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Cyclic Orders

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A family F of triplets of a set X is a cyclic order if the following axioms are satisfied:

$(a, b, c) \in F \Rightarrow (b, c, a)$ and $(c, a, b) \in F$	(*cyclicity*);
$(a, b, c) \in F \Rightarrow (b, a, c) \notin F$	(*antisymmetry*);
(a, b, c) and $(c, d, a) \in F \Rightarrow (b, c, d)$ and $(d, a, b) \in F$	(*transitivity*).

Such a concept aims to formalize some problems related to points or intervals drawn on a circle or on the plane, or with geometric representations of finite groups ($(a, b, c) \in F$ is to be read as 'b is located between a and c'). We move into the context of the so defined ternary relation some classical questions arising in the theory of partially ordered subsets, and deal, for instance, with extendability problem and with the problem of the minimal partition of a cyclic order into complete cyclic suborders. We also present several applications to such questions as the characterization of cyclic graphs or circular-arc graphs.

Notation

The graph theory notation used throughout this paper is consistent with that which appears in the book by Berge [2]. Throughout the paper N, Z, Q, R will denote, respectively, the set of the positive integers, the set of the relative integers, the set of the rational numbers, and the set of the real numbers. Also, (x, y, z) will denote a triplet, while $\{x, y, z\}$ will denote a set.

1. Definitions

A family F of triplets of a set X is said to be a cyclic order if we have:

if $(x_0, x_1, x_2) \in F$ then $(x_1, x_2, x_0) \in F$	(the addition is modulo 3);
if $(x, y, z) \in F$ then $(y, x, z) \notin F$	(antisymmetry);
if (x, y, z) and $(z, t, x) \in F$, then also (y, z, t) and $(t, x, y) \in F$	(transitivity).

If $A \subset X$, the subfamily $F_A = \{(x, y, z) \in F / x, y, z \in A\}$ is called the cyclic suborder induced by A.

F is said to be complete if for every 3-subset $\{x, y, z\}$ of X, (x, y, z) or (y, x, z) is in F. $A \subset X$ is called a loop if F_A is complete; it is called an antiloop if $F_A = \varnothing$.

The cyclic order X, F is said to be completely extensible if it can be extended into a complete cyclic order on X.

2. Motivations

Many problems prove to be considerably harder when they are set in a cyclic context rather than a linear one. This applies, for instance, to flow and network problems (which can be studied on Z, Q or Z/pZ), some scheduling problems, some homomorphism extension problems (knowing whether a simple graph G can be retracted on a subgraph G_A is easy if G_A is a tree, difficult if G_A is an elementary

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cycle), and recognition problems for graphs which are intended to represent some kind of geometric structure (determining if a graph is the intersection graph of some family of intervals is easier by far when those intervals are taken on the real line than when they are considered as drawn on a circle) (see, for snstance [2], [4], [18] and [19]). The point is that, behind our ability to resolve some problems set in a linear context, stand some important properties of partially ordered sets (for instance, the Helly property, which is satisfied by the intervals of a compact lattice [14]); and we lose these properties when we move into a cyclic context. In order to provide us with some albebraic tools for the study of cyclic context problems, Huntington [8] introduced first, in 1929, an axiomatization of the concept of cyclic order, which was then studied by Gallil and Meggido [6], Heyting [7], Muller [10], Novak and Novotny [12], Redei [16] and Tucker [19]. Few striking results could be obtained until now, most of them being related to the extension problem (the problem which consists of determining if a cyclic order is completely extensible, and which was proved to be NP-complete in [8]). Throughout this paper, we will deal with this extension problem, with some applications to the representation of some cyclic geometric structures, and with the connection which exists between the 'partitioning into antiloops' problem and the perfect graph concept.

3. Additional Definitions and Examples

Let X, F be a cyclic order. We denote by $\bar{F}$ the 3-subset family which supports F and by $G(F) = (X, \vec{E}(F))$ the oriented graph defined by:

$$\overrightarrow{x, y} \in \vec{E}(F) \text{ iff there exists } z \in X \text{ such that } (x, y, z) \in F.$$

An independent subset of $G(F)$ will be called a strong antiloop of X, F. A family C of elementary circuits of an oriented graph $G = (X, E)$ is said to be completely cyclic if there exists a complete cyclic order F on X such that:

$$\text{If } x, y, z \text{ occur in this order on a circuit } s \text{ of } C \text{ then } (x, y, z) \in F.$$

Let us denote by S_1 the unit circle of the Euclidian plane R^2, provided with the trigonometric orientation. Then S_1 can be provided in an obvious way with a complete cyclic order structure H_1.

A cyclic embedding of an elementary circuit family C of an oriented graph $G = (X, \vec{E})$ will be an injective function r from X to S, such that if we meet x, y, z in this order when walking once along a circuit of C, then $(x, y, z) \in H$. Obviously, we can state the following:

Proposition 1. *The family C is completely cyclic iff there exists a cyclic embedding of C.*

If $G = (X, \vec{E})$ is an oriented graph, we denote by $W(G)$ the set of all the elementary circuits of G, by $Z(G)$ the space $Z^{|W(G)|}$, and by Circ(G) the circuit space of G (the subspace of $Z^{|\vec{E}|}$ defined by the elementary circuits of G). We denote by $W(G)$ the subspace of $Z(G)$ defined by:

$$W(G) = \{u = (u_s, s \in W(G) / \sum_{s \in W(g)} u_s \cdot s = 0 \text{ in } Z^{|\vec{E}|}\}.$$

If $u \in Z(G)$, we denote by Ker_u the kernel of the linear form defined by u on $Z(G)$. If T is a path in G, oriented from its starting point to its endpoint, we denote by T^+ the set of the arcs of T which are oriented the same way as T, and by T^- the set of the arcs of T which are oriented the opposite way. If g is a function from $\vec{E}$ to Q, we set:

$$\mathrm{Som}(T, g) = \sum_{e \in T^+} g(e) - \sum_{e \in T^-} g(e).$$

We say that such a function g is a numerotation of a given family C of elementary circuits of G if:

(1) $\forall T \in W(G)$, $\mathrm{Som}(T, g) \in Z$ and $1 \leqslant \mathrm{Som}(T, g) \leqslant l(T)$, where $l(T)$ is the length of T;
(2) $\forall T \in C$, $\mathrm{Som}(T, g) = 1$.
A path T in G will be said to be a proper path if T^- is empty.

To a cyclic order, X, F corresponds in an obvious way to a family $C(F)$ of 3-circuits of the graph $G(F)$.

If X, F is a complete cyclic order, there exists a unique Hamiltonian circuit T of $G(F)$ such that any cyclic embedding of $C(F)$ is also an embedding of $C(F) \cup \{T\}$. Such a Hamiltonian circuit is called an oriented loop.

Let us denote by $\mathrm{Int}(S_1)$ the set of the closed intervals of the circle S_1. Taking into account the orientation of S_1, we may denote by $d(x)$ and $f(x)$ respectively the starting point and endpoint of an interval x of S_1. Then, we may define a cyclic order structure Fint on $\mathrm{Int}(S_1)$ by:

$(x, y, z) \in \mathrm{Fint}$ iff x, y, z are pairwise not intersecting and if

$(d(x), d(y), f(y), d(z), d(x))$ is an oriented loop of S_1, H_1.

Proposition 2. *A family P of closed intervals of the circle S_1 is globally intersecting iff it contains $x_0, \ldots, x_k$ such that $(d(x_0), f(x_k), d(x_1), \ldots, f(x_{k-1}), d(x_0))$ is an oriented loop of S_1, H_1.*

If $s = (x_0, x_1, \ldots, x_p, x_0)$ is a circuit in $G(\mathrm{Fint})$, then moving once along the circuit $(d(x_0), f(x_0), d(x_1), \ldots, d(x_p), f(x_p), d(x_0))$ of $G(H_1)$ means turning a number k of times around S. This number k is called the Index of s and is denoted by:

$$\text{Index of } s = \mathrm{Index}(s).$$

If $X, <$ is a partially order set (poset), we obtain a cyclic order structure $F(<)$ by setting:

$$(x, y, z) \in F(<) \text{ iff } x < y < z \text{ or } y < z < x \text{ or } z < x < y.$$

If h is a function from a set X to a set Y, and if F is a cyclic order on X, then we may define the inverse image $h^{-1}(F)$ of F by setting:

$$\text{If } x, y, z \in X, \text{ then } (x, y, z) \in h^{-1}(F) \text{ iff } (h(x), h(y), h(z)) \in F.$$

If X, F and Y, G are two cyclic orders, we may define the product $X, Y, F \otimes G$ as follows:

$$\begin{aligned}((x, x'), (y, y'), (z, z')) \in F \otimes G \text{ iff } & (x, y, z) \in F \text{ and } (x', y', z') \in G \\ \text{or } & (x, y, z) \in F \text{ and } x' = y' = z' \\ \text{or } & x = y = z \text{ and } (x', y', z') \in G.\end{aligned}$$

Since a cyclic group Z/pZ can be represented by a set of complex numbers located on the unit circle S_1, and since any commutative group can be written as a product of cyclic groups and instances of Z (linearly ordered), it follows that any commutative group can be provided with a cyclic order structure. (We are talking here about groups with a finite number of generators.) Now we know that for any finitely generated group

G, there exists a commutative group $\mathrm{Com}(G)$ and a homomorphism r from G to $\mathrm{Com}(G)$ such that, if h is a homomorphism from G to another commutative group H, then there exists a homomorphism h from $\mathrm{Com}(G)$ to H which satisfies:

$$r \circ h = h.$$

Since $\mathrm{Com}(G)$ is finitely generated, it follows that (through the inverse image process) we may provide G with a cyclic order structure.

4. The Extension Problem

Since a cyclic order X, F is completely extensible iff the 3-circuit family $C(F)$ is completely cyclic, we will deal here with circuit families, trying to discover when they are completely cyclic or, equivalently, when a cyclic embedding exists for them.

Theorem 1. *Given an elementary circuit family C of an oriented graph $G = (X, \vec{E})$. C is completely cyclic iff for every $u = (u_s, s \in W(G))$ in $Z(G)$, there exists x in* Ker_u *such that:*

$$\forall s \in W(G), \quad 1 \leqslant x_s \leqslant l(s) = \text{length of } s; \qquad \text{if } s \in C,\ x_s = 1. \tag{$*$}$$

Remark. In the above, x has to be in $Z^{|W(G)|}$. A relaxation of this constraint leads to the following existence criterion for x:

$$x \text{ (eventually in } Q^{|W(G)|} \text{ exists satisfying } (*) \text{ iff } \langle w^0, u\rangle \geqslant 0,$$

where w^0 is defined as being equal to $w^0_s = 1$ if $s \in C$ and $w^0_s = l(s)$ elsewhere.

Lemma 1. *Let V be a finite vector family of a Q vector space E, and let $\{[a_v, b_v], v \in V\}$ be a family of bounded intervals of Z. There exists a linear form h on E such that for every $v \in V$, $h(v) \in [a_v, h_v]$ iff for every vector c in $Z^{|V|}$ such that $\sum_{v \in V} c_v \cdot v = 0$ there exists $x \in Z^{|V|}$ which satisfies: $\langle c, x\rangle = 0$;*

$$\forall v \in V, \qquad x_v \in [a_v, b_v].$$

Remark. It is easy to relax within the constraints of this statement, the hypothesis being related with the finiteness of V. But we need the intervals $[a_v, b_v]$ to be bounded.

Proof. The 'only if' part of the above statement is obvious.

Conversely, let us suppose that the family $\{[a_v, b_v], v \in V\}$ is minimal in the sense that if we replace one of the integer intervals above by a smaller one, our existence condition for h is no longer satisfied. Then it will clearly be sufficient to prove that for every $v \in V$, $a_v = b_v$.

Let us assume that $v_0 \in V$ is such that $a_{v_0} < b_{v_0}$. Replacing $[a_{v_0}, b_{v_0}]$ by $[a_{v_0}, b_{v_0} - 1]$ or by $[b_{v_0}, b_{v_0}]$ yields two vectors c and d of $Z^{|V|}$ such that both systems

$$\begin{array}{lll} (\alpha)\ x \in Z^{|V|};\quad \langle c, x\rangle = 0; & & (\beta)\ x \in Z^{|V|};\quad \langle d, x\rangle = 0; \\ \forall v \in v_0,\quad x_v \in [a_v, b_v]; & \text{and} & \forall v \in v_0,\quad x_v \in [a_v, b_v]; \\ x_{v_0} = b_{v_0}; & & x_{v_0} \in [a_{v_0}, b_{v_0} - 1]; \end{array}$$

do not admit any solution.

Then we easily check that if $t \in Z$ is large enough, and solution $x \in Z^{|V|}$ of the system $\langle (tc + d), x\rangle = 0$, $\forall v \in V$, $x_v \in [a_v, b_v]$; will be such that $\langle c, x\rangle = \langle d, x\rangle = 0$, and thus will satisfy (α) and (β), inducing a contradiction. □

Definition. Let X be a set; a partial order $<$ on the product $X \cdot Q$ is said to be periodic if the following implication is true:

$$i, j, k \in Q, x, y \in X, \qquad (x, i) < (y, j) \Rightarrow (x, i+k) < (y, j+x).$$

Lemma 2. *Let X a finite set, $<$ a partial order relation on $X \cdot Q$. Then there exists an injective function r from $X \cdot Q$ to Q which preserves the relation $<$ and which is such that*:

$$\textit{If } x \in X,\ i, k \in Q, \quad r(x, i+k) = r(x, i) + k.$$

We say that r is a periodic embedding of $X \cdot Q$, $<$.

The proof is easy and is left to the reader.

Lemma 3. *If g is a numerotation of the circuit family C of the oriented graph $G = (X, \vec{E})$, and if G is strongly connected, then for every cycle s of G we have* $\mathrm{Som}(s, g) \in Z$.

Proof. Any such cycle s can be decomposed as the concatenation of n proper paths $T_1, \ldots, T_n$. If T is a proper path which connects the endpoint of T_n to its staring point, then we can write $s = (T_n \oplus T) + (T \oplus \cdots \oplus T_{n-1} \oplus (-T))$. ($(-T)$ is the path T taken from its endpoint to its staring point, $\oplus$ is the concatenation, and $+$ is the sum in $Z^{|\vec{E}|}$.) Therefore, we can conclude by induction on the decomposition number n of the cycle s. $\square$

Definition. A numerotation g of an elementary circuit family of an oriented graph $G = (X, \vec{E})$ is a $\{0, 1\}$-numeration if it takes its value in the 2-set $\{0, 1\}$.

Lemma 4. *If C is an elementary circuit family of an oriented graph $G = (X, \vec{E})$ then the statements* (1), (2) *and* (3) *below are equivalent*:
(1) *C is completely cyclic.*
(2) *There exists a $\{0, 1\}$-numerotation of C.*
(3) *There exists a numerotation of C.*

Proof. (1)$\Rightarrow$(2) is immediate; if r is a cyclic embedding of C and if a_0 is in $S_1 - r(X)$, then we set $g(x, y) = 1$ if we meet a_0 when going from x to y according to the orientation of S_1 and $g(x, y) = 0$ in the opposite case. Then g is a $\{0, 1\}$-numerotation.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1): clearly, G can be supposed to be strongly connected, E being covered by the family C. Let us choose $x_0 \in X$. A numerotation g of C allows us to define a graph structure $(X \cdot Q, \vec{E}_g)$ as follows:

$$\overrightarrow{(x, i), (y, j)} \in \vec{E}_g \text{ iff } \overrightarrow{x, y} \in \vec{E} \text{ and } g(x, y) = j - i.$$

This graph is without any circuit, and its transitive closure is an order relation R_g on $X \cdot Q$ which is periodic.

Let us denote by A the connected component of $(x_0, 0)$ in the graph $G_g = (X \cdot Q, \vec{E}_g)$, taken without its orientation. Lemma 3 yields the result that if (x, i) and (x, j) are in A, then $j - i \in Z$. Then a periodic embedding r of R_g induces the following function $\bar{r}$ from A to S:

$$\bar{r}((x, k)) = \mathrm{e}^{\mathrm{i}2\pi \cdot r(x,k)},$$

which in fact is a function from $\bar{A} = \{x \in X/\exists i \in Q \text{ with } (x, i) \in A\}$ to S. The function $\bar{r}$ is clearly injective (since r is injective and periodic), and the strong connectivity of G is such that $\bar{A} = X$. We claim that $\bar{r}$ is a cyclic embedding of C. Thus we have to check that if x, y, z appear in this order on a given circuit T of C, then $(\bar{r}(x), \bar{r}(y), \bar{r}(z)) \in H$. Let us denote by T_{xy}, T_{yz} and T_{zx} the proper path restrictions of T between x and y, y and z, and z and x, and let us set $p = \mathrm{Som}(T_{xy}, g)$, $q = \mathrm{Som}(T_{yz}, g)$ and $m = \mathrm{Som}(T_{zx}, g)$. If $r(x, i) \in [0, 1[$, then $r(y, i+p)$, $r(z, i+p+q)$ and $r(x, i+1) = r(x, i)+1$ come after it in this order on Q.

If $r(y, i+p)$ and $r(z, i+p+q)$ are in $[0, 1[$ the result follows.

If $r(z, i+p+q) > 1$ and not $r(y, i+p)$, then $r(z, i+p+q-1)$ is before $r(x, i)$ on $[0, 1[$ and the result follows.

If $r(y, i+p) > 1$ then we have $0 \leqslant r(y, i+p-1) < r(z, i+p+q-1) < r(x, i)$ in Q, and we conclude. This is the end of the proof of Theorem 1, but note the following.

Only if part. If C is supposed to be completely cyclic, then there exists a $\{0, 1\}$-numerotation g which can be extended into a linear form g^* on

$$\mathrm{Circ}(G) \text{ by} = \text{if } s \in W(G), \quad g^*(s) = \mathrm{Som}(s, g).$$

Lemma 1 can be applied to the vector family $W(G)$ of $\mathrm{Circ}_Q(G)$ and the Z-interval family $\{[a_s, b_s],\ s \in W(G)\}$ defined by:

$$\text{If } s \in C, \text{ then } a_s - b_s = 1 \text{ else } a_s = 1 \text{ and } b_s = l(s);$$

and it produces the right side of the equivalence asserted in Theorem 1.

If part. If the assertion contained in the right side of the equivalence that we have to prove is supposed to be true then, according to Lemma 1, there exists a linear form h on $\mathrm{Circ}_Q(G)$ such that:

$$\text{If } s \in C \text{ then } h(s) = 1 \text{ and if } s \in W(G) - C \text{ then } 1 \leqslant h(s) \leqslant l(s).$$

A numerotation of C can then be deduced by extending h on the Q-vector space $Q^{|\vec{E}|}$. Then we conclude by Lemma 4. □

5. Algorithmic Interpretation of Theorem 1

The following procedure, EMBEDD, takes the graph $G = (X, \vec{E})$ and the circuit family C and exhibits a $\{0, 1\}$-numerotation g as soon as such a numerotation exists (of course, EMBEDD does not work polynomially).

EMBEDD

(1) Determine a maximal free subset V of C (taken as a subset of $\mathrm{Circ}_Q(G)$), and check that if $s \in C - V$ can be written as $\sum_{v \in V} t_v \cdot v$ in $\mathrm{Circ}_Q(G)$ then $\sum_{v \in V} t_v = 1$.

(2) Resolve the following integer linear program, PROG:

$$\forall e \in E, \qquad z_e \in \{0, 1\};$$

$$\forall s \in V, \qquad \sum_{e \in s} z_e = 1.$$

(3) If PROG is without any solution, then FAILURE and go to (4). Else: look for a circuit T in the graph $G_0 = (X, \vec{E}_0)$, where $\vec{E}_0 = \{e \in \vec{E}/z_e = 0\}$. If T does not exist then SUCCESS and go to (4). Else: set $V := V \cup \{T\}$, add to PROG the equation $\sum_{e \in T} z_e = 1$ and return to (2).

(4) End. (In the event of success, set $\forall e \in \vec{E}$, $g(e) = z_e$.)

The proof of the validity of EMBEDD is left to the reader.

The following procedure, REPRESENT, takes a numerotation g of the circuit family C of the graph $G=(X, \vec{E})$ and exhibits, in polynomial time, a cyclic embedding of C.

REPRESENT

(1) Set $A := \phi(A \subset X)$ and for every $x \in X$, $r(x) =$ unknown (r is from A to Q).

(2) If $A = X$ then go to (6); else choose $x_0 \in A$.

(3) By using, for example, the Dijkstra procedure [13], compute, for every $x \in X$:

$i(x) =$ smallest number $\mathrm{Som}(T, g)$, T belonging to the set of the proper paths from x_0 to x;

$j(x) =$ smallest number $\mathrm{Som}(T, g)$, T belonging to the set of the proper paths from x to x_0.

(4) For every $x \in A$, set $q(x) := r(x) - j(x)$, $p(x) := r(x) + i(x)$. Compute $Q = \mathrm{Sup}\, q(x)$, $x \in A$ and $P = \mathrm{Inf}\, p(x)$, $x \in A$. Add x_0 to A. Choose $r(x_0)$ in $[Q, P]$ in such a way that no $(r(x) - r(x_0))$, $x \in A$ be in Q.

(5) Go back to (2).

(6) For $x \in X$, set $\bar{r}(x) = \mathrm{e}^{\mathrm{i}2\pi\cdot\mathrm{Frac}(r(x))}$, where $\mathrm{Frac}(r(x))$ is the rational part of $r(x)$. $\bar{r}$ is the output embedding.

6. Applications

The previous theory can be extended when, in addition to the elementary circuit family C of the oriented graph $G=(X, \vec{E})$, we consider a function f from C to N^+. Such a pair (C, f) will be called a partial circuit indexation of G. Let us denote by $G_2 = (X, \vec{E}_2)$ the graph which we obtain from G by writing:

$$\vec{E}_2 = \vec{E} \cup \{\overrightarrow{x, y} \text{ for } x, y \in X \text{ and } x \neq y\}.$$

A numerotation ($\{0, 1\}$-numerotation) of C, f will be a function g from $\vec{E}_2$ to $Q(\{0, 1\})$ such that for every circuit s of G_2 we have:

If $a \in C$, then $\mathrm{Som}(s, g) = f(s)$; else $1 \leqslant \mathrm{Som}(s, g) \leqslant l(s) - 1$;
$\mathrm{Som}(s, g)$ being in any case an integer.

A cyclic embedding of (C, f) will be an injective function r from X to S such that if $s = (x_0, x_1, x_2, \ldots, x_{p-1}, x_p, x_0)$ is a circuit in C, moving from $r(x_0)$ to $r(x_1)$, then to $r(x_2), \ldots$, then to $r(x_p)$, and then to $r(x_0)$ again, according to the orientation of S_1, means turning $f(s)$ times around S. Then, by proceeding in the same way as in the proof of Theorem 1, we obtain the following:

Theorem 1A. *A partial circuit indexation (C, f) of an oriented graph $G = (X, \vec{E})$ admits a cyclic embedding iff it admits a numerotation, iff it admits a $\{0, 1\}$-numerotation, and iff for every u in $W(G_2)$ there exists x in Ker_u such that*:

$$\forall s \in W(G_2),\ 1 \leqslant x_s \leqslant l(s); \qquad \textit{if } s \in C,\ x_s = f(s).$$

Then we deduce several applications.

(*a*) *Good orientation for a triangle family.* Given a family F of triangles (3-subsets) of a finite set X, we say that a good orientation of F is a function L which to every triangle $u \in F$ makes correspond one triplet $(L(u))$ among the six triplets supported by u, in such a way that the family $\{L(u), u \in F\}$ can be extended into a complete cyclic order.

In order to determine if there exists a good orientation of F we define the oriented graph $G_F = (X, \vec{E}_F)$ by:

$$\overrightarrow{x, y} \in \vec{E}_F \text{ iff there exists } u \in F \text{ which contains } x \text{ and } y.$$

Next we define the following partial circuit indexation C_F, f_F of G_F:

$s \in W(G_F)$ is in C_F if $l(s) = 2$ and in this case $f_F(s) = 1$
of if $s = (x, y, z, x)$ such that $\{x, y, z\}$
and $\{z, t, x\} \in F$ and $\{x, y, t\}$ or $\{y, t, z\} \notin F$.

THEOREM 2. *There exists a good orientation of F iff there exists a cyclic embedding of the partial circuit indexation (C_F, f_F).*

The proof is left to the reader, who has only to check that a good orientation of F is a cyclic embedding of (C_F, f_F), and conversely.

(*b*) *A circular graph.* This is a simple graph $G = (X, E)$ which may be considered as the intersection graph of a finite family of closed intervals of the circle S_1. A circular representation of G is a function h from X to $\mathrm{Int}(S_1)$ such that two vertices x and y are adjacent in G iff $h(x) \cap h(y) \neq \emptyset$. Such a representation may be chosen (if it exists) in such a way that all the points $d(h(x))$ and $f(h(x))$, $x \in X$ (starting point and endpoint of $h(x)$) be distinct. Circular graphs have been studied by Tucker [19], Quilliot [14] and Trotter [18], etc., the basic problem consisting, of course, of determining if a given graph is a circular graph.

The graph $G = (X, E)$ being given, we create a set $W_x = \{\mathrm{de}_x, \mathrm{fn}_x, x \in X\}$, and we call K_X the complete oriented graph defined on W_X. To every pair x, y in X with $x \neq y$, we may associate a circuit $(\mathrm{de}(x), \mathrm{fn}(x), \mathrm{de}(y), \mathrm{fn}(y), \mathrm{de}(x))$ of K_X. We denote this circuit by $s_{x,y}$, and we set $C_X = \{s_{x,y}, x, y \in X/x \neq y\}$.

Next we say that an edge $[x, y]$ of G covers G if for every pair z, t or adjacent of identical vertices of G, there exists at least one of the two vertices x and y which are adjacent (or identical) to both z and t. Then we set:

$$f_G(s_{x,y}) = 1 \quad \text{if } [x, y] \notin E;$$
$$f_G(s_{x,y}) = 2 \quad \text{if } [x, y] \in E \text{ and does not cover } G;$$
$$f_G(s_{x,y}) = 3 \quad \text{if } [x, y] \in E \text{ and covers } G.$$

We obtain the following:

THEOREM 3. *The simple graph $G = (X, E)$ is circular iff the partial circuit indexation (C_X, f_G) of the complete oriented graph K_X defined above admits a cyclic embedding.*

PROOF. It is known [14] that if G is a circular graph, then a representation h of G may be chosen in such a way that:

If $[x, y] \in E$ covers G, then $h(x) \cup h(y) = S_1$;
all the points $d(h(x)), f(h(x))$ are distinct

We shall say that such a representation is a proper representation. But if h is a proper circular representation of G, we set:

$$r(\mathrm{de}(x)) = d(h(x)) \text{ and } r(\mathrm{fn}(x)) = f(h(x)) \qquad \text{for every } x \in X$$

and we obtain a cyclic embedding of the partial circuit indexation (C_X, f_G). Conversely, if r is a cyclic embedding of (C_X, f_G), the reader will easily check that by setting $h(x) = \text{Interval}[r(\text{de}(x)), r(\text{fn}(x))]$ of S_1, for every $x \in X$, we obtain a proper circular representation of G. □

(*c*) *Chord graphs*. A simple graph $G = (X, E)$ is a chord graph if it can be considered as the intersection graph of a finite family of chords of the circle S_1, the extremities of which are all distinct. Let us denote by $\text{Cor}(S_1)$ the set of all the chords of S_1; any $u \in \text{Cor}(S_1)$ can be defined by its starting point $\text{st}(u)$ and its endpoint $\text{ed}(u)$. A chord representation of G will be a function h from X to $\text{Cor}(S_1)$ such that all the points $\text{st}(h(x))$, $\text{ed}(h(x))$ for $x \in X$ are distinct, and that two chords $h(x)$ and $h(y)$ are intersecting iff $[x, t] \in E$. Chord graphs have been studied by Naji [11], Fournier [5] and Read [15], and can be recognized in polynomial time.

$G = (X, E)$ being a simple graph, we consider the complete oriented graph K_X defined in (b) and we set:

$$\begin{aligned} \bar{C}_G = \{\text{Circuits } &(\text{de}(x), \text{de}(y), \text{fn}(y), \text{fn}(x), \text{de}(x)),\\ &(\text{de}(x), \text{fn}(x), \text{de}(y), \text{fn}(y), \text{de}(x)),\\ &(\text{de}(x), \text{fn}(x), \text{fn}(y), \text{de}(y), \text{de}(x)),\\ &(\text{de}(x), \text{fn}(y), \text{de}(y), \text{fn}(x), \text{de}(x))\\ &\text{of } K_X, \text{ obtained for every } [x, y] \in E \text{ and } x \neq y\}. \end{aligned}$$

For every $s \in \bar{C}_G$, $\bar{f}_G(s) = 2$.

We obtain the following:

THEOREM 4. *The graph $G = (X, E)$ is a chord graph iff the partial circuit indexation $(\bar{C}_G, \bar{f}_G)$ admits a cyclic embedding.*

PROOF. If r is a cyclic embedding of $(\bar{C}_G, \bar{f}_G)$ then we can easily check that by setting $h(x)$ equal to a chord between $r(\text{de}(x))$ and $r(\text{fn}(x))$ for every $x \in X$, we define a chord representation of G. Conversely, if h is such a representation, we obtain a cyclic embedding of $(\bar{C}_G, \bar{f}_G)$ by setting for every $x \in X$:

$$r(\text{de}(x)) = \text{staring point of } h(x);$$

$$r(\text{fn}(x)) = \text{endpoint of } h(x).$$

(Verification left to the reader.) □

7. THE ANTILOOP PROBLEM: TWO PROBLEMS ABOUT INTERVALS

Let X, F be a cyclic order: we say that X, F is regular if every 3-clique of the graph $G(F)$ (taken without its orientation) supports some triplet of F. We set:

$$v(F) = \text{maximal cardinality of a strong antiloop of } F;$$

$$w(F) = \text{minimal cardinality of a partition of } X \text{ into loops.}$$

Recall perfect graphs. A simple graph $G = (X, E)$ is said to be perfect if the chromatic number of any subgraph of G is equal to its clique number. A conjecture of Berge [2] proposes that a graph is perfect iff neither it, nor its complementary graph, contains an elementary cycle with odd length $\geqslant 5$ and without any chord.

A theorem of Lovacz [9] says that a graph is perfect iff its complementary graph is perfect.

We shall say that the cyclic order X, F is perfect if it is regular, and if the graph $G(F)$ (taken without its orientation) is perfect; it is clear that if X, F is perfect, then $v(F) = w(F)$.

Tucker [20] proved the validity of Berge's conjecture for the case of circular graphs: he did this by using come counting tricks. We are going to give another proof of this result, which makes a connection with the concepts that we have presented above.

New Proof of Tucker's Theorem. Let us consider $X \subset \mathrm{Int}(S_1)$: we suppose that neither the intersection graph H^X nor the complementary $\bar{H}^X$ of H^X admit any odd cycle with length $\geqslant 5$ and without any chord. We want to prove $v(\mathrm{Fint}_X) = w(\mathrm{Fing}_X)$ or, equivalently, that the chromatic number of H^X is equal to its clique number. We may clearly suppose that all the points $\{d(x), f(x)$ for $x \in X\}$ are distinct. Proceeding by induction on $|X|$, and using the fact that a graph is perfect iff its complementary graph is perfect, we see that we may suppose that there do not exist $x, y \in X$ such that $x \subset y$. If S_1 can be written $S_1 = \bigcup_{i \in 0 \cdots 2} x_i$, then any $x \in X$ will contain at least one intersection $x_i \cap x_j$ $(i, j \in 0 \cdots 2)$, and H^X will be the union of three cliques. If the stability number of G is 3, we easily conclude (Lovacz's theorem); else the complementary graph $\bar{H}^X$ of H^X is bipartite and the result becomes obvious. That means that we may suppose that we can not find $x_0, x_1, x_2 \in X$ such that $S_1 = \bigcup x_i$, $i \in 0 \cdots 2$ (α).

(α) implies that if $A \subset X$ is a clique of H^X, then $\bigcap x$, $x \in A$ is a non-empty interval $I(A)$ of S_1: let us set $M = \{$maximal cliques of $H^X\}$, and choose $x_0 \in X$. Starting from x_0 and following the orientation of S_1, we meet $I(A_0)$ such that $I(A_0) \cap x_0 = \varnothing$: because of (α) there exists x_1 which contains $I(A_0)$ and is such that $x_0 \cap x_1 = \varnothing$. Proceedings the same way from x_1, we deduce $I(A_1)$, x_2 and so on. There is a time when x_k is equal to some x_i, $i < k$. Then the circuit $T = (x_i, x_{i+1}, \ldots, x_i = x_k)$ of $G(\mathrm{Fint})$ is such that:

$$v(\mathrm{Fint}_X) = v(\mathrm{Fint}_{X\text{-vertices of } T}) + \mathrm{Index}(T).$$

If $\mathrm{Index}(T) = 1$ we obtain our result; else we say that T is (a, b)-decomposable if there exists $j \in i+1, \ldots, k-1$ such that

$$\begin{aligned} s &= (x_i, \ldots, x_j, x_i) \in W(G(\mathrm{Fint})) \text{ and } \mathrm{Index}(s) = a;\\ s' &= (x_{j+1}, \ldots, x_{k-1}, x_{j+1}) \in W(G(\mathrm{Fint})) \text{ and } \mathrm{Index}(s') = b;\\ a + b &= \mathrm{Index}(T). \end{aligned}$$

In fact, we may suppose here that $i = 0$, and we set $p = \mathrm{Index}(T)$. If T is (a, b)-decomposable with $0 < a < p$, then we conclude by induction on p. Else we set:

if $0 \leqslant l \leqslant p$, $i \in 0 \cdots k-1$ then $Q_l(i) =$ largest $j \in 0 \cdots k-1$ such that if $s = (x_i, x_{i+1}, \ldots, x_{i+j-1}, x_i)$ then $\mathrm{Som}(s, g) = l$ (the addition is taken modulo k, and of course $Q_0(i) = Q_p(i) = i$);

and we remark that for any $i \in 0 \cdots k-1$, $l \in 1 \cdots p-1$, the odd circuit $s_{i,l} = (x_i, x_{Q_l(i+1)}, x_{i+1}, \ldots, x_{Q_l(Q(i))}, x_{Q(i)}, x_i)$ is without any chord unless x_i and $x_{Q_l(Q(i))}$ intersect, which will also mean that $Q_{l+1}(i) = Q_l(Q(i))$. But this last relation will contradict the relation $Q_p(i) = i$ when $l = p-1$. □

A Cyclic Scheduling Problem. One can ask how long a circle (basic period) needs to be in order to make it possible for a planner to schedule some intervals (periodic tasks) with given length in a way that is compatible with some given cyclic order on those intervals. We are going to sovle the following problem:

Let C be an elementary circuit family of an oriented graph $G = (X, \vec{E})$, g a numerotation of C, p a function from X to R^+. Is it possible to find a function f from X

to $\mathrm{Int}(S_1)$, *such that for any circuit s in G, the index of the image of s through f be* $\mathrm{Som}(s, g)$, *and the length of any* $(f(x), x \in X$, *be* $p(x)$? *If it exists, such a function f will be called a g-schedule of the data C, p.*

We can state the following:

THEOREM 5. *With the above notation, a g-schedule of C, p will exist iff for any s in* $W(G)$, *we have*:

$$p^*(s) = \sum_{x \in s} p(x)/\mathrm{Som}(s, g) \leq 2\pi.$$

REMARK. Of course, the solution of the general cyclic scheduling problem is given when an optimal choice of the numerotation g is made, C being given, and this is where the true difficulty of the problem is located. Also, the task of checking that the search of s maximizing $p(s)$ is polynomial is left to the reader.

PROOF OF THEOREM 5. We may replace S_1 by an oriented circle S_1^l with length l, minimal with the property that there exists a g-schedule of C, p on S_1^l. Of course, l will be at least equal to $\max p^*(s)$, $s \in W(G)$. Conversely, let us take away a small piece of S_1^l with length $\delta > 0$, and let us try to schedule C, p on $S_1^{l-\delta}$, by pushing some intervals $f(x)$ according to the orientation of $S_1^{l-\delta}$. If $h(x)$ intersects the removed interval, we translate the endpoint of $h(x)$ by δ and we mark $h(x)$; if, at some time, the endpoint of some marked interval $h(y)$ is between the starting point and the endpoint of some interval $h(z)$, which is not marked and is such that $\overrightarrow{y, z} \in \vec{E}$, then we mark $h(z)$ and we move both the starting point and the endpoint of $h(z)$ by δ. (The word between has to be taken here in the sense of the cylic order $H_1^{l-\delta}$ on $s_1^{l-\delta}$. The minimality of l implies that there is a time when we find $h(y)$, $h(z) \in \mathrm{Int}(S_1^{l-\delta})$, both marked, such that $\overrightarrow{y, z} \in \vec{E}$ and $(d(h(z)), f(h(y)), f(h(z))) \in H_1^{l-\delta}$. It will mean the existence of a circuit s of G such that $p^*(s) \geq l - \delta$. Then we conclude that δ converges to 0.

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