# The reachability problem for Vector Addition Systems with one zero-test

Rémi Bonnet

LSV, CNRS, ENS Cachan

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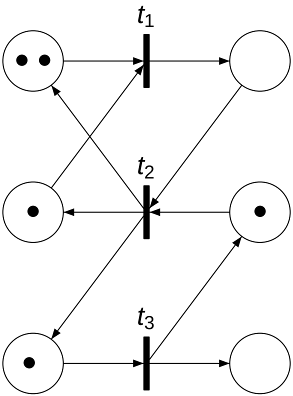
# Petri Nets and Vector Addition Systems

Petri Net:




Vector Addition System:

Initial State:

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 1 | 0 |

Addition Vectors:

| | | | | | |
|---|---|---|---|---|---|
| -1 | -1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | -1 | -1 | 0 |
| 0 | 0 | -1 | 0 | 1 | 1 |

Current State:

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 1 | 0 |

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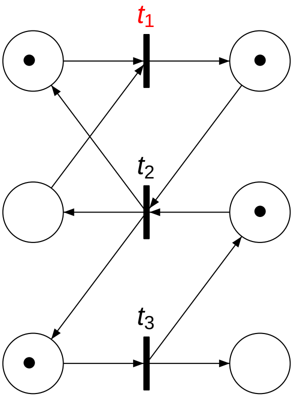


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Current State:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 0 |

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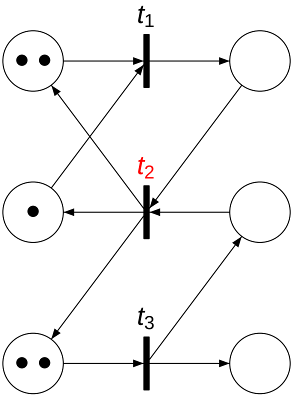


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| 1 | 1 | 1 | -1 | -1 | 0 |
| 0 | 0 | -1 | 0 | 1 | 1 |

Current State:

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 2 | 0 | 0 | 0 |

# Petri Nets and Vector Addition Systems

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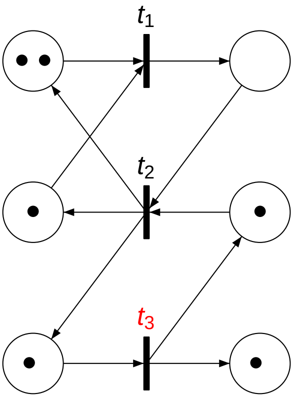


Vector Addition System:

Initial State:

| | | | | | |
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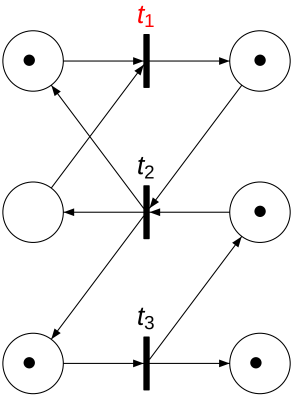


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Current State:

| | | | | | |
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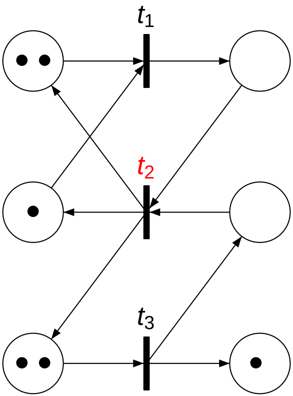


Vector Addition System:

Initial State:

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 1 | 0 |

Addition Vectors:

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|---|---|---|---|---|---|
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| 1 | 1 | 1 | -1 | -1 | 0 |
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| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 2 | 0 | 0 | 1 |

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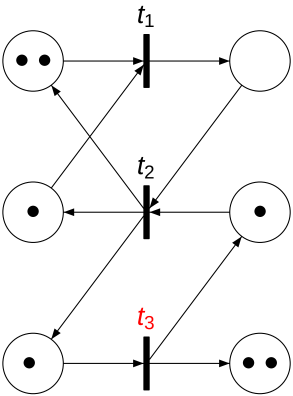


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Addition Vectors:

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|---|---|---|---|---|---|
| -1 | -1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | -1 | -1 | 0 |
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Current State:

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 1 | 2 |

# Definitions

## Definition : VAS$_0$

A Vector Addition System with one zero-test is a tuple $\langle A_z, \delta, a_z \rangle$ where:

- $A_z = A \cup \{a_z\}$ is the set of transition labels
- $\delta$ a function from $A_z$ to $\mathbb{Z}^d$.
- $a_z$ is the special zero-test transition.

- For $a \in A_z$, $\xrightarrow{a}$ is defined by:

$$
\begin{array}{rcll}
x \xrightarrow{a} y & \Longleftrightarrow & y = x + \delta(a) & \qquad a \neq a_z \\
x \xrightarrow{a_z} y & \Longleftrightarrow & \begin{cases} y = x + \delta(a_z) \\ x(1) = 0 \end{cases} &
\end{array}
$$

# Reachability

Notation:

$$x \xrightarrow{L} y \iff \exists u \in L, x \xrightarrow{u} y$$

## The reachability problem

Given a $\mathrm{VAS}_0$, an initial vector $x \in \mathbb{N}^d$ and a final vector $y \in \mathbb{N}^d$, do we have $x \xrightarrow{A_z^*} y$?

# A partial bibliography of the reachability problem

- [Mayr '81] *An Algorithm for the General Petri Net Reachability Problem*: An algorithm for decidability of reachability for VAS.
- [Kosaraju '82] *Decidability of Reachability in Vector Addition Systems*: A similar algorithm for decidability of reachability for VAS.
- [Reinhardt '08] *Reachability in Petri Nets with Inhibitor Arcs*: First proof of decidability of reachability for $VAS_0$.
- [Leroux '09] *The General Vector Addition System Reachability Problem by Presburger Inductive Invariants*: Another take of reachability for VAS, introducing new tools, but still dependant of earlier work.
- [Leroux '11] *The Vector Addition System Reachability Problem*: A new proof of reachability for VAS, independant from earlier proofs.
- [B. '11] *Reachability for Vector Addition Systems with one zero-test*: A new proof of reachability for $VAS_0$, based on the principles introduced by Leroux.

# Reachability

## The reachability problem

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Example:

$x$ :

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 1 | 0 |

$\delta$ :

| | | | | | | |
|---|---|---|---|---|---|---|
| $a_1 \mapsto$ | -1 | -1 | 0 | 1 | 0 | 0 |
| $a_2 \mapsto$ | 1 | 1 | 1 | -1 | -1 | 0 |
| $a_3 \mapsto$ | 0 | 0 | -1 | 0 | 1 | 1 |

y :

| | | | | | |
|---|---|---|---|---|---|
| 3 | 1 | 1 | 0 | 1 | 0 |

## The reachability problem

Given a $VAS_0$, an initial vector $x \in \mathbb{N}^d$ and a final vector $y \in \mathbb{N}^d$, do we have $x \xrightarrow{A_Z^*} y$?

Example:

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| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 1 | 0 |

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| | | | | | | |
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| $a_3 \mapsto$ | 0 | 0 | -1 | 0 | 1 | 1 |

y :

| | | | | | |
|---|---|---|---|---|---|
| 3 | 1 | 1 | 0 | 1 | 0 |

Not Reachable!
$v[1] + v[2] + v[3] + 2v[4] + v[5] = 5$ is an invariant!

# Witnesses of reachability and non-reachability

- If an instance of the reachability problem has a positive answer, there is a witness of reachability : the sequence of vectors going from the initial vector the final one.
- If an instance of the reachability problem has a negative answer, is there a "nice" witness of non-reachability?

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**Theorem [Leroux '11]**

If $\mathcal{S}$ is a VAS such that $v' \in \mathbb{N}^d$ is not reachable from $v \in \mathbb{N}^d$, there is a Presburger invariant (for $\mathcal{S}$) $X$ such that $v \in X$ and $v' \notin X$.

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**Our claim**

This is also true for $\mathrm{VAS}_0$.

# Showing the existence of an invariant

**Theorem [Leroux' 11]**

Let $R$ be a relation such that $R^*$ is *Petri*[a]. If there exists $x$ and $y$ such that $(x, y) \notin R^*$, then there exists a semilinear set $X$ that is an invariant for $R$ with $x \in X$ and $y \notin X$.

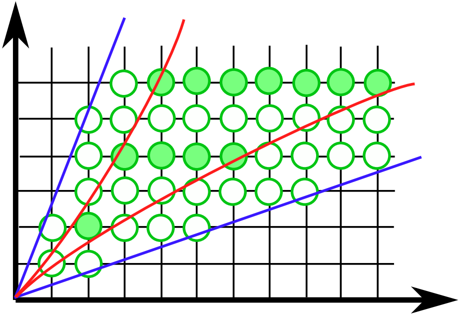
[a] To be defined in next slides

# Definition: Polytope sets

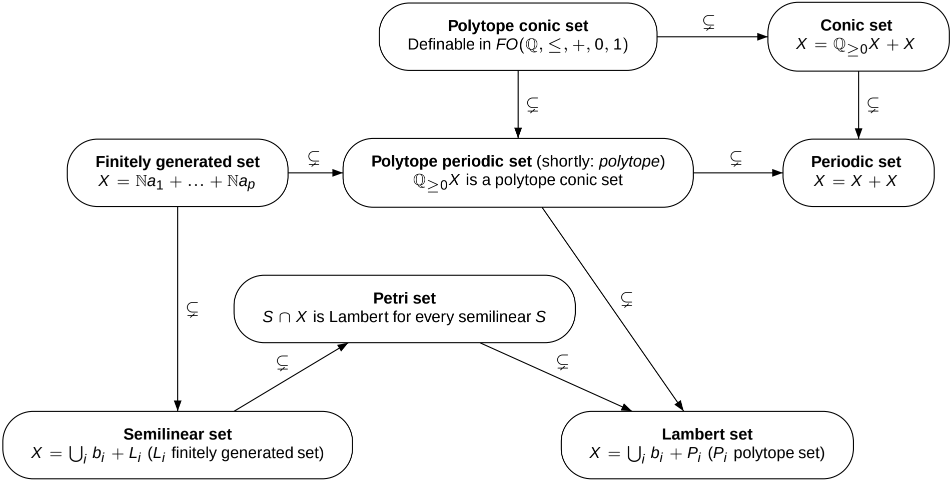
$$\begin{tikzcd}[column sep=large, row sep=large]
\begin{array}{c}\textbf{Polytope conic set}\\ \text{Definable in } FO(\mathbb{Q}, \leq, +, 0, 1)\end{array} \arrow[r, "\subsetneq"] \arrow[d, "\subsetneq"] & \begin{array}{c}\textbf{Conic set}\\ X = \mathbb{Q}_{\geq 0}X + X\end{array} \arrow[d, "\subsetneq"] \\
\begin{array}{c}\textbf{Polytope periodic set} \text{ (shortly: \textit{polytope})}\\ \mathbb{Q}_{\geq 0}X \text{ is a polytope conic set}\end{array} \arrow[r, "\subsetneq"] & \begin{array}{c}\textbf{Periodic set}\\ X = X + X\end{array}
\end{tikzcd}$$

Polytope periodic set

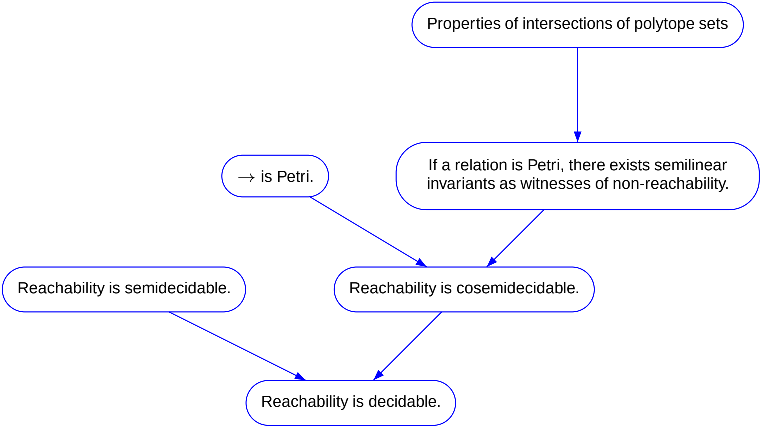
Conic closure (Polytope conic set)



# Definition: Polytope sets



# Structure of the proof



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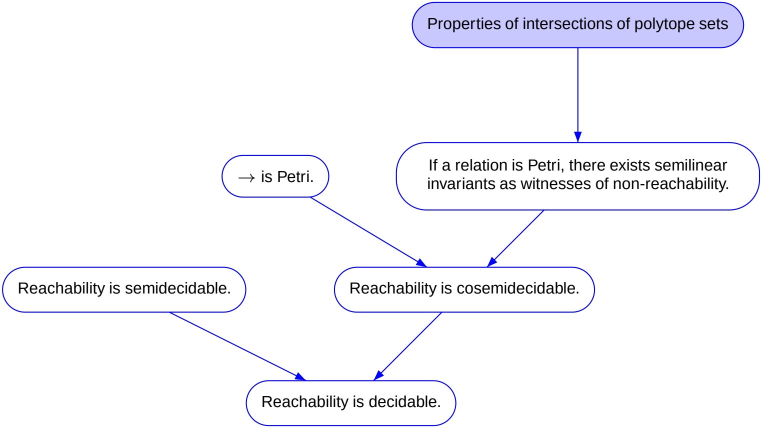


# Properties of polytope sets

## Definition : Linerization

Let $P \subseteq \mathbb{Z}^d$ be a periodic set.

$$lin(P) = (P - P) \cap \overline{\mathbb{Q}_{\geq 0} P}$$

This is a finitely generated set.

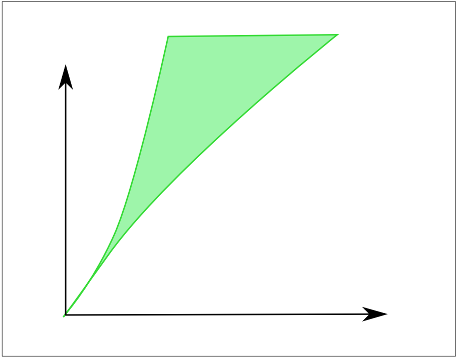
There exists a notion of dimension for polytope periodic sets such that:

## Theorem [Leroux '11]

Let $b_1, b_2 \in \mathbb{Z}^d$ and let $P_1, P_2$ be two polytope periodic sets such that the intersection $(b_1 + P_1) \cap (b_2 + P_2)$ is empty. The intersection $X = (b_1 + lin(P_1)) \cap (b_2 + lin(P_2))$ satisfies:

$$dim(X) < max(b_1 + dim(P_1), b_2 + dim(P_2))$$

# Intersection of linearized disjoint sets : Example



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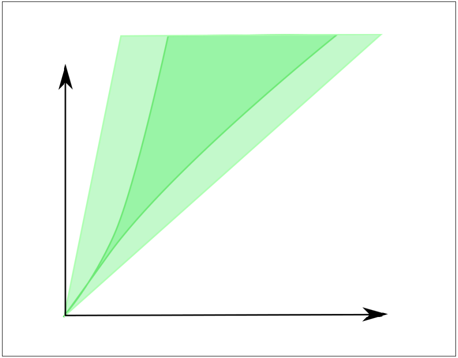

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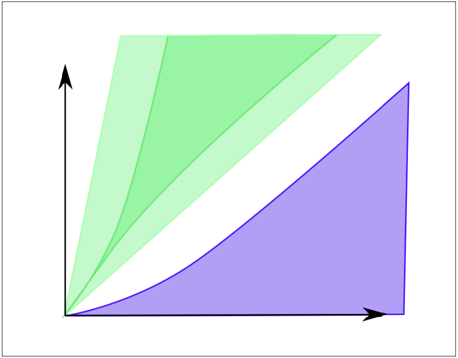

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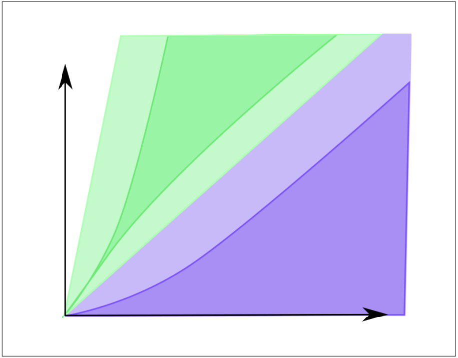

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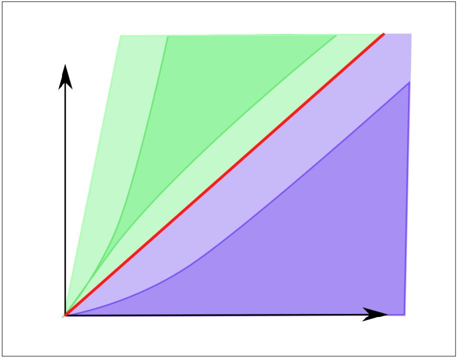

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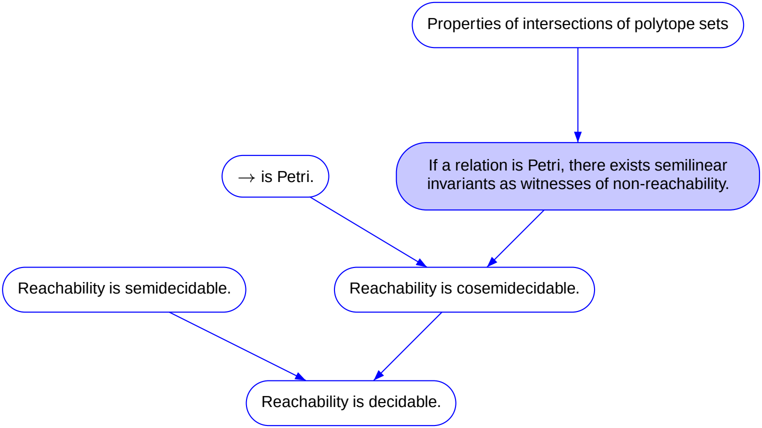


# Existence of a Presburger invariant

**Theorem [Leroux '11]**

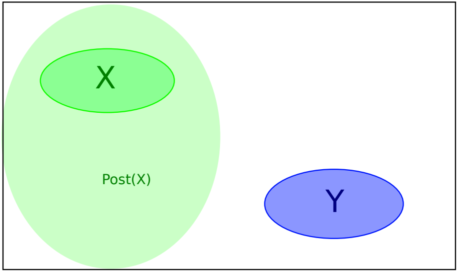
Let $R$ be a relation such that $R^*$ is Petri. If there exists $x$ and $y$ such that $(x, y) \notin R^*$, then there exists a semilinear set $X$ that is an invariant for $R$ with $x \in X$ and $y \notin X$.

Proof:

A couple $(X, Y)$ of Presburger sets is a *separator* if $(X \times Y) \cap R^*$ is empty.

We are interested in minimizing the "gap" $D(X, Y) = \mathbb{Z}^d \setminus (X \cup Y)$. If $D(X, Y) = \emptyset$, then $X$ is a Presburger invariant.

# Existence of a Presburger invariant : reducing the gap




- $D = \mathbb{Z}^d \setminus (X \cup Y)$ (the original "gap")

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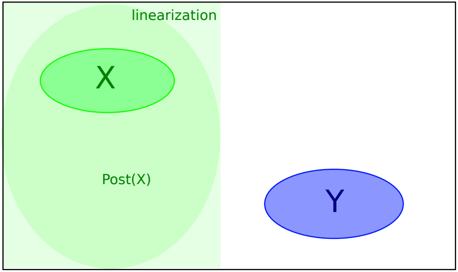
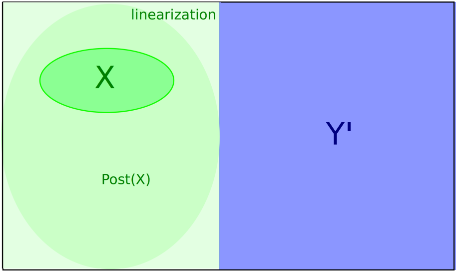



- $D = \mathbb{Z}^d \setminus (X \cup Y)$ (the original "gap")
- $S = lin(post(X) \cap D)$ (a Presburger set)

# Existence of a Presburger invariant : reducing the gap




- $D = \mathbb{Z}^d \backslash (X \cup Y)$ (the original "gap")
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- $Y' = Y \cup (D \backslash S)$ (hence $post(X) \cap Y' = \emptyset$)

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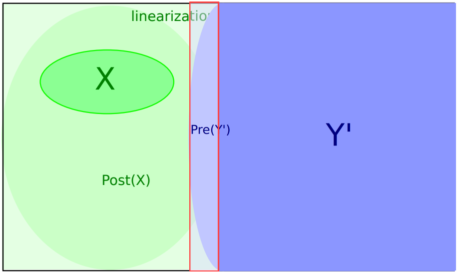


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- $T = lin(pre(Y') \cap D)$ (the other direction)

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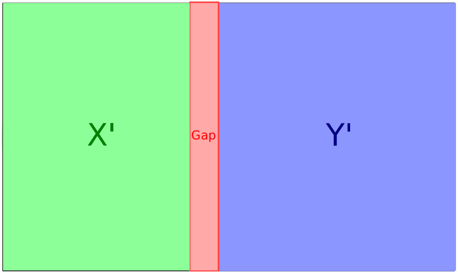


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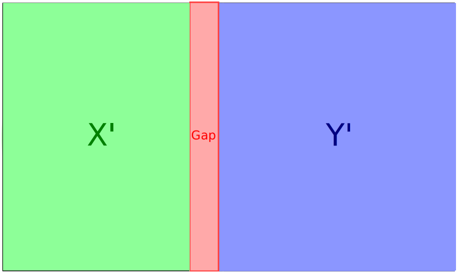


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- $X' = X \cup (D \backslash T)$ ($pre(Y') \cap X' = \emptyset$)
- $D' = \mathbb{Z}^d \backslash (X' \cup Y')$ is a separator.

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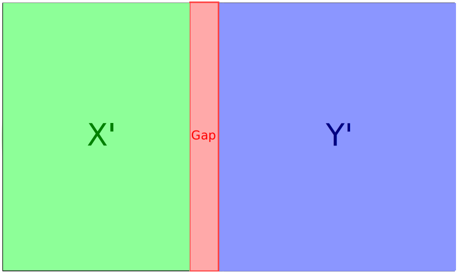


- $D = \mathbb{Z}^d \setminus (X \cup Y)$ (the original "gap")
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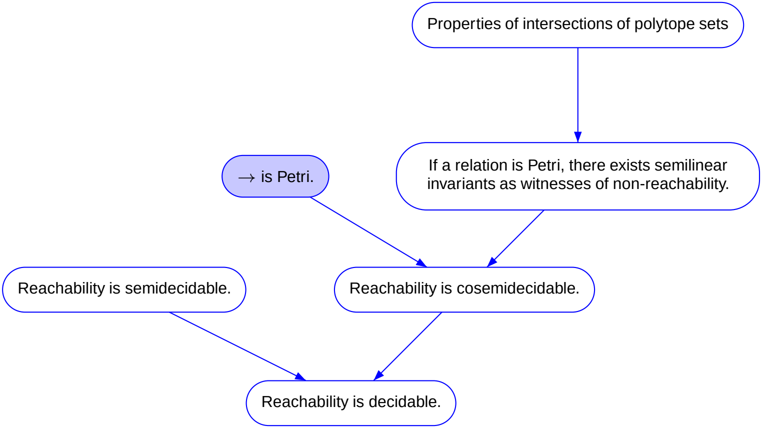
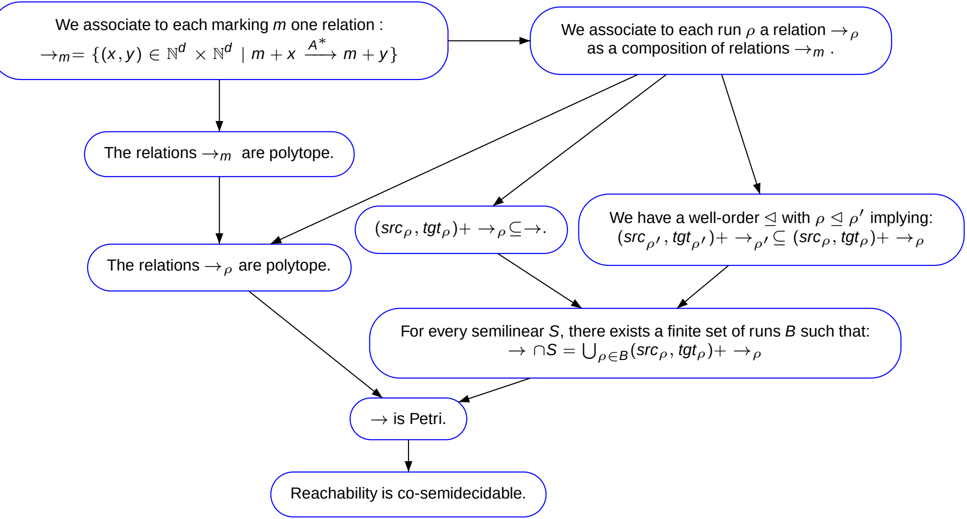
$$dim(D') < dim(D)$$

# Structure of the proof



# Structure of the proof (zoom on $\to$ is Petri)



# Structure of the proof (zoom on $\rightarrow$ is Petri)

We associate to each marking $m$ two relations:

$$\rightarrow_m = \{(x,y) \in \mathbb{N}^d \times \mathbb{N}^d \mid m + x \xrightarrow{A^*} m + y\}$$

$$\rightsquigarrow_m = \{(x,y) \in \mathbb{N}_0^d \times \mathbb{N}_0^d \mid m + x \xrightarrow{A_Z^*} m + y\}$$

We associate to each run $\rho$ a relation $\rightarrow_\rho$ as a composition of relations $\rightarrow_m$ and $\rightsquigarrow_m$.

The relations $\rightarrow_m$ and $\rightsquigarrow_m$ are polytope.

$(src_\rho, tgt_\rho) + \rightarrow_\rho \subseteq \rightarrow$.

We have a well-order $\trianglelefteq$ with $\rho \trianglelefteq \rho'$ implying: $(src_{\rho'}, tgt_{\rho'}) + \rightarrow_{\rho'} \subseteq (src_\rho, tgt_\rho) + \rightarrow_\rho$

The relations $\rightarrow_\rho$ are polytope.

For every semilinear $S$, there exists a finite set of runs $B$ such that: $\rightarrow \cap S = \bigcup_{\rho \in B} (src_\rho, tgt_\rho) + \rightarrow_\rho$

$\rightarrow$ is Petri.

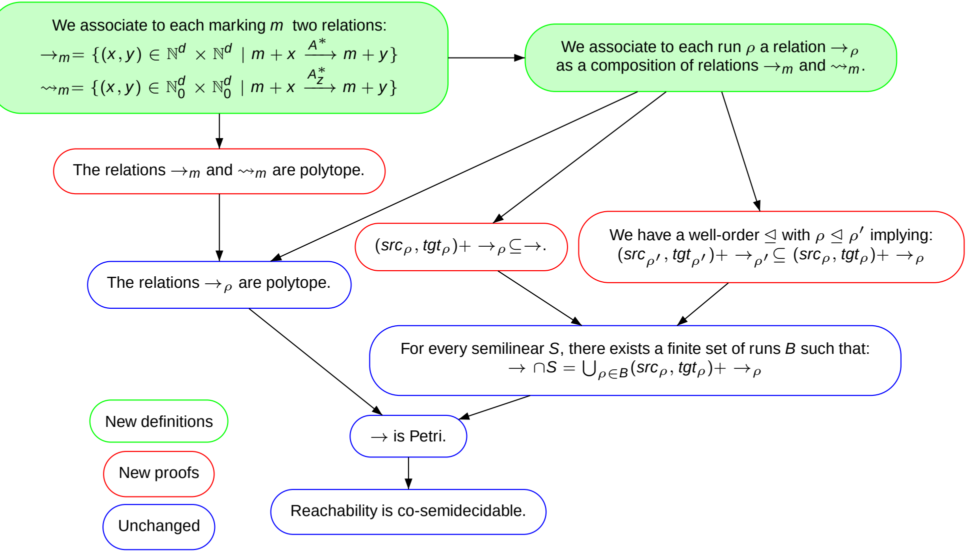
Reachability is co-semidecidable.

New definitions

New proofs

Unchanged

# Structure of the proof (zoom on $\to$ is Petri)



# Production relations for runs

- $\rightarrow_m = \{(x,y) \in \mathbb{N}^d \times \mathbb{N}^d \mid m + x \rightarrow m + y\}$

## Definition [Leroux' 11]

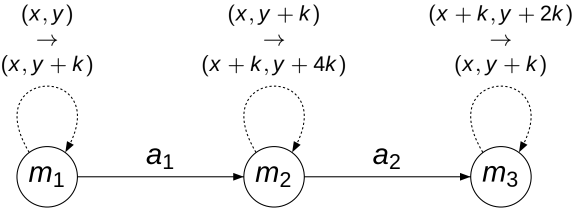
Let $\rho = \left[m_0 \xrightarrow{a_1} m_1 \cdots \xrightarrow{a_n} m_n\right]$ be a run without $a_z$. We have:

$$\rightarrow_\rho = \rightarrow_{m_0} \circ \rightarrow_{m_1} \circ \cdots \circ \rightarrow_{m_n}$$

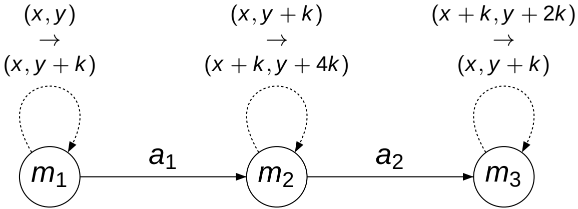
Production relations have nice properties:

- They contain the identity. $m + x \rightarrow m + x$
- They are periodic. $\begin{cases} m + x \rightarrow m + x' \\ m + y \rightarrow m + y' \end{cases} \implies m + x + y \rightarrow m + x + y' \rightarrow m + x' + y'$
- They have monotonic behavior.
  $\begin{cases} m \leq m' \\ (x,y) \in \rightarrow_{m'} \end{cases} \implies (x + (m' - m), y + (m' - m)) \in \rightarrow_m$
- (To be shown) They are polytopes.

# Production relations for runs

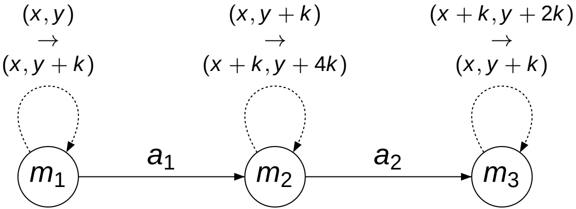


# Production relations for runs



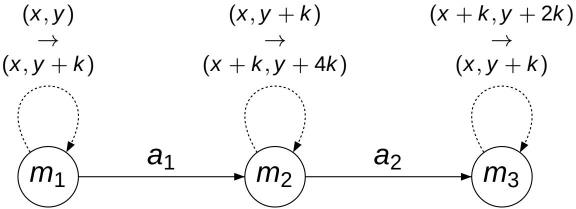

$$m_1 + (0,0) \xrightarrow{a_1} m_2 + (0,0) \xrightarrow{a_2} m_3 + (0,0)$$

# Production relations for runs




$$m_1 \to m_1 + (0,2) \xrightarrow{a_1} m_2 + (0,2) \to m_2 + (2,8) \xrightarrow{a_2} m_3 + (2,8) \to m_3 + (0,4)$$

# Production relations for runs




$$m_1 \to m_1 + (0,12) \xrightarrow{a_1} m_2 + (0,12) \to m_2 + (15,72) \xrightarrow{a_2} m_3 + (15,72)$$

# Production relation for runs

- $\rightarrow_m = \{(x, y) \in \mathbb{N}^d \times \mathbb{N}^d \mid m + x \xrightarrow{A^*} m + y\}$
- $\rightsquigarrow_m = \{(x, y) \in \mathbb{N}_0^d \times \mathbb{N}_0^d \mid m + x \xrightarrow{A_z^*} m + y\}$

We split a run $\mu = \left[m_0 \xrightarrow{a_1} \ldots \xrightarrow{a_n} m_n\right]$ of a $\mathrm{VAS}_0$ in sequences without $a_z$:

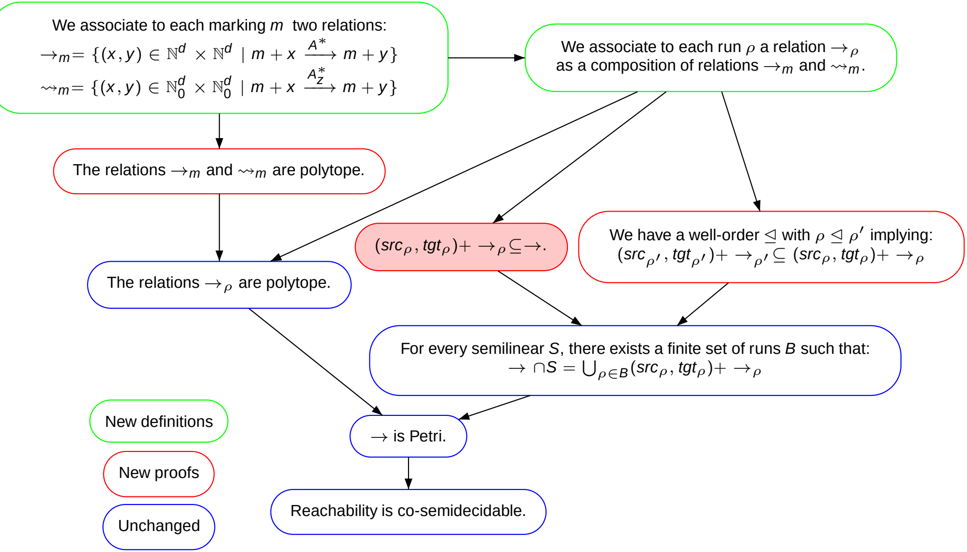
$$\mu = [\rho_0] \xrightarrow{a_z} [\rho_1] \xrightarrow{a_z} \cdots \xrightarrow{a_z} [\rho_p]$$

## Definition

Assuming $\mu$ is decomposed as above, we define the relation $\rightarrow_\mu$ by:

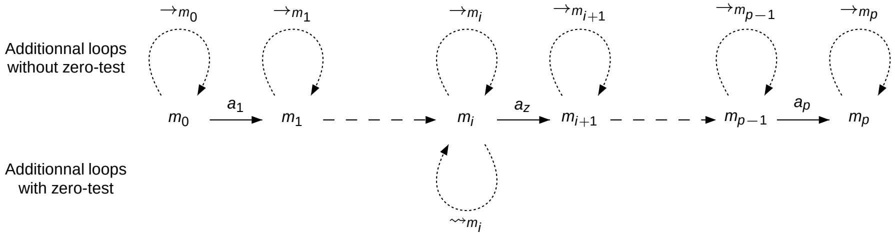
$$\rightarrow_\mu = \rightarrow_{\rho_0} \circ \rightsquigarrow_{tgt_{\rho_0}} \circ \rightarrow_{\rho_1} \cdots \circ \rightsquigarrow_{tgt_{\rho_{p-1}}} \circ \rightarrow_{\rho_p}$$

# Structure of the proof (zoom on $\rightarrow$ is Petri)



# Soundness of production relations

A production relation represents the possible "additions" to a run.




## Proposition

$$(m_0, m_p) + \rightarrow_{m_0} \circ \rightarrow_{m_1} \circ \cdots \rightarrow_{m_i} \circ \rightsquigarrow_{m_i} \circ \rightarrow_{m_{i+1}} \circ \cdots \rightarrow_{m_{p-1}} \circ \rightarrow_{m_p} \subseteq \xrightarrow{A_z^*}$$

## Ideas behind the proof:

- Loops can be composed.
- $\rightsquigarrow_m \in \mathbb{N}_0^d \times \mathbb{N}_0^d$, hence $a_z$ can be fired after such a loop.

# Structure of the proof (zoom on $\to$ is Petri)

We associate to each marking $m$ two relations:

$$\to_m = \{(x,y) \in \mathbb{N}^d \times \mathbb{N}^d \mid m + x \xrightarrow{A^*} m + y\}$$

$$\rightsquigarrow_m = \{(x,y) \in \mathbb{N}_0^d \times \mathbb{N}_0^d \mid m + x \xrightarrow{A_Z^*} m + y\}$$

We associate to each run $\rho$ a relation $\to_\rho$ as a composition of relations $\to_m$ and $\rightsquigarrow_m$.

The relations $\to_m$ and $\rightsquigarrow_m$ are polytope.

$(src_\rho, tgt_\rho) + \to_\rho \subseteq \to$.

We have a well-order $\trianglelefteq$ with $\rho \trianglelefteq \rho'$ implying: $(src_{\rho'}, tgt_{\rho'}) + \to_{\rho'} \subseteq (src_\rho, tgt_\rho) + \to_\rho$

The relations $\to_\rho$ are polytope.

For every semilinear $S$, there exists a finite set of runs $B$ such that: $\to \cap S = \bigcup_{\rho \in B} (src_\rho, tgt_\rho) + \to_\rho$

$\to$ is Petri.

Reachability is co-semidecidable.

New definitions

New proofs

Unchanged

# Well-orders

**Definition: Well-order**

$X$ is well-ordered (by $\leq$) iff:

- $\leq$ is a partial order on $X$.
- It admits no infinite strictly decreasing sequence.
- It admits no infinite set of pairwise incomparable elements.

# Word embedding and Higman's lemma

## Definition : Word embedding

Let $\preceq$ be an ordering on $X$. The $\preceq^{emb}$ (word embedding) ordering is defined on $X^*$ by ($a_i, b_i \in X$):

$$a_1 ... a_n \preceq^{emb} b_1 ... b_m$$
$$\iff$$
$$\exists f : \{1, \ldots, n\} \mapsto \{1, \ldots, m\}, \text{ strictly increasing,}$$
$$\forall i,\ a_i \preceq b_{f(i)}$$

## Higman's lemma

If $X$ is well-ordered by $\preceq$, then $X^*$ is well-ordered by $\preceq^{emb}$.

# A well-ordering on runs

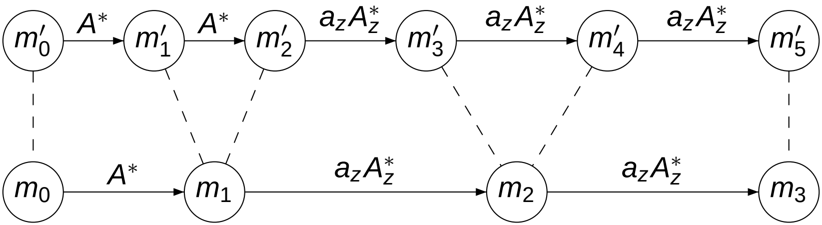
- If $\rho = m_0 \xrightarrow{a_1} m_1 \ldots \xrightarrow{a_p} m_p$ and $\rho' = m'_0 \xrightarrow{a'_1} m'_1 \ldots \xrightarrow{a'_q} m'_q$ are runs without zero-tests, we have $\rho \trianglelefteq \rho'$ if:
  - $m_0 \leq m'_0$ and $m_p \leq m'_q$
  - $\prod_{1 \leq i \leq p}(a_i, m_i) \leq^{emb} \prod_{1 \leq i \leq q}(a'_i, m'_i)$
- For $\mu = [\rho_0] \xrightarrow{a_z} [\rho_1] \ldots \xrightarrow{a_z} [\rho_p]$ and $\mu' = [\rho'_0] \xrightarrow{a_z} [\rho'_1] \xrightarrow{a_z} \ldots \xrightarrow{a_z} [\rho'_q]$ runs (with $\rho_i, \rho'_i$ runs without zero-tests), we have $\mu \trianglelefteq \mu'$ if:
  - $\rho_0 \trianglelefteq \rho'_0$ and $\rho_p \trianglelefteq \rho'_q$
  - $\prod_{1 \leq i \leq p} \rho_i \trianglelefteq^{emb} \prod_{1 \leq i \leq q} \rho'_i$

# A well-ordering on runs

## Proposition

Let $\mu$ and $\mu'$ be two runs with $\mu \trianglelefteq \mu'$. We have:

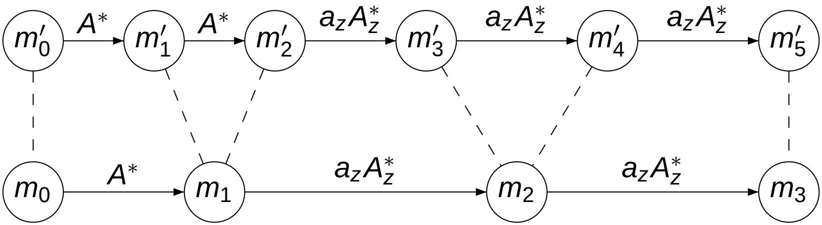
$$(src_{\mu'}, tgt_{\mu'})+ \rightarrow_{\mu'} \subseteq (src_{\mu}, tgt_{\mu})+ \rightarrow_{\mu}$$



## Proposition

Let $\mu$ and $\mu'$ be two runs with $\mu \trianglelefteq \mu'$. We have:

$$(src_{\mu'}, tgt_{\mu'}) + \rightarrow_{\mu'} \subseteq (src_{\mu}, tgt_{\mu}) + \rightarrow_{\mu}$$



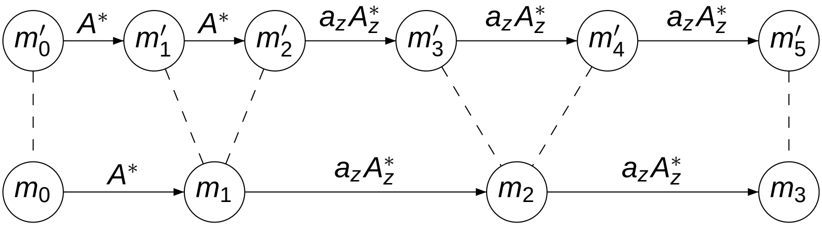

$(m'_0, m'_0) + \rightarrow_{m'_0} \subseteq (m_0, m_0) + \rightarrow_{m_0}$

# A well-ordering on runs

## Proposition

Let $\mu$ and $\mu'$ be two runs with $\mu \trianglelefteq \mu'$. We have:

$$(src_{\mu'}, tgt_{\mu'})+ \rightarrow_{\mu'} \subseteq (src_{\mu}, tgt_{\mu})+ \rightarrow_{\mu}$$



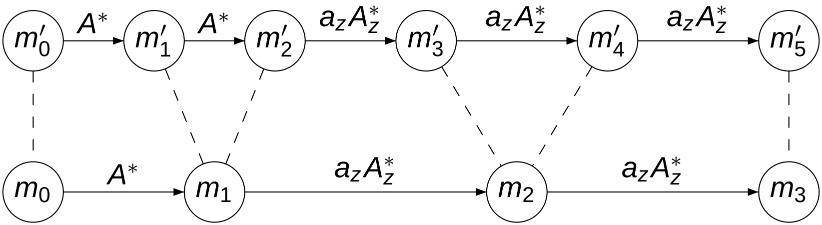

$$\begin{array}{rcl} (m'_1, m'_1)+ \rightarrow_{m'_1} & \subseteq & (m_1, m_1)+ \rightarrow_{m_1} \\ (m'_2, m'_2)+ \rightarrow_{m'_2} & \subseteq & (m_1, m_1)+ \rightarrow_{m_1} \\ (m'_1, m'_2) & \in & (m_1, m_1)+ \rightarrow_{m_1} \end{array}$$

# A well-ordering on runs

## Proposition

Let $\mu$ and $\mu'$ be two runs with $\mu \trianglelefteq \mu'$. We have:

$$(src_{\mu'}, tgt_{\mu'})+ \rightarrow_{\mu'} \subseteq (src_{\mu}, tgt_{\mu})+ \rightarrow_{\mu}$$

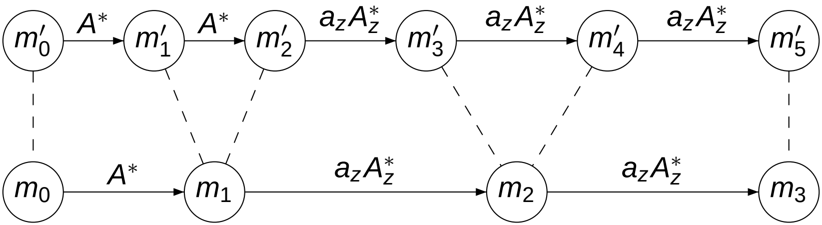



$$\begin{aligned}
(m'_3, m'_3)+ \rightarrow_{m'_3} &\subseteq (m_2, m_2)+ \rightarrow_{m_2} \\
(m'_4, m'_4)+ \rightarrow_{m'_3} &\subseteq (m_2, m_2)+ \rightarrow_{m_2} \\
(m'_3, m'_4) &\in (m_2, m_2)+ \rightsquigarrow_{m_2}
\end{aligned}$$

## Proposition

Let $\mu$ and $\mu'$ be two runs with $\mu \trianglelefteq \mu'$. We have:

$$(src_{\mu'}, tgt_{\mu'})+ \rightarrow_{\mu'} \subseteq (src_{\mu}, tgt_{\mu})+ \rightarrow_{\mu}$$




$(m'_5, m'_5)+ \rightarrow_{m'_5} \subseteq (m_3, m_3)+ \rightarrow_{m_3}$

# Structure of the proof (zoom on $\rightarrow$ is Petri)

We associate to each marking $m$ two relations:
$\rightarrow_m = \{(x,y) \in \mathbb{N}^d \times \mathbb{N}^d \mid m+x \xrightarrow{A^*} m+y\}$
$\rightsquigarrow_m = \{(x,y) \in \mathbb{N}_0^d \times \mathbb{N}_0^d \mid m+x \xrightarrow{A_Z^*} m+y\}$

We associate to each run $\rho$ a relation $\rightarrow_\rho$ as a composition of relations $\rightarrow_m$ and $\rightsquigarrow_m$.

The relations $\rightarrow_m$ and $\rightsquigarrow_m$ are polytope.

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The relations $\rightarrow_\rho$ are polytope.

For every semilinear $S$, there exists a finite set of runs $B$ such that: $\rightarrow \cap S = \bigcup_{\rho \in B} (src_\rho, tgt_\rho) + \rightarrow_\rho$

$\rightarrow$ is Petri.

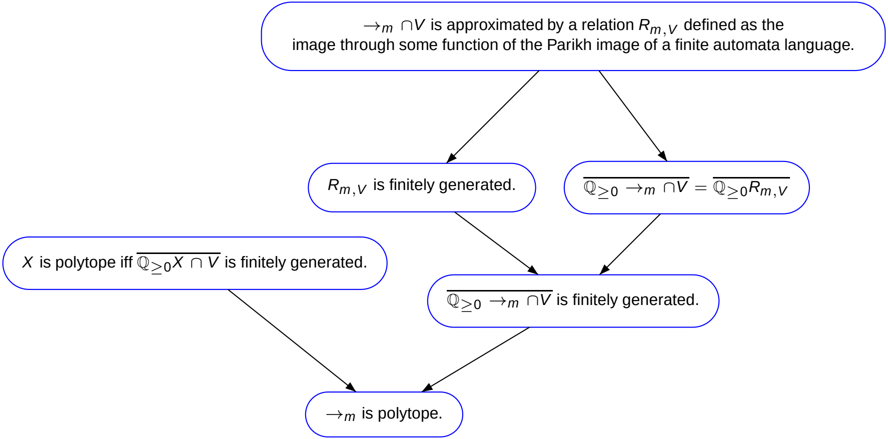
Reachability is co-semidecidable.

New definitions

New proofs

Unchanged

# Structure of the proof (zoom on $\to_m$ is polytope)



# Structure of the proof (zoom on $\to_m$ is polytope)

$\to_m \cap V$ is approximated by a relation $R_{m,V}$ defined as the image through some function of the Parikh image of a finite automata language.

$R_{m,V}$ is finitely generated.

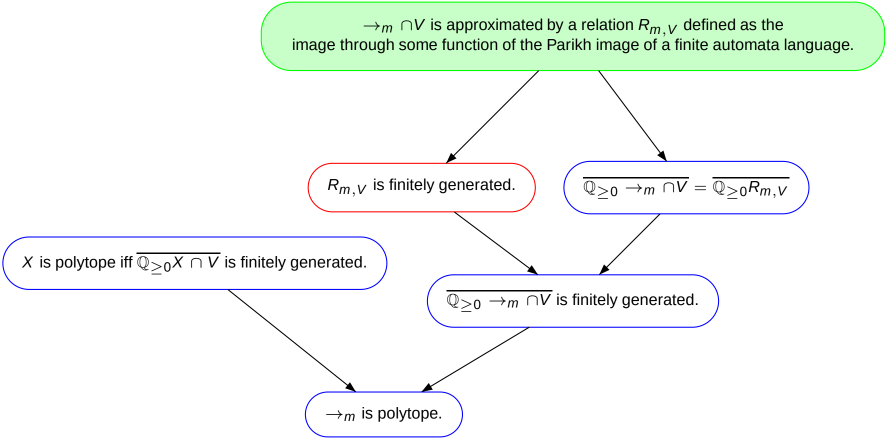
$\overline{\mathbb{Q}_{\geq 0} \to_m \cap V} = \overline{\mathbb{Q}_{\geq 0} R_{m,V}}$

$X$ is polytope iff $\overline{\mathbb{Q}_{\geq 0} X \cap V}$ is finitely generated.

$\overline{\mathbb{Q}_{\geq 0} \to_m \cap V}$ is finitely generated.

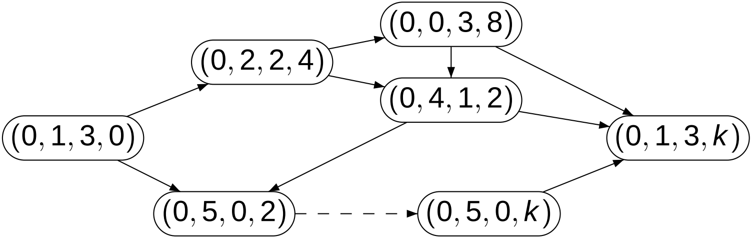
$\to_m$ is polytope.

# Structure of the proof (zoom on $\to_m$ is polytope)



# An approximation of the production relation $\rightsquigarrow_m$

- $Q = \{x \in \mathbb{N}_0^d \mid \exists (r,s) \in V,\ m + r \xrightarrow{A_z^*} x \xrightarrow{A_z^*} m + s\}$
- $I = \{i \in \{1, \dots, d\} \mid \{q(i) \mid q \in Q\} \text{ infinite}\}$



# An approximation of the production relation $\rightsquigarrow_m$

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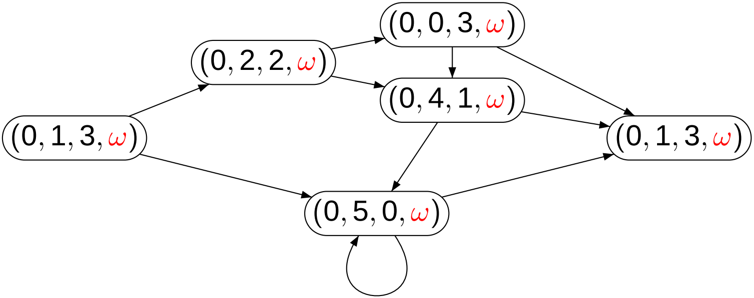
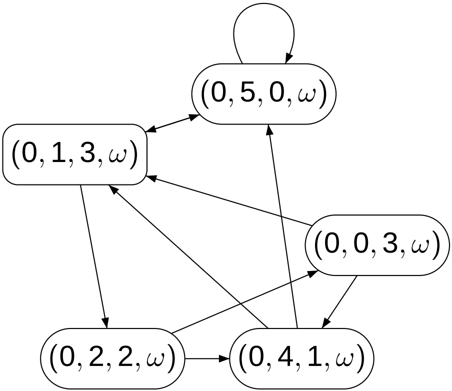
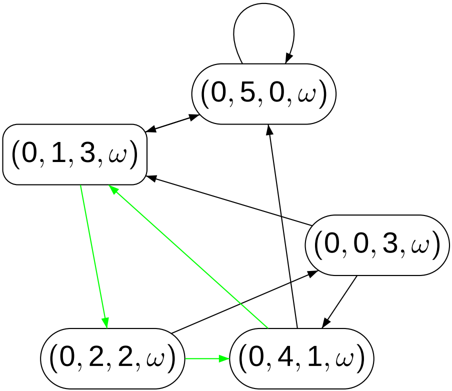



We forget about the indices in $I$ for the transition relation.

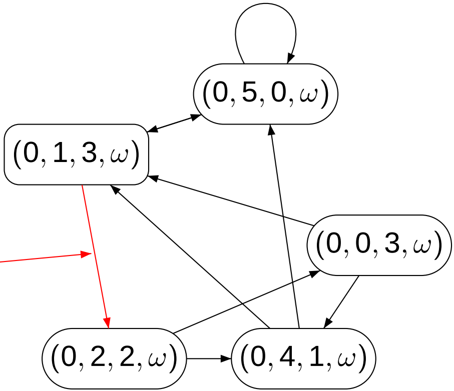
# An approximation of the production relation $\rightsquigarrow_m$






- $\rightsquigarrow_{(0,1,3,.)}$ is approximated by cycles on $(0,1,3,\omega)$.

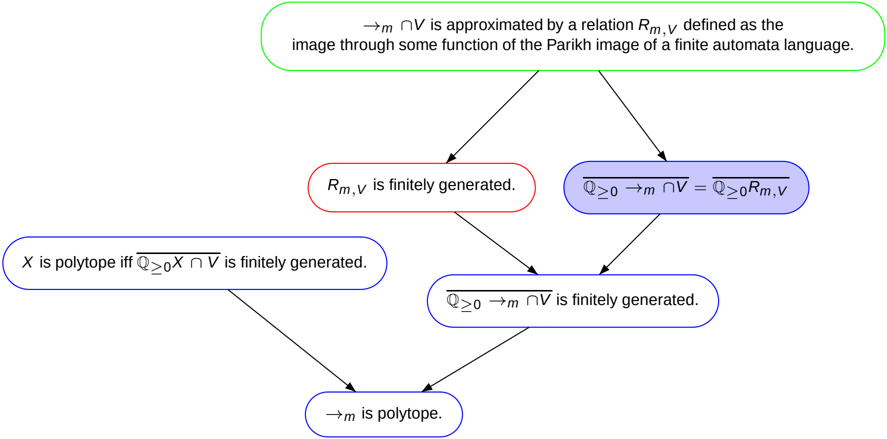
# An approximation of the production relation $\rightsquigarrow_m$




$\xrightarrow{a_z A^*} \cap \{(0,1,3)\} \times \mathbb{N} \times \{(0,2,2)\} \times \mathbb{N}$

- $\rightsquigarrow_{(0,1,3,.)}$ is approximated by cycles on $(0,1,3,\omega)$.
- Using a transition $(0,1,3,\omega) \to (0,2,2,\omega)$ can add any $\delta(u)$ where $(0,1,3,\omega) \xrightarrow{u} (0,2,2,\omega)$.

# Structure of the proof (zoom on $\to_m$ is polytope)



# Approximating $\rightarrow_m$

- $Q = \{x \in \mathbb{N}_0^d \mid \exists (r,s) \in V,\ m + r \xrightarrow{A_z^*} x \xrightarrow{A_z^*} m + s\}$
- $I = \{i \in \{1, \ldots, d\} \mid \{q(i) \mid q \in Q\} \text{ infinite}\}$

We want to suppress components in $I$.

$$m + r \xrightarrow{\quad u \quad} x \xrightarrow{\quad v \quad} m + s$$

$$m + r' \xrightarrow{\quad u' \quad} x + \delta \xrightarrow{\quad v' \quad} m + s'$$

# Approximating $\rightarrow_m$

- $Q = \{x \in \mathbb{N}_0^d \mid \exists (r,s) \in V,\ m + r \xrightarrow{A_z^*} x \xrightarrow{A_z^*} m + s\}$
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We want to suppress components in $I$.

$$m + r \xrightarrow{\quad u \quad} x \xrightarrow{\quad v \quad} m + s$$

$$m + r' \xrightarrow{\quad u' \quad} x + \delta \xrightarrow{\quad v' \quad} m + s'$$

$m + r + r'$

# Approximating $\rightarrow_m$

- $Q = \{x \in \mathbb{N}_0^d \mid \exists (r,s) \in V,\ m + r \xrightarrow{A_z^*} x \xrightarrow{A_z^*} m + s\}$
- $I = \{i \in \{1, \ldots, d\} \mid \{q(i) \mid q \in Q\} \text{ infinite}\}$

We want to suppress components in $I$.

$$m + r \xrightarrow{\quad u \quad} x \xrightarrow{\quad v \quad} m + s$$

$$m + r' \xrightarrow{\quad \color{red}{u'} \quad} x + \delta \xrightarrow{\quad v' \quad} m + s'$$

$$m + r + r' \xrightarrow{\color{red}{u'}} r + x + \delta$$

# Approximating $\rightarrow_m$

- $Q = \{x \in \mathbb{N}_0^d \mid \exists (r,s) \in V,\ m + r \xrightarrow{A_z^*} x \xrightarrow{A_z^*} m + s\}$
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We want to suppress components in $I$.

$$m + r \xrightarrow{\quad u \quad} x \xrightarrow{\quad \color{red}{v} \quad} m + s$$

$$m + r' \xrightarrow{\quad u' \quad} x + \delta \xrightarrow{\quad v' \quad} m + s'$$

$$m + r + r' \xrightarrow{u'} r + x + \delta \xrightarrow{\color{red}{v}} m + r + s + \delta$$

# Approximating $\rightarrow_m$

- $Q = \{x \in \mathbb{N}_0^d \mid \exists (r,s) \in V,\ m + r \xrightarrow{A_z^*} x \xrightarrow{A_z^*} m + s\}$
- $I = \{i \in \{1, \dots, d\} \mid \{q(i) \mid q \in Q\} \text{ infinite}\}$

We want to suppress components in $I$.

$$m + r \xrightarrow{\quad u \quad} x \xrightarrow{\quad v \quad} m + s$$

$$m + r' \xrightarrow{\quad u' \quad} x + \delta \xrightarrow{\quad v' \quad} m + s'$$

$$m + r + r' \xrightarrow{u'} r + x + \delta \xrightarrow{v} m + r + s + \delta \xrightarrow{u} x + s + \delta$$

# Approximating $\rightarrow_m$

- $Q = \{x \in \mathbb{N}_0^d \mid \exists (r,s) \in V,\ m + r \xrightarrow{A_z^*} x \xrightarrow{A_z^*} m + s\}$
- $I = \{i \in \{1, \dots, d\} \mid \{q(i) \mid q \in Q\} \text{ infinite}\}$

We want to suppress components in $I$.

$$m + r \xrightarrow{\quad u \quad} x \xrightarrow{\quad v \quad} m + s$$

$$m + r' \xrightarrow{\quad u' \quad} x + \delta \xrightarrow{\quad {\color{red} v'} \quad} m + s'$$

$$m + r + r' \xrightarrow{u'} r + x + \delta \xrightarrow{v} m + r + s + \delta \xrightarrow{u} x + s + \delta \xrightarrow{{\color{red} v'}} m + s + s'$$

# Approximating $\to_m$

We have shown we can find $\delta$, with $\delta(i) > 0$ for $i \in I$ such that:

$$m + r \to m + \delta \to m + s$$

Assume $(r', s')$ in the approximated relation. We have a run $u$ such that:

$$m + r' + \omega^I \xrightarrow{u} m + s' + \omega^I$$

# Approximating $\rightarrow_m$

We have shown we can find $\delta$, with $\delta(i) > 0$ for $i \in I$ such that:

$$m + r \rightarrow m + \delta \rightarrow m + s$$

Assume $(r', s')$ in the approximated relation. We have a run $u$ such that there exists $p \in \mathbb{N}$:

$$m + r' + p * \delta \xrightarrow{u} m + s' + p * \delta$$

# Approximating $\rightarrow_m$

We have shown we can find $\delta$, with $\delta(i) > 0$ for $i \in I$ such that:

$$m + r \rightarrow m + \delta \rightarrow m + s$$

Assume $(r', s')$ in the approximated relation. We have a run $u$ such that there exists $p \in \mathbb{N}$:

$$m + r' + p * \delta \xrightarrow{u} m + s' + p * \delta$$

By iterating the sequence $u$ $n$ times:

$$m + n * r' + p * \delta \xrightarrow{u^n} m + n * s' + p * \delta$$

# Approximating $\rightarrow_m$

We have shown we can find $\delta$, with $\delta(i) > 0$ for $i \in I$ such that:

$$m + r \rightarrow m + \delta \rightarrow m + s$$

Assume $(r', s')$ in the approximated relation. We have a run $u$ such that there exists $p \in \mathbb{N}$:

$$m + r' + p * \delta \xrightarrow{u} m + s' + p * \delta$$

By iterating the sequence $u$ $n$ times:

$$m+p*r+n*r' \rightarrow m+n*r'+p*\delta \xrightarrow{u^n} m+n*s'+p*\delta \rightarrow m+p*s+n*s'$$

# Approximating $\rightarrow_m$

We have shown we can find $\delta$, with $\delta(i) > 0$ for $i \in I$ such that:

$$m + r \rightarrow m + \delta \rightarrow m + s$$

Assume $(r', s')$ in the approximated relation. We have a run $u$ such that there exists $p \in \mathbb{N}$:

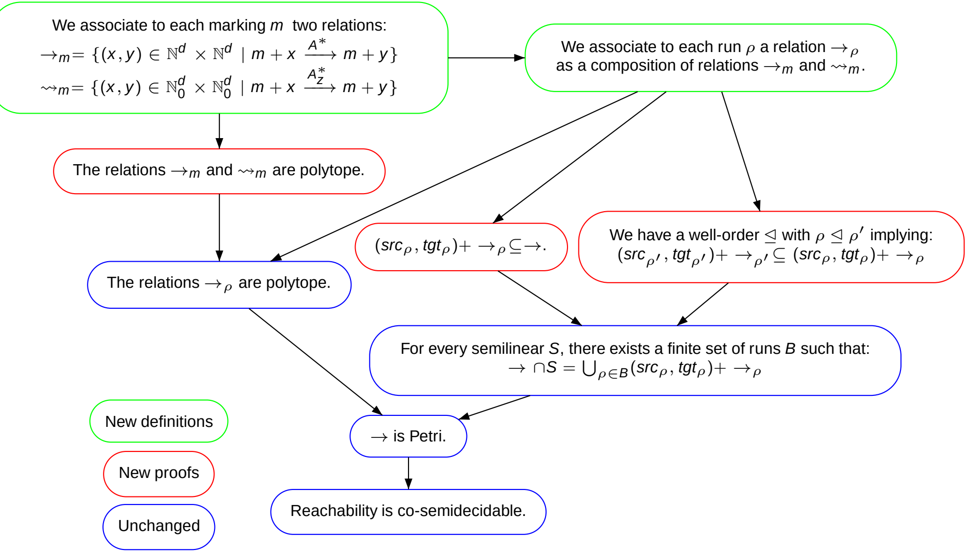
$$m + r' + p * \delta \xrightarrow{u} m + s' + p * \delta$$

By iterating the sequence $u$ $n$ times:

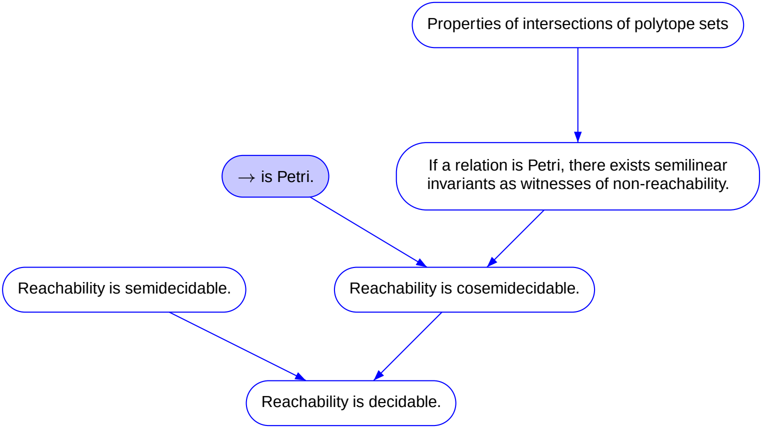
$$m+p*r+n*r' \rightarrow m+n*r'+p*\delta \xrightarrow{u^n} m+n*s'+p*\delta \rightarrow m+p*s+n*s'$$

$$\exists p \in \mathbb{N}, \forall n \in N, (p * r + n * r', p * s + n * s') \in \rightarrow_m$$

# Structure of the proof (zoom on $\to$ is Petri)



# Structure of the proof



# Overview of decidable problems on $VAS_0$

| | VAS | $VAS_0$ |
|---|---|---|
| Boundedness | decidable (EXPSPACE) [Karp and Miller '69, Rackoff '78] | decidable [Finkel and Sangnier '10] |
| Coverability | decidable (EXPSPACE) [Karp and Miller '69, Rackoff '78] | decidable [Abdulla and Mayr '09] |
| Reachability | decidable [Mayr '81, Kosaraju '82, Leroux '11] | decidable [Reinhardt '08, B. '11] |
| Cover | effective [Karp and Miller '69] | effective [B., Finkel, Leroux, Zeitoun '10] |
| LTL on actions | decidable (EXPSPACE) [Esperza '94, Habermehl '97] | decidable [B. '11] |
| LTL on states, CTL | undecidable [Esperza '94] | undecidable |

# Conclusion

**Reachability in $VAS_0$**

- We proposed an alternative proof for the decidability of reachability for $VAS_0$.
- It shows the existence of witnesses of non-reachability as Presburger invariants.
- This proof is an extension of the work by Leroux on VAS.
- The main change from the proof of Leroux is the use of a more complex production relations on runs, and a new well-ordering on them.

**Decidability of problems in $VAS_0$**

- It seems all problems that are decidable in VAS are decidable for $VAS_0$.
- No results on complexity.