Arch. math. Logik 13 (1970), 39–51.

HIERARCHIES OF NUMBER-THEORETIC FUNCTIONS. I.*

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Herrn Professor Dr. Kurt Schütte zum 60. Geburtstag gewidmet

Introduction

The present paper is concerned with a method of classifying number-theoretic functions by means of hierarchies.

Previous related results are contained in Grzegorczyk [5], giving a hierarchy classification of the primitive recursive functions, and Péter [9], giving a hierarchy classification of the multiple recursive functions, refined in Robbin [12] by extending the method of [5]. Our results may also be viewed as an approach to the general problems discussed by Péter [10], regarding the question of extending the class of constructively describable recursive functions beyond those previously considered.

In Section 2 we introduce a general procedure for generating hierarchies, which is applicable to a wide variety of classes of number-theoretic functions, including such which also contain non-recursive functions. It is proved that the hierarchies may be extended through the ordinals of Cantor's second number class without collapsing. In particular, our procedure provides a proper extension of Grzegorczyk's hierarchy [5]. In fact our hierarchies coincide with the latter at level ω.

In Section 3 we show that restricting the ordinals appropriately to the constructive ordinals yields hierarchies of recursive functions.

Section 4 presents a simplification of our general method for the case where the ordinals range over those below ε_0. We conjecture that, in the latter case, the class of functions obtained is co-extensive with the class of ordinal recursive functions of Kreisel [7].

In Section 5 we show that, at level ω^ω, we can obtain precisely the class of multiple recursive functions, thus providing an alternative scheme to [8] and [9] for introducing these functions.

Our procedure depends on a particular method of diagonalization which, at each non-limit stage, is analogous to the steps in the Grzegorczyk hierarchy.

1. Notation

Let N denote the set of natural numbers $0, 1, 2, \ldots,$ and for any fixed $k \in N$, let N^k denote the set of all k-tuples of natural numbers.

This paper is concerned with functions whose arguments and values belong to N. All our functions are totally defined, i.e. if a function has k argument-places, then its domain will be N^k.

* Eingegangen am 3. 12. 68

Lower-case italics $a, b, \ldots, x, y, z$ (with the exception of f, g and h), with or without subscripts, denote natural numbers.
A sequence $x_1, x_2, \ldots, x_k$ will sometimes be denoted by $\boldsymbol{x}$.
Lower-case Greek letters, other than λ, ϱ, and μ, with or without subscripts, will denote countable ordinals.
The letters f, g, and h, with or without subscripts and superscripts, are used as function-variables. Capital letters are also used to denote particular functions.
μ denotes the least-number operator.
If T is a numerical term with the free variables $x_1, \ldots, x_r$ then $\lambda x_1 \ldots x_r \cdot T$ denotes the function whose value, for any particular r-tuple $\langle \boldsymbol{a} \rangle$ is the result of substituting a_1 for $x_1, \ldots, a_r$ for x_r respectively, in T.
Members of N will be identified with the finite ordinals.
Suppose that C is any class of functions.
Then $E(C)$ is the smallest class of functions which contains C and is closed under the operations of

(i) *Substitution*

$$f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n)) .$$

(ii) *Limited Recursion*

$$f(0, x_1, \ldots, x_n) = h_1(x_1, \ldots, x_n) ,$$
$$f(y + 1, x_1, \ldots, x_n) = h_2(y, x_1, \ldots, x_n, f(y, x_1, \ldots, x_n)) ,$$
$$f(y, x_1, \ldots, x_n) \leqq h_3(y, x_1, \ldots, x_n) .$$

Thus, by the work of Grzegorczyk in [5], the class $\mathscr{E}$ of Csillag-Kalmar elementary functions can be characterized as follows:

$$\mathscr{E} = E(\lambda x \cdot 0, \lambda x \cdot x + 1, \lambda \boldsymbol{x} \cdot x_i, \lambda x y \cdot x^y) .$$

For any class C of functions, we let $P(C)$ be the smallest class of functions which contains C and is closed under the operations of Substitution and Primitive Recursion.
Thus $P(\lambda x \cdot 0, \lambda x \cdot x + 1, \lambda \boldsymbol{x} \cdot x_i)$ is the class of primitive recursive functions, which we denote by $\mathscr{P}$.

2. Extending the Grzegorczyk Hierarchy

In [5] Grzegorczyk defined a sequence of classes of functions $\mathscr{E}^i (i \in N)$ such that

(i) for each $i \in N$, $\mathscr{E}^i \subset \mathscr{E}^{i+1}$,

(ii) $\mathscr{E}^3 = \mathscr{E}$,

(iii) $\bigcup_{i \in N} \mathscr{E}^i = \mathscr{P}$.

Robbin, in [12], extended this result by constructing a sequence of classes $E_\alpha (\alpha < \omega^\omega)$ such that

(i) whenever $\alpha < \beta < \omega^\omega$, $E_\alpha \subset E_\beta$,

(ii) for each $k > 0$, $\bigcup_{\alpha < \omega^k} E_\alpha = \mathfrak{R}_k$,

where $\mathfrak{R}_k$ denotes the class of k-recursive functions defined by Péter in [9].
We shall use methods similar to those of Robbin in order to develop a framework within which various extensions of the Grzegorczyk hierarchy can be constructed.

Definition 2.0

By a *fundamental sequence* to a limit ordinal α, we mean an ω-sequence $\{\alpha_i\}_{i \in N}$ of ordinals, such that

(i) for each $i \in N$, $\alpha_i < \alpha_{i+1} < \alpha$,

(ii) $\lim\limits_{i \in N} \alpha_i = \alpha$.

Now, in [12], Robbin specifies, for each limit ordinal less than ω^ω, a particular fundamental sequence to that limit ordinal.

On the other hand, the development of our general framework is based on arbitrary, but fixed, fundamental sequences to all the limit ordinals under consideration. Since every countable limit ordinal has such a fundamental sequence, we are able to define classes $\mathfrak{E}_\alpha$ where α now ranges over all the countable ordinals.

Definition of the Functions F_α^n:

For each countable limit ordinal β, let $\{\beta\}\,(i)$, $i \in N$, denote a (arbitrarily chosen) fixed fundamental sequence to β, i.e. $\{\beta\}$ is a function mapping N into β with the properties

(i) for each $i \in N$, $\{\beta\}\,(i) < \{\beta\}\,(i+1) < \beta$,

(ii) $\lim\limits_{i \in N} \{\beta\}\,(i) = \beta$.

We now define F_α^n as follows:

Def. 1. $F_0^n(x) = (n+1) \cdot (x+1)$,

Def. 2. $F_{\alpha+1}^0(x) = F_\alpha^x(x)$,

Def. 3. $F_\beta^0(x) = F_{\{\beta\}(x)}^0(\varrho_\beta(x))$, β a limit ordinal,

Def. 4. $F_\gamma^{n+1}(x) = F_\gamma^0(F_\gamma^n(x))$, $\gamma \neq 0$,

where, for β a limit ordinal,

$$\begin{cases} \varrho_\beta(0) = 0 \\ \varrho_\beta(m+1) = \mu_z\big(z > \varrho_\beta(m) \;\&\; (i)_{\leq m}\,(F_{\{\beta\}(m+1)}^0(z) > F_{\{\beta\}(i)}^0(z))\big) . \end{cases}$$

We shall now exhibit some of the basic properties of these functions.
First, however, we need the following

Definition 2.1

A function f is *eventually majorized* (e.m.) by a function g if f and g are totally defined and there is a number p such that for all $x \geqq p$, $f(x) < g(x)$.

Lemma 2.2

For every countable ordinal σ, if $\lambda x.\, F^0_\alpha(x)$ is eventually majorized by $\lambda x.\, F^0_\beta(x)$ whenever $\alpha < \beta < \sigma$, then

(i) $\lambda n x.\, F^n_\sigma(x)$ is totally defined,

(ii) for all n and x, $F^n_\sigma(x) > \max(n, x)$.

Proof

We proceed by transfinite induction over the countable ordinals.
Clearly, the result holds when $\sigma = 0$.
Suppose that the result holds for η, and that $\sigma = \eta + 1$.
Then if $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\beta(x)$ whenever $\alpha < \beta < \sigma$, we know that $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\beta(x)$ whenever $\alpha < \beta < \eta$, and hence (i) and (ii) must hold for η.
But $\lambda n x \cdot F^n_\sigma(x)$ is defined from $\lambda n x \cdot F^n_\eta(x)$ as follows:

$$\begin{cases} F^0_\sigma(x) = F^x_\eta(x) \\ F^{n+1}_\sigma(x) = F^0_\sigma(F^n_\sigma(x))\,. \end{cases}$$

Therefore, since, by (i), $\lambda n x \cdot F^n_\eta(x)$ is totally defined, so must be $\lambda n x \cdot F^n_\sigma(x)$.
Also, since (ii) holds for η, we have by Def. 2, $F^0_\sigma(x) = F^x_\eta(x) > x$, for all x.
From this, it can easily be proved by induction that (ii) holds for $\sigma = \eta + 1$.
Hence, if $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\beta(x)$ whenever $\alpha < \beta < \sigma$, (i) and (ii) hold for $\sigma = \eta + 1$.
Finally, suppose that σ is a limit ordinal and that the result holds for every ordinal less than σ.
Then if $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\beta(x)$ whenever $\alpha < \beta < \sigma$ it is clear, by the definition of ϱ_σ, that ϱ_σ is totally defined, and that for every x we can apply the induction hypothesis to deduce that $F^0_{\{\sigma\}(x)}(\varrho_\sigma(x))$ is defined and greater than x.
Hence, by Def. 3, we have for all x

$$F^0_\sigma(x) = F^0_{\{\sigma\}(x)}(\varrho_\sigma(x)) > x.$$

Now, by Def. 4, $F^{n+1}_\sigma(x) = F^0_\sigma(F^n_\sigma(x))$, and so it can easily be proved by induction, that (i) and (ii) must hold for σ.
Hence, if $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\beta(x)$ whenever $\alpha < \beta < \sigma$, (i) and (ii) must hold for σ a limit ordinal.
This completes the induction step, and so Lemma 2.2 is proved.

Lemma 2.3

For every countable ordinal σ, if $\lambda x \cdot F^0_\alpha(x)$ is eventually majorized by $\lambda x \cdot F^0_\beta(x)$ whenever $\alpha < \beta < \sigma$, then

(i) for all n and x, $F_\sigma^n(x+1) > F_\sigma^n(x)$,

(ii) for all n and x, $F_\sigma^{n+1}(x) > F_\sigma^n(x)$.

Proof

Again we proceed by transfinite induction over the countable ordinals.
First, the result clearly holds when $\sigma = 0$.
Suppose, now, that the result holds for η and that $\sigma = \eta + 1$.
Then if $\lambda x \cdot F_\alpha^0(x)$ is e.m. by $\lambda x . F_\beta^0(x)$ whenever $\alpha < \beta < \sigma$ we can apply the induction hypothesis and deduce that for all n and x,

$$F_\eta^n(x) < F_\eta^n(x+1) \quad \text{and} \quad F_\eta^n(x) < F_\eta^{n+1}(x) .$$

Hence, by Def. 2, we have, for every x,

$$F_\sigma^0(x) = F_\eta^x(x) < F_\eta^{x+1}(x) < F_\eta^{x+1}(x+1) = F_\sigma^0(x+1) .$$

Therefore, if $x < y$, $F_\sigma^0(x) < F_\sigma^0(y)$.
Now, if we assume that for every x, $F_\sigma^n(x) < F_\sigma^n(x+1)$, then we have, by Def. 4,

$$F_\sigma^{n+1}(x) = F_\sigma^0(F_\sigma^n(x)) < F_\sigma^0(F_\sigma^n(x+1)) = F_\sigma^{n+1}(x+1) ,$$

which holds for every x.
Hence, by induction, (i) holds for $\sigma = \eta + 1$.
Also, by Lemma 2.2, $F_\sigma^0(x) > x$, for all x, and so we have, by Def. 4,

$$F_\sigma^{n+1}(x) = F_\sigma^0(F_\sigma^n(x)) > F_\sigma^n(x) ,$$

for all n and x.
Hence, by induction, (ii) holds for $\sigma = \eta + 1$.
Thus, if $\lambda x \cdot F_\alpha^0(x)$ is e.m. by $\lambda x \cdot F_\beta^0(x)$ whenever $\alpha < \beta < \sigma$, then (i) and (ii) must hold for $\sigma = \eta + 1$.
Finally, suppose that σ is a limit ordinal, and that the result holds for every ordinal less than σ.
Then if $\lambda x \cdot F_\alpha^0(x)$ is e.m. by $\lambda x \cdot F_\beta^0(x)$ whenever $\alpha < \beta < \sigma$, it is clear that ϱ_σ is totally defined, and that for every x, we can apply the induction hypothesis to get

$$F^0_{\{\sigma\}\,(x)}(y) > F^0_{\{\sigma\}\,(x)}(z) \text{ whenever } y > z.$$

But, from the definition of ϱ_σ, we have, for all x, $\varrho_\sigma(x+1) > \varrho_\sigma(x)$.
Hence, for every x, we have the following:

$$\begin{aligned} F_\sigma^0(x+1) &= F^0_{\{\sigma\}\,(x+1)}(\varrho_\sigma(x+1)) \text{ by Def. 3.} \\ &> F^0_{\{\sigma\}\,(x)}(\varrho_\sigma(x+1)) \text{ by definition of } \varrho_\sigma . \\ &> F^0_{\{\sigma\}\,(x)}(\varrho_\sigma(x)) \\ &= F_\sigma^0(x) \text{ by Def. 3.} \end{aligned}$$

From this result it again easily follows that (i) and (ii) hold for σ a limit ordinal.
Hence, if $\lambda x \cdot F_\alpha^0(x)$ is e.m. by $\lambda x \cdot F_\beta^0(x)$ whenever $\alpha < \beta < \sigma$, then (i) and (ii) hold for σ a limit ordinal.
This completes the induction step, and hence the proof of Lemma 2.3.

Lemma 2.4

For all countable ordinals α, β, if $\alpha < \beta$ then $\lambda x \cdot F^0_\alpha(x)$ is eventually majorized by $\lambda x \cdot F^0_\beta(x)$.

Proof

We use transfinite induction to prove that for every σ, if $\alpha < \sigma$ then $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\sigma(x)$.
The result is trivial when $\sigma = 0$.
Suppose, now, that $\sigma > 0$, and that for every $\delta < \sigma$, if $\alpha < \delta$ then $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\delta(x)$.

We consider two cases:
(a) If $\sigma = \eta + 1$ then by the induction hypothesis, $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\beta(x)$ whenever $\alpha < \beta < \eta$.
Hence, by Lemma 2.3, for all n and x, $F^{n+1}_\eta(x) > F^n_\eta(x)$.
Therefore, for every $x \geqq 1$ we have

$$F^0_\sigma(x) = F^x_\eta(x) > F^0_\eta(x) \, .$$

Thus $\lambda x \cdot F^0_\eta(x)$ is e.m. by $\lambda x \cdot F^0_\sigma(x)$.
But, by the induction hypothesis, if $\alpha < \eta$, then $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\eta(x)$.
Hence, for every $\alpha < \sigma$, $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\sigma(x)$.
This completes case (a).

(b) If σ is a limit ordinal, then for any $\alpha < \sigma$ there is a number p such that $\alpha < \{\sigma\}(p)$
Now, by the induction hypothesis, there must be a number q such that for every $x \geqq q$, $F^0_{\{\sigma\}(p)}(x) > F^0_\alpha(x)$.
Also, by the induction hypothesis, if $\beta < \gamma < \sigma$ then $\lambda x \cdot F^0_\beta(x)$ is e.m. by $\lambda x \cdot F^0_\gamma(x)$.
Hence by definition of ϱ_σ, ϱ_σ is totally defined and, if $y > z$, $\varrho_\sigma(y) > \varrho_\sigma(z)$.
Thus, by definition of ϱ_σ, we have for every $x \geqq \max(p, q)$,

$$F^0_\sigma(x) = F^0_{\{\sigma\}(x)}\left(\varrho_\sigma(x)\right) \geqq F^0_{\{\sigma\}(p)}\left(\varrho_\sigma(x)\right)$$

and
$$F^0_{\{\sigma\}(p)}\left(\varrho_\sigma(x)\right) \geqq F^0_{\{\sigma\}(p)}(x) > F^0_\alpha(x) \, ,$$

since $\varrho_\sigma(x) \geqq x$, and since, by Lemma 2.3 and the induction hypothesis, if $y \geqq z$, then $F^0_{\{\sigma\}(p)}(y) \geqq F^0_{\{\sigma\}(p)}(z)$.
Hence $\lambda x \cdot F^0_\alpha(x)$ is e.m. by $\lambda x \cdot F^0_\sigma(x)$, and this holds for any $\alpha < \sigma$.
This completes case (b).
Cases (a) and (b) together constitute the induction step, and so we have proved Lemma 2.4.
With the aid of this last result, it is now easy to obtain the next Lemma, which we state without proof.

Lemma 2.5

For all countable ordinals α, β, if $\alpha < \beta$ then for each n, $\lambda x \cdot F^n_\alpha(x)$ is eventually majorized by $\lambda x \cdot F^n_\beta(x)$.
Combining Lemmas 2.2, 2.3, 2.4 and 2.5, we obtain the following:

Lemma 2.6

(i) For each α, $\lambda n x \cdot F_\alpha^n(x)$ is totally defined.

(ii) For each α, and all n, x, $F_\alpha^n(x) > \max(n, x)$.

(iii) For each α and all n, x, y, if $x > y$, $F_\alpha^n(x) > F_\alpha^n(y)$.

(iv) For each α and all m, n, x, if $m > n$, $F_\alpha^m(x) > F_\alpha^n(x)$.

(v) For all α, β, if $\alpha < \beta$ then for each n, $\lambda x \cdot F_\alpha^n(x)$ is eventually majorized by $\lambda x \cdot F_\beta^n(x)$.

Lemma 2.7

For each α and all n and x, if $\max(n, x) \geqq 1$ then $F_{\alpha+1}^n(x) > F_\alpha^n(x)$.

Proof

Suppose that $x \geqq 1$.
Then by Lemma 2.6,

$$F_{\alpha+1}^0(x) = F_\alpha^x(x) > F_\alpha^0(x)\ .$$

Now assume that $F_{\alpha+1}^m(x) > F_\alpha^m(x)$.
Then by Def. 4 and Lemma 2.6,

$$F_{\alpha+1}^{m+1}(x) = F_{\alpha+1}^0(F_{\alpha+1}^m(x)) > F_{\alpha+1}^0(F_\alpha^m(x))\ .$$

$$\text{But } F_\alpha^m(x) \geqq 1, \text{ so } F_{\alpha+1}^0(F_\alpha^m(x)) > F_\alpha^0(F_\alpha^m(x))\ .$$

$$\text{Hence } F_{\alpha+1}^{m+1}(x) > F_\alpha^0(F_\alpha^m(x)) = F_\alpha^{m+1}(x)\ .$$

Thus we have proved, by induction, that for all n and all $x \geqq 1$, $F_{\alpha+1}^n(x) > F_\alpha^n(x)$.
Now $F_\alpha^0(0) \geqq 1$ and so by Def. 2 and Def. 4,

$$F_{\alpha+1}^1(0) = F_{\alpha+1}^0(F_{\alpha+1}^0(0)) = F_{\alpha+1}^0(F_\alpha^0(0)) > F_\alpha^0(F_\alpha^0(0))\ .$$

But, again by Def. 4, $F_\alpha^0(F_\alpha^0(0)) = F_\alpha^1(0)$ and so we have $F_{\alpha+1}^1(0) > F_\alpha^1(0)$.
Furthermore, if we assume that $F_{\alpha+1}^m(0) > F_\alpha^m(0)$ for $m \geqq 1$, we can similarly prove that $F_{\alpha+1}^{m+1}(0) > F_\alpha^{m+1}(0)$.
Hence, by induction, $F_{\alpha+1}^n(0) > F_\alpha^n(0)$ for all $n \geqq 1$.
Thus $F_{\alpha+1}^n(x) > F_\alpha^n(x)$ whenever $\max(n, x) \geqq 1$.
Now, by Lemma 2.7 and the fact that for each α, $F_{\alpha+1}^0(0) = F_\alpha^0(0)$ by Def. 2, we get

Lemma 2.8

For each α and every $k \geqq 1$ we have, for all n and x,

$$F_{\alpha+k}^n(x) \geqq F_\alpha^n(x)\ ,$$

with equality holding only when $n = x = 0$.
We now consider a method of extending the Grzegorczyk hierarchy (which is uniform in the choice of fundamental sequences to limit ordinals).
The results contained in Lemmas 2.6 and 2. 8are of basic importance to the work which follows, and we shall use them without referring to them explicitly.

Definition 2.9

For each α, define $\mathfrak{E}_\alpha$ as follows:

$$\mathfrak{E}_\alpha = E(\{\lambda x \cdot 0, \lambda xy \cdot x + y, \lambda \boldsymbol{x} \cdot x_i\} \cup \{\lambda x \cdot F^0_\beta(x) | \beta \leqq \alpha\}).$$

From Def. 2.9 it is obvious that whenever $\alpha \leqq \beta$, $\mathfrak{E}_\alpha \subsetneqq \mathfrak{E}_\beta$.
We now prove that the hierarchy $\{\mathfrak{E}_\alpha\}$ does not collapse.

Theorem 2.10

Let α be any countable ordinal > 0.
Then for every function $f \in \mathfrak{E}_\alpha$, there is a number p such that, for all $x_1, \ldots, x_n$,

$$f(x_1, \ldots, x_n) < F^p_\alpha(\max(x_1, \ldots, x_n)).$$

Proof

First of all, notice that $F^0_\alpha(x) > 0$ for all x, and that, for all $x_1, \ldots, x_n$ and each i $(1 \leqq i \leqq n)$,

$$F^0_\alpha(\max(x_1, \ldots, x_n)) > x_i.$$

Also, $F^0_1(\max(x, y)) \geqq F^x_0(y) > x + y$, for all x and y.
Now take any $\beta < \alpha$.
Since $\lambda x \cdot F^0_\beta(x)$ is c.m. by $\lambda x \cdot F^0_\alpha(x)$, it is clear that there is a number p such that

$$F^0_\beta(x) \leqq F^0_\beta(x + p) < F^0_\alpha(x + p),$$

for all x.
But $F^0_\alpha(x + p) < F^0_\alpha(F^p_\alpha(x)) = F^{p+1}_\alpha(x)$.
Hence $F^0_\beta(x) < F^{p+1}_\alpha(x)$, for all x.
Also, it is clear that $F^0_\alpha(x) < F^1_\alpha(x)$, for all x.
Hence, if I is any initial function of $\mathfrak{E}_\alpha$, there must be a number q such that, for all $x_1, \ldots, x_n$,

$$I(x_1, \ldots, x_n) < F^q_\alpha(\max(x_1, \ldots, x_n)).$$

Now suppose there are numbers $q, p_1, \ldots, p_m$ such that

$$g(\boldsymbol{y}) < F^q_\alpha(\max(\boldsymbol{y})) \text{ for all } \boldsymbol{y},$$

and for each $i = 1, \ldots, m$,

$$h_i(\boldsymbol{x}) < F^{p_i}_\alpha(\max(\boldsymbol{x})) \text{ for all } \boldsymbol{x}.$$

Suppose also that f is defined from $g, h_1, \ldots, h_m$ by substitution, as follows:

$$f(\boldsymbol{x}) = g(h_1(\boldsymbol{x}), \ldots, h_m(\boldsymbol{x})).$$

Then if $p = \max(p_1, \ldots, p_m)$ we have, for all $\boldsymbol{x}$,

$$\begin{aligned} f(\boldsymbol{x}) &< F^q_\alpha(\max(h_1(\boldsymbol{x}), \ldots, h_m(\boldsymbol{x}))) \\ &< F^q_\alpha\big(\max(F^{p_1}_\alpha(\max(\boldsymbol{x})), \ldots, F^{p_m}_\alpha(\max(\boldsymbol{x})))\big) \\ &\leqq F^q_\alpha(F^p_\alpha(\max(\boldsymbol{x}))) \\ &= F^{p+q+1}_\alpha(\max(\boldsymbol{x})) \end{aligned}$$

since, for any n, $F_\alpha^n(x)$ is just the $n+1$ – st. iterate of F_α^0, applied to x.
Finally, suppose f is defined by limited recursion from functions h_1, h_2, h_3 for which there are numbers p_1, p_2, p_3 such that, for each $i = 1, 2, 3$,

$$h_i(\boldsymbol{x}) < F_\alpha^{p_i}(\max(\boldsymbol{x})).$$

Then, since f is bounded by h_3, we have, for all $\boldsymbol{x}$,

$$f(\boldsymbol{x}) < F_\alpha^{p_3}(\max(\boldsymbol{x})).$$

We have now considered all possible ways of defining functions in $\mathfrak{E}_\alpha$.
Hence the Theorem is proved.

Theorem 2.11

Let α be any countable ordinal > 0.
Then for every function $f \in \mathfrak{E}_\alpha$ there is a number p such that whenever $\max(\boldsymbol{x}) \geqq p$,

$$f(\boldsymbol{x}) < F_{\alpha+1}^0(\max(\boldsymbol{x})).$$

Proof

Take any function $f \in \mathfrak{E}_\alpha$.
Then by Theorem 2.10, there is a number p such that, for all $\boldsymbol{x}$,

$$f(\boldsymbol{x}) < F_\alpha^p(\max(\boldsymbol{x})).$$

Suppose $\max(\boldsymbol{x}) \geqq p$. Then we have

$$f(\boldsymbol{x}) < F_\alpha^{\max(\boldsymbol{x})}(\max(\boldsymbol{x})) = F_{\alpha+1}^0(\max(\boldsymbol{x})).$$

Theorem 2.12

For any two countable ordinals α and β, if $\alpha < \beta$ then $\mathfrak{E}_\alpha \subset \mathfrak{E}_\beta$.

Proof

First, by the results of Grzegorczyk in [5], it is clear that

$$E(\lambda x \cdot 0, \lambda xy \cdot x + y, \lambda \boldsymbol{x} \cdot x_i, \lambda x \cdot x + 1) = \mathscr{E}^1$$

is strictly contained in

$$E(\lambda x \cdot 0, \lambda xy \cdot x + y, \lambda \boldsymbol{x} \cdot x_i, \lambda x \cdot x + 1, \lambda x \cdot (x+1)^2) = \mathscr{E}^2.$$

But the first class is just $\mathfrak{E}_0$, and the second $\mathfrak{E}_1$, and hence $\mathfrak{E}_0 \subset \mathfrak{E}_1$.

Now let α be any countable ordinal > 0.
Then by Theorem 2.11, $\lambda x \cdot F_{\alpha+1}^0(x)$ eventually majorizes every unary function in $\mathfrak{E}_\alpha$, and so cannot be a member of $\mathfrak{E}_\alpha$.
However, $\lambda x \cdot F_{\alpha+1}^0(x) \in \mathfrak{E}_\beta$ for every countable ordinal $\beta > \alpha$.
Hence, for every $\beta > \alpha$, $\mathfrak{E}_\alpha \subset \mathfrak{E}_\beta$.
This completes the proof of Theorem 2.12.
We shall now show that the hierarchy $\{\mathfrak{E}_\alpha\}$ is a proper extension of the Grzegorczyk hierarchy.

First we define two ω-sequences of binary functions:

$$A_0(x, y) = y + 1 ,$$
$$A_1(x, y) = x + y ,$$
$$A_2(x, y) = x \cdot y ,$$
$$A_{n+3}(0, y) = 1 ,$$
$$A_{n+3}(x + 1, y) = A_{n+2}(A_{n+3}(x, y), y) .$$

The function $\lambda nxy \cdot A_n(x, y)$ is a slight variation of the Ackermann function used in [1].

$$G_0(x, y) = y + 1 ,$$
$$G_1(x, y) = x + y ,$$
$$G_2(x, y) = (x + 1) \cdot (y + 1) ,$$
$$G_{n+3}(0, y) = G_{n+2}(y + 1, y + 1) ,$$
$$G_{n+3}(x + 1, y) = G_{n+3}(x, G_{n+3}(x, y)) .$$

The function $\lambda nxy \cdot G_n(x, y)$ is that used by Grzegorczyk in [5].
Ritchie [11] and Cleave and Rose [3] have obtained various "monotonicity" properties of the above functions.

We shall make use of the following properties which can be proved fairly easily.

(1) For all n, x, y, z, if $x \leqq z$, $G_n(x, y) \leqq G_n(z, y)$.

(2) For all n, x, y, z, if $y \leqq z$, $G_n(x, y) \leqq G_n(x, z)$.

(3) For all n, y, and all $x \geqq 1$,

$$A_{n+2}(A_{n+3}(x, y), A_{n+3}(x, y)) \geqq A_{n+2}(A_{n+3}(x, y), y) .$$

We can now prove the following relationships between the functions A, F, G.

Lemma 2.13

For all n, x, y, $F_n^x(y) \leqq G_{n+2}(x, y)$.

Proof

By definition, $F_0^x(y) = G_2(x, y)$ for all x, y.

Now assume that for all x, y,

$$F_n^x(y) \leqq G_{n+2}(x, y) .$$

Then for all y, we have

$$F_{n+1}^0(y) = F_n^y(y) \leqq G_{n+2}(y, y) \leqq G_{n+2}(y + 1, y + 1) = G_{n+3}(0, y) .$$

Suppose now that, for all y, $F_{n+1}^x(y) \leqq G_{n+3}(x, y)$.

Then we have the following:

$$\begin{aligned} F_{n+1}^{x+1}(y) &= F_{n+1}^{0}(F_{n+1}^{x}(y)) \text{ by Def. 4} \\ &\leqq F_{n+1}^{0}(G_{n+3}(x, y)) \\ &\leqq G_{n+3}(0, G_{n+3}(x, y)) \\ &\leqq G_{n+3}(x, G_{n+3}(x, y)) \\ &= G_{n+3}(x+1, y), \end{aligned}$$

and this holds for every y.
Hence, by induction, $F_{n+1}^{x}(y) \leqq G_{n+3}(x, y)$ for all x and y.
Therefore, again by induction, we have for all n, x, y, $F_{n}^{x}(y) \leqq G_{n+2}(x, y)$.
This completes the proof.

Lemma 2.14

For all n, x, y, $A_{n+2}(x, y) \leqq F_{n}^{x}(y)$.

Proof

Clearly $A_2(x, y) = x \cdot y \leqq (x+1) \cdot (y+1) = F_0^x(y)$, for all x, y.
Now assume that for all x and y,

$$A_{n+2}(x, y) \leqq F_n^x(y) .$$

Then for all y, $A_{n+3}(0, y) = 1 \leqq F_{n+1}^{0}(y)$, and also, for all y, we have

$$A_{n+3}(1, y) = A_{n+2}(1, y) = \cdots = A_2(1, y) = y \leqq F_{n+1}^{1}(y) .$$

Suppose, now, that $x \geqq 1$ and that for all y,

$$A_{n+3}(x, y) \leqq F_{n+1}^{x}(y) .$$

Then we have the following:

$$\begin{aligned} A_{n+3}(x+1, y) &= A_{n+2}(A_{n+3}(x, y), y) \\ &\leqq A_{n+2}(A_{n+3}(x, y), A_{n+3}(x, y)) \text{ by (3)} \\ &\leqq F_n^{A_{n+3}(x,y)}(A_{n+3}(x, y)) \\ &= F_{n+1}^{0}(A_{n+3}(x, y)) \text{ by Def. 2} \\ &\leqq F_{n+1}^{0}(F_{n+1}^{x}(y)) \\ &= F_{n+1}^{x+1}(y) , \end{aligned}$$

and this holds for every y.
Hence, by induction, $A_{n+3}(x, y) \leqq F_{n+1}^{x}(y)$ for all x, y.
Therefore, again by induction, we have, for all n, x, y,

$$A_{n+2}(x, y) \leqq F_n^x(y) .$$

This completes the proof.

Definition 2.15

We define classes $\mathscr{F}^n$ and $\mathscr{E}^n$ as follows:

$$\mathscr{F}^n = E(\lambda x \cdot 0, \lambda x \cdot x + 1, \lambda \boldsymbol{x} \cdot x_i, \lambda xy \cdot A_n(x, y))$$

$$\mathscr{E}^n = E(\lambda x \cdot 0, \lambda x \cdot x + 1, \lambda \boldsymbol{x} \cdot x_i, \lambda xy \cdot G_n(x, y)) .$$

The classes $\mathscr{E}^n$ are those considered by Grzegorczyk in [5].

Lemma 2.16

For each n, $\mathscr{F}^{n+1} \subseteqq \mathfrak{E}_n$.

Proof

We know that, for every n, $\lambda x \cdot 0$, $\lambda \boldsymbol{x} \cdot x_i$ and $\lambda xy \cdot x + y$ all belong to $\mathfrak{E}_n$.
Also $F_0^0(x) = x + 1$, and so $\lambda x \cdot x + 1 \in \mathfrak{E}_n$ for every n.
Hence, all we need do is show that for each n,

$$\lambda xy \cdot A_{n+1}(x, y) \in \mathfrak{E}_n .$$

Clearly, $\lambda xy \cdot A_1(x, y) \in \mathfrak{E}_0$.
Assume that $\lambda xy \cdot A_{n+1}(x, y) \in \mathfrak{E}_n$.
Then $\lambda xy \cdot A_{n+1}(x, y) \in \mathfrak{E}_{n+1}$, since $\mathfrak{E}_n \subset \mathfrak{E}_{n+1}$.
Now $\lambda xy \cdot A_{n+2}(x, y)$ is defined by a simple primitive recursion from $\lambda xy \cdot A_{n+1}(x, y)$.
Also, by Lemma 2.14, we have

$$A_{n+2}(x, y) \leqq F_n^x(y) \leqq F_n^{x+y}(x + y) = F_{n+1}^0(x + y) .$$

Therefore, $\lambda xy \cdot A_{n+2}(x,y)$ is defined by limited recursion from $\lambda xy \cdot A_{n+1}(x,y) \in \mathfrak{E}_{n+1}$ and $\lambda xy \cdot F_{n+1}^0(x + y) \in \mathfrak{E}_{n+1}$.
Hence $\lambda xy \cdot A_{n+2}(x, y) \in \mathfrak{E}_{n+1}$.
Thus, by induction, $\lambda xy \cdot A_{n+1}(x, y) \in \mathfrak{E}_n$ for every n, and this completes the proof.

Lemma 2.17

For each n, $\mathfrak{E}_n \subseteqq \mathscr{E}^{n+1}$.

Proof

Clearly, $\mathfrak{E}_0 = \mathscr{E}^1$.
Assume that $\mathfrak{E}_n \subseteqq \mathscr{E}^{n+1}$, so that $\mathfrak{E}_n \subseteqq \mathscr{E}^{n+2}$ since $\mathscr{E}^{n+1} \subset \mathscr{E}^{n+2}$.
Hence $\lambda x \cdot F_n^0(x) \in \mathscr{E}^{n+2}$.

Now $\lambda xy \cdot F_n^x(y)$ is defined by a simple primitive recursion from $\lambda x \cdot F_n^0(x)$, and by Lemma 2.13, $F_n^x(y) \leqq G_{n+2}(x, y)$ for all x, y.
Therefore, $\lambda xy \cdot F_n^x(y)$ can be defined by limited recursion from $\lambda x \cdot F_n^0(x) \in \mathscr{E}^{n+2}$ and $\lambda xy \cdot G_{n+2}(x, y) \in \mathscr{E}^{n+2}$.
Hence $\lambda xy \cdot F_n^x(y) \in \mathscr{E}^{n+2}$.
But $F_{n+1}^0(x) = F_n^x(x)$, and so $\lambda x \cdot F_{n+1}^0(x) \in \mathscr{E}^{n+2}$.

Thus, all the initial functions of $\mathfrak{E}_{n+1}$ belong to $\mathscr{E}^{n+2}$, and so it is clear that $\mathfrak{E}_{n+1} \subseteqq \mathscr{E}^{n+2}$.
Hence, by induction, we have, for every n, $\mathfrak{E}_n \subseteqq \mathscr{E}^{n+1}$.
Now Ritchie [11] and Cleave and Rose [3] have proved that, for every n, $\mathscr{F}^n = \mathscr{E}^n$.
Hence, by Lemmas 2.16 and 2.17, we have the following result:

Theorem 2.18

(i) For every n, $\mathfrak{E}_n = \mathscr{E}^{n+1}$.

(ii) $\bigcup_{n \in N} \mathfrak{E}_n = \mathscr{P}$.

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