Positive logic is not elementary

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(joint work with Aleksy Schubert and Paweł Urzyczyn)

Highlights 2013
Paris, September 19, 2013

First-order intuitionistic logic

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In classical logic:

Every formula is classically equivalent to one of the form:

$$Q_1 x_1 \, Q_2 x_2 \dots Q_k x_k . \, Body(x_, x_2, \dots, x_k),$$

where *Body* has no quantifiers

The language we study

First-order formulas

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This fragment is known to be undecidable

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Mints hierarchy (1968): consider the quantifier prefix
a formula *would get*, if classically normalized

$\circ$

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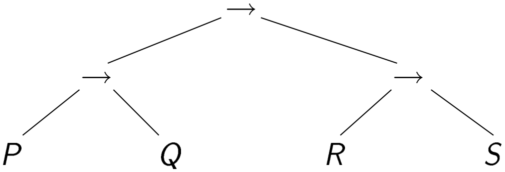

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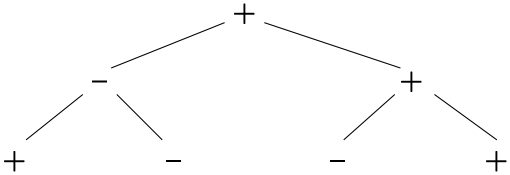

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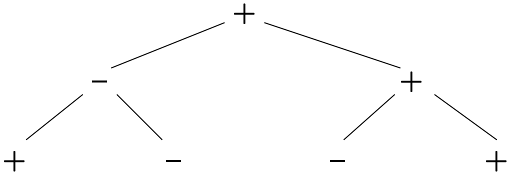

Examples:

$$(\forall x P(x)) \to Q \qquad \text{not positive}$$

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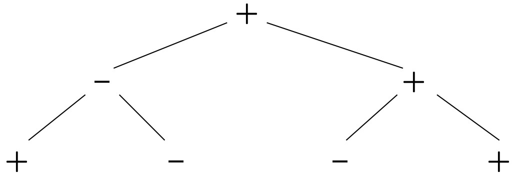

Examples:

$$\begin{array}{ll} (\forall x P(x)) \to Q & \text{not positive} \\ ((\forall x P(x)) \to Q) \to R & \text{positive} \end{array}$$

Positive first-order intuitionistic logic is decidable:
Mints (1968), Dowek, Jiang (2006), Rummelhoff (2007)

Proof construction seen as automaton

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$$\Gamma \vdash \varphi$$

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$$\begin{aligned} \text{Proofs} \quad &\approx \quad \text{Automaton} \\ \varphi \quad &\approx \quad \text{state} \\ \Gamma \quad &\approx \quad \text{memory} \\ \text{proof-search process} \quad &\approx \quad \text{computation} \end{aligned}$$

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Proofs	$\approx$	Automaton
φ	$\approx$	state
Γ	$\approx$	memory
proof-search process	$\approx$	computation
proof completed	$\approx$	accept

Eden automata

Memory: tree of knowledge:

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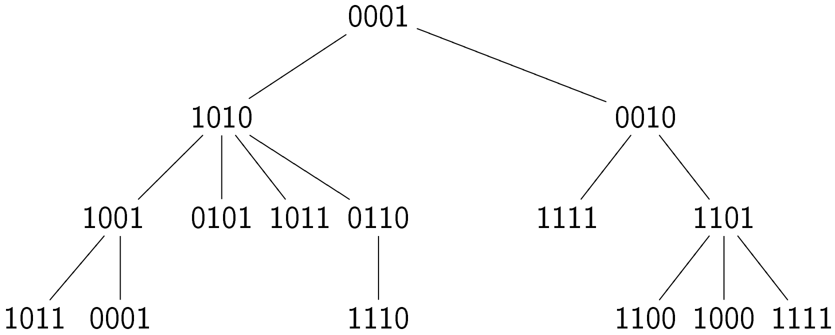
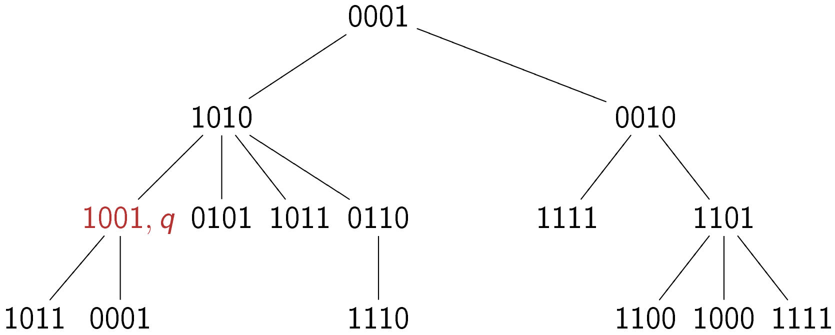


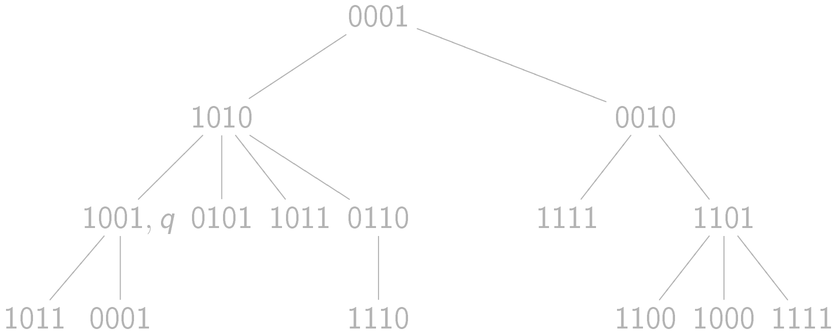
Eden automata

Automata configuration: <tree, node, state>

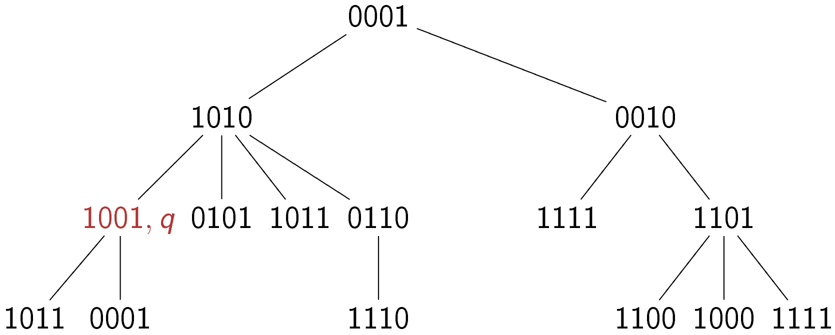


Eden automata

No input, initially one node 0000, q_I



Eden automata—instructions



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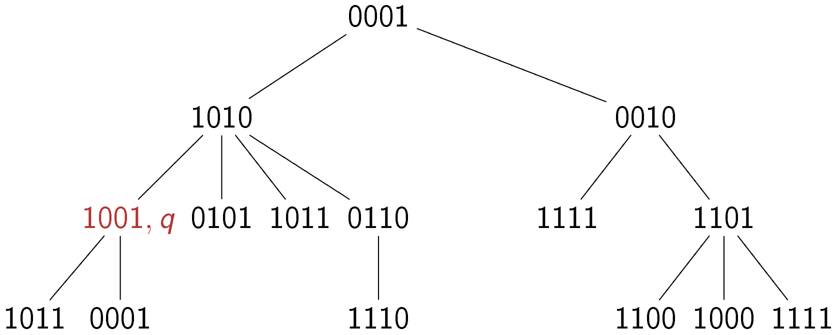
Move up to the father



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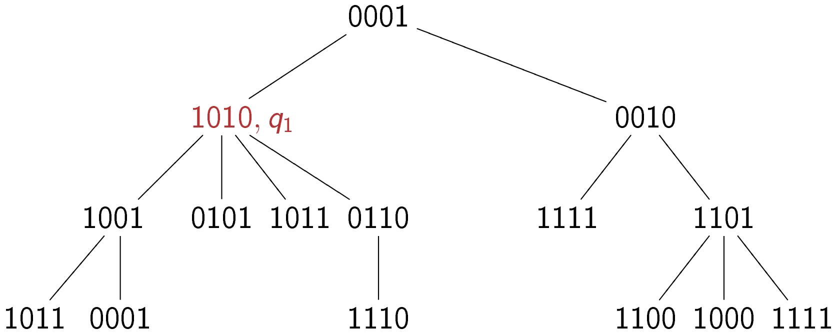

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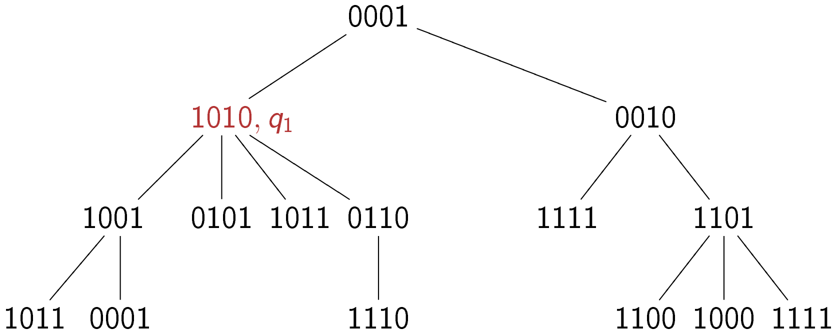

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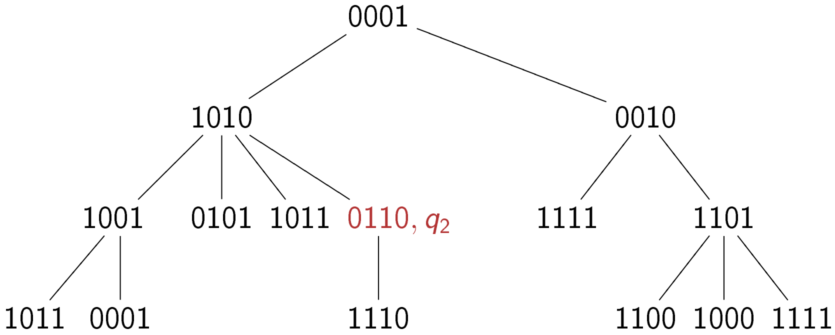
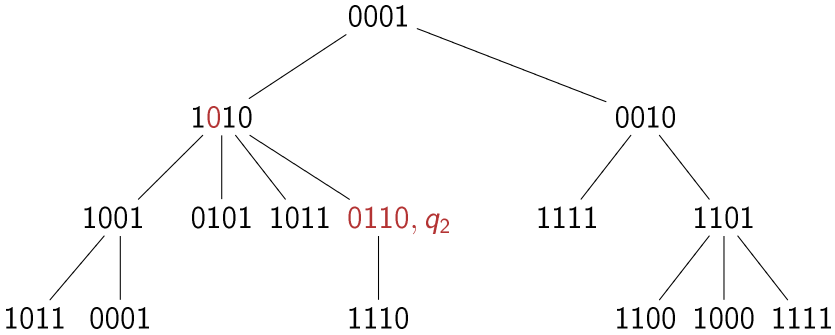


Eden automata—instructions

Write “1” in a given register at a given ancestor



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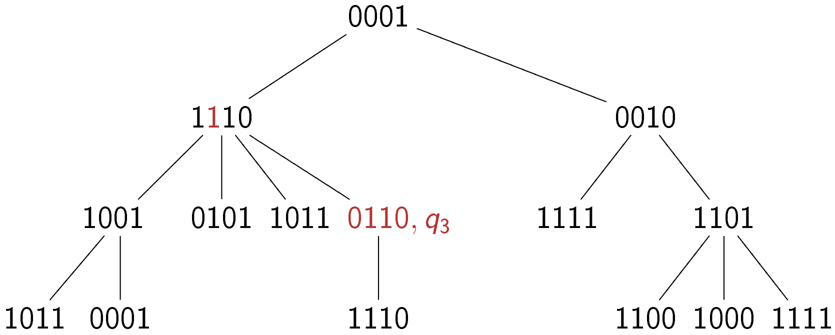

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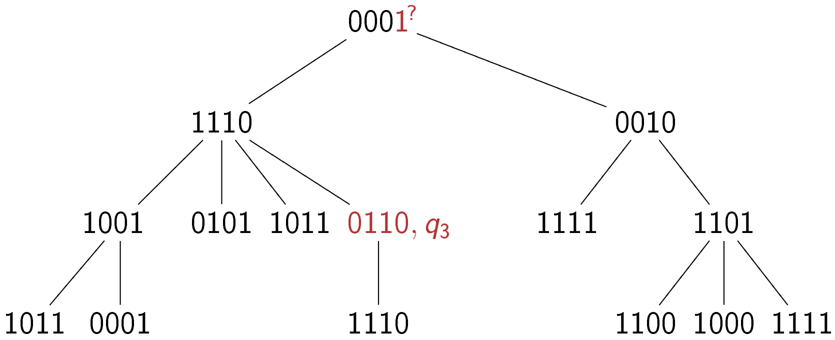

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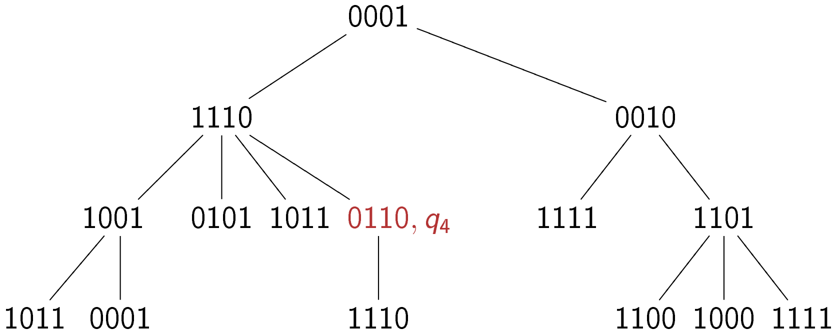

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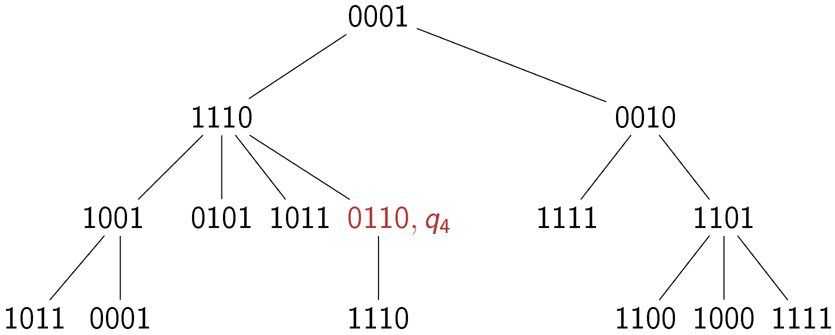

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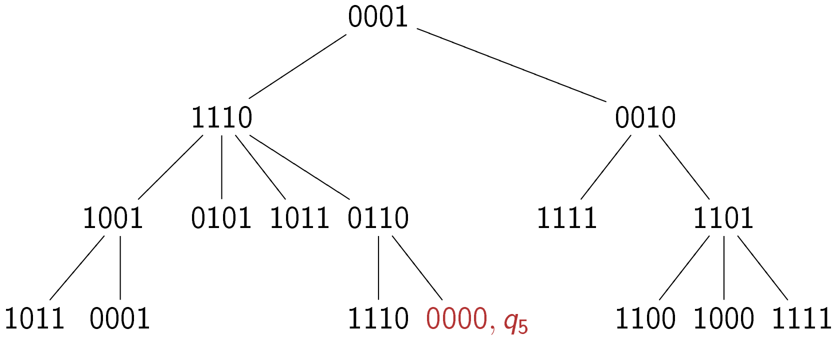

Eden automata—limitations

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Blind-access to children: one enters a random child (nondeterminism)

Bounded memory: one can access only bounded memory, even though the tree is unbounded

Alternation

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Computation is a tree

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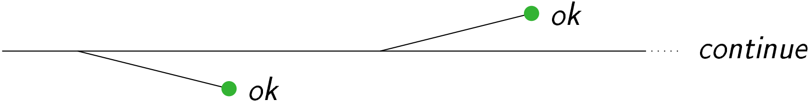
Typical use



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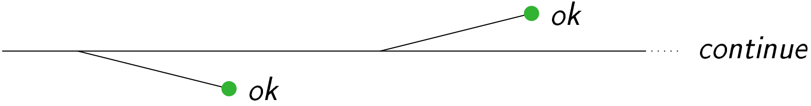
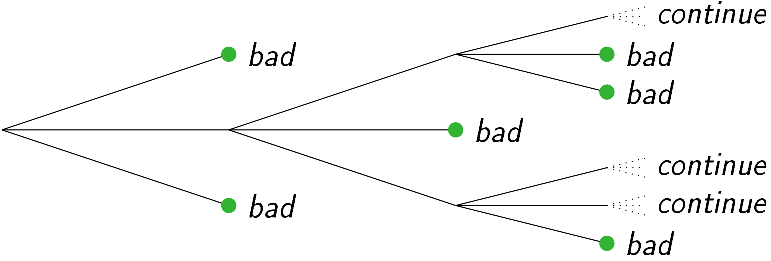


New (for us) use:



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Thm. 2: *The halting problem for Eden Automata is logspace-reducible to positive logic.*

Moral: *The positive logic is not elementary.*

Lessons learned

- Automata are good in proof theory

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- Universal loops are useful
- Positive logic is extremely strong