Logarithmic Weisfeiler-Leman and Treewidth*

Michael Levet¹, Puck Rombach², and Nicholas Sieger³

¹Department of Computer Science, University of Colorado Boulder ²Department of Mathematics and Statistics, University of Vermont

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Abstract

In this paper, we show that the (3k+4)-dimensional Weisfeiler–Leman algorithm can identify graphs of treewidth k in $O(\log n)$ rounds. This improves the result of Grohe & Verbitsky (ICALP 2006), who previously established the analogous result for (4k+3)-dimensional Weisfeiler–Leman. In light of the equivalence between Weisfeiler–Leman and the logic FO + C (Cai, Fürer, & Immerman, Combinatorica 1992), we obtain an improvement in the descriptive complexity for graphs of treewidth k. Precisely, if G is a graph of treewidth k, then there exists a (3k+5)-variable formula φ in FO + C with quantifier depth $O(\log n)$ that identifies G up to isomorphism.

³Department of Mathematics, University of California San Diego

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1 Introduction

The Graph Isomorphism problem (GI) takes as input two graphs G and H, and asks if there exists an isomorphism $\varphi:V(G)\to V(H)$. It is known that $\mathrm{GI}\in\mathsf{NP}\cap\mathsf{coAM}$. The GI problem is in particular conjectured to be NP-intermediate. That is, belonging to NP but neither in P nor NP-complete [Lad75]. Algorithmically, the best known upper-bound on its complexity is $n^{\Theta(\log^2 n)}$, and is due to Babai [Bab16]. It remains open as to whether GI belongs to P. There is considerable evidence suggesting that GI is not NP-complete [Sch88, BH92, IPZ01, Bab16, KST92, AK06, Mat79]. In contrast, the best known lower bound on the complexity of GI is DET [Tor04], which contains NL and is a subclass of TC^1 .

The k-dimensional Weisfeiler–Leman algorithm (k-WL) serves as a key combinatorial tool in GI. It works by iteratively coloring k-tuples of vertices in an isomorphism-invariant manner. On its own, Weisfeiler–Leman serves as an efficient polynomial-time isomorphism test for several families of graphs, including trees [Edm65, IL90], planar graphs [KPS19, GK21], graphs of bounded rank width [GN21], graphs of bounded genus [Gro00, GK19], and graphs for which a specified minor H is forbidden [Gro12]. It is also worth noting that 1-WL identifies almost all graphs [Ros09] and 2-WL identifies almost all regular graphs [Bol82, Kuc87]. In the case of graphs of bounded treewidth [GV06] and planar graphs [GV06, GK21], Weisfeiler–Leman serves even as an NC isomorphism test. Despite the success of WL as an isomorphism test, it is insufficient to place GI into P [CFI92, NS18]. Nonetheless, WL remains an active area of research. For instance, Babai's quasipolynomial-time algorithm [Bab16] combines $O(\log n)$ -WL with group theoretic techniques.

Graphs of bounded treewidth have received significant interest, both for general isomorphism testing and via WL. Bodlaender [Bod90] exhibited the first polynomial-time isomorphism test for this family. In 1999, Grohe & Mariño [GM99] showed that graphs of treewidth k are identified by (k+2)-WL. Subsequently, Grohe & Verbitsky [GV06] showed that the (4k+3)-WL algorithm identifies graphs of treewidth k in $O(\log n)$ rounds, yielding a TC^1 - and the first NC- isomorphism test. Wagner [Wag11] subsequently showed that graphs of bounded treewidth can be canonized in AC^1 . Das, Torán, & Wagner [DTW12] further improved the bound to $\mathsf{LogCFL} = \mathsf{SAC}^1$ for isomorphism testing. Elberfeld & Schweitzer [ES17] finally exhibited a logspace canonization procedure for this family, which implies that isomorphism testing is L-complete under many-to-one $\mathsf{AC}^0 = \mathsf{FO}$ reductions for graphs of bounded treewidth.

Main Results. In this paper, we investigate the power of the Weisfeiler-Leman algorithm in deciding isomorphism for graphs of bounded treewidth.

Theorem 1.1. The (3k + 4)-dimensional Weisfeiler-Leman algorithm identifies graphs of treewidth k in $O(\log n)$ rounds.

In order to prove Theorem 1.1, we utilize a result of [Bod89] that graphs of treewidth k admit a binary tree decomposition of width $\leq 3k + 2$ and height $O(\log n)$. With this decomposition in hand, we leverage a pebbling strategy that is considerably simpler than that of both Kiefer & Neuen [KN22, Theorem 6.4] (who follow the strategy of Grohe & Mariño [GM99]) and Grohe & Verbitsky [GV06].

Remark 1.2. Grohe & Verbitsky [GV06] previously showed that the (4k + 3)-WL identifies graphs of treewidth k in $O(\log n)$ rounds. As a consequence, they obtained the first TC^1 isomorphism test for this family. In light of the close connections between Weisfeiler–Leman and $\mathsf{FO} + \mathsf{C}$ [IL90, CFI92], they also obtained that if G has treewidth k, then there exists a (4k + 4)-variable formula φ in $\mathsf{FO} + \mathsf{C}$ with quantifier depth $O(\log n)$ such that whenever $H \not\cong G$, $G \models \varphi$ and $H \not\models \varphi$. In light of Theorem 1.1, we obtain the following improvement in the descriptive complexity for graphs of bounded treewidth.

Corollary 1.3. Let G be a graph of treewidth k and $H \not\cong G$. Then there exists a formula $\varphi_G \in \mathcal{C}_{3k+5,O(\log n)}$ that identifies G up to isomorphism. That is, for any $H \not\cong G$, $G \models \varphi_G$ and $H \not\models \varphi_G$.

2 Preliminaries

2.1 Weisfeiler-Leman

We begin by recalling the Weisfeiler-Leman algorithm for graphs, which computes an isomorphism-invariant coloring. Let Γ be a graph, and let $k \geq 2$ be an integer. The k-dimension Weisfeiler-Leman, or k-WL,

algorithm begins by constructing an initial coloring $\chi_0: V(\Gamma)^k \to \mathcal{K}$, where \mathcal{K} is our set of colors, by assigning each k-tuple a color based on its isomorphism type. That is, two k-tuples (v_1, \ldots, v_k) and (u_1, \ldots, u_k) receive the same color under χ_0 if and only if the map $v_i \mapsto u_i$ (for all $i \in [k]$) is an isomorphism of the induced subgraphs $\Gamma[\{v_1, \ldots, v_k\}]$ and $\Gamma[\{u_1, \ldots, u_k\}]$ and for all $i, j, v_i = v_j \Leftrightarrow u_i = u_j$.

For $r \geq 0$, the coloring computed at the rth iteration of Weisfeiler–Leman is refined as follows. For a k-tuple $\overline{v} = (v_1, \dots, v_k)$ and a vertex $x \in V(\Gamma)$, define

$$\overline{v}(v_i/x) = (v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_k).$$

The coloring computed at the (r+1)st iteration, denoted χ_{r+1} , stores the color of the given k-tuple \overline{v} at the rth iteration, as well as the colors under χ_r of the k-tuples obtained by substituting a single vertex in \overline{v} for another vertex x. We examine this multiset of colors over all such vertices x. This is formalized as follows:

$$\chi_{r+1}(\overline{v}) = (\chi_r(\overline{v}), \{ (\chi_r(\overline{v}(v_1/x)), \dots, \chi_r(\overline{v}(v_k/x)) | x \in V(\Gamma) \}),$$

where $\{\cdot\}$ denotes a multiset.

Note that the coloring χ_r computed at iteration r induces a partition of $V(\Gamma)^k$ into color classes. The Weisfeiler–Leman algorithm terminates when this partition is not refined, that is, when the partition induced by χ_{r+1} is identical to that induced by χ_r . The final coloring is referred to as the *stable coloring*, which we denote $\chi_{\infty} := \chi_r$.

Remark 2.1. Grohe & Verbitsky [GV06] previously showed that for fixed $k \ge 2$, the classical k-dimensional Weisfeiler-Leman algorithm for graphs can be effectively parallelized. Precisely, each iteration (including the initial coloring) can be implemented using a logspace uniform TC^0 circuit.

As we are interested in both the Weisfeiler–Leman dimension and the number of rounds, we will use the following notation.

Definition 2.2. Let $k \geq 2$ and $r \geq 1$ be integers. The (k, r)-WL algorithm is obtained by running k-WL for r rounds. Here, the initial coloring counts as the first round.

2.2 Pebbling Game

We recall the bijective pebble game introduced by [Hel89, Hel96] for WL on graphs. This game is often used to show that two graphs X and Y cannot be distinguished by k-WL. The game is an Ehrenfeucht–Fraïssé game (c.f., [EFT94, Lib04]), with two players: Spoiler and Duplicator. We begin with k+1 pairs of pebbles, which are placed beside the graph. Each round proceeds as follows.

- 1. Spoiler picks up a pair of pebbles (p_i, p'_i) .
- 2. We check the winning condition, which will be formalized later.
- 3. Duplicator chooses a bijection $f: V(X) \to V(Y)$.
- 4. Spoiler places p_i on some vertex $v \in V(X)$. Then p'_i is placed on f(v).

Let v_1, \ldots, v_m be the vertices of X pebbled at the end of step 1, and let v'_1, \ldots, v'_m be the corresponding pebbled vertices of Y. Spoiler wins precisely if the map $v_\ell \mapsto v'_\ell$ does not extend to an isomorphism of the induced subgraphs $X[\{v_1, \ldots, v_m\}]$ and $Y[\{v'_1, \ldots, v'_m\}]$. Duplicator wins otherwise. Spoiler wins, by definition, at round 0 if X and Y do not have the same number of vertices. We note that X and Y are not distinguished by the first r rounds of k-WL if and only if Duplicator wins the first r rounds of the (k+1)-pebble game [Hel89, Hel96, CFI92].

Remark 2.3. In our work, we explicitly control for both pebbles and rounds. In our theorem statements, we state explicitly the number of pebbles on the board. If Spoiler can win with k pebbles on the board, then we are playing in the (k+1)-pebble game. Note that k-WL corresponds to k-pebbles on the board.

2.3 Logics

We recall key notions of first-order logic. We have a countable set of variables $\{x_1, x_2, \ldots, \}$. Formulas are defined inductively. For the basis, we have that $x_i = x_j$ is a formula for all pairs of variables. Now if φ is a formula, then so are the following: $\varphi \wedge \varphi, \varphi \vee \varphi, \neg \varphi, \exists x_i \varphi$, and $\forall x_i \varphi$. In order to define logics on graphs, we add a relation E(x, y), where E(x, y) = 1 if and only if $\{x, y\}$ is an edge of our graph. In keeping with the conventions of [CFI92], we refer to the first-order logic with relation R as \mathcal{L} and its k-variable fragment as \mathcal{L}_k . We refer to the logic \mathcal{C} as the logic obtained by adding counting quantifiers $\exists^{\geq n} x \varphi$ (there exist at least n elements x that satisfy φ) and $\exists! n x \varphi$ (there exist exactly n elements x that satisfy φ) as \mathcal{C} and its k-variable fragment as \mathcal{C}_k .

The quantifier depth of a formula φ (belonging to either \mathcal{L} or \mathcal{C}) is the depth of its quantifier nesting. We denote the quantifier depth of φ as $\operatorname{qd}(\varphi)$ This is defined inductively as follows.

- If φ is atomic, then $qd(\varphi) = 0$.
- $qd(\neg \varphi) = qd(\varphi)$.
- $\operatorname{qd}(\varphi_1 \vee \varphi_2) = \operatorname{qd}(\varphi_1 \wedge \varphi_2) = \max\{\operatorname{qd}(\varphi_1), \operatorname{qd}(\varphi_2)\}.$
- $qd(Qx\varphi) = qd(\varphi) + 1$, where Q is a quantifier in the logic.

We denote the fragment of \mathcal{L}_k (respectively, \mathcal{C}_k) where the formulas have quantifier depth at most r as $\mathcal{L}_{k,r}$ (respectively, $\mathcal{C}_{k,r}$). We note that the graphs X and Y are distinguished by (k,r)-WL if and only if there exists a formula $\varphi \in \mathcal{C}_{k+1,r}$ such that $X \models \varphi$ and $Y \not\models \varphi$. Furthermore, X is identified by (k,r)-WL if and only if there exists a formula $\varphi \in \mathcal{C}_{k+1,r}$ such that (i) $X \models \varphi$, and (ii) for any $Y \not\cong H$, $Y \not\models \varphi$ [IL90, CFI92].

3 Detecting Separators via Weisfeiler-Leman

In this section, we extend [KN22, Section 4] to show that Weisfeiler–Leman can detect separators in $O(\log n)$ rounds.

Lemma 3.1. Let G, H be graphs. Suppose that $(u, v) \mapsto (x, y)$ have been pebbled. Suppose that $dist(u, v) \neq dist(x, y)$. Then Spoiler can win with 1 additional pebble and $O(\log n)$ rounds.

Proof. Let $f: V(G) \to V(H)$ be the bijection that Duplicator selects. Let m be a midpoint on a shortest u-v path. Spoiler pebbles m. Necessarily, we have that either $\operatorname{dist}(u,m) < \operatorname{dist}(x,f(m))$ or $\operatorname{dist}(m,v) < \operatorname{dist}(y,f(v))$. Without loss of generality, suppose that $\operatorname{dist}(u,m) < \operatorname{dist}(x,f(m))$. We iterate on the above argument, starting from $(u,m) \mapsto (x,f(m))$ and reusing the pebble on v. As $\operatorname{dist}(u,m) \leq \operatorname{dist}(u,v)/2$, we eventually reach in at most $\lceil \log_2 n \rceil$ rounds an instance where $um \in E(G)$, but $f(u)f(m) \notin E(H)$. The result now follows.

Lemma 3.2. Let $G(V, E, \chi)$, $H(V, E, \chi)$ be vertex-colored graphs. Let S be a set of colors. Suppose that $(u, v) \mapsto (x, y)$ have been pebbled. If there exists a u - v path of length d using only vertices from S, but no such x - y path of length d exists in H, then Spoiler can win with 1 additional pebble and $O(\log n)$ additional rounds.

Proof. The pebble game works identically in the case of colored graphs. The only modification is that we check whether the map on the pebbled vertices satisfies $\chi(g_i) = \chi(f(g_i))$ for all i. In light of Lemma 3.1, the result now follows.

Lemma 3.3 (Compare rounds c.f. [KN22, Lemma 4.2]). Let G be a connected graph, and let $\{w_1, w_2\}$ be a 2-separator of G. Let C be a connected component of $G - \{w_1, w_2\}$ such that |C| < n/2, and let $v \in V(C)$. Let $u \in V(G)$ such that the following conditions hold:

- (a) $dist(u, v) \leq 2$, and
- (b) $dist(u, w_i) \leq dist(v, w_i) 2$ for both i = 1, 2.

Suppose that $(w_1, w_2, u) \mapsto (w_1, w_2, v)$ have been pebbled. We have that Spoiler can win with 2 additional pebbles and $O(\log n)$ additional rounds.

Proof. We apply Lemma 3.1 to the pairs (w_1, u) and (w_1, v) . As $\operatorname{dist}(u, w_1) \leq \operatorname{dist}(v, w_1) - 2$, Spoiler can distinguish u from v with 2 additional pebbles and $O(\log(n))$ additional rounds. As w_2 must remain pebbled, we have that Spoiler actually requires 2 additional pebbles rather than the 1 additional pebble prescribed by Lemma 3.1.

Lemma 3.4 (Compare rounds c.f. [KN22, Lemma 4.3]). Let G = (U, V, E) be a 2-connected bipartite graph with n vertices. Let $r \ge \Omega(\log n)$, and let χ_r be the coloring computed by (4, r)-WL. Suppose that:

- (a) $\chi_r(u, u) = \chi_r(u', u')$ for all $u, u' \in U$,
- (b) $\chi_r(v,v) = \chi_r(v',v')$ for all $v,v' \in V$, and
- (c) G has a 2-separator $\{w_1, w_2\}$, with $w_1 \in U$ and $w_2 \in V$.

Let $d := dist(w_1, w_2)$, and let C be the vertex set of a connected component of $G - \{w_1, w_2\}$ with $|C| \le (n-2)/2$. Then $dist(v, w_i) \le d+1$ for all $v \in C$ and all $i \in \{1, 2\}$.

Proof. We use Lemma 3.3 in place of [KN22, Lemma 4.2]. The proof now goes through, $mutatis\ mutandis$. \square

Lemma 3.5 (Compare rounds c.f. [KN22, Lemma 4.4]). Let G = (U, V, E) be a 2-connected bipartite graph with n vertices. Let $k \ge 4, r \ge \Omega(\log n)$, and let χ_G be the coloring computed by (k, r)-WL. Suppose that the following holds:

- 1. $\chi_r(u, u) = \chi_r(u', u')$ for all $u, u' \in U$,
- 2. $\chi_r(v,v) = \chi_r(v',v')$ for all $v,v' \in V$, and
- 3. G has a 2-separator $\{w_1, w_2\}$.

Then $w_1w_2 \notin E(G)$.

Proof. We use Lemma 3.4 in place of [KN22, Lemma 4.3]. The proof now goes through mutatis mutandis.

Kiefer & Neuen use [KPS19, Corollary 7] to distinguish cut from non-cut vertices. We may apply Lemma 3.1 instead, in the following manner.

Lemma 3.6. Let G, H be connected graphs. Suppose v is a cut vertex, and v' is not a cut vertex. We have that $(4, O(\log n))$ -WL will distinguish v from v'.

Proof. Suppose that $v \mapsto v'$ has been pebbled. We may thus treat $v \mapsto v'$ as having been individualized. Now as v is a cut vertex for G, there exist vertices u, w that belong to different components of $G - \{v\}$. At the next two rounds, Spoiler pebbles (u, w). Let $x, y \in V(H)$ be the corresponding pebbled elements. As H is connected and v' is not a cut vertex, x, y belong to the same connected component. We now apply Lemma 3.2 to the color class specified by individualizing $v \mapsto v'$.

Theorem 3.7 (Compare rounds c.f. [KN22, Theorem 4.1]). Let G be a 2-connected graph with a 2-separator $\{w_1, w_2\}$. Now let $k \geq 4, r \geq \Omega(\log n)$. Let $\chi_{G,r}$ be the coloring computed by (k, r)-WL. Suppose that for every $v \in V(G)$, either $\chi_G(v, v) = \chi_G(w_1, w_1)$ or $\chi_G(v, v) = \chi_G(w_2, w_2)$. Then G is a cycle.

Proof. We follow the strategy of [KN22, Theorem 4.1]. By [KN22, Lemma 3.8], we may assume without loss of generality that $\chi_{G,r}(w_1,w_1) \neq \chi_{G,r}(w_2,w_2)$. The proof is by induction on n := |V(G)|. When $n \leq 4$, a case analysis of the possible graphs G yields the result.

Let $n \geq 5$. It suffices to prove the statement for an *n*-vertex graph G with maximum edge set that satisfies the assumptions of the theorem. Define:

$$U := \{ v \in V(G) : \chi_{G,r}(v,v) = \chi_{G,r}(w_1, w_1) \}$$

$$V := \{ v \in V(G) : \chi_{G,r}(v,v) = \chi_{G,r}(w_2, w_2) \}.$$

Let U_1, \ldots, U_k be the connected components of the induced subgraph G[U], and let V_1, \ldots, V_ℓ be the connected components of the induced subgraph G[V]. Without loss of generality, suppose that $w_1 \in U_1$ and $w_2 \in V_1$. Let C be the vertex set of a connected component of $G - \{w_1, w_2\}$ of size at most (n-2)/2. Let $C' := C - (C \cup \{w_1, w_2\})$.

Claim 1 [(c.f. [KN22, Claim 4.7]]. Suppose that $C \subseteq U_1 \cup V_1$ or $C' \subseteq U_1 \cup V_1$. Then G is a cycle.

Proof (Sketch). The proof is virtually identical to [KN22, Claim 4.7]. We outline the changes. First, Kiefer & Neuen use the fact that 2-WL can distinguish arcs (u, v) where u, v belong to the same connected component, from arcs (u', v') where u', v' belong to different connected components. By Lemma 3.1, we have that $(4, O(\log n))$ -WL will distinguish (u, v) from (u', v'). Furthermore, we make use of Lemma 3.6 instead of [KPS19, Corollary 7] to obtain that $(4, O(\log n))$ -WL detects cut vertices. Thus, we may use the coloring computed by $(4, O(\log n))$ -WL in place of the stable coloring computed by 2-WL. The remainder of the proof from Kiefer & Neuen [KN22] goes through $mutatis\ mutandis$.

So we now assume that $C \nsubseteq U_1 \cup V_1$ and $C' \nsubseteq U_1 \cup V_1$. Now as $w_1 \in U_1, w_2 \in V_1$, and G[U], G[V] do not have any cut vertices, we have that G[U] and G[V] contain at least two connected components. Let G' be the graph with $V(G') = \{U_1, \ldots, U_k, V_1, \ldots, V_\ell\}$. Now the edges of G' are precisely the pairs $U_i V_j$ where there exist $u \in U_i, v \in V_j$ such that $uv \in E(G)$. We now claim that G' satisfies the assumptions of this theorem. We first note that $G' - \{U_1, V_1\}$ is not connected. Thus, $\{U_1, V_1\}$ forms a 2-separator of G'. Now the coloring computed by (k, r)-WL on G refines the coloring computed by (k, r)-WL on G[U]. Furthermore, as every vertex of U has the same color as w_1 , we have that every component of G[U] has the same color as U_1 . By a similar argument, every component of G[V] has the same color as V_1 .

Since G is connected, we have that G' is connected. It remains to be shown that G' is 2-connected. The argument is identical to that provided by Kiefer & Neuen [KN22], except that we use Lemma 3.6 instead of [KPS19, Corollary 7]. We have already established that the U_i, U_j have the same color for all i, j, and similarly V_i, V_j have the same color for all i, j.

Suppose that |V(G')| < |V(G)|. Kiefer & Neuen [KN22, Theorem 4.1] previously established that G' is a cycle. Their claim holds, though we use our Claim 1 in place of their [KN22, Claim 4.7].

We note that $|V(G')| \leq |V(G)|$. So if $|V(G')| \not< V(G)|$, then we have that |V(G')| = |V(G)|. In this case, G is bipartite with bipartition $\{U, V\}$, where all the U_i, V_j sets are singletons. By Lemma 3.5, we have that $w_1w_2 \not\in E(G)$. Let $d := \operatorname{dist}(w_1, w_2)$. Note that d is odd, and so $d \geq 3$.

Claim 2 [c.f. [KN22, Claim 4.8]]. Let $u, v \in V(G)$ such that $\operatorname{dist}(u, v) < d$. Then there is a unique shortest path from u to v.

Proof. The proof is similar to [KN22, Claim 4.8]. We outline the changes. Kiefer & Neuen defined an auxiliary graph G'', where V(G'') = V(G) and

$$E(G'') = E(G) \cup \{u''v'' : \chi_G(u, v) = \chi_G(u'', v'')\}.$$

(Kiefer & Neuen refer to G'' as G'. However, to avoid overloading variables, we will refer to the auxiliary graph as G'' so as not to avoid confusion with the bipartite graph G' under consideration leading up to Claim 2.) We have previously established that under the coloring computed by $(4, O(\log n))$ -WL, we have that U_i, U_j have the same color for all i, j, and similarly V_i, V_j have the same color for all i, j. But the U_i, V_j sets are all singletons. Furthermore, the vertices in U all have the same color as w_1 , and the vertices in V all have the same color as w_2 . So for every $v \in V(G)$, v receives the same color as either w_1 or w_2 . The remainder of the proof from Kiefer & Neuen [KN22] goes through mutatis mutandis.

The remainder of the proof is virtually identical to that of Kiefer & Neuen, with the exception of using some of our results in place of theirs. We briefly describe the changes. In [KN22, Claim 4.9], we use (i) our Claim 2 instead of their [KN22, Claim 4.8], and (ii) our Lemma 3.3 instead of their [KN22, Lemma 4.2]. The modifications to [KN22, Claims 4.10 and 4.11] follow similarly, using as well our modifications to [KN22, Claim 4.9]. We note that the proof of [KN22, Claim 4.11] makes use of 2-WL (without control on rounds) via [KN22, Theorem 3.15]. However, since we are considering vertices with distance ≤ 4 from $\{w_1, w_2\}$, we may instead use the coloring computed by (4, O(1))-WL. The result now follows.

4 Detecting Decompositions via Weisfeiler-Leman

Let S be a set of colors. We say that a path u_0, u_1, \ldots, u_ℓ avoids S if $\chi_G(u_i, u_i) \notin S$ for all $i \in [\ell - 1]$. Note that no restrictions are imposed on the endpoints. By Lemma 3.2, $(4, O(\log n))$ -WL detects whether a path avoids S. Let $r \geq 0$ and define $G|_{S,r}(V, E)$ to be the graph with vertex set $V = \{v \in V(G) : \chi_{G,r}(v,v) \in S\}$ and edge set:

 $E := \{uv : \text{ there exists an } S\text{-avoiding path from } u \text{ to } v\}.$

Similarly to the work in Section 3, we extend several results of Kiefer & Neuen [KN22]. The main change is to show that we can use $(4, O(\log n))$ -WL, where Kiefer & Neuen instead use 2-WL without controlling for rounds. The main differences between their approach and ours are most apparent in Corollary 4.3.

Lemma 4.1 (Compare c.f. [KN22, Lemma 5.1]). Let G be a graph, $r \ge \Omega(\log n)$, and let $S \subseteq \{\chi_{G,r}(v,v) : v \in V(G)\}$. Then $\chi_{G|_S,r}(v,v) = \chi_{G|_S,r}(u,u)$ for all $u,v \in V(G)$ satisfying $\chi_{G,r}(v,v) = \chi_{G,r}(u,u)$.

Proof. By Lemma 3.2, $(4, O(\log n))$ -WL detects whether there is a path between two vertices that avoid S. So there exists a set of colors T such that $uv \in E(G|S)$ if and only if $\chi_G(u,v) \in T$. It follows that every refinement step in G|S will also be performed in G.

Theorem 4.2 (Compare rounds c.f. [KN22, Theorem 5.2]). Let G, H be connected graphs, and suppose G is 2-connected. Let $r \geq \Omega(\log n)$. Let $w_1, w_2 \in V(G)$ s.t. $\{w_1, w_2\}$ forms a 2-separator. Let $v_1, v_2 \in V(H)$ such that $\chi_{G,r}(w_1, w_2) = \chi_{H,r}(v_1, v_2)$. Then $\{v_1, v_2\}$ forms a 2-separator in H.

Proof. The argument is identical as in [KN22, Theorem 5.2], with the exception that we use the coloring computed by (4, r)-WL rather than the stable coloring computed by 2-WL. The result now follows.

Corollary 4.3 (Compare rounds c.f. [KN22, Corollary 5.3]). Let $k \geq 2$ and $r \geq \Omega(\log n)$. Let $\chi_{G,r}$ be the coloring computed by $(\max\{k+2,4\},r)$ -WL applied to G, and define $\chi_{H,r}$ analogously. Let G,H be graphs, and suppose $\{w_1,\ldots,w_k\}\subseteq V(G)$ is a k-separator in G. If $\chi_{G,r}(w_1,\ldots,w_k)=\chi_{H,r}(v_1,\ldots,v_k)$, then $\{v_1,\ldots,v_k\}$ forms a k-separator in H.

Proof. Our strategy follows that of [KN22, Corollary 5.3]. Suppose first that k = 2. As G has a 2-separator by assumption, we have that the vertex connectivity number $\kappa(G) \leq 2$. We consider several cases.

- Case 1: Suppose first that G and H are both 2-connected. The result follows from Theorem 4.2.
- Case 2: Suppose instead that one such graph is connected, but not 2-connected. Without loss of generality, suppose that $\kappa(G) = 2$ and $\kappa(H) = 1$. Then H contains a cut vertex, while G does not. So regardless of the bijection $f: V(G) \to V(H)$ that Duplicator selects, Spoiler begins by pebbling a cut vertex $w' \in V(H)$. Let $w = f^{-1}(w') \in V(G)$. By Lemma 3.6, Spoiler wins with 3 additional pebbles and $O(\log n)$ additional rounds.
- Case 3: Suppose now that $\kappa(G) \leq 2$ and $\kappa(H) > 2$. By Case 2, we may assume that $\kappa(G) = 2$. Now w_1, w_2 must lie on a common 2-connected component of G. Otherwise, they do not form a 2-separator. Let C_G denote the vertex set of this component. Suppose that v_1, v_2 do not lie on the same 2-connected component of H. We will show that Spoiler has a winning strategy in the pebble game, starting from the configuration $((w_1, w_2), (v_1, v_2))$. Spoiler pebbles a 2-separator (x_1, x_2) other than w_1, w_2 of C_G . Let (y_1, y_2) be the corresponding pebbled vertices in H. Regardless of Duplicator's choice of bijections at each round, we will have that either (i) $H \{y_1, y_2\}$ is connected or (ii) v_1, v_2 are on different components of $H \{y_1, y_2\}$. In either case, we have by Lemma 3.1 that Spoiler wins with 2 additional pebbles and $O(\log n)$ rounds.

So we may now assume that v_1, v_2 lie on the same 2-connected component, which we call C_H . By an argument similar to the one above, we have that if $w, w' \in C_G$ and $v \in C_H, v' \notin C_H$, then $(4, O(\log n))$ -WL will distinguish (w, w') from (v, v'). We claim that this implies that $(4, O(\log n))$ -WL applied to $(G[C_G], H[C_H])$ will fail to distinguish (w_1, w_2) from (v_1, v_2) . Suppose for a contradiction that this is not the case. In a similar manner as Kiefer & Neuen [KN22], we consider the pebble game on $(G[C_G], H[C_H])$ from the initial configuration $((w_1, w_2), (v_1, v_2))$. We first observe that if Duplicator selects a bijection $f: V(G) \to V(H)$ that does not map $f(C_G) = C_H$ setwise, then Spoiler can

(without loss of generality) pebble some vertex $w \in C_G$ such that $f(w) \notin C_H$. As $((w_1, w_2), (v_1, v_2))$ have been pebbled, we have by Lemma 3.1 that Spoiler wins with 2 additional pebbles and $O(\log n)$ additional rounds. Therefore, a winning strategy in the pebble game on $(G[C_G], H[C_H])$ from the initial configuration $((w_1, w_2), (v_1, v_2))$ can be lifted to a winning strategy in the pebble game on (G, H). As Spoiler wins with a total of 4 pebbles and $O(\log n)$ rounds, this contradicts the assumption that (4, r)-WL applied to (G, H) fails to distinguish (w_1, w_2) from (v_1, v_2) .

By Theorem 4.2, it follows that (v_1, v_2) is a 2-separator in $H[C_H]$, and hence a 2-separator in H.

We now consider the case when k > 2. Let G, H be graphs, and suppose that $\{w_1, \ldots, w_k\}$ is a k-separator in G. Let $\{v_1, \ldots, v_k\} \subseteq V(H)$ such that $(\max\{k+2,4\}, r)$ -WL fails to distinguish (w_1, \ldots, w_k) from (v_1, \ldots, v_k) . Thus, Duplicator has a winning strategy in the (k+3)-pebble, r-round game, starting from the configuration $((w_1, \ldots, w_k), (v_1, \ldots, v_k))$. Let $G' := G - \{w_1, \ldots, w_k\}$ and $H' := H - \{v_1, \ldots, v_k\}$. We claim that (4, r)-WL applied to (G', H') fails to distinguish (w_{k-1}, w_k) from (v_{k-1}, v_k) . Indeed, if this is not the case, Spoiler can win in the 4-pebble, r-round game starting from $((w_{k-1}, w_k), (v_{k-1}, v_k))$. Spoiler's strategy can be lifted to a winning strategy in the (k+3)-pebble, r-round game on (G, H) by never moving the pebbles on $(w_1, \ldots, w_{k-2}) \mapsto (v_1, \ldots, v_{k-2})$, contradicting the assumption that $(\max\{k+2,4\}, r)$ -WL fails to distinguish (w_1, \ldots, w_k) from (v_1, \ldots, v_k) .

Now we may assume that G', H' both fail to be connected; otherwise, by Lemma 3.1, Spoiler can win in the pebble game on G', H' using 3 pebbles and $O(\log n)$ rounds. The remainder of the argument proceeds identically as in [KN22, Corollary 5.3]. We note that in the case when G', H' have at most two vertices, Kiefer & Neuen appeal to [KPS19, Corollary 7], which implicitly uses the full power of 2-WL. However, as G', H' have at most 2 vertices, 2-WL terminates in at most 2 rounds when applied to G', H'. The result now follows.

5 Proof of Main Result

Our goal in this section is to prove the following.

Theorem 5.1. Let G be a graph with treewidth k, and let $H \not\cong G$. We have that $(3k + 4, O(\log n))$ -WL distinguishes G from H.

One key ingredient is that WL can detect separators with only $O(\log n)$ rounds. To this end, we recall the following notation introduced in [KN22]. Let (v_1, \ldots, v_{k+1}) be a (k+1)-tuple of vertices in a graph G. Define $s_G(v_1, \ldots, v_{k+1}) := |C|$, where C is the vertex set of the unique connected component of $G - \{v_1, \ldots, v_k\}$ containing v_{k+1} . If $v_{k+1} \in \{v_1, \ldots, v_k\}$, then $s_G(v_1, \ldots, v_{k+1}) = 0$.

Lemma 5.2. Let G, H be graphs that are connected, but not 2-connected. Let $(v_1, v_2, v_3) \in V(G)^3, (w_1, w_2, w_3) \in V(H)^3$. If $s_G(v_1, v_2, v_3) \neq s_H(v_1, v_2, v_3)$, then Spoiler can win with 3 additional pebbles and $O(\log n)$ rounds starting from the configuration $((v_1, v_2, v_3), (w_1, w_2, w_3))$.

Proof. Without loss of generality, we may assume that $v_3 \notin \{v_1, v_2\}$ and $w_3 \notin \{w_1, w_2\}$. Let C be the connected component of $G - \{v_1\}$ containing v_3 , and let C' be the connected component of $H - \{w_1\}$ containing w_1 . If $|C| \neq |C'|$, then by Lemma 3.1, Spoiler can win with 3 additional pebbles and $O(\log n)$ rounds. By a similar argument, Spoiler can win if $v_2 \in C$ and $w_2 \notin C'$ (or vice-versa).

Now suppose that both v_1, v_2 are cut vertices. We may assume that w_1, w_2 are also both cut vertices. Otherwise, by Lemma 3.1, Spoiler can win with 3 additional pebbles and $O(\log n)$ rounds. By an argument similar to the one in the preceding paragraph, Spoiler can win with 3 additional pebbles and $O(\log n)$ rounds if for some $i \in \{1, 2\}$, one of the following conditions fails to hold:

- v_3 lies in the same connected component of v_{2-i} in $G \{v_i\}$ if and only if w_3 lies in the same connected component of w_{2-i} in $H \{w_i\}$.
- The size of the connected component in $G \{v_i\}$ containing v_3 is the same as the size of the connected component in $H \{w_i\}$ containing w_3 .

The above conditions determine s_G, s_H in this case. Thus, if $s_G \neq s_H$, then Spoiler can win with 3 additional pebbles and $O(\log n)$ additional rounds.

Lemma 5.3. Let G, H be 2-connected graphs on n vertices. Let $(v_1, v_2, v_3) \in V(G)^3, (w_1, w_2, w_3) \in V(H)^3$. If $s_G(v_1, v_2, v_3) \neq s_H(v_1, v_2, v_3)$, then Spoiler can win with 3 additional pebbles and $O(\log n)$ rounds starting from the configuration $((v_1, v_2, v_3), (w_1, w_2, w_3))$.

Proof. We follow the strategy of [KN22, Theorem 5.8]. If neither $\{v_1, v_2\}$ nor $\{w_1, w_2\}$ are 2-separators, then $G - \{v_1, v_2\}$ and $H - \{w_1, w_2\}$ are connected. As |G| = |H| by assumption, the result vacuously holds. If $\{v_1, v_2\}$ is a 2-separator and $\{w_1, w_2\}$ is not (or vice-versa), then by Theorem 4.2, Spoiler can win with 3 additional pebbles (reusing the pebble pair on $v_3 \mapsto w_3$) and $O(\log n)$ rounds.

So now suppose that $\{v_1, v_2\}$ is a 2-separator in G and $\{w_1, w_2\}$ is a 2-separator in H. Let $S = \{\chi_{G,r}(v_1, v_2), \chi_{G,r}(v_2, v_2)\}$ and $A = \{\chi_{H,r}(w_1, w_1), \chi_{H,r}(w_2, w_2)$. Define G' to be the graph with vertex set $V(G') := \{v \in V(G) : \chi_{G,r}(v,v) \in S\}$ and E(G') to be the pairs uv where there is a u-v path in G that avoids S. Define H' analogously with respect to the set A. By the proof of [KN22, Theorem 5.2], G', H' are 2-connected. We have several cases to consider.

- Case 1: Suppose that |V(G')| = 2. This case is handled identically as in the proof of [KN22, Theorem 5.8].
- Case 2: Suppose instead that there is a vertex set C such that $V(G') \subseteq C \cup \{g_1, g_2\}$. This case is handled identically as in the proof of [KN22, Theorem 5.8].
- Case 3: If neither Case 1 nor Case 2 are satisfied, then $\{v_1, v_2\}$ is a 2-separator in G'. By Lemma 4.1 and Theorem 3.7, we have that G' is a cycle. As $\{v_1, v_2\}$ is a 2-separator for G', we have that $|V(G')| \ge 4$ and $v_1v_2 \notin E(V')$. By similar argument, we have that H' is a cycle, $|V(H')| \ge 4$, and $w_1w_2 \notin E(H')$. The remainder of the argument follows identically as in the proof of [KN22, Theorem 5.8].

Lemma 5.4 (Compare rounds c.f. [KN22, Lemma 6.2]). Let $k \ge 2$. Let G, H be graphs. Let $(v_1, \ldots, v_{k+1}) \in V(G)^{k+1}, (w_1, \ldots, w_{k+1}) \in V(H)^{k+1}$. If $s_G(v_1, \ldots, v_{k+1}) \ne s_H(w_1, \ldots, w_{k+1})$, then Spoiler can in the $(\max\{k+3,4\})$ -pebble game using at most $O(\log n)$ rounds, starting from the initial configuration $((v_1, \ldots, v_{k+1}), (w_1, \ldots, w_{k+1}))$.

Proof. We follow the strategy of [KN22, Lemma 6.2]. Without loss of generality, we may assume that $v_{k+1} \notin \{v_1, \ldots, v_k\}$, $w_{k+1} \notin \{w_1, \ldots, w_k\}$, and that G and H are connected graphs of the same order. By Corollary 4.3, we may assume that G if 2-connected if and only if H is 2-connected; otherwise, Spoiler wins with $\max\{k+3,4\}$ pebbles in $O(\log n)$ rounds. Consider first the case when k=2.

- Case 1: Assume first that both graphs are connected, but not 2-connected. This case is handled by Lemma 5.2.
- Case 2: Suppose instead that G and H are both 2-connected. This case is handled by Lemma 5.3.

Finally, consider the case in which k > 2. Let $\hat{G} := G - \{v_1, \dots, v_{k-2}\}$ and $\hat{H} := H - \{w_1, \dots, w_{k-2}\}$. Then $s_{\hat{G}}(v_{k-1}, v_k, v_{k+1}) \neq s_{\hat{H}}(w_{k-1}, w_k, w_{k+1})$. So Spoiler can win with 3 additional pebbles and $O(\log n)$ rounds in the pebble game on (\hat{G}, \hat{H}) starting from the configuration $((v_{k-1}, v_k, v_{k+1}), (w_{k-1}, w_k, w_{k+1}))$. But then Spoiler can win in the max $\{k+2, 4\}$ -pebble game with $O(\log n)$ rounds starting from the initial configuration $((v_1, \dots, v_{k+1}), (w_1, \dots, w_{k+1}))$ by never moving the first k-2 pebbles.

We now prove Theorem 5.1.

Proof of Theorem 5.1. Let G be a graph of treewidth k, and let H be a graph not isomorphic to G. Let (T,β) be a tree decomposition for G of width $\leq 3k+2$ and height $O(\log n)$, with T a binary tree, as prescribed by [Bod89]. Let s be the root node of T. Spoiler begins by pebbling the vertices of $\beta(s)$, using at most 3k+2 pebbles. Now as (T,β) is a tree decomposition, we have that for any edge $uv \in E(T)$, $\beta(u) \cap \beta(v)$ is a separator of G. Let $f:V(G) \to V(H)$ be the bijection that Duplicator selects. By Corollary 4.3, if there

exists a subset S of the pebbled vertices in G, such that f(S) is not a separator of the same size, then Spoiler can win with 2 additional pebbles and $O(\log n)$ additional rounds. This yields a total of 3k + 4 pebbles. So suppose that f preserves separators of pebbled vertices.

Now let ℓ be the left child of s in T, and let r be the right child of s in T. Denote the separators $S_{\ell} := \beta(\ell) \cap \beta(s)$ and $S_r := \beta(r) \cap \beta(s)$. By Lemma 5.4, we may assume that if v belongs to an m-vertex component of $G - S_{\ell}$ (respectively, $G - S_r$), then f(v) belongs to an m-vertex component of $H - f(S_{\ell})$ (respectively, $H - f(S_r)$). Otherwise, Spoiler may win with $O(\log n)$ additional rounds. While Lemma 5.4 prescribes 3 additional pebbles, we note that there exists a vertex in $\beta(s) \setminus \beta(\ell)$ (respectively, $\beta(s) \setminus \beta(r)$). So we may reuse one such pebble in $\beta(s)$, resulting in only 2 additional pebbles (for a total of 3k + 4 pebbles).

Without loss of generality, suppose that $G \setminus S_{\ell} \not\cong H \setminus f(S_{\ell})$. Spoiler moves the pebbles in $\beta(s) \setminus \beta(\ell)$ into $\beta(\ell)$. We now iterate on this argument starting from ℓ as the root of our subtree in the tree decomposition (T,β) . As $G \ncong H$, we will eventually reach a stage (such as when all of $\beta(v)$ is pebbled for some leaf node $v \in V(T)$) where the map induced by the pebbled vertices does not extend to an isomorphism.

It remains to analyze the number of pebbles and rounds. At each level of the tree, we use 3k+2 rounds to pebble the vertices in a given bag. We may use 2 additional pebbles and $O(\log n)$ rounds at a given level as prescribed by Corollary 4.3 or Lemma 5.4. However, invoking Corollary 4.3 or Lemma 5.4 results in Spoiler winning. As T has height $O(\log n)$ and k is bounded, this results in $O(\log n)$ rounds, as desired. The result now follows.

6 Conclusion

We showed that the (3k + 4)-WL identifies graphs of treewidth k in $O(\log n)$ rounds, improving upon the work of Grohe & Verbitsky [GV06], who established the analogous result for (4k + 3)-WL. As a corollary, we obtained that graphs of treewidth k are identified by FO + C formulas with (3k + 5)-variables and quantifier depth $O(\log n)$. We contrast this with the work of Kiefer & Neuen [KN22, Theorem 6.4], who showed that k-WL identifies graphs of treewidth k, though they did not control for rounds. Naturally, it would be of interest to close the gap between our upper bound of (3k+4) on the Weisfeiler-Leman dimension required to achieve $O(\log n)$ rounds and the best known upper bound of k on the Weisfeiler-Leman dimension (without controlling for rounds) of graphs of treewidth k. One approach would be to improve the result on [Bod89], to provide a tree decomposition of width $\leq 3k + 2$ and height $O(\log n)$ for graphs of treewidth k. It would also be of interest to examine special families of graphs with bounded treewidth that possess additional structure, which Weisfeiler-Leman can exploit to achieve $O(\log n)$ rounds with only k + O(1) pebbles.

Kiefer & Neuen [KN22, Theorem 6.1] also established a lower bound of $\lceil k/2 \rceil - 2$ on the Weisfeiler–Leman dimension (again, without controlling for rounds) of graphs of treewidth k. It would be of interest to strengthen their lower bound on the Weisfeiler–Leman dimension when restricting to $O(\log n)$ rounds.

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