The Origin of Mathematics

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I. Some background

1. The two traditions. There are two great traditions, easily discernible, in the history of mathematics: the geometric or constructive and the algebraic or computational. If it could be shown that each of these has a single source – and there are many rather familiar facts that suggest that this is so – and if, moreover, in both cases the sources turn out to be the *same*, it would be plausible to claim that we have found the unique origin of mathematics. That is what I propose to do.

The student who has taken our usual high-school courses in mathematics will recognize the two traditions easily enough. In the second year he studies plane geometry, where he encounters something quite different from his studies in algebra. Previously he computed, whereas now he constructs – perpendiculars, angle bisectors, *etc.*; and he deals with *theorems* and *proofs*. In algebra he learned that "minus times minus is plus," but this wasn't proved or even called a theorem. At best there was a rough explanation, or more probably he was simply told that that's the way it is.

From a strictly mathematical point of view there can hardly be any reason why proof should enter geometry and not algebra, and one might speculate that the notion of proof first arose in the geometric tradition (and then passed to the algebraic). Such a guess will hardly come as a surprise to the readers accustomed to look to Classical Greece for the origin of the constructive tradition.

Although proof is postponed to high-school, actually the student is introduced to some elements of geometry already in an earlier grade. There he is equipped with ruler and compass, and with protractor. He is taught some geometric constructions, for example, how to construct the circumscribed circle of a given triangle: this belongs to Tradition I. If he bisects an angle with ruler and compass, this is Tradition I; if he uses the protractor, Tradition II comes in. The teacher explains that the circumference of a circle is a little over three, and nearly $3^1/_7$, times the diameter; this is in Tradition II. EUCLID's *Elements* has no corresponding theorem.

The Elements, Book I, Prop. 41, says that "if a parallelogram have the same base with a triangle and be in the same parallels, then the parallelogram is double the triangle." Using this theorem, the student will easily prove that the locus of a point P in a given plane, and on a given side of a given line AB in the plane, and such that the triangle ABP has a given area, is a straight line. Later, when he has learned some Analytic Geometry, he will be able to do the same thing basing himself on the grounds that the area of a triangle is one half base times altitude. Here Tradition II comes in: EUCLID has no theorem saying that the area of a triangle is one half base times altitude. He could not very well have, since he never attaches numbers to line segments.

Now, having glanced at the curriculum in the light of the two traditions, let us do the same for the history of mathematics in the modern era. Recall that VIETA, in the sixteenth century, introduced literal notation. Curiously, he maintained a principle of homogeneity for his equations: all the terms in an equation should have the same weight. For example, in a quadratic equation, if the unknown represents a length, and if the coefficient of the square of the unknown is (say) a number, then the coefficient of the unknown should be a length; and the constant term, an area. This restriction will seem strange to a student of high-school algebra, and it would have puzzled an Old-Babylonian of 1700 B.C., too, who freely added areas to lengths (and even men to days).[1] It would have seemed right to the Greeks, though. They had a kind of algebra, but it was an algebra of line segments, areas, and volumes, rather than numbers. In this algebra one added segments to segments, areas to areas, volumes to volumes; but it would not have made much sense to add areas to segments. It looks then as if VIETA was laboring under the tradition of this ancient Greek algebra.

DESCARTES dispensed with the homogeneity. He takes an arbitrary segment as unit length and associates with every other segment a number in accordance with its length. He then explains in geometric terms, following the lines of Tradition I, how to add, subtract, multiply, and divide, and how to take a square root. For example, to multiply a by b, one finds the fourth proportional x to 1, a, b, by a geometric construction involving similar triangles. And now one can multiply the product ab, which appears as a segment, by a third number (or segment) c, and the result by a fourth d, *etc.*

This is so simple, one wonders why the Greeks didn't do it. An obvious suggestion is that the Greeks did not have the concept of a real number; but then neither did DESCARTES. The Greeks, however, did have a definition of number

[1] O. NEUGEBAUER, *The Exact Sciences in Antiquity*, 2nd ed., pp. 51 and 42.

– roughly, a *number* was what we would call a positive integer – and adhered to it; DESCARTES has no definition and spends no words to explain what he means by number. The problem of the ratio of line segments, which appears to be so easily (though tacitly) resolved in DESCARTES' *Geometry*, had a long history in Greece, culminating in the theory of general quantities exposed in *The Elements*, Book V. According to PLATO (*The Republic* 525E), the "experts" refused to "cut up the 'one'" and laughed at anyone who tried. They would have laughed at DESCARTES. Was then DESCARTES an ignoramus? One can hardly entertain such a notion. DESCARTES was a very learned man and surely had read Book V of *The Elements*. But Greek mathematics had died some thousand years earlier, Tradition I did not lie heavily on DESCARTES, and he missed the import of Book V. Moreover, he was heir to Tradition II also, which since Old-Babylonian times associated numbers to line segments. One may criticize DESCARTES if one wishes, or wryly wonder why progress so often depends on misunderstanding, or partial understanding, but the fusion of the two traditions had been accomplished; or, as it is usually put, DESCARTES combined geometry and algebra. The 'one' had been put into (or, as some might say, back into) geometry.

2. View on the origin of geometry in 1900. Not so long ago, say about 75 years, the thesis that mathematics had a single origin would have been taken as a foregone conclusion, since with some minor exceptions, or what were taken to be minor exceptions, there were no competitors to Classical Greece. For example, W.W.R. BALL could easily bring himself to write:[2]

> The history of mathematics cannot with certainty be traced back to any school or period before that of the Ionian Greeks,

and this statement corresponds to a large extent with what was known about ancient mathematics in 1900.[3] Not entirely, however, for there were the *Śulvasūtras*, ancient Indian sacred works on altar constructions. BALL does not mention the *Śulvasūtras* and it is hard to say whether he had ever heard of them, but M. CANTOR, a leading historian of mathematics of the day, had. In 1875 G. THIBAUT had translated a large part of the Śulvasūtras, and these showed that the Indian priests possessed no little mathematical knowledge.[4] In 1877 CANTOR, realizing the importance of THIBAUT's work, began a comparative study of Greek and Indian mathematics.[5] He concluded that the Indian geometry was a derivative of Alexandrian knowledge, an opinion he held for some twenty-five years before finally renouncing it.

THIBAUT was a Sanskrit scholar and in translating the Śulvasūtras his principal object was to make available to the learned world the mathematical knowledge of

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[2] *A Short Account of the History of Mathematics*, 3rd ed. (1901), p. 1.

[3] For what extent, see my paper in the *Archive for History of Exact Sciences*, vol. **14** (1975), p. 285.

[4] "On the Śulvasūtras," *J. Asiatic Society of Bengal*, vol. 44:1. For a translation in full of the Baudhāyana *Śulvasūtra*, see *The Pandit*, vol. **9** (1874), vol. **10** (1875), n.s. vol. **1** (1876–77).

[5] "Gräco-indische Studien," *Zeitschrift für Mathematik und Physik* (Historisch-literarische Abt.), vol. **22** (1877); and "Über die älteste indische Mathematik," *Archiv für Mathematik und Physik*, vol. **8** (1904).

the Vedic Indians; but that wasn't his only object. After commenting that a good deal of Indian knowledge could be traced back to requirements of ritual, THIBAUT adds:[6]

> These facts have a double interest. They are in the first place valuable for the history of the human mind in general; they are in the second place important for the mental history of India and for answering the question relative to the originality of Indian science. For whatever is closely connected with the ancient Indian religion must be considered as having sprung up among the Indians themselves, unless positive evidence of the strongest kind point to a contrary conclusion.
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> We have been long acquainted with the progress which the Indians made in later times in arithmetic, algebra, and geometry; but as the influence of Greek science is clearly traceable in the development of their astronomy, and as their treatises on algebra, etc., form but parts of astronomical text books, it is possible that the Indians may have received from the Greeks also communications regarding the methods of calculation. I merely say possible, because no direct evidence of such influence has been brought forward as yet, and because the general impression we receive from a comparison of the methods employed by the Greeks and Indians respectively seems rather to point to an entirely independent growth of this branch of Indian science. The whole question is unsettled, and new researches are required before we can arrive at a final decision.
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> While therefore unable positively to assert that the treasure of mathematical knowledge contained in the Lilávati, the Vijagaṉita, and similar treatises, has been accumulated by the Indians without the aid of foreign nations, we must search whether there are not traces left pointing to a purely Indian origin of these sciences. And such traces we find in a class of writings, commonly called S'ulvasútras, that means 'sutras of the cord,' which prove that the earliest geometrical and mathematical investigations among the Indians arose from certain requirements of their sacrifices ...

My object at the moment is not to enter into a critique of THIBAUT's views, but merely to display them. THIBAUT himself never belabored, or elaborated, these views; nor did he formulate the obvious conclusion, namely, that the Greeks were not the inventors of plane geometry, rather it was the Indians. At least this was the message that the Greek scholars saw in THIBAUT's paper. And they didn't like it.

R.J. BRAIDWOOD, the well-known archeologist, has remarked that a hazard of his profession is for the archeologist to think that the place he himself has dug up, especially if it's older than anything else that's been dug up, represents the beginning of things.[7] The same holds true of the antiquarian. If the Indians invented plane geometry, what was to become of Greek "genius" or of the Greek "miracle"?

Most of the "refutations" were mere haughty dismissals, but CANTOR at least examined the evidence, and even gave arguments. He made many acute observations and concluded that Indian geometry and Greek geometry, especially of

[6] "On the Śulvasūtras," p. 228.

[7] "Jericho and its Setting in Near Eastern History," *Antiquity*, **31** (1957), p. 73.

HERON, are related. For CANTOR there remained only the question: Who borrowed from whom? He expresses the opinion that the Indians were, in geometry, the pupils of the Greeks.[8]

THIBAUT in 1875 had assigned no absolute date to the Śulvasūtras, thereby showing proper scholary restraint. Therefore CANTOR felt free to press his own chronology.[9] CANTOR had been struck by the analogy of the Indian altar problems to the Greek duplication of the altar and grave problems, problems he assigns to the fourth and fifth centuries B.C.[10] Now according to CANTOR HERON's geometry intruded about 100 B.C. into India, where it was given a theological form. This theologic-geometry then left traces in Greece in poetry ascribed (by CANTOR himself) to EURIPEDES (485–406) – a clear contradiction. Anyway, as already remarked, CANTOR eventually renounced his view and conceded a much earlier date to Indian geometry. Even so, he did not believe that PYTHAGORAS got his geometry from India: he preferred to believe it was Egypt.

One should not imagine that the battle lines were clearly drawn, with the Sanskrit scholars on one side and the Greek on the other. Far from it! Thus in 1884 A. WEBER, a Sanskrit scholar, expressed the opinion that there was "nothing of a literary-historical nature standing in the way of the assumption of a use [on the part of the Śulvasūtras] of the teachings of Hero of Alexandria."[11] In assessing this statement one must keep in mind that the Greek scholars, though Greek studies were already 400 years old, had not yet fixed HERON's data with much assurance, opinions varying from about 200 B.C. to about 200 A.D. WEBER himself gave 215 B.C., while CANTOR assumed 100 B.C. (Recently NEUGEBAUER (*Exact Sciences*, p. 171) has assigned 62 A.D. to HERON.) A.B. KEITH, another Sanskrit scholar, disputed a connection between the Śulvasūtras and the older sacred literature (in

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[8] For a summary of CANTOR's paper with comments, see my "Ritual Origin of Geometry," *Archive for History of Exact Sciences*, vol. **1** (1962), pp. 488–527.

[9] CANTOR starts his paper by reminding us that Greek studies were already about 400 years old, whereas Indian studies were only about 100; as a consequence Greek dates could usually be given within a decade, whereas estimates of Indian dates varied by centuries. But the two chronologies are not on the same footing! Greek dates are obtained largely from references of the ancients to their predecessors and contemporaries whereas Sanskrit chronology is of a "literary-historical" kind (sometimes astronomical considerations have played a part). Dating in terms of intervals of centuries is quite in order in the context of Sanskrit studies, and a variation in estimates is to be expected. Even so, Sanskrit scholars are pretty much in agreement. When THIBAUT says that the high point of the Vedic sacrificial practice was in the 12th Century, VON SCHROEDER insists that the Vedic peg and cord constructions go back to the 10th, BÜRK places the Samhitā period no later than the 8th, while KEITH puts it no later than the 6th; though four centuries have been mentioned, there is no conflict whatsoever.

[10] On the duplication problem, see M. CANTOR, *Vorlesungen über Geschichte der Mathematik* (1907), vol. 1, p. 212; B.L. VAN DER WAERDEN, *Science Awakening*, p. 159ff.; my "Ritual Origin of Geometry," p. 493f.; and a paper of mine, "The Geometry of the Vedic Rituals," to appear in *Agni, the Vedic Ritual of the Fire Altar*, a volume to be brought out by FRITS STAAL (ed.).

[11] Review of L. VON SCHROEDER's *Pythagoras und die Inder*, Literarisches Zentralblatt, **35** (1884), col. 1564.

particular, the Saṃhitās) and denied that it showed any knowledge of geometry.[12] The relations between the Śulvasūtras and the more ancient sacred literature is a vital point in attempts at a chronology, and I will return to it below (*cf.* note 42 and the associated text).

Finally in 1899 THIBAUT, perhaps prodded into it by estimates which he considered to be way off the mark, as for example the 100 B.C. of CANTOR, ventured to assign the fourth or the third centuries B.C. as the latest possible date for the composition of the Śulvasūtras (it being understood that this refers to a codification of far older materials).[13] But then he added: "There is nothing striking in the independent development of a limited amount of practical knowledge by two different peoples." A terrible statement! I cannot help thinking that it shows battle-weariness rather than a considered opinion. A "limited amount of practical knowledge" is surely not what CANTOR was thinking about, nor could it have been what THIBAUT was thinking about in 1875, else what is the relevance of "the mighty sway of religion" and "the requirements of the sacrifice"?[14] Is sacrifice practical knowledge? The Vedic priests wanted the form of their falcon-shaped brick altar to be geometrically exact (for, obviously, if a bird is not well-made, it will not fly), but did any Vedic priest ever expect a brick altar really to fly? No, THIBAUT was losing the thread. I think he was making a concession to CANTOR because of CANTOR's prestige, but perhaps he really did not know how to answer the scholarly objections.

In 1910–02 A. BÜRK translated the Śulvasūtra of ĀPASTAMBA, prefixing it with a commentary.[15] It was this work, according to CANTOR, that brought about a shift in the situation and led him to change his mind. This is to give BÜRK too much credit. BÜRK's paper is excellent and he does make original points, but the argument occurs in all its essential aspects already in THIBAUT's paper. Anyway, the damage had been done and the Śulvasūtras have never taken the position in the history of mathematics that they deserve. E.T. BELL (*Development of Mathematics*) and B.L. VAN DER WAERDEN (*Science Awakening*) do not so much as even mention them, though their contents are briefly described in CANTOR's *History*, which both cite. NEUGEBAUER mentions them briefly a couple of times (*cf.* n. 16). Perhaps with the date of 1700 B.C. for Old-Babylonia well in hand, the question of whether Vedic geometrical knowledge dates from 200 A.D., or 100 B.C., or 300 B.C., or 500 B.C., or even a 1000 B.C., faded into insignificance.

What then was the view on the origin of geometry in 1900, or even in 1904? The Greeks themselves had supposed, or conjectured, that they had received their intellectual capital, especially in geometry, from the more Ancient East, but modern historians have been hard put to corroberate their views. CANTOR with great acuteness conjectured (*op. cit.* 1904) that in very ancient times ("roughly speaking three or four thousand years ago") there already existed a not altogether

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[12] See his review of VON SCHROEDER's *Kathaka Saṃhitā*, *J. Royal* Asiatic Society, 1910, pp. 519–521.

[13] "Astronomie, Astrologie und Mathematik," *Grundriß der Indo-Ärischen Philologie und Altertumskunde*, vol. **3.9**, p. 78.

[14] Nor is it what NEUGEBAUER and VAN DER WAERDEN are thinking about when, especially in view of the information coming in from Old-Babylonia, they go to great lengths to explain why, for highly technical reasons, Greek geometry took the form it did.

[15] *Zeit. der deutschen morgenländischen Ges.*, vol. **55** (1901), vol. **56** (1902).

insignificant mathematical knowledge common to the whole cultured area of that time; but this was based on most scanty materials, indeed. So the Ancient East may have made some minor contributions, often referred to in the literature as "empirical," but the prevailing view was (and indeed remains) that we owe geometry as a Science to the genius of the Greeks.

3. View on the origin of geometry in 1950. In 1928 NEUGEBAUER published a paper on the history of the Pythagorean theorem.[16] He was already busy with the translation of Old-Babylonian mathematical texts. The main point of the paper is to disclose the existence of the Theorem of PYTHAGORAS well over a thousand years before PYTHAGORAS. In this connection he mentions the Śulvasūtras and writes that "the difficulties [involved in the view] of a direct borrowing by the Greeks from India fall away on the assumption of a common origin in Babylonia." This is really an excellent suggestion, and the only trouble with it (as we shall see in more detail below) is that many of the common elements of Greek and Indian mathematics, and especially the geometrical constructions at issue, are not found in Old-Babylonia.

The monumental work of NEUGEBAUER published in the thirties quite transformed our notions of ancient mathematics, at least for (Old)-Babylonia. A magnificent mathematics, on a high level indeed, was disclosed. "What is called Pythagorean in the Greek tradition." NEUGEBAUER wrote in 1937, "had better be called Babylonian."[17] Thus NEUGEBAUER was suggesting that Old-Babylonian mathematics was the basis for the Greek developments. Whatever the truth of this suggestion may turn out to be, the old notion that mathematics started in Greece in the sixth century B.C. is gone. But it was not NEUGEBAUER's view that Greek geometrical science *derives* from Old-Babylonia. Quite the contrary! NEUGEBAUER noted the dominant aspect of Old-Babylonian mathematics, namely, its computational character. The Old-Babylonians, according to NEUGEBAUER, applied number to geometry in the same way that they applied number to anything else, for example, to the distribution of bread to workers digging a canal.[18] In a word, NEUGEBAUER's view on the origin of geometry was the prevailing view already described.

In the 19th Century ZEUTHEN remarked that Book II of *The Elements* was a textbook in algebra, with the algebra in geometric formulation. This surely looks right. Proposition II 4 says that: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments." This is our rule: $(a+b)^2 = a^2 + b^2 + 2ab$. Proposition II 11 asks: "To cut a given line so that the rectangle contained by the whole and one of its segments is equal to the square on the remaining segment," and the solution amounts to solving a quadratic equation.[19] Similar problems, but of greater generality, are

[16] "Zur Geschichte des Pythagoräischen Lehrsatzes," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen* (Math.-Phys. Klasse), 1928. For a remark on an approximation to the square root of 2 in the Śulvasūtras, see *Exact Sciences*, p. 35.

[17] *Mathematical Cuneiform Texts*, p. 41. See also VAN DER WAERDEN, *Science Awakening*, p. 5.

[18] *Exact Sciences*, p. 44f. and p. 79f.

[19] This is the problem of dividing a given line "in mean and extreme ratio."

solved in VI 28 and 29. Now NEUGEBAUER found Old-Babylonian texts showing that the Old-Babylonians could solve quadratic equations. He concluded that the Greeks got the solutions of the quadratic from Old-Babylonia. He did not express himself in terms of absolute certainty, but unquestionably this was his view.[20]

Basing himself on the work of ZEUTHEN and NEUGEBAUER, VAN DER WAERDEN expounded the view that Babylonian mathematics was the basis of Greek mathematics with great clarity and cogency.[21] A question is: Why did not the Greeks simply take over the Babylonian algebra as it was? Why the geometric formulation? VAN DER WAERDEN explains this as follows. Originally the Pythagoreans supposed that any two line segments were to each other as two numbers (*i.e.*, two positive integers), something that was in accordance with their saying "Number rules the universe," and which would even have been true if any two segments had a common measure. But they soon discovered that the side of a square and its diagonal do not have a common measure. If the side is taken to be 1, we would express this by saying that $\sqrt{2}$ is irrational. Thus they could not solve the equation $x^2=2$ in the domain of numbers, even if they had allowed themselves rational numbers or some logical equivalent. But they could solve it, and more general quadratic equations, in the domain of geometric magnitudes. Thus it was logical necessity which forced the Pythagoreans to go over to the geometric formulation. The "one" was *expunged* from geometry.

VAN DER WAERDEN's view of Babylonian geometry was more or less, but not quite the same, as NEUGEBAUER's. In the first edition of his book *Science Awakening* (1954)[22] the difference is scarcely perceptible and hardly explicit. In this first edition he mentions the Babylonian problem: to find the length of the line x parallel to the bases a and b of a trapezoid that divides the area of the trapezoid into two equal parts. (The problem, incidentally, also occurs in EUCLID's *Data*.) Since the solution, $x^2=\frac{1}{2}(a^2+b^2)$, is rather neat and can be obtained rather immediately from a geometrical observation in accordance with known habits of Babylonian thought,[23] one might have conjectured that Babylonian mathematics contained a geometrical component; but this would have been speculation, and VAN DER WAERDEN makes no corresponding remark.[24] In the second edition, however, he introduces a reconstruction by P. HUBER of a problem already given in the first edition. Here VAN DER WAERDEN explicitly notes that "the idea of this solution is geometrical, not algebraic." One should note, though, that this geometry is not what ZEUTHEN called "geometric algebra": the hall-mark of geometric algebra is the absence of number (except for small positive integers), whereas number is

[20] *Exact Sciences*, p. 149ff.

[21] See his *Science Awakening*, especially pp. 118–126.

[22] Translated from the Dutch "Ontwakende Wetenschap" (1950) by A. DRESDEN.

[23] Let $ABCD$ be a trapezoid (Fig. 1) with bases $AD=a$, $BC=b$, and let $EF=x$ be the line parallel to the bases dividing the trapezoid into two equal areas. Let AB and CD meet in G, so that the given trapezoid is the difference of triangles GBC and GAD; and trapezoid $AEFD$ is the difference of triangles GEF and GAD. Then, as the Old-Babylonian scribe would have easily seen, area $\Delta\, GEF=\frac{1}{2}$ (area $\Delta\, GAD$ + area $\Delta\, GBC$). Since similar triangles are to each other as the squares of corresponding sides, one gets $x^2=\frac{1}{2}(a^2+b^2)$.

[24] The presence of the Theorem of PYTHAGORAS points in the same direction.

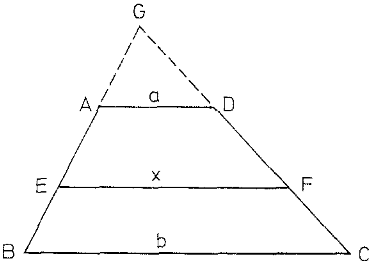

Fig. 1.

present in this problem, as usual for the Babylonians. I would call it "algebraic geometry." Anyway, the presence of this geometrical component had little bearing on VAN DER WAERDEN's view of the origin of geometry, which was the prevailing one but with substance added, for he explained why the Pythagoreans, starting from Babylonian mathematics, hammered it out into Tradition I.

[In *Science Awakening* II, *The Birth of Astronomy*, p. 42, VAN DER WAERDEN makes the change in his judgment on the character of Old-Babylonian mathematics quite explicit.]

There is something very attractive in the theory of NEUGEBAUER and VAN DER WAERDEN on the relation of Greek to Babylonian mathematics. Before, one had to rely on the "genius" of the Greeks as an explanatory device, whereas now one might look to a background of a thousand-year old mathematics. Not everyone finds the vista so inviting, however! Thus M.S. MAHONEY, in a review of NEUGEBAUER's book *Vorgriechische Mathematik* (1934),[25] written on the occasion of its reissue in 1969, would rather invoke the "Greek miracle" than assume that Babylonian mathematics was a basis for Greek developments. "The vast promise of pre-Greek mathematics is, then, to a large extent illusory," he writes, nor can he soften the blow by saying that NEUGEBAUER's translations "[restore] Greek mathematics to a historical context." No, here is a wholesale rejection of NEUGEBAUER's (and VAN DER WAERDEN's) views on the relation of Babylonian to Greek mathematics. What, if anything, he would allow the Greeks to have gotten from Babylonia he does not say.[26]

It is not to be expected that such a vast and complex issue would be exposed without errors, misconceptions, inadequate formulations, anachronisms, *etc.*, and NEUGEBAUER's views have met criticism and queries from the beginning. The main issue, whether Greece borrowed from Babylonia, is clear enough, but even this has to face some weighty objections, some mentioned by MAHONEY, but also others. Thus there is the question of the lines of transmission. The texts in question are Old-Babylonian, dating from about 1700 B.C. Then there is a vast gap of some 1300

[25] "Babylonian Algebra: Form *vs.* Content," *Studies in History and Philosophy of Science*, vol. **1** (1971), pp. 369–380.

[26] I must warn the reader that I may be quite misconstruing MAHONEY's position, that perhaps what he meant was that *although Babylonia furnished the basis* we are still faced with a "miracle" in the Greek developments; and that many of his remarks are intended as nuances on, rather than rejection of, the NEUGEBAUER and VAN DER WAERDEN theses.

years with no mathematical texts from Babylonia![27] Then, in the Seleucid period, about 300 B.C., we begin picking them up again. Where was the Babylonian mathematics all that time, and how could it get to the Greeks? One can only guess, but the similarity of the New problems and the Old shows that somehow the mathematics was handed down.[28] In Old-Babylonia mathematics was systematically taught to schools of scribes. Presumably these schools broke up, but the tradition was nevertheless handed down, though in a trickle. And so, of course, it might have been handed to the Greeks. This is a reconstruction and it emphasizes the lack of direct evidence.

MAHONEY's criticism is of an entirely different kind, however. He of course does not deny that Greece is not far from Babylonia, but he questions, or rather denies, that Babylonian mathematics could possibly lead to Greek mathematics: they are just too different in character.

According to MAHONEY there is a vast gulf between the Greek (and post-Greek) and pre-Greek minds. One is "mythopoeic," the other "rational," with Rationalism on our side (of course). This makes it difficult for us to understand pre-Greek modes of thought, and in particular the "total originality of the Greek mathematical mind ... works to cut us off from the mathematical mind on the other side ... ."

It seems, though, that MAHONEY has gotten the two sides mixed up, at least as far as mathematics is concerned. We may not know how the Old-Babylonians got their procedures, for they do not tell us, but the procedures are entirely straightforward. The Old-Babylonian calmly divides 1 into two equal parts and it does not give us the shivers. For the Greeks, however, 1 was not even a number! Who of us can understand that, or at least understand it without difficulty?

So there is, indeed, a vast gulf between Old-Babylonia and Greece, though its nature is, perhaps, different from what those fixated on Greece imagine it to be. It is surely *a priori* conceivable that Greece shows archaisms long discarded in Old-Babylonia. Using my own terminology, I would say that Old-Babylonian mathematics is purely secular, whereas Greek mathematics operates under the weight of philosophical notions; and the connection of philosophy and theology is so close that it is often imposssible to decide (as THIBAUT remarked, though he was speaking of India) where the one ends and the other begins. If this is what MAHONEY meant, then I, for one, think he was making a valid point.

Anyway, whichever side one takes in this debate, the view on the origin of Tradition I remains the prevailing one. The place: Greece. The time: 600–300 B.C.

4. View on the origin of algebra in 1970. In 1900 and for many years thereafter, as long as the dominant view was that mathematics was a Greek science, with perhaps some petty claims from the Orient, there could not very well have been a debate on the origin of algebra: somehow algebra derived from Tradition I. With the disclosure of the Babylonian mathematics, that impression – one can hardly call it more – evaporated.

According to NEUGEBAUER and VAN DER WAERDEN, Greece got its algebra

[27] *Exact Sciences*, p. 14f. and p. 29.

[28] *Science Awakening*, p. 78.

from Babylonia. The proof is indirect and depends on a comparison of Babylonian and Greek algebra. This method of comparison can be extended to wider fields and leads to the suggestion that algebra had a single source. Thus VAN DER WAERDEN, in commencing a discussion of DIOPHANTUS and after mentioning the kind of popular little algebraic problems to be found in the Greek *Palatine Anthology*, writes (*op. cit.*, p. 280):

> It is probable that the tradition of these algebraic methods was never interrupted so that, along with the scholarly tradition of Greek geometry, there has always existed a more popular tradition of small algebraic problems and methods of solution, a tradition which originates in Babylonian algebra, and which ends in Arabic algebra, with radiations into Greek culture, into China and India.
> We have no real proofs for the existence of such an uninterrupted tradition; too many connecting links are missing for this. It is rather a general impression of relatedness which makes itself felt when one knows the cuneiform texts and then looks through Heron or Diophantus, or the Chinese "classic of the maritime isle," or the Aryabhayṭa of Aryabhata or the Algebra of Alkhwarizmi. According to all Arabic sources, Alkhwarizmi was the first writer on algebra, but his algebra is so mature that we cannot assume that he discovered everything for himself. The algebra of Alkhwarizmi can hardly be accounted for on the basis of the Greek and Indian sources we know; one gets more and more the impression that he has drawn on older sources which in some way or other are connected with Babylonian algebra.

This view on the unique source of algebra is expressed with sufficient tentativeness: it depends not on direct evidence but rather on the comparative method and the presumed relatedness of the instances of algebra wherever we find them. Now the thesis that Greece got its algebra from Babylonia also depends on the comparative method. I think then that we need not be concerned so much with the broader thesis, for the moment at least, but may stick to the main issue. If, after a sufficiently minute comparison of Babylonian algebra and Greek geometric algebra, one agrees that the claimed relatedness is a reality and not an illusion, one will also be inclined (I presume) to accept the broader thesis.

MAHONEY, in the mentioned review, seems to be playing with ideas that he is unwilling to bring to explicit formulation. For example, he writes: "If Babylonian mathematics is algebra, or if it is at least algebraic in approach, then one sees more clearly why 'geometrical algebra' is almost a contradiction in terms." Does he intend to say here that 'geometric algebra' is a contradiction in terms? Or does he mean to deny that Babylonian mathematics is algebra? I think here he means that 'geometric algebra' is a contradiction in terms; the "almost" is just a sign of his reluctance to be unambiguous. Now if 'geometric algebra' is a contradiction in terms, then, of course, there isn't any such thing, and if there isn't any such thing, there is no point in comparing it to Babylonian algebra.

MAHONEY also appears to wish to deny that Babylonian mathematics is algebra. By 'algebra' MAHONEY would like to mean, as he explicitly says, any form of mathematics that fits his characterization of the algebra of the 17^{th} Century A.D. (use of an operational symbolism, preoccupation with mathematical relations, freedom from ontological commitments); and he would prefer to say that Babylonian mathematics shows an "algebraic approach." Now it is well to develop a good terminology; but using terms in common usage in a private way is not conducive to clarity. His characterization of 17th Century A.D. algebra is interesting; but mathematics, in algebra and elsewhere, also involves ingenuity. The Old-Babylonians set the problem:

> Length, width. I have multiplied length and width, thus obtaining the area. Then I added to the area the excess of the length over the width: 183. Moreover I have added length and width: 27. Required: length, width, and area

and show ingenuity in solving it;[29] and the Pythagoreans set the problem: To divide a segment in extreme and mean ratio, and show ingenuity in solving *it*. Now the question is: Are these two expressions of ingenuity related or not?

How would MAHONEY answer this question? As already remarked, I do not know. But I suspect that he was preparing to drag the *deus ex machina* of independent invention, that old stand-by of historical studies, onto the scene.

Let us compare the Babylonian and Greek treatments of quadratic problems. The Babylonian problem already cited can be written in the form of two simultaneous equations:

$$xy+x-y=183,$$
$$x+y=27,$$

where x is the length and y the width. I call this problem *quadratic*, in accordance with current usage, because there is a term (namely, xy) which is the product of two unknowns (and no products of three or more unknowns occur), but of course do not intend to say that the Babylonians had any such classification. The Babylonian treatment of quadratic problems rests on the observation that $xy=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2$; even this statement requires some interpretation of the texts, but that it is a close approximation to the truth would, I believe, be admitted on all sides. The Babylonian scribe appears to have the "normal problem"

$$x+y=a,$$
$$xy=C$$

well in mind: to solve these he would first compute $\left(\frac{x-y}{2}\right)^2$, applying the mentioned identity. Taking the square root, he would find $\frac{x-y}{2}$; now he has $\frac{x-y}{2}$ and $\frac{x+y}{2}$, and adding and subtracting he finds x and y. Actually, as GANDZ has remarked,[30] one nowhere finds this "normal problem" set: there is always another point to the problems found in the texts; and the scribe has the solution of this "normal problem" well in mind before he takes up his scribal studies. Thus in the cited problem the idea is to add the two equations, getting $xy+2x=210$, $x+y=27$;

[29] *Ibid.*, pp. 63–65.
[30] *Osiris*, vol. **3** (1937).

and then (and this is the new point) he introduces a new unknown, $y'=y+2$, whereupon he gets the normal problem $xy'=210$, $x+y'=29$.

The Pythagorean treatment of quadratic problems proceeds on the same lines. Thus *Elements II* 5 reads:

> II 5. If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

In a moment I will transcribe this using literal notation, but one may observe that the proposition says that a certain rectangle (and it can be any rectangle) is the difference of two specified squares; if we think of a rectangle as a product, as the Old-Babylonian did, we see that the proposition tells us how to consider a product as a difference of two specified squares, which is also what the mentioned identity $xy=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2$ does. Moreover, if we call the length of the rectangle x and its width y, then "the straight line between the points of section" is $x-\left(\frac{x+y}{2}\right)$, or $\frac{x-y}{2}$. Then the proposition can be construed to say that $xy+\left(\frac{x-y}{2}\right)^2=\left(\frac{x+y}{2}\right)^2$, which is our familiar identity. There are other possible transliterations of II 5, and I will eventually comment on the ambiguity. Meanwhile, at least provisionally, we may grant that II 5 amounts to the identity $xy=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2$ or a close variant thereof. Now VI 28 solves a quadratic problem; a special case of this can be transliterated to the normal problem $x+y=a$, $xy=C$. In the solution EUCLID appeals to II 5. The diagrams for II 5 and VI 28 (Special Case) are essentially the same (see Fig. 2). EUCLID converts the area C, given as a polygon, into a square (not shown in the figure), builds the square on $\frac{a}{2}$, constructs a square equal to their difference (the "small" square in the figure), places it relative to the square on $\frac{a}{2}$ in the position shown, whence the difference of the square on $\frac{a}{2}$ and the "small" square appears as a gnomon,[31] which as in II 5 is seen to be equal to the rectangle xy, where x is $\frac{a}{2}$ plus the side of the "small" square and y is $\frac{a}{2}$ minus that side.

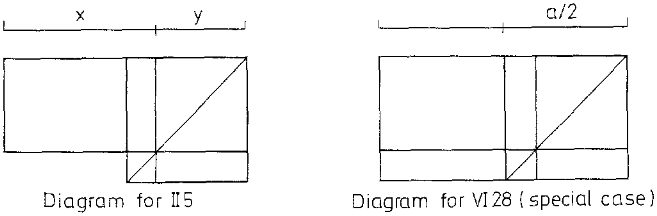

Fig. 2.

[31] A *gnomon* is the L-shaped figure obtained by subtracting from a square a smaller square having with the first a common vertex.

Thus the Old-Babylonian and Greek procedures for solving the normal problem $x+y=a$, $xy=C$ are step-by-step the same. The Old-Babylonian does not, of course, have to convert C into a square, because for him C is just a number, and he does not apply the Theorem of PYTHAGORAS to get the difference of the squares $\left(\frac{a}{2}\right)^2$ and C, for he merely subtracts one number from another. But aside from such general differences between geometric and numerical procedures, the two solutions are the same.[32]

VAN DER WAERDEN, in his book *Science Awakening*, has already given most of the above points, together with similar remarks for the "normal problem" $x-y=b$, $xy=C$. In a recent defense of his views,[33] he has gone over the ground once more; and he has added the important point that the PYTHAGOREAN solutions of these problems are clumsy in comparison with other solutions easily accessible to the PYTHAGOREANS. For details, see his paper. Or consult DESCARTES' *Geometry*.

This new point is of considerable importance for establishing the relatedness of the two solutions, for polemical purposes, anyway. For the solution of the normal problem $x+y=a$, $xy=C$ is a purely logical matter, it might be claimed, and all men are capable of logical thinking, so why should not the Old-Babylonians and the Pythagoreans go through the same logical processes independently to come to the same solution? Skeptical as one may be of this argument, for mathematics is a complex matter, still it is difficult to deny that men, generally speaking, are capable of being logical. The similarity, or coincidence, of logical results may simply be a consequence of the constraints of logic. That is why the *arbitrary* elements, even trivial ones, are of vast importance in the comparative method. It is implausible that the same arbitrary elements appear by coincidence, and this implausibility yields a polemical base for claiming relatedness.

In the same vein, I will mention one more point of comparison between Babylonian algebra and Greek geometric algebra. According to VAN DER WAERDEN *(loc. cit.)*:

> The Babylonians had four standard types of linear and quadratic equations with two unknowns:
>
> $$\text{(I)}\begin{cases} x+y=a \\ xy=C, \end{cases} \quad \text{(II)}\begin{cases} x-y=a \\ xy=C, \end{cases} \quad \text{(III)}\begin{cases} x+y=a \\ x^2+y^2=S, \end{cases} \quad \text{(IV)}\begin{cases} x-y=a \\ x^2+y^2=S. \end{cases}$$
>
> The Greeks formulated four theorems, II 5–6 and II 9–10, by means of which these types can be solved. The solutions thus obtained are the same as the Babylonian solutions, but in geometric language. They differ from all simpler geometrical solutions.

I will add that all four standard types appear to be problems on the rectangle, the first two involving the area and the second two the diagonal of the rectangle. In Babylonia the problems sometimes appear as problems on the rectangle.

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[32] Thus the two procedures are the same not only in content, but also in *form*.

[33] "Defense of a 'Shocking' Point of View," *Archive for History of Exact Sciences*, vol. **15** (1976), 199–210.

Thus the Babylonian treatment of quadratic problems and that of the Greeks are related. VAN DER WAERDEN concludes that the Babylonian procedures were *certainly* the basis for the Greek developments. No! What he should have concluded is that they certainly have a common source, but *not* that the source is certainly Babylonia.

The problem of the independent origins of similar ideas is a touchy one. A view often encountered is that ideas are a spontaneous reaction to environing conditions. One does not expect a mountain folk to invent boats, but people living near the seashore will (so it is claimed) soon invent them. But this could hardly be correct! There are islands in the South Seas where the natives (though they have the material and tools) do not know how to make seaworthy vessels. Their ancestors must have known how. so the techniques have gotten lost, though the seashore was there all along.[34] The bow-and-arrow is a very useful instrument found amongst widely separated and isolated peoples, and so it is supposed that each of these groups has independently invented it. The gun is also a very useful and widespread instrument, but nobody claims the same thing about guns, because the gun was invented in the light of history. It is rather the inventions whose origins are hidden in the mists of antiquity which are claimed to have multiple origins.[35] The invention must not have been difficult, since many untutored people have thought it up. So one tries to imagine the steps leading to the invention. And now curiously, and most embarassingly, even the anthropologists, who are vitally interested in such matters, cannot even imagine how the bow-and-arrow was invented. KROEBER, after noting that the bow is probably a product of Neolithic times, adds:

> The reason for this lateness in the invention of the bow-and-arrow is probably to be sought in the delicacy of the instrument. It is not essentially more complex than the harpoon, certainly not more complex than the harpoon impelled by the spearthrower. But it involves much finer adjustments. A poorly made harpoon is of course inferior to a well-made one, but may be measurably effective. It may retrieve game half the time. But a bow which falls below a certain standard will not shoot at all, or will shoot so feebly as to have a zero efficiency. In fact, one of the things that students of the beginnings of culture have long been puzzled about is how the bow and arrow could have been invented. Most other inventions can be traced through a series of steps, each of which, although incomplete, achieved a certain utility of its own. But, other than toys or musical instruments, no implement has yet been found, or even satisfactorily imagined, which was not yet a bow, which would serve a purpose, and which, by addition or improvement, could give rise to the bow.[36]

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[34] See W.H.R. RIVERS, "The Disappearance of Useful Arts," in WESTERMARCK's *Festskrift*, pp. 109–130.

[35] The question of independent invention in presumed independent cultures and that of independent invention in a single culture are distinct questions; we are concerned here only with the first.

[36] *Anthropology*, p. 167f. KROEBER's remark contains a hint for the solution, though he himself apparently didn't realize this: the solution will come not simply through an examination of the bow-and-arrow itself, but rather through trying to imagine the circumstances in which the bow-and-arrow was desired as a "toy."

Now if the savants cannot even imagine how something was invented, what is the probability that some relatively uncultered group invented it? Figuratively speaking, an infinitesimal. And what's the probability that two such groups would do it independently? Why, an infinitesimal of higher order!

That is the situation with quadratic problems, too. As far as I know, no one has suggested how the Babylonians arrived at their solutions, or why they cared. Babylonian mathematics, the quadratic part anyway, is completely unmotivated. The situation with the Pythagoreans (assuming for a moment independence) is about the same. The emblem of the Pythagoreans was the 5-pointed star, the Pentagram; if they had wanted to construct a regular Pentagram, they could have done so by solving, in effect, the equation $x^2+ax=a^2$, and to that extent, perhaps, we have a motive for the Pythagoreans. This at best refers to a "normal problem" of Type I. Thus the interest of the Pythagoreans in quadratic problems is about as unmotivated as for the Babylonians.

So both on general grounds and by minute inspection of the evidence, the view that Babylonian and Greek algebra are related looks reasonable.

Recently S. UNGURU has given a different and most ingenious resolution of our troubles.[37] According to UNGURU, it's all an illusion! The "modern mathematicians, turned historian" have read quadratic equations and their solutions *into* both the Babylonian mathematics and the Greek mathematics, and then ("no surprise") they read them back out. An amazing trick! How did they do it? With their literal notation, says UNGURU. By assigning letters to the numbers in the cuneiform texts, the modern mathematician sees quadratic equations *there*. By looking at ancient mathematics through modern "literal" eyeglasses, he sees things that aren't there.

Of course, if UNGURU is right, what VAN DER WAERDEN and NEUGEBAUER (and their predecessors TANNERY and ZEUTHEN) said is utter nonsense. But VAN DER WAERDEN has an ingenious – I can't say equally ingenious – refutation. He refers *(loc. cit.)* to THĀBIT IBN QURRA, a contemporary of AL-KHWĀRIZMI, "an excellent geometer and astronomer, fully conversant with the work of Euclid, ... [who] pointed out that the solution of the three types of quadratic equations according to 'the Algebra people' is equivalent to the 'Application of areas with excess and defect' as presented by Euclid." And this, of course, hundreds of years before the invention of literal notation. Beautiful!

To sum up the current views: The view that Classical Greece is the source of Tradition I remains the prevailing one. The source of Tradition II, it is generally held, is Old-Babylonia. On the relation between Old-Babylonia and Greece, discontent is shown in some quarters.

4 bis. The "ambiguity" in the Elements II 5. The following is an aside, but I must return to the promised discussion of II 5, already cited. II 6 says:

> If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added

[37] "On the Need to Rewrite the History of Greek Mathematics," *Archive for History of Exact Sciences*, vol. **15** (1975), pp. 67–114.

straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

VAN DER WAERDEN's first impression was that II 5 and II 6, though they are of course differently worded, and moreover their diagrams are not the same, were saying much the same thing, and was rightly puzzled, for why have two propositions for the same thing? "What was the line of thought of the man who formulated the propositions in this way?" To answer this he followed out, in the *Elements* themselves and EUCLID's other works, the way in which II 5 and II 6 are applied. He found, roughly said, that when there are two unknowns x and y and the sum $x+y$ is given, then II 5 is applied, whereas if $x-y$ is given then II 6 is applied. That is how things go in the *Elements* VI 28, 29 and the *Data* 84, 85. His conclusion is that, "at bottom, II 5 and II 6 are not propositions, but *solutions of problems* … ."

Let us start from this well-founded conclusions. Using the familiar practice of selecting letters toward the end of the alphabet for unknowns, and letters toward the beginning for knowns, I will call the length and width of the rectangle respectively x and y, both in II 5 and II 6; in II 5 I write $x+y=a$; in II 6, for "the straight line [to] be bisected," I write b (so $x-y=b$). Then II 6 is unambiguous and a direct transliteration gives:

$$(b+y)y+\left(\frac{b}{2}\right)^2=\left(\frac{b}{2}+y\right)^2.$$

This identity is adapted to the problem: to find y (and x) when the difference $x-y=b$ and the product $xy=(b+y)y$ are given, and it is *not* adapted to the problem: given $x+y=a$ and $xy=C$, to find x and y. II 5, however, contains an ambiguity. We could transliterate it as:

$$xy+\left(\frac{x-y}{2}\right)^2=\left(\frac{x+y}{2}\right)^2,$$

or as

$$x(a-x)+\left(x-\frac{a}{2}\right)^2=\left(\frac{a}{2}\right)^2.$$

Now the first of these can be adapted equally well to the "normal problems" I and II, so we would be left with the original puzzle, *but the second is adapted to I and not to II.* (Note, too, that the first transliteration requires an implicit, if slight, side calculation, whereas the second is completely explicit.) Therefore I conclude that it is the second identity, and not the first, that "the man who formulated" II 5 had in mind. I have spoken of the Old-Babylonian as having the identity $xy=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2$, but he too, for all I know, had the two identities, and not just the one in mind.

II. The main thesis

5. Comparison of Greek and Vedic mathematics. VAN DER WAERDEN's arguments are so cogent and so clearly set out that it is hard to believe that they do not contain the truth. Moreover I myself feel that all of his comparisons are just and that, in detail, he is 100% or nearly 100% right. Yet I propose to show that his theses (and NEUGEBAUER's) cannot be maintained in their present form.

The main fault in VAN DER WAERDEN's analysis, as I see it, is that at all vital points he takes into account only Old-Babylonia and Greece: if one includes the Vedic mathematics, one will get quite a different perspective on ancient mathematics.

The main issue is the origin of geometric algebra. The Śulvasūtras have geometric algebra, and I will first show that Greece and India have a common heritage that cannot have derived from Old-Babylonia, *i.e.*, the Old-Babylonia of about 1700 B.C. as portrayed in *Science Awakening*.

The Indian priests in their altar rituals had to convert a rectangle into a square. "If you wish to turn an oblong into a square (see Fig. 3), take the tiryaṅmāni, i.e., the

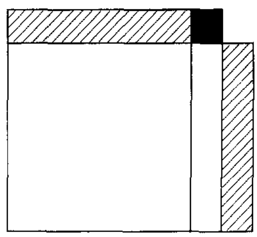

Fig. 3.

shorter side of the oblong, for the side of a square, divide the remainder (that part of the oblong which remains after the square has been cut off) into two parts and inverting [one of them] join these two parts to the sides of the square. (We get then a large square out of which a small square is cut out as it were.) Fill the empty space (in the corner) by adding a small piece (a small square). It has been taught how to deduct it (the added piece)." *Cf.* BAUDHĀYANA *Śulvasūtra* I 54, ĀPASTAMBA *Śulvasūtra* II 7, or KĀTYĀYANA *Śulvasūtra* III 2.[38]

This is entirely in the spirit of *The Elements*, Book II, indeed, I would say it's more in the spirit of Book II than Book II itself. The problem and its solution are precisely that of II 14, except that the diagram of II 6 intervenes instead of that of II 5. In any case the Theorem of PYTHAGORAS and the identity $xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$ are the key facts in the solutions.

[38] There are many versions of the Śulvasūtras; the main three are those of (the schools of) BAUDHĀYANA, of ĀPASTAMBA, and of KĀTYĀYANA. For "how to deduct it", i.e., for how to subtract a square from a square, see BAUD. ŚS. I 51, ĀP. ŚS. II 5, or KĀT, ŚS. III 1. This rests on BAUD. ŚS. I 48 (cf. ĀP. ŚS. I 4, KĀT. ŚS. II 11) which says: "The chord stretched in the diagonal of an oblong produces both (areas) which the cords forming the longer and shorter sides produce separately." This is the Theorem of PYTHAGORAS stated in complete generality except for the isosceles right triangle (or square), which is given separately.

The Old-Babylonians could have had no use for such a procedure: they would simply multiply the two sides and take the square root.

Let us consider now the Theorem of PYTHAGORAS, and under two aspects: in Aspect I, the theorem is used to construct the side of a square equal to the sum or difference of two squares; in Aspect II, the theorem is used (say) to compute the diagonal of a rectangle. Aspect II comes in, for example, when one uses the (3, 4, 5) triangle to construct a right angle.[39] The Śulvasūtras know both aspects. The *Elements* has only Aspect I, but Classical Greek geometry presumably also realized Aspect II since it had Pythagorean number triples. Now the Old-Babylonians had Aspect II, but they would have had no use for Aspect I: they would simply square the lengths of the sides of the given squares, add, and take the square root.

I could give further common elements of the Greek and Indian mathematics not shared by Old-Babylonia, for example, the gnomon; or the problem of squaring the circle. In *Āp. ŚS.* III 9 and in *The Elements* II 4 the gnomon is analysed into two rectangles and a square (see Fig. 4); and the propositions amount to our rule: $(a+b)^2=a^2+b^2+2ab$.

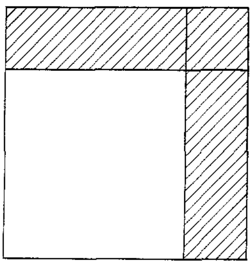

Fig. 4.

The Old-Babylonians know this rule, but they do not have the gnomon (though it lies at hand to conjecture they once did).

The squaring of the circle is a true geometrical problem in Greece and in India; in Babylonia, either it does not exist or is to be considered trivially solvable: the circle has area $3r^2$ and the side of the required square is $\sqrt{3}\,r$.

Conclusion: Either the geometric algebra of Greece came from India or that of India came from Greece or both came from a third source different from Old-Babylonia of 1700 B.C.

There are several grounds on which the second of these alternatives has to be eliminated. First, there is the chronology. I don't want to minimize the difficulty of getting a reliable chronology, but at least the Sanskrit scholars, though they may differ amongst themselves, still do for the most part sufficiently agree for me to get the desired conclusions, namely, that the *Śatapatha Brāhmana*, and hence (as I shall explain) geometric algebra, goes back to before the 6th Century B.C. In my paper "The Ritual Origin of Geometry" (1962) I considered the views on chronology held by the Sanskrit scholars around 1900, but opinion has not changed much since then. Thus *L'Inde Classique*, by L. RENOU & J. FILLIOZAT (a work well thought of

[39] Of course, it is the converse of the Theorem that comes in here.

in Sanskrit circles), places the *Śatapatha Brāhmana* at 1000–800 B.C. (see vol. 1, p. 267).

There are various strata of the Vedic sacred literature. Everybody agrees on their relative ages (at least within the individual schools). Of the strata with which we shall be concerned, the Sūtra period is the most recent: it is generally dated around 500 – 300 B.C. The Śulvasūtras contain (roughly said) all the geometrical details we want. Now they never say they are being original – they always insist that they are doing things as it has been taught, especially in the Saṃhitās and Brāhmaṇas. The Brāhmaṇas purport to give the symbolic meaning of the rituals; the Saṃhitās give the formulae which are uttered at the rites. In the nature of things, the Saṃhitās come before the Brāhmaṇas; and both come before the Śulvasūtras (more generally said, before the Kalpasūtras, which digest the teachings relative to the performance of the rites). The Rig-Veda, a collection of hymns, is still earlier.

One might presume that the geometric constructions given by the Śulvasūtras with reference to the Brāhamaṇas and Saṃhitās must have been known to the composers of those works, and so be coeval with them. But that, however plausible it may be, will here be the issue.

In the Śulvasūtras the construction of altars of various shapes is described, the shape depending on the particular ritual. Thus there are square altars, circular altars, and altars of many other shapes: one, the falcon-shaped altar, was composed of rectangles and does to some extent look like a falcon – or, rather, "the shadow of a falcon about to take wing." "He who desires heaven may construct the falcon-shaped altar; for the falcon is the best flyer amongst the birds; thus he (the sacrificer) having become a falcon himself flies up to the heavenly world."

The altars were, for the most part, composed of five layers of 200 bricks each, which reached together to the height of the knee; for some cases ten or fifteen layers and a corresponding increased height of the altar were prescribed. Most, though not all, of the altars had a level surface, and these were referred to in accordance with the shape and area of the top (or bottom) face. The basic falcon-shaped altar had an area of $7\frac{1}{2}$ square puruṣas: the word "puruṣa" means *man* and is, on the one hand, a linear measure, namely the height of a man (the sacrificer) with his arms stretched upwards (about $7\frac{1}{2}$ feet, say), and, on the other, an areal measure (about $56\frac{1}{4}$ square feet). Aside from secondary modifications or variations, the body of the

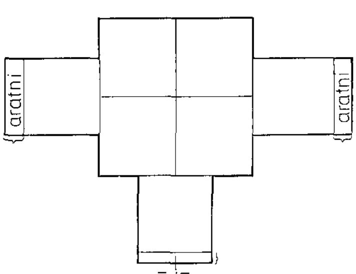

Fig. 5.

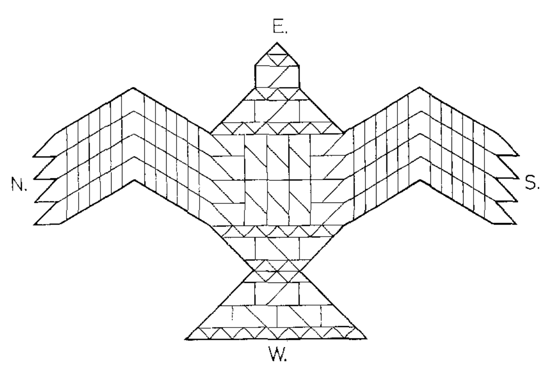

Fig. 6.

falcon-shaped altar was a 2×2 square (4 square puruṣas), the wings and tail were one square puruṣa each; in order that the image might be a closer approach to the real shape of a bird, wings and tail were lengthened, the former by one-fifth of a puruṣa each, the latter by one-tenth (*cf.* Fig. 5; for a variant, Fig. 6). This was the size and shape of the falcon-shaped altar upon its first construction. On the second construction, one square puruṣa was to be added, that is, the area of the second altar constructed would be $8\frac{1}{2}$ square puruṣas; on the next construction another square puruṣa is added, and so on, until one comes to the one-hundred-and-one (and a half)-fold altar. It is clear that the sacrificer is climbing a ladder, the sacrificial rank being determined by, or determining, the area.

In the construction of the larger altars ($8\frac{1}{2}$, $9\frac{1}{2}$, etc.), the same shape as the basic altar is required; the problem of finding a square equal in area to two given squares is actually and explicitly involved: the construction is carried out using the Theorem of PYTHAGORAS. The problem of converting a rectangle into a square is also explicitly involved.

[BAUD. ŚS. II 12 tells how the larger altars are to be constructed:

> That which is different from the original form of the agni (i.e., that area which is to be added to the area of the saptavidha, i.e., sevenfold, agni) is to be divided into fifteen parts and two of these parts are to be added to each vidha (to each of the seven purushas); the one remaining part has consequently to be added to the remaining half purusha; with seven and a half of these (increased vidhas, i.e., units) the agni is to be constructed.

KĀT. ŚS V 4 in effect does the same. ĀP. ŚS. VIII 6 is different in that it speaks of expanding the "7" to "*n*"; and some question remains on what to do about the "half."

BAUDHĀYANA does not explain that the "two parts," which we may suppose to be in the form of a rectangle, are to be converted into a square as explained and then added, as explained, to the smaller vidha (*i.e.*, 1×1 square) to get the increased vidha, or unit, nor should this have been expected as the Śulvasūtras are about as brief as possible. KĀTYĀYANA, however, though the result is the same, follows a slightly different method, in the course of which he is obliged to mention the conversion of a rectangle into a square.[40]]

The Śatapatha Brāhmaṇa speaks about the seven-fold altar and about its augmentation one square puruṣa at a time to the one hundred and one-fold one. Thus in VI, 1, 1, 1–3 we are told that "in the beginning" the Rishis (vital airs) created seven separate persons, who are assimilated to squares. After giving a reason they say: "Let us make these seven persons one Person!" whereupon the seven are composed into the falcon-shaped altar; X, 2, 2, 7 speaks of the lengthening of the wings and X, 2, 2, 8, of the tail. In X, 2, 3, 18 we read:

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[40] *Kāt. Ś.S.* V 4 says: "For the purpose of adding a square puruṣa (to the original falcon-shaped agni), construct a square equivalent to the original agni together with its wings and tail; add to it a square of one puruṣa. Divide the sum (i.e., the resulting square) into fifteen parts and combine two of these into a square. This will be the (new) unit of square puruṣa (for the construction of the enlarged figure)." *Cf.* DATTA, p. 156 or KHADILKHAR, p. 33.

Sevenfold, indeed, was Prajāpati created in the beginning. He went on constructing (developing) his body, and stopped at the one hundred and one-fold one.... Hence one should first construct the seven-fold (agni) and then by increments of one (square purusha) successively up to the one hundred and one-fold one."

Passage X, 2, 3, 11 describes a "ninety-eight"-fold bird:

Now as to the (other) forms of the fire-altar. Twenty-eight (square) purushas and twenty-eight (square) purushas is the body, fourteen the right, and fourteen the left wing, and fourteen the tail. Fourteen *aratnis* he covers (with bricks) on the right, and fourteen the left wing, and fourteen *vitastis* on the tail. Such is the measure of an (altar of) ninety-eight (square) purushas with the additional space for wings and tail.[41]

Śat. Br. X 2, 3, 6 in speaking of the "construction [of the] higher forms," *i.e.* larger altars, and warning the Sacrificer not to enlarge the Gārhapatya altar, mentions that the "fathom" (for measuring the bird-altar) and the "steps" (for measuring the Mahāvedi) are "increased accordingly." Thus, in agreement with the Śulvasūtras, for the larger altars one simply uses "new" units in place of the "old."

From these passages it is clear and, indeed, explicit that the Śatapatha Brāhmaṇa knows the basic $7\frac{1}{2}$ puruṣa bird altar, its augmentation one square puruṣa at a time, and the principle of maintaining similarity of form. Elsewhere the Śatapatha Brāhmaṇa shows that it is concerned with, or rather, takes for granted, exact constructions. Thus in X 2, 1, 1–8 the variation in the wings is spoken of; we read: "He (the sacrificer) thus expands it (the wing) by as much as he contracts it; and thus, indeed, he neither exceeds (its proper size) nor does he make it too small." And X 2, 3, 7 says that those who deprive the agni of its due proportions will suffer the worse for sacrificing. The exact construction of the larger altars (except for the 67 (and a half)-fold one) requires, in effect, the Theorem of PYTHAGORAS. I therefore regard it as certain that the Śatapatha Brāhmaṇa knows the Theorem.

[It is conceivable that the "ninety-eight"-fold altar here spoken about is really the $101\frac{1}{2}$-fold one. According to BAUDHĀYANA, and also the Śatapatha Brāhmaṇa as just cited, the length of the "new" unit x after the m^{th} augmentation satisfies $x^2 = 1 + \frac{2m}{15}$. Here m runs from 1 to 94; for $m = 94$ one gets the largest altar, and $x^2 = 13\frac{8}{15}$. The 14 may be a rounding-off of $13\frac{8}{15}$; and if so, then the $101\frac{1}{2}$-fold altar is being described. The rounding-off, though perhaps reasonable in context, may have given rise to confusion, for a moment later the Śatapatha Brāhmaṇa speaks (it would appear) of adding 3 more puruṣas to the body and proportionate amounts to the wings and tail; and here, at least, there were objections that the agni was being made too large (*cf.* Śat. Br. X 2, 3, 15). In any event note that $\sqrt{13\frac{8}{15}}$, like $\sqrt{14}$, is irrational and that, in either case, the large altar is similar to the basic one.]

The Śatapatha Brāhmaṇa and the Taittirīya Saṃhitā both explicitly give the dimensions of the Mahāvedi: this is an isosceles trapezoid (see Fig. 7) having bases 24 and 30 and width 36. There is a (15, 36, 39) triangle here, and the Śulvasūtras use

[41] For some comments on this translation, see my "Ritual Origin of Geometry," p. 308f.

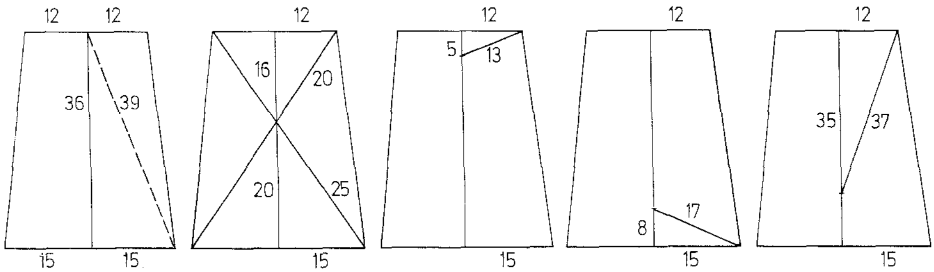

Fig. 7.

this fact to construct the Mahāvedi. Now the question is: did the priests at the time of the Śatapatha Brāhmaṇa and of the Taittirīya Saṃhitā know this Pythagorean triangle? If not, we conclude that the 39 was a later discovery, and that it was just an accident, or let's say, a minor miracle, that this distance was an integral number of units.[42] Now the Mahāvedi is loaded with Pythagorian triples. Not only is (15, 36, 39) there, but also (12, 16, 20), (15, 20, 25), (5, 12, 13), (8, 15, 17), (12, 35, 37), all mentioned in the Āpastamba Śulvasūtra in connection with the construction of the Mahāvedi (see figures). I think these, especially the first two, convert the minor miracle into a major miracle. The conclusion is that it is nearly certain that Aspect II was known at the time of the Taittirīya Samhitā.

Conclusion: If one accepts the mentioned chronology, then one cannot maintain VAN DER WAERDEN's and NEUGEBAUER's thesis on the generation of geometric algebra, for geometric algebra existed in India before the classical period in Greece.

There is another argument, independent of the Indian chronology and which I find very convincing, for rejecting the second alternative, at least when it is combined with the idea that geometric algebra was developed as a way to meet the crisis of the discovery of non-commensurables. For if this is the way things went, then the Indian priests seized on a technical point in mathematics and made of it the central feature of their theologic geometry. This I find too hard to believe.

6. Comparison of Old-Babylonian and Vedic mathematics. O. BECKER (*Geschichte der Mathematik*, with J.E. HOFMANN, pp. 39–41) accepts a date before 600 B.C. for the Theorem of PYTHAGORAS in India. He therefore looks to Babylonia for the source. But he needs to get Aspect I, and Old-Babylonia didn't have it. Could it be that India got Aspect II from Old-Babylonia and transformed it into Aspect I? For Greece we have a theory as to why this might have happened, but no one has ever suggested any such thing for India; and I don't see how it could have happened. Aspects I and II are but two aspects of the same thing and the Śulvasūtras know this. The conclusion is that Old-Babylonia got the Theorem of PYTHAGORAS from India or that both Old-Babylonia and India got it from a third source. Now the

[42] The absence of this 39 is the nub of KEITH's argument (*cf. op. cit.* in note 12).

Sanskrit scholars do not give me a date so far back as 1700 B.C.[43] Therefore I *postulate* a pre-Old-Babylonian (*i.e.*, pre-1700 B.C.) source for the kind of geometric rituals we see preserved in the Sulvasutras, or at least for the *mathematics* involved in these rituals. This sort of hypothesis is made in the physical sciences. Why not in history, too?

There is then the question: Why did Babylonia take over only Aspect II from the source, rejecting Aspect I? I will try to explain this.

But first let us see whether my hypothesis explains anything. For this I want to look at the common elements in the Old-Babylonian and Vedic mathematics. Both appear to know the identity $xy=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2$. In India it is not used for arithmetical purposes, but it is surely close to the surface in the construction from BAUDHĀYANA cited above. Moreover, the KĀTYĀYANA Śulvasātra (VI 5) gives the following construction for a square of n units: Take a line segment $AB=(n-1)$-units (see Fig. 8) and form an isosceles triangle ABC with $AC+CB=(n+1)$-units. Then the "arrow" DC is the side of the desired square. This even looks like an arithmetico-geometric application of the formula for $x=n$, $y=1$.[44]

Note the term "arrow," which was used in a similar, though not quite the same way, at Susa, in the Old-Babylonian period: see my paper in the *Archive for History of Exact Sciences*, vol. 9 (1972), p. 181, note 17.

Now this identity is a definite part of geometric algebra, so it follows that the postulated source knew this. Or, in other words, the Old Babylonians got this identity from a set-up *like* that found in the Śulvasūtras, but, of course, from a pre-1700 source.

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[43] The ancestors of the Vedic Indians are supposed to have been nomads who invaded India around 1500 B.C., and the Rig-Veda is supposed to reflect the culture of these nomads, so the place to look for evidence that the invading Aryan nomads already had the geometrical rituals is the Rig-Veda, but the evidence there is scanty and cannot advance the argument logically. The Vedi is explicitly mentioned, as in I 164, 35; and so are the three fire altars (to be discussed in the Appendix). Agni is often compared to a bird, and in I 58, 5; I 96, 6; and VI 2, 8 is *called* a bird (*cf.* K.F. GELDNER, *Der Rig-Veda*, 4 vols.). In Rig-Veda I 67, 10 (incorrectly given by BÜRK on p. 544 (1910) as I 67, 5) one learns that "skillful men ... measure out ... the seat" of the agni; and X 90 says that "puruṣa is thousand-headed, thousand-eyed, thousand-footed ..." but that is about as definite as one can get. It is better first to establish that Vedic mathematics existed before 1700, and let this bear on the question of whether the invading nomads had the geometric rituals. This subject will be pursued in my paper "The Geometry of the Vedic Rituals" (*cf.* note 10 above). The surmise reached there is that the Vedic Aryans learned the geometrical rituals from the Iranians or from their common ancestors.

[Aside from I 67, 10, there are some poetic references to measurements that it may be well to mention here. Thus RV I 160, 4 says that "the skillfullest among the skillful gods ... measured out the two realms" of Heaven and Earth. RV I 159, 4 is similar and adds: "The enlightened seers are forever stretching a new string to the heaven in the sea." RV III 38, 3 says that "they made both [Heaven and Earth] equal in measure ...." RV VI 8, 2 says that Agni "measured out the air space;" and X 121, 5 is similar. *Cf.* GELDNER, *op. cit.*]

[44] W.R. KNORR (*The Evolution of the Euclidean Elements*, p. 181) has conjectured that THEODORUS constructed $\sqrt{N}$, for any integer $N>1$, by constructing a right triangle of hypotenuse $(N+1)/2$ and leg $(N-1)/2$. Such a construction is, as just mentioned, explicitly given by KĀTYĀYANA.

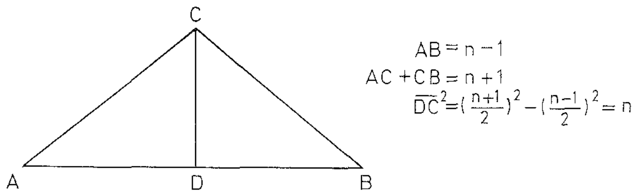

Fig. 8.

In AO 8862, already cited above in full, the scribe writes: "Length, width. I have multiplied length and width, thus obtaining the area. Etc." The problem is purely arithmetical. The Babylonians had purely arithmetical expressions as multiplication, root extraction, *etc.*[45] Then why bring in the rectangle and the geometric terminology? My answer is: the rectangle was already there; the problem arose in a geometric context.

In his "Defense" (1976), § 5, VAN DER WAERDEN speaks of a tendency in Greek arithmetic as well as in Babylonian and Arab algebra to illustrate algebraic notions by means of diagrams. But where does the tendency come from? If it all started as I am postulating, the presence of the tendency is completely intelligible.

The root of the Babylonian treatment of quadratic equations is the identity $xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$, and we have seen how this identity can arise from a geometric problem. So we see that the Babylonian theory of equations could very well have stemmed from geometric knowledge of the kind contained in the Śulvasūtras.

So my hypothesis explains things. Now I want to put forward the view that there was a split: one side expanded the arithmetic methods already present in the source, pushing aside the old geometric constructions; the other side insisted on maintaining the constructions.

The Śulvasūtras have been called manuals for altar construction, and largely and mainly they are that. But there is something else to them, too. Thus in BAUDHĀYANA Śulvasūtra I, 113, the yūpa poles, which are 1 pada in diameter, are said to measure 3 padas in circumference. This information has no application. It's just a piece of erudition.

There are other things of the same kind in the Śulvasūtras. Thus the Śulvasūtras have the circulature of the square and the squaring of the circle. The circulature of the square has a canonical solution in the Śulvasūtras: in Fig. 9, in the given square $ABCD$, let M be the intersection of the diagonals; draw the circle with M as center and MA as radius; let ME be a radius of this circle perpendicular to AD and cutting AD in G; and let $GN = \frac{1}{3}GE$; then MN is taken to be the radius of a circle having the area of $ABCD$. We can say that the solution amounts to taking $\frac{d}{s} = \frac{2+\sqrt{2}}{3}$, where d is the required diameter and s is the side of the given square; but it is to be noted that this solution involves no arithmetic. Now the circulature of the square is *applied* in the Śulvasūtras; but the squaring of the circle is not applied – certainly not explicitly in the Śulvasūtras – and I think never. It's just another piece of erudition.

[45] *Science Awakening*, p. 119.

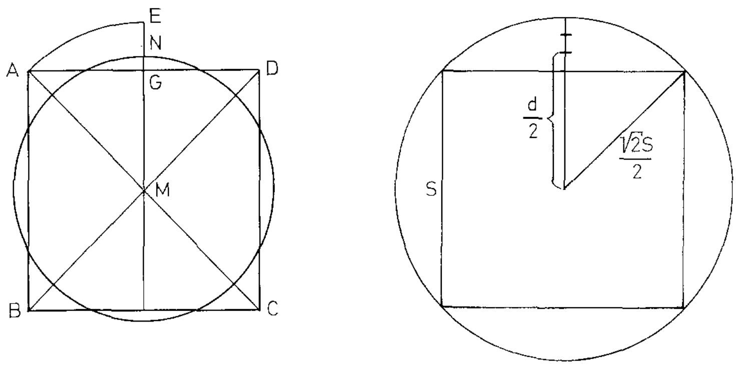

Fig. 9.

As I shall have to refer to it in a moment, I give here BAUDHĀYANA's solution for the squaring of the circle. BAUDHĀYANA Śulvasūtra I, 60 says:

> If you wish to turn a circle into a square, divide the diameter into 8 parts, and again one of these 8 parts into 29 parts; of these 29 parts remove 28, and moreover the sixth part (of the one part left) less the eighth part (of the sixth part).

The meaning is: side of required square $=\frac{7}{8}+\frac{1}{8\cdot 29}-\frac{1}{8\cdot 29\cdot 6}+\frac{1}{8\cdot 29\cdot 6\cdot 8}$ of the diameter of the given circle.

The Śulvasūtras have the rational approximation $\sqrt{2}=1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$ $\left(\text{more precisely: the diagonal of a square}=1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}\text{ of a side}\right)$. ĀPASTAMBA uses it to construct a square, but he had exact ways of doing this, and I can see no advantage to using this approximation for the construction. I think he – I mean his school – was losing the meaning of what was being done. BAUDHĀYANA has the approximation but does not use it for constructing a square.

The only *bona fide* place I can see for the approximation is in the squaring of the circle. In effect, the priests had to get $\frac{3}{2+\sqrt{2}}d$. Now actually I think they knew how to solve geometrically an equation $ax=b$: there is a sutra on this, though it's corrupt.[46] But, even assuming they knew this, they apparently didn't think of it when trying to get $\frac{3}{2+\sqrt{2}}d$. I also think a pure surd in the denominator would not have stopped them, but they did not know how to deal algebraically with the denominator $2+\sqrt{2}$ (*i.e.*, to rationalize it). So they went over to an arithmetical solution: the $\sqrt{2}$ in $\frac{2+\sqrt{2}}{3}$ is approximated with a

[46] This will be discussed in the Appendix.

rational number and then the reciprocal is arithmetically transformed. This is the source of the expression for $\frac{s}{d}$ mentioned just a moment ago.

The approximation to $\sqrt{2}$ is the only clear approximation to a square root in the Śulvasūtras.[47]

The Śulvasūtras have Theorem II 4 of *The Elements*, and they even explain that a square of side $1\frac{1}{2}$ puruṣas has area $2\frac{1}{4}$ square puruṣas, and a square of side $2\frac{1}{2}$ has area $6\frac{1}{4}$. But, although they compute the area of the Mahāvedi, they never have to apply Theorem II 4, or ĀPASTAMBA Śulvasūtra III 9, at least there's no visible application. The only way I can see it coming into any other part of the Śulvasūtras is in the finding of the approximation to $\sqrt{2}$. THIBAUT has already explained how, starting from $1+\frac{1}{3}+\frac{1}{3\cdot 4}$, one could get the (closer) approximation $1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$ (though he missed the point that the same method leads from $1+\frac{1}{3}$ to $1+\frac{1}{3}+\frac{1}{3\cdot 4}$).[48]

Thus ĀP.Ś.S III 9 and $\sqrt{2}=1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$ are both parts of squaring the circle, and all three are erudition. ĀP.Ś.S. III 9, which is, indeed, purely arithmetical, could very well have arisen from a geometric problem.

So we see in the Śulvasūtras the roots for going over to arithmetic methods. Now rites tend to fall away but parts of them can go on having a separate existence. (Building a brick altar is expensive!) Can one not easily enough imagine these

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[47] There are slight indications of one or two other approximations to square roots. The Śulvasūtras also square the circle by taking 13/15 of the diameter $2r$ as the side $2a$ of the desired square. According to a suggestion of C. MÜLLER (*Abhd. Math. Sem. Hamburg*, vol. **7** (1929)), the 13/15 is obtained as follows: One starts with $3r^2$ as the area of the circle. Then $a=\sqrt{3}\,r/2$. An approximation to $\sqrt{3}$ in accordance with note 48 below is $1+2/3+1/15$, or 26/15. Thus twice the 13/15 is an approximation to $\sqrt{3}$. Note that this approximation also occurs in the context of squaring the circle. In another example, BAUDHĀYANA constructs a square of side $3-1/3$ (see Appendix). This looks as though it may have been an approximation to $\sqrt{7}$.

[48] Start with the approximation $x=1+1/3$ to $\sqrt{2}$. Consider a 1×1 and a $4/3\times 4/3$ square. Divide the 1×1 square into 9 little squares each of side $a=1/3$; and the $4/3\times 4/3$ will be divided into 16 squares of area a^2. The large square lacks $2a^2$ from being a square of area 2. So take 2 strips of length 4/3 and area a^2, therefore of width $1/3\cdot 4$, and adjoin them to the $4/3\times 4/3$ square. This gives a $(1+1/3+1/3\cdot 4)\times(1+1/3+1/3\cdot 4)$ square except that a $1/3\cdot 4\times 1/3\cdot 4$ square is missing; and so $1+1/3+1/3\cdot 4>\sqrt{2}$. Now divide this larger square, which is $17/12\times 17/12$, into 17×17 squares of side $b=1/3\cdot 4$ (and the 1×1 square into 12×12 squares of area b^2). So the large square exceeds a square of area 2 by b^2. Proceeding as before, one will cut off 2 strips of length 17/12 and area $1/2\,b^2$, therefore of width $1/3\cdot 4\cdot 34$. This gives a $(1+1/3+1/3\cdot 4-1/3\cdot 4\cdot 34)\times(1+1/3+1/3\cdot 4-1/3\cdot 4\cdot 34)$ square, which however has area >2, since the overlap of the two strips has been subtracted only once instead of twice. For another remark connecting gnomon and square root, see my paper "On the Area of a Semi-Circle," *Archive for History of Exact Sciences*, vol. **9** (1972), p. 180.

arithmetic methods developing into Old-Babylonian mathematics? By the time we come to 1700 B.C., we get a purely secular mathematics that no longer has any use for the exact geometric constructions.

One may wonder whether the ritualists realized that their circulature of the square was not exact, but there is no reason to doubt that they saw that the approximation to $\sqrt{2}$ was not exact. The commentators could see it easily enough and know that there is an excess. If the square were built on the diagonal, it would not exceed twice the given square "even by the smallest part of the minute *nīrvāra* grain falling from the mouth of a parrot." A couple of commentators use this figure (see DATTA, pp. 198, 200). The excess, though minute, is just the kind of difference that would loom large in the eyes of a modern mathematician and of an ancient ritualist. The approximation didn't have to enter the rites, and one side may have shunned it as unnecessary. Thus BAUDHĀYANA, though he knows the approximation, does not use it to construct a square.[49]

So there was a split: on one side we get Old-Babylonian mathematics, on the other, the Vedic mathematics of the Śulvasūtras.

Now I want to go back to VAN DER WAERDEN's theory on the relation between the Babylonian treatment of quadratics and the treatment as found in *The Elements*, or, rather, the simpler version known to the Pythagoreans (see § 11 of "Defense"). This theory is rather compelling. One might wonder about the lines of transmission, but the mathematical comparisons cannot be dismissed. On the other hand the theory is embedded in a theory on the generation of geometric algebra that we have found reason to reject. So my attitude will be to save the phenomenon.

In the 19th Century and the early years of the 20th, there was some talk about the Vedic priests knowing the irrationality of the $\sqrt{2}$. Many people denied it; but they were inclined to deny everything. Anyway, I'm not persuaded that the Vedic priests knew the irrationality. But they did know (I'm thinking) that their approximation was not exact. And this was enough! This inexactitude (and perhaps the realization that closer approximations obtained by the same method would not remove the inexactitude) led one side to avoid arithmetic methods if it could. So the $\sqrt{2}$ (in the picture I'm developing) plays a role something like the one VAN DER WAERDEN assigns to it.

Now let us also imagine that the side expanding the arithmetic methods still retained for a while the ritual apparatus. Just as later in India there were many schools, so also earlier (perhaps in or near Old-Babylonia) there were schools developing in somewhat different ways. And since their objects were really the same, they may have maintained some contact. Now one side, the side that was to become Babylonian mathematics, invented or acquired the simultaneous problem $x+y=a$, $xy=C$ and the other normal problems previously mentioned. Just why the priests invented these problems I cannot say, but that they should consider the rectangle a worthy object of thought is hardly surprising; and all these problems, as we have seen, are problems on the rectangle. Another school – I won't say the other side, rather it is in the middle – wanted these developments, and kept them in geometric formulation. It is from this school that the Pythagoreans got their theory of equations.

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[49] One may note that ĀPASTAMBA does not use the approximation to square the circle.

So the picture I'm drawing is very much like VAN DER WAERDEN's. What I mean to achieve by it is a more satisfactory picture of the lines of transmission.

It may be said that I'm using my imagination and that I cannot point to the postulated source. That's true. But it is not so long ago that one thought that PYTHAGORAS invented the Theorem of PYTHAGORAS, though one knows now that it preceeded PYTHAGORAS by over 1100 years, so it can't be outrageous to suggest that some of the other things attributed to the Pythagoreans could also be that old or older. Anyway, there is no evidence conflicting with the suggestion.

As to the common source of Babylonian and Vedic mathematics, though at one point in the argument I used the word *postulate*, I now regard my thesis as *proved*.

To sum up the argument:

A comparison of Pythagorean and Vedic mathematics together with some chronological considerations showed that the current view on the the generation of geometric algebra is not tenable. A common source for the Pythagorean and Vedic mathematics is to be sought either in the Vedic mathematics or in an older mathematics very much like it. The view that Vedic mathematics is a derivative of Old-Babylonian having been rejected, a common source for these mathematics, different from Old-Babylonia of 1700 B.C., was indicated. Thus what are regarded as the two main sources of Western mathematics, namely Pythagorean mathematics and Old-Babylonian mathematics, both flow from a still older source.

What was this older, common source like? I think its mathematics was very much like what we see in the Śulvasūtras. In the first place, it was associated with ritual.[50] Second, there was no dichotomy between number and magnitude: it did not assign a number to the diagonal of a unit square, not because it refused to do so, but because it didn't know what the number was. Traditions I and II were still in the future, though they arose from tendencies present in the source. In geometry it knew the Theorem of PYTHAGORAS (especially Aspect I) and how to convert a rectangle into a square. It knew the isosceles trapezoid and how to compute its area. It had a facility with (whole) numbers and knew some number theory centered on the existence of Pythagorean triples. It had a canonical solution for the circulature of the square; and in its attempts to square the circle, it learned how to compute a square root. The arithmetical tendencies here encountered were expanded, and in connection with observations on the rectangle, led to Babylonian mathematics. A contrary tendency, namely, a concern for exactness in thought (or the myth of its importance), together with a recognition that arithmetic methods are not exact, led to Pythagorean mathematics.

Appendix: Vedic Mathematics

The object of this section is to provide a somewhat broader view of Vedic mathematics. I will also make some chronological remarks, but as to chronology I can do no better than what I've already done with the bird altars.

[50] Since in the foregoing I have confined myself to mathematical comparisons, the ritual aspects have been slighted; and the thesis that not only the mathematics of the rituals but the rituals themselves go back to before 1700 B.C. needs more argumentation. This has been done in my "Ritual Origin of Geometry" and will be done still more amply in my "Geometry of the Vedic Rituals." In the Appendix, below, I show that the problem of the squaring of the circle has a ritual origin.

The *Taittirīya Saṃhitā* V 4, 11 speaks of a number of altars (the so-called Kāmya, or "optional," altars) to be employed for special desires. The BAUDHĀYANA *Śulvasūtra* goes through the list seriatim; and except for the first three altars, which are somewhat more complicated to describe than the others, the ĀPASTAMBA *Śulvasūtra* does the same, introducing each altar with a citation from the Taittirīya Saṃhitā. From this alone it is plausible to suppose that the altars at the time of the Taittirīya Saṃhitā had the same shapes and sizes as later and that their constructions were the same (or essentially the same), but this will be made still more plausible.

A perusal of Taitt. Saṃ. V 4, 11 (cited in "Ritual Origin of Geometry," p. 507) shows that there are some Brāhmaṇa-like explanations there, but absolutely nothing on the sizes, and almost nothing on the shapes, of the altars. Nor should this have been expected: the place for such details is in the Śulvasūtras (or some other part of the Kalpasūtras) and we do, in many cases, find them there.

Āp.Ś.S. XII 3 speaks of the "modifications" of the basic $7\frac{1}{2}$-fold agni for the "special desires" and since, in many cases explicitly and I believe in all, these altars have (upon initial construction) an area of $7\frac{1}{2}$ puruṣas, it is clear that the "modifications" refer to shape and not to area. Underlying these transformations is the view that the shape for some reason may be changed, but the area is to remain constant.

The seven (and a half) -fold bird altars, even the complicated ones involving bricks of many shapes, do not involve the Theorem of PYTHAGORAS (though the eight (and a half) -fold do). On the other hand, even the simple triangular altar, as constructed in the Śulvasūtras, involves the theorem. The triangle is obtained from a square of area 15 square puruṣas (see Fig. 10). To construct the square, the Theorem of PYTHAGORAS is applied. This *suggests* that Aspect I of the Theorem was already known and applied at the time of the Taittirīya Samhita. We have seen that Aspect II was "nearly certain[ly]" known at that time.

The trough mentioned in the *Taittirīya Saṃhitā*, as described in the Śulvasūtras, is a figure having the shape of two juxtaposed rectangles (see Fig. 11). In the *Baud. Ś.S.* the large rectangle is a square; "its side is 3 puruṣas less one third" (*cf.* III, 219–224). The small rectangle is "one puruṣa less one third" by "half a puruṣa plus ten aṅgulis ($=\frac{1}{12}$ of a puruṣa)." Thus its area is $7\frac{1}{2}$ square puruṣas. *Āp.* S.S. XIII 10 says that the handle is "one-tenth of the whole area." From the description of the bricks one sees that the two rectangles are squares. XIII 10 continues: "Its subtraction ... has been taught." This would appear to mean the following: One starts with a

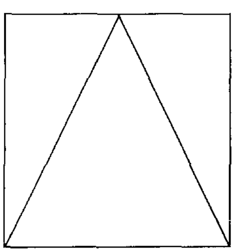
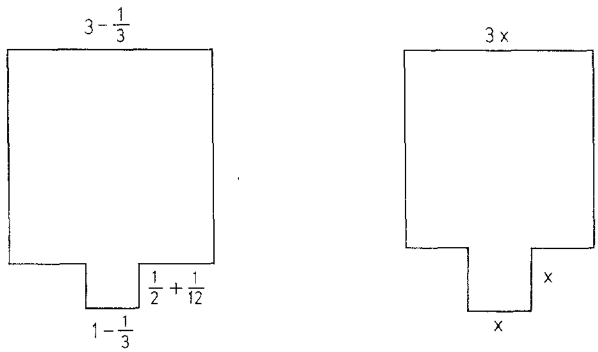


Fig. 10. Construction of a triangle of area $7\frac{1}{2}$ square puruṣas.



Trough according to BAUDHĀYANA Trough according to ĀPASTAMBA

Fig. 11.

square of $7\frac{1}{2}$ square puruṣas. Then one divides this into two rectangles, one of which is one-tenth of the whole. Then one converts the smaller of these rectangles into a square. *Etc.* Note that the side of the large square is exactly three times that of the small; if the size were not (understood to be) specified, the altar could be described in this simpler way.

The construction of the trough as described by ĀPASTAMBA is a problem in converting a polygon of given area into a polygon of given shape. As described by BAUDHĀYANA, it is not; but BAUD. Ś.S. III, 237–242, has a second trough involving a similar problem.

Āp. Ś.S. XII 11 says: "It is taught: 'He who has enemies should pile the *agni* in the form of a chariot wheel'," and XII 12 says: "One transforms a square area equal to the *agni saratni prādeśa* into a circle." Thus the canonical circulature of the square of III 2 finds an application.

The circulature of the square is not a practical problem; the squaring of the circle, yes, but not the circulature of the square. The circulature of the square arises as a purely theological problem: equivalent altars must have the same area, the area is given via rectangles, and sometimes the altar is to be circular. The reverse problem, the squaring of the circle, is derivative. Thus we see in an utterly clear light the ritual origin of a practical problem.

BAUDHĀYANA, like ĀPASTAMBA citing the Taittirīya Saṃhitā, gives the form of altar for those who desire success in the world of the fathers, as follows: It is an isosceles trapezoid with the eastern base 3 units, the western 2; and the width 6 (Fig. 12). The unit here is not the (ordinary) puruṣa, but is given by the side of a square of area $\frac{1}{2}$ puruṣa. Thus one sees that this trapezoid does, indeed, have area $7\frac{1}{2}$ square puruṣas.[51]

Not only can the Vedic priests compute the area of a trapezoid – the Old-Babylonians could do this, too – they can also tell us how to get the answer. Recall that the Mahavedi is an isosceles trapezoid with its eastern base 24 units, its western 30, its width 36 (see Fig. 13).

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[51] Note that the Mahāvedi has its shorter side toward the east, whereas the Śmaśānacit has the shorter side toward the west. Perhaps this is so because at death things should be turned around.

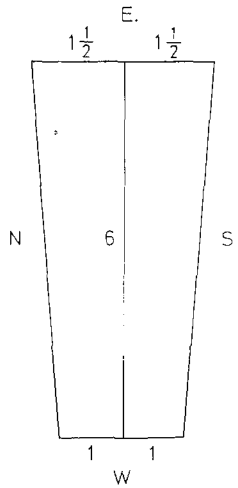

Fig. 12.

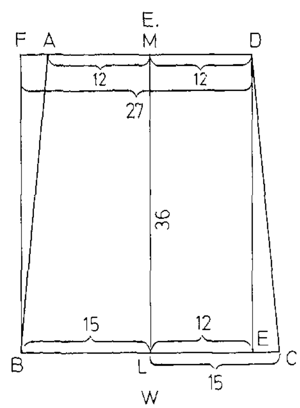

Fig. 13.

Āp. Ś.S. V 7 says that its area is 972 square units.

> (To establish this), one draws (a line) from the southern *amsa* (*D* in Fig.) toward the southern *srōni* (*C*), (namely) to (the point *E* which is) 12 (*padas* from the point *L* of the *prsthya*). Thereupon one turns the piece cut off (i.e., the triangle *DEC*) around and carries it to the other side (i.e., to the north). Thus the *vedi* obtains the form of a rectangle. In this form (*FBED*) one computes its area.

The striking thing here is that we have a proof. One will look in vain for such things in Old-Babylonia. The Old-Babylonians, or their predecessors, must have had proofs of their formulae, but one does not find them in Old-Babylonia.

The isosceles trapezoid is, as we have seen, of great importance in India. I have the impression that this is so in Old-Babylonia, too. In Greece it is at best a minor figure. It does not even occur in the first six books of *The Elements.*

Āp. Ś.S. V 8 says that the vedi at the Sautrāmani sacrifice is to be one-third that of the vedi at the soma sacrifice, *i.e.*, of the Mahāvedi previously described. One may either use a new unit equal to the side of a square of area $\frac{1}{3}$ of a puruṣa, using the numbers 24, 36, 30 as before; or one can use a unit equal to the side of a square of 3 puruṣas, using respectively the numbers 8, 12, 10. The vedi at the horse sacrifice is to be twice as large as the Mahāvedi. Here one uses a new unit equal to the diagonal of a unit square. (*Cf. Āp. Ś.S.* V 10 and VI 1.)

BAUDHĀYANA describes the construction of a chariot wheel having sixteen spokes. One starts by constructing a square of area equal to 1/30 of a square puruṣa. Then a square of $7\frac{1}{2}$ square puruṣas can be exactly covered by 225 of these bricks arranged in 15 columns and 15 rows. To this, 64 such squares are to be added, to get a 17×17 square. One is directed first to place bricks about the 15×15 square to get a 16×16 square. "Thirty-three bricks still remain" and these are now placed so as to yield a 17×17 square. THIBAUT was puzzled by these directions: why not simply form the 17×17 square at once? THIBAUT cites the explanation of a commentator, which, however, he does not find altogether satisfactory. I would suggest the following. The construction of the 17×17 square is so-to-speak "paper work." The 17×17 square is not an altar, and no sacred reason is to be sought for the directions. BAUDHĀYANA is merely trying to convey that if a square number, say m^2, is the sum of s successive odd numbers, say $2n+1$, $2n+3$, *etc.*, then an area of m^2 square units can be added to an $n \times n$ square by successively adjoining gnomons of area $2n+1$, $2n+3$, *etc.*, to give a larger square (in fact, an $(n+s) \times (n+s)$ square). Thus 64 is a square ($=8^2$) and is the sum of the successive odd numbers 31 and 33; and $31 = 2 \times 15 + 1$. So $15^2 + 8^2 = 17^2$. This is what BAUDHĀYANA wished to convey.

Now BAUDHĀYANA draws a 12×12 and a 4×4 square, both centrally placed inside the 17×17 square (see Fig. 14):

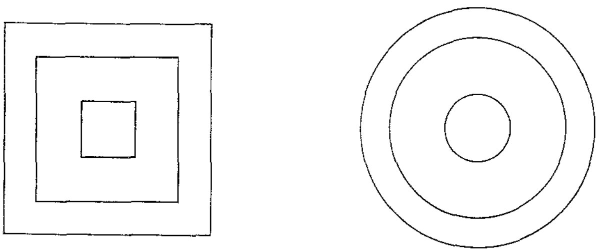

Fig. 14.

The difference $(289 - 144)$ of the large square and the next larger square will be the area of the felloe of the wheel. One converts the 17×17 square into a circle, and removes the concentric circle equal to the 12×12 square; this gives the outer and inner boundaries of the felloe. Then one converts the 4×4 square into a circle and this will be the hub. The space between the felloe and the hub has area 128 ($= 144 - 16$). Now one-half of 128 is 64. Then 64 units are the spaces between the spokes!

So altogether the area of the wheel is $289-64$ or 225 bricks each a 1/30 of a puruṣa, or $7\frac{1}{2}$ square puruṣas.

There is, perhaps, nothing very profound here, but I think if the reader will play a bit with the above facts, he will agree that the Vedic priests had a noteworthy facility with numbers.

On the problem of converting a square into a rectangle of given side, *Baud. Ś.S.* I 53 and *Āp. Ś.S.* III 1 say essentially the same thing.

Āp. Ś.S. says:

> In order to turn a square into an oblong, make a side as long as you wish the oblong to be (i.e., cut off from the square an oblong one side of which is equal to one side of the desired oblong); then join to that the remaining portion as it fits.

This is not intelligible. Turning to the commentators for help, we learn:

> Given, for instance, a square the side of which is five, and required an oblong one side of which is equal to three. Cut off from the square an oblong the sides of which are five and three. There remains an oblong the sides of which are five and two; from this we cut off an oblong of three by two, and join it to the oblong of five and three. There remains a square of two by two, instead of which we take an oblong of 3 by $1\frac{1}{3}$. Joining this oblong to the two oblongs joined previously we get altogether an oblong of 3 by $8\frac{1}{3}$, the area of which is equal to the square 5 by 5.

But this is ridiculous! The hermetic character of the *Śulvasūtras* often forces us to rely on the commentators; on the other hand, the commentators are not always reliable. The arithmetic form of the commentary shows that the commentators (who are already in the algebraic tradition) have not gotten at the meaning of the sutra.

The commentary involves, incidentally, a begging of the question, but the commentators are led into this by following the first part of the sūtra. They themselves would simply have given the other side at once to be $8\frac{1}{3}$. Actually, what they have done is even worse than begging the question, for the sūtra is considering the case that the given side is less than the side of the square, whereas the commentators, after some preliminary maneuvers, run into the case that the given side is *greater* than the side of the given square, and they do not show us how to take a single step in that case.

One commentator does, indeed, give a correct solution to the problem.[52] On the side BC of the given square $ABCD$, lay off the given (shorter) side CH of the proposed rectangle. (See Fig. 15.) Complete the rectangle $HCDG$ and let CG meet BA in E. Then BE is the other side of the required rectangle, and $HCFJ$ is one such rectangle. In fact, subtracting the congruent triangles CHG, CDG and the congruent triangles GAE, GJE from the congruent triangles CBE, CFE (much as in the *Elements* I 44), one sees that rectangle $BHGA=$ rectangle $GDFJ$ and hence square $ABCD=$ rectangle $HCFJ$.

[52] The commentator considers the case that the given side is *longer than* the side of the given square, whereas I will consider the case that it is *shorter*, though strictly in the spirit of his solution. The reason I proceed this way is to facilitate a comparison with the sūtra.

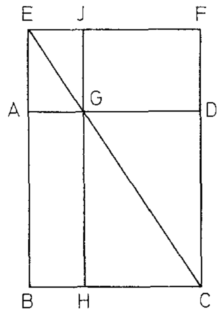

Fig. 15.

The question is: Did the Vedic priests have the indicated solution? BÜRK thinks not, since the method is too scientific. "There can be no doubt about it." He thinks, rather, that the author must have had some more primitive method in mind, probably the arithmetic one indicated. But that is almost certainly out. He notes, too, that the author does not start as the commentator does with the larger rectangle *EBCF* but definitely with the smaller rectangle *HCDG*. Apparently he didn't realize that the commentator was dealing with the case in which the given side (*CF*) is larger than the side of the square, whereas the sūtra is considering the case in which the given side (*CH*) is smaller.

Of course, we cannot without more ado ascribe the commentator's solution to the Vedic ritualists, but if we are obliged to give some *geometric* meaning to the sūtra, if we take into account that the sūtra to some extent coincides with a correct solution, even up to directing us to make rectangle *GDFJ* equal rectangle *ABHG*, and if we observe that this solution is in character with the rest of the Śulvasūtras, then we shall be inclined to credit the ritualists with a correct solution. The sūtra is corrupt and there is nothing to do but try to find the more plausible interpretation.

A solution, whether correct or not, is nowhere explicitly applied in the Śulvasūtras, but, as we shall see, there are one or two places where it might have been applied.

We have seen that the Śulvasūtras can construct a right angle using the Theorem of PYTHAGORAS. They can also do this without that theorem. They have other elementary constructions, for example, they can construct a perpendicular to a line at a given point; or a rectangle of given sides; or a right triangle with given hypotenuse and side, which they need in subtracting one square from another. They speak of inscribing a square as large as possible in a circle, and presumably know how to do this. They also can construct the perpendicular bisector of a given segment along Euclidean lines; but the constructions are for the most part with pegs and cord (sometimes with bamboo rods), and I dare say that to bisect a cord of length *AB*, they bring the ends of the cord together and draw the doubled-up cord taut.

Sometimes a circle had to be divided into a number of equal parts by diameters; for example, in the chariot wheel mentioned, into 32 parts. Though the Śulvasūtras do not give any details, the priests would have had no difficulty in bisecting an angle, so the 32 parts could be constructed easily enough. In another case it is required to divide the circle similarly into 6 equal parts (*cf. Baud. Ś.S.* II 77). In one construction (*cf. Āp. Ś.S.* IV 5) they do construct, at least tacitly, an equilateral triangle, but whether they realized that 6 of these would exactly fit around a point, I cannot say.

The Śulvasūtras sometimes have to divide a segment into n equal parts (*e.g.*, $n = 3$, $n = 7$), but they say nothing on how to do it. The *Śatapatha Brāhmana* mentions that a cord is to be divided into 7 equal pieces: this is done by making the cord overlap itself 7 times. This could be done by successive approximations, but such a method goes against the spirit of exact constructions. The Śulvasūtras might have solved the problem by using the construction on turning a square into a rectangle of specified side (if they had it); but there is not a word on this. If they did have a solution, they had a complete system. If not, there is a yawning gap.[53]

There is some confusion in the rules for augmenting a bird-altar. The sacred injunctions seem clear enough: the shape is to remain constant and the successive increments are one square puruṣa. BAUDHĀYANA and KĀTYĀYANA follow these injunctions in a straightforward way: with them the "n"-fold bird-altar is obtained simply by expanding the basic altar of $7\frac{1}{2}$ square puruṣas up to $n+\frac{1}{2}$ square puruṣas. But ĀPASTAMBA in VIII 6 expands the 7 (*i.e.*, the bird without the additional space for wings and tail) up to n; and some question remains as to what is to be done about the "half." ĀPASTAMBA himself gives two different rules for the *agni* at the horse sacrifice. Thus XXI 6 says: "For the horse sacrifice it is taught: 'The agni is 3 times as great'," and XXI 7 adds: "Here everything is enlarged (not only the *puruṣas*, but also the *aratnis* and *prādeśa*), because nothing special has been prescribed;" while XXI 9 says: "For the horse sacrifice it is taught: "The agni is 21-fold," and XXI 10 adds: "Here, because of the agreement with the number (21), (only) the *puruṣas* are enlarged, not (however) the two *aratnis* and the *prādeśa*." Probably originally some one definite thing was intended. Now XXI 7 leads to an altar having area $22\frac{1}{2}$ square puruṣas and one could object that this is not 21-fold. According to THIBAUT, the alternative was to maintain widths of the additional parts for the wings and tail: this gives an area of $21+\sqrt{3}\cdot\frac{1}{2}$, which is less than 22, and so can be considered as still 21-fold.

There is some difficulty about the words "aratni" and "prādeśa" (or "vitasti"). These are, it would appear, linear measurements, being respectively $\frac{1}{5}$ and $\frac{1}{10}$ of a puruṣa. But the addition to a wing of the basic altar is always spoken of as an aratni, and in context could be taken to mean $\frac{1}{5}$ square puruṣas. Indeed, that is how the word is used by the *Śatapatha Brāhmana* in describing the "98"-fold altar. If, then, in XXI 10 we may take "aratni" and "prādeśa" in the indicated sense, then a possible interpretation of XXI 10 is to use the rule for converting a square into an oblong of given side to augment each wing by exactly $\frac{1}{5}$ square puruṣa and the tail by exactly $\frac{1}{10}$.

[53] I was amazed recently to learn that the rituals in question, which I had thought were extinct, are still being conducted in India. Professor FRITS STAAL, of the University of California at Berkeley, filmed (parts of) the 12-day Atiratra Agnicayana ritual, of which I have seen about 8 hours worth. A beautiful production! He is planning on bringing out a 1-hour documentary. Upon my request Professor STAAL asked a priest how a cord is divided into 5 equal parts. According to the answer, one makes the cord overlap itself 5 times approximately and then makes a couple or so of adjustments.

On the other hand, if *aratni* and *prādeśa* must be taken as linear measurements, and if the altar at the horse sacrifice is to be $21\frac{1}{2}$, then, as has been noted by B. DATTA,[54] the length x of the "new" puruṣa must satisfy $7x^2+\frac{1}{2}x=21\frac{1}{2}$. This brings us to the knotty problem of whether the Vedic priests could solve (complete) quadratic equations. Since there is nothing in the Śulvasūtras on this, there are grounds for skepticism. Nevertheless it is conceivable that contemplation of the bird altar generated questions and answers that did not make their way into the Śulvasūtras.

It has previously been mentioned that the altars were, for the most part, made up of five layers of 200 bricks each. A large part of the Śulvasūtras is taken up in describing the shapes of the bricks and the positions in which they are to be placed. An examination of this material will show that none of the theorems we have spoken about so far, except that of constructing a rectangle, is tied up with the bricks. There remains the question of whether there is *anything* new in the material on the bricks.[55]

The statement or implication that bricks of specified shapes can be assembled into a figure of specified shape does indeed involve some mathematics, but beyond that it is all only implicit in the Śulvasūtras; and even this implicit mathematics is for the most part contained in the other parts of the Śulvasūtras. Here are a few points on the bricks:

The Śulvasūtras know that if a rectangular brick is divided into two along a diagonal, then the two parts are congruent; but this, I would say, is also recognized in the construction of the triangular Kāmya altar previously mentioned. BAUDHĀYANA knows that if an oblong brick is divided into four by both diagonals (see Fig. 16), then one gets two pairs of congruent triangles; since the two types even

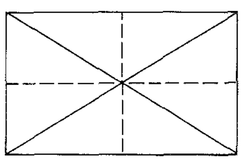

Fig. 16.

have names, this is quite clear (cf. *Baud. S.S.* III 168–9, 178). He also knows that a (certain) half of one type is congruent to a half of the other. Since the other parts of the Śulvasūtras show an understanding of symmetry, and since these observations could follow upon regarding the oblong relative to two axes of symmetry, one could perhaps claim that there is nothing much new here.[56]

54 *Op. cit.*, p. 174.

55 There are some indications that the Vedic Indians got their bricks from an indigenous source (*cf.* H.S. CONVERSE, "The Agnicayana Rite: Indigenous Origin?" *History of Religion*, vol. **14** (1974), 81–95). This makes it relevant to see what mathematics, if any, is specifically tied up with the bricks. This subject will be pursued in my paper "The Geometry of the Vedic Rituals."

56 In one of the constructions of a square (*cf. Āp. Ś.S.* I 7), the two indicated axes of symmetry explicitly occur.

In a third example, BAŪDHAYANA, after constructing an isosceles triangle (of base c and height $3c$), has a sūtra (III 256) that says: "This triangle is divided into ten parts." A commentator explains why this must mean the following: The base and the sides are each divided by three marks into four equal parts. The first mark on the base is joined to the nearest mark on the adjacent side, the second with the next, the third with the remaining mark; and similarly with the base and the other side. In this way the triangle is broken up into 4 congruent triangles and 6 congruent "double triangles." Here we may credit BAUDHĀYANA with realizing that the line joining the midpoints of two sides of a triangle is one-half of the third. (The phenomenon occurs in the head of the bird of Fig. 6, though in special form.)

These examples illustrate the kind of geometrical knowledge one may read out of the material on the bricks.

There was also the requirement that no edge of a brick lie over the edge of another, except at the boundary. Some care was needed, then, in dividing the layers up into the required number of parts. Some of the solutions in the Śulvasūtras are rather neat and all show a good perception of space. DATTA (*op. cit.*, Chap. 14) has shown how some of these solutions can be found by a Diophantine analysis; but, of course, it does not follow that the Vedic priests found them that way. Sometimes they appear to have proceeded as follows: Consider the case that the figure is to be divided into 200 parts. One first divides the area up fairly neatly into about 200 parts, say 198. Now one can always increase the number of bricks by one by splitting a brick in two. Doing this twice, one gets the 200. Or if the number were greater than 200, say 201, one may merge two bricks, to get the required 200. This may show ingenuity, but hardly qualifies as mathematics.

Chronological remarks. The philosophy, or theology, underlying the Kāmya altars in the Śulvasūtras is that equivalent altars are to have the same area: with this requirement one gets the geometric problems, without it one doesn't. If it could be proved that this philosophy was known at the time of the Taittirīya Saṃhitā, this would make it quite plausible that the geometry of the Kāmya altars as seen in the Śulvasūtras was also known at that time. Now the Taitt. S. knows the Kāmya altars, but says nothing about their relative areas; this is the difficulty.

How far are we already from a proof? The Śulvasūtras refer to the Taitt. S. for the Kāmya altars; in Taitt. S. V, 2, 5, 1*ff.* the shape and size of the basic bird altar can be recognized – thus the basic size of the Kāmya altars was known; the Taitt. S. knows at least one application of the Theorem of PYTHAGORAS, the main theorem needed for the transformations of the Kāmya altars. Thus the claim that the Taitt. S. knows the philosophy has a great deal of plausibility to start with.

I will first try to show that the Śatapatha Brāhmana knows the philosophy. Since the Brāhmaṇas can be considered as a kind of commentary on the Saṃhitās, or more exactly said, on the associated rituals, this will make it plausible that the Saṃhitās knew the philosophy, too.

The mathematics of the bird altar, especially that involved in its augmentation, is the same as that of the Kāmya altars: in both one has to construct a figure similar to one given figure and equal in area to another. The object in showing that the Śat. Br. knows the philosophy underlying the constructions of the Kāmya altars is, however, not to show that it knows the mathematics of the Śulvasūtras, for we already know that; rather, the issue for the moment is simply to show that the Śat. Br., which does not speak of the variations in the Kāmya altars, knows the philosophy of equivalence through area. Indeed, the fact that the Śat. Br. knows the mathematics and that the bird altar looks like just a special case of the Kāmya altars is another reason for thinking that the Sat. Br. knows the philosophy.

In the Agnicayana rite there is a large area in trapezoidal form called the Mahāvedi. To the west of this is a smaller rectangular area, the Prācīnavamsa, containing three fire altars in specified positions, the Gārhapatya, the Āhavanīya, and the Dakṣiṇāgni (see Fig. 17). Speaking of Śulvasūtra times, we can say that the

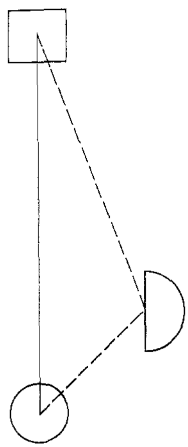

Fig. 17. The Three Fires according to KĀTYĀYANA

Āhavanīya altar was a 1×1 square; the Gārhapatya was, in one version, a circle; and the Dakṣiṇāgni a semicircle (see *Kāty. S.S.* 7, 37 and 7, 38). The three fire altars were already known in Rig-Veda times: they are mentioned, though not by name in *RV* V 11, 2. The Śatapatha Brāhmana refers to the Gārhapatya and the Āhavanīya by name, as in X 2, 3, 1. *Śat. Br.* VII 1, 1, 37 says that the Gārhapatya is circular. EGGELING, in his translation of the Śatapatha Brāhmaṇa gives a "plan of the Sacrificial ground" (in *S.B.E.*, vol. 26, p. 475). This shows the Āhavanīya to be square. Though I have no reference for this in the Śatapatha Brāhmana itself, I will presume that the Āhavanīya was square also in Śatapatha Brāhmana times. Now *Kāty. S.S.* VII 37 gives the Āhavanīya to be a square 24 angulis on a side (24 *angulis* $= \frac{1}{5}$ of a *puruṣa* = 1 *aratni*); the Gārhapatya to be of radius 14; and the Dakṣiṇāgni to be of radius 16. The next sūtra (VII 38) gives the radius of the Dakṣiṇāgni to be $19\frac{1}{2}$ and says that the Dakṣiṇāgni is "semicircular ... of one aratni" (*i.e.*, one square *aratni*). According to KĀTYĀYANA's own circulature of the square (*Kāty. Ś.S.* III 13), which is the same as in the *Baud.* and *Āp. Ś.S.*, the circle (equal to a 24×24 square) should have radius about $13\frac{2}{3}$ $\left(i.e., \frac{1}{2} \times \frac{2+\sqrt{2}}{3} \times 24\right)$ and the Dakṣiṇāgni should have radius about $19\frac{1}{3}$ $\left(i.e., \frac{1}{2} \times \frac{2+\sqrt{2}}{3} \times 24 \times \sqrt{2}\right)$, so the 14 and $19\frac{1}{2}$ are about right. [The $9\frac{1}{2}$ in S.D. KHADILKAR's

Kātyāyana Śulva Sūtra, p. 46, is a misprint for $19\frac{1}{2}$.] Thus KĀTYĀYANA explicitly makes the Āhavanīya and Dakṣiṇāgni to be of the same area; and there are indications that all three were to have the same area.

[The tenth chapter of the Mānava Śrauta-Sūtra, translated by J.M. VAN GELDER in the *Śata-Piṭaka Series*, vol. 27, RAGHU VIRA, Editor, is the Śulvasūtra. Here, in 1, 1, 7 and 1, 1, 8, not only are the measurements 24, $13\frac{2}{3}$, and $19\frac{1}{2}$ given, but, further, the constructions themselves are given. Thus, finally, we have an explicit reference for the equality in area of the Āhavanīya, Gārhapatya, and Dakṣiṇagni.]

In the Śat. Br. (VII, 1, 1, 37) the Gārhapatya is said to "measure one vyāma;" a *vyāma* is the same as a puruṣa. Professor F. STAAL has kindly translated the passage for me as follows:

> It (the Gārhapatya) measures on *vyāma*, for man measures one *vyāma*, and man is Prajāpati, and Prajāpati is Agni. Therefore he makes the womb equal in measure. It is circular for the womb is circular. And the Gārhapatya is this world for this world is indeed circular.

The Āhavanīya is square (*cf.* the "plan" in S.B.E., vol. 26, p. 475). Śat. Br. X, 2, 3, 1 says: "... From the raised (site) of the Gārhapatya he strides seven steps eastward. From there he measures off a fathom (vyāma) towards the east, and having, in the middle thereof, thrown up (the ground) for the Āhavanīya, he sprinkles it with water ..." Thus the Āhavanīya is a square one vyāma on a side.

From these two passages (VII, 1, 1, 27 and X, 2, 3, 1) I would like to conclude that the Gārhapatya and Āhavanīya have equal areas. But the question is whether the Śat. Br. VII, 1, 1, 37 really is referring to area: why could it not be saying that the Gārhapatya is one vyāma in diameter? This is a difficulty, and I think one will not come, without interpretation, to an unambiguous meaning from such passages as Śat. Br. VII, 1, 1, 37, for the simple reason that the ritualists had no single word (as we do) to distinguish square from linear measure. Thus we remain one iota short of a strictly textual proof.

In my paper "The Geometry of the Vedic Rituals," I press the argument further. There I show, or try to show, that the geometry of the Śulvasūtras stems from the philosophy of equivalence through area. The argument, involving as it does a mathematical reconstruction, is largely, but not entirely, textual. If the reconstruction is correct, it would follow that the enlargement of the bird altar is subsequent to that philosophy. Or, to put it another way, the philosophy underlying the mathematics of the Kāmya altars was prior to the enlargements described in the Śatapatha Brāhmaṇa.

The object of the present chronological remarks is to show that the Taitt. S. knows not only Aspect II of the Theorem of PYTHAGORAS, but also Aspect I. Since the full argument has not here been given (and since it is not strictly textual), I would like to emphasize that the point is not needed for my thesis. For my thesis it is sufficient to have a date in India for Aspect I prior to 600 B.C. From the discussion of the Śatapatha Brāhmaṇa, I consider this point beyond dispute.

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