


Geometry of Reachability Sets of Vector Addition Systems

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Abstract

Vector Addition Systems (VAS), aka Petri nets, are a popular model of concurrency. The reachability set of a VAS is the set of configurations reachable from the initial configuration. Leroux has studied the geometric properties of VAS reachability sets, and used them to derive decision procedures for important analysis problems. In this paper we continue the geometric study of reachability sets. We show that every reachability set admits a finite decomposition into disjoint almost hybridlinear sets enjoying nice geometric properties. Further, we prove that the decomposition of the reachability set of a given VAS is effectively computable. As a corollary, we derive a simple algorithm for deciding semilinearity of VAS reachability sets, the first one since Hauschildt’s 1990 algorithm. As a second corollary, we prove that the complement of a reachability set always contains an infinite linear set.

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1 Introduction

Vector Addition Systems (VAS), also known as Petri nets, are a popular model of concurrent systems. The VAS reachability problem consists of deciding if a target configuration of a VAS is reachable from its initial configuration. It was proved decidable in the 1980s [7, 16], but its complexity (Ackermann-complete) could only be determined recently [2], [3], [13].

The *reachability set* of a VAS is the set of all configurations reachable from the initial configuration. Configurations are tuples of natural numbers, and so the reachability set of a VAS is a subset of \mathbb{N}^n for some n called the *dimension* of the VAS. Results on the geometric properties of reachability sets have led to new algorithms in the past. For example, in [11] it was shown that every configuration outside the reachability set \mathbf{R} of a VAS is separated from \mathbf{R} by a semilinear inductive invariant. This immediately leads to an algorithm for the reachability problem consisting of two semi-algorithms, one enumerating all possible paths to certify reachability, and one enumerating all semilinear sets and checking if they are separating inductive invariants. Another example is [12], where it was shown that semilinear reachability sets are flatable. The result led to an algorithm for deciding whether a semilinear set is included in or equal to the reachability set of a given VAS.

The separability and flatability results of [11, 12] are proven not only for VAS reachability sets, but for all sets satisfying some abstract geometric properties, called *Petri sets* in [11]. In this sense, [12] investigated the geometric structure of the semilinear Petri sets. In this paper we study the structure of the *non-semilinear* Petri sets. We introduce hybridization,

or equivalently the class of *almost hybridlinear* sets, a generalization of the hybridlinear sets introduced by Ginsburg and Spanier [4] and studied by Chistikov and Haase [1], and prove the following decomposition:

► **Theorem 1.1.** *Let \mathbf{X} be a Petri set. For every semilinear set \mathbf{S} there exists a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ into pairwise disjoint full linear sets such that for all i either $\mathbf{X} \cap \mathbf{S}_i = \emptyset$, $\mathbf{S}_i \subseteq \mathbf{X}$ or $\mathbf{X} \cap \mathbf{S}_i$ is irreducible with hybridization \mathbf{S}_i . Further, if \mathbf{X} is the reachability set of a VAS, then the partition is computable.*

We use Theorem 1.1 to derive two further corollaries. In 1990, Hauschildt presented in his PhD thesis an algorithm to decide if the reachability set of a given VAS is semilinear [5], and if yes, compute a semilinear representation. The first corollary is an algorithm that essentially coincides with Hauschildt's, but has a simpler correctness proof, a simpler geometric intuition, and can be generalized to larger classes of sets. The second corollary shows that if the complement of a Petri set is infinite, then it contains an infinite linear set. This was left as a conjecture in [6]. This is a first step towards understanding the complements of VAS reachability sets, for which little is known.

The sections of the paper follow the structure of the main theorem. Section 2 contains preliminaries. Section 3 defines full linear sets. Section 4 introduces smooth sets, preparing for the introduction of hybridization and Petri sets in Section 5. Section 6 proves Theorem 1.1. Section 7 proves the corollaries of Theorem 1.1.

2 Preliminaries

We let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}_{\geq 0}$ denote the natural, integer, and (non-negative) rational numbers.

Furthermore, we use uppercase letters except A for sets, with A being used for matrices. We use boldface for vectors and sets of vectors.

Given sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{Q}^n, Z \subseteq \mathbb{Q}$, we write $\mathbf{X} + \mathbf{Y} := \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \mathbf{X}, \mathbf{y} \in \mathbf{Y}\}$ and $Z \cdot \mathbf{X} := \{\lambda \cdot \mathbf{x} \mid \lambda \in Z, \mathbf{x} \in \mathbf{X}\}$. By identifying elements $\mathbf{x} \in \mathbb{Q}^n$ with $\{\mathbf{x}\}$, we define $\mathbf{x} + \mathbf{X} := \{\mathbf{x}\} + \mathbf{X}$, and similarly $\lambda \cdot \mathbf{X} := \{\lambda\} \cdot \mathbf{X}$ for $\lambda \in \mathbb{Q}$. We denote by \mathbf{X}^C the complement of \mathbf{X} . On \mathbb{Q}^n , we consider the usual euclidean norm and its generated topology. We denote the closure of a set \mathbf{X} in this topology by $\overline{\mathbf{X}}$.

A *vector space* $\mathbf{V} \subseteq \mathbb{Q}^n$ is a set such that $\mathbf{V} + \mathbf{V} \subseteq \mathbf{V}$ and $\mathbb{Q} \cdot \mathbf{V} \subseteq \mathbf{V}$. Given a set $\mathbf{F} \subseteq \mathbb{Q}^n$, the vector space generated by \mathbf{F} is the smallest vector space containing \mathbf{F} . Every vector space is finitely generated and can be expressed as $\{\mathbf{x} \in \mathbb{Q}^n \mid A\mathbf{x} = \mathbf{0}\}$ for some integer matrix A .

A set $\mathbf{C} \subseteq \mathbb{Q}^n$ is a *cone* if $\mathbf{0} \in \mathbf{C}$, $\mathbf{C} + \mathbf{C} \subseteq \mathbf{C}$ and $\mathbb{Q}_{>0}\mathbf{C} \subseteq \mathbf{C}$. Given a set $\mathbf{F} \subseteq \mathbb{Q}^n$, the cone generated by \mathbf{F} is the smallest cone containing \mathbf{F} . If \mathbf{C} is a cone, then $\mathbf{C} - \mathbf{C}$ is the vector space generated by \mathbf{C} . Not every cone is finitely generated, instead we have

► **Lemma 2.1.** [17] *Let $\mathbf{C} \subseteq \mathbb{Q}^n$ be a cone. Then \mathbf{C} is finitely generated if and only if $\mathbf{C} = \{\mathbf{x} \in \mathbf{C} - \mathbf{C} \mid A\mathbf{x} \geq \mathbf{0}\}$ for some integer matrix A .*

In particular, finitely generated cones are closed. Finitely generated cones \mathbf{C} have a natural notion of (relative) interior $\text{int}(\mathbf{C})$, namely $\text{int}(\mathbf{C}) = \{\mathbf{x} \in \mathbf{C} - \mathbf{C} \mid A\mathbf{x} > \mathbf{0}\}$ if A is a matrix as above. The boundary of the cone is $\partial(\mathbf{C}) := \overline{\mathbf{C}} \setminus \text{int}(\mathbf{C})$, which in [17] is shown to be a finite union of lower dimensional cones, called facets. In fact, there is a defining matrix A which does not have redundant equations, and then the facets are exactly obtained by choosing an equation to change into $= 0$. For example, in the left of Figure 1, the cone $\{(x, y) \mid x \geq y \geq 0\}$ is depicted, and its facets $\{(x, y) \mid x = y \geq 0\}$ and $\{(x, y) \mid x \geq y = 0\}$ in black form the boundary.

A cone \mathbf{C} is *definable* if it is definable in $\text{FO}(\mathbb{Q}, +, \geq)$, which is equivalent to $\mathbf{C} \setminus \{\mathbf{0}\} = \{\mathbf{x} \in \mathbf{C} - \mathbf{C} \mid A_1 \mathbf{x} > \mathbf{0}, A_2 \mathbf{x} \geq \mathbf{0}\}$ for some integer matrices A_1, A_2 . In this case the closure $\overline{\mathbf{C}}$ is finitely generated. Intuitively, equations changed from ≥ 0 to > 0 remove facets. To obtain for example $\text{int}(\mathbf{C})$, all facets are removed.

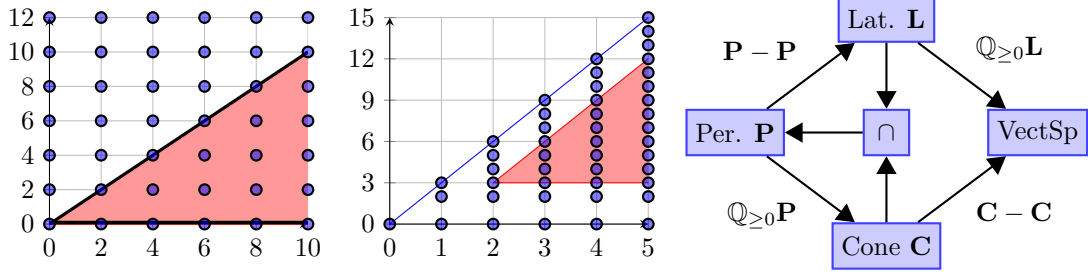


Figure 1 Left: In red with black boundary the cone generated by $\{(1,1), (1,0)\}$, in blue the lattice $(2,0)\mathbb{Z} + (0,2)\mathbb{Z}$. The intersection forms the periodic set $\{(2,0), (2,2)\}^*$. Middle: The periodic set $\mathbf{P} = \{(1,0), (1,2), (1,3)\}^*$ in blue. The vectors $(x,1)$ are not contained in \mathbf{P} , but inside \mathbf{P} we find the red area, where the hole does not occur. I.e. $(2,3) + \text{Fill}(\mathbf{P}) \subseteq \mathbf{P}$. Right: Graph comparing the classes of sets defined in Section 2.

A set $\mathbf{L} \subseteq \mathbb{Z}^n$ is a *lattice* if $\mathbf{L} + \mathbf{L} \subseteq \mathbf{L}$, $-\mathbf{L} \subseteq \mathbf{L}$ and $0 \in \mathbf{L}$. For any finite set $\mathbf{F} = \{\mathbf{x}_1, \dots, \mathbf{x}_s\} \subseteq \mathbb{Z}^n$, the lattice generated by \mathbf{F} is $\mathbb{Z}\mathbf{x}_1 + \dots + \mathbb{Z}\mathbf{x}_s$. Every lattice is finitely generated, and even has a generating set linearly independent over \mathbb{Q} .

A set $\mathbf{P} \subseteq \mathbb{N}^n$ is a *periodic set* if $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$ and $0 \in \mathbf{P}$. For any set $\mathbf{F} \subseteq \mathbb{N}^n$, the periodic set \mathbf{F}^* generated by \mathbf{F} is the smallest periodic set containing \mathbf{F} , explicitly we obtain $\mathbf{F}^* = \{\mathbf{p}_1 + \dots + \mathbf{p}_r \mid r \in \mathbb{N}_0, \mathbf{p}_i \in \mathbf{F} \text{ for all } i\}$. A periodic set \mathbf{P} is *finitely generated* if $\mathbf{P} = \mathbf{F}^*$ for some finite set \mathbf{F} . Finitely generated periodic sets are characterized as follows:

► **Lemma 2.2.** [12, Lemma V.5] *Let $\mathbf{P} \subseteq \mathbb{N}^n$ be a periodic set. Then \mathbf{P} is finitely generated as a periodic set if and only if $\mathbb{Q}_{\geq 0} \mathbf{P}$ is finitely generated as a cone.*

As with any set, periodic sets generate a lattice, a cone and a vector space, however in the case of periodic sets these have simple formulas; namely $\mathbf{P} - \mathbf{P}$, as well as $\mathbb{Q}_{\geq 0} \mathbf{P}$ and $\text{VectSp}(\mathbf{P}) := \mathbb{Q}_{\geq 0}(\mathbf{P} - \mathbf{P}) = \mathbb{Q}_{\geq 0} \mathbf{P} - \mathbb{Q}_{\geq 0} \mathbf{P}$ respectively, as depicted in the right of Figure 1. On the other hand, if \mathbf{C} is a cone and \mathbf{L} is a lattice, then $\mathbf{C} \cap \mathbf{L}$ is a periodic set. We consider the class of periodic sets which can be obtained as such an intersection in Section 3.

A set \mathbf{L} is *linear* if $\mathbf{L} = \mathbf{b} + \mathbf{P}$ with $\mathbf{b} \in \mathbb{N}^n$ and $\mathbf{P} \subseteq \mathbb{N}^n$ a finitely generated periodic set. A set \mathbf{S} is *semilinear* if it is a finite union of linear sets. The semilinear sets coincide with the sets definable via formulas $\varphi \in \text{FO}(\mathbb{N}, +, \geq)$, also called Presburger Arithmetic.

The *dimension* of a vector space defined as its number of generators is a well-known concept. It can be extended to arbitrary subsets of \mathbb{Q}^n as follows.

► **Definition 2.3.** [11] *Let $\mathbf{X} \subseteq \mathbb{Q}^n$. The dimension of \mathbf{X} , denoted $\dim(\mathbf{X})$, is the smallest natural number k such that there exist finitely many vector spaces $\mathbf{V}_i \subseteq \mathbb{Q}^n$ with $\dim(\mathbf{V}_i) \leq k$ and vectors $\mathbf{b}_i \in \mathbb{Q}^n$ such that $\mathbf{X} \subseteq \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{V}_i$.*

This dimension function has the following properties.

► **Lemma 2.4.** *Let $\mathbf{X}, \mathbf{X}' \subseteq \mathbb{Q}^n, \mathbf{b} \in \mathbb{Q}^n$. Then $\dim(\mathbf{X}) = \dim(\mathbf{b} + \mathbf{X})$ and $\dim(\mathbf{X} \cup \mathbf{X}') = \max\{\dim(\mathbf{X}), \dim(\mathbf{X}')\}$. Further, if $\mathbf{X} \subseteq \mathbf{X}'$, then $\dim(\mathbf{X}) \leq \dim(\mathbf{X}')$.*

► **Lemma 2.5.** [11] *Let \mathbf{P} be a periodic set. Then $\dim(\mathbf{P}) = \dim(\text{VectSp}(\mathbf{P}))$.*

Lemma 2.5 for example shows that the lattice and the cone depicted in the left of Figure 1, as well as the periodic set obtained as intersection have dimension 2, because all of them generate the vector space \mathbb{Q}^2 .

3 Full Semilinear Sets

We introduce full linear sets, which were indirectly used already in [12] and [10].

A periodic set \mathbf{P} induces a cone $\mathbb{Q}_{\geq 0}\mathbf{P}$ and a lattice $\mathbf{P} - \mathbf{P}$. Intuitively, the cone gives information about the region of the space containing \mathbf{P} , and the lattice describes “modulo data”, like “only even valued points of the cone are in \mathbf{P} ”. Ideally, one would like to have $\mathbf{P} = (\mathbf{P} - \mathbf{P}) \cap \mathbb{Q}_{\geq 0}\mathbf{P}$, but this is not always the case even for finitely generated periodic sets. Intuitively, the reason is that \mathbf{P} may contain “holes”. For example, consider the set \mathbf{P} in the middle of Figure 1. The points of the form $(x, 1)$ are “holes” in \mathbf{P} .

► **Definition 3.1.** Let \mathbf{P} be a periodic set. The fill of \mathbf{P} is the set $\text{Fill}(\mathbf{P}) := (\mathbf{P} - \mathbf{P}) \cap \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$. A periodic set \mathbf{P} or linear set $\mathbf{b} + \mathbf{P}$ is full if $\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$ is finitely generated and $\mathbf{P} = \text{Fill}(\mathbf{P})$.

The reason for using the closure of $\mathbb{Q}_{\geq 0}\mathbf{P}$ instead of the cone $\mathbb{Q}_{\geq 0}\mathbf{P}$ itself is Lemma 2.2: If the cone is not closed, then the periodic set, in our case $\text{Fill}(\mathbf{P})$, is not finitely generated. If \mathbf{P} was already finitely generated, the definitions coincide.

The following three lemmas collect some useful elementary properties of the fill, similar to properties from [12] or very simple; proofs are in the appendix. Illustrations are in Figure 1.

► **Lemma 3.2.** Let \mathbf{P} be a periodic set. Then $\text{Fill}(\text{Fill}(\mathbf{P})) = \text{Fill}(\mathbf{P})$.

► **Lemma 3.3.** Let \mathbf{P} be a finitely generated periodic set. There exists $\mathbf{x} \in \mathbf{P}$ such that $\mathbf{x} + \text{Fill}(\mathbf{P}) \subseteq \mathbf{P}$.

► **Lemma 3.4.** Let \mathbf{P} be a finitely generated periodic set. For every $\mathbf{x} \in \mathbf{P}$ the set $\mathbf{S} := \mathbf{P} \setminus (\mathbf{x} + \text{Fill}(\mathbf{P}))$ is semilinear and satisfies $\dim(\mathbf{S}) < \dim(\mathbf{P})$.

The lemmas lead to the main result of this section: Every semilinear set is a finite union of full linear sets. In other words, the restriction to full sets does not decrease the “expressive power” of linear sets.

► **Proposition 3.5.** Let \mathbf{S} be a semilinear set. Then there exist full linear sets \mathbf{S}_i such that $\mathbf{S} = \bigcup_{i=1}^r \mathbf{S}_i$. In addition, we can require $\mathbf{S}_i \cap \mathbf{S}_j = \emptyset$ for all $i \neq j$.

Proof. Proof by induction on $\dim(\mathbf{S})$. Write $\mathbf{S} = \bigcup_{i=1}^r \mathbf{L}_i$ with linear sets \mathbf{L}_i such that $\mathbf{L}_i \cap \mathbf{L}_j = \emptyset$ for all i, j . The claim holds for \mathbf{S} if it holds for all \mathbf{L}_i . Hence assume w.l.o.g. \mathbf{S} linear, i.e. $\mathbf{S} = \mathbf{b} + \mathbf{P}$ for $\mathbf{b} \in \mathbb{N}^n$ and \mathbf{P} finitely generated periodic set. By Lemma 3.3 we obtain a vector $\mathbf{x} \in \mathbf{P}$ such that $\mathbf{x} + \text{Fill}(\mathbf{P}) \subseteq \mathbf{P}$. By Lemma 3.4, we obtain that $\mathbf{S} \setminus (\mathbf{x} + \text{Fill}(\mathbf{P}))$ is semilinear with a lower dimension, therefore by induction it can be written as a finite union as required. By simply adding $\mathbf{x} + \text{Fill}(\mathbf{P})$ to this representation, we obtain the required representation for \mathbf{S} . ◀

However, this class is not only visually pleasing, it also has mathematical advantages. Lemma 3.6 provides an equivalent condition for membership in this class, and Lemma 3.7 proves a useful property which unintuitively is false even for finitely generated periodic sets.

► **Lemma 3.6.** Let \mathbf{C} be a finitely generated cone, \mathbf{L} a lattice. Then $\mathbf{P} := \mathbf{C} \cap \mathbf{L}$ is full.

► **Lemma 3.7.** *Let \mathbf{P}, \mathbf{Q} be periodic sets, \mathbf{P} full, and $\mathbf{b}, \mathbf{c} \in \mathbb{Q}^n$ points such that $\mathbf{c} + \mathbf{Q} \subseteq \mathbf{b} + \mathbf{P}$. Then $\mathbf{Q} \subseteq \mathbf{P}$. In particular, the representation $\mathbf{b} + \mathbf{P}$ of a full linear set is unique.*

Observe that if \mathbf{P} is the periodic set in the middle of Figure 1, then $(2, 3) + \{(1, 1)\}^* \subseteq \mathbf{P}$, and the property is violated, since $(1, 1) \notin \mathbf{P}$.

4 Smooth Periodic Sets

Not all periodic sets we need in the paper are finitely generated, but they are smooth, a class introduced by Leroux in [12]. Intuitively, a smooth set \mathbf{P} is at least close to finitely generated, in the sense that $\text{Fill}(\mathbf{P})$ is finitely generated. This result (which was already present in [10] in slightly different form) is proven in Section 4.1. In the second part of this section we identify a novel condition under which smooth sets are closed under intersection and enjoy very good properties (Proposition 4.8).

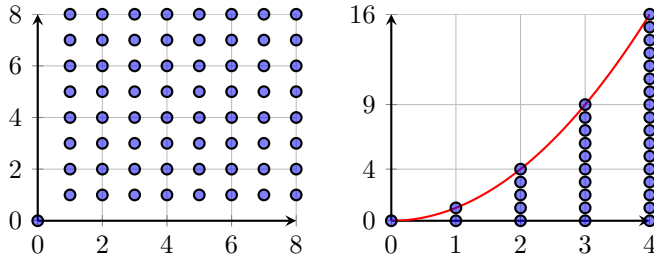
We first introduce the set of directions of a periodic set.

► **Definition 4.1.** [12] *Let \mathbf{P} be a periodic set. A vector \mathbf{d} is a direction of \mathbf{P} if there exists $m \in \mathbb{N}_{>0}$ and a point \mathbf{x} such that $\mathbf{x} + \mathbb{N} \cdot m\mathbf{d} \subseteq \mathbf{P}$, i.e. some line in direction \mathbf{d} is fully contained in \mathbf{P} . The set of directions of \mathbf{P} is denoted $\text{dir}(\mathbf{P})$.*

We can now define smooth periodic sets.

► **Definition 4.2.** [12] *Let \mathbf{P} be a periodic set.*

- \mathbf{P} is asymptotically definable if $\text{dir}(\mathbf{P})$ is a definable cone, i.e. $\text{dir}(\mathbf{P}) \setminus \{0\} = \{\mathbf{x} \in \text{VectSp}(\mathbf{P}) \mid A_1\mathbf{x} > 0, A_2\mathbf{x} \geq 0\}$ for some integer matrices A_1, A_2 .
- \mathbf{P} is well-directed if every sequence $(\mathbf{p}_m)_{m \in \mathbb{N}}$ of vectors $\mathbf{p}_m \in \mathbf{P}$ has an infinite subsequence $(\mathbf{p}_{m_k})_{k \in \mathbb{N}}$ such that $\mathbf{p}_{m_k} - \mathbf{p}_{m_j} \in \text{dir}(\mathbf{P})$ for all $k \geq j$.
- \mathbf{P} is smooth if it is asymptotically definable and well-directed.



■ **Figure 2** The periodic sets $\mathbf{P} = \{(0, 0)\} \cup \mathbb{N}_{>0}^2$ and $\mathbf{P} = \{(x, y) \in \mathbb{N}^2 \mid y \leq x^2\}$ respectively. Neither is finitely generated, both are smooth with $\text{Fill}(\mathbf{P}) = \mathbb{N}^2$.

Figure 2 shows two examples of smooth periodic sets that are not finitely generated.

► **Example 4.3.** Examples of non-smooth sets are $\mathbf{P}_1 = \{(x, y) \mid x \geq \sqrt{2}y\}$ and $\mathbf{P}_2 = (\{(0, 1)\} \cup \{(2^m, 1) \mid m \in \mathbb{N}\})^* = \{(x, n) \in \mathbb{N}^2 \mid x \text{ has at most } n \text{ bits set in binary}\}$. \mathbf{P}_1 is not asymptotically definable, because defining $\text{dir}(\mathbf{P})$ requires irrationals, while \mathbf{P}_2 is not well-directed (see observation 2. below).

Intuitively, the “boundaries” of a smooth periodic set in two dimensions are either straight lines or function graphs “curving outward”, as in the example on the right of Figure 2.

We make a few observations:

1. The set $\text{dir}(\mathbf{P})$ is a cone. Indeed, if two lines in different directions \mathbf{d} and \mathbf{d}' are contained in \mathbf{P} , then by periodicity \mathbf{P} also contains a \mathbf{d}, \mathbf{d}' plane, and so \mathbf{P} contains a line in every direction between \mathbf{d} and \mathbf{d}' .
2. The most important case of Definition 4.2 is when the \mathbf{p}_m are all on the same infinite line $\mathbf{x} + \mathbf{d} \cdot \mathbb{N}$. Then the definition equivalently states that $\mathbf{d} \in \text{dir}(\mathbf{P})$, i.e. some infinite line in direction \mathbf{d} is contained in \mathbf{P} . This makes sets where points are “too scarce” non-smooth. For instance, the set \mathbf{P}_2 of Example 4.3 contains infinitely many points on a horizontal line, but no full horizontal line, which would correspond to an arithmetic progression.

4.1 Fills of Smooth Sets are Finitely Generated

We show that, while a smooth periodic set \mathbf{P} may not be finitely generated, the set $\text{Fill}(\mathbf{P})$ always is. We start with the following lemma, proved in the appendix.

► **Lemma 4.4.** *Let \mathbf{P} be a periodic set. Then $\text{int}(\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}) \subseteq \mathbb{Q}_{\geq 0}\mathbf{P} \subseteq \text{dir}(\mathbf{P}) \subseteq \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$. In particular, all these sets have the same closure.*

► **Example 4.5.** The set on the left of Figure 2 satisfies $\text{int}(\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}) = \mathbb{Q}_{\geq 0}\mathbf{P} \subsetneq \text{dir}(\mathbf{P}) = \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$. Indeed, $\text{int}(\overline{\mathbb{Q}_{\geq 0}\mathbf{P}})$ contains every direction except north and east, but they both belong to $\text{dir}(\mathbf{P})$. The one in the middle satisfies $\text{int}(\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}) \subsetneq \mathbb{Q}_{\geq 0}\mathbf{P} = \text{dir}(\mathbf{P}) \subsetneq \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$, because $\text{int}(\overline{\mathbb{Q}_{\geq 0}\mathbf{P}})$ contains neither north nor east, $\text{dir}(\mathbf{P})$ contains east but not north, and $\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$ contains north and east.

We are now ready to prove the result:

► **Proposition 4.6.** [12] *Let \mathbf{P} be smooth. Then $\text{Fill}(\mathbf{P})$ is full and hence finitely generated.*

Proof. Since \mathbf{P} is smooth, $\text{dir}(\mathbf{P})$ is definable by definition. By Lemma 4.4 we have $\overline{\mathbb{Q}_{\geq 0}\mathbf{P}} = \text{dir}(\mathbf{P})$. So $\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$ is the closure of a definable cone, and hence finitely generated by Lemma 2.1. By Lemma 3.2, $\text{Fill}(\text{Fill}(\mathbf{P})) = \text{Fill}(\mathbf{P})$ is a periodic set with cone $\mathbb{Q}_{\geq 0}\text{Fill}(\mathbf{P}) = \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$, and hence by Lemma 2.2, $\text{Fill}(\mathbf{P})$ is finitely generated. ◀

4.2 Intersection of Smooth Sets

We would like smooth sets to be closed under intersection; further, we would also like that the fill of an intersection of smooth sets, is the intersection of the fills. However, this does not hold in general. The following is a counterexample.

► **Example 4.7.** Define $\mathbf{P} := \{\mathbf{0}\} \cup \mathbb{N}_{>0}^2$, see left of Figure 2, and $\mathbf{P}' = \{(0, 1)\}^*$, the y -axis. We have $\{\mathbf{0}\} = \text{dir}(\mathbf{P} \cap \mathbf{P}') \subsetneq \text{dir}(\mathbf{P}) \cap \text{dir}(\mathbf{P}')$. Also, $\{\mathbf{0}\} = \text{Fill}(\mathbf{P} \cap \mathbf{P}') \subsetneq \text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}') = \mathbf{P}'$.

Fortunately, we can prove: Smooth sets \mathbf{P}, \mathbf{P}' such that $\text{Fill}(\mathbf{P})$, $\text{Fill}(\mathbf{P}')$, and $\text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}')$ have the same dimension behave well under intersection.

► **Proposition 4.8.** *Let \mathbf{P}, \mathbf{P}' be smooth periodic sets such that $\dim(\text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}')) = \dim(\text{Fill}(\mathbf{P})) = \dim(\text{Fill}(\mathbf{P}'))$. Then*

1. $\dim(\mathbf{P} \cap \mathbf{P}') = \dim(\mathbf{P}) = \dim(\mathbf{P}')$.
2. $\text{dir}(\mathbf{P} \cap \mathbf{P}') = \text{dir}(\mathbf{P}) \cap \text{dir}(\mathbf{P}')$.
3. $\text{Fill}(\mathbf{P} \cap \mathbf{P}') = \text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}')$.
4. $\mathbf{P} \cap \mathbf{P}'$ is smooth.

The proofs of all parts are rather technical, and moved to the appendix.

5 Petri sets and Hybridization

We introduce the remaining classes of sets used in our main result: Petri sets and almost hybridlinear sets, as well as hybridization. Petri sets were introduced in [10–12]. Almost hybridlinear sets and hybridization are novel, and play a fundamental role in our main result.

5.1 Petri sets

Leroux introduced almost semilinear sets and developed their theory in [11, 12]. Intuitively, they generalize semilinear sets by replacing linear sets with smooth periodic sets.

► **Definition 5.1.** [11, 12] A set \mathbf{X} is *almost linear* if $\mathbf{X} = \mathbf{b} + \mathbf{P}$, where $\mathbf{b} \in \mathbb{N}^n$ and \mathbf{P} is a smooth periodic set, and *almost semilinear* if it is a finite union of almost linear sets.

It was shown in [11, 12] that VAS reachability sets are almost semilinear. However, it is easy to find almost semilinear sets that are not reachability sets of any VAS. Intuitively, the reason is that the definition of a smooth periodic set only restricts the “asymptotic behavior” of the set, which can be “simple” even if the set itself is very “complex”.

► **Example 5.2.** Let $\mathbf{X} \subseteq \mathbb{N}_{>0}$ be any set, for example $\mathbf{X} := \{m \in \mathbb{N} \mid m \text{ is Gödel-number of non-halting TM}\}$. Then $\mathbf{P} := \{(0, 0)\} \cup (\{1\} \times \mathbf{X}) \cup \mathbb{N}_{\geq 1}^2$ is a smooth periodic set. Indeed, it contains a line in every direction, and is thus well-directed and asymptotically definable.

A way to eliminate at least some of these sets is to require that every intersection of the set with a semilinear set is still almost semilinear, a property enjoyed by all VAS reachability sets. For instance, the intersection of the set \mathbf{X} in Example 5.2 and the linear set $(1, 0) + (0, 1) \cdot \mathbb{N}$ is not almost semilinear. This leads to the notion of a Petri set.

► **Definition 5.3.** [11, 12] A set \mathbf{X} is called a *Petri set* if every intersection $\mathbf{X} \cap \mathbf{S}$ with a semilinear set \mathbf{S} is almost semilinear.

All smooth periodic sets shown so far are also Petri sets, with the exception of Example 5.2. To see that the positive examples are indeed Petri sets we can use the following strong theorem from [12, Theorem IX.1].

► **Theorem 5.4.** *Reachability sets of VAS are Petri sets.*

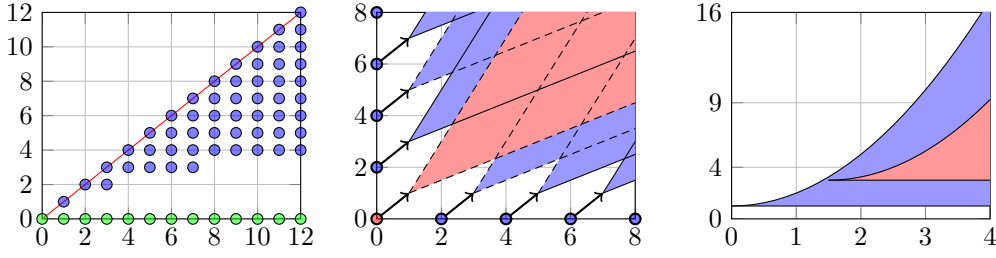
In particular, many expressions $y \leq f(x)$ for convex f , $y \geq f(x)$ for concave f and boolean combinations thereof can be expressed using VAS, and hence form Petri sets.

5.2 Hybridization and almost hybridlinear sets

This section serves as the last crucial buildup for the main results. We introduce the definitions used in the result and furthermore provide important intuition.

The starting point is the following wish: In the proof of Proposition 3.5 we were able to write “w.l.o.g. $\mathbf{S} = \mathbf{b} + \mathbf{P}$ is linear”, in the proofs of our end results we would want to be able to write “w.l.o.g. $\mathbf{X} = \mathbf{b} + \mathbf{P}$ is almost linear”. While this wish is slightly too strong and hence unattainable, our main result shows that we can w.l.o.g. require something negligibly weaker. We start by explaining what type of information is actually contained in “ \mathbf{X} is almost linear”.

Simply on an algebraic level, the main property of almost linear sets is no longer $\mathbf{X} + \mathbf{X} \subseteq \mathbf{X}$ on account of the base point \mathbf{b} . Instead $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$ for a “large” periodic set \mathbf{P} , where large means existence of a point \mathbf{b} such that $\mathbf{X} = \mathbf{b} + \mathbf{P}$. Intuitively, this containment provides



■ **Figure 3** Left: The smooth periodic set $\mathbf{P}_1 := \{(x, y) \mid x \geq y \geq \log_2(x+1)\}$ in blue and $\mathbf{P}_2 := \{(1, 0)\}^*$ in green. The union \mathbf{X} does not have a hybridization, since $\mathbf{P} = \{(0, 0)\}$ is the only possibility to fulfill $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$.

Middle: Let \mathbf{C} be the cone generated by $(2, 1)$ and $(1, 2)$ and assume that $\mathbf{X} + [(1, 1) + \mathbf{C}] \subseteq \mathbf{X}$ holds. Then $(0, 0) \in \mathbf{X}$ implies that the whole red shifted cone is in \mathbf{X} . Importantly, we obtain a similar shifted cone for *every* point $\mathbf{x}' \in \mathbf{X}$, which would fail in the left picture.

Right: Example of an almost linear set with non-zero $\mathbf{b} = (0, 1)$ and $\mathbf{P} = \{(x, y) \mid y \leq x^2\}$. The property $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$ implies that at *any* point in the set, we can imagine another full parabola.

dependencies of the kind “If $\mathbf{x} \in \mathbf{X}$, then the whole set $\mathbf{x} + \mathbf{P} \subseteq \mathbf{X}$ ”. An example is depicted in the middle of Figure 3.

In this section, we slightly weaken our notion of “large” \mathbf{P} , but otherwise require exactly $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$. To give an explanation of just how many such dependencies a periodic set actually provides, we start with a proposition.

► **Proposition 5.5.** *Let \mathbf{P} be a periodic set. Let $\mathbf{F} \subseteq \mathbb{Q}^n$ finite.*

[There exists \mathbf{x} such that $\mathbf{x} + \mathbf{F}^ \subseteq \mathbf{P}$] if and only if $\mathbf{F} \subseteq (\mathbf{P} - \mathbf{P}) \cap \text{dir}(\mathbf{P})$.*

Proof. “ \Rightarrow ”: For all $\mathbf{d} \in \mathbf{F}$, we have $\mathbf{d} = (\mathbf{x} + \mathbf{d}) - \mathbf{x} \in \mathbf{P} - \mathbf{P}$ and $\mathbf{x} + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}$.

“ \Leftarrow ”: We can w.l.o.g. assume that $\mathbf{F} = \{\mathbf{d}\}$ has a single element, because if we find for every $\mathbf{d} \in \mathbf{F}$ an $\mathbf{x}_{\mathbf{d}}$ such that $\mathbf{x}_{\mathbf{d}} + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}$, then $[\sum_{\mathbf{d} \in \mathbf{F}} \mathbf{x}_{\mathbf{d}}] + \mathbf{F}^* \subseteq \mathbf{P}$ by periodicity.

By definition of $\text{dir}(\mathbf{P})$, there exists $m \in \mathbb{N}$ and $\mathbf{x} \in \mathbf{P}$ such that $\mathbf{x} + \mathbb{N} \cdot m\mathbf{d} \subseteq \mathbf{P}$. Since $\mathbf{d} \in \mathbf{P} - \mathbf{P}$, we can write $\mathbf{d} = \mathbf{p} - \mathbf{p}'$ for $\mathbf{p}, \mathbf{p}' \in \mathbf{P}$. We claim that if we define $\mathbf{x}' := \mathbf{x} + (m-1)\mathbf{p}'$, then $\mathbf{x}' + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}$. To see this, let $\lambda \in \mathbb{N}$. Use division with remainder to write $\lambda = qm + r$ with $q, r \in \mathbb{N}, r < m$. We have

$$\begin{aligned} \mathbf{x}' + \lambda \mathbf{d} &= \mathbf{x} + (m-1)\mathbf{p}' + qm\mathbf{d} + r(\mathbf{p} - \mathbf{p}') = [\mathbf{x} + qm\mathbf{d}] + [(m-1)\mathbf{p}' - r\mathbf{p}'] + r\mathbf{p} \\ &\in \mathbf{P} + \mathbf{P} + \mathbf{P} \subseteq \mathbf{P}. \end{aligned}$$

◀

Observe that since \mathbf{P} is periodic, this immediately also shows $\mathbf{P} + \mathbf{x} + \mathbf{F}^* \subseteq \mathbf{P}$.

Next we explain the typical use case of Proposition 5.5, connecting with Figure 3. Start with any full periodic set $\mathbf{Q} = \mathbf{C} \cap (\mathbf{P} - \mathbf{P})$, where \mathbf{C} is finitely generated and $\mathbf{C} \subseteq \text{int}(\overline{\mathbb{Q}_{\geq 0}\mathbf{P}})$. Then, by Lemma 3.6, \mathbf{Q} is finitely generated by some set \mathbf{F} . We can apply Proposition 5.5 on \mathbf{F} to obtain a vector \mathbf{v} (name adapted due to arrow interpretation of the vector) with $\mathbf{P} + \mathbf{v} + \mathbf{Q} \subseteq \mathbf{P}$. We then only draw the cone \mathbf{C} and remember separately that we are only interested in integer points, even valued points, etc.

To simplify proofs, we like to use closure properties, mainly under intersection and union. Since already the class of linear sets is however not closed under intersection, we need to consider an intermediate class of semilinear sets called *hybridlinear sets*, as introduced in [4] by Ginsburg and Spanier and later studied in [1] by Chistikov and Haase. Similar to those papers, we again profit from the nice properties of this class.

► **Definition 5.6.** A set $\mathbf{H} \subseteq \mathbb{N}^n$ is *hybridlinear* if there exists a finitely generated periodic set $\mathbf{P} \subseteq \mathbb{N}^n$ and a finite set $\{\mathbf{b}_1, \dots, \mathbf{b}_r\} \subseteq \mathbb{N}^n$ such that $\mathbf{H} = \{\mathbf{b}_1, \dots, \mathbf{b}_r\} + \mathbf{P}$.

Now we introduce the definition of hybridization, based on the crucial property of $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$. Our notion of “ \mathbf{P} is large” is “Pasting $\text{Fill}(\mathbf{P})$ from finitely many locations covers \mathbf{X} ”. Remember that $\text{Fill}(\mathbf{P})$ can be way larger than \mathbf{P} , for example the parabola (right of Figure 3) fulfills $\text{Fill}(\mathbf{P}) = \mathbb{N}^2$.

► **Definition 5.7.** Let $\mathbf{X} \subseteq \mathbb{N}^n$ be non-empty. A hybridlinear set \mathbf{H} is a *hybridization* of \mathbf{X} if there exists a finite set $\mathbf{B} \subseteq \mathbb{N}^n$ and a smooth periodic set \mathbf{P} such that $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$, $\mathbf{H} = \mathbf{B} + \text{Fill}(\mathbf{P})$ and $\mathbf{X} \subseteq \mathbf{H}$.

► **Remark 5.8.** For the moment, though this will be immediately repaired next section, this property is representation dependent. For example the even numbers $2\mathbb{N}$ have hybridization $H = \{0, 1\} + 2\mathbb{N}$, but not $H = \mathbb{N}$.

We end this section by mentioning that similar to linear and semilinear, also hybridlinear has an interesting “almost”-type generalization, which is closely connected to hybridizations. Furthermore, it is a good source of non-trivial (i.e. not of type \mathbf{P} has hybridization $\text{Fill}(\mathbf{P})$) examples of hybridization: For example $[(0, 1) + \mathbf{P}_1] \cup [(0, 6) + \mathbf{P}_2]$ for $\mathbf{P}_1 = \{(x, y) \in \mathbb{N}^2 \mid y \leq x^2\}$ and $\mathbf{P}_2 = \{(x, y) \in \mathbb{N}^2 \mid y \geq \log_2(x + 1)\}$ has hybridization \mathbb{N}^2 , since $\text{Fill}(\mathbf{P}_1) = \text{Fill}(\mathbf{P}_2) = \mathbb{N}^2$.

► **Definition 5.9.** A non-empty set $\mathbf{X} \subseteq \mathbb{N}^n$ is *almost hybridlinear* if there exist $\mathbf{b}_1, \dots, \mathbf{b}_r \in \mathbb{N}^n$ and smooth $\mathbf{P}_1, \dots, \mathbf{P}_r$ with $\mathbf{X} = \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{P}_i$, such that $\text{Fill}(\mathbf{P}_i) = \text{Fill}(\mathbf{P}_j)$ for all i, j .

► **Theorem 5.10.** A non-empty Petri set $\mathbf{X} \subseteq \mathbb{N}^n$ is almost hybridlinear if and only if it has a hybridization.

6 Proof of Theorem 1.1

In this section we prove Theorem 1.1, which we restate here for convenience.

► **Theorem 1.1.** Let \mathbf{X} be a Petri set. For every semilinear set \mathbf{S} there exists a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ into pairwise disjoint full linear sets such that for all i either $\mathbf{X} \cap \mathbf{S}_i = \emptyset$, $\mathbf{S}_i \subseteq \mathbf{X}$ or $\mathbf{X} \cap \mathbf{S}_i$ is irreducible with hybridization \mathbf{S}_i . Further, if \mathbf{X} is the reachability set of a VAS, then the partition is computable.

The algorithm and its proof will refine the partition in three steps, respectively described in Section 6.1, Section 6.2 and Section 6.3: During the first two steps the sets $\mathbf{X} \cap \mathbf{S}_i$ are not required to be irreducible, and in addition after the first step, the \mathbf{S}_i are allowed to be hybridlinear instead of full linear.

6.1 Existence of a Hybridlinear Partition

We collect five important properties of hybridizations in Proposition 6.2. Then, we use these properties to formulate a procedure for producing a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ of sets, not necessarily full linear, satisfying the properties of Theorem 1.1 except for irreducibility. The procedure is described in Figure 4 below. It is effective for VAS reachability sets, but not in general.

We start by reminding that the class of hybridlinear sets is closed under intersection.

► **Lemma 6.1.** [9, Lemma 7.8] *Let $\mathbf{b}_1 + \mathbf{Q}_1$ and $\mathbf{b}_2 + \mathbf{Q}_2$ be linear sets. Then $(\mathbf{b}_1 + \mathbf{Q}_1) \cap (\mathbf{b}_2 + \mathbf{Q}_2) = \mathbf{B} + (\mathbf{Q}_1 \cap \mathbf{Q}_2)$ for some finite \mathbf{B} .*

We can now collect the properties of hybridizations:

► **Proposition 6.2.** *The following statements hold:*

1. *If \mathbf{H} is a hybridization of \mathbf{X} , then $\dim(\mathbf{X}) = \dim(\mathbf{H})$.*
2. *If \mathbf{H} is a hybridization of \mathbf{X} and $\mathbf{L} = \mathbf{b} + \mathbf{Q}$ a full linear set such that $\dim(\mathbf{H} \cap \mathbf{L}) = \dim(\mathbf{H}) = \dim(\mathbf{L})$, then $\mathbf{H} \cap \mathbf{L}$ is a hybridization for $\mathbf{X} \cap \mathbf{L}$ or $\mathbf{X} \cap \mathbf{L}$ is empty.*
3. *If \mathbf{H} is a hybridization for both \mathbf{X}_1 and \mathbf{X}_2 , then \mathbf{H} is a hybridization for $\mathbf{X}_1 \cup \mathbf{X}_2$.*
4. *For every Petri set \mathbf{X} and semilinear \mathbf{S} there is a partition $\mathbf{X} \cap \mathbf{S} = \mathbf{X}_1 \cup \dots \cup \mathbf{X}_r$ of $\mathbf{X} \cap \mathbf{S}$ such that every \mathbf{X}_i has a full linear hybridization \mathbf{L}_i .*
5. *If \mathbf{X} is the reachability set of a VAS, then the set $\{\mathbf{L}_1, \dots, \mathbf{L}_r\}$ of full linear hybridizations of part 4. is computable.*

Proof. For proofs 1)-2), write $\mathbf{H} := \mathbf{B} + \text{Fill}(\mathbf{P})$, where \mathbf{P} is smooth and $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$.

1): This follows from the properties of dimension in Lemmas 2.4 and 2.5. In particular, $\dim(\mathbf{P}) = \dim(\mathbf{V})$, where \mathbf{V} is the vector space generated by \mathbf{P} , also implies $\dim(\mathbf{P}) = \dim(\text{Fill}(\mathbf{P}))$. Hence $\mathbf{X} \subseteq \mathbf{H}$ implies $\dim(\mathbf{X}) \leq \dim(\mathbf{P})$. Since \mathbf{X} is non-empty, $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$ implies $\dim(\mathbf{X}) \geq \dim(\mathbf{P})$.

2): By Lemma 6.1, $\mathbf{H} \cap \mathbf{L} = \mathbf{F} + (\text{Fill}(\mathbf{P}) \cap \mathbf{Q})$ for some finite set \mathbf{F} . By Proposition 4.8, we have that $\mathbf{P} \cap \mathbf{Q}$ is smooth and $\text{Fill}(\mathbf{P} \cap \mathbf{Q}) = \text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{Q}) = \text{Fill}(\mathbf{P}) \cap \mathbf{Q}$. We have $\mathbf{X} \cap \mathbf{L} \subseteq \mathbf{H} \cap \mathbf{L}$. We also have $(\mathbf{X} \cap \mathbf{L}) + (\mathbf{P} \cap \mathbf{Q}) \subseteq \mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$ and $(\mathbf{X} \cap \mathbf{L}) + (\mathbf{P} \cap \mathbf{Q}) \subseteq \mathbf{L} + \mathbf{Q} \subseteq \mathbf{L}$, hence $\mathbf{H} \cap \mathbf{L}$ is a hybridization of $\mathbf{X} \cap \mathbf{L}$.

3): Write $\mathbf{B}_1 + \text{Fill}(\mathbf{P}_1) = \mathbf{H} = \mathbf{B}_2 + \text{Fill}(\mathbf{P}_2)$, where \mathbf{P}_1 for \mathbf{X}_1 and \mathbf{P}_2 for \mathbf{X}_2 are as in the definition of hybridization. By Lemma 6.1, we have $\mathbf{H} = \mathbf{H} \cap \mathbf{H} = \mathbf{F} + [\text{Fill}(\mathbf{P}_1) \cap \text{Fill}(\mathbf{P}_2)]$ for some finite set \mathbf{F} . Define $\mathbf{P} := \mathbf{P}_1 \cap \mathbf{P}_2$ and $\mathbf{X} := \mathbf{X}_1 \cup \mathbf{X}_2$. By Proposition 4.8, \mathbf{P} is smooth and $\text{Fill}(\mathbf{P}) = \text{Fill}(\mathbf{P}_1) \cap \text{Fill}(\mathbf{P}_2)$. We also have $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$.

4): Since \mathbf{X} is a Petri set, $\mathbf{X} \cap \mathbf{S}$ is almost semilinear, and can hence be written as $\mathbf{X} = \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{P}_i$ for smooth periodic sets $\mathbf{P}_i \subseteq \mathbb{N}^n$ and points $\mathbf{b}_i \in \mathbb{N}^n$. Every $\mathbf{X}_i := \mathbf{b}_i + \mathbf{P}_i$ is by definition almost hybridlinear with hybridization $\mathbf{b}_i + \text{Fill}(\mathbf{P}_i)$, which is a full linear set.

5): 4) can be computed using the Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition, which splits a VAS into finitely many perfect MGTS (the exact name varies, [7], [8], [9]), and for every perfect MGTS, constructs a system of linear equations, called characteristic system, whose solution set is a hybridization [9, Theorem 5.3]. In fact, the definition of a linearization of a pseudo-linear set from [9] is equivalent to hybridization after slightly strengthening one condition. ¹ ◀

► **Proposition 6.3.** *Let \mathbf{X} be a Petri set and let \mathbf{S} be a semilinear set. $\text{Partition}(\mathbf{X}, \mathbf{S})$ produces a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ into pairwise disjoint hybridlinear sets (not necessarily full linear) such that for every i the set $\mathbf{X} \cap \mathbf{S}_i$ is either empty or has hybridization \mathbf{S}_i . Further, if \mathbf{X} is the reachability set of a VAS, then the partition is computable.*

Proof. The procedure is depicted in Figure 4, in addition we give an intuitive description of it: In Step 1) we first partition \mathbf{S} into full linear sets and consider them separately. So

¹ While Hauschildt already used the KLMST decomposition in [5] in 1990, it took until 2019 ([14], [15]) to fully understand the theoretical aspects behind the algorithm and its complexity of Ackermann.

Partition(\mathbf{X}, \mathbf{S}). Input: Petri set \mathbf{X} and semilinear set \mathbf{S} :

1) If \mathbf{S} is empty, return \mathbf{S} . If \mathbf{S} is not full, compute a partition $\mathbf{S}_1, \dots, \mathbf{S}_r$ of \mathbf{S} into full linear sets, return $\bigcup_{i=1}^r \text{Partition}(\mathbf{X}, \mathbf{S}_i)$ and stop.

Otherwise, compute the set $\mathcal{L} = \{\mathbf{L}_1, \dots, \mathbf{L}_r\}$ of full linear hybridizations of the partition $\mathbf{X}_1 \cup \dots \cup \mathbf{X}_r$ of $\mathbf{X} \cap \mathbf{S}$ given by Proposition 6.2(4), and move to step 2).

Remark: This step is not effective for arbitrary Petri sets, but it is effective for VAS reachability sets by Proposition 6.2(5).

If $r = 0$, i.e., if $\mathbf{X} \cap \mathbf{S}$ is empty, then return \mathbf{S} and stop. Otherwise, move to step 2).

2) For every $\mathbf{L}_i \in \mathcal{L}$ compute a decomposition \mathcal{K}_i of $\mathbf{L}_i^C \cap \mathbf{S}$ into full linear sets, where \mathbf{L}_i^C is the complement of \mathbf{L}_i , and move to step 3).

3) Let \mathcal{M} be the set of tuples $(\mathbf{M}_1, \dots, \mathbf{M}_r) \in (\{\mathbf{L}_1\} \cup \mathcal{K}_1) \times \dots \times (\{\mathbf{L}_r\} \cup \mathcal{K}_r)$.

For every $M \in \mathcal{M}$, let $\mathbf{S}_M := \mathbf{S} \cap \mathbf{M}_1 \cap \dots \cap \mathbf{M}_r$.

Remark: $\{\mathbf{S}_M \mid M \in \mathcal{M}\}$ is a partition of \mathbf{S} .

For every $M \in \mathcal{M}$, define P_M as follows: If $\dim(\mathbf{S}_M) < \dim(\mathbf{S})$, then $P_M := \text{Partition}(\mathbf{X}, \mathbf{S}_M)$, otherwise $P_M := \{\mathbf{S}_M\}$. Output $\bigcup_{M \in \mathcal{M}} P_M$.

■ **Figure 4** The procedure Partition(\mathbf{X}, \mathbf{S}).

assume that \mathbf{S} is a full linear set. The procedure uses Proposition 6.2(5) to compute a set of full linear hybridizations $\mathbf{L}_1, \dots, \mathbf{L}_r$ of a partition $\mathbf{X}_1 \cup \dots \cup \mathbf{X}_r$ of $\mathbf{X} \cap \mathbf{S}$. Step 2) considers all possible sets obtained by picking for each $i \in \{1, \dots, r\}$ either the set \mathbf{L}_i or a linear set of its complement (its complement is semilinear, and so a finite union of linear sets), and intersecting all of them. The procedure adds all the sets having full dimension to the output partition, and does a recursive call on the others.

Every step can be performed: The set \mathcal{L} of Step 1 exists by Proposition 6.2(4). To check the dimension of a semilinear set $\mathbf{S} = \bigcup_{j=1}^r \mathbf{b}_j + \mathbf{F}_j^*$, which is needed in step 3), we use Lemma 2.5 to obtain that for \mathbf{F}_j^* this is simply the rank of the generator matrix, and by Lemma 2.4 we have $\dim(\mathbf{S}) = \max_j \dim(\mathbf{F}_j^*)$.

Termination: Partition(\mathbf{X}, \mathbf{S}) only performs a recursive call if \mathbf{S} is not a full linear set or on semilinear sets \mathbf{S}' with $\dim(\mathbf{S}') < \dim(\mathbf{S})$, hence recursion depth is at most $2 \dim(\mathbf{S}) + 1$ and termination immediate.

Correctness: The proof obligation for correctness is that for every $M = (\mathbf{M}_1, \dots, \mathbf{M}_r) \in \mathcal{M}$, where \mathbf{S}_M fulfills $\dim(\mathbf{S}_M) = \dim(\mathbf{S})$, $\mathbf{X} \cap \mathbf{S}_M$ is either empty or has \mathbf{S}_M as hybridization. Therefore fix such M .

Claim: $\dim(\mathbf{M}_j) = \dim(\mathbf{S})$ for all j .

Proof of claim: $\geq \dim(\mathbf{S})$ follows since all these sets contain \mathbf{S}_M , which fulfills $\dim(\mathbf{S}_M) = \dim(\mathbf{S})$. For the other direction, we have $\dim(\mathbf{S}) \geq \dim(\mathbf{X} \cap \mathbf{S}) =_{P.6.3:1,4} \max_j \dim(\mathbf{L}_j)$ to prove “ \leq ” for j where we choose \mathbf{L}_j , and for other j we use $\mathbf{L}_j^C \cap \mathbf{S} \subseteq \mathbf{S}$.

The claim allows us to use Proposition 6.2(2). Let \mathbf{X}_j be such that $\mathbf{X} \cap \mathbf{S} = \bigcup_{j=1}^r \mathbf{X}_j$ and \mathbf{X}_j has hybridization \mathbf{L}_j . By applying (2) enough times, for every j with $\mathbf{M}_j = \mathbf{L}_j$, we have $\mathbf{X}_j \cap \mathbf{S}_M$ has hybridization \mathbf{S}_M . This does not depend on j because intersecting with \mathbf{L}_j twice does not change the set. For all other j we have $\mathbf{X}_j \cap \mathbf{S}_M = \emptyset$, since we intersect with the complement of an overapproximation. Hence $\mathbf{X} \cap \mathbf{S}_M = \bigcup_{j, \mathbf{M}_j = \mathbf{L}_j} (\mathbf{X}_j \cap \mathbf{S}_M)$ has hybridization \mathbf{S}_M by Proposition 6.2(3), or is empty if we never chose $\mathbf{M}_j = \mathbf{L}_j$. ◀

6.2 Existence of a Full Linear Partition

We show that Proposition 6.3 can be strengthened to make the sets \mathbf{S}_i not only hybridlinear, but even full linear. Since the full linear representation is unique by Lemma 3.7, this furthermore removes the representation dependence of Remark 5.8: We always consider the full linear representation.

► **Proposition 6.4.** *Let \mathbf{X} be a Petri set. For every semilinear set \mathbf{S} there exists a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ of \mathbf{S} into pairwise disjoint full linear sets such that for every i the set $\mathbf{X} \cap \mathbf{S}_i$ is either empty or has hybridization \mathbf{S}_i . Further, if \mathbf{X} is the reachability set of a VAS, then the partition is computable.*

Proof. The algorithm uses a subroutine with the same inputs and outputs, which assumes that $\mathbf{X} \cap \mathbf{S}$ has hybridization \mathbf{S} . We first describe the main algorithm, and then the subroutine.

Main algorithm: First apply Proposition 6.3 to obtain a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ into hybridlinear sets otherwise satisfying the conditions. Output $\bigcup_{i=1}^k \text{Subroutine}(\mathbf{X}, \mathbf{S}_i)$.

Subroutine: If \mathbf{S} is already full linear, return \mathbf{S} . Otherwise write $\mathbf{S} = \{\mathbf{c}_1, \dots, \mathbf{c}_r\} + \text{Fill}(\mathbf{P})$. Let $j \sim k \iff \mathbf{c}_j - \mathbf{c}_k \in \text{Fill}(\mathbf{P}) - \text{Fill}(\mathbf{P}) = \mathbf{P} - \mathbf{P}$. Compute a system R of representatives for \sim . For every $i \in R$, define $\mathbf{S}_i := \mathbf{c}_i + \text{Fill}(\mathbf{P})$. Define $\mathbf{S}' := \mathbf{S} \setminus \bigcup_{i \in R} \mathbf{S}_i$ and output $\{\mathbf{S}_i \mid i \in R\} \cup \text{MainAlgorithm}(\mathbf{X}, \mathbf{S}')$.

Termination: We prove that recursion depth $\leq 2 \dim(\mathbf{S}) + 1$ by proving $\dim(\mathbf{S}') < \dim(\mathbf{S})$ in the subroutine. For every equivalence class C of \sim , there exists $\mathbf{c} \in \mathbb{Z}^n$ such that $\mathbf{c}_j - \mathbf{c} \in \mathbf{P}$ for all $j \in C$. To see this, fix some $i \in C$, and write $\mathbf{c}_j - \mathbf{c}_i = \mathbf{p}_j - \mathbf{p}'_j \in \mathbf{P} - \mathbf{P}$. Choose $\mathbf{c} := \mathbf{c}_i - \sum_{j \in C} \mathbf{p}'_j$.

Then $\bigcup_{j \in C} \mathbf{c}_j + \text{Fill}(\mathbf{P}) \subseteq \mathbf{c} + \text{Fill}(\mathbf{P})$, and hence using Lemma 3.4 we obtain $\dim(\bigcup_{j \in C} \mathbf{c}_j + \text{Fill}(\mathbf{P}) \setminus \mathbf{S}_i) \leq \dim(\mathbf{c} + \text{Fill}(\mathbf{P}) \setminus \mathbf{c}_i + \text{Fill}(\mathbf{P})) < \dim(\text{Fill}(\mathbf{P}))$.

Correctness: The main algorithm is clearly correct if the subroutine is. In the subroutine, we have $\mathbf{S}_i \cap \mathbf{S}_j = \emptyset$ since $i \not\sim j$ for $i, j \in R$. All \mathbf{S}_i are full linear by definition. Furthermore, $\mathbf{X} \cap \mathbf{S}_i$ has hybridization $\mathbf{H} \cap \mathbf{S}_i = \mathbf{S}_i$ by Proposition 6.2(2). Observe that Proposition 6.2(2) specifically shows that the intersection of the representations, which is the full linear representation of \mathbf{S}_i , is a hybridization. ◀

6.3 Reducibility of almost hybridlinear Sets

The final ingredient of our main result is reducibility. We name it after its counterpart in Hauschildt's PhD thesis [5].

► **Definition 6.5.** *A set \mathbf{X} with hybridization $\mathbf{c} + \text{Fill}(\mathbf{P})$ is reducible if there exists \mathbf{x} such that $\mathbf{x} + \text{Fill}(\mathbf{P}) \subseteq \mathbf{X}$.*

In other words, \mathbf{X} is reducible if every large enough point of its hybridization is already in \mathbf{X} . Observe that this does not follow from hybridization, as $\text{Fill}(\mathbf{P})$ is larger than \mathbf{P} .

We already know a special case where this property is decidable: Proposition 5.5 in particular shows that in case of $\mathbf{X} = \mathbf{b} + \mathbf{P}$, \mathbf{X} is reducible if and only if $\text{dir}(\mathbf{P}) = \overline{\mathbb{Q}_{\geq 0} \mathbf{P}}$. Similarly, whether an almost hybridlinear set $\mathbf{X} = \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{P}_i$ is reducible only depends on the cones $\text{dir}(\mathbf{P}_i)$. Since matrices for the definable cones $\text{dir}(\mathbf{P}_i)$ can in case of VAS be determined using KLMST-decomposition [5], we obtain the following.

► **Theorem 6.6.** *[5, even without promise] The following problem is decidable.*

Input: Reachability set \mathbf{R} , full linear set \mathbf{S} , with promise that $\mathbf{R} \cap \mathbf{S}$ has hybridization \mathbf{S} .

Output: Is $\mathbf{R} \cap \mathbf{S}$ reducible?

We can now prove our main result.

► **Theorem 1.1.** *Let \mathbf{X} be a Petri set. For every semilinear set \mathbf{S} there exists a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ into pairwise disjoint full linear sets such that for all i either $\mathbf{X} \cap \mathbf{S}_i = \emptyset$, $\mathbf{S}_i \subseteq \mathbf{X}$ or $\mathbf{X} \cap \mathbf{S}_i$ is irreducible with hybridization \mathbf{S}_i . Further, if \mathbf{X} is the reachability set of a VAS, then the partition is computable.*

Proof. Step 1: Use Proposition 6.4 to compute a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ into full linear sets such that $\mathbf{X} \cap \mathbf{S}_i$ has hybridization \mathbf{S}_i if it is non-empty. For every set \mathbf{S}_i with $\mathbf{X} \cap \mathbf{S}_i \neq \emptyset$ do Step 2.

Step 2: Decide whether $\mathbf{X} \cap \mathbf{S}_i$ is reducible using Theorem 6.6. If irreducible, output \mathbf{S}_i . Otherwise there exists \mathbf{x} such that $\mathbf{x} + \mathbf{Q} \subseteq \mathbf{X} \cap \mathbf{S}_i$, where $\mathbf{S}_i = \mathbf{c} + \mathbf{Q}$. Find such an \mathbf{x} , add $\mathbf{x} + \mathbf{Q} \subseteq \mathbf{X}$ to the final partition and do a recursive call on $\mathbf{S}_i \setminus (\mathbf{x} + \mathbf{Q})$.

Termination: We claim that we only perform recursion on \mathbf{S}' with $\dim(\mathbf{S}') < \dim(\mathbf{S})$. To see this, take $\mathbf{S}_i = \mathbf{c} + \mathbf{Q}$ such that $\mathbf{X} \cap \mathbf{S}_i$ is reducible. We have $\dim(\mathbf{S}_i \setminus \mathbf{x} + \mathbf{Q}) = \dim(\mathbf{c} + \mathbf{Q} \setminus \mathbf{x} + \mathbf{Q}) < \dim(\mathbf{Q})$ by Lemma 3.4, wherefore the recursion uses a lower dimensional set, and termination follows from bounded recursion depth.

Correctness: Obvious.

The partition is computable for VAS: We have to be able to find \mathbf{x} with $\mathbf{x} + \mathbf{Q} \subseteq \mathbf{X}$ given the promise that such an \mathbf{x} exists. This is possible since containment of semilinear sets in reachability sets is decidable by [12] using flatability. ◀

7 Corollaries of Theorem 1.1

7.1 VAS semilinearity is decidable

We prove that the semilinearity problem for VAS is decidable. We start with a lemma, whose full proof is in the appendix.

► **Lemma 7.1.** *Let \mathbf{X} be a semilinear Petri set with hybridization $\mathbf{c} + \mathbf{Q}$. Then \mathbf{X} is reducible.*

Proof. Idea: The hybridization describes all “limit directions”, with the problematic ones being for example “north” in case of the parabola $\{(x, y) \mid y \leq x^2\}$, which is a limit but not actually a direction. If \mathbf{X} is semilinear though, then the steepness can only increase finitely often, namely when changing to a different linear component, and all limit directions are actually also directions. Using this for generators of $\text{Fill}(\mathbf{P})$ we find $\mathbf{x} + \text{Fill}(\mathbf{P}) \subseteq \mathbf{X}$. ◀

► **Corollary 7.2.** *The following problem is decidable.*

Input: Reachability set \mathbf{R} of VAS, semilinear \mathbf{S} .

Output: Is $\mathbf{R} \cap \mathbf{S}$ semilinear?

Proof. The algorithm computes the partition of Theorem 1.1 and checks whether the third case does not occur.

Correctness: If $\mathbf{R} \cap \mathbf{S}$ is semilinear, then in particular $\mathbf{R} \cap \mathbf{S}_i$ is semilinear for every part \mathbf{S}_i of the partition. By Lemma 7.1, $\mathbf{R} \cap \mathbf{S}_i$ cannot be irreducible, and so either $\mathbf{R} \cap \mathbf{S}_i = \emptyset$ or $\mathbf{S}_i \subseteq \mathbf{R}$ for all i .

On the other hand, if only the cases $\mathbf{R} \cap \mathbf{S}_i = \emptyset$ and $\mathbf{S}_i \subseteq \mathbf{R}$ occur, then the \mathbf{S}_i such that $\mathbf{S}_i \subseteq \mathbf{R}$ form a semilinear representation. ◀

7.2 On the Complement of a VAS Reachability Set

We show that if the complement of a VAS reachability set is infinite, then it contains an infinite linear set. The main part of the argument was already depicted in the middle of Figure 3: If \mathbf{X} contains enough of the boundary, then it is reducible.

We hence need to formalize the notion of boundary and interior also for full linear sets. If $\mathbf{L} = \mathbf{b} + \mathbf{Q}$ is a full linear set, then $\text{int}(\mathbf{L}) := \mathbf{b} + (\mathbf{Q} \cap \text{int}(\mathbb{Q}_{\geq 0}\mathbf{Q}))$ is the interior of \mathbf{L} and $\partial(\mathbf{L}) := \mathbf{b} + (\mathbf{Q} \cap \partial(\mathbb{Q}_{\geq 0}\mathbf{Q}))$ is the boundary of \mathbf{L} , both are inherited from the cone. These sets are both semilinear, as can be seen by using the definition expressible via $\varphi \in \text{FO}(\mathbb{N}, +, \geq)$, i.e. Presburger Arithmetic. Remember that we consider definable cones, i.e. cones expressible in $\text{FO}(\mathbb{Q}, +, \geq)$. In the appendix, we prove the following proposition, formalizing the first part of the proof.

► **Proposition 7.3.** *Let \mathbf{X} be a set with hybridization $\mathbf{c} + \text{Fill}(\mathbf{P})$. Assume that $|\partial(\mathbf{c} + \text{Fill}(\mathbf{P})) \setminus \mathbf{X}| < \infty$. Then \mathbf{X} is reducible.*

► **Corollary 7.4.** *Let \mathbf{X} be a Petri set. Let \mathbf{S} be a semilinear set such that $\mathbf{S} \setminus \mathbf{X}$ is infinite. Then $\mathbf{S} \setminus \mathbf{X}$ contains an infinite linear set.*

Proof. Proof by induction on $\dim(\mathbf{S})$. If $\dim(\mathbf{S}) = 0$, the property holds vacuously. Else consider the partition of Theorem 1.1. Since $\mathbf{S} \setminus \mathbf{X}$ is infinite, some $\mathbf{S}_i \setminus \mathbf{X}$ is infinite. Fix such an i . If $\mathbf{X} \cap \mathbf{S}_i$ is semilinear, then $\mathbf{S}_i \setminus \mathbf{X}$ also is, and the property holds trivially. Hence $\mathbf{X} \cap \mathbf{S}_i$ is irreducible with hybridization $\mathbf{S}_i = \mathbf{c} + \text{Fill}(\mathbf{P})$. Assume for contradiction that $\mathbf{S}_i \setminus \mathbf{X}$ does not contain an infinite linear set. Then in particular $\partial(\mathbf{S}_i) \setminus \mathbf{X}$ does not. We have $\dim(\partial(\mathbf{S}_i)) < \dim(\mathbf{S}_i)$, since the boundary is contained in the finite union of the facets. Hence $|\partial(\mathbf{S}_i) \setminus \mathbf{X}| < \infty$ by induction. By Proposition 7.3, $\mathbf{X} \cap \mathbf{S}_i$ is reducible. Contradiction. ◀

In the appendix, we even prove another corollary of the partition. The proof is based on the existence of a partition as in Theorem 1.1, which has the properties for two Petri sets \mathbf{X}_1 and \mathbf{X}_2 at once.

► **Corollary 7.5.** *Let \mathcal{V} be a VAS, and \mathbf{X} a Petri set such that $\text{Reach}(\mathcal{V}) \cap \mathbf{X} = \emptyset$. Then there exists a semilinear inductive invariant \mathbf{S}' of \mathcal{V} such that $\text{Reach}(\mathcal{V}) \subseteq \mathbf{S}'$ and $\mathbf{X} \cap \mathbf{S}' = \emptyset$.*

8 Conclusion

We have introduced hybridizations, and used them to prove a powerful decomposition theorem for Petri sets. For VAS reachability sets the decomposition can be effectively computed. We have derived several geometric and computational results.

We think that our decomposition can help to study the computational power of VAS. For example, it leads to this little corollary:

► **Corollary 8.1.** *Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a function whose graph does not contain an infinite line. Then either $\{(x, y) \mid y < f(x)\}$ or $\{(x, y) \mid y > f(x)\}$ is not a Petri set.*

Proof. Assume for contradiction that both are Petri sets. Then, since finite unions of Petri sets are again Petri sets, $\{(x, y) \mid y \neq f(x)\}$ is a Petri set. Its complement is the graph of f , which by assumption does not contain an infinite line. Contradiction to Corollary 7.4. ◀

We plan to study other possible applications of our result, derived from the fact that the reachability relation of a VAS is also a Petri set.

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A

Proofs of Section 2

► **Lemma 2.4.** *Let $\mathbf{X}, \mathbf{X}' \subseteq \mathbb{Q}^n, \mathbf{b} \in \mathbb{Q}^n$. Then $\dim(\mathbf{X}) = \dim(\mathbf{b} + \mathbf{X})$ and $\dim(\mathbf{X} \cup \mathbf{X}') = \max\{\dim(\mathbf{X}), \dim(\mathbf{X}')\}$. Further, if $\mathbf{X} \subseteq \mathbf{X}'$, then $\dim(\mathbf{X}) \leq \dim(\mathbf{X}')$.*

Proof. 1) We prove $\dim(\mathbf{X}) \geq \dim(\mathbf{b} + \mathbf{X})$, the other direction follows by choosing $-\mathbf{b}$. Let \mathbf{b}_i and \mathbf{V}_i such that $\dim(\mathbf{V}_i) \leq \dim(\mathbf{X})$ and $\mathbf{X} \subseteq \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{V}_i$. Then $\mathbf{b} + \mathbf{X} \subseteq \bigcup_{i=1}^r (\mathbf{b} + \mathbf{b}_i) + \mathbf{V}_i$, and hence $\dim(\mathbf{b} + \mathbf{X}) \leq \max_i \dim(\mathbf{V}_i) = \dim(\mathbf{X})$.

2) “ \geq ”: Covering $\mathbf{X} \cup \mathbf{X}'$ covers both \mathbf{X} and \mathbf{X}' .

“ \leq ”: Let \mathbf{b}_i and \mathbf{V}_i such that $\dim(\mathbf{V}_i) \leq \dim(\mathbf{X})$ and $\mathbf{X} \subseteq \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{V}_i$, and \mathbf{c}_i and \mathbf{W}_i such that $\dim(\mathbf{W}_i) \leq \dim(\mathbf{X}')$ and $\mathbf{X}' \subseteq \bigcup_{i=1}^s \mathbf{c}_i + \mathbf{W}_i$. Then

$$\mathbf{X} \cup \mathbf{X}' \subseteq \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{V}_i \cup \bigcup_{i=1}^s \mathbf{c}_i + \mathbf{W}_i.$$

3) Use 2), and observe that $\mathbf{X} \cup \mathbf{X}' = \mathbf{X}'$. ◀

B

 Proofs of Section 3

► **Lemma 3.2.** *Let \mathbf{P} be a periodic set. Then $\text{Fill}(\text{Fill}(\mathbf{P})) = \text{Fill}(\mathbf{P})$.*

Proof. Both $O_1: \mathbf{P} \mapsto \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$ and $O_2: \mathbf{P} \mapsto (\mathbf{P} - \mathbf{P})$ are *hull operations*, i.e. operations on subsets of \mathbb{Q}^n which fulfill for all input sets (in this case periodic sets) $\mathbf{X}, \mathbf{X}' \subseteq \mathbb{Q}^n$ that (1) $\mathbf{X} \subseteq O_i(\mathbf{X})$, (2), if $\mathbf{X} \subseteq \mathbf{X}'$, then $O_i(\mathbf{X}) \subseteq O_i(\mathbf{X}')$, and (3) $O_i(O_i(\mathbf{X})) = O_i(\mathbf{X})$. Intersections of hull operations, in particular $\text{Fill} = O_1 \cap O_2$, are hull operations.

Seeing (1) and (2) is easy, for (3) observe that it is enough to prove $\text{Fill}(\text{Fill}(\mathbf{P})) \subseteq O_i(\mathbf{P})$ for both i . By using first the definition of Fill , then (2) and at last (3) for O_i , we indeed obtain $\text{Fill}(\text{Fill}(\mathbf{P})) \subseteq O_i(\text{Fill}(\mathbf{P})) \subseteq O_i(O_i(\mathbf{P})) = O_i(\mathbf{P})$. ◀

► **Lemma 3.3.** *Let \mathbf{P} be a finitely generated periodic set. There exists $\mathbf{x} \in \mathbf{P}$ such that $\mathbf{x} + \text{Fill}(\mathbf{P}) \subseteq \mathbf{P}$.*

Proof. Let $\mathbf{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_s\}^* \subseteq \mathbb{N}^n$. Let \mathbf{B} be the finite set of minimal non-negative elements of $\mathbf{P} - \mathbf{P}$. Then we have $\mathbf{B}^* = (\mathbf{P} - \mathbf{P}) \cap \mathbb{N}_{\geq 0}^n$. We first consider the set $\mathbf{F} := \{\mathbf{x} \in \text{Fill}(\mathbf{P}) \mid \mathbf{x} - \mathbf{p}_i \notin \mathbb{Q}_{\geq 0}\mathbf{P} \forall i\}$, i.e. we consider all elements of the lattice inside the cone where we cannot subtract any period of \mathbf{P} without leaving the cone. Together with the \mathbf{p}_i , $\mathbf{f} \in \mathbf{F}$ are the minimal elements of $\text{Fill}(\mathbf{P})$, i.e. the generators.

We claim that this set \mathbf{F} is finite, and that every element of $\text{Fill}(\mathbf{P})$ can be written as $\mathbf{f} + \mathbf{p}$ with $\mathbf{f} \in \mathbf{F}$ and $\mathbf{p} \in \mathbf{P}$. The latter is clear, since for any lattice point we can simply subtract periods of \mathbf{P} until we no longer can, and thereby stay in the lattice, i.e. move into \mathbf{F} . To see that \mathbf{F} is finite, for the elements \mathbf{x} of \mathbf{F} consider the non-negative rational linear combination $\mathbf{x} = \sum_{i=1}^s \lambda_i \cdot \mathbf{p}_i$, $\lambda_i \in \mathbb{Q}_{\geq 0}$. Every coefficient $\lambda_i < 1$, otherwise we would be able to remove this period once and stay in the cone. Therefore $\mathbf{x} \leq \sum_{i=1}^s \mathbf{p}_i$. Hence there are only finitely many non-negative integer possibilities left.

Now we finish the proof. Since $\mathbf{B} \subseteq \mathbf{P} - \mathbf{P}$, we can write every element $\mathbf{b} \in \mathbf{B}$ as $\mathbf{b} = \mathbf{p}_b - \mathbf{p}_{b'}$ with $\mathbf{p}_b, \mathbf{p}_{b'} \in \mathbf{P}$. Since $\mathbf{F} \subseteq \text{Fill}(\mathbf{P}) \subseteq \mathbf{B}^*$, we can write every $\mathbf{f} \in \mathbf{F}$ as $\mathbf{f} = \sum_{j=1}^{n_f} \mathbf{b}_j$ with $\mathbf{b}_j \in \mathbf{B}$. Define $\mathbf{x} := \sum_{\mathbf{f} \in \mathbf{F}} \sum_{j=1}^{n_f} \mathbf{p}_{b'}$. This definition guarantees $\mathbf{x} + \mathbf{f} \in \mathbf{P}$ for all $\mathbf{f} \in \mathbf{F}$, because in \mathbf{x} we added the negative parts of an integer combination for \mathbf{f} . Now let $\mathbf{y} \in \mathbf{x} + \text{Fill}(\mathbf{P})$, then we can write $\mathbf{y} = \mathbf{x} + \mathbf{f} + \mathbf{p}$ with $\mathbf{f} \in \mathbf{F}$ and $\mathbf{p} \in \mathbf{P}$. By definition of \mathbf{x} , we have $\mathbf{x} + \mathbf{f} \in \mathbf{P}$, and hence $\mathbf{y} = (\mathbf{x} + \mathbf{f}) + \mathbf{p} \in \mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$. ◀

► **Lemma 3.4.** *Let \mathbf{P} be a finitely generated periodic set. For every $\mathbf{x} \in \mathbf{P}$ the set $\mathbf{S} := \mathbf{P} \setminus (\mathbf{x} + \text{Fill}(\mathbf{P}))$ is semilinear and satisfies $\dim(\mathbf{S}) < \dim(\mathbf{P})$.*

Proof. First of all, since $\text{Fill}(\mathbf{P})$ is semilinear and semilinear sets are closed under all boolean operations, \mathbf{S} is semilinear. Secondly, we have $\mathbf{P} \cap [\mathbf{x} + \mathbb{Q}_{\geq 0}\mathbf{P}] \subseteq \mathbf{x} + \text{Fill}(\mathbf{P})$, i.e. every point of \mathbf{P} which is in the inner cone is removed for \mathbf{S} . To see this, let $\mathbf{y} \in \mathbf{x} + \mathbb{Q}_{\geq 0}\mathbf{P}$. Write $\mathbf{y} = \mathbf{x} + \mathbf{v}$. We then have $\mathbf{v} \in \mathbb{Q}_{\geq 0}\mathbf{P}$ by definition, and $\mathbf{v} \in \mathbf{P} - \mathbf{P}$ since it is the difference of $\mathbf{y} \in \mathbf{P}$ and $\mathbf{x} \in \mathbf{P}$. Therefore $\mathbf{v} \in \text{Fill}(\mathbf{P})$ and $\mathbf{y} \in \mathbf{x} + \text{Fill}(\mathbf{P})$.

Since $\mathbb{Q}_{\geq 0}\mathbf{P}$ is finitely generated, by Lemma 2.1, there exists an integer matrix A such that $\mathbb{Q}_{\geq 0}\mathbf{P} = \{\mathbf{y} \in \text{VectSp}(\mathbf{P}) \mid A\mathbf{y} \geq 0\}$. Let A_i be the i -th row of A . Since by the above, $\mathbf{S} \cap \mathbf{x} + \mathbb{Q}_{\geq 0}\mathbf{P} = \emptyset$, every point $\mathbf{y} \in \mathbf{S}$ fulfills $0 \leq A_i\mathbf{y} \leq A_i \cdot \mathbf{x}$ for at least one i . Since $A_i \cdot \mathbf{y}$ can only take integer values, we define $\mathbf{V}_{i,j} := \{\mathbf{y} \in \text{VectSp}(\mathbf{P}) \mid A_i \cdot \mathbf{y} = j\}$ for all $j \in \{0, \dots, A_i \cdot \mathbf{x}\}$ and obtain that $\mathbf{S} \subseteq \bigcup_{i=1}^r \bigcup_{j=0}^{A_i \cdot \mathbf{x}} \mathbf{V}_{i,j}$. Since every $\mathbf{V}_{i,j}$ has co-dimension 1 in $\text{VectSp}(\mathbf{P})$, we obtain $\dim(\mathbf{S}) \leq \dim(\text{VectSp}(\mathbf{P})) - 1 = \dim(\mathbf{P}) - 1$ as claimed. ◀

► **Lemma 3.6.** *Let \mathbf{C} be a finitely generated cone, \mathbf{L} a lattice. Then $\mathbf{P} := \mathbf{C} \cap \mathbf{L}$ is full.*

Proof. We first claim that $\mathbb{Q}_{\geq 0}\mathbf{P} = \mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{L}$. Using that $\mathbb{Q}_{\geq 0}(\mathbf{L})$ is a vector space, Lemma 2.1 then implies that $\overline{\mathbb{Q}_{\geq 0}\mathbf{P}} = \mathbb{Q}_{\geq 0}\mathbf{P}$ is finitely generated.

Proof of claim: “ \subseteq ” is clear. Hence let $\mathbf{x} \in \mathbf{C} \cap \mathbb{Q}_{\geq 0}\mathbf{L}$. Then there exists $\lambda \in \mathbb{N}$ such that $\lambda\mathbf{x} \in \mathbf{L}$. Then $\lambda\mathbf{x} \in \mathbf{C} \cap \mathbf{L} = \mathbf{P}$ as claimed.

It is left to prove $\mathbb{Q}_{\geq 0}\mathbf{P} \cap (\mathbf{P} - \mathbf{P}) = \mathbf{P}$. “ \supseteq ” is clear, hence let $\mathbf{x} \in \mathbb{Q}_{\geq 0}\mathbf{P} \cap (\mathbf{P} - \mathbf{P})$. It is enough to prove $\mathbf{x} \in \mathbf{C}$ and $\mathbf{x} \in \mathbf{L}$. To prove those inclusions, observe that $\mathbf{x} \in \mathbb{Q}_{\geq 0}\mathbf{P} \subseteq \mathbb{Q}_{\geq 0}\mathbf{C} \subseteq \mathbf{C}$ and $\mathbf{x} \in \mathbf{P} - \mathbf{P} \subseteq \mathbf{L} - \mathbf{L} \subseteq \mathbf{L}$. \blacktriangleleft

► **Lemma 3.7.** *Let \mathbf{P}, \mathbf{Q} be periodic sets, \mathbf{P} full, and $\mathbf{b}, \mathbf{c} \in \mathbb{Q}^n$ points such that $\mathbf{c} + \mathbf{Q} \subseteq \mathbf{b} + \mathbf{P}$. Then $\mathbf{Q} \subseteq \mathbf{P}$. In particular, the representation $\mathbf{b} + \mathbf{P}$ of a full linear set is unique.*

Proof. Since \mathbf{P} is full, it is sufficient to prove $\mathbf{Q} \subseteq \mathbf{P} - \mathbf{P}$ and $\mathbf{Q} \subseteq \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$.

To prove $\mathbf{Q} \subseteq \mathbf{P} - \mathbf{P}$, observe that $\mathbf{Q} = (\mathbf{c} + \mathbf{Q}) - \mathbf{c} \subseteq (\mathbf{b} + \mathbf{P}) - (\mathbf{b} + \mathbf{P}) = \mathbf{P} - \mathbf{P}$.

To prove $\mathbf{Q} \subseteq \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$, write $\overline{\mathbb{Q}_{\geq 0}\mathbf{P}} = \{\mathbf{x} \in \text{VectSp}(\mathbf{P}) \mid A\mathbf{x} \geq 0\}$ for a matrix A , as in Lemma 2.1. Let A_k be the k -th row of A . It hence suffices to show $A_k\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbf{Q}$. To see this, observe that if $A_k\mathbf{x} < 0$, then $A_k(\mathbf{c} + \lambda\mathbf{x}) < A_k\mathbf{b}$ for large enough λ , contradicting $\mathbf{c} + \mathbf{Q} \subseteq \mathbf{b} + \mathbf{P}$.

The remark about uniqueness of the representation follows since if $\mathbf{b}_1 + \mathbf{P}_1 = \mathbf{b}_2 + \mathbf{P}_2$ with full $\mathbf{P}_1, \mathbf{P}_2$, then we can apply the lemma in both directions to obtain $\mathbf{P}_1 = \mathbf{P}_2$. Since $\mathbf{P}_i \subseteq \mathbb{N}^n$ and $\mathbf{0} \in \mathbf{P}_i$, we obtain $\mathbf{b}_1 \leq \mathbf{b}_2$ and $\mathbf{b}_2 \leq \mathbf{b}_1$. \blacktriangleleft

C Proofs of Section 4

► **Lemma 4.4.** *Let \mathbf{P} be a periodic set. Then $\text{int}(\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}) \subseteq \mathbb{Q}_{\geq 0}\mathbf{P} \subseteq \text{dir}(\mathbf{P}) \subseteq \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$.*

In particular, all these sets have the same closure.

Proof. Let $\mathbf{x} \in \text{int}(\overline{\mathbf{C}})$, where $\mathbf{C} := \mathbb{Q}_{\geq 0}\mathbf{P}$. Then there exists $\varepsilon > 0$ such that the open ball $B(\mathbf{x}, \varepsilon)$ of radius ε around \mathbf{x} is contained in $\overline{\mathbf{C}}$ by definition of interior. Hence for every $\mathbf{y} \in B(\mathbf{x}, \frac{\varepsilon}{2})$, there exists $f(\mathbf{y}) \in B(\mathbf{y}, \frac{\varepsilon}{4}) \cap \mathbf{C}$ by definition of closure. We have surrounded \mathbf{x} by points $f(\mathbf{y}) \in \mathbf{C}$, hence by convexity of \mathbf{C} we have $\mathbf{x} \in \mathbf{C}$.

Let $\mathbf{d} \in \mathbb{Q}_{\geq 0}\mathbf{P}$. Then there exists $m \in \mathbb{N}$ such that $m\mathbf{d} \in \mathbf{P}$, in particular $\mathbb{N} \cdot m\mathbf{d} \subseteq \mathbf{P}$.

Let $\mathbf{d} \in \text{dir}(\mathbf{P})$. Then by replacing \mathbf{d} by a multiple $m\mathbf{d}$, there exists \mathbf{x} such that $\mathbf{x} + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}$. We define the sequence $(\mathbf{x}_m)_{m \in \mathbb{N}}$ via $\mathbf{x}_m := \frac{1}{m}(\mathbf{x} + m \cdot \mathbf{d}) \in \mathbb{Q}_{\geq 0}\mathbf{P}$, and observe that its limit is \mathbf{d} , i.e. $\mathbf{d} \in \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$. \blacktriangleleft

► **Proposition 4.8.** *Let \mathbf{P}, \mathbf{P}' be smooth periodic sets such that $\dim(\text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}')) = \dim(\text{Fill}(\mathbf{P})) = \dim(\text{Fill}(\mathbf{P}'))$. Then*

1. $\dim(\mathbf{P} \cap \mathbf{P}') = \dim(\mathbf{P}) = \dim(\mathbf{P}')$.
2. $\text{dir}(\mathbf{P} \cap \mathbf{P}') = \text{dir}(\mathbf{P}) \cap \text{dir}(\mathbf{P}')$.
3. $\text{Fill}(\mathbf{P} \cap \mathbf{P}') = \text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}')$.
4. $\mathbf{P} \cap \mathbf{P}'$ is smooth.

We split the proof of Proposition 4.8 in a lemma for every part, with $\text{Fill}(\mathbf{P} \cap \mathbf{P}')$ having one lemma for the lattice and one lemma for the closed cone.

Proof of Proposition 4.8(1)

► **Lemma C.1.** *Let \mathbf{P}, \mathbf{P}' be smooth periodic sets such that*

$$\dim(\text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}')) = \dim(\text{Fill}(\mathbf{P})) = \dim(\text{Fill}(\mathbf{P}')).$$

Then $\dim(\mathbf{P} \cap \mathbf{P}') = \dim(\mathbf{P}) = \dim(\mathbf{P}')$.

Proof. We only argue $\dim(\mathbf{P} \cap \mathbf{P}') = \dim(\mathbf{P})$, the other equality follows by symmetry. “ \leq ” is immediate.

By Lemma 2.5 we have $\dim(\mathbf{P}) = \dim(\text{VectSp}(\mathbf{P}))$, in particular also $\dim(\mathbf{P}) = \dim(\overline{\mathbb{Q}_{\geq 0}\mathbf{P}}) = \dim(\text{Fill}(\mathbf{P}))$. Hence $\dim(\mathbf{P}) = \dim(\text{Fill}(\mathbf{P})) = \dim(\text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}')) \leq \dim(\overline{\mathbb{Q}_{\geq 0}\mathbf{P} \cap \mathbb{Q}_{\geq 0}\mathbf{P}'})$. The goal is to arrive at $\dim(\mathbb{Q}_{\geq 0}\mathbf{P} \cap \mathbb{Q}_{\geq 0}\mathbf{P}')$. Then, since $\mathbb{Q}_{\geq 0}(\mathbf{P} \cap \mathbf{P}') = \mathbb{Q}_{\geq 0}\mathbf{P} \cap \mathbb{Q}_{\geq 0}\mathbf{P}'$ always holds [10, Lemma 4.5], we would obtain $\dim(\mathbf{P}) \leq \dim(\mathbb{Q}_{\geq 0}(\mathbf{P} \cap \mathbf{P}')) = \dim(\mathbf{P} \cap \mathbf{P}')$, and be done.

One main tool is Lemma 4.4. For definable cones \mathbf{C} , we have $\dim(\partial(\mathbf{C})) < \dim(\mathbf{C})$, since the boundary consists of finitely many facets, which are lower dimensional cones. In addition $\mathbf{C} = \partial(\mathbf{C}) \cup \text{int}(\mathbf{C})$, hence $\dim(\mathbf{C}) = \dim(\text{int}(\mathbf{C}))$ by Lemma 2.4. We will use these two facts for $\mathbf{C} := \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$ and $\mathbf{C}' := \overline{\mathbb{Q}_{\geq 0}\mathbf{P}'}$.

We have $\text{int}(\mathbf{C}) \cap \text{int}(\mathbf{C}') = (\mathbf{C} \setminus \partial(\mathbf{C})) \cap (\mathbf{C}' \setminus \partial(\mathbf{C}')) = (\mathbf{C} \cap \mathbf{C}') \setminus (\partial(\mathbf{C}) \cup \partial(\mathbf{C}'))$, which has dimension $\dim(\mathbf{C} \cap \mathbf{C}')$, since we only remove a lower dimensional set. In total we can finish the chain from above:

$$\dim(\mathbf{P}) \leq \dim(\mathbf{C} \cap \mathbf{C}') = \dim(\text{int}(\mathbf{C}) \cap \text{int}(\mathbf{C}')) \leq \dim(\mathbb{Q}_{\geq 0}\mathbf{P} \cap \mathbb{Q}_{\geq 0}\mathbf{P}') = \dim(\mathbf{P} \cap \mathbf{P}'),$$

where we had already argued the last step earlier. \blacktriangleleft

Proof of Proposition 4.8(2)

We first prove an auxiliary lemma.

► **Lemma C.2.** *Let $\mathbf{P} \subseteq \mathbf{P}'$ be periodic sets with $\dim(\mathbf{P}) = \dim(\mathbf{P}')$. Then for all $\mathbf{p}_1 \in \mathbf{P}'$ there exists a $\mathbf{p}_2 \in \mathbf{P}'$ such that $\mathbf{p}_1 + \mathbf{p}_2 \in \mathbf{P}$.*

Proof. Let $\mathbf{p}_1 \in \mathbf{P}'$. Since $\dim(\mathbf{P}) = \dim(\mathbf{P}')$ and $\mathbf{P} \subseteq \mathbf{P}'$, by Lemma 2.5 they generate the same vector space \mathbf{V} . Hence there exists $\lambda \in \mathbb{N}_{>0}$ such that $\lambda\mathbf{p}_1 \in \mathbf{P} - \mathbf{P}$, namely λ is the common denominator for some rational linear combination of elements from \mathbf{P} . By writing $\lambda\mathbf{p}_1 = \mathbf{p} - \mathbf{p}'$ with $\mathbf{p}, \mathbf{p}' \in \mathbf{P}$ and choosing $\mathbf{p}_2 := (\lambda - 1) \cdot \mathbf{p}_1 + \mathbf{p}' \in \mathbf{P}'$, we obtain $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p} \in \mathbf{P}$. \blacktriangleleft

► **Lemma C.3.** *Let \mathbf{P}, \mathbf{P}' be periodic sets such that*

$$\dim(\mathbf{P} \cap \mathbf{P}') = \dim(\mathbf{P}) = \dim(\mathbf{P}').$$

$$\text{Then } \text{dir}(\mathbf{P} \cap \mathbf{P}') = \text{dir}(\mathbf{P}) \cap \text{dir}(\mathbf{P}').$$

We even have: If $\mathbf{x} + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}$ and $\mathbf{x}' + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}'$ for some \mathbf{x}, \mathbf{x}' , then $\mathbf{x}'' + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P} \cap \mathbf{P}'$ for some \mathbf{x}'' .

Proof. “ \subseteq ” is clear. Therefore let $\mathbf{d} \in \text{dir}(\mathbf{P}) \cap \text{dir}(\mathbf{P}')$. Then there exist $m, m' \in \mathbb{N}_{>0}$ and \mathbf{x}, \mathbf{x}' such that $\mathbf{x} + \mathbb{N} \cdot m\mathbf{d} \subseteq \mathbf{P}$ and $\mathbf{x}' + \mathbb{N} \cdot m'\mathbf{d} \subseteq \mathbf{P}'$. Replacing \mathbf{d} by $mm'\mathbf{d}$, we obtain $\mathbf{x} + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}$ and $\mathbf{x}' + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}'$.

By Lemma C.2, there exists $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{x} + \mathbf{p} \in \mathbf{P} \cap \mathbf{P}'$, and again by the same Lemma $\mathbf{p}' \in \mathbf{P}'$ such that $\mathbf{x}' + \mathbf{p}' \in \mathbf{P} \cap \mathbf{P}'$. We choose $\mathbf{x}'' := (\mathbf{x} + \mathbf{p}) + (\mathbf{x}' + \mathbf{p}') \in \mathbf{P} \cap \mathbf{P}'$, and obtain that $\mathbf{x}'' + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}$, because

$$\mathbf{x}'' + \mathbb{N} \cdot \mathbf{d} = ((\mathbf{x} + \mathbb{N} \cdot \mathbf{d}) + \mathbf{p}) + (\mathbf{x}' + \mathbf{p}') \subseteq \mathbf{P} + (\mathbf{P} \cap \mathbf{P}') \subseteq \mathbf{P}.$$

Symmetrically we also obtain $\mathbf{x}'' + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}'$. \blacktriangleleft

Proof of Proposition 4.8(3)

Recall that $\text{Fill}(\mathbf{P}) := (\mathbf{P} - \mathbf{P}) \cap \overline{\mathbb{Q}_{\geq 0}\mathbf{P}}$. So it suffices to prove that the closed cone and the lattice of an intersection are equal to the intersection of the closed cones and the lattices respectively.

► **Lemma C.4.** *Let \mathbf{P}, \mathbf{P}' be smooth periodic sets with*

$$\dim(\mathbf{P} \cap \mathbf{P}') = \dim(\mathbf{P}) = \dim(\mathbf{P}').$$

$$\text{Then } \overline{\mathbb{Q}_{\geq 0}(\mathbf{P} \cap \mathbf{P}')} = \overline{\mathbb{Q}_{\geq 0}\mathbf{P}} \cap \overline{\mathbb{Q}_{\geq 0}\mathbf{P}'}.$$

Proof. Define $\mathbf{C} := \text{dir}(\mathbf{P})$ and $\mathbf{C}' := \text{dir}(\mathbf{P}')$. Since \mathbf{P} and \mathbf{P}' are asymptotically definable, we have $\mathbf{C} \setminus \{0\} = \{\mathbf{x} \in \text{VectSp}(\mathbf{P}) \mid A_1\mathbf{x} > 0, A_2\mathbf{x} \geq 0\}$ and similarly $\mathbf{C}' \setminus \{0\} = \{\mathbf{x} \in \text{VectSp}(\mathbf{P}') \mid A'_1\mathbf{x} > 0, A'_2\mathbf{x} \geq 0\}$. By Lemma 2.5, we have $\dim(\text{VectSp}(\mathbf{P})) = \dim(\mathbf{C}) = \dim(\mathbf{C}') = \dim(\mathbf{C} \cap \mathbf{C}')$. For equations defining a set of full dimension, taking the closure of the cone is equivalent to changing all equations to ≥ 0 . We hence have $\overline{\mathbf{C}} = \{\mathbf{x} \in \text{VectSp}(\mathbf{P}) \mid A_1\mathbf{x} \geq 0, A_2\mathbf{x} \geq 0\}$, as well as $\overline{\mathbf{C}'} = \{\mathbf{x} \in \text{VectSp}(\mathbf{P}') \mid A'_1\mathbf{x} \geq 0, A'_2\mathbf{x} \geq 0\}$ and $\overline{\mathbf{C} \cap \mathbf{C}'} = \{\mathbf{x} \in \text{VectSp}(\mathbf{P}) \mid A_1\mathbf{x} \geq 0, A_2\mathbf{x} \geq 0, A'_1\mathbf{x} \geq 0, A'_2\mathbf{x} \geq 0\}$. ◀

► **Lemma C.5.** *Let \mathbf{P}, \mathbf{P}' be periodic sets with*

$$\dim(\mathbf{P} \cap \mathbf{P}') = \dim(\mathbf{P}) = \dim(\mathbf{P}').$$

$$\text{Then } (\mathbf{P} \cap \mathbf{P}') - (\mathbf{P} \cap \mathbf{P}') = (\mathbf{P} - \mathbf{P}) \cap (\mathbf{P}' - \mathbf{P}').$$

Proof. “ \subseteq ” is clear, hence let $\mathbf{d} \in (\mathbf{P} - \mathbf{P}) \cap (\mathbf{P}' - \mathbf{P}')$. If lattices coincide on a cone of full dimension like $\text{dir}(\mathbf{P} \cap \mathbf{P}') = \text{dir}(\mathbf{P}) \cap \text{dir}(\mathbf{P}')$, then they coincide everywhere. We can therefore assume $\mathbf{d} \in \text{dir}(\mathbf{P} \cap \mathbf{P}')$.

By Proposition 5.5 (the high number is not a problem, since that proposition does not build on any prior statements), we have $\mathbf{x} + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}$ and $\mathbf{x}' + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P}'$ for some \mathbf{x}, \mathbf{x}' . By the extra remark in Lemma C.3, there exists \mathbf{x}'' such that $\mathbf{x}'' + \mathbb{N} \cdot \mathbf{d} \subseteq \mathbf{P} \cap \mathbf{P}'$. Hence $\mathbf{d} = (\mathbf{x}'' + \mathbf{d}) - (\mathbf{x}'') \in (\mathbf{P} \cap \mathbf{P}') - (\mathbf{P} \cap \mathbf{P}')$. ◀

► **Corollary C.6.** *Let \mathbf{P}, \mathbf{P}' be smooth periodic sets with*

$$\dim(\mathbf{P} \cap \mathbf{P}') = \dim(\mathbf{P}) = \dim(\mathbf{P}').$$

$$\text{Then } \text{Fill}(\mathbf{P} \cap \mathbf{P}') = \text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}').$$

Proof. Follows from Lemmas C.4 and C.5. ◀

Proof of Proposition 4.8(4)

► **Lemma C.7.** *Let \mathbf{P}, \mathbf{P}' be smooth periodic sets with*

$$\dim(\mathbf{P} \cap \mathbf{P}') = \dim(\mathbf{P}) = \dim(\mathbf{P}').$$

$$\text{Then } \mathbf{P} \cap \mathbf{P}' \text{ is smooth.}$$

Proof. $\mathbf{P} \cap \mathbf{P}'$ is asymptotically definable by Lemma C.3. Let $(\mathbf{p}_m + \mathbf{p}'_m)_m$ with $\mathbf{p}_m \in \mathbf{P}$ and $\mathbf{p}'_m \in \mathbf{P}'$ be a sequence. Since \mathbf{P} is well-directed, there exists an infinite set of indices $N_1 \subseteq \mathbb{N}$ such that $\mathbf{p}_m - \mathbf{p}_k \in \text{dir}(\mathbf{P})$ for all $m > k$ in N_1 . Since \mathbf{P}' is well-directed, there exists an infinite set $N_2 \subseteq N_1$ such that moreover $\mathbf{p}'_m - \mathbf{p}'_k \in \text{dir}(\mathbf{P}')$ for all $m > k$ in N_2 . Hence for all $m > k$ in N_2 , we have $(\mathbf{p}_m + \mathbf{p}'_m) - (\mathbf{p}_k + \mathbf{p}'_k) \in \text{dir}(\mathbf{P}) + \text{dir}(\mathbf{P}') \subseteq \text{dir}(\mathbf{P} + \mathbf{P}')$, since $\text{dir}(\mathbf{P} + \mathbf{P}')$ is closed under addition. ◀

D

 Proofs of Section 5

Before we can prove Theorem 5.10, we need a preliminary lemma.

► **Lemma D.1.** *Let \mathbf{P}, \mathbf{P}' be smooth periodic sets. Then $\mathbf{P} + \mathbf{P}'$ is smooth with $\text{dir}(\mathbf{P} + \mathbf{P}') = \text{dir}(\mathbf{P}) + \text{dir}(\mathbf{P}')$.*

Proof. Let $\mathbf{p}_1 + \mathbf{p}'_1, \mathbf{p}_2 + \mathbf{p}'_2 \in \mathbf{P} + \mathbf{P}'$. Then $(\mathbf{p}_1 + \mathbf{p}'_1) + (\mathbf{p}_2 + \mathbf{p}'_2) = (\mathbf{p}_1 + \mathbf{p}_2) + (\mathbf{p}'_1 + \mathbf{p}'_2) \in \mathbf{P} + \mathbf{P}'$, i.e. $\mathbf{P} + \mathbf{P}'$ is a periodic set. Next we show $\text{dir}(\mathbf{P}) + \text{dir}(\mathbf{P}') = \text{dir}(\mathbf{P} + \mathbf{P}')$, where \subseteq is clear. Hence let $\mathbf{d} \in \text{dir}(\mathbf{P} + \mathbf{P}')$. Then there exists $\mathbf{x} \in \mathbf{P} + \mathbf{P}'$ and $\lambda \in \mathbb{N}_{>0}$ such that $\mathbf{x} + \mathbb{N} \cdot \lambda \mathbf{d} \subseteq \mathbf{P} + \mathbf{P}'$. We write $\mathbf{x} + m\lambda \mathbf{d} = \mathbf{p}_m + \mathbf{p}'_m$ with $\mathbf{p}_m \in \mathbf{P}$ and $\mathbf{p}'_m \in \mathbf{P}'$ to obtain the sequences $(\mathbf{p}_m)_m$ and $(\mathbf{p}'_m)_m$. Since \mathbf{P} is well-directed, there exists an infinite set of indices $N_1 \subseteq \mathbb{N}$ such that $\mathbf{p}_m - \mathbf{p}_k \in \text{dir}(\mathbf{P})$ for all $m > k$ in N_1 . Now consider the sequence $(\mathbf{p}'_m)_{m \in N_1}$. Since \mathbf{P}' is well-directed, there exists an infinite set of indices $N_2 \subseteq N_1$ such that furthermore $\mathbf{p}'_m - \mathbf{p}'_k \in \text{dir}(\mathbf{P}')$ for all $m > k$ in N_2 . Choose $m > k \in N_2$. We obtain that

$$\begin{aligned} (m - k)\lambda \mathbf{d} &= (\mathbf{p}_m + \mathbf{p}'_m) - (\mathbf{p}_k + \mathbf{p}'_k) \\ &= (\mathbf{p}_m - \mathbf{p}_k) + (\mathbf{p}'_m - \mathbf{p}'_k) \in \text{dir}(\mathbf{P}) + \text{dir}(\mathbf{P}'). \end{aligned}$$

Hence also $\mathbf{d} \in \text{dir}(\mathbf{P}) + \text{dir}(\mathbf{P}')$. Proving that $\mathbf{P} + \mathbf{P}'$ is well-directed similarly relies upon $N_2 \subseteq N_1 \subseteq \mathbb{N}$. ◀

► **Theorem 5.10.** *A non-empty Petri set $\mathbf{X} \subseteq \mathbb{N}^n$ is almost hybridlinear if and only if it has a hybridization.*

Proof. “ \Rightarrow ”: Write $\mathbf{X} = \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{P}_i$ where the \mathbf{P}_i are smooth periodic sets with $\text{Fill}(\mathbf{P}_i) = \text{Fill}(\mathbf{P}_j)$. Define $\mathbf{P} := \bigcap_{i=1}^r \mathbf{P}_i$, which is smooth by Proposition 4.8, and we have

$$\text{Fill}(\mathbf{P}) = \bigcap_{i=1}^r \text{Fill}(\mathbf{P}_i) = \bigcap_{i=1}^r \text{Fill}(\mathbf{P}_1) = \text{Fill}(\mathbf{P}_1).$$

Since $\mathbf{P}_i + \mathbf{P} \subseteq \mathbf{P}_i$ for all i , we have $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$. We also have $\mathbf{X} \subseteq \mathbf{H} := \{\mathbf{b}_1, \dots, \mathbf{b}_r\} + \text{Fill}(\mathbf{P})$.

“ \Leftarrow ”: Let \mathbf{P} smooth such that $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$ and the hybridization is $\mathbf{H} = \{\mathbf{b}_1, \dots, \mathbf{b}_r\} + \text{Fill}(\mathbf{P})$. Since \mathbf{X} is a Petri set, the sets $\mathbf{X} \cap [\mathbf{b}_i + \text{Fill}(\mathbf{P})]$ are almost semilinear, and so $\mathbf{X} \cap [\mathbf{b}_i + \text{Fill}(\mathbf{P})] = \bigcup_{j=1}^{r_i} \mathbf{b}_{i,j} + \mathbf{P}_{i,j}$ for smooth periodic sets $\mathbf{P}_{i,j}$. Since $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$, we have $\mathbf{X} = \bigcup_{i=1}^r \bigcup_{j=1}^{r_i} \mathbf{b}_{i,j} + (\mathbf{P}_{i,j} + \mathbf{P})$. We prove that this is an almost hybridlinear representation of \mathbf{X} (see Definition 5.9). By Lemma D.1, all sets $(\mathbf{P}_{i,j} + \mathbf{P})$ are smooth. We prove that all their fills are equal to $\text{Fill}(\mathbf{P})$. It suffices to show $\text{Fill}(\mathbf{P}_{i,j} + \mathbf{P}) \subseteq \text{Fill}(\mathbf{P})$, the other inclusion is trivial.

By Lemma 3.7, we have $\mathbf{P}_{i,j} \subseteq \text{Fill}(\mathbf{P})$. By Lemma 3.2, we then also have $\text{Fill}(\mathbf{P}_{i,j}) \subseteq \text{Fill}(\mathbf{P})$. Since $\text{Fill}(\mathbf{P})$ is periodic, $\text{Fill}(\mathbf{P}_{i,j}) + \text{Fill}(\mathbf{P}) \subseteq \text{Fill}(\mathbf{P})$ and so, again by Lemma 3.2, $\text{Fill}(\text{Fill}(\mathbf{P}_{i,j}) + \text{Fill}(\mathbf{P})) \subseteq \text{Fill}(\mathbf{P})$. Since $\mathbf{P}_{i,j} + \mathbf{P} \subseteq \text{Fill}(\mathbf{P}_{i,j}) + \text{Fill}(\mathbf{P})$, we are done. ◀

E

 Proofs of Section 6

E.1 Proof of Theorem 6.6

The starting point for this section is Theorem 5.10, which allows us to consider only the case of almost hybridlinear sets. We prove an equivalent condition of reducibility for this case. As running example for this section we consider $\mathbf{X} := \mathbf{X}_1 \cup \mathbf{X}_2 := [(0, 1) + \mathbf{P}_1] \cup [(0, 6) + \mathbf{P}_2]$

for $\mathbf{P}_1 = \{(x, y) \in \mathbb{N}^2 \mid y \leq x^2\}$ and $\mathbf{P}_2 = \{(x, y) \in \mathbb{N}^2 \mid y \geq \log_2(x+1)\}$. Together, these two almost linear components have the hybridization \mathbb{N}^2 and even though neither of the two components is reducible, the union is.

We will provide an equivalent definition of reducibility in terms of a concept from [12] called complete extraction. This definition will be more suited for an algorithmic check.

► **Definition E.1.** Let $\mathcal{K} = \{\mathbf{K}_1, \dots, \mathbf{K}_r\}$ be a finite set of cones. A complete extraction of \mathcal{K} is a set of finitely generated cones $\{\mathbf{C}_1, \dots, \mathbf{C}_r\}$ such that $\mathbf{C}_i \subseteq \mathbf{K}_i$ for all i and $\bigcup_{i=1}^r \mathbf{C}_i = \bigcup_{i=1}^r \mathbf{K}_i$.

Intuitively, we try to replace the non-finitely generated \mathbf{K}_i by smaller cones \mathbf{C}_i which are finitely generated, but whose union is still the same. This is of course only possible if the cones \mathbf{K}_i have an overlap. The following lemma, proved in [12], in some sense formalizes this intuition and will help us prove that the existence of a complete extraction is decidable.

► **Lemma E.2.** [12, App. E] A finite set $\mathcal{K} = \{\mathbf{K}_1, \dots, \mathbf{K}_r\}$ of cones has a complete extraction if and only if for all vectors $\mathbf{v}_1, \dots, \mathbf{v}_s$ such that $\mathbb{Q}_{>0}\mathbf{v}_1 + \dots + \mathbb{Q}_{>0}\mathbf{v}_s \subseteq \mathbf{K}_i$ for some i , there exists a $j \in \{1, \dots, r\}$ such that $\mathbf{K}_j \cap (\mathbb{Q}_{>0}\mathbf{v}_1 + \dots + \mathbb{Q}_{>0}\mathbf{v}_k) \neq \emptyset$ for all $1 \leq k \leq s$.

► **Example E.3.** The set $\mathbf{X} = \mathbf{X}_1 \cup \mathbf{X}_2$ fulfils $\text{dir}(\mathbf{P}_1) = \{(x, y) \mid x > 0\}$ and $\text{dir}(\mathbf{P}_2) = \{(x, y) \mid y > 0\}$. The set $\{\text{dir}(\mathbf{P}_1), \text{dir}(\mathbf{P}_2)\}$ has a complete extraction, for example $\mathbf{C}_1 := \{(x, y) \mid x \geq y\}$ and $\mathbf{C}_2 := \{(x, y) \mid y \geq x\}$. If $\text{dir}(\mathbf{P}_2)$ were only $\{(x, y) \mid x = 0\}$, then $\mathbf{K}_1 \cup \mathbf{K}_2 = \mathbb{Q}_{\geq 0}^2$ would still hold, but there would be no complete extraction. Intuitively, \mathbf{C}_1 would have to contain the “open border” $x > 0$ in that case. Remember that finitely generated cones have to be closed however.

The complete extraction will translate to

$$(10, 10) + (\mathbf{C}_1 \cap \mathbb{N}^2) \subseteq \mathbf{X}_1 \text{ and } (10, 10) + (\mathbf{C}_2 \cap \mathbb{N}^2) \subseteq \mathbf{X}_2,$$

i.e. $\{(x, y) \in \mathbb{N}^2 \mid x \geq 10, y \geq 10\} \subseteq \mathbf{X}$, proving that \mathbf{X} is reducible.

The equivalent definition of reducibility is as follows.

► **Theorem E.4.** Let \mathbf{X} be almost hybridlinear with hybridization $\mathbf{c} + \text{Fill}(\mathbf{P})$. Write $\mathbf{X} = \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{P}_i$, where the sets $\mathbf{P}_1, \dots, \mathbf{P}_r$ satisfy $\text{Fill}(\mathbf{P}) = \text{Fill}(\mathbf{P}_1) = \dots = \text{Fill}(\mathbf{P}_r)$. Define $\mathbf{K}_i := \text{dir}(\mathbf{P}_i)$. Then \mathbf{X} is reducible if and only if the set of cones $\mathcal{K} = \{\mathbf{K}_1, \dots, \mathbf{K}_r\}$ has a complete extraction.

The theorem is proved in Section E.2.

Once the theorem is proved, the algorithm for checking if an almost hybridlinear set \mathbf{X} is reducible first computes those cones, and afterwards searches for a complete extraction. For correctness, the algorithm also relies upon the following lemma:

► **Lemma E.5.** Let \mathbf{Q} be a full periodic set. Then there exists a smooth periodic set \mathbf{P}' such that $\text{Fill}(\mathbf{P}') = \mathbf{Q}$ and $\text{dir}(\mathbf{P}') = \text{int}(\mathbb{Q}_{\geq 0}\mathbf{Q})$.

Proof. We prove this in the special case of $\mathbf{Q} = \mathbb{N}^n$, via an appropriate linear map we then obtain the result for all full periodic sets whose cone has $\dim(\mathbf{Q})$ many generators, and can extend to the general case.

Hence let $\mathbf{Q} = \mathbb{N}^n$. Define $\mathbf{P}' := \{(x_1, \dots, x_n) \mid x_i \leq 2^{x_j} - 1 \forall i \neq j\}$. First of all, let $\mathbf{v} = (10, \dots, 10)$ be the vector with all components set to 10. Then $\mathbf{v} + \mathbf{e}_i \in \mathbf{P}'$ for all unit vectors \mathbf{e}_i , and hence $\mathbf{e}_i \in \mathbf{P}' - \mathbf{P}'$. Considering the asymptotics, we have that $\overline{\mathbb{Q}_{\geq 0}\mathbf{P}'} = \mathbb{Q}_{\geq 0}^n$. Hence $\text{Fill}(\mathbf{P}') = \mathbb{N}^n$. Since every vector in $\partial(\mathbb{Q}_{\geq 0}^n)$ leaves some component unchanged, none

of those can be directions of \mathbf{P}' . Hence $\text{dir}(\mathbf{P}') = \text{int}(\mathbb{Q}_{\geq 0}^n)$ using Lemma 4.4. $\text{dir}(\mathbf{P}')$ is definable by definition, and well-directed is similarly obvious, since any line parallel to the boundary can only contain finitely many points of \mathbf{P}' , i.e. any infinite sequence $(\mathbf{p}_m)_m \subseteq \mathbf{P}'$ has to contain a subsequence with $p_m - p_k \in \text{int}(\mathbb{Q}_{\geq 0}^n) = \text{dir}(\mathbf{P}')$ for all $m > k$. Hence we have found our choice of \mathbf{P}' . \blacktriangleleft

► **Theorem 6.6.** *[5, even without promise] The following problem is decidable.*

Input: Reachability set \mathbf{R} , full linear set \mathbf{S} , with promise that $\mathbf{R} \cap \mathbf{S}$ has hybridization \mathbf{S} .

Output: Is $\mathbf{R} \cap \mathbf{S}$ reducible?

Proof. Write $\mathbf{X} := \mathbf{R} \cap \mathbf{S}$. The algorithm and its proof are split into two parts: First obtain a representation $\mathbf{X} = \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{P}_i$ with $\text{Fill}(\mathbf{P}_i) = \text{Fill}(\mathbf{P}_j)$ for all i, j , and then check whether $\{\text{dir}(\mathbf{P}_i) \mid 1 \leq i \leq r\}$ has a complete extraction.

By Proposition 6.2(5), we can decompose $\mathbf{X} = \mathbf{X}_1 \cup \dots \cup \mathbf{X}_r$ via KLMST-decomposition. We only used that the full linear hybridizations can be computed, but in fact even more is true: In [5], Hauschildt shows that whether a given vector \mathbf{d} is an element of $\text{dir}(\mathbf{X}_i)$ can be decided. Though we will not explain this in detail, it can be upgraded to compute a representation of the cone $\text{dir}(\mathbf{X}_i)$. Instead we deal with the second problem: This representation $\mathbf{X} = \mathbf{X}_1 \cup \dots \cup \mathbf{X}_r$ might not be an almost hybridlinear representation, i.e. the fills might differ.

Write $\mathbf{S} = \mathbf{c} + \mathbf{Q}$. The algorithmic solution to problem 2 is simple: Add $\text{int}(\mathbf{Q})$ to all cones $\text{dir}(\mathbf{X}_i)$, i.e. consider $\mathcal{K} = \{\mathbf{K}_i := \text{dir}(\mathbf{X}_i) + \text{int}(\mathbf{Q}) \mid 1 \leq i \leq r\}$.

Check for a complete extraction of \mathcal{K} using two semi-algorithms: One to check whether the set of cones \mathcal{K} does not fulfil the property of Lemma E.2, and one that searches for a complete extraction. Output the answer of the semi-algorithm which terminates.

Termination: By Lemma E.2.

Correctness: Let \mathbf{P} smooth such that $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$ and $\text{Fill}(\mathbf{P}) = \mathbf{Q}$. Let \mathbf{P}' be the smooth periodic as in Lemma E.5. Then $\mathbf{P}'' := \mathbf{P} \cap \mathbf{P}'$ is smooth with $\text{Fill}(\mathbf{P}'') = \mathbf{Q}$ and $\text{dir}(\mathbf{P}'') = \text{int}(\mathbb{Q}_{\geq 0}^n \mathbf{Q})$ by Proposition 4.8. Furthermore, we have $\mathbf{X} + \mathbf{P}'' \subseteq \mathbf{X}$. Inspecting the proof of Theorem 5.10, \mathbf{X} has an almost hybridlinear representation with periodic sets $\mathbf{P}_i + \mathbf{P}''$. By Lemma D.1, these fulfil $\text{dir}(\mathbf{P}_i + \mathbf{P}'') = \text{dir}(\mathbf{P}_i) + \text{dir}(\mathbf{P}'')$, i.e. there is an almost hybridlinear representation with the cones considered by the algorithm. Correctness then follows by Theorem E.4. \blacktriangleleft

E.2 Proof of Theorem E.4

We require two geometric properties of almost hybridlinear sets. We start with an auxiliary lemma.

► **Lemma E.6.** *Let \mathbf{P} be a periodic set, and $\mathbf{F} \subseteq \mathbf{P} - \mathbf{P}$ finite. Then there exists $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{p} + \mathbf{F} \subseteq \mathbf{P}$.*

Proof. Write $\mathbf{F} = \{\mathbf{p}_1, \dots, \mathbf{p}_s\}$. Write $\mathbf{p}_i = \mathbf{p}_{i,1} - \mathbf{p}_{i,2}$ with $\mathbf{p}_{i,1}, \mathbf{p}_{i,2} \in \mathbf{P}$ for every i . Define $\mathbf{p} := \sum_{i=1}^s \mathbf{p}_{i,2}$. Then $\mathbf{p} + \mathbf{p}_i \in \mathbf{P}$ for all i . \blacktriangleleft

The first property essentially allows us to ignore the lattice and only consider cones.

► **Proposition E.7.** *Let \mathbf{X} be a set with hybridization $\mathbf{c} + \text{Fill}(\mathbf{P})$. Let $\mathbf{Q} \subseteq \text{Fill}(\mathbf{P})$ be a finitely generated periodic set. If $\mathbf{x} + \lambda \mathbf{Q} \subseteq \mathbf{X}$ for some $\mathbf{x} \in \mathbf{X}, \lambda \in \mathbb{N}_{>0}$, then $\mathbf{x}' + \mathbf{Q} \subseteq \mathbf{X}$ for some $\mathbf{x}' \in \mathbf{X}$.*

Proof. Write $\mathbf{Q} = \{\mathbf{d}'_1, \dots, \mathbf{d}'_s\}^*$ and assume that $\mathbf{x} + \lambda\{\mathbf{d}'_1, \dots, \mathbf{d}'_s\}^* \subseteq \mathbf{X}$. Define $\mathbf{F} := \{0, \dots, \lambda - 1\}\mathbf{d}'_1 + \dots + \{0, \dots, \lambda - 1\}\mathbf{d}'_s \subseteq \text{Fill}(\mathbf{P}) \subseteq \mathbf{P} - \mathbf{P}$. By Lemma E.6 there exists $\mathbf{d} \in \mathbf{P}$ such that $\mathbf{d} + \mathbf{F} \subseteq \mathbf{P}$. Choose $\mathbf{x}' := \mathbf{x} + \mathbf{d}$. Then we claim $\mathbf{x}' + \mathbf{Q} \subseteq \mathbf{X}$. To see this, let $\mathbf{y} = \mathbf{x}' + \lambda_1 \mathbf{d}'_1 + \dots + \lambda_s \mathbf{d}'_s \in \mathbf{x}' + \mathbf{Q}$. Let $\lambda'_i := \lambda_i \bmod \lambda$, and observe that $\mathbf{w}' := \lambda'_1 \mathbf{d}'_1 + \dots + \lambda'_s \mathbf{d}'_s \in \mathbf{F}$, and $\mathbf{y} - \mathbf{w}' \in \mathbf{x} + \mathbf{d} + \lambda\mathbf{Q}$. In total we obtain

$$\mathbf{y} = \mathbf{w}' + (\mathbf{y} - \mathbf{w}') \in \mathbf{F} + \mathbf{x} + \mathbf{d} + \lambda\mathbf{Q} = (\mathbf{x} + \lambda\mathbf{Q}) + (\mathbf{d} + \mathbf{F}) \subseteq \mathbf{X} + \mathbf{P} \subseteq \mathbf{X}.$$

◀

Next the second property. Intuitively, we can move starting points of lines, planes, ... together.

► **Proposition E.8.** *Let \mathbf{X} be a set with hybridization $\mathbf{c} + \text{Fill}(\mathbf{P})$. Assume that there exist finitely many sets $\mathbf{G}_1, \dots, \mathbf{G}_s \subseteq \text{Fill}(\mathbf{P})$ and points $\mathbf{x}_1, \dots, \mathbf{x}_s \in \mathbf{X}$ such that $\mathbf{x}_i + \mathbf{G}_i \subseteq \mathbf{X}$ for all i . Then there exists \mathbf{x}' such that $\mathbf{x}' + (\bigcup_{i=1}^s \mathbf{G}_i) \subseteq \mathbf{X}$.*

Proof. Define $\mathbf{F} := \{\mathbf{c} - \mathbf{x}_1, \dots, \mathbf{c} - \mathbf{x}_s\} \subseteq -\text{Fill}(\mathbf{P}) \subseteq \mathbf{P} - \mathbf{P}$. By Lemma E.6, there exists $\mathbf{p} \in \mathbf{P}$ such that $\mathbf{p} + \mathbf{F} \subseteq \mathbf{P}$. We define $\mathbf{x}' := \mathbf{c} + \mathbf{p}$, and claim that $\mathbf{x}' + (\bigcup_{i=1}^s \mathbf{G}_i) \subseteq \mathbf{X}$. To see this, let $\mathbf{x}' + \mathbf{y} \in \mathbf{x}' + (\bigcup_{i=1}^s \mathbf{G}_i)$. We have $\mathbf{y} \in \mathbf{G}_i$ for some i , and obtain the required

$$\mathbf{x}' + \mathbf{y} = (\mathbf{c} - \mathbf{x}_i + \mathbf{x}_i + \mathbf{p}) + \mathbf{y} = (\mathbf{x}_i + \mathbf{y}) + (\mathbf{p} + (\mathbf{c} - \mathbf{x}_i)) \in \mathbf{X} + \mathbf{P} \subseteq \mathbf{X}.$$

◀

The main use case of Proposition E.7 is obtained from the following lemma.

► **Lemma E.9.** *Let \mathbf{Q}, \mathbf{Q}' be periodic sets with the same finitely generated cone $\mathbf{C} = \mathbb{Q}_{\geq 0}\mathbf{Q} = \mathbb{Q}_{\geq 0}\mathbf{Q}'$. Then $\lambda \cdot \mathbf{Q} \subseteq \mathbf{Q}'$ for some $\lambda \in \mathbb{N}_{>0}$.*

Proof. By Lemma 2.2, both \mathbf{Q} and \mathbf{Q}' are finitely generated. Write $\mathbf{Q} = \{\mathbf{p}_1, \dots, \mathbf{p}_s\}^*$. Since $\mathbf{p}_i \in \mathbb{Q}_{\geq 0}\mathbf{Q}'$, there exists $\lambda_i \in \mathbb{N}_{>0}$ such that $\lambda_i \cdot \mathbf{p}_i \in \mathbf{Q}'$. It follows that $\lambda := \prod_{i=1}^s \lambda_i$ fulfills $\lambda \cdot \mathbf{Q} \subseteq \mathbf{Q}'$. ◀

Now we are finally ready to prove Theorem E.4.

► **Theorem E.4.** *Let \mathbf{X} be almost hybridlinear with hybridization $\mathbf{c} + \text{Fill}(\mathbf{P})$. Write $\mathbf{X} = \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{P}_i$, where the sets $\mathbf{P}_1, \dots, \mathbf{P}_r$ satisfy $\text{Fill}(\mathbf{P}) = \text{Fill}(\mathbf{P}_1) = \dots = \text{Fill}(\mathbf{P}_r)$. Define $\mathbf{K}_i := \text{dir}(\mathbf{P}_i)$. Then \mathbf{X} is reducible if and only if the set of cones $\mathcal{K} = \{\mathbf{K}_1, \dots, \mathbf{K}_r\}$ has a complete extraction.*

Proof. “ \Leftarrow ”: Let $\mathbf{C}_1, \dots, \mathbf{C}_r$ be a complete extraction of \mathcal{K} . We first claim that $\bigcup_{i=1}^r \mathbf{K}_i = \mathbb{Q}_{\geq 0} \text{Fill}(\mathbf{P})$.

Proof of claim: \subseteq is clear, for the other direction first use Lemma 4.4 to obtain that $\text{int}(\mathbb{Q}_{\geq 0} \text{Fill}(\mathbf{P})) = \text{int}(\overline{\mathbb{Q}_{\geq 0} \mathbf{P}_i}) \subseteq \text{dir}(\mathbf{P}_i) = \mathbf{K}_i$. Since every \mathbf{C}_i is finitely generated and hence closed, we have that $\bigcup_{i=1}^r \mathbf{K}_i = \bigcup_{i=1}^r \mathbf{C}_i$ is closed. Therefore we obtain that $\mathbb{Q}_{\geq 0} \text{Fill}(\mathbf{P}) = \overline{\text{int}(\mathbb{Q}_{\geq 0} \text{Fill}(\mathbf{P}))} \subseteq \bigcup_{i=1}^r \mathbf{K}_i$ as claimed.

The idea for the rest is as follows: If $\mathbf{F}_i = \{\mathbf{d}_1, \dots, \mathbf{d}_s\}$ is a set of directions of a periodic set \mathbf{P}_i , then $\mathbf{x}_i + \lambda_i \mathbf{F}_i^* \subseteq \mathbf{P}_i$ for some starting point \mathbf{x}_i . We will remove the factor λ_i , and then use \mathbf{F}_i^* as \mathbf{G}_i as in Proposition E.8 to finish the proof.

Formally: For every $i \in \{1, \dots, r\}$, we do the following. Let \mathbf{F}_i be a finite set of generators of \mathbf{C}_i . Since $\mathbf{F}_i \subseteq \text{VectSp}(\mathbf{P}_i) = \mathbb{Q}_{\geq 0}(\mathbf{P} - \mathbf{P})$, by replacing \mathbf{F}_i by multiples we can assume $\mathbf{F}_i \subseteq \mathbf{P}_i - \mathbf{P}_i$. Then by Proposition 5.5, there exists \mathbf{x}'_i such that $\mathbf{x}'_i + \mathbf{F}_i^* \subseteq \mathbf{P}_i$ and hence $(\mathbf{b}_i + \mathbf{x}'_i) + \mathbf{F}_i^* \subseteq \mathbf{b}_i + \mathbf{P}_i \subseteq \mathbf{X}$.

Observe that \mathbf{F}_i^* is a finitely generated periodic set with the same cone as $\mathbf{C}_i \cap \text{Fill}(\mathbf{P})$. Hence by Lemma E.9 there exists λ_i such that $\lambda_i \cdot (\mathbf{C}_i \cap \text{Fill}(\mathbf{P})) \subseteq \mathbf{F}_i^*$. By Proposition E.7, there exists \mathbf{x}_i such that $\mathbf{x}_i + (\mathbf{C}_i \cap \text{Fill}(\mathbf{P})) \subseteq \mathbf{X}$. By Proposition E.8, there exists \mathbf{x} such that $\mathbf{x} + \bigcup_{i=1}^r (\mathbf{C}_i \cap \text{Fill}(\mathbf{P})) \subseteq \mathbf{X}$. Since $\bigcup_{i=1}^r (\mathbf{C}_i \cap \text{Fill}(\mathbf{P})) = (\bigcup_{i=1}^r \mathbf{C}_i) \cap \text{Fill}(\mathbf{P}) = \mathbb{Q}_{\geq 0} \text{Fill}(\mathbf{P}) \cap \text{Fill}(\mathbf{P}) = \text{Fill}(\mathbf{P})$, we have $\mathbf{x} + \text{Fill}(\mathbf{P}) \subseteq \mathbf{X}$ and hence \mathbf{X} is reducible.

“ \Rightarrow ”: This direction follows from [12, Lemma F.5, F.6]. Their argument was slightly more involved because they did not assume that all \mathbf{P}_i define the same lattice $\mathbf{P}' - \mathbf{P}'$, accordingly they state that all $\mathcal{K}_{V,z}$ fulfill the property of Lemma E.2, i.e. have a complete extraction. In our case there is exactly one $\mathcal{K}_{V,z}$ and that is \mathcal{K} . \blacktriangleleft

F Proofs of Section 7

F.1 Proofs of Section 7.1

► **Lemma 7.1.** *Let \mathbf{X} be a semilinear Petri set with hybridization $\mathbf{c} + \mathbf{Q}$. Then \mathbf{X} is reducible.*

Proof. To complete the proof idea of Section 7.1, we need to argue that our intuitive reasoning of every limit being attained is correct, and that this actually implies reducibility. We start the second part.

Since \mathbf{X} is semilinear, we have $\mathbf{X} = \bigcup_{i=1}^r \mathbf{b}_i + \mathbf{P}_i$ with full periodic sets \mathbf{P}_i . In particular the cones $\mathbb{Q}_{\geq 0} \mathbf{P}_i$ are finitely generated. We cannot simply use Theorem E.4 immediately, since the semilinear representation will almost definitely not fulfill $\text{Fill}(\mathbf{P}_1) = \dots = \text{Fill}(\mathbf{P}_r)$. Instead, as in the proof of Theorem 5.10, we have $\mathbf{X} = \bigcup_{i=1}^r \mathbf{b}_i + (\mathbf{P}_i + \mathbf{P})$, and the $\mathbf{P}'_i := \mathbf{P}_i + \mathbf{P}$ fulfill $\text{Fill}(\mathbf{P}'_i) = \text{Fill}(\mathbf{P})$. By Theorem E.4 it hence suffices to show that $\{\text{dir}(\mathbf{P}'_1), \dots, \text{dir}(\mathbf{P}'_r)\}$ has a complete extraction. We do this by showing $\bigcup_{i=1}^r \mathbb{Q}_{\geq 0} \mathbf{P}_i = \mathbb{Q}_{\geq 0} \text{Fill}(\mathbf{P})$, at which point the cones $\mathbf{C}_i := \mathbb{Q}_{\geq 0} \mathbf{P}_i \subseteq \text{dir}(\mathbf{P}'_i)$ form a complete extraction of the required set. This claim about the cones basically corresponds to the intuition of “every direction is attained”.

Claim 1: $\mathbb{Q}_{\geq 0} \mathbf{P} \subseteq \bigcup_{i=1}^r \mathbb{Q}_{\geq 0} \mathbf{P}_i$.

Proof of claim 1: Let $\mathbf{p} \in \mathbf{P}$. Since $\mathbf{X} \neq \emptyset$, there exists $\mathbf{x} \in \mathbf{X}$. Since $\mathbf{x} \in \mathbf{X}$ and $\mathbf{X} + \mathbf{P} \subseteq \mathbf{X}$, the sequence $(\mathbf{x} + \lambda \mathbf{p})_{\lambda \in \mathbb{N}} \subseteq \mathbf{X}$. Hence all of these points are in some $\mathbf{c}_i + \mathbf{P}_i$. By pigeonhole principle, some \mathbf{P}_i contains infinitely many, in particular some $\lambda_1 \mathbf{p} \in \mathbf{P}_i$ and $\lambda_2 \mathbf{p} \in \mathbf{P}_i$. Then $(\lambda_1 - \lambda_2) \mathbf{p} \in \mathbf{P}_i - \mathbf{P}_i$, and furthermore $\mathbf{p} \in \overline{\mathbb{Q}_{\geq 0} \mathbf{P}_i}$, since infinitely many elements from the sequence are contained in \mathbf{P}_i . In total $(\lambda_1 - \lambda_2) \mathbf{p} \in \text{Fill}(\mathbf{P}_i) = \mathbf{P}_i$, since \mathbf{P}_i is full. Then $\mathbf{p} \in \mathbb{Q}_{\geq 0} \mathbf{P}_i$ as claimed.

Claim 2: $\overline{\mathbb{Q}_{\geq 0} \mathbf{P}} = \bigcup_{i=1}^r \overline{\mathbb{Q}_{\geq 0} \mathbf{P}_i}$, which would finish the proof by observing $\overline{\mathbb{Q}_{\geq 0} \mathbf{P}} = \mathbb{Q}_{\geq 0} \text{Fill}(\mathbf{P})$.

Proof of claim 2: “ \supseteq ” follows from Lemma 3.7. Hence let $\mathbf{d} \in \overline{\mathbb{Q}_{\geq 0} \mathbf{P}}$. Then there exists a sequence $(\mathbf{d}_n)_{n \in \mathbb{N}} \subseteq \mathbb{Q}_{\geq 0} \mathbf{P}$ converging to \mathbf{d} . By Claim 1, $\mathbb{Q}_{\geq 0} \mathbf{P} \subseteq \bigcup_{i=1}^r \mathbb{Q}_{\geq 0} \mathbf{P}_i$, hence infinitely many of the \mathbf{d}_n are in the same $\mathbb{Q}_{\geq 0} \mathbf{P}_i$, and we obtain $\mathbf{d} \in \overline{\mathbb{Q}_{\geq 0} \mathbf{P}_i}$ for some i . Since \mathbf{P}_i is finitely generated, $\mathbb{Q}_{\geq 0} \mathbf{P}_i$ is closed, and hence $\mathbf{d} \in \mathbb{Q}_{\geq 0} \mathbf{P}_i$. \blacktriangleleft

F.2 Proofs of Section 7.2

We want to follow the intuition depicted in the middle of Figure 3. The main difficulty is to define “broad enough cones” via a finite set of generators \mathbf{F} on which to then use Proposition 5.5 to obtain $\mathbf{X} + \mathbf{v} + \mathbf{C} \subseteq \mathbf{X}$. Furthermore, in order to ensure that \mathbf{F} contains directions, we may only choose interior vectors for \mathbf{F} .

► **Lemma F.1.** *Let \mathbf{Q} be a full periodic set. Then there exists a finite set $\mathbf{F} \subseteq \text{int}(\mathbf{Q})$ such that $\mathbf{Q} \subseteq \partial(\mathbf{Q}) + \mathbf{F}^*$.*

Proof. Setup: A relation \leq on \mathbf{X} is called a preorder if it is reflexive and transitive. A preorder is called well-preorder if every subset of \mathbf{X} has finitely many minimal elements. By [12, Lemma V.5], if a periodic set \mathbf{Q} is finitely generated, then \mathbf{Q} is well-preordered by $\leq_{\mathbf{Q}}$ defined via $\mathbf{x} \leq_{\mathbf{Q}} \mathbf{y} \iff \mathbf{y} - \mathbf{x} \in \mathbf{Q}$. Hence in particular the set $\text{int}(\mathbf{Q})$ has a finite set of minimal elements \mathbf{F} w.r.t. $\leq_{\mathbf{Q}}$.

Proof of lemma: Since $\mathbf{C} := \mathbb{Q}_{\geq 0} \mathbf{Q}$ is finitely generated, by Lemma 2.1 there exists an integer matrix A such that $\mathbf{C} = \{\mathbf{x} \in \text{VectSp}(\mathbf{Q}) \mid A\mathbf{x} \geq \mathbf{0}\}$, and the faces are $\mathbf{C}_i := \{\mathbf{x} \in \mathbf{C} \mid A_i\mathbf{x} = 0\}$ for the row A_i .

Since $\mathbf{Q} \subseteq \mathbb{N}^n$, we have $A_i\mathbf{x} \in \mathbb{N}$ for all $\mathbf{x} \in \mathbf{Q}$. To every point $\mathbf{x} \in \mathbf{Q}$, we can hence assign $\text{distance}(\mathbf{x}) := \min_i (A_i\mathbf{x}) \in \mathbb{N}$. This measures distance to the closest boundary. The proof is by induction on this distance.

If $\text{distance}(\mathbf{x}) = 0$, then $\mathbf{x} \in \mathbf{C}_i$ for some i , and hence $\mathbf{x} \in \partial(\mathbf{Q})$ as claimed.

Otherwise $\mathbf{x} \in \text{int}(\mathbf{Q})$, and hence by definition of \mathbf{F} , there exist $\mathbf{f} \in \mathbf{F}$ and $\mathbf{q} \in \mathbf{Q}$ such that $\mathbf{x} = \mathbf{f} + \mathbf{q}$. Since $\mathbf{f} \in \text{int}(\mathbf{Q})$, we have $A_i\mathbf{f} > 0$ for every i . Hence $\text{distance}(\mathbf{q}) < \text{distance}(\mathbf{x})$. By induction, $\mathbf{q} \in \partial(\mathbf{Q}) + \mathbf{F}^*$, and hence $\mathbf{x} = \mathbf{q} + \mathbf{f} \in \partial(\mathbf{Q}) + \mathbf{F}^*$. \blacktriangleleft

► **Proposition 7.3.** *Let \mathbf{X} be a set with hybridization $\mathbf{c} + \text{Fill}(\mathbf{P})$. Assume that $|\partial(\mathbf{c} + \text{Fill}(\mathbf{P})) \setminus \mathbf{X}| < \infty$. Then \mathbf{X} is reducible.*

Proof. Since \mathbf{X} contains almost the whole boundary of $\mathbf{c} + \text{Fill}(\mathbf{P})$, in particular for every facet \mathbf{G}_i of $\text{Fill}(\mathbf{P})$, there exists an \mathbf{x}_i such that $\mathbf{x}_i + \mathbf{G}_i \subseteq \mathbf{X}$. By Proposition E.8, there exists \mathbf{x}' such that $\mathbf{x}' + \partial(\text{Fill}(\mathbf{P})) \subseteq \mathbf{X}$. By Lemma F.1, there exists $\mathbf{F} \subseteq \text{int}(\text{Fill}(\mathbf{P}))$ such that $\text{Fill}(\mathbf{P}) \subseteq \partial(\text{Fill}(\mathbf{P})) + \mathbf{F}^*$. By Lemma 4.4, we have $\text{int}(\text{Fill}(\mathbf{P})) \subseteq \text{dir}(\mathbf{P})$. We furthermore have $\mathbf{F} \subseteq (\mathbf{P} - \mathbf{P})$ by definition. Hence, by Proposition 5.5, there exists a \mathbf{d} such that $\mathbf{d} + \mathbf{F}^* \subseteq \mathbf{P}$. We define $\mathbf{x} := \mathbf{x}' + \mathbf{d}$, and obtain

$$\begin{aligned} \mathbf{x} + \text{Fill}(\mathbf{P}) &\subseteq (\mathbf{x}' + \mathbf{d}) + (\partial(\text{Fill}(\mathbf{P})) + \mathbf{F}^*) \\ &= (\mathbf{x}' + \partial(\text{Fill}(\mathbf{P}))) + (\mathbf{d} + \mathbf{F}^*) \subseteq \mathbf{X} + \mathbf{P} \subseteq \mathbf{X}. \end{aligned}$$

\blacktriangleleft

F.3 Separating a target Petri set

We show that if a VAS reachability set does not intersect a target Petri set, then there exists a semilinear inductive invariant separating them. We start by proving that for two given Petri sets \mathbf{X}_1 and \mathbf{X}_2 and a semilinear set \mathbf{S} , there is a common partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ which fulfills the conditions of Theorem 1.1 with respect to both \mathbf{X}_1 and \mathbf{X}_2 .

► **Corollary F.2.** *Let $\mathbf{X}_1, \mathbf{X}_2$ be Petri sets. For every semilinear set \mathbf{S} there exists a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ into pairwise disjoint full linear sets such that for all $i \in \{1, \dots, k\}$ and $j \in \{1, 2\}$ either $\mathbf{X}_j \cap \mathbf{S}_i = \emptyset$, $\mathbf{S}_i \subseteq \mathbf{X}_j$ or $\mathbf{X}_j \cap \mathbf{S}_i$ is an irreducible almost hybridlinear set with hybridization \mathbf{S}_i . Further, if \mathbf{X}_1 and \mathbf{X}_2 are reachability sets of VASs \mathcal{V}_1 and \mathcal{V}_2 , then the partition is computable.*

Proof. The following procedure computes such a partition.

Step 1: Use Theorem 1.1 with $\mathbf{X} = \mathbf{X}_1$ and $\mathbf{S} = \mathbf{S}$ to compute a partition $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_r$ fulfilling the properties for \mathbf{X}_1 . For every i , we compute a subpartition of \mathbf{S}_i as follows.

Step 2: Use Theorem 1.1 with $\mathbf{X} = \mathbf{X}_2$ and $\mathbf{S} = \mathbf{S}_i$ to compute a partition $\mathbf{S}_i = \bigcup_{j=1}^{k_i} \mathbf{S}_{i,j}$ fulfilling the properties for \mathbf{X}_2 . If $\mathbf{X}_1 \cap \mathbf{S}_i = \emptyset$ or $\mathbf{S}_i \subseteq \mathbf{X}_1$, then end step 2.

Otherwise $\mathbf{X}_1 \cap \mathbf{S}_i$ is irreducible almost hybridlinear. For every j do the following:

Case 1: $\dim(\mathbf{S}_{i,j}) < \dim(\mathbf{S}_i)$: Perform a recursive call with $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{S} = \mathbf{S}_{i,j}$ to obtain an appropriate partition of $\mathbf{S}_{i,j}$.

Case 2: $\dim(\mathbf{S}_{i,j}) = \dim(\mathbf{S}_i)$: Decide whether $\mathbf{X}_1 \cap \mathbf{S}_{i,j}$ is reducible, and whether $\mathbf{X}_1 \cap \mathbf{S}_{i,j} = \emptyset$:

Case 2.1: Irreducible or empty: Then leave $\mathbf{S}_{i,j}$ as is.

Case 2.2: $\dim(\mathbf{S}_{i,j}) = \dim(\mathbf{S}_i)$ and $\mathbf{X}_1 \cap \mathbf{S}_{i,j}$ is reducible (possibly, but not necessarily entire $\mathbf{S}_{i,j}$): Then write $\mathbf{S} = \mathbf{c} + \mathbf{Q}$ and find \mathbf{x} such that $\mathbf{x} + \mathbf{Q} \subseteq \mathbf{X}_1 \cap \mathbf{S}_{i,j}$. Afterwards do a recursive call with $\mathbf{X}_1 = \mathbf{X}_1, \mathbf{X}_2 = \mathbf{X}_2$ and $\mathbf{S} = \mathbf{S}_{i,j} \setminus (\mathbf{x} + \mathbf{Q})$, and use Theorem 1.1 with $\mathbf{X} = \mathbf{X}_2$ and $\mathbf{S} = \mathbf{x} + \mathbf{Q}$ to obtain a partition of $\mathbf{x} + \mathbf{Q}$. Combine the two partitions.

Set the partition of \mathbf{S}_i to the union of the partitions of all the $\mathbf{S}_{i,j}$.

Step 3: Now simply return the union of the subpartitions for all the \mathbf{S}_i .

Termination: By Lemma 3.4, we only perform recursion on sets \mathbf{S}' with $\dim(\mathbf{S}') < \dim(\mathbf{S})$. Hence recursion depth is at most $\dim(\mathbf{S})$ and termination immediate.

Correctness: In case the “Otherwise” does not occur: For \mathbf{X}_2 the properties follow from Theorem 1.1. For \mathbf{X}_1 , we have that $\mathbf{X}_1 \cap \mathbf{S}_{i,j}$ is still empty or $\mathbf{S}_{i,j} \subseteq \mathbf{S}_i \subseteq \mathbf{X}_1$.

Case 1: Correct by induction/recursion.

Case 2: By Proposition 6.2(2), $\mathbf{X}_1 \cap \mathbf{S}_{i,j}$ is almost hybridlinear with hybridization $\mathbf{S}_{i,j}$, hence reducibility is defined.

Case 2.1: By definition of the case, we have $\mathbf{X}_1 \cap \mathbf{S}_{i,j}$ is either empty or irreducible almost hybridlinear with hybridization $\mathbf{S}_{i,j}$. For \mathbf{X}_2 , correctness follows from Theorem 1.1.

Case 2.2: For the partition parts $\mathbf{S}_{i,j,k}$ of $\mathbf{x} + \mathbf{Q}$, we have $\mathbf{S}_{i,j,k} \subseteq \mathbf{X}_1$. For \mathbf{X}_2 , the properties hold by correctness of Theorem 1.1. For the partition parts $\mathbf{S}'_{i,j,k}$ of $\mathbf{S}_{i,j} \setminus (\mathbf{x} + \mathbf{Q})$, we have correctness by induction. \blacktriangleleft

► **Corollary F.3.** *Let \mathbf{X}_1 and \mathbf{X}_2 be Petri sets with $\mathbf{X}_1 \cap \mathbf{X}_2 = \emptyset$. Then there exists a semilinear set \mathbf{S}' such that $\mathbf{X}_1 \subseteq \mathbf{S}'$ and $\mathbf{X}_2 \cap \mathbf{S}' = \emptyset$.*

Proof. Let $\mathbf{S} = \mathbb{N}^n$. Let $\mathbf{S} = \mathbf{S}_1 \cup \dots \cup \mathbf{S}_k$ be the partition of Corollary F.2. Let $I \subseteq \{1, \dots, k\}$ be the set of indices such that $\mathbf{X}_1 \cap \mathbf{S}_i \neq \emptyset$. We claim that $\mathbf{S}' = \bigcup_{i \in I} \mathbf{S}_i$ fulfills the result. $\mathbf{X}_1 \subseteq \mathbf{S}'$ is obvious by construction. Hence let $i \in I$. We are going to show that $\mathbf{S}_i \cap \mathbf{X}_2 = \emptyset$.

If i is an index with $\mathbf{S}_i \subseteq \mathbf{X}_1$, then we clearly have $\mathbf{X}_2 \cap \mathbf{S}_i = \emptyset$ since $\mathbf{X}_1 \cap \mathbf{X}_2 = \emptyset$.

Hence let i be an index such that $\mathbf{X}_1 \cap \mathbf{S}_i$ is almost hybridlinear with hybridization \mathbf{S}_i . We will prove $\mathbf{X}_2 \cap \mathbf{S}_i = \emptyset$ by contradicting the other cases.

If $\mathbf{S}_i \subseteq \mathbf{X}_2$, then we have a contradiction to $\mathbf{X}_1 \cap \mathbf{X}_2 = \emptyset$, since $\mathbf{X}_1 \cap \mathbf{S}_i \neq \emptyset$ by definition of almost hybridlinear.

Now assume for contradiction that $\mathbf{X}_2 \cap \mathbf{S}_i$ is almost hybridlinear with hybridization \mathbf{S}_i . Let $\mathbf{x}_1 \in \mathbf{X}_1 \cap \mathbf{S}_i$ and $\mathbf{x}_2 \in \mathbf{X}_2 \cap \mathbf{S}_i$. Write $\mathbf{S}_i = \mathbf{c} + \mathbf{Q}$. Then let $\mathbf{P}_1, \mathbf{P}_2$ smooth such that $\mathbf{X}_1 + \mathbf{P}_1 \subseteq \mathbf{X}_1, \mathbf{X}_2 + \mathbf{P}_2 \subseteq \mathbf{X}_2$ and $\text{Fill}(\mathbf{P}_1) = \mathbf{Q} = \text{Fill}(\mathbf{P}_2)$. By Proposition 4.8, $\mathbf{P} := \mathbf{P}_1 \cap \mathbf{P}_2$ is smooth and $\text{Fill}(\mathbf{P}) = \mathbf{Q}$. Similar to the proof of Proposition E.8, there exists \mathbf{x}' such that $\mathbf{x}' - \mathbf{x}_i \in \mathbf{P}$ for both i . Hence $\mathbf{x}' = \mathbf{x}_i + (\mathbf{x}' - \mathbf{x}_i) \in \mathbf{X}_i + \mathbf{P}_i \subseteq \mathbf{X}_i$ for both i , contradiction to $\mathbf{X}_1 \cap \mathbf{X}_2 = \emptyset$.

Therefore $\mathbf{S}_i \cap \mathbf{X}_2 = \emptyset$ is the only possibility left, and $\mathbf{S} \cap \mathbf{X}_2$ is empty as claimed. \blacktriangleleft

► **Corollary 7.5.** *Let \mathcal{V} be a VAS, and \mathbf{X} a Petri set such that $\text{Reach}(\mathcal{V}) \cap \mathbf{X} = \emptyset$. Then there exists a semilinear inductive invariant \mathbf{S}' of \mathcal{V} such that $\text{Reach}(\mathcal{V}) \subseteq \mathbf{S}'$ and $\mathbf{X} \cap \mathbf{S}' = \emptyset$.*

Proof. Let $\mathbf{X}_1 := \text{Reach}(\mathcal{V})$. By Corollary F.3, there exists a semilinear set \mathbf{S} with $\mathbf{X}_1 \subseteq \mathbf{S}$ and $\mathbf{X}_2 \cap \mathbf{S} = \emptyset$. Let \mathbf{S}^C be the complement of \mathbf{S} . By [9], there exists a semilinear inductive invariant \mathbf{S}' separating \mathbf{X}_1 from \mathbf{S}^C . \blacktriangleleft