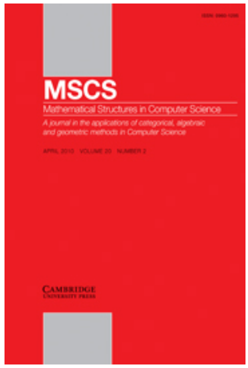

On universal algebra over nominal sets

ALEXANDER KURZ and DANIELA PETRIŞAN

Mathematical Structures in Computer Science / Volume 20 / Special Issue 02 / April 2010, pp 285 - 318
DOI: 10.1017/S0960129509990399, Published online: 25 March 2010

Math. Struct. in Comp. Science (2010), *vol.* 20, *pp.* 285–318.
doi:10.1017/S0960129509990399

On universal algebra over nominal sets

ALEXANDER KURZ and DANIELA PETRIŞAN

Department of Computer Science, University of Leicester, UK
Email: {kurz;petrisan}@mcs.le.ac.uk

Received 26 February 2009; revised 14 November 2009

We investigate universal algebra over the category Nom of nominal sets. Using the fact that Nom is a full reflective subcategory of a monadic category, we obtain an HSP-like theorem for algebras over nominal sets. We isolate a 'uniform' fragment of our equational logic, which corresponds to the nominal logics present in the literature. We give semantically invariant translations of theories for nominal algebra and NEL into 'uniform' theories, and systematically prove HSP theorems for models of these theories.

1. Introduction

Nominal sets were introduced by Gabbay and Pitts (Gabbay and Pitts 1999). This paper describes a step towards universal algebra over nominal sets. There has been some work in this direction, most notably by M. J. Gabbay (Gabbay 2008). The originality of our approach is that we do not start from the analogy between sets and nominal sets. As shown in Gabbay (2008), this is possible, but it requires ingenuity and *ad hoc* constructions. For example, the logic of Gabbay (2008) is not standard equational logic and even fundamental notions such as variables and free algebras have to be revisited.

In contrast, we want to explore how far one can get by using completely standard universal algebra. This is based on the well-known observation that Nom is equivalent to a full reflective subcategory $\mathrm{Sh}(\mathbb{I}^{op})$ of the presheaf category $\mathsf{Set}^{\mathbb{I}}$, where $\mathbb{I}$ is the category of finite subsets of a countable set of names $\mathcal{N}$, with injective maps:

$$\begin{tikzcd} \mathrm{Sh}(\mathbb{I}^{op}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathbb{I}} \arrow[l, bend right] \end{tikzcd} \tag{1}$$

Like any presheaf category, $\mathsf{Set}^{\mathbb{I}}$ is a category of many-sorted algebras: the sorts are the objects of $\mathbb{I}$; the (unary) operations are the arrows in $\mathbb{I}$; and the equations are the commuting diagrams of $\mathbb{I}$. Thus, diagram (1) embeds nominal sets into an equationally defined class of many-sorted algebras. To do the same for algebras, we need a notion of signature. This is provided by endofunctors $\tilde{L} : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathrm{Sh}(\mathbb{I}^{op})$ and $L : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$. The intention here is that the two functors $\tilde{L}$ and L are 'doing the same', or, more precisely they commute with the left adjoint.

The adjunction (1) then lifts to algebras:

$$
\begin{tikzcd}
\mathsf{Alg}(\tilde{L}) \arrow[r, bend right, "\overline{I}"'] \arrow[d, "\tilde{U}"'] \arrow[r, phantom, "\perp"] & \mathsf{Alg}(L) \arrow[l, bend right, "\overline{R}"'] \arrow[d, "U"] \\
\mathrm{Sh}(\mathbb{I}^{op}) \arrow[loop left, "\tilde{L}"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathbb{I}} \arrow[l, bend right, "R"'] \arrow[loop right, "L"]
\end{tikzcd}
\tag{2}
$$

At this point, we are encouraged by a result of Kurz and Rosický (2006) stating that all 'universal-algebraic' endofunctors L have a presentation. In particular, it follows that $\mathsf{Alg}(L)$ is an equationally defined class (variety) of many-sorted algebras *in the standard sense of universal algebra*. For example, we have an equational calculus at hand, which we can use to reason about algebras in $\mathsf{Alg}(L)$ and, hence, to reason about algebras in the full subcategory $\mathsf{Alg}(\tilde{L})$.

Central to our approach is the flexibility provided by the notion of a presentation of a functor (which was introduced in Bonsangue and Kurz (2006) and further developed in Kurz and Rosický (2006) and Kurz and Petrişan (2008)). It allows us to identify the abstract notion of L-algebras with the concrete notion of algebras for a signature and equations. In particular, we may prove results about equational logic in an abstract syntax-free manner. Thus Diagram (2) suggests a program of developing universal algebra over nominal sets, and we will carry out the following first steps.

Section 3 proves an HSP-like theorem for algebras over $\mathrm{Sh}(\mathbb{I}^{op})$. We have Birkhoff's HSP-theorem for $\mathsf{Alg}(L)$, and using a category theoretic formulation of Birkhoff's proof (which is standard, see, for example, Adámek *et al.* (1990, Exercise 16G)), it is straightforward to transfer it to the nominal setting: a class of $\tilde{L}$-algebras is equationally definable if and only if it is closed under products, subalgebras and presheaf-quotients. The usefulness of changing the perspective from Nom to $\mathsf{Set}^{\mathbb{I}}$ is highlighted by Gabbay's remark in Gabbay (2008) that 'An attempt to directly transfer proofs of the HSP theorem to the nominal setting fails'.

Section 4 extracts from the abstract category theoretic treatment of Section 3 a standard equational calculus. To this end, we give presentations by operations and equations of $\mathsf{Set}^{\mathbb{I}}$ and the 'shift' or 'abstraction' functor δ. The functor δ satisfies the conditions set out in Section 3 and we obtain an HSP theorem for algebras with binders. We illustrate the equational calculus with an axiomatisation of the λ-calculus. The observation that this axiomatisation is 'uniform' over the indexing sorts in $\mathbb{I}$ leads to the next section.

Section 5 starts from the observation that the equational logic of the previous sections is too strong in that it does not need to respect a fundamental notion of nominal sets, namely equivariance. We therefore introduce a fragment of our equational logic, called the uniform fragment. We prove an HSPA theorem in the style of Gabbay (2008): a class of $\tilde{L}$-algebras is definable by uniform equations if and only if it is closed under quotients, subalgebras, products and abstraction. This shows that the uniform fragment of equational logic has the same expressiveness as the nominal algebra of Gabbay (2008).

Section 6 shows how to translate theories of nominal algebra (Gabbay and Mathijssen 2009) and nominal equational logic (Clouston and Pitts 2007) into uniform theories. We prove that these translations are semantically invariant, and we obtain an HSPA theorem for nominal equational logic and a new proof of the HSPA theorem of nominal algebra (Gabbay 2008).

2. Preliminaries

2.1. *Sheaves and presheaves on* $\mathbb{I}^{op}$

In the rest of this paper we use $\mathcal{N}$ to denote an infinite countable set of names. We will use $\mathbb{I}$ to denote the category whose objects are finite subsets of $\mathcal{N}$ and whose morphisms are injective maps. A presheaf on $\mathbb{I}^{op}$ is an object in the functor category $\mathsf{Set}^{\mathbb{I}}$, that is, a functor from $\mathbb{I}$ to Set. Note that $\mathsf{Set}^{\mathbb{I}}$ is a category of many-sorted algebras over $\mathsf{Set}^{\mathscr{P}_f(\mathcal{N})}$, where $\mathscr{P}_f(\mathcal{N})$ denotes the set of finite subsets of $\mathcal{N}$. The unary operations are the arrows in $\mathbb{I}$, and the equations are the commuting diagrams of $\mathbb{I}$.

The category of interest throughout this paper is the full subcategory of $\mathsf{Set}^{\mathbb{I}}$ consisting of pullback preserving functors. This category is also known in the literature as the *Schanuel topos* and is denoted by $\mathrm{Sh}(\mathbb{I}^{op})$. The notation is justified by the fact that $\mathrm{Sh}(\mathbb{I}^{op})$ is a Grothendieck topos: a functor $A : \mathbb{I} \to \mathsf{Set}$ preserves pullbacks if and only if it is a sheaf for the atomic topology on $\mathbb{I}^{op}$ (Johnstone 2002). We will now list a few properties of this category relevant for the development of this paper.

Proposition 2.1. The inclusion functor $I : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathsf{Set}^{\mathbb{I}}$ has a left adjoint R, which preserves finite limits.

This is a particular instance of Mac Lane and Moerdijk (1994, Theorem 1, page 128).

Remark 2.2. The functor I preserves filtered colimits and coproducts. This is because both filtered colimits and coproducts commute with pullbacks in Set. Also, I preserves all limits and R preserves all colimits. This follows from the fact that I is right adjoint to R.

Proposition 2.3. A morphism between two presheaves is an epimorphism if and only if it is so pointwise. A sheaf morphism $f : A \to B$ is an epimorphism in $\mathrm{Sh}(\mathbb{I}^{op})$ if and only if for all finite sets of names S and all $y \in B(S)$ there exists an inclusion $l : S \to T$ in $\mathbb{I}$ and $x \in A(T)$ such that $f_T(x) = B(l)(y)$.

Proof. The statement follows from Mac Lane and Moerdijk (1994, Corollary III.7.5.). □

If A is a presheaf, we will call RA the *sheafification* of A.

Remark 2.4. Starting with a presheaf A, the sheaf RA is obtained as follows. For finite sets of names $S \subseteq T$, we will say that an element $x \in A(T)$ is supported by S if and only if whenever two functions $i, j : T \to T'$ coincide when restricted to S, then $A(i)(x) = A(j)(x)$. Two elements supported by S, $x \in A(T)$ and $y \in A(T')$, are said to be equivalent if and only if there exist injective maps $i : T \to T''$ and $j : T' \to T''$ such that $i(s) = j(s)$ for all $s \in S$ and such that $A(i)(x) = A(j)(y)$. We use $\overline{x}$ to denote the equivalence class of x. We define $RA(S)$ to be the set of equivalence classes of elements supported by S. If $i : S \to S'$ is an injective map, then $RA(i)$ maps $\overline{x}$ to $\overline{A(j)(x)}$, where j is a map that makes the square

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "w"] & S' \arrow[d, "w'"] \\ T \arrow[r, "j"] & T' \end{tikzcd} \tag{3}$$

commute for appropriate inclusions w, w'.

2.2. *Finitary presentations for functors on many-sorted varieties*

Let S be a set of sorts. A *signature* is a set of operation symbols together with an arity map. For each signature we consider an endofunctor on Set^S such that for $X \in \mathsf{Set}^S$, we define $\Sigma X \in \mathsf{Set}^S$ by

$$\Sigma X(s) = \coprod \{f_{s_1 \times \cdots \times s_n \to s}\} \times X(s_1) \times \cdots \times X(s_n) \tag{4}$$

where the coproduct is taken after all operation symbols f whose result has sort s. The algebras for a signature are precisely the algebras for the corresponding endofunctor, and form the category denoted by $\mathsf{Alg}(\Sigma)$. The terms over an S-sorted set of variables X are defined in the standard manner, and an equation consists of a pair (τ_1, τ_2) of terms of the same sort, usually denoted by $\tau_1 = \tau_2$. A Σ-algebra A satisfies this equation if and only if, for any interpretation of the variables of X, we obtain equality in A. A full subcategory $\mathscr{A}$ of Σ-algebras is said to be an *equational class* or *variety* if there exists a set of equations E such that an algebra lies in $\mathscr{A}$ if and only if it satisfies all the equations of E. Such an equational class will be denoted by $\mathsf{Alg}(\Sigma, E)$.

The notion of functors with finitary presentations was introduced in Bonsangue and Kurz (2006) for applications to coalgebraic logic. Let $\mathscr{A} = \mathsf{Alg}(\Sigma_{\mathscr{A}}, E_{\mathscr{A}})$ be an equational class. We use $U : \mathscr{A} \to \mathsf{Set}^S$ to denote the forgetful functor and F to denote its left adjoint. Intuitively, an endofunctor L on $\mathscr{A}$ has a finitary presentation by operations Σ_L and equations E_L if, for each object A, we have that LA is uniformly isomorphic to a quotient of $F\Sigma_L UA$ by the equations E_L in a sense made precise in the following definition.

Definition 2.5. A finitary presentation for an endofunctor L on $\mathscr{A}$ is a pair $\langle \Sigma_L, E_L \rangle$ where Σ_L is an endofunctor as in (4) and E_L is a set of equations as follows: for any S-sorted set of variables V, E_V is a subset of $(UF\Sigma UFV)^2$ and E_L is the disjoint union of E_V taken over all finite sets of variables V. The functor L is presented by $\langle \Sigma_L, E_L \rangle$ if, for any $A \in \mathscr{A}$, the algebra LA is the joint coequaliser:

$$\begin{tikzcd} FE_V \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F\Sigma_L UFV \arrow[r, "F\Sigma_L Uv^\sharp"] & F\Sigma_L UA \arrow[r, "q_A"] & LA, \end{tikzcd} \tag{5}$$

which is taken over all finite sets of S-sorted variables V and all valuations $v : V \to UA$. Here $v^\sharp$ denotes the adjoint transpose of a valuation v.

Functors with finitary presentations are characterised in Kurz and Rosický (2006) as exactly those functors that preserve sifted colimits. Recall that sifted colimits are exactly those colimits that commute with finite products in Set. Examples include filtered colimits and reflexive coequalisers.

For an endofunctor L on a category $\mathscr{A}$, we consider the category of *L-algebras*, denoted by $\mathsf{Alg}(L)$, whose objects are defined as pairs (A, α) such that $\alpha : LA \to A$ is a morphism in $\mathscr{A}$. A morphism of L-algebras $f : (A, \alpha) \to (A', \alpha')$ is a morphism $f : A \to A'$ of $\mathscr{A}$ such that $f \circ \alpha = \alpha' \circ Lf$.

If the functor L on an equational class has a finitary presentation, then $\mathsf{Alg}(L)$ is an equational class also, and its presentation can be obtained in a modular way.

Theorem 2.6. Let $\mathscr{A} = \mathsf{Alg}(\Sigma_{\mathscr{A}}, E_{\mathscr{A}})$ be an S-sorted equational class and let $L : \mathscr{A} \to \mathscr{A}$ be a functor presented by operations Σ_L and equations E_L. Then $\mathsf{Alg}(L)$ is concretely isomorphic to $\mathsf{Alg}(\Sigma_{\mathscr{A}} + \Sigma_L, E_{\mathscr{A}} + E_L)$.

Here we understand $\Sigma_{\mathscr{A}} + \Sigma_L$ to be the disjoint union of the signatures, and $E_{\mathscr{A}} + E_L$ to be the disjoint union of the equations $E_{\mathscr{A}}$ and E_L, regarded as equations over $\Sigma_{\mathscr{A}} + \Sigma_L$ (a proof of this theorem is in Kurz and Petrişan (2008)).

3. HSP theorems

In this section we will prove an HSP-theorem for algebras over the topos $\mathrm{Sh}(\mathbb{I}^{op})$ in a systematic way.

In the first subsection we prove general results using categorical techniques. To set the scene, we outline a categorical proof of Birkhoff's HSP theorem. Then, in Theorem 3.5, we show how to obtain an HSP-theorem for a full reflective subcategory $\mathscr{A}$ of a category of algebras $\mathscr{C}$, provided some additional conditions are met. Essentially, this is achieved by 'pushing' the proof of the general HSP-theorem through the adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend right] \end{tikzcd} .$$

This result is interesting because $\mathscr{A}$ might not be a variety. We also prove a general result, Proposition 3.6, concerning a lifting property of an adjunction to categories of algebras for certain functors.

Then, in Section 3.2, we apply these results to the nominal setting. Consider diagram (2), where L is an endofunctor on $\mathsf{Set}^{\mathbb{I}}$ that preserves sifted colimits and $\tilde{L}$ is an endofunctor on $\mathrm{Sh}(\mathbb{I}^{op})$ such that $\tilde{L}R \simeq RL$. Using Proposition 3.6, we prove that the adjunction between $\mathrm{Sh}(\mathbb{I}^{op})$ and $\mathsf{Set}^{\mathbb{I}}$ can be lifted to an adjunction between $\mathsf{Alg}(\tilde{L})$ and $\mathsf{Alg}(L)$. On the right-hand side of this diagram, we have categories monadic over $\mathsf{Set}^{\mathbb{N}}$, for which the classical HSP-theorem holds. We derive an HSP theorem for $\mathsf{Alg}(\tilde{L})$ by applying Theorem 3.5.

3.1. *HSP theorems for full reflective subcategories*

Birkhoff's HSP Theorem. Given a category of algebras $\mathscr{C}$, a full subcategory $\mathscr{B} \subseteq \mathscr{C}$ is closed under quotients (H for homomorphic images), subalgebras (S) and products (P) if and only if $\mathscr{B}$ is definable by equations.

This theorem can be proved at different levels of generality. We assume here that $\mathscr{C}$ is monadic over Set^κ for some cardinal κ. We use $U : \mathscr{C} \to \mathsf{Set}^\kappa$ to denote the forgetful functor and F to denote its left adjoint. Then we can identify a class of equations Φ in variables X with quotients $FX \to Q$. Indeed, given Φ, we let Q be the quotient FX/Φ and, conversely, given $FX \to Q$, we let Φ be the kernel of $FX \to Q$. Furthermore, an algebra $A \in \mathscr{C}$ satisfies the equations if and only if all $FX \to A$ factor through $FX \to Q$ as in the diagram

$$A \models \Phi \quad \Leftrightarrow \quad \begin{tikzcd} FX \arrow[rr] \arrow[dr, "\forall"'] & & FX/\Phi \arrow[dl, dotted, "\exists"] \\ & A & \end{tikzcd} \tag{6}$$

Proof.

'If' We want to show that a subcategory $\mathscr{B}$ defined by equations Φ is closed under HSP. Closure under subobjects $A' \to A$ follows since quotients and subobjects form a factorisation system (see, for example, Adámek *et al.* (1990, 14.1)). Indeed, according to (6), in order to show $A \models \Phi \Rightarrow A' \models \Phi$, we have to find the dotted arrow in

$$\begin{tikzcd} FX \arrow[r] \arrow[d] & FX/\Phi \arrow[d] \arrow[dl, dotted] \\ A' \arrow[r] & A \end{tikzcd}$$

which exists because of the diagonal fill-in property of factorisation systems. A similar argument works for products (because of their universal property) and for quotients (using the fact that free algebras are projective (Adámek *et al.* 1990, 9.27)).

'Only if' Given $\mathscr{B} \subseteq \mathscr{C}$, we first need to find the equations. Since $\mathscr{B}$ is closed under SP, $\mathscr{B}$ is a full reflective subcategory, that is, the inclusion $\mathscr{B} \to \mathscr{C}$ has a left-adjoint H and, moreover, the unit $A \to HA$ is a quotient†. We take as equations all $FX \to HFX$. That all $A \in \mathscr{B}$ satisfy these equations, again using (6), follows immediately from the universal property of the left-adjoint H. Conversely, suppose A satisfies all of these equations. Consider the equations $q : FUA \to HFUA$. Because of (6), the counit $e : FUA \to A$ must factor as $e = f \circ q$. Since e and q are quotients, f is too. Hence A is a quotient of $HFUA$, which is in $\mathscr{B}$. □

† To construct HA given A, consider all arrows $f : A \to B_f$ with codomain in $\mathscr{B}$; factor $f = A \xrightarrow{q_f} \bar{B}_f \xrightarrow{i_f} B$; up to isomorphism, there is a only a proper set of different q_f; now factor $A \xrightarrow{\langle q_f \rangle} \prod_f \bar{B}_f$ as $A \to HA \to \prod_f \bar{B}_f$ to obtain the unit $A \to HA$.

Remark 3.1. Notice that in the proof above we allow quotients $FX \to HFX$ for arbitrary κ-sorted sets X. If the set X is infinite, we allow equations involving infinitely many variables. Therefore we no longer reason within finitary logic. If we impose the condition that the equations involve only finitely many variables, then the HSP theorem is *not* true for arbitrary many-sorted varieties. Indeed, in the many-sorted case, closure under homomorphic images, sub-algebras and products is no longer enough to deduce equational definability (see Adámek *et al.* (draft, Example 10.14.2)). We need an additional constraint, namely closure under directed unions (Adámek *et al.* draft, Theorem 10.12). But in the motivating examples of $\mathsf{Set}^{\mathbb{I}}$, we will prove that closure under HSP implies closure under directed unions.

In the following, we show that it is possible to obtain an HSP theorem for certain subcategories of varieties by pushing the argument above through an adjunction. But first, we should say what we mean by **equationally definable** and **closed under HSP** in this context. We will work in the following setting.

Definition 3.2. Let $\mathscr{C}$ be a category monadic over Set^κ for some cardinal κ. We use U to denote the forgetful functor and F to denote its left adjoint. Let $\mathscr{A}$ be a full subcategory of $\mathscr{C}$ with a factorisation system (E, M) such that morphisms in M are monomorphisms and the inclusion functor I has a left adjoint that preserves the regular factorisation system of $\mathscr{C}$.

We say that $\mathscr{B} \hookrightarrow \mathscr{A}$ is **equationally definable** if there exists a set of equations Φ in $\mathscr{C}$, such that an object A of $\mathscr{A}$ lies in $\mathscr{B}$ if and only if $IA \models \Phi$ (where Φ and $\models$ are as in (6)). We say that $\mathscr{B}$ is **closed under HSP** if and only if:

1 For all morphisms $e : B \to B'$ such that $e \in E$ and Ie is a quotient, we have $B \in \mathscr{B}$ implies $B' \in \mathscr{B}$.
2 For all morphisms $m : B \to B'$ such that $m \in M$, we have $B' \in \mathscr{B}$ implies $B \in \mathscr{B}$.
3 If B_i are in $\mathscr{B}$, their product in $\mathscr{A}$ is an object of $\mathscr{B}$.

Remark 3.3. In general, the inclusion functor I does not preserve epimorphisms. We will assume that the arrows in M are monomorphisms. Being a right adjoint, I preserves products and monomorphisms, but we cannot infer from $B' \to IB$ being a monomorphism in $\mathscr{C}$ that B' is (isomorphic to an object) in $\mathscr{A}$.

Remark 3.4. The third item of Definition 3.2 makes sense only if $\mathscr{A}$ has products. But $\mathscr{A}$ is complete, since $\mathscr{A}$ is a full reflective category of a complete category (Borceux 1994, Proposition 3.5.3).

If $\mathscr{C}$ is a category monadic over Set^κ for some cardinal κ and $\mathscr{A}$ is a full reflective subcategory of $\mathscr{C}$, then $\mathscr{A}$ is complete and is well-powered because $\mathscr{C}$ is. Hence we can equip $\mathscr{A}$ with a strong-epi/mono factorisation system (Borceux 1994, 4.4.3). We can prove the following theorem.

Theorem 3.5. Let $\mathscr{C}$ be a category monadic over Set^κ for some cardinal κ and $\mathscr{A}$ a full reflective subcategory of $\mathscr{C}$, such that the left adjoint of the inclusion functor preserves monomorphisms. Then $\mathscr{B} \subseteq \mathscr{A}$ is closed under HSP in the sense of Definition 3.2 if and only if $\mathscr{B}$ is equationally definable.

Proof. We will use U to denote the forgetful functor $\mathscr{C} \to \mathsf{Set}^\kappa$ and F to denote its left adjoint. I denotes the full and faithful functor $\mathscr{A} \to \mathscr{C}$ and R denotes its left adjoint.

Note that in $\mathscr{A}$ (as in $\mathscr{C}$) strong epis coincide with extremal epis (Borceux 1994, 4.3.7) and with regular epis (Adámek *et al.* 1990, 14.14 and 14.22). The proof of the theorem relies on the following two properties:

$$e \text{ regular epi in } \mathscr{C} \Rightarrow Re \text{ regular epi in } \mathscr{A} \tag{7}$$

$$m \text{ mono in } \mathscr{A} \Rightarrow Im \text{ mono in } \mathscr{C} \tag{8}$$

Property (7) holds because R is a left-adjoint and (8) because I is a right adjoint. Also note that we have the converse of (8) since I is full and faithful.

We begin by proving that equational definability implies closure under HSP. Let $\mathscr{B}$ be an equationally definable subcategory of $\mathscr{A}$. That means that there exists an equationally definable subcategory $\mathscr{B}'$ of $\mathscr{C}$ such that B is an object of $\mathscr{B}$ if and only if IB is an object of $\mathscr{B}'$. The proof of the fact that $\mathscr{B}$ is closed under HSP in the sense of Definition 3.2 follows from the HSP theorem applied for $\mathscr{B}'$ and the following observations:

1 The quotients $e \in E$ considered in Definition 3.2 are exactly those for which Ie is a regular epimorphism in $\mathscr{C}$.
2 I preserves monomorphsims.
3 I preserves products.

Conversely, let $\mathscr{B}$ be a subcategory of $\mathscr{A}$, closed under HSP, as in the previous definition. We will prove that $\mathscr{B}$ is equationally definable. We proceed in three steps:

Step 1: construction of the equations that define $\mathscr{B}$.

Let C be an arbitrary object of $\mathscr{C}$. We will consider all morphisms $f_i : RC \to B_i$ in $\mathscr{A}$ such that B_i is in $\mathscr{B}$. For each i, the corresponding morphism in $\mathscr{C}$, $f_i^\sharp : C \to IB_i$ factors in $\mathscr{C}$:

$$\begin{tikzcd} C \arrow[dd, "f_i^\sharp"'] \arrow[dr, "e_i"] & \\ & \overline{B_i} \arrow[dl, "m_i"] \\ IB_i & \end{tikzcd}$$

We will use η and ϵ to denote the unit and co-unit, respectively, of the adjunction $R \dashv I$. It is easy to show that the following diagram commutes:

$$\begin{tikzcd} RC \arrow[dd, "f_i"'] \arrow[dr, "Re_i"] & \\ & R\overline{B_i} \arrow[dl, "\epsilon_{B_i} \circ Rm_i"] \\ B_i & \end{tikzcd} \tag{9}$$

Since R preserves regular epis and monos, and ϵ_{B_i} is an isomorphism (I is full and faithful), we have that (9) is a factorisation of f_i in $\mathscr{A}$. But $\mathscr{B}$ is closed under subobjects, hence $R\overline{B_i}$ is actually an object of $\mathscr{B}$. Since $\mathscr{C}$ is co-well-powered, there is only a proper set of different e_i up to isomorphism, so we can take the product P of the objects of the form $R\overline{B_i}$, obtained as above. P is again an object of $\mathscr{B}$, and we have a morphism $\alpha : RC \to P$, uniquely determined by the Re_i. We now consider a factorisation in $\mathscr{C}$ of the adjoint map $\alpha^\sharp : C \to IP$:

$$\begin{tikzcd} C \arrow[dd, "\alpha^\sharp"'] \arrow[dr, "e"] & \\ & Q_C \arrow[dl, "m"] \\ IP & \end{tikzcd} \tag{10}$$

Using a similar argument to the above, we deduce that RQ_C is an object of $\mathscr{B}$ and the following diagram commutes:

$$\begin{tikzcd} RC \arrow[rr, "Re"] \arrow[d, "f_i"] & & RQ_C \arrow[d, "\epsilon_P \circ Rm"] \\ B_i & R\overline{B_i} \arrow[l, "\epsilon_B \circ Rm_i"] & P. \arrow[l, "\pi_i"] \end{tikzcd} \tag{11}$$

We consider the class of equations $\mathscr{E}$ of the form $FX \to Q_{FX}$ for all sets X, and use $\mathscr{B}'$ to denote the subcategory of $\mathscr{C}$ defined by these equations.

Step 2: $\mathscr{B}$ is contained in the class defined by the equations $\mathscr{E}$.

We will now show that if an object B of $\mathscr{A}$ lies in $\mathscr{B}$, then IB satisfies the equations in $\mathscr{E}$. Let B be an object of $\mathscr{B}$, and let $u : FX \to IB$ be an arbitrary morphism. For the adjoint morphism $u_\sharp : RFX \to B$, one can construct a morphism $g : RQ_{FX} \to B$, obtained as in diagram (11), such that $g \circ Re = u_\sharp$.

It is easy to see that $g^\sharp : Q_{FX} \to IB$ makes the following diagram commutative, which shows that IB satisfies the equation $e : FX \to Q_{FX}$:

$$\begin{tikzcd} FX \arrow[r, "e"] \arrow[d, "u"'] & Q_{FX} \arrow[dl, "g^\sharp"] \\ IB & \end{tikzcd} \tag{12}$$

Step 3: the subcategory defined by the equations $\mathscr{E}$ is contained in $\mathscr{B}$.

Let B an object in $\mathscr{B}$ such that IB satisfies the equations in $\mathscr{E}$. In particular, IB satisfies $FUIB \to Q_{FUIB}$, so there exists $v : Q_{FUIB} \to IB$ such that $v \circ e = \epsilon'_{IB}$, where ϵ' is the counit of the adjunction $F \dashv U$. Since ϵ' is a regular epi, v is also a regular epi. We have that the composition $\epsilon_B \circ Rv : RQ_{FUIB} \to B$ is a regular epi in $\mathscr{A}$. Since the codomain of v is in the image of I, we have that IRv is also a regular epi, therefore so is $I(\epsilon_B \circ Rv)$. Using the fact that $\mathscr{B}$ is closed under H, and that RQ_{FUIB} is already in $\mathscr{B}$, we can conclude that $B \in \mathscr{B}$. □

The next proposition allows us to lift an adjunction between two categories to an adjunction between categories of algebras for functors satisfying some additional conditions.

Proposition 3.6. Let $\langle R, I, \eta, \epsilon\rangle : \mathscr{A} \rightharpoonup \mathscr{B}$ be an adjunction. Let K and L be endofunctors on $\mathscr{A}$ and $\mathscr{B}$, respectively. Suppose there exist natural transformations $\kappa : RK \to LR$ and $\lambda : LI \to IK$ making the following diagrams commute:

$$
\begin{tikzcd}
LIR \arrow[r, "\lambda R"] & IKR \arrow[r, "I\kappa"] & IRL \\
& L \arrow[ul, "L\eta"] \arrow[ur, "\eta L"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
KRI \arrow[r, "\kappa I"] \arrow[dr, "K\epsilon"'] & RLI \arrow[r, "R\lambda"] & RIK \arrow[dl, "\epsilon K"] \\
& K &
\end{tikzcd}
\tag{13}
$$

(C$_1$) (C$_2$)

Then there exists an adjunction $\langle \overline{R}, \overline{I}, \overline{\eta}, \overline{\epsilon}\rangle : \mathsf{Alg}(K) \rightharpoonup \mathsf{Alg}(L)$ such that $U_{\mathscr{A}}\overline{R} = RU_{\mathscr{B}}$ and $IU_{\mathscr{A}} = U_{\mathscr{B}}\overline{I}$, where $U_{\mathscr{A}}$ and $U_{\mathscr{B}}$ denote the forgetful functors as in the diagram

$$
\begin{tikzcd}
\mathsf{Alg}(K) \arrow[rr, bend right, "\overline{I}"'] \arrow[d, "U_{\mathscr{A}}"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Alg}(L) \arrow[ll, bend right, "\overline{R}"'] \arrow[d, "U_{\mathscr{B}}"] \\
\mathscr{A} \arrow[loop left, "K"] \arrow[rr, bend right, "I"'] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend right, "R"'] \arrow[loop right, "L"]
\end{tikzcd}
\tag{14}
$$

Proof. We begin by defining the functor $\overline{I}$. Let $f : KA \to A$ be a K-algebra. We define $\overline{I}(A, f) := (IA, If \circ \lambda_A)$. For an arbitrary morphism of K-algebras $u : (A, f) \to (A', f')$, we define $\overline{I}(u) = Iu$. The fact that Iu is a morphism of L-algebras follows from the commutativity of the outer square of the diagram

$$
\begin{tikzcd}
LIA \arrow[r, "\lambda_A"] \arrow[d, "LIu"'] & IKA \arrow[r, "If"] \arrow[d, "IKu"'] & IA \arrow[d, "Iu"'] \\
LIA' \arrow[r, "\lambda'_A"] & IKA' \arrow[r, "If'"] & IA'
\end{tikzcd}
\tag{15}
$$

But the small squares commute: the former because λ is a natural transformation and the latter because u is a K-algebra morphism. It is obvious that $\overline{I}$ is a functor and that $IU_{\mathscr{A}} = U_{\mathscr{B}}\overline{I}$.

The functor $\overline{R}$ is defined similarly: if $g : LB \to B$ is an L-algebra, we define $\overline{R}(B, g) = (RB, Rg \circ \kappa_B)$; if $v : (B, g) \to (B', g')$ is an L-algebra morphism, we define $\overline{R}(v) = Rv$. The fact that Rv is indeed a K-algebra morphism is verified easily using the naturality of κ and the fact that v is an L-algebra morphism.

In order to prove that $\overline{R}$ is left adjoint to $\overline{I}$, we will show that the unit η and the counit ϵ of the adjunction $R \dashv I$ are L-algebra and K-algebra morphisms, respectively. This follows from the hypothesis (13) and the naturality of η and ϵ, respectively.

Once this is achieved, η can be lifted to a natural transformation $\overline{\eta} : \mathrm{id} \to \overline{IR}$, and, similarly, ϵ can be lifted to a natural transformation $\overline{\epsilon} : \overline{RI} \to \mathrm{id}$. But η and ϵ are the unit and the counit of the adjunction $R \dashv I$, so they satisfy the usual triangular equalities. Therefore, $\overline{\eta}$ and $\overline{\epsilon}$ satisfy the triangular equalities for $\overline{R}$ and $\overline{I}$. □

Remark 3.7. The proposition has some useful special cases if we impose the additional assumption that I is full and faithful. For each of them, it is straightforward to verify the commutativity of the diagrams (13) using the fact that the counit ϵ is iso.

1 Suppose L is given and we want to find an appropriate K. Then it follows from the theorem that we can do this, provided there is a natural transformation $\alpha : LIR \to IRL$ such that the following diagram commutes:

$$\begin{tikzcd} LIR \arrow[rr, "\alpha"] & & IRL \\ & L \arrow[ul, "L\eta"] \arrow[ur, "\eta L"'] & \end{tikzcd} \tag{16}$$

If this is the case, we define $K = RLI$, $\kappa : KR = RLIR \to RL$ as the composition $\epsilon_{RL} \circ R\alpha$ and $\lambda : LI \to IRLI = IK$ as η_{LI}. Moreover, we have that $LIR \to IRLIR$ is iso and that $R\alpha$ and hence $KR \to RL$ are iso. (If we do not find such an α, we might think of replacing L by $IRLIR$ since we always have $(IRLIR)IR \to IR(IRLIR)$; we still have $K \cong RLI$.)

2 More generally, suppose we are given an iso $\kappa : KR \to RL$. Then we define $\lambda = IK\epsilon \circ I\kappa^{-1}I \circ \eta LI$. (Given K, we can always find such a κ: let $L = IKR$ and $\kappa = (\epsilon KR)^{-1} : KR \cong RIKR$.)

Let Σ be a polynomial functor $\Sigma X = \coprod_{i \in J} X^{n_i}$. If both $\mathscr{A}$ and $\mathscr{B}$ have (co)products, then Σ is defined on both categories, so it makes sense to write $R\Sigma \cong \Sigma R$. The following corollary says that for polynomial functors Σ, the adjunction always lifts from the base categories to the categories of Σ-algebras.

Corollary 3.8. Let $\langle R, I, \eta, \epsilon \rangle : \mathscr{A} \rightharpoonup \mathscr{B}$ be an adjunction such that I is full and faithful. Further assume that both categories have coproducts and finite products and that R preserves finite products. Consider an endofunctor $\Sigma X = \coprod_{i \in J} X^{n_i}$ on $\mathscr{A}$ and on $\mathscr{B}$. Then the adjunction lifts to an adjunction $\langle \bar{R}, \bar{I}, \eta, \epsilon \rangle : Alg(\Sigma) \rightharpoonup Alg(\Sigma)$.

Proof. We use item 2 of the remark above and calculate $R\Sigma A = R(\coprod_{i \in J} A^{n_i}) \cong \coprod_{i \in J} (RA)^{n_i} = \Sigma RA$. □

Remark 3.9. $\overline{R}$ preserves finite limits whenever R does.

Proof. Assume $(B, g) = \varprojlim(B_i, g_i)$ is a finite limit in $\mathsf{Alg}(L)$. Since $U_{\mathscr{B}}$ preserves all limits, we have that $B = \varprojlim(B_i)$ in $\mathscr{B}$, and therefore $RB = \varprojlim(RB_i)$. We use (RB_i, f_i) to denote $\overline{R}(B_i, g_i)$ and $\pi_i : RB \to RB_i$ to denote the morphisms of the limiting cone. For each index i, we have a map $p_i : KRB \to RB_i$ obtained as the composition $f_i \circ K\pi_i$. From the universal property, we obtain a map $f : KRB \to RB$ such that each π_i is a K-algebra morphism from (RB, f) to (RB_i, f_i). We prove next that $(RB, f) = \varprojlim(RB_i, f_i)$ in $\mathsf{Alg}(K)$. Assume that we have a cone $q_i : (C, h) \to (RB_i, g_i)$. Since RB is a limit in $\mathscr{A}$, we get a unique map $k : C \to RB$ such that $\pi_i \circ k = q_i$. We need to show that k is a K-algebra morphism. To this end, we will use the uniqueness of a morphism from $KC \to RB$ that makes the relevant diagrams commutative. □

3.2. *HSP theorem for nominal sets and sheaf algebras*

In this section we will prove an HSP theorem for algebras over $\mathrm{Sh}(\mathbb{I}^{op})$. We will call these algebras 'sheaf algebras'. Some of them, given by particular signatures, correspond, in a sense that will be made precise in Section 6.1, to nominal algebras (Gabbay and Mathijssen 2009). The signature will be given by a functor L on $\mathsf{Set}^{\mathbb{I}}$ that has a finitary presentation. On $\mathrm{Sh}(\mathbb{I}^{op})$ we can define the functor $\tilde{L}$ as RLI. Throughout this section we will use the notations from Diagram (2). The goal of this section is to derive an HSP theorem for $\mathsf{Alg}(\tilde{L})$ from Theorem 3.5. To this end, we will need to impose some reasonable conditions on the functor L.

Theorem 3.10 (HSP theorem for 'nominal algebras'). In the situation of Diagram (2), let L be an endofunctor with a finitary presentation on $\mathsf{Set}^{\mathbb{I}}$ and let $\tilde{L}$ be such that $\tilde{L}R \cong RL$. Then a full subcategory of $\mathsf{Alg}(\tilde{L})$ is closed under HSP if and only if it is equationally definable.

Proof. By Theorem 2.6, we have that $\mathsf{Alg}(L)$ is monadic over $\mathsf{Set}^{\mathscr{P}_f(\mathscr{N})}$, so it has a regular factorisation system. By Proposition 3.6, we can lift the adjunction $R \dashv I$ of Proposition 2.1 to an adjunction $\overline{R} \dashv \overline{I}$ between the categories of $\tilde{L}$-algebras and L-algebras. Since $\overline{R}$ preserves finite limits, it preserves monomorphisms, so we can apply Theorem 3.5. □

Remark 3.11. Notice that in the case of algebras over $\mathsf{Set}^{\mathbb{I}}$ we can assume that the equations defining a subcategory closed under HSP involve only finitely many variables. It is enough to prove that if $\mathscr{A} \subseteq \mathsf{Alg}(L)$ is closed under HSP, then $\mathscr{A}$ is also closed under directed unions – see also Remark 3.1. The idea of the proof is to construct the directed union of a directed family $(X_i)_{i \in J}$ of algebras in $\mathscr{A}$, as a homomorphic image of a subalgebra of a product of algebras X_i, as in the proof of (Adámek and Rosický 1994, Theorem 3.9). The subtlety here is that the product considered there may be empty in the general many-sorted case, even if some of the X_i are not. However, in the case of algebras over $\mathsf{Set}^{\mathbb{I}}$, we can see that if one algebra X_{i_0} has the underlying presheaf non-empty, say, for example $X_{i_0}(S) \neq \varnothing$, then for all $j \geqslant i_0$ and for all sets T of cardinality larger than that of S, we also have that $X_j(T)$ is non-empty. We can also consider the product of the X_j for $j \geqslant i_0$, and this is non-empty.

4. Concrete syntax

This section illustrates the concrete syntax obtained from the abstract category theoretic treatment of Section 3. In the first part of this section we give a presentation for the category $\mathsf{Set}^{\mathbb{I}}$. Then we define a 'shift' functor δ on $\mathsf{Set}^{\mathbb{I}}$, which corresponds to the abstraction operator of Gabbay and Pitts (1999) and to the 'shift' functor on $\mathsf{Set}^{\mathbb{F}}$ from Fiore *et al.* (1999). Then we will illustrate the concrete syntax obtained in our setting by giving a theory for the λ-calculus. We consider an endofunctor L on $\mathsf{Set}^{\mathbb{I}}$ given by

$$LX = \mathcal{N} + \delta X + X \times X \tag{17}$$

where $\mathcal{N}$ denotes the inclusion functor $\mathcal{N} : \mathbb{I} \to \mathsf{Set}$. In order to show that the HSP theorem holds for $\tilde{L}$-algebras, we need to prove that L satisfies the conditions in Theorem 3.10. This is actually a particular case of Proposition 5.4, which will be proved in the next section. In Section 4.2 we prove that the sheaf of λ-terms up to α-equivalence is the initial algebra for $\tilde{L} = RLI$, and we give the equations that characterise the subalgebra of λ-terms modulo $\alpha\beta\eta$-equivalence.

4.1. *A presentation for* $\mathsf{Set}^{\mathbb{I}}$ *and* δ

In order to spell out some equations in our setting, we need a presentation for $\mathsf{Set}^{\mathbb{I}}$. The operation symbols should correspond to morphisms that generate all the arrows in $\mathbb{I}$. One might be tempted to use operation symbols of the form $(a,b)_S$, which correspond to swapping the names a and b of a set S, and $w_{S,a}$, which correspond to inclusions of S into $S \cup \{a\}$. However, swappings and inclusions fail to generate all the bijections in $\mathbb{I}$. For example, if $a \neq b$ and $a, b \notin S$, they cannot generate a bijection from $S \cup \{a\}$ to $S \cup \{b\}$ that maps a to b and acts as identity on the remaining elements of S.

This example suggests the following set $\Sigma_{\mathbb{I}}$ of operation symbols with specified arity:

$$\begin{array}{ll} (b/a)_S : S \cup \{a\} \to S \cup \{b\} & a \neq b,\ a \notin S,\ b \notin S \\ w_{S,a} : S \to S \cup \{a\} & a \notin S \end{array} \tag{18}$$

We will refer to operation symbols of the form $(b/a)_S$ as 'substitutions' and to operation symbols of the form $w_{S,a}$ as 'inclusions'. When the arity can be inferred from the context, or is irrelevant, we will omit S from the subscript.

We consider the set $E_{\mathbb{I}}$ of equations of the form:

$$\begin{array}{rcll} (a/b)_S(b/a)_S(x) & = & x & : S \cup \{a\} \quad (\mathrm{E}_1) \\ (b/a)_{S\cup\{d\}}(d/c)_{S\cup\{a\}}(x) & = & (d/c)_{S\cup\{b\}}(b/a)_{S\cup\{c\}}(x) & : S \cup \{b,d\} \quad (\mathrm{E}_2) \\ (c/b)_S(b/a)_S(x) & = & (c/a)_S & : S \cup \{c\} \quad (\mathrm{E}_3) \\ (b/a)_{S\cup\{c\}}w_{S\cup\{a\},c}(x) & = & w_{S\cup\{b\},c}(b/a)_S & : S \cup \{c,b\} \quad (\mathrm{E}_4) \\ (b/a)_S w_{S,a}(x) & = & w_{S,b}(x) & : S \cup \{b\} \quad (\mathrm{E}_5) \\ w_{S\cup\{b\},a}w_{S,b}(x) & = & w_{S\cup\{a\},b}w_{S,a}(x) & : S \cup \{a,b\} \quad (\mathrm{E}_6) \end{array} \tag{19}$$

Theorem 4.1. $(\Sigma_{\mathbb{I}}, E_{\mathbb{I}})$ is a presentation for $\mathsf{Set}^{\mathbb{I}}$.

We will now define a 'shift' functor δ on $\mathsf{Set}^{\mathbb{I}}$. Assume $P : \mathbb{I} \to \mathsf{Set}$ is a presheaf and $S \subseteq \mathcal{N}$ is a finite set of names. We define an equivalence relation $\equiv$ on $\coprod_{a \notin S} P(S \cup \{a\})$. If $a, b \notin S$, $x \in P(S \cup \{a\})$ and $y \in P(S \cup \{b\})$, we will say that x and y are equivalent if and only if $P((b/a)_S)(x) = y$. We define $(\delta P)(S)$ as the set of equivalence classes of the elements of $\coprod_{a \notin S} P(S \cup \{a\})$. If $x \in P(S \cup \{a\})$ the equivalence class of x is denoted by $\overline{x}^{S,a}$.

If $f : S \to T$ is a morphism in $\mathbb{I}$ and $a \notin S \cup T$, then $f + a : S \cup \{a\} \to T \cup \{a\}$ denotes the function that restricted to S is f and that maps a to a. We define

$$(\delta A)(f)(\overline{x}^{S,a}) = \overline{A(f + a)(x)}^{T,a}$$

for some $a \notin S$. It is easy to check that $(\delta A)(f)$ is well defined and that δ is a functor.

We will now give a presentation for δ. For each finite subset of names $S \subseteq \mathcal{N}$ and for each $a \notin S$, we consider an operation symbol $[a]_S : S \cup \{a\} \to S$, and will use Σ_δ to denote the corresponding functor on $\mathsf{Set}^{\mathscr{P}_f(\mathcal{N})}$. This is given by

$$(\Sigma_\delta X)_S = \coprod_{a \notin S} \{[a]_S\} \times X_{S \cup \{a\}}.$$

We use $U : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathscr{P}_f(\mathcal{N})}$ to denote the forgetful functor and F to denote its left adjoint. For any functor $P : \mathbb{I} \to \mathsf{Set}$, we can give an interpretation of these operation symbols captured by a natural transformation $\rho_P : \Sigma_\delta UP \to U\delta P$ defined by

$$\forall \alpha \in P(S \cup \{a\}) \qquad ([a]_S, \alpha) \mapsto \overline{\alpha}^{S,a} \in (U\delta P)(S).$$

The equations should correspond to the kernel pair of the adjoint transpose $\rho_P^\sharp : F\Sigma_\delta UP \to \delta P$, as in Definition 2.5. We will use the fact that for any $X = (X_S)_{S \in \mathscr{P}_f \mathcal{N}}$ we have

$$(FX)_S = \coprod_{T \in \mathscr{P}_f(\mathcal{N})} X_T \cdot \hom(T, S),$$

where $\cdot$ is the copower. For $f : T \to S$ and $x \in X_T$, we use fx to denote the element of $(FX)_S$ that is the copy of f corresponding to x. The equations E_δ will have the form:

$$\begin{aligned} (c/b)_S [a]_{S \cup \{b\}} t &= [a]_{S \cup \{c\}} (c/b)_{S \cup \{a\}} t && t : S \cup \{a, b\} \\ [a]_S t &= [b]_S (b/a)_S t && t : S \cup \{a\} \\ w_{S,b} [a]_S t &= [a]_{S \cup \{b\}} w_{S \cup \{a\}, b} t && t : S \cup \{a\}. \end{aligned} \tag{20}$$

Theorem 4.2. $(\Sigma_\delta, E_\delta)$ is a presentation for δ.

Notation 4.3. We will use $\{[a]_S \alpha\}_{\delta P}$ to denote the element $\overline{\alpha}^{S,a} \in (\delta P)(S)$.

4.2. *Axioms for the λ-calculus*

The α-equivalence classes of λ-terms over $\mathcal{N}$ form a sheaf Λ_α in $\mathrm{Sh}(\mathbb{I}^{op})$. Indeed, we can define $\Lambda_\alpha(S)$ as the set of α-equivalence classes of λ-terms with free variables in S. On functions, Λ_α acts by renaming the free variables.

We now consider the endofunctor L on $\mathsf{Set}^{\mathbb{I}}$ defined by (17) and the endofunctor $\tilde{L}$ on $\mathrm{Sh}(\mathbb{I}^{op})$ defined as RLI. In a similar fashion to Fiore *et al.* (1999), we will show that Λ_α is isomorphic to the initial algebra $\mathscr{I}_{\tilde{L}}$ for $\tilde{L}$.

First note that the underlying presheaf of $\mathscr{I}_{\tilde{L}}$ is the initial algebra $\mathscr{I}_L$ for L. Indeed, one can prove the following lemma.

Lemma 4.4. We have $I\mathscr{I}_{\tilde{L}} = \mathscr{I}_L$.

Proof. We can check that $\tilde{L}$ preserves ω-chains, so the initial algebra $\mathscr{I}_{\tilde{L}}$ is computed as the colimit of the sequence

$$\tilde{0} \to \tilde{L}\tilde{0} \to \tilde{L}^2\tilde{0} \to \cdots \to \mathscr{I}_{\tilde{L}} \tag{21}$$

where $\tilde{0}$ is just the empty sheaf. We will use 0 to denote the empty presheaf. Similarly, $\mathscr{I}_L$ is the colimit of the initial sequence for L:

$$0 \to L0 \to L^2 0 \to \cdots \to \mathscr{I}_L. \tag{22}$$

Using the observation that L preserves sheaves and the fact that $I\tilde{0} = 0$, we can easily verify that $I\tilde{L}^n\tilde{0} \simeq L^n 0$ for all natural numbers n. But I preserves filtered colimits, so we have $I\mathscr{I}_{\tilde{L}} \simeq \mathscr{I}_L$. □

We consider a functor $\Sigma : \mathsf{Set}^{\mathbb{I}} \to \mathsf{Set}^{\mathbb{I}}$ defined by

$$\Sigma X = \mathscr{N} + \mathscr{N} \times X + X \times X. \tag{23}$$

Notice that Σ preserves sheaves and that the initial algebra for Σ, which we denote by $\mathscr{I}_\Sigma$, is just the presheaf of all λ-terms. Using a similar argument to the above, we can see that $I\mathscr{I}_{\tilde{\Sigma}} = \mathscr{I}_\Sigma$. $\mathscr{I}_{\tilde{\Sigma}}$ is the sheaf of all λ-terms. We will prove the isomorphism between Λ_α and $\mathscr{I}_{\tilde{L}}$ by constructing an epimorphism $\mathscr{I}_{\tilde{\Sigma}} \to \mathscr{I}_{\tilde{L}}$ in $\mathrm{Sh}(\mathbb{I}^{op})$ that identifies exactly α-equivalent terms.

First we will prove the following lemma.

Lemma 4.5. There exists a natural transformation $\theta : \mathscr{N} \times - \to \delta$ such that for any presheaf $X \in \mathrm{Sh}(\mathbb{I}^{op})$ and any finite set of names $S \subseteq \mathscr{N}$, we have that $\theta(a,x) = \theta(b,y)$ for some $a, b \in S$ and $x, y \in X(S)$ if and only if $X(\sigma_{a,b})(x) = y$. Moreover, if X is a sheaf, then θ_X is a sheaf epimorphism.

Proof. We define $\theta_X(S) : \mathscr{N}(S) \times X(S) \to (\delta X)(S)$ by

$$(a,x) \mapsto (\delta X)(w_{S\setminus\{a\},a})(\{[a]_{S\setminus\{a\}}x\}_{\delta X}) \tag{24}$$

where $w_{S\setminus\{a\},a}$ is the inclusion of $S\setminus\{a\}$ into S. It is not difficult to check that this is indeed a natural transformation. Assume now that $(a,x), (b,y) \in \mathscr{N}(S) \times X(S)$ and $c \in \mathscr{N} \setminus S$. We have that $\theta(a,x) = \theta(b,x)$ is equivalent to

$$(\delta X)(w_{S\setminus\{a\},a})(\{[a]_{S\setminus\{a\}}x\}_{\delta X}) = (\delta X)(w_{S\setminus\{b\},b})(\{[b]_{S\setminus\{b\}}y\}_{\delta X}). \tag{25}$$

But

$$\begin{aligned}(\delta X)(w_{S\setminus\{a\},a})(\{[a]_{S\setminus\{a\}}x\}_{\delta X}) &= (\delta X)(w_{S\setminus\{a\},a})(\{[c]_{S\setminus\{a\}}(c/a)_{S\setminus\{a\}}x\}_{\delta X})\\ &= \{[c]_S X(w_{S\setminus\{a\},a}+c)((c/a)_{S\setminus\{a\}}(x))\}_{\delta X}\\ &= \{[c]_S X(w_{S\setminus\{a\}\cup\{c\},a})((c/a)_{S\setminus\{a\}}(x))\}_{\delta X}.\end{aligned} \tag{26}$$

Similarly,

$$(\delta X)(w_{S\setminus\{b\},b})(\{[b]_{S\setminus\{b\}}y\}_{\delta X}) = \{[c]_S X(w_{S\setminus\{b\}\cup\{c\},b})((c/b)_{S\setminus\{b\}}(y))\}_{\delta X}.$$

Therefore (25) is equivalent to

$$\{[c]_S X(w_{S\setminus\{a\}\cup\{c\},a})((c/a)_{S\setminus\{a\}}(x))\}_{\delta X} = \{[c]_S X(w_{S\setminus\{b\}\cup\{c\},b})((c/b)_{S\setminus\{b\}}(y))\}_{\delta X}.$$

Using

$$w_{S\setminus\{a\}\cup\{c\},a} = w_{S\setminus\{b\}\cup\{c\},b}(a/b)_{S\setminus\{a,b\}\cup\{c\}},$$

we can show that (25) is equivalent to

$$X(w_{S\setminus\{b\}\cup\{c\},b})X((a/b)_{S\setminus\{a,b\}\cup\{c\}}(c/a)_{S\setminus\{a\}})(x) = X(w_{S\setminus\{b\}\cup\{c\},b})X((c/b)_{S\setminus\{b\}})(y)).$$

Since X is a sheaf, it preserves monomorphisms, so we have

$$X((a/b)_{S\setminus\{a,b\}\cup\{c\}}(c/a)_{S\setminus\{a\}})(x) = X((c/b)_{S\setminus\{b\}})(y)),$$

or, equivalently,

$$X((b/c)_{S\setminus\{b\}}(a/b)_{S\setminus\{a,b\}\cup\{c\}}(c/a)_{S\setminus\{a\}})(x) = y,$$

which means that $X(\sigma_{a,b})(x) = y$.
In order to prove the last statement of the lemma, we use the characterisation of sheaf epimorphisms given in Proposition 2.3. Let $\{[c]_S y\}_{\delta X}$ be an arbitrary element of $(\delta X)(S)$. We have that $c \notin S$ and $y \in X(S \cup \{c\})$. The conclusion follows from the fact that $\theta_X(S \cup \{c\})(c, y) = (\delta X)(w_{S,c})(\{[c]_S y\}_{\delta X})$. □

Proposition 4.6. The sheaf of α-equivalence classes of λ-terms is isomorphic to the initial $\tilde{L}$-algebra $\mathscr{I}_{\tilde{L}}$.

Proof. We use an inductive argument on the structure of the λ-terms. Using the natural transformation θ defined above, we can construct a natural transformation $\vartheta : \Sigma \to L$ defined as $\vartheta_X = \mathrm{id}_{\mathcal{N}} + \theta + \mathrm{id}_X \times \mathrm{id}_X$. Now we can inductively define a natural transformation $\zeta^{(n)} : \Sigma^n \to L^n$. Explicitly,

$$\begin{aligned} \zeta^{(0)} &= \mathrm{id}_0 \\ \zeta^{(n+1)}_X &= L^n(\vartheta_X)\zeta^{(n)}_{\Sigma X}. \end{aligned}$$

We have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\zeta^{(0)}"] & \Sigma 0 \arrow[r] \arrow[d, "\zeta^{(1)}"] & \Sigma^2 0 \arrow[r] \arrow[d, "\zeta^{(2)}"] & \cdots \arrow[r] & \mathscr{I}_\Sigma \arrow[d, "\zeta"] \\ 0 \arrow[r] & L0 \arrow[r] & L^2 0 \arrow[r] & \cdots \arrow[r] & \mathscr{I}_L \end{tikzcd} \qquad (27)$$

where ζ is obtained by taking the colimit. As seen above, $\mathscr{I}_L$ and $\mathscr{I}_\Sigma$ are the underlying presheaves for the sheaves $\mathscr{I}_{\tilde{L}}$ and $\mathscr{I}_{\tilde{\Sigma}}$, respectively.

Using Lemma 4.5, we can argue inductively that $\zeta^{(n)}_X$ is a sheaf epimorphism for all n and for all sheaves X. One can verify that this implies that ζ is a sheaf epimorphism. If two terms in $\mathscr{I}_\Sigma$ are identified by ζ, they must be identified at some stage n by $\zeta^{(n)}$. Using

Lemma 4.5 again, we can show by induction that two terms are in the kernel of $\zeta^{(n)}$ if and only if they are α-equivalent. $\square$

To illustrate the concrete syntax appearing in our setting, we give a presentation for the functor L and a theory over the signature given by L for $\alpha\beta\eta$-equivalence of λ-terms.

Proposition 4.7. The endofunctor L is presented by a set of operation symbols

$$\begin{aligned} a_S &: S \cup \{a\} \\ \mathsf{app}_S &: S \times S \to S \\ [a]_S &: S \cup \{a\} \to S \end{aligned}$$

where S is a finite set of names and $a \notin S$, and the following set of equations:

$$\begin{aligned}
(b/a)_S a_S &= b_S && && (\mathrm{E}_0)\\
w_{S\cup\{a\},b} a_S &= a_{S\cup\{b\}} && && (\mathrm{E}_1)\\
(c/b)_S([a]_{S\cup\{b\}}t) &= \mathrm{id}_{S\cup\{c\}}([a]_{S\cup\{c\}}(c/b)_{S\cup\{a\}}t) && t : S\cup\{a,b\} && (\mathrm{E}_2)\\
[a]_S t &= [b]_S(b/a)_S t && t : S\cup\{a\} && (\mathrm{E}_3)\\
w_{S,b}[a]_S t &= [a]_{S\cup\{b\}} w_{S\cup\{a\},b} t && t : S\cup\{a\} && (\mathrm{E}_4)\\
w_{S,a}\mathsf{app}_S(t_1,t_2) &= \mathsf{app}_{S\cup\{a\}}(w_{S,a}t_1, w_{S,a}t_2) && t1, t_2 : S && (\mathrm{E}_5)\\
(b/a)_S\mathsf{app}_{S\cup\{a\}}(t_1,t_2) &= \mathsf{app}_{S\cup\{b\}}((b/a)_S t_1, (b/a)_S t_2) && t1, t_2 : S\cup\{a\}. && (\mathrm{E}_6)
\end{aligned}$$

Example 4.8. The subalgebra of $\mathsf{Alg}(\tilde{L})$ of λ-terms modulo $\alpha\beta\eta$-equivalence is definable by the following equations, which are similar to those in Clouston and Pitts (2007, Figure 4):

$$\begin{aligned}
X : S;\ Y : S &\vdash \mathsf{app}_S([a]_S(w_a Y), X) = Y \ : S && (\beta\text{-}1)\\
X : S &\vdash \mathsf{app}_S([a]_S(a_S), X) = X \ : S && (\beta\text{-}2)\\
X : S\cup\{a,b\};\ Y : S &\vdash \mathsf{app}_S([a]_S([b]_{S\cup\{a\}}(X)), Y) = \\
&\qquad [b]_S(\mathsf{app}_{S\cup\{b\}}([a]_{S\cup\{b\}}(X), w_a Y)) \ : S && (\beta\text{-}3)\\
X, Y : S\cup\{a\};\ Z : S &\vdash \mathsf{app}_S([a]_S(\mathsf{app}_{S\cup\{a\}}(X, Y)), Z) = \\
&\qquad \mathsf{app}_S(\mathsf{app}_S([a]_S(X), Z), \mathsf{app}_S([a]_S(Y), Z)) \ : S && (\beta\text{-}4)\\
X : S\cup\{a\} &\vdash \mathsf{app}_{S\cup\{b\}}(w_b[a]_S(X), b_S) = (b/a)X \ : S\cup\{b\} && (\beta\text{-}5)\\
X : S &\vdash [a]_S(\mathsf{app}_{S\cup\{a\}}(w_a X, a_S)) = X \ : S. && (\eta)
\end{aligned}$$

5. Uniform theories

Gabbay (2008) proved the following HSP (or, rather, HSPA) theorem for nominal algebras: a class of nominal algebras is definable by a theory of nominal algebra if and only if it is closed under HSP and under abstraction.

Our equational logic is more expressive than Gabbay's in the sense that more classes are equationally definable, namely all those closed under HSP where H refers not to closure under all quotients as in Gabbay (2008), but to the weaker property of closure under support-preserving quotients (that is, quotients in the presheaf category). Of course, one might ask whether this additional expressivity is wanted. We therefore isolate a fragment of standard equational logic, which we call the *uniform fragment*, and define notions of uniform signature, uniform terms and uniform equations. The main idea is that a uniform equation $t = u : T$, for T a finite subset of $\mathcal{N}$, has an interpretation uniform in all sorts S containing T.

For this uniform fragment, we are able to extend Theorem 3.11 to an HSPA theorem in the style of [9]: classes of sheaf algebras are definable by uniform equations if and only if they are closed under quotients, subalgebras and products, and under abstraction.

We begin with the observation that the theory of the λ-calculus up to $\alpha\beta\eta$-equivalence (Example 4.8) only uses particular operations: names (atoms in Gabbay (2008)), abstraction and operations $f_S : A^n(S) \to A(S)$ that are 'uniform' in S. This motivates us to consider sheaf algebras for signatures given by a particular class of functors, and specified by 'uniform' equations. Throughout this section we will assume that the endofunctor L on $\mathsf{Set}^{\mathbb{I}}$ is presented by a set of operation symbols that form a presheaf in $\mathsf{Set}^{\mathbb{I}}$, say $\mathcal{O}$, and such that all the operation symbols in $\mathcal{O}(T)$ have arity of the form $T \times \cdots \times T \to T'$, for some finite subset $T' \subseteq T$. We will use the notation

$$\mathsf{bind}(f) = T \setminus T'. \tag{28}$$

Additionally, we assume that if an operation symbol $f_T \in \mathcal{O}(T)$ has arity $T \times \cdots \times T \to T'$ and $j : T \to S$ is an injective map, then $\mathcal{O}(j)(f_T)$ has arity $S \times \cdots \times S \to S \setminus j[\mathsf{bind}(f)]$, where $j[\mathsf{bind}(f)]$ denotes the direct image. For an injective map $j : T \to S$, we write $j \bullet f_T$ for $\mathcal{O}(j)(f_T)$. For simplicity, if $j : T \to S$ is an inclusion, we will simply write f_S for $j \bullet f_T$. For example, $f_{T\cup\{a\}}$ stands for $w_a \bullet f_T$.

Definition 5.1. We will say that L has a *uniform presentation* if L is presented by operation symbols forming a presheaf as above, and a set of equations containing the following:

$$\begin{aligned} w_a f_T(x_1,\dots,x_n) &= f_{T\cup\{a\}}(w_a x_1,\dots,w_a x_n) \\ f_T(x_1,\dots,x_n) &= ((b/a)_{T\setminus\{a\}} \bullet f_T)((b/a)_{T\setminus\{a\}} x_1,\dots,(b/a)_{T\setminus\{a\}} x_n) \\ (b/a)_{T'\setminus\{a\}} f_T(x_1,\dots,x_n) &= ((b/a)_{T\setminus\{a\}} \bullet f_T)((b/a)_{T\setminus\{a\}} x_1,\dots,(b/a)_{T\setminus\{a\}} x_n) \end{aligned} \tag{29}$$

where in the second equation $a \in \mathsf{bind}(f)$ and $b \notin T$, while in the third $a \in T'$ and $b \notin T$.

Intuitively, the equations in this definition state that the operations are 'equivariant'. If X is a presheaf, elements of $LX(T')$ will be denoted by $\{f_T(x_1,\dots,x_n)\}_X$, where $x_1,\dots,x_n \in X(T)$ and f_T has arity $T \times \cdots \times T \to T'$.

Remark 5.2. From the first and third equations of Definition 5.1, we can deduce that

$$L(i)(\{f_T(x_1,\dots,x_n)\}_X) = \{((i+\mathrm{id}) \bullet f_T)(L(i+\mathrm{id})(x_1),\dots,L(i+\mathrm{id})(x_n)\}_X,$$

provided $i : T' \to U$ is an injective map, $U \cap \mathsf{bind}(f) = \varnothing$ and id is the identity map on $\mathsf{bind}(f)$.

Example 5.3.

1 The functor δ has a uniform presentation, with the operation symbols given in Section 4.1 structured as a presheaf as follows:

$$\begin{aligned} [a]_S &\in \mathcal{O}(S \cup \{a\}) \\ \mathcal{O}(w_b)([a]_S) &= [a]_{S\cup\{b\}} \\ \mathcal{O}((b/a)_S)([a]_S) &= [b]_S. \end{aligned}$$

We have that $\mathsf{bind}([a]_S) = \{a\}$, so the equations (20) are of the form (29).

2 The presentation of the functor used for the axiomatisation of λ-calculus, defined in (17), is uniform too. Indeed, the equations appearing in Proposition 4.7 are of the form (29) because $\mathsf{bind}(a_S) = \mathsf{bind}(\mathsf{app}_S) = \emptyset$.
3 More generally, functors constructed from $\mathcal{N}$, $+$, $\times$ and δ have uniform presentations.

Proposition 5.4. If L has a uniform presentation, then:

1 there exists a natural transformation $\alpha : LIR \to IRL$ such that the following diagram commutes:

$$\begin{tikzcd} LIR \arrow[rr, "\alpha"] & & IRL \\ & L \arrow[ul, "L\eta"] \arrow[ur, "\eta L"'] & \end{tikzcd} \tag{30}$$

2 $\tilde{L} = RLI$ preserves sheaf epimorphisms.

Proof. Recall Remark 2.4 for the notation used for sheafifcation. We need to define a natural transformation α making diagram (30) commutative. Let X be a presheaf. Elements of $LIRX(T')$ will be of the form $\{f_T(\overline{x_1}, \ldots, \overline{x_n})\}_{IRX}$, where we can assume, without loss of generality, that $x_1, \ldots, x_n$ are elements supported by T of the same set $X(S)$, for some $T \subseteq S$.

We put

$$\alpha_{T'}(\{f_T(\overline{x_1}, \ldots, \overline{x_n})\}_{IRX}) = \overline{\{f_S(x_1, \ldots, x_n)\}_X}.$$

In order to show that this is well defined, we have to prove that $\{f_S(x_1, \ldots, x_n)\}_L$ is supported by T'. Let $i, j : S \setminus \mathsf{bind}(f) \to U$ be two injective maps that agree on T'. We have to show that

$$L(i)(\{f_S(x_1, \ldots, x_n)\}_X) = L(j)(\{f_S(x_1, \ldots, x_n)\}_X).$$

If $U \cap \mathsf{bind}(f) = \emptyset$, this follows easily from Remark 5.2 and the fact that the x_k are supported by T. Otherwise, say, for example, if $U \cap \mathsf{bind}(f) = \{a\}$, we can apply the second equation of Definition 5.1 for some name $b \notin S \cup U$:

$$\{f_S(x_1, \ldots, x_n)\}_X = \{\mathcal{O}((b/a)_{S\setminus\{a\}})(f_S)((b/a)_{S\setminus\{a\}}x_1, \ldots, (b/a)_{S\setminus\{a\}}x_n)\}_X$$

and we can use Remark 5.2 again, plus the fact that $(b/a)_{S\setminus\{a\}}x_k$ is supported by $T \setminus \{a\} \cup \{b\}$.

It is now easy to see that α makes diagram (16) commutative. We still need to check that $\tilde{L}$ preserves sheaf epimorphisms. As noted in Remark 3.7, we know that $LIR \simeq IRLIR$, and this actually means that L preserves sheaves. Therefore, it is enough to prove that whenever $e : X \to Y$ is a sheaf epimorphism, $LIe : LIX \to LIY$ has the property stated in Proposition 2.3. Let $y = \{f_S(y_1, \ldots, y_n)\}_L$ be an element in $(LIY)(S')$, for some operation symbol $f_S : S \times \cdots \times S \to S'$ and $y_1, \ldots, y_n \in Y(S)$. We prove that there exists an inclusion $w' : S' \to T'$ and $x \in LIX(T')$ such that $LIY(w')(y) = (LIe)_{T'}(x)$. Because $e : X \to Y$ is a sheaf epimorphism, there exists an inclusion $w : S \to T$ and $x_k \in X(T)$

$$\frac{t_1 : T, \; \ldots, \; t_n : T}{f(t_1, \ldots t_n) : T'} \quad (f : T \times \cdots \times T \to T' \in \mathcal{O}(T))$$

$$\frac{t : T}{w_a t : T \uplus \{a\}} \qquad \frac{t : T \uplus \{a\}}{(b/a)t : T \uplus \{b\}} \qquad \frac{}{X : T_X}$$

Fig. 1. Uniform terms

for all $1 \leqslant k \leqslant n$ such that $Y(w)(y_k) = e_T(x_k)$ for all k. Let w' denote the inclusion of S' into $T' = T \setminus (S \setminus S')$, and let $x \in LIX(T')$ be $\{f_T(x_1, \ldots, x_n)\}_{LX}$. Using the first equation of Definition 5.1, we can derive $LIY(w')(y) = (LIe)_{T'}(x)$. □

Corollary 5.5. If L has a uniform presentation, then the HSP theorem (Theorem 3.10) holds for $\tilde{L}$-algebras.

Looking at the theory of the λ-calculus in Example 4.8, we find that all operations are equivariant and that the equations are 'uniform' in S. To formalise the uniformity of an equational specification, we first describe uniform terms, given by the set of rules in Figure 1.

Figure 1 shows four rule schemas: one for each operation $f : T \times \cdots \times T \to T'$, two for the operations in $\mathbb{I}$ (weakenings, substitutions), and one for variables. Each rule can be instantiated in an infinite number of ways: T ranges over finite sets of names and a, b over names. The notation $T \uplus \{a\}$ indicates that an instantiation of the schema is only allowed for those sets T and those atoms a where $a \notin T$.

Remark 5.6. The rule for operations $f(t_1, \ldots t_n)$ requires all arguments to be of the same type. This can be achieved by applying weakenings.

Definition 5.7. A signature given by a functor with a uniform presentation is said to be a **uniform signature**. A **uniform term** $t : T$ for a uniform signature is a term t of type T formed according to the rules in Figure 1. A **uniform equation** is a pair of uniform terms of the same sort $u = v : T$ such that any variable X appears with the same type T_X in both u and v. A **uniform theory** consists of a set of uniform equations.

Example 5.8. A uniform theory for λ-calculus consists of the following uniform equations in the uniform signature given by the functor $\delta + \mathcal{N} \times - + - \times -$:

$$\begin{aligned}
\mathsf{app}([a](w_a Y), X) &= Y \ : \varnothing && (\beta\text{-}1)\\
\mathsf{app}([a]a, X) &= X \ : \varnothing && (\beta\text{-}2)\\
\mathsf{app}([a]([b]_{\{a\}}(X)), Y) &= [b](\mathsf{app}_{\{b\}}([a]_{\{b\}}(X), w_a Y)) \ : \varnothing && (\beta\text{-}3)\\
\mathsf{app}([a](\mathsf{app}_{\{a\}}(X, Y)), Z) &= \mathsf{app}(\mathsf{app}([a](X), Z), \mathsf{app}([a](Y), Z)) \ : \varnothing && (\beta\text{-}4)\\
\mathsf{app}_{\{b\}}(w_b[a](X), b) &= (b/a)X \ : \{b\} && (\beta\text{-}5)\\
[a](\mathsf{app}_{\{a\}}(w_a X, a)) &= X \ : \varnothing && (\eta)
\end{aligned}$$

where app, $[a]$ and a stand for $\mathsf{app}_\varnothing$, $[a]_\varnothing$ and $a_\varnothing$, respectively.

The idea is that a uniform equation $u = v : T$ translates to a set of equations in the sense of standard many-sorted universal algebra: $u_S = v_S : T \cup S$ where S ranges over the finite subsets of $\mathcal{N}$ with $S \cap T = \emptyset$. If we want to extend the sort of the equation, we might also have to change the sort of the variables. There is a subtlety here: do we raise the type of the variables or do we add weakenings? We prefer the former if, for example, we want to raise the type of the equation $X = Y\ : \emptyset$ by a set S. This becomes $X_S = Y_S : S$, where X_S, Y_S have type S.

Similarly, if we want to translate the equation (in which $X : \emptyset$ has type $\emptyset$)

$$[a]w_a X = X : \emptyset \tag{31}$$

by a set $\{b\}$, where b is a different name than a, we should get

$$[a]w_a X_{\{b\}} = X_{\{b\}}$$

with $X_{\{b\}}$ a variable of sort $\{b\}$. However, we should be able to translate (31) to a standard equation of sort $\{a\}$. We expect all the appearances of X within the translated equation to have the same sort. If, as above, we change the sort of X from $\emptyset$ to $\{a\}$, then on the left-hand side we would get $[a]w_a X_{\{a\}}$, and this does not match the arity of w_a. In this example, the left-hand side of the equation has an implicit freshness constraint on the variable X. Because of the weakening w_a appearing in front of X, we will not be able to instantiate X with elements whose sorts contain a. So a is 'fresh' for X. The solution is to define the translation of this equation as

$$w_a[a]w_a X_\emptyset = w_a X_\emptyset : \{a\}$$

So we have to distinguish between the cases when we simply need to add some weakenings and the cases when we have to extend the sort of the variable. We formalise these observations in the following definitions.

Definition 5.9. The **freshness set** of a variable X appearing with sort T_X in an equation E of the form $u = v : T$ is the set

$$\mathsf{Fr}_E(X) = \bigcup_{t:T} T \setminus T_X$$

where the union is taken over all sub-terms t of either u or v that contain the variable X.

Example 5.10. In the following uniform equation for the uniform signature given by the functor defined in (17), X has type $\emptyset$ and $\mathsf{Fr}_E(X) = \{a\}$:

$$[a]_\emptyset \mathsf{app}_{\{a\}}(w_a X, a_\emptyset) = X : \emptyset. \tag{32}$$

Definition 5.11. The translation of an equation E of the form $u = v : T_E$ by a finite set S, disjoint from T_E, is $u_S = v_S : T \cup S$, where the $u_S = tr_S\,(u)$, $v_S = tr_S\,(v)$ and translation $tr_S\,(t : T)$ of a sub-term t of either u or v is defined by

$$\begin{array}{rcll} tr_S\,(f_T(t_1,\ldots,t_n) : T') &=& w_{a_1}\ldots w_{a_k} f_{T\cup S}(tr_{S\setminus T}\,(t_1),\ldots,tr_{S\setminus T}\,(t_n)) & \\ tr_S\,(w_a t : T \uplus \{a\}) &=& w_{S\cup T,a}\, tr_S\,(t : T) & \\ tr_S\,((b/a)t : T \uplus \{b\}) &=& (b/a)_{S\cup T}\, tr_S\,(t : T \uplus \{a\}) & \text{if } a \notin S \\ tr_S\,((b/a)t : T \uplus \{b\}) &=& w_a (b/a)_{S\setminus\{a\}\cup T}\, tr_{S\setminus\{a\}}\,(t : T \uplus \{a\}) & \text{if } a \in S \\ tr_S\,(X : T_X) &=& w_{a_1}\ldots w_{a_k} X_{T_X \cup S \setminus \mathsf{Fr}_E(X)} & \end{array} \tag{33}$$

where in the first condition f_T has arity $T \times \cdots \times T \to T'$ and $\{a_1, \ldots, a_k\} = S \cap \mathsf{bind}(f)$, see (28). In the last condition, $\{a_1, \ldots, a_k\} = \mathsf{Fr}_E(X) \cap S$ and $X_{T_X \cup S \setminus \mathsf{Fr}_E(X)}$ is a variable of sort $T_X \cup S \setminus \mathsf{Fr}_E(X)$.

Remark 5.12. We only define $tr_S\,(t : T)$ for finite sets S such that $S \cap T = \varnothing$. The above definition is sound because we initially chose a set S disjoint from T_E, and then we can prove inductively that whenever we compute $tr_{S'}\,(t : T')$, we have $S' \cap T' = \varnothing$.

Notice that the translation $tr_S\,(t)$ depends on the equation for which t is a sub-term, and if t has sort T, then the sort of $tr_S\,(t)$ is $S \cup T$. In the first condition of (33), the set S is disjoint from T', but may contain elements of $\mathsf{bind}(f)$.

Example 5.13. For $b \neq a$, the translation by a set $\{b\}$ of the uniform equation (32) is

$$[a]_{\{b\}}\mathsf{app}_{\{a,b\}}(w_a X_{\{b\}}, a_{\{b\}}) = X_{\{b\}} \;: \{b\}.$$

But the translation of the same equation by $\{a,b\}$ is

$$w_a[a]_{\{b\}}\mathsf{app}_{\{a,b\}}(w_a X_{\{b\}}, a_{\{b\}}) = w_a X_{\{b\}} \;: \{a,b\}.$$

We can do this translation because the set $\{a,b\}$ is disjoint from the type of the uniform equation, which is the empty set. On the left-hand side we use the weakening w_a because $\{a,b\} \cap \mathsf{bind}([a]_\varnothing) = \{a\}$. On the right-hand side we use the weakening w_a because $\mathsf{Fr}_E(X) \cap \{a,b\} = \{a\}$. Also note that there is no connection between the variables $X, X_{\{a\}}$ and $X_{\{a,b\}}$ appearing in the equations (32) and its translations by $\{a\}$ and $\{a,b\}$, respectively. In particular, we should emphasise that we do not have $w_a X = X_{\{a\}}$.

Example 5.14. Translating the uniform theory given in Example 5.8, we get a standard many-sorted theory equivalent to that given in Example 4.8. We appear to get more equations because, for instance, if we translate the (β-1) equation by the set $\{a\}$ we get the equation

$$\mathsf{app}_{\{a\}}(w_a[a]w_a Y, X_{\{a\}}) = w_a Y.$$

This does not appear among the equations of Example 4.8, but it can be derived from them.

The above definition of $tr_S\,(X : T_X)$ is also justified by the next property that one expects for the set of standard equations obtained from a uniform equation.

Lemma 5.15. Let E be a uniform equation $u = v : T$ in which the variable X appears with sort T_X, and let S be a finite set of names disjoint from T. All the occurrences of X in the standard equation $u_S = v_S : T \cup S$ have the same sort $T_X \cup S \setminus \mathsf{Fr}_E(X)$.

Proof. Note that, while applying the algorithm described in Definition 5.11, the subscript of tr may change as we traverse the syntax tree of the terms. Explicitly, it will diminish by elements of $\mathsf{bind}(f)$ when we reach a term $f_T(t_1, \ldots, t_n)$ or by $\{a\}$ if we reach $(b/a)t$. So, we may have to evaluate $tr_{S'}\,(X : T_X)$ for some $S' \subseteq S$. We have to prove that $T_X \cup S' \setminus \mathsf{Fr}_E(X) = T_X \cup S \setminus \mathsf{Fr}_E(X)$. It suffices to show that $S \setminus S' \subseteq T_X \cup \mathsf{Fr}_E(X)$. Let a be a name in $S \setminus S'$. Then there exists a term t containing X such that either $(b/a)t$ or $f_T(t_1, \ldots, t, \ldots, t_n)$, for some operation symbol f_T with $a \in \mathsf{bind}(f)$, is a subterm of either

u or v. In both cases, t is a sub-term within the equation E that contains X and such that $S' \cup \{a\}$ is included in the sort of t. Using Definition 5.9, we either have $a \in T_X$ or $a \in \mathsf{Fr}_E(X)$. □

Notation 5.16. A sheaf algebra $\mathbb{A}$ for a uniform signature given by a functor L is an L-algebra $\alpha : LA \to A$. In subsequent proofs and calculations, we will use the following notation. For $f_T \in \Sigma$ having arity $T \times \cdots \times T \to T'$, α maps $(x_1, \ldots x_n) \in A^n(T)$ to $f_T^{\mathbb{A}}(x_1, \ldots x_n) \in A(T')$. For each algebra $\mathbb{A}$ and each valuation v sending variables $X : T_X$ to elements of $A(T_X)$, a term $t : T$ of type T evaluates to an element $[\![t]\!]_{\mathbb{A},v,T}$ in $A(T)$.

Definition 5.17. An algebra $\mathbb{A}$ satisfies the uniform equation $t = u : T$ if and only if for all $S \cap T = \varnothing$ and all valuations v of variables, we have $\mathbb{A}, v \models t_S = u_S$, that is, $[\![t]\!]_{\mathbb{A},v,S\cup T}$ and $[\![u]\!]_{\mathbb{A},v,S\cup T}$ denote the same element of $A(S \cup T)$.

In the remainder of this section we are going to show that classes of sheaf algebras defined by uniform equations are precisely those closed under sheaf quotients, sub-algebras, products and abstraction. In our setting, abstraction (which corresponds to atoms-abstraction (Gabbay 2008)) maps an algebra with carrier A to an algebra with carrier δA. To describe this notion, we need to recall the definition of δ from Section 4.1. For $c \notin S$, there is an isomorphism

$$\begin{aligned} A(S \cup \{c\}) &\to \delta A(S) \\ x &\mapsto \{[c]_S x\}_{\delta A}. \end{aligned} \tag{34}$$

Definition 5.18. Given a nominal algebra $\mathbb{A}$ for a uniform signature with structure $LA \to A$, its abstraction $\delta\mathbb{A}$ with structure $L(\delta A) \to \delta A$ is given by

$$f_T^{\delta\mathbb{A}}(\{[c]x_1\}_{\delta\mathbb{A}}, \ldots \{[c]x_n\}_{\delta\mathbb{A}}) = \{[c]f_{T\cup\{c\}}\mathbb{A}(x_1, \ldots x_n)\}_{\delta\mathbb{A}}$$

where $c \notin T$.

The following lemmas establish a connection between the evaluation of a uniform term $t : T$ in $\delta\mathbb{A}$ and the evaluation of $t_{\{a\}} : T \cup \{a\}$ in $\mathbb{A}$, for $a \notin T$. Note that this is possible for uniform terms, but not for terms. Recall from Definition 5.7 that a uniform term is not a term (in the sense of set-based universal algebra) but a family of terms.

Lemma 5.19. Consider a uniform subterm $t : T$ within an equation E. For all atoms $a \notin T$ and for all valuations $v_{\mathbb{A}}$ in $\mathbb{A}$ of the variables in $t_{\{a\}} = tr_{\{a\}}(t)$, there exists a valuation $v_{\delta\mathbb{A}}$ in $\delta\mathbb{A}$ of the variables in t such that

$$[\![t]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},T} = \{[a]_T\ [\![t_{\{a\}}]\!]_{\mathbb{A},v_{\mathbb{A}},T\cup\{a\}}\}_{\delta A}. \tag{35}$$

Proof. Note that a variable $X : T_X$ in t will have sort $T_X \cup \{a\} \setminus \mathsf{Fr}_E(X)$ in $t_{\{a\}}$. We know that $v_{\mathbb{A}}(X) \in A(T_X \cup \{a\} \setminus \mathsf{Fr}_E(X))$. We define $v_{\delta\mathbb{A}}(X) \in \delta\mathbb{A}(T_X)$ by

$$v_{\delta\mathbb{A}}(X) = \begin{cases} \{[a]_T v_{\mathbb{A}}(X)\}_{\delta A} & \text{if } a \notin T_X \cup \mathsf{Fr}_E(X) \\ \{[b]A(w_b)v_{\mathbb{A}}(X)\}_{\delta A} & \text{if } a \in T_X \cup \mathsf{Fr}_E(X). \end{cases}$$

We can prove that if a belongs to the sort U of a subterm $u : U$ of t, then

$$[\![u]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},U} = \{[b]_U w_b\ [\![u]\!]_{\mathbb{A},v_{\mathbb{A}},U}\}_{\delta A}. \tag{36}$$

Now we can prove (35) by induction on the structure of terms. For example, let $f_T : T \times \cdots \times T \to T'$ be an operation symbol such that $a \notin T'$. If $a \notin T$, the proof follows by induction. But, if $a \in T \setminus T'$, we have

$$\begin{aligned} [\![f_T(t_1,\ldots,t_n)]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},T'} &= f_T^{\delta\mathbb{A}}([\![t_1]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},T},\ldots,[\![t_n]\!]_{\delta\mathbb{A},v_{\delta\mathbb{A}},T}) \\ &= f_T^{\delta\mathbb{A}}(\{[b]w_b[\![t_1]\!]_{\mathbb{A},v_{\mathbb{A}},T}\}_{\delta A},\ldots,\{[b]w_b[\![t_n]\!]_{\mathbb{A},v_{\mathbb{A}},T}\}_{\delta A}) \\ &= \{[b]w_b f_T^{\mathbb{A}}([\![t_1]\!]_{\mathbb{A},v_{\mathbb{A}},T},\ldots,[\![t_n]\!]_{\mathbb{A},v_{\mathbb{A}},T})\}_{\delta A} \\ &= \{[a]w_a f_T^{\mathbb{A}}([\![t_1]\!]_{\mathbb{A},v_{\mathbb{A}},T},\ldots,[\![t_n]\!]_{\mathbb{A},v_{\mathbb{A}},T})\}_{\delta A} \\ &= \{[a] f_{T\cup\{a\}}^{\mathbb{A}}([\![t_1]\!]_{\mathbb{A},v_{\mathbb{A}},T},\ldots,[\![t_n]\!]_{\mathbb{A},v_{\mathbb{A}},T})\}_{\delta A} \\ &= \{[a][\![t_{\{a\}}]\!]_{\mathbb{A},v_{\mathbb{A}},T\cup\{a\}}\}_{\delta A}. \end{aligned}$$

□

As an illustration, consider the uniform term $t = \mathsf{app}_\varnothing(X,Y)$ within an equation E, such that X, Y are variables of sort $\varnothing$, with $\mathsf{Fr}_E(X) = \{a\}$ and $\mathsf{Fr}_E(Y) = \varnothing$. We have $t_{\{a\}} = \mathsf{app}_{\{a\}}(w_a X, Y_{\{a\}})$. So $v_{\delta\mathbb{A}}(X) \in \delta\mathbb{A}(\varnothing)$ is defined as $v_{\delta\mathbb{A}}(X) = \{[b]A(w_b)v_{\mathbb{A}}(X)\}_{\delta A}$, for some fresh b, whilst $v_{\delta\mathbb{A}}(Y) \in \delta\mathbb{A}(\varnothing)$ is defined as $v_{\delta\mathbb{A}}(Y) = \{[a]v_{\mathbb{A}}(Y_{\{a\}})\}_{\delta A}$.

Lemma 5.20. Consider a uniform term $t : T$ within an equation E and let a be a name such that $a \notin T_X \cup \mathsf{Fr}_E(X)$ for all variables X occurring in E. For all valuations $v_{\delta\mathbb{A}}$ in $\delta\mathbb{A}$ of the variables in t, there exists a valuation in $\mathbb{A}$ of the variables in $tr_{\{a\}}(t)$ such that (35) holds.

Proof. Note that if X has type T_X in t, it has type $T_X \cup \{a\} \setminus \mathsf{Fr}_E(X) = T_X \cup \{a\}$ in $tr_{\{a\}}(t)$. We define $v_{\mathbb{A}}(X)$ as the unique element of $A(T_X \cup \{a\})$ such that $v_{\delta\mathbb{A}}(X) = \{[a]_{T_X} v_{\mathbb{A}}(X)\}_{\delta A}$. The proof is then by induction on the structure of terms. □

Proposition 5.21. If a class $\mathscr{B}$ of nominal algebras is defined by a uniform set of equations, then $\mathscr{B}$ is closed under abstraction.

Proof. Assume that the nominal algebra $\mathbb{A}$ satisfies a uniform equation $t = u : T$. Consider a valuation v of the variables of t_S, u_S in the algebra $\delta\mathbb{A}$. We need to show $\delta\mathbb{A}, v \models t_S = u_S$ for all finite sets of names S, disjoint from T. Choose a name a such that $a \notin S \cup T$ and $a \notin T_X \cup \mathsf{Fr}_E(X)$ for all variables X. Consider the valuation $v_{\mathbb{A}}$ as in Lemma 5.20. Since $\mathbb{A}$ satisfies $t_{S\cup\{a\}} = u_{S\cup\{a\}} : T \cup S \cup \{a\}$, we have

$$[\![t_{S\cup\{a\}}]\!]_{\mathbb{A},v_{\mathbb{A}},S\cup T\cup\{a\}} = [\![u_{S\cup\{a\}}]\!]_{\mathbb{A},v_{\mathbb{A}},S\cup T\cup\{a\}}.$$

So

$$[\![t_S]\!]_{\delta\mathbb{A},v,S\cup T} = [\![u_S]\!]_{\delta\mathbb{A},v,S\cup T}$$

follows from Lemma 5.20 applied for t_S and u_S. □

Proposition 5.22. If a class $\mathscr{B}$ of nominal algebras is defined by a uniform set of equations, then $\mathscr{B}$ is closed under quotients.

Proof. Consider a quotient of sheaves $f : \mathbb{A} \to \mathbb{B}$ such that $\mathbb{A}$ satisfies the uniform equations. Consider the uniform equation $t = u : T$ and choose S disjoint from T and a valuation v in $\mathbb{B}$ of the variables in t_S and u_S. We have to show $[\![t_S]\!]_{\mathbb{B},v,S\cup T} = [\![u_S]\!]_{\mathbb{B},v,S\cup T}$. If a variable X has sort T_X in the uniform equation, its translation has sort $T_X \cup S \setminus \mathsf{Fr}_E(X)$ in $t_S = u_S$, so $v(X) \in \mathbb{B}(T_X \cup S \setminus \mathsf{Fr}_E(X))$. Using Proposition 2.3, we can find a finite set of names S' such that $S \subseteq S'$ and for all variables X appearing in the equation there exists $v_{\mathbb{A}}(X) \in A(T_X \cup S' \setminus \mathsf{Fr}_E(X))$ such that $f_{T_X \cup S' \setminus \mathsf{Fr}_E(X)}(v_{\mathbb{A}}(X)) = \mathbb{B}(w_X)(v(X))$, where w_X denotes the inclusion

$$w_X : T_X \cup S \setminus \mathsf{Fr}_E(X) \to T_X \cup S' \setminus \mathsf{Fr}_E(X).$$

$v_{\mathbb{A}}$ is a valuation of the variables in $t_{S'} = u_{S'}$. From this, we prove by induction on the structure of t that $f_{S'\cup T}([\![t_{S'}]\!]_{\mathbb{A},v_{\mathbb{A}},S'\cup T}) = B(w)([\![t_S]\!]_{\mathbb{B},v,S\cup T})$. Since $B(w)$ is injective, this concludes the proof. □

Theorem 5.23. A class $\mathscr{B}$ of sheaf algebras for a uniform signature is definable by uniform equations if and only if it is closed under sheaf quotients, sub-algebras, products and abstraction.

Proof. Assume that a class of nominal algebras for a uniform signature is defined by uniform equations. Using Corollary 5.5, we can derive closure under subalgebras and products. Closure under abstraction and sheaf quotients follows from Propositions 5.21 and 5.22, respectively. Conversely, from closure under HSPA we derive closure under presheaf epimorphisms, subalgebras and products, hence the class of sheaf-algebras is definable by a set of equations $\mathscr{E}$ in the sense of standard many-sorted universal algebra. We have to show that these equations come from a uniform theory. It is enough to show that whenever $t = u : T$ is in $\mathscr{E}$, we have $B \models t_{\{a\}} = u_{\{a\}} : T \cup \{a\}$ for all $a \notin T$ and $B \in \mathscr{B}$. Since this follows from closure under abstraction and Lemma 5.19, we are done. □

6. Comparison with other nominal logics

Preliminaries on nominal sets

We briefly recall the definition of a nominal set. Intuitively, this is a set equipped with an additional structure that allows well-behaved name swapping in elements of the set. A left action of the group $\mathsf{S}(\mathscr{N})$ of all finitely supported permutations of the set of names $\mathscr{N}$ is a pair $(|\mathbb{X}|, \cdot)$ consisting of a set $|\mathbb{X}|$ and a function $\cdot : \mathsf{S}(\mathscr{N}) \times |\mathbb{X}| \to |\mathbb{X}|$ satisfying

$$\begin{aligned} \mathsf{id}_{\mathscr{N}} \cdot x &= x \\ (\sigma\tau) \cdot x &= \sigma \cdot (\tau \cdot x) \end{aligned} \tag{37}$$

for all $x \in |\mathbb{X}|$ and $\sigma,\tau \in \mathsf{S}(\mathscr{N})$. Let x be an element of $|\mathbb{X}|$. We say that a subset $S \subseteq \mathscr{N}$ *supports* x if and only if for all $a, b \in \mathscr{N} \setminus S$ we have $(a,b) \cdot x = x$, where (a,b) denotes the transposition that swaps a and b. The element x is said to be *finitely supported* if and only if there exists a finite set S that supports x.

Definition 6.1. A nominal set is a left $\mathsf{S}(\mathscr{N})$-action $(|\mathbb{X}|, \cdot)$ such that each element of x is finitely supported.

One can check that for each element x of a nominal set there exists a smallest set, in the sense of inclusion, which supports x. This set is called the *support* of x and will be denoted by $\mathsf{supp}(x)$. We say that $a \in \mathcal{N}$ is *fresh* for x if $a \notin \mathsf{supp}(x)$. A morphism of nominal sets $f : (|\mathbb{X}|, \cdot) \to (|\mathbb{Y}|, \circ)$ is an *equivariant* function between the carrier sets, meaning that f behaves well with respect to permutations of names: $f(\sigma \cdot x) = \sigma \circ f(x)$ for all $x \in |\mathbb{X}|$.

We will use Nom to denote the category of nominal sets and equivariant maps.

Example 6.2. The set $\mathcal{N}$ equipped with the action given by evaluation, $\sigma \cdot a = \sigma(a)$, is a nominal set.

Remark 6.3. If $\mathcal{N}$ and $|\mathbb{X}|$ are equipped with the discrete topology, then $\mathsf{S}(\mathcal{N})$ can be equipped with the topology induced by the product topology on $\mathcal{N}^{\mathcal{N}}$. So a nominal set $(|\mathbb{X}|, \cdot)$ is just a continuous $\mathsf{S}(\mathcal{N})$-action, that is, $\cdot$ is a continuous function.

The equivalence between the nominal sets and the Schanuel topos is a corollary of Mac Lane and Moerdijk (1994, Theorem III.9.1). Spelling out the proof of this theorem, we get the following remark.

Remark 6.4. If $(|\mathbb{X}|, \cdot)$ is a nominal set, the corresponding sheaf $\mathfrak{X} \in \mathrm{Sh}(\mathbb{I}^{op})$ is obtained by taking

$$\begin{aligned} \mathfrak{X}(S) &= \{x \in |\mathbb{X}| \mid \mathsf{supp}(X) \subseteq S\} & & \\ \mathfrak{X}(w_a)(x) &= x & & x \in \mathfrak{X},\ w_a : S \to S \cup \{a\} \\ \mathfrak{X}(b/a)_S(x) &= (a,b) \cdot x & & x \in S \cup \{a\}. \end{aligned} \tag{38}$$

Note that the inclusion functor $\mathbb{I} \hookrightarrow \mathsf{Set}$ corresponds precisely to the nominal set $(\mathcal{N}, \cdot)$ described in Example 6.2.

6.1. *Comparison with nominal algebra*

In this subsection we show how to translate the syntax and theories of the nominal algebra (Gabbay and Mathijssen 2009) to uniform signatures and uniform theories. Then we prove semantic invariance, that is, there is a correspondence between models for a nominal algebra theory and models for the uniform theory obtained via this translation.

Translation of syntax

See Gabbay and Mathijssen (2009) for the syntax and semantics of nominal algebra. To each nominal algebra signature, there corresponds a uniform signature given by the functor $\mathcal{N} + \delta + \Sigma$, where δ is as in Section 4 and Σ is a polynomial functor on $\mathsf{Set}^{\mathbb{I}}$, given by

$$\Sigma A = \coprod \{f_{n_i}\} \times A^{n_i}$$

where the coproduct is taken after all the operation symbols f_{n_i} with arity n_i in the nominal algebra signature.

Translations of equational judgements

Assume $\Delta \vdash t = u$ is an equality judgement in the sense of Gabbay and Mathijssen (2009). It is reasonable to require that the uniform equation obtained by translating such an equality judgement has to satisfy the following:

1 All occurrences of X in the uniform equation have the same sort.
2 If $a\#X$ is in Δ, then in the uniform equation we can only instantiate X with elements whose support does not contain a.
3 We can prove semantic invariance of this translation, that is, a nominal set satisfies an equational judgement if and only if the corresponding sheaf satisfies the translated uniform equation.

In order to address requirements 1 and 2, for each unknown X appearing in this judgement, we have to consider the following sets: $\mathsf{anc}(X)$ defined as the set of names a for which there is an occurrence of πX, for some permutation π, such that $[a]$ is an ancestor of πX in the syntax tree of the equation†, and

$$\mathsf{fresh}(X) = \{a \in \mathcal{N} \mid a\#X \in \Delta\}.$$

Before giving the actual translation, we will first find the type T_E of the uniform equation E obtained by translating $\Delta \vdash t = u$. This is done recursively:

$$\begin{aligned}
\mathsf{type}(t = u) &= \mathsf{type}(t) \cup \mathsf{type}(u) \\
\mathsf{type}(f(t_1, \dots, t_n)) &= \cup \mathsf{type}(t_i) \\
\mathsf{type}([a]t) &= \mathsf{type}(t) \setminus \{a\} \\
\mathsf{type}(a) &= \{a\} \\
\mathsf{type}(\pi X) &= (\mathsf{anc}(X) \setminus \mathsf{fresh}(X)) \cup \mathsf{supp}(\pi).
\end{aligned} \tag{39}$$

We define

$$T_E = \mathsf{type}(t = u) \cup (\bigcup_{X \in E} (\mathsf{fresh}(X) \setminus \mathsf{anc}(X)).$$

The reason for adding $\bigcup_{X \in E} (\mathsf{fresh}(X) \setminus \mathsf{anc}(X)$ is that we want to be able to retrieve the names in $\mathsf{fresh}(X)$ from the uniform equation obtained, even if they do not appear in any subterms. For example, the type of the translation of $b\#X \vdash X = Y$ should be $\{b\}$, and not the empty set.

The actual translation is the uniform equation $\mathcal{T}_{T_E}(t) = \mathcal{T}_{T_E}(u) : T_E$, where $\mathcal{T}_T(t) : T$ is a uniform term of type T defined recursively by

$$\begin{aligned}
\mathcal{T}_T(f(t_1, \dots, t_n)) &= f(\mathcal{T}_T(t_1), \dots, \mathcal{T}_T(t_n)) : T && \\
\mathcal{T}_T(a) &= a_T \ : T && \\
\mathcal{T}_T([a]t) &= [a]_T \mathcal{T}_{T \cup \{a\}}(t) : T && \text{if } a \notin T \\
\mathcal{T}_T([a]t) &= w_a [a]_T \mathcal{T}_T(t) : T && \text{if } a \in T \\
\mathcal{T}_T(\pi X) &= \pi_T w_{a_1} \dots w_{a_k} X_{T \setminus \mathsf{fresh}(X)} \ : T &&
\end{aligned} \tag{40}$$

† We say that $[a]$ is an ancestor of πX rather than of X, because, in the definition of nominal terms, X is not a nominal subterm of the moderated unknown πX.

where, in the last condition, $\{a_1,\dots,a_k\} = T \cap \mathsf{fresh}(X)$. On the right-hand side of the above equations, we have nominal terms for the uniform signature given by $\mathcal{N} + \delta + \Sigma$, obtained according to the rules in Figure 1.

Example 6.5. Consider the following judgement in nominal algebra:

$$a\#X \vdash [a]\mathsf{app}(X,a) = X.$$

We have $\mathsf{fresh}(X) = \{a\}$ and $\mathsf{anc}(X) = \{a\}$ and that the type of the translated uniform equation is $\varnothing$. The translation is the uniform equation

$$[a]_{\varnothing}\mathsf{app}_{\{a\}}(w_a X, a_{\varnothing}) = X,$$

which corresponds to the set of equations (η) of Example 4.8, which is indexed after all finite sets S that do not contain a.

Lemma 6.6. If X is an unknown appearing in the equality judgement E, then all the instances of the variable X have the same sort $(T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$ in the translated uniform equation.

Proof. X may appear more than once in the equality judgement. When traversing the syntax tree, the subscript of $\mathcal{T}$ may change, so we have to prove that whenever we have to translate $\mathcal{T}_T(\pi X)$, the set T has the property that $T \setminus \mathsf{fresh}(X) = (T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$. Note that first we apply the translation with index T_E, and as we traverse the tree, this sort will only increase by a name a when we reach a subterm of the form $[a]t$. If we eventually reach a leaf containing the unknown X, such an a must be in the set $\mathsf{anc}(X)$. Therefore, $T \subseteq T_E \cup \mathsf{anc}(X)$, so we know that $T \setminus \mathsf{fresh}(X) \subseteq (T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$. Conversely, let $a \in (T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$. If $a \in T_E \setminus \mathsf{fresh}(X)$, then $a \in T \setminus \mathsf{fresh}(X)$ because $T_E \subseteq T$. We still need to consider the case when $a \in \mathsf{anc}(X) \setminus \mathsf{fresh}(X)$. We distinguish two cases, depending on whether this particular instance of X has $[a]$ as an ancestor. If this is the case, the set T must contain the name a. If this is not the case, we have $a \in \mathsf{type}(t = u) \subseteq T_E$, and thus $a \in T$. □

Lemma 6.7. $\mathsf{Fr}_E(X) = \mathsf{fresh}(X)$.

Proof. Consider $a \in \mathsf{fresh}(X)$. We will use T_X to denote the sort of the variable X in $\mathcal{T}(E)$. We know that $T_X = (T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$, so $a \notin T_X$. We have two cases:

1 If $a \in \mathsf{anc}(X)$, then there exists a subterm $[a]v$, such that X occurs in v. The sort of $\mathcal{T}(v)$ must contain the name a, so $a \in \mathsf{Fr}_E(X)$.
2 If $a \in \mathsf{fresh}(X) \setminus \mathsf{anc}(X)$, then $a \in T_E$, so again we get $a \in \mathsf{Fr}_E(X)$.

Conversely, if $a \in \mathsf{Fr}_E(X)$, there exists a subterm v in E containing πX, for some permutation π, such that the variable $X : T_X$ occurs in $\mathcal{T}_T(v) : T$ and $a \in T \setminus T_X$. If the sort of $\mathcal{T}(\pi X)$ is S, we have $T \subseteq S$, so we get $a \in S \setminus T_X = S \cap \mathsf{fresh}(X)$, and thus $a \in \mathsf{fresh}(X)$. □

Translation of semantics

Let $\mathbb{X} = (|\mathbb{X}|, \cdot, \mathbb{X}_{atm}, \mathbb{X}_{abs}, \{\mathbb{X}_f \mid f \in \mathsf{S}\})$ be a nominal algebra for a nominal signature S. Let $\mathscr{N} + \delta + \Sigma$ be the functor corresponding to this signature. We consider the sheaf $\mathfrak{X}$ obtained from the nominal set $(|\mathbb{X}|, \cdot)$ as in (38). The translation of $\mathbb{X}$ is the sheaf algebra $\mathscr{N} + \delta\mathfrak{X} + \Sigma\mathfrak{X} \to \mathfrak{X}$ given by

$$\begin{array}{rcl} a_S & \mapsto & \mathbb{X}_{atm}(a) \\ \{[a]_S x\}_{\delta\mathfrak{X}} & \mapsto & \mathbb{X}_{abs}(\mathbb{X}_{atm}(a), x) \\ (f, x_1, \ldots, x_n) & \mapsto & \mathbb{X}_f(x_1, \ldots, x_n). \end{array}$$

That this is well defined follows from the equivariance of $\mathbb{X}_{atm}, \mathbb{X}_{abs}, \mathbb{X}_f$.

Theorem 6.8 (semantic invariance). Let $\mathbb{X}$ be a nominal algebra for a nominal signature. Let E be the uniform equation of type T_E obtained by translating an equality judgement $\Delta \vdash u = v$. Then $\mathbb{X}$ satisfies $\Delta \vdash u = v$ if and only if $\mathfrak{X}$ satisfies the uniform equation E.

Proof. First assume that $[\![\Delta \vdash u = v]\!]^{\mathbb{X}}$ holds. We need to prove that $\mathfrak{X} \models tr_S\, E$ for all finite sets S that are disjoint from T_E. Consider a valuation ς of the variables appearing in $tr_S\,(E)$ in $\mathfrak{X}$. If X is a variable of sort T_X in E, then X has sort $S \setminus \mathsf{Fr}_E(X) \cup T_X$ in $tr_S\,(E)$. So $\varsigma(X) \in \mathfrak{X}(S \setminus \mathsf{Fr}_E(X) \cup T_X)$ is an element of the nominal set $\mathbb{X}$ supported by $S \setminus \mathsf{Fr}_E(X) \cup T_X$. Using Lemmas 6.6 and 6.7, we know that $S \setminus \mathsf{Fr}_E(X) \cup T_X$ is disjoint from $\mathsf{fresh}(X)$, so whenever the freshness primitive $a\#X$ is in Δ, we have $a\#\varsigma(X)$. Let ς' be a valuation in $\mathbb{X}$ of the unknowns in $\Delta \vdash u = v$ that maps X to $\varsigma(X)$. We have to prove that $[\![tr_S\,(\mathscr{T}u)]\!]^{\mathfrak{X}}_{\varsigma} = [\![tr_S\,(\mathscr{T}v)]\!]^{\mathfrak{X}}_{\varsigma}$. This follows from the following claim, which can be proved by induction on the structure of the terms.

Claim 6.9. For all subterms t of either u or v, we have $[\![t]\!]^{\mathbb{X}}_{\varsigma'} = [\![\mathscr{T}t]\!]^{\mathfrak{X}}_{\varsigma}$.

Conversely, assume that $\mathfrak{X}$ satisfies the uniform equation E. Consider a valuation ς' in $\mathbb{X}$ of the unknowns of $\Delta \vdash u = v$ such that $a\#\varsigma'(X)$ whenever $a\#X \in \Delta$. Now consider the finite set of atoms $S := \bigcup_X \mathsf{supp}(\varsigma'(X)) \setminus T_E$. We can define a valuation ς of the variables occurring in $tr_S\,(E)$ in $\mathfrak{X}$ simply by taking $\varsigma(X) = \varsigma'(X)$. In order to prove that this is well defined, we can check that $\varsigma'(X)$ is supported by $S \setminus \mathsf{Fr}_E(X) \cup T_X$. Since ς' can be obtained from ς as before, we can finalise the proof by applying Claim 6.9 again. $\square$

Corollary 6.10. Theorems 6.8 and 5.23 give a new proof for Gabbay's HSPA theorem (Gabbay 2008, Theorem 9.3).

6.2. *Comparison with NEL*

This section compares uniform theories to a fragment of the nominal equational logic of Clouston and Pitts (2007). For simplicity, we will only consider the one-sorted version of NEL†, although extending our work to many-sortedness over sheaves is not difficult. We

† Note that we will not need the sorting environments of Clouston and Pitts (2007) in this case.

only consider theories for which the axioms are of the form

$$\begin{array}{rcl} \{X_1,\dots,X_n\} & \vdash & \bar{a} \mathrel{\#\!\!\!\!\approx} f(X_1,\dots,X_n) \qquad \bar{a} \subseteq \mathsf{supp}(f) \\ \Delta & \vdash & t \approx t'. \end{array} \tag{41}$$

Translation of syntax

Recall that a signature for NEL is given by a nominal set $\mathsf{Op} = (|\mathsf{Op}|, \cdot)$ of operation symbols. Consider a theory for this signature consisting of axioms as in (41). We will construct a presheaf of operations $\mathcal{O}$ as in Definition 5.1, which is almost the sheaf corresponding to Op via the isomorphism between nominal sets and $\mathrm{Sh}(\mathbb{I}^{op})$: for all finite sets of names S, $\mathcal{O}(S)$ contains the operation symbols whose support is contained in S. But we also add more information about the arity of these operation symbols.

Definition 6.11. Consider a NEL theory for a signature that contains an operation symbol f. The set $\mathsf{bind}(f)$ is defined as the set of names a such that there is an axiom in the theory of the form $\{X_1,\dots,X_n\} \vdash \bar{b} \mathrel{\#\!\!\!\!\approx} (\pi \cdot f)(X_1,\dots,X_n)$ for a finite set of names $\bar{b}$ and a permutation π such that $\pi(b) = a$ for some $b \in \bar{b}$.

If $f \in \mathsf{Op}$ is an n-ary operation symbol such that $\mathsf{supp}(f) = T$ and $T \subseteq S$, we consider an operation symbol in $f_S \in \mathcal{O}(S)$, with arity $f_S : S \times \cdots \times S \to S \setminus \mathsf{bind}(f)$. The definition above implies that $\mathsf{bind}(\pi \cdot f) = \pi[\mathsf{bind}(f)]$. We also get $\mathsf{bind}(f) \subseteq \mathsf{supp}(f)$. So for any injective map $j : S \to S'$, we can show that $\mathcal{O}(j)(f_S)$ has the arity $S' \times \cdots \times S' \to S' \setminus j[\mathsf{bind}(f)]$. So $\mathcal{O}$ is a presheaf as in Definition 5.1. Note that the arity of an operation symbol in $\mathcal{O}$ depends not only on the nominal signature, but also on the theory, because of the way freshness constraints are expressed in NEL – see (41). The translation of a NEL signature is the uniform signature given by the functor L with a uniform presentation given by $\mathcal{O}$ – see Definition 5.1.

Example 6.12. If L_a is an operation symbol as in the NEL signature for λ-calculus of Clouston and Pitts (2007, Example 3.1), then $\mathsf{bind}(L_a) = \{a\}$. The translation of this NEL signature is the uniform signature given by the functor defined in (17).

Translation of a theory

From each axiom in a theory in the sense of Clouston and Pitts (2007) having the form

$$\Delta \vdash t \approx t',$$

we will obtain a uniform equation E of sort T_E. As in the previous section, we first describe a way of finding the sort T_E. Again, all occurrences of a variable X are expected to have the same sort in the translation, so we need to pay attention to the bound names of the terms that contain X and to the names that should be fresh for X. To this end, we define the set $\mathsf{anc}(X)$ by

$$\mathsf{anc}(X) = \bigcup \mathsf{bind}(f)$$

taken after all operations f such that X appears in a subterm of either t or t' of the form $f(t_1, \ldots, t_n)$. Similarly, we define

$$\mathsf{fresh}(X) = \bar{a} \quad \text{if and only if} \quad \bar{a} \# X \in \Delta.$$

The fact that $a \# X$ is in the freshness environment will be expressed in the uniform equation by adding a weakening w_a in front of X.

In order to find T_E, we define a function type recursively:

$$\begin{array}{rcl} \mathsf{type}(t = u) & = & \mathsf{type}(t) \cup \mathsf{type}(u) \\ \mathsf{type}(f(t_1, \ldots, t_n)) & = & (\cup \mathsf{type}(t_i) \cup \mathsf{supp}(f)) \setminus \mathsf{bind}(f) \\ \mathsf{type}(\pi X) & = & (\mathsf{anc}(X) \setminus \mathsf{fresh}(X)) \cup \mathsf{supp}(\pi). \end{array} \tag{42}$$

We define $T_E = \mathsf{type}(t = u) \cup (\bigcup\limits_{X \in E} (\mathsf{fresh}(X) \setminus \mathsf{anc}(X))$.

The translation of the axiom $\Delta \vdash t \approx t'$ is the uniform equation $\mathscr{T}_{T_E}(t) = \mathscr{T}_{T_E}(u) : T_E$, where $\mathscr{T}_T(t)$ is a term of sort T, defined recursively by

$$\begin{array}{rcl} \mathscr{T}_T(f(t_1, \ldots, t_n)) & = & w_{T \cap \mathsf{bind} f} f_{T \cup \mathsf{bind}(f)}(\mathscr{T}_{T \cup \mathsf{bind}(f)}(t_1), \ldots, \mathscr{T}_{T \cup \mathsf{bind}(f)}(t_n)) \\ \mathscr{T}_T(\pi X) & = & \pi_T w_{T \cap \mathsf{fresh}(X)} X_{T \setminus \mathsf{fresh}(X)}. \end{array} \tag{43}$$

The permutation π has its support included in T, and π_T is the restriction of π to T. As in the previous section, we can prove that all instances of a variable X have the same sort in the uniform equation, namely $(T_E \cup \mathsf{anc}(X)) \setminus \mathsf{fresh}(X)$. The proof of this is analogous to that of Lemma 6.6. The only difference is that now instead of only reasoning about abstractions $[a]$, we allow more general operation symbols. In a similar way to Lemma 6.7, we get $\mathsf{Fr}_{\mathscr{T}(E)}(X) = \mathsf{fresh}(X)$.

Example 6.13. The η rule of the NEL theory for $\alpha\beta\eta$-equivalence of untyped λ-terms (Clouston and Pitts 2007, Example 6.2)

$$a \# x \vdash L_a(A \ x \ V_a) \approx x$$

translates to

$$[a](\mathsf{app}(w_a X, a)) = X \quad : \varnothing.$$

Translation of semantics

Consider a NEL theory as in (41) for a signature Op. Let $\mathbb{X}$ be an algebra for this theory, that is, a nominal set $|\mathbb{X}|$, equipped with equivariant functions $\mathsf{Op}_n \times |\mathbb{X}|^n \to |\mathbb{X}|$ for all arities n. (Op_n is the set of operation symbols of arity n, and is a nominal subset of Op.) We construct a sheaf $\mathfrak{X}$ from $|\mathbb{X}|$ as in (38). The sheaf algebra corresponding to $\mathbb{X}$ is an algebra for the functor L with a uniform presentation obtained from Op – see the syntax translation above. This sheaf algebra $L\mathfrak{X} \to \mathfrak{X}$ maps

$$\{f_T(x_1, \ldots, x_n)\}_{\mathfrak{X}} \mapsto \mathbb{X}[\![f]\!](x_1, \ldots, x_n)$$

where $\mathbb{X}[\![f]\!]$ is as in Clouston and Pitts (2007). This map is well defined because of the equivariance of the functions $\mathsf{Op}_n \times |\mathbb{X}|^n \to |\mathbb{X}|$.

Theorem 6.14 (semantic invariance). A structure $\mathbb{X}$ for a NEL signature is a structure for a NEL theory as in (41) if and only if the sheaf algebra $\mathfrak{X}$ obtained as above is an algebra for the translated uniform theory.

Proof. First we can check that a structure $\mathbb{X}$ for a nominal signature satisfies a judgement $\Delta \vdash t \approx t'$ if and only if the sheaf algebra $\mathfrak{X}$, constructed as above, satisfies the uniform equation $E : T_E$ obtained as the translation of $\Delta \vdash t \approx t'$. The proof for this follows along the same lines as the proof for Theorem 6.8. From a valuation ς' in $\mathbb{X}$ of the variables in the freshness environment Δ, we get a valuation ς in $\mathfrak{X}$ of the variables in some $tr_S(E)$ for $S \cap T_E = \emptyset$, and *vice versa.* We still need to check that $[\![t]\!]^{\mathbb{X}}_{\varsigma'} = [\![\mathscr{T} t]\!]^{\mathfrak{X}}_{\varsigma}$, and this goes by induction on the structure of the terms. The following equalities hold in the underlying nominal set of $\mathbb{X}$:

$$
\begin{aligned}
[\![\mathscr{T}_T(f(t_1,\dots,t_n))]\!]^{\mathfrak{X}}_{\varsigma} &= [\![w_{T\cap \mathsf{bind} f} f_{T\cup \mathsf{bind}(f)}(\mathscr{T}_{T\cup \mathsf{bind}(f)}(t_1),\dots,\mathscr{T}_{T\cup \mathsf{bind}(f)}(t_n))]\!]^{\mathfrak{X}}_{\varsigma}\\
&= \mathfrak{X}(w_{T\cap \mathsf{bind} f})([\![f_{T\cup \mathsf{bind}(f)}(\mathscr{T}_{T\cup \mathsf{bind}(f)}(t_1),\dots,\mathscr{T}_{T\cup \mathsf{bind}(f)}(t_n))]\!]^{\mathfrak{X}}_{\varsigma})\\
&= [\![f_{T\cup \mathsf{bind}(f)}(\mathscr{T}_{T\cup \mathsf{bind}(f)}(t_1),\dots,\mathscr{T}_{T\cup \mathsf{bind}(f)}(t_n))]\!]^{\mathfrak{X}}_{\varsigma}\\
&= \{f_{T\cup \mathsf{bind}(f)}([\![t_1]\!]^{\mathfrak{X}}_{\varsigma},\dots,[\![t_n]\!]^{\mathfrak{X}}_{\varsigma})\}_{\mathfrak{X}}\\
&= \{f_{T\cup \mathsf{bind}(f)}([\![t_1]\!]^{\mathbb{X}}_{\varsigma'},\dots,[\![t_n]\!]^{\mathbb{X}}_{\varsigma'})\}_{\mathfrak{X}}\\
&= \mathbb{X}[\![f]\!]([\![t_1]\!]^{\mathbb{X}}_{\varsigma'},\dots,[\![t_n]\!]^{\mathbb{X}}_{\varsigma'})\\
&= [\![f(t_1,\dots,t_n)]\!]^{\mathbb{X}}_{\varsigma'}.
\end{aligned}
$$

For axioms of the form $X_1,\dots,X_n \vdash a \,\#\, f(X_1,\dots,X_n)$, semantical invariance follows since the operation symbol f corresponds on the side of uniform signatures to operation symbols whose arities have the property that a does not belong to the result. So, for any valuation of the variables X_i in $\mathfrak{X}$, a translation of $f(X_1,\dots,X_n)$ is evaluated to an element y of $\mathfrak{X}(\mathfrak{S})$ for a finite set S, with $a \notin S$. This means that, if $\mathfrak{X}$ comes from a nominal set $|\mathbb{X}|$, we have that a is fresh for y in $|\mathbb{X}|$. $\square$

Corollary 6.15. Theorems 5.23 and 6.14 give an HSPA theorem for models of NEL.

7. Conclusions

We have studied universal algebra over nominal sets based on the observation that the category Nom of nominal sets is a full reflective subcategory of an equationally definable class of many-sorted algebras, namely, the presheaf category $\mathsf{Set}^{\mathbb{I}}$. As an application, we have proved two versions of Birkhoff's HSP theorem over nominal sets.

Section 3 took a category theoretic approach and investigated how to push the standard category theoretic proof of Birkhoff's HSP theorem through the adjunction relating Nom and $\mathsf{Set}^{\mathbb{I}}$. Theorem 3.5 summarises the category theoretic assumptions needed to prove such an HSP theorem, and Theorem 3.12 specialises it to nominal sets.

Section 5 is based on the particular structure of $\mathsf{Set}^{\mathbb{I}}$ and introduced the notion of a uniform equational theory: uniform equations are invariant under shifting to larger contexts. This notion enabled us to extend Theorem 3.12 to prove a universal-algebra analogue (Theorem 5.18) of Gabbay's HSPA theorem (Gabbay 2008).

Finally, Section 6 showed that, moving from Nom to $\mathsf{Set}^{\mathbb{I}}$, the theories of Gabbay and Mathijsen (Gabbay and Mathijssen 2009) and Clouston and Pitts (Clouston and

Pitts 2007) translate into uniform equational theories. This gives a new way of comparing the two different approaches, and also new proofs of HSPA theorems for nominal algebra (Gabbay 2008) and nominal equational logic (Clouston and Pitts 2007).

There is a range of possibilities for future development. Obviously, further universal algebra results should be made available for algebras over nominal sets. Related to this, our approach could be useful in 'nominalising' other areas of theoretical computer science based on universal algebra, such as the theory of automata and formal languages. Another possibility is to profit from the fact that, unlike in the work of Gabbay and Mathijssen (2009) and Clouston and Pitts (2007), on the universal algebra side, we are not forced to remain inside the uniform fragment; this could be of interest, for example, in the investigation of bounded variable fragments of λ-calculus or first-order logic. It would also be interesting to consider an enriched categorical perspective, in particular, to investigate connections between our many-sorted equational logic and the synthetic nominal equational logic of Fiore and Hur (2008). Finally, one motivation for our study was to give a foundation for the work on logics for nominal calculi in the style of Bonsangue and Kurz (2007), where much still remains to be explored. For example, the models for the π-calculus of Stark (2008) should fit within the realm of universal algebra over nominal sets.

Acknowledgements

We would like to thank Jiří Velebil and Jiří Adámek for useful discussions and the anonymous referees for very helpful comments.

References

Adámek, J., Herrlich, H. and Strecker, G. E. (1990) *Abstract and Concrete Categories*, John Wiley and Sons.

Adámek, J. and Rosický, J. (1994) *Locally Presentable and Accessible Categories*, Cambridge University Press.

Adámek, J., Rosický, J. and Vitale, E. M. (draft) Algebraic Theories: a Categorical Introduction to General Algebra. Available at `http://www.iti.cs.tu-bs.de/~adamek/algebraic.theories.pdf`.

Bonsangue, M. and Kurz, A. (2007) Pi-calculus in logical form. *22nd Annual IEEE Symposium on Logic in Computer Science (LICS 2007)* 303–312.

Bonsangue, M. M. and Kurz, A. (2006) Presenting functors by operations and equations. In: FoSSaCS. *Springer-Verlag Lecture Notes in Computer Science* **3921**.

Borceux, F. (1994) *Handbook of Categorical Algebra*, Cambridge University Press.

Clouston, R. and Pitts, A. (2007) Nominal equational logic. In: Computation, Meaning and Logic: Articles dedicated to Gordon Plotkin. *Electronic Notes in Theoretical Computer Science* **172**.

Fiore, M., Plotkin, G. and Turi, D. (1999) Abstract syntax and variable binding. *Proceedings of the 14th Annual IEEE Symposium on Logic in Computer Science* 193–202.

Fiore, M. P. and Hur, C.-K. (2008) Term equational systems and logics (extended abstract). *Electronic Notes in Theoretical Computer Science* **218** 171–192.

Gabbay, M. (2008) Nominal algebra and the HSP theorem. *Journal of Logic and Computation* **19** (2) 341–367.

Gabbay, M. and Pitts, A. (1999) A new approach to abstract syntax involving binders. *Proceedings of the 14th Annual IEEE Symposium on Logic in Computer Science (LICS'99)* 214–224.

Gabbay, M. J. and Mathijssen, A. (2009) Nominal (universal) algebra: equational logic with names and binding. *Journal of Logic and Computation* (in press).

Johnstone, P. T. (2002) *Sketches of an Elephant: A Topos Theory Compendium, vol. 1*, Oxford Logic Guides **43**, Oxford University Press.

Kurz, A. and Petrişan, D. (2008) Functorial coalgebraic logic: The case of many-sorted varieties. In: Adámek, J. and Kupke, C. (eds.) Proceedings of the Ninth Workshop on Coalgebraic Methods in Computer Science (CMCS 2008). *Electronic Notes in Theoretical Computer Science* **203** (5) 175–194

Kurz, A. and Rosický, J. (2006) Strongly complete logics for coalgebras (submitted).

Mac Lane, S. and Moerdijk, I. (1994) *Sheaves in Geometry and Logic: A First Introduction to Topos Theory (Universitext)*, Springer-Verlag.

Stark, I. (2008) Free-algebra models for the pi -calculus. *Theoretical Computer Science* **390** (2-3) 248–270.