Head linear reduction and pure proof net extraction*

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Abstract

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Proof net calculus introduced in Girard (1987) has been extended in the paradigm of Scott's domains equation $D = D \rightarrow D$ which generates a logical point of view for pure λ-calculus in Donos (1989). Methodologically speaking, in this paper the proof theoretic counterpart of Böhm's theorem given in Böhm (1968) for pure λ-calculus, is proposed as extension of the Curry–Howard paradigm. Technically speaking, as the extraction of a subterm using the β-reduction is possible also subnet extraction can be internalized by cut-elimination, using proof nets realizes managing and better understanding the procedure of extraction.

1. Introduction

Two main traditions of logic in computer science are:

(1) The fixpoint theory tradition: a program is a solution of a fixpoint equation.

(2) The proof theory tradition: a (functional) program satisfying a specification corresponds to a proof of a formula.

Pure λ-calculus appears somewhere in the middle of these two traditions.

● Models of pure λ-calculus can be obtained by fixpoint domain equations.

● β-reduction can be expressed by a normalization process of pure nets in a version of the multiplicative fragment of linear logic with exponentials.

One of the basic syntactical results of pure λ-calculus is the Böhm theorem. The starting point of this result is the so-called Böhm-out lemma. In this paper we propose an extension of this lemma according to the proof theory tradition in the framework of linear logic. More precisely we first define the head linear reduction strategy on

pure proof nets and show the correctness of such a strategy w.r.t. reduction of pure
λ-terms. Then we define some pure nets transformations which are the natural
generalization of Böhm transformations. Finally we obtain the proof theoretic gener-
alization of a finite version of Böhm out lemma.

Pure nets extraction lemma. *For every cut free pure proof net N, for every subnet N' of
N there exists an internal pure proof net transformation ϕ such that $\phi(N) = N'$.*

Moreover we obtain an improvement of such extraction technique by means of the
appropriate choice of a strategy of reduction. The translation of λ-calculus in **PN** and
the previous lemma give a sharper way for treating the problem of extracting subterms
in λ-calculus. Such a result seems to provide some confidence with the extension of the
Curry–Howard isomorphism to pure λ-calculus, opening new promising directions in
the questions of separability and equations in pure λ-calculus.

1. Pure nets calculus

1.1. Pure nets

Let us recall the definition of pure proof nets (or pure nets) given in [7].

Definition 1.1 (*Formulae*). We have only the four formulae:
- I (as input),
- O (as output) and
- $?I$ and $!O$ are formulae.
  We shall say that O and I are *dual* formulae (resp. $?I$ and $!O$).

Definition 1.2. We inductively define the set of pure nets R with conclusions
$X_1, \ldots, X_n$ where $X_1, \ldots, X_n$ are formulae.
  Identities
  (a) *Axiom-link*:

$$\overline{\quad I \qquad O \quad}$$

is a pure net with conclusions I, O.
  (b) *Cut-link*: If R_1 (resp. R_2) is a pure net with conclusions $X, X_1, \ldots, X_n$ and
$Y, Y_1, \ldots, Y_m$, such that X and Y are dual, then R is a pure net with conclusions
$X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$, where R is

$$\begin{array}{cc} R_1 & R_2 \\ \vdots & \vdots \\ \underline{X} & \underline{Y} \end{array}$$

We call X, Y the premises of this link.

Connectives

(a) *Par-link*: If R_1 is a pure net with conclusions $X_1, \ldots, X_n, ?I, O$ then R is a pure net with conclusions $X_1, \ldots, X_n, O$ where R is

$$
\begin{array}{c}
R_1 \\
\vdots \\
\underline{?I \qquad O} \\
O
\end{array}
$$

We call $?I, O$ the premises of this link.

(b) *Times-link*: If R_1 (resp. R_2) is a pure net with conclusions $X_1, \ldots, X_n, !O$ (resp. $Y_1, \ldots, Y_m, I$) then R is a pure net with conclusions $X_1, \ldots, X_n, Y_1, \ldots, Y_m, I$, where R is

$$
\begin{array}{cc}
R_1 & R_2 \\
\vdots & \vdots \\
\underline{!O \quad} & I \\
& I
\end{array}
$$

We call $!O, I$ the premises of this link.

Structural rules

(a) *We-link*: If R_1 is a pure net with conclusions $X_1, \ldots, X_n$, then R is a pure net with conclusions $X_1, \ldots, X_n, ?I$, where R is

$$
\begin{array}{cc}
R_1 & \\
\vdots & \overline{?I}
\end{array}
$$

(b) *Der-link*: If R_1 is a pure net with conclusions $X_1, \ldots, X_n, I$, then R is a pure net with conclusions $X_1, \ldots, X_n, ?I$, where R is

$$
\begin{array}{c}
R_1 \\
\vdots \\
\underline{I} \\
?I
\end{array}
$$

We call I the premises of this link.

(c) *Co-link*: If R_1 is a pure net with conclusions $X_1, \ldots, X_n, ?I, ?I$ then R is a pure net with conclusions $X_1, \ldots, X_n, ?I$, where R is

$$
\begin{array}{c}
R_1 \\
\vdots \\
\underline{?I \qquad ?I} \\
?I
\end{array}
$$

We call $?I, ?I$ the premises of this link.

(d) *!Box*: If R_1 is a pure net with conclusions $?I, \ldots, ?I, O$, then R is a pure net with conclusions $?I, \ldots, ?I, !O$, where R is

$$
\begin{array}{|ccc|}
\hline
 & R_1 & \\
 & \vdots & \\
?I \ldots ?I & & O \\
\hline
\end{array}
$$
$$?I \ldots ?I \qquad !O$$

O is the premise of $!O$ and each occurence of $?I$ in the conclusion of R_1 is the premise of the corresponding of $?I$ in the conclusin of R. We call principal port (resp. auxiliary port) or pal (resp. pax) for short, the $!O$ conclusion (resp. $?I$) of this !Box; the content of this !Box is the pure net R_1.

Remark. We can use as abbreviation of many co-links:

$$
\begin{array}{c}
R_1 \\
\vdots \\
\underline{?I \ldots ?I \ ?I} \\
?I
\end{array}
$$

It is not a trivial notation because for it to make sense, we have to define an equivalence relation on pure nets, to avoid boxes over-crossing; such relation was introduced in [19] and even if we do not want to use it, w.r.t. our strategy it is not relevant to preserve explicitly this order.
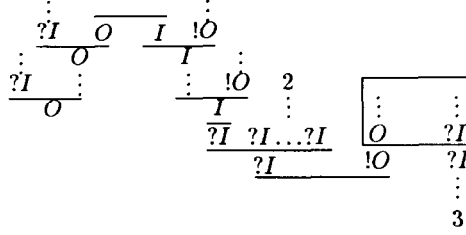
Proposition 1.3. *Let be a pure net of conclusions* $X_1, \ldots, X_n$, *there exists one and only one conclusion* X_i *such that or* $X_i = O$ *either* $X_i = !O$ *(called the* pal *of R).*

Proof. By induction. $\square$

From Proposition 1.3 we know that for every pure net there exists a particular conclusion (i.e. O or $!O$). This remark allows us to define the head structure of pure nets. This structure consists in three sequences of different links. To find out this structure, it is sufficient to look at the conclusion O of the net: it can be conclusion of a par-link or an axiom-link. In the first case the iteration of this argument to the premise O of the par-link gives the conclusion of the sequence of par-links with an axiom-link on the top. The conclusion I of this axiom-link can be premise of a der-link or of a times-link. Iterating the latter case we get the sequence of times-links, delimited on the bottom by a dereliction with conclusion $?I$, which can be premise of a contraction, a par-link, a cut-link or conclusion of the whole net. Therefore, we have three sequences: par-links, times-links and co-links with axiom-link and der-link as

"markers". This structure will be called *head block* and links of the head block will be named *head links* (so, head dereliction, head axiom, etc).

Definition 1.4. *Given a pure net R*

$$
\begin{array}{c}
\vdots \qquad\qquad \vdots \\
?I \quad \overline{\;O\;} \quad \overline{\;I\;} \quad !O \\
\vdots \quad \overline{\;O\;} \qquad \overline{\;I\;} \qquad \vdots \\
?I \qquad\quad \vdots \quad !O \quad 2 \\
\overline{\;O\;} \qquad\qquad \overline{\;I\;} \quad \vdots \\
\qquad\qquad \overline{?I \;\; ?I\ldots ?I} \quad \boxed{\begin{array}{cc}\vdots & \vdots \\ O & ?I\end{array}} \\
\qquad\qquad \overline{?I} \qquad\quad !O \quad ?I \\
\qquad\qquad\qquad\qquad\qquad \vdots \\
\qquad\qquad\qquad\qquad\qquad 3
\end{array}
$$

(1) We call *head-axiom-link* of R the axiom-link that has as conclusion O, the premise of the sequence of par-links with the same conclusion O of the net.

(2) We call *head redex* the cut-link such that its premise $?I$ is the conclusion of the sequence of times-link (we can have a contraction as a last rule) with formula I of the head axiom-links as premise.

(3) We call *head co-link* the co-link with premise the conclusion $?I$ of the der-link under the head axiom-link.
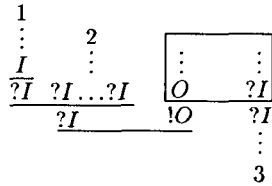
1.2. Head linear reduction

We introduce the head linear reduction strategy of pure nets $(\twoheadrightarrow_{HL})$, which is a natural extension to pure nets of the leftmost outermost strategy for the λ-calculus. The soundness of this notion of reduction was given [16] on computational encoding of pure nets.

The $\twoheadrightarrow_{HL}$ is obtained from the original notion of reduction in [8] by modifying the [cc]-cut rule notion. So the soundness of our reduction strategy is obtained from the original one.

Definition 1.5. The *head linear reduction* $(\twoheadrightarrow_{HL})$ of the pure net R is defined as follows.
 Head reduction: Let R be a pure net with head redex:

$$
\begin{array}{c}
1 \\
\vdots \qquad 2 \\
I \qquad \vdots \\
\overline{?I \;\; ?I\ldots ?I} \quad \boxed{\begin{array}{cc}\vdots & \vdots \\ O & ?I\end{array}} \\
\overline{\quad ?I \quad} \qquad !O \quad ?I \\
\qquad\qquad\qquad\qquad \vdots \\
\qquad\qquad\qquad\qquad 3
\end{array}
$$

if the premise $?I$ of the cut-link is not contracted, skip to Step 2 else:

Step 1. *Head linearization-cut* [hl]: R is reduced to R'

```
1   ┌·····┐ ┌·····┐        2
·   │·    ·│ │·    ·│       ·
·   │·    ·│ │·    ·│       ·
I   │O   ?I│ │?I   O│?I...?I
?I  !O    ─────?I─────  !O   ?I
          ·
          ·
          3
```

Step 2. *Dereliction-cut* [dc]: R' is reduced to R''

```
1   ·          ·  ┌·····┐        2
·   ·          ·  │·    ·│       ·
·   ·          ·  │·    ·│       ·
I   O         ?I  │?I   O│?I...?I
                 ────?I────  !O   ?I
                      ·
                      ·
                      3
```

(Q1) While R'' is such that the premise I and the premise O of the cut-link are not the conclusions of an axiom-link:

```
·   ·   ·   ·      ·  ┌·····┐        2
·   ·   ·   ·      ·  │·    ·│       ·
?I  O  !O   I     ?I  │?I   O│?I...?I
   O       I          ────?I────  !O   ?I
                           ·
                           ·
                           3
```

repeat:
R'' is reduced to
Step 3. *Multiplicative-cut* [mf]:

```
·   ·   ·   ·      ·  ┌·····┐        2
·   ·   ·   ·      ·  │·    ·│       ·
?I  !O  O   I     ?I  │?I   O│?I...?I
                     ────?I────  !O   ?I
                           ·
                           ·
                           3
```

Otherwise (i.e. the premise I or the premise O of the cut-link is the conclusion of an axiom-link):

```
┌──┐
O  │         ·  ┌·····┐        2
·  │         ·  │·    ·│       ·
·  │I   O   ?I  │?I   O│?I...?I
               ────?I────  !O   ?I
                     ·
                     ·
                     3
```

R'' is reduced to:

Step 4. Axiom-cut [ax]:

5
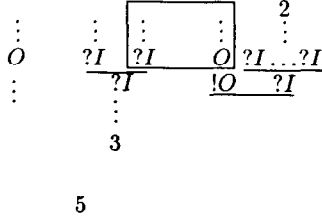
(the axiom cut of second type is reduced in the same way swapping the roles of the formulae *I*, *O*).

(Q2) Apply the previous steps until a net without head redex is reached.

Remark. Let us observe that the while iteration (Q1) is performed until an axiom-link is found; this can happen in three different ways:

- Complete annihilation: two axiom-links on the formule *I*, *O*.
- Head lambda gain: an axiom-link on the formula *I*.
- Arguments gain: an axiom-link on the formula *O*.

Definition 1.6. The *head linear reduction strategy* on the pure net *R* is defined by applying sequentially the following steps:

(1) If *R* contains head redex then apply *HL*-reduction (in Definition 1.5).

(2) If *R* is a pure net without head redex:
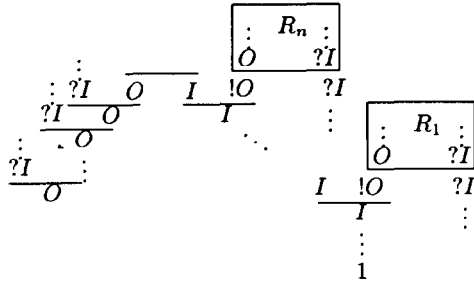
where the links 1 are contractions and/or a par-link and $n \neq 0$. Then:

R is reduced by applying (recursively) the *HL*-strategy to each pure net R_i', which is the greatest subnet of *R* containing the box R_i.

Whenever there exist *i* and *j* such that the pure nets R_i' and R_j' are contracted on the same cut-link, the contraction is to be broken creating two copies of the cut-link (like in the step of linearization).

(3) If *R* is a pure net such that there exists a we-link with conclusion which is premise of a cut-link:

$$
\begin{array}{c}
\boxed{\begin{array}{cc} \vdots & \vdots \\ O & ?I \end{array}} \\
\overline{?I \quad !O \quad ?I} \\
\vdots \\
2
\end{array}
$$

then R is reduced to (*weakening cut*)

$$
\begin{array}{c}
\overline{?I} \\
\vdots \\
2
\end{array}
$$

Now, we give the following important proposition about $\rightarrow_{HL}$.

Proposition 1.7. *The HL-reduction strategy is normalizing.*

Refering to [8], to every pure λ-term M is associated a pure net M^*.

Corollary 1.8. *For every* $M, N \in \Lambda$ *if* $M \rightarrow_\beta N$ *and* N *is a normal form for* M *then* $M^* \rightarrow_{HL} N^*$.

1.3. Translating proofs in lambda terms

The construction of a translation from λ-terms to pure net would take a large amount of space, anyway the λ-calculist can take advantage from a backward translation from pure nets to λ-terms. The translation that follows, is given in such a way to simplify understanding of links between arguments given in pure nets and corresponding Böhm's results given in λ-calculus.

1.3.1. Head translation

As preliminary we state some fundamentals in λ-calculus which are connected to matters of head structures:

Proposition 1.9. *Every λ-term t can be written in only one way in the following form:* $\lambda x_1 \ldots \lambda x_n (v) t_1 \ldots t_k$, *for some* n, k.

Proof. See [13]. □

A λ-term is said to be in head normal form if v in Proposition 1.9 is a variable. If you suppose that t is *solvable* (i.e. it admits a head normal form) then its head normal form has the structure $\lambda x_1 \ldots \lambda x_n (h) t_1 \ldots t_k$ for some n, k and the following property holds:

Proposition 1.10. *Let t be a solvable λ-terms and $\lambda x_1, \ldots, \lambda x_n(h)t_1 \ldots t_k$ its head normal form then numbers n, k are independent from the head noral form reached starting from t, i.e. if $\tilde{t}$ is another hnf of t then $\tilde{t} = \lambda x_1 \ldots \lambda x_n(h)t'_1 \ldots t'_k$ and $t_i \simeq_\beta t'_i$ for $i \in [1, k]$.*

Proof. See [13]. □

In pure nets, head block plays the same role of the head normal form of λ-terms. In Proposition 1.10 we state that $x_1, \ldots, x_n$ are the same for every head normal form and also all terms t_i are the same (i.e. for every fixed i they are β-equivalent). The translation from pure nets into corresponding λ-terms works inductively on the head block of the net; for every step of induction a head block is used to build the λ-term; three basic cases of head block can be distisguished:

(1) $(axiom, \wp)$,

(2) $(axiom, \wp, \otimes)$,

(3) $(axiom, \wp, \otimes, cut)$

with an additional hypothesis of no contraction on conclusions.

Notation. In the following definition variables X, Y (in case indexed) will be used to indicate occurrences of the formula $?I$. Ovals in pictures represent the greatest pure net containing explicitly drawn links.

Definition 1.11. Correspondence $\tau_h : \mathrm{PN} \to \Lambda$ is defined as follows on structure of head blocks in pure nets:

• If the pure net R is
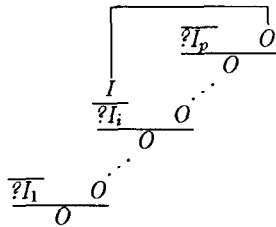
then

$$\vdash \tau_h[R] = \lambda y_1 \ldots \lambda y_p.(y_i) : O$$

● If the pure net R is

$$\frac{\overline{?I_p} \quad O \quad \overline{\dfrac{I}{?I}}}{O}$$

$$\cdot\cdot$$

$$\frac{\overline{?I_1} \quad O}{O}$$

then

$$\overline{\vdash x\!:\!?I, \tau_h[R] = \lambda y_1 \ldots \lambda y_p(x)\!:\!O}$$

● When R is
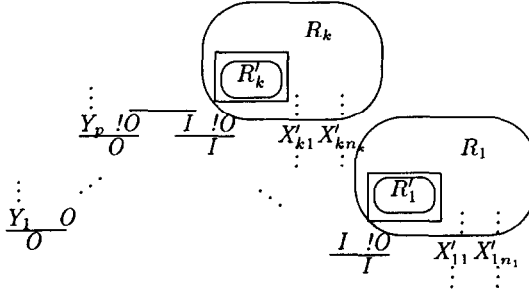
$$\frac{\vdash x'_{i1}\!:\!X'_{i1}, \ldots, x'_{in_i}\!:\!X'_{in_i}, \tau_h[R_i] = M_i\!:\!O \quad i=1,\ldots,k}{\vdash x_1\!:\!X_1, \ldots, x_m\!:\!X_m, \tau_h[R] = \lambda y_1 \ldots \lambda y_p.(h)M_1 \ldots M_k\!:\!O}$$

where X_js are between conclusions X'_{rs} of proof nets R_s, which are different from premises $Y_1 \ldots Y_p$ of head par-links.

Remark. Without loss of generality we can suppose that for every i and j, R_i and R_j are not contracted on the same cut-link in case of the following rule of translation is to be applied:

$$\frac{\vdash x'_{i1}\!:\!X'_{i1}, \ldots, x'_{in_i}\!:\!X'_{in_i}, \tau_h[R_i] = M_i\!:\!O \quad i=1,\ldots,k}{\vdash x_1\!:\!X_1, \ldots, x_m\!:\!X_m, \tau_h[R] = \lambda y_1 \ldots \lambda y_p.(\lambda z.(h)M_1 \ldots M_k)\bar{M}\!:\!O}$$

where $\bar{M} = \tau[\bar{R}]$, $\bar{R}$ is the pure net premise of the cut-link and X_js are between conclusions X'_{rs} of proof nets R_s, which are different from premises $Y_1 \ldots Y_p$ of head par-links and from the premises of the cut-link.

● If the pure net R has an head cut-link and no contraction on conclusions
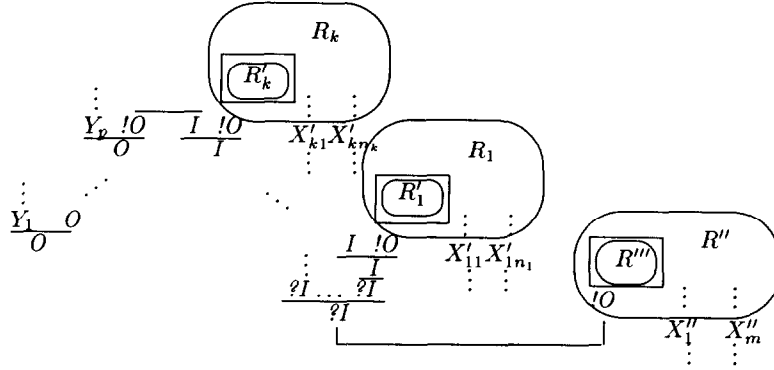
$(R''$ and R''' may coincide) then

$$\dfrac{\vdash x'_1:X'_1,\ldots,x'_{i1}:X'_{in_i},\tau_h[R_i]=M_i:O \quad i=1,\ldots,k \qquad \vdash x''_1:X''_1,\ldots,x''_m:X''_m,\tau_h[R'']=M'':O}{\vdash x_1:X_1,\ldots,x_s:X_s,\tau_h[R]=\lambda y_1\ldots\lambda y_p.(\lambda z.(z)M_1\ldots M_k)M'':O}$$

where $X_1,\ldots,X_s$ are conclusions of $R_1,\ldots,R_k$, R'' which are not premises of head
par-links in R. In case, the latter remark should be applied.
● If one or more conclusions of R are contracted:
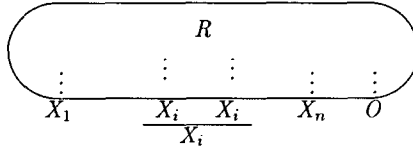
then

$$\dfrac{\vdash x_1:X_1,\ldots,x_i:X_i,x_i:X_i,\ldots,x_n:X_n,\ \tau_h[R]=M:O}{\vdash x_1:X_1,\ldots,x_i:X_i,\ldots,x_n:X_n,\tau_h[R']=M:O}$$

1.3.2. Lambda head linear reduction

In order to prove soundness of translation τ_h, a λ-calculesque version of head linear
reduction will be defined on Λ. Every single step of head linear reductin cannot be
simulated on λ-terms: it is easily verified that there exist elementary steps of cut-
elimination on proof nets which produce no effec on corresponding λ-terms. Apart
from these logical steps devoid of algorithmic sense, head linear reduction can be
simulated on λ-terms reaching cut-free proofs and normal forms correctly related by
the translation defined above.

The notion of head redex given in Proposition 1.9 as the leftmost outermost redex,
can be generalized to that of *principal redex*.

Definition 1.12. Given $t \in \Lambda$ the *principal variable* of t is h:
- if t is $\lambda y_1 \ldots \lambda y_p(h) t_1 \ldots t_k$,
- if t is $\lambda y_1 \ldots \lambda y_p(v) t' t_1 \ldots t_k$ and h is the principal variable of the term v.

The notion of general head redex is defined in the following definition.

Definition 1.13. Given $t \in \Lambda$, a redex of t is said *general head redex* of t if it is head redex of t or it is general head redex of v when t is $\lambda y_1 \ldots \lambda y_p(v) t' t_1 \ldots t_k$.

Remark. A redex in a term $t = \ldots (\lambda x . u) v \ldots \in \Lambda$ is said to be a *redex* on the variable x.

Definition 1.14. Given $t \in \Lambda$, a general head redex of t on the principal variable h of t is said *principal redex of t*.

So, in analogy with Definition 1.5 we give the following definition.

Definition 1.15. The *head linear reduction* of the λ-term is defined as follows.
- Let t be a λ-term with principal redex:

$$\lambda y_1 \ldots \lambda y_p(\ldots (\lambda h \ldots (\lambda x_1 \ldots \lambda x_m(h) v_1 \ldots v_k) \ldots) w \ldots) t' t_1 \ldots t_s$$

Step 1. *Head linearization*:
if the principal variable h has no occurrence in the subterm $\ldots (\lambda x_1 \ldots \lambda x_m(h) v_1 \ldots v_k) \ldots$
    then let be $u = t$ and skip to Step 2
    else $t \to_{\lambda HL} u$ where

$$u = \lambda y_1 \ldots \lambda y_p. (\ldots (\lambda h'(\lambda h \ldots (\lambda x_1 \ldots \lambda x_m(h') v_1 \ldots v_k) \ldots) w) w' \ldots) t' t_1 \ldots t_s$$

and w' is equal w.
Step 2. *Head substitution*: u is readuced to u' where

$$u' = \lambda y_1 \ldots \lambda y_p(\ldots (\lambda h \ldots (\lambda x_1 \ldots \lambda x_m(w') v_1 \ldots v_k) \ldots) w \ldots) t' t_1 \ldots t_s$$

- Apply the previous steps until a term without head redex is reached.

1.3.3. Soundness of the head translation

In order to obtain a theorem of soundness with this definition of $_\lambda$HL-reduction and HL-reduction for pure nets, we show that this notion of $_\lambda$HL-reduction is correct with regard to head reduction in the sense of head normal form as tested in Proposition 1.10 and in that of the notion of *op*-reduction as defined in [19].

Proposition 1.16. *Given* $t \in \Lambda$, *if* $t \to_h \mathrm{hnf}(t)$, *with* $\mathrm{hnf}(t) = \lambda x_1 \ldots \lambda x_n(h) t_1 \ldots t_k$ *head normal form of t and* $t \to_{\lambda HL} t_{hl}$ *with the head linear normal form of t* $t_{hl} \to_{op} \lambda y_1 \ldots \lambda y_p(\lambda z_1 \ldots (\lambda z_m(z) v_1 \ldots v_s) w_m \ldots) w_1$ *and* $z_i \neq z$ *for* $i = 1, \ldots, m$, *we have that:*

(1) *the length of head linear reduction is not greater than the length of head reduction, in particular if the head reduction terminates so does the head linear,*

(2) *the following conditions are satisfied:*

$$h = z, \qquad n = p, \qquad k = s, \qquad y_i = x_i \quad for \ i = 1, \dots, n,$$

$$t_i \cong_\beta (\lambda z_1 \dots (\lambda z_m . v_i) w_m \dots) w_1 \quad for \ i = 1, \dots, k$$

In the following we note $\rightarrow_*$ for reduction of principal redex. Single steps of reduction are marked with $\dot\rightarrow$.

Proof (*sketch*). Single steps of head reduction and λ-head linear reduction communtes, i.e. if $t \dot\rightarrow_h t' \dot\rightarrow_* t''$ then $t \dot\rightarrow_* \tilde{t}'' \rightarrow_h t'$. Let be $t = \lambda y_1 \dots \lambda y_p (\lambda z . v) t t_1 \dots t_k$ and

$$t \dot\rightarrow_h \lambda y_1 \dots \lambda y_p v[t/z] t_1 \dots t_k$$

to perform an head linear step of reduction v has to be in the form:

$$v = (\dots (\lambda h \dots (\lambda x_1 \dots \lambda x_m (h) v_1 \dots v_k) \dots) w \dots)$$

After step of reduction

$$v[t/z] \dot\rightarrow_* (\dots (\dots (\lambda x_1 \dots \lambda x_m (h) v_1 \dots v_k) \dots) [t/z] [w[t/z]/h] \dots)$$

by α-equivalence $h \neq z$ and by commutation of substitutions:

$$(\dots (\lambda x_1 \dots \lambda x_m (h) v_1 \dots v_k) \dots) [w/h] [t/z] \dots)$$

So, reordering the sequence of head reduction, steps of reduction that are not on principal variables can be pushed at the end of the sequence. Formal proof is given in [17]. □

Next proposition states that translation from pure nets to λ-terms is good enough with respect to the notion of head linear reduction as defined on λ-terms; the soundness of the translation with regard to β-calculus is trivially a consequence of Proposition 1.16.

Proposition 1.17. *Let $R \in \mathrm{PN}$, $\tau_h : \mathrm{PN} \rightarrow \Lambda$ the head translation, $\rightarrow_{HL}$ the head linear reduction of pure nets and $\rightarrow_{\lambda HL}$ the head linear reduction of λ-terms, the following diagram commutes:*

$$
\begin{array}{ccc}
R & \rightarrow_{HL} & R' \\
\tau_h \downarrow & & \downarrow \tau_h \\
\tau_h[R] & \rightarrow_{\lambda HL} & \tau_h[R']
\end{array}
$$

Proof (*sketch*). The proof is by induction on the length of the HL-reduction. For more details see [17]. □

In conclusion it was obtained that $_\lambda$HL is the reduction which reach head normal form of λ-terms with the minimal number of duplications of subterms, this property will be useful in the following to manage the extraction technique.

2. Proof net extraction

The object of this section is the answer to this question: "Given a normal pure net, is it possible to extract a subnet internally to the calculus by using cut-elimination?".

As it is well known from the pure λ-calculus, by the Böhm-out technique it is necessary to give up something in the operation: the extraction is not clean (modulo substitutions). By using the pure proof net calculus it is possible to improve such a situation, adding some kind of control on these substitutions.

2.1. Nets transformations

We consider transformation of nets consisting in "manipulating links". We obtain an *internal* procedure (by cut-elimination) to break and create links. Essentially, we shall use the basic property of duality for the connectives of linear logic, obtaining a purely logical view of Böhm transformations.

Definition 2.1. Let $R \in PN$:

$$
\begin{array}{cccc}
 & \vdots & & \\
\vdots & ?I & \overline{O} & I \\
?I & & O & \vdots \\
\vdots & \overline{O} & & 1 \\
?I & \vdots & \vdots & \vdots \\
\overline{O} & & ?I_1 \ldots ?I_m &
\end{array}
$$

the *closure* of R is a pure net $\bar{R}$ obtained from R by linking every *pax* with the *pal* by head par-links.

$$
\begin{array}{cccc}
 & \vdots & & \\
\vdots & ?I & \overline{O} & I \\
?I & \overline{O} & & \vdots \\
\vdots & \overline{O} & & 1 \\
\vdots & ?I & \vdots & \\
^{\sigma_1}?I & & \overline{O} & \\
\vdots & \overline{O} & & \\
^{\sigma_m}?I & \vdots & & \\
\overline{O} & & &
\end{array}
$$

where $\sigma_1 \ldots \sigma_m$ is a permutation of indices $1 \ldots m$ of paxs.

Remark. Changing the order of linking paxs the closure of a net is not unique.

Definition 2.2. (1) A PN-*transformation* is an application

$$\phi : \text{PN} \to \text{PN}.$$

(2) Given $R \in \text{PN}$ and a PN-transformation π such that $\pi(R) = R'$: π is called *internal* PN-transformation iff there exists $T_R \in \text{PN}$ representing π such that $CUT(\bar{R}, T_R)$ is reducible to R' via a certain cut-elimination, where $CUT(\bar{R}, T_R)$ is a PN obtained joining $\bar{R}$ and T_R by a cut-link.

We shall call the pure net T_R of Definition 2.2 *internal representation* of π.

Proposition 2.3. *Given two internal PN-transformations π_1 and π_2 there exists a pure net T representation of the composition $\pi_1 \circ \pi_2$.*

Proof. Let T_1 and T_2 be the internal representations of π_1 and π_2. We suppose that $\pi_1(\pi_2(R)) = R'$ and $CUT(\overline{CUT(\bar{R}, T_2)}, T_1)$ is reduced to R'. So, trivially from the Church–Rosser property for pure net reduction we have that the net $CUT(\bar{R}, CUT(\bar{T}_2, T_1))$ is reduced to R' (e.g. $CUT(\bar{T}_2, T_1)$ is the representation of $\pi_1 \circ \pi_2$). □

2.2. Applicative transformations

Let us observe that the so-called *applicative transformation* which associates to the λ-term M, the λ-term $(M)x$ (denoted by $(\ )x$ in [3]), gives rise to two distinct PN-transformations depending on being $(M)x$ a λ-redex or not: in first case the effect of such a transformation is to make free a bound variable, in the second it is a true application. In PN we obtain, respectively, the anti-par transformation which destroys a par-link, and the times transformation which creates a time-link.

Definition 2.4.
● Given $R \in \text{PN}$ and a par-link l of R,

$$
\begin{array}{cc}
1 & 2 \\
\vdots & \vdots \\
\multicolumn{2}{c}{\underline{(l)?I \quad\quad O}} \\
\multicolumn{2}{c}{O} \\
\multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c}{3}
\end{array}
$$

we shall call *anti-par* transformation the PN-transformation $\phi_\wp^{l(R)}$ such that

$$\phi_\wp^{l(R)}(R) = R',$$

where R' is obtained from R breaking the par-link l:

```
1    2
⋮    ⋮
?I   O
     ⋮
     3
```

• An internal anti-par transformation $\pi_\wp^{l(R)}$ is called *Böhm anti-par* transformation.

Proposition 2.5. *Given a cut free pure net R and a par-link $l(R)$ there exists a pure net T_R which represents the Böhm anti-par transformation $\pi_\wp^{l(R)}$.*

Proof. We show the proposition for a par-link in the head block of R, the general case being treated in a similar way. So, we suppose given the normal pure net R:

```
     ⋮         ⋮
  ⋮  ₙ?I  O  I  !O  ⋮
 ₙ₋₁?I    O      I   !O
  ⋮       O      I    ⋮
 ₁?I      ⋮      ⋮   !O
     O            I
          ⋮
          1
```

and we define the representation T_R of $\pi_\wp^{l(R)}$ as

where m is the number of pax in R.

When we form a cut-link between the pal of the net $\bar{R}$ and the $?I$ conclusion of the head block of the net T_R, the HL-reduction starts with $m+n$-sequence of [mf] with times-links in T_R and we obtain complete annihilation of the par-links in $\bar{R}$. Once the annihilation is completed there are $m+n$ new exponential-cuts: $n-1$ with the premises $?I$ of the par-links of R and $m+1$ with a pax: m from the closure of R plus the ith par-link of R. Finishing the reduction, we get elimination of the ith par-link and the reconstruction of the rest of the net R. In other words, as we shall see also with other kinds of Böhm transformations, the procedure for breaking a link consists in first breaking any possible previous link to it and then accessing to it. $\square$

Definition 2.6.

- Given $R \in \mathrm{PN}$,
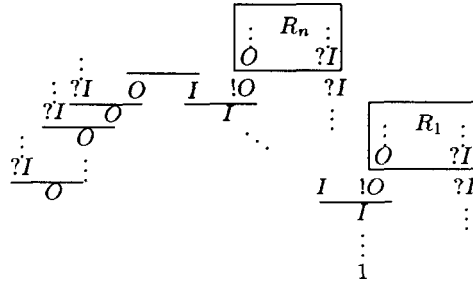
We shall call *times* transformation the PN-transformation $\phi_\otimes$ such that

$$\phi_\otimes(R) = R'$$

is obtained from R, by adding under the head-axiom a new times-link such that in the !-box over its $!O$ premise there is only an axiom-link and the conclusion $?I$ of the !-box is premise of no co-link.
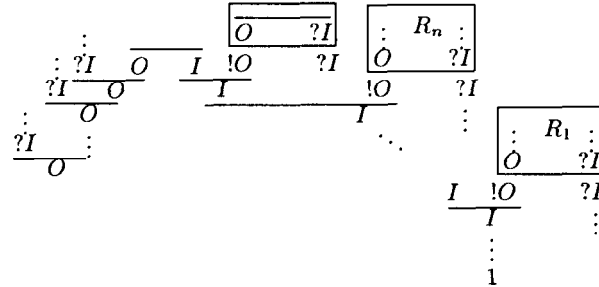
- An internal times transformation $\pi_\otimes$ is called *Böhm times* transformation.

Proposition 2.7. *Given a cut free pure net R there exists a pure net T_R which represents the Böhm times transformation $\pi_\otimes$.*

Proof. Given the cut-free pure net R:
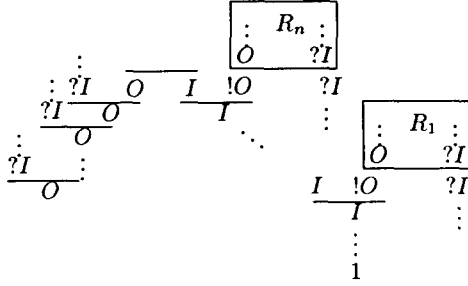
we define the net T_R representation of $\pi_\otimes$ as
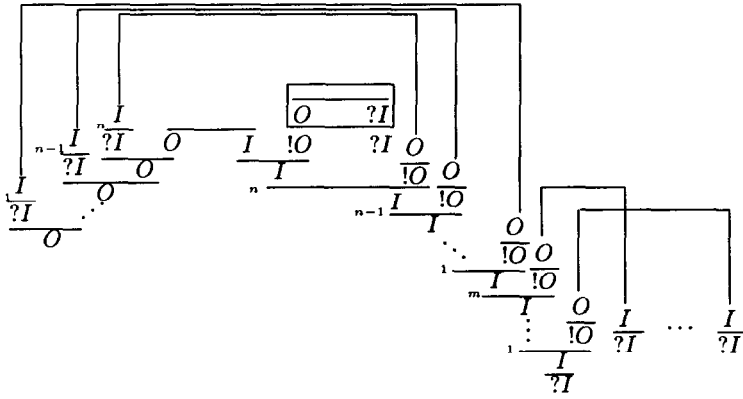
where m is the number of pax of R.

When we form a cut-link between the pal of the net $\bar{R}$ and the head pax of T_R, the HL-reduction starts with an $m+n$ sequence of [mf] with times-links in T_R and we obtain a case of argument gain: the new times-link is now at the right place. Once this reduction of [mf] is terminated, there are $n+m$ exponential cuts to reduce, to restore the m paxs and the n par-links of R. $\square$

2.3. Substitutive transformations

The next step concerns the definition of a third type of Böhm transformation: *substitutive transformation* which associates to a λ-term M the λ-term $M[x:=N]$ for some variable x and λ-term N.

Before defining it, we have to give the definition of *clean instance* of a pure net R.

Definition 2.8. Let R, $R^* \in PN$, we shall say R^* a *clean instance* of R if:
● there exists an integer k such that the head block of R^* has k par-links and k times-links more than the head block of R, and every !-box on the $!O$ premise of

times-links contains an axiom linkes with par-link corresponding to the times-links;

$$
\begin{array}{c}
\cdots \\
\dfrac{I}{?I} \quad \overline{O} \quad \dfrac{I}{O} \\
\dfrac{?I}{O} \qquad \dfrac{!O}{I} \\
\boxed{\quad\quad R \quad\quad}
\end{array}
$$
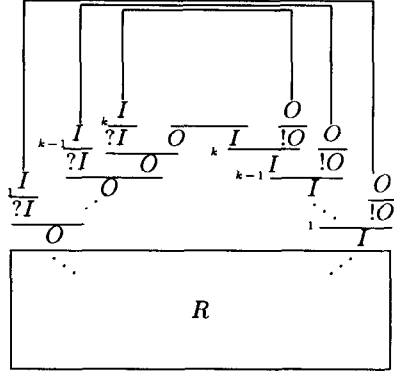
- every subnet of R^* is a clean instance of the corresponding subnet of R.

Definition 2.9.

- Given $R \in \mathrm{PN}$ and the head co-link l of R, we shall call *head* transformation the PN-transformation ϕ_h such that

$$\phi_h(R) = R'$$

is the pure net R where the head co-link l:

$$
\begin{array}{c}
\dfrac{?I \quad \overline{O} \quad I \quad !O}{O} \\
\vdots \qquad \dfrac{}{I} \\
?I \qquad \vdots \quad !O \quad 2 \\
\dfrac{}{O} \qquad \dfrac{}{I} \qquad \vdots \\
(l)\dfrac{\overline{?I} \;\; ?I \ldots ?I}{?I} \quad \dfrac{!O}{} \quad \boxed{\begin{array}{cc} \vdots & \vdots \\ O & ?I \end{array}} \\
\qquad\qquad\qquad\qquad\qquad ?I \\
\qquad\qquad\qquad\qquad\qquad \vdots \\
\qquad\qquad\qquad\qquad\qquad 3
\end{array}
$$

has been broken

$$
\begin{array}{c}
\dfrac{?I \quad \overline{O} \quad I \quad !O}{O} \\
\vdots \qquad \dfrac{}{I} \\
?I \qquad \vdots \quad !O \quad 2 \\
\dfrac{}{O} \qquad \dfrac{}{I} \\
\dfrac{\overline{?I} \;\; ?I \ldots ?I}{?I} \quad \dfrac{!O}{} \quad \boxed{\begin{array}{cc} \vdots & \vdots \\ O & ?I \end{array}} \\
\qquad\qquad\qquad\qquad\qquad ?I \\
\qquad\qquad\qquad\qquad\qquad \vdots \\
\qquad\qquad\qquad\qquad\qquad 3
\end{array}
$$
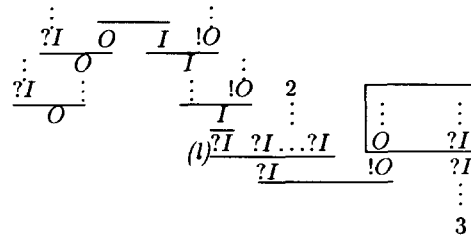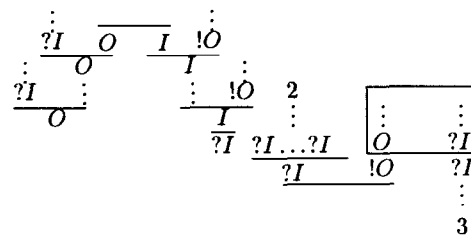
- A *weak head* transformation is a PN-transformation ϕ_h^* such that there exists an head transformation $\phi_h(R) = R'$, and $\phi_h(R)^* = R'^*$ where R'^* is a clean instance of R'.

● We shall call *Böhm head* transformation: an internal weak head transformation π_h.

Proposition 2.10. *Given a cut free pure net R, there exists a pure net T_R which represents the Böhm head transformation π_h.*

Proof. We suppose given the normal pure net R:

$$
\begin{array}{ccc}
\vdots & \quad & \vdots \\
\,_n?I \quad \overline{O} \quad \overline{I} \quad !O & & \\
\,_{n-1}?I \overline{\quad O \quad} \quad \overline{I} \; !O & & \\
\vdots \quad \overline{O} & & \overline{I} \quad \vdots \\
\,_1?I \quad \vdots & & \vdots \; !O \\
\quad \overline{O} & & \overline{I} \\
& & \vdots \\
& & 1
\end{array}
$$
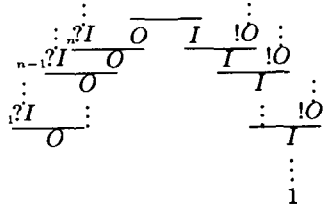
we define the representation T_R of π_h as follows:

Applying a Böhm times transformation to R we obtain the net $\pi_\otimes(R)$ with an head times-link more than R:

$$
\begin{array}{c}
\boxed{\;O \qquad ?I\;} \\
\vdots \quad \overline{\quad} \quad \\
\,_n?I \quad \overline{O} \quad \overline{I} \; !O \qquad ?I \quad \vdots \\
\,_{n-1}?I \overline{\quad O \quad} \quad \overline{I} \qquad \quad !O \; \vdots \\
\vdots \quad \overline{O} \qquad \overline{\quad\quad\quad\quad} \quad \overline{I} \; !O \\
\,_1?I \quad \vdots \qquad \overline{I} \\
\quad \overline{O} \qquad \qquad \vdots \; !O \\
\qquad \qquad \overline{I} \\
\qquad \qquad \vdots \\
\qquad \qquad 1
\end{array}
$$
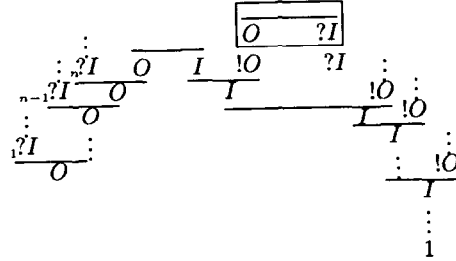
Now, let q be defined as follows:

$$q = \max\{\text{lengths of times link sequences over contractions in } 1\}, \tag{1}$$

We apply $q - t$ times the $\pi_\otimes(R)$ where t is the number of head times-links in R. In such a way we can obtain clean instances of subnets of R.

We define the representation of the Böhm head transformation which works starting from this net and then, by composition, we obtain the representation to be found.
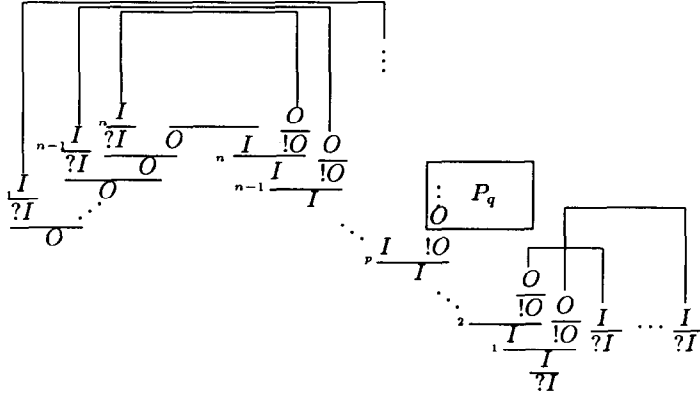
- where p is the position of the par-link of $\bar{R}$ where is linked the head conclusion $?I$, it can appear either between the n par-links of R or between the m par-links of the closure of R;
- where the net P_q is
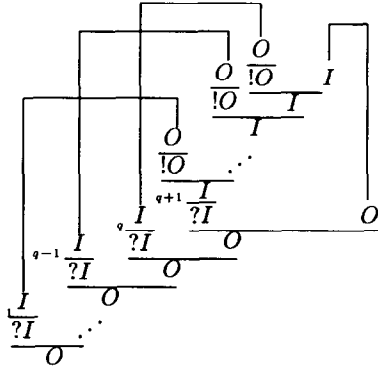
The idea is that the substitution of the head axiom of R with a new axiom not contracted (the one added by the Böhm times transformation) can be effectuated by a pure net P_q without twisting the structure of R (for this the choice of q is very important, see [3]). When we form a cut between the pal of $\pi_{\otimes}^{q-i+1}(R)$ and the head pax of T_R, $n+m$ [mf] are performed by HL-reduction and by the choice of p a new head redex is founded in the head block of $\pi_{\otimes}^{q-i+1}(R)$ with the net P_q, now because we added exactly q times-links to R the HL-reduction performs a complete annihilation of [mf]. So, as announced the axiom in the last !-box added to R, becomes the head axiom without contractions. Three kinds of cut links remains to be performed, before terminating the reduction; we distinguish them, as

- cuts involved in the reconstruction of par-links of R;
- cuts involved in the reconstruction of times-links of R;

● cuts which arrive from the [hl] step of reduction and represent the remaining head contractions of R.

Cuts of first type can be called *insignificant* (in analogy with [5]): if reduced consists in an η-expansion of identity applied to a ?I pax of R or premise of an head par-link of R; the second type consists again of an identity applied to a !-box; and the last type is exactly the cut link which makes the extraction of a subnet *not clean*: here, hl-reduction strategy adds a control on the extraction, we can decide if and when reducing this cut with a procedure that will be described later on. Finally after composition of the above described nets, and after their reduction we obtain a clean instance of R without head contractions.   □

Remark. Notice that the notion of clean instance is not exactly that of instance in [3]: treating the finitary case (cut-free proof nets) we can calculate exactly the value of q in equation (1), that is not the case for infinitary Böhm trees, in fact if the net R is not cut-free, some of cuts to reduce can produce an argument gain (eventually infinite) not allowing the definition of the value q.

2.4. Ready nets

We shall consider a subclass of pure nets: ready pure nets. For such nets, extraction of subnet is simply obtained applying a selector net, but not every net is ready.

Definition 2.11. A normal pure net is called *ready* if it is of the following form:

$$
\begin{array}{c}
\vdots \\
\overline{0 \quad I \quad !O} \;\; \vdots \\
\overline{\quad I \quad} \, !O \\
\overline{\quad I \quad} \;\; \vdots \\
\vdots \quad !O \\
\overline{\quad I \quad} \\
?I
\end{array}
$$

(i.e. without par-links, and without contractions in the head block).

We have this basic proposition.

Proposition 2.12. *For every cut-free pure net R there exists an internal transformation π such that $\pi(R)$ is a ready net instance of R.*

Proof. We show that for every normal pure net R:

$$\begin{array}{ccccccc}
\vdots & & & & & \vdots & \\
\vdots & ?I & \overline{\quad O\quad} & I & !O & \vdots & \\
?I & \overline{\quad\quad O\quad\quad} & & \overline{\quad\quad I\quad\quad} & !O & & \\
& \overline{\quad O\quad} & & & \overline{\quad I\quad} & \vdots & \\
\vdots & & & & & \vdots & \\
\vdots & & & & & \vdots & \\
?I & \vdots & & & \vdots & !O & \\
\overline{\quad O\quad} & & & & \overline{\quad\quad I\quad\quad} & & \\
& & & & & \vdots &
\end{array}$$

There exists some internal transformations such that their composition applied to R gives a ready net whose subnets are instances of the corresponding subnets of R.

We distinguish two steps:

(1) From Proposition 2.5 there exists an internal transformation ψ which performs the elimination of a par-link under the head axiom-link of R, so by composition we obtain an internal transformation ψ_1 which eliminates every par-link of the head of R, in the following we consider the normal form of representants of ψ_1.

(2) From Proposition 2.10 there exists an internal transformation ψ_2 which provides the elimination of contraction under head-axiom-link by instantiating some subnet of R.

Finally, the composition of these two internal transformations gives π. $\quad\square$

2.5. Proof net extraction

We give now an extension of Böhm-out technique still obtaining only an instance of the subnet to be extracted.

Definition 2.13.

- Given $R \in \mathrm{PN}$ and a subnet R_i of R,

$$\begin{array}{c}
\vdots \quad ?I \;\overline{\quad O\quad}\; I \;\overline{!O}\; \left[\begin{array}{cc} \vdots & R_i \quad \vdots \\ \dot O & ?I \end{array}\right] \\
?I \overline{\quad O\quad} \qquad \overline{\quad I\quad} \qquad \\
\vdots \qquad \overline{\quad O\quad} \qquad \vdots \quad !O \qquad ?I \\
?I \qquad \vdots \qquad \overline{\quad\quad I\quad\quad} \qquad ?I \\
\overline{\quad O\quad} \qquad \vdots \qquad \vdots \\
\qquad\qquad I \quad !O \\
\qquad\qquad \overline{\quad I\quad} \\
\qquad\qquad \vdots \\
\qquad\qquad 1
\end{array}$$

we shall call *out* transformation the PN-transformation ϕ_{out} such that

$$\phi_{out}^{R_i}(R) = R_i.$$

- A *weak out* transformation is a PN-transformation $\phi_{out}^{*R_i}$ such that there exists an out transformation $\phi_{out}^{R_i}(R) = R_i$ and $\phi_{out}^{R_i}(R)^* = R_i^*$ where R_i^* is a clean instance of R_i.
- We shall call *Böhm out* transformation: an internal weak out transformation $\pi_{out}^{R_i}$.

Proposition 2.14. *Given a cut free pure net R and a subnet R_i, there exists a pure net T which represents the Böhm out transformation $\pi_{out}^{R_i}$.*

Proof. From Proposition 2.12 there exists an internal transformation which gives a ready instance of R, so it suffices to give a representant of $\pi_{out}^{R_i}$ which works for a ready net R:
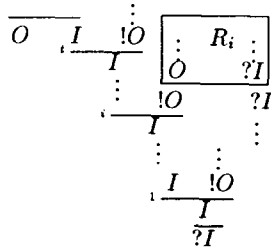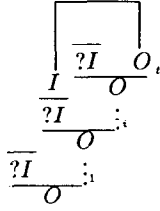

we define T_R as
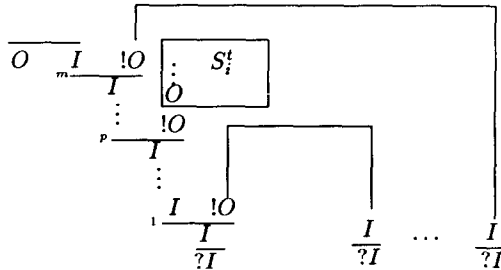
- where m is the number of pax of R;
- where the pth times-link corresponds to the head pax of R_i;
- where the net S_i^t is the *pure net selector* (the translation of the corresponding λ-term):

$$\begin{array}{c} \boxed{} \\ \dfrac{I\ \ \overline{?I}\ \ O}{O}\,_{l} \\ \dfrac{\overline{?I}}{O}\,_{:_{i}} \\ \dfrac{\overline{?I}}{O}\,_{:_{1}} \end{array}$$

When we form a cut-link between the pal of $\bar{R}$ and T_R there is a complete annihilation of the m par-links of $\bar{R}$, which generates m exponential-cuts: $m-1$ of type insignificant (reconstruction of the pax of R), and one, corresponding to the pth par-link in $\bar{R}$ (i.e. the head pax of R) with the net S_i^t, which becomes the new head redex. After HL-reduction of this last cut, we have again a complete annihilation of t [mf], which generates again t exponential-cuts: $t-1$ between a weakened $?I$ and a !-box R_j $(j \neq i)$ and one between the new head axiom and R_i, so obtaining the extraction of the subnet R_i. $\square$

Definition 2.15. A *Böhm tranformation* is inductively defined as follows:

(1) Böhm anti-par, times, head and out transformations are Böhm transformations;

(2) If π_1 and π_2 are Böhm transformations then $\pi_1 \circ \pi_2$ is a Böhm transformation.

We conclude with the following proposition on subnet extraction in the PN calculus which represents an improvement of the Böhm-out technique in pure λ-calculus. An appropriate choice of the net representing the Böhm transformation which performs the extraction, allows us to obtain the subnet to be extracted in a much more efficient way by considering the head linear reduction as in Definition 1.5 instead of the HL-strategy.

Proposition 2.16. (1) *For every cut free pure net R and for every! - Box in R associated to a subnet R', there exists a Böhm transformation π such that $\pi(R)$ is an instance of R'.*

(2) *Given a net T_R representing π, the head linear reduction of $CUT(\bar{R}, T_R)$ terminates on a net $\tilde{R}$, which is the net R' plus some cut-link.*

Proof. (1) We show it by induction on the level of the !-box of R':

● if the !-box is at level 0 we can apply Proposition 2.14, obtaining an instance of R',

● supposing that there exists a Böhm transformation π'' which performs the extraction of a subnet R'' at level n, containing the subnet R' to be extracted, we define the Böhm transformation π by composition of π'' with π_{out}^k, where k is the position of the !-box of R' in the times-links sequences.

(2) The reduction of the net obtained applying the net representing the Böhm transformation π, with the net $\bar{R}$ runs until the subnet R' to be extracted is reached: so, cuts from extraction which are in boxes at level greater than R' are not reduced. $\square$

Remark. Moreover the treatment of extraction of subnet from not cut-free nets, can be improved in a similar way: we can obtain the subnet to be exracted, plus such cuts which are "freezed", by using a modified version of the HL-reduction strategy. *Pure nets with explicit cuts* correspond to pure λ-terms with explicit substitutions. The *freezed head linear reduction strategy* is a variant of HL-reduction strategy for pure nets with explicit cuts obtained by freezing the rest of contractions in the [hl] step of reduction.

3. Related works and the future directions

The work presented in this paper is related to several issues of logic in computer science.

- In proof theory the work of Girard [10] provides the basis. Moreover the issue of "head linear reduction" is mentioned in [11]. An improvement of proof net extraction technique should be possible in a fragment of LU.
- Linear logic approach to pure λ-calculus is given by the calculus of pure nets developed in [7–9, 15, 18, 19]. An algebraic characterization of the proof extraction in the framework of dynamic algebras should be interesting.
- The Böhm-out technique and Böhm theorem, first introduced in [6] is treated in full generality in [3, 13]. The issue of equations in λ-calculus should be revisited in the pure net calculus by means of the proof nets extraction techniques.
- A modified notion of cut-elimination is somewhat related to [1].
- Programming language issues of linear logic concern both functional programming (e.g. [12, 14]), and logic programming (e.g. [2]). Let us mention that the calculus of pure nets (by means HL-reduction) has been implemented in the C language by the second author. Some primary steps are being taken towards the construction of unified functional and logic programming environment based on proof net calculi: evaluation of functional programs as net normalization and answer of logic programming queries as net extraction.
- Concurrency issues related to Milner's π-calculus ([4]) are also very promising.

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