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Survey Paper

Model predictive control: Recent developments and future promise[☆]

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ABSTRACT

This paper recalls a few past achievements in Model Predictive Control, gives an overview of some current developments and suggests a few avenues for future research.

1. Introduction

Model predictive control has had an exceptional history with early intimations in the academic literature coupled with an explosive growth due to its independent adoption by the process industries where it proved to be highly successful in comparison with alternative methods of multivariable control. Its phenomenal success in the process industries is well described in Qin and Badgwell (2003) and was mainly due to its conceptual simplicity and its ability to handle easily and effectively complex systems with hard control constraints and many inputs and outputs. MPC, by applying at state x the first control in a finite sequence of control actions obtained by solving online a constrained, discrete-time, optimal control problem, traded arduous *off-line* computation of a control law $u = \kappa(x)$ for repeated *on-line* solution of a constrained dynamic optimal control problem; this trade-off was perfectly acceptable for control applications for which the optimal control problem could be solved within one sampling interval. The optimal control problem had, for computational reasons, to have a finite horizon so that the resultant controller did not guarantee stability but stability was achieved in most process applications since the horizon of the optimal control problem was normally sufficiently long. Understandably, conditions that ensured stability did not receive attention in the industrial literature but, somewhat more surprisingly, the early academic literature restricted attention to investigating empirically the effect on stability of control and cost horizons and cost parameters. But by 2000, with a belated use of Lyapunov theory, consensus on the form of these conditions was achieved (Mayne, Rawlings, Rao, & Scokaert, 2000); achieving nominal stability of model predictive controlled linear or nonlinear systems with hard state and control constraints, either by adding a terminal cost and constraint or by extension of the horizon of the optimal control problem solved online, was, by then, well understood.

There is a big gulf in the literature on MPC between that dealing with control of deterministic systems and that dealing with control of uncertain systems. While major aspects of nominal MPC were well understood by 2000, the presence of uncertainty, whether in the form of additive disturbances, state estimation error or model error, and the associated topic of robustness against uncertainty, is a major challenge that is still receiving considerable attention. This challenge arises because optimal control in the presence of uncertainty requires feedback so that determination of a control law by solving an open-loop optimal control problem, as is done in nominal MPC, is not optimal. The natural extension of MPC to the control of uncertain systems would be, in the optimal control problem solved online, to optimize over a sequence of control *laws*, as is done in Dynamic Programming, rather than over a sequence of control *actions*. There were some early proposals along these lines, for example on $\mathcal{H}_\infty$ MPC, and the resultant controllers provided insight into the nature of the problem but were conceptual, rather than implementable, because of the complexity of the resultant optimal control problem. As we shall see, this dichotomy has forced the literature on MPC of uncertain systems to differ markedly from that for nominal or deterministic MPC. Because of this, we discuss these two literatures below in separate sections, deterministic MPC and uncertain MPC.

The current century has witnessed an explosion of activity in MPC; during this period at least five books (Camacho & Alba, 2013; Grüne & Pannek, 2011; Kwon & Han, 2005; Maciejowski, 2002; Rawlings & Mayne, 2009) and thousands of papers have

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been published. The field of interest has widened enormously. A comprehensive overview is well nigh impossible; instead we present an admittedly subjective impression of the highlights, an outline of what we consider to be the most significant areas of research together with a few suggestions for future research in this most interesting and fruitful endeavour. Hence, this overview should be read as an introduction to important topics in model predictive control and not necessarily as a survey of the most important papers.

Notation: The Euclidean norm of x is $|x|$; $|x|_Q$ denotes $(x'Qx)^{1/2}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}$ function if $\alpha(0) = 0$ and $\alpha(\cdot)$ is strictly increasing; it is a class $\mathcal{K}_\infty$ function if, in addition, it is unbounded. $\mathbb{I}_{\geq 0}$ denotes the non-negative integers and $\mathbb{I}_{a:b}$ denotes the integers $\{a, a+1, \ldots, b\}$. $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function if, for each t, $s \mapsto \beta(s, t)$ is class $\mathcal{K}$ and if, for each s, $t \mapsto \beta(s, t)$ tends to 0 as $t \to \infty$. A control sequence $\{u(0), u(1), \ldots, u(N-1)\}$ is denoted by $\mathbf{u}$; similarly $\mathbf{x}$ denotes the control sequence $\{x(0), x(1), \ldots, x(N)\}$. The sup norm of a sequence $\mathbf{u}$ is $\|\mathbf{u}\|_\infty$. A function $\rho : Y \to \mathbb{R}$ is positive definite with respect to some point $x_s \in Y$ if it is continuous, $\rho(x_s) = 0$ and $\rho(x) > 0$ for all $x \neq x_s$.

2. MPC of deterministic systems

It might have been thought that most research on MPC during the current century would have been directed to the previous relatively undeveloped area of MPC of uncertain systems but there has also been a surprising amount of research on MPC of deterministic systems both on topics that had previously received attention but also on new areas. We discuss briefly recent research on, *inter alia*, stability of nominal MPC, performance of nominal MPC, explicit MPC, tracking, distributed MPC, economic MPC, algorithms for MPC and embedded MPC.

2.1. Introduction

Most attention has been devoted to MPC of deterministic, nonlinear, discrete-time systems with state x and control u described by:

$$x^+ = f(x, u), \qquad y = h(x) \tag{2.1}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$; x^+ denotes the successor state and $y \in \mathbb{R}^p$ an output. In deterministic MPC the state is assumed known. The controlled system is required to satisfy the state and control constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ respectively where, usually $\mathbb{X}$ is a closed subset of $\mathbb{R}^n$ and $\mathbb{U}$ is a compact subset of $\mathbb{R}^m$; more generally, the state and control are required to satisfy $(x, u) \in \mathbb{Z} \subset \mathbb{R}^{n\times m}$. If, without loss of generality, the control objective is stabilization of the origin, each set is assumed to contain the origin in its interior. In MPC, at each event (x, t) (state x, time t) a control sequence is computed by solving an optimal control problem, and the first control in this sequence applied to the plant. Because of time invariance in most situations, time t can be taken to be zero when solving the optimal control problem. A control sequence $\{u(0), u(1), \ldots, u(N-1)\}$ or $\{u(0), u(1), \ldots\}$ is denoted by $\mathbf{u}$. For any control sequence $\mathbf{u}$, the solution at time k of (2.1) with initial state x (at time 0) is $x^{\mathbf{u}}(k; x)$; $x^{\mathbf{u}}(k; x, t)$ denotes the solution at time k if the initial state is x at time t. If the current state of the system is x, model predictive control solves a finite horizon optimal control problem $\mathbb{P}_N(x)$ defined by:

$$V_N^0(x) = \min_{\mathbf{u}}\{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} \tag{2.2}$$

in which the objective function $V_N : \mathbb{R}^n \times \mathbb{U}^N \to \mathbb{R}$ is defined by

$$V_N(x, \mathbf{u}) \triangleq V_f(x^{\mathbf{u}}(N; x)) + \sum_{i=0}^{N-1} \ell(x^{\mathbf{u}}(i; x), u(i)) \tag{2.3}$$

and $\mathcal{U}_N(x)$ is the set of admissible control sequences satisfying the state, control and terminal constraints:

$$\mathcal{U}_N(x) \triangleq \{\mathbf{u} \in \mathbb{U}^N \mid x^{\mathbf{u}}(i; x) \in \mathbb{X},\ u(i) \in \mathbb{U},\ i \in \mathbb{I}_{0:N-1},\ x^{\mathbf{u}}(N; x) \in X_f\}. \tag{2.4}$$

The set $X_N \triangleq \{x \mid \mathcal{U}_N(x) \neq \emptyset\}$ is the feasible set for $\mathbb{P}_N$. The solution of $\mathbb{P}_N(x)$ is an optimal control sequence $\mathbf{u}^0(x) = \{u^0(0; x), u^0(1; x), \ldots, u^0(N-1; x)\}$ and the associated optimal state sequence is $\mathbf{x}^0(x)$. The control applied to the system being controlled is $u^0(0; x)$; hence, the *implicit* MPC law is

$$\kappa_N(x) \triangleq u^0(0; x). \tag{2.5}$$

Let $x^{\kappa_N}(i; x)$ denote the solution at time i of the controlled system $x^+ = f(x, \kappa_N(x))$ with initial state x at time 0. Sequences $\mathbf{u}^0(x)$ and $\mathbf{x}^0(x)$ are often referred to in the literature as predicted sequences; if the current state-time is (x, t), $x^0(i; x)$ is a prediction of the state $x^{\kappa_N}(t + i; x)$ of the controlled system at time $t + i$. Because of the finite horizon in the optimal control problem, the predicted state $x^0(i; x)$ can differ considerably from the actual state $x^{\kappa_N}(t + i; x)$. Ideally, the horizon N in the optimal control problem solved online should be infinite in which case the predicted state would equal the actual state. For practical reasons, N is usually finite with the result that the controlled system $x^+ = f(x, \kappa_N(x))$ is not necessarily stable or optimal nor is $\mathbb{P}_N$ recursively feasible (recursive feasibility is the property that $x \in X_N$ implies $x^+ = f(x, \kappa_N(x)) \in X_N$ so that the optimal control problem $\mathbb{P}_N(x^{\kappa_N}(i; x))$ is feasible for all $i \in \mathbb{I}_{\geq 0}$ if the initial state x is feasible). Stability and recursive feasibility is usually achieved in two different ways: the first imposes conditions on the terminal cost $V_f(\cdot)$ and constraint set X_f; the second discards X_f and, possibly, $V_f(\cdot)$ and achieves stability by the choice of a sufficiently large horizon N. There have been significant developments in both areas. The finite horizon also adversely affects performance which is, ideally, $V_\infty^0(x)$, the value function of the infinite horizon problem. The actual performance is the infinite horizon cost of the controlled system:

$$V_\infty^{\kappa_N}(x) = \sum_{i=0}^{\infty} \ell(x^{\kappa_N}(i; x), \kappa_N(x^{\kappa_N}(i; x))) \tag{2.6}$$

and is larger than $V_\infty^0(x)$.

2.2. Nominal MPC

2.2.1. Stability and recursive feasibility of nominal MPC

Recursive feasibility and stability are easily established (Mayne et al., 2000) if the stabilizing 'ingredients' $V_f(\cdot)$ and X_f are employed; $V_f(\cdot)$ and X_f have the property that, for all $x \in X_f \subset \mathbb{X}$, there exists a $u \in \mathbb{U}$ such that $V_f(f(x, u)) \leq V_f(x) - \ell(x, u)$ and $f(x, u) \in X_f$; it follows that, for all $N \in \mathbb{I}_{\geq 0}$, all $x \in X_N$, (i) $V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - \ell(x, \kappa_N(x))$ and, (ii) $V_{N+1}^0(x) \leq V_N^0(x)$. Under reasonable conditions it then follows that the value function $V_N^0(\cdot)$ satisfies:

$$V_N^0(x) \in [\alpha_1(|x|), \alpha_2(|x|)] \tag{2.7}$$

$$V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - \alpha_1(|x|) \tag{2.8}$$

for all $x \in X_N$; in (2.7) $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions. The lower bound in (2.7) is a consequence of (i) above and a lower bound on $\ell(\cdot)$, the upper bound follows from an upper bound on $V_f(\cdot)$ and (ii) above; the inequality in (2.8) follows from the properties of $V_f(\cdot)$ and X_f. It follows from (2.7) and (2.8) that the origin is an asymptotically or exponentially stable equilibrium state for the system $x^+ = f(x, \kappa_N(x))$ with a region of attraction X_N, the set of x for which $\mathbb{P}_N(x)$ is feasible. Recursive feasibility (the property that

x feasible for $\mathbb{P}_N(x)$ implies $x^+ = f(x, \kappa_N(x))$ is feasible) is easily proved.

A significant development of the last decade has been the thorough investigation and extension (Grüne & Pannek, 2011) of conditions that ensure recursive feasibility and stability of model predictive control without a terminal stability constraint; this reference provides a wealth of insights into this variant of model predictive control. A helpful introduction to these results is given in the paper (Grüne, 2012) which also provides references to significant earlier papers including Primbs and Nevistić (2000). Under the assumption that there exists a $\gamma > 0$ such that $V_N^0(x) \le \gamma \ell^*(x)$ for all $x \in \mathbb{X}$, all $N \ge 2$ with $\ell^*(x) \triangleq \min_{u \in \mathbb{U}} \ell(x, u)$, the value function $V_N^0(\cdot)$ satisfies

$$V_N^0(f(x, \kappa_N(x))) \le V_N^0(x) - \alpha \ell(x, \kappa_N(x)) \tag{2.9}$$

for all $x \in X_N$, $N \ge 2$, and some $\alpha \in (0, 1]$. This result, in turn, with some further minor assumptions, implies asymptotic stability of the origin for the closed loop system $x^+ = f(x, \kappa_N(x))$ and for all $x \in \mathbb{X}$. Establishing recursive feasibility is more difficult; see Grüne (2012, Section 5).

Convincing reasons for avoiding the use of a terminal constraint are given in Grüne and Pannek (2011). On the other hand, it is shown in Mayne (2013) that, if it is desired that the feasible sets $X_N \triangleq \{x \mid \mathcal{U}_N(x) \neq \emptyset\}$ are positively invariant for $x^+ = f(x, \kappa_N(x))$ and satisfy the nested property $X_N \supset X_{N-1} \supset \cdots$, properties that ensure recursive feasibility, it is necessary that $\mathbb{P}_N(x)$ has an explicit or implicit terminal constraint set that is control invariant.

2.2.2. Inherent robust stability of nominal MPC

Suppose the system is described by $x^+ = f(x, u, w)$, w an additive disturbance. The corresponding nominal system is $x^+ = f_{\text{nom}}(x, u) \triangleq f(x, u, 0)$. For any $x \in \mathbb{R}^n$ and any control sequence $\mathbf{u}$, let $\mathbf{x}^{\mathbf{u}}(x)$ denote the state sequence that is the solution of $x^+ = f_{\text{nom}}(x, u)$ with initial state x and control sequence $\mathbf{u}$. An important question is whether or not the actual system $x^+ = f(x, u, w)$ under MPC *designed for the nominal system* is input-to-state stable (ISS) (see Section 3.2.1 for a definition of ISS). The question is pertinent because it has been shown (Grimm, Messina, Tuna, & Teel, 2004; Teel, 2004), in a dramatic contribution to the study of nominal (inherent) robustness, that a system controlled in this way can be destabilized by an arbitrarily small disturbance w. Kellet and Teel revealed the underlying reason when they proved (Kellet & Teel, 2004) that the system $x^+ = f_{\text{nom}}(x, \kappa_N(x))$ is inherently robustly stable if and only if it admits a *continuous* Lyapunov function. In MPC, the value function $V_N^0(\cdot)$ of the optimal control problem solved online is employed as a Lyapunov function (the ease of obtaining a Lyapunov function for a constrained nonlinear system is an important strength of MPC) and, as is well known, continuity of the value function cannot usually be established when state constraints (including terminal constraints) are present; an exception is the constrained LQR problem when the state and control constraints are affine (Rawlings & Mayne, 2009, Proposition 7.13). If we use $V_N^0(\cdot)$ as a Lyapunov function, we cannot expect nominal MPC to be robustly stable if a terminal constraint (which is a state constraint) is employed. However, a recent result (Yu, Reble, Chen, & Allgöwer, 2011) shows that nominal MPC without state constraints but with a conventional terminal constraint and penalty *is* (despite what is said above) input-to-state stable.

To show input-state stability, suppose that the terminal penalty $V_f(\cdot)$ and the constraint set $X_f := \{x \mid V_f(x) \le a\} \subset \mathbb{X}$ satisfy the normal stabilizing condition for the nominal system so that there exists a control law $\kappa_f(\cdot)$ satisfying $\kappa_f(x) \in \mathbb{U}$, $f(x, \kappa_f(x), 0) \in X_f$ and $V_f(f(x, \kappa_f(x), 0)) - V_f(x) \le -\ell(x, \kappa_f(x))$ for all $x \in X_f$. Suppose also that $\mathbb{X} = \mathbb{R}^n$, i.e. there are no state constraints. The crucial observation in Marruedo, Alamo, and Camacho (2002) and Yu et al. (2011) is that, if $f(\cdot)$, $\ell(\cdot)$ and $V_f(\cdot)$ are sufficiently smooth, then there exists a $b < a$ such that $\kappa_f(\cdot)$ steers any state x of the nominal system lying in X_f to a state $f_{\text{nom}}(x, \kappa_f(x))$ that lies in the set $X_f^* \triangleq \{x \mid V_f(x) \le b\}$; the set X_f^* lies in the strict interior of X_f (there exists an $\varepsilon > 0$ such that $X_f^* + \varepsilon \mathcal{B} \subset X_f$ where $\mathcal{B}$ is the unit ball in $\mathbb{R}^n$). Now let $x \in X_N$ be the state of the system at time t. Hence $x^0(N; x)$ (obtained by solving the nominal optimal control problem) lies in X_f. In view of the observation above, the control sequence $\tilde{\mathbf{u}} \triangleq \{u^0(1; x), u^0(2; x), \ldots, u^0(N-1, x), \kappa_f(x^0(N, x))\}$ steers the nominal system from the nominal state $x^+_{\text{nom}} = x^0(1; x) = f(x, u^0(0, x), 0)$ (at time $t+1$) to the state $x^{\tilde{\mathbf{u}}}(N; x_{\text{nom}}) \in X_f^*$ and also steers the nominal system from the actual state $x^+ = f(x, u^0(0, x), w)$ to $x^{\tilde{\mathbf{u}}}(N; x^+)$. If $f(\cdot)$ and $\ell(\cdot)$ are sufficiently smooth, there exists a $c > 0$ such that $|x^+ - x^+_{\text{nom}}| \le c|w|$ and $\|\mathbf{x}^{\tilde{\mathbf{u}}}(x^+) - \mathbf{x}^{\tilde{\mathbf{u}}}(x^+_{\text{nom}})\|_\infty \le c|w|$ for all $(x, w) \in X_N \times \mathbb{W}$ where $\mathbb{W}$ is a compact subset of $\mathbb{R}^p$ ($\|\cdot\|_\infty$ denotes the sup norm). Hence, $|x^{\tilde{\mathbf{u}}}(N; x^+) - x^{\tilde{\mathbf{u}}}(N; x^+_{\text{nom}})| \le c|w| \le \varepsilon$ so that $x^{\tilde{\mathbf{u}}}(N; x^+) \in X_f$ for all $w \in \mathbb{W}$ if $\mathbb{W}$ is sufficiently small; recursive feasibility and robust positive invariance of X_N follow. Similarly $V_N^0(x^+) \le V_N(x^+, \tilde{\mathbf{u}}) \le V_N(x^+_{\text{nom}}, \tilde{\mathbf{u}}) + c|w| \le V_N^0(x) - \ell(x, \kappa_N(x)) + c|w|$ where the last inequality is a consequence of the assumed properties of $V_f(\cdot)$ and X_f. Hence, under the usual assumptions, the controlled system $x^+ = f(x, \kappa_N(x), w)$ is input-to-state stable (ISS) (with w regarded as the input) if $\mathbb{W}$ is sufficiently small.

2.2.3. Performance of nominal MPC

A useful measure of performance (Grüne & Pannek, 2011) is the difference between $V_\infty^{\kappa_N}(x)$, the infinite horizon cost using the model predictive controller $\kappa_N(\cdot)$, and the ideal infinite horizon performance $V_\infty^0(x)$ (see Grüne & Pannek, 2011, Chapter 4 for conditions that ensure $V_\infty^0(x)$ is well defined). Suppose the system being controlled is linear with the usual quadratic cost $\ell(\cdot)$ and that a terminal cost $V_f(\cdot)$ (but no terminal constraint) is employed in the optimal control problem $\mathbb{P}_N(x)$ with $V_f(\cdot)$ equal to the value function of the unconstrained, infinite horizon, optimal control problem. Let $u = Kx$ be the corresponding unconstrained, infinite horizon, control law and let $a > 0$ be such that the sublevel set $X_f \triangleq \{x \mid V_f(x) \le a\}$ is a subset of $\mathbb{X}$ and KX_f is a subset of $\mathbb{U}$. Then, as shown in Limon, Alamo, Salas, and Camacho (2006b), there exists a $b > 0$ such that $x^0(N; x) \in X_f$ for all $x \in \hat{X}^N \triangleq \{x \in \mathbb{R}^n \mid V_N^0(x) \le a + Nb\}$ so that $V_N^0(x) = V_\infty^0(x) = V_\infty^{\kappa_N}(x)$. For the case when both stabilizing 'ingredients' ($V_f(\cdot)$ and X_f) are employed, Grüne and Pannek (2011) shows that, under the same conditions that guarantee asymptotic stability, $V_\infty^{\kappa_N}(x) \to V_\infty^0(x)$ as $N \to \infty$. For MPC without terminal constraint and cost, Grüne and Pannek (2011) show that satisfaction of (2.9) also implies that $V_\infty^{\kappa_N}(x) \to V_\infty^0(x)$ as $N \to \infty$.

2.2.4. Extension of the region of attraction

The region of attraction for the model predictive control system discussed above, in which the control objective includes steering the state to the origin, is X_N. This set might be undesirably small. To investigate the possibility of increasing this set, a more general control objective, which includes steering the state to an equilibrium state x_r (satisfying $y = h(x_r) = r$) rather than to the origin, is examined; x_r is an equilibrium state if there exists a control $u_r \in \mathbb{U}$ satisfying $x_r = f(x_r, u_r)$. We refer to r as the reference (output), to x_r as the target state and to (x_r, u_r) as the setpoint. The modified optimal control problem where $\ell(\cdot)$, $X_f(\cdot)$ and $V_f(\cdot)$ are chosen to satisfy this objective is $\mathbb{P}_N(x, r)$; in this problem $\ell(x_r, u_r, r) = 0$, $V_f(x_r, r) = 0$ and $x_r \in X_f(r)$. One criticism of employing the stabilizing 'ingredients' $V_f(\cdot, r)$ and $X_f(r)$, is that the region of attraction which is known to be the feasible set $X_N(r)$ of the optimal control problem $\mathbb{P}_N(\cdot, r)$ is restricted since $X_N(r)$ is the set of states that can be steered into $X_f(r)$ along admissible trajectories in N steps or less. Recall that $V_f(\cdot)$ and X_f are required to satisfy: for all $x \in X_f(r) \subset \mathbb{X}$, there exists a $u \in \mathbb{U}$ such that $x^+ = f(x, u) \in X_f(r)$ and $V_f(x^+, r) \le V_f(x, r) - \ell(x, u, r)$; it is often convenient to set $X_f(r)$ equal to a sub-level set of $V_f(\cdot, r)$. In recent literature (Falugi & Mayne, 2013a; Limon, Alvorado, Alamo, & Camacho, 2008; Limon, Alvarado, Alamo, & Camacho, 2010), devoted to tracking a time-varying reference, the restriction on the size of the region of attraction has been alleviated. Instead of steering the state to a particular target x_r, these papers propose that the state be steered initially towards an *artificial* target x_{r^*}, $r^* \in \mathcal{R}$, lying in the set $\mathbb{X}_{\mathcal{R}} \triangleq \cup_{r\in\mathcal{R}}\{x_r\}$ of permissible target states ($\mathcal{R}$ is either a finite set or a continuum of reference outputs) and then using an additional cost to enforce the convergence of r^* to r (and, hence, the convergence of x_{r^*} to x_r) while maintaining feasibility of $\mathbb{P}_N(x, r^*)$. In these papers, convergence of r^* to r is ensured by the addition of a term to $V_N(\cdot)$ that costs the deviation of the artificial reference r^* from the actual reference r; the optimal control problem $\mathbb{P}_N(x)$ is now minimization of $V_N(x, \mathbf{u}, r^*)$ with respect to an augmented decision variable $(\mathbf{u}, r)$. The state x_r is shown to be asymptotically stable with a region of attraction $X_N(\mathcal{R}) \triangleq \cup_{r\in\mathcal{R}} X_N(r)$; $X_N(\mathcal{R})$ can be considerably larger than $X_N(r)$.

An alternative approach is taken in Fagiano and Teel (2012); here an equilibrium set-point (x_r, u_r) that minimizes $\ell(x, u)$ is chosen; $\ell(x_r, u_r)$ is not necessarily zero. The decision variable $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}$ is extended to include $u(N)$ and the cost function $V_N(\cdot)$ is defined as in (2.3) with $V_f(\cdot)$ replaced by $\beta\ell(x^{\mathbf{u}}(N; x), u(N))$ with $\beta > 0$. Since $\ell(\cdot)$ achieves its minimum on the set of equilibrium set-points at (x_r, u_r), this term encourages convergence of the state to x_r. Finally, optimization of $V_N(\cdot)$ is subject to the usual state and control constraints as well as the following terminal constraints: the terminal state $x^{\mathbf{u}}(N; x)$ is required to be an equilibrium state (i.e. to lie in $\mathbb{X}_{\mathcal{R}}$, thus implying the existence of a $u \in \mathbb{U}$ such that $x^{\mathbf{u}}(N; x) = f(x^{\mathbf{u}}(N; x), u)$); also $\ell(x^{\mathbf{u}}(N; x), u(N))$ is constrained to be less than or equal to its previous value. The resultant controller is shown to perform as well as conventional model predictive controllers while possessing a larger region of attraction and requiring a shorter horizon.

2.2.5. Hybrid MPC

In Mayne et al. (2000) the following comment was made: *all processes contain discrete components such as valves, switches, speed selectors and overrides, in addition to continuous components that are described by difference or differential equations. Consideration of hybrid systems that possess both types of components opens up a rich area of research relevant to a range of important problems such as control and supervisory schemes in the process industries. Many system theoretic concepts, as well a control strategies like MPC, require re-examination in this setting*. The current century has indeed seen significant activity in this area. The early paper (Bemporad & Morari, 1999) dealing with control of systems consisting of logical and conventional elements has had substantial impact. The paper (Camacho, Ramirez, Limon, Muñoz de la Peña, & Alamo, 2010) reviews the literature, describes many practical contributions and states that piecewise affine systems, 'which are equivalent to sets of affine systems combined with finite automata' and mixed logical and dynamic systems, proposed in Bemporad and Morari (1999), are the most studied in the model predictive literature with more attention devoted to piecewise affine systems.

Continuous piecewise affine systems have probably received the most attention because continuity of $f(\cdot)$ in the system description $x^+ = f(x, u)$ facilitates stability analysis. Discontinuity in $f(\cdot)$ gives rise to subtle difficulties that are explored in the thesis (Lazar, 2006). The paper (Lazar, Heemels, Weiland, & Bemporad, 2006) extends the sufficiency conditions for stability provided by a terminal constraint and cost to discontinuous systems and provides methods for computing suitable terminal constraints sets and cost functions for continuous piecewise affine systems. As discussed below, input-to-state stability is now widely used for establishing robust stability in model predictive control but must be carefully employed when the system is discontinuous (Lazar, Hemeels, & Teel, 2009, 2013).

Many studies of hybrid systems, such as the mixed logic dynamic description (Bemporad & Morari, 1999), partition the state into two components, a continuous component lying in $\mathbb{R}^n$ and discrete component lying in $\{0, 1\}^{n_d}$. In Goebel, Sanfelice, and Teel (2009) a powerful alternative is presented; a hybrid system is described by

$$\dot{x} \in F(x) \quad x \in C$$
$$x^+ \in G(x) \quad x \in D.$$

The flow map $x \mapsto F(x)$ and the jump map $x \mapsto G(x)$ are set valued; C is the flow set and D the jump set. This model describes a wide variety of phenomena while its simple structure facilitates the development of both a stability theory for hybrid systems and procedures for the design of hybrid control systems. This elegant description models situations where the state has distinct continuous and discrete components as in piecewise affine systems as well as in situations such as a bouncing billiard ball where the state is continuous on some intervals but is also subject to jumps. There is a substantial theoretical literature on hybrid systems; the paper (Goebel et al., 2009) provides a very useful and unifying introduction to this complex area and deals with, *inter alia*, existence of solutions, uniqueness conditions, asymptotic stability, converse theorems, invariance and simulation. It is bound to influence further research on hybrid MPC.

Switched systems is one of the main concerns in the literature on hybrid systems but has different meanings not always explicitly specified in the literature The system being controlled may have different dynamics in different regions of the state-control space so that the switching sequence depends on the state-control trajectory, i.e. is a function of the initial state and the control trajectory. An example is the continuous piecewise affine system; the state (or state-control) space is partitioned into regions in each of which the system is affine; a convergent algorithm for solving the online optimal control problem (in which the cost function $V_N(\cdot)$ is non-differentiable) and the associated model predictive controller are described in Mayne and Raković (2003). In Magni, Scattolini, and Tanelli (2008), the system is nonlinear and the state space is partitioned into regions Γ_i in each of which the stage cost $\ell(\cdot)$ has a specific value $\ell_i(\cdot)$. If the current state x lies in Γ_i region, the standard optimal control problem with stage cost $\ell_i(\cdot)$ is solved to provide a candidate $\kappa_i(x)$ for the current control. In Müller and Allgöwer (2012) the system is continuous-time and nonlinear; the state space is not partitioned into regions but the controller can choose which stage cost $\ell_i(\cdot)$ in a finite set $\{\ell_p(\cdot) \mid p \in \mathcal{P}\}$ to employ in a standard optimal control problem to obtain a candidate $\kappa_i(x)$ for the current control. In both of the latter papers, the current control can change rapidly if the candidate is automatically selected which can cause instability; known techniques, hysteresis and dwell-time conditions from the adaptive control literature (Hespanha & Morse, 2002) have to be satisfied to ensure stability. Therefore Magni et al. (2008) employs hysteresis switching and selects the new candidate $\kappa_i(x)$ only if it gives a finite improvement in the value function of the optimal control problem. And Müller and Allgöwer (2012) accepts the new candidate only if an average dwell-time condition is satisfied. In these two papers, the switching sequence is a design parameter. A different problem arises if the switching sequence is not know a-priori, is not generated by trajectories of the system being controlled and is not a design parameter. The controller has to be designed so that it is stabilizing for a large class of possible switching signals. In Müller, Martius, and Allgöwer (2012) the problem of controlling a continuous-time, nonlinear plant $\dot{x} = f_\sigma(x, u)$ where $\sigma(\cdot)$ is the switching signal is considered. A controller is presented that is stabilizing for all switching sequences satisfying an average dwell-time condition.

A recent paper (Wongpiromsarn, Topcu, & Murray, 2012) opens a new and interesting area in which model predictive control is employed to reduce the complex problem of automatically synthesizing a control protocol expressed in temporal logic into a set of significantly smaller problems.

2.3. Economic MPC

In many control design procedures, the parameters of $\ell(\cdot)$, the stage or running cost, are regarded as being able to be adjusted in order to achieve traditional design objectives such as rapid response with limited overshoot to a step input. In the process industries, on the other hand, profitability is often the primary objective. Consider the system (2.1) that is subject to the state-control constraint $(x, u) \in \mathbb{Z}$. To improve profitability, the following two-stage procedure is conventionally adopted. In the first stage, given a desired output r, chosen for economic reasons, the corresponding set point (x_r, u_r) is chosen (by solving a minimization problem) to satisfy $x_r = f(x_r, u_r)$, $h(x_r) = r$ and $(x_r, u_r) \in \mathbb{Z}$ so that (x_r, u_r) is an equilibrium point. Model predictive control then employs, in the optimal control problem solved online, a stage cost $\ell(\cdot)$ and terminal cost $V_f(\cdot)$ that are zero at (x_r, u_r) and x_r respectively and strictly positive elsewhere. There are disadvantages to this two-stage procedure. Firstly, the most profitable operation of the plant may not occur at an equilibrium set point; the most profitable regime may be, for example, a periodic cycle. Secondly, $\ell(\cdot)$ and $V_f(\cdot)$ are not chosen to reflect economic cost so the transient cost incurred in the passage from the initial state to the target state x_r may not be optimal. Thirdly, the steady state model employed to determine an optimal equilibrium is more accurate than the dynamic model employed by the model predictive controller so that an equilibrium point determined by the former may not be feasible for the latter. Solving the latter problem motivated the first papers (Rawlings & Amrit, 2009; Rawlings, Bonne, Jorgensen, Venkat, & Jorgensen, 2008) on economic MPC.

Economic MPC, in its current stage, addresses these limitations by choosing $\ell(x, u)$ to be the *economic net cost* of operating the plant at (x, u). The best operating point (x_r, u_r) is then chosen to minimize $\ell(x, u)$ with respect to (x, u) subject to $x = f(x, u)$ and $(x, u) \in \mathbb{Z}$. In contrast to standard MPC, $\ell(x_r, u_r) \neq 0$; standard results on MPC do not treat this case so special techniques have to be employed. Ideally, the controller should minimize the average cost $\lim_{N\to\infty}(1/N)\sum_{i=0}^{N-1}\ell(x(i), u(i))$ subject to the usual state-control constraint $(x(i), u(i)) \in \mathbb{Z}$. In practice, a standard finite horizon optimal control problem $\mathbb{P}_N(x)$ of minimizing

$$V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x^{\mathbf{u}}(i; x), u(i)) + V_f(x^u(N; x))$$

subject to $(x^{\mathbf{u}}(i), u(i)) \in \mathbb{Z}$ for all $i \in \mathbb{I}_{0:N-1}$ and the terminal constraint $x^{\mathbf{u}}(i; x) \in X_f$ is employed. The terminal cost $V_f(\cdot)$ and constraint set X_f are required to satisfy milder conditions than those normally employed, namely, $x_r \in X_f$ and, for all $x \in X_f$, there exists a u satisfying $(x, u) \in \mathbb{Z}$ and $f(x, u) \in X_f$; these conditions include the terminal equality constraint case when $V_f = 0$ and $X_f = \{x_r\}$. Let $\mathbf{u}^0(x)$ denote the solution of $\mathbb{P}_N(x)$; $\kappa_N(x) \triangleq u^0(0; x)$, the first element of $\mathbf{u}^0(x)$, is the MPC law. Because $\ell(\cdot)$ does not have the properties required for conventional MPC, the value function $V_N^0(\cdot)$ for Problem $\mathbb{P}_N(x)$, when the terminal ingredients $V_f(\cdot) = 0$ and $X_f = \{x_r\}$ are employed, now satisfies

$$V_N^0(f(x, \kappa_N(x))) \le V_N^0(x) - \ell(x, \kappa_N(x)) + \ell(x_r, u_r) \tag{2.10}$$

and thus cannot be used to establish stability since, in contrast to conventional MPC, it is no longer necessarily true that $\ell(x, \kappa_N(x)) - \ell(x_r, u_r) \ge \alpha(|x - x_r|)$ for some $\mathcal{K}_\infty$ function $\alpha(\cdot)$ ($\ell(x_r, u_r)$ is usually zero in conventional MPC). However, in Diehl, Amrit, and Rawlings (2011), it is shown that, under the assumption of strong duality of the steady state problem (determination of (x_r, u_r)), there exists a stage cost $L(\cdot)$ satisfying $L(x_r, u_r) = 0$ and $L(x, u) \ge \alpha(|x - x_r|)$ where $\alpha(\cdot)$ is a $\mathcal{K}_\infty$ function. If now $L(\cdot)$ replaces $\ell(\cdot)$ in the optimal control problem $\mathbb{P}_N(x)$, then the same solution $\mathbf{u}^0(x)$ is obtained (and, hence, the same model predictive control law $\kappa_N(\cdot)$) but the new value function $\hat{V}_N(\cdot)$ now satisfies

$$\hat{V}_N(f(x, \kappa_N(x))) \le \hat{V}_N(x) - L(x, \kappa_N(x)).$$

Using $\hat{V}_N(\cdot)$ as a Lyapunov function enables asymptotic stability of x_r for the closed loop system $x^+ = f(x, \kappa_N(x))$ to be established. It can also be shown (Angeli, Amrit, & Rawlings, 2012) that the average asymptotic performance of the closed loop system $x^+ = f(x, u^0(0; x))$ is no worse than that of the best admissible steady state in the sense that

$$\limsup_{T\to\infty}(1/T)\sum_{i=0}^{T-1}\ell(x(i), u(i)) \le \ell(x_r, u_r).$$

The assumption of strong duality of the steady-state problem holds if the system is linear, the constraints are convex and $\ell(\cdot)$ is strictly convex and the Slater condition is satisfied. Significantly improved sufficient conditions for asymptotic stability are given in Amrit, Rawlings, and Angeli (2011) and Angeli et al. (2012) under a dissipativity assumption; the system $x^+ = f(x, u)$ is said to be dissipative with respect to the supply rate $s(\cdot)$ if there exists a function $\lambda(\cdot)$ such that

$$\lambda(f(x, u)) - \lambda(x) \le s(x, u)$$

for all $(x, u) \in \mathbb{Z}$ and is strictly dissipative if there exists a positive definite function $\rho(\cdot)$ such that

$$\lambda(f(x, u)) - \lambda(x) \le s(x, u) - \rho(x - x_r).$$

It is shown in Angeli et al. (2012) that if (x_r, u_r) is defined as above (i.e. (x_r, u_r) minimizes $\ell(x, u)$ with respect to (x, u) subject to $x = f(x, u)$ and $(x, u) \in \mathbb{Z}$) and if the system $x^+ = f(x, u)$ is strictly dissipative with respect to the supply rate $s(\cdot)$ defined by

$$s(x, u) = \ell(x, u) - \ell(x_r, u_r)$$

then x_r is an asymptotically stable equilibrium point of the closed loop system $x^+ = f(x, \kappa_N(x))$. A modified stage cost $L(\cdot)$ is defined by $L(x, u) = \ell(x, u) + \lambda(x) - \lambda(f(x, u))$ and is shown to satisfy $\min\{L(x, u) \mid (x, u) \in \mathbb{Z}\} \le \ell(x_r, u_r)$. As before, the optimal control problem with $\ell(\cdot)$ replaced by $L(\cdot)$ has the same solution $\mathbf{u}^0(x)$ as the original problem (and hence the same model predictive control law $\kappa_N(\cdot)$) but a value function $\hat{V}_N(\cdot)$ that satisfies

$$\hat{V}_N(f(x, \kappa_N(x))) \le \hat{V}_N(x) - \rho(x - x_r).$$

Using $\hat{V}_N(\cdot)$ as a Lyapunov function establishes the asymptotic stability of x_r. It has recently been shown (Müller, Angeli, & Allgöwer, 2013b) that the dissipativity condition is not necessary but 'nearly necessary' in a certain sense. Extensions of economic MPC to outperform optimal periodic control or to ensure satisfaction of asymptotic average constraints are also developed in Angeli et al. (2012). Drawing on results in Fagiano and Teel (2013) that show how the terminal constraint set in conventional and economic may be enlarged, Müller, Angeli, and Allgöwer (2013a) presents a version of economic model predictive control that employs a generalized terminal state constraint and cost, namely that the terminal state is any equilibrium point satisfying a parameterized additional constraint on the terminal stage cost and the terminal cost is a parameterized version of the terminal stage cost. The paper compares the different advantages that accrue from the use of six alternative rules for updating the two parameters. In Grüne (2013) Grüne points out that, precisely because the objective is minimization of economic cost (maximization of profit), approximate optimality of the resultant controlled system is of prime importance. Since the choice of suitable terminal conditions is difficult, Grüne considers the use of MPC without terminal conditions. Analysis is difficult and Grüne proceeds by assuming certain turnpike and controllability properties hold. These may be difficult to verify in general but are verified in the paper for two motivating examples that illustrate the turnpike property. It is proved that, under these assumptions, MPC without a terminal constraint yields an approximately optimal infinite horizon performance. In Chen, Heidarinejad, Liu, and Christofides (2012) the authors define a new economic measure for a continuous time benzine process and use it to obtain the optimal economic steady state (x_r, u_r). The authors then formulate a Lyapunov based controller for distributed economic MPC. It is therefore assumed that a control Lyapunov function $V(\cdot)$ that has its minimum at x_r together with its domain of attraction Ω and associated control law is known. This is employed, as in the related controllers described in Section 2.5, in an added constraint in the optimal control problem that enforces stability; the difficulties encountered in the economic model predictive controllers described above that employ terminal stability conditions are therefore avoided; the relative performance of the various types of economic MPC is not yet known.

The topic of economic MPC, though very new, has attracted considerable attention; for a recent review of work in this area and further references see Rawlings, Angeli, and Bates (2012).

2.4. Explicit MPC

MPC replaces offline determination of a *control law* or sequence of control laws by online solution of an optimal control problem whose solution yields the *control action* for the current state; it is therefore useful in situations where an explicit solution of the optimal control problem for all initial states cannot be easily obtained. Since so few optimal control problems admit an explicit offline solution, it is this feature that makes MPC so useful for many problems. To the restricted list of problems, notably LQR and $\mathcal{H}_\infty$, that have explicit solutions, a new problem, namely the *constrained*, finite horizon, discrete-time, LQR problem, has been added; explicit solutions for these problems were independently proposed in Bemporad, Morari, Dua, and Pistikopoulos (2002) and Seron, De Doná, and Goodwin (2000). This problem is a parametric quadratic program; the optimal control sequence is required for each value of the parameter x.

The basic idea underlying solutions of parametric quadratic programs is easily understood. The problem $\mathbb{P}_N(x)$ may be expressed as minimization of a quadratic function $V(x, \mathbf{u})$ of $(x, \mathbf{u})$ subject to an affine constraint $M\mathbf{u} \leq Nx + p$. At given state x, a subset of the affine inequality constraints will be active at the optimal control $\mathbf{u}^0(x)$; these constraints will remain active in a neighbourhood of x and may therefore be expressed as equality constraints. Minimization of $V(x, \mathbf{u})$ subject to these equality constraints is easily done yielding the optimal affine control law $\mathbf{u}_I^0(\cdot)$ where I indexes the active constraints. The local control law $\mathbf{u}_I^0(\cdot)$ is optimal for the problem $\mathbb{P}$ in the region in $\mathbb{R}^n$ in which $\mathbf{u}_I^0(\cdot)$ satisfies the affine state and control constraint $M\mathbf{u} \leq Nx + p$ and in which a local optimality condition (the cost increases in any feasible direction) that may be expressed as $L_I \nabla_{\mathbf{u}} V(x, \mathbf{u}_I^0(x)) \geq 0$ for some matrix L_I is also satisfied. Hence the affine control law $\mathbf{u}_I^0(\cdot)$ is optimal in a polyhedron defined by these inequalities. Repeating this procedure for every possible set of active constraints yields a control law $x \mapsto \mathbf{u}^0(x)$ that is piecewise affine on a polyhedral partition of $\mathbb{X}$; MPC employs only the first component $x \mapsto \kappa_N(x) \triangleq u^0(0; x)$ of $x \mapsto \mathbf{u}^0(x)$.

Instead of determining explicitly the function $x \mapsto \mathbf{u}^0(x) = \{u^0(0; x), u^0(1; x), \ldots, u^0(N-1; x)\}$ (a formidable task given its dimension) it is possible to employ dynamic programming to determine, sequentially, $u^0(N-1; x), u^0(N-2; x), \ldots, u^0(0; x)$ by solving a sequence of single stage problems each of the form $V_j^0(x) = \min_u\{\ell(x, u) + V_{j+1}^0(f(x, u))\}$, $j = 0, 1, \ldots, N-1$, $V_N^0(x) \triangleq V_f(x)$, each subject to the state and control constraints and the constraint $x^+ = f(x, u) \in X_{j+1}$ where X_{j+1} is the feasible set for $\mathbb{P}_{j+1}$. The sets $\{X_j\}$ satisfy $X_j = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U} \text{ such that } f(x, u) \in X_{j+1}\}$, $X_N \triangleq X_f$. For each j, $V_j^0(\cdot)$ is piecewise quadratic, $\kappa_j(\cdot)$ is piecewise affine and X_j is polyhedral necessitating an extension to parametric quadratic programming (Mayne, Rakovíc, & Kerrigan, 2007).

Although the underlying concept of parametric quadratic programming is simple, implementation is by no means easy; while computing a single polyhedron on which the control law is affine is straightforward, computing every polyhedron while avoiding overlaps and gaps in neighbouring polyhedra is numerically difficult. Considerable effort has therefore been devoted to obtaining reliable algorithms (Jones & Morari, 2006; Patrinos & Sarimveis, 2010; Spjøtvold, Kerrigan, Jones, Tøndel, & Johansen, 2006; Tøndel, Johansen, & Bemporad, 2003). Excellent surveys of this research appear in Alessio and Bemporad (2008) and Jones, Baric, and Morari (2007). An upper bound to the number of regions is 2^q where q, the number of constraints, increases with the horizon N and with state dimension n. A control law $\kappa_N^i(x) = K_i x + k_i$ must be stored for each region R_i; implementation requires repeated determination of the region in which the current state lies; this problem, the point location problem, is non-trivial. Useful software packages (Bemporad, 2004; Kvasnica, Grieder, & Baotić, 2006) exist and enable problems with state dimension up to, say, 10 to be solved. The method appears to be potentially useful for embedded control of systems with modest state dimension. Another area of activity is concerned with the development of approximate solutions that preserve stability by the use of relaxed dynamic programming (Lincoln & Rantzer, 2006). The chapter (Jones, 2014, chap. 14) provides a very useful review of "approximate explicit control laws of desired complexity that provide certificates of recursive feasibility and stability".

A simple form of explicit MPC or, more generally, parametric programming, occurs when the decision variable is quantized (Quevedo, Goodwin, & De Dona, 2004). For the linear, quadratic optimal control problem, the cost function to be minimized is $V(x, \mathbf{u}) = (1/2)[x'W_{xx}x + 2\mathbf{u}'W_{ux}x + \mathbf{u}'W_{uu}\mathbf{u}]$ in which $\mathbf{u}$ is constrained to take values (ignoring other constraints) in the finite set $\mathcal{U} = U^N \triangleq \{\mathbf{v}_1, \ldots, \mathbf{v}_J\}$. Assuming $\mathbf{u}^0(x) = \mathbf{v}_j$, we obtain $V_j^0(x) = (1/2)[x'W_{xx}x + 2\mathbf{v}_j'W_{ux}x + \mathbf{v}_j'W_{uu}\mathbf{v}_j]$. Hence $\mathbf{v}_j$ is the minimizing control in the set $X_j = \{x \mid V_j^0(x) \leq V_k^0(x),\ k \neq j\} = \{x \mid F_j x \leq f_j\}$ for some F_j, f_j. The partition of $\mathbb{R}^n$ into the polytopic sets $X_1, X_2, \ldots, X_J$ becomes a Voronoi partition in the transformed space in which W_{uu} is the identity matrix.

2.5. Distributed MPC

There exist many complex systems, such as traffic, electrical and transport networks and large processes, that require control but are unsuitable, because of their complexity, for traditional centralized control. The complexity of these systems gives rise to computational and communication problems, raises modelling and data collection issues, and can make centralized control, assumed elsewhere in this review, impractical. For this reason a very large literature on distributed control and distributed model predictive control (DMPC) has emerged; DMPC is particularly popular because of its ability to handle control and state constraints in multi-input, multiple output systems; it is easier to implement than centralized MPC if it is possible to decompose the original control problem into a set of smaller problems of controlling a set of subsystems of the original system, each subsystem having its own controller (or agent) and, possibly, its own objective. Potential advantages are computational (subproblems easier to solve), robustness (failure of a subsystem may not cause remaining subsystems to fail), scalability and simplification of system maintenance.

Very many approaches have been proposed for distributed MPC. Indeed this area of research, though recent, has its own survey papers, e.g. Christofides, Scattolini, Muñoz de la Peña, and Liu (2013) and Scattolini (2009); the latter survey cites more than 200 references. The sheer diversity of the literature makes it virtually impossible to review the research on this topic concisely. Fortunately, a recent book (Maestre & Negenbaum, 2014) serves a very useful purpose in providing a 'coherent overview' of the literature. Separate chapters provide 35 differing approaches, each written in conformity with the editors' requirements to ensure consistency of notation, specification of the process (linear, nonlinear, hybrid; deterministic or random, control objective, coupling of subsystems), control architecture (coordination of local controllers, information structure, cooperative or non-cooperative, communication between controllers, timing, optimization variables) and theoretical properties of the controlled system (optimality, stability, robustness). Space does not permit such a thorough exposition here; instead we discuss a small subset of the literature chosen (subjectively) to illustrate a few of the proposed approaches to DMPC.

A controller is said to be centralized when it consists of a set of local controllers, each controlling a local subsystem and there is no communication between the local controllers; interaction between the subsystems can result in poor performance or, even, instability. These issues have been explored in the extensive literature on decentralized control and will not be discussed further. When the local controllers communicate the control system is said to be distributed; the limitations of decentralized control and the increased availability of network communication between subsystems, has stimulated interest in the development of control strategy, distributed control, that has the potential to ensure plant wide stability coupled with good performance. If each controller has its own objective, the distributed controller is said to be non-cooperative; if all controllers share a common objective the distributed controller is cooperative.

An important class of problems occurs (Christofides et al., 2013) when the subsystems are decoupled but some subsystem constraints and/or objectives are coupled; this class of problems includes the problem of maneuvering a group of vehicles while maintaining relative position or avoiding collision. Thus Dunbar and Murray (2006) study the control of decoupled, second order, nonlinear systems with coupled objectives and constraints and give a clear description of the approximation needed to decompose the overall objective and constraints into decoupled objectives and constraints (needed for distributed MPC) in such a way that stability is ensured. To decouple the objective, the controller for each subsystem employs an estimate of the state of each other subsystem; a compatibility constraint ensures that the actual state trajectory of each subsystem does not deviate too much from its estimate. A similar procedure is used in other distributed controllers although Keviczky, Borrelli, and Balas (2004) employ, for each controller, worst case (instead of estimated) state trajectories for the other subsystems. In Richards and How (2007) a similar problem is considered; the decoupled subsystems are linear. At each sample time, the subproblems are solved in sequence; in solving the optimal control problem for system p, the actual state trajectories for subsystems $\{1, \ldots, p-1\}$ are employed while estimated trajectories are used for the remaining subsystems $\{p+1, \ldots, J\}$. Results from robust MPC (Chisci, Rossiter, & Zappa, 2001) are employed to cope with the uncertainty induced by the use of estimates. An extension (Trodden & Richards, 2007) employs properties of the robust MPC controller in Mayne, Seron, and Raković (2005) to bound the estimation error; see also Maestre and Negenbaum (2014, Chapter 3). In Müller, Reble, and Allgöwer (2012) the problem considered is cooperative control of a set of decoupled nonlinear subsystems; as in Richards and How (2007), the subproblems are solved in sequence at each sample time. Interestingly, estimates of state trajectories of the subsystems not yet updated are obtained using the same technique employed in conventional MPC for estimating the optimal control sequence for the successor state; see also Maestre and Negenbaum (2014, Chapter 5). In a similar study (Müller, Schurmann, & Allgower, 2012), in which the decoupled systems are linear, robustness of the distributed control system is established using a tube-based technique (Mayne et al., 2005) to bound the estimation error.

If M subsystems are dynamically coupled, the overall system is described by

$$x^+ = f(x, u)$$

in which $x = (x_1, x_2, \ldots, x_M)$ is the state of the composite system and x_i is the state of the ith subsystem. Similarly $u = (u_1, u_2, \ldots, u_M)$ in which $u_i \in \mathbb{U}_i$ is the control or input to the ith subsystem. In non-cooperative control, each subsystem i has its own objective function $V_i(x_i, \mathbf{u}_i)$, $\mathbf{u}_i \triangleq \{u_i(0), u_i(1), \ldots, u_i(N-1)\}$ such as:

$$V_i(x_i, \mathbf{u}_i) = \sum_{j=0}^{N-1} \ell_i(x_i(j), u_i(j)) + V_i^f(x_i(N))$$

in which $V_i^f(\cdot)$ is the terminal cost. Several methods for taking into account the interaction between subsystems in non-cooperative control have been proposed. For example, Farina and Scattolini (2011) propose that each controller chooses a reference trajectory that it transmits to the other controllers with a guarantee that its actual state trajectory will lie within a known neighbourhood of this trajectory. The other controllers can therefore treat this deviation from the known reference trajectory as a bounded disturbance. Known methods for robust MPC, discussed in Section 3.2, can then be employed. A difficulty is the assumption that it is possible to provide the guarantee specified above. Alternative distributed and hierarchical architectures for MPC are described in Scattolini (2009).

Cooperative MPC in which each local controller optimizes a common objective function $V(x, \mathbf{u})$, $\mathbf{u} \triangleq \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M\}$ which may, but not necessarily, have the form $\sum_{i=1}^{M} \rho_i V_i(x_i)$, was introduced in Venkat, Rawlings, and Wright (2005) and developed *inter alia* in Stewart, Wright, and Rawlings (2011). In the context of process control, cooperative control seems natural; convincing reasons for its adoption, given in Stewart et al. (2011), include robustness to loss of communication between individual controllers. At each sample time, each controller receives the current state x of the whole plant, as well as the control sequence computed by each other controller at the previous sample time. For each $i = 1, 2, \ldots, M$, controller i applies one iteration of a gradient projection algorithm to minimize $V(x, \mathbf{u})$ with respect to $\mathbf{u}_i$ keeping $\mathbf{u}_j, j \neq i$, constant yielding a control sequence $\mathbf{v}_i$. The controllers then share their improved controls and obtain a candidate plant wide control sequence $\mathbf{u}^+ = \{\mathbf{u}_1^+, \ldots, \mathbf{u}_M^+\}$ by forming a convex combination of $(\mathbf{u}_0, \ldots \mathbf{u}_{i-1}, \mathbf{v}_i, \mathbf{u}_{i+1}, \ldots \mathbf{u}_M)$, $i = 1, 2, \ldots, M$. If the optimal control problem is convex (e.g. $f(\cdot)$ is linear and $\ell(\cdot)$ is quadratic and positive definite), an arbitrary convex combination suffices to obtain a cost reduction at each stage; otherwise the convex combination can be iteratively adjusted by dropping less effective improvements until a cost reduction is obtained; this is guaranteed to be possible. The procedure is then repeated for a specified number of iterations. It is shown in Stewart et al. (2011) that, with minor modifications to the optimal control problem solved online to satisfy the conditions for suboptimal stability (Scokaert, Mayne, & Rawlings, 1999, Theorem 1), the origin is asymptotically stable for the controlled system with a region of attraction equal to the feasible region of the modified optimal control problem.

Alternative methods for cooperative MPC include sequential MPC (Liu, Chen, Muñoz de la Peña, & Christofides, 2010; Liu, Muñoz de la Peña, & Christofides, 2009) and iterative MPC (Liu et al., 2009) that are well described in the review paper (Christofides et al., 2013). The latter employs *Lyapunov based MPC* in which it is assumed that a stabilizing controller, together with its associated Lyapunov function, is known for the system. Model predictive control, modified to ensure stability by the additional constraint that the Lyapunov function is decreased, is then employed to enhance performance and satisfaction of control and state constraints, either in a decentralized or in a distributed manner, making use of additional networked sensors and actuators. A useful feature of this approach is its potential to improve existing control systems. A disadvantage may be unavailability of a global Lyapunov function. An 'almost decentralized' Lyapunov-based model predictive controller is described in Maestre and Negenbaum (2014, Chapter 14). A comprehensive investigation of the problem of designing cooperative control for a team of agents with decoupled, nonlinear discrete-time dynamics but with objective functions that have individual and cooperative components is carried out in Franco, Magni, Parisini, Polycarpou, and Raimondo (2008). Each agent employs a locally computed control that takes into account delayed information from neighbouring agents. A cooperative distributed model predictive controller in which each controller has very limited global information and limited communication and yet has good performance (in the studied examples) is described in Maestre, Muñoz de la Peña, and Camacho (2011).

A related topic is networked MPC. Admittedly distributed MPC requires communication networks but the focus of networked MPC is different its main concern being the development of control strategies that are robust against network induced uncertainties such as time varying transmission delays in both sensor-to-controller and controller-to-actuator links and packet dropouts as well as model uncertainty. This complex control problem is addressed in Pin and Parisini (2011) and the references therein. In this paper robust stability of the closed loop system is established using a characterization of regional input-to-state stability in terms of time-varying Lyapunov functions.

In Section 2.5 we have omitted any mention of the use of distributed optimization in DMPC; for a more complete overview of distributed MPC see Maestre and Negenbaum (2014).

2.6. Optimization for MPC

2.6.1. Algorithms for the optimal control problem $\mathbb{P}(x)$

Although MPC is often used to control dynamic systems described by differential equations the control generated by the computer is usually piecewise constant so that it is convenient, for analysis and for optimization, to replace the differential equation description $\dot{x} = f_c(x, u)$ by the difference equation $x^+ = f(x, u)$ that has the same solution at sample times. The difference equation may be explicitly obtained if the system is linear; otherwise it is implicitly obtained via the differential equation solver employed in the optimization algorithm. The optimization algorithm may be used to solve the problem $\mathbb{P}_N(x)$, defined in Section 2.1, of minimizing $V_N(x, \mathbf{u})$ with respect to $\mathbf{u}$ subject to the control, state and terminal constraints that may be expressed as $g(x, \mathbf{u}) \leq 0$; this implies that the difference equation (2.1) is solved at each iteration of the algorithm. Alternatively, the problem may be formulated as $\mathbb{P}^*_N(x)$ of minimizing $V^*_N(\mathbf{x}, \mathbf{u})$ defined by

$$V^*_N(\mathbf{x}, \mathbf{u}) = V_f(x(N)) + \sum_{i=0}^{N-1} \ell(x(i), u(i)) \tag{2.11}$$

with respect to $(\mathbf{x}, \mathbf{u})$ subject to the control, state and terminal constraints that may be expressed as $g^*((\mathbf{x}), \mathbf{i}) \leq 0$ and, in addition, the equality constraint $h(\mathbf{x}, \mathbf{u}) = 0$ that enforces satisfaction of the difference equation (2.1). The advantages of the first option are, firstly, that it yields, at each iteration of the algorithm, a pair $(\mathbf{x}, \mathbf{u})$ that satisfies the difference equation, so, if the algorithm is terminated early, the solution may be still employed in sub-optimal MPC as, for example, in cooperative distributed MPC and, secondly, that the dimension of the decision variable $\mathbf{u}$ is less than the dimension of the decision variable $(\mathbf{x}, \mathbf{u})$ of the second option. Its disadvantage is that numerical errors become significant if the difference equation is solved over a long horizon. This disadvantage may be mitigated by the use of multiple shooting in which the horizon is decomposed into a set of intervals I_j (satisfying, therefore, $\{0, 1, \ldots, N\} = I_1 \cup I_2, \ldots, \cup I_J$); the difference equation is satisfied in each interval and a set of equality constraints are added to the optimization problem to ensure that the state at the end of each interval is equal to the state at the beginning of the next. Multiple shooting is intermediate between the first and second options. Conversely, the second option suffers less from numerical error arising from solving a difference equation over long intervals but, if terminated early, yields state and control sequences that do not satisfy the difference equation.

In the optimal control literature, including the early literature circa 1950–1970 (Bryson & Ho, 1969; Polak, 1971), optimality conditions and gradient computations employed an adjoint system that allowed the structure of the optimal control problem to be exploited at a high level. However, the development of finite dimensional optimization permits powerful numerical techniques to be employed if the structure of the problem is exploited at the level of Karush–Kuhn–Tucker (KKT) optimality conditions. Thus it has been shown (Binder et al., 2001; Wright, 1993) that if the decision variables $u(i)$, the adjoint variables $\lambda(i)$ (associated with the difference equation equality constraints) and the Lagrange multipliers $p(i)$ (associated with the state-control inequality constraints) are re-ordered so that, for each i, $u(i)$, $\lambda(i)$ and $p(i)$ are contiguous, then the KKT conditions takes the form $Hv = h$ where H is a block band matrix that can be efficiently factorized with a computational cost of order N compared with order N^3 for a naive ordering.

If $f(\cdot)$ is linear, $\ell(\cdot)$ and $V_f(\cdot)$ are quadratic and positive definite, and the control, state and terminal constraints are polyhedral, then $\mathbb{P}$ and $\mathbb{P}^*$ are strictly convex quadratic programs permitting global solutions to be effectively computed. The usual algorithms employed in this case are (i) active set methods (Kirches, Bock, Schlöder, & Sager, 2010), (ii) interior point methods (Rao, Wright, & Rawlings, 1998) and (iii) gradient projection methods all of which are described in standard texts. A development of the latter algorithm, the fast gradient method (Nesterov, 2004) that has a guaranteed convergence rate, has been proposed for use in constrained MPC (Richter, Morari, & Jones, 2011). A recent paper (Korda & Jones, 2014) establishes verification of stability under tight constraints on computation time when the optimal control problem is solved using the fast gradient method.

Similar considerations apply when the system being controlled is nonlinear; the optimization problem may be expressed as $\mathbb{P}_N$ (in which the difference equation is exactly solved at each iteration leaving $\mathbf{u}$ as the decision variable) or as $\mathbb{P}^*_N$ (in which the decision variable is $(\mathbf{x}, \mathbf{u})$ and satisfaction of the difference equation is asymptotically enforced through the addition of an equality constraint). The two most used algorithms employed for the nonlinear problems are sequential quadratic programming in which a local quadratic approximation to the optimal control problem is solved at each iteration using standard quadratic programs and interior point algorithms. An excellent, open source implementation of the interior point algorithm is the code IPOPT (Wächter & Biegler, 2006). A recent survey of optimization algorithms for nonlinear model predictive control and moving horizon estimation is given in Biegler (2013).

Because of non-convexity, most algorithms find local minima or stationary points rather than global minima; stability may still be achieved using suboptimal (approximate) MPC. In cooperative MPC, the need to maintain exact satisfaction of the underlying difference equation at each iteration of the shared solution of the optimal control problem motivated the use of a first order gradient projection algorithm and suboptimal MPC. A phase I–phase II method of feasible directions algorithm was employed in the complex pursuit–evasion problem discussed in Section 2.7. Further information on these algorithms and the special issues arising from their use in MPC and MHE may be found in the reviews (Biegler, 2013; Binder et al., 2001; Diehl, Ferreau, & Haverbeke, 2009).

2.6.2. Embedded MPC

MPC is typically used in the process industries where the plant being controlled is 'slow'. An exciting recent development is the attention recently being given to model predictive control of small but fast dynamic systems using an embedded system, i.e. a computer system, usually a micro-controller or microprocessor, with a dedicated function. Researchers have developed code to assist implementation of MPC in embedded systems. Thus Bemporad, Oliveri, Poggi, and Storace (2011) describe a hardware-targeted method for ultra-fast MPC via efficient computation of an approximate explicit solution of the optimal control problem on an embedded processor (e.g. a field programmable gate array or an FPGA application-specific integrated circuit), while still guaranteeing closed-loop stability. Houska, Ferreau, and Diehl (2011) have developed software that allows one to define symbolically an optimal control or estimation problem and automatically generates efficient C++ code to implement the solution in an embedded computer. Domahidi, Chu, and Boyd (2013) describe software that allows one to define a sparse second-order cone program for convex optimization problems that is more general than a standard quadratic program and generates very short C++ code that does not rely on any external linear algebra libraries and can be installed in an embedded control system.

Existing formulations of predictive control with off-the-shelf optimization solvers are not guaranteed to work when implemented on embedded systems because the majority of microprocessors in embedded systems do not support IEEE-754 double precision floating-point. Kerrigan, Jerez, Longo, and Constantinides (2013) highlight the importance of the choice of number representation in embedded systems and of co-design of software and hardware. They show how to reduce computational requirements using low precision fixed point arithmetic without unduly compromising performance. Theoretical results are supported by numerical results from implementations on a Field Programmable Gate Array.

2.7. Other topics

There are some topics in MPC that have currently attracted relatively few researchers but have the potential to develop, perhaps considerably. Some of these are briefly described below.

Pursuit–evasion MPC

MPC involving more than one player has concentrated on cooperative 'games' discussed above in Section 2.5. The use of MPC for pursuit–evasion games is explored in the papers (Lee, Polak, & Walrand, 2013; Walrand, Polak, & Chung, 2011) that deal with the problem of defending a harbour in a rectangular channel of width W. Simple linear models are employed for the defender (pursuer) and invader (evader). The objective function is the distance x_i^1 of the invader from the harbour which the invader tries to minimize and the defender to maximize (subscript i specifies invader, subscript d specifies defender and the superscript 1 specifies the first component of a vector). The control and state constraints for the defender are $|u_d| \leq \alpha_d, x_d^1 \geq 0$ and $x_d^2 \in [0, W]$ where W is the width of the channel. The invader has similar constraints ($|u_i| \leq \alpha_i$, $x_i^1 \geq 0$ and $x_i^2 \in [0, W]$) but, in addition, the constraint $|x_i - x_d| \geq \delta$ since the defender can otherwise destroy the invader. The set of defender (invader) control sequences that satisfy all constraints are $\mathcal{U}_N^d(x)$ ($\mathcal{U}_N^i(x, \mathbf{u}_d)$); that $\mathcal{U}_N^i(x, \mathbf{u}_d)$ depends on $\mathbf{u}_d$ is a consequence of the constraint $|x_i - x_d| \geq \delta$. The objective function is

$$\max_{(\mathbf{u}_d \in \mathcal{U}_d(x))} \min_{(\mathbf{u}_i \in \mathcal{U}_i(x, \mathbf{u}_d), k \in \mathbb{I}_{0:N})} \{x_i^1(k)\}.$$

This is a *generalized* max–min problem because the invader's constraint depends on the defender's control; as a consequence, a corresponding min–max problem does not exist nor can a duality gap be defined. The problem above can be solved only using outer-approximations and an exact penalty function (Polak & Royset, 2005) that transforms the problem into a standard min–max problem. Successful results are shown in Lee et al. (2013) for a single defender and two uncoordinated defenders.

MPC with horizon length $N = 1$

It is typical in Power Electronics and Signal Processing that the decision variable is constrained to lie in a finite or countably infinite set necessitating quantization. While the optimal control problem employed in MPC can, in principle, be solved if the decision variable is quantized, its complexity increases exponentially with the horizon N. The requirement for high speed has motivated the use of short horizons such as $N = 1$; this is especially true in the power electronics area where MPC unit horizon length is widely employed. The paper (Aguilera & Quevedo, 2013; Muller, Quevedo, & Goodwin, 2011) provides a significant result, i.e. necessary and sufficient conditions for optimality of quantized model predictive control with horizon 1. The paper also shows that small deviations in model parameters cause a small change from optimality. A few special cases are explored.

Linking MPC to an existing control law

The vast experience gained from controller design for linear systems suggests that it might be beneficial to design, or tune, model predictive controllers to have the same performance as a well designed linear controller when constraints are inactive so that it will inherit its small signal properties. This problem has been explored in several papers, e.g. Di Cairano and Bemporad (2010) and Hartley and Maciejowski (2009). The recent paper (Kong, Goodwin, & Seron, 2013) goes a step further and provides a method for morphing smoothly from an existing control law ($\lambda = 0$) to a full MPC strategy ($\lambda = (1)$). When the control problem is unconstrained, $\lambda = 0$ corresponds to the existing unconstrained controller; otherwise the unconstrained controller, and those corresponding to any value of $\lambda \in [0, 1]$, are suitably modified. The paper (Bemporad, Teel, & Zaccarian, 2004) shows how MPC may be employed in an anti-windup controller to choose compensation signals to minimize the difference between the response of a system with and without control; the system being controlled has unmodelled dynamics.

Feedforward MPC

The paper (Carrasco & Goodwin, 2011) takes previous work by the authors on this topic a stage further and provides, for constrained linear systems, a clean, easily implementable, version of model predictive control that incorporates feedforward. To determine the feedforward component of the control an optimal control problem with horizon N costing the deviation of the output of the nominal system from the predicted reference output and the change in control from its previous value and subject to control and state constraints is solved. The resultant optimal control and output sequences, $\mathbf{u}_{\mathrm{ff}}^0$ and $\mathbf{y}_{\mathrm{ff}}^0$ respectively, are then used in a second optimal control problem to determine the feedback component of the control; the optimal control problem is similar but includes a prediction of the unmeasured output disturbance and a modified control constraint that takes into account the feedforward control sequence $\mathbf{u}_{\mathrm{ff}}^0$. The resultant optimal control sequence is $\mathbf{u}_{\mathrm{fb}}^0$. The control applied to the plant is the first element of the sequence $\mathbf{u}^0 = \mathbf{u}_{\mathrm{ff}}^0 + \mathbf{u}_{\mathrm{fb}}^0$. Examples show the potential advantage of this form of MPC; the advantage depends on the importance of constraints in limiting performance.

Passivity based MPC

The simplification that can be obtained if the system to be controlled is strictly passive is demonstrated in Raff, Ebenbauer, and Allgöwer (2007). Since a strictly passive system is globally asymptotically stable if its storage function is radially unbounded, the storage function, which is a control Lyapunov function, may be employed as a terminal cost function to ensure global asymptotic stability of a system under model predictive control. The horizon length N may have any value and a terminal constraint is not needed.

Suboptimal MPC

The difficulty of obtaining global solutions to nonlinear optimal control problems, the requirement in some applications for rapid computation, and the sharing of the optimization process in co-operative MPC, all support the need for a version of MPC that does not require full optimality in the optimal control problem solved online. Extending earlier work (Scokaert et al., 1999), the paper (Pannocchia, Rawlings, & Wright, 2011) establishes nominal robustness of suboptimal MPC.

3. MPC for uncertain systems

3.1. Introduction

As we indicated in the introduction, there is a big gulf in the literature between that dealing with control of deterministic systems and that dealing with control of uncertain systems. The gulf arises from the fact that, for a deterministic control problem, feedback is not necessary. For a deterministic problem, the optimal control for a given state may be determined *either* by solving an open-loop control problem for the given initial state *or* by solving the dynamic programming equations over the horizon length N or T (the dynamic programming solution gives, of course, the optimal control for any admissible state and any horizon). This is not the case for uncertain systems if the decision variable is chosen, as it usually is, to be a control sequence. For the optimal control problem employed in the controller to give the same control for a given state as that obtained by dynamic programming it is necessary for the decision variable to be a control *policy*, i.e. a sequence $\{\nu_0(\cdot), \nu_1(\cdot), \dots, \nu_{N-1}(\cdot)\}$ of control laws. This is often impractical so it is usual to resort to optimization over control sequences.

If, for practical reasons, the decision variable is chosen to be a control sequence, what options are available? The first option is to ignore the disturbance in solving the optimal control problem $\mathbb{P}(x)$ by determining the control action for a given state using the nominal model. This option is discussed in Section 2.2.2; under certain conditions, the resultant controlled system is robustly stable against a sufficiently small additive disturbance. A second option is to take the disturbance into account by requiring that the constraints are satisfied in the optimal control problem $\mathbb{P}(x)$ solved online for all possible realizations of the disturbance sequence. Several variants of this version are discussed in the next subsection. A significant disadvantage of this type of model control is that the 'diameter' of the tube that contains the nominal trajectory as well as the trajectories due to all the disturbance sequences can become very large, especially if the system being controlled is unstable. This may result in the optimal control problem becoming infeasible i.e. no control sequence exists such that the constraints are satisfied for all disturbance sequences. A second disadvantage, a consequence of the fact that the decision variable is an open-loop control sequence, is that the tube generated, in principle, in solving $\mathbb{P}(x)$ deviates considerably from that generated by an optimal infinite horizon feedback controller that tends to keep all trajectories in a 'small' neighbourhood of the nominal trajectory; closed loop behaviour differs considerably from predicted behaviour. This is consequence of the fact that the decision variable is an open-loop control sequence.

Tube-based MPC was developed in order to mitigate these disadvantages of employing an (open-loop) control sequences as the decision variable. This method of robust MPC employs local feedback around a nominal or reference trajectory thereby keeping the trajectories resulting from the disturbance sequences in a possibly small neighbourhood of the nominal trajectory and is similar, in this respect to the optimal infinite horizon feedback controller. This version of MPC is discussed in the next subsection.

In addition to the literature on robustness of model predictive control to external disturbances and to the similar problem of robustness to parametric uncertainty, there is a very limited literature on the more complex problem of robustness against unstructured uncertainty; this literature is also discussed below as are the problems of tracking random references, output model predictive control in which the uncertainty is due to state estimation error, stochastic MPC in which transgression of constraints with low probability is permitted and adaptive model predictive control which remains a largely open problem.

3.2. Robust MPC

A distinguishing feature of robust MPC is that the state and control constraints have to be satisfied by the controlled system for *all* realizations of the uncertainty.

3.2.1. Robustness against additive disturbances

Introduction:

In the case of additive disturbances the system is described by

$$x^+ = f(x, u, w) \tag{3.1}$$

in which x and u are as specified in (2.1) and are required to satisfy the same constraints and $w \in \mathbb{R}^n$ is an additive disturbance that is assumed to lie in a compact subset $\mathbb{W}$ of $\mathbb{R}^n$. Boundedness of w is required to permit satisfaction of the constraints in the optimal control problem for all possible realizations of the sequence $\mathbf{w} = \{w(0), w(1), \dots, w(N-1)\} \in \mathcal{W}_N \triangleq \mathbb{W}^N$. To establish robust stability the earlier literature on robust MPC employed the value function $V_N^0(\cdot)$ of the nominal optimal control problem $\mathbb{P}$ as a Lyapunov function. If, for example, nominal MPC is employed, then $V_N^0(f(x, \kappa_N(x), 0)) \leq V_N^0(x) - \alpha_1(|x|)$ and $V_N^0(f(x, \kappa_N(x), w)) \leq V_N^0(x) - \alpha_1(|x|) + \delta(x, w)$, $\delta(x, w) \triangleq V_N^0(f(x, \kappa_N(x), w)) - V_N^0(f(x, \kappa_N(x), 0))$. If it is possible to bound $\delta(x, w)$ by c, say, and c is sufficiently small then it is possible to show, as in Khalil (2002) and Rawlings and Mayne (2009), that a level set of the value function $V_N^0(\cdot)$ is robustly asymptotically stable and positively invariant for the closed-loop system $x^+ = f(x, u, w)$ with a region of attraction that is a larger level set of $V_N^0(\cdot)$; a set X is robustly positive invariant for $x^+ = f(x, w)$ if, for each $x \in X$, $f(x, w) \in X$ for all $w \in \mathbb{W}$. Recently, robust stability has been studied using the theory of input-to-state stability (ISS) (Jiang & Wang, 2001; Sontag & Wang, 1995). The system $x^+ = f(x, w)$ is globally input-to-state stable if there exists a $\beta \in \mathcal{KL}$ and a $\gamma \in \mathcal{K}$ such that, for each $x_0 \in \mathbb{R}^n$, each $\mathbf{u} \in \ell_\infty$,

$$|x(k; x_0, \mathbf{w})| \leq \beta(|x_0|, k) + \gamma(\|\mathbf{w}_{0:k-1}\|)$$

for all $k \in \mathbb{Z}_{\geq 0}$ in which $\|\mathbf{w}_{a:b}\| \triangleq \sup_i\{|w(i)| \mid a \leq i \leq b\}$. A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be an ISS-Lyapunov function for system (3.1) if there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot)$ and $\alpha_3(\cdot)$ and a $\mathcal{K}$ function $\sigma(\cdot)$ satisfying $V(x) \in [\alpha_1(|x|), \alpha_2(|x|)]$ and $V(f(x, w)) - V(x) \leq -\alpha_3(|x|) + \sigma(|w|)$ for all $x, w \in \mathbb{R}^n$. It has been established in Jiang and Wang (2001) that a system is ISS if and only if it admits an ISS-Lyapunov function. There exist local versions of the result in which x, u and w lie in bounded sets. Examples of the use of ISS theory include Lazar, Muñoz de la Peña, Hemeels, and Alamo (2008), Limon et al. (2009) and Pin, Raimondo, Magni, and Parisini (2009). There are subtle differences between the classical Lyapunov and input-to-state definitions of stability especially when the system is discontinuous; these and related issues including implications for MPC are thoroughly explored in Lazar et al. (2009, 2013). Incremental ISS is a useful tool for MHE and robust MPC (Bayer, Bürger, & Allgöwer, 2013).

Whichever method is used, the real problem in establishing stability for robust MPC is showing that $|\delta(x, w)|$ is bounded by a constant or by a $\mathcal{K}$ function of $|w|$; continuity of the value function cannot usually be established if there are constraints on the state. Input-to-state stability has the advantage over Lyapunov stability in that it merely requires that $\delta(x, w)$ is bounded by a $\mathcal{K}$ function rather than by a constant; however Lyapunov theory permits characterization of the robust positively invariant set to which trajectories converge, an important performance issue. Also, as cogently pointed out in Riggs and Bitmead (2012), if the optimal control problem is recursively feasible and has state constraints, the system state is automatically bounded for all $k \geq 0$, independently of performance considerations.

The very large literature on robust model predictive control makes an exhaustive survey impractical. Instead we illustrate the field with a few examples that provide further references.

Robust MPC: decision variable a control sequence

The case of inherent stability of nominal MPC in which the disturbance is ignored and the decision variable is a control is considered in Section 2.2. Here we consider the case when the decision variable is also a control sequence but the disturbance is taken into account in the optimal control problem solved online. The state and control constraints, and the terminal constraint if employed, are required to be satisfied for all possible disturbance sequences. There are several options for the cost function $V_N(\cdot)$; the usual choices are the nominal cost function (the cost of the nominal state and control sequences) as defined in (3.2)) or a max cost, defined in (3.3):

$$V_N(x, \mathbf{u}) \triangleq V_f(x^{\mathbf{u},\mathbf{0}}(N; x)) + \sum_{i=0}^{N-1} \ell(x^{\mathbf{u},\mathbf{0}}(i; x), u(i)) \tag{3.2}$$

$$V_N(x, \mathbf{u}) \triangleq \max_{\mathbf{w}} \Bigg\{ V_f(x^{\mathbf{u},\mathbf{w}}(N; x)) \tag{3.3}$$

$$+ \sum_{i=0}^{N-1} \ell(x^{\mathbf{u},\mathbf{w}}(i; x), u(i)) \mid \mathbf{w} \in \mathbb{W}^N \Bigg\} \tag{3.4}$$

in which $\mathbf{0}_N \triangleq \{0, 0, \ldots, 0\}$ is the sequence of N zeros and $x^{\mathbf{u},\mathbf{w}}(i; x)$ is the state at time i if the state at time 0 is x, the control sequence is $\mathbf{u}$ and the disturbance sequence is $\mathbf{w}$. With the latter choice, the optimal control problem $\mathbb{P}_N(x)$ becomes a min–max problem; we consider the first choice (use of the nominal cost). To ensure recursive feasibility it is not enough merely to tighten the state constraint as was observed as early as 1993. A strategy for ensuring recursive feasibility, using a robust invariant terminal set and a terminal penalty, was first obtained in Marruedo et al. (2002). An extension that *inter alia*, does not require a terminal set and penalty is given in Grimm, Messina, Tuan, and Teel (2007).

The problem considered in Marruedo et al. (2002) is control of the system $x^+ = f(x, u, w)$ subject to the state and control constraints $x(i) \in \mathbb{X}$ and $u(i) \in \mathbb{U}$ that have to be satisfied for all permissible realizations of the disturbance sequence. In order to ensure recursive feasibility, the optimal control problem solved online is minimization of $V_N(x, \mathbf{u})$ as defined in (3.2) but subject to tightened state constraints $x^{\mathbf{u},\mathbf{w}}(i; x) \in \mathbb{X}_i$, $i = 0, 1, \ldots, N$ (that have to be satisfied for all disturbance sequences $\mathbf{w} \in \mathbb{W}^N$) as well as the usual control constraints $u(i) \in \mathbb{U}$, $i = 0, 1, \ldots, N - 1$. The modified state constraint sets are determined as follows. Let $\mathbf{u}$ be an arbitrary control sequence in $\mathbb{U}^N$ and $\mathbf{x}^{\mathbf{u},\mathbf{0}}(x) = \{x^{\mathbf{u},\mathbf{0}}(0; x), x^{\mathbf{u},\mathbf{0}}(1; x), \ldots, x^{\mathbf{u},\mathbf{0}}(N; x)\}$ the associated nominal state sequence with initial state x. If $f(\cdot)$ is Lipschitz continuous, the state trajectory $x^{\mathbf{u},\mathbf{w}}(x)$ due to control sequence $\mathbf{u}$ and an arbitrary disturbance sequence $\mathbf{w} \in \mathbb{W}^N$ satisfies $x^{\mathbf{u},\mathbf{w}}(i; x) \in x^{\mathbf{u},\mathbf{0}}(i; x) + \sigma_i|\mathbb{W}|\mathcal{B}$ in which $\{\sigma_i \mid i = 1, 2, \ldots\}$ is an increasing sequence easily computed from knowledge of the Lipschitz constant for $f(\cdot)$, $\mathcal{B}$ is the unit ball in $\mathbb{R}^n$ and $|\mathbb{W}| \triangleq \max_w\{|w| \mid w \in \mathbb{W}\}$; hence $x^{\mathbf{u},\mathbf{w}}(x)$ lies in an expanding *tube* $\{x, x(1) + \sigma_1|\mathbb{W}|\mathcal{B}, \ldots, x(N) + \sigma_N|\mathbb{W}|\mathcal{B}\}$.

The optimal control problem $\mathbb{P}_N(x)$ solved online can now be defined; it is minimization of $V_N(x, \mathbf{u})$ defined in (3.2) subject to the constraints that $x^{\mathbf{u},\mathbf{0}}(i; x) \in \mathbb{X}_i$, $u(i) \in \mathbb{U}$, $i = 0, 1, \ldots, N - 1$ and $x^{\mathbf{u},\mathbf{0}}(N; x) \in X_f^*$; $\{\mathbb{X}_i \mid i = 0, 1, \ldots, N\}$ is a decreasing sequence of sets defined, for each i, by $\mathbb{X}_i \triangleq \mathbb{X} \sim \sigma_i|\mathbb{W}|\mathcal{B}$ ($\sim$ denotes Pontryagin set subtraction so that $x \in \mathbb{X}_i$ implies $x + \sigma_i|\mathbb{W}|\mathcal{B} \in \mathbb{X}$). Also, the terminal cost $V_f(\cdot)$ and terminal constraint set X_f^* are required to satisfy the following conditions: there exists a local controller $\kappa_f(\cdot)$ such that, for all $x \in X_f \triangleq \{x \mid V_f(x) \leq a\} \subset \mathbb{X}$, $x^+ \triangleq f(x, u, 0) \in X_f^* \triangleq \{x \mid V_f(x) \leq b\}$, $V_f(x^+) \leq V_f(x) - \ell(x, \kappa_f(x))$ and $X_f^* + \sigma_{N+1}|\mathbb{W}|\mathcal{B} \subset X_f$. It is possible, under reasonable assumptions, to satisfy these conditions if $|\mathbb{W}|$ is sufficiently small.

Suppose then that $|\mathbb{W}|$ is sufficiently small to ensure the existence of X_N, the feasible set for $\mathbb{P}_N$, and of X_f^*. Let x be the current state (at time t) and suppose that $\mathbb{P}_N(x)$ is feasible. Then $\mathbf{u}^0(x)$, the solution of $\mathbb{P}_N(x)$, and $\mathbf{x}^0(x)$, the associated state sequence, exist and satisfy the tightened state and control constraints; also $V_N^0(f(x, \kappa_N(x), 0)) \leq V_N^0(x) - \ell(x, \kappa_N(x))$. Due to the disturbance, the actual successor state at time $t + 1$ is $x^+ \triangleq f(x, \kappa_N(x), w)$. The extended control sequence $(\mathbf{u}^0(x), \kappa_f(x^0(N; x))) = \{u^0(0; x), u^0(1; x), \ldots, u^0(N-1; x), \kappa_f(x^0(N; x))\}$ steers the nominal system from x to $x^0(N; x) \in X_f^*$ in N steps and, by definition of X_f^*, to $z \triangleq f(x^0(N; x), \kappa_f(x^0(N; x)), 0) \in X_f^*$ in $N + 1$ steps; hence this extended control sequence steers the actual system with disturbance sequence $\{w, \mathbf{0}_N\}$ from x to a point in X_f. This is equivalent to saying that the control sequence $\tilde{\mathbf{u}} \triangleq \{u^0(1; x), \ldots, u^0(N - 1; x), \kappa_f(x^0(N; x))\}$ steers the nominal system from state $x^+ = f(x, \kappa_N(x), w)$ to X_f in $N - 1$ steps and, thus, to X_f^* in N steps. Thus the optimal control problem is recursively feasible.

Since $x^+ \triangleq f(x, \kappa_N(x), w) = x^0(1; x) + f(x, \kappa_N(x), w) - f(x, \kappa_N(x), 0)$ in which $|f(x, \kappa_N(x), w) - f(x, \kappa_N(x), 0)| \leq c_1|w|$ for some $c_1 \in (0, \infty)$, it follows from the discussion above, and the assumption of Lipschitz continuity of $f(\cdot)$ and $\ell(\cdot)$ that $V_N^0(x^+) \leq V_N(x^+, \tilde{\mathbf{u}}) \leq V_N(x^0(1; x), \tilde{\mathbf{u}}) + c_2|w|$ for some $c_2 \in (0, \infty)$. The stability conditions ensure that $V_N(x^0(1; x), \tilde{\mathbf{u}}) \leq V_N^0(x) - \ell(x, \kappa_N(x))$ so that

$$V_N^0(x^+) \leq V_N^0(x) - \alpha(|x|) + \sigma(|w|)$$

in which, with the usual assumption on $\ell(\cdot)$, $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$. The other requirement, that $V_N^0(\cdot)$ is bounded above and below by $\mathcal{K}_\infty$ functions, for $V_N^0(\cdot)$ to be a ISS-Lyapunov function, is usually satisfied. Hence the closed-loop system $x^+ = f(x, \kappa_N(x), w)$ is locally input-to-state stable in X_N, the positively invariant feasible set for $\mathbb{P}_{\mathbb{N}}$.

Although this solution is ingenious, and has motivated further research, it should be recognized that the sequence $\{\sigma_i\}$ can increase very rapidly, especially if $f(\cdot)$ is unstable, so that closed-loop stability is obtained only if $|\mathbb{W}|$ is very small. This is a consequence of optimizing over control sequences.

This method of achieving robust MPC of nonlinear systems is extended in Pin et al. (2009). The most important contribution of this paper is the provision of a method to handle state-dependent uncertainties. If the state is required to lie in a bounded set, the method in Marruedo et al. (2002), in which the tightened state constraint sets are computed offline, can be extended but in a conservative fashion. To overcome this difficulty, Pin et al. in Pin et al. (2009) reduce conservatism by computing the tightened sets online thereby increasing the feasible set for the optimal control problem $\mathbb{P}_N$; the modification of the constraint sets depends on the state and is simple enough to permit online implementation. The controller employs optimization over control sequences whose horizon is less than the horizon of the optimal control problem as in Magni, Nicolao, and Scattolini (2001).

Several papers (Kerrigan & Maciejowski, 2004; Lazar et al., 2008; Limon, Alamo, Salas, & Camacho, 2006a; Scokaert & Mayne, 1998) address the problem of min–max MPC in which the cost function employed in the optimal control problem $\mathbb{P}_N$ is defined by (3.3). These papers employ optimization over control policies rather than control sequences. This methodology is theoretically preferable for reasons discussed above but its computational complexity inhibits its practical use. However, Kerrigan and Maciejowski (2004) shows that if the 1-norm is used to define the cost, optimization over policies may be achieved by solving a linear program albeit one whose complexity increases exponentially with horizon length.

Tube-based MPC

The major disadvantage of using a control sequence as the decision variable when the system being controlled is subject to disturbance is its conservatism; the tube of trajectories generated in solving $\mathbb{P}_N(x)$ is a poor prediction of closed-loop behaviour because of the open-loop nature of the problem $\mathbb{P}_N(x)$. The ideal would be to optimize over control policies; since this is impractical, tube-based MPC employs a simply parameterized local policy.

The useful concept of tubes originated with the seminal papers (Bertsekas & Rhodes, 1971a,b) and remains an active area of research (Kurzhanski & Filippova, 1993). It was initially used in MPC in Chisci et al. (2001) and Mayne and Langson (2001). In contrast to the methods discussed above that enforce, in solving $\mathbb{P}_N$, satisfaction of the constraints, by the *open-loop trajectories* (nominal and perturbed), tube-based methods ensure the *closed-loop trajectories* (trajectories of $x^+ = f(x, \kappa_N(x), w)$) all lie in a tube that satisfies the constraints.

If the system $x^+ = f(x, u, w) = Ax + Bu + w$ is linear, as considered in Chisci et al. (2001), Mayne and Langson (2001) and Mayne et al. (2005) then, employing the control parametrization $u = Kx + v$, first proposed in Rossiter, Kouvaritakis, and Rice (1998), yields $x^+ = A_K x + Bv + w$, $A_K \triangleq A + BK$. Defining the nominal or reference system by $z^+ = A_K z + Bv$ and setting $x = z + e$ yields the difference equation $e^+ = A_K e + w$ for $e = x - z$. If $\mathbb{W}$ is polytopic, there exists a polytopic set S that is positively invariant for $e^+ = A_K e + w$ (Kolmanovsky & Gilbert, 1998). Methods for computing positively invariant outer approximations of S are given in Raković, Kerrigan, Kouramas, and Mayne (2005). Thus, given S or an outer approximation $\hat{S}$, and nominal control and state sequences $\{v(0), v(1), v(2), \ldots\}$, $\{z(0) = z, z(1), z(2), \ldots\}$ satisfying $z^+ = Az + Bu = Ax + Bv + BKe$, $z(0) = x$, it follows that the perturbed control and state sequences $\{u(0), u(1), \ldots\}$ and $\{x(0), x(1), \ldots\}$ satisfy $x(t) \in z(t) + S$ and $u(t) \in v(t) + KS$ for all $t \in \mathbb{I}_{\geq 0}$. If we choose the nominal, or reference, trajectories to satisfy the tightened constraints $z \in \mathbb{X} \sim S$ and $v \in \mathbb{U} \sim KS$ and such that the nominal state $z(t)$ converge to 0, using model predictive control so that $v = \kappa_N(z)$, then the actual trajectories will satisfy $x \in \mathbb{X}$ and $u \in \mathbb{U}$ as desired and $x(t)$ will converge to S. The tightened constraints may be determined by solving an optimization problem thereby avoiding computation of the sets S, $\mathbb{X} \sim S$ and $\mathbb{U} \sim KS$. The state trajectory $\{x(i)\}$ of the *controlled* system therefore lies in the tube $T_x \triangleq \{x, z(1) + S, z(2) + S, \ldots\}$ in which S is bounded (even if A is not a stability matrix); the tube T_x is robustly positively invariant for the controlled system $x^+ = Ax + B(\kappa_N^*(z) + K(x - z)) + w$; if $x \in T_x$, then $x^+ \in T_x$ for all $w \in \mathbb{W}$. If the nominal state $z(t)$ converges to 0, the actual state $x(t)$ converges to S. The procedure in Chisci et al. (2001) differs slightly; K and S and the tightened constraints are determined as above but the nominal trajectory is determined differently by solving an optimal control problem using the actual current actual state x instead of the current nominal state z as the initial state for the nominal optimal control problem; choosing the stage cost in the optimal problem to be a positive definite function of v (rather than of (z, v)) enables the nominal control $\mathbf{v}$ to decrease the nominal value function despite the deviation of x^+ from z^+ enabling stability to be established.

The procedures above cost, for simplicity, the nominal trajectory. Raković et al. in a more sophisticated analysis, present in Raković (2012) and Rakovič, Kouvaritakis, Cannon, Panos, and Findeisen (2012) an ingenious solution to the 'ideal' online optimal control problem, viz minimization of a min–max cost function where the maximization is over disturbance sequences and the minimization is over control policies. This is normally an intractable problem overcome in this paper by the novel parametrization. Employment of a cost function in which the stage and terminal costs take the form $\ell(x, u) = |x|_{\mathcal{Q}} + |u|_{\mathcal{R}}$ and $V_f(x) = |x|_{\mathcal{P}}$, in which the sets $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{P}$ are polytopic and $|x|_{\mathcal{Q}}$ denotes the distance of x from the set $\mathcal{Q}$, transforms the optimal control problem into a linear program; the novel form for $\ell(\cdot)$ facilitates the stability proof. The paper establishes that the solution obtained is equivalent to the dynamic programming solution in, at least, three cases.

The tube-based procedure has been extended to control nonlinear systems; the difficulty is finding an alternative to the local linear stabilizing feedback $u = v + Kz$. In Mayne, Kerrigan, van Wyk, and Falugi (2011), the local linear feedback $u = v + Kz$ is replaced by nonlinear MPC; the cost function in the associated optimal control problem is a measure of the deviation of trajectories of $x^+ = f(x, u, w)$ (with initial state equal to the current state x) from a nominal or reference trajectory satisfying $z^+ = f(z, v, 0)$. The value function for this optimal control problem is used to define a control and a tube, 'centred' on the nominal trajectory, in which all of the perturbed trajectories of the *controlled* system lie. However, the tube does not have the simple structure $\{z(i) + S \mid i = 0, 1, \ldots, N\}$ that is obtained when the system is linear. Consequently offline stochastic optimization is needed to determine the amount by which the constraints, employed in determining the reference trajectory, need to be tightened. A similar cost function that measures deviation from a simple exponential trajectory is already employed in industrial MPC (Qin & Badgwell, 2003).

3.2.2. *Robustness against unstructured uncertainty*

There has been relatively little work on robustness against unstructured uncertainty. A pioneering study appears in Løvaas, Serón, and Goodwin (2008); in order to utilize known insights, the standard $\mathcal{H}_\infty$ model is employed, that is, the uncertain open-loop system is described by a feedback connection of a known linear time-invariant system and an unknown, but norm-bounded, causal operator Δ that may, therefore, be nonlinear, time-varying and infinite dimensional provided only that it has finite l_2 gain; the disturbance due to Δ can therefore be arbitrarily large permitting a global region of attraction when the system has control constraints. When the linear system is unstable, it is assumed that Δ also has finite l_∞ gain. The unstructured uncertainty is specified in the frequency domain using weighting functions. The problem studied is output MPC so that the proposed method uniquely handles two types of uncertainty, state estimation error and unstructured uncertainty. The paper also provides a good guide to related research.

A related approach to the problem is described in Falugi and Mayne (2014) which considers a nonlinear system $x^+ = f(x, u, w)$ where $w = \Delta(y(\cdot))$ and $y = h(x, u)$; Δ is an unknown, causal operator with finite l_∞ gain. The system is subject to control and state constraints $u \in \mathbb{U}$ and $x \in \mathbb{X}$. It is shown that by restricting the input y to the uncertainty operator Δ, i.e. by adding a hard constraint on y, the problem is converted into a simpler, well studied problem of robustness against a bounded disturbance. The latter problem is solved using a tube-based model predictive controller for nonlinear systems (Mayne et al., 2011) described above.

3.2.3. *MPC for tracking a randomly varying reference*

The deterministic tracking problem, i.e. tracking a fixed reference (desired output) is discussed in Section 2.2.4. The tracking problem when the reference is time-varying but converging asymptotically to a constant has been examined in, *inter alia* (Chisci & Zappa, 2003; Limon et al., 2008, 2010; Maeder, Borrelli, & Morari, 2009; Maeder & Morari, 2010; Muske & Badgwell, 2002; Pannocchia & Kerrigan, 2005; Pannocchia & Rawlings, 2003). In Limon et al. (2008) the problem of designing a controller for a linear system to track a piecewise constant reference r is addressed; the interesting feature of this problem is that large random jumps in the value of r are permitted. The system is described by $x^+ = Ax + Bu$, $y = Cx$ and the control is required to stabilize the system, ensure the control and state constraint $(x, u) \in \mathbb{Z}$ is satisfied, and that the tracking error $e \triangleq y - r$ converges to zero if r is ultimately constant. For each constant reference r, it is possible, under reasonable conditions, to design a model predictive controller such that the equilibrium set point $(x_r, u_r) \in \mathbb{Z}$, which satisfies $x_r = Ax_r + Bu_r$ is asymptotically stable for the closed-loop system $x^+ = Ax + B\kappa_N(x, r)$ with a region of attraction $X_N(r)$ that is the feasible set for the optimal control problem. The nature of the tracking problem when r varies can be seen by considering the situation when the closed-loop system is tracking r_1, say, and the reference suddenly changes to r_2. At the time of the change, the current state x lies in $X_N(r_1)$ and may not lie in $X_N(r_2)$, so the control cannot switch from $\kappa_N(r_1)$ to $\kappa_N(r_2)$. To deal with this, Limon et al. (2008) proposes the use of an artificial reference r^* such that the optimal control problem $\mathbb{P}_N(x, r^*)$ is always feasible and $r^* \to r$ if r is constant; the strategy is similar to the reference governor strategy proposed in Gilbert and Kolmanovsky (1999). The strategy proposed in Limon et al. (2008) is to employ a modified optimal control problem $\mathbb{P}_N(x, r)$ in which the cost function is

$$V_N(x, \mathbf{u}, r^*) = \sum_{i=0}^{N-1} \ell(x^{\mathbf{u}}(i; x), u(i), r^*) + V_f(x^{\mathbf{u}}(N, x), r^*) + V(r, r^*).$$

The functions $\ell(x, u, r)$, $V_f(x, r)$ and $V(r, r^*)$ all achieve their minimum value of zero when $x = x_r$, $u = u_r$ and $r^* = r$. The optimization problem is minimization of $V_N(x, \mathbf{u}, r^*)$ with respect $(\mathbf{u}, r^*)$ subject to the state-control constraint, the constraint $r^* \in \mathcal{R}$ (the set of admissible references) and a suitable terminal constraint. Without $V(\cdot)$ in the cost function $V_N(\cdot)$, the state x of the controlled system converges to the set $X^{\mathcal{R}} \triangleq \{x_r \mid r \in \mathcal{R}\}$. Due to the term $V(\cdot)$ in $V_N(\cdot)$, the state x of the controlled system converges to x_r; indeed, under standard assumptions, x_r is asymptotically stable for the controlled system. The reference r can change suddenly from one value in $\mathcal{R}$ to another; the problem $\mathbb{P}_N$ remains feasible and the state remains in $X_N^{\mathcal{R}} \triangleq \cup_{r\in\mathcal{R}} X_N(r)$.

If the reference varies randomly, a different strategy is required. In Falugi and Mayne (2013b) control of the system $x^+ = Ax + Bu$, $y = Cx$ is considered. The output y is required to track the random reference r whose evolution satisfies $r^+ = r + \xi$ where ξ is a random variable taking values in the compact set Ξ. The proposed control strategy also employs an artificial reference r^*, not as a control variable but rather as a state variable that can be altered. At each state (x, r^*), a standard optimal control problem $\mathbb{P}_N(x, r^*)$ is solved; the problem is minimization of

$$V_N(x, \mathbf{u}, r^*) = \sum_{i=0}^{N-1} \ell(x^{\mathbf{u}}(i; x), u(i), r^*) + V_f(x^{\mathbf{u}}(N, x), r^*)$$

with respect to $\mathbf{u}$ subject to the control and state constraints and a terminal constraint $x^{\mathbf{u}}(N; x) \in X_f(r)$. For each $r \in \mathcal{R}$, the pair $(V_f(\cdot, r), X_f(r))$ satisfies the normal stability conditions. A simplified version of the strategy employed in Falugi and Mayne (2013b) follows. At time 0, r^* is chosen to minimize $|r - r^*|$ subject to the constraint that $\mathbb{P}_N(x, r^*)$ is feasible. Subsequently, r^* is left unaltered from its previous value when $x \notin X_f^{\mathcal{R}} \triangleq \cup_{r\in\mathcal{R}} X_f(r)$; when $x \in X_f^{\mathcal{R}}$, r^* is chosen to minimize $|r - r^*|$ with respect to r^* subject to the constraint $x \in X_f(r^*)$. It can be shown, under reasonable conditions, that the state x converges to, and then remains in, the set $X_f^{\mathcal{R}}$, which is positive invariant for the controlled system, $x^+ = f(x, \kappa_N(x, r^*))$ and that the tracking error is bounded.

3.2.4. *Output MPC*

Despite the fact that the state of a system is not usually accessible, most of the literature on MPC deals with the case when the state is accessible. The reason for this neglect is the fact that, if the state is estimated, the controller has to be designed so that the controlled system is robust against estimation error. In practice nominal MPC is employed by replacing the state with the estimated state in the optimal control problem solved online.

Output MPC of the system $\dot{x}_1 = Ax + Bu + \phi(x, u), \dot{x}_2 = \psi(x, u)$, $y = (Cx_1, x_2)$ is considered in the early paper (Imsland, Findeisen, Bullinger, Allgöwer, & Foss, 2003). The control u is subject to a hard constraint. A high gain observer is employed and the control determined using the nominal model to define the cost function that includes a terminal cost and is minimized over permissible control sequences subject to satisfaction of a terminal constraint. Lipschitz continuity of the implicit model predictive control law is assumed. Robustness against a sector bounded static nonlinearity is established.

A simple alternative (Mayne, Raković, Findeisen, & Allgöwer, 2006), when the system to be controlled is linear, is to employ model predictive control to control the estimator. The system to be controlled is described by $x^+ = Ax + Bu + w, y = Cx + \xi$ where the disturbances w and ξ lie in known compact sets. The standard estimator $\hat{x}$ of the state x satisfies the difference equation $\hat{x}^+ = A\hat{x} + Bu + Le$ where $e \triangleq y - C\hat{x} + \xi = C\tilde{x} + \xi$ is the innovation and $\tilde{x} \triangleq x - \hat{x}$ is the state estimation error. The parameter L can be chosen to be the optimal gain for the steady state Kalman estimator. The control objective is to steer the state x to as small a neighbourhood of the origin as possible while satisfying the hard constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Since w and ξ lie in known compact sets, it is possible to use tube-based MPC, as described in

Section 3.2.1, to generate nominal state and control trajectories, $\mathbf{z}$ and $\mathbf{v}$ respectively, such that $\hat{x} \in z+S_1$ and $x \in \hat{x}+S_2 = z+S_1+S_2$ where S_1 and S_2 are computable subsets of $\mathbb{R}^n$; S_1 bounds $\hat{x}-z$ and S_2 bounds $x - \hat{x}$. If the tightened constraint sets on z and v in the optimal control problem are chosen appropriately, the state x and control u of the *controlled* system satisfy $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Since the controller ensures the nominal (reference) state z and control v converge to the origin, $\hat{x}$ converges to the set S_1 and x to the set $S_1 + S_2$ and then remain in these sets.

The tube based solution rests on the assumption that the disturbances w and ξ are bounded. If, instead, the disturbances are Gaussian, satisfaction of hard state and terminal constraints is impossible. Thus these constraints have to be relaxed as, for example, in Yan and Bitmead (2005). This paper is a contribution to stochastic model control and is therefore discussed in Section 3.4.

When the system is nonlinear moving horizon estimation may be used to estimate the state; see Section 3.3. However, given its complexity, it is difficult to see how the estimator can be controlled in a manner analogous to the linear case. The stability problem of nonlinear output MPC that employs moving horizon estimation does not appear to have received any attention apart from early simplistic treatment.

A recent paper (Copp & Hespanha, 2014a) proposes an original and convincing method for solving the problem of output MPC. Simplifying to the time-invariant case, the system considered is

$$x^+ = f(x, u, w), \qquad y = h(x) + v$$

where the control u is constrained to lie in the set $\mathbb{U} \subset \mathbb{R}^m$, the disturbance w lies in the set $\mathbb{W} \subset \mathbb{R}^p$ and the measurement noise $v \in \mathbb{R}^r$. The novel proposal in Copp and Hespanha (2014a,b) is to replace separate solutions of the control and estimation problems by solving a single min–max problem that combines estimation and control. Given any sequences $\{a(t)\}$ let $\mathbf{a}_{p:q}$ denote the restricted sequence $\{a(p), a(p+1), \ldots, a(q)\}$. Simplifying again, the cost function employed in the infinite horizon optimal control/estimation problem at time t is:

$$\begin{aligned} &V_\infty((\mathbf{y}_{0:t}, \mathbf{u}_{0:t}), (x(0), \mathbf{w}_{0:\infty}, \mathbf{u}_{t:\infty})) \\ &\quad = \sum_{s=t}^{\infty} \ell(x(s), u(s)) - \sum_{s=0}^{t} |y(s) - h(x(s))|_R^2 - \sum_{s=0}^{\infty} |w(s)|_Q^2 \end{aligned}$$

in which $(\mathbf{y}_{0:t}, \mathbf{u}_{0:t})$ is the (hyper-)state at time t and $(x(0), \mathbf{w}_{0:\infty}, \mathbf{u}_{t:\infty})$ is the decision variable. The infinite horizon optimal control/estimation problem is

$$V_\infty^0(\mathbf{y}_{0:t}, \mathbf{u}_{0:t}) = \min_{u_{t:\infty}} \max_{(x(0), \mathbf{w}_{0:\infty})} V_\infty((\mathbf{y}_{0:t}, \mathbf{u}_{0:t}), (x(0), \mathbf{w}_{0:\infty}), \mathbf{u}_{t:\infty})$$

subject to the constraints on u and w and satisfaction of the difference equation. A finite horizon version that has an ISS terminal cost $V_f(\cdot)$ is similarly defined. It is shown that this version reduces to conventional MPC if the state is measured perfectly and reduces to conventional MHE if $\ell(\cdot)$ and $V_f(\cdot)$ are replaced by 0. Let $(\hat{x}(0), \hat{\mathbf{w}}_{0:\infty}, \mathbf{u}^*_{t:\infty})$ denote a solution of the min–max problem; the model predictive control is the first element of the sequence $\mathbf{u}^*_{t:\infty}$. Under the assumption that the infinite horizon min–max problem always has a saddle point solution, the authors show that, if w and v are uniformly bounded, the state of the controlled system is uniformly bounded; moreover, if $w(t)$ and $v(t)$ converge to the origin, the state of the controlled system also converges to the origin. Under extra assumptions, similar results hold for the finite horizon case. An example, control of the tip of a flexible beam, illustrates the performance of the controller. In Copp and Hespanha (2014b) the authors provide an efficient interior point algorithm for solving the min–max problem.

3.3. Moving horizon estimation

Conventional observers or the Kalman filter may, of course, be employed to estimate the inaccessible state of a linear system. When the system is nonlinear, there are several alternatives such as the extended Kalman filter, the unscented Kalman filter and the particle filter. For use in MPC, in which an optimal control problem has to be solved at each sampling time, it is natural to consider the use of moving horizon estimation which requires the online solution of a similar optimal control problem. There is some experimental evidence (Haseltine & Rawlings, 2005) to support this choice. A useful, early, review of moving horizon estimation is given in Findeisen, Imsland, Allgöwer, and Foss (2003).

Moving horizon estimation of the state of a system described by

$$x^+ = f(x, u, w), \qquad y = h(x) + v$$

is considered in Rao, Rawlings, and Mayne (2003). An estimate of the actual state sequence $\mathbf{x}_t \triangleq \{x(t-T), x(t-T+1), \ldots, x(t)\}$ given an input/output record $I_t \triangleq (\mathbf{y}, \mathbf{u})_{t-1} \triangleq \{y(t-T), y(t-T+1), \ldots, y(t-1), u(t-T), u(t-T+1), \ldots, u(t-1)\}$ and a function $V^0_{t-T}(\cdot)$ (that summarizes information on $x(t-T)$ prior to time $t-T$ or an estimate thereof) is obtained by minimizing with respect to $(x, \mathbf{w}_t)$ the cost $V_t(x, \mathbf{w}_t)$ defined by

$$\begin{aligned} V_t(x, \mathbf{w}_t) &= V^0_{t-T}(x) + \sum_{t-T}^{t-1} |w(i)|_Q^2 \\ &\qquad + |y(i) - h(x^{\mathbf{u}_t, \mathbf{w}_t}(i; x, t-T))|_R^2 \end{aligned}$$

in which $x^{\mathbf{u}_t, \mathbf{w}_t}(i; x, t-T)$ denotes the solution of $x^+ = f(x, u, w)$ at time i given that the initial state is x at time $t-T$, the control sequence is $\mathbf{u}_t$ and the disturbance sequence is $\mathbf{w}_t$. Let $(\hat{x}(t), \hat{\mathbf{w}}_t)$ denote the solution to this problem. Then $x^{\mathbf{u}_t, \hat{\mathbf{w}}_t}(\hat{x}(t), t)$ is the estimate of the actual state sequence $\mathbf{x}_t$, and its last element, $x^{\mathbf{u}_t, \hat{\mathbf{w}}_t}(t; \hat{x}_t, t-T)$, is the estimate of $x(t)$.

A difficulty with this procedure is determination of $V^0_{t-T}(\cdot)$ or an estimate of it. This difficulty can be avoided by replacing $V^0_i(\cdot)$ by zero for all i (i.e. no prior information); this reduces the accuracy of the estimate. Alternatively, $V^0_t(z)$ at any (z, t) can be determined by solving the optimal control problem defined above with the *extra* constraint $x^{\mathbf{u}_t, \mathbf{w}_t}(i; x, t-T) = z$. These issues, and stability properties of the moving horizon estimator, are discussed in Ji, Rawlings, Hu, Wynn, and Diehl (2013), Rawlings and Ji (2012) and Rawlings and Mayne (2009); the stability properties of the full information estimator ($T \equiv t$) are understood; there remain unsolved problems in the moving horizon case (constant T).

3.4. Stochastic MPC

Soft constraints

In Section 3 on robust MPC, the disturbances were assumed to be bounded and constraints were required to be satisfied for all possible realizations of the disturbance process. In stochastic MPC, on the other hand, disturbances are assumed to be stochastic and not necessarily bounded and (at least some) constraints are softened, i.e. not required to be satisfied for all realizations of the disturbance. Several methods for softening constraints have been considered in the literature. The first, employed for example in Primbs and Sung (2009), is to replace a hard constraint $z \in Z$ by $E_x z \in Z$ where $E_x(\cdot) \triangleq E(\cdot \mid x(0) = x)$ and E denotes 'expectation'. Softening of a control constraint would not be appropriate in many applications but appears to be acceptable in some. A second method, more widely employed, is to replace $z \in Z$ by $\mathcal{P}_x(z \in Z) \geq p$ where $p \in [0, 1]$, $\mathcal{P}$ denotes probability and $\mathcal{P}_x(\cdot) \triangleq \mathcal{P}(\cdot \mid x(0) = x)$. One argument for employing stochastic MPC is that satisfaction of constraints for all possible realization of the disturbance is unnecessarily conservative and, indeed, impossible if the probability distributions of the uncertain variables possess infinite tails. A second argument is that stochastic disturbances arise in many applications and that performance evaluation should reflect the distributions of these disturbances.

Several issues arise. Because of the difficulty in analysing nonlinear stochastic systems, most of the current literature deals with control of linear systems. It is common, even in stochastic model predictive control, to determine the current control action by minimizing a cost function, subject to constraints, over control sequences. Theoretically, in a random environment, minimization over control policies is much to be preferred but is usually computationally too expensive; nevertheless we report one significant result employing this version of stochastic MPC. Optimizing over control sequences results in difficulties similar to those encountered in robust MPC. If the system considered is linear, one of the most important of these problems, namely the expansion with time of the tube of potential trajectories associated with each open-loop control sequence, can be contained by using the control parametrization $u = Kx + v$ with K such that $A_K \triangleq A + BK$ is a stability matrix. More subtle difficulties also arise that impinge on the specification of the online optimal control problem; we discuss these below.

Much of the literature, deals with the problem on controlling a stochastic linear system satisfying $x^+ = Ax + Bu + w$, $y = Cx + \xi$ where the disturbance w and measurement noise ξ are often, but not always, assumed to be sequences of independent, identically distributed random variables with known distribution. It is usual, as in tube-based methods, to express x as the sum of a deterministic component z and a random component e. With $u \triangleq Kx + v$ and $A_K \triangleq A + BK$ this yields

$$z^+ = A_K z + Bv \qquad e^+ = A_K e + w, \qquad z \triangleq x + e.$$

The cost function is $V_N(x, \mathbf{u}, \mathbf{w})$, the usual quadratic cost of the state x and control u, over horizon N; $\mathbf{u}$ denotes the sequence $\{u(0), u(1), \ldots, N-1\}$ with similar definitions for $\mathbf{w}$ and $\mathbf{v}$ but $\mathbf{x} \triangleq \{x, x(1), \ldots, x(N)\}$. Often a terminal cost $V_f(\cdot)$ is added. The optimal control problem $\mathbb{P}_N(x)$ is minimization of $E_x V_N(x, \mathbf{u}, \mathbf{w})$ given the current state is x and subject to constraints which vary from paper to paper because constraints in stochastic problems cause unique difficulties. Given the current state x and control sequence $\mathbf{v}$, the resultant state sequence is $\mathbf{x}^{\mathbf{v},\mathbf{w}}(x) = \mathbf{z}^{\mathbf{v}}(x) + \mathbf{e}^{\mathbf{w}}(x)$; $e(i) = e^{\mathbf{w}}(i; x)$ a random vector of zero mean and computable variance $\Sigma(i)$ enabling the cost $E_x V_N(x, \mathbf{u}, \mathbf{w})$ to be redefined as a quadratic function of $V_N^*(z, \mathbf{v})$ and $\mathbb{P}_N(x)$ redefined as the deterministic problem $\mathbb{P}_N^*(z)$ of minimizing $V_N^*(z, \mathbf{v})$ subject to satisfaction of $z^+ = A_K z + Bv$ and tightened constraints as discussed below; satisfaction by z of the tightened constraints ensures satisfaction by x of the original constraints. The solution $\mathbf{v}^0(x)$ of $\mathbb{P}_N^*(x)$ yields the current control action $\kappa_N^*(x) + Kx$, with $\kappa_N^*(x) \triangleq v^0(0; x)$, for the systems $x^+ = Ax + Bu + w$.

If the state is not accessible, a similar procedure is possible as shown in Yan and Bitmead (2005) where, in the optimal control problem, u is set equal to v (i.e. $K = 0$) and z is set equal to the state of an open-loop predictor and the state constraints are probabilistic. As above, an equivalent deterministic optimal control problem can be defined using the predicted variance Σ. Because the predicted variances increase with time, making satisfaction of constraints problematic and providing a poor model of actual closed-loop behaviour, the authors use the one step ahead variance to determine the tightened constraints on z; they apply their results successfully to a network traffic control problem. This paper is one of very few dealing with output MPC despite its wide use in industry.

If the problem has a soft state constraint of the form $\mathcal{P}(c'x \leq d) \geq p$ and w is Gaussian, it is possible, using $x = z + e$ and $e(i) \sim N(0, \Sigma(i))$, to compute a tightened hard constraint $c'x \leq d' \leq d$ such that $c'z^{\mathbf{v}}(i; x) \leq d'$ implies $\mathcal{P}(c'x(i) \leq d) \geq p$, $x(i) = z^{\mathbf{v}}(i; x) + e^{\mathbf{w}}(i; x)$ for all $i = 0, 1, \ldots, N-1$ and all admissible disturbance sequences. If w is not Gaussian but has known distribution, it is still possible, with more complex offline computation, to determine how to tighten the constraint (Kouvaritakis, Cannon, Raković, & Cheng, 2010) to ensure the same result.

To get recursive feasibility is more difficult and also requires imposition of a suitable terminal constraint. The difficulty is made explicit in Primbs and Sung (2009) where soft constraints of the form $E_x c'x(N) \leq d$ are employed and it is shown that if a soft constraint is satisfied at (x, t) (i.e. state x at time t), say, it is not necessarily satisfied at (x^+, t^+). A similar difficulty arises when soft constraints of the form $\mathcal{P}_x(c'x(N) \leq d) \geq p$ is employed as discussed in Kouvaritakis et al. (2010) where it is concluded that the soft constraints should be satisfied for all realizations of the disturbance sequence necessitating the requirement that the disturbance variable w is bounded.

In Kouvaritakis et al. (2010) use is made of dual mode MPC to control the system $z^+ = A_K z + Bv$ using, in the optimal control problem $\mathbb{P}_N^*(z)$, tightened constraint sets and a terminal constraint set X_f such that $x \in X_f$ implies $A_K x \in X_f$ and that all constraints are satisfied. An asymptotic bound on the cost $V_N(\cdot)$ is claimed.

The same problem is treated very differently in Chatterjee, Hokayem, and Lygeros (2011) and Hokayem, Chatterjee, and Lygeros (2009); Hokayem, Chatterjee, Ramponi, Chaloulos, and Lygeros (2010). The system considered in the latter paper is, again, $x^+ = Ax + Bu + w$ where (A, B) is stabilizable, A is Lyapunov stable and $\{w(t)\}$ is a sequence of independent random variables with bounded fourth moment but not necessarily bounded. Unlike the previous papers, the control u is constrained to lie in a compact set $\mathbb{U}$. The cost $E_x V_N(x, \pi, \mathbf{w})$ to be minimized at each current state x is, as usual, a positive definite quadratic function; almost uniquely, in the stochastic MPC literature, it is proposed to optimize, in the optimal control problem solved iteratively online, $\pi(\cdot)$ over policies $\pi(\cdot)$ rather than sequences $\mathbf{u}$; if x is the current state, $\pi(\cdot)$ is a function of the form $\pi = \{\pi_1(\cdot), \pi_2(\cdot), \ldots, \pi_{N-1}(\cdot)\}$ where, for each i, $\pi(\cdot)$ is an function of $(x(0) = x, x(1), \ldots, x(i))$, i.e. a function of the current state and some previous states. Let Π_N denote the class of measurable functions of this form and satisfying the control constraint. This problem, as stated, is too complex and nonconvex. Motivated by earlier work, in particular, Goulart, Kerrigan, and Maciejowski (2006) and Lofberg (2003), the functions $\pi_i(x(0), \ldots, x(i))$ defined above are replaced by functions of the form $u_i = \eta_i + \sum_{j=0}^{i-1} \theta_{i,j}\phi_j(w(j))$. It was shown in Goulart et al. (2006) that there is a one-to-one nonlinear mapping between the two control mappings when they are both affine. Since $w(j) = x(j+1) - Ax(j) - Bu(j)$, the sequence $\{w(j)\}$ may be determined from available data. The functions $\phi_j(\cdot)$ are chosen to satisfy $|\phi_j(w)|_\infty \leq \phi_{\max}$ for all j, all w. This enables the hard constraint on u to be satisfied which would not be the case if the functions $\phi_j(\cdot)$ were affine because the disturbance w may be unbounded. Making use of the linearity of the dynamic system it is proved in Hokayem et al. (2010) that the optimization problem $\mathbb{P}_N(x)$, i.e. minimization of $E_x \sum_{i=0}^{N-1} |x(i)|_Q^2 + |u(i)|_R^2 + |x(N)|_P^2$ using the control parametrization above and subject to the hard control constraint, is convex. Because w is not necessarily bounded, use of a conventional terminal constraint to ensure closed loop stability is not possible. Instead the authors invoke an abstract assumption to establish mean square boundedness of the state of the closed loop system.

Multiplicative uncertainty is considered in Cannon, Kouvaritakis, and Ng (2009), Cannon, Kouvaritakis, and Wu (2009) and Primbs and Sung (2009). The system to be controlled satisfies

$x^+ = Ax + Bu + \sum_{j \in J} y_j w_j$ where $\{w(i)\}$ is a sequence of independent random variables of zero mean and known variance and $y_j = C_j x + D_j u$. The usual controller parametrization $u = Kx + v$ is employed so that $A_K \triangleq A + BK$ is a stability matrix. In Primbs and Sung (2009) expectation constraints must be satisfied while Cannon, Kouvaritakis, and Ng (2009) and Cannon, Kouvaritakis, and Wu (2009) impose probabilistic constraints. Expressing, as above, the state x as the sum $z + e$ with $z^+ = A_K z + v$ and $e^+ = A_K e + \sum y_j w_j$, the cost $E_x V_N(x, \mathbf{u}, \mathbf{w})$ may be again redefined as a quadratic function $V_N^*(z, \mathbf{v})$ but here the error sequence $\mathbf{e}^{\mathbf{w}}$ is a sequence of zero mean random variables with variance that now depend on $\mathbf{v}$. Hence the optimal control problem is no longer quadratic but is convex and amenable to computation. Stability is addressed differently in these papers; Primbs and Sung (2009) employs the optimal, unconstrained, infinite horizon policy when constraints are not satisfied since recursive feasibility cannot be guaranteed. Because of the nature of the uncertainty it is possible to prove that the state converges to zero with probability 1.

Scenario based optimization

Because of the stochastic nature of the optimal control problem it is natural to consider the use of Monte Carlo and related procedures as in Bernardini and Bemporad (2009) and Kantas, Maciejowski, and Lecchini-Visintini (2009). There has recently been a significant development in the area of stochastic optimization for convex problems that, with a high degree of confidence, gives bounds on the number of scenarios required in order to achieve a solution that satisfies the constraints in the problem with a specified probability. A good overview of this development is given in Calafiore and Campi (2006); the problem considered is $\min_v \{c'v\}$ subject to a stochastic constraint $f(v, \delta) \le 0$ where $v \in \mathbb{R}^d$ is the decision variable and $\delta \in \Delta$ is the uncertainty. It is assumed that, for each $\delta \in \Delta, f(\cdot)$ is continuous and convex in v and that the probability that the uncertainty takes a particular value δ in Δ is defined. Given M independent, identically distributed samples $\delta_1, \ldots, \delta_M$ of the uncertainty, the scenario design procedure is solution of a finite dimensional convex program

$$\min_v \{c'v \mid f(v, \delta_i) \le 0,\ i = 1, 2, \ldots, M\}.$$

The highly useful result obtained in Calafiore and Campi (2006) is: If the number of samples M is larger than $(2/\varepsilon)\ln(1/\beta) + 2d + (2d/\varepsilon)\ln(2/\varepsilon)$ (d is the dimension of v) then, with probability $1 - \beta$, either the problem is infeasible or it is feasible and its solution $\hat{v}_N$ satisfies the constraint $f(v, \delta) \le 0$ with probability $1 - \varepsilon$, i.e. $\text{Prob}\{f(\hat{v}_N, \delta) \le 0\} \le 1 - \varepsilon$. A second result gives upper and lower bounds for the value function. For example with $d = 10$, $\varepsilon = 0.01$ and $\beta = 10^{-5}$, if 8725 or more samples are chosen, then, with probability $1-10^{-5}$, $f(\hat{v}_N, \delta) \le 0$ with probability 0.99. Other methods typically require a much larger number of samples; a feature of the lower bound on the number of samples is that β can be very small (e.g. 10^{-6} or 10^{-9}) without significantly increasing M.

These results are employed for MPC in Calafiore and Fagaino (2013). The system considered is $x^+ = A(\theta)x + B(\theta)u + B_\gamma(\theta)\gamma$ where $\theta \in \Theta$ is a random parameter and $\{\gamma(i)\}$ is a sequence of independent, identically distributed random variables taking values in Γ. The usual control parametrization $u = Kx + v$ is employed. Let $\boldsymbol{\delta} \triangleq (\theta, \gamma(0), \gamma(1), \ldots, \gamma(N-1))$ denote a sample of the uncertainty. The cost of a single sample is

$$J(x, \mathbf{v}, \boldsymbol{\delta}) = \sum_{j=0}^{N-1} d(x^{\mathbf{v},\boldsymbol{\delta}}(i; x), X_f) + |v(i)|_R^2$$

where $d(x, X_f)$ is the distance of x from the terminal constraint set X_f. The optimal control problem is minimization of the maximum of $J(x, \mathbf{v}, \boldsymbol{\delta})$ over M independent samples of δ subject to possibly uncertain control and state constraints and the terminal constraint set that is robustly positively invariant under $u = K_f(x)$. This problem can be recast in the form considered above with the addition of slack variables; with the number of samples M chosen to satisfy the inequality, the resultant random control sequence $\mathbf{v}^*$ satisfies the state, control and terminal constraints with the (β, ϵ) guarantee defined above. The control sequence $\mathbf{v}^*$ is employed in a model predictive controller with an interesting modification that ensures (in the (β, ϵ) sense) that either the state converges asymptotically to X_f or reaches it in finite time.

An interesting application of the use of scenario based optimization appears in Goodwin and Medioli (2013).

Adaptive MPC

The problem of adaptive MPC has received very little attention. A neural network approach is advocated in Sha (2008). Adaptive control of the continuous time nonlinear system $\dot{x} = f(x, u) + g(x, u)\theta$ where θ is constant but unknown and the state x is assumed accessible is considered in Adetola, DeHaan, and Guay (2009). The system is subject to state and control constraints. It is assumed that the unknown parameter θ lies in a known compact set Θ^0. An estimation procedure that yields, at each time t, an estimate $\hat{\theta}(t)$ of θ and an uncertainty set $\Theta(t)$ (a neighbourhood of $\hat{\theta}(t)$) satisfying $\theta \in \Theta(t)$ where $t_2 > t_1$ implies $\Theta(t_2) \subset \Theta(t_1)$. The cost function to be minimized online over control *policies* is the maximum, subject to control, state and terminal constraints, with respect to $\theta \in \Theta(t)$ of the sum of the integral, with horizon T, of the stage cost and a terminal cost; the terminal cost and constraint set are robust with respect to $(\theta, \hat{\theta}) \in \Theta^0 \times \Theta^0$. Note that a robust control problem is solved at each t. If $x(0)$ is feasible for the optimal control problem, then $x(t) \to 0$ as $t \to \infty$.

The adaptive control problem is more difficult if uncertainty in the form of additive disturbance and measurement noise are present. This is the case considered in Marafiore, Bitmead, and Hovd (2013). The system to be controlled is described by $x^+ = f(x, u, w, \theta)$, $y = h(x, u, v, \theta)$ where $\{w(k)\}$ and $\{v(k)\}$ are sequences of independent identically distributed random variables and θ is a constant unknown parameter; the system is subject to state and control constraints. Persistence excitation for non-linear system identification is not well understood; the *assumption* is therefore made in this paper that sufficient richness of the input sequence suffices for persistent excitation. It is shown that sufficient richness of the input (control) sequence is obtained if a simple extra constraint is added to the optimal control problem solved online; the extra constraint at time k is

$$\alpha u(k)u(k)' + \beta u(k)' + [\beta u(k)']' + \Gamma \ge 0$$

where the scalar α, vector β and matrix Γ are functions of the previous $P \ge N$ values of u. This constraint is incorporated in the optimal control problem solved online; it ensures that the control does not tend to zero thereby incurring a cost. The optimal control problem retains feasibility in the absence of state constraints. Persistent excitation is obtained if the system being controlled is linear and is 'output reachable'.

Applications to finance

An interesting new area in which stochastic MPC seems poised to make a useful contribution is finance; applications to dynamic hedging of options, to constrained index tracking and to portfolio optimization are described in Bemporad, Bellucci, and Gabriellini (2012), Primbs (2007, 2009) and Primbs and Sung (2008).

4. What does the future hold?

If it is rash to attempt an overview of past contributions, it is even rasher to predict the future.

Stabilizing conditions: The use of an implicit terminal constraint (choosing the set of permissible initial states and horizon N so that the explicit terminal constraint is automatically satisfied as shown in Limon et al., 2006b) makes the difference between the two schools of stabilizing with or without a terminal constraint marginal provided that an appropriate terminal penalty is employed. Use of an explicit rather than an implicit terminal constraint yields a larger region of attraction but at the expense of performance as measured by $V_\infty^0 - V_\infty^{\kappa_N}$. It is the size of the region of attraction that is a major argument against using a terminal constraint, explicit or implicit but this is now undermined by the recent development of techniques for extending this region (see Section 2.2.4). I anticipate further study of methods for extending the region of attraction.

Process control: The process industries remain by far the largest user of model predictive control so it is reasonable to expect further research in those areas of most relevance to this industrial sector such as cooperative and economic model predictive control and model predictive control of hybrid systems. The relative conceptual simplicity of model predictive control is one of its attractions for this industrial sector; state and terminal constraints, for example, are eschewed probably because the control algorithm would have to incorporate extra recovery manoeuvres to cope with transgression of these constraints. The increasing complexity of many research contributions may militate against their industrial adoption.

Embedded MPC: The progress made in embedded MPC and in associated areas such as explicit MPC and software tools certainly exceeded my expectations. I anticipate further considerable research in this area.

Inherent robustness versus robustness by design: The recent advance in inherent robustness (Yu et al., 2011) questions the need for special techniques, discussed in Section 3, to ensure robustness. One answer is that inherent robustness currently relies on the absence of state constraints. Another is that the magnitude of the permissible disturbance decreases with N, the horizon length, but this is also true for many of the proposed techniques to achieve robustness. Interestingly this restriction does not hold for tube-based robust MPC for linear systems; the reason for this is that the reference trajectory is generated using the nominal system and the local feedback in a neighbourhood of the reference trajectory is provided by a controller that is asymptotically stabilizing.

Robust MPC: Because of the complexity of the problem, an exact solution is too complex for implementation so that, as in the early days of adaptive control, there are a plethora of solutions; a 'winner' has yet to emerge. Acceptance by the process industries requires simplicity as can be seen by the limited use of current research literature.

Tracking and disturbance rejection: Existing theory for the case when the reference that is tracked or the disturbance that is rejected vary randomly is not yet in a satisfactory state. For example, for zero offset disturbance rejection, the estimator for the disturbance is designed using a stochastic model but stability properties for the controlled system are derived using a deterministic model. Similarly, in tracking problems it is often assumed that the reference being tracked converges to a constant value. A start has been made to develop a more comprehensive theory (see Section 3.2.3).

Output MPC: An open, but very difficult problem, is output MPC when the state is estimated using moving horizon estimation. A very promising new direction is proposed in Copp and Hespanha (2014a).

Scenario based stochastic optimization: The recent results on scenario based optimization (Calafiore & Campi, 2006) seem to have considerable potential for advancing research on robust MPC for nonlinear systems because it gives probabilistic guarantees that are meaningful for control design. Even though the use of scenario based methods for online MPC have been investigated, it seems that most benefit will be derived from these results if they are employed for *off-line* design. For example, if it is decided, in the interest of simplicity, that the important parameters in a robust model predictive controller are those that define the tightened constraints as in Mayne et al. (2011) then scenario based optimization can be used in an offline study to determine these parameters.

Adaptive MPC: The problem of adaptive MPC is still open. A useful start has been made in that it has been shown (Marafiore et al., 2013) that the introduction of persistent excitation is possible in that it does not destroy feasibility of the optimal control problem solved online but much remains to be done.

Applications: It seems the sky is the limit in applications. The versatility and simplicity of MPC enable it to be widely used. Surprising applications mentioned in this overview include emergency vehicle scheduling (Goodwin & Medioli, 2013), automatic synthesis of a control protocol expressed in temporal logic (Wongpiromsarn et al., 2012), pursuit–evasion (Lee et al., 2013) and portfolio optimization (Primbs, 2007). Many more applications are anticipated.

Notation: Papers would be clearer, if standard notation for the solution of difference and differential equations, used to good effect in, for example, Grüne and Pannek (2011), were employed in place of the ambiguous notation $x_{j|k}$.

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