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A New Approach to Catalan Numbers Using Differential Equations

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Abstract. In this paper, we introduce two differential equations arising from the generating function of the Catalan numbers which are 'inverses' to each other in a certain sense. From these differential equations, we obtain some new and explicit identities for Catalan and higher-order Catalan numbers. In addition, by other means than differential equations, we also derive some interesting identities involving Catalan numbers which are of arithmetic and combinatorial nature.

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1. INTRODUCTION

The Catalan numbers C_n were first introduced by the Mongolian mathematician Ming Antu in around 1730, even though they were named after the Belgian mathematician Eugène Charles Catalan (1814–1894). Indeed, Ming Antu obtained a number of trigonometric expressions involving Catalan numbers such as

$$\sin 2\theta = 2\sin\theta - \sum_{n=1}^{\infty} \frac{C_{n-1}}{4^{n-1}} \sin^{2n+1}\theta = 2\sin\theta - \sin^3\theta - \tfrac{1}{4}\sin^5\theta - \tfrac{1}{8}\sin^7\theta - \cdots$$

(see [2–4, 6, 10–14]). The Catalan numbers can be given explicitly in terms of binomial coefficients. Namely, for $n \geqslant 0$,

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \prod_{k=2}^{n} \frac{n+k}{k}. \tag{1.1}$$

They satisfy the recurrence relations

$$C_0 = 1, \quad C_n = \sum_{m=0}^{n-1} C_m C_{n-1-m} \quad (n \geqslant 1). \tag{1.2}$$

The Catalan numbers are also given by the generating function

$$\frac{2}{1+\sqrt{1-4t}} = \sum_{n=0}^{\infty} C_n t^n = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} t^n. \tag{1.3}$$

The Catalan numbers form a sequence of positive integers

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796,\ 58786,\ 208012,\ \cdots$$

which is asymptotic to $4^n/n^{\frac{3}{2}}\sqrt{\pi}$, as n tends to ∞, and appears in various counting problems. For example, C_n is the number of Dyck words of length $2n$, the number of balanced n pairs of parentheses, the number of mountain ranges you can form with n upstrokes and downstrokes that all stay above the original line, the number of diagonal-avoiding paths of length $2n$ from the upper left corner to the lower right corner in a grid of $n \times n$ squares, and the number of ways in which $n+1$ factors can be completely parenthesized (see [1–4, 10, 11]).

It is also the number of ways an $(n+2)$-gon can be cut into n triangles, the number of permutations of $\{1, 2, \cdots, n\}$ that avoid the pattern 123, the number of ways to tile a stairstep shape of height n with n rectangles, etc. (see [10–13]).

In [8], T. Kim put forward the fascinating idea of using ordinary differential equations as a method for obtaining new identities for special polynomials and numbers. Namely, a family of nonlinear differential equations were derived, which are indexed by positive integers and satisfied by the generating function of the Frobenius–Euler numbers. Then they were used to obtain an interesting identity, expressing higher-order Frobenius–Euler numbers in terms of (ordinary) Frobenius–Euler numbers (see [5, 7, 9, 15]).

This method turned out to be very fruitful and can be applied to many interesting special polynomials and numbers (see [7–9]). For example, linear differential equations are derived for Bessel polynomials, Changhee polynomials, actuarial polynomials, Meixner polynomials of the first kind, Poisson–Charlier polynomials, Laguerre polynomials, Hermite polynomials, and Stirling polynomials, while nonlinear ones are obtained for Bernoulli numbers of the second kind, Boole numbers, Chebyshev polynomials of the first, second, third, and fourth kind, degenerate Euler numbers, degenerate Eulerian polynomials, Korobov numbers, and Legendre polynomials (see [1, 5, 7, 9, 15]).

In this paper, we introduce two differential equations arising from the generating function of the Catalan numbers which are ‘inverse’ to each other in a certain sense. From these differential equations, we obtain some new and explicit identities for Catalan and higher-order Catalan numbers. In addition, using other tools (which are not differential equations), we also derive some interesting identities involving Catalan numbers which are of arithmetic and combinatorial nature.

2. DIFFERENTIAL EQUATIONS ASSOCIATED WITH CATALAN NUMBERS

Let

$$C = C(t) = \frac{2}{1+\sqrt{1-4t}}. \tag{2.1}$$

Then, by (2.1), we have

$$C^{(1)} = \frac{d}{dt}C(t) = (1-4t)^{-\frac{1}{2}}4(1+\sqrt{1-4t})^{-2} = (1-4t)^{-\frac{1}{2}}C^2, \tag{2.2}$$

and

$$\begin{aligned} C^{(2)} &= \frac{d}{dt}C^{(1)} = 2(1-4t)^{-\frac{3}{2}}C^2 + 2(1-4t)^{-\frac{1}{2}}CC^{(1)} \\ &= 2(1-4t)^{-\frac{3}{2}}C^2 + 2(1-4t)^{-\frac{1}{2}}C\{(1-4t)^{-\frac{1}{2}}C^2\} = 2(1-4t)^{-\frac{3}{2}}C^2 + 2(1-4t)^{-\frac{2}{2}}C^3. \end{aligned} \tag{2.3}$$

So, we are led to put

$$C^{(N)} = \sum_{i=1}^{N} a_i(N)(1-4t)^{-\frac{2N-i}{2}}C^{i+1}, \tag{2.4}$$

where $N = 1, 2, 3, \cdots$ From (2.4), we obtain

$$
\begin{aligned}
C^{(N+1)} &= \frac{d}{dt} C^{(N)} \\
&= \sum_{i=1}^{N} 2(2N-i) a_i(N)(1-4t)^{-\frac{2N+2-i}{2}} C^{i+1} + \sum_{i=1}^{N} (i+1) a_i(N)(1-4t)^{-\frac{2N-i}{2}} C^i C^{(1)} \\
&= \sum_{i=1}^{N} 2(2N-i) a_i(N)(1-4t)^{-\frac{2N+2-i}{2}} C^{i+1} \\
&\quad + \sum_{i=1}^{N} (i+1) a_i(N)(1-4t)^{-\frac{2N-i}{2}} C^i \{(1-4t)^{-\frac{1}{2}} C^2\} \\
&= \sum_{i=1}^{N} 2(2N-i) a_i(N)(1-4t)^{-\frac{2N+2-i}{2}} C^{i+1} + \sum_{i=1}^{N} (i+1) a_i(N)(1-4t)^{-\frac{2N+1-i}{2}} C^{i+2} \\
&= \sum_{i=1}^{N} 2(2N-i) a_i(N)(1-4t)^{-\frac{2N+2-i}{2}} C^{i+1} + \sum_{i=2}^{N+1} i a_{i-1}(N)(1-4t)^{-\frac{2N+2-i}{2}} C^{i+1}.
\end{aligned} \tag{2.5}
$$

On the other hand, replacing N by $N+1$ in (2.4), we obtain

$$
C^{(N+1)} = \sum_{i=1}^{N+1} a_i(N+1)(1-4t)^{-\frac{2N+2-i}{2}} C^{i+1}. \tag{2.6}
$$

From (2.5) and (2.6), we can derive the following recurrence relations:

$$
a_1(N+1) = 2(2N-1) a_1(N), \tag{2.7}
$$

$$
a_{N+1}(N+1) = (N+1) a_N(N), \tag{2.8}
$$

$$
a_i(N+1) = i a_{i-1}(N) + 2(2N-i) a_i(N) \quad (2 \leqslant i \leqslant N). \tag{2.9}
$$

In addition, from (2.2) and (2.4), we observe that

$$
a_1(1)(1-4t)^{-\frac{1}{2}} C^2 = C^{(1)} = (1-4t)^{-\frac{1}{2}} C^2. \tag{2.10}
$$

Thus, by (2.10), we see that

$$
a_1(1) = 1. \tag{2.11}
$$

Below, for any positive integer N, $(2N-1)!!$ will denote

$$
(2N-1)!! = (2N-1)(2N-3) \cdots 1. \tag{2.12}
$$

Using (2.7) and (2.11), we can write

$$
\begin{aligned}
a_1(N+1) &= 2(2N-1) a_1(N) = 2^2 (2N-1)(2N-3) a_1(N-1) \\
&= \cdots = 2^N (2N-1)(2N-3) \cdots 1 a_1(1) = 2^N (2N-1)!!
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
a_{N+1}(N+1) &= (N+1) a_N(N) = (N+1) N a_{N-1}(N-1) \\
&= \cdots = (N+1) N \cdots 2 a_1(1) = (N+1)!.
\end{aligned} \tag{2.14}
$$

In the following, we use the notation

$$(x;\alpha)_n = x(x-\alpha)\cdots(x-(n-1)\alpha) \quad \text{for } n \geqslant 1, \tag{2.15}$$

and $(x;\alpha)_0 = 1$.

For $i = 2$ in (2.9), we have

$$\begin{aligned}
a_2(N+1) &= 2a_1(N) + 2(2N-2)a_2(N) \\
&= 2a_1(N) + 2(2N-2)\big(2a_1(N-1) + 2(2N-4)a_2(N-1)\big) \\
&= 2\big(a_1(N) + 2(2N-2)a_1(N-1)\big) + 2^2(2N-2)(2N-4)a_2(N-1) \\
&= 2\big(a_1(N) + 2(2N-2)a_1(N-1)\big) \\
&\quad + 2^2(2N-2)(2N-4)\big(2a_1(N-2) + 2(2N-6)a_2(N-2)\big) \\
&= 2\big(a_1(N) + 2(2N-2)a_1(N-1)\big) + 2^2(2N-2)(2N-4)a_1(N-2)\big) \\
&\quad + 2^3(2N-2)(2N-4)(2N-6)a_2(N-2) \\
&= \ \cdots \\
&= 2\sum_{k=0}^{N-2} 2^k(2N-2;2)_k a_1(N-k) + 2^{N-1}(2N-2;2)_{N-1}a_2(2) \\
&= 2\sum_{k=0}^{N-1} 2^k(2N-2;2)_k a_1(N-k).
\end{aligned} \tag{2.16}$$

Proceeding analogously with the case of $i = 2$, $i = 3$, and $i = 4$, we obtain

$$a_3(N+1) = 3\sum_{k=0}^{N-2} 2^k(2N-3;2)_k a_2(N-k), \tag{2.17}$$

$$a_4(N+1) = 4\sum_{k=0}^{N-3} 2^k(2N-4;2)_k a_3(N-k). \tag{2.18}$$

Continuing this process, we can deduce that

$$a_i(N+1) = i\sum_{k=0}^{N-i+1} 2^k(2N-i;2)_k a_{i-1}(N-k) \quad \text{for } 2 \leqslant i \leqslant N. \tag{2.19}$$

Now, we give explicit expressions for $a_i(N+1)$ $(2 \leqslant i \leqslant N)$. From (2.13) and (2.16), we have

$$\begin{aligned}
a_2(N+1) &= 2\sum_{k_1=0}^{N-1} 2^{k_1}(2N-2;2)_{k_1} a_1(N-k_1) = 2\sum_{k_1=0}^{N-1} 2^{k_1}(2N-2;2)_{k_1} 2^{N-k_1-1}(2N-2k_1-3)!! \\
&= 2!2^{N-1}\sum_{k_1=0}^{N-1}(2N-2;2)_{k_1}(2N-2k_1-3)!!.
\end{aligned} \tag{2.20}$$

Also, from (2.17) and (2.20), we obtain

$$\begin{aligned}a_3(N+1) &= 3\sum_{k_2=0}^{N-2} 2^{k_2}(2N-3;2)_{k_2} a_2(N-k_2) = 3\sum_{k_2=0}^{N-2} 2^{k_2}(2N-3;2)_{k_2} 2^{N-k_2-1}\\ &\quad\times \sum_{k_1=0}^{N-2-k_2} (2N-2k_2-4;2)_{k_1}(2N-2k_1-2k_1-5)!!\\ &= 3!2^{N-2}\sum_{k_2=0}^{N-2}\sum_{k_1=0}^{N-2-k_2}(2N-3;2)_{k_2}(2N-2k_2-4;2)_{k_1}\times(2N-2k_1-2k_1-5)!!.\end{aligned} \tag{2.21}$$

Continuing this process, we see that

$$\begin{aligned}a_i(N+1) =& 2^{N-i+1} i! \sum_{k_{i-1}=0}^{N-i+1}\sum_{k_{i-2}=0}^{N-i+1-k_{i-1}}\cdots\sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2}(2N-i;2)_{k_{i-1}}\\ &\times(2N-2k_{i-1}-i-1;2)_{k_{i-2}}\times\cdots\times(2N-2k_{i-1}-\cdots-2k_2-2i+2;2)_{k_1}\\ &\times(2N-2k_{i-1}-\cdots-2k_1-2i+1)!!\\ =& 2^{N-i+1} i! \sum_{k_{i-1}=0}^{N-i+1}\sum_{k_{i-2}=0}^{N-i+1-k_{i-1}}\cdots\sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2}\\ &\times\prod_{l=1}^{i-1}(2N-2\sum_{j=l+1}^{i-1}k_j-2i+1+l;2)_{k_l}\times(2N-2\sum_{j=1}^{i-1}k_j-2i+1)!!\end{aligned} \tag{2.22}$$

for $2\leqslant i\leqslant N$.

Remark. We note here that (2.22) is also valid for $i=N+1$.

Therefore, from (2.13) and (2.22), we obtain the following theorem.

Theorem 2.1. *The family of differential equations*

$$C^{(N)}=\sum_{i=1}^{N} a_i(N)(1-4t)^{-\frac{2N-i}{2}}C^{i+1}\quad (N=1,2,3,\cdots) \tag{2.23}$$

has the solution

$$C=C(t)=\tfrac{2}{1+\sqrt{1-4t}},$$

where

$$\begin{aligned}&a_1(N)=2^{N-1}(2N-3)!!,\\ &a_i(N)=2^{N-i}i!\sum_{k_{i-1}=0}^{N-i}\sum_{k_{i-2}=0}^{N-i-k_{i-1}}\cdots\sum_{k_1=0}^{N-i-k_{i-1}-\cdots-k_2}\times\prod_{l=1}^{i-1}\left(2N-2\sum_{j=l+1}^{i-1}k_j-2i-1+l;2\right)_{k_l}\\ &\qquad\qquad\times(2N-2\sum_{j=1}^{i-1}k_j-2i-1)!!.\end{aligned}$$

We recall that the Catalan numbers C_n are defined by the generating funcion

$$C=C(t)=\frac{2}{1+\sqrt{1-4t}}=\sum_{n=0}^{\infty}C_n t^n. \tag{2.24}$$

More generally, the higher-order Catalan numbers $C_n^{(r)}$ of order r are given by

$$\left(\frac{2}{1+\sqrt{1-4t}}\right)^r = \sum_{n=0}^{\infty} C_n^{(r)} t^n. \tag{2.25}$$

On the one hand, from (2.24), we have

$$C^{(N)} = \sum_{n=N}^{\infty} C_n (n)_N t^{n-N} = \sum_{n=0}^{\infty} C_{n+N}(n+N)_N t^n, \tag{2.26}$$

where

$$(x)_n = x(x-1)\cdots(x-n+1) \quad \text{for } n \geqslant 1,\ (x)_0 = 1. \tag{2.27}$$

On the other hand, by Theorem 2.1, we have

$$\begin{aligned} C^{(N)} &= \sum_{i=1}^{N} a_i(N)(1-4t)^{-\frac{2N-i}{2}} C^{i+1} = \sum_{i=1}^{N} a_i(N) \sum_{m=0}^{\infty} \binom{\frac{2N-i}{2}+m-1}{m} 4^m t^m \sum_{l=0}^{\infty} C_l^{(i+1)} t^l \\ &= \sum_{i=1}^{N} a_i(N) \sum_{n=0}^{\infty} \sum_{m=0}^{n} 4^m \binom{\frac{2N-i}{2}+m-1}{m} C_{n-m}^{(i+1)} t^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=1}^{N} \sum_{m=0}^{n} 4^m \binom{\frac{2N-i}{2}+m-1}{m} a_i(N) C_{n-m}^{(i+1)} \right) t^n. \end{aligned} \tag{2.28}$$

Comparing (2.26) with (2.28), we obtain the following theorem.

Theorem 2.2. *For $n = 0, 1, 2, \cdots$, and $N = 1, 2, 3, \cdots$,*

$$C_{n+N} = \frac{1}{(n+N)_N} \sum_{i=1}^{N} \sum_{m=0}^{n} 4^m \binom{\frac{2N-i}{2}+m-1}{m} a_i(N) C_{n-m}^{(i+1)},$$

where the $a_i(N)$'s are as in Theorem 2.1.

3. INVERSE DIFFERENTIAL EQUATIONS ASSOCIATED WITH CATALAN NUMBERS

Here we shall derive "inverse" differential equations to the ones obtained in Section 2. With $C = C(t)$ as in (2.1), we have

$$C^{(1)} = (1-4t)^{-\frac{1}{2}} C^2 \tag{3.1}$$

and

$$C^2 = (1-4t)^{\frac{1}{2}} C^{(1)}. \tag{3.2}$$

Differentiating both sides of (3.2), we obtain

$$2CC^{(1)} = -2(1-4t)^{-\frac{1}{2}} C^{(1)} + (1-4t)^{\frac{1}{2}} C^{(2)}. \tag{3.3}$$

Substituting (3.1) into (3.3), we obtain

$$2C^3 = -2C^{(1)} + (1-4t)C^{(2)}. \tag{3.4}$$

Differentiating both sides of (3.4), we can write

$$3!C^2C^{(1)} = -6C^{(2)} + (1-4t)C^{(3)}. \tag{3.5}$$

Substituting (3.1) into (3.5), we obtain

$$3!C^4 = -6(1-4t)^{\frac{1}{2}}C^{(2)} + (1-4t)^{\frac{3}{2}}C^{(3)}. \tag{3.6}$$

So we are led to put

$$N!C^{N+1} = \sum_{i=0}^{\left[\frac{N}{2}\right]} b_i(N)(1-4t)^{\frac{N}{2}-i}C^{(N-i)} \quad (N=1,2,3,\cdots). \tag{3.7}$$

Here $[x]$ stands for the greatest integer not exceeding x. Differentiating both the sides of (3.7) gives

$$\begin{aligned}(N+1)!C^N C^{(1)} &= \sum_{i=0}^{\left[\frac{N}{2}\right]} -4(\tfrac{N}{2}-i)b_i(N)(1-4t)^{\frac{N}{2}-i-1}C^{(N-i)} + \sum_{i=0}^{\left[\frac{N}{2}\right]} b_i(N)(1-4t)^{\frac{N}{2}-i}C^{(N+1-i)} \\ &= \sum_{i=1}^{\left[\frac{N}{2}\right]+1} -4(\tfrac{N}{2}+1-i)b_{i-1}(N)(1-4t)^{\frac{N}{2}-i}C^{(N+1-i)} + \sum_{i=0}^{\left[\frac{N}{2}\right]} b_i(N)(1-4t)^{\frac{N}{2}-i}C^{(N+1-i)}.\end{aligned} \tag{3.8}$$

Substituting (3.1) into (3.8), we obtain

$$\begin{aligned}(N+1)!C^{N+1} = &\sum_{i=1}^{\left[\frac{N}{2}\right]+1} -4(\tfrac{N}{2}+1-i)b_{i-1}(N)(1-4t)^{\frac{N+1}{2}-i}C^{(N+1-i)} \\ &+ \sum_{i=0}^{\left[\frac{N}{2}\right]} b_i(N)(1-4t)^{\frac{N+1}{2}-i}C^{(n+1-i)}.\end{aligned} \tag{3.9}$$

Also, by replacing N by $N+1$ in (3.7), we can write

$$(N+1)!C^{N+2} = \sum_{i=0}^{\left[\frac{N+1}{2}\right]} b_i(N+1)(1-4t)^{\frac{N+1}{2}-i}C^{(N+1-i)}. \tag{3.10}$$

Comparing (3.9) with (3.10), we have the following recurrence relations. Here we need to consider the even and odd cases of N separately. The details are left to the reader.

$$b_0(N+1) = b_0(N), \qquad b_i(N+1) = -4(\tfrac{N}{2}+1-i)b_{i-1}(N) + b_i(N) \quad \text{for } 1 \leqslant i \leqslant \left[\tfrac{N+1}{2}\right]. \tag{3.11}$$

From (3.2) and (3.7), we obtain

$$C^2 = b_0(1)(1-4t)^{\frac{1}{2}}C^{(1)} = (1-4t)^{\frac{1}{2}}C^{(1)}. \tag{3.12}$$

Thus, from (3.12), we can write

$$b_0(1) = 1. \tag{3.13}$$

From (3.11), we easily obtain

$$b_0(N+1) = b_0(N) = \cdots = b_0(1) = 1. \tag{3.14}$$

The equation in (3.11) can be rewritten as

$$b_i(N+1) = -2(N+2-2i)b_{i-1}(N) + b_i(N). \tag{3.15}$$

To proceed further, we define

$$\begin{aligned} S_{N,1} &= N + (N-1) + \cdots + 1, \\ S_{N,j} &= N S_{N+1,j-1} + (N-1) S_{N,j-1} + \cdots + 1 S_{2,j-1} \quad (j \geqslant 2). \end{aligned} \tag{3.16}$$

Now,

$$\begin{aligned} b_1(N+1) &= -2N b_0(N) + b_1(N) = -2N + b_1(N) = -2N - 2(N-1) + b_1(N-1) \\ &= \cdots = -2(N + (N-1) \cdots + 1) + b_1(1) = -2S_{N,1}, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} b_2(N+1) = &-2(N-2) b_1(N) + b_2(N) = (-2)^2 (N-2) S_{N-1,1} + b_2(N) \\ &= (-2)^2 \big((N-2) S_{N-1,1} + (N-3) S_{N-2,1} \big) + b_2(N-1) = \cdots \\ &= (-2)^2 \big((N-2) S_{N-1,1} + (N-3) S_{N-2,1} + \cdots + 1 S_{2,1} \big) + b_2(3) = (-2)^2 S_{N-2,2}. \end{aligned} \tag{3.18}$$

Similarly to the cases of $i = 1$ and 2, for $i = 3$, we obtain

$$b_3(N+1) = (-2)^3 S_{N-4,3}. \tag{3.19}$$

Thus we can deduce that, for $1 \leqslant i \leqslant \left[\frac{N+1}{2}\right]$,

$$b_i(N+1) = (-2)^i S_{N+2-2i,i}. \tag{3.20}$$

Here, from (3.14) and (3.20), we obtain the following theorem.

Theorem 3.1. *The following family of differential equations*

$$N! C^{N+1} = \sum_{i=0}^{\left[\frac{N}{2}\right]} b_i(N) (1-4t)^{\frac{N}{2}-i} C^{(N-i)} \quad (N = 1, 2, 3, \cdots) \tag{3.21}$$

has a solution

$$C = C(t) = \frac{2}{1 + \sqrt{1-4t}}, \tag{3.22}$$

where

$$b_0(N) = 1, b_i(N) = (-2)^i S_{N+1-2i,i} \quad \left(1 \leqslant i \leqslant \left[\tfrac{N}{2}\right]\right). \tag{3.23}$$

Now, we shall give an application of the result in Theorem 3.1. It follows from (3.21) that

$$\begin{aligned} N! \sum_{k=0}^{\infty} C_k^{(N+1)} t^k &= \sum_{i=0}^{\left[\frac{N}{2}\right]} b_i(N) \sum_{l=0}^{\infty} \binom{\frac{N}{2} - i}{l} (-4t)^l \times \sum_{m=0}^{\infty} C_{m+N-i} (m+N-i)_{N-i} t^m \\ &= \sum_{i=0}^{\left[\frac{N}{2}\right]} b_i(N) \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{\frac{N}{2} - i}{k-m} (-4)^{k-m} \times C_{m+N-i} (m+N-i)_{N-i} t^k \\ &= \sum_{k=0}^{\infty} \Big(\sum_{i=0}^{\left[\frac{N}{2}\right]} \sum_{m=0}^{k} \binom{\frac{N}{2} - i}{k-m} (m+N-i)_{N-i} \\ &\quad \times (-4)^{k-m} b_i(N) C_{m+N-i} \Big) t^k. \end{aligned} \tag{3.24}$$

Thus, from (3.24), we obtain the following theorem.

Theorem 3.2. *For $k = 0, 1, 2, \cdots$, and $N = 1, 2, 3, \cdots$, we have*

$$C_k^{(N+1)} = \frac{1}{N!} \sum_{i=0}^{\left[\frac{N}{2}\right]} \sum_{m=0}^{k} \binom{\frac{N}{2} - i}{k - m} (m + N - i)_{N-i} (-4)^{k-m} b_i(N) C_{m+N-i},$$

where the $b_i(N)'s$ are as in Theorem 3.1.

Remark. Combining (2.23) with (3.21), we can show that

$$(1-4t)^{\frac{N}{2}} C^{N+1} = \sum_{i=0}^{[N/2]} \sum_{j=1}^{N-i} \frac{a_j(N-i)}{N!} b_i(N) (1-4t)^{\frac{j}{2}} C^{j+1} = \sum_{j=1}^{N} \sum_{i=0}^{\min(N-j,\,[N/2])} \frac{a_j(N-i)}{N!} b_i(N) (1-4t)^{\frac{j}{2}} C^{j+1}. \quad (3.25)$$

Equivalently, (3.25) can be expressed as

$$\sum_{i=0}^{\min(N-j, \left[\frac{N}{2}\right])} \frac{a_j(N-i)}{N!} b_i(N) = \delta_{j,N}, \quad (1 \leqslant j \leqslant N) \quad (3.26)$$

when $\delta_{j,N}$ is the Kronecker delta.

4. FURTHER REMARKS

We start our discussion here with the following expansion of $\sqrt{1+y}$:

$$\sqrt{1+y} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n-1}}{4^n(2n-1)} y^n. \quad (4.1)$$

Integrating both sides of (4.1) from 0 to 1, we immediately obtain

$$\sum_{n=0}^{\infty} C_n \frac{(-1)^{n-1}}{4^n(2n-1)} = \frac{1}{3}\left(4\sqrt{2} - 2\right). \quad (4.2)$$

Next, we integrate the generating function of the Catalan numbers from 0 to $\frac{1}{4}$.

$$\int_0^{\frac{1}{4}} \frac{2}{1 + \sqrt{1-4t}} dt = \int_0^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} t^n dt. \quad (4.3)$$

The left hand side of (4.3), after making the change of variable $t = \frac{1}{4}(1 - y^2)$, is equal to

$$\int_0^1 \frac{y}{1+y} dy = [y - \log(1+y)]_0^1 = 1 - \log 2. \quad (4.4)$$

Thus, from (4.3) and (4.4), we derive the following identity:

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \binom{2n}{n} \left(\frac{1}{4}\right)^{n+1} = 1 - \log 2. \quad (4.5)$$

Finally, again from the generating function of Catalan numbers and (4.1), we have

$$
\begin{aligned}
2 &= \Big(\sum_{l=0}^{\infty} C_l t^l\Big)\Big(1+\sqrt{1-4t}\Big) = \Big(\sum_{l=0}^{\infty} C_l t^l\Big)\Big(1-\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{2m-1}t^m\Big) \\
&= \sum_{l=0}^{\infty} C_l t^l - \Big(\sum_{l=0}^{\infty} C_l t^l\Big)\Big(\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{2m-1}t^m\Big) \\
&= \sum_{n=0}^{\infty} C_n t^n - \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{2m}{m}\frac{1}{2m-1}C_{n-m}\right)t^n \\
&= \sum_{n=0}^{\infty}\left(C_n - \sum_{m=0}^{n} C_m C_{n-m}\frac{m+1}{2m-1}\right)t^n.
\end{aligned} \tag{4.6}
$$

Therefore, from (4.6), we obtain the recurrence relation:

$$
C_n - \sum_{m=0}^{n} C_m C_{n-m}\frac{m+1}{2m-1} = \begin{cases} 2 & \text{for } n=0, \\ 0 & \text{for } n>0. \end{cases} \tag{4.7}
$$

Noting that $C_n = \frac{1}{n}\binom{2n}{n+1}$, we see that (4.7) for $n>0$ is equivalent to the following identity:

$$
\binom{2n}{n+1} = n\sum_{m=0}^{n}\frac{m+1}{m(n-m)(2m-1)}\binom{2m}{m+1}\binom{2n-2m}{n-m+1} \quad (n>0). \tag{4.8}
$$

Further, separating terms corresponding to $m=0$ and $m=n$ from (4.7) for $n>0$ and after rearranging the terms, we get the following recurrence relations for the Catalan numbers:

$$
C_0 = C_1 = 1, \quad C_n = \frac{2n-1}{3(n-1)}\sum_{m=1}^{n-1} C_m C_{n-m}\frac{m+1}{2m-1} \quad (n \geqslant 2). \tag{4.9}
$$

Compare (4.9) with the recurrence relation in (1.2).

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