Theoretical Computer Science 98 (1992) 199–247
Elsevier

Fundamental Study

The context-freeness of the languages associated with vector addition systems is decidable*

Sylviane R. Schwer

C.N.A.M., Département de Mathématiques et informatique, 292 rue Saint-Martin, 75141 Paris Cedex 03, France

Communicated by M. Nivat
Received May 1987
Revised December 1988, February 1990

Abstract

Schwer, S.R., The context-freeness of the languages associated with vector addition systems is decidable, Theoretical Computer Science 98 (1992) 199–247.

This paper introduces new tools designed for the study of the languages associated with vector addition systems or, equivalently, Petri nets. With these tools, we prove that the problem of deciding the context-freeness of such a language is solvable.

Contents

INTRODUCTION 200
PART I: TOOLS 200
0. Notations 200
1. VAS languages and iterable factors 202
2. The covering automaton 207
3. Study of iterating systems 217
4. Dominating coordinates 220
PART II: DECIDABILITY OF THE CONTEXT-FREENESS OF A VAS LANGUAGE 223
0. Introduction 223
1. A first necessary condition 224
2. Other necessary conditions 229

* This work was done when the author was a member of LaBRI (URA 726 DU CNRS), at the University of Bordeaux I.

3. More necessary conditions. 231
4. Last necessary conditions. 235
5. All the conditions together are sufficient. 239
6. Decidability and further comments. 242
7. Conclusion. 245
Acknowledgment. 246
References. 246

INTRODUCTION

Vector addition systems (VASs) or equivalent formalisms like Petri nets (PNs) are widespread tools for the modelization of systems [10, 16]. These systems are characterized by the set of all possible sequences of actions described by the associated language. So, there have been a lot of studies in the field of VAS languages or PN languages [4, 7, 16, 22]. It has been shown that all these languages are context-sensitive [16], and that some are not context-free. Conversely, there are context-free languages that are not VAS languages. In [15], Peterson raises the problem of the characterization of those VAS languages that are context-free. The main result of this paper is to give a characterization of (free labelled) vector addition systems that have a context-free language associated with them, and to show that this characteristic property is a decidable one.

Since it has been proved by Ginzburg and Yoeli [6], and also by Valk and Vidal-Naquet [21] (in terms of Petri nets) that it is decidable whether the language associated with a vector addition system is rational, our result completes the knowledge of the relations between VAS languages and the classical Chomsky's hierarchy.

To get this result, we develop several tools that we believe to be of interest by themselves.

The paper is divided into two parts: the first one is devoted to the presentation of these tools, while in the second one, we prove our decidability result. Most of the notions and results of this paper were first presented in terms of PNs in [18].

PART I: TOOLS

In this part, we present several tools for the study of the languages associated with vector addition systems.

0. Notations

$\mathbf{N}$ is the set of nonnegative integers. ω is the cardinal of $\mathbf{N}$, and $\mathbf{N}\omega$ is the set $\mathbf{N}\cup\{\omega\}$. The relations $<$ **and** $\leqslant$ over $\mathbf{N}$ are extended to $\mathbf{N}\omega$ in the usual way: $\forall n\in\mathbf{N}$ $n<\omega$ and $\omega\leqslant\omega$. The functions $+$ and $-$ from $\mathbf{N}\times\mathbf{N}$ to $\mathbf{N}$ are extended to functions from $\mathbf{N}\omega \times \mathbf{N}$ to $\mathbf{N}\omega$, again in the usual way; if $n \in \mathbf{N}$, $\omega + n = \omega$ and $\omega - n = \omega$. $\mathbf{Z}$ is the set of integers, $\mathbf{Q}$ the set of rational numbers and $\mathbf{R}$ the set of real numbers. $[\underline{k}]$ denotes the following set of integers: $\{i \in \mathbf{N} \mid 1 \leqslant i \leqslant k\}$.

For k in $\mathbf{N}$, an element of V^k, where V stands for $\mathbf{N}$ or $\mathbf{N}\omega$ or $\mathbf{Z}$ or $\mathbf{Q}$ or $\mathbf{Z}\omega$, is a k-tuple of elements of V. $v[i]$ denotes the ith component of the k-tuple v. We denote by $\mathbf{0}$ the k-tuple with all components equal to 0, and $\mathbf{1}$ the k-tuple with all components equal to 1. V^k is a monoid with respect to the addition, which is done componentwise, and $\mathbf{0}$ is its unit. If $L \subseteq V^k$, then $v + L = \{v + l \mid l \in L\}$. The relations $<$ and $\leqslant$ are also extended from V to V^k componentwise.

For any subsequence S of $(1, \ldots, k)$, $S = (i_1, i_2, \ldots, i_r)$, the projection of a k-tuple v on S is $\Pi_S(v) = (v(i_1), v(i_2), \ldots, v(i_r))$. If S is reduced to a singleton $\{s\}$, we denote it by $\Pi_s(v)$. For a subset E of V^k, we note $\Pi_S(E) = \{\Pi_S(v) \mid v \in E\}$.

Formal languages

The reader is supposed to have a basic knowledge of the theory of rational and context-free languages [2, 5, 8]; in what follows, we only make precise our notations.

Let T be a finite set, called an alphabet, T^* is the set of all finite strings (or words) on T. For u in T^*, $|u|$ denotes the length of u. λ is the empty word and $T^+ = T^* \setminus \{\lambda\}$.

A word u is a prefix of v if there exists some w such that $v = uw$. The set of all prefixes of v is denoted by $Pref(v)$. This notation is extended to any set of strings: $Pref(L) = \{u \mid \exists v \in L,\ u \in Pref(v)\}$. If $L = Pref(L)$, then L is said to be prefix-closed (in this paper, we deal mainly with prefix-closed languages). The relation "to be a prefix" is a partial order over T^*, which is denoted as $<$. A word u is a factor of v if there exist some w_1 and w_2 such that $v = w_1 u w_2$. A factor u is a proper factor of v if $u \neq \lambda$ and $u \neq v$. The set of all subwords of v is denoted as $Sub(v)$. This notation is extended to any set of words: $Sub(L) = \{u \mid \exists v \in L,\ u \in Sub(v)\}$.

Let u and v be two words over T, the shuffle of u and v is the set

$$u \mathbin{⧢} v = \{u_1 v_1 \ldots u_n v_n \mid u_1, \ldots, u_n, v_1, \ldots, v_n \in T^*,\ u = u_1 u_2 \ldots u_n,\ v = v_1 v_2 \ldots v_n\}.$$

Let L and M be two languages, the shuffle of L and M is the set

$$L \mathbin{⧢} M = \{w \mid \exists u \in L,\ \exists v \in M,\ w \in u \mathbin{⧢} v\}.$$

Linear and stratified sets

We give some definitions [5] concerning linear and stratified sets.

Definition 0.1. Given subsets C and P of $\mathbf{N}^k$, let $L(C, P)$ denote the set of all elements in $\mathbf{N}^k$ which can be represented in the form $c_0 + x_1 + x_2 + \cdots + x_n$ for some c_0 in C and some (possibly empty) sequence of elements $x_1, x_2, \ldots, x_n$ of P. C is called the set of constants, and P the set of periods of $L(C, P)$.

Definition 0.2. A subset A of $\mathbf{N}^k$ is said to be a *linear set* if $A = L(C, P)$ for a set $C = \{c\}$ which is a singleton, and a finite set $P = \{p_1, p_2, \ldots, p_r\}$. It is a *semilinear set* if it is a finite union of linear sets.

We write $L(c, p_1, p_2, \ldots, p_r)$ instead of $L(\{c\}, \{p_1, p_2, \ldots, p_r\})$.

Definition 0.3. A subset A of $\mathbf{N}^k$ is said to be *stratified* if the following two conditions hold:

(i) Each v in A has at most two components $v[i]$ and $v[j]$ not equal to 0.

(ii) There are not two elements v and v' of A, each with two nonzero components: $v[i] \neq 0$, $v[j] \neq 0$, $v'[m] \neq 0$ and $v'[n] \neq 0$, such that $1 \leqslant i < m < j < n \leqslant k$.

Remark. A linear stratified set is a linear set having a stratified period.

Definition 0.4. A language $L \subseteq T^*$ is a *bounded language* if there exist words $w_1, w_2, \ldots, w_n$ in T^* such that $L \subseteq w_1^* w_2^* \ldots w_n^*$.

Bounded languages are a natural representation of sets of vectors of $\mathbf{N}^n$: for each n-tuple of words $w = \langle w_1, w_2, \ldots, w_n \rangle$, we denote by ψ_w the mapping of $\mathbf{N}^n$ into $w_1^* w_2^* \ldots w_n^*$ defined by $\psi_w(i_1, i_2, \ldots, i_n) = w_1^{i_1} w_2^{i_2} \ldots w_n^{i_n}$. Under these conditions, semilinear stratified sets are represented by special bounded languages.

Let us define the following operation on languages: for a subset Z of T^* and two words x and y in T^*, let $(x, y) * Z = \bigcup_{k \in \mathbf{N}} x^k Z y^k$.

Let $\mathscr{B}$ be the smallest family of bounded languages containing all finite bounded languages and closed under union, product and $*$-operation.

Theorem 0.5 (Ginsburg [5]). *Let $L \subseteq w_1^* w_2^* \ldots w_n^*$ be a bounded language, and let $w = \langle w_1, w_2, \ldots, w_n \rangle$. The following propositions are equivalent:*

(i) *L is context-free.*

(ii) *L is in $\mathscr{B}$.*

(iii) *$\psi_w^{-1}(L)$ is a finite union of linear stratified sets.*

(iv) *$L = \psi_w(M)$, where M is a semilinear set, finite union of stratified sets.*

1. VAS languages and iterable factors

Iterable factors

We now introduce the notion of iterable factor for a language L.

Definition 1.1. For a language L of T^* and a word u in T^+, we say that u is an *iterable factor* of L iff $(\forall n \in \mathbf{N})$ $(T^* u^n T^* \cap L \neq \emptyset)$. We denote by $Iter(L)$ the set of all iterable factors of L. u is an *elementary iterable factor* of L iff u is an iterable factor of L and no proper factor of u is an iterable factor of L.

Note that for a prefix-closed language L, u is an iterable factor of L iff $(\forall n\in\mathbf{N})$ $(T^*u^n\cap L\neq\emptyset)$.

Example and counterexample

- every infinite rational or context-free language has an iterable factor (this is easily proved using the pumping lemma in any of its different forms).
- the language of square-free words does not contain any iterable factor.

The following lemma, although obvious, is useful.

Lemma 1.2. *Let L be a language $\subseteq T^*$ and let u be an iterable factor of L. If w is a sesqui-power of u, i.e. $w=(yx)^r$ with $u=xy$ and $r>0$, then w is also an iterable factor of L.*

Vector addition systems

We first recall the basic notions [10, 16] about vector addition systems and their associated languages.

Definition 1.3. A *k-vector addition system* is a triple $A=(T,\varphi,\boldsymbol{a})$, where T is an alphabet, $\varphi: T^*\to\mathbf{Z}^k$ is a morphism, and $\boldsymbol{a}$ is a k-tuple of $\mathbf{N}^k$.

We omit the integer k whenever it can be understood from the context.

Definition 1.4. A string w in T^* is *legal* in $A=(T,\varphi,\boldsymbol{a})$ iff every prefix v of w is such that $\boldsymbol{a}+\varphi(v)\geqslant\mathbf{0}$. The *language associated* with A, denoted by $L(A)$, is the set of all legal strings in A.

It follows from the definition that $L(A)$ is prefix-closed and we can write

$$L(A)=\{w\in T^*\mid \boldsymbol{a}+\varphi(Pref(w))\subset\mathbf{N}^k\}.$$

Example. With $k=4$, let $A=(T,\varphi,\boldsymbol{a})$, where $T=\{t_1,t_2,t_3,t_4,t_5\}$, $\boldsymbol{a}=[1,0,0,0]$ and $\varphi(t_1)=[-1,1,0,0]$, $\varphi(t_2)=[0,-1,1,0]$, $\varphi(t_3)=[2,0,-1,0]$, $\varphi(t_4)=[-3,0,0,1]$, $\varphi(t_5)=[7,0,0,-1]$.

We have $\boldsymbol{a}+\varphi(t_1)=[0,1,0,0]\geqslant\mathbf{0}$, $\boldsymbol{a}+\varphi(t_1t_2)=[0,0,1,0]\geqslant\mathbf{0}$, $\boldsymbol{a}+\varphi(t_1t_2t_3)=[2,0,0,0]\geqslant\mathbf{0}$, so that $t_1t_2t_3$ is a legal word in A, $\boldsymbol{a}+\varphi(t_4t_5)=[5,0,0,0]\geqslant\mathbf{0}$, but $\boldsymbol{a}+\varphi(t_4)=[-2,0,0,1]$ and hence, is not $\geqslant\mathbf{0}$, so, t_4t_5 is not a legal word in A.

Calling $R(A)$ the subset of $\mathbf{N}^k$ of all vectors that can be reached from $\boldsymbol{a}$ by a finite sequence of additions of vectors in $\varphi(T)$, with the condition that, after each addition, the result is in $\mathbf{N}^k$, we have $R(A)=\{\boldsymbol{a}+\varphi(v)\mid v\in L(A)\}$. The well-known reachability problem is: given a k-VAS $A=(T,\varphi,\boldsymbol{a})$ and a k-tuple $\boldsymbol{b}$ of $\mathbf{N}^k$, is $\boldsymbol{b}$ in $R(A)$? Kosaraju [12] proved that this problem is decidable, and Mayr [14] has given a proof of this result in terms of Petri nets. A first partial result was given by Hopcroft and Pansiot [9].

If we restrict our attention to one coordinate, say the ith, and if we define the language $L_i(A)=\{w\in T^*\mid \text{for all prefix } v \text{ of } w\ \boldsymbol{a}[i]+\varphi(v)[i]\geqslant 0\}$, it is clear that $L_i(A)$

is an iterated counter language (it follows from the definition: an iterated counter language is a language recognized by a pushdown automaton with only one stack symbol) and $L(A)$ is, thus, the intersection of k iterated counter languages.

A coordinate i is *bounded* if there exists an integer n_0 such that $\forall v \in L(A)$ $\boldsymbol{a}[i]+\varphi(v)[i]<n_0$. The ith coordinate is *unbounded* otherwise. We note that $Unb(A)=\{i \in [\underline{k}] \mid i$ is an unbounded coordinate of the k-VAS $A\}$, and setting $L_{\text{Unb}}(A)=\bigcap_{i \in Unb(A)} L_i(A)$ and $L_{\text{Boun}}(A)=\bigcap_{i \notin Unb(A)} L_i(A)$, we have $L(A)=L_{\text{Unb}}(A) \cap L_{\text{Boun}}(A)$, and it is well-known [18] that $L_{\text{Boun}}(A)$ is a rational language; hence, the context-freeness depends primarily on the study of $L_{\text{Unb}}(A)$.

It is very easy to see that every factor u of a word of a VAS language $L=L(A)$ with $\varphi(u) \geqslant \mathbf{0}$ is an iterable factor of L (see Lemma 2.13). But in a VAS language $L(A)$, there may of course be iterable factors u such that $\mathbf{not}(\varphi(u) \geqslant \mathbf{0})$ (in what follows, we bring them to evidence). To be more precise, we introduce the following definition.

Definition 1.5. Let $A=(T, \varphi, \boldsymbol{a})$ be a k-VAS and w be an iterable factor of $L(A)$. We set
$\| w \|^{+}=\{p \in Unb(A) \mid \varphi(w)[p]>0\}$ is the *positive support* of w.
$\| w \|^{-}=\{p \in Unb(A) \mid \varphi(w)[p]<0\}$ is the *negative support* of w.
$\| w \|^{0}=\{p \in Unb(A) \mid \varphi(w)[p]=0\}$ is the *zero support* of w.

Iterable factors of a VAS language

We can give now more precise details about iterable factors of VAS languages, proving some technical lemmas that will be useful in the sequel.

We begin with iterable factors having no negative support.

Lemma 1.6. *Let $A=(T, \varphi, \boldsymbol{a})$ be a k-VAS and u, w_0, w_1, w_2 be words in T^* with $u \neq \lambda$. If $\varphi(u) \geqslant \mathbf{0}$ then $w_0 u w_1 u w_2 \in L(A) \Rightarrow w_0 u u w_1 w_2 \in L(A)$.*

We call this the *hoardation principle*: if you have begun to give something you can give all at once, instead of giving from time to time.

Proof. Let $f < w_0 u u w_1 w_2$. Then

(a) $f < w_0 u$ which is in $L(A)$, hence $\boldsymbol{a}+\varphi(f) \geqslant \mathbf{0}$; or

(b) $f = w_0 u u'$ with $u' < u$: $\varphi(f)=\varphi(w_0)+\varphi(u)+\varphi(u')=\varphi(w_0 u')+\varphi(u)$ as $w_0 u' < w_0 u$, $\boldsymbol{a}+\varphi(w_0 u') \geqslant \mathbf{0}$, and $\varphi(u) \geqslant \mathbf{0}$ by hypothesis, hence $\boldsymbol{a}+\varphi(f) \geqslant \mathbf{0}$; or

(c) $f = w_0 u^2 w_1'$ with $w_1' < w_1$: $\boldsymbol{a}+\varphi(f)=\boldsymbol{a}+\varphi(w_0 u w_1')+\varphi(u)$, $\boldsymbol{a}+\varphi(w_0 u w_1') \geqslant \mathbf{0}$ because $w_0 u w_1 \in L(A)$, and $\varphi(u) \geqslant \mathbf{0}$ by hypothesis, hence $\boldsymbol{a}+\varphi(f) \geqslant \mathbf{0}$; or

(d) $f = w_0 u^2 w_1 w_2'$ with $w_2' < w_2$: $\boldsymbol{a}+\varphi(f)=\boldsymbol{a}+\varphi(w_0 u^2 w_1 w_2')=\boldsymbol{a}+\varphi(w_0 u w_1 w_2')+\varphi(u)$, $\boldsymbol{a}+\varphi(w_0 u w_1 u w_2') \geqslant \mathbf{0}$ because $w_0 u w_1 u w_2 \in L(A)$, and $\varphi(u) \geqslant \mathbf{0}$ by hypothesis, hence $\boldsymbol{a}+\varphi(f) \geqslant \mathbf{0}$.

We have shown that for all prefixes f of $w_0 u u w_1 w_2$, $\boldsymbol{a}+\varphi(f) \geqslant \mathbf{0}$. Hence, $w_0 u u w_1 w_2 \in L(A)$. $\square$

Lemma 1.7. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and u, w_0, w_1 be words in T^* with $u \neq \lambda$. If $\varphi(u) \geqslant \mathbf{0}$ then*

(i) $w_0 u \in L(A) \Rightarrow w_0 u^* \subseteq L(A)$,

(ii) $w_0 u w_1 \in L(A) \Rightarrow w_0 u^+ w_1 \subseteq L(A)$.

Proof. (i) is a consequence of (ii) and of the prefix closure of $L(A)$.

(ii) (a) $f < w_0 u$ with $w_0 u$ *in* $L(A)$, hence $\boldsymbol{a}+\varphi(f) \geqslant \mathbf{0}$; or

(b) $f = w_0 u^n u'$ with $u' < u$ and $n > 0$: $\varphi(f) = \varphi(w_0) + n \cdot \varphi(u) + \varphi(u') = \varphi(w_0 u') + n \cdot \varphi(u)$ as $w_0 u' < w_0 u$, $\boldsymbol{a} + \varphi(w_0 u') \geqslant \mathbf{0}$, and $\varphi(u) \geqslant \mathbf{0}$ by hypothesis, hence $\boldsymbol{a}+\varphi(f) \geqslant \mathbf{0}$; or

(c) $f = w_0 u^n w_1'$ *with* $w_1' < w_1$ *and* $n > 0$: $\boldsymbol{a}+\varphi(f) = \boldsymbol{a}+\varphi(w_0 u w_1') + (n-1)\varphi(u)$, $\boldsymbol{a}+\varphi(w_0 u w_1') \geqslant \mathbf{0}$ because $w_0 u w_1 \in L(A)$, and $\varphi(u) \geqslant \mathbf{0}$ by hypothesis, hence $\boldsymbol{a}+\varphi(f) \geqslant \mathbf{0}$. □

Remark. $w_0 u w_1 \in L(A)$ does not imply that $w_0 w_1 \in L(A)$, as shown by the following counterexample:

Let $A=(T,\varphi,\boldsymbol{a})$ with $T=\{a,b,c\}$; $\varphi(a)=(-1,1,1)$, $\varphi(b)=(1,-1,1)$, $\varphi(c)=(0,0,-1)$ and $a=(1,0,0)$.

Let $w_0=\lambda$, $w_1=c$ and $u=ab$. $w_0 u w_1 = abc \in L(A)$ but $w_0 w_1 = c \notin L(A)$.

Lemma 1.8. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and let u be a word in T^+. If u is an elementary iterable factor of $L(A)$, then every proper factor v of u satisfies* $\mathbf{not}(\varphi(v) \geqslant \mathbf{0})$.

Proof. Suppose that $u = u_0 v u_1$ with $\varphi(v) \geqslant \mathbf{0}$.

As u is an iterable factor of $L(A)$, there is w_0 in T^* such that $w_0 u = w_0 u_0 v u_1$ is in $L(A)$. Hence, by Lemma 1.7(ii), $w_0 u_0 v^+$ is included in $L(A)$, i.e. v is also an iterable factor of $L(A)$, which is inconsistent with the assumption that u is an elementary iterable factor of $L(A)$. □

The next lemmas show that there exist iterable factors having a nonempty negative support, and give some information about their relative place.

Lemma 1.9. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and aubv be a word of $L(A)$. If $\varphi(u) \geqslant \mathbf{0}$ and $\|v\|^- \subseteq \|u\|^+$, then v is an iterable factor of $L(A)$. More precisely: there exists $(x,y) \geqslant (1,1)$ such that $\varphi(u^x v^y) \geqslant \mathbf{0}$ and for every (r,s) in $\mathbf{N}^2$ such that $yr \geqslant xs$, we have $(au^r bv^s \in L(A)) \Rightarrow ((\forall n, m \in \mathbf{N},\ n \geqslant m,\ n \neq 0) \Rightarrow (au^{n \cdot r} bv^{m \cdot s} \in L(A)))$.*

Proof. Choose $z = \mathrm{Sup}_{p \in \|v\|} - \{(-\varphi(v)[p]/\varphi(u)[p])\}$; and choose (x,y) such that $x = y \cdot z$; hence, for all p in $\|v\|^-$

$$\begin{aligned}\varphi(u^x v^y)[p] &= x \cdot \varphi(u)[p] + y \cdot \varphi(v)[p] \\ &= y \cdot ((x/y)\varphi(u)[p] + \varphi(v)[p]) = y \cdot (z \cdot \varphi(u)[p] + \varphi(v)[p]) \\ &\geqslant y \cdot ((-\varphi(v)[p]/\varphi(u)[p]) \cdot \varphi(u)[p] + \varphi(v)[p]) = 0.\end{aligned}$$

Thus, we have $\varphi(u^r v^s) \geqslant \mathbf{0}$ whenever $yr \geqslant xs$.

Let f be a prefix of $au^{n \cdot r} bv^{m \cdot s}$. We show that $\boldsymbol{a} + \varphi(f) \geqslant \mathbf{0}$.

Either f is a prefix of $au^{n \cdot r} bv$, and by Lemma 1.7(ii), $\boldsymbol{a} + \varphi(f) \geqslant \mathbf{0}$, or

$$f = au^{n \cdot r} bv^{m' \cdot s + s'} v' \text{ with } m' < m,\ s' < s \text{ and } v' < v.$$

$$f = au^{m' \cdot r + r + (n - m' - 1) \cdot r} bv^{m' \cdot s + s'} v',$$

$$\boldsymbol{a} + \varphi(f) = \boldsymbol{a} + \varphi(au^r bv^{s'} v') + m' \cdot \varphi(u^r v^s) + (n - m' - 1) \cdot r \cdot \varphi(u),$$

$$au^r bv^{s'} v' < au^r bv^s;$$

hence, $\boldsymbol{a} + \varphi(au^r bv^{s'} v') \geqslant \mathbf{0}$; we have $\varphi(u^r v^s) \geqslant \mathbf{0}$ and $\varphi(u) \geqslant \mathbf{0}$, and by hypothesis $n > m'$. So, $\boldsymbol{a} + \varphi(f) \geqslant \mathbf{0}$. $\square$

The two following lemmas are just generalizations of the preceding one.

Lemma 1.10. *Let $A = (T, \varphi, \boldsymbol{a})$ be a k-VAS and $u_1, u_2, v_1, v_2 \in T^+$; $w_0, w_1, w_2, w_3, w_4 \in T^*$. For $i = 1, 2$ if $\varphi(u_i) \geqslant \mathbf{0}$ and if $\| v_i \|^- \subseteq \| u_i \|^+$, then there exists $(x_i, y_i) \geqslant (1, 1)$ such that $\varphi(u^x i v^y i) \geqslant \mathbf{0}$ and for every (r_i, s_i) in $\mathbf{N}^2$ such that $y_i r_i \geqslant x_i s_i$, we have*

(i) $(w_0 u_1^{r_1} w_1 u_2^{r_2} w_2 v_1^{s_1} w_3 v_2^{s_2} w_4 \in L(A)) \Rightarrow (\forall n_1, n_2, m_1, m_2 \geqslant 1)\quad (n_1 \geqslant m_1)\quad (n_2 \geqslant m_2)$ $(w_0 u_1^{r_1 n_1} w_1 u_2^{r_2 n_2} w_2 v_1^{s_1 m_1} w_3 v_2^{s_2 m_2} w_4 \in L(A))$

(ii) $(w_0 u_1^{r_1} w_1 u_2^{r_2} w_2 v_2^{s_2} w_3 v_1^{s_1} w_4 \in L(A)) \Rightarrow (\forall n_1, n_2, m_1, m_2 \geqslant 1)\quad (n_1 \geqslant m_1)\quad (n_2 \geqslant m_2)$ $(w_0 u_1^{r_1 n_1} w_1 u_2^{r_2 n_2} w_2 v_2^{s_2 m_2} w_3 v_1^{s_1 m_1} w_4 \in L(A))$

(iii) $(w_0 u_1^{r_1} w_1 v_1^{s_1} w_2 u_2^{r_2} w_3 v_2^{s_2} w_4 \in L(A)) \Rightarrow (\forall n_1, n_2, m_1, m_2 \geqslant 1)\quad (n_1 \geqslant m_1)\quad (n_2 \geqslant m_2)$ $(w_0 u_1^{r_1 n_1} w_1 v_1^{s_1 m_1} w_2 u_2^{r_2 n_2} w_3 v_2^{s_2 m_2} w_4 \in L(A))$

Proof. (i) Suppose that $w_0 u_1^{r_1} w_1 u_2^{r_2} w_2 v_1^{s_1} w_3 v_2^{s_2} w_4 \in L(A)$. Then, by Lemma 1.9,

$$(\forall n_1, m_1 \geqslant 1)\ (n_1 \geqslant m_1)\ (n_1 \neq 0)\ (w_0 u_1^{r_1 n_1} (w_1 u_2^{r_2} w_2) v_1^{s_1 m_1} (w_3 v_2^{s_2} w_4) \in L(A));$$

then for n_1 and m_1 fixed,

$$(\forall n_2, m_2 \geqslant 1)\ (n_2 \geqslant m_2)\ (n_2 \neq 0)\ ((w_0 u_1^{r_1 n_1} w_1) u_2^{r_2 n_2} (w_2 v_1^{s_1 m_1} w_3) v_2^{s_2 m_2} w_4 \in L(A)).$$

(ii) and (iii) are proved in a similar manner. $\square$

Lemma 1.11. *Let $A = (T, \varphi, \boldsymbol{a})$ be a k-VAS and let $w_1 u_1 w_2 u_2 \ldots w_n u_n \in L(A)$ with $u_1, u_2, \ldots, u_n \in T^+$; $w_1, w_2, \ldots, w_n \in T^*$.*

If $\| u_1 \|^- = \emptyset$ and if for $1 \leqslant i \leqslant n$, $\| u_i \|^- \subseteq \bigcup_{j<i} \| u_j \|^+$, $\exists r_1, r_2, \ldots, r_n \in \mathbf{N}$ such that

$$w_1 u_1^{r_1} w_2 u_2^{r_2} \ldots w_n u_n^{r_n} \in L(A) \Rightarrow (\forall m_1, m_2, \ldots, m_n \geqslant 1)\ (m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n)$$
$$(w_1 u_1^{r_1 m_1} w_2 u_2^{r_2 m_2} \ldots w_n u_n^{r_n m_n} \in L(A)).$$

Proof. This lemma is proved by induction on n. For $n = 2$, it is true because of Lemma 1.9. Suppose that it is true for all integers $\leqslant n$. We prove it for $n + 1$. So, $\exists r'_1, r'_2, \ldots, r'_n \in \mathbf{N}$ such that $w_1 u_1^{r'_1} w_2 u_2^{r'_2} \ldots w_n u_n^{r'_n} \in L(A) \Rightarrow (\forall m_1, m_2, \ldots, m_n \geqslant 1)$ $(m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n)$ $(w_1 u_1^{r'_1 m_1} w_2 u_2^{r'_2 m_2} \ldots w_n u_n^{r'_n m_n} \in L(A))$. Calling $v = u_{n+1}$ and

$u=u_1^{r_1'}u_2^{r_2'}\dots u_n^{r_n'}$, we have $\|v\|^- \subseteq \|u\|^+$, so that there exists $(x,y)\geqslant(1,1)$ such that $\varphi(u^x v^y)\geqslant\mathbf{0}$ and for every (r,s) in $\mathbf{N}^2$ such that $yr\geqslant xs$. We have, calling $r_i=r\cdot r_i'$ for $i\leqslant n$ and $r_{n+1}=s$,

$$(w_1 u_1^{r_1} w_2 u_2^{r_2}\dots w_n u_n^{r_n} w_{n+1} u_{n+1}^{r_{n+1}} \in L(A)) \Rightarrow ((\forall n, m\in\mathbf{N},\ n\geqslant m,\ n\neq 0)$$

$$\Rightarrow (w_1 u_1^{r_1 m_1} w_2 u_2^{r_2 m_2}\dots w_n u_n^{r_n m_n} w_{n+1} u_{n+1}^{m r_{n+1}} \in L(\mathrm{A}))). \qquad \square$$

Corollary 1.12. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and let $u_1,u_2,\dots,u_{n+1}\in T^+$ be words such that $\|u_1\|^-=\emptyset$ and, for $1\leqslant i\leqslant n+1$, $\|u_i\|^-\subseteq\bigcup_{j<i}\|u_j\|^+$. If $w_1,w_2,\dots,w_{n+1}\in T^*$ are words such that $w_1u_1^+w_2u_2^+\dots w_nu_n^+w_{n+1}u_{n+1}\cap L(A)\neq\emptyset$ then*

$$(\forall z\in\mathbf{N})\ (\mathrm{Card}(w_1u_1^+w_2u_2^+\dots w_nu_n^+w_{n+1}u_{n+1}^z\cap L(A))=\infty). \qquad \square$$

2. The covering automaton

In this part we introduce two new notions: the first one is the notion of strong loop for a language in a finite automaton, and the second is the notion of iterating system relative to a loop. With these two tools, we construct from the well-known Karp and Miller's coverability automaton, a new automaton with stronger properties (this result was presented in [20]).

Finite automata and strong loops

A finite automaton $\mathscr{A}=\langle T,Q,\delta,q_0,F\rangle$ consists of an alphabet T, a finite set Q of states, a transition function $\delta: Q\times T\to\mathscr{P}(Q)$, an initial state q_0 of Q, and a subset F, included in Q, of final states. δ is extended to a function $\delta:\mathscr{P}(Q)\times T^*\to\mathscr{P}(Q)$ the canonical way: $\forall G\subseteq Q$, $\forall u\in T^*$: $\delta(G,u)=\bigcup_{q\in G}\delta(q,u)$ where $\delta(q,\lambda)=q$, $\delta(q,ux)=\delta(\delta(q,u),x)$ for all q in Q, all u in T^* and all x in T. A word u is accepted by $\mathscr{A}$ if $\delta(q_0,u)\cap F\neq\emptyset$. $L(\mathscr{A})$, the set of all words accepted by $\mathscr{A}$, is called the language accepted by $\mathscr{A}$. If all the states of an automaton $\mathscr{A}$ are final states, i.e. $F=Q$, then $L(\mathscr{A})$ is prefix-closed; we write then $\mathscr{A}=\langle T,Q,\delta,q_0\rangle$.

A finite automaton $\mathscr{A}=\langle T,Q,\delta,q_0\rangle$ can be viewed as a pointed, labelled graph with T as set of labels, Q as set of vertices and δ as set of edges labelled by elements of T, the graph being pointed at q_0. In the sequel, we apply the usual (basic) terminology of graph theory to finite automata: it must be understood that we refer in fact to the corresponding graph. In particular, we note $Acc(q)=\{q'\in Q\mid\exists u\in T^*,\ q'\in\delta(q,u)\}$ is the set of states accessible from q, and $Coacc(q)=\{q'\in Q\mid\exists u\in T^*,\ q\in\delta(q',u)\}$ is the set of states coaccessible from q. A cycle in an automaton is a path going from a state q to itself: $(q,x_1,q_1)(q_1,x_2,q_2)\dots(q_{n-1},x_n,q)$; $w=x_1x_2\dots x_n$ is then the label of the cycle. A simple path is a nonempty path that contains no factor that is a cycle, and a simple cycle is a cycle that contains no proper factor that is a cycle.

A finite automaton $\langle T, Q, \delta, q_0, F\rangle$ is *reduced* if all its states are accessible from q_0 and coaccessible from F.

We shall make an intensive use of the following notions.

Definition 2.1. Let $\mathscr{A} = \langle T, Q, \delta, q_0, F\rangle$ be a finite automaton. A *loop* in $\mathscr{A}$ is a couple (q, w) such that

(i) $\exists u \in T^* \mid q \in (q_0, u)$ (q is accessible from q_0),

(ii) $q \in \delta(q, w)$ (there a cycle with label w going from q to q).

The first condition is of course true if $\mathscr{A}$ is reduced.

Definition 2.2. Let $L \subseteq T^*$ be a language and $\mathscr{A} = \langle T, Q, \delta, q_0, F\rangle$ be a finite automaton such that $L \subseteq L(\mathscr{A})$. $(q, w) \in Q \times T^*$ is a *strong loop* for L iff

(i) (q, w) is a loop in $\mathscr{A}$,

(ii) $(\forall n \in \mathbf{N})$, $(\exists f_n \in Pref(L))$, $(\delta(q_0, f_n) = q)$ and $(f_n w^n \in Pref(L))$.

It should be noted that a loop is related to a finite automaton, but that a strong loop is not only related to an automaton but also to a language. Every time that L is understood, we speak of strong loop.

Now, given a language L and a finite automaton $\mathscr{A}$ such that $L \subseteq L(\mathscr{A})$, it is an interesting property of $\mathscr{A}$ to have all its loops to be strong.

Definition 2.3. Let $L \subseteq T^*$ be a language and $\mathscr{A} = \langle T, Q, \delta, q_0, F\rangle$ be a finite automaton such that $L \subseteq L(\mathscr{A})$. We say that $\mathscr{A}$ has the *loop accessibility property for L* iff every loop in $\mathscr{A}$ is a strong loop for L.

Karp and Miller's coverability automaton

We now recall the construction of Karp and Miller [10] (see also [7, 11]), which yields a finite tree, called the coverability tree (also: reachability tree in [16]), from which is derived a finite deterministic automaton recognizing a superset of $L(A)$.

The coverability tree of the k-VAS $A = (T, \varphi, \boldsymbol{a})$ is a labelled tree whose vertices are labelled by elements of $(\mathbf{N}\omega)^k$ and whose edges are labelled on T. This tree and the labelling function l are defined simultaneously recursively in the following way.

(1) there is a vertex, labelled by $\boldsymbol{a}$, which is the root of the tree,

(2) for each label $\boldsymbol{s}$ of a vertex e and for each letter t, define $\boldsymbol{s}'$ by

$$\begin{array}{ll} \boldsymbol{s}'[i] = \boldsymbol{s}[i] + \varphi(t)[i] & \text{if there is no vertex, ancestor of } e, \text{ having label } \boldsymbol{h} \\ & \text{with } \boldsymbol{h} \leqslant \boldsymbol{s} + \varphi(t) \text{ and } \boldsymbol{h}[i] < \boldsymbol{s}[i] + \varphi(t)[i], \\ \boldsymbol{s}'[i] = \omega & \text{otherwise,} \end{array}$$

there exists a vertex e', labelled by $\boldsymbol{s}'$, and an edge (e, t, e'), going from e to e' and labelled by t, iff

(i) $\boldsymbol{s} + \varphi(t) \geqslant \mathbf{0}$,

(ii) there is no vertex g, ancestor of e, having label $\boldsymbol{h}$ with $\boldsymbol{h} = \boldsymbol{s}$.

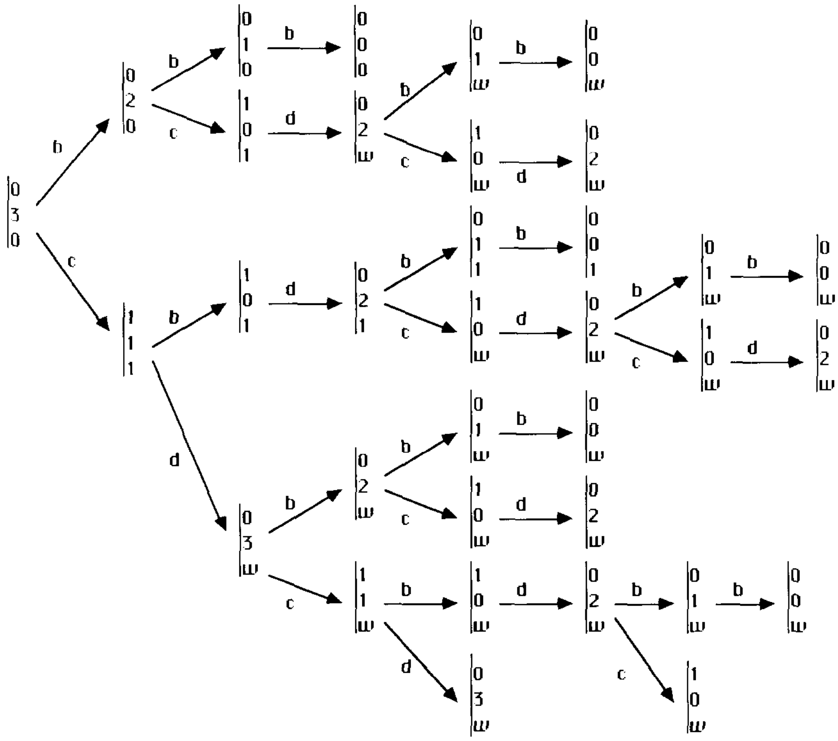


Fig. 1.

Example. Let $A=(T,\varphi,\boldsymbol{a})$ be a 3-VAS with $T=\{b,c,d\}$, $\boldsymbol{a}=[0,3,0]$ and

$$\varphi(b)=[0,-1,0],\ \varphi(c)=[1,-2,1],\ \varphi(d)=[-1,2,0].$$

This 3-VAS has a coverability tree shown in Fig. 1.

It has been proved by Karp and Miller [10] that this algorithm yields a finite tree. They also proved numerous properties, some of which are recalled below. It is easy to see that, if there is a vertex labelled by s such that $s[i]=\omega$, then the ith coordinate is unbounded.

From the coverability tree, one can derive a finite deterministic automaton, the graph of which is obtained by identifying a vertex having the same label as one of its ancestors with this ancestor. Formally, the graph of the automaton is obtained by the preceding algorithm except that whenever there exists a vertex g, ancestor of e, having label $\boldsymbol{h}$ with $\boldsymbol{h}=s'$, instead of creating a new vertex e', labelled by s' (that would be a leaf), and an edge (e,t,e'), going from e to e' and labelled by t, one simply creates an edge (e,t,g).

The finite automaton $\langle T,Q,\delta,q_0\rangle$, where Q is the set of vertices of this graph, q_0 is the vertex corresponding to the root of the tree, δ is completely defined by the edges of

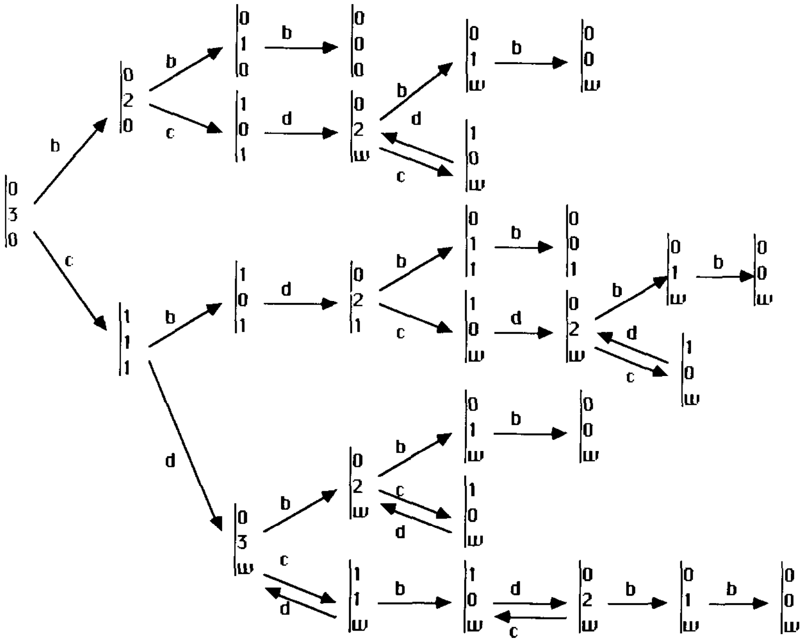


Fig. 2.

the graph, and all states are final states, is called, in the sequel, the coverability automaton of a k-VAS A, and denoted by $\mathscr{G}(A)$.

Example. Using the same example, we get Fig. 2.

Clearly, this automaton is a finite deterministic automaton, and l labels the states of this automaton. In the sequel, we write, for a state q, $q[i]$ as shorthand for $l(q)[i]$. If $\mathscr{G}(A)=\langle T,Q,\delta,q_0\rangle$ is the coverability automaton of a k-VAS A, then

(i) $L(A)\subseteq L(\mathscr{G}(A))$.

(ii) $[(\forall i\in[\underline{k}])\,(\exists q\in Q)\,(q[i]=\omega)]\Rightarrow(i$ is an unbounded coordinate).

Moreover, this automaton has the following properties.

Proposition 2.4 (Karp and Miller [10]). *Let $\mathscr{G}(A)=\langle T,Q,\delta,q_0\rangle$ be the coverability automaton of a k-VAS $A=(T,\varphi,\boldsymbol{a})$, and let $\boldsymbol{b}$ be a k-dimensional vector of nonnegative integers, then the two following statements are equivalent:*

(i) *there is f in $L(A)$ such that $\varphi(f)\geqslant\boldsymbol{b}$*

(ii) *there is q in Q such that $l(q)\geqslant\boldsymbol{b}$.*

Lemma 2.5. *Let $\mathscr{G}(A)=\langle T,Q,\delta,q_0\rangle$ be the coverability automaton of a k-VAS $A=(T,\varphi,\boldsymbol{a})$, if $\delta(q,u^n)=q$ for some n, then $\delta(q,u)=q$.*

Proof. If $l(q)+n\cdot\varphi(u)=l(q)$, then on all coordinates where $l(q)$ is finite, $\varphi(u)=0$ and, so, $l(q)+\varphi(u)=l(q)$.

Corollary 2.6. *If (q,u) is an elementary loop, then u is a primitive word.*

The reduced coverability automaton

Consider the following example.

Let $A=(T,\varphi,\boldsymbol{a})$ be a 3-VAS with $T=\{a,b,c,d\}$, $\boldsymbol{a}=[1,0,0]$ and $\varphi(a)=[-1,1,0]$, $\varphi(b)=[0,-1,1]$, $\varphi(c)=[2,0,-1]$, $\varphi(d)=[-3,1,0]$.

This 3-VAS has a coverability automaton $\mathcal{G}(A)$ shown in Fig. 3.

It can be seen that the transition (q_3,d,q_6) cannot be fired (i.e. there is no word f such that $\delta(q_0,f)=q_3$ and fd is in $L(A)$), as well as the transition (q_4,d,q_4).

Proposition 2.7. *Let $\mathcal{G}(A)=\langle T,Q,\delta,q_0\rangle$ be the coverability automaton of a k-VAS A, for all q in Q and all u in T^+, it is decidable whether there exists a word f such that $\delta(q_0,f)=q$ and fu is in $L(A)$.*

Proof. It is enough to prove it for all t in T, to get the result for all u in T^+. The question being: is it possible to fire the transition t when we are in q? We assume that q is accessible from q_0. Let $x_1,x_2,\ldots,x_s$ be the labels of all the elementary loops that can be found on the elementary path from q_0 to q and w be the label of this elementary path. We assume that these elementary loops are strong loops (see below). Then there exists a word f such that $\delta(q_0,f)=q$ and ft is in $L(A)$ if and only if there exist integers λ_i such that $\varphi(w)+\varphi(t)+\sum_{1\leqslant i\leqslant s}\lambda_i\cdot\varphi(x_i)\geqslant\mathbf{0}$. As there is a finite number of elementary loops, the decidability of this problem is due to the decidability of the problem of the existence of a positive solution to a finite set of integer linear inequalities [13]. □

Thus, it is possible to remove all transitions that cannot be fired, and get a *reduced coverability automaton*, sharing the same properties as the coverability automaton. From now on, we shall suppose that the coverability automaton is reduced.

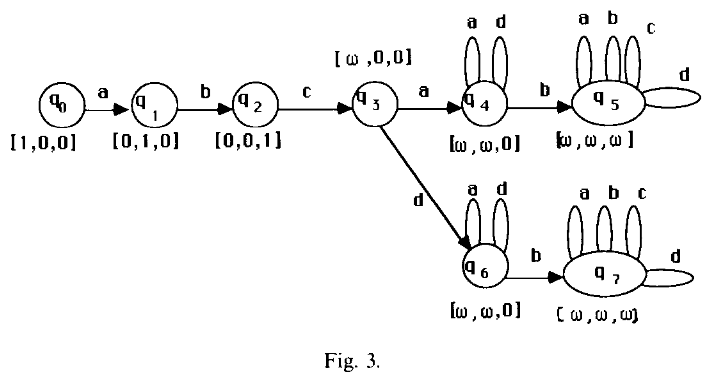


Fig. 3.

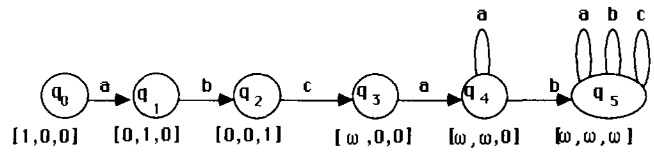


Fig. 3′.

Strong loops for $L(A)$ in $\mathscr{G}(A)$

It must be noted that not all loops in $\mathscr{G}(A)$ are strong loops for $L(A)$ as shown in the same example (where we drop the letter d) (Fig. 3′).

We can see that (q_4, a) is a loop, but not a strong loop: if $(w \in T^*)$ and $(\delta(q_0, w) = q_4)$, then $w \in abca(a)^*$ but $abca(a)^* \cap L(A) = \{abca, abcaa\}$.

Hence, for $n > 1$ there is no f in $L(A)$ such that $\delta(q_0, f) = q_4$ and $fa^n \in L(A)$. (q_5, a) is a strong loop because $(\forall n \in \mathbb{N})$, $(\delta(q_0, (abc)^n) = q_5)$ and $(abc)^n a^n \in L(A)$.

However, the coverability automaton $\mathscr{G}(A) = \langle T, Q, \delta, q_0 \rangle$ has the following property.

Proposition 2.8. *If (q, u) is a loop of $\mathscr{G}(A)$, then there is $q' \in Acc(q_0)$ such that (q', u) is a strong loop.*

This proposition is a consequence of Proposition 2.9 below, stating the close relation between iterable factors of $L(A)$ and loops in $\mathscr{G}(A)$.

Proposition 2.9. *Let $A = (T, \varphi, \boldsymbol{a})$ be a k-VAS and let $\mathscr{G}(A) = \langle T, Q, \delta, q_0 \rangle$ be its coverability automaton. A word u is an iterable factor of $L(A)$ if and only if u is the label of a loop (q, u) in $\mathscr{G}(A)$.*

Proof. The "if" part is a consequence of Proposition 2.4. Let (q, u) be a loop in $\mathscr{G}(A) = \langle T, Q, \delta, q_0 \rangle$. We have: $\delta(q, u) = q$, and this implies by construction of $\mathscr{G}(A)$: $l(q) + \varphi(u) = l(q)$. On the coordinates j such that $l(q)[j] < \omega$, we have $\varphi(u)[j] = 0$ because, otherwise, $l(q) + \varphi(u)$ would be different from $l(q)$. So, we only have to take care of the infinite coordinates of $l(q)$. Now, $(\forall n \in \mathbb{N})$ $(\exists \boldsymbol{b} \in \mathbb{N}^k$ with $\boldsymbol{b}[j] = 0$ for all j such that $l(q)[j] < \omega)$ and $(\boldsymbol{b} + n \cdot \varphi(u) \geqslant 0)$. Clearly, $l(q) \geqslant \boldsymbol{b}$ and, from Proposition 2.4, there is f in $L(A)$ such that $\varphi(f) \geqslant \boldsymbol{b}$ and, so, $\varphi(f) + n \cdot \varphi(u) \geqslant 0$, i.e. $fu^n \in L(A)$. As this is true for all n, u is an iterable factor of $L(A)$.

Let us prove the "only if" part. Since $L(A) \subseteq L(\mathscr{G}(A))$, every iterable factor of $L(A)$ is also an iterable factor of $L(\mathscr{G}(A))$. Let u be an iterable factor of $L(A)$. We have $(\forall n \in \mathbb{N})$ $(\exists f_n \in T^*)$ $(f_n u^n \in L(A))$. Let $\mathscr{G}(A) = \langle T, Q, \delta, q_0 \rangle$.

For each n, considering the sequence of states $\delta(q_0, f_n)$, $\delta(q_0, f_n u), \ldots, \delta(q_0, f_n u^r)$ with $r = \mathrm{Card}(Q)$, two of them have the same value $q_{(n)}$ and there is $i_n < r$ such that $\delta(q_{(n)}, u^{i_n}) = q_{(n)}$. So, there are infinitely many n's for which the same state q is so obtained as $q_{(n)}$ with the same value i as i_n, such that $\delta(q, u^i) = q$, and (q, u^i) is a loop in $\mathscr{G}(A)$.

Now, if $\delta(q, u^i) = q$, we have $l(q) + i \cdot \varphi(u) = l(q)$. So, on the bounded coordinates (coordinates j such that $l(q)[j] < \omega$), $\varphi(u) = 0$. Let q' be the state such that $\delta(q, u) = q'$, we have $l(q) + \varphi(u) = l(q')$. As on the bounded coordinates $\varphi(u) = 0$, $l(q) = l(q')$. As q is an ancestor of q', it follows from the construction that this equality implies $q = q'$. So, (q, u) is a loop in $\mathscr{G}(A)$.

Let us now prove Proposition 2.8. Let (q, u) be a loop; from Proposition 2.9, u is an iterable factor, and we have $(\forall n \in \mathbb{N})$ $(\exists f_n \in T^*)$ such that $f_n u^n \in L(A)$.

Given n, let us define the state q_n such that $\delta(\boldsymbol{a}, f_n) = q_n$ and $f_n u^n \in L(A)$. As the number of states is finite, there is an infinite subsequence of $(q_n)_{n \in \mathbb{N}}$ which is constant with q' as common value; so, (q', u^n) is a strong loop; hence, (q', u) is also a strong loop, in view of Lemma 2.5. □

Corollary 2.10. *For every iterable factor u of $L(A)$, there is a strong loop (q, u) in $\mathscr{G}(A)$.*

Proof. The proof is straightforward from Propositions 2.8 and 2.9. □

Strong loops and iterating systems related to a loop

We have seen that in the coverability automaton of Karp and Miller there may exist loops that are not strong loops. In order to be able to prove that a loop is a strong loop, we introduce a new tool: the iterating system's notion. We prove that a loop is a strong loop iff there exists an iterating system related to this loop.

Definition 2.11. Let $A = (T, \varphi, \boldsymbol{a})$ be a k-VAS and $\mathscr{G}(A) = \langle T, Q, \delta, \boldsymbol{a} \rangle$ be its coverability automaton. Let (q, w) be a loop. An *iterating system* of length p related to (q, w) is a finite sequence $(\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, \alpha_{p-1}, q_p, u_p, \alpha_p, q)$ with

(i) for $0 \leqslant i \leqslant p$, $\alpha_i \in T^*$, $q_i \in Q$, and for $1 \leqslant i \leqslant p$, $u_i \in T^+$,

(ii) for $0 \leqslant i \leqslant p-1$, $\delta(q_i, \alpha_i) = q_{i+1}$; $\delta(q_p, \alpha_p) = q$, and for $1 \leqslant i \leqslant p$, $\delta(q_i, u_i) = q_i$,

(iii) $\alpha_0 u_1^+ \alpha_1 u_2^+ \ldots \alpha_{p-1} u_p^+ \alpha_p w \cap L(A) \neq \emptyset$,

satisfying the following property:

$$(*) \qquad \text{for } 1 \leqslant i \leqslant p, \quad \|u_i\|^- \subseteq \bigcup_{0 \leqslant j < i} \|u_j\|^+ \quad \text{and} \quad \|w\|^- \subseteq \bigcup_{0 \leqslant j < p} \|u_j\|^+.$$

Lemma 2.12. *Let $A = (T, \varphi, \boldsymbol{a})$ be a k-VAS and $\mathscr{G}(A) = \langle T, Q, \delta, \boldsymbol{a} \rangle$ be its coverability automaton. Let (q, w) be a loop, there is an iterating system of length p related to (q, w) if and only if there is an iterating system of length p' related to (q, w) with $p' \leqslant k$.*

Proof. Let $(\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, \alpha_{p-1}, q_p, u_p, \alpha_p, q)$ be an iterating system of length p related to a loop (q, w). Call Γ_i the set: $\bigcup_{0 \leqslant j < i} \|u_j\|^+$. The Γ_i's form an increasing chain of subsets of $[\underline{k}]$. If $\Gamma_i = \Gamma_{i-1}$, then $(\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, \alpha_{i-1} u_i \alpha_i, \ldots, \alpha_{p-1},$

$q_p, u_p, \alpha_p, q)$ is an iterating system of length $p-1$ related to (q, w). As $[\underline{k}]$ is finite, there cannot be a strictly increasing chain of subsets of $[\underline{k}]$ of length greater than k. □

Lemma 2.13. *The property* $(*)$ *involved in Definition* 2.11 *is equivalent to the following assertion*:
$\exists \tau_1, \ldots, \tau_p \in \mathbf{N}$ *such that, for* $1 \leqslant i < p$, $\varphi(u_1^{\tau_1 \tau_2 \cdots \tau_i} u_2^{\tau_2 \cdots \tau_i} \ldots u_i^{\tau_i} u_{i+1}) \geqslant \mathbf{0}$ *and* $\varphi(u_1^{\tau_1 \tau_2 \cdots \tau_p} u_2^{\tau_2 \cdots \tau_p} \ldots u_p^{\tau_p} w) \geqslant \mathbf{0}$.

Proof. This assertion is a sufficient condition to have the property $(*)$: $\varphi(u_1^{\tau_1 \tau_2 \cdots \tau_{i-1}} u_2^{\tau_2 \cdots \tau_{i-1}} \ldots u_{i-1}^{\tau_{i-1}} u_i) \geqslant \mathbf{0}$ and $\varphi(u_1^{\tau_1 \tau_2 \cdots \tau_i} u_2^{\tau_2 \cdots \tau_i} \ldots u_i^{\tau_i} u_{i+1}) \geqslant \mathbf{0}$ implies $\| u_{i+1} \|^- \subseteq \bigcup_{0 \leqslant j \leqslant i} \| u_j \|^+$.

It is a necessary condition: This is due to the fact that

$$\varphi(u_1^{\tau_1 \tau_2 \cdots \tau_i} u_2^{\tau_2 \cdots \tau_i} \ldots u_i^{\tau_i} u_{i+1}) = \tau_1 \varphi(u_1) + \tau_2 \varphi(u_1 u_2) + \cdots + \tau_i \varphi(u_1 u_2 \ldots u_i) + \varphi(u_{i+1}).$$

If $\| u_{i+1} \|^- \subseteq \bigcup_{0 \leqslant j \leqslant i} \| u_j \|^+$, then $\exists \tau_i$ such that for all coordinate p in $\| u_{i+1} \|^-$, $\tau_i \varphi(u_1 u_2 \ldots u_i)[p] \geqslant \varphi(u_{i+1})[p]$. By induction on i, we get the result. □

The following lemma is then obvious.

Lemma 2.14. *If* $(\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, \alpha_{p-1}, q_p, u_p, q_p, \alpha_p, q)$ *is an iterating system related to* (q, w), *then* $(\forall n \geqslant 0)$ $(\mathrm{Card}(\alpha_0 u_1^+ \alpha_1 u_2^+ \ldots \alpha_{p-1} u_p^+ \alpha_p w^n \cap L(A)) = \infty)$.

Let us first give a characterization of a strong positive loop, i.e. strong loop with positive support.

Lemma 2.15 (Schwer [20]). *Let* (q, w) *be a loop, then* (q, w) *is a strong positive loop iff there is an iterating system of length* 0 *related to* (q, w).

Proof. Suppose that (α_0, q) is an iterating system of length 0 related to w. By definition $\varphi(w) \geqslant \mathbf{0}$ and as $\alpha_0 w \in L(A)$ then $\alpha_0 w^* \subseteq L(A)$, i.e. (q, w) is a positive loop which is a strong loop.

Suppose now that (q, w) is a strong positive loop, then by definition $(\exists \alpha_0 \in T^*)$ $(\delta(\boldsymbol{a}, \alpha_0) = q)$ and $(\alpha_0 w \in L(A))$.

As $\varphi(w) \geqslant \mathbf{0}$, by Lemma 1.7(i), $\alpha_0 w^* \subseteq L(A)$, i.e. $(\alpha_0, \delta(\boldsymbol{a}, \alpha_0))$ is an iterating system of length 0 related to w. □

Proposition 2.16 (Schwer [20]). *Let* $A = (T, \varphi, \boldsymbol{a})$ *be a k-VAS and* $\mathscr{G}(A) = \langle T, Q, \boldsymbol{a}, \delta \rangle$ *be its coverability automaton. Let* (q, w) *be in* $Q \times T^+$, *then the following two sentences are equivalent*:

(i) (q, w) *is a strong loop.*

(ii) *There exists an iterating system related to* (q, w).

Proof. (ii)$\Rightarrow$(i): follows from Lemma 2.14.

(i)$\Rightarrow$(ii): Suppose that (q, w) is a strong loop, then by definition $(\forall n \in \mathbf{N})$ $(\exists f_n \in T^*)$ $(\delta(\boldsymbol{a}, f_n) = q)$ $(\delta(q, w) = q$ and $f_n w^n \in L(A))$.

We choose for each n a word f_n of minimal length. There are two possibilities for the infinite sequence $(|f_n|)_{n \geqslant 0}$ of $\mathbf{N}$: either it is bounded or not. In the former case, the same word may be chosen infinitely often: there is then an iterating system of length 0 related to (q, w). In the latter case, we can consider, without loss of generality, that $|f_n| < |f_{n+1}|$ and set $f_n = h_n e_n$ with $|h_n| = n$ and $h_{n+1} = h_n x_{n+1}$, with $x_{n+1} \in T$. From the infinite sequence, we can choose f_{n_0} and f_{n_1} with $n_0 < n_1$ such that n_0 is the minimal integer satisfying $\varphi(h_{n_0}) < \varphi(h_{n_1})$. Let us set $\alpha_0 = h_{n_0}$ and $u_1 = x_{n_0+1} \ldots x_{n_1}$. Then $\alpha_0 u_1 = h_{n_1}$ and $\varphi(u_1) > \mathbf{0}$; hence, $(\delta(\boldsymbol{a}, \alpha_0), u_1)$ is a positive loop.

If $\|w\|^- \subseteq \|u_1\|^+$ then the proposition is proved with the iterating system $(\alpha_0, \delta(\boldsymbol{a}, \alpha_0), u_1, \delta(\boldsymbol{a}, \alpha_0), e_{n_1}, q)$.

If not, we have to consider the set of coordinates $([\underline{k}] \setminus \|u_1\|^+)$ in which remains some negative coordinate of $\|w\|^-$. This set has a cardinality strictly smaller than $\mathrm{Card}(\|w\|^-)$. So, we repeat the same argument until we eliminate $\|w\|^-$. The number of times we repeat the argument gives the length of the so-built iterating system related to (q, w). □

Proposition 2.17. *Let $A = (T, \varphi, \boldsymbol{a})$ be a k-VAS and $\mathscr{G}(A) = \langle T, Q, \delta, \boldsymbol{a} \rangle$ be its coverability automaton. Let (q, w) be in $Q \times T^+$; one can decide whether (q, w) is a strong loop of $\mathscr{G}(A)$ or not.*

Proof. From Proposition 2.9 and Lemma 2.12, one has to decide if there exists an iterating system $(\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, \alpha_{p-1}, q_p, u_p, \alpha_p, q)$ of length p (with $p \leqslant k$) related to (q, w). This is achieved in the following way: Take one state q_1, u_1 is such that

(1) $\delta(q_1, u_1) = q_1$

(2) $\|u_1\|^- = \emptyset$ if and only if u_1 is in the shuffle of elementary loops labelled $x_1, x_2 \ldots, x_t$ such that $\varphi(u_1) = \sum_{1 \leqslant i \leqslant t} \lambda_i \cdot \varphi(x_i) \geqslant \mathbf{0}$. As there is a finite number of elementary loops, the decidability of the existence of u_1 such that $\delta(q_1, u_1) = q_1$ with $\|u_1\|^- = \emptyset$ is due to the decidability of the problem of the existence of a positive solution to a finite set of integer linear inequalities [13]. One has now only to care about the coordinates j that are not in $\|u_1\|^+$. Then take a state q_2 in $Acc(q_1)$, and repeat the same procedure, and so on. As $p \leqslant k$, there is only a finite number of sequences $q_1, q_2, \ldots, q_p$ to check. □

The covering automaton

The fact that in the coverability automaton of $L(A)$ some loops are not strong loops for $L(A)$ leads us to define a new automaton.

Proposition 2.18. *Let $A = (T, \varphi, \boldsymbol{a})$ be a k-VAS and $\mathscr{G}(A)$ be its coverability automaton. There is an automaton, called covering automaton, $\mathscr{C}(A) = \langle T, Q, \delta, q_0 \rangle$, that we can* *construct from $\mathscr{G}(A)$, which satisfies*

(i) *refinement property*: $L(A) \subseteq L(\mathscr{C}(A)) \subseteq L(\mathscr{G}(A))$,

(ii) *loop accessibility property*: $(\forall (q,w) \in Q \times T^+$ *such that* (q,w) *is a loop of* $\mathscr{C}(A))$ $(\forall n \in \mathbf{N})$ $(\exists f_n \in T^*)$ $(\delta(\boldsymbol{a}, f_n) = q$ *and* $\delta(q,w) = q$ *and* $f_n w^n \in L(A))$.

This automaton $\mathscr{C}(A)$ is constructed from $\mathscr{G}(A)$ in the following way: Let (q,u) be an elementary loop in $\mathscr{G}(A)$ that is not a strong loop for $L(A)$. This means that $(\exists n \in \mathbf{N})$, $(\forall f$ such that $\delta(\boldsymbol{a},f)=q)$ $(fw^n \notin L(A))$. Let r be the smallest integer such that $(\forall f \in T^*)$ $(\delta(\boldsymbol{a},f)=q$ and $fu^n \in L(A)) \Rightarrow (n \leqslant r)$. The state q is renamed into $q^{(0)}$, r new states $q^{(1)}, \ldots, q^{(r)}$ are created, and for all (q,t,q') in δ, this transition is replaced by the transitions: $(q^{(i)},t,q')$ with $0 \leqslant i \leqslant r$. Moreover, a path labelled by u is created going from $q^{(i)}$ to $q^{(i+1)}$ for $0 \leqslant i < r$, all the states visited while doing the loop (q,u) being duplicated as many times as necessary, as well as all the parts of the graph accessible and coaccessible from these states.

Example. In our example (Fig. 3′) the loop (q_4, a, q_4) is not a strong loop: it can be fired only one time. The integer r is, so, 1, and we get the automaton $\mathscr{C}(A)$ shown in Fig. 3″.

This automaton has the properties (i) and (ii) by construction, and this construction is effective since one can test for all elementary loops if it is a strong loop or not, and compute a maximal value for the iteration of a loop that is not a strong loop. For full details, we refer to [20].

Moreover, the automaton $\mathscr{C}(A)$ shares with $\mathscr{G}(A)$ several properties, among which we state here one that will be intensively used in the following.

Proposition 2.19. *Let $A = (T, \varphi, \boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A) = \langle T, Q, \delta, q_0 \rangle$ be its covering automaton; let (q,u) and (q',v) be two loops in $\mathscr{C}(A)$ and let w be a word such that $\delta(q,w) = q'$. If u and v are powers of two conjugated words ($u = (xy)^n$ and $v = (yx)^p$), then either w is not such that $w = (xy)^m x$, or $\delta(q,x) = q'$ and $\delta(q',y) = q$.*

Proof. In view of Lemma 2.5, we can take $n = p = 1$. Let (q,u) and (q',v) be two loops in $\mathscr{C}(A)$ such that $u = xy$ and $v = yx$ (u and v are conjugated) and there exists $w = (xy)^m x$

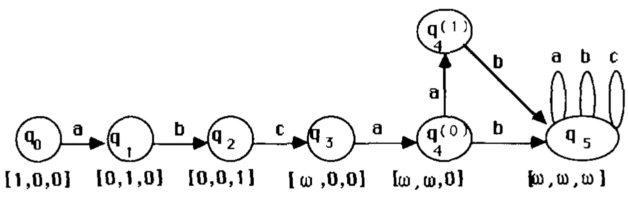


Fig. 3″.

such that $\delta(q,w)=q'$. As $u=xy$ is a loop and $\mathscr{C}(A)$ is deterministic, $\delta(q,xy)=q$; so, $\delta(q,(xy)^m)=q$; hence, $\delta(q,x)=q'$ and $\delta(q',y)=q$. □

Corollary 2.20. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,q_0\rangle$ be its covering automaton; let (q,u) and (q',v) be two elementary loops in $\mathscr{C}(A)$ and let f and w be words such that $\delta(q_0,f)=q$ and $\delta(q,w)=q'$. Call l the set $l=\{(n,p)\mid fu^nwv^p\in L(A)\}$. If the two loops are distinct loops, then there is a bijective rational transduction between $L(A)\cap fu^*wv^*$ and the language $\{a^nb^p\mid(n,p)\in l\}$ where a and b are letters.*

Proof. The proof is straightforward from Proposition 2.19. □

Corollary 2.20′. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,q_0\rangle$ be its covering automaton; let $(q_1,u_1),(q_2,u_2),\dots,(q_p,u_p)$ be loops in $\mathscr{C}(A)$ and let f and $w_1,w_2,\dots,w_{p-1}$ be words such that $\delta(q_0,f)=q$ and $\delta(q_i,w_i)=q_{i+1}$. Call l the set $l=\{(n_1,n_2,\dots,n_p)\mid fu_1^{n_1}w_1u_2^{n_2}w_2\dots u_p^{n_p}\in L(A)\}$. If two consecutive loops follow distinct cycles, then there is a bijective rational transduction between $L(A)\cap fu_1^*w_1u_2^*w_2\dots u_p^*$ and the language $\{a_1^{n_1}a_2^{n_2}\dots a_p^{n_p}\mid(n_1,n_2,\dots,n_p)\in l\}$ where $a_1,a_2,\dots,a_p$ are letters.*

Proof. The proof is straightforward from Proposition 2.19. □

3. Study of iterating systems

We show how iterating systems can be decomposed or reduced, and we introduce generalized iterating systems. In this part, we establish mainly a decidability property, that is necessary for the proofs of Part II.

Definition 3.1. Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. Let (q,w) be a loop, and let $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\dots,\alpha_{p-1},q_p,u_p,\alpha_p,q)$ be an interating system of length p related to (q,w). We call *width of* $\mathscr{I}$ the integer

$$W(\mathscr{I})=\sum_{1\leqslant i\leqslant p}3^{|u_i|}.$$

Definition 3.2. Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. Let $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\dots,\alpha_{p-1},q_p,u_p,\alpha_p,q)$ be an iterating system of length p related to (q,w). $\mathscr{I}$ is an *elementary iterating system* if for all i (q_i,u_i) is an elementary loop.

Recall (Corollary 2.6) that an elementary loop (q,u) is a loop such that u is a primitive word.

Definition 3.3. Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. Let (q,w) be a loop, and let $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\dots,\alpha_{p-1},q_p,u_p,\alpha_p,q)$ be an iterating system of length p related to (q, w) and suppose that (q_i, u_i) is not an elementary loop.

Let $u_i = xu_i'y$ and $q_i' = \delta(q_i, x)$ with (q_i', u_i') a loop. If $\|u_i'\|^- \subseteq \bigcup_{0 \leqslant j < i} \|u_j\|^+$, then $(\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, q_{i-1}, u_{i-1}, \alpha_{i-1}\,x, q_i', u_i', \lambda, q_i', yx, y\alpha_i, q_{i+1}, \ldots, \alpha_{p-1}, q_p, u_p, \alpha_p, q)$ is an iterating system related to (q, w) called a *decomposition* of $\mathscr{I}$.

Remark that a decomposition increases the length of the iterating system, but strictly reduces its width (because $|u_i| = |u_i'| + |yx|$ and for all positive integers: $n = m + l \Rightarrow 3^m + 3^l < 3^n$).

Example. Let $A = (T, \varphi, \boldsymbol{a})$ be a 2-VAS with $T = \{b, c\}$, $\varphi(b) = [-1, 1]$, $\varphi(c) = [2, -1]$ and $\boldsymbol{a} = [1,0]$. A has the covering graph $\mathscr{C}(A)$ shown in Fig. 4. $\mathscr{I} = (q_0, bcb, q_3, bcb, \lambda, q_3)$ is an iterating system of length 1 related to (q_3, b). It is possible to decompose it in: $\mathscr{I}' = (q_0, bcb, q_3, bc, \lambda, q_3, b, \lambda, q_3)$, which is an iterating system related to (q_3, b) that cannot be decomposed: only bc is not an elementary loop, but as $\|b\|^- \subseteq \|c\|^+$ and $\|c\|^- \subseteq \|b\|^+$, there is no way to decompose it.

As this example shows, it is not always possible to get by decomposition an elementary iterating system from an iterating system. However, it is possible to decide whether or not an iterating system can be decomposed in a finite number of steps into an elementary iterating system (as there is a finite number of simple cycles in a cycle, one can try all the possible decompositions).

Definition 3.4. Let $A = (T, \varphi, \boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A) = \langle T, Q, \delta, \boldsymbol{a} \rangle$ be its covering automaton. Let (q, w) be a loop, and let $\mathscr{I} = (\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, \alpha_{p-1}, q_p, u_p, \alpha_p, q)$ be an iterating system of length p related to (q, w).

Suppose that, for some i, the loop (q_i, u_i) is not really necessary to iterate w, more precisely there exists some fixed value n such that $(\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, q_{i-1}, u_{i-1}, \alpha_{i-1}\, u_i^n \alpha_i, q_{i+1}, \ldots, \alpha_{p-1}, q_p, u_p, \alpha_p, q)$ is an iterating system related to (q, w). This new system, of length $p-1$, is called a *reduce* of $\mathscr{I}$.

Remark that a reducing of an iterating system reduces both its length and its width, and that it is easy to decide by inspection whether an iterating system can be reduced or not. In particular, if there is an i such that $\|u_i\|^+ \subseteq \bigcup_{0 \leqslant j < i} \|u_j\|^+$, then the iterating system can be reduced. Remark also that, as the unions $\bigcup_{0 \leqslant j < i} \|u_j\|^+$ forms an

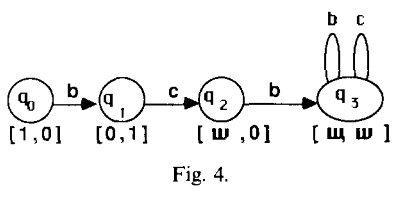


Fig. 4.

increasing chain when i is increasing, an iterating system of length greater than k can always be reduced.

Definition 3.5. Let $\mathscr{I}$ be an iterating system related to (q, w). $\mathscr{I}$ is a *minimal iterating system* if it is not possible to reduce it.

Iterating the operation of reducing, every iterating system can be transformed into a minimal iterating system, but, depending on the way the reductions are done, this does not lead to a unique minimal iterating system in general, as shown by the following example.

Example. Let $A=(T,\varphi,\mathbf{0})$ be a 2-VAS with $T=\{a,b,c,d\}$ and $\varphi(a)=[1,0]$, $\varphi(b)=[1,2]$, $\varphi(c)=[-1,1]$, $\varphi(d)=[0,-1]$.

A has the covering graph $\mathscr{C}(A)$ shown in Fig. 5. $\mathscr{I}=(q_0, a, q_1, a, b, q_2, b, \lambda, q_2, c, \lambda, q_2)$ is an iterating system related to (q_2, d). From this iterating system, one can get $\mathscr{I}'=(q_0, a, q_1, a, b, q_2, c, \lambda, q_2)$ which is a minimal iterating system of length 2 related to (q_2, d), but one can get also $\mathscr{I}''=(q_0, ab, q_2, b, \lambda, q_2)$ which is a minimal iterating system of length 1 related to (q_2, d).

Proposition 3.6. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T, Q, \delta, \boldsymbol{a}\rangle$ be its covering automaton. Let (q, w) be a loop, and let $\mathscr{I}=(\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, \alpha_{p-1}, q_p, u_p, \alpha_p, q)$ be an iterating system of length p related to (q, w). It is decidable, for any integer k, whether $\mathscr{I}$ can be transformed or not into an elementary minimal iterating system of length at most k, by operations of decomposition or reducing.*

Proof. As the width of the iterating system strictly decreases with the two operations, and at each step there is a finite number of possibilities, there is only a finite tree of possibilities to explore. □

To end this section, we introduce the notion of a generalized iterating system, that plays the role of iterating system, but in relation with a sequence of loops.

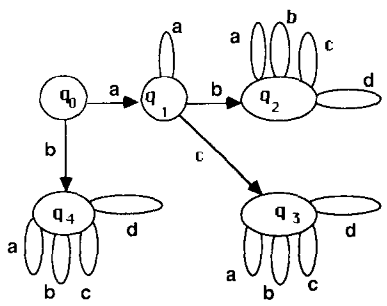


Fig. 5.

Definition 3.7. Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. Let $\mathscr{L}=((p_1,w_1),(p_2,w_2),\ldots,(p_r,w_r))$ be a sequence of r loops such that $p_{i+1}\in Acc(p_i)$ for all $i<r$. An *r-generalized iterating system* related to $\mathscr{L}$ is an ordered set $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\ldots,\alpha_{p-1},q_p,u_p,\alpha_p,q;\beta_1,\ldots,\beta_r)$ satisfying $(\forall i\in[\underline{r}])$ $\delta(q,\beta_i)=p_i$ and $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\ldots,\alpha_{p-1},q_p,u_p,\alpha_p\beta_i,p_i)$ is an iterating system related to (p_i,w_i).

4. Dominating coordinates

We introduce in this paragraph our last tool: the notion of a dominating coordinate.

Definition 4.1. Let $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t$ be t k-tuples of $\mathbf{Z}^k$. For all p in $[\underline{k}]$, we define the linear form $\varphi_p:\mathbf{R}^t\mapsto\mathbf{R}$ by $\forall\boldsymbol{x}=(x_1,x_2,\ldots,x_t)\in\mathbf{R}^n$, $\varphi_p(\boldsymbol{x})=\sum_{1\leqslant j\leqslant t}x_j\boldsymbol{u}_j[p]$, and we call *inequality associated* (*equation associated*) with the coordinate p with respect to $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t\}$ the inequality $\varphi_p(\boldsymbol{x})\geqslant 0$ (the equation $\varphi_p(\boldsymbol{x})=0$). The set $H_p=\{(x_1,x_2,\ldots,x_t)\in\mathbf{R}^t\mid\varphi_p(\boldsymbol{x})\geqslant 0\}$ is called the corresponding *positive half-plane*, while the set $K_p=\{(x_1,x_2,\ldots,x_t)\in\mathbf{R}^t\mid\varphi_p(\boldsymbol{x})=0\}$ is called the corresponding *kernel*.

In the sequel, we are interested in the positive integer solutions of these equations and inequalities. The set $\{(x_1,x_2,\ldots,x_t)\in\mathbf{N}^t\mid\varphi_p(\boldsymbol{x})\geqslant 0\}$ is called the corresponding *natural positive half-plane*, while the set $\{(x_1,x_2,\ldots,x_t)\in\mathbf{N}^t\mid\varphi_p(\boldsymbol{x})=0\}$ is called the corresponding *natural kernel*.

Definition 4.2. Let $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t$ be t k-tuples of $\mathbf{Z}^k$. The coordinate p *dominates* the coordinate q with respect to $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t\}$ if and only if $\forall(x_1,x_2,\ldots,x_t)\in\mathbf{N}^t$ $\sum_{1\leqslant j\leqslant t}x_j\boldsymbol{u}_j[p]\geqslant 0\Rightarrow\sum_{1\leqslant j\leqslant t}x_j\boldsymbol{u}_j[q]\geqslant 0$. The coordinate p is a *dominating coordinate* with respect to $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t\}$ if it dominates all coordinates.

Clearly, the relation of domination between coordinates is a pre-order. There is a dominating coordinate iff there is a maximum element for the order associated.

Example. Let $\boldsymbol{a}=[5,4]$, $\boldsymbol{b}=[-3,-4]$ and $\boldsymbol{c}=[-5,-3]$. The inequalities associated with the first and second coordinates are $5\cdot x_1-3\cdot x_2-5\cdot x_3\geqslant 0$ and $4\cdot x_1-4\cdot x_2-3\cdot x_3\geqslant 0$, respectively. But while $(3,5,0)$ is a solution of the first inequality, it is not a solution of the second. Conversely, $(3,0,4)$ is a solution of the latter but not of the former. Consequently, there is no dominating coordinate with respect to $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$.

Let $\boldsymbol{a}=[6,4]$, $\boldsymbol{b}=[-2,1]$ and $\boldsymbol{c}=[-4,-1]$. The inequalities associated with the first and second coordinates are $6\cdot x_1-2\cdot x_2-4\cdot x_3\geqslant 0$ and $4\cdot x_1+x_2-x_3\geqslant 0$, respectively. Now every 3-tuple of integers (x_1,x_2,x_3) satisfying the first inequality satisfies the second. Hence, the first coordinate dominates the second one with respect to $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$. As there are only these two coordinates, the first coordinate is a dominating coordinate with respect to $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$.

Let $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t$ be t k-tuples of $\mathbf{Z}^k$, and let H_p and K_p be the sets defined as above. For each coordinate p, we set $H_p^+=H_p\cap\{\boldsymbol{x}\in\mathbf{R}^t\mid \boldsymbol{x}\geqslant\mathbf{0}\}$ and $K_p^+=K_p\cap\{\boldsymbol{x}\in\mathbf{R}^t\mid \boldsymbol{x}=\mathbf{0}\}$.

Lemma 4.3. *Let $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t$ be t k-tuples of $\mathbf{Z}^k$. The coordinate p dominates the coordinate q with respect to $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t\}$ if and only if $H_p^+\subseteq H_q^+$. In other words, the inclusion $\{(x_1,x_2,\ldots,x_t)\in\mathbf{R}_+^t \mid \sum_{1\leqslant j\leqslant t} x_j\boldsymbol{u}_j[p]\geqslant 0\}\subseteq\{(x_1,x_2,\ldots,x_t)\in\mathbf{R}_+^t \mid \sum_{1\leqslant j\leqslant t} x_j\boldsymbol{u}_j[q]\geqslant 0\}$ holds in $\mathbf{R}_+^t$ if and only if it holds in $\mathbf{N}^t$.*

Remark 4.4. Let $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t$ be t k-tuples of $\mathbf{Z}^k$. If p is a coordinate such that for all j, $1\leqslant j\leqslant t$: $\boldsymbol{u}_j[p]\geqslant 0$, the inequality associated with the coordinate p with respect to $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t\}$: $\sum_{1\leqslant j\leqslant t} x_j\boldsymbol{u}_j[p]\geqslant 0$ holds for all $(x_1,x_2,\ldots,x_t)\in\mathbf{N}^t$. In these conditions, p is dominated by any coordinate.

Remark 4.5. Let $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t$ be t k-tuples of $\mathbf{Z}^k$. If p is a dominating coordinate with respect to $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t\}$, then p is a dominating coordinate with respect to any subset of $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t\}$.

Lemma 4.6. *Let $\boldsymbol{u},\boldsymbol{v}$ be two k-tuples of $\mathbf{Z}^k$. If $\boldsymbol{u}\geqslant\mathbf{0}$, then there exists a dominating coordinate with respect to $\{\boldsymbol{u},\boldsymbol{v}\}$.*

Proof. If $\boldsymbol{v}\geqslant\mathbf{0}$, then according to Remark 4.4, every coordinate is a dominating one. If **not** ($\boldsymbol{v}\geqslant\mathbf{0}$), let J be the subset of $[\underline{k}]$ of all integers i such that $\boldsymbol{v}[i]<0$. There is p in $[\underline{k}]$ such that the quantity $-\boldsymbol{v}[p]/\boldsymbol{u}[p]$ is maximal. Then p is a dominating coordinate. $\square$

It must be noticed that to have a dominating place is a very strong property to ask for, and there are simple examples where there is no such dominating place.

Example 4.7. For each $k\geqslant 2$, let $W_k=\{\boldsymbol{u},\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ be a set of $k+1$ k-tuples of $\mathbf{Z}$ with for all p in $[\underline{k}]$, $\boldsymbol{u}[p]=+1$, and for all p in $[\underline{k}]$, $\boldsymbol{u}_i[p]=-1$ if $i\neq p$ and $\boldsymbol{u}_p[p]=-2$.

There is no dominating place related to $\{\boldsymbol{u},\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$: let us show that for all p and q with $p\neq q$, the coordinate p does not dominate the coordinate q. Consider the element of $\mathbf{N}^{k+1}$: $(x,x_1,x_2,\ldots,x_k)$ with $x=2\cdot k$ and $x_i=2$ if $i\neq p$ and $x_p=1$. We have $x\boldsymbol{u}[p]+\sum_{1\leqslant i\leqslant k}x_i\boldsymbol{u}_i[p]=0$ and $x\boldsymbol{u}[q]+\sum_{1\leqslant i\leqslant k}x_i\boldsymbol{u}_i[q]=-1$.

Definition 4.8. Let $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t$ be t k-tuples of $\mathbf{Z}^k$. We denote by $\mathbb{C}3$ $(\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t)$, the condition: there exists a coordinate p which is a dominating coordinate with respect to $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t\}$, and by $\mathbb{C}3(\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t;p,q)$, the condition: the coordinate p dominates the coordinate q with respect to $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_t\}$.

Proposition 4.9. *Let $u_1, u_2, \ldots, u_t$ be t k-tuples of $\mathbf{Z}^k$, and let p and q be integers in $[\underline{k}]$. It is decidable whether $\mathbb{C}3(u_1, u_2, \ldots, u_t; p, q)$ holds or not, and it is decidable whether $\mathbb{C}3(u_1, u_2, \ldots, u_t)$ holds or not.*

Proof. To decide $\mathbb{C}3(u_1, u_2, \ldots, u_t)$, it suffices to check $\mathbb{C}3(u_1, u_2, \ldots, u_t; p, q)$ for all p and q in $[\underline{k}]$. It remains to prove that $\mathbb{C}3(u_1, u_2, \ldots, u_t; p, q)$ is decidable. This is equivalent to compute the set $\{(x_1, x_2, \ldots, x_t) \in \mathbf{R}^t \mid \sum_{1 \leqslant j \leqslant t} x_j u_j[p] \geqslant 0\} \cap \{(x_1, x_2, \ldots, x_t) \in \mathbf{R}^t \mid \sum_{1 \leqslant j \leqslant t} x_j u_j[q] \geqslant 0\} \bigcap_{1 \leqslant j \leqslant t} \{(x_1, x_2, \ldots, x_t) \in \mathbf{R}^t \mid x_j \geqslant 0\}$ and compare it with the set $\{(x_1, x_2, \ldots, x_t) \in \mathbf{R}^t \mid \sum_{1 \leqslant j \leqslant t} x_j u_j[p] \geqslant 0\}$. $\square$

To end this section, we prove a proposition stating that, given a set of t k-tuples of $\mathbf{Z}^k$, such that all these t k-tuples have for all coordinates the same sign, then either there is a dominating place or there are two coordinates such that the equations associated with these two coordinates have a nonzero positive common solution. (Though this condition may seem a very particular case, we will have to deal with it in the next part.)

Let n be the number of k-tuples for which the coordinates are all positive and m the number of k-tuples for which the coordinates are all negative (so that $n+m=t$). Considering only the positive k-tuples, we get an $n \times k$ matrix A, where $A[i,j]$ is the jth (positive) element of the ith k-tuple. Considering now only the negative k-tuples, we get an $m \times k$ matrix B, where $B[i,j]$ is the opposite of the jth (negative) element of the ith k-tuple. For every coordinate, say the jth, we first define two linear forms: $\varphi_{A_j}: \mathbf{R}^n \mapsto \mathbf{R}$ and $\varphi_{B_j}: \mathbf{R}^m \mapsto \mathbf{R}$.

φ_{A_j} is defined by: $\forall x \in \mathbf{R}^n$, $\varphi_{A_j}(x) = \sum_{1 \leqslant i \leqslant n} x_i A[i,j]$, and in the same way,

φ_{B_j} is defined by: $\forall x \in \mathbf{R}^m$, $\varphi_{B_j}(x) = \sum_{1 \leqslant i \leqslant m} x_i B[i,j]$.

Moreover, we define $\varphi_j: \mathbf{R}^{n+m} \mapsto \mathbf{R}$ by:

$$\forall x \in \mathbf{R}^{n+m}, \quad \varphi_j(x) = \sum_{1 \leqslant i \leqslant n} x_i A[i,j] - \sum_{n+1 \leqslant i \leqslant n+m} x_i B[i,j].$$

Note that the positive half-plane and kernel of Definition 4.1 can be written with this notation: $H_j = \{x \in \mathbf{R}^{n+m} \mid \varphi_j(x) \geqslant 0\}$ and $K_j = \{x \in \mathbf{R}^{n+m} \mid \varphi_j(x) = 0\}$, respectively.

Proposition 4.10. *With the notations as above, if p and q are two integers such that $1 \leqslant p$, $q \leqslant t$, $K_p^+ \cap K_q^+ = \{\mathbf{0}\} \Rightarrow$ either $H_p^+ \subseteq H_q^+$ or $H_q^+ \subseteq H_p^+$.*

Proof. Suppose that $K_p^+ \cap K_q^+ = \{\mathbf{0}\}$.

Recall the following corollary of the Hahn–Banach theorem (see e.g. [17, Theorem 5.19, p. 114]): Let M be a linear subspace of a normed linear space X, and let $x_0 \in X$. Then x_0 is in the topological closure of M if and only if there is no bounded linear functional f on X such that $f(x) = 0$ for all $x \in M$ but $f(x_0) \neq 0$. Equivalently, this theorem says that there is a bounded linear form that separates the convex M and any point outside M. This theorem is applied here with $X = \mathbf{R}^{n+m}$ which is of finite dimension; hence, all linear forms are bounded, $M = K_p^+$ and x_0 is any element of K_p^+ different from $\mathbf{0}$.

Hence, we have

$$K_p^+ \cap K_q^+ = \{\mathbf{0}\} \Leftrightarrow \exists \lambda, \mu \in \mathbf{R} \text{ such that } \forall \boldsymbol{x} \in \mathbf{R}_+^n,\ \forall \boldsymbol{y} \in \mathbf{R}_+^m,$$

$$\lambda \cdot \varphi_p((\boldsymbol{x},\boldsymbol{y})) + \mu \cdot \varphi_q((\boldsymbol{x},\boldsymbol{y})) > 0.$$

This is equivalent to:

$$\exists \lambda, \mu \in \mathbf{R} \text{ such that } \forall \boldsymbol{x} \in \mathbf{R}_+^n,\ \lambda \cdot \varphi_{A_p}(\boldsymbol{x}) + \mu \cdot \varphi_{A_q}(\boldsymbol{x}) > 0 \quad \text{and}$$

$$\forall \boldsymbol{y} \in \mathbf{R}_+^m,\ \lambda \cdot \varphi_{B_p}(\boldsymbol{y}) + \mu \cdot \varphi_{B_q}(\boldsymbol{y}) < 0.$$

As these two inequalities are true for all $\boldsymbol{x}$ and $\boldsymbol{y}$, λ and μ do not have the same sign. If λ is positive and μ negative, setting $k = -\mu$, we have $\lambda \cdot \varphi_p((\boldsymbol{x},\boldsymbol{y})) > k \cdot \varphi_q((\boldsymbol{x},\boldsymbol{y}))$. In this case $H_p^+ \subseteq H_q^+$. If λ is negative and μ positive, setting $k = -\lambda$, we have $\mu \cdot \varphi_q((\boldsymbol{x},\boldsymbol{y})) > k \cdot \varphi_p((\boldsymbol{x},\boldsymbol{y}))$. In this case $H_q^+ \subseteq H_p^+$. $\square$

PART II: DECIDABILITY OF THE CONTEXT-FREENESS OF A VAS LANGUAGE

0. Introduction

The proof of the decidability of the context-freeness of a VAS language is somewhat complicated since many conditions of different nature are interfering. Let us summarize the way this proof is achieved.

(1) A characterization of context-free VAS languages is given. This characterization is a conjunction of various conditions such as the existence of a dominating coordinate with respect to specified sets of elementary loops, or the fact that some sets have to be stratified sets, etc. Each condition is itself a "simple" condition that has to be verified by a specified set of elements. All these specified sets are derived from the covering automaton of the VAS language, defined in Section I.2.

(2) The proof of the decidability of this characteristic property is done. Since it has been established before (part I) that these "simple" conditions involved are all decidable, it only remains to prove that for each condition, these simple conditions have to be checked for finite sets of elements.

In the sequel, we will have to prove that a number of languages are not context-free. This is done using Ogden's lemma, which is a necessary condition.

Ogden's lemma (Harrison [8]). *If G is a context-free grammar, there exists an integer N such that for all nonterminal S, and for all word f generated by S with at least N marked letters, there is a factorization $\alpha x \beta y \gamma$ of f such that*:

(1) $S \to^* \alpha T \gamma$, $T \to^* x T y$, $T \to^* \beta$ *are derivations in G (so that $\{\alpha x^n \beta y^n \gamma \mid n \geqslant 0\} \subset L$)*

(2) *either each of α, x and β, or each of β, y and γ contains at least one marked letter.*

From a language-theoretical point of view, all the proofs of noncontext-freeness of a language that we will have to do, are very easy to achieve: using the closure under intersection with regular sets of the family of context-free languages, in every case but one, it suffices to prove that a bounded language is not context-free.

In our proofs, we shall take advantage of the covering automata constructed in part I. If A is a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ is its covering automaton, we let for all q in Q, $\mathscr{C}(A)_q$ be the automaton $\mathscr{C}(A)$ reduced to the states accessible from $\boldsymbol{a}$ and coaccessible from q, q being the only final state of $\mathscr{C}(A)_q$. As $\mathscr{C}(A)$ is deterministic, $L(\mathscr{C}(A))$ is the disjoint union of all the languages $L(\mathscr{C}(A)_q)$. Since $L(A)\subset L(\mathscr{C}(A))$, $L(A)$ is context-free if and only if, for each q in Q, $L(A)\cap L(\mathscr{C}(A)_q)$ is context-free.

1. A first necessary condition

We are now ready to start our first necessary condition for a k-VAS to be context-free. This first condition deals with the nature of iterating systems related to an elementary loop.

Lemma 1.1. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. Let (q,u) be a nonpositive elementary loop. If $L(A)$ is context-free, then there is an iterating system of length* 1 *related to it.*

Proof. Let R be the rational set recognized by $\mathscr{C}(A)$ with q as single accepting state, i.e. by the automaton: $\langle T,Q,\delta,\boldsymbol{a},\{q\}\rangle$. As $L(A)$ is context-free, so is $L(A)\cap R.u^*$. Let N be Ogden's constant for $L(A)\cap R.u^*$. As (q,u) is a loop, there is f in R such that fu^N is in $L(A)\cap R.u^*$. We mark in fu^N N times the first letter of u. According to Ogden's lemma, there is a factorization $\alpha x\beta y\gamma$ of fu^N with either α, x and β or β, y and γ containing at least one marked letter, such that $\{\alpha x^n\beta y^n\gamma \mid n\geqslant 0\}\subset L(A)\cap R.u^*$. We shall examine these two cases, but in both cases, it follows from this inclusion that $\varphi(x)\geqslant\mathbf{0}$ and $\varphi(xy)\geqslant\mathbf{0}$. Recall (Corollary I.2.6) that u is a primitive word.

First case: Suppose that α, x and β contain each at least one marked letter. There is then an overlap between x and u^i, and as $\{\alpha x^n\beta y^n\gamma \mid n\geqslant 0\}\subset R.u^*$, αx^n is, in fact, a left factor of $f.u^*$; so, x^* is included in the set *Factors* (u^*). As u is a primitive word, it follows from this inclusion that x must be a sesqui-power of u, and there is a contradiction between $\varphi(x)\geqslant\mathbf{0}$ and u is not a positive loop (cf. Lemma I.1.2).

Second case: $f=\alpha x\beta'$ with β' left factor of β, and y is a factor of u^N containing at least one marked letter. Here again the inclusion $\{\alpha x^n\beta y^n\gamma \mid n\geqslant 0\}\subset L(A)\cap R.u^*$ implies that y is an iterable factor of $R.u^*$. Here we take advantage of the particular form of the rational set recognized by $\mathscr{C}(A)$: as $\varphi(x)\geqslant\mathbf{0}$, x is the label of a loop in $\mathscr{C}(A)$, and $\{\alpha x\beta y^n\gamma \mid n\geqslant 0\}\subset R.u^*$. As u is elementary, u cannot be decomposed; so, y has to be a sesqui-power of u. So, we have that $\varphi(y)$ is not positive, which implies $\varphi(x)>\mathbf{0}$ (strictly). As $\{\alpha x^n\beta y^n\gamma \mid n\geqslant 0\}\subset L(A)$, there is a sesqui-power

$z=(x_2x_1)^i$ of $x=x_1x_2$ and a state q_1 of $\mathscr{C}(A)$ such that $\delta(q_1,z)=q_1$. From Lemma I.2.5 the properties of construction of $\mathscr{C}(A)$, this implies $\delta(q_1,x_2x_1)=q_1$. So, $(ax_1,q_1,x_2x_1,x_2\beta',q)$ is an iterating system of length 1 related to (q,u). $\square$

Recall (Definition I.3.5) that an iterating system related to (q,u) is a minimal iterating system if it is not possible to reduce it.

Lemma 1.2. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. Let (q,w) be a loop. If $L(A)$ is context-free, then every minimal iterating system related to it is of length at most* 1.

Proof. Let $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\ldots,\alpha_{p-1},q_p,u_p,\alpha_p,q)$ be a minimal iterating system related to (q,w) of length $p\geqslant 2$. If $\varphi(w)\geqslant\mathbf{0}$, $(\alpha_0u_1^{i_1}\alpha_1\ldots\alpha_{p-1}u_p^{i_p}\alpha_p,q)$ is, for some $i_1,\ldots,i_p$, an iterating system related to (q,w), and $\mathscr{I}$ is not minimal. So, we only have to consider the case when $\mathbf{not}(\varphi(w)\geqslant\mathbf{0})$.

If $L(A)$ is a context-free language, so is $L=L(A)\cap\alpha_0u_1^*\alpha_1\ldots\alpha_{p-1}u_p^*\alpha_pw^*$. In such a bounded language L, for every decomposition $f=\alpha x\beta y\gamma$ of a word f in L such that $\{\alpha x^n\beta y^n\gamma\mid n\geqslant 0\}\subseteq L$, x and y are sesqui-powers of u_i's. Using Ogden's lemma for L, marking every first letter of each w in a word $f=\alpha_0u_1^{i_1}\alpha_1\ldots\alpha_{p-1}u_p^{i_p}\alpha_pw^N$ of L, this word can be decomposed in $f=\alpha x\beta y\gamma$ such that $\{\alpha x^n\beta y^n\gamma\mid n\geqslant 0\}\subseteq L$, with y containing marked letters. As L is itself contained in the bounded language $\alpha_0u_1^*\alpha_1\ldots\alpha_{p-1}u_p^*\alpha_pw^*$, this implies that y is a sesqui-power of w, and x a sesqui-power of w or of one u_j. The former case leads to $\|w\|^-=\emptyset$, and (q,w) is a loop, while the latter case leads to $\|w\|^-\subseteq\|u_j\|^+$, and $(\alpha_0u_1^{i_1}\alpha_1\ldots\alpha_{j-1},q_j,u_j,\alpha_j\ldots\alpha_{p-1}u_p^{i_p}\alpha_p,q)$ is, for some $i_1,\ldots,i_p$, an iterating system (of length one) related to (q,w). In both cases, $\mathscr{I}$ is not minimal. $\square$

Recall (Definition I.3.3) that an iterating system is a decomposition of another iterating system if one of its constituting loops can be split into two parts in such a way that one of the parts can be iterated before any iterating of the other.

Lemma 1.3. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\mathbf{a}\rangle$ be its covering automaton. Let (q,w) be a loop. If $L(A)$ is context-free, then every minimal undecomposable iterating system of length* 1 *related to it is elementary.*

Proof. Let $\mathscr{I}=(\alpha_0,q_1,u,\alpha_1,q)$ be a minimal undecomposable iterating system of length 1 related to (q,w). We have: $\mathbf{not}(\varphi(w)\geqslant\mathbf{0})$, $\varphi(u)\geqslant\mathbf{0}$ and $\|w\|^-\subseteq\|u\|^+$. Suppose that $\mathscr{I}$ is not elementary, i.e. u is not an elementary loop. This means that u can be decomposed into $u=u_1vu_2$, where v and u_2u_1 are two loops. We set $\alpha=\alpha_0u_1$, $\beta=u_2\alpha_1$ and $x=u_2u_1$. Two cases arise.

First case: u contains a positive factor. This means that $u=u_1v'u_2$ with $\varphi(v')>\mathbf{0}$. We cannot have $\|w\|^-\subseteq\|v'\|^+$ because $\|w\|^-\subseteq\|v'\|^+$ implies that $\mathscr{I}$ is decomposable, which is contrary to the hypothesis.

If $\varphi(u_2u_1)\geqslant\mathbf{0}$, here again we cannot have $\|w\|^-\subseteq\|u_2u_1\|^+$ because $\|w\|^-\subseteq\|u_2u_1\|^+$ implies that $\mathcal{I}$ is decomposable, which is contrary to the hypothesis. To be able to iterate w we have to iterate both u_2u_1 and v. Then, we have

$$L(A)\cap\alpha_0u_1v^*(u_2u_1)^*u_2\alpha_1w^*=\{\alpha_0u_1v^n(u_2u_1)^mu_2\alpha_1w^p\mid p\leqslant\min(n,m)\}.$$

If **not** $(\varphi(u_2u_1)\geqslant\mathbf{0})$, to be able to iterate u_2u_1 we have to first iterate v. So, we have

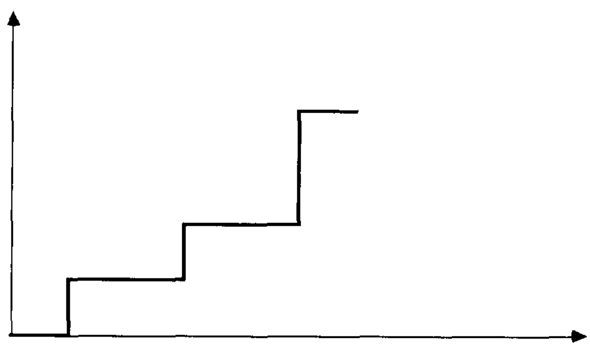
$$L(A)\cap\alpha_0u_1v^*(u_2u_1)^*u_2\alpha_1w^*=\{\alpha_0u_1v^n(u_2u_1)^mu_2\alpha_1w^p\mid p\leqslant m\leqslant n\}.$$

As the loops involving v, u_2u_1 and w are pairwise distinct loops, we may apply Corollary I.2.20′; saying that these two languages are in bijection with the languages $\{a^nb^mc^p\mid p\leqslant\min(n,m)\}$ and $\{a^nb^mc^p\mid p\leqslant m\leqslant n\}$ (where a,b and c are letters), respectively. And it is well known that these two languages are not context-free (see [5]).

Second case: u does not contain any positive factor. This means that **not**$(\varphi(v)>\mathbf{0})$ and **not**$(\varphi(x)\geqslant\mathbf{0})$. Then $\|v\|^-\subseteq\|x\|^+$ and $\|x\|^-\subseteq\|v\|^+$. It now only suffices to prove that the language $L=L(A)\cap\alpha(v^*x^*)^*$ is not context-free. For that we use the following sequence of words f_n in L: $f_n=\alpha v^{k(1)}x^{l(1)}v^{k(2)}x^{l(2)}\ldots v^{k(n)}x^{l(n)}$, with $k(i)$ and $l(i)$ two strictly increasing functions defined by simultaneous recurrence by: $k(i)$ is the largest integer r such that $\alpha v^{k(1)}x^{l(1)}v^{k(2)}x^{l(2)}\ldots x^{l(i-1)}v^r\in L$, and $l(i)$ is the largest integer s such that $\alpha v^{k(1)}x^{l(1)}v^{k(2)}x^{l(2)}\ldots v^{k(i)}x^s\in L$. It is easy to see that this last word is the shortest word of L containing $x^{l(i)}$ as a factor.

Let us give an intuitive vision of these words: We depict the words of L by a (two-dimensional) broken line, a horizontal segment of length 1 standing for each v and a vertical one for each x, as for example the word $\alpha vxvvxvvxxv$ depicted in Fig. 6.

L is then exactly the set of words depicted by those broken lines lying between two half-straight lines, the gradient of the first one defined by the smallest quotient between the positive coordinates of v and the corresponding negative coordinates of x,



The word avxvvxvvxxv

Fig. 6.

and the gradient of the second one by the smallest quotient between the positive coordinates of x and the corresponding negative coordinates of v.

Coming back to our proof that L is not context-free, we consider the words f_n of L depicted by a broken line going from one half-straight line to the other, and ending with a maximal vertical line.

Supposing that L satisfies Ogden's lemma with a constant N, the word $f_N = \alpha v^{k(1)} x^{l(1)} v^{k(2)} \ldots v^{k(N)} x^{l(N)}$, with every first letter of the last $k(N)$ subwords v as marked letters, has a factorization $\mu y \beta z \gamma$ satisfying $\{\mu y^p \beta z^p \gamma \mid p \geqslant 0\} \subset L$ with either μ, y and β or β, z and γ containing at least one marked letter. In the former case, taking $p=0$, we get in L a shorter word containing $x^{l(N)}$, which is impossible, and in the latter case, taking $p=2$, we get a word in L beginning with $\alpha v^{k(1)} x^{l(1)} v^{k(2)} \ldots v^{k(N)+1}$, which is contrary to the maximality of $k(N)$. $\square$

Let us illustrate this case by an example.

Example. Let $A=(T, \varphi, \boldsymbol{a})$ be a 2-VAS with $T=\{b, c\}$, $\boldsymbol{a}=[1,0]$ and $\varphi(b)=[-1,1]$, $\varphi(c)=[2,-1]$.

This 2-VAS has a covering automaton $\mathscr{C}(A)$ shown in Fig. 7. The words $f_n = bcb^2c^2 \ldots b^{k_i}c^{k_i} \ldots b^{k_n}c^{k_n}$ with $k_i = 2^i$ are depicted by a broken line going from one half-straight line to the other (see Fig. 8).

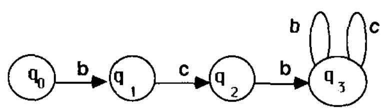


Fig. 7.

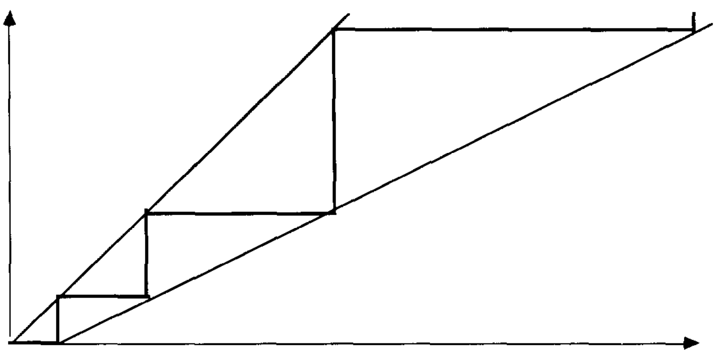


Fig. 8.

This case is the only case, throughout the paper, where a nonbounded language is used to prove that $L(A)$ is not context-free.

We can now state our first necessary condition.

Proposition 1.4. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. If $L(A)$ is context-free, then the following condition holds:*

For every iterating system $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\ldots,\alpha_{p-1},q_p,u_p,\alpha_p,q)$ related to a loop (q,w), every minimal undecomposable iterating system related to it obtained from $\mathscr{I}$ by reductions and decompositions is of length at most 1 *and elementary.*

Proof. The proof is straightforward from the preceding lemmas. $\square$

Definition 1.5. Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. Let (q,w) be a loop, and let $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\ldots,\alpha_{p-1},q_p,u_p,\alpha_p,q)$ be an iterating system of length p related to (q,w). We note $\mathbb{C}1$ $(\mathscr{I},(q,w))$, the condition: the iterating system $\mathscr{I}$ can be transformed into an elementary minimal iterating system, of length at most 1, by operations of decomposition or reduction.

Corollary 1.6. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. If $L(A)$ is context-free, then the following condition holds:*

(P1) *For every iterating system $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\ldots,\alpha_{p-1},q_p,u_p,\alpha_p,q)$ related to an elementary loop (q,u), $\mathbb{C}1$ $(\mathscr{I},(q,u))$ holds.*

This condition (P1) is not a sufficient one, as can be seen from the following example.

Let $A=(T,\varphi,\mathbf{0})$ be a 2-VAS with $T=\{a,b,c,d\}$, $\mathbf{0}=[0,0]$ and $\varphi(a)=[1,0]$, $\varphi(b)=[0,1]$, $\varphi(c)=[-1,0]$, $\varphi(d)=[0,-1]$.

This 2-VAS has a covering automaton $\mathscr{C}(A)$ shown in Fig. 9.

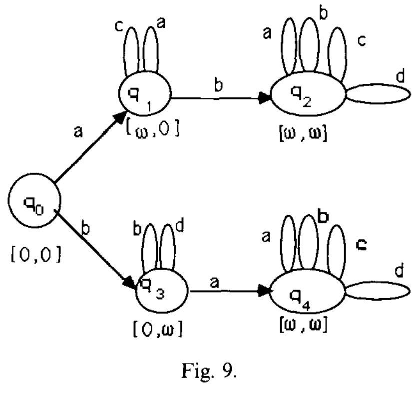


Fig. 9.

Let us examine the iterating systems related to loops labelled by c. As $\|c\|^- \subseteq \|a\|^+$ and $\|c\|^- \cap \|b\|^+ = \emptyset$, $\|c\|^- \cap \|d\|^+ = \emptyset$, any iterating system related to such a loop must contain a loop with an occurrence of a letter a. It is easy to see that all iterating systems related to (q_1, c) can be reduced to $(a, q_1, a, \lambda, q_1)$, that all iterating systems related to (q_2, c) can be reduced to (a, q_1, a, b, q_2) or $(ab, q_2, a, \lambda, q_2)$ and that all iterating systems related to (q_4, c) can be reduced to $(ba, q_4, a, \lambda, q_4)$. The situation is symmetrical for loops labelled by d, and as $\varphi(a) \geqslant 0$ and $\varphi(b) \geqslant 0$, loops labelled by a or b have their iterating systems related to them that can be reduced to systems of length 0, the conclusion of the corollary holds. However, the language $L(A) = \{f \in T^* \mid \forall g\, [f: |g|_a \geqslant |g|_c$ and $|g|_b \geqslant |g|_d\}$ is not context-free.

Remark. This example is not a counterexample to the converse of the Proposition 1.4, but only to the converse of the Corollary 1.5.

2. Other necessary conditions

The counterexample above shows that we have to add conditions to (P1). In the sequel, we consider only k-VAS satisfying (P1). The iterative systems are then reduced to iterating pairs (in the worst case). As it is the case in the counterexample, if there are two iterating pairs such that the second element of the first one is between the two elements of the second pair, the language cannot be context-free. So, our next condition deals with the relative place of the iterating pairs in the iterative systems.

Let A be a k-VAS and $\mathscr{C}(A)$ be its covering automaton, such that (P1) holds.

We consider sets of elementary positive loops and of elementary nonpositive loops. We note $\varepsilon^+ = \{(q,u) \mid u$ is elementary and $\varphi(u) > \mathbf{0}\}$, we note $\varepsilon^- = \{(q,v) \mid v$ is elementary and $\mathbf{not}(\varphi(v) \geqslant \mathbf{0})\}$ and we define the sets for $(q,v) \in \varepsilon^-$, $\mathscr{U}_{(q,v)} = \{(q',u) \in \varepsilon^+ \mid q \in Acc(q')$ and $\|v\|^- \subseteq \|u\|^+\}$ and for $(q,u) \in \varepsilon^+$, $\mathscr{V}_{(q,u)} = \{(q',v) \in \varepsilon^- \mid q' \in Acc(q)$ and $\|v\|^- \subseteq \|u\|^+\}$.

Definition 2.1. We note $\mathbb{C}2((p,u), (p',u'), (q,v), (q',v'))$, the condition: (q,v) and (q',v') are two loops in ε^-, and (p,u) and (p',u') are two loops in ε^+, such that there exist $\alpha_0, \alpha_1, \alpha_2$ and α_3 with $\delta(q_0, \alpha_0) = p$, $\delta(p, \alpha_1) = p'$, $\delta(p', \alpha_2) = q$ and $\delta(q, \alpha_3) = q'$ and such that $\|v\|^- \subseteq \|u\|^+$ and $\|v'\|^- \subseteq \|u'\|^+$, and such that either $\|v\|^-$ is not included in $\|u'\|^+$, or $\|v'\|^-$ is not included in $\|u\|^+$ (or both).

We can state our second necessary condition.

Proposition 2.2. *Let $A = (T, \varphi, \mathbf{a})$ be a k-VAS and $\mathscr{C}(A) = \langle T, Q, \delta, \mathbf{a} \rangle$ be its covering automaton such that* (P1) *holds. If $L(A)$ is context-free, than the following condition holds*:

(P2) *For no $(q,v), (q',v')$ in ε^-, $(p,u), (p',u')$ in ε^+, $\mathbb{C}2((p,u), (p'u'), (q,v), (q',v'))$ holds.*

Proof. It suffices to prove that, for all (q,v) and (q',v') in ε^-, and all (p,u) and (p',u') in ε^+, if $\mathbb{C}2$ $((p,u),(p',u'),(q,v),(q',v'))$ holds, then $L(A)$ is not context-free.

So, let $(p,u),(p',u'),(q,v)$ and (q',v') be four loops such that $\mathbb{C}2((p,u),(p',u'),(q,v),(q',v'))$ holds, and suppose that $\|v\|^-$ is not included in $\|u'\|^+$. Call β the lowest ratio $u[p]/v[p]$ for all p in $\|v\|^-\setminus\|u'\|^+$. If $\|v'\|^-$ is not included in $\|u\|^+$, then let us call γ the lowest ratio $u'[p]/v'[p]$ for all p in $\|v'\|^-\setminus\|u\|^+$. Then the language $L=L(A)\cap\alpha_0u^+\alpha_1u'^+\alpha_2v^+\alpha_3v'^+$ is equal to $\{\alpha_0u^n\alpha_1u'^m\alpha_2v^i\alpha_3v'^j \mid \beta\cdot n\geqslant i$ and $\gamma\cdot m\geqslant j\}$. If $\|v'\|^-$ is included in $\|u\|^+$, then let us call γ the lowest ratio $u'[p]/v[p]$ for all p in $\|v\|^-$, and δ the lowest ratio $u[p]/v'[p]$ for all p in $\|v'\|^-$. Then the language L is equal to $\{\alpha_0u^n\alpha_1u'^m\alpha_2v^i\alpha_3v'^j \mid \beta\cdot n\geqslant i$ and $\delta(\gamma\cdot m+\beta\cdot n-i)\geqslant j\}$. In the two cases, the language L is not context-free (see e.g. Ginsburg [5]). Hence, $L(A)$ is not context-free. $\square$

Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. We note $\sim$ the equivalence on Q such that: $q\sim q'$ iff $q\in Acc(q')$ and $q'\in Acc(q)$, $[q]$ the class of q modulo this equivalence, and $\leqslant$ the order on the set of classes defined by $[q]\leqslant[q']$ iff $q'\in Acc(q)$. It follows from the way $\mathscr{C}(A)$ is constructed that this order $\leqslant$ defines a tree. So, if we restrict our attention to one particular $\mathscr{C}(A)_s$, where s is any state (i.e. to $\mathscr{C}(A)$ reduced to the states that are coaccessible from s), and to the states concerned by this automaton, $\leqslant$ becomes a total order on the classes of states. It is then possible to make an indexation of these sets. We call $[j]$ the jth class in this order.

Lemma 2.3. *For all (q,v) and (q',v') in ε^- with $q\sim q'$, we have $\mathscr{U}_{(q,v)}=\mathscr{U}_{(q',v')}$.*

Proof. Let (q,v) and (q',v') be two loops in ε^- such that $q\sim q'$ and $\mathscr{U}_{(q,v)}\neq\mathscr{U}_{(q',v')}$. Either there is (p,u) in $\mathscr{U}_{(q,v)}$ such that $\|v'\|^-$ is not included in $\|u\|^+$, or there is (p',u') in $\mathscr{U}_{(q',v')}$ such that $\|v\|^-$ is not included in $\|u'\|^+$. Suppose that $(p,u)\in\mathscr{U}_{(q,v)}\setminus\mathscr{U}_{(q',v')}$, and let $(p',u')\in\mathscr{U}_{(q',v')}$ (by (P1), $\mathscr{U}_{(q',v')}$ is not empty).

If $p'\in Acc(p)$, then there exist $\alpha_0,\alpha_1,\alpha_2$ and α_3 such that $\delta(q_0,\alpha_0)=p$, $\delta(p,\alpha_1)=p'$, $\delta(p',\alpha_2)=q$ and $\delta(q,\alpha_3)=q'$. For these loops, $\mathbb{C}2((p,u),(p',u'),(q,v),(q',v'))$ holds. If $p\in Acc(p')$, then there exist $\alpha_0,\alpha_1,\alpha_2$ and α_3 such that $\delta(q_0,\alpha_0)=p'$, $\delta(p',\alpha_1)=p$, $\delta(p,\alpha_2)=q'$ and $\delta(q',\alpha_3)=q$, $\mathbb{C}2((p',u'),(p,u),(q',v'),(q,v))$ holds. By applying Proposition 2.2 in both cases, we conclude that the language associated is not context-free. $\square$

Lemma 2.4. *For all (p,u) and (p',u') in ε^+ with $p\sim p'$, we have $\mathscr{V}_{(p,u)}=\mathscr{V}_{(p',u')}$.*

Proof. As for Lemma 2.3. $\square$

We note: $\mathscr{V}_i=\{(q,v)\in\varepsilon^- \mid q\in[i]\}$ and $\mathscr{U}_j=\{(p,u)\in\varepsilon^+ \mid p\in[j]$ and $\mathscr{V}_{(p,u)}\neq\emptyset\}$. It follows from Lemmas 2.3 and 2.4 that if $\mathscr{U}_j\cap\mathscr{U}_{(q,v)}\neq\emptyset$ then $\mathscr{U}_j\subseteq\mathscr{U}_{(q,v)}$ and, conversely, $\mathscr{V}_i\cap\mathscr{V}_{(p,u)}\neq\emptyset$ implies $\mathscr{V}_i\subseteq\mathscr{V}_{(p,u)}$.

We define the sets $\mathscr{I}^-(j)=\{i \mid \mathscr{V}_i \subseteq \mathscr{V}_{(p,u)}$ and $p\in[j]\}$ and $\mathscr{I}^+(i)=\{j \mid \mathscr{U}_j \subseteq \mathscr{U}_{(q,v)}$ and $q\in[i]\}$.

Corollary 2.5. *If i, i', j and j' are integers such that $i<i'$ and $j<j'$ and such that either $i\in\mathscr{I}^-(j)$ and $i'\in\mathscr{I}^-(j')$, or $j\in\mathscr{I}^+(i)$ and $j'\in\mathscr{I}^+(i')$, then $L(A)$ is not context-free.*

Proof. The proof is straightforward from the Proposition 2.2 and the preceding definitions. □

Condition (P2), together with condition (P1), is not a sufficient one, as can be seen from the following example.

Let $A=(T,\varphi,\boldsymbol{a})$ be a 2-VAS with $T=\{b,c,d\}$, $\boldsymbol{a}=[0,0]$ and $\varphi(b)=[1,1]$, $\varphi(c)=[-1,0]$, $\varphi(d)=[0,-1]$.

This 2-VAS has a covering automaton $\mathscr{C}(A)$ shown in Fig. 10.

We have $\varepsilon^+=\{(q_1,b)\}$ and $\varepsilon^-=\{(q_1,c),(q_1,d)\}$. Clearly, every iterating system related to (q_1,c) or to (q_1,d) can be transformed by reduction and decomposition into (b,q_1,b,λ,q_1) which is of length 1, and (P1) holds. As ε^+ has only one element, (P2) holds trivially. However, the language $L(A)$, which is such that $L(A)\cap b^*c^*d^*=\{b^nc^pd^r \mid n\geqslant \sup(p,r)\}$, is not context-free.

3. More necessary conditions

The necessary conditions obtained so far are "rough" since they do not take into account the values of the vectors, but only their signs. From now on, we will consider these values. In this section, we establish a new necessary condition dealing with dominating coordinates. The intuitive idea behind this condition is the following: if we take a set of vectors corresponding to a set of elementary loops that need no loop outside the set to be iterated, the set of all words that are in the associated language, and obtained by iteration of these loops, corresponds to a set of coefficients that has to satisfy an inequality for each coordinate. So, if we have (at least) three vectors and two independent coordinates, we come to a set of words of the form: $\{\alpha u^i\beta v^j\gamma w^k \mid i,j,k$ satisfy two independent inequations$\}$, a set which is in general (and in our cases) not context-free.

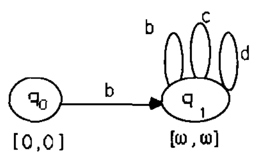


Fig. 10.

In the sequel, we consider only k-VASs satisfying both (P1) and (P2). Let F be a set of loops, we note *label*(F) the set of labels of the loops of F, i.e. $label(F)=\{w \mid \exists q\in Q, (q,w)\in F\}$.

For all class $[i]$ of states, we define $\mathscr{W}_i$ as the set $\mathscr{W}_i=\mathscr{V}_i\cup(\bigcup_{j\in\mathscr{I}^+(i)}\mathscr{U}_j)$, i.e. the union of $\mathscr{V}_i$, nonpositive loops with state in $[i]$, and of the set of all loops (p,u) such that the states of $[i]$ are accessible from p, and such that for each loop (q,v) of $\mathscr{V}_i$, $\|v\|^- \subseteq \|u\|^+$ (this last fact is denoted by $\|\mathscr{V}_i\|^- \subseteq \|u\|^+$).

Symmetrically, $\mathscr{X}_i$ is the set $\mathscr{X}_i=\mathscr{U}_i\cup(\bigcup_{j\in\mathscr{I}^-(i)}\mathscr{V}_j)$.

Let us consider a class $[i]$ of states, such that $\mathscr{V}_i$ is not empty, and let $\mathscr{V}$ be a nonempty subset of $\mathscr{V}_i$. Let $\mathscr{U}$ be a nonempty subset of $\mathscr{U}_j$ with $j\in\mathscr{I}^+(i)$ (this means that, for (p,u) loop in $\mathscr{U}$, $\|\mathscr{V}_i\|^- \subseteq \|u\|^+$), and call V_u the set of labels: $V_u=label(\mathscr{V}\cup\mathscr{U})$.

Lemma 3.1. *For all $\mathscr{V}$, subset of $\mathscr{V}_i$, and for all $\mathscr{U}$, nonempty subset of $\mathscr{U}_j$* with *$j\in\mathscr{I}^+(i)$, if there exist $p\in\|\mathscr{V}_i\|^-$ such that for all v in label $(\mathscr{V})$: $\varphi(v)[p]<0$, then either there exists (among the coordinates satisfying this condition) a dominating coordinate relative to V_u, or the language associated with the k-VAS is not context-free.*

Proof. We define $\|\mathscr{V}\|^{--}$ as the nonempty set of all coordinates satisfying the condition: $\|\mathscr{V}\|^{--}=\{p \mid \forall v\in label(\mathscr{V}),\ \varphi(v)[p]<0\}$. It is clear that if there exists a dominating coordinate, it must be among the coordinates in $\|\mathscr{V}\|^{--}$ (if a vector has a nonnegative coordinate besides its negative coordinates, this nonnegative coordinate cannot dominate them; hence, cannot dominate a coordinate in $\|\mathscr{V}\|^{--}$).

Ab absurdo, let us suppose that the language associated with the k-VAS is context-free and that there is no dominating coordinate relative to V_u. We distinguish two cases, depending on whether every coordinate in $\|\mathscr{V}_i\|^-$ is dominated by a coordinate in $\|\mathscr{V}\|^{--}$, or not.

First case: every coordinate in $\|\mathscr{V}_i\|^-$ is dominated by a coordinate in $\|v\|^{--}$. In this case, we have to prove that, if the language associated with the k-VAS is context-free, there is a coordinate in $\|\mathscr{V}\|^{--}$ dominating all other coordinates in this set (relative to V_u). So, we only have to consider the restriction of the vectors of V_u to the coordinates in $\|\mathscr{V}\|^{--}$, and for it, we are in the situation described in the hypothesis of Proposition I.4.10, i.e. for each vector, the sign is the same for every coordinate in $\|\mathscr{V}\|^{--}$. So, we know from this proposition that either there is a dominating coordinate, or there is a nonzero positive solution to the set of inequalities: $\{\sum_{1\leqslant j\leqslant r} x_j u_j[p]+\sum_{1\leqslant j\leqslant s} y_j v_j[p]\geqslant 0 \mid \text{for all } p\}$. To complete the proof, if suffices to show that in this last case, the language associated with the k-VAS is not context-free.

Let p and q be two coordinates, none of them dominating the other, let $(x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_s)$ be such a nonzero solution to the set of inequalities, and suppose that $L(A)$ is context-free. Let then N be the Ogden's constant of the language $L=L(A)\cap\alpha u_1^* u_2^*\ldots u_r^* v_1^* v_2^*\ldots v_s^*$. The tuple $(Nx_1,Nx_2,\ldots,Nx_r,Ny_1,Ny_2,\ldots,Ny_s)$ is also a nonzero solution of the set of inequalities because we deal with linear inequalities. Call f the word of L corresponding to this last tuple, i.e. $f=\alpha u_1^{Nx_1}u_2^{Nx_2}\dots u_r^{Nx_r}v_1^{Ny_1}v_2^{Ny_2}\dots v_s^{Ny_s}$, and call image of f the point in the $(r+s)$-dimensional space of coordinates $(Nx_1, Nx_2, \dots, Nx_r, Ny_1, Ny_2, \dots, Ny_s)$. Let us mark N times the first letter of a factor, say v_i, in the word f. The Ogden lemma says that there must be an iterating pair, and that when this pair is iterated, the words obtained have to be in L. So the $(r+s)$-tuples corresponding to these words obtained from f by iteration have to be solutions to the inequalities relative to all coordinates and, in particular, to both the inequality relative to p and to the inequality relative to q. When we iterate the pair, all the words obtained have images in the $(r+s)$-dimensional space in a straight line since we always add the same values. But, if all words obtained by iteration are such that the corresponding $(r+s)$-tuples are solutions to the inequality relative to q, then, the straight line must be parallel to the hyper-plane defined by the equation associated to q, and this line crosses the hyper-planes defined by the equation associated to p. So, some word f_1 obtained by iteration has an image which fails to be a solution to the inequality relative to p. Conversely, if all words obtained by iteration are such that the corresponding $(r+s)$-tuples are solutions to the inequality relative to p, some f_2 among them has an image which fails to be a solution to the inequality relative to q, as shown in Fig. 11.

So, one word obtained by iteration, which according to Ogden's lemma should be in the language, does not verify one of the inequality defining the language; hence, a contradiction.

Second case: some coordinate in $\|\mathscr{V}_i\|^-$ is not dominated by a coordinate in $\|\mathscr{V}\|^{--}$.

In this case, we consider the nonempty set of coordinates that are not in $\|\mathscr{V}\|^{--}$, and are not dominated by a coordinate in $\|\mathscr{V}\|^{--}$. In this set, there is a coordinate q which is not strictly dominated by any other coordinate of this set. Call $\{v_1, v_2, \dots, v_s\}$ the subset of all elements v of $\mathscr{V}$ such that $\varphi(v)[q]<0$, and call v' an element of $\mathscr{V}$ not in this subset (v' exists because, otherwise, q would be in $\|\mathscr{V}\|^{--}$).

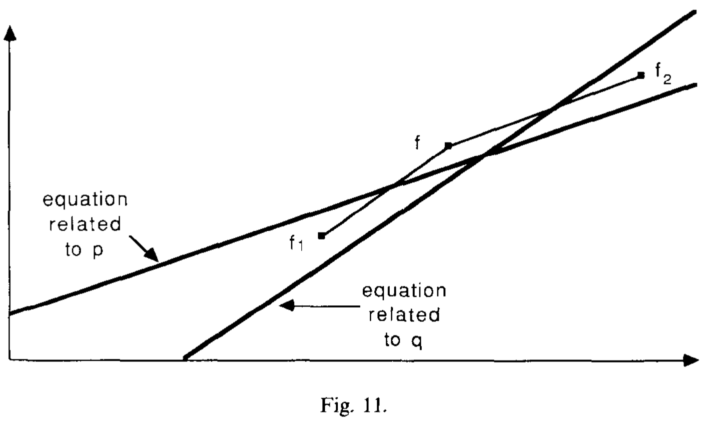


Fig. 11.

Let $\{u_1, u_2, \ldots, u_r\}$ be a subset of elements of $\mathscr{U}_j$ such that $\| \mathscr{V} \|^- \subseteq \| \{u_1, u_2, \ldots, u_r\} \|^+$, there are $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$ such that, for all coordinates p $\sum_{1 \leqslant k \leqslant r} x_k u_k[p] + \sum_{1 \leqslant k \leqslant s} y_k v_k[p] \geqslant 0$, and $\sum_{1 \leqslant k \leqslant r} x_k u_k[q] + \sum_{1 \leqslant k \leqslant s} y_k v_k[q] = 0$. For a coordinate p that does not dominate q, we have $\sum_{1 \leqslant k \leqslant r} x_k u_k[p] + \sum_{1 \leqslant k \leqslant s} y_k v_k[p] > 0$; hence, there exist x and y such that, for all coordinates p $x.(\sum_{1 \leqslant k \leqslant r} x_k u_k[p] + \sum_{1 \leqslant k \leqslant s} y_k v_k[p]) + y.v'[p] \geqslant 0$. Let $L = L(A) \cap \alpha u_1^* u_2^* \ldots u_r^* v_1^* v_2^* \ldots v_s^* v'^*$. The set of exponents $\{(i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_s, t) \mid \alpha u_1^{i_1} u_2^{i_2} \ldots u_r^{i_r} v_1^{j_1} v_2^{j_2} \ldots v_s^{j_s} v'^t \in L\}$ has to satisfy simultaneously the two conditions.

(i) $\sum_{1 \leqslant j \leqslant r} i_j u_j[q] + \sum_{1 \leqslant j \leqslant s} j_j v_j[q] = 0$, corresponding to a left factor.

(ii) $(\sum_{1 \leqslant j \leqslant r} i_j u_j[p] + \sum_{1 \leqslant j \leqslant s} j_j v_j[p]) + t.v'[p] \geqslant 0$.

So, L is not context-free. To prove this fact, it suffices to apply Ogden's lemma, marking every occurrence of the first letter of v' in the word $\alpha u_1^{N.i_1} u_2^{N.i_2} \ldots u_r^{N.i_r} v_1^{N.j_1} v_2^{N.j_2} \ldots v_s^{N.j_s} v'^{N.t}$ which also satisfies the two conditions. As v' cannot be iterated alone, it must be iterated with some u_j, and erasing both terms of the pair would lead to a contradiction with the condition (i).

Hence, $L(A)$ is not context-free. □

Corollary 3.2. *If $\exists p \in \| \mathscr{V}_i \|^-$ such that $\forall v \in label(\mathscr{V}_i)$, $\varphi(v)[p] < 0$, then either there exists (among the coordinates satisfying this condition) a dominating coordinate relative to label $(\mathscr{W}_i)$, or the language associated with the k-VAS is not context-free.*

Proof. In the preceding lemma, we can choose for $\mathscr{V}$ any subset of $\mathscr{V}_i$, and for $\mathscr{U}$ any set of elements u of $\mathscr{U}_j$ with $j \in \mathscr{I}^+(i)$ (i.e. such that $\| \mathscr{V}_i \|^- \subseteq \| u \|^+$ with the states in $[i]$ accessible from the loop labelled by u). As $\mathscr{W}_i = \mathscr{V}_i \cup (\bigcup_{j \in \mathscr{I}^+(i)} \mathscr{U}_j)$, one only need to apply the lemma choosing $\mathscr{V} = \mathscr{V}_i$ and $\mathscr{U} = \bigcup_{j \in \mathscr{I}^+(i)} \mathscr{U}_j$. □

Now we will show that the condition that we stated in Lemma 3.1 must be verified in a context-free VAS language.

Lemma 3.3. *If the language L associated with the k-VAS is context-free, then for all classes of states $[i]$, $\exists p \in \| \mathscr{V}_i \|^-$ such that $\forall v \in label(\mathscr{V}_i)$, $\varphi(v)[p] < 0$.*

Proof. Let u be the label of a positive loop such that $\| \mathscr{V}_i \|^- \subseteq \| u \|^+$. For all v, label of a nonpositive loop in $\mathscr{V}_i$, and all V, set of labels of loops in $\mathscr{V}_i$ not containing v, we consider the language $L \cap u^* V^* v^*$. To this language is attached an integer given by the Ogden's Lemma. Let N be the maximum of all Ogden's lemma integers attached to such languages $L \cap u^* V^* v^*$. For the sake of clarity, in this proof we drop the (fixed) words; this permits us to reach the loops considered.

Let $V = \{v_1, v_2, \ldots, v_r\}$ be the set of labels of the loops of $\mathscr{V}$, where $\mathscr{V}$ is a subset of $\mathscr{V}_i$ of maximum cardinality r, such that for at least one coordinate p, $\varphi(v_i)[p] < 0$ for all v_i. Since $\mathscr{V}_i$ is not empty, $\mathscr{V}$ is not empty either. Suppose that $\forall p \in \| \mathscr{V}_i \|^-$, $\exists (q, v) \in \mathscr{V}_i$, $\varphi(v)[p] \geqslant 0$. This means that there exist (q, v) such that v is in $label(\mathscr{V}_i) \backslash label(\mathscr{V})$. We know from the preceding lemma that there is a dominating coordinate p_v with respect to V_u. We are then in the following situation: for every coordinate p such that $\forall v_i$: $\varphi(v_i)[p]<0$, we have $\varphi(v)[p]\geqslant 0$ (because $\mathscr{V}$ is supposed to be maximal), and for every coordinate p such that $\varphi(v)[p]<0$, there exist $x, x_1, x_2, \ldots, x_r$ such that $x \cdot \varphi(u)[p]+\sum_{1\leqslant i\leqslant r} x_i\varphi(v_i)[p]>0$ when $x \cdot \varphi(u)[p_V]+\sum_{1\leqslant i\leqslant r} x_i\varphi(v_i)[p_V]\geqslant 0$ is true. So, taking x minimum such that this last inequality holds, we can then multiply all these coefficients by a number m (that we choose minimal) such that $m(x \cdot \varphi(u)+\sum_{1\leqslant i\leqslant r} x_i\varphi(v_i))+N\cdot\varphi(v)>0$.

This leads to $L\cap u^*\text{label}(\mathscr{V})^*v^*$ not context-free; hence, L not context-free, contrary to our hypothesis, the following way: taking $u^{x\cdot m}wv^N$ (with w a word corresponding to $\prod v_i^{x_i m}$), since $\mathbf{not}(\varphi(v)\geqslant\mathbf{0})$, v^i is the second element of an iterative pair, the first element of which is u^j, for some i and $j>0$. If we put off these two elements, the remaining word should belong to L (according to Ogden's lemma), but it does not since it has $u^n w$ with $n<x\cdot m$ as a left factor, and $u^n w\notin L$, by minimality of x. □

We can state our third necessary condition.

Proposition 3.4. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton such that* (P1) *and* (P2) *hold. If $L(A)$ is context-free, then the following condition holds:*

(P3) *For all classes $[i]$ of states, $\mathbb{C}3(label(\mathscr{W}_i))$ and $\mathbb{C}3(label(\mathscr{X}_i))$ hold.*

Proof. It means that if the language associated with the k-VAS is context-free, then for each class $[i]$ of states, there exists a dominating coordinate related to *label*($\mathscr{W}_i$), and symmetrically, there exists a dominating coordinate related to *label*($\mathscr{X}_i$).

The proof is straightforward from the preceding lemmas. □

4. Last necessary conditions

The three necessary conditions brought to evidence in the preceding paragraphs are not sufficient, as can be seen in the following example.

Let $A=(T,\varphi,\boldsymbol{a})$ be a 10-VAS with $T=\{a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2,\alpha,\beta,\gamma\}$, where

$\boldsymbol{a}=[0,0,1,0,0,0,0,0,0,0]$ and

$\varphi(a_1)=[1,2,-1,1,0,0,0,0,0,0]$,

$\varphi(a_2)=[0,0,1,-1,0,0,0,0,0,0]$,

$\varphi(b_1)=[2,1,0,0,-1,1,0,0,0,0]$,

$\varphi(b_2)=[0,0,0,0,1,-1,0,0,0,0]$,

$\varphi(c_1)=[-1,-2,0,0,0,0,-1,1,0,0]$,

$\varphi(c_2)=[0,0,0,0,0,0,1,-1,0,0]$

$$\varphi(d_1)=[-2,-1,0,0,0,0,0,0,-1,1],$$

$$\varphi(d_2)=[0,0,0,0,0,0,0,0,1,-1],$$

(we shall call $u_1=a_1a_2$, $u_2=b_1b_2$, $v_1=c_1c_2$, $v_2=d_1d_2$)

$$\varphi(\alpha)=[0,-1,-1,0,1,0,0,0,0,0],$$

$$\varphi(\beta)=[0,1,0,0,-1,0,1,0,0,0],$$

$$\varphi(\gamma)=[0,0,0,0,0,0,-1,0,1,0].$$

Only the first two coordinates are unbounded.
This 10-VAS has a covering automaton $\mathscr{C}(A)$ shown in Fig. 12.

For $q=q_8$, we have

$$L(A)\cap L(\mathscr{C}(A)_q)=\{u_1^n\alpha u_2^m\beta v_1^r\gamma v_2^s \mid n+2\cdot m\geqslant r+4\cdot s \text{ and } 2\cdot n+m\geqslant 2\cdot r+2\cdot s\}.$$

This language is not context-free. This is to be contrasted with what happens if we slightly change the k-VAS by changing only $\varphi(\beta)$ and $\varphi(\gamma)$ to

$$\varphi(\beta)=[0,1,0,0,-1,0,0,0,1,0],$$

$$\varphi(\gamma)=[0,0,0,0,0,0,1,0,-1,0].$$

The new 10-VAS A' has a covering automaton $\mathscr{C}(A')$ as shown in Fig. 13.

For $q=q_8$, we have

$$L(A')\cap L(\mathscr{C}(A')_q)=\{u_1^n\alpha u_2^m\gamma v_2^s\beta v_1^r \mid n+2\cdot m\geqslant r+4\cdot s \text{ and } 2\cdot n+m\geqslant 2\cdot r+2\cdot s\}.$$

The reader may verify that this language is context-free (and so is $L(A')$).

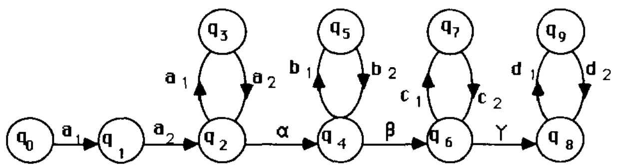


Fig. 12.

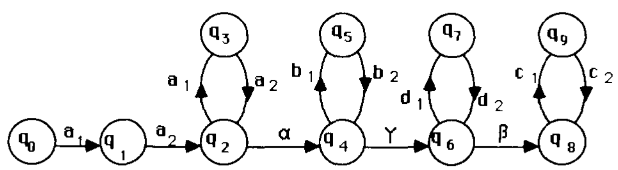


Fig. 13.

We recall here some notations introduced in part I, Section 4: For all sets $W=(w_1, w_2, \ldots, w_r)$ of k-tuples of $\mathbf{Z}$, and for each coordinate j, φ_j is the linear form defined by: $\forall \boldsymbol{x}=(x_1, x_2, \ldots, x_r)$, $\varphi_j(\boldsymbol{x})=\sum_{1 \leqslant i \leqslant r} x_i w_i[j]$; H_j and K_j are the sets: $H_j=\{x \mid \varphi_j(\boldsymbol{x}) \geqslant 0\}$, $K_j=\{x \mid \varphi_j(\boldsymbol{x})=0\}$. We set $H=\bigcap_{1 \leqslant j \leqslant k} H_j$.

Definition 4.1. For any set W of k-tuples, denote Δ_w the set: $\Delta_w=\{\boldsymbol{x} \in H \mid \exists r, s\ \varphi_r$ and φ_s are not colinear and $\boldsymbol{x} \in K_r \cap K_s\}$. We note $\mathbb{C}4(W)$, the condition: the set Δ_w is a stratified set.

Consider $\mathscr{C}(A)_q$ for some state q, and do a numbering of the classes of states from 1 to m. For J a set of integers in $[1, m]$, let W_J denote the set of vectors that are images by φ of the labels w of all elementary loops $(q, w) \in \mathscr{V}_j \cup \mathscr{U}_j$ (with q in some class $[j]$) such that $j \in J$. Recall that $\mathscr{I}^-(j)=\{i \mid \mathscr{V}_i \subseteq \mathscr{V}_{(p,u)}$ and $p \in [j]\}$ and that $\mathscr{I}^+(i)=\{j \mid \mathscr{U}_j \subseteq \mathscr{U}_{(q,v)}$ and $q \in [i]\}$.

For j an integer in $[1, m]$, we define the set $I^*(j)$ of integers in $[1, m]$ as the smallest set of integers in $[1, m]$ containing j and closed for the following operations.

If $i \in I^*(j)$ is such that there exists $(q, v) \in \mathscr{V}_i$, then $\mathscr{I}^+(i) \subseteq I^*(j)$.

If $i \in I^*(j)$ is such that there exists $(p, u) \in \mathscr{U}_i$, then $\mathscr{I}^-(i) \subseteq I^*(j)$.

Intuitively, $I^*(j)$ contains all elementary positive loops that permit to iterate an elementary nonpositive loop of $[j]$, as well as all elementary nonpositive loops which can be iterated with a positive elementary loop already got.

Lemma 4.2. *If W is such that $W=W_J$ for some $J=I^*(j)$ with $j \in [1, m]$, and if $\mathbb{C}4(W)$ does not hold, then $L(A)$ is not context-free.*

Proof. Let $j \in [1, m]$, $J=I^*(j)$, and $W=W_J$ be such that $\mathbb{C}4(W)$ does not hold, i.e. such that $\Delta_w=\{\boldsymbol{x} \in H \mid \exists r, s\ \varphi_r$ and φ_s are not colinear and $\boldsymbol{x} \in K_r \cap K_s\}$ is not a stratified set. Let $\alpha_0 u_1 \alpha_1 u_2 \ldots \alpha_{t-1} u_t$ be a word in $L(A)$, where $u_1, u_2, \ldots, u_t$ are the labels in $\bigcup_{i \in J} \mathscr{V}_i \cup \mathscr{U}_i$. We know (Theorem I.0.5) that $L(A) \cap \alpha_0 u_1^* \alpha_1 u_2^* \ldots \alpha_{t-1} u_t^*$ is context-free if and only if the corresponding set, which is precisely Δ_w, is stratified. Hence, $L(A)$ is not context-free. □

We can state our fourth necessary condition.

Proposition 4.3. *Let $A=(T, \varphi, \boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T, Q, \delta, \boldsymbol{a}\rangle$ be its covering automaton such that* (P1), (P2) and (P3) *hold. If $L(A)$ is context-free, then the following condition holds*:

(P4) *For all $j \in [1, m]$, the set $W=W_{I^*(j)}$ satisfies $\mathbb{C}4(W)$.*

Proof. The proof is straightforward from the preceding lemma. □

We come now to our last necessary condition, following from the consideration of positive elementary iterable factors which can introduce a perturbation in the equations corresponding to a set of vectors W.

The following example provides two VAS each having a positive elementary loop (labelled with c) that cannot iterate any elementary nonpositive loop. Because of the value of the nonpositive elementary loop (labelled with d) the first associated language is not context-free, while the second one is context-free.

Let $A=(T,\varphi,\boldsymbol{a})$ be a 2-VAS with $T=\{b,c,d\}$, $\boldsymbol{a}=[0,0]$ and $\varphi(b)=[1,1]$, $\varphi(c)=[0,1]$, $\varphi(d)=[-1,-2]$, $L(A)\cap b^*c^*d^*=\{b^nc^pd^r \mid n\geqslant r \text{ and } n+p\geqslant 2\cdot r\}$, is not context-free, whereas for $A'=(T,\varphi',\boldsymbol{a})$ a 2-VAS with $T=\{b,c,d\}$, $\boldsymbol{a}=[0,0]$ and $\varphi'(b)=[1,1]$, $\varphi'(c)=[0,1]$, $\varphi'(d)=[-1,-1]$, $L(A)\cap b^*c^*d^*=\{b^nc^pd^r \mid n\geqslant r$ and $n+p\geqslant r\}=\{b^nc^pd^r \mid n\geqslant r\}$, is context-free.

Supposing that there is a dominating coordinate with respect to W_J, we know from Lemma 3.1 that this dominating coordinate is in the set: $\bigcap_{j\in J}\|\mathscr{V}_j\|^{--}$, the complete negative part of J; we note $DOMIN(J)$ the set of all dominating coordinates with respect to W_J.

In the example, for A $DOMIN(J)=\{2\}$, while for A' $DOMIN(J)=\{1,2\}$.

Let $\mathscr{R}_J=\{(q,u)\in\mathscr{E} \mid J\subseteq Acc(q) \text{ and } DOMIN(J)\subseteq\|u\|^+\}$ be the set of all elementary loops that enable to iterate loops with negative coordinates in $DOMIN(J)$.

In the example, for A, $\mathscr{R}_J=\{(q,b),(q,c)\}$, while for A', $\mathscr{R}_J=\{(q,b)\}$.

Let $\mathscr{H}_J=\mathscr{R}_J\backslash(\bigcup_{j\in J}\mathscr{U}_j\cup\bigcup_{j\in J}\mathscr{V}_j)$ be the set of elementary loops that introduce a perturbation in the set of inequalities related to $DOMIN(J)$. These elementary loops (even positive) do not enable to iterate any nonpositive elementary loop in $\bigcup_{j\in J}\mathscr{V}_j$.

In the example, for A, $\mathscr{H}_J=\{(q,c)\}$, while for A', $\mathscr{H}_J=\emptyset$.

Last, call $Z_J=\{\varphi(u) \mid \exists(q,u)\in\mathscr{H}_J\}$ the set of vectors associated with elementary loops of $\mathscr{H}_J$, and $ZERO(J)=\bigcap_{u\in\mathscr{H}_J}\|u\|^0$ the set of coordinates on which all perturbating loops have a null value.

In the example, for A, $ZERO(J)=\{1\}$, while for A', $ZERO(J)=\emptyset$.

Lemma 4.4. *If W is such that $W=W_J$ for some $J=I^*(j)$ with $j\in[1,m]$, and if W has a dominating coordinate, then there is one in $ZERO(J)$ or $L(A)$ is not context-free.*

Proof. Suppose that $L(A)$ is context-free, and that a set $W=W_J$, for some $J=I^*(j)$ with $j\in[1,m]$, has a dominating coordinate s. Suppose then that there is no dominating coordinate in $ZERO(J)$.

Let $u_1,u_2,\ldots,u_t$ be the labels in $\bigcup_{i\in J}\mathscr{V}_i\cup\mathscr{U}_i$. There are coefficients $x_1,x_2,\ldots,x_t$ such that $\sum_{1\leqslant i\leqslant t}x_i\varphi(u_i)[r]\geqslant 0$ for all r in $ZERO(J)$, and $\sum_{1\leqslant i\leqslant t}x_i\varphi(u_i)[r]=0$ for some r in $ZERO(J)$ because the semilinear set of solutions of inequalities is nonempty. Note that the set of solutions of these inequalities is strictly bigger than the set of solutions obtained with the set of all coordinates because all dominating coordinates are outside $ZERO(J)$. The set of coordinates $R=\{r\in[1,k]$ such that $\sum_{1\leqslant i\leqslant t}x_i\varphi(u_i)[r]<0\}$ is not empty since s is in R.

For each r in R there exists $u_{(r)}$ in $\bigcup_{i\in J}\mathcal{U}_i$ such that $\varphi(u_{(r)})[r]>0$ by definition of the sets $\mathcal{U}_i$. Let t' be the number of coordinates in R. t' is not zero; t is at least 2: one positive loop and one nonpositive loop. Then, the set $\{(y_1, y_2, \ldots, y_{t'}, x_1, x_2, \ldots, x_t) \mid \sum_{1\leqslant i\leqslant t'} y_i\varphi(u_{(i)}) + \sum_{1\leqslant i\leqslant t} x_i\varphi(u_i)[r]\geqslant 0\}$ is not a stratified set. Hence, $L(A)$ is not context-free, and we have a contradiction. □

We can state our fifth necessary condition.

Proposition 4.5. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton such that* (P1), (P2), (P3) *and* (P4) *hold. If $L(A)$ is context-free, then the following condition holds*:

(P5) *For all $j\in[1,m]$, if $W_{I^*(j)}$ has a dominating coordinate, then there is one in $ZERO(J)$.*

Proof. The proof is straightforward from the preceding lemma. □

5. All the conditions together are sufficient

In this section, we show that if a k-VAS A satisfies all the necessary conditions proved above, then the language $L(A)$ associated is context-free. So, we get our characterization theorem. We recall first the definitions of the elementary conditions involved.

Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. Let (q,w) be a loop, and let $\mathscr{I}=(\alpha_0, q_1, u_1, \alpha_1, q_2, \ldots, \alpha_{p-1}, q_p, u_p, \alpha_p, q)$ be an iterating system of length p related to (q,w). We note $\mathbb{C}1(\mathscr{I},(q,w))$, the condition: the iterating system $\mathscr{I}$ can be transformed into an elementary minimal iterating system of length at most one, by operations of decomposition or reduction.

We note $\mathbb{C}2((p,u),(p',u'),(q,v),(q',v'))$, the condition: (q,v) and (q',v') are two loops in ε^-, and (p,u) and (p',u') are two loops in ε^+, such that there exist $\alpha_0, \alpha_1, \alpha_2$ and α_3 with $\delta(q_0,\alpha_0)=p$, $\delta(p,\alpha_1)=p'$, $\delta(p',\alpha_2)=q$ and $\delta(q,\alpha_3)=q'$ and such that $\|v\|^-\subseteq\|u\|^+$ and $\|v'\|^-\subseteq\|u'\|^+$, and such that either $\|v\|^-$ is not included in $\|u'\|^+$, or $\|v'\|^-$ is not included in $\|u\|^+$ (or both).

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_t$ be t k-tuples of $\mathbf{Z}^k$. We note $\mathbb{C}3(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_t)$, the condition: there exists a coordinate p which is a dominating coordinate with respect to $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_t\}$, and $\mathbb{C}3(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_t;\ p, q)$, the condition: the coordinate p dominates the coordinate q with respect to $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_t\}$.

For any set W of k-tuples, denote Δ_W the set: $\Delta_W=\{x\in H \mid \exists r, s\ \varphi_r$ and φ_s are not colinear and $x\in K_r\cap K_s\}$. We note $\mathbb{C}4(W)$, the condition: the set Δ_W is a stratified set.

Theorem 5.1. *Let $A=(T,\varphi,\boldsymbol{a})$ be a k-VAS and $L(A)$ be its associated language. Let $\mathscr{C}(A)=\langle T,Q,\delta,\boldsymbol{a}\rangle$ be its covering automaton. $L(A)$ is context-free if and only if the following conditions hold:*

(P1) *For every iterating system $\mathscr{I}=(\alpha_0,q_1,u_1,\alpha_1,q_2,\ldots,\alpha_{p-1},q_p,u_p,\alpha_p,q)$ related to an elementary loop (q,u), $\mathbb{C}1(\mathscr{I},(q,u))$ is satisfied.*

(P2) *For no (q,v), (q',v') in ε^-, (p,u), (p',u') in ε^+ $\mathbb{C}2((p,u),(p',u'),(q,v),(q',v'))$ is satisfied.*

(P3) *For all classes $[i]$ of states $\mathbb{C}3(label(\mathscr{W}_i))$ and $\mathbb{C}3(label(\mathscr{X}_i))$ are satisfied.*

(P4) *For all $j\in[1,m]$, the set $W=W_{I^*(j)}$ satisfies $\mathbb{C}4(W)$.*

(P5) *For all $j\in[1,m]$, if $W_{I^*(j)}$ has a dominating coordinate, then there is one in $ZERO(J)$.*

It remains to prove that the condition is sufficient. For that, we construct a pushdown automaton recognizing $L(A)$. We rename $\mathscr{C}$ the covering automaton $\mathscr{C}(A)$ and K the rational set $L(\mathscr{C}(A))$. As we did in part II.2, we use the accessibility equivalence $\sim$ on the set of states Q of $\mathscr{C}$: $q\sim q'$ if and only if $q\in Acc(q')$ and $q'\in Acc(q)$, and the order $\leqslant$ on the set of classes defined by $[q]\leqslant[q']$ if and only if $q'\in Acc(q)$.

Let us first analyze what it means, for a word f, to belong to $L(A)\subseteq K$. As f is a word recognized by the finite automaton $\mathscr{C}$, f has a decomposition $f=f_1\alpha_1 f_2\ldots f_{k-1}\alpha_{k-1}f_k$, where α_i is the letter of f that leads, when f is read by $\mathscr{C}$, from a state in one class to a state in the next class. Each f_i is a word such that, while $\mathscr{C}$ recognizes f, the states encountered reading the factor f_i are all in the same class. Each f_i can itself be decomposed into elements that are labels of elementary loops (f_i being obtained from a short word with no loops in it, by insertions of them).

So, for every left factor g of f, we have $\varphi(g)=\boldsymbol{c}+\sum x_j\,.\,\varphi(u_j)$, where u_j is the label of an elementary loop, and $\boldsymbol{c}$ is bounded a priori: if we extract from g all elementary loops, we are left with a word of length smaller than the number S of states of the automaton, and $\boldsymbol{c}[p]\leqslant\max\{\varphi(w)[p]\mid |w|\leqslant S\}$. If we have $\varphi(g)=\boldsymbol{c}+\sum x_j\,.\,\varphi(u_j)$ such that $\boldsymbol{a}+\varphi(g)\geqslant\boldsymbol{0}$, then we can write $\varphi(g)=\boldsymbol{c}'+\sum x'_j\,.\,\varphi(u_j)$ with $\sum x'_j\,.\,\varphi(u_j)\geqslant\boldsymbol{0}$, and $\boldsymbol{c}'$ is also bounded a priori. We call $\boldsymbol{c}'$ the residue of g and note it $RES(g)$. We shall take advantage of this boundedness in the construction of the pushdown automaton in the next section.

In fact, we only consider the coordinates such that there exists at least one elementary loop in $\mathscr{C}$ which has a negative value on this coordinate. We call M this set of coordinates, and $\varepsilon=\bigcup_{q\in Q}(\mathscr{U}_q\cup\mathscr{V}_q)$ the set of all elementary nonpositive loops and the set of all positive elementary loops that enable to iterate one elementary nonpositive loop.

Informal description of the pda: The states of the automaton are used for the following purposes:

- verify that the word belongs to K,
- note, for each class $[i]$ of states of $\mathscr{C}$, the set of elementary loops already encountered while the word to recognize is read,

– remember and update the value of the current residue $\boldsymbol{c}'$ and check that this $\boldsymbol{c}'$ is such that $\boldsymbol{a}+\boldsymbol{c}'\geqslant 0$.

We shall use one pushdown store symbol for each subset of M: If $D \subset M$, we call z_D the corresponding symbol, and one element of D is chosen as the representative of this subset.

Every elementary loop not in ε is ignored, while every elementary loop of ε has the following contribution in the store of the pda.

Suppose that the loop being processed has its state in a class $[l]$, and call $\mathscr{U}_i$ the set of loops of ε already encountered belonging to the class $[i]$ (the loop being processed is included in $\mathscr{U}_l$). $\mathscr{U}_i$ is included in a set $\mathscr{U}$ such that there exists a set D of dominating coordinates with respect to *label*($\mathscr{U}$).

If the considered loop is a positive loop, its contribution is to increase the number of z_D by a factor which is its (positive) contribution to the representative of D.

If the considered loop is not a positive loop, its contribution is to decrease the number of stack symbols proportionally to its (negative) contribution to the representative of D (note that D may be smaller than the actual set D' such that $z_{D'}$ is on the top of the stack, and the proportionality factor is introduced because the representative of D' may be not the one chosen for D). Before this, it may be the case that there remain several symbols on the top of the stack pushed for elementary loops not in $\mathscr{U}$; these remaining symbols are popped.

Claim 5.2. *The pushdown automaton $\mathscr{A}$ recognizes $L(A)$.*

Proof. Let f be a word in $L(A)$. From the analysis above, it implies several conditions, and $\mathscr{A}$ has been precisely constructed, to check these conditions. It follows that $\mathscr{A}$ recognizes f.

Conversely, let f be a word recognized by $\mathscr{A}$. We shall prove by induction on the length of the left factors g of f, that $\boldsymbol{a}+\varphi(g)\geqslant \boldsymbol{0}$. The word g can be written: $g=g'x$ with x a letter, and we have $\boldsymbol{a}+\varphi(g')\geqslant\boldsymbol{0}$ by induction hypothesis. Taking off every loop in g', we get a short word h with $\varphi(h)=\boldsymbol{c}'$, and $\varphi(g')=\boldsymbol{c}'+\sum x_i\,.\,\varphi(u_j)$ with $\sum x_i\,.\,\varphi(u_j)\geqslant\boldsymbol{0}$. Then either x is the last letter of an elementary loop in $\mathscr{C}$, or not. In the behaviour of the automaton, when x is read, either it involves a move in the stack or not, respectively.

In the latter case, $\varphi(g'x)=\boldsymbol{c}+\sum x_i\,.\,\varphi(u_j)$ with $\boldsymbol{c}=\boldsymbol{c}'+\varphi(x)$. $\boldsymbol{c}$ is among the values checked by $\mathscr{A}$ for which $\boldsymbol{a}+\boldsymbol{c}\geqslant\boldsymbol{0}$. So, we do have $\boldsymbol{a}+\varphi(g)\geqslant\boldsymbol{0}$. In the former case, $\varphi(g)=\varphi(g'')+\varphi(u)$ with x last letter of the label of an elementary loop (q,u). Of course, q belongs to some class $[k]$. For the set of vectors $\varphi(u)$ concerned, there must be a dominating coordinate p, and, if $\varphi(g'')=\boldsymbol{c}+\sum x_i\,.\,\varphi(u_j)$ with $\boldsymbol{a}+\boldsymbol{c}+\sum x_i\,.\,\varphi(u_j)\geqslant\boldsymbol{0}$, the fact that the automaton accepts f implies that $(\boldsymbol{a}+\boldsymbol{c}+\sum x_i\,.\,\varphi(u_j)+\varphi(u))[p]\geqslant 0$. As p is a dominating coordinate, this inequality is true for all other coordinates, and we do have $\boldsymbol{a}+\varphi(g)\geqslant\boldsymbol{0}$.

As f is such that all its left factors are such that $\boldsymbol{a}+\varphi(g)\geqslant\boldsymbol{0}$, f is in $L(A)$. $\square$

Corollary 5.3. *The language $L(A)$ is deterministic.*

Proof. The pushdown automaton constructed in the proof above is a deterministic pda. $\square$

6. Decidability and further comments

To achieve the proof of the decidability of the context-freeness of the language associated with a k-VAS it suffices now to show that each of the conditions involved in our characterization theorem is decidable.

Let us recall what tasks have to be done to check the characteristic condition.

We begin by some preliminary treatments:

(1) Construct the covering automaton $\mathscr{C}(A)=\langle T, Q, \delta, \boldsymbol{a}\rangle$.

(2) Compute the equivalence $\sim$ on Q: $q\sim q'$ if and only if $q\in Acc(q')$ and $q'\in Acc(q)$, and the order $\leqslant$ on the set of classes defined by $[q]\leqslant[q']$ if and only if $q'\in Acc(q)$.

(3) Compute the set: $\varepsilon=\{(q,u)\mid u \text{ is elementary}\}$.

(4) Compute for all (q,u) in ε the set $\mathscr{I}(q,u)$ of all elementary iterating system of length $p\leqslant\|Q\|$ related to (q,u).

We now come to the characteristic property:

(5) Check that, for every (q,u) in ε, and for every $\mathscr{I}$ in $\mathscr{I}(q,u)$, $\mathbb{C}1(\mathscr{I}(q,u))$ holds.

This verifies the first necessary condition (*Lemma* II.1.5.)

Only in the case of a positive answer at step (5), we have to do:

(6) Compute the sets $\varepsilon^+=\{(q,u)\mid\varphi(u)>\mathbf{0}\}$, $\varepsilon^-=\{(q,v)\mid \mathbf{not}(\varphi(v)\geqslant\mathbf{0})\}$ and the sets

$$\mathscr{U}_{(q,v)}=\{(q',u)\in\varepsilon^+\mid q\in Acc(q') \text{ and } \|v\|^-\subseteq\|u\|^+\}\quad\text{for all } (q,v)\in\varepsilon^-$$

$$\mathscr{V}_{(q,u)}=\{(q',v)\in\varepsilon^-\mid q'\in Acc(q) \text{ and } \|v\|^-\subseteq\|u\|^+\}\quad\text{for all } (q,u)\in\varepsilon^+.$$

(7) Check that, for no $(q,v),(q',v')$ in ε^-, (p,u), (p',u') in ε^+ $\mathbb{C}2((p,u),(p',u'),(q,v),(q',v'))$ holds.

This verifies the second necessary condition (*Proposition* II.2.2.)

Only in the case of a positive answer at step (7), we have to do:

For all states s in Q:

(8) *Following Section* 11.3, compute $\mathscr{C}(A)_s$, the covering automaton reduced to the states coaccessible from s, and do a numbering of the classes of states from 1 to k_s.

(9) Compute for all i,j in $[1,k_s]$ the sets $\mathscr{V}_i=\{(q,v)\in\varepsilon^-\mid q\in[i]\}$ and $\mathscr{U}_j=\{(p,u)\in\varepsilon^+\mid p\in[j] \text{ and } \mathscr{V}_{(p,u)}\neq\emptyset\}$; compute also the sets $\mathscr{I}^-(j)=\{i\mid\mathscr{V}_i\subseteq\mathscr{V}_{(p,u)}$ and $p\in[j]\}$ and $\mathscr{I}^+(i)=\{j\mid\mathscr{U}_j\subseteq\mathscr{U}_{(q,v)}$ and $q\in[i]\}$, as well as the sets $\mathscr{W}_i=\mathscr{V}_i\cup(\bigcup_{j\in\mathscr{I}^+(i)}\mathscr{U}_j)$ and $\mathscr{X}_i=\mathscr{U}_i\cup(\bigcup_{j\in\mathscr{I}^-(i)}\mathscr{V}_j)$.

(10) Check that, for all integer i such that $[i]$ is a class of states, $\mathbb{C}3(\text{label}(\mathscr{W}_i))$ and $\mathbb{C}3(\text{label}(\mathscr{X}_i))$ hold.

This verifies the third necessary condition (Proposition II.3.4.)

Only in the case of a positive answer at step (10), we have to do:

(11) Compute the sets W such that $W = W_J$ for some $J = I^*(j)$ with j in $[1, m]$.
(12) Check, for each of these sets W, whether $\mathbb{C}4(W)$ holds.

This verifies the fourth necessary condition (Proposition II.4.3)

(13) For all $W = W_J$ for some $J = I^*(j)$ with $j \in [1, m]$ such that W has a dominating coordinate, compute the sets Z_J and $ZERO(J)$, and check that there is a dominating coordinate in $ZERO(J)$.

This verifies the fifth and last necessary condition (Proposition II.4.6) *and ends our algorithm.*

Let us now check that each of these tasks is computable.

Task 1: Construct the covering automaton $\mathscr{C}(A) = \langle T, Q, \delta, \boldsymbol{a} \rangle$.

The constructibility of the covering automaton was the aim of part I, Section 2 (see Proposition I.2.18).

Task 2: Compute the equivalence $\sim$ *on* Q: $q \sim q'$ *if and only if* $q \in Acc(q')$ *and* $q' \in Acc(q)$, *and the order* $\leqslant$ *on the set of classes defined by* $[q] \leqslant [q']$ *if and only if* $q' \in Acc(q)$.

This is a very basic operation of graph theory.

Task 3: Compute the set $\varepsilon = \{(q, u) \mid u \text{ is elementary}\}$.

As u is elementary, its length is bounded; so, ε is a finite set.

Task 4: Compute for all (q, u) *in* ε *the set* $\mathscr{I}(q, u)$ *of all elementary iterating system of length* $p \leqslant \| Q \|$ *related to* (q, u).

$\mathscr{I}(q, u)$ is a finite set.

Task 5: Check that, for every (q, u) *in* ε, *and for every* $\mathscr{I}$ *in* $\mathscr{I}(q, u)$, $\mathbb{C}1(\mathscr{I}, (q, u))$ *holds.*

Proposition I.3.6, asserts that, for all $\mathscr{I}$ related to (q, u), we can decide whether $\mathbb{C}1(\mathscr{I}, (q, u))$ holds.

Task 6: Compute the sets $\varepsilon^+ = \{(q, u) \mid \varphi(u) > \mathbf{0}\}$, $\varepsilon^- = \{(q, v) \mid \mathbf{not}(\varphi(v) \geqslant \mathbf{0})\}$ *and the sets*

$$\mathscr{U}_{(q,v)} = \{(q', u) \in \varepsilon^+ \mid q \in Acc(q') \text{ and } \| v \|^- \subseteq \| u \|^+\} \text{ for all } (q, v) \in \varepsilon^-$$

$$\mathscr{V}_{(q,u)} = \{(q', v) \in \varepsilon^- \mid q' \in Acc(q) \text{ and } \| v \|^- \subseteq \| u \|^+\} \text{ for all } (q, v) \in \varepsilon^+.$$

As u and v are elementary, all these sets are finite. The conditions to be checked are easy.

Task 7: Check that, for no $(q, v), (q', v')$ *in* ε^-, $(p, u), (p', u')$ *in* ε^+, $\mathbb{C}2((p, u), (p', u'), (q, v), (q', v'))$ *holds.*

The decidability of $\mathbb{C}2((p, u), (p', u'), (q, v), (q', v'))$ is obvious.

Task 8: Compute $\mathscr{C}(A)_s$.

This automaton is simply the covering automaton reduced to the states coaccessible from s.

Task 9: Compute for all i,j in $[1,k_s]$ *the sets* $\mathscr{V}_i=\{(q,v)\in\varepsilon^-\mid q\in[i]\}$ *and* $\mathscr{U}_j=\{(p,u)\in\varepsilon^+\mid p\in[j]$ *and* $\mathscr{V}_{(p,u)}\neq\emptyset\}$; *compute also the sets* $\mathscr{I}^-(j)=\{i\mid \mathscr{V}_i\subseteq\mathscr{V}_{(p,u)}$ *and* $p\in[j]\}$ *and* $\mathscr{I}^+(i)=\{j\mid \mathscr{U}_j\subseteq\mathscr{U}_{(q,v)}$ *and* $q\in[i]\}$, *as well as the sets* $\mathscr{W}_i=\mathscr{V}_i\cup(\bigcup_{j\in\mathscr{I}^+(i)}\mathscr{U}_j)$ *and* $\mathscr{X}_i=\mathscr{U}_i\cup(\bigcup_{j\in\mathscr{I}^-(i)}\mathscr{V}_j)$.

The computation is easy. All these sets are finite.

Task 10: Check that, for all integer i such that $[i]$ *is a class of states,* $\mathbb{C}3(label(\mathscr{W}_i))$ *and* $\mathbb{C}3(label(\mathscr{X}_i))$ *hold.*

Proposition I.4.9, asserts that, for all sets S of k-tuples of $\mathbf{Z}^k$, $\mathbb{C}3(S)$ is a decidable property.

Task 11: Compute the sets W such that $W=W_J$ *for some* $J=I^*(j)$ *with j in* $[1,m]$.

The computation is trivial. There is only a finite number of such sets W.

Task 12: Check, for each of these sets W, whether $\mathbb{C}4(W)$ *holds.*

By (P3) we know that there are dominating coordinates related to every label$(\mathscr{W}_i)$ and label$(\mathscr{X}_i)$ for i in J. Let D be a minimal set such that $H=\bigcap_{j\in D}H_j$. D is composed of dominating coordinates related to some $label(\mathscr{W}_i)$ or $label(\mathscr{X}_i)$ and $\Delta_W=\{x\in H\mid \exists r,s$ in D φ_r and φ_s are not colinear and $x\in K_r\cap K_s\}$ where linear forms φ_r (for r in D) have nonzero coefficients. It is decidable whether such a set is stratified or not (see e.g. [5]).

Task 13: For all $W=W_J$ *for some* $J=I^*(j)$ *with* $j\in[1,m]$ *such that W has a dominating coordinate, compute the sets* Z_J *and* $ZERO(J)$, *and check that there is a dominating coordinate in* $ZERO(J)$.

Again, $\mathbb{C}3(W)$ is a decidable property. To compute sets Z_J is an elementary problem in graph theory, and it is straightforward to see if a coordinate is in $ZERO(J)$.

We have proved our main Theorem 6.1.

Theorem 6.1. *It is decidable whether the language associated with a k-VAS is context-free or not.*

We can remark that in all the proofs we made that a language is not a context-free one, we used Ogden's lemma. More precisely, we use a consequence of this lemma:

If L is a context-free language, there exists an integer N such that for all words f in L with at least N marked letters, there is a factorization $\alpha x\beta y\gamma$ of f satisfying $\{\alpha x^n\beta y^n\gamma\mid n\geqslant 0\}\subset L$ with either α, x and β or β, y and γ containing at least one marked letter, and either $x\beta y$ containing at most N marked letters, or there is a factorization $\alpha' x'\beta' y'\gamma'$ of β satisfying $\{\alpha x^n\alpha' x'^p\beta' y'^p\gamma' y^n\gamma\mid n,p\geqslant 0\}\subset L$ with either α', x' and β' or β', y' and γ' containing at least one marked letter.

Calling Ogden-like a language satisfying this property, a context-free language is Ogden-like and so are all its intersections with rational sets. It is known [3] that the converse is not true.

We did prove, in fact, the two following results.

Proposition 6.2. *It is decidable whether the language associated with a k-VAS is such that every intersection with a rational set is Ogden-like.*

Proposition 6.3. *A language associated with a k-VAS is context-free if and only if every intersection of this language with a rational set is Ogden-like.*

On the other hand, the proof of the sufficient condition of context-freeness has been made using a deterministic automaton. From Corollary 5.3, we get a proposition to be put aside with the preceding one.

Proposition 6.4. *If the language associated with a k-VAS is context-free, then it is a deterministic language.*

7. Conclusion

We have proved that it is solvable whether the language associated with VAS or with nonlabelled PNs is context-free or not. We now know the position of any of these languages in the Chomsky hierarchy, which was the aim of the question asked by Peterson in [15]. There exist several families of Petri nets, depending on whether the transitions are labelled or not, allowing the empty word to be a label or not, and whether one specifies the final configurations or not. It is known [21] that it is unsolvable for labelled PNs with final configurations whether their language associated is rational. One can conjecture that it is probably so for context-freeness (even though the proof of [21] uses the stability of the rational sets under complementation). For other classes, the solvability of the problem of context-freeness could be carried out with the tools introduced here.

Our proof has been achieved by giving several conditions among which we distinguish two categories:

The first one deals with a superficial knowledge of the behaviour of the loops, based only on the algebraic signs of the coordinates of the associated vectors. This led us to develop two tools: strong loops and iterative systems. The notion of strong loop has allowed to sharpen the Karp and Miller's graph to keep only the loops that may be useful. The notion of iterative system has allowed to get rid of simple cases of noncontext-freeness.

The second one deals with a more accurate knowledge of the behaviour of the loops, taking into account the values of the coordinates of the associated vectors. This led us to develop the notion of a dominating coordinate related to a set of vectors, that plays a crucial role with regard to context-freeness.

We believe that the tools developed here are not just ad hoc tools, but are deeply related to VAS and PNs. They help to formalize, and hopefully give answers to natural questions that arise when dealing with VAS or PNs:

Finding dominating places (or coordinates) enables to reduce the number of places in the net. This is one of the usual concerns when dealing with PNs in practice, because they are often of a big size for a modelization of a system.

Trying to give a sharp approximation of the associated language by means of a rational language is justified by the fact that finite automata are much more simpler to handle than VAS or PNs, and have been studied extensively. To know the dependency relations between loops is essential for the knowledge of the net. These two points re-enforce the interest of the notions of strong loops and iterative systems. Along the same lines we proved for instance that for VAS languages, rationality is equivalent to the so-called 1-*ll* condition [1].

We believe that these notions can also be fruitfully used in the studies of other kinds of languages. For example, the bounded languages have natural rational covers, and it is possible to apply to them the theory of iterative systems.

Acknowledgment

This paper would not have been written without the constant help and encouragement of Jean-Michel Autebert and Bruno Courcelle.

References

[1] J.-M. Autebert, L. Boasson and G. Cousineau, A note on one-locally-linear languages, *Inform. and Control* **37** (1978) 1–4.
[2] J. Berstel, *Transductions And Context-Free Languages* (Springer, Stuttgart, 1979).
[3] L. Boasson and S. Horvatz, On languages satisfying Ogden's lemma, *RAIRO. Inform. Théor. Appl.* **12** (1978) 201–202.
[4] S. Crespi-Reghizzi and D. Mandrioli, Petri nets and Szilard languages, *Inform. and Control* **33** (1977) 177–192.
[5] S. Ginsburg, *The Mathematical Theory of Context-Free Languages* (McGraw-Hill, New York, 1966).
[6] A. Ginzburg and M. Yoeli, Vector addition systems and regular languages, *J. Comput. System Sci.* **20**, (1980) 277–284.
[7] M. Hack, Decidability questions for Petri nets, Ph.D. dissertation, M.I.T., Cambridge, Massachusetts, 1975.
[8] M. Harrison, *Introduction to Formal Language Theory* (Addison-Wesley, Reading, MA, 1978).
[9] J. Hopcroft and J.J. Pansiot, On the reachability problem for five dimensional vector addition systems, *Theoret. Comput. Sci.* **8** (1979) 135–159.
[10] R.M. Karp and R.E. Miller, Parallel program schemata, *J. Comput. System Sci.* **3** (1969) 167–195.
[11] R. Keller, Vector replacement systems: a formalism for modeling asynchronous systems, Tech. Report 117, Comp. Sci. Lab., Princeton University, Princeton, NJ, 1972.
[12] S.R. Kosaraju, Decidability of reachability in vector addition systems, in: *Proc. 14th Ann. ACM STOC*, (1982) 147–195.
[13] H.W. Lenstra, Integer programming with a fixed number of variables, *Math. Oper. Res.* **8** (1983) 538–548.
[14] E. Mayr, An algorithm for the general Petri net reachability problem, *SIAM J. Comput.* **13** (1984) 441–460.
[15] J.L. Peterson, Computation sequence sets, *J. Comput. System Sci.* **13** (1976) 1–24.

[16] J.L. Peterson, *Petri Net Theory And the Modeling of Systems* (Prentice-Hall, Englewood Cliffs, NJ, 1981).

[17] W. Rudin, *Real and Complex Analysis* (Tata McGraw-Hill, New Delhi, 1974).

[18] S.R. Schwer, Décidabilité de l'algébricité des langages associés aux réseaux de Petri, Thèse de 3$^{\text{ème}}$ Cycle de l'Université Paris 7, Paris, 1985; also internal report n°85 04 of the University Bordeaux I.

[19] S.R. Schwer, On the rationality of Petri net languages, *Inform. Process. Lett.* **22** (1986) 145–146.

[20] S.R. Schwer, Fine cover of a VAS language, *Theoret. Comput. Sci.* **95** (1992) 159–168.

[21] R. Valk and G. Vidal-Naquet, Petri nets and regular lanugages, *J. Comput. System Sci.* **23** (1981) 299–325.

[22] M. Yoeli and A. Ginzburg, Petri net languages and their applications, Research Report CS-78-45, Dept. of Comp. Sci. Haifa, 1978.