Infinitary lambda calculi and Böhm models

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Abstract. In a previous paper we have established the theory of transfinite reduction for orthogonal term rewriting systems. In this paper we perform the same task for the lambda calculus. This results in several new Böhm-like models of the lambda calculus, and new descriptions of existing models.

1 Introduction

Infinitely long rewrite sequences of possibly infinite terms are of interest for several reasons. Firstly, infinitary rewriting is a natural generalisation of finitary rewriting which extends it with the notion of computing towards a possibily infinite limit. Such limits naturally arise in the semantics of lazy functional languages, in which it is possible to write and compute with expressions which intuitively denote infinite data structures, such as a list of all the integers. If the limit of a reduction sequence still contains redexes, then it is natural to consider sequences whose length is longer than ω — in fact, sequences of any ordinal length. The question of the computational meaning of such sequences will be dealt with below. Secondly, computations with terms implemented as graphs allow the possibility of using cyclic graphs, which correspond in a natural way to infinite terms. Finite computations on cyclic graphs correspond to infinite computations on terms. Finally, the infinitary theory suggests new ways of dealing with some of the concepts that arise in the finitary theory, such as notions of undefinedness of terms. In this connection, Berarducci and Intrigila ([Ber, BI94]) have independently developed an infinitary lambda calculus and applied it to the study of consistency problems in the finitary lambda calculus.

In [KKSdV–] we developed the basic theory of transfinite reduction for orthogonal term rewrite systems. In this paper we perform the same task for the

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lambda calculus. In contrast to the situation for term rewriting, in lambda calculus there turn out to be several different possible domains of infinite terms which one might study. These give rise to different Böhm-like models of the calculus.

2 Basic definitions

2.1 Finitary lambda calculus

We assume familiarity with the lambda calculus, or as we shall refer to it here, the finitary lambda calculus. [Bar84] is a standard reference. The syntax is simple: there is a set Var of variables; an expression or term E is either a variable, an abstraction $\lambda x.E$ (where x is called the bound variable and E the body), or an application $E_1 E_2$ (where E_1 is called the rator and E_2 the rand). This is the pure lambda calculus — we do not have any built-in constants nor any type system.

As customary, we identify α-equivalent terms with each other, and consider bound variables to be silently renamed when necessary to avoid name clashes.

2.2 What is an infinite term?

Drawing lambda expressions as syntax trees gives an immediate and intuitive notion of infinite terms: they are just infinite trees. Formally, we can define this set as the metric completion of the space of finite trees with a well-known (ultra-)metric. The larger the common prefix of two trees, the more similar they are, and the closer together they may be considered to be. First, some terminology. A *position* or *occurrence* is a finite string of positive integers. Given a term M and a position u, the term $M|u$, when it exists, is a subterm of M defined inductively thus:

$$\begin{aligned} M|\langle\rangle &= M \\ (\lambda x.M)|1\cdot u &= M|u \\ (MN)|1\cdot u &= M|u \\ (MN)|2\cdot u &= N|u \end{aligned}$$

$M|u$ is called the subterm of M at u, and when this is defined, u is called a position of M. The *syntactic depth* of u is its length.

Two positions u and v are *disjoint* if neither is a prefix of the other. Two redexes are disjoint if their positions are. A set of positions or redexes is disjoint if every two distinct members are.

Given two distinct terms M and N, let l be the length of the shortest position u such that $M|u$ and $N|u$ are both defined, and are either of different syntactic types or are distinct variables. Then the larger l is, the more similar are M and N. The distance between M and N is defined to be 2^{-l}. Denote this measure by $d^s(M,N)$. $d^s(M,M)$ is defined to be 0. This is the *syntactic metric.* It is easily proved that it is a metric on the set of finite terms. In fact, it is an ultrametric, i.e. $d^s(M,N) \leq max(d^s(M,P), d^s(P,N))$, although this will not be important. The completion of this metric space adds the infinite terms. We call this set Λ^s.

The above is the definition of infinite terms which we used in our study of transfinite term rewriting, but for lambda calculus the situation is a little more complicated. Consider the term $(((\ldots I)I)I)I$ where $I = \lambda x.x$. See Fig. 1. This

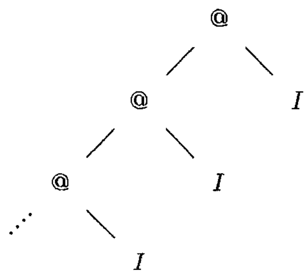

Fig. 1.

term has a combination of properties which is rather strange from the point of view of finitary lambda calculus. By the usual definition of head normal form — being of the form $\lambda x_1 \ldots \lambda x_n.y t_1 \ldots t_m$ — it is not in head normal form. By an alternative formulation, trivially equivalent in the finitary case, it is in head normal form — it has no head redex. It is also a normal form, yet it is unsolvable (that is, there are no terms $N_1, \ldots, N_n$ such that $MN_1 \ldots N_n$ reduces to I). The problem is that application is strict in its first argument, and so an infinitely left-branching chain of applications has no obvious meaning. We can say much the same for an infinite chain of abstractions $\lambda x_1.\lambda x_2.\lambda x_3.\ldots$

Another reason for reconsidering the definition of infinite terms arises from analogy with term rewriting. In a term such as $F(x,y,z)$, the function symbol F is at syntactic depth 0. If it is curried, that is, represented as $Fxyz$, or explicitly $@(@(@(F,x),y),z)$ (as it would be if we were to translate the term rewrite system into lambda calculus), the symbol F now occurs at syntactic depth 3. We could instead consider it to be at depth zero; more generally, we can define a new measure of depth which deems the left argument of an application to be at the same depth as the application itself, and the body of an abstraction to be at the same depth as the abstraction.

Definition 1. Given a term M and a position u of M, the *applicative depth* of the subterm of M at u, if it exists, is defined by:

$$\begin{aligned} D^a(M, \langle\rangle) &= 0 \\ D^a(\lambda x.M, 1 \cdot u) &= D^a(M, u) \\ D^a(MN, 1 \cdot u) &= D^a(M, u) \\ D^a(MN, 2 \cdot u) &= 1 + D^a(N, u) \end{aligned}$$

The associated measure of distance is denoted d^a, and the space of finite and infinite terms Λ^a.

In general, we can specify for each of the three contexts $\lambda x.[]$, $[]M$, and $M[]$ whether the depth of the hole is equal to or one greater than the depth of the whole expression. Syntactic depth sets all three equal to 1. For applicative depth, the three depths are 0, 0, and 1 respectively. This suggests a general definition.

Definition 2. Given a term M a position u of M, and a string of three binary digits abc, there is an associated measure of depth D^{abc}:

$$\begin{aligned}
D^{abc}(M, \langle\rangle) &= 0 \\
D^{abc}(\lambda x.M, 1 \cdot u) &= a + D^{abc}(M, u) \\
D^{abc}(MN, 1 \cdot u) &= b + D^{abc}(M, u) \\
D^{abc}(MN, 2 \cdot u) &= c + D^{abc}(N, u)
\end{aligned}$$

The associated measure of distance is denoted d^{abc} and the space of finite and infinite terms Λ^{abc}.

We write Λ^∞, D, or d when we do not need to specify which space of infinite terms, measure of depth, or metric we are referring to.

We have already seen that $d^s = d^{111}$ and $d^a = d^{001}$. Some of the other measures also have an intuitive significance. d^{101} (*weakly applicative* depth, or d^w) may be associated with the lazy lambda calculus [AO93], in which abstraction is considered lazy — $\lambda x.M$ is meaningful even when M is not. Denote the corresponding set of finite and infinite terms by Λ^w. d^{000} is the discrete metric, the trivial notion in which the depth of every subterm of a term is zero. This gives the discrete metric space of finite terms, no infinite terms, and no reduction sequences converging to infinite terms — the usual finitary lambda calculus.

Many of our results will apply uniformly to all eight infinitary lambda calculi, and we will only specify the depth measure when necessary. In the final section we will find that two of them — Λ^{010} and Λ^{011} — have unsatisfactory technical properties.

Lemma 3. *Considered as a set, Λ^{abc} is the subset of Λ^{111} consisting exactly of those terms which do not contain an infinite sequence of nodes in which each node is at the same abc-depth as its parent. (Its metric and topology are not the subspace metric and topology.)* □

Both Λ^s and Λ^w contain unsolvable normal forms, such as $\lambda x_1.\lambda x_2.\lambda x_3 \ldots$ In Λ^a every normal form is solvable.

2.3 What is an infinite reduction sequence?

We have spoken informally of convergent reduction sequences but not yet defined them. The obvious definition is that a reduction sequence of length ω converges if the sequence of terms converges with respect to the metric. However, this proves to be an unsatisfactory definition, for the same reasons as in [KKSdV–]. There are two problems. Firstly, a certain property which is important for attaching computational meaning to reduction sequences longer than ω fails.

Definition 4. A reduction system admitting transfinite sequences satisfies the *Compression Property* if for every reduction sequence from a term s to a term t, there is a reduction sequence from s to t of length at most ω.

A counterexample to the Compression Property is easily found in Λ^s. Let $A_n = (\lambda x.A_{n+1})(B^n(x))$ and $B = (\lambda x.y)z$. Then $A_0 \rightarrow^\omega C$ where $C = (\lambda x.C)$ (B^ω), and $C \rightarrow (\lambda x.C)(yB^\omega)$. A_0 cannot be reduced to $(\lambda x.C)(yB^\omega)$ in ω or fewer steps. (We do not know if the Compression Property holds for the above notion of convergence in Λ^a or Λ^w.)

The second difficulty with this notion of convergence is that taking the limit of a sequence loses certain information about the relationship between subterms of different terms in the sequence. Consider the term I^ω of Λ^a, and the infinite reduction sequence starting from this term which at each stage reduces the outermost redex: $I^\omega \rightarrow I^\omega \rightarrow I^\omega \rightarrow \ldots$ All the terms of this sequence are identical, so the limit is I^ω. However, each of the infinitely many redexes contained in the original term is eventually reduced, yet the limit appears to still have all of them. It is not possible to say that any redex in the limit term arises from any of the redexes in the previous terms in the sequence.

A third difficulty arises when we consider translations of term rewriting systems into the lambda calculus. Even when such a translation preserves finitary reduction, it may not preserve Cauchy convergent reduction. Consider the term rewrite rule $A(x) \rightarrow A(B(x))$. This gives a Cauchy convergent term rewrite sequence $A(C) \rightarrow A(B(C)) \rightarrow A(B(B(C)))\ldots$ If one tries to translate this by defining $A_\lambda = Y(\lambda f.\lambda x.f(Bx))$ (for some λ-term B), where Y is Church's fixed point operator $\lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$, then the resulting sequence will have an accumulation point corresponding to the term $A(B^\omega)$, but will not be Cauchy convergent. The reason is that what is a single reduction step in the term rewrite system becomes a sequence of several steps in the lambda calculus, and while the first and last terms of that sequence may be very similar, the intermediate terms are not, destroying convergence.

The remedy for all these problems is the same as in [KKSdV–]: besides requiring that the sequence of terms converges, we also require that the depths of the redexes which the sequence reduces must tend to infinity.

Definition 5. A *pre-reduction* sequence of length α is a function ϕ from an ordinal α to reduction steps of Λ^∞, and a function τ from $\alpha + 1$ to terms of Λ^∞, such that if $\phi(\beta)$ is $a \rightarrow^r b$ then $a = \tau(\beta)$ and $b = \tau(\beta + 1)$. Note that in a pre-reduction sequence, there need be no relation between the term $\phi(\beta)$ and any of its predecessors when β is a limit ordinal.

A pre-reduction sequence is a *Cauchy convergent reduction sequence* if τ is continuous with respect to the usual topology on ordinals and the metric on Λ^∞.

It is a *strongly convergent reduction sequence* if it is Cauchy convergent and if, for every limit ordinal $\lambda \leq \alpha$, $\lim_{\beta\rightarrow\lambda} d_\beta = \infty$, where d_β is the depth of the redex reduces by the step $\phi(\beta)$. (The measure of depth is the one appropriate to each version of Λ^∞.)

If α is a limit ordinal, then an *open* pre-reduction sequence is defined as above, except that the domain of τ is α. If τ is continuous, the sequence is *Cauchy continuous*, and if the condition of strong convergence is satisfied at each limit ordinal less than α, it is *strongly continuous*.

When we speak of a reduction sequence, we will mean a strongly continuous reduction sequence unless otherwise stated. Different measures of depth give different notions of strong continuity and convergence.

3 Descendants and residuals

3.1 Descendants

When a reduction $M \to N$ is performed, each subterm of M gives rise to certain subterms of N — its descendants — in an intuitively obvious way. Everything works in almost exactly the same way as for finitary lambda calculus.

Definition 6. Let u be a position of t, and let there be a redex $(\lambda x.M)N$ of t at v, reduction of which gives a term t'. The set of *descendants* of u by this reduction, u/v, is defined by cases.

- If $u \not\geq v$ then $u/v = \{u\}$.
- If $u = v$ or $u = v \cdot 1$ then $u/v = \emptyset$.
- If $u = v \cdot 2 \cdot w$ then $u/v = \{v \cdot y \cdot w \mid y \text{ is a free occurrence of } x \text{ in } M\}$. If $u = v \cdot 1 \cdot w$ then $u/v = \{v \cdot w\}$.

The *trace* of u by the reduction at v, $u/\!/v$, is defined in the same way, except for the second case: if $u = v$ or $u = v \cdot 1$ then $u/\!/v = \{v\}$.

For a set of positions U, $U/v = \bigcup\{u/v \mid u \in U\}$ and $U/\!/v = \bigcup\{u/\!/v \mid u \in U\}$.

The notions of descendant and trace can be extended to reductions of arbitrary length, but first we must define the notion of the limit of an infinite sequence of sets.

Definition 7. Let $S = \{S_\beta \mid \beta < \alpha\}$ be a sequence of sets, where α is a limit ordinal. Define

$$\liminf S = \bigcup_{\beta \to \alpha} \bigcap_{\beta < \gamma < \alpha} S_\gamma \qquad \limsup S = \bigcap_{\beta \to \alpha} \bigcup_{\beta < \gamma < \alpha} S_\gamma$$

When $\liminf S = \limsup S$, write $\lim S$ or $\lim_{\beta \to \alpha} S_\beta$ for both.

Definition 8. Let U be a set of positions of t, and let S be a reduction sequence from t to t'. For a reduction sequence of the form $S \cdot r$ where r is a single step, $U/(S \cdot r) = (U/S) \cdot r$. If the length of S is a limit ordinal α then $U/S = \lim_{\beta \to \alpha} U/S_\beta$.

$U/\!/S$ is defined similarly.

Strong convergence of S ensures that the above limit exists.

Lemma 9. *Let U be a set of positions of redexes of t, and let S be a reduction from t to t'. Then there is a redex at every member of U/S.* □

Definition 10. The redexes at U/S in the preceding lemma are the *residuals* of the redexes at U.

Definition 11. Let u and v be positions in the initial and final terms respectively of a sequence S. If $v \in u//S$, we also say that u *contributes* to v (via S). If there is a redex at v, then u *contributes* to that redex if u contributes to v or $v \cdot 1$.

We do not define descendants, traces, residuals, and contribution for Cauchy convergent reductions, which is not surprising given the examples of section 2.3.

Theorem 12. *For any strongly convergent sequence $t_0 \to^\alpha t_\alpha$ and any position u of t_α, the set of all positions of all terms in the sequence which contribute to u is finite, and the set of all reduction steps contributing to u is finite.*

Proof. For each t_β in the sequence, we construct the set U_β of positions of t_β contributing to u, and prove that it is finite. We also show that there are only finitely many different such sets, hence their union is finite.

Suppose $U_{\beta+1}$ is finite, and $t_\beta \to t_{\beta+1}$ reduces a redex at position v. Let $w \in U_{\beta+1}$. If w and v are disjoint, or $w < v$, then w is the only position of t_β contributing to v in $t_{\beta+1}$. If $w = v$, then v, $v \cdot 1$, $v \cdot 1 \cdot 1$, and possibly $v \cdot 2$ (if the redex has the form $(\lambda x.x)N$) are the only such positions. If $w > v$, and the redex at v is $(\lambda x.M)N$, then there is a unique position in either M or N which contributes to w. In each case, the set of positions is finite, hence U_β, which is the union of those sets for all $w \in U_{\beta+1}$, is finite.

Suppose U_β is defined and finite for a limit ordinal β. By strong convergence and the finiteness of U_β, there is a final segment of $t_0 \to^\beta t_\beta$, say from t_γ to t_β, in which every step is at a depth more than 2 greater than the depth of every member of U. It follows that each U_δ for $\gamma \leq \delta < \beta$ is equal to U_β, and is therefore finite.

Finitely many repetitions of the above argument suffice to calculate U_β for all β, demonstrating that there are only finitely many different such sets, and all of them are finite.

Each reduction step contributing to u takes place at a prefix of a position in some U_β. By strong convergence, only finitely many steps can take place at any one position, therefore there are only finitely many such steps. □

3.2 Developments

Definition 13. A *development* of a set of redexes R of a term M is a sequence in which every step reduces some residual of some member of R by the previous steps of the sequence. It is *complete* if it is strongly convergent and the final term contains no residual of any member of R.

Not every set of redexes has a complete development. In Λ^{--1}, an example is the term $I^\omega = (\lambda x.x)((\lambda x.x)((\lambda x.x)(\ldots)))$. Every attempt to reduce all the redexes in this term must give a reduction sequence containing infinitely many reduction steps at the root of the term, which, by every notion of depth, is not strongly convergent. Note that the set consisting of every redex at odd syntactic depth has a complete development, as does the set consisting of every redex at even syntactic depth, but their union does not. In every other version of Λ^∞ except 000 (the finitary calculus) the term $(\lambda x.((\lambda x.((\lambda x.(\ldots))z))z))z$ behaves in a similar manner.

Theorem 14. *Complete developments of the same set of redexes end at the same term.*

Proof. (Outline.) In the finitary case one proves this by showing that (1) it is true for a set of pairwise disjoint redexes, (2) it is true for any pair of redexes, and (3) all developments are finite. The result then follows by an application of Newman's Lemma.

In the infinitary case, (1) and (2) are still true, and indeed obvious, but (3) is of course false. The situation is complicated by the fact that a set of redexes can have a strongly convergent complete development without all its developments being strongly convergent.

One proceeds instead by picking out one particular development of the given set of redexes, analogous to the "standard" development defined in finitary rewriting, such that the set has a strongly convergent complete development if and only if its standard development is complete. Properties of the standard development then allow one to use (1) and (2) to construct a "tiling diagram" for the standard development and any other complete development, and to show that the right and bottom edges of the diagram are empty. This shows that they converge to the same limit. □

4 The truncation theorem

Some results about the finitary lambda calculus can be transferred to the infinitary setting by using finite approximations to infinite terms.

Definition 15. A $\Lambda_\perp$ term is a term of the version of lambda calculus obtained by adding $\perp$ as a new symbol. $\Lambda_\perp^\infty$ is defined from $\Lambda_\perp$ as Λ^∞ is from Λ.

The terms of $\Lambda_\perp^\infty$ have a natural partial ordering, defined by stipulating that $\perp \leq t$ for all t, and that application and abstraction are monotonic.

A *truncation* of a term t is any term t' such that $t' \leq t$. We may also say that t' is weaker than t, or t is stronger than t'.

Theorem 16. *Let $t_0 \to^\alpha t_\alpha$ be a reduction sequence. Let s_α be a prefix of t_α, and for $\beta < \alpha$, let s_β be the prefix of t_β contributing to s_α. Then for any term r_0 such that $s_0 \leq r_0$ there is a reduction sequence $r_0 \to^{\leq\alpha} r_\alpha$ such that:*

1. *For all β, s_β is a prefix of r_β.*

2. *If $t_\beta \to t_{\beta+1}$ is performed at position u and contributes to s_α, then $r_\beta \to r_{\beta+1}$ by reduction at u.*
3. *If $t_\beta \to t_{\beta+1}$ is performed at position u and does not contribute to s_α, then $r_\beta = r_{\beta+1}$.* □

As an example of the use of this theorem, we demonstrate that Λ^∞ is conservative over the finitary calculus, for terms having finite normal forms.

Corollary 17. *If $t \to^\infty s$ and s' is a finite prefix of s, then t is reducible in finitely many steps to a term having s' as a prefix. In particular, if t is reducible to a finite term, it is reducible to that term in finitely many steps.*

Proof. From Theorems 16 and 12. □

Corollary 18. *If a finite term is reducible to a finite normal form, it is reducible to that normal form in the finitary lambda calculus.* □

5 The Compressing Lemma

One of our justifications for the interest of infinite terms and sequences is to see them as limits of finite terms and sequences. From this point of view, the computational meaning may be obscure of a sequence of length longer than ω — which performs an infinite amount of work and then doing some more work. We therefore wish to be assured that every reduction sequence of length greater than ω is equivalent to one of length no more than ω, in the sense of having the same initial and final term. This allows us to freely use sequences longer than ω without losing computational relevance.

Theorem 19. (Compressing Lemma.) *In Λ^∞, for every strongly convergent sequence there is a strongly convergent sequence with the same endpoints whose length is at most ω.*

Proof. The corresponding theorem of [KKSdV–] shows that the case of a sequence of length $\omega+1$ implies the whole theorem, and the proof is not dependent on the details of rewriting — it is valid for any abstract transfinite reduction system (as defined in [Ken92]).

Suppose we have a reduction of the form $S_{\omega+1} = s_0 \to^\omega s_\omega \to_d s_{\omega+1}$, where the final step rewrites a redex at depth d. By strong convergence of the first ω steps, the sequence must have the form $s_0 \to^* C[(\lambda x.M)N, M_1, \ldots, M_n] \to^\omega_{d+1} C[(\lambda x.M')N', M'_1, \ldots, M'_n] \to_d C[M'[x := N']]$. where the context $C[\ldots]$ is a prefix of every term of the sequence from some point onwards, and all its holes are at depth d. The reduction of $C[(\lambda x.M)N, M_1, \ldots, M_n]$ to $C[(\lambda x.M')N', M'_1, \ldots, M'_n]$ consists of an interleaving of reductions of M to M', N to N', and each M_i to M'_i of length at most ω. Conversely, any reductions of lengths at most ω starting from M, N, and each M_i can be interleaved to give a reduction of length at most ω starting from $C[(\lambda x.M)N, M_1, \ldots, M_n]$. The theorem will therefore be established if, given reductions of M to M' and N to N' of length at most ω, we can construct a reduction from $(\lambda x.M)N$ to $M'[x := N']$ of length at most ω. This can be done by first reducing $(\lambda x.M)N$ to $M[x := N]$, and then interleaving a reduction of M to M' and reductions of all the copies of N to N' in a strongly convergent way. The details are simple to work out. □

Remark. The Compressing Lemma is false for $\beta\eta$-reduction. For a counterexample, let $M = Y(\lambda f.\lambda x.I(fx))$. Then $\lambda x.Mxx \twoheadrightarrow^{\omega} \lambda x.I(I(I(...)))x \rightarrow_{\eta} I(I(I(...)))$. However, $\lambda x.Mxx$ is not reducible in ω steps or fewer to $I(I(I(...)))$.

This is not surprising. The η-rule requires testing for the absence of the bound variable in the body of the abstraction; if the abstraction is infinite, this is an infinite task, and such discontinuities are to be expected.

6 Head normal forms and Böhm trees

In the context of infinitary lambda calculus, the Böhm tree of a term can be seen as being simply its normal form with respect to transfinite reduction with respect to the β rule together with an additional rule for erasing subterms having no head normal form. More generally, we find that with each notion of depth there can naturally be associated a notion of head normal form. The classical notion of head normal form is, in the current setting, a term having no redexes at applicative depth 0. A weak head normal form is a term having no redexes at weakly applicative depth 0. A normal form is a term having no redexes at discrete depth 0 (i.e. no redexes anywhere).

Definition 20. A term of Λ^{∞} is *0-stable* if it cannot be reduced to a term having a redex at depth 0. It is *potentially* 0-stable if it can be reduced to a 0-stable term. It is *0-active* if it is not potentially 0-stable.

We shall demonstrate that for six of the eight notions of depth, the class of 0-active terms satisfies the axioms of [AKK+94] for a set of undefined terms. These axioms are (1) both the set and its complement are closed under reduction, and (2) the set includes all the terms which cannot be reduced to root-stable form. (A root-stable term is one which cannot be reduced to a redex, i.e. is 0-stable with respect to syntactic depth.) This immediately gives rise to a number of Böhm-like models of lambda-calculus, in which the value of a term is its unique normal form with respect to a notion of reduction which allows 0-active terms to be replaced by a symbol $\perp$.

The second of the axioms is immediate from the definition. If a term cannot be reduced to root-stable form, then it cannot be reduced to a 0-stable form, since a redex at the root of a term is at depth 0 for every notion of depth.

Half of the first axiom is immediate: the set of 0-active terms is certainly closed under reduction. It only remains to prove that the set of potentially 0-stable terms is also closed. To do this we must develop some theory of Böhm reduction.

Definition 21. *Böhm reduction* is reduction in $\Lambda_\perp^\infty$ by the β rule and the $\perp$ rule, viz. $M \to \perp$ if M is 0-active and not $\perp$. We write $\to_{\mathcal{B}}$ for Böhm reduction and $\to_\perp$ for reduction by the $\perp$-rule alone.

A *Böhm tree* is a normal form of $\Lambda_\perp^\infty$ with respect to Böhm reduction.

We extend the notions of 0-stability etc. to terms containing $\perp$ thus. A term of $\Lambda_\perp^\infty$ is 0-stable if it cannot be reduced to a term containing a Böhm redex or an occurrence of $\perp$ at depth 0. Potential 0-stability and 0-activeness are similarly extended.

0-stability and 0-activeness were defined in terms of reduction, but now we have defined a new notion of reduction in terms of these concepts, which in turn gives us new notions of 0-stability and 0-activeness. It is important to check that the new notions agree with the old on terms of Λ^∞. This turns out not to be the case for two of the eight possible notions of depth, which we regard as sufficient grounds for excluding them.

Theorem 22. *A term of Λ^∞ is 0-stable with respect to beta reduction if and only if it is 0-stable with respect to Böhm reduction.*

Proof. We shall write $0\mathcal{B}$-stable to abbreviate 0-stable with respect to Böhm reduction. Let t be a term of Λ^∞. Clearly, if t is beta-reducible to a term containing a 0-redex, the same reduction sequence is a Böhm reduction to a term containing a 0-redex. Hence if t is not 0-stable, it is not $0\mathcal{B}$-stable.

Conversely, suppose t is Böhm reducible to a term s which contains either a redex or an occurrence of $\perp$ at depth 0. Omitting all the reduction steps by the $\perp$ rule in the obvious way yields a beta reduction of t to a term r which differs from s only by having 0-active subterms where s has occurrences of $\perp$. If s contains a 0-redex, then so does r. If s has an occurrence of $\perp$ at depth 0, r has a 0-active subterm at depth 0. That subterm is reducible to a term containing a 0-redex, which gives a reduction of r to a term containing a 0-redex. Hence t is not 0-stable. □

Theorem 23. *Except in Λ^{010} and Λ^{011}, a term of Λ^∞ is potentially 0-stable with respect to beta reduction if and only if it is potentially 0-stable with respect to Böhm reduction.*

Proof. For potential 0-stability, one direction is again trivial. If t is potentially 0-stable, then it is potentially $0\mathcal{B}$-stable. Conversely, suppose that t is potentially $0\mathcal{B}$-stable. Then it is Böhm reducible to a $0\mathcal{B}$-stable term s. By the same transformation used previously, there is a beta reduction of t to a term r which differs from s only by having 0-active terms where r has $\perp$. Suppose r is not 0-stable. Then r is beta reducible to a term r' containing a 0-redex, and therefore (since the property of having a 0-redex is determined by some finite prefix) it is so reducible in finitely many steps. We shall construct a Böhm reduction of $s \to_{\mathcal{B}}^* s'$ which imitates the beta reduction of r.

Suppose we have reached a step $r_i \to r_{i+1}$ in the reduction of r, and the reduct of s corresponding to r_i is s_i. The condition on the depth measure says that either the depth of M in $\lambda x.M$ is 1 or the depth of M in MN is 0.

If the former is true, we take as an inductive hypothesis that r_i differs from s_i only in that r_i has 0-active subterms where s_i has occurrences of $\perp$. If the redex reduced in r_i is also present in s_i, we reduce it in s_i to obtain s_{i+1}. If the redex is in a 0-active subterm of r_i corresponding to an occurrence of $\perp$ in s_i, then we ignore it and take $s_{i+1} = s_i$. The only other possibility is where the redex is a subterm $(\lambda x.M)N$ of r_i corresponding to a subterm $\perp N'$ of s_i. But the hypothesis about the depth measure implies that $\lambda x.M$ is 0-stable, therefore by the inductive hypothesis cannot correspond to an occurrence of $\perp$ in s_i. Thus the construction can be carried out along the whole sequence, yielding a reduction of s to a term s' such that wherever s' has an occurrence of $\perp$, r' has a 0-active term. Since r' has a 0-redex, s' must have either a beta redex or an occurrence of $\perp$ at depth 0, hence s' is not $0\mathcal{B}$-stable.

Otherwise, the depth of M in $\lambda x.M$ and in MN is 0. Here the argument is similar, but now we cannot exclude the third case, where a redex $(\lambda x.M)N$ in r_i corresponds to a subterm $\perp N'$ in s_i. We deal with this by relaxing the inductive hypothesis, to require that r_i differs from s_i in having 0-active subterms where s_i has subterms of the form $\perp N_1 \dots N_n$, where $n \geq 0$. We can then carry through a similar construction, to obtain a Böhm reduction of s to s', where s' stands in that relation to r'. Since r' has a 0-redex, s' has either a beta redex or a subterm of the form $\perp N_1 \dots N_n$ at depth zero. But in the latter case, the occurrence of $\perp$ is also at depth 0, so s' is not $0\mathcal{B}$-stable. □

These theorems allow us to drop the notation $0\mathcal{B}$-stable, and to speak (potential) 0-stability and 0-activeness with respect to beta reduction or Böhm reduction interchangeably.

The two depth measures which Theorem 23 excludes are those in which the depth of M in $(\lambda x.M)N$ is 0 and in MN is 1. These appear intuitively to be unnatural. One may associate depth with strictness. These two measures regard abstraction as strict, and application as non-strict in its first argument. The rest of this section deals only with depth measures to which Theorem 23 applies.

Lemma 24. *1. For any Böhm reduction sequence $t \to^{\infty}_{\mathcal{B}} t'$, there are sequences $t' \to^{\infty}_{\perp} t''$ and $t \to^{\infty}_{\mathcal{B}} t''$, such that the latter sequence consists of alternating segments in which first a reduction is performed, no step of which is contained in any 0-active subterm, and then a reduction to normal form with respect to the $\perp$ rule is performed.*

2. For any Böhm reduction sequence $t \to^{\infty}_{\mathcal{B}} t'$, there is a sequence $t \to^{\infty}_{\beta} t'' \to^{\infty}_{\perp} t'$.

Proof. (Outline.) For the first, the proof consists in showing how any Böhm reduction sequence can be transformed step by step into the required form, by inserting $\perp$-reductions as necessary. For the second, the proof proceeds by a step-by-step transformation which postpones all $\perp$-reductions until the end. □

Theorem 25. *Böhm reduction is Church-Rosser.*

Proof. (Outline.) Given two coinitial Böhm reduction sequences, we transform them as described by Lemma 24(1). For sequences of that form, the Church-Rosser property can be proved by a tiling argument analogous to that commonly used in proving the finitary Church-Rosser property. From this the Church-Rosser property for arbitrary Böhm reductions follows. □

Theorem 26. *The set of potentially 0-stable terms is closed under reduction.*

Proof. If t is 0-active, it has the Böhm normal form $\perp$. Suppose t reduces to a non-0-active term t'. Then t' reduces to a 0-stable term, which cannot be Böhm reduced to $\perp$. But by the Church-Rosser property for Böhm reduction, this is impossible. □

Theorem 27. *Every term has exactly one Böhm normal form.*

Proof. From the previous theorem, every term has at most one Böhm normal form.

Given any term t, construct a Böhm reduction from t thus. If t is 0-active, reduce it to $\perp$. Otherwise, it is reducible to a 0-stable term. Perform such a reduction, and then repeat this construction on the maximal subterms of the term at depth 1. This generates a strongly convergent sequence whose limit contains no Böhm redexes. □

Theorem 28. *Beta reduction is Church-Rosser up to identification of 0-active terms.*

Proof. Given two beta reductions $t \to^{\infty} t_0$ and $t \to^{\infty} t_1$, Theorem 25 gives Böhm reductions $t_0 \to_{\mathcal{B}}^{\infty} t_2$ and $t_1 \to_{\mathcal{B}}^{\infty} t_2$. By Lemma 24(2), there are beta-reduct of t_0 and t_1 which are $\perp$-reducible to t_2; such reducts are identical up to identification of 0-active terms. □

We thus have a model of lambda calculus, where the objects are the Böhm normal forms, ordered according to Def. 15. The usual Böhm model is the model associated with applicative depth. The larger model described by Berarducci ([Ber]) is the one associated with syntactic depth. In this model the 0-stable terms are the root-stable terms, and the 0-active terms are the terms which Berarducci calls mute. The Böhm model for weakly applicative depth is related to Ong and Abramsky's models for lazy lambda calculus [AO93]. Discrete depth results in the trivial model (since the 0-active terms are the terms with no normal form, and identifying all of these together results in the equality of all terms). The other two depth measures which satisfy the conditions of Theorem 23 give two more models, whose relation to existing models of the lambda calculus remains to be studied.

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