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On Structural Descriptions of Lower Ideals of Series Parallel Posets

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Abstract In this paper we give an algorithm to determine, for any given suborder closed class of series parallel posets, a structure theorem for the class. We refer to these structure theorems as structural descriptions. This work builds on work of Robertson, Seymour, Thomas, and especially Nigussie on trees. Stated differently, this paper gives an analogue of Nigussie's Tree Algorithm for series parallel posets.

Keywords Structural description · Poset · Series parallel · Structure theorem · Algorithm · Partial order · Bit · Lower ideal · Well quasi order · WQO

1 Introduction

1.1 Background

Many important theorems in combinatorics characterize a class by forbidden subobjects of some kind. This is a description of the class "from the outside", by what is not inside it. An example is Wagner's reformulation [9] of Kuratowski's Theorem [3] stating that a graph is planar iff it has no K_5 minor and no $K_{3,3}$ minor. To be a good characterization, the list of forbidden objects should be finite. Well quasi order theorems such as the Graph Minor Theorem [6] state that for certain classes of objects, there is always such a finite description "from the outside".

Just as important are those theorems that characterize a class "from the inside" by giving some set of starting objects and some set of construction rules. As a simple example, consider (graph theoretic) trees. Each tree is either a single point graph or may be obtained from two smaller, disjoint trees by adding an edge between the trees. Therefore a simple structure theorem for this class would have the single point

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graph as the only starting graph and joining two disjoint graphs by an edge as the sole construction rule.

To be a good characterization, we again hope that it is in some sense finite. First, there should be only finitely many construction rules. We can not necessarily demand there are only finitely many starting objects. We may however demand that at least we start with only finitely many families, and that each such family has some sort of finite description as well.

Analogous to the Graph Minor Theorem and other well quasi order theorems stating that in many cases there is always a finite description from the outside, it was asked if it could be shown in an equally general setting that there is always a finite description from the inside with finitely many starting families, each itself finitely described, and finitely many construction rules.

As it turns out, this appears to be far more difficult. This line of research was first pursued by Robertson et al. in [7] for trees under the topological minor relation. In [5], Nigussie and Robertson build on [7] and correct some technical errors contained therein. In [4], Nigussie gives an algorithm that finds a structure theorem for an arbitrary topological minor closed property of trees. Nigussie's algorithm is efficient enough in practice that structure theorems can be computed by hand with pen and paper that are not at all obvious without the algorithm. We follow the convention of referring to these structure theorems as structural descriptions. The distinction we make is that we use the term structure theorem informally, while we see structural description as a technical term defined in [4] for trees under topological minor and below for series-parallel orders under suborder.

Attempts have been made by various researchers to generalize these results to other classes of graphs, in particular series-parallel graphs. Thus far, no such attempt has succeeded. While many specific graph structure theorems are known, the tree result is to date the only one that allows the automatic computation of a structure theorem for any graph property in a nontrivial, infinite class of properties.

It is key that rooted trees are used in [4, 5, 7]. Rooted trees are as much partial orders as they are graphs, and we view Nigussie's algorithm not just as a graph algorithm, but as a partial order algorithm. It is thus natural to ask for algorithms similar to Nigussie's for classes of partial orders larger than the class of trees. In this paper, we prove an analogous result for series-parallel partial orders by giving a finite structural description for each suborder closed class of series-parallel orders. More precisely, we give an algorithm that takes as input a suborder closed class of series-parallel orders described by forbidden suborders, and which gives as output a finite structural description for that class.

In our context, a structural description will turn out to be a finite set of labeled partial orders. The labels will be families already constructed. Each labeled partial order in the structural description for a class will represent one family or construction rule. Roughly, the labels tell what we are allowed to put in and the partial orders themselves tell us how we are allowed to piece together what we do put in.

2 Basic Definitions and Conventions

A partial order is a (possibly empty) set P together with a reflexive, antisymmetric, transitive binary relation $\leq$ on P. All partial orders in this paper are assumed to be finite. (The only exception to this is that *classes of* partial orders we consider are usually infinite, and this class together with the suborder relation is in fact a partial order. This exception causes no confusion as it is clear in each case whether we are dealing with a partial order or an infinite family of them.) Points x, y in a partial order $(P, \leq)$ are comparable if $x \leq y$ or $y \leq x$. Otherwise x and y are called incomparable, which we write as $x|y$. A chain is a partial order such that any two points are comparable. An antichain is a partial order such that any two points are incomparable.

A lower ideal of partial orders is a family of partial orders that is closed under taking suborders. Given partial orders P and Q, we say that P is Q-free if P has no suborder isomorphic to Q. Given a set F of partial orders, we say that P is F-free if P is Q-free for each Q in F. A lower ideal L is said to be Q-free or F-free if each partial order in L is Q-free or F-free, respectively. A forbidden suborder of a lower ideal L is a suborder minimal partial order P such that L is P-free.

The papers [4, 5, 7] use tree sums to construct new trees from old. For our purposes, tree sums are not sufficient. The correct generalization to our context is partial order lexicographic sums. We call partial orders $(P_i, \leq_i)$ and $(P_j, \leq_j)$ disjoint if P_i and P_j are disjoint.

Definition 1 Let $\{(P_i, \leq_i)\}_{i\in I}$ be a family of pairwise disjoint partial orders and let $(I, \leq_I)$ be a partial order on I. Then the lexicographic sum $\bigoplus_{\leq_I} P_i$ is defined as the unique partial order $(\bigcup_{i\in I} P_i, \leq)$ such that the following conditions hold:

1. Given i in I and x and y in P_i, we have $x \leq y$ iff $x \leq_i y$.
2. Given distinct i, j in I, if $i \leq_I j$, then $x \leq y$ for all x in P_i and y in P_j.
3. Given distinct i, j in I, if i and j are $\leq_I$ incomparable, then x and y are $\leq$ incomparable for all x in P_i and y in P_j.

It is a simple exercise to show that the above three conditions indeed uniquely determine a partial order on $\bigcup_{i\in I} P_i$. We call $(I, \leq_I)$ the outer partial order of the lexicographic sum. Each P_i is called the inner partial order corresponding to i. The lexicographic sum is therefore a partial order on the union of the inner partial orders. We call the partition of $\bigoplus_{\leq_I} P_i$ into the inner partial orders P_i a lexicographic partition. It is simple to show that a partition of a partial order is lexicographic iff for any two distinct cells C_1 and C_2 of the partition, either all elements of C_1 precede all elements of C_2, all elements of C_2 precede all elements of C_1, or all elements of C_1 and C_2 are incomparable. In this case, the outer partial order is uniquely determined in the obvious way.

We call a lexicographic partition nontrivial if there are at least two cells and each cell is nonempty. We call a lexicographic partition a chain partition if the corresponding outer partial order is a chain. Similarly for antichain partitions. We call a lexicographic sum a chain sum or antichain sum if the corresponding partition is nontrivial and the outer partial order is a chain or antichain, respectively. We denote by $P_1 \prec \cdots \prec P_n$ the chain sum of partial orders $P_1, \ldots, P_n$ such that for $1 \leq i < j \leq n$, every x in P_i is less than every y in P_j. We denote by $P_1 \oplus \cdots \oplus P_n$ the antichain sum of partial orders $P_1, \ldots, P_n$ such that for all $i \neq j$, every x in P_i is incomparable to every y in P_j.

The comparability graph of a partial order P is the graph whose vertices are the points of P and such that two points x and y are adjacent iff they are comparable in

P. A component of P is a component of the comparability graph. An anticomponent is a component of the similarly defined incomparability graph. If P is a chain sum, we note that P then has a unique finest chain partition, which is just the partition into anticomponents. If $P = P_1 \prec \cdots \prec P_n$ and $\{P_1, \ldots, P_n\}$ is a finest chain partition with $n \geq 2$, then we call $P_1 \prec \cdots \prec P_n$ a finest chain representation of P. A similar statement holds for antichain sums and components, and we then similarly call $P_1 \bigoplus \cdots \bigoplus P_n$ a finest antichain representation of P for $n \geq 2$.

A partial order is a series-parallel partial order, or SP order, if it is contained in the smallest class of partial orders containing the empty and single point partial orders and closed under chain and antichain sums. We note that for each SP order P, exactly one of the following holds: P is empty, P is a single point, P is a chain sum, or P is an antichain sum. We will make use of the simple but important fact that a suborder of an SP order is also an SP order. It is also worth noting that a finite partial order is an SP order iff it is N-free, where N is the partial order on points a, b, c, d such that $a < b, b > c, c < d$, and all others pairs of points are incomparable [1], though we do not make use of this fact.

Since all our ideals in this paper are lower ideals of SP orders, from now on we simply call these lower ideals. A proper lower ideal is a lower ideal that is strictly contained in the set of all SP orders. A nontrivial lower ideal is one that contains at least one nonempty partial order. Our goal in this paper is to give a structural description for an arbitrary nontrivial, proper lower ideal. More precisely, we give a recursive procedure that takes as input a nontrivial, proper lower ideal, which gives as output a structural description for that lower ideal. This procedure is entirely constructive, and a program could be written to implement it, though algorithmic questions are not our focus.

A structural description, for us, will turn out to be a finite set of labeled SP orders. The labels tell us which objects we may use to construct, and the orders themselves tell us in which ways we may put these together. We now start to make this intuition more precise.

A labeled partial order is a triple $(I, \leq_I, f)$, where $(I, \leq_I)$ is a partial order and f is a function with domain I. We define a label in a labeled partial order $(I, \leq_I, f)$ to be an element of the range $\{f(i) : i \in I\}$ of the function f. We think of f as the labeling function. We sometimes write I_f for this labeled partial order when $\leq_I$ is clear from context. A bit is a labeled SP order such that each label is a lower ideal or the symbol R. We call a point i in a bit $(I, \leq_I, f)$ an ideal labeled point if $f(i)$ is a lower ideal. We call i an R labeled point if $f(i) = R$. A recursive bit is a bit with at least one R labeled point. A nonrecursive bit is a bit with no R labeled points. The two point chain with both points labeled R is denoted by R_C. The two point antichain with both points labeled R is denoted by R_A.

We now tell how to assign to each set S of bits the lower ideal $L(S)$ that S is said to generate. Given a set S of bits and a set X of partial orders, we say that X is S-bit closed if X contains all lexicographic sums of the form $\bigoplus_{\leq_I} P_i$ such that $(I, \leq_I, f)$ is a bit in S, the partial order P_i is contained in the lower ideal $f(i)$ for each ideal labeled point i in I, and P_i is contained in X itself for each R labeled point i in I. The S-bit closure of X is the smallest S-bit closed set containing X. Given a set S of bits, we define the lower ideal $L(S)$ generated by S as the S-bit closure of the set containing the empty partial order, the one point partial order, and no other partial orders.

Given a bit $(I, \leq_I, f)$, we say that X is $(I, \leq_I, f)$-bit closed if X is $\{(I, \leq_I, f)\}$-bit closed. We will have many occasions to use the following simple lemma, whose proof is immediate from the definition.

Lemma 2 *If S is a set of bits and X is a set of partial orders, then X is S-bit closed iff X is $(I, \leq_I, f)$-bit closed for each bit $(I, \leq_I, f)$ in S.*

We now define structural descriptions. We do so by recursively defining structural descriptions of each nonnegative integer *rank*. The empty set, thought of as an empty set of labeled SP orders, is the only structural description of rank 0. Assume the structural descriptions of ranks $0, \ldots, n$ are known. A structural description of rank $n+1$ is a finite set S of finite, labeled SP orders such that each label of each order in S is either the special symbol "R" or a structural description of rank at most n. A structural description is a structural description of some finite rank. Note that since we require finiteness at each step, each of our structural descriptions would be considered a "finite structural description" in the informal sense of the term.

A structural description D generates a lower ideal $L(D)$ analogously to the previous definition for bits. We say this is a structural description *for* $L(D)$ or *of* $L(D)$.

Our recursive procedure will take a lower ideal as input and give a finite structural description as output. We just made precise what form the output takes. To state the form of the input, we first need several definitions. A quasi order is a set Q together with a transitive, reflexive relation $\leq$. A quasi order is a well quasi order, or WQO, if for all infinite sequences $q_1, q_2, \ldots$ of points in Q, there are positive integers $i < j$ such that $q_i \leq q_j$. A class $\mathcal{C}$ of partial orders is then said to be well quasi ordered under suborder if for each infinite sequence $(P_1, \leq_1), (P_2, \leq_2), \ldots$, of partial orders in $\mathcal{C}$, there are positive integers $i < j$ such that $(P_i, \leq_i)$ is a suborder of $(P_j, \leq_j)$.

Given an SP order P, we let $\mathrm{Forb}(P)$ be the set of SP orders forbidding P as a suborder. Given a set $F = \{Q_1, \ldots, Q_k\}$ of SP orders, we denote the set of SP orders forbidding each P in F as a suborder by $\mathrm{Forb}(F)$ or $\mathrm{Forb}(Q_1, \ldots, Q_k)$. It can be shown that finite SP orders form a WQO under the suborder relation. Basic WQO theory then implies that for each lower ideal L, there is a finite set F of SP orders such that $L = \mathrm{Forb}(F)$ [2]. With these facts stated, we may now express the main result of this paper more precisely; we give an algorithm that takes a finite set F of SP orders as input and outputs a structural description D such that $L(D) = \mathrm{Forb}(F)$.

Since our main focus is combinatorial structure theory, we do not concern ourselves with algorithmic or complexity theoretic questions. Though such questions may be interesting, they are simply not our focus here. We thus present our algorithms in the same informal style that is common in mathematics.

3 Technical Lemmas

We note that the reader familiar with SP orders can likely skim or skip much of this section. Even readers unfamiliar with SP orders may find it useful to proceed to the next section and refer back to this section as needed.

We call an SP order connected if its comparability graph is connected. An SP order is anticonnected if its incomparability graph is connected.

Lemma 3 *Every chain sum is connected. Similarly, every antichain sum is anticonnected.*

Proof Let $P_1 \prec \cdots \prec P_n$ be a chain sum. By definition, we have $n \geq 2$ and each P_i is nonempty. For $i \neq j$, each point x of P_i is comparable to each point y in P_j and hence x and y are adjacent in the comparability graph. If two points x and y are contained in the same P_i, then choose $i \neq j$ and z in P_j. Then x and y are both adjacent to z and hence in the same component. Therefore given any points x and y in $P_1 \prec \cdots \prec P_n$, there is a path of length one or two between x and y in the comparability graph of P, and the first claim of the lemma holds. For the second claim, repeat the same proof with $P_1 \bigoplus \cdots \bigoplus P_n$ and the incomparability graph. □

Lemma 4 *Each component of $P_1 \bigoplus \cdots \bigoplus P_n$ is contained in some P_i. Each anticomponent of $P_1 \prec \cdots \prec P_n$ is contained in some P_i.*

Proof A component of $P_1 \bigoplus \cdots \bigoplus P_n$ is connected in the comparability graph. Since there are no edges from P_i to P_j for $i \neq j$ in the comparability graph, we see that each component is contained in some P_i. The proof of the second claim is analogous. □

Lemma 5 *If Q is a chain sum and P_i is Q-free for i in $\{1, \ldots, n\}$, then $P_1 \bigoplus \cdots \bigoplus P_n$ is Q-free.*

Proof Let Q be a chain sum. It is enough to show that if $P_1 \bigoplus \cdots \bigoplus P_n$ contains Q, then P_i contains Q for some i. Since Q is a chain sum, we know by Lemma 3 that Q is connected. By Lemma 4, Q must therefore be contained in some P_i. □

The next lemma is analogous to the previous lemma, and the same proof goes through mutatis mutandis.

Lemma 6 *If Q is an antichain sum and P_i is Q-free for i in $\{1, \ldots, n\}$, then $P_1 \prec \cdots \prec P_n$ is Q-free.*

We need several technical lemmas.

Lemma 7 *If $P_1 \prec \cdots \prec P_n$ is a finest chain representation of an SP order P, then each P_i is an antichain sum or a one point partial order.*

Proof For each i, since P_i is a suborder of an SP order, P_i itself is an SP order. Since $P_1 \prec \cdots \prec P_n$ is a finest chain representation by hypothesis, it follows by definition of finest chain representation that P_i is not itself a chain sum, and P_i is therefore a single point or an antichain sum as claimed. □

The same holds for finest antichain representations. We omit the entirely analogous proof.

Lemma 8 *If $P_1 \oplus \cdots \oplus P_n$ is a finest antichain representation of an SP order P, then each P_i is a chain sum or a one point partial order.*

Lemma 9 *Let $P_1 \oplus \cdots \oplus P_n$ be a finest antichain representation of a partial order P and let $Q_1 \oplus \cdots \oplus Q_k$ be an arbitrary antichain sum. If $P_1 \oplus \cdots \oplus P_n$ is a suborder of $Q_1 \oplus \cdots \oplus Q_k$, then for each i with $1 \leq i \leq n$ there is j with $1 \leq j \leq k$ such that P_i is a suborder of Q_j.*

Proof Choose i. Note that P_i is a chain sum or a one point partial order by Lemma 8. If P_i is a single point, then P_i is of course contained in some Q_i. If P_i is a chain sum, then it is connected and therefore contained in a component of $Q_1 \oplus \cdots \oplus Q_k$. Since each component of $Q_1 \oplus \cdots \oplus Q_k$ is contained in some Q_i, the result follows. □

The following lemma has a similar proof.

Lemma 10 *Let $P_1 \prec \cdots \prec P_n$ be a finest chain representation of a partial order P and let $Q_1 \prec \cdots \prec Q_k$ be an arbitrary chain sum. If $P_1 \prec \cdots \prec P_n$ is a suborder of $Q_1 \prec \cdots \prec Q_k$, then for each i with $1 \leq i \leq n$ there is j with $1 \leq j \leq k$ such that P_i is a suborder of Q_j.*

Lemma 11 *Let $P_1 \prec \cdots \prec P_n$ be a finest chain representation of an SP order P that is contained in the partial order $Q_1 \prec Q_2$. If the P_i of $P_1 \prec \cdots \prec P_n$ is contained in Q_1 then so is $P_1 \prec \cdots \prec P_i$. Similarly, if the P_i of $P_1 \prec \cdots \prec P_n$ is contained in Q_2 then so is $P_i \prec \cdots \prec P_n$.*

Proof We prove the first claim. The second is similar. By hypothesis, the P_i of $P_1 \prec \cdots \prec P_n$ is a suborder of Q_1. Since every point of $P_1 \prec \cdots \prec P_i$ is less than or equal some point of P_i, and since Q_1 is a downward closed subset of $Q_1 \prec Q_2$ containing P_i, it follows that $P_1 \prec \cdots \prec P_i$ is a suborder of Q_1. □

Lemma 12 *If $P_1 \prec \cdots \prec P_n$ is a finest chain representation that is contained in the partial order $Q_1 \prec Q_2$, then one of the following three conditions holds:*

1. *$P_1 \prec \cdots \prec P_n$ is a suborder of Q_1.*
2. *$P_1 \prec \cdots \prec P_n$ is a suborder of Q_2.*
3. *There is i with $1 \leq i < n$ such that $P_1 \prec \cdots \prec P_i$ is a suborder of Q_1 and $P_{i+1} \prec \cdots \prec P_n$ is a suborder of Q_2.*

Proof Since $P_1 \prec \cdots \prec P_n$ is a finest chain representation by hypothesis, we know that each P_i is contained in Q_1 or Q_2 by Lemma 10. If $P_1 \prec \cdots \prec P_n$ is a suborder of Q_1 or Q_2 then we are done. Suppose not. Take the largest i such that P_i is a suborder of Q_1. By Lemma 11, we see that $P_1 \prec \cdots \prec P_i$ is a suborder of Q_1. Since $P_1 \prec \cdots \prec P_n$ is not a suborder of Q_1 by hypothesis, we know that $i < n$. Therefore P_{i+1} is a suborder of Q_2. Again by Lemma 11, we see that $P_{i+1} \prec \cdots \prec P_n$ is a suborder of Q_2, which completes the proof. □

4 The Main Lemmas

For $1 \leq i \leq n$, let X_i be a lower ideal or the symbol R. We let the notation $X_1 \prec \cdots \prec X_n$ denote the n point labeled chain with bottom point labeled X_1, next least point labeled X_2, and so on. Note that $P_1 \prec \cdots \prec P_n$ defined previously is the chain sum of n partial orders $P_1, \ldots, P_n$ (which is of course itself a partial order). On the other hand, $X_1 \prec \cdots \prec X_n$ is a bit $(I, \leq_I, f)$ such that the partial order $(I, \leq_I)$ is an n point chain. As long as the reader keeps this distinction in mind, no confusion arises. Similarly for the expression $X_1 \oplus \cdots \oplus X_n$.

Definition 13 Let $n \geq 2$. The chain bit set $\mathrm{BS}(P)$ corresponding to a chain sum P with finest chain representation $P_1 \prec \cdots \prec P_n$ is defined to be the set of bits B such that one of the following conditions hold:

1. $B = R \prec \mathrm{Forb}(P_n)$.
2. $B = \mathrm{Forb}(P_1) \prec R$.
3. There is i with $1 < i < n$ such that

$$B = \mathrm{Forb}(P_1 \prec \cdots \prec P_i) \prec \mathrm{Forb}(P_i \prec \cdots \prec P_n)$$

We note that since the finest chain representation is uniquely determined, the notation $\mathrm{BS}(P)$ is well defined for chain sums P.

Lemma 14 *Let $n \geq 2$. If P is an SP order with finest chain representation $P_1 \prec \cdots \prec P_n$, then*

$$Forb(P_1 \prec \cdots \prec P_n) = L(BS(P) \cup \{R_A\}).$$

Proof Let $S = \mathrm{BS}(P_1 \prec \cdots \prec P_n) \cup \{R_A\}$. We must show that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is the S-bit closure of the doubleton containing the empty and one point partial orders. Since $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ trivially contains the empty and one point partial orders, it is enough to show that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is S-bit closed and that every S-bit closed set containing the empty and one point partial orders has $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ as a subset.

We first show that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is S-bit closed. By Lemma 2, it is enough to show that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is $(I, \leq_I, f)$-bit closed for each bit $(I, \leq_I, f)$ in S. We consider four cases.

First, if $(I, \leq_I, f)$ is R_A, then to show that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is $(I, \leq_I, f)$-bit closed is simply to show that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is closed under antichain sums. But this is exactly Lemma 5.

Second, if $(I, \leq_I, f)$ is a two point chain with bottom point labeled R and top point labeled $\mathrm{Forb}(P_n)$, then to show that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is $(I, \leq_I, f)$-bit closed is to show that if Q_1 is a partial order in $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ and Q_2 is a partial order in $\mathrm{Forb}(P_n)$, then $Q_1 \prec Q_2$ forbids $P_1 \prec \cdots \prec P_n$. Suppose not. Since $Q_1 \prec Q_2$ contains $P_1 \prec \cdots \prec P_n$, in particular $Q_1 \prec Q_2$ contains the top inner part P_n of the chain sum. By Lemma 10, we see that P_n is a suborder of Q_1 or Q_2. Since Q_2 forbids P_n, we know that P_n is a suborder of Q_1. By Lemma 11, it follows that $P_1 \prec \cdots \prec P_n$ is a suborder of Q_1, contrary to hypothesis. This contradiction shows that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is $(I, \leq_I, f)$-bit closed as claimed.

The third case, that $(I, \leq_I, f)$ is a two point chain with top point labeled R and bottom point labeled $\mathrm{Forb}(P_1)$, is completely analogous to the second case, and the proof goes through mutatis mutandis.

Fourth, if there is i with $1 < i < n$ such that $(I, \leq_I, f)$ is a two point chain with bottom point labeled $\mathrm{Forb}(P_1 \prec \cdots \prec P_i)$ and top point labeled $\mathrm{Forb}(P_i \prec \cdots \prec P_n)$, then to show that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is $(I, \leq_I, f)$-bit closed, we must show that if

Q_1 is a partial order forbidding $P_1 \prec \cdots \prec P_i$ and Q_2 is a partial order forbidding $P_i \prec \cdots \prec P_n$, then $Q_1 \prec Q_2$ forbids $P_1 \prec \cdots \prec P_n$. We prove the contrapositive statement, namely, that if $Q_1 \prec Q_2$ has a $P_1 \prec \cdots \prec P_n$ suborder then Q_1 has a $P_1 \prec \cdots \prec P_i$ suborder or Q_2 has a $P_i \prec \cdots \prec P_n$ suborder. Since $P_1 \prec \cdots \prec P_n$ is a suborder of $Q_1 \prec Q_2$, in particular P_i is also. By Lemma 10, P_i is therefore a suborder of Q_1 or Q_2. By Lemma 11, if P_i is a suborder of Q_1 then $P_1 \prec \cdots \prec P_i$ is as well. Lemma 11 similarly implies that if P_i is a suborder of Q_2 then $P_i \prec \cdots \prec P_n$ is as also. The contrapositive is thus proved, which completes the proof that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is $(I, \leq_I, f)$-bit closed in this final case.

We now know that $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is S-bit closed. Next, we show that every S-bit closed set X containing the empty and one point partial orders has $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ as a subset.

Suppose not. Then the S-bit closure X of the set containing the empty and one point partial orders is a proper subset of the S-bit closed set $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$. Take a minimum cardinality SP order Q in $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ that is not in X. Then Q has at least two elements by choice of X. Since Q is an SP order, it follows that Q is a chain or antichain sum.

If Q is an antichain sum, then we may write $Q = Q_1 \oplus Q_2$, where Q_1 and Q_2 each have fewer elements than Q. Since Q is a minimum size partial order in $\mathrm{Forb}(P_1 \prec \cdots \prec P_n) - X$ by hypothesis, we see that Q_1 and Q_2 are in X. Since X is $(I, \leq_I, f)$-bit closed for $(I, \leq_I, f)$ the two point antichain R_A with both points labeled R, it follows that the antichain sum of two orders in X is in X as well. In particular, Q is in X, contrary to hypothesis. This contradiction shows that Q can not be an antichain sum.

Since Q is not an antichain sum, Q must be a chain sum $Q = Q_1 \prec Q_2$. By choice of Q as minimal, we know that Q_1 and Q_2 are in X. Suppose Q_2 is in $\mathrm{Forb}(P_n)$. Since Q_1 is in X and Q_2 is in $\mathrm{Forb}(P_n)$, and since X is $(I, \leq_I, f)$-bit closed for $(I, \leq_I, f)$ the two point chain with top labeled $\mathrm{Forb}(P_n)$ and bottom labled R, we see that $Q_1 \prec Q_2$ must be in X, contrary to hypothesis. Therefore Q_2 is not in $\mathrm{Forb}(P_n)$. By similar reasoning, Q_1 is not in $\mathrm{Forb}(P_1)$.

Choose the least i such that Q_1 does not have a $P_1 \prec \cdots \prec P_i$ suborder. Then Q_1 has a $P_1 \prec \cdots \prec P_{i-1}$ suborder. If Q_2 has a $P_i \prec \cdots \prec P_n$ suborder, then $Q_1 \prec Q_2$ has a $P_1 \prec \cdots \prec P_n$ suborder, contrary to hypothesis. Therefore Q_2 has no $P_i \prec \cdots \prec P_n$ suborder. Therefore Q_1 is in $\mathrm{Forb}(P_1 \prec \cdots \prec P_i)$ and Q_2 is in $\mathrm{Forb}(P_i \prec \cdots \prec P_n)$. Since the two point chain with top labeled $\mathrm{Forb}(P_i \prec \cdots \prec P_n)$ and bottom labeled $\mathrm{Forb}(P_1 \prec \cdots \prec P_i)$ is a bit in S and X is S-bit closed, it follows that $Q_1 \prec Q_2 = Q$ is in X, contrary to hypothesis.

In all cases, the assumption that X is a proper subset of $\mathrm{Forb}(P_1 \prec \cdots \prec P_n)$ is a contradiction. Equality therefore holds, thus completing the proof. □

To give a similar result for excluding a set of chain sums, we first need some definitions.

Definition 15 Fix $k \geq 1$. For $1 \leq i \leq k$ let P_i be a chain sum. A chain bit choice function for $(P_1, \ldots, P_k)$ is a function c mapping each P_i to a chain bit in $\mathrm{BS}(P_i)$.

Given a chain bit $(I, \leq_I, f)$, we let $\mathrm{Bottom}((I, \leq_I, f))$ and $\mathrm{Top}((I, \leq_I, f))$ denote the labels of the bottom and top points, respectively, of $(I, \leq_I, f)$.

In the next definition, we must intersect labels of bits. If all labels are ideals, then no comment is necessary, but in general some labels may be the symbol R, so we must extend the notion of intersection to include this symbol. We make the convention that in the definition of bit set corresponding to $(P_1, \ldots, P_k)$ below, the symbol R is taken to mean $\mathrm{Forb}(P_1, \ldots, P_k)$. In other words, the intersection of R with a set is the intersection of $\mathrm{Forb}(P_1, \ldots, P_k)$ and that set. Moreover, if a rule tells us that a point should be labeled $\mathrm{Forb}(P_1, \ldots, P_k)$, we label that point R. Without this convention, stating the following definition would be quite lengthy.

Definition 16 Fix $k \geq 1$. For $1 \leq i \leq k$, let P_i be a chain sum. The chain bit set $\mathrm{BS}(P_1, \ldots, P_k)$ corresponding to the tuple $(P_1, \ldots, P_k)$ is the set of two point chain bits of the form

$$\bigcap_{1 \leq i \leq k} \mathrm{Bottom}(c(P_i)) \prec \bigcap_{1 \leq i \leq k} \mathrm{Top}(c(P_i)).$$

such that c is a chain bit choice function for $(P_1, \ldots, P_k)$.

We note that the previous definition is consistent with Definition 13 for the case $k = 1$. The following lemma generalizes Lemma 14 to the case of excluding an arbitrary finite set of chain sums.

Lemma 17 *Let $k \geq 1$. If the SP orders $P_1, \ldots, P_k$ are chain sums, then*

$$\mathit{Forb}(P_1, \ldots, P_k) = L(\mathit{BS}(P_1, \ldots, P_k) \cup \{R_A\}).$$

Proof For $k = 1$, this is just Lemma 14, so we assume without loss of generality that $k \geq 2$.

Let $S = \mathrm{BS}(P_1, \ldots, P_k) \cup \{R_A\}$. We must show that $\mathrm{Forb}(P_1, \ldots, P_k)$ is the S-bit closure of the doubleton containing the empty and one point partial orders. Since $\mathrm{Forb}(P_1, \ldots, P_k)$ trivially contains the empty and one point partial orders, it is enough to show that $\mathrm{Forb}(P_1, \ldots, P_k)$ is S-bit closed and that every S-bit closed set containing the empty and one point partial orders has $\mathrm{Forb}(P_1, \ldots, P_k)$ as a subset.

We first show that $\mathrm{Forb}(P_1, \ldots, P_k)$ is S-bit closed. By Lemma 2, it is enough to show that $\mathrm{Forb}(P_1, \ldots, P_k)$ is $(I, \leq_I, f)$-bit closed for each bit $(I, \leq_I, f)$ in S.

First, if $(I, \leq_I, f)$ is R_A, then to show that $\mathrm{Forb}(P_1, \ldots, P_k)$ is $(I, \leq_I, f)$-bit closed is simply to show that $\mathrm{Forb}(P_1, \ldots, P_k)$ is closed under antichain sums. This follows easily from Lemma 5.

If $(I, \leq_I, f) \neq R_A$, then $(I, \leq_I, f)$ has the form

$$\bigcap_{1 \leq i \leq k} \mathrm{Bottom}(c(P_i)) \prec \bigcap_{1 \leq i \leq k} \mathrm{Top}(c(P_i))$$

for some chain bit choice function c for $(P_1, \ldots, P_k)$. To show that $\mathrm{Forb}(P_1, \ldots, P_k)$ is $(I, \leq_I, f)$-bit closed is thus to show that for each chain bit choice function c for $(P_1, \ldots, P_k)$, if Q_1 and Q_2 are SP orders in $\mathrm{Forb}(P_1, \ldots, P_k)$ such that Q_1 is in $\bigcap_{1 \leq i \leq k} \mathrm{Bottom}(c(P_i))$ and Q_2 is in $\bigcap_{1 \leq i \leq k} \mathrm{Top}(c(P_i))$, then $Q_1 \prec Q_2$ is in $\mathrm{Forb}(P_1, \ldots, P_k)$ as well. To show that $Q_1 \prec Q_2$ is in $\mathrm{Forb}(P_1, \ldots, P_k)$, we must show that $Q_1 \prec Q_2$ forbids P_i for $1 \leq i \leq k$, so choose i. Since Q_1 is in $\bigcap_{1 \leq i \leq k} \mathrm{Bottom}(c(P_i))$, in particular Q_1 is in $\mathrm{Bottom}(c(P_i))$. Similarly Q_2 is in $\mathrm{Top}(c(P_i))$. Since c is a chain bit choice function for $(P_1, \ldots, P_k)$, we see that

$\text{Bottom}(c(P_i)) \prec \text{Top}(c(P_i))$ is a chain bit in $\text{BS}(P_i)$. Both Q_1 and Q_2 are in $\text{Forb}(P_i)$. Therefore $Q_1 \prec Q_2$ is in $\text{Forb}(P_i)$ as needed. This completes the proof that $\text{Forb}(P_1, \ldots, P_k)$ is S-bit closed.

We now know that $\text{Forb}(P_1, \ldots, P_k)$ is S-bit closed. Next, we show that every S-bit closed set X containing the empty and one point partial orders has $\text{Forb}(P_1, \ldots, P_k)$ as a subset.

Suppose not. Then the S-bit closure X of the set containing the empty and one point partial orders is a proper subset of the S-bit closed set $\text{Forb}(P_1, \ldots, P_k)$. Take a minimum cardinality SP order Q in $\text{Forb}(P_1, \ldots, P_k)$ that is not in X. Then Q has at least two elements by choice of X. Since Q is an SP order, it follows that Q is a chain or antichain sum.

If Q is an antichain sum, then we may write $Q = Q_1 \oplus Q_2$, where Q_1 and Q_2 each have fewer elements than Q. Since Q is a minimum size partial order in $\text{Forb}(P_1, \ldots, P_k) - X$ by hypothesis, we see that Q_1 and Q_2 are in X. Since X is $(I, \leq_I, f)$-bit closed for $(I, \leq_I, f)$ the two point antichain R_A with both points labeled R, it follows that the antichain sum of two orders in X is in X as well. In particular, Q is in X, contrary to hypothesis. This contradiction shows that Q can not be an antichain sum.

Since Q is not an antichain sum, Q must be a chain sum $Q = Q_1 \prec Q_2$. By choice of Q as minimal, we know that Q_1 and Q_2 are in X. For each i, since $Q_1 \prec Q_2$ is in $\text{Forb}(P_i) = L(\text{BS}(P_i) \cup \{R_A\})$, we know there is a two point chain bit B_i in $\text{BS}(P_i)$ such that Q_1 is in $\text{Bottom}(B_i)$ and Q_2 is in $\text{Top}(B_i)$. Define the chain bit choice function c for $(P_1, \ldots, P_k)$ by letting $c(P_i) = B_i$ for each i. Then Q_1 is in $\bigcap_{1 \leq i \leq k} \text{Bottom}(c(P_i))$ and Q_2 is in $\bigcap_{1 \leq i \leq k} \text{Top}(c(P_i))$. Moreover, Q_1 and Q_2 are in $\text{Forb}(P_1, \ldots, P_k)$ and

$$\bigcap_{1 \leq i \leq k} \text{Bottom}(c(P_i)) \prec \bigcap_{1 \leq i \leq k} \text{Bottom}(c(P_i))$$

is in $\text{BS}(P_1, \ldots, P_k)$. It follows that $Q = Q_1 \prec Q_2$ is in $\text{Forb}(P_1, \ldots, P_k)$, contrary to hypothesis. This contradiction completes the proof. □

We now move onto excluding sets of antichain sums. As a motivating example, we may wish to compute $\text{Forb}(P_1 \oplus P_2, P_2 \oplus P_3)$. We would then let Γ be the family of subsets of $\{1, 2, 3\}$ consisting of $\{1, 2\}$ and $\{2, 3\}$ and think of $\text{Forb}(P_1 \oplus P_2, P_2 \oplus P_3)$ as

$$\bigcap_{F \in \Gamma} \text{Forb}\left(\bigoplus_{i \in F} P_i\right).$$

This example motivates us to define, given a sequence $P_1, \ldots, P_k$ of SP orders and a family Γ of nonempty subsets of $\{1, \ldots, k\}$, the lower ideal

$$\text{Forb}(\Gamma; P_1, \ldots, P_k) := \bigcap_{F \in \Gamma} \text{Forb}\left(\bigoplus_{i \in F} P_i\right).$$

We need several definitions. A *splitting* of a set X is an ordered pair (A, B) such that the sets A and B partition X. We denote the set of splittings of X by $\text{spl}(X)$. A *splitting function* for X is a function $h : \text{spl}(X) \to \{1, 2\}$.

Let Γ be a family of subsets of $\{1, \ldots, k\}$. An *antichain bit choice function*, or ABCF, for Γ is a function g with domain Γ such that $g_F := g(F)$ is a splitting function for F for each set F in Γ. We define the left cell ideal set $\mathrm{lcis}(g)$ of g as the set of all pairs (A, F) such that F is in Γ with $A \subseteq F$ and $g_F(A, F - A) = 1$. The right cell ideal set $\mathrm{rcis}(g)$ is defined similarly but with $g_F(A, F - A) = 2$.

We define the left cell label $\mathrm{lcl}(g; P_1, \ldots, P_k)$ as the lower ideal

$$\mathrm{lcl}(g; P_1, \ldots, P_k) := \mathrm{Forb}(\Gamma; P_1, \ldots, P_k) \cap \bigcap_{(A,F)\in \mathrm{lcis}(g)} \mathrm{Forb}\left(\bigoplus_{i\in A} P_i\right)$$

and the right cell label $\mathrm{rcl}(g; P_1, \ldots, P_k)$ as the lower ideal

$$\mathrm{rcl}(g; P_1, \ldots, P_k) := \mathrm{Forb}(\Gamma; P_1, \ldots, P_k) \cap \bigcap_{(A,F)\in \mathrm{rcis}(g)} \mathrm{Forb}\left(\bigoplus_{i\in F-A} P_i\right).$$

We now define $\mathrm{BS}(\Gamma; P_1, \ldots, P_k)$ as the set of labeled antichains that have the form

$$\mathrm{lcl}(g; P_1, \ldots, P_k) \oplus \mathrm{rcl}(g; P_1, \ldots, P_k)$$

for some ABCF g for Γ.

We need to use finest antichain partitions in the next lemma. This amounts to assuming that our summands $P_1, \ldots, P_k$ are not themselves antichain sums.

Lemma 18 *Let $k \geq 1$. If the SP orders $P_1, \ldots, P_k$ are not antichain sums, then*

$$Forb(\Gamma; P_1, \ldots, P_k) = L(BS(\Gamma; P_1, \ldots, P_k) \cup \{R_C\}).$$

Proof Let $S = \mathrm{BS}(\Gamma; P_1, \ldots, P_k) \cup \{R_C\}$. We must show that $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$ is the S-bit closure of the doubleton containing the empty and one point partial orders. Since $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$ trivially contains the empty and one point partial orders, it is enough to show that $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$ is S-bit closed and that every S-bit closed set containing the empty and one point partial orders has $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$ as a subset.

We first show that $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$ is S-bit closed. By Lemma 2, it is enough to show that $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$ is $(I, \leq_I, f)$-bit closed for each bit $(I, \leq_I, f)$ in S.

First, if $(I, \leq_I, f)$ is R_C, then $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$ is $(I, \leq_I, f)$-bit closed by Lemma 6. Otherwise, by definition of S and $\mathrm{BS}(\Gamma; P_1, \ldots, P_k)$, we see that $(I, \leq_I, f)$ must have the form $\mathrm{lcl}(g; P_1, \ldots, P_k) \oplus \mathrm{rcl}(g; P_1, \ldots, P_k)$ for some ABCF g for Γ, so choose such a g. To show that $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$ is $(I, \leq_I, f)$-bit closed for

$$(I, \leq_I, f) = \mathrm{lcl}(g; P_1, \ldots, P_k) \oplus \mathrm{rcl}(g; P_1, \ldots, P_k),$$

we must show that if Q_1 is in $\mathrm{lcl}(g; P_1, \ldots, P_k)$ and Q_2 is in $\mathrm{rcl}(g; P_1, \ldots, P_k)$ then $Q_1 \oplus Q_2$ is in $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$. Equivalently, we may show that if $Q_1 \oplus Q_2$ is not in $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$, then Q_1 is not in $\mathrm{lcl}(g; P_1, \ldots, P_k)$ or Q_2 is not in $\mathrm{rcl}(g; P_1, \ldots, P_k)$.

Suppose $Q_1 \oplus Q_2$ is not in

$$\operatorname{Forb}(\Gamma; P_1, \ldots, P_k) = \bigcap_{F\in\Gamma} \operatorname{Forb}\left(\bigoplus_{i\in F} P_i\right).$$

Then there is F in Γ such that $Q_1 \oplus Q_2$ is not in Forb $\left(\bigoplus_{i\in F} P_i\right)$. Therefore $Q_1 \oplus Q_2$ contains a $\bigoplus_{i\in F} P_i$ suborder. We may then choose a one to one order preserving map $h : \bigoplus_{i\in F} P_i \to Q_1 \oplus Q_2$ embedding $\bigoplus_{i\in F} P_i$ into $Q_1 \oplus Q_2$. Since no P_i is an antichain sum, we know by Lemma 5 that $h(P_i)$ is contained in Q_1 or Q_2 for each i. Let $A = \{i \in F : h(P_i) \subseteq Q_1\}$. Then $F - A = \{i \in F : h(P_i) \subseteq Q_2\}$. If A is empty then $\bigoplus_{i\in F} P_i$ is a suborder of Q_2. Therefore Q_2 is not in Forb $\left(\bigoplus_{i\in F} P_i\right)$, which implies Q_2 is not in

$$\bigcap_{F\in\Gamma} \operatorname{Forb}\left(\bigoplus_{i\in F} P_i\right).$$

By the definition of $\operatorname{rcl}(g; P_1, \ldots, P_k)$, this in turn implies that Q_2 is not in $\operatorname{rcl}(g; P_1, \ldots, P_k)$. This proves our claim in the case that A is empty. Similarly if $F - A$ is empty. We may thus assume that A and $F - A$ are nonempty.

Either $g_F(A, F - A) = 1$ or $g_F(A, F - A) = 2$. If $g_F(A, F - A) = 1$, then (A, F) is in lcis(g). Certainly $\bigoplus_{i\in A} P_i$ is not in Forb $\left(\bigoplus_{i\in A} P_i\right)$, and Q_1 contains $\bigoplus_{i\in A} P_i$, which implies Q_1 is not in Forb $\left(\bigoplus_{i\in A} P_i\right)$. Therefore Q_1 is not in

$$\bigcap_{(A,F)\in \operatorname{lcis}(g)} \operatorname{Forb}\left(\bigoplus_{i\in A} P_i\right).$$

By definition of $\operatorname{lcl}(g; P_1, \ldots, P_k)$, we thus see that Q_1 is not in $\operatorname{lcl}(g; P_1, \ldots, P_k)$. Similarly, if $g_F(A, F - A) = 2$ then Q_2 is not in $\operatorname{rcl}(g; P_1, \ldots, P_k)$, as was to be shown. This completes the proof of the claim that $\operatorname{Forb}(\Gamma; P_1, \ldots, P_k)$ is S-bit closed.

We must now show that every S-bit closed set containing the empty and one point partial orders has $\operatorname{Forb}(\Gamma; P_1, \ldots, P_k)$ as a subset. Suppose not. Then the S-bit closure X of the set containing the empty and one point partial orders is a proper subset of the S-bit closed set $\operatorname{Forb}(\Gamma; P_1, \ldots, P_k)$. So take a minimum cardinality SP order Q in $\operatorname{Forb}(\Gamma; P_1, \ldots, P_k)$ that is not in X. Then Q has at least two elements by choice of X. Since Q is an SP order, it follows that Q is a chain or antichain sum. If Q is a chain sum $Q_1 \prec Q_2$ then Q_1 and Q_2 are in X by choice of Q as minimal. Since R_C is in S and X is S-bit closed, it then follows that $Q = Q_1 \prec Q_2$ is in X, contrary to hypothesis. This contradiction shows that Q is an antichain sum.

We write $Q = Q_1 \oplus Q_2$. We wish to get a contradiction in this case as well by showing in fact that Q is in X. Since Q_1 and Q_2 are in X by minimality of Q, and since X is $(I, \leq_I, f)$-bit closed for

$$(I, \leq_I, f) = \operatorname{lcl}(g; P_1, \ldots, P_k) \oplus \operatorname{lcl}(g; P_1, \ldots, P_k),$$

we see it is enough to show there is an ABCF g for Γ such that Q_1 is in $\operatorname{lcl}(g; P_1, \ldots, P_k)$ and Q_2 is in $\operatorname{rcl}(g; P_1, \ldots, P_k)$. Since Q is in the lower ideal $\operatorname{Forb}(\Gamma; P_1, \ldots, P_k)$, the suborders Q_1 and Q_2 are in $\operatorname{Forb}(\Gamma; P_1, \ldots, P_k)$ as well.

By definition of $\mathrm{lcl}(g; P_1, \ldots, P_k)$ and $\mathrm{rcl}(g; P_1, \ldots, P_k)$, it is therefore enough to exhibit an ABCF g for Γ such that Q_1 is in

$$\bigcap_{(A,F)\in \mathrm{lcis}(g)} \mathrm{Forb}\left(\bigoplus_{i\in A} P_i\right)$$

and Q_2 is in

$$\bigcap_{(A,F)\in \mathrm{rcis}(g)} \mathrm{Forb}\left(\bigoplus_{i\in F-A} P_i\right).$$

Choose F in Γ. Since $Q_1 \oplus Q_2$ is in $\mathrm{Forb}(\Gamma; P_1, \ldots, P_k)$, we see that $Q_1 \oplus Q_2$ forbids $\bigoplus_{i\in F} P_i$. Therefore for each splitting (A, B) of F, the SP order Q_1 must forbid $\bigoplus_{i\in A} P_i$ or Q_2 must forbid $\bigoplus_{i\in B} P_i$. Consider the ABCF g for Γ such that for each F in Γ and each splitting (A, B) of F, we have $g_F(A, B) = 1$ if Q_1 forbids $\bigoplus_{i\in A} P_i$ and $g_F(A, B) = 2$ otherwise.

To show that Q_1 is in

$$\bigcap_{(A,F)\in \mathrm{lcis}(g)} \mathrm{Forb}\left(\bigoplus_{i\in A} P_i\right),$$

it is enough to show that Q_1 is in $\mathrm{Forb}\left(\bigoplus_{i\in A} P_i\right)$ for each F in Γ and each nonempty $A \subseteq F$ such that $g_F(A, F - A) = 1$. This is immediate from the definition of g_F. Similarly, it follows immediately from the definition of g_F that Q_2 is in

$$\bigcap_{(A,F)\in \mathrm{rcis}(g)} \mathrm{Forb}\left(\bigoplus_{i\in F-A} P_i\right).$$

This completes the proof of the lemma. □

Lemma 19 *If A and B are nonempty sets of chain sums and antichain sums, respectively, then $Forb(A \cup B) = L(BS(A) \cup BS(B))$.*

Proof We know that $\mathrm{Forb}(A)$ is $(I, \leq_I, f)$-bit closed for each bit $(I, \leq_I, f)$ in $\mathrm{BS}(A)$. We also know by Lemma 5 that $\mathrm{Forb}(A)$ is closed under arbitrary antichain sums, and since each bit in $\mathrm{BS}(B)$ is an antichain, we see that $\mathrm{Forb}(A)$ is $(I, \leq_I, f)$-bit closed for each $(I, \leq_I, f)$ bit in $\mathrm{BS}(B)$. Therefore $\mathrm{Forb}(A)$ is $(I, \leq_I, f)$-bit closed for each bit $(I, \leq_I, f)$ in $\mathrm{BS}(A) \cup \mathrm{BS}(B)$. By similar reasoning, $\mathrm{Forb}(B)$ is $(I, \leq_I, f)$-bit closed for each bit $(I, \leq_I, f)$ in $\mathrm{BS}(A) \cup \mathrm{BS}(B)$ as well. This implies that $\mathrm{Forb}(A \cup B) = \mathrm{Forb}(A) \cap \mathrm{Forb}(B)$ is $(I, \leq_I, f)$-bit closed for each bit $(I, \leq_I, f)$ in $\mathrm{BS}(A) \cup \mathrm{BS}(B)$, and hence $\mathrm{Forb}(A \cup B)$ is $\mathrm{BS}(A) \cup \mathrm{BS}(B)$ closed. Therefore $L(\mathrm{BS}(A) \cup \mathrm{BS}(B)) \subseteq \mathrm{Forb}(A \cup B)$.

If $\mathrm{Forb}(A \cup B) = L(\mathrm{BS}(A) \cup \mathrm{BS}(B))$, we are done. Suppose not. Then $L(\mathrm{BS}(A) \cup \mathrm{BS}(B))$ is a proper subset of $\mathrm{Forb}(A \cup B)$. Choose a minimum cardinality SP order Q in $\mathrm{Forb}(A \cup B)$ that is not in $L(\mathrm{BS}(A) \cup \mathrm{BS}(B))$. Since Q has at least two points, Q is a chain sum or an antichain sum. We assume that Q is a chain sum. The case that Q is an antichain sum is entirely similar.

Since $Q \in \mathrm{Forb}(A \cup B) \subseteq \mathrm{Forb}(A)$, we see that Q is in $\mathrm{Forb}(A) = L(\mathrm{BS}(A) \cup \{R_A\})$. Therefore there is a bit $(I, \leq_I, f)$ in $\mathrm{BS}(A) \cup \{R_A\}$ that generates Q from proper suborders. Since Q is a chain sum, we know that Q is not an antichain sum. Therefore $(I, \leq_I, f) \neq R_A$, which implies $(I, \leq_I, f)$ is in $\mathrm{BS}(A)$. In particular, the $\mathrm{BS}(A) \cup \mathrm{BS}(B)$-bit closure of the set of proper suborders of Q contains Q. Since each proper suborder of Q is in $L(\mathrm{BS}(A) \cup \mathrm{BS}(B))$ and $L(\mathrm{BS}(A) \cup \mathrm{BS}(B))$ is $\mathrm{BS}(A) \cup \mathrm{BS}(B)$-bit closed, we see that Q is in $L(\mathrm{BS}(A) \cup \mathrm{BS}(B))$, contrary to assumption. This contradiction completes the proof. □

5 The Main Theorem

Theorem 20 *There is a structural description for each nontrivial proper lower ideal L.*

Proof It is well known that a quasi order is a well quasi order iff its downward closed sets are well founded under containment. Since finite SP orders are well quasi ordered under the suborder relation [8], it follows that lower ideals are well founded under containment. Thus if there is a nontrivial proper lower ideal with no structural description, there is a minimal one. Suppose such a minimal nontrivial proper lower ideal with no structural description exists. Call it L. We know since finite SP orders are well quasi ordered under suborder that L has the form $\mathrm{Forb}(P_1, \ldots, P_k)$ for some SP orders $P_1, \ldots, P_k$ [8]. Depending on whether $P_1, \ldots, P_k$ is one chain sum, a set of chain sums, a set of antichain sums, or a set of both chain and antichain sums, we use Lemmas 14, 17, 18, or 19, respectively, to obtain a set S of bits such that $L = L(S)$. The reader may check directly from the definitions of those bit sets that all labels in all bits are either the symbol R or lower ideals properly contained in L. Let O be the lower ideal containing only the empty poset. Some of the points in SP orders in S may contain O labeled points. Let S' be obtained from S by removing all O labeled points from all SP orders in S. Note that $L = L(S) = L(S')$. All ideal labeled points in all SP orders in S' are nontrivial proper lower ideals, properly contained in L. By the induction hypothesis, each of these lower ideals thus has its own structural description. Let S'' be the set of labeled SP orders obtained by replacing each ideal labeled point in each SP order in S' with a structural description for that ideal. The set S'' is then a structural description for L, contrary to choice of L. This contradiction shows that every nontrivial proper lower ideal has a structural description as claimed. □

We stress that these results are not just theoretical; they can be applied by hand in practice to obtain specific structure theorems quickly. Though the above inductive proof is written in terms of a minimal L for simplicity, only a slight change in point of view reveals the algorithm. Starting with an arbitrary nontrivial proper lower ideal L, one uses the previous lemmas to obtain a set of bits generating that L. The ideal labels of those sets of bits are properly contained in L, and the same procedure may be applied to them. This is clearly algorithmic at each step. The only issue is whether the process eventually terminates. We may define a sequence of labeled trees as follows. The tree T_0 has one point labeled L. Given T_n, define T_{n+1} as follows. For each leaf x of T_n, the label of x is a nontrivial proper lower ideal L_x. If the one point poset is the only nonempty poset contained in L_x, do not add any vertices to T_n at x. Otherwise use the previous lemmas to find a set of bits S such that $L_x = L(S)$. For each nontrivial lower ideal L' that is the label of some point in some SP order in S, add a vertex x' with label L' adjacent to x. Adding all such labeled vertices to all such leaves gives the tree T_{n+1}. Let T be the union of all these trees. Then T is a finitely branching tree since each set of bits in our previous lemmas consisted of finitely many finite labeled posets. We see also that T has no infinite branch since an infinite branch would yield an infinite decreasing sequence of lower ideals of SP orders, which does not exist by [8]. Therefore by König's Lemma this tree is finite.

We may partially order T with the tree order such that $x \leq y$ iff the path from x to the root contains y. We may then enumerate the vertices of T as $x_1, \ldots, x_n$ such that for each i, the set of predecessors of x_i in the tree order is contained in $\{x_j : j < i\}$. If structural descriptions have been found for the ideal label L_j of x_j for all $j < i$, then we may obtain a structural description for L_i by taking a set S_i of bits such that $L_i = L(S_i)$ and replacing all ideal labels with a structural description for that ideal. (The sets S_i have already been found in the process described in the last paragraph in computing T.) We thus eventually find a structural description for $L_n = L$.

To illustrate using these results to compute structure theorems, we characterize the diamond free SP orders. The diamond is the unique poset on points a, b, c, d such that $a < b < d$, $a < c < d$, and b and c are incomparable. An SP order is called diamond free if there is no diamond suborder. A (partial order theoretic) tree is a poset such that for each x, there are no incomparable elements less than x. A forest is a tree or an antichain sum of trees. An upside down tree (forest) is a poset such that the reverse order is a tree (forest). A forest on top of an upside down forest is a chain sum of a forest and upside down forest with the outer poset a two point chain, the top poset a forest, and the bottom poset an upside down forest. With these definitions, we prove the following corollary, and in the process demonstrate how to obtain structure theorems by hand using the results of this paper.

Corollary 21 *A finite SP order is diamond free iff it is an antichain sum*

$$\bigoplus_{i \in I} P_i,$$

where P_i is a forest on top of an upside down forest for each i.

Proof If P is a diamond, then $P = P_1 \prec P_2 \prec P_3$, where P_1 and P_3 are single point SP orders and P_2 is a two point antichain. Therefore

$$\mathrm{Forb}(P) = \mathrm{Forb}(P_1 \prec P_2 \prec P_3) = L(\mathrm{BS}(P) \cup \{R_A\})$$

by Lemma 14 and by Definition 13 we know that

$$\mathrm{BS}(P) = \big\{R \prec \mathrm{Forb}(P_3), \mathrm{Forb}(P_1) \prec R, \mathrm{Forb}(P_1 \prec P_2) \prec \mathrm{Forb}(P_2 \prec P_3)\big\}.$$

This implies that

$$\mathrm{Forb}(P) = L(R \prec \mathrm{Forb}(P_3), \mathrm{Forb}(P_1) \prec R, \mathrm{Forb}(P_1 \prec P_2) \prec \mathrm{Forb}(P_2 \prec P_3), R_A).$$

Note though that since P_1 and P_3 are single point SP orders, Forb(P_1) and Forb(P_3) each contain only the empty poset. The bits $R \prec \mathrm{Forb}(P_3)$ and $\mathrm{Forb}(P_1) \prec R$ thus generate no additional SP orders. More precisely,

$$\mathrm{Forb}(P) = L(\mathrm{Forb}(P_1 \prec P_2) \prec \mathrm{Forb}(P_2 \prec P_3), R_A).$$

We may ask which SP orders forbid $P_2 \prec P_3$. Since $P_2 \prec P_3$ consists of one maximum point greater than two incomparable points, a poset forbids $P_2 \prec P_3$ as a suborder iff no point has two incomparable predecessors, or equivalently, iff the set of predecessors of each point is a chain. It is well known that a poset is a forest iff the set of predecessors of each point is a chain. Therefore $\mathrm{Forb}(P_2 \prec P_3)$ is exactly the set of forests. Since $P_2 \prec P_3$ is isomorphic to the reverse order of $P_1 \prec P_2$, we similarly see that $\mathrm{Forb}(P_1 \prec P_2)$ is exactly the set of upside down forests. The bit $\mathrm{Forb}(P_1 \prec P_2) \prec \mathrm{Forb}(P_2 \prec P_3)$ thus generates exactly the forests on top of upside down forests. Using the bit R_A as well thus generates exactly the antichain sums of forests on top of upside down forests. Thus

$$\mathrm{Forb}(P) = L(\mathrm{Forb}(P_1 \prec P_2) \prec \mathrm{Forb}(P_2 \prec P_3), R_A)$$

is the set of antichain sums of forests on top of upside down forests as claimed. □

Note that the structural descriptions for ideals are not at all in general unique. Our procedure simply finds one of them. The one found may in fact have redundant rules. Note also that since Lemmas 14, 17, 18, and 19, only involve the two point chain and antichain R_A and R_C, it follows that each lower ideal has a structural description only involving two point posets at any depth. At least to the author, this fact was initially surprising.

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