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Orderings on Graphs and Game Coloring Number

H. A. KIERSTEAD and DAQING YANG
Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287-1804, USA

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Abstract. Many graph theoretic algorithms rely on an initial ordering of the vertices of the graph which has some special properties. We discuss new ways to measure the quality of such orders, give methods for constructing high quality orders, and provide applications for these orders. While our main motivation is the study of game chromatic number, there have been other applications of these ideas and we expect there will be more.

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1. Introduction

Many graph theoretic algorithms depend on a suitable linear ordering of the vertices of the graph. For example a common way to properly color a graph is to first order the vertices and then apply First-Fit. Arbitrary orderings can give very bad results. On the other hand there is always an ordering for which this process gives an optimal coloring. So choosing a suitable linear order to use is critical. The usual technique is to choose a linear order that minimizes the number of predecessors to which any vertex is adjacent. Let us formalize this idea.

For a graph G, let $\Pi(G)$ be the set of linear orderings on the vertex set of G. Let $L \in \Pi(G)$. The orientation $G_L = (V, E_L)$ of G with respect to L is obtained by setting $E_L = \{(v, u) : \{v, u\} \in E \text{ and } v > u \text{ in } L\}$. (If this definition seems backwards, think of $>$ as the head of an arrow.) For a vertex u we denote the neighborhood of u in G by $N_G(u)$ and the out-neighborhood of u in G_L by $N^+_{G_L}(u)$. Also $N_G[u] = N_G(u) \cup \{u\}$ and $N^+_{G_L}[u] = N^+_{G_L}(u) \cup \{u\}$. The *coloring number* of a graph G, denoted by $\mathrm{col}(G)$, is defined by

$$\mathrm{col}(G) = \min_{L\in\Pi(G)} \max_{v\in V(G)} |N^+_{G_L}[v]|.$$

Let $L \in \Pi(G)$ satisfy $\mathrm{col}(G) = \max_{v\in V(G)} |N^+_{G_L}[v]|$. If we color the vertices of G in the order L then any uncolored vertex is adjacent to at most $\mathrm{col}(G) - 1$ colored vertices. It follows that the chromatic number of G is at most $\mathrm{col}(G)$. The usefulness of this bound stems from the fact that while calculating the chromatic number is NP-complete, calculating the coloring number is polynomial. Indeed, the following easy algorithm produces an ordering $L = x_1 \dots x_n$ that witnesses the coloring number of G: Let x_n be a vertex with minimum degree. Next suppose that we have constructed the final segment $x_{i+i} \dots x_n$ of L. Let x_i be a vertex with minimal degree in $G - \{x_{i+i}, \dots, x_n\}$.

If G is chordal this bound is tight; on the other hand bipartite graphs have arbitrarily large coloring number. A better application of coloring number involves list coloring. The list chromatic number of a graph is also bounded by the coloring number and Alon [1] has shown that the coloring number of a graph is bounded by a function of its list chromatic number.

Recently linear orders with more complicated properties have been used for various applications by Chen and Schelp [3], Kierstead and Trotter [8], and Kierstead [6]. The quality of these orders have been measured by parameters such as *arrangability* [3] *admissability* [8], and *rank* [6]. These parameters are very closely related; slight variances arise from attacking different optimization problems. Here we try to unify the discussion by introducing the 2-coloring number of a graph. In our examples we will state results in terms of this parameter even if they were originally proved for one of the other parameters and even if the original statements give somewhat tighter bounds.

For a graph G and order $L \in \Pi(G)$, let $V^+(G_L, u)$ be the set of vertices of G that precede u in L,

$$R_2(G_L, u) = \{v \in V^+(G_L, u) : \exists z \in V(G)\ u \leftarrow z \rightarrow v\}$$

and $R_2[G_L, u] = R_2(G_L, u) \cup \{u\}$. We define the 2-coloring number of G, denoted by $\mathrm{col}_2(G)$ by

$$\mathrm{col}_2(G) = \min_{L \in \Pi(G)} \max_{v \in V(G)} |R_2[G_L, v]|.$$

Chen and Schelp [3] proved the Ramsey theoretic statement that for every k there exists a constant c_k such that for every graph G on n vertices with $\mathrm{col}_2(G) \leqslant k$ and every partition $\{E_1, E_2\}$ of the edges of $H = K_{c_k n}$ either $H[E_1]$ or $H[E_2]$ contains a copy of G. They then proved that any planar graph G satisfies $\mathrm{col}_2(G) \leqslant 761$. Kierstead and Trotter [8] proved that the game chromatic number of a planar graph G is at most $\mathrm{col}_2(G)\chi(G) + 1$. They also proved that any planar graph G satisfies $\mathrm{col}_2(G) \leqslant 10$.

Next we illustrate applications of the $\mathrm{col}_2(G)$ by showing that the *acyclic chromatic number* of G can be bounded in terms of the $\mathrm{col}_2(G)$ in much the same way that the chromatic number of G can be bounded by $\mathrm{col}(G)$. The acyclic chromatic number of G, denoted by $\chi_a(G)$, is the least t such that the vertices of G can be colored with t colors so that every color class is independent and every pair of color classes induces a forest.

THEOREM 1. *Every graph G satisfies $\chi_a(G) \leqslant \mathrm{col}_2(G)$.*

Proof. Let $L \in \Pi(G)$ satisfy $\mathrm{col}_2(G) = \max_{v \in V} |R_2[G_L, v]|$. Color the vertices of G in the order L using First-Fit subject to the condition that if $x \in R_2[G_L, y]$

then y cannot receive the same color as x. This can be done with $\mathrm{col}_2(G)$ colors and accomplishes two things. First it insures that each color class is independent. Second it insures that no uncolored vertex z is ever adjacent to two colored vertices $x <_L y$ which have been colored with the same color: Otherwise z would witness that $x \in R_2[G_L, y]$ contradicting the choice of color for y. Using the acyclic orientation of G_L, it follows easily that any two color classes induce a forest. □

In Section 2 we introduce a more general version of the coloring number denoted by $\mathrm{col}_k(G)$. Then we prove that these generalized coloring numbers are bounded on any topologically closed class of graphs with bounded coloring number. This is the first of two main results of this paper.

In Section 3 we consider the following competitive situation. Two players (processors) Alice and Bob are creating a linear order L on the vertices of a graph G as follows. Alice and Bob take turns marking the yet unmarked vertices of G with Alice playing first. The process ends when all vertices are marked. The ordering L is defined by $x <_L y$ if x is marked before y. Alice's goal is to minimize the score

$$s = \max_{v \in V(G)} |N^+_{G_L}[v]|,$$

while Bob tries to maximize the score. The *game coloring number*, denoted by $\mathrm{gcol}(G)$, is the least s such that Alice can obtain a score of at most s regardless of how Bob plays. This is the competitive version of coloring number. The game coloring number was first explicitly introduced by Zhu [13] to give an improved bound on the game chromatic number of planar graphs, but it had already been considered implicitly by Faigle *et al.* [5] to bound the game chromatic number of trees. Kierstead [6] showed that $\mathrm{gcol}(G)$ can be bounded in terms of the $\mathrm{col}_2(G)$. Previously the game coloring number of G has been denoted by $\mathrm{col}_g(G)$, but we prefer to reserve the subscript for more general versions of game coloring number. In Section 3 we introduce the k-game coloring number, denoted by $\mathrm{gcol}_k(G)$ and bound $\mathrm{gcol}_k(G)$ in terms of $\mathrm{col}_{2k}(G)$. This is our second main result.

In Section 4 we discuss several applications of game coloring numbers to game chromatic numbers. In Section 5 we give several counter examples.

Finally we will need the following important theorem (phrased in our terms) that was proved independently by Bollobás and Thomason [2] and Komlós and Szemerédi [11]. It shows that graphs that do not contain a topological minor of a given clique have bounded coloring number.

THEOREM 2. *There exists a constant $c \in \mathbb{R}$ such that for every $r \in \mathbb{N}$ every graph G with* $\mathrm{col}(G) \geqslant cr^2$ *contains the clique K_r as a topological minor.*

2. General Coloring Numbers

In this section we introduce the k-coloring number. The proofs of our main results concerning the k-coloring number will require a secondary notion called the weak k-coloring number, which we also introduce now.

Let G be a graph and $L \in \Pi(G)$. Let x and y be two vertices of G. We say that x is *weakly k-accessible from y* if $x <_L y$ and there exists a $y - x$ path P of length at most k such that every internal vertex z of P satisfies $x <_L z$. If in addition every internal vertex z of P satisfies $y <_L z$ then x is *k-accessible from y*. Let $R_k(y)$ be the set of vertices that are k-accessible from y and $Q_k(y)$ be the set of vertices that are weakly k-accessible from y. Also let $R_k[y] = R_k(y) \cup \{y\}$ and $Q_k[y] = Q_k(y) \cup \{y\}$. If we want to specify the graph G_L we will write $R_k(G_L, y)$, $R_k[G_L, y]$, $Q_k(G_L, y)$, and $Q_k[G_L, y]$. Define the *k-coloring number* of G, denoted by $\mathrm{col}_k(G)$, and the *weak k-coloring number* of G, denoted by $\mathrm{wcol}_k(G)$, by

$$\mathrm{col}_k(G) = \min_{L\in\Pi(G)} \max_{v\in V(G)} |R_k[G_L, v]| \quad \text{and}$$
$$\mathrm{wcol}_k(G) = \min_{L\in\Pi(G)} \max_{v\in V(G)} |Q_k[G_L, v]|.$$

Note that the $\mathrm{col}(G) = \mathrm{col}_1(G) = \mathrm{wcol}_1(G)$ and $\mathrm{col}_k(G) \leqslant \mathrm{wcol}_k(G)$. The next lemma shows that $\mathrm{wcol}_k(G)$ is bounded in terms of $\mathrm{col}_k(G)$.

LEMMA 3. *Every graph G satisfies* $\mathrm{wcol}_k(G) \leqslant (\mathrm{col}_k(G))^k$.

Proof. Let $L \in \Pi(G)$ satisfy $\mathrm{col}_k(G) = \max_{v\in V(G)} R_k[G_L, v]$. We show by induction on k that $|Q_k[y]| \leqslant (\mathrm{col}_k(G))^k$ for all $y \in V(G)$. The base step $k = 1$ is trivial, so consider the induction step. For each $x \in Q_k(y)$ let P_{yx} be the shortest $y - x$ path such that every vertex w of P_{yx} satisfies $x \leqslant_L w$. Let z be the first vertex on P_{yx} such that $z <_L y$ and let i be the distance from y to z. Then $z \in R_i(y) - R_{i-1}(y)$ and $x \in Q_{k-i}[z]$. It follows that

$$|Q_k(y)| \leqslant \sum_{i=1}^{k} |R_i(y) - R_{i-1}(y)||Q_{k-i}[z]| \leqslant |R_k(y)|\mathrm{wcol}_{k-1}(G).$$

Thus, using the induction hypothesis,

$$|Q_k(y)| \leqslant 1 + R_k(y)\mathrm{wcol}_{k-1}(G) \leqslant (1 + |R_k(y)|)\mathrm{wcol}_{k-1}(G) \leqslant (\mathrm{col}_k(G))^k. \quad \square$$

The following theorem is our first main result. It shows in particular that for any positive integer k, the k-coloring number is bounded on the class of planar graphs.

THEOREM 4. *There exists a function f such that for all positive integers d and k, if $\mathcal{C}$ is a topologically closed class of graphs such that* $\mathrm{col}(G) \leqslant d$ *for every graph $G \in \mathcal{C}$ then* $\mathrm{col}_k(G) \leqslant f(d, k)$ *for every $G \in \mathcal{C}$.*

Proof. Define f by

$$f(d,1) = d,$$
$$f(d,t+1) = df(d,t)^{2t^2}.$$

Let $\mathcal{C}$ be a topologically closed class of graphs such that $\mathrm{col}(G) \leqslant d$ for every graph $G = (V, E) \in \mathcal{C}$.

We first recursively construct an ordering $L = x_1x_2\ldots x_n$ of V as follows. Suppose that we have constructed the final sequence $x_{i+1}\ldots x_n$ of L. (If $i = n$ then this sequence is empty.) Let $M = \{x_{i+1}, \ldots, x_n\}$ be the set of vertices that have already been ordered and $U = V - M$ be the set of vertices that have not yet been ordered. Notice that even though we have not finished constructing L, we have determined $R_k(y)$ for any $y \in M$. However we have not necessarily determined $Q_k(y)$. Our next task is to choose x_i from U. To do this we first define a probability space Ω, where each point in Ω is a graph $H = (U, F) \in \mathcal{C}$. For each pair $\{u, v\} \subseteq U$ for which there exists a $u-v$ path of length at most k whose internal vertices are all in M, choose a shortest such path and denote it by P_{uv}. For each vertex $z \in M$ let

$$S_z = \{\{u, v\} \subseteq U : z \in P_{uv}\}.$$

Label each $z \in M$ with a random element chosen from S_z; if $S_z = \emptyset$ then leave z unlabeled. Let F be the set of edges uv such that every internal vertex of P_{uv} is labeled with $\{u, v\}$. Then H is a topological minor of $\bigcup_{uv\in F} P_{uv} \subseteq G$ and so $H \in \mathcal{C}$. If P_{uv} is defined then the probability that $uv \in F$ is

$$\Pr(uv \in F) = \prod_{z\in M\cap V(P_{uv})} \frac{1}{|S_z|}.$$

In particular, if $uv \in E$ then $\Pr(uv \in F) = 1$. Let $E[d_H(u)]$ be the expected value of the degree of u in H. Choose $x_i \in U$ so that $E[d_H(x_i)]$ is minimal. Since $\Omega \subseteq \mathcal{C}$, $E[d_H(x_i)] < d$. This completes the construction of L.

We now argue by induction on $s \leqslant k$ that $|R_s(y)| < f(d, s)$ for all vertices y. The base step $s = 1$ is trivial so consider the induction step $s = t + 1$. Fix the time that y was added to the final sequence of L. For each $z \in M$ and $\{u, v\} \in S_z$ both u and v are in $Q_t(z)$. Thus, using Lemma 3, $|S_z| < |Q_t(z)|^2 < f^{2t}(d, t)$. It follows that

$$\begin{aligned} d > E[d_H(y)] &= \sum_{x\in R_s(y)} \Pr(xy \in F) \\ &= \sum_{x\in R_s(y)} \prod_{z\in M\cap V(P_{xy})} \frac{1}{|S_z|} \geqslant |R_s(y)| f(d,t)^{-2t^2}. \end{aligned}$$

So

$$|R_s(y)| < df(d,t)^{2t^2} = f(d,s). \qquad \square$$

3. Game Coloring Numbers

In this section we introduce the generalized game coloring numbers $\mathrm{gcol}_k(G)$ and show that $\mathrm{gcol}_k(G)$ is bounded in terms of $\mathrm{col}_{2k}(G)$. A marking game is played by two players Alice and Bob with Alice playing first. At the start of the game all vertices are unmarked. A play by either player consists of marking an unmarked vertex. The game ends when all the vertices have been marked. Together the players create an order $L \in \Pi(G)$ defined by $x <_L y$ iff x is marked before y. Different versions of the game differ in the way the quality of L is computed. In the k-marking game the score is

$$s = \max_{v \in V(G)} |R_k[G_L, v]|.$$

Alice's goal is to minimize the score, while Bob's goal is to maximize the score. We define the *game k-coloring number*, denoted by $\mathrm{gcol}_k(G)$, of G to be the least s such that Alice has a strategy that results in a score of at most s in the k-marking game. When $i = 1$ we refer to the game coloring number of G and when $i \geqslant |V|$ we refer to the *complete game coloring number*, $\mathrm{gcol}_\infty(G)$.

Note that the score of the k-marking game is the maximum number of marked vertices connected to an unmarked vertex u by paths of length at most k whose internal vertices are unmarked *at the time that u is marked.* A similar notion was developed by Kierstead and Trotter in [9], but there the maximum was taken over all vertices and *all* times. The definition given here seems more natural to applications. In agorithmic applications marking u corresponds to choosing to process u. It is at this time that we care about marked vertices adjacent to some local unmarked neighborhood of u.

In order to prove upper bounds on the $\mathrm{gcol}_k(G)$ we need an effective strategy for Alice. The following robust *activation strategy* has been developed over a long sequence of papers including [5, 13, 14, 6], and [9]. Suppose we are playing the marking game on a graph G and H is a graph with the same vertex set as G. Let $L \in \Pi(G)$ $(= \Pi(H))$. The activation strategy with respect to H_L is defined as follows.

Strategy $A(H_L)$. Let U denote the set of unmarked vertices. Initially $U = V(H)$. Alice maintains a subset $A \subset V(H)$ of *active* vertices. Initially $A := \emptyset$. When a new vertex y is put into A we say that y is *activated.* Once a vertex is activated it will remain active forever. Every time a vertex is marked it will be activated, if it is not already active. On her first turn Alice activates and marks the least vertex in the ordering L. Now suppose that Bob has just marked the vertex b. Alice uses the following algorithm to update A and choose the next vertex to mark. First Alice activates b if b is inactive. Whenever she activates a vertex y she defines $s(y)$ to be the least vertex in $N^+_{H_L}[y] \cap (U \cup \{b\})$. If $s(y)$ is inactive she activates it and continues. Eventually she chooses $s(y)$ such that $s(y)$ is active (possibly $s(y) = y$). If $s(y) \neq b$, then she marks $s(y)$; otherwise she marks the L-least unmarked vertex, after activating it if it was inactive.

The following lemma is the key to analyzing the activation strategy. We include its short proof for completeness.

LEMMA 5. *Suppose that Alice uses the strategy $A(G_L)$ for the marking game on G. Then at any time any unmarked vertex x has at most $2|R_2[x]|$ marked inneighbors in G_L.*

Proof. It suffices to show that x has at most $2|R_2[x]|$ active neighbors. Every time an inneighbor y of x is activated Alice activates or marks $s(y)$. Since x is a candidate for $s(y)$, it follows that $s(y) \leqslant_L x$. Thus the path $xys(y)$ witnesses that $s(y) \in R_2[x]$. Since Alice can activate a vertex at most once and mark it at most once, Alice will mark $x \in R_2[x]$ before x can accumulate more than $2|R_2[x]|$ active inneighbors. □

The following theorem is our second main result.

THEOREM 6. *Every graph $G = (V, E)$ satisfies* $\mathrm{gcol}_k(G) \leqslant 3\mathrm{wcol}_{2k}(G)^2 \leqslant 3(\mathrm{col}_{2k}(G))^{4k}$.

Proof. Let $L \in \Pi(G)$ such that $\mathrm{wcol}_{2k}(G) = \max_{v\in V} |Q_{2k}[G_L, v]|$. Define a graph $H = (V, F)$ on V by

$$F = \{xy : x \in Q_k(G_L, y)\}.$$

Then $\mathrm{col}_2(L, H) \leqslant \mathrm{wcol}_{2k}(G)$. Let Alice use the $A(H_L)$ activation strategy. During the game Alice and Bob will create an order $M \in \Pi(G)$. Consider a time when an unmarked vertex y is about to be marked. At this time the initial segment of M ending in y has been created. So $R_k(G_M, y)$ has been defined. We must show that $|R_k(G_M, y)| \leqslant 3\mathrm{wcol}_{2k}(G)^2$. For each $x \in R_k(G_M, y)$ fix an $y - x$ path P_x of length at most k whose internal vertices are all unmarked, i.e., greater than y in M. Let z_x be the L-least vertex in P_x. Partition $R_k(G_M, y)$ into sets $S = \{x \in R_k(G_M, y) : z_x = y\}$ and $S' = \{x \in R_k(G_M, y) : z_x <_L y\}$. Note that each element of S is an inneighbor of y in H. Thus by Lemma 5, $|S| \leqslant 2\mathrm{wcol}_{2k}(G)$. Now consider S'. If $x \in S'$ then $z_x \in Q_k(G_L, y)$ and either x is an inneighbor of z_x in H or $x = z_x$. Using Lemma 5 again, we have $|S'| \leqslant 2\mathrm{wcol}_{2k}(G)|Q_k(G_L, y)|$. It follows that $|R_k(G_M, y)| \leqslant 3\mathrm{wcol}_{2k}(G)^2 \leqslant 3(\mathrm{col}_{2k}(G))^{4k}$. □

Combining Theorems 2, 4, and 6 we have the following corollary.

COROLLARY 7. *There exists a function g such that for all positive integers r and k, if $\mathcal{C}$ is a topologically closed class of graphs that does not contain the clique K_r then* $\mathrm{gcol}_k(G) < g(r, k)$ *for every $G \in \mathcal{C}$.*

4. Applications to Game Chromatic Number

The *coloring game* is played on a graph G by two players Alice and Bob using a set of colors X. The players take turns playing with Alice playing first. A play consists of choosing an uncolored vertex and legally coloring it with a color from X. Alice wins if eventually the entire graph is legally colored; otherwise Bob wins when there comes a time in the game when one of the uncolored vertices cannot be legally colored. Different versions of the game arise from different definitions of legal color. In the usual game a vertex is legally colored if it is properly colored, i.e., none of its neighbors have the same color. In this case the game chromatic number, denoted by $\chi_g(G)$, is the least integer t such that Alice can win the coloring game using a set X of t colors.

The marking game was originally introduced as a tool for bounding the game chromatic number of a graph. Suppose $\mathrm{gcol}(G) = t$. Then Alice can play so that at any time any uncolored vertex u has at most $t-1$ colored (marked) neighbors. Thus if $|X| = t$ there will always be a legal color to color any uncolored vertex with. Usually the best and most interesting upper bounds on game chromatic number are proved by bounding the game coloring number [6]. However a notable exception is the work of Dinski and Zhu [4] on graphs with bounded acyclic chromatic number. Their work was extended by Kierstead [7].

Nešetřil and Sopena [12] introduced the *oriented coloring game*. In this version of the game G is an oriented graph and X is a tournament. A color $\alpha \in X$ is legal for an uncolored vertex u if

- for all colored neighbors v of u the color β of v satisfies $\alpha \to \beta$ iff $u \to v$ and
- if $P = uvw$ is a directed (in either direction) 2-path such that v is uncolored then w is not colored with α.

The second condition is needed to avoid trivial wins for Bob. The oriented game chromatic number, denoted $\mathrm{ogcn}(G)$, of an oriented graph G is the least t such that there exists a tournament X such that Alice can win the oriented coloring game using the colors in X. Nešetřil and Sopena showed (in our terminology) that $\mathrm{ogcn}(G)$ can be bounded in terms of $\mathrm{gcol}_2(G)$. Then Kierstead and Trotter [9] showed that the oriented game chromatic number of planar graphs is bounded by proving the special case of Theorem 6 that $\mathrm{gcol}_2(G)$ is bounded in terms of $\mathrm{col}_4(G)$. Kierstead and Tuza [10] proved that the oriented game chromatic number of chordal graphs is bounded in terms of their clique size by proving the complete coloring number of chordal graphs is bounded in terms of their clique size.

5. Counter Examples

In this section we present two examples that show that in certain ways our results cannot be improved. The first example shows that $\mathrm{gcol}_k(G)$ cannot be bounded in terms of $\mathrm{wcol}_{2k-1}(G)$.

EXAMPLE 8. *For all positive integers k and s there exists a graph G such that* $\mathrm{wcol}_{2k-1}(G) \leqslant 2k+1$ *and* $\mathrm{gcol}_k(G) > s$.

Proof. Let $t = 4^s$ and $G = G_s$ be the graph obtained from the clique K_t by subdividing each edge $2k - 1$ times. More precisely, replace every edge $e = xy$ of K_t with a new $x - y$ path $P_e = xz_1 \ldots z_{2k-1}y$. Denote the central vertex z_k of P_e by c_e. Let $L \in \Pi(G)$ be such that every vertex of K_t precedes every subdivision vertex. If x is a vertex of K_t then $Q_{2k-1}[G_L, x] = \{x\}$. If y is subdivision vertex of the edge e of K_t then $Q_{2k-1}[G_L, y] \subseteq P_e$. So

$$\mathrm{wcol}_{2k-1}(G) \leqslant \max_{v \in V(G)} |Q_{2k-1}[G_L, v]| \leqslant 2k + 1.$$

Next we show by induction on s that Bob has a strategy that will force $|R_k[G_L, x]| > s$ for some vertex in K_t even if Alice is allowed to pass on some turns. The base step $s = 1$ is trivial so consider the inductive step $s = s' + 1$. Let $t' = 4^{s'}$ and G' be the result of replacing every edge e of $K_{t'}$ with the path P_e. Bob fixes a perfect matching $M = \{e_i : i \in [t/2]\}$ in K_t. On his i-th turn with $i \in [t/2]$ Bob marks the center vertex c_{e_i} of P_i if it is not yet marked. Otherwise he marks any vertex. It is easy to see that after the players have made a combined total of t moves there exists a subclique $K_{t'} \subseteq K_t$ such that

- for every $v' \in V(K_{t'})$ there exists $v \in V(K_t - K_{t'})$ such that $c_{vv'}$ is marked and
- for every edge $e \in E(K_{t'})$ no vertex of P_e is marked.

Let $L \in \Pi(G)$ be the order in which the vertices of G are marked and $L' \in \Pi(G')$ be L restricted to $V(G')$. Then at the end of the first t plays $|R_k[G_L, x]| \geqslant 1$ for every vertex $x \in V(K_{t'})$. Moreover, by the induction hypothesis Bob can play in G' so that $|R_k[G'_{L'}, y]| > s'$ for some vertex $y \in V(K_{t'})$. Thus $|R_k[G_L, y]| > s$. □

Our second example shows that the complete game chromatic number of planar graphs is unbounded.

EXAMPLE 9. *Let G be the $k \times k$ grid. Then* $\mathrm{gcol}_k(G) \geqslant k - 2$.

Proof. Let L be the linear order that is created by the players in the k-coloring game on G. We must give a strategy for Bob that insures that some vertex x satisfies $|R_k(G_L, x)| \geqslant k - 3$. Let $l = \lceil k/2 \rceil$. On each of his first $k - 3$ turns Bob marks a vertex in row l of G if possible; otherwise he marks any unmarked vertex. After $2k - 5$ plays there are at least $k - 3$ marked vertices in row l and at most $k - 2$ other rows contain a marked vertex. Thus there is a row j that has no marked vertex. On his $k - 2$cnd move Bob marks the vertex x in row j and column l. Then $R_k(G_L, x)$ contains a vertex from each of the $k - 3$ columns that contain marked vertices in row l: Suppose the vertex y in row l and column i is marked. Consider the $x - y$ path P from x to y that is contained in the union of row j and column i. Its length is at most $2\lfloor k/2 \rfloor \leqslant k$. Let z be the first marked vertex on P. So $z \in R_k(G_L, x)$. Since row j is unmarked, z is in column i. □

References

1. Alon, N.: Restricted colorings of graphs, In: K. Walker (ed.), *Surveys in Combinatorics* (Proc. 14th British Combinatorial Conf.), Cambridge Univ. Press, Cambridge, 1993, pp. 1–33.
2. Bollobás, B. and Thomason, A. G.: Proof of a conjecture of Mader, Erdős, and Hajnal on topological complete subgraphs, Manuscript, 1994.
3. Chen and Schelp: Graphs with linearly bounded Ramsey numbers, *J. Combin. Theory Ser. B* **57** (1993), 138–149.
4. Dinski, T. and Zhu, X.: A bound for the game chromatic number of graphs, *Discrete Math.* **196** (1999), 109–115.
5. Faigle, U., Kern, W., Kierstead, H. A. and Trotter, W. T.: On the game chromatic number of some classes of graphs, *Ars Combin.* **35** (1993), 143–150.
6. Kierstead, H. A.: A simple competitive graph coloring algorithm, *J. Combin. Theory (B)* **78** (2000), 57–68.
7. Kierstead, H. A.: *Weak Acyclic Coloring and Asymmetric Coloring Games*, Manuscript, 2002.
8. Kierstead, H. A. and Trotter, W. T.: Planar graph coloring with an uncooperative partner, *J. Graph Theory* **18** (1994), 569–584.
9. Kierstead, H. A. and Trotter, W. T.: Competitive colorings of oriented graphs. In honor of Aviezri Fraenkel on the occasion of his 70th birthday, *Electron. J. Combin.* **8** (2001), Research Paper 12, 15 pp. (electronic).
10. Kierstead, H. A. and Tuza, Zs.: Marking games and the oriented game chromatic number of partial k-trees, *Graphs and Combinatorics* (2003), 121–129.
11. Komlós, J. and Szemerédi, E.: Topological cliques in graphs II, *Combinatorics, Probability, and Computing* **5** (1996), 79–90.
12. Nešetřil, J. and Sopena, E.: On the oriented game chromatic number. In honor of Aviezri Fraenkel on the occasion of his 70th birthday, *Electron. J. Combin.* **8** (2001), Research Paper 14, 13 pp. (electronic).
13. Zhu, X.: Game coloring number of planar graphs, *J. Combin. Theory (B)* **75** (1999), 245–258.
14. Zhu, X.: The game coloring number of pseudo partial k-trees, *Discrete Math*. **215** (2000), 245–262.