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ORIGINAL PAPER

Torical modification of Newton non-degenerate ideals

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To Professor Heisuke Hironaka on the occasion of his 80th birthday

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Abstract We give a definition of Newton non-degeneracy independent of the system of generators defining the variety. This definition extends the notion of Newton non-degeneracy to varieties that are not necessarily complete intersection. As in the previous definition of non-degeneracy for complete intersection varieties, it is shown that the varieties satisfying our definition can be resolved with a toric modification. Using tools of both toric and tropical geometry we describe the toric modification in terms of the Gröbner fan of the ideal defining the variety. The first part of the paper is devoted to introducing the classical concepts and the proof for the hypersurface case.

Keywords Newton non-degenerate · Tropical geometry · Singularity

Mathematics Subject Classification 14J17 · 14M25 · 14Txx

1 Introduction

Newton non-degenerate singularities were introduced in the 1970s [7,20,21]. For hypersurfaces, the definition is given in terms of the Newton polyhedron of the function defining the hypersurface. For complete intersection singularities they are characterized in terms of the Newton polyhedra of a given set of generators of the ideal. The reference book on the subject is [27].

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Newton non-degenerate singularities of hypersurfaces and of complete intersections have been widely studied (see for example [4,5,19,21,26,29]). A good resolution of a non-degenerate singularity may be constructed from the dual fan of the Newton boundaries.

In this paper we extend the definition of Newton non-degenerate to non-necessarily complete intersection singularities. Our definition is new and does not depend on the system of generators chosen and is given in terms of initial ideals.

The Gröbner fan of an ideal is the extension to non-principal ideals of the concept of fan dual to the Newton polyhedron. The tropical variety associated to a hypersurface H is the $(dim\ H - 1)$-skeleton of the fan dual to the Newton polyhedron of the function defining the hypersurface. Reference books on Tropical Geometry are [10,17,28].

In [1] an extension of the Newton–Puiseux method to compute parametrizations of plane curves is extended to arbitrary codimension replacing the Newton polyhedron by the tropical variety.

We prove that a regular refinement of the Gröbner fan of the ideal defining the non-degenerate variety gives a resolution. We also prove that the strict transform intersects the orbit associated to a cone if and only if the cone is contained in the tropical variety. Both results are original statements.

2 Cones and fans

In this section we introduce some basic concepts of convex geometry. These concepts may be found in several books (see for example [9]).

Given vectors $u^{(1)}, \ldots, u^{(k)} \in \mathbb{R}^n$. The *polyhedral cone* generated by $u^{(1)}, \ldots, u^{(k)}$ is the set

$$\sigma = \langle u^{(1)}, \ldots, u^{(k)} \rangle := \{\lambda_1 u^{(1)} + \cdots + \lambda_k u^{(k)}; \lambda_i \in \mathbb{R}_{\geq 0}, i = 1, \ldots, k\} \subset \mathbb{R}^n.$$

The vectors $u^{(i)}$'s are called the *generators* of the cone. A polyhedral cone is said to be *rational* if it has a set of generators in $\mathbb{Z}^n$.

We will denote by $e^{(1)}, \ldots, e^{(n)}$ the vectors in the standard base of $\mathbb{R}^n$. With this notation the first orthant is the cone $(\mathbb{R}_{\geq 0})^n = \langle e^{(1)}, \ldots, e^{(n)} \rangle$.

The cone generated by the columns of a matrix M will be denoted by $\langle M \rangle$.

The *dimension* of σ is the dimension of the minimal linear subspace $\mathcal{L}(\sigma)$ containing σ and is denoted by $dim(\sigma)$. The dimension of $\langle M \rangle$ is equal to the rank of the matrix M.

The *relative interior* of a cone σ is the interior of σ as a subset of $\mathcal{L}(\sigma)$. That is

$$Int_{rel}\langle u^{(1)}, \ldots, u^{(s)} \rangle = \{\lambda_1 u^{(1)} + \cdots + \lambda_s u^{(s)}; \lambda_i \in \mathbb{R}_{>0}\}.$$

The *dual* $\sigma^{\surd}$ of a cone σ is the cone given by

$$\sigma^{\surd} := \{v \in \mathbb{R}^n; v \cdot u \geq 0, \text{ for all } u \in \sigma\}$$

where $u \cdot v$ stands for the inner product $u \cdot v := u_1 v_1 + \cdots + u_n v_n$ of the vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$.

Let M be an unimodular matrix. We have

$$\langle M \rangle^{\surd} = \langle (M^{-1})^t \rangle \tag{1}$$

where M^t stands for the transpose of the matrix M.

A rational polyhedral cone is said to be *strongly convex* if it does not contain any non-trivial linear subspace. A cone contained in the first orthant is strongly convex. The dual of a cone of maximal dimension is strongly convex.

A vector in $\mathbb{Z}^n$ is said to be *primitive* when the maximum common divisor of its coordinates is 1.

The set of vectors $\{u^{(1)}, \ldots, u^{(k)}\} \subset \mathbb{Z}^n$ is the set of *vertices* of a rational strongly convex cone σ when

- $u^{(i)}$ is primitive for $i \in \{1, \ldots, k\}$.
- $\sigma = \langle u^{(1)}, \ldots, u^{(k)} \rangle$.
- $\langle u^{(1)}, \ldots, u^{(i-1)}, u^{(i+1)}, \ldots, u^{(k)} \rangle \subsetneq \sigma$ for $i = 1, \ldots, k$.

By $\sigma = Cone(u^{(1)}, \ldots, u^{(k)})$ we will denote the rational convex cone with vertices $u^{(1)}, \ldots, u^{(k)}$. We will also write $\sigma = Cone(M)$ where M is the matrix that has as columns the vertices of σ.

A strongly convex rational polyhedral cone $\sigma = Cone(u^{(1)}, \ldots, u^{(k)})$ is said to be *regular* when the group $\mathcal{L}(\sigma) \cap \mathbb{Z}^n$ is of rank k and is generated by the vertices of σ.

Remark 2.1 The vertices of a regular cone are linearly independent over $\mathbb{Q}$. An n-dimensional rational cone in $\mathbb{R}^n$ is regular if and only if $\sigma = Cone(M)$ where $M \in GL(\mathbb{Z}, n)$ is an unimodular matrix.

Remark 2.2 Let $\sigma = Cone(u^{(1)}, \ldots, u^{(k)})$ be a regular cone. The faces of σ are the cones $Cone(u^{(i_1)}, \ldots, u^{(i_s)})$ with $\{i_1, \ldots, i_s\} \subset \{1, \ldots, k\}$.

A collection of cones Σ in $\mathbb{R}^n$ is called a *polyhedral fan* if it satisfies the following properties:

(i) Every face of a cone in Σ is a cone in Σ;
(ii) The intersection of any two cones $\sigma, \tau \in \Sigma$ is a face of both σ and τ.

A polyhedral fan is said to be *regular* if all of its cones are regular.

Remark 2.3 Let Σ be a regular fan and let τ and σ be cones in Σ. By Remark 2.2, the cone τ is a face of σ if and only if the set of vertices of τ is a subset of the set of vertices of σ. That is $\tau = Cone(T)$ and $\sigma = Cone(M)$ where $M = (T, S)$.

A fan Σ' is a *refinement* of a fan Σ if every $\sigma \in \Sigma$ is a union of cones in Σ'. A refinement Σ' is called *regular* if every cone in Σ is regular.

Proposition 2.4 *Any fan has a regular refinement.*

The proof is left as an exercise in section 2.6 of [9].

3 Newton polyhedron

Let $\mathbb{K}$ be an algebraically closed field of any characteristic.

A polynomial in n variables with coefficients in $\mathbb{K}$ is an expression of the form

$$f(x) = \sum_{\mu \in \Omega \subset \mathbb{Z}^n_{\geq 0}} a_\mu x^\mu; \quad \#\Omega < \infty, \quad a_\mu \in \mathbb{K} \tag{2}$$

where $x^\mu := x_1^{\mu_1} \cdots x_n^{\mu_n}$.

The ring of polynomials in n variables with coefficients in $\mathbb{K}$ will be denoted by $\mathbb{K}[x_1, \ldots, x_n]$.

The *support* or *set of exponents* of f is defined by

$$\varepsilon(f) := \{\mu \in {\mathbb{Z}_{\geq 0}}^n; a_\mu \neq 0\}.$$

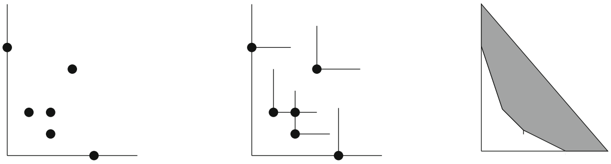


Fig. 1 $\varepsilon(f), \varepsilon(f)+\mathbb{R}^2_{\geq 0}, NP(f)$

Fig. 2 $NP(f), NP(h)$ where $f=x^3y^2h$

Remark 3.1 Given $f \in \mathbb{K}[x_1,\ldots,x_n]$, as in (2), we have $f(\underline{0})=a_{\underline{0}}$, and, then,

$$f(\underline{0}) \neq 0 \Longleftrightarrow \underline{0} \in \varepsilon(f).$$

The *Newton polyhedron* of $f \in \mathbb{K}[x_1,\ldots,x_n]$ is the convex hull (Fig. 1)

$$NP(f) := Conv(\{\mu + (\mathbb{R}_{\geq 0})^n;\ \mu \in \varepsilon(f)\}).$$

Remark 3.2 Given $f \in \mathbb{K}[x_1,\ldots,x_n]$, by Remark 3.1, $f(\underline{0}) \neq 0$ if and only if the Newton polyhedron of f is the first orthant.

Remark 3.3 Let f be a polynomial in $\mathbb{K}[x_1,\ldots,x_n]$. The Newton polyhedron of f has only one vertex if and only if

$$f = x^\alpha h$$

where h is a polynomial in $\mathbb{K}[x_1,\ldots,x_n]$ with $h(\underline{0}) \neq 0$, and α are the coordinates of the vertex of $NP(f)$ (see Fig. 2).

Let f be a polynomial as in (2) and let F be a face of the Newton polyhedron of f. The *restriction* of f to the set $F \subset \mathbb{Z}^n$ is defined as

$$f|_F := \sum_{\mu \in \varepsilon(f) \cap F \subset \mathbb{Z}^n} a_\mu x^\mu.$$

Remark 3.4 Let f be a polynomial in $\mathbb{K}[x_1,\ldots,x_n]$. We have

$$f|_{\langle e^{(1)},\ldots,e^{(s)}\rangle} = f(x_1,\ldots,x_s,0,\ldots,0).$$

4 The dual fan

Given $\omega \in (\mathbb{R}_{\geq 0})^n$ the *ω-order* of a non-zero polynomial in $\mathbb{K}[x_1,\ldots,x_n]$ is defined as

$$\nu_\omega(f) := \min\{\omega \cdot \mu;\ \mu \in \varepsilon(f)\}. \tag{3}$$

Let $f \in \mathbb{K}[x_1,\ldots,x_n]$ be a polynomial. Given a vector ω in the first orthant set

$$\pi_\omega(f) := \{x \in \mathbb{R}^n;\ \omega \cdot x = \nu_\omega(f)\}.$$

The hyperplane $\pi_\omega(f)$ is a supporting hyperplane for $NP(f)$.

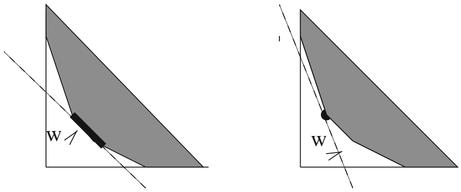


Fig. 3 $face_w(f)$

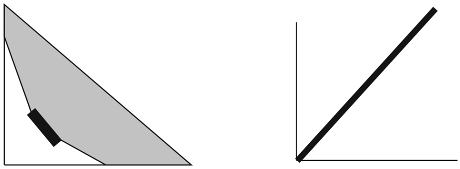


Fig. 4 F, C_F

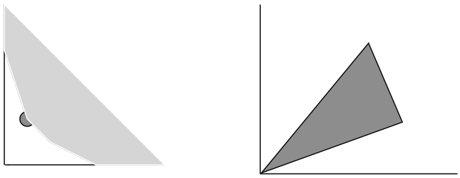


Fig. 5 $\mathbf{P}, C_{\mathbf{P}}$

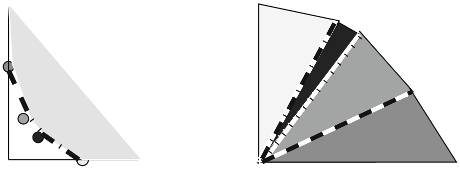


Fig. 6 $NP(f), \Sigma(f)$

The intersection

$$face_\omega(f) := \pi_\omega(f) \cap NP(f)$$

is a face of $NP(f)$ (see Fig. 3).

Given a face F of $NP(f)$ set (Figs. 4, 5)

$$C_F := \{\omega \in (\mathbb{R}_{\geq 0})^n;\ F \subset face_\omega(f)\}.$$

The collection of cones (Fig. 6)

$$\Sigma(f) := \{C_F;\ F \text{ is a face of } NP(f)\}$$

forms a fan.

Remark 4.1 Given $f \in \mathbb{K}[x_1, \ldots, x_n]$, if $f(\underline{0}) \neq 0$ then, by Remark 3.2, we have

$$\Sigma(f) = \{\langle e^{(i_1)}, \ldots, e^{(i_s)} \rangle;\ \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}\}.$$

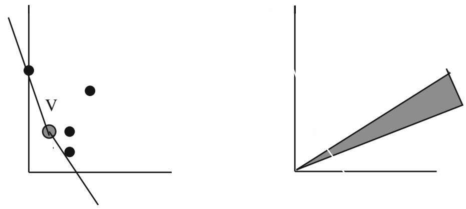
The mapping

$$\begin{aligned} \{\text{faces of } NP(f)\} &\longrightarrow \Sigma(f) \\ F &\mapsto C_F \end{aligned}$$

gives a duality.

For ω in the relative interior of the cone C_F we have $face_\omega(f) = F$.

Fig. 7 $\mathbf{P}$, $C_{\mathbf{P}}$



Remark 4.2 Given ω and ω' in the relative interior of the cone $\langle e^{(i_1)}, \ldots, e^{(i_s)} \rangle$. The equality $face_\omega f = face_{\omega'} f$ holds if and only if ω and ω' belong to the relative interior of the same cone of $\Sigma(f)$.

Remark 4.3 Let $\mathbf{P}$ be a vertex of $NP(f)$. We have (Fig. 7)

$$\varepsilon(f) \subset \mathbf{P} + C_{\mathbf{P}}{}^{\surd}.$$

Definition 4.4 We will say that σ is a *good cone* for f if it is contained in a cone of $\Sigma(f)$.

Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial and let σ be a good cone for f. We have that $face_\omega f = face_{\omega'} f$ for $\omega, \omega' \in Int_{rel}\sigma$. We define the σ-*face* of f as

$$face_\sigma f := face_\omega f \quad \text{where} \quad \omega \in Int_{rel}(\sigma).$$

Remark 4.5 Let σ be a good cone for f of maximal dimension. We have

$$\varepsilon(f) \subset \mathbf{P} + \sigma^{\surd}$$

where $\mathbf{P}$ is the vertex dual to the cone of $\Sigma(f)$ containing σ.

5 Monomial transformations

Given an unimodular matrix with integer entries, $M \in GL(n, \mathbb{Z})$, we will denote by ϕ_M the morphism given by

$$\begin{aligned} \phi_M : (\mathbb{K}^*)^n &\longrightarrow (\mathbb{K}^*)^n \\ z \quad &\longmapsto (z^{u^{(1)}}, z^{u^{(2)}}, \ldots, z^{u^{(n)}}) \end{aligned}$$

where $u^{(1)}, u^{(2)}, \ldots, u^{(n)}$ are the columns of the matrix M and $\mathbb{K}^* := \mathbb{K}\backslash\{0\}$.

The morphism ϕ_M is a bi-rational morphism on $\mathbb{K}^n$. It is bi-regular on the torus $(\mathbb{K}^*)^n \simeq \mathbb{T}^n$ and the following equalities are satisfied:

$$\phi_M \circ \phi_{M'} = \phi_{M'M} \tag{4}$$

$$(\phi_M)^{-1} = \phi_{M^{-1}}. \tag{5}$$

Given a vector $\mu \in \mathbb{Z}^n$, we have

$$\phi_M(x)^\mu = x^{\mu_1 u^{(1)}} \cdots x^{\mu_n u^{(n)}} = x_1{}^{\sum_{i=1}^n \mu_i u_1^{(i)}} \cdots x_n{}^{\sum_{i=1}^n \mu_i u_n^{(i)}} = x^{M\mu}. \tag{6}$$

We will denote by L_M, the linear map given by

$$\begin{aligned} L_M : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x \; &\longmapsto Mx. \end{aligned}$$

Let f be a polynomial in n variables as in (2), Eq. (6) implies

$$f \circ \phi_M(x) = \sum_{\mu \in \varepsilon(f)} a_\mu x^{M\mu} \tag{7}$$

then

$$\varepsilon(f \circ \phi_M) = M \cdot \varepsilon(f) = L_M(\varepsilon(f)). \tag{8}$$

Let π be a supporting hyperplane for $NP(f)$, then $L_M(\pi)$ is a supporting hyperplane for $NP(f \circ \phi_M)$, and

$$f \circ \phi_M|_{L_M(\pi) \cap NP(f \circ \phi_M)} = f|_{\pi \cap NP(f)} \circ \phi_M. \tag{9}$$

Lemma 5.1 *Let M be an unimodular matrix with integer entries, we have*

$$L_M((cone(M^t))^{\surd}) = (\mathbb{R}_{\geq 0})^n.$$

Proof The dual $cone(M^t)^{\surd}$ is generated by $\langle v^{(1)}, \ldots, v^{(n)} \rangle$, where, by (1), $v^{(1)}, \ldots, v^{(n)}$ are the columns of the matrix M^{-1}.

$$\begin{aligned} L_M((cone(M^t))^{\surd}) &= \{L_M(t_1 v^{(1)} + \cdots + t_n v^{(n)})\,;\ t_i \in \mathbb{R}_{\geq 0}\} \\ &= \{t_1(Mv^{(1)}) + \cdots + t_n(Mv^{(n)})\,;\ t_i \in \mathbb{R}_{\geq 0}\} \\ &= \{t_1(1, 0, \ldots, 0) + \cdots + t_n(0, \ldots, 0, 1)\,;\ t_i \in \mathbb{R}_{\geq 0}\} \\ &= \{(t_1, \ldots, t_n)\,;\ t_i \in \mathbb{R}_{\geq 0}\} = (\mathbb{R}_{\geq 0})^n. \end{aligned}$$

Hence we have the required result.

Proposition 5.2 *Let $M \in GL(n, \mathbb{Z})$ be such that $\sigma = cone(M^t)$ is good for f, then $NP(f \circ \phi_M)$ has only one vertex (i.e. $\mathbb{R}^n_{\geq 0}$ is good for $f \circ \phi_M$).*

Proof Suppose that $\sigma = cone(M^t)$ is good for f, then, by Remark 4.5,

$$\varepsilon(f) \subset \mathbf{P} + \sigma^{\surd} = \mathbf{P} + (cone(M^t))^{\surd} \tag{10}$$

with $\mathbf{P} \in \varepsilon(f)$.

By (8) and (10),

$$\varepsilon(f \circ \phi_M) \subset L_M(\mathbf{P}) + L_M((cone(M^t))^{\surd}) = L_M(\mathbf{P}) + (\mathbb{R}_{\geq 0})^n$$

where the equality follows from Lemma 5.1.

Now, since $\mathbf{P}$ is in the support of f, the point $L_M(\mathbf{P})$ is in the support of $f \circ \phi_M$. Then, $L_M(\mathbf{P})$ is the only vertex of $NP(f \circ \phi_M)$, and we have the result.

Proposition 5.3 *Let $M \in GL(n, \mathbb{Z})$ be such that $\sigma = Cone(M)$ is good for f. Given $\lambda \subset \{1, \ldots, n\}$, let τ be the face of σ generated by the i^{th} columns of M, with $i \in \lambda$. We have*

$$f|_{face_\tau(f)} \circ \phi_{M^t} = f \circ \phi_{M^t}|_{\mathbf{P} + Cone(\{e^{(i)}\}_{i \in \lambda C})}$$

where $\mathbf{P}$ is the only vertex of $NP(f \circ \phi_{M^t})$.

Proof Consequence of Eq. (9) and Proposition 5.2.

6 Toric modification

In this section we will recall how to construct the modification associated to a regular fan Σ with support in the first orthant (see for example [27, Cap.2]).

Let $\mathfrak{M}$ be the set of cones in Σ of maximal dimension. Let $\sigma \in \Sigma$ be a cone in $\mathfrak{M}$. We will associate to σ one copy of the affine space $\mathbb{K}^n$ and we will denote it by U_σ. Consider the disjoint union

$$\mathfrak{C} = \bigsqcup_{\sigma \in \mathfrak{M}} U_\sigma. \tag{11}$$

Let $\sigma = Cone(M)$ and $\sigma' = Cone(M')$ be cones in $\mathfrak{M}$.

Consider

$$\phi_{(M^{-1})^t} : (\mathbb{K}^*)^n \longrightarrow U_\sigma \quad \text{and} \quad \phi_{(M'^{-1})^t} : (\mathbb{K}^*)^n \longrightarrow U_{\sigma'}.$$

The composition $\phi_{(M'^{-1})^t} \circ \phi^{-1}_{(M^{-1})^t} = \phi_{M^t \cdot (M'^{-1})^t} : U_\sigma \dashrightarrow U_{\sigma'}$ is a bi-rational morphism.

An equivalence relation is defined in $\mathfrak{C}$ as follows: given two points $u_\sigma \in U_\sigma$ and $u'_{\sigma'} \in U_{\sigma'}$, $u_\sigma \sim u'_{\sigma'}$ if and only if $\phi_{M^t (M'^{-1})^t}$ is bi-regular on u_σ and $\phi_{M^t \cdot (M'^{-1})^t}(u_\sigma) = u'_{\sigma'}$.

The quotient of $\mathfrak{C}$ under this equivalence relation is a smooth variety, X_Σ, called the *toric variety* associated to Σ. Each U_σ is a *chart* of X_Σ.

For each $\sigma = Cone(M) \in \mathfrak{M}$ the mapping

$$\begin{array}{rccc} \pi|_{U_\sigma} : & U_\sigma & \longrightarrow & \mathbb{K}^n \\ & x & \mapsto & \phi_{M^t}(x) \end{array}$$

is a regular morphism, compatible with the gluing.

The induced regular morphism

$$\pi : X_\Sigma \longrightarrow \mathbb{K}^n$$

is called the *toric modification* associated to Σ. The morphism π is a proper bi-rational morphism and it is bi-regular in the torus $(\mathbb{K}^*)^n$.

Given a variety $V \subset \mathbb{K}^n$, the inverse image $\pi^{-1}(V)$ is called the *total transform* of V under the projection π.

Let $V^* \subset V$ be the set of points in V lying outside the coordinate hyperplanes, that is,

$$V^* := V - V(x_1 \cdots x_n).$$

The *strict transform* of V under the projection π is defined by

$$\widetilde{V} := \overline{\pi^{-1}(V^*)},$$

where $\overline{A}$ denotes the Zariski closure of A.

Given $\sigma \in \mathfrak{M}$, there is a natural action

$$\begin{array}{ccc} (\mathbb{K}^*)^n \times U_\sigma & \longrightarrow & U_\sigma \\ (\lambda_1, \dots, \lambda_n) \times (z_1, \dots, z_n) & \mapsto & (\lambda_1 z_1, \dots, \lambda_n z_n) \end{array}$$

that is compatible with the gluing in X_Σ.

Given $\lambda \subset \{1, \dots, n\}$ and $\sigma = Cone(M) \in \mathfrak{M}$, let τ be the cone generated by the ith columns of M with $i \in \lambda$.

The *orbit* associated to τ is the $(n - \#\lambda)$-dimensional torus given by

$$\mathcal{O}(\tau) := \{(x_1, \dots, x_n); x_i = 0 \text{ for } i \in \lambda \text{ and } x_i \neq 0 \text{ for } i \notin \lambda\}.$$

The set $\mathcal{O}(\tau)$ is well defined (it does not depend on the choice of σ) and is an orbit of the natural action on X_Σ.

Remark 6.1 Let $\sigma = Cone(M) \in \Sigma$ be of maximal dimension. Let τ be the cone in Σ generated by the first s columns of M. And let $h : U_\sigma \cap X_\Sigma \longrightarrow \mathbb{K}$ be a regular function.

By Remark 3.4, we have

$$h|_{\mathcal{O}(\tau)\cap U_\sigma} = h|_{Cone(e^{(s+1)},\dots,e^{(n)})}.$$

7 Newton non-degenerate hypersurfaces

A hypersurface singularity $\underline{0} \in H = V(f)$ is said to be *Newton non-degenerate* when, for any face F of $NP(f)$, the hypersurface $V(f|_F)$ has no singularities outside the coordinate hyperplanes. In this section we will prove that a Newton non-degenerate hypersurface singularity has a toric resolution. This result is proved in several texts (see for example [24,32]).

Let Σ be a regular refinement of $\Sigma(f)$, let $\{\underline{0}\} \neq \tau$ be a cone in Σ, and let y be a point in the strict transform of H under the toric modification given by Σ with

$$y \in \widetilde{H} \cap \mathcal{O}(\tau) \subset X_\Sigma.$$

Choose $\sigma = Cone(M) \in \mathfrak{M}$ such that the first s columns of M generate τ. We have

$$y = (0, \dots, 0, y_{s+1}, \dots, y_n) \in U_\sigma$$

with $y_i \neq 0$ for $i \geq s+1$.

Since $Cone(M)$ is good for f, by Proposition 5.2, $NP(f \circ \phi_{M^t})$ has only one vertex $\mathbf{P}$. By Remark 3.3, we have

$$f \circ \phi_{M^t} = x^{\mathbf{P}} h$$

with $h(\underline{0}) \neq 0$.

Then

$$\widetilde{H} \cap U_\sigma = V(h)$$

and, since $\underline{0} \notin V(h)$, $\tau \neq \sigma$.

By Remark 6.1

$$h|_{\mathcal{O}(\tau)\cap U_\sigma} = h|_{Cone(e^{(s+1)},\dots,e^{(n)})}, \tag{12}$$

thus, the function $h|_{Cone(e^{(s+1)},\dots,e^{(n)})}$ does not depend on the first s variables, then

$$h(y) = h|_{Cone(e^{(s+1)},\dots,e^{(n)})}(1, \dots, 1, y_{s+1}, \dots, y_n).$$

By Proposition 5.3,

$$x^{\mathbf{P}} h|_{Cone(e^{(s+1)},\dots,e^{(n)})} = f \circ \phi_{M^t}|_{\mathbf{P}+Cone(e^{(s+1)},\dots,e^{(n)})} = f|_{face_\tau(f)} \circ \phi_{M^t}.$$

The hypersurface, $V(f|_{face_\tau(f)})$ has no singularities outside the coordinate hyperplanes, then, $V(h|_{Cone(e^{(s+1)},\dots,e^{(n)})})$ is non-singular in $(1, \dots, 1, y_{s+1}, \dots, y_n)$. That is, there exists $j \in \{1, \dots, n\}$ such that

$$\frac{\partial h|_{Cone(e^{(s+1)},\dots,e^{(n)})}}{\partial x_j}(1, \dots, 1, y_{s+1}, \dots, y_n) \neq 0.$$

Since $h|_{Cone(e^{(s+1)},\dots,e^{(n)})}$ does not depend on the first s coordinates, we have $j \geq s+1$.

By Eq. (12)

$$\frac{\partial h}{\partial x_j}(y) = \frac{\partial h|_{Cone(e^{(s+1)},\ldots,e^{(n)})}}{\partial x_j}(y) = \frac{\partial h|_{Cone(e^{(s+1)},\ldots,e^{(n)})}}{\partial x_j}(1,\ldots,1,y_{s+1},\ldots y_n) \neq 0$$

that is, y is not a singular point of $\widetilde{H}$ and $\widetilde{H}$ is transversal to $\mathcal{O}(\tau)$.

8 The Gröbner fan

The Gröbner fan is the extension of the concept of dual fan to non-principal ideals. All this may be computed using the program GFAN [18].

Given $\omega \in (\mathbb{R}_{\geq 0})^n$, the *$\omega$-initial part* of f is

$$In_\omega f := f|_{face_\omega(f)}.$$

That is, given f as in (2),

$$In_\omega f = \sum_{\{\mu \in \varepsilon(f);\, \omega\cdot\mu = \nu_\omega f\}} a_\mu x^\mu.$$

Remark 8.1 Given $f \in \mathbb{K}[x_1,\ldots,x_n]$, if $f(\underline{0}) \neq 0$ then $\nu_\omega(f) = 0$ for all $\omega \in (\mathbb{R}_{\geq 0})^n$. And $In_\omega f = f(\underline{0})$ for all $\omega \in (\mathbb{R}_{>0})^n$.

Let $I \subset \mathbb{K}[x_1,\ldots,x_n]$ be an ideal. Given $\omega \in (\mathbb{R}_{\geq 0})^n$, the *$\omega$-initial ideal* of I is the ideal generated by the ω-initial part of every polynomial in I, that is,

$$\mathfrak{In}_\omega I := \langle In_\omega f;\, f \in I\rangle. \tag{13}$$

The variety $V(\mathfrak{In}_{(1,\ldots,1)} I)$ is classically called the *tangent cone* of $V(I)$.

In the light of Remark 4.2, there is a natural way to associate a polyhedral fan to an ideal. In $\mathbb{R}^n$, we define an equivalence relation as follows:

$$\omega \sim \omega' \Leftrightarrow \mathfrak{In}_\omega I = \mathfrak{In}_{\omega'} I.$$

Given $\omega \in \mathbb{R}^n$, the closure in $\mathbb{R}^n$ of its equivalence class, that is,

$$C_\omega(I) := \overline{\{u \in \mathbb{R}^n;\, \mathfrak{In}_u I = \mathfrak{In}_\omega I\}},$$

is a polyhedral cone and the collection

$$\Sigma'(I) := \{C_\omega(I);\, \omega \in \mathbb{R}^n\}$$

forms a fan ([25]).

Since we are interested in a local study of a variety, we will use the fan formed by the intersection of this fan and the first orthant, that is,

$$\Sigma(I) = \{\Sigma'(I) \bigcap E_J;\, J \subset \{1,2,\ldots,n\}\} \subset (\mathbb{R}_{\geq 0})^n$$

where $E_J = Cone(\{e^{(j)};\, j \in J\})$ and $e^{(j)}$ is the jth element of the standard basis for $\mathbb{R}^n$. We will refer to $\Sigma(I)$ as the *Gröbner fan* of the ideal I. The cones in the Gröbner fan of I will be called *Gröbner cones* of I.

Given $f \in \mathbb{K}[x_1,\ldots,x_n]$, we have

$$\Sigma(f) = \Sigma(\langle f\rangle).$$

Remark 8.2 Given $\omega, \omega' \in Int_{rel}\langle e^{(i_1)}, \ldots, e^{(i_s)}\rangle$ we have that the initial ideals $\mathfrak{In}_\omega I$ and $\mathfrak{In}_{\omega'} I$ are equal if and only if $C_\omega(I) = C_{\omega'}(I)$.

Remark 8.3 Let σ be a cone in $\Sigma(I)$ not contained in the coordinate hyperplanes. The cone σ is of maximal dimension if and only if $\mathfrak{In}_\sigma I$ is a monomial ideal.

Definition 8.4 We will say that σ is a *good cone* for I if it is contained in a Gröbner cone of I.

Let $I \subseteq \mathbb{K}[x_1, x_2, \ldots, x_n]$ be an ideal and let σ be a good cone for I. We define

$$\mathfrak{In}_\sigma I := \mathfrak{In}_\omega I \quad \text{where} \quad \omega \in Int_{rel}(\sigma).$$

Remark 8.5 Let $I \subseteq \mathbb{K}[x_1, x_2, \ldots, x_n]$ be an ideal, let σ be a good cone for I and let τ be a face of σ. Then, τ is a good cone for I and

$$\mathfrak{In}_\sigma \mathfrak{In}_\tau I = \mathfrak{In}_\sigma I.$$

9 Reduced Gröbner basis

Given a term order, $\prec$, we define the *initial term*, $In_\prec f$, of a non-zero polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$, as the unique minimal term with respect to $\prec$.

Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. The initial ideal of I with respect to $\prec$ is the ideal generated by the initial terms of the polynomials in I, that is,

$$\mathfrak{In}_\prec I := \langle In_\prec f; \, f \in I\rangle.$$

The closure of the equivalence class:

$$C_\prec(I) := \overline{\{u \in \mathbb{R}^n; \, \mathfrak{In}_u I = \mathfrak{In}_\prec I\}} \tag{14}$$

is a Gröbner cone of maximal dimension.

Proposition 9.1 *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal and let $\prec$ be a term order. There exist a set $G_\prec(I) = \{g_1, \ldots, g_r\} \subset I$ (called reduced Gröbner basis) such that*

(i) *$C_\prec(I)$ is a good cone for g_i, $i \in \{1, \ldots, r\}$.*
(ii) *$\mathfrak{In}_\upsilon(I) = \langle In_\upsilon(g_1), \ldots, In_\upsilon(g_r)\rangle$, for all $\upsilon \in C_\prec(I)$.*

Proof (i) Direct consequence of [8, Prop 2.6].
(ii) Direct consequence of [8, Coro 2.14].

Given $\omega \in (\mathbb{R}_{\geq 0})^n$, we define a total order $\prec_\omega$ in $\mathbb{K}[x_1, \ldots, x_n]$ by,

$$x^\alpha \prec_\omega x^\beta \Leftrightarrow \alpha \cdot \omega < \omega \cdot \beta \quad \text{or} \quad \alpha \cdot \omega = \omega \cdot \beta \quad \text{and} \quad \alpha \prec_{lex} \beta, \tag{15}$$

where $\prec_{lex}$ is the lexicographical order.

Proposition 9.2 *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. Given $\sigma \in \Sigma(I)$ with $dim(\sigma) = n$ and $\omega \in Int_{rel}(\sigma)$ we have that $\sigma = C_{\prec_\omega}(I)$.*

Proof Take $\omega \in Int_{rel}(\sigma)$ and let $\prec_\omega$ be as in (15), then, by definition, we have $\mathfrak{In}_{C_{\prec_\omega}}(I) = \mathfrak{In}_{\prec_\omega}(I)$ and $\mathfrak{In}_\omega(I) = \mathfrak{In}_\sigma(I)$. Also, $\mathfrak{In}_{\prec_\omega}(I) = \mathfrak{In}_{\prec_\omega}\mathfrak{In}_\omega(I)$ (by Fukuda et al. [8, Lemma 2.13]) and $\mathfrak{In}_{\prec_\omega}\mathfrak{In}_\omega(I) = \mathfrak{In}_\omega(I)$ (by Remark 8.3).
All above equations imply $\mathfrak{In}_{C_{\prec_\omega}}(I) = \mathfrak{In}_\sigma(I)$.

Proposition 9.3 *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal and let σ be a good cone for I of maximal dimension. There exists a system of generators $G_\sigma = \{g_1, \ldots, g_r\}$ of I such that σ is a good cone for g_i for $i \in \{1, \ldots, r\}$ and*

$$\mathfrak{In}_\upsilon(I) = \langle In_\upsilon g_1, \ldots, In_\upsilon g_r \rangle$$

for all $\upsilon \in \sigma$.

Proof Let $\sigma' \in \Sigma(I)$ be such that $\sigma \subset \sigma'$ (exists by definition of good cone). Take $\omega \in Int_{rel}\sigma'$ and consider the term order $\prec_\omega$. By Proposition 9.2 we have

$$\sigma \subset \sigma' = C_{\prec_\omega}(I). \tag{16}$$

By Proposition 9.1, there exist $g_1, \ldots, g_s$, a system of generators of I, such that $C_{\prec_\omega}(I)$ is good for each g_i. The result follows from (16).

10 Tropical variety

Definition 10.1 Given an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$, *the tropical variety associated to I* is

$$\mathbf{TV}(I) := \{\omega \in \mathbb{R}^n;\ \mathfrak{In}_\omega(I) \text{ contains no monomial}\}.$$

Proposition 10.2 *An ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ contains no monomials if and only if $V(I) \cap (\mathbb{K}^*)^n \neq \emptyset$.*

Proof $\Leftarrow$] If $ax^\alpha \in I$ and $z \in V(I)$, then $az_1^{\alpha_1} \cdots z_n^{\alpha_n} = 0$ so there exist i such that $z_i = 0$. Hence, $z \notin (\mathbb{K}^*)^n$.
$\Rightarrow$] Suppose that $V(I) \cap (\mathbb{K}^*)^n = \emptyset$. Then $V(I) \subset \bigcup_{i=1}^n \{x_i = 0\}$. The monomial $x^{(1,\ldots,1)}$ satisfies $V(x^{(1,\ldots,1)}) \supset V(I)$ and, by the Nullstellensatz, there exists k such that $x^{(k,\ldots,k)} \in I$, i.e. I contains a monomial.

Proposition 10.2 implies

$$\mathbf{TV}(I) = \{\omega \in \mathbb{R}^n;\ V(\mathfrak{In}_\omega(I)) \cap (\mathbb{K}^*)^n \neq \emptyset\}. \tag{17}$$

Proposition 10.3 *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. Given $\omega \in Int_{rel}(\sigma)$ with $\sigma \in \Sigma(I)$ a cone. We have*

$$\omega \in \mathbf{TV}(I) \Longleftrightarrow \sigma \subset \mathbf{TV}(I).$$

Proof $\Leftarrow$ Suppose that σ is contained in $\mathbf{TV}(I)$ then $\omega \in Int_{rel}(\sigma) \subset \sigma$, implies $\omega \in \mathbf{TV}(I)$.
$\Rightarrow$ Let $\sigma \subset \Sigma(I)$ be a cone. Let $\omega \in Int_{rel}(\sigma)$ be such that $\omega \in \mathbf{TV}(I)$.

- Take $\omega' \in Int_{rel}(\sigma)$, by Remark 8.2 $\omega' \in \mathbf{TV}(I)$.
- Take $\nu \in \sigma \backslash Int_{rel}(\sigma)$. There exist $\tau \in \Sigma(I)$ face of σ such that $\nu \in Int_{rel}(\tau)$. By duality we have

$$\varepsilon(In_\omega(f)) = \varepsilon(In_\sigma(f)) \subset \varepsilon(In_\tau(f)) = \varepsilon(In_\nu(f)).$$

Then, if $In_\omega f$ is not a monomial neither is $In_\nu f$.

From the last proposition, it follows that the tropical variety is a union of cones in the Gröbner fan.

Theorem 10.4 Bieri–Groves [3,30] *Let $\mathbb{K}$ be algebraically closed field and let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be a monomial-free prime ideal. Then the tropical variety $\mathbf{TV}(I)$ can be written as a finite union of polyhedra of dimension $dim\, V(I)$.*

Proposition 10.5 *Let $\mathbb{K}$ be an algebraically closed field and let $V = V(I) \subset \mathbb{K}^n$ be a pure dimensional variety not contained in the coordinate hyperplanes. Given $\omega \in \mathbf{TV}(I)$, we have*

$$dim\, \mathbf{TV}(I) = dim\, \mathbf{TV}(\mathfrak{In}_\omega I).$$

Proof Suppose that $\nu \notin \mathbf{TV}(I)$ then, there exists $f \in I$ such that $In_\nu f$ is a monomial and, then, $In_\nu In_\omega f$ is also a monomial. Since $In_\omega f$ is in $\mathfrak{In}_\omega I$ we have $\nu \notin \mathbf{TV}(\mathfrak{In}_\omega I)$. This gives the inclusion

$$\mathbf{TV}(\mathfrak{In}_\omega I) \subset \mathbf{TV}(I).$$

Now, let d be the dimension of $\mathbf{TV}(I)$. Given ω in the tropical variety $\mathbf{TV}(I)$, by Theorem 10.4, there exists a cone σ that is a good cone for I with $\omega \in \sigma$ and $dim\, \sigma = d$. By Remark 8.5, $\mathfrak{In}_\sigma \mathfrak{In}_\omega I = \mathfrak{In}_\sigma I$. Since $\mathfrak{In}_\sigma I$ is not a monomial we have $\sigma \subset \mathbf{TV}(\mathfrak{In}_\omega I)$.

Then

$$dim\, \mathbf{TV}(\mathfrak{In}_\omega I) \geq \mathbf{TV}(I).$$

And the result is proved.

Proposition 10.6 *Let $\mathbb{K}$ be an algebraically closed field and let $V = V(I) \subset \mathbb{K}^n$ be a pure dimensional variety not contained in the coordinate hyperplanes. Given $\omega \in \mathbf{TV}(I)$, we have*

$$dim\, (V \cap \mathbb{K}^{*n}) = dim\, \mathbf{TV}(I) = dim\, \mathbf{TV}(\mathfrak{In}_\omega I).$$

Proof Direct consequence of Theorem 10.4 and Proposition 10.5.

11 Newton non-degenerate varieties

In this section we will extend the concept of Newton non-degenerate variety to non-principal ideals. For complete intersections the concept was extended by Khovanskii [19,20] in 1976.

Definition 11.1 Let $V = V(f_1, \ldots, f_k) \subset \mathbb{K}^n$ be a variety of dimension $n-k$. The variety V is *Newton non-degenerate* if for any $\omega \in (\mathbb{R}_{\geq 0})^n$ the variety

$$V(In_\omega f_1, In_\omega f_2, \ldots, In_\omega f_k)$$

is of dimension $n-k$ and has no singularities in $(\mathbb{K}^*)^n$.

Let I be the ideal generated by $f_1, \ldots, f_k$, then, $V(f_1, \ldots, f_k) = V(I)$. Given an other set of generators of I, say, $f'_1, \ldots, f'_k$. The ideals

$$\langle In_\omega f_1, In_\omega f_2, \ldots, In_\omega f_k \rangle \quad \text{and} \quad \langle In_\omega f'_1, In_\omega f'_2, \ldots, In_\omega f'_k \rangle$$

are not necessarily the same. Definition 11.1 depends strongly on the generators of the ideal I chosen.

The definition we propose extends the definition above to non-complete intersection singularities and does not depend on the generators.

Definition 11.2 A singularity $\underline{0} \in V(I) \subset \mathbb{K}^n$, where $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is an ideal, is *Newton non-degenerate* if for every $\omega \in (\mathbb{R}_{\geq 0})^n$, the variety defined by $\mathfrak{In}_\omega I$ does not have singularities in $(\mathbb{K}^*)^n$.

To check if an ideal is Newton non-degenerate or not, it is enough to check the condition for one vector in the relative interior of each cone of the Gröbner fan that is contained in the tropical variety associated to I. These computations may be done using the software GFAN [18].

Let $I = \langle f_1, f_2, f_3 \rangle \subset \mathbb{C}[x, y, z, w]$ where

$$f_1 = xy + xw - yw, \quad f_2 = xz - w^2 \quad \text{and} \quad f_3 = yz - yw - w^2.$$

The zero set $V(I)$ is a surface that has a rational singularity of multiplicity three (see [22] or [34]).

In [11] the Gröbner fan and the different initial ideals are computed using GFAN [18]. Then, their singularities are computed using SINGULAR [6] concluding that it is a Newton non-degenerate singularity in the sense of Definition 11.2.

Let $V = V(f_1, \ldots, f_k) \subset \mathbb{K}^n$ be Newton non-degenerate variety in the sense of Definition 11.1. Let Σ be a regular refinement of the dual fan $\Sigma(f_i)$ defined by f_i for $i = 1, \ldots, k$ and let $\pi : X_\Sigma \longrightarrow \mathbb{K}^n$ be the torical modification associated to Σ. Let $\widetilde{V}$ be the strict transform of V. It is well-known [27] that the restriction $\pi : \widetilde{V} \longrightarrow V$ is a good resolution of V. The following sections are devoted to show the corresponding result for Newton non-degenerate varieties in the sense of our definition (Definition 11.2).

12 Tropical variety and strict transform

Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal, let Σ be a regular refinement of the Gröbner fan $\Sigma(I)$, and let $\widetilde{V(I)}$ be the strict transform of $V(I)$ under the modification

$$\pi : X_\Sigma \longrightarrow \mathbb{K}^n.$$

Let $\sigma = Cone(M) \in \Sigma$ be a cone of maximal dimension and let G_σ be a system of generators as in Proposition 9.3.

We have

$$\phi_{M^t *}(I) = \langle \{g \circ \phi_{M^t}\}_{g \in G_\sigma} \rangle. \tag{18}$$

Take $g \in G_\sigma$. Since $Cone(M)$ is good for g, by Proposition 5.2 and Remark 3.3, we have

$$g \circ \phi_{M^t} = x^{\mathbf{P}} h \tag{19}$$

with $h(\underline{0}) \neq 0$.

Set

$$H_{G_\sigma} := \{h \in \mathbb{K}[x_1, \ldots, x_n];\, g \circ \phi_{M^t} = x^{\mathbf{P}} h \text{ for some } g \in G_\sigma \text{ and } h(\underline{0}) \neq 0\}. \tag{20}$$

We have

$$\phi_{M^t *}(I) \subset \langle H_{G_\sigma} \rangle \quad \text{and} \quad V(\phi_{M^t *}(I)) \cap \mathbb{K}^{*n} = V(H_{G_\sigma}) \cap \mathbb{K}^{*n},$$

then

$$U_\sigma \cap \widetilde{V(I)} \subseteq V(H_{G_\sigma}) \overset{(23)}{\subseteq} U_\sigma \cap \pi^{-1}(V(I)). \tag{21}$$

In the following section we will see that, when I is Newton non-degenerate, $V(H_{G_\sigma})$ is non-singular and transversal to $\pi^{-1}(\underline{0})$. As a consequence, the first inclusion is actually an equality.[1]

Proposition 12.1 *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal, let Σ be a regular refinement of the Gröbner fan $\Sigma(I)$, and let $\widetilde{V(I)}$ be the strict transform of $V(I)$ under the modification given by Σ. Given $\tau \in \Sigma$ we have*

$$\widetilde{V(I)} \cap \mathcal{O}(\tau) \neq \emptyset \Rightarrow \tau \subset \mathbf{TV}(I).$$

Proof Given $\tau \in \Sigma$, take $\sigma = Cone(M) \in \Sigma$, of maximal dimension, such that τ is generated by the first s columns of M.

Take $y \in \widetilde{V(I)} \cap \mathcal{O}(\tau)$, we have

$$y = (0, \ldots, 0, y_{s+1}, \ldots, y_n) \in U_\sigma, \quad \text{with } \; y_i \neq 0 \text{ for } \; i \in \{s+1, \ldots, n\}.$$

Set

$$y' = (1, \ldots, 1, y_{s+1}, \ldots, y_n) \in U_\sigma \cap (\mathbb{K}^*)^n.$$

Let G_σ be a system of generators as in Proposition 9.3. Let H_{G_σ} be as in (20). Take $g \in G_\sigma$ and $h \in H_{G_\sigma}$ with $g \circ \phi_{M^t} = x^{\mathbf{P}} h$. By Remark 3.4, the function $h|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}$ does not depend on the first s variables, then

$$h|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}(y') = h|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}(y) = h(y) \overset{(21)}{=} 0.$$

where the second inequality follows from Remark 6.1.

By Proposition 5.3,

$$x^{\mathbf{P}} h|_{Cone(e^{(s+1)}, \ldots, e^{(n)})} = g \circ \phi_{M^t}|_{\mathbf{P}+Cone(e^{(s+1)}, \ldots, e^{(n)})} = g|_{face_\tau(g)} \circ \phi_{M^t},$$

then

$$\phi_{M^t}(y') \in V(In_\tau(g)) \quad \text{for all } \; g \in G_\sigma.$$

By Proposition 9.3,

$$\mathfrak{In}_\tau(I) = \langle \{In_\tau g\}_{g \in G_\sigma} \rangle.$$

then

$$\phi_{M^t}(y') \in V(\mathfrak{In}_\tau(I)).$$

Since $y' \in (\mathbb{K}^*)^n$ then $\phi_{M^t}(y') \in (\mathbb{K}^*)^n$, and the implication follows from (17).

13 Toric resolution

Theorem 13.1 *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal and let Σ be a regular refinement of the Gröbner fan of I. If $\underline{0} \in V = V(I)$ is a Newton non-degenerate singularity, then the strict transform $\widetilde{V}$ of V under the toric modification*

$$\pi : X_\Sigma \longrightarrow \mathbb{K}^n$$

associated to Σ is non-singular.

[1] Since G_σ is a Gröbner basis, it is in particular a normalised standard basis and the equality holds always (see for example [15,16] or, for the analytic case [2]).

Proof Let Σ be a regular refinement of $\Sigma(I)$, let $\{\underline{0}\} \neq \tau$ be a cone in Σ, and let y be a point in the strict transform of $V(I)$ under the toric modification given by Σ with

$$\widetilde{V(I)} \cap \mathcal{O}(\tau) \subset X_{\Sigma}.$$

Choose $\sigma = Cone(M)$ of maximal dimension such that the first s columns of M generate τ. We have

$$y = (0, \ldots, 0, y_{s+1}, \ldots, y_n) \in U_\sigma$$

with $y_i \neq 0$ for $i \geq s+1$.

Set

$$y' = (1, \ldots, 1, y_{s+1}, \ldots, y_n) \in U_\sigma \cap (\mathbb{K}^*)^n.$$

Let $G_\sigma = \{g_1, \ldots, g_r\}$ be a system of generators as in Proposition 9.3. The cone σ is good for each $g \in G_\sigma$ and

$$\mathfrak{In}_\tau(I) = \langle \{In_\tau g\}_{g \in G_\sigma} \rangle. \tag{22}$$

The set G_σ is a system of generators of I, then

$$\phi_{M^t *}(I) = \langle \{g \circ \phi_{M^t}\}_{g \in G_\sigma} \rangle \tag{23}$$

Let $H_{G_\sigma} = \{h_1, \ldots, h_r\}$ be as in (20). We have

$$\widetilde{V} \cap U_\sigma \subseteq V(h_1, \ldots, h_r).$$

By Remark 6.1,

$$h_i|_{\mathcal{O}(\tau) \cap U_\sigma} = h_i|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}. \tag{24}$$

The functions $h_i|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}$ do not depend on the first s variables, hence

$$h_i(y) = h_i|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}(y').$$

By Proposition 5.3,

$$x^{\mathbf{P}_i} h_i|_{Cone(e^{(s+1)}, \ldots, e^{(n)})} = g_i \circ \phi_{M^t}|_{\mathbf{P}_i + Cone(e^{(s+1)}, \ldots, e^{(n)})} = In_\tau g_i \circ \phi_{M^t}.$$

Continuing as in the proof of Proposition 12.1, $\tau \subset \mathbf{TV}(I)$. Then, by Proposition 10.6

$$\begin{aligned} dim\, V &= dim\, \mathbf{TV}(I) = dim\, \mathbf{TV}(\mathfrak{In}_\tau(I)) \\ &\overset{(22)}{=} dim\, V(In_\tau g_1, \ldots, In_\tau g_r) \\ &= dim\, V(h_1|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}, \ldots, h_r|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}). \end{aligned}$$

The variety $V(In_\tau g_1, \ldots, In_\tau g_r)$ has no singularities outside the coordinate hyperplanes, then, $V(h_1|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}, \ldots, h_r|_{Cone(e^{(s+1)}, \ldots, e^{(n)})})$ is non-singular in y'. That is,

$$Rank \left(\frac{\partial h_j|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}}{\partial x_i} \right)(y') = dim\, V.$$

Since $h_j|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}$ does not depend on the first s variables

$$Rank \left(\frac{\partial h_j|_{Cone(e^{(s+1)}, \ldots, e^{(n)})}}{\partial x_i} \right)_{j=1,\ldots,r,\, i=s+1,\ldots,n} (y') = dim\, V$$

Now

$$\left(\frac{\partial h_j|_{Cone(e^{(s+1)},\ldots,e^{(n)})}}{\partial x_i}\right)_{j=1,\ldots,r,\ i=s+1,\ldots,n}(y')$$
$$=\left(\frac{\partial h_j|_{Cone(e^{(s+1)},\ldots,e^{(n)})}}{\partial x_i}\right)_{j=1,\ldots,r,\ i=s+1,\ldots,n}(y)$$
$$=\left(\frac{\partial h_j}{\partial x_i}\right)_{j=1,\ldots,r,\ i=s+1,\ldots,n}(y).$$

Then $V(H_{G_\sigma})$ is non-singular and transversal to $\pi^{-1}(\underline{0})$. Since

$$V(H_{G_\sigma}) \cap (\mathbb{K}^*)^n = \pi^{-1}(V(I)) \cap U_\sigma \cap (\mathbb{K}^*)^n,$$

we have that

$$\widetilde{V(I)} = V(H_{G_\sigma})$$

and the result is proved.

Theorem 13.2 *Let $I \subset \mathbb{K}[x_1,\ldots,x_n]$ be an ideal, let Σ be a regular refinement of the Gröbner fan $\Sigma(I)$, and let $\widetilde{V(I)}$ be the strict transform of $V(I)$ under the modification given by Σ. Given $\tau \in \Sigma$ we have*

$$\widetilde{V(I)} \cap \mathcal{O}(\tau) \neq \emptyset \Leftrightarrow \tau \subset \mathbf{TV}(I).$$

Proof One implication is Proposition 12.1.

Given $\tau \in \Sigma$, take $\sigma = Cone(M) \in \Sigma$ of maximal dimension such that τ is generated by the first s columns of M.

Let G_σ be a system of generators as in Proposition 9.3. Let H_{G_σ} be as in (20). Take $y \in (\mathbb{K}^*)^n$ such that $In_\tau g(y) = 0$ for all $g \in G_\sigma$.

For each $g \in G_\sigma$ write, the same way we did in (19),

$$g \circ \phi_{M^t} = x^{\mathbf{P}_g} h_g \quad \text{with } h_g(\underline{0}) \neq 0.$$

We have

$$\widetilde{V(I)} = V(\{h_g\}_{g\in G_\sigma}).$$

Set

$$z = (z_1,\ldots,z_n) = \phi^{-1}_{M^t}(y),$$

we have

$$0 = In_\tau g(z) = z^{\mathbf{P}_g}(h_g|_{Cone(e^{(s+1)},\ldots,e^{(n)})})(z)$$

then

$$0 = h_g|_{Cone(e^{(s+1)},\ldots,e^{(n)})}(z) = h_g(0,\ldots,0,z_{s+1},\ldots,z_n).$$

Therefore, we conclude

$$(0,\ldots,0,z_{s+1},\ldots,z_n) \in \mathcal{O}(\tau) \cap \widetilde{V(I)}.$$

Note added in proof We received a communicaton from J. Tevelev saying that a similar notion was introduced under the name of "schon variety" in his paper [33]. A significant further work on this concept was done in [12–14,23,31].

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