THE n-TH PRIME IS GREATER THAN $n \log n$

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1. *Summary of results and outline of proof.*

Counting 2 as the first prime, denote the n-th prime by $p(n)$. For large n, $p(n) = n \log n + n \log\log n - n + O(n \log\log n / \log n)$, so that, for large n, $p(n) > n \log n$. In this paper it is shown that, for all positive n, $p(n) > n \log n$. As a consequence of this, one can prove that, if B_n, C_n, and D_n are defined for $n \geqslant 2$ by

$$\sum_{m=1}^{n} \frac{1}{p(m)} = \log\log n + B_n, \qquad \prod_{m=1}^{n} \left(1 - \frac{1}{p(m)}\right) = \frac{C_n}{\log n},$$

and
$$\prod_{m=2}^{n} \left(1 - \frac{2}{p(m)}\right) = \frac{D_n}{\log^2 n},$$

then $B_n > B_{n+1}$, $C_{n+1} > C_n$, and $D_{n+1} > D_n$. It is readily shown that all three sequences approach limits, and the numerical values of these limits are given to ten places of decimals.

These results will be used in the author's paper "An improvement of Brun's method in number theory".

For completeness, the following theorems are also proved: *If* $n > 3$, $p(n) < n \log n + 2n \log\log n$. *If* $n > 1$,

$$n \log n + n \log\log n - n - 9n < p(n) < n \log n + n \log\log n - n + 9n.$$

The general procedure of the proof that $p(n) > n \log n$ is as follows. By use of Lehmer's table of primes, it is shown that, for $n \leqslant 1071$, $p(n) > n \log n$. If $p(n) > n \log n$ for all $n \leqslant m$, then

$$\psi\big(p(m)\big) \geqslant \theta\big(p(m)\big) = \sum_{r=1}^{m} \log\big(p(r)\big) > \sum_{r=1}^{m} \log(r \log r).$$

We can get a lower bound for the latter expression by straightforward methods; in particular, we can show that, if $m > 1000$,

$$\psi\big(p(m)\big) > (m+1)\log(m+1) \times\left[1+\frac{\log\log m}{\log m}-\frac{1}{\log m}-\frac{1}{\log^2 m}-\frac{2}{\log^3 m}-\frac{2}{m}\right].$$

If $H(x)$ is never zero, and is so small that $x\big(1+H(x)^{-1}\big) \geqslant \psi(x)$, then

$$p(m)\left(1+\frac{1}{H\big(p(m)\big)}\right) \geqslant \psi\big(p(m)\big) > (m+1)\log(m+1) \times\left[1+\frac{\log\log m}{\log m}-\frac{1}{\log m}-\frac{1}{\log^2 m}-\frac{2}{\log^3 m}-\frac{2}{m}\right],$$

and so, if

$$\frac{1}{H\big(p(m)\big)} \leqslant \frac{\log\log m}{\log m}-\frac{1}{\log m}-\frac{1}{\log^2 m}-\frac{2}{\log^3 m}-\frac{2}{m},$$

then $p(m+1) > p(m) > (m+1)\log(m+1)$. Hence $p(n) > n\log n$ for all $n \leqslant m+1$. This gives the means of proving $p(n) > n\log n$ by induction, provided that $x\big(1+H(x)^{-1}\big) \geqslant \psi(x)$ can be proved for a sufficiently large $H(x)$. For most x, the standard theory of the approximation of $\psi(x)$ can be used by evaluating the various constants which appear. For the smaller values of x $(< e^{3000})$, a special method was devised, which gave a proof of $x\big(1+H(x)^{-1}\big) \geqslant \psi(x)$ for a somewhat larger $H(x)$ than could be used under the standard theory.

2. *Preliminary lemmas.*

The word "Ingham" refers to Ingham's "The distribution of prime numbers", *Cambridge Tract*, No. 30, the word "Landau" to the first volume of Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, and the word "Davis" to the first volume of H. T. Davis's *Tables of the higher mathematical functions*.

Throughout the paper s denotes a complex number, and σ and t denote its real and imaginary parts.

LEMMA 1. *If $t > 12$, then* $\log\Gamma(s) = (s-\frac{1}{2})\log s - s + \frac{1}{2}\log(2\pi) + \theta/(3t)$, *where* $|\theta| < 1$.

Proof. For all s except on the negative real axis,

$$\log \Gamma(s) = (s-\tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log(2\pi) + \int_0^\infty \frac{P_1(x)\,dx}{s+x},$$

where* $P_1(x) = [x] - x + \frac{1}{2}$.

$$\int_0^\infty \frac{P_1(x)\,dx}{s+x} = \sum_{n=0}^{\infty} \int_0^1 \frac{P_1(n+\theta)\,d\theta}{s+n+\theta}$$

$$= \sum_{n=0}^{\infty} \left[(\tfrac{1}{2}+s+n) \log\left(\frac{s+n+1}{s+n}\right) - 1 \right].$$

If $t > 12$, $|s+n| > 12$, and $\log\big((s+n+1)/(s+n)\big)$ can be expanded into a power series in $(s+n)^{-1}$. Multiplying this by $\frac{1}{2}+s+n$ and arranging according to powers of $s+n$, we have

$$\int_0^\infty \frac{P_1(x)\,dx}{s+x}$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{12(s+n)^2} - \frac{1}{12(s+n)^3} + \frac{3}{40(s+n)^4} + \dots + \frac{(r-1)(-1)^r}{2r(r+1)(s+n)^r} + \dots \right].$$

Since $|s+n| > 12$,

$$\left| \int_0^\infty \frac{P_1(x)\,dx}{s+x} \right| < \sum_{n=0}^{\infty} \left| \frac{1}{11(s+n)^2} \right| < \sum_{-\infty}^{\infty} \left| \frac{1}{11(s+n)^2} \right|$$

$$< \frac{1}{11t^2} + \frac{1}{11} \int_{-\infty}^{\infty} \frac{dx}{t^2+x^2} < \frac{1}{3t}.$$

LEMMA 2. *If* $0 < \sigma \leqslant 2$, *and* $1400 \leqslant t$, *then*

$$|\Gamma(s)| < (1{\cdot}0003)\, t^{\sigma-\frac{1}{2}} (2\pi)^{\frac{1}{2}} e^{-\frac{1}{2}\pi t}.$$

Proof. Take $0 < \sigma \leqslant 2$ and $1400 \leqslant t$.

$$\log |\Gamma(s)| = \mathfrak{R}\big(\log \Gamma(s)\big)$$

$$< (\sigma - \tfrac{1}{2}) \log(\sigma^2+t^2)^{\frac{1}{2}} - t \arctan \frac{t}{\sigma} - \sigma + \tfrac{1}{2} \log(2\pi) + 1/(3t).$$

However,

$$\arctan x = \tfrac{1}{2}\pi - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots$$

* Bieberbach, *Lehrbuch der Funktionentheorie*, 1 (1921), 308 and 298.

for $x>1$, and $\log(\sigma^2+t^2)^{\frac{1}{2}}<\log t+\sigma^2/(2t^2)$. So

$$\log|\Gamma(s)|<(\sigma-\tfrac{1}{2})\log t-\tfrac{1}{2}\pi t+\tfrac{1}{2}\log(2\pi)+1/(3t)+4/t^2.$$

The lemma follows.

LEMMA 3. *If* $0\leqslant\sigma\leqslant 2$, *and* $700\leqslant t$, *then*

$$\mathfrak{R}\left(\frac{\Gamma'(s)}{\Gamma(s)}\right)<\log t+\frac{4}{t^2}.$$

Proof. Take $0\leqslant\sigma\leqslant 2$ and $700\leqslant t$. Put

$$f(s)=\log\Gamma(s)-(s-\tfrac{1}{2})\log s+s-\tfrac{1}{2}\log(2\pi).$$

Then
$$f'(s)=\frac{1}{2\pi i}\int_C\frac{f(w)\,dw}{(w-s)^2}.$$

Take C to be a circle with centre at s and radius $\frac{1}{3}t$. Then, by Lemma 1, $|f(w)|<1/(2t)$ on C. Also $|w-s|=\frac{1}{3}t$ on C. Hence $|f'(s)|<3/(2t^2)$. Hence

$$\frac{\Gamma'(s)}{\Gamma(s)}=\log s-\frac{1}{2s}+\frac{3\theta}{2t^2},$$

where $|\theta|<1$. Hence

$$\mathfrak{R}\left(\frac{\Gamma'(s)}{\Gamma(s)}\right)<\log(\sigma^2+t^2)^{\frac{1}{2}}-\frac{\sigma}{2\sigma^2+2t^2}+\frac{3}{2t^2}.$$

The lemma follows because $\log(\sigma^2+t^2)^{\frac{1}{2}}<\log t+\sigma^2/(2t^2)$.

LEMMA 4. *If* $\frac{1}{13}\leqslant\sigma$, *and* $1400\leqslant t$, *then* $|\zeta(s)|\leqslant 7t^{\frac{1}{2}}$.

Proof. Take $\frac{1}{13}\leqslant\sigma$ and $1400\leqslant t$. By Ingham, formula (7), p. 27,

$$|\zeta(s)|<\sum_{n\leqslant X}\frac{1}{n^\sigma}+\frac{1}{tX^{\sigma-1}}+\frac{1}{X^\sigma}+|s|\int_X^\infty\frac{dx}{x^{\sigma+1}}.$$

Put $X=t$. Then

$$|\zeta(s)|<\sum_{n\leqslant t}\frac{1}{n^\sigma}+\frac{2}{t^\sigma}+\frac{t+\sigma}{\sigma t^\sigma}.$$

For $0<\sigma$, the right-hand side of this inequality is a decreasing function of σ, and so

$$|\zeta(s)|<\sum_{n\leqslant t}\frac{1}{n^{\frac{1}{2}}}+\frac{2}{t^{\frac{1}{2}}}+\frac{t+\frac{1}{2}}{\frac{1}{2}t^{\frac{1}{2}}}$$

if $\frac{1}{2}\leqslant\sigma$. Now, for $0<\sigma<1$,

$$\sum_{n\leqslant t}\frac{1}{n^\sigma}+\frac{2}{t^\sigma}+\frac{t+\sigma}{\sigma t^\sigma}<\int_0^t\frac{dx}{x^\sigma}+\frac{3}{t^\sigma}+\frac{t^{1-\sigma}}{\sigma}\leqslant t^{1-\sigma}\left(\frac{1}{1-\sigma}+\frac{1}{\sigma}+\frac{1}{400}\right).$$

Hence, for $\frac{1}{2} \leqslant \sigma$,

$$|\zeta(s)| < (4\tfrac{1}{400})\, t^{\frac{1}{2}} < 7t^{\frac{1}{2}}.$$

Now, by Landau, p. 285,

$$\zeta(1-s) = \frac{2}{(2\pi)^s}\,\Gamma(s)\cos\tfrac{1}{2}\pi s\,\zeta(s).$$

Now $|1/(2\pi)^s| = 1/(2\pi)^\sigma$. Also, since $1400 \leqslant t$,

$$|\cos\tfrac{1}{2}\pi s| < \tfrac{1}{2}(1{\cdot}00001)\, e^{\frac{1}{2}\pi t}.$$

So, for $\frac{1}{2} \leqslant \sigma < 1$, we have, by Lemma 2,

$$|\zeta(1-s)| < (1{\cdot}0004)\left(\frac{t}{2\pi}\right)^{\sigma-\frac{1}{2}} |\zeta(s)|.$$

However, it has just been shown that, for $\frac{1}{2} \leqslant \sigma < 1$,

$$|\zeta(s)| < t^{1-\sigma}\left(\frac{1}{1-\sigma}+\frac{1}{\sigma}+\frac{1}{400}\right).$$

Hence $$|\zeta(1-s)| < (1{\cdot}0004)\,\frac{(2\pi t)^{\frac{1}{2}}}{(2\pi)^\sigma}\left(\frac{1}{1-\sigma}+\frac{1}{\sigma}+\frac{1}{400}\right).$$

So, for $\frac{1}{2} \leqslant \sigma \leqslant \frac{12}{13}$, $|\zeta(1-s)| \leqslant 7t^{\frac{1}{2}}$. This, together with the inequality for $|\zeta(s)|$ when $\frac{1}{2} \leqslant \sigma$, proves the lemma.

Whenever ρ, β, and γ are used henceforth, it is to be understood that ρ is a complex zero of $\zeta(s)$ and β and γ are the real and imaginary parts of ρ. $N(T)$ denotes the number of ρ's for which $0 < \gamma \leqslant T$.

LEMMA 5. *If* $1450 \leqslant T$, *then*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \tfrac{7}{8} + \theta[1{\cdot}12\log T + 9{\cdot}5],$$

where $|\theta| \leqslant 1$.

Proof. If $1 < \sigma$, then

$$|\zeta(s)| = \prod_p |1-1/p^s|^{-1}$$

[Ingham, formula (12), p. 17]. However,

$$|1-1/p^s|^{-1} \geqslant (1+1/p^\sigma)^{-1} = \frac{(1-1/p^{2\sigma})^{-1}}{(1-1/p^\sigma)^{-1}},$$

Hence $|\zeta(s)| \geqslant \zeta(2\sigma)/\zeta(\sigma)$. However, $\zeta(1{\cdot}25) = 4{\cdot}595$ and $\zeta(2{\cdot}5) = 1{\cdot}341$*.

Hence
$$\frac{1}{|\zeta(\frac{5}{4}+it)|} \leqslant \frac{\zeta(1{\cdot}25)}{\zeta(2{\cdot}5)} = 3{\cdot}425.$$

Hence either

$$\frac{1}{\left|\mathfrak{R}\left(\zeta(\frac{5}{4}+it)\right)\right|} \leqslant 4{\cdot}844 \quad \text{or} \quad \frac{1}{\left|\mathfrak{I}\left(\zeta(\frac{5}{4}+it)\right)\right|} \leqslant 4{\cdot}844.$$

Let $g(s, T)$ be $\frac{1}{2}\left(\zeta(s+iT)+\zeta(s-iT)\right)$ if

$$\frac{1}{\left|\mathfrak{R}\left(\zeta(\frac{5}{4}+iT)\right)\right|} \leqslant 4{\cdot}844,$$

and let it be $\frac{1}{2}\left(\zeta(s+iT)-\zeta(s-iT)\right)$ otherwise. Then, for all T,

$$\frac{1}{|g(\frac{5}{4}, T)|} \leqslant 4{\cdot}844.$$

On pp. 68–69 of Ingham, it is shown that

$$N(T) = \frac{1}{\pi}\left[\pi - \tfrac{1}{2}T\log\pi + \mathfrak{I}\left(\log\Gamma(\tfrac{1}{4}+\tfrac{1}{2}Ti)\right) + [\arg\zeta(s)]_L\right],$$

where by $[\arg\zeta(s)]_L$ is meant the increase in $\arg\zeta(s)$ as s traverses the pair of straight lines going from 2 to $2+Ti$ and from $2+Ti$ to $\frac{1}{2}+Ti$. By Lemma 1, if $1450 \leqslant T$, then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \tfrac{7}{8} + \frac{\theta}{3T} + \frac{1}{\pi}[\arg\zeta(s)]_L,$$

where $|\theta| \leqslant 1$. Now the change in $\arg\zeta(s)$ along the straight line from 2 to $2+Ti$ is less than $\frac{1}{2}\pi$, since $\arg\zeta(2) = 0$ and $\mathfrak{R}\left(\zeta(2+ti)\right) > \frac{1}{4}$. Hence, if m is the number of zeros of $g(s, T)$ on the line $\frac{1}{2} \leqslant s \leqslant 2$,

$$|[\arg\zeta(s)]_L| \leqslant (m+\tfrac{3}{2})\pi$$

(Ingham, p. 69). Now apply Theorem D (Ingham, p. 49) to $g(s, T)$ and the circles $|s-\frac{5}{4}| \leqslant \frac{61}{52}$ and $|s-\frac{5}{4}| \leqslant \frac{3}{4}$. If $1450 \leqslant T$, then, by Lemma 4, $|g(s, T)| < 7(T+2)^{\frac{1}{2}}$ on the larger circle. Also

$$\frac{1}{|g(\frac{5}{4}, T)|} \leqslant 4{\cdot}844.$$

* J. P. Gram, " Tafeln für die Riemannsche Zetafunktion, herausgegeben von N. E. Nörlund ", *D. Kgl. Danske Vidensk. Selsk. Skrifter, Kopenhagen* (8), 10 (1925–26), 313–325.

So, for $1450 \leqslant T$, $(\frac{6}{3}\frac{1}{9})^r \leqslant 34T^{\frac{1}{2}}$, where r is the number of zeros of $g(s, T)$ in the circle $|s-\frac{5}{4}| \leqslant \frac{3}{4}$. Hence $|[\arg \zeta(s)]_L| \leqslant (1{\cdot}12 \log T + 9{\cdot}4)\pi$. The lemma follows.

Denote
$$\sum_{1468 \leqslant \gamma \leqslant e^8} \frac{1}{\gamma^2} \text{ by } S_8,$$

and, if $r > 8$,
$$\sum_{e^{r-1} \leqslant \gamma \leqslant e^r} \frac{1}{\gamma^2} \text{ by } S_r.$$

LEMMA 6. $$S_8 < 0{\cdot}000{,}332.$$

Proof. By Lemma 5, if $1450 \leqslant T \leqslant e^8$,

$$N(T) \leqslant \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \tfrac{7}{8} + 1{\cdot}12 \log e^8 + 9{\cdot}5$$

$$< \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + 20.$$

Put
$$F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}.$$

Then $F(1468) = 1040{\cdot}58$ and $F(e^8) = 2449{\cdot}09$. Since $F(T)$ is a concave function, $F(1468 + 1513n/1409) < 1041 + n$ for $0 \leqslant n \leqslant 1409$. Hence

$$N(1468 + 1513n/1409) < 1061 + n \quad \text{for} \quad 0 \leqslant n \leqslant 1409.$$

If we let γ_m stand for the m-th positive γ in order of magnitude and count duplications, it follows that $1468 + 1513n/1409 < \gamma_{n+1061}$ for $0 \leqslant n \leqslant 1409$. However, the first 1041 ρ's all have $\beta = \frac{1}{2}$ and $0 < \gamma \leqslant 1468$, and the 1042-nd ρ has $\gamma \geqslant 1468$, and $\gamma_{1040} < 1468$*. Hence

$$S_8 = \sum_{1468 \leqslant \gamma \leqslant e^8} \frac{1}{\gamma^2}$$

$$< \frac{21}{(1468)^2} + \sum_{n=1}^{1409} \frac{1}{(1468 + 1513n/1409)^2}$$

$$< \frac{21}{(1468)^2} + \int_0^{1409} \frac{dx}{(1468 + 1513x/1409)^2}$$

$$= \frac{21}{(1468)^2} + \frac{1409}{1513}\left[\frac{1}{1468} - \frac{1}{2981}\right]$$

$$= 0{\cdot}000{,}331{,}72.$$

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* Titchmarsh, "The zeros of the Riemann zeta-function", *Proc. Royal Soc. London* A) 157 (1936), 261–263.

LEMMA 7. *If* $r > 8$, $S_r < 16\cdot6(r+9)e^{-2r}+0\cdot274(r-2\cdot25)e^{-r}$.

Proof. $F(e^{r-1}) = \frac{e^{r-1}}{2\pi}[r-2-\log 2\pi]$ and $F(e^r) = \frac{e^r}{2\pi}[r-1-\log 2\pi]$.

So
$$F\left(e^{r-1}+\frac{2\pi n}{r-1-\log 2\pi+1/(e-1)}\right) \leqslant F(e^{r-1})+n,$$
for $0 \leqslant n \leqslant F(e^r)-F(e^{r-1})$. Hence
$$e^{r-1}+\frac{2\pi n}{r-1-\log 2\pi+1/(e-1)} < \gamma_M,$$
with $M = n+F(e^{r-1})+11+1\cdot12r$, for $0 \leqslant n \leqslant F(e^r)-F(e^{r-1})$. However,
$$\gamma_m < e^{r-1}$$
for $m \leqslant F(e^{r-1})-9-1\cdot12(r-1)$. Hence
$$\sum_{e^{r-1}\leqslant\gamma\leqslant e^r}\frac{1}{\gamma^2} < \frac{19+2\cdot24r}{(e^{r-1})^2}+\sum_{n=1}^{F(e^r)-F(e^{r-1})}\left(e^{r-1}+\frac{2\pi n}{r-1-\log 2\pi+1/(e-1)}\right)^{-2}$$
So
$$S_r < \frac{2\cdot24(r+9)}{e^{2r-2}}+\frac{r-1-\log 2\pi+1/(e-1)}{2\pi}\left[\frac{1}{e^{r-1}}-\frac{1}{e^r}\right]$$
$$< 16\cdot6(r+9)e^{-2r}+0\cdot274(r-2\cdot25)e^{-r}.$$

COROLLARY. $S_9 < 0\cdot000{,}233$.

COROLLARY. $S_{10} < 0\cdot000{,}097{,}1$.

COROLLARY. $S_{11} < 0\cdot000{,}040{,}2$.

COROLLARY. $S_{12} < 0\cdot000{,}016{,}5$.

Clearly the upper bound for S_{r+1} is less than half of the upper bound for S_r for $r > 8$, and so:

COROLLARY. $S_{13}+S_{14}+\ldots < 0\cdot000{,}016{,}5$.

LEMMA 8. $\sum_\rho \frac{1}{\gamma^2} < 0\cdot0463$.

Proof. $-\frac{\zeta'(s)}{\zeta(s)} = -b+\frac{1}{s-1}+\frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)}-\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$

Ingham, p. 58). However,
$$\lim_{s\to1}\left(\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{s-1}\right) = C$$

(Landau, p. 165). Hence

$$\sum_\rho \left(\frac{1}{1-\rho}+\frac{1}{\rho}\right) = -b+C+\tfrac{1}{2}\frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})}.$$

Now $2^{2x}\Gamma(x)\Gamma(x+\frac{1}{2}) = 2\pi^{\frac{1}{2}}\Gamma(2x)$ (Davis, p. 180). Differentiating logarithmically, we get

$$2\log 2+\frac{\Gamma'(x)}{\Gamma(x)}+\frac{\Gamma'(x+\frac{1}{2})}{\Gamma(x+\frac{1}{2})} = 2\frac{\Gamma'(2x)}{\Gamma(2x)}.$$

Putting $x = 1$ and using $\Gamma'(1)/\Gamma(1) = -C$ and $\Gamma'(2)/\Gamma(2) = 1-C$ (Davis, pp. 277–278), we get

$$\frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} = 2-C-2\log 2.$$

Hence

$$\sum_\rho \left(\frac{1-\beta}{(1-\beta)^2+\gamma^2}+\frac{\beta}{\beta^2+\gamma^2}\right) = -b+C+1-\tfrac{1}{2}C-\log 2$$

$$= 2+C-\log(4\pi),$$

since $b = \log(2\pi)-1-\frac{1}{2}C$ (Ingham, p. 58). However (Ingham, p. 48) the complex zeros of $\zeta(s)$ lie symmetrically about the lines $t = 0$ and $\sigma = \frac{1}{2}$, and so for every zero $\rho = \beta+i\gamma$, there is a zero $\rho = (1-\beta)+i\gamma$. Hence

$$\sum_\rho \frac{\beta}{\beta^2+\gamma^2} = \tfrac{1}{2}\sum_\rho \left(\frac{1-\beta}{(1-\beta)^2+\gamma^2}+\frac{\beta}{\beta^2+\gamma^2}\right)$$

$$= 1+\tfrac{1}{2}C-\tfrac{1}{2}\log(4\pi)$$

$$= 0{\cdot}023{,}095{,}709.$$

Now, for $\gamma < 1468$, we have $\beta = \frac{1}{2}$, and the smallest value of γ is $14{\cdot}1$*. Hence

$$\frac{1}{\gamma^2} < 1{\cdot}0013\,\frac{1}{\beta^2+\gamma^2}$$

for all ρ. Hence

$$\sum_\rho \frac{\beta}{\gamma^2} < 1{\cdot}0013\sum_\rho \frac{\beta}{\beta^2+\gamma^2}.$$

When $\beta = \frac{1}{2}$,
$$\frac{1}{\gamma^2} = 2\frac{\beta}{\gamma^2}.$$

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* J. I. Hutchinson, "On the roots of the Riemann zeta-function", *Trans. American Math. Soc.*, 27 (1925), 49–60.

When $\beta \neq \frac{1}{2}$, $\beta+i\gamma$ and $(1-\beta)+i\gamma$ are both zeros and

$$\frac{1}{\gamma^2}+\frac{1}{\gamma^2}=2\left[\frac{\beta}{\gamma^2}+\frac{1-\beta}{\gamma^2}\right].$$

Hence we can conclude that

$$\sum_\rho \frac{1}{\gamma^2}=2\sum_\rho \frac{\beta}{\gamma^2}<2\cdot 0026 \sum_\rho \frac{\beta}{\beta^2+\gamma^2}<0\cdot 0463.$$

LEMMA 9. *If* $1400 \leqslant \gamma$, *then*

$$\beta<1-\frac{1}{19 \log \gamma}.$$

The proof is just like Landau's proof that, for large γ,

$$\beta<1-\frac{1}{18\cdot 52 \ldots \log \gamma}$$

(Landau, pp. 318-324), except that certain constants are evaluated numerically.

Proof.

$$-\frac{\zeta'(s)}{\zeta(s)}=-0\cdot 549+\frac{1}{s-1}+\frac{1}{2} \frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)}-\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$$

(Ingham, p. 58). By Lemma 3, if $1 \leqslant \sigma \leqslant 2$ and $1400 \leqslant t$, then

$$-\mathfrak{R}\left(\frac{\zeta'(s)}{\zeta(s)}\right)<-0\cdot 549+\frac{1}{t^2}+\frac{1}{2} \log \frac{1}{2}t+\frac{8}{t^2}-\sum_\rho \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}-\sum_\rho \frac{\beta}{\beta^2+\gamma^2}.$$

With $1 \leqslant \sigma$, $\sigma-\beta$ and β are always positive, so that we can deduce that

$$-\mathfrak{R}\left(\frac{\zeta'(s)}{\zeta(s)}\right)<-0\cdot 89+\frac{1}{2} \log t.$$

Also, if we choose a $\gamma \geqslant 1400$ and put $t=\gamma$, we get

$$\frac{1}{\sigma-\beta}<\mathfrak{R}\left(\frac{\zeta'(\sigma+i\gamma)}{\zeta(\sigma+i\gamma)}\right)-0\cdot 89+\frac{1}{2} \log \gamma.$$

For $\frac{3}{2} \leqslant \sigma \leqslant \frac{61}{40}$,

$$\mathfrak{R}\left(\frac{\Gamma'(\sigma)}{\Gamma(\sigma)}\right)=\frac{\Gamma'(\sigma)}{\Gamma(\sigma)} \leqslant \frac{\Gamma'(\frac{61}{40})}{\Gamma(\frac{61}{40})}=0\cdot 0596$$

(Davis, p. 324), so, for $1 < \sigma \leqslant 1{\cdot}05$,

$$-\mathfrak{R}\left(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right) < -0{\cdot}549 + \frac{1}{\sigma-1} + \tfrac{1}{2}\frac{\Gamma'(\frac{1}{2}\sigma+1)}{\Gamma(\frac{1}{2}\sigma+1)} \leqslant -0{\cdot}51 + \frac{1}{\sigma-1}.$$

So for $1400 \leqslant \gamma$ and $1 < \sigma \leqslant 1{\cdot}05$,

$$\frac{1}{\sigma-\beta} < -0.89 + \tfrac{1}{2}\log\gamma + \tfrac{5}{8}\left(-0{\cdot}51 + \frac{1}{\sigma-1}\right)$$
$$+\tfrac{1}{2}\Big(-0{\cdot}89 + \tfrac{1}{2}\log(2\gamma)\Big) + \tfrac{1}{8}\Big(-0{\cdot}89 + \tfrac{1}{2}\log(3\gamma)\Big)$$
$$< \tfrac{13}{16}\log\gamma + \tfrac{5}{8}\frac{1}{\sigma-1}.$$

Put $\sigma = 1 + 1/(5\log\gamma)$. Then $1 < \sigma < 1{\cdot}05$. Thus

$$\beta < 1 - \frac{1}{19\log\gamma}.$$

Take
$$\theta(x) = \sum_{p \leqslant x} \log p, \qquad \psi(x) = \sum_{p^m \leqslant x} \log p,$$

and
$$\psi_1(x) = \int_1^x \psi(u)\,du$$

(Ingham, p. 30).

LEMMA 10. *If*

$$K_1(x) = \sum_{r=8}^{\infty} S_r x^{-1/(19r)} + 0{\cdot}0463x^{-\frac{1}{2}} + 1{\cdot}9x^{-1} + 3x^{-2},$$

then, for $x > 1$,

$$\psi(x) < x[1 + 2K_1(x)^{\frac{1}{2}} + 2K_1(x) + 2K_1(x)\,K_1(x)^{\frac{1}{2}}].$$

Proof.

$$\psi_1(x) = \tfrac{1}{2}x^2 - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} - x\frac{\zeta'(0)}{\zeta(0)} + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)}$$

(Ingham, p. 73). Now $\zeta'(0)/\zeta(0) = 1{\cdot}837{,}877$ and*

$$\zeta'(-1)/\zeta(-1) = 1{\cdot}985{,}054.$$

Also
$$\left|\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)}\right| < \sum_{\rho}\frac{x^{\beta+1}}{\gamma^2}.$$

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* A. Walther, "Anschauliches zur Riemannschen Zetafunktion", *Acta Math.*, 48 (1926), 393–400.

Now
$$\sum_{\beta\leqslant\frac{1}{2}} \frac{x^{\beta+1}}{\gamma^2} \leqslant x^{\frac{3}{2}} \sum_{\beta\leqslant\frac{1}{2}} \frac{1}{\gamma^2} \leqslant x^{\frac{3}{2}} \sum_{\rho} \frac{1}{\gamma^2} < 0{\cdot}0463 x^{\frac{3}{2}}$$

by Lemma 8. Also

$$\sum_{\beta>\frac{1}{2}} \frac{x^{\beta+1}}{\gamma^2} = 2x^2 \sum_{\substack{\beta>\frac{1}{2}\\ \gamma>0}} \frac{x^{\beta-1}}{\gamma^2} = 2x^2 \sum_{\substack{\beta>\frac{1}{2}\\ \gamma\geqslant 1468}} \frac{x^{\beta-1}}{\gamma^2},$$

since $\beta=\frac{1}{2}$ for $\gamma<1468$. Hence, by Lemma 9,

$$\sum_{\beta>\frac{1}{2}} \frac{x^{\beta+1}}{\gamma^2} < 2x^2 \sum_{\substack{\beta>\frac{1}{2}\\ \gamma\geqslant 1468}} \frac{x^{-1/(19\log\gamma)}}{\gamma^2}.$$

However, for every $\beta>\frac{1}{2}$, $(1-\beta)+i\gamma$ is also a zero. Thus

$$\sum_{\beta>\frac{1}{2}} \frac{x^{\beta+1}}{\gamma^2} < x^2 \sum_{\gamma\geqslant 1468} \frac{x^{-1/(19\log\gamma)}}{\gamma^2} < x^2 \sum_{r=8}^{\infty} S_r x^{-1/(19r)}.$$

So, for $x>1$, $\psi_1(x)=\frac{1}{2}x^2+\theta x^2 K_1(x)$, where $|\theta|<1$. Denote $2xK_1(x)^{\frac{1}{2}}$ by h. Since $\psi(x)$ is a non-decreasing function of x,

$$\psi(x) \leqslant \frac{1}{h}\int_x^{x+h} \psi(y)\,dy = \frac{\psi_1(x+h)}{h} - \frac{\psi_1(x)}{h}$$

$$< x+\tfrac{1}{2}h+\frac{(x+h)^2 K_1(x+h)}{h}+\frac{x^2 K_1(x)}{h}.$$

However, $K_1(y)$ is a decreasing function of y, and so the lemma results after replacing $K_1(x+h)$ by $K_1(x)$ and h by $2xK_1(x)^{\frac{1}{2}}$ in the above inequality.

For particular choices of $x>e^{50}$, it is possible to get smaller upper bounds for $\psi(x)$ than those given in Lemma 10, by giving the proper values to a in

LEMMA 11. *If $x>e^{50}$, and*

$$K_2(x) = 0{\cdot}000{,}000{,}2\, x^{-\frac{1}{152}} + \sum_{r=9}^{\infty} \frac{S_r}{e^{r-1}}\, x^{-1/(19r)},$$

and a is a positive constant, then

$$\psi(x) < x\left[1+2aK_2(x)^{\frac{1}{2}}+\frac{2}{a^2}+\frac{9K_2(x)^{\frac{1}{2}}}{a}+21K_2(x)+18aK_2(x)\,K_2(x)^{\frac{1}{2}}\right].$$

Proof. By arguing as in the proof of Lemma 10, we have

$$\sum_{\rho} \frac{x^{\beta-1}}{|\gamma|^3} < 0{\cdot}0463 x^{-\frac{1}{2}} + \sum_{1468\leqslant\gamma\leqslant e^8} \frac{1}{\gamma^3}\, x^{-\frac{1}{152}} + \sum_{r=9}^{\infty} \left[\sum_{e^{r-1}\leqslant\gamma\leqslant e^r} \frac{1}{\gamma^3}\right] x^{-1/(19r)}.$$

However,
$$\sum_{e^{r-1} \leqslant \gamma \leqslant e^r} \frac{1}{\gamma^3} < \frac{1}{e^{r-1}} S_r.$$

By arguing as in the proof of Lemma 6, we get

$$\sum_{1468 \leqslant \gamma \leqslant e^8} \frac{1}{\gamma^3} < 0{\cdot}000{,}000{,}18.$$

Hence, for $x > e^{50}$,

$$0.0463 x^{-\frac{1}{2}} + \sum_{1468 \leqslant \gamma \leqslant e^8} \frac{1}{\gamma^3} x^{-\frac{1}{152}} < 0{\cdot}000{,}000{,}2 x^{-\frac{1}{152}}.$$

Thus
$$\sum_\rho \frac{x^{\beta-1}}{|\gamma|^3} < K_2(x).$$

Denote

$$\sum_\rho \frac{x^{\beta-1}}{|\gamma|^3} \text{ by } l(x), \quad 2axl(x)^{\frac{1}{2}} \text{ by } h(x), \quad \text{and} \quad axl(x)^{\frac{1}{2}} \text{ by } k(x),$$

and write $h(x)$, $k(x)$, and $l(x)$ as h, k, and l when no confusion will be caused thereby. Then

$$\int_x^{x+k} \left(\sum_\rho \frac{y^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(y+h)^{\rho+1}}{\rho(\rho+1)} \right) dy$$

$$= \sum_\rho \frac{(x+k)^{\rho+2}}{\rho(\rho+1)(\rho+2)} - \sum_\rho \frac{x^{\rho+2}}{\rho(\rho+1)(\rho+2)} - \sum_\rho \frac{(x+h+k)^{\rho+2}}{\rho(\rho+1)(\rho+2)} + \sum_\rho \frac{(x+h)^{\rho+2}}{\rho(\rho+1)(\rho+2)}$$

$$< \sum_\rho \frac{x^{\beta+2}}{|\gamma|^3} + \sum_\rho \frac{(x+k)^{\beta+2}}{|\gamma|^3} + \sum_\rho \frac{(x+h)^{\beta+2}}{|\gamma|^3} + \sum_\rho \frac{(x+h+k)^{\beta+2}}{|\gamma|^3}.$$

However,
$$(x+a)^{\beta+2} = x^{\beta+2} \left(1 + \frac{a}{x}\right)^{\beta+2} < x^{\beta+2} \left(1 + \frac{a}{x}\right)^3.$$

Hence the integral is less than

$$\sum_\rho \frac{x^{\beta+2}}{|\gamma|^3} [1 + 1 + 3al^{\frac{1}{2}} + 3a^2 l + a^3 ll^{\frac{1}{2}} + 1 + 6al^{\frac{1}{2}} + 12a^2 l$$
$$+ 8a^3 ll^{\frac{1}{2}} + 1 + 9al^{\frac{1}{2}} + 27a^2 l + 27a^3 ll^{\frac{1}{2}}]$$

$$= x^3 l [4 + 18al^{\frac{1}{2}} + 42a^2 l + 36a^3 ll^{\frac{1}{2}}].$$

Let z be the value of y between x and $x+k$ for which

$$\sum_\rho \frac{y^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(y+h)^{\rho+1}}{\rho(\rho+1)}$$

takes its minimum value. Then

$$k\left[\sum_\rho \frac{z^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(z+h)^{\rho+1}}{\rho(\rho+1)}\right] \leqslant \int_x^{x+k} \left(\sum_\rho \frac{y^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(y+h)^{\rho+1}}{\rho(\rho+1)}\right) dy.$$

Hence

$$\sum_\rho \frac{z^{\rho+1}}{\rho(\rho+1)} - \sum_\rho \frac{(z+h)^{\rho+1}}{\rho(\rho+1)} < x^2 l^{\frac{1}{2}} \left[\frac{4}{a} + 18 l^{\frac{1}{2}} + 42al + 36a^2 ll^{\frac{1}{2}}\right].$$

Now

$$\begin{aligned} h\psi(x) &\leqslant \int_z^{z+h} \psi(y)\, dy \\ &= \frac{(z+h)^2}{2} - \frac{z^2}{2} - \sum_\rho \frac{(z+h)^{\rho+1}}{\rho(\rho+1)} + \sum_\rho \frac{z^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)} h \\ &\qquad - \sum_{r=1}^{\infty} \frac{(z+h)^{1-2r}}{2r(2r-1)} + \sum_{r=1}^{\infty} \frac{z^{1-2r}}{2r(2r-1)}. \end{aligned}$$

However, since $h > 0$,

$$\begin{aligned} \frac{1}{z^{2r-1}} - \frac{1}{(z+h)^{2r-1}} &= \left(\frac{1}{z} - \frac{1}{z+h}\right)\left(\frac{1}{z^{2r-2}} + \dots + \frac{1}{(z+h)^{2r-2}}\right) \\ &< \frac{h(2r-1)}{z^{2r}}. \end{aligned}$$

Since $\zeta'(0)/\zeta(0) = 1{\cdot}837{,}877$, we have

$$-\frac{\zeta'(0)}{\zeta(0)} h - \sum_{r=1}^{\infty} \frac{(z+h)^{1-2r}}{2r(2r-1)} + \sum_{r=1}^{\infty} \frac{z^{1-2r}}{2r(2r-1)} < 0$$

if $z \geqslant 2$. So, since $x > e^{50}$,

$$\psi(x) < z + \tfrac{1}{2}h + x\left[\frac{2}{a^2} + \frac{9 l^{\frac{1}{2}}}{a} + 21 l + 18 a l l^{\frac{1}{2}}\right].$$

However, $z \leqslant x+k = x + axl^{\frac{1}{2}}$ and $h = 2axl^{\frac{1}{2}}$, and so the lemma follows from the equality $l(x) < K_2(x)$.

LEMMA 12. *If $m > 1000$, and if $p(n) > n \log n$ for all $n \leqslant m$, and if $H(x)$ is a positive function such that $x\left(1 + H(x)^{-1}\right) \geqslant \psi(x)$, then*

$$p(m+1) > (m+1)\log(m+1)$$

if

$$\log m \leqslant H\big(p(m)\big)\left[\log\log m - 1 - \frac{1}{\log m} - \frac{2}{\log^2 m} - \frac{2\log m}{m}\right].$$

Proof.

$$\psi\big(p(m)\big) \geqslant \theta\big(p(m)\big) = \sum_{r=1}^{m} \log\big(p(r)\big)$$

$$= \log 2 + \log 3 + \log 5 + \log 7 + \log 11 + \sum_{r=6}^{m} \log\big(p(r)\big)$$

$$> 7{\cdot}745 + \sum_{r=6}^{m} \log(r\log r) > 7{\cdot}745 + \int_5^m (\log x + \log\log x)\,dx$$

$$= 7{\cdot}745 + m\log m - m - 5\log 5 + 5 + m\log\log m$$

$$-5\log\log 5 - \int_5^m \frac{dx}{\log x}$$

$$\geqslant 2{\cdot}318 + m\log m - m + m\log\log m - \frac{m}{\log m} + \frac{5}{\log 5} - \int_5^m \frac{dx}{\log^2 x}$$

$$\geqslant 5{\cdot}425 + m\log m - m + m\log\log m - \frac{m}{\log m} - \int_5^{15} \frac{dx}{\log^2 5}$$

$$-\int_{15}^{25} \frac{dx}{\log^2 15} - \int_{25}^{35} \frac{dx}{\log^2 25} - \int_{35}^{45} \frac{dx}{\log^2 35} - \int_{45}^{55} \frac{dx}{\log^2 45} - \int_{55}^{m} \frac{dx}{\log^2 x}.$$

Now, for $x \geqslant 55$, $\log x \geqslant 4$, and so

$$\frac{2}{\log^2 x} - \frac{4}{\log^3 x} \geqslant \frac{1}{\log^2 x}.$$

Hence

$$\psi\big(p(m)\big) > m\log m - m + m\log\log m - \frac{m}{\log m} - 2{\cdot}245$$

$$-\int_{55}^{m} \left[\frac{2}{\log^2 x} - \frac{4}{\log^3 x}\right] dx$$

$$= m\log m - m + m\log\log m - \frac{m}{\log m} - \frac{2m}{\log^2 m} - 2{\cdot}245 + \frac{110}{\log^2 55}$$

$$> m\log m - m + m\log\log m - \frac{m}{\log m} - \frac{2m}{\log^2 m}.$$

However, since $m > 1000$,

$$\psi\big(p(m)\big) > m\log m + m\log\log m - m - \frac{m}{\log m} - \frac{2m}{\log^2 m} - 2\log m + \log m + \log\log m + 2 + \frac{2\log\log m}{\log m}$$

$$> (m\log m + \log m + 2)\left[1 + \frac{\log\log m}{\log m} - \frac{1}{\log m} - \frac{1}{\log^2 m} - \frac{2}{\log^3 m} - \frac{2}{m}\right]$$

$$> (m+1)\log(m+1)\left[1 + \frac{\log\log m}{\log m} - \frac{1}{\log m} - \frac{1}{\log^2 m} - \frac{2}{\log^3 m} - \frac{2}{m}\right].$$

If $x\big(1 + H(x)^{-1}\big) \geqslant \psi(x)$, then

$$p(m)\left(1 + \frac{1}{H\big(p(m)\big)}\right) \geqslant \psi\big(p(m)\big)$$

$$> (m+1)\log(m+1)\left[1 + \frac{\log\log m}{\log m} - \frac{1}{\log m} - \frac{1}{\log^2 m} - \frac{2}{\log^3 m} - \frac{2}{m}\right].$$

Hence $p(m) > (m+1)\log(m+1)$ if

$$\frac{1}{H\big(p(m)\big)} \leqslant \frac{\log\log m}{\log m} - \frac{1}{\log m} - \frac{1}{\log^2 m} - \frac{2}{\log^3 m} - \frac{2}{m}.$$

The lemma follows on multiplying both sides by $\log m\, H\big(p(m)\big)$.

LEMMA 13. *If* $1000 < m_1 < m_2$, *and if* $p(n) > n\log n$ *for all* $n \leqslant m_1$, *and if* $H(x)$ *is a positive function such that* $x\big(1 + H(x)^{-1}\big) \geqslant \psi(x)$, *and if*

$$\log n \leqslant H\big(p(n)\big)\left[\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}\right]$$

for all n *between* $m_1 - 1$ *and* m_2, *then* $p(n) > n\log n$ *for all* $n \leqslant m_2$.

We take m to be $m_1, m_1+1, m_1+2, \ldots, m_2-1$ successively in Lemma 12.

LEMMA 14. $p(n) > n\log n$ *for all* $n \leqslant 1071$.

Proof. For $n \geqslant 1$, $p(n)$ and $n\log n$ are both increasing functions of n, so that from $p(26) > 29\log 29$ we infer that $p(26) > 26\log 26$, $p(27) > 27\log 27$, $p(28) > 28\log 28$, and $p(29) > 29\log 29$. Using this idea, the lemma follows from the inequalities stated below, which can be verified by computation. In making the computation, one must bear in mind that Lehmer gives 1 as the first prime, whereas in this paper 2 is counted as the first prime.

$$p(1) = 2 > 1 \log 1. \quad p(2) = 3 > 2 \log 2. \quad p(3) = 5 > 3 \log 3.$$

$$p(4) = 7 > 4 \log 4. \quad p(5) = 11 > 6 \log 6.$$

$$p(7) = 17 > 8 \log 8. \quad p(9) = 23 > 9 \log 9.$$

$$p(10) = 29 > 11 \log 11. \quad p(12) = 37 > 14 \log 14. \quad p(15) = 47 > 16 \log 16.$$

$$p(17) = 59 > 19 \log 19. \quad p(20) = 71 > 22 \log 22. \quad p(23) = 83 > 25 \log 25.$$

$$p(26) = 101 > 29 \log 29. \quad p(30) = 113 > 32 \log 32. \quad p(33) = 137 > 37 \log 37.$$

$$p(38) = 163 > 43 \log 43. \quad p(44) = 193 > 49 \log 49. \quad p(50) = 229 > 56 \log 56.$$

$$p(57) = 269 > 64 \log 64. \quad p(65) = 313 > 72 \log 72. \quad p(73) = 367 > 83 \log 83.$$

$$p(84) = 433 > 95 \log 95. \quad p(96) = 503 > 107 \log 107.$$

$$p(108) = 593 > 123 \log 123. \quad p(124) = 683 > 138 \log 138.$$

$$p(139) = 797 > 157 \log 157. \quad p(158) = 929 > 179 \log 179.$$

$$p(180) = 1069 > 201 \log 201. \quad p(202) = 1231 > 226 \log 226.$$

$$p(227) = 1433 > 258 \log 258. \quad p(259) = 1637 > 288 \log 288.$$

$$p(289) = 1879 > 324 \log 324. \quad p(325) = 2153 > 364 \log 364.$$

$$p(365) = 2467 > 410 \log 410. \quad p(411) = 2833 > 461 \log 461.$$

$$p(462) = 3271 > 522 \log 522. \quad p(523) = 3761 > 589 \log 589.$$

$$p(590) = 4297 > 661 \log 661. \quad p(662) = 4951 > 744 \log 744.$$

$$p(745) = 5657 > 840 \log 840. \quad p(841) = 6481 > 945 \log 945.$$

$$p(946) = 7477 > 1071 \log 1071.$$

3. *Theorems.*

THEOREM 1. *For each positive integer n, the n-th prime is greater than $n \log n$, if 2 is counted as the first prime.*

Proof. By Lemma 14, the theorem is true for $n \leqslant 1071$. We proceed at first by combining Lemma 13 and Lemma 10. For $m \geqslant 1071$, $p(m) \geqslant 8599$. Since the $K_1(x)$ of Lemma 10 is a decreasing function of x, we have $K_1\big(p(m)\big) \leqslant 0{\cdot}001{,}46$. Thus, if we take $H = 12{\cdot}5$, then

$$p(m)(1+H^{-1}) \geqslant \psi\big(p(m)\big)$$

for $m \geqslant 1071$. Now

$$\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}$$

is an increasing function of n, so that, for $1071 \leqslant n$,

$$0{\cdot}763 < \log\log 1071 - 1 - \frac{1}{\log 1071} - \frac{2}{\log^2 1071} - \frac{2\log 1071}{1071}$$
$$\leqslant \log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}.$$

Hence, for $1071 \leqslant n \leqslant e^{9{\cdot}5}$,

$$\log n < (12{\cdot}5)(0{\cdot}763)$$
$$< H\big(p(n)\big)\left[\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}\right].$$

We can now put $n = e^{9{\cdot}5}$ in the right-hand side of the inequality and deduce that the inequality is satisfied for $e^{9{\cdot}5} \leqslant n \leqslant e^{14}$. Proceeding in this way, we show that the inequality is satisfied for $1071 \leqslant n \leqslant e^{28}$. Hence, by Lemma 13, $p(n) > n\log n$ for $n \leqslant e^{28}$.

For $m \geqslant e^{28}$, $p(m) > 28e^{28} > e^{31}$. So for $m \geqslant e^{28}$, $K_1\big(p(m)\big) \leqslant 0{\cdot}000,736$. So, if we take $H = 17{\cdot}8$, then $p(m)(1+H^{-1}) \geqslant \psi\big(p(m)\big)$ for $m \geqslant e^{28}$. Then we can show that

$$\log n \leqslant H\left[\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}\right]$$

for $e^{28} \leqslant n \leqslant e^{51}$. Hence $p(n) > n\log n$ for $n \leqslant e^{51}$.

We now use Lemma 11 and Lemma 13. The $K_2(x)$ of Lemma 11 is a decreasing function of x. If $m \geqslant e^{51}$, $p(m) \geqslant e^{54}$. Then

$$K_2\big(p(m)\big) \leqslant 0{\cdot}000,000,232.$$

Take $a = 16$. Then if $H = 42$, $p(m)(1+H^{-1}) \geqslant \psi\big(p(m)\big)$ for $m \geqslant e^{51}$. Then

$$\log n \leqslant H\left[\log\log n - 1 - \frac{1}{\log n} - \frac{2}{\log^2 n} - \frac{2\log n}{n}\right]$$

for $e^{51} \leqslant n \leqslant e^{174}$. So $p(n) > n\log n$ for $n \leqslant e^{174}$.

If $m \geqslant e^{174}$, $p(m) \geqslant e^{179}$. So $K_2\big(p(m)\big) \leqslant 0{\cdot}000,000,103$. Take $a = 18$. Then if $H = 55$, $p(m)(1+H^{-1}) \geqslant \psi\big(p(m)\big)$ for $m \geqslant e^{174}$. Then $p(n) > n\log n$ for $n \leqslant e^{247}$.

We continue this process, taking successively:

$$m \geqslant e^{247}, \quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}065{,}2, \quad a = 20, \quad H = 65;$$

$$m \geqslant e^{306}, \quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}045, \quad a = 21, \quad H = 73;$$

$$m \geqslant e^{355}, \quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}033{,}1, \quad a = 22, \quad H = 81;$$

$$m \geqslant e^{404}, \quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}024{,}3, \quad a = 23, \quad H = 90;$$

$$m \geqslant e^{461}, \quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}017, \quad a = 25, \quad H = 102;$$

$$m \geqslant e^{539}, \quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}010{,}5, \quad a = 27, \quad H = 119;$$

$$m \geqslant e^{651}, \quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}005{,}27, \quad a = 30, \quad H = 151;$$

$$m \geqslant e^{870}, \quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}001{,}44, \quad a = 37, \quad H = 233;$$

$$m \geqslant e^{1460}, \quad K_2\big(p(m)\big) \leqslant 0{\cdot}000{,}000{,}000{,}2, \quad a = 50, \quad H = 450.$$

We have then proved that $p(n) > n \log n$ for $n \leqslant e^{3000}$.

We now return to Lemma 10 and use the inequalities

$$S_8 < 0{\cdot}000{,}332 < 0{\cdot}274(8e^{-8}),$$

and, if $r > 8$,

$$S_r < 16{\cdot}6(r+9)\,e^{-2r} + 0{\cdot}274(r - 2{\cdot}25)\,e^{-r} < 0{\cdot}274re^{-r}.$$

Hence, for $x \geqslant e^{3000}$,

$$K_1(x) < \sum_{r=8}^{\infty} 0{\cdot}3re^{-r}x^{-1/(19r)}.$$

However, for $r \geqslant 8$,

$$0{\cdot}2(\log x)^{\frac{1}{2}}\, e^{-2\sqrt{(\log x/19)}} \geqslant 0{\cdot}3re^{-r}x^{-1/(19r)}.$$

We can prove this by taking the logarithm of both sides and showing that

$$r + \frac{\log x}{19r} + \log\Big(0{\cdot}2(\log x)^{\frac{1}{2}}\Big) \geqslant 2\sqrt{(\log x/19)} + \log\,(0{\cdot}3r).$$

However, the minimum value of $r + \log x/19r$ is $2\sqrt{(\log x/19)}$, so that the inequality is satisfied for $r \leqslant \frac{2}{3}(\log x)^{\frac{1}{2}}$. If $r \geqslant \frac{2}{3}(\log x)^{\frac{1}{2}}$,

$$\begin{aligned} r &\geqslant 2\sqrt{(\log x/19)} + 0{\cdot}3r \\ &\geqslant 2\sqrt{(\log x/19)} + \log\,(0{\cdot}3r). \end{aligned}$$

So the sum of the first $(\log x)^{\frac{1}{2}}$ terms of the summation is less than

$$\tfrac{1}{5}\log x\, e^{-2\sqrt{(\log x/19)}}.$$

Since the summation starts with $r=8$, the sum of the remaining terms is less than

$$\int_{\sqrt{\log x}}^{\infty} re^{-r}\,dr = (\log x)^{\frac{1}{2}}\, e^{-\sqrt{\log x}} + e^{-\sqrt{\log x}}.$$

For $x \geqslant e^{3000}$, this is less than $0{\cdot}05 \log x\, e^{-2\sqrt{(\log x/19)}}$. So, for $x \geqslant e^{3000}$,

$$K_1(x) < \sum_{r=8}^{\infty} 0{\cdot}3r\, e^{-r} x^{-1/(19r)} < \tfrac{1}{4}\log x\, e^{-2\sqrt{(\log x/19)}}.$$

Now, for $m \geqslant e^{3000}$,

$$\log m \leqslant (\log m)^{-\frac{1}{2}}\, e^{\sqrt{(\log m/19)}}$$

and so

$$\log m \leqslant (\log m)^{-\frac{1}{2}}\, e^{\sqrt{(\log m/19)}}\left[\log\log m - 1 - \frac{1}{\log m} - \frac{2}{\log^2 m} - \frac{2\log m}{m}\right].$$

Hence we can conclude by Lemma 10 and Lemma 13 that, for all $n \geqslant e^{3000}$, $p(n) > n\log n$.

THEOREM 2. *If* $3 < n$, *then* $p(n) < n\log n + 2n\log\log n$.

Proof. Lemmas analogous to Lemmas 10–14 are proved and then this theorem is proved in a manner precisely analogous to the way Theorem 1 was proved.

Theorems 1 and 2 make more precise the ordinary theorem $p(n) \sim n\log n$ (Landau, p. 214). The sharper result $p(n) = n\big(\log n + \log\log n + O(1)\big)$ (Laudau, p. 215) is made more precise in Theorem 3. The still sharper result $p(n) = n\log n + n\log\log n - n + O(n\log\log n/\log n)$ (which can be proved by the method of Landau, p. 215) could also be made more precise by the methods of this paper, which accounts for the "$-n$" which occurs in Theorem 3.

THEOREM 3. *If* $n > 1$,

$$n\log n + n\log\log n - n - 9n < p(n) < n\log n + n\log\log n - n + 9n.$$

Proof. For $$1 < n \leqslant e^{e^{10}},$$

$n\log n + n\log\log n - n - 9n < p(n)$ by Theorem 1. For

$$e^{e^{10}} \leqslant n,$$

the method of proof used in Theorem 1 will work. The other half of the theorem is proved analogously.

Let B_n, C_n, and D_n be defined for $n > 1$ by

$$\sum_{m=1}^{n} \frac{1}{p(m)} = \log\log n + B_n, \quad \prod_{m=1}^{n}\left(1-\frac{1}{p(m)}\right) = \frac{C_n}{\log n},$$

and

$$\prod_{m=2}^{n}\left(1-\frac{2}{p(m)}\right) = \frac{D_n}{\log^2 n}.$$

THEOREM 4. *For* $n > 1$, $B_n > B_{n+1}$, $C_n < C_{n+1}$, *and* $D_n < D_{n+1}$.

Proof.

$$\begin{aligned} B_{n+1} + \log\log(n+1) &= \sum_{m=1}^{n+1} \frac{1}{p(m)} = \sum_{m=1}^{n} \frac{1}{p(m)} + \frac{1}{p(n+1)} \\ &< B_n + \log\log n + \frac{1}{(n+1)\log(n+1)} \\ &< B_n + \log\log n + \int_n^{n+1} \frac{dx}{x\log x} \\ &< B_n + \log\log(n+1). \end{aligned}$$

Thus $B_{n+1} < B_n$.

$$\begin{aligned} \frac{C_n}{C_{n+1}} &= \frac{\log n}{\log(n+1)\big(1-1/p(n+1)\big)} < \frac{\log(n+1)+\log\big(n/(n+1)\big)}{\log(n+1)\Big(1-1\Big/\big((n+1)\log(n+1)\big)\Big)} \\ &< \frac{\log(n+1)-1/(n+1)}{\log(n+1)\Big(1-1\Big/\big((n+1)\log(n+1)\big)\Big)} = 1. \end{aligned}$$

Thus $C_n < C_{n+1}$.

$$\begin{aligned} \frac{D_n}{D_{n+1}} &= \frac{\log^2 n}{\log^2(n+1)\big(1-2/p(n+1)\big)} < \frac{\Big[\log(n+1)+\log\big(n/(n+1)\big)\Big]^2}{\log^2(n+1)\Big(1-2\Big/\big((n+1)\log(n+1)\big)\Big)} \\ &< \frac{\Big[\log(n+1)-1/(n+1)-1\Big/\big(2(n+1)^2\big)\Big]^2}{\log^2(n+1)\Big(1-2\Big/\big((n+1)\log(n+1)\big)\Big)} \\ &< \frac{\log^2(n+1)-2\log(n+1)/(n+1)}{\log^2(n+1)\Big(1-2\Big/\big((n+1)\log(n+1)\big)\Big)}, \end{aligned}$$

if $n>2$. Thus $D_n/D_{n+1}<1$ for $n>2$. Moreover, $D_2=0{\cdot}16$ and $D_3=0{\cdot}24$.

THEOREM 5.

$$\lim_{n\to\infty} B_n = 0{\cdot}261{,}497{,}212{,}8.$$

$$\lim_{n\to\infty} C_n = 0{\cdot}561{,}459{,}483{,}6.$$

$$\lim_{n\to\infty} D_n = 0{\cdot}832{,}429{,}065{,}7.$$

Proof.
$$\prod_{p<x}\left(1-\frac{1}{p}\right)=\frac{e^{-C}}{\log x}+O\left(\frac{1}{\log^2 x}\right)$$

(Landau, p. 139). Since $p(n)\sim n\log n$, it follows that

$$\lim_{n\to\infty} C_n = e^{-C} = \frac{1}{1{\cdot}781{,}072{,}417{,}990} = 0{\cdot}561{,}459{,}483{,}567.$$

Now, if $n>6$,

$$\begin{aligned}\log C_n &= \log\log n+\sum_{m=1}^{n}\log\left(1-\frac{1}{p(m)}\right)\\ &= \log\log n+\log(1-\tfrac{1}{2})+\log(1-\tfrac{1}{3})+\log(1-\tfrac{1}{5})+\log(1-\tfrac{1}{7})\\ &\qquad +\log(1-\tfrac{1}{11})+\log(1-\tfrac{1}{13})-\sum_{m=7}^{n}\frac{1}{p(m)}\\ &\qquad -\tfrac{1}{2}\sum_{m=7}^{n}\frac{1}{\left(p(m)\right)^2}-\ldots-\frac{1}{r}\sum_{m=7}^{n}\frac{1}{\left(p(m)\right)^r}-\ldots\\ &= \log\log n-\sum_{m=1}^{n}\frac{1}{p(m)}+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{13}+6\log 2\\ &\qquad +\log 3-\log 7-\log 11-\log 13\\ &\qquad -\tfrac{1}{2}\sum_{m=7}^{n}\frac{1}{\left(p(m)\right)^2}-\ldots-\frac{1}{r}\sum_{m=7}^{n}\frac{1}{\left(p(m)\right)^r}-\ldots.\end{aligned}$$

However, for $r>1$,

$$\sum_{m=7}^{n}\frac{1}{\left(p(m)\right)^r}<\sum_{m=17}^{\infty}\frac{1}{m^r}<\frac{1}{(r-1)\,16^{r-1}}.$$

Thus we can conclude that

$$\lim_{n\to\infty} B_n = -\log\left(\lim_{n\to\infty} C_n\right)+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{13}$$

$$+6\log 2+\log 3-\log 7-\log 11-\log 13$$

$$-\tfrac{1}{2}\sum_{m=7}^{\infty}\frac{1}{\left(p(m)\right)^2}-\ldots-\frac{1}{r}\sum_{m=7}^{\infty}\frac{1}{\left(p(m)\right)^r}-\ldots.$$

However,

$$-\log\left(\lim_{n\to\infty} C_n\right) = C = 0{\cdot}577{,}215{,}664{,}901{,}329,$$

and

$$\sum_{m=7}^{\infty}\frac{1}{\left(p(m)\right)^r}$$

can be computed for $r>1$ from tables* of

$$\sum_{m=1}^{\infty}\frac{1}{\left(p(m)\right)^r}.$$

This gives $\lim_{n\to\infty} B_n = 0{\cdot}261{,}497{,}212{,}847{,}643.$

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* Such a table is given in the second volume of H. T. Davis's *Tables of the higher mathematical functions*, 249–250. The present author checked this table for $n > 23$ by the relation

$$\Sigma_n = 1+S_n-S_{2n}-S_{3n}-\frac{1}{6^n}S_n-\frac{1}{10^n}-\frac{1}{14^n}-\frac{1}{15^n}-\ldots,$$

and found that the last digits of the values given in the table for Σ_{30}, Σ_{32}, Σ_{38}, and Σ_{48} were all too large by one unit, and for Σ_{23}, Σ_{25}, Σ_{29}, Σ_{33}, Σ_{35}, and Σ_{37} were all too small by one unit. This would make it seem likely that, while the last digit is unreliable, it is probably never very much in error. As a further confirmation of this conclusion, it should be noted that the values check, except for the last digit, in the relation,

$$C-\lim_{x\to\infty}\left(\sum_{p<x}\frac{1}{p}-\log\log x\right)=\tfrac{1}{2}\log S_2+\tfrac{1}{3}\log S_3+\tfrac{1}{5}\log S_5-\tfrac{1}{6}\log S_6+\ldots$$

$$=\tfrac{1}{2}\Sigma_2+\tfrac{1}{3}\Sigma_3+\tfrac{1}{4}\Sigma_4+\tfrac{1}{5}\Sigma_5+\ldots.$$

This relation can be proved by combining modern results in the theory of primes with the argument given by J. W. Glaisher, "On the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+$etc.", *Quart. J. of Math.*, 25 (1891), 369–375. The numerical value of this constant works out to be 0·315,718,452,053,890,076,851,085 (last digit doubtful).

For $n > 6$,

$$\log D_n = 2 \log \log n + \sum_{m=2}^{n} \log \left(1 - \frac{2}{p(m)}\right)$$

$$= -2B_n - \log 7 + \log 9 - \log 13 + \tfrac{2}{2} + \tfrac{2}{3} + \tfrac{2}{5} + \tfrac{2}{7} + \tfrac{2}{11} + \tfrac{2}{13}$$

$$- \tfrac{4}{2} \sum_{m=7}^{n} \frac{1}{\big(p(m)\big)^2} - \dots - \frac{2^r}{r} \sum_{m=7}^{n} \frac{1}{\big(p(m)\big)^r} - \dots.$$

Letting n approach infinity, we get

$$\lim_{n \to \infty} \log D_n = -0{\cdot}183{,}407{,}267{,}169{,}9$$

(last digit doubtful). So

$$\lim_{n \to \infty} D_n = \frac{1}{1{\cdot}201{,}303{,}559{,}967} = 0{\cdot}832{,}429{,}065{,}662.$$

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