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THE POLYNOMIAL SOLVABILITY OF CONVEX QUADRATIC PROGRAMMING*

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AN ACCURATE quadratic programming algorithm is constructed, in which the amount of work is bounded by a polynomial of the length of the recording of the problem in the binary number system.

Consider the quadratic programming problem

$$f(x)=\frac{1}{2}(x,Cx)+(d,x)=\frac{1}{2}\sum_{i,j=1}^{n}c_{ij}x_ix_j+\sum_{j=1}^{n}d_jx_j\to\min,$$
$$(A_i,x)=\sum_{j=1}^{n}a_{ij}x_j\leqslant b_i,\qquad i=1,2,\ldots,m,\quad x\in R^n, \tag{1}$$

where C is an integer-valued symmetric positive-semidefinite matrix, and the vectors d, A_i and the scalars b_i are also integer-valued.

The input length of the quadratic programming problem is defined as the quantity

$$L=L_1+L_2=\left[\sum_{i,j=1}^{n}\log_2(|c_{ij}|+1)+\sum_{j=1}^{n}\log_2(|d_j|+1)+1\right]$$
$$+\left[\sum_{i,j=1}^{m,n}\log_2(|a_{ij}|+1)+\sum_{i=1}^{m}\log_2(|b_i|+1)+\log_2 nm+1\right], \tag{2}$$

which determines the number of binary symbols necessary for recording the input information of the problem.

By an accurate solution of a quadratic programming problem we mean the following:

a) determination of the consistency of the linear inequalities;

b) in case of consistency, establishing whether the functional $f(x)$ is underbounded in the set $X\subseteq R^n$ of solutions of the system of inequalities;

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c) if the constraints are consistent and the functional is bounded on X, finding its extremal value t/s (t and s are relatively prime integers), and also of the point $x^0 \in X$ where the extremum is attained.

In this paper we construct an algorithm for the accurate solution of the quadratic programming problem whose amount of work is bounded by a polynomial of the input length L, that is, it is shown that quadratic programming belongs to the class P of problems solvable on deterministic Turing machines after a time which is a polynomical of the input [1]. The proof uses [2], where an algorithm is constructed for a linear programming polynomial in the input length.

1. In [2] an algorithm T was constructed to determine the consistency in R^n of the systems of m linear inequalities

$$(A_i, x) \leqslant b_i, \qquad i=1, 2, \ldots, m,$$

with an amount of work which is a polynomical of the input length L_2, where L_2 is defined by formula (2). The characteristics of the algorithm T are as follows: a memory of $O(n^2 + nm)$ numbers, each of which has, in fixed-point binary notation, $O(L_2)$ digits; on these numbers $O(n^3(n+m)L_2)$ elementary operations, $+$, $-$, $\times$, $/$, $\sqrt{\ }$, max, min are performed with an accuracy of $O(L_2)$ digits.

Therefore, the consistency of the system of linear inequalities of the quadratic programming problem is checked by the algorithm T, so that clause a) is exhausted. Below it is assumed that $X \neq \phi$.

2. The underboundedness of the convex quadratic functional on X is equivalent to the consistency of the system of $n + m$ linear inequalities and equations in the variables $(x, \lambda) \in R^{n+m}$:

$$Cx+d+\sum_{i=1}^{m} \lambda_i A_i = 0, \qquad \lambda_i \geqslant 0, \qquad i=1, 2, \ldots, m. \tag{3}$$

The input of the system (3) does not exceed $2L$, where L is the input of the quadratic programming problem. Therefore, the consistency of system (3), and consequently also the underboundedness of $f(x)$ on X, is checked by an algorithm T with an amount of work which is a polynomial in L. This requires $O((n+m)^4 L)$ elementary operations on $O(L)$-digit numbers. Therefore, clause b) of the accurate solution of the quadratic programming problem is also exhausted. In what follows we assume that $f(x)$ is underbounded on X. We note that on these assumptions the minimum of the quadratic functional is attained (see, for example, [3], p. 36).

3. Let x^* be the arbitrary point where the minimum is attained:

$$\min_{x \in X} f(x) = f(x^*), \qquad x^* \in X.$$

We distinguish the constraints active at the point x^*, that is, we consider the subset of indexes $I = I(x^*) \subseteq \{1, 2, \ldots, m\}$, such that $(A_i, x^*) = b_i$ for $i \in I$; we denote by $r = |I|$ the number of active constraints, $0 \leqslant r \leqslant m$. It follows from the Kuhn–Tucker theorem that scalars λ_i, $i \in I$ can be found such that the pair $\langle x^*, \lambda^* \rangle \in R^{n+r}$ satisfies the system of $n+m+r$ linear inequalities and equations:

$$\begin{gathered}(A_i, x) = b_i, \quad i \in I, \qquad (A_i, x) \leqslant b_i, \qquad i \notin I, \\ Cx + d + \sum_{i \in I} \lambda_i A_i = 0, \qquad \lambda_i \geqslant 0, \quad i \in I.\end{gathered} \tag{4}$$

Conversely, every solution (x^0, λ^0) of the consistent system (4) gives an extremal point, since the Kuhn–Tucker conditions are sufficient. It follows from the principle of boundary solutions for systems of linear inequalities (see [4], p. 32) that a solution (x^0, λ^0) exists whose components have the form

$$x_j^0 = \Delta_j/\Delta, \qquad \lambda_i^0 = \delta_i/\Delta, \qquad j = 1, 2, \ldots, n, \quad i \in I,$$

where Δ_j, δ_i and $\Delta \neq 0$ are some determinants from the integer-valued augmented matrix of system (4). The modulus of any determinant of the augmented matrix of (4) is overbounded by the quantity $2^{2L}/nm$, where L is defined by formula (2). Moreover, because of integrality, $|\Delta| \geqslant 1$. Therefore the following assertions hold.

Assertion 1

There exists a solution x^0 of the quadratic programming problem (1) from the Euclidean sphere $S = \{x \mid \|x\| \leqslant 2^{2L}\}$.

Assertion 2

There exists a subset of indexes $I \subseteq \{1, 2, \ldots, m\}$, such that at the point x^0 all the constraints $i \in I$ are active, and the gradient of $f(x)$ is expressed in terms of the vectors A_i in the form

$$\nabla f(x^0) = Cx^0 + d = -\sum_{i \in I} \lambda_i^0 A_i,$$

where

$$0 \leqslant \sum_{i \in I} \lambda_i^0 \leqslant 2^{2L}, \qquad \lambda_i^0 \geqslant 0 \quad \text{if} \quad i \in I.$$

Assertion 3

The optimal value $f^0 = f(x^0)$ of the functional is rational and has the form

$$f^0 = \left(\sum_{i,j=1}^{n} c_{ij} \Delta_i \Delta_j + 2\Delta \sum_{j=1}^{n} d_j \Delta_j \right) (2\Delta^2)^{-1} = \frac{t}{s},$$

where t and s are relatively prime integers, $|t| \leqslant 2^{5L}$, $|s| \leqslant 2^{4L}$.

4. We now show that for the accurate discovery of the optimal value $f^0 = t/s$ of the functional it is sufficient to determine the consistency in R^n of not more than $13L+3$ systems P_k of the form

$$(A_i, x) \leqslant b_i, \quad i=1, 2, \ldots, m, \quad f(x) \leqslant t_k / s_k,$$

the integers t_k and s_k defining the system P_k, $k = 1, 2, \ldots, 13L+3$, not exceeding in modulus $|t_k| \leqslant 2^{13L+2}$, $|s_k| \leqslant 2^{8L+2}$. Indeed, from Assertion 3 it follows that $f^0 \in [-2^{5L}, 2^{5L}]$. Therefore, by the method of bisection of the interval $[-2^{5L}, 2^{5L}]$, checking at each step the consistency of the system P_k with the corresponding t_k and s_k, we can find after $13L+3$ steps the approximate value of f^0 with accuracy 2^{-8L-2}, that is, we can find a number $f = t_{13L+3}/s_{13L+3}$, such that

$$|f - f^0| = |f - t/s| \leqslant 2^{-8L-2}, \quad |s| \leqslant 2^{4L}. \tag{5}$$

From (5) and Theorem 19 in [5] it follows that t/s is a convergent of the number f and can be found by expanding f in a continued fraction. Therefore the accurate value f^0 is found from the approximate f after a time polynomial in L: this requires not more than $O(L)$ operations $+, -, \times, /$, max, performed with an accuracy of $O(L)$ digits on $O(L)$-digit numbers.

5. We now construct an algorithm, polynomial in L, to determine the consistency of the arbitrary system P_k, $k = 1, 2, \ldots, 13L+3$. We will first prove two lemmas.

Lemma 1

If the system P_k is consistent (that is, $t_k/s_k \geqslant t/s$), then a solution of it from the Euclidean sphere $S = \{x \mid \|x\| \leqslant 2^{2L}\}$ exists.

Proof. As a solution of P_k we can take the vector x^0 defined in Assertion 1. The lemma is proved.

Let

$$\theta(x) = \max\{0, f(x) - t_k/s_k, (A_i, x) - b_i\}, \quad i=1, 2, \ldots, m,$$

be the error of the system P_k at the point $x \in R^n$. In particular, $\theta(x) = 0$ if and only if x is a solution of P_k.

Lemma 2

If the system P_k is inconsistent (that is, $t_k/s_k < t/s$), then for any $x \in R^n$ the error $\theta(x) \geqslant 2.2^{-15L}$.

Proof. We denote by $\theta_1(x)$ and $\theta_2(x)$ respectively, the error for the functional and the error for the linear inequalities:

$$\theta_1(x) = \max\{0, f(x) - t_k/s_k\}, \quad \theta_2(x) = \max\{0, (A_i, x) - b_i\}, \quad i=1, 2, \ldots, m,$$

then the total error for the system P_k is $\theta(x)=\max\{\theta_1(x),\ \theta_2(x)\}$. Let $x^0 \in X$ be the optimal solution of the quadratic programming problem defined in clause 3. Since the system P_k is inconsistent,

$$\theta_1(x^0)=f(x^0)-\frac{t_k}{s_k}=\frac{t}{s}-\frac{t_k}{s_k}\geqslant\frac{1}{|ss_k|}\geqslant 2^{-12L-2}. \tag{6}$$

For the convex functional $\theta_1(x)$ with gradient Cx^0+d at the point x^0 the following inequality holds:

$$\theta_1(x)-\theta_1(x^0)\geqslant(Cx^0+d,\ x-x^0),\qquad x\in R^n.$$

Therefore from (6) and Assertion 2 we obtain the following chain of inequalities:

$$\theta_1(x)\geqslant\theta_1(x^0)-\sum_{i\in I}\lambda_i^0(A_i,x-x^0)=\theta_1(x^0)-\sum_{i\in I}\lambda_i^0((A_i,x)-b_i)$$
$$\geqslant\theta_1(x^0)-\theta_2(x)\sum_{i\in I}\lambda_i^0\geqslant 2^{-12L-2}-\theta_2(x)\cdot 2^{2L}.$$

Consequently,

$$\theta(x)=\max\{\theta_1(x),\theta_2(x)\}\geqslant\max\{2^{-12L-2}-\theta_2(x)2^{2L},\theta_2(x)\}$$
$$\geqslant\min_{0\leqslant\tau\leqslant\infty}\max\{2^{-12L-2}-\tau\cdot 2^{2L},\tau\}=\frac{2^{-12L-2}}{2^{2L}+1}\geqslant 2\cdot 2^{-15L},$$

which proves the lemma.

Let θ_S be the minimum of the error $\theta(x)$ on the sphere S. Lemmas 1 and 2 imply that to determine the consistency of the system P_k it is sufficient to find in R^n a vector x such that

$$\theta(x)\leqslant\theta_S+2^{-15L}. \tag{7}$$

Indeed, in this case either $\theta(x)\leqslant 2^{-15L}$ and the system P_k is consistent, or $\theta(x)\geqslant 2.2^{-15L}$ and the system P_k is inconsistent.

In [2] the algorithm of ellipsoids [6, 7] for finding the point x satisfying (7) is described. The characteristics of this algorithm for our case are as follows: a memory of $O(L)$ digits; the number of elementary operations $O(L)$ digits; the number of elementary operations $O(n^3(n+m)L)$ with an accuracy of $O(L)$ digits.

Therefore, in sections 3–5 we have constructed an algorithm, polynomial in L, for the determination of the extremal value of a convex quadratic functional with linear constraints.

6. Suppose the optimal value of the functional $f^0=t/s$, $|t|\leqslant 2^{5L}$, $|s|\leqslant 2^{4L}$, has been found. To find some optimal point of the quadratic programming problem it is sufficient to find a vector satisfying the system U_1

$$(A_i,\ x)\leqslant b_i,\quad i=1,\ 2,\ \dots,\ m,\quad f(x)\leqslant t/s.$$

The solution of this consistent system can be found from the algorithm constructed above.

Let $X_1 \neq \phi$ be the set of solutions of system U_1. We replace in it the first in order linear inequality $(A_1, x) \leqslant b_1$ by the equation $(A_1, x) = b_1$ and determine the consistency of the system obtained as a result of this replacement. If it is inconsistent, then the first linear constraint in U_1 is redundant (that is, the hyperplane defined by this linear constraint does not intersect X_1) and it is discarded. Otherwise the equation in the first row is fixed. The system U_2 thus obtained has a non-empty set of solutions $X_2 \subseteq X_1$, thereby the total number of constraints in U_2 is not increased, and the number of linear constraint-inequalities is reduced by unity. Then, we replace in U_2 the first in order linear inequality $(A_2, x) \leqslant b_2$ by an equation and determine the consistency of the system resulting after the replacement. In case of inconsistency the inequality is discarded, in case of consistency it is replaced by an equation, and so on.

After m steps we obtain a consistent system U_{m+1} consisting of only some linear inequalities and the quadratic inequality $f(x) \leqslant t/s$. Considering that the point of attainment of the minimum of a quadratic functional in r, $0 \leqslant r \leqslant m$, of linear integer-valued constraint-inequalities is found in a time polynomial in L, we obtain the accurate solution of the quadratic programming problem (1).

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