Math. Proc. Camb. Phil. Soc. (2020), **169**, 639–647 © *Cambridge Philosophical Society* 2019 639
doi:10.1017/S0305004119000380
First published online 31 October 2019

Bisecting measures with hyperplane arrangements

BY ALFREDO HUBARD

Université Paris-Est Marne-la-Vallée, LIGM.
UMR 8049, *CNRS, ENPC, ESIEE, UPEM, F*-77454, *Marne-la-Vallée, France.*
e-mail: alfredo.hubard@u-pem.fr

AND ROMAN KARASEV

Moscow Institute of Physics and Technology,
Institutskiy per. 9, *Dolgoprudny, Russia* 141700.
Institute for Information Transmission Problems RAS,
Bolshoy Karetny per. 19, *Moscow, Russia* 127994.
e-mail: r_n_karasev@mail.ru

(*Received* 14 *May* 2018; *revised* 24 *January* 2019)

Abstract

We show that provided that n is a power of two, any nD measures in $\mathbb{R}^n$ can be bisected by an arrangement of D hyperplanes.

2010 Mathematics Subject Classification: 52C35, 55M20, 55M35

1. *Introduction*

Let $\mathcal{H} = \{H_1, H_2, \dots H_D\}$ be a finite set of hyperplanes, $\{A_1, A_2, \dots A_D\}$ affine functions such that the zero set of A_i is H_i, and $P^{\mathcal{H}} = A_1 A_2 \dots A_D$ the product of these affine functions. If μ is a measure in $\mathbb{R}^n$, we will say that $\mathcal{H}$ *bisects* μ if

$$\mu\left\{v \in \mathbb{R}^n : P^{\mathcal{H}}(v) > 0\right\} \le \frac{\mu(\mathbb{R}^n)}{2} \quad \text{and} \quad \mu\left\{v \in \mathbb{R}^n : P^{\mathcal{H}}(v) < 0\right\} \le \frac{\mu(\mathbb{R}^n)}{2}.$$

THEOREM 1. *Let n and D be integers such that $D > 0$ and $n > 1$ is a power of two. Given nD finite measures $\mu_1, \mu_2, \dots \mu_{nD}$ in $\mathbb{R}^n$, there exists an arrangement of at most D hyperplanes that bisects each of the measures.*

Observe that a family of $nD + 1$ delta masses based at a set of points, no $n + 1$ of which lie on the same hyperplane cannot be simultaneously bisected by less than $D + 1$ hyperplanes. Barba and Schnider [2] conjectured that the previous theorem holds for any n and confirmed this conjecture for the case of four measures in the plane ($n = D = 2$). Notice that the case $D = 1$ of this conjecture corresponds to the classical ham sandwich theorem (see [4] for many other ham sandwich type results).

Let us sketch informally the proof using the definitions that we'll introduce later. In Section 3 we show that the theorem follows from the case in which measures are *smoothings of odd number of delta masses in general position*. In Section 4 we show that if such a family of measures is *well separated* then the number of bisecting arrangements is odd. Finally in Section 5, we show that when we can deform general smoothings of oddly supported measures in general position to ones that are well separated, and through this deformation the parity of the number of bisecting arrangements remains invariant.

2. *Parametrisation of arrangements*

Parametrise hyperplanes in $\mathbb{R}^n$ by elements of $\mathbb{S}^n$ mapping $(a_0, a_1, \dots, a_n) \in \mathbb{S}^n$ to the affine function

$$A(x) = a_0 + a_1 x_1 + \cdots + a_n x_n,$$

and considering the zero set $A^{-1}(0)$. Parametrise hyperplane arrangements by elements of $(\mathbb{S}^n)^D$ mapping an element of $(\mathbb{S}^n)^D$ which corresponds to D affine functions $A_1, \dots, A_D$ to the polynomial of degree D given by

$$P^{\mathcal{H}}(x) = A_1(x) \cdots A_D(x),$$

with zero set $\mathcal{H} = \{A_1^{-1}(0), A_2^{-1}(0), \dots A_D^{-1}(0)\}$.

Let $\mathfrak{S}_D$ be the symmetric group of permutations of D elements, and $\mathbb{Z}/2^D$ be the D-fold product of the abelian group on two elements. Let $G = \mathfrak{S}_D \ltimes \mathbb{Z}/2^D$ be their semi-direct product. The group $\mathbb{Z}/2^D$ acts on $(\mathbb{S}^n)^D$ by the antipodal map $A \mapsto -A$ on each $\mathbb{S}^n$ factor, this action is free. The group $\mathfrak{S}_D$ acts on $(\mathbb{S}^n)^D$ by permuting the factors. Their semi-direct product acts by permuting and applying antipodal maps on some of the factors. The action of G is not free, its non-free part Σ corresponds to D-tuples $(A_1, \dots, A_D)$ such that $A_i = A_j$ or $A_i = -A_j$ for some $i \neq j$.

Notice that the the affine maps $A(x) = 1$ and $A(x) = -1$ correspond to degenerate hyperplanes "at infinity", which turn out to be irrelevant for us since they cannot bisect any measure.

3. *Approximation of measures*

In this section we construct a subspace of measures, which is a dense subset of $\mathcal{P}$, the space of Borel probability measures with the weak topology; and show that the theorem follows from its validity on this subspace by approximation. Denote by $\mathcal{P}^k$ the k-fold Cartesian product, whose elements are sets $\{\mu_1, \mu_2 \dots \mu_k\}$ of Borel probability measures in $\mathbb{R}^n$. The material that we need from measure theory is covered in many analysis books, see for instance [**5**, **7**].

LEMMA 2. *For $k, D > 0$, the set of ordered k-tuples of Borel measures that are not bisectable by D hyperplanes is open in $\mathcal{P}^k$.*

Proof. Assume that $M = (\mu_1, \dots, \mu_k) \in \mathcal{P}^k$ is a k-tuple of Borel probability measures that cannot be bisected by an arrangement of D hyperplanes $\mathcal{H}$. For any polynomial $P^{\mathcal{H}}$, there exists a sign $\pm$ and an $i \in \{1, \dots, k\}$ such that

$$\mu_i \left\{ \pm P^{\mathcal{H}} > 0 \right\} > 1/2.$$

From continuity of the measure μ_i we can choose an open set W whose closure is compact and is contained in $\{\pm P^{\mathcal{H}} > 0\}$ such that,

$$\mu_i(W) > 1/2.$$

By definition, the ordered k-tuples of Borel probability measures $M' = (\mu'_1, \ldots, \mu'_k)$ such that $\mu'_i(W) > 1/2$ constitute a neighbourhood $\mathcal{U} \ni M$ in the weak topology. The arrangements of hyperplanes $\mathcal{H}'$ such that $P^{\mathcal{H}'}$ is positive on the closure of W constitute a neighbourhood $\mathcal{V} \ni \mathcal{H}$ in the topology on the space of arrangements. Any pair of $M' \in \mathcal{U}$ and $\mathcal{H}' \in \mathcal{V}$ have the property that $\mathcal{H}'$ does not bisect M'.

Since the space of arrangements $(\mathbb{S}^n)^D$ is compact, a finite number of such $\mathcal{V}_1, \ldots, \mathcal{V}_N$ cover the whole space of arrangements. The intersection of the respective $\mathcal{U}_1, \ldots, \mathcal{U}_N$ produce a neighbourhood of M every member of which cannot be bisected with any arrangement of hyperplanes.

COROLLARY 3. *Theorem 1 for Borel measures follows from its validity on any dense subset of* $\mathcal{P}^k$.

Denote by δ_v the Dirac delta mass at the point v, i.e. for a Borel set X, $\delta_v(X) = 1$ if $v \in X$ and $\delta_v(X) = 0$ otherwise. We call measures of the form $1/N \sum_{k=1}^{N} \delta_{v_k}$ with odd N and $v_1, \ldots, v_N$ in general position, *oddly supported measures*. We say that a finite family of measures is in *general position* if no hyperplane intersects $n+1$ connected components of the union of their supports.

LEMMA 4. *Oddly supported measures in general position are dense in* $\mathcal{P}$. *Ordered* k*-tuples of oddly supported measures in general position are dense in* $\mathcal{P}^k$.

Proof. Assume the contrary, then there is a Borel probability measure μ whose weak neighborhood $\mathcal{V}$ contains no oddly supported measure. It is sufficient to consider $\mathcal{V}$ from the base of the weak topology given by a finite set of inequalities

$$\mu'(U_1) > m_1, \ldots, \mu'(U_\ell) > m_\ell$$

for an arbitrary finite collection of open sets $\mathcal{U} = \{U_1, U_2, \ldots U_\ell\}$ and real numbers $m_1, m_2, \ldots m_\ell$ such that $m_i < \mu(U_i)$. Let N be an odd number. Sample N points $v_1, v_2 \ldots v_N$ independently, distributed according to μ and consider the random measure

$$\nu_N = \frac{1}{N} \sum_{k=1}^{N} \delta_{v_k}.$$

The random variable $\nu_N(U_i)$ is given by,

$$\nu_N(U_i) = \frac{\#\{k = 1, \ldots, N : v_k \in U_i\}}{N}.$$

This is a sum of N independent Bernoulli random variables with expectation $\mu(U_i)$. By the law of large numbers $\nu_N(U_i)$ converges almost surely to $\mu(U_i)$. Hence for sufficiently large N the probability of satisfying the inequalities $\nu_N(U_i) > m_i$ simultaneously is arbitrarily close to 1. To make sure that the point set is in general position, we might perturb the points v_k so that none of them leave any U_i it belonged to and so that the perturbed measure $\nu_{N,\mathcal{U}}$

is still in $\mathcal{V}$ and yields the first statement. The second statement follows immediately, we do the same for k measures and take a single sufficiently large odd N. After that we perturb the total Nk support points so that none of them leaves any U (from the definition of a weak neighbourhood) it belonged to.

We now define η_v to be an ε-smoothing of the delta mass at v. Specifically η_v is a Borel probability measure centrally symmetric around v, which is supported inside a ball $B_v(\varepsilon)$ of radius ε centered at v and has a smooth density on all $\mathbb{R}^n$. Now take points in general position $v_1, \dots, v_N$ and consider a sum of ε-smoothings

$$\mu = \frac{1}{N}\sum_{k=1}^{N} \eta_{v_k}.$$

If N is an odd number and no $n+1$ tuple of the balls $B_{v_k}(\varepsilon)$ are intersected by a hyperplane, then we include μ in the set $\mathcal{M}_\varepsilon$. Finally we put $\mathcal{M} := \cup_{\varepsilon>0}\mathcal{M}_\varepsilon$, this is the set of measures we will work with.

LEMMA 5. *The set $\mathcal{M}$ is dense in the space of probability measures with the weak topology, moreover the set of ordered k-tuples of measures in general position in $\mathcal{M}^k$ is dense in $\mathcal{P}^k$*

Proof. For any oddly supported measure, we weakly approximate every delta mass δ_{v_k} by its respective η_{v_k} supported in the respective $B_{v_k}(\varepsilon)$. If ε is sufficiently small then no $n+1$ of the balls will be intersected by a single hyperplane. So $\mathcal{M}$ is dense in the space of oddly supported measures which by Lemma 4, is dense in $\mathcal{P}$. Similarly, $\mathcal{M}^k$ is dense in $\mathcal{P}^k$.

4. *Bisecting well separated sets of measures*

We say that a family of sets $X_1, X_2, \dots X_m$ in $\mathbb{R}^n$ is *well separated* if no n-tuple of their convex hulls $conv(X_1), conv(X_2), \dots conv(X_m)$ is intersected by an $(n-2)$-dimensional affine space. A family of measures is *well separated* if their supports are well separated. The following lemma was shown in [**1**] for absolutely continuous measures.

LEMMA 6. *For any family of n well separated measures in general position $\mathbb{R}^n$, each of them contained in $\mathcal{M}$, there exists a unique hyperplane H that bisects each of the measures.*

Proof. The existence of this hyperplane is provided by the ham sandwich theorem, we only need to show the uniqueness. Assume that each of the hyperplanes H and H' bisect every measure. Since the measures are well-separated, for some i, the intersection $H \cap H'$ does not intersect the convex hull of the support of μ_i. Both hyperplanes must intersect the interior of the support of μ_i, since μ_i was constructed from an oddly supported measure. Hence one of the half spaces $H_- \cap conv \operatorname{supp} \mu_i$ or $H_+ \cap conv \operatorname{supp} \mu_i$ strictly contains $H'_- \cap conv \operatorname{supp} \mu_i$ or $H'_+ \cap conv \operatorname{supp} \mu_i$, and therefore either H or H' cannot bisect μ_i.

The following lemma describes the bisecting arrangements of hyperplanes in the case when the measures are well separated.

LEMMA 7. *For any family of nD well separated measures in general position in $\mathbb{R}^n$, an arrangement of D unordered hyperplanes $\mathcal{H}$ is bisecting, if and only if, there is bijection*

φ between the elements of $\mathcal{H}$ and a partition of $[nD]$ into unordered n-tuples such that the hyperplane $H \in \mathcal{H}$ bisects the n-tuple of measures with indices in $\varphi(H)$.

Proof. Let $Y = \{Y_1, Y_2, \ldots Y_D\}$ be a partition of $[nD]$ into n-tuples. By Lemma 6, for each n-tuple Y_i, the corresponding measures are bisected by a unique ham sandwich cut, this defines a hyperplane arrangement $\mathcal{H}$ and a bijection $\varphi^{-1}: Y \to \mathcal{H}$. Since the measures are well separated, any measure with index not in $\varphi(H)$ is not intersected by H. So the arrangement bisects the measures. Conversely, since the supports are well separated, each hyperplane of a bisecting arrangement must intersect the supports of precisely n of the measures, otherwise at least one measure cannot be bisected. In this situation each hyperplane bisects n measures and does not touch the convex hulls of the supports of the remaining measures. By Lemma 6, such a hyperplane must be the unique ham sandwich cut of the corresponding n-tuple of measures.

Let $N(n, D)$ be the number of unordered partitions of a set of nD elements into D sets of n elements each. Lemma 7 implies that for a family of nD measures in general position there are exactly $N(n, D)$ unordered bisecting arrangements with D hyperplanes.

Clearly

$$N(n, D) = \frac{(nD)!}{D!(n!)^D},$$

but we will not use this formula. It is important to consider unordered partitions and unordered families of hyperplanes to have the following parity result:

LEMMA 8. *If n is a power of two then $N(n, D)$ is odd.*

Proof. Consider the action of a fixed 2-Sylow subgroup $S \subset \mathfrak{S}_{nD}$ on these partitions. To describe this Sylow subgroup consider a binary tree with 2^m leaves, where 2^m is the smallest power of two not smaller than nD. Fix a right descendant and a left descendant at every vertex that is not a leaf, now we can assign to each leaf a binary word of length m. Interpret these words as numbers in base 2, and drop the leaves that correspond to numbers strictly greater than nD. Drop also every vertex of the tree has no descendant leafs strictly greater than nD. Let S be the symmetry group of the remaining tree. Its embedding into $\mathfrak{S}_{nD}$ is obtained by looking at how S permutes the leaves of the tree. It is a 2-Sylow subgroup of $\mathfrak{S}_{nD}$ because by construction its order equals

$$2^{\sum_{k\geq 1} \lfloor nD/2^k \rfloor},$$

which is the largest power of two that divides $(nD)! = |\mathfrak{S}_{nD}|$.

For each unordered partition into n-tuples Y and each element $\sigma \in \mathfrak{S}_{nD}$, there is a natural unordered partition $\sigma(Y)$, defined by following the image of each element in $[nD]$. We will restrict this action to the subgroup S to count the parity of the number of partitions. Consider the unordered partition $Y = Y_1 \cup \cdots \cup Y_D$ for which each Y_i consists of consecutive elements. When n is a power of two, each Y_i corresponds to a full binary sub-tree. In this case the group S permutes transitively each Y_i while fixing all elements of the other Y_j, $j \neq i$. An unordered partition into n-tuples that is fixed by S must coincide with the chosen partition $Y_1 \cup \cdots \cup Y_D$, any other unordered partition is not stabilized under the S action, hence for any $Y' \neq Y$, we have that the orbit $orb_S(Y')$ equals $|S|$, which is even since S is a 2-group. Summing the sizes of all orbits we obtain that $N(n, D)$ must be odd.

5. *Proof of the Theorem*

By Lemmas 5 and Corollary 3 it is sufficient to prove the theorem for measures in $\mathcal{M}$ (smoothed oddly supported measures) in general position. Denote by $\mathcal{A}_i$ the support of the measure μ_i and by $C_i := \{c_{i_1}, c_{i_2}, \ldots c_{i_N}\}$ the set of centers of balls whose union is the support $\mathcal{A}_i$.

Denote by M the family $\{\mu_1, \mu_2, \ldots, \mu_{nD}\}$. Arguing similarly to Lemma 6 observe that for any family of nD measures in general position in $\mathcal{M}$ (not necessarily well-separated) a bisecting arrangement has to be a union of D hyperplanes each of which intersects a set of n connected components no two of which correspond to te same measure. We only need to count such arrangements.

We deform the measures μ_i smoothly to a situation where we can easily count the number of bisecting arrangements of the family. We use measures in $\mathcal{M}$ throughout, so we might prescribe the trajectories of the measures by prescribing the trajectory of the set of centers $\{C_i\}$ and choose $\varepsilon > 0$ later so that the smooth measures behave like points in general position. In the following all the objects that we deal with depend on $t \in [0, 1]$ which we call time, and denote this time with a subscript t. For each $t \in [0, 1]$ we consider a measure $\mu_{i,t} \in \mathcal{M}$ that depends continuously on t such that $\mu_{i,0} := \mu_i$ and the family at time $t = 1$, denoted $M_1 := \{\mu_{1,1}, \mu_{2,1}, \ldots \mu_{nD,1}\}$, is well separated and in general position. By Lemma 7 we know that the family M_1 has exactly $N(n, D)$ bisecting arrangements.

Let us describe the motion of M_t in more detail. We want to describe a *generic smooth trajectory* of measures in general position. Consider a set of points $b_1, b_2, \ldots b_{nD}$ in general position and choose $\alpha > 0$ so that the balls $B(b_i, \alpha)$ are well separated. Then move each of the points of the set of centers C_i towards b_i in such a way that each set C_i is always in general position within itself and, at the end the support of $\mu_{i,1}$ is contained in $B(b_i, \alpha)$. For example, the deformation could follow a homothecy with center b_i. By perturbing the speed of the trajectories if necessary, we can assume that at no moment of time there exist two $(n+1)$-tuples of connected components of the supports each of which is intersected by a hyperplane. In particular, at no time t, an $(n+2)$-tuple of connected components is intersected by a single hyperplane. To put it short, in a generic trajectory the events when some $n+1$ supporting balls of the measures can be intersected by a hyperplane come one by one.

Denote by Z_t the subset of points of $(\mathbb{S}^n)^D$ corresponding to bisecting arrangements of the family M_t. Our *crucial observation* is that Z_t does not touch the non-free part $\Sigma \subset (\mathbb{S}^n)^D$ because an assumed G-fixed point of Z_t corresponds to a set of hyperplanes in which two of the hyperplanes coincide. From the assumption on the generic trajectory it follows that we thus have at most $D-1$ distinct hyperplanes that intersect at least nD supporting balls of the measures in the set M_t. But there is a unique $(n+1)$-tuple of such balls that can be intersected by a single hyperplane, in all other situations the hyperplanes intersect at most n balls each. The inequality $n(D-1)+1 < nD$ thus implies that the non-free part of the space of arrangements is not touched during the motion.

Let us show that the parity of the number of bisecting arrangements stays invariant during the motion; then Lemma 8 delivers the result in the case we are interested in.

Consider the G-equivariant map $f : (\mathbb{S}^n)^D \times [0, 1] \to \mathbb{R}^{nD}$ given by

$$(f_t(x))_i = \mu_{i,t}\{P > 0\} - \mu_{i,t}\{P < 0\},$$

where P is the polynomial we associate to $x \in (\mathbb{S}^n)^D$. Since the measures have been smoothed f is a smooth equivariant map. Put $Z_t = f_t^{-1}(0) \subset (\mathbb{S}^n)^D \setminus \Sigma$ at time t, and observe that the theorem corresponds to $Z_0 \neq \emptyset$ so let us assume the contrary, that $Z_0 = \emptyset$.

The union of all such Z_t for $t \in [0, 1]$ is the preimage of zero $Z = f^{-1}(0)$, a closed subset of the product $(\mathbb{S}^n)^D \times [0, 1]$ not touching the non-free part of this product $\Sigma \times [0, 1]$. Denote the free part by

$$F = \left((\mathbb{S}^n)^D \setminus \Sigma\right) \times [0, 1]$$

for brevity. The quotient of F by the G-action is an open manifold F/G. The G-equivariant function f can be reinterpreted as a section of the vector bundle $F \times_G \mathbb{R}^{nD}$ over F/G. Now we approximate this section f in a neighbourhood of the quotient Z/G (not touching Σ) to obtain a smooth generic section f', and let $Z' := f'^{-1}(0)$. We mean "generic" in the sense of Thom's transversality theorem (see [**6**] or the textbook [**3**, pp. 68–69] or our comments in Appendix 6), that is f' is transversal to the zero section of the bundle and therefore Z' is a smooth submanifold of F/G. Notice that by construction at time 1 the set Z was already smooth and transversal to the zero section so we might assume $Z'_1 = Z_1$. When we unravel the action of G, we obtain a smooth G-equivariant function, which abusing notation we still denote by f'. Therefore $Z' := f'^{-1}(0)$ is a smooth equivariant one-dimensional compact submanifold with boundary, that is, a union of smooth topological circles and intervals with endpoints in $(\mathbb{S}^n)^D \times \{0, 1\}$. By Lemma 8 there is an odd number of G-orbits in $Z'_1 = Z_1$. From transversality it follows that a point in Z'_1 is the end point of exactly one topological interval in Z'. If $\gamma \subset Z'$ is a topological interval and there exists $g \in G$ such that $g(\gamma(0)) = \gamma(1)$, then g must map γ to itself and therefore $g(\gamma(t)) = \gamma(t)$ for some t, contradicting the fact that the action was free. Therefore the number of G-orbits of Z'_0 has the same parity as the number of G-orbits of Z'_1, in particular it is not zero, and Z'_0 is not empty. Since f' is arbitrary close to f, this implies that Z_0 is not empty.

Remark 9. The previous version of this paper (as a preprint on arxiv.org) incorrectly claimed Theorem 1 for any n. It was claimed that the cohomology class that was denoted there by e_i vanished on the complement of the set of arrangements of D hyperplanes bisecting a single measure. Actually the argument given there with the curve γ_i provides this fact for the class $\sum_{i=1}^{D} e_i$, the modulo two Euler class of the one-dimensional representation of $(\mathbb{Z}/2)^D$, on which each generator of every $\mathbb{Z}/2$ acts antipodally. The vanishing lemma implies that if $(e_1 + \cdots + e_D)^k$ is nonzero in the cohomology ring of the product of projective spaces, then for every k measures there exist an arrangement of D hyperplanes bisecting the measures. This in turn amounts to finding an odd multinomial coefficient, $\binom{k}{k_1\ k_2\ \ldots\ k_D} = \binom{k}{k_1} \cdot \binom{k-k_1}{k_2} \ldots \binom{k-k_1-\cdots-k_{D-1}}{k_D}$ with $k_1, \ldots, k_d \le n$. For such a coefficient to be odd, when we add the numbers in the sum $k_1 + \cdots + k_D$ in binary representation then no carry should occur. Consider the largest m such that, $2^m \le n$, then, we need $k_1 + \cdots + k_D \le 2^{m+1} - 1$. There is an example of such a sum with no carry if we put for $D \le m + 2$, $k_1 = 2^m$, $k_2 = 2^{m-1}, \ldots, k_{D-1} = 2^{m-D+2}$, $k_D = 2^{m-D+2} - 1$, and for $D \ge m + 2$, $k_1 = 2^m$, $k_2 = 2^{m-1}, \ldots, k_m = 2$, $k_{m+1} = 1$, $k_{m+2} = \cdots = k_D = 0$. From which we can conclude that we can bisect $2^{m+1} - 1 \le 2n - 1$ measures with at most 2 hyperplanes, and taking more hyperplanes, does not yield anything new with this technique (not using the permutations $\mathfrak{S}_D$).

On the other hand, if $2^m \le n$ and we have $2^m D$ measures in $\mathbb{R}^n$, we can project linearly to $\mathbb{R}^{2^m}$, apply Theorem 1 to obtain a bisecting arrangement of D hyperplanes in $\mathbb{R}^{2^m}$ and look at their inverse image, an arrangement of D hyperplanes in $\mathbb{R}^n$ that bisects the original measures. Since $2^m D > 2^{m+1} - 1$ in the nontrivial case $D \ge 2$, Theorem 1 always provides a better result then the above cohomological argument.

6. *Appendix. The usage of Thom's tranvsersality theorem*

In the book [4] a proof of the Borsuk–Ulam theorem is given that uses a similar technique to the one given in this paper. There, the parity argument is formalised using piece-wise linear topology. The crucial idea is the same as in this paper, we approximate the map we are interested in by a map for which the preimage of zero on the cylinder $M \times [0, 1]$ is a 1-manifold. We have chosen the more classical approach using the smooth category and relied on classical results on transversality. In this appendix we provide more details about this approach.

The version of Thom's transversality theorem we use corresponds to the case where the boundary is empty in [3, pp. 68–69], namely: if a smooth map between manifolds $f : M \times P \to N$ is transversal to a submanifold $L \subset N$ then, for almost every $p \in P$, the restriction of f to $M \times \{p\}$ is also transversal to L. Here f is called *transversal to L* if, for any $x \in M \times P$ such that $f(x) \in L$, the push-forward of the tangent space, $df\,(T_x(M \times P)) \subseteq T_{f(x)}N$, maps surjectively onto the quotient $T_{f(x)}N/T_{f(x)}L$. An important consequence of the transvsersality assumption is that $f^{-1}(L)$ is a submanifold of $M \times P$.

Thom's theorem is an easy consequence of Sard's theorem on regular values [3, appendix 1], which asserts that for any smooth map f between manifolds, if C is the set of critical points of f (the subset of the domain on which df is not surjective) then $f(C)$ has measure zero. In the proof of Thom's theorem, Sard's theorem is applied to the natural projection of the submanifold $f^{-1}(L) \subset M \times P \to P$.

In a situation like ours, we have a continuous section $s : B \to E$ of a smooth vector bundle $\xi : E \to B$, whose image intersects the zero section $L \subset E$ in a compact set Z (which we frequently consider as a subset of B), the standard way to invoke Thom's theorem is to modify the section s so that the new section s' becomes smooth and transversal to L and the intersection $s(B) \cap L$ remains compact. We can first smoothen the map s so that the zeros still remain a compact subset of B. To do so, we first choose a locally finite at most countable open cover $\{U_i\}$ of B by coordinate neighbourhoods, this can be done along with choosing compact $C_i \subset U_i$ so that $\{C_i\}$ is still a cover of B. Then, for all i, we work in coordinates and modify s inside U_i so that its modification is supported inside U_i, s remains smooth where it was smooth, becomes smooth over C_i, and does not acquire zeros on this or on a neighbouring set C_j if it had no zeros there previously. Notice that the last assumption is satisfied when the modification is sufficiently small. Such a modification can be obtained approximating s by a convolution with a smooth kernel and then multiplying the modification by a bump function supported in U_i and equal to 1 over C_i. In total, the infinite sequence of such modifications is locally finite and makes s smooth, keeping its zero set in a compact subset of B.

Now we work with a smooth s with zeros inside a compact $Z \subseteq B$. We use compactness of Z to find a finite number of compactly supported sections $\sigma_i : B \to E$ such that in a neighborhood $U \supset Z$, for any point $x \in U$, the values $\sigma_i(x)$ span the fiber of the bundle ξ. We may also assume that the union of the supports of the σ_i has a compact closure $C \supset U$. Assuming that the vector bundle has some Euclidean metric on the fibers, we choose the minimal $|s(x)|$ over the compactum $C \setminus U$, this minimum is not zero because $U \supset Z$, let this minumum be $\varepsilon K > 0$. Assume also that $|\sigma_i(x)| \le 1$ everywhere.

Now consider the combination

$$s(x) + p_1\sigma_1(x) + \cdots + p_K\sigma_K(x)$$

as a smooth map $S : B \times (-\varepsilon, \varepsilon)^K \to E$, that is $-\varepsilon < p_i < \varepsilon$ are variables. From the assumption on the σ_i and ε it follows that S is transversal to L. Indeed, for $x \in B \setminus C$, the section s is nonzero while $\sigma_i(x)$ is zero for each i, hence the image of S does not intersect L. From the choice of ε, it follows that for $x \in C \setminus U$, the image of S does not intersect L. For $x \in U$, the map S is transversal to L by the choice of the compactly supported sections σ_i. Applying Thom's transversality theorem we conclude that for almost every choice of the parameters $p_1, \dots, p_K$ the corresponding section $s' : B \to E$ is transversal to L. Moreover, its zero set is contained in a compactum C and it is therefore compact itself.

Acknowledgements. We acknowledge and thank Pavle V. M. Blagojević for corrections and useful comments. R.K. was supported by the Federal professorship program grant 1.456.2016/1.4 and the Russian Foundation for Basic Research grants 18-01-00036 and 19-01-00169.

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