O(m log n) Time Algorithms for DFA Minimization and More

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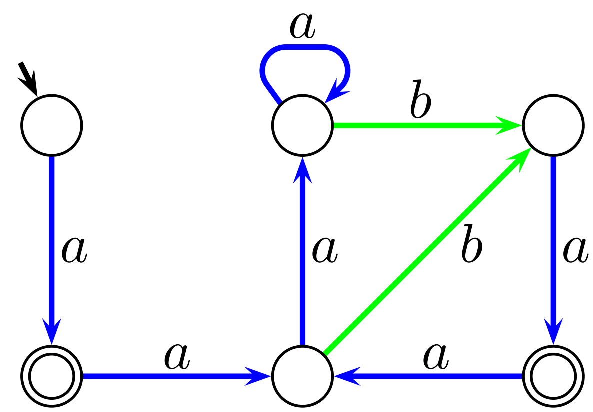
Part I:

DFA Minimization

1 Deterministic Finite Automata

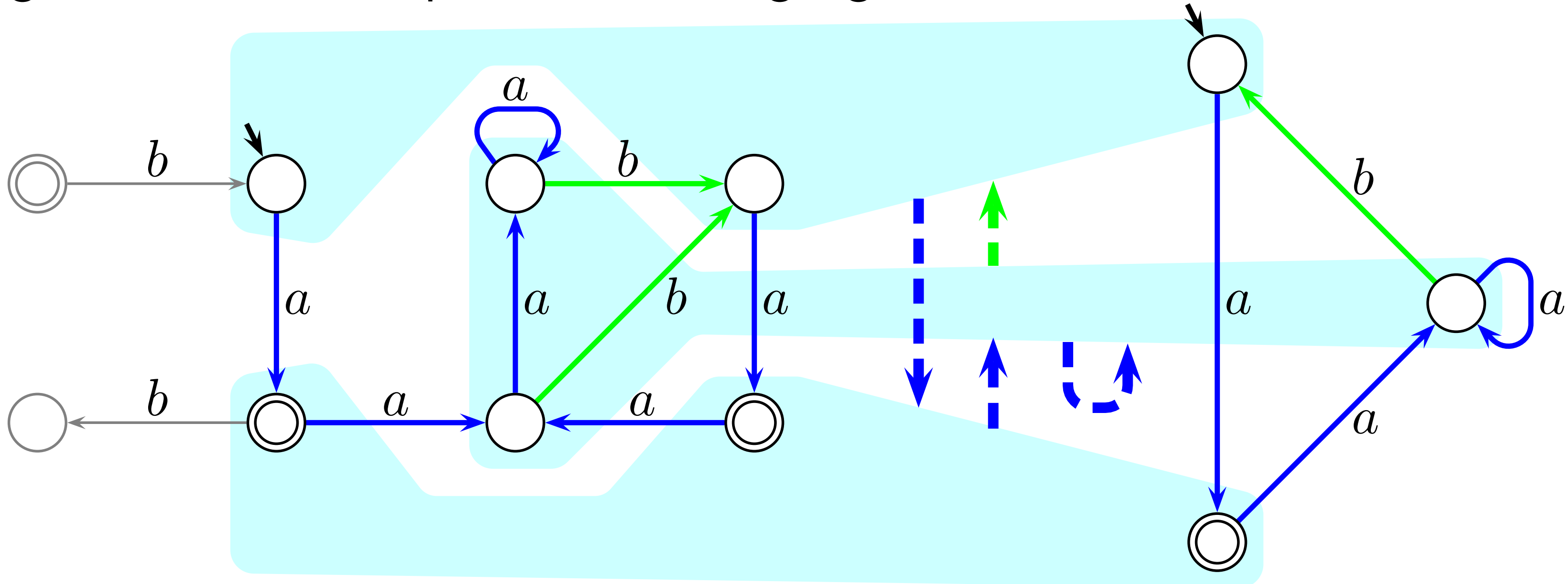


- $D = (Q, \Sigma, \delta, \hat{q}, F)$
  - Q = **states**
  - Σ = **labels** (the alphabet)
  - δ = **transitions** (*partial function!*)
  - $\hat{q}$ = **initial state**
  - F = **final states**
- Let $n = |Q|$, $m = |\delta| = |$*defined* transitions$|$, $\alpha = |\Sigma| = |$*available* labels$|$
  - technical convenience asumption: $n = O(m)$ (e.g., $n \leq 2m + 1$)
- Let $q \in Q$
- The **language** accepted by q is the set of strings of labels on the paths from q to final states
  - e.g., bottom middle state: $\{aba, aaba, \ldots, ba, baaba, \ldots\}$
  - e.g., bottom right state: $\{\varepsilon, aba, aaba, aaaba, \ldots, \ldots\}$
- Denote it with $\mathcal{L}(q)$
- The language accepted by D is $\mathcal{L}(D) = \mathcal{L}(\hat{q})$

2 Minimization of Deterministic Finite Automata

- **The minimization problem:**
  Find the smallest DFA that accepts the same language as the given DFA.

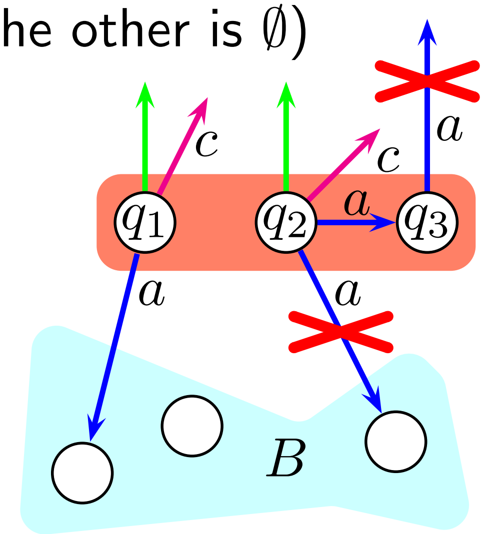
- solution:
  1. Remove **irrelevant** states and transitions. (textbook stuff)
     - those that are not reachable from $\hat{q}$
     - those from which no final state can be reached (except $\hat{q}$)
  2. Merge states that accept the same language.



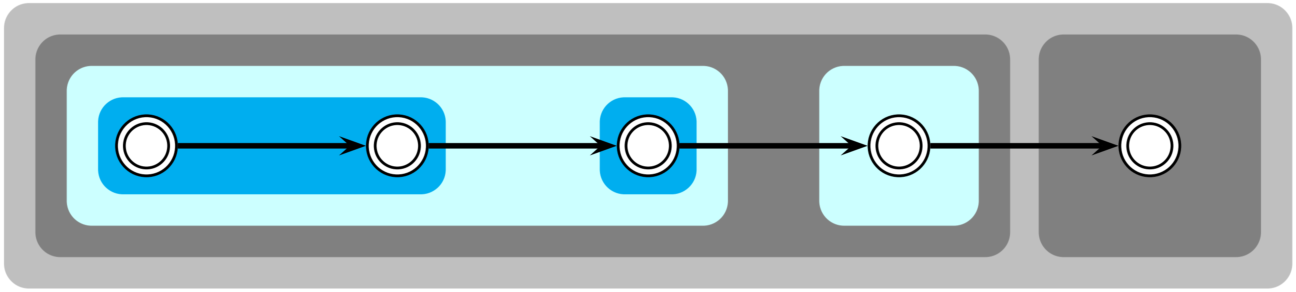
- How can we test if $\mathcal{L}(q_1) = \mathcal{L}(q_2)$?

3 Block Splitting

- States are partitioned into **blocks**
- q_1 and q_2 go to different blocks only when it is certain that $\mathcal{L}(q_1) \neq \mathcal{L}(q_2)$
- Initially blocks are F and $Q \setminus F$ (or just one of them, if the other is $\emptyset$)
- Blocks are split as long as possible
- Reason for putting q_1 and q_2 to different blocks: for some label a and block B, q_1 has and q_2 does not have an a-transition to a state in B



- At most $n - 1$ successful splittings ($n = |\text{States}|$)
- **Problem:** vulnerable to lots of work

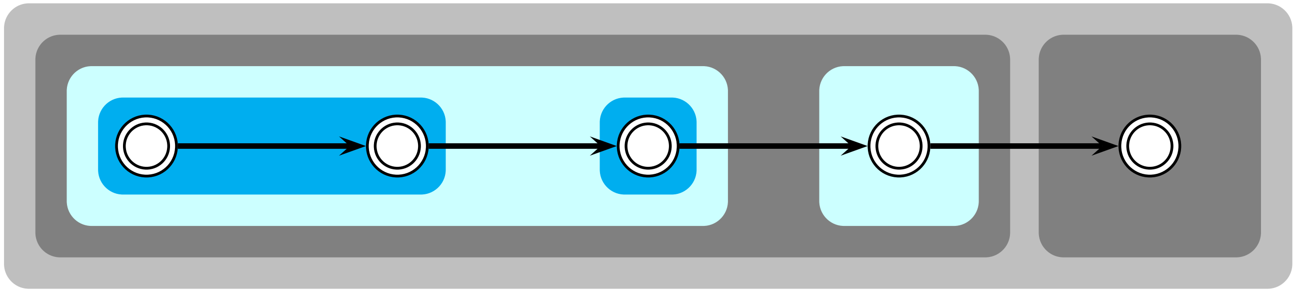


  - $O(n^2)$ even if $\alpha = 1$ and $m = O(n)$
- **Cost of "useless" "little" work may be important**

4 Hopcroft's Ideas [1971]

- Clarified and improved by Gries [1973] (and, e.g., Knuutila [2001])
- Assumes that δ is full
- **Idea:** Traverse transitions backwards
  - **splitter** = (block, label) = (B, a)
  - process one splitter at a time
  - find the q such that $\delta(q, a) \in B$, move them to tentative new blocks
  - each block splits to backwards-encountered and others (if both non-empty)

  $\Rightarrow$ no futile scanning of states without relevant output transitions



- **Idea:** If (B, a) has been used and B splits to B_1 and B_2,
  then it suffices to use one of (B_1, a) and (B_2, a)
  - use the "smaller" one (meaning of "smaller" is less trivial than it seems!)

  $\Rightarrow$ each state is used as a splitter state at most $\log_2 n$ instead of n times
- Both ideas $\Rightarrow$ running time in $O(\alpha n \log n)$ $\qquad$ $m \leq \alpha n$, often $m \ll \alpha n$

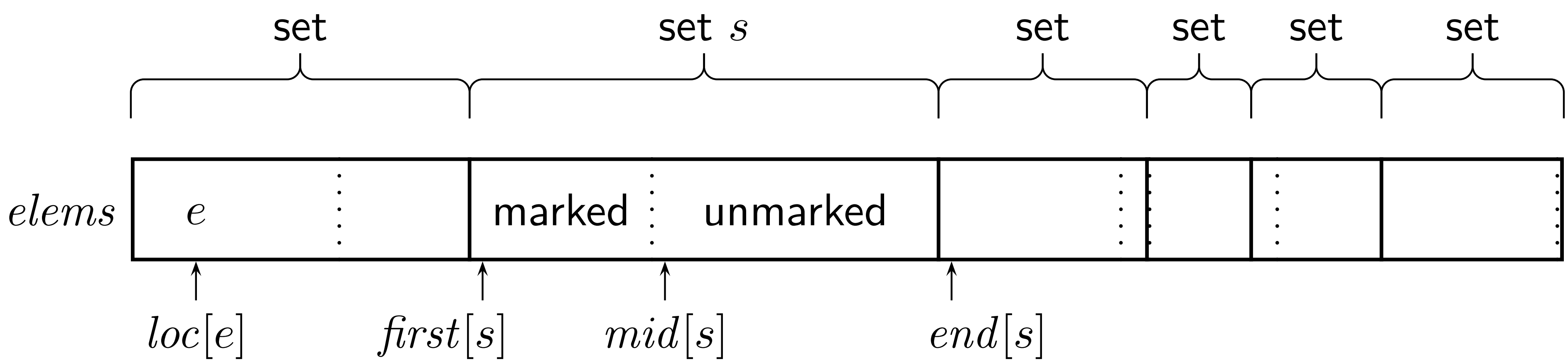
5 Gries' Data Structures (Roughly)

- Partition of Q
  - for each block, there is a doubly linked list of the states in it
  - the block has a pointer to its main list and "tentative new" list
  - each state has a pointer to its block
  - the block knows its size (and the size of the "tentative new" list)
- Inverse transitions: for each q and a, the states q' such that $\delta(q', a) = q$
- Worksets for temporary storage
  - unprocessed splitters, and pointers to them for each (B, a)
  - backwards-encountered states — don't cut the branch on which you sit!
  - backwards-encountered blocks $=$ **touched blocks**
- $\Theta(\alpha n)$ memory

while there are unprocessed splitters (B, a) **do**
  choose any and remove it from the workset
  compute its backwards-encountered states
  use backwards-encountered states to tentatively split blocks
  for each touched block B' **do**
    split or reset back to earlier status
    if B' is split **then for** $b \in \Sigma$ **do** update the (B', b)

6 More Recent Refinable Partition Data Structure

- Maintains a partition of $\{1, 2, \ldots, N\}$, for some N
- Like Gries, constant time $Mark(e)$ and amortized constant time $Split(s)$



- Also $sidx[e]$
- Of course, all arrays must be updated appropriately in each operation
- $Mark(e)$ swaps e with the first unmarked element and increments $mid[s]$
  - **Trick 1 for the future:** returns set number iff all elements were unmarked
- **Trick 2 for the future:** $Split(s)$ gives new block number to smaller half
  - earlier papers: marked states become the new block
  - does not affect amortized speed, as long as new $\neq$ unmarked bigger half

7 DFA Minimization in $O(m \log n)$ Time 1/3

- Valmari & Lehtinen [2008], improvements Valmari [2010]
- **Problem:** How to avoid spending excessive time scanning empty splitters?
  - empty (B, a) = no a-transition ends at B
  - and how to avoid using and initializing $\Theta(\alpha n)$ memory?

  $\Rightarrow$ cannot use, e.g., `in_trans[ state, label ]`
- **Idea:** Non-empty splitters constitute a partition of transitions that can be maintained similarly to blocks

$\Rightarrow$ Two refinable partitions in the same program

  - $\mathcal{B}$ = blocks, partition of states
  - $\mathcal{C}$ = **cords**, partition of transitions

- Inverse transitions: $In\_trans[q] = \{ (q_1, a, q_2) \in \delta \mid q_2 = q \}$
  - numbers of input transitions of q *in arbitrary order*

  $\Rightarrow$ no need for $\Theta(\alpha n)$ data structures, easy to initialize
- Worksets: only one, W
  - e.g., array of integers used as a stack

8 DFA Minimization in $O(m \log n)$ Time 2/3

- Algorithm *in great detail*

$c := 1$; $b := 2$; $W := \emptyset$;
while $c \leq |\mathcal{C}|$ **do**
  use cord $\#c$ to split blocks
  $c := c + 1$; $W := \emptyset$
  while $b \leq |\mathcal{B}|$ **do**
    use block $\#b$ to split cords
    $b := b + 1$; $W := \emptyset$

for $\ell := \mathcal{C}.\mathit{first}[c]$ **to** $\mathcal{C}.\mathit{end}[c] - 1$ **do**
  $b' := \mathcal{B}.\mathit{Mark}(\ \mathit{tail}[\ \mathcal{C}.\mathit{elems}[\ell]\ ]\ )$
  if $b' > 0$ **then** $W := W \cup \{b'\}$
for $b' \in W$ **do** $\mathcal{B}.\mathit{Split}(b')$

for $\ell := \mathcal{B}.\mathit{first}[b]$ **to** $\mathcal{B}.\mathit{end}[b] - 1$ **do**
  for $t \in \mathit{In\_trans}(\ \mathcal{B}.\mathit{elems}[\ell]\ )$ **do**
    $c' := \mathcal{C}.\mathit{Mark}(t)$
    if $c' > 0$ **then** $W := W \cup \{c'\}$
for $c' \in W$ **do** $\mathcal{C}.\mathit{Split}(c')$

- Earlier Trick 1
  $\Rightarrow$ b' and c' may be added to W without testing if they are already there

- Why is there no workset for unprocessed block-splitters and cord-splitters?
  - Hopcroft, Gries: either smaller or new half must be added to unprocessed
  - Trick 2: new number is given to smaller half $\Rightarrow$ these cases are the same
  - unprocessed are chosen for processing in first in – first out order

  $\Rightarrow$ the unprocessed are always $\{b, b+1, \ldots, |\mathcal{B}|\}$ and $\{c, c+1, \ldots, |\mathcal{C}|\}$

9 DFA Minimization in $O(m \log n)$ Time 3/3

- **Problem:** Not enough time to initially sort the transitions! $O(m \log m)$
  - we aim at so fast an algorithm that ordinary things become too slow
  - solution: counting sort + classification trick of Aho & al. [1974] exercise
  - practical average-time solution: hash table
  - engineer's: don't bother, $O(m \log m)$ is not much worse than $O(m \log n)$
- A prototype implementation
  - 590 lines of C++ (+ libraries for formatted i/o and error messages)
  - refinable partition data structure: 50 lines, other data structures: 20 lines
  - block splitting: 40 lines
  - Aho & al. [1974] & heapsorting as an alternative: 60 lines
  - removal of irrelevant states: 70 lines, other initialization: 80 lines
  - input and output: 130 lines
  - heapsort: 40 lines, range-checking array: 60 lines, main comment: 40 lines

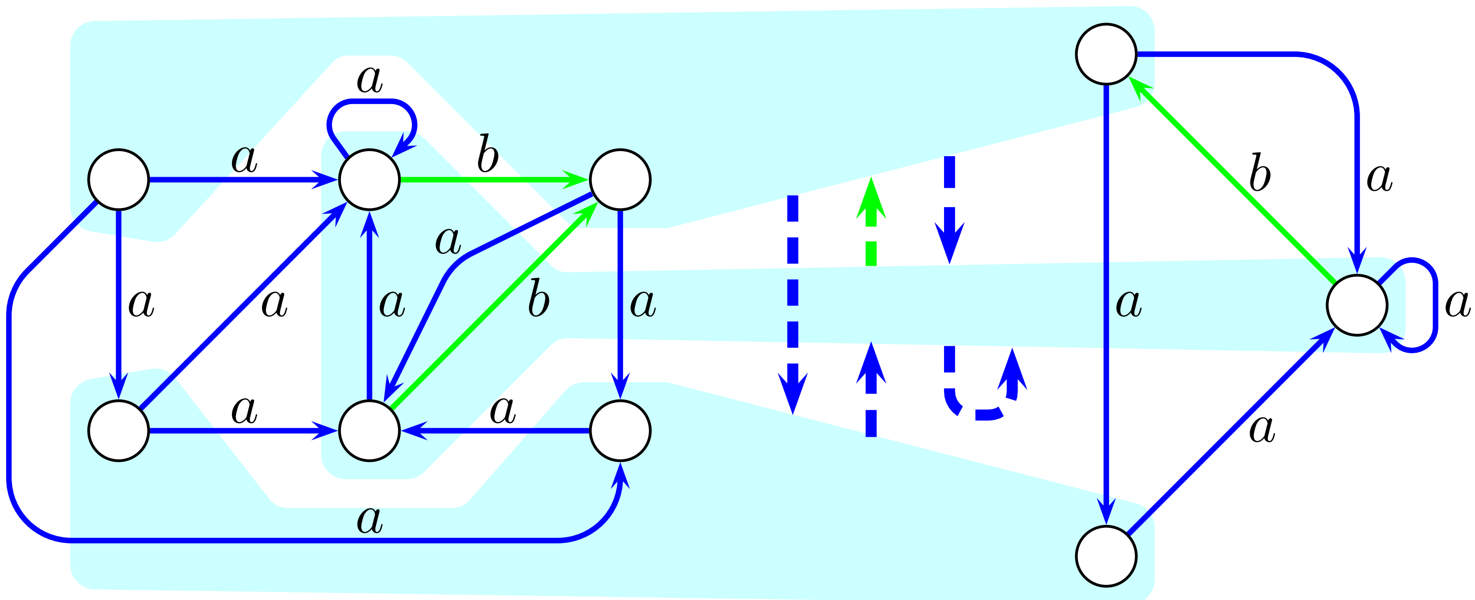
$\Rightarrow$ Compared to general idea of the difficulty of programming fast DFA minimization, this is very simple

- 50 sec on a laptop when $n = 10^5$, $\alpha = 1000$, $m = 5 \cdot 10^6$ (includes i/o)

Part II:

and More

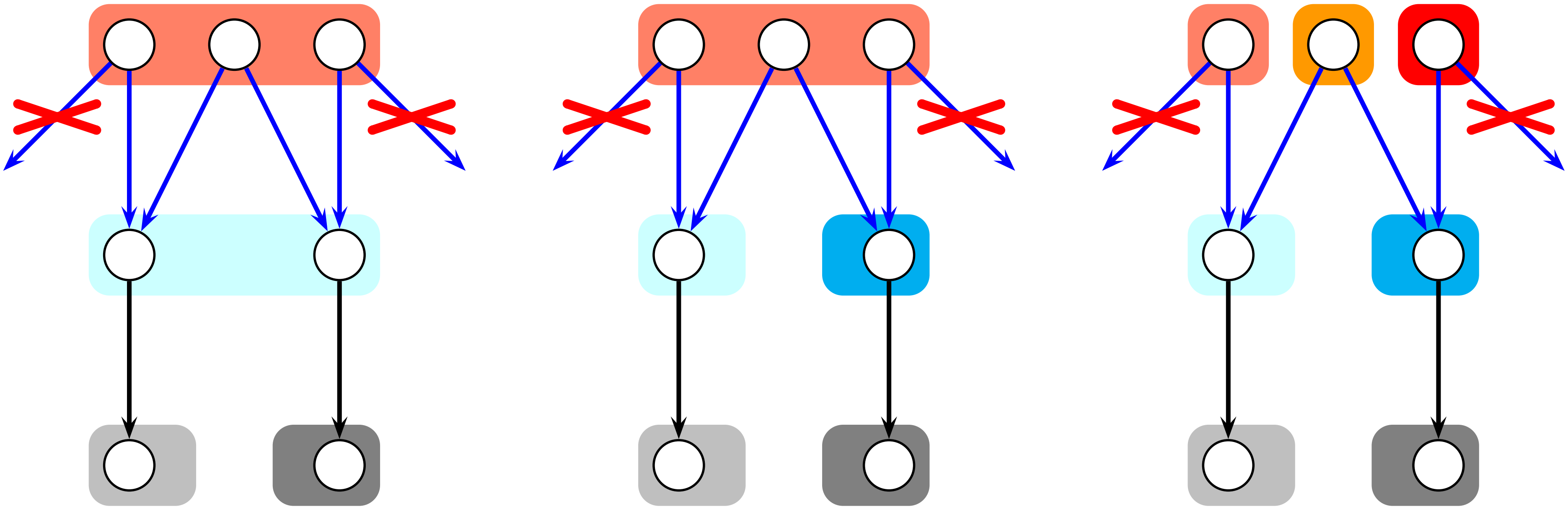
10 Bisimilarity



- Weakest relation that satisfies:
  - $q_1 \sim q_2 \quad \Rightarrow \quad \text{label}(q_1) = \text{label}(q_2)$
  - $q_1 \sim q_2 \wedge (q_1, a, q_1') \in \Delta_1 \quad \Rightarrow \quad \exists q_2' : q_1' \sim q_2' \wedge (q_2, a, q_2') \in \Delta_2$
  - $q_1 \sim q_2 \wedge (q_2, a, q_2') \in \Delta_2 \quad \Rightarrow \quad \exists q_1' : q_1' \sim q_2' \wedge (q_1, a, q_1') \in \Delta_1$
  - $q_1^{\text{init}} \sim q_2^{\text{init}}$, or $\forall q_1 \in \text{Initials}_1 : \exists q_2 \in \text{Initials}_2 : q_1 \sim q_2$, and ...
- Fundamental relation in concurrency theory
- Often much better "strong" equivalence than isomorphism
  - unifies states that only differ on x, if the next thing on x is $x := 0$

11 Minimal Bisimilar Graph

- Nondeterministic version of DFA minimization
- Remove what is not reachable from initial states
- Fuse bisimilar states
- Applications
  - smaller graph for further processing
  - bisimilarity test
- **Problem** due to nondeterminism: **three-way splitting**

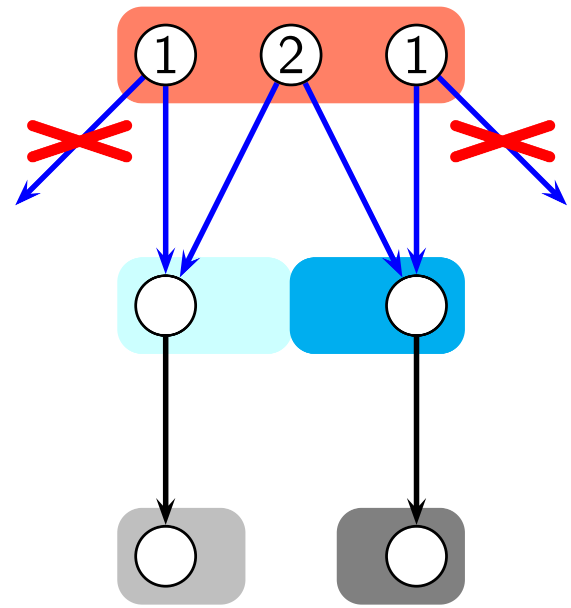


12 Paige–Tarjan $O(m \log n)$ RCP Algorithm [1987]

- **Relational coarsest partition problem**:
  Nondeterministic graph, only one label ($\alpha = 1$)

- **Idea: Compound blocks**, **"q-B-counters"**, and **"q-counters"**

- Compound block $\approx$
  union of blocks that has been used for splitting
  - let $\widehat{B}$ be the compound block that covers block B

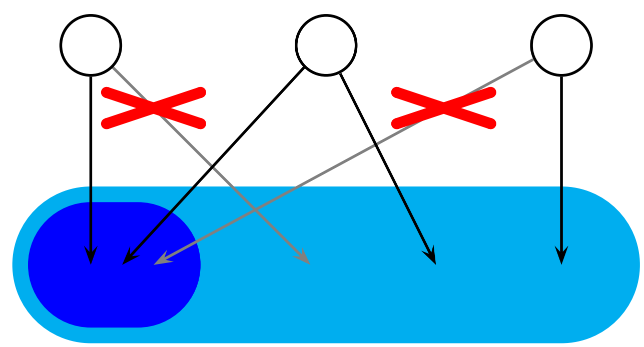


- Biggest block in a compound block
  need not be used in further splitting
  $\Rightarrow$ each state is used in a splitter at most $\log_2 n$ times

- **Problem:** How to implement three-way splitting?

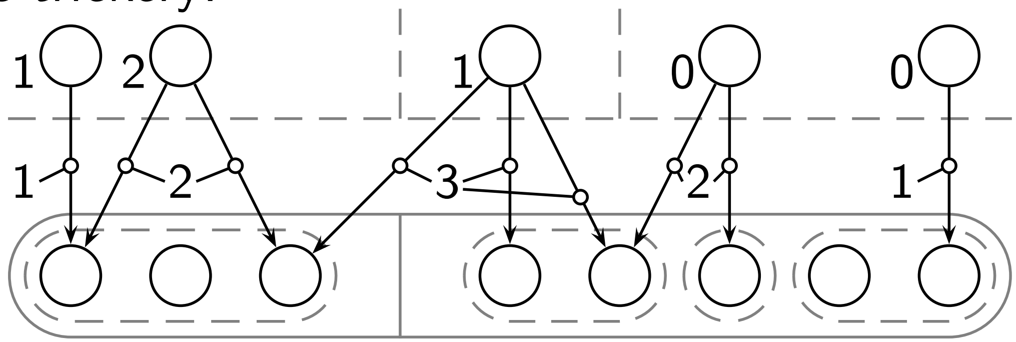
$\Rightarrow$ Maintain, for each q and B, the
number of transitions from q to B

- Each B' has (at most) three kinds of sub-blocks
  - **left block**: $\#(q, B) > 0 = \#(q, \widehat{B} \setminus B)$
  - **middle block**: $\#(q, B) > 0 < \#(q, \widehat{B} \setminus B)$
  - **right block**: $\#(q, B) = 0$



13 Extension to Bisimulation [2009, 2010]

- (Still RCP) cannot afford to represent counters that store 0
  $\Rightarrow$ data structure trickery!



- Extend q-B-counters to q-a-B-counters

- Use cords (and the year 2010 DFA tricks)

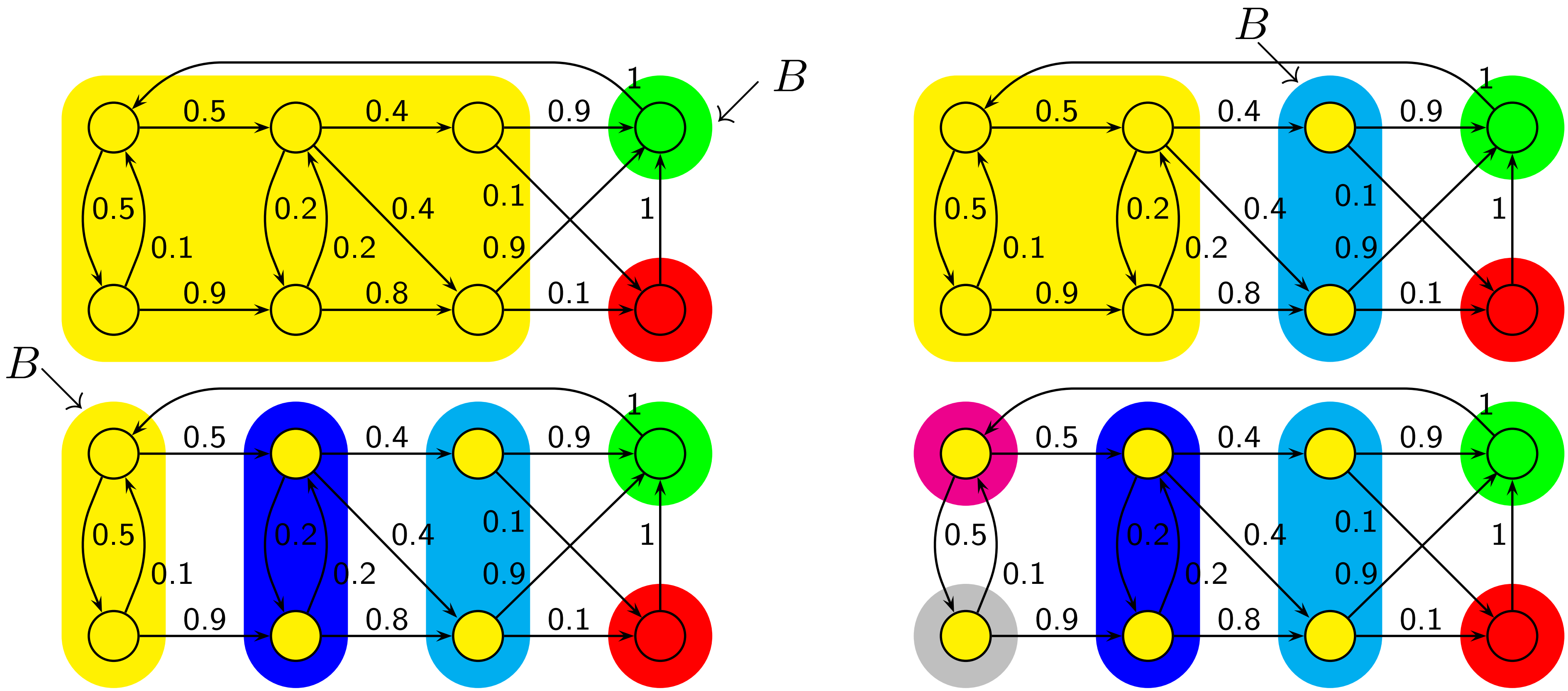
- Initialization: sort each cord according to start states
  - $O(m_a \log m_a)$, where $m_a \leq n^2$

- Splitting
  - distinguish left blocks from middle blocks by counter values
  - extract left blocks like with DFAs
  - update old and create new counter when extracting middle block
  - main loop and splitting of cords are like with DFAs

- Tricky details here and there and in the correctness proof, but it works!

14 Markov Chains and Markov Decision Processes

- These ideas have also been applied to state lumping of Markov Chains by Valmari and Franceschinis [2010]
  - would be a story of its own
  - finds use for the elegant but mostly useless majority candidate algorithm



- Everything has been put together in a program for minimizing Markov Decision Processes

Part III:

Conclusions

15 Conclusions

- $O(m \log n)$ time DFA, bisimulation, and MDP minimization are possible
  - such an algorithm for DFAs was found amazingly late, 1971 $\leftrightarrow$ 2008
  - the other two problems are strictly more general
  - $O(m \log n)$ time Markov chain lumping was solved in 2003, but our results simplify it
- The breakthrough was the use of another partition, this time of transitions
- In the end the programs are relatively simple
  - we got rid of some data structures in earlier algorithms
- However, many tricky ideas had to be fine-tuned to make it all work
  - algorithms, correctness proofs, and programs
  - details **are** important for obtaining the promised performance!
- Hidden theme: representing a mapping where some result value is far more common than others
  - sparse mapping
  - avoid explicitly representing that value

Part IV:

Thank you for attention.

Questions?