

Resultant-based Elimination in Ore Algebra

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Abstract

We consider resultant-based methods for elimination of indeterminates of **Ore polynomial systems in Ore algebra**. We start with defining the concept of **resultant** for bivariate Ore polynomials then compute it by the Dieudonné determinant of the polynomial coefficients. Additionally, we apply noncommutative versions of evaluation and interpolation techniques to the computation process to improve the efficiency of the method.

The implementation of the algorithms will be performed in Maple to evaluate the performance of the approaches.

Keywords: Ore algebra, elimination, resultant, symbolic computation, modular method, noncommutative algebra.

1 Introduction

Solving polynomial systems in general, whether in the commutative or noncommutative case, is a fundamental mathematical interest. In addition to computer algebra, it is used for numerous applications in sciences, for example in engineering [9], computer vision [38], economics [37], cryptology [53], algebraic geometry [61]. Also seen in many noncommutative cases, such as in cryptography [8], linear codes [6], robotics [30], etc.

However, despite its popularity, the challenges associated in the computation of large polynomial systems have limited their potential until the advent of computer algebra systems where it became possible to solve a large class of different type of polynomial systems. But the problem of high computational time [59] of such polynomial systems remains an important concern for both researchers and industries. Optimization techniques (such as modular methods) have been increasingly used to improve the efficiency as the algorithms become computationally intensive and time

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consuming. This motivates the present work to develop modular algorithms for solving noncommutative polynomial systems (by elimination of the indeterminates).

The *modular approach* is a powerful method for controlling the intermediate expression growth during the computation process that improves the efficiency of solving the system of polynomial equations. This method is well studied in commutative algebra, for example in [14, 3, 54] in which the idea behind these techniques is to employ resultant and gcd (greatest common divisor) approaches to solve polynomial systems by means of *triangular sets* (a sequence of polynomial equations with strictly increasing main indeterminates) and then considering evaluation and interpolation techniques to improve the efficiency of the computation process. In this work, we propose methods to use this approach effectively for solving multivariate (initially bivariate) Ore polynomial systems in Ore algebra.

The transition from a commutative algebra to an Ore algebra is not a straightforward extension, not only because of the noncommutative nature of the Ore algebra which comes with its own challenges to navigate (compared to the commutative case), but also because of some necessary key components in this study are not even available on multivariate Ore polynomials in its general form (such as multivariate resultant). Thus, we have to define noncommutative analogues of these components to address these gaps. The main challenges in this transition include:

- *Unavailability of multivariate resultant:* One of the difficulties encountered in this study is **unavailability of multivariate resultant in Ore algebra** that we could not find it in the literature.
- *Evaluation map is not common:* The most common evaluation map one thinks of (namely by plug-in values) is no longer a valid process in noncommutative world because it does not preserve multiplications.
- *Determinant is different:* One of the primary challenges when it comes to computing determinants in noncommutative algebra is the lack of a standard definition of determinant. Many different definitions of noncommutative determinant have been formulated and studied in the literature.
- *More conditions are needed for interpolation:* Selecting valid values for the interpolation method is different than the commutative case because simply using pairwise different values is not suitable anymore, more conditions are required.

Although most of these challenges are known in the noncommutative algebras, their solutions are different depending on the study.

The ultimate aim of this study is to design, optimize, and implement symbolic algorithms for solving Ore polynomial systems. The contributions include:

- New methods to eliminate indeterminates from bivariate Ore polynomial systems using two different ways:
 - By using a direct resultant computation from the polynomial coefficients.
 - By applying noncommutative evaluation and interpolation techniques.

The proof of the methods together with auxiliary lemmas will be provided.

- New definitions and properties; these include
 - An analogous version of Sylvester’s resultant for bivariate Ore polynomials.
 - An extended version of the operator evaluation for bivariate Ore polynomials.
- Design and implement the algorithms to demonstrate the methods.

Why bivariate case? The importance of bivariate case in this study resides in the recursive evaluation and interpolation procedure, which allows us to reduce the general case of an $n \times n$ system to $(n - 1) \times (n - 1)$ system by evaluating one of the indeterminates then, up to details, this process can be repeated for each case until reaching the bivariate system which then can use the bivariate techniques to solve the original system.

2 Related work

This study is based on the concept of *characteristic set (polynomials in triangular form)*. This is based on elimination theory, which is one of the core concepts in differential and difference algebras.

Eliminations via Gröbner bases have been studied in [18] for linear differential and difference equations, followed by works in [45] for the case where the indeterminates do not commute with each other, and more recently in [39] for operator algebra. However, these studies do not use modular approaches toward the indeterminates.

Resultant theory is yet another technique that can be viewed as an elimination method. Many elimination algorithms are based on resultant computations such as; in [60] for resultant elimination using interpolation of implicit equation, and in [27] for computing multivariate sparse resultant, also in [35] for computing sparse resultants using the computations of Chow forms. However, none of these resultant methods use Ore polynomials.

The resultant for univariate skew polynomials over a division ring has been studied in [28], and for univariate Ore polynomials over commutative rings in [41]. The subresultant theory was also well studied for differential operators in [13], thereafter generalized to the univariate Ore polynomials in [42], and then further studied in [33] to provide an analogous version of the subresultant expressed in terms of the solutions. Additionally, an improved version of the subresultant sequence was studied in [34]. All these (sub)resultant works are studied in the univariate case of Ore polynomials.

Differential algebra was introduced by Joseph Ritt in 1950 [56], where differential resultant was first studied for differential operators by Ore [51] and for nonlinear differential equations by [55]. Since then, a variety of studies have shown the importance of differential resultant for example [48, 29, 62, 5] also for multivariate differential resultants such as [57, 58, 64, 11]. That being said, only the *difference* case is considered here in this study.

Modular techniques are frequently used in computer algebra to reduce the cost of polynomial coefficient growth. Modular algorithms for multivariate polynomials

have been most studied in the commutative case, for example the cases of bivariate and trivariate polynomials have been studied in [54], and in a more general settings when the solution set splits addressed in [14]. However, modular methods for Ore algebra have received less attention in the literature. In [43], a modular approach was used towards the coefficients of Ore polynomials over the commutative polynomial ring $\mathbb{Z}[t]$. Additionally, in [15] and in its Maple implementation [16] modular approaches have used for a matrix of Ore polynomials also over $\mathbb{Z}[t]$. Recently, a modular technique on a probabilistic algorithm was provided by [23] for computing Gröbner bases in G-algebra defined over the field of rational numbers \mathbb{Q} . These modular methods are applied toward coefficients over commutative rings. None of these studies use noncommutative evaluation and interpolation techniques.

There are some fairly recent research developments on noncommutative evaluation and interpolation methods such as in [63] where the Newton interpolation for skew polynomials was studied, as well as discussed in [44] with a different setup of evaluation values. Moreover, a Lagrange-type interpolation with an evaluation map that vanishes at every points over multivariate skew polynomial rings was studied in [46].

With the current missing studies about modular methods over Ore polynomials, we consider extending the modular algorithms in [54] to (bivariate) Ore polynomial rings.

Tools: There are some packages that are implemented for Ore polynomials. Maple package OreTools [1] uses a modular algorithm for the coefficients of univariate Ore polynomials over $\mathbb{Z}[t]$. This package, however, does not deal with the multivariate case. The packages Ore_algebra [17] in Maple and ore_algebra in SageMath [36] can process multivariate Ore polynomials but they do not use modular techniques.

3 Background

In this section, we provide a brief review of the Ore algebra, operator evaluation and Dieudonné determinant which will be used in later sections.

3.1 Ore algebra

An Ore algebra is an algebra of noncommutative polynomials [18]. It can describe linear differential, difference, and difference-differential equations in a unified framework that collectively models a large subclass of noncommutative rings.

In the following we recall the definition of Ore ring [52] and followed by the definition of Ore algebra [18]. Note that throughout this study \mathbb{A} is a (skew) field, unless otherwise stated.

Definition 3.1 (Ore polynomial ring) *Let $\sigma : \mathbb{A} \rightarrow \mathbb{A}$ be an automorphism of \mathbb{A} (called conjugate operator). The noncommutative polynomial ring $\mathbb{A}[x; \sigma]$ is the set of polynomials in indeterminate x over \mathbb{A} with the usual polynomial addition (+) and noncommutative multiplication defined as*

$$xa = \sigma(a)x, \quad \forall a \in \mathbb{A}. \quad (1)$$

This ring is called the Ore polynomial ring (or left Ore ring) with conjugate operator σ over \mathbb{A} . Elements of $\mathbb{A}[x; \sigma]$ are called Ore polynomials.

A nonzero polynomial in $\mathbb{A}[x; \sigma]$ has *degree* n if n is the highest power of x with a nonzero coefficient, or $-\infty$ for the zero polynomial.

Remark 3.2 *Note that the typical commutative polynomials are a special case of Ore polynomials. This can be shown by taking the σ to be the identity map of a commutative ring \mathbb{A} , where the commutation rule (1) simply becomes $xa = ax$ for all $a \in \mathbb{A}$. Thus, this study can be applied to the commutative case as well, and it will work as usual.*

To manage more complicated types of Ore rings than just the ring of polynomials in one indeterminate, the process of Ore polynomial ring can be iterated with different suitable σ to construct Ore polynomial rings with $n > 1$ indeterminates. We state the following definition of *Ore algebra* [18] in the bivariate case.

Definition 3.3 (Ore algebra) An **Ore algebra** is the iterated Ore polynomial ring

$$\mathbb{S} = \mathbb{A}[x_1; \sigma_1][x_2; \sigma_2],$$

with two commuting indeterminates x_1, x_2 over \mathbb{A} and two automorphisms σ_1, σ_2 of \mathbb{S} that satisfy the relations

$$\sigma_j(x_i) = x_i \ (i \neq j), \sigma_j \sigma_i = \sigma_i \sigma_j \text{ and } x_i a = \sigma_i(a) x_i, \quad (2)$$

for all $a \in \mathbb{A}$ and $1 \leq i, j \leq 2$. Elements of \mathbb{S} are called *bivariate Ore polynomials*.

3.2 Operator evaluation

One of the main differences when it comes to the evaluations in noncommutative rings is that evaluation maps behave quite differently comparing to the commutative case (plug-in values) mainly because noncommutative evaluation maps generally do not preserve the products.

We are looking for a suitable evaluation map that will not only be a ring homomorphism but also correctly handles the commutation between indeterminates (since in this study the indeterminates are assumed to commute with each other). For example, in $\mathbb{C}[x_1; \sigma_1][x_2; \sigma_2]$ over complex numbers \mathbb{C} , substituting i for x_1 in a multivariate expression such as $x_1 x_2$ will be $i x_2$, while the same substitution for its commuting expression $x_2 x_1$ leads to $x_2 i = \sigma_2(i) x_2$, and the two results are not the same in general.

Although the basic concept of evaluation is the same, there are many different noncommutative evaluation maps that serve different purposes. In addition to the evaluating formula, there are also remainder theorem [19], product formula [40], operator evaluation [7], σ -evaluation [4], etc.

One of the essential parts of Ore polynomials is the presence of operators; such as σ , in fact, all the elements of Ore polynomial rings can be viewed as operators which naturally suggests to think of a map that can evaluate at operators where an interesting target would be evaluating at σ because we know σ is a ring homomorphism and this will have its influence on the evaluation process. Thus, we consider what is called the *operator evaluation* [7].

At this point, it is convenient to state the following ring [7]

$$\mathbb{A}[\sigma; \circ] = \left\{ \sum_{i=0}^n a_i \sigma^i : a_i \in \mathbb{A} \right\}, \quad (3)$$

which is a left Ore ring [52] (also see [47, Section 3]) with the coefficient-wise addition and the composition of operators as multiplication. Thus, any $f = \sum_{i=0}^n a_i \sigma^i$ and $g = \sum_{j=0}^n b_j \sigma^j$ in $\mathbb{A}[\sigma; \circ]$ are added as

$$f + g = \sum_{k=0}^n (a_k + b_k) \sigma^k,$$

and multiplied by the operator composition (denoted by $f \circ g$ or simply fg) as

$$fg = \sum_{i=0}^n \sum_{j=0}^n a_i \sigma^i (b_j) \sigma^{i+j}.$$

We call the elements of $\mathbb{A}[\sigma; \circ]$ polynomials of the indeterminate σ over the (skew) field \mathbb{A} .

Now we can state the following Lemma related to operator evaluation for univariate skew polynomials [7].

Lemma 3.4 *Let $f = \sum_{i=0}^n a_i x^i$ be an Ore polynomial in $\mathbb{A}[x; \sigma]$. The evaluation map $\text{Eval}_{(x-\sigma)}(f)$ defined as*

$$\text{Eval}_{(x-\sigma)} : \begin{array}{l} \mathbb{A}[x; \sigma] \rightarrow \mathbb{A}[\sigma; \circ] \\ f = \sum_{i=0}^n a_i x^i \mapsto f(\sigma) = \sum_{i=0}^n a_i \sigma^i. \end{array}$$

is a morphism of rings.

Accordingly, the following concepts of *operator evaluation* and *solution* can be defined [7].

Definition 3.5 (Operator evaluation) *Let $f = \sum_{i=0}^n a_i x^i$ be an Ore polynomial in $\mathbb{A}[x; \sigma]$, and let L_f denote the map $f(x)$. The operator evaluation $\text{Eval}_{(x-\sigma)}(f)$ at $a \in \mathbb{A}$ is defined as*

$$\text{Eval}_{(x-\sigma)(a)} : \begin{array}{l} \mathbb{A}[x; \sigma] \rightarrow \mathbb{A} \\ f = \sum_{i=0}^n a_i x^i \mapsto f(\sigma)(a) = \sum_{i=0}^n a_i \sigma^i(a). \end{array}$$

If $\text{Eval}_{(x-\sigma)(a)}(L_f) = 0$ then $y = a$ is a solution of $L_f(y) = 0$. When the operator under evaluation is clear from context, we denote $f(\sigma)(a)$ by $f^(a)$.*

3.3 Dieudonné determinant

Before describing the definition of Dieudonné determinant [24], we would like to mention some notations about elementary row operations in noncommutative case

(mostly analogous to the commutative case), that will be used throughout this study. For more details on this please see, for example, [21, §9.2].

Let the notation $M_n(\mathbb{S})$ denotes the set of all $n \times n$ matrices over a (skew) field \mathbb{S} . Let e_{ij} be a matrix whose elements are all zeros except the value 1 in the (i, j) -th coordinate. Also, let E_{ij} represents interchanging rows i and j . Additionally, denote by $E_i(u) = I + (u - 1)e_{ii}$ the elementary operation of multiplying row i by a scalar unit u . Finally, the *elementary matrix* of adding (q times row i) to row j is denoted by

$$E_{ij}(q) = I + qe_{ji} \quad (i \neq j, q \in \mathbb{S}). \quad (4)$$

Let P be a general *permutation matrix* obtained from the $n \times n$ identity matrix I by permuting some rows (or equivalently some columns) according to a permutation map p of n elements

$$p : \{1, \dots, n\} \rightarrow \{1, \dots, n\}.$$

Note that the determinant $\text{Det}(P)$, the Dieudonné determinant which will be described in Definition 3.6 and Remark 3.7, is always either 1 or -1 depending on the number of permutations whether even or odd. Sometimes, it is needed to keep the value of $\text{Det}(P)$ to be always 1, and in this case, it is convenient to define what is called *signed permutation*, which is the same as the general permutation matrix except whenever we permute two rows (or columns) then we change the sign of one of them [21, p. 350]. Furthermore, it is known that the signed permutation of two rows i and j can be accomplished by $E_{ij}(u)$, in three steps in the form

$$P = E_{ij}(1)E_{ji}(-1)E_{ij}(1), \quad (5)$$

which replaces row j by row i and row i by $(-1$ times row $j)$. For example, if we apply it to the 2×2 identity matrix I , we obtain $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, noting that the determinant value stays invariant.

We can use these row operations to reduce a matrix into a simpler form without compromising its properties and that is what we will be practicing throughout this study. For more details about these notations and their properties, please see [21, §9.2] or [2, Chapter IV].

In the following section we recall the definition of the Dieudonné determinant via diagonal matrices, for example see [2, Chapter IV] or [26, §20].

Definition 3.6 (Dieudonné determinanant) *Let \mathbb{S} be a (skew) field, $M_n(\mathbb{S})$ be the $n \times n$ matrices over \mathbb{S} , and $\text{GL}_n(\mathbb{S})$ be the multiplicative group of invertible $n \times n$ matrices with entries in \mathbb{S} . Additionally, let $[\mathbb{S}^\times, \mathbb{S}^\times]$ be the (normal) multiplicative commutator subgroup of \mathbb{S} . The Dieudonné determinant (denoted by $\text{Det}(A)$) of a square diagonal matrix $A \in M_n(\mathbb{S})$ with diagonal entries d_i ($i = 1, \dots, n$) is defined as*

$$\text{Det}(A) = \begin{cases} 0 & A \notin \text{GL}_n(\mathbb{S}) \\ [\text{sign}(p) \prod_{i=1}^n d_i] & A \in \text{GL}_n(\mathbb{S}) \end{cases} \quad (6)$$

where p is a permutation map and $[\cdot]$ is the canonical projection from \mathbb{S}^\times to $\mathbb{S}^\times/[\mathbb{S}^\times, \mathbb{S}^\times]$.

The Ore polynomial ring $\mathbb{A}[x; \sigma]$ can be embedded into a skew field $\mathbb{S} = \mathbb{A}(x; \sigma)$ which is the field of left fractions of $\mathbb{A}[x; \sigma]$ [50] (also see [20, Corollary 0.7.2]).

Accordingly, we can find the determinant $\mathcal{D}et(A)$ of any invertible matrix A over $\mathbb{A}[x; \sigma]$ where in Lemma 4.1 we will show that this determinant can be represented by a polynomial belong to $\mathbb{A}[x; \sigma]$ modulo commutators.

Note that the meaning of $\mathcal{D}et(A)$ in (6) is that the Dieudonné determinant of an invertible matrix A is a value in an abelian group $\mathbb{S}^\times/[\mathbb{S}^\times, \mathbb{S}^\times]$, on the other hand if A is not invertible then this simply means $\mathcal{D}et(A) = 0$. For more details on Dieudonné determinant and its properties please see, for example, [49, 2, 24].

Remark 3.7 *It is known [21, p. 352] that for any invertible matrix A , a signed permutation matrix P satisfies the property*

$$\mathcal{D}et(PA) = \mathcal{D}et(A). \quad (7)$$

Thus, using signed permutation, we can clear $\text{sign}(p)$ in formula (6) for invertible diagonal matrices and write

$$\mathcal{D}et(A) = [\prod_{i=1}^n d_i]. \quad (8)$$

One of the main properties of Dieudonné determinant is that it satisfies the elementary row operations [2], analogous to the commutative case. Therefore, Dieudonné determinant can be computed by a procedure similar to the Gauss-Jordan elimination method aiming to diagonalize the matrix first and then applying the determinant formula (8).

Furthermore, the method of triangularization of a square matrix may also be performed to compute the Dieudonné determinant, especially if we would like to have the determinant value to stay in the same ring of the matrix entries, which is the focus of Section 4.1.

Note that Dieudonné determinant is unique only up to commutators. We can illustrate this with a 2×2 matrix A over a (skew) field \mathbb{A} as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Assume $a_{11} \neq 0$ and $a_{21} \neq 0$ (otherwise the case is trivial).

If we fix a_{11} then by using row reduction properties, we can have

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} - a_{21}a_{11}^{-1}a_{12} \end{pmatrix},$$

the Dieudonné determinant is

$$[a_{11}a_{22} - a_{11}a_{21}a_{11}^{-1}a_{12}]. \quad (9)$$

On the other hand, if we fix a_{21} then

$$A = \begin{pmatrix} 0 & a_{12} - a_{11}a_{21}^{-1}a_{22} \\ a_{21} & a_{22} \end{pmatrix},$$

thus the Dieudonné determinant in this case is

$$[a_{21}a_{11}a_{21}^{-1}a_{22} - a_{21}a_{12}]. \quad (10)$$

Now we need to prove that (9) and (10) are equivalent. The following diagram illustrates the proof:

$$\begin{array}{ccc}
[a_{11}a_{22} - a_{11}a_{21}a_{11}^{-1}a_{12}] & \stackrel{?}{=} & [a_{21}a_{11}a_{21}^{-1}a_{22} - a_{21}a_{12}] \\
\Downarrow & & \Downarrow \\
[a_{11}a_{22} - a_{11}a_{21}a_{11}^{-1}a_{21}^{-1}a_{21}a_{12}] & & [a_{21}a_{11}a_{21}^{-1}a_{11}^{-1}a_{11}a_{22} - a_{21}a_{12}] \\
\Downarrow & & \Downarrow \\
[a_{11}a_{22} - xa_{21}a_{12}] & & [x^{-1}(a_{11}a_{22} - xa_{21}a_{12})]
\end{array}$$

where $x = a_{11}a_{21}a_{11}^{-1}a_{21}^{-1}$ which is a commutator. Thus, it shows the equivalence and this means the determinant is well defined (modulo commutators).

Remark 3.8 *Note that in this study, all computations of Dieudonné determinant being performed modulo commutators even if this is not explicitly stated.*

4 Elimination

In this section we discuss two different but related new methods for eliminating indeterminates in a system of Ore polynomials, one based on the direct computation of resultant from the polynomials coefficients, and the other is based on the evaluation and interpolation techniques. First, we need to have a property that Dieudonné determinant is a unique *polynomial* (modulo commutators) when the entries are Ore polynomials as in the following section.

4.1 Dieudonné determinant over Ore polynomial rings

From the perspective of normal forms of matrices, one can think of diagonalizing an invertible matrix A over $\mathbb{A}[x; \sigma]$ to be converted to the form:

$$D = UAV = \text{diag}(1, \dots, 1, d), \quad (11)$$

where U and V are *unimodular matrices*, D is a diagonal matrix whose diagonal entries are all 1's except the last entry which is equal to $d \in \mathbb{A}(x; \sigma)$, that means we can identify the matrix D by an element d in $\mathbb{A}(x; \sigma)$.

Using formula (8), the Dieudonné determinant of D as in (11) may be obtained by simply multiplying the diagonal entries;

$$\mathcal{D}et(D) = [d]. \quad (12)$$

However, one problem of this process for our study is that, in general, the value d becomes a rational function in $\mathbb{A}(x; \sigma)$ rather than staying a polynomial in $\mathbb{A}[x; \sigma]$, something that we would like to address in this section.

Knowing that the Dieudonné determinant of *an upper (or lower) triangular matrix* is the product of the diagonal entries (see for example, [10] p. 822, [25] p. 3 Example 2), we can reduce a matrix to a triangular form in a manner that all the diagonal entries become polynomials so that their product is again a polynomial in $\mathbb{A}[x; \sigma]$. This can be obtained by using elementary row operations over a Euclidean domain, as in Lemma 4.1, that we prove it similar to [32], but using different techniques. The next lemma is perhaps well known but we couldn't find a precise reference for it; we prove it anyway, for completeness and verification purposes.

Lemma 4.1 *Let $\mathbb{A}[x; \sigma]$ be an Ore polynomial ring. Dieudonné determinant of an invertible matrix $A \in \mathbb{A}[x; \sigma]^{n \times n}$ can be represented by a unique Ore polynomial (modulo commutators).*

Proof. Let A be an invertible matrix in $\mathbb{A}[x; \sigma]^{n \times n}$, suppose $B = A$. For each $i = 1, \dots, n-1$, we can assume i -th column of the matrix B is non-zero (otherwise the determinant is 0), select a non-zero entry below the diagonal element with the lowest degree in the same i -th column (move to the next i if all entries below the diagonal element are zeros), then by using the row operation (5) we can interchange it with the (i, i) -th value (if not already). As our ring of polynomials is a Euclidean domain [52], for each element b_{ki} ($k > i$) of B if $\deg(b_{ki}) > \deg(b_{ii})$ then there exist $q, r \in \mathbb{A}[x; \sigma]$ such that $b_{ki} - qb_{ii} = r$ where $\deg(r) < \deg(b_{ii})$, and thus we can use the row operation (4) to replace the entry b_{ki} below the diagonal by a smaller degree r , continue in this manner until eventually the zero degree (and then the zero value) will be reached for all $k > i$ in the i -th column (other than the one exactly on the diagonal line).

Repeating this process until all the i -columns ($i = 1, \dots, n-1$) are checked yields to the desired matrix. Note that Euclidean domain allow us to keep the diagonal elements b_{ii} of B remain polynomials in $\mathbb{A}[x; \sigma]$.

Therefore, by applying (8) we conclude that the Dieudonné determinant can be written as the product of the diagonal entries

$$\mathcal{D}et(A) = \mathcal{D}et(B) = [\prod_{i=1}^n b_{ii}].$$

□

Remark 4.2 *There is no specific reason for selecting an upper triangular form in Lemma 4.1. Indeed, we can use as well a lower triangular form with a similar proof by applying elementary row operations from the right (column operations). Which means any invertible matrix with Ore polynomial entries can be diagonalized with Ore polynomial entries on the main diagonal, simply by making it upper and lower triangular.*

Another useful property of the Dieudonné determinant is that it does not depend on the choice of elementary row operations, neither on the order in the product of diagonal entries of the matrix (see for example, [10] p. 822).

4.2 Elimination by direct resultant

Resultant is a powerful tool in the theory of elimination which is commonly used for solving systems of polynomial equations (due to its lower computational complexity compared to Gröbner-based methods).

We provide the following new definition of resultant of two bivariate Ore polynomials in an Ore algebra that can compute the resultant directly from the polynomials coefficients (generalizing the commutative case of [Sylvester's determinant](#)).

Definition 4.3 *Let $\mathbb{S} = \mathbb{A}[x_1; \sigma_1][x_2; \sigma_2]$ be an Ore algebra. Consider two bivariate polynomials $f = \sum_{i=0}^n a_i(x_1)x_2^i$ and $g = \sum_{j=0}^m b_j(x_1)x_2^j$ in $(\mathbb{A}[x_1; \sigma_1])[x_2; \sigma_2]$ where*

$a_i(x_1), b_j(x_1) \in \mathbb{A}[x_1; \sigma_1]$. We define the resultant of f and g with respect to x_2 (denoted by $\text{Res}_{x_2}(f, g)$), by the following Dieudonné determinant

$$\text{Res}_{x_2}(f, g) = \begin{vmatrix} x_2^{m-1}f & a_n^{[m-1]}(x_1) & a_{n-1}^{[m-1]}(x_1) & \cdots & \cdots & a_0^{[m-1]}(x_1) \\ x_2^{m-2}f & a_n^{[m-2]}(x_1) & a_{n-1}^{[m-2]}(x_1) & \cdots & \cdots & a_0^{[m-2]}(x_1) \\ \vdots & & & & & \ddots \\ f & & & & a_n^{[0]}(x_1) & a_{n-1}^{[0]}(x_1) & \cdots & \cdots & a_0^{[0]}(x_1) \\ x_2^{n-1}g & b_m^{[n-1]}(x_1) & b_{m-1}^{[n-1]}(x_1) & \cdots & b_0^{[n-1]}(x_1) & & & & \\ & b_m^{[n-2]}(x_1) & b_{m-1}^{[n-2]}(x_1) & \cdots & b_0^{[n-2]}(x_1) & & & & \\ \vdots & & & & & & & & \ddots \\ x_2 g & & & & b_m^{[1]}(x_1) & b_{m-1}^{[1]}(x_1) & \cdots & b_0^{[1]}(x_1) & \\ g & & & & b_m^{[0]}(x_1) & b_{m-1}^{[0]}(x_1) & \cdots & b_0^{[0]}(x_1) \end{vmatrix},$$

where the i -th row ($i = 1, \dots, m$) contains the coefficient sequence of the multiplication $x_2^{m-i}f$, the coefficients of this multiplication are denoted by $a_j^{[m-i]}(x_1)$ ($j = n, \dots, 0$). Similarly, the $(m+i)$ -th row ($i = 1, \dots, n$), contains the coefficients of $x_2^{n-i}g$, these coefficients are denoted by $b_j^{[n-i]}(x_1)$ ($j = m, \dots, 0$). Thus, for notational simplicity, we can write the resultant $\text{Res}_{x_2}(f, g)$ in the form

$$\text{Res}_{x_2}(f, g) = \mathcal{D}et(x_2^{m-1}f, \dots, x_2 f, f, x_2^{n-1}g, \dots, x_2 g, g). \quad (13)$$

Recall that the indeterminates x_1 and x_2 do not commute with the coefficients but rather act according to the ring automorphisms σ_1 and σ_2 such that for each $a \in \mathbb{A}$,

$$x_1 a = \sigma_1(a) x_1 \quad \text{and} \quad x_2 a = \sigma_2(a) x_2, \quad (14)$$

which means the noncommutative properties in the determinants' entries are preserved, since the rows are multiplied by a power of x_2 to the left.

Remark 4.4 *The definition of Dieudonné determinant 3.6 requires entries to be in a (skew) field, this can be obtained by embedding $\mathbb{A}[x_1, \sigma_1]$ in a (skew) field [20, Corollary 0.7.2] (since $\mathbb{A}[x_1, \sigma_1]$ is an Ore ring).*

4.3 Elimination by evaluation and interpolation resultant

In this section we describe another method to compute resultant with Ore polynomial entries by evaluation and interpolation techniques (generalizing the commutative case of [54]). We prove a theorem which establishes a relation between bivariate resultants and evaluation maps.

Additionally, we apply the operator evaluation methods in [31] or [12] to improve the efficiency of the polynomial multiplications during the computation of the resultant, but we proceed slightly differently by considering the operator σ_1 itself as an element of the base ring (as in Remark 4.5), which provides a more convenient way to determine the commutators and the conjugacy classes that will be needed later.

Remark 4.5 Let $\mathbb{A}[\sigma_1, \circ]$ be the left Ore ring as in (3), then we can have the ring $\mathbb{A}[\sigma_1, \circ][x_1; \sigma_1]$ which is also a left Ore ring [18, Corollary 1] with the properties

$$x_1\sigma_1 = \sigma_1x_1, \sigma_1a = \sigma_1(a)\sigma_1 \text{ and } x_1a = \sigma_1(a)x_1, \forall a \in \mathbb{A}. \quad (15)$$

In the same manner, this process can be iterated to have the Ore algebra

$$\mathbb{A}[\sigma_1; \circ][x_1; \sigma_1][x_2; \sigma_2].$$

Such that, in addition to the relations (2) and (15), it satisfies

$$\sigma_1x_2 = x_2\sigma_1. \quad (16)$$

Since $\mathbb{A}[\sigma_1; \circ]$ is a left Ore ring, it can be embedded into a (skew) field $\tilde{\mathbb{A}} = \mathbb{A}(\sigma_1, \circ)$ [20, Corollary 0.7.2], for which (later in Definition 4.10) we need a multiplicative group, denoted by $\tilde{\mathbb{A}}^\times$, that contains nonzero elements of $\mathbb{A}(\sigma_1, \circ)$. The multiplication in $\tilde{\mathbb{A}}^\times$ is defined similarly to the multiplication with rational functions in Ore algebra where for any two left fractions $f_2^{-1}f_1$ and $g_2^{-1}g_1$ in $\tilde{\mathbb{A}}^\times$, the multiplication is defined as

$$f_2^{-1}f_1 \cdot g_2^{-1}g_1 = (d_2f_2)^{-1}d_1g_1,$$

such that $d_1g_2 = d_2f_1$, where d_1 and d_2 are from left Ore conditions¹.

From the Remark 4.5, we can conclude the following:

Lemma 4.6 Let $\tilde{\mathbb{A}}[x_1; \sigma_1][x_2; \sigma_2]$ be an Ore algebra where $\tilde{\mathbb{A}} = \mathbb{A}(\sigma_1, \circ)$. For all nonnegative integers n, m and for all $a \in \mathbb{A}$, the following holds:

- (i) $\sigma_1^n x_2^m = x_2^m \sigma_1^n$
- (ii) $\sigma_1^n a \sigma_1^m = \sigma_1^n(a) \sigma_1^{n+m}$.

Note that, in this Lemma 4.6, the product (i) is commutative while (ii) is not.

Example 4.7 Consider the Ore algebra $\tilde{\mathbb{A}}[x_1; \sigma_1][x_2; \sigma_2]$ where $\tilde{\mathbb{A}} = \mathbb{A}(\sigma_1, \circ)$, we can find $(\sigma_1x_2)(a)$ as following:

$$(\sigma_1x_2)(a) = (x_2\sigma_1)(a) = x_2\sigma_1(a) = \sigma_2(\sigma_1(a))x_2.$$

Similarly for all $a, b \in \mathbb{A}$, we can describe $(\sigma_1a)(b)$ as:

$$(\sigma_1a)(b) = (\sigma_1(a)\sigma_1)(b) = \sigma_1(a)\sigma_1(b) = \sigma_1(ab).$$

In the following, we extend the evaluation notations described in Section 3.2 to bivariate Ore polynomials as defined below:

Definition 4.8 Let $\tilde{\mathbb{A}}[x_1; \sigma_1][x_2; \sigma_2]$ be a bivariate Ore polynomial ring. Since we have commuting indeterminates x_1 and x_2 , we can consider the Ore ring in the form $\mathbb{S} = \mathbb{E}[x_1; \sigma_1]$ where $\mathbb{E} = \tilde{\mathbb{A}}[x_2; \sigma_2]$, that is, polynomials are regarded with respect to

¹By the left Ore condition we can find $d_1, d_2 \in \tilde{\mathbb{A}}$ such that $d_1g = d_2f$, for all nonzero elements $f, g \in \tilde{\mathbb{A}}$.

x_1 . Then for each polynomial $f = \sum_{i=0}^n \alpha_i x_1^i$, $\alpha_i \in \mathbb{E}$, we define the evaluation map $\text{Eval}_{(x_1-\sigma_1)}(f)$ as

$$\begin{aligned} & \mathbb{E}[x_1; \sigma_1] \rightarrow \mathbb{E} \\ \text{Eval}_{(x_1-\sigma_1)} : & f = \sum_{i=0}^n \alpha_i x_1^i \mapsto f(x_2, \sigma_1) = \sum_{i=0}^n \alpha_i \sigma_1^i. \end{aligned}$$

In the following, we show that the map $\text{Eval}_{(x_1-\sigma_1)}$ is a ring homomorphism (morphism of rings) for *bivariate* Ore polynomials.

Lemma 4.9 *The map $\text{Eval}_{(x_1-\sigma_1)}$ is a ring homomorphism.*

Proof. Since we have commuting indeterminates, the polynomials can be regarded with respect to x_1 in $\mathbb{S} = (\tilde{\mathbb{A}}[x_2; \sigma_2])[x_1; \sigma_1]$. Let $f = \sum_{i=0}^n \alpha_i x_1^i$ and $g = \sum_{j=0}^m \beta_j x_1^j$ be two polynomials in \mathbb{S} with $\alpha_i, \beta_j \in \tilde{\mathbb{A}}[x_2; \sigma_2]$.

It is easy to check that the property of ring homomorphism holds for the addition and identity. Now we check the multiplication:

$$\begin{aligned} \text{Eval}_{(x_1-\sigma_1)}(fg) &= \text{Eval}_{(x_1-\sigma_1)}\left(\sum_{i=0}^n \alpha_i x_1^i \sum_{j=0}^m \beta_j x_1^j\right) \\ &= \text{Eval}_{(x_1-\sigma_1)}\left(\sum_{i=0}^n \sum_{j=0}^m \alpha_i x_1^i \beta_j x_1^j\right) \\ &= \text{Eval}_{(x_1-\sigma_1)}\left(\sum_{i=0}^n \sum_{j=0}^m \alpha_i \sigma_1^i (\beta_j) x_1^{i+j}\right) \\ &= \sum_{i=0}^n \sum_{j=0}^m \alpha_i \sigma_1^i (\beta_j) \sigma_1^{i+j} \\ &= \sum_{i=0}^n \sum_{j=0}^m \alpha_i \sigma_1^i (\beta_j \sigma_1^j) \\ &= \sum_{i=0}^n \alpha_i \sigma_1^i \sum_{j=0}^m \beta_j \sigma_1^j \\ &= \text{Eval}_{(x_1-\sigma_1)}\left(\sum_{i=0}^n \alpha_i x_1^i\right) \text{Eval}_{(x_1-\sigma_1)}\left(\sum_{j=0}^m \beta_j x_1^j\right) \\ &= \text{Eval}_{(x_1-\sigma_1)}(f) \text{Eval}_{(x_1-\sigma_1)}(g) \end{aligned}$$

□

In the following, we define Eval when the input polynomial is regarded as modulo commutators.

Definition 4.10 *Let $\tilde{\mathbb{A}}(x; \sigma)$ be a (skew) field and let $\tilde{\mathbb{A}}^\times(x; \sigma)$ denotes multiplicative group of $\tilde{\mathbb{A}}(x; \sigma)$ containing nonzero elements of $\tilde{\mathbb{A}}(x; \sigma)$. Let $f = f_2^{-1} f_1$ be an element in $\tilde{\mathbb{A}}^\times(x; \sigma) / [\tilde{\mathbb{A}}^\times(x; \sigma), \tilde{\mathbb{A}}^\times(x; \sigma)]$. Additionally, let \mathfrak{D} denotes the normal*

subgroup generated by the multiplicative commutators of $\tilde{\mathbb{A}}^\times(x; \sigma)$, similarly let \mathfrak{D}' be the commutators subgroup of $\tilde{\mathbb{A}}^\times$. The modular evaluation map $\text{Eval}_{(x-\sigma)}(f)$ is defined as:

$$\text{Eval}_{(x-\sigma)} : \begin{array}{l} \tilde{\mathbb{A}}^\times(x; \sigma) / [\tilde{\mathbb{A}}^\times(x; \sigma), \tilde{\mathbb{A}}^\times(x; \sigma)] \rightarrow \tilde{\mathbb{A}}^\times / [\tilde{\mathbb{A}}^\times, \tilde{\mathbb{A}}^\times] \\ f = f_2^{-1} f_1 \pmod{\mathfrak{D}} \mapsto f(\sigma) = (f_2(\sigma))^{-1} f_1(\sigma) \pmod{\mathfrak{D}'}, f_2(\sigma) \neq 0. \end{array}$$

In particular, if $f = \sum_{i=0}^n a_i x^i$ is an Ore polynomial in $\tilde{\mathbb{A}}^\times(x; \sigma) / [\tilde{\mathbb{A}}^\times(x; \sigma), \tilde{\mathbb{A}}^\times(x; \sigma)]$ then

$$\text{Eval}_{(x-\sigma)} : f = \sum_{i=0}^n a_i x^i \pmod{\mathfrak{D}} \mapsto f(\sigma) = \sum_{i=0}^n a_i \sigma^i \pmod{\mathfrak{D}'}$$

Remark 4.11 Note that in this study, we assume the evaluation map Eval becomes modular (by default) as in Definition 4.10 when the input argument computed modulo commutators.

In the following, we show that the map $\text{Eval}_{(x-\sigma)}$ is well defined.

Lemma 4.12 The evaluation map $\text{Eval}_{(x-\sigma)}$ as in Definition 4.10 is well defined.

Proof. Since any quotient group by its commutator subgroup is abelian, $\tilde{\mathbb{A}}^\times / [\tilde{\mathbb{A}}^\times, \tilde{\mathbb{A}}^\times]$ has an abelian structure where the *universal property of abelianization* follows as in Figure 1, such that

$$\text{eval}_{(x-\sigma)} = \text{Eval}_{(x-\sigma)} \circ \pi,$$

$$\begin{array}{ccc} \tilde{\mathbb{A}}^\times(x; \sigma) & \xrightarrow{\text{eval}_{(x-\sigma)}} & \tilde{\mathbb{A}}^\times / [\tilde{\mathbb{A}}^\times, \tilde{\mathbb{A}}^\times] \\ & \searrow \pi & \uparrow \text{Eval}_{(x-\sigma)} \\ & & \tilde{\mathbb{A}}^\times(x; \sigma) / [\tilde{\mathbb{A}}^\times(x; \sigma), \tilde{\mathbb{A}}^\times(x; \sigma)] \end{array}$$

Figure 1: Universal property of abelianization.

where π is the canonical map and $\text{eval}_{(x-\sigma)}$ is homomorphism between multiplicative groups defined as

$$\text{eval}_{(x-\sigma)} : f \mapsto f(\sigma) \pmod{\mathfrak{D}'}, \forall f \in \tilde{\mathbb{A}}^\times(x; \sigma).$$

We know that

$$\ker \pi = [\tilde{\mathbb{A}}^\times(x; \sigma), \tilde{\mathbb{A}}^\times(x; \sigma)] \leq \ker \text{eval}_{(x-\sigma)}.$$

Now, to show that $\text{Eval}_{(x-\sigma)}$ is well defined, suppose

$$f \pmod{\mathfrak{D}} = g \pmod{\mathfrak{D}},$$

implies that

$$fg^{-1} \in \mathfrak{D} = [\tilde{\mathbb{A}}^\times(x; \sigma), \tilde{\mathbb{A}}^\times(x; \sigma)] \leq \ker \text{eval}_{(x-\sigma)}.$$

Thus,

$$\text{eval}_{(x-\sigma)}(fg^{-1}) = \text{eval}_{(x-\sigma)}(f) \text{eval}_{(x-\sigma)}(g)^{-1} = 1 \pmod{\mathfrak{D}'},$$

that is $\text{eval}_{(x-\sigma)}(f) = \text{eval}_{(x-\sigma)}(g)$, implies that

$$(\text{Eval}_{(x-\sigma)} \circ \pi)(f) = (\text{Eval}_{(x-\sigma)} \circ \pi)(g),$$

which means $\text{Eval}_{(x-\sigma)}(f \pmod{\mathfrak{D}}) = \text{Eval}_{(x-\sigma)}(g \pmod{\mathfrak{D}})$. □

Lemma 4.13 Let $\mathbb{S} = \tilde{\mathbb{A}}[x_1; \sigma_1][x_2; \sigma_2]$ be an Ore algebra. For any polynomial $f = \sum_{i=0}^n a_i(x_1)x_2^i$ in \mathbb{S} where $a_i(x_1) \in \tilde{\mathbb{A}}[x_1; \sigma_1]$, we have

$$\text{Eval}_{(x_1-\sigma_1)}(f) = \sum_{i=0}^n a_i(\sigma_1)x_2^i.$$

Proof. Let $f = \sum_{i=0}^n a_i(x_1)x_2^i$ be a polynomial in $\tilde{\mathbb{A}}[x_1; \sigma_1][x_2; \sigma_2]$. Since we have commuting indeterminates x_1 and x_2 , we can consider the ring $\mathbb{S} = (\tilde{\mathbb{A}}[x_2; \sigma_2])[x_1; \sigma_1]$ where the polynomial f can be arranged in the form $f = \sum_{j=0}^m \alpha_j(x_2)x_1^j$, $\alpha_j(x_2) \in \tilde{\mathbb{A}}[x_2; \sigma_2]$. By Definition 4.8 of $\text{Eval}_{(x_1-\sigma_1)}$, we can have $\text{Eval}_{(x_1-\sigma_1)}(f) = \sum_{j=0}^m \alpha_j(x_2)\sigma_1^j$. By Lemma 4.6-(i), we can re-arrange the evaluated polynomial in the form

$$\text{Eval}_{(x_1-\sigma_1)}(f) = \sum_{i=0}^n a_i(\sigma_1)x_2^i.$$

□

A similar technique of Definition 4.3 for resultant when the entries contain automorphisms can be described as following.

Definition 4.14 Let $\mathbb{S} = \tilde{\mathbb{A}}[x_1; \sigma_1][x_2; \sigma_2]$ be an Ore algebra. Consider two bivariate polynomials $f = \sum_{i=0}^n a_i(x_1)x_2^i$ and $g = \sum_{j=0}^m b_j(x_1)x_2^j$ in $(\tilde{\mathbb{A}}[x_1; \sigma_1])[x_2; \sigma_2]$ where $a_i(x_1), b_j(x_1) \in \tilde{\mathbb{A}}[x_1; \sigma_1]$. By Lemma 4.13, we can let $f' = \text{Eval}_{(x_1-\sigma_1)}(f) = \sum_{i=0}^n a_i(\sigma_1)x_2^i$ and $g' = \text{Eval}_{(x_1-\sigma_1)}(g) = \sum_{j=0}^m b_j(\sigma_1)x_2^j$. Accordingly, we define the resultant of f' and g' with respect to x_2 (denoted by $\text{Res}_{x_2}(f', g')$) by the following Dieudonné determinant;

$$\text{Res}_{x_2}(f', g') = \begin{vmatrix} x_2^{m-1} f' & a_n^{[m-1]}(\sigma_1) & a_{n-1}^{[m-1]}(\sigma_1) & \cdots & \cdots & a_0^{[m-1]}(\sigma_1) \\ x_2^{m-2} f' & & a_n^{[m-2]}(\sigma_1) & a_{n-1}^{[m-2]}(\sigma_1) & \cdots & \cdots & a_0^{[m-2]}(\sigma_1) \\ & & & & & \ddots & \\ \vdots & & & & & & \\ x_2 f' & & & & a_n^{[0]}(\sigma_1) & a_{n-1}^{[0]}(\sigma_1) & \cdots & \cdots & a_0^{[0]}(\sigma_1) \\ x_2^{n-1} g' & b_m^{[n-1]}(\sigma_1) & b_{m-1}^{[n-1]}(\sigma_1) & \cdots & b_0^{[n-1]}(\sigma_1) & & & & \\ \vdots & & b_m^{[n-2]}(\sigma_1) & b_{m-1}^{[n-2]}(\sigma_1) & \cdots & b_0^{[n-2]}(\sigma_1) & & & \\ & & & & & \ddots & & & \\ x_2 g' & & & & b_m^{[1]}(\sigma_1) & b_{m-1}^{[1]}(\sigma_1) & \cdots & b_0^{[1]}(\sigma_1) \\ g' & & & & & b_m^{[0]}(\sigma_1) & b_{m-1}^{[0]}(\sigma_1) & \cdots & b_0^{[0]}(\sigma_1) \end{vmatrix},$$

where the i -th row ($i = 1, \dots, m$) contains the coefficient sequence of the multiplication $x_2^{m-i} f'$, the coefficients of this multiplication are denoted by $a_j^{[m-i]}(\sigma_1)$ ($j = n, \dots, 0$). Similarly, the $(m+i)$ -th row ($i = 1, \dots, n$), contains the coefficients of $x_2^{n-i} g'$, these coefficients are denoted by $b_j^{[n-i]}(\sigma_1)$ ($j = m, \dots, 0$). An interesting observation here is that the noncommutative property is preserved, since the rows are multiplied by a power of x_2 to the left, which means the coefficients have to

follow the commutation rule. Thus, we can write the resultant $\text{Res}_{x_2}(f', g')$ in the form

$$\text{Res}_{x_2}(f', g') = \mathcal{D}et(x_2^{m-1}f', \dots, x_2f', f', x_2^{n-1}g', \dots, x_2g', g'),$$

which implies that

$$\begin{aligned} \text{Res}_{x_2}(\text{Eval}_{(x_1-\sigma_1)}(f), \text{Eval}_{(x_1-\sigma_1)}(g)) &= \mathcal{D}et(x_2^{m-1}\text{Eval}_{(x_1-\sigma_1)}(f), \dots, x_2\text{Eval}_{(x_1-\sigma_1)}(f), \text{Eval}_{(x_1-\sigma_1)}(f), \\ &\quad x_2^{n-1}\text{Eval}_{(x_1-\sigma_1)}(g), \dots, x_2\text{Eval}_{(x_1-\sigma_1)}(g), \text{Eval}_{(x_1-\sigma_1)}(g)). \end{aligned} \quad (17)$$

Similar to its commutative counterpart [22], we can prove the following theorem.

Theorem 4.15 *Let $\mathbb{S} = \tilde{\mathbb{A}}[x_1; \sigma_1][x_2; \sigma_2]$ be an Ore algebra. For all polynomials $f, g \in \mathbb{S}$, if $\deg_{x_2}(f) = \deg_{x_2}(\text{Eval}_{(x_1-\sigma_1)}(f))$ and $\deg_{x_2}(g) = \deg_{x_2}(\text{Eval}_{(x_1-\sigma_1)}(g))$, where \deg_{x_2} is the largest power of x_2 whose coefficient is not zero, then the following formula holds:*

$$\text{Eval}_{(x_1-\sigma_1)}(\text{Res}_{x_2}(f, g)) = \text{Res}_{x_2}(\text{Eval}_{(x_1-\sigma_1)}(f), \text{Eval}_{(x_1-\sigma_1)}(g)). \quad (18)$$

Proof. Let $f, g \in \mathbb{S}$ be two bivariate Ore polynomials of positive degrees n and m respectively. By its definition, we can write the right side of Equation(18) as:

$$\begin{aligned} \text{Res}_{x_2}(\text{Eval}_{(x_1-\sigma_1)}(f), \text{Eval}_{(x_1-\sigma_1)}(g)) &= \mathcal{D}et(x_2^{m-1}\text{Eval}_{(x_1-\sigma_1)}(f), \dots, x_2\text{Eval}_{(x_1-\sigma_1)}(f), \text{Eval}_{(x_1-\sigma_1)}(f), \\ &\quad x_2^{n-1}\text{Eval}_{(x_1-\sigma_1)}(g), \dots, x_2\text{Eval}_{(x_1-\sigma_1)}(g), \text{Eval}_{(x_1-\sigma_1)}(g)) \\ &= \mathcal{D}et(\text{Eval}_{(x_1-\sigma_1)}(x_2^{m-1}f), \dots, \text{Eval}_{(x_1-\sigma_1)}(x_2f), \text{Eval}_{(x_1-\sigma_1)}(f), \\ &\quad \text{Eval}_{(x_1-\sigma_1)}(x_2^{n-1}g), \dots, \text{Eval}_{(x_1-\sigma_1)}(x_2g), \text{Eval}_{(x_1-\sigma_1)}(g)) \\ &= \text{Eval}_{(x_1-\sigma_1)}(\mathcal{D}et(x_2^{m-1}f, \dots, x_2f, f, x_2^{n-1}g, \dots, x_2g, g)) \\ &= \text{Eval}_{(x_1-\sigma_1)}(\text{Res}_{x_2}(f, g)). \end{aligned}$$

Note that in the last step we have assumed that the degrees of the polynomials stay the same since it is given that the degrees will be preserved after evaluation. This completes the proof. \square

Using Theorem 4.15, we can conclude that the two methods $\text{Eval}_{(x_1-\sigma_1)}(\text{Res}_{x_2}(f, g))$ and $\text{Res}_{x_2}(\text{Eval}_{(x_1-\sigma_1)}(f), \text{Eval}_{(x_1-\sigma_1)}(g))$ are the same (viewed as operators). Thus, for all a in \mathbb{A} we have:

$$\text{Eval}_{(x_1-\sigma_1)}(\text{Res}_{x_2}(f, g))(a) = \text{Res}_{x_2}(\text{Eval}_{(x_1-\sigma_1)}(f), \text{Eval}_{(x_1-\sigma_1)}(g))(a). \quad (19)$$

The left side of Equation (19) describes the operator evaluation of direct resultant of two bivariate Ore polynomials f and g at a value a in \mathbb{A} , while the right side provides a way how to obtain the resultant via operator evaluation of its entries which follows by applying evaluation at a .

Remark 4.16 *An advantage of using Dieudonné determinant in Theorem 4.15 is that the case can be reduced to a triangular determinant with diagonal entries of*

polynomials $d'_i(x_1)$ ($i = 1, \dots, k$; $k = n + m$) for the direct method of the left side of Equation (19), while the right side will be in the form $d_i(\sigma_1)$ ($i = 1, \dots, k$; $k = n + m$) which can be computed by the following product:

$$\begin{aligned} \text{Res}_{x_2}(\text{Eval}_{(x_1-\sigma_1)}(f), \text{Eval}_{(x_1-\sigma_1)}(g))(a) &= \left(\prod_{i=1}^k d_i(\sigma_1) \right)(a) \\ &= (d_1(\sigma_1) d_2(\sigma_1) \cdots d_k(\sigma_1))(a) \\ &= d_1^*(d_2^*(\cdots d_k^*(a))), \end{aligned} \quad (20)$$

where $d_i^* = d_i(\sigma_1)$ for all $i = 1, \dots, k$.

It is evident that over some finite division rings (finite fields), computing the product (20) by evaluation and interpolation methods [31, 12] is asymptotically more efficient than its corresponding product by the direct method. Thus, we can consider the evaluation and interpolation method for the bivariate Ore polynomial rings over finite fields.

The idea of the evaluation and interpolation method is that instead of computing resultant directly from two bivariate Ore polynomials $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ with respect to x_2 , we propose to choose enough distinct values a_i ($i = 0, \dots, k$) to compute the evaluation of f_1 and f_2 with respect to x_1 at the values a_i ($i = 0, \dots, k$). Then, we will compute what we call *partial resultants* of these already evaluated f_1 and f_2 with respect to x_2 . Finally, we can deduce the original resultant by combining all these partial resultants using an interpolation technique. However, this process is not straightforward in Ore algebra, some challenges need to be overcome:

- (i) The current process of the right side of Equation (19) is to evaluate a polynomial at σ_1 and then applying the whole operator polynomial to a selected value of a in \mathbb{A} , which may not reflect the true evaluation of the original resultant at the left side of the equation unless applied in the same exact manner, and this makes it difficult to recover the original polynomial through a Lagrange like interpolation technique without modifications or by using another evaluation and interpolation method. One way to overcome this difficulty is using the coefficient comparison method which will always work in this case.
- (ii) In some cases, we may not even have enough values to check for valid evaluation points, which make it not possible to continue without extending the available choices. Thus, we extend the domain to enough elements for the interpolation process. Recall, we have the freedom to choose these evaluation points (as long as they belong to distinct conjugacy classes).
- (iii) The assumption of preserving the degrees of the input polynomials before and after evaluation in Theorem 4.15, is to avoid what is called *bad evaluation* where an evaluation value could cause the leading coefficient to vanish, hence altering the degree of the polynomial. Therefore, we need to check whether an evaluation value is a bad evaluation which can easily be determined from the leading term.
- (iv) We need to make sure all the selected evaluation values belong to pairwise distinct conjugacy classes in order to be able to use evaluation and interpolation techniques. For this, we can choose primitive elements and show that they belong to different conjugacy classes.

5 Conclusion

In this work, we used an elimination technique for bivariate Ore polynomials by using new resultant computations directly from the polynomial coefficients, as well as by using evaluation and interpolation methods. The methodology uses modular approaches to further optimize the algorithms.

Also, to conclude that the challenges mentioned in the introduction section can be overcome as following:

- (i) *No resultant for bivariate Ore polynomials*: We have defined a new resultant for bivariate Ore polynomials (including the case where the entries are automorphisms).
- (ii) *Evaluation map is not common*: In general, it is known that most of the non-commutative evaluation maps are not ring homomorphism with the exception of operator evaluation [7] in which we had to extend it to the bivariate case.
- (iii) *Noncommutative determinant is different*: We have used Dieudonné determinant to compute the resultant which is unique modulo commutators.
- (iv) *Interpolation requires more conditions*: The selected values for the noncommutative interpolation need to be in distinct conjugacy classes. We can extend the domain to distinct primitive elements which then can be used for the the evaluation and interpolation method.

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