decomposition, Spoiler plays an ordered split pair (\bar{a}, \bar{b}) for the set $\gamma(t)$ and identifies some component C (with respect to some flip function) that is different from the corresponding component (specified by the bijection chosen by Duplicator) in the second graph. In order to distinguish these components, Spoiler continues to play along the rank decomposition going down the tree. A crucial step to realise this strategy is to ensure that we can remove the pebbles from an ordered split pair of t once Spoiler has pebbled ordered split pairs of the children of t . Towards this end, we introduce the notion of *nice* (triples of) split pairs.

For sets X, X_1, X_2 we write $X = X_1 \uplus X_2$ to denote that X is the disjoint union of X_1 and X_2 , that is, $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$.

Definition IV.1. Let G be a graph and $X, X_1, X_2 \subseteq V(G)$ such that $X = X_1 \uplus X_2$. Let (A, B) be a split pair of X

and let (A_i, B_i) be split pairs for X_i , $i \in \{1, 2\}$. We say that (A_i, B_i) , $i \in \{1, 2\}$, are *nice* (with respect to (A, B)) if

- 1) $A \cap X_i \subseteq A_i$, and
- 2) $B_{3-i} \cap X_i \subseteq A_i$

for both $i \in \{1, 2\}$.

Naturally, a triple of ordered split pairs is nice if the underlying unordered triple of split pairs is nice.

Lemma IV.2. *Let G be a graph and $X, X_1, X_2 \subseteq V(G)$ such that $X = X_1 \uplus X_2$. Let (A, B) be a split pair of X . Then there are nice split pairs (A_i, B_i) for X_i , $i \in \{1, 2\}$, such that additionally $B_i \cap \bar{X} \subseteq B$.*

Proof. We first pick A_i for both $i \in \{1, 2\}$. Since $X_i \subseteq X$ we can choose A_i in such a way that $A \cap X_i \subseteq A_i$ by Lemma III.1.

The set $\text{vec}_{X_{3-i}}(A_{3-i})$ spans every element in the set $\text{vec}_{X_{3-i}}(X_{3-i}) \subseteq \mathbb{F}_2^{X_i \cup \bar{X}}$. Hence, $\text{vec}_{\bar{X}_i}(A_{3-i})$ spans every element in the set $\text{vec}_{\bar{X}_i}(X_{3-i}) \subseteq \mathbb{F}_2^{X_i}$.

Moreover, the set $\text{vec}_{\bar{X}}(B)$ spans every element in the set $\text{vec}_{\bar{X}}(\bar{X}) \subseteq \mathbb{F}_2^{X_1 \cup X_2}$. So $\text{vec}_{\bar{X}_i}(B)$ spans every element in the set $\text{vec}_{\bar{X}_i}(\bar{X}) \subseteq \mathbb{F}_2^{X_i}$.

Together this means that $\text{vec}_{\bar{X}_i}(B \cup A_{3-i})$ spans every element in the set $\text{vec}_{\bar{X}_i}(\bar{X}_i) \subseteq \mathbb{F}_2^{X_i}$. We choose $B_i \subseteq B \cup A_{3-i}$ inclusionwise maximal such that $\text{vec}_{\bar{X}_i}(B_i)$ is linearly independent. \square

Also, we shall need the following simple observation.

Observation IV.3. *Let G, H be two non-isomorphic graphs and let $\sigma: V(G) \rightarrow V(H)$ be any bijection. Then there is some $v \in V(G)$ such that $G[A] \not\equiv G[B]$ where A is the connected component of G such that $v \in A$ and B is the connected component of H such that $\sigma(v) \in B$.*

Theorem IV.4 (Theorem I.1 restated). *The $(3k + 4)$ -dimensional Weisfeiler-Leman algorithm identifies every graph of rank width at most k .*

Proof. Let $G = (V(G), E(G), \chi_G)$ be a graph such that $\text{rw}(G) \leq k$ and moreover let $H = (V(H), E(H), \chi_H)$ be a second graph such that $G \not\equiv H$. Let (T, γ) be a rank decomposition of width k for the graph G .

We argue that Spoiler wins the bijective ℓ -pebble game played over graphs G and H where $\ell = 3k + 5$. In combination with Theorem II.2 this proves the theorem. Actually, we first give a winning strategy for Spoiler that requires $\ell = 6k + 5$ many pebbles. Then we proceed to argue how to realise this strategy using only $3k + 5$ many pebbles.

For a node $t \in V(T)$ a tuple (\bar{a}, \bar{b}) is an *ordered split pair* for t if (\bar{a}, \bar{b}) is an ordered split pair for $\gamma(t)$.

Now suppose the play is at a position $((\bar{a}, \bar{b}, v), (\bar{a}', \bar{b}', v'))$ such that the following conditions are satisfied:

- There is a node $t \in V(T)$ such that (\bar{a}, \bar{b}) is an ordered split pair for t .
- $v \in \gamma(t)$.
- Let f be the flip function obtained from Lemma III.6 with respect to $X = \gamma(t)$. Let $C \in \text{Comp}((G, \chi_{(\infty)}^{(\bar{a}, \bar{b})}), f)$ such

that $v \in C$. Similarly let $C' \in \text{Comp}((H, \chi_{(\infty)}^{(\bar{a}', \bar{b}')}), f)$ such that $v' \in C'$. Then

$$(G[C], \chi_{(\infty)}^{(\bar{a}, \bar{b}, v)}) \not\cong (H[C'], \chi_{(\infty)}^{(\bar{a}', \bar{b}', v')}).$$

Note that initially it is easy for Spoiler to reach such a position for the root node of T . Also observe that in such a position the number of pebbles is at most $2k + 1$. We now prove by induction on $|\gamma(t)|$ that Spoiler wins from such a position.

For the base step suppose that $|\gamma(t)| = 1$. In this case $C = \{v\}$ and Spoiler easily wins using two additional pebbles. Note that the sets C and C' can be recognised by colour refinement since one of the vertices in each set is individualised (cf. Corollary III.9).

So for the inductive step suppose $|\gamma(t)| > 1$. Let t_1 and t_2 be the children of t . Let $X = \gamma(t)$ and $X_i = \gamma(t_i)$ for $i \in \{1, 2\}$. Note that $X = X_1 \uplus X_2$.

Let (\bar{a}_i, \bar{b}_i) , $i \in \{1, 2\}$, be nice ordered split pairs for t_i (cf. Lemma IV.2). On an intuitive level, the advantage of pebbling nice ordered split pairs is that, for $i \in \{1, 2\}$, we can remove the pebbles (\bar{a}, \bar{b}) and $(\bar{a}_{3-i}, \bar{b}_{3-i})$ without unpebbling some element from X_i .

Now Spoiler plays $(\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2)$ and let $(\bar{a}'_1, \bar{b}'_1, \bar{a}'_2, \bar{b}'_2)$ be Duplicator's answer. Let $\bar{\alpha} = (\bar{a}, \bar{b}, \bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2, v)$ and $\bar{\alpha}' = (\bar{a}', \bar{b}', \bar{a}'_1, \bar{b}'_1, \bar{a}'_2, \bar{b}'_2, v')$.

Let f_i be the flip function obtained from Lemma III.6 with respect to the ordered split pair (\bar{a}_i, \bar{b}_i) and the set X_i . Now Spoiler wishes to play another pebble and let $\sigma: V(G) \rightarrow V(H)$ be the bijection chosen by Duplicator. Without loss of generality we can assume that

- (a) $\sigma(\bar{\alpha}) = \bar{\alpha}'$ and
- (b) $\sigma(C) = C'$.

Additionally, we can assume without loss of generality that $v \in X_1$ (otherwise we swap the roles of X_1 and X_2).

Let $G' = (G[C], \chi_{(\infty)}^{\bar{\alpha}})$ and $H' = (H[C'], \chi_{(\infty)}^{\bar{\alpha}'})$. Observe that $G' \not\cong H'$ and σ induces a bijection from $V(G') = C$ to $V(H') = C'$. First consider the flip function f_1 . By Observation IV.3 and Lemma III.7 there is some $w \in C$ such that $G'[M] \not\cong H'[M']$ where $M \in \text{Comp}(G', f_1)$ such that $w \in M$ and $M' \in \text{Comp}(H', f_1)$ such that $\sigma(w) \in M'$.

We distinguish two cases. First suppose $M \subseteq X_1$. Let $C_1 \in \text{Comp}((G, \chi_{(\infty)}^{(\bar{a}_1, \bar{b}_1)}), f_1)$ such that $M = C_1 \cap C$ and let $C'_1 \in \text{Comp}((H, \chi_{(\infty)}^{(\bar{a}'_1, \bar{b}'_1)}), f_1)$ such that $M' = C'_1 \cap C'$. Observe that $\chi_{(\infty)}^{\bar{\alpha}}(u) \neq \chi_{(\infty)}^{\bar{\alpha}'}(u')$ for all $u \in C$ and $u' \in V(G) \setminus C$. This is clear for the graph G^f since $v \in C$ and C forms a connected component in G^f and thus, it also holds for G by Corollary III.9. Hence, we get that

$$(G[C_1], \chi_{(\infty)}^{\bar{\alpha}}) \not\cong (H[C'_1], \chi_{(\infty)}^{\bar{\alpha}'}).$$

Now Spoiler plays the next pebble as follows: if $v \in C_1$ and $v' \in C'_1$ then he plays $z = v$ and $z' = v'$, otherwise Spoiler plays $z = w$ and $z' = \sigma(w)$. Clearly,

$$(G[C_1], \chi_{(\infty)}^{(\bar{\alpha}, z)}) \not\cong (H[C'_1], \chi_{(\infty)}^{(\bar{\alpha}', z')})$$

Now consider again the flip function f_1 . In G^{f_1} the set C_1 forms a connected component and similarly, in H^{f_1}

the set C'_1 forms a connected component. Hence, removing any pebbles from vertices outside C_1 (resp. C'_1) does not affect the stable colouring restricted to the component C_1 (resp. C'_1) by Corollary III.9. Since all pebbles $(\bar{a}, \bar{b}, \bar{a}_2, \bar{b}_2, v)$ (resp. $(\bar{a}', \bar{b}', \bar{a}'_2, \bar{b}'_2, v')$) are either outside of C_1 or the corresponding vertices are also pebbled using $(\bar{a}_1, \bar{b}_1, z)$ (resp. $(\bar{a}'_1, \bar{b}'_1, z')$), we can remove the pebbles $(\bar{a}, \bar{b}, \bar{a}_2, \bar{b}_2, v)$ and $(\bar{a}', \bar{b}', \bar{a}'_2, \bar{b}'_2, v')$ and still get that

$$(G[C_1], \chi_{(\infty)}^{(\bar{a}_1, \bar{b}_1, z)}) \not\cong (H[C'_1], \chi_{(\infty)}^{(\bar{a}'_1, \bar{b}'_1, z')})$$

by Corollary III.10 (or Spoiler wins using two additional pebbles). But now we can apply the induction hypothesis to t_1 . As a result, Spoiler wins from the current position and hence, Spoiler wins from position $((\bar{a}, \bar{b}, v), (\bar{a}', \bar{b}', v'))$.

For the second case $M \subseteq X_2$. Let us first remark that this case is not symmetric to the first case since the set M is defined with respect to the flip function f_1 .

First Spoiler plays the next pebble on w and $w' = \sigma(w)$. Observe that $\chi_{(\infty)}^{(\bar{\alpha}, w)}(u) \neq \chi_{(\infty)}^{(\bar{\alpha}, w)}(u')$ for all $u \in M$ and $u' \in V(G) \setminus M$.

Now consider the flip function f_2 . Spoiler wishes to play the next pebble. Let $\sigma': V(G) \rightarrow V(H)$ be the bijection chosen by Duplicator. Without loss of generality we can assume that

- (a) $\sigma(\bar{\alpha}) = \bar{\alpha}'$,
- (b) $\sigma(w) = w'$,
- (c) $\sigma(M) = M'$.

Let $G'' = (G[M], \chi_{(\infty)}^{(\bar{\alpha}, w)})$ and $H'' = (H[M'], \chi_{(\infty)}^{(\bar{\alpha}', w')})$. Observe that $G'' \not\cong H''$ and σ' induces a bijection from $V(G'') = M$ to $V(H'') = M'$. Consider the flip function f_2 . By Observation IV.3 and Lemma III.7 there is some $z \in M$ such that $G''[N] \not\cong H''[N']$ where $N \in \text{Comp}(G'', f_2)$ such that $z \in N$ and $N' \in \text{Comp}(H'', f_2)$ such that $\sigma'(z) \in N'$.

Observe that $N \subseteq X_2$. Let $C_2 \in \text{Comp}((G, \chi_{(\infty)}^{(\bar{a}_2, \bar{b}_2)}), f_2)$ such that $N = C_2 \cap M$ and let $C'_2 \in \text{Comp}((H, \chi_{(\infty)}^{(\bar{a}'_2, \bar{b}'_2)}), f_2)$ such that $N' = C'_2 \cap M'$. We get that

$$(G[C_2], \chi_{(\infty)}^{(\bar{\alpha}, w)}) \not\cong (H[C'_2], \chi_{(\infty)}^{(\bar{\alpha}', w')}).$$

Now Spoiler plays the next pebble as follows: if $w \in C_2$ and $w' \in C'_2$ then he plays $x = w$ and $x' = w'$, otherwise Spoiler plays $x = z$ and $x' = \sigma'(z)$. Clearly,

$$(G[C_2], \chi_{(\infty)}^{(\bar{\alpha}, w, x)}) \not\cong (H[C'_2], \chi_{(\infty)}^{(\bar{\alpha}', w', x')})$$

Now consider again the flip function f_2 . In G^{f_2} the set C_2 forms a connected component and similarly, in H^{f_2} the set C'_2 forms a connected component. Hence, removing any pebbles from vertices outside C_2 (resp. C'_2) does not affect the stable colouring restricted to the component C_2 (resp. C'_2) by Corollary III.9. Since all pebbles $(\bar{a}, \bar{b}, \bar{a}_1, \bar{b}_1, v, w)$ (resp. $(\bar{a}', \bar{b}', \bar{a}'_1, \bar{b}'_1, v', w')$) are either outside of C_2 (recall that $v \in X_1$ and hence, $v \notin C_2$) or the corresponding vertices are also pebbled using $(\bar{a}_2, \bar{b}_2, x)$ (resp. $(\bar{a}'_2, \bar{b}'_2, x')$), we can remove the pebbles $(\bar{a}, \bar{b}, \bar{a}_1, \bar{b}_1, v, w)$ and $(\bar{a}', \bar{b}', \bar{a}'_1, \bar{b}'_1, v', w')$ and still get that

$$(G[C_2], \chi_{(\infty)}^{(\bar{a}_2, \bar{b}_2, x)}) \not\cong (H[C'_2], \chi_{(\infty)}^{(\bar{a}'_2, \bar{b}'_2, x')})$$

by Corollary III.10 (or Spoiler wins using two additional pebbles). But now we can apply the induction hypothesis to t_2 . As a result, Spoiler wins from the current position and hence, Spoiler wins from position $((\bar{a}, \bar{b}, v), (\bar{a}', \bar{b}', v'))$.

Overall, by the induction principle, this gives us a winning strategy for Spoiler in the pebble game played over the graphs G and H . It remains to analyse the number of pebbles required to implement this strategy. Looking at Spoiler strategy, it is not difficult to see that it requires at most $6k+5$ many pebbles. More precisely, Spoiler needs $6k$ pebbles to pebble the three ordered split pairs (\bar{a}, \bar{b}) and (\bar{a}_i, \bar{b}_i) for $i \in \{1, 2\}$. The base step requires three additional pebbles. In the inductive step, five additional pebbles suffice, three for pebbling v, w and x and two pebbles to simulate colour refinement in case the bijections chosen by Duplicator do not match up.

However, taking a closer look, some vertices are always pebbled multiple times due to the nice ordered split pairs. In particular, we get that $\bar{a} \subseteq \bar{a}_1 \cup \bar{a}_2$. Since there is no need to pebble any vertex multiple times, we conclude that Spoiler can also win using only $5k+5$ many pebbles.

But even this number can be further improved. Indeed, Spoiler can also find nice ordered split pairs (\bar{a}, \bar{b}) and (\bar{a}_i, \bar{b}_i) for $i \in \{1, 2\}$ such that additionally $\bar{b}_i \cap \bar{X} \subseteq \bar{b}$ (cf. Lemma IV.2). Then $\bar{b}_i \subseteq \bar{b} \cup \bar{a}_{3-i}$ for both $i \in \{1, 2\}$. Again, there is no need to pebble any vertex multiple times and hence, Spoiler actually requires only $3k+5$ many pebbles. \square

V. CAPTURING PTIME ON GRAPHS OF BOUNDED RANK WIDTH

In this section, we prove Theorem I.5. We start with a quick introduction to the necessary background from descriptive complexity theory.

A. Preliminaries from Descriptive Complexity Theory

We assume that the reader has a solid background in logic and, in particular, is familiar with the standard fixed-point logics used in finite model theory. For background and precise definitions, we refer the reader to the textbooks [12], [14], [18], [29], [35].

Relational Structures: We work with finite structures over a relational vocabulary τ . The universe of a structure A is denoted by $V(A)$, and the interpretation of a k -ary relation symbol R is denoted by $R(A)$. In particular, we view graphs as structures of vocabulary $\{E\}$ for a binary relation symbol E . For a structure A and a subset $U \subseteq V(A)$, the *induced substructure* $A[U]$ is the structure with universe $V(A[U]) := U$ and relations $R(A[U]) := R(A) \cap U^k$ for every k -ary relation symbol in the vocabulary. Two structures A, B of the same vocabulary τ are *isomorphic* (we write: $A \cong B$) if there is a bijective mapping, called an *isomorphism*, $f : V(A) \rightarrow V(B)$ such that for all k -ary $R \in \tau$ and all $\bar{a} \in V(A)^k$ we have $\bar{a} \in R(A) \iff f(\bar{a}) \in R(B)$. We extend isomorphism to structures with *individualised elements*: for tuples $\bar{v} = (v_1, \dots, v_\ell) \in V(A)^\ell$, $\bar{w} = (w_1, \dots, w_\ell) \in V(B)^\ell$, an *isomorphism* from (A, \bar{v}) to (B, \bar{w}) is an isomorphism f from A to B such that $f(v_i) = w_i$ for all $i \in [\ell]$. We write $(A, \bar{v}) \cong (B, \bar{w})$ if such an isomorphism exists.

Fixed-Point Logic with Counting: *Inflationary fixed-point logic* is the extension of first-order logic by a fixed-point operator with an inflationary semantics. Instead of giving a formal definition of its syntax (where we follow [18]) and semantics, we give one illustrative example.

Example V.1. The IFP-sentence

$$\text{conn} := \forall x_1 \forall x_2 \text{ ifp } \left(X \leftarrow (x_1, x_2) \mid x_1 = x_2 \vee E(x_1, x_2) \vee \exists x_3 (X(x_1, x_3) \wedge X(x_3, x_2)) \right) (x_1, x_2)$$

states that a graph is connected.

IFP-formulas have individual variables, ranging over the elements of the universe of a structure, and relation variables, each with a prescribed arity, ranging over relations of this arity over the universe. We write $\varphi(X_1, \dots, X_k, x_1, \dots, x_\ell)$ to denote that the free relation variables of a formula are among X_1, \dots, X_k and the free individual variables are among x_1, \dots, x_ℓ . For a structure A , relations R_1, \dots, R_k of the appropriate arities, and elements v_1, \dots, v_ℓ , we write $A \models \varphi(R_1, \dots, R_k, v_1, \dots, v_\ell)$ to denote that A satisfies φ if the X_i are interpreted by R_i and the x_j are interpreted by v_j .

Inflationary fixed-point logic with counting, FP+C, is the extension of IFP by counting operators that allow it to speak about cardinalities of definable sets and relations. To define FP+C, we interpret the logic IFP over two sorted extensions of structures by a numerical sort. For a structure A , we let $N(A)$ be the initial segment $\{0, \dots, |V(A)|\}$ of the nonnegative integers. We let A^+ be the two-sorted structure $A \cup (N(A), \leq)$, where \leq is the natural linear order on $N(A)$. To avoid confusion, we always assume that $V(A)$ and $N(A)$ are disjoint.

In a structure A , individual variables of the logic FP+C range either over the set $V(A)$ (*vertex variables*) or over the set $N(A)$ (*number variables*). Relation variables may range over mixed relations, having certain places for vertices and certain places for numbers. The logic FP+C has all the constructors of IFP, and in addition *counting terms* of the form $\#x \varphi$, where x is a vertex variable and φ some formula. The value of this term in a structure A is the number of $v \in V(A)$ such that A satisfies φ if x is interpreted by v (under some fixed assignment to the other free variables of φ).

Example V.2. We start by giving an a FP+C-formula $\text{even}(y)$ with one free number variable y stating that y is an even number:

$$\text{even}(y) := \text{ifp } \left(Y \leftarrow y \mid \forall y' y \leq y' \vee \exists y'' \exists y' (Y(y') \wedge \text{succ}(y', y'') \wedge \text{succ}(y'', y)) \right) (y),$$

where $\text{succ}(z, z') := z \leq z' \wedge \neg z = z' \wedge \forall z'' (z'' \leq z \vee z' \leq z'')$. Then the following FP+C-sentence defines the class of Eulerian graphs (that is, graphs with a cyclic walk that traverses all edges, which are well-known to be exactly the connected graphs in which all vertices have even degree):

$$\text{eulerian} := \text{conn} \wedge \forall x \text{ even}(\#x' E(x, x')),$$

where *conn* is the sentence from Example V.1.

We like to think of definitions in the logic FP+C in an algorithmic way, where formulas are “programs” computing an input-output relation. Rather than writing out syntactical details, we describe these programs in a high-level form, as we would do with any type of algorithms and leave the “FP+C-implementation” to the reader. A thorough technical treatment of the issues involved in this can be found in [18, Chapters 2 and 3].

We will make assertions about the existence of FP+C-formulas, or “programs”, with a specified input-output behaviour. In the most basic setting of an FP+C-sentence like *eulerian*, the input is a structure (a graph) and the output a Boolean value. For a formula like *even*(*y*) the input is a structure *A* and the output is a subset of $N(A)$. But the input can be more complicated. For example, the input may be a triple (G, U, d) , where *G* is a graph, $U \subseteq V(G)$, $d \in N(G)$, and our task is to write an FP+C-formula that computes the set of all vertices of *G* that have distance at most *d* from a vertex in *U*. Formally, this means that we have to write an FP+C-formula $\varphi(X, y, x)$ such that for all graphs *G*, all subsets $U \subseteq V(G)$, and all $d \in N(G)$ we have $G \models \varphi(U, d, v)$ if and only if the distance of *v* to *U* in *G* is at most *d*. An even more complicated type of assertion we will frequently see is of the following form: *given a tuple (G, U, v, H) , where G is a graph, $U \subseteq V(G)$, $v \in V(G) \setminus U$, and H is a graph with vertex set $V(H) \subseteq N(G)$, in FP+C we can decide if the connected component of $G \setminus U$ that contains *v* is isomorphic to H .* Here our task is to define an FP+C-formula $\varphi(X, x, Y, Z)$, where *X* is a unary relation ranging over the vertex sort, *x* is a vertex variable, and *Y* and *Z* are a unary and a binary relation symbol both ranging over the number sort, such that for all graphs *G*, $U \subseteq V(G)$, $v \in V(G) \setminus U$, $P \subseteq N(G)$, $Q \subseteq N(G)^2$ we have $G \models \varphi(U, v, P, Q)$ if and only if the connected component of $G \setminus U$ that contains *v* is isomorphic to the graph *H* with $V(H) = P$ and $E(H) = Q$. Actually, it is known that such a formula can not exist (see [5]). However, in this work we will only require such formulas for specific graph classes \mathcal{C} , that is, when the input is restricted to graphs $G \in \mathcal{C}$.

Lemma V.3 (cf. [38]). *Let \mathcal{C} be a hereditary graph class (i.e. \mathcal{C} is closed under induced subgraphs) such that *k*-dimensional Weisfeiler-Leman identifies all graphs $G \in \mathcal{C}$ for some constant number *k*. Then there is an FP+C-formula $\varphi(X, Y, Z)$, where *X* is a unary relation ranging over the vertex sort and *Y* and *Z* are a unary and a binary relation symbol both ranging over the number sort, such that for all graphs $G \in \mathcal{C}$, $U \subseteq V(G)$, $P \subseteq N(G)$, $Q \subseteq N(G)^2$ we have $G \models \varphi(U, P, Q)$ if and only if $G[U] \cong (P, Q)$.*

In a setting where we have several input objects, such as the tuple (G, U, v, H) above, we always have one *main input structure*, which will be listed first. In the example (G, U, v, H) , this is the graph *G*. All other objects are defined relative to this main structure and its numerical extension.

In the example, *U* was a subset of $V(G)$, *v* an element of $V(G)$, and *H* a structure with universe $N(G)$. Sometimes, we will have to deal with whole families of structures. They will always be indexed by tuples of elements of the main structure. For example, we may be given a pair $(A, (H_{(v,p)})_{(v,p) \in V(A) \times N(A)})$ where the $H_{(v,p)}$ are graphs with universe $V(H_{(v,p)}) \subseteq N(A)$. Formally, we can represent such a family by the ternary relation $R = \{(v, p, q) \in V(A) \times N(A) \times N(A) \mid q \in V(H_{(v,p)})\}$ and the quaternary relation $S = \{(v, p, q, q') \in V(A) \times N(A) \times N(A) \times N(A) \mid (q, q') \in E(H_{(v,p)})\}$.

Definable Canonisation: Recall from the introduction that a logic captures polynomial time on a class \mathcal{C} of structures if each polynomial time decidable property of structures in \mathcal{C} is expressible by a sentence of the logic. By the Immerman-Vardi Theorem [28], [43], IFP captures polynomial time on the class of all ordered structures.¹ A straightforward way of applying this theorem to a class \mathcal{C} of unordered structures is to define a linear order on this class: if there is a formula *ord*(*x, y*) of the logic IFP that defines a linear order on all structures in \mathcal{C} , then IFP still captures polynomial time on \mathcal{C} . Unfortunately, this observation is rarely applicable, because usually it is impossible to define linear orders. For example, it is impossible to define a linear order on a structure that has a nontrivial automorphism.

A much more powerful idea, going back to [30], [38] and known as *definable canonisation*, is to define an *ordered copy* of the input structure. To implement this idea, FP+C is particularly well-suited, because we can take the numerical part $N(A)$ of a structure *A* as the universe of the ordered copy of *A*. Technically, definable canonisation is based on syntactical interpretations (called transductions in [18]). Instead of introducing the unwieldy machinery of syntactical interpretations in full generality, we just focus on a special case that suffices for our purposes. Suppose, we have a structure *A* of vocabulary τ . To define an ordered copy of *A*, we need a formula $\varphi_R(y_1, \dots, y_k)$ with free number variables y_i for every *k*-ary $R \in \tau$. A family $\Phi = (\varphi_R(\bar{y}) \mid R \in \tau)$ of such formulas defines a structure A^Φ with universe $V(A^\Phi) := \{0, \dots, |V(A)| - 1\} \subseteq N(A)$ and relations $R(A^\Phi) := \{(p_1, \dots, p_k) \in V(A^\Phi)^k \mid A \models \varphi_R(p_1, \dots, p_k)\}$. We say that Φ defines an *ordered copy* of *A* if $A^\Phi \cong A$. Observe that if Φ defines an ordered copy of *A*, then this ordered copy is *canonical* in the sense that for all $B \cong A$ it holds that $B^\Phi = A^\Phi$, because we have $N(B) = N(A) = \{0, \dots, n\}$ for $n = |V(A)| = |V(B)|$, and definitions in the logic FP+C are isomorphism invariant. We say that a class \mathcal{C} of τ -structures *admits FP+C-definable canonisation* if there is a family $\Phi = (\varphi_R(\bar{y}) \mid R \in \tau)$ of FP+C-formulas such that for all $A \in \mathcal{C}$ it holds that $A^\Phi \cong A$. The following lemma is a direct consequence of the Immerman-Vardi Theorem (for a proof, see [18, Lemma 3.3.8]).

¹Originally, the Immerman-Vardi Theorem states that *least fixed-point logic* LFP captures polynomial time on the class of all ordered structures. However, it is known that LFP and IFP have the same expressive power [26], [33].

Lemma V.4. *Let \mathcal{C} be a class of τ -structure that admits FP+C-definable canonisation. Then FP+C captures polynomial time on \mathcal{C} .*

Sometimes, we need to define ordered copies of substructures of a structure. To define an ordered copy of a substructure of a τ -structure we use a family Ψ of formulas that in addition to formulas $\psi_R(\bar{y})$ for the relations contains a formula $\psi_V(y)$ that specifies the universe of the ordered copy. Given a pair (A, B) , where A is a τ -structure and $B \subseteq A$ a substructure, such a family Ψ defines a structure B' with universe $V(B') := \{p \in N(A) \mid A \models \psi_V(p)\}$ and relations $R(B') := \{(p_1, \dots, p_k) \in V(B')^k \mid A \models \psi_R(p_1, \dots, p_k)\}$. If $B' \cong B$, we say that Ψ defines an ordered copy of B in A .

We will also see more complicated assertions such as the following: given a tuple (G, U, v) , where G is a graph, $U \subseteq V(G)$, $v \in V(G) \setminus U$, in FP+C we can compute an ordered copy of the connected component of $G \setminus U$ that contains v . This means that we can construct FP+C-formulas $\psi_V(X, x, y)$ and $\psi_E(X, x, y_1, y_2)$ such that for all $G, U \subseteq V(G)$, and $v \in V(G) \setminus U$, the graph with universe $V' := \{p \in N(G) \mid G \models \psi_V(U, v, p)\}$ and edge relation $E' := \{(p_1, p_2) \in (V')^2 \mid G \models \psi_E(U, v, p_1, p_2)\}$ is isomorphic to the connected component of v in $G \setminus U$.

We will routinely have to compare ordered copies of substructures of our input graphs. To do this, we define a *lexicographical order* on τ -structures whose universe is an initial segment of the nonnegative integers. First, we fix a linear order of the relation symbols in τ . Say, $\tau = \{R_1, \dots, R_\ell\}$ and we order the R_i by their indices. Now let A, B be two τ -structures such that $V(A) = \{0, \dots, n_A - 1\}$ and $V(B) = \{0, \dots, n_B - 1\}$. Structure A is *lexicographically smaller than or equal to* structure B (we write $A \leq_{\text{lex}} B$) if either $A = B$ or $A \neq B$ and $n_A = n_B$ and for the least $i \in [\ell]$ such that $R_i(A) \neq R_i(B)$ the lexicographically first tuple \bar{p} in the symmetric difference of $R_i(A)$ and $R_i(B)$ is contained in $R_i(B)$ or $n_A < n_B$.

B. Definable Canonisation of Graphs of Bounded Rank Width

Recall that our goal is to prove that FP+C captures PTIME on the class of graphs of rank width at most k . Towards this end we prove the following theorem.

Theorem V.5. *For every $k \geq 1$, the class of all graphs of rank width at most k admits FP+C-definable canonisation.*

Observe that, combined with Lemma V.4, this theorem implies Theorem I.5.

The rest of this section is devoted to a proof of Theorem V.5. Let us fix $k \geq 1$. Our strategy to define an ordered copy of a graph G of rank width at most k is similar to the proof strategy for showing that the Weisfeiler-Leman algorithm identifies such a graph. For ordered split pairs (\bar{a}, \bar{b}) , flip functions f , and components C of the flipped graph we recursively define an ordered copy of the induced subgraph $(G[C \cup \bar{a} \cup \bar{b}], \bar{a}, \bar{b})$. The first hurdle towards implementing this strategy is that we need to have explicit access to the flip function (this is different

from the previous section where we only needed the existence of such a function in order to describe a strategy for Spoiler). However, we can not simply list all of the flip functions as there may be exponentially many. We remedy this by altering the definition of a flip so that, for every fixed k , there is only a polynomial number of flips.

Throughout this section let $k \geq 1$ be a fixed natural number. Let $G = (V, E)$ be a graph of rank width at most k . In this section an *ordered split pair of order at most k* is simply a pair (\bar{a}, \bar{b}) where $\bar{a}, \bar{b} \in V^{\leq k}$. For $v, w \in V$ we say that $v \approx_{(\bar{a}, \bar{b})} w$ if $N(v) \cap (\bar{a}, \bar{b}) = N(w) \cap (\bar{a}, \bar{b})$. Clearly, $\approx_{(\bar{a}, \bar{b})}$ defines an equivalence relation on V . For tuples $\bar{a}, \bar{b} \in V^{\leq k}$ we denote by $2^{\bar{a} \cup \bar{b}}$ the set of all subsets of $\bar{a} \cup \bar{b} \subseteq V$ where we interpret the tuples \bar{a} and \bar{b} as subsets of V . A *flip extension* of an ordered split pair (\bar{a}, \bar{b}) is a tuple

$$\bar{s} = (\bar{a}, \bar{b}, f: (2^{\bar{a} \cup \bar{b}})^2 \rightarrow [n] \cup \{\perp\})$$

such that for all $M, N \in 2^{\bar{a} \cup \bar{b}}$ with $M \neq N$, either $f(M, N) = \perp$ or $f(N, M) = \perp$. For $v, w \in V$ we say that $v \approx_{\bar{s}} w$ if $v \approx_{(\bar{a}, \bar{b})} w$. We denote by $[v]_{\approx_{\bar{s}}}$ the equivalence class of v with respect to $\approx_{\bar{s}}$. Moreover, we define the graph $G^{\bar{s}} = (V, E^{\bar{s}}, \bar{a}, \bar{b})$ where

$$\begin{aligned} E^{\bar{s}} = & \{vw \in E \mid f(N(v) \cap (\bar{a}, \bar{b}), N(w) \cap (\bar{a}, \bar{b})) = d \in [n] \\ & \wedge |N(v) \cap [w]_{\approx_{\bar{s}}}| < d\} \\ & \cup \{vw \notin E \mid f(N(v) \cap (\bar{a}, \bar{b}), N(w) \cap (\bar{a}, \bar{b})) = d \in [n] \\ & \wedge |N(v) \cap [w]_{\approx_{\bar{s}}}| \geq d\}. \end{aligned}$$

Finally we let $\text{Comp}(G, \bar{s}) \subseteq 2^V$ be the set of vertex sets of the connected components of the graph $G^{\bar{s}}$ and for $v \in V$ we define $\text{Comp}(G, \bar{s}, v)$ to be the unique $C \in \text{Comp}(G, \bar{s})$ such that $v \in C$.

The following lemma is similar in nature to Lemma III.6.

Lemma V.6. *Let G be a graph and let $X \subseteq V(G)$. Furthermore, let (\bar{a}, \bar{b}) be an ordered split pair for X . Then there is a flip extension $\bar{s} = (\bar{a}, \bar{b}, f)$ such that $C \subseteq X$ or $C \subseteq \bar{X}$ for every $C \in \text{Comp}(G, \bar{s})$.*

Definition V.7. Let $G = (V, E)$ be a graph. Let $\bar{s} = (\bar{a}, \bar{b}, f)$ be a flip extension and let $C \subseteq V(G)$. An *\bar{s} -anchored (ordered) copy* of C is a tuple $C_{\text{anc}} = (V_{\text{anc}}, E_{\text{anc}}, \bar{a}_{\text{anc}}, \bar{b}_{\text{anc}}, \eta)$ such that

- 1) V_{anc} is an initial segment of the nonnegative integers,
- 2) $\eta: V_{\text{anc}} \rightarrow (2^{\bar{a} \cup \bar{b}} \rightarrow \mathbb{N})$, and
- 3) there is an isomorphism $\sigma: (G[C \cup \bar{a} \cup \bar{b}], \bar{a}, \bar{b}) \cong (V_{\text{anc}}, E_{\text{anc}}, \bar{a}_{\text{anc}}, \bar{b}_{\text{anc}})$ such that $(\eta(i))(M) = |N(\sigma^{-1}(i)) \cap \{w \in V(G) \mid N(w) \cap (\bar{a}, \bar{b}) = M\}|$ for all $i \in V_{\text{anc}}$.

In the following, this definition is typically applied to sets $C \in \text{Comp}(G, \bar{s})$. More precisely, our aim is to define \bar{s} -anchored copies of C for all $C \in \text{Comp}(G, \bar{s})$ and all suitable flip extensions \bar{s} in FP+C in an inductive fashion. An important feature of an \bar{s} -anchored copy of C is that, in addition to being an ordered copy of C , it records some “context information”

on how the set C is connected to the rest of the graph. This is the purpose of the function η . The context information will play a vital role in the proofs since it gives the relevant information to perform flips also in the \bar{s} -anchored copy of C .

Let us start by discussing how to represent the relevant objects in the logical framework. Since k is a constant we can view a flip extension as a tuple $\bar{s} \in V(G)^{2k} \times \{0, \dots, n\}^{2^{2k}}$ of fixed length. The first $2k$ components represent the split pair (\bar{a}, \bar{b}) and the function $f: 2^{\bar{a} \cup \bar{b}} \rightarrow [n] \cup \{\perp\}$ can be seen as a tuple in $\{0, \dots, n\}^{2^{2k}}$ where a 0-entry is interpreted as \perp . Similar to the previous section, components (with respect to some flip extension) are represented by a single vertex from that component. To be more precise, a set $C \in \text{Comp}(G, \bar{s})$ is represented by \bar{s} and some $v \in C$. Observe that there is an FP+C-formula $\varphi(\bar{x}, y, z)$ such that $G \models \varphi(\bar{s}, v, w)$ if and only if $w \in \text{Comp}(G, \bar{s}, v) = C$ (see Example V.1 for how to define reachability in FP+C).

To represent an \bar{s} -anchored copy $C_{\text{anc}} = (V_{\text{anc}}, E_{\text{anc}}, \bar{a}_{\text{anc}}, \bar{b}_{\text{anc}}, \eta)$ of a set $C \subseteq V(G)$, we represent the function η by a relation $P_\eta \subseteq N(G)^{1+2^{2k}}$ containing elements (p, \bar{p}_η) for every $p \in V_{\text{anc}}$. The tuple $\bar{p}_\eta \in N(G)^{2^{2k}}$ represents the function $\eta(p)$ and has an entry for each subset $M \subseteq \bar{a} \cup \bar{b}$ which specifies $(\eta(p))(M)$. Typically, we will denote the set $V_{\text{anc}} \subseteq N(G)$ by P_V , the relation $E(G)_{\text{anc}} \subseteq N(G)^2$ by P_E , the tuple \bar{a}_{anc} by \bar{p}_a , the tuple \bar{b}_{anc} by \bar{p}_b , and the relation representing η by P_η . Slightly abusing notation, we will write $C_{\text{anc}} = (P_V, P_E, \bar{p}_a, \bar{p}_b, P_\eta)$. We can define such an anchored copy by FP+C-formulas $\psi_V(z), \psi_E(z_1, z_2), \psi_{\bar{a}}(\bar{z}), \psi_{\bar{b}}(\bar{z}), \psi_\eta(z, \bar{z}')$, where z, z_1, z_2 are number variables and \bar{z}, \bar{z}' are tuples of number variables of lengths $k, 2^{2k}$, respectively.

We now start by constructing various FP+C formulas. They will form the basic building blocks of the formulas defining \bar{s} -anchored copies of C for sets $C \in \text{Comp}(G, \bar{s})$.

Lemma V.8. *Let G be a graph of rank width at most k . Also let \bar{s} be a flip extension, $D \in \text{Comp}(G, \bar{s})$, and $D^* = (P_V, P_E, \bar{p}_a, \bar{p}_b, P_\eta)$ where $P_V \subseteq N(G)$, $P_E \subseteq N(G)^2$, $\bar{p}_a, \bar{p}_b \in N(G)^k$ and $P_\eta \subseteq N(G)^{1+2^{2k}}$.*

There is an FP+C-sentence that, given access to the objects (G, \bar{s}, D, D^) , decides if D^* is an \bar{s} -anchored copy of D .*

Lemma V.9. *Let G be a graph, $X_1 \subseteq X \subseteq V(G)$, (\bar{a}, \bar{b}) an ordered split pair for X and (\bar{a}_1, \bar{b}_1) an ordered split pair for X_1 . Let $\bar{s} = (\bar{a}, \bar{b}, f)$ and $\bar{s}_1 = (\bar{a}_1, \bar{b}_1, f_1)$ be flip extensions. Moreover, suppose that $X_1 \cap \bar{a} \subseteq \bar{a}_1$ and let $D \in \text{Comp}(G, \bar{s}_1)$ such that $D \subseteq X_1$. Also let D_{anc} be an \bar{s}_1 -anchored copy of D and let σ be an isomorphism according to Definition V.73.*

Given access to the objects $(G, \bar{s}, \bar{s}_1, D, D_{\text{anc}})$, the following queries can be defined using FP+C-formulas:

- 1) *given $p \in V(D_{\text{anc}})$, determine $N(v) \cap (\bar{a}, \bar{b})$ where $v = \sigma^{-1}(p)$,*
- 2) *given $p \in V(D_{\text{anc}})$ and $v' \in V(G)$, determine whether $v \approx_{\bar{s}} v'$ where $v = \sigma^{-1}(p)$,*
- 3) *given $p \in V(D_{\text{anc}})$ and $w \in V(G)$, determine $|N(v) \cap [w]_{\approx_{\bar{s}}}|$ where $v = \sigma^{-1}(p)$, and*

- 4) *given $p, q \in V(D_{\text{anc}})$, determine whether $vw \in E(G^{\bar{s}})$ where $v = \sigma^{-1}(p)$ and $w = \sigma^{-1}(q)$.*

Just to be on the safe side, let us again explain the exact technical meaning of the assertions of the lemma, taking assertion 1 as an example. We need to construct an FP+C-formula $\varphi(\bar{x}, \bar{x}_1, y, Z_V, Z_E, \bar{z}_a, \bar{z}_b, Z_\eta, z, y')$, where Z_V is a unary relation symbol of type 'number', Z_E is a binary relation symbol of type 'number \times number', Z_η is a $(1 + 2^{2k})$ -ary relation symbol of type ('number') $^{(1+2^{2k})}$, y, y' are vertex variables, z is a number variable, \bar{z}_a, \bar{z}_b are k -tuples of number variables, and \bar{x}, \bar{x}_1 are tuples of individual variables of the type appropriate for representing flip extensions. The formula is supposed to have the following meaning. Suppose that $D_{\text{anc}} = (P_V, P_E, \bar{p}_a, \bar{p}_b, P_\eta)$. Then for all $u \in D$, $p \in P_V$, and $w \in V(G)$,

$$G \models \varphi(\bar{s}, \bar{s}_1, u, P_V, P_E, \bar{p}_a, \bar{p}_b, P_\eta, p, w)$$

if and only if $w \in N(v) \cap (\bar{a}, \bar{b})$. We should think of the relations P_V, P_E, P_η , which determine the core of the structure D_{anc} , as being defined earlier in some inductive process. Note that we do not specify D explicitly in the definition, but only implicitly by giving the flip extension \bar{s}_1 and the vertex u .

Recall the definition of nice triples of ordered split pairs (see Definition IV.1)

Lemma V.10. *Let G be a graph and $X, X_1, X_2 \subseteq V(G)$ such that $X = X_1 \uplus X_2$. Let (\bar{a}, \bar{b}) be an ordered split pair for X and let (\bar{a}_i, \bar{b}_i) be ordered split pairs for X_i , $i \in \{1, 2\}$, that are nice with respect to (\bar{a}, \bar{b}) . Moreover, let $\bar{s} = (\bar{a}, \bar{b}, f)$ and $\bar{s}_i = (\bar{a}_i, \bar{b}_i, f_i)$, $i \in \{1, 2\}$, be flip extensions. Let $C \in \text{Comp}(G, \bar{s})$ such that $C \subseteq X$, and let $\mathcal{D}_i \subseteq \text{Comp}(G, \bar{s}_i)$ such that*

$$X_i \subseteq \bigcup_{D \in \mathcal{D}_i} D.$$

For every $D \in \mathcal{D}_i$ let D_{anc} be an \bar{s}_i -anchored copy of D and let $\mathcal{D}_i^{\text{anc}}$ be the set of all of those copies for $i \in \{1, 2\}$.

Then in FP+C, given access to $(G, \bar{s}, \bar{s}_1, \bar{s}_2, C, \mathcal{D}_1, \mathcal{D}_1^{\text{anc}}, \mathcal{D}_2, \mathcal{D}_2^{\text{anc}})$, we can define an \bar{s} -anchored copy C_{anc} of C .

Let us again discuss the precise meaning of this statement. Specifically, we need to elaborate on how to represent the families of sets \mathcal{D}_i and the families of \bar{s}_i -anchored copies $\mathcal{D}_i^{\text{anc}}$. We index the two families by elements $u \in V(G)$. For every $u \in V(G)$ we let $D_{i,u} \in \text{Comp}(G, \bar{s}_i)$ be the unique component with $u \in D_{i,u}$. If $D_{i,u} \in \mathcal{D}_i$, we denote its anchored copy in $\mathcal{D}_i^{\text{anc}}$ by $D_{i,u}^{\text{anc}}$. Note, however, that we do not necessarily have $D_{i,u} \in \mathcal{D}$ for all u . The only requirement is that X_i is a subset of the union of all $D_{i,u}$ in \mathcal{D} . We represent the family $\mathcal{D}_i^{\text{anc}}$ by a binary relation $P_{i,V} \subseteq V(G) \times N(G)$, a ternary relation $P_{i,E} \subseteq V(G) \times N(G)^2$, $(k+1)$ -ary relations $P_{i,\bar{a}}, P_{i,\bar{b}} \subseteq V(G) \times N(G)^k$, and a $(2^{2k} + 2)$ -ary relation $P_{i,\eta}$ such that if $u \in V(G)$ with $D_{i,u} \in \mathcal{D}$ then $D_{i,u}^{\text{anc}} = (V, E, \bar{a}, \bar{b}, \eta)$ where

- $V = \{p \in N(G) \mid (u, p) \in P_{i,V}\};$
- $E = \{(p, q) \in N(G)^2 \mid (u, p, q) \in P_{i,E}\};$

- $\bar{a} \in N(G)^k$ is the unique tuple with $(u, \bar{a}) \in P_{i,\bar{a}}$;
- $\bar{b} \in N(G)^k$ is the unique tuple with $(u, \bar{b}) \in P_{i,\bar{b}}$;
- η is represented by $P_\eta \subseteq N(G)^{1+2^k}$ where $P_\eta = \{\bar{p} \in N(G)^{1+2^k} \mid (u, \bar{p}) \in P_{i,\eta}\}$.

To define C_{anc} , we need to construct FP+C-formulas $\varphi_V, \varphi_E, \varphi_{\bar{a}}, \varphi_{\bar{b}}, \varphi_\eta$. They all have free variables $\bar{x}, \bar{x}_1, \bar{x}_2$ for the flip extensions $\bar{s}, \bar{s}_1, \bar{s}_2$, a free vertex variable y for an element of the component C , and for $i = 1, 2$ free relation variables $Z_{i,V}, Z_{i,E}, Z_{i,a}, Z_{i,b}, Z_{i,\eta}$ for the family $\mathcal{D}_i^{\text{anc}}$. The family \mathcal{D}_i is only specified implicitly: \mathcal{D}_i consists of all components $D \in \text{Comp}(G, \bar{s}_i)$ such that a $u \in D$ appears as an index of a structure in $\mathcal{D}_i^{\text{anc}}$, that is, as the first component of a tuple in the relations $Z_{i,V}, Z_{i,E}, Z_{i,a}, Z_{i,b}, Z_{i,\eta}$. In addition, the formula φ_V has a free number variable z for the elements of $V(C_{\text{anc}})$. The formula φ_E has two free number variables z_1, z_2 for the elements of $E(C_{\text{anc}})$. The formula $\varphi_{\bar{a}}, \varphi_{\bar{b}}$ have a k -tuple of free number variables \bar{z} for $\bar{a}_{\text{anc}}, \bar{b}_{\text{anc}}$, respectively. And finally, the formula φ_η has a $(1+2^k)$ -tuple of free number variables \bar{z} to specify the function η of C_{anc} .

With this, we are ready to prove our theorem.

Idea of proof of Theorem V.5. Let G be a graph of rank width at most k . We will inductively construct \bar{s} -anchored copies C_{anc} for flip extensions \bar{s} and components $C \in \text{Comp}(G, \bar{s})$, using Lemma V.10 in the inductive step.

To describe the proof, we fix G , but of course the FP+C-formulas we shall construct will not depend on the specific graph G and will work for every graph of rank width at most k . Let $F(G) \subseteq V(G)^{2k} \times N(G)^{2^k}$ be the set of all flip extensions.

In our main induction, to be implemented by an FP+C-formula, we will define an increasing collection of anchored copies of components $C \in \text{Comp}(G, \bar{s})$ for flip extensions \bar{s} . To achieve this we simultaneously define six relations

- $R^{(\ell)} \subseteq F(G) \times V(G)$,
- $R_V^{(\ell)} \subseteq F(G) \times V(G) \times N(G)$,
- $R_E^{(\ell)} \subseteq F(G) \times V(G) \times N(G)^2$,
- $R_{\bar{a}}^{(\ell)}, R_{\bar{b}}^{(\ell)} \subseteq F(G) \times V(G) \times N(G)^k$,
- $R_\eta^{(\ell)} \subseteq F(G) \times V(G) \times N(G)^{1+2^k}$.

for every $\ell \geq 1$. The first relation contains all flip extensions \bar{s} and $u \in V(G)$ such that we have already defined an \bar{s} -anchored copy of $\text{Comp}(G, \bar{s}, u)$. Such a copy is specified by the other five relation similar to the descriptions given above.

In the FP+C-formula that we construct, the relations will be represented by relation variables $X, X_V, X_E, X_{\bar{a}}, X_{\bar{b}}, X_\eta$ of appropriate types. $R^{(\ell)}$ will be the value of X after the ℓ -th iteration of the main fixed-point iteration (and similarly for the other variables).

In the base step of our induction, we define $R^{(1)}$ to consist of all tuples $(\bar{s}, u) \in F(G)$ such that $\{u\} \in \text{Comp}(G, \bar{s})$. Defining anchored copies for these components is easy.

So let us turn to the inductive step. We have already defined relations $R^{(\ell)}, R_V^{(\ell)}, \dots, R_\eta^{(\ell)}$. We look at a flip extension \bar{s} and a vertex $u \in V(G)$ such that $(\bar{s}, u) \notin R^{(\ell)}$, that is, we have not yet defined an anchored copy of $\text{Comp}(G, \bar{s}, u)$. For all

$\bar{s}_1, \bar{s}_2 \in F(G)$, we do the following. We let \mathcal{D}_i be the set of all $D \in \text{Comp}(G, \bar{s}_i)$ such that $(\bar{s}_i, v) \in R^{(\ell)}$ for some $v \in D$. By induction, this means that actually $(\bar{s}_i, v) \in R^{(\ell)}$ for all $v \in D$ and that we have already computed an anchored copy D_{anc} of D , which is represented by the (\bar{s}_i, v) -entries of the relations in $R_V^{(\ell)}, \dots, R_\eta^{(\ell)}$. We let $\mathcal{D}_i^{\text{anc}}$ be the set of all these anchored copies D_{anc} . Now we apply the FP+C-formulas of Lemma V.10 to $(G, \bar{s}, \bar{s}_1, \bar{s}_2, C, \mathcal{D}_1, \mathcal{D}_1^{\text{anc}}, \mathcal{D}_2, \mathcal{D}_2^{\text{anc}})$. We obtain a structure $C_{\bar{s}_1, \bar{s}_2} = (P_V, P_E, \bar{p}_a, \bar{p}_b, P_\eta)$. Note that $C_{\bar{s}_1, \bar{s}_2}$ is not necessarily an \bar{s} -anchored copy of C , because we do not know whether there are sets X, X_1, X_2 such that $\bar{s}, \bar{s}_1, \bar{s}_2$ satisfy the assumptions of Lemma V.10. However, using Lemma V.8, we can check if $C_{\bar{s}_1, \bar{s}_2}$ is an anchored copy, regardless of whether the assumptions of Lemma V.10 are satisfied. If $C_{\bar{s}_1, \bar{s}_2}$ is an \bar{s} -anchored copy of C , we call (\bar{s}_1, \bar{s}_2) *good* for \bar{s} .

To define an \bar{s} -anchored copy of C we take the lexicographically smallest of all structures $C_{\bar{s}_1, \bar{s}_2}$ where (\bar{s}_1, \bar{s}_2) is good for \bar{s} .

To prove that this inductive process eventually defines an ordered copy of the complete graph, we utilize the existence of a rank decomposition of width at most k for G . One can prove by induction that for every node $t \in V(T)$, every flip extension \bar{s} for the set $X = \gamma(t)$, and every component $C \in \text{Comp}(G, \bar{s})$ such that $C \subseteq X$, there is an $\ell \geq 1$ such that $(\bar{s}, u) \in R^{(\ell)}$ for all $u \in C$. Indeed, we can choose ℓ to be 1 plus the depth of t in the tree, that is, the maximum length of a (directed) path from t to leaf. \square

VI. CONCLUSIONS

In this paper we considered the isomorphism and canonisation problem for graphs of bounded rank width. The first main result is that the Weisfeiler-Leman dimension of graphs of rank width at most k is at most $3k+4$, that is, the $(3k+4)$ -dimensional Weisfeiler-Leman algorithm identifies all graphs of rank width at most k . This implies that isomorphism testing and canonisation for graphs of rank width at most k can be done in time $n^{O(k)}$.

The second main result is that fixed-point logic with counting captures polynomial time on the class of graphs of rank width at most k .

We remark that it is not difficult to obtain an $\Omega(k)$ lower bound on the Weisfeiler-Leman dimension of graphs of rank width k . Thus our upper bound is tight up to a constant factor. But still it would be nice to close or narrow the gap between the upper and lower bound.

A more important question is whether isomorphism testing is also fixed-parameter tractable when parameterized by rank width. We remark that fpt algorithms for isomorphism testing parameterized by tree width are known [36], [22].

An interesting open question on the logical side is whether rank decompositions can be defined in monadic second order logic. A partial result for graphs of bounded linear clique width has been obtained in [4]. We believe the techniques developed in this paper might also prove helpful for resolving the general question.

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