Positional Determinacy of Games with Infinitely Many Priorities*

Erich Grädel
Aachen University

Igor Walukiewicz
LaBRI

Abstract

We study two-player games of infinite duration that are played on finite or infinite game graphs. A winning strategy for such a game is positional *if it only depends on the current position, and not on the history of the play. A game is* positionally determined, *if from each position, one of the two players has a positional winning strategy.*

The theory of such games is well studied for winning conditions that are defined in terms of a mapping that assigns to each position a priority *from a finite set C. Specifically, in Muller games the winner of a play is determined by the set of those priorities that have been seen infinitely often; an important special case are parity games where the least (or greatest) priority occurring infinitely often determines the winner. It it is well-known that parity games are positionally determined whereas Muller games are determined via finite-memory strategies.*

In this paper, we extend this theory to the case of games with infinitely many priorities. Such games arise in several application areas, for instance in pushdown games with winning conditions depending on stack contents.

For parity games there are several generalisations to the case of infinitely many priorities. While max-parity games over ω or min-parity games over larger ordinals than ω require strategies with infinite memory, we can prove that min-parity games with priorities in ω are positionally determined. Indeed, it turns out that the min-parity condition over ω is the only infinitary Muller condition that guarantees positional determinacy on all game graphs.

The discrepancy between (min-)parity games and max-parity games has an intersting application to an old problem posed by Jonathan Swift.

*This research has been partially supported by the European Research Training Network "Games and Automata for Synthesis and Validation" (GAMES)

1 Motivation

The problem of computing winning positions and winning strategies in infinite games has numerous applications in computing, most notably for the synthesis and verification of reactive controllers and for the model-checking of CTL and other logics. Of special importance are parity games, due to several reasons [15, 16]. Many complicated games, such as games modeling reactive systems, with winning conditions specified in some temporal logic, can be reduced to parity games (over larger game graphs) [1, 10]. Further, parity games arise as the model checking games for fixed point logics [6]. In particular the model checking problem for the modal μ-calculus can be solved in polynomial time if, and only if, winning regions for parity games can be decided in polynomial time. Finally, parity games are positionally determined [5, 13]. This is a game theoretical result of fundamental importance in all of the constructions mentioned above.

In most traditional applications of games in computer science, the arena, and therefore also the number of priorities appearing in the winning condition, are finite. However, due to applications in the verification of infinite-state systems and other areas where infinite structures become increasingly important, there is now considerable interest on games with infinite arenas that admit some kind of finite presentation. The best studied class of such games are *pushdown games* [9, 17], where the arena is the configuration graph of a pushdown automaton. Other relevant classes of infinite, but finitely presented (game) graphs, include prefix-recognizable graphs, HR- and VR-equational graphs, graphs in the Caucal hierarchy, and so on. On all these classes of graphs, monadic second-order logic can be evaluated effectively, which implies, for instance, that winning regions of parity games (with a finite number of priorities) are decidable. However, once we move to infinite game graphs, winning conditions depending on infinitely many priorities arise naturally. In pushdown games, stack height and stack contents are natural parameters that may take infinitely many values. In [3], Cachat, Duparc, and Thomas study pushdown games with an infinity condition on stack contents, and Bouquet, Serre, and Walukiewicz [2] consider more general winning conditions for pushdown games, combining a parity condition on the states of the underlying pushdown automaton with an unboudedness condition on stack heights.

To establish positional determinacy or finite-memory determinacy is a fundamental first step in the analysis of an infinite game, and is also crucial for the algorithmic construction of winning strategies. In the case of parity games with finitely many priorities the positional determinacy immediately implies that winning regions can be decided in NP $\cap$ Co-NP; with a little more effort it follows that the problem is in fact in UP $\cap$ Co-UP [8]. Further, although it is not known yet whether parity games can be solved in polynomial time, the known approaches towards an efficient algorithmic solution make use of positional determinacy.

In general, the positional determinacy of a game may depend on specific properties of the arena and on the winning condition. For instance, the previously known results on pushdown games make use of the fact that the arena is a pushdown graph. However, this is not always the case. As we show here, there are interesting cases, where positional determinacy is a consequence of the winning condition only. Most notably this is the case for the parity condition (little endian style) on ω. In fact, we completely classify the infinitary Muller conditions with this property and show that they are equivalent to a parity condition. This result gives a general, arena-independent explanation of the positional determinacy of certain pushdown games. We hope and expect that it will be the first step for algorithmic solutions for other infinite games with finitely presented arenas.

For the reader interested in literature and history, there is another motivation for this work. Quite a while ago Jonathan Swift reported on serious disagreements between the Big-Endians and the Little-Endians, two factions within the society of Lilliput, on a matter of fundamental importance: Big-Endians break their eggs at the larger end ("the primitive way"), while Little-Endians break it at the smaller end, and doing so support the view of the Emperor who had *"published an edict, commanding all his subjects, upon great penaltys, to break the smaller end of their eggs."* The issue was very serious indeed because the Big-Endians *"so highly resented this law, that our historys tell us there have been six rebellions raised on that account; wherein one Emperor lost his life, and another his crown."* [14, Part I, Chapter IV].

The fights between Big-Endians and Little-Endians continued and became even more vigorous when, with the advent of modern computing machines, new issues arose: should numbers be stored most or least significant byte first? Big-endians insist that the "big end" is stored first because this is the way God intended integers to be represented, most important part first. Those in the big-endian camp include the Java VM virtual computer, the Java binary file format, the IBM 360 and and most mainframes. Little-Endians assert that putting the low-order part first is more natural because when you do arithmetic, you start at the least significant part and work toward the most significant part. In the little-endian camp are the Intel 8080, 8086, 80286, Pentium and follow ons and the MOS 6502.

A yet more serious battle between the two factions arose when the people of Lilliput abandonded their traditional games like chess, soccer, Russian roulette, etc. and started to play *parity games*. The little-endian rules for playing parity games assert, quite reasonably, that the smallest priority occurring infinitely often in a play determines the winner. The Big-Endians on the other side insist, strangely enough, that it is the largest priority seen infinitely often that decides who wins the game.

The goal of this paper is to settle this issue once and for all and to prove that the Little-Endians are right. One method to do so would of course be to show the existence of an egg that can only be broken at the smaller end. However, it can be proved that the size of such an egg would necessarily have to be infinite; hence at least a constructive proof seems out of reach for present methodology. Instead we resolve this issue by way of parity games with infinitely many priorities. Whatever arguments there are between Little-Endians and Big-Endians, both factions agree that the right way to win parity games is via *positional strategies* or, if these are unavailable, *finite-memory strategies*, because the use of infinite memory *"is just not fair"*. We prove here that any parity game with priorities in ω, defined in the little-endian style, can indeed be won be means of a positional winning strategy. The Big-Endians, however, miserably fail on their variant of parity games, as soon as they admit infinitely many priorities.

2 Introduction

2.1 Games and strategies

We study two-player games of infinite duration on arenas with infinitely many priorities. An *arena* $\mathcal{G} = (V, V_0, V_1, E, C, \Omega)$, consists of a directed graph (V, E), with a partioning $V = V_0 \cup V_1$ of the nodes into positions of Player 0 and positions of Player 1. The possible moves are described by the edge relation $E \subseteq V \times V$. The function $\Omega : V \to C$ assigns to every position a *priority*. Occasionally we encode the priority function by the collection $(P_c)_{c \in C}$ of unary predicates where $P_c = \{v \in V : \Omega(v) = c\}$.

In case $(v, w) \in E$ we call w a successor of v and we denote the set of all successors of v by vE. To avoid tedious case distinctions, we assume that every position has at least one successor. A *play* of $\mathcal{G}$ is an infinite path $v_0 v_1 \ldots$ formed by the two players starting from a given initial position v_0. Whenever the current position v_n belongs to V_σ, then Player σ chooses a successor $v_{n+1} \in v_n E$. A *game* is given by an arena and a *winning condition* that describes which of the plays $v_0 v_1 \ldots$ are won by Player 0, in terms of the sequence $\Omega(v_0)\Omega(v_1)\ldots$ of priorities appearing in the play. Thus, a winning condition is a set $W \subseteq C^\omega$ of infinite sequences of priorities.

A *(deterministic) strategy* for Player σ is a partial function $f : V^* V_\sigma \to V$ that assigns to finite paths through $\mathcal{G}$ ending in a position $v \in V_\sigma$ a successor $w \in vE$. A play $v_0 v_1 \cdots \in V^\omega$ is *consistent* with f if, for each initial segment $v_0 \ldots v_i$ with $v_i \in V_\sigma$, we have that $v_{i+1} = f(v_0 \ldots v_i)$. We say that such a strategy f is winning from position v_0 if every play that starts at v_0 and that is consistent with f is won by Player σ. The *winning region* of Player σ, denoted W_σ, is the set of positions from which Player σ has a winning strategy. A game $\mathcal{G}$ is *determined* if $W_0 \cup W_1 = V$, i.e., if from each position one of the two players has a winning strategy.

Winning strategies can be rather complicated. Of special interest are simple strategies, in particular *finite memory strategies* and *positional strategies*. While positional strategies only depend on the current position, not on the history of the play, finite memory strategies have access to bounded amount of information on the past. Finite memory strategies can be defined as strategies that are realisable by finite automata.

More formally, a strategy with memory M for Player σ is given by a triple (m_0, U, F) with initial memory state $m_0 \in M$, a memory update function $U : M \times V \to M$ and a next-move function $F : V_\sigma \times M \to V$. Initially, the memory is in state m_0 and after the play has gone through the sequence $v_0 v_1 \ldots v_m$ the memory state is $u(v_0 \ldots v_m)$, defined inductively by $u(v_0 \ldots v_m v_{m+1}) = U(u(v_0 \ldots v_m), v_{m+1})$. In case $v_m \in V_\sigma$, the next move from $v_1 \ldots v_m$, according to the strategy, leads to $F(v_m, u(v_0 \ldots, v_m))$. In case $M = \{m_0\}$, the strategy is positional; it can be described by a function $F : V_\sigma \to V$.

Definition 2.1. A game is *positionally determined*, if it is determined, and each player has a positional winning strategy on his winning region.

Clearly, if the arena is a forest, then all strategies are positional, so the game is positionally determined if, and only if, it is determined.

2.2 Games with infinitely many priorities

In most papers on finite-memory determinacy or positional determinacy of infinite games it is assumed that the range of the priority function is finite, and the winning condition is defined by a formula on infinite paths (from S1S or LTL, say) referring to the predicates $(P_c)_{c \in C}$, or by an automata-theoretic condition like a Muller, Rabin, Streett, or parity (Mostowski) condition (see e.g. [7, 4, 18]). In Muller games the winner of a play depends only the set of priorities that have been seen infinitely often; it has been proved by Gurevich and Harrington that Muller games are determined via finite-memory strategies. An important special case of Muller games are *parity games* where the least (or greatest) priority occurring infinitely often determines the winner.

Here we will extend the study of positional determinacy to games with infinitely many priorities. Specifically we are interested in games with priority assignments $\Omega : V \to \omega$. Besides the theoretical interest (and the relevance for the problem mentioned in Section 1), such games arise in several areas. The winning conditions pushdown games are specific instances of abstract winning conditions in games with infinitely many priorities. It is interesting to study these games in a general setting, and to isolate the winning conditions that lead to positional determinacy on arbitrary arenas, not just on specific ones like pushdown games.

Based on priority assigments $\Omega : V \to \omega$ we will first consider the following classes of games.

Infinity games are games where Player 0 wins those infinite plays in which no priority appears infinitely often.

Parity games are games where Player 0 wins the infinite plays where the least priority seen infinitely often is even, or where all priorities appear only finitely often.

Max-parity games are games where Player 0 wins if the maximal priority occurring infinitely often is even, or does not exist.

Note that we have chosen the definitions so that in case no priority appears infinitely often, the winner is always Player 0. It is clear that these games are determined, because the winning consitions are Borel sets, and it is a celebrated result due to Martin [11] that all Borel games are determined. To be more precise, the infinity and parity winning conditions are on the $\mathbf{\Pi}^0_3$-level of the Borel hierarchy, and the max-parity condition is on the $\mathbf{\Delta}^0_4$-level.

For games with only finitely many priorities, min-parity and max-parity winning conditions can be (and are) used interchangeably. This is not the case when we have infintely many priorities.

Proposition 2.2. *Max-parity games with infinitely many priorities in general do not admit finite memory winning strategies.*

Proof. Consider the max-parity game with positions $V_0 = \{0\}$ and $V_1 = \{2n+1 : n \in \mathbb{N}\}$ (where the name of a position is also its priority), such that Player 0 can move from 0 to any position $2n+1$ and Player 1 can move back from $2n+1$ to 0. Clearly Player 0 has a winning strategy from each position but no winning stategy with finite memory. $\square$

However, we will see that min-parity games with priorities in ω are positionally determined.

2.3 Strategy forests

Let f be a strategy for Player σ in the game $\mathcal{G} = (V, V_0, V_1, E, \Omega)$. For any initial position v_0 of the game, we can associate with f the *strategy tree* $\mathcal{T}_f$, the tree of all plays that start at v_0 and that are consistent with f. In the obvious way, $\mathcal{T}_f$ can itself be considered as a game graph, with a canonical homomorphism $h : \mathcal{T}_f \to \mathcal{G}$. For every position v of $\mathcal{G}$, we call the nodes $s \in h^{-1}(v)$ the *occurrences of* v in $\mathcal{T}_f$. If f is a deterministic strategy, then every occurence of a node $v \in V_\sigma$ has precisely one successor in $\mathcal{T}_f$, whereas every occurrence of a node $v \in V_1$ has precisely as many successors in $\mathcal{T}_f$ as v has in $\mathcal{G}$. If f is a winning strategy from v_0, then every path through $\mathcal{T}_f$ is a winning play for Player σ. If we consider a set of initial positions (like the entire winning region W_σ) then $\mathcal{T}_f$ is a *strategy forest* with a separate tree for each initial position.

By moving from game graphs to strategy forests we can eliminate the interaction between the two players and thus simplify the analysis. We already know that the games that we study are determined. To prove positional determinacy we proceed as follows.

We take a winning strategy and define a collection of well-founded pre-orders on its strategy forest. We then define a positional winning strategy for the original game, by copying for each position in the winning region, the winning stategy from a minimal occurrence of the position in the strategy tree. We then show that the resulting positional strategy is indeed winning.

To simplify the exposition we first discuss infinity games. Note that these can be seen as a special case of parity games. Indeed, if we change the priorities of an infinity game $\mathcal{G}$ so that all priorities become odd, by setting $\Omega'(v) := 2\Omega(v) + 1$, and replace the infinity winning condition by the parity condition, then the resulting parity game $\mathcal{G}'$ is equivalent to $\mathcal{G}$.

3 Infinity Games

We start with some remarks on arbitrary transition systems. We will then apply them to strategy forests.

Given any transition system $\mathcal{K} = (S, E, P)$ with set of states S, transition relation E and atomic proposition P, we assign to each state s an ordinal $\alpha(s)$ or ∞. Informally, $\alpha(s)$ tells us how often a path from s can hit P. To define this precisely, we proceed inductively. For any ordinal α, let X^α be the set of all $s \in S$ such that whenever a path from s hits a node $t \in P$, then all successors of t belong to $\bigcup_{\beta<\alpha} X^\beta$. Finally, let $\alpha(s) = \min\{\alpha : s \in X^\alpha\}$. If s is not contained in any X^α, the we put $\alpha(s) = \infty$.

Remark. We can equivalently define $\alpha(s)$ in terms of closure ordinals in the modal μ-calculus. Consider the formula $\mu X.\varphi(X)$, with $\varphi(X) := \nu Y.(P \rightarrow \Box X) \wedge \Box Y$. It expresses that on all paths, there are only finitely many occurrences of P. We define the stage X^α of the least fixed point induction via $\varphi(X)$ by $X^\alpha = \{s : \mathcal{K}, s \models \varphi(X^{<\alpha})\}$ where $X^{<\alpha} := \bigcup_{\beta<\alpha} X^\beta$. It is easily seen that this coincides with the definition given above.

Remark. Note that although $\mu X.\varphi(X)$ expresses that on every path there are only finitely many occurrences of P the closure ordinals need not be finite. For a simple example, consider an infinite path $v_0 v_1 v_2 \ldots$ without occurences of P and attach to each v_n another infinite path on which P is seen precisely n times. On all these attached paths, $\alpha(s)$ will take only finite values, but $\alpha(v_n) = \omega$ for all n.

The following lemma is a direct consequence of the definitions.

Lemma 3.1. *Suppose that every path in $\mathcal{K}$ contains only finitely many occurrences of P (i.e., $\mathcal{K}, s \models \mu X.\varphi(X)$ for all s). Then $\alpha(s) \geq \alpha(t)$ for all edges (s,t) of $\mathcal{K}$, and the inequality is strict for $s \in P$.*

Assume next that we have a transition system $\mathcal{K} = (S, E, P_0, P_1, P_2, \ldots)$ with infinitely many atomic propositions P_n. Proceeding as above for P_n instead of P, we obtain, for each n, a function α_n mapping states $s \in S$ to ordinals. The *signature* of s is $\mathrm{sig}(s) := \langle \alpha_n(s) : n < \omega \rangle$; we compare signatures lexicographically. Further, for each $n < \omega$, let $\mathrm{sig}_n(s) = \langle \alpha_0(s), \ldots, \alpha_n(s) \rangle$ and let $s <_n t$ denote that $\mathrm{sig}_n(s) < \mathrm{sig}_n(t)$ (i.e., that the signature of s is strictly smaller than the signature of t on the first $n+1$ positions). Similarly, let $s \leq_n t$ denote that $\mathrm{sig}_n(s) \leq \mathrm{sig}_n(t)$.

Note that $s <_n t$ implies $s <_{n+1} t$ and that each pre-order $<_n$ is well-founded (i.e., all descending chains are finite). On the other side, when we have infinitely many P_n, the lexicographic order of unrestricted signatures admits infinite descending chains.

Theorem 3.2. *Infinity games are positionally determined*

Proof. Let W_0 and W_1 be the winning regions of the two players for the infinity game on the arena $\mathcal{G}$. Note that the situation for the two players is not symmetric, so we have to consider them separately.

Let f be any winning strategy for Player 0 on W_0. If f is positional then we are done. Otherwise, we consider the strategy forest $\mathcal{T}_f$ and the canonical homomorphism $h : \mathcal{T}_f \rightarrow \mathcal{G}$. On $\mathcal{T}_f$ every path is winning for Player 0, and thus hits each P_n only finitely often. Hence the functions $\alpha_n(s)$ are defined and satisfy the properties of Lemma 3.1.

We define a positional strategy f' for Player 0 as follows. Select a function $s : W_0 \rightarrow \mathcal{T}_f$ that associates with each vertex $v \in W_0$ of priority n a $<_n$-minimal element $s(v) \in h^{-1}(v)$ (i.e., a $<_n$-minimal occurrence of v in the strategy forest). If v, and hence also $s(v)$, is a node of Player 0, then there is a unique successor t of $s(v)$ in $\mathcal{T}_f$; define $f'(v) := h(t)$. Further, we define values of α_n (and hence sig_n) on W_0 by $\alpha_n(v) := \alpha_n(s(v))$.

We claim that f' is winning from each node $v_0 \in W_0$. Otherwise there exists a play $v_0 v_1 v_2 \ldots$ that is consistent with f' and winning for Player 1. Let n be the least priority seen infinitely often on this play; take a suffix of the play on which priorities smaller than n do no longer occur. We claim that the values of sig_n never increase on this suffix.

To see this, consider a move from v to w in this suffix and the corresponding moves in $\mathcal{T}_f$ from $s := s(v)$ to t with $h(t) = w$. By construction, and since v and w have priorities $\geq n$, we have $\mathrm{sig}_n(v) = \mathrm{sig}_n(s)$ and $\mathrm{sig}_n(w) \leq \mathrm{sig}_n(t)$. By Lemma 3.1, we have $\alpha_m(s) \geq \alpha_m(t)$ for all m and the inequality is strict if m is the priority of s. It follows that

$$\mathrm{sig}_n(v) = \mathrm{sig}_n(s) \geq \mathrm{sig}_n(t) \geq \mathrm{sig}_n(w)$$

and $\mathrm{sig}_n(v) > \mathrm{sig}_n(w)$ in case $v \in P_n$. Since there are infinitely many nodes $v_{i_1}, v_{i_2}, \ldots$ of priority n in the suffix, we obtain an infinite descending chain

$$\mathrm{sig}_n(v_{i_1}) > \mathrm{sig}_n(v_{i_2}) > \cdots$$

which is impossible. Hence f' is indeed a winning strategy.

We now consider the case of Player 1. Let g be a strategy for Player 1 on W_1, with strategy forest $\mathcal{T}_g$ and canonical homomorphism $h : \mathcal{T}_g \rightarrow \mathcal{G}$. We define the 0-*ancestor* of a node $s \in \mathcal{T}_g$ to be the closest ancestor of s that has priority 0. Note that 0-ancestors may be undefined. More generally, the m-*ancestor* of s is the closest ancestor of priority m, provided it lies between s and the j-ancestor of s, for all $j < m$ for which j-ancestor is defined. We can thus associate with every node s of priority

m an $(m+1)$-tuple $a(s) = \langle a_0(s), \ldots, a_m(s) \rangle \in (\mathcal{T}_g \cup \{\bot\})^{m+1}$ of ancestors, where $a_i(s) = \bot$ means that i-th ancestor of s is not defined. Observe that $a_m(s) = s$ as s is an ancestor of itself.

We fix a well-order $<$ on $\mathcal{T}_g \cup \{\bot\}$ (with maximal element $\bot$) and we compare tuples of ancestors via the lexicographical order that is induced by $<$. We can then associate with every $v \in W_1$ of priority m the node $s(v) \in h^{-1}(v)$ with minimal tuple of ancestors. Note that $s(v)$ is well-defined, because every position $v \in W_1$ has at least one occurrence $s \in \mathcal{T}_g$, and $a_m(s) = s$ if m has priority m (so at least one ancestor is defined). We extend the ancestor function to W_1 by setting $a(v) := a(s(v))$; this assigns to every node in W_1 a tuple of ancestors in $\mathcal{T}_g$. To define the positional strategy g', we select for any $v \in V_1 \cap W_1$ the unique successor t of $s(v)$ and set $g'(v) := h(t)$.

We claim that this strategy is winning for Player 1. Suppose conversely that there is a losing play respecting the strategy. Then no priority appears infinitely often on this play. Consider the suffix of the play after the last appearance of priority 0. Let us look at the 0-ancestors of the positions in this suffix. These ancestors can only get smaller as the play proceeds. Indeed a move from v to w in such a play corresponds to a move from $s(v)$ to t in $\mathcal{T}_g$ with $h(t) = w$. Since w does not have priority 0, $a_0(s) = a_0(t)$ and therefore $a_0(v) = a_0(s) = a_0(t) \geq a_0(w)$. This means that from some moment on all positions in the play will have the same 0-ancestor. Consider the suffix of the play consisting only of these vertices. Next do the same with priority 1. We find a position after which the 1-ancestor stabilises. Observe that it is a descendant of the 0-ancestor and that there is no occurrence of priority 0 on the path between the two. Proceeding in this way we construct a path in the strategy tree $\mathcal{T}_g$ on which no priority appears infinitely often. But this is impossible, since g was a winning strategy for Player 1. $\square$

4 Parity games

For parity games we proceed quite similarly, but we have to consider more complicated orderings on the strategy forests.

Consider a transition system $\mathcal{K} = (S, E, P, Q)$ with two atomic propositions P and Q. We assign to each state s an ordinal $\beta(s)$ or ∞ which, informally, tells us how often a path from s can hit P before seeing Q. Let X^0 be the set of all s such that all paths from s hit Q before hitting P, and for $\beta > 0$, let X^β be the set of all s such that whenever a path from s hits a node $t \in P$, then all successors of t belong to $X^{<\beta}$. Finally, let $\beta(s) = \min\{\beta : s \in X^\beta\}$.

Again, we have an equivalent definition in terms of the modal μ-calculus. This time, consider the formula $\mu X.\varphi(X)$, with $\varphi(X) := \nu Y.(\neg P \vee \Box X) \wedge (Q \vee \Box Y)$. It expresses that on all paths, there are only finitely many occurrences of P before seeing Q. Then $\beta(s)$ is the stage at which the least fixed point induction defined by $\varphi(X)$ becomes true at node s.

Lemma 4.1. *Suppose that every path in $\mathcal{K}$ contains only finitely many occurrences of P before hitting Q. Then $\beta(s) \geq \beta(t)$ for all edges (s,t) of $\mathcal{K}$, and the inequality is strict for $s \in P$.*

For infinity games we have defined ordinals $\alpha_n(s)$ telling us how often a path from s can see priority n, independently for each n. Now we need different bounds β_n which, informally, describe how often a path can hit the odd priority n before seeing a smaller one.

Let $\mathcal{G}$ be a parity game, and let $\mathcal{T}_f = (S, E, P_0, P_1, P_2, \ldots)$ be the strategy forest of a winning strategy f for Player 0. Note that for every odd priority n, each path through $\mathcal{T}_f$ sees only finitely many occurrences of n before seeing a priority $< n$. Hence, proceeding as above for $P := P_n$ and $Q := \bigcup_{m<n} P_m$ we obtain, for each odd n, a function β_n mapping nodes $s \in \mathcal{T}_f$ to ordinals. The *0-signatures* of s are $\mathrm{sig}^0_n(s) := \langle \beta_1(s), \beta_3(s) \ldots, \beta_{n'}(s) \rangle$, where $n' = n$ for odd n and $n' = n-1$ for even n; let $s <^0_n t$ denote that $\mathrm{sig}^0_n(s) < \mathrm{sig}^0_n(t)$. Further, $s \leq^0_n t$ means that $\mathrm{sig}^0_n(s) \leq \mathrm{sig}^0_n(t)$.

For strategy forests $\mathcal{T}_g$ of winning strategies of Player 1, we proceed dually, associating with every node s ordinals $\beta_n(s)$, for even n. We then define 1-signatures $\mathrm{sig}^1_n(s) = \langle \beta_0(s), \ldots, \beta_{n'}(s) \rangle$ (where n' is the largest even number not exceeding n) and the corresponding signature orderings $<^1_n$.

Again, we immediatley see that $s <^i_n t$ implies $s <^i_{n+1} t$ and that each $\leq^i_n$ is a well-founded. Further, these orderings have very useful properties on strategy forests.

Lemma 4.2. *Let $\mathcal{T}_f$ be the strategy forest associated with a winning strategy for Player 0 for a parity game. Then $t \leq^0_{\Omega(s)} s$ for all edges (s,t) of $\mathcal{T}_f$*

and the inequality is strict if $\Omega(s)$ is odd. In a strategy forest $\mathcal{T}_g$ of Player 1, we have $t \leq^1_{\Omega(s)} s$ for all edges (s,t) and the inequality is strict if $\Omega(s)$ is even.

Proof. If (s,t) is a edge in $\mathcal{T}_f$, then by Lemma 4.1, $\beta_m(t) \leq \beta_m(s)$ for $m \leq \Omega(s)$, and, if $n = \Omega(s)$ is odd, and $\beta_n(t) < \beta_n(s)$. Similarly for $\mathcal{T}_g$. $\square$

Theorem 4.3. *Parity games with priorities in ω are positionally determined.*

Proof. The proof for Player 0 is precisely the same as for infinity games, using 0-signatures and the associated orderings $<^0_n$.

For Player 1 we combine the approach for infinity games based on ancestors in the strategy forest with comparisons based on 1-signatures. As in the proof of Theorem 3.2 we associate with every node s of priority m in the strategy tree $\mathcal{T}_g$ the $(m+1)$-tuple $a(s) = \langle a_0(s), \dots, a_m(s)\rangle \in (\mathcal{T}_g \cup \{\bot\})^{m+1}$ of ancestors.

Let s, s' be two nodes of $\mathcal{T}_g$ of the same priority m. We write $s \lhd_m s'$ that there is an $i \leq m$ such that $a_i(s)$ and $a_i(s')$ are comparable with respect to $<^1_i$, and $a_i(s) <^1_i a_i(s')$ for the smallest such i. For each m we then select on the nodes of priority m a well-order $\prec_m$ that extends $\lhd_m$.

For any position $v \in W_1$ of priority m we now take the $\prec_m$-minimal occurrence $s(v)$ in $\mathcal{T}_g$ and define ancestors by $a(v) := a(s(v))$. For $v \in V_1 \cap W_1$ we consider the unique successor t of $s(v)$ and set $g'(v) := h(t)$. This defines a positional strategy g' for Player 1 on $\mathcal{G}$.

We claim that this strategy is winning on W_1. Suppose conversely that there is a losing play respecting the strategy. Then either no priority appears infinitely often on this play, or the smallest priority occurring infinitely often is even.

If no priority occurs infinitely often, then we can proceed as in the proof of Theorem 3.2 to show that all ancestors eventually stabilise on the play, and thus obtain an infinite path in $\mathcal{T}_g$ on which no priority appears infinitely often. This is impossible since g is a winning strategy for Player 1.

If the minimal priority p appearing infinitely often is even, then we consider a suffix of the play that contains only priorities $\geq p$. By the same reasoning as in the first case it follows that all q-ancestors, for $q < p$, eventually stabilise on the play. Consider the suffix of the play after this has happened. A move from v to w on this suffix corresponds to a move from $s(v)$ to t in $\mathcal{T}_g$. By definition, the p-ancestor of v is $a_p(v) = a_p(s(v))$ and $a_p(w) \preceq_p a_p(t)$. On $\mathcal{T}_g$ we obviously have $a_p(t) = t$ if $\Omega(t) = p$ and $a_p(t) = a_p(s(v))$ if $\Omega(t) > p$. Now t is a descendant of $a_p(s)$, so by Lemma 4.2 we have $t <^1_p a_p(s)$; for the case that $\Omega(t) = p$ this means that $a_p(t) = t <^1_p a_p(s)$. Since $\prec_p$ extends $<^1_p$ on nodes of priority p, we have that $a_p(w) \preceq_p a_p(v)$ and that the inequality is strict if $\Omega(w) = p$. But on the suffix we have an inifinite sequence of positions with priority p, and hence an infinite $\prec_p$-decreasing chain of p-ancestors, which is impossible. $\square$

Remark: Parity games over larger ordinals. We can also define parity games with a priority function $\Omega : V \to \alpha$ taking values in a larger set of ordinals than ω. Recall that any ordinal can be written in a unique way as a sum $\lambda + n$ where λ is a limit ordinal and $n < \omega$. We call $\lambda + n$ even if n is. The question arises whether the positional determinacy of parity games over ω extends to larger ordinals. However, a tiny modification of the game in Proposition 2.2 shows that this is not the case. Indeed, if we replace in that game priority 0 by ω, and use the (min-)parity winning condition, then Player 0 has a winning strategy from each position but no winning strategy with finite memory. For larger ordinals, a similar construction applies. This proves that parity games over ordinals $\alpha > \omega$ in general do not admit finite memory winning strategies.

Essentially the same construction shows that finite-memory determinacy also fails for some other variants of parity games over ω, such as

- parity games where the priority function is partial (i.e., not all vertices have a priority),
- parity games with priorities on edges rather than vertices.

5 Muller games

Why do parity games and max parity games behave differently? Both are Muller conditions (i.e. they refer only to the set of priorities seen infinitely often) and the question arises which properties of Muller conditions are responsible for positional determinacy or determinacy with finite memory. In this section we assume that the set of priorities is countable. This is reasonable as on each play one can see only a countable number of them.

Definition 5.1. A Muller condition over a set C of priorities is written in the form $(\mathcal{F}_0, \mathcal{F}_1)$ where

$\mathcal{F}_0 \subseteq \mathcal{P}(C)$ and $\mathcal{F}_1 = \mathcal{P}(C) - \mathcal{F}_0$. A play in a game with Muller winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ is won by Player σ if, and only if, the set of priorities seen infinitely in the play belongs to $\mathcal{F}_\sigma$.

For infinity games, we have $\mathcal{F}_0 = \{\emptyset\}$ and $\mathcal{F}_1 = \mathcal{P}(\omega) - \{\emptyset\}$. For parity games,

$$\mathcal{F}_0 = \{X \subseteq \omega : \min(X) \text{ is even}\} \cup \{\emptyset\}$$
$$\mathcal{F}_1 = \{X \subseteq \omega : \min(X) \text{ is odd}\}$$

For max-parity games, we have

$$\mathcal{F}_0 = \{X \subseteq \omega : \text{ if } X \text{ is finite and non-empty, then } \max(X) \text{ is even}\}$$
$$\mathcal{F}_1 = \{X \subseteq \omega : X \text{ is finite, non-empty, and } \max(X) \text{ is odd}\}$$

Definition 5.2. We say that $(\mathcal{F}_0, \mathcal{F}_1)$ *admits positional winning strategies* if *all* games with winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ are positionally determined.

Following McNaughton [12] and Zielonka [18] we say that $\mathcal{F}_\sigma$ has a *strong split* if there exist sets X_0, X_1 with $X_0 \cap X_1 \neq \emptyset$ and $X_0 \cup X_1 \in \mathcal{F}_{1-\sigma}$. Zielonka [18] has shown that a Muller condition over a *finite* set of priorities admits positional winning strategies if, and only if,

(P0) $\mathcal{F}_0$ and $\mathcal{F}_1$ have no *strong splits*.

Remark. A weak split is a pair of *disjoint* sets with $X_0, X_1 \in \mathcal{F}_\sigma$ and $X_0 \cup X_1 \in \mathcal{F}_{1-\sigma}$. Muller conditions over finite sets of priorities may have weak splits and still admit positional winning strategies. The simplest case is when $\mathcal{F}_0$ consists of the set $\{0, 1\}$, but $\{0\}$ and $\{1\}$ belong to $\mathcal{F}_1$.

We want to find a similar characterisation of Muller conditions with positional winning strategies for the case of infinite sets of priorities.

We observe that for infinity games and parity games $\mathcal{F}_0$ and $\mathcal{F}_1$ are closed under unions and non-empty intersections of chains:

(P1) For every infinite descending chain $X_1 \supseteq X_2 \supseteq \ldots$ of elements of $\mathcal{F}_\sigma$ either $\bigcap_{i<\omega} X_i = \emptyset$ or it is an element of $\mathcal{F}_\sigma$.

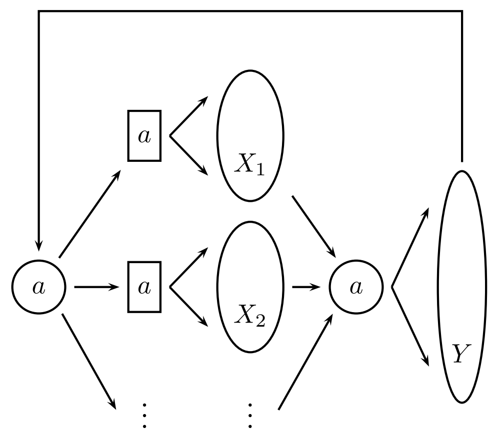
(P2) For every chain $X_1 \subseteq X_2 \subseteq \ldots$ of elements of $\mathcal{F}_\sigma$, also $\bigcup_{i<\omega} X_i$ belongs to $\mathcal{F}_\sigma$.

On the other side, for the max-parity condition, $\mathcal{F}_0$ is not closed under non-empty intersections of chains (take $X_i = \{1\} \cup \{n : n > i\}$) and $\mathcal{F}_1$ is not closed under unions of chains (take $X_i = \{j : j \leq 2i+1\}$). Condition (P1) fails also for min-parity condition for ordinals $\alpha > \omega$. Indeed we have $\mathcal{F}_1 = \{X \subseteq \alpha : \min(X) \text{ is odd}\}$ which is not closed under non-empty intersections of chains (take $X_i = \{\omega\} \cup \{n : 2i+1 \leq n < \omega\}$).

We will show first, that condition (P1) is necessary for the positional determinacy of a Muller condition.

Lemma 5.3. *If there is an infinite sequence $X_1 \supseteq X_2 \supseteq \cdots$ of elements of $\mathcal{F}_{1-\sigma}$ with $\bigcap X_i = Y \neq \emptyset$ and $Y \in \mathcal{F}_\sigma$ then there is game with winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ that Player σ wins, but needs infinite memory to do so.*

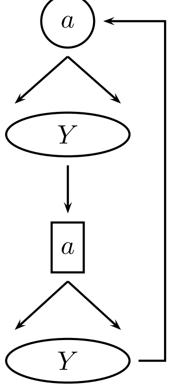
Proof. Consider the following game.



Here a is some arbitrary element of Y. A play in this game is an infinite sequence of subplays; in each subplay Player σ first decides from which X_i the opponent is going to choose next. After Player $(1-\sigma)$ has made his choice, Player σ can select an element from Y.

If Player σ allows her opponent to choose from some X_i infinitely often then Player $(1-\sigma)$ can make all elements of X_i appear infinitely often on the play. This means that in order not to lose, Player σ must permit Player $(1-\sigma)$ to choose from each X_i only finitely often. If she does this then she wins as she can make sure that each element of Y is seen infinitely often thanks to the last part of the each subplay. Thus Player σ has a winning strategy, but none that uses only finite memory. $\square$

To show the necessity of conditions (P0) and (P2) we consider the following game.



Here, Y is a set and $a \in Y$. The arrows to the ovals with Y mean that the player can choose any element of Y. Clearly, if $Y \in \mathcal{F}_\sigma$, then Player σ can win by visiting all elements of Y infinitely often. However, if Player σ plays memoryless then she must select a fixed element b, and her opponent can chose an arbitrary set $X \subseteq Y$ of nodes and make sure that the set of nodes visited infinitely often is $\{a, b\} \cup X$. More generally, if Player σ plays with a finite memory strategy this amounts to selecting a finite set B; Player $(1-\sigma)$ can then win if there exists a set $X \in \mathcal{F}_{1-\sigma}$ with $\{a\} \cup B \subseteq X \subseteq Y$.

Lemma 5.4. *If $\mathcal{F}_{1-\sigma}$ contains a strong split, then there is a game with winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ that is won by Player σ, but not with a positional strategy.*

Proof. Let $X_0 \cup X_1 \in \mathcal{F}_\sigma$ with $X_0, X_1 \in \mathcal{F}_{1-\sigma}$ and $X_0 \cap X_1 \neq \emptyset$. Take the game above with $Y = X_0 \cup X_1$ and $a \in X_0 \cap X_1$. Player σ wins since $Y = X_0 \cup X_1 \in \mathcal{F}_\sigma$. However, she cannot win positionally. Indeed the single element b selected by a positional strategy of Player σ belongs to X_i ($i = 0$ or 1), and Player $1 - \sigma$ can win by making sure that all elements of X_i, and only these, are visited infinitely often. $\square$

Lemma 5.5. *If $\mathcal{F}_{1-\sigma}$ is not closed under unions of chains, then there is a game winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ that Player σ wins, but needs infinite memory to do so.*

Proof. Let $X_1 \subseteq X_2 \subseteq \cdots$ be an infinite ascending chain in $\mathcal{F}_{1-\sigma}$ with $\bigcup_i X_i \in \mathcal{F}_\sigma$. Take the game described above with $Y = \bigcup_i X_i$ and $a \in X_1$. Again Player σ wins since $Y \in \mathcal{F}_\sigma$. But if Player σ plays with finite memory, this amounts to selecting a finite set $B \subseteq Y$ of elements that she visits infinitely often. Since B is finite $B \subseteq X_i$ for some i; hence Player $1 - \sigma$ can make sure that the set of elements visited infinitely often is X_i and wins. $\square$

In the remaining part of the section we will show that any Muller condition satisfying porperties (P0), (P1) and (P2) is isomorphic to a parity condition over an ordinal $\alpha \leq \omega$.

Proposition 5.6. *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller condition satisfying property (P2). For every set $X \in \mathcal{F}_{1-\sigma}$ and every set $Y \in \mathcal{P}(X) \cap \mathcal{F}_\sigma$ there is a $Y' \supseteq Y$ which is maximal in $\mathcal{P}(X) \cap \mathcal{F}_\sigma$.*

Proof. By (P2) the union over any chain $Y_0 \subseteq Y_1 \subseteq \ldots$ in $\mathcal{P}(X) \cap \mathcal{F}_\sigma$ is again contained $\mathcal{P}(X) \cap \mathcal{F}_\sigma$. Hence, by Zorn's Lemma, $\mathcal{P}(X) \cap \mathcal{F}_\sigma$ has maximal elements.

Now let S be the set of elements of $\mathcal{P}(X) \cap \mathcal{F}_\sigma$ that are not below a maximal element. For any $Y \in S$ there exists a set $Y' \supsetneq Y$ which must again belong to S. Further, the union over any chain in S is again contained in S. If S were non-empty, then, again by Zorn's Lemma, S would contain maximal elements, which is absurd. $\square$

Theorem 5.7. *A Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ over a countable set C of priorities admits positional winning strategies if, and only if, it is isomorphic to a parity condition on an ordinal $\alpha \leq \omega$.*

Proof. Without a loss of generality let us assume that the set of all colours belongs to $\mathcal{F}_0$. We construct a finite or infinite sequence of cofinite sets $Z_0 \supseteq Z_1 \supseteq \ldots$ with the following two properties.

1. $Z_{2i} \in \mathcal{F}_0$, $Z_{2i+1} \in \mathcal{F}_1$.
2. If $Y \subseteq Z_{2i}$ and $Y \not\subseteq Z_{2i+1}$, then $Y \in \mathcal{F}_0$. Similarly, if $Y \subseteq Z_{2i+1}$ and $Y \not\subseteq Z_{2i+2}$, then $Y \in \mathcal{F}_1$.

We put $Z_0 = C$, the set of all colours. Suppose that we have constructed $Z_i \in \mathcal{F}_{1-\sigma}$. By Proposition 5.6 we known that there is a maximal element Y in $\mathcal{P}(Z_i) \cap \mathcal{F}_\sigma$. If $Z_i - Y$ were infinite then we could find a descending chain $X_1 \supseteq X_2 \supseteq \ldots$ of sets in $\mathcal{F}_{1-\sigma}$ whose intersection is Y. But this would violate property (P1). Hence Y is cofinite.

Now suppose that there are two maximal elements Y_1, Y_2 in $\mathcal{P}(Z_i) \cap \mathcal{F}_\sigma$. Since Y_2 must also be cofinite, $Y_1 \cap Y_2 \neq \emptyset$. By property (P0), $Y_1 \cup Y_2 \in \mathcal{F}_\sigma$ which is impossible by the maximality of Y_1 and Y_2. This means that we can define Z_{i+1} as the greatest element in $\mathcal{P}(Z_i) \cap \mathcal{F}_\sigma$. By construction we have that the properties (1) and (2) above hold.

By property (P1) we must have that $\bigcap_i Z_i = \emptyset$. Hence for every colour c there is the biggest i such that $c \in Z_i$. Define $\Omega(c) = i$. We have that $X \in \mathcal{F}_0$ iff $\min(\Omega(X))$ is even. $\square$

Remark. In particular, this shows that conditions (P0), (P1), and (P2) actually imply a strengthening of (P0):

(P0') $\mathcal{F}_0$ and $\mathcal{F}_1$ have no (weak or strong) splits.

Theorem 5.7 classifies the Muller conditions that imply positional determinacy on all game graphs. We remark that for Muller games, determinacy itself is an issue that deserves investigation. If either $\mathcal{F}_0$ or $\mathcal{F}_1$ is countable, then the Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ is Borel (on level $\mathbf{\Sigma}_4^0$ or $\mathbf{\Pi}_4^0$), so determinacy follows from Martin's Theorem. In general however, Muller conditions over infinite sets of priorities need not be Borel, and (on the basis of Boolean Prime Ideal Theorem, which is a weak form of the Axiom of Choice) it is not too difficult to construct non-determined Muller games.

6 Further results

6.1 Uncountable sets of priorities

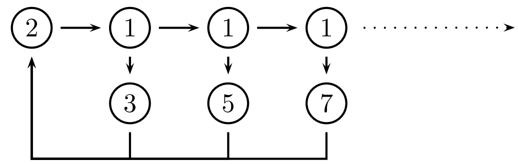
In the previous section we have assumed that the set of all priorities is countable. However, it can be shown that the characterization of the Muller conditions that admit positional winning strategies remains the same for uncountable sets of priorities. As in each play, there appear only countably many priorities, uncountable sets play no role in a Muller condition. Still, the argument is slightly more involved than in Theorem 5.7 because we cannot start the construction from the set C of all priorities. Nevertheless, we can show that if every restriction of the Muller condition to a countable subset of C satisfies (P0), (P1), (P2) then it is equivalent to a parity condition.

6.2 Games of bounded degree

The question arises whether the class of winning conditions that admit positional winning strategies becomes larger if we only consider game graphs of finite or even finite and bounded (out-)degree. In particular this question has been asked for max-parity games and for parity games over larger ordinals than ω, where the counter-example that we have presented has infinite degree. However, it turns out that the classification of positionally determined Muller conditions remains almost the same.

Proposition 6.1. *Max-parity games with infinitely many priorities in general do not admit finite memory strategies, even for solitaire games and even for game graphs of degree two.*

Consider the following game of degree two.



Assuming the max-parity winning condition it is obvious that there is an infinite memory strategy for Player 0 to enforce that the set of priorities seen infinitely often is $\{1, 2\}$, but that any finite memory strategy is losing.

The same construction works for (min-)parity games on ordinals $\alpha > \omega + 1$. Indeed, if we replace priorities 2 and 1 by ω and $\omega + 1$, we obtain a min-parity game that requires an infinite memory winning strategy.

However there is an interesting case where parity games of bounded degree behave differently than games of unbounded degree.

Theorem 6.2. *Parity games of bounded degree with priorities in $\omega + 1$ are positionally determined.*

Informally, a complete characterisation of the Muller conditions that admit positional winning strategies on all game graphs of bounded degree can be described as follows: Either the condition is equivalent to a parity condition over an ordinal $\alpha \leq \omega + 1$, or its Zielonka tree (see [4, 18]) is just a finite path with a weak split at the end. Details will be given in the full version of this paper.

6.3 Finite appearance of priorities

We may also ask whether the characterisation of positionally determined Muller conditions changes if we only consider games where $\Omega^{-1}(c)$ is finite for every priority c. This is not the case. Indeed, the counter-examples for properties (P0) and (P2) are games with this property, and in the counter-example for (P1) we can easily eliminate infinite occurrences of priorities. Consider the figure in the proof of Lemma 5.3. It suffices to omit the sets $X_2, X_3, \ldots$ and redirect, for every $i \geq 2$, each arrow from a to an element $x_i \in X_i$ to the element $x_i \in X_1$.

References

[1] A. ARNOLD, A. VINCENT AND I. WALUKIEWICZ, *Games for synthesis of controllers with partial observation*, Theoretical Computer Science 303 (2003), 7–34.

[2] A. BOUQUET, O. SERRE, AND I. WALUKIEWICZ, *Pushdown games with unboundedness and regular conditions*. Submitted for publication.

[3] T. CACHAT, J. DUPARC, AND W. THOMAS, *Solving pushdown games with a σ_3 winning cndition*, in Computer Science Logic, CSL 2002, Lecture Notes in Computer Science Nr. 2471, Springer 2002, pp. 322–336.

[4] S. DZIEMBOWSKI, M. JURDZIŃSKI, AND I. WALUKIEWICZ, *How much memory is needed to win infinite games?*, in Proceedings of 12th Annual IEEE Symposium on Logic in Computer Science (LICS 97), 1997, pp. 99–110.

[5] A. EMERSON AND C. JUTLA, *Tree automata, mu-calculus and determinacy*, in Proc. 32nd IEEE Symp. on Foundations of Computer Science, 1991, pp. 368–377.

[6] A. EMERSON, C. JUTLA, AND P. SISTLA, *On model checking for the μ-calculus and its fragments*, Theoretical Computer Science 258 (2001), 491–522.

[7] E. GRÄDEL, W. THOMAS, AND T. WILKE, eds., *Automata, Logics, and Infinite Games*, Lecture Notes in Computer Science Nr. 2500, Springer, 2002.

[8] M. JURDZIǸSKI, *Deciding the Winner in Parity Games is in UP $\cap$ Co-UP*, Information Processing Letter 68 (1998), pp. 119-124.

[9] O. KUPFERMAN AND M. VARDI, *An automata-theoretic approach to reasoning about infinite-state systems*, in Proceedings of CAV 2000, Lecture Notes in Computer Science Nr. 1855, Springer, 2000, pp. 36–52.

[10] O. KUPFERMAN AND M. VARDI, *Synthesizing distributed systems*, Proceedings of 16th IEEE Symposium on Logic In Computer Science LICS 2001.

[11] D. MARTIN, *Borel determinacy*, Annals of Mathematics, 102 (1975), pp. 336–371.

[12] R. MCNAUGHTON, *Infinite games played on finite graphs*, Annals of Pure and Applied Logic, 65 (1993), pp. 149–184.

[13] A. MOSTOWSKI, *Games with forbidden positions*, Tech. Rep. Tech. Report 78, University of Gdansk, 1991.

[14] J. SWIFT, *Gulliver's Travels*, George Routledge & Sons, London, 1906. First published in 1726.

[15] W. THOMAS, *On the synthesis of strategies in infinite games*, in Proceedings of STACS 95, Lecture Notes in Computer Science Nr. 900, Springer, 1995, pp. 1–13.

[16] W. THOMAS, *Languages, automata, and logic*, in Handbook of Formal Languages Vol. 3, G. Rozenberg and A. Salomaa, eds., Springer, 1997, pp. 389–455.

[17] I. WALUKIEWICZ, *Pushdown processes: Games and model checking*, Theoretical Computer Science, 164 (2001), pp. 234–263.

[18] W. ZIELONKA, *Infinite games on finitely coloured graphs with applications to automata on infinite trees*, Theoretical Computer Science, 200 (1998), pp. 135–183.