International Journal of Foundations of Computer Science
Vol. 23, No. 7 (2012) 1481–1497

DOI: 10.1142/S0129054112400606

THREE APPLICATIONS TO RATIONAL RELATIONS OF THE HIGH UNDECIDABILITY OF THE INFINITE POST CORRESPONDENCE PROBLEM IN A REGULAR ω-LANGUAGE

OLIVIER FINKEL

Equipe de Logique Mathématique
Institut de Mathématiques de Jussieu
CNRS et Université Paris Diderot Paris 7
UFR de Mathématiques case 7012, site Chevaleret
75205 Paris Cedex 13, France
finkel@logique.jussieu.fr

Received 10 October 2010
Accepted 8 May 2011
Communicated by Maurice Margenstern

It was noticed by Harel in [$Har86$] that "one can define Σ_1^1-complete versions of the well-known Post Correspondence Problem". We first give a complete proof of this result, showing that the infinite Post Correspondence Problem in a regular ω-language is Σ_1^1-complete, hence located beyond the arithmetical hierarchy and highly undecidable. We infer from this result that it is Π_1^1-complete to determine whether two given infinitary rational relations are disjoint. Then we prove that there is an amazing gap between two decision problems about ω-rational functions realized by finite state Büchi transducers. Indeed Prieur proved in [$Pri01, Pri02$] that it is decidable whether a given ω-rational function is continuous, while we show here that it is Σ_1^1-complete to determine whether a given ω-rational function has at least one point of continuity. Next we prove that it is Π_1^1-complete to determine whether the continuity set of a given ω-rational function is ω-regular. This gives the exact complexity of two problems which were shown to be undecidable in [$CFS08$].

Keywords: Decision problems; infinite Post Correspondence Problem; analytical hierarchy; high undecidability; infinitary rational relations; omega rational functions; topology; points of continuity.

1. Introduction

Many classical decision problems arise naturally in the fields of Formal Language Theory and of Automata Theory. It is well known that most problems about regular languages accepted by finite automata are decidable. On the other hand, at the second level of the Chomsky Hierarchy, most problems about context-free languages accepted by pushdown automata or generated by context-free grammars are undecidable. For instance it follows from the undecidability of the Post Correspondence Problem that the universality problem, the inclusion and the equivalence problems for context-free languages are also undecidable. Notice that some few problems about context-free languages remain decidable like the following ones: "Is a given context-free language L empty ?" "Is a given context-free language L infinite ?" "Does a given word x belong to a given context-free language L ?" Senizergues proved in [$Sen01$] that the difficult problem of the equivalence of two deterministic pushdown automata is decidable. Notice that almost all proofs of undecidability results about context-free languages rely on the undecidability of the Post Correspondence Problem which is complete for the class of recursively enumerable problems, i.e. complete at the first level of the arithmetical hierarchy. Thus undecidability proofs about context-free languages provided only hardness results for the first level of the arithmetical hierarchy.

On the other hand, some decision problems are known to be located beyond the arithmetical hierarchy, in some classes of the analytical hierarchy, and are then usually called "highly undecidable". Harel proved in [$Har86$] that many domino or tiling problems are Σ_1^1-complete or Π_1^1-complete. For instance the "recurring domino problem" is Σ_1^1-complete. It is also Σ_1^1-complete to determine whether a given Turing machine, when started on a blank tape, admits an infinite computation that reenters infinitely often in the initial state. Alur and Dill used this latter result in [$AD94$] to prove that the universality problem for timed Büchi automata is Π_1^1-hard. In [$CC89$], Castro and Cucker studied many decision problems for ω-languages of Turing machines. In particular, they proved that the non-emptiness and the infiniteness problems for ω-languages of Turing machines are Σ_1^1-complete, and that the universality problem, the inclusion problem, and the equivalence problem are Π_2^1-complete. Thus these problems are located at the first or the second level of the analytical hierarchy. Using Castro and Cucker's results, some reductions of [$Fin06a, Fin06b$], and topological arguments, we have proved in [$Fin09$] that many decision problems about 1-counter ω-languages, context free ω-languages, or infinitary rational relations, like the universality problem, the inclusion problem, the equivalence problem, the determinizability problem, the complementability problem, and the unambiguity problem are Π_2^1-complete. Notice that the exact complexity of numerous problems remains still unknown. For instance the exact complexities of the universality problem, the determinizability, or the complementability problem for timed Büchi automata which are known to be Π_1^1-hard, see [$AD94, Fin06c$].

We intend to introduce here a new method for proving high undecidability results which seems to be unexplored. It was actually noticed by Harel in [$Har86$] that "one can define Σ_1^1-complete versions of the well-known Post Correspondence Problem", but it seems that this possibility has not been later investigated. We first give a complete proof of this result, showing that the infinite Post Correspondence Problem in a regular ω-language is Σ_1^1-complete, hence located beyond the arithmetical hierarchy and highly undecidable. We infer from this result a new high undecidability result, proving that it is Π_1^1-complete to determine whether two given infinitary rational relations are disjoint. Then we apply this Σ_1^1-complete version of the Post Correspondence Problem to the study of continuity problems for ω-rational functions realized by finite state Büchi transducers, considered by Prieur in $[Pri01, Pri02]$ and by Carton, Finkel and Simonnet in $[CFS08]$. We prove that there is an amazing gap between two decision problems about ω-rational functions. Indeed Prieur proved in $[Pri01, Pri02]$ that it is decidable whether a given ω-rational function is continuous, while we show here that it is Σ_1^1-complete to determine whether a given ω-rational function has at least one point of continuity. Next we prove that it is Π_1^1-complete to determine whether the continuity set of a given ω-rational function is ω-regular. This gives the exact complexity of two problems which were shown to be undecidable in $[CFS08]$.

The paper is organized as follows. We recall basic notions on automata and on the analytical hierarchy in Section 2. We state in Section 3 the Σ_1^1-completeness of the infinite Post Correspondence Problem in a regular ω-language. We prove our main new results in Section 4. Some concluding remarks are given in Section 5.

2. Recall of Basic Notions

We assume now the reader to be familiar with the theory of formal (ω)-languages $[Tho90, Sta97]$. We recall some usual notations of formal language theory.

When Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 1. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ.

The *first infinite ordinal* is ω. An ω-*word* over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)\ldots$, where for all i, $\sigma(i) \in \Sigma$, and $\sigma[n] = \sigma(1)\sigma(2)\ldots\sigma(n)$.

The usual concatenation product of two finite words u and v is denoted $u \cdot v$ and sometimes just uv. This product is extended to the product of a finite word u and an ω-word v: the infinite word $u \cdot v$ is then the ω-word such that: $(u \cdot v)(k) = u(k)$ if $k \leq |u|$, and $(u \cdot v)(k) = v(k - |u|)$ if $k > |u|$.

The *set of* ω-*words* over the alphabet Σ is denoted by Σ^ω. An ω-*language* over an alphabet Σ is a subset of Σ^ω.

Definition 2.1. *A Büchi automaton is a* 5*-tuple* $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$*, where* K *is a finite set of states,* Σ *is a finite input alphabet,* $q_0 \in K$ *is the initial state and* δ *is a mapping from* $K \times \Sigma$ *into* 2^K*. The Büchi automaton* $\mathcal{A}$ *is said to be deterministic iff:* $\delta : K \times \Sigma \to K$.

Let $\sigma = a_1 a_2 \ldots a_n \ldots$ *be an* ω*-word over* Σ*. A sequence of states* $r = q_1 q_2 \ldots q_n \ldots$ *is called an (infinite) run of* $\mathcal{A}$ *on* σ*, starting in state* p*, iff:* (1) $q_1 = p$ *and* (2) *for each* $i \geq 1$, $q_{i+1} \in \delta(q_i, a_i)$*. In case a run* r *of* $\mathcal{A}$ *on* σ *starts in state* q_0*, we call it simply "a run of* $\mathcal{A}$ *on* σ *". For every run* $r = q_1 q_2 \ldots q_n \ldots$ *of* $\mathcal{A}$*,* $In(r)$ *is the set of states in* K *entered by* $\mathcal{A}$ *infinitely many times during run* r*. The* ω*-language accepted by* $\mathcal{A}$ *is:*

$$L(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \text{ there exists a run } r \text{ of } \mathcal{A} \text{ on } \sigma \text{ such that } In(r) \cap F \neq \emptyset\}.$$

An ω*-language* $L \subseteq \Sigma^\omega$ *is said to be regular iff it is accpted by some Büchi automaton* $\mathcal{A}$*.*

We recall that the class of regular ω-languages is the ω-Kleene closure of the class of regular finitary languages, see [$Tho90, Sta97$]: an ω-language $L \subseteq \Sigma^\omega$ is regular iff it is of the form $L = \bigcup_{1 \leq i \leq n} U_i \cdot V_i^\omega$, for some regular finitary languages $U_i, V_i \subseteq \Sigma^\star$. Acceptance of infinite words by other finite machines like pushdown automata, Turing machines, Petri nets, ..., with various acceptance conditions, has also been considered. In particular, the class of context-free ω-languages is the class of ω-languages accepted by Büchi pushdown automata, see [$Tho90, Sta97, EH93$].

The set of natural numbers is denoted by $\mathbb{N}$, and the set of functions from $\mathbb{N}$ into $\mathbb{N}$ is denoted by $\mathcal{F}$. We assume the reader to be familiar with the arithmetical hierarchy on subsets of $\mathbb{N}$. We now recall the definition of classes of the analytical hierarchy which may be found in [$Rog67, Odi89, Odi99$].

Definition 2.2. *Let $k, l > 0$ be some integers. Φ is a partial computable functional of k function variables and l number variables if there exists $z \in \mathbb{N}$ such that for any $(f_1, \ldots, f_k, x_1, \ldots, x_l) \in \mathcal{F}^k \times \mathbb{N}^l$, we have*

$$\Phi(f_1, \ldots, f_k, x_1, \ldots, x_l) = \tau_z^{f_1, \ldots, f_k}(x_1, \ldots, x_l),$$

where the right hand side is the output of the Turing machine with index z and oracles $f_1, \ldots, f_k$ over the input $(x_1, \ldots, x_l)$. For $k > 0$ and $l = 0$, Φ is a partial computable functional if, for some z,

$$\Phi(f_1, \ldots, f_k) = \tau_z^{f_1, \ldots, f_k}(0).$$

The value z is called the Gödel number or index for Φ.

Definition 2.3. *Let $k, l > 0$ be some integers and $R \subseteq \mathcal{F}^k \times \mathbb{N}^l$. The relation R is said to be a computable relation of k function variables and l number variables if its characteristic function is computable.*

We now define analytical subsets of $\mathbb{N}^l$.

Definition 2.4. *A subset R of $\mathbb{N}^l$ is analytical if it is computable or if there exists a computable set $S \subseteq \mathcal{F}^m \times \mathbb{N}^n$, with $m \geq 0$ and $n \geq l$, such that*

$$R = \{(x_1, \ldots, x_l) \mid (Q_1 s_1)(Q_2 s_2) \ldots (Q_{m+n-l} s_{m+n-l}) S(f_1, \ldots, f_m, x_1, \ldots, x_n)\},$$

where Q_i is either $\forall$ or $\exists$ for $1 \leq i \leq m + n - l$, and where $s_1, \ldots, s_{m+n-l}$ are $f_1, \ldots, f_m, x_{l+1}, \ldots, x_n$ in some order.

The expression $(Q_1 s_1)(Q_2 s_2) \ldots (Q_{m+n-l} s_{m+n-l}) S(f_1, \ldots, f_m, x_1, \ldots, x_n)$ is called a predicate form for R. A quantifier applying over a function variable is of type 1, otherwise it is of type 0. In a predicate form the (possibly empty) sequence of quantifiers, indexed by their type, is called the prefix of the form. The reduced prefix is the sequence of quantifiers obtained by suppressing the quantifiers of type 0 from the prefix.

We can now distinguish the levels of the analytical hierarchy by considering the number of alternations in the reduced prefix.

Definition 2.5. *For $n > 0$, a Σ^1_n-prefix is one whose reduced prefix begins with $\exists^1$ and has $n-1$ alternations of quantifiers. A Σ^1_0-prefix is one whose reduced prefix is empty. For $n > 0$, a Π^1_n-prefix is one whose reduced prefix begins with $\forall^1$ and has $n-1$ alternations of quantifiers. A Π^1_0-prefix is one whose reduced prefix is empty.*

A predicate form is a Σ^1_n (Π^1_n)-form if it has a Σ^1_n (Π^1_n)-prefix. The class of sets in some $\mathbb{N}^l$ which can be expressed in Σ^1_n-form (respectively, Π^1_n-form) is denoted by Σ^1_n (respectively, Π^1_n).

The class $\Sigma^1_0 = \Pi^1_0$ is the class of arithmetical sets.

We now recall some well known results about the analytical hierarchy.

Proposition 2.6. *Let $R \subseteq \mathbb{N}^l$ for some integer l. Then R is an analytical set iff there is some integer $n \geq 0$ such that $R \in \Sigma^1_n$ or $R \in \Pi^1_n$.*

Theorem 2.7. *For each integer $n \geq 1$,*

(a) $\Sigma^1_n \cup \Pi^1_n \subsetneq \Sigma^1_{n+1} \cap \Pi^1_{n+1}$.
(b) *A set $R \subseteq \mathbb{N}^l$ is in the class Σ^1_n iff its complement is in the class Π^1_n.*
(c) $\Sigma^1_n - \Pi^1_n \neq \emptyset$ *and* $\Pi^1_n - \Sigma^1_n \neq \emptyset$.

Transformations of prefixes are often used, following the rules given by the next theorem.

Theorem 2.8. *For any predicate form with the given prefix, an equivalent predicate form with the new one can be obtained, following the allowed prefix transformations given below :*

(a) $\ldots\exists^0\exists^0\ldots \to \ldots\exists^0\ldots,$
$\ldots\forall^0\forall^0\ldots \to \ldots\forall^0\ldots;$
(b) $\ldots\exists^1\exists^1\ldots \to \ldots\exists^1\ldots,$
$\ldots\forall^1\forall^1\ldots \to \ldots\forall^1\ldots;$
(c) $\ldots\exists^0\quad\ldots \to \ldots\exists^1\ldots,$
$\ldots\forall^0\quad\ldots \to \ldots\forall^1\ldots;$
(d) $\ldots\exists^0\forall^1\ldots \to \ldots\forall^1\exists^0\ldots,$
$\ldots\forall^0\exists^1\ldots \to \ldots\exists^1\forall^0\ldots;$

We now recall the notions of 1-reduction and of Σ^1_n-completeness (respectively, Π^1_n-completeness). Given two sets $A, B \subseteq \mathbb{N}$ we say A is 1-reducible to B and write $A \leq_1 B$ if there exists a total computable injective function f from $\mathbb{N}$ to $\mathbb{N}$ with $A = f^{-1}[B]$. A set $A \subseteq \mathbb{N}$ is said to be Σ^1_n-complete (respectively, Π^1_n-complete) iff A is a Σ^1_n-set (respectively, Π^1_n-set) and for each Σ^1_n-set (respectively, Π^1_n-set) $B \subseteq \mathbb{N}$ it holds that $B \leq_1 A$.

We now recall an example of a Σ^1_1-complete decision problem which will be useful in the sequel.

Definition 2.9. *A non deterministic Turing machine $\mathcal{M}$ is a 5-tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$, where Q is a finite set of states, Σ is a finite input alphabet, Γ is a* *finite tape alphabet satisfying* $\Sigma \subseteq \Gamma$ *and containing a special blank symbol* $\square \in \Gamma \setminus \Sigma$, q_0 *is the initial state, and* δ *is a mapping from* $Q \times \Gamma$ *to subsets of* $Q \times \Gamma \times \{L, R, S\}$.

Harel proved the following result in $[Har86]$.

Theorem 2.10. *The following problem is* Σ^1_1*-complete: Given a Turing machine* $\mathcal{M}_z$*, of index* $z \in \mathbb{N}$*, does* $\mathcal{M}_z$*, when started on a blank tape, admit an infinite computation that reenters infinitely often in the initial state* q_0 *?*

3. The Infinite Post Correspondence Problem

Recall first the well known result about the undecidability of the Post Correspondence Problem, denoted PCP.

Theorem 3.1 (Post, see [*HMU*01]) *Let* Γ *be an alphabet having at least two elements. Then it is undecidable to determine, for arbitrary n-tuples* $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *of non-empty words in* $\Gamma^\star$*, whether there exists a non-empty sequence of indices* $i_1, i_2, \ldots, i_k$ *such that* $x_{i_1} x_{i_2} \ldots x_{i_k} = y_{i_1} y_{i_2} \ldots y_{i_k}$.

On the other hand, the infinite Post Correspondence Problem, also called ω-PCP, has been shown to be undecidable by Ruohonen in $[Ruo85]$ and by Gire in $[Gir86]$.

Theorem 3.2. *Let* Γ *be an alphabet having at least two elements. Then it is undecidable to determine, for arbitrary n-tuples* $(x_1, \ldots, x_n)$ *and* $(y_1, \ldots, y_n)$ *of non-empty words in* $\Gamma^\star$*, whether there exists an infinite sequence of indices* $i_1, i_2, \ldots, i_k \ldots$ *such that* $x_{i_1} x_{i_2} \ldots x_{i_k} \ldots = y_{i_1} y_{i_2} \ldots y_{i_k} \ldots$

Notice that an instance of the ω-PCP is given by two n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of non-empty words in $\Gamma^\star$, and if there exist some solutions, these ones are infinite words over the alphabet $\{1, \ldots, n\}$.

We are going to consider now a variant of the infinite Post Correspondence Problem where we restrict solutions to ω-words belonging to a given ω-regular language $L(\mathcal{A})$ accepted by a given Büchi automaton $\mathcal{A}$.

An instance of the ω-PCP in a regular ω-language, also denoted ω-PCP(Reg), is given by two n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of non-empty words in $\Gamma^\star$ along with a Büchi automaton $\mathcal{A}$ accepting words over $\{1, \ldots, n\}$. A solution of this problem is an infinite sequence of indices $i_1, i_2, \ldots, i_k \ldots$ such that $i_1 i_2 \ldots i_k \ldots \in L(\mathcal{A})$ and $x_{i_1} x_{i_2} \ldots x_{i_k} \ldots = y_{i_1} y_{i_2} \ldots y_{i_k} \ldots$.

Notice that one can associate in a recursive and injective way an unique integer z to each Büchi automaton $\mathcal{A}$, this integer being called the index of the automaton $\mathcal{A}$. We shall denote $\mathcal{A}_z$ the Büchi automaton of index z. Then each instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A}_z)$ can be also characterized by an index $\bar{I} \in \mathbb{N}$.

We can now state precisely the following result.

Theorem 3.3. *It is* Σ^1_1*-complete to determine, for a given instance* $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A}_z)$*, given by its index* $\bar{I}$*, whether there is an infinite* *sequence of indices* $i_1, i_2, \ldots, i_k \ldots$ *such that* $i_1 i_2 \ldots i_k \ldots \in L(\mathcal{A}_z)$ *and* $x_{i_1} x_{i_2} \ldots x_{i_k} \ldots = y_{i_1} y_{i_2} \ldots y_{i_k} \ldots$.

Proof. We first prove that this problem is in the class Σ_1^1. It is easy to see that there is an injective computable function $\Phi : \mathbb{N} \to \mathbb{N}$ such that for all $\bar{I} \in \mathbb{N}$ the Büchi Turing machine $\mathcal{M}_{\Phi(\bar{I})}$ of index $\Phi(\bar{I})$, where $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A}_z)$, accepts the set of infinite words $i_1 i_2 \ldots i_k \ldots \in L(\mathcal{A}_z)$ such that $x_{i_1} x_{i_2} \ldots x_{i_k} \ldots = y_{i_1} y_{i_2} \ldots y_{i_k} \ldots$. Then the ω-PCP(Reg) of instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A}_z)$ has a solution if and only if the ω-language of the Büchi Turing machine $\mathcal{M}_{\Phi(\bar{I})}$ is non-empty. Thus the ω-PCP(Reg) is reduced to the non-emptiness problem of Büchi Turing machines which is known to be in the class Σ_1^1. Indeed for a given Büchi Turing machine $\mathcal{M}_z$ reading infinite words over an alphabet Σ we can express $L(\mathcal{M}_z) \neq \emptyset$ by the formula

$$\exists x \in \Sigma^\omega \ \exists r \ [r \text{ is an accepting run of } \mathcal{M}_z \text{ on } x].$$

This is a Σ_1^1-formula because the existential second order quantifications are followed by an arithmetical formula, see $[CC89, Fin09]$ for related results. Therefore the ω-PCP(Reg) is also in the class Σ_1^1.

We now prove the completeness part of the theorem. Recall that the following problem (P) is Σ_1^1-complete by Theorem 2.10.

(P): Given a Turing machine $\mathcal{M}_z$, of index $z \in \mathbb{N}$, does $\mathcal{M}_z$, when started on a blank tape, admit an infinite computation that reenters infinitely often in the initial state q_0 ?

We can reduce this problem to the ω-PCP in a regular ω-language in the following way.

Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ be a Turing machine, where Q is a finite set of states, Σ is a finite input alphabet, Γ is a finite tape alphabet satisfying $\Sigma \subseteq \Gamma$ and containing a special blank symbol $\Box \in \Gamma \setminus \Sigma$, q_0 is the initial state, and δ is a mapping from $Q \times \Gamma$ to subsets of $Q \times \Gamma \times \{L, R, S\}$.

We are going to associate to this Turing machine an instance of the ω-PCP(Reg). First we define the two following lists $x = (x_i)_{1 \leq i \leq n}$ and $y = (y_i)_{1 \leq i \leq n}$ of finite words over the alphabet $\Sigma \cup \Gamma \cup Q \cup \{\#\}$, where $\#$ is a symbol not in $\Sigma \cup \Gamma \cup Q$.

x	y	
$\# = x_1$	$\# q_0 \# = y_1$	
#	#	
a	a	for each $a \in \Gamma$
qa	q'b	if $(q', b, S) \in \delta(q, a)$
qa	bq'	if $(q', b, R) \in \delta(q, a)$
cqa	q'cb	if $(q', b, L) \in \delta(q, a)$
q#	bq'#	if $(q', b, R) \in \delta(q, \Box)$
cq#	q'cb#	if $(q', b, L) \in \delta(q, \Box)$
q#	q'b#	if $(q', b, S) \in \delta(q, \Box)$

The integer n is the number of words in the list x and also in the list y. We assume that these two lists are indexed so that $x = (x_i)_{1 \leq i \leq n}$ and $y = (y_i)_{1 \leq i \leq n}$. Let now $E \subseteq \{1, 2, \ldots, n\}$ be the set of integers i such that the initial state q_0 of the Turing machine $\mathcal{M}$ appears in the word y_i. The ω-language $L \subseteq \{1, 2, \ldots, n\}^\omega$ of infinite words over the alphabet $\{1, 2, \ldots, n\}$ which begin by the letter 1 and have infinitely many letters in E is a regular ω-language and it is accepted by a (deterministic) Büchi automaton $\mathcal{A}$. We now consider the instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ of the ω-PCP(Reg). It is easy to check that this instance of the ω-PCP(Reg) has a solution $i_1, i_2, \ldots, i_k \ldots$ if and only if the Turing machine $\mathcal{M}$, when started on a blank tape, admits an infinite computation that reenters infinitely often in the initial state q_0.

Thus the Σ_1^1-complete problem (P) is reduced to the ω-PCP in a regular ω-language and this latter problem is also Σ_1^1-complete. □

4. Applications to Infinitary Rational Relations

4.1. *Infinitary rational relations*

We now recall the definition of infinitary rational relations, via definition by Büchi transducers:

Definition 4.1. *A Büchi transducer is a sextuple $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$, where K is a finite set of states, Σ and Γ are finite sets called the input and the output alphabets, Δ is a finite subset of $K \times \Sigma^\star \times \Gamma^\star \times K$ called the set of transitions, q_0 is the initial state, and $F \subseteq K$ is the set of accepting states.*

A computation $\mathcal{C}$ of the transducer $\mathcal{T}$ is an infinite sequence of consecutive transitions

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \ldots (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \ldots$$

The computation is said to be successful iff there exists a final state $q_f \in F$ and infinitely many integers $i \geq 0$ such that $q_i = q_f$. The input word and output word of the computation are respectively $u = u_1.u_2.u_3\ldots$ and $v = v_1.v_2.v_3\ldots$ The input and the output words may be finite or infinite. The infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ accepted by the Büchi transducer $\mathcal{T}$ is the set of pairs $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ such that u and v are the input and the output words of some successful computation $\mathcal{C}$ of $\mathcal{T}$. The set of infinitary rational relations will be denoted RAT_2.

If $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ is an infinitary rational relation recognized by the Büchi transducer $\mathcal{T}$ then we denote

$$Dom(R(\mathcal{T})) = \{u \in \Sigma^\omega \mid \exists v \in \Gamma^\omega \quad (u, v) \in R(\mathcal{T})\}$$

and

$$Im(R(\mathcal{T})) = \{v \in \Gamma^\omega \mid \exists u \in \Sigma^\omega (u, v) \in R(\mathcal{T})\}.$$

It is well known that, for each infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$, the sets $Dom(R(\mathcal{T}))$ and $Im(R(\mathcal{T}))$ are regular ω-languages and that one can construct, from the Büchi transducer $\mathcal{T}$, some Büchi automata $\mathcal{A}$ and $\mathcal{B}$ accepting the ω-languages $Dom(R(\mathcal{T}))$ and $Im(R(\mathcal{T}))$.

To each Büchi transducer $\mathcal{T}$ can be associated in an injective and recursive way an index $z \in \mathbb{N}$ and we shall denote in the sequel $\mathcal{T}_z$ the Büchi transducer of index z.

We proved in $[Fin09]$ that many decision problems about infinitary rational relations are highly undecidable. In fact many of them, like the universality problem, the equivalence problem, the inclusion problem, the cofiniteness problem, the unambiguity problem, are Π_2^1-complete, hence located at the second level of the analytical hierarchy.

We can now use the Σ_1^1-completeness of the ω-PCP in a regular ω-language to obtain a new result of high undecidability.

Theorem 4.2. *It is Π_1^1-complete to determine whether two given infinitary rational relations are disjoint, i.e. the set $\{(z, z') \in \mathbb{N}^2 \mid R(\mathcal{T}_z) \cap R(\mathcal{T}_{z'}) = \emptyset\}$ is Π_1^1-complete.*

Proof. We are going to show that the complement of this set is Σ_1^1-complete, i.e. that the set $\{(z, z') \in \mathbb{N}^2 \mid R(\mathcal{T}_z) \cap R(\mathcal{T}_{z'}) \neq \emptyset\}$ is Σ_1^1-complete.

Firstly, it is easy to see that, for two given Büchi transducers $\mathcal{T}_z$ and $\mathcal{T}_{z'}$, one can define a Büchi Turing machine $\mathcal{M}_{\Phi(z,z')}$ of index $\Phi(z, z')$ accepting the ω-language $R(\mathcal{T}_z) \cap R(\mathcal{T}_{z'})$. Moreover one can construct the function $\Phi : \mathbb{N}^2 \to \mathbb{N}$ as an injective computable function. This shows that the set $\{(z, z') \in \mathbb{N}^2 \mid R(\mathcal{T}_z) \cap R(\mathcal{T}_{z'}) \neq \emptyset\}$ is reduced to the set $\{z \in \mathbb{N} \mid L(\mathcal{M}_z) \neq \emptyset\}$ which is in the class Σ_1^1, since the non-emptiness problem for ω-languages of Turing machines is in the class Σ_1^1. Thus the set $\{(z, z') \in \mathbb{N}^2 \mid R(\mathcal{T}_z) \cap R(\mathcal{T}_{z'}) \neq \emptyset\}$ is in the class Σ_1^1.

Secondly, we have to show the completeness part of the theorem. We are going to reduce the ω-PCP in a regular ω-language to the problem of the non-emptiness of the intersection of two infinitary rational relations. Let then $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ be an instance of the ω-PCP(Reg), where the x_i and y_i are words over an alphabet Γ. We can then construct Büchi transducers $\mathcal{T}_{\psi_1(\bar{I})}$ and $\mathcal{T}_{\psi_2(\bar{I})}$ such that the infinitary rational relation $R(\mathcal{T}_{\psi_1(\bar{I})}) \subseteq \{1, 2, \ldots, n\}^\omega \times \Gamma^\omega$ is the set of pairs of infinite words in the form $(i_1 i_2 i_3 \cdots ; x_{i_1} x_{i_2} x_{i_3} \cdots)$ with $i_1 i_2 i_3 \cdots \in L(\mathcal{A})$. And similarly $R(\mathcal{T}_{\psi_2(\bar{I})}) \subseteq \{1, 2, \ldots, n\}^\omega \times \Gamma^\omega$ is the set of pairs of infinite words in the form $(i_1 i_2 i_3 \cdots ; y_{i_1} y_{i_2} y_{i_3} \cdots)$ with $i_1 i_2 i_3 \cdots \in L(\mathcal{A})$. Thus it holds that $R(\mathcal{T}_{\psi_1(\bar{I})}) \cap R(\mathcal{T}_{\psi_2(\bar{I})})$ is non-empty iff there is an infinite sequence $i_1 i_2 \cdots i_k \cdots \in L(\mathcal{A})$ such that $x_{i_1} x_{i_2} \cdots x_{i_k} \cdots = y_{i_1} y_{i_2} \cdots y_{i_k} \cdots$. The reduction is now given by the injective computable function $\Psi : \mathbb{N} \to \mathbb{N}^2$ given by $\Psi(z) = (\Psi_1(z), \Psi_2(z))$. $\square$

4.2. *Continuity of ω-rational functions*

Recall that an infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ is said to be functional iff it is the graph of a function, i.e. iff

$$[\forall x \in Dom(R(\mathcal{T})) \;\; \exists! y \in Im(R(\mathcal{T})) \;\; (x, y) \in R(\mathcal{T})].$$

Then the functional relation $R(\mathcal{T})$ defines an ω-rational (partial) function $F_{\mathcal{T}} : Dom(R(\mathcal{T})) \subseteq \Sigma^\omega \to \Gamma^\omega$ by: for each $u \in Dom(R(\mathcal{T}))$, $F_{\mathcal{T}}(u)$ is the unique $v \in \Gamma^\omega$ such that $(u, v) \in R(\mathcal{T})$.

Recall the following previous decidability result.

Theorem 4.3 ([*Gir*86]) *One can decide whether an infinitary rational relation recognized by a given Büchi transducer $\mathcal{T}$ is a functional infinitary rational relation.*

One can then associate in a recursive and injective way an index to each Büchi transducer $\mathcal{T}$ accepting a functional infinitary rational relation $R(\mathcal{T})$. In the sequel we consider only these Büchi transducers and we shall denote $\mathcal{T}_z$ the Büchi transducer of index z (such that $R(\mathcal{T}_z)$ is functional).

It is very natural to consider the notion of continuity for ω-rational functions defined by Büchi transducers.

We assume the reader to be familiar with basic notions of topology which may be found in [Kec95, Tho90, Sta97, PP04]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ which is called the prefix metric and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $d(u, v) = 2^{-l_{pref(u,v)}}$ where $l_{pref(u,v)}$ is the least integer n such that the $(n+1)^{th}$ letter of u is different from the $(n+1)^{th}$ letter of v. This metric induces on Σ^ω the usual Cantor topology for which open subsets of Σ^ω are in the form $W \cdot \Sigma^\omega$, where $W \subseteq \Sigma^\star$.

We recall that a function $f : Dom(f) \subseteq \Sigma^\omega \to \Gamma^\omega$, whose domain is $Dom(f)$, is said to be continuous at point $x \in Dom(f)$ if :

$$\forall n \geq 1 \quad \exists k \geq 1 \quad \forall y \in Dom(f) \quad [\; d(x, y) < 2^{-k} \Rightarrow d(f(x), f(y)) < 2^{-n} \;].$$

The continuity set $C(f)$ of the function f is the set of points of continuity of f. The function f is said to be continuous if it is continuous at every point $x \in Dom(f)$, i.e. if $C(f) = Dom(f)$.

Prieur proved the following decidability result.

Theorem 4.4 (Prieur [*Pri*01, *Pri*02]) *One can decide whether a given ω-rational function is continuous.*

On the other hand the following undecidability result was proved in [CFS08].

Theorem 4.5 (see [*CFS*08]) *One cannot decide whether a given ω-rational function f has at least one point of continuity.*

We can now give the exact complexity of this undecidable problem.

Theorem 4.6. *It is Σ_1^1-complete to determine whether a given ω-rational function f has at least one point of continuity, i.e. whether the continuity set $C(f)$ of f is non-empty. In other words the set $\{z \in \mathbb{N} \mid C(F_{\mathcal{T}_z}) \neq \emptyset\}$ is Σ_1^1-complete.*

We first prove the following lemma.

Lemma 4.7. *The set $\{z \in \mathbb{N} \mid C(F_{\mathcal{T}_z}) \neq \emptyset\}$ is in the class Σ_1^1.*

Proof. Let F be a function from $Dom(F) \subseteq \Sigma^\omega$ into Γ^ω. For some integers $n, k \geq 1$, we consider the set

$$X_{k,n} = \{x \in Dom(F) \mid \forall y \in Dom(F) \quad [\ d(x,y) < 2^{-k} \Rightarrow d(F(x), F(y)) < 2^{-n}\ ]\}\,.$$

For $x \in Dom(F)$ it holds that:

$$x \in C(F) \Longleftrightarrow \forall n \geq 1 \quad \exists k \geq 1 \quad [\ x \in X_{k,n}\ ]\,.$$

We shall denote $X_{k,n}(z)$ the set $X_{k,n}$ corresponding to the function $F_{\mathcal{T}_z}$ defined by the Büchi transducer of index z. Then it holds that:

$$x \in C(F_{\mathcal{T}_z}) \Longleftrightarrow \forall n \geq 1 \quad \exists k \geq 1 \quad [\ x \in X_{k,n}(z)\ ]\,.$$

And we denote $R_{k,n}(x,z)$ the relation given by:

$$R_{k,n}(x,z) \Longleftrightarrow [x \in X_{k,n}(z)]\,.$$

We now prove that this relation is a Π_3^0-relation.

For $x \in \Sigma^\omega$ and $k \in \mathbb{N}$, we denote $B(x, 2^{-k})$ the open ball of center x and of radius 2^{-k}, i.e. the set of $y \in \Sigma^\omega$ such that $d(x,y) < 2^{-k}$. We know, from the definition of the distance d, that for two ω-words x and y over Σ, the inequality $d(x,y) < 2^{-k}$ simply means that x and y have the same $(k+1)$ first letters. Thus $B(x, 2^{-k}) = x[k+1] \cdot \Sigma^\omega$. But by definition of $X_{k,n}(z)$ it holds that:

$$x \in X_{k,n}(z) \Longleftrightarrow (x \in Dom(F_{\mathcal{T}_z}) \text{and} F_{\mathcal{T}_z}[B(x, 2^{-k}) \cap Dom(F_{\mathcal{T}_z})] \subseteq B(F_{\mathcal{T}_z}(x), 2^{-n})).$$

We claim that there is an algorithm which, given $x \in \Sigma^\omega$ and $z \in \mathbb{N}$, can decide whether

$$F_{\mathcal{T}_z}[B(x, 2^{-k}) \cap Dom(F_{\mathcal{T}_z})] \subseteq w \cdot \Gamma^\omega,$$

for some finite word $w \in \Gamma^\star$ such that $|w| = n+1$.

Indeed the ω-language $B(x, 2^{-k}) \cap Dom(F_{\mathcal{T}_z}) = x[k+1] \cdot \Sigma^\omega \cap Dom(F_{\mathcal{T}_z})$ is the intersection of two regular ω-languages and one can construct a Büchi automaton accepting it. The graph of the restriction of the function $F_{\mathcal{T}_z}$ to the set $x[k+1] \cdot \Sigma^\omega \cap Dom(F_{\mathcal{T}_z})$ is also an infinitary rational relation and one can then also find a Büchi automaton $\mathcal{B}$ accepting $F_{\mathcal{T}_z}[x[k+1] \cdot \Sigma^\omega \cap Dom(F_{\mathcal{T}_z})]$. One can then find the set of prefixes of length $n+1$ of infinite words in $L(\mathcal{B})$. If there is only one such prefix w then $F_{\mathcal{T}_z}[B(x, 2^{-k}) \cap Dom(F_{\mathcal{T}_z})] \subseteq w \cdot \Gamma^\omega$ and otherwise we have $F_{\mathcal{T}_z}[B(x, 2^{-k}) \cap Dom(F_{\mathcal{T}_z})] \nsubseteq w' \cdot \Gamma^\omega$ for every word $w' \in \Gamma^\star$ such that $|w'| = n+1$. We now write $S(x,k,n,z)$ iff $F_{\mathcal{T}_z}[B(x, 2^{-k}) \cap Dom(F_{\mathcal{T}_z})] \subseteq w \cdot \Gamma^\omega$, for some finite word $w \in \Gamma^\star$ such that $|w| = n + 1$. As we have just seen the relation $S(x, k, n, z)$ is computable, i.e. a Δ_1^0 relation.

On the other hand, we have

$$x \in X_{k,n}(z) \Longleftrightarrow (x \in Dom(F_{\mathcal{T}_z}) \text{ and } S(x, k, n, z))\,.$$

But $Dom(F_{\mathcal{T}_z})$ is a regular ω-language accepted by a Büchi automaton $\mathcal{A}$ which can be constructed effectively from $\mathcal{T}_z$ and hence from the index z. And the relation $(x \in L(\mathcal{A}))$ is known to be an arithmetical Π_3^0 (and also a Σ_3^0) relation, see [LT94]. Thus "$x \in X_{k,n}(z)$" can be expressed also by a Π_3^0 (and also a Σ_3^0) formula because the relation S is a Δ_1^0 relation.

Now we have the following equivalences:

$$C(F_{\mathcal{T}_z}) \neq \emptyset \Longleftrightarrow \exists x \quad [x \in C(F_{\mathcal{T}_z})] \Longleftrightarrow \exists x \quad [\forall n \geq 1 \quad \exists k \geq 1 \quad x \in X_{k,n}(z)]\,.$$

Clearly the formula $\exists x \quad [\forall n \geq 1 \quad \exists k \geq 1 \quad x \in X_{k,n}(z)]$ is a Σ_1^1-formula where there is a second order quantification $\exists x$ followed by an arithmetical Π_5^0-formula in which the quantifications $\forall n \geq 1 \quad \exists k \geq 1$ are first order quantifications on integers. □

End of Proof of Theorem 4.6. To prove the completeness part of the theorem we use some ideas of [CFS08] but we shall modify the constructions of [CFS08] in order to use the Σ_1^1-completeness of the ω-PCP(Reg) instead of the undecidability of the PCP. We are now going to reduce the ω-PCP in a regular ω-language to the non-emptiness of the continuity set of an ω-rational function.

Let then $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ be an instance of the ω-PCP(Reg), where the x_i and y_i are words over an alphabet Γ. We can construct an ω-rational function F in the following way.

Firstly, the domain $Dom(F)$ will be a set of ω-words over the alphabet $\{1, \ldots, n\} \cup \{a, b\}$, where a and b are new letters not in $\{1, \ldots, n\}$. For $x \in (\{1, \ldots, n\} \cup \{a, b\})^\omega$ we denote $x(/\{a, b\})$ the (finite or infinite) word over the alphabet $\{1, \ldots, n\}$ obtained from x when removing every occurrence of the letters a and b. And $x(/\{1, \ldots, n\})$ is the (finite or infinite) word over the alphabet $\{a, b\}$ obtained from x when removing every occurrence of the letters $1, \ldots, n$. Then $Dom(F)$ is the set of ω-words x over the alphabet $\{1, \ldots, n\} \cup \{a, b\}$ such that $x(/\{a, b\}) \in L(\mathcal{A})$ (so in particular $x(/\{a, b\})$ is infinite) and $x(/\{1, \ldots, n\})$ is infinite. It is clear that this domain is a regular ω-language.

Secondly, for $x \in Dom(F)$ such that $x(/\{a, b\}) = i_1 i_2 \cdots i_k \cdots \in L(\mathcal{A})$ we set:

- $F(x) = x_{i_1} x_{i_2} \cdots x_{i_k} \cdots$ if $x(/\{1, \ldots, n\}) \in (\{a, b\}^\star \cdot a)^\omega$, and
- $F(x) = y_{i_1} y_{i_2} \cdots y_{i_k} \cdots$ if $x(/\{1, \ldots, n\}) \in \{a, b\}^\star \cdot b^\omega$.

The ω-language $(\{a, b\}^\star.a)^\omega$ is the set of ω-words over the alphabet $\{a, b\}$ having infinitely many letters a. The ω-language $\{a, b\}^\star.b^\omega$ is the complement in $\{a, b\}^\omega$ of the ω-language $(\{a, b\}^\star.a)^\omega$: it is the set of ω-words over the alphabet $\{a, b\}$ containing only finitely many letters a. The two ω-languages $(\{a, b\}^\star.a)^\omega$ and $\{a, b\}^\star.b^\omega$ are ω-regular, and one can easily construct Büchi automata accepting them. Then it is easy to see that the function F is ω-rational and that one can construct a Büchi transducer $\mathcal{T}$ accepting the graph of the function F. Moreover one can construct an injective computable function $\psi : \mathbb{N} \to \mathbb{N}$ such that $\mathcal{T} = \mathcal{T}_{\psi(\bar{I})}$ and so $F = F_{\mathcal{T}_{\psi(\bar{I})}}$.

We now prove that if $x \in Dom(F_{\mathcal{T}_{\psi(\bar{I})}})$ is a point of continuity of the function $F_{\mathcal{T}_{\psi(\bar{I})}}$ then the ω-PCP(Reg) of instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ has a solution $i_1 i_2 \cdots i_k \cdots$, i.e. an ω-word $i_1 i_2 \cdots i_k \cdots \in L(\mathcal{A})$ such that $x_{i_1} x_{i_2} \cdots x_{i_k} \cdots = y_{i_1} y_{i_2} \cdots y_{i_k} \cdots$.

To simplify the notations we denote by F the function $F_{\mathcal{T}_{\psi(\bar{I})}}$. We now distinguish two cases.

First Case. Assume firstly that $x(/\{1, \ldots, n\}) \in (\{a, b\}^\star \cdot a)^\omega$ and that $x(/\{a, b\}) = i_1 i_2 \cdots i_k \cdots \in L(\mathcal{A})$. Then by definition of F it holds that $F(x) = x_{i_1} x_{i_2} \cdots x_{i_k} \cdots$. We denote $z = x(/\{1, \ldots, n\})$. Notice that there is a sequence of elements $z_p \in \{a, b\}^\star \cdot b^\omega$, $p \geq 1$, such that the sequence $(z_p)_{p \geq 1}$ is convergent and $lim(z_p) = z = x(/\{1, \ldots, n\})$. This is due to the fact that $\{a, b\}^\star . b^\omega$ is dense in $\{a, b\}^\omega$. We call t_p the infinite word over the alphabet $\{1, \ldots, n\} \cup \{a, b\}$ such that, for each integer $i \geq 1$, we have $t_p(i) = x(i)$ if $x(i) \in \{1, \ldots, n\}$ and $t_p(i) = z_p(k)$ if $x(i)$ is the k th letter of z. Then the sequence $(t_p)_{p \geq 1}$ is convergent and $lim(t_p) = x$. But by definition of F it holds that $F(t_p) = y_{i_1} y_{i_2} \cdots y_{i_k} \cdots$ for every integer $p \geq 1$ while $F(x) = x_{i_1} x_{i_2} \cdots x_{i_k} \cdots$. Thus if x is a point of continuity of the function F then it holds that $x_{i_1} x_{i_2} \cdots x_{i_k} \cdots = y_{i_1} y_{i_2} \cdots y_{i_k} \cdots$ and the ω-PCP(Reg) of instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ has a solution $i_1 i_2 \cdots i_k \cdots$.

Second Case. Assume now that $x(/\{1, \ldots, n\}) \in \{a, b\}^\star \cdot b^\omega$ and that $x(/\{a, b\}) = i_1 i_2 \cdots i_k \cdots \in L(\mathcal{A})$. Notice that $(\{a, b\}^\star . a)^\omega$ is also dense in $\{a, b\}^\omega$. Then reasoning as in the first case we can prove that if x is a point of continuity of F then $x_{i_1} x_{i_2} \cdots x_{i_k} \cdots = y_{i_1} y_{i_2} \cdots y_{i_k} \cdots$ and the ω-PCP(Reg) of instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ has a solution $i_1 i_2 \cdots i_k \cdots$.

Conversely assume that the ω-PCP in a regular ω-language of instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ has a solution $i_1 i_2 \cdots i_k \cdots$, i.e. an ω-word $i_1 i_2 \cdots i_k \cdots \in L(\mathcal{A})$ such that $x_{i_1} x_{i_2} \cdots x_{i_k} \cdots = y_{i_1} y_{i_2} \cdots y_{i_k} \cdots$. We now show that each $x \in Dom(F)$ such that $x(/\{a, b\}) = i_1 i_2 \cdots i_k \cdots$ is a point of continuity of the function F. Consider an infinite sequence $(t_p)_{p \geq 1}$ of elements of $Dom(F)$ such that $lim(t_p) = x$. It is easy to see that the sequence $(t_p(/\{a, b\}))_{p \geq 1}$ is convergent and that its limit is the ω-word $x(/\{a, b\}) = i_1 i_2 \cdots i_k \cdots$. This implies easily that the sequence $F(t_p)_{p \geq 1}$ is convergent and that its limit is the ω-word $F(x) = x_{i_1} x_{i_2} \cdots x_{i_k} \cdots = y_{i_1} y_{i_2} \cdots y_{i_k} \cdots$. Thus x is a point of continuity of F and this ends the proof. □

We consider now the continuity set of an ω-rational function and its possible complexity. The following undecidability result was proved in [$CFS08$].

Theorem 4.8 (see [$\boldsymbol{CFS08}$]) *One cannot decide whether the continuity set of a given ω-rational function f is a regular* (*respectively, context-free*) *ω-language.*

We can now give the exact complexity of the first above undecidable problem.

Theorem 4.9. *It is Π^1_1-complete to determine whether the continuity set $C(f)$ of a given ω-rational function f is a regular ω-language. In other words the set $\{z \in \mathbb{N} \mid C(F_{\mathcal{T}_z})$ is a regular ω-language$\}$ is Π^1_1-complete.*

We first prove the following lemma.

Lemma 4.10. *The set $\{z \in \mathbb{N} \mid C(F_{\mathcal{T}_z})$ is a regular ω-language$\}$ is in the class Π^1_1.*

Proof. Recall that $\mathcal{A}_z$ denotes the Büchi automaton of index z. We can express the sentence "$C(F_{\mathcal{T}_z})$ is a regular ω-language" by the sentence:

$$\exists z' \;\; C(F_{\mathcal{T}_z}) = L(\mathcal{A}_{z'}).$$

On the other hand we have seen in the proof of Lemma 4.7 that

$$x \in C(F_{\mathcal{T}_z}) \Longleftrightarrow [\forall n \geq 1 \;\; \exists k \geq 1 \;\; x \in X_{k,n}(z)]$$

and then that $x \in C(F_{\mathcal{T}_z})$ can be expressed by an arithmetical Π^0_5-formula. We can now express $C(F_{\mathcal{T}_z}) = L(\mathcal{A}_{z'})$ by:

$$\forall x \;\; [(x \in C(F_{\mathcal{T}_z}) \text{ and } x \in L(\mathcal{A}_{z'})) \text{ or } (x \notin C(F_{\mathcal{T}_z}) \text{ and } x \notin L(\mathcal{A}_{z'}))]$$

which is a Π^1_1-formula because there is one universal second order quantification $\forall x$ followed by an arithmetical formula (recall that $x \in L(\mathcal{A}_{z'})$ can be expressed by an arithmetical Π^0_3-formula).

Finally the sentence

$$\exists z' \;\; C(F_{\mathcal{T}_z}) = L(\mathcal{A}_{z'})$$

can be expressed by a Π^1_1-formula because the quantification $\exists z'$ is a first-order quantification bearing on integers and the formula $C(F_{\mathcal{T}_z}) = L(\mathcal{A}_{z'})$ can be expressed by a Π^1_1-formula. □

End of Proof of Theorem 4.9. To prove the completeness part of the theorem we reduce the ω-PCP in a regular ω-language to the problem of the non-regularity of the continuity set of an ω-rational function.

As in the proof of the above Theorem 4.8 in [CFS08], we shall use a particular instance of Post Correspondence Problem. For two letters c, d, let PCP_1 be the Post Correspondence Problem of instance $((t_1, t_2, t_3), (w_1, w_2, w_3))$, where $t_1 = c^2$, $t_2 = t_3 = d$ and $w_1 = w_2 = c$, $w_3 = d^2$. It is easy to see that its solutions are the sequences of indices in $\{1^i \cdot 2^i \cdot 3^i \mid i \geq 1\} \cup \{3^i \cdot 2^i \cdot 1^i \mid i \geq 1\}$. In particular, this language over the alphabet $\{1, 2, 3\}$ is not context-free and this will be useful in the sequel.

Let then $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ be an instance of the ω-PCP(Reg), where the x_i and y_i are words over an alphabet Γ. We can construct an ω-rational function F' in the following way.

Let $D = \{d_1, d_2, d_3\}$ such that D and $\{1, \ldots, n\} \cup \{a, b\}$ are disjoint. The domain $Dom(F')$ will be a set of ω-words in $D^+ \cdot Dom(F)$, where, as in the proof of Theorem 4.6, $Dom(F)$ is the set of ω-words x over the alphabet $\{1, \ldots, n\} \cup \{a, b\}$ such that $x(/\{a, b\}) \in L(\mathcal{A})$ (so in particular $x(/\{a, b\})$ is infinite) **and** $x(/\{1, \ldots, n\})$ is infinite. It is clear that the domain $Dom(F')$ is a regular ω-language.

Now, for $x \in Dom(F')$ such that $x = d_{j_1} \cdots d_{j_p} \cdot y$ with $y \in Dom(F)$ and $y(/\{a, b\}) = i_1 i_2 \cdots i_k \cdots \in L(\mathcal{A})$ we set:

- $F'(x) = t_{j_1} \cdots t_{j_p} x_{i_1} x_{i_2} \cdots x_{i_k} \cdots$ if $y(/\{1, \ldots, n\}) \in (\{a, b\}^\star \cdot a)^\omega$, and
- $F'(x) = w_{j_1} \cdots w_{j_p} y_{i_1} y_{i_2} \cdots y_{i_k} \cdots$ if $y(/\{1, \ldots, n\}) \in \{a, b\}^\star \cdot b^\omega$.

Then it is easy to see that the function F' is ω-rational and that one can construct a Büchi transducer $\mathcal{T}'$ accepting the graph of the function F'. Moreover one can construct an injective computable function $\Theta : \mathbb{N} \to \mathbb{N}$ such that $\mathcal{T}' = \mathcal{T}_{\Theta(\bar{I})}$ and so $F' = F_{\mathcal{T}_{\Theta(\bar{I})}}$.

Reasoning as in the preceding proof we can prove that the function F' is continuous at point $x = d_{j_1} \cdots d_{j_p} \cdot y$, where $y \in Dom(F)$, if and only if the sequence $j_1, \ldots, j_p$ is a solution of the Post Correspondence Problem PCP_1 and $y(/\{a, b\}) = i_1 i_2 \cdots i_k \cdots$ is a solution of the ω-PCP(Reg) of instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$.

Thus if the ω-PCP(Reg) of instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ has no solution, then the continuity set $C(F')$ is empty, hence it is ω-regular.

On the other hand assume that the ω-PCP(Reg) of instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ has some solutions. In that case the continuity set $C(F')$ is in the form $T \cdot R$ where $T = \{d_1^i \cdot d_2^i \cdot d_3^i \mid i \geq 1\} \cup \{d_3^i \cdot d_2^i \cdot d_1^i \mid i \geq 1\}$ and R is a set of infinite words over the alphabet $\{1, \ldots, n\} \cup \{a, b\}$. In that case the continuity set $C(F')$ can not be ω-regular because otherwise the language T should be regular (since $D = \{d_1, d_2, d_3\}$ and $\{1, \ldots, n\} \cup \{a, b\}$ are disjoint) and it is not even context-free.

This shows that the ω-PCP(Reg) of instance $I = ((x_1, \ldots, x_n), (y_1, \ldots, y_n), \mathcal{A})$ has a solution if and only if the continuity set $C(F')$ is not ω-regular. This ends the proof. □

It is natural to ask whether the set $\{z \in \mathbb{N} \mid C(F_{\mathcal{T}_z})$ is a context-free ω-language$\}$ is also Π_1^1-complete. But one cannot extend directly Lemma 4.10, replacing regular by context-free. If we replace the Büchi automaton $\mathcal{A}_z$ of index z by the Büchi pushdown automaton $\mathcal{B}_z$ of index z, we get only that the set $\{z \in \mathbb{N} \mid C(F_{\mathcal{T}_z})$ is a context-free ω-language$\}$ is in the class Π_2^1 because the "$x \in L(\mathcal{B}_{z'})$" can only be expressed by a Σ_1^1-formula. On the other hand, the second part of the proof of Theorem 4.9 proves in the same way that the set $\{z \in \mathbb{N} \mid C(F_{\mathcal{T}_z})$ is a context-free ω-language$\}$ is Π_1^1-hard. Thus we can now state the following result.

Theorem 4.11. *The set $\{z \in \mathbb{N} \mid C(F_{\mathcal{T}_z})$ is a context-free ω-language$\}$ is Π_1^1-hard and in the class $\Pi_2^1 \setminus \Sigma_1^1$.*

5. Concluding Remarks

We have given a complete proof of the Σ_1^1-completeness of the ω-PCP in a regular ω-language, also denoted ω-PCP(Reg). Then we have applied this result and obtained the exact complexity of several highly undecidable problems about infinitary rational relations and ω-rational functions. In particular, we have showed that there is an amazing gap between two decision problems about ω-rational functions realized by finite state Büchi transducers: it is decidable whether a given ω-rational function is continuous, while it is Σ_1^1-complete to determine whether a given ω-rational function has at least one point of continuity.

We hope that this paper will attract the reader's attention on a new highly undecidable problem, the ω-PCP(Reg), which could be very useful to study other decision problems.

References

[AD94] R. Alur and D. L. Dill. A theory of timed automata. *Theoretical Computer Science*, 126(2):183–235, 1994.

[CC89] J. Castro and F. Cucker. Nondeterministic ω-computations and the analytical hierarchy. *Journal Math. Logik und Grundlagen d. Math*, 35:333–342, 1989.

[CFS08] O. Carton, O. Finkel, and P. Simonnet. On the continuity set of an omega rational function. *Theoretical Informatics and Applications*, 42(1):183–196, 2008.

[CG78] R.S. Cohen and A.Y. Gold. ω-computations on Turing machines. *Theoretical Computer Science*, 6:1–23, 1978.

[CR03] D. A. Cenzer and J. B. Remmel. Index sets for omega-languages. *Mathematical Logic Quaterly*, 49(1):22–33, 2003.

[EH93] J. Engelfriet and H. J. Hoogeboom. X-automata on ω-words. *Theoretical Computer Science*, 110(1):1–51, 1993.

[Fin06a] O. Finkel. Borel ranks and Wadge degrees of omega context free languages. *Mathematical Structures in Computer Science*, 16(5):813–840, 2006.

[Fin06b] O. Finkel. On the accepting power of two-tape Büchi automata. In *Proceedings of the 23rd International Symposium on Theoretical Aspects of Computer Science, STACS 2006*, volume 3884 of *Lecture Notes in Computer Science*, pages 301–312. Springer, 2006.

[Fin06c] O. Finkel. Undecidable problems about timed automata. In *Proceedings of the 4th International Conference on Formal Modeling and Analysis of Timed Systems, FORMATS 2006*, volume 4202 of *Lecture Notes in Computer Science*, pages 187–199. Springer, 2006.

[Fin09] O. Finkel. Highly undecidable problems for infinite computations. *RAIRO-Theoretical Informatics and Applications*, 43(2):339–364, 2009.

[FL09] O. Finkel and D. Lecomte. Decision problems for Turing machines. *Information Processing Letters*, 109(23–24):1223–1226, 2009.

[FS00] R. Freund and L. Staiger. Acceptance of ω-languages by communicating deterministic Turing machines. In *Words, Sequences, Grammars, Languages: where Biology, Computer Science, Linguistics and Mathematics Meet*, volume 1, pages 115–125. 2000.

[Gir86] F. Gire. Two decidability problems for infinite words. *Information Processing Letters*, 22(3):135–140, 1986.

[Har86] D. Harel. Effective transformations on infinite trees, with applications to high undecidability, dominoes, and fairness. *Journal of the ACM*, 33(1):224–248, 1986.

[HH06] V. Halava and T. Harju. Undecidability of infinite Post correspondence problem for instances of size 9. *Theoretical Informatics and Applications*, 40(4):551–557, 2006.

[HHK06] V. Halava, T. Harju, and J. Karhumäki. Undecidability in omega-regular languages. *Fundamamenta Informaticae*, 73(1–2):119–125, 2006.

[HMU01] J. E. Hopcroft, R. Motwani, and J. D. Ullman. *Introduction to automata theory, languages, and computation*. Addison-Wesley Publishing Co., Reading, Mass., 2001. Addison-Wesley Series in Computer Science.

[Kec95] A. S. Kechris. *Classical descriptive set theory*. Springer-Verlag, New York, 1995.

[LT94] H. Lescow and W. Thomas. Logical specifications of infinite computations. In J. W. de Bakker, W. P. de Roever, and G. Rozenberg, editors, *A Decade of Concurrency*, volume 803 of *Lecture Notes in Computer Science*, pages 583–621. Springer, 1994.

[Odi89] P.G. Odifreddi. *Classical Recursion Theory, Vol I*, volume 125 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1989.

[Odi99] P.G. Odifreddi. *Classical Recursion Theory, Vol II*, volume 143 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[PP04] D. Perrin and J.-E. Pin. *Infinite words, automata, semigroups, logic and games*, volume 141 of *Pure and Applied Mathematics*. Elsevier, 2004.

[Pri01] C. Prieur. How to decide continuity of rational functions on infinite words. *Theoretical Computer Science*, 250(1–2):71–82, 2001.

[Pri02] C. Prieur. How to decide continuity of rational functions on infinite words. *Theoretical Computer Science*, 276(1–2):445–447, 2002.

[Rog67] H. Rogers. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, New York, 1967.

[Ruo85] K. Ruohonen. Reversible machines and Post's correspondence problem for biprefix morphisms. *Elektronische Informationsverarbeitung und Kybernetik*, 21(12):579–595, 1985.

[Sen01] G. Senizergues. L(A)=L(B)? decidability results from complete formal systems. *Theoretical Computer Science*, 251(1-2):1–166, 2001.

[Sta93] L. Staiger. Recursive automata on infinite words. In P. Enjalbert, A. Finkel, and K. W. Wagner, editors, *Proceedings of the 10th Annual Symposium on Theoretical Aspects of Computer Science, STACS 93, Würzburg, Germany, February 25-27, 1993*, volume 665 of *Lecture Notes in Computer Science*, pages 629–639. Springer, 1993.

[Sta97] L. Staiger. ω-languages. In *Handbook of formal languages, Vol. 3*, pages 339–387. Springer, Berlin, 1997.

[Sta00] L. Staiger. On the power of reading the whole infinite input tape. In *Finite Versus Infinite: Contributions to an Eternal Dilemma*, Discrete Mathematics and Theoretical Computer Science, pages 335–348. Springer-Verlag London, 2000.

[SW78] L. Staiger and K. Wagner. Rekursive folgenmengen I. *Z. Math Logik Grundlag. Math.*, 24:523–538, 1978.

[Tho90] W. Thomas. Automata on infinite objects. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, Formal models and semantics, pages 135–191. Elsevier, 1990.