MATHEMATICS

WELL-PARTIAL ORDERINGS AND HIERARCHIES

BY

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§ 1. INTRODUCTION

Hierarchies in recursive function theory are typically constructed in the following manner. One starts with a basic stock I of functions, say the successor function, the zero function, etc. One also has some operations like composition, primitive recursion, etc. which yield functions from (one or more) given functions. Given a class C of functions already at hand, and an operation F, one can extend C to a new class FC, the closure of C under F. If G is another operation, then unless G and F commute FC will not be closed under G even if C itself was. Hence one can extend FC to GFC etc. Thus one gets a family $\mathscr{F}$ of classes obtained from the original stock I and some subfamily of $\mathscr{F}$ is our hierarchy. Hierarchies of Grzegorczyk, Ritchie, Cleave etc. follow this pattern.

What is interesting about this situation is that we are really playing with a partial ordering of strings, already investigated by Higman [H], and applying Higman's results to the present situation one can immediately conclude that the hierarchy will necessarily be well-ordered. Moreover, it follows from our theorem 3.11 that if we have k closure operations, then the maximum ordinal for such a hierarchy is $\omega^{\omega^{k-2}}$.

The present paper is divided into three sections. In this section, below, we shall define well-partial orderings and explain how the results of Higman and our own results apply to hierarchies. Section 2 is principally devoted to proving (theorem 2.13) that the ordinal $o(X, \leqslant)$ associated with a well-partial ordering $\leqslant$ on X is actually reached by an extension $\leqslant'$ of $\leqslant$. In section 3, we go into the question of calculating $o(X, \leqslant)$ in some specific cases, including the case of strings mentioned above, which is needed for our application to hierarchies.

Some further results on well-partial orderings will appear in separate paper by the Jongh.

DEF. 1.1. Let $\langle X, \leqslant\rangle$ be a partially ordered set. A subset $Y \subseteq X$ is said to be *closed* if $a \in Y$ and $a \leqslant b$ imply $b \in Y$. The *closure* $Cl(Z)$ of a set Z is the smallest $Y \supseteq Z$ such that Y is closed.

THEOREM 1.2. The following conditions on a partially ordered set $\langle X, \leqslant\rangle$ are equivalent:

(1) Every closed subset of X is the closure of a finite subset of X.

(2) The ascending chain condition holds for closed subsets of X.

(3) If B is any subset of X, then there is a finite $B_0 \subseteq B$ such that $B \subseteq Cl(B_0)$.

(4) Every infinite sequence in X has a (weakly) increasing infinite subsequence.

(5) If $a_1, a_2, \ldots$ is a sequence in X, then there are i and j such that $i<j$ and $a_i \leqslant a_j$.

(6) $\leqslant$ is well-founded and every subset of X of mutually incomparable elements is finite.

(7) If $\leqslant'$ is a linear order on X, extending $\leqslant$, then $\leqslant'$ is a well-order.

Theorem 1.2 is essentially theorem 2.1 of Higman [H], (cf. also e.g., Kruskal [K]).

Def. 1.3. A partial order $\langle X, \leqslant\rangle$ will be called a *well-partial order* (wpo) if it satisfies any (and hence all) of the conditions of theorem 1.2 above.

Notation. If $\langle X, \leqslant\rangle$ is a well-order, then $|X, \leqslant| = |\leqslant| =$ the order type of $\leqslant$.

We shall now associate a natural ordinal with all the well-founded partial orderings.

Def. 1.4. Let $\langle X, \leqslant\rangle$ be well-founded. Then $o(X, \leqslant) = \sup(|X, \leqslant'| : \leqslant'$ is a well-order on X extending $\leqslant)$. If the context provides $\leqslant$, then $o(X, \leqslant)$ will be written $o(X)$.

Def. 1.5. Let $\Sigma^* =$ the set of all finite strings on the alphabet Σ. We define the embedding order $\preceq$ on Σ^* by letting $x \preceq y$ if there exist strings $x_i (1 \leqslant i \leqslant n)$ and $y_i (1 \leqslant i \leqslant n+1)$ such that $x = x_1 x_2 \ldots x_n$ and $y = y_1 x_1 y_2 \ldots x_n y_{n+1}$. (The y_i, of course, may be empty and thus $x \preceq x$ holds).

Theorem 1.6 (Higman). If Σ is finite, then the embedding order $\preceq$ on Σ^* is a *wpo*.

We now indicate the connection between strings and hierarchies. If F is an operation on functions yielding functions and C is a class of functions then FC is the closure of C under F.

Lemma 1.7. (a) $C \subseteq FC$. (b) If $C \subseteq D$, then $FC \subseteq FD$. (c) $FFC = FC$.

Proof. Trivial.

Now let $F_1, \ldots, F_k$ be any operations from classes of functions to classes of functions which satisfy conditions 1.7 (a), (b) above. (It is not necessary that the F_i come from operations on functions themselves, nor even is it essential that the classes are classes of functions.) Let $\Sigma = \{F_1, \ldots, F_k\}$. For a class C and $x \in \Sigma^*$ we define C_x by induction on x.

DEF. 1.8. (a) $C_\wedge = C$ where $\wedge$ is the empty string. (b) If $x = F_t y$, then $C_x = F_t C_y$.

LEMMA 1.9. If $x,\ y \in \Sigma^*$, $x \leqslant y$ and $C \subseteq D$, then $C_x \subseteq D_y$, (i.e. the operation $C,\ x \to C_x$ is monotone in both C and x).

PROOF. By induction on x.

If $x = \wedge$, then $C_x = C \subseteq D$ and by 1.7(a), for all y, $D \subseteq D_y$. Hence $C_x \subseteq D_y$.

Suppose now that $x = x_1 x_2 \ldots x_n$ and $y = y_1 x_1 x_2 \ldots x_n y_{n+1}$. We can assume that $x_1 \neq \wedge$. Let $x' = x_2 \ldots x_n$, $y' = y_2 x_2 \ldots x_n y_{n+1}$ and $y'' = x_1 y_2 \ldots x_n y_{n+1}$. By induction hypothesis, $C_{x'} \subseteq D_{y'}$. Applying 1.7(b) several times we get $C_x \subseteq D_{y''}$. But by 1.7(a), $D_{y''} \subseteq D_y$. Hence $C_x \subseteq D_y$.

Now let I be some fixed initial class of functions. We are interested in classes I_x for $x \in \Sigma^*$.

THEOREM 1.10. Let $\mathscr{F}$ be a linearly ordered family of classes I_x. Then $\mathscr{F}$ is well-ordered and $|\mathscr{F}, \subseteq| \leqslant o(\Sigma^*, \preceq)$.

PROOF. Choose a set $Y \subseteq \Sigma$ such that the map $x \to I_x$ is $1-1$ from Y onto $\mathscr{F}$.

For $x,\ y \in Y$ let $x \leqslant y$ iff $I_x \subseteq I_y$. Then $\leqslant$ is a linear order on Y extending $\preceq$. By cor. 2.3., $\preceq$ is a *wpo* on Y and $o(Y, \leqslant) \leqslant o(\Sigma^*, \preceq)$. Thus $\mathscr{F}$ is well-ordered by $\subseteq$ and $|\mathscr{F}, \subseteq| = |Y, \leqslant| \leqslant o(Y, \leqslant) \leqslant o(\Sigma^*, \preceq)$.

This gives us by theorem 3.11 a bound of $\omega^{\omega^{k-1}}$ for hierarchies with k closure operations. However, for such hierarchies, condition 1.7(c) holds as well and any two successive applications of the same element of Σ collapse into one. It is not hard to see that this reduces the bound to $\omega^{\omega^{k-2}}$.

COR. 1.11. Any hierarchy obtained by means of a set of initial functions and closure operations has an ordinal less than $\omega^{\omega^{\omega}}$.

§ 2. SOME BASIC PROPERTIES OF WPO'S

Throughout this section, unless otherwise stated, $\leqslant$, $\leqslant'$, $\leqslant_1$ etc. are well-partial orders. The main result of this section is theorem 2.13, that the ordinal $o(X, \leqslant)$ defined in 1.4 is actually attained by a well-order $\leqslant'$ extending $\leqslant$.

DEF. 2.1. If $x \in X$, then $L_X(x) = \{y \in X | x \nleqslant y\}$, $U_X(x) = \{y \in X | x \leqslant y\} = X - L_X(x)$, $l_X(x) = o(L_X(x))$, $u_X(x) = o(U_X(x))$. The subscript X will be dropped, if X is clear from the context.

LEMMA 2.2. Let Y be partially ordered by $\leqslant_1$, Z a subset of Y and $\leqslant_1'$ the restriction of $\leqslant_1$ to Z. Let $\leqslant_2$ be a partial order on Z extending $\leqslant_1'$. Then there is a partial order $\leqslant$ on Y which extends both $\leqslant_1$ and $\leqslant_2$.

PROOF. Let $\leqslant$ be the transitive closure of $\leqslant_1 \cup \leqslant_2$. It is reflexive, transitive and includes both $\leqslant_1$ and $\leqslant_2$. So, suppose it is not antisymmetric. Then there is a chain $x_1, x_2, \ldots, x_n, x_1$ such that each element is related to the next one by $\leqslant_1$ or $\leqslant_2$. Choose n least possible. Then, since $<_1$, $<_2$ are transitive, neither was used twice in succession, and they must have been used alternately. Hence, n is even, and our chain looks like: $a_1 <_2 b_1 <_1 a_2 <_2 b_2 \ldots b_m <_1 a_1$, where $a_1 = x_1$ and $n = 2m$. Since, for each $i (1 \leqslant i \leqslant m)$, $a_i <_2 b_i$, a_i and b_i are both in Z and the whole chain is a chain of $<_2$ inside Z. This is a contradiction. Thus $\leqslant$ is antisymmetric and the required partial order.

Cor. 2.3. If $\langle Y, \leqslant \rangle$ is a *wpo* and $Z \subseteq Y$, then $\langle Z, \leqslant \rangle$ is a *wpo* (as noted in [H]) and $o(Z, \leqslant) \leqslant o(Y, \leqslant)$.

Proof. Let $\alpha < o(Z, \leqslant)$. Then there is a linear order $\leqslant'$ extending $\leqslant$ on Z whose ordinal is $> \alpha$. By 2.2 above and the fact that each partial order can be extended to a linear order, one can extend $\leqslant'$ to a linear order $\leqslant''$ on Y which extends $\leqslant$ on Y. Then $\alpha < |\leqslant'| \leqslant |\leqslant''| \leqslant o(Y, \leqslant)$. As $\alpha < o(Z, \leqslant)$ is arbitrary, $o(Z, \leqslant) \leqslant o(Y, \leqslant)$.

Cor. 2.4. $l_X(x) \leqslant o(X)$, $u_X(x) \leqslant o(X)$.

Proof. Immediate from 2.3 above.

Notation. If $\leqslant'$ is a well-order on X and $x \in X$, then

$$\mathrm{seg}_{\leqslant'}(x) = \{y \in X \mid y <' x\}$$

and

$$|x|_{\leqslant'} = |\mathrm{seg}_{\leqslant'}(x), \ \leqslant' \restriction \mathrm{seg}_{\leqslant'}(x)|.$$

Lemma 2.5. If $x \in X$ and $\leqslant'$ is a well-order extending $\leqslant$, then

$$|x|_{\leqslant'} \leqslant l(x).$$

Proof. Trivial.

Lemma 2.6. If $o(X)$ is a limit number, then $o(X) = \sup_{x \in X} l(x)$.

Proof. By 2.4. above $o(X) \geqslant \sup_{x \in X} l(x)$.
Let $\alpha < o(X)$. Then, for some linear order $\leqslant'$ extending $\leqslant$ on X, $|X, \leqslant'| > \alpha$ and, therefore, for some $x \in X$, $|x|_{\leqslant'} = \alpha$. By 2.5 above, then $\alpha \leqslant l(x)$. Since $\alpha < o(X)$ is arbitrary, this implies $o(X) \leqslant \sup_{x \in X} l(x)$.

Lemma 2.7. (a) If $A \subseteq \omega^\alpha$ and $|A, \leqslant| < \omega^\alpha$, then $|\omega^\alpha - A, \leqslant| = \omega^\alpha$ (where $-$ is set substraction). (b) If $A \subseteq \omega^\alpha.\ n$ and $|A, \leqslant| < \omega^\alpha \cdot m$; then $|\omega^\alpha \cdot n - A, \leqslant| > \omega^\alpha (n - m)$.

Proof. The proof is by simultaneous induction on α for both (a), (b) Obvious, when $\alpha = 0$.

(1) 2.7 (a) for $\alpha \Rightarrow$ 2.7 (b) for α. Let, for each $i (1 \leqslant i \leqslant n)$,
$X_i = \omega^\alpha \cdot i - \omega^\alpha (i-1)$, $A_i = A \cap X_i$ and $B_i = X_i - A_i$.

Then, since $|A, \leqslant| < \omega^{\alpha}\cdot m$, at most $m-1$ of the A_i have type ω^{α} and hence at least $n-m+1$ of the B_i have type ω^{α}, by 2.7. (a) for α. But $\omega^{\alpha}-A = B_1 + \ldots B_n$. Hence $|\omega^{\alpha}-A, \leqslant| \geqslant \omega^{\alpha}(n-m+1) > \omega^{\alpha}(n-m)$.

(2) 2.7 (b) for $\alpha \Rightarrow$ 2.7 (a) for $\alpha+1$.

Suppose $A \subseteq \omega^{\alpha+1} = \omega^{\alpha}\cdot\omega$. Let A_i, B_i, X_i be as in (1) above. Since $|A, \leqslant| < \omega^{\alpha+1}$, there is an m such that $|A, \leqslant| < \omega^{\alpha}\cdot m$. Hence $|\omega^{\alpha}(n+m)-A, \leqslant|$ exceeds $\omega^{\alpha}\cdot n$ for all n. But then $|\omega^{\alpha+1}-A, \leqslant| \geqslant |\omega^{\alpha}(n+m)-A, \leqslant| > \omega^{\alpha}\cdot n$ for all n and hence is at least $\omega^{\alpha+1}$.

(3) If γ is a limit number and 2.7. (a) holds for all $\alpha<\gamma$, then 2.7. (a) holds for γ. For let $|A, \leqslant| < \omega^{\gamma}$. Then there is an $\alpha<\gamma$ such that $|A, \leqslant| < \omega^{\alpha}$. Hence, for all $\beta(\alpha<\beta<\gamma)$, $|\omega^{\beta}-A, \leqslant| = \omega^{\beta}$. Hence, for all $\beta(\alpha<\beta<\gamma)$, $|\omega^{\gamma}-A, \leqslant| \geqslant \omega^{\beta}$. Thus $|\omega^{\gamma}-A, \leqslant|$ is at least ω^{γ}.

Def. 2.8. An element $x \in X$ such that $l(x)=o(X)$ will be called *superfluous*. We shall show later that superfluous elements do not exist in *wpo*'s, but for the moment we have to take into account the possibility that they do exist.

Lemma 2.9. For each $\langle X, \leqslant\rangle$ there is a $Y \subseteq X$ such that $o(Y)=o(X)$ and Y contains no superfluous elements.

Proof. We define sequences $X_0, X_1, \ldots$ and $x_0, x_1, \ldots$ as follows. $X_0 = X$ and for each n, if X_n contains a superfluous element, then let x_n be such an element and let $X_{n+1} = L_{X_n}(x_n)$. It is clear that, for each n, $o(X_n) = X$; moreover, the construction has to break off after a finite number of steps, since $i<j$ implies $x_i \nleqslant x_j$. Hence, for some m, we will obtain an X_m with no superfluous elements and we can choose $Y = X_m$.

Lemma 2.10. If $o(X) = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k} (\alpha_1 \geqslant \ldots \geqslant \alpha_k)$, $x \in X$ and $u(x) < \omega^{\alpha_k}$, then x is superfluous.

Proof. Let x and X be as assumed. We distinguish between the cases that α_k is a limit number and a successor.

Case I. α_k is a limit number. Then $u(x) < \omega^{\beta}$ for some $\beta < \alpha^k$. There will be a well-ordering $\leqslant'$ extending $\leqslant$ such that

$$|X, \leqslant'| \geqslant \omega^{\alpha_1} + \ldots + \omega^{\alpha_{k-1}} + \omega^{\beta}.$$

By the assumptions $|U_X(x), \leqslant'| < \omega^{\beta} < \omega^{\alpha_j}$ for each $j(1 \leqslant j \leqslant k-1)$ and therefore, by lemma 2.7. (a).

$$|L_X(x), \leqslant'| = |X - U_X(x), \leqslant'| \geqslant \omega^{\alpha_1} + \ldots + \omega^{\alpha_{k-1}} + \omega^{\beta}.$$

In this argument each $\gamma(\beta \leqslant \gamma < \alpha_k)$ can be substituted for β. Therefore $l(x) = o(X)$.

Case II. $\alpha_k = \beta+1$. In this case $\omega^{\alpha_k} = \bigcup_{i\in\omega} \omega^{\beta}\cdot i$. Let $u(x) \leqslant \omega\cdot j$ and let $i > j (i, j \in \omega)$. There is a well-order $\leqslant'$ extending $\leqslant$ such that

$$|X, \leqslant'| \geqslant \omega^{\alpha_1} + \ldots + \omega^{\alpha_k} + \omega^{\beta} \cdot i.$$

Applying Lemma 2.7. b) and a) we obtain

$$|L_X(x), \leqslant'| \geqslant \omega^{\alpha_1} + \ldots + \omega^{\alpha_{k-1}} + \omega^{\beta}(i-j).$$

Again $i > j$ is arbitrary, so $l(x) = o(X)$.

The next two lemmas are needed only in the proof of theorem 2.13 in the case that $o(X)$ is uncountable.

LEMMA 2.11. If X contains no superfluous elements, then there is a $Y \subseteq X$ such that $\cup \{l(y) | y \in X\} = o(X)$, but for each $z \in Y$,

$$\cup \{l(y) | y \in L_Y(z)\} < o(X).$$

PROOF. As in the proof of lemma 2.9 we construct sequences $X_0, X_1, \ldots$ and $x_0, x_1, \ldots$ Again $X_0 = X$ and $X_{n+1} = L_{X_n}(x_n)$. Here however we take x_n to be an element of X_n such that $\cup \{l(y) | y \in L_{X_n}(x_n)\} = o(X)$ if such an element exists. This procedure has to break off again for a certain m and we can take $Y = X_m$.

LEMMA 2.12. Let $Y \subseteq X$ fullfill the conditions of lemma 2.11. Let $\eta = \operatorname{cof}(o(X))$, and, for $\theta < \eta$, θ a limit, let $\{y_\xi\}_{\xi < \theta}$ be a strictly increasing sequence of elements in Y. Then there is an element $y_\theta \in Y$ such that $y_\xi < y_\theta$ for all $\xi < \theta$.

PROOF. It is obvious that, in Y, $x \leqslant y$ iff $L_Y(x) \subseteq L_Y(y)$ and iff $U_Y(y) \subseteq U_Y(x)$. Therefore, to show that under the conditions of the lemma y_θ exist in Y, it is sufficient to show that

$$\bigcap_{\xi < \theta} U_Y(y_\xi) \neq \phi \text{ or that } \bigcup_{\xi < \theta} L_Y(y_\xi) \neq Y.$$

For that purpose, define, for each $\xi < \theta$ the ordinal

$$\mu_\xi = \cup \{l_X(y) | y \in L_Y(y_\xi)\}$$

Then, by the conditions of lemma 2.11, $\mu_\xi < o(X)$.

Furthermore, since $\theta < \eta = \operatorname{cof}(o(X))$, $\bigcup_{\xi < \theta} \mu_\xi < o(X)$. But this implies

$$\cup \{l_X(y) | y \in \bigcup_{\xi < \theta} L_Y(y_\xi)\} = \bigcup_{\xi < \theta} \cup \{l_X(y) | y \in L_Y(y_\xi)\} =$$

$$\bigcup_{\xi < \theta} \eta_\xi < o(X) = \bigcup_{y \in Y} l_X(y).$$

So, indeed, $\bigcup_{\xi < \theta} L_Y(y_\xi) \neq Y$.

THEOREM 2.13. For each *wpo* $\langle X, \leqslant \rangle$, there is a well-ordering $\leqslant'$ of X extending $\leqslant$ such that $|X, \leqslant'| = o(X, \leqslant)$.

PROOF. By induction on $o(X)=\omega^{\alpha_1}+\ldots+\omega^{\alpha_k}$ $(\alpha_1\geqslant\ldots\geqslant\alpha_k)$. If $o(X)$ is a successor the result is trivial. So we can assume $\alpha_k\geqslant 1$. Using lemma 2.2 a well-ordening of length $o(x)$ of any subset of X as guaranteed by lemma 2.9 can be extended to the whole of X. This means that X can be assumed to be free of superfluous elements. Then by lemma 2.10, for each $x\in X$, $u(x)\geqslant\omega^{\alpha_k}$.

Let $\leqslant'$ be a well-order extending $\leqslant$ such that $|X, \leqslant'|>\omega^{\alpha_1}+\ldots+\omega^{\alpha_{k-1}}$ and let $|x|_{\leqslant'}=\omega^{\alpha_1}+\ldots+\omega^{\alpha_{k-1}}$. From the obvious fact that $|x|_{\leqslant'}+u(x)\leqslant o(X)$, we obtain $u(x)\leqslant\omega^{\alpha_k}$ and therefore $u(x)=\omega^{\alpha_k}$. This implies that it is sufficient to extend $\leqslant$ to a well-ordering $\leqslant_1$ on $U(x)$ such that $|U(x), \leqslant_1|=\omega^{\alpha_k}$. In other words, without loss of generality we can assume that $o(X)=\omega^\alpha$ for some $\alpha\geqslant 1$.

Finally let us assume that Y fulfills the conditions of 2.12. We distinguish the cases that α is a successor and a limit ordinal.

CASE I. $\alpha=\beta+1$. In this case $\omega^\alpha=\omega^\beta\cdot\omega$ and $\operatorname{cof}(\omega^\alpha)=\omega$. Let $\{\gamma_i\}_{i\in\omega}$ be a strictly increasing sequence of ordinals such that $\gamma_0=0$ and $\bigcup_{i\in\omega}\gamma_i=\omega^\alpha$. We will define a strictly increasing sequence $\{x_i\}_{i\in\omega}$ in X and a well-ordering $\leqslant'$ extending $\leqslant$ such that, for each $i\in\omega$ $|L(x_i), \leqslant'|\geqslant\gamma_i$. This will obviously be sufficient. We can take x_0 to be a minimal element of X.

Assume x_n and $\leqslant'$ on $L(x_n)$ have been defined in such a way that $|L(x_n), \leqslant'|\geqslant\gamma_n$. Note that $u(x_n)=\omega^\alpha$. Hence, there is a well-order $\leqslant_1$ extending $\leqslant$ on $U(x_n)$ such that $|U(x_n), \leqslant_1|>\gamma_{n+1}$. We can now take x_{n+1} such that $|x_{n+1}|_{\leqslant_1}=\gamma_{n+1}$ and we extend $\leqslant'$ to $L(x_{n+1})-L(x_n)$ by identifying $\leqslant'$ with $\leqslant_1$ on that set. Now, indeed $|L(x_{n+1}), \leqslant'|\geqslant\gamma_{n+1}$.

CASE II. α is a limit. Let $\eta=\operatorname{cof}(\omega^\alpha)$ and let $\{\gamma_\xi\}_{\xi<\eta}$ be a sequence of ordinals such that $\gamma_0=0$ and for each limit number $\delta<\eta$, $\gamma_\delta=\bigcup_{\xi<\delta}\gamma_\xi$ and $\bigcup_{\xi<\eta}\gamma_\xi=\omega^\alpha$. (Note that, in case $\operatorname{cof}(\omega^\alpha)=\omega$, as is the case for all countable ordinals, we could use the same proof as in case I).

We will define a strictly increasing sequence $\{y_\xi\}_{\xi<\eta}$ of elements of Y and a well-order $\leqslant'$ of X such that, for each $\xi<\eta$, $|L_X(y_\xi), \leqslant'|\geqslant\gamma_\xi$. We can take y_0 to be a minimal element of Y. If $\delta<\eta$ is a limit number, then we just have to insure that there is a $y_\delta\in Y$ such that, for each $\xi<\delta$, $y_\xi<y_\delta$, but this follows from 2.12. If $\xi=\zeta+1$, it is sufficient to find a $y_\xi>y_\zeta$ and a well-ordering $\leqslant_1$ extending $\leqslant$ such that

$$|L_X(y_\xi)-L_X(y_\zeta), \leqslant_1|\geqslant\gamma_\xi.$$

We first note that, by the properties of Y

$$\cup\{l_X(y)|y\in L_Y(y_\xi)\}<\omega^\alpha, \text{ so } \cup\{l_X(y)|y\in U_Y(y_\xi)\}=\omega^\alpha.$$

Since α is a limit number this means that there exist a y_ξ and a β such that $l_X(y_\xi)\geqslant\omega^\beta>l_X(y_\zeta)$ and $\omega^\beta\geqslant\gamma_\xi$. Let $\leqslant_1$ be a well-ordering of $L_X(y_\xi)$ such that $|L_X(y_\xi), \leqslant_1|\geqslant\omega^\beta$. Since $l(y_\zeta)<\omega^\beta$, by 2.7 (a),

$$|L_X(y_\xi)-L_X(y_\zeta), \leqslant_1|\geqslant\omega^\beta\geqslant\gamma_\xi, \text{ as required.}$$

COR. 2.14. For each $x \in X$, $l(x)+u(x) \leqslant o(X)$ and hence $l(x)<o(X)$, i.e. superfluous elements do not exist.

PROOF. Apply theorem 2.13 to $L(x)$ and $U(x)$ to obtain $\leqslant_1$ and $\leqslant_2$ extending $\leqslant$ on $L(x)$ and $U(x)$ respectively such that $|L(x), \leqslant_1|=l(x)$ and $|L(x), \leqslant_2|=u(x)$. Finally define $\leqslant'$ to agree with $\leqslant_1$ on $L(x)$, with $\leqslant_2$ on $U(x)$ and such that $y \leqslant' z$ for each $y \in L(x)$, $z \in U(x)$.
$\leqslant'$ is a well-order of length $l(x)+u(x)$ extending $\leqslant$ on X.

COR. 2.15. If $o(X)=\omega^{\alpha_1}+\ldots+\omega^{\alpha_k}(\alpha_1 \geqslant \ldots \geqslant \alpha_k)$ and $x \in X$, then $u(x) \geqslant \omega^{\alpha_k}$.

PROOF. Immediate from 2.14 and 2.10.

COR. 2.16. If $o(X)=\omega^{\alpha_1}+\ldots+\omega^{\alpha_k}$, then there is an $x \in X$ such that $u(x)=\omega^{\alpha_k}$ and $\omega^{\alpha_1}+\ldots+\omega^{\alpha_{k-1}} \leqslant l(x)$.

PROOF. Let $\leqslant'$ be a well-order on X as provided by theorem 2.13 and let $|x|_{\leqslant'}=\omega^{\alpha_1}+\ldots+\omega^{\alpha_{k-1}}$, then x fulfills the required conditions.

COR. 2.17. If $o(X)=\omega^{\alpha_1}+\ldots+\omega^{\alpha_k}$, then, for some $m \in \omega$, there are $x_1, \ldots, x_m \in X$ such that $o(U(x_1) \cup \ldots \cup U(x_m))=\omega^{\alpha_k}$ and

$$o(L(x_1) \cap \ldots \cap L(x_m))=\omega^{\alpha_1}+\ldots+\omega^{\alpha_{k-1}}.$$

PROOF. We define a sequence $x_1, x_2, \ldots$. Choose x_1 as x in the proof of 2.16. Assume $x_1, \ldots, x_n$ are such that $o(U(x_1) \cup \ldots \cup U(x_n))=\omega^{\alpha_k}$ and $o(L(x_1) \cap \ldots \cap L(x_n))>\omega^{\alpha_1}+\ldots+\omega^{\alpha_{k-1}}$. Then there is a well-ordering $\leqslant'$ of $L(x_1) \cap \ldots \cap L(x_n)$ and a $y \in L(x_1) \cap \ldots \cap L(x_n)$ such that $|y|_{\leqslant'}= =\omega^{\alpha_1}+\ldots+\omega^{\alpha_{k-1}}$.

Choose x_{n+1} to be y. Then $o(U(x_1) \cup \ldots \cup U(x_{n+1}))=\omega^{\alpha_k}$. It is clear that, for some $m \in \omega$, $o(L(x_1) \cap \ldots \cap L(x_m))$ will be equal to $\omega^{\alpha_1}+\ldots+\omega^{\alpha_{k-1}}$, otherwise we would obtain an infinite sequence of the wrong kind in X.

§ 3. COMPUTATION OF $o(X)$

In this section we will compute the value of $o(X)$ for some specific *wpo*'s $\langle X, \leqslant\rangle$. In the sequel we will always assume that X and Y are disjoint.

DEF. 3.1. $X+Y$ and $X \times Y$ denote the disjoint union and the cartesian product respectively of X and Y. If $\leqslant_1$, $\leqslant_2$ are *wpo*'s on X, Y respectively, then $\leqslant_1+\leqslant_2$ is the disjoint union of $\leqslant_1$, $\leqslant_2$ and $\leqslant_1 \times \leqslant_2$ is the order defined on $X \times Y$ by letting $(x, y) \leqslant (x', y')$ iff $x \leqslant_1 x'$ and $y \leqslant_2 y'$. If $n \in \omega$, X^n will denote the obvious cartesian product and $X^*=\bigcup_{n \in \omega} X^n$. We define $\leqslant_1{}^*$ on X^* by letting $a \leqslant_1{}^* b$ iff $a=(a_1, \ldots, a_m)$,

$b=(b_1, \ldots, b_n)$ and there is a strictly increasing ϕ from $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$ such that, for all $i (1 \leqslant i \leqslant m)$, $a_i \leqslant b_{\phi(i)}$.

THEOREM 3.2. $o(X)$ is a successor ordinal iff, for some $x \in X$, $U_X(x)=\{x\}$, in which case $o(X)=l(x)+1$.

PROOF. $\Rightarrow$ Let $o(X)=\alpha+1$. For some well-order $\leqslant'$ extending $\leqslant$, $|X, \leqslant'|=\alpha+1$. Let $x \in X$ be the last element of this well-order. It is clear that $U(x)=\{x\}$.

$\Leftarrow$ Let $x \in X$ and $U(x)=\{x\}$. By Cor. 2.14, $l(x)+1 \leqslant o(X)$. For some well-order $\leqslant'$ extending $\leqslant$, $|X, \leqslant'|=o(X)$. By Lemma 2.5, $|X-\{x\}, \leqslant'| \leqslant l(x)$. Since to this last ordering of $X-\{x\}$ we can add x as a greatest element, also $o(x) \leqslant l(x)+1$.

Our next purpose will be to show that $o(X+Y)=o(X) \# o(Y)$ and $o(X \times Y)=o(X) \otimes o(Y)$, where $\#$ and $\otimes$ are the natural sum and product of Hessenberg (cf. Bachmann [B]): if

$$\alpha=\omega^{\alpha_1}+\ldots+\omega^{\alpha_k}(\alpha_1 \geqslant \ldots \geqslant \alpha_k)$$

and

$$\beta=\omega^{\beta_1}+\ldots+\omega^{\beta_l}(\beta_1 \geqslant \ldots \geqslant \beta_l),$$

then

$$\alpha \# \beta=\omega^{\gamma_1}+\ldots+\omega^{\gamma_{k+l}},$$

where $(\gamma_1, \ldots, \gamma_{k+l})$ is a rearrangement of $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l)$ such that $\gamma_1 \geqslant \ldots \geqslant \gamma_{k+l}$ and $\alpha \otimes \beta=\Sigma_{(i,j) \leqslant (k,l)}\, \omega^{\alpha_i \# \beta_j}$, where Σ stands for a natural sum of an arbitrary finite number of factors.

LEMMA 3.3. For all ordinals α, β, γ,

a) $\alpha \# \beta=\beta \# \alpha$ and $\alpha \otimes \beta=\beta \otimes \alpha$,

b) $\alpha \# (\beta+1)=(\alpha \# \beta)+1$,

c) if $(\alpha, \beta)<(\gamma, \delta)$ then $\alpha \# \beta<\gamma \# \delta$ and $\alpha \otimes \beta<\gamma \otimes \delta$,

d) if α and β are limit numbers, then $\alpha \# \beta=\bigcup_{(\gamma,\delta)<(\alpha,\beta)} (\gamma \# \delta)$,

e) if $\beta<\omega^\alpha$ and $\gamma<\omega^\alpha$; then $\beta \# \gamma<\omega^\alpha$,

f) $\alpha \otimes (\beta \# \gamma)=(\alpha \otimes \beta) \# (\alpha \otimes \gamma)$,

g) if $\beta<\omega^{\omega^\alpha}$ and $\gamma<\omega^{\omega^\alpha}$, then $\beta \otimes \gamma<\omega^{\omega^\alpha}$.

PROOF. (a), (b) and (c) are trivial. (d) Let

$$\alpha=\omega^{\alpha_1}+\ldots+\omega^{\alpha_k}, \beta=\omega^{\beta_1}+\ldots+\omega^{\beta_l}, \alpha_1 \geqslant \ldots \geqslant \alpha_k>0$$

and $\beta_1 \geqslant \ldots \geqslant \beta_l>0$. W.l.o.g. we can assume that $\alpha_k \geqslant \beta_l$. In that case

$$\alpha \# \beta=\bigcup_{\eta<\omega} {}_{\beta}(\alpha \# (\omega^{\beta_1}+\ldots+\omega^{\beta_{l-1}}+\eta)).$$

So $\alpha \# \beta \leqslant \bigcup_{(\gamma,\delta)<(\alpha,\beta)} (\gamma \# \delta)$. The reverse inequality is immediate from (b); (e) and (f) are trivial and (g) follows from (e).

THEOREM 3.4. $o(X+Y)=o(X)\mathbin{\#}o(Y)$.

PROOF. Let $o(X)=\alpha$, $o(Y)=\beta$. The theorem will be proved by induction on (α, β).

I. Either α or β is a successor. Without loss of generality assume β is, i.e. $\beta=\gamma+1$. By theorem 3.2, for some $y\in Y$, $U_Y(y)=\{y\}$ and $l_Y(y)=\gamma$. By the induction hypothesis, $o(X+L_Y(y))=\alpha\mathbin{\#}\gamma$.

Applying theorem 3.2 again gives

$$o(X+Y)=(\alpha\mathbin{\#}\gamma)+1=(\text{by Lemma 3.3(b)})\ \alpha\mathbin{\#}(\gamma+1)=\alpha\mathbin{\#}\beta.$$

II. Both α and β are limit numbers. By Lemma 3.3 (d), it is sufficient to prove that

$$o(X+Y)=\bigcup_{(\gamma,\delta)<(\alpha,\beta)}(\gamma\mathbin{\#}\delta).$$

By Lemma 2.6 (and theorem 3.2), $o(X+Y)=\bigcup_{z\in X\cup Y} l_{X+Y}(z)$. By the induction hypothesis, $l_{X+Y}(z)=\gamma\mathbin{\#}\delta$ for some $(\gamma,\delta)<(\alpha,\beta)$. This implies that $o(X+Y)\leqslant\bigcup_{(\gamma,\delta)<(\alpha,\beta)}(\gamma\mathbin{\#}\delta)$. On the other hand, assume that $(\gamma, \delta)<(\alpha, \beta)$. Without loss of generality we can assume that $\delta<\beta$. For some $y\in Y$, $l_Y(y)>\delta$. By the induction hypothesis and Lemma 3.3 (c) now

$$o(X+L_Y(y))>\gamma\mathbin{\#}\delta,\ \text{whence also}\ o(X+Y)\geqslant\bigcup_{(\gamma,\delta)<(\alpha,\beta)}(\gamma\mathbin{\#}\delta).$$

THEOREM 3.5. $o(X\times Y)=o(X)\divideontimes o(Y)$.

PROOF. Let $o(X)=\alpha$, $o(Y)=\beta$. The proof will be by induction on (α, β).

I. $\alpha=\omega^{\alpha_1}+\ldots+\omega^{\alpha_k}$, $\beta=\omega^{\beta_1}+\ldots+\omega^{\beta_l}$ with either $k>1$ or $l>1$. For example, assume $l>1$. In that case, for some $m\in\omega$, there are $y_1, \ldots, y_m\in Y$ such that for $Y_1=U_Y(y_1)\cup\ldots\cup U_Y(y_m)$ and

$$Y_2=L_Y(y_1)\cap\ldots\cap L_Y(y_m),\ \ Y=Y_1\cup Y_2,\ \ o(Y_1)=\omega^{\beta_1}+\ldots+\omega^{\beta_{l-1}}$$

and $o(Y_2)=\omega^{\beta_l}$.

We now have

$$\begin{aligned}o(X\times Y)&\leqslant o(X\times(Y_1+Y_2))=o((X\times Y_1)+(X\times Y_2))=\\&=(\alpha\divideontimes(\omega^{\beta_1}+\ldots+\omega^{\beta_{l-1}}))\mathbin{\#}(\alpha\divideontimes\omega^{\beta_l})=\alpha\divideontimes\beta\end{aligned}$$

(induction hypothesis, lemma 3.3 (f)).

Since it is clear that $o(X\times Y)\geqslant o(\alpha\times\beta)$ it is now sufficient to prove that $o(\alpha\times\beta)\geqslant\alpha\divideontimes\beta$. Define, for (i, j), $(m, n)\leqslant(k, l)$, $(i, j)<'(m, n)$ iff $\alpha_i\mathbin{\#}\beta_j<\alpha_m\mathbin{\#}\beta_n$. Next extend $<'$ in an arbitrary way to a linear ordering $\leqslant_b$; by Lemma 3.3 (c), $\leqslant_b$ is an extension of $\leqslant$.

Let us write, for all A, $B\subseteq\gamma$ (where γ is an ordinal), $A\lll B$ for $\forall\alpha\in A\,\forall\beta\in B(\alpha<\beta)$. Finally define $A_1, \ldots, A_k, B_1, \ldots, B_l$ in such a way that $\alpha = A_1 \cup \ldots \cup A_k$, $\beta = B_1 \cup \ldots \cup B_l$,

$$V_{ij}(1 \leqslant i \leqslant j \leqslant k \to A_i \lll A_j),\ V_{ij}(1 \leqslant i < j \leqslant l \to B_i \lll B_j),$$
$$V_{ij}(1 \leqslant i \leqslant k \wedge 1 \leqslant j \leqslant l \to |A_i, \leqslant| = \omega^{\alpha_i} \wedge |B_j, \leqslant| = \omega^{\beta_j}).$$

Then $\alpha \times \beta = \bigcup_{(i,j) \leqslant (k,l)} (A_i \times B_j)$.

Applying the induction hypothesis for each $(i, j) \leqslant (k, l)$ we can extend $\leqslant$ to a well-order $\leqslant''$ on $A_i \times B_j$ such that $|A_i \times B_j, \leqslant''| = \omega^{\alpha_i} ⋇ \omega^{\beta_j} = \omega^{\alpha_i \# \beta_j}$.

$\leqslant''$ is extended to be a well-order of the whole set $A \times B$ by requiring that, if $z \in A_i \times B_j$ and $w \in A_m \times B_n$, then $z <'' w$ iff $(i, j) <_b (m, n)$. Clearly $|\alpha \times \beta, \leqslant''| = \alpha ⋇ \beta$, whence $o(\alpha \times \beta) \geqslant \alpha ⋇ \beta$.

II. $\alpha = \omega^{\alpha_1}$ and $\beta = \omega^{\beta_1}$. Take an arbitrary $(x, y) \in X \times Y$.

$L((x, y)) = (L_X(x) \times L_Y(y)) \cup (L_X(x) \times U_Y(y)) \cup (U_X(x) \times L_Y(y))$, whence, by Theorem 3.4,

$$l((x, y)) \leqslant o(L_X(x) \times L_Y(y)) \# o(L_X(x) \times U_Y(y)) \# o(U_X(x) \times L_Y(y)) =$$

(by the induction hypothesis) $(\gamma ⋇ \delta) \# (\omega^{\alpha_1} ⋇ \delta) \# (\gamma ⋇ \omega^{\beta_1})$ for some $\gamma < \alpha$, $\delta < \beta$. Each of the three terms in this natural sum is $< \omega^{\alpha_1 \# \beta_1}$, so, by Lemma 3.3 (e), $l((x, y)) < \omega^{\alpha_1 \# \beta_1}$, whence $o(X \times Y) \leqslant \omega^{\alpha_1 \# \beta_1} = \omega^{\alpha_1} ⋇ \omega^{\beta_1}$.

Again it is sufficient to show $o(\alpha \times \beta) \geqslant \alpha ⋇ \beta$. To prove this by induction we consider two subcases.

IIa. Either α_1 or β_1 is a successor. For example let $\beta_1 = \gamma + 1$. Then $\omega^{\beta_1} = \omega^{\gamma} \cdot \omega$ and $\omega^{\beta_1} = \bigcup_{i \in \omega} B_i$, where, for each i, $j (i < j < \omega)$, $B_i < B_j$ and $|B_i| = \omega^{\gamma}$. By the induction hypothesis, $o(\omega^{\alpha_1} \times \omega^{\gamma}) = \omega^{\alpha_1 \# \gamma}$. Therefore, $o(\omega^{\alpha_1} \times \omega^{\beta_1}) \geqslant \omega^{\alpha_1 \# \gamma} \cdot \omega = \omega^{(\alpha \# \gamma) + 1} = \omega^{\alpha_1 \# \beta_1}$ (Lemma 3.3 (b)).

IIb. Both α_1 and β_1 are limitnumbers. In that case, by Lemma 3.3 (d),

$$\omega^{\alpha_1 \# \beta_1} = \bigcup_{(\gamma, \delta) < (\alpha_1, \beta_1)} \omega^{\gamma \# \delta}.$$

An application of the induction hypothesis now gives the desired result.

Def. 3.6. $\alpha^{\bar{0}} = 1$, $\alpha^{\overline{\beta+1}} = \alpha^{\bar{\beta}} ⋇ \alpha$ and, if γ is a limit number, then $\alpha^{\bar{\gamma}} = \bigcup_{\beta < \gamma} \alpha^{\bar{\beta}}$.

Lemma 3.7. If $o(X) = \alpha$, then, for each $n \in \omega$, $o(X^n) = \alpha^{\bar{n}}$.

Proof. Immediate from Theorem 3.5.

Lemma 3.8. $(\omega^{\omega^\beta})^{\bar{\lambda}} = (\omega^{\omega^\beta})^{\lambda}$ for all λ.

Proof. By induction on λ.

$(\omega^{\omega^\beta})^{\overline{\lambda+1}} = (\omega^{\omega^\beta})^{\bar{\lambda}} ⋇ \omega^{\omega^\beta} =$ (induction hypothesis)

$$(\omega^{\omega^\beta})^{\lambda} ⋇ \omega^{\omega^\beta} = \omega^{\omega^\beta . \lambda} ⋇ \omega^{\omega^\beta} = \omega^{\omega^\beta . \lambda \# \omega^\beta} = \omega^{\omega^\beta (\lambda + 1)} = (\omega^{\omega^\beta})^{\lambda + 1}.$$

Def 3.9. A sequence $\{x_\xi\}_{\xi < \alpha}$ over X is a *majorizing sequence* for X, if, for each $x \in X$ and each $\beta < \alpha$, there is a $\xi (\beta \leqslant \xi < \alpha)$ such that $x \leqslant x_\xi$.

LEMMA 3.10. There are majorizing sequences for X and, if $\{x_\xi\}_{\xi<\alpha}$ is a majorizing sequence for X, then $\bigcup_{\xi<\alpha} l(x_\xi)=o(X)$ provided only that $o(X)$ is a limit number.

PROOF. Trivial.

THEOREM 3.11. If $o(X)=n$, then $o(X^*)=\omega^{\omega^{n-1}}$.

PROOF. By induction on n. For $o(X)=1$, clearly $o(X^*)=\omega=\omega^{\omega^0}$. Assume the theorem to be valid for n and let $o(X)=n+1$. By theorem 3.2, for some $x\in X$, $U(x)=\{x\}$ and $L(x)=X-\{x\}$. By the induction hypothesis, $o((L(x))^*)=\omega^{\omega^{n-1}}$.

Define, for any $m\in\omega$, A_m to be the set of elements of X^* with exactly m occurrences of x. If $a\in A_m$, then a can be written uniquely as $a_0xa_1xa_2x\ldots xa_m$ with $a_0,\ldots,a_m\in(L(x))^*$. If a and b are both elements of A_m, then $a\leqslant^* b$ iff $(a_0,\ldots,a_m)\leqslant(b_0,\ldots,b_m)$ in $((L(x))^*)^m$. Lemmas 3.7 and 3.8 imply then that $o(A_m)=(\omega^{\omega^{n-1}})^m$, whence $o(X^*)\geqslant(\omega^{\omega^{n-1}})^\omega=\omega^{\omega^n}$.

For the proof of the reverse inequality assume $y_0, y_1, \ldots$ to be an enumeration of $L(x)$ in which each element of $L(x)$ occurs infinitely often. Then $d_0=y_0$, $d_1=d_0xy_1$, $d_2=d_1xy_2$, ... will be a majorizing sequence for X^*. By Lemma 3.9, it will be sufficient to show that, for each $m\in\omega$, $l_{X^*}(d_m)<\omega^{\omega^n}$. Since $L_{X^*}(d_0)=(L_X(y_0))^*$, we have, by the induction hypothesis of the theorem $l_{X^*}(d_0)\leqslant\omega^{\omega^{n-1}}<\omega^{\omega^n}$. Consider an arbitrary $e\in L_{X^*}(d_{m+1})$. If an occurrence of x in e is such that $e=e_1xe_2$, then we call this occurrence *left sided* if $d_m\not\leqslant^* e_1$, *right sided* if $y_{m+1}\not\leqslant^* e_2$.

Note that each occurrence of x in e is left sided or right sided, that occurrences of x to the left of a leftsided occurrence of x are left sided and that occurrences of x to the right of a rightsided occurrence of x are right sided. This means that e can be written in the form $e_1xe_2xe_3$, where the two explicit occurrences of x are neigthboring left sided and right sided occurrences, $e_1\in L_{X^*}(d_m)$, $e_2\in(L_X(x))^*$ and $e_3\in(L_X(y_m))^*$. (Degenerate cases, where e contains no left sided occurrences of x, no right sided occurrences of x, or no occurrences of x at all can be subsumed in the following argument by writing e in the form e_2xe_3, e_1xe_2 or e_2 respectively). From this it follows that

$$l_{X^*}(d_{m+1})\leqslant o(L_{X^*}(d_m)\times(L_X(x))^*\times(L_X(y_m))^*).$$

By the induction hypothesis (for n) and theorem 3.5 this implies that

$$l_{X^*}(d_{m+1})\leqslant l_{X^*}(d_m)\divideontimes\omega^{\omega^{n-1}}\divideontimes\omega^{\omega^{n-1}},$$

Finally the induction hypothesis (for m) and lemma 3.3 (g) give the required $l_{X^*}(d_{m+1})<\omega^{\omega^n}$.

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