

A Proof of the CSP Dichotomy Conjecture

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Many natural combinatorial problems can be expressed as constraint satisfaction problems. This class of problems is known to be NP-complete in general, but certain restrictions on the form of the constraints can ensure tractability. The standard way to parameterize interesting subclasses of the constraint satisfaction problem is via finite constraint languages. The main problem is to classify those subclasses that are solvable in polynomial time and those that are NP-complete. It was conjectured that if a constraint language has a weak near-unanimity polymorphism then the corresponding constraint satisfaction problem is tractable; otherwise, it is NP-complete.

In the article, we present an algorithm that solves Constraint Satisfaction Problem in polynomial time for constraint languages having a weak near unanimity polymorphism, which proves the remaining part of the conjecture.¹

CCS Concepts: • **Theory of computation** → **Constraint and logic programming**; *Complexity theory and logic*; • **Mathematics of computing** → *Combinatorial algorithms*;

Additional Key Words and Phrases: Constraint satisfaction problem, computational complexity, CSP dichotomy, relational clones, weak near-unanimity

ACM Reference format:

Dmitriy Zhuk. 2020. A Proof of the CSP Dichotomy Conjecture. *J. ACM* 67, 5, Article 30 (August 2020), 78 pages.

<https://doi.org/10.1145/3402029>

1 INTRODUCTION

The *Constraint Satisfaction Problem (CSP)* is the problem of deciding whether there is an assignment to a set of variables subject to some specified constraints. Formally, the *Constraint Satisfaction Problem* is defined as a triple $\langle X, D, C \rangle$, where

- $X = \{x_1, \dots, x_n\}$ is a set of variables,
- $D = \{D_1, \dots, D_n\}$ is a set of the respective domains,
- $C = \{C_1, \dots, C_m\}$ is a set of constraints,

¹ An extended abstract of this paper appeared at the 58th Annual Symposium on Foundations of Computer Science (FOCS 2017) [65].

The author has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 771005) and from Russian Foundation for Basic Research (grant 19-01-00200).

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0004-5411/2020/08-ART30 \$15.00

<https://doi.org/10.1145/3402029>

where each variable x_i can take on values in the nonempty domain D_i , every constraint $C_j \in \mathcal{C}$ is a pair (t_j, ρ_j) , where t_j is a tuple of variables of length m_j , called the *constraint scope*, and ρ_j is an m_j -ary relation on the corresponding domains, called the *constraint relation*.

The question is whether there exists a *solution* to $\langle X, D, C \rangle$, that is a mapping that assigns a value from D_i to every variable x_i such that for each constraint C_j the image of the constraint scope is a member of the constraint relation.

In this article, we consider only CSP over finite domains. The general CSP is known to be NP-complete [46, 50]; however, certain restrictions on the allowed form of constraints involved may ensure tractability (solvability in polynomial time) [17, 22, 28, 38, 39, 41]. Below we provide a formalization to this idea.

To simplify the formulation of the main result we assume that the domain of every variable is a finite set A . Later, we will assume that the domain of every variable is a unary relation from the constraint language Γ (see below). By R_A , we denote the set of all finitary relations on A , that is, subsets of A^m for some m . Thus, all the constraint relations are from R_A .

For a set of relations $\Gamma \subseteq R_A$ by $\text{CSP}(\Gamma)$ we denote the Constraint Satisfaction Problem where all the constraint relations are from Γ . The set Γ is called a *constraint language*. Another way to formalize the Constraint Satisfaction Problem is via conjunctive formulas. Every h -ary relation on A can be viewed as a predicate, that is, a mapping $A^h \rightarrow \{0, 1\}$. Suppose $\Gamma \subseteq R_A$, then $\text{CSP}(\Gamma)$ is the following decision problem: given a formula

$$\rho_1(v_{1,1}, \dots, v_{1,n_1}) \wedge \dots \wedge \rho_s(v_{s,1}, \dots, v_{s,n_s}),$$

where $\rho_1, \dots, \rho_s \in \Gamma$, and $v_{i,j} \in \{x_1, \dots, x_n\}$ for every i, j ; decide whether this formula is satisfiable.

It is well known that many combinatorial problems can be expressed as $\text{CSP}(\Gamma)$ for some constraint language Γ . Moreover, for some sets Γ the corresponding decision problem can be solved in polynomial time; while for others it is NP-complete. It was conjectured that $\text{CSP}(\Gamma)$ is either in P or NP-complete [29].

CONJECTURE 1. *Suppose $\Gamma \subseteq R_A$ is a finite set of relations. Then $\text{CSP}(\Gamma)$ is either solvable in polynomial time or NP-complete.*

We say that an operation $f : A^n \rightarrow A$ preserves the relation $\rho \in R_A$ of arity m if for any tuples $(a_{1,1}, \dots, a_{1,m}), \dots, (a_{n,1}, \dots, a_{n,m}) \in \rho$ the tuple $(f(a_{1,1}, \dots, a_{n,1}), \dots, f(a_{1,m}, \dots, a_{n,m}))$ is in ρ . We say that an operation preserves a set of relations Γ if it preserves every relation in Γ . A mapping $f : A \rightarrow A$ is called an *endomorphism* of Γ if it preserves Γ .

THEOREM 1.1 [37]. *Suppose $\Gamma \subseteq R_A$. If f is an endomorphism of Γ , then $\text{CSP}(\Gamma)$ is polynomially reducible to $\text{CSP}(f(\Gamma))$ and vice versa, where $f(\Gamma)$ is a constraint language with domain $f(A)$ defined by $f(\Gamma) = \{f(\rho) : \rho \in \Gamma\}$.*

A constraint language is a *core* if every endomorphism of Γ is a bijection. It is not hard to show that if f is an endomorphism of Γ with minimal range, then $f(\Gamma)$ is a core. Another important fact is that we can add all singleton unary relations to a core constraint language without increasing the complexity of its CSP. By $\sigma_{=a}$, we denote the unary relation $\{a\}$.

THEOREM 1.2 [17]. *Let $\Gamma \subseteq R_A$ be a core constraint language, and $\Gamma' = \Gamma \cup \{\sigma_{=a} \mid a \in A\}$. Then $\text{CSP}(\Gamma')$ is polynomially reducible to $\text{CSP}(\Gamma)$.*

Therefore, to prove Conjecture 1, it is sufficient to consider only the case when Γ contains all unary singleton relations. In other words, all the predicates $x = a$, where $a \in A$, are in the constraint language Γ .

Schaefer [54] classified all tractable constraint languages over a two-element domain. Bulatov [19] generalized the result for a three-element domain. His dichotomy theorem was formulated in terms of a G -set. Later, the dichotomy conjecture was formulated in several different forms (see Reference [17]).

The result of McKenzie and Maróti [48] allows us to formulate the dichotomy conjecture in the following nice way. An operation f on a set A is called a *weak near-unanimity operation (WNU)* if it satisfies $f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, x, \dots, x, y)$ for all $x, y \in A$. An operation f is called *idempotent* if $f(x, x, \dots, x) = x$ for all $x \in A$.

CONJECTURE 2. *Suppose $\Gamma \subseteq R_A$ is a finite set of relations. Then $\text{CSP}(\Gamma)$ can be solved in polynomial time if there exists a WNU preserving Γ ; $\text{CSP}(\Gamma)$ is NP-complete otherwise.*

It is not hard to see that the existence of a WNU preserving Γ is equivalent to the existence of a WNU preserving a core of Γ and also equivalent to the existence of an idempotent WNU preserving the core. Hence, Theorems 1.1 and 1.2 imply that it is sufficient to prove Conjecture 2 for a core and an idempotent WNU.

One direction of this conjecture follows from Reference [48].

THEOREM 1.3 [48]. *Suppose $\Gamma \subseteq R_A$ and $\{\sigma_{=a} \mid a \in A\} \subseteq \Gamma$. If there exists no WNU preserving Γ , then $\text{CSP}(\Gamma)$ is NP-complete.*

The dichotomy conjecture was proved for many special cases: for CSPs over undirected graphs [34], for CSPs over digraphs with no sources or sinks [4], for constraint languages containing all unary relations [18], and many others. More information about the algebraic approach to CSP can be found in Reference [6].

In this article, we present an algorithm that solves $\text{CSP}(\Gamma)$ in polynomial time if Γ is preserved by an idempotent WNU and therefore prove the dichotomy conjecture.

THEOREM 1.4. *Suppose $\Gamma \subseteq R_A$ is a finite set of relations. Then $\text{CSP}(\Gamma)$ can be solved in polynomial time if there exists a WNU preserving Γ ; $\text{CSP}(\Gamma)$ is NP-complete otherwise.*

Another proof of the dichotomy conjecture was announced by Andrei Bulatov [20, 21]. Even though both algorithms appeared at the same time, they are significantly different. Bulatov's algorithm uses full strength of the few subpowers algorithm [35], and uses Maróti's trick for trees on top of Mal'tsev [47], while this one just checks some local consistency and solves linear equations over prime fields. Also Bulatov's algorithm works for infinite constraint languages, which is not the case for the algorithm presented in this article. But its slight modification works even for infinite constraint languages [63].

The article is organized as follows. In Section 2, we explain the algorithm informally and give an example showing how the algorithm works for a system of linear equations in \mathbb{Z}_4 . In Section 3, we give main definitions, and in Section 4 we give a formal description of the algorithm showing a pseudocode for most functions and explain the meaning of every function. In Section 5, we formulate all theorems that are necessary to prove the correctness of the algorithm. Then, we prove that on every algebra (domain) with a WNU operation there exists a subuniverse of one of four types, which is the main ingredient of the algorithm. Additionally, in this section we prove that some functions of the algorithm work properly and the algorithm actually works in polynomial time.

In Section 6, we give the remaining definitions. The proof of the main theorems is divided into three sections. In Section 7, we study properties of subuniverses of each of four types (absorbing, central, PC, and linear subuniverses). In Section 8, we prove all the auxiliary statements, and in the last section we prove the main theorems of this article formulated in Section 5.

In Section 10, we discuss open questions and consequences of this result. In particular, we consider generalizations of the CSP such as Valued CSP, Infinite Domain CSP, Quantified CSP, Promise CSP, and so on.

2 OUTLINE OF THE ALGORITHM

In this section, we give an informal description of the algorithm and show how it works for a system of linear equations in \mathbb{Z}_4 . The algorithm is based on the following three ingredients:

- Each domain has either one of three kinds of proper strong subsets (absorbing, central, polynomially complete) or an equivalence relation modulo where the domain is essentially a product of prime fields (Theorem 5.1).
- If a sufficient level of consistency (cycle consistency + irreducibility; see Section 3.6) is enforced, then we do not lose all the solutions when we reduce the domain to a proper strong subset (that is, if the original instance has a solution, then the reduced instance has a solution as well), which is guaranteed by Theorems 5.5 and 5.6.
- If we cannot reduce the domain in such a way, then we are left with an instance whose each domain has an equivalence relation modulo whereby it is a product of prime fields, and all relations are affine subspaces. Now we have the following:

A = the set of all solutions of the instance factorized by the equivalences;

B = the set of all solutions of the factorized instance (where all domains and relations are factorized).

Both A and B are affine subspaces, $A \subseteq B$. We would like to know whether A is empty, and what we can efficiently compute is B (using Gaussian elimination). The algorithm gradually makes B smaller (of smaller dimension), while maintaining the property $A \subseteq B$.

First, for some solution from B we check whether A has the same solution, which can be done by a recursive call of the algorithm for smaller domains. If A has it, then we are done. If A has not, then $A \neq B$. In this case, we can make (see Theorem 5.7) the instance weaker maintaining the property $A' \subsetneq B$ (here A' is A for the weaker instance) until the moment when

- (i) A' is a subspace of B of codimension one,
- (ii) or $A = A' = \emptyset$,
- (iii) or the obtained instance is not linked (it splits into several instances on smaller domains, hence A' can be calculated using recursion).

In (i) and (ii) A' can be computed by linearly many recursive calls of the algorithm for smaller domains. In fact, A' can be defined by a linear equation $c_1x_1 + \dots + c_hx_h = c_0$ in a prime field \mathbb{Z}_p . Then the coefficients c_0, c_1, \dots, c_h can be learned (up to a multiplicative constant) by $(p \cdot h + 1)$ queries of the form “ $(a_1, \dots, a_h) \in A'?$ ” (see Subsection 4.3 for more details). To check each query we just need to call the algorithm recursively for the smaller domains that are the equivalence classes corresponding to a_1, \dots, a_h .

We update $B = A'$, return back to the original instance and continue tightening B . We eventually stop when $B = A$, which gives us the answer to our question: If $B \neq \emptyset$, then the original instance has a solution; if $B = \emptyset$, then it has no solutions.

We demonstrate the work of the algorithm on a system of linear equations in \mathbb{Z}_4 :

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 2 \\ 2x_1 + x_2 + x_3 + x_4 = 2 \\ \quad \quad \quad x_1 + x_2 = 2 \\ \quad \quad \quad x_1 + x_2 + 2x_4 = 2 \end{cases} \quad (1)$$

All the relations (equations) are invariants of the WNU $x_1 + \dots + x_5 \pmod{4}$, and, therefore, this system of equations is an instance of $\text{CSP}(\Gamma)$, where Γ is the set of all relations of arity at most 4 preserved by the WNU. Hence, we can apply the algorithm.

First, \mathbb{Z}_4 does not have a proper strong subset, which is why for every domain there should be an equivalence relation modulo that it is just a product of prime fields. In our example, it is the modulo 2 equivalence relation.

We factorize our instance modulo 2 and obtain a system of linear equations in \mathbb{Z}_2 , where $x'_i = x_i \pmod{2}$ for every i ,

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 + x'_2 = 0 \end{cases} . \quad (2)$$

Using Gaussian elimination, we solve this system of equations in a field, choose independent variables x'_1 and x'_3 , and write the general solution (the set B in the informal description): $x'_1 = x'_1, x'_2 = x'_1, x'_3 = x'_3, x'_4 = x'_1 + x'_3$.

We choose any solution from B . Let it be $(0, 0, 0, 0)$ for $x'_1 = x'_3 = 0$. Then we check whether (1) has a solution corresponding to $(0, 0, 0, 0)$ by restricting every domain to the set $\{0, 2\}$ ($x_i \pmod{2} = 0$). We recursively call the algorithm for smaller domain and find out that (1) has no solutions inside $\{0, 2\}$. This means that $(0, 0, 0, 0)$ does not belong to A from the informal description and therefore $A \subsetneq B$.

Then, we try to make the instance weaker so that $A' \subsetneq B$, where A' is the intersection of B with the set of all solutions of the new instance factorized by the equivalences. Let us remove the last equation from (1) to obtain a new solution set A' ,

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 2 \\ 2x_1 + x_2 + x_3 + x_4 = 2 \\ x_1 + x_2 = 2 \end{cases} . \quad (3)$$

Again, by solving an instance on the 2-element domain $\{0, 2\}$, we find out that (3) has no solutions corresponding to $(0, 0, 0, 0)$. Therefore, we have $A' \subsetneq B$.

We need to check that if we remove one more equation from (3), then we get $A' = B$. Thus, for every weaker instance we need to check that for any $a_1, a_3 \in \mathbb{Z}_2$ there exists a solution corresponding to $(x'_1, x'_3) = (a_1, a_3)$. Since A' is an affine subspace, it is sufficient to check this for $(x'_1, x'_3) = (0, 0)$, $(x'_1, x'_3) = (1, 0)$, and $(x'_1, x'_3) = (0, 1)$, i.e., for $h + 1$ tuples, where h is the dimension of B . Again, to check a concrete solution from B , we recursively call the algorithm for 2-element domains.

Since (3) is linked, Theorem 5.7 guarantees that the dimension of A' equals the dimension of B minus one or A' is empty. Hence, we need exactly one equation to describe all pairs (a_1, a_3) such that (3) has a solution corresponding to $(x'_1, x'_3) = (a_1, a_3)$. Let the equation be $c_1 x'_1 + c_3 x'_3 = c_0$. We need to find c_1, c_3 , and c_0 . Recursively calling the algorithm for smaller domains, we find out that (3) has a solution $(3, 3, 0, 1)$ corresponding to $(x'_1, x'_3) = (1, 0)$ (the solution $(1, 1, 0, 1)$ from B) but does not have a solution corresponding to $(x'_1, x'_3) = (0, 1)$ (the solution $(0, 0, 1, 1)$ from B). We have

$$\begin{cases} c_1 \cdot 0 + c_3 \cdot 0 \neq c_0 \\ c_1 \cdot 1 + c_3 \cdot 0 = c_0 \\ c_1 \cdot 0 + c_3 \cdot 1 \neq c_0 \end{cases} ,$$

which implies that $c_1 = 1, c_3 = 0, c_0 = 1$, and the equation we are looking for is $x'_1 = 1$. Thus, we found A' .

We add this equation to (2) (update $B = A'$) and solve the new system of linear equations in \mathbb{Z}_2 ,

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 = 1 \end{cases} \quad (4)$$

The general solution of this system (the new set B) is $x'_1 = 1, x'_2 = 1, x'_3 = x'_4 = x'_3 + 1$, where x'_3 is an independent variable. Thus, we decreased the dimension of the solution set B by 1 and we still have the property that $A \subseteq B$. We go back to (1) and check whether it has a solution corresponding to $x'_3 = 0$ (the solution $(1, 1, 0, 1)$ from B). Again, by solving an instance on the 2-element domain we find out that $(1, 1, 0, 1) \notin A$. Therefore, $A \subsetneq B$.

The remaining part of the procedure looks trivial, but we want to follow the algorithm until the end to make it clear. Again, we try to make the instance weaker so that $A' \subsetneq B$. Let us remove the third equation from (1),

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 2 \\ 2x_1 + x_2 + x_3 + x_4 = 2 \\ x_1 + x_2 + 2x_4 = 2 \end{cases} \quad (5)$$

By solving this instance on smaller domains, we find out that (5) has no solutions corresponding to $x'_3 = 0$ (the solution $(1, 1, 0, 1)$ of B). Therefore, we obtained a new set $A' \subsetneq B$.

Then, we try to remove one more equation from (5) maintaining the property $A' \subsetneq B$. We check for every weaker instance that for any $a_3 \in \mathbb{Z}_2$ there exists a solution corresponding to $x'_3 = a_3$.

Again, the instance (5) is linked, and by Theorem 5.7 we need exactly one equation to describe all elements a_3 such that (5) has a solution corresponding to $x'_3 = a_3$. Let the equation be $c_3 x'_3 = c_0$. We already checked that it does not hold for $x'_3 = 0$. By solving an instance on 2-element domains, we find out that (5) has a solution $(3, 3, 1, 0)$ corresponding to the solution $(1, 1, 1, 0)$ from B and $x'_3 = 1$. Thus, we have

$$\begin{cases} c_3 \cdot 0 \neq c_0 \\ c_3 \cdot 1 = c_0 \end{cases},$$

which implies $c_3 = 1, c_0 = 1$, and the equation we are looking for is $x'_3 = 1$ (we calculated A').

We add this equation to (4) (update $B = A'$) and solve the new system of linear equations in \mathbb{Z}_2 ,

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 = 1 \\ x'_3 = 1 \end{cases} \quad (6)$$

The only solution of this system is $(x'_1, x'_2, x'_3, x'_4) = (1, 1, 1, 0)$. Thus, we decreased the dimension of the solution set B to 0, and we still have the property that $A \subseteq B$. It remains to check whether the original system (1) has a solution corresponding to the solution $(1, 1, 1, 0)$ of B . Again, by solving an instance on the 2-element domain we find a solution $(3, 3, 1, 0)$ of the original instance. Therefore, $(1, 1, 1, 0) \in A$, and we finally reached the condition $A = B$.

3 DEFINITIONS

A set of operations is called a *clone* if it is closed under composition and contains all projections. For a set of operations M by $\text{Clo}(M)$, we denote the clone generated by M .

An idempotent WNU w is called *special* if $x \circ (x \circ y) = x \circ y$, where $x \circ y = w(x, \dots, x, y)$. It is not hard to show that for any idempotent WNU w on a finite set there exists a special WNU $w' \in \text{Clo}(w)$ (see Lemma 4.7 in Reference [48]).

A relation $\rho \subseteq A_1 \times \dots \times A_n$ is called *subdirect* if for every i the projection of ρ onto the i th coordinate is A_i . For a relation ρ by $\text{pr}_{i_1, \dots, i_s}(\rho)$, we denote the projection of ρ onto the coordinates i_1, \dots, i_s .

3.1 Algebras

An *algebra* is a pair $\mathbf{A} := (A; F)$, where A is a finite set, called *universe*, and F is a family of operations on A , called *basic operations of \mathbf{A}* . In the article, we always assume that we have a special WNU w preserving all constraint relations. Therefore, every domain D , which is from the constraint language, can be viewed as an algebra $(D; w)$. By $\text{Clo}(\mathbf{A})$, we denote the clone generated by all basic operations of \mathbf{A} .

An equivalence relation σ on the universe of an algebra \mathbf{A} is called a *congruence* if it is preserved by every operation of the algebra. A congruence (an equivalence relation) is called *proper* if it is not equal to the full relation $A \times A$. A subuniverse is called *nontrivial* if it is proper and nonempty. We use standard universal algebraic notions of term operation, subalgebra, factor algebra, and product of algebras, see Reference [8]. We say that a subalgebra $\mathbf{R} = (R; F_R)$ is a *subdirect subalgebra* of $\mathbf{A} \times \mathbf{B}$ if R is a subdirect relation in $A \times B$.

3.2 Polynomially Complete Algebras

An algebra $(A; F_A)$ is called *polynomially complete (PC)* if the clone generated by F_A and all constants on A is the clone of all operations on A (see References [36, 45]).

3.3 Linear Algebra

An idempotent finite algebra $(A; w_A)$ is called *linear* (similar to affine in Reference [31]) if it is isomorphic to $(\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_s}; x_1 + \dots + x_m)$ for prime numbers p_1, \dots, p_s . Since $\mathbf{A}/(\sigma \cap \tau)$ is always isomorphic to a subalgebra of $\mathbf{A}/\sigma \times \mathbf{A}/\tau$, and since linear algebras are closed under products and subalgebras by Corollary 7.20.1, for every idempotent finite algebra $(B; w_B)$ there exists a least congruence σ , called the *minimal linear congruence*, such that $(B; w_B)/\sigma$ is linear.

3.4 Absorption

Let B be a (probably empty) subuniverse of $\mathbf{A} = (A; F_A)$. We say that B *absorbs \mathbf{A}* if there exists $t \in \text{Clo}(\mathbf{A})$ such that $t(B, B, \dots, B, A, B, \dots, B) \subseteq B$ for any position of A . In this case, we also say that B is an *absorbing subuniverse of \mathbf{A} with a term operation t* . If the operation t can be chosen binary, then we say that B is a binary absorbing subuniverse of \mathbf{A} . For more information about absorption and its connection with CSP; see Reference [3].

3.5 Center

Suppose $\mathbf{A} = (A; w_A)$ is a finite algebra with a special WNU operation. $C \subseteq A$ is called a *center* if there exists an algebra $\mathbf{B} = (B; w_B)$ with a special WNU operation of the same arity and a subdirect subalgebra $(R; w_R)$ of $\mathbf{A} \times \mathbf{B}$ such that there is no nontrivial binary absorbing subuniverse in \mathbf{B} and $C = \{a \in A \mid \forall b \in B : (a, b) \in R\}$. This notion was motivated by central relations defining maximal

clones on finite sets (see Section 5.2.5 in Reference [44]), and it is very similar to ternary absorption (see Corollary 7.10.2).

3.6 CSP Instance

An instance of the constraint satisfaction problem is called a *CSP instance*. Sometimes we use the same letter for a CSP instance and for the set of all constraints of this instance. For a variable z by D_z , we denote the domain of the variable z .

We say that $z_1 - C_1 - z_2 - \dots - C_{l-1} - z_l$ is a *path* in a CSP instance Θ if z_i, z_{i+1} are in the scope of C_i for every i . We say that a *path* $z_1 - C_1 - z_2 - \dots - C_{l-1} - z_l$ *connects* b and c if there exists $a_i \in D_{z_i}$ for every i such that $a_1 = b, a_l = c$, and the projection of C_i onto z_i, z_{i+1} contains the tuple (a_i, a_{i+1}) .

A CSP instance is called *1-consistent* if every constraint of the instance is subdirect. A CSP instance is called *cycle-consistent* if it is 1-consistent and for every variable z and $a \in D_z$ any path starting and ending with z in Θ connects a and a . Other types of local consistency and its connection with the complexity of CSP are considered in Reference [43]. A CSP instance Θ is called *linked* if for every variable z occurring in the scope of a constraint of Θ and every $a, b \in D_z$ there exists a path starting and ending with z in Θ that connects a and b .

Suppose $X' \subseteq X$. Then we can define a projection of Θ onto X' , that is, a CSP instance where variables are elements of X' and constraints are projections of the constraints of Θ onto the intersection of their scopes with X' , ignoring any constraint whose scope does not intersect X' . We say that an instance Θ is *fragmented* if the set of variables X can be divided into two disjoint sets X_1 and X_2 such that each of them contains a variable from the scope of a constraint of Θ , and the constraint scope of any constraint of Θ either has variables only from X_1 or only from X_2 . Thus, if an instance is fragmented, then it can be divided into several nontrivial instances.

A CSP instance Θ is called *irreducible* if any instance $\Theta' = (X', D', C')$ such that $X' \subseteq X, D'_x = D_x$ for every $x \in X'$, and every constraint of Θ' is a projection of a constraint from Θ on some set of variables is fragmented, or linked, or its solution set is subdirect.

We say that a constraint $C_1 = ((y_1, \dots, y_t); \rho_1)$ is *weaker or equivalent* to a constraint $C_2 = ((z_1, \dots, z_s); \rho_2)$ if $\{y_1, \dots, y_t\} \subseteq \{z_1, \dots, z_s\}$ and C_2 implies C_1 . In other words, the second condition says that the solution set to $\Theta_1 := (\{z_1, \dots, z_s\}, (D_{z_1}, \dots, D_{z_s}), C_1)$ contains the solution set to $\Theta_2 := (\{z_1, \dots, z_s\}, (D_{z_1}, \dots, D_{z_s}), C_2)$. We say that C_1 is *weaker than* C_2 if C_1 is weaker or equivalent to C_2 but C_1 does not imply C_2 .

The following remark justifies weakening constraints of the instance in the algorithm (this remark follows from Lemma 6.1).

Remark 1. Suppose $\Theta = \langle X; D; C \rangle$ and $\Theta' = \langle X'; D'; C' \rangle$ are CSP instances such that $X' \subseteq X, D'_x = D_x$ for every $x \in X'$, and every constraint of Θ' is weaker or equivalent to a constraint of Θ . If Θ is cycle-consistent and irreducible, then so is Θ' .

We say that a variable y_i of the constraint $((y_1, \dots, y_t); \rho)$ is *dummy* if ρ does not depend on its i th variable.

Remark 2. Adding a dummy variable to a constraint and removing of a dummy variable do not affect the property of being cycle-consistent and irreducible.

Let $D'_i \subseteq D_i$ for every i . A constraint C of Θ is called *crucial* in (D'_1, \dots, D'_n) if it has no dummy variables, Θ has no solutions in (D'_1, \dots, D'_n) but the replacement of $C \in \Theta$ by all weaker constraints gives an instance with a solution in (D'_1, \dots, D'_n) . A CSP instance Θ is called *crucial* in (D'_1, \dots, D'_n) if it has at least one constraint and every constraint of Θ is crucial in (D'_1, \dots, D'_n) .

Remark 3. Suppose Θ has no solutions in (D'_1, \dots, D'_n) . We can replace each constraint by its projection onto its non-dummy variables. Then we iteratively replace every constraint by all weaker constraints having no dummy variables until it is crucial. Finally, we get a CSP instance that is crucial in (D'_1, \dots, D'_n) .

4 ALGORITHM

4.1 Main Part

Suppose we have a constraint language Γ_0 that is preserved by an idempotent WNU operation. As it was mentioned before, Γ_0 is also preserved by a special WNU operation w . Let k_0 be the maximal arity of the relations in Γ_0 . By Γ we denote the set of all relations of arity at most k_0 that are preserved by w . Obviously, $\Gamma_0 \subseteq \Gamma$, therefore every instance of $\text{CSP}(\Gamma_0)$ is an instance of $\text{CSP}(\Gamma)$.

In this section, we provide an algorithm that solves $\text{CSP}(\Gamma)$ in polynomial time. Suppose we have a CSP instance $\Theta = \langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$, where $\mathbf{X} = \{x_1, \dots, x_n\}$ is a set of variables, $\mathbf{D} = \{D_1, \dots, D_n\}$ is a set of the respective domains, $\mathbf{C} = \{C_1, \dots, C_q\}$ is a set of constraints. Let the arity of the WNU w be equal to m .

The main part of the algorithm (function SOLVE) is an iterative loop; in each pass through the loop, the algorithm calls a subroutine ANSWERORREDUCE whose job is to find a reduction of a domain or to terminate with the final answer. The reduction returned by the function should satisfy the following property: If Θ has a solution, then it has a solution after the reduction. If the reduction was found, then we apply the function REDUCE, which takes an instance $\Theta = \langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$ and a domain set $\mathbf{D}' = (D'_1, \dots, D'_n)$ and returns a new instance $\langle \mathbf{X}, \mathbf{D}', \mathbf{C}' \rangle$, where $\mathbf{C}' = \{((x_{i_1}, \dots, x_{i_s}), \rho \cap (D'_{i_1} \times \dots \times D'_{i_s})) \mid ((x_{i_1}, \dots, x_{i_s}), \rho) \in \mathbf{C}\}$.

```

1: function SOLVE( $\Theta$ )
2:   Input:  $\text{CSP}(\Gamma)$  instance  $\Theta = \langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$ ,  $\mathbf{X} = (x_1, \dots, x_n)$ ,  $\mathbf{D} = (D_1, \dots, D_n)$ 
3:   repeat
4:      $\text{Output} := \text{ANSWERORREDUCE}(\Theta)$ 
5:     if  $\text{Output} = \text{"Solution"}$  then return "Solution"
6:     if  $\text{Output} = \text{"No solution"}$  then return "No solution"
7:     if  $\text{Output} = (x_i, U)$  then  $\triangleright \emptyset \neq U \subset D_i$ 
8:        $\Theta := \text{REDUCE}(\Theta, (D_1, \dots, D_{i-1}, U, D_{i+1}, \dots, D_n))$   $\triangleright \text{Set } D_i = U$ 
9:   until Done

```

The function ANSWERORREDUCE (see the pseudocode) checks different types of consistency such as cycle-consistency and irreducibility and reduces a domain if the instance is not consistent. If it is consistent, then either it reduces a domain to a proper strong subset, or it uses SOLVELINEARCASE to solve the remaining case.

First, the function ANSWERORREDUCE checks whether the instance Θ is cycle-consistent (function CHECKCYCLECONSISTENCY). If it is not cycle-consistent, then either some domain can be reduced, or the instance has no solutions. In both cases, we terminate the function and return the result. If it is cycle-consistent, then we go on.

If the size of every domain is one, then it returns that a solution was found.

Then we check whether the instance is irreducible (function CHECKIRREDUCIBILITY). If it is not irreducible, then we return how to reduce some domain or return that there is no solutions; otherwise, we go on.

After that, we check a different type of consistency (function CHECKWEAKERINSTANCE). We make a copy of Θ and simultaneously replace every constraint by all weaker constraints without dummy variables. Recursively calling the algorithm, we check that the obtained instance has a

```

1: function ANSWERORREDUCE( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta = (X, D, C), X = (x_1, \dots, x_n), D = (D_1, \dots, D_n)$ 
3:   Output := CHECKCYCLECONSISTENCY( $\Theta$ )
4:   if Output  $\neq$  "Ok" then return Output
5:   if  $|D_i| = 1$  for every  $i$  then return "Solution"
6:   Output := CHECKIRREDUCIBILITY( $\Theta$ )
7:   if Output  $\neq$  "Ok" then return Output
8:   Output := CHECKWEAKERINSTANCE( $\Theta$ )
9:   if Output  $\neq$  "Ok" then return Output
10:  if  $B_i$  is a nontrivial binary absorbing subuniverse of  $D_i$  then return  $(x_i, B_i)$ 
11:  if  $C_i$  is a nontrivial center of  $D_i$  then return  $(x_i, C_i)$ 
12:  if  $\sigma$  is a proper congruence on  $D_i$  and  $(D_i; w)/\sigma$  is polynomially complete then
13:    Choose an equivalence class  $E$  of  $\sigma$ 
14:  return  $(x_i, E)$ 
  return SOLVELINEARCASE( $\Theta$ )

```

solution with $x_i = b$ for every $i \in \{1, 2, \dots, n\}$ and $b \in D_i$. If not, then reduce D_i to the projection onto x_i of the solution set of the obtained instance. Otherwise, go on.

By Theorem 5.5, we cannot pass from an instance having solutions to an instance having no solutions when reduce a domain to a nontrivial binary absorbing subuniverse or to a nontrivial center. Thus, if D_i has a nontrivial binary absorbing subuniverse $B_i \subsetneq D_i$ for some i , then we reduce D_i to B_i . Similarly, if D_i has a nontrivial center $C_i \subsetneq D_i$ for some i , then we reduce D_i to C_i .

By Theorem 5.6 we cannot pass from an instance having solutions to an instance having no solutions when reduce a domain to an equivalence class of a proper congruence σ such that $(D_i; w)/\sigma$ is polynomially complete. Thus, if such a congruence on D_i exists, then we reduce D_i to its equivalence class.

By Theorem 5.1, it remains to consider the case when on every domain D_i of size greater than 1 there exists a proper congruence σ such that $(D_i; w)/\sigma$ is isomorphic to $(\mathbb{Z}_p; x_1 + \dots + x_m)$ for some p . In this case, the problem is solved by the function SOLVELINEARCASE, which will be described in the next subsection. A detailed description and a pseudocode for the functions CHECKCYCLECONSISTENCY, CHECKIRREDUCIBILITY, and CHECKWEAKERINSTANCE will be given in Subsection 4.4.

4.2 Linear Case

In this section, we define the function SOLVELINEARCASE (see the pseudocode). For every i , let σ_i be the minimal linear congruence on D_i , which is the smallest congruence σ such that $(D_i; w)/\sigma$ is linear. Then $(D_i; w)/\sigma_i$ is isomorphic to $(\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_l}; x_1 + \dots + x_m)$ for prime numbers p_1, \dots, p_l . Recall that we apply the function SOLVELINEARCASE only if σ_i is proper for every i such that $|D_i| > 1$. We will show that modulo these congruences the instance can be viewed as a system of linear equations in fields.

We denote D_i/σ_i by L_i and define a new CSP instance Θ_L with domains L_1, \dots, L_n as follows. To every constraint $((x_{i_1}, \dots, x_{i_s}); \rho) \in \Theta$ we assign a constraint $((x'_{i_1}, \dots, x'_{i_s}); \rho')$, where $\rho' \subseteq L_{i_1} \times \dots \times L_{i_s}$ and $(E_1, \dots, E_s) \in \rho' \Leftrightarrow (E_1 \times \dots \times E_s) \cap \rho \neq \emptyset$. The constraints of Θ_L are all constraints that are assigned to the constraints of Θ . The function generating the instance Θ_L from Θ is called FACTORIZEINSTANCE in the pseudocode. Note that Θ_L is a CSP instance but not necessarily an instance in the constraint language Γ .

Since each L_i is isomorphic to some $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s}$, we may define a natural bijective mapping $\psi : \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r} \rightarrow L_1 \times \cdots \times L_n$, and assign a variable z_i to every \mathbb{Z}_{p_i} . Since every relation on $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$ preserved by $x_1 + \cdots + x_m$ is known (see Lemma 7.20) to be a conjunction of linear equations, the instance Θ_L can be viewed as a system of linear equations over z_1, \dots, z_r . Note that every equation is an equation in \mathbb{Z}_p but p can be different for different equations, and only variables with the same domain \mathbb{Z}_p may appear in one equation.

```

1: function SOLVELINEARCASE( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta = (X, D, C)$ ,  $X = (x_1, \dots, x_n)$ ,  $D = (D_1, \dots, D_n)$ 
3:    $\Theta_L := \text{FACTORIZEINSTANCE}(\Theta)$ 
4:    $Eq := \emptyset$  ▷ The equations we add to  $\Theta_L$ 
5:   repeat
6:      $\phi := \text{SOLVELINEARSYSTEM}(\Theta_L \cup Eq)$  ▷  $\phi(\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k})$  is the solution set of  $\Theta_L \cup Eq$ 
7:     if  $\phi = \emptyset$  then return “No solution”
8:     if  $\text{SOLVE}(\text{REDUCE}(\Theta, \phi(0, 0, \dots, 0))) = \text{“Solution”}$  then return “Solution”
9:     else if  $k=0$  then return “No solution” ▷  $\Theta_L$  has just one solution
10:     $\Theta' := \text{REMOVETRIVIALITIES}(\Theta)$ 
11:    repeat ▷ Try to weaken  $\Theta'$ 
12:       $\text{Changed} := \text{false}$ 
13:      for  $C \in \Theta'$  do
14:         $\Omega := \text{REMOVETRIVIALITIES}(\text{WEAKENCONSTRAINT}(\Theta', C))$ 
15:        if  $\neg \text{CHECKALLTUPLES}(\Omega, \phi)$  then
16:           $\Theta' := \Omega$ 
17:           $\text{Changed} := \text{true}$ 
18:          break
19:    until  $\neg \text{Changed}$  ▷  $\Theta'$  cannot be weakened anymore
20:    if  $\Theta'$  is not linked then
21:       $Eq := Eq \cup \text{FINDEQUATIONSNONLINKED}(\Theta')$ 
22:    else
23:       $Eq := Eq \cup \{\text{FINDONEEQUATIONLINKED}(\Theta', \phi)\}$ 
24:  until Done

```

As it was described in Section 2, we consider the set A , which is the solution set of Θ factorized by the congruences $\sigma_1, \dots, \sigma_n$, and the set B , which is the solution set of Θ_L . We know that $A \subseteq B$ and we want to check whether A is empty. We iteratively add new equations to the set Θ_L maintaining the property that $A \subseteq B$ and therefore reduce the dimension of B . We start with the empty set of equations Eq (line 4 of the pseudocode).

Then we apply the function `SOLVELINEARSYSTEM` that solves the system of linear equations $\Theta_L \cup Eq$ using Gaussian elimination. If the system has no solutions, then Θ has no solutions and we are done. Otherwise, we choose independent variables y_1, \dots, y_k , then the general solution (the set B) can be written as an affine mapping $\phi : \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k} \rightarrow L_1 \times \cdots \times L_n$. Denote $Z = \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k}$, then any solution of $\Theta_L \cup Eq$ can be obtained as $\phi(a_1, \dots, a_k)$ for some $(a_1, \dots, a_k) \in Z$.

Note that for any tuple $(a_1, \dots, a_k) \in Z$ we can check recursively whether Θ has a solution in $\phi(a_1, \dots, a_k)$ (i.e., whether $\phi(a_1, \dots, a_k) \in A$). To do this, we just need to reduce the domains to the solution (function `REDUCE`) and solve an easier CSP instance (on smaller domains). Similarly, we can check whether Θ has a solution in $\phi(a_1, \dots, a_k)$ for every $(a_1, \dots, a_k) \in Z$ (i.e., whether $A = B$). Since A and B are subuniverses of $L_1 \times \cdots \times L_n$ (almost subspaces), we just need to check

the existence of a solution in $\phi(0, \dots, 0)$ and $\phi(0, \dots, 0, 1, 0, \dots, 0)$ for any position of 1. See the pseudocode of the function `CHECKALLTUPLES` for the last procedure.

```

1: function CHECKALLTUPLES( $\Theta, \phi$ )
2:   Input: CSP(T) instance  $\Theta$ , a solution of a linear system of equations  $\phi$ 
3:   if SOLVE(REDUCE( $\Theta, \phi(0, \dots, 0)$ )) = "No solution" then return false
4:   for  $i = 1, 2, \dots, k$  do
5:      $t := (0, \dots, 0, 1, 0, \dots, 0)$ 
6:     if SOLVE(REDUCE( $\Theta, \phi(t)$ )) = "No solution" then return false
   return true;

```

Let us go back to the function `SOLVELINEARCASE`. After solving the linear system, we check whether there exists a solution of Θ corresponding to the solution $\phi(0, 0, \dots, 0)$ of $\Theta_L \cup Eq$. If $k = 0$, i.e., $\Theta_L \cup Eq$ has only one solution, then we denote this solution by $\phi(0, 0, \dots, 0)$. If Θ has a solution in $\phi(0, \dots, 0)$, then it remains to return the result "Solution." If it has no solutions and $k = 0$, then return the result "No solution."

At this point (line 10 of the pseudocode of `SOLVELINEARCASE`), we have the property that the set B is of dimension at least 1, and $A \neq B$, since we found a solution $\phi(0, \dots, 0)$ of the system of linear equations without the corresponding solution of Θ .

Then we iteratively remove from Θ all constraints that are weaker than some other constraints of Θ , remove all constraints without non-dummy variables, and replace every constraint by its projection onto non-dummy variables. This procedure we denote by the function `REMOVETRIVIALITIES`. In the pseudocode of `SOLVELINEARCASE`, we denote the obtained instance by Θ' .

Then we try to make the constraints of Θ' weaker maintaining the property that $A' \neq B$, where A' is the solution set of Θ' factorized by the congruences $\sigma_1, \dots, \sigma_n$. Precisely, we choose a constraint C , replace it by all weaker constraints without dummy variables (function `WEAKENCONSTRAINT`), apply `REMOVETRIVIALITIES`, and check using the function `CHECKALLTUPLES` whether $A' = B$. If not, then we replace Θ' by the new weaker instance.

Suppose we cannot make any constraint weaker maintaining the property $A' \neq B$. Then Θ' has no solutions in $\phi(b_1, \dots, b_k)$ for some $(b_1, \dots, b_k) \in Z$, but if we replace any constraint $C \in \Theta'$ by all weaker constraints, then we get an instance that has a solution in $\phi(a_1, \dots, a_k)$ for every $(a_1, \dots, a_k) \in Z$. Therefore, Θ' is crucial in $\phi(b_1, \dots, b_k)$. Note that by Lemma 6.1 the instance Θ' is still cycle-consistent and irreducible. Also, Θ' is not fragmented, because it is crucial.

Then, in line 20 of the function `SOLVELINEARCASE`, we have two options.

If Θ' is not linked, then using the function `FINDEQUATIONSNONLINKED` we calculate its solution set factorized by the congruences (the set A'). This solution set can be defined by a set of linear equations, which we add to Eq and therefore replace B by $A' \cap B$. Thus, we made B smaller and we still have the property $A \subseteq B$, since A' is the factorized solution set of the instance Θ' , which is weaker than Θ .

If Θ' is linked, then by Theorem 5.7 either $A' = \emptyset$, or the dimension of A' is equal to the dimension of B minus 1, which allows us to find a new linear equation by polynomially many queries "Does there exist a solution of Θ' in $\phi(a_1, \dots, a_k)$?" We calculate this new equation by the function `FINDONEEQUATIONLINKED`, which will be defined in the next section as well as the function `FINDEQUATIONSNONLINKED`. Note that the new equation can be " $0 = 1$ " if $A' = \emptyset$.

After new equations found, we go back to line 6 of the function `SOLVELINEARCASE` and solve a system of linear equations again. Since every time we reduce the dimension of B by at least one, the procedure will stop in at most r steps.

In this section, we define the functions `FINDONEEQUATIONLINKED`, `FINDONEEQUATIONNONLINKED`, and `FINEQUATIONSNONLINKED`, which allow us to find new equations defining the set A' .

First, we explain how the function `FINDONEEQUATIONLINKED` works. Suppose V is an affine subspace of \mathbb{Z}_p^k of dimension $k - 1$, and thus V is the solution set of a linear equation $c_1y_1 + \dots + c_ky_k = c_0$. Then the coefficients c_0, c_1, \dots, c_k can be learned (up to a multiplicative constant) by $(p \cdot k + 1)$ queries of the form “ $(a_1, \dots, a_k) \in V$?” as follows. First, we need at most $(k + 1)$ queries to find a tuple $(t_1, \dots, t_k) \notin V$. To do this we just check all tuples with 0s and at most one 1 (lines 4–11 of the pseudocode). Then, to find this equation, it is sufficient to check for every a and every i whether the tuple $(t_1, \dots, t_{i-1}, a, t_{i+1}, \dots, t_k)$ satisfies this equation (lines 13–19 of the pseudocode). Here the query is performed by the reduction of all domains to the corresponding solution (the function `REDUCE`) and a recursive call of the main function `SOLVE`.

We start with an empty set of equations E and claim that the first variable is independent, and by I we denote the set of independent variables (see the pseudocode of `FIND_EQUATIONS_NONLINKED`). Assume that we already found all the equations over z_1, \dots, z_{j-1} , i.e., we described the projection of A' onto z_1, \dots, z_{j-1} .

Journal of the ACM, Vol. 67, No. 5, Article 30. Publication date: August 2020.

```

1: function FINDEQUATIONSNONLINKED( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ 
3:    $I := \{1\}$   $\triangleright I$  is the set of independent variables
4:    $E := \emptyset$   $\triangleright$  We start with an empty set of equations
5:   for  $j = 1, 2, \dots, r$  do
6:      $e := \text{FINDONEEQUATIONNONLINKED}(\Theta, I \cup \{j\})$ 
7:     if  $e = "0 = 0"$  then  $\triangleright j$ th variable is independent
8:        $I := I \cup \{j\}$ 
9:     else if  $e = "0 = 1"$  then return "No solution"
10:    else
11:       $E := E \cup e$   $\triangleright$  Add the equation we found
  return  $E$ 

```

check a query: In `FINDEQUATIONSNONLINKED` we use the function `CHECKTUPLE` instead of `REDUCE` and `SOLVE` (see the pseudocode).

If the new equation was found and this equation is not trivial, then we add this equation to E and claim that z_j is not independent. If the equation we found is " $0 = 0$," then add x_j to the set of independent variables and go to the next variable.

It remains to explain how the function `CHECKTUPLE` works. As an input it takes an instance Θ , a set of variables I , and a tuple t of length $|I|$. The restriction of the variables from I to the tuple t implies the restrictions L'_1, \dots, L'_n of the domains L_1, \dots, L_n . Put $D'_i = \bigcup_{E \in L'_i} E$ for every i . Then we add unary constraints $x_i \in D'_i$ to Θ and solve the obtained instance by the function `SOLVENONLINKED`, which works only for non-linked instances and will be defined in the next section.

4.4 Remaining Functions

In this subsection, we define the functions `CHECKCYCLECONSISTENCY`, `CHECKIRREDUCIBILITY`, and `CHECKWEAKERINSTANCE` which were used in Subsection 4.1 and function `SOLVENONLINKED` from Subsection 4.3.

First, we define the function `CHECKCYCLECONSISTENCY`. To check cycle-consistency it is sufficient to use constraint propagation providing a variant of (2,3)-consistency (see the pseudocode). First, for every pair of variables (x_i, x_j) , we consider the intersections of projections of all constraints onto these variables. The corresponding relation we denote by $\rho_{i,j}$. Then, for every $i, j, k \in \{1, 2, \dots, n\}$, we replace $\rho_{i,j}$ by $\rho'_{i,j}$ where $\rho'_{i,j}(x, y) = \exists z \rho_{i,j}(x, y) \wedge \rho_{i,k}(x, z) \wedge \rho_{k,j}(z, y)$.

We repeat this procedure while we can change some $\rho_{i,j}$. If in the end we get a relation $\rho_{i,j}$ that is not subdirect in $D_i \times D_j$, then we can either reduce D_i or D_j or, if $\rho_{i,j}$ is empty, state that there are no solutions. If every relation $\rho_{i,j}$ is subdirect in $D_i \times D_j$, then we claim (see Lemma 5.3) that the original CSP instance is cycle-consistent.

Let us explain how `CHECKIRREDUCIBILITY` works. For every $k \in \{1, 2, \dots, n\}$ and every maximal congruence σ_k on D_k , we do the following. We start with the partition σ_k of the k th variable, so we put $I = \{k\}$ (line 4 of the pseudocode), which is the set of variables with a partition. Then we try to extend the partition of D_k to other domains. We choose a constraint having x_k in the scope, choose another variable x_j , and consider the projection of C onto x_k, x_j , which we denote by δ . Since σ_k is maximal, we may have two possibilities: Either all equivalence classes of σ_k are connected in δ or none of the equivalence classes are connected in δ . In the second case, the partition of D_k generates a partition of D_j with the same number of classes, and we add j to I (lines 10–15 of the pseudocode).

```

1: function FINDONEEQUATIONNONLINKED( $\Theta, I$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta, I = \{i_1, \dots, i_h\}$  a set of variables
3:    $t := \emptyset$  ▷ We search for a tuple  $t$  outside of the solution set
4:   if  $\neg \text{CHECKTUPLE}(\Theta, I, (0, \dots, 0))$  then
5:      $t := (0, \dots, 0)$ 
6:   else
7:     for  $j = 1, 2, \dots, h$  do
8:        $t' := \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_j$ 
9:       if  $\neg \text{CHECKTUPLE}(\Theta, I, t')$  then
10:         $t := t'$ 
11:      break
12:   if  $t = \emptyset$  then return “0 = 0”
13:   for  $j = 1, 2, \dots, h$  do
14:      $b_j := 0$ 
15:     for  $a \in \mathbb{Z}_{p_{i_j}} \setminus \{t(j)\}$  do
16:        $t' := t$ 
17:        $t'(j) := a$ 
18:       if  $\text{CHECKTUPLE}(\Theta, I, t')$  then
19:          $b_j := 1/(a - t(j))$ 
20:   return “ $b_1(z_{i_1} - t(1)) + \dots + b_h(z_{i_h} - t(r)) = 1$ ”

```

```

1: function CHECKTUPLE( $\Theta, I, t$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta, I$  a subset of variables,  $t$  a tuple of length  $|I|$ 
3:    $R := \{\alpha \in \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_r} \mid \text{pr}_I(\alpha) = t\}$  ▷ We do not really calculate  $R$ 
4:   for  $i = 1, 2, \dots, n$  do
5:      $D'_i := \bigcup_{E \in \text{pr}_i(\psi(R))} E$  ▷ We calculate  $D'_i$ 
6:   if  $\text{SOLVENONLINKED}(\Theta \wedge (x_1 \in D'_1) \wedge \dots \wedge (x_n \in D'_n)) = \text{“Solution”}$  then return true
7:   elsereturn false

```

We continue this procedure while we can add new variables to I . As a result, we get a set I and a partition of D_i for every $i \in I$. Put $\mathbf{X}' = \{x_i \mid i \in I\}$. Then, the projection of Θ onto \mathbf{X}' can be split into several instances on smaller domains, and each of them can be solved using recursion. Thus, we can check whether the solution set of the projection of the instance onto \mathbf{X}' is subdirect or empty. If it is empty, then we state that there are no solutions. If it is not subdirect, then we can reduce the corresponding domain. If it is subdirect, then we go to the next $k \in \{1, 2, \dots, n\}$ and the next maximal congruence σ_k on D_k and repeat the procedure. If for all k and all maximal congruences the solution set of the obtained instance is subdirect, then the instance is irreducible (see Lemma 5.4).

Define the function **CHECKWEAKERINSTANCE**, which checks that if we simultaneously weaken every constraint then the solution set of the obtained instance is subdirect. Thus, we weaken every constraint of Θ (function **WEAKENEVERYCONSTRAINT** in the pseudocode), that is, we make a copy of Θ and replace each constraint by all weaker constraints without dummy variables. Recursively calling the algorithm, check that the obtained instance has a solution with $x_i = b$ for every $i \in \{1, 2, \dots, n\}$ and $b \in D_i$. If not, then reduce D_i to the projection onto x_i of the solution set of the obtained instance. Otherwise, go on.

It remains to define the function **SOLVENONLINKED**, which solves an instance that is not linked and not fragmented (see the pseudocode). Such an instance can be split into several instances

```

1: function CHECKCYCLECONSISTENCY( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ 
3:   for  $i, j \in \{1, 2, \dots, n\}$  do ▷ Calculate binary projections  $\rho_{i,j}$ 
4:      $\rho_{i,j} := D_i \times D_j$ 
5:     for  $C \in \Theta$  do
6:        $\rho_{i,j} := \rho_{i,j} \cap \text{pr}_{x_i, x_j} C$  ▷  $\text{pr}_{x_i, x_j} C$  is the projection of  $C$  onto  $x_i, x_j$ 
7:   repeat ▷ Propagate constraints to reduce  $\rho_{i,j}$ 
8:      $\text{Changed} := \text{false}$ 
9:     for  $i, j, k \in \{1, 2, \dots, n\}$  do
10:       $\rho'_{i,j}(x, y) := \exists z \rho_{i,j}(x, y) \wedge \rho_{i,k}(x, z) \wedge \rho_{k,j}(z, y)$ 
11:      if  $\rho_{i,j} \neq \rho'_{i,j}$  then
12:         $\rho_{i,j} := \rho'_{i,j}$ 
13:         $\text{Changed} := \text{true}$ 
14:   until  $\neg \text{Changed}$  ▷ We cannot reduce  $\rho_{i,j}$  anymore
15:   for  $i, j \in \{1, 2, \dots, n\}$  do
16:     if  $\rho_{i,j} = \emptyset$  then return "No solution"
17:     if  $\text{pr}_1(\rho_{i,j}) \neq D_i$  then return  $(x_i, \text{pr}_1(\rho_{i,j}))$ 
18:     if  $\text{pr}_2(\rho_{i,j}) \neq D_j$  then return  $(x_j, \text{pr}_2(\rho_{i,j}))$ 
   return "Ok"

```

on smaller domains. First, we consider the set X' of all variables appearing in the constraints of the instance and take the projection of the instance onto X' . Then we consider each linked component, that is, elements that can be connected by a path in the instance. Since the instance is cycle-consistent, the division into linked components defines a congruence on every domain (see Lemma 6.2), and each block of this congruence is a subuniverse of the domain. Thus, each linked component can be viewed as a CSP instance in a constraint language Γ on smaller domains, which can be solved using the recursion. If at least one of them has a solution, then the original instance has a solution.

5 CORRECTNESS OF THE ALGORITHM

5.1 Rosenberg Completeness Theorem

The main idea of the algorithm is based on a beautiful result obtained by Ivo Rosenberg in 1970, who found all maximal clones on a finite set. Applying this result to the clone generated by a WNU together with all constant operations, we can show that every algebra with a WNU operation has a nontrivial binary absorbing subuniverse, or a nontrivial center, or it is polynomially complete or linear modulo some proper congruence.

THEOREM 5.1. *Suppose $\mathbf{A} = (A; w)$ is a finite algebra, where w is a special WNU of arity m . Then one of the following conditions holds:*

- (1) *there exists a nontrivial binary absorbing subuniverse $B \subsetneq A$,*
- (2) *there exists a nontrivial center $C \subsetneq A$,*
- (3) *there exists a proper congruence σ on A such that $(A; w)/\sigma$ is polynomially complete,*
- (4) *there exists a proper congruence σ on A such that $(A; w)/\sigma$ is isomorphic to $(\mathbb{Z}_p; x_1 + \dots + x_m)$.*

PROOF. Let us prove this statement by induction on the size of A . If we have a nontrivial binary absorbing subuniverse in A , then there is nothing to prove. Assume that A has no nontrivial binary

```

1: function CHECKIRREDUCIBILITY( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ 
3:   for  $k = 1, \dots, n$  do
4:     for  $\sigma_k = \{E_k^1, \dots, E_k^t\}$  is a maximal congruence on  $D_k$  do
5:        $I := \{k\}$ 
6:       repeat
7:          $\text{Changed} := \text{false}$ 
8:         for  $C \in \Theta, i \in I, j \notin I$  such that  $x_i$  and  $x_j$  are in the scope of  $C$  do
9:            $\delta := \text{pr}_{x_i, x_j} C$   $\triangleright \text{pr}_{x_i, x_j} C$  is the projection of  $C$  onto  $x_i, x_j$ 
10:          for  $u = 1, 2, \dots, t$  do  $\triangleright$  Calculate the partition on  $D_j$ 
11:             $E_j^u := \{b \in D_j \mid \exists a \in E_i^u : (a, b) \in \delta\}$ 
12:            if  $E_j^1, \dots, E_j^t$  are disjoint then
13:               $I := I \cup \{j\}$ 
14:               $\text{Changed} := \text{true}$ 
15:              break
16:       until  $\neg \text{Changed}$ 
17:       for  $i \in I$  do
18:          $D'_i := \emptyset$ 
19:         for  $a \in D_i$  do
20:           Choose  $u$  such that  $a \in E_i^u$ 
21:           for  $j = 1, 2, \dots, n$  do
22:             if  $j = i$  then
23:                $E_j := \{a\}$ 
24:             else if  $j \in I$  then
25:                $E_j := E_j^u$ 
26:             else
27:                $E_j := D_j$ 
28:            $X' := \{x_i \mid i \in I\}$ 
29:           if  $\text{SOLVE}(\text{pr}_{X'}(\text{REDUCE}(\Theta, (E_1, \dots, E_n)))) = \text{"Solution"}$  then
30:              $D'_i := D'_i \cup \{a\}$ 
31:           if  $D'_i = \emptyset$  then return "No solution"
32:           else if  $D'_i \neq D_i$  then return  $(x_i, D'_i)$ 
return "Ok"

```

absorbing subuniverse. Let M be the clone generated by w and all constant operations on A . If M is the clone of all operations, then $(A; w)$ is polynomially complete.

Otherwise, by Rosenberg's Theorem [53], M belongs to one of the following maximal clones.

- (1) Maximal clone of monotone operations, that is, the clone of operations preserving a partial order relation with the greatest and the least element;
- (2) Maximal clone of autodual operations, that is, the clone of operations preserving the graph of a permutation of a prime order without a fixed element;
- (3) Maximal clone defined by an equivalence relation;
- (4) Maximal clone of quasi-linear operations;
- (5) Maximal clone defined by a central relation;
- (6) Maximal clone defined by an h -regularly generated (or h -universal) relation.

Let us consider all the cases.

- (1) As we assumed, there is no nontrivial binary absorbing subuniverse on A . Hence, the least element of the partial order can be viewed as a center by letting $\mathbf{B} = \mathbf{A}$ and using the partial order relation as a subdirect subuniverse of $\mathbf{A} \times \mathbf{B}$ (the least element is connected with all other elements in the partial order relation). Thus, we have a nontrivial center in A .
- (2) Constants are not autodual operations. This case cannot happen.
- (3) Let δ be a maximal congruence on \mathbf{A} . We consider a factor algebra $(A; w)/\delta$ and apply the inductive assumption.
 - (a) If \mathbf{A}/δ has a binary absorbing subuniverse $B' \subseteq A/\delta$, then $\bigcup_{E \in B'} E$ is a binary absorbing subuniverse of A with the same term operation.
 - (b) If \mathbf{A}/δ has a nontrivial center $C' \subseteq A/\delta$ witnessed by a subdirect relation $R' \subseteq A/\delta \times B$, then $\bigcup_{E \in C'} E$ is a nontrivial center of A witnessed by $R = \bigcup_{(E,b) \in R'} E \times \{b\}$.
 - (c) Suppose $(\mathbf{A}/\delta)/\sigma$ is polynomially complete. Since δ is a maximal congruence, σ is the equality relation and \mathbf{A}/δ is polynomially complete.
 - (d) Suppose $(\mathbf{A}/\delta)/\sigma$ is isomorphic to $(\mathbb{Z}_p; x_1 + \dots + x_m)$. Since δ is a maximal congruence, σ is the equality relation and \mathbf{A}/δ is isomorphic to $(\mathbb{Z}_p; x_1 + \dots + x_m)$.
- (4) By Lemma 6.4 from Reference [62], we know that $w(x_1, \dots, x_m) = x_1 + \dots + x_m$, where $+$ is the operation in an Abelian group. We assume that \mathbf{A} has no nontrivial congruences; otherwise, we refer to case (3). Then the algebra \mathbf{A} is simple and isomorphic to $(\mathbb{Z}_p; x_1 + \dots + x_m)$ for a prime number p .
- (5) Let ρ be a central relation of arity k preserved by w . It is not hard to see that the existence of a nontrivial binary absorbing subuniverse on $\underbrace{\mathbf{A} \times \dots \times \mathbf{A}}_{k-1}$ implies the existence of a nontrivial binary absorbing subuniverse on \mathbf{A} (see Lemma 7.3). Since there is no nontrivial binary absorbing subuniverse on \mathbf{A} and the relation ρ contains all tuples (b_1, \dots, b_k) such that b_1 is from the center of ρ , the center of ρ is a center of A by letting $\mathbf{B} = \underbrace{\mathbf{A} \times \dots \times \mathbf{A}}_{k-1}$.
- (6) By Corollary 5.10 from Reference [62] this case cannot happen. \square

5.2 The Algorithm is Polynomial

LEMMA 5.2. *The depth of the recursion in the algorithm is less than $|A| + |\Gamma|$.*

PROOF. We use the recursion in the functions SOLVELINEARCASE, FINDONEEQUATIONLINKED, CHECKALLTUPLES, CHECKIRREDUCIBILITY, CHECKWEAKERINSTANCE, SOLVENONLINKED. In each of them but CHECKWEAKERINSTANCE, we reduce all domains of size greater than 1 before using the recursion, and we never increase the domain. Therefore, every path in the recursion tree contains at most $|A|$ calls of the function SOLVE in the above functions.

```

1: function CHECKWEAKERINSTANCE( $\Theta$ )
2:   Input: CSP( $T$ ) instance  $\Theta$ 
3:    $\Theta' = \text{WEAKENEVERYCONSTRAINT}(\Theta)$ 
4:   for  $i = 1, \dots, n$  do
5:      $D'_i := \emptyset$ 
6:     for  $a \in D_i$  do
7:        $\text{Output} := \text{SOLVE}(\text{REDUCE}(\Theta', (D_1, \dots, D_{i-1}, \{a\}, D_{i+1}, \dots, D_n)))$ 
8:       if  $\text{Output} = \text{"Solution"}$  then
9:          $D'_i := D'_i \cup \{a\}$ 
10:    if  $D'_i = \emptyset$  then return "No solution"
11:    else if  $D'_i \neq D_i$  then return  $(x_i, D'_i)$ 
return "Ok"

```

```

1: function SOLVENONLINKED( $\Theta$ )
2:   Input: CSP( $\Gamma$ ) instance  $\Theta$ 
3:    $X' := \text{Var}(\Theta)$  ▷ Choose variables that appear in  $\Theta$ 
4:    $\Theta' := \text{pr}_{X'}(\Theta)$  ▷ Remove variables that never occur
5:   for a linked component  $(D'_1, \dots, D'_{n'})$  of  $\Theta'$  do
6:     if SOLVE(REDUCE( $\Theta', (D'_1, \dots, D'_{n'})$ )) = "Solution" then return "Solution"
   return "No solution"

```

Let us consider the function CHECKWEAKERINSTANCE. First, we introduce a partial order on the set of relations in Γ . We say that $\rho_1 \leq \rho_2$ if one of the following conditions hold:

- (1) the arity of ρ_1 is less than the arity of ρ_2 .
- (2) the arities of ρ_1 and ρ_2 are equal, $\text{pr}_i(\rho_1) \subseteq \text{pr}_i(\rho_2)$ for every i , $\text{pr}_j(\rho_1) \neq \text{pr}_j(\rho_2)$ for some j .
- (3) the arities of ρ_1 and ρ_2 are equal, $\text{pr}_i(\rho_1) = \text{pr}_i(\rho_2)$ for every i , and $\rho_1 \supseteq \rho_2$.

We can check that in the algorithm we never make any relation bigger, and every time we use recursion in CHECKWEAKERINSTANCE we make every constraint relation strictly smaller. Since our constraint language Γ is finite, every path in the recursion tree contains at most $|\Gamma|$ calls of the function SOLVE in CHECKWEAKERINSTANCE. Therefore the depth of the recursion tree is bounded by $|A| + |\Gamma|$. \square

COROLLARY 5.2.1. *The algorithm is polynomial.*

PROOF. Since the depth of the recursive tree is bounded by $|A| + |\Gamma|$, it remains to show that each loop in each function is polynomial.

In the function SOLVE we go through the loop at most $n \cdot |A|$ times, which is polynomially many.

In the function SOLVELINEARCASE, we go through the external **repeat** loop at most r times, where r is the dimension of $L_1 \times \dots \times L_n$. Therefore, r is bounded by $|A| \cdot n$. We go through the inner **repeat** loop at most $|\Gamma| \cdot N$ times, where N is the number of constraints of the instance.

In the function CHECKCYCLECONSISTENCY, we go through the **repeat** loop at most $|\Gamma| \cdot n^2$ times, because every time we change at least one relation $\rho_{i,j}$, which is from Γ , and we have n^2 of them.

In the function CHECKIRREDUCIBILITY, we go through the **repeat** loop at most n times, since we always add an element to I .

All other loops are **for** loops, and polynomial bounds for them follow from the description of the algorithm. Therefore, the algorithm is polynomial. \square

5.3 Correctness of the Auxiliary Functions

LEMMA 5.3. *If the function CHECKCYCLECONSISTENCY returns "Ok," then the instance is cycle-consistent, if it returns "No solution" then the instance has no solutions, if it returns (x_i, D) then any solution of the instance has $x_i \in D$.*

PROOF. Assume that the function returned "Ok." Since every relation $\rho_{i,j}$ in the end of the algorithm is subdirect, the instance is 1-consistent. Consider a path $x_{i_1} - C_1 - x_{i_2} - \dots - x_{i_{l-1}} - C_{l-1} - x_{i_l}$ starting and ending with $x_{i_1} = x_{i_l}$. Since the projection of C_j onto $x_{i_j}, x_{i_{j+1}}$ contains $\rho_{i_j, i_{j+1}}$ for every j , to show that the instance is cycle-consistent, it is sufficient to prove that the formula

$$\delta(x_{i_1}) = \exists x_{i_2} \dots \exists x_{i_{l-1}} \rho_{i_1, i_2}(x_{i_1}, x_{i_2}) \wedge \dots \wedge \rho_{i_{l-1}, i_l}(x_{i_{l-1}}, x_{i_l})$$

defines D_{i_1} . This follows from the fact that we terminated the function when for all i, j, k

$$\rho_{i,j}(x, y) = \exists z \rho_{i,j}(x, y) \wedge \rho_{i,k}(x, z) \wedge \rho_{k,j}(z, y).$$

The remaining part follows from the fact that all the constraints $\rho_{i,j}(x_i, x_j)$ were derived from the original constraints, and therefore they should hold for any solution. \square

LEMMA 5.4. *If the function CHECKIRREDUCIBILITY returns “Ok,” then the instance is irreducible; if it returns “No solution,” then the instance has no solutions; if it returns (x_i, D) , then any solution of the instance has $x_i \in D$.*

PROOF. Assume that CHECKIRREDUCIBILITY returned “Ok,” but the instance is not irreducible. Then, there exists an instance Θ' such that every constraint of Θ' is a projection of a constraint from the original instance Θ on some set of variables, and Θ' is not fragmented, not linked, and its solution set is not subdirect. Let X' be the set of all variables occurring in Θ' . Choose a variable $x_k \in X'$. If we consider the set of all pairs $(a, b) \in D_k^2$ such that a and b can be connected by a path in Θ' , then we get a congruence (see Lemma 6.2). Since Θ' is not linked, there should be a maximal congruence σ_k containing the congruence. This congruence was chosen in the line 4 of the pseudocode.

Since Θ' is not fragmented, there exists a path in Θ' from x_k to any other variable from X' . Following this path, we can always define a partition on the next variable using the partition on the previous one. Since every constraint of Θ' is a projection of a constraint from Θ , we could define the same partitions on Θ (see the pseudocode of the function). We just need to show that on every domain D_i , we can generate a unique partition using σ_k (the order in which we add elements to I and the way how we choose constraints is not important). Consider two paths from x_k to x_i defining two partitions. We glue together the beginnings of these paths and get a path from x_i to x_i connecting these partitions. Since the instance is cycle-consistent, these partitions should be equal. Thus, we showed that starting from the congruence σ_k (in the pseudocode) we get a unique partition on every variable $x_i \in X'$. Therefore, we actually checked in the algorithm that the solution set of Θ' is subdirect, which gives us a contradiction. Hence, Θ is irreducible.

The remaining part follows from the fact that D'_i is the set of all possible evaluations of x_i in solutions of a weaker instance. \square

5.4 Main Theorems without a Proof

To explain the correctness of the algorithm in Section 4, we used the following main facts, which will be proved in Section 9.

THEOREM 5.5. *Suppose Θ is a cycle-consistent irreducible CSP instance and B is a nontrivial binary absorbing subuniverse or a nontrivial center of D_i . Then Θ has a solution if and only if Θ has a solution with $x_i \in B$.*

THEOREM 5.6. *Suppose Θ is a cycle-consistent irreducible CSP instance, there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on D_j for every j , $(D_i; \omega)/\sigma$ is a polynomially complete algebra, and E is an equivalence class of σ . Then Θ has a solution if and only if Θ has a solution with $x_i \in E$.*

THEOREM 5.7. *Suppose the following conditions hold:*

- (1) Θ is a linked cycle-consistent irreducible CSP instance with domain set (D_1, \dots, D_n) ;
- (2) there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on D_j for every j ;
- (3) if we replace every constraint of Θ by all weaker constraints then the obtained instance has a solution with $x_i = b$ for every i and $b \in D_i$ (the obtained instance has a subdirect solution set);
- (4) $L_i = D_i/\sigma_i$ for every i , where σ_i is the minimal linear congruence on D_i ;

- (5) $\phi : \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k} \rightarrow L_1 \times \cdots \times L_n$ is a homomorphism, where q_1, \dots, q_k are prime numbers;
- (6) if we replace any constraint of Θ by all weaker constraints, then for every $(a_1, \dots, a_k) \in \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k}$ there exists a solution of the obtained instance in $\phi(a_1, \dots, a_k)$.

Then $\{(a_1, \dots, a_k) \mid \Theta \text{ has a solution in } \phi(a_1, \dots, a_k)\}$ is empty, full, or an affine subspace of $\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k}$ of codimension 1 (the solution set of a single linear equation).

6 THE REMAINING DEFINITIONS

6.1 Variety of Algebras

We consider the variety of all algebras $\mathbf{A} = (A; w)$ such that w is a special WNU operation of arity m . As mentioned in Section 3, every domain D will be viewed as a finite algebra $(D; w)$ from this variety. Note that in the remainder of this article any claim or assumption “ ρ is a relation” should be understood as “ ρ is a subalgebra of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ ” for the corresponding finite algebras $\mathbf{A}_1, \dots, \mathbf{A}_n$ from this variety.

6.2 Additional Notations

For a relation $\rho \subseteq A_1 \times \cdots \times A_n$ and a congruence σ on A_i , we say that the i th variable of the relation ρ is *stable under σ* if $(a_1, \dots, a_n) \in \rho$ and $(a_i, b_i) \in \sigma$ imply $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$. We say that a relation is *stable under σ* if every variable of this relation is stable under σ .

We say that a congruence σ is *irreducible* if it is proper and it cannot be represented as an intersection of other binary relations $\delta_1, \dots, \delta_s$ stable under σ . For an irreducible congruence σ on a set A by σ^* , we denote the minimal binary relation $\delta \supseteq \sigma$ stable under σ .

For a relation ρ by $\text{Con}(\rho, i)$, we denote the binary relation $\sigma(y, y')$ defined by

$$\exists x_1 \dots \exists x_{i-1} \exists x_{i+1} \dots \exists x_n \rho(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \wedge \rho(x_1, \dots, x_{i-1}, y', x_{i+1}, \dots, x_n).$$

For a constraint $C = \rho(x_1, \dots, x_n)$ by $\text{Con}(C, x_i)$, we denote $\text{Con}(\rho, i)$. For a set of constraints Ω by $\text{Con}(\Omega, x)$, we denote the set $\{\text{Con}(C, x) \mid C \in \Omega\}$.

A congruence σ on \mathbf{A} is called a *PC congruence* if \mathbf{A}/σ is a PC algebra without a nontrivial binary absorbing subuniverse or center. For an algebra \mathbf{A} by $\text{ConPC}(\mathbf{A})$, we denote the intersection of all PC congruences. A subuniverse $A' \subseteq A$ is called a *PC subuniverse* if $A' = E_1 \cap \cdots \cap E_s$, where each E_i is an equivalence class of a PC congruence. Note that a PC subuniverse can be empty or full.

A congruence σ on \mathbf{A} is called *linear* if \mathbf{A}/σ is a linear algebra. For an algebra \mathbf{A} by $\text{ConLin}(\mathbf{A})$, we denote the minimal linear congruence. A subuniverse of \mathbf{A} is called a *linear subuniverse* if it is stable under $\text{ConLin}(\mathbf{A})$. Note that we could not define a PC subuniverse in the same way, because not every subuniverse stable under $\text{ConPC}(\mathbf{A})$ is a PC subuniverse of \mathbf{A} (see Subsection 7.3).

A subuniverse $B \subseteq A$ is called a *one-of-four subuniverse* if it is a binary absorbing subuniverse, a center, a PC subuniverse, or a linear subuniverse. We say that B is a one-of-four subuniverse of *absorbing type*, *central type*, *PC type*, or *linear type*, respectively. A subuniverse of type \mathcal{T} is called *minimal* if it is a minimal nontrivial subuniverse of this type. Note that a minimal PC/linear subuniverse is a block of $\text{ConPC}(\mathbf{A})/\text{ConLin}(\mathbf{A})$.

6.3 pp-formula, Subconstraint, and Coverings

Every variable x appearing in the article has its domain, which we denote by D_x . In the article, we usually identify a CSP instance and a set of constraints. For an instance Ω by $\text{Var}(\Omega)$, we denote the set of all variables occurring in constraints of Ω (the set of all variables X is not important, all the properties of the instance depend only on the variables that actually occur in the instance).

For an instance Ω and two sets of variables x_1, \dots, x_n and y_1, \dots, y_n by $\Omega_{x_1, \dots, x_n}^{y_1, \dots, y_n}$, we denote the instance obtained from Ω by replacement of every variable x_i by y_i .

Sometimes we write an instance $\{C_1, \dots, C_n\}$ as a conjunctive formula $C_1 \wedge \dots \wedge C_n$. We say that an instance is a *tree-formula* if there is no path $z_1 - C_1 - z_2 - \dots - z_{l-1} - C_{l-1} - z_l$ such that $l \geq 3$, $z_1 = z_l$, and all the constraints C_1, \dots, C_{l-1} are different.

An expression $\exists y_1 \dots \exists y_s (C_1 \wedge \dots \wedge C_n)$ is called a *positive primitive formula (pp-formula)*. To simplify, we use a notation $\Omega(x_1, \dots, x_n)$ to write the pp-formula $\exists y_1 \dots \exists y_s \Omega$, where Ω is an instance (or a conjunction of constraints) and y_1, \dots, y_s are all variables occurring in Ω except for x_1, \dots, x_n . Then, we say that a pp-formula $\Omega(x_1, \dots, x_n)$ defines a relation ρ if $\rho(x_1, \dots, x_n) = \exists y_1 \dots \exists y_s \Omega$. Sometimes, if it is convenient, we write $\Omega(x_1, \dots, x_n)$ meaning the relation defined by the pp-formula. A pp-formula $\Omega(x_1, \dots, x_n)$ is called a *subconstraint of Θ* if $\Omega \subseteq \Theta$ and Ω and $\Theta \setminus \Omega$ do not have common variables except for x_1, \dots, x_n . Note that all relations that can be defined by a pp-formula are preserved by the WNU (see References [13, 14, 32]).

For a formula Ω by $\text{Coverings}(\Omega)$, we denote the set of all formulas Ω' such that there exists a mapping $S : \text{Var}(\Omega') \rightarrow \text{Var}(\Omega)$ satisfying the following conditions:

- (1) the domain of any variable x from Ω' is equal to the domain of $S(x)$ in Ω ;
- (2) for every constraint $((x_1, \dots, x_n); \rho)$ of Ω' , $((S(x_1), \dots, S(x_n)); \rho)$ is a constraint of Ω ;
- (3) if a variable x appears in both Ω and Ω' then $S(x) = x$.

Similarly, by $\text{ExpCov}(\Omega)$ (*Expanded Coverings*), we denote the set of all formulas Ω' such that there exists a mapping $S : \text{Var}(\Omega') \rightarrow \text{Var}(\Omega)$ satisfying the following conditions:

- (1) the domain of any variable x from Ω' is equal to the domain of $S(x)$ in Ω ;
- (2) for every constraint $((x_1, \dots, x_n); \rho)$ of Ω' either the variables $S(x_1), \dots, S(x_n)$ are different and the constraint $((S(x_1), \dots, S(x_n)); \rho)$ is weaker or equivalent to some constraint of Ω , or $S(x_1) = \dots = S(x_n)$ and $\{(a, a, \dots, a) \mid a \in D_{x_1}\} \subseteq \rho$;
- (3) if a variable x appears in both Ω and Ω' then $S(x) = x$.

For a variable x , we say that $S(x)$ is the *parent of x* .

The following easy facts about coverings can be derived from the definition:

- (1) every time we replace some constraints by weaker constraints we get an expanded covering of the original instance;
- (2) any solution of the original instance can be naturally expanded to a solution of a covering (expanded covering);
- (3) suppose Ω is a covering (expanded covering) of a 1-consistent instance and Ω is a tree-formula, then the solution set of Ω is subdirect;
- (4) the union (union of all constraints) of two coverings (expanded coverings) is also a covering (expanded covering);
- (5) a covering (expanded covering) of a covering (expanded covering) is a covering (expanded covering).

Another important property is formulated in the following lemma.

LEMMA 6.1. *Suppose Θ is a cycle-consistent irreducible CSP instance and $\Theta' \in \text{ExpCov}(\Theta)$. Then Θ' is cycle-consistent and irreducible.*

PROOF. Let us prove that Θ' is cycle-consistent. Consider a path in Θ' starting and ending with z . Since Θ' is an expanded covering, for every constraint of Θ' either there exists a corresponding constraint in Θ , or this constraint is reflexive (contains all tuples (a, a, \dots, a)). Thus, to transform the path in Θ' to a path in Θ it is sufficient to replace every variable x in the path by $S(x)$ (from

the definition of expanded coverings), remove all reflexive constraints, and replace the remaining constraints by the corresponding constraints from Θ . Since Θ is cycle-consistent, the obtained path connects a with a for any $a \in D_z$. Since constraints in the path in Θ' are weaker or equivalent to constraints in the path in Θ and relations, we removed are reflexive, the path in Θ' also connects a with a for every $a \in D_z$.

Let us show that Θ' is irreducible. Assume the converse, then there exists an instance Ω' consisting of projections of constraints from Θ' that is not linked, not fragmented, and its solution set is not subdirect. By Ω , we denote the set of corresponding projections of constraints from Θ corresponding to the constraints of Ω' (we ignore reflexive constraints from Ω'). To be more accurate, suppose a constraint $C'' \in \Omega'$ is equal to $\text{pr}_X(C')$ for a constraint $C' \in \Theta'$ and a set of variable X , and C' is weaker or equivalent to a constraint $C \in \Theta$. Then, we add the constraint $\text{pr}_{S(X)}(C)$ to Ω .

Let us show that Ω is not linked. Assume the contrary. For any path in Ω connecting elements a and b of D_x we can build a path connecting a and b in Ω' in the following way. We replace every constraint of Ω by the corresponding constraint of Ω' , and glue them with any path in Ω' starting and ending with the corresponding variables having the same parent. Since Ω' is not fragmented, we can always do this. Since Ω is cycle-consistent, the obtained path connects a and b in Ω' . Thus, Ω is not linked. Any solution of Ω can be naturally extended to a solution of Ω' , and hence the solution set of Ω cannot be subdirect. Since Ω' is not fragmented, Ω is also not fragmented. Thus, Ω is not linked and not fragmented, and its solution set is not subdirect, which contradicts the fact that Θ is irreducible. \square

For an instance Θ and its variable x by $\text{LinkedCon}(\Theta, x)$, we denote the binary relation on the set D_x defined as follows: $(a, b) \in \text{LinkedCon}(\Theta, x)$ if there exists a path in Θ that connects a and b .

LEMMA 6.2. *Suppose Θ is a cycle-consistent CSP instance, $x \in \text{Var}(\Theta)$. Then there exists a path in Θ connecting all pairs $(a, b) \in \text{LinkedCon}(\Theta, x)$ and $\text{LinkedCon}(\Theta, x)$ is a congruence.*

PROOF. Since the instance is cycle-consistent, gluing all the paths starting and ending at x we can build a path connecting all pairs $(a, b) \in \text{LinkedCon}(\Theta, x)$. The set of all pairs (a, b) connected by this path can be defined by a pp-formula, and therefore it is an invariant relation, which is also reflexive (by cycle-consistency) and transitive (we can glue paths). \square

6.4 Critical, Key Relations, and Parallelogram Property

We say that a relation ρ has the *parallelogram property* if any permutation of its variables gives a relation ρ' satisfying

$$\forall \alpha_1, \beta_1, \alpha_2, \beta_2 : (\alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2 \in \rho' \Rightarrow \alpha_1\alpha_2 \in \rho').$$

Note that the parallelogram property plays an important role in universal algebra (see Reference [40] for more details).

We say that the *i th variable of a relation ρ is rectangular* if for every $(a_i, b_i) \in \text{Con}(\rho, i)$ and $(a_1, \dots, a_n) \in \rho$ we have $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$. We say that a relation is *rectangular* if all of its variables are rectangular. The following facts can be easily seen: If the i th variable of a subdirect relation ρ is rectangular, then $\text{Con}(\rho, i)$ is a congruence; if a relation has the parallelogram property, then it is rectangular.

A relation $\rho \subseteq A_1 \times \dots \times A_n$ is called *essential* if it cannot be represented as a conjunction of relations with smaller arities. It is easy to see that any relation ρ can be represented as a conjunction of essential relations that are projections of ρ on some sets of variables (see Lemma 4.2 in Reference [59]).

A relation $\rho \subseteq A_1 \times \cdots \times A_n$ is called *critical* if it cannot be represented as an intersection of other subalgebras of $A_1 \times \cdots \times A_n$ and it has no dummy variables. This notion was introduced in Reference [40] but appeared in References [57, 61] by the name maximal. For a critical relation ρ the minimal relation ρ' (a subalgebra of $A_1 \times \cdots \times A_n$) such that $\rho' \supsetneq \rho$ is called *the cover of ρ* .

Suppose $\rho \subseteq A_1 \times \cdots \times A_h$. A tuple $\Psi = (\psi_1, \psi_2, \dots, \psi_h)$, where $\psi_i : A_i \rightarrow A_i$, is called a *unary vector-function*. We say that Ψ *preserves ρ* if $\Psi \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix} \right) := \begin{pmatrix} \psi_1(a_1) \\ \psi_2(a_2) \\ \vdots \\ \psi_h(a_h) \end{pmatrix} \in \rho$ for every $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix} \in \rho$. We say that ρ is a *key relation* if there exists a tuple $\beta \in (A_1 \times \cdots \times A_h) \setminus \rho$ such that for every $\alpha \in (A_1 \times \cdots \times A_h) \setminus \rho$ there exists a vector-function Ψ , which preserves ρ and gives $\Psi(\alpha) = \beta$. A tuple β is called a *key tuple* for ρ . The notion key relation was introduced in Reference [62], where such relations were characterized for all algebras having a WNU term operation.

A constraint is called *critical/essential/key* if the constraint relation is critical/essential/key. The notions critical, crucial, essential, and key relation are related to each other; namely, we can observe:

- (1) if C is a constraint in a CSP instance and C is crucial in some (D_1, \dots, D_n) , then the constraint relation of C is critical;
- (2) every critical relation of arity greater than 1 is essential;
- (3) every critical relation of arity greater than 1 is a key relation (see Lemma 2.4 in Reference [62]).

The notions essential, critical, and key relations (see Reference [62] for their comparison) proved their efficiency in clone theory and universal algebra (see Reference [40, 57–61]). Instead of considering all relations we consider only relations with one of these properties, and this is still the general case, because any relation can be represented as a conjunction of essential/key/critical relations. For instance, we can always assume that all constraint relations are critical.

6.5 Reductions

Suppose the domain set of an instance Θ is $D = (D_1, \dots, D_n)$. A domain set $D' = (D'_1, \dots, D'_n)$ is called a *reduction of Θ* if D'_i is a subuniverse of D_i for every i . Note that to avoid unnecessary bold font starting in this subsection, we do not use it for domain sets. Thus, every time we write D without a subscript we mean a domain set or a reduction. Note that any reduction of Θ can be naturally extended to a covering (expanded covering) of Θ , thus we assume that any reduction is automatically defined on any covering (expanded covering).

A reduction $D' = (D'_1, \dots, D'_n)$ is called *1-consistent* if the instance obtained after reduction of every domain is 1-consistent.

We say that D' is an *absorbing reduction* if there exists a term operation t such that D'_i is a binary absorbing subuniverse of D_i with the term operation t for every i . We say that D' is a *central reduction* if D'_i is a center of D_i for every i . We say that D' is a *PC/linear reduction* if D'_i is a PC/linear subuniverse of D_i and D_i does not have a nontrivial binary absorbing subuniverse or a nontrivial center for every i . Additionally, we say that D' is a *minimal central/PC/linear reduction* if D' is a minimal center/PC/linear subuniverse of D_i for every i . We say that D' is a *minimal absorbing reduction* for a term operation t if D' is a minimal absorbing subuniverse of D_i with t for every i .

A reduction is called *nonlinear* if it is an absorbing, central, or PC reduction. A reduction D' is called *one-of-four reduction* if it is an absorbing, central, PC, or linear reduction such that $D' \neq D$.

We usually denote reductions by $D^{(j)}$ for some j (or by $D^{(\top)}$). In this case, by $C^{(j)}$ we denote the constraint obtained after the reduction of the constraint C . Similarly, by $\Theta^{(j)}$ we denote the instance obtained after the reduction of every constraint of Θ . For a relation ρ by $\rho^{(j)}$, we denote the relation ρ restricted to the corresponding domains of $D^{(j)}$. Sometimes we write $(a_1, \dots, a_n) \in D^{(j)}$ meaning that every a_i belongs to the corresponding $D_x^{(j)}$.

A *strategy* for a CSP instance Θ with a domain set D is a sequence of reductions $D^{(0)}, \dots, D^{(s)}$, where $D^{(j)} = (D_1^{(j)}, \dots, D_n^{(j)})$, such that $D^{(0)} = D$ and $D^{(j)}$ is a one-of-four 1-consistent reduction of $\Theta^{(j-1)}$ for every $j \geq 1$. A strategy is called *minimal* if every reduction in the sequence is minimal.

6.6 Bridges

Suppose σ_1 and σ_2 are congruences on D_1 and D_2 , respectively. A relation $\rho \subseteq D_1^2 \times D_2^2$ is called a *bridge* from σ_1 to σ_2 if the first two variables of ρ are stable under σ_1 , the last two variables of ρ are stable under σ_2 , $\text{pr}_{1,2}(\rho) \supseteq \sigma_1$, $\text{pr}_{3,4}(\rho) \supseteq \sigma_2$, and $(a_1, a_2, a_3, a_4) \in \rho$ implies

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$

An example of a bridge is the relation $\rho = \{(a_1, a_2, a_3, a_4) \mid a_1, a_2, a_3, a_4 \in \mathbb{Z}_4 : a_1 - a_2 = 2a_3 - 2a_4\}$. We can check that ρ is a bridge from the equality relation (0-congruence) and (*mod* 2) equivalence relation. For example, we have $\text{pr}_{1,2} \rho$ is (*mod* 2)-equivalence relation, $\text{pr}_{3,4} \rho$ is full relation.

The notion of a bridge is strongly related to other notions in Universal Algebra and Tame Congruence Theory such as similarity and centralizers (see Reference [56] for the detailed comparison).

For a bridge ρ by $\tilde{\rho}$ we denote the binary relation defined by $\tilde{\rho}(x, y) = \rho(x, x, y, y)$.

The following lemma shows how we can compose bridges.

LEMMA 6.3. *Suppose $\sigma_1, \sigma_2, \sigma_3$ are irreducible congruences, ρ_1 is a bridge from σ_1 to σ_2 , ρ_2 is a bridge from σ_2 to σ_3 . Then the formula*

$$\rho(x_1, x_2, z_1, z_2) = \exists y_1 \exists y_2 \rho_1(x_1, x_2, y_1, y_2) \wedge \rho_2(y_1, y_2, z_1, z_2)$$

defines a bridge from σ_1 to σ_3 . Moreover, $\tilde{\rho} = \tilde{\rho}_1 \circ \tilde{\rho}_2$.

PROOF. Stability of the first two variables under σ_1 and of the last two variables under σ_3 follows from the definition.

Let us prove that $\text{pr}_{1,2}(\rho) \supseteq \sigma_1$ (the inclusion $\text{pr}_{3,4}(\rho) \supseteq \sigma_3$ can be proved in the same way). By the definition, for every a there exists b such that $(a, a, b, b) \in \rho_1$, and for every b there exists c such that $(b, b, c, c) \in \rho_2$. Then $(a, a, c, c) \in \rho$, and since the first two variables of ρ_1 are stable under σ_1 we obtain $\text{pr}_{1,2}(\rho) \supseteq \sigma_1$. Since σ_2 is irreducible, $\text{pr}_{3,4}(\rho_1) \supseteq \sigma_2^*$ and $\text{pr}_{1,2}(\rho_2) \supseteq \sigma_2^*$. Choose $(b_1, b_2) \in \sigma_2^*$, then there exist a_1, a_2, c_1, c_2 such that $(a_1, a_2, b_1, b_2) \in \rho_1$ and $(b_1, b_2, c_1, c_2) \in \rho_2$. Then $(a_1, a_2, c_1, c_2) \in \rho$, which means that $\text{pr}_{1,2}(\rho) \supseteq \sigma_1$.

Suppose $(a_1, a_2, c_1, c_2) \in \rho$. If $(a_1, a_2) \in \sigma_1$, then, since ρ_1 is a bridge, the corresponding values of y_1 and y_2 are equivalent modulo σ_2 . Since ρ_2 is a bridge we obtain that c_1 and c_2 are equivalent modulo σ_3 .

The equation $\tilde{\rho} = \tilde{\rho}_1 \circ \tilde{\rho}_2$ follows directly from the definition of ρ . □

A bridge $\rho \subseteq D^4$ is called *reflexive* if $(a, a, a, a) \in \rho$ for every $a \in D$.

We say that two congruences σ_1 and σ_2 on a set D are *adjacent* if there exists a reflexive bridge from σ_1 to σ_2 .

Remark 4. Since we can always put $\rho(x_1, x_2, x_3, x_4) = \sigma(x_1, x_3) \wedge \sigma(x_2, x_4)$, any proper congruence σ is adjacent with itself.

A reflexive bridge ρ from an irreducible congruence σ_1 to an irreducible congruence σ_2 is called *optimal* if there does not exist a reflexive bridge ρ' from σ_1 to σ_2 such that $\tilde{\rho}' \supsetneq \tilde{\rho}$. Suppose ρ is a

reflexive bridge from σ_1 to σ_2 . Then we can build a new bridge,

$$\rho'(x_1, x_2, y_1, y_2) = \exists x'_1 \exists x'_2 \exists y'_1 \exists y'_2 [\rho(x_1, x_2, y'_1, y'_2) \wedge \rho(x'_1, x'_2, y'_1, y'_2) \wedge \rho(x'_1, x'_2, y_1, y_2)]$$

from σ_1 to σ_2 such that $\tilde{\rho}' = \tilde{\rho} \circ \tilde{\rho}^{-1} \circ \tilde{\rho}$. Note that because of the reflexivity, $\tilde{\rho}$ contains the equality relation. Thus, if ρ is optimal, then $\tilde{\rho}$ is a congruence. For an irreducible congruence σ by $\text{Opt}(\sigma)$, we denote the congruence $\tilde{\rho}$ for an optimal bridge ρ from σ to σ . Since we can compose two reflexive bridges, $\text{Opt}(\sigma)$ is unique and therefore well defined. For a set of irreducible congruences \mathbb{C} put $\text{Opt}(\mathbb{C}) = \{\text{Opt}(\sigma) \mid \sigma \in \mathbb{C}\}$.

LEMMA 6.4. *Suppose σ_1 and σ_2 are irreducible adjacent congruences. Then $\text{Opt}(\sigma_1) = \text{Opt}(\sigma_2)$.*

PROOF. Let ρ_1 be an optimal bridge from σ_1 to σ_1 , ρ_2 be an optimal bridge from σ_2 to σ_2 , and ρ be a reflexive bridge from σ_1 to σ_2 .

Assume that $\text{Opt}(\sigma_2) \not\subseteq \text{Opt}(\sigma_1)$, that is, $\tilde{\rho}_2 \not\subseteq \tilde{\rho}_1$. Using Lemma 6.3, we compose bridges ρ_1 , ρ , ρ_2 , and ρ (in this order) to obtain a reflexive bridge ρ'_1 from σ_1 to σ_1 . Since $\tilde{\rho}'_1 \supseteq \tilde{\rho}_1 \cup \tilde{\rho}_2$, we get a contradiction with the fact that ρ_1 is optimal. \square

We say that two rectangular constraints C_1 and C_2 are *adjacent* in a common variable x if $\text{Con}(C_1, x)$ and $\text{Con}(C_2, x)$ are adjacent. A formula is called *connected* if every constraint in the formula is critical and rectangular, and the graph, whose vertexes are constraints and edges are adjacent constraints, is connected. Note that this connectedness is not related to the paths from one variable to another connecting two elements. Recall that if for every a, b there exists a path that connects a and b , then the instance is called *linked* (see Section 3.6).

It can be shown (see Corollary 8.22.1) that every two constraints with a common variable in a connected instance are adjacent.

7 ABSORPTION, CENTER, PC CONGRUENCE, AND LINEAR CONGRUENCE

7.1 Binary Absorption

LEMMA 7.1. [1] *Suppose ρ is defined by a pp-formula $\Omega(x_1, \dots, x_n)$ and Ω' is obtained from Ω by replacement of some constraint relations $\sigma_1, \dots, \sigma_s$ by constraint relations $\sigma'_1, \dots, \sigma'_s$ such that σ'_i absorbs σ_i with a term operation t for every i . Then the relation defined by $\Omega'(x_1, \dots, x_n)$ absorbs ρ with the term operation t .*

COROLLARY 7.1.1. *Suppose θ is a congruence of A .*

- (1) *If B is an absorbing subuniverse of A , then $\{b/\theta \mid b \in B\}$ is an absorbing subuniverse of A/θ with the same term.*
- (2) *If A has no nontrivial (binary) absorbing subuniverse, then neither does A/θ .*

COROLLARY 7.1.2. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a relation such that $\text{pr}_1(\rho) = A_1$ and $C = \text{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$, where C_i is an absorbing subuniverse in A_i with a term t for every i . Then C is an absorbing subuniverse in A_1 with the term t .*

PROOF. It is not hard to see that the sets C and A_1 can be defined by the following pp-formulas:

$$(x_1 \in C) = \exists x_2 \dots \exists x_n [(x_1 \in C_1) \wedge \dots \wedge (x_n \in C_n) \wedge \rho(x_1, \dots, x_n)],$$

$$(x_1 \in A_1) = \exists x_2 \dots \exists x_n [(x_1 \in A_1) \wedge \dots \wedge (x_n \in A_n) \wedge \rho(x_1, \dots, x_n)].$$

It remains to apply Lemma 7.1. \square

LEMMA 7.2. *Suppose $\kappa_A \subseteq A \times A$ is the equality relation, $\sigma \supseteq \kappa_A$, and ω is a nontrivial binary absorbing subuniverse in σ . Then $\omega \cap \kappa_A \neq \emptyset$.*

PROOF. We prove the lemma by induction on the size of A . Suppose ω absorbs σ with a binary absorbing term operation f .

Assume that there exists a nontrivial binary absorbing subuniverse $B \subsetneq A$ with the absorbing operation f . For any $(b_1, b_2) \in \omega$ and $b \in B$, we have $(f(b_1, b), f(b_2, b)) \in \omega \cap (B \times B)$. Then by Lemma 7.1, $\omega \cap (B \times B)$ is a nontrivial absorbing subuniverse in $\sigma \cap (B \times B)$, and we can restrict σ and ω to B and apply the inductive assumption.

Thus, we assume that there does not exist a nontrivial binary absorbing subuniverse $B \subsetneq A$ with the absorbing operation f . By Lemma 7.1, $\text{pr}_1(\omega)$ and $\text{pr}_2(\omega)$ binary absorb A , then $\text{pr}_1(\omega) = \text{pr}_2(\omega) = A$. Now, the statement of the lemma could be derived from Reference [5, Theorem 6], but we will finish the argument because it is simple.

For every $b \in A$, we consider $A_b = \{a \mid (a, b) \in \sigma\}$ and $C_b = \{a \mid (a, b) \in \omega\}$. Since $\text{pr}_2(\omega) = A$, $C_b \neq \emptyset$ for every b . By Lemma 7.1, C_b is a binary absorbing subuniverse in A_b with f . Therefore, $A_b \neq A$ or $A_b = C_b = A$. In the latter case, we have $(b, b) \in \omega$, which completes this case.

Assume that $A_b \neq A$ for some b . Since $\sigma \supseteq \kappa_A$, we have $b \in A_b$ and $(A_b \times A_b) \cap \omega \supseteq (C_b \times \{b\}) \cap \omega \neq \emptyset$. Then, we restrict σ and ω to A_b and apply the inductive assumption. \square

LEMMA 7.3. *Suppose ρ is a nontrivial absorbing subuniverse of $A_1 \times \cdots \times A_n$. Then, for some i there exists a nontrivial absorbing subuniverse B_i in A_i with the same term.*

PROOF. We prove this lemma by induction on the arity of ρ . If the projection of ρ onto the first coordinate is not A_1 , then by Lemma 7.1 this projection is an absorbing subuniverse with the same term. Otherwise, we choose any element $a \in A_1$ such that ρ does not contain all tuples starting with a , and consider $\rho' = \{(a_2, \dots, a_n) \mid (a, a_2, \dots, a_n) \in \rho\}$, which, by Lemma 7.1, is a nontrivial absorbing subuniverse in $A_2 \times \cdots \times A_n$ with the same term. It remains to apply the inductive assumption. \square

A relation $\rho \subseteq A^n$ is called *C-essential* if $\rho \cap (C^{i-1} \times A \times C^{n-i}) \neq \emptyset$ for every i but $\rho \cap C^n = \emptyset$. A relation $\rho \subseteq A_1 \times \cdots \times A_n$ is called (C_1, \dots, C_n) -essential if $\rho \cap (C_1 \times \cdots \times C_{i-1} \times A_i \times C_{i+1} \times \cdots \times C_n) \neq \emptyset$ for every i but $\rho \cap (C_1 \times \cdots \times C_n) = \emptyset$.

LEMMA 7.4 [1]. *Suppose C is a subuniverse of A . Then C absorbs A with an operation of arity n if and only if there does not exist a C -essential relation $\rho \subseteq A^n$.*

LEMMA 7.5. *Suppose $D^{(1)}$ is an absorbing reduction of a CSP instance Θ and a relation $\rho \subseteq D_{i_1} \times \cdots \times D_{i_n}$ is subdirect, where D_{i_1}, \dots, D_{i_n} are domains of variables from Θ . Then $\rho^{(1)}$ is not empty.*

PROOF. It is sufficient to apply the binary absorbing term operation t to all the tuples of ρ using term $t(x_1, t(x_2, t(x_3, \dots, t(x_{s-1}, x_s))))$, where $s = |\rho|$. The resulting tuple will be from $\rho^{(1)}$, which means that $\rho^{(1)}$ is not empty. \square

7.2 Center

LEMMA 7.6. *Suppose ρ is defined by a pp-formula $\Omega(x_1, \dots, x_n)$ and Ω' is obtained from Ω by replacement of some constraint relations $\sigma_1, \dots, \sigma_s$ by constraint relations $\sigma'_1, \dots, \sigma'_s$ such that σ'_i is a center of σ_i for every i . Then the relation defined by $\Omega'(x_1, \dots, x_n)$ is a center of ρ .*

PROOF. Suppose $\Omega'(x_1, \dots, x_n)$ defines a relation ρ' . Suppose B_i and R_i are the corresponding algebra and binary relation such that $\sigma'_i = \{c \mid \forall b \in B_i : (c, b) \in R_i\}$. Let $|B_i| = n_i$ for every i . Let Υ be obtained from Ω by replacement of every constraint $\sigma_i(y_1, \dots, y_t)$ by

$$R_i((y_1, \dots, y_t), z_{i,1}) \wedge \cdots \wedge R_i((y_1, \dots, y_t), z_{i,n_i}).$$

Suppose $\Upsilon((x_1, \dots, x_n), (z_{1,1}, \dots, z_{s,n_s}))$ defines a relation R . It is not hard to see that $\rho' = \{c \mid \forall b \in (B_1^{n_1} \times \cdots \times B_s^{n_s}) : (c, b) \in R\}$. By Lemma 7.3, there is no nontrivial binary absorbing subuniverse on $B_1^{n_1} \times \cdots \times B_s^{n_s}$. This proves that ρ' is a center of ρ . \square

COROLLARY 7.6.1. *Suppose θ is a congruence of A*

- (1) *If B is a center of A , then $\{b/\theta \mid b \in B\}$ is a center of A/θ .*
- (2) *If A has no nontrivial center, then neither does A/θ .*

COROLLARY 7.6.2. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a relation such that $\text{pr}_1(\rho) = A_1$ and $C = \text{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$, where C_i is a center in A_i for every i . Then C is a center in A_1 .*

COROLLARY 7.6.3. *Suppose C_i is a center of D_i for every i . Then $C_1 \times \cdots \times C_n$ is a center of $D_1 \times \cdots \times D_n$.*

COROLLARY 7.6.4. *Suppose C_1 and C_2 are centers of D . Then $C_1 \cap C_2$ is a center of D .*

LEMMA 7.7. *Suppose ρ is a nontrivial center of $A_1 \times \cdots \times A_n$. Then for some i there exists a nontrivial center C_i of A_i .*

PROOF. We prove by induction on the arity of ρ . If the projection of ρ onto the first coordinate is not A_1 , then by Lemma 7.6 this projection is a center.

Otherwise, we choose any element $a \in A_1$ such that ρ does not contain all tuples starting with a . Then we consider $\rho' = \{(a_2, \dots, a_n) \mid (a, a_2, \dots, a_n) \in \rho\}$, which, by Lemma 7.6, is a nontrivial center of $A_2 \times \cdots \times A_n$. It remains to apply the inductive assumption. \square

In the proof of the following two lemmas we assume that a center C is defined by $C = \{a \in A \mid \forall b \in B : (a, b) \in R\}$ for a subalgebra R of $\mathbf{A} \times \mathbf{B}$. For an element $a \in A$, we put $a^+ = \{b \mid (a, b) \in R\}$. Also, we introduce a quasi-order on elements of A . We say that $y_1 \leq y_2$ if $y_1^+ \subseteq y_2^+$, and $y_1 \sim y_2$ if $y_1^+ = y_2^+$. Note that if $b_1, b_2, \dots, b_m \geq c$, then $w(b_1^+, \dots, b_m^+) \supseteq w(c^+, \dots, c^+) \supseteq c^+$, and therefore $w(b_1, \dots, b_m) \geq c$.

LEMMA 7.8. *Suppose $(c_1, \dots, c_m) \in A^m$, $c_i \in C$ for every $i \neq j$, and $c_j \notin C$. Then $w(c_1, \dots, c_m) > c_j$.*

PROOF. Assume the contrary, then $w(c_1, \dots, c_m) \sim c_j$ and $w(\underbrace{B, \dots, B}_{i-1}, c_j^+, \underbrace{B, \dots, B}_{m-i}) \subseteq c_j^+$. This is enough to imply that c_j^+ is a binary absorbing subuniverse with the term $x \circ y = w(x, x, \dots, x, y)$. In fact, if $b_1 \in B$ and $b_2 \in c_j^+$, then we can write $b_1 \circ b_2 = w(b_1, \dots, b_1, b_2, b_1, \dots, b_1)$ with b_2 in the j th spot; if $b_1 \in c_j^+$ and $b_2 \in B$, then we can write $b_1 \circ b_2 = w(b_1, \dots, b_1, b_2, b_1, \dots, b_1)$ with one of the b_1 's in the j th spot. In both cases we obtain $b_1 \circ b_2 \in c_j^+$. Contradiction. \square

LEMMA 7.9. *Suppose w is a special WNU of arity m , C is a nontrivial center in A , $\delta \subseteq A^s$ is C -essential. Then $s < m^{|A|}$.*

PROOF. Choose $\alpha_1, \dots, \alpha_s \in \delta$ such that $\alpha_i \in C^{i-1} \times A \times C^{s-i}$ for every i . We start with the matrix M_1 whose columns are tuples $\alpha_1, \dots, \alpha_s$. Then, we build a matrix M_2 whose columns are tuples $w(\alpha_1, \dots, \alpha_m), w(\alpha_{m+1}, \dots, \alpha_{2m}), w(\alpha_{2m+1}, \dots, \alpha_{3m}), \dots$. Then we apply the WNU w to the corresponding columns of the previous matrix to define a new matrix M_3 . We continue this way until we get a matrix with less than m columns. Note that the next matrix has m times less columns than the previous one. It is not hard to see that every row of every matrix has at most one element that is not from the center. Moreover, by Lemma 7.8, the noncentral element in the i th row of the $(j+1)$ th matrix is greater than the noncentral element in the i th row of the j th matrix. This means that the $|A|$ th matrix, if it exists, has only central elements, which contradicts our assumptions. Hence, it does not exist and $s < m^{|A|}$. \square

Combining this result with Lemma 7.4, we obtain the following corollary.

COROLLARY 7.9.1. *Suppose C is a center of A . Then C is an absorbing subuniverse of A .*

The following lemma is a stronger version of an original lemma suggested by Marcin Kozik.

LEMMA 7.10. *Suppose $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ are centers, B is a subuniverse of D , and a relation $\rho \subseteq A_1 \times D^l \times A_2$ is (C_1, B, \dots, B, C_2) -essential. Then there exists a relation $\rho' \subseteq A_1 \times D^{2l} \times A_1$ that is (C_1, B, \dots, B, C_1) -essential.*

PROOF. Assume that ρ is a minimal relation (with respect to inclusion) that is (C_1, B, \dots, B, C_2) -essential. Put $E = \text{pr}_{l+2}(\rho \cap (C_1 \times B^l \times A_2))$. Since ρ is minimal, for any $b \in E$ the algebra generated by $\{b\} \cup C_2$ contains $\text{pr}_{l+2}(\rho)$ (otherwise, we would restrict the $(l+2)$ th variable of ρ to this algebra). Fix $b \in E$.

Let σ be the subalgebra of $A_2 \times A_2$ generated by $\{b\} \times C_2 \cup C_2 \times C_2 \cup C_2 \times \{b\}$. Since our algebras are idempotent, for any $c \in \text{pr}_{l+2}(\rho)$ we have $\{c\} \times C_2 \subseteq \sigma$. Put

$$\rho'(x, y_1, \dots, y_l, y'_1, \dots, y'_l, x') = \exists z \exists z' \rho(x, y_1, \dots, y_l, z) \wedge \rho(x', y'_1, \dots, y'_l, z') \wedge \sigma(z, z').$$

Let us show that ρ' is (C_1, B, \dots, B, C_1) -essential. Since ρ is (C_1, B, \dots, B, C_2) -essential, for any $i \in \{1, \dots, l+1\}$ there exists a tuple (a_1, \dots, a_{l+2}) such that only its i th element is not from the corresponding set of (C_1, B, \dots, B, C_2) . Since $b \in E$, there exists c_1, \dots, c_{l+1} such that $(c_1, \dots, c_{l+1}, b) \in \rho \cap (C_1 \times B^l \times A_2)$. Then $(a_1, \dots, a_{l+1}, c_2, \dots, c_{l+1}, c_1) \in \rho'$ (it is sufficient to put $z = a_{l+2}$ and $z' = b$). Thus, for any $i \in \{1, \dots, l+1\}$, we build a tuple from ρ' such that only its i th element is not from the corresponding set of (C_1, B, \dots, B, C_1) . In the same way we can build such a tuple for each $i \in \{l+2, \dots, 2l+2\}$.

To prove that ρ' is (C_1, B, \dots, B, C_1) -essential it remains to show that $(C_1 \times B^{2l} \times C_1) \cap \rho' = \emptyset$. Assume the converse, let a tuple from the intersection be obtained by sending z to d and z' to d' . Clearly, $d, d' \in E$ and $\{e \in A_2 \mid (e, d') \in \sigma\} \supseteq \{d\} \cup C_2$, and, therefore, $\{e \in A_2 \mid (e, d') \in \sigma\} \supseteq \text{pr}_{l+2}(\rho)$. Hence, $\{e \in A_2 \mid (b, e) \in \sigma\} \supseteq \{d'\} \cup C_2$ and $\{e \in A_2 \mid (b, e) \in \sigma\} \supseteq \text{pr}_{l+2}(\rho)$.

Thus, $(b, b) \in \sigma$ and there exists an n -ary term t such that

$$t(b, b, \dots, b, c_1, \dots, c_i) = b, \quad t(c'_1, \dots, c'_j, b, b, \dots, b) = b,$$

where $i+j \geq n$ and $c_1, \dots, c_i, c'_1, \dots, c'_j \in C_2$. Suppose $R \subseteq A_2 \times G$ is a binary relation from the definition of the center C_2 , $b^+ = \{a \mid (b, a) \in R\}$. Since t preserves R , we have

$$t(b^+, b^+, \dots, b^+, \underbrace{G, \dots, G}_i) \subseteq b^+, \quad t(\underbrace{G, \dots, G}_j, b^+, b^+, \dots, b^+) \subseteq b^+,$$

and therefore b^+ absorbs G with the binary term $t(\underbrace{x, \dots, x}_j, y, \dots, y)$. This contradiction completes the proof. \square

COROLLARY 7.10.1. *Suppose $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ are centers and $B \subseteq D$ is an absorbing subuniverse. Then there does not exist (C_1, B, C_2) -essential relation $\rho \subseteq A_1 \times D \times A_2$.*

PROOF. Assume that such a relation ρ exists. Iteratively applying Lemma 7.10 to ρ , we can obtain a (C_1, B, \dots, B, C_1) -essential relation $\rho_l \subseteq A_1 \times D^l \times A_1$ for $l = 2, 4, 8, \dots$. If we restrict the first and the last variables of ρ_l to C_1 and consider the projection onto the remaining variables, then we get a B -essential relation of arity l . Since we can make l as large as we need, we get a contradiction with Lemma 7.4 and the fact that B is an absorbing subuniverse. \square

COROLLARY 7.10.2. *Suppose C is a center of A . Then C is a ternary absorbing subuniverse of A .*

PROOF. Assume that C is not a ternary absorbing subuniverse then by Lemma 7.4, there exists a C -essential relation of arity 3. By Corollary 7.9.1, C is an absorbing subuniverse of A , then by Corollary 7.10.1 such a relation cannot exist. \square

COROLLARY 7.10.3. *Suppose C_i is a center of A_i for $i \in \{1, 2, \dots, k\}$ and $k \geq 3$. Then there does not exist a (C_1, \dots, C_k) -essential relation $\rho \subseteq A_1 \times \dots \times A_k$.*

PROOF. If such a relation ρ exists, then restricting all but the first three variables of ρ to the corresponding centers and projecting the result onto the first three variables we obtain (C_1, C_2, C_3) -essential relation, which cannot exist by Corollary 7.10.1. \square

7.3 PC Subuniverse

LEMMA 7.11. *Suppose A is a PC algebra and $\rho \subseteq A^n$ is a relation containing all the constant tuples (a, \dots, a) . Then ρ can be represented as a conjunction of binary relations of the form $x_i = x_j$.*

PROOF. All constant operations preserve ρ , and together with the constant operations the algebra A generates all operations on the set A . Then ρ is preserved by all operations on A , and, therefore, ρ is diagonal (see Theorem 2.9.3 from Reference [44]), and it can be represented as a conjunction of binary relations of the form $x_i = x_j$. \square

LEMMA 7.12. *Suppose $\rho \subseteq A \times B$ is a subdirect relation and A is a PC algebra. Then either for every $b \in B$ there exists a unique $a \in A$ such that $(a, b) \in \rho$, or there exists $b \in B$ such that $(a, b) \in \rho$ for every $a \in A$.*

PROOF. Put $\sigma_l(x_1, x_2, \dots, x_l) = \exists y \rho(x_1, y) \wedge \dots \wedge \rho(x_l, y)$. It is not hard to see that σ_l contains all constant tuples. Therefore, Lemma 7.11 implies that σ_2 is either full or the equality relation.

If σ_2 is the equality relation, then for every $b \in B$ there exists a unique $a \in A$ such that $(a, b) \in \rho$.

Suppose σ_2 is full. Then, we consider the minimal l , if it exists, such that σ_l is not full. Since σ_{l-1} is full, the relation σ_l contains all tuples whose elements are not different. Then Lemma 7.11 implies that σ_l is a full relation, which means that σ_l is a full relation for every l . Substituting $l = |A|$ and $\{x_1, \dots, x_l\} = A$ in the definition of σ_l , we obtain that there exists b such that $(a, b) \in \rho$ for every $a \in A$. \square

LEMMA 7.13. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a subdirect relation, A_i is a PC algebra for every $i \in \{2, \dots, n\}$, and there is no nontrivial binary absorbing subuniverse or nontrivial center on A_i for every $i \in \{1, \dots, n\}$. Then ρ can be represented as a conjunction of binary relations $\delta_1, \dots, \delta_k$ such that $\text{Con}(\delta_l, j)$ is the equality relation whenever the domain of the j th variable of δ_l is a PC algebra.*

This lemma says that the relation ρ can be represented by constraints from the first coordinate to an i th coordinate such that the i th coordinate is uniquely determined by the first (also we can define the corresponding PC congruence on the first coordinate using this relation) and by bijective binary constraints between pairs of coordinates other than first. Also, it says that in a subdirect product of PC algebras without a nontrivial binary absorbing subuniverse or center (even A_1 is a PC algebra) we can choose some essential coordinates that can have any value, each other coordinate is uniquely determined by exactly one of them (in a bijective way).

PROOF. We prove by induction on the arity of ρ . If ρ is binary, then Lemma 7.12 implies that there exists a nontrivial binary absorbing subuniverse on A_2 , or there exists a nontrivial center on A_1 witnessed by ρ , or the second coordinate of ρ is uniquely determined by the first, or ρ is full. The first two conditions contradict our assumptions, and the last two conditions are what we need.

Assume that ρ is not essential, then it can be represented as a conjunction of essential relations satisfying the same properties. By the inductive assumption, each of them can be represented as a conjunction of binary relations. It remains to join these binary relations to complete the proof for this case.

Assume that ρ is essential. The projection of ρ onto any proper set of variables gives a relation of a smaller arity satisfying the same properties. By the inductive assumption, the relation of a smaller arity can be represented as a conjunction of binary relations $\delta_1, \dots, \delta_k$ such that $\text{Con}(\delta_l, j)$ is the equality relation whenever the domain of the j th variable of δ_l is a PC algebra. In each relation δ_i one variable (let it be the u th variable of ρ) is uniquely determined by another, and therefore the relation ρ can be represented as a conjunction of δ_i and the projection of ρ onto all variables but u th, which cannot happen with an essential relation. Therefore, each projection of ρ onto any proper set of variables is a full relation.

Let us consider the relation $\rho \subseteq (A_1 \times \dots \times A_{n-1}) \times A_n$ as a binary relation. By Lemma 7.12, we have one of the following two situations.

Case 1: There exist b_1, \dots, b_{n-1} such that $(b_1, \dots, b_{n-1}, a) \in \rho$ for every $a \in A_n$. We consider the maximal s such that $\rho(b_1, \dots, b_s, x_{s+1}, \dots, x_n)$ is not a full relation. It is easy to see that $s \leq n-2$ and s exists. Let $R(x_{s+1}, \dots, x_n) = \rho(b_1, \dots, b_s, x_{s+1}, \dots, x_n)$. Since the projection of ρ onto any proper subset of variables is full, R is a subdirect relation. By Lemma 7.3, there is no nontrivial binary absorbing subuniverse on $A_{s+2} \times \dots \times A_n$, and then we get a nontrivial center C on A_{s+1} defined by $C = \{a_{s+1} \in A_{s+1} \mid \forall a_{s+2} \dots \forall a_n : (a_{s+1}, a_{s+2}, \dots, a_n) \in R\}$ and witnessed by R .

Case 2: For every a_1, \dots, a_{n-1} there exists a unique b such that $(a_1, \dots, a_{n-1}, b) \in \rho$. We can show in the same way that for any (a_1, a_3, \dots, a_n) there exists a unique b such that $(a_1, b, a_3, \dots, a_n) \in \rho$. Let us consider the relation ζ defined by

$$\begin{aligned} \zeta(z_1, z_2, z_3, z_4) = & \exists x_1 \exists x_2 \dots \exists x_{n-1} \exists x'_1 \exists x'_2 \rho(x_1, x_2, x_3, \dots, x_{n-1}, z_1) \wedge \\ & \rho(x_1, x'_2, x_3, \dots, x_{n-1}, z_2) \wedge \rho(x'_1, x_2, x_3, \dots, x_{n-1}, z_3) \wedge \rho(x'_1, x'_2, x_3, \dots, x_{n-1}, z_4). \end{aligned}$$

Since any projection of ρ onto any proper subset of variables is a full relation, any projection of ζ onto three variables is a full relation. Since ρ is subdirect, ζ contains all constant tuples. Then Lemma 7.11 implies that ζ is a full relation. Suppose $a \neq b$ and $(a, a, a, b) \in \zeta$ witnessed by $x_1, \dots, x_{n-1}, x'_1, x'_2$. Since $z_1 = z_2 = a$, we have $x_2 = x'_2$ and therefore $z_3 = z_4$, that is, $a = b$. Contradiction. \square

COROLLARY 7.13.1. *Suppose $\sigma_1, \dots, \sigma_k$ are all PC congruences on A . Put $A_i = A/\sigma_i$, and define $\psi : A \rightarrow A_1 \times \dots \times A_k$ by $\psi(a) = (a/\sigma_1, \dots, a/\sigma_k)$. Then*

- (1) ψ is surjective, hence $A/\text{ConPC}(A) \cong A_1 \times \dots \times A_k$;
- (2) the PC subuniverses are the sets of the form $\psi^{-1}(S)$, where $S \subseteq A_1 \times \dots \times A_k$ is a relation definable by unary constraints of the form $x_j = a_j$;
- (3) for each nonempty PC subuniverse B of A there is a congruence θ of A such that B is an equivalence class of θ and A/θ is isomorphic to a product of PC algebras having no nontrivial binary absorbing subuniverse or center.

PROOF. Consider the image $\psi(A)$, which is a subdirect subuniverse of $A_1 \times \dots \times A_k$. By Lemma 7.13, this relation can be represented as a conjunction of binary relations whose one coordinate uniquely determines another (in a bijective way). This means that congruences σ_i corresponding to these coordinates should be equal, which contradicts the definition. Then $\psi(A)$ is a full relation and ψ is surjective.

Claim (2) follows directly from the definition of a PC subuniverse.

To prove Claim (3), consider the intersection of all congruences whose equivalence classes we intersected to define the PC subuniverse. Then, in the same way as in Claim (1) we can prove the isomorphism. \square

COROLLARY 7.13.2. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a subdirect relation, there is no nontrivial binary absorbing subuniverse or nontrivial center on A_1 , and $C = \text{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$, where C_i is a PC subuniverse in A_i for every i . Then C is a PC subuniverse in A_1 .*

PROOF. By the previous corollary for every i we choose PC algebras $A_{i,1}, \dots, A_{i,k_i}$ and a mapping $\psi_i : A_i \rightarrow A_{i,1} \times \cdots \times A_{i,k_i}$ such that $A_i / \text{ConPC}(A_i) \cong A_{i,1} \times \cdots \times A_{i,k_i}$. Define $\phi : A_1 \times \cdots \times A_n \rightarrow A_1 \times \prod_{i,j} A_{i,j}$ by $\phi(a_1, \dots, a_n) = (a_1, \psi_1(a_1), \dots, \psi_n(a_n))$. Let $\gamma = C_1 \times \cdots \times C_n$, $\rho' = \phi(\rho)$, $\gamma' = \phi(\gamma)$. We can check that $\text{pr}_1(\rho \cap \gamma) = \text{pr}_1(\rho' \cap \gamma')$, and then it is sufficient to show that $\text{pr}_1(\rho' \cap \gamma')$ is a PC subuniverse of A_1 .

Since ρ' is subdirect, by Lemma 7.13 it can be represented by binary constraints from the first coordinate to an i th coordinate such that the i th coordinate is uniquely determined by the first and by bijective binary constraints between pairs of coordinates other than first. The relation γ' can be represented by constraints of the form $x_{i,j} = a_{i,j}$ and canonical constraints saying that the j th element of $\psi_1(x_1)$ is equal to $x_{1,j}$. To calculate $\text{pr}_1(\rho' \cap \gamma')$, we join constraints of these two representations. Let us explain how any constraint from x_1 to $x_{i,j}$ in this representation looks like. There exists a congruence σ on A_1 such that A_1/σ is a PC algebra isomorphic to $A_{i,j}$, and then the constraint assigns to all elements of each equivalence class of σ the corresponding element of $A_{i,j}$. All other constraints of this representations are of the form $x_{i,j} = a_{i,j}$ or bijective constraints between two coordinates. This implies that $\text{pr}_1(\rho' \cap \gamma')$ is an intersection of equivalence classes of PC congruences, that is, $\text{pr}_1(\rho' \cap \gamma')$ is a PC subuniverse. \square

COROLLARY 7.13.3. *Suppose C_i is a PC subuniverse of A_i for $i \in \{1, 2, \dots, n\}$ and $n \geq 3$. Then there does not exist a subdirect (C_1, \dots, C_n) -essential relation $\rho \subseteq A_1 \times \cdots \times A_n$.*

PROOF. Assume that such a relation ρ exists. By Corollary 7.13.1, for every i we choose PC algebras $A_{i,1}, \dots, A_{i,k_i}$ and a mapping $\psi_i : A_i \rightarrow A_{i,1} \times \cdots \times A_{i,k_i}$ such that $A_i / \text{ConPC}(A_i) \cong A_{i,1} \times \cdots \times A_{i,k_i}$. Define $\phi : A_1 \times \cdots \times A_n \rightarrow \prod_{i,j} A_{i,j}$ by $\phi(a_1, \dots, a_n) = (\psi_1(a_1), \dots, \psi_n(a_n))$. Let $\gamma_i = C_1 \times \cdots \times C_{i-1} \times A_i \times C_{i+1} \times \cdots \times C_n$ for every i and $\gamma = C_1 \times \cdots \times C_n$. Put $\rho' = \phi(\rho)$, $\gamma' = \phi(\gamma)$, and $\gamma'_i = \phi(\gamma_i)$ for every i .

Since ρ' is subdirect, by Lemma 7.13 ρ' can be represented by bijective binary constraints between pairs. By Corollary 7.13.1 γ' can be represented by constraints of the form $x_{i,j} = a_{i,j}$. If $\rho \cap \gamma = \emptyset$, then $\rho' \cap \gamma' = \emptyset$, which can only happen if two unary constraints defining γ' assign contradictory values to variables with respect to the binary constraints defining ρ' . Since $k \geq 3$, we can choose l such that γ'_l includes the two contradictory unary constraints. Then $\rho' \cap \gamma'_l = \emptyset$ and $\rho \cap \gamma_l = \emptyset$, which gives a contradiction. \square

LEMMA 7.14. *Suppose $\sigma \supseteq \sigma_1 \cap \cdots \cap \sigma_n$, where $\sigma_1, \dots, \sigma_n$ are PC congruences on D and σ is a proper congruence on D . Then there exists $I \subseteq \{1, 2, \dots, n\}$ such that $\sigma = \bigcap_{i \in I} \sigma_i$.*

PROOF. Consider a $2n$ -ary relation $R \subseteq D/\sigma_1 \times \cdots \times D/\sigma_n \times D/\sigma_1 \times \cdots \times D/\sigma_n$ consisting of all tuples $(a/\sigma_1, \dots, a/\sigma_n, b/\sigma_1, \dots, b/\sigma_n)$, where $(a, b) \in \sigma$. By Lemma 7.13, R can be represented as a conjunction of binary bijective relations. Since $(a/\sigma_1, \dots, a/\sigma_n, a/\sigma_1, \dots, a/\sigma_n) \in R$ for every $a \in D$, we conclude that all these binary relations are equalities. This implies that $\sigma = \bigcap_{i \in I} \sigma_i$ for some $I \subseteq \{1, 2, \dots, n\}$. \square

LEMMA 7.15. *For every D the algebra $D / \text{ConPC}(D)$ has no nontrivial binary absorbing subuniverse or center.*

PROOF. By Corollary 7.13.1, $D / \text{ConPC}(D) \cong A_1 \times \cdots \times A_k$, where A_i is a PC algebra without a nontrivial binary absorbing subuniverse or center. Then Lemmas 7.3 and 7.7 imply that there cannot be a nontrivial binary absorbing subuniverse or center on D . \square

LEMMA 7.16. *Suppose σ is a PC congruence on $A_1 \times A_2$, there is no nontrivial binary absorbing subuniverse or center on A_1 and A_2 . Then there exist $i \in \{1, 2\}$ and a PC congruence σ_i on A_i such that $\sigma = \{(\alpha, \beta) \mid (\text{pr}_i(\alpha), \text{pr}_i(\beta)) \in \sigma_i\}$.*

PROOF. First, consider $S \subseteq A_1 \times (A_1 \times A_2)/\sigma$ consisting of all pairs $(a_1, (a_1, a_2)/\sigma)$ such that $a_1 \in A_1, a_2 \in A_2$. Since there is no nontrivial binary absorbing subuniverse or center on A_1 , Lemma 7.13 implies that either S is a full relation, or $\text{Con}(S, 2)$ is the equality relation. In the latter case, the congruence σ depends only on the first coordinate, that is, there exists a PC congruence σ_1 on A_1 such that $\sigma = \{(\alpha, \beta) \mid (\text{pr}_1(\alpha), \text{pr}_1(\beta)) \in \sigma_1\}$, which completes this case.

Thus, we assume that S is a full relation and for every $a_1 \in A_1$ and every equivalence class E of σ there exists $a_2 \in A_2$ such that $(a_1, a_2) \in E$. In the same way we assume that for every $a_2 \in A_2$ and every equivalence class E of σ there exists $a_1 \in A_1$ such that $(a_1, a_2) \in E$.

Choose an element $c_1 \in A_1$. By σ_2 we denote the congruence $\{(a_2, a'_2) \mid ((c_1, a_2), (c_1, a'_2)) \in \sigma\}$. As it follows from the above assumptions, $A_2/\sigma_2 \cong (A_1 \times A_2)/\sigma$. Consider the ternary relation $\rho \subseteq A_1 \times (A_1 \times A_2)/\sigma \times A_2/\sigma_2$ consisting of all the tuples $(a_1, (a_1, a_2)/\sigma, a_2/\sigma_2)$, where $a_1 \in A_1$ and $a_2 \in A_2$. As we already know, the projection of ρ onto any two coordinates is a full relation. Then Lemma 7.13 implies that ρ is a full relation, which contradicts the fact that $(c_1, (c_1, a_2)/\sigma, b_2/\sigma_2) \notin \rho$ for any $(a_2, b_2) \notin \sigma_2$. \square

COROLLARY 7.16.1. *Suppose σ is a PC congruence on $A_1 \times A_2 \times \cdots \times A_n$, there is no nontrivial binary absorbing subuniverse or center on A_i for every i . Then there exist $i \in \{1, 2, \dots, n\}$ and a PC congruence σ_i on A_i such that $\sigma = \{(\alpha, \beta) \mid (\text{pr}_i(\alpha), \text{pr}_i(\beta)) \in \sigma_i\}$.*

PROOF. We prove this corollary by induction on n . For $n = 2$ it follows from Lemma 7.16. By Lemmas 7.3, 7.7, there is no nontrivial binary absorbing subuniverse or center on $A_2 \times \cdots \times A_n$. We apply Lemma 7.16 to $A_1 \times (A_2 \times \cdots \times A_n)$ to get a PC congruence on A_1 or on $A_2 \times \cdots \times A_n$. In the latter case, we apply the inductive assumption to complete the proof. \square

LEMMA 7.17. *Suppose B is a PC subuniverse on $A_1 \times \cdots \times A_n$, and there is no nontrivial binary absorbing subuniverse or center on A_i for every i . Then there exists a PC subuniverse B_i on A_i for every i such that $B = B_1 \times \cdots \times B_n$.*

PROOF. Assume that $B = E_1 \cap \cdots \cap E_t$, where E_i is an equivalence class of a PC congruence σ_i on $A_1 \times \cdots \times A_n$ for every i . By Corollary 7.16.1, for every i there exists s_i and a congruence σ'_i on A_{s_i} such that $\sigma_i = \{(\alpha, \beta) \mid (\text{pr}_{s_i}(\alpha), \text{pr}_{s_i}(\beta)) \in \sigma'_i\}$. Then there exists an equivalence class E'_i of σ'_i such that $E_i = A_1 \times \cdots \times A_{s_i-1} \times E'_i \times A_{s_i+1} \times \cdots \times A_n$. Hence, the intersection $E_1 \cap \cdots \cap E_t$ is equal to $B_1 \times \cdots \times B_n$ for PC subuniverses B_1, \dots, B_n . \square

LEMMA 7.18. *Suppose $\rho \subseteq A \times B$ is a subdirect relation, A is a PC algebra without nontrivial binary absorbing subuniverse or center, and $C = \{b \in B \mid \forall a \in A : (a, b) \in \rho\}$. Then C binary absorbs B .*

PROOF. Suppose $A = \{a_1, \dots, a_k\}$. Let us consider the matrix M whose rows are the tuples $(\underbrace{a, a, \dots, a}_{k+1}, b, a_1, \dots, a_k)$ and $(b, a_1, \dots, a_k, \underbrace{a, a, \dots, a}_{k+1})$ for all $a, b \in A$. The $2k + 2$ columns of this matrix we denote by $\alpha_1, \dots, \alpha_{2k+2}$. By β we denote the tuple of length $2k^2$ such that the i th element of β equals b from the corresponding row. By Lemma 7.13, the relation generated by $\alpha_1, \dots, \alpha_{2k+2}$ is a full relation. Hence, there exists a term operation f such that $f(\alpha_1, \dots, \alpha_{2k+2}) = \beta$. Let us show that C absorbs B with the term operation defined by $h(x, y) = f(\underbrace{x, \dots, x}_{k+1}, y, \dots, y)$. Suppose $d \in B$, $c \in C$. Assume that $h(d, c) = e \notin C$. Choose elements $a, a' \in A$ such that $(a, e) \notin \rho$ and $(a', d) \in \rho$.

Consider the row $(a', \dots, a', a, a_1, \dots, a_k)$ from the matrix. We know that f returns a on this tuple and $f(\underbrace{d, \dots, d}_{k+1}, c, \dots, c) = e$, which contradicts the fact that f preserves ρ . Thus, $h(d, c) \in C$.

In the same way, we can prove that $h(c, d) \in C$ for every $d \in B, c \in C$. \square

LEMMA 7.19. *Suppose $\rho \subseteq A \times B \times B$ is a subdirect relation, A is a PC algebra without a nontrivial binary absorbing subuniverse or center, and for every $b \in B$ there exists $a \in A$ such that $(a, b, b) \in \rho$. Then for every $a \in A$ there exists $b \in B$ such that $(a, b, b) \in \rho$.*

PROOF. We prove the lemma by induction on the size of B .

By Lemma 7.12, only two situations are possible: Either there exist $c_1, c_2 \in B$ such that $(a, c_1, c_2) \in \rho$ for every $a \in A$, or for each $(b_1, b_2) \in \text{pr}_{2,3}(\rho)$ there exists a unique $a \in A$ such that $(a, b_1, b_2) \in \rho$.

Case 1. There exist $c_1, c_2 \in B$ such that $(a, c_1, c_2) \in \rho$ for every $a \in A$. Put $D = \{(b, c) \mid \forall a \in A : (a, b, c) \in \rho\}$. By Lemma 7.18, D is a binary absorbing subuniverse in the projection of ρ onto the last two variables. By Lemma 7.2, there exists $(b, b) \in D$. This completes this case.

Case 2. For each $(b_1, b_2) \in \text{pr}_{2,3}(\rho)$ there exists a unique $a \in A$ such that $(a, b_1, b_2) \in \rho$. Let δ_1 be the projection of ρ onto the first two variables. By Lemma 7.12 we have one of two situations.

Case 2A. For every $b \in B$ there exists a unique a such that $(a, b) \in \delta_1$. Since ρ is subdirect, for every a there exists $(a, b, b') \in \rho$, which implies that $(a, b, b) \in \rho$ and completes this case.

Case 2B. There exists an element b such that $(a, b) \in \delta_1$ for every $a \in A$. Consider the relation $\delta_2(x, y_2) = \rho(x, b, y_2)$. If $\text{pr}_2(\delta_2) \neq B$, then we restrict the last two variables of ρ to $\text{pr}_2(\delta_2)$ and apply the inductive assumption. Assume that $\text{pr}_2(\delta_2) = B$. By the definition of the second case, we know that for every $c \in B$ there exists a unique a such that $(a, c) \in \delta_2$. Then $\sigma = \text{Con}(\delta_2, 2)$ is a proper congruence such that $B/\sigma \cong A$. If σ is the equality relation, then $B \cong A$, and, by Lemma 7.13, ρ can be represented by binary bijective constraints. If the first coordinate of ρ is uniquely defined by the second or the third, then it is equivalent to the case 2A, which we already considered. If the first coordinate of ρ does not depend on the others, then the claim is trivial.

If σ is not the equality relation, then we consider the relation ρ' obtained from ρ by factorization of the last two variables by σ , that is, $\rho' \subseteq A \times B/\sigma \times B/\sigma$ contains all tuples $(a, b/\sigma, b'/\sigma)$ such that $(a, b, b') \in \rho$. By the inductive assumption for any $a \in A$ there exists $E \in B/\sigma$ such that $(a, E, E) \in \rho'$. By Lemma 7.12, we have one of the following situations. Case 1. There exists $E \in B/\sigma$ such that for every $a \in A$ we have $(a, E, E) \in \rho'$. Then we restrict the last two variables of ρ to E and apply the inductive assumption. Case 2. For every $E \in B/\sigma$ there exists a unique $a \in A$ such that $(a, E, E) \in \rho'$. In this case, for any $a \in A$ we choose E such that $(a, E, E) \in \rho'$. By the uniqueness of a we have $(a, b, b) \in \rho$ for any $b \in E$, which completes the proof. \square

7.4 Linear Subuniverse

We have the following well-known fact from linear algebra [33].

LEMMA 7.20. *Suppose $\rho \subseteq (\mathbb{Z}_{p_1})^{n_1} \times \dots \times (\mathbb{Z}_{p_k})^{n_k}$, where p_1, \dots, p_k are distinct prime numbers dividing $m - 1$ and $\mathbb{Z}_{p_i} = (\mathbb{Z}_{p_i}; x_1 + \dots + x_m)$ for every i . Then $\rho = L_1 \times \dots \times L_k$ where each L_i is an affine subspace of $(\mathbb{Z}_{p_i})^{n_i}$.*

COROLLARY 7.20.1. *The set of linear algebras is closed under taking subalgebras, quotients, and finite products.*

LEMMA 7.21. *A linear algebra has no nontrivial absorbing subuniverse, nontrivial center, or nontrivial PC subuniverse.*

PROOF. Let us prove that a linear algebra A has no nontrivial absorbing subuniverse, which by Corollary 7.9.1 implies that A has no nontrivial center. By Lemma 7.3, it is sufficient to show that

\mathbb{Z}_p has no nontrivial absorbing subuniverse. Every term operation in \mathbb{Z}_p can be represented as $a_1x_1 + \dots + a_lx_l$, and for each $a_l \neq 0$ fixing all variables but x_l to some values gives a bijective mapping, which means that this term cannot witness an absorption.

Since linear algebras (by Corollary 7.20.1) are closed under quotients, to prove that it does not have a nontrivial PC subuniverse it is sufficient to prove that a linear algebra A cannot be a PC algebra. Assume that A is isomorphic to $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_k}$ for prime numbers p_1, \dots, p_k , and $\psi : A \rightarrow \mathbb{Z}_{p_1}$ is the canonical mapping. Let ρ be the set of all tuples (a, b, c, d) such that $\psi(a) + \psi(b) = \psi(c) + \psi(d)$. We can check that ρ is preserved by w and all constants but not all operations, therefore A cannot be polynomially complete. \square

LEMMA 7.22. *Suppose $\rho \subseteq A_1 \times A_2$ is a subdirect relation, A_2 is a linear algebra, and there is no nontrivial binary absorbing subuniverse on A_1 . Then for all $a, b \in A_1$ we have*

$$|\{c \mid (a, c) \in \rho\}| = |\{c \mid (b, c) \in \rho\}|.$$

PROOF. Assume the contrary, then we choose all elements a with the maximal $|\{c \mid (a, c) \in \rho\}|$. Denote the set of such elements by C .

Since $w(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_m)$ is a bijection on A_2 for every $a_1, \dots, a_m \in A_2$, we have $w(A_1, \dots, A_1, C, A_1, \dots, A_1) \subseteq C$. Hence $w(x, \dots, x, y)$ is a binary absorbing operation and C is a binary absorbing subuniverse. \square

LEMMA 7.23. *Suppose A is a linear algebra. Then $w(a, b, \dots, b) = a$ for every $a, b \in A$.*

PROOF. Suppose $A \cong \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_k}$. Since the WNU w is special and idempotent, each p_i divides $m - 1$. Therefore, $w(a, b, \dots, b) = a$ for every $a, b \in A$. \square

LEMMA 7.24. *Suppose $\rho \subseteq A_1 \times A_2$ is a subdirect relation, A_2 is a linear algebra, and there is no nontrivial binary absorbing subuniverse on A_1 . Then ρ has the parallelogram property.*

PROOF. First, we define a relation σ_k for every $k \geq 2$ by

$$\sigma_k(y_1, \dots, y_k) = \exists x \rho(x, y_1) \wedge \dots \wedge \rho(x, y_k).$$

By Lemma 7.23, $w(a, b, \dots, b, b) = a$ and $w(b, b, \dots, b, c) = c$ for any $a, b, c \in A_2$, therefore $(a, b), (b, c), (b, b) \in \sigma_2$ implies $(a, c) \in \sigma_2$. Since σ_2 is reflexive and symmetric, it is a congruence.

Let us show by induction on k that $\sigma_k(y_1, \dots, y_k) = \bigwedge_{i=2}^k \sigma_2(y_1, y_i)$. For $k = 2$, it is obvious. Consider a tuple (a_1, \dots, a_k) such that $(a_i, a_j) \in \sigma_2$ for any i, j . By the inductive assumption for $k - 1$ we have $(a_1, a_1, a_3, \dots, a_k), (a_1, a_2, a_1, a_4, \dots, a_k), (a_1, a_1, a_1, a_4, \dots, a_k) \in \sigma_k$. If we apply the term operation $g(x, y, z) = w(x, y, z, \dots, z)$ to these three tuples (in the same order), then we obtain (a_1, \dots, a_k) , which means that $(a_1, \dots, a_k) \in \sigma_k$. Thus, $\sigma_k(y_1, \dots, y_k) = \bigwedge_{i=2}^k \sigma_2(y_1, y_i)$ for every k .

Substituting $\{y_1, \dots, y_k\} = E$ in the definition of σ_k for an equivalence class E of σ_2 , we derive that there exists $c \in A_1$ such that $(c, d) \in \rho$ for any $d \in E$. Then it follows from Lemma 7.22 that ρ has the parallelogram property. \square

COROLLARY 7.24.1. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a relation such that $\text{pr}_1(\rho) = A_1$, there is no nontrivial binary absorbing subuniverse on A_1 , and $C = \text{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$, where C_i is a linear subuniverse of A_i for every i . Then C is a linear subuniverse of A_1 .*

PROOF. Let $\psi : A_1 \times \dots \times A_n \rightarrow A_1 \times A_2 / \text{ConLin}(A_2) \times \dots \times A_n / \text{ConLin}(A_n)$ be a natural homomorphism. Put $\gamma = C_1 \times \dots \times C_n$, $\rho' = \psi(\rho)$, $\gamma' = \psi(\gamma)$. We can check that $\text{pr}_1(\rho \cap \gamma) = \text{pr}_1(\rho' \cap \gamma')$. The relation ρ' can be viewed as a subdirect subalgebra of $A_1 \times B$, where $B = \text{pr}_{2, \dots, n}(\rho')$ is a linear algebra by Corollary 7.20.1. Then $D = \text{pr}_{2, \dots, n}(\gamma') \cap B$ can be viewed as a subalgebra of B . We need to show that $\text{pr}_1(\rho' \cap (C_1 \times D))$ is a linear subuniverse of A_1 . By

Lemma 7.24, the binary relation ρ' has the parallelogram property, then ρ' induces an isomorphism $A_1/\sigma_1 \cong B/\sigma_2$, where $\sigma_1 = \text{Con}(\rho', 1)$, $\sigma_2 = \text{Con}(\rho', 2)$. Note that σ_1 is a linear congruence, since B is a linear algebra. Hence, $D_1 = \{a \in A_1 \mid \exists d \in D : (a, d) \in \rho'\}$ is stable under σ_1 (and under $\text{ConLin}(A_1)$). Then $\text{pr}_1(\rho \cap \gamma) = \text{pr}_1(\rho' \cap \gamma') = C_1 \cap D_1$ is stable under $\text{ConLin}(A_1)$, which completes the proof. \square

7.5 Common Properties

In this subsection, we list some properties that are common for all types of one-of-four subuniverses.

LEMMA 7.25. *Suppose $R \subseteq D_1 \times \dots \times D_n$ is a subdirect relation, B_i is a one-of-four subuniverse of D_i of type \mathcal{T} for every $i \in \{1, \dots, n\}$; if \mathcal{T} is the absorbing type, then the absorbing subuniverses are witnessed by the same term operation. Then $R \cap (B_1 \times \dots \times B_n)$ is a one-of-four subuniverse of R of type \mathcal{T} .*

PROOF. If \mathcal{T} is the absorbing type, then the statement follows from Lemma 7.1, if \mathcal{T} is the central type, then the statement follows from Lemma 7.6.

Suppose \mathcal{T} is the linear type, then put $\sigma_i = \text{ConLin}(D_i)$ for each $i \in \{1, \dots, n\}$. First, extend every σ_i naturally on $D = D_1 \times \dots \times D_n$ and denote the obtained congruence σ'_i so that $D/\sigma'_i \cong D_i/\sigma_i$. Since linear algebras are closed under taking subalgebras and quotients (Corollary 7.20.1), $\sigma = \sigma'_1 \cap \dots \cap \sigma'_n$ is a linear congruence and $B_1 \times \dots \times B_n$ is stable under this congruence. Therefore, $\sigma \cap (R \times R)$ is a linear congruence and $R \cap (B_1 \times \dots \times B_n)$ is stable under it. This completes this case.

It remains to consider the case when \mathcal{T} is the PC type. Let $\delta_1, \dots, \delta_t$ be the set of all PC congruences on D_1, \dots, D_n we need to define B_1, \dots, B_n . For every $i \in \{1, 2, \dots, t\}$ by δ'_i we denote δ_i naturally extended on $D = D_1 \times \dots \times D_n$, by E_i we denote the equivalence class of δ'_i containing $B_1 \times \dots \times B_n$. Since R is subdirect, $R/\delta'_i \cong D/\delta'_i$ and R/δ'_i is a PC algebra without a nontrivial binary absorbing subuniverse or center. Since $R \cap (B_1 \times \dots \times B_n) = R \cap (E_1 \cap \dots \cap E_t)$, the set $R \cap (B_1 \times \dots \times B_n)$ is a PC subuniverse of R . \square

LEMMA 7.26. *Suppose σ is a congruence on D , B is a one-of-four subuniverse of D stable under σ . Then $\{b/\sigma \mid b \in B\}$ is a one-of-four subuniverse of D/σ of the same type as B .*

PROOF. For a binary subuniverse and a center it follows from Corollaries 7.1.1 and 7.6.1, respectively. Suppose B is a linear subuniverse. Let δ be the minimal congruence containing both σ and $\text{ConLin}(D)$. By Corollary 7.20.1, D/δ is a linear algebra. Since B is stable under δ , $\{b/\sigma \mid b \in B\}$ is a linear subuniverse of D/σ .

It remains to consider the case when B is a PC subuniverse of D , that is, $B = E_1 \cap \dots \cap E_s$, where E_i is an equivalence class of a PC congruence σ_i for every i . Let δ be the minimal congruence containing σ and $\sigma_1 \cap \dots \cap \sigma_s$. By Lemma 7.14, δ is an intersection of PC congruences $\delta_1, \dots, \delta_t$. Since B is stable under δ and B is an equivalence class of $\sigma_1 \cap \dots \cap \sigma_s$, B is an equivalence class of δ . Hence, $\{b/\sigma \mid b \in B\}$ is an intersection of the equivalence classes of congruences on D/σ corresponding to $\delta_1, \dots, \delta_t$. \square

The following corollaries (proved earlier) state that if we restrict all coordinates of a relation to one-of-four subuniverses of type \mathcal{T} then we restrict its projection onto the first coordinate to a subuniverse of type \mathcal{T} . The only difference is that for PC subuniverse we require the relation to be subdirect and without nontrivial binary absorbing subuniverse or center on every coordinate, and for linear subuniverse the first coordinate should be without a nontrivial binary absorbing subuniverse.

COROLLARY 7.1.2. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a relation such that $\text{pr}_1(\rho) = A_1$ and $C = \text{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$, where C_i is an absorbing subuniverse in A_i with a term t for every i . Then C is an absorbing subuniverse in A_1 with the term t .*

COROLLARY 7.6.2. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a relation such that $\text{pr}_1(\rho) = A_1$ and $C = \text{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$, where C_i is a center in A_i for every i . Then C is a center in A_1 .*

COROLLARY 7.13.2. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a subdirect relation, there is no nontrivial binary absorbing subuniverse or nontrivial center on A_1 , and $C = \text{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$, where C_i is a PC subuniverse in A_i for every i . Then C is a PC subuniverse in A_1 .*

COROLLARY 7.24.1. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a relation such that $\text{pr}_1(\rho) = A_1$, there is no nontrivial binary absorbing subuniverse on A_1 , and $C = \text{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$, where C_i is a linear subuniverse of A_i for every i . Then C is a linear subuniverse of A_1 .*

Another common property is that we cannot have (C_1, \dots, C_k) -essential relation of arity greater than 2 if C_1, \dots, C_k are subuniverses of a fixed type (any but linear). Note that for PC subuniverses we additionally require the relation to be subdirect. From these claims it can be derived that for the nonlinear case (see Corollary 9.2.1) it is sufficient to check cycle consistency (all calculations are on binary relations) to guarantee a solution.

LEMMA 7.27. *Suppose C_i is a nontrivial binary absorbing subuniverse of A_i with a term t for $i \in \{1, 2, \dots, k\}$, $k \geq 2$. Then there does not exist a (C_1, \dots, C_k) -essential relation $\rho \subseteq A_1 \times \cdots \times A_k$.*

PROOF. Assume that such a relation exists. To get a contradiction it is sufficient to apply term t to a tuple from $A_1 \times C_2 \times \cdots \times C_k$ and a tuple from $C_1 \times \cdots \times C_{k-1} \times A_k$. \square

The following two corollaries were proved earlier.

COROLLARY 7.10.3 *Suppose C_i is a center of A_i for $i \in \{1, 2, \dots, k\}$, $k \geq 3$. Then there does not exist a (C_1, \dots, C_k) -essential relation $\rho \subseteq A_1 \times \cdots \times A_k$.*

COROLLARY 7.13.3. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a subdirect relation, $n \geq 3$, C_i is a PC subuniverse in A_i . There does not exist a (C_1, \dots, C_n) -essential relation.*

7.6 Interaction

Here we explain how one-of-four subuniverses of different types interact with each other.

LEMMA 7.28. *Suppose B_1 is a binary absorbing, central, or linear subuniverse of D , and B_2 is a subuniverse of D . Then $B_1 \cap B_2$ is a binary absorbing, central, or linear subuniverse of B_2 , respectively.*

PROOF. If B_1 is a binary absorbing subuniverse or a center, then the claim follows from Lemmas 7.1 and 7.6, respectively. If B_1 is a linear subuniverse, then by Corollary 7.20.1 $B_2 / \text{ConLin}(D)$ is a linear algebra, and hence $B_1 \cap B_2$ is a linear subuniverse of B_2 . \square

LEMMA 7.29. *Suppose B_1 and B_2 are nonempty one-of-four subuniverses of D , $B_1 \cap B_2 = \emptyset$. Then B_1 and B_2 are subuniverses of the same type.*

PROOF. Assume the converse. Consider all possible cases.

Case 1. B_1 is a linear subuniverse, and B_2 is a binary absorbing subuniverse. By Corollary 7.1.1 $\{b / \text{ConLin}(D) \mid b \in B_2\}$ is a binary absorbing subuniverse on $D / \text{ConLin}(D)$. By Lemma 7.21, this subuniverse should be trivial, which contradicts the fact that $B_1 \cap B_2 = \emptyset$ and B_1 is stable under $\text{ConLin}(D)$.

Case 2. B_1 is a linear subuniverse, and B_2 is a center. By Corollary 7.6.1 $\{b / \text{ConLin}(D) \mid b \in B_2\}$ is a center of $D / \text{ConLin}(D)$. By Lemma 7.21, this subuniverse should be trivial, which contradicts the fact that $B_1 \cap B_2 = \emptyset$ and B_1 is stable under $\text{ConLin}(D)$.

Case 3. B_1 is a linear subuniverse, and B_2 is a PC subuniverse. Let $S \subseteq (D/\text{ConLin}(D)) \times D$ consist of all the tuples $(c/\text{ConLin}(D), c)$, where $c \in D$. By Lemma 7.21, there is no nontrivial binary absorbing subuniverse or center on $D/\text{ConLin}(D)$. Hence, by Corollary 7.13.2, the restriction of the second variable to B_2 implies the restriction of the first variable to a PC subuniverse. Since $B_1 \cap B_2 = \emptyset$, this restriction is nontrivial. Thus, there exists a nontrivial PC subuniverse on $D/\text{ConLin}(D)$, which contradicts Lemma 7.21.

Case 4. B_1 is a PC subuniverse, and B_2 is a binary absorbing subuniverse. By Corollary 7.1.1 the set $\{b/\text{ConPC}(D) \mid b \in B_2\}$ is a binary absorbing subuniverse of $D/\text{ConPC}(D)$. By Lemma 7.15, this subuniverse should be trivial, which contradicts the fact that $B_1 \cap B_2 = \emptyset$ and B_1 is a PC subuniverse.

Case 5. B_1 is a PC subuniverse, and B_2 is a center. By Corollary 7.6.1 $\{b/\text{ConPC}(D) \mid b \in B_2\}$ is a center of $D/\text{ConPC}(D)$. By Lemma 7.15 this subuniverse should be trivial, which contradicts the fact that $B_1 \cap B_2 = \emptyset$ and B_1 is a PC subuniverse.

Case 6. B_1 is a binary absorbing subuniverse, and B_2 is a center. Suppose $R \subseteq D \times G$ is the binary relation from the definition of the center B_2 , and denote $b^+ = \{a \mid (b, a) \in R\}$ for every $b \in D$. We prove this case by induction on the size of D . Assume that $b_1^+ \neq b_2^+$ for some $b_1, b_2 \in B_1$. Choose an element $c \in b_1^+ \setminus b_2^+$ (or in $b_2^+ \setminus b_1^+$). Put $D' = \{a \mid (a, c) \in R\}$. Note that $D' \subsetneq D$, $D' \cap B_1 \neq \emptyset$, $D' \cap B_2 = B_2$. Thus, we obtain subuniverses $B_1 \cap D'$ and B_2 of a smaller set D' that are a binary absorbing subuniverse and a center (by Lemma 7.28), respectively. It remains to apply the inductive assumption to $B_1 \cap D'$ and B_2 . Let us consider the case when $b_1^+ = b_2^+$ for any $b_1, b_2 \in B_1$. Since $B_1 \cap B_2 = \emptyset$, $b^+ \neq G$ for every $b \in B_1$. Let f be the binary absorbing operation. Choose $b \in B_1$ and $e \in B_2$. Then $f(b, e) = b_1 \in B_1$ and $f(e, b) = b_2 \in B_1$, which means that $f(b^+, G) \subseteq b_1^+ = b^+$, $f(G, b^+) \subseteq b_2^+ = b^+$. This contradicts the definition of a center, saying that there is no nontrivial binary absorbing subuniverse on G . \square

THEOREM 7.30. *Suppose B_1 and B_2 are one-of-four subuniverses of D of types \mathcal{T}_1 and \mathcal{T}_2 , respectively. Then $B_1 \cap B_2$ is a one-of-four subuniverse of B_2 of type \mathcal{T}_1 .*

PROOF. If B_1 is not a PC subuniverse, then the claim follows from Lemma 7.28. Assume that B_1 is a PC subuniverse of D .

Let $\sigma_1, \dots, \sigma_t$ be the set of all PC congruences on D . Assume that B_2 is not a PC subuniverse. Every equivalence class E of σ_i is a PC subuniverse. Then Lemma 7.29 implies that E has a nonempty intersection with B_2 . Therefore, $B_2/\sigma_i \cong D/\sigma_i$ and $\sigma_i \cap (B_2 \times B_2)$ is a PC congruence on B_2 for every i . Hence, $B_1 \cap B_2$ is a PC subuniverse of B_2 , which completes this case.

If B_2 is also a PC subuniverse, then by Corollary 7.13.1, $B_1 \cap B_2$ is a PC subuniverse of B_2 . \square

LEMMA 7.31. *Suppose $D = A_0 = B_0$, $s \geq 1$, $t \geq 0$, A_i is a one-of-four subuniverse of A_{i-1} for every $i \in \{1, \dots, s\}$, and B_i is a one-of-four subuniverse of B_{i-1} for every $i \in \{1, \dots, t\}$. Then $A_s \cap B_t$ is a one-of-four subuniverse of $A_{s-1} \cap B_t$ of the same type as A_s .*

PROOF. We prove this lemma by induction on $s + t$. Let A_s be a one-of-four subuniverse of A_{s-1} of type \mathcal{T} . For $t = 0$, the claim follows from the statement. Assume that $t \geq 1$. By the inductive assumption, $A_{s-1} \cap B_t$ and $A_s \cap B_{t-1}$ are one-of-four subuniverses of $A_{s-1} \cap B_{t-1}$, and the second of them is of type \mathcal{T} . Then by Theorem 7.30, their intersection $A_s \cap B_t$ is a one-of-four subuniverse of $A_{s-1} \cap B_t$ of type \mathcal{T} . \square

LEMMA 7.32. *Suppose $R \subseteq A_0 \times B_0$ is a subdirect relation, B_i is a one-of-four subuniverse of B_{i-1} for every $i \in \{1, \dots, t\}$, A_1 is a one-of-four subuniverse of A_0 . Then $\text{pr}_2(R \cap (A_1 \times B_t))$ is a one-of-four subuniverse of $\text{pr}_2(R \cap (A_1 \times B_{t-1}))$ of the same type as B_t .*

PROOF. By Lemma 7.25, $R \cap (A_0 \times B_i)$ is a one-of-four subuniverse of $R \cap (A_0 \times B_{i-1})$ of the same type as B_i , and $R \cap (A_1 \times B_0)$ is a one-of-four subuniverse of R . By Lemma 7.31, $R \cap (A_1 \times B_t)$

is a one-of-four subuniverse of $R \cap (A_1 \times B_{t-1})$ of the same type as B_t . Let σ be the congruence on $R \cap (A_1 \times B_0)$ such that two elements are equivalent whenever their projections onto the second coordinate are equal. Then $R \cap (A_1 \times B_t)$ is stable under σ for every i . By Lemma 7.26, $\text{pr}_2(R \cap (A_1 \times B_t))$ is a one-of-four subuniverse of $\text{pr}_2(R \cap (A_1 \times B_{t-1}))$ of the same type as B_t . \square

THEOREM 7.33. *Suppose B_1, \dots, B_n are one-of-four subuniverses of D , and $B_1 \cap \dots \cap B_n = \emptyset$. Then there exists $I \subseteq \{1, \dots, n\}$ with $\bigcap_{i \in I} B_i = \emptyset$ satisfying one of the following conditions:*

- (1) $|I| \leq 2$ and all subuniverses B_i , where $i \in I$, are of the same type;
- (2) B_i is a linear subuniverse for every $i \in I$;
- (3) B_i is a binary absorbing subuniverse for every $i \in I$.

PROOF. Let us prove by induction on n . For $n = 1$, it is trivial. For $n = 2$, it follows from Lemma 7.29. If $\bigcap_{i \in I} B_i = \emptyset$ for some $I \subsetneq \{1, 2, \dots, n\}$, then applying the inductive assumption to $\bigcap_{i \in I} B_i$ we obtain the required property. Thus, we assume that if we remove one one-of-four subuniverse from the intersection $B_1 \cap \dots \cap B_n$, then we get a nonempty set.

Let us show that all subuniverses should be of the same type. Put $C_i = B_i \cap B_n$ for every $i \in \{1, 2, \dots, n-1\}$. By Lemma 7.30, C_i is a one-of-four subuniverse of B_n of the same type as B_i . Applying the inductive assumption to $C_1 \cap \dots \cap C_{n-1} = \emptyset$, we derive that C_1, \dots, C_{n-1} are of the same type, and hence B_1, \dots, B_{n-1} are of the same type. Similarly, we can show that B_2, \dots, B_n are of the same type, and, therefore, since $n \geq 3$, all of them are of the same types.

Assume that all subuniverses B_1, \dots, B_n are centers or PC subuniverses. Let R be the n -ary relation consisting of all tuples (a, a, \dots, a) . Then R is a (B_1, \dots, B_n) -essential relation, which contradicts Corollary 7.10.3 for centers and Corollary 7.13.3 for PC subuniverses. \square

8 PROOF OF THE AUXILIARY STATEMENTS

8.1 One-of-Four Reductions

LEMMA 8.1. *Suppose $D^{(1)}$ is a one-of-four reduction for an instance Θ of type \mathcal{T} , which is not the PC type. Then $\Theta^{(1)}(z)$ is a one-of-four subuniverse of $\Theta(z)$ of type \mathcal{T} for every variable z .*

PROOF. Let $\text{Var}(\Theta) = \{x_1, \dots, x_t\}$ and $\Theta(x_1, \dots, x_t)$ define the relation R . By Lemma 7.28, $D_{x_i}^{(1)} \cap \text{pr}_i(R)$ is a one-of-four subuniverse of $\text{pr}_i(R)$ of type \mathcal{T} for every i . Considering R as a subdirect relation on smaller domains and applying Corollaries 7.1.2, 7.6.2, and 7.24.1, we conclude that $\Theta^{(1)}(z)$ is a one-of-four subuniverse of $\Theta(z)$ of type \mathcal{T} . \square

LEMMA 8.2. *Suppose $D^{(1)}$ is a PC reduction for a 1-consistent instance Θ , for every variable y appearing at least twice in Θ the pp-formula $\Theta(y)$ defines D_y , and $\Theta(z)$ defines D_z for a variable z . Then $\Theta^{(1)}(z)$ is a PC subuniverse of D_z .*

PROOF. First, we rename the variables in Θ so that every variable occurs just once and denote the obtained instance by Θ_0 . Then we identify variables back to obtain the original instance step by step. Thus, we get a sequence $\Theta_0, \Theta_1, \Theta_2, \dots, \Theta_s$ such that Θ_{i+1} is obtained from Θ_i by identifying of two variables and $\Theta_s = \Theta$. Let us show by induction on i that for every variable z the set $\Theta_i(z) \cap D_z^{(1)}$ is a PC subuniverse of $\Theta_i(z)$. For $i = 0$ it follows from the fact that Θ is 1-consistent, and, therefore, $\Theta_0(z)$ defines the full D_z .

Assume that Θ_{i+1} is obtained from Θ_i by identifying of y and y' , and the variable in Θ corresponding to y and y' is y . We know that for every variable z appearing at least twice in Θ , $\Theta(z)$ defines D_z . Hence $\Theta_{i+1}(y)$ also defines D_y . Thus, we just need to show that for any variable z different from y and y' the set $\Theta_{i+1}(z) \cap D_z^{(1)}$ is a PC subuniverse of $\Theta_{i+1}(z)$. By the inductive assumption $\Theta_i(z) \cap D_z^{(1)}$ is a PC subuniverse of $\Theta_i(z)$. Then $\Theta_i(z) \cap D_z^{(1)} = E_1 \cap \dots \cap E_t$, where E_j is

an equivalence class of a PC congruence σ_j on $\Theta_i(z)$ for every j . Let $S \subseteq \Theta_i(z)/\sigma_j \times D_y \times D_y$ be the relation consisting of all tuples $(a/\sigma_j, b, b')$ such that Θ_i has a solution with $z = a$, $y = b$, $y' = b'$. Since the variable y appears at least twice in Θ , $\Theta(y)$ defines a full relation. Hence, the relation S is subdirect, and for every $b \in D_y$ there exists E such that $(E, b, b) \in S$. Lemma 7.19 implies that for every equivalence class E of σ_j there exists b such that Θ_i has a solution with $z \in E$ and $y = y' = b$, which means that there exists a solution of Θ_{i+1} with $z \in E$. Therefore, $\Theta_i(z)/\sigma_j \cong \Theta_{i+1}(z)/\sigma_j$, which implies that $\Theta_{i+1}(z) \cap D_z^{(1)}$ is a PC subuniverse of $\Theta_{i+1}(z)$. This completes the inductive step.

Since $\Theta = \Theta_s$, we proved that $\Theta(z) \cap D_z^{(1)}$ is a PC subuniverse of $\Theta(z)$ for every variable z of Θ .

Suppose $\text{Var}(\Theta) = \{x_1, \dots, x_t\}$, $\Theta(x_1, \dots, x_t)$ defines a relation R . Then R can be viewed as a subdirect relation if we reduce the domain of every variable x_i to $\Theta(x_i)$. By Corollary 7.13.2, for any variable z with $\Theta(z) = D_z$ we obtain that $\Theta^{(1)}(z)$ is a PC subuniverse of D_z . \square

LEMMA 8.3. *Suppose $D^{(1)}$ is a minimal absorbing, central, or linear reduction for an instance Θ , and $\Theta(x_1, \dots, x_n)$ defines a full relation. Then $\Theta^{(1)}(x_1, \dots, x_n)$ defines a full relation or an empty relation.*

PROOF. If $\Theta^{(1)}(x_1, \dots, x_n)$ defines an empty relation, then there is nothing to prove. Assume that $\Theta^{(1)}(x_1, \dots, x_n)$ is not empty.

We prove by induction on n . For $n = 1$ by Lemma 8.1 $\Theta^{(1)}(x_1)$ is a subuniverse of $\Theta(x_1)$ of the corresponding type. By the minimality of the reduction $D^{(1)}$ the pp-formula $\Theta^{(1)}(x_1)$ defines $D_{x_1}^{(1)}$.

Let us prove the induction step. For each $i \in \{1, \dots, n-1\}$, choose $a_i \in D_{x_i}^{(1)}$. By the inductive assumption, $\Theta^{(1)}(x_1, \dots, x_{n-1})$ defines a full relation, and hence there exists a solution of $\Theta^{(1)}$ having $x_i = a_i$ for every $i \in \{1, \dots, n-1\}$.

Add the constraint $x_i = a_i$ to Θ for every $i \in \{1, \dots, n-1\}$ and denote the obtained instance by Ω . By the condition of this lemma $\Omega(x_n)$ defines D_{x_n} . By Lemma 8.1, $\Omega^{(1)}(x_n)$ defines a one-of-four subuniverse of D_{x_n} of the corresponding type, which by the minimality of the reduction $D^{(1)}$ implies that $\Omega^{(1)}(x_n)$ defines $D_{x_n}^{(1)}$. Since we chose a_1, \dots, a_{n-1} arbitrary, this means that $\Theta^{(1)}(x_1, \dots, x_n)$ defines a full relation. \square

LEMMA 8.4. *Suppose $D^{(1)}$ is a minimal PC reduction for a 1-consistent instance Θ , for every variable y appearing at least twice in Θ the pp-formula $\Theta(y)$ defines D_y , and $\Theta(x_1, \dots, x_n)$ defines a full relation. Then $\Theta^{(1)}(x_1, \dots, x_n)$ defines a full relation or an empty relation.*

PROOF. If $\Theta^{(1)}(x_1, \dots, x_n)$ defines an empty relation, then there is nothing to prove. Assume that $\Theta^{(1)}(x_1, \dots, x_n)$ is not empty.

First, we join variables x_1, \dots, x_n into one variable X with domain $D_{x_1} \times \dots \times D_{x_n}$. We replace x_1, \dots, x_n by X and change all constraints containing one of the variables x_1, \dots, x_n correspondingly. The obtained instance we denote by Ω . Since $\Theta(x_1, \dots, x_n)$ defines a full relation, the instance Ω is 1-consistent.

Second, we define a reduction $D^{(1)}$ on the domain of the new variable X by $D_X^{(1)} = D_{x_1}^{(1)} \times \dots \times D_{x_n}^{(1)}$. Let us show that this is a PC reduction. By Lemma 7.25, $D_X^{(1)}$ is a PC subuniverse of D_X . By Lemmas 7.3 and 7.7, there is no nontrivial binary absorbing subuniverse or center on D_X . Thus, $D^{(1)}$ is a PC reduction for Ω . By Lemma 8.2, $\Omega^{(1)}(X)$ is a PC subuniverse of D_X . By Lemma 7.17, $\Omega^{(1)}(X) = B_1 \times \dots \times B_n$, where B_i is a PC subuniverse of D_{x_i} for every i . By the minimality of $D^{(1)}$ on Θ we obtain that $B_i = D_{x_i}^{(1)}$. Hence, $\Theta^{(1)}(x_1, \dots, x_n)$ defines a full relation. \square

LEMMA 8.5. *Suppose $D^{(1)}$ is a one-of-four minimal reduction of an instance Θ , $\rho(x_1, \dots, x_n)$ is a subdirect constraint of Θ , and $\rho^{(1)}$ is not empty. Then $\rho^{(1)}$ is subdirect.*

PROOF. We need to show that $\text{pr}_i(\rho \cap (D_{x_1}^{(1)} \times \cdots \times D_{x_n}^{(1)})) = D_{x_i}^{(1)}$. By Corollaries 7.1.2, 7.6.2, 7.13.2, and 7.24.1, $B_i = \text{pr}_i(\rho \cap (D_{x_1}^{(1)} \times \cdots \times D_{x_n}^{(1)}))$ is a one-of-four subuniverse of D_{x_i} of the same type. Since $\rho^{(1)}$ is not empty, B_i is not empty. Since $D_{x_i}^{(1)}$ is a minimal subuniverse of this type, we have $B_i = D_{x_i}^{(1)}$. \square

LEMMA 8.6. *Suppose $D^{(1)}$ is a one-of-four minimal reduction for a cycle-consistent irreducible CSP instance Θ , and $\Theta^{(1)}$ has a solution. Then $\Theta^{(1)}$ is cycle-consistent and irreducible.*

PROOF. Consider a path P in Θ starting and ending with one variable x . By Ω we denote its covering $z_1 - Q_1 - z_2 - \cdots - Q_{l-1} - z_l$ (which is also a covering of Θ) that is obtained from P by renaming the variables so that every variable except for z_2, \dots, z_{l-1} occurs just once, z_2, \dots, z_{l-1} occur twice. Thus, z_1 and z_l are different but $S(z_1) = S(z_l) = x$ in the definition of the covering. By Ω' , we denote the formula obtained from Ω by substituting z_1 for z_l .

First, we prove that P connects a with a in $\Theta^{(1)}$ for every $a \in D_x^{(1)}$. Since Θ is cycle-consistent, $\Omega'(z_1)$ defines D_x . Since $\Theta^{(1)}$ has a solution, $\Omega'^{(1)}(z_1)$ defines a nonempty relation. By Lemmas 8.3 and 8.4, $\Omega'^{(1)}(z_1)$ defines $D_x^{(1)}$, which means that P connects a with a in $\Theta^{(1)}$ for every $a \in D_x^{(1)}$. Hence, $\Theta^{(1)}$ is cycle-consistent.

Assume that P connects any two elements of D_x , which means that $\Omega(z_1, z_l)$ defines a full relation. Since $\Theta^{(1)}$ has a solution, $\Omega^{(1)}(z_1, z_l)$ defines a nonempty relation. By Lemmas 8.3 and 8.4, $\Omega^{(1)}(z_1, z_l)$ also defines a full relation, which means that P connects any two elements of $D_x^{(1)}$ in $\Theta^{(1)}$.

Let us prove that $\Theta^{(1)}$ is irreducible. Consider an instance $\Upsilon_1 = \{C'_1, \dots, C'_s\}$ consisting of projections of constraints from $\Theta^{(1)}$ such that it is not fragmented and not linked. Let $\text{Var}(\Upsilon_1) = \{x_1, \dots, x_n\}$. By the definition for each constraint C'_i we can find a constraint $C_i \in \Theta$ such that C'_i is a projection of $C_i^{(1)}$ onto some variables. Let Υ_2 consist of the projections of C_1, \dots, C_s onto the same variables as in Υ_1 , and $\Upsilon \in \text{Coverings}(\Theta)$ is obtained from $\{C_1, \dots, C_s\}$ by renaming variables so that each variable except for x_1, \dots, x_n appears just once. Then the pp-formulas $\Upsilon_1(x_1, \dots, x_n)$ and $\Upsilon^{(1)}(x_1, \dots, x_n)$ define the same relation, $\Upsilon_2(x_1, \dots, x_n)$ and $\Upsilon(x_1, \dots, x_n)$ define the same relation. Since Υ_1 is not fragmented, both Υ and Υ_2 are not fragmented. Also, by Lemma 6.1, both Υ and Υ_2 are cycle-consistent and irreducible.

Assume that Υ_2 is linked. By Lemma 6.2 there exists a path that connects any two elements of D_{x_1} in Υ_2 . Then there exists a corresponding path within the variables x_1, \dots, x_n of Υ connecting any two elements of D_{x_1} . As we showed earlier this path, reduced to $D^{(1)}$, also connects any two elements of $D_{x_1}^{(1)}$ in $\Upsilon^{(1)}$. The same path can be used to connect any two elements of $D_{x_1}^{(1)}$ in Υ_1 , which contradicts our assumption that Υ_1 is not linked.

Suppose Υ_2 is not linked. Since Υ is irreducible, the solution set of Υ_2 is subdirect. Thus, for each variable x_i (these are only variables appearing more than once in Υ) we have $\Upsilon(x_i) = \Upsilon_2(x_i) = D_{x_i}$. Then by Lemmas 8.3 and 8.4, $\Upsilon^{(1)}(x_i)$ defines $D_{x_i}^{(1)}$ or an empty set. It cannot be empty because $\Theta^{(1)}$ has a solution; therefore, we have $\Upsilon_1(x_i) = \Upsilon^{(1)}(x_i) = D_{x_i}^{(1)}$ for every i , and the solution set of Υ_1 is subdirect, which completes the proof. \square

LEMMA 8.7. *Suppose $D^{(1)}$ is a minimal absorbing or central reduction for Θ , the solution set of Θ is subdirect, $D_{x_1} = D_{x_2}$, $D_{x_1}^{(1)} = D_{x_2}^{(1)}$, both $\Theta(x_1, x_2)$ and $\Theta^{(1)}(x_1, x_2)$ define reflexive symmetric relations, and $\Theta(x_1, x_2)$ contains $(a, b) \in D_{x_1}^{(1)} \times D_{x_2}^{(1)}$. Then a and b are linked in the relation defined by $\Theta^{(1)}(x_1, x_2)$.*

PROOF. Let $\text{Var}(\Theta) = \{x_1, x_2, y_1, \dots, y_t\}$, $\Theta(x_1, x_2, y_1, \dots, y_t)$ define a relation R . The relation R can be viewed as a ternary relation $R \subseteq D_{x_1} \times D_{x_2} \times (D_{y_1} \times \cdots \times D_{y_t})$. By Lemmas 7.1 and 7.6,

$G := D_{y_1}^{(1)} \times \cdots \times D_{y_t}^{(1)}$ is a one-of-four subuniverse of $D_{y_1} \times \cdots \times D_{y_t}$ of the same type as the reduction $D^{(1)}$. Let

$$R'(Y, Y', Y'') = \exists x_1 \exists x_2 R(a, x_1, Y) \wedge R(x_1, x_2, Y') \wedge R(x_2, b, Y'') \wedge x_1 \in D_{x_1}^{(1)} \wedge x_2 \in D_{x_1}^{(1)}.$$

Since $\Theta(x_1, x_2)$ contains (a, b) and $\Theta^{(1)}(x_1, x_2)$ defines a reflexive relation, there exist $B_1, B'_1 \in G$ such that $(B_1, B'_1, B''_1) \in R'$ (put $x_1 = x_2 = a$). Similarly, there exist $B'_2, B''_2 \in G$ such that $(B_2, B'_2, B''_2) \in R'$ (put $x_1 = x_2 = b$) and $B_3, B'_3 \in G$ such that $(B_3, B'_3, B''_3) \in R'$ (put $x_1 = a, x_2 = b$). By Lemma 7.27 and Corollary 7.10.3 R' cannot be G -essential, which means that $R \cap (G \times G \times G) \neq \emptyset$. Hence a and b are linked (by a path of length 3) in $\Theta^{(1)}(x_1, x_2)$. \square

LEMMA 8.8. *Suppose $D^{(1)}$ is a minimal PC reduction for Θ , the solution set of Θ is subdirect, $D_{x_1} = D_{x_2}$, $D_{x_1}^{(1)} = D_{x_2}^{(1)}$, both $\Theta(x_1, x_2)$ and $\Theta^{(1)}(x_1, x_2)$ define reflexive symmetric relations, and $\Theta(x_1, x_2)$ contains $(a, b) \in D_{x_1}^{(1)} \times D_{x_2}^{(1)}$. Then a and b are linked in the relation defined by $\Theta^{(1)}(x_1, x_2)$.*

PROOF. Suppose y is a variable of Θ and σ is a PC congruence on D_y . Consider a relation $\rho \subseteq D_{x_1} \times D_y/\sigma$ consisting of all the tuples (c, C) such that there exists a solution of Θ with $x_1 = c$ and $y \in C$. Since there is no nontrivial binary absorbing subuniverse or center on D_{x_1} , by Lemma 7.13, either ρ is a full relation, or $\text{Con}(\rho, 2)$ is the equality relation. In the first case, it does not matter what value we substitute for the variable x_1 the variable y can be at any equivalence class of σ . In the second case the equivalence class is uniquely determined by the variable x_1 . Moreover, since $D_{x_1}^{(1)}$ is a minimal PC subuniverse, the equivalence class is the same for all elements of $D_{x_1}^{(1)}$. Since $\Theta^{(1)}$ has a solution, this equivalence class is the class containing $D_y^{(1)}$. Later, we will specify whether a PC congruence is of the *first type* (from the first case) or of the *second type* (from the second case).

Let $\text{Var}(\Theta) = \{x_1, x_2, y_1, \dots, y_t\}$. Let R be the relation defined by $\Theta(x_1, x_2, y_1, \dots, y_t)$. By Y denote the following formula:

$$R(a, x_2, y_1, \dots, y_t) \wedge R(x_1, x_2, y'_1, \dots, y'_t) \wedge R(x_1, x'_2, z_1, \dots, z_t) \wedge R(b, x'_2, z'_1, \dots, z'_t) \wedge x_1 \in D_{x_1}^{(1)}.$$

Consider a congruence σ of the first type on the domain of any variable y of Y .

Assume that $y \in \{x_2, y_1, \dots, y_t, y'_1, \dots, y'_t\}$. It follows from the definition of the first type that for any equivalence class E of σ there exists a solution of Y such that $x_1 = a, x'_2 = b, y_i = y'_i$ for every i , and $y \in E$.

Similarly, assume that $y \in \{x'_2, z_1, \dots, z_t, z'_1, \dots, z'_t\}$. For any equivalence class E of σ there exists a solution of Y such that $x_1 = x_2 = b, z_i = z'_i$ for every i , and $y \in E$.

Thus, we showed that in both cases $Y(y)/\sigma \cong D_y/\sigma$. Let E be the equivalence class of σ containing $D_y^{(1)}$. By δ , we denote the extension of σ onto the solution set of Y , and by E_σ we denote the equivalence class of δ corresponding to E . Since $Y(y)/\sigma \cong D_y/\sigma$ and D_y/σ is a PC algebra without a nontrivial binary absorbing subuniverse or center, E_σ is a PC subuniverse of the solution set of Y .

Consider the intersection of E_σ for all PC congruences σ of the first type. If this intersection is not empty, then there exists a solution of Y such that any element of this solution is in the equivalence class containing $D_y^{(1)}$ for any PC congruence of the first type. Since $a, b \in D_{x_1}^{(1)}$ and $x_1 \in D_{x_1}^{(1)}$ in the definition of Y , the same is true for any PC congruence of the second type. Since $D_y^{(1)}$ is the intersection of all equivalence classes containing $D_y^{(1)}$ of all PC congruences for any variable y , the solution is in $D^{(1)}$, which means that a and b are linked in $\Theta^{(1)}(x_1, x_2)$.

Assume that the intersection of E_σ for all PC congruences σ of the first type is empty. By Theorem 7.33 there should be two congruences σ and σ' such that $E_\sigma \cap E_{\sigma'} = \emptyset$. Let y and y' be the variables of Y corresponding to σ and σ' . Consider several cases.

Case 1. $y, y' \in \{x_2, x'_2, y'_1, \dots, y'_t, z_1, \dots, z_t, z'_1, \dots, z'_t\}$. Since $\Theta^{(1)}(x_1, x_2)$ defines a reflexive relation, Υ has a solution with $x_2 = x_1 = x'_2 = b$ and all the variables $y'_1, \dots, y'_t, z_1, \dots, z_t, z'_1, \dots, z'_t$ are from $D^{(1)}$. This contradicts the fact that $E_\sigma \cap E_{\sigma'} = \emptyset$.

Case 2. $y, y' \in \{x_2, x'_2, y_1, \dots, y_t, y'_1, \dots, y'_t, z_1, \dots, z_t\}$. Similarly, Υ has a solution with $x_1 = x_2 = x'_2 = a$ and all the variables $y_1, \dots, y_t, y'_1, \dots, y'_t, z_1, \dots, z_t$ are from $D^{(1)}$. This contradicts the fact that $E_\sigma \cap E_{\sigma'} = \emptyset$.

Case 3. $y \in \{y_1, \dots, y_t\}, y' \in \{z'_1, \dots, z'_t\}$. Similarly, Υ has a solution with $x_1 = x_2 = a, x'_2 = b$ and all the variables $y_1, \dots, y_t, y'_1, \dots, y'_t, z'_1, \dots, z'_t$ are from $D^{(1)}$. Again, this contradicts the fact that $E_\sigma \cap E_{\sigma'} = \emptyset$. \square

LEMMA 8.9. *Suppose $D^{(1)}$ is a minimal nonlinear reduction for Θ , the solution set of Θ is subdirect, $\Theta^{(1)}$ is not empty, $\Theta(x_1, x_2)$ defines a relation containing $(a, b) \in D_{x_1}^{(1)} \times D_{x_2}^{(1)}$. Then a and b are linked in the relation defined by $\Theta^{(1)}(x_1, x_2)$.*

PROOF. By Lemma 8.5, the solution set of $\Theta^{(1)}$ is subdirect. Let $\text{Var}(\Theta) = \{x_1, x_2, y_1, \dots, y_t\}$, R be the relation defined by $\Theta(x_1, x_2, y_1, \dots, y_t)$. Let Ω be the following instance

$$R(x_1, x_2, y_1, \dots, y_t) \wedge R(x'_1, x_2, y'_1, \dots, y'_t).$$

Since the solution sets of Θ and $\Theta^{(1)}$ are subdirect, the solution sets of Ω and $\Omega^{(1)}$ are also subdirect. Also, there should be a solution of $\Theta^{(1)}$ with $x_2 = b$. Let $x_1 = b'$ in this solution. Then $\Omega(x_1, x'_1)$ contains (a, b') . Since both $\Omega(x_1, x'_1)$ and $\Omega^{(1)}(x_1, x'_1)$ define symmetric reflexive relations, Lemmas 8.7 and 8.8 imply that a and b' are linked in $\Omega^{(1)}(x_1, x'_1)$. Since (b', b) is in $\Theta^{(1)}(x_1, x_2)$, we derive that a and b are linked in $\Theta^{(1)}(x_1, x_2)$, which completes the proof. \square

8.2 Properties of $\text{Con}(\rho, x)$

LEMMA 8.10. *Suppose ρ is a critical rectangular relation of arity $n \geq 2$, ρ' is the cover of ρ . Then $\text{Con}(\rho', 1) \supsetneq \text{Con}(\rho, 1)$, for $n > 2$ we also have $\text{Con}(\text{pr}_{1,2}(\rho), 1) \supsetneq \text{Con}(\rho, 1)$.*

PROOF. For every $i \in \{1, 2, \dots, n\}$ we define $\rho_i(x_1, \dots, x_n) = \exists x'_i \rho(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$. Since ρ is critical, it has no dummy variables, and therefore $\rho \subsetneq \rho_i$ for every i . Also $\rho \subsetneq \bigcap_i \rho_i$. Choose a tuple $(a_1, \dots, a_n) \in \rho' \setminus \rho$. Since ρ' is the cover of ρ , we have $\rho' \subseteq \bigcap_i \rho_i$. Since this tuple is in ρ_i , for every i there is b_i such that $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$. Then $(a_1, b_1) \in \text{Con}(\rho', 1)$, which means by the rectangularity of ρ that $\text{Con}(\rho', 1) \supsetneq \text{Con}(\rho, 1)$. For $n > 2$, we have $(b_1, a_2, \dots, a_n), (a_1, \dots, a_{n-1}, b_n) \in \rho$, hence $(a_1, b_1) \in \text{Con}(\text{pr}_{1,2}(\rho), 1)$ and therefore $\text{Con}(\text{pr}_{1,2}(\rho), 1) \supsetneq \text{Con}(\rho, 1)$. \square

LEMMA 8.11. *Suppose ρ is a critical subdirect relation and the i th variable of ρ is rectangular. Then $\text{Con}(\rho, i)$ is an irreducible congruence.*

PROOF. To simplify notations assume that $i = 1$. Put $\sigma = \text{Con}(\rho, 1)$. As we mentioned in Section 6.4, σ should be a congruence. Assume that it is not an irreducible congruence. Consider binary relations $\delta_1, \dots, \delta_s \supsetneq \sigma$ stable under σ such that $\delta_1 \cap \dots \cap \delta_s = \sigma$. Put

$$\rho_j(x_1, \dots, x_n) = \exists x'_1 \rho(x'_1, x_2, \dots, x_n) \wedge \delta_j(x_1, x'_1).$$

Consider a tuple (x_1, \dots, x_n) in the intersection of ρ_1, \dots, ρ_s . Since δ_j is stable under $\sigma = \text{Con}(\rho, 1)$, we may assume that x'_1 takes the same value in the definition of every ρ_j . Then (x_1, x'_1) should be in δ_j for every j , which implies that $(x_1, x'_1) \in \sigma$ and $(x_1, \dots, x_n) \in \rho$. Hence, the intersection of ρ_1, \dots, ρ_s gives ρ . Since $\rho \subsetneq \rho_j$ for every j , this contradicts the fact that ρ is critical. \square

For a relation ρ of arity n by $\text{UnPol}(\rho)$, we denote the set of all unary vector-functions preserving the relation ρ .

LEMMA 8.12. Suppose a pp-formula $\Omega(x_1, \dots, x_n)$ defines a relation ρ , $\alpha \in D_{x_1} \times \dots \times D_{x_n}$, and $\rho' = \{f(\alpha) \mid f \in \text{UnPol}(\rho)\}$. Then there exists $\Omega' \in \text{Coverings}(\Omega)$ such that $\Omega'(x_1, \dots, x_n)$ defines ρ' .

PROOF. Suppose $\alpha = (a_1, \dots, a_n)$. We introduce new variables x_i^a for every $i \in \{1, 2, \dots, n\}$ and $a \in D_{x_i}$. By Υ , we denote the following formula $\bigwedge_{(b_1, \dots, b_n) \in \rho} \rho(x_1^{b_1}, \dots, x_n^{b_n})$. This formula can be understood in the following way. If we encode a unary vector function by variables so that $f(b_1, \dots, b_n) = (x_1^{b_1}, \dots, x_n^{b_n})$ for every b_1, \dots, b_n , then the formula says that the vector function preserves ρ . Then ρ' can be defined by a pp-formula $\Upsilon(x_1^{a_1}, \dots, x_n^{a_n})$. To obtain the formula Ω' it is sufficient to replace each $\rho(x_1^{b_1}, \dots, x_n^{b_n})$ by a copy of Ω (replacing x_1, \dots, x_n with $x_1^{b_1}, \dots, x_n^{b_n}$) and then replace $x_1^{a_1}, \dots, x_n^{a_n}$ with x_1, \dots, x_n . \square

COROLLARY 8.12.1. Suppose a pp-formula $\Omega(x_1, \dots, x_n)$ defines a relation without a tuple $\alpha \in D_{x_1} \times \dots \times D_{x_n}$, Σ is the set of all relations defined by $\Upsilon(x_1, \dots, x_n)$, where $\Upsilon \in \text{Coverings}(\Omega)$, and ρ is an inclusion-maximal relation in Σ without the tuple α . Then α is a key tuple for ρ .

PROOF. For every tuple $\beta \notin \rho$, we consider $\rho_\beta := \{f(\beta) \mid f \in \text{UnPol}(\rho)\}$. Since f can be a constant mapping to a tuple from ρ and an identity, we have $\rho_\beta \supsetneq \rho$ for every β . By Lemma 8.12, ρ_β should be in Σ . Since ρ is inclusion-maximal, $\alpha \in \rho_\beta$. Therefore, any β can be mapped to α by a unary vector-function preserving ρ , which means that α is a key tuple for ρ . \square

The next lemma shows that we can apply the operation Con and a nonlinear reduction $D^{(1)}$ to a pp-formula $\Upsilon(x_1, \dots, x_n)$ in any order, the result will be the same. For the linear reduction a slight modification of the statement is required (see Lemma 8.14).

LEMMA 8.13. Suppose $D^{(1)}$ is a minimal nonlinear reduction for an instance Υ , the solution set of Υ is subdirect, and $\Upsilon^{(1)}(x_1, \dots, x_n)$ defines a subdirect rectangular relation. Then for every i

$$(\text{Con}(\Upsilon(x_1, \dots, x_n), i))^{(1)} = \text{Con}(\Upsilon^{(1)}(x_1, \dots, x_n), i).$$

PROOF. Put $\sigma_0 = \text{Con}(\Upsilon(x_1, \dots, x_n), i)$, $\sigma_1 = \text{Con}(\Upsilon^{(1)}(x_1, \dots, x_n), i)$. Let $\{x_1, \dots, x_n, y_1, \dots, y_s\}$ be the set of all variables of Υ . Let $\Xi = \Upsilon \wedge \Upsilon_{x_i, y_1, \dots, y_s}^{x_i', y_1', \dots, y_s'}$. We can check that σ_0 is defined by $\Xi(x_i, x_i')$, and σ_1 is defined by $\Xi^{(1)}(x_i, x_i')$. Since $\Upsilon^{(1)}(x_1, \dots, x_n)$ defines a rectangular relation, σ_1 is a congruence. It follows from the definition that $\sigma_0^{(1)} \supseteq \sigma_1$. Let us prove the backward inclusion. Choose a pair $(a, b) \in \sigma_0^{(1)}$. Since σ_0 is defined by $\Xi(x_i, x_i')$, by Lemma 8.9, a and b should be linked in $\Xi^{(1)}(x_i, x_i')$. Since σ_1 is a congruence, a and b can be linked only if $(a, b) \in \sigma_1$, which means that $\sigma_0^{(1)} = \sigma_1$. \square

LEMMA 8.14. Suppose $D^{(1)}$ is a minimal linear reduction for Υ , $\Upsilon^{(1)}(x_1, \dots, x_n)$ defines a subdirect rectangular relation, $\text{Var}(\Upsilon) = \{x_1, \dots, x_n, v_1, \dots, v_r\}$, and $\Omega = \Upsilon \wedge \bigwedge_{i=1}^r \sigma_i(v_i, u_i)$, where $\sigma_i = \text{ConLin}(D_{v_i})$. Then $(\text{Con}(\Omega(x_1, \dots, x_n, u_1, \dots, u_r), j))^{(1)} = \text{Con}(\Upsilon^{(1)}(x_1, \dots, x_n), j)$ for every j .

PROOF. Without loss of generality assume that $j = 1$. Since the reduction $D^{(1)}$ is minimal, we have the following inclusion

$$(\text{Con}(\Omega(x_1, \dots, x_n, u_1, \dots, u_r), 1))^{(1)} \supseteq \text{Con}(\Upsilon^{(1)}(x_1, \dots, x_n), 1).$$

Let us prove the backward inclusion. Suppose $\Omega(x_1, \dots, x_n, u_1, \dots, u_r)$ and $\Upsilon^{(1)}(x_1, \dots, x_n)$ define the relations ρ' and ρ , respectively. Choose $a, b \in D_{x_1}^{(1)}$ such that $(a, b) \in \text{Con}(\rho', 1)$. For some β we have $a\beta, b\beta \in \rho'$. Since ρ is subdirect, there exist α_a and α_b in $D^{(1)}$ such that $a\alpha_a, b\alpha_b \in \rho$. Since

w preserves ρ' ,

$$\begin{aligned} w(a, a, \dots, a)w(\alpha_a, \beta, \dots, \beta) &\in \rho', \\ w(a, b, \dots, b)w(\alpha_a, \beta, \dots, \beta) &\in \rho', \\ w(b, \dots, b, a)w(\alpha_b, \beta, \dots, \beta) &\in \rho', \\ w(b, b, \dots, b)w(\alpha_b, \beta, \dots, \beta) &\in \rho'. \end{aligned}$$

By Lemma 7.23, $w(\alpha_a, \beta, \dots, \beta)$ and $w(\alpha_b, \beta, \dots, \beta)$ belong to $D^{(1)}$. Then, for $c = w(a, b, \dots, b) = w(b, \dots, b, a)$, we have $(a, c), (c, b) \in \text{Con}(\rho, 1)$. Since ρ is rectangular, we have $(a, b) \in \text{Con}(\rho, 1)$. \square

8.3 Adding Linear Variable

Below we formulate few statements from Reference [62] that will help us to prove the main property of a bridge. This property will be the main ingredient of the proof of the fact that A' from the informal description of the algorithm should be of codimension 1.

A relation $\rho \subseteq A^n$ is called *strongly rich* if for every tuple (a_1, \dots, a_n) and every $j \in \{1, \dots, n\}$ there exists a unique $b \in A$ such that $(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) \in \rho$. We will need two statements from Reference [62].

Recall that for any bridge ρ by $\tilde{\rho}$, we denote the binary relation defined by $\tilde{\rho}(x, y) = \rho(x, x, y, y)$.

THEOREM 8.15 [62]. *Suppose $\rho \subseteq A^n$ is a strongly rich relation preserved by an idempotent WNU. Then there exists an Abelian group $(A; +)$ and bijective mappings $\phi_1, \phi_2, \dots, \phi_n : A \rightarrow A$ such that*

$$\rho = \{(x_1, \dots, x_n) \mid \phi_1(x_1) + \phi_2(x_2) + \dots + \phi_n(x_n) = 0\}.$$

LEMMA 8.16 [62]. *Suppose $(G; +)$ is a finite Abelian group, the relation $\sigma \subseteq G^4$ is defined by $\sigma = \{(a_1, a_2, a_3, a_4) \mid a_1 + a_2 = a_3 + a_4\}$, and σ is preserved by an idempotent WNU f . Then $f(x_1, \dots, x_n) = t \cdot x_1 + t \cdot x_2 + \dots + t \cdot x_n$ for some $t \in \{1, 2, 3, \dots\}$.*

THEOREM 8.17. *Suppose $\sigma \subseteq A^2$ is a congruence, ρ is a bridge from σ to σ such that $\tilde{\rho}$ is a full relation, $\text{pr}_{1,2}(\rho) = \omega$, ω is a minimal relation stable under σ such that $\omega \supsetneq \sigma$. Then there exists a prime number p and a relation $\zeta \subseteq A \times A \times \mathbb{Z}_p$ such that $\text{pr}_{1,2}\zeta = \omega$ and $(a_1, a_2, b) \in \zeta$ implies that $(a_1, a_2) \in \sigma \Leftrightarrow (b = 0)$.*

PROOF. Since the relations ρ and ω are stable under σ , we consider A/σ instead of A and assume that σ is the equality relation.

Without loss of generality we assume that $\rho(x_1, x_2, y_1, y_2) = \rho(y_1, y_2, x_1, x_2)$ and $(a, b, a, b) \in \rho$ for any $(a, b) \in \omega$. Otherwise, we consider the relation ρ' instead of ρ , where

$$\rho'(x_1, x_2, y_1, y_2) = \exists z_1 \exists z_2 \rho(x_1, x_2, z_1, z_2) \wedge \rho(y_1, y_2, z_1, z_2).$$

We prove by induction on the size of A . Assume that for some subuniverse $A' \subsetneq A$ we have $(A' \times A') \cap (\omega \setminus \sigma) \neq \emptyset$. By σ' , we denote the equality relation on A' . By ω' , we denote a minimal relation such that $\sigma' \subsetneq \omega' \subseteq (A' \times A') \cap \omega$. Since $\text{pr}_{1,2}(\rho \cap (\omega' \times \omega')) = \omega' \supsetneq \sigma'$, the relation $\rho \cap (\omega' \times \omega')$ is a bridge from σ' to σ' . The inductive assumption for $\rho \cap (\omega' \times \omega')$ implies that there exists a relation $\zeta' \subseteq A' \times A' \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta' \Leftrightarrow (x_1, x_2) \in \sigma'$ and $\text{pr}_{1,2}(\zeta') = \omega'$. Put

$$\zeta(x_1, x_2, z) = \exists y_1 \exists y_2 \rho(x_1, x_2, y_1, y_2) \wedge \zeta'(y_1, y_2, z).$$

By the minimality of ω , we have $\text{pr}_{1,2}(\zeta) = \omega$. The remaining property of ζ follows from the fact that ρ is a bridge and the properties of ζ' .

Thus, we may assume that for any subuniverse $A' \subsetneq A$ we have $(A' \times A') \cap (\omega \setminus \sigma) = \emptyset$.

Consider a pair $(a_1, a_2) \in \omega \setminus \sigma$. Let $A' = \{a \mid (a_1, a) \in \omega\}$. Since $\omega \supsetneq \sigma$, we have $a_1 \in A'$, and therefore $(a_1, a_2) \in (A' \times A') \cap (\omega \setminus \sigma) \neq \emptyset$ and $A' = A$. Thus, $\{a \mid (a_1, a) \in \omega\} = \{a \mid (a, a_2) \in \omega\} = A$. Hence, any element connected in ω to some other element is connected to all elements. Therefore, $(a_1, a), (a, a_2) \in \omega$ for every $a \in A \setminus \{a_1, a_2\}$, which for $|A| > 2$ implies that $\omega = A \times A$.

If $|A| = 2$ and $\omega \neq A \times A$, then $\omega = \{(a, a), (a, b), (b, b)\}$ and ρ is uniquely defined. We know [52] that any clone on a 2-element domain containing an idempotent WNU operation contains majority operation, conjunction, disjunction, or minority operation. None of them preserve ρ , which contradicts our assumptions.

Thus, we proved that $\omega = A \times A$ and A has no proper subuniverses of size at least 2.

Note the remaining part of the proof could also be derived from known facts of commutator theory. In fact, it follows from the properties of ρ that σ (the equality) is an equivalence block of a congruence on A^2 , which means that A is Abelian. Using Abelianess for Taylor varieties (since we have a WNU), we could also define the required ternary relation ζ (see Reference [8] for more details). Nevertheless, we do not want to introduce new algebraic notions and give a proof based on two claims from Reference [62].

Let us show that for any $a_1, a_2, a_3 \in A$ there exists a unique a_4 such that $(a_1, a_2, a_3, a_4) \in \rho$. For every $a \in A$ put $\lambda_a(x_1, x_2) = \exists y_2 \rho(x_1, x_2, a, y_2)$. It is easy to see that $\sigma \subsetneq \lambda_a \subseteq \omega$. Therefore, $\lambda_a = \omega = A \times A$ for every a . We consider the unary relation defined by $\delta(x) = \rho(a_1, a_2, a_3, x)$. By the above fact, δ is not empty. Since ρ is a bridge, δ is not full. If δ contains more than one element, then we get a contradiction with the fact that there are no proper subuniverses of size at least 2.

Then ρ is a strongly rich relation. By Theorem 8.15, there exist an Abelian group $(A; +)$ and bijective mappings $\phi_1, \phi_2, \phi_3, \phi_4 : A \rightarrow A$ such that

$$\rho = \{(a_1, a_2, b_1, b_2) \mid \phi_1(a_1) + \phi_2(a_2) + \phi_3(b_1) + \phi_4(b_2) = 0\}.$$

Without loss of generality, we can assume that $\phi_1(x) = x$. We know that $(a, a, b, b) \in \rho$ for any $a, b \in A$, then $\phi_1(x) + \phi_2(x) + \phi_3(0) + \phi_4(0) = 0$, which means that $\phi_2(x) = -x - \phi_3(0) - \phi_4(0)$. Since $(a, b, a, b) \in \rho$ for any $a, b \in A$, we have $\phi_1(x) + \phi_2(0) + \phi_3(x) + \phi_4(0) = 0$, which means that $\phi_3(x) = -\phi_1(x) - \phi_2(0) - \phi_4(0) = -x + \phi_3(0)$. Similarly, since $\phi_1(0) + \phi_2(0) + \phi_3(x) + \phi_4(x) = 0$, we have $\phi_4(x) = x - \phi_3(0) - \phi_2(0) - \phi_1(0) = x + \phi_4(0)$. Substituting this into the definition of ρ we obtain

$$\rho = \{(a_1, a_2, b_1, b_2) \mid a_1 - a_2 - a_3 + a_4 = 0\}.$$

It follows from Lemma 8.16 that w on A is defined by $t(x_1 + \dots + x_m)$. Since w is special, $t \cdot (t - 1)$ should be divided by the order of any element of A . By the idempotency, t and the order of any element are coprime. Hence, $t - 1$ should be divided by the order of any element, and we may put $t = 1$. Therefore, the relation $\zeta \subseteq A \times A \times A$ defined by $\zeta = \{(b_1, b_2, b_3) \mid b_1 - b_2 + b_3 = 0\}$ is preserved by w . If $(A; +)$ is not simple, then any equivalence class of a congruence is a proper subuniverse of size at least 2, which contradicts our assumption. Therefore, $(A; +)$ is a simple Abelian group. \square

COROLLARY 8.17.1. *Suppose $\sigma \subseteq A^2$ is an irreducible congruence and ρ is a bridge from σ to σ such that $\tilde{\rho}$ is a full relation. Then there exists a prime number p and a relation $\zeta \subseteq A \times A \times \mathbb{Z}_p$ such that $\text{pr}_{1,2} \zeta = \sigma^*$ and $(a_1, a_2, b) \in \zeta$ implies that $(a_1, a_2) \in \sigma \Leftrightarrow (b = 0)$.*

LEMMA 8.18. *Suppose $\rho \subseteq A^4$ is an optimal bridge from σ_1 to σ_2 , and σ_1 and σ_2 are different irreducible congruences. Then $\tilde{\rho} \supsetneq \sigma_2$.*

PROOF. Since the first two variables are stable under σ_1 and the last two variables are stable under σ_2 , we have $\sigma_1 \subseteq \tilde{\rho}$ and $\sigma_2 \subseteq \tilde{\rho}$. Assume that the lemma does not hold, then $\tilde{\rho} = \sigma_2$.

Since σ_1 and σ_2 are different, we obtain $\sigma_1 \subsetneq \sigma_2$. First, we want (a, d) to be from σ_2 for every $(a, b, c, d) \in \rho$. Put $\rho_1(x_1, x_2, y_1, y_2) = \rho(x_1, x_2, y_1, y_2) \wedge \sigma_2(x_1, y_2)$. If ρ_1 is a bridge, then we replace

ρ by ρ_1 . Assume that ρ_1 is not a bridge, then for every $(a, b, c, d) \in \rho$ with $(a, d) \in \sigma_2$ we have $(a, b) \in \sigma_1$. Put $\rho_2(x_1, x_2, y_1, y_2) = \exists z \rho(x_1, x_2, z, y_1) \wedge \sigma_2(x_1, y_2)$. Let us show that ρ_2 is a bridge. Suppose $(x_1, x_2) \in \sigma_1$ and $(x_1, x_2, y_1, y_2) \in \rho_2$. Then $(x_1, x_2, z, y_1) \in \rho$. Since ρ is a bridge from σ_1 to σ_2 , this implies that $(z, y_1) \in \sigma_2$ and $(x_1, y_1) \in \tilde{\rho}$. Since $\tilde{\rho} = \sigma_2$, we have $(x_1, y_1) \in \sigma_2$ and therefore $(y_1, y_2) \in \sigma_2$. If $(y_1, y_2) \in \sigma_2$ and $(x_1, x_2, y_1, y_2) \in \rho_2$, then $(x_1, y_1) \in \sigma_2$ and by the above assumption we have $(x_1, x_2) \in \sigma_1$. It remains to show that $\text{pr}_{1,2}(\rho_2) \supsetneq \sigma_1$. Consider any tuple $(a_1, a_2, a_3, a_4) \in \rho$ such that $(a_1, a_2) \notin \sigma_1$, then $(a_1, a_2, a_4, a_1) \in \rho_2$. Thus, ρ_2 is a bridge with the required property, so we replace ρ by ρ_2 .

Second, we want $\text{pr}_{1,2}(\rho)$ to be equal to σ_1^* , and $\text{pr}_{3,4}(\rho)$ to be equal to σ_2^* . To achieve this we replace ρ by the relation defined by $\rho(x_1, x_2, y_1, y_2) \wedge \sigma_1^*(x_1, x_2) \wedge \sigma_2^*(y_1, y_2)$, which has the same properties.

Recall that a *polynomial* is an operation that can be defined by a term over the basic operations of an algebra and constant operations. In our case, a polynomial is an operation defined by a term over the WNU w and constants. Let D be a minimal subset (not necessarily a subuniverse) of A such that

- (1) there exists a unary polynomial h such that $h(h(x)) = h(x)$ and $h(A) = D$, and
- (2) $(\sigma_2^* \setminus \sigma_2) \cap D^2 \neq \emptyset$.

Since constants preserve a reflexive bridge ρ and congruences σ_1 and σ_2 , the unary polynomial h also preserves ρ , σ_1 and σ_2 . It is not hard to see that $h(w(x_1, \dots, x_m))$ is an idempotent WNU on D , and then by w^D we denote a special WNU on D that can be derived from the idempotent WNU on D . For any relation δ , by δ^D we denote its restriction to D (that is $h(\delta)$). It is not hard to see that ρ^D is a bridge from σ_1^D to σ_2^D .

The idea of the proof is to define a bridge ϵ^D from σ_2^D to σ_2^D such that $\epsilon^D \not\subseteq \sigma_2$. Then we define a bridge ϵ from σ_2 to σ_2 having the same property and use this bridge to make $\tilde{\rho}$ bigger, which gives us a contradiction, because ρ is optimal.

Consider $(b_1, b_2) \in (\sigma_2^*)^D \setminus \sigma_2^D$ and the unary operation $g_{b_1}(x) = w^D(b_1, \dots, b_1, x)$. Since w^D is a special WNU, $g_{b_1}(g_{b_1}(x)) = g_{b_1}(x)$. Let us show that $(\sigma_2^* \setminus \sigma_2) \cap (D')^2 \neq \emptyset$, where $D' = g_{b_1}(D)$.

Since $\text{pr}_{3,4}(\rho) = \sigma_2^*$, there are a_1, a_2 such that $(a_1, a_2, b_1, b_2) \in \rho$. Since $(a_1, b_2) \in \sigma_2$, and $(b_1, b_2) \notin \sigma_2$, we have $(a_1, b_1) \notin \sigma_2$. Consider the relation $\delta(x, y)$ defined by $\exists x_1 \exists x_2 \exists y_2 \sigma_2(x, x_1) \wedge \rho(x_1, x_2, y, y_2)$. It follows from the definition that δ is stable under σ_2 . Also $(a_1, b_1) \in \delta$, therefore $\delta \supsetneq \sigma_2$. From irreducibility of σ_2 , we obtain that $(b_1, b_2) \in \delta$. Then by the definition of δ there exist $(c_1, c_2, b_2, c_3) \in \rho$ such that $(b_1, c_1) \in \sigma_2$. Put $d_i = h(c_i)$ for $i = 1, 2, 3$. Then $(d_1, d_2, b_2, d_3) \in \rho$ (we have $h(b_2) = b_2$). Since h preserves σ_2 and $h(b_1) = b_1$, we have $(d_1, b_1) \in \sigma_2$. Therefore, $(d_1, b_2) \notin \sigma_2$. Since $\tilde{\rho} = \sigma_2$, we have $(d_1, d_2) \notin \sigma_1$ and $(b_2, d_3) \notin \sigma_2$. Let E be the equivalence class of σ_2^D containing d_1 and d_2 (they are in one class because $\text{pr}_{1,2}(\rho) = \sigma_1^* \subseteq \sigma_2$). By w' we denote w^D restricted to E , put $\rho' = \rho \cap (E^2 \times D^2)$ and $\sigma'_1 = \sigma_1 \cap E^2$.

Since $(d_1, d_2) \in (\sigma_1^* \setminus \sigma_1) \cap E^2$, we can find a minimal relation $\omega \subseteq \sigma_1^* \cap E^2$ stable under σ'_1 such that $\omega \supsetneq \sigma'_1$. It is not hard to check that the formula

$$\rho''(x_1, x_2, x'_1, x'_2) = \exists y_1 \exists y_2 \rho'(x_1, x_2, y_1, y_2) \wedge \rho'(x'_1, x'_2, y_1, y_2) \wedge \omega(x_1, x_2) \wedge \omega(x'_1, x'_2)$$

defines a reflexive bridge ρ'' from σ'_1 to σ'_1 . Since $\tilde{\rho} = \sigma_2$ and E is an equivalence class of σ_2^D , we have $\tilde{\rho}'' = E^2$. Then by Theorem 8.17, there exists a prime number p and a relation $\zeta \subseteq E \times E \times \mathbb{Z}_p$ such that $\text{pr}_{1,2}(\zeta) = \omega$ and $(a_1, a_2, b) \in \zeta$ implies $(a_1, a_2) \in \sigma'_1 \Leftrightarrow (b = 0)$. We want to show that for each $(e_1, e_2) \in \omega \setminus \sigma'_1$ we have $(w^D(d_1, \dots, d_1, e_1), w^D(d_1, \dots, d_1, e_2)) \notin \sigma_1$. In fact, choose $b \in \mathbb{Z}_p$ such that $(e_1, e_2, b) \in \zeta$. Note that $b \neq 0$ and $(d_1, d_1, 0) \in \zeta$. Applying w' to n tuples $(d_1, d_1, 0)$ and one tuple (e_1, e_2, b) , we get, by Lemma 7.23, a tuple $(w^D(d_1, \dots, d_1, e_1), w^D(d_1, \dots, d_1, e_2), b) \in \zeta$.

Since $b \neq 0$, we have $(w^D(d_1, \dots, d_1, e_1), w^D(d_1, \dots, d_1, e_2)) \notin \sigma_1$. We can find $(e_3, e_4) \in \sigma_2^* \setminus \sigma_2$ such that $(e_1, e_2, e_3, e_4) \in \rho$. Since h preserves ρ , w^D preserves ρ^D , and ρ^D is a bridge, we can derive that $(w^D(d_1, \dots, d_1, h(e_3)), w^D(d_1, \dots, d_1, h(e_4))) \notin \sigma_2$. Since $(d_1, b_1) \in \sigma_2$ we also have $(w^D(b_1, \dots, b_1, h(e_3)), w^D(b_1, \dots, b_1, h(e_4))) \notin \sigma_2$. Thus, we proved that $(\sigma_2^* \setminus \sigma_2) \cap (D')^2 \neq \emptyset$, where $D' = g_{b_1}(D)$ and $g_{b_1}(x) = w^D(b_1, \dots, b_1, x)$. If $D' \neq D$, then we found a smaller set D' and the corresponding polynomial $g_{b_1}(h(x))$, which contradicts the minimality of D .

Thus, for any $(b_1, b_2) \in (\sigma_2^*)^D \setminus \sigma_2^D$, we have $w^D(b_1, \dots, b_1, x) = x$. Let us show that $\text{Con}(\rho^D, 1) = \sigma_1^D$ and $\text{Con}(\rho^D, 3) = \sigma_2^D$. Choose $(a_1, a_2, a_3, a_4) \in \rho^D$. We consider two cases.

Case 1: $(a_1, a_2) \in \sigma_1$ and $(a_3, a_4) \in \sigma_2$. Then, for any tuple $(a'_1, a_2, a_3, a_4) \in \rho^D$, we have $(a_1, a'_1) \in \sigma_1$. Similarly, for any tuple $(a_1, a_2, a'_3, a_4) \in \rho^D$, we have $(a_3, a'_3) \in \sigma_2$.

Case 2: $(a_1, a_2) \notin \sigma_1$ and $(a_3, a_4) \notin \sigma_2$. Since $(a_1, a_4) \in \sigma_2$ and $(a_1, a_2) \in \sigma_1^* \subseteq \sigma_2$, we have $(a_1, a_3), (a_2, a_3), (a_3, a_4) \in (\sigma_2^*)^D \setminus \sigma_2^D$. Also notice that σ_2^* is symmetric. Assume that $(a'_1, a_2, a_3, a_4) \in \rho^D$. Since w^D preserves ρ^D , we have

$$(w^D(a'_1, a_1, \dots, a_1), w^D(a_2, \dots, a_2, a_1), w^D(a_3, \dots, a_3, a_1), w^D(a_4, \dots, a_4, a_1)) \in \rho^D.$$

As we showed before, this tuple equals (a'_1, a_1, a_1, a_1) , which means that $(a_1, a'_1) \in \sigma_1$. Similarly, if $(a_1, a_2, a'_3, a_4) \in \rho^D$, then we consider a tuple

$$(w^D(a_1, \dots, a_1, a_3), w^D(a_2, \dots, a_2, a_3), w^D(a'_3, a_3, \dots, a_3, a_3), w^D(a_4, \dots, a_4, a_3)) \in \rho^D$$

that equals (a_3, a_3, a'_3, a_3) , which means that $(a_3, a'_3) \in \sigma_2$. Thus we proved that $\text{Con}(\rho^D, 1) = \sigma_1^D$ and $\text{Con}(\rho^D, 3) = \sigma_2^D$.

Consider $(a_1, a_2, b_1, b_2) \in \rho^D$ with $(b_1, b_2) \notin \sigma_2$ and the formula

$$\Theta = \rho(z, x_1, x_2, x_3) \wedge \rho(z', x_1, x'_2, x'_3) \wedge \rho(z, x_4, x_5, x_6) \wedge \rho(z', x_4, x'_5, x'_6).$$

Let ϵ be the relation defined by $\Theta(x_2, x'_2, x_5, x'_5)$. Since $h(h(x)) = h(x)$ and h preserves ρ , ϵ^D is defined by the same formula but with ρ^D instead of ρ everywhere. Let us prove that ϵ^D is a bridge from σ_2^D to σ_2^D . Assume that $(x_2, x'_2) \in \sigma_2^D$. Since $(x_1, z), (x_1, z') \in \sigma_1^* \subseteq \sigma_2$, we have $(z, z') \in \sigma_2^D$. Recall that $(a, d) \in \sigma_2$ whenever $(a, b, c, d) \in \rho$, hence $(x_3, x'_3), (x_6, x'_6) \in \sigma_2^D$. Since $\text{Con}(\rho^D, 1) = \sigma_1^D$, we have $(z, z') \in \sigma_1$. Since $\text{Con}(\rho^D, 3) = \sigma_2^D$, we have $(x_5, x'_5) \in \sigma_2$. In the same way we can show that if $(x_5, x'_5) \in \sigma_2^D$, then $(x_2, x'_2) \in \sigma_2^D$. Since all the variables of ϵ are stable under σ_2 and ϵ^D is reflexive, we have $\text{pr}_{1,2}(\epsilon^D) \supseteq \sigma_2^D$. By sending (z, x_1, x_2, x_3) to (a_1, a_2, b_1, b_2) , (z', x_1, x'_2, x'_3) to (a_2, a_2, a_2, a_2) , (z, x_4, x_5, x_6) to (a_1, a_2, b_1, b_2) , and (z', x_4, x'_5, x'_6) to (a_2, a_2, a_2, a_2) , we show that $(b_1, a_2, b_1, a_2) \in \epsilon$. Since $(a_1, a_2), (a_1, b_2) \in \sigma_2$ and $(b_1, b_2) \notin \sigma_2$, we have $(b_1, a_2) \notin \sigma_2$, and therefore $\text{pr}_{1,2}(\epsilon^D) \supsetneq \sigma_2^D$. In the same way, we can show that $\text{pr}_{3,4}(\epsilon^D) \supsetneq \sigma_2^D$. Thus, ϵ^D is a bridge.

Let us show that ϵ is a bridge from σ_2 to σ_2 . Assume the contrary. Then, without loss of generality, we assume that there exists $(d_0, d_0, d_1, d_2) \in \epsilon$ such that $(d_1, d_2) \notin \sigma_2$. Put $\delta_0(y, z) = \exists x \epsilon(x, x, y, z)$. The relation δ_0 is stable under σ_2 and strictly larger than σ_2 , hence $\delta_0 \supseteq \sigma_2^*$ and $(b_1, b_2) \in \delta_0$. Then there exists d such that $(d, d, b_1, b_2) \in \epsilon$, which means that $(h(d), h(d), b_1, b_2) \in \epsilon^D$. This contradicts the fact that ϵ^D is a bridge. Hence, ϵ is also a bridge. By sending $(z, x_1, x_2, x'_2, x_3, x'_3)$ to $(a_1, a_2, b_1, b_1, b_2, b_2)$ and $(z', x_4, x_5, x'_5, x_6, x'_6)$ to $(a_1, a_1, a_1, a_1, a_1, a_1)$, we can show that $(b_1, a_1) \in \tilde{\epsilon}$. If we compose bridges ρ and ϵ , then we get a bridge ϵ' from σ_1 to σ_2 containing (b_1, b_1, a_1, a_1) . Hence $\tilde{\epsilon}' \supsetneq \tilde{\rho}$, which contradicts the fact that ρ is optimal. \square

8.4 Existence of a Bridge

In this subsection, we show four ways to build a bridge: from congruences with an additional property; from a rectangular relation, by composing bridges appearing in the instance; and from a pp-formula.

LEMMA 8.19. *Suppose σ, σ_1 , and σ_2 are congruences on A , $\sigma \cap \sigma_1 = \sigma \cap \sigma_2$, and $\sigma \setminus \sigma_1 \neq \emptyset$. Then σ_1 and σ_2 are adjacent.*

PROOF. Let us define a relation ρ by

$$\rho(x_1, x_2, y_1, y_2) = \exists z_1 \exists z_2 \sigma_1(x_1, z_1) \wedge \sigma_2(z_1, y_1) \wedge \sigma_1(x_2, z_2) \wedge \sigma_2(z_2, y_2) \wedge \sigma(z_1, z_2).$$

It is clear that the first two variables of ρ are stable under σ_1 and the last two variables are stable under σ_2 .

Let us show that for any $(a_1, a_2, a_3, a_4) \in \rho$ that $(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2$. In fact, if $(x_1, x_2) \in \sigma_1$, then $(z_1, z_2) \in \sigma_1$. Since $\sigma \cap \sigma_1 = \sigma \cap \sigma_2$, we have $(z_1, z_2) \in \sigma_2$. Therefore, $(y_1, y_2) \in \sigma_2$.

Also $(a, a, a, a) \in \rho$ for any $a \in A$. Choose $(a, b) \in \sigma \setminus \sigma_1$. Then $(a, b, a, b) \in \rho$ (put $z_1 = a, z_2 = b$), which proves that ρ is a reflexive bridge. \square

LEMMA 8.20. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a subdirect relation, the first and the last variables of ρ are rectangular, and there exist $(b_1, a_2, \dots, a_n), (a_1, \dots, a_{n-1}, b_n) \in \rho$ such that $(a_1, a_2, \dots, a_n) \notin \rho$. Then there exists a bridge δ from $\text{Con}(\rho, 1)$ to $\text{Con}(\rho, n)$ such that $\tilde{\delta} = \text{pr}_{1,n}(\rho)$.*

PROOF. The required bridge can be defined by

$$\delta(x_1, x_2, y_1, y_2) = \exists z_2 \dots \exists z_{n-1} \rho(x_1, z_2, \dots, z_{n-1}, y_1) \wedge \rho(x_2, z_2, \dots, z_{n-1}, y_2).$$

In fact, since the first and the last variables of ρ are rectangular, we have $(x_1, x_2) \in \text{Con}(\rho, 1)$ if and only if $(y_1, y_2) \in \text{Con}(\rho, n)$. It remains to notice that $(b_1, a_1, a_n, b_n) \in \delta$ and $(b_1, a_1) \notin \text{Con}(\rho, 1)$. \square

Recall that by Lemma 8.11 $\text{Con}(\rho, i)$ is an irreducible congruence for every critical subdirect rectangular relation ρ and its coordinate i .

LEMMA 8.21. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a critical subdirect rectangular relation. Then*

- (1) *there exists a bridge δ from $\text{Con}(\rho, 1)$ to $\text{Con}(\rho, n)$ such that $\tilde{\delta} = \text{pr}_{1,n}(\rho)$. Moreover, if $n = 2$, then $\text{Con}(\tilde{\delta}, 1) = \text{Con}(\rho, 1)$ and $\text{Con}(\tilde{\delta}, 2) = \text{Con}(\rho, n)$; if $n > 2$, then $\text{Con}(\tilde{\delta}, 1) \supsetneq \text{Con}(\rho, 1)$ and $\text{Con}(\tilde{\delta}, 2) \supsetneq \text{Con}(\rho, n)$.*
- (2) *if $\text{Opt}(\text{Con}(\rho, n)) \neq \text{Con}(\rho, n)$, then there exists a bridge δ from $\text{Con}(\rho, 1)$ to $\text{Con}(\rho, n)$ such that δ contains the projection of the cover of ρ onto the first and the last coordinates.*

PROOF. Using the argument from Lemma 8.10, we find tuples $(b_1, a_2, \dots, a_n)(a_1, \dots, a_{n-1}, b_n)$ satisfying the conditions of Lemma 8.20. Then, to prove the first part it is sufficient to use the formula from Lemma 8.20 to define a bridge δ . If $n = 2$, then $\tilde{\delta} = \rho$ and we have the required property. If $n > 2$, then by Lemma 8.10 we have $\text{Con}(\tilde{\delta}, 1) = \text{Con}(\text{pr}_{1,n}(\rho), 1) \supsetneq \text{Con}(\rho, 1)$ and $\text{Con}(\tilde{\delta}, 2) = \text{Con}(\text{pr}_{1,n}(\rho), 2) \supsetneq \text{Con}(\rho, n)$.

Let us prove the second part of the claim. Let ξ be an optimal bridge from $\text{Con}(\rho, n)$ to $\text{Con}(\rho, n)$. Define a bridge $\delta(x_1, x_2, y_1, y_2)$ by

$$\exists z_2 \dots \exists z_{n-1} \exists u_1 \exists u_2 \rho(x_1, z_2, \dots, z_{n-1}, u_1) \wedge \rho(x_2, z_2, \dots, z_{n-1}, u_2) \wedge \xi(u_1, u_2, y_1, y_2).$$

Note that δ is just a composition of the bridge constructed in Lemma 8.20 and ξ . Then we have $\tilde{\delta}(x, y) = \exists z_2 \dots \exists z_{n-1} \exists u \rho(x, z_2, \dots, z_{n-1}, u) \wedge \tilde{\xi}(u, y)$.

Put $\rho'(x_1, \dots, x_n) = \exists x'_n \rho(x_1, \dots, x_{n-1}, x'_n) \wedge \tilde{\xi}(x'_n, x_n)$. Since $\tilde{\xi} \supsetneq \text{Con}(\rho, n)$, the relation ρ' contains the cover of ρ . Since $\text{pr}_{1,n}(\rho') = \tilde{\delta}$, $\tilde{\delta}$ contains the projection of the cover of ρ onto the first and the last coordinates, which completes the proof. \square

THEOREM 8.22. *Suppose Θ is a cycle-consistent connected instance. Then for every constraints C, C' with variables x, x' there exists a bridge δ from $\text{Con}(C, x)$ to $\text{Con}(C', x')$ such that δ contains all pairs*

of elements linked in Θ . Moreover, if $\text{Con}(C'', x'') \neq \text{LinkedCon}(\Theta, x'')$ for some constraint $C'' \in \Theta$ and a variable x'' , then δ can be chosen so that δ contains all pairs of elements linked in Θ' , where Θ' is obtained from Θ by replacing every constraint relation by its cover.

PROOF. Since C and C' are connected, there exists a path $z_0 C_1 z_1 C_2 z_2 \dots C_{t-1} z_{t-1} C_t z_t$, where $z_0 = x$, $z_t = x'$, $C_1 = C$, $C_t = C'$, $z_{i-1} \neq z_i$, and C_i and C_{i+1} are adjacent in z_i for every i .

By Lemma 8.11, every relation defined by $\text{Con}(C_0, x_0)$ for some C_0 and x_0 is an irreducible congruence. Suppose ζ_i is an optimal bridge from $\text{Con}(C_i, z_i)$ to $\text{Con}(C_{i+1}, z_{i+1})$, δ_i is a bridge from $\text{Con}(C_i, z_{i-1})$ to $\text{Con}(C_i, z_i)$ from the first item of Lemma 8.21 for every i . Then we compose all bridges together and define a new bridge $\delta(u_0, u'_0, v_t, v'_t)$ from $\text{Con}(C, x)$ to $\text{Con}(C', x')$ by

$$\exists u_1 \exists u'_1 \exists v_1 \exists v'_1 \dots \exists u_{t-1} \exists u'_{t-1} \exists v_{t-1} \exists v'_{t-1} \delta_1(u_0, u'_0, v_1, v'_1) \wedge \bigwedge_{i=1}^{t-1} (\zeta_i(v_i, v'_i, u_i, u'_i) \wedge \delta_{i+1}(u_i, u'_i, v_{i+1}, v'_{i+1})) . \quad (7)$$

Since $\tilde{\delta}$ can be defined as a composition of $\tilde{\zeta}$'s and $\tilde{\delta}$'s, and $\tilde{\zeta}$'s are reflexive, it follows that $\tilde{\delta}$ contains all pairs of elements linked by this path. Since Θ is cycle-consistent, if $x = x'$, then δ is a reflexive bridge from $\text{Con}(C, x)$ to $\text{Con}(C', x)$. Thus we proved that any two constraints with a common variable are adjacent.

Using Lemma 6.2, we can show that there exists a path in Θ starting at x and ending at x' that connects any pair of elements linked in Θ . Since any two constraints with a common variable are adjacent, we can assume that the above path $z_0 C_1 z_1 C_2 z_2 \dots C_{t-1} z_{t-1} C_t z_t$ connects any pair of elements linked in Θ . Again, it follows that $\tilde{\delta}$ contains all pairs of elements linked in Θ .

To prove the remaining part of the theorem, assume that $\text{Con}(C'', x'') \neq \text{LinkedCon}(\Theta, x'')$ for some constraint $C'' \in \Theta$ and a variable x'' . First, observe that any bridge ρ from σ_1 to σ_2 defined by the first item of Lemma 8.21 satisfies one of the following properties:

- (1) $\text{Con}(\tilde{\rho}, 1) = \sigma_1$ and $\text{Con}(\tilde{\rho}, 2) = \sigma_2$,
- (2) $\text{Con}(\tilde{\rho}, 1) \supsetneq \sigma_1$ and $\text{Con}(\tilde{\rho}, 2) \supsetneq \sigma_2$.

If $\sigma_1 \neq \sigma_2$, then by Lemma 8.18 an optimal bridge from σ_1 to σ_2 satisfies property (2). If $\sigma_1 = \sigma_2$, then an optimal bridge from σ_1 to σ_2 obviously satisfies one of the two properties. Thus, every bridge in (7) satisfies one of the above properties.

Let us show that if a bridge ρ from σ_1 to σ_2 satisfies property (1), then for all $(a_1, b_1), (a_2, b_2) \in \tilde{\rho}$ we have $(a_1, a_2) \in \sigma_1 \Leftrightarrow (b_1, b_2) \in \sigma_2$. In fact, if $(a_1, a_2) \in \sigma_1$, then since the first two variables of ρ are stable under σ_1 , we have $(a_1, b_2) \in \tilde{\rho}$, and hence $(b_1, b_2) \in \text{Con}(\tilde{\rho}, 2) = \sigma_2$. Now we want to show that if we compose bridges ρ_1, \dots, ρ_s together as in (7) and at least one of the bridges satisfies property (2), then the obtained bridge satisfies property (2). Let ρ_j be a bridge from σ_{j-1} to σ_j for every j , and then the composition ρ is a bridge from σ_0 to σ_s . Consider the first bridge ρ_i in the sequence having property (2). Then $(a_{i-1}, a_i), (b_{i-1}, b_i) \in \tilde{\rho}_i$ for some $(a_{i-1}, b_{i-1}) \notin \sigma_{i-1}$ and $a_i = b_i$. Choose a_j and b_j so that $(a_{j-1}, a_j), (b_{j-1}, b_j) \in \tilde{\rho}_j$ for every j and $a_j = b_j$ for every $j \geq i$. Then $(a_0, b_0) \in \text{Con}(\tilde{\rho}, 1) \setminus \sigma_0$ and $(a_0, a_s), (b_0, b_s) \in \tilde{\rho}$. Since $a_s = b_s$, we get $(a_0, b_0) \in \text{Con}(\tilde{\rho}, 1)$ and $\text{Con}(\tilde{\rho}, 1) \supsetneq \sigma_0$. To prove that $\text{Con}(\tilde{\rho}, 2) \supsetneq \sigma_s$, we consider the last bridge in the sequence having property (2) and do exactly the same.

By the first part of the theorem $\text{Opt}(\text{Con}(C'', x'')) \supsetneq \text{Con}(C'', x'')$, hence an optimal bridge from $\text{Con}(C'', x'')$ to $\text{Con}(C'', x'')$ satisfies property (2). We may assume that any path goes through the variable x'' and through the optimal bridge from $\text{Con}(C'', x'')$ to $\text{Con}(C'', x'')$, which guarantees that every bridge we obtain satisfies property (2). Consider a constraint $C_0 \in \Theta$ and a variable x_0 in it. Considering a path from x_0 to x_0 going through x'' , we can build a reflexive bridge having

property (2), which means that $\text{Opt}(\text{Con}(C_0, x_0)) \supsetneq \text{Con}(C_0, x_0)$ for any constraint $C_0 \in \Theta$ and any variable x_0 in it.

To complete the proof, we replace every δ_i in (7) by the corresponding bridge δ'_i obtained using the second item of Lemma 8.21 and replace the path by the corresponding path connecting any pair of linked elements in Θ' . Since $\widetilde{\delta}'_i$ contains the projection of the cover of C_i onto the variables z_{i-1} and z_i , $\widetilde{\delta}$ contains all pairs of elements linked in Θ' . \square

COROLLARY 8.22.1. *Suppose Θ is a cycle-consistent connected instance. Then, for every constraint C, C' with a common variable x there exists a bridge δ from $\text{Con}(C, x)$ to $\text{Con}(C', x)$ such that $\widetilde{\delta}$ contains the relation $\text{LinkedCon}(\Theta, x)$.*

LEMMA 8.23. *Suppose $D^{(1)}$ is a minimal one-of-four reduction for an instance Υ , the solution set of Υ is subdirect, $\Upsilon^{(1)}(x_1, \dots, x_n)$ defines a subdirect key rectangular relation ρ . For $i = 1, 2$ the variable x_i of every constraint of Υ containing x_i is stable under an irreducible congruence σ_i , and there exist tuples $(a_1, a_2, \dots, a_n), (b_1, b_2, b_3, \dots, b_n) \in \rho$, $(a'_1, a_2, \dots, a_n), (b_1, b'_2, b_3, \dots, b_n) \notin \rho$ such that $(a_1, a'_1) \in \sigma_1^* \setminus \sigma_1$, $(b_2, b'_2) \in \sigma_2^* \setminus \sigma_2$. Then there exists a bridge δ from σ_1 to σ_2 such that $\widetilde{\delta}$ contains $\Upsilon(x_1, x_2)$.*

PROOF. If $D^{(1)}$ is a nonlinear reduction, then by ω we denote the relation defined by $\Upsilon(x_1, \dots, x_n)$. If $D^{(1)}$ is a linear reduction, then by ω we denote the relation defined by $\Omega(x_1, \dots, x_n, u_1, \dots, u_r)$, where $\text{Var}(\Upsilon) = \{x_1, \dots, x_n, v_1, \dots, v_r\}$, $\Omega = \Upsilon \wedge \bigwedge_{i=1}^r \delta_i(v_i, u_i)$, and $\delta_i = \text{ConLin}(D_{v_i})$ (see Lemma 8.14).

We know from Lemmas 8.13 and 8.14 that $\text{Con}(\omega, j)^{(1)} = \text{Con}(\rho, j)$ for every $j \in \{1, 2\}$. Since ρ is rectangular, we have $(a_1, a'_1) \notin \text{Con}(\rho, 1)$, and therefore $(a_1, a'_1) \notin \text{Con}(\omega, 1)^{(1)}$. Since x_1 in every constraint containing x_1 is stable under σ_1 , the relation $\text{Con}(\omega, 1)$ is stable under σ_1 . Therefore, $\text{Con}(\omega, 1)$ should be equal σ_1 , since otherwise $\text{Con}(\omega, 1) \supsetneq \sigma_1^*$, which contradicts $(a_1, a'_1) \notin \text{Con}(\omega, 1)^{(1)}$. In the same way, we can show that $\text{Con}(\omega, 2) = \sigma_2$.

Since ρ is a key relation, there should be a key tuple $\beta \in (D_{x_1}^{(1)} \times \dots \times D_{x_n}^{(1)}) \setminus \rho$ such that for every $\alpha \in (D_{x_1}^{(1)} \times \dots \times D_{x_n}^{(1)}) \setminus \rho$ there exists a vector-function Ψ , which preserves ρ and gives $\Psi(\alpha) = \beta$. First, we put $\alpha_a = (a'_1, a_2, \dots, a_n)$ and apply the corresponding unary vector-function Ψ_a to (a_1, a_2, \dots, a_n) to get a tuple β_a . Second, we put $\alpha_b = (b_1, b'_2, b_3, \dots, b_n)$ and apply the corresponding unary vector-function Ψ_b to $(b_1, b_2, b_3, \dots, b_n)$ to get a tuple β_b . As a result, we get two tuples β_a and β_b from ρ that differ from the key tuple β just in the first and second coordinates, respectively. Since $\text{Con}(\omega, j)^{(1)} = \text{Con}(\rho, j)$ for every $j \in \{1, 2\}$, we have $\beta \notin \omega$ if $D^{(1)}$ is nonlinear, and $\beta \notin \text{pr}_{1, \dots, n}(\omega^{(1)})$ if $D^{(1)}$ is linear.

Then by applying Lemma 8.20 to ω and β_a, β_b, β (if $D^{(1)}$ is a linear reduction we extend these tuples), we get a bridge δ from σ_1 to σ_2 such that $\widetilde{\delta}$ contains $\text{pr}_{1,2}(\omega)$, which is equal to the relation defined by $\Upsilon(x_1, x_2)$. \square

8.5 Expanded Coverings of Crucial Instances

In this subsection, we prove two properties of expanded coverings of crucial instances.

LEMMA 8.24. *Suppose Θ is a crucial instance in $D^{(1)}$, $\Theta' \in \text{ExpCov}(\Theta)$ via the map $S : \text{Var}(\Theta') \rightarrow \text{Var}(\Theta)$, and Θ' has no solution in $D^{(1)}$. Then for every constraint $C = \rho(x_1, \dots, x_n)$ in Θ there exists a constraint C' in Θ' whose image in Θ is C (i.e., $C' = \rho(y_1, \dots, y_n)$ and $S(y_i) = x_i$ for $i = 1, 2, \dots, n$).*

PROOF. Let Θ'' be obtained from Θ by replacing every variable y by $S(y)$. Obviously, Θ'' still does not have a solution in $D^{(1)}$. By the definition of expanded coverings every relation in the obtained instance is either unary (and full) or weaker or equivalent to a constraint from Θ' . Since

Θ is crucial in $D^{(1)}$ and Θ'' has no solutions in $D^{(1)}$, there should be a constraint C' in Θ' such that its image C'' in Θ'' is weaker or equivalent to C but not weaker than C . Since Θ is crucial, all variables of C are not dummy. Since C'' cannot have more variables than C we obtain that $C'' = C$, which means that $C' = \rho(y_1, \dots, y_n)$ and $S(y_i) = x_i$ for every $i \in \{1, 2, \dots, n\}$. \square

LEMMA 8.25. *Suppose Θ is a crucial instance in $D^{(1)}$, $\Theta' \in \text{ExpCov}(\Theta)$ has no solutions in $D^{(1)}$, every constraint relation of Θ is a critical rectangular relation, and Θ' is connected. Then Θ is connected.*

PROOF. Let Θ'' be obtained from Θ' by replacing every variable y by $S(y)$ from the definition of the expanded covering.

Let us show that any two constraints C_1 and C_2 with a common variable x of Θ are adjacent. By Lemma 8.24, there exist constraints C'_1 and C'_2 of Θ' whose images in Θ are C_1 and C_2 . Since Θ' is connected, the instance Θ'' is also connected. By Corollary 8.22.1 constraints C_1 and C_2 of Θ'' are adjacent in x . Therefore, C_1 and C_2 are adjacent in Θ . Thus, we proved that any two constraints of Θ with a common variable are adjacent. Since Θ is crucial in $D^{(1)}$, it is not fragmented, which implies that Θ is connected. \square

8.6 Strategies

THEOREM 8.26. *Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for Ω , the solution set of $\Omega^{(i)}$ is subdirect for every $i \in \{0, 1, \dots, s\}$, $j < s$, $D^{(s+1)}$ is a one-of-four reduction, at least one of the two reductions $D^{(j+1)}$, $D^{(s+1)}$ is nonlinear, and $(\Omega^{(j)}(x_1, \dots, x_n))^{(s+1)}$ defines a nonempty relation. Then $(\Omega^{(j+1)}(x_1, \dots, x_n))^{(s+1)}$ defines a nonempty relation.*

PROOF. Let $\text{Var}(\Omega) = \{x_1, \dots, x_n, y_1, \dots, y_t\}$, $\Omega^{(j)}(x_1, \dots, x_n, y_1, \dots, y_t)$ define a relation R . Let the reduction $D^{(j+1)}$ be of type \mathcal{T}_1 , the reduction $D^{(s+1)}$ be of type \mathcal{T}_2 .

Assume that $D^{(s+1)}$ is an absorbing reduction. Since $\Omega^{(s)}(x_1, \dots, x_n, y_1, \dots, y_t)$ defines a subdirect relation, Lemma 7.5 implies that $\Omega^{(s+1)}(x_1, \dots, x_n, y_1, \dots, y_t)$ defines a nonempty relation. From now on we assume that \mathcal{T}_2 is not the absorbing type.

For $i \in \{j, \dots, s\}$ and $k \in \{1, \dots, n\}$ put

$$\begin{aligned} B_i &= R \cap (D_{x_1}^{(i)} \times \dots \times D_{x_n}^{(i)} \times D_{y_1}^{(j)} \times \dots \times D_{y_t}^{(j)}), \\ B'_i &= R \cap (D_{x_1}^{(i)} \times \dots \times D_{x_n}^{(i)} \times D_{y_1}^{(j+1)} \times \dots \times D_{y_t}^{(j+1)}), \\ B^k &= R \cap (D_{x_1}^{(s)} \times \dots \times D_{x_{k-1}}^{(s)} \times D_{x_k}^{(s+1)} \times D_{x_{k+1}}^{(s)} \times \dots \times D_{x_n}^{(s)} \times D_{y_1}^{(j)} \times \dots \times D_{y_t}^{(j)}). \end{aligned}$$

By Lemma 7.25 B_{j+1} and B'_j are one-of-four subuniverses of $R = B_j$ of type \mathcal{T}_1 . Similarly, B_{i+1} is a one-of-four subuniverse of B_i for every i and B^k is a one-of-four subuniverse of B_s of type \mathcal{T}_2 for every k (here we may need to reduce the domain of the last t variables to achieve the subdirectness of B_i).

Let us show by induction on i that B'_i is a one-of-four subuniverse of B_i of type \mathcal{T}_1 . For $i = j$, we already know this. Assume that B'_i is a one-of-four subuniverse of B_i . By Lemma 7.30, $B_{i+1} \cap B'_i = B'_{i+1}$ is a one-of-four subuniverse of B_{i+1} of type \mathcal{T}_1 . Therefore, B'_s is a one-of-four subuniverse of B_s of type \mathcal{T}_1 .

We need to prove that $B^1 \cap \dots \cap B^n \cap B'_s \neq \emptyset$. Since $(\Omega^{(j)}(x_1, \dots, x_n))^{(s+1)}$ defines a nonempty relation, $B^1 \cap \dots \cap B^n \neq \emptyset$. Since the solution set of $\Omega^{(s)}$ is subdirect, $B^k \cap B'_s \neq \emptyset$ for every $k \in \{1, \dots, n\}$. Note that B^k is of type \mathcal{T}_2 and B'_s is of type \mathcal{T}_1 , and they cannot be both linear. Since B^k is not a binary absorbing subuniverse, Lemma 7.33 implies that $B^1 \cap \dots \cap B^n \cap B'_s \neq \emptyset$. \square

COROLLARY 8.26.1. *Suppose Θ is a cycle-consistent CSP instance, $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for Θ , $\Upsilon \in \text{ExpCov}(\Theta)$ is a tree-formula, x is a parent of x_1 and x_2 , and either (i) B is a center of $D_x^{(s)}$,*

or (ii) B is a PC subuniverse of $D_x^{(s)}$ and $D_y^{(s)}$ has no nontrivial binary absorbing subuniverse or center for every y . Then the pp-formula $\Upsilon^{(s)}(x_1, x_2)$ defines a binary relation with a nonempty intersection with $B \times B$.

PROOF. Since every reduction in a strategy is 1-consistent and Υ is a tree-formula, the solution set of $\Upsilon^{(i)}$ is subdirect for every i . If $B = D_x^{(s)}$, then the claim follows from the definition of a strategy (every reduction is 1-consistent). Otherwise, let us define a reduction $D^{(s+1)}$ by $D_x^{(s+1)} = D_{x_1}^{(s+1)} = D_{x_2}^{(s+1)} = B$, $D_y^{(s+1)} = D_y^{(s)}$ for the remaining variables. Thus, we have a nonlinear reduction $D^{(s+1)}$. Since the instance Θ is cycle-consistent, and x is a parent of x_1 and x_2 , $\Upsilon(x_1, x_2)$ defines a reflexive relation. Hence, $(\Upsilon(x_1, x_2))^{(s+1)}$ defines a nonempty relation. By Theorem 8.26, we obtain that $(\Upsilon^{(1)}(x_1, x_2))^{(s+1)}$ defines a nonempty relation. Repeatedly applying Theorem 8.26, we show that $(\Upsilon^{(2)}(x_1, x_2))^{(s+1)}, (\Upsilon^{(3)}(x_1, x_2))^{(s+1)}, \dots, (\Upsilon^{(s)}(x_1, x_2))^{(s+1)}$ define nonempty relations, which means that $\Upsilon^{(s)}(x_1, x_2)$ has a nonempty intersection with $B \times B$. \square

LEMMA 8.27. Suppose $R \subseteq A_0 \times B_0$ is a subdirect relation, and

- (1) $A_0 \supseteq A_1 \supseteq \dots \supseteq A_{s+1}$ and A_{i+1} is a one-of-four subuniverse of A_i for $i \in \{0, 1, 2, \dots, s\}$;
- (2) $B_0 \supseteq B_1 \supseteq \dots \supseteq B_{t+1}$ and B_{i+1} is a one-of-four subuniverse of B_i for $i \in \{0, 1, 2, \dots, t\}$;
- (3) A_{s+1} and B_{t+1} are linear subuniverses of A_s and B_t , respectively;
- (4) there exist $a \in A_{s+1}$, $b \in B_{s+1}$, $a' \in A_0$, $b' \in B_0$ such that $(a, b'), (a', b), (a', b') \in R$;
- (5) $R \cap (A_s \times B_t) \neq \emptyset$.

Then $R \cap (A_{s+1} \times B_{t+1}) \neq \emptyset$.

PROOF. Denote $a'' = w(a, a', \dots, a')$, $b'' = w(b, b', \dots, b') = w(b', \dots, b', b)$.

We prove by induction on $s + t$. Assume that $s + t = 0$, which implies $s = t = 0$. By Lemma 7.23, we have $a'' \in A_1$ and $b'' \in B_1$. Since w preserves R , $(a'', b'') \in R$, which completes this case.

Let us prove the induction step. Assume that $s + t > 0$. Without loss of generality we assume that $s > 0$. Put

$$R'(x_1, x_2, y_1, y_2) = R(x_1, y_2) \wedge R(y_1, x_2) \wedge R(y_1, y_2).$$

Put $P_i = R' \cap (A_i \times B_0 \times A_0 \times B_0)$, $Q_i = R' \cap (A_0 \times B_i \times A_0 \times B_0)$, $T = R' \cap (A_0 \times B_0 \times A_1 \times B_0)$. Since R' is subdirect, by Lemma 7.25 P_{i+1} is a one-of-four subuniverse of P_i , Q_{i+1} is a one-of-four subuniverse of Q_i for every i , and T is a one-of-four subuniverse of $R' = P_0 = Q_0$. We want to prove that $P_{s+1} \cap Q_{t+1} \cap T \neq \emptyset$. Since $(a, b, a', b') \in P_{s+1} \cap Q_{t+1}$, Lemma 7.31 implies that $P_{s+1} \cap Q_t$ and $P_s \cap Q_{t+1}$ are one-of-four subuniverses of $P_s \cap Q_t$. Since $R \cap (A_s \times B_t) \neq \emptyset$, we have $T \cap P_s \cap Q_t \neq \emptyset$. Lemma 7.31 implies that $P_0 \supseteq P_1 \supseteq \dots \supseteq P_s \supseteq P_s \cap Q_1 \supseteq \dots \supseteq P_s \cap Q_t$ (here all inclusions mean one-of-four subuniverses), which by the same lemma implies that $T \cap P_s \cap Q_t$ is a one-of-four subuniverse of $P_s \cap Q_t$.

If A_1 is not a linear subuniverse of A_0 , then by Theorem 7.33 the intersection of one-of-four subuniverses $P_{s+1} \cap Q_t$, $P_s \cap Q_{t+1}$, and $T \cap P_s \cap Q_t$ of different types cannot be empty, that is, $P_{s+1} \cap Q_{t+1} \cap T \neq \emptyset$, which completes this case.

Assume that A_1 is linear. Since $(a, b, a', b'), (a', b', a', b'), (a', b', a, b') \in R'$ and w preserves R' , we obtain $(a, b, a', b'), (a'', b'', a', b'), (a'', b'', a'', b') \in R'$. Note that by Lemma 7.23 $a'' \in A_1$ and $b'' \in B_1$. We look at R' as a binary relation $R' \subseteq (A_0 \times B_0) \times (A_0 \times B_0)$ to apply the inductive assumption. Put $\mathcal{A}_i = \text{pr}_{1,2}(R') \cap (A_{i+1} \times B_1)$ for $0 \leq i \leq s$ and $\mathcal{A}_i = \text{pr}_{1,2}(R') \cap (A_{s+1} \times B_{i-s+1})$ for $s+1 \leq i \leq s+t$. Combining Lemma 7.25 and Lemma 7.31 we derive that \mathcal{A}_{i+1} is a one-of-four subuniverse of \mathcal{A}_i for $i = 0, 1, \dots, s+t-1$. Put $\mathcal{B}_i = \text{pr}_{3,4}(R' \cap (A_1 \times B_1 \times A_i \times B_0))$ for $i = 0, 1$. Combining Lemmas 7.25, 7.31, and 7.32, we derive that \mathcal{B}_1 is a linear subuniverse of \mathcal{B}_0 . Then we apply the inductive assumption to $R' \cap (A_0 \times B_0)$, $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \dots \supseteq \mathcal{A}_{s+t}$ and $\mathcal{B}_0 \supseteq \mathcal{B}_1$ and show that $R' \cap (A_{s+1} \times B_{t+1} \times A_1 \times B_0) = P_{s+1} \cap Q_{t+1} \cap T \neq \emptyset$.

Put $B'_i = \text{pr}_2(R \cap (A_1 \times B_i))$ for $i = 0, 1, \dots, s+1$. By Lemma 7.32, B'_{i+1} is a one-of-four subuniverse of B'_i for every i . Applying the inductive assumption to $R \cap (A_1 \times B_0)$, $A_1 \supseteq A_2 \supseteq \dots \supseteq A_{s+1}$, and $B'_0 \supseteq B'_1 \supseteq \dots \supseteq B'_{t+1}$, we obtain that $A_{s+1} \cap B_{t+1} \neq \emptyset$. \square

LEMMA 8.28. Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for a subdirect constraint $\rho(x_1, \dots, x_n)$, $D^{(s+1)}$ is a linear reduction, and

$$\begin{aligned} (b_1, \dots, b_t, a_{t+1}, \dots, a_n) &\in \rho, \\ (a_1, \dots, a_t, b_{t+1}, \dots, b_n) &\in \rho, \\ (b_1, \dots, b_t, b_{t+1}, \dots, b_n) &\in \rho, \\ (a_1, \dots, a_t, a_{t+1}, \dots, a_n) &\in D^{(s+1)}. \end{aligned}$$

Then there exists $(d_1, d_2, \dots, d_n) \in \rho^{(s+1)}$.

PROOF. For $i = 0, 1, 2, \dots, s+1$ put

$$\begin{aligned} A_i &= \text{pr}_{1,2,\dots,t} \left(\rho \cap \left(D_{x_1}^{(i)} \times \dots \times D_{x_t}^{(i)} \times D_{x_{t+1}} \times \dots \times D_{x_n} \right) \right), \\ B_i &= \text{pr}_{1,2,\dots,t} \left(\rho \cap \left(D_{x_1} \times \dots \times D_{x_t} \times D_{x_{t+1}}^{(i)} \times \dots \times D_{x_n}^{(i)} \right) \right). \end{aligned}$$

Since $\rho^{(i)}$ is subdirect for every $i \in \{0, 1, \dots, s\}$, Lemma 7.25 implies that A_{i+1} is a one-of-four subuniverse of A_i and B_{i+1} is a one-of-four subuniverse of B_i for every $i \in \{0, 1, \dots, s\}$. Since $(a_1, \dots, a_t) \in A_{s+1}$ and $(a_{t+1}, \dots, a_n) \in B_{s+1}$, Lemma 8.27 implies that $A_{s+1} \cap B_{s+1} \neq \emptyset$, which completes the proof. \square

8.7 Growing Population Divides into Colonies

In this section, we prove a theorem that clarifies the inductive strategy used in the proof of Theorem 9.6. To simplify explanation we decided to avoid our usual terminology. Instead, we argue in terms of organisms, reproduction, and friendship.

We consider a set X whose elements we call *organisms*. At the moment 1, we had a set of organisms X_1 . At every moment some organisms give a birth to new organisms, and as a result we get a sequence of organisms $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$, where $\bigcup_i X_i = X$, $X_i \subseteq X$, and $|X_i| < \infty$ for every i . We assume that each organism from $X \setminus X_1$ has exactly one parent.

Every organism has a characteristic that we call *strength*. Thus, we have a mapping $\xi : X \rightarrow \{1, 2, \dots, S\}$ that assigns a characteristic to every organism. Also we have a binary reflexive symmetric relation F on the set X , which we call *friendship*. For an organism x by $\text{BirthDate}(x)$, we denote the minimal i such that $x \in X_i$. A sequence of organisms x_1, \dots, x_n such that x_i is a friend of x_{i+1} for every i is called a *path*.

THEOREM 8.29. Suppose $X_1, X_2, X_3, \dots, \xi$, and F satisfy the following conditions:

- (1) **A child is always weaker than its parent.** If y is the parent of x , then $\xi(y) > \xi(x)$.
- (2) **Older friends are parents' friends.** If $\text{BirthDate}(y) < \text{BirthDate}(x)$ and x is a friend of y , then the parent of x is a friend of y (or the parent of x is y).
- (3) **Only friends' kids can be friends.** If $\text{BirthDate}(x) = \text{BirthDate}(y)$ and x is a friend of y , then the parents of x and y are friends.
- (4) **No one can have infinitely many friends.** $|\{y \in X \mid (x, y) \in F\}| < \infty$ for every $x \in X$.
- (5) **Reproduction never stops.** $|\bigcup_i X_i| = \infty$.

Then there exists N such that X_N can be divided into two nonempty disjoint sets X'_N and X''_N such that there is no friendship between X'_N and X''_N .

PROOF. Assume the contrary. Then there exists a path between any two organisms.

For every moment t and every organism x by x^t , we denote the predecessor of x from X_t with the maximal BirthDate, that is, a closest predecessor who already lived at the moment t . For example, $x^t = x$ for $t \geq \text{BirthDate}(x)$, and $x^{\text{BirthDate}(x)-1}$ is the parent of x .

Suppose we have a path of organisms x_1, \dots, x_n . We claim that x_1^t, \dots, x_n^t is also a path for any t . We will prove by induction starting with sufficiently large t such that $x_1, \dots, x_n \in X_t$ and therefore $(x_1^t, \dots, x_n^t) = (x_1, \dots, x_n)$. As the inductive step, we assume that this is a path for $t = t_0$ and show that this is a path for $t = t_0 - 1$. The induction step follows from hypotheses (2) and (3). The path x_1^t, \dots, x_n^t will be called a *path at the moment t* . Note that organisms of the path at the moment t are not weaker than the corresponding organisms of the original path.

Choose a maximal strength s such that we have infinitely many organisms of strength s . Then infinitely many of them have the same parent, and hence, there exists a parent reproducing infinitely many times.

For every x and a strength s by $\text{KIDs}(x, s)$, we denote the set of all children y of x such that there exists a path from x to y with all the organisms in the path stronger than s . We consider the maximal s_0 such that $\text{KIDs}(x, s_0)$ is infinite for some organism x . Since we can always put $s_0 = 0$ for a parent reproducing infinitely many times, s_0 exists. Note that this implies that x is stronger than $s_0 + 1$.

By Y , we denote the set of all organisms y such that there exists a path from x to y with all the organisms in the path stronger than $s_0 + 1$. Note that Y includes x . Let us show that Y is finite. Assume the opposite. Let s be the maximal strength such that we have infinitely many organisms of this strength in Y . Consider an organism v from Y with strength s such that $\text{BirthDate}(v) > \text{BirthDate}(x)$ (we still have infinitely many of them). Considering the path from x to v at the moment $\text{BirthDate}(v) - 1$, we get a path from x to the parent of v , which means that the parent of v is also in Y . Since parents are stronger than children and we may have only finitely many organisms stronger than s in Y , we have only finitely many such parents in Y . Therefore, there exists a parent $z \in Y$ with infinitely many children from Y . Since we can glue a path from the parent to x and a path from x to its kid, this implies that $\text{KIDs}(z, s_0 + 1)$ is infinite. This contradicts the maximality of s_0 and proves that Y is finite.

Let t be the first moment such that X_t contains all friends of friends of organisms from Y . Consider an organism y from $\text{KIDs}(x, s_0)$ with $\text{BirthDate}(y) > t$. Choose a path from x to y with all organisms stronger than s_0 . We consider the last organism u in the path such that $\text{BirthDate}(u) < \text{BirthDate}(y)$. Taking the fragment of this path from y to u at the moment $\text{BirthDate}(y) - 1$, we obtain a path from x to u with all organisms but u stronger than $s_0 + 1$. This means that all the organisms but u in this path are from Y . Thus u has a friend from Y , which means that all friends of u were born before the moment t . This contradicts the fact that an organism next to u in the original path from x to y was born after the moment $\text{BirthDate}(y) - 1$. \square

9 PROOF OF THE MAIN THEOREMS

9.1 Existence of a Next Reduction

The next lemma has its roots in Theorem 20 from Reference [29], where the authors proved that bounded width 1 is equivalent to tree duality.

LEMMA 9.1. Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for a 1-consistent CSP instance Θ , and $D^{(\top)}$ is a reduction of $\Theta^{(s)}$.

- (1) If there exists a 1-consistent reduction contained in $D^{(\top)}$ and $D^{(s+1)}$ is maximal among such reductions, then for every variable y of Θ there exists a tree-formula $\Upsilon_y \in \text{Coverings}(\Theta)$ such that $\Upsilon_y^{(\top)}(y)$ defines $D_y^{(s+1)}$.
- (2) Otherwise, there exists a tree-formula $\Upsilon \in \text{Coverings}(\Theta)$ such that $\Upsilon^{(\top)}$ has no solutions.

PROOF. The proof is based on the constraint propagation procedure. We consider the instance $\Theta^{(s)}$. We start with an empty set Υ_y (empty tree-formula) for every y , these tree-formulas define the reduction $D^{(\top)}$.

Then we introduce a recursive algorithm that gives a correct tree-formula Υ_y for every variable y . If at some step the reduction defined by these tree-formulas is 1-consistent, then we are done. Otherwise, we consider a constraint C that breaks 1-consistency. Then the current restrictions of the variables z_1, \dots, z_l in the constraint $C = \rho(z_1 \dots, z_l)$ imply a stronger restriction of some variable z_i and the corresponding domain $D_{z_i}^{(s)}$. We change the tree-formula Υ_{z_i} describing the reduction of the variable z_i in the following way $\Upsilon_{z_i} := C \wedge \Upsilon_{z_1} \wedge \dots \wedge \Upsilon_{z_l}$.

Note that we have to be careful with all the variables appearing in different Υ_y to avoid collisions. Every time we join Υ_u and Υ_v , we rename the variables so that they do not have common variables.

Obviously, this procedure will eventually stop. If $\Upsilon_y^{(\top)}(y)$ defines an empty set for some y , then Υ_y can be taken as Υ to witness condition (2). Otherwise, these tree-formulas define a 1-consistent reduction, which is a maximal 1-consistent reduction, since it is defined by tree-formulas. \square

THEOREM 9.2. Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for a cycle-consistent CSP instance Θ .

- If $D_x^{(s)}$ has a nontrivial binary absorbing subuniverse B , then there exists a 1-consistent absorbing reduction $D^{(s+1)}$ of $\Theta^{(s)}$ with $D_x^{(s+1)} \subseteq B$.
- If $D_x^{(s)}$ has a nontrivial center B , then there exists a 1-consistent central reduction $D^{(s+1)}$ of $\Theta^{(s)}$ with $D_x^{(s+1)} \subseteq B$.
- If $D_y^{(s)}$ has no nontrivial binary absorbing subuniverse or center for every y but there exists a nontrivial PC subuniverse B in $D_x^{(s)}$ for some x , then there exists a 1-consistent PC reduction $D^{(s+1)}$ of $\Theta^{(s)}$ with $D_x^{(s+1)} \subseteq B$.

PROOF. Without loss of generality we assume that B is a minimal one-of-four subuniverse of this type. Let us define a reduction $D^{(\top)}$ by $D_x^{(\top)} = B$ and $D_y^{(\top)} = D_x^{(s)}$ for $y \neq x$ and apply Lemma 9.1. We consider two cases corresponding to two cases of Lemma 9.1.

Case 1. There exists a 1-consistent reduction $D^{(s+1)}$ of $\Theta^{(s)}$ such that $D_y^{(s+1)}$ is defined by $\Upsilon_y(y)$ for a tree-formula Υ_y for every variable y . Let R be the solution set of $\Upsilon_y^{(s)}$. Since Υ_y is a tree-formula and $\Theta^{(s)}$ is 1-consistent, the solution set R is subdirect. Applying Corollaries 7.1.2, 7.6.2, and 7.13.2 to R , we derive that $D_y^{(s+1)}$ is a one-of-four subuniverse of the corresponding type.

Case 2. There exists a tree-formula $\Upsilon \in \text{Coverings}(\Theta)$ such that $\Upsilon^{(\top)}$ has no solutions. We consider the minimal set of variables $\{x_1, \dots, x_k\}$ from Υ whose parent is x such that $\Upsilon^{(s)}(x_1, \dots, x_k)$ does not have any tuple in B^k . Since $\Theta^{(s)}$ is 1-consistent and Υ is a tree-formula, $k \geq 2$. If B is a binary absorbing subuniverse, then we get a contradiction with Lemma 7.5. For other cases with $k = 2$, we get a contradiction from Corollary 8.26.1. If $k \geq 3$ and B is a center, then we get a contradiction with Corollary 7.10.3. If $k \geq 3$ and B is a PC subuniverse, then we get a contradiction with Corollary 7.13.3. \square

As a corollary, we can derive that cycle-consistency is a sufficient condition to guarantee the existence of a solution of an instance whose domains avoid linear algebras (so-called bounded width case). Note that this corollary follows from the result by Marcin Kozik in Reference [43].

COROLLARY 9.2.1. *Suppose Θ is a cycle-consistent CSP instance, for every domain D_x there is no $B \subseteq D_x$ and a congruence σ on B such that B/σ is a nontrivial linear algebra. Then Θ has a solution.*

PROOF. We recursively build a strategy $D^{(0)}, D^{(1)}, \dots, D^{(s)}$. We start with $s = 0$. If every domain $D_x^{(s)}$ is of size 1, then we already have a solution, because $\Theta^{(s)}$ is 1-consistent. Otherwise, by Theorem 5.1 on every domain $D_x^{(s)}$ of size greater than 1 there exists a nontrivial one-of-four subuniverse. Note that this subuniverse cannot be linear, because this contradicts the assumption that there is no $B \subseteq D_x$ and a congruence σ on B such that B/σ is linear. If we found a binary absorbing subuniverse or a center, then by Theorem 9.2 we can always find a next 1-consistent absorbing or central reduction $D^{(s+1)}$. Otherwise, by the same theorem we can find a 1-consistent PC reduction. Since the strategy cannot be infinite, we eventually stop with the instance whose variable domains are of size 1. \square

THEOREM 9.3. *Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for a cycle-consistent CSP instance Θ , and $D^{(\tau)}$ is a nonlinear 1-consistent reduction of $\Theta^{(s)}$. Then there exists a 1-consistent minimal reduction $D^{(s+1)}$ of $\Theta^{(s)}$ of the same type such that $D_x^{(s+1)} \subseteq D_x^{(\tau)}$ for every variable x .*

PROOF. Let the reduction $D^{(\tau)}$ be of type \mathcal{T} . Let us consider a minimal by inclusion 1-consistent reduction $D^{(s+1)}$ of $\Theta^{(s)}$ of type \mathcal{T} such that $D_x^{(s+1)} \subseteq D_x^{(\tau)}$ for every variable x .

Assume that for some z the domain $D_z^{(s+1)}$ is not a minimal one-of-four subuniverse of type \mathcal{T} . Then choose a minimal one-of-four subuniverse B of $D_z^{(s)}$ of this type contained in $D_z^{(s+1)}$. We define a reduction $D^{(\perp)}$ of $\Theta^{(s)}$ by $D_z^{(\perp)} = B$, $D_y^{(\perp)} = D_y^{(s+1)}$ if $y \neq z$, and apply Lemma 9.1. Since $D_y^{(s+1)}$ is a minimal by inclusion reduction, there exists a tree-formula $\Upsilon \in \text{Coverings}(\Theta)$ such that $\Upsilon^{(\perp)}$ has no solutions. Again, we consider a minimal set of variables $\{z_1, \dots, z_k\}$ from Υ whose parent is z such that $\Upsilon^{(s+1)}(z_1, \dots, z_k)$ does not have any tuple in B^k . Since the reduction $D^{(s+1)}$ is 1-consistent, $B \subsetneq D_z^{(s+1)}$, and Υ is a tree-formula, we have $k \geq 2$. If $D^{(\tau)}$ is an absorbing or central reduction of $\Theta^{(s)}$, then it is also an absorbing or central reduction of $\Theta^{(s+1)}$. Then we can get a contradiction just as we did in the proof of Theorem 9.2 using Lemma 7.5, Corollary 8.26.1, or Corollary 7.10.3.

It remains to consider the case when B is a PC subuniverse. Choose a minimal set of variables y_1, \dots, y_t of Υ different from z_1, \dots, z_k such that $(\Upsilon^{(s)}(z_1, \dots, z_k, y_1, \dots, y_t))^{(s+1)}$ does not have tuples with the first k elements from B . If $t = 0$ and $k = 2$, then $\Upsilon^{(s)}(z_1, z_2)$ has an empty intersection with $B \times B$, which contradicts Corollary 8.26.1. If $t + k \geq 3$, then the relation defined by $\Upsilon^{(s)}(z_1, \dots, z_k, y_1, \dots, y_t)$ is $(B, \dots, B, D_{y_1}^{(s+1)}, \dots, D_{y_t}^{(s+1)})$ -essential relation, which contradicts Corollary 7.13.3. \square

THEOREM 9.4. *Suppose $D^{(\tau)}$ is a 1-consistent PC reduction for a cycle-consistent irreducible CSP instance Θ , and Θ is not linked and not fragmented. Then there exist a reduction $D^{(1)}$ of Θ and a minimal strategy $D^{(1)}, \dots, D^{(s)}$ for $\Theta^{(1)}$ such that the solution set of $\Theta^{(1)}$ is subdirect, the reductions $D^{(2)}, \dots, D^{(s)}$ are nonlinear, $D_x^{(s)} \subseteq D_x^{(\tau)}$ for every variable x .*

PROOF. Since Θ is not linked, there exists a maximal congruence σ_x on D_x for a variable x of Θ such that $\text{LinkedCon}(\Theta, x) \subseteq \sigma_x$. Choose an equivalence class $D_x^{(1)}$ of σ_x with a nonempty intersection with $D_x^{(\tau)}$. For every variable y by $D_y^{(1)}$, we denote the set of all elements of D_y linked to an element of $D_x^{(1)}$. Note that for every y there is a congruence σ_y on D_y such that $D_x/\sigma_x \cong D_y/\sigma_y$. Then $D_y^{(1)}$ is an equivalence class of σ_y . By Corollaries 7.1.1 and 7.6.1, there is no nontrivial binary absorbing subuniverse or center on D_x/σ_x . Then by Theorem 5.1, σ_x is either PC congruence or linear congruence, which means that $D^{(1)}$ is a PC reduction or linear reduction.

Let us show that $D_y^{(1)} \cap D_y^{(\top)} \neq \emptyset$ for every y . Since Θ is not fragmented, we may consider a path starting at x and ending at y . Since the reduction $D^{(\top)}$ is 1-consistent, this path connects an element of $D_x^{(1)} \cap D_x^{(\top)}$ with some element of $D_y^{(\top)}$, which is also in $D_y^{(1)}$.

Since Θ is irreducible, the solution set of $\Theta^{(1)}$ is subdirect. We build the remaining part of the strategy in the following way. Suppose we already have $D^{(0)}, D^{(1)}, \dots, D^{(t)}$, where the reductions $D^{(2)}, \dots, D^{(t)}$ are absorbing or central. If there exists a nontrivial binary absorbing subuniverse or a nontrivial center on $D_y^{(t)}$ for some y , then by Theorems 9.2 and 9.3 we can find the next minimal 1-consistent absorbing or central reduction $D^{(t+1)}$.

Suppose there is no binary absorbing subuniverse or center on $D_y^{(t)}$ for every y . Put $D_y^{(\perp)} = D_y^{(\top)} \cap D_y^{(t)}$ for every variable y . By Lemma 7.31 $D_y^{(\perp)}$ is a PC subuniverse of $D_y^{(t)}$ for every variable y . Hence, $D^{(\perp)}$ is a PC reduction of $\Theta^{(t)}$.

Then we apply Lemma 9.1 to find a 1-consistent reduction of $\Theta^{(t)}$ smaller than $D^{(\perp)}$. If we cannot find it, then there exists a tree-formula Υ such that $\Upsilon^{(\perp)}$ has no solutions. Let R be the solution set of Υ . Note that $R^{(i)}$ is a subdirect relation for $i = 0, 1, \dots, t$, because Υ is a tree-formula and $D^{(i)}$ is a 1-consistent reduction. By Lemma 7.25, $R^{(\top)}$ is a PC subuniverse of R . Since $D^{(\top)}$ is 1-consistent, the intersection $R^{(1)} \cap R^{(\top)}$ is not empty.

Let us prove by induction on i that $R^{(i)} \cap R^{(\top)}$ is a nonempty PC subuniverse of $R^{(i)}$ for $i = 1, 2, \dots, t$. By the inductive assumption, we assume that $R^{(i-1)} \cap R^{(\top)}$ is a nonempty PC subuniverse of $R^{(i-1)}$ (for $i = 1$ it follows from the definition). By Lemma 7.25, $R^{(i)}$ is a one-of-four subuniverse of $R^{(i-1)}$. For $i \geq 2$, it is not a PC subuniverse, then by Theorem 7.33, the intersection of $R^{(i-1)} \cap R^{(\top)}$ and $R^{(i)}$, that is, $R^{(i)} \cap R^{(\top)}$, cannot be empty. For $i = 1$, we already know that $R^{(1)} \cap R^{(\top)} \neq \emptyset$. Applying Theorem 7.30 to $R^{(i-1)} \cap R^{(\top)} \subseteq R^{(i-1)}$ and $R^{(i)} \subseteq R^{(i-1)}$, we derive that $R^{(i)} \cap R^{(\top)}$ is a nonempty PC subuniverse of $R^{(i)}$. Thus, we proved that $R^{(t)} \cap R^{(\top)}$ is not empty, which contradicts the assumption about the tree-formula Υ .

Hence, there exists a 1-consistent reduction $D^{(\Delta)}$ of $\Theta^{(t)}$ smaller than $D^{(\perp)}$ such that for every variable y the new domain $D_y^{(\Delta)}$ can be defined by a tree-formula $\Upsilon_y^{(\perp)}$. Since the solution set of $\Upsilon_y^{(t)}$ is subdirect, by Corollary 7.13.2, the domain $D_y^{(\Delta)}$ is a PC subuniverse of $D_y^{(t)}$. Hence $D^{(\Delta)}$ is a PC reduction of $\Theta^{(t)}$. It remains to apply Theorem 9.3 to find a minimal PC reduction $D^{(t+1)}$ smaller than $D_y^{(\Delta)}$, put $s = t + 1$, and finish the strategy. \square

9.2 Existence of a Linked Connected Component

In this subsection, we prove that all constraints in a crucial instance have the parallelogram property, show that we can always find a linked connected component with required properties, and prove that we cannot pass from an instance having solutions to an instance having no solutions while applying a nonlinear reduction.

THEOREM 9.5. *Suppose $D^{(1)}$ is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance Θ , $\Omega(x_1, \dots, x_n)$ is a subconstraint of Θ , the solution set of $\Omega^{(1)}$ is subdirect, $\Theta \setminus \Omega$ has a solution in $D^{(1)}$, and Θ has no solutions in $D^{(1)}$. Then there exist instances $\Upsilon_1, \dots, \Upsilon_t \in \text{Coverings}(\Omega)$ such that $\Phi = (\Theta \setminus \Omega) \cup \Upsilon_1 \cup \dots \cup \Upsilon_t$ has no solutions in $D^{(1)}$, each $\Upsilon_i(x_1, \dots, x_n)$ is a subconstraint of Φ , and $\Upsilon_i^{(1)}(x_1, \dots, x_n)$ defines a subdirect key relation with the parallelogram property for every i .*

THEOREM 9.6. *Suppose $D^{(1)}$ is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance Θ , Θ is crucial in $D^{(1)}$ and not connected. Then there exists an instance $\Theta' \in \text{ExpCov}(\Theta)$ that is crucial in $D^{(1)}$ and contains a linked connected component whose solution set is not subdirect.*

THEOREM 9.7. *Suppose $D^{(1)}$ is a 1-consistent nonlinear reduction of a cycle-consistent irreducible CSP instance Θ . If Θ has a solution, then it has a solution in $D^{(1)}$.*

THEOREM 9.8. *Suppose $D^{(0)}, \dots, D^{(s)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ , and a constraint $\rho(x_1, \dots, x_n)$ of Θ is crucial in $D^{(s)}$. Then ρ is a critical relation with the parallelogram property.*

THEOREM 9.9. *Suppose $D^{(0)}, \dots, D^{(s)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ , $\Upsilon(x_1, \dots, x_n)$ is a subconstraint of Θ , the solution set of $\Upsilon^{(s)}$ is subdirect, $k \in \{1, 2, \dots, n-1\}$, $\text{Var}(\Upsilon) = \{x_1, \dots, x_n, u_1, \dots, u_t\}$,*

$$\Omega = \Upsilon_{x_1, \dots, x_k, u_1, \dots, u_t}^{y_1, \dots, y_k, v_1, \dots, v_t} \wedge \Upsilon_{x_{k+1}, \dots, x_n, u_1, \dots, u_t}^{y_{k+1}, \dots, y_n, v_{t+1}, \dots, v_{2t}} \wedge \Upsilon_{x_1, \dots, x_n, u_1, \dots, u_t}^{y_1, \dots, y_n, v_{2t+1}, \dots, v_{3t}},$$

and $\Theta^{(s)}$ has no solutions. Then $(\Theta \setminus \Upsilon) \cup \Omega$ has no solutions in $D^{(s)}$.

To prove these theorems, we need to introduce a partial order on domain sets. To every domain set $D^{(\tau)}$, we assign a tuple of integers $\text{Size}(D^{(\tau)}) = (|D_1|, |D_2|, \dots, |D_s|)$, where D_1, D_2, \dots, D_s is the set of all different domains of $D^{(\tau)}$ ordered by their size starting from the large one. Then the lexicographic order on tuples of integers induces a partial order on domain sets, that is, we say that $(a_1, \dots, a_k) < (b_1, \dots, b_l)$ if there exists $j \in \{1, 2, \dots, \min(k+1, l)\}$ such that $a_i = b_i$ for every $i < j$, and $a_j < b_j$ or $j = k+1$.

It follows from the definition that \leq is transitive and there does not exist an infinite descending chain of reductions. Note that duplicating domains does not affect this partial order, and that is why we do not make the size of a domain set larger if we consider an expanded covering. At the same time, for every minimal (proper) one-of-four reduction $D^{(1)}$ of the instance with a domain set $D^{(0)}$ we have $\text{Size}(D^{(1)}) < \text{Size}(D^{(0)})$. Let us show this for a central reduction. We replace every domain having a nontrivial center by a smaller domain, and we do not change other domains. Let $D_y^{(0)}$ be a domain of the maximal size having a nontrivial center. Then $|D_y^{(0)}|$ will be replaced by smaller numbers in the sequence $\text{Size}(D^{(0)})$ making the sequence smaller.

We prove theorems of this subsection simultaneously by the induction on the size of the domain sets. Let $D^{(\perp)}$ be a domain set. Assume that Theorems 9.5, 9.6, and 9.7 hold on instances Θ with a domain set $D^{(0)}$ if $\text{Size}(D^{(0)}) < \text{Size}(D^{(\perp)})$ and Theorems 9.8 and 9.9 hold if $\text{Size}(D^{(s)}) < \text{Size}(D^{(\perp)})$. Let us prove Theorems 9.5, 9.6, and 9.7 on instances Θ with a domain set $D^{(0)}$ if $\text{Size}(D^{(0)}) = \text{Size}(D^{(\perp)})$ and Theorems 9.8 and 9.9 for $\text{Size}(D^{(s)}) = \text{Size}(D^{(\perp)})$.

THEOREM 9.5 *Suppose $D^{(1)}$ is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance Θ , $\Omega(x_1, \dots, x_n)$ is a subconstraint of Θ , the solution set of $\Omega^{(1)}$ is subdirect, $\Theta \setminus \Omega$ has a solution in $D^{(1)}$, and Θ has no solutions in $D^{(1)}$. Then there exist instances $\Upsilon_1, \dots, \Upsilon_t \in \text{Coverings}(\Omega)$ such that $\Phi = (\Theta \setminus \Omega) \cup \Upsilon_1 \cup \dots \cup \Upsilon_t$ has no solutions in $D^{(1)}$, each $\Upsilon_i(x_1, \dots, x_n)$ is a subconstraint of Φ , and $\Upsilon_i^{(1)}(x_1, \dots, x_n)$ defines a subdirect key relation with the parallelogram property for every i .*

PROOF. Let Σ be the set of all relations defined by $\Upsilon^{(1)}(x_1, \dots, x_n)$, where $\Upsilon \in \text{Coverings}(\Omega)$. To every relation $\rho \in \Sigma$ we assign a constraint $((x_1, \dots, x_n); \rho)$, which we denote by $C(\rho)$. We can find $\Sigma_0 \subseteq \Sigma$ such that the instance $(\Theta^{(1)} \setminus \Omega^{(1)}) \cup C(\Sigma_0)$ has no solutions, but if we replace any relation of Σ_0 by all bigger relations from Σ (weaker in terms of constraints), then we get an instance with a solution.

Let $\Sigma_0 = \{\rho_1, \dots, \rho_t\}$. For each ρ_i and each $\alpha \notin \rho_i$ we consider an inclusion-maximal relation $\rho_{i,\alpha} \supseteq \rho_i$ from Σ such that $\alpha \notin \rho_{i,\alpha}$. Since $\rho_i = \bigcap_{\alpha \notin \rho_i} \rho_{i,\alpha}$, if $\rho_i \neq \rho_{i,\alpha}$ for each α , then ρ_i could be replaced by bigger relations that are still in Σ , which contradicts our assumptions. Then for each ρ_i there exists a tuple α_i such that ρ_i is an inclusion-maximal relation without α_i in Σ .

By Corollary 8.12.1, ρ_i is a key relation for every i . Therefore, we get a sequence of instances $\Upsilon_1, \dots, \Upsilon_t \in \text{Coverings}(\Omega)$ such that $\Upsilon_i^{(1)}$ defines ρ_i for every i . Put $\Phi = (\Theta \setminus \Omega) \cup \Upsilon_1 \cup \dots \cup \Upsilon_t$. We choose variables in the instance so that the only common variables of $\Upsilon_1, \dots, \Upsilon_t$ are x_1, \dots, x_n , which guarantees that $\Upsilon_i(x_1, \dots, x_n)$ is a subconstraint of Φ .

Since Φ is a covering of Θ , by Lemma 6.1, Φ is cycle-consistent and irreducible. Assume that ρ_i does not have the parallelogram property. Without loss of generality, we assume that the failing partition is $\{x_1, \dots, x_k\}, \{x_{k+1}, \dots, x_n\}$. Define the instance Ω_i from Υ_i using the construction from Theorem 9.9. Then the relation defined by $\Omega_i^{(1)}(x_1, \dots, x_n)$ is bigger than ρ_i and $\Omega_i \in \text{Coverings}(\Omega)$, which means that $(\Phi \setminus \Upsilon_i) \cup \Omega_i$ has a solution in $D^{(1)}$ and contradicts the inductive assumption for Theorem 9.9. Hence, ρ_i has the parallelogram property for every i . \square

To prove the next theorem, we will need additional definitions and few auxiliary lemmas. First, we assign a characteristic to every variable of an instance, and then we introduce a partial order on the set of characteristics. After that, we define three transformations of the instance giving an expanded covering of the original instance. We will prove that these transformations change the characteristics in a good way, so they can be used to generate an instance required in Theorem 9.6.

Let us assign a characteristic to every variable of an instance Φ whose constraints are critical and rectangular. For a variable x , let \mathbb{C}_1 be the set of all minimal congruences among the set $\text{Con}(\Phi, x)$. Then let \mathbb{C}_2 be the set of all minimal congruences among the congruences of $\text{Con}(\Phi, x)$ that are not adjacent with any congruence from \mathbb{C}_1 . Thus, we assign a pair $(\mathbb{C}_1, \mathbb{C}_2)$ to every variable x , which we denote $\xi(\Phi, x)$ and call *characteristic*.

Let us introduce a partial order on the set of all characteristics. For two sets of irreducible congruences \mathbb{C}_1 and \mathbb{C}_2 , we write $\mathbb{C}_1 \leq \mathbb{C}_2$ if for every $\sigma \in \mathbb{C}_1$ there exists $\delta \in \mathbb{C}_2$ such that $\delta \subseteq \sigma$. We write $\mathbb{C}_1 < \mathbb{C}_2$ if $\mathbb{C}_1 \leq \mathbb{C}_2$ and $\mathbb{C}_2 \not\leq \mathbb{C}_1$. It is easy to see that \leq is a transitive relation.

By $\uparrow \text{Opt}(\mathbb{C})$, we denote the set of all congruences σ such that $\sigma \supseteq \delta$ for some $\delta \in \text{Opt}(\mathbb{C})$. We write $(\mathbb{C}_1, \mathbb{C}_2) \lesssim (\mathbb{C}'_1, \mathbb{C}'_2)$ if one of the following conditions holds:

- (1) $\mathbb{C}_1 < \mathbb{C}'_1$;
- (2) $\mathbb{C}_1 = \mathbb{C}'_1$ and $\mathbb{C}_2 \leq \mathbb{C}'_2$;
- (3) $\mathbb{C}_1 = \mathbb{C}'_1$, $\mathbb{C}_2 \not\leq \mathbb{C}'_2$, $\mathbb{C}'_2 \not\leq \mathbb{C}_2$, $\mathbb{C}_2 \setminus (\uparrow \text{Opt}(\mathbb{C}_1)) < \mathbb{C}'_2 \setminus (\uparrow \text{Opt}(\mathbb{C}_1))$.

LEMMA 9.10. \lesssim is a transitive relation.

PROOF. Assume that $(\mathbb{C}_1, \mathbb{C}_2) \lesssim (\mathbb{C}'_1, \mathbb{C}'_2)$ and $(\mathbb{C}'_1, \mathbb{C}'_2) \lesssim (\mathbb{C}''_1, \mathbb{C}''_2)$.

If $\mathbb{C}_1 < \mathbb{C}'_1$ or $\mathbb{C}'_1 < \mathbb{C}''_1$, then $\mathbb{C}_1 < \mathbb{C}''_1$, which completes this case.

Thus, we assume that $\mathbb{C}_1 = \mathbb{C}'_1 = \mathbb{C}''_1$. It follows from (2) and (3) that

$$\mathbb{C}_2 \setminus (\uparrow \text{Opt}(\mathbb{C}_1)) \leq \mathbb{C}'_2 \setminus (\uparrow \text{Opt}(\mathbb{C}_1)) \leq \mathbb{C}''_2 \setminus (\uparrow \text{Opt}(\mathbb{C}_1)).$$

If $\mathbb{C}_2 \leq \mathbb{C}'_2 \leq \mathbb{C}''_2$, then $\mathbb{C}_2 \leq \mathbb{C}''_2$, which completes this case. Thus, we assume that at least one of the above comparisons is strict (comes from (3)). Hence, $\mathbb{C}_2 \setminus (\uparrow \text{Opt}(\mathbb{C}_1)) < \mathbb{C}''_2 \setminus (\uparrow \text{Opt}(\mathbb{C}_1))$. Therefore, $\mathbb{C}''_2 \not\leq \mathbb{C}_2$ and (2) or (3) holds for $(\mathbb{C}_1, \mathbb{C}_2)$ and $(\mathbb{C}''_1, \mathbb{C}''_2)$, which completes the proof. \square

Remark 5. Note that \leq is not a partial order in general, but it is a partial order on sets of mutually non-inclusive congruences. Similarly, \lesssim is not a partial order in general, but it is a partial order if we consider only pairs $(\mathbb{C}_1, \mathbb{C}_2)$ such that all the congruences of \mathbb{C}_i are not included into each other for $i = 1, 2$. Thus, as it follows from the definition of the characteristic, we defined a partial order on the set of all characteristics.

A variable x of an instance Θ is called *stable* if all the congruences in $\text{Con}(\Theta, x)$ are adjacent. We say that variables y_1 and y_2 are *friends* in Θ if they appear in the scope of some constraint of Θ .

Transformation $T_1(\Theta)$: make an instance crucial in $D^{(1)}$. Using Remark 3, we replace constraints by all weaker constraints until we get a CSP instance that is crucial in $D^{(1)}$.

Note that $T_1(\Theta) \in \text{ExpCov}(\Theta)$.

Below we assume that the instance Θ is crucial in $D^{(1)}$, which by the inductive assumption for Theorem 9.8 means that every constraint in Θ has the parallelogram property and is critical.

Transformation $T_2(\Theta, \sigma_1, \sigma_2, x)$: split a variable. Let Ω_i be the set of all constraints $C \in \Theta$ such that $\text{Con}(C, x) = \sigma_i$ for $i \in \{1, 2\}$. Let Ω_0 be the set of all constraints $C \in \Theta \setminus (\Omega_1 \cup \Omega_2)$ containing x . We transform our instance in the following way:

- (1) Choose 2 new variables x_1 and x_2 ;
- (2) Rename x by x_1 in all constraints from Ω_0 and Ω_1 ;
- (3) Rename x by x_2 in all constraints from Ω_2 ;
- (4) Add the constraints $\sigma_1^*(x_1, x_2)$ and $\sigma_2^*(x_1, x_2)$;
- (5) For every $\sigma \in \text{Con}(\Omega_0, x)$ add the constraint $\sigma(x_1, x_2)$.

Note that $T_2(\Theta, \sigma_1, \sigma_2, x)$ is an expanded covering of Θ , where the parent of x_1 and x_2 is x .

LEMMA 9.11. *Suppose $D^{(1)}$ is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance Θ , Θ is crucial in $D^{(1)}$, and congruences $\sigma_1, \sigma_2 \in \text{Con}(\Theta, x)$ are not adjacent. Then the instance $T_2(\Theta, \sigma_1, \sigma_2, x)$ has no solutions in $D^{(1)}$.*

PROOF. Let $\Theta' = T_2(\Theta, \sigma_1, \sigma_2, x)$, σ be the intersection of all congruences from $\text{Con}(\Omega_0, x)$.

Assume that Θ' has a solution in $D^{(1)}$. Suppose $(x_1, x_2) = (a_1, a_2)$ in this solution. Put $\Upsilon = \sigma_1(x_1, x) \wedge \sigma_2(x_2, x) \wedge \sigma(x_2, x)$. Consider the instance $\Theta' \wedge \Upsilon$. Since $(x_1, x_2) \in \sigma$ (by the definition of the transformation) and $(x_2, x) \in \sigma$, we have $(x, x_1) \in \sigma$. Then each solution of $\Theta' \wedge \Upsilon$ can be taken as a solution of Θ (we just ignore x_1 and x_2). Hence, the instance $\Theta' \wedge \Upsilon$ has no solutions in $D^{(1)}$. We apply Theorem 9.5 to the subconstraint $\Upsilon(x_1, x_2)$ to obtain a sequence of formulas $\Omega_1, \dots, \Omega_t \in \text{Coverings}(\Upsilon)$ such that $\Theta' \cup \Omega_1 \cup \dots \cup \Omega_t$ has no solutions in $D^{(1)}$, and $\Omega_i^{(1)}(x_1, x_2)$ defines a subdirect key relation ρ_i with the parallelogram property for every i . Note that the relation ρ_i is reflexive; therefore, ρ_i is a congruence on $D_{x_1}^{(1)}$. If the reduction $D^{(1)}$ is nonlinear, then by ω_i we denote the relation defined by $\Omega_i(x_1, x_2)$. If the reduction $D^{(1)}$ is linear, then by ω_i we denote the relation defined by $\Omega_i'(x_1, x_2, u_1, \dots, u_r)$ from Lemma 8.14. We know from Lemmas 8.14 and 8.13 that $\text{Con}(\omega_i, 1)^{(1)} = \text{Con}(\rho_i, 1) = \rho_i$. Every constraint in Ω_i , which contains x_1 must have σ_1 for its constraint relation; thus, the first variable of ω_i is stable under σ_1 and $\text{Con}(\omega_i, 1) \supseteq \sigma_1$. Consider the following two cases:

Case 1. Assume that $\rho_i \neq \sigma_1^{(1)}$ for every i , then $\text{Con}(\omega_i, 1) \supseteq \sigma_1^*$. Hence $\rho_i \supseteq (\sigma_1^*)^{(1)}$ for every i . Then we may put $x_1 = a_1$ and $x = x_2 = a_2$ to get a solution of $\Theta' \cup \Omega_1 \cup \dots \cup \Omega_t$ in $D^{(1)}$, which contradicts the properties of the sequence $\Omega_1, \dots, \Omega_t$.

Case 2. Assume that $\rho_i = \sigma_1^{(1)}$ for some i . Since $(a_1, a_2) \in (\sigma_1^*)^{(1)} \setminus \sigma_1$ and $\text{Con}(\omega_i, 1)^{(1)} = \rho_i$, we have $\text{Con}(\omega_i, 1) \not\supseteq \sigma_1^*$. Hence $\text{Con}(\omega_i, 1) = \sigma_1$. Suppose $D^{(1)}$ is a nonlinear reduction. $\Upsilon(x_1, x_2)$ contains $\sigma_2 \cap \sigma$, and therefore $\sigma_2 \cap \sigma \subseteq \text{Con}(\omega_i, 1) = \sigma_1$. The symmetric conclusion $\sigma_1 \cap \sigma \subseteq \sigma_2$ can be obtained by a symmetric argument, switching the roles of σ_1 and σ_2 . Since, $(a_1, a_2) \in \sigma \setminus \sigma_1$, by Lemma 8.19 σ_1 and σ_2 are adjacent, which contradicts our assumptions. Similarly, if $D^{(1)}$ is a linear reduction, then we can show that $\sigma_2 \cap \sigma \cap \text{ConLin}(D_x) \subseteq \text{Con}(\omega_i, 1) = \sigma_1$. Indeed, suppose $(c, d) \in \sigma_2 \cap \sigma \cap \text{ConLin}(D_x)$. To witness that $(c, d) \in \text{Con}(\omega_i, 1)$, we need to define two tuples from ω_i that differ only in the first component. To obtain the first tuple, we assign c to every variable of Ω_i' . To obtain the second tuple, we assign d to all variables whose parent is x_1 or x , and c to the remaining variables (including u_1, \dots, u_r). Thus, we can show that $\sigma_2 \cap \sigma \cap \text{ConLin}(D_x) \subseteq \sigma_1$

and $\sigma_1 \cap \sigma \cap \text{ConLin}(D_x) \subseteq \sigma_2$. Since $(a_1, a_2) \in (\sigma \cap \text{ConLin}(D_x)) \setminus \sigma_1$, Lemma 8.19 implies that σ_1 and σ_2 are adjacent, which contradicts our assumptions. \square

Informally speaking, the following lemma states that when we apply $T_1(T_2(\Theta, \sigma_1, \sigma_2, x))$ the characteristic of every new variable is less than the characteristic of its parent, the characteristic of old variables does not change, and if a stable variable gets a new friend then the friend's parent is not its friend anymore.

LEMMA 9.12. *Suppose $D^{(1)}$ is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance Θ , Θ is crucial in $D^{(1)}$, congruences σ_1, σ_2 are minimal congruences among $\text{Con}(\Theta, x)$, σ_1 and σ_2 are not adjacent, $\Theta' = T_1(T_2(\Theta))$. Then*

- (1) $\xi(\Theta', y') < \xi(\Theta, y)$, if y is a parent of y' and $y' \neq y$;
- (2) $\xi(\Theta', y) = \xi(\Theta, y)$ if $y \in \text{Var}(\Theta) \cap \text{Var}(\Theta')$;
- (3) if y is stable in Θ , $y' \in \text{Var}(\Theta') \setminus \text{Var}(\Theta)$, then y cannot be a friend of both y' and the parent of y' in Θ' ;
- (4) Θ and Θ' have a common variable.

PROOF. By Lemma 9.11 and the definition of T_1 , Θ' is crucial, then by Lemma 8.24 for every constraint C in Θ there exists a constraint C' in Θ' whose image in Θ is C . Therefore, when we apply T_1 we weaken only binary constraints we added in T_2 but not the constraints from Θ . Then Claim (2) follows from the definition of the transformation.

$\text{Con}(\Theta', x_1)$ has all the congruences of $\text{Con}(\Theta, x)$ but σ_2 . Additionally, it may contain congruences δ such that $\delta \supseteq \sigma_1^*$, $\delta \supseteq \sigma_2^*$, or $\delta \supseteq \sigma$ for $\sigma \in \text{Con}(\Omega_0, x)$. None of these congruences are minimal, so they cannot affect the first coordinate of $\xi(\Theta', x_1)$. Thus, $\xi(\Theta', x_1) < \xi(\Theta, x)$. Similarly, we can show that $\xi(\Theta', x_2) < \xi(\Theta, x)$, which completes Claim (1).

Claim (3) follows from the fact that x , which is the only parent of variables from $\text{Var}(\Theta') \setminus \text{Var}(\Theta)$, is not in Θ' .

Since a crucial instance cannot have just one variable, Θ and Θ' have a common variable, which is Claim (4). \square

For an instance $\Omega \subseteq \Theta$ by $\text{MinVar}(\Omega, \Theta)$, we denote the set of all variables x such that there exists $\sigma \in \text{Con}(\Omega, x)$ that is minimal among $\text{Con}(\Theta, x)$.

Transformation $T_3(\Theta, \Omega)$ for a connected component Ω . Let $\text{MinVar}(\Omega, \Theta) = \{x_1, \dots, x_s\}$, where $s \geq 1$. Let us define the new instance in the following way:

- (1) Choose new variables x'_1, \dots, x'_s ;
- (2) Rename the variables x_1, \dots, x_s by x'_1, \dots, x'_s in $\Theta \setminus \Omega$;
- (3) Add the covers of all constraints from Ω with x'_1, \dots, x'_s instead of x_1, \dots, x_s ;
- (4) For every j and every $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$ add the constraint $\sigma^*(x_j, x'_j)$;
- (5) For every j and $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$ such that $\text{LinkedCon}(\Omega, x_j) \not\subseteq \sigma$ add the constraint $\delta_j(x_j, x'_j)$, where $\{\delta_j\} = \text{Opt}(\text{Con}(\Omega, x_j))$.

Note that by Corollary 8.22.1 all congruences of $\text{Con}(\Omega, x_j)$ are adjacent. Then by Lemma 6.4 $\text{Opt}(\text{Con}(\Omega, x_j))$ contains just one element and δ_j is well defined in (5). Also, $T_3(\Theta, \Omega)$ is an expanded covering of Θ , where the parent of every x'_i is x_i .

LEMMA 9.13. *Suppose $D^{(1)}$ is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance Θ , Θ is crucial in $D^{(1)}$, Ω is a connected component of Θ , the solution set of Ω is subdirect, Ω has a solution in $D^{(1)}$, and for every $x \in \text{Var}(\Omega)$ any two congruences that are minimal among $\text{Con}(\Theta, x)$ are adjacent. Then the instance $T_3(\Theta, \Omega)$ has no solutions in $D^{(1)}$.*

PROOF. Suppose $\text{Var}(\Omega) \setminus \text{MinVar}(\Omega, \Theta) = \{z_1, \dots, z_n\}$. Since the solution set of Ω is subdirect, by Lemma 8.5 we know that the solution set of $\Omega^{(1)}$ is subdirect. We consider two cases:

Case 1: $\text{LinkedCon}(\Omega, y) = \text{Con}(C, y)$ for every variable y and every constraint $C \in \Omega$ having y in the scope. This means that $\text{Con}(\Omega, y)$ contains exactly one congruence for every variable $y \in \text{Var}(\Omega)$. Since $\text{Con}(C, x_j)$ is minimal among $\text{Con}(\Theta, x_j)$ for every j and every constraint $C \in \Omega$ containing x_j , we have $\text{LinkedCon}(\Omega, x_j) \subsetneq \sigma$ for every j and every $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$. Notice that for any constraint $C \in \Omega$ having y_1 and y_2 in the scope we have $\text{LinkedCon}(\Omega, y_1) \supseteq \text{Con}(\text{pr}_{y_1, y_2}(C), y_1)$. Since all constraints of Ω are rectangular and critical, Lemma 8.10 together with $\text{LinkedCon}(\Omega, y_1) = \text{Con}(C, y_1)$ imply that the constraint C should be binary. Thus, all the constraint relations are binary.

Assume that $n = 0$. Since Θ is crucial in $D^{(1)}$, the instance Ω , viewed as a graph whose vertexes are variables, cannot have a cycle (otherwise, removing a constraint(edge) from the cycle would not affect the solution set, which contradicts the fact that Θ is crucial). Hence, we can choose a constraint $C \in \Omega$ with a variable x_j that appears just once in Ω . We replace the variable x_j in $\Theta \setminus \{C\}$ by x'_j and add the constraint $\sigma_0^*(x_j, x'_j)$, where $\sigma_0 = \text{Con}(C, x_j)$. The obtained instance we denote by Θ' . Since the constraint C is crucial in $D^{(1)}$, Θ' has a solution in $D^{(1)}$. Since $\sigma_0^* \subseteq \sigma$ for every $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$, the solution of Θ' gives a solution of Θ in $D^{(1)}$, which contradicts our assumptions.

Suppose $n > 0$. By Ω' , we denote the copy of Ω with covers instead of constraints we introduced in (3). For every variable y of Ω by σ_y we denote the minimal congruence such that $\sigma_y \supsetneq \text{LinkedCon}(\Omega, y)$. Since $\text{LinkedCon}(\Omega, y)$ is an irreducible congruence, σ_y is well defined. For every constraint $C = \rho(u, v)$ of Ω , by C' we denote the constraint $\rho'(u, v)$, where $\rho'(u, v) = \exists u' \exists v' \rho(u', v') \wedge \sigma_u(u, u') \wedge \sigma_v(v, v')$. Let us show that ρ' is a rectangular relation such that $\text{Con}(\rho', 1) = \sigma_u$ and $\text{Con}(\rho', 2) = \sigma_v$, that is a bijective mapping between equivalence classes of σ_u and σ_v . Since ρ is rectangular, the congruence σ_u generates a congruence on D_v that is strictly greater than $\text{Con}(\rho, 2)$, and therefore containing σ_v . Therefore, σ_v has at least as many equivalence classes as σ_u . The same is true for σ_u , which means that the congruence generated on D_v from σ_u using ρ is equal to σ_v . Therefore, ρ' is a rectangular relation such that $\text{Con}(\rho', 1) = \sigma_u$ and $\text{Con}(\rho', 2) = \sigma_v$. Note that $\rho' \supsetneq \rho$.

Since $n > 0$ and Ω is not fragmented, there exists a path in Ω connecting a variable z_i with a variable x_j for every i and j . Then we glue a path going from x_j to z_i in Ω with a path going from z_i to x'_j in Ω' . For every constraint $C \in \Omega$ the constraint C' (defined above) is weaker or equivalent to its cover in Ω' . Therefore, every constraint C in the obtained path from x_j to x'_j is not weaker than C' , which means (by the properties of C') that x_j and x'_j should be equivalent modulo σ_{x_j} for every j in any solution of $T_3(\Theta, \Omega)$.

Assume that $T_3(\Theta, \Omega)$ has a solution in $D^{(1)}$ with

$$(x_1, \dots, x_s, x'_1, \dots, x'_s) = (b_1, \dots, b_s, b'_1, \dots, b'_s).$$

Since $\sigma_{x_j} \subseteq \sigma$ for every $\sigma \in \text{Con}(\Theta \setminus \Omega)$, we have $(b_i, b'_i) \in \sigma$. Therefore, we can assign

$$(x_1, \dots, x_s, x'_1, \dots, x'_s) = (b_1, \dots, b_s, b_1, \dots, b_s)$$

to get a solution of $\Theta^{(1)}$ (the remaining variables take on the same values). This contradiction completes this case.

Case 2: $\text{LinkedCon}(\Omega, z) \neq \text{Con}(C_z, z)$ for some variable z and some constraint $C_z \in \Omega$.

Assume that $n = 0$ and $\text{LinkedCon}(\Omega, x_j) \subseteq \sigma$ for every j and every $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$. We rename the variable z in C_z by z' and add the constraint $\sigma_L(z, z')$, where $\sigma_L = \text{LinkedCon}(\Omega, z)$. Since Θ is crucial in $D^{(1)}$, the new instance has a solution β in $D^{(1)}$. Let z be equal to c in β . Since the solution set of $\Omega^{(1)}$ is subdirect, there exists a solution γ of $\Omega^{(1)}$ with $z = c$. Note that the

corresponding elements of β and γ are linked in Ω . Since $\text{LinkedCon}(\Omega, x_j) \subseteq \sigma$ for every j and every $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$, we can build a solution of $\Theta^{(1)}$ with the values for x_j from γ and the values for the remaining variables from β , which gives us a contradiction.

Thus, we assume that $n > 0$ or $\text{LinkedCon}(\Omega, x_h) \not\subseteq \sigma$ for some h and $\sigma \in \text{Con}(\Theta \setminus \Omega, x_h)$. In this case, we consider a different transformation defined as follows:

- (1) Choose new variables x'_1, \dots, x'_s and x''_1, \dots, x''_s .
- (2) Add a copy of Ω to Θ with all the variables x_1, \dots, x_s replaced by x'_1, \dots, x'_s . We denote the copy by Ω' .
- (3) Rename x_1, \dots, x_s in $\Theta \setminus \Omega$ by x''_1, \dots, x''_s .
- (4) For every i and every $\sigma \in \text{Con}(\Theta \setminus \Omega, x_i)$ add a new variable y and add the constraints $\sigma(x'_i, y)$ and $\sigma(x''_i, y)$.
- (5) For every i and every $\sigma \in \text{Con}(\Theta \setminus \Omega, x_i)$ add the constraint $\sigma^*(x_i, x''_i)$.
- (6) For every j and $\sigma \in \text{Con}(\Theta \setminus \Omega, x_j)$ such that $\text{LinkedCon}(\Omega, x_j) \not\subseteq \sigma$ add the constraint $\delta_j(x_j, x'_j)$, where $\{\delta_j\} = \text{Opt}(\text{Con}(\Omega, x_j))$.

Since here we just copied Ω , any solution of the obtained instance would give a solution to Θ (we use values of $x'_1, \dots, x'_s, z_1, \dots, z_n$ to generate a solution), and hence the obtained instance has no solutions in $D^{(1)}$. We replace constraints from Ω' containing at least one of the variables x'_1, \dots, x'_s by their covers step by step. Thus, in one step we replace just one constraint from Ω' . We consider two cases.

Assume that after all replacements we get an instance Θ_0 without solutions in $D^{(1)}$. Any solution of $T_3(\Theta, \Omega)$ gives a solution of Θ_0 : If $x'_i = a_i$ in the solution of $T_3(\Theta, \Omega)$, then we put $x'_i = x''_i = y = a_i$ in Θ_0 for every i and the corresponding y 's (the remaining variables take the same values). Therefore, $T_3(\Theta, \Omega)$ has no solutions in $D^{(1)}$, which completes this case.

Assume that after some replacement the instance gets a solution in $D^{(1)}$. Suppose the instance before this replacement is Θ' and the corresponding constraint to be replaced is C . Choose a variable $x'_i \in \text{Var}(C)$.

Let $\delta = \text{Con}(C, x'_i)$, ρ be an optimal bridge from δ to δ . Let us define a new bridge by

$$\rho'(u_1, u_2, u_3, u_4) = \exists v_1 \exists v_2 \rho(u_1, u_2, v_1, v_2) \wedge \rho(u_3, u_4, v_1, v_2) \wedge \rho(u_1, u_1, u_3, u_3) \wedge \delta^*(u_3, u_4).$$

Since $\rho'(x, x, y, y) = \rho(x, x, y, y)$, ρ' is also an optimal bridge. Additionally, ρ' has the following property: If $(a, b, c, d) \in \rho'$, then $(a, c) \in \bar{\rho}$ and $(c, d) \in \delta^*$.

Then we change Θ' in the following way. We add three new variables u_1, u_2, x'''_i , replace x'_i in C by x'''_i , and add the constraint $\rho'(x'_i, x'''_i, u_1, u_2)$ and the constraint $\delta(u_1, u_2)$. We denote the new instance by Θ'' . By the definition of a bridge, Θ'' has no solutions in $D^{(1)}$.

By Υ we denote all constraints of Θ'' containing x'_j for some j or x'''_i . Let $\{y_1, \dots, y_t\}$ be the set of all variables of Υ except for $z_1, \dots, z_n, x_1, \dots, x_s, x'_1, \dots, x'_s, u_1, u_2$, and x'''_i . Suppose that the variable x_{i_j} is the corresponding variable and σ_j is the corresponding congruence for y_j (see step (4) of the transformation).

If we remove the constraint $\delta(u_1, u_2)$ from Θ'' , then it is equivalent to making a constraint C of Θ' weaker, which means that we get a solution of Θ'' in $D^{(1)}$ after the removal. Let

$$(x_1, \dots, x_s, x'_1, \dots, x'_s, x''_1, \dots, x''_s, y_1, \dots, y_t, z_1, \dots, z_n, u_1, u_2) = \\ (a_1, \dots, a_s, a'_1, \dots, a'_s, a''_1, \dots, a''_s, d_1, \dots, d_t, b_1, \dots, b_n, c_1, c_2)$$

in this solution.

First, we want to show that $(a_l, a_l, c_1, c_1) \in \rho'$. By the definition of ρ' and Θ'' , we have $(a'_l, c_1) \in \bar{\rho}$. We consider two subcases.

Case 2A. Suppose $n > 0$. Gluing a path from x_l to z_1 in Ω and a path from z_1 to x'_1 in Ω' , we show that a_l and a'_l are linked in Ω' . We apply Theorem 8.22 to get a bridge from δ to δ containing (a_l, a_l, a'_l, a'_l) . Then, we compose this bridge with the bridge ρ to obtain a bridge from δ to δ containing (a_l, a_l, c_1, c_1) . Since the bridge ρ' is optimal, we have $(a_l, a_l, c_1, c_1) \in \rho'$.

Case 2B. $\text{LinkedCon}(\Omega, x_h) \not\subseteq \sigma$ for some h and $\sigma \in \text{Con}(\Theta \setminus \Omega, x_h)$. Let $\zeta \in \text{Con}(\Omega, x_h)$ and ξ_0 be an optimal bridge from ζ to ζ . Note that by step (6) of the new transformation $(a_h, a'_h) \in \tilde{\zeta}$. We know that a_l and a_h are linked in Ω , a'_h and a'_l are linked in Ω' . We apply Theorem 8.22 to get a bridge ξ_1 from δ to ζ containing (a_l, a_l, a_h, a_h) and a bridge ξ_2 from ζ to δ containing (a'_h, a'_h, a'_l, a'_l) . Then we compose ξ_1 , ξ_0 , ξ_2 , and ρ (in this order) to obtain a bridge from δ to δ containing (a_l, a_l, c_1, c_1) . Since the bridge ρ' is optimal, we have $(a_l, a_l, c_1, c_1) \in \rho'$.

Consider a subconstraint $\Upsilon(y_1, \dots, y_t, x_1, \dots, x_s, z_1, \dots, z_n, u_1, u_2)$. The constraint $\delta(u_1, u_2)$ is isolated in $\Theta'' \setminus \Upsilon$, hence $\Theta'' \setminus \Upsilon$ has a solution in $D^{(1)}$. Using Theorem 9.5, we find $\Upsilon_1, \dots, \Upsilon_v \in \text{Coverings}(\Upsilon)$ such that $\Upsilon_i^{(1)}(y_1, \dots, y_t, x_1, \dots, x_s, z_1, \dots, z_n, u_1, u_2)$ defines a key relation ρ_i with the parallelogram property for every i . Since $(\Theta'' \setminus \Upsilon) \cup \Upsilon_1 \cup \dots \cup \Upsilon_v$ has no solutions in $D^{(1)}$, we can choose k such that ρ_k omits the tuple $(d_1, \dots, d_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_1)$. By the definition of Υ , we can substitute the value c_1 instead of u_1 and u_2 and put $x_i = x'_i = x''_i = a_i$ and $y_j = x_{ij}$ for every i and j to get a solution of Υ . Precisely, for every j we put $d'_j = a_{ij}$, then $(d'_1, \dots, d'_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_1) \in \rho_k$ (here we used that $(a_l, a_l, c_1, c_1) \in \rho'$). Also we know that $(d_1, \dots, d_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_1) \notin \rho_k$ and $(d_1, \dots, d_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_2) \in \rho_k$. Then we consider the minimal j such that $(d'_1, \dots, d'_j, d_{j+1}, \dots, d_t, a_1, \dots, a_s, b_1, \dots, b_n, c_1, c_1) \in \rho_k$. It follows from the definition of Θ' that $(d_j, d'_j) \in \sigma_j^*$, and from the definition of ρ' that $(c_1, c_2) \in \delta^*$. Then by Lemma 8.23 there exists a bridge ζ_1 from δ to σ_j such that $\tilde{\zeta}_1$ contains $\Upsilon(u_2, y_j)$, and therefore it contains $\Omega(x_l, x_{ij})$.

Suppose $\delta_0 \in \text{Con}(\Omega, x_{ij})$. Applying Theorem 8.22 to Ω and the variables x_{ij} and x_l , we get a bridge ζ_2 from δ_0 to δ such that $\tilde{\zeta}_2$ contains all elements linked in Ω , and therefore it contains $\Omega(x_{ij}, x_l)$. Composing the bridges ζ_2 and ζ_1 we get a bridge from δ_0 to σ_j . Since $\Omega(x_l, x_{ij})$ is subdirect, the obtained bridge is reflexive. Hence, δ_0 and σ_j are adjacent, which contradicts the fact that $\sigma_j \in \text{Con}(\Theta \setminus \Omega, x_{ij})$. \square

Below we prove a property of the transformation T_3 similarly to the property of T_2 proved in Lemma 9.12.

LEMMA 9.14. *Suppose $D^{(1)}$ is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance Θ , Θ is crucial in $D^{(1)}$, Ω is a connected component, the solution set of Ω is subdirect, Ω has a solution in $D^{(1)}$, for every $x \in \text{Var}(\Omega)$ any two congruences that are minimal among $\text{Con}(\Theta, x)$ are adjacent, and $\Theta' = T_1(T_3(\Theta, \Omega))$. Then*

- (1) $\xi(\Theta', y') < \xi(\Theta, y)$, if y is a parent of y' and $y' \neq y$;
- (2) $\xi(\Theta', y) \leq \xi(\Theta, y)$, if $y \in \text{Var}(\Theta) \cap \text{Var}(\Theta')$;
- (3) $\xi(\Theta', y) < \xi(\Theta, y)$, if $y \in \text{MinVar}(\Omega, \Theta)$ and y not stable in Θ ;
- (4) if y is stable in Θ , $y' \in \text{Var}(\Theta') \setminus \text{Var}(\Theta)$, then y cannot be a friend of both y' and the parent of y' in Θ' ;
- (5) Θ and Θ' have a common variable.

PROOF. By Lemma 9.13 Θ' is crucial, then by Lemma 8.24 for every constraint C in Θ there exists a constraint C' in Θ' whose image in Θ is C . Therefore, when we apply T_1 we weaken only binary constraints and covers we added in T_3 but not the constraints from Θ .

First, let us show that $\xi(\Theta, z) = \xi(\Theta', z)$ for every $z \in \text{Var}(\Theta) \setminus \{x_1, \dots, x_s\}$. The only constraints with z we added are the constraints we obtained from the covers of constraints from Ω using T_1 .

Let C'' be the cover of a constraint C' from Ω . Let C''' be a constraint obtained from C'' using T_1 . By Lemma 8.10, $\text{Con}(C'', z) \supsetneq \text{Con}(C', z)$. By the definition of T_1 , $\text{Con}(C''', z) \supseteq \text{Con}(C'', z)$. Hence, $\text{Con}(C''', z) \supsetneq \text{Con}(C', z)$. Since $\text{Con}(C', z)$ is not adjacent with a minimal congruence among $\text{Con}(\Theta, z)$, $\text{Con}(C''', z)$ cannot affect the characteristic of z . Therefore, $\xi(\Theta, z) = \xi(\Theta', z)$.

Second, let us show that $\xi(\Theta', x_i) \leq \xi(\Theta, x_i)$ for every i . Since any two congruences that are minimal among $\text{Con}(\Theta, x_i)$ are adjacent and the congruence we add in (5) cannot be a new minimal congruence among $\text{Con}(\Theta, x_i)$, the first components of $\xi(\Theta', x_i)$ and $\xi(\Theta, x_i)$ are equal. If the variable x_i is stable, then we do not add anything in (4) and (5), hence the second components of $\xi(\Theta', x_i)$ and $\xi(\Theta, x_i)$ are empty, which completes Claim (2) for this case. Assume that x_i is not stable. Then the second component of $\xi(\Theta', x_i)$ has congruences appeared in (4) and (5) instead of congruences from $\text{Con}(\Theta \setminus \Omega, x_i)$. The congruences we added in (4) are bigger than the corresponding congruences from $\text{Con}(\Theta \setminus \Omega, x_i)$. Hence if we added nothing in (5), then $\xi(\Theta', x_i) < \xi(\Theta, x_i)$ because of the second components. Otherwise, consider a minimal congruence $\sigma \in \text{Con}(\Theta \setminus \Omega, x_i)$ such that $\text{LinkedCon}(\Omega, x_i) \not\subseteq \sigma$. By Corollary 8.22.1, congruences we obtain using (5) are greater than $\text{LinkedCon}(\Omega, x_i)$, and hence they cannot be smaller than σ . Therefore, σ belongs to the second component of $\xi(\Theta, x_i)$, and the second component of $\xi(\Theta', x_i)$ does not have a congruence that is equal to or smaller than σ . We conclude that either second components of $\xi(\Theta, x_i)$ and $\xi(\Theta', x_i)$ are incomparable or the second component of $\xi(\Theta', x_i)$ is smaller. Note that all the congruences we obtain in (5) are from $\uparrow \text{Opt}(\text{Con}(\Omega, x_i))$, which means that $\xi(\Theta', x_i) < \xi(\Theta, x_i)$ in this case. Thus we proved Claims (2) and (3).

To prove Claim (1) we need to show that $\xi(\Theta', x'_i) < \xi(\Theta, x_i)$ for every i . Every congruence from $\text{Con}(\Theta \setminus \Omega, x_i)$ is bigger than some congruence from $\text{Con}(\Omega, x_i)$. Hence, $\xi(\Theta', x'_i) < \xi(\Theta, x_i)$ because of the first components.

To prove Claim (4) consider two cases.

Case 1. Suppose x_i is a stable variable. Since x_i cannot be a friend of x'_j , we obtain the necessary condition.

Case 2. Suppose $z \in \text{Var}(\Theta) \setminus \{x_1, \dots, x_s\}$ is a stable variable. By the definition of being stable we conclude that $z \notin \text{Var}(\Omega)$. Hence z cannot be a friend of x_j , which proves Claim (4).

The Claim (5) follows from the fact that x_1 should be in both Θ and Θ' . \square

THEOREM 9.6 *Suppose $D^{(1)}$ is a minimal 1-consistent one-of-four reduction of a cycle-consistent irreducible CSP instance Θ , Θ is crucial in $D^{(1)}$ and not connected. Then there exists an instance $\Theta' \in \text{ExpCov}(\Theta)$ that is crucial in $D^{(1)}$ and contains a linked connected component whose solution set is not subdirect.*

PROOF. We build a sequence of instances $\Theta_1, \Theta_2, \Theta_3, \dots$ such that $\Theta_{i+1} \in \text{ExpCov}(\Theta_i)$, and every Θ_i is crucial in $D^{(1)}$. Recall that by the inductive assumption for Theorem 9.8 all constraint relations of each Θ_i are critical relations with the parallelogram property. We start with $\Theta_1 = \Theta$. We want the final element of this sequence to contain a linked connected component whose solution set is not subdirect. Suppose we already defined Θ_i .

If there exist congruences $\sigma_1, \sigma_2 \in \text{Con}(\Theta_i, x)$ for some variable x that are not adjacent and minimal among $\text{Con}(\Theta, x)$, then put $\Theta_{i+1} := T_1(T_2(\Theta_i, \sigma_1, \sigma_2, x))$. By Lemma 9.11, Θ_{i+1} is crucial in $D^{(1)}$.

Otherwise, we know that any two minimal congruences in $\text{Con}(\Theta, x)$ for every variable x are adjacent. By Lemma 8.25, Θ_i is not connected. Since Θ_i is crucial, it is also not fragmented. Then there exist a variable x that is not stable. Choose “an oldest” nonstable variable in Θ_i , that is, a variable x with the minimal number j such that $x \in \text{Var}(\Theta_j)$. Choose the connected component Ω containing a minimal congruence of $\text{Con}(\Theta, x)$. Put $\Theta_{i+1} := T_1(T_3(\Theta_i, \Omega))$.

If Ω is not linked, then irreducibility of Θ implies that the solution set of Ω is subdirect. If Ω is linked and the solution set of Ω is not subdirect, then the theorem is proved and we stop the process. Thus, we assume that the solution set of Ω is subdirect. Since Θ_i is crucial in $D^{(1)}$ and not connected, $\Omega^{(1)}$ has a solution. Then by Lemma 9.13, Θ_{i+1} is crucial in $D^{(1)}$.

Thus, the next element of the sequence is defined either by $T_1(T_2(\Theta_i, \sigma_1, \sigma_2, x))$ or by $T_1(T_3(\Theta_i, \Omega))$. Now, we want to prove that the sequence $\Theta_1, \Theta_2, \Theta_3, \dots$ cannot be infinite, which means that the last element with the required property exists. To prove this we are going to use Theorem 8.29. First, we extend a partial order \lesssim on characteristics to a linear order \leq such that $(\Omega_1, \Omega_2) \lesssim (\Omega'_1, \Omega'_2)$ implies $(\Omega_1, \Omega_2) \leq (\Omega'_1, \Omega'_2)$. Second, we consider the set of all pairs $(x, \xi(\Theta_i, x))$, where $x \in \text{Var}(\Theta_i)$, as the set of organisms. Two organisms are friends if the corresponding variables were friends in Θ_i for some i (if they've ever been friends). If $x \in \text{Var}(\Theta_i) \cap \text{Var}(\Theta_{i+1})$ and $\xi(\Theta_{i+1}, x) < \xi(\Theta_i, x)$, then we say that $(x, \xi(\Theta_i, x))$ is the parent of $(x, \xi(\Theta_{i+1}, x))$. Also, if $x \in \text{Var}(\Theta_{i+1}) \setminus \text{Var}(\Theta_i)$, and x' is a parent of x in Θ_i , then $(x', \xi(\Theta_i, x'))$ is the parent of $(x, \xi(\Theta_{i+1}, x))$. The characteristic $\xi(\Theta_i, x)$ is considered as the strength of the organism $(x, \xi(\Theta_i, x))$. Then the set of organisms X_i is the set of all pairs $(x, \xi(\Theta_j, x))$ for $j \leq i$.

Let us check all the assumptions we have in Theorem 8.29. Condition (1) follows from Lemma 9.12 (claims 1,2) and Lemma 9.14 (claims 1,2). Conditions (2) and (3) follow from the fact that each Θ_{i+1} is from $\text{ExpCov}(\Theta_i)$.

Since the transformation T_2 (followed by T_1) replace a variable by two variables with smaller characteristic and does not change the characteristic of other variables, for the sequence to be infinite, we need to apply the transformation T_3 infinitely many times. By Lemma 9.14 (claim 3), we always reduce the characteristic of the chosen nonstable variable when we apply T_3 , which means that every variable will be stable at some moment.

It remains to show that condition (4) holds. As we noticed above, every variable will be stable at some moment. It remains to show that a variable z stable in Θ_i cannot get infinitely many friends (here we care only about the variables but not about the organisms). By Lemma 9.12 (claim 3) and Lemma 9.14 (claim 4), the variable z cannot be a friend of some variable y appeared in Θ_j for $j > i$ and the parent of y . Thus, if we consider the set of friends of z in Θ_j for $j > i$, then we see that going from Θ_j to Θ_{j+1} we can replace an old friend by new friends (that are weaker) but we cannot add a new friend keeping its parent. Therefore, after getting stable a variable cannot get infinitely many friends and condition (4) holds.

Since Θ_i is crucial in $D^{(1)}$, it is not fragmented. By Lemma 9.12 (claim 4) and Lemma 9.14 (claim 5), Θ_i and Θ_{i+1} have at least one common variable for every i . Therefore, the set of all organisms cannot be divided into two disjoint sets with no friendship between them. Thus, condition (5) of Theorem 8.29 cannot hold, which proves that the process will stop at some Θ_i having a linked connected component whose solution set is not subdirect. \square

THEOREM 9.7 Suppose $D^{(1)}$ is a 1-consistent nonlinear reduction of a cycle-consistent irreducible CSP instance Θ . If Θ has a solution, then it has a solution in $D^{(1)}$.

PROOF. Assume the contrary, that is, Θ has a solution but $\Theta^{(1)}$ has no solutions. By Theorem 9.3, there exists a minimal 1-consistent nonlinear reduction such that Θ has no solutions in it.

First, we consider the set of all minimal 1-consistent nonlinear reductions of Θ , which we denote by \mathfrak{R} . Then, we consider an instance $\Theta' \in \text{ExpCov}(\Theta)$ with the minimal positive number of reductions $D^{(\Delta)} \in \mathfrak{R}$ such that Θ' has no solutions in $D^{(\Delta)}$. Note that this transformation of Θ to Θ' can be omitted if $D^{(1)}$ is not a PC reduction. Then we weaken the instance Θ' (replace any constraint by all weaker constraints), while we still have a reduction $D^{(\Delta)} \in \mathfrak{R}$ such that Θ' has no solutions in $D^{(\Delta)}$. After that, we remove all dummy variables from constraints and denote the obtained instance by Θ'' . Note that Θ'' is not fragmented (since it is crucial in some $D^{(\Delta)}$), $\Theta'' \in \text{ExpCov}(\Theta)$,

and for any reduction $D^{(\Delta)} \in \mathfrak{R}$ the instance Θ'' is either crucial in $D^{(\Delta)}$ or has a solution in $D^{(\Delta)}$. The last property also holds for any expanded covering if Θ'' , which is crucial in some reduction $D^{(\Delta)}$. Choose a reduction $D^{(\Delta)}$ from \mathfrak{R} such that Θ'' is crucial in it.

Assume that Θ'' is not linked. If $D^{(\Delta)}$ is a PC reduction, then we apply Theorem 9.4 to find a reduction $D^{(1)}$ (it is a different reduction $D^{(1)}$) and a strategy $D^{(1)}, \dots, D^{(s)}$ for $\Theta''^{(1)}$ such that the solution set of $\Theta''^{(1)}$ is subdirect, the strategy has only nonlinear reductions, $D_y^{(s)} \subseteq D_y^{(\Delta)}$ for every y . Then $\Theta''^{(1)}$ is cycle-consistent and irreducible. By the inductive assumption $\Theta''^{(2)}$ has a solution, then by Lemma 8.6 $\Theta''^{(2)}$ is cycle-consistent and irreducible, by the inductive assumption $\Theta''^{(3)}$ has a solution, and so on. Thus we can prove that $\Theta''^{(s)}$ has a solution, which means that $\Theta''^{(\Delta)}$ has a solution and contradicts our assumption.

If $D^{(\Delta)}$ is an absorbing or central reduction, then we choose a variable x of Θ'' and an element $c \in D_x^{(\Delta)}$, and for every variable y by $D_y^{(\tau)}$ we denote the set of all elements of D_y linked to c . Since Θ'' is irreducible, the solution set of $\Theta''^{(\tau)}$ is subdirect. Therefore, $\Theta''^{(\tau)}$ is irreducible and cycle-consistent. By Lemmas 7.1, 7.6 the reduction $D^{(\perp)}$, defined by $D_y^{(\perp)} = D_y^{(\tau)} \cap D_y^{(\Delta)}$ for every variable y , is an absorbing or central reduction for $\Theta''^{(\tau)}$. Since $D^{(\Delta)}$ is a 1-consistent reduction and $D^{(\tau)}$ is just a linked component, the reduction $D^{(\perp)}$ is also 1-consistent. By the inductive assumption, $\Theta''^{(\perp)}$ has a solution, which gives a contradiction.

Thus, we assume that Θ'' is linked. Recall that by the inductive assumption for Theorem 9.8, every constraint of Θ'' is critical and has the parallelogram property. If Θ'' is not connected, then by Theorem 9.6, there exists an instance $\Upsilon \in \text{ExpCov}(\Theta'')$ that is crucial in $D^{(\Delta)}$ and contains a linked connected subinstance Ω . If Θ'' is connected, then Θ'' is a linked connected component itself and we put $\Upsilon = \Omega = \Theta''$. At the moment, we have $\Upsilon \in \text{ExpCov}(\Theta'')$ that is crucial in $D^{(\Delta)}$ and a linked connected subinstance Ω .

Let x_1 be the first variable in a constraint $C \in \Omega$. By Lemma 8.11, $\text{Con}(C, x_1)$ is irreducible. By Corollary 8.22.1, there exists a bridge δ from $\text{Con}(C, x_1)$ to $\text{Con}(C, x_1)$ such that $\delta(x, x, y, y)$ is a full relation. By Corollary 8.17.1, there exists a relation $\zeta \subseteq D_{x_1} \times D_{x_1} \times \mathbb{Z}_p$ such that $(y_1, y_2, 0) \in \zeta \Leftrightarrow (y_1, y_2) \in \text{Con}(C, x_1)$ and $\text{pr}_{1,2}(\zeta) = \text{Con}(C, x_1)^*$. Let us replace the variable x_1 of C in Υ by x'_1 and add the constraint $\zeta(x_1, x'_1, z)$. The obtained instance we denote by Υ' . Let $\text{Var}(\Upsilon) = \{x_1, \dots, x_n\}$, $\Upsilon'(x_1, \dots, x_n, z)$ define the relation S , which is the projection of the solution set of Υ' onto all variables but x'_1 . Let $C = R(x_1, x_{i_1}, \dots, x_{i_s})$, $R'(x_1, x_{i_1}, \dots, x_{i_s}) = \exists x'_1 R(x'_1, x_{i_1}, \dots, x_{i_s}) \wedge (x_1, x'_1) \in \text{Con}(C, x_1)^*$. The projection of S onto the first n variables is the solution set of the instance Υ whose constraint C is replaced by the weaker constraint $R'(x_1, x_{i_1}, \dots, x_{i_s})$. Since Υ is crucial in $D^{(\Delta)}$, the solution set S contains a tuple whose first n elements are from $D^{(\Delta)}$. Moreover, the last element of all such tuples is not equal to 0, since otherwise this would imply that Υ has a solution in $D^{(\Delta)}$.

By the assumption, Θ has a solution, and therefore Υ has a solution, which means that Υ' has a solution with $z = 0$ and, equivalently, S has a tuple whose last element is 0. Since \mathbb{Z}_p does not have proper subalgebras of size greater than 1, we have $\text{pr}_{n+1}(S) = \mathbb{Z}_p$.

Let us show for $i \in \{1, 2, \dots, n\}$ that $(\text{pr}_i(S))^{(\Delta)}$ is a one-of-four subuniverse of $\text{pr}_i(S)$ of the same type as $D^{(\Delta)}$. For absorbing and central reductions it follows from Lemma 7.28. For the PC type we consider a PC congruence σ on D_{x_i} . By Theorems 9.2, 9.3, for every equivalence class U of σ there exists a minimal 1-consistent PC reduction $D^{(\nabla)} \in \mathfrak{R}$ such that $D_{x_i}^{(\nabla)} \subseteq U$. As we assumed earlier, for any reduction from \mathfrak{R} the instance Υ is either crucial in it, or has a solution in it. Therefore, Υ' has a solution in any reduction from \mathfrak{R} , and Υ' has a solution with $x_i \in U$. Hence, σ restricted to $\text{pr}_i(S)$ is still a PC congruence. Moreover, $(\text{pr}_i(S))^{(\Delta)}$ is an intersection of equivalence classes of the corresponding PC congruences on $\text{pr}_i(S)$. Thus, we showed that $(\text{pr}_i(S))^{(\Delta)}$ is a one-of-four subuniverse of $\text{pr}_i(S)$ of the same type as $D^{(\Delta)}$.

By Lemma 7.25, $S^{(\Delta)}$ is a nonlinear one-of-four subuniverse of S (here we do not reduce the last variable). Also, by Lemma 7.25, the set of all tuples from S whose last element is 0 is a linear subuniverse of S , we denote this subuniverse by S_0 . By Lemma 7.29, the intersection $S^{(\Delta)} \cap S_0$ is not empty, which means that Υ has a solution in $D^{(\Delta)}$ and contradicts our assumptions. \square

Note that Theorem 9.8 could be derived from Theorem 9.9, but we decided to keep the original proof of Theorem 9.8, because it demonstrates the idea for both theorems in an easier way.

THEOREM 9.8. *Suppose $D^{(0)}, \dots, D^{(s)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ , and a constraint $\rho(x_1, \dots, x_n)$ of Θ is crucial in $D^{(s)}$. Then ρ is a critical relation with the parallelogram property.*

PROOF. Since $\rho(x_1, \dots, x_n)$ is crucial, ρ is a critical relation. Let Θ' be obtained from Θ by replacement of $\rho(x_1, \dots, x_n)$ by all weaker constraints. Since Θ is crucial in $D^{(s)}$, Θ' has a solution in $D^{(s)}$. By Lemma 6.1, Θ' is cycle-consistent and irreducible.

Assume that $|D_x^{(s)}| = 1$ for every variable x . Since the reduction $D^{(s)}$ is 1-consistent, we get a solution, which contradicts the fact that Θ has no solutions in $D^{(s)}$.

If we have a nontrivial binary absorbing subuniverse, or a nontrivial center, or a nontrivial PC subuniverse on some domain $D_x^{(s)}$, then by Theorems 9.2 and 9.3, there exists a minimal nonlinear 1-consistent reduction $D^{(s+1)}$ for Θ . As we explained before, $\text{Size}(D^{(s+1)}) < \text{Size}(D^{(s)})$.

Then, by Lemma 8.6, $\Theta'^{(s)}$ is cycle-consistent and irreducible. By Theorem 9.7, Θ' has a solution in $D^{(s+1)}$. Hence, $\rho(x_1, \dots, x_n)$ is crucial in $D^{(s+1)}$. By the inductive assumption ρ has the parallelogram property.

It remains to consider the case when $\text{ConLin}(D_x^{(s)})$ is proper for every x such that $|D_x^{(s)}| > 1$. Let α be a solution of Θ' in $D^{(s)}$. Let the projection of α onto the variables x_1, \dots, x_n be (a_1, \dots, a_n) .

Assume that ρ does not have the parallelogram property. Without loss of generality, we can assume that there exist c_1, \dots, c_n and d_1, \dots, d_n such that

$$\begin{aligned} (c_1, \dots, c_k, c_{k+1}, \dots, c_n) &\notin \rho, \\ (c_1, \dots, c_k, d_{k+1}, \dots, d_n) &\in \rho, \\ (d_1, \dots, d_k, c_{k+1}, \dots, c_n) &\in \rho, \\ (d_1, \dots, d_k, d_{k+1}, \dots, d_n) &\in \rho. \end{aligned}$$

Put

$$\begin{aligned} \rho'(x_1, \dots, x_n) &= \exists y_1 \dots \exists y_n \rho(x_1, \dots, x_k, y_{k+1}, \dots, y_n) \wedge \\ &\rho(y_1, \dots, y_k, x_{k+1}, \dots, x_n) \wedge \rho(y_1, \dots, y_k, y_{k+1}, \dots, y_n). \end{aligned}$$

Obviously, $\rho \subsetneq \rho'$ and $\rho' \in \Gamma$, and therefore $(a_1, \dots, a_n) \in \rho'$. Hence, there exist b_1, \dots, b_n such that

$$\begin{aligned} (a_1, \dots, a_k, b_{k+1}, \dots, b_n) &\in \rho, \\ (b_1, \dots, b_k, a_{k+1}, \dots, a_n) &\in \rho, \\ (b_1, \dots, b_k, b_{k+1}, \dots, b_n) &\in \rho. \end{aligned}$$

By Lemma 8.28, there exists a tuple $(e_1, \dots, e_n) \in \rho$ such that $(a_i, e_i) \in \text{ConLin}(D_{x_i}^{(s)})$ for every i .

Consider the minimal linear reduction $D^{(s+1)}$ of $\Theta^{(s)}$ such that $\alpha \in D^{(s+1)}$. Then $(e_1, \dots, e_n) \in \rho^{(s+1)}$, and by Lemma 8.5, $D^{(s+1)}$ is a 1-consistent reduction of $\Theta^{(s)}$. Since Θ' has a solution in $D^{(s+1)}$, $\rho(x_1, \dots, x_n)$ is crucial in $D^{(s+1)}$. We get a longer minimal strategy with smaller $\text{Size}(D^{(s+1)})$, hence by the inductive assumption the relation ρ is a critical relation with the parallelogram property. \square

THEOREM 9.9. Suppose $D^{(0)}, \dots, D^{(s)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ , $\Upsilon(x_1, \dots, x_n)$ is a subconstraint of Θ , the solution set of $\Upsilon^{(s)}$ is subdirect, $k \in \{1, 2, \dots, n-1\}$, $\text{Var}(\Upsilon) = \{x_1, \dots, x_n, u_1, \dots, u_t\}$,

$$\Omega = \Upsilon_{x_1, \dots, x_k, u_1, \dots, u_t}^{y_1, \dots, y_k, v_1, \dots, v_t} \wedge \Upsilon_{x_{k+1}, \dots, x_n, u_1, \dots, u_t}^{y_{k+1}, \dots, y_n, v_{t+1}, \dots, v_{2t}} \wedge \Upsilon_{x_1, \dots, x_n, u_1, \dots, u_t}^{y_1, \dots, y_n, v_{2t+1}, \dots, v_{3t}},$$

and $\Theta^{(s)}$ has no solutions. Then $(\Theta \setminus \Upsilon) \cup \Omega$ has no solutions in $D^{(s)}$.

PROOF. Put $\Theta' = (\Theta \setminus \Upsilon) \cup \Omega$. Since Ω is a covering of Υ , Lemma 6.1 implies that $\Theta \cup \Omega$ is cycle-consistent and irreducible. Assume that Θ' has a solution in $D^{(s)}$.

We recursively build a strategy $D^{(s)}, D^{(s+1)}, \dots, D^{(q)}$ for $\Theta \cup \Omega = \Theta' \cup \Upsilon$ satisfying the following conditions:

- (1) $D^{(s)}, D^{(s+1)}, \dots, D^{(q)}$ is a minimal strategy for $\Theta'^{(s)}$;
- (2) if $s \leq j < q$ and $D^{(j+1)}$ is a linear reduction, then for each $i \in \{1, 2, \dots, t\}$

$$D_{u_i}^{(j+1)} = \text{pr}_{n+i} \left(\rho' \cap \left(D_{x_1}^{(j+1)} \times \dots \times D_{x_n}^{(j+1)} \times D_{u_1}^{(j)} \times \dots \times D_{u_t}^{(j)} \right) \right),$$

where ρ' is the relation defined by $\Upsilon(x_1, \dots, x_n, u_1, \dots, u_t)$;

- (3) the solution set of $\Upsilon^{(j)}$ is subdirect for $s \leq j \leq q$;
- (4) Θ' has a solution in $D^{(q)}$.

Note that here we allow $D^{(j)}$ to be equal to $D^{(j+1)}$ in a strategy, which can happen if $D^{(j+1)}$ is a proper reduction for $\Upsilon^{(j)}$ but not proper for $\Theta'^{(j)}$.

We will prove that we can make this sequence longer while $|D_{x_i}^{(q)}| > 1$ for some i . By Theorem 5.1, there exists a nontrivial one-of-four subuniverse on $D_x^{(q)}$ if $|D_x^{(q)}| > 1$. We consider two cases:

Case 1. There exists a nontrivial binary absorbing subuniverse, or a nontrivial center, or a nontrivial PC congruence on some domain $D_x^{(q)}$. Then applying Theorems 9.2, 9.3 to the strategy $D^{(0)}, D^{(1)}, \dots, D^{(q)}$ of $\Theta \cup \Omega$, we conclude that there exists a minimal 1-consistent nonlinear reduction $D^{(q+1)}$ for $(\Theta \cup \Omega)^{(q)}$. By Lemma 8.6, $\Theta'^{(q)}$ and $\Upsilon^{(q)}$ are cycle-consistent and irreducible. By Theorem 9.7, Θ' has a solution in $D^{(q+1)}$ and Υ has a solution in $D^{(q+1)}$. By Lemma 8.5, the solution set of $\Upsilon^{(q+1)}$ is subdirect. Thus, we made the sequence longer.

Case 2. $\text{ConLin}(D_x^{(q)})$ is proper for every x such that $|D_x^{(q)}| > 1$. Let α be a solution of Θ' in $D^{(q)}$. We define the new linear reduction $D^{(q+1)}$ as follows. For all variables but u_1, \dots, u_t , we choose an equivalence class of $\text{ConLin}(D_x^{(q)})$ containing the corresponding element of the solution α . For the variable u_i we define $D_{u_i}^{(q+1)}$ by the formula in (2) from the above list for $j = q$. By Lemma 8.5, $D^{(q+1)}$ is 1-consistent for Θ' . Note that it does not follow from the definition that $D_{u_i}^{(q+1)}$ is not empty and we will prove this later.

Let the projection of α onto the variables x_1, \dots, x_n be (a_1, \dots, a_n) . Suppose $\Upsilon^{(s)}(x_1, \dots, x_n)$ defines a relation ρ . Since α is a solution of $\Theta'^{(s)}$, there exist b_1, \dots, b_n such that

$$\begin{aligned} (a_1, \dots, a_k, b_{k+1}, \dots, b_n) &\in \rho, \\ (b_1, \dots, b_k, a_{k+1}, \dots, a_n) &\in \rho, \\ (b_1, \dots, b_k, b_{k+1}, \dots, b_n) &\in \rho. \end{aligned}$$

Since the solution set of $\Upsilon^{(j)}$ is subdirect for $s \leq j \leq q$, we can apply Lemma 8.28 to ρ and the strategy $D^{(s)}, \dots, D^{(q)}$. Hence, there exists a tuple $(d_1, \dots, d_n) \in \rho$ such that $(a_i, d_i) \in \text{ConLin}(D_{x_i}^{(q)})$ for every i . Therefore, $(\Upsilon^{(s)}(x_1, \dots, x_n))^{(q+1)}$ is not empty.

Let us show by induction on $j = s, s + 1, \dots, q$ that $(\Upsilon^{(j)}(x_1, \dots, x_n))^{(q+1)}$ is not empty. For $j = s$, we already know this. Assume that $(\Upsilon^{(j)}(x_1, \dots, x_n))^{(q+1)}$ is not empty. If the reduction $D^{(j+1)}$ is not linear, then we apply Theorem 8.26 to $(\Upsilon^{(j)}(x_1, \dots, x_n))^{(q+1)}$ and the strategy $D^{(s)}, \dots, D^{(q)}$, and obtain that $(\Upsilon^{(j+1)}(x_1, \dots, x_n))^{(q+1)}$ is not empty. If the reduction $D^{(j+1)}$ is linear, then it follows from the definition of $D_{u_i}^{(j+1)}$ that $(\Upsilon^{(j+1)}(x_1, \dots, x_n))^{(q+1)}$ is not empty. Thus, we can prove that $(\Upsilon^{(q)}(x_1, \dots, x_n))^{(q+1)}$ is not empty, and therefore $D_{u_i}^{(q+1)}$ is not empty for every i . Considering the solution set of Υ and applying Corollary 7.24.1, we derive that $D_{u_i}^{(q+1)}$ is a linear subuniverse of $D_{u_i}^{(q)}$. Hence, the reduction $D^{(q+1)}$ is a 1-consistent linear reduction for $\Upsilon^{(q)}$.

By Lemma 8.5, $(\Upsilon^{(q)}(x_1, \dots, x_n))^{(q+1)}$ is subdirect. From the definition of $D_{u_i}^{(q+1)}$ we derive that the projection of the solution set of $\Upsilon^{(q+1)}$ onto u_i is $D_{u_i}^{(q+1)}$ for every i , which means that the solution set of $\Upsilon^{(q+1)}$ is subdirect. Hence, we get a longer strategy having all the necessary properties.

Thus, we showed that we can make the sequence longer until $|D_{x_i}^{(q)}| = 1$ for every i . Assume that we reached this final state. Since both Υ and Θ' have a solution in $D^{(q)}$ and x_1, \dots, x_n are their only common variables, Θ has a solution in $D^{(q)}$, which contradicts the fact that Θ has no solutions in $D^{(s)}$. \square

9.3 Theorems from Section 5

In this subsection, we assume that the variables of the instance Θ are x_1, \dots, x_n , and the domain of x_i is D_i for every i . The first two theorems are proved together.

THEOREM 5.5. *Suppose Θ is a cycle-consistent irreducible CSP instance, and B is a nontrivial binary absorbing subuniverse or a nontrivial center of D_i . Then Θ has a solution if and only if Θ has a solution with $x_i \in B$.*

THEOREM 5.6. *Suppose Θ is a cycle-consistent irreducible CSP instance, there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on D_j for every j , $(D_i; w)/\sigma$ is a polynomially complete algebra, and E is an equivalence class of σ . Then Θ has a solution if and only if Θ has a solution with $x_i \in E$.*

PROOF. By Theorems 9.2, 9.3, there exists a minimal 1-consistent nonlinear reduction $D^{(1)}$ such that $D_{x_i}^{(1)} \subseteq B$ for Theorem 5.5, and $D_{x_i}^{(1)} \subseteq E$ for Theorem 5.6. By Theorem 9.7, there exists a solution in $D^{(1)}$. \square

The next theorem will be used in the proof of Theorem 5.7 from Section 5.

THEOREM 9.15. *Suppose the following conditions hold:*

- (1) Θ is a linked cycle-consistent irreducible CSP instance;
- (2) there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on D_j for every j ;
- (3) if we replace every constraint of Θ by all weaker constraints then the obtained instance has a solution with $x_i = b$ for every i and $b \in D_i$ (the obtained instance has a subdirect solution set);
- (4) $D^{(1)}$ is a minimal linear reduction for Θ ;
- (5) Θ is crucial in $D^{(1)}$.

Then there exists a constraint $\rho(x_{i_1}, \dots, x_{i_s})$ in Θ and a subuniverse ζ of $D_{i_1} \times \dots \times D_{i_s} \times \mathbb{Z}_p$ such that the projection of ζ onto the first s coordinates is bigger than ρ but the projection of $\zeta \cap (D_{i_1} \times \dots \times D_{i_s} \times \{0\})$ onto the first s coordinates is equal to ρ .

PROOF. We consider two cases. Case 1. Assume that Θ contains just one constraint $\rho(x_1, \dots, x_n)$. By Corollary 7.24.1, $D'_n = \text{pr}_n(\rho \cap (D_1^{(1)} \times \dots \times D_{n-1}^{(1)} \times D_n))$ is a linear subuniverse of D_n . By Lemma 7.20, $D_n^{(1)}$ and D'_n can be viewed as products of affine subspaces and can be defined by linear equations. Since $D_n^{(1)} \cap D'_n = \emptyset$ and $D_n^{(1)}$ is a minimal reduction, we can take an equation defining D'_n that does not hold on $D_n^{(1)}$ to get a maximal linear congruence σ on D_n such that $D_n^{(1)}$ and D'_n are in different equivalence classes of σ . Note that $D_n/\sigma \cong \mathbb{Z}_p$ for some p . Let ψ be the corresponding homomorphism from D_n to \mathbb{Z}_p . Put

$$\zeta(x_1, \dots, x_n, z) = \exists x'_n \rho(x_1, \dots, x_{n-1}, x'_n) \wedge (\psi(x_n) = \psi(x'_n) + z),$$

where the expression $(\psi(x_n) = \psi(x'_n) + z)$ defines ternary subalgebra of $D_n \times D_n \times \mathbb{Z}_p$. Thus, we have ρ and ζ with the required properties.

Case 2. Θ contains more than one constraint. Then by condition (5), every constraint $C^{(1)}$ is not empty, which by Lemma 8.5 implies that $C^{(1)}$ is subdirect. Then $D^{(1)}$ is a minimal 1-consistent linear reduction. By Theorem 9.8, every constraint in Θ is critical and has the parallelogram property. If Θ is not connected, then by Theorem 9.6 there exists an instance $\Theta' \in \text{ExpCov}(\Theta)$ that is crucial in $D^{(1)}$ and contains a linked connected component Ω such that the solution set of Ω is not subdirect. By condition (3), since the solution set of Ω is not subdirect, Ω should contain a constraint relation from the original instance Θ . If Θ is connected, then Θ is a linked connected component itself and we put $\Omega = \Theta$. Thus, in both cases we have a linked connected instance Ω having a constraint relation ρ from Θ . Let $\rho(x_{i_1}, \dots, x_{i_s})$ be a constraint of Θ .

By Lemma 8.11, $\text{Con}(\rho, 1)$ is an irreducible congruence. By Corollary 8.22.1, there exists a bridge δ from $\text{Con}(\rho, 1)$ to $\text{Con}(\rho, 1)$ such that δ is a full relation. By Corollary 8.17.1, there exists a relation $\xi \subseteq D_{i_1} \times D_{i_1} \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \xi \Leftrightarrow (x_1, x_2) \in \text{Con}(\rho, 1)$ and $\text{pr}_{1,2}(\xi) = \text{Con}(\rho, 1)^*$.

It remains to put $\zeta(x_{i_1}, \dots, x_{i_s}, z) = \exists x'_{i_1} \rho(x'_{i_1}, x_{i_2}, \dots, x_{i_s}) \wedge \xi(x_{i_1}, x'_{i_1}, z)$. \square

THEOREM 5.7. *Suppose the following conditions hold:*

- (1) Θ is a linked cycle-consistent irreducible CSP instance with domain set (D_1, \dots, D_n) ;
- (2) there does not exist a nontrivial binary absorbing subuniverse or a nontrivial center on D_j for every j ;
- (3) if we replace every constraint of Θ by all weaker constraints then the obtained instance has a solution with $x_i = b$ for every i and $b \in D_i$ (the obtained instance has a subdirect solution set);
- (4) $L_i = D_i/\sigma_i$ for every i , where σ_i is the minimal linear congruence on D_i ;
- (5) $\phi: \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k} \rightarrow L_1 \times \dots \times L_n$ is a homomorphism, where q_1, \dots, q_k are prime numbers;
- (6) if we replace any constraint of Θ by all weaker constraints then for every $(a_1, \dots, a_k) \in \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k}$ there exists a solution of the obtained instance in $\phi(a_1, \dots, a_k)$.

Then $\{(a_1, \dots, a_k) \mid \Theta \text{ has a solution in } \phi(a_1, \dots, a_k)\}$ is either empty, or is full, or is an affine subspace of $\mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k}$ of codimension 1 (the solution set of a single linear equation).

PROOF. Put $B = \{(a_1, \dots, a_k) \mid \Theta \text{ has a solution in } \phi(a_1, \dots, a_k)\}$. If B is full, then there is nothing to prove. Assume that B is not full, then consider $(b_1, \dots, b_k) \notin B$. It follows from condition (6) that Θ is crucial in $\phi(b_1, \dots, b_k)$. Note that $\phi(b_1, \dots, b_k)$ defines a minimal linear reduction for Θ .

By Theorem 9.15 there exists a constraint $\rho(x_{i_1}, \dots, x_{i_s})$ in Θ and a subuniverse ζ of $D_{i_1} \times \dots \times D_{i_s} \times \mathbb{Z}_p$ such that the projection of ζ onto the first s coordinates is bigger than ρ but the projection of $\zeta \cap (D_{i_1} \times \dots \times D_{i_s} \times \{0\})$ onto the first s coordinates is equal to ρ .

Then we add a new variable z with domain \mathbb{Z}_p and replace $\rho(x_{i_1}, \dots, x_{i_s})$ by $\zeta(x_{i_1}, \dots, x_{i_s}, z)$. We denote the obtained instance by Υ . Let L be the set of all tuples $(a_1, \dots, a_k, b) \in \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_k} \times \mathbb{Z}_p$ such that Υ has a solution with $z = b$ in $\phi(a_1, \dots, a_k)$. We know that the projection of L onto the first k coordinates is a full relation and $(b_1, \dots, b_k, 0) \notin L$. Therefore, L is defined by one linear equation. If this equation is $z = b$ for some $b \neq 0$, then B is empty. Otherwise, we put $z = 0$ in this equation and get an equation describing all (a_1, \dots, a_k) such that Θ has a solution in $\phi(a_1, \dots, a_k)$. \square

10 CONCLUSIONS

Even though the main problem has been resolved, there are many important questions that are still open. In this section, we will discuss some consequences of this result, as well as some open questions and generalizations of the CSP.

10.1 A General Algorithm for the CSP

The algorithm presented in the article, as well as the algorithm of Andrei Bulatov [20, 21], uses detailed knowledge of the algebra and depends exponentially on the size of the domain. Is there a “truly polynomial algorithm”? By CSP-WNU, we denote the following decision problem: Given a formula

$$\rho_1(v_{1,1}, \dots, v_{1,n_1}) \wedge \dots \wedge \rho_s(v_{s,1}, \dots, v_{1,n_s}),$$

where all relations ρ_1, \dots, ρ_s are preserved by a WNU (we just know it exists); decide whether this formula is satisfiable.

PROBLEM 1. *Does there exist a polynomial algorithm for CSP-WNU?*

If the domain is fixed, then CSP-WNU can be solved by the algorithm presented in this article. In fact, we know from Reference [2, Theorem 4.2] that from a WNU on a domain of size k we can always derive a WNU (and also a cyclic operation) of any prime arity greater than k . Thus, we can find finitely many WNU operations on domain of size k such that any constraint language preserved by a WNU is preserved by one of them. It remains to apply the algorithm for each WNU and return a solution if one of them gave a solution.

10.2 A Simplification of the Algorithm

We believe that the algorithm presented in the article can be simplified. For instance, we strongly believe that the function `WEAKENEVERYCONSTRAINT` can be removed from the main function `SOLVE` without any consequences.

PROBLEM 2. *Would the algorithm still work if the function `WEAKENEVERYCONSTRAINT` was removed from the function `SOLVE`?*

This would reduce the complexity of the algorithm significantly (the depth of the recursion would be $|A|$ instead of $|A| + |\Gamma|$, see Lemma 5.2).

10.3 A Generalization for the nonWNU Case

Another important question is whether some results and ideas introduced in this article can be applied for constraint languages not preserved by a WNU. For example, it is not clear what assumptions are sufficient to reduce safely a domain to a binary absorbing subuniverse.

PROBLEM 3. *What are the weakest assumptions for Theorems 5.5 and 5.6 to hold.*

10.4 Infinite Domain CSP

If we allow the domain to be infinite, then the situation is changing significantly. As it was shown in Reference [10] every computational problem is equivalent (under polynomial-time Turing reductions) to a problem of the form $\text{CSP}(\Gamma)$. In Reference [12], the authors gave a nice example of a constraint language Γ such that $\text{CSP}(\Gamma)$ is undecidable. Let Γ consists of three relations (predicates) $x + y = z$, $x \cdot y = z$ and $x = 1$ over the set of all integers \mathbb{Z} . Then the Hilbert's 10th problem can be expressed as $\text{CSP}(\Gamma)$, which proves undecidability of $\text{CSP}(\Gamma)$.

A reasonable assumption on Γ that sends the CSP back to the class NP is that Γ is a reduct of a finitely bounded homogeneous structure. A nice result for such constraint languages is the full complexity classification of the CSPs over the reducts of $(\mathbb{Q}; <)$ [11]. This additional assumption allows to formulate a statement of the algebraic dichotomy conjecture for the complexity of the infinite domain CSP [7]. For more information about the infinite domain CSP and the algebraic approach see References [9, 12]. For a method of reducing an infinite domain CSP to CSPs over finite domains see Reference [51].

10.5 Valued CSP

A natural generalization of the Constraint Satisfaction Problem is the *Valued Constraint Satisfaction Problem* (VCSP), where constraint relations are replaced by mappings to the set of rational numbers, and conjunctions are replaced by sum [55]. For a finite set A and a set Γ of mappings $A \rightarrow \mathbb{Q} \cup \{\infty\}$ by $\text{VCSP}(\Gamma)$, we denote the following problem: Given a formula

$$f(x_1, \dots, x_n) = f_1(v_{1,1}, \dots, v_{1,n_1}) + \dots + f_s(v_{s,1}, \dots, v_{s,n_s}),$$

where all the mappings f_1, \dots, f_s are from Γ and $v_{i,j} \in \{x_1, \dots, x_n\}$ for every i, j ; find an assignment (a_1, \dots, a_n) that minimizes $f(x_1, \dots, x_n)$.

In Reference [42, Theorem 21], the authors proved that the dichotomy conjecture for CSP would imply the dichotomy conjecture for the Valued CSP and described all sets of mappings Γ such that $\text{VCSP}(\Gamma)$ is tractable (modulo the CSP Dichotomy Conjecture). Thus, the result obtained in this article implies the characterization of the complexity of $\text{VCSP}(\Gamma)$ for all Γ .

10.6 Quantified CSP

An equivalent definition of $\text{CSP}(\Gamma)$ is to evaluate a sentence $\exists x_1 \dots \exists x_n (\rho_1(\dots) \wedge \dots \wedge \rho_s(\dots))$, where ρ_1, \dots, ρ_s are from the constraint language Γ . Then a natural generalization of CSP is the *Quantified Constraint Satisfaction Problem* (QCSP), where we allow to use both existential and universal quantifiers. For a constraint language Γ , $\text{QCSP}(\Gamma)$ is the problem to evaluate a sentence of the form $\forall x_1 \exists y_1 \dots \forall x_n \exists y_n (\rho_1(\dots) \wedge \dots \wedge \rho_s(\dots))$, where ρ_1, \dots, ρ_s are relations from the constraint language Γ (see References [15, 25, 26, 49]).

It was conjectured by Hubie Chen [24, 26] that for any constraint language Γ the problem $\text{QCSP}(\Gamma)$ is either solvable in polynomial time, or NP-complete, or PSpace-complete. Recently, this conjecture was disproved in Reference [64], where the authors found constraint languages Γ such that $\text{QCSP}(\Gamma)$ is coNP-complete (on 3-element domain), DP-complete (on 4-element domain), and Θ_2^P -complete (on 10-element domain). Also the authors classified the complexity of the Quantified Constraint Satisfaction Problem for constraint languages on 3-element domain containing all unary singleton relations (so called idempotent case), that is, they showed that for such languages $\text{QCSP}(\Gamma)$ is either tractable, or NP-complete, or coNP-complete, or PSpace-complete. Nevertheless, for higher domain as well as for the nonidempotent case the complexity is not known.

PROBLEM 4. *What can be the complexity of $\text{QCSP}(\Gamma)$?*

Now it is hard to believe that there will be a simple answer to this question, and that is why it is interesting to start with 3-element domain (nonidempotent case) and 4-element domain. Another natural question is how many complexity classes can be expressed by $\text{QCSP}(\Gamma)$ up to polynomial equivalence. Probably more important problem is to describe all tractable cases.

PROBLEM 5. *Describe all constraint languages Γ such that $\text{QCSP}(\Gamma)$ is tractable.*

10.7 Promise CSP

Another natural generalization of the CSP is *the Promise Constraint Satisfaction Problem*, where a promise about the input is given (see References [16, 23]). Let $\Gamma = \{(\rho_1, \sigma_1), \dots, (\rho_t, \sigma_t)\}$, where ρ_i and σ_i are relations of the same arity over the domains A and B , respectively. Then $\text{PCSP}(\Gamma)$ is the following decision problem: Given two formulas

$$\begin{aligned} &\rho_{i_1}(v_{1,1}, \dots, v_{1,n_1}) \wedge \dots \wedge \rho_{i_s}(v_{s,1}, \dots, v_{s,n_s}), \\ &\sigma_{i_1}(v_{1,1}, \dots, v_{1,n_1}) \wedge \dots \wedge \sigma_{i_s}(v_{s,1}, \dots, v_{s,n_s}), \end{aligned}$$

where $(\rho_{i_j}, \sigma_{i_j})$ are from Γ for every i and $v_{i,j} \in \{x_1, \dots, x_n\}$ for every i, j , distinguish between the case when both of them are satisfiable and when both of them are not satisfiable. Thus, we are given two CSP instances and a promise that if one has a solution then another has a solution. Usually it is also assumed that there exists a mapping (homomorphism) $h : A \rightarrow B$ such that $h(\rho_i) \subseteq \sigma_i$ for every i . In this case, the satisfiability of the first formula implies the satisfiability of the second one. To make sure that the promise can actually make an NP-hard problem tractable, see example 2.8 in Reference [23].

The most popular example of the Promise CSP is graph (k, l) -colorability, where we need to distinguish between k -colorable graphs and not even l -colorable, where $k \leq l$. This problem can be written as follows.

PROBLEM 6. *Let $|A| = k$, $|B| = l$, $\Gamma = \{(\neq_A, \neq_B)\}$. What is the complexity of $\text{PCSP}(\Gamma)$?*

Recently, it was proved [23] that (k, l) -colorability is NP-hard for $l = 2k - 1$ and $k \geq 3$ but even the complexity of $(3, 6)$ -colorability is still not known.

Even for two element domain the problem is widely open, but recently a dichotomy for symmetric Boolean PCSP was proved [30].

PROBLEM 7. *Let $A = B = \{0, 1\}$. Describe the complexity of $\text{PCSP}(\Gamma)$ for all Γ .*

10.8 Surjective CSP

Another modification of the CSP is *the Surjective Constraint Satisfaction Problem*. For a constraint language Γ over a domain A , $\text{SurjCSP}(\Gamma)$ is the following decision problem: Given a formula

$$\rho_1(\dots) \wedge \dots \wedge \rho_s(\dots),$$

where all relations ρ_1, \dots, ρ_s are from Γ , decide whether there exists a surjective solution, that is, a solution with $\{x_1, \dots, x_n\} = A$. Only few results are known about the complexity of the Surjective CSP [27]. That is why we suggest to start studying this question with a very concrete constraint language on a 3-element domain.

PROBLEM 8. *Suppose $A = \{a, b, c\}$, $R = \{(x, y, z) \mid \{x, y, z\} \neq A\}$. What is the complexity of $\text{SurjCSP}(\{R\})$?*

After this problem (called *no-rainbow problem*) we can move to the general question.

PROBLEM 9. *Describe the complexity of $\text{SurjCSP}(\Gamma)$ for all constraint languages Γ .*

ACKNOWLEDGMENTS

I dedicate this article to my father, Nikolay Zhuk, who was a great chemist and a great person. He explained me trigonometric functions when I was in primary school, he gave me a programmable calculator on which I wrote my first program, and he taught me to love science.

I am very grateful to Zarathustra Brady, whose comments and remarks allowed me to fill many gaps in the original proof and to significantly improve the text. Also, I want to thank my colleagues and friends for very fruitful discussions, especially Andrei Bulatov, Marcin Kozik, Libor Barto, Ross Willard, Jakub Opršal, Jakub Bulín, Valeriy Kudryavtsev, Alexey Galatenko, Stanislav Moiseev, and Grigoriy Bokov. Additionally, I want to thank the referees of the article, whose suggestions helped me to rewrite the whole article. They deserve to be the authors of this article.

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Received December 2017; revised May 2020; accepted May 2020