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Countermodels from Sequent Calculi in Multi-Modal Logics

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***Abstract*—A novel countermodel-producing decision procedure that applies to several multi-modal logics, both intuitionistic and classical, is presented. Based on backwards search in labeled sequent calculi, the procedure employs a novel termination condition and countermodel construction. Using the procedure, it is argued that multi-modal variants of several classical and intuitionistic logics including K, T, K4, S4 and their combinations with D are decidable and have the finite model property. At least in the intuitionistic multi-modal case, the decidability results are new. It is further shown that the countermodels produced by the procedure, starting from a set of hypotheses and no goals, characterize the atomic formulas provable from the hypotheses.**

***Keywords*-Multi-modal logic; countermodels; labeled sequent calculus; decidability**

I. Introduction

Modal logics are widely used in several fields of Computer Science and their decidability is a subject of deep interest to the academic community. The subject has been investigated through various techniques, notably semantic filtrations [5, 8], semantic tableaux [4, 11, 12], and translation into decidable fragments of first-order logic [2], yet many areas related to decidability of modal logics remain open. Two such areas are: (1) Decidability of multi-modal intuitionistic logics, especially when modalities interact with each other, and (2) Decision procedures based on sequent calculi that can be directly implemented. Both these areas are challenging. Decidability of intuitionistic modal logics is challenging because standard techniques like semantic filtrations and tableaux have not been studied extensively in the intuitionistic setting, whereas sequent calculi are difficult to use for decision procedures in modal logic because of a well-known problem of looping [11, 17, 26], which is exacerbated by the interaction between modalities and intuitionistic implication.

Spanning both these areas, we present a uniform decision procedure for several propositional multi-modal logics (both intuitionistic and classical), based on backwards search in labeled sequent calculi. Our decision procedure is constructive, which means that for any given formula it either produces a derivation which shows that the formula is true in all (Kripke) models or produces a finite set of finite countermodels on all of which the formula is false.

The decision procedure is also general; it applies to any intuitionistic modal logic without possibility (diamond) modalities and any classical modal logic (even with possibility modalities), provided the logic satisfies a specific technical condition, namely the existence of what we call a suitable closure relation or SCR. As examples, we show that the classical and intuitionistic variants of the following multi-modal logics are constructively decidable by our method: K (the basic normal modal logic), T (reflexive frames), K4 (transitive frames), S4 (reflexive and transitive frames) and their combinations with D (serial frames). We further show that several interaction axioms between modalities such as I $((\Box_A \alpha) \rightarrow (\Box_B \Box_A \alpha))$ [9, 14], unit $(\alpha \rightarrow (\Box_A \alpha))$ [7] and subsumption $((\Box_A \alpha) \rightarrow (\Box_B \alpha))$ result in decidable logics. Constructive decidability also implies the finite model property, so our results also show this property for all the logics listed earlier. For multi-modal intuitionistic logics, not only our method, but also the decidability results are new.

Technical approach and challenges: We present our method separately for classical and intuitionistic logics due to minor technical differences between the two. Our method uses the labeled approach to proof theory of modal logic as developed, among others, by [17, 24, 26], and more specifically [17], to produce labeled sequent calculi with strong analyticity properties. We define a multi-modal intuitionistic (classical) logic MMI$^\chi$ (MM$^\chi$) as the set of all formulas that are valid in all intuitionistic (classical) Kripke frames satisfying stipulated conditions, represented as a set χ. Conditions in χ can be arbitrary, but are restricted in two ways: (1) They must be of the form $\forall \vec{x}.((\wedge_{i=1,\ldots,n} \ x_i R_i x_i') \rightarrow (xRx'))$, where R, R_i range over the relations of a Kripke frame, and (2) The conditions χ must have a SCR, which is a relation over Kripke frames satisfying some stipulated properties, as discussed later in the paper. Briefly, the existence of the SCR implies that frame relations can be deduced from the conditions χ only in some specific ways. We then build a non-terminating, standard labeled sequent calculus for the logic, which we refine in two steps to obtain a constructive decision procedure that, for a given sequent, either produces a proof of it, or a finite set of finite countermodels for it. Next, we build an extension of our method that works for any logic on which our original method works, extended with seriality. Finally, we prove an interesting property of our decision procedure: The set of countermodels it produces for a given hypotheses without a specific goal completely characterizes the atomic formulas that can be proved from the hypotheses. Thus the set of

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countermodels produced is, in a sense, complete. We call this property *comprehensiveness*.

The first challenge for our work is to find a general termination condition for backwards search in labeled sequent calculi. Our termination condition is based on containment of the sets of formulas labeling worlds, which we show to be complete so long as the logic has an SCR. This idea is a non-trivial generalization of existing work on logic-specific termination conditions for many uni-modal classical logics [6, 11, 17]. Finding an appropriate definition for SCRs that is both sufficient to obtain termination and general enough, is the main technical challenge of our work and also its main technical contribution.

The second challenge in our work is to actually build the countermodel when we know that backwards proof search has unsuccessfully terminated. To this end, we observe that a straightforward extension of the model inherent in the sequent at which search terminates (with a few more relations) is actually a countermodel to the sequent. As far as we know, this construction is novel.

Contributions: Our work makes the following contributions: (1) It proves, by uniform method, the decidability of the necessitation-only, multi-modal intuitionistic variants of the logics K, T, K4 and S4 and their combinations with the logic D. Our decision procedure produces countermodels, and also establishes the finite model property for these logics. (2) We provide the first sequent calculus based constructive decision procedure for multi-modal logics. (3) At a technical level, we provide a general method for forcing termination in labeled sequent calculi for modal logics and a sufficient condition (the existence of SCRs) under which it works without loss of completeness. We also present a simple method to extract countermodels when search terminates. (4) As far as we know, ours is the first decision procedure which produces a set of countermodels that is comprehensive in the sense described above.

Limitations: There are existing undecidability results for modal logics with frame conditions such as symmetry and transitivity [3]. Consequently, we cannot hope for a method that proves decidability for all logics MMI$^\chi$ or all logics MM$^\chi$. Nonetheless, there are some classes of frame conditions which are not known to immediately result in undecidability, but to which our method does not apply, either because they do not fit our definition of χ or because they do not have SCRs. First, due to an interaction with intuitionistic implication, our method cannot handle possibility modalities in intuitionistic logic, even though it works fine with them in classical logic. Second, we do not know whether our method can handle "label-producing" conditions like density or confluence. However, it can be easily extended to work with seriality, as discussed in Section III-F. Finally, like many other methods in this domain, an analysis of our proofs does not necessarily produce good upper bounds on the actual complexity of modal logics.

Organization: Since our constructive decision procedure and its correctness proof are almost identical for intuitionistic and classical modal logics, in this paper, we present our results only for the intuitionistic case. The case of classical modal logic is briefly discussed in Section IV and its details are deferred to our online technical report (TR in the sequel) [10]. In Section II, we define the syntax, semantics and a standard, but non-terminating labeled sequent calculus for the intuitionistic multi-modal logic MMI$^\chi$. Section III presents the main technical work. Starting from an informal introduction, we proceed to a description of the decision procedure (Sections III-A – III-D), its comprehensiveness (Section III-E), its extension with seriality (Section III-F), and its instantiation to several known intuitionistic multi-modal logics (Section III-G). Section IV briefly lists modifications needed to adapt the method to classical logic. Related work is discussed in Section V and Section VI concludes the paper. Proofs of theorems and complete descriptions of all calculi are presented in our TR [10].

II. MMI$^\chi$: Multi-Modal Intuitionistic Logic

We start by defining formally the family of intuitionistic multi-modal logics we consider in this paper. The family is parametrized by a set χ of conditions on Kripke frames that must have a specific (standard) form, as described later in this section. The logic obtained by instantiating our definition of syntax and semantics with a specific set χ is called MMI$^\chi$.

Let $\mathcal{I} = \{A, B, \ldots\}$ be a finite set of indices for modalities and p denote an atomic formula, drawn from a countable set of such formulas. Then, the syntax of formulas of the logic MMI$^\chi$ is:

$$\varphi, \alpha, \beta \quad ::= \quad p \mid \top \mid \bot \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha \rightarrow \beta \mid A \;\mathsf{nec}\; \alpha$$

Connectives $\top$, $\bot$, $\wedge$, and $\vee$ have their usual meanings. Implication $\rightarrow$ is interpreted intuitionistically. $A \;\mathsf{nec}\; \alpha$ is the necessitation modality of index A. This is commonly written $\Box_A \alpha$, but we prefer the more descriptive notation. Negation is not primitive, but may be defined in a standard way as $\neg\alpha = \alpha \rightarrow \bot$. We do not consider possibility modalities in the intuitionistic setting, since they are incompatible with our method.

Semantics: We provide Kripke (frame) semantics to formulas of MMI$^\chi$ and assume in our presentation that the reader has basic familiarity with this style of semantics. Several different Kripke semantics for intuitionistic modal logic have been proposed [1, 24, 27]. In what follows, we use semantics that are similar to those of Wolter et al. [27]. This choice makes the technical development easier.

Definition II.1 (Kripke model)**.** An intuitionistic model, Kripke model or, simply, model $\mathcal{M}$ is a tuple $(W, \leq, \{N_A\}_{A \in \mathcal{I}}, h)$ where,

- $(W, \leq)$ is a preorder. Elements of W are called worlds and written x, y, z, w. Since $\leq$ is a preorder, it satisfies the following conditions:

$$\forall x.(x \leq x) \qquad \text{(refl)}$$
$$\forall x, y, z.(((x \leq y) \wedge (y \leq z)) \rightarrow (x \leq z)) \qquad \text{(trans)}$$

- Each N_A is a binary relation on W that satisfies the condition $(\leq \circ N_A) \subseteq N_A$, i.e.

$$\forall x,y,z.(((x \le y) \wedge (yN_A z)) \to (xN_A z)) \quad \text{(mon-N)}$$

- h assigns to each atom p the set of worlds $h(p) \subseteq W$ where p holds. We require h to be monotone w.r.t. $\le$, i.e., if $x \in h(p)$ and $x \le y$ then $y \in h(p)$.

A model without the assignment, i.e., the tuple $(W, \le, \{N_A\}_{A \in \mathcal{I}})$ is also called a *frame* and the conditions on relations above, e.g., (mon-N), are called *frame conditions*.

The frame conditions χ: In addition to the frame conditions (refl), (trans) and (mon-N), we allow a countable number of additional frame conditions as rules of the following form: $\forall \vec{x}.((\wedge_{i=1,\ldots,n}\, x_i R_i x_i') \to (xRx'))$, where $R_1, \ldots, R_n, R$ are from the set $\{N_A \mid A \in \mathcal{I}\} \cup \{\le\}$ and all variables x_i, x_i', x, x' are in $\vec{x}$. A set of such additional frame conditions is denoted χ. MMI$^\chi$ is the logic whose valid formulas are exactly those that are valid (in the sense defined below) in frames that satisfy *all* conditions in χ.

Definition II.2 (Satisfaction)**.** Given a model $\mathcal{M} = (W, \le, \{N_A\}_{A \in \mathcal{I}}, h)$ and a world $w \in W$, we define the satisfaction relation $\mathcal{M} \models w : \alpha$, read "the world w satisfies formula α in model $\mathcal{M}$" by induction on α (we omit standard definitions for $\vee$, $\wedge$ and $\top$):

- $\mathcal{M} \models w : p$ iff $w \in h(p)$
- $\mathcal{M} \models w : \alpha \to \beta$ iff for every w' such that $w \le w'$ and $\mathcal{M} \models w' : \alpha$, we have $\mathcal{M} \models w' : \beta$.
- $\mathcal{M} \models w : A \;\mathsf{nec}\; \alpha$ iff for every w' such that $wN_A w'$, we have $\mathcal{M} \models w' : \alpha$.

We say that $\mathcal{M} \not\models w : \alpha$ if it is not the case that $\mathcal{M} \models w : \alpha$. In particular, for every $\mathcal{M}$ and every w, $\mathcal{M} \not\models w : \bot$.

A formula α is true in a model $\mathcal{M}$, written $\mathcal{M} \models \alpha$, if for every world $w \in \mathcal{M}$, $\mathcal{M} \models w : \alpha$. A formula α is *valid* in MMI$^\chi$, written $\models \alpha$, if $\mathcal{M} \models \alpha$ for every model $\mathcal{M}$ satisfying all conditions in χ.

Valid axioms: We list below some valid axioms and admissible rules that are common to all logics MMI$^\chi$. If a rule/axiom is standard in literature, its common name is mentioned to the extreme right. ($\alpha \equiv \beta$ means $(\alpha \to \beta) \wedge (\beta \to \alpha)$.)

(all tautologies of intuitionistic propositional logic) (IPC)

If $\models \alpha$, then $\models A \;\mathsf{nec}\; \alpha$ (nec)

$\models (A \;\mathsf{nec}\; (\alpha \to \beta)) \to ((A \;\mathsf{nec}\; \alpha) \to (A \;\mathsf{nec}\; \beta))$ (K)

$\models ((A \;\mathsf{nec}\; \alpha) \wedge (A \;\mathsf{nec}\; \beta)) \equiv (A \;\mathsf{nec}\; (\alpha \wedge \beta))$

The frame conditions χ can be used to force additional axioms in a standard way, which has been explored in great detail in literature on correspondence theory [5]. For example, the condition $\forall x, y.((xN_A y) \to (x \le y))$ corresponds to the axiom $\alpha \to (A \;\mathsf{nec}\; \alpha)$, commonly called (unit) and of central importance in lax logic [7]. Similarly, the condition $\forall x, y, z.(((xN_A y) \wedge (yN_A z)) \to (xN_A z))$ corresponds to the axiom $(A \;\mathsf{nec}\; \alpha) \to (A \;\mathsf{nec}\; A \;\mathsf{nec}\; \alpha)$.

The following is a fundamental property of Kripke models in intuitionistic modal logics, proved by induction on α.

Lemma II.3 (Monotonicity)**.** *If $\mathcal{M} \models w : \alpha$ and $w \le w' \in \mathcal{M}$, then $\mathcal{M} \models w' : \alpha$.*

A. Seq-MMI$^\chi$: A Labeled Sequent Calculus for MMI$^\chi$

We present a sound, complete, cut-free sequent calculus for MMI$^\chi$. Following the work of Negri [17], our calculus is presented in what is known as the "labeled" style of calculi for modal logics, which means that the calculus proves formulas labeled with symbolic worlds. A labeled formula contains a symbol x, y, z, w, u denoting a world and a formula α, written together as $x : \alpha$. A sequent in our calculus has the form $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$, where: Σ is a finite set of world symbols appearing in the rest of the sequent (world symbols are also called *labels*); $\mathbb{M}$ is a finite multi-set of relations of the forms $x \le y$ and $xN_A y$; Γ and Δ are finite multi-sets of labeled formulas. Intuitively, if $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$ is valid, then every model with a world set containing at least Σ, satisfying all relations in $\mathbb{M}$ and all labeled formulas in Γ must satisfy at least one labeled formula in Δ. This is formalized in the following definition.

$$\frac{}{\Sigma; \mathbb{M}, x \le y; \Gamma, x : p \Rightarrow^\chi y : p, \Delta}\,\text{init}$$

$$\frac{\Sigma, y; \mathbb{M}, x \le y; \Gamma, y : \alpha \Rightarrow^\chi y : \beta, x : \alpha \to \beta, \Delta}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi x : \alpha \to \beta, \Delta}\,{\to}\text{R}$$

$$\frac{\begin{array}{c}\Sigma; \mathbb{M}, x \le y; \Gamma, x : \alpha \to \beta \Rightarrow^\chi y : \alpha, \Delta \\ \Sigma; \mathbb{M}, x \le y; \Gamma, x : \alpha \to \beta, y : \beta \Rightarrow^\chi \Delta\end{array}}{\Sigma; \mathbb{M}, x \le y; \Gamma, x : \alpha \to \beta \Rightarrow^\chi \Delta}\,{\to}\text{L}$$

$$\frac{\Sigma, y; \mathbb{M}, xN_A y; \Gamma \Rightarrow^\chi y : \alpha, x : A \;\mathsf{nec}\; \alpha, \Delta}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi x : A \;\mathsf{nec}\; \alpha, \Delta}\,\mathsf{necR}$$

$$\frac{\Sigma; \mathbb{M}, xN_A y; \Gamma, x : A \;\mathsf{nec}\; \alpha, y : \alpha \Rightarrow^\chi \Delta}{\Sigma; \mathbb{M}, xN_A y; \Gamma, x : A \;\mathsf{nec}\; \alpha \Rightarrow^\chi \Delta}\,\mathsf{necL}$$

$$\frac{\Sigma; \mathbb{M}, x \le y, yN_A z, xN_A z; \Gamma \Rightarrow^\chi \Delta}{\Sigma; \mathbb{M}, x \le y, yN_A z; \Gamma \Rightarrow^\chi \Delta}\,\text{mon-N}$$

$$\frac{\begin{array}{c}(\forall \vec{x}.((\wedge_i\, (x_i R_i x_i')) \to (xRx'))) \in \chi \\ x_i R_i x_i' \in \mathbb{M} \qquad \Sigma; \mathbb{M}, xRx'; \Gamma \Rightarrow^\chi \Delta\end{array}}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta}\,\chi$$

Fig. 1. Seq-MMI$^\chi$: A labeled sequent calculus for MMI$^\chi$, selected rules

Definition II.4 (Sequent satisfaction and validity)**.** A model $\mathcal{M}$ and a mapping ρ from elements of Σ to worlds of $\mathcal{M}$ satisfy a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$, written $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$, if at least one of the following holds: (1) There is an $xRy \in \mathbb{M}$ with $R \in \{\le\} \cup \{N_A \mid A \in \mathcal{I}\}$ such that $\rho(x)\; R\; \rho(y) \notin \mathcal{M}$; (2) There is an $x : \alpha \in \Gamma$ such that $\mathcal{M} \not\models \rho(x) : \alpha$; (3) There is an $x : \alpha \in \Delta$ such that $\mathcal{M} \models \rho(x) : \alpha$.

A model $\mathcal{M}$ satisfies a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$, written $\mathcal{M} \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$, if for every mapping ρ, we have $\mathcal{M}, \rho \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$. Finally, a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta$ is valid, written $\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$, if for every model $\mathcal{M}$ we have $\mathcal{M} \models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta)$.

Rules of the sequent calculus: Selected rules of the sequent calculus for MMI$^\chi$ are shown in Figure 1. Following standard approach in labeled calculi, the rules for each connective mimic the (Kripke) semantic definition of the connective.

The rules ($\rightarrow$R) and (necR) introduce fresh worlds into Σ, consistent with the semantic definition (Definition II.2). We say that $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$ if $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$ has a proof in the calculus. The sequent calculus is both sound and complete with respect to the semantics.

Theorem II.5 (Soundness)**.** *If* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$*, then* $\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$.

The converse of Theorem II.5, completeness, also holds but we do not need this result in the rest of our development so we do not present it here. The following property is critical to the design and correctness of our constructively complete decision procedure.

Theorem II.6 (Weak subformula property)**.** *If a formula* φ *appears in any proof tree (possibly infinite) obtained by applying the rules of Figure 1 backwards starting from a concluding sequent* $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$*, then* φ *is a subformula of some formula in either* Γ *or* Δ.

III. Constructively Complete Decision Procedure

As argued by Negri [17], and also illustrated in our TR, constructing a decision procedure based on the sequent calculus of Figure 1 is non-trivial because backwards search in the calculus can loop forever due to unbounded creation of new worlds in the rules ($\rightarrow$R) and (necR). Although prior work such as that of Negri describes loop-detection methods for many *specific* uni-modal classical logics individually, there is no known general solution. In this section we present a *general* technique that not only detects such loops in a wide variety of logics MMI$^{\chi}$, but also produces Kripke countermodels witnessing the non-validity of the end-sequent when such loops are detected. Although the end-result of our technique, i.e., the decision procedure itself, is quite straightforward, building up to it requires some non-standard machinery, which we motivate here by presenting an informal outline of our method.

To control the use of rules ($\rightarrow$R) and (necR) in backwards proof search, we generalize a technique from existing work on tableaux calculi for classical uni-modal logics [11]. The technique prevents loops by checking for containment of formulas that label a world in those labeling another. We start by observing that in any sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$ obtained during backwards proof search starting from a single goal formula, all worlds in $\mathbb{M}$ lie on a rooted, directed tree, whose edges are relations in $\mathbb{M}$ that were introduced by the rules ($\rightarrow$R) and (necR) in earlier steps of the search. We call the reflexive-transitive closure of this tree $\ll$. Next, we define a function $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ that lists, *approximately*, all formulas labeled by x in Γ and Δ (the exact definition of Sfor depends on χ, and is one of our key technical contributions). This function satisfies a pivotal property whose proof requires induction on $\ll$: If there is a world y ($y \neq x$) such that $y \ll x$ and $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x) \subseteq \mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, y)$, then it is *useless* to apply any of the rules ($\rightarrow$R) and (necR) on any principal formula labeled by x in the sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$ in backwards proof search. It only remains to show that this condition forces termination. This follows from the fact that $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ increases monotonically for each x in a backwards proof search and the fact that the number of possible values of Sfor is finite, which, in turn, is a consequence of the weak subformula property (Theorem II.6).

We further show that if no rule applies backwards to a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$ (after imposing our termination checks), then we can obtain a *countermodel* to the sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$ by adding an edge $x \leq y$ whenever $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x) \subseteq \mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, y)$. This forms the basis of our countermodel extraction. (As further explained in Section V, this method of extracting countermodels is inspired by Negri's proof of completeness of labeled sequent calculi for uni-modal logic with respect to Kripke semantics [18].)

The definition of the function Sfor depends on the conditions χ that define the logic. In our formal development, we define a suitable Sfor for every logic for which there exists what we call a *suitable closure relation* (SCR). Technically, a SCR is a family of relations on frames, which satisfies some stipulated properties. Our entire method applies to any logic MMI$^{\chi}$ for whose χ a SCR exists. This begs the question of how general the existence of a SCR is. As we show in Section III-G, several (multi-)modal logics with reflexivity, transitivity, and modality interaction conditions, including the multi-modal logics K, K4, S4, T, and I [9, 14] have SCRs. We further show in Section III-F that our method can be extended to any logic with a SCR plus the seriality condition on its accessibility relations.

Our technical presentation consists of the following steps: (1) We define the term "SCR for χ". Its existence is the only condition that must hold for our method to apply to the logic MMI$^{\chi}$. (2) We define the function Sfor using SCRs. We also define a predicate on sequents, whose elements (sequents) are called *saturated histories*. Roughly, a saturated history is a sequent on which backwards application of all rules is useless, i.e., applying any rule other than ($\rightarrow$R) and (necR) backwards on the sequent does not add any new labeled formulas, and the application of these two rules is blocked by the containment condition on Sfor. We prove, by construction, the key property of our entire method: If a sequent is a saturated history, then it has a finite countermodel. (3) We define an intermediate sequent calculus Seq-MMI$^{\chi}_{\mathrm{CM}}$ with judgments $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$. Here, S is a (possibly empty) finite set of finite countermodels. The correctness property of this calculus is: If $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\}$ has a proof, then $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$ and if $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$ has a proof for $S \neq \{\}$, then every $\mathcal{M} \in S$ satisfies: $\mathcal{M} \not\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$. Backwards search in this calculus does not necessarily terminate. However, we find the calculus a useful intermediate step to prove many properties. (4) We observe that a specific strategy for proof search in Seq-MMI$^{\chi}_{\mathrm{CM}}$ terminates. This strategy is presented as a sequent calculus Seq-MMI$^{\chi}_{\mathrm{T}}$, which is a countermodel producing decision procedure.

In what follows, we present the technical details of each of these four steps. In particular, Section III-D describes our decision procedure.

A. Suitable Closure Relations (SCRs)

Our constructive decision procedure applies to any logic MMI^{χ} whose χ has a SCR. Call a frame $\mathbb{M}$ *closed* if it is closed under the conditions (refl), (trans), (mon-N) and χ. Given a frame $\mathbb{M}$, let $\overline{\mathbb{M}}$ denote its closure obtained by closing the frame under the conditions (refl), (trans), (mon-N) and χ, obtained as the least fixed point of the application of these rules. Informally, a SCR is a family of relations $(R(A))_{A\in\mathcal{I}}$ that, given any closed frame $\mathbb{M}$ and any extension $\mathbb{M}'$ of it with additional edges of the form $\leq$, characterizes all relations N_A in $\overline{\mathbb{M}'}$ in terms of the relations in $\mathbb{M}$ and the difference $\mathbb{M}'-\mathbb{M}$.

Definition III.1 (Suitable closure relation (SCR))**.** A family of binary relations $(R(A))_{A\in\mathcal{I}}$ is called a suitable closure relation or SCR for χ if the following hold: (0) Each $R(A)$ is definable in first-order logic in terms of the relations $\leq \cup \{N_A \mid A \in \mathcal{I}\}$. (1) For a finite frame $\mathbb{M}$ and $x, y \in \mathbb{M}$, it can be decided whether $x(R(A))y$ or not. (2) $((R(A))^* \circ N_A) \subseteq N_A$ can be derived from the frame conditions (refl), (trans), (mon-N) and χ. (3) For any *closed* frame $\mathbb{M}$ and any $C \subseteq \{x \leq y \mid x, y \in \mathbb{M}\}$, if $xN_Ay \in \overline{\mathbb{M} \cup C}$, then $x((R(A) \cup C)^* \circ N_A \circ \leq)y$, where all relations $R(A)$ and N_A on the right are in $\mathbb{M}$. (4) For any closed frame $\mathbb{M}$ and any $C \subseteq \{x \leq y \mid x, y \in \mathbb{M}\}$, if $x \leq y \in \overline{\mathbb{M} \cup C}$, then $x(\leq \cup C)^*y$, where all $\leq$ relations on the right are in $\mathbb{M}$. Condition (4) depends only on χ, not on the choice of $R(A)$, but we include it here for uniformity.

Example III.2 (SCR for transitivity)**.** Let $\mathtt{trans}(A)$ be the frame condition $\forall x, y, z.(((xN_Ay) \wedge (yN_Az)) \rightarrow (xN_Az))$ and let $\chi = \{\mathtt{trans}(A) \mid A \in \mathcal{I}\}$. Then, the relation $R(A) = N_A \cup \leq$ is a SCR for χ. To prove this, we verify each of the conditions (0)–(4) in the definition of SCR. Conditions (0) and (1) are trivially true. (2) is equivalent to $((N_A \cup \leq)^* \circ N_A) \subseteq N_A$, which follows from the frame conditions (mon-N) and χ. To prove (3), suppose that $xN_Ay \in \overline{\mathbb{M} \cup C}$. Then, because the only way to derive a relation N_A is to use either (mon-N) or $\mathtt{trans}(A)$, it follows that in $\mathbb{M} \cup C$, we have $x((N_A \cup \leq)^* \circ N_A)y$. So, we also have $x((N_A \cup \leq \cup C)^* \circ N_A)y$, where all $\leq$ and N_A relations are in $\mathbb{M}$, i.e., $x((R(A) \cup C)^* \circ N_A)y$. Finally, since $\leq$ is reflexive, we have: $x((R(A) \cup C)^* \circ N_A \circ \leq)y$, as required. The proof of (4) is similar to that of (3).

B. Saturated Histories

Our method applies only to those logics MMI^{χ} whose χ has a SCR, so in the sequel we fix a set of frame conditions χ and assume there is a SCR $(R(A))_{A\in\mathcal{I}}$ for this χ. Although we present the technical material generically with respect to an abstract χ and SCR, we advise the reader to choose a specific χ and its SCR (for instance, those from Example III.2), and instantiate our definitions and theorems on them.

A *history* is a tuple $\Sigma; \mathbb{M}; \Gamma; \Delta$ (equivalently, a sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$) such that all labels in $\mathbb{M}$, Γ and Δ occur in Σ. Let $T(\varphi)$ and $F(\varphi)$ be two uninterpreted unary relations on formulas. Informally, we read $T(\varphi)$ as "φ should be true" and $F(\varphi)$ as "φ should be false". Given a history $\Sigma; \mathbb{M}; \Gamma; \Delta$ and $x \in \Sigma$, the *signed formulas* of x, written $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$, are defined as follows:

$$\begin{gathered}\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x) = \\ \{T(\varphi) \mid x : \varphi \in \Gamma\} \cup \{F(\varphi) \mid x : \varphi \in \Delta\} \cup \\ \{T(A \;\mathsf{nec}\; \varphi) \mid \exists y.\; y(R(A))^* x \in \mathbb{M} \text{ and } y : A \;\mathsf{nec}\; \varphi \in \Gamma\} \cup \\ \{T(\varphi \rightarrow \psi) \mid \exists y.\; y \leq x \in \mathbb{M} \text{ and } y : \varphi \rightarrow \psi \in \Gamma\} \cup \\ \{T(p) \mid \exists y.\; y \leq x \in \mathbb{M} \text{ and } y : p \in \Gamma\}\end{gathered}$$

When $\Sigma, \mathbb{M}, \Gamma, \Delta$ are clear from context, we abbreviate $\mathrm{Sfor}(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$ to $\mathrm{Sfor}(x)$. We say that $x \preccurlyeq y$ iff $\mathrm{Sfor}(x) \subseteq \mathrm{Sfor}(y)$.

We call a frame $\mathbb{M}$ *tree-like* if it can be derived from a finite tree of the relations $\leq$ and N_A and (possibly partial) closure by frame rules. This tree is called the underlying tree of $\mathbb{M}$ and we say that $x \ll y$ (in $\mathbb{M}$) iff there is a directed path from x to y in the tree underlying $\mathbb{M}$.

The key concept of our method is a *saturated history*, a generalization of Hintikka sets to labeled, intuitionistic sequents. Intuitively, a history $\Sigma; \mathbb{M}; \Gamma; \Delta$ is saturated if we can directly define a countermodel for the sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$. (The definition of this countermodel is given immediately after the definition of a saturated history.)

Definition III.3 (Saturated history)**.** A history $\Sigma; \mathbb{M}; \Gamma; \Delta$ is called saturated if the following hold: (1) $\mathbb{M}$ is tree-like and saturated with respect to the conditions (refl), (trans), (mon-N) and χ. In particular, because $\mathbb{M}$ is tree-like, it has a relation $\ll$ defined on it. (2) If $x : p \in \Gamma$, then there is no y such that $x \leq y \in \mathbb{M}$ and $y : p \in \Delta$. (3) There is no x such that $x : \top \in \Delta$. (4) There is no x such that $x : \bot \in \Gamma$. (5) If $x : \alpha \wedge \beta \in \Gamma$, then $x : \alpha \in \Gamma$ and $x : \beta \in \Gamma$. (6) If $x : \alpha \wedge \beta \in \Delta$, then either $x : \alpha \in \Delta$ or $x : \beta \in \Delta$. (7) If $x : \alpha \vee \beta \in \Gamma$, then either $x : \alpha \in \Gamma$ or $x : \beta \in \Gamma$. (8) If $x : \alpha \vee \beta \in \Delta$, then $x : \alpha \in \Delta$ and $x : \beta \in \Delta$. (9) If $x : \alpha \rightarrow \beta \in \Gamma$ and $x \leq y \in \mathbb{M}$, then either $y : \alpha \in \Delta$ or $y : \beta \in \Gamma$. (10) If $x : \alpha \rightarrow \beta \in \Delta$, then either: (10a) There is a y such that $x \leq y \in \mathbb{M}$, $y : \alpha \in \Gamma$ and $y : \beta \in \Delta$, or (10b) There is a y such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$. (11) If $x : A \;\mathsf{nec}\; \alpha \in \Gamma$ and $xN_Ay \in \mathbb{M}$, then $y : \alpha \in \Gamma$. (12) If $x : A \;\mathsf{nec}\; \alpha \in \Delta$, then either: (12a) There is a y such that $xN_Ay \in \mathbb{M}$ and $y : \alpha \in \Delta$, or (12b) There is a y such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.

Definition III.4 (Countermodel of a saturated history)**.** For a saturated history $\Sigma; \mathbb{M}; \Gamma; \Delta$, the *countermodel* of the history, $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ is defined as follows: (1) The worlds of $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ are exactly those in Σ. (2) The relations of $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ are $\overline{\mathbb{M} \cup C}$, where $C = \{x \leq y \mid x \preccurlyeq y\}$. (3) $h(p) = \{x \mid \exists y.\; (y \leq x \in \mathbb{M}) \wedge (y : p \in \Gamma)\}$.

It is not obvious that $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ is a model. It does satisfy all frame conditions. However, we must also show monotonicity: If $x \leq y \in \mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ and $x \in h(p)$, then $y \in h(p)$. The following lemma states that this is the case.

Lemma III.5. *If $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history, then $\mathit{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ has a monotonic valuation h, i.e., $x \in h(p)$ and $x \leq y \in \mathit{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ imply $y \in h(p)$.*

The next Lemma states the central property of our method. In particular, the Lemma immediately implies that if $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history, then $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ is a countermodel to the sequent $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$.

Lemma III.6. *The following hold for any saturated history* $\Sigma; \mathbb{M}; \Gamma; \Delta$*: (A) If* $T(\varphi) \in$ *Sfor*$(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$*, then CM*$(\Sigma; \mathbb{M}; \Gamma; \Delta) \models x : \varphi$*. (B) If* $F(\varphi) \in$ *Sfor*$(\Sigma; \mathbb{M}; \Gamma; \Delta, x)$*, then CM*$(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models x : \varphi$*.*

Proof: We prove both properties simultaneously by lexicographic induction, first on φ, and then on the partial (tree-like) order $\ll$ of $\mathbb{M}$. ∎

Corollary III.7 (Existence of countermodel)**.** *If* $\Sigma; \mathbb{M}; \Gamma; \Delta$ *is a saturated history, then CM*$(\Sigma; \mathbb{M}; \Gamma; \Delta) \not\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$*.*

C. *Seq-MMI$^{\chi}_{CM}$: Countermodels for MMI$^{\chi}$*

Having defined a saturated history, i.e., a sequent for which a countermodel exists (Corollary III.7), we now define a sequent calculus Seq-MMI$^{\chi}_{\mathrm{CM}}$, written $\Rightarrow^{\chi}_{\mathrm{CM}}$, which uses this fact to emit countermodels from unprovable sequents. Although this calculus is not a decision procedure, we find it a useful step in proving several results, in particular, the results of Section III-E.

Sequents of Seq-MMI$^{\chi}_{\mathrm{CM}}$ have the form $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$, where S is a finite set of finite models. We write $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S)$ if $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$ has a proof. The meaning of $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$ depends on S. If $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\})$, then $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$ and if $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S)$ with $S \neq \{\}$, then every model $\mathcal{M} \in S$ is a countermodel to $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta$ in the sense of (the converse of) Definition II.4.

Selected rules of the sequent calculus Seq-MMI$^{\chi}_{\mathrm{CM}}$ are shown in Figure 2. First, every rule in the ordinary sequent calculus (Figure 1) is modified to have in the conclusion the union of the sets of (counter)models in the premises. This is sound because the rules of the sequent calculus are invertible (i.e., the conclusion of each rule holds iff the premises hold). Second, there is a new rule (CM) that produces the countermodel $\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)$ when $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history.

We emphasize again that this calculus is not necessarily a decision procedure because it includes all rules of $\Rightarrow^{\chi}$ and hence admits all of the latter's infinite backwards derivations as well.

Theorem III.8 (Soundness 1)**.** *If* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} \Delta \searrow \{\})$*, then* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$*.*

Theorem III.9 (Soundness 2)**.** *If* $\vdash (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{CM} \Delta \searrow S)$*, then for every model* $\mathcal{M} \in S$*,* $\mathcal{M} \not\models (\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$*.*

Proof: By induction on the given derivation of $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S$. The case of rule (CM) follows by Corollary III.7. ∎

$$\frac{\Sigma; \mathbb{M}; \Gamma; \Delta \text{ is a saturated history}}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow \{\mathrm{CM}(\Sigma; \mathbb{M}; \Gamma; \Delta)\}}\,\mathrm{CM}$$

$$\frac{}{\Sigma; \mathbb{M}, x \leq y; \Gamma, x : p \Rightarrow^{\chi}_{\mathrm{CM}} y : p, \Delta \searrow \{\}}\,\mathrm{init}$$

$$\frac{\Sigma, y; \mathbb{M}, x \leq y; \Gamma, y : \alpha \Rightarrow^{\chi}_{\mathrm{CM}} y : \beta, x : \alpha \to \beta, \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x : \alpha \to \beta, \Delta \searrow S}\,{\to}\mathrm{R}$$

$$\frac{\begin{array}{c}\Sigma; \mathbb{M}, x \leq y; \Gamma, x : \alpha \to \beta \Rightarrow^{\chi}_{\mathrm{CM}} y : \alpha, \Delta \searrow S_1 \\ \Sigma; \mathbb{M}, x \leq y; \Gamma, x : \alpha \to \beta, y : \beta \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S_2\end{array}}{\Sigma; \mathbb{M}, x \leq y; \Gamma, x : \alpha \to \beta \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S_1, S_2}\,{\to}\mathrm{L}$$

$$\frac{\Sigma, y; \mathbb{M}, x N_A y; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} y : \alpha, x : A \text{ nec } \alpha, \Delta \searrow S}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} x : A \text{ nec } \alpha, \Delta \searrow S}\,\mathrm{necR}$$

$$\frac{\Sigma; \mathbb{M}, x N_A y; \Gamma, x : A \text{ nec } \alpha, y : \alpha \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma; \mathbb{M}, x N_A y; \Gamma, x : A \text{ nec } \alpha \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\,\mathrm{necL}$$

$$\frac{\Sigma; \mathbb{M}, x \leq y, y N_A z, x N_A z; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}{\Sigma; \mathbb{M}, x \leq y, y N_A z; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\,\text{mon-N}$$

$$\frac{\begin{array}{c}(\forall \vec{x}.((\wedge_i\ (x_i R_i x_i')) \to (x R x'))) \in \chi \\ x_i R_i x_i' \in \mathbb{M} \qquad \Sigma; \mathbb{M}, x R x'; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S\end{array}}{\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S}\,\chi$$

Fig. 2. Seq-MMI$^{\chi}_{\mathrm{CM}}$: Countermodel producing sequent calculus for MMI$^{\chi}$, selected rules

D. *Seq-MMI$^{\chi}_{T}$: Termination and Countermodel Extraction*

Next, we describe a particular backwards proof search strategy in Seq-MMI$^{\chi}_{\mathrm{CM}}$ that always terminates without losing completeness, thus obtaining a countermodel producing decision procedure for MMI$^{\chi}$. This strategy is described as a calculus Seq-MMI$^{\chi}_{\mathrm{T}}$, with sequents of the form $\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S$. The rules of the calculus can be interpreted backwards as a decision procedure with inputs Σ, $\mathbb{M}$, Γ, and Δ and output S. For a given Σ, $\mathbb{M}$, Γ, and Δ, $(\Sigma; \mathbb{M}; \Gamma \Rightarrow^{\chi} \Delta)$ is provable iff $S = \{\}$, else every model in S is a countermodel to the sequent.

Selected rules of the calculus Seq-MMI$^{\chi}_{\mathrm{T}}$ are shown in Figure 3. Each rule in the calculus corresponds to a rule of the same name in Seq-MMI$^{\chi}_{\mathrm{CM}}$ (Figure 2). The only significant difference between the two calculi is that the premise of the rule (CM) in Seq-MMI$^{\chi}_{\mathrm{CM}}$ requires that $\Sigma; \mathbb{M}; \Gamma; \Delta$ be a saturated history, but the rule (CM) applies in Seq-MMI$^{\chi}_{\mathrm{T}}$ only when no other rule applies. To ensure that "no other rule applies" implies that $\Sigma; \mathbb{M}; \Gamma; \Delta$ is a saturated history, we spread the negations of the conditions (1) and (5)–(12) from the definition of saturated history to rules other than (CM) as pre-conditions, called *applicability conditions* (written in $\boxed{\text{boxes}}$ in Figure 3). Conditions (2), (3) and (4) obviously hold when the rules (init), ($\top$R) and ($\bot$L) do not apply, respectively. Hence, when no rule other than (CM) applies, all 12 conditions of the definition of saturated history must hold, so $\Sigma; \mathbb{M}; \Gamma; \Delta$ must be a saturated history. It only remains to show that the calculus with these applicability conditions does not admit

$$\frac{\boxed{\text{No other rule applies}}}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow \{\mathrm{CM}(\Sigma;\mathbb{M};\Gamma;\Delta)\}}\ \mathrm{CM}$$

$$\frac{}{\Sigma;\mathbb{M}, x \leq y;\Gamma, x:p \Rightarrow^{\chi}_{\mathrm{T}} y:p,\Delta \searrow \{\}}\ \mathrm{init}$$

$$\frac{\begin{array}{c}\boxed{\forall y \in \Sigma.(x \leq y \in \mathbb{M}) \Rightarrow (y:\alpha \notin \Gamma \text{ or } y:\beta \notin \Delta)} \\ \boxed{\forall y \in \Sigma.(y \ll x) \Rightarrow (x = y \text{ or } x \not\preccurlyeq y)} \\ \Sigma, y;\mathbb{M}, x \leq y;\Gamma, y:\alpha \Rightarrow^{\chi}_{\mathrm{T}} y:\beta, x:\alpha \to \beta, \Delta \searrow S\end{array}}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:\alpha \to \beta, \Delta \searrow S}\ \to\mathrm{R}$$

$$\frac{\begin{array}{c}\boxed{y:\alpha \notin \Delta \text{ and } y:\beta \notin \Gamma} \\ \Sigma;\mathbb{M}, x \leq y;\Gamma, x:\alpha \to \beta \Rightarrow^{\chi}_{\mathrm{T}} y:\alpha, \Delta \searrow S_1 \\ \Sigma;\mathbb{M}, x \leq y;\Gamma, x:\alpha \to \beta, y:\beta \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S_2\end{array}}{\Sigma;\mathbb{M}, x \leq y;\Gamma, x:\alpha \to \beta \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S_1, S_2}\ \to\mathrm{L}$$

$$\frac{\begin{array}{c}\boxed{\forall y \in \Sigma.(xN_Ay \in \mathbb{M}) \Rightarrow y:\alpha \notin \Delta} \\ \boxed{\forall y \in \Sigma.(y \ll x) \Rightarrow (x = y \text{ or } x \not\preccurlyeq y)} \\ \Sigma, y;\mathbb{M}, xN_Ay;\Gamma \Rightarrow^{\chi}_{\mathrm{T}} y:\alpha, x:A \ \mathsf{nec}\ \alpha, \Delta \searrow S\end{array}}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} x:A \ \mathsf{nec}\ \alpha, \Delta \searrow S}\ \mathrm{necR}$$

$$\frac{\boxed{y:\alpha \notin \Gamma} \qquad \Sigma;\mathbb{M}, xN_Ay;\Gamma, x:A \ \mathsf{nec}\ \alpha, y:\alpha \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma;\mathbb{M}, xN_Ay;\Gamma, x:A \ \mathsf{nec}\ \alpha \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\ \mathrm{necL}$$

$$\frac{\boxed{xN_Az \notin \mathbb{M}} \qquad \Sigma;\mathbb{M}, x \leq y, yN_Az, xN_Az;\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}{\Sigma;\mathbb{M}, x \leq y, yN_Az;\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\ \mathrm{mon\text{-}N}$$

$$\frac{\begin{array}{c}(\forall \vec{x}.((\wedge_i\ (x_iR_ix'_i)) \to (xRx'))) \in \chi \\ x_iR_ix'_i \in \mathbb{M} \qquad \boxed{xRx' \notin \mathbb{M}} \qquad \Sigma;\mathbb{M}, xRx';\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S\end{array}}{\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S}\ \chi$$

Fig. 3. Seq-MMI$^{\chi}_{\mathrm{T}}$: Terminating, countermodel producing sequent calculus for MMI$^{\chi}$, selected rules. Applicability conditions are written in $\boxed{\text{boxes}}$. Wherever mentioned, the relation $\preccurlyeq$ is the equivalence relation of the contexts $\Sigma;\mathbb{M};\Gamma;\Delta$ in the conclusion of the rule. Similarly, $\ll$ is the order of the underlying tree of $\mathbb{M}$.

infinite backwards derivations, as in Theorem III.12.

Lemma III.10 (Correctness of CM)**.** *Let Σ, $\mathbb{M}$, Γ and Δ be such that $\mathbb{M}$ is tree-like and no rule except (CM) applies backwards to $\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{T} \Delta \searrow \ldots$. Then, $\Sigma;\mathbb{M};\Gamma;\Delta$ is a saturated history.*

Lemma III.11 (Tree-like $\mathbb{M}$)**.** *Let $\mathbb{M}$ be tree-like. Then, the $\mathbb{M}'$ in any sequent $\Sigma';\mathbb{M}';\Gamma' \Rightarrow^{\chi}_{T} \Delta' \searrow \ldots$ appearing in a backwards search starting from $\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{T} \Delta \searrow \ldots$ is tree-like.*

Note that the underlying tree of $\mathbb{M}$ in any sequent of a backward proof search starting from a single formula consists of exactly those edges that are introduced in one of the rules ($\to$R) and (necR).

Theorem III.12 (Termination)**.** *The following hold:*

1) *Any backwards search in Seq-MMI$^{\chi}_{T}$ starting from a sequent $\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{T} \Delta$ with $\mathbb{M}$ tree-like terminates.*
2) *For any $\Sigma;\mathbb{M};\Gamma;\Delta$ with $\mathbb{M}$ tree-like, there is an S such that $\vdash (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{T} \Delta \searrow S)$ and such an S can be finitely computed.*

Proof: By a straightforward counting argument. ∎

Note that Theorem III.12(2) does not stipulate that the S be unique. Indeed, depending on the order in which the rules of the calculus $\Rightarrow^{\chi}_{\mathrm{T}}$ are applied to a given sequent, S may be different. However, the fact that at least one such S exists and can be computed is enough to get decidability for MMI$^{\chi}$.

Lemma III.13 (Simulation)**.** *If $\mathbb{M}$ is tree-like and $\vdash (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{T} \Delta \searrow S)$, then $\vdash (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{CM} \Delta \searrow S)$.*

Proof: By induction on the given derivation of $\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{T}} \Delta \searrow S$. The case of rule (CM) follows from Lemma III.10. The rest of the cases are immediate from the i.h. The only fact to take care of is that the tree-like property holds for each i.h. application. This follows from Lemma III.11. ∎

Theorem III.14 (Decidability)**.** *For a tree-like $\mathbb{M}$, suppose that S is such that $\vdash (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{T} \Delta \searrow S)$ (such an S must exist and can be computed using Theorem III.12). Then: (1) If $S = \{\}$, then $\models (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi} \Delta)$. (2) If $S \neq \{\}$, then every model $\mathcal{M}$ in S is a countermodel to the sequent, i.e., $\mathcal{M} \not\models (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi} \Delta)$.*

Proof: By Lemma III.13, we have that $\vdash (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S)$. Now, (1) follows from Theorems III.8 and II.5 and (2) follows from Theorem III.9. ∎

Corollary III.15 (Decidability and finite model property)**.** *If a SCR exists for χ, then MMI$^{\chi}$ is decidable, has the finite model property and has a constructive decision procedure.*

E. Comprehensiveness of Constructed Countermodels

Countermodels generated by Seq-MMI$^{\chi}_{\mathrm{CM}}$ (and Seq-MMI$^{\chi}_{\mathrm{T}}$) have an interesting property: If $\vdash (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{\mathrm{CM}} \Delta \searrow S)$, then $\vdash (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi} x:p, \Delta)$ if and only if $\forall \mathcal{M} \in S.\ \mathcal{M} \models x:p$. Thus, if we can produce a set of countermodels S by running without an actual goal (like $x:p$), then the set of atoms that are actually true are exactly those that are in the intersection of the valuation of all models in the set S. Further, because the result applies to derivations in Seq-MMI$^{\chi}_{\mathrm{CM}}$, it also applies to derivations in Seq-MMI$^{\chi}_{\mathrm{T}}$ due to Lemma III.13 and the latter can be used to actually produce the set S. We call this result comprehensiveness.

Theorem III.16 (Comprehensiveness)**.** *Suppose $\vdash (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi}_{CM} \Delta \searrow S)$. Then $\vdash (\Sigma;\mathbb{M};\Gamma \Rightarrow^{\chi} x:p, \Delta)$ iff $\forall \mathcal{M} \in S.\ \mathcal{M} \models x:p$.*

F. Adding Seriality

In this section we show that if χ has a SCR, then our method applies not only to the logic MMI$^{\chi}$ (Corollary III.15), but also to the logic which, in addition, forces *seriality* with respect to some of its relations N_A. Seriality for index A is the condition $\forall x.\exists y.(xN_Ay)$. This corresponds to the axiom $\neg(A\ \mathsf{nec}\ \bot)$, also called D in literature [5]. Note that seriality does not fit our definition of χ because frame conditions in

χ cannot contain existentials, so it cannot be handled in the method described so far. Consequently, we must modify our method slightly to include seriality as a frame condition. The only new challenge is to control creation of worlds due to the seriality condition during backwards search; for this we use an approach similar to that for controlling the use of rules ($\rightarrow$R) and (necR). Proofs do not change significantly.

Suppose we wish to make relations N_A for $A \in \mathcal{J} \subseteq \mathcal{I}$ serial. We first add the following rule to our sequent calculus Seq-MMI$^\chi$ (Figure 1) for every $A \in \mathcal{J}$:

$$\frac{\Sigma, x, y; \mathbb{M}, xN_Ay; \Gamma \Rightarrow^\chi \Delta}{\Sigma, x; \mathbb{M}; \Gamma \Rightarrow^\chi \Delta}\,\mathrm{D}$$

Next, we change clause (1) of the definition of saturated history *not* to require closure under this new frame condition, which would cause infinite models, but instead new conditions based on $\preccurlyeq$:

(1) $\mathbb{M}$ is tree-like and saturated with respect to the conditions (refl), (trans), (mon-N) and χ. In addition, at least one of the following must hold for each $x \in \Sigma$ and each index $A \in \mathcal{J}$: (1a) There is a $y \in \Sigma$ such that $xN_Ay \in \mathbb{M}$, or (1b) There is a $y \in \Sigma$ such that $y \neq x$, $y \ll x$ and $x \preccurlyeq y$.

With this new clause (1), we can show by induction on $\ll$ that CM($\Sigma; \mathbb{M}; \Gamma; \Delta$) is closed under seriality for $A \in \mathcal{J}$, hence it is a model of our (modified) logic. Next, we add the following rule for every $A \in \mathcal{J}$ to the terminating calculus Seq-MMI$^\chi_{\mathrm{T}}$ and a corresponding rule without the applicability conditions to Seq-MMI$^\chi_{\mathrm{CM}}$.

$$\frac{\boxed{\forall y \in \Sigma.\ (xN_Ay \notin \mathbb{M})} \quad \boxed{\forall y \in \Sigma.(y \ll x) \Rightarrow (y = x \text{ or } x \not\preccurlyeq y)} \quad \Sigma, x, y; \mathbb{M}, xN_Ay; \Gamma \Rightarrow^\chi_{\mathrm{T}} \Delta \searrow S}{\Sigma, x; \mathbb{M}; \Gamma \Rightarrow^\chi_{\mathrm{T}} \Delta \searrow S}\,\mathrm{D}$$

With these changes, our entire development works with only minor changes to the proofs (interestingly, the proof of Lemma III.6 does not change).

Theorem III.17 (Constructive decidability with seriality)**.** *Let D contain seriality conditions for some set of indices and let the frame conditions χ have a SCR. Then the logic MMI$^{\chi,D}$ is constructively decidable by our method.*

G. Application to Common Intuitionistic Logics

In this Section, we list some common sets of frame conditions with their SCRs, thus showing that the intuitionistic logics corresponding to each of them is constructively decidable by our method. Unfortunately, SCRs are not modular: We cannot combine the SCRs for frame conditions χ_1 and χ_2 to get a SCR for the modal logic $\chi_1 \cup \chi_2$. As a result, we must explicitly construct a SCR for every modal logic of interest.

Figure 4 lists some common intuitionistic logics, their frame conditions χ, the corresponding axioms they admit and the corresponding SCRs. In cases where the name of the logic is not common, we have cited the source of the logic. We note two things: (1) This list is not exhaustive, but merely representative, and (2) Our method also applies to any of these logics combined with seriality from Section III-F due to Theorem III.17.

Theorem III.18 (Decidability of Common Logics)**.** *The intuitionistic logics shown in Figure 4 have the SCRs also shown in that figure. Consequently, all these logics (and their combination with the seriality condition from Section III-F) are constructively decidable by our method.*

IV. Classical Logic

Our decision procedure can be modified to apply to classical multi-modal logics as well. The overall approach of using SCRs and the structure of the calculi remains the same. However, because Kripke frames in classical logic do not require the preorder $\leq$, we must change the definition of CM($\Sigma; \mathbb{M}; \Gamma; \Delta$) (Definition III.4) that relies on $\leq$. This is not difficult: Instead of adding $x \leq y$ when $x \preccurlyeq y$, we add the relation xN_Az when yN_Az and $x \preccurlyeq y$. Our TR describes in detail our method as it applies to classical logics, together will all relevant proofs. In classical logic, the possibility modality of index A can be defined as the DeMorgan dual of the necessitation modality of the same index, so on classical logic, our method applies to possibility modalities as well.

V. Related Work

The applicability conditions of rules ($\rightarrow$R) and (necR) in Figure 3, based on the relation $\preccurlyeq$, are inspired by the work of Gasquet et al. [11] in which tableaux-based decision procedures are given for classical uni-modal logics with the following frame conditions: Transitivity, reflexivity, symmetry, Euclideanness, seriality and confluence. Our method is based on labeled sequent calculi and it applies to both classical and intuitionistic modal logics with any number of modalities. As a consequence, we had to develop new proof techniques to establish our results, particularly in the intuitionistic setting.

Our method of extracting countermodels is inspired by Negri's proof of completeness of labeled sequent calculi for uni-modal logic with respect to their Kripke semantics [18][19, Chapter 11]. In that proof, it is shown how to extract (possibly infinite) countermodels from non-terminating branches of a failed proof search taking the union of all $\mathbb{M}$ occurring along the branch. Here, instead, countermodels are built in the context of a decision procedure and *finite* countermodels are built by adding additional edges based on the relation $\preccurlyeq$ and saturating with respect to the frame conditions.

Boretti and Negri [6] develop a countermodel producing decision procedure similar to ours for a Priorean linear time fixed point calculus (a variant of linear time temporal logic, LTL), which also includes two rules like seriality. They also use a notion of saturation and construct countermodels from histories. The main difference between this work and [6] is that this work handles general frame conditions and, additionally, intuitionistic connectives. Boretti and Negri also discuss previous tableaux-style approaches to the generation of countermodels for LTL, such as [23].

Logic	Frame conditions χ	Additional Axioms	SCR
K	$\{\}$	–	$R(A) = (\leq)$
T	$\forall A, x.\ xN_Ax$	$(A\ \mathsf{nec}\ \alpha) \to \alpha$	$R(A) = (\leq)$
K4	$\forall A, x, y, z.\ ((xN_Ay) \wedge (yN_Az)) \to (xN_Az)$	$(A\ \mathsf{nec}\ \alpha) \to (A\ \mathsf{nec}\ A\ \mathsf{nec}\ \alpha)$	$R(A) = (N_A \cup \leq)$
S4	Conditions of K4 and T	Axioms of K4 and T	$R(A) = (N_A \cup \leq)$
I [9, 14]	$\forall A, B, x, y, z.\ ((xN_By) \wedge (yN_Az)) \to (xN_Az)$	$(A\ \mathsf{nec}\ \alpha) \to (B\ \mathsf{nec}\ A\ \mathsf{nec}\ \alpha)$	$R(A) = ((\cup_{B\in\mathcal{I}} N_B) \cup \leq)$
unit [7]	$\forall A, x, y.\ (xN_Ay) \to (x \leq y)$	$\alpha \to (A\ \mathsf{nec}\ \alpha)$	$R(A) = (\leq)$
–	$\forall x, y.\ (xN_Ay) \to (xN_By)$	$(B\ \mathsf{nec}\ \alpha) \to (A\ \mathsf{nec}\ \alpha)$	$R(A) = (\leq)$

Fig. 4. SCRs for some multi-modal intuitionistic logics. All these logics are constructively decidable by our method.

Countermodel producing sequent calculi, also known in the literature as "refutation calculi", have been given for intuitionistic logic, bi-intuitionistic logic, and provability logics [13, 15, 20] and in a way closer to the present paper's approach in [21]. One of the peculiarities of our method in relation to previous work is that the countermodel construction is made part of the calculus itself.

Gasquet and Said [12] introduce a technique called dynamic filtration to establish complexity bounds for the satisfiability problem of classical *layered modal logics* (LMLs), i.e., "logics characterized by semantic properties only stating the existence of possible worlds that are in some sense further than the other". Typically, such logics include confluence-like conditions, but they do not include transitivity-like conditions. Our work provides constructive decision procedures for a different and disjoint class of logics to which the techniques in [12] do not apply. In fact, with the exception of seriality, none of the frame conditions considered in this paper fall in the class of LMLs. Moreover, because LMLs cannot be applied with transitivity conditions it is not clear whether the techniques in [12] are suitable in the intuitionistic setting.

Simpson [24] presents decision procedures based on labeled sequent calculi for the intuitionistic uni-modal logics K, D, T and B together with S5. He leaves open the decidability of intuitionistic S4, K4 and KD4. Our method shows that the necessitation-only fragments of all three logics are decidable, not only in the uni-modal case, but also in the multi-modal case and, further, that the logics have constructive decision procedures.

Schmidt and Tishkovsky [22] present a general method for synthesizing sound and complete tableaux calculi given a semantic filtration [5] for the underlying classical modal logic. Although semantic filtrations are a powerful and general technique, their definition is not clear for many intuitionistic and multi-modal logics, so our method handles several logics that cannot be handled by Schmidt and Tishkovsky. To the best of our knowledge, the only work on filtrations for intuitionistic logics is limited to the uni-modal case [16] (the author of the paper notes in the conclusion that generalizing to the multi-modal case is not trivial). Filtration-based methods are also technically different from our syntactic approach. Whereas filtrations manipulate Kripke models, our method is purely syntactic and SCRs only work with sets of formulas generated during a specific backwards search. A consequence of this difference is that, for any of the logics considered in this paper, we have not been able to find a suitable filtration on the obvious model whose worlds are equivalence classes of $\preccurlyeq \cap \succcurlyeq$. In particular, it seems extremely difficult to satisfy the "back condition" of a filtration.

Alechina and Shakatov [2] present a general technique to prove decidability of intuitionistic (multi)-modal logics by embedding the relational definition of the semantics into Monadic Second Order Logic (MSO). As noted by the authors themselves and unlike our method, this approach does not give good decision procedures since it proceeds by reduction to satisfiability of formulas in SkS (monadic second-order theory of trees with constant branching factor k), which has non-elementary complexity. Moreover, the method applies to a logic only if its frame conditions can be expressed as acyclic closure conditions in MSO; this makes the method inapplicable to logics with frame conditions mentioning more than one modal relation, e.g., the logic (I) from Figure 4.

Negri [17] and Viganó [25] provide sound, complete and terminating labeled sequent calculi for uni-modal classical logics K, T, K4 and S4. Out method extends these results for a wider class of modal logics including axiom D, multiple modalities and intuitionistic logics.

Goré and Nguyen [14] present *non-labeled* tableaux calculi for seven types of classical multi-modal logics to reason about epistemic states of agents in distributed systems. The introduced tableaux require formulas of the logic to be first translated into a clausal form. We observe that one of the axioms presented in [14] is positive introspection for beliefs and corresponds to axiom (I) in Figure 4. To the best of our knowledge, [14] is the only work to provide decision procedures for logics including axiom (I).

VI. Conclusion and Future Work

We have presented a labeled sequent calculus-based, constructive decision procedure for several multi-modal logics, both intuitionistic and classical. Besides a novel construction of countermodels and a novel termination condition, we show, by uniform method, that several standard multi-modal logics without possibility modalities, as well as several logics with interactions between modalities, are decidable. We also show that our procedure has a novel comprehensiveness property.

We believe that our method can be extended to label-producing frame conditions more general than seriality. In particular, it should be possible to extend the technique of Section III-F to all layered modal logics of Gasquet and

Said [12], which we discussed in Section V. Second, we would like to either extend our method, or find a new one that can handle possibility modalities in intuitionistic logic.

Another direction of research is to exploit the embedding of intuitionistic modal logics into classical bi-modal logics to establish decidability results for the former, a direction pursued in [27] but with semantic rather than syntactic methods.

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